1. $\boxed{\text{5 marks}}$ Consider the LP:

$$\begin{array}{lccccl} \text{Maximize} & 12x_1 & +20x_2 & +21x_3 & +18x_4 & \\ & 24x_1 & +40x_2 & +46x_3 & +44x_4 & \leq 1200 \\ & x_1 & +x_2 & +x_3 & +x_4 & \leq 30 \\ & 3x_1 & +6x_2 & +6x_3 & +6x_4 & \leq 150 \end{array} \qquad x_1, x_2, x_3, x_4 \geq 0$$

Someone claims a final dictionary has

$$z = 540 - x_2 - 3x_4 - 4x_6 - 3x_7$$

Explain what does this imply to the dual optimal solution and why there must have been an error in the final dictionary. **You must not solve this by doing the simplex method iterations yourself; you will get zero mark if you do so. Instead, use relevant theorems and deduce your conclusion from them.**

Solution:

- From the Strong Duality Theorem, an optimal basic solution to the dual has

$$y_i^* = -\text{coefficient of the } i\text{th slack of the primal}$$

and so $y_1^* = 0, y_2^* = 4, y_3^* = 3$.

- But our supposedly optimal solution to the primal has $z = 540$ and yet $1200y_1^* + 30y_2^* + 150y_3^* = 570$ which violates Strong Duality.

- So some error must have been made.

2. $\boxed{\text{5 marks}}$ Find an example where both primal and dual LP's are infeasible. The better the simpler example. You have to justify your answer, especially why they are infeasible.

Solution: Consider the LP

$$\begin{aligned} &\text{maximize } z = 2x_1 - x_2 \\ &\text{subject to } \begin{array}{rrcr} x_1 & -x_2 & \leq & 1 \\ -x_1 & +x_2 & \leq & -2 \end{array} \\ &x_1, x_2 \geq 0 \end{aligned}$$

and its dual LP

$$\begin{aligned} &\text{minimize } w = y_1 - 2y_2 \\ &\text{subject to } \begin{array}{rrcr} y_1 & -y_2 & \geq & 2 \\ -y_1 & +y_2 & \geq & -1 \end{array} \\ &y_1, y_2 \geq 0 \end{aligned}$$

The primal problem is not feasible because when $x_1 - x_2 \leq 1$, the value of $-x_1 + x_2 = -(x_1 - x_2)$ should be ≥ -1, so it cannot satisfy ≤ -2.

Similarly, the dual problem is not feasible because when $y_1 - y_2 \geq 2$, the value of $-y_1 + y_2 = -(y_1 - y_2)$ should be ≤ -2, so it cannot satisfy ≥ -1.

3. $\boxed{\text{5 marks}}$ Solve the linear programming problem

$$\begin{array}{rrrrrrcr} \text{minimize } w = 40y_1 + 30y_2 + 20y_3 + 10y_4 + 10y_5 \\ & y_1 & & +2y_3 & +y_4 & & \geq & 300 \\ \text{subject to} & y_1 & +y_2 & & & +2y_5 & \geq & 400 \\ & & 2y_2 & & +2y_4 & +y_5 & \geq & 1000 \\ & y_1, \dots, y_5 \geq 0 \end{array}$$

by solving its dual problem (by the simplex method, following Anstee's rule) and then using Complementary Slackness; **you have to follow this to earn credits.**

Solution: The dual problem is

$$\text{maximise } z = 300x_1 + 400x_2 + 1000x_3$$

$$\begin{array}{rrrrcr} & x_1 & +x_2 & & \leq & 40 \\ & & +x_2 & +2x_3 & \leq & 30 \\ \text{subject to non-negativity and} & 2x_1 & & & \leq & 20 \\ & x_1 & & +2x_3 & \leq & 10 \\ & & 2x_2 & +x_3 & \leq & 10 \end{array}$$

Write in dictionary form:

$$\begin{array}{rcrrrr} z & = & 0 & +300x_1 & +400x_2 & +1000x_3 \\ \hline x_4 & = & 40 & -x_1 & -x_2 & \\ x_5 & = & 30 & & -x_2 & -2x_3 \\ x_6 & = & 20 & -2x_1 & & \\ x_7 & = & 10 & -x_1 & & -2x_3 \\ x_8 & = & 10 & & -2x_2 & -x_3 \end{array}$$

So x_3 enters and x_7 leaves

$$\begin{array}{rcrrrr} z & = & 5000 & -200x_1 & +400x_2 & -500x_7 \\ \hline x_3 & = & 5 & -(1/2)x_1 & & -(1/2)x_7 \\ x_4 & = & 40 & -x_1 & -x_2 & \\ x_5 & = & 20 & +x_1 & -x_2 & +x_7 \\ x_6 & = & 20 & -2x_1 & & \\ x_8 & = & 5 & +(1/2)x_1 & -2x_2 & +(1/2)x_7 \end{array}$$

So x_2 enters and x_8 leaves

$$\begin{array}{rcrrrr} z & = & 6000 & -100x_1 & -400x_7 & -200x_8 \\ \hline x_2 & = & (5/2) & +(1/4)x_1 & +(1/4)x_7 & -(1/2)x_8 \\ x_3 & = & 5 & -(1/2)x_1 & -(1/2)x_7 & \\ x_4 & = & (75/2) & -(5/4)x_1 & -(1/4)x_7 & +(1/2)x_8 \\ x_5 & = & (35/2) & +(3/4)x_1 & +(3/4)x_7 & +(1/2)x_8 \\ x_6 & = & 20 & -2x_1 & & \end{array}$$

Hence the solution is $z = 6000$ and

$$(x_1, x_2, x_3) = (0, 5/2, 5)$$
$$(x_4, \ldots, x_8) = (75/2, 35/2, 20, 0, 0)$$

We need to map this back to the original problem. We use complementary slackness to do so. It tells us that

$$x_j^* > 0 \implies \sum a_{ij} y_i^* = c_j$$
$$\sum a_{ij} x_j^* < b_i \implies y_i^* = 0$$

So since $x_2, x_3 > 0$ we have

$$y_1^* + y_2^* + 2y_5^* = 400$$
$$2y_2^* + 2y_4^* + y_5^* = 1000$$

Subsituting the x_j^* into the inequalities we see (in order)

$$\begin{aligned} 2.5 &< 40 \\ 12.5 &< 30 \\ 0 &< 20 \\ 10 &= 10 \\ 10 &= 10 \end{aligned}$$

Hence $y_1^* = y_2^* = y_3^* = 0$. The two equations above then imply $y_5^* = 200$ and $y_4^* = 400$. We can check that this does also give $w = 6000$.

4. $\boxed{\text{5 marks}}$ The optimal solution to the linear program:

$$\begin{array}{lrrrl}
\text{Maximize} & 10x_1 & +14x_2 & +20x_3 & \\
 & 2x_1 & +3x_2 & +4x_3 & \leq 220 \\
 & 4x_1 & +2x_2 & -x_3 & \leq 385 \\
 & x_1 & & +4x_3 & \leq 160
\end{array} \qquad x_1, x_2, x_3 \geq 0$$

is $x_1 = 60, x_2 = 0, x_3 = 25$. Write down the dual problem. Use this information above and relevant theorems to find an optimal solution to the dual problem; you have to explain your work (including name the precise theorems you used). **Do not use the simplex method iterations as you will get zero mark if you do so.** Explain also how this confirms that the primal optimal solution I claimed is in fact an optimal solution.

Solution:

- $$\text{Dual:} \quad \begin{array}{lrrrl}
\text{Minimize} & 220y_1 & +385y_2 & +160y_3 & \\
 & 2y_1 & +4y_2 & +y_3 & \geq 10 \\
 & 3y_1 & +2y_2 & & \geq 14 \\
 & 4y_1 & -y_2 & +4y_3 & \geq 20
\end{array} \qquad y_1, y_2, y_3 \geq 0$$
- Now $x_1 = 60 > 0$ implies $2y_1^* + 4y_2^* + y_3^* = 10$ by Complementary Slackness.
- Also $x_3 = 25 > 0$ implies $4y_1^* - y_2^* + 4y_3^* = 20$ by Complementary Slackness.
- Also $4x_1 + 2x_2 - x_3 = 215 < 385$ implies $y_2^* = 0$ by Complementary Slackness.
- The optimal solution to the dual can be determined by solving three equations in 3 unknowns to obtain
$$y_1^* = 5, y_2^* = 0, y_3^* = 0$$
- We check feasibilty of our primal and dual solutions and then, since Complementary Slackness is satisfied, the Theorem of Complementary Slackness shows that the primal (and dual) solution is optimal.
- An alternate way is to note you have a feasible solution to the primal $(60, 0, 25)$ with objective function value $10 \times 60 + 14 \times 0 + 20 \times 25 = 1100$ and a feasible solution to the dual $(5, 0, 0)$ with objective function value $220 \times 5 + 385 \times 0 + 160 \times 0 = 1100$ and so by Weak Duality, both must be optimal.
- Comment: It is somewhat lucky that the three equations determine an optimal dual solution but that is how the question was chosen. The value of the question is in testing your hands on understanding of Complementary Slackness.

5. $\boxed{\text{5 marks}}$ Prove the following:

Theorem Let $\mathbf{A}$ and $\vec{b}$ be given. Then either

- there exists an $\vec{x}$ such that $\vec{x} \geq 0$ and $\mathbf{A}\vec{x} \leq \vec{b}$, or
- there exists a $\vec{y}$ such that $\mathbf{A}^T y \geq 0, \vec{y} \geq 0$ and $\vec{b}^T \vec{y} < 0$,

but not both. Note the strict inequality in the second.

Hint: Use both weak and strong duality theorems. You will also need the fundamental theorem of linear programming.

Solution:

Let us write this as an LP problem and use duality. Now we have constraints, but no objective function, so let us just use $z = \vec{0}^T \vec{x}$.

$$\begin{aligned} \max \vec{0}^T \vec{x} \\ \mathbf{A}\vec{x} \leq \vec{b} \\ \vec{x} \geq 0 \end{aligned} \qquad\qquad \begin{aligned} \min \vec{b}^T \vec{y} \\ \mathbf{A}^T \vec{y} \geq 0 \\ \vec{y} \geq 0 \end{aligned}$$

Since the primal problem is bounded (the objective function is zero), the fundamental theorem of linear programming tells us that the primal problem either has an optimal solution or is infeasible.

- If the primal has an optimal solution (option 1), $\vec{x}^*$ then the objective function is 0. Weak duality tells us that any feasible solution of the dual problem must give $0 \leq \vec{b}^T \vec{y}$. Hence option 2 cannot happen if option 1 does happen.
- Assume the primal is infeasible. We see that the dual is always feasible since $\vec{y} = 0$ is always a solution. Again, the fundamental theorem of LP tells that the dual is either has an optimal solution or unbounded.
  - If the dual has an optimal solution, $\vec{y}^*$ then strong duality implies that there is an optimal and so feasible solution to the primal — which we have assumed does not exist.
  - Hence the dual must be unbounded. Since it is unbounded, then there exists a feasible solution of the dual, $\vec{y}'$ with objective function $\vec{b}^T \vec{y}' = -1$. Hence there exists a $\vec{y}$ such that $\vec{b}^T \vec{y} < 0$.
- The fundamental theorem of linear programming tells us that either the primal has an optimal solution or is infeasible. If optimal, then option 1 happens but not option 2. If the primal is infeasible, option 1 cannot happen, but option 2 must.

The following two problems are OPTIONAL for your practice with complementary slackness. Do not hand in.

6. This question is Question 5.3(a) from the book of Chvatal.

$$\begin{array}{lrrrrrcl}
\text{Maximise} & 7x_1 & +6x_2 & +5x_3 & -2x_4 & +3x_5 & & \\
\text{subject to} & x_1 & +3x_2 & +5x_3 & -2x_4 & +2x_5 & \leq & 4 \\
 & 4x_1 & +2x_2 & -2x_3 & +x_4 & +x_5 & \leq & 3 \\
 & 2x_1 & +4x_2 & +4x_3 & -2x_4 & +5x_5 & \leq & 5 \\
 & 3x_1 & +x_2 & +2x_3 & -x_4 & -2x_5 & \leq & 1 \\
 & & & & & & \multicolumn{2}{r}{x_1, x_2, x_3, x_4, x_5 \geq 0}
\end{array}$$

Is $(0, 4/3, 2/3, 5/3, 0)$ optimal?

Solution:

- The dual problem is

$$\begin{array}{lrrrrl}
\text{minimise} & 4y_1 & +3y_2 & +5y_3 & +y_4 & \\
\text{subject to} & y_1 & +4y_2 & +2y_3 & +3y_4 & \geq 7 \\
 & 3y_1 & +2y_2 & +4y_3 & +y_4 & \geq 6 \\
 & 5y_1 & -2y_2 & +4y_3 & +2y_4 & \geq 5 \\
 & -2y_1 & +y_2 & -2y_3 & -y_4 & \geq -2 \\
 & 2y_1 & +y_2 & +5y_3 & -2y_4 & \geq 3 \\
 & & & & \multicolumn{2}{r}{y_1, y_2, y_3, y_4 \geq 0}
\end{array}$$

- Since $x_2^*, x_3^*, x_4^* > 0$ it follows that the corresponding dual constraints are equalities:

$$\begin{aligned}
3y_1 + 2y_2 + 4y_3 + y_4 &= 6 \\
5y_1 - 2y_2 + 4y_3 + 2y_4 &= 5 \\
-2y_1 + y_2 - 2y_3 - y_4 &= -2
\end{aligned}$$

- Substitute x^* into the primal constraints gives

$$\begin{aligned}
4 + 10/3 - 10/3 &= 4 \\
8/3 - 4/3 + 5/3 &= 3 \\
16/3 + 8/3 - 10/3 &= 14/3 < 5 \\
4/3 + 4/3 - 5/3 &= 1
\end{aligned}$$

  Since the third constraint is a strict inequality, $y_3^* = 0$.

- So we have to solve the following 3 equations

$$\begin{aligned} 3y_1 + 2y_2 + 4y_3 + y_4 &= 6 \\ 5y_1 - 2y_2 + 4y_3 + 2y_4 &= 5 \\ -2y_1 + y_2 - 2y_3 - y_4 &= -2 \end{aligned}$$

These have solution $y_1^* = y_2^* = y_3^* = 1$.

- So complementary slackness implies the dual optimal solution is $(1, 1, 0, 1)$.
- This gives optimal value $4 + 3 + 1 = 8$. The optimal value of the primal was $24/3 + 10/3 - 10/3 = 8$.
- All the $y_i^* > 0$, sub into the constraints gives

$$\begin{aligned} &1 + 4 + 3 = 8 > 7 \\ &3 + 2 + 1 = 6 \\ &-2 + 4 + 2 = 4 < 5 \qquad \text{problem} 1 + 5 - 2 = 4 > 3 \end{aligned}$$

- The dual solution implied by complementary slackness is not feasible. Hence the orginal values of x^* is not optimal.

7. This question is Question 5.3(b) from the book of Chvatal.

maximise	$4x_1$	$+5x_2$	$+x_3$	$+3x_4$	$-5x_5$	$+8x_6$		
st non-neg and	1	0	-4	3	1	1	$\leq$	1
	5	3	1	0	-5	3	$\leq$	4
	4	5	-3	3	-4	1	$\leq$	4
	0	-1	0	2	1	-5	$\leq$	5
	-2	1	1	1	2	2	$\leq$	7
	2	-3	2	-1	4	5	$\leq$	5

Is $(0, 0, 5/2, 7/2, 0, 1/2)$ optimal?

Solution:

- Since $x_3, x_4, x_6 > 0$ the corresponding dual constraints are equalities:

$$\begin{aligned} -4y_1 + y_2 - 3y_3 + y_5 + 2y_6 &= 1 \\ 3y_1 + 3y_3 + 2y_4 + y_5 - y_6 &= 3 \\ y_1 + 3y_2 + y_3 - 5y_4 + 2y_5 + 5y_6 &= 8 \end{aligned}$$

- Sub the x^* into the constraints to get

$$\begin{aligned} -20/2 + 21/2 + 1/2 &= 1 \\ 5/2 + 3/2 &= 4 \\ -15/2 + 21/2 + 1/2 &= 7/2 < 4 \\ 14/2 - 5/2 &= 9/2 < 5 \\ 5/2 + 7/2 + 2/2 &= 7 \\ 10/2 - 7/2 + 5/2 &= 4 < 5 \end{aligned}$$

  Since the third, fourth and sixth constraints are strict inequalities, it follows that $y_3^*, y_4^*, y_6^* = 0$.

- Hence we need to solve

$$\begin{aligned} -4y_1 + y_2 + y_5 &= 1 \\ 3y_1 + y_5 &= 3 \\ y_1 + 3y_2 + 2y_5 &= 8 \end{aligned}$$

  This solve to give $y_1^* = 1/2, y_2^* = 3/2, y_5^* = 3/2$.

- The dual objective function is $z = 1/2 + 12/2 + 21/2 = 17$. The primal is $5/2 + 21/2 + 8/2 = 17$.

- The dual variables are all non-negative. Need to check dual constraints.

$$\begin{aligned} [1, 5, 4, 0, -2, 2][1/2, 3/2, 0, 0, 3/2, 0]^T &= 1/2 + 15/2 - 6/2 = 5 > 4\checkmark \\ [0, 3, 5, -1, 1, -3][1/2, 3/2, 0, 0, 3/2, 0]^T &= 9/2 + 3/2 = 12/2 > 1\checkmark \\ [-4, 1, -3, 0, 1, 2][1/2, 3/2, 0, 0, 3/2, 0]^T &= -4/2 + 3/2 + 3/2 = 1\checkmark \\ [3, 0, 3, 2, 1, -1][1/2, 3/2, 0, 0, 3/2, 0]^T &= 3/2 + 3/2 = 3\checkmark \\ [1, -5, -4, 1, 2, 4][1/2, 3/2, 0, 0, 3/2, 0]^T &= 1/2 - 15/2 + 6/2 = -8/2 > -5\checkmark \\ [1, 3, 1, -5, 2, 5][1/2, 3/2, 0, 0, 3/2, 0]^T &= 1/2 + 9/2 + 6/2 = 8\checkmark \end{aligned}$$

- So the dual solution is feasible and gives same objective function.

- Hence the original solution is optimal.