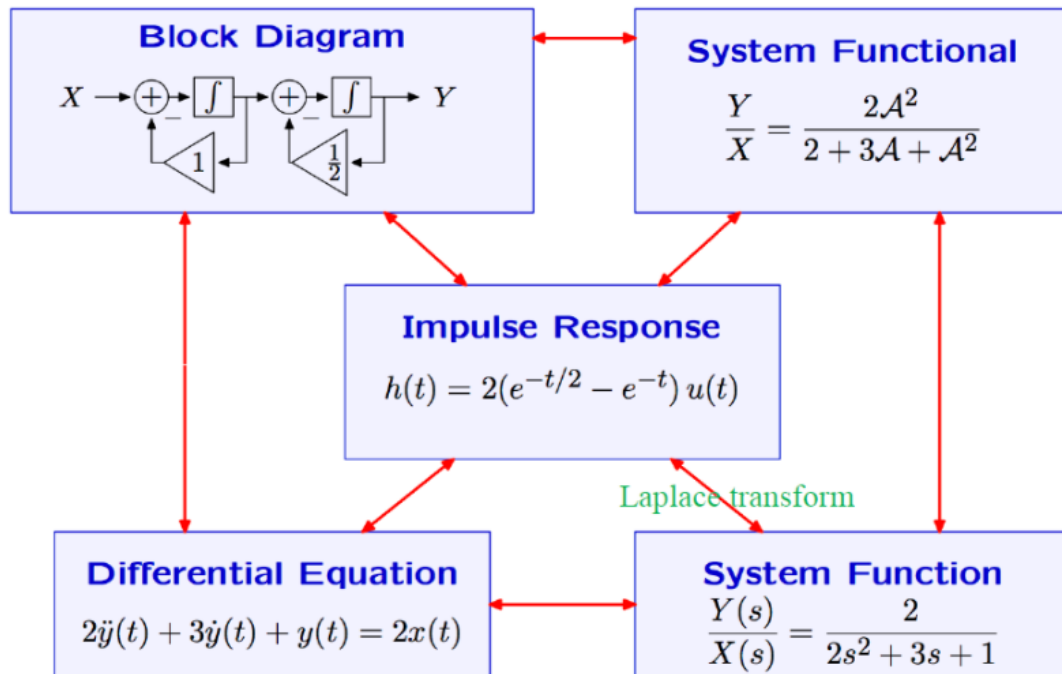# VE216 Lecture 6

Laplace Transform

Concept Map

Laplace Transform Definition

$X(s) = \int x(t)e^{-st}\,dt$

Two versions

- Unilateral: $X(s) = \int_0^\infty x(t)e^{-st}\,dt$
- Bilateral: $X(s) = \int_{-\infty}^\infty x(t)e^{-st}\,dt$

(We focus on bilateral version.)

Region of Convergence (ROC)

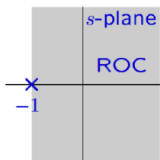This is based on the $X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt$, we can see from the examples.

Example

Exercise 1

$x_1(t) = \begin{cases} e^{-t} & t \geq 0 \\ 0 & t < 0 \end{cases}$ Laplace transform.

$X_1(s) = L[x_1(t)] = \int_0^{\infty} e^{-t}e^{-st}\,dt = -\frac{1}{1+s}\int_0^{\infty} -(1+s)e^{-(1+s)t}\,dt = -\frac{1}{1+s}e^{-(1+s)t}\Big|_0^{\infty} = -\frac{1}{1+s}$

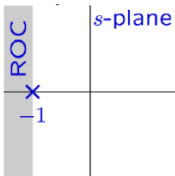Since $e^{-(1+s)t}$ is convergent when $1+s < 0$, or $Re(s) > -1$, so ROC shows

Exercise 2

$x_2(t) = \begin{cases} -e^{-t} & t \leq 0 \\ 0 & t > 0 \end{cases}$ Laplace transform.

$X_2(s) = \int_{-\infty}^{0} -e^{-t}e^{-st}\,dt = \frac{1}{1+s}\int_{-\infty}^{0} -(1+s)e^{-(1+s)t}\,dt = \frac{1}{1+s}$

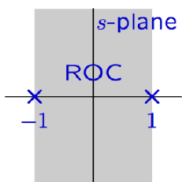Since $e^{-(1+s)t}$ is convergent for $x \in (-\infty, 0)$, so $-(1+s) > 0$ or $Re(s) < -1$ with ROC shows

Exercise 3

$x_3(t) = e^{-|t|}$ Laplace transform.

$X_3(s) = \int_{-\infty}^{\infty} e^{-|t|-st}\,dt = \int_0^{\infty} e^{-(1+s)t}\,dt + \int_{-\infty}^{0} e^{(1-s)t}\,dt = \frac{e^{(1-s)t}}{1-s}\Big|_{-\infty}^{0} - \frac{e^{-(1+s)t}}{1+s}\Big|_0^{\infty} = \frac{1}{1-s} + \frac{1}{1+s}$

So $1-s > 0$ and $1+s > 0$, $1 > s > -1$, with ROC shows

Exercise 4

$\frac{2s}{s^2-4}$ Laplace inverse transform, how many possible solution and ROCs?

$\frac{2s}{s^2-4} = \frac{1}{s+2} + \frac{1}{s-2}$, with ± 2 as poles.

- $\frac{1}{s+2}$ has 2 forms with 2 ROCs.

    ○ $\frac{1}{s+2} = \int_{-\infty}^{\infty} e^{-st}e^{-2t}u(t)dt = \int_{0}^{\infty} e^{-(2+s)t}dt = -\frac{1}{s+2}e^{-(2+s)t}\Big|_{0}^{\infty}$, so $2+s > 0$ with $Re(s) > -2$.

    ○ $\frac{1}{s+2} = \int_{-\infty}^{\infty} -e^{-st}e^{-2t}u(-t)dt = \int_{-\infty}^{0} -e^{-(s+2)t} = \frac{1}{s+2}e^{-(s+2)t}\Big|_{-\infty}^{0}$, so $2+s < 0$ with $Re(s) < -2$.

- $\frac{1}{s-2}$ has 2 forms with 2 ROCs

    ○ $\frac{1}{s-2} = \int_{-\infty}^{\infty} e^{-st}e^{2t}u(t)dt = \int_{0}^{\infty} e^{(2-s)t}dt = \frac{1}{2-s}e^{(2-s)t}\Big|_{0}^{\infty}$, so $2-s < 0$ with $Re(s) > 2$.

    ○ $\frac{1}{s-2} = \int_{-\infty}^{\infty} -e^{-st}e^{2t}u(-t)dt = \int_{-\infty}^{0} -e^{(2-s)t}dt = \frac{1}{s-2}e^{(2-s)t}\Big|_{-\infty}^{0}$, so $2-s > 0$ with $Re(s) < 2$.

So there are totally 3 solutions:

- $x(t) = e^{-2t}u(t) + e^{2t}u(t)$ with $Re(s) > 2$
- $x(t) = -e^{-2t}u(-t) - e^{2t}u(-t)$ with $Re(s) < -2$
- $x(t) = e^{-2t}u(-t) - e^{2t}u(-t)$ with $2 > Re(s) > -2$

Laplace Transform of a Derivative

$X_d(s) = \int_{-\infty}^{\infty} x'(t)e^{-st}dt = x(t)e^{-st}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} x(t)(-s)e^{-st}dt$

Since $X(s)$ is convergent, so $x(t)e^{-st}\Big|_{-\infty}^{\infty} = 0$, thus $X_d(s) = sX(s)$.

Laplace Transform Properties

Property	$x(t)$	$X(s)$	ROC
Linearity	$ax_1(t) + bx_2(t)$	$aX_1(s) + bX_2(s)$	$\supset (R_1 \cap R_2)$
Delay by T	$x(t-T)$	$X(s)e^{-sT}$	R
Multiply by t	$tx(t)$	$-\frac{dX(s)}{ds}$	R
Multiply by $e^{-\alpha t}$	$x(t)e^{-\alpha t}$	$X(s+\alpha)$	shift R by $-\alpha$
Differentiate in t	$\frac{dx(t)}{dt}$	$sX(s)$	$\supset R$
Integrate in t	$\int_{-\infty}^{t} x(\tau)\,d\tau$	$\frac{X(s)}{s}$	$\supset \left(R \cap (Re(s) > 0)\right)$
Convolve in t	$\int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)\,d\tau$	$X_1(s)X_2(s)$	$\supset (R_1 \cap R_2)$