Announcement : • HW

• Midterm

Lecture 9

- Limiting probabilities
- Mean return times

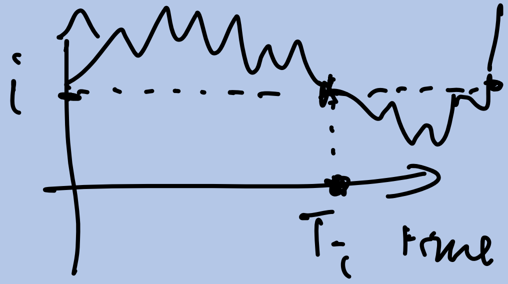
One more classification

Def: Given a recurrent state i, let T_i be the time to reach i (starting after first transition):

$T_i = \min\{n \geq 1 \mid X_n = i\}$

Set m_i to be the mean return time to state i:

$m_i = \mathbb{E}(T_i \mid X_0 = i)$

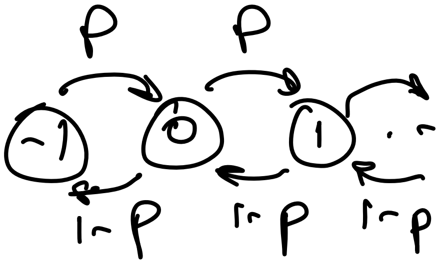


We say i is

- positive recurrent if $m_i < \infty$,
- null recurrent if $m_i = \infty$. $\longrightarrow$ only possible if infinite MC.

Prop: Positive recurrent and null recurrence are class properties.

Example: 1-d random walk $\left(p = \frac{1}{2}\right)$..



Goal: show that $\mathbb{E}(T_0 / X_0 = 0] = \infty$

$$\to \mathbb{E}(T_0 | X_0 = 0) = \frac{1}{2}\left[1 + \mathbb{E}(T_0 | X_0 = 1)\right] + \frac{1}{2}\left[1 + \mathbb{E}(T_0 | X_0 = -1)\right]$$

$$= 1 + \cancel{\frac{1}{2}}\left[\underbrace{\mathbb{E}(T_0 | X_0 = 1)}_{\mathbb{E}(T_1 | X_0 = 0)} + \cancel{\mathbb{E}(T_0 | X_0 = -1)}\right]$$

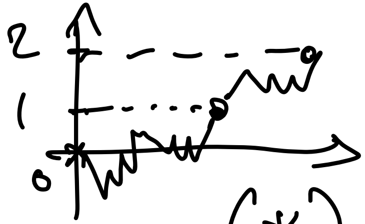
$\leftarrow$ we focus on this.

$= \mathbb{E}(T_0 | X_0 = 1)$ (by symmetry)

$$\to \mathbb{E}(T_1 | X_0 = 0) = \mathbb{E}(T_1 | X_0 = 0)$$

$$= \underbrace{\mathbb{E}(T_1 | X_0 = 0, X_1 = 1)}_{= 1} \cdot \frac{1}{2} + \mathbb{E}(T_1 | X_0 = 0, X_1 = -1) \cdot \frac{1}{2}$$

$$= \frac{1}{2} + \frac{1}{2}\left(1 + \underbrace{\mathbb{E}(T_1 | X_0 = -1)}_{= \mathbb{E}(T_2 | X_0 = 0) = 2\,\mathbb{E}(T_1 | X_0 = 0)}\right)$$



$$(*) \quad \boxed{\mathbb{E}(T_1 | X_0 = 0) = 1 + 2\,\mathbb{E}(T_1 | X_0 = 0)}$$

(*) has no finite solution!

$\Rightarrow E(T_1 \mid X_0 = 0) = \infty$

$\Rightarrow E(T_0 \mid X_0 = 0) = \infty$

So the chain is <u>null-recurrent</u>

(<u>Alternatively</u>:

use result from previous lecture

$P(T = 2n)$

$= (\dots)$

and show

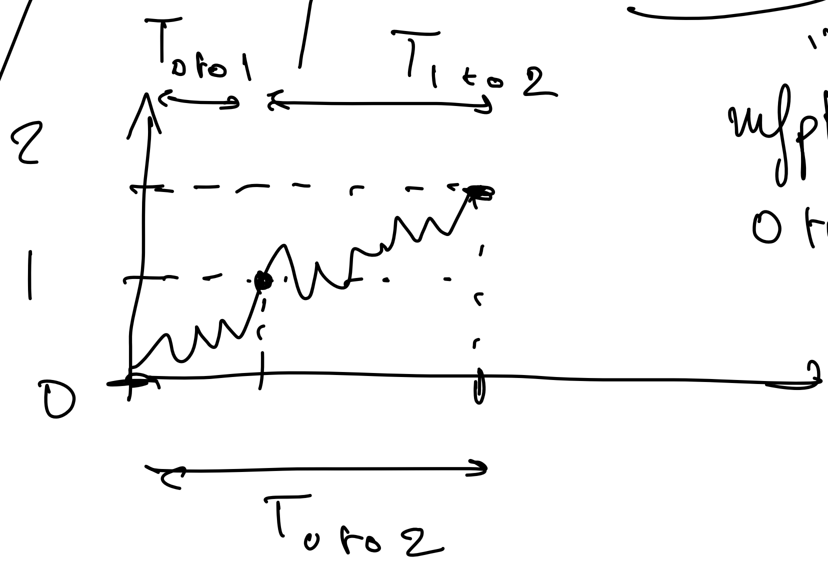
$\sum P(T = 2n) \underset{+\infty}{=}$)

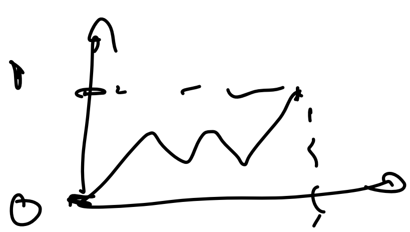
$E(T_2 \mid X_0 = 0)$

" mean 1st passage time (mfpt) from 0 to 2

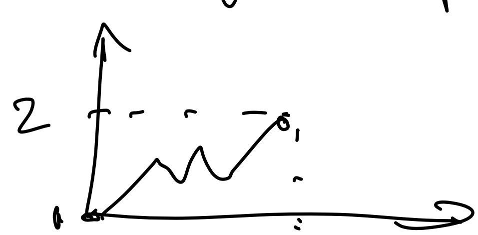
" mfpt from 0 to 1 + <u>mfpt from 1 to 2</u>

" mfpt from 0 to 1





$\Updownarrow$ same probability



Limiting probabilities: Warm-up

Recall:

- Initial distribution for a MC $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)$ $\Big(P(X_0 = i) = \alpha_i \Big)$
- Distribution after n steps:
  - Set $\alpha_i^n := P(X_n = i)$
  - $\alpha^n := (\alpha_1^n, \ldots, \alpha_N^n)$

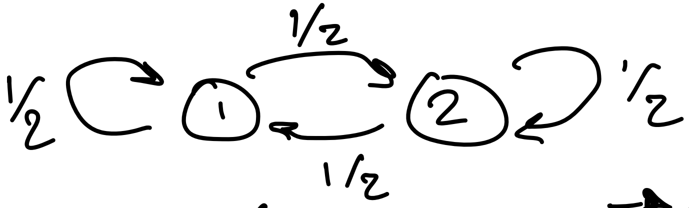
How to compute α^n? $\alpha^n = \alpha * \tilde{P}^n$

Limiting probabilities: What can we say about $\lim_{n\to\infty} \alpha^n$?

- Does it exist?
- Does it depend on α?

Warm-up continued $\left(\lim_{n\to\infty} \alpha^n \,?\right)$

Example: 2-state Markov chains

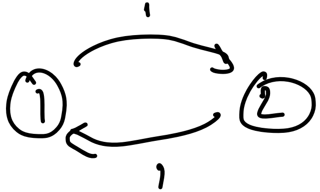


$$\alpha = (\alpha_1, \alpha_2)$$

$$\alpha^1 = (\alpha_1, \alpha_2)\tilde{P}$$
$$= (\alpha_1, \alpha_2)\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$$
$$= \Big(\tfrac{1}{2}(\underbrace{\alpha_1 + \alpha_2}_{1}), \tfrac{1}{2}(\alpha_1 + \alpha_2)\Big)$$
$$= \left(\tfrac{1}{2}, \tfrac{1}{2}\right)$$

$$\to \alpha^n = \left(\tfrac{1}{2}, \tfrac{1}{2}\right) \quad \forall n > 1$$

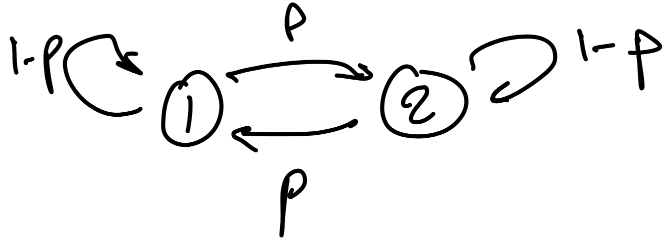
$$\alpha = (\alpha_1, \alpha_2)$$
$$\alpha^1 = (\alpha_1, \alpha_2)$$
$$\alpha^n = (\alpha_1, \alpha_2)$$
$$n \to \infty \downarrow \qquad \downarrow$$
$$= (\alpha_1, \alpha_2)$$



$$\alpha = (\alpha_1, \alpha_2)$$
$$\alpha^1 = (\alpha_2, \alpha_1)$$
$$\alpha^2 = (\alpha_1, \alpha_2)$$

$\alpha = (\alpha_1, \alpha_2)$ if $\alpha_1 \neq 0.5$

$\lim_{n\to\infty} \alpha^n$ does not exist



$$\tilde{P} = \begin{pmatrix} 1-p & p \\ p & 1-p \end{pmatrix}$$

Exercise : Show that

$$\tilde{P}^n = \frac{1}{2}\begin{pmatrix} 1+(2p-1)^n & 1-(2p-1)^n \\ 1-(2p-1)^n & 1+(2p-1)^n \end{pmatrix}$$

$(2p-1)^n \xrightarrow[n\to+\infty]{} 0$

$$\downarrow n \to +\infty$$

$$\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

$$\Rightarrow (\alpha_1, \alpha_2)\tilde{P}^n \xrightarrow[n\to+\infty]{} \frac{1}{2}\Big(\underbrace{\alpha_1+\alpha_2}_{=1}, \underbrace{\alpha_1+\alpha_2}_{=1}\Big)$$

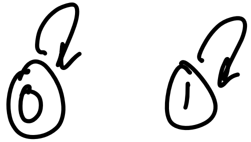
$$\Rightarrow \lim_{n\to+\infty} \alpha^n = \left(\tfrac{1}{2}, \tfrac{1}{2}\right)$$

When a limiting distribution exists, what form does it take?

Def: A vector $\pi = (\pi_1, \pi_2, \dots)$ is called a stationary distribution if it satisfies:

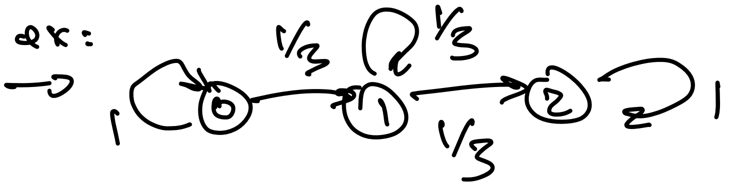
- $\pi = \pi \tilde{P}$, $\rightarrow$ "the distribution is the same after 1 step."
- $\sum_i \pi_i = 1$,
- $0 \leq \pi_i \leq 1 \quad$ for $i \geq 1$

Q: Can there be more than 1 stationary distribution? A: Yes $\rightarrow$



(see previous example)

ex:



$\rightarrow \pi = (3/4, 0, 1/4)$

$\pi = (1/3, 0, 1/3)$

Can we guarantee the existence of a limiting distribution?

"Good Markov chains"

<u>Recall</u>: Positive recurrent: $m_i < \infty$
where m_i is the mean time to return to i

Def: State i of an MC is called ergodic if it is aperiodic and positive recurrent. An MC is called ergodic if all states are ergodic.

Q) Do conditions simplify for finite-state MC?

<u>Remark</u>: For a finite M-C, positive recurrent $\Leftrightarrow$ recurrent

$\Rightarrow$ ergodic $\Leftrightarrow$ aperiodic and recurrent

The big theorem

(recall : m_i = mean time to return)

for a finite state M-C, we only need aperiodic and recurrent.

Thm: For an irreducible, ergodic MC:

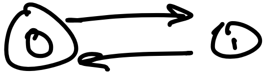
1. There is a unique stationary distribution, in fact, there is only one vector satisfying

   $a)\ \pi \tilde{P} = \pi$ and b) $\sum \pi_i = 1$.

2. $\lim_{n\to\infty} \alpha \tilde{P}^n = \pi$ (it exists, it does not depend on α, it is π)
3. $\pi_j = 1/m_j$ (thus, $\pi_j > 0$)
4. $\pi_j = \lim_{n\to\infty} \frac{\#\text{ of visits to } j \text{ by time } n}{n} =$ long run proportion of time spend in state j.

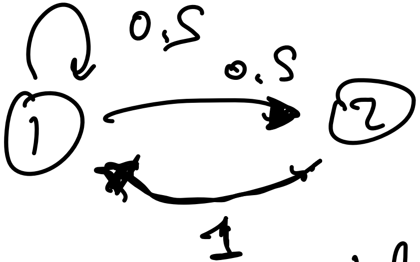
- Note: If we remove aperiodicity assumption, three of the four properties still hold. Which ones?

cf. last week.

example: 

→ Limiting probabilities do not exist, so 2 doesn't hold.

- Note: If we lose irreducibility, what do we lose? → we lose uniqueness of π

Example: 

- irreducible ✓ (1 communication class)
- recurrence ✓ (finite & irreducible)
- aperiodicity ✓ ($p_{11} > 0$)

a) Is the M.C irreducible and ergodic? → A:

b) If so, what is the limiting distribution?

From the theorem, we know that the limiting distribution exists, and we know that it satisfies

$$\begin{cases} \pi \tilde{P} = \pi \\ \pi_1 + \pi_2 = 1 \end{cases}, \quad \text{where } \tilde{P} = \begin{array}{c} \\ \end{array}\begin{array}{cc} 1 & 2 \end{array}$$

$$\tilde{P} = \begin{array}{cc|c} 1 & 2 & \\ \hline \end{array}$$

$$\tilde{P} = \begin{pmatrix} 0.5 & 0.5 \\ 1 & 0 \end{pmatrix}\begin{matrix} 1 \\ 2 \end{matrix}$$

$$\Leftrightarrow \begin{cases} (\pi_1, \pi_2)\begin{pmatrix} 0.5 & 0.5 \\ 1 & 0 \end{pmatrix} = (\pi_1, \pi_2) \\ \pi_1 + \pi_2 = 1 \end{cases}$$

$$\Rightarrow \pi_1 \cdot 0.5 + \pi_2 \cdot 1 = \pi_1 \Leftrightarrow \pi_1 = 2 \cdot \pi_2$$

and $\pi_1 = 1 - \pi_2$

so we obtain $\boxed{(\pi_1, \pi_2) = \left(\frac{2}{3}, \frac{1}{3}\right)}$

<u>Example 2</u>

MC with transition matrix $\tilde{P} = \begin{pmatrix} .5 & .4 & .1 \\ .3 & .4 & .3 \\ .2 & .3 & .5 \end{pmatrix}$

Q: find the long run proportion of time spent in each state.

A: The chain is ergodic and irreducible, so we solve

$$\begin{cases} \pi \tilde{P} = \pi \\ \pi_1 + \pi_2 + \pi_3 = 1 \end{cases} \Leftrightarrow \begin{cases} \pi_1 = 0.5\pi_1 + 0.4\pi_2 + 0.1\pi_3 \\ \pi_2 = 0.3\pi_1 + 0.4\pi_2 + 0.3\pi_3 \\ \pi_3 = 0.2\pi_1 + 0.3\pi_2 + 0.5\pi_3 \\ \pi_1 + \pi_2 + \pi_3 = 1 \end{cases}$$

(exercise) After doing Gaussian elimination,

we obtain $\pi = \left(\frac{21}{62}, \frac{23}{62}, \frac{18}{62}\right)$

So the proportion of time spent in $\begin{cases} 0 \text{ is } 21/62 \\ 1 \text{ is } 23/62 \\ 2 \text{ is } 18/62 \end{cases}$

$\longrightarrow$

The thm has the following (important) consequence

<u>Prop</u>: For an irreducible M-C.

(i) If there is no solution of $\begin{cases} \sum \pi_i = 1 \\ \pi P = \pi \end{cases}$, then the MC is transient or null-recurrent and $\pi_j = 0$

(ii) if there is a solution, then the MC is positive recurrent.

In practice, if the MC is irreducible, we can try to solve $\begin{cases} \pi P = \pi \\ \sum \pi_i = 1 \end{cases}$

If we solve it (or guess a solution that works), then we know that it is stationary and the chain is positive recurrent.