- Only part of the problems may be graded. But, you have to submit all the problems.

1. $\boxed{\text{5 marks}}$ Give an example of a dictionary (with more than one decision variables) for which the current basic feasible solution is optimal and yet one of the coefficients of the non-basic variables in the z row (the objective function part) is positive. You have to justify your answer.

Solution:

An easy example is

$$\begin{array}{lrrl} \max & x_1 & & \\ & x_1 & +x_2 & \leq 0 \qquad x_1, x_2 \geq 0 \\ & x_1 & -2x_2 & \leq 1 \end{array}$$

which has the initial dictionary

$$\begin{array}{rclll} z & = & & x_1 & \\ x_3 & = & 0 & -x_1 & -x_2 \\ x_4 & = & 1 & -x_1 & +2x_2 \end{array}$$

Notice that the basic feasible solution $x_1 = 0, x_2 = 0$ (and $x_3 = 0, x_4 = 1$) is the only feasible solution, thus it is the optimal solution.

As in this example, optimal dictionaries are not necessarily the final dictionary of the simplex algorithm. The above dictionary gives the optimal solution, but, we can still do pivot (with x_1 entering and x_3 leaving) to get the next dictionary.

$$\begin{array}{rclll} z & = & & -x_3 & -x_2 \\ x_1 & = & 0 & -x_3 & -x_2 \\ x_4 & = & 1 & +x_3 & +3x_2 \end{array}$$

This is the final dictionary of the simplex algorithm, with optimal solution $x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 1$.

2. $\boxed{\text{5 marks}}$ Let m, n be two natural numbers. Let $a_{ij}, y_i, w_j, c_i \in \mathbf{R}$, be given for $i = 1, \cdots, m$, $j = 1, \cdots, n$.
Compute the following.

$$\max_{x_1,\cdots,x_n \in \mathbf{R}} \left[\sum_{i=1}^{m} y_i \left[\sum_{k=1}^{n} a_{ik}x_k + c_i \right] + \sum_{j=1}^{n} w_j x_j \right]$$

Here, the max is the maximum value of the function of $x_1, ..., x_n$ on the right, while the variables $x_1, ..., x_n$ satisfy the conditions below it, that is, $x_1, ..., x_n \in \mathbf{R}$.
You must justify your answer clearly.

Solution:

- First,

$$\sum_{i=1}^{m} y_i \left[\sum_{k=1}^{n} a_{ik}x_k + c_i \right] + \sum_{j=1}^{n} w_j x_j = \sum_{i=1}^{m} y_i c_i + \sum_{i=1}^{m} \left[\sum_{k=1}^{n} a_{ik}x_k \right] + \sum_{j=1}^{n} w_j x_j.$$

- Note that $\max_{x_1,\cdots,x_n \in \mathbf{R}}$ only applies for the term with x_i's.
- Therefore,

$$\begin{aligned} &\max_{x_1,\cdots,x_n \in \mathbf{R}} \left[\sum_{i=1}^{m} y_i \left[\sum_{k=1}^{n} a_{ik}x_k + c_i \right] + \sum_{j=1}^{n} w_j x_j \right] \\ &= \sum_{i=1}^{m} y_i c_i + \max_{x_1,\cdots,x_n \in \mathbf{R}} \left[\sum_{i=1}^{m} \left[\sum_{k=1}^{n} a_{ik}x_k \right] + \sum_{j=1}^{n} w_j x_j \right]. \end{aligned}$$

- We see that

$$\begin{aligned} &\sum_{i=1}^{m} y_i \left[\sum_{k=1}^{n} a_{ik}x_k \right] + \sum_{j=1}^{n} w_j x_j \\ &= \sum_{k=1}^{n} \left[\sum_{i=1}^{m} y_i a_{ik} \right] x_k + \sum_{j=1}^{n} w_j x_j \text{ (by changing the order of summation)} \\ &= \sum_{j=1}^{n} \left[\sum_{i=1}^{m} y_i a_{ij} \right] x_j + \sum_{j=1}^{n} w_j x_j \text{ (by changing the name of index from } k \text{ to } j.\text{)} \\ &= \sum_{j=1}^{n} \left[\left[\sum_{i=1}^{m} y_i a_{ij} \right] + w_j \right] x_j \text{ (by combining the two sums).} \end{aligned}$$

- Therefore,

$$\max_{x_1,\cdots,x_n\in\mathbf{R}}\left[\sum_{i=1}^m y_i\left[\sum_{k=1}^n a_{ik}x_k\right]+\sum_{j=1}^n w_jx_j\right]=\max_{x_1,\cdots,x_n\in\mathbf{R}}\left[\sum_{j=1}^n\left[\left[\sum_{i=1}^m y_ia_{ij}\right]+w_j\right]x_j\right].$$

- The last expression is of the form

$$\max_{x_1,\cdots,x_n\in\mathbf{R}}(b_1x_1+\cdots+b_nx_n) \text{ for } b_j=\textstyle\sum_{i=1}^m y_ia_{ij}+w_j.$$

- If one of b_j is nonzero, say fors $j=l$ for some l, then we can consider $x_1=0,..,x_{l-1}=0,x_l=tb_l,x_{l+1}=0,...,x_n=0$ for $0\le t\in\mathbf{R}$, then, $b_1x_1+\cdots+b_nx_n=t|b_l|^2\to\infty$ as $t\to\infty$, because $|b_l|^2>0$. This implies that $\max_{x_1,\cdots,x_n\in\mathbf{R}}(b_1x_1+\cdots+b_nx_n)=\infty$ unless all $b_i=0$, for $i=1,...,n$.

- On the other hand, if all $b_i=0$, for $i=1,...,n$, it trivially holds $\max_{x_1,\cdots,x_n\in\mathbf{R}}(b_1x_1+\cdots+b_nx_n)=0$.

- In summary, we have the result:

$$\max_{x_1,\cdots,x_n\in\mathbf{R}}\left[\sum_{i=1}^m y_i\left[\sum_{k=1}^n a_{ik}x_k\right]+\sum_{j=1}^n w_jx_j\right]$$
$$=\begin{cases}0 & \text{if } \sum_{i=1}^m y_ia_{ij}+w_j=0 \text{ for all } i=1,...,n\ ,\\ +\infty & \text{otherwise.}\end{cases}$$

- Finally,

$$\max_{x_1,\cdots,x_n\in\mathbf{R}}\left[\sum_{i=1}^m y_i\left[\sum_{k=1}^n a_{ik}x_k+c_i\right]+\sum_{j=1}^n w_jx_j\right]$$
$$=\begin{cases}\sum_{i=1}^m y_ic_i & \text{if } \sum_{i=1}^m y_ia_{ij}+w_j=0 \text{ for all } i=1,...,n\ ,\\ +\infty & \text{otherwise.}\end{cases}$$

Simpler version

- We can use vector/matrix notation. Let $\vec{x}=(x_1,...,x_n)$, $\vec{y}=(y_1,...,y_m)$, $\vec{w}=(w_1,...,w_n)$, $\vec{c}=(c_1,...,c_m)$, and let $A=[a_{ij}]$ be the m-by-n matrix of the entries a_{ij}'s, that is its entry at the i-th row and j-th column is a_{ij}.

- Let us view the vectors as column vectors.
Then, the expression $\sum_{i=1}^m y_i\left[\sum_{k=1}^n a_{ik}x_k+c_i\right]+\sum_{j=1}^n w_jx_j$ is nothing but $\vec{y}^T[A\vec{x}+\vec{c}]+\vec{w}^T\vec{x}=\vec{y}^T\vec{c}+\vec{y}^TA\vec{x}+\vec{w}^T\vec{x}$.

- So

$$\max_{x_1,\cdots,x_n\in\mathbf{R}}\left[\sum_{i=1}^m y_i\left[\sum_{k=1}^n a_{ik}x_k\right]+\sum_{j=1}^n w_jx_j\right]=\max_{\vec{x}\in\mathbf{R}^n}\left[\vec{y}^T\vec{c}+\vec{y}^TA\vec{x}+\vec{w}^T\vec{x}\right].$$

- The last expression is the same as $\vec{y}^T\vec{c}+\max_{\vec{x}\in\mathbf{R}^n}\left[\left[\vec{y}^TA+\vec{w}^T\right]\vec{x}\right]$.

- If $\vec{y}^TA+\vec{w}^T\neq\vec{0}$, then, we choose $\vec{x}=t(\vec{y}^TA+\vec{w}^T)$ for $t\in\mathbf{R}$, then, $\left[\vec{y}^TA+\vec{w}^T\right]\vec{x}=t|\vec{y}^TA+\vec{w}^T|^2$. Since $\vec{y}^TA+\vec{w}^T\neq\vec{0}$, we have $\vec{y}^TA+\vec{w}|^T|^2>0$, therefore, $t|\vec{y}^TA+\vec{w}^T|^2\to\infty$ as $t\to\infty$. This implies $\max_{\in\mathbf{R}^n}\left[\vec{y}^TA\vec{x}+\vec{w}^T\vec{x}\right]=\infty$ if $\vec{y}^TA+\vec{w}^T\neq\vec{0}$.

- On the other hand, if $\vec{y}^TA+\vec{w}^T=\vec{0}$, then we trivially have that $\max_{\in\mathbf{R}^n}\left[\vec{y}^TA\vec{x}+\vec{w}^T\vec{x}\right]=0$.

- Therefore,

$$\max_{\vec{x}\in\mathbf{R}^n}\left[\vec{y}^TA\vec{x}+\vec{w}^T\vec{x}\right]=\begin{cases}0 & \text{if } \vec{y}^TA+\vec{w}^T=\vec{0},\\ \infty & \text{otherwise.}\end{cases}$$

- Combining these with the identity

$$\max_{x_1,\cdots,x_n\in\mathbf{R}}\left[\sum_{i=1}^m y_i\left[\sum_{k=1}^n a_{ik}x_k\right]+\sum_{j=1}^n w_jx_j\right]=\vec{y}^T\vec{c}+\max_{\vec{x}\in\mathbf{R}^n}\left[\vec{y}^TA\vec{x}+\vec{w}^T\vec{x}\right]$$

we see that

$$\max_{x_1,\cdots,x_n\in\mathbf{R}}\left[\sum_{i=1}^m y_i\left[\sum_{k=1}^n a_{ik}x_k+c_i\right]+\sum_{j=1}^n w_jx_j\right]=\begin{cases}\vec{y}^T\vec{c} & \text{if } \vec{y}^TA+\vec{w}^T=\vec{0},\\ +\infty & \text{otherwise.}\end{cases}$$

3. $\boxed{\text{5 marks}}$ For a given $\vec{c} \in \mathbf{R}^n$, define the function $F(\vec{y})$ of $\vec{y} \in \mathbf{R}^n$, as

$$F(\vec{y}) = \max_{\vec{x} \in \mathbf{R}^n \,\&\, \vec{x} \geq \vec{0}} [\vec{y} \cdot \vec{x} + \vec{c} \cdot \vec{x}] \, .$$

(a) $\boxed{\text{2 marks}}$ Evaluate $F(\vec{y})$, whose value, of course, depends not only on $\vec{y}$ but also on $\vec{c}$. You have to justify your answer. *Hint: Be careful with the constraint* $\vec{x} \geq \vec{0}$.

Solution:

- Notice that

$$F(\vec{y}) = \max_{\vec{x} \in \mathbf{R}^n \,\&\, \vec{x} \geq \vec{0}} [\vec{y} \cdot \vec{x} + \vec{c} \cdot \vec{x}] = \max_{\vec{x} \in \mathbf{R}^n \,\&\, \vec{x} \geq \vec{0}} [(\vec{y} + \vec{c}) \cdot \vec{x}]$$

- Now to consider the values of $(\vec{y} + \vec{c}) \cdot \vec{x}$ under the condition $\vec{x} \geq \vec{0}$, notice that if any component of $\vec{y} + \vec{c} = (y_1 + c_1, y_2 + c_2, ..., y_n + c_n)$ has a positive entry, namely, at the l-th entry for some l, $y_l + c + l > 0$, then we can choose $\vec{x}$ such that $x_i = 0$ for all $i \neq l$ and $x_l = t$, $t \geq 0$. Then, the resulting product $(\vec{y} + \vec{c}) \cdot \vec{x}$ with these choices becomes $t(y_l + c_l)$; since $y_l + c_l > 0$, it diverges to $+\infty$ as $t \to \infty$.

- So, if any of $y_i + c_i > 0$ for some i, then

$$\max_{\vec{x} \in \mathbf{R}^n \,\&\, \vec{x} \geq \vec{0}} [(\vec{y} + \vec{c}) \cdot \vec{x}] = +\infty.$$

- On the other hand, if we have $y_i + c_i \leq 0$ for all $i = 1, ..., n$, then, $(\vec{y} + \vec{c}) \cdot \vec{x} \leq 0$ for those $\vec{x} \leq \vec{0}$, simply because $(\vec{y} + \vec{c}) \cdot \vec{x} = (y_1 + c_1)x_1 + \cdots + (y_n + c_n)x_n$. And when $\vec{x} = \vec{0}$, this quantity becomes 0. So, in this case, the maximum value is 0, that is,

$$\max_{\vec{x} \in \mathbf{R}^n \,\&\, \vec{x} \geq \vec{0}} [(\vec{y} + \vec{c}) \cdot \vec{x}] = 0.$$

- Therefore, combining these two cases we see that

$$F(\vec{y}) = \max_{\vec{x} \in \mathbf{R}^n \,\&\, \vec{x} \geq \vec{0}} [(\vec{y} + \vec{c}) \cdot \vec{x}] = \begin{cases} 0 & \text{if } \vec{y} + \vec{c} \leq \vec{0} \, , \\ +\infty & \text{otherwise, that is, if } \vec{y} + \vec{c} \not\leq \vec{0} \, . \end{cases}$$

(b) $\boxed{\text{3 marks}}$ Continuation from (a).
Solve the following optimization problem:

$$\text{Minimize } F(\vec{y})$$
$$\text{subject to } \vec{y} \geq \vec{0}.$$

You have to justify your answer. *Hint: The answer depends on $\vec{c}$. You also have to find the condition on $\vec{c}$ for this problem to have an optimal solution. Notice that solving an optimization problem is to do one of the following: a) find that there is no feasible solution so that problem is infeasible, b) find the optimal solution(s), or c) find that the problem is feasible but still there is no optimal solution.*

Solution:

- Now back to the minimization problem of $F(\vec{y})$ under the constraint $\vec{y} \geq \vec{0}$.
- Because $F(\vec{y}) = \infty$ if $\vec{y}+\vec{c} \not\leq \vec{0}$, we can exclude this case in the minimization.
  That is, we can consider only those $\vec{y}$ that satisfies $\vec{y} + \vec{c} \leq \vec{0}$, that is, $\vec{y} \leq -\vec{c}$. And for those $\vec{y}$, we saw that $F(\vec{y}) = 0$.
- Therefore, the original problem that is,
  $$(Prob1) \qquad \begin{aligned} &\text{Minimize } F(\vec{y}) \\ &\text{subject to } \vec{y} \geq \vec{0}. \end{aligned}$$
  is equivalent to
  $$(Prob2) \qquad \begin{aligned} &\text{Minimize } 0 \\ &\text{subject to } \vec{y} \leq -\vec{c} \\ &\qquad\qquad\quad \vec{y} \geq \vec{0} \end{aligned}$$
  in the sense that the case the second problem is infeasible corresponds to the first problem having the $+\infty$ minimum.
- For the latter problem (Prob2) the objective function is constant 0, so, what only matters is feasibility: Any feasible solution $\vec{y}$ is also optimal. What matters is whether we have any feasible solution $\vec{y}$.
- We must have $\vec{y} \geq \vec{0}$, so all $y_1, ..., y_n \geq 0$. On the other hand, it should satisfy $\vec{y} \leq -\vec{c}$, so $y_1 \leq -c_1,\ y_2 \leq -c_2, ...,\ y_n \leq -c_n$. For these constraints to be satisfied we should have that $-c_1, -c_2, ..., -c_n \geq 0$, in other words $\vec{c} \leq \vec{0}$; without such a condition, no $\vec{y}$ will satisfy the constraints. And of course, if $\vec{c} \leq \vec{0}$, equivalently $-\vec{c} \geq \vec{0}$, than there are $\vec{y}$ that satisfies both $\vec{0} \leq \vec{y}$ and $\vec{y} \leq -\vec{c}$.
- Therefore, the problem is feasible if and only if $\vec{c} \leq \vec{0}$.
- We have this conclusion:

  The problem (Prob2) is feasible if and only if $\vec{c} \leq \vec{0}$,
  and in the case it is feasible, any feasible solution is optimal,
  with the optimal objective value 0.

- Back to the original problem (Prob1), it means,

  The problem (Prob1) has a finite minimum objective value if and only if $\vec{c} \leq \vec{0}$, and in the case it has a finite minimum value, any solution $\vec{y} \geq \vec{0}$ that satisfies $\vec{y} \leq -\vec{c}$, is optimal, with the optimal objective value 0.

4. **Setting up an LP from a practical problem.** In the fine tradition of bad puns on mathematical assignments, my colleagues and I are starting the soon-to-be-famous Opple Rubber Company. Our company makes three different products:

   - fashionable rubberised slippers sold under the "iMoc" name
   - rubberised protectors for fruit called "oPods", and
   - a small annoying musical instrument called the "oPhone".

   Our company receives the rubber it needs in 200cm long ribbons. Each product requires a certain amount of rubber.

   - 1 pair of iMoc slippers needs 90cm of a ribbon
   - 1 oPod needs 70cm of a ribbon
   - 1 oPhone needs 50cm of ribbon

   A large order has come in and the company needs to make at least

   - 300 pairs of iMocs
   - 400 oPods
   - 1000 oPhones

   We would like to work out how to cut up the sheets so as to minimise waste. This problem can be broken down into smaller parts:

   (a) [2 marks] There are 6 ways to cut a 200cm rubber sheet into pieces of length 90cm, 70cm and 50 cm with minimal waste — what are they and how much rubber does each one waste? Please list them in order of most waste to least. *Note* Do not include ways such as $(70, 70)$ since this leaves 60cm and one could cut a 50cm segment from it.

   Solution: The 6 different ways are

   1. $(90, 70)$ wasting 40
   2. $(70, 50, 50)$ wasting 30
   3. $(90, 90)$ wasting 20
   4. $(90, 50, 50)$ wasting 10
   5. $(70, 70, 50)$ wasting 10
   6. $(50, 50, 50, 50)$ wasting 0

(b) $\boxed{\text{4 marks}}$ Each of the ways of cutting a sheet wastes a certain amount of rubber. Obviously we would like to minimise this waste while still producing enough iMocs, oPods and oPhones. For some reason our cutting machine is unable to cut the ribbon in four equal pieces, so ignore this possibility — this leaves the other five cutting options. Write this as a linear programming problem. You have to explain your answer. *Note* Please label your variables $y_1, \ldots, y_5$ so that the corresponding amount ribbon wasted is ordered from greatest to least.

Solution:

- Let $y_1, y_2, \ldots, y_5$ be the number of ribbons that are cut according to the 5 different ways above (excluding the possibility of $(50, 50, 50, 50)$.
- The wasted rubber is therefore

$$w = 40y_1 + 30y_2 + 20y_3 + 10y_4 + 10y_5$$

- The number of 90cm segments needs to be at least 300, so

$$y_1 + 2y_3 + y_4 \geq 300.$$

- The number of 70cm segments needs to be at least 400 so

$$y_1 + y_2 + 2y_5 \geq 200.$$

- The number of 50cm segments needs to be at least 1000 so

$$2y_2 + 2y_4 + y_5 \geq 1000$$

- So the LP problem is

$$\begin{array}{lllllll} \text{minimize } w = 40y_1 + 30y_2 + 20y_3 + 10y_4 + 10y_5 & & & & & & \\ & y_1 & & +2y_3 & +y_4 & & \geq & 300 \\ \text{subject to} & y_1 & +y_2 & & & +2y_5 & \geq & 400 \\ & & 2y_2 & & +2y_4 & +y_5 & \geq & 1000 \\ & y_1, \ldots, y_5 \geq 0 & & & & & \end{array}$$

Note: To be more precise, there is an additional condition that $y_1, ...y_5$ are integers. In part (c) when you write the Jupyter notebook with PuLP package, the decision variables should be integer variables.

(c) $\boxed{\text{4 marks}}$ Write and run a Jupyter notebook to solve the LP problem from (b). You must attach the screenshot of your Jupyter notebook and results.

Solution: See the Jupyter notebook file in the Canvas.

(d) $\boxed{\text{5 marks}}$ Write down the dual problem to the problem of (b). Then write and run a Jupyter notebook to solve it. You must attach the screenshot of your Jupyter notebook and results. Also check whether the optimal objective value of this problem is the same as the optimal objective value of the problem in (b).

Solution: See the Jupyter notebook file in the Canvas. You should be able to see that the optimal objective values are equal.