SOLITONS AND INSTANTONS

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1 Solitary Wave Solutions [1]

Field theories, classical and quantized, admit non-singular, non-dissipative, finite energy solutions termed, most generally, as solitary waves. These waves are special solutions to non-linear, partial differential wave equations. For example we consider the following familiar linear and non-dissipative relativistic wave equation

$$\Box\phi(x,t) = \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)\phi(x,t) = 0 \tag{1.1}$$

Where $\Box \equiv \partial^\mu \partial_\mu$ and $\phi(x,t)$ is a real scalar field in $(1+1)$ dimensions. This wave equation admits solutions with two relevant properties

1. Any well-behaved function of the form $f(x \pm ct)$ is a solution to Eq.(1.1). Given a localised function f we can construct a wave packet solution which propagates with phase velocity[1] $\pm c$ with no distortion through time. This is related to the fact that $\cos(kx \pm \omega t)$ and $\sin(kx \pm \omega t)$ where[2] $\omega = kc$ form a complete set of solutions to the Eq.(1.1) as in any well-behaved localised function satisfying the wave equation, e.g. $f(x-ct)$, can be rewritten as a superposition of sine and cosine

$$f(x-ct) = \int dk\, [a_1(k)\cos(kx-\omega t) + a_2(k)\sin(kx - \omega t)] \tag{1.3}$$

2. Given two localised wave packet solutions, $f_1(kx \pm ct)$ and $f_2(kx \pm ct)$, their sum $f_3(x,t) = f_1(kx \pm ct) + f_2(kx \pm ct)$ is also a solution. At $t \to -\infty$ f_3 consists of two largely separated wave packets travelling towards one another, during time $-t$ to t the wave packets collide and as $t \to \infty$ the wave packets continue travelling undistorted away from one another with their original velocities. In one dimension we can consider two wave packets travelling towards one another.

[1] The phase velocity of a wave is determined by the speed of a point on the wave with constant phase ($kx - \omega t = \text{const.}$)

$$\frac{d(kx-\omega t)}{dt} = 0 \to k\dot{x} - \omega = 0 \to v = \frac{\omega}{k} \tag{1.2}$$

[2] Consider $\phi = \sin(kx - \omega t)$, plugging this into Eq.(1.1) we obtain the relation

$$-\frac{\omega^2}{c^2} + k^2 = 0 \rightsquigarrow \omega^2 = k^2 c^2$$

These packets will interfere and continue straight past one another when, of course, we consider an ideal string.

Summarized, the wave equation yields solutions which (1) retain shape and velocity of wave packets through time and (2) retain shape and velocity of several packets after collisions. These properties are of course easily soiled with the addition of non-linear or dispersive terms within the wave equation. For example if we consider the Klein-Gordon equation in two dimensions

$$(\Box + m^2c^2)\phi(x,t) \equiv \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + m^2c^2\right)\phi(x,t) = 0 \tag{1.4}$$

The equation is still linear and its solutions can still be written as a superposition of plane waves, however, the dispersion relation is now given by $\omega^2 = k^2c^2 + m^2c^4$ indicating that plane waves at different wavelengths will now travel at different speeds and as a result disperse ($v = \omega(k)/k$). Including non-linear terms into the wave equation also result in dispersive plane wave solutions.

Wave equations which contain both dispersive and non-linear terms may enjoy a mutual cancellation and yield special solutions which retain feature (1). Such solutions, called ***solitary waves***, exist and have been found in one, two, and three dimensions. Also existing but in a much smaller quantity, are solitary wave solutions which also exhibit feature (2), these solutions are called ***solitons***.

The conditions stated above aren't exactly useful in the business of *defining* solitary wave solutions. Thus, we must be a bit more precise in our definition. From here on, we restrict ourselves to those field equations which have an associated energy density $\mathcal{E}(x,t)$ that is some function of the fields $\phi_a(x,t)$. We also require that the spatial integral of this energy density is a conserved total energy functional $E[\phi_a]$. Solutions to this class of field equations are termed "localised" if their energy density $\mathcal{E}(x,t)$ at any finite time t is finite in some region of space and falls to zero at spatial infinity fast enough as to be integrable. Given this definition of localised with respect to the energy density we define a solitary wave as that localised, non-singular solution of any non-linear field equation whose energy density, in addition to being localised, has a space-time dependence of the form

$$\mathcal{E}(\mathbf{x},t) = \mathcal{E}(\mathbf{x} - \mathbf{u}t) \tag{1.5}$$

Where $\mathbf{u}$ is some velocity vector. The above condition indicates that the energy density should move undistorted with constant velocity. This definition also indicates that any static (time-independent) localised solution is automatically a solitary wave with $\mathbf{u} = 0$. Thus, for systems with particular spacetime symmetries we can first obtain static solutions and perform an arbitrary boost to obtain the general non-zero velocity solution.

1.1 Solitary waves in two-dimensions

As an example we consider a system of scalar fields in two-dimensions (1+1). Described by the Lagrangaign density

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \mathcal{U}(\phi), \qquad \mu = 0, 1 \tag{1.6}$$

Where $\mathcal{U}$ is the 'potential' and is any semi-definite function of ϕ reaching a minimum value of zero for some configuration or configurations of ϕ. The Euler-Lagrange equation yields

$$\Box\phi(x,t) = -\frac{\partial\mathcal{U}}{\partial\phi}(x,t) \tag{1.7}$$

For static solutions the equations of motion reduce to

$$\frac{\partial^2\phi}{\partial x^2} = \frac{\partial\mathcal{U}}{\partial\phi} \tag{1.8}$$

The dynamics are of course determined completely by the form of the potential $\mathcal{U}$. The absolute minima of $\mathcal{U}$ occur at N points ($N \geq 1$).

$$\mathcal{U}(\phi) = 0 \quad \text{for} \quad \phi = g^{(i)} \quad i = 1, \cdots, N \tag{1.9}$$

We are interested in finding static vacuum field configurations i.e. field configurations which minimize the total energy fucntional. Considering an arbitrary infinitesimal translation in Minkowski space

$$x^\mu \to x^\mu + \epsilon^\mu, \qquad \phi(x) \to \phi(x) + \epsilon^\mu\partial_\mu\phi(x) \tag{1.10}$$

Likewise, the Lagrangian density is transformed as

$$\mathcal{L} \to \mathcal{L} + \epsilon^\mu\partial_\mu\mathcal{L} \tag{1.11}$$

Noether's theorem tells us that there is a conserved current given by the energy-momentum tensor

$$T^\mu_\nu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\nu\phi - \delta^\mu_\nu\mathcal{L} \tag{1.12}$$

The conserved charge associated with time translations is the total energy functional

$$E[\phi] = \int dx T_0^0 = \int dx \left[\frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} \partial_0 \phi - \mathcal{L} \right] \tag{1.13}$$

Given the Lagrangian density in Eq.(1.6) we obtain

$$E = \int_{-\infty}^{\infty} dx \left[\frac{1}{2}(\dot{\phi})^2 + \frac{1}{2}(\phi')^2 + \mathcal{U}(\phi) \right] \tag{1.14}$$

Considering only static field configurations ($\dot{\phi} = 0$) the total energy becomes

$$E[\phi] = \int_{-\infty}^{\infty} dx \left[\frac{1}{2}(\phi')^2 + \mathcal{U}(\phi) \right] \tag{1.15}$$

For a solitary wave solution the energy density must be localised and the energy finite. To remain finite the field configuration must go to one of the zeroes of $\mathcal{U}$ at spatial infinity[3]. This requires that as $x \to \pm\infty$ the field $\phi \to g^{(i)}$. If $N = 1$ the field must tend to this value at both $\pm\infty$ however, if $N > 1$ the field can tend to any of the adjacent values $g^{(i)}$ at $\pm\infty$. To find field configurations which satisfy these boundary conditions we can rewrite our potential in the following form

$$\mathcal{U}[\phi] = \frac{1}{2} \left(\frac{dW(\phi)}{d\phi} \right)^2 \tag{1.16}$$

This can always be done, as long as $\mathcal{U}$ is non-negative. The energy becomes

$$E = \frac{1}{2} \int_{-\infty}^{\infty} dx \left[(\phi')^2 + \left(\frac{dW}{d\phi} \right)^2 \right] \tag{1.17}$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} dx \left[(\phi')^2 + \left(\frac{dW}{d\phi} \right)^2 + \phi' \frac{dW}{d\phi} - \phi' \frac{dW}{d\phi} \right] \tag{1.18}$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} dx \left[\phi' \mp \frac{dW}{d\phi} \right]^2 \pm \int_{-\infty}^{\infty} dx \frac{d\phi}{dx} \frac{dW}{d\phi} \tag{1.19}$$

[3] Otherwise the energy functional is divergent. This can be seen by rewriting the scalar field ϕ in terms of creation and annihilation operators and expressing the energy as $E = \int \frac{d^3p}{(2\pi)^3} \omega_{\mathbf{p}} \left[a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + \frac{1}{2}(2\pi)^3 \delta^{(3)}(0) \right]$. The delta function is evaluated at it's infinite spike and the integral over $\omega_{\mathbf{p}}$ diverges at large p. See Sect. 2.2 of [2]

$$= \frac{1}{2}\int_{-\infty}^{\infty} dx \left[\phi' \mp \frac{dW}{d\phi}\right]^2 \pm \int_{-\infty}^{\infty} dW \tag{1.20}$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} dx \left[\phi' \mp \frac{dW}{d\phi}\right]^2 \pm \left[W(\phi(\infty)) - W(\phi(-\infty))\right] \tag{1.21}$$

The second term is fixed by our boundary condition, thus the energy is minimized with the choice

$$\phi' = \pm\frac{dW}{d\phi} \tag{1.22}$$

Or

$$\phi' = \pm\sqrt{2\mathcal{U}(\phi)} \tag{1.23}$$

And the energy is given by

$$E = W(\phi(\infty)) - W(\phi(-\infty)) \equiv W(\phi_+) - W(\phi_-) \tag{1.24}$$

To continue further we must make a choice for our scalar potential, choosing

$$\mathcal{U}(\phi) = -\frac{1}{2}m^2\phi^2 + \frac{\lambda}{4}\phi^4 + \frac{\lambda}{4}v^4 = \frac{\lambda}{4}(\phi^2 - v^2)^2 \tag{1.25}$$

Where m^2 and λ are both positive and

$$v = \sqrt{\frac{m^2}{\lambda}} \tag{1.26}$$

The potential has two degenerate minima at $\phi = \pm v$ and the constant term in the potential has been chosen such that $\mathcal{U}(\phi) = 0$ at these minima. The *static* equations of motion are given by

$$\phi'' = \lambda(\phi^2 - v^2)\phi \tag{1.27}$$

Given the boundary conditions $\lim_{x\to\pm\infty}\phi(x) = \pm v$ and the knowledge that our solitary wave solution will satisfy the derived Bogomolny condition in Eq.(1.23) the field equation can be solved without hesitation. We have

$$\phi' = \pm\sqrt{\frac{\lambda}{2}}(\phi^2 - v^2) \tag{1.28}$$

For a positive right hand side we have

$$\frac{1}{\phi^2 - v^2}d\phi = \sqrt{\frac{2}{\lambda}}dx \tag{1.29}$$

Integration yields our kink solution

$$\phi(x) = v \tanh\left[\frac{m}{\sqrt{2}}(x - x_0)\right] \tag{1.30}$$

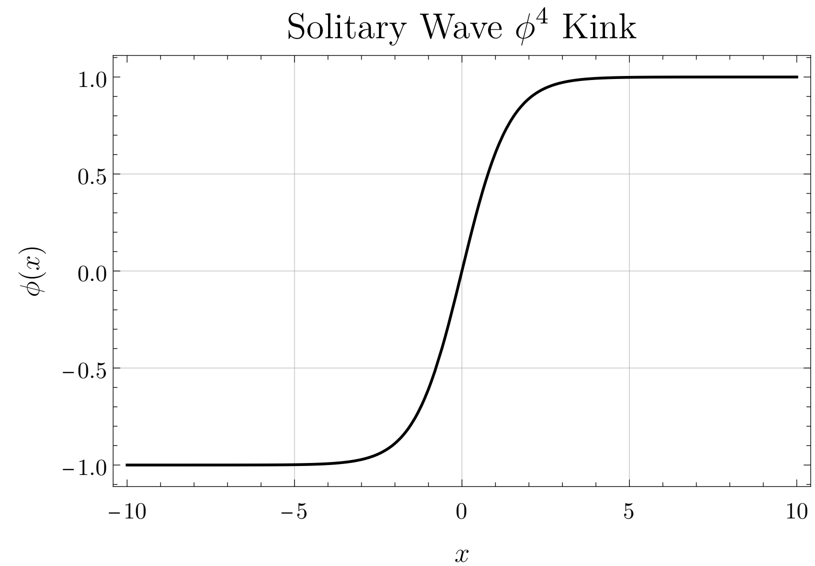

Figure 1: Solitary wave solution for ϕ^4 theory

Swapping the boundary values i.e. $\lim_{x\to\pm\infty} \phi(x) = \mp v$ we obtain the anti-kink solution

$$\phi(x) = -v \tanh\left[\frac{m}{\sqrt{2}}(x - x_0)\right] \tag{1.31}$$

The energy density of the kink is given by

$$\mathcal{E} = \frac{1}{2}(\phi')^2 + \frac{\lambda}{4}(\phi^2 - v^2)^2 = \frac{1}{4}v^2\left(m^2 + \lambda v^2\right)\operatorname{sech}^4\left(\frac{m(x - x_0)}{\sqrt{2}}\right) \tag{1.32}$$

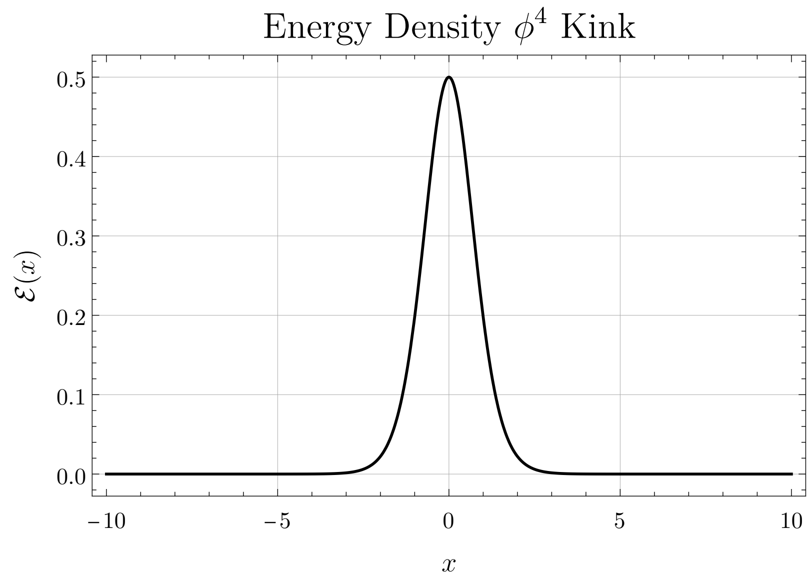

Figure 2: Solitary wave energy density for ϕ^4 kink solution.

With total energy given by

$$E = \int_{-\infty}^{\infty} \mathcal{E}(x) dx = \frac{\sqrt{2} v^2 \left(m^2 + \lambda v^2\right)}{3m} = \frac{2\sqrt{2} m^3}{3\lambda} \tag{1.33}$$

For many systems of equations it is possible to make a topological classification of solutions. For this system we can define a topological current

$$J^\mu = \frac{1}{2v} \epsilon^{\mu\nu} \partial_\nu \phi \tag{1.34}$$

Where $\epsilon^{\mu\nu}$ is the antisymmetric symbol ($\epsilon^{\mu\nu} = -\epsilon^\nu \mu$) and $\epsilon^0 1 = 1$. The corresponding topological charge is given by

$$Q = \int_{-\infty}^{\infty} dx J^0 = \frac{1}{2v} \int_{-\infty}^{\infty} dx \partial_1 \phi \tag{1.35}$$

$$= \frac{1}{2v} \left[\phi(\infty) - \phi(-\infty)\right] \tag{1.36}$$

This is equal to 1 for the kink and -1 for the antikink. The charge is conserved due to the fact that no finite energy process can change the asymptotic value of the field and is the fundamental reason for the stability of the solution. For this reason the kink solution is considered a topological solitary wave.

2 Instantons

2.1 Classical Instantons

As an example we will consider self-coupled SU(2) gauge fields A_μ^a. We define the matrix valued vector field

$$\mathbf{A}_\mu(x) \equiv \sum_a g \frac{\sigma^a}{2i} A_\mu^a(x) \tag{2.1}$$

Where the three matrices σ^a are the Pauli sigma matrices and $\sigma^a/2$ form the fundamental 2-dimensional representation of the group SU(2). The gauge field strength tensor is defined as

$$\mathbf{G}_{\mu\nu} \equiv \sum_a g \frac{\sigma^a}{2i} G_{\mu\nu}^a = \partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu + [\mathbf{A}_\mu, \mathbf{A}_\nu] \tag{2.2}$$

Under SU(2) gauge transformations the fields fields transform as

$$\mathbf{A}_\mu \to U \mathbf{A}_\mu U^{-1} + U \partial_\mu U^{-1} \tag{2.3}$$

$$\mathbf{G}_{\mu\nu} \to U \mathbf{G}_{\mu\nu} U^{-1} \tag{2.4}$$

Now we consider the following Euclidean action

$$S = -\frac{1}{2g^2} \int d^4x \mathrm{Tr}[\mathbf{G}^{\mu\nu} \mathbf{G}_{\mu\nu}] \tag{2.5}$$

The field equations are obtained straightforwardly

$$D^\mu \mathbf{G}_{\mu\nu} \equiv \partial^\mu \mathbf{G}_{\mu\nu} + [\mathbf{A}^\mu, \mathbf{G}_{\mu\nu}] \tag{2.6}$$

Instantons are finite-energy solutions to the field equations above. To find these solutions we can start by considering the zero-action field configurations. From Eq.(2.5) we see the only zero-action solution occurs when $\mathbf{G}_{\mu\nu} = 0$. This conditions can be satisfied by $\mathbf{A}_\mu = 0$ or by any gauge transformed field obtained from $\mathbf{A}_\mu$ i.e. pure gauge fields

$$\mathbf{A}_\mu = U(x) \partial_\mu (U^{-1}(x)) \tag{2.7}$$

We can verify this by plugging the pure gauge into Eq.(2.2)

$$\mathbf{G}_{\mu\nu} = \partial_\mu [U \partial_\nu U^{-1}] - \partial_\nu [U \partial_\mu U^{-1}] + [U \partial_\mu U^{-1}, U \partial_\nu U^{-1}] \tag{2.8}$$

Using the identity

$$\partial_\mu (U U^{-1}) = 0 = (\partial_\mu U) U^{-1} + U (\partial_\mu U^{-1}) \tag{2.9}$$

For stationary finite-action field configurations, we require $\mathbf{G}_{\mu\nu} \to 0$ as $r \equiv |r|^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2 \to \infty$. In other words, $\mathbf{G}_{\mu\nu}$ must vanish on the boundary of Euclidean four-space i.e. the three-dimensional spherical surface S_{E}^3 at $r = \infty$ where r is the radius in four-dimensions. This condition implies a boundary condition on $\mathbf{A}_\mu$

$$\lim_{r\to\infty} \mathbf{A}_\mu(x) = \lim_{r\to\infty} U(x)\partial_\mu U^{-1}(x) \tag{2.10}$$

Thus, for each finite-action field configuration $\mathbf{A}_\mu$ we would like to associate a group function U defined on S_{E}^3. The surface S_{E}^3 can be parametrized by three variables θ_1, θ_2, and θ_3 which could be, for example, the polar angles in four-dimensions. We are left with the task of classifying and distinguishing the infinite number of mappings between the hyperspherical surface S_{E}^3 into the group space of SU(2) functions $U(\theta_1, \theta_2, \theta_3)$. This can be accomplished with homotopy considerations. To start we can take a closer look at the topology of SU(2). The matrices U can be written uniquely in the form

$$U = a^\mu s_\mu \tag{2.11}$$

Where $s_4 \equiv \mathbb{1}_{2\times 2}$, $s_i \equiv i\sigma_i$, and a^μ are any four real numbers satisfying

$$a^\mu a_\mu = 1 \tag{2.12}$$

Thus, the group is parametrized by the four real numbers a^μ subject to the condition stated in Eq.(2.12). The group space can then be described as the three-dimensional unit-sphere in four-dimensions which we will call $S_{\mathrm{SU}_2}^3$. This indicates that the matrix function $U(\theta_1, \theta_2, \theta_3)$ is a mapping of $S_E^3 \to S_{\mathrm{SU}_2}^3$. The corresponding homotopy group is isomorphic to the group of integers

$$\pi_3(S^3) = \mathbb{Z} \tag{2.13}$$

Each mapping can be divided into a discrete infinity of homotopy classes, each characterized by an integer Q, known as the Pontryagin index. Each class is topologically distinct in that mappings from one class cannot be continuously deforms into mappings from another class. This also indicates that our finite-action field configurations can also be categorized into homotopy sectors characterized by the same Q. A field $\mathbf{A}_\mu$ sitting in a sector Q cannot be continuously deformed into another field $\mathbf{A}'_\mu$ belonging to sector Q' without violating the finiteness of action. We can express the Pontryagin index Q as an integral over the fields

$$Q \equiv \int d^4x Q(x) = \frac{1}{16\pi^2} \int d^4x \mathrm{Tr}[\tilde{\mathbf{G}}_{\mu\nu} \mathbf{G}^{\mu\nu}] \tag{2.14}$$

$$\tilde{\mathbf{G}}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\mathbf{G}^{\rho\sigma} \tag{2.15}$$

We now set out to prove that the expression above for the homotopy index Q is essentially the 'winding number' of S^3_E into $S^3_{\mathrm{SU}_2}$ i.e. the number of times the group space wraps around S^3_{E}.

We can better understand the winding number by considering mappings of circles to circles. We consider a circle S^1 (parametrized by an angle θ) mapped into another circle S^1 via a mapping $\Lambda(\theta)$. We consider two mapping belonging to the same homotopy class.

$$\Lambda(\theta) = 0 \tag{2.16}$$

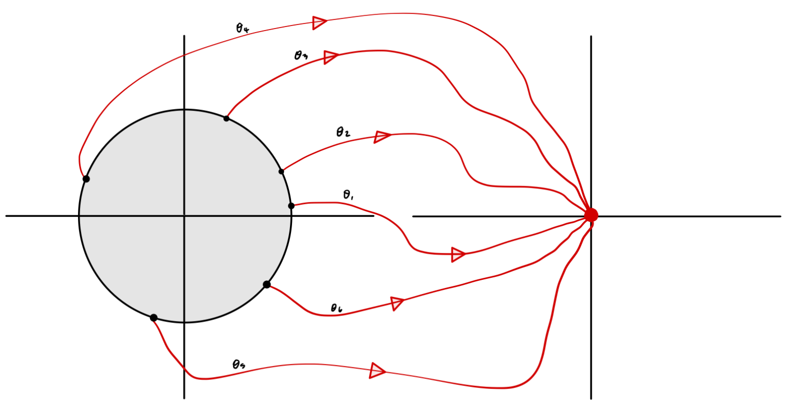

Figure 3: $\Lambda(\theta) = 0$ map.

$$\Lambda'(\theta) = \begin{cases} t\theta, & \text{for } 0 \leq \theta \leq \pi \\ t(2\pi - \theta), & \text{for } \pi \leq \theta \leq 2\pi \end{cases} \tag{2.17}$$

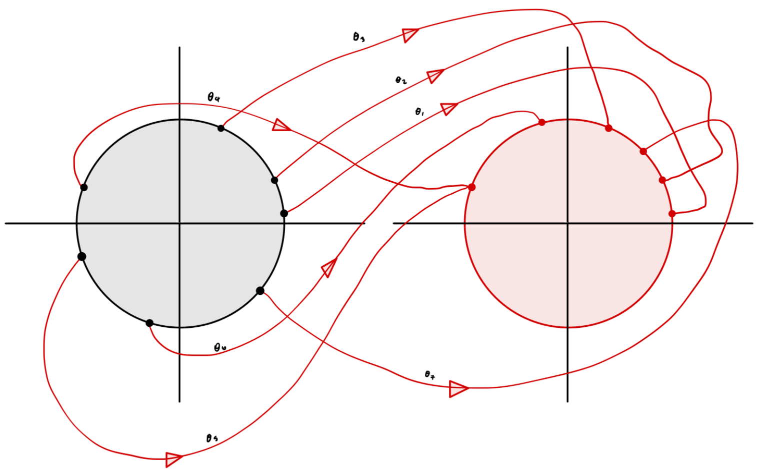

Figure 4: $\Lambda'(\theta)$ map with $t = 1$.

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Back to the calculation at hand, we want to show that the right hand side of Eq.(2.14) is equivalent to the winding number. The first step is to rewrite right-hand side as a surface integral over S^3_{E}. We begin by rewriting the covariant derivative of the dual field strength tensor

$$D_\mu \tilde{\mathbf{G}}_{\mu\nu} \equiv \partial_\mu \tilde{\mathbf{G}}_{\mu\nu} + [\mathbf{A}_\mu, \tilde{\mathbf{G}}_{\mu\nu}] \tag{2.18}$$

$$= \epsilon_{\mu\nu\rho\sigma} \left\{ \partial_\mu \left(\partial_\rho \mathbf{A}_\sigma - \partial_\sigma \mathbf{A}_\rho + [\mathbf{A}_\rho, \mathbf{A}_\sigma] \right) + [\mathbf{A}_\mu, \partial_\rho \mathbf{A}_\sigma - \partial_\sigma \mathbf{A}_\rho + [\mathbf{A}_\rho, \mathbf{A}_\sigma]] \right\} \tag{2.19}$$

Before we proceed further we note,

$$\tilde{\mathbf{G}}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \left\{ \partial_\rho \mathbf{A}_\sigma - \partial_\sigma \mathbf{A}_\rho + [\mathbf{A}_\rho, \mathbf{A}_\sigma] \right\} \tag{2.20}$$

$$= \frac{1}{2} \left\{ \epsilon_{\mu\nu\rho\sigma} \partial_\rho \mathbf{A}_\sigma - \epsilon_{\mu\nu\rho\sigma} \partial_\rho \mathbf{A}_\sigma + \epsilon_{\mu\nu\rho\sigma} \mathbf{A}_\rho \mathbf{A}_\sigma - \epsilon_{\mu\nu\sigma\rho} \mathbf{A}_\rho \mathbf{A}_\sigma \right\} \tag{2.21}$$

Where I have renamed $\rho \leftrightarrow \sigma$ in the second and fourth terms.

$$= \epsilon_{\mu\nu\rho\sigma} \left\{ \partial_\rho \mathbf{A}_\sigma + \mathbf{A}_\rho \mathbf{A}_\sigma \right\} \tag{2.22}$$

Plugging back into Eq.(2.19)

$$= \epsilon_{\mu\nu\rho\sigma} \left\{ \partial_\mu \left(\partial_\rho \mathbf{A}_\sigma + \mathbf{A}_\rho \mathbf{A}_\sigma \right) + [\mathbf{A}_\mu, \partial_\rho \mathbf{A}_\sigma + \mathbf{A}_\rho \mathbf{A}_\sigma] \right\} \tag{2.23}$$

$$= \epsilon_{\mu\nu\rho\sigma}\{\partial_\mu(\partial_\rho \mathbf{A}_\sigma) + \partial_\mu(\mathbf{A}_\rho \mathbf{A}_\sigma) + [\mathbf{A}_\mu, \partial_\rho \mathbf{A}_\sigma] + [\mathbf{A}_\mu, \mathbf{A}_\rho \mathbf{A}_\sigma]\} \tag{2.24}$$

$$\begin{aligned} = \epsilon_{\mu\nu\rho\sigma}\{&\partial_\mu(\partial_\rho \mathbf{A}_\sigma) + (\partial_\mu \mathbf{A}_\rho)\mathbf{A}_\sigma + \mathbf{A}_\rho(\partial_\mu \mathbf{A}_\sigma) + \mathbf{A}_\mu(\partial_\rho \mathbf{A}_\sigma) - (\partial_\rho \mathbf{A}_\sigma)\mathbf{A}_\mu \\ &+ \mathbf{A}_\mu \mathbf{A}_\rho \mathbf{A}_\sigma - \mathbf{A}_\rho \mathbf{A}_\sigma \mathbf{A}_\mu\} \end{aligned} \tag{2.25}$$

$$\begin{aligned} = \epsilon_{\mu\nu\rho\sigma}\{&\partial_\mu\partial_\rho \mathbf{A}_\sigma + \left[1 - (-1)(-1)^2(-1)^3\right](\partial_\mu \mathbf{A}_\rho)\mathbf{A}_\sigma \\ &+ [1 + (-1)]\,\mathbf{A}_\rho(\partial_\mu \mathbf{A}_\sigma) + [1 - (-1)^2(-1)^2]\mathbf{A}_\mu \mathbf{A}_\rho \mathbf{A}_\sigma\} \end{aligned} \tag{2.26}$$

$$= \epsilon_{\mu\nu\rho\sigma}\partial_\mu\partial_\rho \mathbf{A}_\sigma = -\epsilon_{\mu\nu\rho\sigma}\partial_\mu\partial_\rho \mathbf{A}_\sigma = 0 \tag{2.27}$$

Where in the last line I have used the antisymmetry of $\epsilon_{\mu\nu\rho\sigma}$ and the commutativity of $\partial_\mu\partial_\rho$. Note that this result does not require the $\tilde{\mathbf{G}}_{\mu\nu}$ satisfy the equation of motion. Next we see that the topological charge density can be written as a total divergence

$$-16\pi^2 Q(x) = \mathrm{Tr}[\tilde{\mathbf{G}}_{\mu\nu}\mathbf{G}_{\mu\nu}] \tag{2.28}$$

$$= \mathrm{Tr}\left[(\partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu)\tilde{\mathbf{G}}_{\mu\nu} + [\mathbf{A}_\mu, \mathbf{A}_\nu]\tilde{\mathbf{G}}_{\mu\nu}\right] \tag{2.29}$$

$$= \mathrm{Tr}\left[(\partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu)\tilde{\mathbf{G}}_{\mu\nu} + (\mathbf{A}_\mu \mathbf{A}_\nu - \mathbf{A}_\nu \mathbf{A}_\mu)\tilde{\mathbf{G}}_{\mu\nu}\right] \tag{2.30}$$

$$= \mathrm{Tr}\left[(\partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu)\tilde{\mathbf{G}}_{\mu\nu} + (\mathbf{A}_\mu \mathbf{A}_\nu - \mathbf{A}_\nu \mathbf{A}_\mu)\tilde{\mathbf{G}}_{\mu\nu}\right] \tag{2.31}$$

$$= \mathrm{Tr}\left[(\partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu)\tilde{\mathbf{G}}_{\mu\nu} + (\mathbf{A}_\mu \mathbf{A}_\nu \tilde{\mathbf{G}}_{\mu\nu} - \mathbf{A}_\mu \tilde{\mathbf{G}}_{\mu\nu}\mathbf{A}_\nu)\right] \tag{2.32}$$

$$= \mathrm{Tr}\left[(\partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu)\tilde{\mathbf{G}}_{\mu\nu} + \mathbf{A}_\mu[\mathbf{A}_\nu, \tilde{\mathbf{G}}_{\mu\nu}]\right] \tag{2.33}$$

$$= \mathrm{Tr}\left[(\partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu)\tilde{\mathbf{G}}_{\mu\nu} - \mathbf{A}_\mu \partial_\nu \tilde{\mathbf{G}}_{\mu\nu}\right] \tag{2.34}$$

$$= \mathrm{Tr}\left[(\partial_\mu \mathbf{A}_\nu)\tilde{\mathbf{G}}_{\mu\nu} - \partial_\nu(\mathbf{A}_\mu \tilde{\mathbf{G}}_{\mu\nu})\right] \tag{2.35}$$

$$= \epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}\left[(\partial_\mu \mathbf{A}_\nu)(\partial_\rho \mathbf{A}_\sigma - \partial_\sigma \mathbf{A}_\rho + [\mathbf{A}_\rho, \mathbf{A}_\sigma]) - \partial_\nu\left(\mathbf{A}_\mu\left(\partial_\rho \mathbf{A}_\sigma - \partial_\sigma \mathbf{A}_\rho + [\mathbf{A}_\rho, \mathbf{A}_\sigma]\right)\right)\right] \tag{2.36}$$

$$= \epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}\left[(\partial_\mu \mathbf{A}_\nu)\partial_\rho \mathbf{A}_\sigma + (\partial_\mu \mathbf{A}_\nu)\mathbf{A}_\rho \mathbf{A}_\sigma - \partial_\nu\left(\mathbf{A}_\mu \partial_\rho \mathbf{A}_\sigma + \mathbf{A}_\mu \mathbf{A}_\rho \mathbf{A}_\sigma\right)\right] \tag{2.37}$$

Note,

$$\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}[\partial_\mu(\mathbf{A}_\nu \mathbf{A}_\rho \mathbf{A}_\sigma)] = \epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}\left[(\partial_\mu \mathbf{A}_\nu)\mathbf{A}_\rho \mathbf{A}_\sigma + \mathbf{A}_\nu \partial_\mu(\mathbf{A}_\rho \mathbf{A}_\sigma)\right] \tag{2.38}$$

$$= \epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}\left[(\partial_\mu \mathbf{A}_\nu)\mathbf{A}_\rho \mathbf{A}_\sigma + \mathbf{A}_\nu(\partial_\mu \mathbf{A}_\rho)\mathbf{A}_\sigma + \mathbf{A}_\nu \mathbf{A}_\rho(\partial_\mu \mathbf{A}_\sigma)\right] \tag{2.39}$$

$$= \epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}\left[(\partial_\mu \mathbf{A}_\nu)\mathbf{A}_\rho \mathbf{A}_\sigma + (-1)^2(\partial_\mu \mathbf{A}_\nu)\mathbf{A}_\rho \mathbf{A}_\sigma + (-1)(-1)(\partial_\mu \mathbf{A}_\nu)\mathbf{A}_\rho \mathbf{A}_\sigma\right] \tag{2.40}$$

$$= 3\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}\left[(\partial_\mu \mathbf{A}_\nu)\mathbf{A}_\rho \mathbf{A}_\sigma\right] \tag{2.41}$$

Or,

$$\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}\left[(\partial_\mu \mathbf{A}_\nu)\mathbf{A}_\rho\mathbf{A}_\sigma\right] = \frac{1}{3}\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}[\partial_\mu(\mathbf{A}_\nu\mathbf{A}_\rho\mathbf{A}_\sigma)] \tag{2.42}$$

Using this identity in Eq.(2.37) (after relabeling indicies)

$$= \epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}\left[(\partial_\mu \mathbf{A}_\nu)\partial_\rho\mathbf{A}_\sigma + \partial_\mu\left(\mathbf{A}_\nu\partial_\rho\mathbf{A}_\sigma + \frac{4}{3}\mathbf{A}_\nu\mathbf{A}_\rho\mathbf{A}_\sigma\right)\right] \tag{2.43}$$

Also note,

$$\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}\left[\partial_\mu(\mathbf{A}_\nu\partial_\rho\mathbf{A}_\sigma)\right] = \epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}\left[(\partial_\mu\mathbf{A}_\nu)\partial_\rho\mathbf{A}_\sigma + \mathbf{A}_\nu(\partial_\mu\partial_\rho\mathbf{A}_\sigma)\right] = \epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}\left[(\partial_\mu\mathbf{A}_\nu)\partial_\rho\mathbf{A}_\sigma\right] \tag{2.44}$$

Where I have used the fact that the sum of a symmetric and antisymmetric tensor is equivalently zero. This leaves us with

$$-16\pi^2 Q(x) = \epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}\left[2\partial_\mu\left(\mathbf{A}_\nu\partial_\rho\mathbf{A}_\sigma + \frac{2}{3}\mathbf{A}_\nu\mathbf{A}_\rho\mathbf{A}_\sigma\right)\right] \tag{2.45}$$

Or,

$$Q(x) = \partial_\mu j_\mu \tag{2.46}$$

Where

$$j \equiv \frac{1}{8\pi^2}\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}\left[\left(\mathbf{A}_\nu\partial_\rho\mathbf{A}_\sigma + \frac{2}{3}\mathbf{A}_\nu\mathbf{A}_\rho\mathbf{A}_\sigma\right)\right] \tag{2.47}$$

Using the divergence theorem[4] we can express Q as a surface integral over S^3_{E}.

$$Q = \int d^4x\,\partial_\mu j_\mu = \oint_{S^3_{\mathrm{E}}} d\sigma_\mu j_\mu \tag{2.48}$$

$$= \frac{1}{8\pi^2}\oint_{S^3_{\mathrm{E}}} d\sigma_\mu\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}\left[\left(\mathbf{A}_\nu\partial_\rho\mathbf{A}_\sigma + \frac{2}{3}\mathbf{A}_\nu\mathbf{A}_\rho\mathbf{A}_\sigma\right)\right] \tag{2.49}$$

At the boundary of S^3_{E} we require $\tilde{\mathbf{G}}_{\mu\nu} = 0$ and thus, $\epsilon_{\mu\nu\rho\sigma}\partial_\rho\mathbf{A}_\sigma = -\epsilon_{\mu\nu\rho\sigma}\mathbf{A}_\rho\mathbf{A}_\sigma$ leaving us with

$$= -\frac{1}{24\pi^2}\oint_{S^3_{\mathrm{E}}} d\sigma_\mu\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}\left[\mathbf{A}_\nu\mathbf{A}_\rho\mathbf{A}_\sigma\right] \tag{2.50}$$

[4]Given a vector field in Euclidean space $\mathbf{A}_\mu$

$$\int_{\mathcal{V}} dV(\partial_\mu\mathbf{A}^\mu) = \oint_{\mathcal{S}} d\mathbf{a}_\mu\mathbf{A}^\mu$$

Inputting the asymptotic behavior of the gauge fields

$$= -\frac{1}{24\pi^2} \oint_{S^3_{\mathrm{E}}} d\sigma_\mu \epsilon_{\mu\nu\rho\sigma} \mathrm{Tr}\left[U\partial_\nu U^{-1} U\partial_\rho U^{-1} U\partial_\sigma U^{-1}\right] \tag{2.51}$$

$$= \frac{1}{24\pi^2} \oint_{S^3_{\mathrm{E}}} d\sigma_\mu \epsilon_{\mu\nu\rho\sigma} \mathrm{Tr}\left[(\partial_\nu U)U^{-1}(\partial_\rho U)U^{-1}(\partial_\sigma U)U^{-1}\right] \tag{2.52}$$

We have now expressed the topological charge in terms of SU(2) group-element-valued functions U on S^3_{E}.

Now we need to identify an expression equivalent to the volume of group space i.e. the measure of SU(2). As we saw above the group can be uniquely identified by four real parameters a^μ subject to the constraint in Eq.(2.12) or equivalently parametrized by three independent variables ξ_1, ξ_2, and ξ_3 corresponding, for example, to the three polar coordinates of the hyperspherical geometry. Some group element U is then identified uniquely by the coordinates (ξ_1, ξ_2, ξ_3). The measure $d\mu(U)$ can then be be identified by considering the density of group elements infinitesimally close to $U(\xi_1, \xi_2, \xi_3)$. In the vicinity of U the measure can be expressed as

$$d\mu(U) = \rho(\xi_1, \xi_2, \xi_3) d\xi_1 d\xi_2 d\xi_3 \tag{2.53}$$

The density $\rho(\xi_1, \xi_2, \xi_3)$ is subject to one condition, it must be such that the group measure is invariant under group translations. This stems from the fact that the number of group elements does not change when they are all multiplied by the same group element. Explicitly, we define

$$U' = \tilde{U}U \tag{2.54}$$

U' corresponds to the parameters (ξ_1', ξ_2', ξ_3'). Let the set of elements in the infinitesimal volume $d\xi_1 d\xi_2 d\xi_3$, when multiplied by the same $\tilde{U}$ fall in some new volume $d\xi_1' d\xi_2' d\xi_3'$. The density must be such that

$$\begin{aligned} d\mu(U) &= \rho(\xi_1, \xi_2, \xi_3) d\xi_1 d\xi_2 d\xi_3 \\ &= d\mu(U') \\ &= \rho(\xi_1', \xi_2', \xi_3') d\xi_1' d\xi_2' d\xi_3' \end{aligned} \tag{2.55}$$

A particular choice of $\rho(\xi_1, \xi_2, \xi_3)$ which satisfies this rrequirement is given by

$$\rho(\xi_1, \xi_2, \xi_3) = \epsilon_{ijk} \mathrm{Tr}\left[U^{-1}\frac{\partial U}{\partial \xi_i} U^{-1}\frac{\partial U}{\partial \xi_j} U^{-1}\frac{\partial U}{\partial \xi_k}\right] \tag{2.56}$$

Where the group elements are written in their matrix representation. Multiplying on the left by $\tilde{U}$ gives

$$U(\xi_1, \xi_2, \xi_3) \to U'(\xi'_1, \xi'_2, \xi'_3) \equiv \tilde{U}U(\xi_1, \xi_2, \xi_3) \tag{2.57}$$

$$U = \tilde{U}^{-1}U', \qquad U^{-1} = (U')^{-1}\tilde{U} \tag{2.58}$$

Plugging into Eq.(2.56)

$$= \epsilon_{ijk}\mathrm{Tr}\left[(U')^{-1}\tilde{U}\frac{\partial(\tilde{U}^{-1}U')}{\partial\xi_i}(U')^{-1}\tilde{U}\frac{\partial(\tilde{U}^{-1}U')}{\partial\xi_j}(U')^{-1}\tilde{U}\frac{\partial(\tilde{U}^{-1}U')}{\partial\xi_k}\right] \tag{2.59}$$

$$= \epsilon_{ijk}\mathrm{Tr}\left[(U')^{-1}\tilde{U}\tilde{U}^{-1}\frac{\partial U'(\vec{\xi'})}{\partial\xi_i}(U')^{-1}\tilde{U}\tilde{U}^{-1}\frac{\partial U'(\vec{\xi'})}{\partial\xi_j}(U')^{-1}\tilde{U}\tilde{U}^{-1}\frac{\partial U'(\vec{\xi'})}{\partial\xi_k}\right] \tag{2.60}$$

$$= \epsilon_{ijk}\mathrm{Tr}\left[(U')^{-1}\frac{\partial U'(\vec{\xi'})}{\partial\xi'_\lambda}\frac{\partial\xi'_\lambda}{\partial\xi_i}(U')^{-1}\frac{\partial U'(\vec{\xi'})}{\partial\xi'}\frac{\partial\xi'_\rho}{\partial\xi_j}(U')^{-1}\frac{\partial U'(\vec{\xi'})}{\partial\xi'_\sigma}\frac{\partial\xi'_\sigma}{\partial\xi_k}\right] \tag{2.61}$$

$$= \epsilon_{ijk}\frac{\partial\xi'_\lambda}{\partial\xi_i}\frac{\partial\xi'_\rho}{\partial\xi_j}\frac{\partial\xi'_\sigma}{\partial\xi_k}\mathrm{Tr}\left[(U')^{-1}\frac{\partial U'}{\partial\xi'_\lambda}(U')^{-1}\frac{\partial U'}{\partial\xi'_\rho}(U')^{-1}\frac{\partial U'}{\partial\xi'_\sigma}\right] \tag{2.62}$$

Noting the following identity

$$\epsilon_{i_1,i_2,\ldots,i_n}A_{i_1j_1}A_{i_2j_2}\ldots A_{i_nj_n} = \epsilon_{j_n,j_2,\ldots,j_n}\det(\mathbf{A}) \tag{2.63}$$

Eq.(2.62) becomes

$$= \epsilon_{\lambda\rho\sigma}\det\left(\frac{\partial\xi'}{\partial\xi}\right)\mathrm{Tr}\left[(U')^{-1}\frac{\partial U'}{\partial\xi'_\lambda}(U')^{-1}\frac{\partial U'}{\partial\xi'_\rho}(U')^{-1}\frac{\partial U'}{\partial\xi'_\sigma}\right] \tag{2.64}$$

Where, $\det(\partial\xi'/\partial\xi)$ is just the Jacobian determinant associated with the transformation from coordinates ξ to ξ'. Thus, the group-element density under the transformation becomes

$$\rho(\xi_1, \xi_2, \xi_3) = \rho(\xi'_1, \xi'_2, \xi'_3)\det\left(\frac{\partial\xi'}{\partial\xi}\right) \tag{2.65}$$

Thus,

$$\rho(\xi_1, \xi_2, \xi_3)d\xi_1 d\xi_2 d\xi_3 = \rho(\xi'_1, \xi'_2, \xi'_3)\det\left(\frac{\partial\xi'}{\partial\xi}\right)d\xi_1 d\xi_2 d\xi_3 \tag{2.66}$$

$$= \rho(\xi'_1, \xi'_2, \xi'_3)d\xi'_1 d\xi'_2 d\xi'_3 \tag{2.67}$$

Our last job is to show that the surface integral for Q reduces to an integral over the group measure $d\mu(U)$. It is appropriate to state again that the surface integral in Eq.(2.52) deals with a given field configuration, whose boundary conditions introduce a gauge function U, defined on S^3_{E}. Here, the group matrices U as well as the corresponding group parameters, are functions of the spatial coordinates x_μ on S^3_{E}. With this in mind we can rewrite Eq.(2.52) as

$$= \frac{1}{24\pi^2} \oint_{S^3_{\mathrm{E}}} d\sigma_\mu \epsilon_{\mu\nu\rho\sigma} \frac{\partial \xi_i}{\partial x_\nu} \frac{\partial \xi_j}{\partial x_\rho} \frac{\partial \xi_k}{\partial x_\sigma} \mathrm{Tr} \left[U^{-1} \frac{\partial U}{\partial \xi_i} U^{-1} \frac{\partial U}{\partial \xi_j} U^{-1} \frac{\partial U}{\partial \xi_k} \right] \tag{2.68}$$

We are free to choose the shape of the boundary surface at $r = \infty$, as the divergence theorem holds regardless of the surface we choose. Thus, to make our calculation as easy as possible we can choose to deform the spherical surface at $r = \pm\infty$ into a four-dimensional hypercube. The eight sides of the cube are surfaces at $x_\mu = \pm\infty$ for $\mu = 1, 2, 3, 4$. We can now consider the contribution of one of these surfaces, say $x_4 = \infty$, to the surface integral in Eq.(2.68). The contribution can be written as

$$\frac{1}{24\pi^2} \int dx_1 dx_2 dx_3 \epsilon_{\mu\nu\rho\sigma} \frac{\partial \xi_i}{\partial x_\nu} \frac{\partial \xi_j}{\partial x_\rho} \frac{\partial \xi_k}{\partial x_\sigma} \mathrm{Tr} \left[U^{-1} \frac{\partial U}{\partial \xi_i} U^{-1} \frac{\partial U}{\partial \xi_j} U^{-1} \frac{\partial U}{\partial \xi_k} \right] \tag{2.69}$$

Again noting,

$$dx_1 dx_2 dx_3 \epsilon_{\mu\nu\rho\sigma} \frac{\partial \xi_i}{\partial x_\nu} \frac{\partial \xi_j}{\partial x_\rho} \frac{\partial \xi_k}{\partial x_\sigma} = d\xi_1 d\xi_2 \xi_3 \epsilon_{\mu ijk} \tag{2.70}$$

We have then,

$$= \frac{1}{24\pi^2} \int d\xi_1 d\xi_2 \xi_3 \epsilon_{\mu ijk} \mathrm{Tr} \left[U^{-1} \frac{\partial U}{\partial \xi_i} U^{-1} \frac{\partial U}{\partial \xi_j} U^{-1} \frac{\partial U}{\partial \xi_k} \right] \tag{2.71}$$

Which is exactly equal to an integral over the group measure over the surface at $x_4 = \infty$ (up to an overal constant)! A similar contribution comes from all eight surfaces of the hypercube. Their sum can be written approximately as

$$Q \propto \int d\xi_1 d\xi_2 d\xi_3 \rho(\xi_1, \xi_2, \chi_3) \propto \int d\xi_2 d\xi_3 d\xi_4 \rho(\xi_2, \xi_3, \chi_4) \propto \cdots \tag{2.72}$$

$$\propto \int d\mu(U) \tag{2.73}$$

Thus, we see that Q reduces to an integral over group space of the group measure. Roughly then, Q is proportional to the volume of group space spanned by the gauge function U as it varies on S^3_{E}. Homotopy theory tells us is that when we integrate over S^3_{E} once, the group space $S^3_{\mathrm{SU(2)}}$ may be spanned an integral number of times and Q will be proportional to this integer. The constant $(1/16\pi^2)^{-1}$ has been chosen so that Q will in fact equal this integer. The constant can be determined by looking at a test case, for example we consider the group function

$$U_1(x) = \frac{(x_4 + ix_j\sigma_j)}{|x|} = \hat{x}_\mu s_\mu \tag{2.74}$$

This function corresponds to $a_\mu = \hat{x}_\mu$. That is, every point on S^3_{E} is mapped onto the 'corresponding' point, at the same polar angles on $S^3_{\mathrm{SU(2)}}$. This indicates that the Pontryagin index must be equal to one. Inserting Eq.(2.74) into Eq.(2.52) we find (using Mathematica)

Topological Charge Example

We are interested in working out an explicit example of the topological charge given the group-element-function

$$U = \frac{\sigma_4 + i\, x_j\, \sigma_j}{|x|}$$

To compute the charge we must calculate

$$Q = \frac{1}{24\,\pi^2} \oint d\sigma_\mu\, \epsilon_{\mu\nu\rho\sigma}\, \mathrm{Tr}\left[(\partial_\nu U)\, U^{-1} (\partial_\rho U)\, U^{-1} (\partial_\sigma U)\, U^{-1}\right]$$

In[1]:=
```
σ1 := (0  1
       1  0);
σ2 := (0  -i
       i   0);
σ3 := (1   0
       0  -1);
σ4 := (1  0
       0  1);
```

In[5]:=
```
U := 1/X (Subscript[x, 4] σ4 + i Subscript[x, 1] σ1 + i Subscript[x, 2] σ2 +
      i Subscript[x, 3] σ3) /. X → Sqrt[x1^2 + x2^2 + x3^2 + x4^2];
U // MatrixForm
```

Out[6]//MatrixForm=

$$\begin{pmatrix} \frac{i\, x_3 + x_4}{\sqrt{x_1^2+x_2^2+x_3^2+x_4^2}} & \frac{i\, x_1 + x_2}{\sqrt{x_1^2+x_2^2+x_3^2+x_4^2}} \\ \frac{i\, x_1 - x_2}{\sqrt{x_1^2+x_2^2+x_3^2+x_4^2}} & \frac{-i\, x_3 + x_4}{\sqrt{x_1^2+x_2^2+x_3^2+x_4^2}} \end{pmatrix}$$

In[7]:=
```
Ui := Inverse[U] // FullSimplify ;
Ui // MatrixForm
```

Out[8]//MatrixForm=

$$\begin{pmatrix} \frac{-i\, x_3 + x_4}{\sqrt{x_1^2+x_2^2+x_3^2+x_4^2}} & \frac{-i\, x_1 - x_2}{\sqrt{x_1^2+x_2^2+x_3^2+x_4^2}} \\ \frac{-i\, x_1 + x_2}{\sqrt{x_1^2+x_2^2+x_3^2+x_4^2}} & \frac{i\, x_3 + x_4}{\sqrt{x_1^2+x_2^2+x_3^2+x_4^2}} \end{pmatrix}$$

In[9]:=
```
DU := Table[D[U, Subscript[x, i]], {i, 4}] // MatrixForm ;
DU // MatrixForm
```

Out[10]//MatrixForm=

$$\begin{pmatrix} \begin{pmatrix} -\frac{x_1 (i x_3+x_4)}{(x_1^2+x_2^2+x_3^2+x_4^2)^{3/2}} \\ -\frac{x_1 (i x_1+x_2)}{(x_1^2+x_2^2+x_3^2+x_4^2)^{3/2}} + \frac{i}{\sqrt{x_1^2+x_2^2+x_3^2+x_4^2}} \end{pmatrix} & \begin{pmatrix} -\frac{x_1 (i x_1-x_2)}{(x_1^2+x_2^2+x_3^2+x_4^2)^{3/2}} + \frac{i}{\sqrt{x_1^2+x_2^2+x_3^2+x_4^2}} \\ -\frac{x_1 (-i x_3+x_4)}{(x_1^2+x_2^2+x_3^2+x_4^2)^{3/2}} \end{pmatrix} \\ \begin{pmatrix} -\frac{x_2 (i x_3+x_4)}{(x_1^2+x_2^2+x_3^2+x_4^2)^{3/2}} \\ -\frac{x_2 (i x_1+x_2)}{(x_1^2+x_2^2+x_3^2+x_4^2)^{3/2}} + \frac{1}{\sqrt{x_1^2+x_2^2+x_3^2+x_4^2}} \end{pmatrix} & \begin{pmatrix} -\frac{(i x_1-x_2) x_2}{(x_1^2+x_2^2+x_3^2+x_4^2)^{3/2}} - \frac{1}{\sqrt{x_1^2+x_2^2+x_3^2+x_4^2}} \\ -\frac{x_2 (-i x_3+x_4)}{(x_1^2+x_2^2+x_3^2+x_4^2)^{3/2}} \end{pmatrix} \\ \begin{pmatrix} -\frac{x_3 (i x_3+x_4)}{(x_1^2+x_2^2+x_3^2+x_4^2)^{3/2}} + \frac{i}{\sqrt{x_1^2+x_2^2+x_3^2+x_4^2}} \\ -\frac{(i x_1+x_2) x_3}{(x_1^2+x_2^2+x_3^2+x_4^2)^{3/2}} \end{pmatrix} & \begin{pmatrix} -\frac{(i x_1-x_2) x_3}{(x_1^2+x_2^2+x_3^2+x_4^2)^{3/2}} \\ -\frac{x_3 (-i x_3+x_4)}{(x_1^2+x_2^2+x_3^2+x_4^2)^{3/2}} - \frac{i}{\sqrt{x_1^2+x_2^2+x_3^2+x_4^2}} \end{pmatrix} \\ \begin{pmatrix} -\frac{x_4 (i x_3+x_4)}{(x_1^2+x_2^2+x_3^2+x_4^2)^{3/2}} + \frac{1}{\sqrt{x_1^2+x_2^2+x_3^2+x_4^2}} \\ -\frac{(i x_1+x_2) x_4}{(x_1^2+x_2^2+x_3^2+x_4^2)^{3/2}} \end{pmatrix} & \begin{pmatrix} -\frac{(i x_1-x_2) x_4}{(x_1^2+x_2^2+x_3^2+x_4^2)^{3/2}} \\ -\frac{x_4 (-i x_3+x_4)}{(x_1^2+x_2^2+x_3^2+x_4^2)^{3/2}} + \frac{1}{\sqrt{x_1^2+x_2^2+x_3^2+x_4^2}} \end{pmatrix} \end{pmatrix}$$

In[11]:=
```
ϵ := LeviCivitaTensor [4, List]
```

For example,

In[12]:=
```
ϵ[[1]]
```

Out[12]=
```
{{{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}},
 {{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 1}, {0, 0, -1, 0}},
 {{0, 0, 0, 0}, {0, 0, 0, -1}, {0, 0, 0, 0}, {0, 1, 0, 0}},
 {{0, 0, 0, 0}, {0, 0, 1, 0}, {0, -1, 0, 0}, {0, 0, 0, 0}}}
```

In[13]:=
```
ϵ[[1, 1]]
```

Out[13]=
```
{{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}
```

In[14]:=
```
ϵ[[1, 1, 1]]
```

Out[14]=
```
{0, 0, 0, 0}
```

In[15]:=
```
ϵ[[1, 1, 1, 1]]
```

Out[15]=
```
0
```

In[16]:=
```
Λ :=
  Table[∑_{ν=1}^{4} ∑_{ρ=1}^{4} ∑_{σ=1}^{4} ϵ[[i, ν, ρ, σ]] × DU[[1, ν]].Ui.DU[[1, ρ]].Ui.DU[[1, σ]].Ui,
    {i, 4}] // FullSimplify ;
Λ // MatrixForm
```

Out[17]//MatrixForm=

$$\begin{pmatrix} \begin{pmatrix} -\frac{6 x_1}{(x_1^2+x_2^2+x_3^2+x_4^2)^2} \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ -\frac{6 x_1}{(x_1^2+x_2^2+x_3^2+x_4^2)^2} \end{pmatrix} \\ \begin{pmatrix} -\frac{6 x_2}{(x_1^2+x_2^2+x_3^2+x_4^2)^2} \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ -\frac{6 x_2}{(x_1^2+x_2^2+x_3^2+x_4^2)^2} \end{pmatrix} \\ \begin{pmatrix} -\frac{6 x_3}{(x_1^2+x_2^2+x_3^2+x_4^2)^2} \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ -\frac{6 x_3}{(x_1^2+x_2^2+x_3^2+x_4^2)^2} \end{pmatrix} \\ \begin{pmatrix} -\frac{6 x_4}{(x_1^2+x_2^2+x_3^2+x_4^2)^2} \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ -\frac{6 x_4}{(x_1^2+x_2^2+x_3^2+x_4^2)^2} \end{pmatrix} \end{pmatrix}$$

In[18]:=
```
TrΛ := Table[Tr[Λ[[i]]], {i, 4}];
TrΛ /. x1^2 + x2^2 + x3^2 + x4^2 → x^2 // MatrixForm
```

Out[19]//MatrixForm=

$$\begin{pmatrix} -\frac{12 x_1}{x^4} \\ -\frac{12 x_2}{x^4} \\ -\frac{12 x_3}{x^4} \\ -\frac{12 x_4}{x^4} \end{pmatrix}$$

This can then be placed into the integral with $d\sigma_\mu \to d\Omega\, x_\mu \,|\, x\,|^2$ and leaves us with

$$Q = \frac{1}{2\pi^2} \int d\Omega$$

The final step is to compute the solid angle in four-dimensions, we have

In[20]:=
```
x1 := r Sin[ψ] Sin[ϕ] Cos[θ];
x2 := r Sin[ψ] Sin[ϕ] Sin[θ];
x3 := r Sin[ψ] Cos[ϕ];
x4 := r Cos[ψ];
```

$$\texttt{hyperVar} := \begin{pmatrix} r \\ \theta \\ \phi \\ \psi \end{pmatrix};$$

In[25]:=
```
J := Table[D[Subscript[x, i], hyperVar[[j]]], {i, 4}, {j, 4}];
J // MatrixForm
```

Out[26]//MatrixForm=

$$\begin{pmatrix} \mathrm{Cos}[\theta]\,\mathrm{Sin}[\phi]\,\mathrm{Sin}[\psi] & -r\,\mathrm{Sin}[\theta]\,\mathrm{Sin}[\phi]\,\mathrm{Sin}[\psi] & r\,\mathrm{Cos}[\theta]\,\mathrm{Cos}[\phi]\,\mathrm{Sin}[\psi] & r\,\mathrm{Cos}[\theta]\,\mathrm{Cos}[\psi]\,\mathrm{Sin}[\phi] \\ \mathrm{Sin}[\theta]\,\mathrm{Sin}[\phi]\,\mathrm{Sin}[\psi] & r\,\mathrm{Cos}[\theta]\,\mathrm{Sin}[\phi]\,\mathrm{Sin}[\psi] & r\,\mathrm{Cos}[\phi]\,\mathrm{Sin}[\theta]\,\mathrm{Sin}[\psi] & r\,\mathrm{Cos}[\psi]\,\mathrm{Sin}[\theta]\,\mathrm{Sin}[\phi] \\ \mathrm{Cos}[\phi]\,\mathrm{Sin}[\psi] & 0 & -r\,\mathrm{Sin}[\phi]\,\mathrm{Sin}[\psi] & r\,\mathrm{Cos}[\phi]\,\mathrm{Cos}[\psi] \\ \mathrm{Cos}[\psi] & 0 & 0 & -r\,\mathrm{Sin}[\psi] \end{pmatrix}$$

In[27]:=
```
Det[J] // FullSimplify
```

Out[27]= $r^3\,\mathrm{Sin}[\phi]\,\mathrm{Sin}[\psi]^2$

In[28]:= $\int_0^{\pi}\int_0^{\pi}\int_0^{2\pi} \mathrm{Sin}[\psi]^2\,\mathrm{Sin}[\phi]\,d\theta\,d\psi\,d\phi$

Out[28]= $2\,\pi^2$

Thus, we see that $Q = 1$ as expected.

References

[1] Coleman, Sidney. Aspects of symmetry: selected Erice lectures. Cambridge University Press, 1988.

[2] Tong, David. "Quantum field theory. lecture notes." (2007).