Числено интегриране. Квадратурни формули на Нютон-Коутс

Равномерна мрежа:

$\{x_i\}_{i=0}{}^{n}$

$x_i = a + i * h, \quad i = \overline{0, n}, \quad h = \frac{b-a}{n}$

$S_i = h * f(x_i)$

$I_1 = h * \sum_{i=0}^{n-1} f(x_i)$ (формула на левите правоъгълници)

$I_2 = h * \sum_{i=1}^{n} f(x_i)$ (формула на десните правоъгълници)

$S_i = h * \frac{f(x_i) + f(x_{i+1})}{2}$ - лице на трапец

$\frac{h}{2} * \sum_{i=0}^{n-1} [f(x_i) + f(x_{i+1})]$

$I_t = \frac{h}{2} * [f(a) + 2*\sum_{i=1}^{n-1} f(x_i) + f(b)]$ (форму на трапците)

$I_3 = h * \sum_{i=0}^{n-1} f\left(x_i + \frac{h}{2}\right)$ (формула на средните правоъгълници)

Общ подход:

$f(x) = \mathcal{L}_n(x)$

$\int_a^b f(x)\,dx \approx \int_a^b \mathcal{L}_n(x)\,dx = \sum_{i=0}^{n} C_i * f(x_i) \qquad C_i = ?$

За n = 1:

x $\quad x_i \quad x_{i+1}$

y $\quad f(x_i) \quad f(x_{i+1})$

$\mathcal{L}_1(x; f)\,dx = f(x_i)\,\frac{x-x_{i+1}}{x_i-x_{i+1}} + f(x_{i+1})\,\frac{x-x_i}{x_{i+1}-x_i} =$

$= \frac{1}{h}[-f(x_i)(x-x_{i+1}) + f(x_{i+1})(x-x_i)]$

$\int_{x_i}^{x_{i+1}} \mathcal{L}_1(x, f)\,dx =$

$= \frac{1}{h}\int_{x_i}^{x_{i+1}} [-f(x_i)\,(x - x_{i+1}) + f(x_{i+1})\,(x - x_i)]\,dx =$

$= \frac{1}{h}\,[-f(x_i)\int_{x_i}^{x_{i+1}} (x - x_{i+1})\,dx + f(x_{i+1})\int_{x_i}^{x_{i+1}} (x - x_i)\,dx] =$

$= \frac{1}{h}\,[-f(x_i)\int_{x_i}^{x_{i+1}} \left(\frac{x-x_{i+1}}{2}\right)\,dx + f(x_{i+1})\int_{x_i}^{x_{i+1}} \left(\frac{x - x_i}{2}\right)\,dx] =$

$= \frac{1}{h}\,[-f(x_i)\,(0 - (-\frac{h^2}{2})) + f(x_{i+1})(\frac{h^2}{2} - 0) =$

$= \frac{1}{h} * \frac{h^2}{2}[f(x_i) + f(x_{i+1})] =$

$= \frac{h}{2}[f(x_i) + f(x_{i+1})]$ (съвпада с формула на трапците)

За n = 2:

x $\quad x_{i-1} \quad x_i \quad x_{i+1}$

y $f(x_{i-1})$ $f(x_i)$ $f(x_{i+1})$

$\mathcal{L}_2(x;f)$ dx= $f(x_{i-1})\,\frac{(x-x_i)(x-x_{i+1})}{(x_{i-1}-x_i)(x_{i-1}-x_{i+1})} + f(x_i)\,\frac{(x-x_{i-1})(x-x_{i+1})}{(x_i-x_{i-1})(x_i-x_{i+1})} + f(x_{i+1})\,\frac{(x-x_{i-1})(x-x_i)}{(x_{i+1}-x_{i-1})(x_{i+1}-x_i)}$

$\mathcal{L}_2(x;f)$ dx = $\frac{1}{2h^2}[f(x_{i-1}) * (x-x_i)(x-x_{i+1}) - 2f(x_i)(x-x_{i-1})(x-x_{i+1}) + f(x_{i+1})(x-x_{i-1})(x-x_i)]$

In[∘]:= $\int_{xi-h}^{xi+h} (x - xi)\,(x - (xi + h))\,dx$

Out[∘]= $\frac{2\,h^3}{3}$

In[∘]:= $\int_{xi-h}^{xi+h} (x - (xi - h))\,(x - (xi + h))\,dx$

Out[∘]= $-\frac{4\,h^3}{3}$

In[∘]:= $\int_{xi-h}^{xi+h} (x - (xi - h))\,(x - xi)\,dx$

Out[∘]= $\frac{2\,h^3}{3}$

$\mathcal{L}_2(x;f)$ dx = $\frac{1}{2h^2}[f(x_{i-1}) * \frac{2h^3}{3} - 2f(x_i) * (-\frac{4\,h^3}{3}) + f(x_{i+1}) * \frac{2h^3}{3}] =$

$= \frac{1}{2h^2} * \frac{2h^3}{3}[f(x_{i-1}) + 4f(x_i) + f(x_{i+1})] =$

$\frac{h}{3}[f(x_{i-1}) + 4f(x_i) + f(x_{i+1})]$

Изискване за прилагане на тази формула!

n = 2m - четно (брой на подинтервали)

<=> **n + 1** - нечетно (брой на точките)

Формула на Симпсън:

$I_s = \frac{h}{3}[f(a) + 4\sum_{i=1}^{m} f(x_{2i-1}) + 2\sum_{i=1}^{m-1} f(x_{2i}) + f(b)]$