Two Ways of Seesaw Balancing under Disturbance

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ME314 Theory of Machines - Dynamics

***Abstract*—The task of balancing a seesaw is one form of the ball and beam system, which is a classical problem in control theory. Such a system is naturally unstable and nonlinear, which requires external control force to keep it balanced. There are two ways to perform seesaw balancing: 1) Controlling the seesaw tilting angle to keep the ball at the center. 2) Controlling the ball rolling back and forth to keep the seesaw balanced. Both ways are implemented and demonstrated in this project with the ball being replaced by a sliding block. The external control force is calculated by the feed-forward and feedback PID controller based on the dynamics system modeled by the Euler – Lagrangian approach. Also, a disturbance with impact on the seesaw is presented to challenge the control system.**

***Index Terms*—ball and beam system, PID controller, impact**

I. Introduction

A. The Original Proposal

The main task of the project is to balance a seesaw with a rolling polygon (The current plan is a nonagon). At the beginning, the nonagon will appear at the middle of the seesaw. Occasionally, there will be small lightweight mass dropping from the above and hitting the seesaw, which is a minor disturbance. The nonagon needs to move left and right to balance the seesaw. The plan is to use a PID controller to control the nonagon.

B. Changes Made to the Original Proposal

- In the original project proposal, the system consists of a rolling polygon ball on the seesaw. A polygon ball will have discrete contact points with the seesaw and therefore provide only discrete points for balancing the seesaw. This is not ideal for controlling a seesaw with a continuous tilting angle. Also, it adds more complexity to the project given only limited time. Therefore, it is replaced by a sliding block.
- The original six degrees of freedom system is reduced to five degrees of freedom by imposing constraint. The sliding block position was described by x and y positions, but now it is described by the position from the center position along the seesaw upper surface. More information is shown in section II.
- Other than controlling the block to balance the seesaw, controlling the seesaw to keep the ball at the center is also implemented.

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II. Setup

A. Generalized Coordinates

The generalized coordinates of this system are described as the following with the visualization in Fig. 1.

- The seesaw angle θ_{ss}
- The displacement of the sliding block from the center position z_b
- The x-position of the disturbance triangle x_d
- The y-position of the disturbance triangle y_d
- The rotation angle of the disturbance triangle θ_d

$$q = [\theta_{ss}, z_b, x_d, y_d, \theta_d] \tag{1}$$

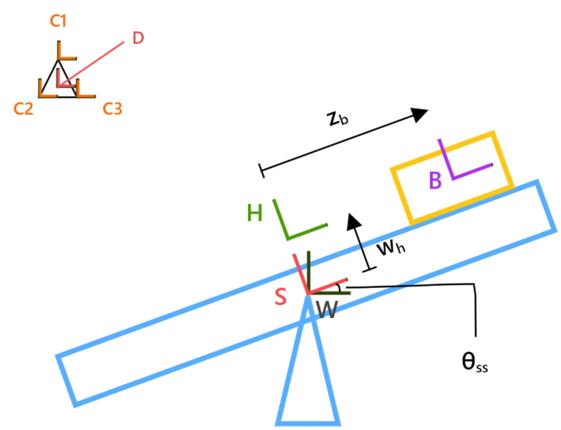

Fig. 1. Drawing of the system with frames and coordinates

B. Rigid Body Transformations

The rigid body transformations of this system are shown as the following with the visualization in Fig. 1.

- The seesaw frame S in the world frame W:

$$g_{WS} = \begin{bmatrix} cos(\theta_{ss}) & -sin(\theta_{ss}) & 0 & 0 \\ sin(\theta_{ss}) & cos(\theta_{ss}) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2}$$

- The sliding block frame B in the world frame W is calculated as the following. The variable b_h is the thickness of the sliding block:

$$g_{WB} = g_{WS} * g_{SH} * g_{HB} \tag{3}$$

$$g_{SH} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{w_h}{2} + \frac{b_h}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4}$$

$$g_{HB} = \begin{bmatrix} 1 & 0 & 0 & z_b \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{5}$$

- The disturbance triangle frame D in the world frame W:

$$g_{WD} = \begin{bmatrix} cos(\theta_d) & -sin(\theta_d) & 0 & x_d \\ sin(\theta_d) & cos(\theta_d) & 0 & y_d \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{6}$$

- The three corners frames of the disturbance triangle C_1, C_2, C_3 in the world frame W are calculated by the following equations. The variable d_{height} is the distance from the center of the triangle to the corner:

$$g_{WC_1} = g_{WD} * g_{DC_1} \tag{7}$$

$$g_{WC_2} = g_{WD} * g_{DC_2} \tag{8}$$

$$g_{WC_3} = g_{WD} * g_{DC_3} \tag{9}$$

$$g_{DC_1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & d_{height} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{10}$$

$$g_{DC_2} = \begin{bmatrix} 1 & 0 & 0 & -d_{height} * cos(pi/6) \\ 0 & 1 & 0 & -d_{height} * sin(pi/6) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{11}$$

$$g_{DC_3} = \begin{bmatrix} 1 & 0 & 0 & d_{height} * cos(pi/6) \\ 0 & 1 & 0 & -d_{height} * sin(pi/6) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{12}$$

III. Euler-Lagrange Equations

In order to calculate the Euler-Lagrange Equations, the Lagrangian of the system needs to be determined first, which is calculated by the kinetic energy and potential energy of each rigid body.

- The seesaw kinetic energy KE_{ss} and potential energy PE_{ss}:

$$\widehat{V_{Sb}} = g_{WS}^{-1} * \dot{g_{WS}} \tag{13}$$

$$KE_{ss} = \frac{1}{2} V_{Sb}^T I_{Sb} V_{Sb} \tag{14}$$

$$PE_{ss} = 0 \ \ (y_{ss} = 0) \tag{15}$$

- The sliding block kinetic energy KE_B and potential energy PE_B:

$$\widehat{V_{Bb}} = g_{WB}^{-1} * \dot{g_{WB}} \tag{16}$$

$$KE_B = \frac{1}{2} V_{Bb}^T I_{Bb} V_{Bb} \tag{17}$$

$$PE_B = m_b g_{gravity} (y \, of \, g_{WB}) \tag{18}$$

- The disturbance triangle kinetic energy KE_D and potential energy PE_D:

$$\widehat{V_{Db}} = g_{WD}^{-1} * \dot{g_{WD}} \tag{19}$$

$$KE_D = \frac{1}{2} v_{Db}^T I_{Db} v_{Db} \tag{20}$$

$$PE_D = m_d g_{gravity} (y \, of \, g_{WD}) \tag{21}$$

The Lagrangian of the system is:

$$L = (KE_{ss} + KE_B + KE_D) - (PE_B + PE_D) \tag{22}$$

The right hand side of the Euler-Lagrange Equations is the constraint and the external force. Since the position and x and y position of the sliding block are replaced by z_b, the sliding block will always attach to the seesaw upper surface, so the constraint has already been considered. The external force are the control force for the seesaw F_t and the sliding block Fz:

$$F_{ext} = \begin{bmatrix} F_t \\ F_z \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{23}$$

The Euler-Lagrange Equations can be calculated by:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = F_{ext} \tag{24}$$

IV. Impact

The impact of this system happens between the corners of the disturbance triangle and the seesaw upper surface. The impact condition can be determined by frame transformations. It is when corner frame is at y=0 in the E frame as shown in Fig. 2. The corner C3 will be used as an example, and all equations are also applied to C1 and C2.

$$g_{SE} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{w_h}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{25}$$

$$g_{EC_3} = (g_{WS} g_{SE})^{-1} g_{WD} g_{DC_3} \tag{26}$$

$$\phi = y \ \ of \ \ (g_{EC3}) \tag{27}$$

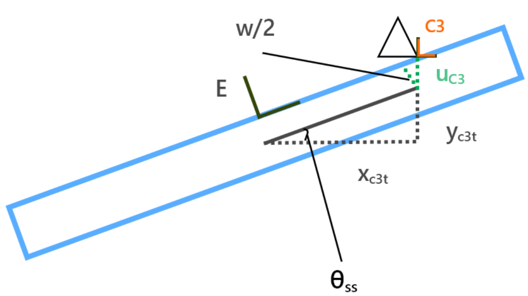

Fig. 2. Drawing of the impact location

However, possibly due to limited computation resources, the jupyter notebook kernel was dead with memory running at the maximum using this ϕ. In order to continue on the project, the calculation is attempted with basic geometry:

$$\phi = (y \ \ of \ \ g_{WC_3}) - y_{c3t} - u_{c3} \tag{28}$$

$$\phi = (y \ \ of \ \ g_{WC_3}) - x_{c3t} tan(\theta_{ss}) - \frac{\frac{w_h}{2}}{cos(\theta_{ss})} = 0 \tag{29}$$

Unfortunately, the kernel was still dead, so the equation is simplified with small angle approximation and ignore the thickness of the seesaw:

$$tan(\theta_{ss}) \approx \theta_{ss} \tag{30}$$

$$\phi = (y \ \ of \ \ g_{WC_3}) - x_{c3} * \theta_{ss} \tag{31}$$

With the simplification, the animation result still looks reasonable. The thickness part of the seesaw contributing to the impact detection $\frac{\frac{w_h}{2}}{cos(\theta_{ss})}$ is added to the position of the disturbance triangle in the animation. To identify if impact is happening, each step of the simulation will check if ϕ value is smaller than a threshold and if the x position of the triangle corner is within the span L of the seesaw. The two ends of the seesaw are determined by:

$$x_{border} = \pm \frac{L}{2} cos(theta_{ss}) \tag{32}$$

With ϕ being defined, the impact update law can be calculated by solving:

$$\left. \frac{\partial L}{\partial \dot{q}} \right|_{\tau^-}^{\tau^+} = \lambda \frac{\partial \phi}{\partial q} \tag{33}$$

$$\left[\frac{\partial L}{\partial \dot{q}} \cdot \dot{q} - L(q, \dot{q}) \right]_{\tau^-}^{\tau^+} = 0 \tag{34}$$

V. External Force

A. The Seesaw Balances the Block

In this case, the seesaw needs to rotate to keep the sliding block at the center position where $z_b = 0$. The seesaw tilted angle θ_{ss} is controlled by the external control force F_t. The external force for keeping the block at the center position is calculated by using one feedforward plus PID controller (f+PID) and one PD controller. The block diagram of the control system is in Fig.3.

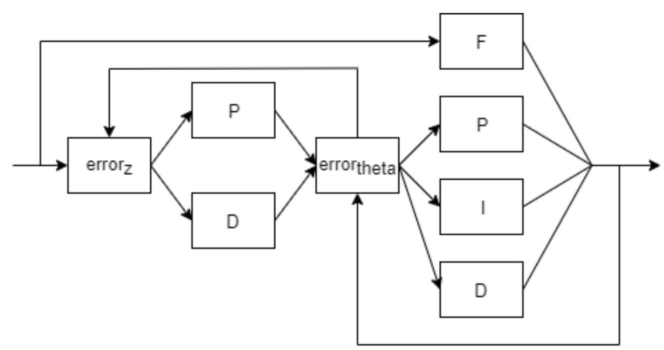

Fig. 3. Block diagram for seesaw balancing the block

First, the current position of the block $z_{b_{current}}$ will be obtained from the dynamic model to compare with the target $z_{b_{target}} = 0$. Therefore, the error is defined by:

$$error_{zb} = z_{b_{current}} \tag{35}$$

In this part, the required tilted angle $\theta_{ss_{target}}$ to keep the block at the center is calculated by the PD controller with the P and D gains showing in Table 1.

TABLE I
GAINS FOR THE PD CONTROLLER

P gain	D gain
0.05	0.005

In order to reach the $\theta_{ss_{target}}$, the f+PID controller will output the required force to rotate the seesaw based on the error. The error for the PID component is defined by:

$$error_{\theta} = \theta_{ss_{current}} - \theta_{ss_{target}} \tag{36}$$

The feedforward component of this controller allows the seesaw counter the weight of the block while rotating, which helps with the converging process. The PID gains of this controller are shown in Table 2.

TABLE II
GAINS FOR THE F+PID CONTROLLER

Feedforward	P gain	I gain	D gain
$15m_b$	48.0	30.1	20.0

B. The Block Balances the Seesaw

The block needs to move back and forth to keep the seesaw at the balance position where $theta_{ss} = 0$. The position of the sliding block z_b will be controlled by the external force F_z to balance the seesaw. Different from the seesaw balances the block case, the control system consists of a PID controller and a PD controller with a block diagram shown in Fig. 4.

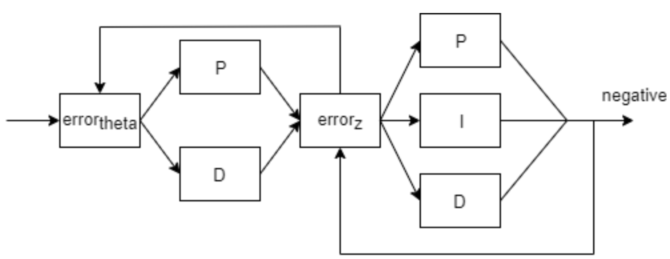

Fig. 4. Block diagram for block balancing seesaw

The system will need to first determine the difference between the current seesaw angle $\theta_{ss_{current}}$ and the target angle, which is zero. Therefore, the error can be defined as:

$$error_{\theta} = \theta_{ss_{current}} \tag{37}$$

Using the error as the input, the PD controller can calculate

TABLE III
GAINS FOR THE PD CONTROLLER

P gain	D gain
0.1	0.001

the required position $z_{b_{target}}$ to balance the seesaw for the sliding block with the gains in Table 3. Then the error of the sliding block position will become the input of the PID controller, which the error is defined as:

$$error_{z_b} = z_{b_{current}} - z_{b_{target}} \tag{38}$$

The PID controller will then output the force to move the sliding block. The output of the PID needs to be negated before it is applied to the sliding block. The gains of the PID controller are in Table 4.

TABLE IV
GAINS FOR THE PID CONTROLLER

P gain	I gain	D gain
25.5	35.5	0.5

VI. SIMULATION RESULT

A. Correctness of the Model

To prove the correctness of the model, different tests without external force are performed.

- Start the seesaw and the block at the equilibrium position without disturbance. It stays equilibrium.
- Start the seesaw at the equilibrium position and the block on the tip of the seesaw without disturbance. The seesaw rotates towards the block side (heavier side). The block also slides away in the correct direction.
- Start the seesaw and the block at the equilibrium position. Make the disturbance triangle hit one side of the seesaw. The seesaw rotates correspond to the impact, the block also slides away in the correct direction.
- Carefully inspect impact update from the animation. After impact, the disturbance triangle bounces at the upper surface of the seesaw, and it bounces away in a reasonable behavior. The seesaw is affected by the impact in the correct direction.
- Test different mass of the disturbance, the seesaw responds to different impact force with proper reaction.

B. The Seesaw Balances the Block

With the controller described in the previous section, the system is able to converge closely to the desired state, which is $z_b = 0$. It is tested with the initial state:

$$\begin{bmatrix} \theta_{ss} = 0.0 \\ z_b = -0.15 \\ x_d = 0.4 \\ y_d = 0.5 \\ \theta_d = 0.1 \end{bmatrix} \tag{39}$$

All initial velocities are zeros. The parameters of the system for testing are:

$$\begin{bmatrix} m_{ss} = 1.0 \\ Length_{ss} = 1.0 \\ Thickness_{ss} = 0.1 \\ m_b = 0.1 \\ Thickness_b = 0.1 \\ m_d = 0.05 \\ d_{height} = 0.05 \end{bmatrix} \tag{40}$$

The simulation ran for 40 seconds with a step size of 0.01 second. The step response plot for the block position z_b is shown in Fig. 5. Even with disturbance at the beginning, the

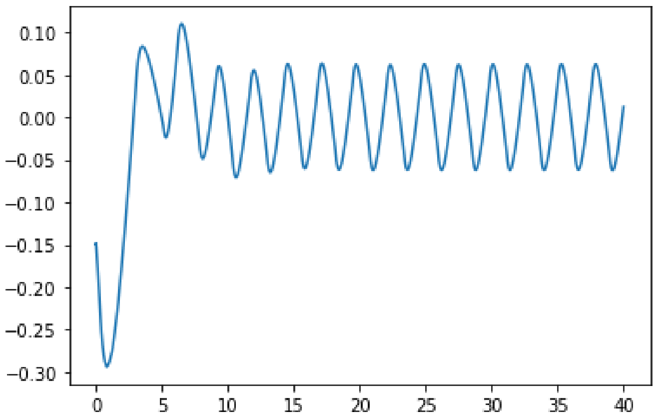

Fig. 5. Step response for the block position

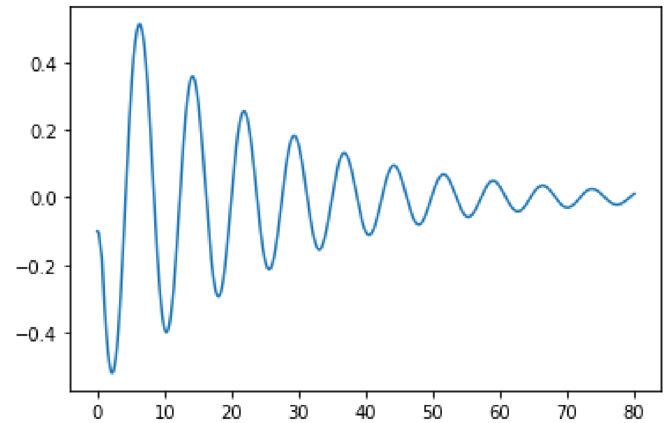

Fig. 6. Step response for the seesaw angle

seesaw can keep the block closed to the target zero position after about 10 seconds. Although the step response shows a lot of small oscillations ± 0.05, they can be reduced by more tuning in the future.

C. The Block Balances the Seesaw

The block can also successfully balance the seesaw, but the converging process is very slow. Also, the allowed initial state is limited compared to the case of seesaw balancing the block. The reason behind these phenomenons could be that the mass of the block is much smaller than the seesaw, so it is more difficult for the block to affect the seesaw. A test has been performed with the following initial state and parameters:

$$\begin{bmatrix} \theta_{ss} = -0.1 \\ z_b = 0.0 \\ x_d = 0.3 \\ y_d = 0.3 \\ \theta_d = 0.1 \end{bmatrix} \tag{41}$$

$$\begin{bmatrix} m_{ss} = 1.0 \\ Length_{ss} = 1.0 \\ Thickness_{ss} = 0.1 \\ m_b = 0.1 \\ Thickness_b = 0.1 \\ m_d = 0.01 \\ d_{height} = 0.05 \end{bmatrix} \tag{42}$$

The disturbance triangle is lighter in this case but it still affects the system. The simulation ran for 80 seconds with a step size of 0.01 second. The step response for the seesaw tilted angle θ_{ss} is shown in Fig. 6. The step response plot clearly shows the converging process even though there are oscillations. More tuning should be performed in the future.

D. Animation Movie

There are three parts in the movie

1) The system without any control force but with disturbance
2) The seesaw balances ball with control force under disturbance
3) The ball balances the seesaw with control force under disturbance

In the animation movie, some parts of the movie are lengthy simulation, so they are sped up. I was not sure how to speed up the animations in plotly. The speed-up parts are labeled.

VII. Supplementary

https://youtu.be/JXpEqwk_3M0

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Thanks to Professor Murphey, Katie, Conor, and Muchen for making this class working well in the online setting. I have learned a lot.