CS 600 Homework 10 Solutions

R-19.2 Anteaters and Bears

In any give game, the Anteaters will beat the Bears with a probability $\frac{2}{3}$, independent of any other games they play. We want a bound on the probability that, in spite of this, the Bears will win a majority of n games.

Let $X_1, X_2, \ldots, X_n$ be $0-1$ indicator random variables such that if $X_i = 1$ then the Bears have won game i and if $X_i = 0$ then the Bears have lost game i. Since the Bears have a $\frac{1}{3}$ change of winning any game, we have $\mu = E[X] = \frac{n}{3}$.

We can use the following Chernoff bound to compute a bound on the probability that the Bears will win a majority of n games,

$$\Pr(X > (1+\delta)\mu) \leq \left[\frac{e^{\delta}}{(1+\delta)^{(1+\delta)}}\right]^{\mu} \text{ with } \delta > 0$$

Since we want $(1+\delta)(\frac{n}{3}) = \frac{n}{2}$, we set $\delta = \frac{1}{2}$. Therefore,

$$\Pr(X > \frac{n}{2}) \leq \left[\frac{e^{\frac{1}{2}}}{(1+\frac{1}{2})^{(1+\frac{1}{2})}}\right]^{\frac{n}{3}}$$

$$\Pr(X > \frac{n}{2}) \leq \left[\frac{1.6487}{1.8371}\right]^{\frac{n}{3}} \approx 0.89645^{\frac{n}{3}}$$

Therefore, the probability that the Bears win a majority of the games is bounded by $0.9^{\frac{n}{3}}$.

C-19.7 Red and Blue Coupon Collection

There is a collection of $3n$ distinct coupons, n of which are red, and $2n$ of which are blue. Each time we go to the ticket window, the clerk randomly decides with a probability of $\frac{1}{2}$, whether he will give a red or blue coupon and then chooses a coupon uniformly at random of the chosen color. We are interested in computing the expected number of times we must visit the ticket window to get all $3n$ coupons.

We assume that the clerk has an infinite supply of red and blue coupons out of which there are only $3n$ distinct coupons. If this assumption isn't made the answer is bounded by $3n$ trips.

We break the problem into two coupon collector problems. Imagine the colored tickets are dispensed from two different windows. In this case, we would need to make nH_n trips to get all the n distinct red coupons, and $2nH_{2n}$ trips to get all the $2n$ distinct blue coupons. In making these many trips, we get all $3n$ distinct coupons. It is clear that we need to make more trips to get all the blue coupons.

Now, coming back to the original problem, we know that there is a probability of $\frac{1}{2}$ that we get a blue coupon. Hence, we need an expected number of $4nH_{2n}$ trips to get all the blue coupons. During these trips, we can expect that half the time, we get a red coupon. That is, the expected number of red coupons we get is $2nH_{2n}$.

Since $nH_n < 2nH_{2n}$, we know that we will get all the required red coupons when we make $2nH_{2n}$ trips. Further, in making the $4nH_{2n}$, we collect all the distinct red coupons in all those trips where we didn't get a blue coupon. Therefore, the expected number of trips we make to get all $3n$ distinct coupons is $4nH_{2n}$.

A-19.2 Mega Millions

In the Mega Millions game, a player picks five lucky numbers in the range from 1 to 56, and one additional Mega number, in the range from 1 to 46. In order to win the jackpot, the player must match all six chosen numbers. We assume that every time a lottery ticket is sold, it is chosen as an independent random pick of five lucky numbers and a Mega number. We are interested in computing the expected number of lottery tickets that must be sold to guarantee with 100% certainty that there is a winner in a given draw.

The number of possible combinations of the first five numbers is $\frac{56!}{5! \times (56-5)!} = 3,819,816$. Combined with the 46 possibilities (the Mega number), we get $46 \times 3,819,816 = 175,711,536$. This represents the total number of distinct tickets we can have.

Notice that this is similar to the coupon collection problem. In order to ensure with a 100% chance that there is a winner, we need to ensure that all n distinct tickets are sold. Since we know that the tickets are chosen as an independent random pick of numbers, we can pose this as a coupon collection problem.

We know that to collect n distinct coupons, we need to make nH_n trips. Therefore, in this case, we have $175,711,536H_{175,711,536}$ which is $175,711,536 \times (\log 175,711,536 + 1)$.