Lecture 4

Questions from students, Jupyter notebook, and more example(s)

Recall: • Transition matrix | diagram • C-K equation

$$P_{ij} = P(X_{n+1} = j \mid X_n = i) \qquad P^{m+n}_{ij} = \sum_{k \in S} \underbrace{P^{m}_{ik} P^{n}_{kj}}_{(\tilde{P}^m \tilde{P}^n)_{ij}}$$

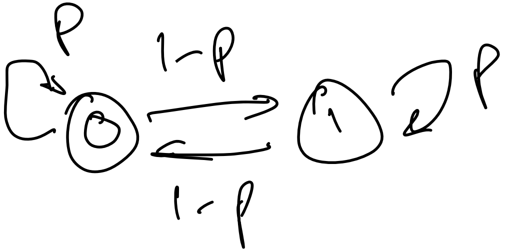
$\tilde{P}$

m+n-step transition

• csq: If $P(X_0 = i) = \alpha_i$

Then $P(X_n = j) = \left((\alpha_1 \dots \alpha_s)\tilde{P}^n\right)_j$ ← matrix product

ex:



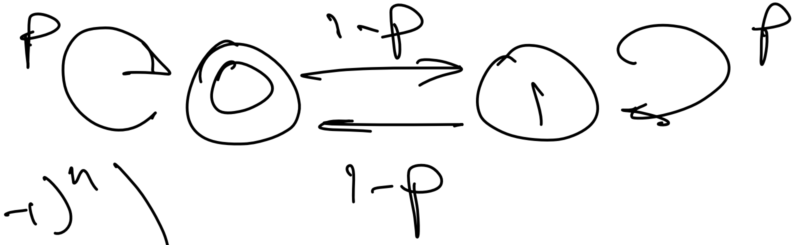
Q: a) Assume $X_0 = 1$. $P(X_n = 0)$? as a function of $\tilde{P}$

b) Prove by induction

$$\tilde{P}^n = \frac{1}{2}\begin{pmatrix} 1+(2p-1)^n & 1-(2p-1)^n \\ 1-(2p-1)^n & 1+(2p-1)^n \end{pmatrix}$$

See Jupyter Notebook ← for simulations.

a) $X_0 = 1$ $\quad P(X_n = 0) = ?$



b) $\tilde{P}^n = \frac{1}{2}\begin{pmatrix} 1+(2p-1)^n & 1-(2p-1)^n \\ 1-(2p-1)^n & 1+(2p-1)^n \end{pmatrix}$

$$\tilde{P} = \begin{array}{cc} \begin{array}{cc} 0 & 1 \end{array} & \\ \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix} & \begin{array}{c} 0 \\ 1 \end{array} \end{array}$$

a) $P(X_n = 0 \mid X_0 = 1) = (1 \ \ 0) \cdot \tilde{P}^n$

b) $n = 1 \rightarrow \frac{1}{2}\begin{pmatrix} \cancel{1}+(2p\cancel{-1})^1 & 1-(2p-1)^1 \\ 1-(2p-1)^1 & \cancel{1}+(2p\cancel{-1})^1 \end{pmatrix} = \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix} = \tilde{P}$

$n > 1 \quad \tilde{P}^{n+1} = \tilde{P} \cdot \tilde{P}^n = \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix} \cdot \frac{1}{2}\begin{pmatrix} 1+(2p-1)^n & 1-(2p-1)^n \\ 1-(2p-1)^n & 1+(2p-1)^n \end{pmatrix}$

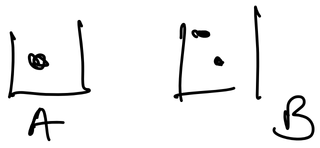
use induction here

$$
= \frac{1}{2} \begin{pmatrix} p + p(2p-1)^n + (1-p)(1-(2p-1)^n & (\ldots) \\ (\ldots) & (\ldots) \end{pmatrix}
$$

$$
= \frac{1}{2} \begin{pmatrix} \cancel{p} + p(2p-1)^n + 1 - (2p-1)^n \cancel{-p} + p(2p-1)^n & (\ldots) \\ (\ldots) & (\ldots) \end{pmatrix}
$$

$$
= \frac{1}{2} \begin{pmatrix} 1 + (2p-1)^{n+1} & (\ldots) \\ (\ldots) & (\ldots) \end{pmatrix}
$$

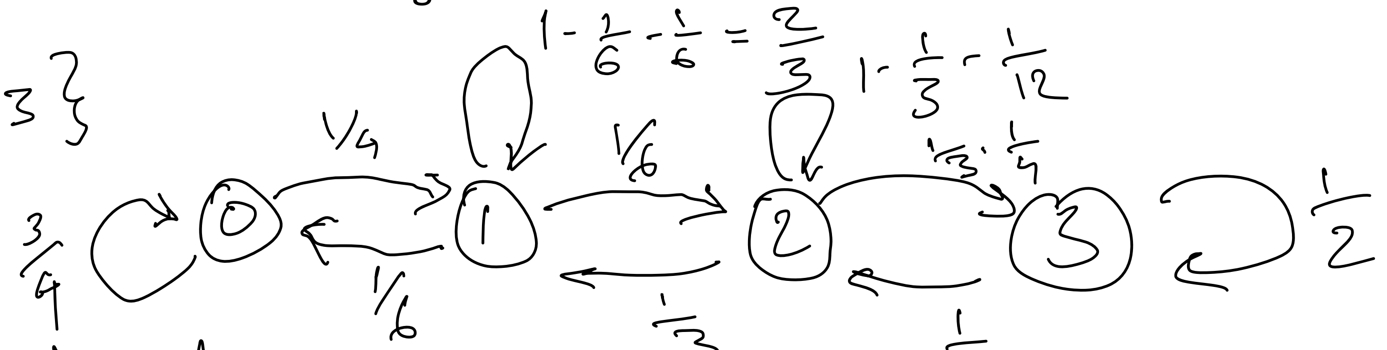
Examples



A total of 3 balls are divided between two urns, A and B. A ball is chosen at random. If it is chosen from urn A then it is placed in urn B with probability ½ and otherwise it is returned to urn A. Similarly, if the ball is chosen from urn B then it is placed in urn A with probability ¼. Let X_n denote the number of balls in urn A after n trials.

What is the state space? Draw the transition diagram.

- $S = \{0, 1, 2, 3\}$
- Transition diagram:



$X_n = 1$ pick in A return in A: $X_{n+1} = 1$

ex: return in B = $X_{n+1} = 0$

(. . .)

one can count

You flip a fair coin over and over. Let X_n be the number of heads in a row at flip n, thus defining a MC.

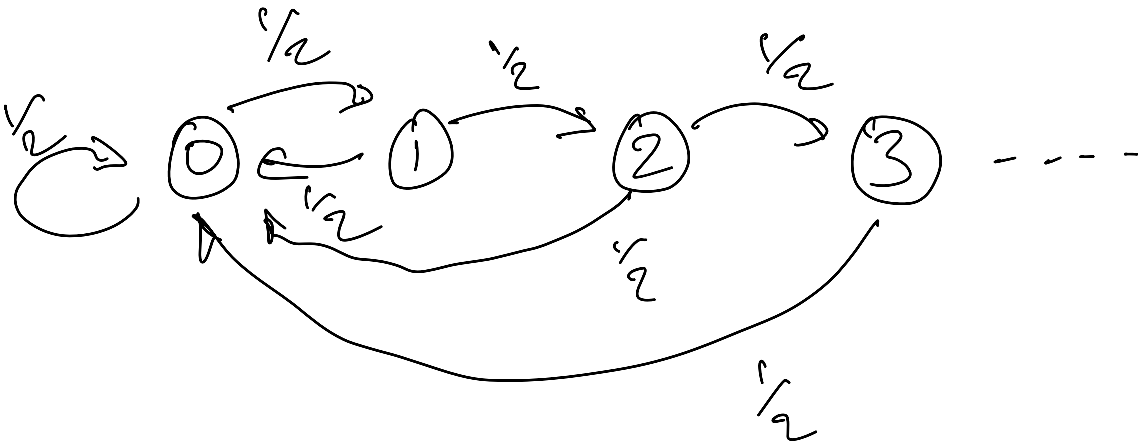
ex: T H H T H H H
1 2 3 4 5 6 7

What is the state space?

What is the transition diagram?

State space $= \{0, 1, 2, \ldots\} = \mathbb{N}$

TD :



T
$X_1 = 0$

TH
$X_2 = 1$

THH
$X_3 = 2$

THHT
$X_4 = 0$

THHTH
$X_5 = 1$

$X_6 = 2$, $X_7 = 3$

Let T_k be the number of flips until ~~2~~ K heads appear in a row. What is $\mathbb{E}(T_2)$?

Hint: Condition on the 1st outcome

ex: $\mathbb{E}(T_1) = \underbrace{\mathbb{E}(T_1 \mid H)}_{=1} \cdot \underbrace{P(H)}_{0.5} + \underbrace{\mathbb{E}(T_1 \mid T)}_{1+\mathbb{E}(T_1)} \cdot \underbrace{P(T)}_{0.5} = \frac{1}{2}\left(1 + 1 + \mathbb{E}(T_1)\right)$

$\Rightarrow \boxed{\mathbb{E}(T_1) = 2}$

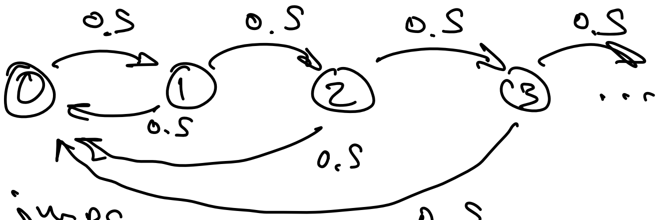
Similarly: $\mathbb{E}(T_2) = \mathbb{E}(T_2 \mid H) \cdot P(H) + \underbrace{\mathbb{E}(T_2 \mid T)}_{1+\mathbb{E}(T_2)} \cdot P(T)$

$= \left[\underbrace{\mathbb{E}(T_2 \mid HH)}_{2} \cdot P(H) + \underbrace{\mathbb{E}(T_2 \mid HT)}_{2+\mathbb{E}(T_2)} \cdot P(T)\right] P(H) + \left(1 + \mathbb{E}(T_2)\right) P(T)$

$(\ldots) \Leftrightarrow \boxed{\mathbb{E}(T_2) = 6}$

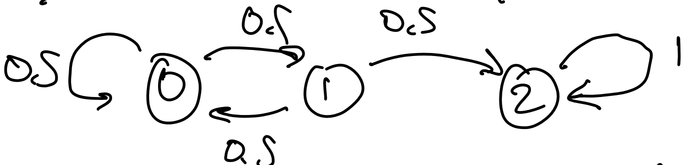
Assume $X_0 = 0$

What is the probability that $T_2 \leq 10$?



= Probability to arrive at 2 in ≤ 10 jumps (steps)

Trick: We modify the M-C and turn 2 into an absorbing state



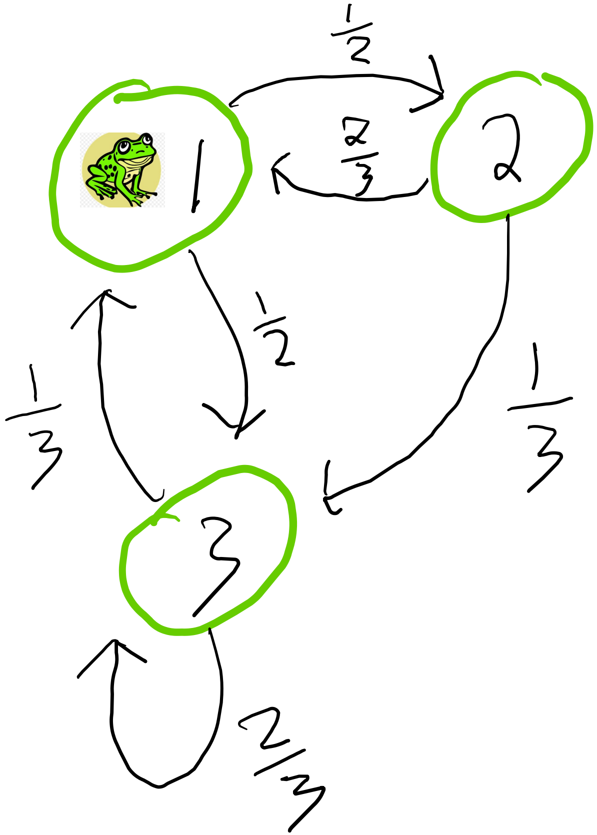
← Once at this state, we don't move anymore

Let's call this M-C Y_n with transition matrix $M = \begin{pmatrix} P_{00} & P_{01} & P_{02} \\ P_{10} & P_{11} & P_{12} \\ P_{20} & P_{21} & P_{22} \end{pmatrix} = \begin{pmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0 & 0.5 \\ 0 & 0 & 1 \end{pmatrix}$

$$P(T_2 \leq 10 \mid X_0 = 0) = P(Y_{10} = 2 \mid Y_0 = 0) = \left((1\ 0\ 0)M^{10}\right)_{s=2}$$

(C-K equation)

Frog on 3 lily pads



Q1: Given that the frog starts on the first lily pad, what is the probability that she is on the second lily pad after 10 jumps?

Q2: Suppose she starts at a uniform at random lily pad. What is the probability that she is on the third lily pad after 20 jumps?

Q3: Now, suppose there is an alligator hiding under the second lily pad and the frog will be eaten if she ever jumps there. Given that the frog starts on the first lily pad, what is the probability she is not eaten after 30 jumps?

Q_1 : The Transition matrix is $$\tilde{P} = \begin{array}{c|ccc|c} & 1 & 2 & 3 & \\ \hline & 0 & 1/2 & 1/2 & 1 \\ & 2/3 & 0 & 1/3 & 2 \\ & 1/3 & 0 & 2/3 & 3 \end{array}$$

$$\rightarrow P(X_{10} = 2 \mid X_0 = 1) = \left(\tilde{P}^{10}\right)_{1,2}$$

Q_2 : Initial state : $\alpha = \frac{1}{3}(1, 1, 1)$

$$P(X_{20} = 3 \mid X_0 \sim \alpha) = \left(\alpha \tilde{P}^{20}\right)_3$$

Q_3 : We make 2 absorbing, so upon considering the new transition matrix $$\tilde{D} = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 0 & 1 & 0 \\ 1/3 & 0 & 2/3 \end{pmatrix}, \quad P = \left(1 - \tilde{D}^{30}\right)_{1,2}$$

probability not being eaten after 30 steps