Lecture 18

Sums of iid exponential rv’s are Gamma rv’s

Recall

Given independent continuous random variables, X, Y with densities f_X and f_Y, what is the density of $Z := X + Y$?

$$f_Z(z) = (f_X * f_Y)(z) = \int_{-\infty}^{+\infty} f_X(x) f_Y(z-x)\, dx$$

pdf of $Z = X+Y$

convolution product

$\rightarrow$ Let's apply this to $\mathrm{Exp}(\lambda_i)$

Sums of iid exponential rv's

Let $X, Y \sim Exp(\lambda)$ be independent. What is the density of $Z := X + Y$?

$$f_Z(z) = \int_{-\infty}^{+\infty} f_X(x) f_Y(z-x)\, dx$$

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{else} \end{cases}$$

$$f_Y(z-x) = \begin{cases} \lambda e^{-\lambda(z-x)} & \text{if } z-x \geq 0 \Downarrow x \leq z \\ 0 & \text{else} \quad x > z \end{cases}$$

$$= \int_0^z \lambda \cancel{e^{-\lambda x}}\, \lambda e^{-\lambda(z-\cancel{x})}\, 1\, dx$$

$$= \lambda^2 e^{-\lambda z} z$$

Q) Let $X \sim Exp(\lambda_1), Y \sim Exp(\lambda_2)$ be independent. What is the density of $Z := X + Y$?

same method (exercise)

$$\boxed{f_Z(z) = \frac{\lambda_1 \lambda_2}{\lambda_1 - \lambda_2} e^{-\lambda_2 z} + \frac{\lambda_2 \lambda_1}{\lambda_2 - \lambda_1} e^{-\lambda_1 z}} \quad (\text{if } z > 0)$$

Gamma distribution

Def: We say X is a gamma (n, λ) rv if it has density

$$f_X(t) = \begin{cases} \lambda e^{-t} \dfrac{(\lambda t)^{n-1}}{(n-1)!} & \text{for } t \geq 0 \\ 0 & \text{for } t \overset{<}{\cancel{\geq}} 0 \end{cases}$$

Notation: $X \sim \Gamma(n, \lambda)$

Gamma (greek)

Rmk: We observe that

$\mathrm{Exp}(\lambda) + \mathrm{Exp}(\lambda) \sim \Gamma(2, \lambda)$

Prop: Let $X_1, X_2, \dots, X_n$ be iid $Exp(\lambda)$ rv's. Then

$X_1 + X_2 + \cdots + X_n \sim \underline{\Gamma(n, \lambda)}$

(Rmk: One can prove it by induction)

Let $X \sim \Gamma(n, \lambda)$

Q: What are EX and $Var(X)$? $\longrightarrow$ Use the prop.

$$A: \quad \Gamma(n,\lambda) = \sum_{i=1}^{n} X_i \qquad X_i \overset{iid}{\sim} Exp(\lambda)$$

$$\Rightarrow E(\Gamma(n,\lambda)) = \sum_{i=1}^{n} E(X_i) = n\, E(Exp(\lambda)) = \boxed{\frac{n}{\lambda}}$$

$$Var(\Gamma(n,\lambda)) = \sum_{i=1}^{n} Var(X_i) = \boxed{\frac{n}{\lambda^2}}.$$