Background/Motivation

I started thinking about this problem while I was swimming. Each day, I would add another lap, and I would keep track of my progress by making a fraction of the laps I'd completed over the total laps for the day and seeing if it simplified. For example, if the total number of laps were 30, every lap number that shares a factor with 30 would have a fraction that simplifies, but if the total laps were 31, no lap number would simplify because 31 is prime. While working out these fractions, I noticed in the case of 30 that the only lap numbers that didn't simplify were themselves prime. This meant that every number less than 30 that doesn't share a factor with 30 is prime. I began to wonder how many such numbers exist. After some thought, I conjectured that there were 9 such numbers, the largest of which is 30. Below is my attempt at a formal proof.

Let

$$S = \{n \in \mathbb{N}\backslash\{1\} : \forall_{m \in \mathbb{N} \cap (1,n)} \mathrm{GCD}(n, m) = 1 \implies m \text{ is prime}\}$$

that is, let S be the set of all natural numbers n, except for 1, such that for all natural numbers m between 1 and n, if m and n are coprime (share no factors other than 1), then m must be prime.

Conjecture:

The largest element of S is 30.

Lemma 1:

The only odd element of S is 3.

Proof of Lemma 1:

Assume $x \in \mathbb{N}\backslash\{1\}$ is odd. Then, $x = 2k + 1$ for some $k \in \mathbb{N}$. Now, $\forall_{k\in\mathbb{N}} \mathrm{GCD}(2k, 2k + 1) = 1$ but $2k$ is only prime for $k = 1$ (since 2 is the only even prime). Therefore, the only possible odd element of S occurs when $k = 1 \implies x = 2k + 1 = 3$. To show that $3 \in S$, we need only look at numbers greater than 1 but less than 3, namely, 2. 2 is coprime with 3, but 2 is prime, so $3 \in S$

Lemma 2:

$$\text{For prime } p,\ p \nmid x \wedge x > p^2 \implies x \notin S.$$

Proof of Lemma 2:

Assume p does not divide x and $x > p^2$. Since the only factors of p^2 are p and 1 and p is not a factor of x, p and x share no common factors other than 1. That is, $\text{GCD}(x, p^2) = 1$, but p^2 is not prime as it has a factor $p > 1$. This means $x \notin S$.

Before considering the Conjecture, we enumerate elements of S between 1 and 30. Because of Lemma 1, we need only consider even natural numbers and 3. Because of Lemma 2, for numbers greater than the square of a prime p, we need only consider multiples of p.

We start with $n = 2$. Since $\{m : m \in \mathbb{N} \cap (1, n)\} = \emptyset$, the condition is vacuously true, so $2 \in S$.

For $n = 4$, $\{m : m \in \mathbb{N} \cap (1, n)\} = \{2, 3\}$. Of these, only 3 is coprime with 4, but 3 is prime, so $4 \in S$.

For $n = 6$, $\{m : m \in \mathbb{N} \cap (1, n)\} = \{2, 3, 4, 5\}$. Of these, 3 and 5 are coprime with 6, but these are both prime, so $6 \in S$.

For $n = 8$, $\{m : m \in \mathbb{N} \cap (1, n)\} = \{2, 3, 4, 5, 6, 7\}$. Of these, 3, 5, and 7 are coprime with 8, but these are all prime, so $8 \in S$.

Now, $n = 10$ is excluded because of Lemma 2 since $10 > 9 = 3^2$ and $3 \nmid 10$. To see this in more detail, consider GCD(9,10)=1, but 9 is not prime, so $10 \notin S$. This means moving forward, we need only consider multiples of 2 and 3, that is, multiples of 6.

For $n = 12$, $\{m : m \in \mathbb{N} \cap (1, n)\} = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$. Of these, 5, 7, and 11 are coprime with 12, but these are all prime, so $12 \in S$.

For $n = 18$, the numbers less than and coprime with 18 are 5, 7, 11, 13, and 17. Since these are all prime, $18 \in S$.

For $n = 24$, the numbers less than and coprime with 24 are 5, 7, 11, 13, 17, 19, and 23. Since these are all prime, $24 \in S$.

For $n = 30$, the numbers less than and coprime with 30 are 5, 7, 11, 13, 17, 19, 23, and 29. Since these are all prime, $30 \in S$.

Now, $n = 36 \notin S$ by Lemma 2 because $36 > 25 = 5^2 \wedge 5 \nmid 36$. This means moving forward, we need only consider multiples of 2, 3, and 5, i.e. 30.

So far, we have $S = \{2, 3, 4, 6, 8, 12, 18, 24, 30, ...\}$

The next multiple of 30 is $n = 60$. However, $60 > 49 = 7^2 \wedge 7 \nmid 60$, so $60 \notin S$ by Lemma 2. This means we need to consider multiples of 2, 3, 5, and 7, i.e. 210. However, $210 > 121 = 11^2 \wedge 11 \nmid 210$, so $210 \notin S$ by Lemma 2. As can be seen, if the product of the first k primes is greater than the square of the k+1th prime, the product will not be in S.

Returning to the initial conjecture, the largest element of S is 30 is equivalent to the following:

$$\forall_{k \in \mathbb{N} \setminus \{1,2,3\}} \prod_{i=1}^{k} P_i > P_{k+1}^2$$

where P_n is the sequence of consecutive primes $P_n = \{2, 3, 5, 7, 11, 13, 17, 19, ...\}$. This is a famous result called Bonse's inequality which has many known proofs. Therefore, the Conjecture is true and

$$S = \{2, 3, 4, 6, 8, 12, 18, 24, 30\}$$