Задача 1: б)

$(x^2+1)y' - (2x+1)y = 0$

$(x^2+1)y' = (2x+1)y \quad :| x^2+1$

$y' = \frac{2x+1}{x^2+1} \cdot y$

$\frac{dy}{dx} = \frac{2x+1}{x^2+1} \cdot y \Rightarrow$ УРП

$\frac{dy}{dx} = \frac{2x+1}{x^2+1} \cdot y \quad :| y \cdot dx \quad \boxed{y \neq 0}$

$\frac{dy}{y} = \frac{2x+1}{x^2+1} \cdot dx$

Интегриране:

$\int \frac{dy}{y} = \int \frac{2x+1}{x^2+1} dx$

$\ln|y| = \int \frac{2x}{x^2+1} dx + \int \frac{1}{x^2+1} dx$

$\ln|y| = 2\int \frac{x}{x^2+1} dx + arctg(x) \quad \int x dx = \frac{x^2}{2}$

$\ln|y| = \cancel{2} \cdot \frac{1}{\cancel{2}} \int \frac{2(x^2+1)}{x^2+1} + arctg(x)$

$\ln|y| = \ln(x^2+1) + arctg(x) + C$ - общо реш. на уравн. в неявен вид

$e^{\ln|y|} = e^{\ln(x^2+1) + arctg(x) + C}$

$|y| = e^{\ln(x^2+1)} \cdot e^{arctg(x)} \cdot e^C$

$|y| = (x^2+1) \cdot e^{arctg(x)} \cdot e^C$

$\varepsilon_1 \cdot y = \varepsilon_2 (x^2+1) \cdot e^{arctg(x)} \cdot e^C \,/\!:\varepsilon_1 \quad \varepsilon_{1,2} = \pm 1$

$y = \dfrac{\varepsilon_2 \cdot e^C}{\varepsilon_1 \text{ и } C_1} \cdot e^{arctg(x)} (x^2+1)$

$y = C_1 \cdot e^{arctg(x)} (x^2+1)$

Проверка:

$y = 0$

$\dfrac{dy}{dx} = \dfrac{2x+1}{x^2+1} \cdot y$

$0 = \dfrac{2x+1}{x^2+1} \cdot 0$

$0 = 0 \Rightarrow y = 0$ е решение

Отговор:

$y = C_1 e^{arctg(x)} (x^2+1)$; $y = 0$

Задача 2: а)

$y' = (x+y-1)^2$

Полагаме: $x + y - 1 = z$, $z = z(x)$, $z' = \dfrac{dz}{dx}$

$y = z - x + 1$

$y' = (z - x + 1)'$

$y' = z' - 1$

Заместване: $y' = z' - 1$

$z' - 1 = z^2$

$z' = z^2 + 1 \Rightarrow$ УРП

$z' = z^2 + 1 \; : \; |z^2 + 1$

$\frac{z'}{z^2+1} = 1$

$\frac{dz}{dx} \cdot \frac{1}{z^2+1} = 1 \; |dx$

$\frac{dz}{z^2+1} = dx$

Интегриране:

$\int \frac{dz}{z^2+1} = \int dx$

$\operatorname{arctg}(z) = x + C$

От това, че $\operatorname{arctg}(z) = a \Rightarrow z = \operatorname{tg}(a)$

$z = \operatorname{tg}(x + C)$

Заместване: $z = x + y - 1$

$x + y - 1 = \operatorname{tg}(x + C)$

$\boxed{y = \operatorname{tg}(x + C) - x + 1}$

Задача 3:

$xy' = y\left(1 + \ln \frac{y}{x}\right) \quad :/x$

$y' = \frac{y}{x}\left(1 + \ln \frac{y}{x}\right) \Rightarrow$ ХДУ

Полагане: $\frac{y}{x} = z, \quad z = z(x), \quad z' = \frac{dz}{dx}$

$y = zx$ (дифф. спрямо x)

$\Rightarrow y' = (zx)' = z'x + z \cdot x' = z'x + z$

$z'x + z = z(1 + \ln z)$

$z'x + \cancel{z} = \cancel{z} + z \cdot \ln z \quad :/x \qquad \boxed{x \neq 0}$

$z' = \frac{z \cdot \ln z}{x}$

$$\frac{dz}{dx} = \frac{z \cdot \ln z}{x} \Rightarrow \text{УРП}$$

$$\frac{dz}{dx} = \frac{z \cdot \ln z}{x} \quad | \cdot dx \; : (z \cdot \ln z)$$

$$\frac{1 dz}{z \ln z} = \frac{dx}{x}$$

<u>Интегриране:</u>

$$\int \frac{dz}{z \cdot \ln z} = \int \frac{dx}{x}$$

$$\int \frac{(\ln z)' \, dz}{\ln z} = \ln|x| + C$$

$$\int \frac{d \ln z}{\ln z} = \ln|x| + C$$

$$\ln|\ln z| = \ln|x| + C$$

$$e^{\ln|\ln z|} = e^{\ln|x| + C}$$

$$|\ln z| = e^{\ln|x|} \cdot e^{C}$$

$$\varepsilon_1 \ln z = \varepsilon_2 |x| \cdot e^{C} \; /: \varepsilon_1 \quad \varepsilon_{1,2} = \pm 1$$

$$\ln z = \underbrace{\frac{\varepsilon_2 \cdot e^{C}}{\varepsilon_1}}_{C_1} \cdot x$$

$$\Rightarrow \ln z = C_1 \cdot x$$

Проверка:

$x = 0$

$$\frac{\partial x}{\partial z} = \frac{x}{z \cdot \ln z}$$

$$0 = \frac{0}{z \cdot \ln z}$$

$0 = 0$, но $\ln \neq 0 \Rightarrow x = 0$ не е реш.

Отговор:

$\ln z = C_1 \cdot x$

Задача 4: б)

$$(y-x)y' = x+y \quad :|y-x$$

$$y' = \frac{x+y}{y-x}$$

$$y' = -\frac{x+y}{x-y}$$

$$y' = -\frac{\cancel{x}(1+\frac{y}{x})}{\cancel{x}(1-\frac{y}{x})}$$

(*)
$$y' = -\frac{1+\frac{y}{x}}{1-\frac{y}{x}} \Rightarrow ХДУ$$

Полагане: $\frac{y}{x} = z, \quad z = z(x), \quad z' = \frac{dz}{dx}$

$y = zx$ (диференциране спрямо x)

$$\Rightarrow y' = (zx)' = z'x + z \cdot x' = z'x + z$$

Заместване в (*):

$$z'x + z = -\frac{1+z}{1-z}$$

$$z'x = -\frac{\frac{1+z}{1-z} - z^{(1-z}}{1-z}$$

$$z'x = -\frac{1+\cancel{z}-\cancel{z}+z^2}{1-z} \quad :|x \quad \boxed{x \neq 0}$$

$$z' = -\frac{1+z^2}{1-z} \cdot \frac{1}{x} \Rightarrow УРП$$

$$\frac{dz}{dx} = \frac{z^2+1}{z-1}\cdot\frac{1}{x}\cdot|dx$$

$$dz = \frac{z^2+1}{z-1}\cdot\frac{dx}{x} \quad \cdot|(z-1) : (z^2+1)$$

$$\frac{z-1}{z^2+1}\,dz = \frac{dx}{x}$$

<u>Интегриране:</u>

$$\int\frac{z-1}{z^2+1}\,dz = \int\frac{dx}{x}$$

$$\int\frac{z}{z^2+1}\,dz - \int\frac{1}{z^2+1}\,dz = \ln|x| + C$$

$$\int z\,dz = \frac{z^2}{2}$$

$$\frac{1}{2}\int\frac{d(z^2+1)}{z^2+1} - arctg(z) = \ln|x| + C$$

$$\frac{1}{2}\ln(z^2+1) - arctg(z) = \ln|x| + C$$

Връщаме се в полагането и получаваме:

$$\frac{1}{2}\ln\left(\left(\frac{y}{x}\right)^2+1\right) - arctg\left(\frac{y}{x}\right) = \ln|x| + C$$

Проверка:

$x = 0$

$$\frac{dx}{dz} = -\frac{1-z}{1+z^2} \cdot x$$

$$0 = -\frac{1-z}{1+z^2} \cdot 0$$

$0 = 0$, но в общото решение x е в знаменател $\Rightarrow x = 0$ не е решение

Отговор:

$$\frac{1}{2} \ln\left(\left(\frac{y}{x}\right)^2 + 1\right) - \operatorname{arctg}\left(\frac{y}{x}\right) = \ln|x| + C$$

Задача 5: а)

$$2x^3 y' = 2x^2 y - 3 \quad | : 2x^3$$

$$y' = \frac{\cancel{2x^2}}{\cancel{2x^3}} y - \frac{3}{2x^3} \Rightarrow \text{лин. уравн. от I-ви ред}$$

$$y' = \frac{1}{x} y - \frac{3}{2x^3}$$

$$y = e^{\int \frac{1}{x} dx}\left(C + \int \frac{3}{2x^3} \cdot e^{-\int \frac{1}{x} dx} dx\right)$$

$$y = e^{\ln|x|}\left(C + \int \frac{3}{2x^3} \cdot e^{-\ln|x|} dx\right)$$

$$y = |x|\left(C + \int \frac{3}{2x^3} \cdot |x|^{-1} dx\right)$$

$$y = |x| \cdot \left(C + \int \frac{3}{2x^3} \cdot \frac{1}{x} \cdot dx \right)$$

$$y = |x| \cdot \left(C + \int \frac{3}{2x^4} \, dx \right)$$

$$y = |x| \cdot \left(C + \frac{3}{2} \int \frac{1}{x^4} \, dx \right)$$

$$y = |x| \left(C + \frac{3}{2} \int x^{-4} \, dx \right)$$

$$y = |x| \cdot \left(C + \frac{3}{2} \cdot \frac{x^{-4+1}}{-4+1} \right)$$

$$y = |x| \left(C + \frac{3}{2} \cdot \frac{x^{-3}}{-3} \right)$$

$$y = |x| \left(C + \frac{\cancel{3}}{2} \cdot \frac{1}{x^3} \left(-\frac{1}{\cancel{3}} \right) \right)$$

$$y = |x| \left(C - \frac{1}{2x^3} \right)$$

<u>Задача 6:</u> а)

$$4xy' + (4x+1)y^2 - 4y = 0$$

$$4xy' = 4y - (4x+1)y^2 \quad | : 4x$$

$$y' = \frac{\cancel{4}y}{\cancel{4}x} - \frac{(4x+1)y^2}{4x}$$ => уравн. на Бернули

$$y' = \frac{y}{x} - \frac{(4x+1)y^2}{4x} \quad \Big| : y^2$$

$$\frac{y'}{y^2} = \frac{y}{xy^2} - \frac{4x+1}{4x}$$

$$\frac{y'}{y^2} = \frac{y^{1-2}}{x} - \frac{4x+1}{4x}$$

$$\frac{y'}{y^2} = \frac{y^{-1}}{x} - \frac{4x+1}{4x}$$

Полагаме: $y^{-1} = z \quad , \quad z = z(x)$

$$z' = (y^{-1})' = (-1) \cdot y^{-1-1} \cdot y' = -y^{-2} \cdot y' = -\frac{y'}{y^2}$$

$$\Rightarrow \frac{y'}{y^2} = -z'$$

$$-z' = \frac{2z}{x} - \frac{4x+1}{4x} \quad \cdot (-1)$$

$$z' = -\frac{2z}{x} + \frac{4x+1}{4x}$$ $\Rightarrow$ лин. уравн. от I-ви ред

$$z = e^{\int -\frac{2}{x} dx}\left(C + \int \frac{4x+1}{4x} \cdot e^{-\int -\frac{2}{x} dx} dx\right)$$

$$z = e^{-2\int \frac{1}{x} dx}\left(C + \int \frac{4x+1}{4x} \cdot e^{2\int \frac{1}{x} dx} dx\right)$$

$$z = e^{-2\ln|x|}\left(C + \int \frac{4x+1}{4x} \cdot e^{2\ln|x|} dx\right)$$

$$z = e^{\ln|x|^{-2}}\left(C + \int \frac{4x+1}{4x} \cdot e^{\ln|x|^2} dx\right)$$

$$z = |x|^{-2}\left(C + \int \frac{4x+1}{4x} \cdot |x|^2 \cdot dx\right)$$

$$z = \frac{1}{x^2} \cdot \left(C + \int \frac{4x+1}{4x} \cdot x^2 \cdot dx\right)$$

$$z = \frac{1}{x^2}\left(C + \frac{1}{4}\int \frac{4x+1}{x} \cdot x^2 \, dx\right)$$

$$z = \frac{1}{x^2}\left(C + \frac{1}{4}\left(\int \frac{4\cancel{x}}{\cancel{x}}\, dx + \int \frac{1}{x}\, dx\right) x^2 \, dx\right)$$

$$z = \frac{1}{x^2}\left(C + \frac{1}{4}\left(\int 4\, dx + \int \frac{1}{x}\, dx\right) \cdot x^2 \cdot dx\right)$$

$$z = \frac{1}{x^2}\left(C + \frac{1}{\cancel{4}}\left(\cancel{4}x + \ln|x|\right) \cdot x^2 \cdot dx\right)$$

$$z = \frac{1}{x^2}\left(C + x + \frac{\ln|x|}{4} \cdot x^2 \, dx\right)$$