EXTRACTING SPARSE EILENBERG-MACLANE COORDINATES VIA SOFT SHEAVES (DRAFT V7.2)

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Abstract. We present in this paper an application of the sheaf theory to the problem of non-linear dimensionality reduction in data analysis. Specifically, we construct Eilenberg-MacLane coordinates — functions from data to Eilenberg-MacLane spaces $K(G, n)$ — that encode nontrivial persistent cohomology classes in arbitrary dimensions n and coefficients G. Using soft sheaf theory, we establish a pipeline from persistent cohomology classes to the homotopy classes of such coordinate functions, extending previous results on circular, projective, and lens coordinates to all dimensions and arbitrary discrete Abelian groups. Crucially, our proof is constructive: we derive explicit formulas via inverses of Čech coboundary operators, yielding a recursive algorithm with polynomial complexity based on matrix computations. We validate our approach through experiments on synthetic datasets, and present novel examples of $K(\mathbb{Z}/2, 2)$-coordinates not addressed in prior work.

1. Introduction

As a new field that has been developing rapidly since the 1990s, topological data analysis focuses on inferring topological features of an unknown topological space $\mathbb{X}$, given a finite sample $X = \{x_i\}_{i=1}^N \subseteq \mathbb{X}$. With the concept of persistent homology, introduced by Edelsbrunner, Letscher, and Zomorodian in 2002 [6], as well as the persistence barcodes (a visualizable invariant of persistent homology) and the efficient algorithm for computing it due to Carlsson and Zomorodian [7], homological inference has become mature. Nowadays, given a finite point cloud X sampled from an underlying space $\mathbb{X}$ (such as $S^1, T^2, \mathbb{RP}^2$, etc.) embedded in a metric space $(\mathbb{M}, d)$, the usual workflow is as follows:

(1) Build simplicial complexes, such as the Rips or Čech complexes:

$$\begin{aligned} R_\alpha(X) &:= \{\sigma \subseteq X \mid \forall i, j \in \sigma,\ d(x_i, x_j) < \alpha\} \\ \check{C}_\alpha(X) &:= \left\{\sigma \subseteq X \mid \bigcap_{x\in\sigma} B_\alpha(x) \neq \varnothing\right\} \end{aligned}$$

where $B_\alpha(x) := \{y \in \mathbb{M} \mid d(x, y) < \alpha\}$ means the ball with radius α centered at x. Both constructions yield a filtered simplicial complex; that is, a family $\mathcal{K} = \{K_\alpha\}_{\alpha\geq 0}$ of simplicial complexes K_α — or K_j, $j \in \mathbb{Z}$, if the index α is discretized to α_j — so that $K_\alpha \subseteq K_{\alpha'}$ whenever $\alpha \leq \alpha'$.

(2) Compute the persistent homology or cohomology of $\mathcal{K}$, denoted by $PH^n(\mathcal{K}; \mathbb{F}) := \bigoplus_{j\in\mathbb{Z}} H^n(K_j; \mathbb{F})$. Since X is finite, $\bigcup \mathcal{K}$ will have finitely many simplices, and thus $PH^n(\mathcal{K}; \mathbb{F})$ can be endowed with the structure of a finitely generated graded module over the polynomial ring $\mathbb{F}[t]$, with

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coefficients in a field $\mathbb{F}$. By the classification theorem of finitely generated modules over PIDs,

$$PH^n(\mathcal{K};\mathbb{F}) \cong \left(\bigoplus_{q=1}^{Q} t^{n_q} \cdot \mathbb{F}[t]\right) \oplus \left(\bigoplus_{l=1}^{L} (t^{m_l} \cdot \mathbb{F}[t])/(t^{m_l+d_l})\right)$$

for unique $n_q, m_l \in \mathbb{Z}, d_l \in \mathbb{N}$. The multiset of intervals

$$[n_1,\infty),[n_2,\infty),...,[n_Q,\infty),[m_1,m_1+d_1),[m_2,m_2+d_2),...,[m_L,m_L+d_L)$$

is referred to as the barcode of $PH^n(\mathcal{K};\mathbb{F})$, and provides a complete discrete invariant for the $\mathbb{F}[t]$-isomorphism type of $PH^n(\mathcal{K};\mathbb{F})$.

(3) The long intervals in step 2 are in correspondence with generators $\eta \in H^n(K_\alpha;\mathbb{F})$, and under appropriate hypotheses recover generators for the homology/cohomology of $\mathbb{X}$.

Although the technology of obtaining generators for the homology/cohomology of $\mathbb{X}$ has become mature, topological data analysis aims to recover more data hidden beneath $\eta \in H^n(K_\alpha;\mathbb{F})$. Indeed, the typical algorithms for persistence take $\mathbb{F} = \mathbb{Z}/q\mathbb{Z}$ for various choices of prime q, and thus η can be interpreted as (or lifted to a) class $\eta \in H^n(K_\alpha;G)$ for G finitely generated and Abelian. The representability of cohomology via Eileberg-MacLane spaces, i.e. the natural bijection $H^n(|K_\alpha|;G) \cong [|K_\alpha|, K(G,n)]$, implies that η is represented by a map $f_\eta : |K_\alpha| \to K(G,n)$, unique up to homotopy. Moreover, if $X^{(\alpha)} := \bigcup_{x\in X} B_\alpha(x)$ denotes the α-neighborhood of X in $\mathbb{M}$ (called the α-offset of X), then when K_α is either $R_\alpha(X)$ or $\check{C}_\alpha(X)$, one has a shadow map $X^{(\alpha)} \to |K_\alpha|$ (which is a homotopy equivalence under appropriate conditions) that after composing with f_η yields $X^{(\alpha)} \to K(G,n)$.

Much research has been done in the last decade working on particular cases of $K(G,n)$. In 2018, Jose A. Perea presented a method for constructing maps to real and complex projective spaces via line bundles (i.e. $X^{(\alpha)} \to \mathbb{RP}^\infty, \mathbb{CP}^\infty$) [5]. In 2019, Polanco and Perea [4] developed lens coordinates for cyclic coefficient groups $\mathbb{Z}/q\mathbb{Z}$ and further developed lens-PCA. In 2020, Jose A. Perea worked out an explicit formula for deriving circular coordinates (i.e. $X^{(\alpha)} \to S^1$) on data with nontrivial 1-dimensional persistent cohomology [1] in the sense that the map is also defined on an open neighborhood of data samples. In 2023, Luis Scoccola et al. identified a formal notion of geometric correlation between circular coordinates and described a systematic procedure for constructing low-energy torus-valued maps (i.e. $X^{(\alpha)} \to T^d$) [3]. These previous works used the same result of bijections between Čech Cohomology groups and classifying spaces via principal bundles:

$$\check{H}^1(B,\mathcal{F}_G) \xrightarrow{\text{Map 2}} \mathrm{Prin}_G(B) \xrightarrow{\text{Map 1}} [B,BG]$$

to obtain *explicit formulas and algorithms*. Here:

- G is a topological Abelian group; its classifying space is denoted as BG.
- B is a paracompact Hausdorff space;
- $\mathcal{F}_G$ is the sheaf of continuous maps onto G: $\mathcal{F}_G(U) := \{f : U \to G \mid f \text{ is continuous}\}, \forall U \underset{\text{open}}{\subseteq} B$;
- $\mathrm{Prin}_G(B)$ denotes the isomorphism classes of principal G-bundles over B;
- $[B,BG]$ denotes the homotopy classes of maps from B to BG;

1.1. **Our Contribution.** The purpose of this project is to generalize previous results to general Eilenberg-MacLane coordinates $X^{(\alpha)} \to K(G,n)$, derived from persistent cohomology $PH^n(\mathcal{R}(X);G)$ for all dimensions $n > 0$. Our contributions are threefold:

1. **Theoretical Framework and Explicit Formulas.**

We establish a complete bijection between the Čech cohomology $\check{H}^n(X^{(\alpha)};G)$ and homotopy classes of maps $X^{(\alpha)} \to K(G,n)$ for arbitrary discrete Abelian groups G and dimensions $n \geq 1$ using principal bundles and soft sheaves. This generalizes previous results:

Coordinate Type	Cohomology Source	Target Space	Concrete Model
Circular [1]	$PH^1(\mathcal{R}(X);\mathbb{Z})$	$K(\mathbb{Z},1)$	S^1
Torodial [3]	$PH^1(\mathcal{R}(X);\mathbb{Z}^l)$	$K(\mathbb{Z}^l,1)$	$T^l := S^1 \times ... \times S^1$
Real Projective [5]	$PH^1(\mathcal{R}(X);\mathbb{Z}_2)$	$K(\mathbb{Z}_2,1)$	$\mathbb{RP}^\infty$
Complex Projective [5]	$PH^2(\mathcal{R}(X);\mathbb{Z})$	$K(\mathbb{Z},2)$	$\mathbb{CP}^\infty$
Lens [4]	$PH^1(\mathcal{R}(X);\mathbb{Z}_q)$	$K(\mathbb{Z}_q,1)$	$L_q^\infty := S^\infty/\mathbb{Z}_q$
Eilenberg-MacLane (Our work)	$PH^n(\mathcal{R}(X);\mathbb{Z}_q)$	$K(G,n)$	B^nG (n-fold classifying space)

While previous work focused on specific low-dimensional cases (circles, projective spaces, lens spaces, tori), our framework handles:

- Arbitrary dimensions $n \geq 1$.
- Arbitrary discrete Abelian groups G, including but not limited to $\mathbb{Z}$, $\mathbb{Z}/q\mathbb{Z}$, and products thereof.
- Higher Eilenberg-MacLane spaces such as $K(G,n)$ for $n \geq 2$, which encompass fundamentally different topological information.

Additionally, we derive explicit formulas via the inverse of the Čech coboundary operator's connecting homomorphism, which further provides an approach for constructing a canonical map $X^{(\alpha)} \to K(G,n)$ — unlike previous work yielding only isomorphism classes.

2. **Algorithmic Implementation: The Sparse Eilenberg-MacLane Algorithm.**

Our constructive proof directly translates into a systematic computational procedure, which we present as the Sparse Eilenberg-MacLane Algorithm (Algorithms 1-5). The algorithm has three key features:

Matrix-based computation: The entire coordinate construction reduces to solving linear systems involving Čech cochain matrices, making the method amenable to standard numerical linear algebra.

Sparse structure: As the same technique Perea [1] did for circular coordinates, by working with a landmark-based cover of the data, we avoid computing coordinates for all data points, yielding a sparse representation that scales efficiently.

Polynomial complexity: We provide a detailed complexity analysis showing that for a landmark set of size N and persistent class in dimension n, the algorithm runs in $O(N^{n+1})$ time, demonstrating computational feasibility for moderate dimensions.

The algorithm employs a recursive gluing construction that systematically extends local sections over an open cover, effectively handling the sheaf-theoretic machinery through elementary matrix operations.

3. **Experimental Validation and Novel Examples.**

We validate our framework through three experiments on synthetic datasets with known topology, demonstrating both correctness and generality:

Experiment 1: Validation against existing methods ($X = S^1$ for $K(\mathbb{Z},1)$-coordinates). We compute $K(\mathbb{Z},1)$-coordinates (circular coordinates) for data sampled from $X = S^1$, reproducing results consistent with Perea's circular coordinate construction [1]. This validates the correctness of our algorithm. Importantly, our explicit formulas—derived via the inverse of the Čech coboundary operator's connecting homomorphism—provide a canonical choice of coordinate map $X^{(\alpha)}ßK(G,n)$, unlike previous work that yields only isomorphism classes of bundles.

Experiment 2: Higher-dimensional iteration ($X = S^2$ for $K(\mathbb{Z},2)$-coordinates). We compute $K(\mathbb{Z},2)$-coordinates for data sampled from S^2, demonstrating that our algorithm successfully iterates the inverse Čech coboundary operator from dimension 2 to dimension 1 to dimension 0. This experiment confirms the algorithm's applicability to arbitrary dimensions n. The resulting coordinates exhibit structure analogous to complex projective coordinates [5], and we visualize the coordinate functions via projective principal component analysis.

Experiment 3: Novel coordinate types ($X = M(\mathbb{Z}_2,2)$ for $K(\mathbb{Z}_2,2)$-coordinates). We present the first computational examples of coordinates valued in $K(\mathbb{Z}_2,2)$. These $K(\mathbb{Z}_2,2)$-coordinates have not been addressed in prior work on topological coordinates, illustrating our framework's ability to extract fundamentally new types of topological features beyond one-dimensional persistent cohomology.

Together, these experiments demonstrate that our method (i) correctly reproduces known constructions, (ii) scales to higher dimensions through recursive iteration, and (iii) enables computation of genuinely novel coordinate types, expanding the toolkit available to practitioners of topological data analysis.

1.2. **The Sparse Eilenberg-MacLane Algorithm.** Let us describe next the steps needed to construct such an Eilenberg-MacLane coordinate. The rest of the paper is devoted to the theory behind these computations.

(1) Let X be the input data set with metric d; i.e., a finite metric space. Select a set of landmarks $L = \{\ell_1 \ldots, \ell_N\} \subset X$, e.g. at random or via `maxmin` sampling, and let
$$r_L := \max_{x\in X} \min_{\ell\in L} d(x,\ell)$$
be the radius of coverage. In particular, r_L is the Hausdorff distance between L and X.
(2) Suppose that an n-dimensional persistent cohomology class for the Rips filtration on the landmark set L, namely $\eta' \in PH^n(\mathcal{R}(L);\mathbb{Z}/q)$ is detected, with coefficients in $\mathbb{Z}/q$ for a prime q. Let $\mathsf{dgm}(L)$ be the resulting persistence diagram.
(3) If there exists $(a,b) \in \mathsf{dgm}(L)$ so that $\max\{a, r_L\} < \frac{b}{2}$, then let
$$\alpha = t\cdot \max\{a, r_L\} + (1-t)\frac{b}{2} \quad , \quad \text{for some} \quad 0 < t < 1$$
Let $\eta' \in Z^n(R_{2\alpha}(L);\mathbb{Z}/q)$ be a cocycle representative for the persistent cohomology class corresponding to $(a,b) \in \mathsf{dgm}(L)$.
(4) Lift $\eta' \in Z^n(R_{2\alpha}(L);\mathbb{Z}/q)$ to a cocycle $\eta \in Z^n(R_{2\alpha}(L);G)$. That is, one for which $\kappa(\eta) = \eta'$:
$$\ldots \to H^n(R_{2\alpha}(L);H) \xrightarrow{\iota_\#} H^n(R_{2\alpha}(L);G) \xrightarrow{\kappa_\#} H^n(R_{2\alpha}(L);\mathbb{Z}/q) \xrightarrow{\Delta} H^{n+1}(R_{2\alpha}(L);H) \to \ldots$$
Compute the Bockstein algorithm to determine if $\Delta(\eta) = 0$ for the following series of exact sequences. If it returns a "success" at 0, take $G = \mathbb{Z}$. Otherwise, stop at the last exact sequence that returns "success".

	$0 \to H \xrightarrow{\iota} G \xrightarrow{\kappa} \mathbb{F} \to 0$
0	$0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/q \to 0$
1	$0 \to \mathbb{Z} \to \mathbb{Z}/q \to \mathbb{Z}/q \to 0$
2	$0 \to \mathbb{Z}/q \to \mathbb{Z}/q^2 \to \mathbb{Z}/q \to 0$
3	$0 \to \ldots \to \ldots \to \ldots \to 0$

(5) Use the following algorithm to compute the coordinate function $f_\eta : B \to K(G, n)$

Algorithm 1 From $\eta \in \check{H}^n(\mathcal{U}, \mathcal{F}_G)$ to $f_\eta : \bigcup \mathcal{U} \to K(G, n)$

Input: $\eta^{(d)} = (\eta_{i_0 i_1 \dots i_d})_{i_0 < \dots < i_d}$, where $\eta_{i_0 i_1 \dots i_d} : U^\alpha_{i_0 i_1 \dots i_d} \to G$

for $n = d, d-1, \dots, 1$ **do**

  Compute $i_\# \eta^{(n)} := (\eta_{i_0 i_1 \dots i_n}, 1) : U^\alpha_{i_0 i_1 \dots i_n} \to EB^{d-n}G$

  Choose $\beta < \alpha$ such that $\mathcal{U}^\beta$ still covers B.

  Define $t_0 \in \check{C}^{n-1}(\mathcal{U}^\alpha, \mathcal{F}_{EB^{d-n}G})(U^\alpha_0)$ by $t_{0, i_0 \dots i_{n-1}}(x) := (\eta_{0 i_0 \dots i_{n-1}}(x), 1)$, $\forall x \in U^\alpha_0 \cap U^\alpha_{i_0 \dots i_{n-1}}$.

  for $i = 1, 2, \dots, N-1$ **do**

    Choose γ_i such that $\beta < \gamma_i < \gamma_{i-1}$ (By convention, $\gamma_0 = \alpha$).

    Define $t_i \in \check{C}^{n-1}(\mathcal{U}^\alpha, \mathcal{F}_{EB^{d-n}G})(\tilde{U}^{\gamma_i}_i)$ by

$$t_{i, i_0 \dots i_{n-1}}(x) := \begin{cases} \eta_{i i_0 \dots i_{n-1}}(x) + \sum_{j=0}^{n-1} (-1)^j h(t_{i-1, i i_0 \dots \hat{i_j} \dots i_{n-1}}(x), g_i(x)) & \text{if } x \in (U^{\gamma_i}_i - \tilde{U}^{\gamma_i}_{i-1}) \cap U^\alpha_{i_0 \dots i_{n-1}} \\ t_{i-1, i_0 \dots i_{n-1}}(x) & \text{if } x \in \tilde{U}^{\gamma_i}_{i-1} \cap U^\alpha_{i_0 \dots i_{n-1}} \end{cases}$$

  end for

  Set $Q^n(\eta^{(n)}) \in \check{C}^{n-1}(\mathcal{U}^\alpha, \mathcal{F}_{EB^{d-n}G})(B)$ by $Q^n(\eta^{(n)})_{i_0 \dots i_{n-1}}(x) := t_{N, i_0 \dots i_{n-1}}(x), \forall x \in B$.

  Update $\eta^{(n-1)} := j_\#(Q^n(\eta^{(n)})) := [Q^n(\eta^{(n)})] \in \check{C}^{n-1}(\mathcal{U}^\alpha, \mathcal{F}_{B^{d-n+1}G})(B)$.

end for

Recall that the loop stops when $\eta^{(0)} = (\eta_{i_0})_{i_0=0}^N$ is assigned with values.

Set coordinate function $f_\eta : B \to B^d G$ by $f_\eta(x) := \begin{cases} \eta_0(x) & \text{if } x \in U^\beta_0 \\ \eta_1(x) & \text{if } x \in U^\beta_1 - U^\beta_0 \\ \dots \end{cases}$

return f_η.

1.3. **Organization.** We start in Section 2 with a few preliminaries on classifying spaces, highlighting the main theorems needed in later parts of the paper. We assume familiarity with persistent cohomology, as well as the definition of Čech cohomology with coefficients in a presheaf. In section 3,

2. Preliminaries

2.1. **A good model for $K(G, n)$.** We present here a way to construct a good model for $K(G, n)$.

Definition 2.1 (Classifying Spaces)**.** For a topological group G, there exists a weakly contractible space EG with a free continuous action $EG \times G \to EG$. Moreover, defining $BG := EG/G$ by group action will yield a principal G-bundle $G \to EG \to BG$, called a *universal bundle* of G. Any space BG satisfying such a property above is called a *classifying space of G*.

Therefore, the existence of a derived classifying space of a group is always guaranteed. Although there is no unique way of constructing such ones, it turns out that all classifying spaces of the same group G are homotopic equivalent to each other. Several classical construction methods include Milnor construction, Bar construction, Segal construction, etc. For the sake of the construction of higher dimensions $B^n(G) = B(B(\dots B(G)))$, however, we are required to equip an Abelian group structure for BG (which is not inherited from Milnor construction), hence bar construction is adapted in this project. Here is how it is defined:

Definition 2.2 (Bar Construction)**.** Define $EG := |E_*(G)|$ and $BG := |B_*(G)|$ where the $E_*(G)$ and $B_*(G)$ are the simplicial sets given by

$$E_n(G) := G^{n+1} \qquad B_n(G) := G^n \qquad p_n : G^{n+1} \to G^n; (g_1, g_2, ..., g_n, g_{n+1}) \mapsto (g_1, g_2, ..., g_n)$$

$$\begin{tikzcd}
E_*(G): & G \arrow[r, bend right, "s_0"'] \arrow[d, "p_0"] & G^2 \arrow[l, bend right, "d_0;d_1"'] \arrow[r, bend right, "s_0;s_1"'] \arrow[d, "p_1"] & G^3 \arrow[l, bend right, "d_0;d_1;d_2"'] \arrow[r, bend right, "s_0;s_1;s_2"'] \arrow[d, "p_2"] & G^4 \arrow[l, bend right, "d_0;d_1;d_2;d_3"'] \arrow[r, bend right, "..."'] \arrow[d, "p_3"] & ... \arrow[l, bend right, "..."'] \\
B_*(G): & 0 \arrow[r, bend right, "\sigma_0"'] & G \arrow[l, bend right, "\delta_0;\delta_1"'] \arrow[r, bend right, "\sigma_0;\sigma_1"'] & G^2 \arrow[l, bend right, "\delta_0;\delta_1;\delta_2"'] \arrow[r, bend right, "\sigma_0;\sigma_1;\sigma_2"'] & G^3 \arrow[l, bend right, "\delta_0;\delta_1;\delta_2;\delta_3"'] \arrow[r, bend right, "..."'] & ... \arrow[l, bend right, "..."']
\end{tikzcd}$$

with the face maps and degeneracy maps defined as follows

$$d_i(g_1, g_2, ..., g_{n+1}) := \begin{cases} (g_2, ..., g_{n+1}) & \text{if } i = 0 \\ (g_1, ..., g_{i-1}, g_i g_{i+1}, g_{i+2}, ..., g_{n+1}) & \text{if } 1 \le i \le n \end{cases}$$

$$s_i(g_1, g_2, ..., g_{n+1}) := (g_1, ..., g_i, 1_G, g_{i+1}, ..., g_{n+1}) \qquad \forall 0 \le i \le n$$

$$\delta_i(g_1, g_2, ..., g_n) := \begin{cases} (g_2, ..., g_n) & \text{if } i = 0 \\ (g_1, ..., g_{i-1}, g_i g_{i+1}, g_{i+2}, ..., g_n) & \text{if } 1 \le i \le n-1 \\ (g_1, ..., g_{n-1}) & \text{if } i = n \end{cases}$$

$$\sigma_i(g_1, g_2, ..., g_n) := (g_1, ..., g_{i-1}, 1_G, g_i, ..., g_n) \qquad \forall 0 \le i \le n$$

Definition 2.3 (Bar Construction, Equivalent)**.** For a topological group G with identity $e \in G$, let

$$EG := \bigsqcup_{n \ge 0} G^{n+1} \times \Delta_n / \sim$$

with the equivalence relation $\sim$ given by

- $((g_0, g_1, ..., g_n), (0, t_0, t_1, ..., t_{n-1})) \sim ((g_1, ..., g_n), (t_0, t_1, ..., t_{n-1}))$
- $((g_0, ..., g_n), (t_0, ..., t_{i-1}, 0, t_i, ..., t_{n-1})) \sim (((g_0, ..., g_{i-2}, g_{i-1} g_i, g_{i+1}, ..., g_n), (t_0, ..., t_{n-1})))$
- $((g_0, ..., g_n), (t_0, ..., t_{i-1}, t_i + t_{i+1}, ..., t_{n+1})) \sim (((g_0, ..., g_{i-1}, e, g_i, ..., g_n), (t_0, ..., t_{n+1})))$

Remark: Equivalently, a concrete way to see these relations is

$$\begin{cases} ((g_0, g_1, ..., g_n), (0, t_0, t_1, ..., t_{n-1})) \sim ((g_1, ..., g_n), (t_0, t_1, ..., t_{n-1})) \\ ((g_0, g_1, ..., g_n), (t_0, 0, t_1, ..., t_{n-1})) \sim ((g_0 g_1, g_2, ..., g_n), (t_0, t_1, ..., t_{n-1})) \\ \cdots \\ ((g_0, g_1, ..., g_n), (t_0, t_1, ..., t_{n-1}, 0)) \sim ((g_0, g_1, ..., g_{n-2}, g_{n-1} g_n), (t_0, t_1, ..., t_{n-1})) \end{cases}$$

$$\begin{cases} ((g_0, g_1, ..., g_n), (t_0 + t_1, t_2, ..., t_{n+1})) \sim ((e, g_0, g_1, ..., g_n), (t_0, t_1, ..., t_{n+1})) \\ ((g_0, g_1, ..., g_n), (t_0, t_1 + t_2, ..., t_{n+1})) \sim ((g_0, e, g_1, ..., g_n), (t_0, t_1, ..., t_{n+1})) \\ \cdots \\ ((g_0, g_1, ..., g_n), (t_0, t_1, t_2, ..., t_n + t_{n+1})) \sim ((g_0, g_1, ..., e, g_n), (t_0, t_1, ..., t_{n+1})) \end{cases}$$

Here $\Delta_n := \{(t_0, ..., t_n) \in \mathbb{R}^n \mid t_i \ge 0, t_0 + ... + t_n = 1\}$ denotes the standard n-simplex.

There is naturally a G-action on EG induced by a simplex-wise definition:

$$\begin{array}{rcl} G^{n+1} \times G & \to & G^{n+1} \\ (g_0, ..., g_n) \cdot g & := & (g_0, ..., g_{n-1}, g_n g) \end{array}$$

In this context, we may realize that $B_*(G) = E_*(G)/G$ where the quotient is taken by group action. Therefore, the construction here forms a principal G-bundle (called the bar construction of G)

$$G \to EG \xrightarrow{p\ :=\ |p_*|} BG$$

which admits a group structure on BG by

$$BG \times BG = |B_*(G)| \times |B_*(G)| \cong |B_*(G) \times B_*(G)| \cong |B_*(G \times G)| = B(G \times G) \to BG$$

Here, the first isomorphism is given by the geometric realization that admits the product of two simplicial sets; the second isomorphism is derived naturally from the reordering of the coordinates in $G \times G$.

It is important to point out that bar construction preserves the Abelian-ness by [14] (BG is Abelian if G is Abelian).

We will briefly show that $B^n(G)$ is a $K(G,n)$ under the assumption that G is discrete Abelian.

Definition 2.4 (Eilenberg-MacLane Spaces)**.** Let G be a group and $n \geq 1$ be an integer. An *Eilenberg-MacLane space* $K(G,n)$ is a connected space whose homotopy groups are only nontrivial (as G) at dimension n, i.e. $\pi_i(K(G,n)) = \begin{cases} G & (\text{if } i = n) \\ 0 & (\text{else}) \end{cases}$

By [8], [15], EG is contractible, hence $\pi_i(EG) = 0$ for all $i \geq 0$. The long exact sequence of homotopy groups derived from $G \to EG \to BG$ is

$$... \to \pi_{i+1}(EG) \to \pi_{i+1}(BG) \to \pi_i(G) \to \pi_i(EG) \to \pi_i(BG) \to ...$$

which splits into isomorphisms $\pi_{i+1}(BG) \cong \pi_i(G)$ for all $i \geq 0$. Therefore,

$$\pi_i(B^n(G)) \cong \pi_{i-1}(B^{n-1}(G)) \cong ... \cong \pi_{i-n}(B^0(G)) = \pi_{i-n}(G) = \begin{cases} G & (\text{if } i = n) \\ 0 & (\text{if } i \neq n) \end{cases}$$

whereas the last equality needs G to be discrete.

Proposition 2.5 (Contractibility of EG)**.** *The bar construction of EG, as defined above, is contractible given by the homotopy* $h : EG \times [0,1] \to EG$

$$\begin{array}{rccc} h: & \left(\bigsqcup_{n\geq 0} G^{n+1} \times \Delta_n / \sim\right) \times [0,1] & \to & \bigsqcup_{n\geq 0} G^{n+1} \times \Delta_n / \sim \\ & ((g_0,...,g_n),(t_0,...,t_n),t) & \mapsto & (((g_0...g_n)^{-1}, g_0,...,g_n),(1-t, tt_0,...,tt_n)) \end{array}$$

Proof. We first show that h is a well-defined continuous map.

As a geometric realization, EG is a quotient of a disjoint union of topological simplices. Therefore, it suffices to show that h is continuous after restricting to each simplex in the disjoint union, and that h is well-defined, i.e., respects the equivalence classes.

On a particular simplex $G^{n+1} \times \Delta_n$, with a fixed n, h locally is a map from and to product spaces $G^{n+1} \times \Delta_n \times [0,1] \to G^{n+2} \times \Delta_{n+1}$, which is continuous if and only if the function on the components $G^{n+1} \to G^{n+2}; (g_0,...,g_n) \mapsto ((g_0...g_n)^{-1}, g_0,...,g_n)$ and $\Delta_n \times [0,1] \to \Delta_{n+1}; (t_0,...,t_n), t \mapsto (1-t, tt_0,...,tt_n)$ are both continuous. Since the group law and inverse of G are continuous, and multiplication within $[0,1]$ is continuous, we know these two maps are continuous, therefore, h is continuous on each simplex.

The well-definedness of h can be verified throughout each of the equivalence relations of EG:

(1) Whenever $x_1 = ((g_0, ..., g_n), (t_0, ..., t_{i-1}, 0, t_i, ..., t_{n-1})) \sim ((g_0, ..., g_{i-2}, g_{i-1}g_i, g_{i+1}, ..., g_n), (t_0, ..., t_{n-1})) = x_2$, their images $h(x_1, t) = (((g_0...g_n)^{-1}, g_0, ..., g_n), (1-t, tt_0, ..., tt_{i-1}, 0, tt_i, ..., tt_{n-1})) \sim (((g_0...g_n)^{-1}, g_0, ..., g_{i-2}, g_{i-1}g_i, g_{i+1}, ..., g_n), (1-t, tt_0, ..., tt_{n-1})) = h(x_2, t)$.

(2) Whenever $y_1 = ((g_0, g_1, ..., g_n), (0, t_0, t_1, ..., t_{n-1})) \sim ((g_1, ..., g_n), (t_0, t_1, ..., t_{n-1})) = y_2$, their images $h(y_1, t) = (((g_0...g_n)^{-1}, g_0, g_1, ..., g_n), (1-t, 0, tt_0, tt_1, ..., tt_{n-1})) \sim (((g_1...g_n)^{-1}, g_1, ..., g_n), (1-t, tt_0, tt_1, ..., tt_{n-1})) = h(y_2, t)$.

(3) Whenever $z_1 = ((g_0, ..., g_n), (t_0, ..., t_{i-1}, t_i+t_{i+1}, ..., t_{n+1})) \sim ((g_0, ..., g_{n-1}, e, g_i, ..., g_n), (t_0, ..., t_{n+1})) = z_2$, their images $h(z_1, t) = (((g_0...g_n)^{-1}, g_0, ..., g_n), (1-t, tt_0, ..., tt_{i-1}, tt_i + tt_{i+1}, ..., tt_{n+1})) \sim (((g_0...g_n)^{-1}, g_0, ..., g_{i-1}, e, g_i, ..., g_n), (1-t, tt_0, ..., tt_{n+1})) = h(z_2, t)$.

As we can see, h also defines a homotopy:

$$\begin{aligned} h((g_0, ..., g_n), (t_0, ..., t_n), 0) &= ((g_0...g_n)^{-1}, g_0, ..., g_n), (1, 0, ..., 0)) \\ &= ((g_0, ..., g_n)^{-1} \cdot g_0 \cdot ... \cdot g_n, 1) \\ &= (e, 1) \end{aligned}$$

$$\begin{aligned} h((g_0, ..., g_n), (t_0, ..., t_n), 1) &= ((g_0...g_n)^{-1}, g_0, ..., g_n), (0, t_0, ..., t_n)) \\ &= ((g_0, ..., g_n), (t_0, ..., t_n)) \end{aligned}$$

Q.E.D. ■

2.2. Principal Bundles.

For the following setting, let G be an Abelian topological group.

Definition 2.6 (Principal G-Bundle)**.** A *principal G-bundle* is a fiber bundle $G \to E \xrightarrow{\pi} B$ together with a continuous right group action $E \times G \to E$ such that

- G preserves the fibers of E, i.e. $\forall b \in B, \pi^{-1}(b) \cdot G \subseteq \pi^{-1}(b)$.
- G acts freely and transitively on them, and $\forall e \in E$, $e \cdot G : G \to \pi^{-1}(b); g \mapsto e \cdot g$ is a homeomorphism.

Examples of principal bundles are $\mathbb{Z} \to \mathbb{R} \to S^1$, $\mathbb{Z}_n \to S^\infty \to L_n$, $S^1 \to S^\infty \to \mathbb{CP}^\infty$, etc.

Definition 2.7 (Bundle Isomorphisms)**.** Two principal G-bundles $G \to E_1 \xrightarrow{\pi_1} B$ and $G \to E_2 \xrightarrow{\pi_2} B$ are *isomorphic*, denoted by $\cong$, if there is a homeomorphism $\Phi : E_1 \to E_2$ such that

- Φ is G-equivariant, i.e. $\forall e_1 \in E_1, \forall g \in G, \Phi(e_1 \cdot g) = \Phi(e_1) \cdot g$.
- $\pi_2 \circ \Phi = \pi_1$.

The isomorphism class of principal G-bundles is defined to be $\mathrm{Prin}_G(B) := \{G \to E \xrightarrow{\pi} B$: principal bundles over B$\}/\cong$.

2.3. Čech Resolutions.

Definition 2.8. Let B be a topological space and $\mathcal{U}$ be an open cover of B. $\mathcal{F}$ is a sheaf of Abelian groups over B. Then,

$$\check{C}^n(\mathcal{U}, \mathcal{F})(V) := \prod_{i_0 < ... < i_n} \mathcal{F}(U_{i_0...i_n} \cap V), \qquad \forall V \underset{\text{open}}{\subseteq} B$$

defines a sheaf over B, along with a sheaf map $d^n : \check{C}^n(\mathcal{U}, \mathcal{F})(B) \to \check{C}^{n+1}(\mathcal{U}, \mathcal{F})(B)$ given by

$$\begin{array}{rcl} d^n : \check{C}^n(\mathcal{U}, \mathcal{F})(V) & \to & \check{C}^{n+1}(\mathcal{U}, \mathcal{F})(V) \\ (s_{i_0...i_n})_{i_0<...<i_n} & \mapsto & \left(\sum_{j=0}^{n+1} (-1)^j s_{i_0...\hat{i_j}...i_{n+1}}|_{U_{i_0...i_{n+1}} \cap V} \right)_{i_0<...<i_{n+1}} \end{array}$$

called the boundary map of Čech cochains on dimension n.

Lemma 2.9 (Čech Resolution)**.** *The maps d^n form an exact sequence of sheaves as follows*

$$0 \to \mathcal{F} \xrightarrow{e} \check{C}^0(\mathcal{U},\mathcal{F}) \xrightarrow{d^0} \check{C}^1(\mathcal{U},\mathcal{F}) \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} \check{C}^n(\mathcal{U},\mathcal{F}) \xrightarrow{d^n} \cdots$$

Proof. We could decompose this long exact sequence into short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F} \arrow[r, "e"] & \check{C}^0(\mathcal{U},\mathcal{F}) \arrow[rr, "d^0"] \arrow[dr, "p^0"] & & \check{C}^1(\mathcal{U},\mathcal{F}) \arrow[rr, "d^1"] \arrow[dr, "p^1"] & & \check{C}^2(\mathcal{U},\mathcal{F}) \arrow[r, "d^2"] & ... \\
& & & \ker d^1 \arrow[ur, "e^1"] \arrow[dr] & & \ker d^2 \arrow[ur, "e^2"] \arrow[dr] & & ... \\
& & 0 \arrow[ur] & & 0 \arrow[ur] & & 0 & ...
\end{tikzcd}$$

So that the exactness of the long sequence is equivalent to proving that all the associated short sequences

$$0 \to \mathcal{F} \xrightarrow{e} \check{C}^0(\mathcal{U},\mathcal{F}) \xrightarrow{p^0} \ker d^1 \to 0$$

$$0 \to \ker d^1 \xrightarrow{e^1} \check{C}^1(\mathcal{U},\mathcal{F}) \xrightarrow{p^1} \ker d^2 \to 0$$

$$\cdots$$

$$0 \to \ker d^n \xrightarrow{e^n} \check{C}^n(\mathcal{U},\mathcal{F}) \xrightarrow{p^n} \ker d^{n+1} \to 0$$

$$\cdots$$

are exact. Here, $e^n : \ker d^n \to \check{C}^n(\mathcal{U},\mathcal{F})$ is defined to be the inclusion, i.e., $\forall V \underset{\text{open}}{\subseteq} B, \forall \eta \in \ker d^n_V, e^n_V(\eta) := \eta$. The map $p^n : \ker d^n \to \check{C}^n(\mathcal{U},\mathcal{F})$ is defined as the same assignment with the boundary map, with codomain restricted, i.e.,

$$p^n_V\left((\eta_{i_0\dots i_n})_{i_0<\dots<i_n}\right) := \sum_{j=0}^{n+1}(-1)^j \eta_{i_0\dots\hat{i_j}\dots i_{n+1}}|_{U_{i_0\cdots i_{n+1}}\cap V} \in \ker d^{n+1}$$

This is always possible, since $d^{n+1}\circ d^n = 0$ for all $n \geq 0$.

Now we are going to prove that p^n is surjective, and its kernel is the same as $\ker d^n$ (which is by definition true). Let $V \underset{\text{open}}{\subseteq} B$, then it suffices to prove that $p_{V\cap U_i}$'s are surjective, as $\{V\cap U_i\}_{i\in I}$ forms an open cover of V:

Take $\eta \in (\ker d^n)(V\cap U_i)$, then we may define $\tau \in \check{C}^{n-1}(\mathcal{U},\mathcal{F})(V\cap U_i)$ by

$$\tau(i_0,...,i_{n-1}) := \eta(i, i_0, ..., i_{n-1}), \quad \forall (i_0,...,i_{n-1})$$

The verification of $d^{n-1}_{V\cap U_i}\tau = \eta$ follows the same strategy as the theorem of local fineness of Čech Cohomology, 2.11, as depicted as follows.

A remark is that this proof also applies to the statement that $\forall W \underset{\text{open}}{\subseteq} B$, and $W \subseteq U_i$ for some i, then p_W is surjective. Q.E.D. ■

Definition 2.10. Let B be a topological space and $\mathcal{U}$ be an open cover of B. $\mathcal{F}$ is a sheaf of Abelian groups over B. The Čech Cohomology groups are defined as the homology of the global sections of the Čech Resolution, i.e.,

$$\check{H}^n(\mathcal{U},\mathcal{F}) := H_n(\mathcal{C}^\bullet) := \ker d_B^n / \mathrm{im}\; d_B^{n-1}$$

where

$$\mathcal{C}^\bullet : 0 \to \mathcal{F}(B) \xrightarrow{e_B} \check{C}^0(\mathcal{U},\mathcal{F})(B) \xrightarrow{d_B^0} \check{C}^1(\mathcal{U},\mathcal{F})(B) \xrightarrow{d_B^1} \cdots \xrightarrow{d_B^{n-1}} \check{C}^n(\mathcal{U},\mathcal{F})(B) \xrightarrow{d_B^n} \cdots$$

Lemma 2.11 (Local fineness of Čech Cohomology)**.** *$\check{H}^n(\mathcal{U},\mathcal{F}) = 0$ if $B \in \mathcal{U}$.*

Proof. Since $B \in \mathcal{U}$, we know that $B = U_i$ for some $i \in I$. Take a cocycle $\eta \in \check{C}^n(\mathcal{U},\mathcal{F})(B)$ so that $d_B^n \eta = 0$. Then, we claim that $\tau \in \check{C}^{n-1}(\mathcal{U},\mathcal{F})(B)$, defined by

$$\tau(i_0,...,i_{n-1}) := \eta(i,i_0,...,i_{n-1}), \quad \forall (i_0,...,i_{n-1})$$

is a $(n-1)$-cochain whose boundary is η.

Firstly, τ is well-defined since $U_{ii_0\dots i_{n-1}} = U_i \cap U_{i_0\dots i_{n-1}} = B \cap U_{i_0\dots i_{n-1}} = U_{i_0\dots i_{n-1}}$.

Secondly, for any n-simplex $(i_0,...,i_n)$,

$$\begin{aligned}(d_B^{n-1}\tau)(i_0,...,i_n) &= \sum_{j=0}^{n} (-1)^j \cdot \tau(i_0,...,\hat{i_j},...,i_n)|_{U_{i_0 i_1 \dots i_n}} \\ &= \sum_{j=0}^{n} (-1)^j \cdot \eta(i,i_0,...,\hat{i_j},...,i_n)|_{U_{i_0 i_1 \dots i_n}}\end{aligned}$$

- If $i \in \{i_0,...,i_n\}$, then $\exists k \in I$ such that $i = i_k$. Then,

$$\begin{aligned}(d_B^{n-1}\tau)(i_0,...,i_n) &= \sum_{j\neq k} (-1)^j \cdot \eta(i,i_0,...,\hat{i_j},...,i_n)|_{U_{i_0 i_1 \dots i_n}} + (-1)^k \cdot \eta(i,i_0,...,\hat{i_k},...,i_n)|_{U_{i_0 i_1 \dots i_n}} \\ &= 0 + (-1)^k \cdot \eta(i,i_0,...,\hat{i_k},...,i_n)|_{U_{i_0 i_1 \dots i_n}} \text{(Since each term has two } i\text{'s)} \\ &= (-1)^k \cdot \eta(i,i_0,...,i_{k-1},i_{k+1},...,i_n)|_{U_{i_0 i_1 \dots i_n}} \\ &= (-1)^k \cdot \mathrm{sgn}\begin{pmatrix} i & i_0 & \dots & i_{k-2} & i_{k-1} \\ i_0 & i_1 & \dots & i_{k-1} & i \end{pmatrix} \cdot \eta(i_0,...,i_{k-1},i,i_{k+1},...,i_n)|_{U_{i_0 i_1 \dots i_n}} \\ &= (-1)^k \cdot (-1)^k \cdot \eta(i_0,...,i_n)|_{U_{i_0 i_1 \dots i_n}} \\ &= \eta(i_0,...,i_n)\end{aligned}$$

- If $i \notin \{i_0,...,i_n\}$, then $(i,i_0,...,i_n)$ forms a $(n+1)$-cocycle, hence can be applied to cocycle condition given by η: Since $d_B^n \eta = 0$, we have $(d_B^n\eta)(i,i_0,...,i_n) = 0$. Then,

$$\eta(i_0,...,i_n)|_{U_{ii_0\dots i_n}} - \sum_{j=0}^{n} (-1)^j \cdot \eta(i,i_0,...,\hat{i_j},...,i_n)|_{U_{ii_0\dots i_n}} = 0$$

  Since $U_{ii_0\dots i_n} = U_{i_0\dots i_n}$, we have

$$\eta(i_0,...,i_n)|_{U_{i_0\dots i_n}} - \sum_{j=0}^{n} (-1)^j \cdot \eta(i,i_0,...,\hat{i_j},...,i_n)|_{U_{i_0\dots i_n}} = 0$$

Therefore, the calculation of the boundary gives $(d_B^{n-1}\tau)(i_0,...,i_n) = \eta(i_0,...,i_n)|_{U_{i_0\cdots i_n}} = \eta(i_0,...,i_n)$.

As a summary, for all n-simplex $(i_0,...,i_n)$, $(d_B^{n-1}\tau)(i_0,...,i_n) = \eta(i_0,...,i_n)$, so $d_B^{n-1}\tau = \eta$. Q.E.D. ■

3. Methodology

This section provides the mathematical foundation we use throughout the paper. As explained in section 2, we aim to attain an isomorphism $\check{H}^n(B,\mathcal{F}_G) \cong [B, B^n(G)]$ given by the following sequences:

$$\check{H}^n(B,\mathcal{F}_G) \xrightarrow[\cong]{\text{Map 3}} \check{H}^{n-1}(B,\mathcal{F}_{BG}) \xrightarrow[\cong]{\text{Map 3}} ... \xrightarrow[\cong]{\text{Map 3}} \check{H}^1(B,\mathcal{F}_{B^{n-1}(G)})$$

$$\check{H}^1(B,\mathcal{F}_{B^{n-1}(G)}) \xrightarrow[\cong]{\text{Map 2}} \mathrm{Prin}_{B^{n-1}(G)}(B) \xrightarrow[\cong]{\text{Map 1}} [B, B^n(G)] \xrightarrow{\cong} [B, K(G,n)]$$

To meet the requirement for the construction to work, later on, we will present step by step that ultimately we need G to be assumed to be a discrete Abelian topological group, and B to be assumed paracompact Hausdorff.

$$\begin{tikzcd}
\check{C}^{n-1}(\mathcal{U},\mathcal{F}_G)(B) \arrow[r, dashed, "i_\#"] \arrow[d, dashed, "d_B^{n-1}"] & \check{C}^{n-1}(\mathcal{U},\mathcal{F}_{EG})(B) \arrow[r, "j_\#"] \arrow[d, bend left, "d_B^{n-1}"] & \check{C}^{n-1}(\mathcal{U},\mathcal{F}_{BG})(B) \arrow[d, dashed, "d_B^{n-1}"] \\
\check{C}^{n}(\mathcal{U},\mathcal{F}_G)(B) \arrow[r, "i_\#"] & \check{C}^{n}(\mathcal{U},\mathcal{F}_{EG})(B) \arrow[r, dashed, "j_\#"] \arrow[u, bend left, "Q^n"] & \check{C}^{n}(\mathcal{U},\mathcal{F}_{BG})(B)
\end{tikzcd}$$

3.1. Map 1: $\mathbf{Prin}_G(B) \cong [B, BG]$.

We claim that there is an isomorphism between the classifying maps and isomorphism classes of principal G-bundles, given by:

$$[B, BG] \cong \mathrm{Prin}_G(B)$$
$$[f : B \to BG] \mapsto [f^*(EG)]$$
$$[f : B \to BG] \leftarrow\!\shortmid [G \to E \xrightarrow{q} B]$$

<u>Forward Direction:</u> Take a map $f : B \to BG$, the pullback bundle $G \to f^*(EG) \xrightarrow{p^*} B$ of the universal bundle $G \to EG \xrightarrow{p} BG$ forms a representative of the class of principal G-bundles over B. It is given by

$$f^*(EG) := \{(b, e') \in B \times EG \mid f(b) = p(e')\}$$

with $p^*(b,e') := b, f'(b,e') := e', \forall b \in B, \forall e' \in EG$. The following diagram shows how to construct $f^*(EG)$.

$$\begin{tikzcd}
G \arrow[r] & EG \arrow[r, "p"] & BG \\
G \arrow[r] & f^*(EG) \arrow[r, dashed, "p^*"] \arrow[u, dashed, "f'"] & B \arrow[u, "f"]
\end{tikzcd}$$

<u>Backward Direction:</u> Given a principal G-bundle $G \to E \xrightarrow{q} B$, we may compare it with the universal bundle derived from G: $G \to EG \xrightarrow{p} BG$.

Thm ([16], 4.12.2) For each numerable principal G-bundle $G \to E \xrightarrow{q} B$ there exists a map $f : B \to BG$ such that $(G \to E \xrightarrow{q} B) \cong f^*(EG)$.

Remark: In our case, B is assumed to be paracompact Hausdorff, so $G \to E \xrightarrow{q} B$ is numerable.

The backward map is constructed as follows: Take an open cover $\mathcal{U} = \{U_i\}_{i\in I}$ of B with I equipping a total order and a system of local trivializations $\{h_i : q^{-1}(U_i) \to U_i \times G\}$. Since B is paracompact Hausdorff, there is a partition of unity $\{\phi_i : B \to [0,1]\}_{i\in I}$ with respect to $\mathcal{U}$. We are going to define a map $g : E \to EG$, by such a procedure: Take $\forall e \in E$, and denote $\sigma := \{i \in I \mid q(e) \in U_i\} = \{\sigma_0, \sigma_1, ..., \sigma_n\}$ (By paracompactness, $\mathcal{U}$ is locally finite, hence σ is always finite).

$$\begin{tikzcd} U_i \times G \arrow[d, "\mathrm{proj}_2"'] \arrow[drr, "\mathrm{proj}_1"'] & & q^{-1}(U_i) \arrow[ll, "h_i"', "\cong"] \arrow[d, "q"] \\ G & & U_i \end{tikzcd}$$

We may observe that for any $i \in \sigma$, $(\mathrm{proj}_2 \circ h_i)(e)$ returns an element in G, while $\phi_i(q(e))$ returns an element in $[0,1]$ satisfying the condition that $\sum_{i\in\sigma} \phi_i(q(e)) = \sum_{i\in I} \phi_i(q(e)) = 1$. Therefore,

$$(\mathrm{proj}_2 \circ h_{\sigma_0}(e), \mathrm{proj}_2 \circ h_{\sigma_1}(e), ..., \mathrm{proj}_2 \circ h_{\sigma_n}(e), \phi_{\sigma_0} \circ q(e), \phi_{\sigma_1} \circ q(e), ..., \phi_{\sigma_n} \circ q(e)) \in G^{n+1} \times \Delta^n$$

gives a representative in the nth skeleton in $EG := \coprod_{m\geq 0} G^{m+1} \times \Delta_m / \sim$. The class it represents is defined to be the image $g(e)$.

Hence, given a point $b \in B$, consider any element $e \in q^{-1}(b)$ in its fiber and define $f(b) := p \circ g(e)$. The claim is that the image $f(b)$ is independent of the choice of the preimage $e \in q^{-1}(b)$.

$$\begin{tikzcd} G \arrow[r] & E \arrow[r, "q"] \arrow[d, "g"] & B \arrow[d, dashed, "f"] \\ G \arrow[r] & EG \arrow[r, "p"] & BG \end{tikzcd}$$

3.2. **Map 2: $\check{H}^1(B, \mathcal{F}_G) \cong \mathbf{Prin}_G(B)$.**

Let B be a topological space and $\mathcal{U} = \{U_i\}_{i\in I}$ be an open cover of B, with a total order set I. Let $\mathcal{F}$ be a sheaf of Abelian groups over B (Remark: It suffices to define the Čech cohomology on presheaves, however, the sheaf of maps "maps$(U, -)$" that we are working on in this project is a sheaf).

3.2.1. *Transition Functions give a Čech 1-cocycle.*

The punchline of constructing the bijection between principal bundles and the first order Čech cohomology is by looking at the transition functions of a principal bundle.

We now restrict our view of $\mathcal{F}$ to be a particular sheaf: the sheaf of maps, denoted by $\mathcal{F}_G$. Given G as an Abelian topological group, $\mathcal{F}_G$ gives a sheaf over B where

$$\forall U \subseteq_{\text{open}} B, \mathcal{F}_G(U) := \text{maps}(U, G) = \{f : U \to G \mid f \text{ is continuous}\}$$

$\mathcal{F}_G(U)$ also inherits an Abelian group structure by "pointwise" group law in G: $\forall f, g \in \mathcal{F}_G(U)$, define $f + g : U \to G$ by $\forall x \in U, (f + g)(x) = f(x) + g(x)$.

Let $G \to E \xrightarrow{p} B$ be a principal-G bundle chosen from $\mathrm{Prin}_G(B)$. We may construct a cocycle $\eta \in \check{H}^1(\mathcal{U}, \mathcal{F}_G)$ as follows:

$\forall \sigma \in \mathcal{N}(\mathcal{U})^{(1)}$, by definition, we know $\sigma = \{j, k\}$ for some $j, k \in I$ such that $U_j \cap U_k \neq \varnothing$. Since $G \to E \xrightarrow{p} B$ is a principal G-bundle, its local trivializations give homeomorphisms $h_j : U_j \times G \xrightarrow{\cong} p^{-1}(U_j)$ and $h_k : U_k \times G \xrightarrow{\cong} p^{-1}(U_k)$. Then $\forall b \in U_j \cap U_k$, $1 \in G$, we have $h_j(b, 1), h_k(b, 1) \in p^{-1}(b)$. Since G preserves the fibers of E and acts freely and transitively on them, there uniquely exists an element $g \in G$ such that $h_j(b, 1) = h_k(b, 1) \cdot g$. We denote this element by $h_{jk}(b) := g$, then $h_{jk} : U_j \cap U_k \to G; b \mapsto \exists! g$ gives a function, so $h_{jk} \in \mathcal{F}_G(U_\sigma)$. Defining $\eta = \{h_{jk}\}_{j,k \in I}$ will give $\eta \in \check{C}^1(\mathcal{U}; \mathcal{F}_G)$.

One can also check (for example, in [1]) the cochain η given by the construction above meets the cocycle condition, i.e. $\forall b \in U_{\{j,k,l\}} = U_j \cap U_k \cap U_l, h_{jk}(b) + h_{kl}(b) = h_{jl}(b)$. Therefore, η is a Čech 1-cocycle.

3.2.2. *Gluing transition functions gives a principal G-bundle.* Given $[\eta] \in \check{H}^1(\mathcal{U}; \mathcal{F}_G)$ a Čech 1-cocycle, by definition, $\forall \sigma \in \mathcal{N}(\mathcal{U})^{(1)}$, i.e., $\forall j, k \in I$ such that $U_j \cap U_k \neq \varnothing$, we have $\eta_{jk} \in \mathcal{F}_G(U_{\{j,k\}})$. In other words, $\eta_{jk} : U_j \cap U_k \to G$ is a continuous function. Using this information, we may reconstruct a principal G-bundle by gluing η_{jk}'s:

$$E_\eta := \left(\bigsqcup_{j \in J} U_j \times \{j\} \times G \right) / \sim$$

where $(b, j, g) \sim (b, k, g + \eta_{jk}(b))$ for $\forall j, k \in I, \forall b \in U_j \cap U_k, \forall g \in G$.

As one may check, E_η along with the projection map $p : E_\eta \to B;\ (b, j, g) \mapsto b$, forms a principal G-bundle $G \to E_\eta \xrightarrow{p} B$. It is shown from [1] that the operations obtained from (2) and (3) are inverses of each other up to Čech cohomology and bundle isomorphisms respectively. These yield an isomorphism on the level of topological spaces, i.e. $\check{H}^1(B, \mathcal{F}_G) \cong \mathrm{Prin}_G(B)$.

3.3. **Map 3: $\check{H}^n(B, \mathcal{F}_G) \cong \check{H}^{n-1}(B, \mathcal{F}_{BG}) \cong \ldots \cong \check{H}^1(B, \mathcal{F}_{B^{n-1}G})$.** Map 3 is a sequence of isomorphisms that can be obtained recursively by

$$\check{H}^n(B, \mathcal{F}_G) \cong \check{H}^{n-1}(B, \mathcal{F}_{BG})$$

Our idea of making this happen is to derive a long exact sequence of Čech cohomology groups by changing sheaves and prove that the Čech cohomology groups on sheaves of maps given by EG all vanish, splitting the long exact sequence into isomorphisms.

3.3.1. *Long exact sequence of Čech Cohomology.* Starting from this section, we need to assume that B is paracompact Hausdorff.

Theorem 3.1 (Miranda)**.** *([12], IX.3.18; [13], I.3.25) Let B be a paracompact space, and $0 \to \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H} \to 0$ is a short exact sequence of sheaves on B, where α and β are sheaves of maps.*

Then there exist connecting homomorphisms $\Delta : \check{H}^n(B,\mathcal{H}) \to \check{H}^{n+1}(B,\mathcal{F})$ *for every* $n \geq 0$ *such that the sequence of Čech cohomology groups*

$$0 \to \check{H}^0(B,\mathcal{F}) \to ... \to \check{H}^n(B,\mathcal{F}) \xrightarrow{\alpha^*} \check{H}^n(B,\mathcal{G}) \xrightarrow{\beta^*} \check{H}^n(B,\mathcal{H}) \xrightarrow{\Delta} \check{H}^{n+1}(B,\mathcal{F}) \to ...$$

is exact.

As we discussed in 3.1(2), given G as an Abelian topological group, we may generate its universal bundle $G \to EG \xrightarrow{p} BG$, with BG another Abelian topological group. Therefore, there is a series of principal G-bundles given by the bar construction

$$G \to EG \to BG$$

$$BG \to E(BG) \to B(BG) = B^2(G)$$

$$B^2(G) \to EB^2(G) \to B^3(G)$$

$$...$$

$$B^{n-1}(G) \to EB^{n-1}(G) \to B^n(G)$$

Then, they induce the following short exact sequences of sheaves of maps:

$$0 \to \mathcal{F}_G \to \mathcal{F}_{EG} \to \mathcal{F}_{BG} \to 0$$

$$0 \to \mathcal{F}_{BG} \to \mathcal{F}_{EBG} \to \mathcal{F}_{B^2(G)} \to 0$$

$$0 \to \mathcal{F}_{B^2(G)} \to \mathcal{F}_{EB^2(G)} \to \mathcal{F}_{B^3(G)} \to 0$$

$$...$$

$$0 \to \mathcal{F}_{B^{n-1}(G)} \to \mathcal{F}_{EB^{n-1}(G)} \to \mathcal{F}_{B^n(G)} \to 0$$

Lemma 3.2. $0 \to \mathcal{F}_G \xrightarrow{i_\#} \mathcal{F}_{EG} \xrightarrow{j_\#} \mathcal{F}_{BG} \to 0$ *is a short exact sequence of sheaves.*

Proof. Let $\forall x \in \forall U \underset{\text{open}}{\subseteq} X$ be an arbitrary point with its neighborhood, and take a section $\forall s : U \to BG$ over U. Then it suffices to show that $\exists V \underset{\text{open}}{\subseteq} X$, $x \in V \subseteq U$, and a section $\exists \tilde{s} : V \to EG$ such that $(j_\#)_V(\tilde{s}) = s|_V$.

Since $s(x) \in BG$, by the local trivialization of principal bundle $G \xrightarrow{i} EG \xrightarrow{j} BG$, $\exists W \underset{\text{open}}{\subseteq} BG, s(x) \in W$, such that there is a homomorphism $\varphi : j^{-1}(W) \to W \times G$ satisfying the property $j = \text{proj}_1 \circ \varphi$. Since $s : U \to BG$ is continuous, and $W \underset{\text{open}}{\subseteq} BG$, there exists $s^{-1}(W) \underset{\text{open}}{\subseteq} X$, so we define $V := s^{-1}(W)$. It is clear that $x \in V$, since $s(x) \in W$. We can also define that $\tilde{s} : W \to EG$ by $\tilde{s}(x) := \varphi^{-1}(s(x), e) \in j^{-1}(W) \subseteq EG$.

Since $\forall x \in V$, $j \circ \tilde{s}(x) = j \circ \varphi^{-1}(s(x), e) = \text{proj}_1 \circ \varphi \circ \varphi^{-1}(s(x), e) = s(x)$, we know that $j \circ \tilde{s} = s|_V$, hence $j_\#$ is a surjective sheaf map.

Furthermore,

$$\begin{aligned}(\ker j_\#)(V) &= \ker(j_\#)_V \\ &= \{f : V \to EG \mid j \circ f = 0\} \\ &= \{f : V \to EG \mid \forall x \in V, j(f(x)) = (e,1) \in EG\} \\ &= \{f : V \to EG \mid \forall x \in V, \exists g \in G, f(x) = (g,1)\} \\ &\cong \{f : V \to G \mid \forall x \in V, \exists g \in G, f(x) = g\} \\ &= \{f : V \to G\} \\ &= \mathcal{F}_G(V)\end{aligned}$$

Therefore, $\ker j_\# \cong \mathcal{F}_G$, so that $0 \to \mathcal{F}_G \xrightarrow{i_\#} \mathcal{F}_{EG} \xrightarrow{j_\#} \mathcal{F}_{BG} \to 0$ is a short exact sequence. Q.E.D. ■

Then we get a series of long exact sequences on the level of Čech cohomology:

$$... \to \check{H}^i(B, \mathcal{F}_{EG}) \to \check{H}^i(B, \mathcal{F}_{BG}) \to \check{H}^{i+1}(B, \mathcal{F}_G) \to \check{H}^{i+1}(B, \mathcal{F}_{EG}) \to ...$$

$$... \to \check{H}^i(B, \mathcal{F}_{EBG}) \to \check{H}^i(B, \mathcal{F}_{B^2(G)}) \to \check{H}^{i+1}(B, \mathcal{F}_{BG}) \to \check{H}^{i+1}(B, \mathcal{F}_{EBG}) \to ...$$

$$... \to \check{H}^i(B, \mathcal{F}_{EB^2(G)}) \to \check{H}^i(B, \mathcal{F}_{B^3(G)}) \to \check{H}^{i+1}(B, \mathcal{F}_{B^2(G)}) \to \check{H}^{i+1}(B, \mathcal{F}_{EB^2(G)}) \to ...$$

$$...$$

$$... \to \check{H}^i(B, \mathcal{F}_{EB^{n-1}(G)}) \to \check{H}^i(B, \mathcal{F}_{B^n(G)}) \to \check{H}^{i+1}(B, \mathcal{F}_{B^{n-1}(G)}) \to \check{H}^{i+1}(B, \mathcal{F}_{EB^{n-1}(G)}) \to ...$$

Given the theorem that we will prove in later sections, which states as follows, we may split the long exact sequences into isomorphisms that we wish to derive at the beginning of this section.

Theorem 3.3 (Čech Cohomology vanishes over $\mathcal{F}_E$)**.** *Let B be a paracompact space, and E is a contractible topological Abelian group, then the sheaf of maps $\mathcal{F}_E$ over B gives trivial Čech cohomology groups on positive dimensions, i.e., $\check{H}^n(B; \mathcal{F}_E) = 0$ for all $n > 0$.*

The rest part of this section aims to prove the theorem (*).

3.3.2. *Soft sheaves and Sheaf Cohomology.*

Definition 3.4. Let X be a topological space, and $\mathcal{F}$ be a sheaf of Abelian groups over X. For any subspace $Y \subseteq X$, define

$$\mathcal{F}(Y) := \varinjlim_{\substack{Y \subseteq U \subseteq X \\ \text{open}}} \mathcal{F}(U)$$

Definition 3.5. Let B be a paracompact space. A sheaf $\mathcal{F}$ on B is said to be *soft* if $\forall K \underset{\text{closed}}{\subseteq} B$, the restriction map $\mathcal{F}(X) \to \mathcal{F}(K)$ is surjective, i.e., every section of $\mathcal{F}$ over K can be extended to a global section.

Lemma 3.6 ($\mathcal{F}_E$ is soft)**.** *([9], 4.4.6) Let B be a paracompact space, and E is contractible topological Abelian group. Then the sheaf represented by E over B, i.e., $\mathcal{F}_E := maps(-, E)$, is soft.*

Proof. Let $h : E \times [0,1] \to E$ be a contracting homotopy. In other words, $h(-,1) = \mathbb{1}_E$, $h(-,0) =$ constant map on $p \in E$. Take $\forall K \underset{\text{closed}}{\subseteq} B$, with $s : U \to E$ representing a section of E over B for some U, where $K \subseteq U \underset{\text{open}}{\subseteq} B$. Since B is paracompact, we may choose another open set $K \subseteq V \underset{\text{open}}{\subseteq} B$ such that $\overline{V} \subseteq U$. It suffices to extend $s|_V$ over all of B. To make this happen, choose a map $g : B \to [0,1]$ with $g|_{\overline{V}} = 1$ and $g|_{B \setminus U} = 0$. Define $\tilde{s} : B \to E$ by

$$\tilde{s}(b) := \begin{cases} h(s(b), g(b)) & b \in U \\ p & b \notin U \end{cases}$$

This extend $s|_V$ to all of B, i.e., $\tilde{s}|_V = s|_V$. Q.E.D. ∎

Lemma 3.7 (Čech Sheaves are soft over $\mathcal{F}_E$)**.** *Let B be a paracompact space, and E is a contractible topological Abelian group. Let $\mathcal{U}$ be an open cover of B, then the Čech sheaves $\check{C}^n(\mathcal{U}, \mathcal{F}_E)$ are soft for all $n \geq 0$.*

Proof. Take an arbitrary $\forall K \underset{\text{closed}}{\subseteq} B$ and $[s] \in \check{C}(\mathcal{U}, \mathcal{F}_E)(K)$. Then, by definition of sections on closed sets, there exists V, $K \subseteq V \underset{\text{open}}{\subseteq} B$, such that

$$s = (s_{i_0 i_1 \ldots i_n})_{i_0 < i_1 < \ldots < i_n} \in \prod_{i_0 < i_1 < \ldots < i_n} \mathrm{maps}(U_{i_0 i_1 \ldots i_n} \cap V, E)$$

In other words, these coordinates $s_{i_0 i_1 \ldots i_n}$ are continuous maps $s_{i_0 i_1 \ldots i_n} : U_{i_0 i_1 \ldots i_n} \cap V \to E$.

Since the closed set K satisfies $K \subseteq V \underset{\text{open}}{\subseteq} B$, by paracompactness of B, there exists an open set $\exists W \underset{\text{open}}{\subseteq} B$ such that $K \subseteq \overline{W} \subseteq V$, as well as a continuous map $g : B \to [0,1]$ such that $g|_{\overline{W}} = 1$ and $g|_{B-V} = 0$. Since E is contractible, there is a homotopy $h : E \times [0,1] \to E$ such that $h(-,1) = \mathbb{1}_E$ and $h(-,0)$ is a constant at some point $p \in E$.

For each coordinate indexed by n-simplex $(i_0, \ldots, i_n)$, now we may define $\tilde{s}_{i_0 i_1 \ldots i_n} : U_{i_0 i_1 \ldots i_n} \to E$ by

$$\tilde{s}_{i_0 i_1 \ldots i_n}(b) := \begin{cases} h(s_{i_0 i_1 \ldots i_n}(b), g(b)) & \forall b \in U_{i_0 i_1 \ldots i_n} \cap V \\ p & \forall b \in U_{i_0 i_1 \ldots i_n} - V \end{cases}$$

To verify that $\tilde{s} := (\tilde{s}_{i_0 i_1 \ldots i_n})_{i_0 < i_1 < \ldots < i_n}$ forms an extension of s, we see that $\forall b \in W$, in particular $\forall b \in K$,

$$\tilde{s}_{i_0 i_1 \ldots i_n}(b) = h(s_{i_0 i_1 \ldots i_n}(b), 1) = s_{i_0 i_1 \ldots i_n}(b)$$

Q.E.D. ∎

3.4. **Sheaf Theory.**

Lemma 3.8 (Gluing over 2 closed sets)**.** *Let B be a topological space and $\mathcal{F}$ be a sheaf of Abelian groups on B. Assume $B = Z_1 \cup Z_2$, with $Z_1, Z_2 \underset{\text{closed}}{\subseteq} B$, $s_1 \in \mathcal{F}(Z_1), s_2 \in \mathcal{F}(Z_2)$ such that $s_1|_{Z_1 \cap Z_2} = s_2|_{Z_1 \cap Z_2}$, then $\exists s \in \mathcal{F}(B)$, such that $s|_{Z_1} = s_1$ and $s|_{Z_2} = s_2$.*

Proof. Consider s_1 and s_2 are represented by U_1 and U_2, respectively, with $Z_i \subseteq U_i \underset{\text{open}}{\subseteq} B$ $(i = 1,2)$. Then, by definition, $s_1|_{W_{12}} = s_2|_{W_{12}}$ for some $Z_1 \cap Z_2 \subseteq W_{12} \subseteq U_1 \cap U_2$ with $W_{12} \underset{\text{open}}{\subseteq} B$.

Define $V_1 := Z_2^c \cup W_{12}$, and $V_2 := Z_1^c \cup W_{12}$. Firstly, $V_i \subseteq U_i$ and $Z_i \subseteq V_i$ for $i = 1,2$. Secondly, $V_1 \cup V_2 = B$. Thirdly, $V_i \underset{\text{open}}{\subseteq} B$. Fourthly, $V_1 \cap V_2 = (Z_1^c \cup W_{12}) \cap (Z_2^c \cup W_{12}) = (Z_1^c \cap Z_2^c) \cup W_{12} =$

$(Z_1 \cup Z_2)^c \cup W_{12} = B^c \cup W_{12} = \varnothing \cup W_{12} = W_{12}$, so $s_1|_{V_1 \cap V_2} = s_2|_{V_1 \cap V_2}$. Therefore, the gluing axiom of open sets applies to s_1, s_2 over V_1, V_2.

Therefore, $\exists s \in \mathcal{F}(B)$ such that $s|_{V_1} = s_1, s|_{V_2} = s_2$. As consequences, $s|_{Z_1} = s_1$ and $s|_{Z_2} = s_2$. Q.E.D. ■

Lemma 3.9 (Gluing on finitely many closed sets)**.** *Let B be a topological space and $\mathcal{F}$ be a sheaf of Abelian groups on B. Assume $B = Z_1 \cup Z_2 \cup ... \cup Z_r$ with each $Z_i \underset{closed}{\subseteq} B$, and $s_i \in \mathcal{F}(Z_i)$ such that $s_i|_{Z_i \cap Z_j} = s_j|_{Z_i \cap Z_j}$ for all i, j, then $\exists s \in \mathcal{F}(B)$ such that $s|_{Z_i} = s_i$.*

Proof. We prove this by induction on r. The base case is vacuously true when $r = 1$. In the inductive step, we denote $Y := Z_1 \cup Z_2 \cup ... \cup Z_{r-1}$, and assume $Z_i \underset{\text{closed}}{\subseteq} Y$, so that whenever $s_i \in \mathcal{F}(Z_i)$ such that $s_i|_{Z_i \cap Z_j} = s_j|_{Z_i \cap Z_j}$ for $\forall i, j$, we have $\exists s \in \mathcal{F}(Y)$ such that $s|_{Z_i} = s_i$.

Since $Z_i \underset{\text{closed}}{\subseteq} B, Z_i \underset{\text{closed}}{\subseteq} Y$ by subspace topology. By hypothesis, $\exists s \in \mathcal{F}(Y)$ such that $s|_{Z_i} = s_i$. For the conditions of the gluing axiom in 3.8 (over Y and Z_r) to be satisfied, we need to check the following:

(1) $Y \underset{\text{closed}}{\subseteq} B$, because $Z_i \underset{\text{closed}}{\subseteq} B$.
(2) $Z_r \underset{\text{closed}}{\subseteq} B$.
(3) $B = Y \cup Z_r$.
(4) The last thing needed to show is that $s|_{Y \cap Z_r} = s_r|_{Y \cap Z_r}$.

The locality theorem of finitely many closed sets, which is not shown here because of its triviality, can be applied to complete the proof of condition 4. The conditions it needs can be checked as follows:

(1) $Y \cap Z_r = (Z_1 \cup ... \cup Z_{r-1}) \cap Z_r = (Z_1 \cap Z_r) \cup ... \cup (Z_{r-1} \cap Z_r)$ where $Z_i \cap Z_r \underset{\text{closed}}{\subseteq} Y \cap Z_r$
(2) $s|_{Y \cap Z_r}, s_r|_{Y \cap Z_r} \in \mathcal{F}(Y \cap Z_r)$
(3) For each component $Z_i \cap Z_r$, $s_i|_{Z_i \cap Z_r} = s_r|_{Z_i \cap Z_r}$, $s|_{Z_i} = s_i$, so $s|_{Z_i \cap Z_r} = s_r|_{Z_i \cap Z_r}$.

Therefore, by 3.8, $\exists \tilde{s} \in \mathcal{F}(B)$ such that $\tilde{s}|_Y = s, \tilde{s}|_{Z_r} = s_r$.

Therefore, $\tilde{s}|_{Z_i} = s|_{Z_i} = s_i$. Q.E.D. ■

Lemma 3.10 (Gluing on locally finite closed sets)**.** *Let B be a topological space and $\mathcal{F}$ be a sheaf of Abelian groups on B. Consider $B = \bigcup_i Z_i$ with each $Z_i \underset{closed}{\subseteq} B$. Assume $\{Z_i\}_{i \in I}$ is a locally finite (closed) cover. If $s_i \in \mathcal{F}(Z_i)$ such that $s_i|_{Z_i \cap Z_j} = s_j|_{Z_i \cap Z_j}$ for all i, j, then $\exists s \in \mathcal{F}(B)$ such that $s|_{Z_i} = s_i$.*

Proof. By definition of local finiteness, $\forall x \in B, \exists x \in N_x \underset{\text{open}}{\subseteq} B$ such that N_x only intersects finitely many Z_i's. In other words, if we denote

$$I_x := \{i \in I \mid N_x \cap Z_i \neq \varnothing\}$$

then I_x is finite.

(1) Since $Z_i \underset{\text{closed}}{\subseteq} B$, $N_x \cap Z_i \underset{\text{closed}}{\subseteq} N_x$.
(2) $\bigcup_{i \in I_x} (N_x \cap Z_i) = N_x$.
(3) $s_i|_{(N_x \cap Z_i) \cap (N_x \cap Z_j)} = s_i|_{(N_x \cap Z_i) \cap (N_x \cap Z_j)}$ where $s_i|_{N_x \cap Z_i} \in \mathcal{F}(N_x \cap Z_i)$ since $s_i \in \mathcal{F}(Z_i)$.

These fit the conditions of the gluing theorem on finitely many closed sets, therefore by 3.9, $\exists s_x \in \mathcal{F}(N_x)$ such that $s_x|_{N_x \cap Z_i} = s_i|_{N_x \cap Z_i}$.

In other words, we have proved a partial result stating that $\forall x \in B, \exists x \in N_x \underset{\text{open}}{\subseteq} B$. $\exists s_x \in \mathcal{F}(N_x)$, such that $s_x|_{N_x \cap Z_i} = s_i|_{N_x \cap Z_i}$.

Let's consider $\{N_x\}_{x \in B}$ which forms an open cover of B: $B = \bigcup_{x \in B} N_x$ with each $s_x \in \mathcal{F}(N_x)$.

By the locality axiom on finitely many closed sets, it is true that $s_x|_{N_x \cap N_y} = s_y|_{N_x \cap N_y}$ because

$$s_x|_{(N_x \cap N_y) \cap (Z_i \cap Z_j)} = s_i|_{(N_x \cap N_y) \cap (Z_i \cap Z_j)} = s_j|_{(N_x \cap N_y) \cap (Z_i \cap Z_j)} = s_y|_{(N_x \cap N_y) \cap (Z_i \cap Z_j)}$$

Then by gluing axiom on $\{N_x\}_{x \in B}$, $\exists s \in \mathcal{F}(B)$ such that $s|_{N_x} = s_x$.

Finally, to show that $s|_{Z_i} = s_i$, it suffices to show that $s|_{N_x \cap Z_i} = s_i|_{N_x \cap Z_i}$.

(1) $Z_i = \bigcup_{x \in B} (N_x \cap Z_i)$
(2) Since $N_x \underset{\text{open}}{\subseteq} B$, $N_x \cap Z_i \underset{\text{open}}{\subseteq} Z_i$
(3) $s|_{Z_i}, s_i \in \mathcal{F}(Z_i)$
(4) $s|_{N_x \cap Z_i} = s_x|_{N_x \cap Z_i} = s_i|_{N_x \cap Z_i}$

Therefore, by the locality axiom on arbitrary open sets, $s|_{Z_i} = s_i$. Q.E.D. ■

Proposition 3.11 (Soft sheaf preserves exactness over global sections)**.** *Let B be a paracompact space and $\mathcal{A}, \mathcal{A}', \mathcal{A}''$ be sheaves of Abelian groups on B. If $0 \to \mathcal{A}' \to \mathcal{A} \xrightarrow{\varphi} \mathcal{A}'' \to 0$ is a short exact sequence with $\mathcal{A}'$ being soft, then $0 \to \mathcal{A}'(B) \to \mathcal{A}(B) \xrightarrow{\varphi_B} \mathcal{A}''(B) \to 0$ is also exact.*

Proof. Recall that φ is surjective if $\forall x \in U \underset{\text{open}}{\subseteq} B, \exists x \in V \underset{\text{open}}{\subseteq} B, V \subseteq U$ such that φ_V is surjective. Therefore, $\forall x \in B$, there exists U_x such that $x \in U_x \underset{\text{open}}{\subseteq} B$ and φ_{U_x} is surjective.

Then, $\forall s \in \mathcal{A}''(B)$, $\exists s_x \in \mathcal{A}(U_x)$ such that

$$\varphi_{U_x}(s_x) = s|_{U_x}, \forall x \in B$$

Since B is paracompact, $\exists \{V_i\}_{i \in I}$ an open refinement of $\{U_x\}_{x \in B}$ such that $\{V_i\}_{i \in I}$ is locally finite (i.e. $V_i \subseteq U_{x(i)}$, where $x(\cdot)$ here indicates the refinement map). Also, there exists a shrinking $\{W_i\}_{i \in I}$ of $\{V_i\}_{i \in I}$, in other words, $B = \bigcup_{i \in I} W_i$ and $\overline{W_i} \subseteq V_i, \forall i \in I$.

Then, $s_{x(i)}|_{\overline{W_i}} \in \mathcal{A}(\overline{W_i})$ and $\varphi_{\overline{W_i}}(s_{x(i)}|_{\overline{W_i}}) = s|_{\overline{W_i}}$.

The rest of the proof aims to glue those $s_{x(i)}$ according to the extension of their difference on the intersection. For $\forall J \subseteq I$, denote $F_J := \bigcup_{j \in J} \overline{W_j}$ as the set that have already been glued.

Since $\{\overline{W_j}\}_{j \in J}$ is locally finite, and the union of a locally finite collection of closed sets is closed, we know that F_J is closed for all $J \subseteq I$.

Consider $\mathcal{C} := \{(J, t) \mid J \subseteq I, t \in \mathcal{A}(F_J), \varphi_{F_J}(t) = s|_{F_J}\}$.

We claim that $\mathcal{C} \neq \varnothing$, since $B \neq \varnothing$, $(\overline{W_i}, s_{x(i)}|_{\overline{W_i}}) \in \mathcal{C}$ at least for some i. Therefore, we can put an order to $\mathcal{C}$:

$$(J, t) \leq (J', t') \text{ iff } J \subseteq J', t'|_{F_J} = t$$

The punchline here is that $(\mathcal{C}, \leq)$ satisfies the two conditions of Zorn's lemma, hence has a maximal element:

(1) "$\leq$" is a partial order on $\mathcal{C}$.
(2) Any chain in $\mathcal{C}$ has an upper bound.

The proof of (2) is furthermore explained as follows. Consider a chain $\{(J_k,t_k)\}_{k\in\mathcal{K}}\subseteq\mathcal{C}$ with a partially ordered set $(\mathcal{K},\leq)$. Then, we may define

$$J:=\bigcup_{k\in\mathcal{K}}J_k$$

and aim to construct a section $t\in\mathcal{A}(F_J)$ to form an upper bound $(J,t)\in\mathcal{C}$, by 3.10. In order to make this happen, we need to prove the four conditions needed for gluing those t_k's:

(1) $\{\overline{W_j}\}_{j\in J}$ is locally finite. This is true, since $\{\overline{W_j}\}_{j\in J}$ is a subcollection of $\{\overline{W_i}\}_{i\in I}$ as a locally finite cover.
(2) $F_J=\bigcup_{j\in J}\overline{W_j}$, with each $\overline{W_j}\underset{\text{closed}}{\subseteq}F_J$. The first equation is true because of the definition of J; Moreover, since $\overline{W_j}\underset{\text{closed}}{\subseteq}B$, as subspace topology $\overline{W_j}=\overline{W_j}\cap F_J\underset{\text{closed}}{\subseteq}F_J$.
(3) $\exists s_j\in\mathcal{A}(\overline{W_j})$ for any $j\in J$.
For any $j\in J$, by definition $j\in\bigcup_{k\in\mathcal{K}}J_k$, so $\exists k=k(j)\in\mathcal{K}$ such that $j\in J_k$ (by axiom of choice). Since $(J_k,t_k)\in\mathcal{C}$, by definition, $J_k\subsetneq I, t_k\in\mathcal{A}(F_{J_k})$, and $\varphi_{F_{J_k}}(t_k)=s|_{F_{J_k}}$. Since $\overline{W_j}\subseteq F_{J_k}$, $t_k|_{\overline{W_j}}\in\mathcal{A}(\overline{W_j})$ is defined, so there exists $s_j:=t_{k(j)}|_{\overline{W_j}}\in\mathcal{A}(\overline{W_j})$.
(4) $\forall j,l\in J, s_j|_{\overline{W_j}\cap\overline{W_l}}=s_l|_{\overline{W_j}\cap\overline{W_l}}$.
Take $\forall j,l\in J$, then there are $k(j),k(l)\in\mathcal{K}$ such that $j\in J_{k(j)},l\in J_{l(j)}$. Since $(\mathcal{K},\leq)$ is a total order, without loss of generality, we may assume $k(j)\leq k(l)$. Then, $J_{k(j)}\subseteq J_{k(l)}$, and $t_{k(l)}|_{F_{J_{k(j)}}}=t_{k(l)}$. We may further restrict sections to $\overline{W_j}$, i.e., $t_{k(j)}|_{\overline{W_j}}=t_{k(l)}|_{\overline{W_j}}$, so that $s_j|_{\overline{W_j}\cap\overline{W_l}}=s_l|_{\overline{W_j}\cap\overline{W_l}}$.

By 3.10, $\exists t\in\mathcal{A}(F_J)$ such that $t|_{\overline{W_j}}=s_j:=t_{k(j)}|_{\overline{W_j}}$ for each $j\in J$. Then, by the locality axiom for any collection of closed sets, $\varphi_{F_J}(t)=s|_{F_J}$. Therefore, we created an upper bound $(J,t)\in\mathcal{C}$ of the chain $\{(J_k,t_k)\}_{k\in\mathcal{K}}$.

By Zorn's lemma, there exists a maximal element $(J,t)\in\mathcal{C}$. The next goal is to argue that $J=I$, which will be done by proof by contradiction:

Assume $J\subsetneqq I$, then we can select $i\in I-J$. Note that we have two sections $s_{x(i)}|_{\overline{W_i}}\in\mathcal{A}(\overline{W_i})$ and $t\in\mathcal{A}(F_J)$. Denote $\overline{s_i}:=s_{x(i)}|_{\overline{W_i}}\in\mathcal{A}(\overline{W_i})$, then by definition

$$\begin{cases}\varphi_{\overline{W_i}}(\overline{s_i})=s|_{\overline{W_i}}\\ \varphi_{F_J}(t)=s|_{F_J}\end{cases}$$

Therefore,

$$\varphi_{\overline{W_i}\cap F_J}(\overline{s_i}|_{\overline{W_i}\cap F_J}-t|_{\overline{W_i}\cap F_J})=0$$

so

$$\overline{s_i}|_{\overline{W_i}\cap F_J}-t|_{\overline{W_i}\cap F_J}\in\ker(\varphi_{\overline{W_i}\cap F_J})=(\ker\varphi)_{\overline{W_i}\cap F_J}=\mathcal{A}'(\overline{W_i}\cap F_J)$$

Since $\mathcal{A}'$ is soft, $\exists\tilde{s}\in\mathcal{A}'(F_J)$ such that $\tilde{s}|_{\overline{W_i}\cap F_J}=\overline{s_i}|_{\overline{W_i}\cap F_J}-t|_{\overline{W_i}\cap F_J}$, then

$$(\tilde{s}+t)|_{\overline{W_i}\cap F_J}=\overline{s_i}|_{\overline{W_i}\cap F_J}$$

By gluing axiom on finitely many closed sets (3.9), $\exists\tilde{t}\in\mathcal{A}(\overline{W_i}\cup F_J)=\mathcal{A}(F_{J\cup\{i\}})$ such that

$$\begin{cases}\tilde{t}|_{F_J}=\tilde{s}+t\\ \tilde{t}|_{\overline{W_i}}=\overline{s_i}\end{cases}$$

Want to show: $\varphi_{F_{J\cup\{i\}}}(\tilde{t}) = s|_{F_{J\cup\{i\}}}$. We know that

$$\varphi_{F_{J\cup\{i\}}}(\tilde{t})|_{\overline{W_i}} = \varphi_{\overline{W_i}}(\tilde{t}|_{\overline{W_i}}) = \varphi_{\overline{W_i}}(\overline{s_i}) = s|_{\overline{W_i}}$$

By locality axiom on closed sets,

$$\varphi_{F_{J\cup\{i\}}}(\tilde{t}) = s|_{F_{J\cup\{i\}}}$$

so $(J \cup \{i\}, \tilde{t}) \in \mathcal{C}$, contradicting the assumption that (J, t) is maximal.

Therefore, $J = I$, so $\exists t \in \mathcal{A}(F_I)$ such that $\varphi_{F_I}(t) = s|_{F_I}$. Also, $F_I = \bigcup_{j\in I} \overline{W_j} = B$, so $\exists t \in \mathcal{A}(B)$ such that $\varphi_B(t) = s|_B = s$.

The exactness of the rest part of the chain complex can be easily checked:

(1) $\ker(\varphi_B) \subseteq \mathrm{im}(i_B)$: Take $\forall s \in \mathcal{A}(B)$ such that $\varphi_B(s) = 0$. Then $s \in \ker(\varphi_B) = (\ker\varphi)_B = \mathcal{A}'(B)$.

(2) i_B is injective: i being injective means that $\forall U \underset{\text{open}}{\subseteq} B$, i_U is injective, so in particular i_B is injective. Q.E.D. ■

Proposition 3.12 (Softness passes over on short exact sequences)**.** *Let B be a paracompact space and $\mathcal{A}, \mathcal{A}', \mathcal{A}''$ be sheaves of Abelian groups on B. If $0 \to \mathcal{A}' \to \mathcal{A} \xrightarrow{\varphi} \mathcal{A}'' \to 0$ is a short exact sequence with $\mathcal{A}'$ being soft, then $\mathcal{A}$ is soft if and only if $\mathcal{A}''$ is soft.*

Proof. Take any $K \underset{\text{closed}}{\subseteq} B$. Consider the following commutative diagram given by the sections of the exact sequence, as well as restriction maps:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{A}'(B) & \longrightarrow & \mathcal{A}(B) & \xrightarrow{\varphi_B} & \mathcal{A}''(B) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle (1)} & & \downarrow{\scriptstyle (2)} & & \downarrow{\scriptstyle (3)} & & \\ 0 & \longrightarrow & \mathcal{A}'(K) & \longrightarrow & \mathcal{A}(K) & \xrightarrow{\varphi_K} & \mathcal{A}''(K) & \longrightarrow & 0 \end{array}$$

The exactness of the first row is given by 3.11, but with a little more handling of the paracompactness of K, the same argument yields the exactness of the second row. Here is a detailed explanation:

We aim to show that $\varphi_K : \mathcal{A}(K) \to \mathcal{A}''(K)$ is surjective.

To do so, take any section $[s] \in \mathcal{A}''(K)$, which, by definition, means that $s \in \mathcal{A}''(V)$ for some V, $K \subseteq V \underset{\text{open}}{\subseteq} B$. Since $0 \to \mathcal{A}' \to \mathcal{A} \xrightarrow{\varphi} \mathcal{A}'' \to 0$ is exact, there is an open cover $\{V_i\}_{i\in I}$ of V, i.e., $V_i \underset{\text{open}}{\subseteq} B, V \subseteq \bigcup_{i\in I} V_i$, and $s_i \in \mathcal{A}(V_i)$ such that $\varphi_{V_i}(s_i) = s|_{V_i}$. Since B is paracompact, K is paracompact as a closed subset, therefore a shrinking of the open cover $\{V_i\}_{i\in I}$ of K exists, as denoted by $\{W_i\}_{i\in I}$, where $W_i \underset{\text{open}}{\subseteq} B$ and $K \subseteq \bigcup_{i\in I} W_i$, such that $\overline{W_i} \subseteq V_i$. Denote $W := \bigcup_{i\in I} W_i$, then $K \subseteq W \underset{\text{open}}{\subseteq} B$.

The rest of the proof follows the same argument as 3.11 for gluing those $s_i|_{\overline{W_i}} \in \mathcal{A}(\overline{W_i})$ according to their differences on the intersection. As an upshot, $\exists \tilde{s} \in \mathcal{A}(\overline{W})$ such that $\varphi_{\overline{W}}(\tilde{s}) = s|_{\overline{W}}$, so $\varphi_K(\tilde{s}|_K) = s|_K$.

Since both rows are exact, and map (1) is surjective by softness of $\mathcal{A}'$, it can be shown that map (2) is surjective if and only if map (3) is surjective, by simply diagram chasing.

Remark: A typical proof for this theorem in most of the literature uses the restriction of sheaves over K, i.e., the exactness of $0 \to \mathcal{A}'|_K \to \mathcal{A}|_K \xrightarrow{\varphi|_K} \mathcal{A}''|_K \to 0$. The following commutative diagram

$$
\begin{tikzcd}
0 \arrow[r] & \mathcal{A}'(B) \arrow[r] \arrow[d, "(1)"] & \mathcal{A}(B) \arrow[r, "\varphi_B"] \arrow[d, "(2)"] & \mathcal{A}''(B) \arrow[r] \arrow[d, "(3)"] & 0 \\
0 \arrow[r] & \mathcal{A}'(K) \arrow[r] \arrow[d, equal] & \mathcal{A}(K) \arrow[r, "\varphi_K"] \arrow[d, equal] & \mathcal{A}''(K) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{A}'|_K(K) \arrow[r] & \mathcal{A}|_K(K) \arrow[r, "\varphi|_K"] & \mathcal{A}''|_K(K) \arrow[r] & 0
\end{tikzcd}
$$

has map (1) as a surjection by softness of $\mathcal{A}'$, hence it can be shown that map (2) is surjective if and only if map (3) is surjective. The proof of this theorem in the paper gives the exactness of φ_K explicitly, but it essentially still does the same argument as the typical proof. Q.E.D. ■

Proposition 3.13 (Čech Cohomology vanishes over $\mathcal{F}_E$)**.** *Let B be a paracompact space and $\mathcal{F}$ be a soft sheaf over B. If $\mathcal{U}$ is an open cover of B such that $\check{C}^n(\mathcal{U},\mathcal{F})$ are soft for all $n \geq 0$, then $\check{H}^n(\mathcal{U},\mathcal{F}) = 0$ for all $n \geq 0$.*

Proof. As proven in 2.9, the Čech resolution of B

$$0 \to \mathcal{F} \xrightarrow{e} \check{C}^0(\mathcal{U},\mathcal{F}) \xrightarrow{d^0} \check{C}^1(\mathcal{U},\mathcal{F}) \xrightarrow{d^1} \dots \xrightarrow{d^{n-1}} \check{C}^n(\mathcal{U},\mathcal{F}) \xrightarrow{d^n} \dots$$

is exact and naturally breaks down into short exact sequences of sheaves

$$0 \to \mathcal{F} \xrightarrow{e} \check{C}^0(\mathcal{U},\mathcal{F}) \xrightarrow{p^0} \ker d^1 \to 0$$

$$0 \to \ker d^1 \xrightarrow{e^1} \check{C}^1(\mathcal{U},\mathcal{F}) \xrightarrow{p^1} \ker d^2 \to 0$$

$$\dots$$

$$0 \to \ker d^n \xrightarrow{e^n} \check{C}^n(\mathcal{U},\mathcal{F}) \xrightarrow{p^n} \ker d^{n+1} \to 0$$

where $\ker d^n$ is the kernel sheaf of map d^n, and as a subsheaf of $\check{C}^n(\mathcal{U},\mathcal{F})$, e^n is defined to be the inclusion to $\check{C}^n(\mathcal{U},\mathcal{F})$. p^n is taken as the boundary map d^n with the codomain restricted to $\ker d^{n+1}$.

Since $\mathcal{F}$ is soft, and $\check{C}^n(\mathcal{U},\mathcal{F})$ is soft for all $n \geq 0$. By 3.12, $\ker d^n$ is soft for all $n \geq 0$. By 3.11, the exactness of these short exact sequences is preserved after taking the global section, hence

$$0 \to \mathcal{F}(B) \xrightarrow{e_B} \check{C}^0(\mathcal{U},\mathcal{F})(B) \xrightarrow{p^0_B} \ker d^1_B \to 0$$

$$0 \to \ker d^1_B \xrightarrow{e^1_B} \check{C}^1(\mathcal{U},\mathcal{F})(B) \xrightarrow{p^1_B} \ker d^2_B \to 0$$

$$\dots$$

$$0 \to \ker d^n_B \xrightarrow{e^n_B} \check{C}^n(\mathcal{U},\mathcal{F})(B) \xrightarrow{p^n_B} \ker d^{n+1}_B \to 0$$

are all exact, showing that the Čech resolution is exact after taking the global section:

$$0 \to \mathcal{F}(B) \xrightarrow{e_B} \check{C}^0(\mathcal{U},\mathcal{F})(B) \xrightarrow{d^0_B} \check{C}^1(\mathcal{U},\mathcal{F})(B) \xrightarrow{d^1_B} \dots \xrightarrow{d^{n-1}_B} \check{C}^n(\mathcal{U},\mathcal{F})(B) \xrightarrow{d^n_B} \dots$$

The Čech cohomology groups are exactly the homology of the Čech resolution, hence $\check{H}^n(\mathcal{U},\mathcal{F}) = 0$ for all $n \geq 0$. Q.E.D. ■

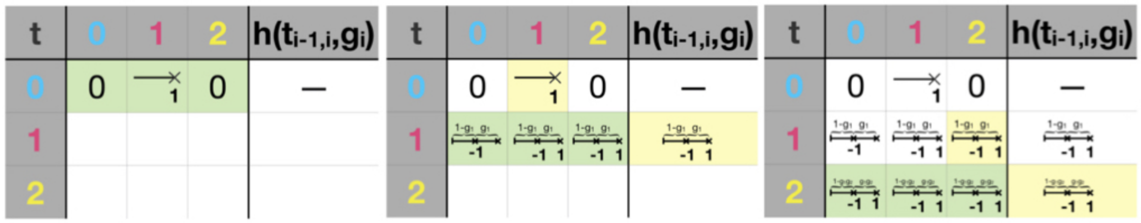


FIGURE 1. An illustration of Q^1—how the Eilenberg-MacLane Algorithm works at $n = 1$

4. Algorithm & Experiment

4.1. $Q^1 : \ker d_B^1 \to \check{C}^0(\mathcal{U}, \mathcal{F})$**.** Combining the proofs of 3.13, 3.11 and all proofs they further quote, we may derive an algorithm for proving $\check{H}^1(\mathcal{U}, \mathcal{F}) = 0$, i.e., an explicit form of preimage of η in the Čech Resolution given a cocycle $\eta \in \ker d_B^1$:

Consider the space we are interested in is $B := X^{(\alpha)}$, with $\mathcal{U}^\alpha := \{B_\alpha(x)\}_{x\in X}$ denoting its open cover. Given $\eta = (\eta_{i_0 i_1})_{i_0<i_1} \in \ker d_B^1$, we will use 3.11 on this particular exact sequence

$$0 \to \mathcal{F} \xrightarrow{e} \check{C}^0(\mathcal{U}, \mathcal{F}) \xrightarrow{p^0} \ker d^1 \to 0$$

Since the surjectivity of p^0 is given by 2.11, it sets $\{U_x\}_{x\in B}$ in 3.11 to be $\mathcal{U}^\alpha$ with the corresponding sections $\eta_i := (\eta_{ii_0})_{i_0\in I}$ over U_i^α. The refinement $\{V_x\}_{x\in B}$ in 3.11 is taken to be the same as $\mathcal{U}^\alpha$ since we consider B as a finite set. The shrinking will be chosen as $\mathcal{U}^\beta$, with some $\beta < \alpha$, that provides the platform where all the sections are glued on. The part of Zorn's lemma in 3.11 is not needed in creating the algorithm here since $\mathcal{U}^\alpha$ has only finitely many open sets.

In the gluing process, we start from two sections $\eta_0 := (\eta_{0i_0})_{i_0\in I}$ and $\eta_1 := (\eta_{1i_0})_{i_0\in I}$, over U_0^α and U_1^α respectively. Taking their difference $\eta_1|_{U_{01}^\alpha} - \eta_0|_{U_{01}^\alpha}$ over U_{01}^α gives a section in $\mathcal{F}(U_{01}^\alpha)$, serving as a representative of $\mathcal{F}(\overline{U_{01}^\beta})$. By 3.7, we can extend the section $\eta_1|_{U_{01}^\alpha} - \eta_0|_{U_{01}^\alpha}$ from over $\overline{U_{01}^\beta}$ to over $\overline{U_1^\beta}$, which, in the sense of representatives, extend from over U_{01}^α to U_1^γ, where $\mathcal{U}^\gamma$ serves as the intermediate shrinking needed in 3.7 ($\beta < \gamma < \alpha$). The extension is of the form

$$\overline{t_1}(x) := \begin{cases} h(\eta_{0i_x}(x) - \eta_{1i_x}(x), g_1(x)) & \text{if } x \in U_0^\alpha \cap U_1^\alpha \\ (e, 1) & \text{if } x \in U_1^\alpha - U_0^\alpha \end{cases}$$

where $g_1 : B \to [0, 1]$ needs to be a map satisfying $g_1|_{\overline{U_{01}^\gamma}} = 1, g_1|_{B-U_{01}^\alpha} = 0$; i_x indicates any index i such that $x \in U_i^\alpha$.

Since $\overline{t_1} + \eta_1$ agrees with η_0 on the intersection $\overline{U_{01}^\gamma}$ (by 3.7), we may choose the preimage $t_1 \in \check{C}^0(\mathcal{U}^\alpha, \mathcal{F})(U_0^\gamma \cup U_1^\gamma)$ to be set as:

$$t_1(x) := \begin{cases} \eta_1(x) + \overline{t_1}(x) & \text{if } x \in U_1^\gamma - U_0^\gamma \\ t_0(x) & \text{if } x \in U_0^\gamma \end{cases}$$

In general, if we denote $\tilde{U}_i^\alpha := \bigcup_{j\leq i} U_j^\alpha$ as the partial set we glue on, the gluing process brings in a $t_{i-1} \in \check{C}^0(\mathcal{U}^\alpha, \mathcal{F})(\tilde{U}_{i-1}^{\gamma_{i-1}})$ and a $\eta_i \in \mathcal{F}(U_i^\alpha)$, where $\beta < \gamma_{i-1} < ... < \gamma_0 = \alpha$. Choose γ_i so that

$\beta < \gamma_i < \gamma_{i-1}$, then the extension $\overline{t_i} : U_i^{\gamma_{i-1}} \to EG$ is given by 3.7:

$$\overline{t_i}(x) := \begin{cases} h(t_{i-1,i_x}(x) - \eta_{ii_x}(x), g_i(x)) & \text{if } x \in \tilde{U}_{i-1}^{\gamma_{i-1}} \cap U_i^{\gamma_{i-1}} \\ (e,1) & \text{if } x \in U_i^{\gamma_{i-1}} - \tilde{U}_{i-1}^{\gamma_{i-1}} \end{cases}$$

where $g_i : B \to [0,1]$ such that $g_i|_{\overline{\tilde{U}_{i-1}^{\gamma_i}} \cap \overline{U_i^{\gamma_i}}} = 1$ and $g_i|_{B - \tilde{U}_{i-1}^{\gamma_{i-1}} \cap U_i^{\gamma_{i-1}}} = 0$. So that the gluing of $\overline{t_i} + \eta_i$ and t_{i-1} in the proof 3.11 gives the form of $t_i \in \check{C}^0(\mathcal{U}^\alpha, \mathcal{F})(\tilde{U}_i^{\gamma_i})$ by:

$$t_{i,i_0}(x) := \begin{cases} \eta_{ii_0}(x) + \overline{t_i}(x) & \text{if } x \in (U_i^{\gamma_i} - \tilde{U}_{i-1}^{\gamma_i}) \cap U_{i_0}^\alpha \\ t_{i-1,i_0}(x) & \text{if } x \in \tilde{U}_{i-1}^{\gamma_i} \cap U_{i_0}^\alpha \end{cases}$$

The process includes a recursive gluing until the biggest indexed open set in $\mathcal{U}^\alpha$ is glued. Consider $\mathcal{U}^\alpha$ has open sets as $\{U_0^\alpha, U_1^\alpha, ..., U_N^\alpha\}$, then this gluing process ends with $t_N \in \check{C}^0(\mathcal{U}^\alpha, \mathcal{F})(\tilde{U}_N^{\gamma_N})$. Since $\tilde{U}_N^{\gamma_N} = \bigcup_{i_0=0}^N U_{i_0}^{\gamma_N} \supseteq \bigcup_{i_0=0}^N U_{i_0}^\beta \supseteq B$, we may take $Q^1(\eta) := t_N|_B$ as a restriction on the sheaf $\check{C}^0(\mathcal{U}^\alpha, \mathcal{F})$, i.e., $Q^1(\eta)_{i_0}(x) := t_{N,i_0}(x), \forall x \in B$.

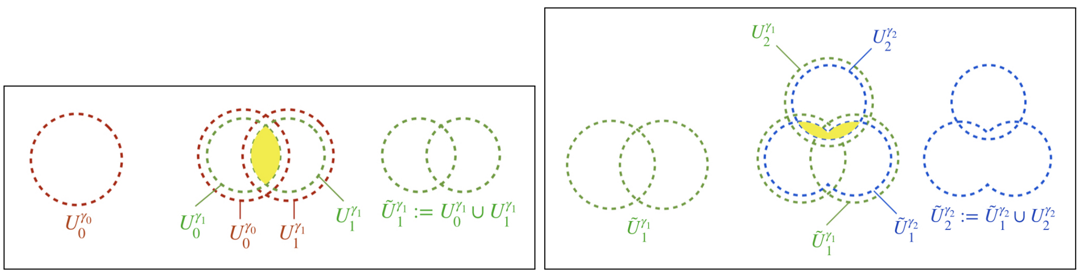


FIGURE 2. The first two loops of the recursive procedure: Taking the difference, Extending, and Gluing

The algorithm in a precise and neat form is summarized as follows:

Algorithm 2 $\check{H}^1(\mathcal{U}, \mathcal{F}) = 0$ given by 3.13

Require: $\eta = (\eta_{i_0 i_1})_{i_0 < i_1} \in \ker d_B^1 \subseteq \check{C}^1(\mathcal{U}^\alpha, \mathcal{F})(B)$

Choose $\beta < \alpha$ such that $\mathcal{U}^\beta$ still covers B.

Define $t_0 \in \check{C}^0(\mathcal{U}^\alpha, \mathcal{F})(U_0^\alpha)$ by $t_{0,i_0}(x) := \eta_{0,i_0}(x)$, $\forall x \in U_0^\alpha \cap U_{i_0}^\alpha$.

for $i = 1, 2, ..., N$ **do**

  Choose γ_i such that $\beta < \gamma_i < \gamma_{i-1}$ (By convention, $\gamma_0 = \alpha$).

  Define $t_i \in \check{C}^0(\mathcal{U}^\alpha, \mathcal{F})(\tilde{U}_i^{\gamma_i})$ by

$$t_{i,i_0}(x) := \begin{cases} \eta_{ii_0}(x) + h(t_{i-1,i_x}(x) - \eta_{ii_x}(x), g_i(x)) & \forall x \in (U_i^{\gamma_i} - \tilde{U}_{i-1}^{\gamma_i}) \cap U_{i_0}^\alpha \\ t_{i-1,i_0}(x) & \forall x \in \tilde{U}_{i-1}^{\gamma_i} \cap U_{i_0}^\alpha \end{cases}$$

end for

Set $Q^1(\eta) \in \check{C}^0(\mathcal{U}^\alpha, \mathcal{F})(B)$ by $Q^1(\eta)_{i_0}(x) := t_{N,i_0}(x), \forall x \in B \subseteq \tilde{U}_N^{\gamma_N}$.

We claim that $d^0(Q^1(\eta)) = \eta$.

The justification of this form $Q^1(\eta)$ is guaranteed from the correctness of the proofs above. Besides, it can be verified by taking the boundary of t_N: $\forall (i_0, i_1)$, $\forall x \in B$,

$$\begin{aligned}
d_B^0(Q^1(\eta))_{i_0 i_1}(x) &= Q^1(\eta)_{i_1}(x) - Q^1(\eta)_{i_0}(x) \\
&= t_{N,i_1}(x) - t_{N,i_0}(x) \\
&= \begin{cases} \left(\eta_{N i_1}(x) + \overline{t_N}(x)\right) - \left(\eta_{N i_0}(x) + \overline{t_N}(x)\right) & \forall x \in U_N^{\gamma_N} - \tilde{U}_{N-1}^{\gamma_N} \\ t_{N-1,i_1}(x) - t_{N-1,i_0}(x) & \forall x \in \tilde{U}_{N-1}^{\gamma_N} \end{cases} \\
&= \begin{cases} \eta_{N i_1}(x) - \eta_{N i_0}(x) & \forall x \in U_N^{\gamma_N} - \tilde{U}_{N-1}^{\gamma_N} \\ t_{N-1,i_1}(x) - t_{N-1,i_0}(x) & \forall x \in \tilde{U}_{N-1}^{\gamma_N} \end{cases} \\
&= \begin{cases} \eta_{i_0 i_1}(x) & \forall x \in U_N^{\gamma_N} - \tilde{U}_{N-1}^{\gamma_N} \\ t_{N-1,i_1}(x) - t_{N-1,i_0}(x) & \forall x \in \tilde{U}_{N-1}^{\gamma_N} \end{cases}
\end{aligned}$$

Since $\forall x \in U_0^\alpha \cap U_{i_0 i_1}^\alpha, t_{0,i_1}(x) - t_{0,i_0}(x) = \eta_{0 i_1}(x) - \eta_{0 i_0}(x) = \eta_{i_0 i_1}(x)$, and the difference $t_{i,i_1} - t_{i,i_0}$ is either $\eta_{i_0 i_1}$ or the same with the difference of the previous index $t_{i-1,i_1} - t_{i-1,i_0}$, it follows that $t_{N,i_1}(x) - t_{N,i_0}(x) = \eta_{i_0 i_1}(x)$ for all $x \in B \subseteq U_N^{\gamma_N}$, which means

$$\forall (i_0, i_1), \forall x \in B, d_B^0(Q^1(\eta))_{i_0 i_1}(x) = \eta_{i_0 i_1}(x)$$

Therefore, $d_B^0(Q^1(\eta)) = \eta$.

Q.E.D. ■

4.2. $Q^2 : \ker d_B^2 \to \check{C}^1(\mathcal{U}, \mathcal{F})$**.** An algorithm for dimension 2 is illustrated as follows.

Given $\eta = (\eta_{i_0 i_1 i_2})_{i_0 < i_1 < i_2} \in \ker d_B^2$, we will use 3.11 on the exact sequence

$$0 \to \ker d^1 \xrightarrow{e^1} \check{C}^1(\mathcal{U}, \mathcal{F}) \xrightarrow{p^1} \ker d^2 \to 0$$

Since the surjectivity of p^1 is given by 2.11, it sets $\{U_x\}_{x \in B}$ in 3.11 to be $\mathcal{U}^\alpha$ with the corresponding sections $\eta_i := (\eta_{i i_0 i_1})_{i_0 < i_1}$ over U_i^α. The refinement $\{V_x\}$ in 3.11 is taken to be the same as $\mathcal{U}^\alpha$ since we consider B as a finite set. The shrinking will be chosen as $\mathcal{U}^\beta$, with some $\beta < \alpha$, that provides the platform where all the sections are glued on. The part of Zorn's lemma in 3.11 is not needed in creating the algorithm here since $\mathcal{U}^\alpha$ has only finitely many open sets.

The gluing process of Q^2 is the only part that differs from that of Q^1. As before, we set our $t_0 := \eta_0 \in \check{C}^1(\mathcal{U}^\alpha, \mathcal{F})(U_0^\alpha)$, and see how the process takes in a $t_{i-1} \in \check{C}^1(\mathcal{U}^\alpha, \mathcal{F})(\tilde{U}_{i-1}^{\gamma_{i-1}})$ and a $\eta_i \in \check{C}^1(\mathcal{U}^\alpha, \mathcal{F})(U_i^\alpha)$, where $\beta < \gamma_{i-1} < ... < \gamma_0 = \alpha$, and outputs a $t_i \in \check{C}^1(\mathcal{U}^\alpha, \mathcal{F})(\tilde{U}_i^{\gamma_i})$. First, we take the difference between them over the intersection, which, by 3.11, gives a section in $\ker d^1$:

$$\sigma_i := t_{i-1}|_{\tilde{U}_{i-1}^{\gamma_{i-1}} \cap U_i^\alpha} - \eta_i|_{\tilde{U}_{i-1}^{\gamma_{i-1}} \cap U_i^\alpha} \in (\ker d^1)(\tilde{U}_{i-1}^{\gamma_{i-1}} \cap U_i^\alpha)$$

To give an extension of σ_i, i.e., to show that $\ker d^1$ is soft, we need to use the result from 3.12 on the short exact sequence $0 \to \mathcal{F} \xrightarrow{e} \check{C}^0(\mathcal{U}, \mathcal{F}) \xrightarrow{p^0} \ker d^1 \to 0$, specifically:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{F}(B) & \xrightarrow{e_B} & \check{C}^0(\mathcal{U}, \mathcal{F})(B) & \xrightarrow{p_B^0} & \ker d_B^1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{F}(K) & \xrightarrow{e_K} & \check{C}^0(\mathcal{U}, \mathcal{F})(K) & \xrightarrow{p_K^0} & \ker d_K^1 & \longrightarrow & 0
\end{array}$$

on the condition that $K := \overline{\tilde{U}_{i-1}^\beta \cap U_i^\beta}$, with the representative σ_i over $\tilde{U}_{i-1}^{\gamma_{i-1}} \cap U_i^\alpha$. An important observation is that $p_{U_i^\alpha}^0$ is surjective by 2.11, hence the process of breaking down the open set into pieces which have preimages, described in 3.12, is not necessary (since p^0 is surjective on the whole set). From 2.11, we know that the preimage of σ_i is $(\sigma_i)_{ii_0}$, i.e.,

$$(t_{i-1,ii_0})_{i_0\in I} \in \check{C}^0(\mathcal{U},\mathcal{F})(\tilde{U}_{i-1}^{\gamma_{i-1}} \cap U_i^\alpha)$$

Then, 3.7 gives a way of extending a section over K to B. For implementing the details, we choose γ_i so that $\beta < \gamma_i < \gamma_{i-1}$. Then the extension $\overline{t_i} : U_{\gamma_{i-1}} \to EG$ would be $\tilde{\eta} := (\tilde{\eta}_{i_0})_{i_0\in I}$ where $\tilde{\eta}_{i_0} : U_{i_0}^\alpha \to EG$ is given by

$$\tilde{\eta}_{i_0}(x) := \begin{cases} h(t_{i-1,ii_0}(x), g_i(x)) & \forall x \in U_{i_0}^\alpha \cap (\tilde{U}_{i-1}^{\gamma_{i-1}} \cap U_i^{\gamma_{i-1}}) \\ (e,1) & \forall x \in U_{i_0}^\alpha - (\tilde{U}_{i-1}^{\gamma_{i-1}} \cap U_i^{\gamma_{i-1}}) \end{cases}$$

Lastly, we may apply p_B^0 to get the extension of σ, namely, $\overline{t_i} := p_B^0(\tilde{\eta}) = (\tilde{\eta}_{i_1}|_{U_{i_0i_1}} - \tilde{\eta}_{i_0}|_{U_{i_0i_1}})_{i_0<i_1}$, so

$$\overline{t}_{i,i_0i_1}(x) := (\tilde{\eta}_{i_1}(x) - \tilde{\eta}_{i_0}(x)) = h(t_{i-1,ii_1}(x), g_i(x)) - h(t_{i-1,ii_0}(x), g_i(x))$$

After extension, the rest of the process will remain the same as that of $n = 1$. We have $\overline{t_i} + \eta_i$ and t_{i-1} agree on $\tilde{U}_{i-1}^{\gamma_i} \cap U_i^{\gamma_i}$, therefore we may define

$$t_i(x) := \begin{cases} \eta_{ii_0i_1}(x) + \overline{t}_{i,i_0i_1}(x) & \text{if } x \in (U_i^{\gamma_i} - \tilde{U}_{i-1}^{\gamma_i}) \cap U_{i_0i_1}^\alpha \\ t_{i-1,i_0i_1}(x) & \text{if } x \in \tilde{U}_{i-1}^{\gamma_i} \cap U_{i_0i_1}^\alpha \end{cases}$$

Or, more explicitly,

$$t_{i,i_0i_1}(x) := \begin{cases} \eta_{ii_0i_1}(x) + h(t_{i-1,ii_1}(x), g_i(x)) - h(t_{i-1,ii_0}(x), g_i(x)) & \text{if } x \in (U_i^{\gamma_i} - \tilde{U}_{i-1}^{\gamma_i}) \cap U_{i_0i_1}^\alpha \\ t_{i-1,i_0i_1}(x) & \text{if } x \in \tilde{U}_{i-1}^{\gamma_i} \cap U_{i_0i_1}^\alpha \end{cases}$$

Algorithm 3 $\check{H}^2(\mathcal{U},\mathcal{F}) = 0$ given by 3.13

Require: $\eta = (\eta_{i_0i_1i_2})_{i_0<i_1<i_2} \in \ker d_B^2 \subseteq \check{C}^2(\mathcal{U}^\alpha,\mathcal{F})(B)$

Choose $\beta < \alpha$ such that $\mathcal{U}^\beta$ still covers B.
Define $t_0 \in \check{C}^1(\mathcal{U}^\alpha,\mathcal{F})(U_0^\alpha)$ by $t_{0,i_0i_1}(x) := \eta_{0i_0i_1}(x)$, $\forall x \in U_0^\alpha \cap U_{i_0i_1}^\alpha$.
for $i = 1, 2, ..., N$ **do**
    Choose γ_i such that $\beta < \gamma_i < \gamma_{i-1}$ (By convention, $\gamma_0 = \alpha$).
    Define $t_i \in \check{C}^1(\mathcal{U}^\alpha,\mathcal{F})(\tilde{U}_i^{\gamma_i})$ by
$$t_{i,i_0i_1}(x) := \begin{cases} \eta_{ii_0i_1}(x) + h(t_{i-1,ii_1}(x), g_i(x)) - h(t_{i-1,ii_0}(x), g_i(x)) & \forall x \in (U_i^{\gamma_i} - \tilde{U}_{i-1}^{\gamma_i}) \cap U_{i_0i_1}^\alpha \\ t_{i-1,i_0i_1}(x) & \forall x \in \tilde{U}_{i-1}^{\gamma_i} \cap U_{i_0i_1}^\alpha \end{cases}$$
end for
Set $Q^2(\eta) \in \check{C}^1(\mathcal{U}^\alpha,\mathcal{F})(B)$ by $Q^2(\eta)_{i_0i_1}(x) := t_{N,i_0i_1}(x), \forall x \in B$.
We claim that $d^1(Q^2(\eta)) = \eta$.

The justification of this form $Q^2(\eta)$ can be further verified by taking the boundary of t_N: $\forall(i_0,i_1,i_2)$, $\forall x\in B$, we have that

$$\begin{aligned}
&d_B^1(Q^2(\eta))_{i_0i_1i_2}(x)\\
=&Q^2(\eta)_{i_1i_2}(x)-Q^2(\eta)_{i_0i_2}(x)+Q^2(\eta)_{i_0i_1}(x)\\
=&t_{N,i_1i_2}(x)-t_{N,i_0i_2}(x)+t_{N,i_0i_1}(x)\\
=&\begin{cases}
\begin{array}{l}(\eta_{Ni_1i_2}(x)+h(t_{N-1,Ni_2}(x),g_N(x))-h(t_{N-1,Ni_1}(x),g_N(x)))\\
-(\eta_{Ni_0i_2}(x)+h(t_{N-1,Ni_2}(x),g_N(x))-h(t_{N-1,Ni_0}(x),g_N(x)))\\
+(\eta_{Ni_0i_1}(x)+h(t_{N-1,Ni_1}(x),g_N(x))-h(t_{N-1,Ni_0}(x),g_N(x)))\end{array} & \forall x\in U_N^{\gamma_N}-\tilde U_{N-1}^{\gamma_N}\\
\\
t_{N-1,i_1i_2}(x)-t_{N-1,i_0i_2}(x)+t_{N-1,i_0i_1}(x) & \forall x\in \tilde U_{N-1}^{\gamma_N}
\end{cases}\\
=&\begin{cases}
\eta_{Ni_1i_2}(x)-\eta_{Ni_0i_2}(x)+\eta_{Ni_1i_2}(x) & \forall x\in U_N^{\gamma_N}-\tilde U_{N-1}^{\gamma_N}\\
t_{N-1,i_1i_2}(x)-t_{N-1,i_0i_2}(x)+t_{N-1,i_0i_1}(x) & \forall x\in \tilde U_{N-1}^{\gamma_N}
\end{cases}\\
=&\begin{cases}
\eta_{i_0i_1i_2}(x) & \forall x\in U_N^{\gamma_N}-\tilde U_{N-1}^{\gamma_N}\\
t_{N-1,i_1i_2}(x)-t_{N-1,i_0i_2}(x)+t_{N-1,i_0i_1}(x) & \forall x\in \tilde U_{N-1}^{\gamma_N}
\end{cases}
\end{aligned}$$

Since $\forall x\in U_0^\alpha\cap U_{i_0i_1i_2}^\alpha, t_{0,i_1i_2}(x)-t_{0,i_0i_2}(x)+t_{0,i_0i_1}(x)=\eta_{0i_1i_2}(x)-\eta_{0i_0i_2}(x)+\eta_{0i_0i_1}(x)=\eta_{i_0i_1i_2}(x)$, and the sum $t_{i,i_1i_2}-t_{i,i_0i_2}+t_{i,i_0i_1}$ is either $\eta_{i_0i_1i_2}$ or the same with the difference of the previous index $t_{i-1,i_1i_2}-t_{i-1,i_0i_2}+t_{i-1,i_0i_1}$, it follows that $t_{N,i_1i_2}(x)-t_{N,i_0i_2}(x)+t_{N,i_0i_1}(x)=\eta_{i_0i_1i_2}(x)$ for all $x\in B\subseteq U_N^{\gamma_N}$, which means

$$\forall(i_0,i_1,i_2),\forall x\in B, d_B^0(Q^1(\eta))_{i_0i_1i_2}(x)=\eta_{i_0i_1i_2}(x)$$

Therefore, $d_B^1(Q^2(\eta))=\eta$.

Q.E.D. ■

4.3. $Q^n:\ker d_B^n\to\check C^{n-1}(\mathcal U,\mathcal F)$. Note that the same analogy of $n=2$ applies to any dimension n. A general algorithm, as we computed, is shown as follows:

Algorithm 4 $\check H^n(\mathcal U,\mathcal F)=0$ given by 3.13

Require: $\eta=(\eta_{i_0\dots i_n})_{i_0<\dots<i_n}\in\ker d_B^n\subseteq\check C^n(\mathcal U^\alpha,\mathcal F)(B)$

Choose $\beta<\alpha$ such that $\mathcal U^\beta$ still covers B.

Define $t_0\in\check C^{n-1}(\mathcal U^\alpha,\mathcal F)(U_0^\alpha)$ by $t_{0,i_0\dots i_{n-1}}(x):=\eta_{0i_0\dots i_{n-1}}(x)$, $\forall x\in U_0^\alpha\cap U_{i_0\dots i_{n-1}}^\alpha$.

for $i=1,2,\dots,N$ **do**

  Choose γ_i such that $\beta<\gamma_i<\gamma_{i-1}$ (By convention, $\gamma_0=\alpha$).

  Define $t_i\in\check C^{n-1}(\mathcal U^\alpha,\mathcal F)(\tilde U_i^{\gamma_i})$ by

$$t_{i,i_0\dots i_{n-1}}(x):=\begin{cases}\eta_{ii_0\dots i_{n-1}}(x)+\sum_{j=0}^{n-1}(-1)^jh(t_{i-1,ii_0\dots\hat{i_j}\dots i_{n-1}}(x),g_i(x)) & \forall x\in(U_i^{\gamma_i}-\tilde U_{i-1}^{\gamma_i})\cap U_{i_0\dots i_{n-1}}^\alpha\\ t_{i-1,i_0\dots i_{n-1}}(x) & \forall x\in\tilde U_{i-1}^{\gamma_i}\cap U_{i_0\dots i_{n-1}}^\alpha\end{cases}$$

end for

Set $Q^n(\eta)\in\check C^{n-1}(\mathcal U^\alpha,\mathcal F)(B)$ by $Q^n(\eta)_{i_0\dots i_{n-1}}(x):=t_{N,i_0\dots i_{n-1}}(x),\forall x\in B$.

We claim that $d^{n-1}(Q^n(\eta))=\eta$.

4.4. **Construction of g:** All construction of g_i's in different dimensions can be defined to be the same, since the only requirements for $g_i:B\to[0,1]$ are $g_i|_{\overline{\tilde U_{i-1}^{\gamma_i}}\cap\overline{U_i^{\gamma_i}}}=1$ and $g_i|_{B-\tilde U_{i-1}^{\gamma_{i-1}}\cap U_i^{\gamma_{i-1}}}=0$.

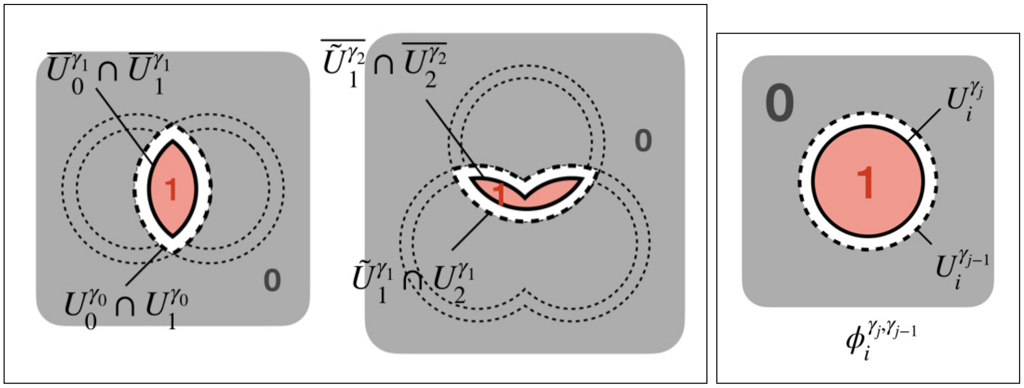


FIGURE 3. Regions where the bump functions g_is value 0 (colored in gray) and 1 (colored in red). Dashed lines represent an open boundary.

One way to define it is by:

$$g_i(x) := \Phi_i(\min_{y \in X} d(x, y)), \forall x \in B$$

where

$$\Phi_i : \mathbb{R}_+ \to [0, 1]; \Phi_i(x) := \begin{cases} 1 & \forall 0 < x \leq \gamma_i \\ \frac{\gamma_{i-1} - x}{\gamma_{i-1} - \gamma_i} & \forall \gamma_i \leq x \leq \gamma_{i-1} \\ 0 & \forall x \geq \gamma_{i-1} \end{cases}$$

is a function normalizing the radius.

4.5. **Total Formula:** 2

5. Experiment

Now that there is an algorithm showing how the one-to-one correspondence works, in the following we are going to show an example of case $B = S^1, G = \mathbb{Z}_q, n = 2$ to show you how

$$H^2(S^1; \mathbb{Z}_q) \to [S^1, B^2\mathbb{Z}_q]$$

works.

An experiment is done on the case $X = S^2, n = 2, G = \mathbb{Z}/q\mathbb{Z}$, with an open cover
$\mathcal{U}^\beta = \{U_0, U_1, U_2, U_3\}$
containing four sets. Given a generic representative
$\eta := (\eta_{i_0 i_1 i_2})_{i_0 < i_1 < i_2} \in \check{C}^2(\mathcal{U}^\beta, \mathcal{F}_G)$
, with

$$\eta_{i_0 i_1 i_2} := \begin{cases} k & \text{if } i_0 i_1 i_2 = 012 \\ 0 & \text{else} \end{cases}$$

The result is:

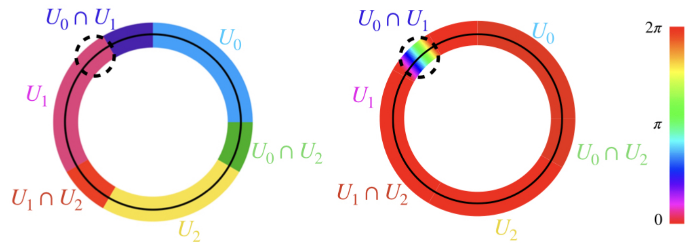
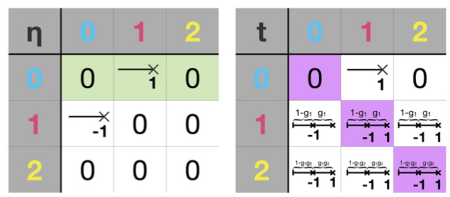


FIGURE 4. The space $X = S^1$ covered by three open sets $\{U_0, U_1, U_2\}$, and its Eilenberg-MacLane coordinates $f_\eta : X \to S^1$ derived from $\eta = (1, 0, 0)$ is labeled by the color map.



$$f_\eta(b) = \begin{cases} 0 & \text{if } x \in U_0 \\ (-1, (1 - g_1(x), g_1(x))) & \text{if } x \in U_1 - U_0 \\ (-1, (1 - g_1(x)g_2(x), g_1(x)g_2(x))) & \text{if } x \in U_2 - U_0 \cup U_1 \end{cases}$$

FIGURE 5. The coordinates given by the Eilenberg-MacLane Algorithm computed in matrices

$\forall x \in X^{(\beta)}$, the coordinate function $X^{(\beta)} \to B^2\mathbb{Z}_q$

$$f_\eta(x) = \begin{cases} \text{if } x \in U_0^\beta : & 0 \\ \text{if } x \in U_1^\beta : & ((-k, (1 - g_1(x), g_1(x))), (\varphi_0(x) + \varphi_2(x), \varphi_3(x))) \\ \text{if } x \in U_2^\beta : & (((-k, k), (1 - g_2(x), g_2(x) - g_1(x)g_2(x), g_1(x)g_2(x))), \\ & (k, (1 - g_1(x)g_2(x), g_1(x)g_2(x)))), (\varphi_0(x), \varphi_1(x), \varphi_3(x)) \\ \text{if } x \in U_3^\beta : & ((k, (1 - g_2(x)g_3(x), g_2(x)g_3(x))), (\varphi_0(x), \varphi_1(x) + \varphi_2(x))) \end{cases}$$

Algorithm 5 $\eta = (\eta_{i_0 \dots i_n})_{i_0 < \dots < i_n} \in \ker d_B^n \subseteq \check{C}^n(\mathcal{U}^\alpha, \mathcal{F})(B) \Longrightarrow Q^n(\eta) \in \check{C}^{n-1}(\mathcal{U}^\alpha, \mathcal{F})(B)$

Choose $\beta < \alpha$ such that $\mathcal{U}^\beta$ still covers B.
Define $t_0 \in \check{C}^{n-1}(\mathcal{U}^\alpha, \mathcal{F})(U_0^\alpha)$ by $t_{0, i_0 \dots i_{n-1}}(x) := \eta_{0 i_0 \dots i_{n-1}}(x)$, $\forall x \in U_0^\alpha \cap U_{i_0 \dots i_{n-1}}^\alpha$.
for $i = 1, 2, ..., N$ **do**
    Choose γ_i such that $\beta < \gamma_i < \gamma_{i-1}$ (By convention, $\gamma_0 = \alpha$).
    Define $t_i \in \check{C}^{n-1}(\mathcal{U}^\alpha, \mathcal{F})(\tilde{U}_i^{\gamma_i})$ by

$$t_{i, i_0 \dots i_{n-1}}(x) := \begin{cases} \eta_{i i_0 \dots i_{n-1}}(x) + \sum_{j=0}^{n-1} (-1)^j h(t_{i-1, i i_0 \dots \hat{i_j} \dots i_{n-1}}(x), g_i(x)) & \forall x \in (U_i^{\gamma_i} - \tilde{U}_{i-1}^{\gamma_i}) \cap U_{i_0 \dots i_{n-1}}^\alpha \\ t_{i-1, i_0 \dots i_{n-1}}(x) & \forall x \in \tilde{U}_{i-1}^{\gamma_i} \cap U_{i_0 \dots i_{n-1}}^\alpha \end{cases}$$

end for
Set $Q^n(\eta) \in \check{C}^{n-1}(\mathcal{U}^\alpha, \mathcal{F})(B)$ by $Q^n(\eta)_{i_0 \dots i_{n-1}}(x) := t_{N, i_0 \dots i_{n-1}}(x), \forall x \in B$.
We claim that $d^{n-1}(Q^n(\eta)) = \eta$.

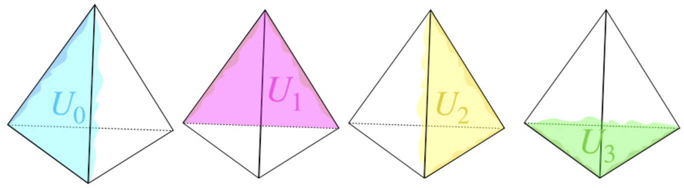


FIGURE 6. The space $X = S^2$ covered by four open sets $\{U_0, U_1, U_2, U_3\}$.

Stability Theory: To develop a stability theory on the previous construction. To be specific, we will prove that the map we derive in step 2 is stable when giving a little perturbation on the given data set X. It is known that the persistence homology computation is stable with respect to the Gromov-Hausdorff distance when giving a small perturbation. Our goal is to further extend the stability theory on the "Eilenberg-MacLane coordinates" derived from X.

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