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CSE150
HW02a

Problem 1

For each of the following statements about sets determine whether it is always true (also provide an example), or only sometimes true (also provide an example and counterexample). Please provide an explanation.

1. $A \in P(A)$

   Always true.

   True Example: $\emptyset \in P(\emptyset) \iff \emptyset \in \{\emptyset\}$
   The power set is the set of all subsets, so A will always be an element in P(A).

2. $A \subseteq P(A)$

   Sometimes True.

   False Example: $\{1, 2\}$ is not a subset of $\{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$

3. $(|A| \leq |B| \Rightarrow (A \subseteq B))$

   Sometimes True.

   True Example: $|\{4, 5, 6\}| \leq |\{4, 5, 6, 7\}|$ and $\{4, 5, 6\} \subseteq \{4, 5, 6, 7\}$ is true.
   False Example: $|\{1, 2, 3\}| \leq |\{4, 5, 6, 7\}|$ and $\{1, 2, 3\} \subseteq \{4, 5, 6, 7\}$ is not true.

4. $(A \subseteq B) \Rightarrow (|A| \leq |B|)$ Always true.

   True Example: $\{1, 2\} \subseteq \{1, 2, 3\}$ and $|\{1, 2\}| \leq |\{1, 2, 3\}|$
   If A is a subset of B, then the number of elements in A is less than or equal to B.

Problem 2

Find the smallest two finite sets A and B for each of the four conditions.

Note: The smallest sets may not be unique.

1. $A \in B, A \subseteq B$, and $P(A) \subseteq B$

   $A = \emptyset, B = \{\emptyset\}$

2. $(\mathbb{N} \cap A) \in A, B \subset A$, and $P(B) \subseteq A$.

   $A = \{\emptyset, \{\emptyset\}\}, B = \emptyset$

3. $A \subseteq (P(P(B)) - P(A)$.

   $A = \emptyset, B = \emptyset$

4. $A \supseteq P(P(B)) - P(A)$.

   $A = \{\emptyset\}, B = \emptyset$

Problem 3

Prove or disprove (by providing a counterexample) each of the following properties of binary relations:

Let S(A) be the symmetric closure of set A. Let T(A) be the transitive closure of set A. For every binary relation R,

1. $T(S(R)) \subseteq S(T(R))$

   False.

   $R = \{(1,3)\}$

   $S(R) = \{(3,1)\}$

   $T(R) = \{(1,3)\}$

   $T(S(R)) = \{(1,3),(3,1),(1,1),(3,3)\}$

   $S(T(R)) = \{(1,3),(3,1)\}$

2. $S(T(R)) \subseteq T(S(R))$

   True

   $T(R) = \{(x,y),(y,z) \in R \mid (x,z) \in R\}$
   $S(T(R)) = \{(x,y),(y,z)(x,z) \in R \mid (y,x),(z,y),(z,x) \in R\}$

   $S(R) = \{(x,y),(y,z) \in R \mid (y,x),(z,y) \in R\}$
   $T(S(R)) = \{(x,y),(y,z)(y,x),(z,y) \in R \mid (x,z),(z,x) \in R\}$

   These cases will end up being equivalent to each other, except when T(R) fails to add an edge to R. That's when S(T(R)) becomes a subset of T(S(R)).

Problem 4

How many reflexive binary relations are there on S x S? How many symmetric relations? Explain.
$Bonus$: How many equivalence relations are there on S x S? Explain.

Reflexive Relations

S x S = $\{x \in S$ and $y \in S \mid (x, y) \in R\}$

The reflexive relations in S x S can be shown with a table:

X	0	1	2	3	...	n - 1	n
0	(0, 0)	(0, 1)	(0, 2)	(0, 3)	(0, ...)	(0, n - 1)	(0, n)
1	(1, 0)	(1, 1)	(1, 2)	(1, 3)	(1, ...)	(1, n - 1)	(1, n)
2	(2, 0)	(2, 1)	(2, 2)	(2, 3)	(2, ...)	(2, n - 1)	(2, n)
3	(3, 0)	(3, 1)	(3, 2)	(3, 3)	(3, ...)	(3, n - 1)	(3, n)
...	(..., 0)	(..., 1)	(..., 2)	(..., 3)	(..., ...)	(..., n - 1)	(..., n)
n - 1	(n - 1, 0)	(n - 1, 1)	(n - 1, 2)	(n - 1, 3)	(n - 1, ...)	(n - 1, n - 1)	(n - 1, n)
n	(n, 0)	(n, 1)	(n, 2)	(n, 3)	(n, ...)	(n, n - 1)	(n, n)

The number of elements in all possible ordered pairs that can be reflexive is the length of S x S minus the length of the set containing the ordered pairs, (x, y), such that x = y because any set can be reflexive so long as the set with x = y is a subset. So the power set of the length of the set would be 2^{n^2-n}

Symmetric Relations

S x S = $\{x \in S$ and $y \in S \mid (x, y) \in R\}$

The symmetric relations in S x S can be shown with a table:

X	0	1	2	3	...	n - 1	n
0	(0, 0)	(0, 1)	(0, 2)	(0, 3)	(0, ...)	(0, n - 1)	(0, n)
1	(1, 0)	(1, 1)	(1, 2)	(1, 3)	(1, ...)	(1, n - 1)	(1, n)
2	(2, 0)	(2, 1)	(2, 2)	(2, 3)	(2, ...)	(2, n - 1)	(2, n)
3	(3, 0)	(3, 1)	(3, 2)	(3, 3)	(3, ...)	(3, n - 1)	(3, n)
...	(..., 0)	(..., 1)	(..., 2)	(..., 3)	(..., ...)	(..., n - 1)	(..., n)
n - 1	(n - 1, 0)	(n - 1, 1)	(n - 1, 2)	(n - 1, 3)	(n - 1, ...)	(n - 1, n - 1)	(n - 1, n)
n	(n, 0)	(n, 1)	(n, 2)	(n, 3)	(n, ...)	(n, n - 1)	(n, n)

The number of elements in all possible ordered pairs that can be in a symmetric relation is the sum of the length of the set containing (x,y) where x = y, and the length of the set containing an (y, x) for every (x, y) where x != y divided by 2. For every (x, y), there must be a (y, x), so dividing by 2 would get rid of redundancies.

To calculate how many cells are in the highlighted area given a set S, you would use Σn which is the same as $\frac{n(n+1)}{2}$. The total number of symmetric relations would then be the length of the power set of the set, which would be $2^{\frac{n(n+1)}{2}}$.

***Collaborated with Sean Chu, Raymond Wu, and David Song