Lecture 11 (TuTh)

Weak duality and application.

Strong duality. Vanderbei sec 5.4. statement and application

Lecture 12 Complementary slackness (Vanderbei, sec 5.5)

From Lec 10

$g_{\vec y,\vec w}(\vec x) = \boxed{\vec y^T}(\vec b - A\vec x) + \boxed{\vec w^T}\vec x$

$\vec c^T\vec x + g_{\vec y,\vec w}(\vec x) \geq \vec c^T\vec x$, if $\vec x \geq \vec 0$ & $A\vec x \leq \vec b$

$F(\vec y,\vec w) = \max_{\vec x}\left[\vec c^T\vec x + g_{\vec y,\vec w}(\vec x)\right]$

$$F(\vec y,\vec w) \geq \max_{\substack{A\vec x\leq \vec b\\ \vec x\geq\vec 0}} c^T x$$

$$\max_{\vec x}(\vec a^T\vec x) = \begin{cases} 0 & \text{if } \vec a = \vec 0 \\ +\infty & \text{if } \vec a \neq 0\end{cases}$$

$$F(\vec y,\vec w) = \max_{\vec x}\left[\vec c^T\vec x + g_{\vec y,\vec w}(\vec x)\right] = \underbrace{\vec y^T\vec b}_{(=\vec b\cdot\vec y)} + \max_{\vec x}\left[\underbrace{[\vec c^T + \vec w^T - \vec y^T A]\,\vec x}_{(\vec c+\vec w - A^T\vec y)^T\vec x}\right]$$

* Note that

$$\max_{\vec x}\left[\,[\vec c + \vec w - A^T\vec y]\,\vec x\,\right] = \begin{cases} 0 & \text{if } \vec c + \vec w - A^T\vec y = \vec 0 \\ +\infty & \text{otherwise}\end{cases}$$

can choose any $\vec x$.

$\boxed{\text{Your exercise}}$ Show this

** When we $\boxed{\text{minimize}}$ $F(\vec y,\vec w)$, we don't need to consider the case it takes $+\infty$.

Therefore.

minimize $F(\vec y,\vec w)$
subject to $\vec y, \vec w \geq \vec 0$.

$\Longleftrightarrow$

minimize $\vec b^T\vec y$
subject to $\vec c + \boxed{\vec w} - A^T\vec y \boxed{=\vec 0}$
$\vec y, \boxed{\vec w} \geq \vec 0$.

Here $\vec w$ is the slack variable of this

$\vec w = A^T\vec y - \vec c$

* In all those they have the same minimum objective value.

$\Longleftrightarrow$

minimize $\vec b^T\vec y$
subject to $A^T\vec y \geq \vec c$
$\vec y \geq \vec 0$.

Note. This is the dual problem of

maximize $\vec c^T\vec x$
subject to $A\vec x \leq \vec b$, $\vec x \geq \vec 0$

- All these three are equivalent.

(giving the same optimal solution and the same optimal objective value).

Lec 11. ↓

Weak duality

For any $\vec{y}, \vec{w} \geq \vec{0}$

Note that $F(\vec{y}, \vec{w}) \geq \max_{\substack{A\vec{x} \leq \vec{b} \\ \vec{x} \geq \vec{0}}} \left(\vec{c}^T \vec{x} \right)$

and $\min_{\vec{y}, \vec{w} \geq 0} F(\vec{y}, \vec{w}) = \min_{\substack{A^T\vec{y} \geq \vec{c} \\ \vec{y} \geq \vec{0}}} \vec{b}^T \vec{y}$.

Therefore,

"weak duality"

$$\boxed{\min_{\substack{A^T\vec{y} \geq \vec{c} \\ \vec{y} \geq 0}} \vec{b} \cdot \vec{y} \geq \max_{\substack{A\vec{x} \leq \vec{b} \\ \vec{x} \geq \vec{0}}} \left(\vec{c}^T \vec{x} \right)}$$

primal	dual
maximize $\vec{c} \cdot \vec{x}$	minimize $\vec{b} \cdot \vec{y}$
subject to $A\vec{x} \leq \vec{b}$. $\vec{x} \geq \vec{0}$.	subject to $A^T\vec{y} \geq \vec{c}$, $\vec{y} \geq \vec{0}$

So we have:

Thm (Weak duality)

max of primal $\leq$ min of dual

More precisely if $\vec{x}$ is a solution to primal constraints.

$$A\vec{x} \leq \vec{b}$$
$$\vec{x} \geq 0$$

and $\vec{y}$ a solution to dual constraint

$$A^T \vec{y} \geq \vec{c}$$
$$\vec{y} \geq 0$$

Then, $\vec{c} \cdot \vec{x} \leq \vec{b} \cdot \vec{y}$.

For primal feasible $\vec{x}$, dual feasible $\vec{y}$,

$\vec{c} \cdot \vec{x} \leq$ max of primal objective

$\leq$ min of dual objective

$\leq \vec{b} \cdot \vec{y}$

A) any primal feasible objective value $\leq \vec{b}^T\vec{y}^*$.

$\vec{c}^T\vec{x}^*$ $\quad$ $\vec{b}^T\vec{y}^*$ $\quad$ $\mathbb{R}$

Consider the primal problem

B) any dual feasible value $\geq \vec{c}^T\vec{x}^*$.

$$\boxed{\max_{A\vec{x}\leq\vec{b}\ \&\ \vec{x}\geq\vec{0}} \quad \vec{c}\cdot\vec{x}} \quad \text{and its dual} \quad \boxed{\min_{A^T\vec{y}\geq\vec{c}\ \&\ \vec{y}\geq\vec{0}} \quad \vec{b}\cdot\vec{y}.}$$

$\leq$ Weak duality.

Assume that these problems are both feasible. Let $\vec{x}^*$ be a feasible solution of the primal problem and $\vec{y}^*$ be a feasible solution of the dual problem. **In this situation, choose a wrong statement:**

A) The primal problem must be bounded. True.

B) The dual problem must be bounded. True.

C) It is possible (in some examples) that $\vec{c}\cdot\vec{x}^* < \vec{b}\cdot\vec{y}^*$. True.

D) If $\vec{c}\cdot\vec{x}^* = \vec{b}\cdot\vec{y}^*$, then $\vec{x}^*$ is an optimal solution of the primal problem and $\vec{y}^*$ is an optimal solution of the dual problem. True

E) When $\vec{x}^*$ and $\vec{y}^*$ are optimal solutions of the primal and dual problems, respectively, it may happen (in some examples) that $\vec{c}\cdot\vec{x}^* < \vec{b}\cdot\vec{y}^*$.

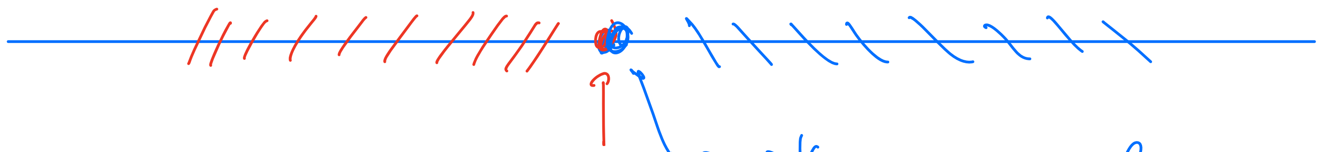
WRONG due to 'strong duality' we learn below.

We assumed primal feasible region and dual feasible region are $\neq \emptyset$.

So there are feasible primal objective values

and feasible dual objective values

D). True.



If $\vec{c}^T\vec{x}^* = \vec{b}^T\vec{y}^*$ and feasible

then $\vec{b}^T\vec{y}^* = \vec{c}^T\vec{x}^* \leq \vec{b}^T\vec{y}$ for all dual feasible $\vec{y}$,

That is $\vec{y}^*$ is a minimum for the dual.

Also, $\vec{c}^T\vec{x}^* = \vec{b}^T\vec{y}^* \geq \vec{c}^T\vec{x}$ for all primal feasible $\vec{x}$,

so $\vec{x}^*$ is a maximum for the primal.

- Some consequences of weak duality.

* Any feasible solution to the dual problem gives an upper bound of the objective function for the primal problem.

* If the dual LP is unbounded (objective function $\to -\infty$) then the primal LP is not feasible.

If the primal LP is unbounded (objective function $\to +\infty$) then the dual LP is not feasible.

Optimality and weak duality:

Suppose

$\vec{x}_0$ is a feasible solution of an LP: $\displaystyle\max_{\substack{\vec{x}\geq 0\\ A\vec{x}\leq\vec{b}}} \vec{c}^T\vec{x}$

$\vec{y}_0$ is a " " of the dual LP: $\displaystyle\min_{\substack{\vec{y}\geq 0\\ A^T\vec{y}\geq\vec{c}}} \vec{b}^T\vec{y}$

If $\vec{c}^T\vec{x}_0 = \vec{b}^T\vec{y}_0$

then $\vec{x}_0$ is optimal for the primal problem

$\vec{y}_0$ is optimal for the dual problem.

Proof $\vec{c}^T\vec{x}_0 \leq$ max of primal $\leq$ min of dual $\leq \vec{b}^T\vec{y}_0$.

weak duality,

If $\vec{c}^T\vec{x}_0 = \vec{b}^T\vec{y}_0$

then $\vec{c}^T\vec{x}_0 =$ max of primal $=$ min of dual $= \vec{b}^T\vec{y}_0$.

So, $\vec{x}_0$ is optimal for the primal LP

$\vec{y}_0$ is optimal " the dual LP. $\square$

- An optimal final dictionary is a feasible dictionary whose z row has all non-positive coefficient

objective

e.g.

$$\boxed{\begin{array}{l} z = 3 - x_1 + 0 \cdot x_2 \\ \hline x_3 = 5 + x_1 - x_2 \end{array}}$$

optimal final

e.g.

$$\boxed{\begin{array}{l} z = 3 + x_1 \\ \hline x_3 = -x_1 - x_2 \end{array}}$$

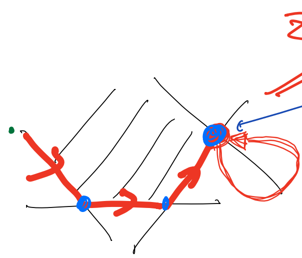
optimal but not final

having optimal basic solution

$$\boxed{\begin{array}{l} z = 3 - x_3 - x_2 \\ \hline x_1 = -x_3 - x_2 \end{array}}$$

optimal and final

- When there is degeneracy, an optimal dictionary may NOT be the final dictionary



optimal solution

Pivots may possibly do not change basic solutions

Strong duality

primal	dual
maximize $\vec{c}\cdot x$ subject to $A\vec{x} \le \vec{b}$, $\vec{x} \ge \vec{0}$.	minimize $\vec{b}\cdot\vec{y}$ subject to $A^T\vec{y} \ge \vec{c}$, $\vec{y} \ge \vec{0}$

Thm If an LP problem has an optimal solution, so does its dual LP, and their respected optimal values are equal.

$$\max \vec{c}^T\vec{x} = \min \vec{b}^T\vec{y}$$
$A\vec{x} \le \vec{b}$, $\vec{x} \ge \vec{0}$ $\quad$ $A^T y \ge \vec{c}$, $\vec{y} \ge \vec{0}$ $\quad$ if one of them has an optimal solution.

That is, if the primal problem has an optimal solution $\vec{x}^* = (x_1^*, \dots, x_n^*)$

then • the dual has an optimal solution $\vec{y}^* = (y_1^*, \dots, y_m^*)$

and

• $\vec{c}^T\vec{x}^* = \vec{b}^T\vec{y}^*$ (i.e. $\sum_{j=1}^{n} c_j x_j^* = \sum_{i=1}^{m} b_i y_i^*$).

Moreover, from the row of the objective function of the optimal final dictionary of the primal problem

$$z = \underbrace{z^*}_{\text{the optimal value}} + \sum_{k=1}^{n+m} \underbrace{c_k^*}_{\text{coefficient}} x_k$$

$$z = z^* + c_1^* x_1 + c_2^* x_2 + \cdots + c_n^* x_n + c_{n+1}^* x_{n+1} + \cdots + c_{n+m}^* x_{n+m}$$

decision variables $\quad$ slack variables.

(note $c_k^* = 0$ if x_k basic)

We can find a dual optimal solution

$\vec{y}^* = (y_1^*, \dots, y_m^*)$.

as $\quad y_i^* = -c_{n+i}^*$

• We will prove this later.

Ex

$$\vec{c} = \begin{bmatrix} 4 \\ 3 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 2 & 1 & -1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 1 \\ 0 & -1 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

$$\begin{array}{l} \max \ \vec{c}\cdot\vec{x} \\ A\vec{x} \le \vec{b} \\ \vec{x} \ge \vec{0} \end{array} \quad \xleftrightarrow{\text{dual}} \quad \begin{array}{l} \min \ \vec{b}\cdot\vec{y} \\ A^T\vec{y} \ge \vec{c} \\ \vec{y} \ge \vec{0} \end{array}$$

primal

max $4x_1 + 3x_2 + x_3 + x_4$

subj. to
$$x_1 + 2x_2 - x_4 \le 3$$
$$2x_1 + x_2 - x_3 + x_4 \le 2$$
$$x_2 + x_3 \le 2$$
$$x_1, x_2, x_3, x_4 \ge 0$$

dual

minimize $3y_1 + 2y_2 + 2y_3$

subject to
$$y_1 + 2y_2 \ge 4$$
$$2y_1 + y_2 + y_3 \ge 3$$
$$-y_2 + y_3 \ge 1$$
$$-y_1 + y_2 \ge 1$$
$$y_1, y_2, y_3 \ge 0.$$

For primal: initial dictionary

$n = 4, \quad m = 3.$

$$
\begin{array}{l}
z = 4x_1 + 3x_2 + x_3 + x_4 \\
\hline
x_5 = 3 - x_1 - 2x_2 + x_4 \\
x_6 = 2 - 2x_1 - x_2 + x_3 - x_4 \\
x_7 = 2 - x_2 - x_3
\end{array}
$$

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 1 & -1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$$\vec{b} = \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}$$

Run a simplex algorithm:

→ optimal final dictionary z^*

$$z = 10 - 0 \cdot x_1 - 2x_2 - 0 \cdot x_3 - x_4 - 0 \cdot x_5 - 2x_6 - 3x_7$$

$$
\begin{array}{l}
z = 10 - 2x_2 - x_4 - 2x_6 - 3x_7 \\
\hline
x_1 = 2 - x_2 - (1/2)x_4 - (1/2)x_6 - (1/2)x_7 \\
x_3 = 2 - x_2 - x_7 \\
x_5 = 1 - x_2 + (3/2)x_4 + (1/2)x_6 + (1/2)x_7
\end{array}
$$

slack variable!

$c^*_{4+1} = 0$

$c^*_{4+2} = -2$

$c^*_{4+3} = -3$

Primal optimal solution $\vec{x}^* = (\underbrace{2, 0, 2}_{\text{decision}}, \underbrace{0, 1, 0, 0}_{\text{slack}})$

$\vec{c}^* = (0, -2, 0, -1, \boxed{0, -2, -3})$

$$
\begin{cases}
y_1^* = 0 \\
y_2^* = -c^*_{4+2} = 2 \\
y_3^* = -c^*_{4+3} = 3
\end{cases}
$$

So $\vec{y}^* = (y_1^*, y_2^*, y_3^*) = (0, 2, 3)$ is optimal solution to the dual

Check. $\vec{b} \cdot \vec{y}^* = 10 = \vec{c}^T \vec{x}^*$.

$A^T \vec{y}^* = \begin{pmatrix} 4 \\ 5 \\ 1 \\ 2 \end{pmatrix} \geq \begin{bmatrix} 4 \\ 3 \\ 1 \\ 1 \end{bmatrix}$ So $A^T \vec{y}^* \geq \vec{b}$. !

And $\vec{y}^* \geq \vec{0}$.

$*$ primal $\xrightarrow{\text{dual}}$ dual $\xrightarrow{\text{dual}}$ primal

So, if you find an optimal dual solution (by simplex method) After changing it into a standard form

then you can use the final dictionary

to find the optimal primal solution.

- $\boxed{\text{Sometimes,}}$ e.g. # of constraints $\gg$ # of variables
  including $A\vec{x} \leq \vec{b}$.

  It is much easier to solve the dual problem.

Recall

- Possibilities for an LP.

Fundamental theorem of LP

$\Longrightarrow$ an LP has only three possibilities

feasible

unbounded

optimal.

infeasible

Roughly,

- Weak duality: "min of dual $\geq$ max of primal"
- Strong duality: "If optimal sol. for primal exists then,
  - min of dual = max of primal
  - optimal dual solution can be found using the final dictionary of the primal."

- possiblities for (primal, dual) pair

dual \ primal	finite optimum	unbounded	infeasible
finite optimum	Yes (strong duality)	NO	NO.
unbounded	NO	NO.	Yes
infeasible	NO	Yes (due to weak duality)	? ← possible.

- Exercise Find an example of LP where the primal and dual problem are both infeasible; Write down them in standard form, and justify your answer.

← Lec 11 (Tu Th) ✓