Final Project

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1 Assumption

- There are 2 main frame: inertial frame fixed on ground $\{s\}$, body-fixed frame $\{b\}$
- Consider that the booster is a uniform cylinder, as shown Fig.1, with height H, radius r, mass $\boldsymbol{m}$ and inertia tensor $\boldsymbol{I_0}$ w.r.t the fixed-body frame.

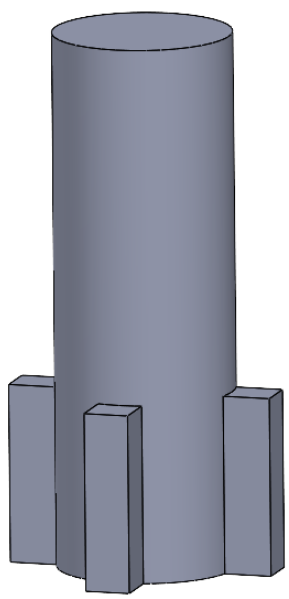

Figure 1: Booster model.

- The booster is considered a perfect rigid body, meaning its mass and inertia tensor remain constant throughout the fuel-burning process.
- The state of system q (position and intrinsic Euler angles) and their velocity $\dot{q}$ (linear and angular velocity) are fully observed by the tracking system.
- The simulation environment is set up to navigate the booster from random position in mid air to the initial launcher as shown in Fig.2.

Figure 2: Simulation setup.

2 Kinematic Presentation

This section explains the a kinematic model describing the position, velocity, orientation rotation of the booster in three-dimensional space. This state is measured by sensors system. The motion of booster is decomposed into linear motion of Center of Mass (COM) and rotation motion around COM.

- **Position Vector**: Define the position of COM w.r.t s space

$$\mathbf{r}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$$

- **Linear Velocity**: Linear velocity of COM in s space

$$\mathbf{v}(t) = \begin{bmatrix} v_x(t) \\ v_y(t) \\ v_z(t) \end{bmatrix}$$

- **Orientation Representation (Euler Angles)**: Use intrinsic Euler angles (ϕ, θ, ψ)

to describe the booster's orientation

$$\begin{cases} \phi(t) & \text{(Roll)} \\ \theta(t) & \text{(Pitch)} \\ \psi(t) & \text{(Yaw)} \end{cases}$$

From euler angles, we can compute the rotation matrix

$$R_{sb} = R_z(\psi) R_y(\theta) R_x(\phi)$$

- **Angular Velocity**: angular velocity in body frame measured from IMU

$$\omega^{\mathbf{b}}(t) = \begin{bmatrix} \omega_x^b(t) \\ \omega_y^b(t) \\ \omega_z^b(t) \end{bmatrix}$$

3 Forward Dynamic and Simulation

This section introduces the method for updating the state of a dynamic system given the applied forces and torques. Newton–Euler equations are used to derive the equation of motion for rigid body in the inertial fixed frame $\{s\}$. Moreover, the Euler method is applied to solve the ODEs and update the system's state.

3.1 Translation Motion

- The translational motion of the booster in 3D space is controlled using external thrust forces applied along the primary axes of the body-fixed frame $\{b\}$.

$$F^b = F_{thrust} = \begin{bmatrix} F_x^b \\ F_y^b \\ F_z^b \end{bmatrix}$$

- Equation of motion in the fixed frame $\{s\}$:

$$\begin{aligned} F^s + mg &= \frac{dp}{dt} \\ \implies \Delta p &= (F + mg)\Delta t \\ \implies p_{t+1} &= p_t + \Delta p \\ \implies v_{t+1} &= \frac{p_{t+1}}{m} \\ \implies x_{t+1} &= x_t + v_{t+1}\Delta t \end{aligned}$$

whereas p is the linear moment and $F^s = R_{sb} F^b$ is thrust force in fixed frame.

3.2 Rotation Motion

- The rotational motion of the booster is controlled by moments along body-fixed frame, which is a combination of torque generated by aerodynamic surfaces like grid fins and force generated by thrust from engines.

$$M^b = M_{ext} + r_{thrust} \times F_{thrust} = \begin{bmatrix} M_{xb} \\ M_{yb} \\ M_{zb} \end{bmatrix}$$

  where r_{thrust} is the vector from COM to thrust generation point in b frame.

- Equation of motion in the fixed frame $\{s\}$:

$$\begin{aligned} M^s &= \frac{dH}{dt} \\ &\implies \Delta H = M_s \Delta t \\ &\implies H_{t+1} = H_t + \Delta H = R_t I_0 R_t^T \omega \\ &\implies \omega_{t+1} = R_t I_0^{-1} R_t^T H_{t+1} \\ &\implies \dot{R}_{t+1} = \ \omega_{t+1} \times R_t = [\omega_{t+1}] R_t \\ &\implies R_{t+1} = R_t + \dot{R}_{t+1} \Delta t \end{aligned}$$

  where H is the angular momentum at C.O.M, ω is angular velocity, and $M^s = R_{sb} M^b$ is external moment at C.O.M observed in inertial fixed frame.

4 Inverse Dynamics and Control

In this problem, the full-state feedback LQR controller combined with g-compensation is used to navigate the booster. The parameters is tuned by given dynamics model to obtain optimal trajectory. The strategy for controlling rigid body includes the following steps:

- Step1: Measure error between current state and target pose.
- Step2: Calculated the required wrench (force + torque) in $\{s\}$ frame by controller.
- Step3: Map required wrench to control input (thrust + moment) in $\{b\}$ frame.

4.1 LQR Formula

- The LQR structure is demonstrated in Fig.3. We can apply LQR for 6 general coordinates system.

$$\mathbf{x} = \mathbf{A}\dot{\mathbf{x}} + \mathbf{B}\mathbf{u}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

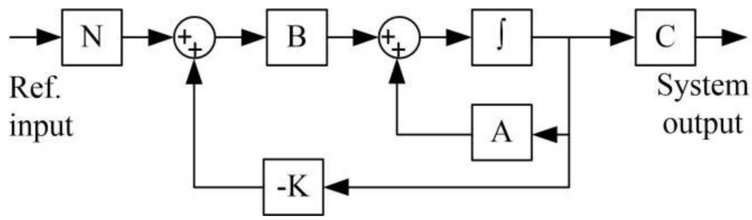

Figure 3: Structure of LQR controlled system.

$$\mathbf{B} = \begin{bmatrix} 0 \\ \frac{1}{m} \text{ or } \frac{1}{I} \end{bmatrix}$$

$$\mathbf{K} = \begin{bmatrix} K_1 & K_2 \end{bmatrix}$$

$$\mathbf{N} = \frac{\mathbf{-1}}{\mathbf{C(A-BK)^{-1}B}} = \begin{bmatrix} K_1 & 0 \end{bmatrix}$$

$$\mathbf{u} = \mathbf{NR} - \mathbf{KX} = K_1(x - x) - K_2\dot{x} = K_1 e(t) - K_2 v(t)$$

4.2 Translation

- The purpose is moving COM of booster to reference position on launcher. Control input in s frame is calculated:

$$F^s = u + G_{com} = K_1[r^s_d(t) - r(t)] - K_2 v(t) + mg$$

- The control input is then mapped to b frame

$$F^b = R^T_{sb} F^s$$

4.3 Rotation

- The purpose is keep booster in vertical direction. The control input is caculated directly in $\{b\}$ frame.

$$M^b = K_1(0 - \text{euler angle}) - K_2 w^b(t)$$

4.4 Wrench mapping

- The desirable control input in b frame is mapped to thrust and moment command of booster:

$$\mathbf{T} = \mathbf{J}^\dagger \begin{bmatrix} F^b \\ M^b \end{bmatrix}$$

where $\mathbf{J} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & r_{thrust} & 0 & 1 & 0 & 0 \\ -r_{thrust} & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ end $\mathbf{J}^\dagger$ is the pseudo inverse of $\mathbf{J}$

- Finally the command is clipped to lower and upper bound to compensate saturation.

5 Result

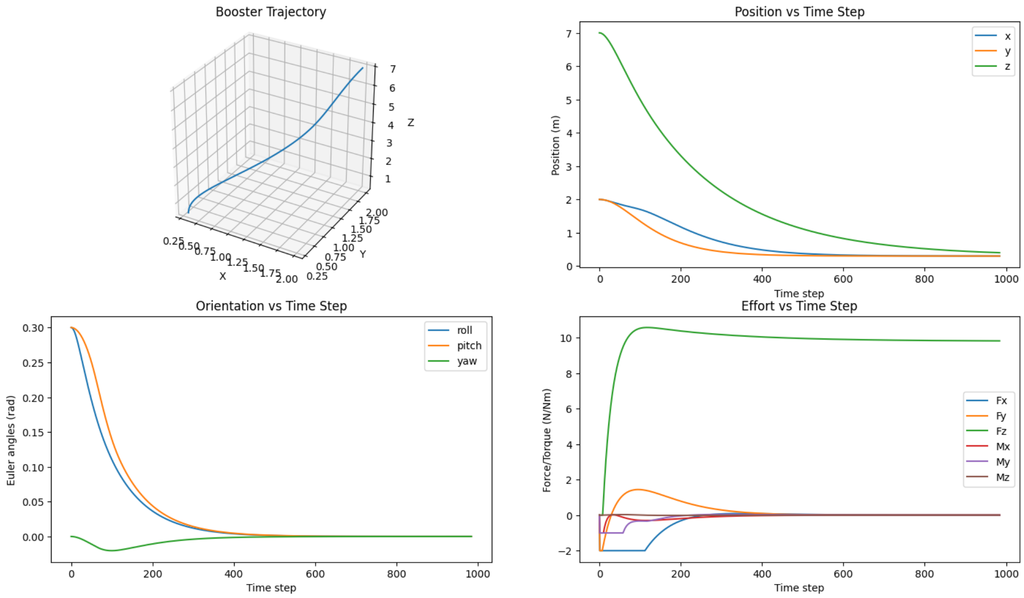

Figure 4: Booster trajectory and control effort.

Fig.4 shows the result of simulation control. The top left is trajectory of the booster; the top right is position vs time; the bottom left is Euler angles vs time; the bottom right is control effort. Because the system behaves as an invert pendulum, the orientation should be prior to rapidly obtain the stable position. The position control is considered to prioritize the control efforts, which minimized required fuel for booster. Therefore, the convergence time of position control is slower compared to that of orientation.