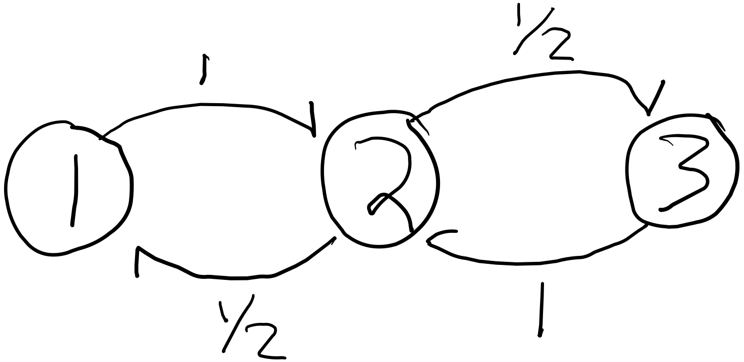
Lecture 12

More examples of (potentially) reversible MC

3-state (non-circular) random walk



Remark: One can show that the chain has period 2

$\Rightarrow$ One cannot use the ergodic ("big") theorem to say that there are limiting probabilities

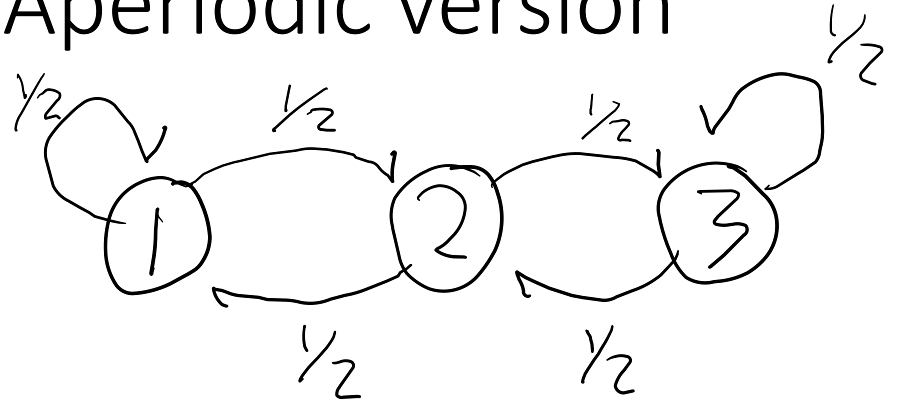
But we know that there is a stationary distrib. (the chain is irreducible)

$\rightarrow$ Let's find π using detailed balance eqs.

$$\begin{cases} x_1 p_{12} = x_2 p_{21} \Rightarrow x_1 = \frac{1}{2} x_2 \Rightarrow x_2 = 2x_1 \\ x_2 p_{23} = x_3 p_{32} \Rightarrow x_2 \cdot \frac{1}{2} = x_3 \Rightarrow x_3 = x_1 \\ x_1 + x_2 + x_3 = 1 \Rightarrow x_1 = \frac{1}{4} \end{cases}$$

$\rightarrow$ $\boxed{\pi = \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right)}$

Aperiodic version



- We already know that Π exists
- Since it's now aperiodic

$$\rightarrow \lim_{n\to\infty} \alpha P^n = \Pi$$

for any distribution α

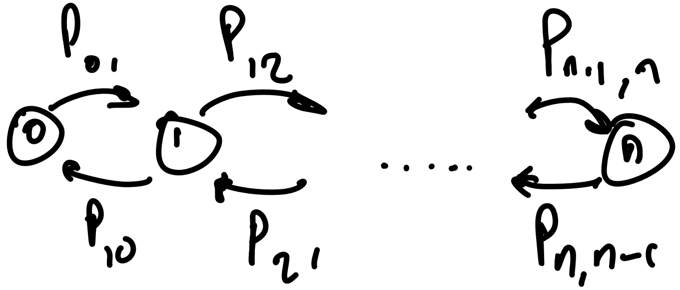
Let's find Π.

Doubly stochastic

$$\tilde{P} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \longrightarrow \boxed{\Pi = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)}$$

Arbitrary non-circular, finite RW

- n states,
- $p_{i,i+1}, p_{i+1,i} > 0$ for $i = 0,1, \ldots n-1$.
- $p_{i,j} = 0$, if $|i-j| \geq 2$.



Detailed balance eqs.

$$x_0 p_{01} = p_{10} x_1$$

$$\vdots$$

$$x_i p_{i,i+1} = p_{i+1,i} x_{i+1}$$

$$\vdots$$

$$x_{n-1} p_{n-1,n} = p_{n,n-1} x_n$$

$$\sum_i x_i = 1$$

$$\Rightarrow x_{i+1} = x_i \frac{p_{i,i+1}}{p_{i+1,i}}$$

$$= x_i \alpha_{i+1}$$

where $\alpha_{i+1} := \dfrac{p_{i,i+1}}{p_{i+1,i}}$

By induction $\rightarrow$ $x_i = \alpha_i \alpha_{i-1} \cdots \alpha_1 x_0 = \gamma_i x_0$,

where $\gamma_i := \alpha_i \alpha_{i-1} \cdots \alpha_1$ and $\gamma_0 = 1$

$\rightarrow \sum_i x_i = 1$ yields $1 = \sum_{i=0}^{n} x_i = \sum_{i=0}^{n} \gamma_i x_0 \Rightarrow x_0 = \dfrac{1}{\sum_i \gamma_i}$

Solution: $\boxed{x = \dfrac{(1, \gamma_1, \dots, \gamma_n)}{\sum_{i=0}^{n} \gamma_i}}$ $(*)$

$\Rightarrow$ Any noncircular 1-D RW with finite states and nonzero transition probabilities has the stationary distribution $(*)$

Exercise (or see 202 section)

Identify the Ehrenfest chain seen last week to a noncircular RW as above and check how the stationary distribution we found matches the one above.

Umbrella problem

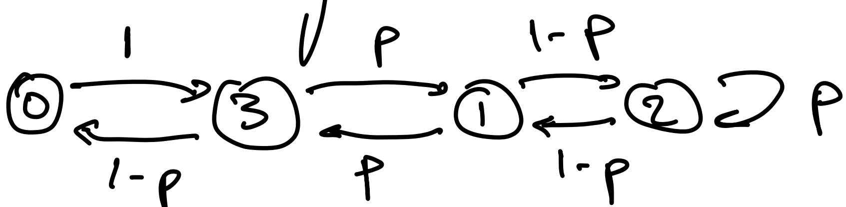
Smith has 3 umbrellas (total) at home and at casino.

- Each time Smith switches location, she takes an umbrella if she has one and if it is raining.
  - If it’s not raining, she doesn’t take an umbrella.
- Probability that it is raining is p (assumed independence each time she switches location).

Q: What fraction of the time does Smith have a wet walk?

A: Let X_n = # umbrellas at current location at time n

- State space = $\{0, 1, 2, 3\}$
- Transition diagram



- The chain is ergodic and irreducible so we know that the chain converges to a stationary distribution π
- The answer to the question is then $\pi_0 \cdot p$, prob. of rain.

  π_0: prob. of having no umbrella
- Assuming that X_n is time reversible we solve the detailed balance eqs.

$$\begin{cases} \pi_0 \cancel{P_{03}}^{\,1} = \pi_3 P_{30}^{\;1-p} \\ \pi_3 P_{13}^{\;p} = \pi_1 P_{13}^{\;p} \\ \pi_1 P_{12}^{\;1-p} = \pi_2 P_{21}^{\;1-p} \\ \pi_1 + \pi_2 + \pi_3 = 1 \end{cases} \iff \begin{cases} \pi_0 = \pi_3 (1-p) \\ \pi_1 = \pi_2 = \pi_3 \\ \pi_0 + \pi_1 + \pi_2 + \pi_3 = 1 \end{cases}$$

and we get $\pi_3 = \dfrac{1}{4-p}$, $\pi_0 = \dfrac{1-p}{4-p}$

so the answer is $\boxed{p \cdot \dfrac{1-p}{4-p}}$

MC on a graph → (Ross, example 4.38)

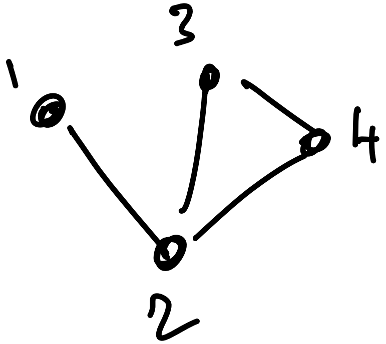
bidirected

Consider a graph (i.e., a network). The MC jumps from node to node, at each step choosing an edge at random. What is the stationary distribution?

Graph Definition: $G = (V, E)$

vertices or nodes

edges

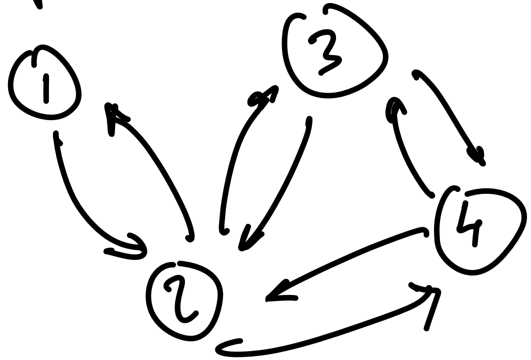


$V = \{1, 2, 3, 4\}$

$E = \{\{12\}, \{23\}, \{34\}, \{24\}\}$

We define the M.C on G:

Transition diagram



$$P_{ij} = \begin{cases} 0 & \text{if there is no edge between } i \text{ and } j \\ \dfrac{1}{d_i} & \text{if there is at least 1 edge} \end{cases}$$

$d_i :=$ degree of the i^{th} node $=$ # edges leaving i^{th} node

Detailed balance:

If there is an edge between i and j: $\dfrac{x_i}{d_i} = \dfrac{x_j}{d_j}$

<u>Exercise</u>: By assuming x_i proportional to d_i (i.e. $x_i = \lambda d_i$ for some λ), find the stationary distribution $x_i = \dfrac{d_i}{\sum d_i}$ $\boxtimes$