Introduction to Statistical Method

Hypothesis Testing

A second major statistical method for gaining information on a probability.

The goal is to reject or fail to reject statements (hypotheses) based on statistical data.

Hypothesis Definition

A statement about a population parameter θ. The hypothesis will compare θ to a null value donated θ_0.

Fisher's Null Hypothesis Test

This hypothesis will be donated by H_0 and is null hypothesis.

Three forms: $H_0 : \theta = \theta_0$ or $H_0 : \theta \leq \theta_0$ or $H_0 : \theta \geq \theta_0$.

A hypothesis test is based on rejecting a hypothesis.

One-Tailed Test

The test of a hypothesis of the form $H_0 : \theta \leq \theta_0$ or $H_0 : \theta \geq \theta_0$ is said to be one-tailed tests.

P-Value for a One-Tailed Test

- Apply an example first

  We want to find evidence that a new car design has a mean mileage greater than 26 mpg. Therefore, we set up the null hypothesis: $H_0 : \mu \leq 26$.

  The goal is to reject the null hypothesis.

- Example explanation

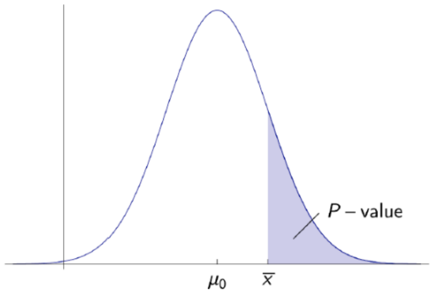
  We take a random sample and calculate $\overline{X}$, if it is much greater than 26, then there is reason to believe that H_0 is false.

  Take a random sample of size n and find the value $\overline{x}$ for the sample mean.

  The probability of obtaining the measured value of $\overline{x}$ or a larger result if H_0 is true is the **significance** or **P-Value** of the test.

  (TM还是说中文吧：就是说，猜测$\mu \leq 26$，然后我们得到了样本的平均的观测值$\overline{x}$，根据样本个数和标准差，我们可以对样本的平均值$\overline{X}$得到基于$\mu \leq 26$的test，然后可以放宽到$\mu = 26$，然后可以用$Z = \frac{\overline{X} - \mu_0}{\sigma / \sqrt{n}}$算出$P$的值来判断这个样本是否符合这个假设)

  $P[\overline{X} \geq \overline{x} | \mu \leq 26] \leq P[\overline{X} \geq \overline{x} | \mu = 26]$



$H_0: \mu \le \mu_0$ shows the case if $\mu = \mu_0$, the curve shift left if $\mu < \mu_0$

.

The shaded area shows the probability of obtaining $\overline{X} \ge \overline{x}$ if $\mu = \mu_0$.

- The P-value is therefore an upper bound of the probability of obtaining the data if H_0 is true.
- $P = P[D|H_0]$ if D represents the statistical data, we will reject H_0 if it is small.

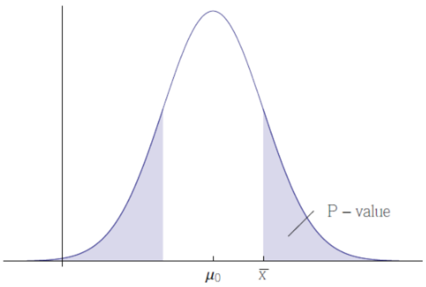
  So either:

  - fail to reject H_0 at P level of significance
  - reject the H_0 at P level of significance

  The statistic on which the P is based is **test statistic**.

Two-Tailed Test

If we are testing a hypothesis of the form $H_0 : \theta = \theta_0$, we say we are performing a two-tailed test.



$H_0: \mu = \mu_0$ The P is twice the value of one-tailed test.

Does a Small P-Value Provide Evidence that H0 is False

Since we know the fact that the $P = P[D|H_0]$, but some researcher want $P[H_0|D]$.

We can derive the fact from the Bayes's theorem:

$P[D|H_0] = P[D \cap H_0]/P[H_0]$, then we can derive $P[H_0|D] = P[D \cap H_0]/P[D]$.

$$
\begin{aligned}
P[H_0|D] &= \frac{P[D \cap H_0]}{P[D]} = \frac{P[D|H_0] \cdot P[H_0]}{P[D]} = \frac{P[D|H_0] \cdot P[H_0]}{P[D|H_0] \cdot P[H_0] + P[D|\neg H_0] \cdot (1 - P[H_0])} \\
&= \frac{P[D|H_0]}{P[D|H_0] + P[D|\neg H_0] \cdot (\frac{1-P[H_0]}{P[H_0]})}
\end{aligned}
$$

Is Hypothesis Testing Logical?

Since we get the $P[H_0|D]$ representation, we can let it be close to 1 depending on $P[H_0]$.

Hence, it is possible that: given H_0 and the data is very unlikely, but given the data H_0 is very likely.

- In the classic argument:

  If P then Q; not Q therefore not P
- In hypothesis testing, we want to argue that

  If P then Q; Q is unlikely therefore P is unlikely

  Actually this is wrong.

Bayesian & Frequentist Statistics

Bayesian

Claim to understand the **logical inconsistencies** and intend to compensate for them with **prior and posterior probability** distributions.

Theoretically true, difficult to implement in practice.

Frequentist

Mainly ignore the problems mentioned here or claim that they are not relevant in their specific research.

Neyman-Pearson Decision Theory

Two competing hypothesis: H_0, H_1.

Seek to reject H_0 to accept H_1.

- H_0 is **null hypothesis**
- H_1 is **research hypothesis** or **alternative hypothesis**.

So there are four possible outcomes of the decision-making process:

- We reject H_0 when H_0 is untrue.
- **Type I Error**: We reject H_0 even though H_0 is true.
- **Type II Error**: We fail to reject H_0 even though H_0 is untrue.
- We fail to reject H_0 when H_0 true.

Type I and Type II error should be as small as possible.

Power, Type I & II Error Probabilities

$\alpha = P[\text{Type I Error}] = P[\text{reject } H_0 \mid H_0 \text{ true}] = P[\text{accept } H_1 \mid H_1 \text{ false}]$

$\beta = P[\text{Type II Error}] = P[\text{fail to reject } H_0 \mid H_0 \text{ false}]$

$\text{Power} = 1 - \beta$

The power shows how likely our experiment is successful.