Lecture 9 (TuTh)

- The fundamental theorem of linear programming. Vanderbei section 3.5
- Uniqueness and non-uniqueness of optimal solutions.

Duality. Vanderbei sec 5.1 , 5.2, 5.3

- Next lecture: Where does duality come from? Related to Vanderbei sec 5.9 and 5.10

Choose a WRONG statement:

A) If an LP has no optimal solution, then it is either infeasible or unbounded.

B) If a standard form LP problem has a feasible solution, then it has a basic feasible solution.

C) If a standard form LP problem has an optimal solution, then it has a basic optimal solution.

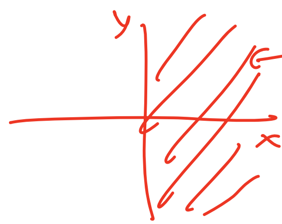
D) For a standard form LP problem, all its optimal solutions must to be basic optimal solutions.

E) There are LP problems that have infinitely many optimal solutions.

A) Correct In other words, if an LP is feasible and bounded then it must have an optimal (feasible) solution.

B) Correct Feasible regions are polyhedrons (n-dimensional polytopes), and have vertices (corner points), which are basic feasible solutions.

[EXCEPT]

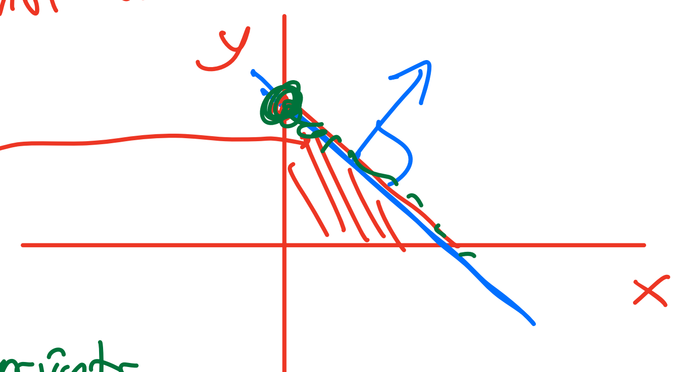


If this is your feasible region there is no vertex.

C) Correct The optimal solutions found from the simplex method are basic optimal solutions

D) Not correct. Some optimal solutions are not basic.

E) Yes : ⟶ all the points on this side are optimal points.

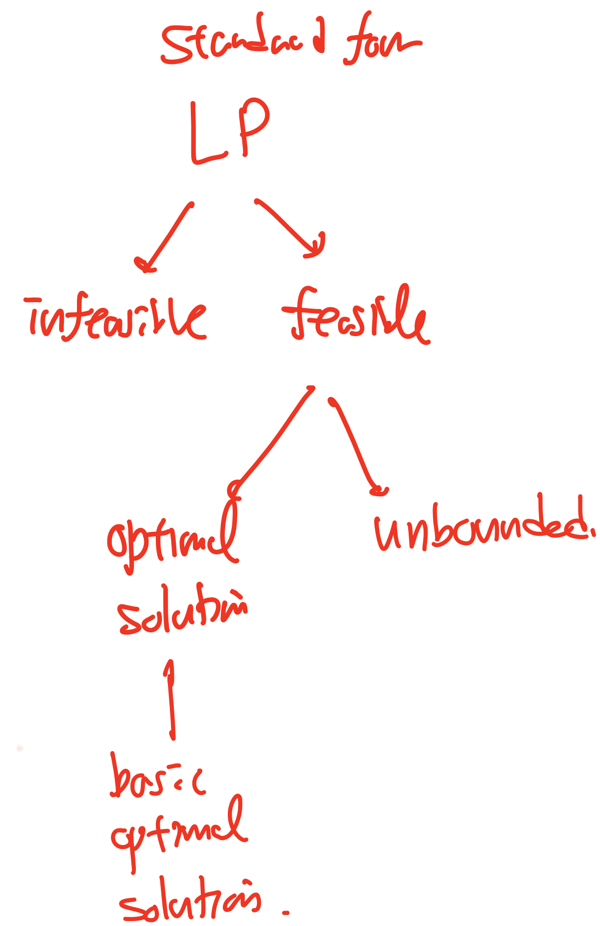


We discuss a fundamental result that combines the two phase simplex method and Bland's theorem.

The fundamental theorem of LP.

Every linear programming problem in standard form has the following properties:

- If it has no optimal solution, then it is either infeasible or unbounded.
- If it has a feasible solution, then it has a basic feasible solution.
- If it has an optimal solution then it has a basic optimal solution.



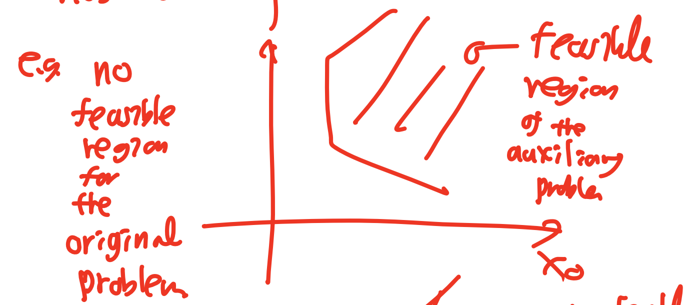
proof

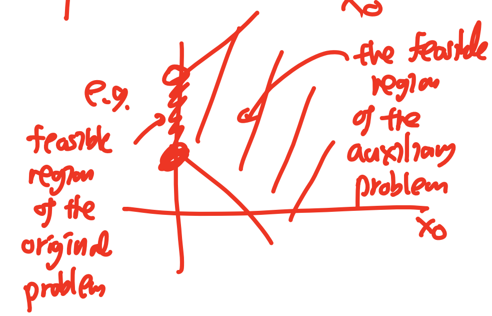
- Simplex algorithm will always terminate (by applying the smallest subscript rule [Bland's theorem])

- The 1^{st} phase of the two phase simplex method always terminate. (And the auxiliary LP always has an optimal solution.

  And will either give a basic feasible solution of the original LP.

  OR show the original problem is not feasible.





- The 2^{nd} phase of the two phase simplex method will terminate,

  and will either give a basic optimal solution,

  or it will show that the LP is unbounded.

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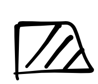
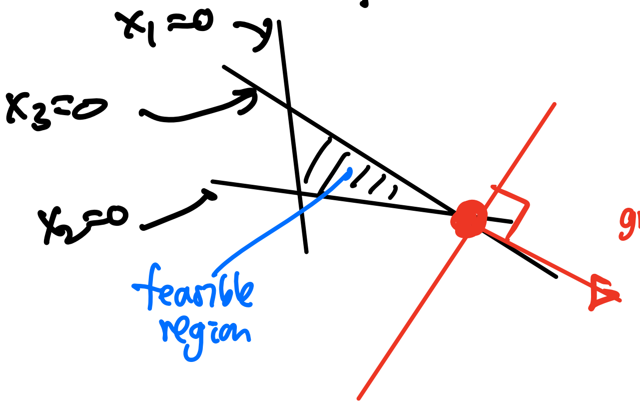


- Discussion on uniqueness of optimal solution.

e.g. unique optimal solution!



dictionary.

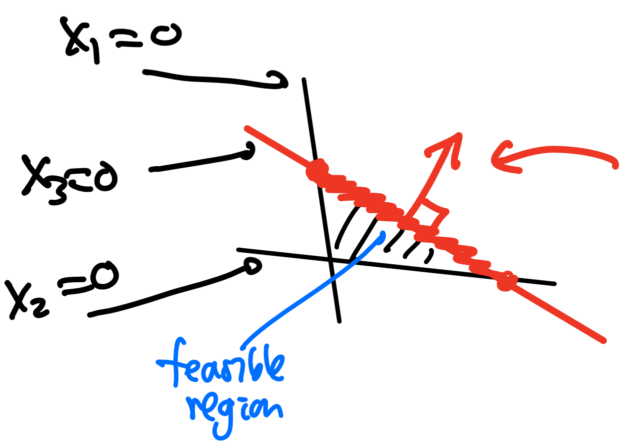
e.g.

$$\begin{array}{l} z = -5 - x_2 - x_3 \\ \hline x_1 = 3 - x_2 + x_3 \end{array}$$

has unique optimal solution

$\underline{x_1=3,\ x_2=0,\ x_3=0}$.

e.g. **nonunique** optimal solution! **(Infinitely many optimal solutions)**



dictionary.

e.g.

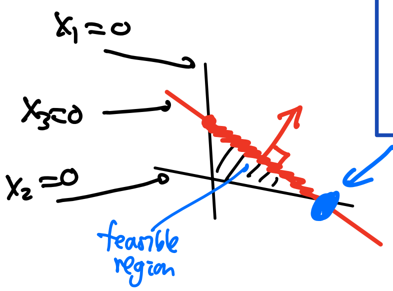
$$\boxed{\begin{array}{l} z = -5 + 0x_2 - x_3 \\ \hline x_1 = 3 - x_2 + x_3 \end{array}}$$

is an optimal dictionary, and has an optimal solution

$\underline{x_2=0,\ x_3=0,\ x_1=3}$

- If we pivot on an optimal dictionary with an objective function with a zero coefficient, with the entering variable of that zero coefficient,

then we do not increase the objective function.
but we may find another optimal dictionary (basic optimal solution).

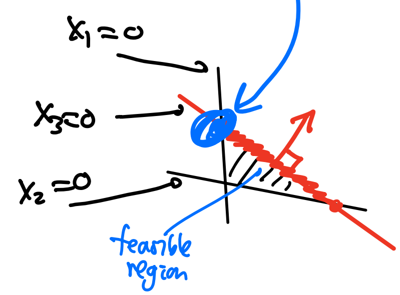


$$\boxed{\begin{aligned} z &= -5 + 0x_2 - x_3 \\ \hline x_1 &= 3 - x_2 + x_3 \end{aligned}}$$

$\xrightarrow[x_1 \text{ leave}]{x_2 \text{ enter}}$

$$\boxed{\begin{aligned} z &= -5 + 0 \cdot x_1 - x_3 \\ \hline x_2 &= 3 - x_1 + x_3 \end{aligned}}$$

an optimal dictionary.



Another example

$$\begin{aligned} z &= 10 - x_4 + 0\boxed{x_2} + 0x_3 \\ \hline \boxed{x_1} &= 10 - x_4 - x_2 - x_3 \end{aligned}$$

optimal dictionary

nonbasic : x_4, x_2, x_3

basic x_1

$\xrightarrow[x_1 \text{ leave}]{x_2 \text{ enter}}$

$$\begin{aligned} z &= 10 - x_4 + 0x_1 + 0\boxed{x_3} \\ \hline \boxed{x_2} &= 10 - x_4 - x_1 - x_3 \end{aligned}$$

optimal dictionary

nonbasic : x_4, x_1, x_3

basic x_2

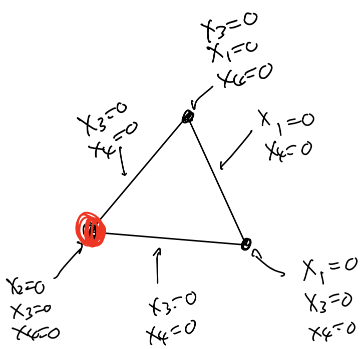
$\xrightarrow[x_2 \text{ leave}]{x_3 \text{ enter}}$

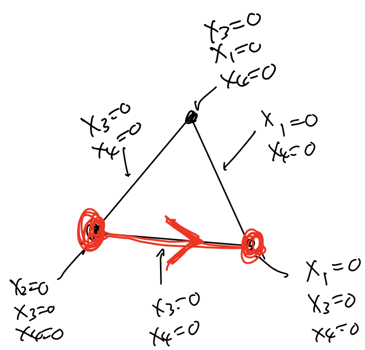
$$\begin{aligned} z &= 10 - x_4 + 0x_1 + 0x_2 \\ \hline x_3 &= 10 - x_4 - x_1 - x_2 \end{aligned}$$

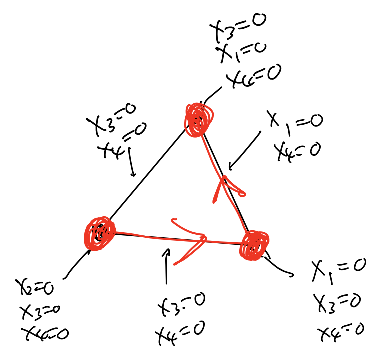
optimal dictionary

nonbasic : x_4, x_1, x_2

basic x_3







In the objective function.

So pivoting with zero coefficient variables, from an optimal dictionary

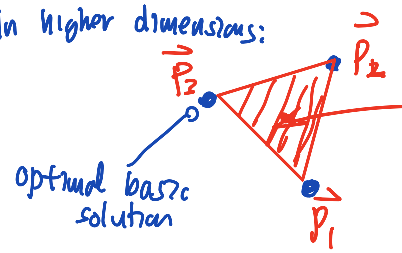
we can get all basic optimal solutions among optimal solutions.

optimal basic solution

= basic feasible solution (corner point) that is also optimal

. Other optimal solutions are obtained by "convex combination" of optimal basic solutions.

e.g. In higher dimensions:



$\vec{v}$ in this triangle can be written as

$\vec{v} = \lambda_1 \vec{P_1} + \lambda_2 \vec{P_2} + \lambda_3 \vec{P_3}$

for some real numbers

$\lambda_1, \lambda_2, \lambda_3 \geq 0$

and $\lambda_1 + \lambda_2 + \lambda_3 = 1$.

- For vectors $\vec{P_1}, \vec{P_2}, \cdots, \vec{P_N}$ in $\mathbb{R}^d$

  their convex combination is of the form

  $$\sum_{i=1}^{N} \lambda_i \vec{P_i}$$

  for some $\lambda_i \geq 0$, $i = 1, \cdots, N$

  and $\lambda_1 + \lambda_2 + \cdots + \lambda_N = 1$.

- Duality (standard form)

Primal LP	Dual LP
maximize $c_1x_1 + c_2x_2 + \cdots + c_nx_n$	minimize $b_1y_1 + \cdots + b_my_m$
subject to $a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \le b_1$	subject to $a_{11}y_1 + a_{21}y_2 + \cdots + a_{m1}y_m \ge c_1$
$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \le b_2$	$a_{12}y_1 + a_{22}y_2 + \cdots + a_{m2}y_m \ge c_2$
$\vdots$	$\cdots$
$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \le b_m$	$a_{1n}y_1 + a_{2n}y_2 + \cdots + a_{mn}y_m \ge c_n$
$x_1, x_2, \cdots, x_n \ge 0.$	$y_1, y_2, \cdots, y_m \ge 0.$

- Matrix form.

$m\times n$ matrix $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & \ddots & \vdots \\ a_{m1} & \cdots & \cdots & a_{mn} \end{bmatrix}$, $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$, $\vec{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \in \mathbb{R}^n$ $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \in \mathbb{R}^m$. $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} \in \mathbb{R}^m$.

$\vec{c}^T\vec{x} = \vec{c}\cdot\vec{x}$

primal LP	dual LP.
maximize $\vec{c}^T\vec{x}$	minimize $\vec{b}^T\vec{y}$
subject to $A\vec{x} \le \vec{b}$	subject to $A^T\vec{y} \ge \vec{c}$
$\vec{x} \ge \vec{0}$.	$\vec{y} \ge \vec{0}$

A^T = transpose of A

$= \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & & \ddots & \vdots \\ a_{1n} & \cdots & \cdots & a_{mn} \end{bmatrix}$ $n\times m$ matrix e.g.

(primal) maximize $4x_1 + 3x_2 + x_3 + x_4$

subject to

$$\begin{aligned} x_1 + 2x_2 \quad\quad - x_4 &\le 3 \\ 2x_1 + x_2 - x_3 + x_4 &\le 2 \\ x_2 + x_3 \quad\quad &\le 2 \end{aligned}, \quad x_1, x_2, x_3, x_4 \ge 0$$

(Dual:) minimize $3y_1 + 2y_2 + 2y_3$

subject to

$$\begin{aligned} y_1 + 2y_2 \quad\quad &\ge 4 \\ 2y_1 + y_2 + y_3 &\ge 3 \\ -y_2 + y_3 &\ge 1 \\ -y_1 + y_2 \quad\quad &\ge 1 \end{aligned}, \quad y_1, y_2, y_3 \ge 0$$

- The primal $\xrightarrow{\text{dual}}$ the dual $\xrightarrow{\text{dual}}$ the primal.

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dual (in the standard form)

maximize $-\vec{b}^T \vec{y}$

subject to $-A^T \vec{y} \le -\vec{c}$

$\vec{y} \ge \vec{0}$

$\xrightarrow{\text{dual}}$ minimize $-\vec{c}^T \vec{x}$

subject to $(-A^T)^T \vec{x} \ge -\vec{b}$

$\vec{x} \ge \vec{0}$

$(A^T)^T = A$

In standard form (noting $(A^T)^T = A$)

maximize $\vec{c}^T \vec{x}$

subject to $A\vec{x} \le \vec{b}$

$\vec{x} \ge \vec{0}$.

Lec 9. TuTh.