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Problem Set 5

1. Let α =(1 5) (9 6) (4 7) (3 10) (2 8), β = (1 9) (2 3) (8 7) (6 10) (4 5), and

γ = (1 2 9 3) (5 4 7 A) (6 B E) (C 8 D) be permutations.

1. Find the representation of αβ as a product of disjoint cycles, and verify that the lengths of the cycles come in pairs.
2. Find two permutations σ and π on the set {1,2,...,9,A,B,C,D,E,F} each of which is a product of seven disjoint cycles and such that γ = πσ.

Solution:

1. If we write βα as a product of disjoint cycles, we have that:

* 1 maps to 5 in α, 5 maps to 4 in β
* 4 maps to 7 in α, 7 maps to 8 in β
* 8 maps to 2 in α, 2 maps to 3 in β
* 3 maps to 10 in α, 10 maps to 6 in β
* 6 maps to 9 in α, 9 maps to 1 in β
* 2 maps to 8 in α, 8 maps to 7 in β
* 7 maps to 4 in α, 4 maps to 5 in β
* 5 maps to 1 in α, 1 maps to 9 in β
* 9 maps to 6 in α, 6 maps to 10 in β
* 10 maps to 3 in α, 3 maps to 2 in β

This gives the following product of disjoint cycles: (1 4 8 3 6) (2 7 5 9 10). We can verify that that the lengths of the cycles come in pairs because we have two cycles, each of length 5.

1. To find π and σ such that γ = πσ, we can begin by guessing the first cycle in π, and use it to figure out the corresponding cycle in σ:

* If 1 maps to 4 in σ, then 4 must map to 2 in π, since 1 maps to 2 in γ.
* 5 must map to 2 in σ since 2 maps to 4 in π. This gives 5 maps to 4 in γ.
* Since 2 maps to 5 in σ, then 5 maps to 9 in π. This gives 2 maps to 9 in γ.
* A must map to 9 in σ, since 9 maps to 5 in π. This gives A maps to 5 in γ.
* Since 9 maps to A in σ, then A maps to 3 in π. This gives 9 maps to 3 in γ.
* 7 must map to 3 in σ, since 3 maps to A in π. This gives 7 maps to A in γ.
* If 3 maps to 7 in σ, then 7 must map to 1 in π. This gives 3 maps to 1 in γ.

So far, we have that σ = (14) (52) (9A) (73) and π = (42) (59) (A3) (71). Next ,we can guess the next cycle for σ:

* If 6 maps to 8 in σ then 8 must map to B in π, since 6 maps to B in γ.
* C must map to B in σ, since B maps to 8 in π. This gives C maps to 8 in γ.
* Since B maps to C in σ, then C must map to E in π. This gives B maps to E in γ.
* D must map to E in σ, since E maps to C in π. This gives D maps to C in γ.
* Since E maps to D in σ, D must map to 6 in π. This gives E maps to 6 in γ.

Now we have that σ = (6 8)(C B) (D E) and π = (8 B)(C E)(D 6). By combining the two, we get the final permutations:

σ = (14) (52) (9A) (73) (6 8) (C B) (D E)

π = (42) (59) (A3) (71) (8 B) (C E)(D 6)

1. Suppose σ = (1 7 4) (5 8 9 2) (6 3) and ϮσϮ-1 = (5 6 3) (1 4 2 7) (8 9). Find two possible values for Ϯ.

Since we know that σ maps j to k, and that ϮσϮ-1 maps Ϯ(j) to Ϯ(k), we have that

ϮσϮ-1 = ( Ϯ(1) Ϯ(7) Ϯ(4) ) ( Ϯ(5) Ϯ(8) Ϯ(9 ) Ϯ(2) ) ( Ϯ(6) Ϯ(3) )

= (5 6 3) (1 4 2 7) (8 9).

Therefore, one solution gives

* 1 maps to 5
* 5 maps to 1
* 2 maps to 7
* 7 maps to 6
* 6 maps to 8
* 8 maps to 4
* 4 maps to 3
* 3 maps to 9
* 9 maps to 2

This gives the following permutation: Ϯ = (1 5) (2 7 6 8 4 3 9)

To find another permutation, we can rewrite ϮσϮ-1 as (6 3 5) (1 4 2 7) (8 9). Therefore, we have that:

ϮσϮ-1 = ( Ϯ(1) Ϯ(7) Ϯ(4) ) ( Ϯ(5) Ϯ(8) Ϯ(9 ) Ϯ(2) ) ( Ϯ(6) Ϯ(3) )

= (6 3 5) (1 4 2 7) (8 9).

Therefore, the next solution gives:

* 1 maps to 6
* 6 maps to 8
* 8 maps to 4
* 4 maps to 5
* 5 maps to 1
* 2 maps to 7
* 7 maps to 3
* 3 maps to 9
* 9 maps to 2

This gives the permutation Ϯ = (1 6 8 4 5) (2 7 3 9)

Extra credit: Since the first cycle in ϮσϮ-1 is of length 3, there are 3 ways we can rewrite this cycle. This gives us 3 possible mappings for the first cycle. The next cycle is of length 4, so we can rewrite the cycle in 4 ways. This gives us 4 mappings for the second cycle. The last cycle is of length 2, so we can rewrite it in 2 different ways. This gives us to mappings for the last cycle.

Therefore, there are 3 x 4 x 2 = 24 possible permutations Ϯ.