Parametric Equations of Circles on Spheres

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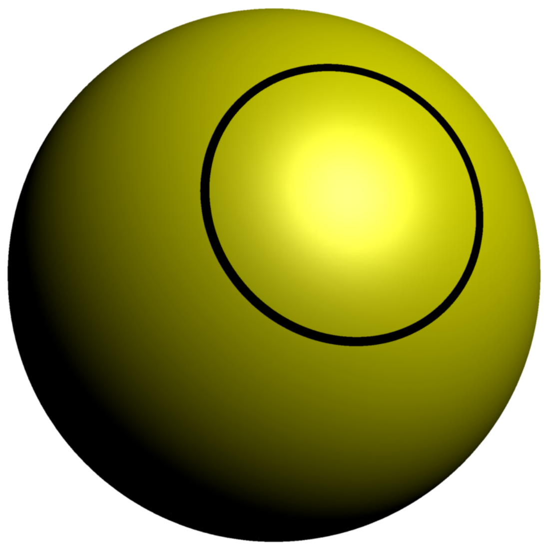


In this exploration, we will attempt to find the form of parametric equations of a circle on the surface of a sphere. The parametric equations will be based only on the spherical coordinates of the center of the circle, the radius of the circle, and the radius of the sphere. We will first find the parametric equations for circles around the z-axis. We will then extend this to all circles on spheres by finding rotation matrices. Multiplying those with the parametric equations earlier will yield the parametric equations of any circle on a sphere.

1 Introduction

On the surface of a sphere, both lines and curved paths can be modeled by circles. There are many practical applications of this because it allows us to model the path of objects, such as airplanes and satellites. It is important that we find the parametric equations for circles on spheres because the x, y, and z values are defined based on a parameter in the equations. This allows us to model the circles at any point in time and lets us find a point on the path at any given time.

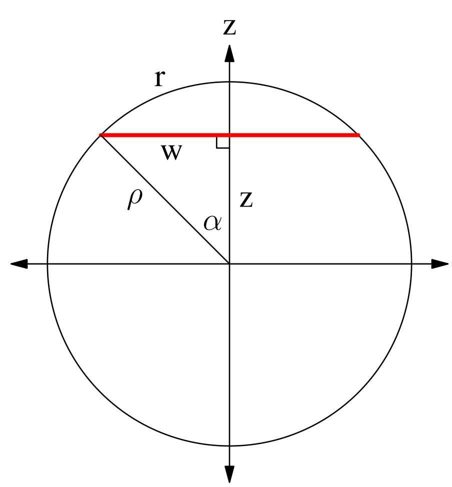
In order to find the form of the parametric equations of circles on spheres, we need to base them on certain variables. We will attempt to find the parametric equations given the radius of the circle, the radius of the sphere, and the center of the circle.

We will define the radius r of the circle as the distance from the center of the circle to the circle along the surface of the sphere. We let the radius of the sphere be ρ. The center of the circle will be in spherical coordinates because spherical coordinates are based on angles and the distance from the center of the earth, which is very similar to the longitude and latitude system.

2 Circles around the z Axis

We will first find the parametric equations for circles on a sphere centered around the z-axis. The parametric equations will be based on the radius of the circle only because all of these circles have the same center at the point $(0, 0, \rho)$. This will only find the parametric equations for all possible radii. We will later generalize the parametric equations for any circle on a sphere by rotating this circle with rotation matrices. This will allow the parametric equations to cover all possible centers in addition to all possible radii.

We let α be the angle the center of the sphere to the circle makes with the positive z-axis, w be the perpendicular distance from the z axis to the circle, and z be the z coordinate of each point on the circle. The cross section of the sphere is shown below. In the graph, the origin is the center of the sphere and the red line is the circle.

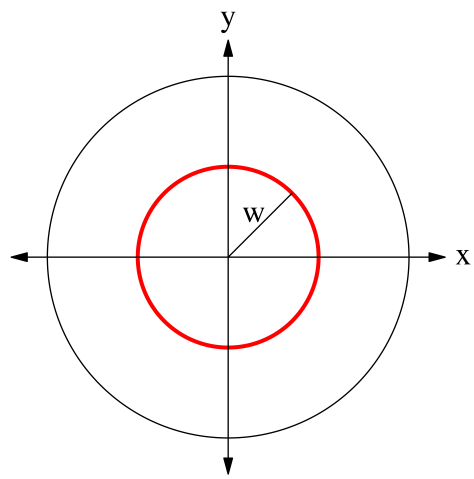


Since nothing changes if the cross section is rotated around the z-axis, the other axis is not labeled.

Using the formula for the length of an arc, we get $\alpha\rho = r$. Dividing both sides by ρ gives $\alpha = \frac{r}{\rho}$. Since we constructed a right triangle, we know that $\cos\alpha = \frac{z}{\rho}$ and $\sin\alpha = \frac{w}{\rho}$. Rearranging and using the fact that $\alpha = \frac{r}{\rho}$ yields

$$z = \rho\cos\frac{r}{\rho} \qquad \text{and} \qquad w = \rho\sin\frac{r}{\rho}.$$

We now take the cross section of the sphere in the xy plane. The red line is the circle like before.



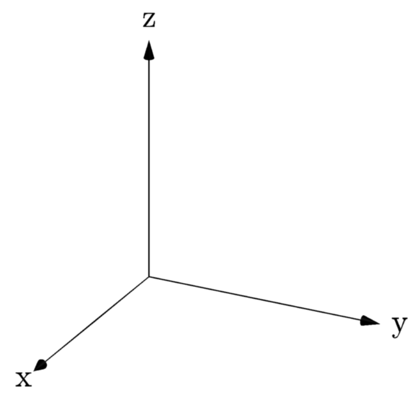
The parametric equations of this circle are $x = w\cos t$ and $y = w\sin t$. We can confirm this by squaring the equations and adding them. This gives $x^2 + y^2 = w^2(\sin^2 t + \cos^2 t)$. Since $\sin^2 t + \cos^2 t = 1$, we can substitute this in to get $x^2 + y^2 = w^2$, which is the equation of a circle centered at the origin with a radius of w.

Earlier we found that $w = \rho\sin\frac{r}{\rho}$, so we can substitute this into the parametric equations of this circle. Combining this with our z value that we found earlier, we have

$$x = \rho\sin\frac{r}{\rho}\cos t$$
$$y = \rho\sin\frac{r}{\rho}\sin t$$
$$z = \rho\cos\frac{r}{\rho}$$

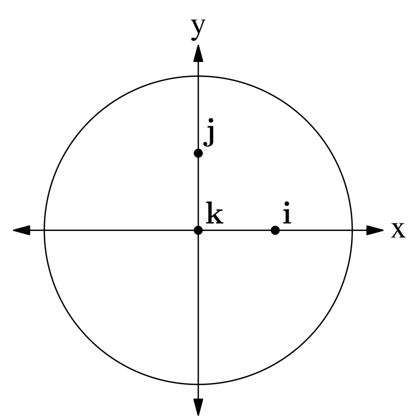
as the parametric equations for any circle on a sphere centered around the z-axis.

3 Rotation around the y and z Axis



To generalize the parametric equations for any circle on a sphere, we need to find a way to rotate the parametric equations for circles around the x-axis. Rotation matrices are a good way to accomplish this.

We first find the rotation matrix for rotating counterclockwise around the z-axis. We let the rotation matrix be $\mathbf{A}$. The unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are shown below in the xy plane.



From this graph, we see that

$$\mathbf{Zi} = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix}$$

because Z rotates $\mathbf{Z}$ rotates $\mathbf{i}$ by θ. Similarly, we know that

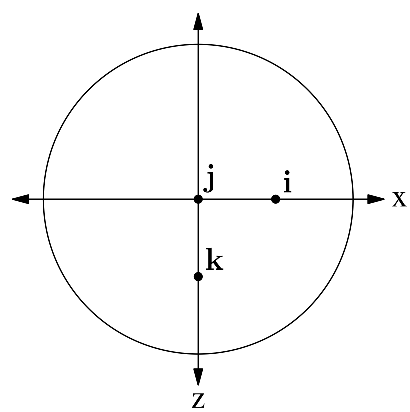
$$\mathbf{Zj} = \begin{bmatrix} \cos(\theta + \frac{\pi}{2}) \\ \sin(\theta + \frac{\pi}{2}) \\ 0 \end{bmatrix} = \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix}$$

because it is the same thing as $\mathbf{Zi}$ except rotated $\frac{\pi}{2}$. Since $\mathbf{k}$ is on the z-axis, rotating around the z-axis does not affect it. So, $\mathbf{Zk} = \mathbf{k}$.

Since multiplying a matrix by $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ produces the first, second, and third columns respectively, we combine the three vectors to get the rotation matrix around the z-axis:

$$\mathbf{Z} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We now find the rotation matrix for rotating counterclockwise around the y-axis. We let the rotation matrix be $\mathbf{Y}$. The unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are shown below in the xz plane. The positive z-axis is on the bottom because the orientation of the axes was kept the same as last time. This is important because we defined the rotation matrices as matrices for rotating counterclockwise.



Since the positive z axis is "pointing" down instead of up and since $\sin\phi$ is the y value in the Cartesian plane, this is similar to the usual Cartesian plane, except every time we use $\sin$, it will have the opposite sign and become $-\sin$. So,

$$\mathbf{Yi} = \begin{bmatrix} \cos\phi \\ 0 \\ -\sin\phi \end{bmatrix}.$$

Since $\mathbf{j}$ is on the y-axis, rotating around the y-axis does not affect it. So, $\mathbf{Zj} = \mathbf{j}$. Since $\mathbf{k}$ is a $-\frac{\pi}{2}$ rotation of $\mathbf{i}$, we have

$$\mathbf{Yk} = \begin{bmatrix} \cos(\phi + \frac{\pi}{2}) \\ 0 \\ -\sin(\phi - \frac{\pi}{2}) \end{bmatrix} = \begin{bmatrix} \sin\phi \\ 0 \\ \cos\phi \end{bmatrix}.$$

Combining these three vectors produces the three columns of the rotation matrix, so

$$\mathbf{Y} = \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix}.$$

4 Parametric Equations

Now that we have parametric equations for circles on a sphere around the z axis and rotation matrices, we can multiply them together to get the parametric equations for any circle on a sphere. We want the center of the sphere to be defined in spherical coordinates because that is similar to the longitude and latitude system.

First, we let the center of the circle be (r, θ, ϕ) in spherical coordinates. Since the parametric equations we found earlier were centered at $(\rho, 0, 0)$ in spherical coordinates, we do not need to rotate anything back there. We first rotate by $\mathbf{Y}$ to get the ϕ value and then rotate by $\mathbf{Z}$ to get the θ value. So, the desired parametric equations are

$$\mathbf{Z}\left(\mathbf{Y}\begin{bmatrix}\rho\sin\frac{r}{\rho}\cos t\\ \rho\sin\frac{r}{\rho}\sin t\\ \rho\cos\frac{r}{\rho}\end{bmatrix}\right).$$

Since matrix multiplication is associative, we can first find $\mathbf{ZY}$.

$$\mathbf{ZY}=\begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\cos\phi & 0 & \sin\phi\\ 0 & 1 & 0\\ -\sin\phi & 0 & \cos\phi\end{bmatrix}=\begin{bmatrix}\cos\theta\cos\phi & -\sin\theta & \cos\theta\sin\phi\\ \sin\theta\cos\phi & \cos\theta & \sin\theta\sin\phi\\ -\sin\phi & 0 & \cos\phi\end{bmatrix}.$$

Now we can multiply this by the parametric equations we found earlier.

$$\begin{bmatrix}\cos\theta\cos\phi & -\sin\theta & \cos\theta\sin\phi\\ \sin\theta\cos\phi & \cos\theta & \sin\theta\sin\phi\\ -\sin\phi & 0 & \cos\phi\end{bmatrix}\begin{bmatrix}\rho\sin\frac{r}{\rho}\cos t\\ \rho\sin\frac{r}{\rho}\sin t\\ \rho\cos\frac{r}{\rho}\end{bmatrix}=$$

$$\begin{bmatrix}\rho\sin\frac{r}{\rho}\cos t\cos\theta\cos\phi-\rho\sin\frac{r}{\rho}\sin t\sin\theta+\rho\cos\frac{r}{\rho}\cos\theta\sin\phi\\ \rho\sin\frac{r}{\rho}\cos t\sin\theta\cos\phi+\rho\sin\frac{r}{\rho}\sin t\cos\theta+\rho\cos\frac{r}{\rho}\sin\theta\sin\phi\\ -\rho\sin\frac{r}{\rho}\cos t\sin\phi+\rho\cos\frac{r}{\rho}\cos\phi\end{bmatrix}.$$

So the parametric equations of circles on spheres are:

$$\begin{aligned}
x&=\rho\sin\frac{r}{\rho}\cos t\cos\theta\cos\phi-\rho\sin\frac{r}{\rho}\sin t\sin\theta+\rho\cos\frac{r}{\rho}\cos\theta\sin\phi\\
y&=\rho\sin\frac{r}{\rho}\cos t\sin\theta\cos\phi+\rho\sin\frac{r}{\rho}\sin t\cos\theta+\rho\cos\frac{r}{\rho}\sin\theta\sin\phi\\
z&=-\rho\sin\frac{r}{\rho}\cos t\sin\phi+\rho\cos\frac{r}{\rho}\cos\phi.
\end{aligned}$$

5 Real World Applications

We will now show an example of how useful these parametric equations are. I have been to China many times, so I will model the path between Maryland, USA and Beijing, China.

We first need to find the coordinates of the locations. The spherical coordinates of Maryland are $(6371, -77°, 39°)$ and the spherical coordinates of Beijing are $(6371, 116°, 40°)$. The first coordinate is in kilometers. The second and third coordinates are longitude and latitude, respectively.

There are infinite possible paths between two points, but we will model the shortest possible path between them. To do that, we need to find the vector orthogonal to the vectors from the points to the center of the sphere. We first convert the coordinates into spherical coordinates. The coordinates of Maryland are

$$\begin{bmatrix} 6371\cos(-77°)\sin(39°) \\ 6371\sin(-77°)\sin(39°) \\ 6371\cos(39°) \end{bmatrix} = \begin{bmatrix} 902 \\ -3907 \\ 4951 \end{bmatrix}.$$

The coordinates of Beijing are

$$\begin{bmatrix} 6371\cos(116°)\sin(40°) \\ 6371\sin(116°)\sin(40°) \\ 6371\cos(40°) \end{bmatrix} = \begin{bmatrix} -1795 \\ 3681 \\ 4880 \end{bmatrix}.$$

The cross product of two vectors yields a vector orthogonal to them. So,

$$\begin{bmatrix} 902 \\ -3907 \\ 4951 \end{bmatrix} \times \begin{bmatrix} -1795 \\ 3681 \\ 4880 \end{bmatrix} = \begin{bmatrix} -37290309 \\ -13290262 \\ -3693538 \end{bmatrix}.$$

However, this is not on the sphere. We can scale it onto the sphere by dividing by the magnitude and then multiplying by 6371. The magnitude of the cross product is

$$\sqrt{(-37290309)^2 + (-13290262)^2 + (-3693538)^2} = 39759784.$$

Multiplying the cross product by $\dfrac{6371}{39759784}$ yields

$$\frac{6371}{39759784}\begin{bmatrix} -37290309 \\ -13290262 \\ -3693538 \end{bmatrix} = \begin{bmatrix} -5975 \\ -2130 \\ -592 \end{bmatrix}.$$

We now need to convert back to spherical coordinates. The ρ value is clearly 6371 because the point is on the sphere. Since $\tan\theta = \dfrac{y}{x} = \dfrac{-2130}{-5975}$, we have $\theta = 20°$. Since $\cos\phi = \dfrac{z}{\rho} = \dfrac{-592}{6371}$, we have $\phi = 95°$.

So, the spherical coordinates of the orthogonal vector are $(6371, 20°, 95°)$. We also need to know the "radius" of the circle. Since the shortest path between two points lies on one of the great circles, we need to find one fourth of the circumference of the circle. So, the radius of the circle would be

$$\frac{1}{4}(2\pi r) = \frac{6371}{2}\pi = 10008.$$

We can now plug everything in to the equation of a circle on a sphere that we derived earlier.

$$x = 6371 \sin\frac{10008}{6371}\cos t \cos 20° \cos 95° - 6371 \sin\frac{10008}{6371}\sin t \sin 20° + 6371 \cos\frac{10008}{6371}\cos 20° \sin 95°$$
$$y = 6371 \sin\frac{10008}{6371}\cos t \sin 20° \cos 95° + 6371 \sin\frac{10008}{6371}\sin t \cos 20° + 6371 \cos\frac{10008}{6371}\sin 20° \sin 95°$$
$$z = -6371 \sin\frac{10008}{6371}\cos t \sin 95° + 6371 \cos\frac{10008}{6371}\cos 95°.$$

Simplifying yields the final result:

$$x = -557\cos t - 2139 \sin t$$

$$y = -199\cos t + 6001 \sin t$$

$$z = -6343\cos t.$$

This process can be repeated for any two points to find the path between them. These equations will allow a computer to model paths over the surface of the earth, which has numerous useful applications.