Recall: • We introduced the generating function of a r.v. X as

$$G_X(s) = \mathbb{E}(s^X)$$

• $G_X(0) = P(X=0)$

• For the b.p. $(Z_n)_{n\geq 0}$ with reprod. law X

$$\underset{\substack{\uparrow \\ \text{g. fct of } Z_n}}{G_n(s)} = \underset{\substack{\uparrow \\ \text{n-th iteration of } G_1}}{G_1^{(n)}(s)} = G_X^{(n)}(s)$$

Lecture 16

Goal today: Extinction probabilities for branching processes

(RmK: Solution shown on Feb 13 contained errors
→ fixed in updated slides)

Warm-up: What is the probability of extinction by Generation n?

$\{\text{Extinct at Generation } n\} = \{Z_n = \underline{\ \ 0\ \ }\}$

$\{\text{Extinct by Generation } n\} = \bigcup_{k=0}^{n} \{Z_k = 0\} = \{Z_n = 0\}$

since $\{Z_n = 0\} \subset \{Z_{n+1} = 0\}$

In terms of the generating function:

$P(\text{extinction by } n) = P(\{Z_n = 0\}) = G_n(0) = G_X^{(n)}(0)$

(using previous results → recall)

Eventual extinction

$\{\text{Eventual extinction}\} = \left\{ \text{For } n \text{ large enough } Z_n = 0 \right\} = \bigcup_{n\in\mathbb{N}} \{Z_n = 0\}$

$= \lim_{N\to+\infty} \bigcup_{n=0}^{N} \{Z_n = 0\}$

$P\{\text{Eventual extinction}\} \neq 1$

$$= P\left(\bigcup_{n\in\mathbb{N}} \{Z_n = 0\} \right) = \lim_{n\to+\infty} P(Z_n = 0)$$

$$= \boxed{\lim_{n\to+\infty} G_n(0)}$$

(from previous slide)

Determining $P\{\text{eventual extinction}\}$

Recall: $\mu := \mathbb{E}\xi,\ \sigma^2\ := Var(\xi)$.

"smallest fixed point of G"

Thm: $P\{\text{eventual extinction}\} := \eta$ satisfies the following:

1. η is the smallest non-negative root of the equation $G(s) = s$.
2. If $\mu < 1$, then $\eta = 1$ (subcritical regime)
3. If $\mu > 1$, then $\eta < 1$ (supercritical regime)
4. If $\mu = 1$, then (critical regime)
   a. If $\sigma^2 = 0, \eta =$ 0 → only happens when $Z_n = 1\ \forall n$
   b. If $\sigma^2 > 0, \eta =$ 1

Example, $\xi \sim Bin(2, p)$ $\left(\sim \xi_1 + \xi_2 \text{ where } \xi_i \overset{iid}{\sim} Bern(p)\right)$

Rmk: $\mathbb{E}(\xi) = 2p \qquad Var(\xi) = 2p(1-p)$

- From the thm, we have that $\eta = \begin{cases} 1 & \text{if } p < \frac{1}{2} \\ < 1 & \text{if } p > \frac{1}{2} \end{cases}$

- Assume $p > \frac{1}{2}$ and let's find η

$$G_\xi(s) = G_{\xi_1+\xi_2}(s) = G_{\xi_1}^2(s) = (1-p+ps)^2$$

$$\Rightarrow G_\xi(s) = s \iff (1-p+ps)^2 = s$$

$$\iff (1-p)^2 + 2(1-p)ps + p^2s^2 = s$$

$$\iff (1-p)^2 + (2(1-p)p-1)s + p^2 s^2 = 0$$

$$\iff a s^2 + b s + c = 0$$

where $a = p^2$, $b = 2(1-p)p - 1$, $c = (1-p)^2$

$$\Delta = b^2 - 4ac = (2(1-p)p-1)^2 - 4p^2(1-p)^2$$

$$= \cancel{4(1-p)^2 p^2} - 4(1-p)p + 1 - \cancel{4p^2(1-p)^2}$$

$$= 1 - 4p(1-p) > 0$$ (exercise : show this is true)

$\rightarrow$ 2 roots :
$$\frac{1 - 2(1-p)p \pm \sqrt{1-4p(1-p)}}{2p^2}$$

Since $\Delta = b^2 - \underbrace{4ac}_{>0} < b^2$

$$\sqrt{1-4p(1-p)} < 1 - 2(1-p)p$$

and the smallest non negative root is

$$\boxed{\eta = \frac{1 - 2(1-p)p - \sqrt{1-4p(1-p)}}{2p^2}}.$$

Proof that η is smallest non-neg root of $G(s) = s$.

Lemma 1: $\eta = G(\eta)$.

Fact: $\eta = \lim_{n \to +\infty} G_n(0)$

Proof: $\hookrightarrow G(\eta) = \lim_{n \to +\infty} G(G_n(0)) = \lim_{n \to +\infty} (G_{n+1}(0)) = \eta$ $\boxtimes$

(for $s \geq 0$)

Lemma 2: $G(s)$ is an increasing function. $(\Leftrightarrow G' \geq 0)$

Proof: $G(s) = \sum_{k=0}^{+\infty} s^k P(X=k)$

$$G'(s) = \sum_{k=1}^{+\infty} \underbrace{k}_{\geq 0} \underbrace{s^{k-1}}_{\geq 0} \underbrace{P(X=k)}_{\geq 0} \geq 0$$ $\boxtimes$

Fact: $G(\beta) = \beta$

Lemma 3: $\eta \leq \beta$ where β is any root of $G(s) = s$.

Proof: From lemma 2, $G(0) \leq G(\beta)$ for $\beta > 0$

$$\Rightarrow G(G(0)) \leq G(G(\beta)) = \beta \text{ (since } G \nearrow)$$

$$\vdots \qquad \vdots$$

$$G_n(0) \leq G_n(\beta) = \beta$$

$$\downarrow n \to +\infty \qquad \downarrow n \to +\infty$$

$$\eta \leq \beta$$

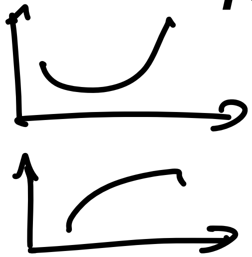
$\boxtimes$

Proof of relationship between η, μ, σ^2

Lemma 4: G is convex.

convex

concave



$\Leftarrow G'' \geq 0$

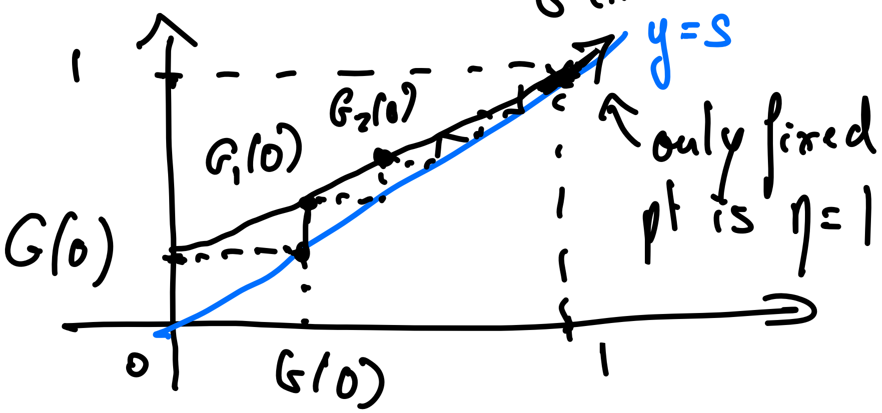
(exercise: show that G'' is convex)

Graph of solutions to $G(s) = s$:

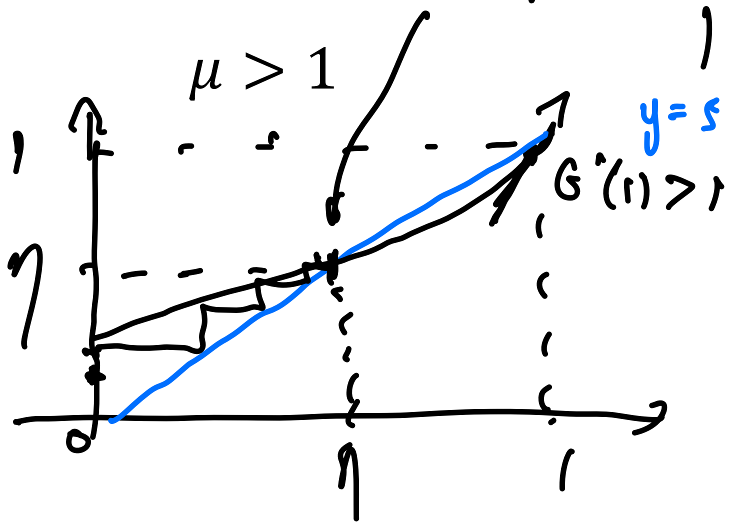
Fact: $G'(1) = \mu$

$G_n(0) \xrightarrow[n\to\infty]{} \eta < 1$

$\mu < 1$



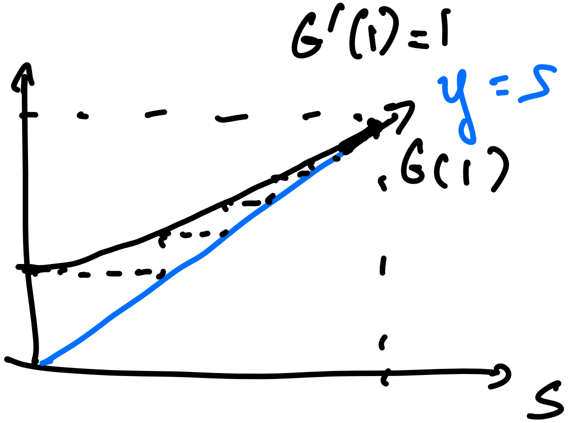
$\mu > 1$



Final case: $\mu = 1$ $\quad \sigma^2 = \mathrm{Var}(\xi) = \mathbb{E}\left(\left(\xi - \mathbb{E}(\xi)\right)^2\right) = \mathbb{E}\left((\xi-1)^2\right)$

If $\sigma^2 = 0$: $\quad \xi = 1$ w.p. $1 \implies Z_n = 1 \;\forall n \implies \eta = 0$

If $\sigma^2 > 0$:



Fact:

$$G''(1) = \mathrm{Var}(\xi) - \mathbb{E}(\xi) - \mathbb{E}(\xi)^2$$

$$G''(1) = \mathrm{Var}(\xi) = \sigma^2 > 0$$

(The only fixed point is at 1)

$\implies \eta = 1$.

$\boxtimes$

Example: $\xi \sim \text{Unif}\{0,1,2\}$.

What is the probability of eventual extinction?

$\mu = 1$ $\qquad\qquad$ $\sigma^2 = \frac{1}{3}(1+4) - 1 > 0$

$\Rightarrow \boxed{\eta = 1}$

Example: $\xi \sim 3 \cdot Bern(p)$ $(p \neq 0)$

- $\mu = 3p$
- $\sigma^2 = 9p - 9p^2 > 0$

- $p < \frac{1}{3} \to \eta = 1 \quad (\mu < 1)$
- $p = \frac{1}{3} \to \eta = 1 \quad (\mu = 1, \sigma^2 > 0)$
- $p > \frac{1}{3}$ $\quad G_\xi(s) = 1 - p + ps^3$

$$G_\xi(s) = s \iff 1 - p - s + ps^3 = 0$$

Exercise: Factorize $\underbrace{1 - p - s + ps^3}_{f(s)}$ (hint: use $f(1) = 0$)

Answer: $\eta = \dfrac{-1 + \sqrt{\frac{4}{p} - 3}}{2}$.

Conclusion of Chap 1

- Testable learning outcomes for midterm.
  - Derive and use transition matrices & diagrams
  - Application to compute probabilities and expectations
  - Classify M-C states
  - Studying limiting probabilities and their applications
  - Properties of time reversed M-C
  - Branching processes

- To go further. (cf. Ross textbook)
  - → Applications to optimization, CS, stat.
    - Markov Decision process
    - MCMC (cf. Metropolis Hastings in HW)

      Markov Chain Monte Carlo
    - HMM (Viterbi Algorithm)

      Hidden Markov models

    ⟹ Many applications in inference, Machine learning
- Review Notebooks for getting familiar with computational implementation / simulation