Optimization – Exercises

Day 1

Let $(H,\ \langle\cdot,\cdot\rangle)$ be a real Hilbert space. We denote $\|\cdot\|$ the norm derived by the scalar product.

Exercise 1 (Necessary and sufficient optimality conditions).
Let $f\colon H \longrightarrow \mathbb{R}$ be a twice differentiable function. Show that if x is a local minimizer of f, then

$$\nabla f(x) = 0$$
$$\nabla^2 f(x) \succeq 0$$

Is the first order condition a sufficient condition for x to be a local minimizer ? If no, give an example. What assumption can you make for this condition to be an equivalence ?

Exercise 2 (Caracterizations of convex functions).
Let $f\colon H \longrightarrow \mathbb{R}$ be a twice differentiable function. Show the following equivalences :

1. f is convex if, and only if,
$$\forall (x,y) \in H \times H,\ f(y) \geqslant f(x) + \langle \nabla f(x) \mid y - x\rangle.$$
2. f is convex if, and only if,
$$\forall x \in H,\ \nabla^2 f(x) \succeq 0,$$

where $\nabla^2 f(x)$ is the hessian of f at x.

Exercise 3 (Squared distance function).
Let A be a nonempty closed convex subset of H. We consider the function “squared distance to A” defined for all $x \in H$ by
$$g(x) = \inf_{y \in A} \|x - y\|^2.$$

1. Show that g is convex.
2. Show that g is Fréchet differentiable, with $\nabla g(x) = 2(x - \mathrm{p}_A(x))$, where p_A denotes the projection on A.

Exercise 4 (Minimization of a quadratic function).
Let $A \in \mathcal{S}_n^{++}(\mathbb{R})$ (set of symmetric positive definite matrices of $\mathbb{R}^{n\times n}$) and $b \in \mathbb{R}^n$. Let f be defined for all $x \in \mathbb{R}^n$ by
$$f(x) = \frac{1}{2}\langle Ax, x\rangle - \langle b, x\rangle.$$
Show that f admits a unique minimizer and give an expression of this minimizer.

Exercise 5 (Convex optimization exam 2019).
Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a convex, differentiable and bounded function on $\mathbb{R}^n$. Show f is constant.

Exercise 6 (About ε-minimizers).
Let $f\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ be a continuous function bounded from below on $\mathbb{R}^n$. Let $\varepsilon > 0$ and u a ε-minimizer of f, i.e. u satisfies
$$f(u) \leqslant \inf_{x \in \mathbb{R}^n} f(x) + \varepsilon.$$

Let $\lambda > 0$ and consider

$$g\colon x \in \mathbb{R}^n \mapsto g(x) := f(x) + \frac{\varepsilon}{\lambda}\|x - u\|.$$

1. Show there exists $v \in \mathbb{R}^n$ which minimizes g on $\mathbb{R}^n$. Show this point v satisfies the following conditions :
   (i) $f(v) \leqslant f(u)$,
   (ii) $\|u - v\| \leqslant \lambda$,
   (iii) $\forall x \in \mathbb{R}^n$, $f(v) \leqslant f(x) + \frac{\varepsilon}{\lambda}\|x - v\|$.
2. Suppose in addition that f is differentiable on $\mathbb{R}^n$. Show that for all $\epsilon > 0$, there exists $x_\epsilon \in \mathbb{R}^n$ such that
$$\|\nabla f(x_\epsilon)\| \leqslant \epsilon.$$

Exercise 7.
Let $\mathcal{O} = \mathcal{S}_n^{++}(\mathbb{R})$ be the (open) set of symmetric positive definite matrices of $\mathbb{R}^{n\times n}$. $\mathcal{O}$ is endowed with the scalar product $\langle U, V\rangle = \mathrm{Tr}(UV)$. Let $A \in \mathcal{O}$ and f be defined for all $X \in \mathcal{O}$ by

$$f(X) = \mathrm{Tr}(X^{-1}) + \mathrm{Tr}(AX).$$

1. Show there exists a minimizer to f on $\mathcal{O}$. *Hint : you may use the inequality* $\mathrm{Tr}(UV) \geqslant \sum_{i=1}^n \lambda_i(U)\lambda_{n-i+1}(V)$*, where all eigenvalues* $\lambda_1, \ldots, \lambda_n$ *are in descending order ; i.e.,* $\lambda_1 \geqslant \cdots \geqslant \lambda_n$*.*
2. Find the minimizer and the optimal value of f.

Exercise 8 (Penalty method).
Let $F\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ be a lower semi-continuous function, coercive on $\mathbb{R}^n$. Let C be a closed set of $\mathbb{R}^n$ with $\mathrm{dom}(f) \cap C \neq \varnothing$. We seek to solve the constrained problem

$$\begin{aligned} \underset{x\in\mathbb{R}^n}{\text{minimize}} \quad & F(x) \\ \text{s.t.} \quad & x \in C. \end{aligned} \tag{$\mathcal{P}$}$$

Let $R\colon \mathbb{R}^n \longrightarrow \mathbb{R}^+$ be a lower semi-continuous function such that

$$R(x) = 0 \quad \Longleftrightarrow \quad x \in C.$$

R is called penalty function as it assigns a positive cost to any point that is not in the constraint set C. Let $(\gamma_k)_{k\in\mathbb{N}}$ be a nondecreasing sequence of positive reals satisfying $\lim_{k\to+\infty} \gamma_k = +\infty$. We denote by $(\mathcal{P}_k)$ the following penalized problem :

$$\underset{x\in\mathbb{R}^n}{\text{minimize}} \quad F_{\gamma_k}(x) := F(x) + \gamma_k R(x). \tag{$\mathcal{P}_k$}$$

Show that :

1. For all $k \in \mathbb{N}$, $(\mathcal{P}_k)$ has at least one solution x_k.
2. The sequence $(x_k)_{n\in\mathbb{N}}$ is bounded.
3. Any cluster point of $(x_k)_{k\in\mathbb{N}}$ is a solution to $(\mathcal{P})$.
4. What can we say if F is strictly convex ?