Midterm: 6:30-7:30 pm, Tuesday (tomorrow), IRC 2

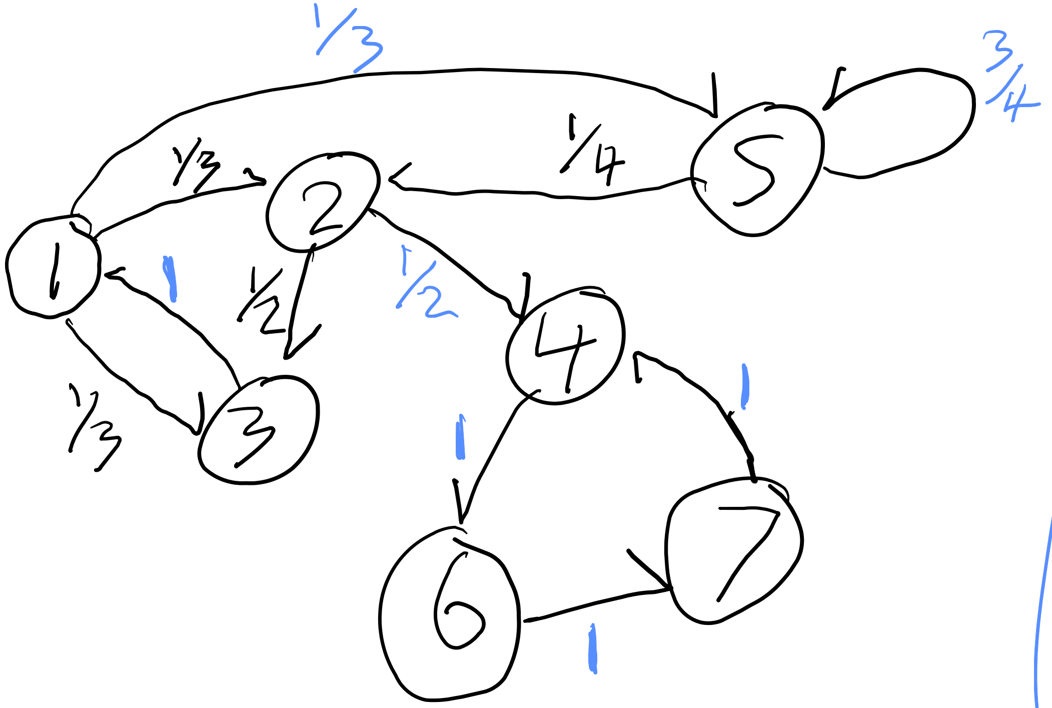
Wednesday: no class!

Lecture 19

Review

Problem 1

Fill in the missing probabilities on the following transition diagram, then determine the transition matrix.



$$
\begin{array}{c|ccccccc|c}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \\
\hline
 & & \frac{1}{3} & \frac{1}{3} & & \frac{1}{3} & & & 1 \\
 & & & \frac{1}{2} & \frac{1}{2} & & & & 2 \\
 & 1 & & & & & & & 3 \\
 & & & & & & 1 & & 4 \\
 & & \frac{1}{4} & & & \frac{3}{4} & & & 5 \\
 & & & & & & & 1 & 6 \\
 & & & & 1 & & & & 7
\end{array}
$$

Classify states:

Communicating classes: not closed

$\{1, 2, 3, 5\}$ transient, period = 1

$\{4, 6, 7\}$ recurrent, period = 3

closed

Starting in State 5, what is the probability that the MC visits State 6 within 6 steps?

$A = \{\text{visit } 6 \text{ w/in } 6 \text{ steps}\}$

$= \{\text{visit } 4 \text{ w/in } 5 \text{ steps}\}$

$P(A) = \sum \text{prob. of path that get to } 4$
w/in 5 steps

$= P(5 \to 2 \to 4) + P(5 \to 5 \to 2 \to 4) + P(5 \to 5 \to 5 \to 2 \to 4)$

$+ P(5 \to 2 \to 3 \to 1 \to 2 \to 4) + P(5 \to 5 \to 5 \to 2 \to 4)$

$$= \boxed{\frac{1}{4} \cdot \frac{1}{2} + \frac{3}{4} \cdot \frac{1}{4} \cdot \frac{1}{2} + \left(\frac{3}{4}\right)^2 \cdot \frac{1}{4} \cdot \frac{1}{2} + \left(\frac{3}{4}\right)^3 \cdot \frac{1}{4} \cdot \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{2}}$$

Smith drives a taxi that serves the city and the airport. A trip that originates in the city has a destination in the city with probability 0.9, and has the airport as destination with probability 0.1. A trip that originates in the airport always goes to the city.

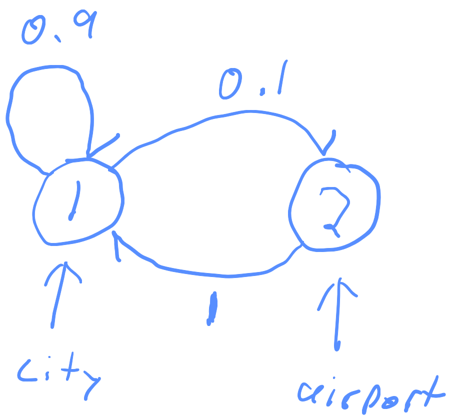
(a) (2 marks) Let X_n denote Smith's location (city or airport) after his n^{th} trip. This defines a Markov chain. What is its transition matrix?

(b) (2 marks) Determine the stationary distribution of the Markov chain.

(c) (1 mark) What fraction of trips originate in the city, in the long run?

(d) (2 marks) What fraction of all trips are trips from the city to the airport?

(e) (3 marks) Smith makes an average profit of \$8 for trips that remain in the city, and an average profit of \$12 for trips that involve the airport. What is his overall average profit per trip?



a)

$$\begin{array}{ccc} 1 & 2 & \\ \left[\begin{array}{cc} 0.9 & 0.1 \\ 1 & 0 \end{array}\right] & & \begin{array}{c} 1 \\ 2 \end{array} \end{array}$$

b) Reversible, Balance eqns:

$\pi_1 \cdot 0.1 = \pi_2 \cdot 1$, $\quad 1 = \pi_1 + \pi_2 = \pi_1 + 0.1\pi_1 = 1.1\pi_1$

$$\Rightarrow \boxed{\pi_1 = \frac{1}{1.1} = \frac{10}{11}, \quad \pi_2 = \frac{1}{11}}$$

c) Note MC is irreducible & finite state $\Rightarrow$ recurrent

By "big thm", frac of trips originating in ① $= \pi_1 = \frac{10}{11}$

d) (proportion of trips from city) $\cdot$ P(city to airport)

$= \pi_1 \cdot P_{12} = \frac{10}{11} \cdot 0.1 = \boxed{\frac{1}{11}}$

e) $8 \cdot$ P(city to city) $+ 12 \cdot$ P(city $\to$ airport or airport $\to$ city)

$= 8 \cdot \pi_1 \cdot P_{11} + 12 \cdot (\pi_1 \cdot P_{12} + \pi_2 \cdot P_{2,1})$

$= \boxed{8 \cdot \frac{10}{11} \cdot \frac{9}{10} + 12 \cdot \left(\frac{10}{11} \cdot \frac{1}{10} + \frac{1}{11} \cdot 1\right)}$

One hundred balls, some of them black and some of them white, are in an urn. At each time step, a ball is chosen from the urn uniformly at random, and is replaced by a white ball with probability p and by a black ball with probability $1-p$ (independently for each time). Here $0 < p < 1$. Let X_n denote the number of white balls after the n^{th} replacement.

(a) (3 marks) Find the transition probabilities for this Markov chain.

(b) (5 marks) Using any method, determine the stationary distribution of the Markov chain.

(c) (2 marks) Suppose that there are initially only black balls in the urn. How long will it take, on average, until there are again only black balls in the urn?

$P_{k,k+1} = P(X_1 = k+1 \mid X_0 = k) = P(\#\text{ of white balls goes up by 1} \mid X_0 = k)$

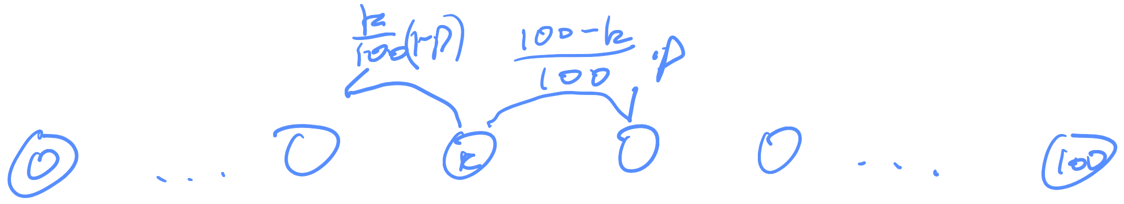
$= P(\text{choosing a black ball, replacing w/ a white ball} \mid X_0 = k)$

$= \underbrace{\frac{100-k}{100}}_{\text{choose black}} \cdot \underbrace{p}_{\text{replace w/ white}}$

$P_{k,k-1} = \frac{k}{100} \cdot 1-p$

$$P_{k,k} = 1 - P_{k,k+1} - P_{k,k-1} = 1 - \frac{100-k}{100} \cdot p - \frac{k}{100} \cdot (1-p)$$

b)



Reversible. Detailed balance:

$$\pi_k \cdot P_{k,k+1} = \pi_{k+1} P_{k+1,k}$$

$$\pi_k \cdot \frac{100-k}{100} \cdot p = \pi_{k+1} \cdot \frac{k+1}{100} \cdot (1-p)$$

$$\sum \pi_k = 1$$

$$\Rightarrow \pi_{k+1} = \frac{100-k}{k+1} \cdot \frac{p}{1-p} \cdot \pi_k$$

$$\pi_1 = \frac{100-0}{1} \cdot \frac{p}{1-p} \cdot \pi_0 = 100 \cdot \frac{p}{1-p} \cdot \pi_0$$

$$\pi_2 = \frac{100-1}{2}\cdot\frac{p}{1-p}\cdot\pi_1 = \frac{100\cdot 99}{2}\cdot\left(\frac{p}{1-p}\right)^2\cdot\pi_0$$

$$\vdots$$

$$\pi_k = \frac{100!}{(100-k)!\,k!}\cdot\left(\frac{p}{1-p}\right)^k\cdot\pi_0$$

Now, $$1 = \sum \pi_k = \sum \binom{100}{k}\cdot p^k\,\frac{1}{(1-p)^k}\cdot\pi_0$$

$$= \sum \binom{100}{k}\cdot p^k\,\underbrace{(1-p)^{100-k}}_{\frac{(1-p)^{100}}{(1-p)^k}}\cdot\cancelto{1}{\frac{\pi_0}{(1-p)^{100}}}$$

so series sums to 1

$$= \sum P(\mathrm{Bin}(100,p)=k) = 1$$

$$\Rightarrow \boxed{\pi_k = P(\mathrm{Bin}(100,p)=k) = \binom{100}{k}\cdot p^k(1-p)^{100-k}}$$

c) Chain is irreducible and recurrent, thus, by "big thm"

$$\mathbb{E}\left[\begin{array}{c}\text{return time to} \\ 0\end{array}\right] = \frac{1}{\pi_0} = \frac{1}{\binom{100}{0} \cdot p^0 (1-p)^{100}}$$

$$= \boxed{\frac{1}{(1-p)^{100}}}$$

Note: Recurrent ⟹ pos recurrent since MC is finite state

Sheila goes to an alien casino with n coins. In the first round of play, she

1. Flips each of the n coins three times, and gives all of the coins that weren't HHH to the casino.
2. For each coin that was HHH, she roles a q-sided die (with sides labeled 1, 2, …, q). For each roll, the casino gives her that many coins. (She also keeps HHH coin)

After the first round, she keeps playing again and again, always with all of her coins. At most, how many sides does the die have?

Suppose $n=1$. Then this a branching process. Recall: $P(\text{extinction}) = 1 \iff \underset{\substack{\uparrow \\ \text{descendant law}}}{\mathbb{E}[\xi]} \le 1$

$$\mathbb{E}\,\xi = \mathbb{E}[\xi \mid HHH]\cdot P(HHH) + \cancel{\mathbb{E}[\xi \mid HHH^c]}^{\,0}\cdot P(HHH^c)$$

$$= (1+\mathbb{E}[\text{die}])\,\frac{1}{8} = \left(1+\frac{q}{2}\right)\frac{1}{8} \underset{\substack{\uparrow \\ \text{casino}}}{\le} 1$$

$$1+\frac{q+1}{2} \le 8 \Rightarrow \frac{q+1}{2} \le 7 \Rightarrow q+1 \le 14 \Rightarrow \boxed{q \le 13}$$

For $n > 1$ each coin she starts w/ becomes its own independent branching process.

$$P(\text{all go extinct}) = P(1 \text{ goes extinct})^n$$

$\Rightarrow$ As before $P(\text{all go extinct}) = 1 \iff \mathbb{E}\,\xi \le 1$

Again; $\boxed{q \le 13}$