Introduction to Statistical Method

Hypothesis Testing

A second major statistical method for gaining information on a probability.

The goal is to reject or fail to reject statements (hypotheses) based on statistical data.

Hypothesis Definition

A statement about a population parameter θ. The hypothesis will compare θ to a null value donated θ_0.

Fisher's Null Hypothesis Test

This hypothesis will be donated by H_0 and is null hypothesis.

Three forms: $H_0 : \theta = \theta_0$ or $H_0 : \theta \leq \theta_0$ or $H_0 : \theta \geq \theta_0$.

A hypothesis test is based on rejecting a hypothesis.

One-Tailed Test

The test of a hypothesis of the form $H_0 : \theta \leq \theta_0$ or $H_0 : \theta \geq \theta_0$ is said to be one-tailed tests.

P-Value for a One-Tailed Test

- Apply an example first

  We want to find evidence that a new car design has a mean mileage greater than 26 mpg. Therefore, we set up the null hypothesis: $H_0 : \mu \leq 26$.

  The goal is to reject the null hypothesis.

- Example explanation

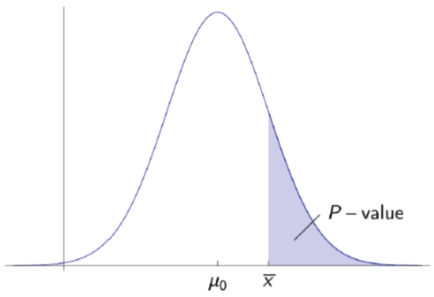
  We take a random sample and calculate $\overline{X}$, if it is much greater than 26, then there is reason to believe that H_0 is false.

  Take a random sample of size n and find the value $\overline{x}$ for the sample mean.

  The probability of obtaining the measured value of $\overline{x}$ or a larger result if H_0 is true is the **significance** or **P-Value** of the test.

  (TM还是说中文吧：就是说，猜测$\mu \leq 26$，然后我们得到了样本的平均的观测值$\overline{x}$，根据样本个数和标准差，我们可以对样本的平均值$\overline{X}$得到基于$\mu \leq 26$的test，然后可以放宽到$\mu = 26$，然后可以用$Z = \frac{\overline{X}-\mu_0}{\sigma/\sqrt{n}}$算出$P$的值来判断这个样本是否符合这个假设)

  $P[\overline{X} \geq \overline{x} | \mu \leq 26] \leq P[\overline{X} \geq \overline{x} | \mu = 26]$



$H_0: \mu \leq \mu_0$ shows the case if $\mu = \mu_0$, the curve shift left if $\mu < \mu_0$

.

The shaded area shows the probability of obtaining $\overline{X} \geq \overline{x}$ if $\mu = \mu_0$.

- The P-value is therefore an upper bound of the probability of obtaining the data if H_0 is true.
- $P = P[D|H_0]$ if D represents the statistical data, we will reject H_0 if it is small.

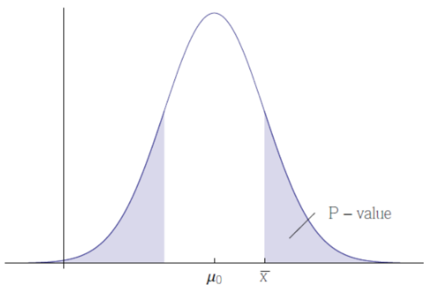
  So either:

  - fail to reject H_0 at P level of significance
  - reject the H_0 at P level of significance

  The statistic on which the P is based is **test statistic**.

Two-Tailed Test

If we are testing a hypothesis of the form $H_0 : \theta = \theta_0$, we say we are performing a two-tailed test.



$H_0: \mu = \mu_0$ The P is twice the value of one-tailed test.

Does a Small P-Value Provide Evidence that H0 is False

Since we know the fact that the $P = P[D|H_0]$, but some researcher want $P[H_0|D]$.

We can derive the fact from the Bayes's theorem:

$P[D|H_0] = P[D \cap H_0]/P[H_0]$, then we can derive $P[H_0|D] = P[D \cap H_0]/P[D]$.

$$
\begin{aligned}
P[H_0|D] &= \frac{P[D \cap H_0]}{P[D]} = \frac{P[D|H_0] \cdot P[H_0]}{P[D]} = \frac{P[D|H_0] \cdot P[H_0]}{P[D|H_0] \cdot P[H_0] + P[D|\neg H_0] \cdot (1 - P[H_0])} \\
&= \frac{P[D|H_0]}{P[D|H_0] + P[D|\neg H_0] \cdot (\frac{1-P[H_0]}{P[H_0]})}
\end{aligned}
$$

Is Hypothesis Testing Logical?

Since we get the $P[H_0|D]$ representation, we can let it be close to 1 depending on $P[H_0]$.

Hence, it is possible that: given H_0 and the data is very unlikely, but given the data H_0 is very likely.

- In the classic argument:

  If P then Q; not Q therefore not P
- In hypothesis testing, we want to argue that

  If P then Q; Q is unlikely therefore P is unlikely

  Actually this is wrong.

Bayesian & Frequentist Statistics

Bayesian

Claim to understand the **logical inconsistencies** and intend to compensate for them with **prior and posterior probability** distributions.

Theoretically true, difficult to implement in practice.

Frequentist

Mainly ignore the problems mentioned here or claim that they are not relevant in their specific research.

Neyman-Pearson Decision Theory

Two competing hypothesis: H_0, H_1.

Seek to reject H_0 to accept H_1.

- H_0 is **null hypothesis**
- H_1 is **research hypothesis** or **alternative hypothesis**.

So there are four possible outcomes of the decision-making process:

- We reject H_0 when H_0 is untrue.
- **Type I Error**: We reject H_0 even though H_0 is true.
- **Type II Error**: We fail to reject H_0 even though H_0 is untrue.
- We fail to reject H_0 when H_0 true.

Type I and Type II error should be as small as possible.

Power, Type I & II Error Probabilities

$\alpha = P[\text{Type I Error}] = P[\text{reject } H_0 \mid H_0 \text{ true}] = P[\text{accept } H_1 \mid H_1 \text{ false}]$

$\beta = P[\text{Type II Error}] = P[\text{fail to reject } H_0 \mid H_0 \text{ false}]$

$\text{Power} = 1 - \beta$

The power shows how likely our experiment is successful.

- By requiring strong evidence before rejecting H_0 (a value of the test statistic that is very different from its null value), α can be made small.
- The range of values for the test statistic that causes us to reject H_0 is **critical region**.

  We choose the critical region in such a way to make α small.
- The more evidence we require to reject H_0 (the smaller the critical region is), the harder it is to actually reject H_0 in the first place.

  The power decreases with β becomes larger.
- For given H_0 and H_1, β can be controlled by increasing the sample size.

Example of Neyman-Pearson Decision Theory

The mean is supposed to be $\mu_0 = 40$ and the standard deviation is $\sigma = 2$.

Two hypothesis $H_0 : \mu = 40$, $H_1 : |\mu - 40| \geq 1$

Then the sample size is $n = 25$.

The probability of committing Type I Error is $\alpha \leq 5\%$. (derive from H_0 and H_1?)

So we apply the test statistic $Z = \dfrac{\overline{X} - \mu_0}{\sigma/\sqrt{n}}$ and let $-z_{\alpha/2} \leq Z \leq z_{\alpha/2}$ to get probability of $1 - \alpha$ and $P[|Z| > z_{\alpha/2}] = \alpha$.

The $\left|\dfrac{\overline{X} - \mu_0}{\sigma/\sqrt{n}}\right| \geq 1.96$ is the critical region, with this critical region there are 5% of Type I Error.

If the sample mean is $\overline{x} = 40.9$, then the test statistic is $z = 2.25 > 1.96$.

Type II Error

$|Z| = \left|\dfrac{\overline{X} - \mu_0}{\sigma/\sqrt{n}}\right| \geq z_{\alpha/2} = 2.575$ when $\alpha = 1\%$. So with $z = 2.25 < 2.575$, there is a Type II Error.

Assume null hypothesis $H_0 : \mu = \mu_0$ and the true value is $\mu = \mu_0 + \delta, \delta \in \mathbb{R}\backslash\{0\}$.

$Z = \dfrac{\overline{X} - \mu_0}{\sigma/\sqrt{n}}$ is the test statistic actually follows a normal distribution with unit variance and mean $\delta\sqrt{n}/\sigma$.

Review that if $-z_{\alpha/2} \leq Z \leq z_{\alpha/2}$ then we cannot reject H_0, deriving the Type II Error.

$$\beta = P[|Z| \leq z_{\alpha/2}] = \frac{1}{\sqrt{2\pi}} \int_{-z_{\alpha/2}}^{z_{\alpha/2}} e^{-(t-\delta\sqrt{n}/\sigma)^2/2} dt \approx \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z_{\alpha/2}-\delta\sqrt{n}/\sigma} e^{-t^2/2} dt$$

We donate $-z_\beta \approx z_{\alpha/2} - \delta\sqrt{n}/\sigma$ or $n \approx \frac{(z_{\alpha/2}+z_\beta)^2\sigma^2}{\delta^2}$

Given a critical region determined by α, we can find a sample size n so that the probability of committing a Type II Error is β.

Remark

In order for the statistical procedure to be valid, **a critical region must be fixed before any data obtained**.