1. $\boxed{\text{6 marks}}$ Consider the problem [Vanderbei 5th edition, Exercise 1.1].

 (a) $\boxed{\text{3 marks}}$ Write this as a Linear Programming problem (in the standard form). You must explain your notation and variables.

**Solution:** Set the decision variables

$$\begin{aligned} x_1 &= \text{ the amount of Bands to be produced in tons} \\ x_2 &= \text{ the amount of Coils to be produced in tons} \end{aligned}$$

We want to maximize the objective function

$$\text{Profit in \$} = 25x_1 + 30x_2$$

Before writing the constrains, we first translate the condition on production rates into the condition on time to produce $x_1$ amount of Bands, and $x_2$ amount of Coils.

$$\begin{aligned} &\text{producing } x_1 \text{ tons will require } \tfrac{1}{200}x_1 \text{ hours} \\ &\text{producing } x_1 \text{ tons will require } \tfrac{1}{140}x_1 \text{ hours} \end{aligned}$$

Then we have the following constraints:

$$\begin{aligned} x_1 &\leq 6000 \\ x_2 &\leq 4000 \\ \frac{1}{200}x_1 + \frac{1}{140}x_2 &\leq 40 \\ x_1, x_2 &\geq 0 \end{aligned}$$

The final LP problem is

$$\begin{aligned} \max\ & 25x_1 + 30x_2 \\ \text{subject to. } & x_1 \leq 6000 \\ & x_2 \leq 4000 \\ & \frac{1}{200}x_1 + \frac{1}{140}x_2 \leq 40 \\ & x_1, x_2 \geq 0 \end{aligned}$$

 (b) $\boxed{\text{3 marks}}$ Solve the LP by writing down a code in the Python language using the Jupyter notebook; login to UBC syzygy website and the Jupyter notebook.
 Attach the screenshots.

**Solution:** See the file HW2-1b.ipynb.

2. 11 marks Consider the problem [Vanderbei. 5th edition, Exercise 1.2].

(a) 5 marks Write this as a Linear Programming problem (in the standard inequality form). You must explain your notation and variables.

**Solution:** Notice that the flight is to fly from Ithaca to Newark, then to Boston. Set the decision variables. First name the routes:

R1 Ithaca to Newark

R2 Newark to Boston

R3 Ithaca to Boston

We use the letters Y, B, M, for the available classes. Then, the decision variables are, for each $i = 1, 2, 3$

$$\begin{aligned} x_{iY} &= \text{ the number of Y class tickets for route } i \\ x_{iB} &= \text{ the number of Y class tickets for route } i \\ x_{iM} &= \text{ the number of Y class tickets for route } i \end{aligned}$$

Notice that there are 9 such variables.

We want to maximize the objective function (the revenue in \$)

$$300x_{1Y} + 220x_{1B} + 100x_{1M} + 160x_{2Y} + 130x_{1B} + 80x_{2M} + 360x_{3Y} + 280x_{3B} + 140x_{3M}$$

Now constraints. First, the obvious constraints: all variables are nonnegative and integer. Then, the upper bounds from the forecasted maximum demand.

$$x_{1Y} \leq 4, x_{1B} \leq 8, x_{1M} \leq 22, x_{2Y} \leq 8, x_{2B} \leq 13, x_{2M} \leq 20, x_{3Y} \leq 3, x_{3B} \leq 10, x_{3M} \leq 18.$$

Now the constraint from not allowing overbooking. The aircraft has 30 seats. So, we have

$$\begin{aligned} x_{1Y} + x_{1B} + x_{1M} + x_{3Y} + x_{3B} + x_{3M} &\leq 30 \\ x_{2Y} + x_{2B} + x_{2M} + x_{3Y} + x_{3B} + x_{3M} &\leq 30 \end{aligned}$$

The first line is from the route between Ithaca and Newark, where there are route 1 (Ithaca to Newark) route 3 (Ithaca to Boston) passengers are to be onboard. The second line is from the route between Newark and Boston, where there are route 2 (Newark to Boston) route 3 (Ithaca to Boston) passengers are to be onboard.

The final LP problem is

max
$300x_{1Y} + 220x_{1B} + 100x_{1M} + 160x_{2Y} + 130x_{1B} + 80x_{2M} + 360x_{3Y} + 280x_{3B} + 140x_{3M}$
subject to

$x_{1Y} \leq 4,$
$x_{1B} \leq 8,$
$x_{1M} \leq 22,$
$x_{2Y} \leq 8,$
$x_{2B} \leq 13,$
$x_{2M} \leq 20,$
$x_{3Y} \leq 3,$
$x_{3B} \leq 10,$
$x_{3M} \leq 18,$
$x_{1Y} + x_{1B} + x_{1M} + x_{3Y} + x_{3B} + x_{3M} \leq 30,$
$x_{2Y} + x_{2B} + x_{2M} + x_{3Y} + x_{3B} + x_{3M} \leq 30,$
$x_{1Y}, x_{1B}, x_{1M}, x_{2Y}, x_{1B}, x_{2M}, x_{3Y}, x_{3B}, x_{3M} \geq 0,$
and $x_{1Y}, x_{1B}, x_{1M}, x_{2Y}, x_{1B}, x_{2M}, x_{3Y}, x_{3B}, x_{3M}$ are integers.

(b) 6 marks Solve the LP by writing down a code in the Python language using the Jupyter notebook; login to UBC syzygy website and the Jupyter notebook.
Attach the screenshots.
[Hint1: You will need integer variables, those taking only integer values. For that you can add the command cat='Integer' like in the following example:

ticketvars = LpVariable.dicts("ticket", ticket, lowBound=0, cat='Integer')

Here the last part cat='Integer' restrict the variables to be integer variables. ]
[Hint2: It is not necessary, but, to practice with 'for-loops' and 'dictionary', you can try to use dictionary variables as done in the blending (cat food) example. ]

**Solution:** See the file HW2-2b.ipynb.

3. $\boxed{\text{3 marks}}$ For a nonempty set $S \subset \mathbf{R}^n$ and a positive real number $r \in \mathbf{R}$ (and $r > 0$), define the set $rS$ as follows:

$$rS \ := \{z \in \mathbf{R}^n \mid z = rx,\, x \in S\}.$$

Here $rx$ is the multiplication of the vector $x \in \mathbf{R}^n$ by the scalar $r \in \mathbf{R}$; in your more familiar notation, $r\vec{x}$. The set $rS$ is the set of all points that are obtained by multiplying $r$ with the vectors $x \in S$.

For a given nonempty set $S \subset \mathbf{R}^n$ and a given positive number $r > 0$, prove that if $S$ is a convex set then $rS$ is a convex set as well.

**Solution:**

- Assume $S$ is convex.
- Suppose $x, y \in rS$, then $x = rx'$, $y = ry'$ for some $x', y' \in S$.
- Now, consider for each $t \in [0, 1]$, $(1-t)x+ty = (1-t)rx'+try' = r[(1-t)x'+ty']$.
- Notice that since $S$ is convex and $x', y' \in S$, we see that for that $t \in [0, 1]$, $(1-t)x' + ty' \in S$.
- Therefore by the definition of $rS$, $(1-t)x + ty = r[(1-t)x' + ty'] \in rS$.
- This verifies that $rS$ is convex.

4. $\boxed{\text{5 marks}}$ For given two nonempty sets $S_1, S_2 \subset \mathbf{R}^n$, define the operation $S_1 + S_2$ as follows:

$$S_1 + S_2 := \{z \in \mathbf{R}^n \mid \text{ there exist some } x \in S_1 \text{ and some } y \in S_2 \text{ such that } z = x + y \quad \}$$

that is, each point $z \in S_1 + S_2$ is the one that can be expressed as the sum $x + y$ for some $x \in S_1$, and $y \in S_2$; here the sum $x + y$ is the vector sum between the two vectors. One does this for all $x \in S_1$ and $y \in S_2$ and get the set $S_1 + S_2$.

(a) Consider $S_1 = \{(x_1, x_2) \in \mathbf{R}^2 \mid |x_1 - 1| \le 1 \,\&\, |x_2 - 2| \le 1\}$ and $S_2 = \{(x_1, x_2) \in \mathbf{R}^2 \mid |x_1| \le 2 \,\&\, |x_2| \le 1\}$. Sketch the set $S_1 + S_2$. You do not need to explain your solution for this question. But, your sketch should be neat and very clear, indicating all the relevant coordinate values.

> **Solution:** The resulting set is
> $S_1 + S_2 = \{(x_1, x_2) \in \mathbf{R}^2 \mid |x_1 - 1| \le 3 \ \&\ |x_2 - 2| \le 2\}$.
> We skip the sketch. Your sketch should clearly show that it is the square shape centered at $(1, 2)$ with horizontal side length 6 and vertical side length 4.

(b) Is it true that $S_1 + S_2$ must be convex for **any** nonempty convex sets $S_1$ and $S_2$ in $\mathbf{R}^n$? Justify your answer carefully. [**This problem is independent of part (a). The sets $S_1, S_2$ are arbitrary convex sets in this question, not the particular example given in part (a). If you do this problem only for the sets of part (a) or a particular example, you will get zero mark.** ]

> **Solution:** Such a sum $S_1 + S_2$ is called the Minkovski sum of $S_1$ and $S_2$. It is convex if $S_1$ and $S_2$ are.
>
> - To see this, let $x, y$ be given two points in $S_1 + S_2$. It means there are points $x_1, y_1 \in S_1$ and $x_2, y_2 \in S_2$ such that $x = x_1 + x_2$, and $y = y_1 + y_2$.
> - Now, consider for each $0 \le t \le 1$,
>
> $$z_t = (1 - t)x + ty$$
>
> - We claim that $z_t \in S_1 + S_2$ which will finish the proof.
> - Notice that
>
> $$\begin{aligned} z_t &= (1 - t)x + ty \\ &= (1 - t)(x_1 + x_2) + t(y_1 + y_2) \\ &= [(1 - t)x_1 + ty_1)] + [(1 - t)x_2 + ty_2] \text{ by rearranging the terms.} \end{aligned}$$
>
> - Notice that $[(1 - t)x_1 + ty_1)] \in S_1$ from convexity of $S_1$, since $x_1, y_2 \in S_1$ and $0 \le t \le 1$.

- Likewise, $[(1-t)x_2 + ty_2] \in S_2$ from convexity of $S_2$, since $x_1, y_2 \in S_1$ and $0 \leq t \leq 1$.
- As $z_t$ is given by their sum, we conclude that $z_1 \in S_1 + S_2$ by the definition of the sum $S_1 + S_2$.
- Therefore we proved our claim, so this finishes the proof for convexity of $S_1 + S_2$ when $S_1$ and $S_2$ are convex.