Elements of Probability Theory

Continuous Random Variables

Continuous Random Variables

Definition

- S be a **sample space**.
- **Continuous** random variable: $X : S \to \mathbb{R}$ with $f_X : \mathbb{R} \to \mathbb{R}$.
  - $f_X \geq 0$
  - $\int_{-\infty}^{\infty} f_X(x)dx = 1$.
- f_X is called the **probability density function**, or **density** of the random variable X.
- $P[a \leq X \leq b] = \int_a^b f_X(x)dx$, means the probability that X **assumes values** x **in a given range**.
- The probability that X **assumes any specific value is 0**: $P[X = x] = \int_x^x f_X(y)dy = 0$.
- **Cumulative Distribution**:
  - Let (X, f_X) be continuous random variable.
  - $F : \mathbb{R} \to \mathbb{R}$, $F_X(x) := P[X \leq x] = \int_{-\infty}^{x} f_X(y)dy$ is the **cumulative distribution function**.

Remark

- For two **continuous random variables** (X, f_X) and (Y, f_Y), if f_X and f_Y differ only on **sets of measure zero** (e.g., countable sets), then $\forall a, b \in \mathbb{R}$

  $P[a \leq X \leq b] = P[a \leq Y \leq b] = \int_a^b f_X(x)dx = \int_a^b f_Y(y)dy$.

  We say this $(X, f_X) = (Y, f_Y)$ almost surely.
- $f_X(x) = F_X'(x)$.

Expectation and Variance

- (X, f_X) be continuous random variable
- $H : \mathbb{R} \to \mathbb{R}$
- Expected value of $H \circ X$ is $E[H \circ X] = \int_{-\infty}^{\infty} H(x) \cdot f_X(x)dx$

  (if it converge absolutely)
- $E[X] = \int_{\mathbb{R}} x \cdot f_X(x)dx$
- $Var\ X = E[X^2] - E[X]^2$

Exponential Distribution

Definition

- $\beta \in \mathbb{R}, \beta > 0$
- Continuous random function (X, f_β) with density:

  $$f_\beta(x) = \begin{matrix} \frac{1}{\beta}e^{-x/\beta}, & x > 0 \\ 0, & x \leq 0 \end{matrix}$$

  is said to **follow an exponential distribution with parameter** β.

Expectation and Variance

- $E[X] = \int_{-\infty}^{\infty} xf_\beta(x)dx = \int_0^\infty \frac{x}{\beta}e^{-x/\beta}dx = -xe^{-x/\beta}\Big|_0^\infty + \int_0^\infty e^{-x/\beta}dx$

  thus $E[X] = 0 + (-\beta)\int_{x=0}^{x=\infty} e^{-x/\beta}d(-\frac{x}{\beta}) = (-\beta)\int_0^{-\infty} e^p dp = \beta$
- $E[X^2] = \int_{-\infty}^{\infty} x^2 f_\beta(x)dx = \int_0^\infty \frac{x^2}{\beta}e^{-x/\beta}dx = \int_0^\infty 2xe^{-x/\beta}dx - x^2e^{-x/\beta}\Big|_0^\infty$

  thus $E[X^2] = 2\beta\int_0^\infty \frac{x}{\beta}e^{-x/\beta}dx - 0 = 2\beta^2$
- $Var\ X = \beta^2$

MGP

$m_X(t) = E[e^{tX}] = \int_{-\infty}^{\infty} e^{tx}f_X(x)dx = \int_0^\infty \frac{1}{\beta}e^{-(1/\beta - t)x}dx$ defined only for $t < \frac{1}{\beta}$

so $m_X(t) = \frac{1}{\beta}\frac{-1}{\frac{1}{\beta}-t}\int_0^\infty -(\frac{1}{\beta}-t)e^{-(1/\beta-t)x}dx = \frac{1}{\beta t-1}e^p\Big|_0^{-\infty} = \frac{1}{1-\beta t}$

Connection to Poisson Distribution

The probability of x arrivals in time intervals $[0, t]$ was given by

$p(x) = \frac{(\lambda t)^x}{x!}e^{-\lambda t}, x \in \mathbb{N}$

Then $p(0)$ is the probability for no arrivals in $[0, t]$, or the first and the following arrivals occur at $(t, +\infty)$.

Donate **the first arrival time** as a **continuous random variable** by T.

Then $P[T > t] = p(0) = e^{-\lambda t}, t \geq 0$

If we donate F as the **cumulative distribution** of the density of T, then

$F(t) = P[T \leq t] = 1 - e^{-\lambda t}, t \geq 0$

Since $f_\tau(t) = F'(t)$, the density is $f_\tau(t) = \lambda e^{-\lambda t}, t \geq 0$.

So the time between **two arrivals**, or **successive arrivals** of a Poisson-distributed random variables is exponentially distributed with parameter $\beta = \frac{1}{\lambda}$. ($\lambda$ is the density of things happen in a unit time, like 0.0005 failure/hour; β is the rate of a thing to happen, like 50000 hours / failure)

An Example about Poisson Distribution Connection

> An electronic component is known to have a useful life represented by an exponential density with failure rate of $\lambda = 10^{-5}$ failures per hour, i.e., $1/\beta = 10^{-5}$. The mean time to failure, E[X], is thus $\beta = 10^5$ hours.

So $P[T \leq \beta] = \int_0^\beta \beta^{-1} e^{-x/\beta} dx = 1 - e^{-1}$.

So this is irrelevant to the value of β.

Location of Continuous Distributions

The location is supposed to give the **center** of the distribution:

- **median** M_X, defined by $P[X \leq M_X] = 0.5$.
- **mean** $E[X]$.
- **mode** x_0, the location of max f_X.

Memoryless Property of the Exponential Distribution

For exponential distribution, we have: $P[X > x + s | X > x] = P[X > s]$

Use the words to describe, we say: the distribution of a object to fail is same for both duration s from time 0 or from time x.

Since $P[X > x] = \int_x^\infty f(t)dt = \int_x^\infty \lambda e^{-\lambda t} dt = e^{-\lambda x}$

Then $P[X > x + s | X > x] = \frac{P[(X > x+s) \cap (X > x)]}{P[X > x]} = \frac{P[X > x+s]}{P[X > x]} = e^{-\lambda s} = P[X > s]$

Time to Serval Arrivals

A generalization for the **probability density function** of the first arrival in Poisson process.

Suppose T_j random variable describes the time needed for $j \in \mathbb{N} \backslash \{0\}$, then the cumulative distribution function:

$F_{T_j} = P[T_j < t] = 1 - P[T_j > t] = 1 - P[\text{strictly less than } j \text{ arrivals before } t]$

$F_{T_j} = 1 - \sum_{n=0}^{j-1} \frac{(\lambda t)^n}{n!} e^{-\lambda t}$ for $t > 0$.

Then we differentiate the F_{T_j} to get f_{T_j}:

$f_{T_j} = \lambda e^{-\lambda t} \sum_{n=0}^{j-1} \frac{(\lambda t)^n}{n!} - \lambda e^{-\lambda t} \sum_{n=1}^{j-1} \frac{(\lambda t)^{n-1}}{(n-1)!} = \lambda e^{-\lambda t} \frac{(\lambda t)^{j-1}}{(j-1)!}$

Gamma Distribution

Definition

- $\alpha, \beta \in \mathbb{R}$ with $\alpha, \beta > 0$.
- Continuous random variable $(X, f_{\alpha,\beta})$ with density

  $$f_{\alpha,\beta}(x) = \begin{cases} \dfrac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}, & x > 0 \\ 0, & x \le 0 \end{cases}$$

  is said to follow an gamma distribution with parameter α and β.

  $\Gamma(\alpha) = \int_0^\infty z^{\alpha-1} e^{-z} dz$, $\alpha > 0$ is the Euler gamma function.

  And the $\Gamma(\alpha+1) = \int_0^\infty z^\alpha e^{-z} dz = (-e^{-z} z^\alpha)\Big|_0^\infty + \int_0^\infty \alpha z^{\alpha-1} e^{-z} dz = 0 + \alpha\Gamma(\alpha)$

  So to say $n! = \Gamma(n+1)$ for $n \in \mathbb{N}$.

Mean, Variance and MGF

The time needed for next j arrivals in Poisson process with rate λ is determined by Gamma distribution with $\alpha = j$ and $\beta = 1/\lambda$.

Let $(X, f_{\alpha,\beta})$ be a Gamma distributed random variable with $\alpha, \beta > 0$.

- The MGF of X is given by $m_X : (-\infty, 1/\beta) \to \mathbb{R}, m_X(t) = (1 - \beta t)^{-\alpha}$.

  $m_X(t) = E[e^{tX}] = \int_0^\infty \frac{e^{tx}}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} dx = \frac{1}{\Gamma(\alpha)\beta^\alpha} \int_0^\infty x^{\alpha-1} e^{-(1/\beta - t)x} dx$

  let $y = x(1/\beta - t)$, then

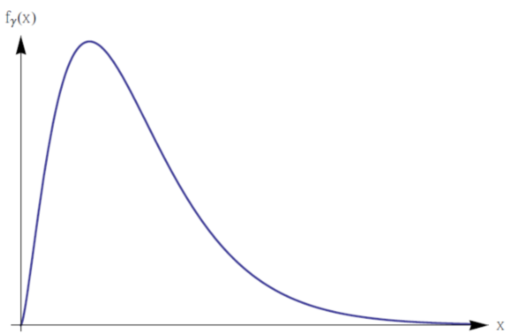
  $m_X(t) = \frac{1}{\Gamma(\alpha)\beta^\alpha} (1/\beta - t)^{-1} \int_0^\infty (\frac{y}{1/\beta - t})^{\alpha-1} e^{-y} dy = \frac{(1/\beta - t)^{-\alpha}}{\Gamma(\alpha)\beta^\alpha} \int_0^\infty y^{\alpha-1} e^{-y} dy = \frac{(1/\beta - t)^{-\alpha}}{\beta^\alpha}$

  Thus $m_X(t) = (1 - \beta t)^{-\alpha}$
- $E[X] = \alpha\beta$ and $Var\ X = \alpha\beta^2$.

Special Cases of Gamma Distribution: Chi Squared Distribution

Let X be a gamma random variable with $\beta = 2$ and $\alpha = \gamma/2$ for $\gamma \in \mathbb{N}$.

Then $X = \chi^2_\gamma$ is said to have a **chi squared distribution** with γ **degree of freedom**.



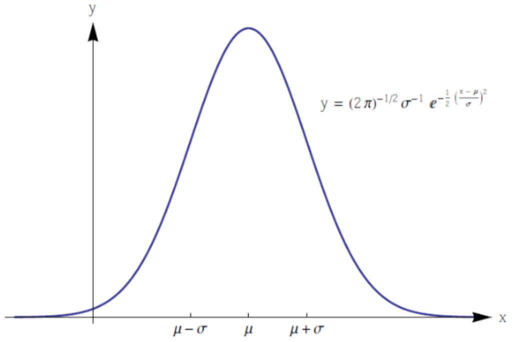
Normal (Gaussian) Distribution

Definition

- $\mu \in \mathbb{R}, \sigma > 0$.
- Continuous random variable (X, f_X) with density

  $f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{((x-\mu)/\sigma)^2}{2}}$

  is said to follow a normal distribution with parameter μ and σ.



Mean, Variance and MGF

- The MGP of X is given by $m_X : \mathbb{R} \to \mathbb{R}, m_X(t) = e^{\mu t + \sigma^2 t^2/2}$

  $m_X(t) = E[e^{tX}] = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{xt - \frac{(x-\mu)^2}{2\sigma^2}} dx$

  Since $tx - \frac{(x-\mu)^2}{2\sigma^2} = -\frac{(x-(\mu+\sigma^2 t))^2}{2\sigma^2} + \mu t + \frac{1}{2}\sigma^2 t^2$

  Thus $m_X(t) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\frac{(x-(\mu+\sigma^2 t))^2}{2\sigma^2} + \mu t + \frac{1}{2}\sigma^2 t^2} dx = \frac{1}{\sqrt{2\pi}\sigma} e^{\mu t + \frac{1}{2}\sigma^2 t^2} \int_{-\infty}^{\infty} e^{-\frac{(x-(\mu+\sigma^2 t))^2}{2\sigma^2}} dx$

  Since $\frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\frac{(x-(\mu+\sigma^2 t))^2}{2\sigma^2}} dx = 1$

  Thus $m_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$
- $E[X] = \mu$ and $Var\, X = \sigma^2$.

Standard Normal Distribution

Definition

A **normal distributed random variable** with parameters $\mu = 0$ and $\sigma = 1$ is called **standard normal** random variable. Donated by Z.

> Any **normally distributed** random variable can be transformed into a **standard-normally distributed** one.

Transform from Normally Distributed to Standard Normal Distribution

Let X be **normally distributed** random variable with:

- mean μ
- standard deviation σ

Then $Z := \frac{X-\mu}{\sigma}$ has standard normal distribution.

Transformation of Random Variable

- X is a continuous random variable
- f_X is the density
- $Y = \varphi \circ X$, $\varphi : \mathbb{R} \to \mathbb{R}$ is **strictly monotonic and differentiable**.

Thus the density for Y is given by:

$f_Y(y) = f_X(\varphi^{-1}(y)) \cdot \left| \frac{d\varphi^{-1}(y)}{dy} \right|$ for $y \in \mathbf{ran}\ \varphi$.

$f_Y(y) = 0$ for $y \notin \mathbf{ran}\ \varphi$.

Proof

$F_Y(y) = P[Y \leq y] = P[\varphi(X) \leq y]$

WLOG we assume φ is strictly decreasing, φ is strictly increasing is vise versa.

So $F_Y(y) = P[\varphi(X) \leq y] = P[\varphi^{-1}(\varphi(X)) \geq \varphi^{-1}(y)] = P[X \geq \varphi^{-1}(y)] = 1 - F_X(\varphi^{-1}(y))$

Thus $f_Y(y) = F_Y'(y) = -f_X(\varphi^{-1}(y))\frac{d\varphi^{-1}(y)}{dy} = f_X(\varphi^{-1}(y)) \cdot \left| \frac{d\varphi^{-1}(y)}{dy} \right|$

Considering cases $F_Y(y) = \begin{cases} 0 & y < \forall x \in \mathbf{ran}\ \varphi \\ 1 & y > \forall x \in \mathbf{ran}\ \varphi \end{cases}$, then if $y \notin \mathbf{ran}\ \varphi$, hence $f_Y = F_Y' = 0$.

Cumulative Distribution Function

Donated by Φ, $\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-t^2/2} dt$

Estimate on Variability

$P[-\sigma < X - \mu < \sigma] = 0.68$

$P[-2\sigma < X - \mu < 2\sigma] = 0.95$

$P[-3\sigma < X - \mu < 3\sigma] = 0.997$

Chebyshev's Inequality

- (X, f_X) be a discrete or continuous random variable.
- $k > 0$ is a positive number.

$P[-k\sigma < X - \mu < k\sigma] \geq 1 - \frac{1}{k^2}$, or equivalently, $P[|X - \mu| \geq k\sigma] \leq \frac{1}{k^2}$.

Proof

$\sigma^2 = Var\ X = E[(X-\mu)^2] = \int_{\mathbb{R}} (x-\mu)^2 f_X(x)dx$

$\sigma^2 = \int_{-\infty}^{\mu-\sqrt{K}} (x-\mu)^2 f_X(x)dx + \int_{\mu-\sqrt{K}}^{\mu+\sqrt{K}} (x-\mu)^2 f_X(x)dx + \int_{\mu+\sqrt{K}}^{\infty} (x-\mu)^2 f_X(x)dx, \forall K > 0$

Hence $\sigma^2 \geq \int_{-\infty}^{\mu-\sqrt{K}} (x-\mu)^2 f_X(x)dx + \int_{\mu+\sqrt{K}}^{\infty} (x-\mu)^2 f_X(x)dx$.

Since $(x-\mu)^2 \geq K\ iff.\ |x-\mu| \geq \sqrt{K}$, equivalently, $x \geq \mu + \sqrt{K}$ or $x \leq \mu - \sqrt{K}$.

$$\begin{aligned}
\sigma^2 &\geq \int_{-\infty}^{\mu-\sqrt{K}} (x-\mu)^2 f_X(x)dx + \int_{\mu+\sqrt{K}}^{\infty} (x-\mu)^2 f_X(x)dx \\
&\geq K \int_{-\infty}^{\mu-\sqrt{K}} f_X(x)dx + K \int_{\mu+\sqrt{K}}^{\infty} f_X(x)dx \\
&= K(P[X \leq \mu - \sqrt{K}] + P[X \geq \mu + \sqrt{K}])
\end{aligned}$$

simplified, $P[|X-\mu| \geq \sqrt{K}] \leq \frac{\sigma^2}{K}$.

If let $K = k^2 \cdot \sigma^2$, then $P[|X-\mu| \geq k\sigma] \leq \frac{1}{k^2}$.

Approximating the Binomial Distribution

> De Moivre-Laplace Theorem behind the approximation of Binomial Distribution by Normal Distribution.

> Galton Board: a board with n rows of nails then simulates a binomial experiment with $p = 0.5$ and parameter n.
>
> The proportion of balls approximates the probability density of binomial distribution.
>
> If n is large and sufficient number of balls are used, the density begins to resemble a normal distribution.

$n! \sim \sqrt{2\pi}e^{-n}n^{n+1/2}$, or so to say $\lim_{n\to\infty} \frac{n!-n!\sim\sqrt{2\pi}e^{-n}n^{n+1/2}}{n!} = 0$

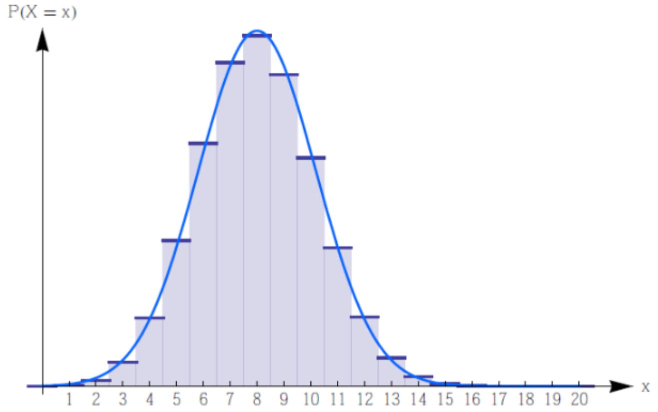
Eventually, the binomial random variable with parameters n and p can show that:

$P[X = x] = \frac{n!}{x!(n-x)!}p^x q^{n-x} \simeq \frac{1}{\sqrt{npq}\sqrt{2\pi}}e^{-(x-np)^2/(2npq)}$

So mean $\mu = np$ and variance $\sigma^2 = npq$.

The approximation will be good if p is close to $\frac{1}{2}$ and $n > 10$.

Otherwise, we require $n \cdot min\{p, 1-p\} > 5$.



We can see if we want to sum over $x \leq y$ corresponds to the area of the bars to the left of y, then the approximation need to integrate to $y + 1/2$.

Thus $P[X \leq y] = \sum_{x=0}^{y} \binom{n}{x} p^x (1-p)^{n-x} \simeq \Phi(\frac{y+1/2-np}{\sqrt{np(1-p)}})$.

The additional term $1/2$ is known as **half-unit correction**.

Reliability

> Concerned with assessing whether or not a system functions adequately under the conditions for which it was designed.
>
> Interest centers on describing the behavior of the random variable X, the time to failure of a system that can not be repaired once it fails to operate.

Focus on three functions:

- the failure density f

  $f(t) = \lim_{\Delta t \to 0} \frac{P[t \leq X \leq t+\Delta t]}{\Delta t} = \lim_{\Delta t \to 0} \frac{F(t+\Delta t) - F(t)}{\Delta t}$ where F is cumulative distribution function of X.
- the reliability function R

  $R(t) = 1 - P[\text{component fails before time } t] = 1 - \int_0^t f(x)dx = 1 - F(t)$.
- the failure or hazard rate ϱ

  $\varrho(t) := \lim_{\Delta t \to 0} \frac{P[t \leq X \leq t+\Delta t | t \leq X]}{\Delta t} = \lim_{\Delta t \to 0} \frac{P[t \leq X \leq t+\Delta t]}{P[X \geq t] \cdot \Delta t}$

  $\varrho(t) = \frac{f(t)}{R(t)}$

Finding the Reliability Function

- X be a random variable failure density f, reliability R and hazard rate ϱ

$R(t) = e^{-\int_0^t \varrho(x)dx}$

Proof

$\varrho(x) = \frac{f(x)}{R(x)} = \frac{F'(x)}{R(x)} = -\frac{R'(x)}{R(x)}$, thus $R(t) = e^{-\int_0^t \varrho(x)dx}$

Weibull Density

- if $\varrho(t) = \alpha\beta t^{\beta-1}$ with $t, \alpha, \beta > 0$
- then $R(t) = e^{-\int_0^t \alpha\beta x^{\beta-1}dx} = e^{-\alpha t^\beta}$

failure density $f_X(t) = f(t) = \begin{cases} \varrho(t)R(t) = \alpha\beta t^{\beta-1}e^{-\alpha t^\beta} & x > 0 \\ 0 & otherwise \end{cases}$