Math 303
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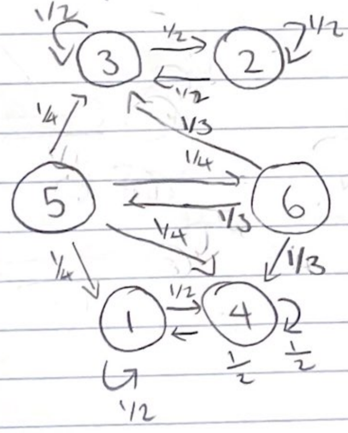
1) a) complete transition matrix, with MC defined on $\{1, \dots, 6\}$, so that next state of MC is chosen uniformly at random for which there is non-zero 1-step transition probability > 0

$$\begin{array}{cccccc|c} 1 & 2 & 3 & 4 & 5 & 6 & \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & 1 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 2 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 3 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & 4 \\ \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & 5 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 6 \end{array}$$

b) Draw transition diagrams and find the communicating classes



Let C_k be communicating classes $k = 1, 2, 3$
$C_1 = \{1, 4\}$
$C_2 = \{2, 3\}$
$C_3 = \{5, 6\}$

Two states i and j communicate,
if $i \to j$ (→ i accessible to j), such $p_{i,j}^{(n)} > 0$ for some $n \geq 0$
and $j \to i$, such $p_{j,i}^{(n)} > 0$ for some $n \geq 0$
or simply $i \leftrightarrow j$

c) find the period of states, if transient or recurrent
period of state S_i = $d(i)$ = greatest common divisor of $\{n \in \{1, 2, \dots\} : p_{i,i}^{n} > 0\}$

$d(1) \overset{gcd}{=} \{1, 2, 3\} = 1 = d(2) = d(3) = d(4)$
$d(5) = d(6) = 2$
same communicating class → C_1 = Recurrent → period 1 C_3 = transient period 2
share same recurrence/transience C_2 = Recurrent → period 1

d) Choose one transient state i (if any) and assuming $X_0 = i$, find mean time spent in i starting from i and probability of return to i

Known C_3 is the class $\{5,6\}$ transient, so we could choose any of $X_0 = 5$ or $X_0 = 6$, this case use $X_0 = 5$ for simplicity, i.e. $i = 5$

$$P_T = \begin{array}{c|cc|c} & 5 & 6 & \\ & 0 & 1/4 & 5 \\ & 1/3 & 0 & 6 \end{array}$$

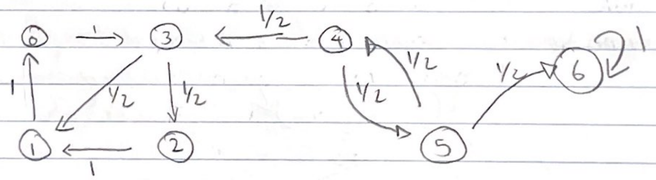
Mean time spent in 5 starting from 5 $= S_{55}$, where $S = (I - P_T)^{-1}$

$$\Rightarrow S = (I - P_T)^{-1}$$

$$= \begin{bmatrix} 1 & -1/4 \\ -1/3 & 1 \end{bmatrix}^{-1} = \frac{1}{1-(1/12)} \begin{bmatrix} 1 & 1/4 \\ 1/3 & 1 \end{bmatrix} = \begin{array}{cc|c} 5 & 6 & \\ 12/11 & 3/11 & 5 \\ 4/11 & 12/11 & 6 \end{array}$$

So $S_{55} = 12/11$ and $P(\text{return to } 5 \mid X_0 = 5) = f_5 = 1 - \frac{1}{S_{55}} = 1 - (\frac{11}{12}) = 1/12$

2) consider MC $\{X_n : n \geq 0\}$ with state-space $\{0,1,2,3,4,5,6\}$ and transition diagram



a) Determine communicating classes (No justify)

Class 1: $\{6\}$
class 2: $\{4,5\}$
class 3: $\{0,1,2,3\}$

b) Find period of states, if transient or recurrent

$d(0) = \gcd\{3,4\} = 1 \Rightarrow d(1) = d(2) = d(3) = d(0) = 1$ since in same class
$d(4) = 2 = d(5)$ since in same class
$d(6) = 1$

class 1 recurrent, class 2 transient, class 3 recurrent

c) What is $P(X_4 = 1 \mid X_0 = 4)$?

We could compute n-step transition matrix

In this case $n = 4$

$$\Rightarrow \quad \overset{n=0}{P} = \begin{array}{c|ccccccc|c} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & \\ \hline & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ & 0 & 1/2 & 1/2 & 0 & 0 & 0 & 0 & 3 \\ & 0 & 0 & 0 & 1/2 & 0 & 1/2 & 0 & 4 \\ & 0 & 0 & 0 & 0 & 1/2 & 0 & 1/2 & 5 \\ & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 6 \end{array}$$

And get $P^4 =$

skipped computation

$$\begin{array}{ccccccc|c} 0 & 1 & 2 & 3 & 4 & 5 & 6 & \rightarrow \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 1/2 & 1/2 & 0 & 0 & 0 & 0 & 3 \\ 0 & 1/16 & 1/16 & 1/2 & 1/16 & 0 & 5/16 & 4 \\ 1/4 & 0 & 0 & 1/16 & 0 & 1/16 & 5/8 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 6 \end{array}$$

And find $P^4(4,1) = 1/16 = P(X_4 = 1 \mid X_0 = 4)$

d) What is $P(X_{16} = 1, X_4 = 1 \mid X_0 = 4)$?

$$P(X_{16} = 1, X_4 = 1 \mid X_0 = 4) = \frac{P(X_{16} = 1, X_4 = 1, X_0 = 4)}{P(X_0 = 4)}$$

$$= P(X_{16} = 1 \mid X_4 = 1, X_0 = 4)\, P(X_4 = 1 \mid X_0 = 4)$$

By markov property $= P(X_{16} = 1 \mid X_4 = 1)\, P(X_4 = 1 \mid X_0 = 4)$

$= P(X_{12} = 1 \mid X_0 = 1) \cdot \left(\frac{1}{16}\right)$ → from c)

$= \frac{3}{16}\left(\frac{1}{16}\right) = \boxed{\frac{3}{256}}$

$P^{(12)}_{(1,1)} = 3/16$

↳ skipped calculation

Problem 3

a) let $p(n) = P(\text{ruin} \mid X_0 = n)$ $(0 \le n \le N)$ what are $p(0)$, $p(N)$

find relation between $p(n)$, $p(n+1)$, $p(n-1)$

Recall :

wins game with probability p

loses a dollar with probability $1-p = q$

$p(0) = P(\text{ruin} \mid X_0 = 0) = 1$

↳ loss, given they're 0 dollars now

$p(N) = p(\text{ruin} \mid X_0 = N) = 0$

↳ loss gave give you've ~~reached~~ N dollars

$$\Rightarrow p(n) = \overset{\text{win}}{p(P(n+1))} + \overset{\text{loss}}{q(p(n-1))}$$

$$\Rightarrow p(p(n)) + q(p(n)) = p(P(n+1)) + q(P(n-1))$$

$$\underbrace{(p+q)}_{=1} p(n) = p(P(n+1)) + q(P(n-1))$$

b) We know from class, either X_i reaches 0 or N with probability 1.

Defined associated random time $T = \min\{k \ge 0 \text{ s.t. } X_k = 0 \text{ or } N\}$

so mean duration of whole game, given initial wealth n $(0 \le n \le N)$

is $E_n(T) = E(T \mid X_0 = n)$

let $X_n = E_n(T)$

what are $X(0)$ and $X(N)$

Hint: when $p \neq 0.5$

$$E_n(T) = \frac{1}{1-2p}\left(n - N\left(\frac{1-(a-2)^n}{1-(a-2)^N}\right)\right) \quad \text{and } a = \frac{1}{p} - 1$$

$$X(0) = E_0(T) = E(T \mid X_0 = 0)$$

$$= E(\min\{k \ge n \text{ s.t. } X_k = 0 \text{ or } N \mid X_0 = 0\})$$

$$= E(0) = 0$$

Since you started broke, so you are already at the ruin

$x(N) = E_N(T) = E(T \mid X_0 = N)$ with $p \neq 0.5$

$= \frac{1}{1-2p}\left(N - N\left(\cdots \right)\right)$ $\rightarrow 0$

$x(N) = E_N(T) = \left(\frac{1}{1-2p}\right) 0$

$x(N) = E_N(T) = 0$

since you already won the game

=

$x(n) = E(T \mid X_0 = n)$

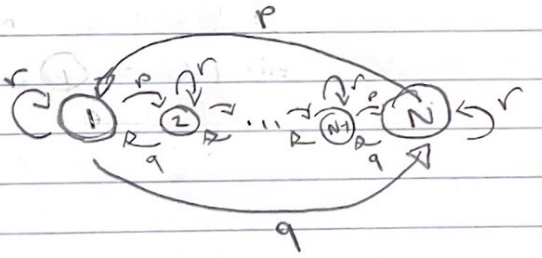
$= E(T \mid X_1 = n+1, X_0 = n)\, \overset{\rightarrow p}{P(X_1 = n+1 \mid X_0 = n)} +$

$E(T \mid X_1 = n-1, X_0 = n)\, \underbrace{P(X_1 = n-1 \mid X_0 = n)}_{1-p = q}$

$= (1 + E(T \mid X_0 = n+1))p + (1 + E(T \mid X_0 = n-1))\, q$

$= 1 + p \cdot x(n+1) + q \cdot x(n-1)$

4) Consider MC on $\{1, \cdots, N\}$



a) write transition matrix for $N = 4$

$$P = \begin{array}{c} \\ \\ \\ \\ \end{array}\begin{array}{cccc|c} 1 & 2 & 3 & 4 & \\ r & p & 0 & q & 1 \\ q & r & p & 0 & 2 \\ 0 & q & r & p & 3 \\ p & 0 & q & r & 4 \end{array}$$

$p = q = r = 1/3$ in this case

b) We suppose $p, q > 0$. What are communicating classes?

$\text{class}_1 = \{1, 2, 3, 4\}$ $\rightarrow$ $p^n_{41}\ p^n_{42}\ p^n_{43}, p^n_{44} \geq 0$ for $n \geq 0$

since $p^n_{11}, p^n_{12}, p^n_{13}, p^n_{14} \geq 0$ for $n \geq 0$ $\Rightarrow 1 \leftrightarrow 2, 2 \leftrightarrow 3, \ldots, N-1 \leftrightarrow N$

so only one class

for generic N, this still holds, so one communicating class $\{1, 2, \ldots, N-1, N\}$

c) period of each state

Since in same communicating class, so they all same period

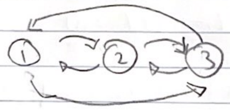
$d(1) = d(2) = d(3) = d(4) = gcd(1, 2, 3, 4) = 1$

And this could be generalized to N states not just N = 4

d) We suppose $p, q > 0, r = 0$. If N odd, show chain aperiodic

Let $N = 3$

Then chain is



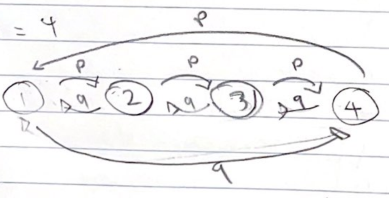
→ Again it is of same communication class
⇒ so they have same period

$d(1) = d(2) = d(3)$
$= gcd\{2, 3, 4, \ldots\}$
$= 1$

So if state has $d(i) = 1$ ⇒ state aperiodic
⇒ all states in same class ⇒ all state same period
⇒ all state are aperiodic
⇒ MC is aperiodic

e) We supose $p, q > 0, r = 0$. N even show chain has period 2

$N = 4$



Again they are all accessible to each other, so in one class ⇒ should have same period

$d(1) = d(2) = d(3) = d(4)$
$= gcd(2, 4 6, \ldots\}$
$= 2$

So whole chain has period 2

f) suppose $N = 4$, $p = 0.3$, $q = 0.2$. N is absorbing state. Get mean time spent in 3 starting from 2. And starting from 2, what is probability that chain never returns to 2?

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PT = np.matrix( [ [0.5, 0.3, 0],
                  [0.2, 0.5, 0.3],
                  [0, 0.2, 0.5]])
S = np.linalg.inv(np.identity(PT.shape[0]) -PT)

# The output as normal matrix, not np.matrix
```

$$S = \begin{array}{cc} \begin{bmatrix} 2.923 & 2.308 & 1.385 \\ 1.538 & 3.846 & 2.308 \\ 0.615 & 1.538 & 2.923 \end{bmatrix} & \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} \end{array}$$

Mean time spent in 3 from 2 $= S_{2,3} = 2.308$

$P(\text{never return to } 2 \mid X_0 = 2) = 1 - f_2$

And $S_{2,2} = \frac{1}{1-f_2} \Rightarrow P(\text{never return to } 2 \mid X_0 = 2) = \frac{1}{S_{2,2}}$

$$= \frac{1}{3.846}$$

$P(\text{never return to } 2 \mid X_0 = 2) = 0.26$