Lecture 14.

- Economic meaning of dual problem.
    - Dual optimal solution as shadow price (marginal value)
- Geometric discussion about the small change of \vec b.
- Non-degeneracy at primal optimal solution and uniqueness of dual optimal solution.

Example

A company produces trucks and vans.

- A truck requires one engine, 1 hour to assemble, and two heaters.
- A van requires one engine, 2 hours to assemble, and one heater.
- The company can get only 5 engines and 8 heaters per day from the supplier.
- They work only up to 8 hours per day.

They make \$4 profit on each truck and \$5 profit on each van.

Question: $\underbrace{\text{How many trucks}}_{x_1}$ and $\underbrace{\text{vans}}_{x_2}$ should they make per day, to maximize the profit?

The profit they make /day

objective function

decision variables

x_1 = # of trucks

x_2 = # of vans

they manufactor per day.

max. $4x_1 + 5x_2$

subj to.

$x_1 + x_2 \le 5$ # of engine/day

$x_1 + 2x_2 \le 8$ # of labor /day

$2x_1 + x_2 \le 8$ # of heater hours /day

Constraints

$x_1, x_2 \geq 0$

dual: min $5y_1 + 8y_2 + 8y_3$

Subj. to

$y_1 + y_2 + 2y_3 \geq 4$

$y_1 + 2y_2 + y_3 \geq 5$

$y_1, y_2, y_3 \geq 0$

primal

max $\vec{c} \cdot \vec{x}$

subj. to $A\vec{x} \le \vec{b}$

$\vec{x} \geq \vec{0}$.

dual

min $\vec{b} \cdot \vec{y}$

subj. to $A^T \vec{y} \geq \vec{c}$

$\vec{y} \geq \vec{0}$

• Meaning of the dual problem.

"resources"

$i = 1, 2, 3$

$i=1$: engine

$i=2$: time

$i=3$: heater.

the net value of the total resources to the company

dual: $\min \quad 5y_1 + 8y_2 + 8y_3$

Subj. to $\quad y_1 + y_2 + 2y_3 \geq 4$

$\quad y_1 + 2y_2 + y_3 \geq 5$

$\quad y_1, y_2, y_3 \geq 0$

profit of making a unit of product j. (j = truck)

the net value of resources the company spend for manufacturing a unit of truck.

The net value of the resources the company spend for manufacturing a van.

y_i = the $\boxed{\text{possible}}$ net value of resource i to the company (when it does certain economic activities with it).

= [a $\boxed{\text{possible}}$ value of the resource i to the company] − [the price the company pays to buy the resource i].

= How much the resource i would be valuable to the company to buy and use it − (the price they pay).

• Can assume $y_i \geq 0$ as the company (if reasonable) would not buy the resource i if the possible net value is < 0. In that case, zero value and zero price.

$$
\begin{aligned}
y_1 + y_2 + 2y_3 &\geq 4 \\
y_1 + 2y_2 + y_3 &\geq 5
\end{aligned}
$$

- **Explanation on $\geq$ in the dual constraint.**

* Note that the company may use the resources to do other activities and there might be more profitable activities than making the products. So, the value of the resources that are used to make item j should be $\geq$ the profit to manufacture item j.

(because there might be other more profitable activities to use these resources)

- The dual problem determines for the company how much they are willing to add to the payment for the resources when the supplier would like to increase the price.

the dual optimal solution $\vec{y}^* = (y_1^*, \cdots, y_m^*)$

- $\vec{y}_*$ are the net value of the resources associated to the given manufacturing activity.

(In our case, making trucks and vans)

y_i^* = the net value of the resource i, for the given activity.

* From Strong duality, we have

$$\boxed{\text{the minimum total net resource value}} = \boxed{\text{the maximum total profit}}$$

$$\vec{b} \cdot \vec{y}^* = \vec{c} \cdot \vec{x}^*$$

for optimal solution $\vec{x}^*$, dual optimal solution $\vec{y}^*$.

- The y_i^*'s are those net values of resources that give the same amount as the possible profit for the given particular (making trucks, vans) economic activity,

Below is another way to see the economic meaning of the dual optimal solution $\vec{y}^*$.

Behaviour under small change of $\vec{b}$

Theorem

Consider $\max\limits_{A\vec{x}\leq\vec{b}\,\&\,\vec{x}\geq\vec{0}} \vec{c}\cdot\vec{x}$. *Let*

- $\vec{x}^*$ *be a basic optimal solution, and*
- z^* *the optimal objective value.*
- $\vec{y}^*$ *an optimal dual solution.*

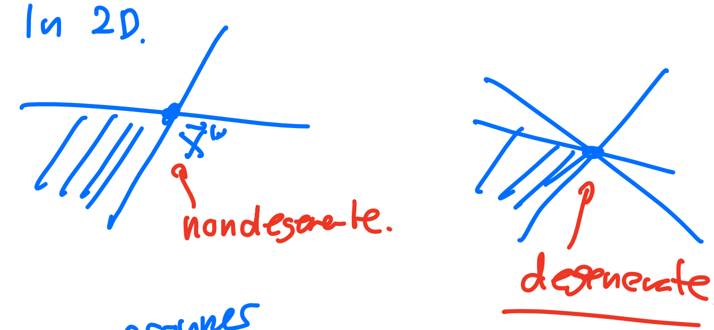
Assume that $\vec{x}^*$ *is* **non-degenerate.**
Then the following holds:
There is $\epsilon > 0$ *such that*
if $|t_i| \leq \epsilon$ *for* $\vec{t} = (t_1, \ldots t_m)$, *then the LP problem*

e.g. $z = 5 - 3x_2 - x_1$

$x_3 = 3 - \ldots$

$x_4 = 4 + \ldots$

e.g. In 2D.



$$\max_{A\vec{x}\leq(\vec{b}+\vec{t})\,\&\,\vec{x}\geq\vec{0}} \vec{c}\cdot\vec{x}$$

The supply of resources is changed by the amount of $(t_1, \cdots, t_m)$.

has its optimal value:

the optimal value of

$$z^{**} = z^* + \vec{y}^*\cdot\vec{t}.$$

NOTE: $\vec{b}+\vec{t} = (b_1+t_1, \cdots, b_m+t_m)$ and $\vec{y}^*\cdot\vec{t} = \sum_{i=1}^m y_i^* t_i$.

• Economic meanings of the dual optimal solution $\vec{y}^*$.

• Small change t_i to the amount of resource i

$\Rightarrow$ the profit changes by $y_i^* t_i$

• y_i^* = the rate of increase/decrease of profit per increase/decrease of the amount of the resource i = marginal value of resource i (shadow price)

The net value of the resource i can be measured by how much profit it may

* In particular,

if the supply of resource i is changed by t_i

assuming other conditions (e.g. price of resources) are the same.,

then the total profit

is changed by $y_i^* t_i$,

So, the manufacturer is additionally willing to pay at most y_i^* (not more than this) for those additional amount t_i

. The additional profit to the company with t_i more resource i at the increased price for that t_i amount.

$$= t_i\, y_i^* - t_i\,(\text{the additional price}) = t_i\left(y_i^* - \text{the additional price}\right)$$

Explanation of the above theorem:

Rough idea (but not completely correct.)

$z^* = \vec{c} \cdot \vec{x}^* = \vec{b} \cdot \vec{y}^*$ by strong duality.

change $\vec{b} \longrightarrow \vec{b} + \vec{t}$

$\rightarrow z^{**} \text{ "}=\text{" } (\vec{b} + \vec{t}) \cdot \vec{y}^*$

the new optimal value

if $\vec{y}^*$ is still an optimal dual solution to this modified problem

$= \underbrace{\vec{b} \cdot \vec{y}^*} + \vec{t} \cdot \vec{y}^*$

$= z^* + \vec{t} \cdot \vec{y}^*$

So we get $z^{**} = z^* + \vec{t} \cdot \vec{y}^*$.

. This is <u>not</u> rigorous, as $\vec{y}^*$ may not in general be an optimal dual solution to the modified problem.

We will see that the conditions $\begin{cases} \cdot\ \vec{t} \text{ being small} \\ \cdot\ \text{non-degeneracy of } \vec{x}^* \end{cases}$, will ensure $\vec{y}^*$ be still an optimal dual solution to the modified problem.

Lec 14 (TuTh)

Your exercise

Question:

- Consider $\max_{A\vec{x}\leq\vec{b},\, \vec{x}\geq\vec{0}} \vec{c}\cdot\vec{x}$.

Choose a correct statement:

A) If $\vec{c}$ changes then the optimal primal solution $\vec{x}^*$ must change.

B) If $\vec{b}$ changes then the optimal dual solution $\vec{y}^*$ must change.

C) There are examples where changing $\vec{b}$ changes the feasible region of **the dual problem**.

D) A, B, C are all wrong.