Introduction to Statistical Method

Hypothesis Testing

A second major statistical method for gaining information on a probability.

The goal is to reject or fail to reject statements (hypotheses) based on statistical data.

Hypothesis Definition

A statement about a population parameter θ. The hypothesis will compare θ to a null value donated θ_0.

Fisher's Null Hypothesis Test

This hypothesis will be donated by H_0 and is null hypothesis.

Three forms: $H_0 : \theta = \theta_0$ or $H_0 : \theta \leq \theta_0$ or $H_0 : \theta \geq \theta_0$.

A hypothesis test is based on rejecting a hypothesis.

One-Tailed Test

The test of a hypothesis of the form $H_0 : \theta \leq \theta_0$ or $H_0 : \theta \geq \theta_0$ is said to be one-tailed tests.

P-Value for a One-Tailed Test

- Apply an example first

  We want to find evidence that a new car design has a mean mileage greater than 26 mpg. Therefore, we set up the null hypothesis: $H_0 : \mu \leq 26$.

  The goal is to reject the null hypothesis.

- Example explanation

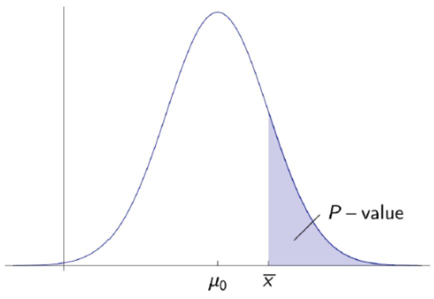
  We take a random sample and calculate $\overline{X}$, if it is much greater than 26, then there is reason to believe that H_0 is false.

  Take a random sample of size n and find the value $\overline{x}$ for the sample mean.

  The probability of obtaining the measured value of $\overline{x}$ or a larger result if H_0 is true is the **significance** or **P-Value** of the test.

  (TM还是说中文吧：就是说，猜测$\mu \leq 26$，然后我们得到了样本的平均的观测值$\overline{x}$，根据样本个数和标准差，我们可以对样本的平均值$\overline{X}$得到基于$\mu \leq 26$的test，然后可以放宽到$\mu = 26$，然后可以用$Z = \frac{\overline{X}-\mu_0}{\sigma/\sqrt{n}}$算出$P$的值来判断这个样本是否符合这个假设)

  $P[\overline{X} \geq \overline{x} | \mu \leq 26] \leq P[\overline{X} \geq \overline{x} | \mu = 26]$



$H_0: \mu \leq \mu_0$ shows the case if $\mu = \mu_0$, the curve shift left if $\mu < \mu_0$

.

The shaded area shows the probability of obtaining $\overline{X} \geq \overline{x}$ if $\mu = \mu_0$.

- The P-value is therefore an upper bound of the probability of obtaining the data if H_0 is true.
- $P = P[D|H_0]$ if D represents the statistical data, we will reject H_0 if it is small.

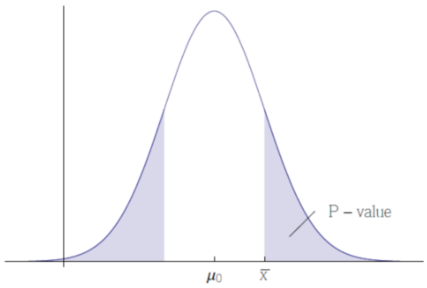
  So either:

  - fail to reject H_0 at P level of significance
  - reject the H_0 at P level of significance

  The statistic on which the P is based is **test statistic**.

Two-Tailed Test

If we are testing a hypothesis of the form $H_0 : \theta = \theta_0$, we say we are performing a two-tailed test.



$H_0: \mu = \mu_0$ The P is twice the value of one-tailed test.

Does a Small P-Value Provide Evidence that H0 is False

Since we know the fact that the $P = P[D|H_0]$, but some researcher want $P[H_0|D]$.

We can derive the fact from the Bayes's theorem:

$P[D|H_0] = P[D \cap H_0]/P[H_0]$, then we can derive $P[H_0|D] = P[D \cap H_0]/P[D]$.

$$
\begin{aligned}
P[H_0|D] &= \frac{P[D \cap H_0]}{P[D]} = \frac{P[D|H_0] \cdot P[H_0]}{P[D]} = \frac{P[D|H_0] \cdot P[H_0]}{P[D|H_0] \cdot P[H_0] + P[D|\neg H_0] \cdot (1 - P[H_0])} \\
&= \frac{P[D|H_0]}{P[D|H_0] + P[D|\neg H_0] \cdot (\frac{1-P[H_0]}{P[H_0]})}
\end{aligned}
$$

Is Hypothesis Testing Logical?

Since we get the $P[H_0|D]$ representation, we can let it be close to 1 depending on $P[H_0]$.

Hence, it is possible that: given H_0 and the data is very unlikely, but given the data H_0 is very likely.

- In the classic argument:

  If P then Q; not Q therefore not P
- In hypothesis testing, we want to argue that

  If P then Q; Q is unlikely therefore P is unlikely

  Actually this is wrong.

Bayesian & Frequentist Statistics

Bayesian

Claim to understand the **logical inconsistencies** and intend to compensate for them with **prior and posterior probability** distributions.

Theoretically true, difficult to implement in practice.

Frequentist

Mainly ignore the problems mentioned here or claim that they are not relevant in their specific research.

Neyman-Pearson Decision Theory

Two competing hypothesis: H_0, H_1.

Seek to reject H_0 to accept H_1.

- H_0 is **null hypothesis**
- H_1 is **research hypothesis** or **alternative hypothesis**.

So there are four possible outcomes of the decision-making process:

- We reject H_0 when H_0 is untrue.
- **Type I Error**: We reject H_0 even though H_0 is true.
- **Type II Error**: We fail to reject H_0 even though H_0 is untrue.
- We fail to reject H_0 when H_0 true.

Type I and Type II error should be as small as possible.

Power, Type I & II Error Probabilities

$\alpha = P[\text{Type I Error}] = P[\text{reject } H_0 \mid H_0 \text{ true}] = P[\text{accept } H_1 \mid H_1 \text{ false}]$

$\beta = P[\text{Type II Error}] = P[\text{fail to reject } H_0 \mid H_0 \text{ false}]$

$\text{Power} = 1 - \beta$

The power shows how likely our experiment is successful.

- By requiring strong evidence before rejecting H_0 (a value of the test statistic that is very different from its null value), α can be made small.
- The range of values for the test statistic that causes us to reject H_0 is **critical region**.

  We choose the critical region in such a way to make α small.
- The more evidence we require to reject H_0 (the smaller the critical region is), the harder it is to actually reject H_0 in the first place.

  The power decreases with β becomes larger.
- For given H_0 and H_1, β can be controlled by increasing the sample size.

Example of Neyman-Pearson Decision Theory

The mean is supposed to be $\mu_0 = 40$ and the standard deviation is $\sigma = 2$.

Two hypothesis $H_0 : \mu = 40$, $H_1 : |\mu - 40| \geq 1$

Then the sample size is $n = 25$.

The probability of committing Type I Error is $\alpha \leq 5\%$. (derive from H_0 and H_1?)

So we apply the test statistic $Z = \dfrac{\overline{X} - \mu_0}{\sigma/\sqrt{n}}$ and let $-z_{\alpha/2} \leq Z \leq z_{\alpha/2}$ to get probability of $1 - \alpha$ and $P[|Z| > z_{\alpha/2}] = \alpha$.

The $\left|\dfrac{\overline{X} - \mu_0}{\sigma/\sqrt{n}}\right| \geq 1.96$ is the critical region, with this critical region there are 5% of Type I Error.

If the sample mean is $\overline{x} = 40.9$, then the test statistic is $z = 2.25 > 1.96$.

Type II Error

$|Z| = \left|\dfrac{\overline{X} - \mu_0}{\sigma/\sqrt{n}}\right| \geq z_{\alpha/2} = 2.575$ when $\alpha = 1\%$. So with $z = 2.25 < 2.575$, there is a Type II Error.

Assume null hypothesis $H_0 : \mu = \mu_0$ and the true value is $\mu = \mu_0 + \delta, \delta \in \mathbb{R}\backslash\{0\}$.

$Z = \dfrac{\overline{X} - \mu_0}{\sigma/\sqrt{n}}$ is the test statistic actually follows a normal distribution with unit variance and mean $\delta\sqrt{n}/\sigma$.

Review that if $-z_{\alpha/2} \leq Z \leq z_{\alpha/2}$ then we cannot reject H_0, deriving the Type II Error.

$$\beta = P[|Z| \leq z_{\alpha/2}] = \frac{1}{\sqrt{2\pi}} \int_{-z_{\alpha/2}}^{z_{\alpha/2}} e^{-(t-\delta\sqrt{n}/\sigma)^2/2} dt \approx \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z_{\alpha/2}-\delta\sqrt{n}/\sigma} e^{-t^2/2} dt$$

We donate $-z_\beta \approx z_{\alpha/2} - \delta\sqrt{n}/\sigma$ or $n \approx \frac{(z_{\alpha/2}+z_\beta)^2\sigma^2}{\delta^2}$

Given a critical region determined by α, we can find a sample size n so that the probability of committing a Type II Error is β.

Remark

In order for the statistical procedure to be valid, **a critical region must be fixed before any data obtained**.

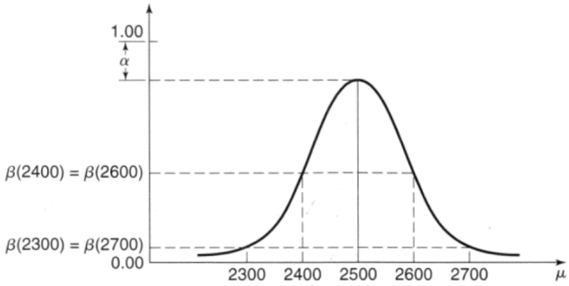
Operating Characteristic (OC) Curves

The previous probability of Type II Error is calculated directly. For distribution other than normal distribution with known variance is different. These probability are often provided as OC curves.

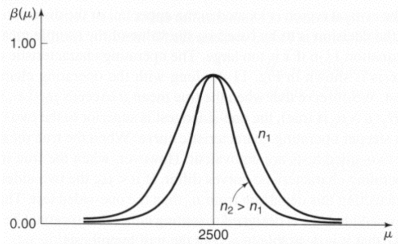
According to **central limit theorem**, $\overline{X}$ follows a normal distribution with mean μ.

Through our **choice of critical region** and sample size, we can effectively fix α.

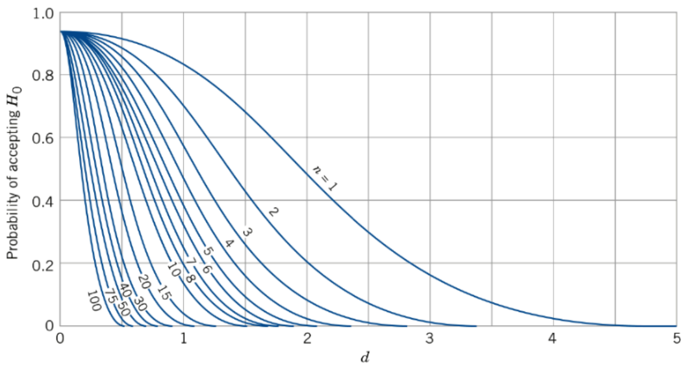
Depending on the true value of μ, β can be large or small.

We represent β as function of μ through a curve: 

$\beta(\mu) = P[\text{fail to reject } H_0 \mid \mu]$ and when $\mu \to \mu_0$ the $\beta(\mu) \to P[\text{fail to reject } H_0 \mid \mu = \mu_0] = 1 - \alpha$

Also, the β depends on the sample size, a larger sample size reduces the variance of $\overline{X}$ and **make it less likely to reject H_0 if $\mu \neq \mu_0$.** 

OC Curves for Normal Distribution

The OC curve for $\alpha = 0.05$ to find β: 

where $d := \frac{|\mu - \mu_0|}{\sigma}$

One-Tailed Tests

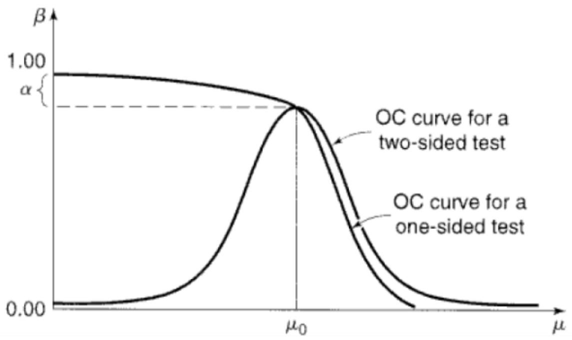
The one tailed hypothesis of form $H_0 : \theta \leq \theta_0, H_1 : \theta \geq \theta_1$ or $H_0 : \theta \geq \theta_0, H_1 : \theta \leq \theta_1$ also has OC curve.

The $\alpha = \alpha(\theta)$ depends on the actual value of θ. $\alpha(\theta_0)$ gives an upper bound for all $\alpha(\theta)$.

OC for One-Sided Hypotheses

One sided test with null hypothesis $H_0 : \mu \leq \mu_0, H_1 : \mu > \mu_0$ then we can only define $\beta(\mu)$ for $\mu > \mu_0$.

Since $\beta(\mu_0) = 1 - \alpha(\mu_0)$ and we have a $\alpha = \alpha(\mu)$ for $\mu \leq \mu_0$, we just use $\beta(\mu) = 1 - \alpha(\mu)$ for $\mu \leq \mu_0$.



Selecting Appropriate Hypotheses