- Lecture 13. TuTh

Proof of Strong Duality and Complementary Slackness.

(Lecture 14 Economic significance of dual variables .)

- Proof of Strong duality & CS.

<u>Recall</u> Can write the initial dictionary

$$z = \sum_{j=1}^{n} c_j x_j$$

$$\boxed{x_{n+i}} = b_i - \sum_{j=1}^{n} a_{ij} x_j \quad , \quad i=1,\cdots,m$$

slack variables

$$\vec{x}_{slack} = \vec{b} - A\vec{x}$$

Question: **Choose a wrong statement:**

A) ~~Consider a standard form LP problem with n decision variables $x_1, x_2, \cdots, x_n$, and m slack variables $x_{n+1}, x_{n+2}, \cdots, x_{n+m}$. After applying the simplex method starting from a feasible dictionary, we found its optimal solution, moreover, in the final dictionary, the objective function is written in the form $z = z^* + \sum_{k=1}^{n+m} c_k^* x_k$. Then, $c_k^* \leq 0$ for all k, and z^* is the optimal value.~~

True

B) Let $\vec{c} \in \mathbb{R}^n$ and $d \in \mathbb{R}$ are given. If $\vec{c} \cdot \vec{x} = d$ for all $\vec{x} \geq \vec{0}$, then $\vec{c} = \vec{0}$ and $d = 0$.

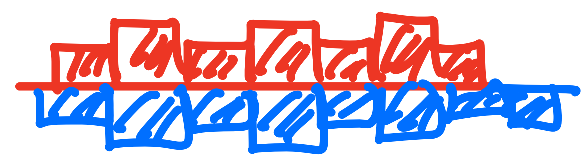
True

C) Let $p_1, \cdots, p_m$, $q_1, \cdots, q_n$ are given and $p_1, \cdots, p_m \geq 0$ and $q_1, \cdots, q_n \leq 0$.

If $\sum_{i=1}^{m} p_i = \sum_{j=1}^{n} q_j$, then $p_1 = \cdots = p_m = q_1 = \cdots = q_n = 0$.

True

D) $$\sum_{j=1}^{n} c_j x_j + \sum_{i=1}^{m} \left[y_i \left(b_i - \sum_{j=1}^{n} a_{ij} x_j \right) \right]$$
$$= \sum_{i=1}^{m} b_i y_i + \sum_{j=1}^{n} \left[x_j \left(c_j - \sum_{i=1}^{m} a_{ij} y_i \right) \right]$$



If $p \geq 0$, $q \leq 0$,
and $p = q$
then $p = 0$ and $q = 0$.

WRONG

E) One of ~~A~~, B, C, D is wrong.

D). $\vec{c}^T\vec{x} + \vec{y}^T(\vec{b} - A\vec{x})$

True. $= \vec{c}^T\vec{x} + \vec{y}^T\vec{b} - \vec{y}^TA\vec{x}$

$= \vec{y}^T\vec{b} + \underbrace{\vec{c}^T\vec{x} - \vec{y}^TA\vec{x}}$

$= \vec{y}^T\vec{b} + (\underbrace{\vec{c}^T - \vec{y}^TA})\vec{x}$

$\vec{c}^T - \vec{y}^TA$

$= (\vec{c} - A^Ty)^T$

$= \vec{b}^T\vec{y} + (\vec{c} - A^T\vec{y})^T\vec{x}$

$$\vec{c} - A^T\vec{y} = \begin{bmatrix} c_1 - \sum_{i=1}^{m} a_{ij}y_1 \\ c_2 - \sum_{i=1}^{m} a_{ij}y_2 \\ \vdots \end{bmatrix}$$

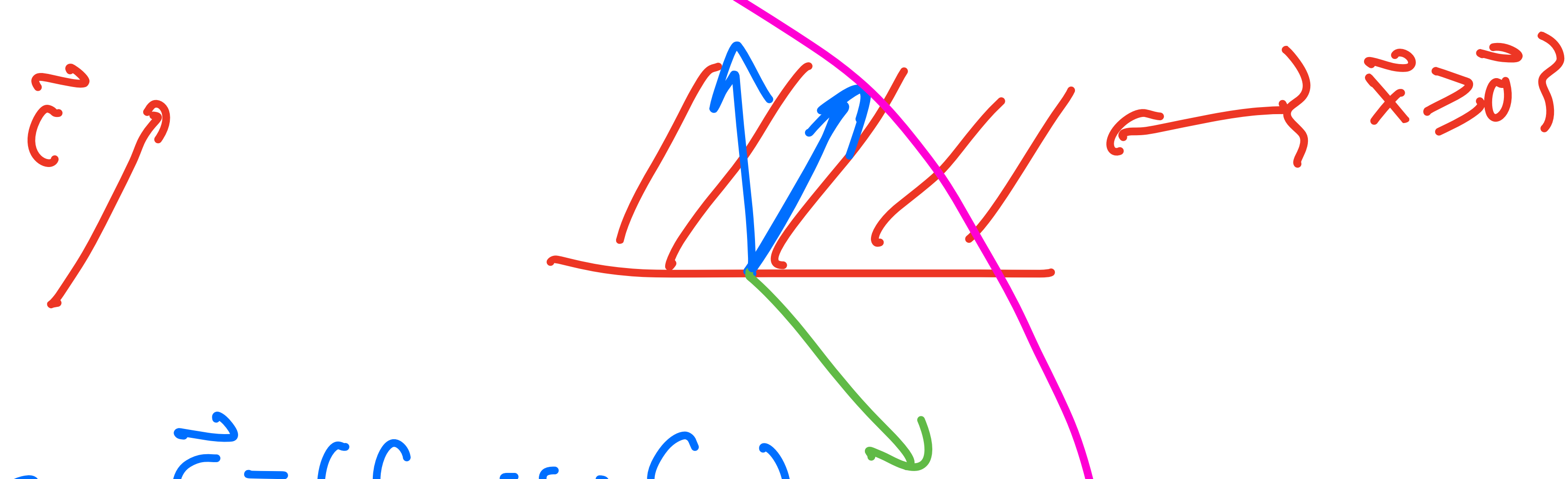
✓

B) $d = 0$ because we can take

$\vec{x} = \vec{0} \geq \vec{0}$

$\vec{c} \cdot \vec{0} = 0.$

True



Suppose $\vec{c} = (c_1, \dots, c_n)$.

If $\vec{c} \neq \vec{0}$, then $c_i \neq 0$ for some i.

So we can choose $\vec{x} = (0, \dots, 0, \underset{\uparrow \; i\text{-th}}{|c_i|}, 0, \dots, 0)$

For this $\vec{x}$, $\vec{x} \geq \vec{0}$,

and $\vec{c} \cdot \vec{x} = c_i |c_i| \neq 0$.

But $\vec{c} \cdot \vec{x} = \boxed{d = 0}$. Therefore $\underbrace{\vec{c} = \vec{0}}$.

Strong duality

Thm If an LP problem has an optimal solution, so does its dual, and the respective optimal objective values are equal.

That is, if the primal problem maximize $\vec{c}\cdot\vec{x}$ subject to $A\vec{x} \le \vec{b}$. $\vec{x} \geq \vec{0}$.

has an optimal solution $(x_1^*, x_2^*, \dots, x_n^*)$

then • the dual has an optimal solution $(y_1^*, \dots, y_m^*)$

and

• $\sum_{j=1}^{n} c_j x_j^* = \sum_{i=1}^{m} b_i y_i^*$

where there is no entering variable.

Moreover, from the z row of the optimal final dictionary of the primal problem

the optimal value; coefficient, $c_k^* \le 0$ for optimal final dictionary

$$z = z^* + \sum_{k=1}^{n+m} c_k^* x_k \quad \left(\text{note } c_k^* = 0 \text{ if } x_k \text{ basic}\right)$$

We can find

$y_i^* = -c_{n+i}^*$, $i = 1, \dots, m$, the coefficients for slack variable x_{n+i}

for a dual optimal basic solution $\vec{y}^*$.

optimal dictionary of the primal can also determine optimal solution for dual.

from these $(c_1^*, c_2^*, \dots, c_n^*, c_{n+1}^*, \dots, c_{n+m}^*)$ in the final dictionary, a dual optimal basic solution $\vec{y}^*$ is determined as $y_i^* = -c_{n+i}^*$, $i = 1, \dots, m$

$y_{m+j}^* = -c_j^*$ $j = 1, \dots, n$

proof of strong duality

- Let $\vec{x} = (x_1, \cdots, x_n)$ ← the decision variable.

  $\vec{x}_{slack} = (x_{n+1}, \cdots, x_{n+m}) = \vec{b} - A\vec{x}$

- Let $\vec{x}^* = (x_1^*, \cdots, x_n^*)$ be an optimal solution.

- From the final optimal dictionary, get expression of $z = \vec{c} \cdot \vec{x}$

as $$z = z^* + \sum_{k=1}^{n+m} c_k^* x_k = z^* + \vec{c}^* \cdot \vec{x} + \vec{c}^*_{slack} \cdot \vec{x}_{slack}$$

- Note. $\vec{c}^* \leq \vec{0}$, $\vec{c}^*_{slack} \leq \vec{0}$ as from the final optimal dictionary.
  - $z^* = \vec{c} \cdot \vec{x}^*$ the optimal value.

So, $\vec{c}\cdot\vec{x} = \vec{c}\cdot\vec{x}^* + \vec{c}^*\cdot\vec{x} + \vec{c}^*_{slack}\cdot\underbrace{\vec{x}_{slack}}$ ← they are equal as functions. (i.e. for all $\vec{x}$).

$$= \vec{c}\cdot\vec{x}^* + \vec{c}^*\cdot\vec{x} + \vec{c}^*_{slack}\cdot(\vec{b} - A\vec{x})$$

$\vec{x}_{slack} = \vec{b} - A\vec{x}$.

$$= \vec{c}\cdot\vec{x}^* + \vec{c}^*\cdot\vec{x} + \vec{c}^*_{slack}\cdot\vec{b} - \vec{c}^*_{slack}\cdot A\vec{x}$$

$$= \vec{c}\cdot\vec{x}^* + \vec{c}^*\cdot\vec{x} + \vec{c}^*_{slack}\cdot\vec{b} - (A^T\vec{c}^*_{slack})\cdot\vec{x}$$

$$\underbrace{(\vec{c}^*_{slack})^T A\vec{x}} = (A^T\vec{c}^*_{slack})^T\vec{x}.$$

$\vec{a}\cdot\vec{b} = \vec{a}^T\vec{b} = \vec{b}^T\vec{a}$

Rearrange

$$(\vec{c} - \vec{c}^* + A^T\vec{c}^*_{slack})\cdot\vec{x} = \vec{c}\cdot\vec{x}^* + \vec{c}^*_{slack}\cdot\vec{b}$$

This implies

- $\vec{c} - \vec{c}^* + A^T\vec{c}^*_{slack} = \vec{0}$ & $\vec{c}\cdot\vec{x}^* + \vec{c}^*_{slack}\cdot\vec{b} = 0$
- Define $\vec{y}^* = -\vec{c}^*_{slack}$ (i.e. $y^*_i = -c^*_{n+i}$, $i=1,\dots,m$)

Also. $\vec{c} - A^T\vec{y}^* = \vec{c}^* \leq \vec{0}$

So $\vec{c} - A^T\vec{y}^* \leq \vec{0}$

i.e. $A^T\vec{y}^* \geq \vec{c}$.

- Note $\vec{y}^* \geq \vec{0}$ (as $\vec{c}^*_{slack} \leq 0$)

for the optimal final dictionary

So, $\vec{y}^*$ is dual feasible.

- Moreover, $\vec{c}\cdot\vec{x}^* = \vec{b}\cdot\vec{y}^*$

So, from weak duality, $\vec{y}^*$ is (dual) optimal.

$$\vec{y}^* = (-c^*_{n+1}, -c^*_{n+2}, \dots, -c^*_{n+m}).$$

$\boxed{///}$

Complementary Slackness.

$\left\{ \begin{array}{l} \vec{x}^* \text{ is optimal for primal.} \\ \vec{y}^* \text{ is optimal for dual.} \end{array} \right.$

$\iff$ $\left[\begin{array}{l} \vec{x}^* = (x_1^*, \ldots, x_n^*) \quad \text{feasible for primal} \\ \vec{y}^* = (y_1^*, \ldots, y_m^*) \quad \text{feasible for dual.} \\ \\ \text{and} \quad \text{CS} \cdots \left[\begin{array}{ll} y_i^* \cdot x_{n+i}^* = 0 & i = 1, \ldots, m \\ x_j^* \; y_{m+j}^* = 0 & j = 1, \ldots, n. \end{array} \right. \end{array} \right.$

Here $x_{n+i}^* = i\text{-th slack of } \vec{x}^* = b_i - \sum_{j=1}^{n} a_{ij} x_j^*$

$y_{m+j}^* = j\text{-th slack of } \vec{y}^* = \sum_{i=1}^{m} a_{ij} y_i^* - c_j$

Another (equivalent) form
of complementary slackness.

Complementary slackness

Thm Suppose • $\vec{x}^* = (x_1^*, \ldots, x_n^*)$ be feasible for primal

$\vec{y}^* = (y_1^*, \ldots, y_m^*)$ be feasible for dual

Then,

• $(\vec{x}^*, \vec{y}^*)$ optimal for (primal, dual)

$\iff$ CS : $y_i^*\left(b_i - \sum_{j=1}^{n} a_{ij} x_j^*\right) = 0$ for each i

& $x_j^*\left(c_j - \sum_{i=1}^{m} a_{ij} y_i^*\right) = 0$ for each j

Proof of complementary slackness

$$(*)\quad \vec{c}\cdot\vec{x} + \vec{y}\cdot(\vec{b}-A\vec{x}) = \vec{c}\cdot\vec{x} + \vec{y}\cdot\vec{b} - \vec{y}\cdot A\vec{x} = \vec{c}\cdot\vec{x} + \vec{y}\cdot\vec{b} - (A^T\vec{y})\cdot\vec{x}$$

$$= \vec{b}\cdot\vec{y} + \vec{x}\cdot(\vec{c} - A^T\vec{y})$$

- optimality of $\vec{x}^*, \vec{y}^*$ for primal / dual, respectively.

$\overset{\text{by strong and weak duality}}{\Longleftrightarrow}$ $\begin{cases} \bullet \text{ feasibility of } \vec{x}^*, \vec{y}^* \text{ (i.e. feasibility of } \vec{x}^*\text{, dual feasibility of } \vec{y}^*) \\ \bullet\ \vec{c}\cdot\vec{x}^* = \vec{b}\cdot\vec{y}^* \end{cases}$ ($\Longleftarrow$ by weak duality, $\Longrightarrow$ by strong duality).

$\Longleftrightarrow$ $\begin{cases} \bullet \text{ feasibility of } \vec{x}^*, \vec{y}^*. \\ \bullet\ \underbrace{\underbrace{\vec{y}^*}_{\geq 0}\cdot(\underbrace{\vec{b}-A\vec{x}^*}_{\vec{x}^*_{slack}\ \geq 0})}_{\geq 0} = \underbrace{\underbrace{\vec{x}^*}_{\geq 0}\cdot(\underbrace{\vec{c}-A^T\vec{y}^*}_{-\vec{y}^*_{slack}\ \leq 0})}_{\leq 0} \end{cases}$ (by $(*)$).

$\vec{y}^*_{slack} = A^T\vec{y}^* - \vec{c}$

$i = 1, \dots, n$

$j = 1, \dots, m$

$\Longleftrightarrow$ $\begin{cases} \bullet \text{ feasibility of } \vec{x}^*, \vec{y}^* \\ \bullet\ \vec{y}^*\cdot(\underbrace{\vec{b}-A\vec{x}^*}_{\vec{x}^*_{slack}}) = 0 \ \&\ \underbrace{\vec{x}^*\cdot(\overbrace{\vec{c}-A^T\vec{y}^*}^{-\vec{y}^*_{slack}})}_{} = 0. \end{cases}$ $\Longleftarrow$ $\begin{cases} \bullet \text{ feasibility of } \vec{x}^*, \vec{y}^* \\ \bullet\ y_i^*\cdot x^*_{n+i} = 0,\ i = 1, \dots, m \\ \quad x_j^*\ y^*_{m+j} = 0,\ j = 1, \dots, n. \end{cases}$ CS.

i.e. $\sum_{i=1}^{m} \underbrace{y_i^*\ x^*_{n+i}}_{\geq 0} = 0\ \&\ \sum_{j=1}^{n} \underbrace{x_j^*}_{\geq 0}\ \underbrace{\vec{y}^*_{m+j}}_{\geq 0} = 0.$

We used:

$$\boxed{\text{For } p_i \geq 0,\ q_j \leq 0 \quad \sum_{i=1}^{m} p_i = \sum_{j=1}^{n} q_j \Rightarrow p_i = 0 = q_j \quad \forall\, i=1,\dots,m,\ j=1,\dots,n}$$

$\blacksquare$