Elements of Probability Theory

Joint Distribution

Discrete Bivariate Random Variables

Definition

- S be a sample space
- $\Omega \subset \mathbb{Z}^2$
- Bivariate random variable $(X, Y) : S \to \Omega$, with $f_{XY} : \Omega \to \mathbb{R}$
    - $f_{XY}(x, y) \geq 0 \, \forall (x, y) \in \Omega$
    - $\sum_{(x,y)\in\Omega} f_{XY}(x, y) = 1$

Discrete Marginal Density

Definition

- Let $((X, Y), f_{XY})$ be a discrete random variable
- Marginal density $f_X(x) = \sum_y f_{XY}(x, y)$.

Continuous Bivariate Random Variables

Definition

- S be a sample space.
- Continuous bivariate random variable $(X, Y) : S \to \mathbb{R}^2$ with $f_{XY} : \mathbb{R}^2 \to \mathbb{R}$
    - $f_{XY}(x, y) \geq 0 \, \forall (x, y) \in \mathbb{R}^2$
    - $\displaystyle\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x, y) dy dx = 1$
- $\displaystyle P[(X, Y) \in \Omega] = \int \int_{\Omega} f_{XY}(x, y) d(x, y)$

Continuous Marginal Density

Definition

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y) dy$$

Independence

Definition

- $((X, Y), f_{XY})$ be bivariate random variable
- marginal densities f_X and f_Y
- $\mathbf{dom} f_{XY} = (\mathbf{dom} f_X) \times (\mathbf{dom} f_Y)$
- $f_{XY}(x, y) = f_X(x) f_Y(y) \, \forall (x, y) \in \mathbf{dom} f_{XY}$
- Then (X, f_X) and (Y, f_Y) are independent random variables.

Conditional Densities

Definition

- $((X, Y), f_{XY})$ be bivariate random variable
- marginal densities f_X and f_Y.
- The conditional density for X given $Y = y$ is defined to be $f_{X|Y} = \dfrac{f_{XY}(x, y)}{f_{Y(y)}}$

Expectation for Discrete Bivariate Random Variables

- $((X, Y), f_{XY})$ be a discrete bivariate random variable
- $H : \Omega \to \mathbb{R}$
- $E[H \circ (X, Y)] = \sum_{(x,y) \in \Omega} H(x, y) \cdot f_{XY}(x, y)$
- $E[X] = \sum_{(x,y) \in \Omega} x \cdot f_{XY}(x, y)$
- $E[X + Y] = E[X] + E[Y]$

Expectation for Continuous Bivariate Random Variables

- $((X, Y), f_{XY})$ be a continuous bivariate random variable
- $H : \mathbb{R}^2 \to \mathbb{R}$
- $E[H \circ (X, Y)] = \int \int_{\mathbb{R}^2} H(x, y) \cdot f_{XY}(x, y) dx dy$.
- $E[X] = \int \int_{\mathbb{R}^2} x \cdot f_{XY}(x, y) dx dy$

Conditional Expectation

Discrete Definition

- $((X, Y), f_{XY})$ be a discrete bivariate random variable
- $E[Y|x] := \sum_y y \cdot f_{Y|X}(y)$

Continuous Definition

- $((X, Y), f_{XY})$ be a continuous bivariate random variable
- $E[Y|x] := \int_{\mathbb{R}} y \cdot f_{Y|X}(y) dy$

Covariance

$$
\begin{aligned}
Var(X+Y) &= E[((X+Y)-E[X+Y])^2] \\
&= E[(X+Y)^2 - 2(X+Y)E[X+Y] + E[X+Y]^2] \\
&= E[(X-E[X])^2 + (Y-E[Y])^2 + 2(X-E[X])(Y-E[Y])] \\
&= VarX + VarY + 2E[(X-E[X])(Y-E[Y])]
\end{aligned}
$$

Definition

- $((X,Y), f_{XY})$ be a bivariate random variable
- $\mu_X = E[X]$
- Covariance of (X,Y) is $\mathrm{Cov}(X,Y) = \sigma_{XY} = E[(X-\mu_X)(Y-\mu_Y)]$
- We can see $\mathrm{Cov}\ (X,Y) = E[XY] - E[X]E[Y]$
- $\mathrm{Cov}(X,X) = VarX$

Theorem

- $((X,Y), f_{XY})$ be bivariate random variable
- $\mathrm{Cov}(X,Y) = 0$, $E[XY] = E[X]E[Y]$