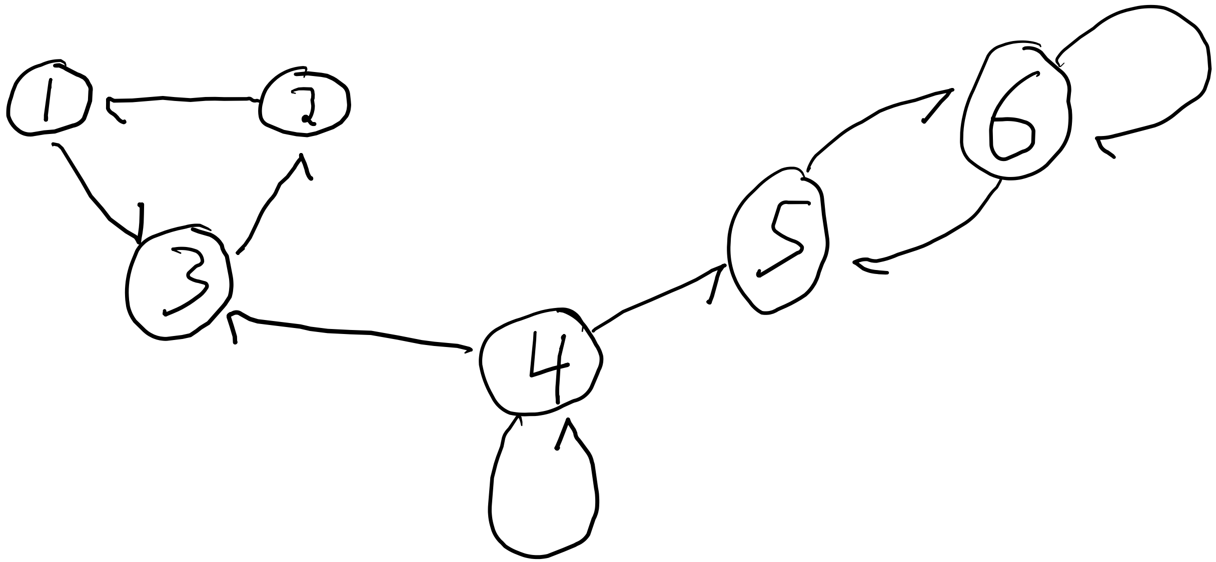
Lecture 5

Graph properties of MC

- Communicating classes (and accessibility)
- Periodicity
- Transience/recurrence

Ex)



We will learn to make the following kind of classifications:

Communicating classes:

$\{1, 2, 3\} \leftarrow$ recurrent, period 3

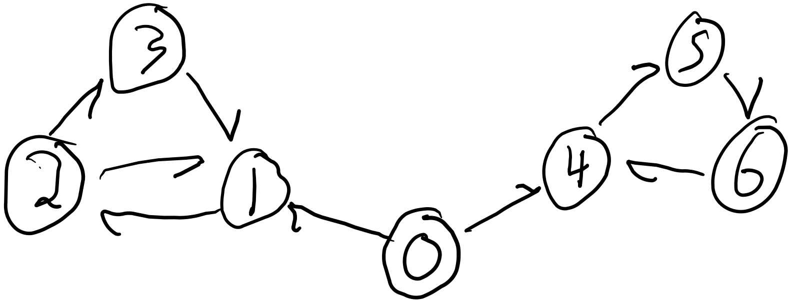
$\{4\} \leftarrow$ transient, aperiodic

$\{5, 6\} \leftarrow$ recurrent, aperiodic

Defs:

- State s_i is <u>accessible</u> from s_j if $p^n_{i,j} > 0$ for some $n \geq 0$.
- States s_i and s_j <u>communicate</u> if s_i is accessible from s_j and s_j is accessible from s_i. This is denoted by $i \leftrightarrow j$.

Ex)



True or False:

- All states are accessible from 0 (T or F)
- All states communicate with 0 (T or F)
- $1 \leftrightarrow 2$ (T or F)
- $1 \leftrightarrow 4$ (T or F)

Proposition: Communication is an *equivalence relation, i.e., it is*

(i) • *Reflexive:* $i \leftrightarrow i$ for all states i

(ii) • *Symmetric:* $i \leftrightarrow j \Rightarrow j \leftrightarrow i$

(iii) • *Transitive:* $i \leftrightarrow j$ and $j \leftrightarrow k \Rightarrow i \leftrightarrow k$

Proof: (i) $P_{ii}^{0} = 1$ so $i \leftrightarrow i$ $(n=0)$

(ii) $i \leftrightarrow j$ then $\exists\, m, n$ $P_{ij}^{n} > 0$ and $P_{ji}^{m} > 0 \Rightarrow j \leftrightarrow i$

(iii) Suppose $\exists\, m \geq 0 \mid P_{ij}^{m} > 0$ and $\exists\, n \geq 0$ $P_{jk}^{n} > 0$

and let's show $P_{ik}^{m+n} > 0$ (so $i \rightarrow k$)

Using the Chapman Kolmogorov equation:

$$P_{ik}^{m+n} = \sum_{l \in S} \underbrace{P_{il}^{m}}_{\geq 0} \cdot \underbrace{P_{lk}^{n}}_{\geq 0} \geq \underbrace{P_{ij}^{m}}_{>0} \cdot \underbrace{P_{jk}^{n}}_{>0} > 0$$

$$\Rightarrow \quad i \rightarrow k$$

and similarly one can show $k \rightarrow i \quad \Rightarrow \quad \boxed{i \leftrightarrow k}$ ⌧

Consequence (from equivalence relation in general)

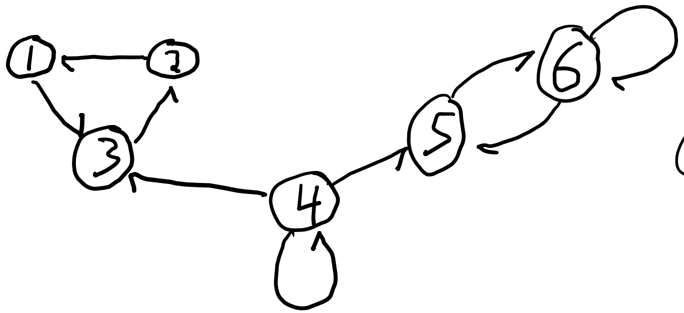
We can partition (split) the state space into communicating classes

Partitioning

Any equivalence relation on a set partitions the set into *equivalence classes*.

Thus, the state space always partitions into communicating classes.

Ex:



$S = \{1, 2, 3, 4, 5, 6\}$

$C_1 = \{1, 2, 3\}$ $C_2 = \{4\}$

$C_3 = \{5, 6\}$

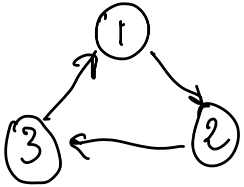
Def: A MC is irreducible if it has only one communicating class.

Def: The period of state s_i, denoted $d(i)$, is the greatest common divisor of $\{n \in \{1, 2, \dots\}: p_{i,i}^n > 0\}$.

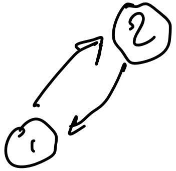
I.e., starting in s_i the MC can only return to s_i at numbers of steps that are multiples of d, and d is the largest such integer.

What is $d(1)$?

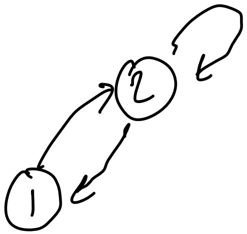
Examples:



$d(1) = 3 \; (= d(2) = d(3))$



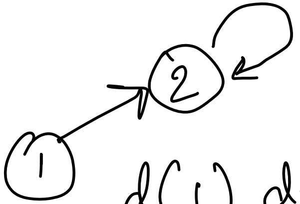
$d(1) = 2$



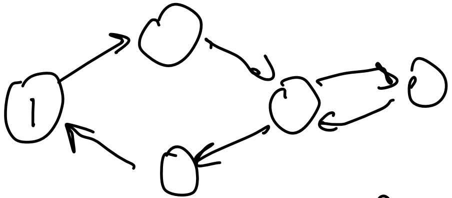
121 $n=2$

1221 $n=3$

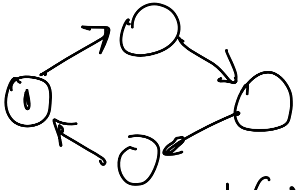
$\Longrightarrow \begin{cases} d(1) \mid 2 \\ d(1) \mid 3 \end{cases} \Rightarrow d(1) = 1$



$d(1)$ does not exist



$d(1) = 2$ $\quad \{n \mid p_{11}^n > 0\} = \{4, 6, 8 \dots\}$



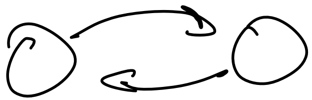
$d(1) = 4$

Rmk: A state can have no period

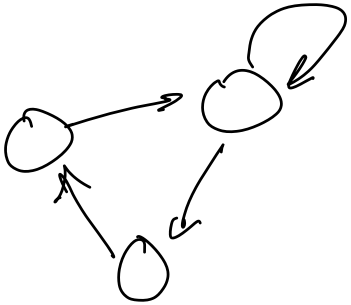
Def: We say that a state is aperiodic if it has period 1, and we say that an MC is aperiodic if all states have period 1.

Examples: Aperiodic T F

irreducible



F



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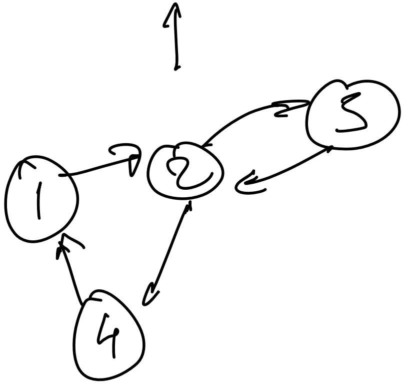


T

$$\left(\left.\begin{matrix} d(2)\mid 2 \\ d(2)\mid 3 \end{matrix}\right\} \Rightarrow d(2)=1\right)$$

Proposition: Period is a class property, i.e., all states in the same communicating class have the same period.

↳ Rmk : In practice, to study the periodicity of a M.C, we just need to find the periodicity associated with 1 state in each communicating class.

Proof:

↳ Suppose $i \leftrightarrow j$ $\left(\begin{array}{l} \exists m \mid P_{ij}^{m} > 0 \\ \exists n \mid P_{ji}^{n} > 0 \end{array} \right)$ and let's show that $d(i) = d(j)$

• First, we show $d(i) \leq d(j)$

$$P_{ii}^{m+n} \underset{\text{C-K}}{=} \sum_k P_{ik}^{m} P_{ki}^{n} \geqslant \underbrace{P_{ij}^{m}}_{>0} \underbrace{P_{ji}^{n}}_{>0} > 0 \Rightarrow d(i) \mid m+n$$

Now consider $\ell \mid P^{\ell}_{jj} > 0$ (we know that such ℓ exists since $d(j)$ exists)

$$P^{m+n+\ell}_{ii} \geq \underbrace{P^{m}_{ij}}_{>0} \underbrace{P^{\ell}_{jj}}_{>0} \underbrace{P^{n}_{ji}}_{>0} > 0 \Rightarrow d(i) \mid m+n+\ell$$

So $d(i) \mid m+n+\ell - (m+n) \Rightarrow d(i) \mid \ell \Rightarrow \boxed{d(i) \leq d(j)}$

$d(i)$ is a common divisor $X = \{n \mid P^{n}_{jj} > 0\}$

$d(j)$ is the gcd of X

By symmetry we also have $d(j) \geq d(i)$

So $\boxed{d(i) = d(j)}$

$\boxtimes$

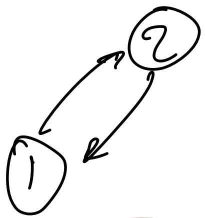
Def: Let $f_i := P(X_n = i \text{ for some } n \geq 1 | X_0 = i)$. Then

- if $f_i = 1$, State s_i is recurrent,
- if $f_i < 1$, State s_i is transient.

In words: $f_i = 1$ if you are always sure to visit i again

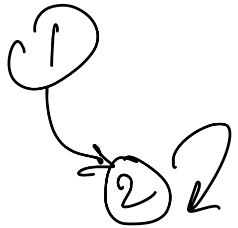
Proposition: Recurrence and transience are class properties.

Examples:



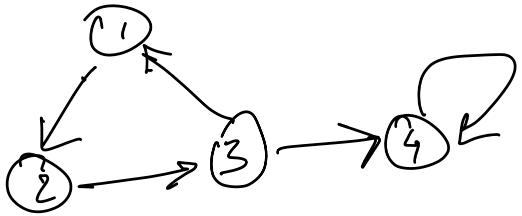
1 recurrent or transient

$(f_1 \geq P_{11}^2 = 1)$



2 recurrent

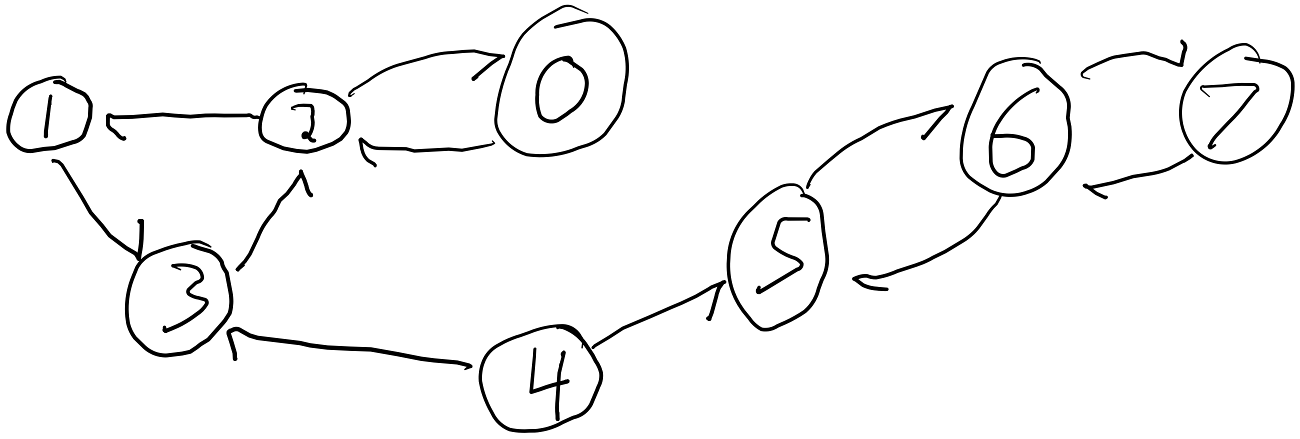
1 transient $(f_1 = 0)$



$\{1, 2, 3\}$ transient

4 recurrent

Ex)



Communication classes : $\{0, 1, 2, 3\}$, $\{4\}$ $\{5, 6, 7\}$

- $\{0, 1, 2, 3\}$ $\downarrow$ recurrent
- $\{4\}$ $\downarrow$ transient
- $\{5, 6, 7\}$ $\downarrow$ recurrent