Lecture 13

- Branching processes
- Generating functions

THE

EDUCATIONAL TIMES,

AND

Journal of the College of Preceptors.

Vol. XXVI.] New Series, No. 148. AUGUST 1, 1873. {Registered for Transmission Abroad} PRICE 6d. BY POST 7d.

4001. (Proposed by FRANCIS GALTON.)—A large nation, of whom we will only concern ourselves with the adult males, N in number, and who each bear separate surnames, colonise a district. Their law of population is such that, in each generation, a_0 per cent. of the adult males have no male children who reach adult life; a_1 have one such male child; a_2 have two; and so on up to a_5 who have five. Find (1) what proportion of the surnames will have become extinct after r generations; and (2) how many instances there will be of the same surname being held by m persons.

Solution by the Rev. H. W. WATSON, M.A.

Therefore, if a series of functions of x be formed, such that

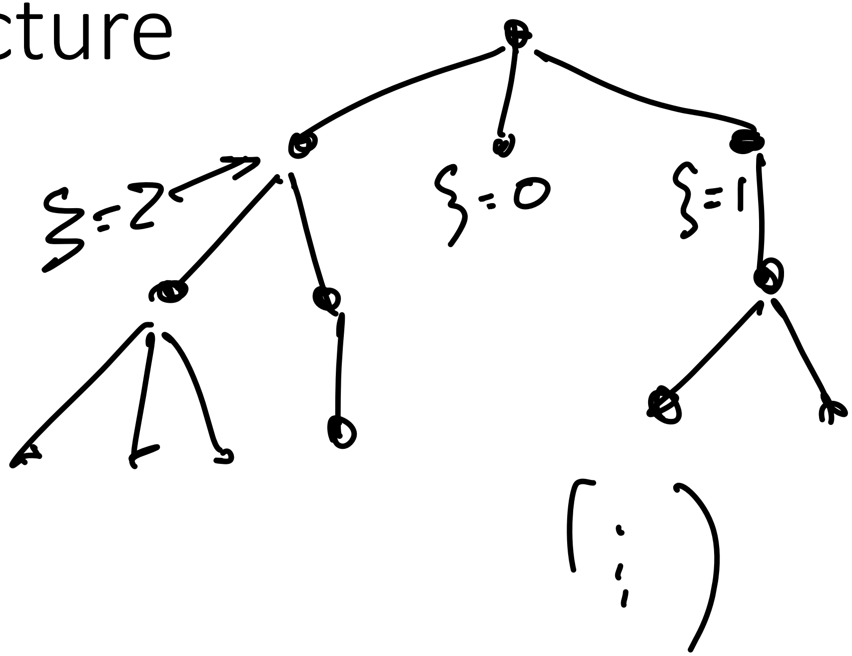
$$f_1(x) = t_0 + t_1 x \ldots + t_q x^q, \quad \text{and} \quad f_r(x) = f_{r-1}(t_0 + t_1 x \ldots + t_q x^q),$$

then the number of groups of ...

Branching process

- Population ~~involves~~ evolves in generations
    - Z_n = # of individuals in generation n
    - Assume $Z_0 = 1$
- Each individual has a random number of offspring
    - Independent of each other individual
    - Each with the same distribution
- Let ξ be a random variable having distribution of the offspring of one individual
    - $p_j := P(\xi = j),\ j = 0, 1, \ldots$
    - Assume $p_0 > 0$

Picture



generation 0 $\quad Z_0 = 1$

generation 1 $\quad Z_1 = 3$

generation 2 $\quad Z_2 = 3$

$\vdots$

Applications in: Genetics, ecology, epidemiology, etc.

Markov Chain representation

Where is the MC? $(Z_n)_{n\geq 0}$

- State space: $\{0,1,2,\ldots\} = \mathbb{N}$
- The absorbing (recurrent) state: $\{0\}$
- The transient states: $\{1,2,3,\ldots\}$ → (they can all go to 0)

Two possibilities: Since any finite set of transient states can only be visited finitely often (with probability 1), either

1) $Z_n = 0$ at a certain time
↪ means the population gets extinct

or 2) $Z_n \xrightarrow[n\to+\infty]{} +\infty$ → We'll see how to compute the probability of extinction

What are the transition probabilities?

Recall:

- $p_j := P(\xi = j),\ j = 0, 1, \ldots$

of the branching process

In terms of the transition matrix, this gives:

$$P_{1j} = p_j$$

for example P_{2j}

we would have to consider all the possible ways to get j offsprings from 2 individuals

Other transition probabilities are more complicated. Galton used generating function and so will we.

for ex:

$\ldots\ (j-3, 3)\ (j-2, 2),\ (j-1, 1)$

Generating functions

Def: For a r.v. $\xi \in \{0,1,\ldots\}$, its generating function G_ξ satisfies

$$G(s) = \cancel{\mathbb{E}x^\xi} = \sum_{j \geq 0} s^j P(\xi = j).$$

$$\mathbb{E}(s^\xi)$$

Relation to moment generating function:

$$\left(\phi_\xi(s) = \mathbb{E}e^{s\xi} = \mathbb{E}(e^s)^\xi = G(e^s)\right)$$

Bernoulli Example :

$$\xi = \begin{cases} 1 & \text{w.p. } p \\ 0 & \text{w.p. } 1-p \end{cases} \qquad (\xi \sim \text{Bernoulli}(p))$$

$$G(s) = \mathbb{E}(s^\xi) = P(\xi=0)s^0 + P(\xi=1)\cdot s^1 = \boxed{1-p+ps}$$

ex1: X, Y iid with law Bernoulli(p)

$Q = X + Y \quad G_Q(s) = ?$

$$Q = \begin{cases} 0 & \text{w.p. } (1-p)^2 \\ 1 & \text{w.p. } 2p(1-p) \\ 2 & \text{w.p. } p^2 \end{cases}$$

$$G_Q(s) = (1-p)^2 s^0 + 2p(1-p)s^1 + p^2 s^2 = (1-p)^2 + 2p(1-p)s + p^2 s^2$$

$$\boxed{G_Q(s) = (1-p+ps)^2}$$

ex2: $X_1, \dots, X_t$ iid with same law as ξ

$Q = \sum_{i=1}^{t} X_i \quad G_Q(s) = ?$

$$G_Q(s) = \mathbb{E}(s^Q) = \mathbb{E}(s^{X_1 + X_2 + \dots + X_t}) = \mathbb{E}(\underbrace{s^{X_1} \cdot s^{X_2} \cdots s^{X_t}}_{\text{independent}})$$

$$= \underbrace{\mathbb{E}(s^{X_1})}_{G_\xi(s)} \cdot \underbrace{\mathbb{E}(s^{X_2})}_{G_\xi(s)} \cdots \underbrace{\mathbb{E}(s^{X_t})}_{G_\xi(s)} = \boxed{(G_\xi(s))^t}$$

$\leftarrow$ t-th power of G_ξ

ex3: $Q = X_1 + \dots + X_N$

where N is a random variable with generating function $G_N(s)$ and X_i are iid with same law as ξ

$$G_Q(s) = \mathbb{E}(s^Q) = \mathbb{E}_N(\underbrace{\mathbb{E}(s^Q \mid N)}_{= (G_\xi(s))^N \text{ from ex. 2}}) = \mathbb{E}(G_\xi(s)^N)$$

$$= G_N(G_\xi(s))$$

Rmk: • The radius of convergence of $G(s)$ (largest value $s > 0$ s.t. $G(s) < +\infty$) is greater than or equal to 1 since $G(1) = \sum_{i=0}^{+\infty} P(\xi = i) = 1 \implies G(s)$ is well defined for $0 \leq s \leq 1$

Properties of generating functions

1. (Sum of r.v.s) If X, Y are independent r.v.'s, then

$G_{X+Y}(s) =$ $G_X(s) \cdot G_Y(s)$ (rmk: If $X_1, \dots, X_n$ are independent, then $G_{\sum_i X_i}(s) = \prod_i G_{X_i}(s)$)

2. (Random sum of r.v.'s)

- Let $X_1, X_2, \dots$ are independent copies of a r.v. X,
- N be independent of $X_1, X_2, \dots$
- Let $T = X_1 + X_2 + \cdots + X_N$

Then $G_T(x) =$ $G_N(G_X(s))$ (i.e. $G_T = G_N \circ G_X$)

$$G_X(s) = G(s) = E(s^X) = P(X=0)\cdot s^0 + P(X=1)\cdot s^1 + \cdots + P(X=k)s^k + \cdots$$

$G(0), G(1)$, Relation to moments

1. $G(0) = \boxed{P(X=0)}$

2. $G(1) = P(X=0) + P(X=1) + \cdots = \sum_i P(X=i) = \boxed{1}$

3. $G'(s) = \dfrac{d \sum_{k=0}^{+\infty} s^k P_k}{ds}$, where $P_k = P(X=k)$

$$= \sum_{k=0}^{+\infty} P_k \frac{ds^k}{ds} = \sum_{k=1}^{+\infty} k \cdot s^{k-1} P_k \quad . \quad (\text{for } 0 \le s \le 1)$$

$$\Rightarrow \boxed{G'(1) = E(X)}$$

3. $G''(s) = \frac{d}{ds}\left(G'(s)\right) = \sum_{k=2}^{+\infty} k(k-1)\, s^{k-2} p_k$ (same as above)

$$\Rightarrow G''(1) = \sum_k (k^2 - k)\, p_k = \underbrace{\sum_k k^2 p_k}_{\mathbb{E}(X^2)} - \underbrace{\sum_k k p_k}_{\mathbb{E}(X)}$$

$$\Rightarrow \mathrm{Var}(X) = \mathbb{E}(X^2) - \left(\mathbb{E}(X)\right)^2$$

$$= G''(1) + \underbrace{\mathbb{E}(X)}_{G'(1)} - \left(\underbrace{\mathbb{E}(X)}_{G'(1)}\right)^2$$

$$\boxed{\mathrm{Var}\, X = G''(1) + G'(1) - G'(1)^2}$$

(more generally, any n-th order moment of X can be expressed using first n^{th} derivatives of G)