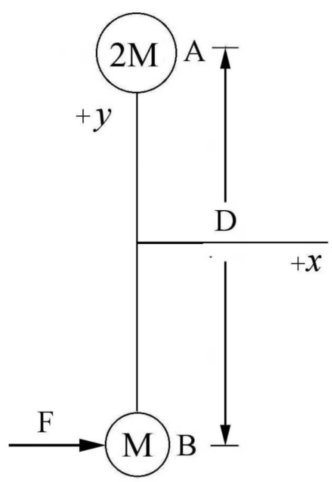
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics

Physics 8.01 **Fall 2012**

Problem Set 10 Rotational and Translational Motion Solutions

Problem 1 Angular Impulse

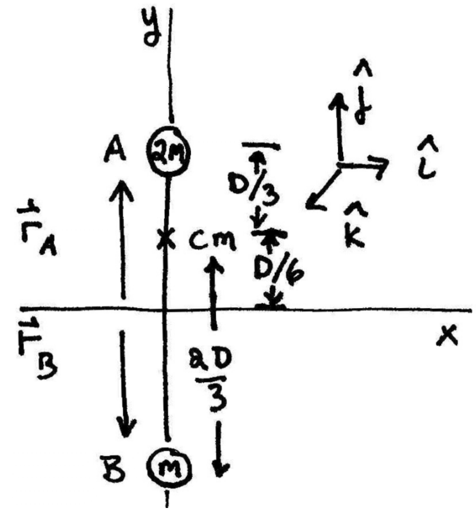
Two point-like objects are located at the points A, and B, of respective masses $M_A = 2M$, and $M_B = M$, as shown in the figure below. The two objects are initially oriented along the y-axis and connected by a rod of negligible mass of length D, forming a rigid body. A force of magnitude $F = \left|\vec{\mathbf{F}}\right|$ along the x direction is applied to the object at B at $t = 0$ for a short time interval Δt. Neglect gravity. Give all your answers in terms of M, F, Δt and D as needed. What is the magnitude of the angular velocity of the system after the collision?



Solution:

An impulse of magnitude $F\,\Delta t$ is applied in the $+x$ direction, and the center of mass of the system will move in this direction. The two masses will rotate about the center of mass, counterclockwise in the figure above. Before the force is applied we can calculate the position of the center of mass with respect to the origin.

$$\vec{R}_{cm} = \frac{M_A \vec{r}_A + M_B \vec{r}_B}{M_A + M_B} = \frac{2M(D/2)\hat{\mathbf{j}} + M(D/2)(-\hat{\mathbf{j}})}{3M} = (D/6)\hat{\mathbf{j}}\,.$$



The center of mass is a distance $(2/3)\mathrm{D}$ from the object at B and is a distance $(1/3)\mathrm{D}$ from the object at A. Because

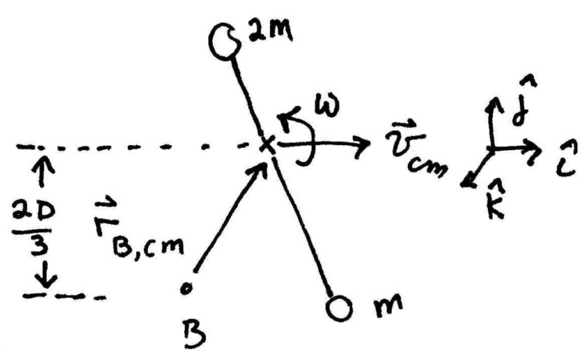
$$F\Delta t\hat{\mathbf{i}} = 3M\vec{\mathbf{V}}_{cm},$$

the magnitude of the velocity of the center of mass is then

$$V_{cm} = F\Delta t / 3\mathrm{M}$$

and the direction is in the positive-$\hat{\mathbf{i}}$ direction (to the right).

Because the force is applied at the point B, there is no torque about the point B, hence the angular momentum is constant about the point B. The initial angular momentum about the point B is zero.



The angular momentum about the point B after the impulse is applied is the sum of two terms,

$$\vec{\mathbf{0}} = \vec{\mathbf{L}}_{B,f} = \vec{\mathbf{r}}_{B,f} \times 3M\vec{\mathbf{V}}_{cm} + \vec{\mathbf{L}}_{cm} = (2D/3)\hat{\mathbf{j}} \times F\Delta t\hat{\mathbf{i}} + \vec{\mathbf{L}}_{cm}$$

$$\vec{\mathbf{0}} = (2DF\Delta t/3)(-\hat{\mathbf{k}}) + \vec{\mathbf{L}}_{cm}$$

The angular momentum about the center of mass is given by

$$\vec{\mathbf{L}}_{cm} = I_{cm}\omega\,\hat{\mathbf{k}} = (2M(D/3)^2 + M(2D/3)^2)\omega\,\hat{\mathbf{k}} = (2/3)MD^2\omega\,\hat{\mathbf{k}}$$

Thus the angular momentum about the point B after the impulse is applied is

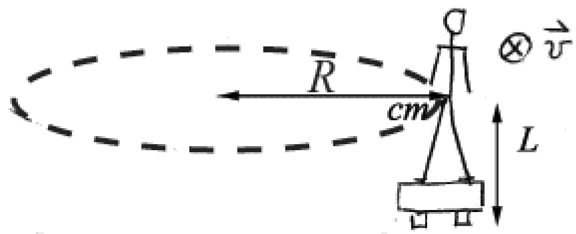
$$\vec{\mathbf{0}} = (2DF\Delta t/3)(-\hat{\mathbf{k}}) + (2/3)MD^2\omega\,\hat{\mathbf{k}}\,.$$

We can solve this equation for the angular speed

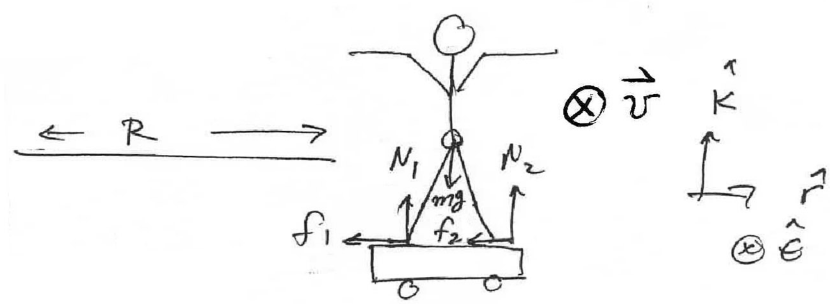
$$\omega = \frac{F\Delta t}{MD}\,.$$

Problem 2 Standing on a Moving Railroad Car on a Circular Track

A person of mass M is standing on a railroad car, which is rounding an unbanked turn of radius R at a speed v. His center of mass is at a height of L above the car midway between his feet, which are separated by a distance of d. The man is facing the direction of motion. What is the magnitude of the normal force on each foot?



Solution: We begin by choosing a cylindrical coordinate system and drawing a free-body force diagram, shown below.



We decompose the contact force between the foot closest to the center of the circular motion and the ground into a tangential component corresponding to static friction $\vec{\mathbf{f}}_1$ and a perpendicular component, $\vec{\mathbf{N}}_1$. In a similar fashion we decompose the contact force between the foot furthest from the center of the circular motion and the ground into a tangential component corresponding to static friction $\vec{\mathbf{f}}_2$ and a perpendicular component, $\vec{\mathbf{N}}_2$. We do not assume that the static friction has its maximum magnitude nor due we assume that $\vec{\mathbf{f}}_1 = \vec{\mathbf{f}}_2$ or $\vec{\mathbf{N}}_1 = \vec{\mathbf{N}}_2$. The gravitational force acts at the center of mass.

We shall use our two dynamical equations of motion,

$$\vec{\mathbf{F}} = m\vec{\mathbf{a}}_{cm} \tag{1}$$

for translational motion and

$$\vec{\boldsymbol{\tau}}_{cm} = I_{cm}\vec{\boldsymbol{\alpha}}_{\mathrm{cm}} \tag{2}$$

for the rotational motion about the center of mass noting that we are considering the special case that $\vec{\boldsymbol{\alpha}}_{cm} = 0$ because the object is not rotating about the center of mass.

In order to apply Eq. (1), we treat the person as a point-like particle located at the center of mass and all the eternal forces act at this point. The radial component of Newton's Second Law is given by

$$\hat{\mathbf{r}} : -f_1 - f_2 = -m\frac{v^2}{R}. \tag{3}$$

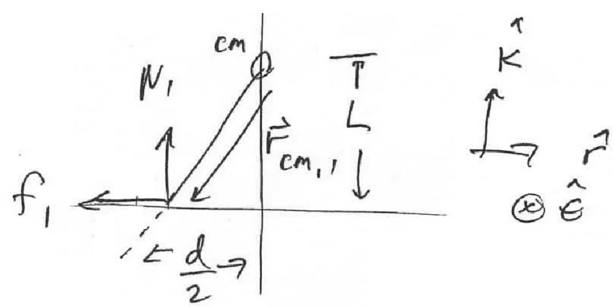
The vertical component of Newton's Second Law is given by

$$\hat{\mathbf{k}} : N_1 + N_2 - mg = 0. \tag{4}$$

The rotational equation of motion (Eq. (2)) is

$$\vec{\boldsymbol{\tau}}_{\text{cm}}^{\text{total}} = 0. \tag{5}$$

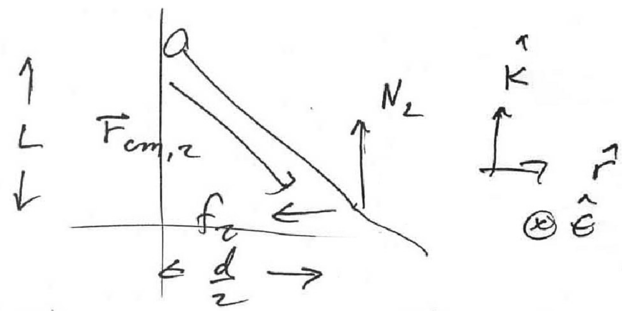
We begin our calculation of the torques about the center of mass by noting that the gravitational force does not contribute to the torque because it is acting at the center of mass. We draw a torque diagram in the figure below showing the location of the point of application of the forces, the point we are computing the torque about (which in this case is the center of mass), and the vector $\vec{\mathbf{r}}_{\text{cm},1}$ from the point we are computing the torque about to the point of application of the forces.



The torque on the inner foot is given by

$$\vec{\boldsymbol{\tau}}_{\text{cm},1} = \vec{\mathbf{r}}_{\text{cm},1} \times (\vec{\mathbf{f}}_1 + \vec{\mathbf{N}}_1) = \left(-\frac{d}{2}\hat{\mathbf{r}} - L\hat{\mathbf{k}}\right) \times (-f_1\hat{\mathbf{r}} + N_1\hat{\mathbf{k}}) = \left(\frac{d}{2}N_1 + Lf_1\right)\hat{\boldsymbol{\theta}}. \tag{6}$$

We draw a similar torque diagram for the forces applied to the outer foot.



The torque on the outer foot is given by

$$\vec{\boldsymbol{\tau}}_{\mathrm{cm},2} = \vec{\mathbf{r}}_{\mathrm{cm},2} \times (\vec{\mathbf{f}}_2 + \vec{\mathbf{N}}_2) = \left(+\frac{d}{2}\hat{\mathbf{r}} - L\hat{\mathbf{k}}\right) \times (-f_2\hat{\mathbf{r}} + N_2\hat{\mathbf{k}}) = \left(-\frac{d}{2}N_2 + Lf_2\right)\hat{\boldsymbol{\theta}}. \tag{7}$$

Notice that the forces $\vec{\mathbf{f}}_1$, $\vec{\mathbf{N}}_1$, and $\vec{\mathbf{f}}_2$ all contribute torques about the center of mass in the positive $\hat{\boldsymbol{\theta}}$-direction while $\vec{\mathbf{N}}_2$ contribute torques about the center of mass in the negative $\hat{\boldsymbol{\theta}}$-direction while $\vec{\mathbf{N}}_2$. According to Eq. (5) the sum of these torques about the center of mass must be zero. Therefore

$$\begin{aligned} \vec{\boldsymbol{\tau}}_{\mathrm{cm}}^{\mathrm{total}} &= \vec{\boldsymbol{\tau}}_{\mathrm{cm},1} + \vec{\boldsymbol{\tau}}_{\mathrm{cm},2} = \left(\frac{d}{2}N_1 + Lf_1\right)\hat{\boldsymbol{\theta}} + \left(-\frac{d}{2}N_2 + Lf_2\right)\hat{\boldsymbol{\theta}} \\ &= \left(\frac{d}{2}(N_1 - N_2) + L(f_1 + f_2)\right)\hat{\boldsymbol{\theta}} = \vec{\mathbf{0}} \end{aligned}. \tag{8}$$

Notice that the magnitudes of the two friction forces appear together as a sum in Eqs. (3) and (8). We now can solve Eq. (8) for $f_1 + f_2$ and substitute the result into Eq. (3) yielding the condition that

$$\frac{d}{2}(N_1 - N_2) + Lm\frac{v^2}{R} = 0. \tag{9}$$

We can rewrite this equation as

$$N_2 - N_1 = \frac{2Lmv^2}{dR}. \tag{10}$$

We also rewrite the vertical equation of motion, Eq. (4), in the form

$$N_2 + N_1 = mg. \tag{11}$$

We now can solve for N_2 by adding together Eqs. (10) and (11), then divide by two,

$$N_2 = \frac{1}{2}\left(\frac{2Lmv^2}{dR} + mg\right). \tag{12}$$

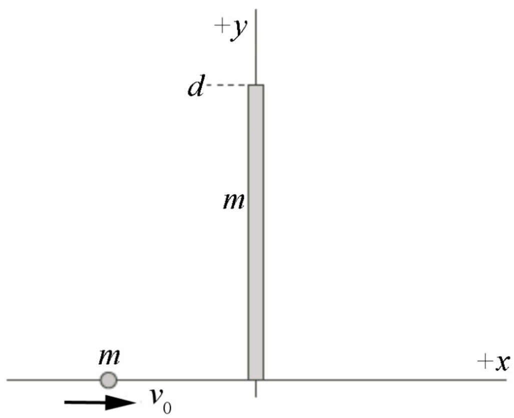
We now can solve for N_1 by subtracting Eqs. (10) from (11), then divide by two,

$$N_1 = \frac{1}{2}\left(mg - \frac{2Lmv^2}{dR}\right). \tag{13}$$

Check your result: We see that the normal force acting on the outer foot is greater in magnitude than the normal force acting on the inner foot. We expect this result because as we increase the speed v, we find that at a maximum speed $v_{\max}$, the normal force on the inner foot goes to zero and we start to rotate in the positive $\hat{\boldsymbol{\theta}}$-direction, tipping outward. We can find this maximum speed by setting $N_1 = 0$ in Eq. (13) resulting in

$$v_{\max} = \sqrt{\frac{gdR}{2L}}\,. \tag{14}$$

Problem 3 Collision Rotation and Translation A slender uniform rod of length d and mass m rests along the y-axis on a frictionless, horizontal table. A particle of equal mass m is moving along the x-axis at a speed v_0. At $t = 0$ the particle strikes the end of the rod and sticks to it.



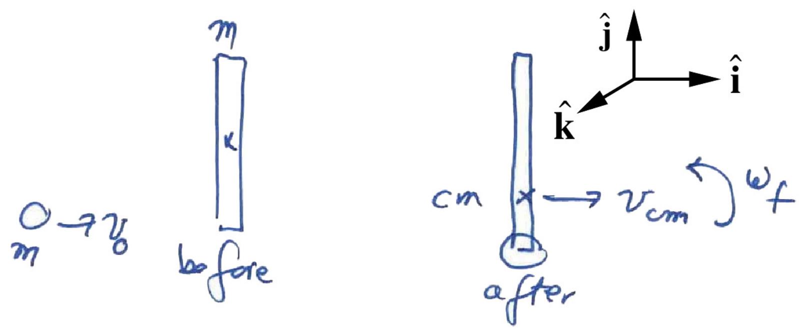
a) Find a vector expression for the position of the center of mass of the system as a function of time, $\vec{\mathbf{R}}_{cm}(t)$. Express your answer in terms of some or all of the parameters m, d and v_0 as needed.

b) What is the magnitude of the angular velocity ω of the rod about its new center of mass immediately after the collision? Express your answer in terms of some or all of the m, d and v_0 as needed.

c) What is the magnitude of the velocity v of the attached particle immediately after the collision?

Solution:

In this problem we will calculate the center of mass of the puck-stick system after the collision. There are no external forces or torques acting on this system so the momentum of the center of mass is constant before and after the collision and the angular momentum about the center of mass of the puck-stick system is constant before and after the collision. We shall use these relations to compute the final angular velocity of the puck-stick about the center of mass. We note that the mechanical energy is not constant because collision of the puck with the stick is completely inelastic.



a) With respect to the center of the stick, the center of mass of the stick-puck combination is (neglecting the radius of the puck) at time $t = 0$ is

$$\vec{\mathbf{R}}_{\text{cm}}(t=0) = \frac{m_{\text{stick}}(d/2)}{m_{\text{stick}} + m_{\text{puck}}}\hat{\mathbf{j}} = \frac{m(d/2)}{m+m}\hat{\mathbf{j}} = \frac{d}{4}\hat{\mathbf{j}}. \tag{1}$$

the collision, the only net forces on the system (the stick-puck combination) are the internal forces between the stick and the puck (transmitted through the putty) so the velocity of the center of mass of the stick-puck combination does not change and is

$$\vec{\mathbf{V}}_{\text{cm}} = \frac{m v_0}{m+m}\hat{\mathbf{i}} = \frac{v_0}{2}\hat{\mathbf{i}} \tag{2}$$

Therefore

$$\vec{\mathbf{R}}_{cm}(t) = \vec{\mathbf{R}}_{cm}(t=0) + \vec{\mathbf{V}}_{cm}t = \frac{d}{4}\hat{\mathbf{j}} + \frac{v_0}{2}t\hat{\mathbf{i}} \tag{3}$$

We could also have determined this because the linear momentum is constant. Initially only the puck had linear momentum with magnitude given by $p_0 = m v_0$. After the collision, the center of mass of the system is moving with speed v_f. Equating initial and final magnitude of the linear momenta yields

$$m v_0 = (2m) v_f \;\Rightarrow\; v_f = \frac{v_0}{2}\;. \tag{4}$$

The direction of the velocity is the same as the initial direction of the puck's velocity.

Note that the forces that deform the putty do negative work (the putty is compressed somewhat), and so mechanical energy is not conserved; the collision is totally inelastic.

b) Choose the center of mass of the stick-puck combination, as found in part a) at time $t = 0$, as the point about which to find angular momentum. This choice means that after the collision there is no angular momentum due to the translation of the center of mass. Before the collision, the angular momentum was entirely due to the motion of the puck,

$$\vec{\mathbf{L}}_0 = \vec{\mathbf{r}}_{\text{puck}} \times \vec{\mathbf{p}}_0 = (d/4)(m v_0)\hat{\mathbf{k}}\,, \tag{5}$$

where $\hat{\mathbf{k}}$ is directed out of the page in the figure above. After the collision, the angular momentum is

$$\vec{\mathbf{L}}_f = I_{\text{cm}'}\omega_f\,\hat{\mathbf{k}}\,, \tag{6}$$

where $I_{\text{cm}'}$ is the moment of inertia about the center of mass of the stick-puck combination. Let C denote the center of the stick. The moment of inertia of the stick about the new center of mass is found from the parallel axis theorem, and the moment of inertia of the puck is $m(d/4)^2$, and so the moment of inertial of the stick-puck system is

$$I_{\text{cm}'} = I_{\text{cm}',\,\text{C}} + I_{\text{cm}',\,\text{puck}} = (I_C + m(d/4)^2) + m(d/4)^2 = I_{\text{C}} + \frac{m d^2}{8}\,. \tag{7}$$

Inserting this expression into Equation (6), equating the expressions for $\vec{\mathbf{L}}_0$ and $\vec{\mathbf{L}}_f$ and solving for ω_f yields

$$\omega_f = \frac{m(d/4)}{I_C + m d^2/8} v_0\,. \tag{8}$$

If the stick is uniform, $I_C = m d^2/12$ and Equation (8) reduces to

$$\omega_f = \frac{6}{5}\frac{v_0}{d}\,. \tag{9}$$

Alternative Calculation: It may be tempting to try to calculate angular momentum about the contact point S, where the puck hits the stick. If this is done, there is no initial angular momentum,

$$\vec{\mathbf{L}}_{S,i} = \vec{\mathbf{0}}\,. \tag{10}$$

After the collision the puck still contributes zero angular momentum about the point S. Therefore we only need to calculate the angular momentum of the stick about the contact point S. Treating the stick as our object, consider all the mass of the stick placed at its center point C. Then the orbital angular momentum about S is given by

$$\vec{\mathbf{L}}^{\text{orbital}}_{S,f} = \vec{\mathbf{r}}_{S,C} \times \vec{\mathbf{p}}_C = (d/2)\hat{\mathbf{j}} \times m\vec{\mathbf{v}}_C\,. \tag{11}$$

The center point C of the stick is rotating about the center of mass of the system with velocity

$$\vec{\mathbf{v}}'_C = \frac{d}{4}\omega_f(-\hat{\mathbf{i}}), \tag{12}$$

By the law of addition of velocities, the velocity of the center point C of the stick in a reference frame fixed to the ground is then given by

$$\vec{\mathbf{v}}_C = \vec{\mathbf{V}}_{\text{cm}} + \vec{\mathbf{v}}'_C. \tag{13}$$

Substituting Eqs. (2) and (12) into Eq. (13) yields

$$\vec{\mathbf{v}}_C = (v_0 / 2 - d\omega_f / 4)\hat{\mathbf{i}}. \tag{14}$$

Therefore the orbital angular momentum of the stick about the point S is

$$\vec{\mathbf{L}}_{S,f}^{\text{orbital}} = (d/2)\hat{\mathbf{j}} \times m(v_0/2 - d\omega_f/4)\hat{\mathbf{i}} = (d/2)m(v_0/2 - d\omega_f/4)(-\hat{\mathbf{k}}). \tag{15}$$

Because the stick is rotating about its center of mass with angular velocity

$$\vec{\boldsymbol{\omega}}_f = \omega_f \hat{\mathbf{k}} \tag{16}$$

there is also another contribution to the angular momentum about S, the spin angular momentum which is independent of the point S and given by

$$\vec{\mathbf{L}}_f^{\text{spin}} = I_{\text{cm}}\vec{\boldsymbol{\omega}}_f = I_C\omega_f\hat{\mathbf{k}}. \tag{17}$$

Hence the angular momentum of the stick about S is the sum of these two contributions (Eqs. (15) and (17))

$$\vec{\mathbf{L}}_{S,f} = \vec{\mathbf{L}}_{S,f}^{\text{orbital}} + \vec{\mathbf{L}}_f^{\text{spin}} = (d/2)m(v_0/2 - d\omega_f/4)(-\hat{\mathbf{k}}) + I_C\omega_f\hat{\mathbf{k}}. \tag{18}$$

Because there are no torques about the point S, the angular momentum is constant about the point S. Therefore

$$-(d/2)m(v_0/2 - \omega_f d/4) + I_C\omega_f = 0. \tag{19}$$

We can solve Eq. (19) for the angular speed ω_f:

$$\omega_f = \frac{mv_0 d/4}{(I_C + md^2/8)}. \tag{20}$$

The moment of inertial $I_C = md^2/12$. Therefore

$$\omega_f = \frac{6v_0}{5d}. \tag{21}$$

in agreement with our earlier calculation (Eq. (9)).

This alternative derivation should serve two purposes. One is that it doesn't matter which point we use to find angular momentum. The second is that use of foresight, in this case choosing the center of mass of the system so that the final velocity does not contribute to the angular momentum, can prevent extra calculation. It's often a matter of trial and error ("learning by misadventure") to find the "best" way to solve a problem.

c) What is the magnitude of the velocity v of the attached particle immediately after the collision?

We again use the law of addition of velocities to determine the velocity $\vec{\mathbf{v}}$ of the attached particle immediately after the collision in a reference frame fixed to the ground,

$$\vec{\mathbf{v}} = \vec{\mathbf{V}}_{\text{cm}} + \vec{\mathbf{v}}'. \tag{22}$$

The particle is rotating about the center of mass of the system with velocity

$$\vec{\mathbf{v}}' = \frac{d}{4}\omega_f \hat{\mathbf{i}}, \tag{23}$$

Substituting Eqs. (2) and (23) into Eq. (22) yields

$$\vec{\mathbf{v}} = (v_0 / 2 + d\omega_f / 4)\hat{\mathbf{i}}. \tag{24}$$

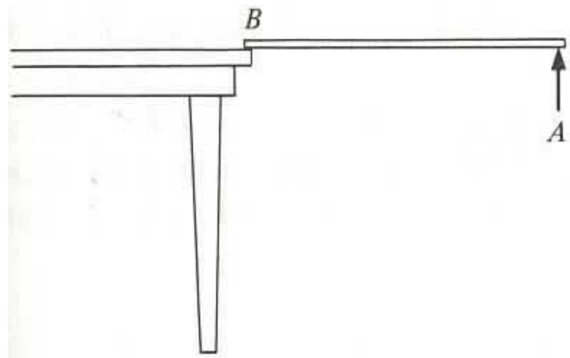
So the magnitude of the velocity is

$$v = v_0 / 2 + d\omega_f / 4. \tag{25}$$

Remark: The time of one rotation will be the same for all observers, independent of reference frame. This fact is crucial in solving problems, in that the angular velocity will be the same (this was used in the alternate derivation for part d) above). The time for one rotation is the period $T = 2\pi / \omega_f$ and the distance the center of mass of the system moves is

$$x_{\text{cm}} = v_{\text{cm}} T = 2\pi \frac{v_{\text{cm}}}{\omega_f} = 2\pi \frac{v_0 / 2}{6v_0 / 5d} = \frac{5}{6}\pi d. \tag{26}$$

Problem 4: Falling Stick A uniform stick of mass m and length l is suspended horizontally with end B at the edge of a table and the other end A is held by hand. Point A is suddenly released. At the instant after release:



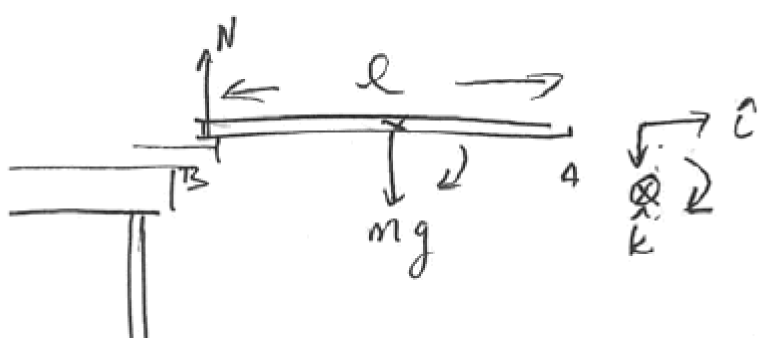
a) What is the torque about the end B on the table?

b) What is the angular acceleration about the end B on the table?

c) What is the vertical acceleration of the center of mass?

d) What is the vertical component of the hinge force at B? Does the hinge force have a horizontal component at the instant after release?

Solution: The torque diagram is shown in the figure below.



The toque about the point B is

$$\vec{\boldsymbol{\tau}}_B = \vec{\mathbf{r}}_{B,mg} \times m\vec{\mathbf{g}} = \frac{l}{2} mg\hat{\mathbf{k}} . \quad (1)$$

The moment of inertia about the axis passing perpendicularly through the end of the stick is

$$I_z = I_{cm} + m(l/2)^2 = (1/12)ml^2 + m(l/2)^2 = (1/3)ml^2 . \quad (2)$$

Therefore the torque equation $\vec{\tau}_B = I_B \vec{\alpha}$ becomes

$$(l/2)mg = (1/3)ml^2\alpha . \tag{3}$$

So the angular acceleration is

$$\alpha = 3g/2l . \tag{4}$$

Newton's Second Law in the vertical direction is

$$mg - N = ma_z . \tag{5}$$

The component of the angular acceleration about the end and the linear acceleration of the center of mass are related by

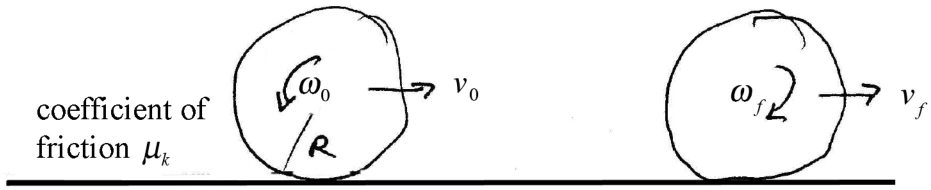
$$a_z = (l/2)\alpha = (l/2)(3g/2l) = 3g/4 . \tag{6}$$

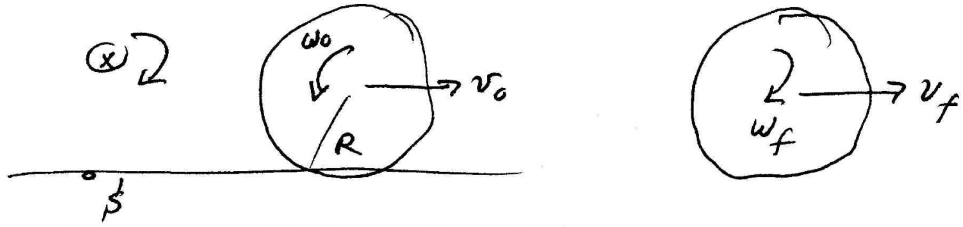
Therefore the vertical component of the hinge force at B (the normal force)

$$N = mg - ma_z = mg/4 . \tag{7}$$

Problem 5: Rotating Drum on a Surface with Friction

A solid uniform drum of radius R and mass m skids across a frictionless surface with an initial speed v_0. It is spinning backward with an initial angular speed ω_0. The drum skids into a rough area with coefficient of sliding friction μ_k. Eventually it rolls without slipping. Determine the change in kinetic energy ΔK of the drum.





$$L_0 = -I\omega_0 + R m v_0$$

$$L_f = I\omega_f + R m v_f$$

$$v_f = R\omega_f \qquad I = \tfrac{1}{2} m R^2$$

$$-\tfrac{1}{2} m R^2 \omega_0 + R m v_0 = \left(\tfrac{1}{2} m R^2 + m R^2\right)\omega_f$$

$$\omega_f = \frac{R m v_0 - \tfrac{1}{2} m R^2 \omega_0}{\tfrac{3}{2} m R^2}$$

$$= \frac{v_0 - \tfrac{1}{2} R\omega_0}{\tfrac{3}{2} R}$$

$$\Delta K = \left(\tfrac{1}{2} I \omega_f^2 + \tfrac{1}{2} m v_f^2\right) - \left(\tfrac{1}{2} I \omega_0^2 + \tfrac{1}{2} m v_0^2\right)$$

$$= \tfrac{1}{2} I (\omega_f^2) + \tfrac{1}{2} m R^2 \omega_f^2 - \tfrac{1}{2} I \omega_0^2 - \tfrac{1}{2} m v_0^2$$

$$= \tfrac{1}{2}\left(\tfrac{1}{2} m R^2 + m R^2\right)\omega_f^2 - \tfrac{1}{2} I \omega_0^2 - \tfrac{1}{2} m v_0^2$$

$$= \tfrac{1}{2}\left(\tfrac{3}{2} m R^2 \omega_f^2\right) - \tfrac{1}{2} I \omega_0^2 - \tfrac{1}{2} m v_0^2$$

$$\Delta K = \frac{1}{2}\left(\frac{3}{2} m R^2\right)\left(\frac{R m v_0 - \frac{1}{2} m R^2 \omega_0}{\frac{3}{2} m R^2}\right)^2 - \frac{1}{2}\left(\frac{1}{2} m R^2 \omega_0^2\right) - \frac{1}{2} m v_0^2$$

$$= \frac{1}{2}\,\frac{R^2 m^2 v_0^2 + \frac{1}{4} m^2 R^4 \omega_0^2 - R m v_0 m R^2 \omega_0}{\frac{3}{2} m R^2} - \frac{1}{2}\left(\frac{1}{2} m R^2 \omega_0^2\right) - \frac{1}{2} m v_0^2$$

$$= \frac{1}{2}\,\frac{2}{3} m v_0^2 + \frac{1}{2}\,\frac{2}{3}\,\frac{1}{4} m R^2 \omega_0^2 - \frac{1}{2}\,\frac{2}{3} R m v_0 \omega_0 - \frac{1}{4} m R^2 \omega_0^2 - \frac{1}{2} m v_0^2$$

$$= -\frac{1}{6} m v_0^2 - \frac{1}{6} m R^2 \omega_0^2 - \frac{1}{3} R m v_0 \omega_0$$

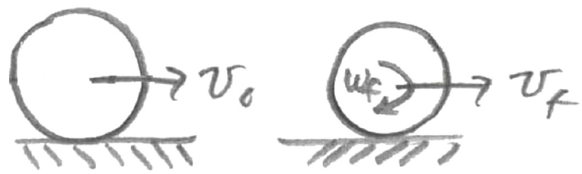
$$= -\frac{1}{6}\left(m v_0^2 + m R^2 \omega_0^2 + 2 R m v_0 \omega_0\right)$$

$$= -\frac{1}{6} m\left(v_0^2 + R^2 \omega_0^2 + 2 R v_0 \omega_0\right)$$

$$\Delta K = -\frac{1}{6} m \left(v_0 + R \omega_0\right)^2$$

Problem 6 Bowling Ball Torque Method About Center of Mass

A bowling ball of mass m and radius R is initially thrown down an alley with an initial speed v_0, and it slides without rolling but due to friction it begins to roll. The moment of inertia of the ball about its center of mass is $I_{\text{cm}} = (2/5)mR^2$. Using the torque method about the center of mass point, find the speed v_f of the bowling ball when it just start to roll without slipping?



Solution: We begin by coordinates for our angular and linear motion. Choose an angular coordinate θ increasing in the clockwise direction. Choose positive $\hat{\mathbf{k}}$ unit vector pointing into the page. Then the angular velocity vector is

$$\vec{\boldsymbol{\omega}} = \omega_z \hat{\mathbf{k}} = \frac{d\theta}{dt}\hat{\mathbf{k}},$$

and the angular acceleration vector is

$$\vec{\boldsymbol{\alpha}} = \alpha_z \hat{\mathbf{k}} = \frac{d^2\theta}{dt^2}\hat{\mathbf{k}}.$$

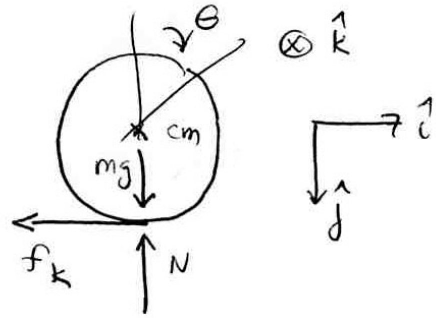
Choose the positive $\hat{\mathbf{i}}$ unit vector pointing to the right in the figure. Then the velocity of the center of mass is given by

$$\vec{\mathbf{v}}_{cm} = v_{cm,x}\hat{\mathbf{i}} = \frac{dx_{cm}}{dt}\hat{\mathbf{i}}$$

and the acceleration of the center of mass is given by

$$\vec{\mathbf{a}}_{cm} = a_{cm,x}\hat{\mathbf{i}} = \frac{d^2x_{cm}}{dt^2}\hat{\mathbf{i}}.$$

The free body force diagram is shown in the figure below.



At $t = 0$, when the ball is released, $\vec{\mathbf{v}}_{cm,0} = v_0\hat{\mathbf{i}}$ and $\vec{\boldsymbol{\omega}}_0 = \vec{\mathbf{0}}$, so the wheel is skidding and hence the friction force on the wheel due to the sliding of the wheel on the surface opposes the motion is kinetic friction and hence acts in the negative $\hat{\mathbf{i}}$-direction.

We will separately apply Newton's Second Law to the translational motion of the center of mass and we will apply our "rule to live by"

$$\vec{\boldsymbol{\tau}}_S = \frac{d\vec{\mathbf{L}}_S}{dt} \tag{1}$$

for the rotational motion about the center of mass.

Gravity exerts no torque about the center of mass, and the normal component of the contact force has a zero moment arm; the only force that exerts a torque is the frictional force, with a moment arm of R (the force vector and the radius vector are perpendicular).

The frictional force should be in the negative direction, to the left in the figure above. From the right-hand rule, the direction of the torque is into the page, and hence in the positive z-direction. Equating the z-component of the torque to the rate of change of angular momentum,

$$\tau_{cm} = R\,f_k = I_{cm}\alpha_z\,, \tag{2}$$

where f_k is the magnitude of the kinetic friction force and α is the z-component of the angular acceleration of the bowling ball. Note that Equation (2) results in a positive angular acceleration, which is consistent with the ball tending to rotate as indicated in the figure.

The friction force is also the only force in the horizontal direction, and will cause an acceleration of the center of mass,

$$a_{cm,x} = -f_k\,/\,m\,; \tag{3}$$

Note that the acceleration will be negative, as expected.

Now we need to consider the kinematics. The bowling ball will increase its angular speed as given in Equation (2) and decrease its linear speed as given in Equation (3);

$$\begin{aligned} \omega_z(t) &= \alpha_z\, t = \frac{Rf_k}{I_{\text{cm}}}\, t \\ v_{cm,x}(t) &= v_{cm,0} - \frac{f_k}{m}\, t. \end{aligned} \tag{4}$$

As soon as the ball stops slipping, the kinetic friction no longer acts, static friction is zero, and the ball moves with constant angular and linear velocity. Denote the time when this happens as t_f at which

$$v_{cm,f} = R\omega_{z,f} \tag{5}$$

and the relations in Equation (4) become

$$\begin{aligned} R^2\frac{f_k}{I_{cm}}t_f &= v_{cm,f} \\ v_{cm,0} - \frac{f_k}{m}t_f &= v_{cm,f} \end{aligned}. \tag{6}$$

We can now solve the first equation in Eq. (6) for t_f and find that

$$t_f = \frac{I_{cm}}{f_k R^2} v_{cm,f}\,. \tag{7}$$

We now substitute Eq. (7) into the second equation in Eq. (6) and find that

$$\begin{aligned} v_{cm,f} &= v_{cm,0} - \frac{f_k}{m}\frac{I_{cm}}{f_k R^2} v_{cm,f} \\ v_{cm,f} &= v_{cm,0} - \frac{I_{cm}}{m\,R^2} v_{cm,f} \end{aligned} \tag{8}$$

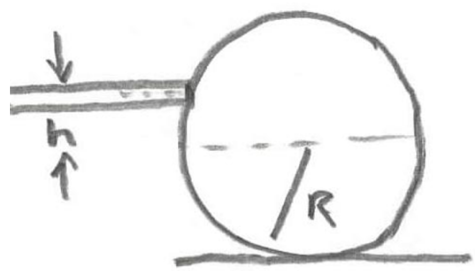
The second equation in (8) is easily solved for

$$v_{cm,f} = \frac{v_0}{1 + I_{cm}/mR^2} = \frac{5}{7}v_{cm,0}\,, \tag{9}$$

where we have used $I_{\text{cm}} = (2/5)m\,R^2$ for a uniform sphere.

Problem 7 Billiards

A spherical billiard ball of uniform density has mass m and radius R, and moment of inertia about the center of mass $I_{\text{cm}} = (2/5)mR^2$. The ball, initially at rest on a table, is given a sharp horizontal impulse by a cue stick that is held an unknown distance h above the centerline (see diagram below). The force applied by the cue to the ball is sufficiently large that you may ignore the friction between the ball and the table during the impulse (as any pool player knows). The ball leaves the cue with a given speed v_0 and an angular velocity ω_0. Because of its initial rotation, the ball eventually acquires a maximum speed of $(9/7)v_0$.



a) Using the fact that the angular impulse on the ball changes the angular momentum, and the linear impulse changes the linear momentum, find an expression for the angular velocity ω_0 of the ball just after the end of the impulse in terms of v_0, R, h and m.

a) Briefly explain why angular momentum is conserved about any point along the line of contact between the ball and the table *after* the impulse.

b) Use conservation of angular momentum about any point along the line of contact between the ball and the table, and your results from part a), to find the ratio h/R.

Solutions:

a) There are several ways to approach this problem. The method presented here avoids any calculation of the force or torque provided by friction, or the details of the force between the cue and the ball. This method will first consider the "collision" between the cue and the ball by taking the collision point as the origin for finding the angular momentum, as the force between the cue and the ball exerts no torque about this point, and we are given that the friction may be ignored during this interaction. After this collision, the angular momentum will be taken about the initial contact point between the ball and the felt. It will be helpful to infer, either from the figure and from the fact that $v_f > v_0$, that the ball is given overspin.

b) With respect to the point where the cue is in contact with the ball, note that the rotational angular momentum and the angular momentum due to the motion of the center of mass have different signs; the former is clockwise and the latter is counterclockwise.

The sum of these contributions to the angular momenta must sum to zero, and hence have the same magnitude;

$$I_{\text{cm}}\omega_0 = mv_0h \, . \tag{1}$$

While the ball is rolling and slipping, angular momentum is conserved about the contact between the ball and the felt. The initial and final angular momenta are

$$\begin{aligned}
L_{\text{initial}} &= mv_0R + I_{\text{cm}}\omega_0 \\
&= mv_0(R+h) \\
L_{\text{final}} &= mv_{\text{f}}R + I_{\text{cm}}\omega_{\text{f}} \\
&= mv_{\text{f}}R + (2/5)(mR^2)(v_{\text{f}}/R) \\
&= (7/5)mv_{\text{f}}R \\
&= (9/5)mv_0R,
\end{aligned} \tag{2}$$

where Equation (1) and the given relations $I_{\text{cm}} = (2/5)mR^2$ and $(9/7)v_0$ have been used. Setting the initial and final angular momenta equal and solving for h/R gives

$$\frac{h}{R} = \frac{4}{5} \, . \tag{3}$$

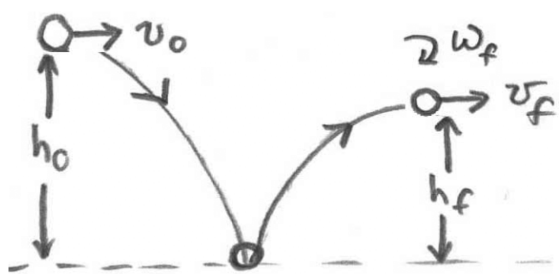
As an alternative, taking the angular momentum after the collision about the center of the ball, note that the time Δt between the moments the ball is struck and when it begins to roll without slipping is $\Delta v / (\mu_{\text{k}} g)$. But, if the angular momentum is taken about the center of the ball, after the ball is struck the angular impulse delivered to the ball by the friction force is

$$(\mu_{\text{k}} mg) R \, \Delta t = I_{\text{cm}} \left(\omega_f - \omega_0\right). \tag{4}$$

Eliminating Δt between these expressions leads to the same result obtained by equating the first and third expressions in (2).

Problem 8 Tennis Ball Bouncing off the Ground

A person, walking at a constant horizontal velocity with magnitude $v_0 = 5.0\times10^{-1}\, m\cdot s^{-1}$, drops a tennis ball of mass $m = 5.67\times10^{-2}\, kg$ from a height of $h_0 = 1.0\, m$. The ball rebounds to maximum height of $h_f = 6.0\times10^{-1}\, m$. The tennis ball has a radius, $R = 3.2\, cm$. When the tennis ball hits the ground, it slows to an unknown final horizontal velocity, v_f, it starts to spin, and it changes the vertical direction of its velocity. The collision is inelastic. After the collision, the ball rotates with a constant angular velocity, ω_f. When the ball rebounds to its maximum height, it has completed exactly one rotation. The moment of inertia of a uniform spherical of mass M and radius R about axis passing through center is $(2/3)MR^2$. You may neglect effects due to air resistance.



The goal of this problem is to calculate the change in the mechanical energy of the ball due to the collision.

a) How long does the ball take to reach its maximum height after the collision?

b) What is the angular velocity of the ball immediately after the collision?

c) What is the angular momentum of the ball about it's center of mass immediately after the collision? What is the kinetic energy of rotation immediately after the collision?

d) Friction exerts an impulse in the horizontal direction during the time, Δt_c, that the ball is in contact with the ground producing a change in the horizontal velocity Does this force do any work on the ball during the collision? Write down an integral expression relating the impulse to the change in horizontal momentum during the collision with the ground.

e) The force in part d) also exerts a torque and hence an angular impulse on the ball during the collision. This angular impulse produces angular rotation of the ball. Find an integral expression for the angular impulse in terms of the change in angular momentum of the ball about the center of mass.

f) Using your results from parts d) and e), what is the magnitude of the horizontal velocity of the ball, v_f, immediately after the collision?

g) What is the change in mechanical energy of the ball during the collision? You may want to compare the change in mechanical energy from the point the ball was released until the moment the ball reaches it's maximum height after the collision. Be sure to include the effects of rotational energy.

a) The time Δt_f for the ball to reach maximum height is equal to the time it takes to fall a height h_f when released from rest (or when moving with a horizontal velocity). Thus

$$h_f = \frac{1}{2} g (\Delta t_f)^2 \Rightarrow \Delta t_f = \left(\frac{2h_f}{g}\right)^{1/2}$$

b) Since the ball makes one full rotation in the time it takes to rise a height h_f

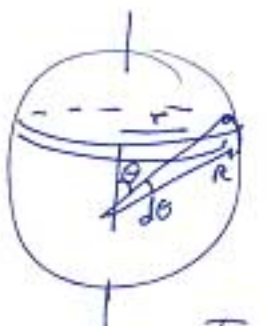
$$\omega_f = \frac{2\pi}{\Delta t_f} = \frac{2\pi}{\left(\frac{2h_f}{g}\right)^{1/2}} = 2\pi\left(\frac{g}{2h_f}\right)^{1/2}$$

$r = R\sin\theta$

$I_{cm,z} = \int dI_{ring}$, $\sigma = \dfrac{m}{4\pi R^2}$

$dI_{ring} = dm\, r^2$, $dm = \sigma\, da$

$da = 2\pi R^2 \sin\theta\, d\theta$ for a ring of thickness $R\,d\theta$



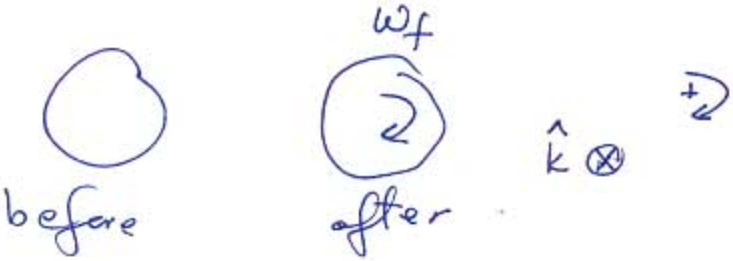
$dI_{ring} = \dfrac{m}{4\pi R^2}\, 2\pi R^2 \sin\theta\, d\theta\, r^2$

$dI_{ring} = \dfrac{m}{2} \sin\theta\, d\theta\, r^2 = \dfrac{m}{2} \sin^3\theta\, R^2\, d\theta$

$$I_{cm,z} = \int_{\theta=0}^{\theta=\pi} \frac{mR^2}{2} \sin^3\theta\, d\theta$$

$$I_{cm,z} = \frac{mR^2}{2}\int_0^{\pi} (\sin\theta)(1-\cos^2\theta)\, d\theta = \frac{mR^2}{2}\left(-\cos\theta\Big|_0^{\pi} + \frac{\cos^3\theta}{3}\Big|_0^{\pi}\right)$$

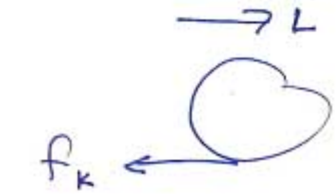
$$= \frac{mR^2}{2}\,\frac{4}{3} = \frac{2}{3} mR^2$$



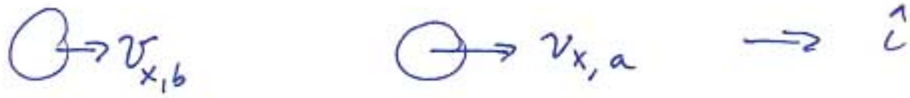
$$\vec{L}_{cm,\,after} = I_{cm}\,\omega_f\,\hat{k} = \left(\frac{2}{3} m R^2\right)\left(2\pi\left(\frac{g}{2h_f}\right)^{1/2}\right)\hat{k}$$

$$K_{rotation,\,f} = \frac{1}{2} I_{cm}\,\omega_f^2 = \left(\frac{1}{2}\right)\left(\frac{2}{3} m R^2\right)\left(\frac{4\pi^2 g}{2h_f}\right) = \frac{2}{3}\frac{m R^2 \pi^2 g}{h_f}$$

d)



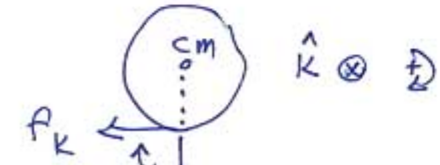
thus friction does work so the horizontal momentum changes



$$\text{Impulse} = \int \vec{F}\,dt = \Delta\vec{p} \quad \Rightarrow$$

$$-\int f_k\,dt\;\hat{i} = m(v_{x,a} - v_{x,b})\,\hat{i}$$

e)



$$\text{angular impulse} = \int \vec{\tau}\,dt = \Delta\vec{L}$$

$$\hat{k}: \quad R\int f_k\,dt = I_{cm}\,\omega_f$$

f) $$-\int f_k \, dt = m(v_{x,a} - v_{x,b})$$

$$R\int f_k \, dt = I_{cm}\,\omega_f$$

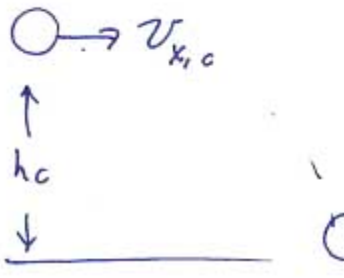
$$\Rightarrow \quad -mR(v_{x,a} - v_{x,b}) = I_{cm}\,\omega_f$$

$$mR\,v_{x,b} - I_{cm}\,\omega_f = mR\,v_{x,a}$$

$$v_{x,a} = v_{x,b} - \frac{I_{cm}\,\omega_f}{mR}$$

$$v_{x,a} = v_{x,b} - \frac{\frac{2}{3}mR^2}{mR}\,2\pi\left(\frac{g}{2h_f}\right)^{1/2}$$

$$= v_{x,b} - \frac{4}{3}R\,\pi\left(\frac{g}{2h_f}\right)^{1/2}$$



since the horizontal velocity of the ball does not change (ignore air resistance) when the ball is dropped

$$v_{x,c} = v_{x,b} \quad \Rightarrow$$

$$v_{x,a} = v_{x,c} - \frac{4}{3}R\pi\left(\frac{g}{2h_f}\right)^{1/2}$$

$$\Delta t_f = \left(\frac{2h_f}{g}\right)^{1/2} = \left(\frac{(2)(.6\,m)}{(9.8\frac{m}{s^2})}\right)^{1/2} = .35\ s$$

$$\omega_f = 2\pi\left(\frac{g}{2h_f}\right)^{1/2} = 18.0\ rad/sec$$

$$I_{cm} = \frac{2}{3} m R^2 = \left(\frac{2}{3}\right)\left(5.67\times10^{-2}\,kg\right)\left(3.2\times10^{-2}\,m\right)^2 = 3.87\times10^{-5}\,kg\text{-}m^2$$

$$L_{cm,a} = \left(\frac{2}{3} m R^2\right)\left(2\pi\left(\frac{g}{2h_f}\right)^{1/2}\right) = 6.9\times10^{-4}\ kg\cdot m^2\cdot s^{-1}$$

$$K_{rot,a} = 6.23\times10^{-3}\ J$$

$$v_{x,a} = v_{x,0} - \frac{4}{3}R\pi\left(\frac{g}{2h_f}\right)^{1/2} = v_{x,0} - \frac{I_{cm}\,\omega_f}{mR}$$

$$v_{x,a} = 1.2\times10^{-1}\ m\cdot s^{-1}$$

$$\Delta E_{mech} = E_f - E_0 = mgh_f + \frac{1}{2} I_{cm}\,\omega_f^2 + \frac{1}{2} m v_{x,f}^2 - \left(mgh_0 + \frac{1}{2} m v_{x,0}^2\right)$$

$$= mg(h_f - h_0) + \frac{1}{2} m\left(v_{x,f}^2 - v_{x,0}^2\right) + \frac{1}{2} I_{cm}\,\omega_f^2$$

$$\Delta E_{mech} = -2.23\times10^{-1}\ J$$