Задача 1:

а) $y'' - 6y' + 8y = 0$

$\lambda^2 - 6\lambda + 8 = 0$

$D = (-6^2) - 4.1.8$

$D = 4$

$\lambda_{1,2} = \dfrac{6 \pm 2}{2}$ $\begin{cases} 4 \\ 2 \end{cases}$

$y = C_1 e^{4x} + C_2 e^{2x}$

б) $y''' - 2y'' + y' = 0$

$\lambda^3 - 2\lambda^2 + \lambda = 0 \quad :|\lambda$

$\lambda^2 - 2\lambda + 1 = 0$

$D = (-2^2) - 4.1.1$

$D = 0$

$\lambda_1 = \lambda_2 = \dfrac{\cancel{2}}{\cancel{2}} = 1$

$y = C_1 e^{x} + C_2 x e^{x}$

Задача 2:

а) $y'' - 2y' + 10y = 0$

$\lambda^2 - 2\lambda + 10 = 0$

$D = (-2)^2 - 4.1.10$

$D = -36 = 6i$

$\lambda_{1,2} = \dfrac{2 \pm 6i}{2} = \dfrac{\cancel{2}(1 \pm 3i)}{\cancel{2}} = 1 \pm 3i$

$\lambda_1 = 1 + 3i \qquad \lambda_2 = 1 - 3i$

$e^{\lambda_1 x} = e^{(1+3i)x} = e^{x+3ix} = e^{x} \cdot e^{3ix} =$

$= e^{x}(\cos 3x + i\sin 3x) = e^{x}\cos 3x + ie^{x}\sin 3x$

$y = C_1 e^{x}\cos 3x + C_2 e^{x}\sin 3x$

б) $y'' - 4y' + 8y = 0$

$\lambda^2 - 4\lambda + 8 = 0$

$D = (-4)^2 - 4 \cdot 1 \cdot 8$

$D = -16 = 4i$

$\lambda_{1,2} = \dfrac{4 \pm 4i}{2} = \dfrac{2(2 \pm 2i)}{2} = 2 \pm 2i$

$\lambda_1 = 2 + 2i \qquad \lambda_2 = 2 - 2i$

$e^{\lambda_1 x} = e^{(2+2i)x} = e^{2x+2ix} = e^{2x} \cdot e^{2ix} =$

$= e^{2x}(\cos 2x + i\sin 2x) = e^{2x}\cos 2x + ie^{2x}\sin 2x$

$y = C_1 e^{2x}\cos 2x + C_2 e^{2x}\sin 2x$

Задача 3:

г) $y'' + 2y' + 2y = \frac{e^{-x}}{\cos x}$

I етап: $\lambda^2 + 2\lambda + 2 = 0$

$D = 2^2 - 4 \cdot 1 \cdot 2$

$D = -4 = 2i$

$\lambda_{1,2} = \frac{-2 \pm 2i}{2} = \frac{\cancel{2}(-1 \pm i)}{\cancel{2}} = -1 \pm i$

$\lambda_1 = -1 + i \qquad \lambda_2 = -1 - i$

$e^{\lambda_1 x} = e^{(-1+i)x} = e^{-x+ix} = e^{-x} \cdot e^{ix} =$
$= e^{-x}(\cos x + i\sin x) = e^{-x}\cos x + ie^{-x}\sin x$

$y_{\text{зол.}} = C_1 e^{-x}\cos x + C_2 e^{-x}\sin x$

II етап: $\eta(x) = C_1(x)e^{-x}\cos x + C_2(x)e^{-x}\sin x$

$$\begin{cases} C_1' e^{-x}\cos x + C_2' e^{-x}\sin x = 0 & \Big| : e^{-x} \\ C_1'(e^{-x}\cos x)' + C_2'(e^{-x}\sin x)' = \frac{e^{-x}}{\cos x} & \Big| : e^{-x} \end{cases}$$

$$\begin{cases} C_1'\cos x + C_2'\sin x = 0 \\ C_1'(\cos x)' + C_2'(\sin x)' = \frac{1}{\cos x} \end{cases}$$

$$\begin{cases} C_1'\cos x + C_2'\sin x = 0 \\ -C_1'\sin x + C_2'\cos x = \dfrac{1}{\cos x} \end{cases} \quad \left.\begin{matrix} |\cdot \sin x \\ |\cdot \cos x \end{matrix}\right) +$$

$$C_2'\sin^2 x + C_2'\cos^2 x = 1$$

$$C_2'(\sin^2 x + \cos^2 x) = 1$$

$$C_2' = 1 \quad \Rightarrow C_2 = \int 1\, dx = x$$

$$\begin{cases} C_1'\cos x + C_2'\sin x = 0 \\ -C_1\sin x + C_2'\cos x = \dfrac{1}{\cos x} \end{cases} \quad \begin{matrix} |\cdot \cos x \\ |\cdot -\sin x \end{matrix}$$

$$C_1'\cos^2 x + C_2'\sin^2 x = -\frac{\sin x}{\cos x}$$

$$C_1(\sin^2 x + \cos^2 x) = -\frac{\sin x}{\cos x}$$

$$C_1' = -\frac{\sin x}{\cos x} \Rightarrow C_1 = \int -\frac{\sin x}{\cos x}\, dx =$$

$$= -\int \frac{\sin x}{\cos x}\, dx = -\int \frac{d\cos x}{\cos x} = -\ln|\cos x|$$

$$\Rightarrow \eta = -\ln|\cos x| e^{-x}\cos x + x e^{-x}\sin x$$

III етап: $y_{нехом} = y_{хом} + \eta$

$$y_{нехом} = C_1 e^{-x}\cos x + C_2 e^{-x}\sin x - \ln|\cos x| e^{-x}\cos x + x e^{-x}\sin x$$

Задача 4:

а) $y'' + 3y' - 4y = e^{-4x} + xe^{-x}$

I этап: $\lambda^2 + 3\lambda - 4 = 0$

$D = 3^2 - 4 \cdot 1 \cdot (-4)$

$D = 25$

$\lambda_{1,2} = \dfrac{-3 \pm 5}{2}$ — -4; 1

$y_{общ.} = C_1 e^{-4x} + C_2 e^{x}$

II этап: $y'' + 3y' - 4y = \overset{q(x)}{1} e^{-4x}$ ← $\lambda = -4$

$\eta_1(x) = x \cdot a \cdot e^{-4x}$ | $\deg q(x) = 0$, $k = 1$

$\eta_1'(x) = a(e^{-4x} - 4e^{-4x} \cdot x)$

$\eta_1''(x) = a(16e^{-4x} x - 8e^{-4x})$

Зам. в урав: $a(16e^{-4x}x - 8e^{-4x}) + 3a(e^{-4x} - 4e^{-4x} \cdot x) - 4(e^{-4x} a \cdot x) = e^{-4x} \;|: e^{-4x}$

$16\cancel{ax} - 8a + 3a - \cancel{12ax} - \cancel{4ax} = 1$

$-5a = 1 \;|:(-5)$

$a = -\dfrac{1}{5}$

$\Rightarrow \eta_1(x) = -\dfrac{1}{5} x e^{-4x}$

III этап: $y'' + 3y' - 4y = \overset{q(x)}{x} e^{-x}$ ← $\lambda = -1$

$\eta_2(x) = (ax + b)e^{-x}$ | $\deg q(x) = 1$, $k = 0$

$\eta_2'(x) = e^{-x}(ax - a - b)$

$\eta_2''(x) = e^{-x}(ax - 2a + b)$

Зам. в урав: $e^{-x}(ax - 2a + b) + 3e^{-x}(ax - a - b) - 4e^{-x}(ax + b) = xe^{-x} \;|: e^{-x}$

$\cancel{ax} - 2a + b + 3\cancel{ax} - 3a - 3b - 4\cancel{ax} - 4b = x$

$-5a - 6b = x$

$$\begin{cases} -5a = 1 \Rightarrow a = -\frac{1}{5} \\ -6b = 0 \Rightarrow b = 0 \end{cases}$$

$\Rightarrow \eta_2(x) = -\frac{1}{5} e^{-x}$

IV етап: $y_{\text{нехом.}} = y_{\text{хом.}} + \eta_1 + \eta_2$

$y_{\text{нехом.}} = C_1 e^{-4x} + C_2 e^{x} - \frac{1}{5} x e^{-x} - \frac{1}{5} e^{-x}$

б) $y'' - 4y' + 4y = x^2 + 2e^{2x}$

I етап: $\lambda^2 - 4\lambda + 4 = 0$

$D = (-4^2) - 4 \cdot 1 \cdot 4$

$D = 0$

$\lambda_1 = \lambda_2 = \frac{4}{2} = 2$

$y_{\text{хом}} = C_1 e^{2x} + C_2 x e^{2x}$

II етап: $y'' - 4y' + 4y = \overset{q(x)}{\overset{\shortparallel}{x^2}} \cdot e^{0x} \leftarrow a = 0$ | $\deg q(x) = 2$, $k = 0$

$\eta_1(x) = ax^2 + bx + c$

$\eta_1'(x) = 2ax + b$

$\eta_1''(x) = 2a$

Зам. в урав: $2a - 4(2ax + b) + 4(ax^2 + bx + c) = x^2$

$2a - 8ax - 4b + 4ax^2 + 4bx + 4c = x^2$

$$\begin{cases} 2a = 1 \Rightarrow a = \frac{1}{2} \\ -4b = 0 \Rightarrow b = 0 \\ 4c = 0 \Rightarrow c = 0 \end{cases}$$

$\Rightarrow \eta_1(x) = \frac{1}{2} x^2$

III етап: $y'' - 4y' + 4y = \overset{q(x)}{2}e^{2x} \quad \leftarrow \alpha = 2$ $\quad\Big|\ \deg q(x) = 0,\ k = 2$

$\eta_2(x) = x^2 \cdot a \cdot e^{2x}$

$\eta_2'(x) = a(2xe^{2x} + e^{2x} \cdot 2x^2)$

$\eta_2''(x) = a(4e^{2x} \cdot x^2 + 8e^{2x} \cdot x + 2e^{2x})$

Зам. в урав: $a(4e^{2x} \cdot x^2 + 8e^{2x} \cdot x + 2e^{2x}) - 4a(2xe^{2x} + e^{2x} \cdot 2x^2) + 4(x^2 \cdot a \cdot e^{2x}) =$

$= 2e^{2x} \ \Big|: e^{2x}$

$a(4x^2 + 8x + 2) - 4a(2x + 2x^2) + 4(x^2 \cdot a) = 2$

$\cancel{4ax^2} + \cancel{8ax} + 2a - \cancel{8ax} - \cancel{8ax^2} + \cancel{4ax^2} = 2$

$2a = 2 \quad |:2$

$a = 1$

$\Rightarrow \eta_2(x) = x^2 e^{2x}$

IV етап: $y_{нехом} = y_{хом} + \eta_1 + \eta_2$

$y_{нехом.} = C_1 e^{2x} + C_2 x e^{2x} + \frac{1}{2}x^2 + x^2 e^{2x}$