1. $\boxed{\text{10 marks}}$ Write an one-page essay about the three founding fathers of linear programming: Kantorovich, Von Neumann, and Dantzig. For example, you can explains their contributions to the development of linear programming and optimization theory in general. The essay should be about all these three people. **You must type your article using font size 12pt.** Your essay should be written in your own words, though you must use the available sources, e.g. from the internet; you must cite your sources. One of the main purposes of this problem is to let you be familiar with the background of the subject.

2. $\boxed{\text{3 marks}}$ [You should study the self-study material on the standard forms of LP, available in the Canvas/Files.] Put the following linear programming problem in standard form, that is, standard inequality form. (Do not solve it.)

$$\begin{array}{rl} \text{minimize} & x_1 - 3x_2 \\ \text{subject to} & x_1 + x_2 = 2 \\ & x_1 \geq 3 \\ & x_2 \text{ unconstrained} \end{array} .$$

Solution: "minimize $x_1 - 3x_2$" $\Leftrightarrow$ "maximize $-x_1 + 3x_2$". "$x_1 + x_2 = 2$" $\Leftrightarrow$ "$x_1 + x_2 \leq 2$ and $-x_1 - x_2 \leq -2$". Also, let $x_2 = x_2' - x_2''$, $x_2', x_2'' \geq 0$ and $x_1 = x_1' + 3$. Then, the objective function becomes $-(x_1' - 3x_2' + 3x_2'' + 3) = -x_1' + 3x_2' - 3x_2'' - 3$ but -3 can be dropped.

$$\begin{array}{rl} \text{maximize} & -x_1' + 3x_2' - 3x_2'' \\ \text{subject to} & x_1' + x_2' - x_2'' \leq -1 \\ & -x_1' - x_2' + x_2'' \leq 1 \\ & x_1', x_2', x_2'' \geq 0 \end{array} .$$

3. $\boxed{\text{5 marks}}$ Consider a finite set of nonzero vectors, $v_1, v_2, ..., v_k \in \mathbf{R}^n$, with $|v_i| > 0$, $i = 1, ..., k$; $k \geq 1$. Define

$$C = \bigcap_{i=1}^{k} H_{v_i}. \tag{1}$$

where we denote

$$H_v = \{x \in \mathbf{R}^n \mid v \cdot x \leq 1\}.$$

a. For the set C defined in the above equation (1), is it possible for some case that $C = \emptyset$? Here, $\emptyset$ denotes the empty set. Justify your answer, either by giving such a case or proving that it is not possible.

b. Give an example of the vectors $v_i \in \mathbf{R}^2$, with $|v_i| > 0$, $i = 1, 2, 3$, (so $n = 2$ and $k = 3$),where the set C defined in the above equation (1), is a bounded set. Justify your answer clearly.

(Here, a set S is said to be bounded, if there exists a positive number K such that $|x| \leq K$ for any $x \in S$.)

Solution:

a.

Note that $x = 0$ always satisfy $v \cdot 0 = 0 \leq 1$, so, $x = 0$ belongs to C, and we see that C must be nonempty.

b. For example, let $v_1 = (1, 0)$, $v_2 = (0, 1)$, $v_2 = (-1, -1)$.

- Then, for $x = (x_1, x_2)$, $v_1 \cdot x = x_1$, $v_2 \cdot x = x_2$ and $v_3 \cdot x = -x_1 - x_2$.
- So, for $x \in C$, we have $v_i \cdot x \leq 1$ for $i = 1, 2, 3$, thus, we get $x_1, x_2 \leq 1$ as well as $-x_1 - x_2 \leq 1$.
- This implies that $x_1 \geq -x_2 - 1$. Together with $x_2 \leq 1$, which is equivalent to $-x_2 \geq -1$, we get $x_1 \geq -1 - 1 = -2$.
- Similarly $x_2 \geq -x_1 - 1 \geq -1 - 1$. Therefore we see that $-2 \leq x_1, x_2 \leq 1$. In particular, $|x_1|, |x_2| \leq 2$, so $|x| = \sqrt{|x_1|^2 + |x_2|^2} \leq \sqrt{2^2 + 2^2} \leq \sqrt{8}$.
- We can choose $K = \sqrt{8}$. This shows the boundedness of C.

4. $\boxed{\text{7 marks}}$ Let $C \subset \mathbf{R}^n$ be a given set and $C \neq \emptyset$. Let $f : \mathbf{R}^n \to \mathbf{R}$ be a given function. Consider the following optimization problem:

$$\text{(Prob1)} \qquad \text{Maximize} \qquad f(x)$$
$$\text{under the constraint: } x \in C.$$

Define $F_r := \{x \in \mathbf{R}^n \mid f(x) \geq r\}$, and consider

$$\text{(Prob2)} \qquad \text{Maximize} \qquad r$$
$$\text{under the constraint: } C \cap F_r \neq \emptyset.$$

Here the constraint $C \cap F_r \neq \emptyset$ is a condition on r.

Assume that an optimal solution of (Prob1) and an optimal solution of (Prob2) both exist. Find and explain the relation between the optimal solution of (Prob1) and that of (Prob2). You have to justify your answer. [Hint: To get an intuition, try first $n = 1$ case and $n = 2$ case, with a linear function f and the set C given by linear constraints. You final answer should be for general n and for general f and C.] [**This problem will be marked strictly: To get a nontrivial mark, please make your explanation very clear in a logical manner.**]

Solution:

Suppose $\bar{x}$ is an optimal solution of (Prob1) and $\bar{r}$ is an optimal solution of (Prob2). Then the following equation hold:

$$\bar{r} = f(\bar{x}).$$

Basically it tells that $\bar{r}$ is nothing but the optimal value of the objective function of (Prob1). So, $\bar{r} = f(\bar{x})$ should hold.) You can do this by explaining why $\bar{r} \geq f(\bar{x})$ and why $\bar{r} \leq f(\bar{x})$.)

Note that $\bar{x}$ is feasible (that is, it satisfies the constraint $x \in C$). Also, the set $F_{f(\bar{x})}$ obviously contains $\bar{x}$. We so see that $\bar{x} \in C \cap F_{f(\bar{x})}$. So, the value $f(\bar{x})$ is feasible for (Prob2). Therefore

$$(*) \qquad \bar{r} \geq f(\bar{x}) \text{ as } \bar{r} \text{ is the maximum objective value of (Prob2).}$$

On the other hand, as $\bar{r}$ is also feasible for (Prob2), we have $C \cap F_{\bar{r}} \neq \emptyset$. This means there exists $\bar{x}' \in C \cap F_{\bar{r}}$, that is, $\bar{x}' \in C$ and $f(\bar{x}') \geq \bar{r}$, where the latter is from the definition of F_r. Since $\bar{x}'$ is feasible for (Prob1), we have that the optimal value $f(\bar{x})$ of (Prob1) should be $f(\bar{x}) \geq f(\bar{x}')$. Therefore, we conclude that $f(\bar{x}) \geq f(\bar{x}') \geq \bar{r}$, thus,

$$(**) \qquad f(\bar{x}) \geq \bar{r}.$$

From (*) and (**), we see that $\bar{r} = f(\bar{x})$.