3) consider finite set of nonzero vectors, $v_1, v_2, \ldots, v_k \in \mathbb{R}^n$
with $|v_i| > 0$, $i = 1, \ldots, k$; $k \geq 1$. Define

$$C = \bigcap_{i=1}^{k} H v_i$$ → intersection of Halfspaces spaces

where,
$$H_v = \{ x \in \mathbb{R}^n \mid v \cdot x \leq 1 \}$$

a) Is it possible for some case that $C = \emptyset$?

Take $\vec{x} = \vec{0}$

then

$$H_v = \{ \vec{v} \cdot \vec{0} \leq 1 \}$$
$$\Rightarrow \{ \vec{0} \} = H_v$$
$$\Rightarrow \vec{0} \in H_v$$

Since $\vec{0} \in H_v \Rightarrow \vec{0} \in H v_i$ for all $i = 1, \ldots, k$
And $C = \bigcap_{i=1}^{k} H v_i$ $i = 1, \ldots, k$ s.t. $k \geq 1$

$$\Rightarrow C = H_1 \cap H v_2 \cap H v_3 \cap \ldots \cap H v_k$$

$$\Rightarrow C = \vec{0} \text{ or } \vec{0} \in C$$
$$\Rightarrow C \neq \emptyset$$

b) Give example of vectors $v_i \in \mathbb{R}^2$, $|v_i| > 0$, $i = 1, 2, 3$
so $(n = 2, k = 3)$, where set C above is bounded set

Hint: set S is bounded, if there exists positive number K such
$|x| \leq K$ for any $x \in S$

$k = 3$, $n = 2 \Rightarrow v_i \in \mathbb{R}^2$

$$C = \bigcap_{i=1}^{3} H v_1 \cap H v_2 \cap H v_3$$

where $H v_1 = \{ x \in \mathbb{R}^2 \mid v_1 \cdot x \leq 1 \}$

$H_{v2} = \{x \in \mathbb{R}^2 \mid v_2 \cdot x \le 1\}$ and $H_{v3} = \{x \in \mathbb{R}^2 \mid v_3 \cdot x \le 1\}$

let $x = [x_1, x_2]^T$

say choose the following:

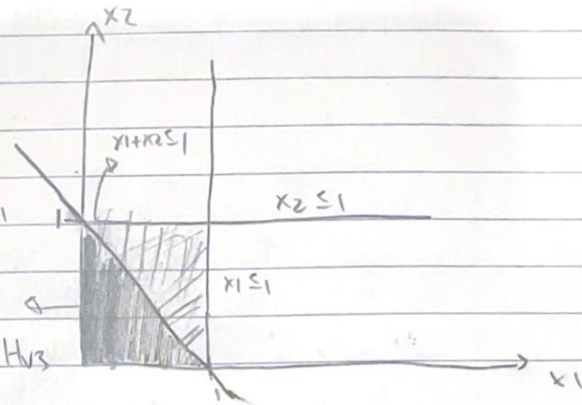
$v_1 = [1, 0] \Rightarrow H_{v1} = x_1 \le 1$

$v_2 = [0, 1] \Rightarrow H_{v2} = x_2 \le 1$

$v_3 = [1, 1] \Rightarrow H_{v3} = x_1 + x_2 \le 1$

check bounded

This is C = $H_{v1} \cap H_{v2} \cap H_{v3}$



$|v_1| = 1 \le 3$

$|v_2| = 1 \le 3$

$|v_3| = \sqrt{2} \le 3 \Rightarrow$ so C is bounded

4) Let $C \subset \mathbb{R}^n$ be given set, $C \ne \phi$. Let $f: \mathbb{R}^n \to \mathbb{R}$ be given function

(Prob 1) max $f(x)$
s.t. $x \in C$

Define $F_r = \{x \in \mathbb{R}^n \mid f(x) \ge r\}$

(Prob 2) max r
$C \cap F_r \ne \phi$ (condition on r)

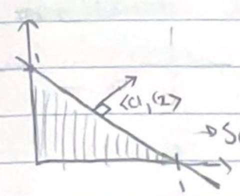
Assume optimal sol of (Prob 1) and optimal sol of (Prob 2) exist.
Find and explain relation between their optimal sol.

$n = 2$

$C \subset \mathbb{R}^2$, $C \ne \phi$

prob 1) max $f(x)$
s.t. $x \in C$

Said f is linear
$\Rightarrow f(x) = x_1 + x_2$
s.t. $x_1 + x_2 \le 1$ ~ $\in C$



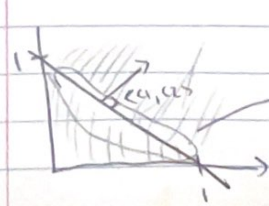
$\Rightarrow$ So max $f(x)$ occurs on here

Prob 2) max r
s.t. $C \cap F_r \ne \phi$

And $F_r = \{x \in \mathbb{R}^2 \mid f(x) \ge r\}$

$F_r = \{x \in \mathbb{R}^2 \mid x_1 + x_2 \ge r\}$

let $r = 1$



$\Rightarrow$ Then $x_1 + x_2 = 1$ is constraint and that $C \cap F_r \ne \phi$ $r \in C$
and $f(x) \ge r$