Tony Cai

Q1) a) mean of 6-sided dice $= \dfrac{1+2+3+4+5+6}{6} = 3.5$

$$\text{Var(one dice roll)} = \frac{(1-3.5)^2+(2-3.5)^2+(3-3.5)^2+(4-3.5)^2+(5-3.5)^2+(6-3.5)^2}{6} = \boxed{\frac{35}{12}}$$

b) From Example 29.29 in LZ Textbook, we observed that $Var(X+X) = 4\,Var(X)$

Since we're essentially trying to roll the same number from the first roll, we can apply this observation such that $Var(X) = \frac{35}{12}$, where X is the event of one dice roll, and

$$4 \cdot \frac{35}{12} = 4\,Var(X) = \boxed{\frac{35}{3}}$$

c) From Theorem 29.30 in the LZ Textbook, we see that $Var(X+Y) = Var(X) + Var(Y)$ if X, Y are independent. We know that the two rolls of a dice are independent since each roll has no influence on a different roll. From Q1 a) we found $Var(X)$, where X is one roll of a dice to be $\frac{35}{12}$. Since we know $Var(Y)$ is also a roll of a dice, then $Var(X) = Var(Y)$ and $Var(X) + Var(Y) = \frac{35}{12} + \frac{35}{12} =$

$$\boxed{\frac{70}{12} = \text{Var(Sum of two rolls of a six-sided die)}}$$

Q2) a) Expected value of first roll: 3.5

Rolling a 4, 5, 6 on first roll → don't reroll

Rolling a 1, 2, 3 on first roll → reroll

Expected value of second roll: 3.5

$$\frac{4+5+6}{3} = 5$$

$0.5 * 5 + 0.5 * 3.5 = 4.25$

Thus, we should reroll if the first die is a 1, 2, 3, and keep the score of our first roll if we roll a 4, 5, 6. The expected score for this optimal strategy is $\boxed{4.25}$

Q2 b) Probability we roll a 1 : $\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^{n-1}$

Expected number of times we roll the die to get a 1 : $\sum_{n=1}^{\infty} n \left(\frac{5}{6}\right)^{n-1}\left(\frac{1}{6}\right) = \frac{1}{6}\left(\frac{1}{1-\frac{5}{6}}\right)^2 = 6$

Expected value of each roll before a 1 : $\frac{2+3+4+5+6}{5} = 4$

we are expecting the last roll to be a 1

Total Sum of the rolls : $\underbrace{4+4+4+4+4+1}_{6 \text{ total rolls}} = \boxed{21}$

Q3) a) A → picked fake coin   B → flipped heads twice

$P(A|B) = ?$

$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)}$

$P(A) = \frac{1}{100}$ $P(B|A) = 1$

$P(\bar{A}) = \frac{99}{100}$ $P(B|\bar{A}) = (0.5)^2 = 0.25$

$Pr(B) = P(B|A)P(A) + P(B|\bar{A})P(\bar{A})$

$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B|A) \cdot P(A) + P(B|\bar{A}) \cdot P(\bar{A})} = \frac{1 \cdot \frac{1}{100}}{1 \cdot \frac{1}{100} + 0.25 \cdot 0.99} = \frac{0.01}{0.01 + 0.2475}$

$= \boxed{0.0388}$

b)

N = # of fake coins

A = probability of fake coin   B = flipped heads twice

$P(A) = \frac{N}{100}$ $P(\bar{A}) = 1 - \frac{N}{100} = \frac{100-N}{100}$

$P(B|A) = 1$ $P(B|\bar{A}) = 0.25$

Thus, at least 20 coins must be fake for at least a 50% chance that the randomly chosen coin is fake.

$P(A|B) = \frac{P(B|A) \cdot P(A)}{\underbrace{P(B|A) \cdot P(A) + P(B|\bar{A}) \cdot P(\bar{A})}_{P(B)}} = \frac{1 \cdot \frac{N}{100}}{1 \cdot \frac{N}{100} + \frac{1}{4} \cdot \frac{(100-N)}{100}} =$

$\frac{\frac{N}{100}}{\frac{N}{100} + \frac{(100-N)}{400}} = \frac{\frac{4N}{400}}{\frac{4N+100-N}{400}} = \frac{4N}{3N+100}$

$\frac{4N}{3N+100} = \frac{1}{2}$

$3N + 100 = 8N$

$100 = 5N$

$N = 20$