Lecture 11

Time reversal

(Reversible M-C)

Recall: Conditional independence $\left(P(A|B) = \frac{P(AB)}{P(B)} \right)$

Consider events A, B, C, with $P(A \cap B \cap\ C) = P(ABC) > 0$

- A is independent of B if $P(AB) = P(A) \cdot P(B)$ $\iff$ $P(A|B) = P(A)$, $P(B|A) = P(B)$
- Conditioned on C, A is independent of B if $P(AB|C) = P(A|C) \cdot P(B|C)$

$$\iff \begin{cases} P(A|B,C) = P(A|C) \\ P(B|A,C) = P(B|C) \end{cases}$$

For a MC conditioned on the present, past and future are independent

Given an MC $\left(X_n\right)_{0\leq n\leq N}$ let $A,\ B,\ C$ encode past, present, and future.

Future: $A =$ $\{X_{n+1}=i_{n+1}, X_{n+2}=i_{n+2}, \ldots\ldots, X_N = i_N\}$

Present: $B =$ $\{X_n = i_n\}$

Past: $C =$ $\{X_0=i_0, X_1=i_1, \ldots, X_{n-1}=i_{n-1}\}$

Markov property

$$P(A \mid B, C) = P(\underbrace{X_{n+1}=i_{n+1},\ldots\ldots, X_N=i_N}_{A} \mid \underbrace{X_n=i_n}_{B}, \underbrace{\cancel{X_{n-1}=i_{n-1}, \ldots, X_0=i_0}}_{C})$$

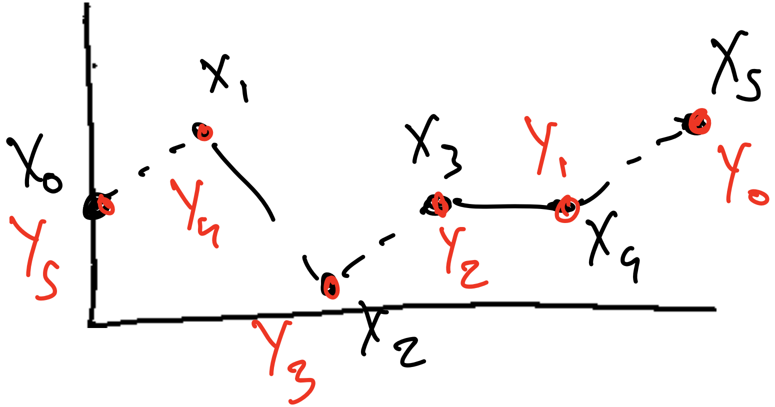
$$= P(A \mid B)$$

in other words, A and C are independent (i.e. past & future are independent)

Implication: MC run backwards is also a MC

Given MC $(X_n)_{0 \le n \le N}$, and a positive integer N, let $Y_n = X_{N-n}$

$$\begin{pmatrix} Y_0 = X_N \\ Y_1 = X_{N-1} \\ \vdots \end{pmatrix}$$



The sequence $(Y_n)_{0 \le n \le N}$ switches past and future

Transition probabilities for $(Y_n)_{0\le n\le N}$?

Thm: Given a MC $(X_n)_{0\le n\le N}$ with initial distribution $\pi =$ (stationary distribution), let $Y_n = X_{N-n}$.

Then $(Y_n)_{0\le n\le N}$ is a homogeneous (i) MC with stationary distribution (ii) π and with transition probabilities

$$q_{i,j} = p_{j,i} \cdot \frac{\pi_j}{\pi_i} \quad \text{(iii)}$$

Notation: Set $\tilde{Q}$ to be the transition matrix for $(Y_n)_{0\le n\le N}$ so

$$\left(\tilde{Q}\right)_{i,j} = q_{i,j}$$

(rmk: The existence of π guarantees that Y_n is homogeneous $\to$ see proof of (iii))

Proof: (i) (to be completed)

(iii) Transition probabilities:

Bayes rule.

$$q_{ij} = P(Y_1 = j \mid Y_0 = i) = \underbrace{P(Y_0 = i \mid Y_1 = j)}_{p_{ji}} \cdot \frac{\overbrace{P(Y_1 = j)}^{P(X_{N-1} = j) = \pi_j}}{\underbrace{P(Y_0 = i)}_{P(X_N = i) = \pi_i}}$$

p_{ji} $\left(= P(X_N = i \mid X_{N-1} = j) = P(X_1 = i \mid X_0 = j) \right)$

$$= p_{ji} \cdot \frac{\pi_j}{\pi_i}$$

(ii) π is stationary:

$$(\pi Q)_j = \sum_i \pi_i q_{ij} = \sum_i \cancel{\pi_i} p_{ji} \frac{\pi_j}{\cancel{\pi_i}} = \pi_j \cdot \underbrace{\sum_i p_{ji}}_{=1}$$

(P stochastic)

$\rightarrow (\pi Q)_j = \pi_j$ so π is stationary

(i) Since we saw past and future are independent given present, one can conclude that Y_n satisfies the Markov property.

More formally, we can show

$$\underbrace{P(Y_n=j \mid Y_{n-1}=i, Y_{n-2}, \dots)}_{\text{LHS}} = \underbrace{P(Y_n=j \mid Y_{n-1}=i)}_{\text{RHS}}$$

$$\text{LHS} = \frac{P(X_{N-n}=j, X_{N-n+1}=i, \overbrace{X_{N-n+2}=\dots, \dots}^{E})}{P(X_{N-n+1}=i, E)}$$

$$= \frac{\cancel{P(E \mid \overset{\text{Markov property}}{\cancel{X_{N-n}=j}}, X_{N-n+1}=i)} \cdot P(X_{N-n}=j, X_{N-n+1}=i)}{P(X_{N-n+1}=i) \cdot \cancel{P(E \mid X_{N-n+1}=i)}}$$

$$= \frac{P(\overbrace{X_{N-n}}^{Y_n}=j, \overbrace{X_{N-n+1}}^{Y_{n-1}}=i)}{P(\underbrace{X_{N-n+1}}_{Y_{n-1}}=i)} = \underbrace{P(Y_n=j \mid Y_{n-1}=i)}_{\text{RHS}}$$

$\boxtimes$

Time reversible MC

Def: An MC is time reversible if $\tilde{Q} = \tilde{P} \iff q_{ij} = p_{ij} \;\; \forall i,j \iff \boxed{p_{ji}\,\dfrac{\pi_j}{\pi_i} = p_{ij}}$

Detailed balance equations:

$$\boxed{\forall i,j \quad p_{ij}\cdot\pi_i = p_{ji}\,\pi_j}$$

This is not always verified by a MC, but if it is we know that the MC is time reversible.

$$\pi_i \;\; i \underset{p_{ji}}{\overset{p_{ij}}{\rightleftarrows}} j \;\; \pi_j$$

If there is a solution to the detailed balance equations, it is a stationary distribution

Prop: Suppose $x = (x_1,\ x_2, \ldots)$ is a distribution and satisfies the detailed balance equations, i.e.,

$$\forall i,j \quad x_i P_{ij} = x_j P_{ji} \quad , \left(\sum x_i = 1 \ , \ 0 \le x_i \le 1 \right)$$

Then x is a stationary distribution.

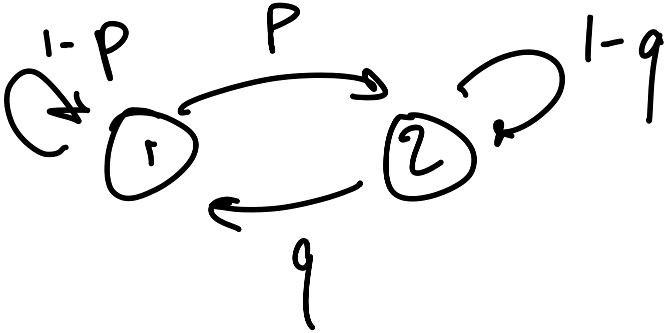
→ This can simplify the search for a stationary distribution

(→ $\pi P = \pi$)

Examples

- Two-state MC
- 3-state unbalanced circular random walk
- Ehrenfest chain

2-state MC



Q: Can we find π that satisfies detailed balance equations?

$\rightarrow \forall\, i,j \quad \pi_i P_{ij} = \pi_j P_{ji}$

A: We only have to solve $x_1 \cdot P_{12} = x_2 P_{21}$, $x_1 + x_2 = 1$

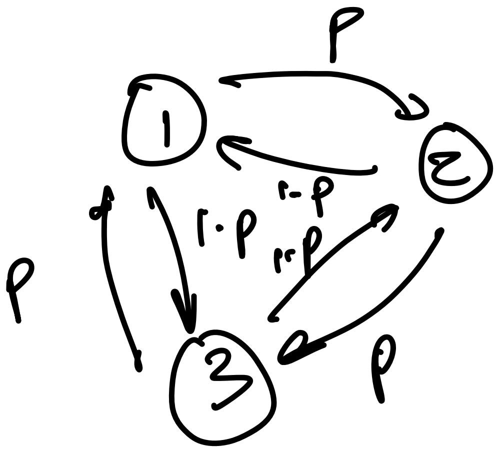
$$\Updownarrow$$

$$x_1 \cdot p = x_2 \cdot q$$

$$\Rightarrow x_2 \cdot \frac{q}{p} + x_2 = 1 \Leftrightarrow x_2\left(1 + \frac{q}{p}\right) = 1$$

so $x_2 = \dfrac{p}{p+q}$ and $x_1 = \dfrac{q}{p+q}$

$$\Rightarrow \boxed{\left(\frac{q}{p+q}, \frac{p}{p+q}\right)} \text{ is stationary}$$



$$\tilde{P} = \begin{array}{c} \\ \\ \\ \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 0 & p & 1-p \\ 1-p & 0 & p \\ p & 1-p & 0 \end{pmatrix} \end{array}\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}$$

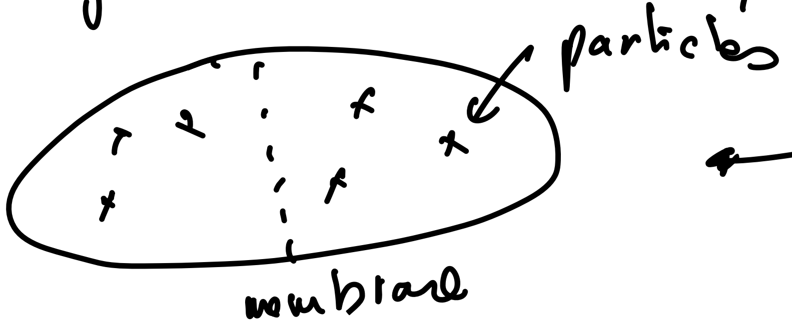
Stationary distribution: $\pi = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$

Is the chain reversible? Are the detailed balance eqs verified?

$\to \pi_1 p = \pi_2 (1-p) \iff \frac{1}{\cancel{3}} p = \frac{1}{\cancel{3}}(1-p) \iff 2p = 1$

The chain is reversible iff $p = \frac{1}{2}$

Ehrenfest chain → (Toy model for gas behaviour)



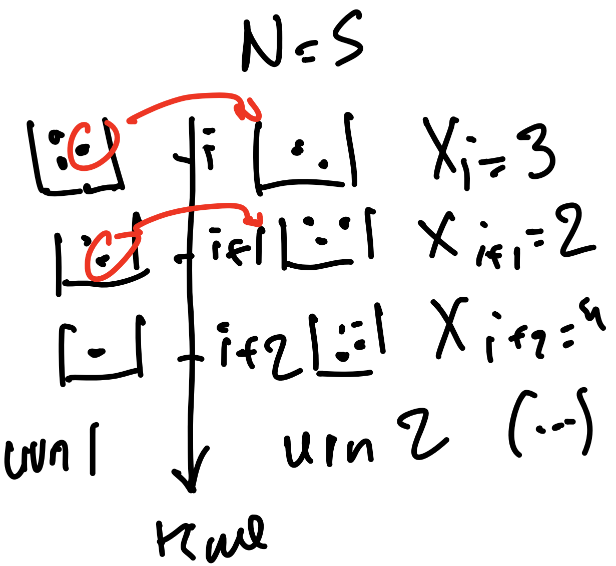
↔ Consider N balls distributed in 2 urns, and at each step we pick a ball at random and we move it to the other urn.

→ Let's study $X_n := \#$ balls in urn 1.

State space $= \{0, \dots, N\}$

$P_{01} = 1$, $P_{N\,N-1} = 1$, $p_{i\,i+1} = \frac{N-i}{N}$ $i \neq 0$

$p_{i,i-1} = \frac{i}{N}$ $i \neq N$



Period : 2 Irreducibility : Y

Does the chain have a stationary distribution?

irreducible + finite state space $\Rightarrow$ Yes.

Q : Find π

A: We'll guess a solution and check if it satisfies detailed balance eqs

→ If we track one ball, in the long run it will spend as much time in urn 1 and urn 2

→ At stationarity, → # balls in urn 1 = # heads in N independent coin flips

$\Rightarrow \pi \sim \text{Binom}(N, \frac{1}{2})$

$\rightarrow$ Our guess is $\pi_n = \binom{N}{n} \cdot \frac{1}{2^N}$

We only have to verify

$$\underbrace{\pi_i P_{i,i+1}}_{LHS} = \underbrace{\pi_{i+1} P_{i+1,i}}_{RHS} \qquad (i = 0, \dots, N-1)$$

$$LHS: \binom{N}{i} \cdot \frac{1}{2^N} \cdot \frac{N-i}{N} = \frac{\cancel{N!}\ (N-1)!}{i!\ \cancel{(N-i)!}\ (N-i-1)!} \cdot \frac{1}{2^N} \cdot \frac{\cancel{N-i}}{\cancel{N}}$$

RHS: (exercise) $\longrightarrow$ = "

Detailed balance eq. are satisfied!

So π is stationary and the process is time reversible