The University of British Columbia

Math 303: Section 201

2018, February 14

Name: ______________________________ Student ID: ________________

Instructions

- This exam consists of **4 questions** worth a total of 40 points.
- Make sure this exam has **6 pages** excluding this cover page.
- Explain your reasoning thoroughly, and justify your answers unless the question indicates otherwise.
- No notes, calculators, or other electronic devices are allowed.
- If you need more space, use the back of the pages.
- Duration: **50** minutes.

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
Total:	40	

10 marks 1. Consider the Markov chain with state space $\{0, 1, 2, 3, 4, 5, 6\}$ and transition matrix

$$\mathbf{P} = \begin{array}{c|ccccccc|} & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 3 & 0 & 0 & 0 & 0 & \frac{4}{5} & \frac{1}{5} & 0 \\ 4 & 0 & \frac{9}{10} & 0 & 0 & 0 & 0 & \frac{1}{10} \\ 5 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 6 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}$$

(a) (2 marks, **no justification needed**) Draw the transition diagram showing the seven states with arrows indicating possible transitions and their probabilities.

(b) (2 marks, **no justification needed**) Determine all the communicating classes of this Markov chain.

(c) (2 marks, **no justification needed**) Determine which states are recurrent and which are transient.

(d) (2 marks, **no justification needed**) Determine the period of each state.

(e) (2 marks) Suppose the Markov chain started in State 0. What is the probability that it will be in State 0 after 4 steps?

10 marks

2. **Challenge** A knight starts at the bottom left of an 8×8 chess board and performs random moves (she can move to a square that is two squares away horizontally and one square vertically, or two squares vertically and one square horizontally). At each state, she picks one of the available legal moves with equal probability, independently of the earlier moves. What is the mean number of moves before she returns to her starting square?

10 marks

3. Consider a branching process Z_n, $n = 0, 1, \ldots$ with offspring distribution ξ with probability mass function

$$P(\xi = k) = (1-p)^k p, \qquad k = 0, 1, \ldots$$

where $p \in (0, 1)$ is a parameter. Assume $Z_0 = 1$.

(a) (2 marks) What is $\mathbb{E}(Z_n)$?

(b) (4 marks) What is the probability that the process is extinct by step 3, i.e., what is $\mathbb{P}(Z_3 = 0)$?

(c) (4 marks) What is the probability of eventual extinction?

10 marks

4. Each of two switches is either on or off during a day. On day n, each switch will independently be on with probability

$$\frac{1 + \text{number of switches during day } (n-1)}{4}.$$

For instance, if both switches are on during day $n-1$, then each will be independently on during day n with probability 3/4. What fraction of days are both switches on? What fraction are both off?