Lecture 3

Chapman-Kolmogorov equation

This lecture: n-step transition probabilities

Recall: Given a Markov chain $(X_n)_{n\geq 0}$ the transition matrix $\tilde{P}$ satisfies

$$(\tilde{P})_{i,j} = P(X_{n+1} = s_j | X_n = s_i) =: p_{i,j}$$

Q: What is $P(X_{n+k} = s_j | X_k = s_i) =: p^n_{i,j}$?

$= P(X_n = s_j | X_0 = s_i)$ by homogeneity

Denote the n-step transition matrix by $\tilde{P}^{(n)}$. It satisfies $\left(\tilde{P}^{(n)}\right)_{i,j} := p^n_{i,j}$.

Q: How can we compute $\tilde{P}^{(n)}$?

Warm-up: $n = 0,1$

$\underline{n = 0}$

$$P_{i,j}^{0} = P(X_{0+0} = s_j \mid X_0 = s_i)$$
$$= P(X_0 = s_j \mid X_0 = s_i) = \begin{cases} 1 & i = j \\ 0 & \text{else} \end{cases}$$

$$\tilde{P} = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{bmatrix} = I$$

$\underline{n = 1}$

$$P_{i,j}^{1} = P(X_1 = s_j \mid X_0 = s_i) = P_{ij}$$

as before

$$\tilde{P}^{(1)} = \tilde{P}$$

$m + n$ steps

$$p_{i,j}^{m+n} = P(X_{m+n}=j \mid X_0=i)$$

$$= \sum_k P(X_{m+n}=j \mid X_m=k, X_0=i) \cdot P(X_m=k \mid X_0=i)$$

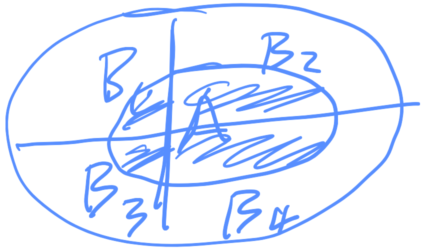
$$= \sum_k P(X_{m+n}=j \mid X_m=k)\, P(X_m=k \mid X_0=i)$$

$$= \sum_k P_{kj}^{n} \cdot P_{ik}^{m}$$

Q1) $P(X_{n+m}=j \mid X_0=i, X_m=k)$

$= P(X_{n+m}=j \mid X_m=k)$

by Markov prop.



$$P(A) = \sum P(A \cap B_i)$$

$$= \sum P(A \mid B_i) \cdot P(B_i)$$

Chapman-Kolmogorov equation: $p_{i,j}^{m+n} = \sum_k p_{i,k}^m \cdot p_{k,j}^n$

Implications:

- $p_{i,j}^2 =$ $\sum_k P_{i,k}^1 \cdot P_{k,j}^1 = \sum_k P_{i,k} \cdot P_{k,j}$

  $\tilde{P}^{(2)} = \tilde{P} \cdot \tilde{P} = \tilde{P}^2$

- $\tilde{P}^{(n)} =$ $\tilde{P} \cdot \tilde{P} \cdots \tilde{P} = \tilde{P}^n$ by induction.

so, n-step transition matrix is nth power of the transition matrix.

The last piece: A random initial state

Suppose X_0 is randomly chosen, i.e.,

$P(X_0 = s_i) =: \alpha_i, \qquad \sum \alpha_i = \underline{\ 1\ }.$

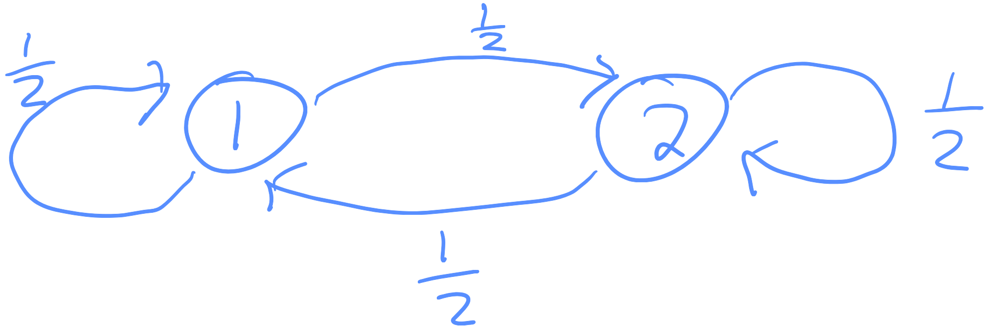
Note: sometimes i is used for shorthand for s_i.

Set $\alpha = \underline{(\alpha_1, \alpha_2, \ldots)}$.

Then, $P(X_n = s_j) = \sum_i \underbrace{P(X_n = s_j \mid X_0 = s_i)}_{P_{ij}^n} \cdot \underbrace{P(X_0 = s_i)}_{\alpha_i}$

$= \sum_i P_{ij}^n \cdot \alpha_i = (\alpha \cdot \tilde{P}^n)_j = \left(\boxed{\alpha}\ \boxed{\tilde{P}}^{\,n}\right)_j$

Ex)



$$\alpha = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \end{bmatrix} = \begin{bmatrix} P(X_0 = 1), & P(X_0 = 2) \end{bmatrix},$$

$$\tilde{P} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\alpha \tilde{P} = \frac{1}{2}\begin{bmatrix} \frac{3}{4} & \frac{1}{4} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \frac{3}{4}\cdot 1 + \frac{1}{4}\cdot 1, & \frac{3}{4}\cdot 1 + \frac{1}{4}\cdot 1 \end{bmatrix}$$

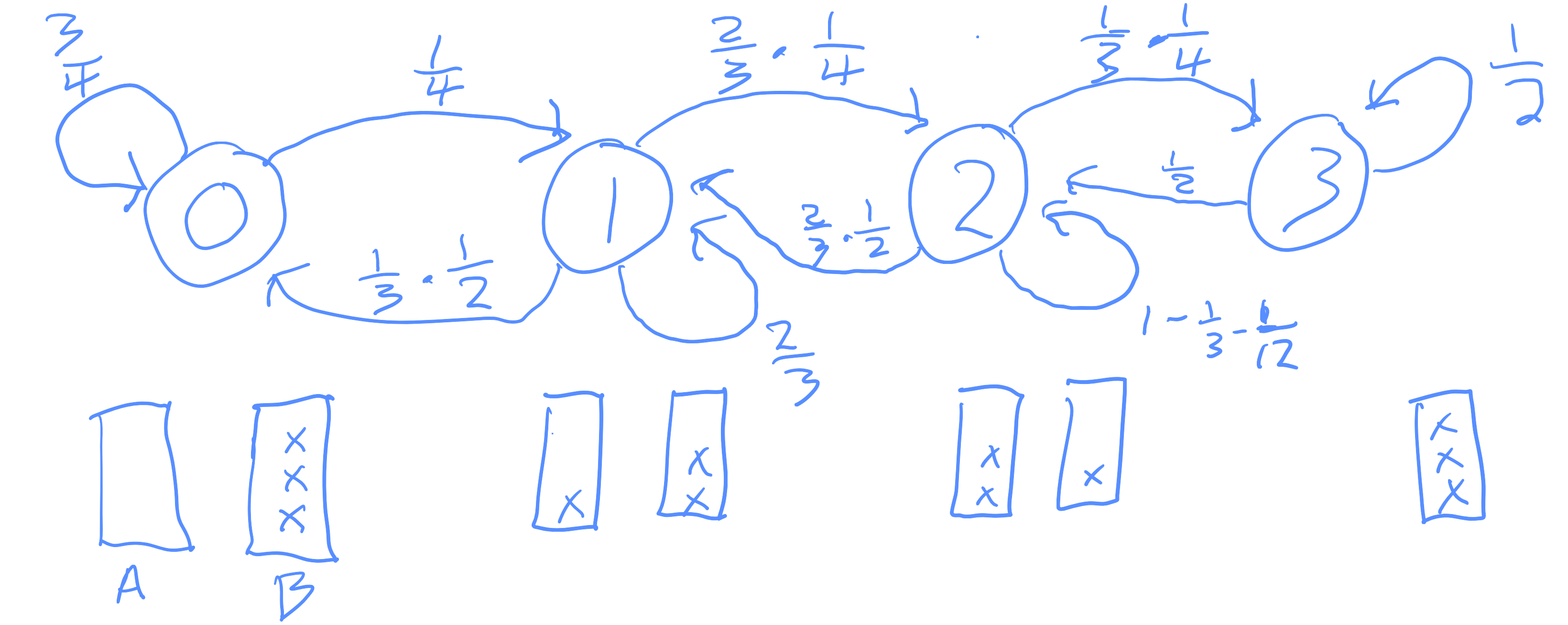
$$= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Examples

A total of $\boxed{\text{3 balls}}$ are divided between two urns, A and B. A ball is chosen at random. If it is chosen from urn A then it is placed in urn B with probability ½ and otherwise it is returned to urn A. Similarly, if the ball is chosen from urn B then it is placed in urn A with probability ¼. Let X_n denote the $\boxed{\text{number of balls in urn } A}$ after n trials.

What is the state space? Draw the transition diagram.

State space $= \{0, 1, 2, 3\} = \{$ possible # of balls $\}$



You flip a fair coin over and over. Let X_n be the number of heads in a row at flip n, thus defining a MC.

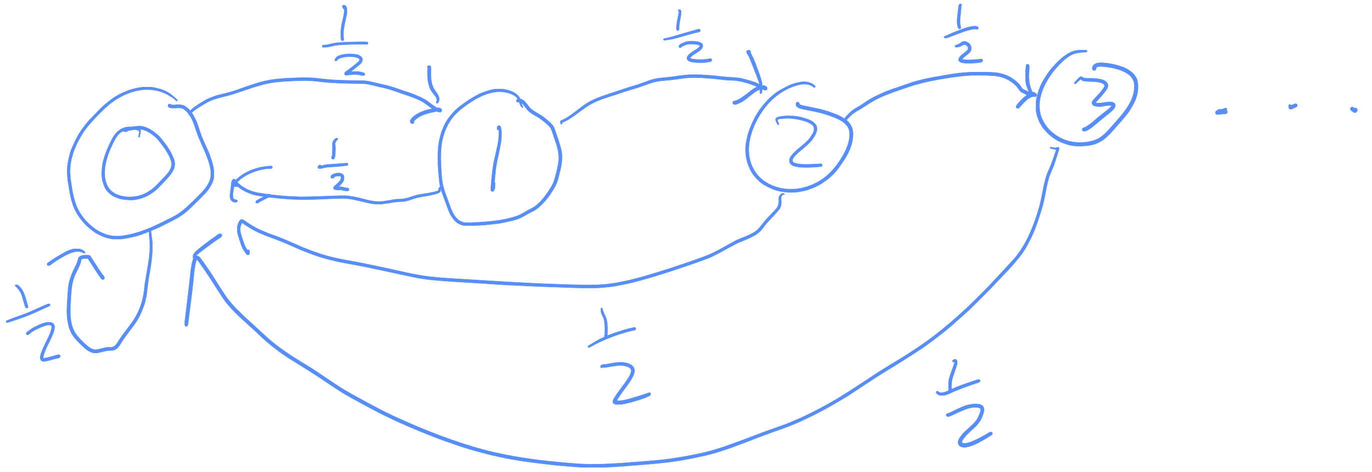
What is the state space?

What is the transition diagram?

THHT

$X_3 = 2$

State space $= \{0, 1, 2, \dots\} = \mathbb{N}$



Let T_k be the number of flips until ~~2~~ k heads appear in a row. What is $\mathrm{E}\, T_2$?

$\underbrace{\underline{T}\ \underline{H}\ \underline{H}}\ \underline{T}\ \ldots \Rightarrow T_2 = 3$

$$\mathbb{E}\, T_2 = \mathbb{E}[T_2 \mid \underbrace{HH}_{\text{1st 2 flips}}] \cdot P(HH) + \mathbb{E}[T_2 \mid HT] \cdot P(HT) + \mathbb{E}[T_2 \mid \underset{\substack{\uparrow \\ \text{1st flip}}}{T}] \cdot P(T)$$

$$= 2 \cdot \frac{1}{4} + (2 + \mathbb{E}\, T_2) \cdot \frac{1}{4} + (1 + \mathbb{E}\, T_2) \cdot \frac{1}{2}$$

$$\mathbb{E}\, T_2 = \frac{1}{2} + \frac{1}{2} + \frac{\mathbb{E}\, T_2}{4} + \frac{1}{2} + \frac{\mathbb{E}\, T_2}{2}$$

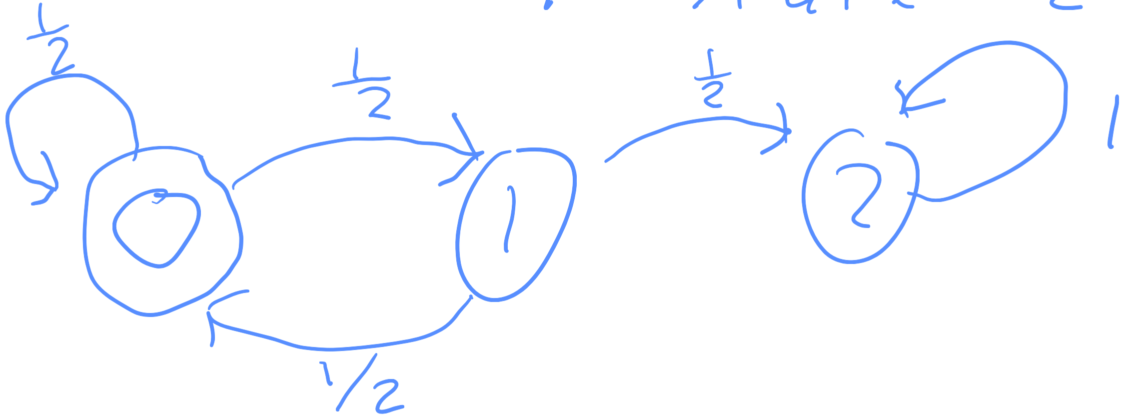
$$\Rightarrow \frac{\mathbb{E}\, T_2}{4} = \frac{3}{2} \Rightarrow \boxed{\mathbb{E}\, T_2 = 6}$$

$\underbrace{HT}_{2}\ \underbrace{\ldots\ldots\ldots}$

What is the probability that $T_2 \le 10$?

$T_2 \le 10 \iff \underbrace{\text{You arrive at state 2 in} \le 10 \text{ jumps}}_{\text{unaffected by changing transition prob. from state 2.}}$

We turn state 2 into an <u>absorbing state</u>



For this MC: $T_2 \leq 10 \iff X_{10} = 2$

$$\tilde{P} = \begin{array}{c} \\ \\ \\ \end{array}\begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 0 & 1 \end{pmatrix} \end{array} \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}$$

$$P(X_{10} = 2 \mid X_0 = 0) = \boxed{\left(\tilde{P}^{10}\right)_{1,3}}$$

$s_1 = 0$, $s_2 = 1$, $s_3 = 2$

$$P(X_{10} = 2 \mid X_0 = 0) = P(X_{10} = s_3 \mid X_0 = s_1)$$