# Math 6150: PDE 1

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**Course Description:** This course emphasizes the "classical" aspects of partial differential equations (PDEs) — analytic methods for linear second-order PDEs and first-order nonlinear PDEs — without relying on more modern tools of functional analysis.

We will mostly follow the Partial Differential Equations by L.C. Evans. I intend to cover in detail representation formulas for linear PDEs [Chapter 2] and the basic theory for first-order non-linear equations (starting with the method of characteristics and concentrating on Hamilton-Jacobi PDEs and conservation laws) [Chapter 3]. Time permitting, we will also discuss assorted topics from Chapter 4 (similarity solutions, transform methods, asymptotics, power series, homogenization), Chapter 8 (calculus of variations), and/or Chapter 10 (control theoretic interpretation of Hamilton-Jacobi PDEs).

Some notes are borrowed from classmates (Kevin) for lectures I missed.

# Contents

# 1 Introduction (8/29/23)

**What are they?** Functional equations (solution is a function) involving an unknown function (e.g., $u(x)$) and its partial derivatives that has to be satisfied at every point $x$ in the domain of interest. Often accompanied by some "boundary conditions" (e.g., $u$ or some of its derivatives prescribed on a part of the domain's boundary.)

**Notation/terminology**

- domain $U$ is an open subset of $\mathbb{R}^2$
- $u : \underbrace{U}_{\text{or } \bar{U}} \to \mathbb{R}$ is the unknown function (of $x = (x_1, \ldots, x_n) \in U \subset \mathbb{R}^n$)

**Partial Derivatives:**

$$u_{x_i}(x) = \frac{\partial u}{\partial x_i}(x) = \lim_{\epsilon \to 0} \frac{u(x + \epsilon \mathbf{e}_i) - u(x)}{\epsilon}$$

where $\mathbf{e}_i$ is the $i$-th canonical unit vector.

**Gradient:**

$$Du(x) = \begin{bmatrix} u_{x_1}(x) \\ \vdots \\ u_{x_n}(x) \end{bmatrix}, \quad Du : U \to \mathbb{R}^2$$

**Hessian:**

$$D^2u(x) = \begin{bmatrix} u_{x_1x_1} & \cdots & u_{x_1x_n} \\ \cdots & \cdots & \cdots \\ u_{x_nx_1} & \cdots & u_{x_nx_n} \end{bmatrix}(x), \quad D^2i : U \to \mathbb{R}^{n^2}$$

**Laplacian:**

$$\Delta u(x) = \sum_{i=1}^{n} u_{x_ix_i}(x) = \operatorname{tr}(D^2u)(x)$$

Order of PDE is the order of the highest derivative used in it.

## 1.1 Classification (increasing order of complexity)

A PDE is *linear* if it is a linear equation in terms of $u$ and all participating derivatives. (Note: the coefficients can still be non-linear functions of $x$!)

*Example* 1.1. A linear first order PDE

$$e^{x_2}u_{x_1} + (x_1^2 - x_2)u_{x_2} + (\tan x_1)u = 1$$

A PDE is *semi-linear* if it is linear in terms of the highest participating derivatives and the coefficients in front of them depend on $x$.

*Example* 1.2. Semi-linear, 2nd order PDE

$$\cos(x_2)u_{x_1x_2} + u_{x_1}u_{x_2} - u^2 = e^{x_1}$$

A PDE is *quasi-linear* if it is linear in terms of the highest participating derivatives, but the coefficients in front of them also depend of the lower-order derivatives and/or $u$.

*Example* 1.3. A quasi-linear, 1st-order

$$u_{x_1} + u^2 u_{x_2} = 0$$

A PDE is *fully non-linear* if it is non-linear in terms of (at least one of) the highest-order participating derivative.

*Example* 1.4. Fully non-linear, 2nd-order

$$(u_{x_1 x_2})^2 = u_{x_1} + u_{x_2} + u$$

*Example* 1.5. Eikonal equation PDE (1st order, fully non-linear)

$$(u_{x_1})^2 + (u_{x_2})^2 = k^2(x)$$

Evans uses a *multi-index notation* for (higher) derivatives: $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index if $\alpha_i \in \mathbb{N}v\{0\}$ $\forall i$. $\alpha_i$ is how many derivatives are take with respect to $x_i$.

*Example* 1.6. If $U \subset \mathbb{R}^2$ and $\alpha = (1, 2, 1)$ then $D^2 u = u_{x_1 x_2 x_2 x_3} = u_{x_3 x_2 x_1 x_2} = \ldots$ (ordering does not matter if $u$ is smooth.)

$|\alpha| \coloneqq \alpha_1 + \cdots + \alpha_n$, (the order of this derivative). $D^k u$ is the set of all partial derivatives of order $k \in \mathbb{N}$. (all $D^\alpha u$ such that $|\alpha| = k$). This can also be viewed as a point in $\mathbb{R}^{n^k}$ (or as a tensor).

**Most general form:** A $k$-th order PDE is an expression of the form

$$G(D^k u(x), D^{k-1} u(x), \ldots, Du(x), u(x), x) = 0, \quad \forall x \in U \subset \mathbb{R}^n,$$

where $G$ is a known function $G : \mathbb{R}^{n^k} \times \mathbb{R}^{n^{k-1}} \times \cdots \times \mathbb{R}^2 \times \mathbb{R} \times U \to \mathbb{R}$.

- Can restate the above classification in terms of $G$.
- Can also extend this definition to systems of PDEs. [1]
- If $u$ is "time-dependent", you can always view time $t$ as one of the components of $x$.

"Well-posedness" is what we hope for in PDE problems. Three parts of well-posedness: [2]

1. existence of a solution (in what class of functions?)
2. uniqueness of that solution (satisfying not just the PDE, but also some affitional criteria?)
3. continuous dependence on problem data (at least on a part of the domain? or with problem data satisfying additional conditions?)

Well-posedness can let us build approximations.

**Even just the existence can be problematic.** Do we have any reason (or moral right) to expect a solution to be real-analytic or even "just" infinitely-differentiable (in $C^\infty(U)$)? A more modest expectation: to have all participating derivatives well-defined and continuous (e.g., $u \in C^k(U)$). This is called a *classical solution*.

[1] See both of these in section 1.1 and many examples in 1.2 of Evans.

[2] Noble aspirations written, dirty tricks and compromises in parens

Alas, still not guaranteed for many important types of problems (e.g.g, Hamilton-Jacobi, conservation laws). Will need to define a notion of *generalized/weak solutions*, with (application-specific) criteria imposed to ensure uniqueness.

Keeping track of where PDEs came from is not just fun (making the subject more lively), but also very useful in

- figuring out what to prove,
- finding with *weak solutions* might have practival relevance, and
- deciding when to give up on a PDE model that produces nonsense despite its cool mathematical properties.

*Example* 1.7. Deriving PDEs from applications: consider a class of "Continuity Equations". $x \in \mathbb{R}^2,\ t \in \mathbb{R},\ \tilde{x} = (x, t) \in \mathbb{R}^{n+1}$. $u(x, t)$ is the density of stuff. Let's look at the flux $F(x, t, u, D_x u, ??)$ as the net flux of that stuff (flow rate per unit area). $F : ?? \to \mathbb{R}^n$.

1.
$$F = -aDu,\ a > 0 \tag{1.1}$$
   (a) $u$ is chemical concentration. Equation 1.1 is Fick's Law of diffusion
   (b) $u$ is temperature. Equation 1.1 is Fourier's Law of heat conduction
   (c) $u$ is electrostatic potential. Equation 1.1 is Ohm's law of electrical conduction

2.
$$F = v(x, t, u)u(x, t),$$
   where $v$ is the velocity of individual (possibly interacting from $u$-dep) particles. This is the conservation of particle density $u$.

Choose an arbitrary compact set $V \subset \mathbb{R}^n$ as piecewise-smooth boundary $\partial V$ and define $\hat{u}(t) =$ the total amount of "stuff" inside $V$ at the time $t$:

$$\hat{u}(t) = \int_V u(x, t)\,\mathrm{d}x.$$

Supposing for simplicity that $u, u_t \in C(\mathbb{R}^2 \times \mathbb{R}$, which allows me to change order of differentiation of integration. Let $|\nu| = 1$ and $\nu \perp \partial V$.

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{u}(t) \underbrace{=}_{\text{Leibniz integral rule}} \int_V u_t(x, t)\,\mathrm{d}x = -\int_{\partial V} \nu(y) \cdot F(y)\,\mathrm{d}S(y),$$

where $\nu(y)$ is the unit normal to $\partial S$, $F(y)$ is the flux, $y$ is a point on $\partial V$, $\nu \cdot F$ represents the outward flow rate through $\partial V$ at $y$ per unit $\partial V$ surface area, and $dS(y)$ is the "area" of infinitesimal part of $\partial V$ centered at $y$. This equation is essentially telling us that the stuff only leaves from/comes to $V$ through $\partial V$.

Then from the divergence

$$(F(x, \dots) \in \mathbb{R}^n, \text{ and } \operatorname{div}(F) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} F_i(x, \dots) = D_x \cdot F)$$

theorem,

$$\int_V \operatorname{div}(F)\,\mathrm{d}x = \int_{\partial V} \nu \cdot F\,\mathrm{d}S,$$

we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{u}(t) = \int_V u_t(x,t)\,\mathrm{d}x = -\int_{\partial V} \nu(y)\cdot F(y)\,\mathrm{d}S(y) = -\int_V \operatorname{div}(F(x))\,\mathrm{d}x.$$

Since we now have 2 integrals with the same boundary, we can combine them and say

$$\int_V \underbrace{u_t(x,t) + \operatorname{div}(F(x,\dots))}_{\text{must be 0 to make this true } \forall t, V}\,\mathrm{d}x = 0,$$

and note continuity of integrand makes integrand 0. This gives us the *continuity equation*

$$u_t + \operatorname{div}(F) = 0. \tag{1.2}$$

*Remark* 1.8. Divergence theorem is just generalization of FTC.

$$\int_a^b f'(x)\,\mathrm{d}x = f(b) - f(a)$$

If $F = -aDu$, where $a > 0$, then $\operatorname{div}(F) = -a\sum_{i=1}^n \frac{\partial}{\partial x_i} u_{x_i} = -a\Delta u$ and Equation 1.2 becomes a *heat equation:* $u_t - a\Delta u = 0$. A "steady-state version" (when $u_t = 0$) yields *Laplace's equation:* $\Delta u = 0$. A non-homogeneous variant of the latter is called *Poisson's equation:* $\Delta u = f(x)$.

*What physical problem would have it as a steady-state solution? E.g. try to explain it in terms of chemical diffusion.* If $F = v(x, y, u)u$, where $v$ is velocity of particles, the Equation 1.2 becomes a statement about the conservation of particles.

- The simplest case: $F = bu$ for a constant speed $b \in \mathbb{R}^n$, yields a *transport equation* (aka advection equation)

$$u_t + b \cdot Du = 0.$$

- A more interesting case: $F(u) = V(u)u$ yields a *hyperbolic conservation law*

$$u_t + F'(u) \cdot Du = 0.$$

- If $n = 1$, it can be used to model the density of cars on a highway; e.g. with

$$v(u) = v_{\max}\left(1 - \frac{u}{u_{\max}}\right)$$

which is known as the "Lighthill-Whitham-Richards" model of car speed.

# 2 (8/31/23)

## 2.1 Transport Equation

From last time this is $F = bu$, gives us

$$u_t + b \cdot Du = 0, \text{ const } b \in \mathbb{R}^n \tag{2.1}$$

with $u(x,t) = u(y)$, $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, $y = (x,t) \in \mathbb{R}^{n+1}$. The lhs of the equation looks like a directional derivative

$$D_y u(y) = \begin{bmatrix} D_x u(x,t) \\ u_t(x,t) \end{bmatrix}$$

then our equation looks like

$$D_y u \cdot \begin{bmatrix} b \\ 1 \end{bmatrix} = 0.$$

More formally. Define a straight line in the time space $y(s) = (x + bs, t + s)$ and a scalar function $z(s) = u(y(s))$. Then use chain rule

$$\dot{z}(s) = \frac{\mathrm{d}}{\mathrm{d}s} z(s) = D_x u(x + bs, t + s) \cdot b + u_t(x + bs, t + s) \cdot 1 = 0$$

since $u$ is assumed to be a solution of Equation 2.1, then it equals 0 so then $z$ is constant. Assume $u \in C^1(\mathbb{R} \times \mathbb{R})$. Then we have

$$u(x,t) = z(0) = z(-t) = u(x - bt, 0) = g(x - bt).$$

**Initial Value Problem (IVP).** $u = g$ on $\mathbb{R}^n \times \{t = 0\}$, and Equation 2.1 holds on $\mathbb{R}^n \times (0, \infty)$. This classical solution only exists if $g$ is smooth.

**Nonhomogeneous.** Let's consider the nonhomogeneous version of the equation

$$u_t + b \cdot D_x u = f(x,t),$$
$$D_y u \cdot \begin{bmatrix} b \\ 1 \end{bmatrix} = f(y).$$

*Remark* 2.1. (What is the physical meaning of being able to get the nonhomogenous version by changing the diffusion equation? Answer: $f$ is telling us the rate chemicals are being added at a specific location. We can see this by making $b = 0$, then $u_t = f(x,t)$).

Then the change is now is

$$\dot{z}(s) = \frac{\mathrm{d}}{\mathrm{d}s} z(s) = D_x u(x + bs, t + s) \cdot b + u_t(x + bs, t + s) \cdot 1 = f(x + sb, t + s).$$

We can convert to integral by fundamental theorem of calculus and change of variable ($\tilde{s} = t + s, \ \mathrm{d}\tilde{s} = \mathrm{d}s$)

$$z(0) - z(-t) = \int_{-t}^{0} \dot{z}(s) \, \mathrm{d}s = \int_{-t}^{0} f(x + bs, t + s) \, \mathrm{d}s = \int_{0}^{t} f(x + b(\tilde{s} - t), \tilde{s}) \, \mathrm{d}\tilde{s}.$$

Then we have the solution

$$u(x,t) = g(x - bt) + \int_0^t f(x + (\tilde{s} - t)b, \tilde{s}) \, \mathrm{d}\tilde{s}.$$

This was the simplest hyperbolic PDE.

## 2.2 Elliptic PDE

**Laplace's Equation.**

$$\Delta u = \sum_{u=1}^{n} u_{x_i x_i}(x) = 0, \tag{2.2}$$

where $x \in \mathbb{R}^n,\ u : \bar{U} \to \mathbb{R}$, and open $U \subset \mathbb{R}^n$. Solutions are called *harmonic*.

**Poisson's Equation.**

$$-\Delta u = f(x) \tag{2.3}$$

**Definition 2.2.** *Superposition principle.* For linear homogeneous equations: If $u_1, u_2$ are solutions, then so is also $\beta_1 u_1 + \beta_2 u_2\ \forall \beta_1, \beta_2 \in \mathbb{R}$. For non-homogeneous case: true if $\beta_1 + \beta_2 = 1$.

**Proposition 2.3.** *For Laplace's: a rotated or "shifted" solution is still a solution.*

Look for special solutions of Equation 2.2 that are rotationaly invariant, meaning

$$u(x) = V(r(x));\ r(x) = \sqrt{\sum_{i=1}^{n} x_i^2},\ V : (0, \infty) \to \mathbb{R}.$$

Assume $u \in C^2(\mathbb{R}^n \setminus \{0\})$. We want to put this into Equation 2.2. First we get the derivatives as

$$\begin{aligned} \frac{\partial r}{\partial x_i} &= \frac{x_i}{r(x)}, \\ \frac{\partial^2 r}{\partial x_i^2} &= \left( \frac{1}{r(x)} + x_i \frac{-1}{r(x)^2} \frac{x_i}{r(x)} \right) = \left( \frac{1}{r(x)} - \frac{x_i^2}{r(x)^3} \right). \end{aligned}$$

Next,

$$\begin{aligned} u_{x_i} &= V'(r(x)) \frac{x_i}{r(x)}, \\ u_{x_i x_i} &= V''(r) \frac{x_i^2}{r^2} + V'(r) \left( \frac{1}{r} - \frac{x_i^2}{r^3} \right). \end{aligned}$$

Now we can write Equation 2.2 as

$$\Delta u = \sum_{i=1}^{n} u_{x_i x_i} = V''(r) \underbrace{\sum_{i=1}^{n} \frac{x_i^2}{r^2}}_{=1} + V'(r) \underbrace{\sum_{i=1}^{n} \left( \frac{1}{r} - \frac{x_i^2}{r^3} \right)}_{\frac{n-1}{r}} = V''(r) + V'(r) \frac{n-1}{r} = 0. \tag{2.4}$$

We can manipulate this to get something that looks like the derivative of a log

$$\left( \ln |V'| \right)' = \frac{V''}{V'} = \frac{1-n}{r}.$$

Solving,

$$\begin{aligned} \ln |V'| &= (1-n) \ln r + K, \\ V' &= r^{1-n} \underbrace{\left( \pm e^K \right)}_{c}, \\ V' &= c r^{1-n}. \end{aligned}$$

Consider the cases

$$V(r) = \begin{cases} cr + b, & n = 1 \\ c \ln r + b, & n = 2 \\ cr^{2-n} + b, & n \geq 3. \end{cases}$$

Notice that the solutions for $n \geq 2$ blow up for $r \to 0$. That is why we usually exclude origin when talking about nice properties of $u$.

Next, we are going to describe the fundamental solutions for particular choice of constants $c, v$.

**Definition 2.4.** The *fundamental solution* to Laplace's equation is

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi} \ln |x|, & n = 2 \\ \frac{1}{n(n-2)\alpha(n)} |x|^{2-n}, & n \geq 3 \end{cases}$$

Evan's uses

$$\alpha(n) = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2} + 1\right)},$$

which is the volume of a unit ball. We take $\Delta\Phi(x) = 0\ \forall x \neq 0$ and $\Phi(x) \to \infty$ as $x \to 0$.

Integrability of $\Phi(x)$ on compact sets ($n = 2$).

$$\begin{aligned} \int_{B(0,1/\delta)} \Phi(x)\, \mathrm{d}x = \frac{-2\pi}{2\pi} \int_0^{1/\delta} (\ln r) r\, \mathrm{d}r &= \lim_{\epsilon \to 0^+} \int_\epsilon^{1/\delta} (-\ln r) r\, \mathrm{d}r \\ &= \lim_{\epsilon \to 0^+} \left( \frac{r^2}{4} - \frac{1}{2} r^2 \ln r \right) \Bigg|_\epsilon^{1\delta} \\ &= \frac{\delta^2}{4} + \frac{1}{2} \delta^{-2} \ln \delta - \lim_{\epsilon \to 0^+} \Bigg[ \underbrace{\frac{\epsilon^2}{4}}_{\to 0} - \underbrace{\frac{1}{2} \epsilon^2 \ln \epsilon}_{\to 0 \text{ by LHopital}} \Bigg], \\ &= \frac{1}{\delta^2} \left[ \frac{1}{4} + \frac{1}{2} \ln \delta \right]. \end{aligned}$$

We used integration by shells in $\mathbb{R}^n$. Look at appendix C. This blows up if $\delta \to 0^+$ but ok if $\delta$ is finite. This means that $\Phi$ is "totally integrable."

# 3 (9/5/23) Properties of Harmonic Functions

Things to review:

- What does it mean to blow up in different dimension?
- Rules to interchange order of operators

*Remark* 3.1. Integrating in "polar" coordinates in $\mathbb{R}^n$

$$\int_{\mathbb{R}^n} f(x)\, \mathrm{d}x = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, \ldots, x_n)\, \mathrm{d}x_1 \ldots \mathrm{d}x_n = \int_0^\infty \left[ \int_{\partial B(x_0, r)} f(y)\, \mathrm{d}S(y) \right] \mathrm{d}r.$$

If $f(x)=f(r)$, then $\int_{\mathbb{R}^n} f\,\mathrm{d}x=\int_0^\infty f(r)\left[\int_{\partial B(x_0,r)}\,\mathrm{d}S(y)\right]\,\mathrm{d}r=\int_0^\infty f(r)r^{n-1}\left[\int_{\partial B(x_0,1)}\,\mathrm{d}S(y)\right]=n\alpha(n)\int_0^\infty f(r)r^{n-1}\,\mathrm{d}r$. Use Divergence Theorem with $F(x)=x$.

$$\int_{\partial B(0,1)}\,\mathrm{d}S=\int_{\partial B(0,1)} F\cdot\nu\,\mathrm{d}S=\int_{B(0,1)}\operatorname{div}(F)\,\mathrm{d}x=n\int_{B(0,1)}\,\mathrm{d}x=n\alpha(n).$$

Back to testing integrability of $\Phi(x)$ with $n=2$. (See above). Also true for $n\geq 3$

$$\begin{aligned}\int_{B(0,1/\delta)}\Phi(x)\,\mathrm{d}x&=\int_{B(0,\delta^{-1})}\frac{1}{n(n-2)\alpha(n)}\frac{1}{|x|^{n-1}}\,\mathrm{d}x=\frac{n\alpha(n)}{n(n-2)\alpha(n)}\int_0^{1/\delta}\frac{1}{r^{n-2}}r^{n-1}\,\mathrm{d}r\\&=\frac{1}{n-2}\int_0^{1/\delta} r\,\mathrm{d}r=\frac{1}{n-2}\frac{1}{2}\left(\frac{1}{\delta}\right)^2.\end{aligned}$$

So, integrable on any bounded $U\subset\mathbb{R}^n$.

Please also check: $|D\Phi(x)|\leq c/|x|^{n-1}$ and $|D^2\Phi(x)|\leq c/|x|^n$ $\forall x\neq 0$. The notation $D^2u=\{D^\alpha u:|\alpha|=2\}=\{u_{x_ix_j}:i,j=1,\dots,n\}$. Where

$$\left|D^k u\right|:=\left(\sum_{\alpha\text{ s.t. }|\alpha|=k}(D^\alpha u)^2\right)^{1/2}.$$

Since the shifts/translations of harmonic are harmonic and linear combination of harmonic are harmonic, then the following misleading arguments might seem credible

1. $\forall y\in\mathbb{R}^n,\ \Phi(x-y)$ is a harmonic function
2. $\forall y,\ y_2\in\mathbb{R}^n$ and $\beta_1,\beta_2\in\mathbb{R}$ then $(\beta_1\Phi(x-y_1)+\beta_2\Phi(x-y_2))$
3. Similarly works $\forall y_1,\dots,y_r\in\mathbb{R}^n$
4. Conjecture: $u(x):=\int_{\mathbb{R}^n} f(y)\Phi(y-x)\,\mathrm{d}y$ should be harmonic.

$$\Delta u(x)\overset{?}{=}\int_{\mathbb{R}^n} f(y)\underbrace{\Delta_x\Phi(x-y)}_{=0}\,\mathrm{d}y=0.$$

This, however, is WRONG since $\Delta\Phi=O(r^{-n})$ is not integrable!

In fact, that $u(x)$ solves Poisson's equation on $\mathbb{R}^n$!

**Theorem 3.2.** *Suppose $f\in C^2(\mathbb{R}^n)$ and has compact support; i.e. $\exists$ compact $U_f\subset\mathbb{R}^n$ such that $f(x)=0$ $\forall x\notin U_f$. Then,*

1. *$u(x)=\int_{\mathbb{R}^n} f(y)\Phi(y-x)\,\mathrm{d}y$ is $C^2(\mathbb{R}^n)$ and*
2. *$-\Delta u(x0=f(x)$ $\forall x\in\mathbb{R}^n$*

*Proof.* **1.** First, convince yourself that $\int_{\mathbb{R}^n} f(y)\Phi(y-x)\,\mathrm{d}y=\int_{\mathbb{R}^n} f(y-x)\Phi(y)\,\mathrm{d}y$. Consider

$$\begin{aligned}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f(y)\Phi(x-y)\,\mathrm{d}y_1,\dots\mathrm{d}y_n\underbrace{=}_{z=x-y}&\int_{\infty}^{-\infty}\cdots\int_{\infty}^{-\infty} f(x-z)\Phi(z)(-\,\mathrm{d}z_1)\dots(-\,\mathrm{d}z_n)\\=&\int_{\mathbb{R}^n} f(x-z)\Phi(z)\,\mathrm{d}z.\end{aligned}$$

**Continuity:** $u \in C(\mathbb{R}^n)$,

$$|u(x) - u(z)| = \left| \int_{\mathbb{R}^n} \Phi(y) \left(f(x-y) - f(z-y)\right) \mathrm{d}y \right| \le \int_{\mathbb{R}^n} |\Phi(y)| \underbrace{|f(x-y) - f(z-y)|}_{A \to 0 \text{ as } |x-z| \to 0} \mathrm{d}y$$

Since $U_f$ is compact, $f$ is *uniformly continuous*, (i.e. $\forall \epsilon > 0\ \exists \delta > 0$ s.t. $|\xi_1 - \xi_2| < \delta \implies |f(\xi_1) - f(\xi_2)| < \epsilon\ \forall \xi_1, \xi_2 \in U_f$ (or $\mathbb{R}^n$).)

Take $\tilde{U} = \{\xi \in \mathbb{R}^n : \mathrm{dist}(\xi, U_f) \le 1\}$. Let $|x - z| < 1$. Then

$$y \notin (x + \tilde{U}_f) \implies \begin{cases} (x-y) \notin \tilde{U}_f \\ (z-y) \notin U_f \end{cases} \implies f(x-y) = 0 = f(z-y).$$

If $\forall \epsilon > 0\ \exists \delta > 0$ such that $|x - z| < \delta \implies A < \epsilon/x$. Using $\int_{B(0,S)} \Phi(y) = \frac{1}{2(n-2)} s^2 = C, \quad (s = 1/\delta$ in past)),

$$\int_{B(0,S) \supset \tilde{U}_f} |\Phi(y)| \underbrace{|f(x-y) - f(z-y)|}_{A} \mathrm{d}y \le \frac{\epsilon}{x} \underbrace{\int |\Phi(y)| \mathrm{d}y}_{c} = \epsilon.$$

**$u$ is $C^1$.**

Use finite difference definition,

$$\lim_{h \to 0^+} \frac{u(x + h\mathbf{e}_i) - u(x)}{h} = \int_{\mathbb{R}^n} \Phi(y) \underbrace{\lim_{h \to 0^+} \frac{f(x + h\mathbf{e}_i - y) - f(x-y)}{h}}_{f_{x_i}(x-y)} \mathrm{d}y$$

Justify exchange of limit and integral. It follows from Lebesgue's Dominatated Convergence Theorem.

**Theorem 3.3.** Lebesgue's Dominatated Convergence. *If $\forall \psi_k, \psi, g$ are integrable and $\psi_k \to \psi$ almost everywhere and $|\psi_k| \le g$ almost everywhere,*

$$\int_{\mathbb{R}^n} \psi_k \, \mathrm{d}x \to \int_{\mathbb{R}^n} \psi \, \mathrm{d}x$$

Let our finite difference be $L(x, y, h_k)$. Then $\psi_k = \Phi(y) L(x, y, h_k)$. We need to know if this is dominated by some $g$. Let $g(y) = \Phi(y)(\|f_{x_i}\|_{L^\infty} + 1)$. So, we can get a continuous derivative.

**Diffs for $u_{x_i x_j}$.**

Similar setup with the limit of a finite difference. Skipped details for the sake of time.

**2.**

$$u(x) = \int_{\mathbb{R}^n} \Phi(y) f(x-y) \, \mathrm{d}y = \int_{B(0,\epsilon)} \Phi(y) f(x-y) \, \mathrm{d}y + \int_{\mathbb{R}^n \setminus B(0,\epsilon)} \Phi(y) f(x-y) \, \mathrm{d}y.$$

Now hit with Laplacian,

$$\Delta u(x) = \underbrace{\int_{B(0,\epsilon)} \Phi(y) \Delta_x f(x-y) \, \mathrm{d}y}_{I_\epsilon} + \underbrace{\int_{\mathbb{R}^n \setminus B(0,\epsilon)} \Phi(y) \Delta_x f(x-y) \, \mathrm{d}y}_{J_\epsilon}.$$

First show

$$|I_\epsilon| \le c\left\|D^2 f\right\|_{L^\infty(\mathbb{R}^n)} \int_{B(0,\epsilon)} |\Phi(y)|\,\mathrm{d}y \le \begin{cases} \tilde{c}\epsilon^2 \ln\epsilon, & n=2, \\ \tilde{c}\epsilon^2, & n \ge 3. \end{cases} \to 0 \text{ as } \epsilon \to 0.$$

Next, for $J_\epsilon$, we can say that $\Delta_x$ is the same operation as $\Delta_y$ since the negative from twice derivatives will cancel. Also use integration by parts ($\int_a^b u'v\,\mathrm{d}x = -\int_a^b uv'\,\mathrm{d}x + uv|_a^b$). Let $V_\epsilon = \mathbb{R}^n \setminus B(0,\epsilon)$.

$$J_\epsilon = \int_{\mathbb{R}^n \setminus B(0,\epsilon)} \Phi(y)\Delta_y f(x-y)\,\mathrm{d}y = -\int_{V_\epsilon} D\Phi(y)\cdot D_y f(x-y)\,\mathrm{d}y = \int_{\partial V_\epsilon} \Phi(y)\frac{\partial f}{\partial \nu}(x-y)\,\mathrm{d}S(y).$$

TODO: Go through integration by parts in $\mathbb{R}^n$.

# 4 (9/7/23)

*Remark* 4.1. Assume $U$ has a piecewise-smooth boundary, $\bar{U}$ is compact and $u \in C^1(U)$. Then from Gauss-Green theorem

$$\int_U u_{x_i}\,\mathrm{d}x = \int_{\partial V} u\nu^i\,\mathrm{d}S,$$

where $\nu^i$ is the $i$-th component of the outward pointing unit vector normal to $\Omega$.

*Proof.* This is true using divergence theorem. Letting $F(x) = (0,\ldots,0,u,0,\ldots,0)$, where $F_i(x) = u(x)$, then let

$$\int_U \underbrace{\operatorname{div} F}_{u_{x_i}}\,\mathrm{d}x = \int_{\partial U} \underbrace{F(y)\cdot\nu(y)}_{u\nu^i}\,\mathrm{d}S(y).$$

□

Let $u, v \in C^1(\nu)$ and use first eqn on $w = uv$.

$$\int_U u_{x_i} v\,\mathrm{d}x = -\int_U u v_{x_i}\,\mathrm{d}x + \int_{\partial U} uv\nu^i\,\mathrm{d}S.$$

Assuming $v \in C^2(U)$, now use the equation above on $v_{x_i}$ instead of $v$ and sum over $i = 1,\ldots,n$ :

$$\int_U Du\cdot Dv\,\mathrm{d}x = -\int_U u\Delta v\,\mathrm{d}x + \int_{\partial U} u(Dv\cdot\nu)\,\mathrm{d}S.$$

Back to the proof of $-\Delta u = f$.

$$\begin{aligned} \Delta u(x) &= \int_{\mathbb{R}^n} \Phi(y)\Delta_x f(x-y)\,\mathrm{d}y, \quad \text{justified since } f \in C^2 \\ &= \underbrace{\int_{B(0,\epsilon)} \Phi(y)\Delta_x f(x-y)\,\mathrm{d}y}_{I_\epsilon} + \underbrace{\int_{\mathbb{R}^n\setminus B(0,\epsilon)} \Phi(y)\Delta_x f(x-y)\,\mathrm{d}y}_{J_\epsilon}. \end{aligned}$$

$I_\epsilon$ is easy since $f$ is nice function, 2nd derivate on compact set, so obtains maximum.

$$|I_\epsilon| \le c\left\|D^2 f\right\|_{L^\infty(\mathbb{R}^n)} \int_{B(0,\epsilon)} |\Phi(y)|\,\mathrm{d}y \le \begin{cases} \tilde{c}\epsilon^2 \ln\epsilon, & n=2, \\ \tilde{c}\epsilon^2, & n \ge 3. \end{cases} \to 0 \text{ as } \epsilon \to 0.$$

Next, for $J_\epsilon$, we can say that $\Delta_x$ is the same operation as $\Delta_y$ since the negative from twice derivatives will cancel. Also use integration by parts ($\int_a^b u'v\,\mathrm{d}x = -\int_a^b uv'\,\mathrm{d}x + uv|_a^b$). Let $V_\epsilon = \mathbb{R}^n \setminus B(0,\epsilon)$.

$$\begin{aligned} J_\epsilon &= \int_{\mathbb{R}^n\setminus B(0,\epsilon)} \Phi(y)\Delta_x f(x-y)\,\mathrm{d}y \underbrace{=}_{\text{since } (-1)^2=1} \int_{\mathbb{R}^n\setminus B(0,\epsilon)} \Phi(y)\Delta_y f(x-y)\,\mathrm{d}y \\ &= \underbrace{-\int_{V_\epsilon} D\Phi(y)\cdot D_y f(x-y)\,\mathrm{d}y}_{K\epsilon} + \underbrace{\int_{\partial V_\epsilon} \Phi(y) D_y f(x-y)\cdot\nu(y)\,\mathrm{d}S(y)}_{L_\epsilon}, \text{ from Green's Identity.} \end{aligned}$$

Note that $\nu(y)\frac{-y}{|y|}$ since $U = \mathbb{R}^n \setminus B(0,\epsilon)$.

$$|L_\epsilon| \le \|DF\|_{L^\infty(\mathbb{R}^n)} \int_{\partial B(0,\epsilon)} |\Phi(y)|\,\mathrm{d}S(y) = \begin{cases} \tilde{c}\epsilon\,|\ln\epsilon|\,, & n=2 \\ \tilde{c}\epsilon, & n=3. \end{cases}$$

This came from that

$$\Phi(y) = \begin{cases} c|\ln\epsilon|, & n=2 \\ c\epsilon^{2-n}, & n\ge 3, \end{cases} \quad \int_{\partial B(0,\epsilon} \mathrm{d}S(y) = n\alpha(n)\epsilon^{n-1}.$$

So, $|L_\epsilon| \to 0$ as $\epsilon \to 0^+$. $K_\epsilon$ is not too bad since $\Phi$ is smooth and $\Delta\Phi = 0$ on $\mathbb{R}^n \setminus B(0,\epsilon)$. Integration by parts:

$$\begin{aligned} K_\epsilon &= -\int_{\mathbb{R}^n\setminus B(0,\epsilon)} D\Phi(y)\cdot D_y f(x-y)\,\mathrm{d}y \\ &= \int_{\mathbb{R}^n\setminus B(0,\epsilon)} \underbrace{\Delta\Phi(y)}_{0} f(x-y)\,\mathrm{d}y - \int_{\partial B(0,\epsilon)} (D\Phi(y)\cdot \underbrace{\nu(y)}_{\frac{-y}{\epsilon}}) d(x-y)\,\mathrm{d}S(y) \end{aligned}$$

From derivation of $\Phi(x) - v(|x|)$, recall that $v'(r) - \frac{1}{n\alpha(n)}\frac{1}{r^{n-1}}$. Note that $D_x(|x|) = x/|x|$. Thus, $D\Phi(y) = \frac{-1}{n\alpha(n)}\frac{y}{|y|^n}$. Evaluating on $\partial B(0,\epsilon)$,

$$D\Phi(y)\cdot\nu(y) = \frac{-1}{n\alpha(n)}\frac{-\epsilon^2}{\epsilon^{n+1}} = \frac{1}{n\alpha(n)\epsilon^{n-1}} = \frac{1}{\int_{\partial B(0,\epsilon)}\mathrm{d}S(y)}.$$

Last step is bc unit ball volume is $\epsilon^n$, so the product is surface area in $n$ dim. So, $K_\epsilon = \fint_{\partial B(0,\epsilon)} -f(x-y)\,\mathrm{d}S(y) \to -f(x)$ as $\epsilon \to 0^+$ since $f \in C$. □

*Remark* 4.2.

$$\fint_U G(y)\,\mathrm{d}y := \frac{\int_U G(y)\,\mathrm{d}y}{\int_U \mathrm{d}y}$$

If we use the "Dirac delta" notation, $-\Delta\Phi = \delta_0$ assigns unit mass to the origin. Thus,

$$-\Delta u(x) = \int_{\mathbb{R}^n} -\Delta_x\Phi(x-y)f(y)\,\mathrm{d}y = f(x)\ \forall x\in\mathbb{R}^n.$$

## 4.1 Mean Value Property

**Theorem 4.3.** *If $u \in C^2(U)$ is harmonic, then*

$$u(x) = \fint_{\partial B(x,r)} u(y)\,\mathrm{d}S(y) = \fint_{B(x,r)} u(y)\,\mathrm{d}y \ \forall x, r \text{ such that } B(x,r) \subset U.$$

*I.e., a harmonic function at the point is an average value for the same function over any ball/sphere centered at that point.*

*Proof.* Fix $x$ and define $\phi(r) \coloneqq \fint_{\partial B(x,r)} u(y)\,\mathrm{d}S(y) = \fint_{\partial B(0,1)} u(x+rz)\,\mathrm{d}S(z)$ by change of variables so we want a domain that does not depend on $r$. Since it is an average, we don't need the extra power of $r^{n-1}$. Then since $u \in C^2$, we can change order of integral and derivative then we change back after differentiation:

$$\begin{aligned}\frac{\mathrm{d}\phi}{\mathrm{d}r} &= \fint_{\partial B(0,1)} \underbrace{Du(x+rz)\cdot z)}_{\frac{\mathrm{d}}{\mathrm{d}r}u(x+rz)}\,\mathrm{d}S(z) = \fint_{\partial B(x,r)} Du(y)\cdot \underbrace{\frac{y-x}{r}\,\mathrm{d}S(y)}_{\text{unit outward normal } \nu(y)} \\ &= \fint_{\partial B(x,r)} \frac{\partial u}{\partial \nu}(y)\,\mathrm{d}S(y) = \frac{1}{n\alpha(n)r^{n-1}} \int_{\partial B(x,r)} \frac{\partial u}{\partial \nu}(y)\,\mathrm{d}S(y).\end{aligned}$$

Note that $\int_{\partial B(x,r)} \frac{\partial u}{\partial \nu}(y)\,\mathrm{d}S(y) = \int_{b(x,r)} \Delta u(y)\,\mathrm{d}y = 0$ since $u$ is harmonic. This is yet another Green's identity, which is obtained from (3) if we use in it $u \equiv 1$.

So, $\phi(r)$ is constant. But $\lim_{r\to 0^+} \phi(r) = u(x)$ since $u \in C$. Thus, $\phi(r) = u(x) \forall r$.

Also, by polar integration

$$\begin{aligned}\int_{B(x,r)} u(y)\,\mathrm{d}y &= \int_0^r \left( \int_{\partial B(x,\rho)} u(y)\,\mathrm{d}S(y) \right) \mathrm{d}\rho \\ &= \int_0^r \rho^{n-1} n\alpha(n) u(x)\,\mathrm{d}\rho = \rho^n \alpha(n) u(x)\big|_{\rho=0}^{\rho=r} = \underbrace{\alpha(n) r^n}_{\int_{B(x,r)} \mathrm{d}y} u(x)\end{aligned}$$

So, $u(x) = \fint_{B(x,r)} u(y)\,\mathrm{d}y$. ◻

The converse is also true!

**Theorem 4.4.** *If $u \in C^2(U)$ satisfies $u(x) = \fint_{\partial B(x,r)} u(y)\,\mathrm{d}S(y)$ $\forall x, r$ such that $B(x,r) \subset U$ the $u$ is harmonic.*

*Proof.* Suppose $\Delta u(x) \neq 0$ for some $x \in U$. Since $u \in X^2$ there exists $r > 0$ such that $\Delta u$ doesn't change sign on $B(x,r)$. Then, using $\phi(r)$ defined above,

$$0 = \phi'(r) = \frac{r}{n} \fint_{B(x,r)} \Delta u(y)\,\mathrm{d}y \neq 0,$$

a contradiction. ◻

# 5 Mean Value Properties Cont. (9/12/23)

**On many uses of the Mean Value Property**

**Theorem 5.1.** Strong maximum principle theorem. *If $u \in C^2(U) \cap C(\bar{U})$ is harmonic on $U$, then*

1. $$\max_{\bar{U}} u = \max_{\partial U} u$$

2. Strong version. *If $U$ is a connected set and $\exists x_0 \in U$ such that $u(x_0) = \max_{\bar{U}} u$, then $u$ is constant on $U$.*

*Also valid for* min, *since $-u$ is also harmonic and* $\max(-u) = -\min(u)$.

*Proof.* You cannot be an average of values smaller than you. More precisely, if $\exists x_0 \in U$ such that $u(x_0) = M = \max_{\bar{U}} u$ and $r \leq \operatorname{dist}(x_0, \partial U)$, then

$$M = u(x_0) = \fint_{B(x_0,r)} u(y)\,\mathrm{d}y \leq M$$

can only hold if $u(y) = M\ \forall y \in B(x_0, r)$. Taking $r = \operatorname{dist}(x_0, \partial U)$ proves (1). Note also that $U_M = \{x \in U | u(x) = M\}$ is both open and relatively closed in $U$. So, if $U$ is connected, then $U_M = U$, proving (2). □

**Corollary 5.2.** *If $U$ is connected, $u \in C^2(U) \cap C(\bar{U})$, $u \geq 0$ on $\partial U$ and $U(\tilde{x}) > 0$ for some $\tilde{x} \in \partial U$, then $u(x) > 0 \forall x \in U$.*

*Remark* 5.3. What does Laplace say to a probablist? Suppose some Brownian motion occurs and $u = g$ on $\partial U$. Then $u(x) = \mathbb{E}[g(X_T)]$, where $\mathrm{d}X_t = \mathrm{d}B$ is a stochastic process. $X_0 = x$ is the starting point. First time of intersection with boundary $T = \min\{t | X \in \partial U\}$. One can prove that $\Delta u = 0$ on $U$ and $u = g$ on $\partial U$. So, I have non-negative payoff on the boundary, then the expectation is strictly positive.

*Proof.* $m = \min_{\bar{U}} u \geq 0$. If $m > 0$ then (1) is enough since $u(x) \geq m > 0\ \forall x \in U$. If $m = 0$, then $u(x) = m \implies u = m$ on $U$, which contradicts $u(\tilde{x}) > 0$. □

**Theorem 5.4.** Uniqueness. *If $f \in C(U)$ and $g \in C(\partial U)$ and $u_1, u_2 \in C^2(U) \cap C(\bar{U})$ both solve $-\Delta u = f$ on $U$ and $u = g$ on $\partial U$, then $u_1 \equiv u_2$ on $U$.*

*Proof.* Apply the maximum principle to both $(u_1 - u_2)$ and $(u_2 - u_1)$. Both the max and min is 0 on the boundary. So $u_1 - u_2$ and $u_2 - u_1$ cannot be bigger than 0 in the interior, thus they are equivalent. □

**Theorem 5.5.** Smoothness of harmonic functions. *If $u \in C(U)$ satisfies the MVP on every $B(x, r) \subset U$, the $u \in C^\infty(U)$. (Note that $u \in C(\bar{U})$ doesn't have to hold though!)*

**Mollifiers.**

**Definition 5.6.** For an open $U$, $U_\epsilon = \{x \in U | \operatorname{dist}(x, \partial U) > \epsilon\}$.

**Definition 5.7.** A *bump function (aka mollifier).*

$$\xi(x) = \begin{cases} c\exp\left(\frac{1}{|x|^2-1}\right), & \text{if } |x| < 1, \\ 0, & \text{if } |x| \geq 1. \end{cases}$$

with $c > 0$ selected such that $\int_{\mathbb{R}^n} \xi(x)\,\mathrm{d}x = 1$. TODO: Verify $\xi \in X^\infty$.

"Standard" mollifier: $\forall \epsilon > 0$, define $\xi_\epsilon(x) = \frac{1}{\epsilon^n}\xi\left(\frac{x}{\epsilon}\right)$. Note support$\xi_\epsilon =$ interior of $B(0,\epsilon)$.

$$\int_{\mathbb{R}^n} \xi_\epsilon\,\mathrm{d}x = \int_{B(0,\epsilon)} \xi_\epsilon\,\mathrm{d}x = 1.$$

**Definition 5.8.** If $f : U \to \mathbb{R}$ is locally integrable, its mollification is defined as $f^\epsilon := \xi_\epsilon * f$ on $U_\epsilon$ with

$$f^\epsilon(x) = \int_U \xi_\epsilon(x-y)f(y)\,\mathrm{d}y = \int_U \xi_\epsilon f(x-y)\,\mathrm{d}y \ \forall x \in U^\epsilon.$$

**Theorem 5.9.** Properties of mollification

1. *$f^\epsilon \in C^\infty(U^\epsilon)$ (Read this one)*
2. *$f^\epsilon \to f$ almost everywhere as $\epsilon \to 0^+$*
3. *If $f \in C(U)$, then $f^\epsilon \to f$ uniformly on compact subsets of $U$.*
4. *If $1 \leq p < \infty$ and $f^\epsilon \in L^p_{loc}(U)$, then $f^\epsilon \to f$ in $L^p_{loc}(U)$.*

*Proof.* See Evans, Appendix C5 □

*Proof.* Proof strategy: creative (radially symmetric averaging). That $u \in C(U)$ and MVP $\implies u \in C^\infty(U)$. Define $u^\epsilon \in C^\infty$ as $u := \xi_\epsilon * u$ on $U^\epsilon$ ($\epsilon$ away from $\partial U$) Then

$$u^\epsilon(x) = \int_U \xi_\epsilon(x-y)u(y)\,\mathrm{d}y = \int_{B(x,\epsilon)} \xi(x-y)u(y)\,\mathrm{d}y \tag{5.1}$$

$$= \int_0^\epsilon \int_{\partial B(x,r)} \frac{1}{\epsilon^n}\xi\left(\frac{|x-y|}{\epsilon}\right)u(y)\,\mathrm{d}S(y)\,\mathrm{d}r \tag{5.2}$$

$$= \frac{1}{\epsilon^n}\int_0^\epsilon \xi\left(\frac{r}{\epsilon}\right)\int_{\partial B(x,r)} u(y)\,\mathrm{d}S(y)\,\mathrm{d}r = u(x)\int_0^\epsilon \int_{\partial B(x,r)} \frac{1}{\epsilon^n}\xi\left(\frac{r}{\epsilon}\right)\mathrm{d}S(y)\,\mathrm{d}r \tag{5.3}$$

$$= u(x)\int_{B(x,\epsilon)} \xi_\epsilon(y)\,\mathrm{d}y = u(x), \tag{5.4}$$

which must be thus $C^\infty$ at every $x \in U^\epsilon$ for every $\epsilon > 0$. □

**Theorem 5.10.** *Assume $u \in C^2(U)$ and MVP holds (thus, $u \in C^\infty(U)$ and $\Delta u = 0$ on $U$). Then, for all $x_0 \in U$,*

1. $$|u(x_0)| \leq \frac{1}{\alpha(n)r^n}\|u\|_{L^1(B(x_0,r))}$$

2. $$|u_{x_i}(x_0)| \leq \frac{2^{n+1}n}{\alpha(n)}\frac{1}{r^{n+1}}\|u\|_{L^1(B(x_0,r))} \ \forall i$$

*Proof.* (1) follows directly from the MVP and triangle inequality. For (2), note that $\frac{\partial}{\partial x_i}\Delta u = \Delta u_{x_i} = 0$. So, MVP also holds for $u_{x_i}$. Then

$$|u_{x_i}(x_0)| \le \frac{1}{\alpha(n)r^n}\left\|u_{x_i}\right\|_{L^1(B(x_0,r))}$$

and

$$\begin{aligned}
|u_{x_i}(x_0)| &= \left|\fint_{B(x_0,r/2)} u_{x_i}(x)\,\mathrm{d}x\right| = \frac{2^n}{\alpha(n)r^n}\left|\int_{B(x_0,r/2)} u_{x_i}(x)\,\mathrm{d}x\right| = \\
&= \frac{2^n}{\alpha(n)r^n}\left|\int_{\partial B(x_0,r/2)} u(y)\nu^i(y)\,\mathrm{d}S(y)\right| \le \frac{2^n}{\alpha(n)r^n}\int_{\partial B(x_0,r/2)} |u(y)|\underbrace{|\nu^i(y)|}_{\le\|\nu\|=1}\,\mathrm{d}S(y) \\
&\le \frac{2^n}{\alpha(n)r^n} n\alpha(n)\left(\frac{r}{2}\right)^{n-1}\|u\|_{L^\infty(B(x_0,r/2))} = \frac{2n}{r}\|u\|_{L^\infty(B(x_0,r/2))}.
\end{aligned}$$

Note that $x \in B(x_0,r/2) \implies B(x,r/2) \subset B(x_0,r) \subset U$. So, by part 1

$$|u(x) \le \frac{1}{\alpha(n)}\left(\frac{2}{r}\right)^n\|u\|_{L^1(B(x_0,r/2))} \le \frac{1}{\alpha(n)}\left(\frac{2}{r}\right)^n\|u\|_{L^1(B(x_0,r))}.$$

$$\implies |u_{x_i}(x_0)| \le \frac{2n}{r}\|u\|_{L^\infty(B(x_0,r/2))} \le \underbrace{\frac{2n}{r}\frac{1}{\alpha(n)}\frac{2^n}{r^n}}_{\frac{n2^{n+1}}{\alpha(n)r^{n+1}}}\|u\|_{L^1(B(x_0,r))}.$$

□

See analytic theorem and proof in Evans.

**Theorem 5.11.** Liouville Theorem. *Suppose $u : \mathbb{R}^n \to \mathbb{R}$ is harmonic and bounded. Then $u$ is constant.*

*Proof.* Fix $x_0 \in \mathbb{R}^n$ and $r > 0$. Note that, for every $i$,

$$\begin{aligned}
|u_{x_i}(x_0)| &\le \frac{2^{n+1}n}{\alpha(n)r^{n+1}}\|u\|_{L^1(B(x_0,r))} \le \frac{2^{n+1}n}{\alpha(n)r^{n+1}}\alpha(n)r^n\|u\|_{L^\infty(B(x_0,r))} \\
&= \frac{n2^{n+1}}{r}\|u\|_{L^\infty(B(x_0,r))} \to 0 \text{ as } r \to \infty
\end{aligned}$$

since $u$ is bounded on $\mathbb{R}^n$. □

Evan's notation: for open $U$ and $V$, $V \subset\subset U$ means that $\bar{V}$ is compact and $\bar{V} \subset U$. ($V$ is "compactly contained" in $U$).

**Theorem 5.12.** Harnack's inequality. *For each connected open $V \subset\subset U$, $\exists C > 0$ depending only on $V$ such that $\sup_V u \le c \inf_V u$ for every non-negative harmonic $u$ on $U$. Thus, $\frac{1}{c}u(y) \le u(x) \le cu(y)$ for all $x, y \in V$ and all such $u$.*

The idea: since $\bar{V}$ is some distance away from $\partial U$, the averaging effect of Laplace's equation prevent $u$ from varying much on $V$

*Proof.* Take $r = \frac{1}{4}\text{dist}(V, \partial U)$ and choose any $x, y \in V$ such that $|x-y| < r$.

$$
\begin{aligned}
u(x) &= \fint_{B(x,2r)} u(z)\,\mathrm{d}z = \frac{1}{2^n r^n \alpha(n)} \int_{B(x,2r)} u(z)\,\mathrm{d}z \underbrace{\geq}_{u\geq 0,\ |x-y|<r} \frac{1}{2^n r^n \alpha(n)} \int_{B(y,r)} u(z)\,\mathrm{d}z \\
&= \frac{1}{2^n} \fint_{B(y,r)} u(z)\,\mathrm{d}z = \frac{1}{2^n} u(y).
\end{aligned}
$$

So, $|x-y| < r \implies u(x) \leq \frac{1}{2^n} u(y)$. $V$ is connected, $\bar{V}$ is compact $\implies$ $\exists$ a finite covering of $\bar{V}$ by balls $\{B_i\}_{i=1}^N$ of radius $r/2$ with $B_i \cap B_{i-1} \neq \emptyset$ for $u = 2, \dots, N$.

$$
u(x) \leq \frac{1}{2^n} u(x_1) \leq \frac{1}{2^{2n}} u(x_2) \leq \dots
$$

So, $u(x) \leq \frac{1}{2^{Nn}} u(y)$ $\forall x, y \in V$. Where $x_1, x_2$ are any point at the same $B_i$ as $x$. and $x_2$ is any point in any adjacent $B_j$. □

# 6 Green's Functions (9/14/23)

*Gokul Nair guest lecture.*

Recall the fundamental solution for Laplace.

$$
\Phi(x) = \begin{cases} -\frac{1}{2\pi} \ln|x|, & n = 2 \\ \frac{1}{n(n-2)\alpha(n)} |x|^{2-n}, & n \geq 3 \end{cases}
$$

where $-\Delta u = f$ on $\mathbb{R}^n$. For Green's functions. Suppose we have $-\Delta u = f$ in $U$ and $u = g$ on $\partial U$. Then

$$
\int_{V_\epsilon} u(y)\Delta\Phi(x-y) - \Phi(x-y)\Delta u = \int_{\partial V_\epsilon} u(y) \frac{\partial \Phi}{\partial \nu}(y-x) - \Phi(y-x)\frac{\partial u}{\partial \nu}(y)\,\mathrm{d}S,
$$

where $V_\epsilon \coloneqq U \setminus B(x,\epsilon)$, $\partial V_\epsilon = \partial U \cup \partial B(x,\epsilon)$. Then

$$
\left| \int_{\partial B(x,\epsilon)} \Phi(y-x) \frac{\partial u}{\partial \nu}(y)\,\mathrm{d}S(y) \right| \leq C \max_{\partial B(x,\epsilon} |\Phi| \epsilon^{n-1},\ to 0 \text{ as } \epsilon \to 0.
$$

Next

$$
\int_{\partial B(x,\epsilon)} u(y) \frac{\partial \Phi}{\partial \nu}(y-x)\,\mathrm{d}S(y) = \fint_{\partial B(x,\epsilon)} u(y)\,\mathrm{d}S(y),\ to 0 \text{ as } \epsilon \to 0.
$$

Now Green's Representation is

$$
u(x) = \int_{\partial U} \Phi(y-x)\frac{\partial u}{\partial \nu}(y) - u(y)\frac{\partial \Phi}{\partial \nu}(y-x)\,\mathrm{d}S(y) - \int_U \Phi(y-x)\Delta u(y)\,\mathrm{d}y,
$$

where $\frac{\partial u}{\partial \nu}$ is unknown to us on $\partial U$. Then we introduce **corrector functions:**

$$
\phi^x(y) = \begin{cases} \Delta \phi^x = 0, & \text{in } U, \\ \phi^x = \Phi(y-x), & \text{on } \partial U. \end{cases}
$$

Then

$$
\begin{aligned}
-\int \phi^x(y)\Delta u(y)\,\mathrm{d}y &= \int_{\partial U} u(y)\frac{\partial \phi^x}{\partial \nu}(y) - \phi^x(y)\frac{\partial u}{\partial \nu}\,\mathrm{d}S(y) \\
&= \int_{\partial U} u(y)\frac{\partial \phi^x}{\partial \nu}(y) - \Phi(y-x)\frac{\partial u}{\partial \nu}\,\mathrm{d}S.
\end{aligned}
$$

Using our $u(x)$ Green's representation with the integral above to get

$$
u(x) = \int_U \phi^x(y)\Delta u(y)\,\mathrm{d}y - \int_{\partial U} u(y)\frac{\partial \Phi}{\partial \nu}(y-x)\,\mathrm{d}S - \int_U \Phi(y-x)\Delta u(y)\,\mathrm{d}y + \int_{\partial U} u(y)\frac{\partial \phi^x}{\partial \nu}(y)\,\mathrm{d}S(y).
$$

So we get the representation formula using Green's function

$$
u(x) = -\int_{\partial U} u(y)\frac{\partial G}{\partial \nu}(x,y)\,\mathrm{d}S(y) - \int_U G(x,y)\Delta u(y)\,\mathrm{d}y,
$$

where the Green's function is

$$
G(x,y) = \Phi(y-x) - \phi^x(y),
$$

and $\phi^x$ is explicitly be computed on $\mathbb{R}^n_+,\ B(0,1)$ only. Another representation is

$$
\begin{cases}
-\Delta G = \delta_x, & \text{in } U, \\
G = 0, & \text{on } \partial U.
\end{cases}
$$

**Theorem 6.1.** Symmetry of $G(x,y)$. *$x, y \in U$ and $x \neq y$. Then $G(x,y) = G(y,x)$.*

*Proof.* Fix $x, y \in U$ with $x \neq y$. Define $v(z) \coloneqq G(x,z)$ and $w(z) \coloneqq G(y,z)$. Then we have

$$
\begin{aligned}
\Delta v(z) &= 0, \quad z \neq x \\
\Delta w(z) &= 0, \quad z \neq y \\
w = v &= 0, \quad \text{on } \partial U \\
V &\coloneqq U \setminus (B(x,\epsilon) \cup B(y,\epsilon))
\end{aligned}
$$

Using the integration by parts formula (one of Green formulae),

$$
\int_V \Delta v W - \Delta w V\,\mathrm{d}z = \int_{\partial V} W\frac{\partial v}{\partial \nu} - v\frac{\partial W}{\partial \nu}\,\mathrm{d}S(z).
$$

$$
\int_{\partial B(x,\epsilon)} \frac{\partial v}{\partial \nu}w - \frac{\partial w}{\partial \nu}v\,\mathrm{d}S(z) = \int_{\partial B(y,\epsilon} \frac{\partial w}{\partial \nu}v - \frac{\partial v}{\partial \nu}w\,\mathrm{d}S(z).
$$

Looking at terms specifically,

$$
\begin{aligned}
\left|\int_{\partial B(x,\epsilon)} \frac{\partial w}{\partial \nu}v\,\mathrm{d}S\right| &\le C\epsilon^{n-1} \to 0 \\
\int_{\partial B(x,\epsilon)} \frac{\partial v}{\partial \nu}w\,\mathrm{d}S &= \fint_{\partial(x,\epsilon)} w(z)\,\mathrm{d}z \to w(x)
\end{aligned}
$$

□

Let's look at examples.
**Green's function on Half space.** $\mathbb{R}^n_+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n > 0\}$. Now consider $\Phi(x-y)$ on $\mathbb{R}^n$. We want to have the potential be 0 on the points $x_n = 0$, this is related to the method of images in electrostatics.

**Definition 6.2.** $x \in \mathbb{R}^n_+$ and $x = (x_1, \ldots, x_{n-1}, x_n)$. The *reflection* of $x$ on $\partial\mathbb{R}^n_+$ is $\tilde{x} = (x_1, \ldots, x_{n-1}, -x_n)$.

Now the corrector function is

$$phi^x(y) := \Phi(y-x).$$

Recall that the corrector function satisfies $\Delta\phi^x = 0$ on $U$ (true because harmonic and $\tilde{x}$ is not in domain) and $\phi^x = \Phi(y-x)$ on $\partial U$. Let's verify the latter case.
$\phi^x(y_) = \Phi(y-\tilde{x}),\ y \int \partial\mathbb{R}^n_+$ so $y = (y_1, \ldots, y_{n-1}, 0)$.

$$\begin{aligned} \phi^x(y) &= \Phi(y_1 - x_1, y_2, -x_2, \ldots, y_{n-1} - x_{n-1}, x_n) \\ &= \Phi(y_1 - x_1, \ldots, y_{n-1} - x_{n-1}, -x_n). \end{aligned}$$

The last step can be seen from definition of fundamental solution.

**Definition 6.3.** Green's function for $\mathbb{R}^n_+$.

$$G(x,y) = \Phi(y-x) - \Phi(y-\tilde{x}).$$

Let's plug this in to see what it looks like. The problem is $\Delta u = 0$ in $\mathbb{R}^n_+$ and $u = g$ on $\partial\mathbb{R}^n_+$. The normal derivatives (on the boundary) look like

$$\frac{\partial G}{\partial \nu}(x,y) = -\frac{\partial}{\partial y_n} G(x,y) = \frac{-2x_n}{n\alpha(n)} \frac{1}{|x-y|^n} \text{ on } \{y_n = 0\}.$$

Then plugging in gives

$$\begin{aligned} u(x) &= -\int_{\partial U} g(y) \frac{\partial G}{\partial \nu}(x,y) \,\mathrm{d}S(y), \\ &= \frac{2x_n}{n\alpha(n)} \int_{\mathbb{R}^n_+} \frac{g(y)}{|x-y|^n} \,\mathrm{d}y. \end{aligned}$$

We assumed that we have a $C^2$ function that satisfies the Laplace. This is proven in Evan's Theorem 14

**Theorem 6.4.** $g \in C^0(\mathbb{R}^{n-1}) \cap L^\infty(\mathbb{R}^{n-1}$. *Then $u$ defined above satisfies*

1. $u \in C^\infty(\mathbb{R}^n_+) \cap L^\infty(\mathbb{R}^n_+)$
2. $\Delta u = 0$ *in* $\mathbb{R}^n_+$
3. $\lim_{x \to \infty^+} u(x) = g(x^o), x^o \in \partial\mathbb{R}^n_+$.

**Definition 6.5.** *Poisson's Kernel for* $\mathbb{R}^n_+$.

$$K(x,y) = \frac{2x_n}{n\alpha(n)|x-y|^n}.$$

Where we use $u(x) = \int_{\partial\mathbb{R}^n_+} K(x,y) g(y) \,\mathrm{d}S$. We need to use the fact that $\int_{\partial\mathbb{R}^n_+} K(x,y) \,\mathrm{d}y = 1$.

**Green's function for a Ball.**

**Definition 6.6.** if $x \in \mathbb{R}^n \setminus \{0\}$. Then the *inversion* of $x$ through the sphere $\partial B(0,1)$ is given by $\tilde{x} := \frac{x}{|x|^2}$. What this does: as $x$ is closer to boundary, $\tilde{x}$ sends to $\infty$. Things that are outside the ball get sent inside. And $\tilde{x} = x$ on $\partial B(0,1)$. This essentially turns the ball "inside out."

We want the corrector function $\Delta \phi^x = 0$ on $B(0,1)$ (open ball), and $\phi^x = \Phi(y-x)$ on $\partial B(0,1)$. Then we claim (this is along the lines of method of images still as $\tilde{x}$ is the $x$ outisde of the sphere and then we have some magnitude correction to it)

$$\begin{aligned}\phi^x(y) &:= \Phi(|x|(y-\tilde{x})),\\ &= \frac{1}{n(n-2)\alpha(n)} \frac{1}{|x|^{n-2}||y-\tilde{x}|^{n-2}},\end{aligned}$$

for $n \geq 3$. See that $y \mapsto \phi^x(y)$ then $\Delta_y \phi^x(y) = 0$. Then for $y \in \partial B(0,1),\ x \neq 0$, we use definition of inversion and parallelogram law

$$|x|^2|y-\tilde{x}|^2 = |x|^2\left(|y|^2 - \frac{2y\cdot x}{|x|^2} + \frac{1}{|x|^2}\right) = |x|^2 - 2yx + 1 = |x-y|^2.$$

So $\phi^x(y) = \Phi(y-x)$ on $\partial B(0,1)$.

**Definition 6.7.** The Green's function on unit ball is

$$G(x,y) = \Phi(y-x) - \Phi(|x|(y-\tilde{x})).$$

Now consider the cases where we are on any size ball. Start with 1 and do change of variable later.

$$\begin{cases}\Delta u = 0, & \text{in } B(0,1)\\ u = g, & \text{on } \partial B(0,1).\end{cases}$$

Then on $\partial B(0,1)$,

$$\frac{\partial G}{\partial y_i} = \frac{1}{n\alpha(n)}\frac{x_i - y_i}{|x-y|^n} + \frac{1}{n\alpha(n)}\frac{y_i|x|^2 - x_i}{|x-y|^n}.$$

To get the normal,

$$\frac{\partial G}{\partial \nu} = \sum_{i=1}^n \frac{\partial G}{\partial y_i} y_i = \frac{-1}{n\alpha(n)}\frac{1-|x|^2}{|x-y|^n}.$$

Then using the gradient formula, we get

$$u(x) = \frac{1-|x|^2}{n\alpha(n)}\int_{\partial B(0,1)} \frac{g(y)}{|x-y|^n}\,\mathrm{d}S(y).$$

Use change of variables to get for a ball of radius $r$:

$$u(x) = \frac{r^2-|x|^2}{n\alpha(n)r}\int_{\partial B(0,r)} \frac{g(y)}{|x-y|^n}\,\mathrm{d}S(y).$$

**Definition 6.8.** *Poisson's Kernel for $B(0,r)$.*

$$K(x,y) = \frac{r^2-|x|^2}{n\alpha(n)r} - \frac{1}{|x-y|^n}.$$

There is a theorem in Evan's that says

**Theorem 6.9.** *If $u$ satisfies the formula above, it solves the Laplace condition.*

# 7 (9/19/23)

**Some recap.** Last time Gokul showed us Green's functions.

$$\begin{cases} -\Delta u = f & \text{on } U, \\ u = g & \text{on } \partial U \end{cases}, \quad u(x) = \int_U G(x,y) f(y)\, \mathrm{d}y - \int_{\partial U} g(y) \underbrace{\frac{\partial G}{\partial \nu}(x,y)}_{-K(x,y)} \mathrm{d}S(y).$$

If let $f = 0$, now just Laplance,

$$u(x) = \int_{\partial U} g(y) K(x,y)\, \mathrm{d}S(y), \quad \text{where} \int_{\partial U} K(x,y)\, \mathrm{d}S(y) = 1,\ K(x,y) > 0\ \forall x \in U,\ y \in \partial U.$$

So $K(x,y)$ looks like a PDF for starting at an $x$ and Brownian motion to hitting a value at the boundary. What is the probabilistic interpretation if $f = 1$?

## 7.1 Energy Methods for Laplace and Poisson

**Theorem 7.1.** Uniqueness Theorem. *Suppose $U$ is open and bounded with $\partial U \in C^1$. $\exists$ at most one $C^2(\bar{U})$ solution of*

$$\begin{cases} -\Delta u = f & \text{on } U, \\ u = g & \text{on } \partial U \end{cases}.$$

*Proof.* Suppose $u_1, u_2 \in C^2(\bar{U})$ that both solve the conditions for $u$. Then $w = u_1 - u_2$ satisfies $\begin{cases} \Delta w = 0 & \text{on } U \\ w = 0 & \text{on } \partial U \end{cases}$. Then

$$0 = \int_U (\Delta w)\, \mathrm{d}x \underbrace{=}_{u=v=w} - \int_U |Dw|^2\, \mathrm{d}x \implies Dw = 0.$$

Then $w = 0$ on $U$. We used Green's formula. Recall that from Green's

$$\int_U (\Delta u) v\, \mathrm{d}x = -\int_U Du \cdot Dv\, \mathrm{d}x + \underbrace{\int_{\partial U} v \frac{\partial u}{\partial \nu}\, \mathrm{d}S}_{=0,\ w=0 \text{ on } \partial U}.$$

$\square$

**Definition 7.2.** *Dirichlet's principle.* (preview of calculus of variations) Define *admissible class* $\mathcal{A} = \{w \in C^2(\bar{U}) | w = g \text{ on } \partial U\}$ and an *energy functional*

$$I[w] = \int_U \left( \frac{1}{2} |Dw|^2 - fw \right) \mathrm{d}x.$$

**Theorem 7.3.** *If $u \in \mathcal{A}$, then $-\Delta u = f$ on $U$ $\iff$ $I[u] = \min_{w \in \mathcal{A}} I[w]$.*

*Proof.* ($\implies$) If $u$ solves Poisson, then for every $w \in \mathcal{A}$,

$$0 = \int_0 \underbrace{(-\Delta u - f)}_{=0 \text{ on } U} \underbrace{(u - w)}_{=0 \text{ on } \partial U} \mathrm{d}x = \int_U Du \cdot D(u-w)\, \mathrm{d}x - \int_U f(u-w)\, \mathrm{d}x.$$

Equality from integrate by parts. No boundary terms since $u = w$ on $\partial U$. Then we have (and use Cauchy-Schwartz)

$$\int_U \frac{1}{2}(|Du|^2 + |Dw|^2)\,\mathrm{d}x \geq \int_U Du \cdot Dw\,\mathrm{d}x = \int_U |Du|^2\,\mathrm{d}x - \int_U fu\,\mathrm{d}x + \int_U fw\,\mathrm{d}x.$$

Subtract $\int_U \frac{1}{2}|Du|^2\,\mathrm{d}x$ from both sides $\implies I[u] \leq I[w]$.

We have a minimum. For a nice, smooth function of finitely many variables, we usually say the gradient is 0. To make it a one variable problem, pass a line through that point to use the notion of a directional derivative. We can use this in functional spaces. Let every directional derivative be 0.

( $\Longleftarrow$ ) Suppose $u \in \mathcal{A}$ and $I[u] = \min_{w\in\mathcal{A}} I[w]$. Choose any $v \in C_c^\infty(\bar{U})$ (which implies $v = 0$ on $\partial U$). Define $i(\tau) = I[u + \tau v]$ for all $\tau \in \mathbb{R}$. Note $\tau v$ is a perturbation of $u$ and $(u + \tau v) \in \mathcal{A}$. Thus, $i(0) \leq i(\tau)\ \forall \tau \in \mathbb{R}$. Since $u \in C^2(\bar{U})$ and $v \in C^\infty(\bar{U})$, easy to show that $i \in C^2(\mathbb{R})$ (so can change order of differentiation with integration) with

$$\begin{aligned}
i'(\tau) &= \frac{\mathrm{d}}{\mathrm{d}\tau}\int_U \left(\frac{1}{2}(Du + \tau Dv)\cdot(Du + \tau Dv) - (u + \tau v)f\right)\mathrm{d}x,\\
&= \frac{\mathrm{d}}{\mathrm{d}\tau}\int_U \left(\frac{1}{2}|Du|^2 + (Du\cdot Dv)\tau + \frac{\tau^2}{2}|Dv|^2 - uf - \tau vf\right)\mathrm{d}x,\\
&= \int_U \big((Du\cdot Dv) + \tau|Dv|^2 - vf\big)\,\mathrm{d}x.
\end{aligned}$$

We know that $i'(0) = 0$. Then use integration by parts, where we know $\int (Du \cdot Dv)\,\mathrm{d}x = -\int (\Delta u)v\,\mathrm{d}x$ since $v = 0$ on $\partial U$

$$0 = i'(0) = \int_U (Du\cdot Dv) - vf)\,\mathrm{d}x = \int_U v(-\Delta u - f)\,\mathrm{d}x.$$

Since this has to hold for every $v \in C_c^\infty(\bar{U})$, it implies $-\Delta u(x) - f(x) = 0\ \forall x \in U$.
(Suppose not, e.g. $-\Delta u(x)f(x) > 0$ for some $x \in U$. Since $u \in C^2$, $\exists r > 0$ such that $-\Delta u - f > 0$ on $B(x, r) \subset U$. Take any positive $v \in C^\infty(\bar{U})$ with support $v \subset B(x, r)$. E.g., you can take $v = \xi_{r/4} * \chi_{B(x,r/2)}$. Then $\int_U v(-\Delta u - f)\,\mathrm{d}x > 0$, a contradiction) $\square$

## 7.2 (Inhomogeneous) heat equation:

It is

$$u_t - \Delta u = f, \quad \begin{cases} u : \bar{U}\times[0,\infty) \to \mathbb{R}\\ \Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}\\ f : \bar{U}\times[0,\infty) \to \mathbb{R}\end{cases}.$$

Transient behavior on the road to Poisson's equation? See derivation in Lecture 1.
Homogeneous (or simply "heat equation"): $f = 0$.

**Ideas for finding special solutions** for $u_t - \Delta u = 0$. If $u(x, t)$ is a solution, the so is also $u(\lambda x, \lambda^2 t)$. So, can try for $u(x, t) = v\left(\frac{r^2}{t}\right)$, where $r = |x|$.

Instead, we will try to make it "invariant under dilation;" i.e. look for $u(x,t) = \lambda^\alpha u(\lambda^\beta x, \lambda t)$ for some (special) $\alpha, \beta$ and $\forall x \int \mathbb{R}^n,\ t > 0,\ \lambda > 0$. if you take $\lambda = \frac{1}{t}$, then

$$u(x,t) = t^{-\alpha} \underbrace{u(t^{-\beta}x, 1)}_{v(y),\ y = t^{-beta}x} = \frac{1}{t^\alpha} v\left(\frac{x}{t^\beta}\right).$$

Use product and chain rules

$$\begin{aligned}
u_t &= (-\alpha)t^{(-\alpha-1)}v(y) + t^{-\alpha}\left(Dv(y)\cdot x\right)(-\beta)t^{(-\beta-1)} = t^{(-\alpha-1)}(-\alpha v(y) - \beta Dv(y)\cdot y),\\
u_{x_i} &= t^{-\alpha}v_{y_i}(y)t^{-\beta} = t^{-\alpha-\beta}v_{y_i}(y),\\
u_{x_ix_i} &= t^{-\alpha-2\beta}v_{y_iy_i}(y),\\
\implies \Delta_x u(x,t) &= t^{-\alpha-2\beta}\Delta_y v(y).
\end{aligned}$$

Now

$$-u_t + \Delta u = 0 = t^{(-\alpha-1)}(\alpha v(y) + \beta Dv(y)\cdot y) + t^{(-\alpha-2\beta)}\Delta_y v(y).$$

To simplify, take $\beta = \frac{1}{2}$ and multiply through by $t^{\alpha+1}$ :

$$\alpha v(y) + \frac{1}{2}Dv(y)\cdot y + \Delta v(y) = 0.$$

**To get an ODE:** look for a radially symmetric $v(y) = w(r) = w(|y|)$. This is similar to derivation of fundamental solution of Laplace.

$$Dv(y) = w'(r)\frac{y}{r}, \quad \Delta v(y) = w''(r) + \frac{n-1}{r}w'(r)w'(r),$$

so, the equation now becomes

$$\alpha w(r) + \frac{r}{2}w'(r) + w''(r) + \frac{n-1}{r}w'(r) = 0.$$

Find an integrating factor (make LHS of equation above to be a derivative of some function; inexact to exact)? Try $\mu(r) = r^{n-1}$ because if we look at the equation above, it looks like it could be the result of a product rule, given some missing factors of $r$.

$$\underbrace{(\alpha r^{n-1}w(r) + \frac{1}{2}r^n w'(r))}_{=\left(\frac{1}{2}r^n w(r)\right)' \text{ if } \alpha = \frac{n}{2}} + \underbrace{(r^{n-1}w''(r) + r^{n-2}(n-1)w'(r))}_{=(r^{n-1}w'(r))'} = 0.$$

# 8 (9/21/23)

Taking $\alpha = \frac{n}{2}$ and integrating

$$r^{n-1}w'(r) + \frac{1}{2}r^n w(r) = \underbrace{a}_{\text{const.}} \in \mathbb{R}$$

**More assumptions.**

$$\lim_{r\to\infty} r^n w(r) = \lim_{r\to\infty} r^{n-1}w'(r) = 0 \implies a = 0.$$

Then solve the separable ODE

$$w' + \frac{1}{2}rw = 0, \quad \frac{w'(r)}{w(r)} = (\ln(w(r)))' = -\frac{1}{2}r,$$
$$\ln(w(r)) = -\frac{r^2}{4} + C, \quad w(r) = be^{-\frac{r^2}{4}} = v(y),$$
$$r = |y|,\ y = t^{-1/2}x, \quad u(x,t) = \frac{1}{t^{\frac{n}{2}}}v(t^{-\frac{1}{2}}x) = \frac{1}{t^{\frac{n}{2}}}be^{\frac{-|x|^2}{4t}}.$$

Where $b$ is a constant.

**Definition 8.1.** *Fundamental Solution of the Heat equation.*

$$\Phi(x,t) = \begin{cases} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}, & t > 0 \\ 0, & t < 0 \end{cases}$$

for $x \in \mathbb{R}^n$

Let's check $\lim_{t\to 0^+} \Phi(x,t) = 0$ and $\lim_{t\to 0^+} \Phi(0,t) = +\infty\ \forall x \neq 0$.

Now, why did we choose $b = \dfrac{1}{(4\pi)^{n/2}}$

**Lemma 8.2.** *For every $t > 0$,*

$$\int_{\mathbb{R}^n} \Phi(x,t)\,\mathrm{d}x = 1.$$

*Proof.*

$$\frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4t}}\,\mathrm{d} = \frac{1}{\pi^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-z^2}\,\mathrm{d}z.$$

Use substitution $z = \frac{x}{\sqrt{4t}}$, $\mathrm{d}z = \left(\dfrac{1}{\sqrt{4t}}\right)^n \mathrm{d}x$, $\mathrm{d}x = (4t)^{n/2}\,\mathrm{d}z$ to say the above expression

$$= \frac{1}{\pi^{n/2}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-z_1^2} e^{-z_2^2} \cdots e^{-z_n^2}\,\mathrm{d}z_1 \ldots \mathrm{d}z_n = \frac{1}{\pi^{n/2}} \prod_{i=1}^{} \overbrace{\left(\int_{-\infty}^{\infty} e^{-z_i^2}\,\mathrm{d}z_i\right)}^{\pi^{1/2}} = 1.$$

$\square$

*Proof.* Standard proof that $\int_{-\infty}^{\infty} e^{-z^2}\,\mathrm{d}z = \sqrt{\pi}$.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,\mathrm{d}x\,\mathrm{d}y = \left(\int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\,\mathrm{d}y\right) = I^2.$$

Then

$$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} = \int_0^{\infty}\int_0^{2\pi} e^{-r^2} r\,\mathrm{d}\theta\,\mathrm{d}r$$
$$= \pi \int_0^{\infty} 2re^{-r^2}\,\mathrm{d}r \underbrace{=}_{s=r^2,\ \mathrm{d}s=r2\,\mathrm{d}r} \pi \int_0^{\infty} e^{-s}\,\mathrm{d}s = -\pi e^{-s}|_0^{\infty} = \pi.$$

$\square$

## 8.1 Initial Value Problem

IVP aka Cauchy problem for homogeneous heat equation

$$\begin{cases} u_t - \Delta u = 0, & \text{on } \mathbb{R}^n \times (0,\infty) \\ u = g, & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}.$$

No boundary conditions for now. Since $\Phi$ solves the heat equation away from $(0,0)$, we will try to use its convolution to satisfy the initial condition.

**Theorem 8.3.** *Assume $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and define*

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t) g(y)\,\mathrm{d}y = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y)\,\mathrm{d}y \ \forall x \in \mathbb{R}^n,\ t>0.$$

*Then*

1. $u \in C^\infty(\mathbb{R}^n \times (0,\infty))$,
2. $u_t - \Delta u = 0$ *on* $\mathbb{R}^n \times (0,\infty)$,
3. $\lim_{(x,t)\to(x_0,0)} u(x,t) = g(x_0)\ \forall x_0 \in \mathbb{R}^n$ *for* $t>0$.

*Proof.* Prove each of the three.

1. For every $\delta > 0$, $\Phi \in C^\infty(\mathbb{R}^n \times (\delta,\infty))$ and all derivatives of $\Phi$ have a uniform (though $\delta$-dependent) bounds on $\mathbb{R}^n \times (\delta,\infty) \implies (1)$.
2. Also, for every $t>0$,

$$u_t(x,t) - \Delta u(x,t) \underbrace{=}_{\text{since } \Phi\in C^\infty, t>0} \int_{\mathbb{R}^n} (\underbrace{[\Phi_t - \Delta_x \Phi]}_{=0,\ t>0}(x-y,t)) g(y)\,\mathrm{d}y = 0.$$

3. Fix $x_0 \in \mathbb{R}^n$ and $\epsilon > 0$, then choose $\delta > 0$ such taht $|y-x_0| < \delta \implies |g(x_0) - g(y)| < \epsilon$. If $|x-x_0| < \delta/2$, then

$$|u(x,t) - g(x,0)| = \left| \int_{\mathbb{R}^n} \Phi(x-y,t) g(y)\,\mathrm{d}y - \underbrace{\int_{\mathbb{R}^n} \Phi(x-y,t) g(x_0)\,\mathrm{d}y}_{=g(x_0)} \right|$$

then split the domain and use triangle inequality as

$$\le \underbrace{\int_{B(x_0,\delta)} \Phi(x-y,t)|g(y)-g(x_0)|\,\mathrm{d}y}_{I} + \underbrace{\int_{\mathbb{R}^n\setminus B(x_0,\delta)} \Phi(x-y,t)|g(y)-g(x_0)|\,\mathrm{d}y}_{J}$$

$$I = \int_{B(x_0,\delta)} \Phi(x-y,t)|g(y)-g(x_0)|\,\mathrm{d}y \underbrace{\le}_{|y-x_0|<\delta} \epsilon \int_{B(x_0,\delta)} \Phi(x-y,t)\,\mathrm{d}y \le \epsilon.$$

Note if $|y-x_0| \ge \delta$ and $|x-x_0| \le \delta/2$, then

$$|y-x_0| \le |y-x| + |x-x_0| \le |y-x| + \frac{\delta}{2} \le |y-x| + \frac{1}{2}|y-x_0|.$$

So $\frac{1}{2}|y-x_0| \leq |y-x|$. We will use this fact Then use this fact in the second line below then use polar integration so say

$$J = \int_{\mathbb{R}^n \setminus B(x_0,\delta)} \Phi(x-y,t)|g(y)-g(x_0)| \,\mathrm{d}y \leq 2\left\|g\right\|_{L^\infty} \int_{\mathbb{R}^n \setminus B(x_0,\delta)} \Phi(x-y,t)\,\mathrm{d}y,$$
$$= \frac{c}{t^{n/2}} \int_{\mathbb{R}^n \setminus B(x_0,\delta)} e^{\frac{-|x-y|^2}{4t}} \,\mathrm{d}y \leq \frac{c}{r^{n/2}} \int_{\mathbb{R}^n \setminus B(x_0,\delta)} e^{-\frac{|x_0-y|^2}{16t}} \,\mathrm{d}y = \frac{c}{r^{n/2}} \int_\delta^\infty e^{-\frac{r^2}{16t}} r^{n-1}\,\mathrm{d}r,$$

Now let $s = \frac{r}{\sqrt{t}}$, $r = s\sqrt{t}$, $\mathrm{d}r = \sqrt{t}\,\mathrm{d}S$

$$J \leq \frac{c}{t^{n/2}} \int_{\delta/\sqrt{t}}^\infty e^{-\frac{s^2}{16}} s^{n-1} t^{(n-1)/2} t^{1/2}\,\mathrm{d}s = c\int_{\delta/\sqrt{t}}^\infty e^{-s^2/16} s^{n-1}\,\mathrm{d}s \to 0 \text{ as } t \to 0^+.$$

Thus, $|x-x_0| < \delta/2$ and $t>0$ small enough $\implies |u(x,t)-g(x_0) \leq 2\epsilon$.

□

**Interpretation of the Fundamental Solution.** It solves

$$\begin{cases} \Phi_t - \Delta\Phi = 0, & \text{on } \mathbb{R}^n \times (0,\infty) \\ \Phi = \delta_0, & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}.$$

Since $u(x,0) = \int_{\mathbb{R}^n} \Phi(y-x,0)g(y)\,\mathrm{d}y = g(x)$.

Also note the "infinite speed of information propogation" for the heat equation: e.g. if $g \in C_c(\mathbb{R}^n)$ is non-negative and bounded, then $u(x,t) > 0\ \forall x \in \mathbb{R}^n,\ t>0$. I.e. regardless of how far $x$ is from support($g$) and how little time has passed. [Definitely note consistent with experimental observations on heat or diffusion. Shows the limits of usefulness/applicability of parabolic PDEs.]

Also worth noting: $g$ is just continuous (and can be relaxed!) while $u \in X^\infty$.

# 9 (9/26/23)

**Theorem 9.1.** Duhamel's principle: linear superposition in time. *Can be used to solve the IVP for the nonhomogeneous heat equation*

$$\begin{cases} u_t - \Delta u = f, & \text{on } \mathbb{R}^n \times (0,\infty) \\ u = 0, & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}.$$

$f(x,t)$ is basically the amount of "stuff" deposited at $x$ at the time $t$.

**The idea:** define $u(x,t;s)$ to encode the *homogeneous* diffusion of stuff deposited at time $s$ for all future times $t>s$:

$$u(x,t;s) = \int_{\mathbb{R}^n} \Phi(x-y,t-s)f(y,s)\,\mathrm{d}y$$

solves

$$\begin{cases} u_t(x,t;s) - \Delta u(x,t;s) = 0 & \text{on } \mathbb{R}^n \times (s,\infty) \\ u(x,t;s) = f(x,s) & \text{on } \mathbb{R}^n \times \{t=s\}. \end{cases}$$

(This is my "single account" that I create at time $s$. Then I have to add up all the accounts made.) Then we integrate in $s$ to add up the density of stuff deposited at different times (up to the current $t$). I.e. define $u(x,t) = \int_0^t u(x,t;s)\,\mathrm{d}s$ for $x \in \mathbb{R}^n,\ t > 0$.

$$u(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s) f(y,s)\,\mathrm{d}y\,\mathrm{d}s = \int_0^t [4\pi(t-s)]^{-n/2} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4(t-s)}} f(y,s)\,\mathrm{d}y\,\mathrm{d}s.$$

**Theorem 9.2.** *Suppose $f \in C_1^2(\mathbb{R}^n \times [0,\infty))$ and has compact support, while $u(x,t)$ is defined as above. Then*

1. $u \in C_1^2(\mathbb{R}^n \times (0,\infty))$ *(i.e. $u_t, u_{x_i}, u_{x_i x_j} \in C\ \forall i,j$.*
2. $u_t - \Delta u = f$ *on* $\mathbb{R}^n \times (0,\infty)$
3. *For $t > 0$, $\lim_{(x,t)\to(x_0,0)} = 0\ \forall x_0 \in \mathbb{R}^n$*

*Proof.* In order.

1. Since $\Phi$ is singular at $(0,0)$, cannot directly interchange the order of integration and differentiation. Need a change of variables first:

$$u(x,t) = \int_0^t \underbrace{\int_{\mathbb{R}^n} \Phi(y,s) f(x-y,t-s)\,\mathrm{d}y}_{a(t,s)}\,\mathrm{d}s.$$

*Remark* 9.3. Recall Leibniz formula: Consider $A(t,s)$, $a(t,s) = \frac{\partial}{\partial s} A(t,s)$ and $v(t) \coloneqq \int_0^t a(t,s)\,\mathrm{d}s = A(t,t) - A(t,0)$. Then

$$v'(t) = A_t(t,t) + \underbrace{A_s(t,t)}_{a(t,t)} - A_t(t,0) = a(t,t) + \int_0^t a_t(t,s)\,\mathrm{d}s.$$

So, using this Leibniz rules,

$$u_t(x,t) = \int_0^t \underbrace{\int_{\mathbb{R}^n} \Phi(y,s) f_t(x-y,t-s)\,\mathrm{d}y}_{a_t(t,s)}\,\mathrm{d}s + \underbrace{\int_{\mathbb{R}^n} \Phi(y,t) f(x-y,0)\,\mathrm{d}y}_{a(t,t)}$$

Similarly,

$$u_{x_i x_j}(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(y,s) f_{x_i x_j}(x-y,t-s)\,\mathrm{d}y\,\mathrm{d}s \quad \text{for } i,j = 1,\ldots,n$$

and $u, u_t, u_{x_i}, u_{x_i x_j} \in C(\mathbb{R}^n \times (0,\infty))$.

2.

$$\begin{aligned} &u_t(x,t) - \Delta u(x,t) \\ &= \int_0^t \int_{\mathbb{R}^n} \underbrace{\Phi(y-s) \left[\frac{\partial}{\partial t} - \Delta_x\right] f(x-y,t-s)}_{Q(x,y,t,s)}\,\mathrm{d}y\,\mathrm{d}s + \underbrace{\int_{\mathbb{R}^n} \Phi(y,t) f(x-y,0)\,\mathrm{d}y}_{K} \\ &= \underbrace{\int_\epsilon^t \int_{\mathbb{R}^n} Q(x,y,t,s)\,\mathrm{d}y\,\mathrm{d}s}_{I_\epsilon} + \underbrace{\int_0^\epsilon \int_{\mathbb{R}^n} Q(x,y,t,s)\,\mathrm{d}y\,\mathrm{d}s}_{J_\epsilon} + K. \end{aligned}$$

We see that $|J_\epsilon| \le (\|f_t\|_\infty + \left\|D_x^2 f\right\|_\infty)\epsilon \to 0$ as $\epsilon \to 0$. Then

$$\begin{aligned}
I_\epsilon &= \int_\epsilon^t \int_{\mathbb{R}^n} \Phi(y,s)\left[\frac{\partial}{\partial t} - \Delta x\right] f(x-y,t-s)\,\mathrm{d}y\,\mathrm{d}s \\
&= \int_\epsilon^t \int_{\mathbb{R}^n} \Phi(y,s)\left[-\frac{\partial}{\partial s} - \Delta_y\right] f(x-y,t-s)\,\mathrm{d}y\,\mathrm{d}s \text{ now integrate by parts, use } f(\cdot,t)\in C_c(\mathbb{R}^n) \\
&= \int_\epsilon^t \int_{\mathbb{R}^n} \underbrace{\left[\left(\frac{\partial}{\partial s} - \Delta_y\right)\Phi(y,s)\right]}_{=0,\ s>\epsilon>0} f(x-y,t-s)\,\mathrm{d}y\,\mathrm{d}s \\
&\quad + \underbrace{\int_{\mathbb{R}^n} \Phi(y,\epsilon) f(x-y,t-\epsilon)\,\mathrm{d}y - \overbrace{\int_{\mathbb{R}^n} \Phi(y,t) f(x-y,0)\,\mathrm{d}y}^{K}}_{\text{boundary terms when integrating by parts in } s}.
\end{aligned}$$

So,

$$I_\epsilon + K = \int_{\mathbb{R}^n} \Phi(y,\epsilon) f(x-y,t-\epsilon)\,\mathrm{d}y = \int_{\mathbb{R}^n} \Phi(x-y,\epsilon) f(y,t-\epsilon)\,\mathrm{d}y \to f(x,t) \text{ as } \epsilon \to 0.$$

3.

$$\begin{aligned}
|u(x,t)| &= \left|\int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s) f(y,s)\,\mathrm{d}y\,\mathrm{d}s\right| \le \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s)|f(y,s)|\,\mathrm{d}y\,\mathrm{d}s \\
&\le \|f\|_\infty \int_0^t \underbrace{\int_{\mathbb{R}^n} \Phi(x-y,t-s)\,\mathrm{d}y}_{=1}\,\mathrm{d}s = t\,\|f\|_\infty \to 0 \text{ as } t \to 0.
\end{aligned}$$

Combining 2 previous theorems: Suppose $f \in C_1^2(\mathbb{R}^n \times [0,\infty))$, $f(x,t)$ has compact support and $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Then

$$u(x,t) \coloneqq \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s) f(y,s)\,\mathrm{d}y\,\mathrm{d}s + \int_{\mathbb{R}^n} \Phi(x-y,t) g(y)\,\mathrm{d}y$$

is $C_2^1$ and solves

$$\begin{cases} u_t - \Delta u = f & \text{on } \mathbb{R}^n \times (0,\infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}.$$

$\square$

**A bit of notation:** $U_T = U \times (0,T]$ is a *parabolic cylinder.* $\Gamma_T = \bar{U}_T \setminus U_T$ is a *parabolic boundary* (does not include the "top").

**Theorem 9.4** (Maximum principle of the heat equation)**.** *Suppose $u \in C_1^2(U_T) \cap C(\bar{U}_T)$ solves $u_t - \Delta u = 0$ on $U_T$.*

*1.* Weak maximum principle:

$$\max_{\bar{U}_T} u = \max_{\Gamma_T} u$$

2. Strong maximum principle: *if $U$ is connected and $(x_0, t_0) \in U_T$ is such that $u(x_0, t_0) = \max_{\bar{U}_T} u$, then*

$$u(x, t) = u(x_0, t_0) \; \forall (x, t) \in \bar{U}_{t_0}.$$

(Intuition on $\mathbb{R}^n$ :

$$u(x, t) = \int_{\mathbb{R}^n} \Phi(x - y, t - s) u(y, s) \, \mathrm{d}y \; \forall s < t$$

means that $u(x, t)$ is the average of $u$ values at an earlier time; so, it should not be bigger than all of them.) An elementary proof of the weak maximum principle on bounded domains:

*Proof.* Suppose $U$ is open and bounded, while $u \in C_1^2(U_T) \cap C(\bar{U}_T)$ satisfies $u_t \leq \Delta u$ on $U_T$. Then $\max_{\bar{U}_T} u = \max_{\Gamma_T} u$. (Assuming $T > 0$ is finite).

Since $\bar{U}_T$ is compact, $u$ attains its maximum at some $(x_0, t_0)$. If we assume $(x_0, t_0) \notin \Gamma_T$, then $u_t(x_0, t_0) \geq 0$ and $\Delta u(x_0, t_0) \leq 0$ (otherwise it wouldn't be a maximum). If we had $u_t < \Delta u$ on $U_T$, this would yield a contradiction, but we only have $u_t \leq \Delta u$. So, we define $v^\epsilon(x, t) := u(x, t) - \epsilon t$ for $\epsilon > 0$. Note that $v_t^\epsilon = u_t - \epsilon$ and $\Delta v^\epsilon = \Delta u$. So $v_t^\epsilon < \Delta v^\epsilon$ and the above argument implies that $\max_{\bar{U}_T} v^\epsilon = \max_{\Gamma_T} v^\epsilon$. Taking $\epsilon \to 0$ completes the proof. □

Can similarity prove the "minimum principle" for every $u \in C_1^2(U_T) \cap C(\bar{U} - T)$ such that $u_t \geq \Delta u$ on $U_T$.

A useful corollary:

**Theorem 9.5** (Uniqueness of solution on bonded domains)**.** *Suppose $U$ is open and bounded, $g \in C(\Gamma_t)$ and $f \in C(U_T)$. If $u_1, u_2 \in C_1^2(U_T) \cap C(\bar{U}_T)$ and both solve*

$$\begin{cases} u_t - \Delta u = f & \text{on } U_T \\ u = g & \text{on } \Gamma_T, \end{cases}$$

*then $u_1 = u_2$ on $U_T$.*

*Proof.* Note that both $v = u_1 - u_2$ and $-v = u_2 - u_1$ solve $w_t - \Delta_w = 0$ on $U_T$ and $W = 0$ on $\Gamma_T$. The weak maximum principles implies

$$\begin{cases} v \leq 0 & \text{on } U_T \\ -v \leq 0 & \text{on } U_T \end{cases} \implies u_1 = u_2 \text{ on } U_T.$$

□

# 10 Numerics of Heat Equation (9/28/23)

We will skip mean value property of heat equation and proof of smoothness of solution on a bounded domain. Also estimates of the derivatives from the mean value properties. Also uniqueness of solution backwards in time.

Today, we are doing numerics for the heat equation (application!). We consider $\mathbb{R}$, and we are on a ring. Let's impose a 2D grid with points $(x_i, t_m)$ with $x_i = ih$, $t_m = mk$ for $h, k > 0$. The notation $u_i^m := u(x_i, t_n)$. The heat equation in 1D is

$$u_t - a u_{xx} = 0,$$

where $a > 0$ is the diffusivity constant. We take $U_i^m \sim$ numerical approximation. Consider the Taylor expansion with remainder expanded about $t_m$:

$$u_i^{m+1} = u(x_i, t_m + k) = u_i^m + k\frac{\partial u}{\partial t}(x_i, t_m) + \frac{k^2}{2!}\frac{\partial^2 u}{\partial t^2}(x_i, \xi), \quad \xi \in (t_m, t_{m+1}).$$

Solving for the time derivative,

$$\frac{\partial u}{\partial t}(x_i, t_m) = \underbrace{\frac{u_i^{m+1} - u_i^m}{k}}_{D_t^+ u_i^m} - \underbrace{\frac{k}{2}\frac{\partial^2 u}{\partial t^2}(x_i, \xi)}_{\text{error}}.$$

This is our approximation of the time derivative, and the + superscript is because we take a forward difference here. We can do the same thing in space, but then we need 2 derivatives.

$$u_{i\pm 1}^m = u(x_i \pm h, t_m) = u_i^m \pm h\frac{\partial u}{\partial x}(x_i, t_m) + \frac{h^2}{2!}\frac{\partial^2 u}{\partial x^2}(x_i, t_m) \pm \frac{h^3}{6}\frac{\partial^3 u}{\partial x^3}(x_i, t_m) + \frac{h^4}{24}\frac{\partial^4 u}{\partial x^4}(\eta_\pm, t_m),$$

with $\eta_\pm \in (x_{i-1}, x_{i+1})$. Trying to simplify,

$$\begin{aligned} u_{i-1}^m + u_{i+1}^m &= 2u_i^m + h^2\frac{\partial^2 u}{\partial x^2}(x_i, t_m) + \frac{h^4}{12}\frac{\partial^4 u}{\partial x^4}(\eta, t_m) \\ \underbrace{\frac{u_{i-1}^m + u_{i+1}^m - 2u_i^m}{h^2}}_{D_x^+ D_x^- u_i^m} &= \frac{\partial^2 u}{\partial x^2}(x_i, t_m) + \frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}(\eta, t_m). \end{aligned}$$

$D_x^- u_i^m = \frac{u_i^m - u_{i-1}^m}{h}$, so then

$$D_x^+ D_x^- u_i^m = \frac{\frac{u_{i+1}^m - u_i^m}{h} - \frac{u_i^m - u_{i-1}^m}{h}}{h}.$$

Now putting it all together,

$$\begin{aligned} u_t - au_{xx} \approx D_t^+ U_i^m - aD_x^+ D_x^- U_i^m &= 0 \ \forall i, m \\ \frac{U_i^{m+1} - U_i^m}{k} - a\frac{U_{i-1}^m - 2U_i^m + U_{i+1}^m}{h^2} &= 0. \end{aligned}$$

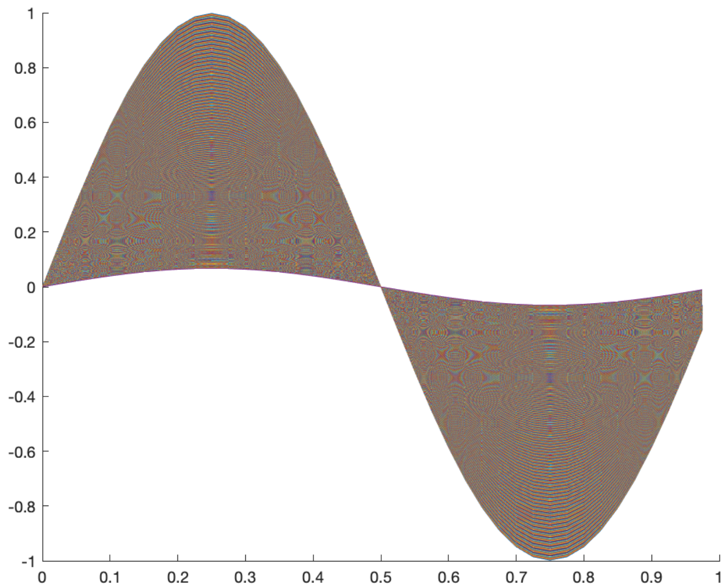
Treat $U^m$ as known and $U^{m+1}$ as unknown. Let $\lambda = ak/h^2$.

$$U_i^{m+1} = \lambda U_{i-1}^m + (1 - 2\lambda)U_i^m + \lambda U_{i+1}^m.$$

---

```
% Solve heat equation
numX = 40;
h = 1/numX;

k = 0.001/16;

a = 1; % Diffusivity
lambda = a*k/(h^2);

T = 2;
M = T/k;
% Circle with circumference 1
x = (0:(numX-1))*h;
```

```matlab

% Initial condition
u = sin(2*pi*x);

close all
hold on
for m = 1:M
    u_left = [u(end), u(1:numX-1)];
    u_right = [u(2:numX), u(1)];
    u = lambda * u_left + (1-2*lambda)*u + lambda*u_right;
    plot(x, u);
    pause(0.1);
end
hold off
```




In example, we can't let $h$ and $k$ go to 0 while fixing the other (this causes blowup in the solution). We have to shrink both in some refinement path together.

The property you want in a numerical scheme is maximum principle. If you have that, things cannot blow up because we know the norm of the solution is always bound by the initial.

$$\left|U_i^{m+1}\right| \leq \lambda \left|U_{i-1}^m\right| + |1-2\lambda| \left|U_i^m\right| + \lambda \left|U_{i+1}^m\right|$$

If $\lambda \leq \frac{1}{2}$, then the middle term is positive, and we can drop the abs.

$$\begin{aligned} \left|U_i^{m+1}\right| &\leq \lambda \underbrace{\left|U_{i-1}^m\right|}_{\leq \|U^m\|_\infty = \max_j |U_j^m|} + (1-2\lambda)\left|U_i^m\right| + \lambda \underbrace{\left|U_{i+1}^m\right|}_{\leq \|U^m\|_\infty = \max_j |U_j^m|}, \\ &\leq \underbrace{(\lambda + (1-2\lambda) + \lambda)}_{=1} \|U^m\|_\infty \implies \left\|U^{m+1}\right\| \infty \leq \|U^m\|_\infty \end{aligned}$$

from the maximum principle. We want $\lambda = a\frac{k}{h^2} \le \frac{1}{2}$. With $\lambda = \frac{1}{2}$,

$$U_i^{m+1} = \frac{1}{2}(U_{i-1}^m + U_{i+1}^m).$$

This is an average like diffusion with random walk. Why is this independent of $a$? (Question on a homework).

# 11 More on Heat Equation (10/3/23)

Remember the definitions. $U_T = U \times (0, T]$ is a *parabolic cylinder*. $\Gamma_T = \bar{U}_T \setminus U_T$ is a *parabolic boundary* (does not include the "top"). **Another proof of uniqueness** (by "energy method") on a bounded $U$.

**Theorem 11.1.** *Suppose $U \subset \mathbb{R}^n$ is open and bounded with $\partial U \in C^1$ (smooth boundary). If $u_1, u_@ \in C_1^2(\bar{U}_T)$ both solve*

$$\begin{cases} u_t - \Delta u = f, & \text{on } U_t \\ u = g & \text{on } \Gamma_T, \end{cases}$$

*then $u_1 \equiv u_2$ on $\bar{U}_T$.*

*Proof.* We know $w = u_1 - u_2$ solves $w_t - \Delta w = 0$ on $U_T$ and $w = 0$ on $\Gamma_T$. Define the "energy"

$$e(t) := \int_U w^2(x,t)\,\mathrm{d}x, \quad t \in [0,T].$$

Then the derivative

$$\begin{aligned} e'(t) &= \int_U 2w(x,t)w_t(x,t)\,\mathrm{d}x \\ &= 2\int_U w\Delta w\,\mathrm{d}x \\ &= -2\int_U |Dw|^2\,\mathrm{d}x \le 0 \quad \text{from integrate by parts} \\ &\le 0. \end{aligned}$$

So, $e(0) = 0$ and $e(t) \ge 0 \implies e(t) = 0\ \forall t \in [0,T] \iff u_1 \equiv u_2$ on $U_T$. □

Remarks:

- Backwards solution to heat equation is ill-posed because it does not have continuous dependence on the data.
- Looking at other norms over time is instructive on solution. We saw infinity norm is not growing. We just saw the $L^2$ norm also is not growing. From boundary conditions, we can interpret the $L^1$ norm of the solution.

Weak maximum principle for a heat equation on $\mathbb{R}^n$ turns out to hold only with an additional growth bound.

**Theorem 11.2** (Weak maximum principle for heat equation)**.** *Suppose $u \in C_1^2(\mathbb{R}^n \times (0,T]) \cap C(\mathbb{R}^n \times [0,T])$ solves*

$$\begin{cases} u_t - \Delta u = 0 & \text{on } \mathbb{R}^n \times (0,T) \\ u = g & \text{on } \mathbb{R}^n \times \{t=0\}, \end{cases}$$

*and satisfies $u(x,t) \le Ae^{a|x|^2}$ for some constants $a, A$ and all $x \in \mathbb{R}^n$, $t \in [0,T]$. Then*

$$\sup_{\mathbb{R}^n \times [0,T]} u = \sup_{\mathbb{R}^n} g.$$

Three key ideas for the proof:

1. It's enough to show the fact for $t \in [0,\delta]$ with some $\delta > 0$ and then iterate (for $t \in [\delta, 2\delta]$, etc.). This works because $u(x,\delta)$ is playing the role of initial condition when $t \in [\delta, 2\delta]$, and we will already have $u(x,\delta) \le \sup_{\mathbb{R}^n} g \; \forall x \in \mathbb{R}^n$.
2. Find a way to use the weak maximum principle on compact sets (already proven!) to prove 1.
3. It's enough to do 2. with some $v^\mu(x,t)$ such that $v^\mu(x,t) \to u(x,t)$ as $\mu \to 0^+$.

*Proof.* Choose $\delta \in (0, \frac{1}{4a})$ and $\epsilon \ge 0$ such that $(\delta + \epsilon) < \frac{1}{4a}$. Our goal is to first show weak max principle for $t \in [0,\delta]$. For any (fixed) $y \in \mathbb{R}^n$, we will now show $u(y,t) \le \sup_{\mathbb{R}^n} g \; \forall t \in [0,\delta]$. Define

$$v(x,t) \coloneqq u(x,t) - \mu \underbrace{\frac{1}{(\delta + \epsilon - t)^{n/2}} e^{\frac{|x-y|^2}{4(\delta+\epsilon-t)}}}_{w(x,t)}.$$

(Treat $\mu > 0$ as fixed.) By direct verification: $w_t - \Delta w = 0$ and thus $v_t - \Delta v = 0$. Now we need to find a compact $\bar{U}$ on which to conclude

$$\max_{\bar{U}_\delta} v = \max_{\Gamma_\delta} v \le \sup_{\mathbb{R}^n} g.$$

Recall parabolic cylinder and boundary, respectively, are $U_\delta = U \times (0,\delta]$ and $\Gamma_\delta = \bar{U}_\delta \setminus U_\delta$.

The idea: take $U \coloneqq B^0(y,r)$ for a sufficiently large $r > 0$. $\Gamma_\delta$ consists of the bottom ($t = 0$, $x \in B^0(y,r)$) and the walls ($t \in [0,\delta]$, $x \in \partial B(y,r)$). For the bottom:

$$v(x,0) = u(x,0) - \mu(\delta+\epsilon)^{-n/2} \exp\left(\frac{|x-y|^2}{4(\delta+\epsilon)}\right) \le u(x,0) = g(x).$$

For the walls:

$$\begin{aligned} v(x,t) &= \underbrace{u(x,t)}_{\le Ae^{a|x|^2}} - \mu(\delta+\epsilon-t)^{-n/2} \exp\left(\frac{r^2}{4(\delta+\epsilon-t)}\right) \quad \text{and note that } |x| \le |y| + r \\ &\le Ae^{a(|y|+r)^2} - \mu(\delta+\epsilon-t)^{-n/2} \exp\left(\frac{r^2}{4(\delta+\epsilon-t)}\right) \\ &\le Ae^{a(|y|+r)^2} - \mu(\delta+\epsilon)^{-n/2} \exp\left(\frac{r^2}{4(\delta+\epsilon)}\right) \\ &\le Ae^{a(|y|+r)^2} - \mu(\delta+\epsilon)^{-n/2} \exp((a+\gamma)r^2) \to -\infty \text{ as } r \to \infty \end{aligned}$$

since the second term grows faster than the first. The last few lines are from $\delta + \epsilon < \frac{1}{4a} \iff a < \frac{1}{4(\delta+\epsilon)}$. So, $a + \gamma = \frac{1}{4(\delta+\epsilon)}$ for some $\gamma > 0$.

So, on the walls, $v(x,t) \le \sup_{\mathbb{R}^n} g \ \forall x \in \partial B(y,r)\ t \in [0,\delta]$ if $r$ is large enough. Thus,

$$v(y,t) \le \max_{\bar{U}_\delta} v \le \max_{\Gamma_\delta} v \le \sup_{\mathbb{R}^n} g \ \forall y \in \mathbb{R}^n,\ t \in [0,\delta].$$

Let $\mu \to 0$ to get the same result for $u$. Now iterate the argument for $t \in [\delta, 2\delta]$, $t \in [2\delta, 3\delta], \ldots$ until reaching $T$. $\square$

*Remark* 11.3 (Analytic solution for heat equation on a bounded domain). If we start at some $(x,t)$ and begin Brownian motion, we can either stay inside the domain for the entire time $T$ or we will end up hitting the parabolic boundary. We can ask about the probability distribution about the parts of the parabolic boundary we will be hitting. If we have $u_t - \Delta u = 0$ on $U_T$ and $u = g$ on $\Gamma_T$, we can represent

$$u(x,t) = \int_{\Gamma_T} K(x,t,y,s)\,\mathrm{d}y\,\mathrm{d}s,\ (y,s) \in \Gamma_T.$$

This $K$ will be the probability, that if I start at $(x,t)$ and run Brownian motion, the first point I touch on the boundary is $(y,s)$. So we see that finding the expression for the $K$ can be quite hard.

*Remark* 11.4. MATLAB demo for melting a penny. Heat equation smooth things out. It can help denoise an image. Photoshop does not use this exactly. Look at Anisotropic diffusion.

## 11.1 Wave equation

We have

$$u_{tt} + \Delta u = f,\ t \ge 0,\ x \in U \subset \mathbb{R}^n.$$

Consider $\square u \coloneqq u_{tt} - \Delta u$ is called the D'Alembertian.

**Physcially.** $n = 1$ is a vibrating string, $n = 2$ is a vibrating membrane, $n = 3$ is an elastic solid. $u(x,t)$ is the physical displacement ("in some direction") of a point (which normally is at rest at $x$) at the time $t$.

**Derivation.** If $V$ is a smooth subset of $U$, its total acceleration is

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int_V u\,\mathrm{d}x = \int_V u_{tt}\,\mathrm{d}x.$$

Assume unit mass density, if $F$ is the force at every point on $V$,

$$\int_V u_{tt}\,\mathrm{d}x = -\int_{\partial V} F \cdot \nu(y)\,\mathrm{d}S(y) \text{ Newton's 2nd Law.}$$

Then use Gauss-Green, to say

$$\int_V u_{tt} + \mathrm{div}(F)\,\mathrm{d}x = 0.$$

Since this holds for any $V$, then the integrand must be identically 0, giving us

$$u_{tt} = -\operatorname{div} F.$$

For the elastic case: $F(Du) \approx -aDu$ for $a > 0$ (Hook's Law). Then

$$u_{tt} = a\Delta u.$$

## 12 Wave Equation (10/5/23)

We start with solving the wave equation with $n = 1$.

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{on } \mathbb{R} \times (0, \infty), \\ u = g, \ u_t = h & \text{on } \mathbb{R} \times \{t = 0\}. \end{cases}$$

"Factoring" $D_{tt} - D_{xx} : u_{tt} - u_{xx} = \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) u = 0$

The idea: define $v(x,t) = u_t(x,t)$. Then,

$$v_t + v_x = (u_{tt} - u_{xt}) + (-u_{xx}) = u_{tt} - u_{xx} = 0.$$

So, instead of solving a single 2nd order equation, we can solve a first-order system:

$$\begin{cases} v_t + v_x = 0 \\ u_t - u_x = v \end{cases},$$

which are homogeneous and non-homogeneous transport equations.

Recall from lecture 2:

$$\begin{cases} w_t + b \cdot Dw = 0 & \text{on } \mathbb{R}^n \times (0, \infty), \\ w = a(x) & \text{on } \mathbb{R}^n \times \{t = 0\}, \end{cases}$$

implies $w(x,t) = a(x - tb)$. So, $v_t + v_x = 0$ implies

$$v(x,t) = a(x-t) = u_t(x-t, 0) - u_x(x-t, 0) = h(x-t) - g'(x-t).$$

Recall again from Lecture 2:

$$\begin{cases} w_t + b \cdot Dw = c(x,t) & \text{on } \mathbb{R}^n \times (0, \infty), \\ w = a(x) & \text{on } \mathbb{R}^n \times \{t = 0\}, \end{cases}$$

implies $w(x,t) = a(x - tb) + \int_0^t c(x + (s-t)b, s)\, \mathrm{d}s$. So, ($b = 0$ then) $u_t - u_x = v(x,t) = h(x-t) - g'(x-t) = c(x,t)$ on $\mathbb{R} \times (0, \infty)$ and $\mathbb{R} \times \{t = 0\}$ implies

$$\begin{aligned} u(x,t) &= g(\underbrace{x+t}_{x-tb}) + \int_0^t c(\underbrace{x+t-s}_{x+(s-t)b}, s)\, \mathrm{d}s, \\ &= g(x+t) + \int_0^t (h(x+t-s-s) - g'(x+t-s-s))\, \mathrm{d}s, \\ &= g(x+t) + \frac{1}{2}\int_{x-t}^{x+t} h(y) - g'(y)\, \mathrm{d}y, \text{ from } y = x+t-2s, \ \ \mathrm{d}y = -2\, \mathrm{d}s, \\ &= \frac{1}{2}(g(x+t) + g(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} h(y)\, \mathrm{d}y. \end{aligned}$$

Last line from FTC is *d'Alembert's formula* for $n = 1$.

**Some interesting things to notice.** For any point $(x,t)$, there is a *cone of dependence*, where the lines of $u$ and $v$ form a cone about the points $(x,t)$, $(x-t, 0)$, $(x+t, 0)$. This is the finite speed of information propagation. With more and more time, we learn about more things that were further from the location $x$ when things started (cone gets bigger). For any point $(y, 0)$, there is a cone of influence that tells us which points $(x, t)$ are affected by $(y, 0)$.

We are not covering general $n$ of wave equation. We will only cover $n = 2, 3$ because the form is basically the same, and we will save time/notational headache.

**Theorem 12.1** (d'Alembert's formula produces a solution)**.** *If $g \in C^2(\mathbb{R}), h \in C^1(\mathbb{R})$ and $u(x,t)$ is defined by the equation above, then*

1. $u \in C^2(\mathbb{R} \times \mathbb{R}_+)$
2. $\Box u = u_{tt} - u_{xx} = 0$ *on* $\mathbb{R} \times \mathbb{R}_+$
3. *For* $t > 0$, $\lim_{x,t \to (x_0,0)} u(x,t) = g(x_0)$ *and* $\lim_{x,t \to (x_0,0)} u_t(x,t) = h(x_0)$ $\forall x_0 \in \mathbb{R}^n$.

*Proof.* 1. By Leibnitz Integration Rule and **??**.

2. Direct verification based on **??**.
3. By part 1 and (**??**) used with $t = 0$.

$\square$

Application/illustration: wave equation of a half-lines $\mathbb{R}_+ = \{x > 0\}$. Suppose

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{on } \mathbb{R}_+ \times \mathbb{R}_+ \\ u = g, \ u_t = h & \text{on } \mathbb{R}_+ \times \{t = 0\}. \end{cases}$$

The boundary condition is that $u = 0$ on $\{x = 0\} \times \mathbb{R}_+$ with $g(0) = h(0) = 0$.

An idea: extend to $\mathbb{R} \times \mathbb{R}_+$ via an *odd reflection.*

$$\tilde{u}(x,t) = \begin{cases} u(x,t), & x \geq 0 \\ -u(-x,t), & x \leq 0 \end{cases}, \quad \tilde{g}(x) = \begin{cases} g(x), & x \geq 0 \\ -g(-x), & x \leq 0 \end{cases}, \quad \tilde{h}(x) = \begin{cases} h(x), & x \geq 0 \\ -h(-x), & x \leq 0 \end{cases}.$$

This ensures the continuity of $u_x$ at $x = 0$. Would also need $g''(0) = 0$ for the continuity of $u_{xx}$. Now from the formula,

$$\tilde{u(x,t)} = \frac{1}{2}(\tilde{g}(x-t) + \tilde{g}(x+t)) + \frac{1}{2}\int_{x-t}^{x+t} \tilde{h}(y)\, \mathrm{d}y.$$

We can skip the $\tilde{}$ when $x \geq t \geq 0$. Otherwise, wit $t \geq x \geq 0$, $\tilde{g}(x-t) = -g(t-x)$ and

$$\int_{x-t}^{x+t} \tilde{h}(y)\, \mathrm{d}y = \int_{x-t}^{0} -h(-y)\, \mathrm{d}y + \int_{0}^{x+t} h(y)\, \mathrm{d}y = \int_{t-x}^{0} h(z)\, \mathrm{d}z + \int_{0}^{t+x} h(y)\, \mathrm{d}y.$$

So,

$$u(x,t) = \begin{cases} \frac{1}{2}(g(x+t) + g(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} h(y)\, \mathrm{d}y, & x \geq t \geq 0 \\ \frac{1}{2}(g(x+t) - g(t-x)) + \frac{1}{2}\int_{t-x}^{x+t} h(y)\, \mathrm{d}y, & t \geq x \geq 0 \end{cases}.$$

This can be seen as "wave reflection" at $x = 0$, which is the wall. Can think about once the domain of dependence reaches the origin. It cannot go past the origin, so what would go past gets mapped across the reflection across the $t$-axis.

The general $n \geq 2$ case: solution by "spherical means." Suppose $u(x,t) \in C^m(\mathbb{R}^n \times [0,\infty))$, $m \geq 2$, $x \in \mathbb{R}^n$ and

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{on } \mathbb{R}_n \times \mathbb{R}_+ \\ u = g,\ u_t = h & \text{on } \mathbb{R}_n \times \{t = 0\}. \end{cases}$$

Define

$$U(x;r,t) := \fint_{\partial B(x,r)} u(y,t)\,\mathrm{d}S(y), \quad G(x;r) := \fint_{\partial B(x,r)} g(y)\,\mathrm{d}S(y), \quad H(x;r) := \fint_{\partial B(x,r)} h(y)\,\mathrm{d}S(y).$$

The idea: Fix $x \in \mathbb{R}^n$ and derive a PDE for $U$ as a function of $(r,t)$, with $G, H$ specifying the boundary conditions.

**Theorem 12.2** (Euler-Poisson-Darboux equation)**.** *Let $u \in C^m(\mathbb{R}^n \times [0,\infty))$ satisfy the above equation. Fix $x \in \mathbb{R}^n$. Then $U(r,t) \in C^m([0,\infty) \times [0,\infty))$ will satisfy*

$$\begin{cases} U_{tt} - U_{rr} - \frac{n-1}{r} U_r = 0, & \text{on } \mathbb{R}_+ \times \mathbb{R}_+ \\ U = G, U_t = H, & \text{on } \mathbb{R}_+ \times \{t = 0\} \end{cases}.$$

*Proof.* Let (when we average over a ball, Jacobian terms does not arrive with change of variables)

$$\begin{aligned} U(r) &= \frac{\partial}{\partial r} \fint_{\partial B(x,r)} u(y,t)\,\mathrm{d}S(y) = \frac{\partial}{\partial r} \fint_{\partial B(0,1)} u(x + rz, t)\,\mathrm{d}S(z) \\ &= \fint_{\partial B(0,1)} Du(x+rz,t) \cdot z\,\mathrm{d}S(z) = \fint_{\partial B(x,r)} Du(y,t) \cdot \underbrace{\frac{y-x}{r}}_{\nu}\,\mathrm{d}S(y) \\ &= \frac{1}{n\alpha(n) r^{n-1}} \int_{B(x,r)} \Delta u(y,t)\,\mathrm{d}S(y), \quad \text{from divergence thm} \\ &= \frac{r}{n} \fint_{B(x,r)} \Delta u(y,t)\,\mathrm{d}y. \end{aligned}$$

Note that

$$U_t(r,t) = \frac{r}{n} \fint_{B(x,r)} \Delta u(y,t)\,\mathrm{d}y \to 0 \text{ as } r \to 0^+ \text{ since } u \in C^2.$$

Next,

$$\begin{aligned} U_{rr}(r,t) &= \frac{1}{n} \fint_{B(x,r)} \Delta u(y,t)\,\mathrm{d}y + \frac{r}{n} \frac{\partial}{\partial r} \left[ \frac{1}{\alpha(n) r^n} \int_0^r \int_{\partial B(x,\tau)} \Delta u(y,t)\,\mathrm{d}S(y)\,\mathrm{d}\tau \right] \\ &= \frac{1}{n} \fint_{B(x,r)} \Delta u(y,t)\,\mathrm{d}y + \frac{r}{n} \left[ \frac{-n}{\alpha(n) r^{n+1}} \int_{B(x,r)} \Delta u(y,t)\,\mathrm{d}y + \frac{1}{\alpha(n) r^n} \int_{\partial B(x,r)} \Delta u(y,t)\,\mathrm{d}S(y) \right] \\ &= \left( \frac{1}{n} - 1 \right) \fint_{B(x,r)} \Delta u(y,t)\,\mathrm{d}y + \fint_{\partial B(x,r)} \Delta u(y,t)\,\mathrm{d}S(y). \end{aligned}$$

So, $U_{rr} \to \Delta u(x,t) \frac{1}{n}$ as $r \to )^=$. (Higher derivatives can be computed similarly up to order $m$.)

$$\begin{aligned} U_{tt} - U_{rr} &= \underbrace{\fint_{\partial B(x,r)} \underbrace{u_{tt}(y,t)}_{=\Delta u(y,t)}\,\mathrm{d}S(y) - \fint_{\partial B(x,r)} \Delta(y,t)\,\mathrm{d}S(y)}_{=0} + \left(1 - \frac{1}{n}\right) \fint_{B(x,r)} \Delta u(y,t)\,\mathrm{d}y \\ &= \frac{n-1}{n} \frac{r}{r} \fint_{B(x,r)} \Delta u(y,t)\,\mathrm{d}y = \frac{n-1}{r} U_r, \end{aligned}$$

yielding Euler-Poisson-Darboux equation. □

# 13 More on Wave Equation (10/12/23)

# 14 (10/17/23)

## 14.1 Nonhomogeneous Wave Equation

$$\begin{cases} u_{tt} - \Delta u = f(x,t) & \text{on } \mathbb{R}^n \times \mathbb{R}_+ \\ u = 0, \ u_t = 0 & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases}$$

**Duhamel's Principle.** Suppose $v(x,t;s)$ ($s$ is a variation of parameter) solves

$$\begin{cases} v_{tt}(x,t;s) - \Delta_x v(x,t;s) = 0 & \text{on } \mathbb{R}^n \times (s,\infty) \\ v(x,t;s) = 0, \ v_t(x,t;s) = f(x;s) & \text{on } \mathbb{R}^n \times \{t = s\}. \end{cases}$$

Then $u(x,t) = \int_0^t v(x,t;s)\,\mathrm{d}s$ solves the nonhomogeneous problem. (Note: assuming $f \in C^{floor(n/2)+1}(\mathbb{R}^n \times \mathbb{R}_+)$, one can show that $u \in C^2(\mathbb{R}^n \times \mathbb{R}_+). See Thm 4, p81)$.

*Proof.*

$$u_t(x,t) = \underbrace{v(x,t;t)}_{=0} + \int_0^t v_t(x,t;s)\,\mathrm{d}s = \int_0^t v_t(x,t;s)\,\mathrm{d}s$$

$$u_{tt}(x,t) = v_t(x,t;t) + \int_0^t v_{tt}(x,t;s)\,\mathrm{d}s = f(x,t) + \int_0^t v_{tt}(x,t;s)\,\mathrm{d}s$$

$$\text{Finally, } \Delta u(x,t) = \int_0^t \Delta_x v(x,t;s)\,\mathrm{d}s = \int_0^t v_{tt}(x,t;s)\,\mathrm{d}s$$

□

Note:

1. $u(x,0) = 0$ and $u_t(x,0) = 0$ follows from the definition of $u$.
2. Solution of the nonhomogeneous wave equation with non-zero initial conditions is the solution of the nonhomogeneous problem with 0 IC + the solution of the homogeneous problem with non-zero coefficients.

**Explicit formulas for Nonhomogeneous.** For n=1:$t - s$ used instead of $t$ since the initial condition is at $t = s$

$$\begin{aligned} v(x,t;s) &= \frac{1}{2}\int_{x-\tau+s}^{x+t-s} f(y,s)\,\mathrm{d}y \\ u(x,t) &= \frac{1}{2}\int_0^t \int_{x-\tau+s}^{x+t+s} f(y,s)\,\mathrm{d}y\,\mathrm{d}s \underbrace{=}_{\xi = t-s, \ \mathrm{d}\xi = -s} -\frac{1}{2}\int_t^0 \int_{x-\xi}^{x+\xi} f(y,t-\xi)\,\mathrm{d}y\,\mathrm{d}\xi \\ &= \frac{1}{2}\int_0^t \int_{x-\xi}^{x+\xi} f(y,t-\xi)\,\mathrm{d}y\,\mathrm{d}\xi. \end{aligned}$$

For $n = 3$ :

$$
\begin{aligned}
v(c, t : s) &= (t-s) \fint_{\partial B(x, t-s)} f(y, s) \, \mathrm{d}S(y) \\
u(x, t) &= \int_0^t (t-s) \left( \fint_{\partial B(x, t-s)} f(y, s) \, \mathrm{d}S(y) \right) \mathrm{d}s \\
&= \int_0^t (t-s) \frac{1}{4\pi (t-s)^2} \left( \fint_{\partial B(x, t-s)} f(y, s) \, \mathrm{d}S(y) \right) \mathrm{d}s \\
&= \frac{1}{4\pi} \int_0^t \int_{\partial B(x, r)} \frac{f(y, t-r)}{r} \, \mathrm{d}S(y) \, \mathrm{d}r, \quad r = t-s, \ \mathrm{d}r = -\mathrm{d}s \\
&= \frac{1}{4\pi} \int_{B(x,t)} \underbrace{\frac{f(y, t-|y-x|)}{|y-x|}}_{\text{“retarded potential”}} \, \mathrm{d}y
\end{aligned}
$$

## 14.2 Energy methods for wave equation

**Theorem 14.1** (Uniqueness). *If $u, \tilde{u} \in C^2(\bar{U}_T)$ and both solve*

$$
\begin{cases}
u_{tt} - \Delta u = f & \text{on } U_t = U \times (0, T] \\
u = g & \text{on } \Gamma_T = \bar{U}_T \setminus U_t \\
u_t = h & \text{on } U \times \{t = 0\}
\end{cases},
$$

*then $u \equiv \tilde{u}$ on $\bar{U}_T$.*

*Proof.* $w = u - \tilde{u}$ solves $w_t - \Delta w = 0$ on $U_T$ with $w = 0 = w_t$ on $\Gamma_T$. Define

$$
\begin{aligned}
e(t) &= \frac{1}{2} \int_U w_t^2(x, t) + Dw(x, t) \cdot Dw(x, t) \, \mathrm{d}x, \ t \in [0, T] \\
e'(t) &\underbrace{=}_{u \in C^2} \int_U (w_t w_{tt} + Dw \cdot Dw_t) \, \mathrm{d}x \underbrace{=}_{\text{by parts, } Dw = 0 \text{ on } \Gamma_T} \int_U (w_t w_{tt} - \Delta w w_t) \, \mathrm{d}x \\
&= \int_U w_t \underbrace{(w_{tt} - \Delta w)}_{0} \, \mathrm{d}x = 0 \implies e(t) = e(0) \underbrace{=}_{w = 0 = w_t \text{ at } t = 0} 0 \underbrace{\implies}_{w_t \equiv 0, |Dw| \equiv 0} w \equiv 0 \text{ on } U_T.
\end{aligned}
$$

□

**Theorem 14.2** (Finite speed of information propagation). *Suppose $u \in C^2(\mathbb{R}^n \times \mathbb{R}_{+,0})$ and $u_{tt} - \Delta u = 0$ on $\mathbb{R}^n \times \mathbb{R}_+$. Define the* domain of dependence

$$
K(x_0, t_0) = \{(x, t) | t \in [0, t_0], \ |x - x_0| \leq t_0 - t\}.
$$

*If $u = u_t = 0$ on $B(x_0, t_0) \times \{t = 0\}$, then $u \equiv 0$ on $K(x_0, t_0)$.*

*Proof.* Define

$$
e(t) = \frac{1}{2} \int_{B(x_0, t_0 - t)} (u_t^2 + Du \cdot Du) \, \mathrm{d}x.
$$

Note that

$$\frac{\partial}{\partial r}\int_{B(x,r)} A(y)\,\mathrm{d}y = \frac{\partial}{\partial r}\int_0^r \int_{\partial B(x,\tau)} A(y)\,\mathrm{d}S(y)\,\mathrm{d}\tau = \int_{\partial B(x,r)} A(y)\,\mathrm{d}S(y).$$

Then

$$\begin{aligned} e'(t) &= \int_{B(x_0,t_0-t)} (u_t u_{tt} + Du\cdot Du_t)\,\mathrm{d}x - \frac{1}{2}\int_{\partial B(x_0,t_0-t)} (u_t^2 + Du\cdot Du)\,\mathrm{d}S \\ &= \int_{B(x_0,t_0-t)} u_t \underbrace{(u_{tt}-\Delta u)}_{0}\,\mathrm{d}x + \underbrace{\int_{\partial B(x_0,t_0-t)} u_t \frac{\partial u}{\partial \nu}\,\mathrm{d}S}_{\text{boundary term from int by parts}} - \frac{1}{2}\int_{\partial B(x_0,t_0-t)} \underbrace{(u_t^2 + Du\cdot Du)}_{\geq 0}\,\mathrm{d}S. \end{aligned}$$

But

$$u_t \frac{\partial u}{\partial \nu} \leq |u_t| \left|\underbrace{\frac{\partial u}{\partial \nu}}_{\nu\cdot Du}\right| \underbrace{\leq}_{\text{Cauchy-Schwartz, } |\nu|=1} |u_t||Du| \underbrace{\leq}_{\text{Cauchy ineq}} \frac{1}{2}(u_t^2 + |Du|^2).$$

So, $e'(t) \leq 0$. Since $e(t) \geq 0$ and $e(0)=0$, this implies $e(t)=0$ $\forall t \in [0,t_0]$. Then

$$e(t) = 0 \implies u_t = 0 = |Du| \text{ on } K(x_0,t_0) \implies u \equiv 0 \text{ on } L(x_0,t_0).$$

□

Showing undergraduate versions of heat, wave, Laplace using separation of variables.

# 15 (10/19/23)

We have finally moved on from Chapter 2 of Evans. Topics follow from Evans 3.2.

## 15.1 Method of characteristics

The plan is to solve the general 1st order PDE on open $U \subseteq \mathbb{R}^n$ given by

$$F(Du(x), u(x), x) = 0$$

where $F\colon \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is "some" function and we have some boundary condition $u = g$ on $\partial U$. For reasons that will become obvious later, we will use the notation

$$p = Du \qquad \text{and} \qquad z = u(x).$$

Then the PDE becomes

$$F(p,z,x) = 0.$$

*Example* 15.1 (Transport equation). Take $F = p\cdot b - c(x,u(x))$. Then this is the transport equation

$$Du \cdot b = c(x,u(x)).$$

The idea of solving this is to look at $u$ along lines parallel to $b$. For a line $\ell$ parallel to $b$, we consider where $\ell$ hits the boundary $\partial U$. We would then use the boundary condition $u = g$ on $\partial U$ and the transport equation to find $u$ on $\ell$.

The **main idea** to solve for general $F$ is to use a similar idea as in the transport equation. We find an ODE space for curves $x(s)$ in $U$ such that the solution $u(x(s))$ can be found on each curve separately by solving some ODEs. In the transport equation, these curves $x(s)$ were the lines $\ell$.

Let $x\colon I \to U$ be a curve in $U$ for some interval $I \subseteq \mathbb{R}$. Define

$$z(s) = u(x(s)) \qquad \text{and} \qquad p(s) = Du(x(s)).$$

The **main idea** is to find a closed system of $2n+1$ ODEs for

$$\dot{x}(s) \coloneqq \frac{dx}{ds}(s), \qquad \dot{p}(s) \coloneqq \frac{dp}{ds}(s), \qquad \dot{z}(s) \coloneqq \frac{dz}{ds}(s).$$

(The reason for $2n+1$ is the $n$ components of $p$, the 1 component of $z$, and the $n$ components of $x$.) We can compute

$$\dot{z}(s) = \frac{d(u \circ x)}{ds}(s) = Du(x(s)) \cdot \dot{x}(s). = p(s) \cdot \dot{x}(s).$$

We have

$$\dot{p}_i(s) = \frac{d(u_{x_i} \circ x)}{ds}(s) = \sum_{j=1}^{n} u_{x_i x_j}(x(s))\dot{x}_j(s). \tag{15.1}$$

We want to get rid of the second partials to make things nicer. To do so, we differentiate $F(Du, u, x) = 0$:

$$\begin{aligned} 0 &= \frac{\partial}{\partial x_i} F(Du, u, x) \\ &= \sum_{j=1}^{n} F_{p_j}(\cdot) u_{x_j x_i} + F_z(\cdot) u_{x_i} + F_{x_i}(\cdot) \\ &= \sum_{j=1}^{n} F_{p_j}(\cdot) u_{x_j x_i} + F_z(\cdot) p_i + F_{x_i}(\cdot). \end{aligned}$$

Suppose $u \in \mathcal{C}^2$ and we take $\dot{x}_j = F_{p_j}(\cdot)$. Then the above becomes

$$0 = \sum_{j=1}^{n} \dot{x}_j u_{x_i x_j} + F_z(\cdot) p_i + F_{x_i}(\cdot).$$

Therefore, Equation 15.1 becomes

$$\dot{p}_i(s) = -F_z(p(s), z(s), x(s)) p_i - F_{x_i}(p(s), z(s), x(s)).$$

The vector form of everything becomes

$$\begin{aligned} \dot{x}(s) &= D_p F(\cdot) \\ \dot{p}(s) &= -D_x F(\cdot) - D_z F(\cdot) p(s) \\ \dot{z}(s) &= p \cdot \dot{x} = p \cdot D_p F(\cdot) \end{aligned}$$

where $\cdot = (p, z, x)$. We now use boundary condition. Say

$$x(0) = \widetilde{x} \in \partial U, \qquad z(0) = \widetilde{z} = g(\widetilde{x}).$$

Then we want to find $p(0) = \widetilde{p}$.

### 15.2 Example 1: Linear PDE

Suppose

$$F(p,z,x) = b(x)\cdot p + c(x)z.$$

Then

$$\begin{aligned} \dot{x} &= D_pF = b(x) \\ \dot{z} &= p\cdot\dot{x} = b(x)\cdot p = -c(x)z. \end{aligned}$$

*Example* 15.2. Suppose

$$x_1u_{x_2} - x_2u_{x_1} = u \qquad \text{on } U = \{x\in\mathbb{R}^2 : x_1, x_2 > 0\}.$$

Say $u = g$ on $\Gamma = \{(x_1, 0) : x_1 > 0\}$. Then find $u$.

*Proof.* We have

$$F = (-x_2, x_1)\cdot p - z = 0.$$

Then

$$\begin{aligned} \dot{x}_1 &= -x_2 \\ \dot{x}_2 &= x_1 \\ \dot{z} &= z. \end{aligned}$$

Then

$$\begin{aligned} x_1(s) &= \widetilde{x}_1\cos s \\ x_2(s) &= \widetilde{x}_1\sin s \\ z(s) &= \widetilde{z}e^s = g(\widetilde{x}_1)e^s. \end{aligned}$$

where $\widetilde{x}$ and $\widetilde{z}$ are the initial values. The characteristic $x(s)$ describes a quarter circle centered at the origin. Given $(x_1, x_2)\in U$, we have

$$\widetilde{x}_1 = \sqrt{x_1^2 + x_2^2}, \qquad s = \tan^{-1}(x_2/x_1).$$

From $z(s) = g(\widetilde{x}_1)e^s$, we find

$$\boxed{u(x_1,x_2) = g\left(\sqrt{x_1^2+x_2^2}\right)e^{\tan^{-1}(x_2/x_1)}}$$

□

### 15.3 Example 2: Quasilinear PDE

Consider

$$F = b(x,z)\cdot p + c(x,z) = 0.$$

Then

$$\begin{aligned} \dot{x} &= b(x,z) \\ \dot{z} &= p\cdot\dot{x} = -c(x,z). \end{aligned}$$

*Example* 15.3. Solve

$$\begin{cases} u_{x_1} + u_{x_2} = u^2 & \text{on } U = \{x : x_2 > 0\} \\ u = g & \text{on } \Gamma = \{x : x_2 = 0\} \end{cases}$$

*Proof.* We have

$$\dot{x} = (1,1) \implies x_1(s) = \widetilde{x}_1 + s \qquad \text{and} \qquad x_2 = s.$$

Also,

$$\dot{z} = z^2 \implies \frac{dz}{z^2} = ds \implies z(s) = \frac{1}{c-s} = \frac{1}{\frac{1}{\widetilde{z}} - s} = \frac{\widetilde{z}}{1 - s\widetilde{z}}.$$

Therefore,

$$\boxed{u(x_1, x_2) = \frac{g(x_1 - x_2)}{1 - x_2 g(x_1, x_2)}.}$$

□

# 16 Characteristics (10/24/23)

Let $f : U \to \mathbb{R}^n$ is $C^k$. $x_0 \in U$, $z_0 = f(x_0)$, and

$$Df = \begin{pmatrix} f^1_{x_1} & \cdots & f^1_{x_n} \\ \vdots & \cdots & \vdots \\ f^n_{x_1} & \cdots & f^n_{x_n} \end{pmatrix}, \quad \mathcal{J}f = \det(Df).$$

Note superscript is which component and subscript is the derivative.

**Theorem 16.1** (Inverse Function Theorem)**.** *If $\mathcal{J}f(x_0) \neq 0$, then there exists an open $V \subset U$ with $x_0 \in V$ and open $W \subset \mathbb{R}^n$ with $z_0 \in W$ such that $f : V \to W$ is bijective and $f^{-1} \in C^k$.*

**Theorem 16.2** (Implicit function Theorem)**.** *$x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $U \subset \mathbb{R}^n \times \mathbb{R}^m$. Then $f : U \to \mathbb{R}^m$ is $C^k$ $(k \geq 1)$. Assume $(x_0, y_0) \in U$ and $z_0 = f(x_0, y_0)$. Define $Df = (D_x f, D_y f)$. Suppose $\mathcal{J}_y f(x_0, y_0) \neq 0$. Then $\exists V \subset U$ with $(x_0, y_0) \in V$ and $W \in \mathbb{R}^n$ with $z_0 \in W$ and a $C^k$ function $g : W \to \mathbb{R}^m$ such that*

1. *$g(x_0) = y_0$.*
2. *$f(x, g(x)) = z_0$ $\forall x \in W$.*
3. *If $(x, y) \in V$ and $f(x, y) = z_0$, then $y = g(x)$.*

From last time, we started with

$$F(\underbrace{Du(x)}_{p}, \underbrace{u(x)}_{z}, x) = 0$$

on $x \in U \subset \mathbb{R}^n$ and $u = g$ on $\Gamma$ (boundary). This our general, first order PDE. The hope is to take the domain $U$ and identify a trajectory from the boundary to the domain of $U$ and formulate a closed system of ODE's. Parametrize $x(s)$ the path and take its value and gradient as we go along the trajectory.

Let $z(s) = u(x(s))$, $p(s) = Du(x(s))$. The system of characteristics is

$$\begin{aligned}
\dot{x}(s) &= D_p F(p(s), z(s), x(s)), \\
\dot{p}(s) &= -D_x F(p(s), z(s), x(s)) - D_z F(p(s), z(s), x(s)) p(s), \\
\dot{z}(s) &= p \cdot \dot{x} = p \cdot D_p F(p(s), z(s), x(s))
\end{aligned}$$

Boundary conditions for PDE give initial conditions for the system of characteristics. $x(0) = x^0 \in \partial U$, $z(0) = g(x^0)$, $p(x) = p^0 \dots$. We have $(n-1)$ equations for $p_0$ from directional derivatives of $g$ and the $n$-th condition is $F(p^0, z^0, x^0) = 0$.

**Let's formalize this.** Locally flatten the boundary. Given domain $U$ and interested in a point $x^0$ on the boundary. Use a function $\Phi$ that transforms into a picture where the neighborhood of $x^0$ is flat. So we are now in $V$ along $y^0$. Let $\Psi$ be the inverse mapping.

How to build this? If $\Gamma \in C^1$, then $\exists \gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ such that near $x^0$,
$\Gamma = \{(x_1, \dots, x_{n-1}, \gamma(x_1, \dots, x_{n-1})\}$ (possibly relabeling $x_1, \dots, x_n$). Then we can write

$$\Phi(x_1, \dots, x_n) = \begin{bmatrix} x_1 \\ \vdots \\ x_{n-1} \\ x_n - \gamma(x_1, \dots, x_{n-1}) \end{bmatrix}.$$

Locally, we can have $\partial U = \mathbb{R}^{n-1} \times \{x = 0\}$. The conditions for $p^0$ now become $p_i^0 = g_{x_i}(x^0),\ i = 1, \dots, n-1$.

We say $(p^0, z^0, x^0)$ is admissible if

1. $x^0 \in \Gamma$
2. $z(0) = g(x^0) = z^0$
3. $p_i^0 = g_{x_i}(x^0)$ and $F(p^0, z^0, x^0) = 0$

**Definition 16.3.** An admissible $(p^0, z^0, x^0)$ is *non-characteristic* if $F_{p_n}(p^), z^0, x^0) \neq 0$.

**Lemma 16.4.** *If $(p^0, z^0, x^0)$ is non-characteristic, then for all $y \in \Gamma$ close enough to $x^0$, $\exists!$ (! means unique) function $q : \Gamma \to \mathbb{R}^n$ such that $q^i(y) = g_{x_i}(y),\ i = 1, \dots, n-1$ and $F(q(y), g(y), y))$.*

This tells us that if you focus on some neighborhood of $x^0$ at the boundary, there is a neighborhood such that for every initial $y$, you will only be able to find exactly one unique solution for $p^0$. *Characteristic should not be tangential to the boundary.* ($x^n$ direction is perpendicular to boundary?)

*Proof.* Define $G(p, y) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ as $G^i(p, y) = p^i - g_{x_i}(y),\ i = 1, \dots, n-1$ and $G^n(p, y) = F(p, g(x), y)$. Note $q$ recovers a zero level set of $G$. $G^i(q(y), y) = q^i(y) - g_{x_i}(y) = 0$, and $G^n(q(y), y) = F(q(y), g(y), y) = 0$. Note that $G(p^0, x^0) = 0$. By the Implicit Function Theorem, $p = g(y)$ is well-defined near $(p^0, x^0)$ provided $\det(D_p G(p^0, x^0)) \neq 0$.

$$D_p G = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & 1 & 0 & 0 \\ \vdots & 0 & \cdots & 1 & 0 \\ F_{p_1} & \cdots & \cdots & \cdots & F_{p_n} \end{bmatrix}, \quad \det D_p G = F_{p_n}(p^0, z^0, x^0) \neq 0$$

by non-characteristic condition. $\square$

Assuming $y \in \Gamma$ is close enough to $x^0$ and $(p^0, z^0, x^0)$ is non-characteristic, then $(q(y), g(y), y)$ is a unique admissible triple with that $y$.

**Notation.** Taking $p^0 = q(y)$,

$$\begin{cases} p(s) \coloneqq p(y,s) \\ z(s) \coloneqq z(y,s) \\ x(s) \coloneqq x(y,s) \end{cases},$$

where $y_n = 0$ since $y \in \Gamma$. $y$ represents the initial condition for $x(s)$.

Previously we start with a PDE and we want to find the solution $u(x)$. Later we say we have some $x^0$ at the boundary and trace the characteristic curve, and hopefully it hits the point $x \in U$. At that point $u(x) = z(s)$. This relied on starting on a specific point and a specific duration to get there. Now, we say we can start at a neighborhood of $x^0$, then we will have multiple places we can go in time $s$.

# 17 More Characteristics (10/26/23)

**Lemma 17.1** (Local Invertibility)**.** *Assume $(p^0, z^0, x^0)$ is non-characteristic. Let $x^0 = (x_1^0, \dots, x_{n-1}^0, 0)$ and $\tilde{x^0} = (x_1^0, \dots, x_{n-1}^0)$. Then $\exists$ a time interval $I \subset \mathbb{R}$ containing 0, $W \subset \Gamma \subset \mathbb{R}^{n-1}$ containing $x^0$, $V \subset \bar{U} \subset \mathbb{R}^n$ containing $\tilde{x}^0$ such that for all $x \in V$, $\exists! y \in W,\ s \in I$ such that $x = x(y,s)$ and this mapping $x \to y$ and $x \to s$ is $C^2$. ($x$ is the arbitrary point in the interior).*

*Proof.* $x(\tilde{x}^0, 0) = x^0$. Inverse function theorem requires $\det(Dx(\tilde{x}^0, 0)) \neq 0$ (take $\tilde{x}^0 \in \mathbb{R}^{n-1}$).

$$x(y,0) = y,\ \forall y \in \Gamma \implies \frac{\partial}{\partial y_i} x^j(\tilde{x}^0, 0) = \delta_{ij}$$

for $j = 1, \dots, n$ and $i = 1, \dots, n-1$. We also have

$$\dot{x}(s) = D_p F(\dots) \implies \frac{\partial}{\partial s} x^j(\tilde{x}^0, 0) = F_{p_j}(p^0, z^0, x^0).$$

The derivative

$$Dx(\tilde{x}^0, 0) = \begin{bmatrix} 1 & 0 & \cdots & 0 & F_{p_1} \\ 0 & 1 & 0 & \cdots & \vdots \\ 0 & 0 & \ddots & 0 & \vdots \\ 0 & 0 & \cdots & 1 & F_{p_{n-1}} \\ 0 & 0 & \cdots & 0 & F_{p_n} \end{bmatrix}$$

is invertible since $F_{p_n} \neq 0$. $\square$

Is this method of characteristics consistent?

**Theorem 17.2** (Local Existence)**.** *Suppose $x = x(y,s)$ is invertible on $V$ and we can write $y = y(x), s = s(x)$. Define*

$$u(x) = z(y(x), s(x)), \quad p(x) = p(y(x), s(x)).$$

*Then $u(x) \in X^2(V)$ and solves $F(Du, u, x) = 0$ on $V$ with $u = g$ on $\gamma \cap V$.*

*Proof.* $f(y,s) = F(p(y,s), z(y,s), x(y,s))$. From the definition of $q(y)$, $f(y,0) = 0$. From the system of characteristic ODEs, $f(y,s) = 0 \ \forall s \in \mathbb{R}$ such that $y \in \Gamma$ close to $x^0$, we have $x(y,s) \in V$. This is because

$$\frac{\partial}{\partial s} f(y,s) = D_p F \cdot \frac{\partial}{\partial s} p(y,s) + D_z F \frac{\partial}{\partial s} z(y,s) + D_x F \cdot \frac{\partial}{\partial s} x(y,s) = 0$$

from the way the system of ODEs were formed. Thus, $F(p(x), u(x), x) = 0$. We want this $p \to Du$.

To prove $p(x) = Du(x)$ for $x \in V$. We have

$$\frac{\partial z}{\partial s} = \sum_j p^j(y,s) \frac{\partial x^j}{\partial s}(y,s). \quad (\#)$$

Fix $y \in \Gamma$ and define $r^i(s) = \frac{\partial z}{\partial y_i} - \sum_j p^j \frac{\partial x^j}{\partial y_i}$ for $i = 1, \ldots, n-1$. It should be the case that $r^i(s) = 0$ if $p = D_u$. From the definition of $q(y)$,

$$r^i(0) = 0, \quad \dot{r}_i(s) = \frac{\partial^2 z}{\partial y_i \partial s} - \sum_{j=1}^{n} \left[ \frac{\partial p^j}{\partial s} \frac{\partial x^j}{\partial y_i} + p^j \frac{\partial^2 x^j}{\partial y_i \partial s} \right].$$

Differentiate (#) wrt $y_i$

$$\frac{\partial^2 z}{\partial y_i \partial s} = \sum_j \left[ \frac{\partial p^j}{\partial y_i} \frac{\partial x^j}{\partial s} + p^j \frac{\partial^2 x^j}{\partial s \partial y_i} \right].$$

Then use equality of mixed partials and the information from characteristic ODEs,

$$\dot{r}_i(s) = \sum_j \left[ \frac{\partial p^j}{\partial y_i} \frac{\partial x^j}{\partial s} - \frac{\partial p^j}{\partial s} \frac{\partial x^j}{\partial y_i} \right] = \sum_{j=1}^{n} \left[ \frac{\partial p^j}{\partial y_i} F_{p_j} - \frac{\partial x^j}{\partial y_i} (-F_{x_j} - F_z p^j) \right].$$

In addition,

$$0 = \frac{\partial}{\partial y_i} F(p(y,s), z(y,s), x(y,s)) = \underbrace{D_p F \cdot \frac{\partial}{\partial y_i} p}_{\sum_j \frac{\partial p^j}{\partial y_i} \frac{\partial F}{\partial p_j}} + F_z \frac{\partial z}{\partial y_i} + \underbrace{D_x F \cdot \frac{\partial x}{\partial y_i}}_{\sum \frac{\partial F}{\partial x_j} \frac{\partial x^j}{\partial y_i}}.$$

Plugging this into our previous expression,

$$\dot{r}_i(s) = -F_z \left[ \frac{\partial z}{\partial y_i} - F_z \sum_{j=1}^{n} p^j \frac{\partial x^j}{\partial y_i} \right] = -F_z r^i(s).$$

Since $r^i(0) = 0$, then $r^i(s) = 0 \ \forall s$.

Take (#) with $\frac{\partial z}{\partial y_i} = \sum_{j=1}^n p^j \frac{\partial x^j}{\partial y_i}$, will give that

$$\frac{\partial u}{\partial x_j} = \cdots = p^j.$$

$\square$

# 18 (10/31/23)

We are solving $F(p, z, x) = 0$ on $U$ with boundary condition $u = g$ on $\Gamma$ where $\Gamma \subseteq \partial U$. The characteristic equation is

$$\begin{aligned} \dot{x} &= D_p F \\ \dot{z} &= p \cdot \dot{x} \\ \dot{p} &= -D_x F - (D_z F)p. \end{aligned}$$

We will give a bunch of applications.

## 18.1 Characteristics of Linear PDE

Let

$$F = p \cdot b(x)$$

where $b\colon \overline{U} \to \mathbb{R}^b$. Then

$$\begin{aligned} \dot{x} &= b(x) \\ \dot{z} &= b(x) \cdot p = 0. \end{aligned}$$

Hence,

$$z(s) = u(x(s)) = g(x_0).$$

## 18.2 Characteristics of hyperbolic conversation law

Consider $u(x, t)$ for $x \in \mathbb{R}$ where

$$u_t + [f(u)]_x = 0.$$

For smooth $u$,

$$u_t + f'(u)u_x = 0.$$

Let $y = (x, t)$. Then

$$G(p, z, y) = p_2 + f'(z)p_1 = 0.$$

We use the initial condition $u(x, 0) = g(x)$. From characteristics,

$$\begin{aligned} &\dot{x}(s) = f'(z(s)) \\ &\dot{t}(s) = 1,\ \ t(0) = 0 \implies t(s) = s. \\ &\dot{z}(s) = (f'(z), 1) \cdot (p_1, p_2) = 0 \implies z(s) = g(x^0). \end{aligned}$$

The first and last line imply

$$x'(s) = f(z(s)) = \text{constant}.$$

Therefore,

$$u(x, t) = g(x^0).$$

We have

$$x = x^0 + tf'(g(x^0)) = x^0 + tf'(u(x, t)).$$

Therefore,

$$\boxed{u = g(x - tf'(u)).}$$

This is the implicit solution for $u$. We can use the implicit function theorem to find $u$ locally. Define

$$K(x,t,u) = u - g(x - tf'(u)).$$

To write $u$ in terms of $x, t$, we need to make sure $\partial_u K \neq 0$. In other words, if

$$\boxed{\partial_u K = 1 + tf''(u)g'(x - tf'(u)) \neq 0,}$$

then we can write $u$ as a function of $x, t$ in some neighborhood.

*Example* 18.1 (Traffic). Take

$$f(u) = v(u)u$$

where

$$v(u) = v_{max}\left(1 - \frac{u}{u_{max}}\right).$$

$v$ represents the (average) velocity of the cars. If $u = 0$, then $v = v_{max}$; and if $u = u_{max}$, then $v = 0$, meaning the cars are not moving. We consider $u \in [0, u_{max}]$.

## 18.3 Characteristics of Hamilton-Jacobi PDEs

Consider the PDE

$$G(D_x u, u_t, x, t) := u_t(x,t) + H(D_x u(x,t), x) = 0$$

where $x \in \mathbb{R}^n$ and $t > 0$. Suppose $u(x, 0) = g(x)$. Let

$$y = (x,t) \qquad \text{and} \qquad q = D_y u = \begin{bmatrix} p \\ q_{n+1} \end{bmatrix}.$$

Then

$$G(q, z, y) = q_{n+1} + H(q_1, \dots, q_n, y_1, \dots, y_n).$$

Then

$$\begin{aligned}
\dot{x}(s) &= D_p G = D_p H(p(s), x(s)) \\
\dot{t}(s) &= D_{q_{n+1}} G = 1, \ \ t(0) = 0 \implies t(s) = s \\
\dot{p}(s) &= -D_x G - (D_p G)z = -D_x H(p(s), x(s)).
\end{aligned}$$

The *Hamilton ODEs* refer to the first and last equations:

$$\begin{cases} \dot{x}(t) = D_p H(p(t), x(t)) \\ \dot{p}(t) = -D_x H(p(t), x(t)) \end{cases}$$

(We can interchange $s$ and $t$ from $t(s) = s$.) We can find $u$ by

$$\dot{z}(s) = \dot{x} \cdot q = u_t + D_p H(D_x u, x) \cdot D_x u.$$

## 18.4 Calculus of variations

Next time, we will cover a version of calculus of variations. We will introduce a Lagrangian $L\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$. Let $w\colon [0,t] \to \mathbb{R}^n$ be a curve from fixed points $y$ to $\widetilde{x}$. Define

$$I[w] := \int_0^t L(\dot{w}(s), w(s))\, ds.$$

This will help us solve the Hamilton-Jacobi matrix. We claim that we can write $L$ in terms of $H$ so that the characteristic $x(t)$ that minimizes $I[w]$ satisfies Hamilton ODEs.

# 19 (11/2/23

I was absent today. Apparently we covered Evans 3.3.1, so here are my notes on it.

## 19.1 Hamilton-Jacobi equations

We wish to study

$$\begin{cases} u_t + H(Du) = 0 & \text{on } \mathbb{R}^n \times (0, \infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\} \end{cases}$$

where $u \colon \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ written as $u(x, t)$. We call $H$ the *Hamiltonian*. Recall the characteristic equations give

$$\begin{aligned} \dot{x} &= D_p H(p, x) \\ \dot{p} &= -D_x H(p, x). \end{aligned}$$

## 19.2 Calculus of variations & Euler-Lagrange equations

Let $L \colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be smooth written as $L(\mathbf{v}, \mathbf{x})$ for $\mathbf{v}, \mathbf{x} \in \mathbb{R}^n$. In particular,

$$D_{\mathbf{v}} L = (L_{v_1}, \ldots, L_{v_n}) \qquad \text{and} \qquad D_{\mathbf{x}} L = (L_{x_1}, \ldots, L_{x_n}).$$

We call $L$ the *Lagrangian*. Define the *action* to be

$$I[\mathbf{w}] = \int_0^t L(\dot{\mathbf{w}}(s), \mathbf{w}(s))\, ds$$

where $\mathbf{w} \colon \mathbb{R} \to \mathbb{R}^n$ is a curve. Define the *admissible class* to be

$$\mathcal{A} = \{\mathbf{w} \in \mathcal{C}^2([0, t], \mathbb{R}^n) : \mathbf{w}(0) = \mathbf{y} \text{ and } \mathbf{w}(t) = \mathbf{x}\}.$$

The *calculus of variations* problem is to find a curve $\mathbf{x}(s) \in \mathcal{A}$ such that

$$I[\mathbf{x}] = \min_{\mathbf{w} \in \mathcal{A}} I[\mathbf{w}].$$

**Theorem 19.1** (Euler-Lagrange equations)**.** *The function $\mathbf{x}(s)$ that solves the calculus of variations problem satisfies*

$$D_{\mathbf{x}} L(\dot{\mathbf{x}}(s), \mathbf{x}(s)) - \frac{d}{ds}(D_{\mathbf{v}} L(\dot{\mathbf{x}}(s), \mathbf{x}(s))) = 0 \qquad \textit{for all } 0 \le s \le t.$$

This is more famously written as

$$\boxed{\frac{\partial L}{\partial \mathbf{x}} - \frac{d}{ds}\frac{\partial L}{\partial \dot{\mathbf{x}}} = 0}$$

or in components,

$$\boxed{\frac{\partial L}{\partial x_i} - \frac{d}{ds}\frac{\partial L}{\partial \dot{x}_i} = 0.}$$

### 19.3 Hamilton's equations

We will now transform the Euler-Lagrange equations into Hamilton's equations. Let $\mathbf{x}(s)$ be the minimizer of $I$. Define the *generalized momentum* to be

$$\mathbf{p}(s) := D_{\mathbf{v}} L(\dot{\mathbf{x}}(s), \mathbf{x}(s)).$$

Define $\mathbf{v}(\mathbf{p}, \mathbf{x})$ so that

$$\mathbf{p} = D_{\mathbf{v}} L(\mathbf{v}(\mathbf{p}, \mathbf{x}), \mathbf{x}).$$

The *Hamiltonian* associated with the Lagrangian $L$ is

$$H(\mathbf{p}, \mathbf{x}) = \mathbf{p} \cdot \mathbf{v}(\mathbf{p}, \mathbf{x}) - L(\mathbf{v}(\mathbf{p}, \mathbf{x}), \mathbf{x}).$$

**Theorem 19.2** (Hamilton's equation)**.** *Let $\mathbf{x}(s)$ be the minimizer of the Lagrangian $L$, and define $\mathbf{p}(s) = D_{\mathbf{v}} L(\dot{\mathbf{x}}(s), \mathbf{x}(s))$. Then $\mathbf{x}(s)$ and $\mathbf{p}(s)$ solve Hamilton's equations given by*

$$\begin{aligned} \dot{\mathbf{x}}(s) &= D_{\mathbf{p}} H(\mathbf{p}(s), \mathbf{x}(s)) \\ \dot{\mathbf{p}}(s) &= -D_{\mathbf{x}} H(\mathbf{p}(s), \mathbf{x}(s)) \end{aligned}$$

*for all $0 \leq s \leq t$. Moreover, the mapping $s \mapsto H(\mathbf{p}(s), \mathbf{x}(s))$ is constant.*

This is also written as

$$\boxed{\frac{d\mathbf{x}}{ds} = \frac{\partial H}{\partial \mathbf{p}} \qquad \text{and} \qquad \frac{d\mathbf{p}}{ds} = -\frac{\partial H}{\partial \mathbf{x}}}$$

or in components,

$$\boxed{\frac{dx_i}{ds} = \frac{\partial H}{\partial p_i} \qquad \text{and} \qquad \frac{dp_i}{ds} = -\frac{\partial H}{\partial x_i}.}$$

## 20 Calculus of Variations (11/7/23)

**Last time:** Given $\tilde{y} = x(0)$, $\tilde{x} = x(t)$, find optimal $x : \mathbb{R} \to \mathbb{R}^n$.

**Today:** find the optimal $u : V \to \mathbb{R}$, $U \subset \mathbb{R}^n$ (Evans 8.1.2?) Assume $U$ is open with a smooth boundary $\partial U$. Assume there is some function $g \in \partial U \to \mathbb{R}$ that is smooth.

**Definition 20.1** (Admissible functions)**.**

$$\mathcal{A} = \{w : \bar{U} \to \mathbb{R} | w \in C^2(\bar{U}), u = g \text{ on } \partial U\}.$$

Let $L : \mathbb{R}^n \times \mathbb{R} \times \bar{U} \to \mathbb{R}$. $L(p, z, x)$.

**Definition 20.2** (Action integral)**.**

$$I[w] = \int_U L(Dw(x), w(x), x) \, \mathrm{d}x.$$

Suppose $u \in \mathcal{A}$ is a minimizer of $I[w]$. Choose any fixed $v \in C_c^\infty(U) \implies v \equiv 0$ on $\partial U$.

**Definition 20.3.** Let $\tau \in \mathbb{R}$.

$$i(\tau) = I[\underbrace{u+\tau v}_{\in \mathcal{A}}] = \int_U L(Du(x)+\tau Dv(x), u(x)+\tau v(x), x)\,\mathrm{d}x,$$

$$i'(\tau) = \int_U \left[\sum_{i=1}^n L_{p_i}(\dots)v_{x_i}(x) + L_z(\dots)v(x)\right]\mathrm{d}x.$$

$u$ is a minimizer $\implies$ $i(0) \le i(\tau)$ $\forall \tau \in \mathbb{R}$. Then

$$\begin{aligned} 0 = i'(0) &= \int_U \left[\sum_{i=1}^n L_{p_i}(Du(x),u(x),x)v_{x_i}(x) + L_z(Du(x),u(x),x)v(x)\right]\mathrm{d}x, \\ &= \int_U \underbrace{\left(-\sum_{i=1}^n [L_{p_i}(Du(x),u(x),x)]_{x_i} + L_z(Du(x),u(x),x)\right)}_{=0} v(x)\,\mathrm{d}x \end{aligned}$$

Integrate by parts the first line. Boundary terms go away because of compact support. The result is known as the *Euler-Lagrange PDE.*

**Definition 20.4** (Euler-Lagrange PDE)**.** The *Euler-Lagrange PDE* is

$$\begin{cases} -\mathrm{div}(D_pL(Du(x),u(x),x)) + L_z(Du(x),u(x),x) = 0 & \text{on } U, \\ u = g & \text{on } \partial U. \end{cases}$$

*Example* 20.5. $L(p,z,x) = \frac{1}{2}p\cdot p$. Then $D_pL = p$. Then $-\Delta u = 0$.

*Example* 20.6. $L(p,z,x) = \frac{1}{2}p\cdot p - F(z)$. $D_pL = p$. Then $-\Delta u - \underbrace{F'(u)}_{f(u)} = 0$ gives $-\Delta u = f(u)$.

*Example* 20.7. $L(p,z,x) = \sqrt{1+p\cdot p}$. So, $I[w]$ is the surface are of $w$'s graph.

$$D_pL = \frac{1}{2}\frac{2p}{\sqrt{1+p\cdot p}} = \frac{p}{L(p,z,x)}.$$

Then the Euler-Lagrange equation is

$$\underbrace{\mathrm{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right)}_{n\times\text{mean curavature of } u\text{'s graph at } x} = 0.$$

Wires in soap bubbles are the minimizers of surface area. It is not guaranteed that what you get after you dip it is actually a function. These functions satisfy Euler-Lagrange.

## 20.1 Brachistochrone example

Origins of Calculus of Variations. Bernoulli gave a challenge to solve *find a wire shape $y(x)$ minimizing the time of travel for a bead starting with zero velocity at a point $A$ and sliding without friction to a point $B$.* $A = (0,0)$ and $B = (\bar{x},\bar{y})$.

Conservation of energy tells how speed changes along path $y(x)$. $mgy = \frac{mv^2}{2}$, then solving for $v$ gives $v(y) = \sqrt{2gy}$. Bernoulli's ingenious idea was that $v(y)$ is piecewise constant. So constant $v$ in each portion. This is given by Snell's law of refraction $\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2}$. Then $\frac{\sin\theta_i}{v_i} = k$ a constant. 2 realizations

1. $y'(0) = \infty$ since $v(0) = 0$
2. $k = \frac{1}{v_{max}} = \frac{1}{\sqrt{2gD}}$, where $D$ is position of this lowest point.

Then

$$\sin\theta_i = \frac{\Delta x_i}{\Delta s_i} = \frac{\Delta x_i}{\sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}} = \frac{1}{\sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2}}.$$

So,

$$\frac{\sin\theta(y)}{v(y)} = \frac{1}{\sqrt{1 + (y')^2}\sqrt{2gy}} = \frac{1}{\sqrt{2gD}}.$$

Then,

$$(1 + (y'(x))^2)y(x) = D \implies \frac{\mathrm{d}x}{\mathrm{d}y} = \sqrt{\frac{y}{D - y}}.$$

It was known that the parametric curve that satisfied the ODE, upside-down cycloid (curve traced by circle rolling without slipping)

$$\begin{cases} x(t) = \frac{D}{2}(t - \sin t) \\ y(t) = \frac{D}{2}(1 - \cos t) \end{cases}.$$

To get rid of unknown $D$, have to differentiate both sides again. This gives

$$y'(x)(1 + y'(x)^2 + 2y(x)y''(x)) = 0$$

# 21 Hamiltonian and Lagrangian (11/9/23)

**Previously.**

$$H(p, x) \coloneqq p \cdot v(p, x) - L(v(p, x), x)$$

with $v(p, x)$ defined implicitly by $p = F_v L(v, x)$. Then $x(\cdot)$ minimizing $I[y(\cdot)] = \int_0^t L(\dot{y}(s), y(s))\,\mathrm{d}s$ among all $y(\cdot)$ such that $y(0) = \tilde{y},\ y(t) = \tilde{x}$, and $p(s) \coloneqq D_v L(\cdot x(s), x(s))$ must satisfy

**Definition 21.1** (Hamilton ODEs)**.**

$$\begin{cases} \dot{x}(s) = D_p H(p(s), x(s)), \\ \dot{p}(s) = -D_x H(P(s), x(s)). \end{cases}$$

Remember this came from the HJB PDE $u_t + H(D_x u, x) = 0$ for $x \in \mathbb{R}^n$, $t > 0$ and $u = g$ on $\mathbb{R}^n \times \{t = 0\}$. We turned the crank on the method of characteristics and got the Hamilton ODE's along with $\dot{z}(s) = p \cdot D_p H - H(p, x)$. Euler-Lagrange is $n$ equations, we can turn it into $2n$ ODE's. We then showed that

$$z(t) = z(0) + \int_0^t L(\dot{x}(s), x(s))\,\mathrm{d}s.$$

Let's get a variational mapping between $L$ and $H$ so we can talk about solutions to HJB globally instead of just locally from the method of characteristics. This will require looking at the weak solution.

**Simple case.** Assume

1. $L(v, x) = L(v)$,
2. $L(v)$ is convex (thus continuous),
3. $L(v)$ growing super fast (superlinear). $\lim_{v\to\infty} \frac{L(v)}{|v|} = +\infty$.

**Definition 21.2** (Legendre Transform)**.** Let $p, v \in \mathbb{R}^n$. The *Legendre Transform* is

$$\boxed{L^*(p) = \sup_{v\in\mathbb{R}^n} \{p \cdot v - L(v)\}.}$$

Seems like an odd definition. Note that the supremum is attained at some $v_*(p)$, so it's actually a max (in textbook $q$ and $v$ are interchanged because of typos). Convexity implies continuity and since $L$ will dominate over $p\cdot v$ for large $v$, then we have a compact set. In order for $v_*$ to be unique, we need strict convexity, but Evans is loose with this. Then we can write

$$L^*(p) = p \cdot v_*(p) - L(v_*(p)) = H(p).$$

Our motivation of this definition is that the Hamiltonian is defined to be the Legendre Transform of the Lagrangian.

**Theorem 21.3** (Convex duality)**.** *If $L$ and $H = L^*$ are defined as before. Then*

1. *$H(p)$ is also convex,*
2. *$H$ also has superlinear growth* $\lim_{p\to\infty} \frac{H(p)}{|p|} = +\infty$,
3. *$H^*(v) = L^{**}(v) = L(v)$.*

*Proof.* We prove each item.

1. By definition,
$$H(p) = \max_{v\in\mathbb{R}^n}\{\underbrace{p \cdot v - L(v)}_{S_{v(p)}}\}.$$
This is linear in $p$. Plot $S_v(p)$ vs $p$. Then the pointwise maximum of these linear functions is convex.

2. WTS for large enough $|p|$, $\frac{H(p)}{|p|} \geq \lambda$ for any $\lambda > 0$. Choose $\lambda > 0$. Define $\tilde{v} = \lambda\frac{p}{|p|}$. Then
$$\begin{aligned} H(p) &\geq p \cdot \tilde{v} - L(\tilde{v}) = \lambda|p| - L\left(\lambda\frac{p}{|p|}\right), \\ &\geq \lambda|p| - \underbrace{\max_{v\in B(0,\lambda)} L(v)}_{C}. \end{aligned}$$
$$\frac{H(p)}{|p|} \geq \lambda - \frac{C}{|p|},$$
$$\liminf_{|p|\to\infty} \frac{H(p)}{|p|} \geq \liminf_{|p|\to\infty}\left(\lambda - \frac{C}{|p|}\right) = \lambda.$$

3. Note $H(p) + L(v) \geq v \cdot p \ \forall v, p \in \mathbb{R}^n$, this is clear from Legendre Transform definition. Then, we can say $L(v) \geq v \dot{-} H(p)$. Since this holds for all $v, p$, we can fix $v$ and take supremum over $p$ to show
$$L(v) \geq \sup_{p \in \mathbb{R}^n} \{v \cdot p - H(p)\} = H^*(v).$$
For the other direction,
$$H^*(v) = \sup_p \{p \cdot b - H(p)\} = \sup_p \{p \cdot v - \sup_r \{p \cdot r - L(r)\}\}.$$
Remember that $-\max_x\{f(x)\} = \min_x\{-f(x)\}$. So,
$$H^*(v) = \sup_p \inf_r \{p \cdot (v - r) + L(r)\}.$$
By convexity of $L$, $\exists s \in \mathbb{R}^n$ such that $L(r) \geq L(v) + s \cdot (r - v) \ \forall r \in \mathbb{R}^n$. $s = DL(v)$ if $L$ is smooth. Take $p = s$,
$$H^*(c) \geq \inf_r \{s \cdot (v - r) + L(r)\} \geq L(v).$$

□

Keeping the assumptions we stated, let's use our variational interpretation of the optimal $x$ and the ODE for $z$. Remember that $z(0) = g(x(0))$. This allows us to modify the definition of
$$I[y(\cdot)] = \int_0^t L(\dot{y}(s)) \, \mathrm{d}s + g(y(0)).$$
Before, we were going from $\tilde{y} = y(0)$ to $\tilde{x} = y(t)$, both fixed. Now, we only fix $\tilde{x}$, and we can choose $y(0)$. This is a slightly more relaxed version of calculus of variations with the penalty specified by $g$. Let
$$\mathcal{A} = \{y(\cdot) \in C^2(\mathbb{R}) | y(t) = \tilde{x}\}, \ x(\cdot) \in \mathcal{A} \text{ such that } I[x(\cdot)] \leq I[y(\cdot)] \forall y(\cdot) \in \mathcal{A}.$$

**Definition 21.4.**
$$u(\tilde{x}, t) := \inf_{y \in \mathcal{A}} I[y].$$

*Remark* 21.5. $u$ solves $u_t + H(Du) = 0$ on $\mathbb{R}^n \times \mathbb{R}_+$, $u = g$ on $\mathbb{R}^n \times \{t = 0\}$ where $H$ is convex ($H = L^*$) and superlinear and $g$ is Lipschitz continuous.

The next step is to simplify the definition of $u$.

**Definition 21.6** (Hopf-Lax Formula)**.**
$$\boxed{u(\tilde{x}, t) = \min_{\tilde{y} \in \mathbb{R}^n} \left\{ tL\left(\frac{\tilde{x} - \tilde{y}}{t}\right) + g(\tilde{y}) \right\}}$$

Can interpret $\frac{\tilde{x} - \tilde{y}}{t}$ as the velocity if we have a straight line case between $\tilde{y}, \tilde{x}$.

Next time: $u(\tilde{x}, t) \geq$ Hopf-Lax.

## 22 (11/14/23)

**Continuing from last time.** (Change notation to remove tilde. $x$ becomes optimal path). Remember $L(v)$ is convex, superlinear growing and $g$ is Lipschitz continuous. Define

$$H(p) := L^*(p) := \max_{v\in\mathbb{R}^n}\{p\cdot v - L(v)\}$$
$$u(x) := \inf_{w(\cdot)\in\mathcal{A}}\left\{\int_0^t L(\dot{w}(s))\,\mathrm{d}s + g(w(0))\right\}$$
$$\mathcal{A} = \{w:\mathbb{R}\to\mathbb{R}^n | w\in C^2 \text{ and } w(t)=x\}.$$

Hopf-Lax formula is

$$v(x,t) = \min_{y\in\mathbb{R}^n}\left\{tL\left(\frac{x-y}{t}\right)+g(y)\right\}.$$

*Remark* 22.1. Hoph-Lax formula is the optimal solution.

*Proof.* From the definition, $u(x,t)\le v(x,t)$. Want to show flipped inequality now.

Suppose $w(0)=y$. $⨍_0^t \dot{w}(s)\,\mathrm{d}s = \frac{x-y}{t}$. Jensen's inequality gives

$$\underbrace{L\left(⨍_0^t \dot{w}(s)\,\mathrm{d}s\right)}_{L\left(\frac{x-y}{t}\right)} \le \frac{1}{t}\int_0^t L(\dot{w}(s))\,\mathrm{d}s$$
$$tL\left(\frac{x-y}{t}\right)+g(y) \le \int_0^t L(\dot{w}(s))\,\mathrm{d}s + g(y).$$

Thus, $v(x,t)\le u(x,t)$. □

### 22.1 Solving Hamilton-Jacobi

Remember if $u\in C^2$ it solves $u_t + H(D_x u) = 0$ for $t>0,\ x\in\mathbb{R}^n,\ u=g$ on $(\mathbb{R}^n\times\{t=0\})$. What if $u$, solution to characteristics of Hamilton-Jacobi are not $C^2$? Notice how Hopf-Lax does not have any requirements on the derivatives. Then, we can use it to formulate a weak solution.

**Lemma 22.2.** *Let* $0\le s<t$. *Then*

$$v(x,t) := \min_{y\in\mathbb{R}^n}\left\{(t-s)L\left(\frac{x-y}{t-s}\right)+u(y,s)\right\} = u(x,t).$$

*Proof.* $v(x,t)$ is over the two piecewise straight paths, but $u(x,t)$ is over only all straight paths. So the minimum over the richer set by definition implies $v(x,t)\le u(x,t)$.
We want to write $\frac{x-z}{t}$ as a linear combination of $\frac{y-z}{s}$ and $\frac{x-y}{t-s}$,

$$\frac{x-z}{t} = \underbrace{\frac{s}{t}}_{a_1}\frac{y-z}{s} + \underbrace{\frac{t-s}{t}}_{a_2}\frac{x-y}{t-s},$$

where $a_1, a_2 \geq 0$ and $a_1 + a_2 = 1$. Then using the convexity of $L$,

$$L\left(\frac{x-z}{t}\right) \leq \frac{s}{t} L\left(\frac{y-z}{s}\right) + \frac{t-s}{t} L\left(\frac{x-y}{t-s}\right)$$

$$u(x,t) \leq g(z) + tL\left(\frac{x-z}{t}\right) \leq \underbrace{sL\left(\frac{y-z}{s}\right) + g(z)}_{u(y,s)} + (t-s)L\left(\frac{x-y}{t-s}\right) = v(x,t).$$

□

*Remark* 22.3. The Hamilton ODE's from characteristics were (using the assumptions made about $L$ and $H$),

$$\dot{x}(s) = D_p H(p(s)), \quad \dot{P} = -D_x G = 0 \implies p(s) = p(0) \forall s.$$

Then $\dot{x}(s)$ is constant. Hopefully, this demystifies the Hopf-Lax.

**Lemma 22.4.** *Hopf-Lax formula implies that $u(x,t)$ is Lipschitz continuous.*

*Proof.* Consider the locations from $(y_*, 0)$ to $(x,t)$ and $y_* + \tilde{x} - x, 0)$ to $(\tilde{x}, t)$. Then

$$u(\tilde{x}, t) \leq \underbrace{u(x,t)}_{tL\left(\frac{x-y_*}{t}\right)+g(y_*)} - g(y_*) + g(y_) + \tilde{x} - x) \leq u(x,t) + L_g|\tilde{x} - x|.$$

where $L_g$ is Lipschitz constant of $g$. Then

$$|u(\tilde{x},t) - u(x,t)| \leq L_g |x - \tilde{x}|.$$

Plug in $y = x$ in Hopf-Lax to get the bound $u(x,t) \leq tL(0) + g(x)$. Then

$$u(x,t) - g(x) \leq tL(0).$$

Next, plug in $g(y) \geq g(x) - L_g|x-y|$ into Hopf-Lax. This gives

$$\begin{aligned} u(x,t) &\geq \min_y \left\{ tL\left(\frac{x-y}{t}\right) - L_g|x-y| \right\} + g(x), \\ &= -\max_y \left\{ L_g|x-y| - tL\left(\frac{x-y}{t}\right) \right\} + g(x), \\ &= g(x) - t\max_z \left\{ L_g|z| - L(z) \right\}. \end{aligned}$$

where we let $z = \frac{x-y}{t}$. This looks like the definition of the Hamiltonian (via Legendre Transform). Continuing,

$$\begin{aligned} u(x,t) &\geq g(x) - t \max_{z\mathbb{R}^n} \max_{\xi \in B(0,L_g)} \{\xi \cdot z - L(z)\}, \\ &= g(x) - t \underbrace{\max_{\xi \in B(0,L_g)} H(\xi)}_{\tilde{c}} \\ g(x) - u(x,t) &\leq \tilde{c}t. \end{aligned}$$

Then,

$$|u(x,t) - g(x)| \leq ct, \quad c = \max(\tilde{c}, L(0)).$$

□

Consider a location $z_*$ between $(y_*, 0), (x, t)$ and time at $z_*$ as the line $x, \hat{t})$.

$$
\begin{aligned}
|u(x,t) - u(x,\hat{t})| &= \left|(t-\hat{t})L\left(\frac{x - z_*}{t - \hat{t}}\right) + u(z_*, \hat{t}) - u(x, \hat{t})\right|, \\
&\leq \left|(t-\hat{t})L\left(\frac{x - z_*}{t - \hat{t}}\right)\right| + |x - z_*|L_g
\end{aligned}
$$

**Theorem 22.5.** *Hopf-Lax defines a "weak" soltuion of HJB equation. I.e. if $D_u(x,t)$ is defined, then*

$$u_t + H(Du(x,t)) = 0.$$

Satisfies for almost everywhere differentiable, but this does not give uniqueness.

# 23 HJ in Control-Theoretic Framework (11/15/23)

Missed.

# 24 HJ Control Continued (11/16/23)

The *vanishing viscosity method.* Suppose $\{\epsilon_i\}$ is a positive sequence converging to zero and and $\{u^{\epsilon_i}\}$ form a **uniformly bounded** and equicontinuous family of solutions to

$$
\boxed{\begin{cases} H(\nabla u^{\epsilon_i}, x) = \max_{a \in A}\{-\nabla^{\epsilon_i} \cdot \hat{f}(x,a) - K(x,a\} = \epsilon_i \Delta u^{\epsilon_i} & \text{on } \Omega \\ u^{\epsilon_i} = q & \text{on } \partial\Omega. \end{cases}}
$$

*Remark* 24.1. Suppose $\phi \in C^\infty(\Omega)$ is such that $(u - \phi)$ has a *strict local maximum* at some $x_0 \in \Omega$. Supposing that $\{u^{\epsilon_i}\}$ is that subsequence of $\{e^{\epsilon_i}\}$, which uniformly converges to $u$, there will exist a sequence $\{x_{\epsilon_j}\} \subset \Omega$ converging to $x_0$ such that each $x_{\epsilon_j}\}$ is a local (not necessarily strict) maximum of $(u^{\epsilon_j} - \phi)$.

*Proof.* Let $B_r \equiv B(x_0, r) \subset \Omega$. For all sufficiently small $\epsilon_j$, $\max_{\partial B_r} \leq u^{\epsilon_j}(x-)) - \phi(x_0)$ from the triangle inequality. Then $(u^{\epsilon_j} - \phi)$ attains a local maximum in the interior of $B_r$. Repeat the argument, letting $r \to 0$. □

Since $u^{\epsilon_j}$ and $\phi$ are smooth, $(u^{\epsilon_j} - \phi)$ has a local maximum at $x_{\epsilon_j} \implies \nabla u^{\epsilon_j}(x_{\epsilon_j}) = \nabla\phi(x_{\epsilon_j})$ and $\Delta u^{\epsilon_j}(x_{\epsilon_j}) \leq \Delta\phi(x_{\epsilon_j})$. So,

$$H(\nabla\phi(x_{\epsilon_j}), x_{\epsilon_j}) = H(\nabla u^{\epsilon_j}(x_{\epsilon_j}), x_{\epsilon_j}) = \epsilon_j \Delta u^{\epsilon_j}(x_{\epsilon_j}) \leq \epsilon_j \Delta\phi(x_{\epsilon_j}).$$

Since this must hold $\forall \epsilon_j$ in our (sub)sequence $\to 0$ and $x_{\epsilon_j} \to x_0$, we must have $H(\nabla\phi(x_0), x_0) \leq 0$.

By a similar argument, $(u - \phi)$ has a local minimum at some $x_0 \in \Omega$ should imply that $H(\nabla\phi(x_0), x_0) \geq 0$.

Gandall and Lions used this argument as a definition: any bounded uniformly continuous function $u : \Omega \to \mathbb{R}$ satsisfying the above properties for all $\phi \in C^\infty(\Omega)$ is called *the viscosity solution.*

Since our Hamiltonian depends on $\nabla u$, but not on $u$ itself, we can add a constant to $\phi$ without changing the relevant properties of $(u - \phi)$ and $\phi$ at and near $x_0$. We can select this constant so that $u(x_0) = \phi(x_0)$ and

- $(u - \phi)$ has a *minimum* at $x_0 \iff$ graph of $\phi$ touches the graph of $u$ from below at $x_0$
- $(u - \phi)$ has a *maximum* at $x_0 \iff$ graph of $\phi$ touches the graph of $u$ from above at $x_0$.

*Remark* 24.2. To be fair, the original Crandall-Lions definition of viscosity solutions was made in terms of *subdifferentials* and *superdifferentials* of $u$ (and in the case of 2nd-order equations, using *semi-jets*). L.C.Evans realized that the same conditions can be stated in terms of inequalities $G(x, \phi, \nabla\phi, \nabla^2\phi) \geq$ or $\leq 0$ for test functions $\phi \in C^\infty(\Omega)$ touching $u$ from above or below.

Also note that if $u$ is a classical solution of $H(\nabla u, x) = 0$, it is also a viscosity solution: if $(u - \phi)$ has an extremum at $x_0 \in \Omega$, then $\nabla u(x_0) = \nabla\phi(x_))$; so $H(\nabla\phi(x_0), x_0) = 0$. In fact, a slightly stronger statement also holds: if $u$ is a viscosity solutions and $u$ is differentiable at $x_0$, then $H(\nabla u(x_0), x_0) = 0$ (i.e. $u$ satisfies that PDE at $x_0$ in the classical sense). The proof is fairly simple and based on the same idea; see Evans 10.1.2.

To recap, $u \in C(\bar{\Omega})$ is

1. a *viscosity subsolution* of (HJBs) if $\forall\phi \in C^\infty(\Omega)$. $(u - \phi)$ has a **local max** at $x_0 \in \Omega \implies H(\nabla\phi(x_0), x_0) \leq 0$ **and** $u \leq q$ on $\partial\Omega$.
2. ...

**A subtety worth noting:** There were 2 different versions of (**):

$$u_t + \tilde{H}(\nabla u, x) = 0, \text{ and } -u_t + H(\nabla u, x) = 0$$

with $H(p, x) = -\tilde{H}(p, x)$. The classical solution $u \in C^1(\mathbb{R}^n \times \mathbb{R})$ satisfies either both of them or none at all. But the same is **not** true for viscosity solutions—our definitions on p.18 covers only the latter. We would have to change around $\leq$ and $\geq$ to cover the former. See also a comment on p. 597 in Evans and one of your HW problems (Evans 10.5 on p.603). Weak solutions are fragile when it comes to "benign transforms" (e.g. multiplying the PDE by a negative constant)!

**A few more general observations.**

1. If $p \mapsto H(p, \dots)$ os convex (or concave), the HJB PDE can be **always** interpreted as an HJB equation for the value funcaiton $u$ of some optimal control problem (see Legendre Transfrom in Evans 3.3.2). Moreover, the viscosity solution $u$ can also be viewed as the (pointwise) maximal viscosity subsolution (or pointwise minimal viscosity supersolution) of that PDE. (Persson's Method fro proving existanence).

2. If $H$ is not convex or concave in $p$, but is still Lipschitz, the HJB PDE can be viewed as an H-J-Isaacs equation for the value function of a suitably chosen differential game.

3. For convex $H$ and assuming the existence of optimal controls, the characteristscs of HJB are the optimal trajectories for the controlled system $\implies$ they are all the (locally) minimal solutions in $(p, z, x)$ space of the associated system of Euler-Lagrange equations. Knowing $u(x)$ and $\nabla u(x)$, we can determine the initial value of optimal control by taking $a^*(x)$ to be the minimizer in the definition of $H(p, x)$. [If $p = \nabla u(x)$ is well-defined, we still would need a few assumptions to ensure that there is **only one** optimal $a^*(x)$. E.g. it is easy to show this if $k = 1$ and the *velocity profile* $S_{\hat{f}}(x) = \{\hat{f}(x, a) | a \in A\}$ is convex $\forall x \in \Omega$. If $\nabla u(x)$ is not defined, this indicates that there are multiple characteristics leading to $x$ from $\partial\Omega$ **and** each of them brings to $x$ the same value $u(x0 = z(s)$; see Evans p. 113-4].

   What if we continue a characteristic beyond such an intersection point? It becomes a **locally** optimal for the control problem. E.g., for the 2D example from p.9, $|\nabla u| = 1$ on $\Omega =$

$[0,1]\times[0,1]$ with $u=0$ on $\partial\Omega$ 2 characteristics are shown below for each of the starting positions $x_1, x_2$ with diagonals representing those points, from which there is more than 1 globlly optimal path to $\partial\Omega$.

4. For the non-convex $H(p,x)$, the characteristics are still the (game-) optimal paths, but they are now the "saddle-type" solutions of the associated Euler-Lagrange system.

# 25 (11/21/23)

On existance.

# 26 Hyperbolic Conservation Laws (11/28/23)

## 26.1 Scalar Case

$$u_t + \operatorname{div}(F(u)) = 0,$$

where this is in $\mathbb{R}^n$, $F$ is the flux, and $u$ is some conserved quantity. We solve this for $t>0$.

$$u_t + (F(u))_x = 0, \quad u(x,0) = g(x),$$

for $x\in\mathbb{R}^n$. If we want to use the method of characteristics, we write

$$G(Du, u_t, u, x, t) = u_t + \operatorname{div}(F(u)) = u_t + F'(u)\cdot Du = 0$$

for $F:\mathbb{R}\to\mathbb{R}^n$. We can introduce $y=(x,t),\ q=(p,p^{n+1})),\ p\in\mathbb{R}^n$. Then

$$G(q,z,y) = p^{n+1} + F'(z)\cdot p = 0.$$

Turn the crank on method of characteristics,

$$\begin{aligned}
\dot{y} &= D_qG = \begin{bmatrix} F'(z) \\ 1 \end{bmatrix}, \\
\dot{x} &= F'(z),\ \dot{t} = 1, t(0) = 0 \implies t = s, \\
\dot{z} &= q\cdot\dot{y} = p\cdot\dot{x} + p^{n+1} = 0 \implies z(s) = g(x(0)).
\end{aligned}$$

This means that $z$ is constant along every characteristic. Thus, $\dot{x}$ is also constant, $\dot{x}(s) = F'(g(x_0))$ and characteristics are straight lines.

*From now on,* $n=1$. In the $xt$ plane, the slope is $1/F'(u)$.

*Example* 26.1 (Burgers Equations). Let $F(u)=u^2/2,\ F'(u)=u$.

$$u_t + \left(\frac{u^2}{2}\right)_x = u_t + uu_x = 0, \quad u(x,0) = g(x).$$

Consider

$$g(x) = \begin{cases} 1, & x\le 0, \\ 1-x, & x\in(0,1], \\ 0, & x>1. \end{cases}$$

This is in Evans. One might assume the behavior is from nonsmoothness of $g$, but if we smooth it out, say using mollifiers, then qualitative behavior will stay the same.

The characteristics from $x >= 1$ are just straight vertical lines because of infinite slope. The ones from $x < 0$ have slope 1. For the interval $(0, 1]$, then we have for starting point $\tilde{x}$, $\tilde{x} = (1-\tilde{x})t = x(t)$. Then $x(1) = 1\ \forall \tilde{x} \in [0, 1]$. So all characteristics starting in this interval go to the point $x = 1, t = 1$. For up to $t = 1$, we can write the solution. First solve for middle interval,

$$u(x,t) = 1 - \tilde{x} = 1 - \frac{x-t}{1-t} = \frac{1-x}{1-t}.$$

Then the whole solution up to $t = 1$ is

$$u = \begin{cases} 1, & x \le t, \\ \frac{1-x}{1-t}, & x \in [t, 1], \\ 0, & x \ge 1. \end{cases}$$

At $t = 1$, $u$ has a discontinuity at $x = 1$. For $t > 1$, we will have 3 competing values from each of the characteristics in the range for $x \in (1, t)$.

## 26.2 Integral (weak) solutions of HCL

Now, we want to develop the theory of the weak integral solutions.

$$u_t + (F(u))_x = 0,\ u(x, 0) = g(x) \quad (*).$$

If $u \in C^1$, multiply $(*)$ by smooth $v(x, t)$ with compact support and integrate by parts:

$$-\int_0^\infty \int_{-\infty}^\infty (u_t+(F(u))_x)v(x,t)\,\mathrm{d}x\,\mathrm{d}t = \boxed{\int_0^\infty \int_{-\infty}^\infty (uv_t + F(u)v_x)\,\mathrm{d}x\,\mathrm{d}t + \int_{-\infty}^\infty g(x)v(x,0)\,\mathrm{d}x = 0\ (**)}$$

This is a common pattern where you take a smooth function and then integrate by parts and throw all derivatives onto it.

**Definition 26.2.** $u \in L^\infty$ is an integral solution if $(**)$ holds for every such $v$.

Suppose $u$ is an integral solution on $V \in \mathbb{R} \times \mathbb{R}_+$ and $u$ is smooth on $V_L, V_R$. The picture is as follows: in the $xt$ plane, there is some curve $C$ (discontinuity line of $u$, often times called the *shock line*) and assume it splits $V$ into left and right pieces $V_L, V_R$. $V \setminus C = V_L \cup V_R$. Suppose $v(x, t)$ is smooth with support in $V$. Then

$$\underbrace{\iint_{V_L} (uv_t + F(u)v_x)\,\mathrm{d}x\,\mathrm{d}t}_{I_1} + \underbrace{\iint_{V_R} (uv_t + F(u)v_x)\,\mathrm{d}x\,\mathrm{d}t}_{I_2} = 0.$$

Looking at the values of each of the integrals, (let $\nu$ be a unit vector normal to $C$ and pointing into $V_R$)

$$I_1 = -\iint_{V_L} \underbrace{[u_t + (F(u))_x]}_{=0} v\,\mathrm{d}x\,\mathrm{d}t + \int_C (u_L\nu^2 + F(u_L)\nu')v\,\mathrm{d}\ell.$$

Recall that

$$\int_\Omega u_{y_i} v\,\mathrm{d}y = -\int_\Omega v_{y_i}\,\mathrm{d}y + \int_{\partial\Omega} uv\nu^i\,\mathrm{d}S(y), \quad y = \begin{pmatrix} x \\ t \end{pmatrix}.$$

$u_L$ is the limit of $u$ on $C$ as you approach from $V_L$. Next,

$$I_2 = -\int_C (u_R\nu^2 + F(u_R)\nu')v \, \mathrm{d}\ell.$$

Putting them both together,

$$\int_C ((\underbrace{F(u_L) - F(u_R))}_{[[F(u)]]}\nu' + \underbrace{(u_L - u_R)}_{[[u]]}\nu^2)v \, \mathrm{d}x = 0,$$

where $[[\cdot]]$ is a jump in $\cdot$. Then,

$$[[F]]\nu' + [[u]]\nu^2 = 0.$$

Then if we define

$$\sigma = \frac{-\nu^2}{\nu'}, \; [[F]] = \sigma[[u]],$$

where $\sigma$ is the speed of discontinuity. (In the Burger's examples, this gives us $\sigma = 1/2$)