Interest rates forecasting: Between Hull and White and the CIR#—How to make a single-factor model work

Article *in* Journal of Forecasting · May 2021

DOI: 10.1002/for.2783

2 authors:

Giuseppe Orlando
Università degli Studi di Bari Aldo Moro

Michele Bufalo
Università degli Studi di Bari Aldo Moro

Interest rates forecasting: between Hull and White and the CIR#. How to make a single factor model work

Giuseppe Orlando[1a], Michele Bufalo[b]

[a] *Università degli Studi di Bari Aldo Moro - Department of Economics and Finance, Via C. Rosalba 53, Bari, I-70124 Italy, Tel. +39 080 5049218, giuseppe.orlando@uniba.it*
[b] *Università degli Studi di Roma "La Sapienza" - Department of Methods and Models for Economics, Territory and Finance, Via del Castro Laurenziano 9, Roma, I-00185, Tel. +39 06 49766903, michele.bufalo@uniroma1.it*

Abstract

In this work we present our findings of the so-called CIR#, which is a modified version of the Cox, Ingersoll & Ross (CIR) model, turned into a forecasting tool for any term structure. The main feature of the CIR# model is its ability to cope with negative interest rates, cluster volatility and jumps. By considering a dataset composed of money market interest rates during turmoil and calmer periods, we show how the CIR# performs in terms of directionality of rates and forecasting error. Comparison is carried out with a revamped version of the CIR model (denoted CIR_{adj}), the Hull and White model and the EWMA which is often adopted whenever no structure in data is assumed. Testing and validation is performed on both historical and *had hoc* data with different metrics and clustering criteria to confirm the analysis.

Keywords: Interest rate forecasting, Hull and White model, CIR model, ARIMA, cluster volatility and jumps fitting
JEL Classification: G12, E43, E47
2010 MSC: 91G30, 91B84, 91G60, 91G70, 62M10

---

1. Introduction

This paper expands on previous research Orlando et al. (2018), (2019a), (2019b) where it was provided a new accessible methodology to forecast future interest rates called CIR#. In this introduction we are not going to list all details of the CIR# model but we will recall the most important characteristics. Above all we mention

[1]Corresponding author.

This article has been accepted for publication and undergone full peer review but has not been through the copyediting, typesetting, pagination and proofreading process which may lead to differences between this version and the Version of Record. Please cite this article as doi: 10.1002/for.2783

that, in order to preserve its analytical tractability and simplicity of single-factor model, the CIR# has been designed in a way that the improvements are obtained within the original Cox, Ingersoll & Ross (CIR) framework. Apart from turning a short rate model used for pricing into a forecasting tool, the novelty of the CIR# consists in an appropriate partitioning of the dataset into sub-groups. To this end, in Orlando et al. (2020), it was shown how the said partitioning enables to capture statistically significant time changes in volatility of interest rates. This, in turn, implies modelling sudden changes/jumps in data. To solve the issue of negative/near-to-zero interest rates, as others did (e.g. Brigo and Mercurio (2000)), we contemplate an opportune shift of interest rates to positive values. Another feature of the CIR# model is the calibration of parameters to actual data. With regard to the latter, in our framework the random part of the numerical simulation scheme is driven not by a standard Brownian motion but by the Gaussian residuals of an "optimal" ARIMA model appropriately chosen. Thanks to that it is obtained an exact trajectory of CIR fitted values which replaces the usual Monte Carlo averaged over 100,000 simulated trajectories, thus reducing considerably the computational cost of the estimating procedure.

As mentioned above the challenge is to forecast interest rates with a single factor model, therefore, in this paper there is a comparison of the CIR# model versus real data, the EWMA, the "revamped" (through partitioning, shifting and calibrating) CIR model (which we call CIR adjusted) and the single-factor Hull and White model. To level the playing field, the latter comparison is performed by using the same data (in terms of shifting and partitioning). The performance of both models has been tested on monthly money market interest rates ranging from 1 day to 12 months over EUR, USD, JPY and CHF currencies. The dataset includes both turmoil and calmer periods on which we compute the predictive power in terms of directionality of rates and error. Apart from the usual techniques available to investors, risk managers and regulators, this paper suggests i) an index to verify the predictive power of the model in terms of directionality of interest rates, ii) the Bland-Altman plot for verifying the similarity between predictions and realizations, iii) a new application of hierarchical clustering for model testing and validation. Those implementations are new in literature.

The remainder of the paper is organized as follows. Section 2 summarizes the literature on the CIR model. Section 3 explains the reasons behind our idea to propose a new approach in the CIR framework and provides a short description of it. Section 4 presents the numerical procedure in full detail and tests the goodness-of-fit of the new methodology to real market data. Section 5 describes how forecasts can be tested and validated. Finally, Section 6 concludes.

2. Literature review

In 1985 Cox, Ingersoll & Ross proposed a term structure model (CIR), to describe the price of discount zero-coupon bonds with various maturities under no-arbitrage condition. The idea was to generalize the Vasicek model (1977) to the case of non constant volatility with underlying short term interest rate assumed to be a diffusion process, i.e. a continuous Markov process. A positive feature at the time was that the paths of the CIR process never reach negative values. Its relatively simplicity and analytical tractability allowed the CIR model to become one of the most widely used short-term structure models in financial institutions. Additional applications are stochastic volatility modelling in option pricing Heston (1993), Mininni et al. (2020) or default intensities in credit risk Duffie (2005).

With time some drawbacks such as failure in calibration were exposed: "the zero coupon curve is quite likely to be badly reproduced, also because some typical shapes, like that of an inverted yield curve, may not be reproduced by the model,...... no matter the values of the parameters in the dynamics that are chosen" (see Brigo & Mercurio (2001)). Cox, Ingersoll and Ross themselves, acknowledged that the model can "produce only normal, inverse or humped shapes" (1985). The seeming contradiction lies in the practical implementation. In fact, as proved by Keller-Ressel and Steiner (2008) the yield-to-maturity curve of any time-homogeneous, affine one-factor model is either normal, humped or inverse. For the CIR model, the yield curve is normal when $r(t) \leq k\theta/(\gamma - 2(k + \lambda))$, it is inverse when $r(t) \geq k\theta/(k + \lambda)$ and for intermediate values is humped.

In any case "tweaking the parameters can produce yield curves with one hump or one dip (a local minimum), but it is very difficult (if not impossible) to calibrate the parameters so that the hump/dip sits where desired. There are not enough parameters to calibrate the models to account for observed features contained in the prices quoted on the markets" (see Carmona and Tehranchi (2006)).

This created the quest for models able to fit to the observed yield curve, which could take into account multiple sources of risks as well as jumps. Among the best known developments we mention: the Hull-White (1990) with time-dependent coefficients; the Chen (1996) three-factor model; the CIR++ model by Brigo & Mercurio (2001); the jump diffusion JCIR model Brigo & Mercurio (2006) and JCIR++ by Brigo & El-Bachir (2006); the CIR2 and CIR2++ two-factor models Brigo & Mercurio (2006). Zhu (2014) proposed a CIR process with Hawkes "clustering effect", Moreno et al. (2015) suggested a square-root model with harmonic oscillators driving the long-run mean and the volatility and Najafi et al. (2017) extended the CIR model with mixed fractional Brownian motion to account for the randomness.

All above mentioned extensions of the CIR model keep the positivity of interest rates but do not fit with the financial crisis of 2007 and the ensuing quantitative easing policies. “Interest rates have been extraordinarily low for an exceptionally long time, in nominal and inflation-adjusted terms, against any benchmark”. “Between December 2014 and end-May 2015, on average around \$2 trillion in global long-term sovereign debt, much of it issued by euro area sovereigns, was trading at negative yields”. “Such yields are unprecedented. Policy rates are even lower than at the peak of the Great Financial Crisis in both nominal and real terms. And in real terms they have now been negative for even longer than during the Great Inflation of the 1970s. Yet, exceptional as this situation may be, many expect it to continue”. “Such low rates are the most remarkable symptom of a broader malaise in the global economy: the economic expansion is unbalanced, debt burdens and financial risks are still too high, productivity growth too low, and the room for manoeuvre in macroeconomic policy too limited. The unthinkable risks becoming routine and being perceived as the new normal” (see Engelen (2015) and BIS (2015)). Bianchi (2020) conducted an empirical test and found that ”the calibration of multi-factor CIR models is affected by the presence of low level (near zero) rates, and these observed patterns complicate the convergence properties of the optimization algorithm”.

Thus, the market needs a model able to cope with changing regimes, shocks and negative rates. To this end, we have proposed within the CIR framework Orlando et al. (2018, 2019a, 2019b, 2020) a new methodology that fits well the structure of interest rates and preserves the analytical tractability of the original CIR model.

3. Methods and material

3.1. Data

Our data consists of LIBOR time series retrieved from Bloomberg and test data built by us for this exercise.

3.1.1. Real time series

Real data consists of weekly interest rates from 31 December, 2009 to 31 January, 2020 as retrieved from Bloomberg at the “LIBOR Index Page BBAM <GO>” and described in Table 1.

3.1.2. Test data

Test data are composed by the EUR Overnight, a copy of the EUR Overnight, the out-of-sample forecast of the EUR Overnight, a random time series, the EUR Overnight plus the random time series above-mentioned and the changed sign of the EUR Overnight as detailed in Table 2.

Table 1: Bloomberg tickers for money market interest rates

Code	Description	Bloomberg Ticker	Tenor
EUR1	EUR Overnight	EE00O/N Curncy	N
EUR2	EUR 1 Month	EE0001M Curncy	1M
EUR3	EUR 2 Month	EE0002M Curncy	2M
EUR4	EUR 3 Month	EE0003M Curncy	3M
EUR5	EUR 6 Month	EE0006M Curncy	6M
EUR6	EUR 12 Month	EE0012M Curncy	12M
USD1	USD Overnight	US00O/N Curncy	N
USD2	USD 1 Month	US0001M Curncy	1M
USD3	USD 2 Month	US0002M Curncy	2M
USD4	USD 3 Month	US0003M Curncy	3M
USD5	USD 6 Month	US0006M Curncy	6M
USD6	USD 12 Month	US0012M Curncy	12M
JPY1	JPY Spot Next	JY00S/N Curncy	N
JPY2	JPY 1 Month	JY0001M Curncy	1M
JPY3	JPY 2 Month	JY0002M Curncy	2M
JPY4	JPY 3 Month	JY0003M Curncy	3M
JPY5	JPY 6 Month	JY0006M Curncy	6M
JPY6	JPY 12 Month	JY0012M Curncy	12M
CHF1	CHF Overnight	SF00S/N Curncy	N
CHF2	CHF 1 Month	SF0001W Curncy	1M
CHF3	CHF 2 Month	SF0001M Curncy	2M
CHF4	CHF 3 Month	SF0003M Curncy	3M
CHF5	CHF 6 Month	SF0006M Curncy	6M
CHF6	CHF 12 Month	SF0012M Curncy	12M

Table 2: Bloomberg tickers for money market interest rates

#	Test data	Description
1	RealEUR1	EE00O/N Curncy
2	Random	Randomly generated time series
3	NoiseEUR1	RealEUR1 + Random
4	NegEUR1	Negative RealEUR1
5	CopyEUR1	Copy of RealEUR1
6	ForEUR1	Out-of-sample forecast of RealEUR1

3.2. Models

3.2.1. The CIR model

The CIR interest rate model was proposed by Cox, Ingersoll & Ross (1985) to solve the problem of pricing discount zero-coupon bonds with various maturities under no-arbitrage condition. The model generalizes the Vasicek model (1977) to the case of non constant volatility by assuming that the evolution of the underlying short-term interest rate is a diffusion process unique solution to the stochastic differential equation (SDE)

$$dr(t) = [k(\theta - r(t)) - \lambda(t, r(t))]dt + \sigma\sqrt{r(t)}dW(t), \tag{1}$$

with initial condition $r(0) = r_0 > 0$. $(W(t))_{t\geq 0}$ denotes a standard Brownian motion and the interest rate process $(r(t))_{t\geq 0}$ is called CIR or square root process. The parameters k, θ and σ, are time-independent. The short interest rate dynamics are driven only by the market price of risk $\lambda(t, r(t)) := \lambda r(t)$, where λ is a constant. The SDE Eq. (1) is then composed by the "mean reverting" drift component $k[\theta - r(t)]$, to ensure the rate $r(t)$ is elastically pulled towards a long-run mean value $\theta > 0$ at speed $k > 0$, and the random component $W(t)$, scaled by the standard deviation $\sigma\sqrt{r(t)}$. The volatility of the instantaneous short rate is denoted by $\sigma > 0$. In the remainder of this paper, and in order to make the comparison even, we show the performance of the CIR over the same data used for the CIR#. This means that the CIR is not calibrated on the original data (which would be difficult in a low/negative interest rate environment) but benefits of the shifting and partitioning embedded in our algorithm. Therefore we denote this version of the classical model as CIR adjusted (CIR_{adj}).

3.2.2. The CIR# model

In this Section we summarize the procedure of the CIR# model as detailed in Orlando et al. (2019b). Let us denote the shifted monthly interest rates as

$$r_{shift} = \{r_{real,h} + \alpha \mid h = 1, ..., n\}$$

and its suitable partition, for $j = 1, ..., J$

$$r_{shift}^{(j)} = \{r_{shift,h}^{(j)} \mid h = n_{j-1} + 1, ..., n_j\} \; (n_0 = 0),$$

where $\sum_{j=1}^{J} n_j = n$ [2] and $J > 1$ is the number of partitioning sub-groups. We denote $r^{(j)}_{shift}$ with $r^{(j)}$.

The fitted values $\widehat{r}^{(j)} = \{\widehat{r}^{(j)}_h \mid h = n_{j-1}+1, , ..., n_j\}$ are given by applying the Milstein discretization scheme to the SDE Eq. (1), for $j = 1, ..., J$

$$\widehat{r}^{(j)}_{h+1} = \widehat{r}^{(j)}_h + \hat{k}_j(\hat{\theta}_j - \widehat{r}^{(j)}_h)\,\Delta + \hat{\sigma}_j\sqrt{\widehat{r}^{(j)}_h\,\Delta}\,\widehat{Z}^{(j)}_h + \frac{(\hat{\sigma}_j)^2}{4}\,[(\sqrt{\Delta}\,\widehat{Z}^{(j)}_h)^2 - \Delta], \qquad (2)$$

where $\Delta = 1/30$,

$$\hat{\theta}_j = \frac{1}{n_j}\sum_{h=n_{j-1}+1,}^{n_j} r^{(j)}_h, \qquad \hat{\sigma}_j = \sqrt{\frac{\sum_{h=n_{j-1}+1,}^{n_j}(r^{(j)}_h - \hat{\theta}_j)^2}{n_j - 1}},$$

$$\widehat{k}_j = \min_{k>0} S_j(k) = \min_{k>0}\sqrt{\frac{\sum_{h=n_{j-1}+1,}^{n_j}(u^{(j)}_h(k))^2}{n_j - 1}},$$

with $u^{(j)}(k) = \{\mathfrak{r}^{(j)}_h(k) - r^{(j)}_h \mid h = n_{j-1}+1, , ..., n_j,\ k > 0\}$ and $\mathfrak{r}^{(j)}_h : \mathbb{R} \to \mathbb{R}$ such that

$$\mathfrak{r}^{(j)}_{h+1}(k) = \mathfrak{r}^{(j)}_h(k) + k(\hat{\theta}_{(j)} - \mathfrak{r}^{(j)}_h)\,\Delta + \hat{\sigma}_j\sqrt{\mathfrak{r}^{(j)}_h\Delta}\,\widehat{Z}^{(j)}_h + \frac{(\hat{\sigma}_j)^2}{4}\,[(\sqrt{\Delta}\,\widehat{Z}^{(j)}_h)^2 - \Delta]. \qquad (3)$$

Note that the elements $\widehat{Z}^{(j)}_h$'s in Eq. (2) and (3) are the Gaussian standardized residuals of an "optimal" ARIMA model suitable chosen as follows. Consider the following set, for $j = 1, ..., J$

$$\{Z^{(j)}_h = f((r^{(j)}_{h+1} - \widehat{r}^{(j)}_{ARIMA,h+1}(p_j, i_j, q_j) - \mu_j)/\eta_j) \mid h = n_{j-1}+1, , ..., n_j,\ (p_j, i_j, q_j) \in \mathcal{I}_{AC}\},$$

where $f : \mathbb{R} \to \mathbb{R}$ represents the Johnson transformation (1949) , $\widehat{r}^{(j)}_{ARIMA,h+1}(p_j, i_j, q_j)$ is the estimate of $r^{(j)}_{h+1}$ by a ARIMA(p_j, i_j, q_j) from a set $\mathcal{I}_{AC}$ of candidate models satisfying some conditions (see Orlando et al. (2019b, Section 4.4.1)), and μ_j, η_j are respectively the mean and the standard deviation of the sample $\{r^{(j)}_h -$

[2] For convenience we use the subscript index j for the constants and the superscript index j for the arrays.

$\widehat{r}^{(j)}_{ARIMA,h}(p_j, i_j, q_j) \mid h = n_{j-1}+1, , ..., n_j\}$. The "optimal" ARIMA$(\widehat{p}_j, \widehat{i}_j, \widehat{q}_j)$ model for the $jth-$ subgroup is then chosen in the set $\mathfrak{I}_{AC}$ minimizing the error

$$\min_{\widehat{r}^{(j)}} \varepsilon_j = \min_{\widehat{r}^{(j)}} \sqrt{\frac{1}{n_j} \sum_{h=n_{j-1}+1}^{n_j} (r_h^{(j)} - \widehat{r}_h^{(j)})^2} \tag{4}$$

whith respect to all the samples of fitted values $\widehat{r}^{(j)} = \{\widehat{r}_h^{(j)} \mid h = n_{j-1}+1, , ..., n_j\}$ computed by Eq. (2). Thus the $\widehat{Z}_h^j$'s are the residuals of the ARIMA$(\widehat{p}_j, \widehat{i}_j, \widehat{q}_j)$. They replace the realizations of a standard Brownian motion allowing us to get an exact trajectory of CIR fitted values instead of a curve averaged over 100,000 simulated trajectories. Consequently, the computational cost is considerably reduced.

To forecast the next interest rates, we have to first calibrate the model, i.e. the six parameters $(k, \theta, \sigma, p, i, q)$ as described above, on a fixed rolling window w of length m of historical data, say $w = \{r_h, ..., r_{h+m-1}\}$, $h \geq 1$, and then the future interest rate value r^F_{h+m+s}, $s \geq 0$, can be computed by the steps described in Orlando et al. (2019b).

3.2.3. The Hull-White model

The Hull-White (HW) model (1990) is one of the most popular one-factor model in financial literature, under which the dynamics of the short interest rate $r(t)$ is given by

$$dr(t) = (\theta(t) - \alpha r(t))dt + \sigma dW(t). \tag{5}$$

In Eq. (5), α is the strength of the mean reversion, σ represents the local volatility and $\theta(t)$ is a function of time. Generally, $\theta(t)$ describes the long-mean rate trend, so it is typically chosen to match the term structure of zero-coupon bond prices (or equivalently, the yields). By virtue of the innovation term $\theta(t)$, the Hull-White model can be considered as an extension of the Vasicek model.
It is easy to see the solution of Eq. (5) is

$$r(t) = e^{-\alpha t} r(0) + \int_0^t e^{\alpha(u-t)} \theta(u)\, du + \sigma e^{-\alpha t} \int_0^t e^{\alpha u}\, dW(u), \tag{6}$$

where

$$r(t) \sim N\biggl(e^{-\alpha t} r(0) + \int_0^t e^{\alpha(u-t)} \theta(u)\, du, \frac{\sigma^2}{2\alpha}\bigl(1 - e^{2\alpha t}\bigr)\biggr). \tag{7}$$

In order to calibrate this model and use it to make forecasting, we choose $\theta(u)$ like the EWMA time series. Then we approximate the integral in Eq. (7) by the trapezoidal rule, i.e.,

$$\int_0^t e^{\alpha(u-t)}\theta(u)\,du \simeq \frac{1}{2}\sum_{i=1}^{N-1}\left(e^{\alpha(u_i-t)}\theta(u_i) + e^{\alpha(u_{i+1}-t)}\theta(u_{i+1})\right)(u_{i+1}-u_i),$$

with $0 = u_1 \leq u_2 \leq ... \leq u_N = t$. Observe that $\theta(u_N)$, which is unknown at time u_N, can be given by the EWMA prediction. The other parameters α, σ are obtained by solving the following optimization problem

$$\min_{(\alpha,\sigma)} \sum_i \left(r_i - r_i^{HW}(\alpha,\sigma)\right)^2,$$

where r_i, r_i^{HW} denote the market interest rates and the Hull-White interest rates (given by Eq. (6)), respectively.
Finally, we predict the future value $r(t)$ by the (conditioning) expected value (see Eq. (7))

$$\mathbb{E}[r(t)|r(s)] = e^{-\alpha(t-s)}r(s) + \int_s^t e^{\alpha(u-t)}\theta(u)\,du, \qquad 0 \leq s < t. \tag{8}$$

3.3. Calibration

Among many approaches existing in the literature to estimate the parameters of the CIR model (see Laurini et al. (2017) and references therein), we considered the MATLAB implementation of the maximum likelihood (ML) estimation method for the CIR process proposed by Kladìviko (2007), and the estimating function approach for ergodic diffusion models introduced in Bibby et al. (2010). As shown in Orlando et al. (2019a), the latter method has turned out to be very useful in obtaining optimal estimators for the parameters of discretely sampled diffusion-type models whose likelihood function is usually not explicitly known.

3.4. Forecasting accuracy

In order to check whether our forecasts are closely matching the data, we use a measure of the amplitude of the error and an indicator of the direction of the interest rate dynamics. The first is quantified by the root mean squared error (RMSE) and its related normalized version (NRMSE), while the second is given by the so-called directional “success” criterion.

3.4.1. Root mean square error

The root mean squared error (RMSE) is a measure of the closeness between the observed data and the predicted values from a given model. Hence, it represents the accuracy of the model in terms of goodness of fit. It is defined as follows

$$RMSE = \sqrt{\frac{1}{n}\sum_{h=1}^{n} e_h^2}, \tag{9}$$

where e_h denote the residuals between the observed data and their predictions, over n times. Hence, a value of 0 (almost never achieved in practice) indicates a perfect fit to the data, and a value lower than 1 represents a good result. We note that the RMSE depends on the scale of observed data, thus it is sensitive to the outliers; however, larger errors have a disproportionately large effect. For that reason, and to facilitate the comparison between data, we adopt the so-called normalized root mean squared error (NRMSE):

$$NRMSE = \frac{RMSE}{r_{\max} - r_{\min}}, \tag{10}$$

where $r_{\max}$ denotes the maximum value and $r_{\min}$ the minimum value of the observed sample data.

3.4.2. Directionality of forecasting

In order to understand whether the forecast predicts correctly a rise or a drop of interest rates, we introduce the index of directionality (IDX). Let us denote r_t as the interest rate at time t and the corresponding forecast as r_t^f. We define the variable $\alpha_{t+1} := r_{t+1} - r_t$ as the difference between two consecutive interest rates, and the variable $\beta_{t+1} := r_{t+1}^f - r_t$ as the difference between the forecast at time $t+1$ and the actual interest rates at time t. Further, we consider the indicator variable $H(t+1)$ assuming only the values 0, 1 as follows

$$\begin{cases} H(t+1) = 1 & \text{if } sgn(\alpha_{t+1}) = sgn(\beta_{t+1}) \\ H(t+1) = 0 & \text{if } sgn(\alpha_{t+1}) \neq sgn(\beta_{t+1}). \end{cases}$$

We attribute the term of forecast "success" (in sign) when $H(t+1) = 1$. Therefore the index IDX is defined as an average of the $H(t+1)$ values on the number of forecasts over a time series of length T that is,

$$\text{IDX} := \frac{1}{T-1}\sum_{t=1}^{T-1} H(t+1). \tag{11}$$

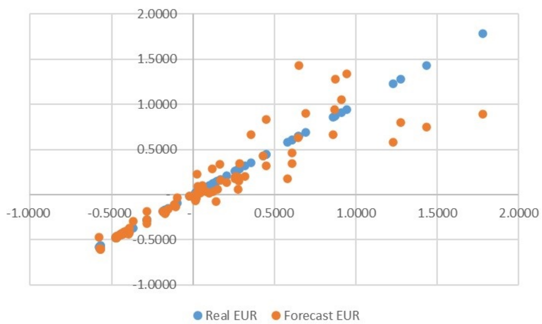
IDX indicates the percentage of correct predictions of interest rate directionality.

4. Results

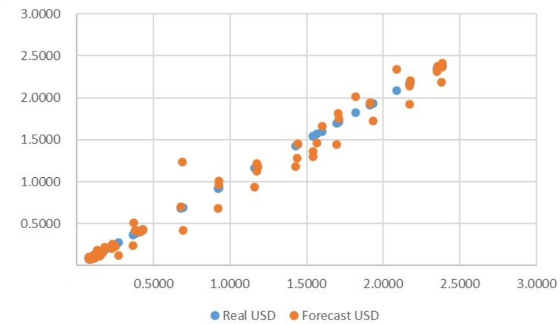
In this section firstly we are going to show the results over the full dataset and, secondly, we will focus on turbulent periods where volatility is above the median.

4.1. Forecasting results over the whole dataset

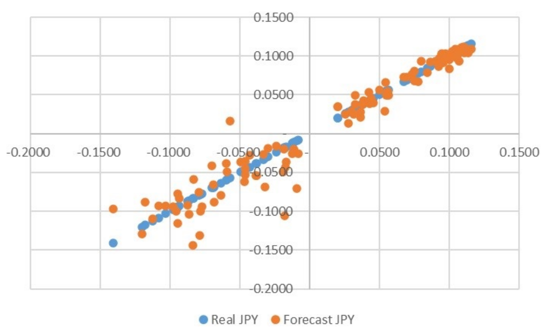
Let us start by plotting the performances versus the forecasts for all the four considered currencies. In Figure 1 we show that the forecasts closely follow the occurrences and they are not spread out.



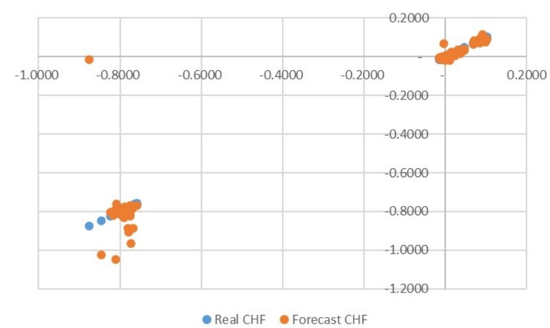
(a) EUR overnight. Plot of the realized interest rate occurrence versus the forecasted value.



(b) USD overnight. Plot of the realized interest rate occurrence versus the forecasted value.



(c) JPY overnight. Plot of the realized interest rate occurrence versus the forecasted value.



(d) CHF overnight. Plot of the realized interest rate occurrence versus the forecasted value.

Figure 1: Multiple comparisons for the overnight interest rate occurrences across currencies versus their corresponding forecasts.

We supplement this visual analysis by considering the Bland–Altman plot (also called difference plot) which is a very popular tool in medicine and chemistry to analyse the agreement between two different methods Altman and Bland (1983), Bland and Altman (1986). What inspired Bland and Altman was the need to know how much the outcomes of a method may differ from another in order to be considered equivalent. In our case, as the real data overlap with forecasts, we want to investigate the said agreement more clearly. Figure 2 shows the agreement between real data and forecasts (see Rik (2020)). Based on this analysis, it is possible to confirm the good agreement between data and forecasts as the maximum number of outliers are 0.885% (=9/113).

After having concluded the graphical analysis, we introduce the results based on the more traditional analysis consisting of the results obtained with the Normalized Root Mean Square Error (NRMSE). In Tables 3, 4, 5 we compare the forecasting error (NRMSE) and the directionality of forecasting (IDX) for the EWMA, CIR_{adj}, Hull and White and CIR#. We have included the EWMA because it is a basic version of the Autoregressive Conditional Heteroscedasticity (ARCH) model, which is a common tool for forecasting time-varying financial data and a simple benchmark whenever no structure in data is assumed. The results show that, generally, the CIR# performs better over the whole dataset.

Table 3: Averaged NRMSE and IDX for the dataset in Table 1.

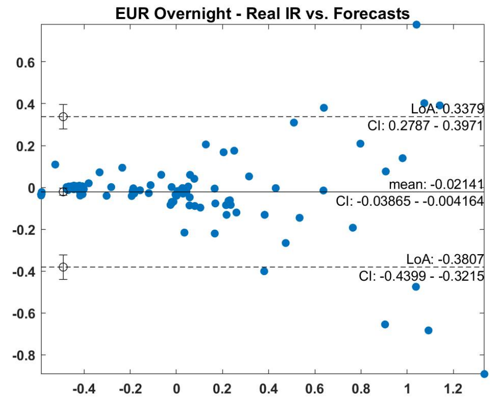
	EUR				USD				JPY				CHF			
	CIR#	**CIR_{adj}**	**EWMA**	**HW**	**CIR#**	**CIR_{adj}**	**EWMA**	**HW**	**CIR#**	**CIR_{adj}**	**EWMA**	**HW**	**CIR#**	**CIR_{adj}**	**EWMA**	**HW**
NRMSE	3.47%	5.57%	11.41%	9.40%	9.15%	13.60%	14.86%	14.04%	5.31%	10.56%	9.41%	14.74%	8.26%	24.06%	11.54%	16.51%
IDX	71.11%	65.60%	28.85%	42.36%	62.77%	63.93%	38.55%	52.55%	75.62%	70.39%	41.54%	36.15%	53.38%	63.12%	57.51%	46.60%

Table 4: NRMSE for different models, tenors and currencies

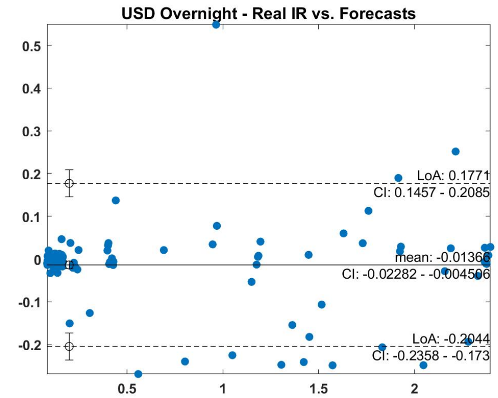
	EUR					
	EE00O/N	EE0001M	EE0002M	EE0003M	EE0006M	EE0012M
CIR #	5.14%	3.63%	3.56%	3.05%	2.71%	2.77%
CIR$_{adj}$	10.94%	5.39%	4.87%	4.39%	4.18%	3.64%
EWMA	11.54%	11.77%	11.61%	11.50%	11.11%	10.95%
HW	9.95%	8.63%	8.62%	8.86%	9.74%	10.60%
	USD					
	US00O/N	US0001M	US0002M	US0003M	US0006M	US0012M
CIR#	7.88%	8.71%	8.53%	14.06%	7.83%	7.90%
CIR$_{adj}$	10.98%	28.15%	16.48%	10.98%	7.60%	7.75%
EWMA	15.69%	14.85%	15.07%	15.69%	14.32%	13.55%
HW	12.89%	16.20%	16.62%	12.89%	15.49%	10.16%
	JPY					
	JY00S/N	JY0001M	JY0002M	JY0003M	JY0006M	JY0012M
CIR#	8.66%	5.20%	5.37%	4.85%	3.92%	3.88%
CIR$_{adj}$	13.55%	11.65%	9.26%	16.64%	6.82%	5.47%
EWMA	9.46%	9.82%	9.91%	9.60%	9.11%	8.56%
HW	13.74%	15.69%	16.09%	16.46%	15.72%	10.74%
	CHF					
	CH00S/N	CH0001M	CH0002M	CH0003M	CH0006M	CH0012M
CIR#	9.29%	9.19%	8.53%	8.51%	7.51%	6.56%
CIR$_{adj}$	8.96%	16.80%	14.15%	12.59%	65.21%	26.73%
EWMA	12.47%	12.23%	12.11%	12.00%	10.95%	9.52%
HW	17.64%	17.03%	16.68%	16.54%	17.05%	14.14%

Table 5: IDX for different models, tenors and currencies

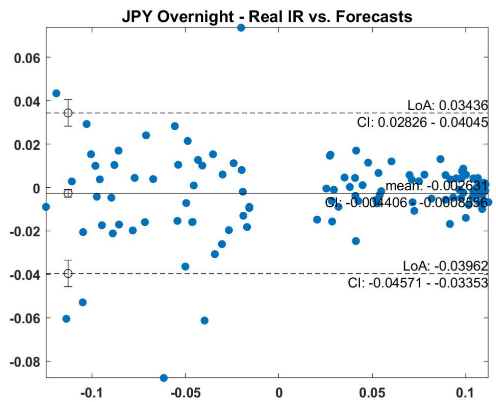
	EUR					
	EE00O/N	EE0001M	EE0002M	EE0003M	EE0006M	EE0012M
CIR #	66.67%	71.67%	75.00%	75.00%	68.33%	70.00%
CIR$_{adj}$	53.73%	68.65%	65.67%	68.65%	65.67%	71.64%
EWMA	38.80%	28.35%	25.37%	29.85%	25.37%	25.37%
HW	47.45%	54.23%	50.84%	49.15%	30.50%	22.03%
	USD					
	US00O/N	US0001M	US0002M	US0003M	US0006M	US0012M
CIR #	60.00%	58.33%	58.33%	66.67%	60.00%	73.33%
CIR$_{adj}$	73.13%	49.25%	50.74%	73.13%	71.64%	65.67%
EWMA	47.76%	40.29%	32.83%	47.76%	26.86%	35.82%
HW	61.01%	50.84%	47.45%	61.10%	45.76%	49.15%
	JPY					
	JY00S/N	JY0001M	JY0002M	JY0003M	JY0006M	JY0012M
CIR #	67.16%	71.64%	74.62%	70.14%	86.56%	83.58%
CIR$_{adj}$	64.17%	77.61%	65.67%	65.67%	77.61%	71.64%
EWMA	56.71%	38.80%	47.76%	47.76%	26.86%	31.34%
HW	45.76%	32.20%	40.67%	42.37%	27.11%	28.83%
	CHF					
	CH00S/N	CH0001M	JY0002M	CH0003M	CH0006M	CH0012M
CIR #	53.09%	53.09%	57.52%	52.21%	51.32%	53.09%
CIR$_{adj}$	69.02%	66.37%	61.06%	67.25%	59.29%	55.75%
EWMA	61.06%	56.63%	61.06%	61.06%	52.21%	53.09%
HW	52.54%	50.84%	49.12%	49.15%	37.28%	40.67%



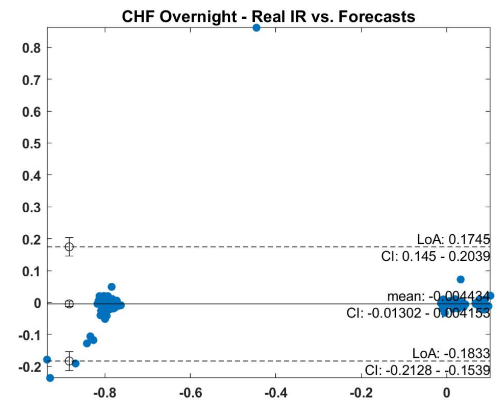
(a) EUR overnight. Over 113 data points outliers are 9 (out of which only 5 outside LoA confidence level).



(b) USD overnight. Over 113 data points outliers are 9 (out of which only 6 outside LoA confidence level).



(c) JPY overnight. Over 113 data points outliers are 5 (all of them outside LoA confidence level).



(d) CHF overnight. Over 113 data points outliers are 2 (none of them outside LoA confidence level).

Figure 2: Bland Altman plot. Multiple comparisons for the overnight interest rates occurrences across currencies versus their corresponding forecasts. Maximum number of outliers are 0.885%.

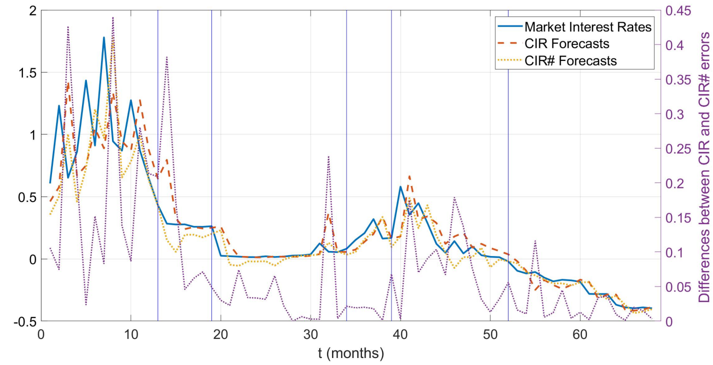
4.2. Forecasting results in turbulent periods

As described in Section 3.2.3, through our procedure we can identify clusters of volatility. In Section 4.1 we have shown the averaged performance over the whole dataset (Table 3) and for each currency (Tables 4, 5). In this Section we illustrate how the CIR# model performs when volatility is high and forecasts are more challenging. This corresponds to considering the overnight (as it is the most exposed to market sentiment), which we have partitioned into clusters of volatility. Then, for each currency, we have selected the two clusters with higher volatility. Table 6 compares the forecasting error and the index of directionality for the EWMA, CIR$_{adj}$, CIR#

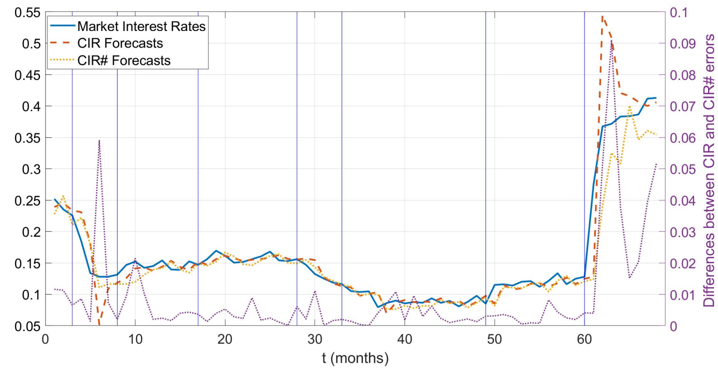
and Hull and White. Note that volatility ratio shows the volatility of the cluster over the median volatility recorded on the whole dataset for the selected currency. As displayed the CIR# performs better in any situation. To understand how the CIR# fits the overnight, for each currency, in Fig. 3 we display the evolution of market interest rates, the partitioning into clusters (vertical bars) and the CIR_{adj} taken as benchmark. The difference in absolute value between the prediction errors of CIR# and CIR_{adj} is also shown.

Table 6: NRMSE and IDX in turbulent periods for the CIR#, CIR_{adj} and EWMA over overnight maturities and different currencies

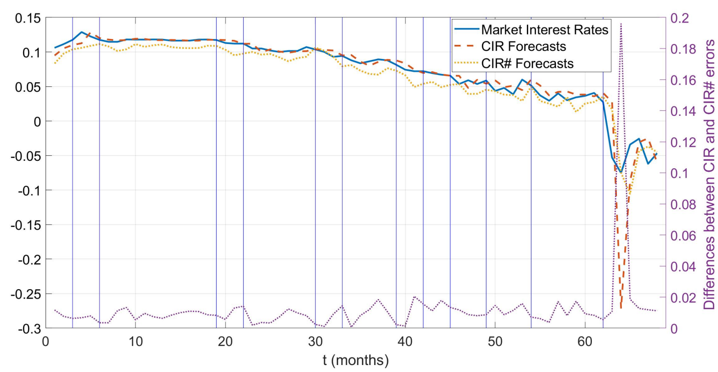
	EUR		USD		JPY		CHF	
Cluster	1-13	39-52	3-8	28-33	49-54	62-68	12-19	29-60
Volatility ratio	$\frac{7.63\%}{10.00\%}$	$\frac{18.33\%}{10.00\%}$	$\frac{4.07\%}{1.26\%}$	$\frac{1.60\%}{1.26\%}$	$\frac{0.83\%}{0.49\%}$	$\frac{3.35\%}{0.49\%}$	$\frac{3.5\%}{1.22\%}$	$\frac{23.63\%}{1.22\%}$
NRMSE CIR#	59.58%	24.80%	27.90%	16.10%	50.00%	38.46%	54.31%	16.73%
NRMSE CIR_{adj}	59.58%	26.50%	43.50%	25.50%	67.10%	81.00%	67.46%	27.80%
NRMSE EWMA	54.50%	27.32%	61.40%	49.2%	56.30%	52.80%	98.12%	26.78%
NRMSE HW	45.65%	29.89%	59.39%	47.08%	79.78%	51.56%	38.60%	23.49%
IDX CIR#	36.30%	76.9%	100.00%	100.00%	80.00%	83.30%	66.67%	66.57%
IDX CIR_{adj}	35.30%	53.84%	80.00%	80.00%	60.00%	83.30%	66.65%	64.28%
IDX EWMA	36.60%	30.70%	40.00%	20.00%	60.00%	66.66%	64.28%	51.42%
IDX HW	50.00%	46.15%	40.00%	20.00%	60.00%	50.00%	71.42%	64.51%



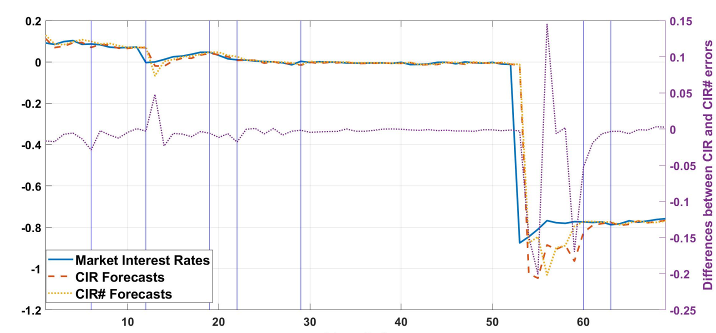
(a) EUR overnight. On the left y-axis EUR interest rates. On the right y-axis the difference in absolute value between the prediction errors of CIR_{adj} and CIR#.



(b) USD overnight. On the left y-axis USD interest rates. On the right y-axis the difference in absolute value between the prediction errors of CIR_{adj} and CIR#.



(c) JPY overnight. On the left y-axis JPY interest rates. On the right y-axis the difference in absolute value between the prediction errors of CIR_{adj} and CIR#.



(d) CHF overnight. On the left y-axis CHF interest rates. On the right y-axis the difference in absolute value between the prediction errors of CIR_{adj} and CIR#.

Figure 3: Multiple comparisons for the overnight interest rates across currencies. Vertical lines identify cluster of volatilities partitioning the sample data in subgroups.

4.3. Correlations between forecasts and real data

In this Section, in Tables 7, 9, 11 and 13, we report the correlation results obtained with Kendall rank correlation between real data and forecasts. The alternative hypothesis, against which p-values are computed, is of no correlation (see Tables 8, 10, 12 and 14). Notice that we obtained similar results with Spearman's rank correlation but for reason of space we do not report them.

Table 7: EUR Kendall correlations

		Real time series					
		EUR1	EUR2	EUR3	EUR4	EUR5	EUR6
Forecasts	ForEUR1	0.898198	0.864369	0.840025	0.828644	0.847929	0.847613
	ForEUR2	0.876561	0.888257	0.881618	0.882883	0.889521	0.884780
	ForEUR3	0.830896	0.862203	0.915962	0.925449	0.914064	0.910902
	ForEUR4	0.812871	0.844494	0.909321	0.927979	0.912167	0.908689
	ForEUR5	0.840341	0.858046	0.887765	0.896617	0.932343	0.942144
	ForEUR6	0.816817	0.842105	0.885096	0.891102	0.930299	0.945788

Table 8: EUR Kendall p-values

		Real time series					
		EUR1	EUR2	EUR3	EUR4	EUR5	EUR6
Forecasts	ForEUR1	3.99E-45	6.41E-42	1.09E-39	1.15E-38	2.09E-40	2.24E-40
	ForEUR2	4.50E-43	3.49E-44	1.50E-43	1.13E-43	2.64E-44	7.49E-44
	ForEUR3	7.52E-39	1.06E-41	7.74E-47	8.92E-48	1.19E-46	2.43E-46
	ForEUR4	2.95E-37	4.48E-40	3.47E-46	5.00E-48	1.83E-46	4.00E-46
	ForEUR5	1.02E-39	2.47E-41	4.01E-44	5.67E-45	1.74E-48	1.79E-49
	ForEUR6	1.25E-37	6.90E-40	6.99E-44	1.86E-44	2.70E-48	7.41E-50

Table 9: USD Kendall correlations

		Real time series					
		USD1	USD2	USD3	USD4	USD5	USD6
Forecasts	ForUSD1	0.888994	0.876977	0.832068	0.733397	0.807084	0.794434
	ForUSD2	0.870514	0.913202	0.886957	0.745613	0.859447	0.843320
	ForUSD3	0.835151	0.886992	0.915758	0.761814	0.891418	0.880354
	ForUSD4	0.797123	0.849917	0.893543	0.762981	0.901446	0.894491
	ForUSD5	0.797724	0.853034	0.892541	0.764855	0.916245	0.914981
	ForUSD6	0.787515	0.839984	0.879494	0.750849	0.909522	0.922797

Table 10: USD Kendall p-values

		Real time series					
		USD1	USD2	USD3	USD4	USD5	USD6
Forecasts	ForUSD1	3.24E-44	4.48E-43	5.98E-39	1.23E-30	9.54E-37	1.17E-35
	ForUSD2	1.76E-42	1.42E-46	4.93E-44	1.28E-31	1.89E-41	5.65E-40
	ForUSD3	2.93E-39	4.61E-44	7.50E-47	5.93E-33	1.74E-44	1.97E-43
	ForUSD4	6.51E-36	1.36E-40	1.10E-44	4.81E-33	1.91E-45	8.94E-45
	ForUSD5	5.57E-36	6.77E-41	1.32E-44	3.26E-33	6.50E-47	8.66E-47
	ForUSD6	4.20E-35	1.06E-39	2.34E-43	4.59E-32	3.02E-46	1.49E-47

Table 11: JPY Kendall correlations

		Real time series					
		JPY1	JPY2	JPY3	JPY4	JPY5	JPY6
Forecasts	ForJPY1	0.850634	0.837342	0.823735	0.810033	0.781963	0.772469
	ForJPY2	0.858410	0.863476	0.872340	0.859270	0.811873	0.784963
	ForJPY3	0.859633	0.838415	0.865016	0.861759	0.809597	0.786796
	ForJPY4	0.850535	0.861313	0.872725	0.885000	0.828978	0.796960
	ForJPY5	0.776996	0.783958	0.785540	0.797469	0.861483	0.862749
	ForJPY6	0.782279	0.772785	0.785760	0.788829	0.864241	0.886393

Table 12: JPY Kendall p-values

		Real time series					
		JPY1	JPY2	JPY3	JPY4	JPY5	JPY6
Forecasts	ForJPY1	1.40E-40	2.24E-39	3.66E-38	5.91E-37	1.47E-34	9.13E-34
	ForJPY2	2.87E-41	9.78E-42	1.46E-42	2.43E-41	4.27E-37	8.66E-35
	ForJPY3	2.35E-41	2.01E-39	7.48E-42	1.52E-41	7.10E-37	6.38E-35
	ForJPY4	1.88E-40	1.94E-41	1.69E-42	1.20E-43	1.63E-38	1.00E-35
	ForJPY5	3.79E-34	9.86E-35	7.25E-35	7.09E-36	1.38E-41	1.06E-41
	ForJPY6	1.38E-34	8.57E-34	7.02E-35	3.90E-35	7.77E-42	6.44E-44

Table 13: CHF Kendall correlations

		Real time series					
		CHF1	CHF2	CHF3	CHF4	CHF5	CHF6
Forecasts	ForCHF1	0.659276	0.604152	0.549661	0.551562	0.569303	0.591796
	ForCHF2	0.691701	0.673323	0.667619	0.668887	0.660965	0.692017
	ForCHF3	0.690049	0.668470	0.744631	0.735111	0.721783	0.726226
	ForCHF4	0.650816	0.650182	0.725014	0.758942	0.705037	0.757356
	ForCHF5	0.695040	0.673840	0.700103	0.723203	0.744087	0.743771
	ForCHF6	0.644845	0.675838	0.689121	0.732764	0.716003	0.814042

Table 14: CHF Kendall p-values

		Real time series					
		CHF1	CHF2	CHF3	CHF4	CHF5	CHF6
Forecasts	ForCHF1	5.38E-25	3.01E-21	7.39E-18	5.70E-18	4.82E-19	1.88E-20
	ForCHF2	2.42E-27	5.38E-26	1.39E-25	1.12E-25	4.14E-25	2.29E-27
	ForCHF3	3.77E-27	1.40E-25	2.42E-31	1.37E-30	1.50E-29	6.80E-30
	ForCHF4	2.31E-24	2.55E-24	7.70E-30	1.54E-32	2.62E-28	2.08E-32
	ForCHF5	1.17E-27	4.28E-26	4.88E-28	8.29E-30	1.86E-31	1.97E-31
	ForCHF6	4.66E-24	2.89E-26	3.04E-27	1.38E-30	2.80E-29	2.36E-37

5. Testing and validation

In this Section the basic question we want to answer is how our forecasts differ from: a) original time series, b) purely random data and c) noise? For reason of space we show the analysis on the EUR Overnight but we got similar results for all considered time series.

5.1. Results on test data

This analysis is carried out with R on data described in Section 3.1.2 and the purpose of creating that data is to get an answer to the following question: is the analysis we intend to run valid and consistent? If yes, time series 1 (EUR1) and 5 (a copy of EUR1) should be identical while time series 1 and 2 (random) should be unrelated. After having passed that check, next question is: does time series 6 (EUR1 forecast) look similar to time series 1 or it does resemble more 2 or 3 (EUR1 + noise)?

Apart from the classical instruments described in Section 3.4 for testing the goodness of fit, and the correlation in Section 4.3, we measure the distance between data with several metrics. We start with the default Euclidean distance and complete clustering (see Warnes and al. (2020)). Figure 4 shows that there is no difference between EUR1 and its copy and that the next similarity is with the forecasts. Noise and random series are correctly grouped together and the changed sign EUR1 is left alone.

Now we want to understand whether the metric or the clustering criterion may influence the results. In particular we adopt the Manhattan distance and we complement our clustering analysis with Ward's criterion (also known as Ward's minimum variance method) by Ward Jr (1963), Everitt et al. (2001).

As expected Figure 5 shows that the EUR1 times series and its copy are identical (which confirms that the analysis is able to correctly identify this feature). The forecasts follow immediately after. Noise and random are recognized as similar and they group with the EUR1 changed sign time series.

A generalization of both the Euclidean and the Manhattan distance is the Minkowski distance (see Giusti and Batista (2013), Batyrshin (2013), Shirkhorshidi et al. (2015)). Fig. 6 shows similar results to those already presented. Hence, we can conclude that our model both spatially and hierarchically is very close to reality and, so, fits well with the intended purposes.

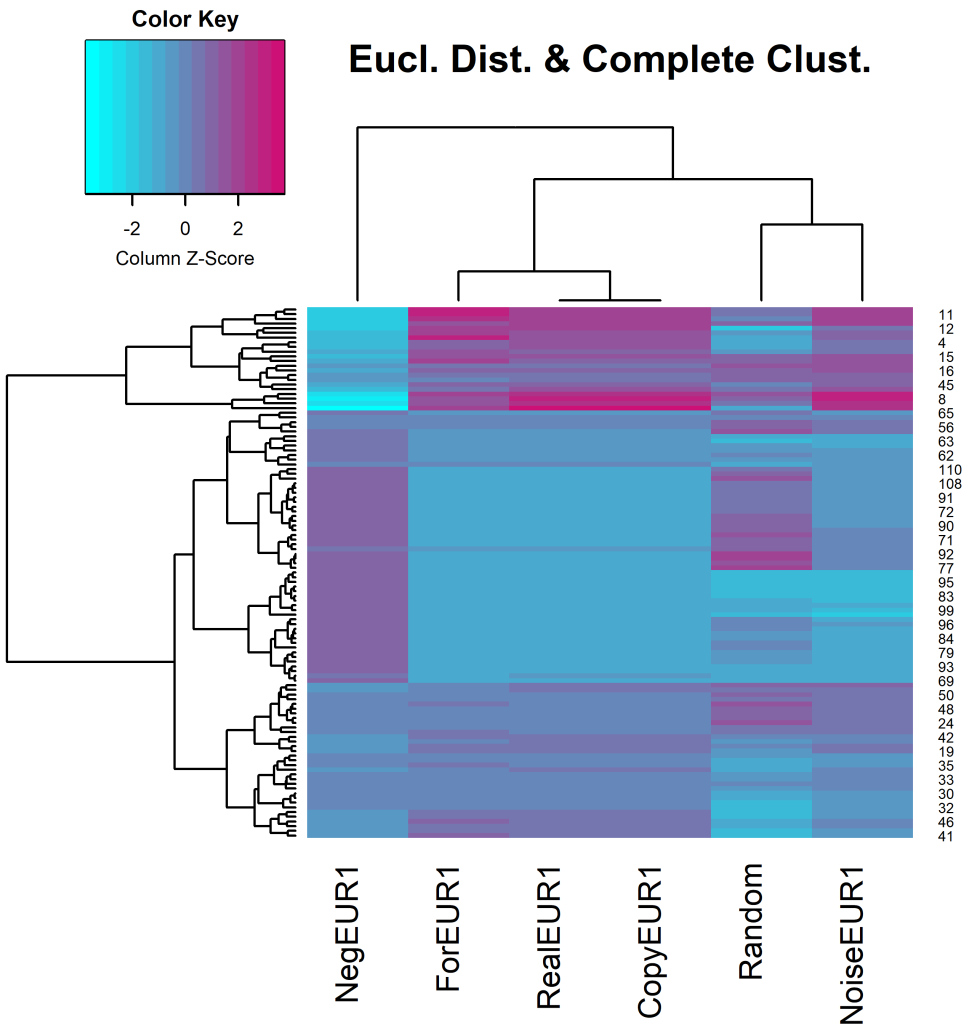


Figure 4: EUR overnight and test data. Euclidean distance and dendrogram based on complete clustering criterion.

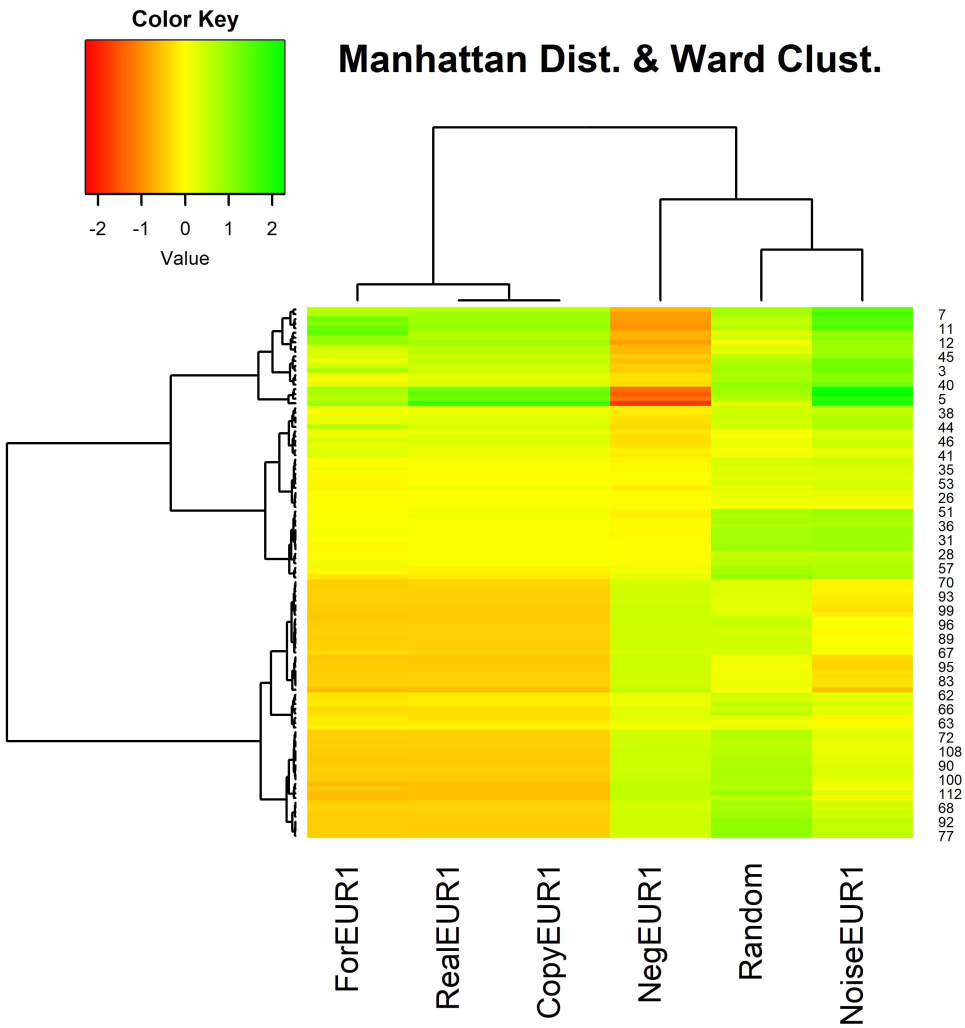


Figure 5: EUR overnight and test data. Manhattan distance and dendrogram based on Ward's criterion.

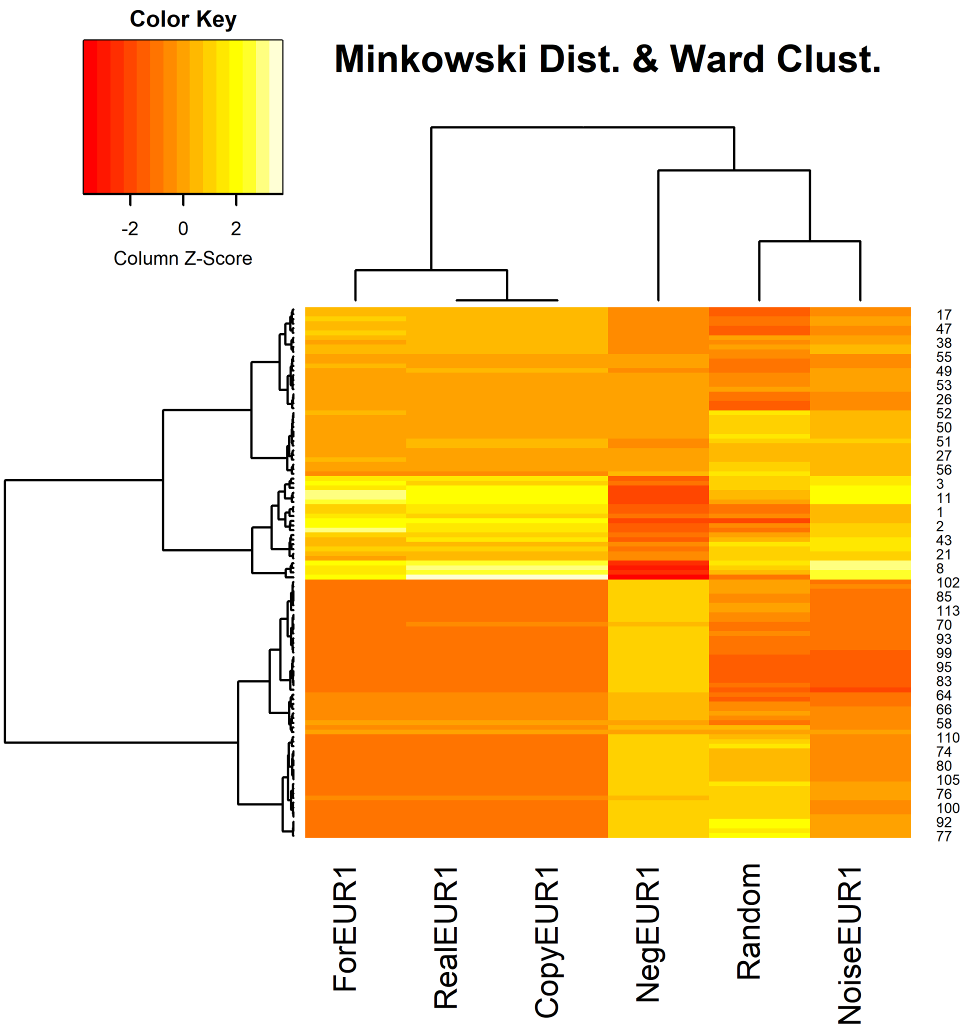


Figure 6: EUR overnight and test data. Minkowski distance and dendrogram based on Ward's criterion.

6. Conclusions

Forecasting interest rates is important for both investment and risk management reasons. To this end we adopted the Cox, Ingersoll & Ross framework that was initially proposed for pricing and short-rates modelling. Several different extensions of the original model have been proposed to date, with the aim of overcoming the limitations of the CIR model: from one-factor models including time-varying coefficients or jump diffusions to multi-factor models. All these extensions preserve the positivity of interest rates but, in some cases, the analytical tractability of the basic model is violated. Moreover, often, the said extensions are not suited for modelling both periods of high volatility and negative rates. In this work we have shown that the CIR# model, instead, while preserving those features, is capable of coping with negative interest rates, cluster volatility and jumps. This has been tested on money market interest rates during turmoil and calmer periods, by measuring the directionality of rates as well as the forecasting error. Rank correlation and related p-values witness a link between real data and out-of-sample forecasts. Besides that, we have demonstrated how the model's results could be compared with real data with the Bland–Altman plot. Hierarchical clustering with different metrics is also suggested for testing and validation.

7. Data availability

The data that support the findings of this study are available from the corresponding author upon reasonable request.

References

Altman, D. G. and Bland, J. M. (1983). Measurement in medicine: the analysis of method comparison studies. *Journal of the Royal Statistical Society: Series D (The Statistician)*, 32(3):307–317.

Batyrshin, I. (2013). Constructing time series shape association measures: Minkowski distance and data standardization. In *2013 BRICS Congress on Computational Intelligence and 11th Brazilian Congress on Computational Intelligence*, pages 204–212. IEEE.

Bianchi, M. L. (2020). Are multi-factor Gaussian term structure models still useful? An empirical analysis on Italian BTPs. *Communications in Statistics-Simulation and Computation*, pages 1–29.

Bibby, B. M., Jacobsen, M., and Sørensen, M. (2010). Estimating functions for discretely sampled diffusion-type models. In Aït-Sahalia, Y. and Hansen, L. P., editors, *Handbook of Financial Econometrics: Tools and Techniques*, volume 1, pages 203–268. North-Holland, Oxford.

BIS (2015). Is the unthinkable becoming routine? Technical report, Bank for International Settlements.

Bland, J. M. and Altman, D. (1986). Statistical methods for assessing agreement between two methods of clinical measurement. *The Lancet*, 327(8476):307–310.

Brigo, D. and El-Bachir, N. (2006). Credit derivatives pricing with a smile-extended jump stochastic intensity model. *ICMA Centre Discussion Papers in Finance DP2006-13. Available at SSRN 950208.*

Brigo, D. and Mercurio, F. (2000). The CIR++ model and other deterministic-shift extensions of short rate models. In *Proceedings of the 4th Columbia-JAFEE conference for mathematical finance and financial engineering*, pages 563–584.

Brigo, D. and Mercurio, F. (2001). A deterministic-shift extension of analytically-tractable and time-homogeneous short rate models. *Finance and Stochastics*, 5:369–388.

Brigo, D. and Mercurio, F. (2006). *Interest rate models-theory and practice: with smile, inflation and credit*. Springer-Verlag, Berlin, Heidelberg.

Carmona, R. A. and Tehranchi, M. R. (2006). *Interest rate models: an infinite dimensional stochastic analysis perspective*. Springer-Verlag: Berlin, Heidelberg.

Chen, L. (1996). *Stochastic Mean and Stochastic Volatility: A Three-Factor Model of the Term Structure of Interest Rates and Its Applications and Its Applications in Derivatives Pricing and Risk Management*. Cambridge, MA :Blackwell Publishers.

Cox, J. C., Ingersoll, J. E., and Ross, S. A. (1985). A theory of the term structure of interest rates. *Econometrica*, 53:385–407.

Duffie, D. (2005). Credit risk modeling with affine processes. *Journal of Banking & Finance*, 29(11):2751–2802.

Engelen, K. C. (2015). The unthinkable as the new normal. *The International Economy*, 29 (Summer 2015)(3):30.

Everitt, B., Landau, S., and Leese, M. (2001). *Cluster Analysis*. Oxford University Press.

Giusti, R. and Batista, G. E. (2013). An empirical comparison of dissimilarity measures for time series classification. In *2013 Brazilian Conference on Intelligent Systems*, pages 82–88. IEEE.

Heston, S. L. (1993). A closed-form solution for options with stochastic volatility with applications to bond and currency options. *The Review of Financial Studies*, 6(2):327–343.

Hull, J. and White, A. (1990). Pricing interest-rate-derivative securities. *The Review of Financial Studies*, 3(4):573–592.

Johnson, N. L. (1949). Systems of frequency curves generated by methods of translation. *Biometrika*, 36(1/2):149–176.

Keller-Ressel, M. and Steiner, T. (2008). Yield curve shapes and the asymptotic short rate distribution in affine one-factor models. *Finance and Stochastics*, 12(2):149–172.

Kladıvko, K. (2007). Maximum likelihood estimation of the Cox-Ingersoll-Ross process: the Matlab implementation. `https://it.mathworks.com/matlabcentral/fileexchange/37297-maximum-likelihood-estimation\-of-the-cox-ingersoll-ross-process-the-matlab-implementation`.

Mininni, M., Orlando, G., and Taglialatela, G. (2020). Challenges in approximating the black and scholes call formula with hyperbolic tangents. *Decisions in Economics and Finance*, pages 1–28.

Moreno, M. and Platania, F. (2015). A cyclical square-root model for the term structure of interest rates. *European journal of operational research*, 241:109–121.

Najafi, A. R. and Mehrdoust, F. (2017). Bond pricing under mixed generalized CIR model with mixed Wishart volatility process. *J. Comput. Appl. Math.*, 319(C):108–116.

Orlando, G., Mininni, R. M., and Bufalo, M. (2018). A new approach to CIR short-term rates modelling. In Mili, M., Medina Samaniego, R., and Filippo, d. P., editors, *New Methods in Fixed Income Modeling - Fixed Income Modeling*, pages 35–44. Springer International.

Orlando, G., Mininni, R. M., and Bufalo, M. (2019a). Interest rates calibration with a CIR model. *The Journal of Risk Finance*, 20(4):370–387.

Orlando, G., Mininni, R. M., and Bufalo, M. (2019b). A new approach to forecast market interest rates through the CIR model. *Studies in Economics and Finance*, 37(2):267–292.

Orlando, G., Mininni, R. M., and Bufalo, M. (2020). Forecasting interest rates through Vasicek and CIR models: a partitioning approach. *Journal of Forecasting*.

Poletti Laurini, M. and Hotta, L. K. (2017). GMC-GEL estimation of stochastic volatility models. *Communications in Statistics - Simulation and Computation*, 46:6828–6844.

Rik (2020). Bland-Altman Plot. MATLAB Central File Exchange. Retrieved June 30, 2020.

Shirkhorshidi, A. S., Aghabozorgi, S., and Wah, T. Y. (2015). A comparison study on similarity and dissimilarity measures in clustering continuous data. *PloS one*, 10(12):e0144059.

Vasicek, O. (1977). An equilibrium characterization of the term structure. *Journal of financial economics*, 5(2):177–188.

Ward Jr, J. H. (1963). Hierarchical grouping to optimize an objective function. *Journal of the American statistical association*, 58(301):236–244.

Warnes, G. and al. (2020). R Enhanced Heat Map - gplots package. https://www.rdocumentation.org/packages/gplots/versions/3.0.4/topics/heatmap.2. Accessed: 2020-07-10.

Zhu, L. (2014). Limit theorems for a Cox-Ingersoll-Ross process with Hawkes jumps. *Journal of Applied Probability*, 51(3):699–712.