Lecture 17

Section 1: Discrete time Markov chains ✓ Chapter 4 of book

Section 2: Exponential distribution and Poisson process Chapter 5

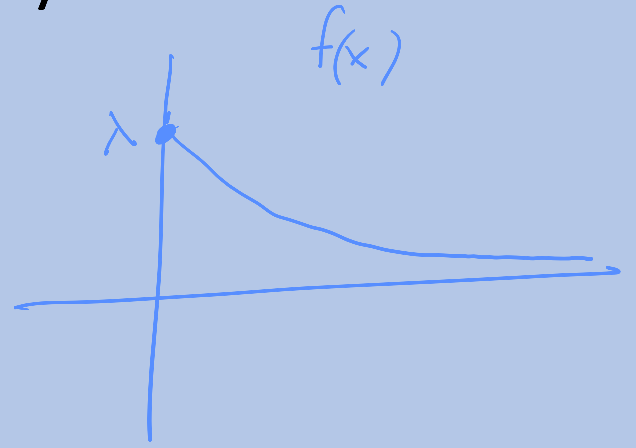
Section 3: Continuous time Markov chains Chapter 6

This lecture: Exponential distribution

Exponential distribution

Def: We call X an exponential random variable (rv), with parameter $\lambda > 0$, i.e., $X \sim$ __$Exp(\lambda)$__, if X has density:

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$



Exponential random variables often model __waiting times__.

Warm-up

Let $X \sim Exp(\lambda)$.

What is $P(X > t)$?

$$P(X > t) \underset{\uparrow}{=} P(X \geq t) = \int_t^\infty \lambda e^{-\lambda x}\, dx = -e^{-\lambda x}\Big|_t^\infty = 0 - -e^{-\lambda t} = \boxed{e^{-\lambda t}}$$

since $P(X=t) = 0$

What is $\mathbb{E}X$?

$$\mathbb{E}X \underset{\uparrow}{=} \int_0^\infty P(X \geq t)\, dt = \int_0^\infty e^{-\lambda t}\, dt = \frac{1}{-\lambda} e^{-\lambda t}\Big|_0^\infty = \boxed{\frac{1}{\lambda}}$$

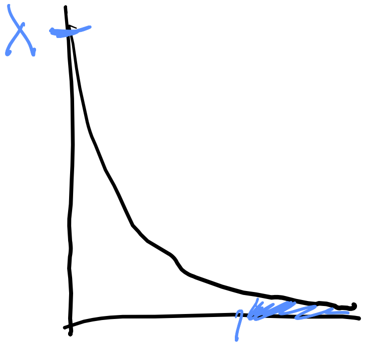
true for any non-neg r.v.

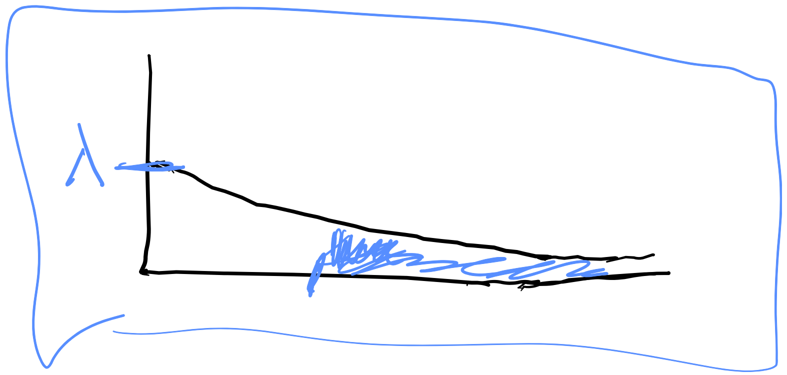
Properties of exponential rv's

Fact: Let $X \sim Exp(\lambda)$. Then,

- $\mathbb{E}X = \underline{\quad 1/\lambda \quad}$
- $Var(X) = \underline{\quad 1/\lambda^2 \quad}$ Exercize for you
- $P(X > t) = \underline{\quad P(X \geq t) = e^{-\lambda t} \quad}$

Which exponential density gives higher expectation value?



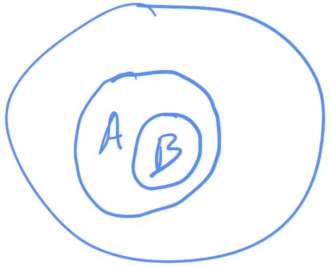


Larger expected value

Example) Suppose $T \sim Exp\left(\frac{1}{10}\right)$ is the time you have to wait for the bus. Given that you have been waiting for 20 minutes, what is the chance that the bus comes within the next 10 minutes? On average, how much longer must you wait?

$$P(\underbrace{T\in[20,30]}\mid \underbrace{T\geq 20}) = \frac{P(\{T\in[20,30]\} \,\&\, \{T\geq 20\})}{P(\{T\geq 20\})}$$

$$= \frac{P(T\in[20,30])}{P(T\geq 20)} = \frac{P(\{T\geq 20\}\setminus\{T>30\})}{P(T\geq 20)} = \frac{P(T\geq 20)-P(T>30)}{P(T\geq 20)}$$



$$= \frac{e^{-\frac{1}{10}\cdot 20} - e^{-\frac{1}{10}\cdot 30}}{e^{-\frac{1}{10}\cdot 20}} = 1 - e^{\frac{1}{10}\cdot 20 - \frac{1}{10}\cdot 30} = \boxed{1-e^{-1}}$$

Notice: $P(\text{bus comes in 1st 10 min})$

$$= P(T\in[0,10]) = 1 - P(T\geq 10) = 1 - e^{-\frac{10}{10}} = \boxed{1-e^{-1}}$$

Memorylessness

Prop: Let $X \sim Exp(\lambda)$. Then,

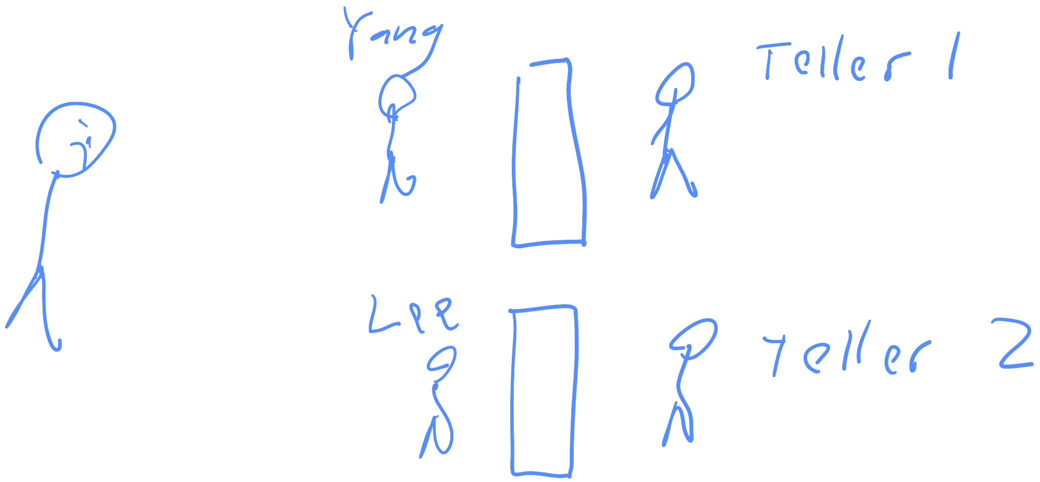
$P(X \geq t + s | X \geq s) =$ $P(X \geq t)$

$[(X - s) | X \geq s] \sim Exp(\lambda)$

Q) $P(X < t + s | X \geq s) =$ $P(X < t)$

$\mathbb{E}[X - s | X \geq s] = \mathbb{E}[Exp(\lambda)] = \frac{1}{\lambda}$

Example) You enter a bank which has two tellers, currently servicing Yang and Lee. Service times are iid $Exp(\lambda)$. What is the chance you leave last?



Once either Yang or Lee exits you start being served. The time for you to be served $\sim Exp(\lambda)$. Remaining time for whoever is still being served (Yang or Lee) is also $Exp(\lambda)$. Thus, $P(\text{you leave last}) = P(X > Y)$ where $X, Y \overset{iid}{\sim} Exp(\lambda)$. By symmetry $P(X>Y) = \frac{1}{2}$.

Minimum of two exponential rv's

Q) Suppose $X \sim Exp(\lambda_1), Y \sim Exp(\lambda_2)$ are independent. What is the distribution of $Z = \min(X, Y)$?

$$P(Z > t) = P(\min(X,Y) > t) = P(\{X > t\} \cap \{Y > t\})$$

$$\underset{\text{independence}}{=} P(X > t) \cdot P(Y > t) = e^{-\lambda_1 t} \cdot e^{-\lambda_2 t} = e^{-(\lambda_1 + \lambda_2) t}$$

$$= P(Exp(\lambda_1 + \lambda_2) > t)$$

So $\boxed{Z \sim Exp(\lambda_1 + \lambda_2)}$

Fastest to finish

Q) Let $X \sim Exp(\lambda_1), Y \sim Exp(\lambda_2)$. What is $\underline{P(X < Y)}$?

Ex) $\lambda_1 = 1$, $\lambda_2 = 2$. Symmetry argument:

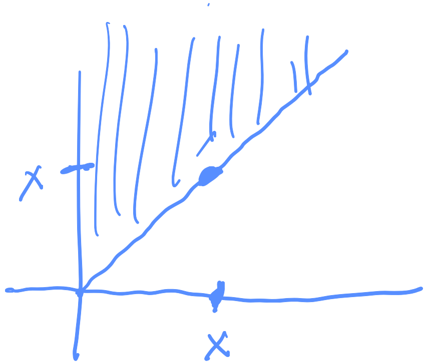
Note: $Y \overset{dist}{=} \min(Y_1, Y_2)$ $Y_1, Y_2 \overset{iid}{\sim} Exp(1)$

$P(X<Y) = P(X < \min(Y_1, Y_2)) = (*)$ Note $X, Y_1, Y_2 \overset{iid}{\sim} Exp(1)$

The chance of any given ordering is the same

$(*) = \frac{1}{3} = \frac{\text{\# of orderings w/ } X \text{ 1st}}{\text{\# of orderings}} = \frac{2}{3!}$

Generally, $P(X<Y) = \iint_{x<y} \lambda_1 e^{-\lambda_1 x} \cdot \lambda_2 e^{-\lambda_2 y}\, dy\, dx$



$$= \int_0^\infty \int_x^\infty \lambda_1 e^{-\lambda_1 x} \cdot \lambda_2 e^{-\lambda_2 y}\, dy\, dx$$

$$= \int_0^\infty \lambda_1 e^{-\lambda_1 x} P(Y > x)\, dx$$

$$= \int_0^\infty \lambda_1 e^{-\lambda_1 x} \cdot e^{-\lambda_2 \cdot x}\, dx$$

$$= \lambda_1 \int_0^\infty e^{-(\lambda_1+\lambda_2)x}\, dx$$

$$= \lambda_1 \cdot \frac{1}{-(\lambda_1+\lambda_2)} e^{-(\lambda_1+\lambda_2)x}\Big|_0^\infty$$

$$= \boxed{\frac{\lambda_1}{\lambda_1+\lambda_2}}$$

Q) You enter a bank which has two tellers, currently servicing Yang and Lee. Service times are iid $Exp(\lambda)$. What is the expected time until you are finished being served?

T = time until you are finished

$= \min(Y, L) + T_2$

$\underline{\underline{\text{dist}}}$ $Exp(2\lambda) + Exp(\lambda)$

indep

$Y, L \sim Exp(\lambda)$

times for Yang and Lee to be served

$T_2 \sim Exp(\lambda)$

time to be served once you reach a teller

$\mathbb{E}T = \mathbb{E}[Exp(2\lambda)] + \mathbb{E}[Exp(\lambda)]$

$= \underline{\frac{1}{2\lambda} + \frac{1}{\lambda}}$