Introduction to Statistical Method

Estimation

- An **estimator** for a population parameter θ is a statistic and donated by $\hat{\theta}$.
- Any given value of $\hat{\theta}$ is an **estimate**.

Desirable Properties of a Point Estimator

- The expected value of $\hat{\theta}$ should be θ.
- $\hat{\theta}$ should have **small variance** for **large sample size**.

Bias

- The difference $\theta - E[\hat{\theta}]$ is the bias of an estimator $\hat{\theta}$ for a population parameter θ.
- $E[\hat{\theta}] = \theta$ means $\hat{\theta}$ is unbiased.

Mean Square Error of Estimator

- The **mean square error** of $\hat{\theta}$ is defined as $\mathrm{MSE}(\hat{\theta}) := E[(\hat{\theta} - \theta)^2]$.
- The mean square error measures the **overall quality of an estimator**.

$$\begin{aligned}\mathrm{MSE}(\hat{\theta}) &= E[(\hat{\theta} - E[\hat{\theta}] + E[\hat{\theta}] - \theta)^2] = E[(\hat{\theta} - E[\hat{\theta}])^2 + (E[\hat{\theta}] - \theta)^2 + 2(\hat{\theta} - E[\hat{\theta}])(E[\hat{\theta}] - \theta)] \\ &= Var(\hat{\theta}) + (bias)^2 + 2E[(\hat{\theta} - E[\hat{\theta}])(E[\hat{\theta}] - \theta)] = Var(\hat{\theta}) + (bias)^2 + 2E[(\hat{\theta} - E[\hat{\theta}])](E[\hat{\theta}] - \theta) \\ &= Var(\hat{\theta}) + (bias)^2 + 2(E[\hat{\theta}] - E[\hat{\theta}])(E[\hat{\theta}] - \theta) = Var(\hat{\theta}) + (bias)^2\end{aligned}$$

Sample Mean Variance and Bias

Theorem: Unbiased Sample Mean

Let $X_1, \cdots, X_n$ be a random sample of size n from a distribution with mean μ. Then the sample mean $\overline{X}$ is an unbiased estimator for μ.

Since $\overline{X} = \dfrac{1}{n}\displaystyle\sum_{k=1}^{n} X_k$, then we see $\mu - E[\overline{X}] = \mu - \frac{1}{n}n\mu = 0$, thus unbiased.

Theorem: Sample Average Variance

Let $\overline{X}$ be the sample of a random sample of size n from a distribution with mean μ and variance σ^2. Then $Var\overline{X} = E[(\overline{X} - \mu)^2] = \frac{1}{n}\sigma^2$

Since $Var\overline{X} = Var(\dfrac{1}{n}\displaystyle\sum_{k=1}^{n} X_k) = \dfrac{1}{n^2}Var(\displaystyle\sum_{k=1}^{n} X_k)$, then check `ve401_note_4`, we can see a fact that $Var(X + Y) = VarX + VarY + \mathrm{Cov}(X, Y)$, where $\mathrm{Cov}(X, Y) = E[(X - \mu_X)(Y - \mu_Y)]$, then we get $Var\overline{X} = \dfrac{1}{n^2}nVarX = \dfrac{1}{n}\sigma^2$

The standard deviation of $\overline{X}$ is given by $\sqrt{Var\overline{X}} = \sigma/\sqrt{n}$ and is called standard error of mean.

Theorem: Sample Variance

Sample variance $S^2 = \frac{1}{n-1}\sum_{k=1}^{n}(X_k - \overline{X})^2$ is an unbiased estimator for σ^2.

$$
\begin{aligned}
E[\frac{1}{n-1}\sum_{k=1}^{n}(X_K - \overline{X})^2] &= \frac{1}{n-1}E[\sum_{k=1}^{n}(X_K - \mu + \mu - \overline{X})^2] \\
&= \frac{1}{n-1}E[\sum_{k=1}^{n}(X_k - \mu)^2 - 2(\overline{X} - \mu)(\sum_{k=1}^{n}X_k - n\mu) + n(\mu - \overline{X})^2] \\
&= \frac{1}{n-1}E[\sum_{k=1}^{n}(X_k - \mu)^2 - n(\mu - \overline{X})^2] \\
&= \frac{1}{n-1}(\sum_{k=1}^{n}E[(X_k - \mu)^2] - nE[(\mu - \overline{X})^2]) = \frac{1}{n-1}(\sum_{k=1}^{n}\sigma^2 - n\frac{\sigma^2}{n}) = \sigma^2
\end{aligned}
$$

Finding Estimator: Method of Moments

We have an unbiased estimator $M_k = \frac{1}{n}\sum_{i=1}^{n}X_i^k$ for the k^{th} moments $E[X^k]$ given random samples $X_1, \cdots, X_n$.

The idea is then that population parameters θ_j can often be expressed in terms of moments of distribution. Replacing the moments in these expressions by their estimators for parameter θ_j.

Estimators obtained in this way are not necessarily unbiased.

Finding Estimator: Method of Maximum Likelihood

- Assume we have a random sample $x_1, \cdots, x_n$ from the distribution of a random variable X with density f and parameter θ.
- Define likelihood function by $L(\theta) = \prod_{i=1}^{n} f(x_i)$.
- Find the θ by maximizing the $L(\theta)$.
- Replace θ with $\hat{\theta}$.