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CSE150
HW02b

Problem 5

Show that each function $f : \mathbb{N} \to \mathbb{N}$ has the listed properties.

1. $f(x) = 2x$ (one-to-one but not onto)

   Injection: $\forall_{a,b} \in A$, $f(a) = f(b) \Rightarrow a = b$
   Take any two arbitrary x: x_1, x_2. If $f(x_1) = f(x_2)$, then $x_1 = x_2$ is true if the function is one-to-one.
   $f(x_1) = 2x_1$
   $f(x_2) = 2x_2$
   $2x_1 = 2x_2$
   $x_1 = x_2$
   So $f(x)$ is one-to-one.

   Onto: $\forall_y \in Y, \exists x \in X,\ f(x) = y$ For this to be onto, every element in the set of $\mathbb{N}$ must have a corresponding element in the set of $\mathbb{N}$.

   Let's take the number 7, from the set $\mathbb{N}$.

   Since only f(3.5) = 7, and 3.5 is not in the set $\mathbb{N}$, **then it's not onto.**

2. $f(x) = x + 1$ (one-to-one but not onto)

   Injection: $\forall_{a,b} \in A$, $f(a) = f(b) \Rightarrow a = b$
   Take any two arbitrary x: x_1, x_2. If $f(x_1) = f(x_2)$, then $x_1 = x_2$ is true if the function is one-to-one.
   $f(x_1) = x_1 + 1$
   $f(x_2) = x_2 + 1$
   $x_1 + 1 = x_2 + 1$
   $x_1 = x_2$
   So $f(x)$ is one-to-one.

   Onto: $\forall_y \in Y, \exists x \in X,\ f(x) = y$ For this to be onto, every element in the set of $\mathbb{N}$ must have a corresponding element in the set of $\mathbb{N}$.

   Let's take the number 0, from the set $\mathbb{N}$.

   Since only f(-1) = 0, and -1 is not in the set $\mathbb{N}$, **then it's not onto.**

3. $f(x) =$ if x is odd then $x - 1$ else $x + 1$ (bijective)

   Injection: $\forall_{a,b} \in A$, $f(a) = f(b) \Rightarrow a = b$
   Take any two arbitrary x for the even case and
   any two arbitrary x for the odd case, where:
   –$2x$ represents the xth even number and
   –$2x + 1$ represents the xth odd number:
   x_1, x_3 for odds.
   x_2, x_4 for evens.
   If $f(2x_1 + 1) = f(2x_3 + 1)$, then $2x_1 + 1 = 2x_3 + 1$
   is true if the function is one-to-one.
   If $f(2x_2) = f(2x_4)$, then $2x_2 = 2x_4$ is true if the function is one-to-one.

   $$f(x_1) = 2x_1 + 1 - 1$$
   $$f(x_3) = 2x_3 + 1 - 1$$
   $$2x_1 + 1 = 2x_3 + 1$$
   $$2x_1 + 1 = 2x_3 + 1$$
   $$f(x_2) = 2x_2 + 1$$
   $$f(x_4) = 2x_4 + 1$$
   $$2x_1 + 1 = 2x_3 + 1$$
   $$2x_2 = 2x_4$$

   So $f(x)$ is one-to-one.

   Onto: $\forall_y \in Y, \exists x \in X$, $f(x) = y$ For this to be onto, every element in the set of $\mathbb{N}$ must have a corresponding element in the set of $\mathbb{N}$.

   For this piecewise function to be onto, both pieces must also be onto.
   For any arbitrary x_1, let x_1 represent the odd inputs, where $x_1 = 2n + 1$ and $n \in \mathbb{N}$.
   For any arbitrary x_2, let x_2 represent the even inputs where $x_2 = 2n$ and $n \in \mathbb{N}$.

   $f(x_1) = x_1 - 1 = (2n + 1) - 1 = 2n$
   $f(x_2) = x_2 + 1 = (2n) + 1 = 2n + 1$

   This proves that for every odd input, there will be a corresponding even input and
   for every even input, there will be a corresponding odd input.

Problem 6

Show that the product $(a + bi)(c + di)$ of two complex numbers can be evaluated using just three real-number multiplications. You may use a few extra additions.

1. $(a + bi)(c + di)$
2. $ac + cbi + adi - bd$ [by foiling]
3. $ac - bd + i(cb + ad)$ [by grouping]
4. $ac - bd + i(cb + ad) + bc - bc - ad + ad$ [adding zero]
5. $ac + bc - bd - ad + i(cb + ad) - bc + ad$ [by grouping]
6. $c(a + b) - d(b + a) + i(cb + ad) - bc + ad$[by grouping]
7. $(a + b)(c - d) + i(cb + ad) - bc + ad$[by grouping]

Problem 7

Given a function $f : A \to A$. An element $a \in A$ is called a *fixed point of* f if $f(a) = a$. Find the set of fixed points for each of the following functions.

1. $f : A \to A$ where $f(x) = x$
   $\{x \mid x \in \mathbb{R}\}$

2. $f : \mathbb{N} \to \mathbb{N}$ where $f(x) = x + 1$
   $\{\}$

3. $f : \mathbb{N}_6 \to \mathbb{N}_6$ where $f(x) = 2x$ mod 6
   $\{0\}$

4. $f : \mathbb{N}_6 \to \mathbb{N}_6$ where $f(x) = 3x$ mod 6
   $\{0,3\}$

Problem 8

Let $f(x) = x^2$ and $g(x, y) = x + y$. Find the compositions that use the functions f and g for each of the following expressions.

1. $(x + y)^2 = f(g(x, y))$

2. $x^2 + y^2 = g(f(x), f(y))$

3. $(x + y + z)^2 = f(g(g(x, y), z))$

4. $x^2 + y^2 + z^2 = g(g(f(x), f(y)), f(z))$

***Collaborated with David Song, Raymond Wu, Sean Chu

****Special thanks to Zane for help with Problem 5