Lecture 2

- Homogenous Markov chain
- Transition matrix
- Transition diagram

Warm up: Two-state example (see Ross, ex. 4.2)

Communication system which transmits digits 0 and 1.

Each digit must pass through several stages, and at each stage the digit remains unchanged with probability p.

Let X_n be the state of the digit after it passes through n stages.

$P(X_{n+1} = 0|\mathrm{X_n} = 0) =$ p, $P(X_{n+1} = 1|\mathrm{X_n} = 1) =$ p,
$P(X_{n+1} = 1|\mathrm{X_n} = 0) =$ $1-p$, $P(X_{n+1} = 0|\mathrm{X_n} = 1) =$ $1-p$

No changes

Q: $P(X_2 = 0|X_0 = 0) =$ $p^2 + (1-p)^2$ ← change twice.

$P(X_n = 0|X_0 = 0) = ?$ → we'll see how to answer next.

mutually exclusive events

A: $= P(X_2=0, (X_1=0) \cup (X_1=1) \mid X_0=0) = P(X_2=0, X_1=0 \mid X_0=0) + P(X_2=0, X_1=1 \mid X_0=0)$

(formal proof)

$= P(X_2=0 \mid X_1=0, X_0=0) \cdot P(X_1=0 \mid X_0=0) + P(X_2=0 \mid X_1=1, X_0=0) \cdot P(X_1=1 \mid X_0=0)$

↓ Markov property

$= P(X_2 \mid X_1=0)\, P(X_1=0 \mid X_0=0) + P(X_2=0 \mid X_1=1) \cdot P(X_1=1 \mid X_0=0) = p^2 + (1-p)^2$

Recall: • We defined a M-C by the Markov property
$P(X_{n+1}=x_{n+1} | X_n = x_n, \dots, X_0 = x_0) = P(X_{n+1}=x_{n+1} | X_n = x_n)$

• We also saw an example → random walk in 1D → Jupyter Notebook
- Transition matrix

Homogeneous Markov chain

Defs:

- A Markov chain (MC) is homogenous if $\forall\ (x, y) \in S^2$, $P(X_{n+1} = x | X_n = y)$ is the same for all n.
- By indexing the states $S = \{s_1, s_2, \dots, s_i, \dots\}$, we can define the transition matrix $\widetilde{\boldsymbol{P}}$ for the MC to satisfy

$$\left(\widetilde{\boldsymbol{P}}\right)_{i,j} = \boldsymbol{P}\left(\boldsymbol{X}_{n+1} = \boldsymbol{s}_j \middle| \boldsymbol{X}_n = \boldsymbol{s}_i\right) =: \boldsymbol{p}_{i,j}$$

We consider homogeneous MC's in this class.

Two-state example: $\widetilde{\boldsymbol{P}} = \begin{array}{cc} \begin{array}{cc} 0 & 1 \end{array} & \\ \begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix} & \begin{array}{c} 0 \\ 1 \end{array} \end{array} = \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix}$

$P(X_{n+1} = 0 | X_n = 0)$ $\qquad$ $P(X_{n+1} = 1 | X_n = 0)$

Transition diagram

A MC may also be represented by a directed graph.

- Each node represents a state.
- We draw an arrow, from x to y if $p_{x,y} > 0$, labeled with weight $p_{x,y}$.
- This graph is called a transition diagram.

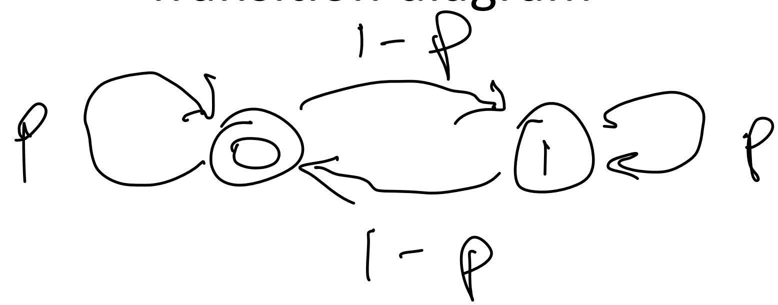
Two-state example:

Transition matrix

$$\tilde{P} = \begin{array}{c} \\ \\ \end{array}\begin{array}{cc} 0 & 1 \\ \end{array}$$

$$\tilde{P} = \begin{array}{cc|c} 0 & 1 & \\ \hline p & 1-p & 0 \\ 1-p & p & 1 \end{array}$$

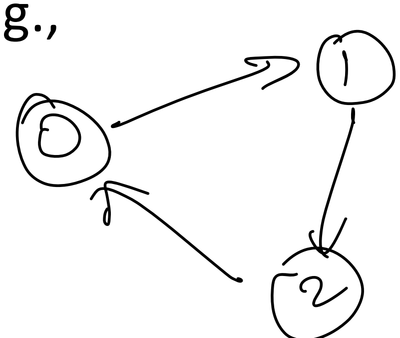
Transition diagram



Remarks

R.W : $\begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & & (0) \\ & \ddots & & \ddots & \\ (0) & & \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$

- Transition matrices and diagrams exist for infinite-state MC
- Given transition matrix, you can find transition diagram and vice versa, e.g.,



$\longleftrightarrow$ $\begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \end{array} \begin{array}{c} 0 \\ 1 \\ 2 \end{array}$

- Transition diagrams are more convenient if the transition matrix is sparse; they also illuminate graph structures…

$$p_{ij} = (\tilde{P})_{ij} = P(X_{n+1} = j \mid X_n = i)$$

Properties of $\tilde{P}$

(i) $\quad \forall\, i, j \in S \quad \underline{\ 0\ } \le p_{i,j} \le \underline{\ 1\ }$ (it's a probability)

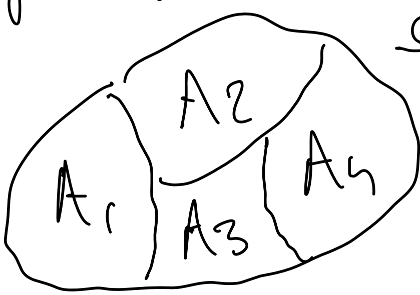
(ii) $\quad \forall\, i \in S \quad \underline{\sum_{j \in S} p_{ij}} = 1$

Note: A matrix that satisfies 1 and 2 is called stochastic.

Proof of (ii):

We use the fact that if a set of events A_i forms a partition of a probability space Ω, then $\sum_i P(A_i) = 1$

$\Omega = \bigcup A_i \qquad 1 = P(\Omega) = P(\dot{\bigcup} A_i) = \sum P(A_i)$



↳ Application: $\sum_{j \in S} p_{ij} = \sum_j P(X_1 = j \mid X_0 = i) = 1$