Lecture 4 (TuTh)

- Basic Geometry
  - Geometry of linear constraints. Vanderei 2.5
  - Convex sets. Vanderbei 10.1.

Next lecture. Simplex method
Pre-lecture reading material:
- Matrix description. Vanderbei 6.1

Simplex method: Vanderei 2.5, 2.1, 2.2
- Geometric motivation:
- Examples
- Dictionary method

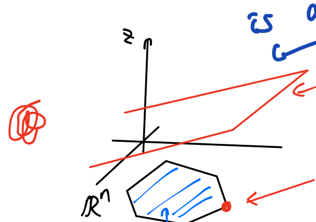
Theoretical considerations

Wisdom for us:
- Use geometry to get intuition for finding effective algebraic algorithms.

Observation to generalize:

- The feasible set is of polygonal shape (polyhedron in general dimensions).
- The maximum occurs at the vertex (corner) of the polygonal shape (polyhedron in general dimensions).

The graph is a hyperplane. In $\mathbb{R}^{n+1}$



Objective function (which is linear)
$f(x_1, \dots, x_n) = c_1 x_1 + c_2 x_2 + \dots + c_n x_n$

The maximum point is a vertex of the polyhedron

feasible region by linear constraints is a polyhedron

A fundamental geometric $\boxed{\text{concept}}$: Convexity.

It is a central notion in "optimization problems":

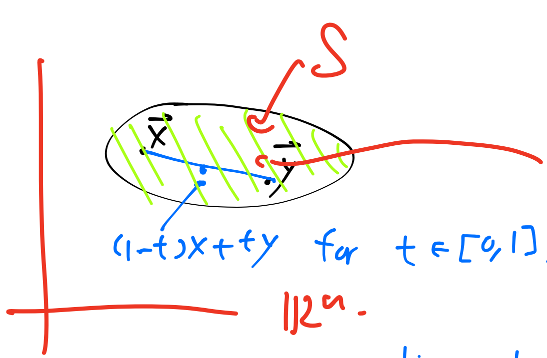
- $\underline{\text{Convex}}$ optimization $\subset$ optimization. (non-convex optimization)

- $\underline{\text{Convex sets}}$

A set $S \subseteq \mathbb{R}^n$ is $\underline{\text{convex}}$

if $\forall\, \vec{x}, \vec{y} \in S$, $t \in [0,1]$,

$(1-t)\vec{x} + t\vec{y}$ is again in S

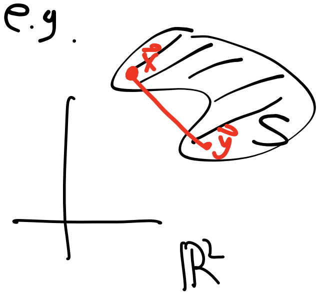


line segment connecting $\vec{x}$ and $\vec{y}$.

that is, for any two points in S, the line segment connecting them does belong to S.

Geometrically.

e.g.

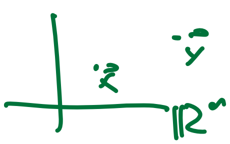


is $\underline{\text{NOT convex.}}$

e.g. $S = \mathbb{R}^n$ is $\underline{\text{convex.}}$

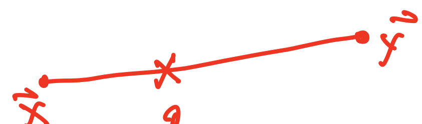
e.g. A point set is convex.

e.g. $S = \{\vec{x}, \vec{y}\}$ Suppose $\vec{x} \neq \vec{y}$.



is $\underline{\text{not}}$ convex.

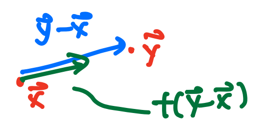
$\underline{\text{Rmk}}$ line segment



each point here can be written as $(1-t)\vec{x} + t\vec{y}$ for some $t \in [0,1]$.

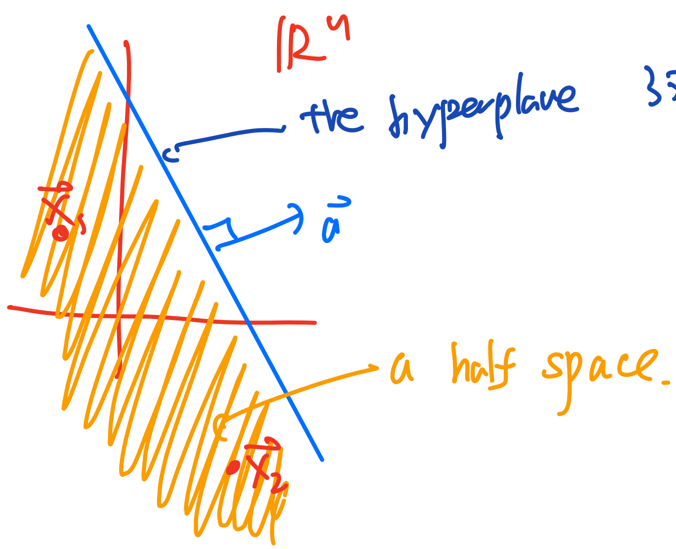
$\underline{\text{Note}}$ $(1-t)\vec{x} + t\vec{y}$

$= \vec{x} + t(\vec{y} - \vec{x})$



Half spaces.

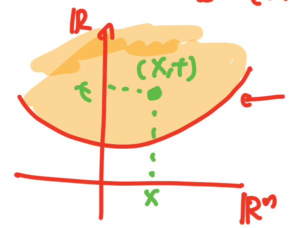
$H = \{ \vec{x} \in \mathbb{R}^n \mid \vec{a} \cdot \vec{x} \leq b \}$ is <u>Convex.</u>



Side Rmk A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be a <u>convex function</u> if the set

$$\underbrace{\{ (x, t) \in \mathbb{R}^n \times \mathbb{R} \mid f(x) \leq t \}}_{\substack{\text{the set of points in } \mathbb{R}^{n+1} \\ \text{"above" the graph of } f.}}$$

is convex.



(Exercise Define concavity of a function similarly.)

proof Suppose $\vec{x}_1, \vec{x}_2 \in H$. ← any $\vec{x}_1, \vec{x}_2 \in H$.

It means $\vec{a} \cdot \vec{x}_1 \leq b$ and $\vec{a} \cdot \vec{x}_2 \leq b$.

The line segment: pick any $t \in [0, 1]$ (i.e. $0 \leq t \leq 1$)

Let $\vec{x}_t = (1-t)\vec{x}_1 + t \vec{x}_2$.

Check the condition:

$$\begin{aligned}
\vec{a} \cdot \vec{x}_t &= \vec{a} \cdot \left((1-t)\vec{x}_1 + t \vec{x}_2 \right). \\
&= \underbrace{(1-t)}_{\geq 0} \underbrace{\vec{a} \cdot \vec{x}_1}_{\leq b} + \underbrace{t}_{\geq 0} \underbrace{\vec{a} \cdot \vec{x}_2}_{\leq b.} \\
&\leq (1-t)\, b + t\, b \\
&= b
\end{aligned}$$

(because $0 \leq t \leq 1$
$0 \leq 1-t \leq 1$
$0 \leq t \leq 1$) →

We checked $\vec{a} \cdot \vec{x}_t \leq b$, for any $t \in [0, 1]$.

So $\underbrace{\vec{x}_t = (1-t)\vec{x}_1 + t\vec{x}_2 \text{ for any } t \in [0,1]}$ belongs to H.

any point in the line segment between $\vec{x}_1$ and $\vec{x}_2$.

Therefore H is convex. ∎

Q. Find a wrong statement:

1) An intersection of two convex sets is again convex. True
2) A union of two convex sets is convex. No
3) A straight line is a convex set. Yes
4) An empty set is convex. Yes

A set $S \subseteq \mathbb{R}^n$ is <u>convex</u> if $\forall \vec{x}, \vec{y} \in S$, $t \in [0,1]$, $(1-t)\vec{x} + t\vec{y}$ is again in S

E.g. An empty set is convex. Why?

E.g. Hyperplanes are convex.

E.g. Intersections of hyperplanes are [redacted]

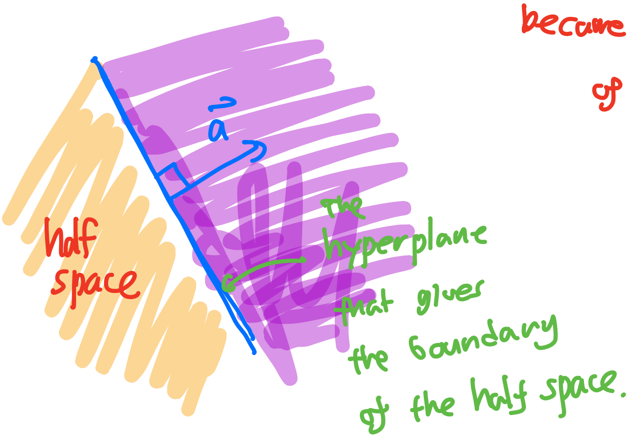
- $L = \{ \vec{x} \in \mathbb{R}^n \mid \vec{a} \cdot \vec{x} = b \}$ is convex.

because L is the intersection of two half spaces.

$H_1 = \{ \vec{x} \in \mathbb{R}^n \mid \vec{a} \cdot \vec{x} \leq b \}$

and $H_2 = \{ \vec{x} \in \mathbb{R}^n \mid \vec{a} \cdot \vec{x} \geq b \}$

· And an intersection of two convex sets is again convex.



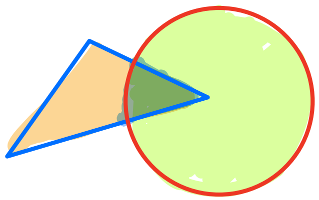
Theorem Let $S_1, S_2 \subset \mathbb{R}^n$.

Suppose S_1 and S_2 are both convex.

Then $S_1 \cap S_2$ is a convex set.

Remark: When $S_1 \cap S_2 = \emptyset$, it is still convex.

e.g.



Proof. Suppose $\vec{x}_1, \vec{x}_2 \in S_1 \cap S_2$.

$\forall t \in [0,1]$, consider

"for every"

$$\vec{x}_t = (1-t)\vec{x}_1 + t\,\vec{x}_2.$$

Note. $\vec{x}_1, \vec{x}_2 \in S_1$ which is convex,

so $\vec{x}_t \in S_1$

. $\vec{x}_1, \vec{x}_2 \in S_2$ which is convex,

so $\vec{x}_t \in S_2$.

Therefore $\vec{x}_t \in S_1 \cap S_2$

This shows $S_1 \cap S_2$ is convex. ▨

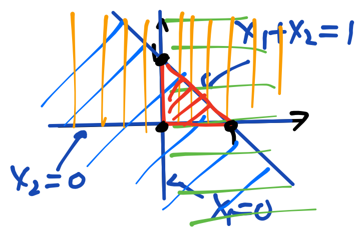
Intersections of half spaces

As any half space is convex,

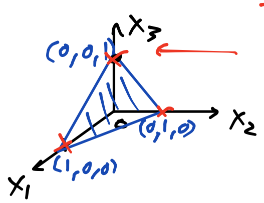
for half spaces $H_i = \{\vec{x} \in \mathbb{R}^n \mid \vec{a}_i \cdot \vec{x} \le b_i\}$, $i = 1, \dots, N$

$H_1 \cap H_2 \cap \cdots \cap H_N$ is convex.

e.g. $\begin{cases} x_1 + x_2 \le 1 \\ x_1 \ge 0, \\ x_2 \ge 0 \end{cases}$ ← three linear constraints.



e.g. $\begin{cases} x_1 + x_2 + x_3 \le 1 \\ x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0 \end{cases}$

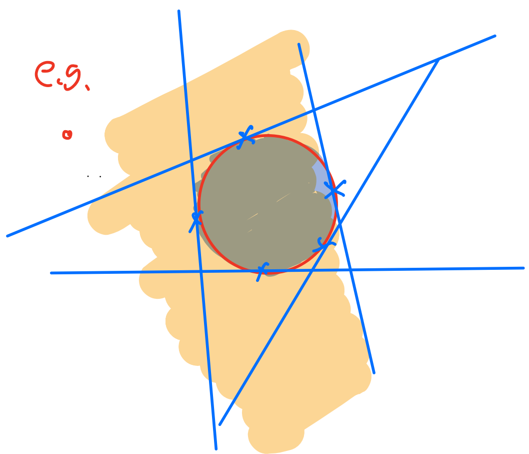


the vertices are where the equality cases of $\underbrace{\text{the constraints}}_{\text{hyperplanes}}$ meet together.

[We will see more about convex sets later in the course.]

Optional remarks

e.g.



- Can you realize a disk as an intersection of half spaces?

Remark A convex set can be written as an intersection of (possibly infinitely many) half spaces.

e.g.