1. $\boxed{\text{7 marks}}$ (a) $\boxed{\text{4 marks}}$ Recall the definition of a convex set: We say a set C of points in $\mathbf{R}^n$ is *convex* if for every pair $\mathbf{x}, \mathbf{y} \in C$, all points on the line segment joining $\mathbf{x}$ and $\mathbf{y}$ are in C. Thus C is a convex set, by definition, if for every pair $\mathbf{x}, \mathbf{y} \in C$ and any $\lambda \in [0, 1]$ we have $\lambda\mathbf{x} + (1 - \lambda)\mathbf{y} \in C$.

   Let A be an $m \times n$ matrix and $\mathbf{b}$ a given vector in $\mathbf{R}^m$. Show that

$$F = \{\mathbf{x} \in \mathbf{R}^n \,:\, A\mathbf{x} \leq \mathbf{b}, \quad \mathbf{x} \geq \mathbf{0}\}$$

   is a convex set.

Solution:

- We check for $\lambda \in [0, 1]$, that $\lambda \geq 0$ and $(1 - \lambda) \geq 0$. Let $\mathbf{x}, \mathbf{y} \in F$.
- Then we have $A\mathbf{x} \leq \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$, $A\mathbf{y} \leq \mathbf{b}$ and $\mathbf{y} \geq \mathbf{0}$.
- We note that if $\mathbf{u}, \mathbf{v}$ are vectors and $c \geq 0$ a scalar (i.e. a real number), then if $\mathbf{u} \geq \mathbf{v}$, we have $c\mathbf{u} \geq c\mathbf{v}$. You can check entry by entry to see this.
- Now $\mathbf{x} \geq \mathbf{0}$ implies $\lambda\mathbf{x} \geq \lambda\mathbf{0} = \mathbf{0}$ and $\mathbf{y} \geq \mathbf{0}$ implies $(1-\lambda)\mathbf{y} \geq (1-\lambda)\mathbf{0} = \mathbf{0}$. Then
$$\lambda\mathbf{x} + (1 - \lambda)\mathbf{y} \geq \mathbf{0}.$$
- Also $A\mathbf{x} \leq \mathbf{b}$ implies $A(\lambda\mathbf{x}) = \lambda(A\mathbf{x}) \leq \lambda\mathbf{b}$ and $A\mathbf{y} \leq \mathbf{b}$ implies $A((1 - \lambda)\mathbf{y}) = (1 - \lambda)(A\mathbf{y}) \leq (1 - \lambda)\mathbf{b}$.
- Then we obtain
$$A(\lambda\mathbf{x} + (1 - \lambda)\mathbf{y}) = A(\lambda\mathbf{x}) + A((1 - \lambda)\mathbf{y}) \leq \lambda\mathbf{b} + (1 - \lambda)\mathbf{b} = \mathbf{b}.$$
  That is,
$$A(\lambda\mathbf{x} + (1 - \lambda)\mathbf{y}) \leq \mathbf{b}.$$
- We now conclude $\lambda\mathbf{x} + (1 - \lambda)\mathbf{y} \in F$ and so F is convex.

(b) $\boxed{\text{3 marks}}$ Consider an LP (written in a matrix form):

$$\begin{aligned} &\max\ \mathbf{c} \cdot \mathbf{x} \\ &\text{subject to } A\mathbf{x} \leq \mathbf{b} \text{ and } \mathbf{x} \geq \mathbf{0}. \end{aligned}$$

Assume the LP has two optimal solutions $\mathbf{u}$ and $\mathbf{v}$. Show that for any choice of $\lambda \in [0, 1]$ (i.e. $0 \leq \lambda \leq 1$), the point $\lambda\mathbf{u} + (1 - \lambda)\mathbf{v}$ is also an optimal solution to the same LP. To do this:

- First show $\lambda\mathbf{u} + (1 - \lambda)\mathbf{v}$ is a feasible solution to the LP.
- Then show that $\lambda\mathbf{u} + (1 - \lambda)\mathbf{v}$ has the same value of the objective function as $\mathbf{u}$.

Solution:

- By the previous question (part (a)), $\lambda \mathbf{u} + (1-\lambda)\mathbf{v}$ is a feasible solution to the LP.
- Since $\mathbf{u}$ and $\mathbf{v}$ are both optimal they have the same value of the objective functions, that is, $\mathbf{c} \cdot \mathbf{u} = \mathbf{c} \cdot \mathbf{v}$.
- We compute

$$\mathbf{c} \cdot (\lambda \mathbf{u} + (1-\lambda)\mathbf{v}) = \lambda \mathbf{c} \cdot \mathbf{u} + (1-\lambda)\mathbf{c} \cdot \mathbf{v} = (\lambda + (1-\lambda))\mathbf{c} \cdot \mathbf{u} = \mathbf{c} \cdot \mathbf{u}$$

  using $\mathbf{c} \cdot \mathbf{u} = \mathbf{c} \cdot \mathbf{v}$.
- We conclude that $\lambda \mathbf{u} + (1-\lambda)\mathbf{v}$ is an optimal solution since it is feasible and has the optimal value of the objective function.

2. $\boxed{\text{5 marks}}$ (a) $\boxed{\text{2 marks}}$ Is it true that every linear programming problem has an optimal solution? Justify your answer: either prove it or give a counterexample.

Solution: No. There are unbounded LPs having no optimal solution.
For example,

$$\begin{aligned} &\text{maximize } x \\ &\text{subject to } x \geq 10, \end{aligned}$$

has no optimal solution with the maximum value of the objective function the value of the objective function $\to \infty$ as $x \to \infty$.

(b) $\boxed{\text{3 marks}}$ Is it true that if the feasible region of a linear programming problem is unbounded, then the linear programming problem does not have an optimal solution? Justify your answer: either prove it or give a counterexample.

Solution:
No. For example,

$$\begin{aligned} &\text{maximize } x \\ &\text{subject to } x \leq 10, \end{aligned}$$

has unbounded feasible region, but, it still has the optimal solution $x = 10$.

3. $\boxed{\text{3 marks}}$ The following dictionary is obtained from solving a standard form LP problem

with the simplex method. Find the next dictionary in the simplex method.

$$\begin{array}{rcrrr} z & = & 7 & -x_1 & +2x_4 \\ \hline x_2 & = & 12 & +x_1 & -x_4 \\ x_3 & = & 5 & +x_1 & -x_4 \\ x_5 & = & 4 & +x_1 & -x_4 \end{array}$$

To get credits for this problem, you should use

Anstee's rule:

- Choose the entering variable with the largest positive coefficient.
- If there is a tie, then choose the one with the smaller subscript.
- If there is a choice of leaving variable, then choose the one with the smallest subscript.

Solution:

The current basic feasible solution is

$$x_1 = x_4 = 0 \qquad x_2 = 2 \qquad x_3 = 5 \qquad x_5 = 4.$$

We only have 1 choice of entering variable: x_4.

It is constrained to be at most 12, 5, or 4, so x_5 is the leaving variable.

From the corresonding row, that is $x_5 = 4 + x_1 - x_4$, we get $x_4 = 4 + x_1 - x_5$. We use this and get the next dictionary as follows:

$$\begin{array}{rcrrr} z & = & 15 & +x_1 & -2x_5 \\ \hline x_2 & = & 8 & & +x_5 \\ x_3 & = & 1 & & +x_5 \\ x_4 & = & 4 & +x_1 & -x_5 \end{array}$$

4. $\boxed{\text{8 marks}}$ Solve the following LP problems using the simplex method. At each step please state the entering and leaving variables and the current basic feasible solution. Clearly state the optimal solution and the optimal value. To get credits for this problem, you should use

Anstee's rule:

- Choose the entering variable with the largest positive coefficient.
- If there is a tie, then choose the one with the smaller subscript.
- If there is a choice of leaving variable, then choose the one with the smallest subscript.

(a) $\boxed{\text{4 marks}}$ Maximise $z = 2x_1 + 3x_2 + 3x_3$, subject to

$$\begin{array}{rrrcl} 3x_1 & +x_2 & & \leq & 60 \\ -x_1 & +x_2 & +4x_3 & \leq & 10 \\ 2x_1 & -2x_2 & +5x_3 & \leq & 15 \end{array}$$

and $x_1, x_2, x_3 \geq 0$. *This problem requires 2 pivots.*

Solution: We first introduce slack variables x_4, x_5 and x_6 and write our first dictionary:

$$\begin{array}{ccrrrr} x_4 & = & 60 & -3x_1 & -x_2 & \\ x_5 & = & 10 & +x_1 & -x_2 & -4x_3 \\ x_6 & = & 15 & -2x_1 & +2x_2 & -5x_3 \\ \hline z & = & & 2x_1 & +3x_2 & +3x_3 \end{array}$$

A quick check shows that this is feasible. The current basic feasible solution is

$$x_1 = x_2 = x_3 = 0 \qquad x_4 = 60 \qquad x_5 = 10 \qquad x_6 = 15 \qquad z = 0$$

We have a choice of 3 entering variables, and x_2 and x_3 both have equally large coefficients, so we pick the one with the smaller subscript — x_2 is the entering variable. Now, x_2 is restricted to be less than 60 and 10 by the first and second equations (the last equation does not constrain it). Hence the leaving variable is x_5.

$$\begin{array}{ccrrrr} z & = & 30 & +5x_1 & -9x_3 & -3x_5 \\ \hline x_2 & = & 10 & +x_1 & -4x_3 & -x_5 \\ x_4 & = & 50 & -4x_1 & +4x_3 & +x_5 \\ x_6 & = & 35 & & -13x_3 & -2x_5 \end{array}$$

So the current feasible solution is

$$x_1 = x_3 = x_5 = 0 \qquad x_2 = 10 \qquad x_4 = 50 \qquad x_6 = 35 \qquad z = 30$$

Now, since there is a positive coefficient in the expression for z we are not yet done. There is only 1 choice of entering variable, x_1. The first and third equations do not restrict x_1, and the second restricts $x_1 \leq 50/4 = 25/2$. Hence x_4 is the leaving variable. Pivoting gives:

$$\begin{array}{ccrrrr} z & = & \frac{185}{2} & -4x_3 & -\frac{5}{4}x_4 & -\frac{7}{4}.x_5 \\ \hline x_1 & = & \frac{25}{2} & +x_3 & -\frac{1}{4}.x_4 & +\frac{1}{4}.x_5 \\ x_2 & = & \frac{45}{2} & -3x_3 & -\frac{1}{4}.x_4 & -\frac{3}{4}.x_5 \\ x_6 & = & 35 & -13x_3 & & -2x_5 \end{array}$$

The current feasible solution is

$$x_1 = \frac{25}{2} \qquad x_2 = \frac{45}{2} \qquad x_3 = x_4 = x_5 = 0 \qquad x_6 = 35 \qquad z = \frac{185}{2}$$

Since there are no longer any positive coefficients in z, this is the optimal solution and $z = \frac{185}{2}$ is the optimal value.

(b) $\boxed{\text{4 marks}}$ Maximise $z = 3x_1 + 2x_2 + 4x_3$, subject to

$$\begin{array}{llll} x_1 & +x_2 & +2x_3 & \leq & 4 \\ 2x_1 & & +3x_3 & \leq & 5 \\ 2x_1 & +x_2 & +3x_3 & \leq & 7 \end{array}$$

and $x_1, x_2, x_3 \geq 0$. *This problem requires 3 pivots.*

Solution: Introduce slack variables and write in dictionary form

$$\begin{array}{ccrrrr} z & = & & 3x_1 & +2x_2 & +4x_3 \\ \hline x_4 & = & 4 & -x_1 & -x_2 & -2x_3 \\ x_5 & = & 5 & -2x_1 & & -3x_3 \\ x_6 & = & 7 & -2x_1 & -x_2 & -3x_3 \end{array}$$

All the variables are potentially entering, but we pick the one with the largest coefficient — so x_3 is the entering variable. It is restricted to be at most $2, 5/3, 7/3$ (respectively). So x_5 is the leaving variable.

Pivoting gives

$$\begin{array}{ccrrrr} z & = & \frac{20}{3} & +\frac{1}{3}x_1 & +2x_2 & -\frac{4}{3}x_5 \\ \hline x_3 & = & \frac{5}{3} & -\frac{2}{3}x_1 & & -\frac{1}{3}x_5 \\ x_4 & = & \frac{2}{3} & +\frac{1}{3}x_1 & -x_2 & +\frac{2}{3}x_5 \\ x_6 & = & 2 & & -x_2 & +x_5 \end{array}$$

The current feasible solution is

$$x_1 = x_2 = x_5 = 0 \qquad x_3 = \frac{5}{3} \qquad x_4 = \frac{2}{3} \qquad x_6 = 2 \qquad z = \frac{20}{3}$$

Choose x_2 to be the entering variable. It is constrained to be at most $2/3, 2$ — so x_4 is the leaving variable. Pivoting gives

$$\begin{array}{ccrrrr} z & = & 8 & +x_1 & -2x_4 & \\ \hline x_2 & = & \frac{2}{3} & +\frac{1}{3}x_1 & -x_4 & +\frac{2}{3}x_5 \\ x_3 & = & \frac{5}{3} & -\frac{2}{3}x_1 & & -\frac{1}{3}x_5 \\ x_6 & = & \frac{4}{3} & -\frac{1}{3}x_1 & +x_4 & +\frac{1}{3}x_5 \end{array}$$

So the current feasible solution is

$$x_1 = x_4 = x_5 = 0 \qquad x_2 = \frac{2}{3} \qquad x_3 = \frac{5}{3} \qquad x_6 = \frac{4}{3} \qquad z = 8$$

We only have 1 choice of entering variable — x_1. It is constrained to be at most $5/2$ or 4, so x_3 is the leaving variable. Pivoting gives

$$\begin{array}{ccrrrr} z & = & \frac{21}{2} & -\frac{3}{2}x_3 & -2x_4 & -\frac{1}{2}x_5 \\ \hline x_1 & = & \frac{5}{2} & -\frac{3}{2}x_3 & & -\frac{1}{2}x_5 \\ x_2 & = & \frac{3}{2} & -\frac{1}{2}x_3 & -x_4 & +\frac{1}{2}x_5 \\ x_6 & = & \frac{1}{2} & +\frac{1}{2}x_3 & +x_4 & +\frac{1}{2}x_5 \end{array}$$

Since there are no negative coefficients in the expression for z we are done. The optimal solution is therefore

$$x_1 = \frac{5}{2} \qquad x_2 = \frac{3}{2} \qquad x_3 = x_4 = x_5 = 0 \qquad x_6 = \frac{1}{2}$$

and the optimal value is $z = \frac{21}{2}$.

5. $\boxed{\text{3 marks}}$ The following dictionary is obtained while solving a standard form LP problem by using the simplex method. What does this dictionary imply to the LP problem? Does it have an optimal solution? **Explain your answer clearly.**

$$\begin{array}{rcrrrr} z & = & 8 & +x_1 & -2x_4 & +2x_6 \\ \hline x_2 & = & 12 & -x_1 & +x_4 & +x_6 \\ x_3 & = & 1 & +x_1 & & +x_6 \\ x_5 & = & 4 & +x_1 & +x_4 & \end{array}$$

Solution:

Let $x_6 = t$.

Then, while keeping $x_1 = x_4 = 0$,

we have $x_2 = 12 + t, x_3 = 1 + t, x_5 = 4 + t$ is feasible and $z = 8 + 2t$,

therefore, as $t \to +\infty$, $z \to +\infty$.

This means that the LP problem is unbounded and has no optimal solution.