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Quiz 4

4) $\cdot\ \lambda = 4$

$$\det(A-\lambda I) = \det\begin{pmatrix} -4 & 1 \\ \alpha & b-4 \end{pmatrix} = -4b+16+\alpha = \alpha-4b+16 \quad (1)$$

$\cdot\ \lambda = 7$

$$\det(A-\lambda I) = \det\begin{pmatrix} -7 & 1 \\ \alpha & b-7 \end{pmatrix} = -7b+49+\alpha = \alpha-7b+49 \quad (2)$$

$$(2) - (1) = -7b+4b+33 = 0 \Rightarrow 3b = 33 \Rightarrow b = 11,\ \alpha = -28$$

Quiz 5

5) $$\det(A-\lambda I) = \det\begin{pmatrix} 1-\lambda & 0 & 2 \\ -1 & 1-\lambda & 3 \\ 0 & 0 & 2-\lambda \end{pmatrix}$$

$$= (1-\lambda)\begin{vmatrix} 1-\lambda & 3 \\ 0 & 2-\lambda \end{vmatrix} - 0\begin{vmatrix} -1 & 3 \\ 0 & 2-\lambda \end{vmatrix} + 2\begin{vmatrix} -1 & 1-\lambda \\ 0 & 0 \end{vmatrix}$$

$$= (1-\lambda)(1-\lambda)(2-\lambda)$$

So $\lambda_2 = 1$, $\lambda_2 = 2$

$\cdot\ \lambda = 1$ $\quad |A-\lambda I| = \begin{pmatrix} 0 & 0 & 2 \\ -1 & 0 & 3 \\ 0 & 0 & 1 \end{pmatrix}$ $\quad$ AM = 2, GM = 1

$\cdot\ \lambda = 2$ $\quad |A-\lambda I| = \begin{pmatrix} -1 & 0 & 2 \\ -1 & -1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad$ AM = 2, GM = 1

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Quiz 6

6) $\det(A-\lambda I) = \begin{pmatrix} 2-\lambda & \alpha \\ -1 & 1-\lambda \end{pmatrix} = (2-\lambda)(1-\lambda) - \alpha =$

$$= \lambda^2 - 3\lambda + 2 - \alpha$$

$$\lambda_{1,2} = \frac{-(-3) \pm \sqrt{(-3)^2 - 4\cdot 1\cdot(2-\alpha)}}{2}$$

The root must be ≥ 0

$$(-3)^2 - 4(2-\alpha) \geq 0 \Rightarrow 9-8+4\alpha \geq 0 \Rightarrow 4\alpha \geq -1 \Rightarrow$$

$$\Rightarrow \alpha \geq -\frac{1}{4}$$

Exercise 7

a.

In [58]:
```
%display latex
```

In [59]:
```
#7.a
```

In [60]:
```
J = diagonal_matrix(RDF, [RDF.random_element(1,5) for _ in range(5)])
B = random_matrix(RDF, 5)
A = B.transpose() * J * B
A[1,1] = 7
A[2,2] = 9
A.is_positive_definite()
```

Out[60]: True

In [61]:
```
A.n(digits=4)
```

Out[61]:

$$\begin{pmatrix} 4.352 & -1.067 & 5.262 & -2.822 & 0.7164 \\ -1.067 & 7.000 & -2.725 & -0.5583 & -0.3226 \\ 5.262 & -2.725 & 9.000 & -3.221 & 0.9947 \\ -2.822 & -0.5583 & -3.221 & 3.887 & -0.6321 \\ 0.7164 & -0.3226 & 0.9947 & -0.6321 & 3.073 \end{pmatrix}$$

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b.

In [62]:
```
# 7.b
```

In [63]:
```
C = A.cholesky()
C.n(digits=4)
```

Out[63]:

$$\begin{pmatrix} 2.086 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ -0.5116 & 2.596 & 0.0000 & 0.0000 & 0.0000 \\ 2.522 & -0.5525 & 1.527 & 0.0000 & 0.0000 \\ -1.353 & -0.4817 & -0.04904 & 1.350 & 0.0000 \\ 0.3434 & -0.05659 & 0.06368 & -0.1420 & 1.711 \end{pmatrix}$$

In [64]:
```
(C * C.transpose()).n(digits=4)
```

Out[64]:

$$\begin{pmatrix} 4.352 & -1.067 & 5.262 & -2.822 & 0.7164 \\ -1.067 & 7.000 & -2.725 & -0.5583 & -0.3226 \\ 5.262 & -2.725 & 9.000 & -3.221 & 0.9947 \\ -2.822 & -0.5583 & -3.221 & 3.887 & -0.6321 \\ 0.7164 & -0.3226 & 0.9947 & -0.6321 & 3.073 \end{pmatrix}$$

In [65]:
```
A.n(digits=4) == (C * C.transpose()).n(digits=4)
```

Out[65]: True

c.

In [66]:
```
# 7.c
```

In [67]:
```
C.SVD()
```

Out[67]:

$$\left(\begin{pmatrix} -0.48729952161831336 & 0.1874264426926893 & -0.06113100053558205 & 0.07039514324868244 & -0.8477723924442482 \\ 0.2963637756695601 & 0.8927257000045578 & -0.03844709326385946 & -0.33723578230577855 & 0.0017847056967345148 \\ -0.7441454076116214 & 0.04814913126928053 & -0.14353128217372277 & -0.5079824093107066 & 0.4065487871793549 \\ 0.32797486448134633 & -0.40500459188481114 & -0.03374463658565982 & -0.7818521799585906 & -0.34054696741618723 \\ -0.11570561481804671 & 0.039561279569383906 & 0.986430378857194 & -0.10944218320417694 & -0.004963069772491442 \end{pmatrix}, \begin{pmatrix} 3.886506814144289 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 2.6206734020743596 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.7081099531862545 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 1.3112357023495487 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.8374217997012765 \end{pmatrix}, \begin{pmatrix} -0.9078920677995661 & 0.2354629423126925 & -0.05005599867849093 & 0.04424887830207018 & -0.3403316467644242 \\ 0.26477504609965846 & 0.9476872018893553 & -0.035165354955055836 & -0.16162825934692163 & -0.06650448052137617 \\ -0.29844257991027 37 & 0.036599103651637425 & -0.09058328991971396 & -0.5677144490866121 & 0.7609779917934567 \\ 0.11816824965252153 & -0.21080703325314099 & -0.10866943688342587 & -0.7932379281771492 & -0.5482346233992631 \\ -0.05093583669785704 & 0.025827708289365942 & 0.9880503104002067 & -0.14280138706960 08 & -0.010139920781880977 \end{pmatrix} \right)$$

d.

In [68]:
```
# 7.d
```

In [69]:
```
U, S, V = C.SVD()
X = U * S * U.transpose()
X.n(digits=4)
```

Out[69]:

$$\begin{pmatrix} 1.630 & -0.1512 & 1.112 & -0.6470 & 0.1290 \\ -0.1512 & 2.582 & -0.5098 & -0.2223 & -0.05711 \\ 1.112 & -0.5098 & 2.670 & -0.5865 & 0.1690 \\ -0.6470 & -0.2223 & -0.5865 & 1.749 & -0.1327 \\ 0.1290 & -0.05711 & 0.1690 & -0.1327 & 1.734 \end{pmatrix}$$

In [72]:
```
A.n(digits=4) == (X*X).n(digits=4)
```

Out[72]: True

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Exercise 8

a.

In [1]:
```
# 8.a
```

In [41]:
```
A = matrix(CDF, 5, 5)
for i in range(5):
    for j in range(5):
        A[i,j] = ((i + 1) * (j + 1))/((i + 1) + 2 * (j + 1) + 2)
A.n(digits=4)
```

Out[41]:

$$\begin{pmatrix} 0.2000 & 0.2857 & 0.3333 & 0.3636 & 0.3846 \\ 0.3333 & 0.5000 & 0.6000 & 0.6667 & 0.7143 \\ 0.4286 & 0.6667 & 0.8182 & 0.9231 & 1.000 \\ 0.5000 & 0.8000 & 1.000 & 1.143 & 1.250 \\ 0.5556 & 0.9091 & 1.154 & 1.333 & 1.471 \end{pmatrix}$$

b.

In [ ]:
```
# 8.b
```

In [42]:
```
U1, S1, V1 = A.SVD()
A.SVD()
```

Out[42]:

$$\left(\begin{pmatrix} -0.17592812201757424 & -0.5472703943604452 & 0.6814367801164214 & -0.43059474769982503 & -0.1406289063379944 \\ -0.3185009147216849 & -0.5502763547924007 & -0.0372945534741662 & 0.6215110863698877 & 0.4561646484311581 7 \\ -0.4369128433229614 & -0.2891207906917357 & -0.4867628265855173 4 & 0.003925228257817825 5 & -0.6989727312040381 \\ -0.5370917468986874 & 0.10129507304788093 & -0.31853528333040526 & -0.5807430334299879 & 0.5123910284008639 \\ -0.6230939623825456 & 0.5512164052157376 & 0.4425497202731269 & 0.3017188334829394 3 & -0.1450168240975156 \end{pmatrix}, \begin{pmatrix} 4.0615186352 38639 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.08838336113677744 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.002854582544775180 7 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 6.152339335723195 \times 10^{-05} & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 5.600188412203899 \times 10^{-07} \end{pmatrix}, \begin{pmatrix} -0.23225552508888156 & -0.6778376296234678 & 0.6433558135468509 & -0.26476583399597126 & -0.05085144300817154 \\ -0.3685600024594068 & -0.47640708731473047 & -0.339770888308623 & 0.661750380669953 & 0.2895548115375707 \\ -0.45876035220462325 & -0.133833445226373 & -0.4878555365423757 & -0.3136729561322241 & -0.6597225176088442 \\ -0.523012018089227 & 0.20345408113860314 & -0.12660527962599516 & -0.4948089882775692 & 0.6513064056900042 \\ -0.5711552011775203 & 0.5042494232318153 & 0.4654214426401225 & 0.38569266073207864 & -0.23267640087994398 \end{pmatrix}\right)$$

In [43]:
```
kr = 2
A2 = matrix(U1[:,:kr] * S1[:kr,:kr] * V1[:, :kr].transpose())
A2.n(digits=4)
```

Out[43]:

$$\begin{pmatrix} 0.1987 & 0.2864 & 0.3343 & 0.3639 & 0.3837 \\ 0.3334 & 0.4999 & 0.6000 & 0.6667 & 0.7143 \\ 0.4295 & 0.6662 & 0.8175 & 0.9229 & 1.001 \\ 0.5006 & 0.7997 & 0.9995 & 1.143 & 1.250 \\ 0.5547 & 0.9095 & 1.154 & 1.333 & 1.470 \end{pmatrix}$$

c.

In [ ]:
```
# 8.c
```

In [45]:
```
U2, S2, V2 = A2.SVD()
```

In [46]:
```
S1 == S2
```

Out[46]: False

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Exercise 9

9) a) $\det(A-\lambda I) = \begin{bmatrix} 3-\lambda & 1 & 0 \\ 1 & 2-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{bmatrix}$

$= (3-\lambda)\begin{vmatrix} 2-\lambda & -1 \\ -1 & 3-\lambda \end{vmatrix} - 1\begin{vmatrix} 1 & -1 \\ 0 & 3-\lambda \end{vmatrix} + 0\begin{vmatrix} 1 & 2-\lambda \\ 0 & -1 \end{vmatrix}$

$= (3-\lambda)\big((2-\lambda)(3-\lambda) - (-1)(-1)\big) - 1\big(1(3-\lambda) - (-1)\cdot 0\big)$

$= (3-\lambda)\big((2-\lambda)(3-\lambda) - 1\big) - (3-\lambda)$

$= (2-\lambda)(3-\lambda)^2 - (3-\lambda) - (3-\lambda)$

$= (3-\lambda)\big((2-\lambda)(3-\lambda) - 2\big)$

$= \underbrace{(3-\lambda)}_{\lambda_1 = 3}\underbrace{(\lambda^2 - 5\lambda - 4)}$

$\hookrightarrow \lambda_{2,3} = \dfrac{-(-5) \pm \sqrt{(-5)^2 - 4\cdot 1\cdot(-4)}}{2\cdot 1}$

$= \dfrac{5 \pm \sqrt{9}}{2} = \dfrac{5 \pm 3}{2}$

$\lambda_2 = 4 \quad , \lambda_3 = 1$

b) $\cdot\ \lambda = 1 \qquad (A-\lambda I)\vec{x} = 0$

$\begin{pmatrix} 3-1 & 1 & 0 \\ 1 & 2-1 & -1 \\ 0 & -1 & 3-1 \end{pmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \Rightarrow \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & -1 \\ 0 & -1 & 2 \end{pmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \Rightarrow$

$\Rightarrow \left.\begin{aligned} 2x_1 + x_2 &= 0 \\ x_1 + x_2 - x_3 &= 0 \\ -x_2 + 2x_3 &= 0 \end{aligned}\right\} \quad \begin{aligned} x_1 &= -\frac{x_2}{2} \\ x_3 &= x_1 + x_2 = \frac{x_2}{2} \\ x_2 &= 2x_3 \end{aligned}$

$\vec{v}_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1 \\ 1/2 \end{bmatrix} x_2$

$\vec{v}_1 = \dfrac{1}{|\vec{v}_1|} = \dfrac{1}{\sqrt{(-1/2)^2 + 1^2 + (1/2)^2}} = \dfrac{1}{\sqrt{\frac{3}{2}}}$

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$$\vec{v}_1 = \begin{bmatrix} -\frac{1}{2\sqrt{\frac{3}{2}}} \\ \frac{1}{\sqrt{\frac{3}{2}}} \\ \frac{1}{2\sqrt{\frac{3}{2}}} \end{bmatrix}$$

• $\lambda = 3 \quad (A - \lambda I)\vec{x} = 0$

$$\begin{pmatrix} 3-3 & 1 & 0 \\ 1 & 2-3 & -1 \\ 0 & -1 & 3-3 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \Rightarrow \begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & 0 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \Rightarrow$$

$$\Rightarrow \begin{matrix} x_2 = 0 \\ -x_2 - x_3 = 0 \\ x_2 = 0 \end{matrix} \qquad \vec{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$ Μηδενικός vector

• $\lambda = 4 \quad (A - \lambda I)\vec{x} = 0$

$$\begin{pmatrix} 3-4 & 1 & 0 \\ 1 & 2-4 & -1 \\ 0 & -1 & 3-4 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \Rightarrow \begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & -1 \\ 0 & -1 & -1 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \Rightarrow$$

$$\Rightarrow \left.\begin{matrix} -x_1 + x_2 = 0 \\ x_1 - 2x_2 - x_3 = 0 \\ -x_2 - x_3 = 0 \end{matrix}\right\} \begin{matrix} x_1 = x_2 \\ x_2 = \frac{x_1 - x_3}{2} \\ x_3 = -x_2 \end{matrix}$$

$$\vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} x_2 = \frac{1}{|\vec{v}_3|} = \frac{1}{\sqrt{1^2 + 1^2 + (-1)^2}} = \frac{1}{\sqrt{3}}$$

$$\vec{v}_3 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ -1/\sqrt{3} \end{bmatrix}$$

C) $$V^T \cdot A \cdot V = \begin{bmatrix} -\frac{1}{2\sqrt{\frac{3}{2}}} & \frac{1}{\sqrt{\frac{3}{2}}} & \frac{1}{2\sqrt{\frac{3}{2}}} \\ 0 & 0 & 0 \\ 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \end{bmatrix} \begin{pmatrix} 3 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 3 \end{pmatrix} \begin{bmatrix} \frac{1}{-2\sqrt{\frac{3}{2}}} & 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{\frac{3}{2}}} & 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{2\sqrt{\frac{3}{2}}} & 0 & -\frac{1}{\sqrt{3}} \end{bmatrix}$$

$$= \begin{pmatrix} -\frac{3}{2\sqrt{\frac{3}{2}}} + \frac{1}{\sqrt{\frac{3}{2}}} & \cancel{-\frac{1}{2\sqrt{\frac{3}{2}}} + \frac{1}{\sqrt{\frac{3}{2}}} + \frac{1}{2\sqrt{\frac{3}{2}}}} \xrightarrow{} 0 & -\frac{1}{\sqrt{\frac{3}{2}}} + \frac{3}{2\sqrt{\frac{3}{2}}} \\ 0 & 0 & 0 \\ 3/\sqrt{3} + 1/\sqrt{3} & \cancel{1/\sqrt{3} + 2/\sqrt{3} - 1/\sqrt{3}} & -1/\sqrt{3} - 3/\sqrt{3} \end{pmatrix} \cdot$$

$$\cdot \begin{bmatrix} \frac{1}{-2\sqrt{\frac{3}{2}}} & 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}/2} & 0 & \frac{1}{\sqrt{3}} \\ 2\frac{1}{\sqrt{\frac{3}{2}}} & 0 & -1/\sqrt{3} \end{bmatrix}$$

$$= \begin{pmatrix} \left(-\frac{1}{2\sqrt{\frac{3}{2}}}\right)\left(-\frac{1}{2\sqrt{\frac{3}{2}}}\right) + 0 \cdot \left(\frac{1}{\sqrt{\frac{3}{2}}}\right) + \left(\frac{1}{\sqrt{\frac{3}{2}}}\right) \cdot \left(\frac{1}{2\sqrt{\frac{3}{2}}}\right) & 0 & \left(\frac{1}{2\sqrt{\frac{3}{2}}}\right)\left(\frac{1}{\sqrt{3}}\right) + 0\left(\frac{1}{\sqrt{\frac{3}{2}}}\right) + \left(\frac{1}{\sqrt{\frac{3}{2}}}\right)\left(\frac{1}{2\sqrt{\frac{3}{2}}}\right) \\ 0 & 0 & 0 \\ \left(\frac{4}{\sqrt{3}}\right)\left(\frac{1}{-2\sqrt{\frac{3}{2}}}\right) + \left(\frac{2}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{\frac{3}{2}}}\right) + \left(-\frac{4}{\sqrt{3}}\right)\left(\frac{1}{2\sqrt{\frac{3}{2}}}\right) & 0 & \left(\frac{4}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}}\right) + \left(\frac{2}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}}\right) + \left(-\frac{4}{\sqrt{3}}\right)\left(-\frac{1}{\sqrt{3}}\right) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{6} + \frac{1}{3} & 0 & \frac{\sqrt{2}}{6} + \frac{1}{3} \\ 0 & 0 & 0 \\ -\cancel{\frac{2\sqrt{2}}{3}} + \cancel{\frac{2\sqrt{2}}{3}} - \frac{2\sqrt{2}}{3} & 0 & \frac{4}{3} + \frac{2}{3} + \frac{4}{3} \end{pmatrix}$$

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$$= \begin{pmatrix} 1/2 & 0 & \frac{\sqrt{2}+2}{6} \\ 0 & 0 & 0 \\ -\frac{2\sqrt{2}}{3} & 0 & 10/3 \end{pmatrix}$$

Obviously $\det(v^T \cdot A \cdot v) = 0$ so it is positive and semi-definite matrix

d) We know that $f(\lambda) = \det(A - \lambda I)$ which is a property of $f(\lambda)$.

$f(A)$ would equal $\det(A - AI)$ which equals to zero as $\det(A - AI) = 0$ also as a property

Another way isby using the polynomial

$f(A) = -A^3 + 8A^2 - 19A + 12I$

$$A^2 = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 3 \end{pmatrix} \begin{pmatrix} 3 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 3 \end{pmatrix} = \begin{pmatrix} 10 & 5 & -1 \\ 5 & 6 & -5 \\ -1 & -5 & 10 \end{pmatrix}$$

$$A^2 \cdot A = \begin{pmatrix} 10 & 5 & -1 \\ 5 & 6 & -5 \\ -1 & -5 & 10 \end{pmatrix} \begin{pmatrix} 3 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 3 \end{pmatrix} = \begin{pmatrix} 35 & 21 & -8 \\ 21 & 22 & -21 \\ -8 & -21 & 35 \end{pmatrix}$$

$$F(A) = -\begin{pmatrix} 35 & 21 & -8 \\ 21 & 22 & -21 \\ -8 & -21 & 35 \end{pmatrix} + 8\begin{pmatrix} 10 & 5 & -1 \\ 5 & 6 & -5 \\ -1 & -5 & 10 \end{pmatrix} -$$

$$-19\begin{pmatrix} 3 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 3 \end{pmatrix} + 12\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$

$$= -\begin{pmatrix} 35 & 21 & -8 \\ 21 & 22 & -21 \\ -8 & -21 & 35 \end{pmatrix} + \begin{pmatrix} 80 & 40 & -8 \\ 40 & 48 & -40 \\ -8 & -40 & 80 \end{pmatrix} - \begin{pmatrix} 57 & 19 & 0 \\ 19 & 38 & -19 \\ 0 & -19 & 57 \end{pmatrix} +$$

$$+ \begin{pmatrix} 12 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 12 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

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Exercise 10

10) a)

$$A^TA = \begin{pmatrix} 0 & 0 \\ 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{pmatrix}$$

$$\det(A^TA - \lambda I) = \begin{pmatrix} -\lambda & 0 & 0 \\ 0 & 4-\lambda & 0 \\ 0 & 0 & 9-\lambda \end{pmatrix} =$$

$$= \dots = -\lambda(4-\lambda)(9-\lambda)$$

So $\lambda_1 = 0$, $\lambda_2 = 4$, $\lambda_3 = 9$

• $\lambda = 9$

$$\begin{pmatrix} -9 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \Rightarrow \begin{array}{l} x_1 = 0 \\ x_2 = 0 \\ x_3 = x_3 \end{array}$$

So we assume that $x_3 = 1$

$$\vec{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

• $\lambda = 4$

$$\begin{pmatrix} -4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \Rightarrow \begin{array}{l} x_1 = 0 \\ x_2 = x_2 \Rightarrow x_2 = 1 \\ x_3 = 0 \end{array}$$

$$\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$\cdot\ \lambda = 0$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \qquad \begin{aligned} x_1 &= x_1 \Rightarrow x_1 = 1 \\ x_2 &= 0 \\ x_3 &= 0 \end{aligned}$$

$$P = [v_1, X_2, v_3] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = V$$

$$P^T = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = V^T$$

For $\sigma_1 = 3$, $\sigma_2 = 2$ we have $\Sigma = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$

$$u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{3} \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$u_2 = \frac{1}{\sigma_2} A v_2 = \frac{1}{2} \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$U = [u_1, u_2] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

So $A = U \Sigma V^T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

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b) Using the matrices we found on (a)

$$U \cdot \Sigma \cdot V^T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 2 & 0 \\ 3 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} = A$$