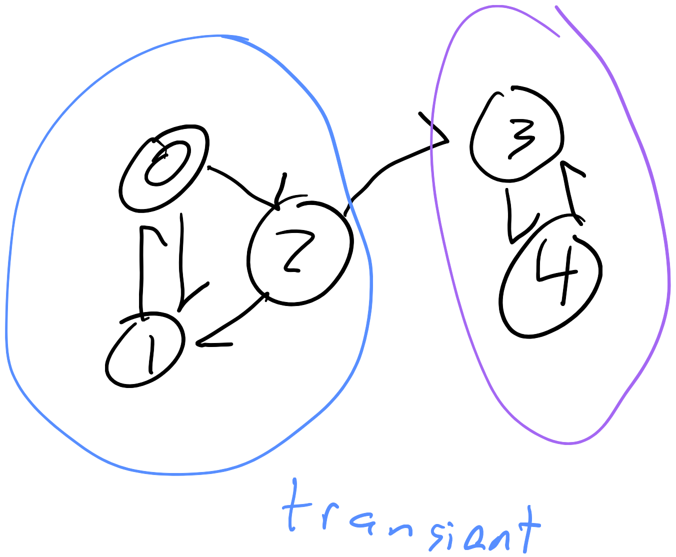
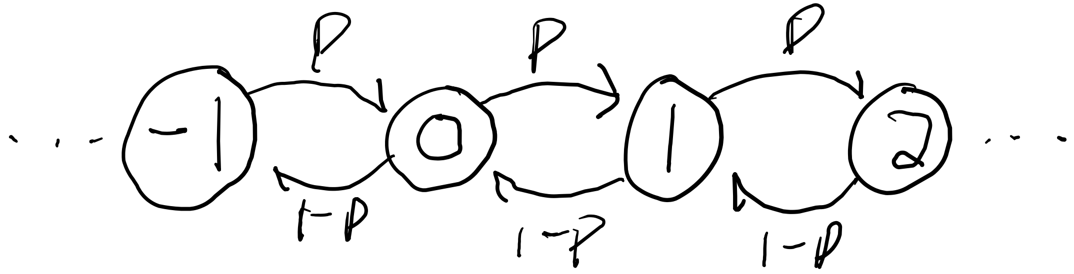
Midterm, Feb 27, 5pm ESB 1013

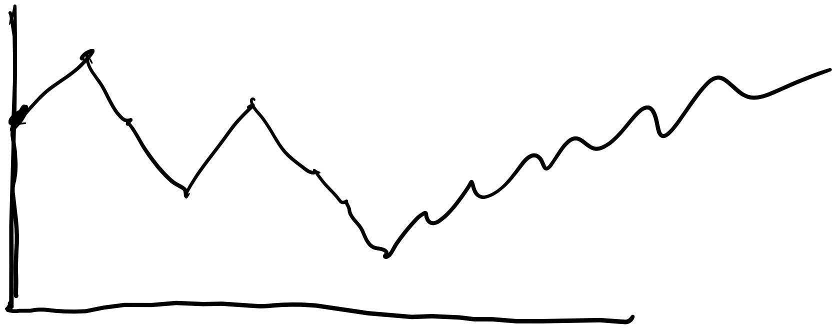


Lecture 8

Transience/recurrence of random walk on $\mathcal{Z}$ (and $\mathcal{Z}^d$)

Random walk on $\mathbb{Z}$





How many communicating classes?

Closed or not closed?

This does not determine transience/recurrence for ∞-state MC.

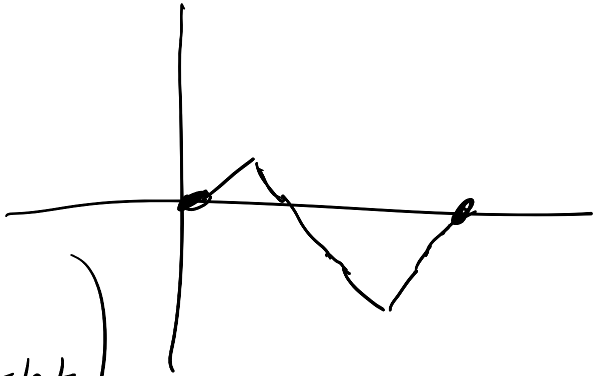
Q: Is the random walk transient or recurrent?

Recall:

State 0 is transient $\Leftrightarrow$ $\sum_{n \geq 0} P_{0,0}^{n} < \infty$

State 0 is recurrent $\Leftrightarrow$ $\sum_{n \geq 0} P_{0,0}^{n} = \infty$

Q: What is $P_{0,0}^n$?



$$P(X_n = 0 \mid X_0 = 0) = P\left(\begin{array}{l}\#\text{ of steps left} \\ = \#\text{ of steps right}\end{array}\right)$$

$$= P\left(\#\text{ of steps left} = \frac{n}{2}\right)$$

$$= P\left(\mathrm{Bin}(n, 1-p) = \frac{n}{2}\right) = 0 \quad \text{if } n \text{ is odd}$$

Thus, consider $P_{0,0}^{2n} = P(\mathrm{Bin}(2n, 1-p) = n) = \binom{2n}{n} \cdot p^n (1-p)^n$

Thus, $\displaystyle\sum_{\underbrace{n\geq 0}_{n\text{ even}}} P_{0,0}^n = \sum_{n\geq 0} P_{0,0}^{2n} = \sum_{n\geq 0} \binom{2n}{n} p^n (1-p)^n$

asymptotic equivalance

Stirling approximation: $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$

$$\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n} = 1$$

Implication:

Lemma: $p_{0,0}^{\cancel{n}\,2n} \sim \dfrac{2^{2n}p^n(1-p)^n}{\sqrt{\pi n}}$

Proof: $p_{0,0}^{2n} = \binom{2n}{n} p^n (1-p)^n$ we will show that

$$\binom{2n}{n} \sim \frac{2^{2n}}{\sqrt{\pi n}}.$$

We have

$$\binom{2n}{n} = \frac{(2n)!}{n!\cdot n!} \underset{\text{stirling approx}}{\sim} \frac{\sqrt{2\pi\cdot 2n}\cdot\left(\frac{2n}{e}\right)^{2n}}{\left(\sqrt{2\pi n}\cdot\left(\frac{n}{e}\right)^n\right)^2}$$

$$= \cancel{\frac{\sqrt{2\pi}\cdot\sqrt{2}}{(\sqrt{2\pi})^2}}\,\frac{1}{\sqrt{\pi}} \cdot \frac{1}{\sqrt{n}}\cdot \frac{2^{2n}\cdot\cancel{\left(\frac{n}{e}\right)^{2n}}}{\cancel{\left(\frac{n}{e}\right)^{2n}}} = \frac{1}{\sqrt{\pi}}\,\frac{1}{\sqrt{n}}\cdot 2^{2n} \quad \checkmark$$

Summing the series

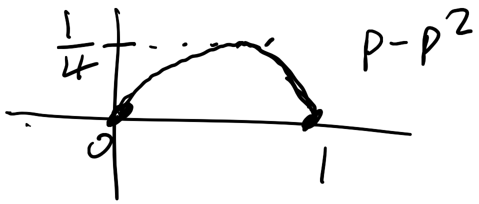
Thus $\sum p_{0,0}^{2n}$ converges $\Leftrightarrow$ $\underline{\sum_{n\geq 0} 2^{2n} (p(1-p))^n \cdot \frac{1}{\sqrt{\pi n}}}$ converges

$$= \sum_{n\geq 0} (4p(1-p))^n \cdot \frac{1}{\sqrt{\pi n}}$$

Note $p(1-p) = p - p^2 = \begin{cases} \frac{1}{4} & p = \frac{1}{2} \\ < \frac{1}{4} & p \neq \frac{1}{2} \end{cases}$

Thus, for $p \neq \frac{1}{2}$

$4p(1-p) < 1$



$a < 1$

<u>Case 1: $p \neq 1/2$</u>

$$\sum_{n\geq 0} (4p(1-p))^n \frac{1}{\sqrt{\pi n}} < \sum_{n\geq 0} (\underbrace{4p(1-p)}_{a})^n = \sum_{n\geq 0} a^n = \frac{1}{1-a} < \infty$$

so state 0 is transient, thus since there is 1 class & transience is class property, MC is transient.

Case 2 $p=\frac{1}{2}$, so $4p(1-p)=1$.

Then $\sum_{n\geq 1} 2^{2n}\cdot p(1-p)\frac{1}{\sqrt{\pi n}} = \sum_{n\geq 1.} \underbrace{\frac{1}{\sqrt{\pi n}}}_{\geq \frac{1}{\sqrt{\pi}}\cdot\frac{1}{n}}$

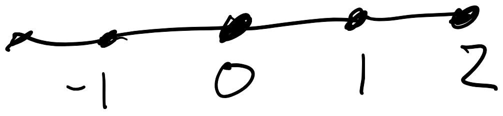
Since $\sum_{n\geq 1} \frac{1}{\sqrt{\pi}}\cdot\frac{1}{n} = \infty$

so does series above.

"well known" sum of harmonic series

i.e. for $p=\frac{1}{2}$, MC is recurrent.

"Proof" 2 for $p = \frac{1}{2}$.



α = prob of returning to 0.

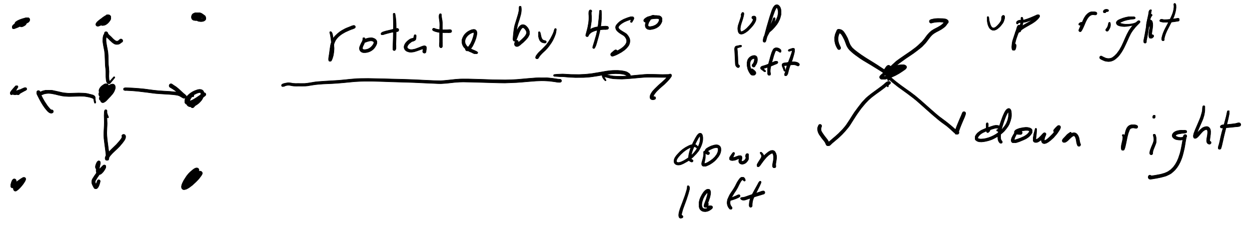
$$\alpha = \frac{1}{2} + \frac{1}{2}\alpha \cdot \alpha = \frac{1}{2} + \frac{1}{2}\alpha^2$$

$$\Rightarrow \alpha^2 - 2\alpha + 1 = 0$$

$$\Rightarrow (\alpha - 1) = 0$$

$$\Rightarrow \boxed{\alpha = 1}$$

Higher dimensions



$d = 2, p = \overset{\frac{1}{4}}{\cancel{\frac{1}{2}}}$: A random walk on $\mathcal{Z}^2$ is recurrent

$d = 3, p = \overset{1/8}{\cancel{\frac{1}{2}}}$: A random walk on $\mathcal{Z}^3$ is transient

Also for $d > 3$, transient.