# Курсова работа 2.2

## Задача 1:

г) $\begin{cases} \dot{x} = 2x - 4y \\ \dot{y} = x + 2y + z \\ \dot{z} = 3y + 2z \end{cases}$ $\lambda_1 = 2,\ \lambda_2 = 2+i,\ \lambda_3 = 2-i$

$$A - \lambda E = \begin{pmatrix} 2-\lambda & -4 & 0 \\ 1 & 2-\lambda & 1 \\ 0 & 3 & 2-\lambda \end{pmatrix}$$

1) $\lambda_1 = 2$

$(A - \lambda_1 E)h_1 = 0$

$$\begin{pmatrix} 0 & -4 & 0 \\ 1 & 0 & 1 \\ 0 & 3 & 0 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -4\beta = 0 \\ \alpha + \gamma = 0 \Rightarrow \alpha = -\gamma \\ 3\beta = 0 \Rightarrow \beta = 0 \end{cases}$$

$h_1 = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} -\gamma \\ 0 \\ \gamma \end{pmatrix}$ Изб. $\gamma = 1 \Rightarrow h_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$$X_1 = h_1 e^{\lambda_1 t} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} e^{2t}$$

2) $\lambda_2 = 2+i$

$(A - \lambda_2 E)h_2 = \Theta$

$$\begin{pmatrix} -i & -4 & 0 \\ 1 & -i & 1 \\ 0 & 3 & -i \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left| \begin{array}{l} i\alpha - 4\beta = 0 \Rightarrow i\alpha = 4\beta \Rightarrow \alpha = \dfrac{4\beta}{i} \\ \alpha - i\beta + \gamma = 0 \\ 3\beta - i\gamma = 0 \Rightarrow i\gamma = 3\beta \Rightarrow \gamma = \dfrac{3\beta}{i} \end{array} \right.$$

$$h_2 = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 4\beta/i \\ \beta \\ 3\beta/i \end{pmatrix} \quad \text{Изб. } \beta = i \Rightarrow h_2 = \begin{pmatrix} 4 \\ i \\ 3 \end{pmatrix}$$

$$X_2 = h_2 e^{\lambda_2 t} = \begin{pmatrix} 4 \\ i \\ 3 \end{pmatrix} e^{(2+i)t} = \begin{pmatrix} 4 \\ i \\ 3 \end{pmatrix} e^{2t} \cdot e^{it} =$$

$$= \begin{pmatrix} 4 \\ i \\ 3 \end{pmatrix} e^{2t}(\cos t + i\sin t) = \begin{pmatrix} 4e^{2t}\cos t + 4ie^{2t}\sin t \\ ie^{2t}\cos t - e^{2t}\sin t \\ 3e^{2t}\cos t + 3ie^{2t}\sin t \end{pmatrix} =$$

$$= \begin{pmatrix} 4e^{2t}\cos t \\ -e^{2t}\sin t \\ 3e^{2t}\cos t \end{pmatrix} + \begin{pmatrix} 4ie^{2t}\sin t \\ ie^{2t}\cos t \\ 3ie^{2t}\sin t \end{pmatrix} = \underbrace{e^{2t}\begin{pmatrix} 4\cos t \\ -\sin t \\ 3\cos t \end{pmatrix}}_{X_2^{(1)}} + i\underbrace{e^{2t}\begin{pmatrix} 4\sin t \\ \cos t \\ 3\sin t \end{pmatrix}}_{X_2^{(2)}}$$

$$\Rightarrow X = c_1 X_1 + c_2 X_2^{(1)} + c_3 X_2^{(2)}$$

$$X = c_1 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} e^{2t} + c_2 e^{2t} \begin{pmatrix} 4\cos t \\ -\sin t \\ 3\cos t \end{pmatrix} + c_3 e^{2t} \begin{pmatrix} 4\sin t \\ \cos t \\ 3\sin t \end{pmatrix}$$

## Задача 2:

б) $\begin{cases} \dot{x} = 2x + 12y - 3z \\ \dot{y} = -x - 5y + z \\ \dot{z} = -x - 12y + 4z \end{cases}$

$\lambda_1 = -1, \ \lambda_{2,3} = 1$

$A - \lambda E = \begin{pmatrix} 2-\lambda & 12 & -3 \\ -1 & -5-\lambda & 1 \\ -1 & -12 & 4-\lambda \end{pmatrix}$

1) $\lambda_1 = -1$

$(A - \lambda_1 E) h_1 = \Theta$

$\begin{pmatrix} 3 & 12 & -3 \\ -1 & -4 & 1 \\ -1 & -12 & 5 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\begin{cases} 3\alpha + 12\beta - 3\gamma = 0 \\ -\alpha - 4\beta + \gamma = 0 \\ -\alpha - 12\beta + 5\gamma = 0 \end{cases}$ $+ \ 2\alpha + 2\gamma = 0 \Rightarrow \alpha = -\gamma$

$\Rightarrow 6\gamma - 12\beta = 0 \Rightarrow \beta = \frac{\gamma}{2}$

$h_1 = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} -\gamma \\ \gamma/2 \\ \gamma \end{pmatrix}$ Изб. $\gamma = 1 \Rightarrow h_2 = \begin{pmatrix} -1 \\ 1/2 \\ 1 \end{pmatrix}$

$X_1 = h_1 e^{\lambda_1 t} = \begin{pmatrix} -1 \\ 1/2 \\ 1 \end{pmatrix} e^{-t}$

2) $\lambda_{2,3} = 1$

$(A - \lambda E) h_2 = \Theta$

$\begin{pmatrix} 1 & 12 & -3 \\ -1 & -6 & 1 \\ -1 & -12 & 3 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\Rightarrow -\alpha - 6\beta + \gamma = 0 \qquad \gamma = \alpha + 6\beta$

$$h_2 = \begin{pmatrix} \alpha \\ \beta \\ \alpha \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \\ \alpha + 6\beta \end{pmatrix} = \begin{pmatrix} \alpha \\ 0 \\ \alpha \end{pmatrix} + \begin{pmatrix} 0 \\ \beta \\ 6\beta \end{pmatrix} =$$

$$= \alpha \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \\ 6 \end{pmatrix}$$

Два линейно незав. вектора

$k = m = 2$

Изб: $h_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $h_3 = \begin{pmatrix} 0 \\ 1 \\ 6 \end{pmatrix}$

$$X_2 = h_2 e^{\lambda t} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{t} \qquad X_3 = h_3 e^{\lambda t} = \begin{pmatrix} 0 \\ 1 \\ 6 \end{pmatrix} e^{t}$$

$$X = c_1 X_1 + c_2 X_2 + c_3 X_3$$

$$X = c_1 \begin{pmatrix} -1 \\ 1/2 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{t} + c_3 \begin{pmatrix} 0 \\ 1 \\ 6 \end{pmatrix} e^{t}$$

Задача 3:

г) $\begin{cases} \dot{x} = 3x + y \\ \dot{y} = -4x - y + \dfrac{e^t}{2\sqrt{t}} \end{cases}$

$A - \lambda E = \begin{pmatrix} 3-\lambda & 1 \\ -4 & -1-\lambda \end{pmatrix}$ $|A - \lambda E| = (3-\lambda)(-1-\lambda) - (-4)\cdot(1) =$

$= -3 - 3\lambda + \lambda + \lambda^2 + 4 =$

$= \lambda^2 - 2\lambda + 1$

$\lambda^2 - 2\lambda + 1 = 0$

$D = (-2^2) - 4 \cdot 1 \cdot 1$

$D = 0$

$\lambda_{1,2} = 1$

$\lambda_{1,2} = 1$

$(A-\lambda E)h_1 = \theta$

$$\begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left|\begin{array}{l} 2\alpha + \beta = 0 \\ -4\alpha - 2\beta = 0 \;:(-2) \end{array}\right| \Rightarrow 2\alpha + \beta = 0 \quad \beta = -2\alpha$$

$$h_1 = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha \\ -2\alpha \end{pmatrix} \quad \text{Изб. } \alpha = 1 \Rightarrow h_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Търсим $h_2$ : $h_1$ и $h_2$ образуват серия

$$\left|\begin{array}{l} (A-\lambda E)h_1 = \theta \\ (A-\lambda E)h_2 = h_1 \end{array}\right. \quad \cdot(A-\lambda E) \text{ отляво} \Rightarrow (A-\lambda E)^2 h_2 = h_1$$

$$(A-\lambda E)^2 = \begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Избиране $h_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\Rightarrow X_{хом} = c_1\begin{pmatrix} 1 \\ -2 \end{pmatrix}e^t + c_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^t$$

Търсим частно решение по метода на Лагранж

$$\eta(t) = c_1(t)\begin{pmatrix} 1 \\ -2 \end{pmatrix}e^t + c_2(t)\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^t$$

$$\left|\begin{array}{l} \dot{c}_1 e^t + \dot{c}_2 e^t = 0 \\ -2\dot{c}_1 e^t + \dot{c}_2 e^t = \dfrac{e^t}{2\sqrt{t}} \end{array}\right) -$$

$$-\dot{c_1}e^t = \frac{e^t}{2\sqrt{t}} \quad | : -e^t$$

$$\dot{c_1} = -\frac{1}{2\sqrt{t}} \Rightarrow c_1 = \int -\frac{1}{2\sqrt{t}}dt = -\frac{1}{2}\int\frac{1}{\sqrt{t}}dt = -\frac{1}{2}\int\frac{1}{t^{\frac{1}{2}}}dt =$$

$$= -\frac{1}{2}\int t^{-\frac{1}{2}}dt = \frac{-1}{2}\cdot\frac{t^{-\frac{1}{2}+1}}{\frac{-1}{2}+1} = \frac{-1}{\cancel{2}}\cancel{2}t^{\frac{1}{2}} = -\sqrt{t}$$

$$\boxed{c_1 = -\sqrt{t}}$$

$$\begin{cases} \dot{c_1}e^t + \dot{c_2}e^t = 0 & \cdot (2) \\ -2\dot{c_1}e^t + \dot{c_2}e^t = \frac{e^t}{2\sqrt{t}} & )+ \end{cases}$$

$$3\dot{c_2}e^t = \frac{e^t}{2\sqrt{t}} \quad | : 3e^t$$

$$\dot{c_2} = \frac{1}{6\sqrt{t}} \Rightarrow c_2 = \int\frac{1}{6\sqrt{t}}dt = \frac{1}{6}\int\frac{1}{\sqrt{t}}dt = \frac{1}{\cancel{6}}\cdot \cancel{2}t^{\frac{1}{2}}$$

$$\Rightarrow \boxed{c_2 = \frac{\sqrt{t}}{3}}$$

$$\Rightarrow \eta = -\sqrt{t}\begin{pmatrix}1\\-2\end{pmatrix}e^t + \frac{\sqrt{t}}{3}\begin{pmatrix}1\\1\end{pmatrix}e^t$$

$$X_{нехом} = X_{хом} + \eta$$

$$X_{нехом} = c_1\begin{pmatrix}1\\-2\end{pmatrix}e^t + c_2\begin{pmatrix}1\\1\end{pmatrix}e^t - \sqrt{t}\begin{pmatrix}1\\-2\end{pmatrix}e^t + \frac{\sqrt{t}}{3}\begin{pmatrix}1\\1\end{pmatrix}e^t$$

Задача 4:

а) $\begin{cases} \dot{x} = -2x - y + 37\sin t \\ \dot{y} = -4x - 5y \end{cases}$ $\quad F = \begin{pmatrix} 37\sin t \\ 0 \end{pmatrix}$

$A - \lambda E = \begin{pmatrix} -2-\lambda & -1 \\ -4 & -5-\lambda \end{pmatrix}$ $\quad |A - \lambda E| = (-2-\lambda)(-5-\lambda) - (-4)(-1)$
$= 10 + 2\lambda + 5\lambda + \lambda^2 - 4 =$
$= \lambda^2 + 7\lambda + 6$

$\lambda^2 + 7\lambda + 6 = 0$

$D = 7^2 - 4 \cdot 1 \cdot 6$

$D = 25$

$\lambda_{1,2} = \dfrac{-7 \pm 5}{2}$ $\begin{matrix} -6 \\ -1 \end{matrix}$

1) $\lambda_1 = -6$

$(A - \lambda_1 E) h_1 = \Theta$

$\begin{pmatrix} 4 & -1 \\ -4 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\begin{cases} 4\alpha - \beta = 0 \\ -4\alpha + \beta = 0 \end{cases} \Rightarrow \beta = 4\alpha$

$h_1 = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha \\ 4\alpha \end{pmatrix}$ избд. $\alpha = 1 \Rightarrow h_2 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$

2) $\lambda_2 = -1$

$(A - \lambda_2 E) h_2 = \Theta$

$\begin{pmatrix} -1 & -1 \\ -4 & -4 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\left|\begin{array}{l} -\alpha-\beta=0 \quad \cdot(-1) \\ -4\alpha-4\beta=0 \quad :(-4) \end{array}\right| \Rightarrow \alpha+\beta=0 \qquad \beta=-\alpha$$

$$h_2=\begin{pmatrix}\alpha\\ \beta\end{pmatrix}=\begin{pmatrix}\alpha\\ -\alpha\end{pmatrix} \text{ изб. } \alpha=1 \Rightarrow h_2=\begin{pmatrix}1\\ -1\end{pmatrix}$$

$$X_{\text{хом.}}=c_1\begin{pmatrix}1\\ 4\end{pmatrix}e^{-6t}+c_2\begin{pmatrix}1\\ -1\end{pmatrix}e^{-t}$$

Търсим частно решение по метода на неопр. коеф.

$$F=\begin{pmatrix}37\sin t\\ 0\end{pmatrix}e^{0.t}=\begin{pmatrix}0.\cos t+37\sin t\\ 0.\cos t+0.\sin t\end{pmatrix}e^{0.t}$$

$$\left.\begin{array}{l} \alpha+i\beta \text{ не е хар. корен} \Rightarrow k=0 \\ p_i(t) \text{ и } q_i(t) \text{ са константи} \Rightarrow m=0 \end{array}\right\} \Rightarrow k+m=0$$

$$\eta=\begin{pmatrix}a\cos t+b\sin t\\ c\cos t+d\sin t\end{pmatrix}e^{0.t}=\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}$$

Заместване в нехом. с-ма: $x \subset \eta_1$ и $y \subset \eta_2$

$$\left|\begin{array}{l} (a\cos t+b\sin t)'=-2(a\cos t+b\sin t)-(c\cos t+d\sin t)+37\sin t \\ (c\cos t+d\sin t)'=-4(a\cos t+b\sin t)-5(c\cos t+d\sin t) \end{array}\right.$$

$$\left|\begin{array}{l} -a\sin t+b\cos t=-2a\cos t-2b\sin t-c\cos t-d\sin t+37\sin t \\ -c\sin t+d\cos t=-4a\cos t-4b\sin t-5c\cos t-5d\sin t \end{array}\right.$$

$$\left|\begin{array}{l} b=-2a-c \\ -a=-2b-d+37 \\ d=-4a-5c \\ -c=-4b-5d \end{array}\right.$$

Заместване $d = -4a - 5c$ и $b = -2a - c$ :

$$\begin{cases} -a = -2(-2a - c) - (-4a - 5c) + 37 \\ -c = -4(-2a - c) - 5(-4a - 5c) \end{cases}$$

$$\begin{cases} -a = 4a + 2c + 4a + 5c + 37 \\ -c = 8a + 4c + 20a + 25c \end{cases}$$

$$\begin{cases} -9a = 7c + 37 \\ -30c = 28a \end{cases} \Rightarrow a = -\frac{7c + 37}{9}$$

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$$\Rightarrow -30c = -\frac{28(7c + 37)}{9}$$

$$-270c = -196c - 1036$$

$$-270c + 196c = -1036$$

$$-74c = -1036 : (-74)$$

$$\boxed{c = 14}$$

$$\Rightarrow a = \frac{(-7)14 - 37}{9}$$

$$a = \frac{-98 - 37}{9}$$

$$a = \frac{-135}{9} =$$

$$\boxed{a = -15}$$

$$\Rightarrow d = -4(-15) - 5 \cdot 14$$

$$d = 60 - 70$$

$$\boxed{d = -10}$$

$$\Rightarrow b = -2\cdot(-15) - 14$$

$$b = 30 - 14$$

$$\boxed{b = 16}$$

$$\eta = \begin{pmatrix} -15\cos t + 16\sin t \\ 14\cos t - 10\sin t \end{pmatrix}$$

$$X_{нехом} = X_{хом} + \eta$$

$$X_{нехом} = c_1 \begin{pmatrix} 1 \\ 4 \end{pmatrix} e^{-6t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + \begin{pmatrix} -15\cos t + 16\sin t \\ 14\cos t - 10\sin t \end{pmatrix}$$