→ Midterm Tu Feb 28 6:30-7:30 PM

→ Midterm sample (2018) → Pbm 2 is hard.

↓ check slides from 207

Lecture 17

Section 1: Discrete time Markov chains

Section 2: Exponential distribution and Poisson process

Section 3: Continuous time Markov chains

This lecture: Exponential distribution

Exponential distribution ($\to$ continuous distribution)

Def: We call X an exponential random variable (rv), with parameter $\lambda > 0$, i.e., $X \sim Exp(\lambda)$, if X has density:

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

Exponential random variables often model waiting times.

Warm-up

$\longrightarrow \int^{s} e^{-\lambda x} dx = \frac{e^{-\lambda s}}{-\lambda}$

Let $X \sim Exp(\lambda)$.

$e^{-\lambda x} \xrightarrow[x \to +\infty]{} 0$

$(P(X=t)=0)$

What is $P(X > t)$?

$$P(X>t) = P(X \geq t) = \int_t^{\infty} \lambda e^{-\lambda x} dx = \left[-e^{-\lambda x}\right]_t^{+\infty}$$
$$= \boxed{e^{-\lambda t}}$$

What is $\mathbb{E}X$?

$$\mathbb{E}(X) = \int_0^{\infty} P(X \geq t) dt = \int_0^{\infty} e^{-\lambda t} dt = \frac{1}{-\lambda}\left[e^{-\lambda t}\right]_0^{+\infty} = \boxed{\frac{1}{\lambda}}$$

(true in general)

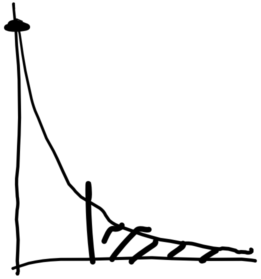
Properties of exponential rv's

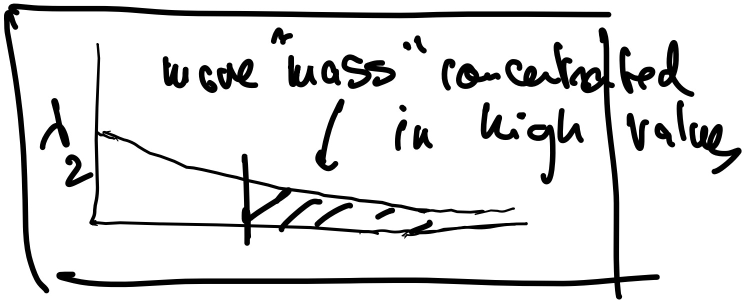
Fact: Let $X \sim Exp(\lambda)$. Then,

- $\mathbb{E}X =$ $1/\lambda$
- $Var(X) =$ $1/\lambda^2$ (exercise)
- $P(X > t) =$ $e^{-\lambda t}$

Which exponential density gives higher expectation value?

λ_1





$\lambda_1 > \lambda_2$

$\Rightarrow \frac{1}{\lambda_1} < \frac{1}{\lambda_2}$

Average waiting time is $\frac{1}{1/10} = 10$ min.

Example) Suppose $T \sim Exp\left(\frac{1}{10}\right)$ is the time you have to wait for the bus. Given that you have been waiting for 20 minutes, what is the chance that the bus comes within the next 10 minutes? On average, how much longer must you wait?

given that we already waited 20 mins.

$$A: P(T \in [20,30] \mid T \geq 20) = \frac{P(\{T \in [20,30]\}, \{T \geq 20\})}{P(T \geq 20)}$$

$\{T \in [20,30]\} \cap \{T \geq 20\}$

$$= \frac{P(T \in [20,30])}{P(T \geq 20)} = \frac{P(\{T \geq 20\} \setminus \{T \leq 30\})}{P(T \geq 20)}$$

$$= \frac{P(T \geq 20) - P(T \geq 30)}{P(T \geq 20)} = \frac{e^{-\frac{20}{10}} - e^{-\frac{30}{10}}}{e^{-20/10}}$$

$$= 1 - e^{-3+2} = \boxed{1 - e^{-1}}$$

Rmk: P(A bus is coming within 10 min) $= P(T \leq 10) = 1 - e^{-10/10} = \boxed{1-e^{-1}}$

Memorylessness

Prop: Let $X \sim Exp(\lambda)$. Then,

$P(X \geq t + s | X \geq s) =$ $P(X \geq t)$

$[X - s \mid X \geq s] \sim Exp(\lambda)$

Rmk: One can show that the Exp distribution is the only memoryless continuous r.v. in 1D.

Q) $P(X < t + s | X \geq s) =$ $P(X < t)$

$$\mathbb{E}(X - s \mid X \geq s) = \mathbb{E}(Exp(\lambda)) = \frac{1}{\lambda}$$

so the answer in the previous slide is 10 min

Example) You enter a bank which has two tellers, currently servicing Yang and Lee. Service times are iid $Exp(\lambda)$. What is the chance you leave last?

Suppose that one of the customers leaves (e.g. Yang)
From this moment and because of the memoryless property,
the remaining time that Lee will have to spend is again
$Exp(\lambda)$. Since your service time is also $Exp(\lambda)$
than the probability to leave after Lee is $\boxed{\frac{1}{2}}$

Answer : $P(\text{leave last}) = \frac{1}{2}$

Minimum of two independent exponential rv's

also Check → week 6 Notebook

Q) Suppose $X \sim Exp(\lambda_1), Y \sim Exp(\lambda_2)$ are independent. What is the distribution of $Z = \min(X, Y)$?

idea: Consider the event $E = \{Z \geq t\}$ for $t \geq 0$

$Z > t \iff X > t$ and $Y > t$ so $E = \{X > t, Y > t\}$

so $P(E) = P(X > t, Y > t) = \underbrace{P(X > t)}_{e^{-\lambda_1 t}} \underbrace{P(Y > t)}_{e^{-\lambda_2 t}}$ (by indep.)

$= e^{-(\lambda_1 + \lambda_2)t}$

1 − cumulative distribution fct of Z

$\Rightarrow F_Z(t) = P(Z \leq t) = \underbrace{1 - e^{-(\lambda_1+\lambda_2)t}}_{\text{cdf of } Exp(\lambda_1+\lambda_2)}$

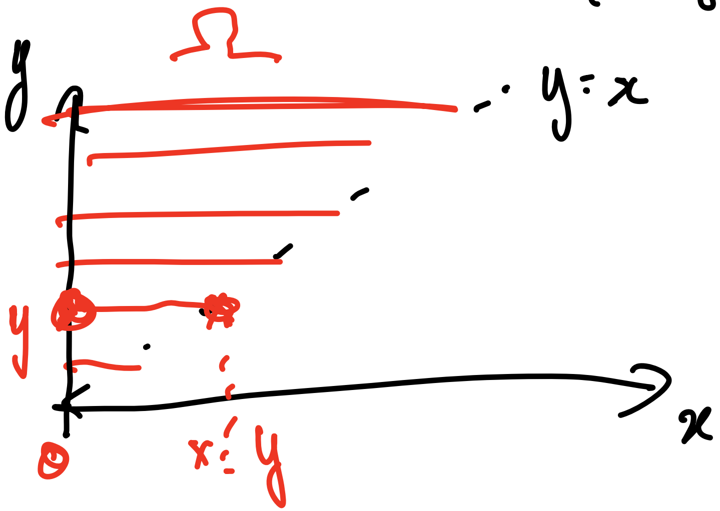
$\Rightarrow \boxed{Z \sim Exp(\lambda_1 + \lambda_2)}$

Fastest to finish

$\left(P(Z \in E) = \int_E f(z)\,dz \right)$

Q) Let $X \sim Exp(\lambda_1), Y \sim Exp(\lambda_2)$. What is $P(X < Y)$?

A: $P(X < Y) = \iint_{\{(x,y) \mid x<y\}} f_X(x) f_Y(y)\,dx\,dy.$ ← Ω



$$= \int_0^{+\infty} dy\, f_Y(y) \int_0^y dx\, f_X(x)$$

$$= \int_0^{+\infty} dy\, \lambda_2 e^{-\lambda_2 y} \int_0^y dx\, \lambda_1 e^{-\lambda_1 x}$$

$$(\ldots) = \boxed{\frac{\lambda_1}{\lambda_1 + \lambda_2}}$$

(exercise)

Q) You enter a bank which has two tellers, currently servicing Yang and Lee. Service times are iid $Exp(\lambda)$. What is the expected time until you are finished being served?

A: $\underbrace{\text{Time to be served}}_{T} = \overbrace{\text{Time for Yang or Lee}}^{\sim Exp(2\lambda)}$ to be served

$+ \text{ Time for } \underbrace{\text{you to be served}}_{\sim Exp(\lambda)}$

$$\Rightarrow \quad \mathbb{E}(T) = \mathbb{E}(Exp(2\lambda)) + \mathbb{E}(Exp(\lambda))$$

$$= \frac{1}{2\lambda} + \frac{1}{\lambda} = \boxed{\frac{3}{2\lambda}}.$$