Extracting Sparse Eilenberg-MacLane Coordinates via Principal Bundles

-- YTM 2025, Stockholm

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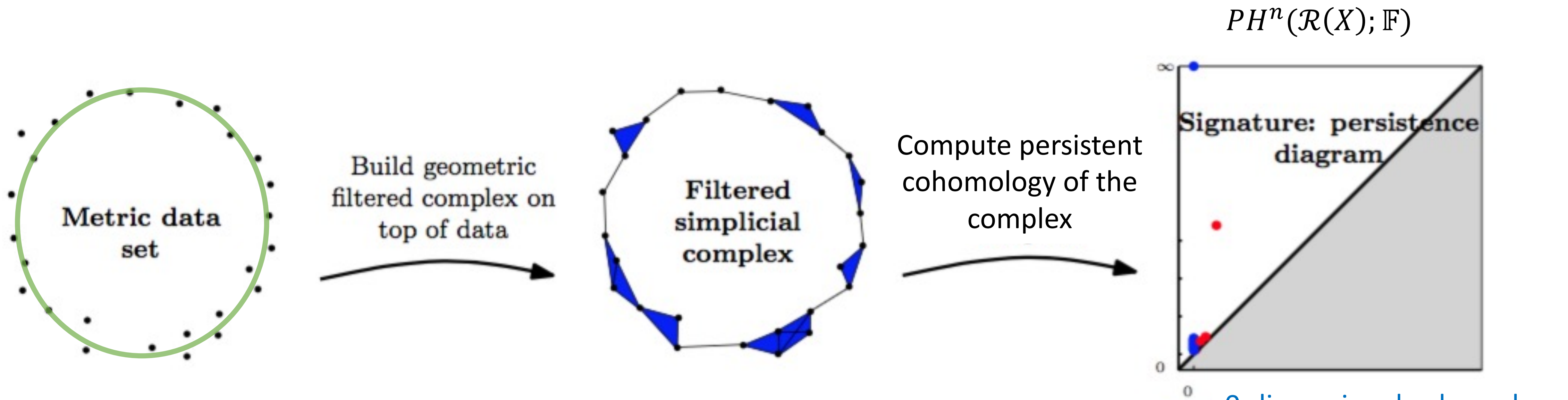
Pipeline of TDA

$\mathbb{X} \subseteq (\mathbb{M}, d)$: metric space

$X = \{x_i\}_{i \in I}$: a finite set sampled from $\mathbb{X}$

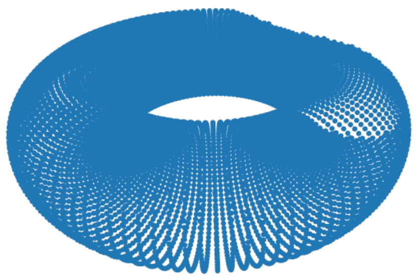
$\mathbb{F}$: a field



$$\mathcal{R}_\alpha(X) \coloneqq \{\sigma \subseteq X \mid \forall i, j \in \sigma, d(x_i, x_j) < \alpha\}$$

More behind the Persistence Diagram

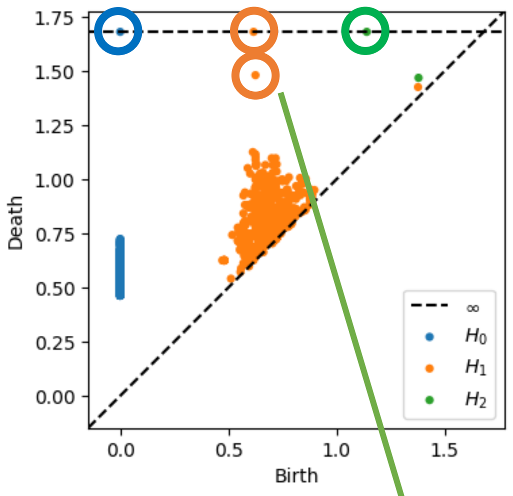
Data Point Cloud



X

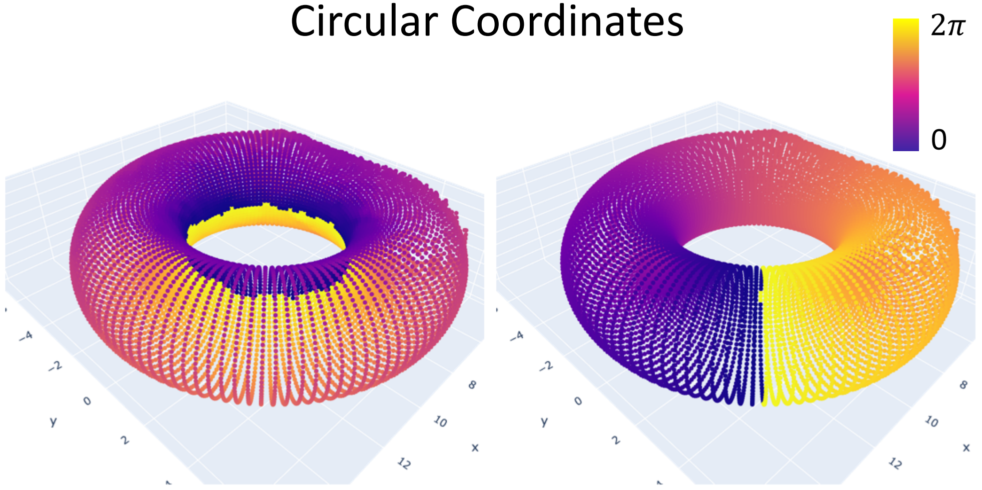
TDA Pipeline

Persistence Diagram



Interpretation

Circular Coordinates



$f_\eta : X^{(\alpha)} \to S^1$

$\mathbb{X} = T^2$

$$H^n(T^2;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}\ (n = 0, 2) \\ \mathbb{Z}^2\ (n = 1) \\ 0\ (\text{other}) \end{cases}$$

$\eta \in PH^1(\mathcal{R}(X); \mathbb{Z}_q)$

α-offset of X:

$$X^{(\alpha)} \coloneqq \bigcup_{x \in X} B_\alpha(x)$$

Sparse circular coordinates: [Jose A. Perea, **Abel Symp**, 2020]

Principle of the circular coordinates

- Choose $\{\eta\} \in PH^1(\mathcal{R}(X); \mathbb{Z}_q)$ and $\text{birth}(\eta) < \alpha < \dfrac{\text{death}(\eta)}{2}$

- Lift $\eta \in H^1(\mathcal{R}_{2\alpha}(X); \mathbb{Z}_q)$ to a class in $H^1(\mathcal{R}_{2\alpha}(X); \mathbb{Z})$ by

$$\ldots \to H^1(\mathcal{R}_{2\alpha}(X); \mathbb{Z}) \to H^1(\mathcal{R}_{2\alpha}(X); \mathbb{Z}) \textcolor{red}{\to} \underset{\textcolor{red}{\eta}}{H^1\big(\mathcal{R}_{2\alpha}(X); \mathbb{Z}_q\big)} \xrightarrow{\Delta} H^2(\mathcal{R}_{2\alpha}(X); \mathbb{Z}) \to \cdots$$

derived from $0 \to \mathbb{Z} \xrightarrow{\cdot q} \mathbb{Z} \to \mathbb{Z}_q \to 0$

$X^{(\alpha)} \to \check{\mathcal{C}}_\alpha(X)$
$\mathcal{R}_\alpha(X) \subseteq \check{\mathcal{C}}_\alpha(X) \subseteq \mathcal{R}_{2\alpha}(X)$

- The map $X^{(\alpha)} \to |\mathcal{R}_{2\alpha}(X)|$ determines a map $H^1(\mathcal{R}_{2\alpha}(X); \mathbb{Z}) \to H^1\big(X^{(\alpha)}; \mathbb{Z}\big)$

- There is an isomorphism between Čech cohomology and singular cohomology

$$H^1\big(X^{(\alpha)}; \mathbb{Z}\big) \cong \check{H}^1\big(X^{(\alpha)}; C_{\mathbb{Z}}\big)$$

Locally constant sheaf

Pipeline of Circular Coordinates

$$PH^1\big(\mathcal{R}(X); \mathbb{Z}_q\big) \to H^1\big(\mathcal{R}_{2\alpha}(X); \mathbb{Z}_q\big) \to H^1(\mathcal{R}_{2\alpha}(X); \mathbb{Z}) \to H^1\big(X^{(\alpha)}; \mathbb{Z}\big) \to \check{H}^1\big(X^{(\alpha)}; C_{\mathbb{Z}}\big) \to \cdots \to [X^{(\alpha)}, S^1]$$

Principle of the circular coordinates

$$\check{H}^1(X^{(\alpha)};\mathcal{F}_{\mathbb{Z}}) \xrightarrow{\cong} \mathrm{Prin}_{\mathbb{Z}}(X^{(\alpha)}) \xrightarrow{\cong} [X^{(\alpha)}, S^1]$$

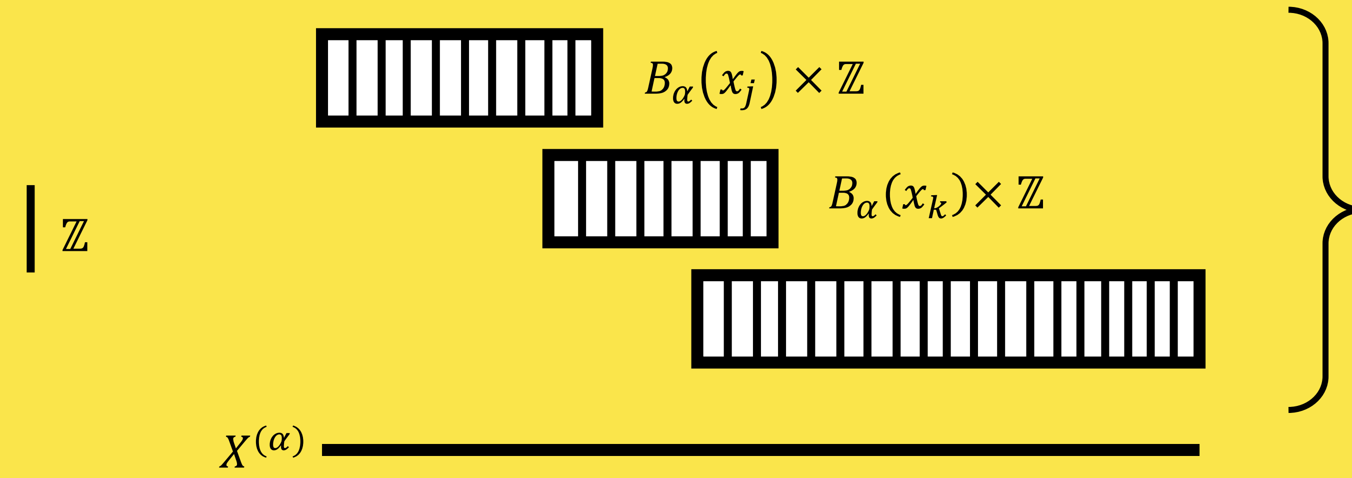
sheaf of maps
$\mathcal{F}_{\mathbb{Z}}(U) \coloneqq \mathrm{maps}(U,\mathbb{Z})$
over $X^{(\alpha)}$

η

Isomorphism classes of principal $\mathbb{Z}$-bundles

Homotopy classes of maps

$\{\eta_{jk}\colon B_\alpha(x_j)\cap B_\alpha(x_k)\to\mathbb{Z}\}_{j,k}$



Principal $\mathbb{Z}$-Bundle

$$\begin{array}{c} \mathbb{Z}\to E_\eta \to X^{(\alpha)} \\ \Big\downarrow \text{Classifying map} \\ \mathbb{Z}\to\mathbb{R}\to S^1 \end{array}$$

Pipeline of Circular Coordinates

$$PH^1(\mathcal{R}(X);\mathbb{Z}_q)\to\cdots\to\check{H}^1(X^{(\alpha)};\mathcal{C}_{\mathbb{Z}}) = \check{H}^1(X^{(\alpha)};\mathcal{F}_{\mathbb{Z}})\to\mathrm{Prin}_{\mathbb{Z}}(X^{(\alpha)})\to[X^{(\alpha)},S^1]$$

Principle of the circular coordinates

$$\check{H}^1(X^{(\alpha)};\mathcal{F}_{\mathbb{Z}}) \overset{\cong}{\to} \mathrm{Prin}_{\mathbb{Z}}(X^{(\alpha)}) \overset{\cong}{\to} [X^{(\alpha)}, S^1]$$

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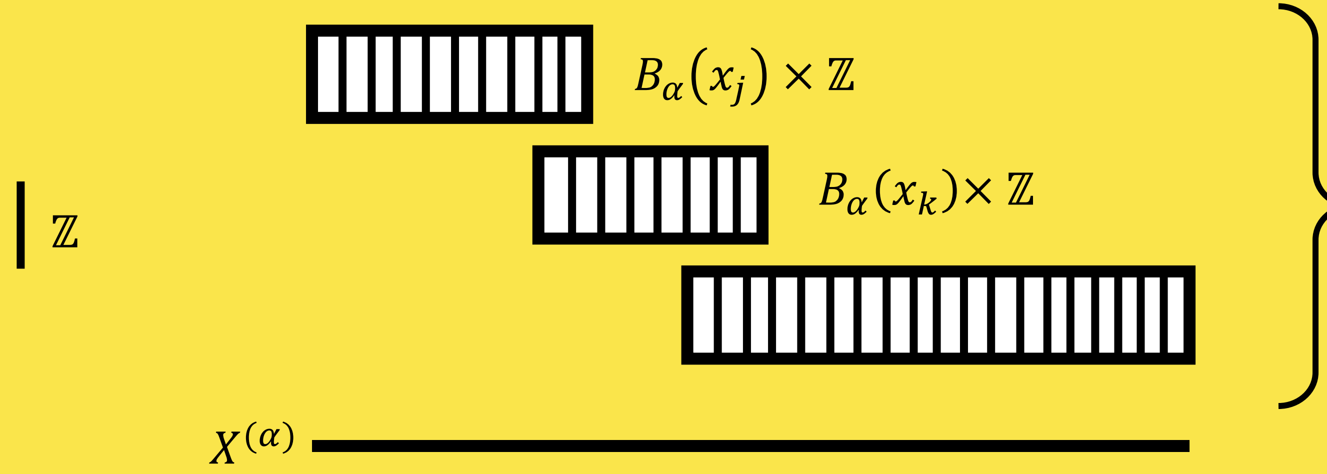


Principal $\mathbb{Z}$-Bundle

$$\begin{array}{c} \mathbb{Z} \to E_\eta \to X^{(\alpha)} \\ \Big\downarrow \text{ Classifying map} \\ \mathbb{Z} \to \mathbb{R} \to S^1 \end{array}$$

Pipeline of Circular Coordinates

$$PH^1(\mathcal{R}(X);\mathbb{Z}_q) \to \cdots \to \check{H}^1(X^{(\alpha)};\mathcal{C}_{\mathbb{Z}}) = \check{H}^1(X^{(\alpha)};\mathcal{F}_{\mathbb{Z}}) \to \mathrm{Prin}_{\mathbb{Z}}(X^{(\alpha)}) \to [X^{(\alpha)}, S^1]$$

Principle of the circular coordinates

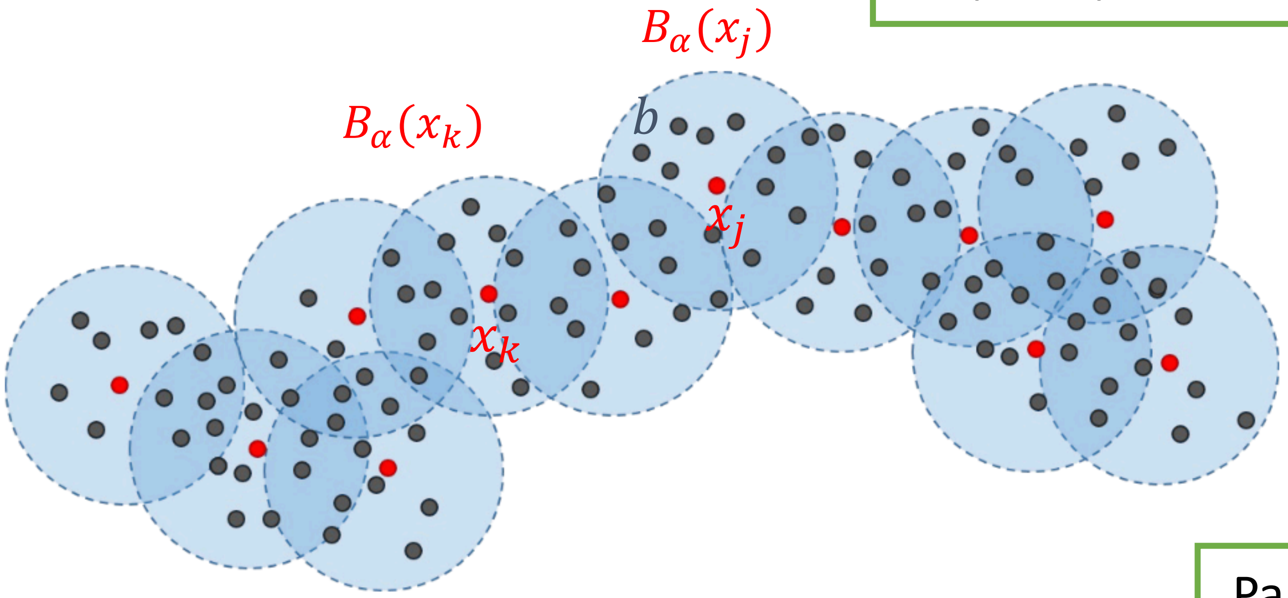
$$\check{H}^1\left(X^{(\alpha)};\mathcal{F}_{\mathbb{Z}}\right) \overset{\cong}{\to} \mathrm{Prin}_{\mathbb{Z}}\left(X^{(\alpha)}\right) \overset{\cong}{\to} \left[X^{(\alpha)},S^1\right]$$

η

sheaf of maps
$\mathcal{F}_{\mathbb{Z}}(U) \coloneqq \mathrm{maps}(U,\mathbb{Z})$
over $X^{(\alpha)}$

Isomorphism classes of principal $\mathbb{Z}$-bundles

Homotopy classes of maps



Final Output

$f_\eta: X^{(\alpha)} \to S^1$

$$b \in B_\alpha(x_j) \mapsto \exp\left\{2\pi i \sum_{k\in I} \varphi_k(b)\eta_{jk}\right\}$$

Partition of unity subordinated to $\{B_\alpha(x_k)\}_{k\in I}$

Pipeline of Circular Coordinates

$$PH^1(\mathcal{R}(X);\mathbb{Z}_q) \to \cdots \to \check{H}^1\left(X^{(\alpha)};\mathcal{C}_{\mathbb{Z}}\right) = \check{H}^1\left(X^{(\alpha)};\mathcal{F}_{\mathbb{Z}}\right) \to \mathrm{Prin}_{\mathbb{Z}}\left(X^{(\alpha)}\right) \to \left[X^{(\alpha)},S^1\right]$$

Previous Work

Sparse circular coordinates: [Jose A. Perea, **Abel Symp**, 2020]

$$\{\eta\} \in PH^1(\mathcal{R}(X); \mathbb{Z}) \quad \Longrightarrow \quad f_\eta : X^{(\alpha)} \to S^1$$

Torodial coordinates: [Luis Scoccola et al, **SoCG**, 2023]

$$\{([\eta_1], \dots, [\eta_l])\} \in PH^1(\mathcal{R}(X); \mathbb{Z}^l) \quad \Longrightarrow \quad f_{\eta_1,\dots,\eta_l} : X^{(\alpha)} \to T^l$$

Projective coordinates: [Jose A. Perea, **D&CG**, 2018]

$$\{\theta\} \in PH^1(\mathcal{R}(X); \mathbb{Z}_2) \quad \Longrightarrow \quad f_\theta : X^{(\alpha)} \to \mathbb{RP}^n$$

$$\{\nu\} \in PH^2(\mathcal{R}(X); \mathbb{Z}) \quad \Longrightarrow \quad f_\nu : X^{(\alpha)} \to \mathbb{CP}^n$$

Lens coordinates: [Luis Polanco, Jose A. Perea, **CCCG**, 2019]

$$\{\mu\} \in PH^1(\mathcal{R}(X); \mathbb{Z}_q) \quad \Longrightarrow \quad f_\mu : X^{(\alpha)} \to S^{2n-1}/(\mathbb{Z}_q)$$

Code: **DREiMac** [Jose A. Perea, Luis Scoccola, Chris Tralie, 2023]

Dimensionality Reduction with Eilenberg-MacLane Coordinates

Idea of this project -- Generalization

G: finitely generated Abelian group

Q. What information can we have from $\{\eta\} \in PH^n(\mathcal{R}(X); G)$?

B: CW-complex

$K(G,n)$: Eilenberg-MacLane Space

$$\pi_i(K(G,n)) \cong \begin{cases} G \ (i = n) \\ 0 \ (i \neq n) \end{cases}$$

$$H^n(B; G) \cong [B, K(G,n)]$$

$\boldsymbol{n}$	1	1	1	1	2
$\boldsymbol{G}$	$\mathbb{Z}$	$\mathbb{Z}^l$	$\mathbb{Z}_2$	$\mathbb{Z}_q$	$\mathbb{Z}$
$\boldsymbol{K(G,n)}$	S^1	T^l	$\mathbb{RP}^\infty$	L_q^∞	$\mathbb{CP}^\infty$

A glimpse of our pipeline

Start: $[\eta] \in PH^n(\mathcal{R}(X);\mathbb{F})$

G: finitely generated Abelian topological group

$$
\begin{tikzcd}
\eta \in PH^n(\mathcal{R}(X);\mathbb{F}) \arrow[r, "\mathbb{F}:\mathit{field}"] & \eta \in H^n(R_{2\alpha}(X);\mathbb{F}) \arrow[r, "\mathit{Select}G"] & H^n(R_{2\alpha}(X);G) \arrow[d, "\cong"] \\
\check{H}^n(X^{(\alpha)};\mathcal{F}_G) \arrow[d, "(2)"] & H^n(X^{(\alpha)};G) \arrow[l, hook', "\mathit{inclusion}"'] & H^n(|R_{2\alpha}(X)|;G) \arrow[l, "(1)"'] \\
\check{H}^{n-1}(X^{(\alpha)};\mathcal{F}_{BG}) \arrow[r, "(2)"] & \dots \arrow[r, "(2)"] & \check{H}^1(X^{(\alpha)};\mathcal{F}_{B^{n-1}G}) \arrow[d, "(3)"] \\
\left[X^{(\alpha)}, K(G,n)\right] & \left[X^{(\alpha)}, B^nG\right] \arrow[l, "\cong"'] & \mathrm{Prin}_{B^{n-1}G}(X^{(\alpha)}) \arrow[l, "(4)"']
\end{tikzcd}
$$

End: $f_\eta : X^{(\alpha)} \to K(G,n)$

Methodology

- Choose $\{\eta\} \in PH^n(\mathcal{R}(X); \mathbb{Z}_q)$ and $\text{birth}(\eta) < \alpha < \dfrac{\text{death}(\eta)}{2}$
- Select your group G:
  - Find appropriate G and H that fit into this short exact sequence

$$0 \to H \to G \to \mathbb{Z}_q \to 0$$

  - For each possible sequence, ask the Bockstein question: "Can I lift η" ?

$$\ldots \to H^n(\mathcal{R}_{2\alpha}(X); H) \to H^n(\mathcal{R}_{2\alpha}(X); G) \to \underset{\eta}{H^n\big(\mathcal{R}_{2\alpha}(X); \mathbb{Z}_q\big)} \xrightarrow{\Delta} H^{n+1}(\mathcal{R}_{2\alpha}(X); H) \to \cdots$$

- The map $X^{(\alpha)} \to |\mathcal{R}_{2\alpha}(X)|$ determines a map $H^n(\mathcal{R}_{2\alpha}(X); G) \to H^n\big(X^{(\alpha)}; G\big)$
- There is an isomorphism between Čech cohomology and singular cohomology

$$H^n\big(X^{(\alpha)}; G\big) \cong \check{H}^n\big(X^{(\alpha)}; C_G\big)$$

Pipeline of Eilenberg-MacLane Coordinates

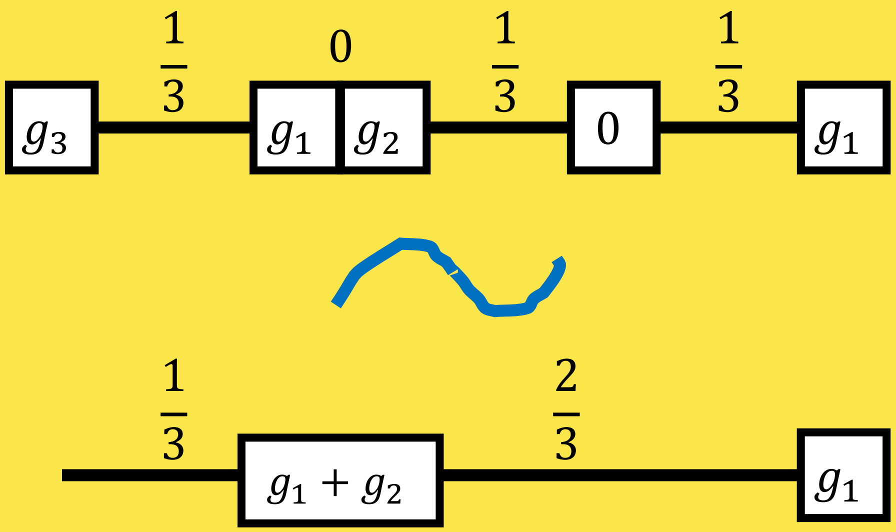
$$PH^n\big(\mathcal{R}(X); \mathbb{Z}_q\big) \to H^n\big(\mathcal{R}_{2\alpha}(X); \mathbb{Z}_q\big) \to H^n(\mathcal{R}_{2\alpha}(X); G) \to H^n\big(X^{(\alpha)}; G\big) \to \check{H}^n\big(X^{(\alpha)}; \mathcal{F}_G\big) \to \cdots \to [X^{(\alpha)}, K(G, n)]$$

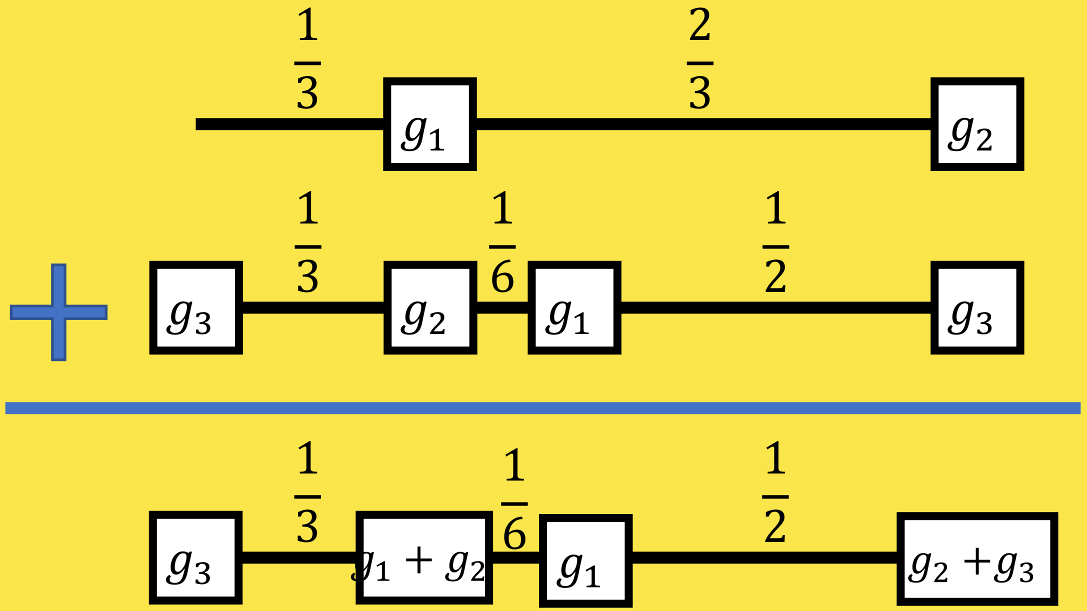
Methodology

$$\check{H}^n\left(X^{(\alpha)};\mathcal{F}_G\right) \xrightarrow{\cong} \check{H}^{n-1}\left(X^{(\alpha)};\mathcal{F}_{BG}\right) \xrightarrow{\cong} \cdots \xrightarrow{\cong} \check{H}^1\left(X^{(\alpha)};\mathcal{F}_{B^{n-1}G}\right)$$

$\mathcal{G}$: Abelian topological group

- We use the “bar construction” for $E\mathcal{G}$ and $B\mathcal{G}$: $\quad E\mathcal{G} \coloneqq {\coprod_{n\geq 0} \mathcal{G}^{n+1} \times \Delta_n}/{\sim}$
- $\mathcal{G}$-action on $E\mathcal{G}$ is defined simplex wise: $\quad (g_0, \dots, g_n) \cdot g \coloneqq (g_0, \dots, g_n + g)$
- $B\mathcal{G}$ is $E\mathcal{G}$’s quotient under the group action: $\quad B\mathcal{G} \coloneqq E\mathcal{G}/\mathcal{G}$





Methodology

$$\check{H}^n\left(X^{(\alpha)};\mathcal{F}_G\right) \overset{\cong}{\to} \check{H}^{n-1}\left(X^{(\alpha)};\mathcal{F}_{BG}\right) \overset{\cong}{\to} \cdots \overset{\cong}{\to} \check{H}^1\left(X^{(\alpha)};\mathcal{F}_{B^{n-1}G}\right)$$

$\mathcal{G}$: Abelian topological group

The universal bundle $\mathcal{G} \to E\mathcal{G} \to B\mathcal{G}$ induces a short exact sequence of sheaves $0 \to \mathcal{F}_{\mathcal{G}} \to \mathcal{F}_{E\mathcal{G}} \to \mathcal{F}_{B\mathcal{G}} \to 0$, which further induces a long exact sequence

$$\dots \to \cancel{\check{H}^n\left(X^{(\alpha)};\mathcal{F}_{E\mathcal{G}}\right)} \to \check{H}^n\left(X^{(\alpha)};\mathcal{F}_{B\mathcal{G}}\right) \overset{\Delta}{\to} \check{H}^{n+1}\left(X^{(\alpha)};\mathcal{F}_{\mathcal{G}}\right) \to \cancel{\check{H}^{n+1}\left(X^{(\alpha)};\mathcal{F}_{E\mathcal{G}}\right)} \to \cdots$$

Proposition. Let B be a paracompact space, and E be a contractible topological Abelian group. Then, $\check{H}^n(B;\mathcal{F}_E) = 0$ for all $n > 0$.

Pipeline of Eilenberg-MacLane Coordinates

$$PH^n\left(\mathcal{R}(X);\mathbb{Z}_q\right) \to \cdots \to \check{H}^n\left(X^{(\alpha)};\mathcal{F}_G\right) \to \check{H}^{n-1}\left(X^{(\alpha)};\mathcal{F}_{BG}\right) \to \cdots \to \check{H}^1\left(X^{(\alpha)};\mathcal{F}_{B^{n-1}G}\right) \to \cdots \to [X^{(\alpha)}, K(G,n)]$$

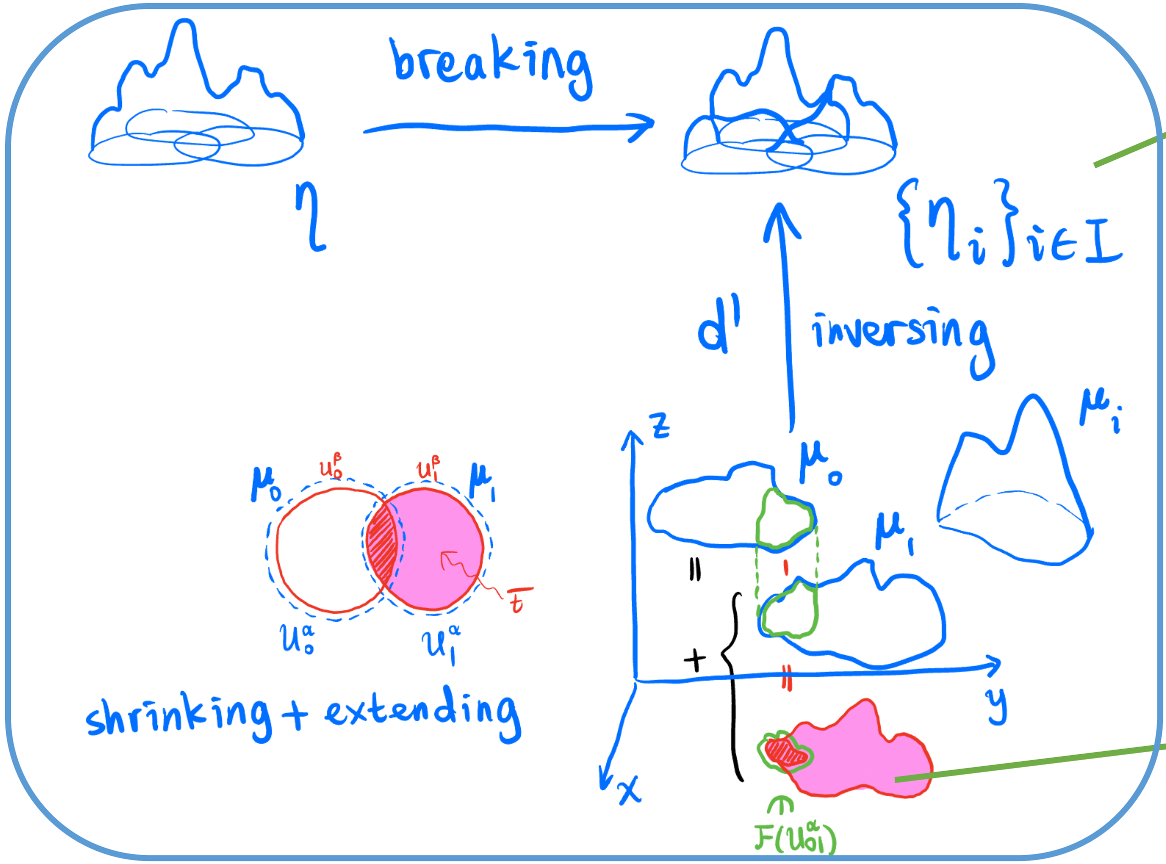
Methodology

Proposition. $\check{H}^n(X^{(\alpha)}; \mathcal{F}_{EG}) = 0$ for all $n > 0$

Solving a coboundary problem – It is very HARD to be written down explicitly!

$$\cdots \to \check{C}^1(\mathcal{U}; \mathcal{F}_{EG}) \xrightarrow{d^1} \check{C}^2(\mathcal{U}; \mathcal{F}_{EG}) \xrightarrow{d^2} \check{C}^3(\mathcal{U}; \mathcal{F}_{EG}) \to \cdots$$

$$?? \longmapsto \eta \longmapsto 0$$



Idea: Breaking, inversing, gluing, shrinking

Proof Punchline – sheaf theory
$\mathcal{F}_{EG}$ is a soft sheaf

$\mathcal{U}^\alpha := \{B_\alpha(x)\}_{x\in X}$ is an open cover of $B := X^{(\alpha)}$
$\mathcal{F} := \mathcal{F}_{E\mathcal{G}}$

Algorithm 2 $\check{H}^2(\mathcal{U},\mathcal{F}) = 0$ given by $\boxed{4.13}$

Require: $\eta = (\eta_{i_0 i_1 i_2})_{i_0<i_1<i_2} \in \ker d_B^2 \subseteq \check{C}^2(\mathcal{U}^\alpha,\mathcal{F})(B)$

1: Choose $\beta < \alpha$ such that $\mathcal{U}^\beta$ still covers B.

2: Define $t_0 \in \check{C}^1(\mathcal{U}^\alpha,\mathcal{F})(U_0^\alpha)$ by $t_{0,i_0 i_1}(x) := \eta_{0 i_0 i_1}(x)$, $\forall x \in U_0^\alpha \cap U_{i_0 i_1}^\alpha$.

3: **for** $i = 1, 2, \ldots, N$ **do**

4: Choose γ_i such that $\beta < \gamma_i < \gamma_{i-1}$ (By convention, $\gamma_0 = \alpha$).

5: Define $t_i \in \check{C}^1(\mathcal{U}^\alpha,\mathcal{F})(\tilde{U}_i^{\gamma_i})$ by

$$t_{i,i_0 i_1}(x) := \begin{cases} \eta_{i i_0 i_1}(x) + h(t_{i-1,i i_1}(x), g_i(x)) - h(t_{i-1,i i_0}(x), g_i(x)) & \forall x \in (U_i^{\gamma_i} - \tilde{U}_{i-1}^{\gamma_i}) \cap U_{i_0 i_1}^\alpha \\ t_{i-1,i_0 i_1}(x) & \forall x \in \tilde{U}_{i-1}^{\gamma_i} \cap U_{i_0 i_1}^\alpha \end{cases}$$

6: **end for**

7: Set $Q^2(\eta) \in \check{C}^1(\mathcal{U}^\alpha,\mathcal{F})(B)$ by $Q^2(\eta)_{i_0 i_1}(x) := t_{N,i_0 i_1}(x), \forall x \in B$.

8: We claim that $d^1(Q^2(\eta)) = \eta$.

Shrinking

Gluing

Pipeline of Eilenberg-MacLane Coordinates

$$PH^n(\mathcal{R}(X);\mathbb{Z}_q) \to \cdots \to \check{H}^n(X^{(\alpha)};\mathcal{F}_G) \to \check{H}^{n-1}(X^{(\alpha)};\mathcal{F}_{BG}) \to \cdots \to \check{H}^1(X^{(\alpha)};\mathcal{F}_{B^{n-1}G}) \to \cdots \to [X^{(\alpha)}, K(G,n)]$$

$\mathcal{U}^{\alpha} \coloneqq \{B_{\alpha}(x)\}_{x \in X}$ is an open cover of $B \coloneqq X^{(\alpha)}$

$\mathcal{F} \coloneqq \mathcal{F}_{E\mathcal{G}}$

Algorithm 3 $\check{H}^n(\mathcal{U}, \mathcal{F}) = 0$ given by $\boxed{4.13}$

Require: $\eta = (\eta_{i_0 \dots i_n})_{i_0 < \dots < i_n} \in \ker d_B^n \subseteq \check{C}^n(\mathcal{U}^{\alpha}, \mathcal{F})(B)$

Choose $\beta < \alpha$ such that $\mathcal{U}^{\beta}$ still covers B.

Define $t_0 \in \check{C}^{n-1}(\mathcal{U}^{\alpha}, \mathcal{F})(U_0^{\alpha})$ by $t_{0, i_0 \dots i_{n-1}}(x) := \eta_{0 i_0 \dots i_{n-1}}(x)$, $\forall x \in U_0^{\alpha} \cap U^{\alpha}_{i_0 \dots i_{n-1}}$.

for $i = 1, 2, \dots, N$ **do**

  Choose γ_i such that $\beta < \gamma_i < \gamma_{i-1}$ (By convention, $\gamma_0 = \alpha$).

  Define $t_i \in \check{C}^{n-1}(\mathcal{U}^{\alpha}, \mathcal{F})(\tilde{U}_i^{\gamma_i})$ by

$$t_{i, i_0 \dots i_{n-1}}(x) := \begin{cases} \eta_{i i_0 \dots i_{n-1}}(x) + \sum_{j=0}^{n-1} (-1)^j h(t_{i-1, i i_0 \dots \hat{i_j} \dots i_{n-1}}(x), g_i(x)) & \forall x \in (U_i^{\gamma_i} - \tilde{U}_{i-1}^{\gamma_i}) \cap U^{\alpha}_{i_0 \dots i_{n-1}} \\ t_{i-1, i_0 \dots i_{n-1}}(x) & \forall x \in \tilde{U}_{i-1}^{\gamma_i} \cap U^{\alpha}_{i_0 \dots i_{n-1}} \end{cases}$$

end for

Set $Q^n(\eta) \in \check{C}^{n-1}(\mathcal{U}^{\alpha}, \mathcal{F})(B)$ by $Q^n(\eta)_{i_0 \dots i_{n-1}}(x) := t_{N, i_0 \dots i_{n-1}}(x), \forall x \in B$.

We claim that $d^{n-1}(Q^n(\eta)) = \eta$.

Shrinking

Gluing

Pipeline of Eilenberg-MacLane Coordinates

$$PH^n(\mathcal{R}(X); \mathbb{Z}_q) \to \cdots \to \check{H}^n(X^{(\alpha)}; \mathcal{F}_G) \to \check{H}^{n-1}(X^{(\alpha)}; \mathcal{F}_{BG}) \to \cdots \to \check{H}^1(X^{(\alpha)}; \mathcal{F}_{B^{n-1}G}) \to \cdots \to [X^{(\alpha)}, K(G, n)]$$

Methodology

Milnor Construction:
Defined by "infinite join"

$\mathcal{G}$: Abelian topological group

$$\check{H}^1\left(X^{(\alpha)}; \mathcal{F}_{\mathcal{G}}\right) \xrightarrow{\cong} \mathrm{Prin}_{\mathcal{G}}\left(X^{(\alpha)}\right) \xrightarrow{\cong} \left[X^{(\alpha)}, \mathcal{B}\mathcal{G}\right]$$

μ

Final Output

$$f_\eta : X^{(\alpha)} \to \mathcal{B}\mathcal{G} = \mathcal{G} * \mathcal{G} * \cdots / \mathcal{G}$$

$$b \in U_j \mapsto \left[\left(\mu_{jk}(b)\right)_{k \in I}, \left(\varphi_k(b)\right)_{k \in I}\right]$$

Partition of unity subordinated to $\{B_\alpha(x_k)\}_k$

G is discrete, so $\mathcal{B}B^{n-1}G$ is a $K(G,n)$

Pipeline of Eilenberg-MacLane Coordinates

$$PH^n\left(\mathcal{R}(X); \mathbb{Z}_q\right) \to \cdots \to \check{H}^1\left(X^{(\alpha)}; \mathcal{F}_{B^{n-1}G}\right) \to \mathrm{Prin}_{B^{n-1}G}\left(X^{(\alpha)}\right) \to \left[X^{(\alpha)}, \mathcal{B}B^{n-1}G\right] \cong \left[X^{(\alpha)}, K(G,n)\right]$$

To summarize

- A “long-enough” persistence generator $\{\eta\} \in PH^n(\mathcal{R}(X); \mathbb{F})$ with a parameter $\text{birth}(\eta) < \alpha < \frac{\text{death}(\eta)}{2}$ gives a coordinate function $f_\eta \colon X^{(\alpha)} \to K(G, n)$

- But! This coordinate function is not as explicit as those can be written in a single formula

- But!! the coordinates can still be computed in an inductive manner, so an algorithm can still be given

- If there is a good model for specific $K(G, n)$ you are interested in, then you may write this coordinate function