Introduction to Statistical Method

Comparing Two Means and Two Variances

Comparing Two Means - A Point Estimator

We have two populations with different means μ_1 and μ_2, the goal is to estimate the difference $\mu_1 - \mu_2$ by taking a sample from each population in independent way.

Natural point estimator: $\mu_1 \hat{-} \mu_2 := \hat{\mu_1} - \hat{\mu_2} = \overline{X}_1 - \overline{X}_2$

To determine confidence intervals and to test hypothesis we need to know the distribution $\overline{X}_1 - \overline{X}_2$

Theorem

The $\overline{X}_1$ and $\overline{X}_2$ be the sample means based on independent random samples of size n_1 and n_2 drawn from normal distributions with mean μ_1 and μ_2 and variance σ_1^2 and σ_2^2.

The $\overline{X}_1 - \overline{X}_2$ is normal with mean $\mu_1 - \mu_2$ and variance $\sigma_1^2/n_1 + \sigma_2^2/n_2$

$\dfrac{\overline{X}_1 - \overline{X}_2 - (\mu_1 - \mu_2)}{\sqrt{\sigma_1^2/n_1 + \sigma_2^2/n_2}}$ is standard normal random variable.

(Central Limit Theorem allows us to apply this result even to non-normal populations if we have really large sample sizes)

OC Curve Application

$d = \dfrac{|\mu_1 - \mu_2|}{\sqrt{\sigma_1^2 + \sigma_2^2}}$, if $n = n_1 = n_2$, unchanged, else $n = \dfrac{\sigma_1^2 + \sigma_2^2}{\sigma_1^2/n_1 + \sigma_2^2/n_2}$

> If the variances are unknown, we need some more sophisticated methods.
>
> The unknown variances are equal, situation is much easier.

Comparing Two Variances

Consider test types of this:

- $H_0 : \sigma_1^2 = \sigma_2^2$, $H_1 : \sigma_1^2 > \sigma_2^2$ (right-tailed test)
- $H_0 : \sigma_1^2 = \sigma_2^2$, $H_1 : \sigma_1^2 \neq \sigma_2^2$ (two-tailed test)

we move the σ to one side to consider only about the quotient.

> $(n-1)S^2/\sigma^2$ follows a chi-squared distribution with $n-1$ degree of freedom.
>
> If the variance is put into quotient, it is easier to handle

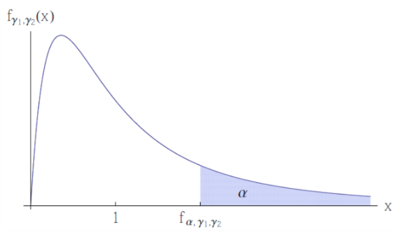
F-Distribution

$X^2_{\gamma_1}$ and $X^2_{\gamma_2}$ are independent chi-squared random variables with γ_1 and γ_2 degrees of freedom.

Random variable $F_{\gamma_1,\gamma_2} = \dfrac{X^2_{\gamma_1}/\gamma_1}{X^2_{\gamma_2}/\gamma_2}$ is said to follow a F-distribution with γ_1 and γ_2 degree of freedom.

$P[F_{\gamma_1,\gamma_2} < x] = P[1/F_{\gamma_1,\gamma_2} > 1/x] = 1 - P[F_{\gamma_2,\gamma_1} < 1/x]$

Then $f_{\gamma_1,\gamma_2}(x) = \gamma_1^{\gamma_1/2}\gamma_2^{\gamma_2/2} \dfrac{\Gamma(\frac{\gamma_1+\gamma_2}{2})}{\Gamma(\frac{\gamma_1}{2})\Gamma(\frac{\gamma_2}{2})} \dfrac{x^{\gamma_1/2-1}}{(\gamma_1 x + \gamma_2)^{(\gamma_1+\gamma_2)/2}}$ for $x \geq 0$



Define $f_{\alpha,\gamma_1,\gamma_2}$ by $P[F_{\gamma_1,\gamma_2} > f_{\alpha,\gamma_1,\gamma_2}] = \alpha$

Then $\begin{aligned} 1-\alpha &= P[F_{\gamma_1,\gamma_2} \geq f_{1-\alpha,\gamma_1,\gamma_2}] \\ &= 1 - P[F_{\gamma_1,\gamma_2} < f_{1-\alpha,\gamma_1,\gamma_2}] \\ &= P[F_{\gamma_2,\gamma_1} < 1/f_{1-\alpha,\gamma_1,\gamma_2}] \\ &= 1 - P[F_{\gamma_2,\gamma_1} \geq 1/f_{1-\alpha,\gamma_1,\gamma_2}] \end{aligned}$ also we can see $\alpha = P[F_{\gamma_2,\gamma_1} \geq f_{\alpha,\gamma_2,\gamma_1}]$.

So $f_{1-\alpha,\gamma_1,\gamma_2} \cdot f_{\alpha,\gamma_2,\gamma_1} = 1$.

Remark

Let S_1^2 and S_2^2 be sample variance based on independent random samples of size n_1 and n_2 from normal populations with means μ_1 and μ_2 and variance σ_1^2 and σ_2^2.

If $\sigma_1^2 = \sigma_2^2$ then the statistic S_1^2/S_2^2 follows F-distribution with $n_1 - 1$ and $n_2 - 1$ distribution.

Since $F_{n_1-1,n_2-1} = \dfrac{[(n_1-1)S_1^2/\sigma_1^2]/(n_1-1)}{[(n_2-1)S_2^2/\sigma_2^2]/(n_2-1)} = \dfrac{\sigma_2^2 S_1^2}{\sigma_1^2 S_2^2}$, so it is trivial to require $\sigma_1^2 = \sigma_2^2$

F-Test

We can derive F-Test from F-distribution that:

$H_0 : \sigma_1 = \sigma_2$ based on $F_{n_1-1,n_2-1} = \frac{S_1^2}{S_2^2}$ is a F-Test

We reject H_0 at significance level α

- in favor of $H_1 : \sigma_1 > \sigma_2$ if $\frac{S_1^2}{S_2^2} > f_{\alpha,n_1-1,n_2-1}$
- in favor of $H_1 : \sigma_1 < \sigma_2$ if $\frac{S_2^2}{S_1^2} > f_{\alpha,n_2-1,n_1-1}$
- in favor of $H_1 : \sigma_1 \neq \sigma_2$ if $\frac{S_1^2}{S_2^2} > f_{\alpha/2,n_1-1,n_2-1}$ or $\frac{S_2^2}{S_1^2} > f_{\alpha/2,n_2-1,n_1-1}$

When testing to see whether two population variances are equal for the purpose of comparing their means, one hopes to not reject H_0.

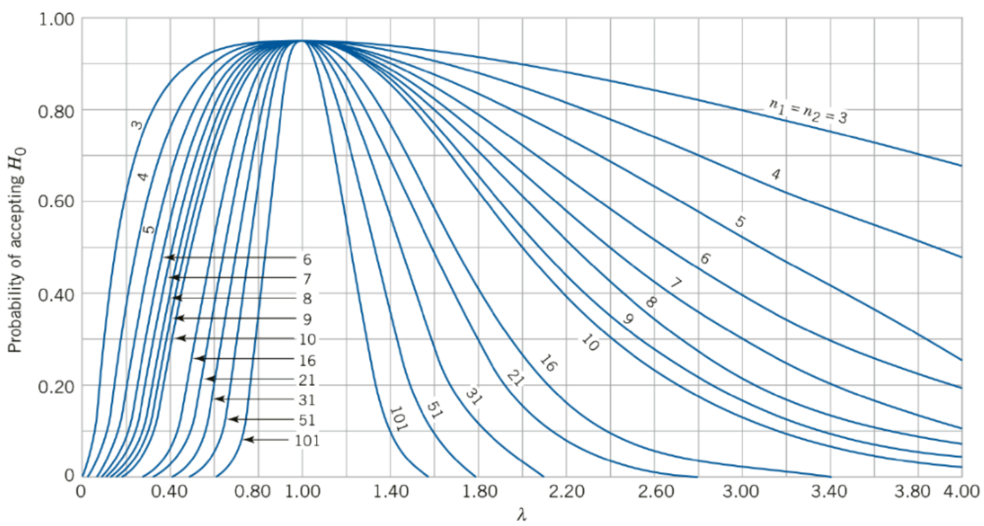
If H_0 is not rejected, one can assume that the variances are in fact equal and continue with the test for equality for means.

In this case, a small Type II error β is more important than α small.

OC Curves for F-Test

For case $n = n_1 = n_2$, the OC curves plotting β against the parameter $\lambda = \frac{\sigma_1}{\sigma_2}$.

The curves are for both one- two- sided alternatives.



Comparing Two Means - Equal Variances