3) For non-empty set $S \subset \mathbb{R}^n$ and positive real number $r \in R$, $r > 0$ define set rS as follow:

$$rS := \{ z \in \mathbb{R}^n \mid z = rx,\ x \in S \}$$

rx is multiplication of vector $x \in \mathbb{R}^n$ by scalar $r \in \mathbb{R}$.
rS is set of all points by multiplying r with vectors $x \in S$

prove if S is convex, then rS is convex set as well

A set $S \subset \mathbb{R}^n$ is convex if $\forall \vec{x}, \vec{y} \in S$, $\forall t \in [0,1]$
s.t. $(1-\lambda)\vec{x} + \lambda\vec{y} = \vec{z} \in S$

Assume S is convex, then

$rS = z \in \mathbb{R}^n \mid z = rx,\ x \in S$

$\Rightarrow (1-\lambda)\vec{x} + t\vec{y} = r\vec{x},\ \vec{x} \in S$
$= \vec{z} \Rightarrow \vec{z} \in S$

so if a non-empty set $S \subset \mathbb{R}^n$ is convex, then rS is also convex set

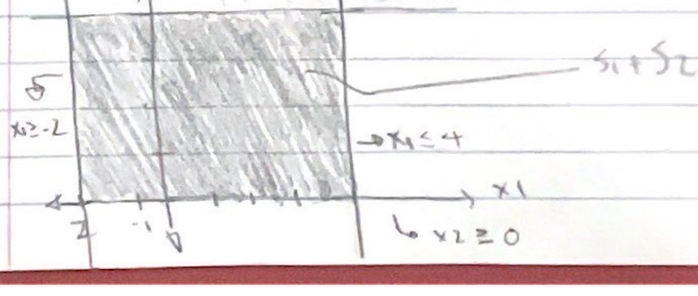
4) For given two non empty sets $S_1, S_2 \subset \mathbb{R}^n$, define $S_1 + S_2$ as follows: $S_1+S_2 = \{ z \in \mathbb{R}^n \mid \exists x \in S_1, \exists y \in S_2 \text{ s.t. } z = x+y \}$

a) $S_1 = \{(x_1, x_2) \in \mathbb{R}^2 \mid |x_1 - 1| \le 1 \ \& \ |x_2 - 2| \le 1\}$ sketch $S_1 + S_2$
$S_2 = \{(x_1, x_2) \in \mathbb{R}^2 \mid |x_1| \le 2 \ \& \ |x_2| \le 1$

for S_1
$\Rightarrow |x_1 - 1| \le 1 \Rightarrow -1 \le x_1 - 1 \le 1 \Rightarrow 0 \le x_1 \le 2$
$\Rightarrow |x_2 - 2| \le 1 \Rightarrow -1 \le x_2 - 2 \le 1 \Rightarrow 1 \le x_2 \le 3$

For S_2
$|x_1| \le 2 \Rightarrow -2 \le x_1 \le 2$
$|x_2| \le 1 \Rightarrow -1 \le x_2 \le 1$

And $S_1 + S_2 \Rightarrow -2 \le x_1 \le 4$ for x_1 || $0 \le x_2 \le 4$ for x_2



b) Is it true $S_1 + S_2$ must be convex for any non empty convex sets S_1, S_2 in $\mathbb{R}^n$? [This problem is independent from a), or S_1 and S_2 are arbitrary convex sets]

Let S_1 be arbitrary convex set
Let S_2 be arbitrary convex set

A set S is convex if:
$\forall \vec{x}, \vec{y} \in S$, $\forall \lambda \in [0, 1]$
such that $(1-\lambda)\vec{x} + \lambda\vec{y} = \vec{z} \in S$

So S_1 convex $\Rightarrow (1-\lambda)\vec{x} + \lambda\vec{y} = \vec{z} \in S_1$
S_2 convex $\Rightarrow (1-\lambda)\vec{u} + \lambda\vec{v} = \vec{w} \in S_2$

$S_1 + S_2 \Rightarrow (1-\lambda)\vec{x} + \lambda\vec{y} + [(1-\lambda)\vec{u} + \lambda\vec{v}] = \vec{c} \in S_1+S_2$

$\vec{x} - \lambda\vec{x} + \lambda\vec{y} + \vec{u} - \lambda\vec{u} + \lambda\vec{v} = \vec{c} \in S_1+S_2$

$\Rightarrow (1-\lambda)(\vec{x}+\vec{u}) + \lambda(\vec{y}+\vec{v}) = \vec{c} \in S_1+S_2$

let $\vec{a} = (\vec{x}+\vec{u})$
$\vec{b} = (y+\vec{v})$

$\Rightarrow (1-\lambda)\vec{a} + \lambda\vec{b} = \vec{c} \in S_1 + S_2$

so S_1+S_2 must be convex for any nonempty convex sets S_1 and S_2.