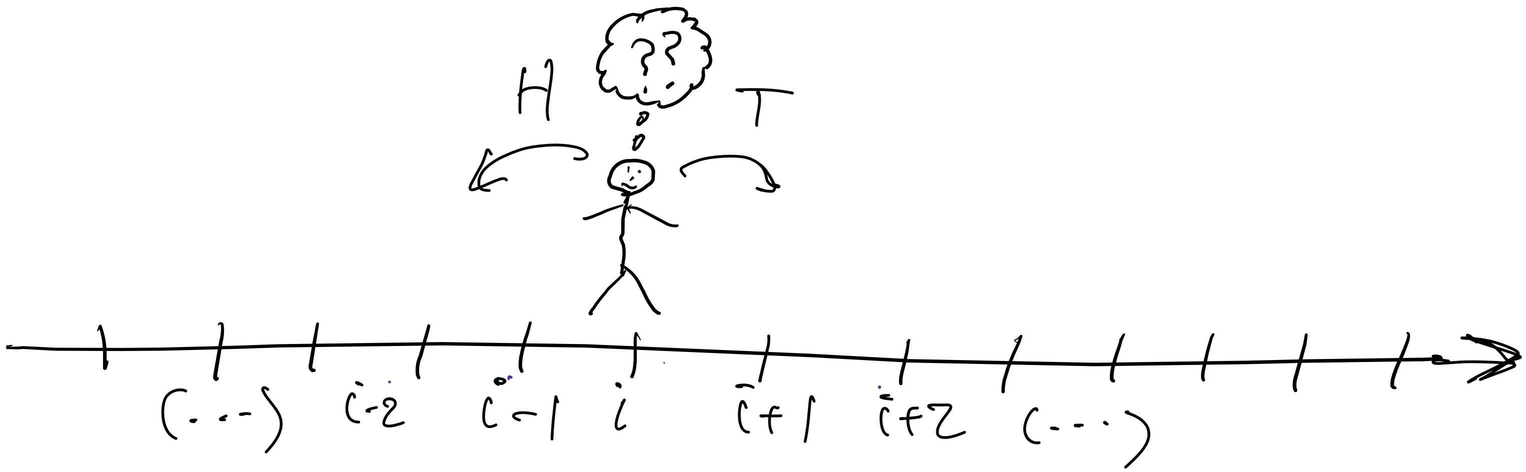
Part 1: Discrete-time Markov Chains

Lecture 1: What is a Markov Chain?

- Motivating example: Random walk
- Formal definition

Example: Symmetric random walk



- Flip a sequence of coins
- For each head, move one step to the left, for each tail, one step to the right
- $X_n =$ position after Flip n,
- Start at $X_0 = i$.

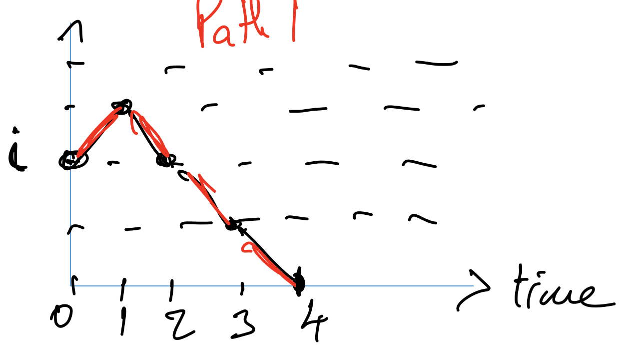
Example: HHT gives $X_0 =$ <u>i</u>, $X_1 =$ <u>i-1</u>, $X_2 =$ <u>i-2</u>, $X_3 =$ <u>i-1</u>

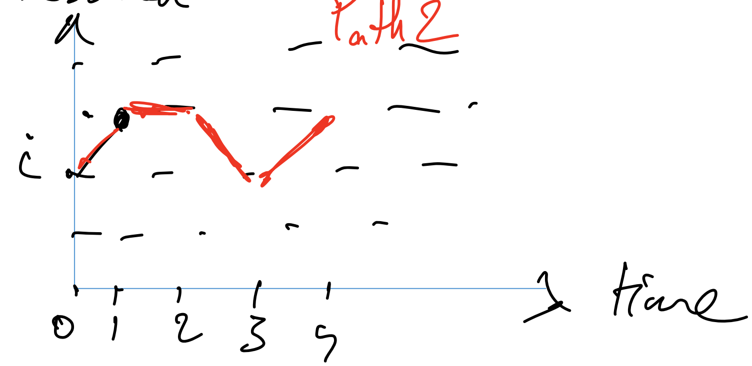
(Assuming that the coin is unbiased)

Warm up questions

What is the probability of the following paths?

- $P(\text{Path } 1) = (0.5)^4$
- $P(\text{Path } 2) = 0$





- $P(X_{n+1} = i+1 | X_n = i) = \underline{0.5}$, $P(X_{n+1} = i-1 | X_n = i) = \underline{0.5}$
- $P(X_2 = i+2 | X_0 = i) = \underline{P(TT) = P(T)P(T) = 0.5 \cdot 0.5 = 0.25}$
- $P(X_2 = i | X_0 = i, X_1 = i+1) = \underline{P(H) = 0.5}$ $(= P(X_2 = i | X_1 = i+1))$
- $P(X_{n+1} = k | X_0 = i, X_1 = i_1, \ldots, X_n = i_n) = \underline{0.5}$ $(= P(X_{n+1} = i | X_n = i_n))$

(we assume that $\forall j$ $i_{j+1} = i_j - 1$ or $i_j + 1$)

we assume that the throws are independent

Definitions

Def: A state space, typically denoted by S, is the set of values that the variables in a Markov chain may take.

- In the random walk, $S = \underline{\mathbb{Z}} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
- In this class, S is *discrete* (finite or countable)

↳ (means that the state space can be indexed by an integer ($\simeq \mathbb{N}$))

Definitions

Def: Let $(X_n)_{n\geq 0}$ be a sequence of random variables (r.v.'s) taking values in state space S. $(X_n)_{n\geq 0}$ is a Markov chain if it satisfies the Markov property:

$$\forall n \in \mathbb{N}, \forall (x_0, x_1, \dots, x_{n+1}) \in S^{n+2}$$

$$P(X_{n+1} = x_{n+1} | X_0 = x_0, X_1 = x_1, \dots, X_n = x_n) = \underline{P(X_{n+1} = x_{n+1}} | X_n = x_n)$$

Interpretation: "What happens at time $n+1$ only depends on the state at time n".

<u>Example</u>: Random walk.

$$P(X_{n+1} = x_{n+1} | X_0 = x_0, \dots, X_n = x_n) = \begin{cases} 0.5 & \text{if } x_{n+1} = x_n + 1 \\ 0.5 & \text{if } x_{n+1} = x_n - 1 \\ 0 & \text{else.} \end{cases}$$