Tony Cai

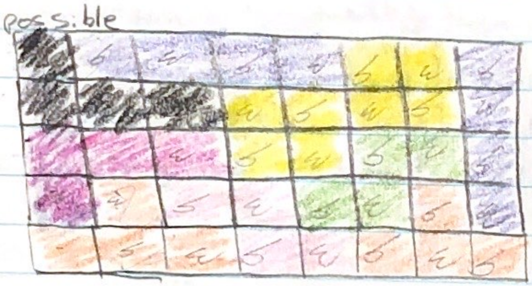
CS 230 - Week 1 Assignment

1) a) Proof: Observe that a 4x5 chess board has 10 white squares and 10 black squares. Each tetris pieces covers exactly 2 black squares and 2 white squares, with the exception of the following, which covers either 3 black squares and 1 white square or 3 white squares and 1 black square:

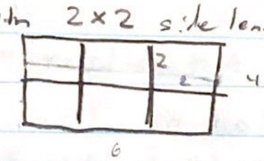
 or 

Thus, there would **not** be a way to fit these pieces on a 4x5 chess board, regardless of rotation, because they would not cover the same amount of white squares as black squares in total. □

b) Yes, it is possible



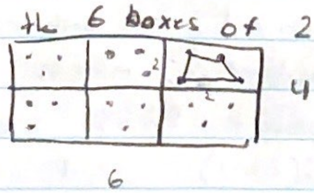
2) Proof) If we divide the 4 x 6 square into 6 sections, the box would look like the following with 2x2 side lengths or an area of 4.



For points on the lines between squares, consider them part of the square above or to the right. If each of these 6 boxes was assigned to the least amount of points, then by the pigeonhole principle, 5 boxes would have 3 points and one box would have 4 points. Specifically, this can be represented by $kn + 1$ (pigeonhole principle equation), where $k = 3$ and $n = 6$, so

$$(3 \cdot 6) + 1 = 19$$

which is represented in the 6 boxes of 2x2 area by the following:



Thus, we see that at least box must have 4 points and since the size of a box has an area of 4, then the 4 points would at most form a quadrilateral of area 4 if they were placed on the corners of the box. □

x, y are real numbers $\quad |x+y| = |x|+|y|$

4) a) $x = -2$, $y = 3$, $|-2+3| = |-2|+|3|$

$1 \neq 5$, so false

because x and y must be greater than or equal to 0

b) $x^2 < x^4$, x is real number

$x = 0.5 \quad x^2 < x^4 \longrightarrow 0.25 < 0.0625$

false because x must be a real number greater than 1

c) x, y are real numbers

If $|x+y| = |x \cdot y|$, then $y = 0$

$x = 0$ $\quad |1| = |-1| \Rightarrow 1 = 1$, but $y \neq 0$ so the conjecture is false

$y = 1$

5) Proof) For each set of n alien socks, such that n is a positive, integer, there exists 3n individual socks being washed. If we create a box for each type of sock in set n, we would have n boxes. By using the general form of the pigeonhole principle, we would need to pull out 2n+1 socks to guarantee a matching triplet

$\square$

6) a) Proposition: If m and n are odd, then $5m-3n$ is even.

Proof) Assume m and n are both odd integers. By the definition of an odd integer, $m=2a+1$ and $n=2b+1$ for some integers a and b. Next, we need to prove that $5m-3n$ is even, thus we can set up the following:

$$5(2a+1) - 3(2b+1) = 2c+1$$

$$10a+5-6b-3 = 10a-6b+2 = 2(5a-3b+1)$$

Since $5a-6b+4$ is an integer that can be represented as k, our result would be $2k$, which resembles the definition of an even $\square$

6 b) Proposition: If m and n are even, then $3mn$ is divisible by 4.

Proof: Assume m and n are both even numbers. By the definition of even numbers, $m = 2a$ and $n = 2b$ for some integers a and b.

Next, we need to prove $3mn$ is divisible by 4, thus we can set up the following

$$3(2a)(2b) = 12ab$$

Since our result is $12ab$ which is an integer divisible by 4, then we can conclude this theorem to be true. □

7) a) Proposition: If n is a positive integer, then 4 divides $1+(-1)^n(2n-1)$

proof: Assume that n is a positive integer that is either even or odd.

Cases

Case 1: n is even. Then $n = 2a$ for some integer a. Thus,

$$1+(-1)^{2a}(2(2a)-1) = 1+1(4a-1) = 4a$$

Since $4a$ is an integer divisible by 4, we know that 4 divides $1+(-1)^n(2n-1)$ when n is even.

Case 2: n is odd. Then $n = 2a+1$ for some integer a. Thus,

$$1+(-1)^{2a+1}(2(2a+1)-1) = 1+-1(4a+2-1) = 1-4a-1 = -4a$$

Since $-4a$ is an integer divisible by 4, we know that 4 divides $1+(-1)^n(2n-1)$ when n is odd.

We have shown that 4 divides $1+(-1)^n(2n-1)$ whether n is even or odd, when n is a positive integer, thus completing the proof. □

8) $3^{302} \bmod 28$? $3^3 = 27$ $27 \bmod 28 \equiv -1 \bmod 28$

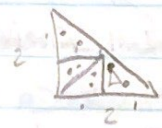
$(3^3)^{100} \bmod 28 \cdot 9 \bmod 28$

$((3^3)^{100} \bmod 28 \cdot 9 \bmod 28) \bmod 28 = (1 \bmod 28 \cdot 9 \bmod 28) \bmod 28$

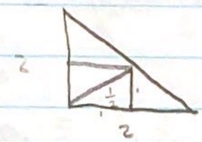
$(-1)^{100} = 1 \bmod 28$ $= 9 \bmod 28$

So remainder of $3^{302} \div 28$ is $\boxed{9}$

3) Sketch)



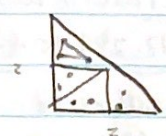
Proof) We can split the triangle into 4 triangles with non-hypotenuse side lengths of 1 and an area of $\frac{1}{2}$, as follows:



For the points on the lines, consider them part of the triangle below or to the right. Each of these points will thus be assigned to one of these triangles or boxes and due to the pigeonhole principle, ≥ 3 boxes will have at least 2 points and one box will have 3 points. Specifically, the pigeonhole principle equation can be represented by, $kn+1$ and applied to this context with $k = 2$ and $n = 4$, so

$$(2 \cdot 4) + 1 = 9$$

These 9 points could be assigned to the boxes, as follows:



Thus, we see that since one box must have at least 3 points, it could form a triangle of less than $\frac{1}{2}$ since the box has an area of $\frac{1}{2}$ and the points would need to be placed in the exact corners for the triangle formed to have an area of $\frac{1}{2}$.

Pset #1 Continued

7b) Every multiple of 4 is equal to $1+(-1)^n(2n+1)$ for some positive integer n

Proof: Assume n is a positive integer that is either even or odd, and a multiple of 4 can be represented by $4k$ for some integer k.

We can use two cases to prove this proposition true:

Case 1: n is even. Then $n = 2a$ for some integer a. Thus,

$$1+(-1)^{2a}(2(2a)+1) = 1 + 4a - 1 = 4a$$

Since $4a$ can also be represented by $4k$, we know that every multiple of 4 will equal $1+(-1)^n(2n+1)$ for an even, positive integer of n.

Case 2: n is odd. Then $n = 2b+1$ for some integer b. Thus,

$$1+(-1)^{2b+1}(2(2b+1)+1) = 1 - (4b+2-1) = -4b$$

Since $-4b$ can also be represented by $4k$, we know that every multiple of 4 will equal $1+(-1)^n(2n+1)$ for an odd, positive integer of n. Thus, we have shown that every multiple of 4 is equal to $1+(-1)^n(2n+1)$ for some positive integer n when n is either even or odd, which completes the proof. $\square$

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a) Proof: Let x be an integer where $x = y \bmod 3$ and $y \in \{0,1,2\}$. First, we can use cases of y to prove that any square must be congruent to $0 \bmod 3$ or $1 \bmod 3$.

Case 1: $y=0$

$x = 0 \bmod 3$, $x^2 \equiv 0 \bmod 3$

Case 2: $y=1$

$x = 1 \bmod 3$, $x^2 \equiv 1 \bmod 3$

Case 3: $y=2$

$x = 2 \bmod 3$, $x^2 \equiv 4 \bmod 3 \equiv 1 \bmod 3$

Thus, $x^2 \bmod 3$ is congruent to either $0 \bmod 3$ or $1 \bmod 3$.

Next, we can use a proof by contradiction to show that if neither a nor b is divisible by 3, then the squares must be $1 \bmod 3$. Assume a and b are integers not divisible by 3:

$a = 1 \bmod 3$, $a^2 \equiv 1 \bmod 3$

$a = 2 \bmod 3$, $a^2 \equiv 4 \bmod 3 \equiv 1 \bmod 3$

$b = 1 \bmod 3$, $b^2 \equiv 1 \bmod 3$

$b = 2 \bmod 3$, $b^2 \equiv 4 \bmod 3 \equiv 1 \bmod 3$

Proof continued on next page →

9 continued) Thus, we can observe that when a, b are not divisible by 3, $a^2, b^2 \equiv 1 \bmod 3$ for every case. Consequently, we can also write $a^2 = 3m+1$ and $b^2 = 3n+1$ where m, n are integers. This can then be rewritten as:

$$(3m+1) + (3n+1) = c^2$$
$$3mn + 2 = c^2$$

c^2 must then be congruent to $2 \bmod 3$ since $3mn$ is a multiple of 3 and the remainder would be 2. However, we initially proved that any square is congruent to $0 \bmod 3$ or $1 \bmod 3$ so therefore $c^2 \not\equiv 2 \bmod 3$ and either a or b must be divisible by 3, thus completing the proof. $\square$

~~a) Proof: Let x be an integer where $x \equiv y \bmod 3$ and $y \in \{0, 1, 2\}$. First, we can use cases for y to prove that $x^2 \bmod 3$ must be congruent to either 0 or 1 mod 3.~~

~~Case 1: $y = 0$~~

~~$x \equiv 0 \bmod 3$ $\quad x^2 \equiv 0 \bmod 3$~~

~~Case 2: $y = 1$~~

~~$x \equiv 1 \bmod 3$ $\quad x^2 \equiv 1 \bmod 3$~~

~~Case 3: $y = 2$~~

16) a) Let $f(n) = n^2 - n + 5$. If n is an integer, then $f(n)$ is a prime number

$n = 5 \rightarrow (5)^2 - 5 + 5 = 25 - 5 + 5 = 25$, has factors of 1, 5, 25,

so $f(n)$ is not a prime number and the theorem is false

b) Suppose a, b and c are pos. integers. If $a|bc$, then $a|b$ or $a|c$.

$a = 6$

$b = 3$ $\quad a|bc = 6|6 = 1$ ✓ 1 is an integer

$c = 2$ $\quad a|b = 6|3 = 0.5$ X 0.5 is not an integer

$\quad a|c = 6|2 = 0.\overline{3}$ X $0.\overline{3}$ is not an integer

so this theorem is false

c) Suppose a and b are integers. If $a|b$ and $b|a$, then $a = b$.

$a = 3$ $\quad a|b \rightarrow 3|-3 = -1$ ✓, -1 is an integer

$b = -3$ $\quad b|a \rightarrow -3|3 = -1$ ✓, -1 is an integer

$a \neq b$, so this theorem is false