2023 Math 340 201. Introduction to Linear Programming

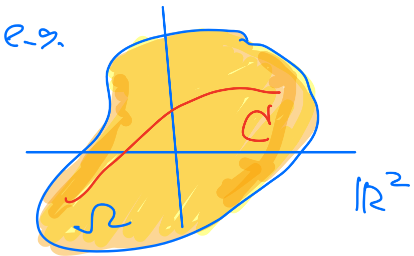
Lec 1.

- What is linear programming (LP)? [Read Vanderbei. Ch.1 and Section 2.5. Do Exercises. 1.1, 1.2,]
  - Optimization problem in general
    - Objective function,

      $\Omega \subseteq \mathbb{R}^n$ $\quad (x_1, \dots, x_n) \in \mathbb{R}^n$

      $f : \Omega \longrightarrow \mathbb{R} \cup \{+\infty\}$. $\quad f(\vec{x}) = f(x_1, \dots, x_n)$.

      e.g.



    - decision variables, $\vec{x} = (x_1, \dots, x_n)$
    - constraints. $\vec{x} \in C$ for a given set C $(\subseteq \Omega)$.
    - feasible set: C ← The given set

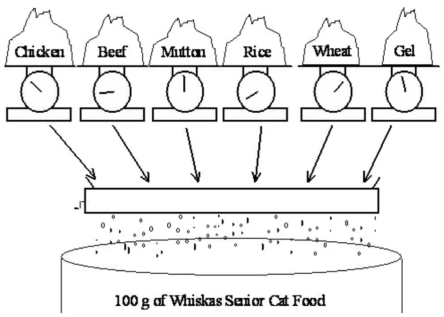
Goal: maximize/minimize $f(\vec{x})$
under the constraint $\vec{x} \in C$

← Find max/min $\vec{x}$ of f in the set C

  - A Motivating example: Blending problem from
    - https://coin-or.github.io/pulp/CaseStudies/a_blending_problem.html#problem-description

A Blending Problem

Problem Description



Nutrients per 100g.
Protein at least 8g
Fat at least 6g
Fibre at most 2g
Salt at most 0.4g

Information. Constraints

Whiskas cat food, shown above, is manufactured by Uncle Ben's. Uncle Ben's want to produce their cat food products as cheaply as possible while ensuring they meet the stated nutritional analysis requirements shown on the cans. Thus they want to vary the quantities of each ingredient used (the main ingredients being chicken, beef, mutton, rice, wheat and gel) while still meeting their nutritional standards.

The costs of the chicken, beef, and mutton are \$0.013, \$0.008 and \$0.010 respectively, while the costs of the rice, wheat and gel are \$0.002, \$0.005 and \$0.001 respectively. (All costs are per gram.) For this exercise we will ignore the vitamin and mineral ingredients. (Any costs for these are likely to be very small anyway.)

Each ingredient contributes to the total weight of protein, fat, fibre and salt in the final product. The contributions (in grams) per gram of ingredient are given in the table below.

Stuff	Protein	Fat	Fibre	Salt
Chicken	0.100	0.080	0.001	0.002
Beef	0.200	0.100	0.005	0.005
Mutton	0.150	0.110	0.003	0.007
Rice	0.000	0.010	0.100	0.002
Wheat bran	0.040	0.010	0.150	0.008
Gel	0.000	0.000	0.000	0.000

Information for the objective function and

Simplified Formulation

First we will consider a simplified problem

Assume Whiskas want to make their cat food out of just two ingredients: Chicken and Beef. We will first define our decision variables:

Identify the Decision Variables

$$
\begin{aligned}
x_1 &= \text{percentage of chicken meat in a can of cat food} \\
x_2 &= \text{percentage of beef used in a can of cat food}
\end{aligned}
$$

The limitations on these variables (greater than zero) must be noted but for the Python implementation, they are not entered or listed separately or with the other constraints.

Q: Formulate an LP problem for this simplified case.
Set up decision variables, objective function, and constraints.

Formulate the Objective Function

The objective function becomes:

$$\textbf{min}\ 0.013x_1 + 0.008x_2$$

The cost of using x_1 grams of chicken and x_2 grams of beef.

The Constraints

The constraints on the variables are that they must sum to 100 and that the nutritional requirements are met:

$$
\begin{aligned}
1.000x_1 + 1.000x_2 &= 100.0 \\
\text{Protein} \rightarrow \quad 0.100x_1 + 0.200x_2 &\geq 8.0 \\
\text{fat} \rightarrow \quad 0.080x_1 + 0.100x_2 &\geq 6.0 \\
\text{fibre} \rightarrow \quad 0.001x_1 + 0.005x_2 &\leq 2.0 \\
\text{Salt} \rightarrow \quad 0.002x_1 + 0.005x_2 &\leq 0.4
\end{aligned}
$$

$x_1 + x_2 = 100.$

and $x_1, x_2 \geq 0$.

Full Formulation

Now we will formulate the problem fully with all the variables. Whilst it could be implemented into Python with little addition to our method above, we will look at a better way which does not mix the problem data, and the formulation as much. This will make it easier to change any problem data for other tests. We will start the same way by algebraically defining the problem:

1. Identify the Decision Variables For the Whiskas Cat Food Problem the decision variables are the percentages of the different ingredients we include in the can. Since the can is 100g, these percentages also represent the amount in g of each ingredient included. We must formally define our decision variables, being sure to state the units we are using.

$$
\begin{aligned}
x_1 &= \text{percentage of chicken meat in a can of cat food} \\
x_2 &= \text{percentage of beef used in a can of cat food} \\
x_3 &= \text{percentage of mutton used in a can of cat food} \\
x_4 &= \text{percentage of rice used in a can of cat food} \\
x_5 &= \text{percentage of wheat bran used in a can of cat food} \\
x_6 &= \text{percentage of gel used in a can of cat food}
\end{aligned}
$$

decision variables.

   Note that these percentages must be between 0 and 100.

2. Formulate the Objective Function For the Whiskas Cat Food Problem the objective is to minimise the total cost of ingredients per can of cat food. We know the cost per g of each ingredient. We decide the percentage of each ingredient in the can, so we must divide by 100 and multiply by the weight of the can in g. This will give us the weight in g of each ingredient:

$$\min \$0.013x_1 + \$0.008x_2 + \$0.010x_3 + \$0.002x_4 + \$0.005x_5 + \$0.001x_6$$

objective function.

To meet the nutritional analysis requirements, we need to have at least 8g of Protein per 100g, 6g of fat, but no more than 2g of fibre and 0.4g of salt. To formulate these constraints we make use of the previous table of contributions from each ingredient. This allows us to ~~formulate the following constraints on the~~ total contributions of protein, fat, fibre and salt from the ingredients:

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 100$$

$$
\begin{aligned}
\text{protein} &\rightarrow 0.100x_1 + 0.200x_2 + 0.150x_3 + 0.000x_4 + 0.040x_5 + 0.0x_60 \geq 8.0 \\
\text{fat} &\rightarrow 0.080x_1 + 0.100x_2 + 0.110x_3 + 0.010x_4 + 0.010x_50 + 0.0x_6 \geq 6.0 \\
\text{fibre} &\rightarrow 0.001x_1 + 0.005x_2 + 0.003x_3 + 0.100x_40 + 0.150x_5 + 0.0x_6 \leq 2.0 \\
\text{salt} &\rightarrow 0.002x_1 + 0.005x_2 + 0.007x_30 + 0.002x_4 + 0.008x_5 + 0.0x_6 \leq 0.4
\end{aligned}
$$

and $x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$.

Constraints.

Definitions:

- linear functions

$f(\vec{x}) = \vec{c} \cdot \vec{x}$ $\vec{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$

$= c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$ $\boxed{\vec{c} = (c_1, \dots, c_n) \in \mathbb{R}^n \text{ constant}}$

e.g. $3x_1 + 2x_2$

- linear constraints

linear functions

$g(\vec{x}) = b$ $b \in \mathbb{R}$

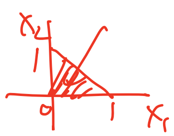
$h(\vec{x}) \leq c,$ $h(\vec{x}) \geq d$ $c, d \in \mathbb{R}.$

e.g. e.g. $3x_1 + 2x_2 = 3,$ $x_1 \geq 0$

$\begin{cases} x_1 + x_2 \leq 1 \\ x_1 \geq 0 \\ x_2 \geq 0 \end{cases}$ $x_1 - 10x_2 + 3x_4 \leq 2,$ $x_2 \geq 0$

e.g. $\begin{cases} x_1 \geq -1 \\ x_1 \leq 2 \end{cases} \Rightarrow$ feasible set

- feasible set and feasibility:
  - Feasible set: = the set of x that satisfy the constraints all together.
  - Feasibility: = there is a feasible x, that is, the feasible set is non-empty.



e.g. "$x_1 \geq 0$, $x_1 \leq -1$," $\leftarrow$ not feasible.

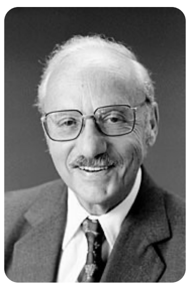
Solving an LP problem: Either find an optimal solution or that no optimal solution exists.
E.g. If the feasible set is empty then there is no optimal solution to an LP.

- **Why LP?**
  - Easy to handle.
    - Linear functions and linear constraints can be handled only by simple operations, like addition and multiplication.
  - Many practical/theoretical (and important) problems can be written as a linear programming problem

- Historical comments.

Kantorovich (a father of linear programming, duality theory, optimal transport theory, Nobel prize in Economics),

Von Neumann (a father of modern computer, Game theory, Minimax theorem)...



Dantzig (a father of operation research. Creator of Simplex method), "matching"

Ex (Dantzig) A planning problem

How to most efficiently assign N people to N tasks
(Allow a person can do multiple tasks)

Assume: • benefit of assigning person i to task j $= c_{ij}$ ← some given number

• x_{ij} = the portion of person i's time spent on doing task j

decision variables

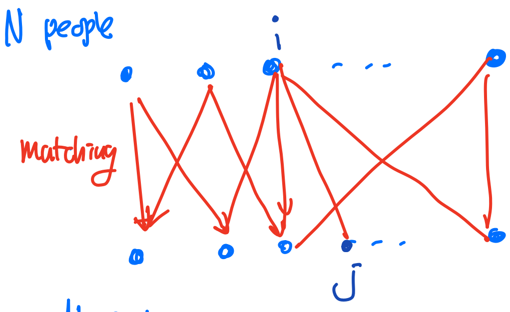
$0 \le x_{ij}$ & $\sum_{j=1}^{N} x_{ij} = 1$ for each i

• each job must be done: $\sum_{i=1}^{N} x_{ij} = 1$. for each j

e.g. For job 2

$x_{12} + x_{22} + \cdots + x_{N2} = 1$

N people



N tasks

e.g. $x_{11} + x_{12} + \cdots + x_{1N} = 1$

maximize $\sum_{i=1}^{N} \sum_{j=1}^{N} c_{ij} x_{ij}$ ← objective function

subject to

constraints that determine the feasible set

$$\sum_{j=1}^{N} x_{1j} = 1 \qquad \sum_{i=1}^{N} x_{i1} = 1$$

$$\sum_{j=1}^{N} x_{2j} = 1 \qquad \sum_{i=1}^{N} x_{i2} = 1$$

$$\vdots \qquad \vdots$$

$$\sum_{j=1}^{N} x_{Nj} = 1 \qquad \sum_{i=1}^{N} x_{iN} = 1$$

$$x_{ij} \ge 0 \quad \text{for } 1 \le i, j \le N$$

$N \times N$ variables x_{ij} $1 \le i \le N$, $1 \le j \le N$
& $N + N + N \times N$ constraints!

For $N = 60$? (e.g. Math dept. of UBC has about 60 regular faculty members)

What about $N = 1000$, 10000, 100000? ...

Key points:

- Practical LP problems are NOT easy to solve due to many variables (high dimensions).

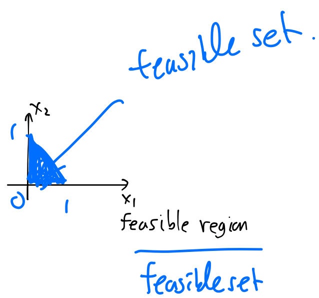
- There are two approaches to solving an LP problem:
  - Geometric

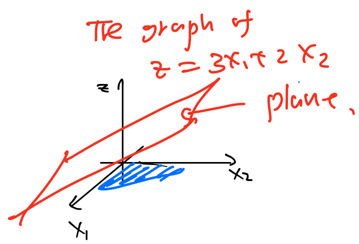
Example. Maximize $3x_1+2x_2$

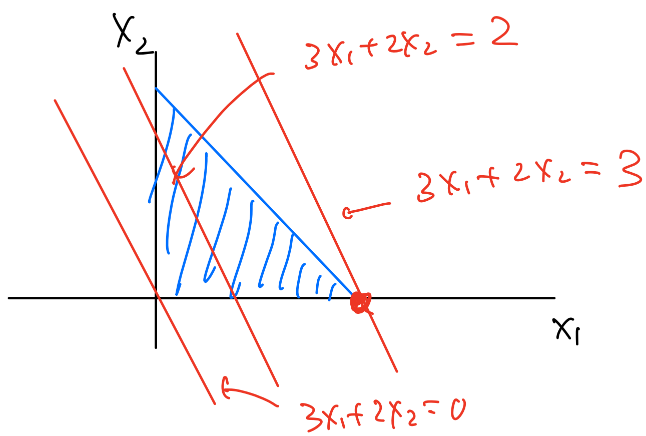
subject to
$$x_1+x_2 \le 1$$
$$x_1 \ge 0$$
$$x_2 \ge 0$$

<sol>

Geometric method:







algebraic method: Maximize $3x_1+2x_2$

subject to
$$x_1+x_2 \le 1$$
$$x_1 \ge 0$$
$$x_2 \ge 0$$

this inequality condition is inconvenient to handle.

Introduce $x_3 = 1-x_1-x_2$ slack variable.

$x_1+x_2 \le 1 \iff x_1+x_2+x_3=1$

Rewrite the problem: maximize $3x_1+2x_2$
$$x_1+x_2+x_3=1$$
$$x_1 \ge 0$$
$$x_2 \ge 0$$

How to handle this?

Use $x_1 = 1-x_2-x_3$

Then
$$3x_1+2x_2 = 3(1-x_2-x_3)+2x_2$$
$$= 3 - x_2 - 3x_3$$

Then the problem is:

maximize $3 - x_2 - 3x_3$
$$x_1+x_2+x_3=1$$
$$x_1, x_2, x_3 \ge 0.$$

So, max 3 at $x_2=0, x_3=0$ at $(x_1,x_2)=(1,0)$ ∎

Lec 1.