Lecture 10

Doubly stochastic transition matrices

How to determine stationary distribution(s)?

Generally, we have to solve:

$$\begin{cases} \pi \tilde{P} = \pi \\ \sum_i \pi_i = 1 \end{cases}$$

→ This can be done using Gaussian elimination, but that can be tedious as the number of states ↗

→ Here, we will see a specific case of MC, where the stationary distribution is straightforward to find.

1. Doubly stochastic transition matrices

Recall: The transition matrix $\tilde{P}$ is always a *stochastic matrix*, i.e.,

The sum of terms in each row = 1

$$\sum_j p_{i,j} = \underline{\sum_j P(X_1 = j \mid X_0 = i)} = 1 \qquad \underline{\ 0\ } \le p_{i,j} \le \underline{\ 1\ }.$$

Def: $\tilde{P}$ is called doubly stochastic if $\sum_i p_{ij} = 1$ for all j

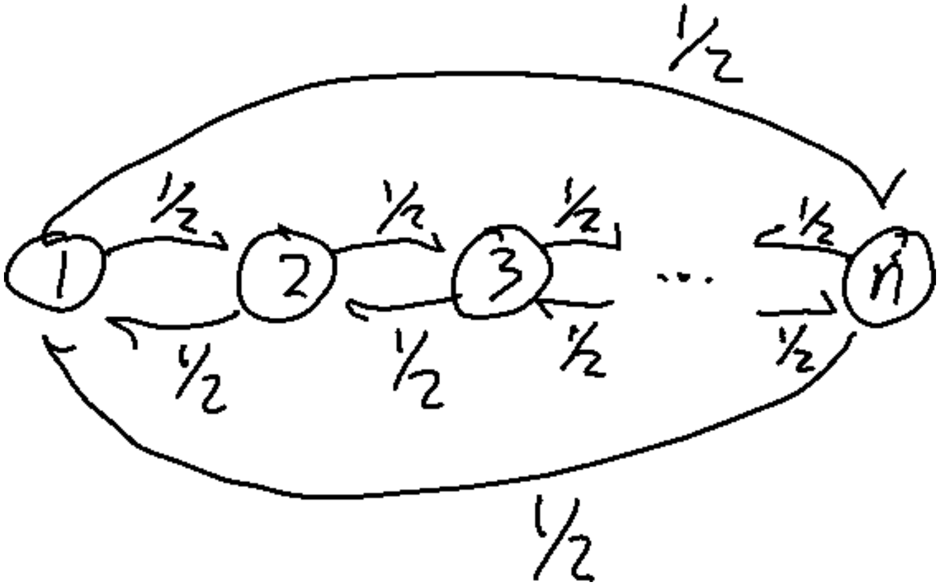
(i.e. the sum of terms in each column is 1)

Rmk: in other words $\tilde{P}$ is doubly stochastic if ${}^t\tilde{P}$ is stochastic.

$$({}^t\tilde{P})_{ij} = \tilde{P}_{ji}$$

cf. Homework 2.

Example: Circular random walk



$$\tilde{P} = \begin{bmatrix} 0 & \frac{1}{2} & & (0) & \frac{1}{2} \\ \frac{1}{2} & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \frac{1}{2} \\ \frac{1}{2} & (0) & & \frac{1}{2} & 0 \end{bmatrix}$$

$\tilde{P}$ is doubly stochastic.

Q: What is the stationary distribution?

$\rightarrow \left(\frac{1}{n}, \ldots\ldots, \frac{1}{n}\right)$ = Unif $\{1, \ldots, n\}$ is stationary

$(\pi \tilde{P} = \pi)$

Stationary dist. is uniform for finite doubly stochastic transition matrices

Prop: If an MC has n states and the transition matrix $\tilde{\boldsymbol{P}}$ is doubly stochastic, then $\pi = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$ is a stationary distribution.

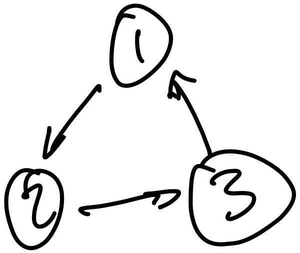
Proof: Let's consider $\pi = (\frac{1}{n}, \ldots, \frac{1}{n})$

$$\underbrace{(\pi P)_j}_{\text{we want to show that it is equal to } \pi_j \ (= \frac{1}{n})} = \sum_i \underbrace{\pi_i}_{\frac{1}{n}} P_{ij} = \frac{1}{n} \underbrace{\sum_i P_{ij}}_{=1} = \frac{1}{n} = \pi_j \qquad \boxtimes$$

doubly stochastic

Examples

(1)

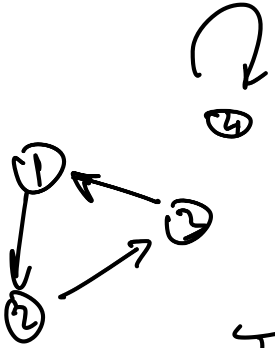


$$\begin{array}{c|ccc|c} & 1 & 2 & 3 & \\ \hline & 0 & 1 & 0 & 1 \\ & 0 & 0 & 1 & 2 \\ & 1 & 0 & 0 & 3 \end{array}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \to \text{Doubly stochastic}$$

$\Rightarrow \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ is stationary.

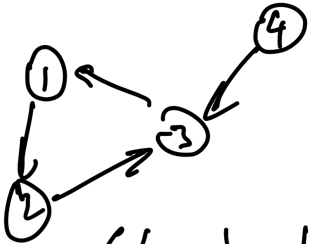
(2)



$$\begin{pmatrix} & 1 & & \\ & & 1 & \\ 1 & & & \\ & & & 1 \end{pmatrix} \text{ Doubly stochastic}$$

$\rightarrow \pi = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$

(3)



$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Not doubly stoch.

$\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0\right), (0, 0, 0, 1)$ are stationary

(Exercise: Are these doubly stochastic? Stationary distribution?)

Summary of the results (How we can study a MC)

Discrete time MC

$\downarrow$

transition probabilities

- transition matrix
  - n-step transition probabilities (C-K equation)
    - P^n
    - $\sum_n P^n_{ii} < \infty$ or $= \infty$ $\longrightarrow$ recurrence or transience
  - stationary distribution ($\pi P = \pi$)
- transition diagram
  - communication class
    - infinite state space $\longrightarrow$ $\sum_n P^n_{ii} < \infty$ or $= \infty$
    - finite state space
      - closed class
        - positive recurrent
        - $\mathbb{E}(\text{time to return}) = \frac{1}{\pi_i}$
      - non-closed class
        - transience (no stationary distribution)
  - period

Unique stationary distribution $\longleftrightarrow$ positive recurrent

Aperiodicity $d = 1$ + Unique stationary distribution

$\downarrow$

ergodicity $\longrightarrow$ $P^n \xrightarrow[n \to \infty]{} \pi$