Aerodynamics of a Volleyball Serve

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1 Introduction

I have been playing volleyball for 4 years now. One of the most crucial aspects of a volleyball game is the serve that happens at the beginning of every point. If there is a good serve, a team can disrupt the other team's attacks or even immediately score a point. We can use mathematics to model the trajectory of a volleyball as it travels through the air. This exploration will use mathematics to model a volleyball serve in order to figure out what the most optimized serve is.

2 Variables

We first define the standard volleyball court. The court is a rectangle that is 9 m by 18 m, with a net in the center, dividing the court into 2 squares. The height of a men's volleyball net is 2.43 m. Extending beyond the sides of the court are imaginary lines that the server must stand between to serve. The server must start before the base line, but can jump over and hit the volleyball in the air. Without loss of generality, we assume the server starts all the way at the right side of the court because that will give the server the most room to serve. We let x_0 be the distance over the service line when the volleyball is hit and y_0 be the height of the volleyball above the ground.

There are 3 variables that govern the motion of the volleyball: the velocity and 2 angles to determine its direction in 3-dimensional space. We let v_0 be the initial velocity, ϕ be the angle the volleyball makes with the right sideline, and θ be the angle the volleyball makes with respect to the plane parallel to the ground. 2 cross-sections of the volleyball are shown below.

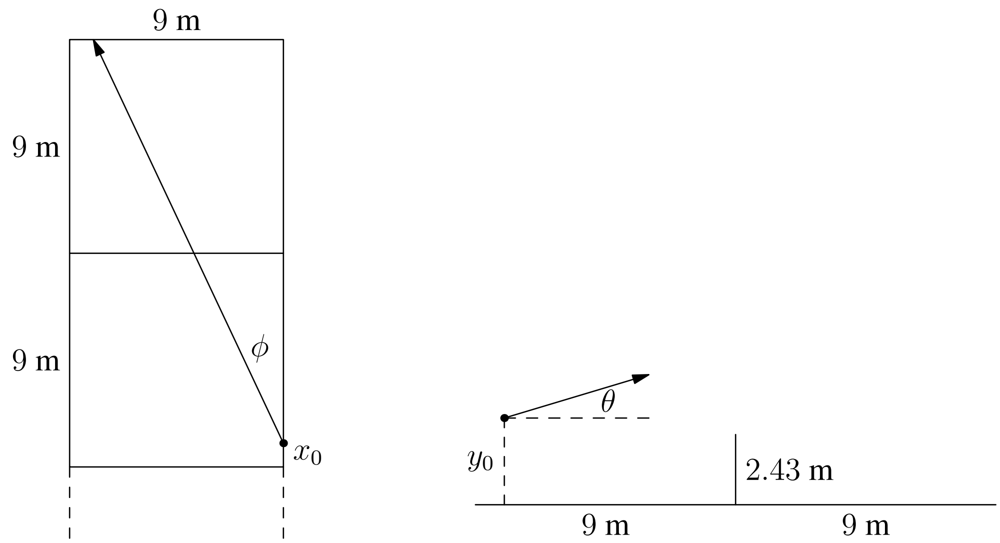


Figure 1: A diagram of a volleyball court from the top and from the side.

A standard volleyball has a circumference of 65-67 cm. Since $c = 2\pi r$, the radius varies between 0.103 and 0.107 m. We will assume the radius is 0.105 m for this exploration.

For this exploration, we will assume that the best serve is one in which the time the volleyball spends in the air is minimized. This is because the less time the volleyball spends in the air, the less time the opponents have to react to the serve, which is desirable.

3 Optimization

In order to simplify this problem, we can set an axis along the path of the volleyball and parallel to the ground. This way, we can effectively combine the x and the z axes. Looking at the right of Figure 1, we see that the net makes a right angle with the sideline, creating a right triangle. We can use trigonometry to figure out the distance from the volleyball to the net and from the net to the place where it lands.

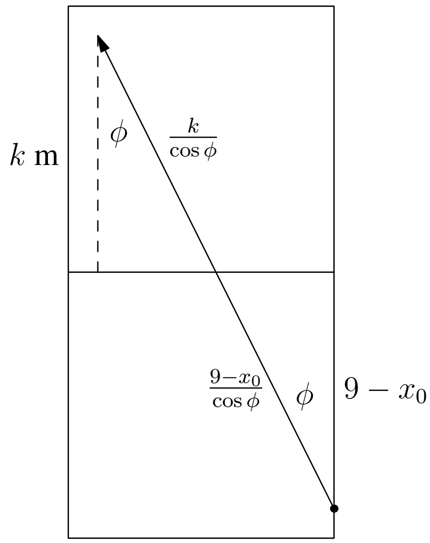
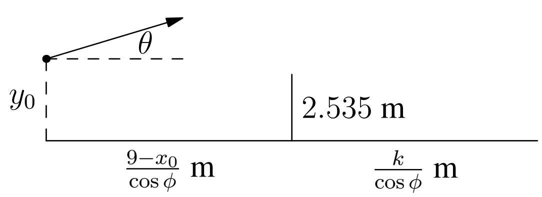


Figure 2: A diagram of a volleyball court with the distances to the net and to the ground.

In Figure 2, the closest distance from the volleyball to the net is $9 - x_0$. Since the angle the volleyball makes with the sideline is ϕ, the distance from the volleyball to the net along the path it travels is $d_1 = \dfrac{9 - x_0}{\cos\phi}$.

We let k be the distance the volleyball lands from the net. Using trigonometry, we see that the distance from the net along the path the volleyball travels is $d_2 = \dfrac{k}{\cos\phi}$, where $0 < k \leq 9$. We let the total distance traveled by the volleyball be $d = d_1 + d_2 = \dfrac{9 + k - x_0}{\cos\phi}$.

Since the volleyball has a radius of 0.105 m, we will represent the height of the net as the sum of the net height and the volleyball radius, 2.43 m $+0.105$ m $= 2.535$ m. We can now represent all possible serves with the following diagram.



4 Gravity Model

If we only consider gravity, the only force that acts on the volleyball once it is served is gravity, which we will assume is $9.8\ \mathrm{m/s^2}$ downwards. This means the motion of the volleyball can be modeled as a parabola.

Since there is no acceleration in the x-direction, the horizontal velocity is constant at $v_0 \cos\theta$, which means that the total distance traveled is $x = v_0 t \cos\theta$, where x is the total horizontal distance traveled and t is the time in the air. Rearranging yields

$$t = \frac{x}{v_0 \cos\theta}.$$

The y movement can be modeled with

$$y = y_0 + v_0 t \sin\theta - \frac{1}{2} g t^2.$$

We set $y = 0$ in order to find the time it takes for the volleyball to hit the ground. Plugging $x = d$ into the first equation gives

$$0 = y_0 + \frac{d v_0 \sin\theta}{v \cos\theta} - \frac{1}{2} g \frac{d^2}{v_0^2 \cos^2\theta} = y_0 + d\tan\theta - \frac{g d^2}{2 v_0^2 \cos^2\theta}.$$

Rearranging gives us the optimized initial velocity

$$\frac{g d^2}{2 v_0^2 \cos^2\theta} = y_0 + d\tan\theta,$$

$$2 v_0^2 \cos^2\theta = \frac{g d^2}{y_0 + d\tan\theta},$$

$$v_0^2 = \frac{g d^2}{(2\cos^2\theta)(y_0 + d\tan\theta)},$$

$$v_0 = \frac{d}{\cos\theta}\sqrt{\frac{g}{2(y_0 + d\tan\theta)}}.$$

The higher the maximum height of the volleyball, the longer it takes to fall, so we want to minimize the height in order to minimize the air time. Thus, the volleyball should pass as close as possible to the net. Plugging in $x = d_1$ into equation 1 to get the time the volleyball takes to get to the net gives us $t = \dfrac{d_1}{v_0 \cos\theta}$. We can then plug in $y = 2.535$ and our expression for t into the equation for y above to get

$$\begin{aligned}
2.535 &= y_0 + v_0 t \sin\theta - \frac{1}{2} g t^2 \\
&= y_0 + \frac{d_1 v_0 \sin\theta}{v_0 \cos\theta} - \frac{1}{2} g \left(\frac{d_1}{v_0 \cos\theta} \right)^2 \\
&= y_0 + d_1 \tan\theta - \frac{g d_1^2}{2 v_0^2 \cos^2\theta} \\
&= y_0 + d_1 \tan\theta - \frac{g d_1^2}{\frac{g d^2}{y_0 + d \tan\theta}} \\
&= y_0 + d_1 \tan\theta - \frac{d_1^2 (y_0 + d \tan\theta)}{d^2} \\
&= y_0 + d_1 \tan\theta - \frac{d_1^2 y_0}{d^2} - \frac{d_1^2 \tan\theta}{d}.
\end{aligned}$$

Rearranging yields

$$2.535 - y_0 + \frac{d_1^2 y_0}{d^2} = \left(d_1 - \frac{d_1^2}{d} \right) \tan\theta,$$

so

$$\begin{aligned}
\tan\theta &= \frac{2.535 - y_0 + \frac{d_1^2 y_0}{d^2}}{d_1 - \frac{d_1^2}{d}} \\
&= \frac{d^2 (2.535 - y_0) + y_0 d_1^2}{d_1 d_2 d} \\
&= \frac{2.535 d^2 + y_0 (d_1^2 - d^2)}{d_1 d_2 d}.
\end{aligned}$$

Taking the inverse tangent of both sides yields the optimized angle

$$\theta = \tan^{-1} \left(\frac{2.535 d^2 + y_0 (d_1^2 - d^2)}{d_1 d_2 d} \right).$$

Thus, we can obtain the optimal time, velocity, and angle given the starting height and the path of the volleyball:

$$\begin{aligned} t &= \frac{d}{v_0 \cos\theta}, \\ v_0 &= \frac{d}{\cos\theta}\sqrt{\frac{g}{2(y_0 + d\tan\theta)}}, \\ \theta &= \tan^{-1}\left(\frac{2.535d^2 + y_0(d_1^2 - d^2)}{d_1 d_2 d}\right). \end{aligned}$$

5 Gravity Model Solutions

We derived 3 equations above for the optimal time, initial velocity, and angle. We notice that we can plug in the third equation into the second equation, and then the second equation into the first equation in order to get a formula for time. Multiplying both sides of the second equation by $\cos\theta$ and then plugging it in the first equation yields

$$t = \frac{d}{v_0\cos\theta} = \frac{d}{d\sqrt{\frac{g}{2(y_0+d\tan\theta)}}} = \sqrt{\frac{2(y_0 + d\tan\theta)}{g}}.$$

We have $\tan\theta$ from the equation for the angle. Plugging it in to the equation above gives us

$$\begin{aligned} t &= \sqrt{\frac{2\left(y_0 + d\left(\frac{2.535d^2+y_0(d_1^2-d^2)}{d_1d_2d}\right)\right)}{g}} \\ &= \sqrt{\frac{2(y_0d_1d_2 + 2.535d^2 + y_0(d_1^2 - d^2))}{gd_1d_2}} \\ &= \sqrt{\frac{2(2.535d^2 + y_0(d_1d_2 + d_1^2 - d^2))}{gd_1d_2}} \\ &= \sqrt{\frac{2d(2.535d - y_0d_2)}{gd_1d_2}}. \end{aligned}$$

Thus, we have a formula for t in terms of the distances. In order to optimize a volleyball serve, we must minimize t. We know that $d_1 = \dfrac{9 - x_0}{\cos\phi}$, $d_2 = \dfrac{k}{\cos\phi}$, and $d = \dfrac{9 + k - x_0}{\cos\phi}$, where $0 < k \leq 9$. Plugging these in and simplifying yields

$$
\begin{aligned}
t &= \sqrt{\frac{2d(2.535d - y_0 d_2)}{g d_1 d_2}} \\
&= \sqrt{\frac{2\left(\frac{9+k-x_0}{\cos\phi}\right)\left(2.535\left(\frac{9+k-x_0}{\cos\phi}\right) - \frac{ky_0}{\cos\phi}\right)}{kg\left(\frac{9-x_0}{\cos^2\phi}\right)}} \\
&= \sqrt{\frac{2(9+k-x_0)(2.535(9+k-x_0) - ky_0)}{kg(9-x_0)}}.
\end{aligned}
$$

We notice something very interesting. The time only depends on the 2 starting values, x_0 and y_0, and not on the angle the volleyball is hit cross court. Intuitively, this means that if the volleyball is hit cross court, there is more distance to travel, but the volleyball can also be hit harder, so the total time remains the same.

6 Optimal Distances

We will now find the optimal distance k away from the net that the volleyball should land. We suspect that the minimized time occurs when $k = 9$, since the volleyball would be able to be served at a flatter angle. So, we will test to see if t is a decreasing function. We let $f(x) = \sqrt{x}$ and $g(k)$ be the radicand of t. Thus, $t = f(g(k))$. Differentiating yields

$$\frac{dt}{dk} = f'(g(k))g'(k) = \frac{1}{2\sqrt{g(k)}}g'(k) = \frac{1}{2t}g'(k).$$

Since t is always positive, $\frac{dt}{dk}$ is always negative if $\frac{d}{dk}g(k)$ is negative for all $0 < k \leq 9$. We will let $a = 9 - x_0$, so $a > 0$, since the distance from the starting spot to the net has to be positive.

Thus, we have

$$
\begin{aligned}
\frac{dg}{dk} &= \frac{d}{dk}\left(\frac{2(a+k)(2.535(a+k)-ky_0)}{agk}\right) \\
&= \frac{1}{ag}\frac{d}{dk}\left(\frac{5.07(a+k)^2-2aky_0-2k^2y_0}{k}\right) \\
&= \frac{k(10.14(a+k)-2ay_0-4ky_0)-5.07(a+k)^2+2aky_0+2k^2y_0}{agk^2} \\
&= \frac{(a+k)(10.14k-5.07(a+k))-2k^2y_0}{agk^2} \\
&= \frac{2.535(a+k)(k-a)-k^2y_0}{\frac{1}{2}agk^2}
\end{aligned}
$$

We want to show that $\frac{dg}{dk} < 0$. Since $\frac{1}{2}agk^2 > 0$, we must prove

$$2.535(a+k)(k-a) < k^2y_0.$$

In order to prove that this inequality is true, we will assume that $x_0 < y_0$. This is a reasonable assumption as the initial vertical height of the volleyball is the sum of the reach of a person and the height they jump, which is greater than the horizontal distance they jump forward. Rearranging the inequality yields

$$
\begin{aligned}
2.535(a+k)(k-a) &< k^2y_0 \\
2.535(k^2-a^2) &< k^2y_0 \\
k^2(y_0-2.535)+2.535a^2 &> 0.
\end{aligned}
$$

This is a quadratic in k. If $y_0 \geq 2,535$, then the left hand side of the inequality is positive, which makes this inequality true.

We now need to prove the inequality for $y_0 < 2.535$. In this case, the quadratic opens down. Since the y-intercept $2.535a^2$ is positive, the x values between the x-intercepts are positive. Since we only need to prove the inequality for $0 < k \leq 9$, the inequality is true if the positive x-intercept is at least 9. To solve for the positive x-intercept, we set the quadratic equal to 0 and solve:

$$\begin{aligned} k^2(2.535 - y_0) &= 2.535a^2 \\ k^2 &= \frac{2.535a^2}{2.535 - y_0} \\ k &= \sqrt{\frac{2.535a^2}{2.535 - y_0}} \\ k &= a\sqrt{\frac{2.535}{2.535 - y_0}}. \end{aligned}$$

We could square root above because the fraction is positive. We can plug in $a = 9 - x_0$. Thus, we need to show that

$$(9 - x_0)\sqrt{\frac{2.535}{2.535 - y_0}} > 9,$$

so

$$\frac{9 - x_0}{9} > \sqrt{\frac{2.535 - y_0}{2.535}}.$$

Since $x_0 < y_0$, we have

$$\frac{9 - x_0}{9} > \frac{2.535 - x_0}{2.535} > \frac{2.535 - y_0}{2.535} > \sqrt{\frac{2.535 - y_0}{2.535}}.$$

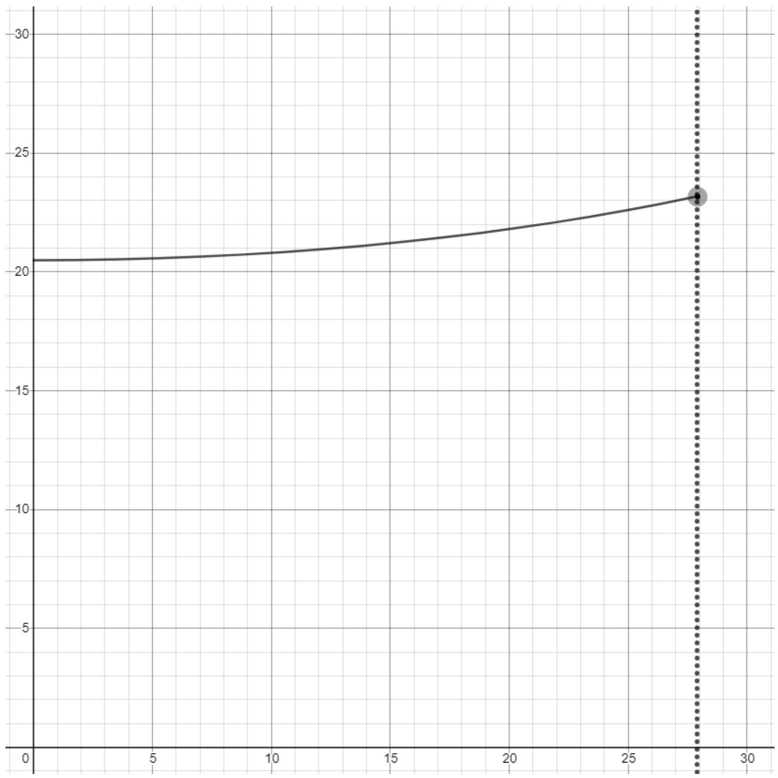
Thus, $\dfrac{dt}{dk}$ is negative for all $0 < k \leq 9$, so t is a strictly decreasing function. This means that the minimum possible time that volleyball spends in the air occurs when the volleyball is aimed for the end-line on the other court, so $k = 9$.

7 Real World Example

We found that the minimum time is not affected by ϕ and that it is optimal to serve at the end-line of the court. However, the velocity still depends on the cross-court angle that the volleyball is hit. We will model a potential real world situation with reasonable values. We let $x_0 = 1$ and $y_0 = 3.2$. Since the volleyball can travel at most 17 m forward and 9 m to the side, the maximum value for ϕ

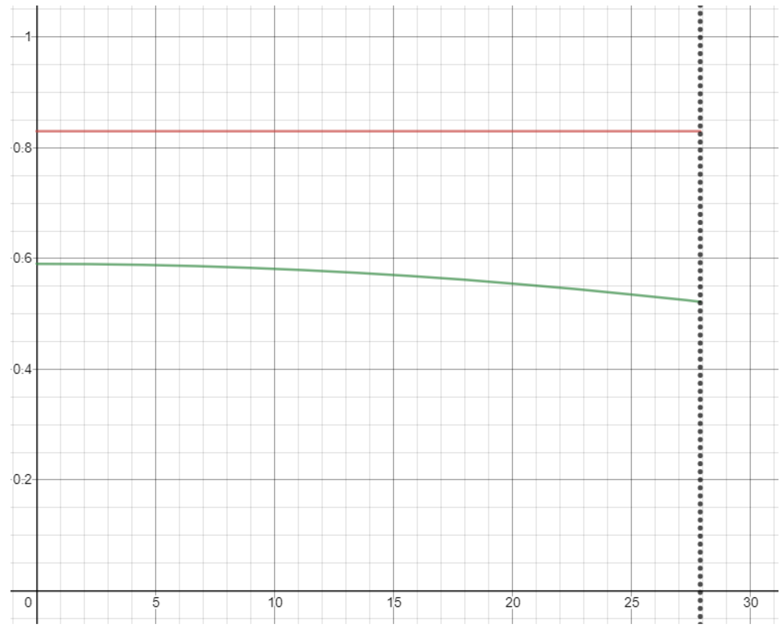
is $\tan^{-1}\left(\dfrac{9}{17}\right)$.

We can plug these values into the equation to get the following graphs



A graph of velocity (m/s) vs ϕ (degrees).

In this graph, as the cross-court angle increases, the velocity increases. At $\phi = \tan^{-1}\left(\dfrac{9}{17}\right)$, we have $v_0 = 23.2$ m/s, which is the maximum possible velocity. We can also graph the time the volleyball is in the air and the vertical angle the volleyball is hit initially.



A graph of time (t) on the top and θ (degrees) on the bottom vs ϕ (degrees).

As we found earlier, the total time remains the same at $t = 0.83$ s. However, the angle hit vertically decreases slightly as the cross-court angle increases. Since the volleyball is hit from a height of 3.2 m, the volleyball can be hit at an almost flat angle of $0.59°$ to $0.52°$.

8 Conclusion

In this exploration, we found the optimal volleyball serve given only the cross-court angle and the starting position of the volleyball. We found that the volleyball should always be served at the end-line and that the cross-court angle does not affect the air time, which is an important metric in determining how good a volleyball serve is. However, serving to the opposite corner increases the speed of the volleyball, which is also beneficial as it is harder to pass a faster volleyball.

This exploration could be improved by considering the effects of drag and spin. In our current model, we do not consider the spin, which means that the volleyball is a float serve. Including these other two variables would allow a more complete analysis of volleyball serves.