Extracting Sparse Eilenberg-MacLane Coordinates via Principal Bundles

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1. Context / Background
2. Framework
3. Methodology
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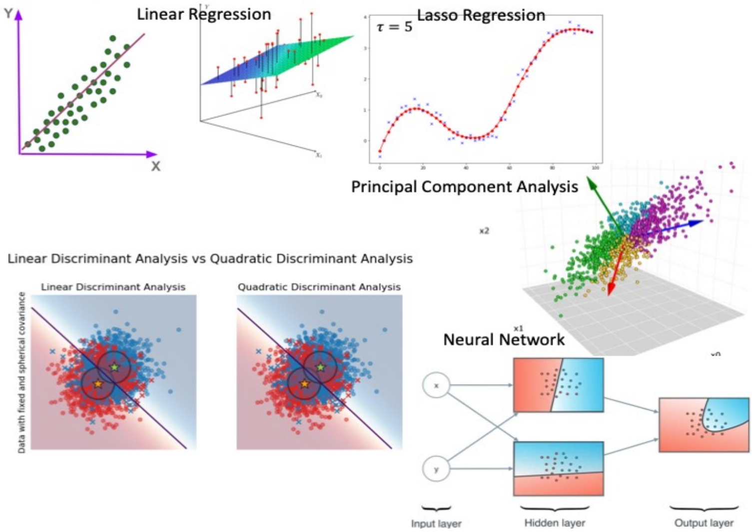
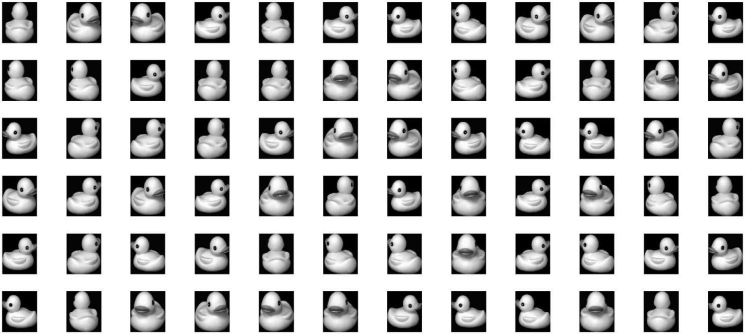
Some common data analysis models



Example 1. Rotating Ducks [Columbia Object Image Library, 1996]

Data $\subset \mathbb{R}^{128\times 128}$



https://www.cs.columbia.edu/CAVE/software/softlib/coil-20.php

Figure 2: Rotating Ducks

Example 2. Planar Equilateral Pentagons

Data $\subseteq \{(z_1, ..., z_5) \in \mathbb{C}^5 \mid |z_1 - z_2| = |z_2 - z_3| = |z_3 - z_4| = |z_4 - z_5| = |z_5 - z_1| = 1\}$

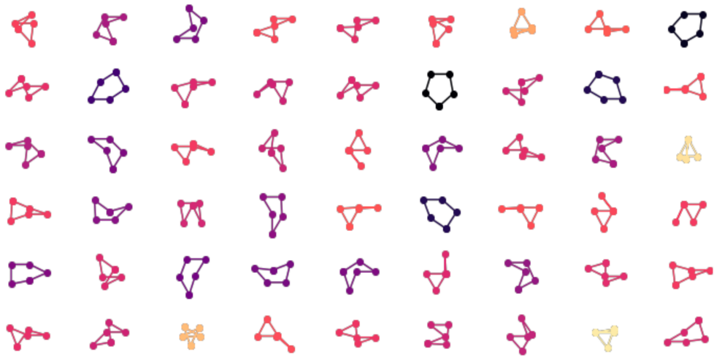
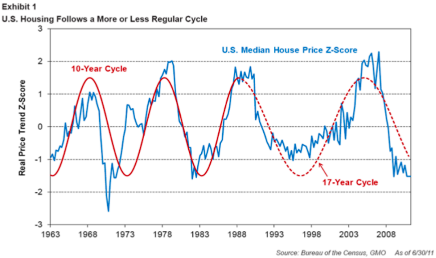
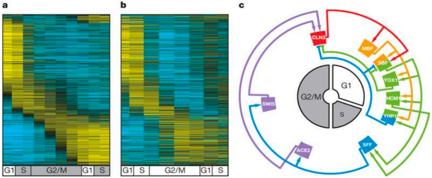
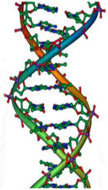


Figure 3: Planar Unilateral Pentagons

Example 3. Recurrent Time Series







Global control of cell-cycle transcription by coupled SDK and network oscillators, D. Orlando et. al., Nature, 2008

Figure 4: Recurrent Time Series

Question. How to analyze these non-contractible spaces?

Goal: To identify the hole in the middle.
Ambient space: $(\mathbb{M}, d) := \mathbb{R}^2$
Underlying space: $\mathbb{X} := S^1$
Dataset: $X := \{(x_i, y_i)\}_{i=1}^N$ a finite set

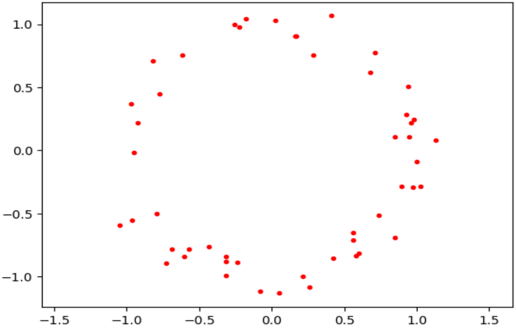


Figure 5: Dataset

Modern workflow of Topological Data Analysis (TDA)

Per TDA's usual convention, let's turn to codes

Upshot of the modern workflow of TDA

- TDA can tell you what your dataset looks like, without making more assumptions
- Homology groups (i.e. number of holes in all dimensions) are guaranteed to be recovered.
- Thm (Guarantee of Recovery)

  [Niyogi, Smale, Weinberger, 2008]

  If $\mathbb{X} \subseteq \mathbb{R}^N$ is a compact, differentiable manifold, and if $X \subseteq \mathbb{R}^N$ satisfies

  $$d_H(X, \mathbb{X}) < \sqrt{\frac{3}{20}}\mathsf{rch}(\mathbb{X})$$

  Then, $\forall \alpha$ s.t. $2d_H(X, \mathbb{X}) < \alpha < \sqrt{\frac{3}{5}}\mathsf{rch}(\mathbb{X})$, we have

  $$X^{(\alpha)} \simeq \mathbb{X}$$

Demand for more recovery

We can tell how many *n*-holes in $\mathbb{X}$! Sounds great!!
Little Test: Identify the homotopy type ("shape") of the Rotating Ducks.

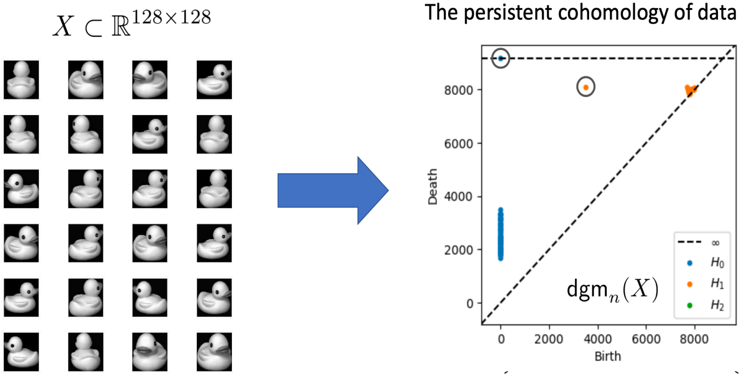
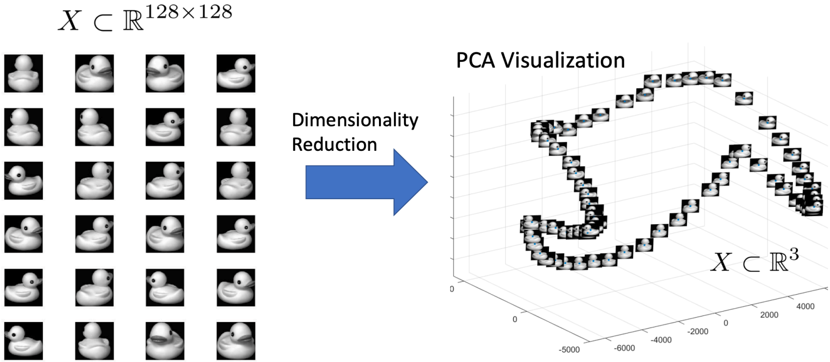


Figure 6: Rotating Ducks (Persistence Diagram)

Demand for more recovery



https://www.cs.columbia.edu/CAVE/software/softlib/coil-20.php

Figure 7: Rotating Ducks (Principal Component Analysis)

Demand for more recovery

Big Test: Identify the homotopy type ("shape") of the moduli space of equilateral pentagons.

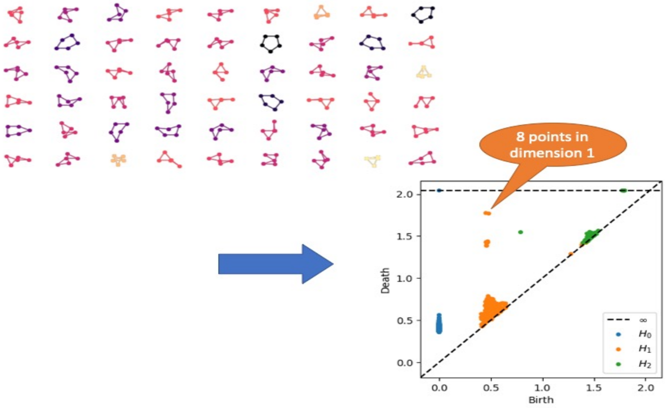


Figure 8: Moduli space of Equilateral Pentagons (Persistence Diagram)

Demand for more recovery

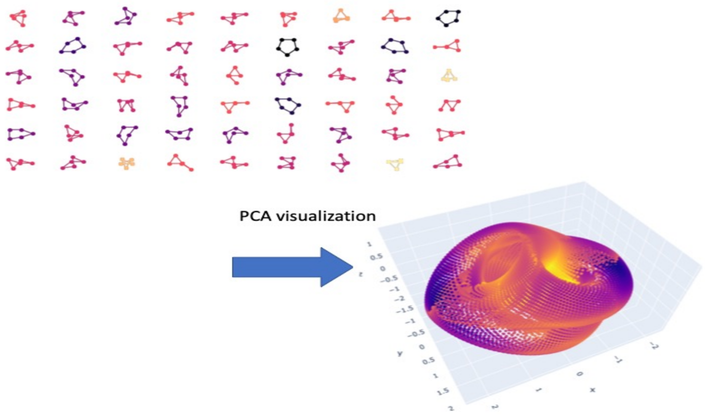


Figure 9: Moduli space of Equilateral Pentagons (Principal Component Analysis)

Demand for more recovery from $\eta \in H^n(|K_\alpha|; G)$

Here is the IDEA:

- We collect all the "important" generators $\eta \in H^n(|K_\alpha|; G)$ from the persistence diagram
- Brown representability theorem $H^n(|K_\alpha|; G) \cong [|K_\alpha|, K(G, n)]$ implies that η represents a map $f_\eta : |K_\alpha| \to K(G, n)$

  (Remark: $f_\eta : |K_\alpha| \to K(G, n)$ is called an Eilenberg-MacLane coordinate)
- When $K_\alpha = \mathcal{R}_\alpha(X)$ or $\check{C}_\alpha(X)$, there is a homotopy equivalence $X^{(\alpha)} \simeq |K_\alpha|$ under appropriate conditions
- Therefore, η yields a map $X^{(\alpha)} \to K(G, n)$
- (Future work) These maps f_η are planned to be recombined together to fully recover $\mathbb{X}$ by Postnikov tower

Eilenberg-MacLane Spaces and Homotopy Groups

Def (Homotopy Groups) Let X be a topological space, with $x_0 \in X$. A homotopy group of dimension n is

$$\pi_n(X, x_0) := \{f : S^n \to X \mid f(s_0) = x_0\} / \simeq$$

Def (Eilenberg-MacLane Spaces) Let G be a group and $n \geq 1$ be an integer. An *Eilenberg-MacLane space* $K(G, n)$ is a connected space whose homotopy groups are only nontrivial (as G) at dimension n, i.e. $\pi_i(K(G, n)) = \begin{cases} G & (\text{if } i = n) \\ 0 & (\text{else}) \end{cases}$

Ex (Eilenberg-MacLane Spaces)

- $K(\mathbb{Z}, 1) = S^1$
- $K(\mathbb{Z}, 2) = \mathbb{CP}^\infty$
- $K(\mathbb{Z}_2, 1) = \mathbb{RP}^\infty$
- $K(\mathbb{Z}^n, 1) = \mathbb{T}^n = S^1 \times \ldots \times S^1$

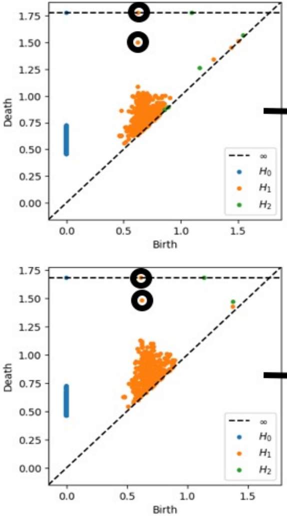
Previous Work

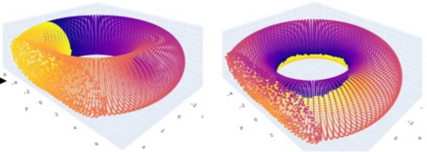
The goal of this project: To derive explicit formulas and algorithms on $X^{(\alpha)} \to K(G,n)$ given by $\eta \in H^n(|K_\alpha|; G)$.

Previous work on particular cases of $K(G,n)$:

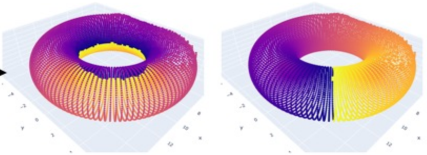
- [Jose A. Perea, 2018]
  Real / complex projective coordinates: $X^{(\alpha)} \to \mathbb{RP}^\infty, \mathbb{CP}^\infty$
- [Jose A. Perea, 2019]
  Circular coordinates: $X^{(\alpha)} \to S^1$
- [Luis Scoccola et al, 2023]
  Toroidal coordinates: $X^{(\alpha)} \to T^d$
- [L. Polanco, 2019]
  Len-space coordinates: $X^\alpha \to L_n$

Expected Results





$$X^{(\alpha)} \to S^1$$
$$b \in B_\alpha(x_j) \mapsto \exp\left\{2\pi i \cdot \left(\tau_j + \sum_{k=1}^{N} \varphi_k(b)\theta_{jk}\right)\right\}$$



Expected Results

Circular coordinate gives us the structure of the moduli space of Equilateral Pentagons!

Primary Stages

- Methodology Stage: Derive an explicit formula for $\check{H}^n(B;\mathcal{F}_G) \cong [B, K(G,n)]$ using the theory of principal G-bundles. The formula is planned to be acquired by:

$$\check{H}^n(B,\mathcal{F}_G) \xrightarrow[\cong]{\text{Map 3}} \check{H}^{n-1}(B,\mathcal{F}_{BG}) \xrightarrow[\cong]{\text{Map 3}} \dots \xrightarrow[\cong]{\text{Map 3}} \check{H}^1(B,\mathcal{F}_{B^{n-1}(G)})$$

$$\check{H}^1(B,\mathcal{F}_{B^{n-1}(G)}) \xrightarrow[\cong]{\text{Map 2}} \mathrm{Prin}_{B^{n-1}(G)}(B) \xrightarrow[\cong]{\text{Map 1}} [B, B^n(G)] \xrightarrow{\cong} [B, K(G,n)]$$

- Algorithm & Experiment Stage: To develop an algorithm to construct $f_\eta : X^{(\alpha)} \to K(G,n)$ from a cocycle representative $[\eta] \in \check{H}^n(|R_\alpha(X)|;\mathcal{F}_G)$. Then, we will have experiments on some examples of spaces to test our algorithm.
- Stability Theory Stage: We will show that the proposed coordinates satisfy certain stability properties, i.e., small perturbations to the input X result in small changes to the output coordinates.

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Overall result

- G: discrete Abelian topological group
- B: paracompact Hausdorff topological space
- $\mathcal{F}_G$: the sheaf of maps over B, i.e.,
$\forall U \subseteq_{\text{open}} B, \mathcal{F}_G(U) := \{f : U \to G \mid f \text{ is continuous}\}$
- BG: the classifying space of G given by bar construction (which is another Abelian topological group)

$$\check{H}^n(B, \mathcal{F}_G) \xrightarrow[\cong]{\text{Map 3}} \check{H}^{n-1}(B, \mathcal{F}_{BG}) \xrightarrow[\cong]{\text{Map 3}} \ldots \xrightarrow[\cong]{\text{Map 3}} \check{H}^1(B, \mathcal{F}_{B^{n-1}(G)})$$

$$\check{H}^1(B, \mathcal{F}_{B^{n-1}(G)}) \xrightarrow[\cong]{\text{Map 2}} \text{Prin}_{B^{n-1}(G)}(B) \xrightarrow[\cong]{\text{Map 1}} [B, B^n(G)] \xrightarrow{\cong} [B, K(G, n)]$$

Map 1: $\mathrm{Prin}_G(B) \cong [B, BG]$

Thm/Def (Classifying Spaces) For a topological group G, there exists a weakly contractible space EG with a free continuous action $EG \times G \to EG$. Moreover, defining $BG := EG/G$ by group action will yield a principal G-bundle $G \to EG \to BG$, called a *universal bundle* of G. Any space BG satisfying such a property above is called a *classifying space of* G.

Therefore, the existence of a derived classifying space of a group is always guaranteed. Although there is no unique way of constructing such ones, it turns out that all classifying spaces of the same group G are homotopic equivalent to each other. Several classical construction methods include Milnor construction, Bar construction, Segal construction, etc. For the sake of the construction of higher dimensions $B^n(G) = B(B(...B(G)))$, however, we are required to equip an Abelian group structure for BG (which is not inherited from Milnor construction), hence bar construction is adapted in this project.

Map 1: $\mathrm{Prin}_G(B) \cong [B, BG]$

Def (Simplicial Sets) A *simplicial set* X_* is a sequence of sets $X_* = \{X_n\}_{n\geq 0}$ consisting of these following information:

- Face maps: $d_{n,i} : X_n \to X_{n-1} \qquad (\forall 0 \leq i \leq n)$.
- Degeneracy maps: $s_{n,i} : X_n \to X_{n+1} \qquad (\forall 0 \leq i \leq n)$.

subject to the following five conditions (called simplicial identities):

- $d_i \circ d_j = d_{j-1} \circ d_i \qquad (\forall i < j)$
- $d_i \circ s_j = \begin{cases} s_{j-1} \circ d_i & (\forall i < j) \\ \mathbb{1} & (\forall i = j \text{ or } i = j+1) \\ s_j \circ d_{i-1} & (\forall i > j+1) \end{cases}$
- $s_i \circ s_j = s_{j+1} \circ s_i \qquad (\forall i \leq j)$

Map 1: $\mathrm{Prin}_G(B) \cong [B, BG]$

Def (Bar Construction) Define $EG := |E_*(G)|$ and $BG := |B_*(G)|$ where the $E_*(G)$ and $B_*(G)$ are the simplicial sets given by

$$E_n(G) := G^{n+1} \qquad B_n(G) := G^n$$

$$p_n : G^{n+1} \to G^n; (g_1, g_2, \dots, g_n, g_{n+1}) \mapsto (g_1, g_2, \dots, g_n)$$

$$\begin{tikzcd}
E_*(G): & G \arrow[r, bend right, "s_0"'] \arrow[d, "p_0"] & G^2 \arrow[l, bend right, "d_0;d_1"'] \arrow[r, bend right, "s_0;s_1"'] \arrow[d, "p_1"] & G^3 \arrow[l, bend right, "d_0;d_1;d_2"'] \arrow[r, bend right, "s_0;s_1;s_2"'] \arrow[d, "p_2"] & G^4 \arrow[l, bend right, "d_0;d_1;d_2;d_3"'] \arrow[r, bend right, "\dots"'] \arrow[d, "p_3"] & \dots \arrow[l, bend right, "\dots"'] \\
B_*(G): & 0 \arrow[r, bend right, "\sigma_0"'] & G \arrow[l, bend right, "\delta_0;\delta_1"'] \arrow[r, bend right, "\sigma_0;\sigma_1"'] & G^2 \arrow[l, bend right, "\delta_0;\delta_1;\delta_2"'] \arrow[r, bend right, "\sigma_0;\sigma_1;\sigma_2"'] & G^3 \arrow[l, bend right, "\delta_0;\delta_1;\delta_2;\delta_3"'] \arrow[r, bend right, "\dots"'] & \dots \arrow[l, bend right, "\dots"']
\end{tikzcd}$$

Map 1: $\mathrm{Prin}_G(B) \cong [B, BG]$

with the face maps and degeneracy maps defined as follows

$$d_i(g_1, g_2, \ldots, g_{n+1}) := \begin{cases} (g_2, \ldots, g_{n+1}) & \text{if } i = 0 \\ (g_1, \ldots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \ldots, g_{n+1}) & \text{if } 1 \le i \le n \end{cases}$$

$$s_i(g_1, g_2, \ldots, g_{n+1}) := (g_1, \ldots, g_i, 1_G, g_{i+1}, \ldots, g_{n+1}) \qquad \forall 0 \le i \le n$$

$$\delta_i(g_1, g_2, \ldots, g_n) := \begin{cases} (g_2, \ldots, g_n) & \text{if } i = 0 \\ (g_1, \ldots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \ldots, g_n) & \text{if } 1 \le i \le n-1 \\ (g_1, \ldots, g_{n-1}) & \text{if } i = n \end{cases}$$

$$\sigma_i(g_1, g_2, \ldots, g_n) := (g_1, \ldots, g_{i-1}, 1_G, g_i, \ldots, g_n) \qquad \forall 0 \le i \le n$$

Map 1: $\mathrm{Prin}_G(B) \cong [B, BG]$

We can furthermore define a G-action on $E_*(G)$ in such a way:

$$\begin{array}{rcl} E_n(G) \times G & \to & E_n(G) \\ (g_1, \dots, g_n, g_{n+1}) \cdot g & := & (g_1, \dots, g_n, g_{n+1}g) \end{array}$$

In this context, we may realize that $B_*(G) = E_*(G)/G$ where the quotient is taken by group action. Therefore, the construction here forms a principal G-bundle (called the bar construction of G)

$$G \to EG \xrightarrow{p \,:=\, |p_*|} BG$$

which admits a group structure on BG by

$$BG \times BG = |B_*(G)| \times |B_*(G)| \cong |B_*(G) \times B_*(G)| \cong |B_*(G \times G)| = B(G \times G) \to BG$$

Here, the first isomorphism is given by the geometric realization that admits the product of two simplicial sets; the second isomorphism is derived naturally from the reordering of the coordinates in $G \times G$.
By [1], BG is Abelian if G is Abelian.

Map 1: $\mathrm{Prin}_G(B) \cong [B, BG]$

Q. How this map works?

$$[B, BG] \cong \mathrm{Prin}_G(B)$$
$$[f : B \to BG] \mapsto [f^*(EG)]$$
$$[f : B \to BG] \leftarrow [G \to E \xrightarrow{q} B]$$

Backward Direction: Given a principal G-bundle $G \to E \xrightarrow{q} B$, we may compare it with the universal bundle derived from G: $G \to EG \xrightarrow{p} BG$.

Thm ([2], 4.12.2) For each numerable principal G-bundle $G \to E \xrightarrow{q} B$ there exists a map $f : B \to BG$ such that $(G \to E \xrightarrow{q} B) \cong f^*(EG)$.

Remark: In our case, B is assumed to be paracompact Hausdorff, so $G \to E \xrightarrow{q} B$ is numerable.

Map 1: $\mathrm{Prin}_G(B) \cong [B, BG]$

Take an open cover $\mathcal{U} = \{U_i\}_{i \in I}$ of B with I equipping a total order and a system of local trivializations $\{h_i : q^{-1}(U_i) \to U_i \times G\}$. Since B is paracompact Hausdorff, there is a partition of unity $\{\phi_i : B \to [0,1]\}_{i \in I}$ with respect to $\mathcal{U}$. We are going to define a map $g : E \to EG$, by such a procedure: Take $\forall e \in E$, and denote $\sigma := \{i \in I \mid q(e) \in U_i\} = \{\sigma_0, \sigma_1, \dots, \sigma_n\}$ (By paracompactness, $\mathcal{U}$ is locally finite, hence σ is always finite).

$$\begin{tikzcd} U_i \times G \arrow[d, "\mathrm{proj}_2"'] \arrow[dr, "\mathrm{proj}_1"] & q^{-1}(U_i) \arrow[l, "h_i"', "\cong"] \arrow[d, "q"] \\ G & U_i \end{tikzcd}$$

We may observe that for any $i \in \sigma$, $(\mathrm{proj}_2 \circ h_i)(e)$ returns an element in G, while $\phi_i(q(e))$ returns an element in $[0,1]$ satisfying the condition that $\sum_{i \in \sigma} \phi_i(q(e)) = \sum_{i \in I} \phi_i(q(e)) = 1$. Therefore,

$$(\mathrm{proj}_2 \circ h_{\sigma_0}(e), \mathrm{proj}_2 \circ h_{\sigma_1}(e), \dots, \mathrm{proj}_2 \circ h_{\sigma_n}(e), \phi_{\sigma_0} \circ q(e), \phi_{\sigma_1} \circ q(e), \dots, \phi_{\sigma_n} \circ q(e)) \in G^{n+1} \times \Delta^n$$

gives a representative in the nth skeleton in $EG := \bigsqcup_{m \geq 0} G^{m+1} \times \Delta_m / \sim$. The class it represents is defined to be the image $g(e)$.

Hence, given a point $b \in B$, consider any element $e \in q^{-1}(b)$ in its fiber and define $f(b) := p \circ g(e)$. The claim is that the image $f(b)$ is independent of the choice of the preimage $e \in q^{-1}(b)$.

$$\begin{tikzcd} G \arrow[r] & E \arrow[r, "q"] \arrow[d, "g"] & B \arrow[d, dashed, "f"] \\ G \arrow[r] & EG \arrow[r, "p"] & BG \end{tikzcd}$$

Map 2: $\check{H}^1(B, \mathcal{F}_G) \cong \mathrm{Prin}_G(B)$

Given $[\eta] \in \check{H}^1(\mathcal{U}; \mathcal{F}_G)$ a Cech 1-cocycle, by definition, $\forall \sigma \in \mathcal{N}(\mathcal{U})^{(1)}$, i.e., $\forall j, k \in I$ such that $U_j \cap U_k \neq \varnothing$, we have $\eta_{jk} \in \mathcal{F}_G(U_{\{j,k\}})$. In other words, $\eta_{jk} : U_j \cap U_k \to G$ is a continuous function. Using this information, we may reconstruct a principal G-bundle by gluing η_{jk}'s:

$$E_\eta := \left(\bigsqcup_{j \in J} U_j \times \{j\} \times G \right) / \sim$$

where $(b, j, g) \sim (b, k, g + \eta_{jk}(b))$ for $\forall j, k \in I, \forall b \in U_j \cap U_k, \forall g \in G$. As one may check, E_η along with the projection map $p : E_\eta \to B;\ (b, j, g) \mapsto b$, forms a principal G-bundle $G \to E_\eta \xrightarrow{p} B$. It is shown from [3] that the operations obtained from (2) and (3) are inverses of each other up to Cech cohomology and bundle isomorphisms respectively. These yield an isomorphism on the level of topological spaces, i.e. $\check{H}^1(B, \mathcal{F}_G) \cong \mathrm{Prin}_G(B)$.

Map 3: $\check{H}^n(B, \mathcal{F}_G) \cong \check{H}^{n-1}(B, \mathcal{F}_{BG}) \cong \ldots \cong \check{H}^1(B, \mathcal{F}_{B^{n-1}G})$

Map 3 is a sequence of isomorphisms that can be obtained recursively by

$$\check{H}^n(B, \mathcal{F}_G) \cong \check{H}^{n-1}(B, \mathcal{F}_{BG})$$

Our idea of making this happen is to derive a long exact sequence of Cech cohomology groups by changing sheaves and prove that the Cech cohomology groups on sheaves of maps given by EG all vanish, splitting the long exact sequence into isomorphisms.

Map 3: $\check{H}^n(B,\mathcal{F}_G) \cong \check{H}^{n-1}(B,\mathcal{F}_{BG}) \cong \ldots \cong \check{H}^1(B,\mathcal{F}_{B^{n-1}G})$

Thm ([4], IX.3.18; [5], I.3.25) Let B be a paracompact space, and $0 \to \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H} \to 0$ is a short exact sequence of sheaves on B, where α and β are sheaves of maps. Then there exist connecting homomorphisms $\Delta : \check{H}^n(B,\mathcal{H}) \to \check{H}^{n+1}(B,\mathcal{F})$ for every $n \geq 0$ such that the sequence of Cech cohomology groups

$$\ldots \to \check{H}^n(B,\mathcal{F}) \xrightarrow{\alpha^*} \check{H}^n(B,\mathcal{G}) \xrightarrow{\beta^*} \check{H}^n(B,\mathcal{H}) \xrightarrow{\Delta} \check{H}^{n+1}(B,\mathcal{F}) \to \ldots$$

is exact.

$$\ldots \to \check{H}^i(B,\mathcal{F}_{EG}) \to \check{H}^i(B,\mathcal{F}_{BG}) \to \check{H}^{i+1}(B,\mathcal{F}_G) \to \check{H}^{i+1}(B,\mathcal{F}_{EG}) \to \ldots$$

$$\ldots \to \check{H}^i(B,\mathcal{F}_{EBG}) \to \check{H}^i(B,\mathcal{F}_{B^2G}) \to \check{H}^{i+1}(B,\mathcal{F}_{BG}) \to \check{H}^{i+1}(B,\mathcal{F}_{EBG}) \to \ldots$$

$$\ldots \to \check{H}^i(B,\mathcal{F}_{EB^2G}) \to \check{H}^i(B,\mathcal{F}_{B^3G}) \to \check{H}^{i+1}(B,\mathcal{F}_{B^2G}) \to \check{H}^{i+1}(B,\mathcal{F}_{EB^2G}) \to \ldots$$

$$\ldots$$

$$\ldots \to \check{H}^i(B,\mathcal{F}_{EB^{n-1}G}) \to \check{H}^i(B,\mathcal{F}_{B^nG}) \to \check{H}^{i+1}(B,\mathcal{F}_{B^{n-1}G}) \to \check{H}^{i+1}(B,\mathcal{F}_{EB^{n-1}G})$$

Map 3: $\check{H}^n(B,\mathcal{F}_G) \cong \check{H}^{n-1}(B,\mathcal{F}_{BG}) \cong \ldots \cong \check{H}^1(B,\mathcal{F}_{B^{n-1}G})$

Thm (*) Let B be a paracompact space, and E is a contractible topological Abelian group, then the sheaf of maps $\mathcal{F}_E$ over B gives trivial Cech cohomology groups on positive dimensions, i.e., $\check{H}^n(B;\mathcal{F}_E) = 0$ for all $n > 0$.

Prop ([6], 4.4.6) Let B be a paracompact space, and E is contractible topological Abelian group. Then the sheaf represented by E over B, i.e., $\mathcal{F}_E := \text{maps}(-, E)$, is soft.

Prop ([7], II.9.11) Let B be a paracompact space, and $\mathcal{F}$ a soft sheaf on B, then the sheaf cohomology $H^n(B;\mathcal{F}) = 0$ for all $n > 0$.

Prop ([8], Page 11) For a presheaf $\mathcal{F}$ over B, let $\mathcal{F}^+$ denote its sheafification. Then we have a natural comparison map $\chi : \check{H}^*(B,\mathcal{F}) \to H^*(B,\mathcal{F}^+)$ from Cech cohomology to sheaf cohomology. Furthermore, if B is paracompact Hausdorff, then χ is an isomorphism.

Map 3: $\check{H}^n(B,\mathcal{F}_G) \cong \check{H}^{n-1}(B,\mathcal{F}_{BG}) \cong \ldots \cong \check{H}^1(B,\mathcal{F}_{B^{n-1}G})$

Def (Soft sheaf) Let B be a paracompact space. A sheaf $\mathcal{F}$ on B is said to be *soft* if $\forall K \subseteq_{\text{closed}} B$, the restriction map $\mathcal{F} \to \text{colim}_{K\subseteq U\subseteq_{\text{open}} B}\mathcal{F}(U)$ is subjective, i.e., every section of $\mathcal{F}$ over K can be extended to the whole B.

Prop ([6], 4.4.6) Let B be a paracompact space, and E is contractible topological Abelian group. Then $\mathcal{F}_E := \text{maps}(-, E)$ is soft.

Proof Let $h : E \times [0,1] \to E$ be a contracting homotopy. In other words, $h(-,1) = \mathbb{1}_E$, $h(-,0) =$ constant map on $p \in E$. Take $\forall K \subseteq_{\text{closed}} B$, with $s : U \to E$ representing a section of E over B for some $K \subseteq U \subseteq_{\text{open}} B$. Since B is paracompact, we may choose another open set $K \subseteq V \subseteq_{\text{open}} B$ such that $\overline{V} \subseteq U$. It suffices to extend $s|_V$ over all of B. To make this happen, choose a map $f : B \to [0,1]$ with $f|_{\overline{V}} = 1$ and $f|_{B\setminus U} = 0$. Define $\tilde{s} : B \to E$ by

$$\tilde{s}(b) := \begin{cases} h(b, f(b)) & b \in U \\ p & b \notin U \end{cases}$$

This extend $s|_V$ to all of B, i.e., $\tilde{s}|_V = s|_V$.

Remarks for $B^n(G)$ being an $K(G, n)$

So far we have proven this natural bijection $\check{H}^n(B, \mathcal{F}_G) \cong [B, B^n(G)]$. We will briefly show that $B^n(G)$ is a $K(G, n)$ under the assumption that G is discrete Abelian.
By [9], [10], EG is contractible, hence $\pi_i(EG) = 0$ for all $i \geq 0$. The long exact sequence of homotopy groups derived from $G \to EG \to BG$ is

$$... \to \pi_{i+1}(EG) \to \pi_{i+1}(BG) \to \pi_i(G) \to \pi_i(EG) \to \pi_i(BG) \to ...$$

which splits into isomorphisms $\pi_{i+1}(BG) \cong \pi_i(G)$ for all $i \geq 0$.
Therefore,

$$\pi_i(B^n(G)) \cong \pi_{i-1}(B^{n-1}(G)) \cong ... \cong \pi_{i-n}(G) = \begin{cases} G & (\text{if } i = n) \\ 0 & (\text{if } i \neq n) \end{cases}$$

whereas the last equality needs G to be discrete.

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