Lecture 20

Poisson processes

Recall (from Math 302)

Poisson r.v.: We say $X \sim Poisson(\lambda)$ if $P(X = k) =$ $\underline{\frac{\lambda^k}{k!} e^{-\lambda}}$ $k = 0, 1, 2, \ldots$

$\mathbb{E}X =$ $\underline{\lambda}$, $Var(X) =$ $\underline{\lambda}$

Relationship to Binomial r.v.:

$$\lim_{n\to\infty} P\left(Bin\left(n, \frac{\lambda}{n}\right) = k\right) = \underline{\frac{\lambda^k}{k!} e^{-\lambda}}$$

$$Bin\left(n, \frac{\lambda}{n}\right) \xrightarrow[n\to\infty]{dist} Poisson(\lambda)$$

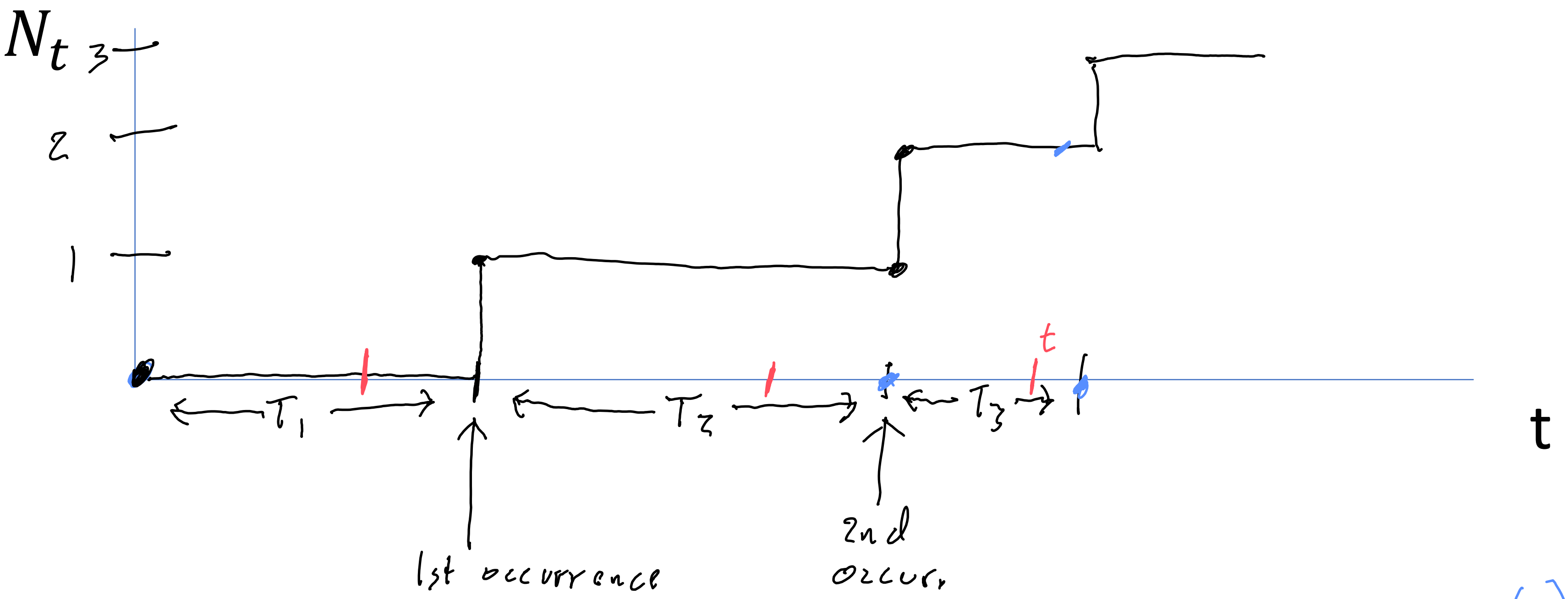
Goal: Random model for the number of occurrences of following as a function of time, starting now.

- Number of high-magnitude earthquakes in certain location
- Number of planes that pass overhead
- Number of sneezes in today's class

Let $(N_t)_{t\geq 0}$ be such a model. We will sometimes write $N_t = N(t)$. What can we say about N_t?

1. $N_0 = N(0) = 0$ Start counting at 0
2. $N_t \in \{0, 1, 2, \dots\}$
3. $t \geq s \geq 0$, $N_t \geq N_s$

These define a counting process



<u>Main assumption:</u> Waiting time between occurrences is $\underline{\ Exp(\lambda)\ }$.

Relationship between N_t and waiting times:

$N_t = 0 \Leftrightarrow$ $\underline{\ T_1 > t\ }$ $\qquad$ $N_t = 1 \Leftrightarrow$ $\underline{\ T_1 \le t,\ T_1 + T_2 > t\ }$

$N_t \geq n \Leftrightarrow$ $\underline{\ T_1 + T_2 + \dots + T_n < t\ }$ $\qquad$ $N_t =$ $\underline{\ \max\{n : \sum_{i=1}^{n} T_i < t\}\ }$

$\{n : \sum_{i=1}^{n} T_i < t\} = \{1, \underset{\substack{\uparrow \\ \max}}{2}\}$

Poisson process (first definition)

Def 1: The (homogeneous) Poisson process with rate λ is a counting process $(N_t)_{t\geq 0}$ satisfying

$$N_t = \max\{n | \sum_{i=1}^{n} T_i \leq t\}$$

waiting times are iid $Exp(\lambda)$

Where $T_i \sim Exp(\lambda)$ are iid.

From now on, unless otherwise specified, $(N_t)_{t\geq 0}$ refers to a Poisson process.

What is the distribution of N_t?

$$P(N_t \geq n) = P(\underbrace{T_1 + T_2 + \dots + T_n}_{\text{sum of } n \ Exp(\lambda) \text{ r.v.s} \ \sim \Gamma(n,\lambda)} \leq t) = P(\Gamma(n,\lambda) \leq t)$$

$$= \int_0^t \lambda e^{-\lambda x} \cdot \frac{(\lambda x)^{n-1}}{(n-1)!} \, dx$$

$u = \lambda x$

$du = \lambda dx$

$$= \int_0^{\lambda t} e^{-u} \frac{u^{n-1}}{(n-1)!} \, du$$

integration by parts. (Exercize)

$$= \dots = 1 - \sum_{i=0}^{n-1} \frac{1}{i!} (t\lambda)^i e^{-\lambda t}$$

Now, $P(N_t = n) = P(N_t \geq n) - P(N_t \geq n+1)$

$$= 1 - \sum_{i=0}^{n-1} (\quad) - \left[1 - \sum_{i=0}^{n} (\quad) \right]$$

$$= \boxed{\frac{1}{n!} (t\lambda)^n e^{-\lambda t}} \Longrightarrow \boxed{N_t \sim Poisson(\lambda t)}$$

Q: Let $(N_t)_{t\geq 0}$ be a Poisson process with rate $\frac{1}{2}$. $\lambda = \frac{1}{2}$

- What is $P(N_{10} = 3)$?

$N_{10} \sim \text{Poisson}(10 \cdot \tfrac{1}{2}) = \text{Poisson}(5)$

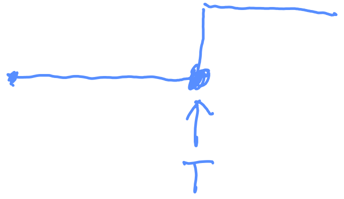
$$\boxed{P(N_{10} = 3) = \frac{5^3}{3!} e^{-5}}$$

- What is $\mathbb{E}N_2$?

$\mathbb{E}N_2 = \mathbb{E}\left[\text{Poisson}(2 \cdot \tfrac{1}{2})\right] = 1.$

- Let $T = \min\{\mathrm{t}|\mathrm{N_t} \geq 1\}$. What is ET?

$\underbrace{\min\{\mathrm{t}|\mathrm{N_t} \geq 1\}}$ waiting time to 1st occurrence $\sim \text{Exp}(\tfrac{1}{2})$



$\mathbb{E}T = \mathbb{E}(\text{Exp}(\tfrac{1}{2})) = \boxed{2.}$

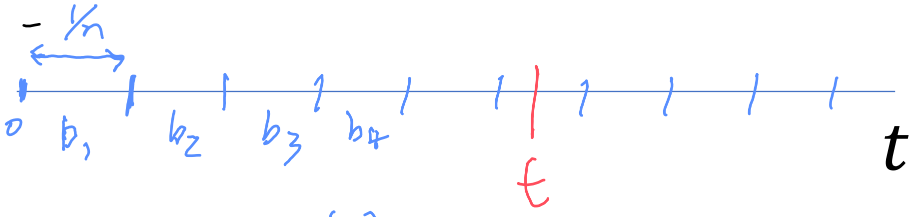
$P(N_t \geq 10) = P(\text{Poisson}(t\lambda) \geq 10)$

$$= \boxed{\sum_{k=10}^{\infty} \frac{(t\lambda)^k}{k!} e^{-t\lambda}}$$

An intuitive derivation of Poisson processes by quantizing time

Again, we want a model for, say, number of sneezes starting at time 0.

- Discretize time to a grid with time steps at length $\frac{1}{n}$ intervals.
- During each interval, there is a sneeze with probability $\frac{\lambda}{n}$.



$b_i \sim \text{Bern}\left(\frac{\lambda}{n}\right)$

sample

For t on the grid, set N_t = # of sneezes by time t.

How many buckets by time t? Ans: nt

$N_t \sim Bin(nt, \frac{\lambda}{n})$ = sum of nt Bern$(\frac{\lambda}{n})$ r.v.'s

$$\lim_{n\to\infty} N_t \xrightarrow{dist} Poisson(nt \cdot \frac{\lambda}{n}) = Poisson(t\lambda)$$

Let T be time until 1st sneeze.

$$P(T \geq t) = P(b_1 = 0, b_2 = 0, \ldots b_{tn} = 0) \underset{\substack{\uparrow \\ \text{by indep.}}}{=} \left(1 - \frac{\lambda}{n}\right)^{tn}$$

$$\lim_{n\to\infty} P(T \geq t) = \lim_{n\to\infty} \left(1 - \frac{\lambda}{n}\right)^{tn} = e^{-\frac{\lambda}{\cancel{n}} \cdot t\cancel{n}} = e^{-\lambda t} = P(Exp(\lambda) > t)$$

i.e. waiting time to 1st occurence converges to $Exp(\lambda)$.

Little o notation

We say a function $f(h)$ is $o(h)$ if $\lim_{h\to 0} \frac{f(h)}{h} = 0$

Meaning: $f(h)$ converges to 0 faster than h.

Ex) $f(h) = h^2$ is $o(h)$ since $\lim_{h\to 0} \frac{f(h)}{h} = \lim_{h\to 0} h = 0$

Ex) $f(h) = h$ is not $o(h)$ since $\lim_{h\to 0} \frac{f(h)}{h} = \lim_{h\to 0} 1 = 1 \neq 0$

Ex) $f(h) = e^h = 1 + h + \underbrace{\frac{h^2}{2} + \frac{h^3}{3!} + \cdots}_{o(h)}$

$= 1 + h + o(h)$

Second definition of Poisson process

Def 2: A counting process $(N_t)_{t\geq 0}$ is a rate-λ Poisson process if:

- Increments are independent, i.e., for $u \leq v \leq s \leq t$
  $N_v - N_u$ is indep of $N_t - N_s$.

- $P(N(t+h) - N(t) = 1) = \lambda h + o(h)$
- $P(N(t+h) - N(t) \geq 2) = o(h)$

Prop:

- Both definitions are equivalent.
- $N(t) - N(s) \sim$ Poisson(λt)