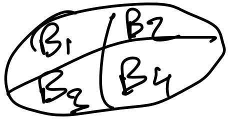
Lecture 7

- A few notes about conditional expectation
- Examples
- Recurrence/transience for finite MC
- Mean time in a transient state
- Gambler’s ruin
- Jupyter
- Questions

Conditional expectation (a few notes)

Ω



Recall:

A partition of the sample space is a set of events which are disjoint and cover Ω, $B_1, B_2, \dots$

Law of total probability: $P(A) = \sum_i P(A \cap B_i) = \sum_i P(A|B_i) \cdot P(B_i)$

Analog for computing expected value: $E[X] =$ ____________

$E(X) = \sum_i E(X|B_i) \cdot P(B_i) \xrightarrow{\text{conditional version}} E(X|A) = \sum_i E(X|A, B_i) \cdot P(B_i|A)$

Example MC version with integer state space:

$E[X_n|X_0 = 1] = \sum_i E(X_n | X_0 = 1, X_1 = i) \cdot P(X_1 = i | X_0 = 1)$

$= \sum_i E(X_n | X_1 = i) P_{1i} = \sum_i E(X_{n-1} | X_0 = i) P_{1i}$

Markov property, homogeneity, and conditioning

Homogeneity implies: $P(X_n = j|X_0 = i) = P(X_{n+s} = j|X_s = i)$

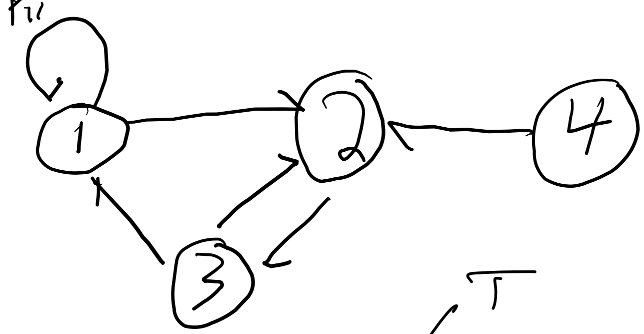
It follows that:

$E[X_{n+s}|X_s = i] = E(X_n | X_0 = i)$

And

$E[X_n|X_0 = i, X_1 = j] = E(X_n | X_1 = j)$

Example



T

Starting in State 1, compute the expected time to reach State 3.

$$\mathbb{E}(T \mid X_0 = 1) = \sum_i \mathbb{E}(T \mid X_0 = 1, X_1 = i)$$

$$= \mathbb{E}(T \mid X_1 = 1) P_{11} + \mathbb{E}(T \mid X_1 = 2) P_{12}$$

(all other $P_{1i} = 0$)

$$= (1 + \mathbb{E}(T \mid X_0 = 1)) P_{11} + 1 \cdot P_{12}$$

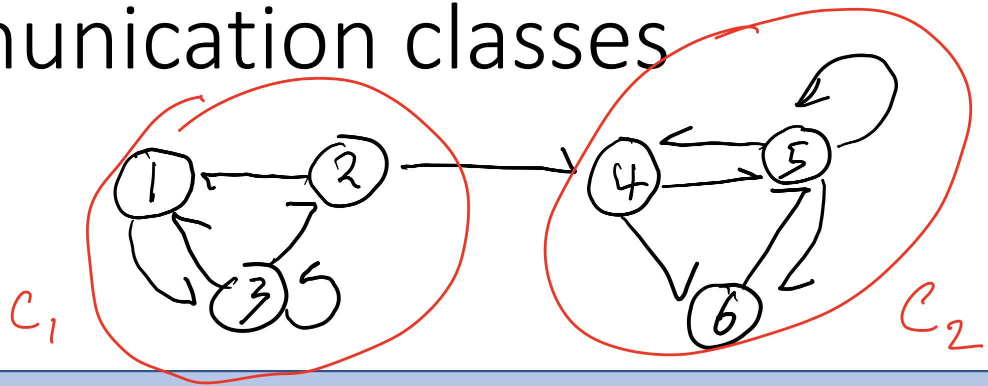
$$\Rightarrow \boxed{\mathbb{E}(T \mid X_0 = 1) = \frac{P_{11} + P_{12}}{1 - P_{11}}}$$

Recurrence/transience for finite communication classes

→ to be used in practice to show recurrence | transience

Example:

$P(X_1=j \mid X_0=i)$



C_1 closed : No ($P_{24}>0$)

C_2 closed : Yes ($P_{ij}=0$)

Defs A communication class C is closed if

$P_{ij}=0$ whenever $i\in C$ $j\notin C$

(" there is no escape path from C)

$i=\{4,5,6\}$

$j=\{1,2,3\}$

Prop:

- A communication class which is not closed is always transient.
- A *finite* communication class which is closed is always recurrent.

Rmk: For infinite communication class, there are counter-examples (see next lectures)

Mean time spent in a transient state

Let $S_{i,j}$ be $E[N_j|X_0 = i] =$ ~~$\mathbb{E}(\#\{n \geq 0, X_n = j\}$~~ $| X_0 = i)$

For notational simplicity, index the states so that the transient states are numbered as $T = \{1, 2, \dots, t\}$, and let's denote R the set of recurrent states

We restrict S to transient states so $S \in$ ~~$R^{t \times t}$~~ $\mathbb{R}^{t \times t}$

How do we determine the values of S? $\mathbb{E}(N_j | X_0 = k)$ if $k \neq j$ $1 + \mathbb{E}(N_j | X_0 = k)$ if $k = j$

Markov property

$$S_{ij} = \mathbb{E}(N_j | X_0 = i) = \sum_{k \in T} \mathbb{E}(N_j | X_1 = k, X_0 = i) \cdot P_{ik}$$

$$+ \sum_{k \in R} \mathbb{E}(N_j | X_1 = k, X_0 = i) P_{ik}$$

($\forall n, p_{kj}^{(n)} = 0$; if not, one can show that j is then recurrent) $0 =$

$$\longrightarrow \boxed{s_{ij} = \delta_{ij} + \sum_{k=1}^{t} P_{ik}\, s_{kj}} \quad \text{where } \delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{else.} \end{cases}$$

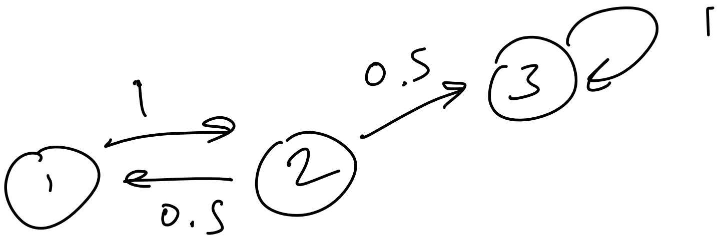
$$\Longleftrightarrow \underbrace{S}_{\text{matrix of } s_{ij}} = \underbrace{I}_{\text{Id matrix}} + P_T \cdot S \qquad \Longleftrightarrow$$

Prop:

$$\boxed{S = (\mathrm{Id} - P_T)^{-1}}$$

P_T: transition probability matrix of transient states $\begin{array}{c|c} & 1 \ \cdots \ t \\ \hline \begin{array}{c} 1 \\ \vdots \\ t \end{array} & \begin{pmatrix} P_{11} & P_{12} & \\ & \ddots & \\ & & \end{pmatrix} \end{array}$

ex :



Transient states : 1 , 2

$$P_T = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0.5 & 0 \end{pmatrix}$$

$$S = \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0.5 & 0 \end{pmatrix}\right]^{-1} = \begin{pmatrix} 1 & -1 \\ -0.5 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix}$$

$$\boxed{\text{Rmk: } \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}$$

ex: $\mathbb{E}(N_2 \mid X_0 = 1) = s_{12} = 1$

Example: Gambler's ruin → see also HW2.

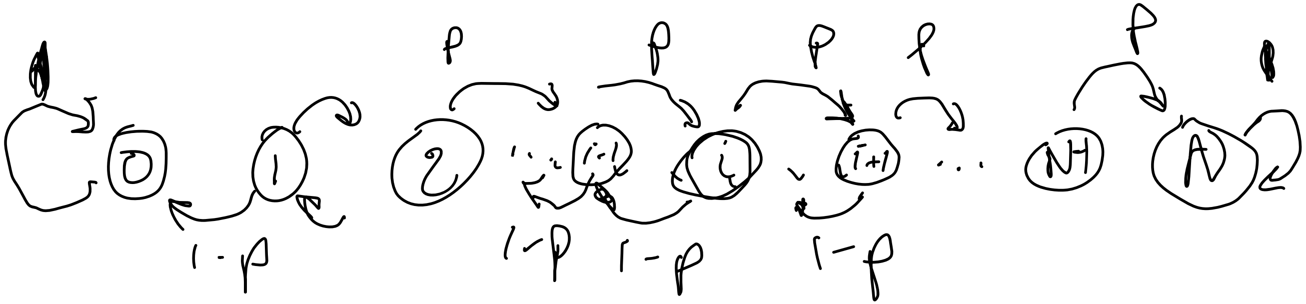
A gambler has n dollars, and plays a game with probability p of winning 1\$ at each round, and $1-p$ of losing 1\$ $(0<p<1)$

The gambler plays until they get broke, or reach a goal of N \$

→ Let us denote X_i the gambler's wealth at time i

It is a Markov chain

Transition diagram



Communicating classes, recurrence, transience ...

{0} {1, ..., N-1} {N}

recurrent transient recurrent

(closed) (Not closed) (closed)

$P_{N-1,N} > 0$ and $N \notin C$

→ Notebook 2