Lecture 15

- Simplified Metropolis-Hastings algorithm
- Jupyter notebook

High-dimensional sampling

We have the following engineering challenge:

- There is a discrete random variable X and we wish to generate samples from it.
- It takes values $\{1,2,3, \ldots, t\}$. The number of possible values, t, is huge.
- We don't know the exact probability mass function, we only know that

$$\boxed{P(X=i)=C \cdot x_i,} \qquad i=1,2,\ldots,t$$

We can't solve $1 = C\sum_i x_i$ to find $C = \frac{1}{\sum x_i}$

How do we generate a sample of X?

Simplified Metropolis Hastings algorithm

Solution: Without knowing C, we will create an irreducible, ergodic MC whose stationary distribution is the distribution of X.

Then what? Run it for several steps, eventually it approaches stat dist, ie dist of X, then sample.

The MC:

- State space: $\{1, 2, \ldots, t\}$
- Stationary distribution: $\pi_i = C \cdot x_i$

What can we compute?

Ratios of π: $\frac{\pi_i}{\pi_j} = \frac{Cx_i}{Cx_j} = \frac{x_i}{x_j}$ ← known } computable

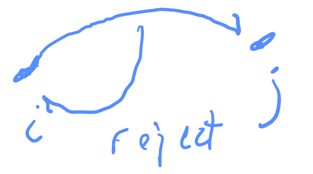
Accept/reject transitioning between states

From State i:

- Choose a potential next state uniformly at random (simplified version)
- Let $\tilde{x}$ be that state so $\tilde{x} \sim Unif\{1, 2, \ldots t\}$, $P(\tilde{x} = k) = \frac{1}{t}$ $k = 1, \ldots, t$
- Accept/reject:
  - Sample r.v. $u \sim Unif[0, 1]$
  - If $\mu \leq \min\left(1, \frac{\pi_{\tilde{x}}}{\pi_i}\right)$ then the MC moves to State $\tilde{x}$, otherwise it stays in State i

$$P(accept \mid \tilde{x} = k) = P\left(u \leq \min\left(1, \frac{\pi_{\tilde{x}}}{\pi_i}\right) \Big| \tilde{x} = k\right)$$

$$= P\left(u \leq \min\left(1, \frac{\pi_k}{\pi_i}\right)\right) = \min\left(1, \frac{\pi_k}{\pi_i}\right)$$



What are the transition probabilities?

$$P_{i,j} = P(X_1 = j \mid X_0 = i) = P(X_1 = j \mid X_0 = i, \tilde{X} = j) \cdot \underline{P(\tilde{X} = j \mid X_0 = i)}$$

$$= P(\text{accept} \mid X_0 = i, \tilde{X} = j) \cdot \frac{1}{t} \leftarrow \text{Uniformly sample } \tilde{X}$$

$$= \min\left(1, \frac{\pi_j}{\pi_i}\right) \cdot \frac{1}{t}$$

Is π stationary?

Check for reversibility:

$$\pi_i P_{ij} \stackrel{?}{=} \pi_j \cdot P_{ji}$$

LHS $\pi_i \cdot P_{ij} = \pi_i \cdot \min\left(1, \frac{\pi_j}{\pi_i}\right) \cdot \frac{1}{t} = \min(\pi_i, \pi_j) \cdot \frac{1}{t}$

RHS $\pi_j P_{ji} = \pi_j \min\left(1, \frac{\pi_i}{\pi_j}\right) \cdot \frac{1}{t} = \min(\pi_j, \pi_i) \cdot \frac{1}{t}$

Balance eqns hold!

π is stationary

A note on Markov Chain Monte Carlo

New goal: Given a function $h: \underbrace{\{1, 2, \dots, t\}} \to \mathbb{R}$, estimate $\underbrace{\sum_i h(i) \cdot \pi_i}_{\mathbb{E}[h(X)]}$.

Less clever soln: Generate N approx. samples of X, called $X_1, X_2, \dots X_N$

$$h(X) \approx \frac{1}{N} \sum_{k=1}^{N} h(X_k)$$

Markov chain Monte Carlo: Run the MC once for N steps $(X_n)_{n=1,\dots N}$

dependent states of MC

Examine: $\frac{1}{N}\sum_{k=1}^{N} h(X_k) = \frac{1}{N}\sum_{i\in\{1,\dots,t\}} \left(\begin{matrix}\text{\# of times} \\ \text{MC visits } i\end{matrix}\right)\cdot h(i)$

$$= \sum_i \left(\begin{matrix}\text{proportion of} \\ \text{time MC is} \\ \text{in state } i\end{matrix}\right)\cdot h(i) \xrightarrow{N\to\infty} \sum_i \pi_i\, h(i)$$