Symmetry Methods in Nonlinear Waves

A brief introduction

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Sophus Marius Lie

The Godfather of symmetry

- Born 1842 in Norway near Bergen.
- Learned about groups from Camille Jordan while traveling with Felix Klein.
- Devoted his entire life to developing and applying the theory of continuous groups, later named Lie groups in his honor by Hermann Weyl.

Lie Groups

Groups of transformations

Consider a domain $D \subseteq \mathbb{R}^N$ and a subset $S \subseteq \mathbb{R}$. The set of transformations,

$$\mathbf{x}^\star = \mathbf{X}(\mathbf{x}; \varepsilon), \quad \mathbf{X} : D \times S \to D$$

forms a **one-parameter group of transformations** on D if it satisfies the following axioms:

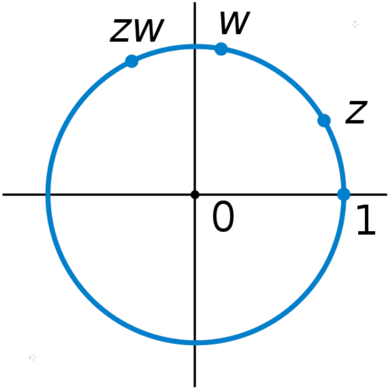
- The mapping $\mathbf{X}$ is a bijection.
- S with the law of composition $\phi(\varepsilon_1, \varepsilon_2)$ is a group with identity e.
- $\mathbf{X}(\mathbf{x}; e) = \mathbf{x}, \quad \forall x \in D.$
- $\mathbf{X}(\mathbf{X}(\mathbf{x}; \varepsilon_1); \varepsilon_2) = \mathbf{X}(\mathbf{x}; \phi(\varepsilon_1, \varepsilon_2)), \quad \forall \mathbf{x} \in D, \quad \forall \varepsilon_1, \varepsilon_2 \in S$

A group of transformations defines a **one-parameter Lie group of transformations** if it satisfies the following axioms:

- The parameter ε is continuous such that S is an interval in R.
- $\mathbf{X}$ is smooth with respect to $\mathbf{x}$ in D and an analytic function (power series) of ε in S.
- The composition function $\phi(\varepsilon_1, \varepsilon_2)$ is analytic (power series) with respect to ε_1 and ε_2 in S.

Lie Groups

Example



Rotations of the unit circle, usually denoted $SO(2)$, S^1, or $\mathbb{T}$.

Lie Groups

Infinitesimal Transformations

Recall from the definition of a one-parameter Lie group of transformations,

$$\mathbf{x}^\star = \mathbf{X}(\mathbf{x};\varepsilon).$$

Since $\mathbf{X}$ is analytic in the variable ε it is possible to take a Taylor series,

$$\mathbf{x}^\star = \mathbf{x} + \varepsilon \frac{\partial \mathbf{X}(\mathbf{x};\varepsilon)}{\partial \varepsilon}|_{\varepsilon=0} + \frac{\varepsilon^2}{2}\frac{\partial^2 \mathbf{X}(\mathbf{x};\varepsilon)}{(\partial \varepsilon)^2}|_{\varepsilon=0} + \ldots$$

By setting $\zeta(\mathbf{x}) = \frac{\partial \mathbf{X}(\mathbf{x};\varepsilon)}{\partial \varepsilon}|_{\varepsilon=0}$ one obtains the **infinitesimal transformation** of the Lie group of transformations,

$$\mathbf{x}^\star = \mathbf{x} + \varepsilon\zeta(\mathbf{x}).$$

Theorem: The Lie group of transformations is equivalent to the solution of the initial value problem for the system of first order differential equations

$$\frac{d\mathbf{x}^\star}{d\varepsilon} = \zeta\left(\mathbf{x}^\star\right), \quad \mathbf{x}^\star(0) = \mathbf{x}.$$

This result connects the essential information characterizing a one-parameter Lie group of transformations with the infinitesimal transformations.

Why study this stuff anyway?

Lie Groups

Relationship to nonlinear Partial Differential Equations

- Lie groups give us a way to characterize the symmetries of a partial differential equation in the same way ordinary groups characterize the symmetries of algebraic equations.
- Provide a cohesive, unified theory for all differential equations; ordinary, partial, linear, nonlinear, etc.
- If the only symmetries we cared about were scalings, translations, or rotations then we wouldn't need Lie groups but we are also interested in compositions of these things.
- Very versatile.
- Completely algorithmic in application but can result in hundreds of equations to handle.

Point Symmetries of PDEs

Point Transformations

Consider a partial differential equation with one dependent variable u and two independent variables, x and t. A **point transformation** is a diffeomorphism,

$$\Gamma : (x, t, u) \to (\tilde{x}(x, t, u), \tilde{t}(x, t, u), \tilde{u}(x, t, u))$$

which maps a solution surface $u(x, t)$ to another solution surface $\tilde{u}(\tilde{x}, \tilde{t})$.

Point Symmetries of PDEs

Prolongations

The surface $\tilde{u}(\tilde{x}, \tilde{t})$ is parametrized by x and t:

$$\tilde{x} = \tilde{x}(x, t, u(x, t)),$$
$$\tilde{t} = \tilde{t}(x, t, u(x, t)),$$
$$\tilde{u} = \tilde{u}(x, t, u(x, t)).$$

We can calculate the **prolongation** of the variables of the given point transformation Γ by differentiating these relations using the total derivative operators:

$$\mathfrak{D}_x = \partial_x + u_x \partial_u + u_{xx} \partial_{u_x} + u_{xt} \partial_{u_t} + \ldots,$$
$$\mathfrak{D}_t = \partial_t + u_t \partial_u + u_{tt} \partial_{u_t} + u_{tx} \partial_{u_x} + \ldots$$

Prolongations allow us to handle terms like $\tilde{u}_{\tilde{x}}, \tilde{u}_{\tilde{t}}$, etc.

The equations,

$$\tilde{x} = \tilde{x}(x, t, u(x, t)),$$
$$\tilde{t} = \tilde{t}(x, t, u(x, t)),$$

can be inverted locally to give x and t as functions of $\tilde{x}$ and $\tilde{t}$ as long as,

$$\mathcal{J} \equiv \det \begin{vmatrix} \mathfrak{D}_x \tilde{x} & \mathfrak{D}_x \tilde{t} \\ \mathfrak{D}_t \tilde{x} & \mathfrak{D}_t \tilde{t} \end{vmatrix} \neq 0.$$

Plugging these expressions into $\tilde{u}$ gives,

$$\tilde{u} = \tilde{u}\left(\tilde{x}, \tilde{t}\right).$$

Differentiating $\tilde{u}$ using the chain rule and total derivative operators gives,

$$\begin{bmatrix} \mathfrak{D}_x \tilde{u} \\ \mathfrak{D}_t \tilde{u} \end{bmatrix} = \begin{bmatrix} \mathfrak{D}_x \tilde{x} & \mathfrak{D}_x \tilde{t} \\ \mathfrak{D}_t \tilde{x} & \mathfrak{D}_t \tilde{t} \end{bmatrix} \begin{bmatrix} \tilde{u}_{\tilde{x}} \\ \tilde{u}_{\tilde{t}} \end{bmatrix}.$$

Using Cramer's rule, then, gives the first prolongations of Γ,

$$\tilde{u}_{\tilde{x}} = \frac{1}{\mathcal{J}} \det \begin{vmatrix} \mathfrak{D}_x \tilde{u} & \mathfrak{D}_x \tilde{t} \\ \mathfrak{D}_t \tilde{u} & \mathfrak{D}_t \tilde{t} \end{vmatrix}, \quad \tilde{u}_{\tilde{t}} = \frac{1}{\mathcal{J}} \det \begin{vmatrix} \mathfrak{D}_x \tilde{x} & \mathfrak{D}_x \tilde{u} \\ \mathfrak{D}_t \tilde{x} & \mathfrak{D}_t \tilde{u} \end{vmatrix}.$$

Point Symmetries of PDEs

Second Prolongations

Higher order prolongations are obtained recursively,

$$\tilde{u}_{\tilde{x}\tilde{x}} = \frac{1}{\mathcal{J}}\det\begin{vmatrix}\mathfrak{D}_x\tilde{u}_{\tilde{x}} & \mathfrak{D}_x\tilde{t}\\ \mathfrak{D}_t\tilde{u}_{\tilde{x}} & \mathfrak{D}_t\tilde{t}\end{vmatrix}, \quad \tilde{u}_{\tilde{t}\tilde{t}} = \frac{1}{\mathcal{J}}\det\begin{vmatrix}\mathfrak{D}_x\tilde{x} & \mathfrak{D}_x\tilde{u}_{\tilde{t}}\\ \mathfrak{D}_t\tilde{x} & \mathfrak{D}_t\tilde{u}_{\tilde{x}}\end{vmatrix}$$

$$\tilde{u}_{\tilde{x}\tilde{t}} = \frac{1}{\mathcal{J}}\det\begin{vmatrix}\mathfrak{D}_x\tilde{u}_{\tilde{t}} & \mathfrak{D}_x\tilde{t}\\ \mathfrak{D}_t\tilde{u}_{\tilde{t}} & \mathfrak{D}_t\tilde{t}\end{vmatrix} = \frac{1}{\mathcal{J}}\det\begin{vmatrix}\mathfrak{D}_x\tilde{x} & \mathfrak{D}_x\tilde{u}_{\tilde{x}}\\ \mathfrak{D}_t\tilde{x} & \mathfrak{D}_t\tilde{u}_{\tilde{x}}\end{vmatrix}$$

Et cetera... It is necessary to prolong up to the order of the PDE one wishes to consider.

Point Symmetries of PDEs

Formal definition

Consider an nth order PDE,

$$\Delta(x, t, u, u^{(1)}, u^{(2)}, ..., u^{(n)}) = 0 \tag{1}$$

where $u^{(k)}$ denotes the set of all kth-order partial derivatives of the dependent variables with respect to the independent variables. A point transformation Γ is a **point symmetry** of the above PDE if

$$\Delta(\tilde{x}, \tilde{t}, \tilde{u}, \tilde{u}^{(1)}, \tilde{u}^{(2)}, ..., \tilde{u}^{(n)}) = 0 \quad \text{when (1) holds.}$$

Typically, this condition is complicated enough to forget about solving directly. It is still useful, however, for checking whether or not a particular point transformation is a point symmetry of a particular PDE.

Point Symmetries of PDEs

Useful invariance theorem

Theorem: Let,

$$\Xi = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \eta(x,t,u)\frac{\partial}{\partial u}$$

be the infinitesimal generator corresponding to a Lie group of point transformations and $\Xi^{(k)}$ be its kth prolongation. A surface $\Delta\left(x,t,u,u^{(1)},\dots,u^{(k)}\right) = 0$ is invariant with respect to the group of transformations under consideration iff,

$$\Xi^{(k)}\Delta\left(x,t,u,u^{(1)},\dots,u^{(k)}\right) = 0 \iff \Delta\left(x,t,u,u^{(1)},\dots,u^{(k)}\right) = 0.$$

This is the condition which is used to actually find point symmetries of PDEs. Recall the recursive nature of prolongations,

$$\Xi^{(1)} = \Xi + \eta^{t}\frac{\partial}{\partial u_t} + \eta^{x}\frac{\partial}{\partial u_x}, \Xi^{(2)} = \Xi^{(1)} + \eta^{tt}\frac{\partial}{\partial u_{tt}} + \eta^{xt}\frac{\partial}{\partial u_{xt}} + \eta^{xx}\frac{\partial}{\partial u_{xx}}$$

Search for prolonged point symmetries of the form,

$$\tilde{x} = x + \varepsilon\xi(x,t,u) + \mathcal{O}\left(\varepsilon^2\right) \qquad \tilde{t} = t + \varepsilon\tau(x,t,u) + \mathcal{O}\left(\varepsilon^2\right)$$

$$\tilde{u} = u + \varepsilon\eta(x,t,u) + \mathcal{O}\left(\varepsilon^2\right) \qquad \tilde{u}_{\tilde{x}} = u_x + \varepsilon\eta^x(x,t,u,u^{(1)}) + \mathcal{O}\left(\varepsilon^2\right)$$

$$\tilde{u}_{\tilde{t}} = u_t + \varepsilon\eta^t(x,t,u,u^{(1)}) + \mathcal{O}\left(\varepsilon^2\right)$$

$$\tilde{u}_{\tilde{x}\tilde{x}} = u_{xx} + \varepsilon\eta^{xx}(x,t,u,u^{(1)},u^{(2)}) + \mathcal{O}\left(\varepsilon^2\right)$$

$$\ldots$$

Using the infinitesimal generator $\Xi^{(k)}$. To do this, it is necessary to find explicit representations of η^x, η^{xx}, etc, which is easy but ugly.

Point Symmetries of PDEs

Explicit representation of prolonged infinitesimals

$$
\begin{aligned}
\eta^{x} &= \eta_x + (\eta_u + \xi_x)\, u_x + \tau_x u_t - \xi_u u_x^2 - \tau_u u_x u_t, \\
\eta^{t} &= \eta_t - \xi_t u_x + (\eta_u - \tau_t)\, u_t - \xi_u u_x u_t - \tau_u u_t^2, \\
\eta^{xx} &= \eta_{xx} + (2\eta_{xu} - \xi_{xx})\, u_x - \tau_{xx} u_t + (\eta_{uu} - 2\xi_{xu})\, u_x^2 \\
&\quad - 2\tau_{xu} u_x u_t - \xi_{uu} u_x^3 - \tau_{uu} u_x^2 u_t + (\eta_u - 2\xi_x)\, u_{xx} \\
&\quad - 2\tau_x u_{xt} - 3\xi_u u_x u_{xx} - \tau_u u_t u_{xx} - 2\tau_u u_x u_{xt}, \\
\eta^{xt} &= \eta_{xt} + (\eta_{tu} - \xi_{xt})\, u_x + (\eta_{xu} - \tau_{xt})\, u_t - \xi_{tu} u_x^2 \\
&\quad + (\eta_{uu} - \xi_{xu} - \tau_{tu})\, u_x u_t - \tau_{xu} u_t^2 - \xi_{uu} u_x^2 u_t - \tau_{uu} u_x u_t^2 \\
&\quad - \xi_t u_{xx} - \xi_u u_t u_{xx} + (\eta_u - \xi_x - \tau_t)\, u_{xt} - 2\xi_u u_x u_{xt} \\
&\quad - 2\tau_u u_t u_{xt} - \tau_x u_{tt} - \tau_u u_x u_{tt}, \\
\eta^{tt} &= \eta_{tt} - \xi_{tt} u_x + (2\eta_{tu} - \tau_{tt})\, u_t - 2\xi_{tu} u_x u_t \\
&\quad + (\eta_{uu} - 2\tau_{tu})\, u_t^2 - \xi_{uu} u_x u_t^2 - \tau_{uu} u_t^3 - 2\xi_t u_{xt} \\
&\quad - 2\xi_u u_t u_{xt} + (\eta_u - 2\tau_t)\, u_{tt} - \xi_u u_x u_{tt} - 3\tau_u u_t u_{tt}.
\end{aligned}
$$

Example - Viscid Burgers Equation

Determining determining equations

Consider the viscid Burgers equation,

$$u_t + uu_x = u_{xx}.$$

Applying the the twice prolonged infinitesimal generator $\Xi^{(2)}$ gives,

$$\eta^t + u\eta^x + u_x\eta = \eta^{xx}.$$

The first step in solving the determining equations (not shown) is to replace u_{xx} everywhere it appears by $u_t + uu_x$. The highest order derivative terms of what remains have a factor u_{xt}

Example - Viscid Burgers Equation

Explicit represenations

$$
\begin{aligned}
u \cdot \eta^{x} &= u \cdot \left(\eta_x + (\eta_u + \xi_x)\, u_x + \tau_x u_t - \xi_u u_x^2 - \tau_u u_x u_t\right), \\
\eta^{t} &= \eta_t - \xi_t u_x + (\eta_u - \tau_t)\, u_t - \xi_u u_x u_t - \tau_u u_t^2, \\
\eta^{xx} &= \eta_{xx} + (2\eta_{xu} - \xi_{xx})\, u_x - \tau_{xx} u_t + (\eta_{uu} - 2\xi_{xu})\, u_x^2 \\
&\quad - 2\tau_{xu} u_x u_t - \xi_{uu} u_x^3 - \tau_{uu} u_x^2 u_t + (\eta_u - 2\xi_x)\, u_{xx} \\
&\quad - 2\tau_x u_{xt} - 3\xi_u u_x u_{xx} - \tau_u u_t u_{xx} - 2\tau_u u_x u_{xt}
\end{aligned}
$$

Example - Viscid Burgers Equation

Solving the determinging equations

Collecting the red terms gives,

$$0 = -2\tau_x u_{xt} - 2\tau_u u_x u_{xt} \implies \tau_x = \tau_u = 0.$$

This kills the purple terms in the determining equations leaving,

$$\eta_t - \xi_t u_x + (\eta_u - \tau_t)\, u_t - \xi_u u_x u_t + u\left(\eta_x + (\eta_x - \xi_x)\, u_x - \xi_u u_x^2\right) + u_x \eta = \eta_{xx} + (2\eta_{xu} - \xi_{xx})\, u_x + (\eta_{uu} - 2\xi_{xu})\, u_x^2 - \xi_{uu} u_x^3 + (\eta_u - 2\xi_x - 3\xi_u u_x)\,(u_t + u u_x)$$

Equating the terms with u_t gives,

$$\begin{aligned} (\eta_u - \tau_t)\, u_t - \xi_u u_x u_t &= (\eta_u - 2\xi_x - 3\xi_u u_x)\, u_t \\ \implies \xi_u = 0, \quad \xi_x &= \frac{1}{2}\tau'(t) \\ \implies \xi &= \frac{1}{2}\tau'(t)x + \alpha(t) \end{aligned}$$

Example - Viscid Burgers Equation

Lie Algebra of point symmetry generators

The remaining equations determine τ and α up to five arbitrary constants, signifying a five dimensional Lie algebra.
After a bit more grinding, one obtains a five dimensional spanning set,

$$\mathbf{X}_1 = \partial_x, \quad \mathbf{X}_2 = \partial_t, \quad \mathbf{X}_3 = t\partial_x + \partial_u,$$
$$\mathbf{X}_4 = x\partial_x + 2t\partial_t - u\partial_u, \quad \mathbf{X}_5 = xt\partial_x + t^2\partial_t + (x - ut)\partial_u.$$

which characterizes the Lie algebra.

Symmetry of Linear Homogeneous PDE

A quick remark

If a PDE is linear and homogeneous it necessarily has an infinite dimensional Lie algebra of point symmetry generators. This comes directly from the fact that a linear operator has linear superposition and so if $u(x,t)$ and $v(x,t)$ are two solutions of $\Delta = 0$ then so is $\tilde{u} = u + \varepsilon v(x,t)$ and therefore,

$$\mathbf{X}_v = v(x,t)\partial_u$$

must be a symmetry generator for any solution $v(x,t)$, of which there are infinitely many.

Symmetry of Linear Homogeneous PDE

When is linearization possible?

If the Lie algebra of point symmetry generators of a given PDE is finite dimensional, the PDE **cannot** be linearized by a point transformation. It might be possible to linearize using other types of transformations, i.e. potential transformations, etc.

Application of Point Symmetries

Burgers Equation

The Burgers equation has a five dimensional Lie algebra and so it cannot be linearized by a point transformation. Recall that Burgers equation can be written as a conservation law,

$$u_t + \left(\frac{u^2}{2} - u_x\right)_x = 0.$$

Then, there exists a potential, v, such that,

$$\begin{aligned} u_x &= v_t + \frac{u^2}{2}, \\ v_x &= u. \end{aligned}$$

This leads to the potential form of Burgers equation,

$$v_t + \frac{v_x^2}{2} = v_{xx}.$$

Application of Point Symmetries

Potential form of Burgers equation

One can find point symmetries of the system,

$$u_x = v_t + \frac{u^2}{2},$$
$$v_x = u.$$

The infinitesimal generators of such a system are of the form,

$$\Xi = \xi(x, t, u, v)\frac{\partial}{\partial x} + \tau(x, t, u, v)\frac{\partial}{\partial t} + \eta(x, t, u, v)\frac{\partial}{\partial u} + \sigma(x, t, u, v)\frac{\partial}{\partial v}.$$

This leads to a linearized symmetry condition of the form,

$$\eta^x = \sigma^t + u\eta$$
$$\sigma^x = \eta$$

Application of Point Symmetries

Point symmetry generators of the potential system

After solving the determining equations, there is an infinite dimensional Lie algebra spanned by,

$$
\begin{aligned}
&\mathbf{X}_1 = \partial_x, \quad \mathbf{X}_2 = \partial_t, \quad \mathbf{X}_3 = \partial_v,\\
&\mathbf{X}_4 = x\partial_x + 2t\partial_t - u\partial_u,\\
&\mathbf{X}_5 = t\partial_x + \partial_u + x\partial_v,\\
&\mathbf{X}_6 = xt\partial_x + t^2\partial_t + (x - tu)\partial_u + \left(\frac{x^2}{2} + t\right)\partial_v,
\end{aligned}
$$

$$
\mathbf{X}_W = \left[W_x(x,t) + \frac{1}{2}W(x,t)u\right] e^{\frac{v}{2}}\partial_u + W(x,t)e^{\frac{v}{2}}\partial_v, \quad W_t = W_{xx}.
$$

Application of Point Symmetries

Restricting the point symmetry generators

Consider the restriction of the Lie algebra to the variables (x, t, u). This gives,

$$\mathbf{X}_1 = \partial_x, \quad \mathbf{X}_2 = \partial_t, \quad \cancel{\mathbf{X}_3 = \partial_v},$$
$$\mathbf{X}_4 = x\partial_x + 2t\partial_t - u\partial_u,$$
$$\mathbf{X}_5 = t\partial_x + \partial_u + \cancel{x\partial_v},$$
$$\mathbf{X}_6 = xt\partial_x + t^2\partial_t + (x - tu)\partial_u + \cancel{\left(\frac{x^2}{2} + t\right)\partial_v},$$

$$\cancel{\mathbf{X}_W = \left[W_x(x,t) + \frac{1}{2}W(x,t)u\right]e^{\frac{v}{2}}\partial_u + W(x,t)e^{\frac{v}{2}}\partial_v, \quad W_t = W_{xx}.}$$

This allows one to recover the old Lie algebra of the point symmetries of Burgers equation.

Consider the restriction of the Lie algebra to the variables (x, t, v). This gives,

$$\mathbf{X}_1 = \partial_x, \quad \mathbf{X}_2 = \partial_t, \quad \mathbf{X}_3 = \partial_v,$$
$$\mathbf{X}_4 = x\partial_x + 2t\partial_t - \cancel{u\partial_u},$$
$$\mathbf{X}_5 = t\partial_x + \cancel{\partial_u} + x\partial_v,$$
$$\mathbf{X}_6 = xt\partial_x + t^2\partial_t + \cancel{(x - tu)\partial_u} + \left(\frac{x^2}{2} + t\right)\partial_v,$$
$$\mathbf{X}_W = \cancel{\left[W_x(x,t) + \frac{1}{2}W(x,t)u\right] e^{\frac{v}{2}}\partial_u} + W(x,t)e^{\frac{v}{2}}\partial_v, \quad W_t = W_{xx}.$$

These are the point symmetry generators for the potential form of Burgers equation. Notice that the generators $\mathbf{X}_W$ depend on arbitrary solutions of the heat equation $W_t = W_{xx}$.

Hopf-Cole Transformation

Slightly less arbitrary

The transformation $W = e^{-\frac{v}{2}}$ maps the potential form of Burger's equation to the heat equation,

$$W_t = W_{xx} \iff -\frac{e^{-\frac{v}{2}}}{2} v_t = -\frac{e^{-\frac{v}{2}}}{2}\left(-\frac{v_x^2}{2} + v_{xx}\right) \iff v_t + \frac{v_x^2}{2} = v_{xx}.$$

from which one can write,

$$\begin{aligned} W_x &= -\frac{1}{2}(v_x)\left(e^{-\frac{v}{2}}\right) = -\frac{1}{2}(u)(W) \\ \implies u &= -2\frac{W_x}{W} = -2\frac{\partial}{\partial x}\ln W \end{aligned}$$

the famous Hopf-Cole transformation.

The End

Thanks for listening