Introduction to Statistical Method

Simultaneous Estimation of the Mean and Variance

Chi Random Variable

Consider a problem:

- $z = (z_1, \cdots, z_n) \in \mathbb{R}^n$ randomly chosen.
- the value z_k is determined by random variable Z_k, following a standard normal distribution.
- think about distribution about $\chi_n := \sqrt{\sum_{i=1}^n Z_i^2}$
- χ_n is a chi random variable, follows chi distribution with n degree of freedom.

Cumulative Distribution Function

$$F_{\chi_n}(y) = P[\chi_n \leq y] = P[\chi_n^2 \leq y^2] = P[\sum_{i=1}^n Z_i^2 \leq y^2] = \int_{\sum_{k=1}^n z_k^2 \leq y^2} f_{Z_1 \dots Z_n}(z_1, \cdots, z_n) dz_1 \dots dz_n$$

Since they are $Z_1, \dots Z_n$ that n independent standard variables, then we see the joint density:

$$f_{Z_1 \dots Z_n}(z_1, \dots z_n) = \frac{1}{(2\pi)^{n/2}} e^{-\sum_{k=1}^n z_k^2/2}$$

Thus $F_{\chi_n}(y) = \int_{\sum_{k=1}^n z_k^2 \leq y^2} (2\pi)^{-n/2} e^{-\sum_{k=1}^n z_k^2/2} dz_1 \dots dz_n$

Apply polar coordinate with $(r, \theta_1, \dots, \theta_0)$ with $r > 0$, $0 < \theta_{n-1} < 2\pi$ and $-\pi/2 < \theta_k < \pi/2$ for $k = 1, \dots n-2$:

$$\begin{aligned}
x_1 &= r \sin\theta_1 \\
x_2 &= r \cos\theta_1 \sin\theta_2 \\
x_3 &= r \cos\theta_1 \cos\theta_2 \sin\theta_3 \\
&\vdots \\
x_{n-1} &= r \cos\theta_1 \cos\theta_2 \dots \cos\theta_{n-2} \sin\theta_{n-1} \\
x_n &= r \cos\theta_1 \cos\theta_2 \dots \cos\theta_{n-2} \cos\theta_{n-1}
\end{aligned}$$

Then the integral become: $$\begin{aligned} F_{\chi_n}(y) = \int_0^{2\pi} \int_{-\pi/2}^{\pi/2} \cdots \int_{-\pi/2}^{\pi/2} \int_0^y (2\pi)^{-n/2} e^{-r^2/2}\, r^{n-1} \\ \times D(\theta_1, \dots, \theta_{n-1})\, dr\, d\theta_1 \dots d\theta_{n-2}\, d\theta_{n-1} \end{aligned}$$

Since $D(\theta_1, \dots \theta_{n-1})$ is independent of r, $C_n = (2\pi)^{-n/2} \int_0^{2\pi} \int_{-\pi/2}^{\pi/2} \cdots \int_{-\pi/2}^{\pi/2} D(\theta_1, \dots \theta_{n-1}) d\theta_1 \cdots d\theta_{n-1}$

we have $F_{\chi_n}(y) = C_n \int_0^y e^{-r^2/2} r^{n-1} dr$.

Obviously, the $1 = \lim_{y \to \infty} C_n \int_0^\infty e^{-r^2/2} r^{n-1} dr = C_n \Gamma(\frac{n}{2}) 2^{n/2-1}$

Thus $C_n = (\Gamma(\frac{n}{2}) 2^{n/2-1})^{-1}$ and $f_{\chi_n}(y) = \frac{2}{\Gamma(\frac{n}{2}) 2^{n/2}} y^{n-1} e^{-y^2/2}$

Chi-Squared Distribution

we hence derive from the $F_{\chi_n^2} = (\Gamma(\dfrac{n}{2})2^{n/2-1})^{-1} \int_0^{\sqrt{y}} e^{-r^2/2} r^{n-1} dr$

$f_{\chi_n^2} = F'_{\chi_n^2}(y) = (\Gamma(\dfrac{n}{2})2^{n/2-1})^{-1} e^{-y/2} \sqrt{y}^{n-1} \cdot \dfrac{d}{dy}\sqrt{y} = \dfrac{1}{2^{n/2}\Gamma(\frac{n}{2})} y^{n/2-1} e^{-y/2}$

Sum of Independent Chi-Squared Variables

Given $\chi_m^2 = \sum_{i=1}^{m} X_i^2$ and $\chi_n^2 = \sum_{j=1}^{n} Y_j^2$, then $\chi_{m+n}^2 = \chi_m^2 + \chi_n^2 = \sum_{i=1}^{m} X_i^2 + \sum_{j=1}^{n} Y_j^2$

It follows a chi-squared distribution, but with $m+n$ degree of freedom.

It extends to multi-addition case, trivial.

Joint Sampling of Mean and Variance

In the previous chapter, we were able to analyze the sample mean, and also its distribution, under the assumption of known variance.

If variance $\sigma^2 = E[(X-\mu)^2]$ is unknown, then we must first see $S^2 = \frac{1}{n-1}\sum_{k=1}^{n}(X_k - \overline{X})^2$

So we are using the random sample $X_1, \dots X_n$ to get $\overline{X}$ and S^2 at same time.

So we are getting the joint distribution of $\overline{X}$ and S^2.

Theorem

Predicate

- $X_1, \dots X_n$ $n \geq 2$ be a random sample of size n.
- Normal distribution with μ and variance σ^2.

Content

- The sample mean $\overline{X}$ is independent of the sample variance S^2.
- $\overline{X}$ is normally distributed with mean μ and σ^2/n.
- $(n-1)S^2/\sigma^2$ is chi-squared distributed with $n-1$ degree of freedom.

Helmert Transformation

The Helmert transformation is a very special kind of orthogonal transformation from a set of $n \geq 2$ normal random variables $X_1, \dots X_n$ to a new set of random variables $Y_1, \dots Y_n$.

$$
\begin{aligned}
Y_1 &= \frac{1}{\sqrt{n}}(X_1 + \cdots + X_n) \\
Y_2 &= \frac{1}{\sqrt{2}}(X_1 - X_2) \\
Y_3 &= \frac{1}{\sqrt{6}}(X_1 + X_2 - 2X_3) \\
&\vdots \\
Y_n &= \frac{1}{\sqrt{n(n-1)}}(X_1 + X_2 + \cdots + X_{n-1} - (n-1)X_n)
\end{aligned}
$$

$$
\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ \vdots \\ Y_n \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \cdots & \frac{1}{\sqrt{n}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & \cdots & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\sqrt{n(n-1)}} & \frac{1}{\sqrt{n(n-1)}} & \frac{1}{\sqrt{n(n-1)}} & \cdots & -\frac{n-1}{\sqrt{n(n-1)}} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ \vdots \\ X_n \end{pmatrix}
$$

The matrix A is orthonormal since $A^{-1} = A^T$. This implies $|\det A| = 1$.

$$\sum_{i=1}^{n} y_i^2 = \langle y, y \rangle = \langle Ax, Ax \rangle = (Ax)^T(Ax) = x^T A^T Ax = \langle A^T Ax, x \rangle = \langle x, x \rangle = \sum_{i=1}^{n} x_i^2$$

Thus the joint distribution:

$$f_{X_1\cdots X_n}(x_1,\dots,x_n)=\prod_{i=1}^{n}(2\pi)^{-1/2}\sigma^{-1}e^{-\frac{1}{2\sigma^2}(x_i-\mu)^2}$$
$$=(2\pi)^{-n/2}\sigma^{-n}e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i^2-2\mu x_i+\mu^2)}$$
$$=(2\pi)^{-n/2}\sigma^{-n}e^{-\frac{1}{2\sigma^2}\left(\sum_{i=1}^{n}x_i^2-2\mu\sum_{i=1}^{n}x_i+n\mu^2\right)}$$

$$f_{Y_1\cdots Y_n}(y_1,\dots,y_n)$$
$$=f_{Y_1\cdots Y_n}(\mathbf{y})=f_{X_1\cdots X_n}(\mathbf{x})_{\mathbf{x}=A^{-1}\mathbf{y}}\cdot\underbrace{|\det DA^{-1}(\mathbf{y})|}_{=1}$$

Apply back into the y_n,

$$=(2\pi)^{-n/2}\sigma^{-n}e^{-\frac{1}{2\sigma^2}\left(\sum_{i=1}^{n}y_i^2-2\mu\sqrt{n}y_1+n\mu^2\right)}$$
$$=(2\pi)^{-n/2}\sigma^{-n}e^{-\frac{1}{2\sigma^2}\left(\sum_{i=2}^{n}y_i^2+(y_1-\sqrt{n}\mu)^2\right)}$$
$$=(2\pi)^{-1/2}\sigma^{-1}e^{-\frac{1}{2\sigma^2}(y_1-\sqrt{n}\mu)^2}\prod_{i=2}^{n}(2\pi)^{-1/2}\sigma^{-1}e^{-\frac{1}{2\sigma^2}y_i^2}$$

Then $f_{Y_1}(y_1)=(2\pi)^{-1/2}\sigma^{-1}e^{-\frac{1}{2\sigma^2}(y_1-\sqrt{n}\mu)^2}$ and $f_{Y_i}(y_i)=(2\pi)^{-1/2}\sigma^{-1}e^{-\frac{1}{2\sigma^2}y_i^2}$ for $2\le i\le n$.

$f_{Y_1\ldots Y_n}(y_1,\ldots y_n)=f_{Y_1}(y_1)\cdot f_{Y_2}(y_2)\ldots f_{Y_n}(y_n)$

So Y_1 is normally distributed with mean $\sqrt{n}\mu$ and variance σ^2, while $Y_2\ldots Y_n$ are having mean 0 and variance σ^2.

Proof for Previous Theorem

So $\overline{X}=n^{-1/2}Y_1$ and

$$(n-1)S^2=\sum_{k=1}^{n}(X_k-\overline{X})^2=\sum_{k=1}^{n}X_k^2-2\sum_{k=1}^{n}X_k\overline{X}+n\overline{X}^2$$
$$=\sum_{k=1}^{n}X_k^2-n\overline{X}^2=\sum_{k=1}^{n}Y_k^2-Y_1^2=\sum_{k=2}^{n}Y_k^2$$

Since $\overline{X}=n^{-1/2}Y_1$ and $f_{Y_1}(y_1)=(2\pi)^{-1/2}\sigma^{-1}e^{-\frac{1}{2\sigma^2}(y_1-\sqrt{n}\mu)^2}$, then according to the rule in `ve401 note 3 page 6` that we get $f_{\overline{X}}(x)=(2\pi)^{-1/2}\sigma^{-1}e^{-\frac{1}{2\sigma^2}(\sqrt{n}x-\sqrt{n}\mu)^2}\sqrt{n}$

So the $\overline{X}$ is normally distributed with mean μ and variance σ^2/n.

$(n-1)S^2/\sigma^2=\frac{1}{\sigma^2}\sum_{k=2}^{n}Y_k^2=\sum_{k=2}^{n}(\frac{Y_k}{\sigma})^2$ is a chi-squared distribution with $n-1$ freedom.

Independence of Sample Mean and Sample Variance in more General Form

The converse result for the previous theorem is also true:

$X_1\ldots X_n$, with $n\ge 2$ be independent identical distributed random variables.

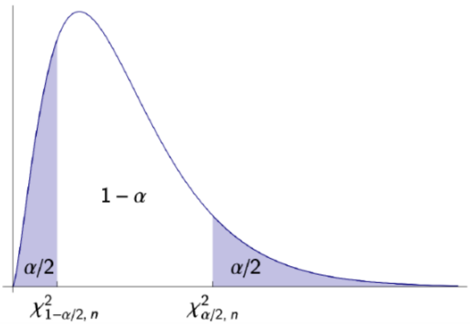
If $\overline{X}$ and S^2 are independent, then X_k with $k=1\ldots n$ follows normal distribution.

This means that the independence of $\overline{X}$ and S^2 is a characteristic property of the normal distribution. If in a given situation we assume that $\overline{X}$ and S^2 are independently distributed we essentially assuming that the population is normally distributed.

Interval Estimation of Variability

We let $0 < \alpha < 1$ and we define $\chi^2_{1-\alpha/2,n} \le \chi^2_{\alpha/2,n} \in \mathbb{R}$

We use the previous theorem to find a confidential interval for the variance based on sample variance S^2.



Thus we get $1-\alpha = P[\chi^2_{1-\alpha/2,n-1} \le \dfrac{(n-1)S^2}{\sigma^2} \le \chi^2_{\alpha/2,n-1}]$

Then $1-\alpha = P[\dfrac{(n-1)S^2}{\chi^2_{\alpha/2,n-1}} \le \sigma^2 \le \dfrac{(n-1)S^2}{\chi^2_{1-\alpha/2,n-1}}]$, which is the $100(1-\alpha)\%$ confidence interval for σ^2.

Interval Estimation for Mean in Variance Unknown

If we know the variance, then $Z = \frac{\overline{X}-\mu}{\sigma/\sqrt{n}}$.

Our main goal is to derive a general formula for a confidence interval on the mean when the value of σ is not known and must be estimated.

So we should know the distribution of $\frac{\overline{X}-\mu}{S/\sqrt{n}}$.

T-distribution

- Z is a standard normal variable.
- χ^2_γ be an independent chi-squared random variable with γ degree of freedom.
- $T_\gamma = \frac{Z}{\sqrt{\chi^2_\gamma/\gamma}}$ is a T-distribution with γ degree of freedom.

The density for a T distribution with γ degrees of freedom is given by:

$f_{T_\gamma}(t) = \dfrac{\Gamma((\gamma+1)/2)}{\Gamma(\gamma/2)\sqrt{\pi\gamma}}(1+\dfrac{t^2}{\gamma})^{-\frac{\gamma+1}{2}}$ check page 332 to 335

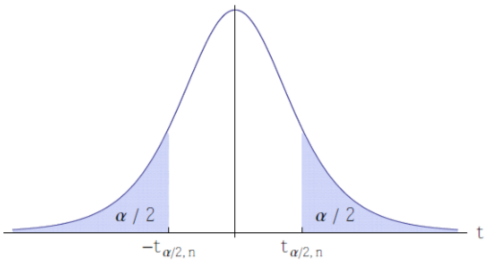
So $X_1 \dots X_n$ be a random sample from a normal distribution with mean μ and variance σ^2.

The random variable $T_{n-1} = \frac{\overline{X}-\mu}{S/\sqrt{n}}$ follows T distribution with $n-1$ degree of freedom.

(Come from the $(\overline{X}-\mu)/(\sigma/\sqrt{n})$ as standard normal and $(n-1)S^2/\sigma^2$ as chi-squared with $n-1$ degree of freedom, put them into T_γ)

Interval Estimation of Mean with Variance Unknown

We define $t_{\alpha/2,n} \geq 0$ by $\int_{t_{\alpha/2,n}}^{\infty} f_{T_n}(t)dt = \alpha/2$



Theorem

$X_1, \ldots X_n$ be a random sample of size n from a normal distribution with mean μ and variance σ^2.

The $100(1-\alpha)\%$ confidential interval on μ is given by $\overline{X} \pm t_{\alpha/2,n-1} S/\sqrt{n}$

Tolerance Limits

A tolerance interval determined from a sample of size n consists of two numbers $L1, L2$, called tolerance limits.

So it goes like $(1-\alpha) \cdot 100\%$ certainty at least $\delta \cdot 100\%$ of population lies between $L1, L2$.

Since we know the fact that 95% of the population lies in the interval $\mu \pm 1.96\sigma$.

The $\overline{X} \pm 1.96S$ will not always cover 95% of the population since it is a random interval.

Theorem: Two-sided Tolerance Limits

X is normally distributed random variable with $\overline{X}$ and S^2 from a sample of size n.

$\exists K = K(n, \alpha, \delta)$ to have an interval $\overline{X} \pm K(n, \alpha, \delta)S$ covers at least $\delta \cdot 100\%$ of population with $(1-\alpha) \cdot 100\%$ confidence. So the K is a two-sided tolerance limit.

Theorem: One-sided Tolerance Limits

X is normally distributed random variable with $\overline{X}$ and S^2 from a sample of size n.

$\exists K = K(n, \alpha, \delta)$ to have interval $(-\infty, \overline{X} + KS)$ and $(\overline{X} - KS, \infty)$ covers at least $\delta \cdot 100\%$ of population with $(1-\alpha) \cdot 100\%$ confidence. So the K is a one-sided tolerance limit.

Non-Parametric Tolerance Limits

A method for finding a tolerance interval without regard to the distribution of the population.

The probability of $\sigma \cdot 100\%$ of the population lying within the values of the sample is

$1 - \alpha = P[(X_{min}, X_{max}) \textbf{ covers at least } \delta \cdot 100\% \textbf{ of the population}] = 1 - n\delta^{n-1} + (n-1)\delta^n$

Also there is a approximate formula $n \simeq \frac{1}{2} + \frac{1+\delta}{1-\delta} \cdot \frac{\chi^2_{\alpha,4}}{4}$