Remarks: $\rightarrow$ HW 4 $\rightarrow$ Lecture 15

Lecture 14

Branching processes and generating functions Part 2:

- Generating function of a branching process: $G_{Z_n}(s)$
- Probability of eventual extinction

HW5 $\longrightarrow$ Result: P_e is a solution of $G_\xi(s) = s$

where ξ is the reproduction law

Recall

- Branching process: $$Z_{n+1} = \sum_{k=1}^{Z_n} X_k \qquad X_k \text{ iid} \sim \underset{\text{reproduction law}}{\xi}$$

- Generating function: $$G_X(s) = \mathbb{E}(s^X) = \sum_{k=0}^{+\infty} s^k P(X=k)$$

X: r.v. $\in \{0, 1, \dots\}$

$$= 1 \cdot P(X=0) + s \cdot P(X=1) + s^2 \cdot P(X=2) + \dots$$

Q: Given G_X, how do you find $P(X = k)$?

(i.e. if I know the generating function $G_X(s)$, can I obtain the p.m.f. of X (i.e. $f(x) = P(X=k)$?))

ex: $G_X(s) = \frac{1}{3} + \frac{1}{3}s + \frac{1}{3}s^2 \longrightarrow P(X=1) = \frac{1}{3}$

$= P(X=0).s^0 + P(X=1).s^1 + P(X=2).s^2 + \dots$

$\Rightarrow$ One can identify $P(X=k)$ with the coefficient associated with s^k in $G_X(s)$.

ex : $G_X(s) = \dfrac{1}{1 - \frac{1}{2}s}$

$= \sum_{k=0}^{+\infty} \left(\frac{1}{2}\right)^k s^k$

$\Rightarrow P(X = k) = \left(\frac{1}{2}\right)^k$

(Rmk: $1 + s + s^2 + \dots + s^N$
$\quad \parallel$
$\dfrac{1 - s^{N+1}}{1 - s} \xrightarrow[N \to \infty]{} \dfrac{1}{1-s}$ if $|s| < 1$)

Another way to obtain $P(X = k)$ from $G_X(s)$

$G_X(s) = \sum_{k=0}^{+\infty} s^k \cdot P_k \Rightarrow \boxed{G_X(0) = P_0}$ $\quad (P_k = P(X = k))$

$\Rightarrow G_X'(s) = \sum_{k=1}^{+\infty} k s^{k-1} P_k \Rightarrow G_X'(0) = 1 \cdot P_1 + 0 \cdot P_2 + 0 \cdot P_3 + \dots$

$\boxed{G_X'(0) = P_1}$

By induction, $\boxed{P(X = k) = \dfrac{G_X^{(k)}(0)}{k!}}$

$G_n(s)$
$\shortparallel$

What is $G_{Z_n}(s)$? $\quad (Z_{n+1} = X_1 + \cdots + X_{Z_n})$

Notation: $G_n(s) := G_{Z_n}(s)$ and $G(s) := G_1(s)$.

Recall: if $Q = X_1 + \cdots + X_N$, r.v. X_i iid with generating function G

$$G_Q = G_N(G(s))$$

$(Z_n = X_1 + \cdots + X_{Z_{n-1}})$

$$\Rightarrow G_{Z_{n+1}} = G_{n+1}(s) = G_{Z_n}(G(s)) = G_n(G(s)) = G_{Z_{n-1}}(G(G(s)))$$

$$= (\cdots) = G(G(G(\cdots(G(s)))))$$

$$= \underbrace{G \circ G \circ \cdots \circ G}_{n+1 \text{ times}}(s)$$

Generating function of Z_n is n compositions of G

(Notation: $G_n := G_{Z_n}$)

Prop: $G_n(s) = \underbrace{G \circ \cdots\cdots \circ G}_{n \text{ times}}(s) = G(G(G\ldots(s)))$

Corollary: $G_{m+n}(s) = G_m \circ G_n(s)$

Fact: $P(Z_n = k) = \begin{cases} \text{coefficient associated with } s^k \text{ for } G_n \\ \dfrac{G_n^{(k)}(0)}{k!} \end{cases} \left(= \dfrac{1}{k!} \dfrac{d^k G_n}{ds^k}(0) \right)$

Ex) Suppose individual offspring distribution satisfies

$\xi \sim$Unif$\{0,1,2\}$.

What are $P(Z_2 = 0), P(Z_2 = 1)$?

$$G_\xi(s) = \frac{1}{3} + \frac{1}{3}s^1 + \frac{1}{3}s^2 = G_1(s)$$

$$\Rightarrow G_2(s) = G_1 \circ G_1(s) = \frac{1}{3} + \frac{1}{3}G_1(s) + \frac{1}{3}G_1(s)^2$$

$$= \frac{1}{3} + \frac{1}{3}\left(\frac{1}{3} + \frac{1}{3}s + \frac{1}{3}s^2\right) + \frac{1}{3}\left(\frac{1}{3} + \frac{1}{3}s + \frac{1}{3}s^2\right)^2$$

$$= \underbrace{\left(\frac{1}{3} + \frac{1}{9} + \frac{1}{27}\right)}_{P(Z_2=0)} + \underbrace{\left(\frac{1}{9} + \frac{1}{3}\,\frac{2}{9}\right)}_{P(Z_2=1)}s + \cdots$$

Alternatively:

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

$$P(Z_2 = 1) = G_2'(0) = (G_1 \circ G_1)'(0)$$
$$= G_1'(G_1(0)) \cdot G_1'(0)$$

$$G_1(s) = \frac{1}{3} + \frac{1}{3}s + \frac{1}{3}s^2 \Rightarrow G_1'(s) = \frac{1}{3} + \frac{2}{3}s$$

$$\Rightarrow P(Z_2 = 1) = \left(\frac{1}{3} + \frac{2}{3}(G_1(0))^2\right) \cdot \left(\frac{1}{3}\right)$$
$$= \left(\frac{1}{3} + \frac{2}{3} \cdot \frac{1}{9}\right) \cdot \frac{1}{3}$$

Mean and variance of Z_n

$= \mathbb{E}(\xi^2) - (\mathbb{E}(\xi))^2$

Prop: Let $\mu = \mathbb{E}\xi$, $\sigma^2 = Var(\xi)$. Then

1. $\mathbb{E}Z_n =$ μ^n

2. $$Var(Z_n) = \begin{cases} n\sigma^2 & \text{if } \mu = 1 \\ \frac{\sigma^2(\mu^n - 1)\mu^{n-1}}{\mu - 1} & \text{if } \mu \neq 1 \end{cases}$$

Proof: Recall from lecture 13: $\bullet\ \mathbb{E}(\xi) = G_\xi'(1)$

$\bullet\ \mathrm{Var}(\xi) = G_\xi''(1) + G'(1) - (G'(1))^2$

First, how do we write μ, σ^2 in terms of G?

$$\mathbb{E}(Z_n) = G_n'(1) = \frac{d}{ds}(G_n)(1)$$

$$= \frac{d}{ds}\Big(\underbrace{G \circ G \circ \cdots \circ G}_{n}\Big)(1)$$

$$= \frac{d}{ds}\big(G(G_{n-1})\big)(1)$$

$$= G'\big(\underbrace{G_{n-1}(1)}\big) \cdot G_{n-1}'(1)$$

$$= \mathbb{E}(1^\xi) = \sum_k 1 \cdot P(X=k) = 1$$

$$= G'(1) \cdot G'(G_{n-2}(1))$$

$$= (\cdots\cdots) = (\underbrace{G'(1)}_{\mu\ (=\mathbb{E}(\xi))})^n = \boxed{\mu^n}$$

Similarly, using $Var(Z_n) = G_n''(1) + G_n'(1) - (G_n'(1))^2$

we can prove the result $\boxtimes$

Example: Suppose $\xi \sim \text{Bin}\left(4, \frac{1}{2}\right)$.

What is $\mathbb{E}Z_n$?

What is $\mathbb{E}Z_n^2$?

$$\mathbb{E}(\xi) = 4 \times \frac{1}{2} = 2$$

$$Var(\xi) = 4 \times \frac{1}{2} \times \frac{1}{2} = 1$$

$$\Rightarrow \quad \mathbb{E}(Z_n) = 2^n$$

$$\mathbb{E}(Z_n^2) = Var(Z_n) + (\mathbb{E}(Z_n))^2$$

$$= 2^{n-1} + 2^{2n}$$