Lecture 6

Properties of Transience and recurrence

Recall:

Def: Let $f_i := P(X_n = i \text{ for some } n \geq 1 | X_0 = i)$. Then

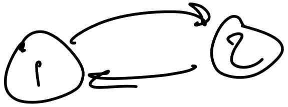
- if $f_i = 1$, State s_i is recurrent, (abbreviated as i is recurrent)
- if $f_i < 1$, State s_i is transient (abbreviated as i is transient)

Number of (return) visits

= Number of times n is visited in a trajectory (X_n) this is a random variable (it depends on the random trajectory)

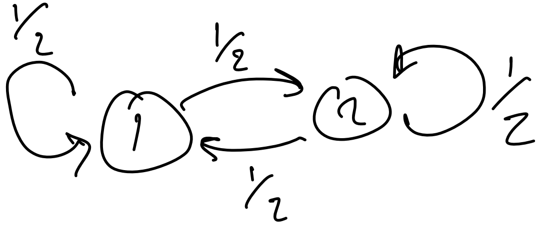
$$N_i = \#\{n \geq 0 : X_n = i\} \in \{0, 1, 2, 3, \dots\} \cup \{\infty\}$$

Examples:

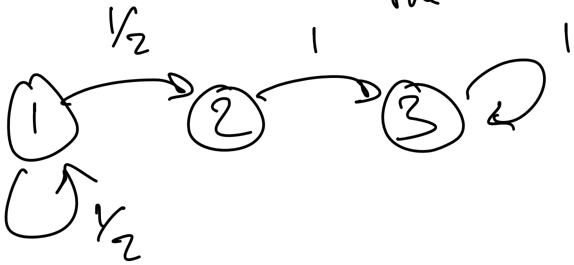


1 2 1 2 1 2 1 2 ....

$N_1 = +\infty$



$N_1 = \infty$



$N_1 \mid X_0 = 1 \sim Geom\left(\frac{1}{2}\right)$

Number of return visits

$\left(f_i = P(X_n = i \text{ for some } n \geq 1 \mid X_0 = i)\right)$

Proposition:

- If i is recurrent, then $P(N_i = \infty | X_0 = i) =$ __1______
- If i is transient, then $E\,[N_i | X_0 = i] =$ __$1 / 1 - f_i$__

Proof: • If i is recurrent we know in a finite time, the MC returns to i. After that the MC starts over and returns to 1 again in finite time and so on

• If transient $P(N_i = 1 | X_0 = i) = 1 - f_i$

$P(N_i = 2 | X_0 = i) = \underset{= Pr(\text{revisiting } i)}{f_i} \cdot \underbrace{(1 - f_i)}_{Pr(\text{Not revisiting } i)}$

$(\ldots)$ $\boxed{P(N_i = k \mid X_0 = i) = f_i^{k-1} \cdot (1 - f_i)}$

end of the proof : $N_i \mid X_0 = i \sim \text{Geom}(1-f_i) \Rightarrow E(N_i \mid X_0 = i) = \frac{1}{1-f_i}$ $\boxtimes$

Expressing N_i as a sum of indicators

$N_i = \sum_{n=0}^{+\infty} \mathbb{1}[X_n = i]$

$\mathbb{1}[X_n = i] = \begin{cases} 1 & \text{if } X_n = i \\ 0 & \text{else.} \end{cases}$

"adding 1 if I see $X = i$ at time n

Implication:

$E[N_i|X_0 = i] = E\left(\sum_{n=0}^{+\infty} \mathbb{1}_{[X_n=i]} \mid X_0 = i\right]$

$= \sum_{n=0}^{+\infty} E[\mathbb{1}_{[X_n=i]} \mid X_0 = i] = \sum_{n=0}^{+\infty} \underbrace{\cancel{1} \cdot P(X_n = i \mid X_0 = i)}_{P_{ii}^n} \cancel{(+0 \cdot P(X_n \neq i \mid X_0 = i)}$

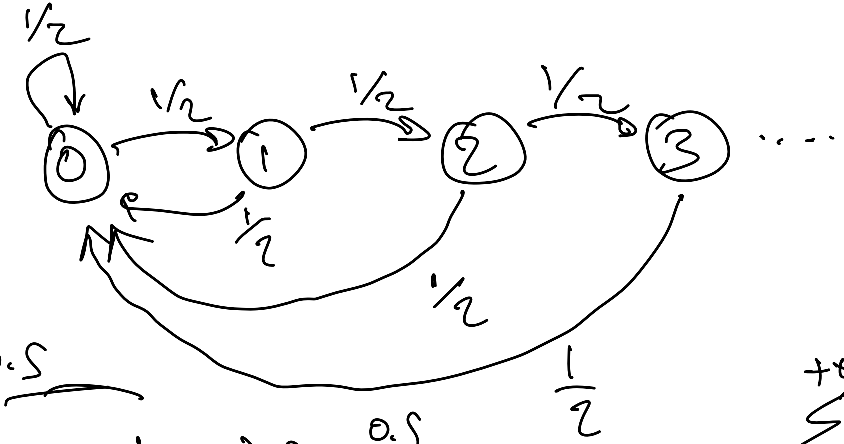
$= \sum_{n=0}^{+\infty} P_{ii}^n$

Proposition:

- If i is recurrent, then $\sum_{n=0}^{\infty} p_{i,i}^{n} = \underline{\ \infty\ }$
- If i is transient, then $\sum_{n=0}^{\infty} p_{i,i}^{n} \ \underline{<\ \infty}$

Example: 1-d random walk (to be continued later) → future application

Ex:



Is 0 transient or recurrent?

$$p_{0,0}^{n} = \frac{1}{2}$$

$$\sum_{n=0}^{+\infty} p_{0,0}^{n} = \sum_{n=0}^{\infty} \frac{1}{2} = +\infty$$

$$p_{00}^{2} = \overbrace{P(X_2=0 \mid X_1=0)}^{0.5}\, p_{00}^{=0.5} + \underbrace{P(X_2=0 \mid X_1=1)}_{0.5}\, p_{01}^{=0.5} = \frac{1}{2}$$

$\Longrightarrow$ 0 is recurrent

transient

If $i \leftrightarrow j$, then they are both recurrent or both transient

Proposition:

Recurrence and transience are class properties.

Proof: Assume i is recurrent, so $\sum_{n=0}^{+\infty} P_{ii}^{n} = +\infty$, and $i \leftrightarrow j$

so $\exists\, n_1, n_2 \mid P_{ij}^{n_1} > 0,\ P_{ji}^{n_2} > 0$

$$\sum_{n=0}^{+\infty} P_{jj}^{n} \geq \sum_{n=0}^{+\infty} P_{jj}^{n+n_1+n_2} \underset{\text{C-K eq.}}{\geq} \sum_{n=0}^{+\infty} \underbrace{P_{ji}^{n_2}}_{>0} \cdot P_{ii}^{n} \cdot \underbrace{P_{ij}^{n_1}}_{>0}$$

$$\geq P_{ji}^{n_2} P_{ij}^{n_1} \cdot \underbrace{\sum_{n=0}^{+\infty} P_{ii}^{n}}_{+\infty} \Rightarrow \sum_{n=0}^{+\infty} P_{jj}^{n} = +\infty$$

$\Rightarrow j$ is recurrent $\otimes$

Summary

Proposition: Let f_i, N_i be as above. Then,

<u>i is recurrent is equivalent to:</u>

- $P(N_i = \infty | X_0 = i) =$ 1
- $\sum_{n=0}^{\infty} p_{i,i}^n =$ ∞
- For any j in the same communicating class, j is recurrent

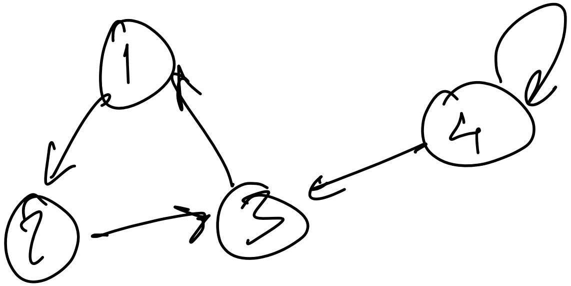
<u>i is transient is equivalent to:</u>

- $E[N_i | X_0 = i] =$ $\frac{1}{1-f_i} < +\infty$
- $\sum_{n=0}^{\infty} p_{i,i}^n$ $< \infty$
- For any j in the same communicating class, j is transient

Finite-state MC

Proposition: For a finite-state MC, at least one state is recurrent. Thus, for an irreducible, finite state MC, all states are recurrent.

Q: For a finite-state MC, can you characterize which states are recurrent based on graph properties?



A: 1, 2, 3 recurrent

4 transient