Title: General Relativity

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Summary:

1 DIFFERENTIAL FORMS

See also Wheeler. From a notational standpoint, I should do all of the exercises in Ch. 14 of MTW. For an true and complete discussion of differential forms I should write notes on MTW Chs.(2.5-2.7)(3.2)(3.5)

1.1 VECTORS

A vector is a geometrical object which extends between two events in spacetime. In other words, a vector is a line with a preferred direction in spacetime which also has a definite magnitude. A very common way to visualize a vector in spacetime is by imagining an arrow which points from one spacetime point to another. We choose to represent this object with boldface characters such as $\boldsymbol{v}$. Vectors are interesting objects because we can represent them in a number of different ways depending on the context. If, for instance, we are interested in writing down a vector which represents the displacement between arbitrary spacetime event A and spacetime event B we can express this as

$$\boldsymbol{v}_{AB} \equiv B - A \tag{1.1}$$

The same arrow could also be written as a parameterized straight line

$$P(\lambda) = A + \lambda(B - A) \tag{1.2}$$

Which satisfies $P(0) = A$ and $P(1) = B$. Taking the derivative yields

$$\frac{d}{d\lambda}P(\lambda) = B - A = P(1) - P(0) \equiv \boldsymbol{v}_{AB} \tag{1.3}$$

The above definition allows us to define the concept of a 1-point object i.e. a tangent vector along a curve

$$\boldsymbol{v}_{AB} = \left(\frac{dP(\lambda)}{d\lambda}\right)_{\lambda=0} \tag{1.4}$$

Most importantly we've written down this vector without any reference as to what frame we are looking at this vector. This illustrates explicitly that the vector exists regardless of the frame it is viewed in. If we choose to analyze the vector in a particular Lorentz frame then we must choose a set of orthogonal coordinates which allow us to express the vector uniquely. Say we choose four arbitrary orthogonal coordinates $\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. The spacetime points A and B as well as the vector $\mathbf{v}_{AB}$ may now be written explicitly

$$A \equiv (a^0\mathbf{e_0}, a^1\mathbf{e}_1, a^2\mathbf{e_2}, a^3\mathbf{e}_3) \tag{1.5}$$

$$B \equiv (b^0\mathbf{e_0}, b^1\mathbf{e}_1, b^2\mathbf{e_2}, b^3\mathbf{e}_3) \tag{1.6}$$

$$\begin{aligned} \mathbf{v}_{AB} = B - A &\equiv (b_0 - a_0)\mathbf{e}_0 + (b_1 - a_1)\mathbf{e}_1 + (b_2 - a_2)\mathbf{e}_2 + (b_3 - a_3)\mathbf{e}_3 \\ &= v^0\mathbf{e}_0 + v^1\mathbf{e}_2 + v^2\mathbf{e}_2 + v^3\mathbf{e}_3 \\ &\equiv v^\mu\mathbf{e}_\mu \end{aligned} \tag{1.7}$$

Again, we can rewrite the vector in terms of a parameterized straight line. Say, for example, that we parameterize the worldline of a particle by its propertime τ, The worldline is then expressed as $P(\tau)$ and exists independently of a coordinate system. If we wanted to express the position on the worldline at some value of τ with respect to some origin $\mathcal{O}$ of a particular frame we would have

$$P(\tau) - \mathcal{O} = x^0(\tau)\mathbf{e}_0 + x^1(\tau)\mathbf{e}_1 + x^2(\tau)\mathbf{e}_2 + x^3(\tau)\mathbf{e}_3 \equiv x^\mu(\tau)\mathbf{e}_\mu \tag{1.8}$$

Given this definition we can define the 4-velocity $\mathbf{u}$

$$\mathbf{u} \equiv \frac{dP(\tau)}{d\tau} = \frac{dx^\mu(\tau)}{d\tau}\mathbf{e}_\mu = u^0(\tau)\mathbf{e}_0 + u^1(\tau)\mathbf{e}_1 + u^2(\tau)\mathbf{e}_2 + u^3(\tau)\mathbf{e}_3 \tag{1.9}$$

1.2 ONE-FORMS

Now that we have introduced vectors we can introduce the concept of a "differential form" or "one-form". A one-form is a different type of geometrical object which exists by itself, independent of a particular frame of coordinates. To understand what a one-form is, imagine an infinite stack of flat surfaces spanning the x-y plane and traversing up and down the z-axis separated by an arbitrary but fixed distance. Now imagine we take a vector and pierce through this stack of planes. The vector, with finite length, will only pierce through a finite number of planes. We can quantify the number of piercings via a machine which take two inputs and outputs a single number. In the first slot we input the one-form and in the second slot we input the vector. The machine will output the number of times the vector pierces the one-form.

$$\langle \boldsymbol{\sigma}, \boldsymbol{v} \rangle = \text{ \# of times the vector } \boldsymbol{v} \text{ pierces the one-form } \boldsymbol{\sigma} \tag{1.10}$$

Vectors and one forms are said to be dual when the following relation holds,

$$\boldsymbol{v} \cdot \boldsymbol{\sigma} = \langle \boldsymbol{\sigma}, \boldsymbol{v} \rangle = \eta_{\mu\nu} v^\mu \sigma^\nu \tag{1.11}$$

1.3 p-FORMS

1.4 WEDGE PRODUCT ($\wedge$)

We define the notation

$$T_{[a_1 \cdots a_l]} = \frac{1}{p!} \sum_\pi \delta_\pi T_{a_{\pi(1)} \cdots a_{\pi(p)}} \tag{1.12}$$

where the sum is taken over all permutations, π, of $1, \ldots, p$ and δ_π is $+1$ for even permutations and -1 for odd permutations. Brackets $[\cdots]$ indicate antisymmeterization over the indicies whereas parentheses $(\cdots)$ indicate symmeterization. For example,

$$T_{(ab)} = \frac{1}{2}(T_{ab} + T_{ba}) \qquad T_{[ab]} = \frac{1}{2}(T_{ab} - T_{ba}) \tag{1.13}$$

The $1/p!$ factor ensures that the symmeterization is of strength one i.e. if we symmetrize or antisymmetrize twice, the tensor remains the same $T_{((a_1 \cdots a_n))} = T_{(a_1 \cdots a_n)}$, $T_{[[a_1 \cdots a_n]]} = T_{[a_1 \cdots a_n]}$. The symmetric and antisymmetric tensors resulting from the tensor T are termed *cotensors* of T. Generally a p-index cotensor W is an object

$$W = W_{\mu_1 \mu_2 \cdots \mu_p} dx^{\mu_1} \otimes dx^{\mu_2} \otimes \cdots \otimes dx^{\mu_p} \tag{1.14}$$

where $W_{\mu_1 \mu_2 \cdots \mu_n}$ are its components with respect to the basis $dx^{\mu_1} \otimes dx^{\mu_2} \otimes \cdots \otimes dx^{\mu_n}$. If the cotensor is antisymmetric in its indices it will make an antisymmetric projection on the tensor product of basis 1-forms dx^μ. To denote a antisymmeterized product of basis 1-forms we introduce the *wedge product* denoted by $\wedge$ and define

$$dx^\mu \wedge dx^\nu = dx^\mu \otimes dx^\nu - dx^\nu \otimes dx^\mu \tag{1.15}$$

$$\begin{aligned} dx^\mu \wedge dx^\nu \wedge dx^\sigma =& dx^\mu \otimes dx^\nu \otimes dx^\sigma + dx^\nu \otimes dx^\sigma \otimes dx^\mu + dx^\sigma \otimes dx^\mu \otimes dx^\nu \\ &- dx^\mu \otimes dx^\sigma \otimes dx^\nu - dx^\sigma \otimes dx^\nu \otimes dx^\mu - dx^\nu \otimes dx^\sigma \otimes dx^\mu \end{aligned} \tag{1.16}$$

$$\vdots$$

Cotensors antisymmetric in p indices are called *p-forms*. Suppose we have a p-form A with components $A_{\mu_1 \mu_2 \cdots \mu_p}$ its expansion is given by

$$A = \frac{1}{p!} A_{\mu_1 \mu_2 \cdots \mu_p} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p} \tag{1.17}$$

A function is a special type of p-form with $p = 0$. We see that if given a p-form A and a q-form B then they must satisfy

$$A \wedge B = (-1)^{pq} B \wedge A \tag{1.18}$$

1.5 EXTERIOR DERIVATIVE (d)

The exterior derivative d is defined to act on a p-form field and produce a $(p+1)$-form field. On functions (0-forms) the exterior derivative acts as the familiar differential operator

$$df = \partial_\mu f dx^\mu \tag{1.19}$$

More generally, on a p-form $\omega = \frac{1}{p!}\omega_{\mu_1\mu_2\cdots\mu_p} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}$, it is defined by

$$d\omega = \frac{1}{p!}(\partial_\nu \omega_{\mu_1\mu_2\cdots\mu_p}) dx^\nu \wedge dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p} \tag{1.20}$$

From our definition of p-forms, the components of the $(p+1)$-form $d\omega$ are given as

$$(d\omega)_{\nu\mu_1\cdots\mu_p} = (p+1)\partial_{[\nu}\omega_{\mu_1\cdots\mu_p]} \tag{1.21}$$

i.e. the expansion of the $(p+1)$-form $d\omega$ in the coordinate basis we are using is given by

$$d\omega = \frac{1}{(p+1)!}(d\omega)_{\mu_1\cdots\mu_p+1} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_{p+1}} \tag{1.22}$$

From these definitions we see that p-form A and a q-form B obey the Leibnitz rule

$$d(A \wedge B) = dA \wedge B + (-1)^p A \wedge dB \tag{1.23}$$

The exterior derivative satisfies

$$dd\Omega = 0 \tag{1.24}$$

where Ω is *any* differential form of any degree p. This follows from the fact that d is an antisymmetric derivative whereas partial derivatives commute. Explicitly

$$\begin{aligned} dd\omega &= \frac{1}{p!}(\partial_\rho\partial_\nu\omega_{\mu_1\cdots\mu_p}) dx^\rho \wedge dx^\nu \wedge dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p} \\ &= -\frac{1}{p!}(\partial_\nu\partial_\rho\omega_{\mu_1\cdots\mu_p}) dx^\nu \wedge dx^\rho \wedge dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p} \\ &= -\frac{1}{p!}(\partial_\rho\partial_\nu\omega_{\mu_1\cdots\mu_p}) dx^\nu \wedge dx^\rho \wedge dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p} \\ &= -dd\omega = 0 \end{aligned} \tag{1.25}$$

As an application of the above ideas in a familiar arena, we can look at Maxwell Theory. The vector potential is a 1-form $A = A_\mu dx^\mu$, and the field strength is a 2-form $F = \frac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu$ and can be constructed from the vector potential via the application of an exterior derivative

$$F = dA = \partial_\mu A_\nu dx^\mu \wedge dx^\nu = \frac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu \tag{1.26}$$

from which we can see $F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]} = \partial_\mu A_\nu - \partial_\nu A_\mu$. This also implies $dF = ddA = 0$, this is nothing other than the Bianchi identity:

$$dF = \frac{1}{2}\partial_\rho F_{\mu\nu}dx^\rho \wedge dx^\mu \wedge dx^\nu = \partial_{[\rho}F_{\mu\nu]} = 0 \tag{1.27}$$

1.6 HODGE DUAL ($\star$)

The Maxwell field equation can also be expressed (in its most elegant form) in terms of differential operators with one additional notational definition. We first define the total antisymmetric tensor density $\varepsilon_{\mu_1\cdots\mu_n}$ in n-dimensions speficifed completely by the statement

$$\varepsilon_{012\cdots n-1} = +1 \tag{1.28}$$

The totally antisymmetric Levi-Cevita tensor is then defined as

$$\epsilon_{\mu_1\cdots\mu_n} = \sqrt{-g}\varepsilon_{\mu_a\cdots\mu_n} \tag{1.29}$$

If we write $n = q + p$, and take the product of two epislon tensors contracted on p indicies

$$\epsilon_{\mu_1\cdots\mu_p\nu_1\cdots\nu_q}\epsilon^{\mu_1\cdots\mu_p\rho_1\cdots\rho_q} = -p!q!\delta^{\rho_1\cdots\rho_q}_{\nu_1\cdots\nu_q} \tag{1.30}$$

where

$$\delta^{\rho_1\cdots\rho_q}_{\nu_1\cdots\nu_q} \equiv \delta^{\rho_1}_{[\nu_1}\delta^{\rho_2}_{\nu_2}\cdots\delta^{\rho_q}_{\nu_q]} \tag{1.31}$$

Note that the minus sign in Eq.(1.30) arises because of the negative eigenvalue of the metric tensor in a spacetime of signature $(-++\cdots+)$.

We can now define the Hodge dual operator $\star$ which maps p-forms to $(n-p)$-forms:

$$\star(dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}) \equiv \frac{1}{(n-p)!}\epsilon_{\nu_1\cdots\nu_{n-p}}{}^{\mu_1\cdots\mu_p}dx^{\nu_1} \wedge \cdots \wedge dx^{\nu_{n-p}} \tag{1.32}$$

Acting on a p-form ω

$$\star\omega = \frac{1}{p!(n-p)!}\epsilon_{\nu_1\cdots\nu_{n-p}}{}^{\mu_1\cdots\mu_p}\omega_{\mu_1\cdots\mu_p}dx^{\nu_1} \wedge \cdots \wedge dx^{\nu_{n-p}} \tag{1.33}$$

such that the $(n-p)$-form $\star\omega$ has the components

$$(\star\omega)_{\mu_1\cdots\mu_q} = \frac{1}{p!}\epsilon_{\nu_1\cdots\nu_{n-p}}{}^{\mu_1\cdots\mu_p}\omega_{\mu_1\cdots\mu_p}\omega_{\mu_1\cdots\mu_p} \tag{1.34}$$

If we act the twice with $\star$ on a p-form we obtain another p-form:

$$\star\star\,\omega = (-1)^{pq+1}\omega \tag{1.35}$$

where again $n = p + q$[1]. It can be shown that the operation $\star d \star \omega$ is a $(p-1)$-form and is related to the divergence of ω via

$$(\star d \star \omega)_{\mu_a\cdots\mu_{p-1}} = (-1)^{pq+p}\nabla_\nu \omega^\nu{}_{\mu_1\cdots\mu_{p-1}} \tag{1.36}$$

where ∇_ν is the usual covariant derivative built using the Christoffel connection. With these definitions the *source-free* Maxwell field equations ($\nabla_\mu F^{\mu\nu} = 0$) can be written as

$$d\star F = 0 \tag{1.37}$$

Take for example the Hodge dual of a 0-form f living in D spacetime dimensions. The dual must be a D-form whose components are the Levi-Cevita tensor

$$\star f = \boldsymbol{\epsilon} = \frac{f}{D!}\epsilon_{\nu_1\cdots\nu_D} dx^{\nu_1}\wedge\cdots\wedge dx^{\nu_D} = \frac{f}{D!}\sqrt{|g|}\varepsilon_{\nu_1\cdots\nu_D} dx^{\nu_1}\wedge\cdots\wedge dx^{\nu_D} \tag{1.38}$$

Note:

$$\varepsilon_{\mu_1\cdots\mu_N} dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_N} = N!(dx^0\otimes\cdots\otimes dx^{N-1}) = d^N x \tag{1.39}$$

or

$$dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_N} = (-1)^t\varepsilon^{\mu_1\cdots\mu_N} d^N x = (-1)^t\epsilon^{\mu_1\cdots\mu_N}\sqrt{|g|}d^N x \tag{1.40}$$

Thus,

$$\star f = d^D x\sqrt{|g|}f \tag{1.41}$$

The dual of a 0-form is simply the coordinate invariant volume element!

$$\int f d(\text{Volume}) = \int \star f \tag{1.42}$$

Now consider two p-forms $\boldsymbol{A}$ and $\boldsymbol{B}$ then,

$$\star\boldsymbol{A}\wedge\boldsymbol{B} = \star\boldsymbol{B}\wedge\boldsymbol{A} = \frac{1}{p!}|\boldsymbol{A}\cdot\boldsymbol{B}|\boldsymbol{\epsilon} = \frac{1}{p!}|\boldsymbol{A}\cdot\boldsymbol{B}|\star f \tag{1.43}$$

where

$$|\boldsymbol{A}\cdot\boldsymbol{B}| \equiv A_{\mu_1\cdots\mu_p}B^{\mu_1\cdots\mu_p} \tag{1.44}$$

verify this

[1] Be careful in spacetimes with time dimensions $t \geq 2$, in which $\star\star\,\omega = (-1)^{pq+t}\omega$

1.7 VIELBEIN

By introducing a vielbein (German for "many legs") we may "take the square root of" a metric $g_{\mu\nu}$. A vielbein is a basis of 1-forms $e^a = e^a_\mu dx^\mu$, with components e^a_μ having the property

$$g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu \tag{1.45}$$

where the indices a, b are a new type called local-Lorentz indices or tangent-space indices and η_{ab} is a "flat" metric, with constant components. The language of local-Lorentz indices stems from the situation when the metric $g_{\mu\nu}$ has the Minkowskian signature. The signature of η_{ab} must be the same as that of $g_{\mu\nu}$ so if we are working with Minkowskian signature

$$\eta_{ab} = \mathrm{diag}(-1, +1, +1, \ldots, +1) \tag{1.46}$$

The choice of vielbeins e^a as the square root of the metric is, to some extent, arbitrary. Particularly we could perform an (psuedo)orthogonal transformation to get another equally-valid vielbein e'^a given by:

$$e'^a = \Lambda^a{}_b e^b \tag{1.47}$$

where $\Lambda^a{}_b$ is a matrix satisfying the (psuedo)orthogonal condition

$$\eta_{ab} \Lambda^a{}_c \Lambda^b{}_d = \eta_{cd} \tag{1.48}$$

Note Λ can be coordinate-dependent. If the n-dimensional manifold has a Euclidean-signature the $\eta = \mathbb{1}$ and Eq.(1.48) is literally the orthogonality condition ($\Lambda^T \Lambda = \mathbb{1}$). Thus, the arbitrariness in choice of the vielbein lies in the freedom to make local $O(1)$ rotations in the tangent space. If the metric signature is Minkowskian, one then has the ability to perform Lorentz transformations in the tangent space. The Lorentz transformation matrix may depend on the spacetime coordinates and thus is termed a "local Lorentz transformation".

1.8 SPIN CONNECTION AND CURVATURE 2-FORM

We introduce the connection 1-forms $\omega^a{}_b = (\omega^b{}_a)_\mu dx^\mu$, and the torsion 2-forms $T^a = \frac{1}{2} T^a_{\mu\nu} dx^\mu \wedge dx^\nu$ defining

$$T^a \equiv de^a + \omega^a{}_b \wedge e^b \tag{1.49}$$

Next, the curvature 2-form is defined via

$$\Theta^a{}_b \equiv d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b \tag{1.50}$$

We note, adopting the matrix notation where local Lorentz transformations are given by $e' = \Lambda e$, then

$$\omega' = \Lambda\omega\Lambda^{-1} + \Lambda d\Lambda^{-1} \tag{1.51}$$

$$T' = \Lambda T \tag{1.52}$$

$$\Theta' = \Lambda\Theta\Lambda^{-1} \tag{1.53}$$

The torsion 2-form and curvature 2-form both transform nicely i.e. in a covariant way under local Lorentz transformations, while the spin-connection does not. The spin connection contains the characteristic inhomogenous term in its transformation rule. To remedy this, we define the Lorentz-covariant exterior derivative D:

$$DV^a{}_b \equiv dV^a{}_b + \omega^a{}_c \wedge V^c{}_b - \omega^c{}_b \wedge V^a{}_c \tag{1.54}$$

where $V^a{}_b$ is some set of p-forms carrying tangent-space indices a and b. We see that if $V^a{}_b$ transforms covariantly under local Lorentz transformation, then so does $DV^a{}_b$. Here, we have taken the example of $V^a{}_b$ with one upstairs and one downstairs tangent space index for simplicity, but the generalization to an arbitrary number of indices can be seen immediately. Insert general rule.

The covariant exterior derivative will commute nicely with the process of contracting tangent-space indices with η_{ab}, provided that we require

$$D\eta_{ab} \equiv d\eta_{ab} - \omega^c{}_a\eta_{cb} - \omega^c{}_b\eta_{ac} = 0 \tag{1.55}$$

Because we take $\eta_{ab} = \mathrm{diag}(-1, +1, \ldots, +1)$ we have $d\eta_{ab} = 0$ and thus we have the condition

$$\omega_{ab} = -\omega_{ba} \tag{1.56}$$

which is known as the *metric compatibility* condition. Note $\omega_{ab} \equiv \eta_{ac}\omega^c{}_b$. By imposing this condition we are able to freely move the local Lorentz metric tensor η_{ab} through the derivative i.e. $[D, \eta_{ab}] = 0$. In addition to the metric compatibility condition, we can choose to work with a vanishing torsion 2-form. In which case we have the following two conditions which determine the spin connection coefficients $\omega^a{}_b$ uniquely:

$$de^a = -\omega^a{}_b \wedge e^b, \quad \omega_{ab} = -\omega_{ba} \tag{1.57}$$

The unique solutions can be obtained by defining the exterior derivative of the vielbeins e^a

$$de^a \equiv -\frac{1}{2}c_{bc}{}^a e^b \wedge e^c \tag{1.58}$$

where the structure constants ${c_{bc}}^a$ are antisymmetric in bc by definition. The solutions for ω_{ab} is then given by

$$\omega_{ab} = \frac{1}{2}(c_{abc} + c_{acb} - c_{bca})e^c \tag{1.59}$$

The procedure then for computing the curvature 2-forms for a metric $g_{\mu\nu}$ with vielbeins e^a is the following:

1. Write down a choice of vielbein
2. Take an exterior derivative and read off the coefficients $c_b c^a$
3. Using these structure coefficients, compute the spin connection using Eq.(1.59)
4. Substitute the newly found spin connection into Eq.(1.50) to compute the curvature 2-forms

Each curvature 2-form has a tensor which is antisymmetric in two coordinate indices. This is the Riemann tensor, defined by

$$\Theta^a{}_b = \frac{1}{2}(R^a{}_b)_{\mu\nu} dx^\mu dx^\nu \tag{1.60}$$

We may always use the vielbein e^a_μ, which is a non-degenerate $n \times n$ matrix in n dimensions, to convert between coordinate indices μ and tangent-space indices a. For this, we need the inverse of the vielbein. sometimes denoted as E^μ_a (not necessary to define a new symbol $E^\mu_a = g^{\mu\nu}\eta_{ab}e^b_\nu$) and satisfying:

$$E^\mu_a e^a_\nu = \delta^\mu_\nu, \qquad E^\mu_a e^b_\mu = \delta^a_b \tag{1.61}$$

With this, we can define the Riemann tensor components entirely within the tangent-frame basis as:

$$R^a{}_{bcd} \equiv E^\mu_c E^\nu_d (R^a{}_b)_{\mu\nu} \tag{1.62}$$

If we choose to use the same symbols for tensors, we must pay close attention to the indices and establish unambiguous notations to keep track of which indices correspond to coordinates and which refer to the tangent-space. With these definitions, the 2-form curvature tensor can be written as

$$\Theta^a{}_b = \frac{1}{2} R^a{}_{bcd} e^c \wedge e^d \tag{1.63}$$

And as usual, from the Riemann tensor we can define the tangent space Ricci tensor R_{ab}

$$R_{ab} = R^c{}_{acb} \tag{1.64}$$

and the Ricci scalar R

$$R = \eta^{ab} R_{ab} \tag{1.65}$$

which have the following symmetry properties

$$R_{abcd} = -R_{bacd} = -R_{abdc} = R_{cdab} \tag{1.66}$$

$$R_{abcd} + R_{acdb} + R_{adbc} = 0 \tag{1.67}$$

$$R_{ab} = R_{ba} \tag{1.68}$$

2 COVARIANT DERIVATIVES

In general relativity the metric is promoted to a dynamical field variable. In the presence of an energy density, the metric (spacetime) acquires off diagonal elements resulting in curvature. We define curvature in terms of the parallel transport of vectors along closed curves. In the presence of curvature, derivative operators do not commute and upon traveling along a closed curve we find that the initial vector V does not remain parallel to the parallel-transported vector V'. When we speak of parallel transport we mean that the vector is moved along the instantaneous tangent plane to the surface along the curve.

We define the (covariant) derivative operator ∇, on a manifold M as a map which takes each smooth tensor field of type (k, l) to a smooth tensor field of type $(k, l+1)$. That is, the derivative operator tacks on one more covariant index to the tensor field with which it acts upon. The derivative operator also satisfies five properties

1. Linearity

$$\nabla_c \left(\alpha A^{a_1 \cdots a_k}{}_{b_1 \cdots b_l} + \beta B^{a_1 \cdots a_k}{}_{b_1 \cdots b_l} \right) = \alpha \nabla_c A^{a_1 \cdots a_k}{}_{b_1 \cdots b_l} + \beta \nabla_c B^{a_1 \cdots a_k}{}_{b_1 \cdots b_l} \tag{2.1}$$

2. Leibnitz rule

$$\nabla_e \left[A^{a_1 \cdots a_k}{}_{b_1 \cdots b_l} B^{a_1 \cdots a_k}{}_{b_1 \cdots b_l} \right] = \left[\nabla_e A^{a_1 \cdots a_k}{}_{b_1 \cdots b_l} \right] B^{a_1 \cdots a_k}{}_{b_1 \cdots b_l} + A^{a_1 \cdots a_k}{}_{b_1 \cdots b_l} \left[\nabla_e B^{a_1 \cdots a_k}{}_{b_1 \cdots b_l} \right] \tag{2.2}$$

3. Commutativity with contraction

$$\nabla_d (A^{a_1 \cdots a_k}{}_{b_1 \cdots b_l}) = \nabla_d A^{a_1 \cdots a_k}{}_{b_1 \cdots b_l} \tag{2.3}$$

4. Consistency with the notion of tangent vector as directional derivatives on scalar fields. For $t \in V_p$ where V_p is the tangent space at a point p of a manifold M and $f \in \mathcal{T}$ where $\mathcal{T}$ is the set of smooth function from manifold M into $\mathbb{R}$. I don't really like this notation, should come back later and build up consistent notation with MTW.

$$t(f) = t^a \nabla_a f \tag{2.4}$$

$$\nabla_u f \tag{2.5}$$

5. Torsion free[2]

$$\nabla_a \nabla_b f = \nabla_b \nabla_a f \tag{2.6}$$

Any "rule" ∇, for producing new vector fields from old and which satisfy the five conditions above, is called by differential geometers a "*symmetric covariant derivative*". There are as many ways of defining a covariant derivative ∇ as there are of rearranging sources of the gravitational field. Different geodesics (free-fall trajectories) result from different distributions of masses and thus different definitions of ∇. This raises the question of uniqueness. From property (4) we know that any two derivative operators must agree in their action on scalar fields. What about the next lowest ranking tensor field? By comparing the difference of two derivative operators ∇ and $\tilde{\nabla}$ when acting on a dual vector field ω_b we find that the difference $(\nabla - \tilde{\nabla})$ defines a map of dual vectors at a point p to tensors of rank $\binom{1}{2}$ at p. Thus, given any any two derivative operators there exists a tensor field $C^c{}_{ab}$ such that

$$\nabla_a \omega_b = \tilde{\nabla}_a \omega_b - C^c{}_{ab} \omega_c \tag{2.7}$$

Where $C^c{}_{ab} = C^c{}_{ba}$. This displays the potential disagreement of ∇ and $\tilde{\nabla}$ on dual vector fields. For arbitrary tensor fields we find

$$\begin{aligned} \nabla_a T^{b_1 \cdots b_k}{}_{c_1 \cdots c_l} = \tilde{\nabla}_a T^{b_1 \cdots b_k}{}_{c_1 \cdots c_l} + \sum_i C^{b_i}{}_{ad} T^{b_1 \cdots d \cdots b_k}{}_{c_1 \cdots c_l} \\ - \sum_j C^d{}_{ac_j} T^{b_1 \cdots b_l}{}_{c_1 \cdots d \cdots c_l} \end{aligned} \tag{2.8}$$

Thus, the difference between the two derivative operators ∇ and $\tilde{\nabla}$ is completely characterized by the tensor field $C^c{}_{ab}$. We see that, given only the manifold structure, there are many distinct choices of derivative operator ∇, none preferred over the

[2] This condition is not necessary, in the case it is not imposed, $\nabla_a \nabla_b f - \nabla_b \nabla_a f = -T^c{}_{ab} \nabla_c f$ where $T^c{}_{ab}$ is antisymmetric in a and b and is called the *torsion tensor*.

other. However, if we are given a metric g_{ab} on the manifold, a natural choice of derivative operator is uniquely picked out. With a metric in hand, it a very natural condition for parallel transport seemingly pops out. Consider two vectors v^a and u^a. If we parallel-transport these two vectors we would demand that their inner product remain invariant. After all, parallel transport should preserve angles and magnitudes between and of these vectors. If we consider a parallel transport in some direction $\mathbf{w}$ in the tangent plane

$$\nabla_{\mathbf{w}}(g_{ab}v^a u^b) = (\nabla_{\mathbf{w}} g_{ab})v^a u^b + g_{ab}(\nabla_{\mathbf{w}} v^a)u^b + g_{ab}v^a(\nabla_{\mathbf{w}} u^b) = 0 \tag{2.9}$$

During parallel transport the components of v and u will remain the same ($\nabla_{\mathbf{w}} v^a = \nabla_{\mathbf{w}} u^b = 0$).

$$v^a u^b \nabla_{\mathbf{w}} g_{ab} = 0 \tag{2.10}$$

Thus, the dot product of two vector which are parallel transported will be preserved if and only if

$$\nabla_{\mathbf{w}} g_{ab} = 0 \tag{2.11}$$

This condition uniquely determines ∇. We can prove this by again considering two derivative operators ∇ and $\tilde{\nabla}$. From Eqs. (2.8) and (2.11) we have

$$0 = \nabla_a g_{bc} = \tilde{\nabla}_a g_{bc} - C^d{}_{ab} g_{dc} - C^d{}_{ac} g_{bd} \tag{2.12}$$

$$\tilde{\nabla}_a g_{bc} = C^d{}_{ab} g_{dc} + C^d{}_{ac} g_{bd} \tag{2.13}$$

$$\tilde{\nabla}_a g_{bc} = C_{cab} + C_{bac} \tag{2.14}$$

By renaming indices we also have

$$\tilde{\nabla}_b g_{ac} = C_{cba} + C_{abc} \tag{2.15}$$

$$\tilde{\nabla}_c g_{ba} = C_{acb} + C_{bca} \tag{2.16}$$

Adding the first two relations

$$C_{cab} + C_{bac} + C_{cba} + C_{abc} = \tilde{\nabla}_a g_{bc} + \tilde{\nabla}_b g_{ac} \tag{2.17}$$

$$2C_{cab} + C_{bac} + C_{abc} = \tilde{\nabla}_a g_{bc} + \tilde{\nabla}_b g_{ac} \tag{2.18}$$

Subtracting Eq.(2.16)

$$2C_{cab} + C_{bac} + C_{abc} - C_{acb} - C_{bca} = \tilde{\nabla}_a g_{bc} + \tilde{\nabla}_b g_{ac} - \tilde{\nabla}_c g_{ba} \tag{2.19}$$

$$2C_{cab} = \tilde{\nabla}_a g_{bc} + \tilde{\nabla}_b g_{ac} - \tilde{\nabla}_c g_{ba} \tag{2.20}$$

$$C^c{}_{ab} = \frac{1}{2} g^{cd} \left(\tilde{\nabla}_a g_{bd} + \tilde{\nabla}_b g_{ad} - \tilde{\nabla}_d g_{ba} \right) \tag{2.21}$$

This choice of $C^c{}_{ab}$ satisfies Eq.(2.11) uniquely. A metric g_{ab} then, naturally defines a derivative operator ∇.

As I said at the beginning of this section, we can use the path dependence of parallel transport to define an intrinsic notion of curvature. We do a similar calculation as above but for the action of two derivative operators. We find that the difference of interchanged orderings of the derivative operators can be expressed as a rank $\binom{1}{3}$ tensor field.

$$\nabla_a \nabla_b \omega_c - \nabla_b \nabla_a \omega_c = R_{abc}{}^d \omega_d \tag{2.22}$$

$R_{abc}{}^d$ is called the *Riemann curvature tensor* and is directly related to the failure of a vector to return to its initial value when parallel transported around a small closed curve. For an arbitrary tensor field

$$\begin{aligned} (\nabla_a \nabla_b - \nabla_b \nabla_a) T^{c_1 \cdots c_k}{}_{d_1 \cdots d_l} = - &\sum_{i=1}^{k} R_{abe}{}^{c_i} T^{c_1 \cdots e \cdots c_k}{}_{d_1 \cdots d_l} \\ &\sum_{j=1}^{l} R_{abd_j}{}^{e} T^{c_1 \cdots c_k}{}_{d_1 \cdots e \cdots d_l} \end{aligned} \tag{2.23}$$

The Riemann tensor has the following properties

1. $R_{abc}{}^d = -R_{bac}{}^d$
2. $R_{[abc]}{}^d = 0$
3. For the derivative operator ∇_a naturally associated with the metric, $\nabla_a g_{bc} = 0$, we have
$$R_{abcd} = -R_{abdc} \tag{2.24}$$
4. The Bianchi identity holds[3]:
$$\nabla_{[a} R_{bc]d}{}^e = 0 \tag{2.27}$$

---

[3]Note the notation

$$T_{[a_1 \cdots a_l]} = \frac{1}{p!} \sum_{\pi} \delta_\pi T_{a_{\pi(1)} \cdots a_{\pi(p)}} \tag{2.25}$$

where the sum is taken over all permutations, π, of $1, \ldots, p$ and δ_π is $+1$ for even permutations and -1 for odd permutations. Brackets $[\cdots]$ indicate antisymmeterization over the indicies whereas parentheses $(\cdots)$ indicate symmeterization. For example,

$$T_{(ab)} = \frac{1}{2}(T_{ab} + T_{ba}) \qquad T_{[ab]} = \frac{1}{2}(T_{ab} - T_{ba}) \tag{2.26}$$

$$\frac{1}{6}(\nabla_a R_{bcd}{}^e - \nabla_b R_{acd}{}^e + \nabla_b R_{cad}{}^e + \nabla_c R_{abd}{}^e - \nabla_c R_{bad}{}^e - \nabla_a R_{cbd}{}^e) = 0 \quad (2.28)$$

It is useful to decompose the Riemann tensor into a 'trace part' and a 'trace free part'. From properties (1) and (3) the trace over its first and last two indices vanishes. However, the trace over the second and fourth (or first and third) define the *Ricci tensor*, R_{ac}

$$R_{ac} = R_{abc}{}^b \quad (2.29)$$

The Ricci tensor is a symmetric tensor $R_{ac} = R_{ca}$. The *scalar curvature*, R, is the trace over the Ricci tensor

$$R = R_a{}^a \quad (2.30)$$

The trace-free part of the Riemann tensor is called the *Weyl tensor* (conformal tensor), C_{abcd}. For manifold of dimensions $n \geq 3$ the Weyl tensor is defined by

$$R_{abcd} = C_{abcd} + \frac{2}{n-2}(g_{a[c}R_{d]b} - g_{b[c}R_{d]a}) - \frac{2}{(n-1)(n-2)} R g_{a[c}g_{d]b} \quad (2.31)$$

Contracting Eq.(2.28) gives an important relation. First contracting a and e

$$\frac{1}{6}(\nabla_a R_{bcd}{}^a - \nabla_b R_{acd}{}^a + \nabla_b R_{cad}{}^a + \nabla_c R_{abd}{}^a - \nabla_c R_{bad}{}^a - \nabla_a R_{cbd}{}^a) = 0 \quad (2.32)$$

$$\frac{1}{6}(\nabla_a R_{bcd}{}^a + \nabla_b R_{cad}{}^a + \nabla_b R_{cad}{}^a - \nabla_c R_{bad}{}^a - \nabla_c R_{bad}{}^a + \nabla_a R_{bcd}{}^a) = 0 \quad (2.33)$$

$$\nabla_a R_{bcd}{}^a + \nabla_b R_{cd} - \nabla_c R_{bd} = 0 \quad (2.34)$$

Raising the d index and contracting b and d gives

$$\nabla_a R_c{}^a + \nabla_b R_c{}^b - \nabla_c R = 0 \quad (2.35)$$

Or,

$$\nabla^a G_{ab} = 0 \quad (2.36)$$

where

$$G_{ab} = R_{ab} - \frac{1}{2} R g_{ab} \quad (2.37)$$

G_{ab} is known as the *Einstein tensor*.

The $1/p!$ factor ensures that the symmeterization is of strength one i.e. if we symmetrize or antisymmetrize twice, the tensor remains the same $T_{((a_1 \cdots a_n))} = T_{(a_1 \cdots a_n)}$, $T_{[[a_1 \cdots a_n]]} = T_{[a_1 \cdots a_n]}$. The symmetric and antisymmetric tensors resulting from the tensor T are termed *cotensors* of T.

Okay, so where does this lead us? Well, we now have the tools to compare arbitrarily ranked tensor fields in different coordinate frames or on manifolds with perturbed metrics. Let's now express the Riemann curvature tensor in terms of arbitrary connection coefficients. We start from Eq.(2.7) where we relate two differential operators acting on dual vector fields

$$\nabla_b \omega_c = \tilde{\nabla}_b \omega_c - C^d{}_{bc}\omega_d \tag{2.38}$$

Using Eq.(2.8)

$$\begin{aligned}\nabla_a \nabla_b \omega_c = &\tilde{\nabla}_a (\tilde{\nabla}_b \omega_c - C^d{}_{bc}\omega_d) \\ &- C^d{}_{ab}\tilde{\nabla}_d \omega_c - C^d{}_{ac}\tilde{\nabla}_b \omega_d \\ &- C^d{}_{ab}C^b{}_{ce}\omega_d + C^d{}_{ab}C^c{}_{de}\omega_c \\ &+ C^d{}_{ac}C^e{}_{bd}\omega_e + C^d{}_{ab}C^b{}_{ce}\omega_d\end{aligned} \tag{2.39}$$

The first term on the third line cancels with the final term leaving us with

$$\begin{aligned}\nabla_a \nabla_b \omega_c = &\tilde{\nabla}_a (\tilde{\nabla}_b \omega_c - C^d{}_{bc}\omega_d) \\ &- C^d{}_{ab}(\tilde{\nabla}_d \omega_c - C^c{}_{de}\omega_c) \\ &- C^d{}_{ac}(\tilde{\nabla}_b \omega_d - C^e{}_{bd}\omega_e)\end{aligned} \tag{2.40}$$

Taking the difference with $\nabla_b \nabla_a$ and using the symmetry properties of $C^a{}_{bc}$ (we see immediately the second line will cancel because it is symmetric in ab)

$$\begin{aligned}(\nabla_a \nabla_b - \nabla_b \nabla_a)\omega_c = &(\tilde{\nabla}_a \tilde{\nabla}_b - \tilde{\nabla}_b \tilde{\nabla}_a)\omega_c - (\tilde{\nabla}_a C^d{}_{bc} - \tilde{\nabla}_b C^d{}_{ac})\omega_d \\ &- C^d{}_{ac}(\tilde{\nabla}_b \omega_d - C^e{}_{bd}\omega_e) \\ &+ C^d{}_{bc}(\tilde{\nabla}_a \omega_d - C^e{}_{ad}\omega_e)\end{aligned} \tag{2.41}$$

Of course we can rewrite $(\tilde{\nabla}_a \tilde{\nabla}_b - \tilde{\nabla}_b \tilde{\nabla}_a)\omega_c = \tilde{R}^d_{abc}\omega_d$. In the last line we swap $bc \to cb$ and rename $b \to a$

$$\begin{aligned}R_{abc}{}^d \omega_d = &\tilde{R}^d_{abc}\omega_d - (\tilde{\nabla}_a C^d{}_{bc} - \tilde{\nabla}_b C^d{}_{ac})\omega_d \\ &+ (C^d{}_{ca}\tilde{\nabla}_b - C^d{}_{ac}\tilde{\nabla}_b)\omega_d \\ &+ (C^e{}_{ac}C^d{}_{be} - C^e{}_{bc}C^d{}_{ae})\omega_d\end{aligned} \tag{2.42}$$

Because this holds for any ω_d we can drop this factor. The second line cancels ($C^d{}_{ca} = C^d{}_{ac}$) and we are left with the following relation

$$R_{abc}{}^d = \tilde{R}^d_{abc} + 2\left[-\tilde{\nabla}_{[a} C^d{}_{b]c} + C^e{}_{c[a} C^d{}_{b]e}\right] \tag{2.43}$$

2.1 CONFORMAL TRANSFORMATIONS

Now we look to put it all together by comparing the quantities we have computed above under conformal transformations. Let M be an n-dimensional manifold with metric g_{ab} of any signature. If W is a smooth, strictly positive function, then the metric $\tilde{g}_{ab} = W^2 g_{ab}$ is said to arise from g_{ab} via a *conformal transformation*. Both metrics obey the same causal structures. Their inverses are given by g^{ab} and $\tilde{g}^{ab}$ and $\tilde{g}^{ab} = W^{-2} g^{ab}$ such that $\tilde{g}^{ab}\tilde{g}_{bc} = W^2 W^{-2} g^{ab} g_{bc} = \delta^a{}_c$. Each metric is accompanied by a derivative operator ∇ with g_{ab} and $\tilde{\nabla}$ with $\tilde{g}_{ab}$ such that $\nabla_a g_{bc} = \tilde{\nabla}_a \tilde{g}_{bc} = 0$. The two operators are related via Eq.(2.8). Interchanging ∇ and $\tilde{\nabla}$ i.e. ($\tilde{\nabla}_a \omega_b = \nabla_a \omega_b - C^c{}_{ab}\omega_c$) we see that the connection coefficients $C^c{}_{ab}$ are now given by Eq.(2.21)

$$C^c{}_{ab} = \frac{1}{2}\tilde{g}^{cd}\left(\nabla_a \tilde{g}_{bd} + \nabla_b \tilde{g}_{ad} - \nabla_d \tilde{g}_{ba}\right) \tag{2.44}$$

$$= \frac{1}{2} W^{-2} g^{cd}\left(\nabla_a (W^2 g_{bd}) + \nabla_b (W^2 g_{ad}) - \nabla_d (W^2 g_{ba})\right) \tag{2.45}$$

Noting $\nabla_a g_{bc} = 0$

$$= W^{-1} g^{cd}\left(g_{bd}\nabla_a W + g_{ad}\nabla_b W - g_{ba}\nabla_d W\right) \tag{2.46}$$

$$= g^{cd}\left(g_{bd}\nabla_a \ln W + g_{ad}\nabla_b \ln W - g_{ba}\nabla_d \ln W\right) \tag{2.47}$$

$$= \delta^c{}_b \nabla_a \ln W + \delta^c{}_a \nabla_b \ln W - g^{cd} g_{ba} \nabla_d \ln W \tag{2.48}$$

$$= 2\delta^c{}_{(b}\nabla_{a)} \ln W - g^{cd} g_{ba} \nabla_d \ln W \tag{2.49}$$

Of particular interest is the relation of curvature $\tilde{R}_{abc}{}^d$ associated with $\tilde{\nabla}$ and $R_{abc}{}^d$ associated with ∇. We have already done this computation! The result is given in Eq.(2.43)

$$\tilde{R}_{abc}{}^d = R_{abc}{}^d + 2\left[-\nabla_{[a} C^d{}_{b]c} + C^e{}_{c[a} C^d{}_{b]e}\right] \tag{2.50}$$

$$= R_{abc}{}^d - \nabla_a C^d{}_{bc} + \nabla_b C^d{}_{ac} + C^e{}_{ac} C^d{}_{be} - C^e{}_{bc} C^d{}_{ae} \tag{2.51}$$

$$= R_{abc}{}^d + (C^e{}_{ac} C^d{}_{be} - \nabla_a C^d{}_{bc}) - (C^e{}_{bc} C^d{}_{ae} - \nabla_b C^d{}_{ac}) \tag{2.52}$$

Plugging in our derived connection coefficients from Eq.(2.49) explicitly term by term

$$\nabla_a C^d{}_{bc} = \nabla_a\left[(\delta^d{}_c \nabla_b + \delta^d{}_b \nabla_c - g^{de} g_{cb} \nabla_e) \ln W\right] \tag{2.53}$$

$$= \delta^d{}_c \nabla_a (\nabla_b \ln W) + \delta^d{}_b \nabla_a (\nabla_c \ln W) - g^{de} g_{cb} \nabla_a (\nabla_e \ln W) \tag{2.54}$$

$$C^e{}_{ac} C^d{}_{be} = (\delta^e{}_c \nabla_a \ln W + \delta^e{}_a \nabla_c \ln W - g^{ef} g_{ca} \nabla_f \ln W)(\delta^d{}_e \nabla_b \ln W + \delta^d{}_b \nabla_e \ln W - g^{dh} g_{eb} \nabla_h \ln W) \tag{2.55}$$

$$
\begin{aligned}
&= \delta^d{}_c(\nabla_a \ln W)\nabla_b \ln W + \delta^d{}_b(\nabla_a \ln W)\nabla_c \ln W - (\nabla_a \ln W) g_{bc} g^{df} \nabla_f \ln W \\
&+ \delta^d{}_a(\nabla_c \ln W)\nabla_b \ln W + \delta^d{}_b(\nabla_c \ln W)\nabla_a \ln W - (\nabla_c \ln W) g_{ab} g^{dh} \nabla_h \ln W \\
&- g_{ca} g^{df}(\nabla_f \ln W)\nabla_b \ln W - g_{ca}\delta^d{}_b g^{ef}(\nabla_f \ln W)\nabla_e \ln W + (\nabla_b \ln W) g_{ca} g^{df}\nabla_f \ln W
\end{aligned}
\tag{2.56}
$$

$$
\nabla_b C^d{}_{ac} = \text{Eq.}(2.53) \text{ with } a \Leftrightarrow b \tag{2.57}
$$

$$
C^e{}_{bc} C^d{}_{ae} = \text{Eq.}(2.56) \text{ with } a \Leftrightarrow b \tag{2.58}
$$

Putting everything together and noting from Eq.(2.52) that the full expression is asymmetric in a and b

$$
\begin{aligned}
\tilde{R}_{abc}{}^d = R_{abc}{}^d + 2\Big[&\delta^d{}_{[a}\nabla_{b]}\nabla_c \ln W - g^{de} g_{c[a}\nabla_{b]}\nabla_e \ln W \\
&+ (\nabla_{[a} \ln W)\delta^d{}_{b]}\nabla_c \ln W - (\nabla_{[a} \ln W) g_{b]c} g^{df}\nabla_f \ln W \\
&- g_{c[a}\delta^d{}_{b]} g^{ef}(\nabla_e \ln W)\nabla_f \ln W\Big]
\end{aligned}
\tag{2.59}
$$

Tracing over b and d gives the Ricci tensor

$$
\begin{aligned}
\tilde{R}_{ac} = R_{ac} &- (n-2)\nabla_a\nabla_c \ln W - g_{ac} g^{de}\nabla_d\nabla_e \ln W \\
&+ (n-2)(\nabla_a \ln W)\nabla_c \ln W - (n-2) g_{ac} g^{de}(\nabla_d) \ln W \nabla_e \ln W
\end{aligned}
\tag{2.60}
$$

Contracting the above equation with $\tilde{g}^{ac} = W^{-2} g^{ac}$ give the scalar curvature

$$
\tilde{R} = W^{-2}\big[R - 2(n-1) g^{ac}\nabla_a\nabla_c \ln W - (n-2)(n-1) g^{ac}(\nabla_a \ln W)\nabla_c \ln W\big] \tag{2.61}
$$

Eqs.(2.59)(2.60)(2.61) describe how curvature is changed by conformal transformations.

2.2 COMPUTING CURVATURE WITH A COORDINATE FRAME

Given a metric in a coordinate basis, the computation of the curvature proceeds by the following scheme:

$$
g_{\mu\nu} \xrightarrow{\Gamma\sim\partial g} \Gamma_{\mu\alpha\beta} \rightarrow \Gamma^\mu_{\alpha\beta} \xrightarrow{R\sim\partial\Gamma+\Gamma\Gamma} R^\mu_{\nu\alpha\beta} \tag{2.62}
$$

In the coordinate basis the components of the Christoffel symbol are given by

$$\Gamma^{\rho}{}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \left[\frac{\partial g_{\nu\sigma}}{\partial x^{\mu}} + \frac{\partial g_{\mu\sigma}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\sigma}} \right] \tag{2.63}$$

$$R_{\mu\nu\rho}{}^{\sigma} = \frac{\partial}{\partial x^{\nu}} \Gamma^{\sigma}{}_{\nu\rho} - \frac{\partial}{\partial x^{\mu}} \Gamma^{\sigma}{}_{\nu\rho} + \Gamma^{\alpha}{}_{\mu\rho} \Gamma^{\sigma}{}_{\alpha\nu} - \Gamma^{\alpha}{}_{\nu\rho} \Gamma^{\sigma}{}_{\alpha\mu} \tag{2.64}$$

$$R_{\mu\rho} = R_{\mu\nu\rho}{}^{\nu} = \frac{\partial}{\partial x^{\nu}} \Gamma^{\nu}{}_{\mu\rho} - \frac{\partial}{\partial x^{\mu}} \Gamma^{\nu}{}_{\nu\rho} + \Gamma^{\alpha}{}_{\mu\rho} \Gamma^{\nu}{}_{\alpha\nu} - \Gamma^{\alpha}{}_{\nu\rho} \Gamma^{\nu}{}_{\alpha\mu} \tag{2.65}$$

$$R = R_{\mu}{}^{\mu} = \frac{\partial}{\partial x^{\nu}} \Gamma^{\nu\mu}{}_{\mu} - \frac{\partial}{\partial x^{\mu}} \Gamma^{\nu\mu}{}_{\nu} + \Gamma^{\alpha\mu}{}_{\mu} \Gamma^{\nu}{}_{\nu\alpha} - \Gamma^{\alpha\mu}{}_{\nu} \Gamma^{\nu}{}_{\alpha\mu} \tag{2.66}$$

We note that,

$$\Gamma^{\nu}{}_{\nu\mu} = \frac{1}{2} g^{\mu\alpha} \frac{\partial g_{\nu\alpha}}{\partial x^{\mu}} \tag{2.67}$$

Using the formula for the inverse of a matrix,

$$g^{\mu\alpha} \frac{\partial g_{\nu\alpha}}{\partial x^{\mu}} = \frac{1}{g} \frac{\partial g}{\partial x^{\mu}} \tag{2.68}$$

And thus,

$$\Gamma^{\nu}{}_{\nu\mu} = \frac{1}{2} \frac{1}{g} \frac{\partial g}{\partial x^{\mu}} \tag{2.69}$$

To compute the curvature via this method use the formula for ds^2 as a table of g_{kl} values. Then compute the six possible different $\Gamma_{jkl} = \Gamma_{jlk}$ (40 in four dimensions) from the formulae above (See MTW Box 14.2).

2.3 FORMING THE EINSTEIN TENSOR

There are a few equivalent ways to calculate the Einstein tensor:

1. Successive contractions of the Reimann tensor

$$R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu}, \qquad R = g^{\mu\nu} R_{\mu\nu} \tag{2.70}$$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \tag{2.71}$$

2. Forming the dual of the Riemann tensor and then contracting

$$G_{\alpha\beta}{}^{\gamma\delta} \equiv (\star R \star)_{\alpha\beta}{}^{\gamma\delta} = \epsilon_{\alpha\beta\mu\nu} R^{|\mu\nu|}{}_{|\rho\sigma|} \epsilon^{\rho\sigma\gamma\delta} \tag{2.72}$$

$$= -\delta^{\rho\sigma\gamma\delta}{}_{\alpha\beta\mu\nu} R^{|\mu\nu|}{}_{|\rho\sigma|} \tag{2.73}$$

$$G_{\beta}{}^{\delta} = G_{\alpha\beta}{}^{\alpha\delta} \tag{2.74}$$

3. Or by using a combination of Eqs.(2.72)(2.74)

$$G_\beta{}^\delta = G^\delta{}_\beta = -\delta^{\delta\rho\sigma}{}_{\beta\mu\nu} R^{|\mu\nu|}{}_{|\rho\sigma|} \tag{2.75}$$

From Eq.(2.75) the Einstein tensor reads

$$\begin{aligned} G^0{}_0 &= -(R^{12}{}_{12} + R^{23}{}_{23} + R^{31}{}_{31}) \\ G^1{}_1 &= -(R^{02}{}_{02} + R^{03}{}_{03} + R^{23}{}_{23}) \\ G^0{}_1 &= R^{02}{}_{12} + R^{03}{}_{13} \\ G^1{}_2 &= R^{10}{}_{20} + R^{13}{}_{23} \end{aligned} \tag{2.76}$$

with every other component given by a similar formula obtained by permutations of indices.

3 VARIATIONAL APPROACH

See Valeria as well as MTW

4 COSMOLOGY

See Ch. 27 of MTW

5 SCHWARZSCHILD GEOMETRY

See MTW Ch. 23.1 To begin we start with a static, spherical system — in special relativity the spherically symmetric metric is given by

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2 \tag{5.1}$$

where

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2 \tag{5.2}$$

The next task is to add the effects of gravitating matter without spoiling the spherical symmetry of the metric. Generically (and typically) we can follow a ‘follow and ask questions later’ approach and just rewrite the metric such that each non-zero component is now becomes a function of the radial component (we’re still considering (for now) the static metric i.e. $\partial g_{\mu\nu}/\partial t = 0$)

$$ds^2 = -e^{2\Phi}dt^2 + e^{2\Lambda}dr^2 + r^2 d\Omega^2 \tag{5.3}$$

where Φ and Λ are functions of the radial coordinate r. These coordinates are called "curvature coordinates" or sometimes "Schwarzschild coordinates". The 'central idea' of these coordinates is that (Schwarzschild r-coordinate) = (proper circumference)/2π.

See MTW Ch. 23.5 Now we are tasked with solving for the unknown functions $\lambda(r)$ and $\Phi(r)$. We need more information, in the form of constraints/boundary conditions in order to pin down the functional forms — consequently we look to the Einstein field equations. To start we need to derive the field equations in an orthonormal frame (Lagrangian "proper" reference frame See MTW Ch. 13.6 or Sect. 1.7) i.e. introduce a vielbein (tetrads)

$$\boldsymbol{\omega}^{\hat{t}} \equiv e^{\Phi} dt, \quad \boldsymbol{\omega}^{\hat{r}} = e^{\Lambda} dr, \quad \boldsymbol{\omega}^{\hat{\theta}} = r d\theta, \quad \boldsymbol{\omega}^{\hat{\phi}} = r \sin\theta d\phi \tag{5.4}$$

Next we must express the Einstein field equation in terms of this velibein basis which amounts to rewriting the Einstein tensor $G^{\mu\nu}$ and the stress-energy tensor $T^{\mu\nu}$ (MTW box 14.5, in fact Ch. 14 itself is an absolute gold mine of useful info).

$$\vdots$$

We find

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - 2M/r} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{5.5}$$

5.1 ORBITS

See MTW 25.5 and Eq.(25.38)

5.2 CONICAL/COORDINATE SINGULARITIES

Generally, spacetime metrics can contain singularities i.e. components which diverge or vanish at any spacetime point. These singularities come in two forms: coordinate singularities and true curvature singularities. The former is a latter singularity associated with the geometry of the space whereas the former is a result of a poor choice of coordinate system. Coordinate singularities can be removed by an appropriate coordinate transformation. To determine what category a singularity falls into boils down to trying to find a coordinate system in which the metric is regular at the point of interest.

As a warm up example let's consider the following metric

$$ds^2 = dr^2 + r^2 d\theta^2 \tag{5.6}$$

we see that the above metric is singular at the point $r = 0$. We don't explode because upon making the transformation $x = r\cos\theta$ and $y = r\sin\theta$ the metric becomes

$$ds^2 = dx^2 + dy^2 \tag{5.7}$$

informing us that it was indeed just a coordinate singularity as the new metric is regular everywhere. However, there was an implicit assumption in the coordinate transformation above! For a well defined transformation, the coordinate θ had to be periodic in 2π. If θ we not periodic the space would contain a conical singularity[4].

Now let's consider the Euclidean Schwarzschild black hole metric:

$$ds^2 = \left(1 - \frac{2M}{r}\right) d\tau^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 d\Omega_2^2 \tag{5.8}$$

The metric appears to have a singularity at $r = 2M$ (τ-component $\to 0$, r-component $\to \infty$) known as the "gravitational radius", "Schwarzschild radius", "Schwarzschild horizon", etc. We can determine the type of the singularity via two methods: the first procedure implements a well chosen coordinate transformation while the second method analyzes orbital properties.

As an example of the first method we perform the following coordinate transformation

$$\rho^2 = 8M(r - 2M) \to r = \frac{\rho^2}{8M} + 2M \tag{5.9}$$

$$ds^2 = \left(\frac{\rho^2}{16M^2 + \rho^2}\right) d\tau^2 + \frac{\rho^2}{16M^2}\left(1 + \frac{16M^2}{\rho^2}\right) d\rho^2 + \left(\frac{\rho^2}{8M} + 2M\right)^2 d\Omega_2^2 \tag{5.10}$$

near $\rho = 0$ we have

$$ds^2 \approx \frac{\rho^2}{(2M)^2} d\tau^2 + d\rho^2 + 4M^2 d\Omega_2^2 \tag{5.11}$$

we see that in order for $\rho = 0$ to remain a coordinate singularity we require the coordinate τ have periodicity $2\pi/\beta$ where

$$\beta^2 = \frac{1}{(2M)^2} \tag{5.12}$$

[4]Consider a sheet of paper in which the angular coordinate is defined as a counter-clockwise motion around the origin. This sheet of paper is regular everywhere and the angle θ is identified as periodic in 2π i.e. $\theta = 0 = 2\pi$. Now if we were to cut a slice out of this sheet of paper and recombine the edges we would left with a cone (i.e. a space where $\sin^2\theta + \cos^2\theta \neq 1$). This apex of this cone is singular ($\theta = 0$ or $x = y = 0$)

in other words we require

$$\tau = \tau + 8\pi M \tag{5.13}$$

Typically, this would be done with some form of ansatz which satisfies two properties:

1. The new coordinate is zero at the horizon (i.e. the singularity point)
2. The new coordinate, we'll call it ρ, substitution leaves the $g_{\rho\rho}$ component of the metric to be equal to one.

So typically the ansatz would look something like $\rho^2 = C(r-2M)$ in the above case. This would be substituted into Eq.(5.8) and expanded near $\rho = 0$ (only leading order pieces kept). Finally, using the constraint on the coefficient of the $g_{\rho\rho}$ metric component we can solve for C.

For a more general analysis, consider a general Euclidean metric given by

$$ds^2 = g(r)\left[f(r)d\tau^2 + d\vec{x}^2\right] + \frac{1}{h(r)}dr^2 \tag{5.14}$$

We assume that the functions $f(r)$ and $h(r)$ have first order zeros at $r = r_0$, which is the location of the horizon, and $g(r_0) \neq 0$. For $r \approx r_0$

$$f(r) \approx f'(r_0)(r-r_0), \quad h(r) \approx h'(r_0)(r-r_0) \tag{5.15}$$

The near-horizon metric can then be written as

$$ds^2 \approx g(r_0)\left[f'(r_0)(r-r_0)d\tau^2 + d\vec{x}^2\right] + \frac{1}{h'(r_0)(r-r_0)}dr^2 \tag{5.16}$$

we follow the same prescription as above and define a new coordinate ρ as

$$\rho^2 = Ch(r) \tag{5.17}$$

Near the horizon we have

$$\rho^2 = Ch'(r_0)(r-r_0) \tag{5.18}$$

and thus

$$r = \frac{\rho^2}{Ch'(r_0)} + r_0, \quad dr = \frac{2\rho}{Ch'(r_0)}d\rho \tag{5.19}$$

the near horizon metric can now be written as

$$ds^2 \approx g(r_0)\left[\frac{f'(r_0)\rho^2}{Ch'(r_0)}d\tau^2 + d\vec{x}^2\right] + \frac{4}{C(h'(r_0))^2}d\rho^2 \tag{5.20}$$

Now we can solve for the constant C by demanding $g_{\rho\rho} = 1$

$$C = \frac{4}{(h'(r_0))^2} \tag{5.21}$$

We now have

$$ds^2 \approx g(r_0)\left[\frac{\rho^2 f'(r_0)h'(r_0)}{4}d\tau^2 + d\vec{x}^2\right]d\rho^2 \tag{5.22}$$

defining

$$\beta^2 = \frac{\rho^2 f'(r_0)h'(r_0)}{4} \tag{5.23}$$

and following the same arguments as above, in order for the $\rho = 0$ singularity to not be a curvature singularity we demand that the Euclidean time coordinate be periodic in $2\pi/\beta$ i.e. we *compactify* τ and identify

$$\tau = \tau + \frac{4\pi}{\sqrt{g(r_0)f'(r_0)h'(r_0)}} \tag{5.24}$$

A more physically motivated method for determining the form of the singularity is to consider the measured spacetime by a traveler nearing the singular radial distance. Assume the explorer is falling freely radially in the Schwarzschild horizon. His trajectory through spacetime is given by (See orbits section, TBD...)

$$\frac{\tau}{2M} = \tag{5.25}$$

6 GRAVITATIONAL COLLAPSE AND BLACK HOLES

7 BLACK HOLE THERMODYNAMICS

Consider an ensemble of a large number N, of microstates μ_α, corresponding to a given macrostate where each microstate occurs with probability p_α. A system at a fixed temperature $T = 1/\beta$ ($k_B = 1$) can be achieved by placing the system in contact with a reservoir. The ensemble of states are termed a *canonical* ensemble. The canonical density matrix is given by

$$\rho(\beta) = \frac{\exp(-\beta\mathcal{H})}{Z(\beta)} \tag{7.1}$$

where $\mathcal{H}$ is the Hamiltonian operator. With the condition $\mathrm{Tr}(\rho) = 1$ the partition function is given by

$$Z = \mathrm{Tr}\left[(\exp(-\beta\mathcal{H})\right] = \sum_n \exp(-\beta E_n) \tag{7.2}$$

The ensemble average of an operator is then given by

$$\langle\mathcal{O}\rangle = \mathrm{Tr}(\rho\mathcal{O}) \tag{7.3}$$

written as a path integral over quantum states Ψ the expectation value is written as

$$\langle\mathcal{O}\rangle \sim \int \mathcal{D}\Psi\, \langle\Psi(t)|\mathcal{O}e^{-\beta\mathcal{H}}|\Psi(t)\rangle \tag{7.4}$$

Being that the Hamiltonian is also the time evolution operator we can write

$$= \int \mathcal{D}\Psi\, \langle\psi(t)|\mathcal{O}|\psi(t+i\beta)\rangle \tag{7.5}$$

and upon rotating to Euclidean (imaginary) time $t \to it_E$ we must make the identification

$$t_E = t_E + \beta \tag{7.6}$$

in order to have a sensible averaging. That is, we must *compactify* (identify as periodic) Euclidean time. Thus we conclude that thermal averages (thermal correlation functions) in a quantum system at finite temperature must be analyzed with compactified imaginary time.

In Sect.(5.2) we developed a prescription to determine the properties of singular metric. In particular we were interested in determining if the singularity was due to a poor choice of coordinates i.e. a coordinate singularity or if the singularity was an innate property of the metric i.e. a curvature singularity. We described a method for determining a coordinate transformation which would leave the metric regular at all values of the coordinates. However, in performing the transformation we found that the metric would remain regular *only if* the Euclidean time component was identified as periodic over an interval specific to the metric geometry. We saw above that the periodic nature of Euclidean time can be associated with the temperature of a canonical ensemble i.e. system in contact with a thermal reservoir. Thus, the identification of the compactification radius of Euclidean time for a given metric can be related to the temperature of a black hole at its horizon. Given the results in Eqs.(5.24) and (7.6) we can make the identification of the black hole temperature (Hawking temperature) as:

$$\frac{1}{T} = \frac{4\pi}{\sqrt{g(r_0)f'(r_0)h'(r_0)}} \tag{7.7}$$

7.1 SCHWARZSCHILD METRIC

We have already computed the relevant condition for the Schwarzchild metric in Eq.(5.13). We now can interpret this result in the context of the temperature of a Schwarzschild black hole:

$$T = \frac{1}{8\pi M} \tag{7.8}$$

Given the temperature we can also compute the entropy of the Schwarzschild black hole after identifying the mass M as the internal energy

$$dM = TdS \tag{7.9}$$

$$dM = \frac{dS}{8\pi M} \tag{7.10}$$

separating variables and integrating we find

$$S = 4\pi M^2 \tag{7.11}$$

7.2 ADS BLACK HOLE

The metric for

References

[1] Misner, Charles W., Kip S. Thorne, and John Archibald Wheeler. Gravitation. Macmillan, 1973.

[2] Wald, Robert M. General relativity. University of Chicago press, 2010.