CS230 - week 3 Tony Ci

1) a) Proposition: $6 \mid (n^3 - n)$

Proof: We proceed by induction, where $n \in \mathbb{N}$.

Base Case. The base case is when $n = 1$ and

$$6 \mid (1^3 - 1)$$
$$6 \mid 0$$

$6 \mid 0$ is true since every number is a factor of 0.

Inductive Hypothesis. Let $k \in \mathbb{N}$, and assume that

$$6 \mid (k^3 - k)$$

Induction Step. We want to prove that the result holds for $k+1$ such that

$$6 \mid ((k+1)^3 - (k+1))$$

We can rewrite this expression to be

$$6 \mid (k+1)(k+1)(k+1) - (k+1)$$
$$6 \mid k^3 + 3k^2 + 3k + 1 - (k+1)$$
$$6 \mid k^3 + 3k^2 + 2k$$
$$6 \mid k(k+1)(k+2)$$

We know that in the expression $k(k+1)(k+2)$ that there either exists an even integer and an integer divisible by 3 since one of k or $k+1$ must be even by the definition of an even integer and one of the integers k, $k+1$, or $k+2$ must be divisible by 3, OR there exists one integer in k, $k+1$, $k+2$ that is both divisible by 3 and even. In either case, we could get $6c$ where $c \in \mathbb{N}$, which would be divisible by 6, so $6 \mid k(k+1)(k+2)$ is true.

Conclusion. Therefore, we can conclude by induction that $6 \mid (n^3 - n)$ for $n \in \mathbb{N}$.

1) b) Proposition: $1^3 + 2^3 + 3^3 + \dots + n^3 = (1+2+3+\dots+n)^2$

Proof: We proceed by Induction, where $n \in \mathbb{N}$.

<u>Base Case</u> The base case is when $n=1$ and

$$1^3 = 1^2$$
$$1 = 1$$

So the base case is true.

<u>Inductive Hypothesis</u>. Let $k \in \mathbb{N}$ and assume that

$$1^3 + 2^3 + 3^3 + \dots + k^3 = (1+2+3+\dots+k)^2$$

<u>Inductive Step</u>. We wish to show that the result holds for $k+1$ such that

$$1^3 + 2^3 + 3^3 + \dots + (k+1)^3 = (1+2+3+\dots+k+1)^2$$

We can rewrite this expression such that

$$1^3 + 2^3 + 3^3 + \dots + k^3 + (k+1)^3 = (1+2+3+\dots+k+k+1)^2 \quad \text{expanding}$$

$$(1+2+3+\dots+k)^2 + (k+1)^3 = (1+2+3+\dots+k+k+1)^2 \quad \text{using inductive hypothesis}$$

$$(k+1)^3 = (1+2+3+\dots+k+k+1)^2 - (1+2+3+\dots k)^2 \quad \text{rearranging expression}$$

$$(k+1)^3 = (k+1)\left(2(1+2+3+\dots+k) + (k+1)\right) \quad \text{difference of squares}$$

$$(k+1)^2 = 2(1+2+\dots+k) + (k+1) \quad \text{Simplifying}$$

We know $(1+2+\dots+k) = \frac{k(k+1)}{2}$ by <u>Proposition 4.2</u> in the "Proofs" textbook which proves that $1+2+3+\dots+n = \frac{n(n+1)}{2}$ is true for $n \in \mathbb{N}$. If we apply this proposition,

$$(k+1)^2 = 2\left(\frac{k(k+1)}{2}\right) + k+1$$
$$(k+1)^2 = (k+1)(k+1)$$
simplifying

Conclusion: Therefore we can conclude by induction that

$$1^3 + 2^3 + 3^3 + \dots + n^3 = (1+2+3+\dots+n)^2 \text{ for all } n \in \mathbb{N}.$$

1) c) Proposition: $1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \ldots + n \cdot n! = (n+1)! - 1$

Proof: We proceed by induction, where $n \in \mathbb{N}$.

Base Case. The base case is when $n = 1$, and

$$1 \cdot 1! = (1+1)! - 1$$

$$1 = 2 - 1$$

$$1 = 1$$

So the base case is true.

Inductive Hypothesis. Let $k \in \mathbb{N}$ and assume that

$$1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \ldots + k \cdot k! = (k+1)! - 1$$

Induction Step. We want to prove that the result holds for $k+1$ such that

$$1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \ldots + (k+1)(k+1)! = ((k+1)+1)! - 1$$

We can rewrite this expression such that

expanding $$1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \ldots + k \cdot k! + (k+1)(k+1)! = (k+2)! - 1$$

using inductive hypothesis $$(k+1)! - 1 + (k+1)(k+1)! = (k+2)! - 1$$

Simplifying by rearranging $$(k+1)(k+1)! = (k+2)! - (k+1)!$$

Simplifying by dividing by $(k+1)!$ $$(k+1) = (k+2) - 1$$

$$k+1 = k+1$$

Conclusion. Thefore, we can conclude by induction that

$$1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \ldots + n \cdot n! = (n+1)! - 1 \quad \text{for } n \in \mathbb{N}.$$

1) d) Proposition: $1+\frac{n}{2} \leq \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\dots+\frac{1}{2^n-1}+\frac{1}{2^n}$

Proof: We proceed by induction, where $n \in \mathbb{N}$.

Base Case: The base case is when $n=1$ and

$$1+\frac{1}{2} \leq \frac{1}{2^1-1}+\frac{1}{2^1}$$

$$1+\frac{1}{2} \leq 1+\frac{1}{2}$$

So the base case is true.

Inductive Hypothesis: Let $n \in \mathbb{N}$, $k \in \mathbb{N}$ and assume that

$$1+\frac{n}{2} \leq \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{2^n-1}+\frac{1}{2^n} \quad \text{or} \quad 1+\frac{n}{2} \leq \sum_{n=1}^{2^n} \frac{1}{k}$$

Induction Step: We wish to show that the result holds for $n+1$ such that

$$1+\frac{n+1}{2} \leq \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{2^{n+1}-1}+\frac{1}{2^{n+1}} \quad \text{or} \quad 1+\frac{n+1}{2} \leq \sum_{n=1}^{2^{n+1}} \frac{1}{k}$$

To begin the induction step, we know that for $n+1$,

$$1+\frac{n}{2}+\sum_{k=2^{n+1}}^{2^{n+1}} \frac{1}{k} \leq \sum_{n=1}^{2^{n+1}} \frac{1}{k}$$ since we are adding $\frac{1}{2^{n+1}}$ to

both sides. With $\sum_{n=1}^{2^{n+1}} \frac{1}{k}$, we can observe that the smallest value will be $\frac{1}{2^{n+1}}$.

By the definition of a summation, a summation must be greater than or equal to its smallest value multiplied by the total number of elements, so

$$\sum_{n=1}^{2^{n+1}} \frac{1}{k} \geq (2^{n+1}-2^n)\left(\frac{1}{2^{n+1}}\right)$$

$$\frac{2^{n+1}-2^n}{2^{n+1}} = \frac{1}{2}$$

This means we could simplify the expression such that $1+\frac{n}{2}+\frac{1}{2} \leq \sum_{n=1}^{2^{n+1}} \frac{1}{k}$ or $1+\frac{n}{2} \leq \sum_{n=1}^{2^n} \frac{1}{k}$.

Conclusion: Therefore by induction, $1+\frac{n}{2} \leq \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\dots+\frac{1}{2^n-1}+\frac{1}{2^n}$ is true for $n \in \mathbb{N}$.

2) a) $\boxed{1} < \boxed{3} > \boxed{2} < \boxed{4} < \boxed{6} > \boxed{5}$

b) Proof: We proceed by induction, where $n \in \mathbb{N}$.

Base Case: The base case is when $n = 1$, which would be true because there would be $n-1$ or 0 inequalities.

Inductive hypothesis: Let $k \in \mathbb{N}$ and assume that there are k boxes with $k-1$ inequalities between the boxes. We can rearrange the numbers for 1 to k such that each box has one number and the boxes satisfy all of the inequalities.

Induction Step: We want to prove that the inequalities hold for $k+1$ boxes.

Case 1: If either the inequality that is added to the left most side is a $>$ symbol or the inequality that is added to the right most side is a $<$ symbol, then we can assign $k+1$ to the new box on the outside of the newly added inequality since $k+1$ is the largest number in the sequence from 1 to $k+1$. This retains the satisfaction of the inequalities, since we know that the previous k boxes with $k-1$ inequalities must also hold from the inductive hypothesis.

Case 2: If either the inequality that is added to the left most side is a $<$ symbol or the inequality that is added to the right most side is a $>$ symbol, then we know we're looking for a smaller number. To guarantee that the inequality holds true for $k+1$, we can increment all existing numbers in the boxes by 1, such that all existing numbers range from 2 to $k+1$. Then, we can assign 1 to the new box on the outside of the newly added inequality since 1 is the smallest number in the sequence from 1 to $k+1$. This retains the satisfaction of the inequalities, since we know that the previous k boxes with $k-1$ must hold by the inductive hypothesis.

Conclusion: If one has n boxes with $n-1$ inequalities between them, then it is always possible to place numbers $1, 2, 3, \dots n$ into these boxes so that the inequalities are all correct for $n \in \mathbb{N}$.

3) Proof: We proceed by induction, where $n \in \mathbb{N}$.

Base Case: The base case is when $n = 1$. This means we would have 4^1 congruent triangle with the top corner removed as shown below:

For the base case, only the shape: is left, which is the shape we want to cover the remaining triangles by, so the base case is true.

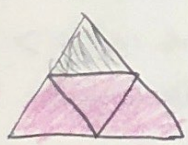
Inductive Hypothesis: Let $k \in \mathbb{N}$ and assume that the remaining number of triangles we want to be covered with the shape: is

$4^k - 1$

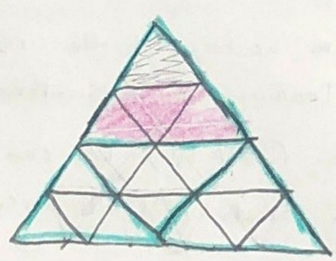
Induction Step: We want to prove that the resulting triangle can be covered with the shape: for $k+1$. Our new total number of triangles expression would be 4^{k+1} or $4 \cdot 4^k$ or 4 times as many triangles and the total number of triangles we would need to cover would be $4^{k+1} - 1$ or $4(4)^k - 1$, shown below:

4^n triangle

(Pink implies covered)

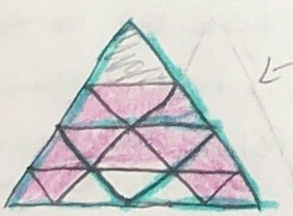


4^{n+1} triangle



We can observe that for the 4^{n+1} triangle, there exists 4 of the 4^n triangles (outlined in blue). If we apply the placement of the top triangle, making sure to rotate the pieces like so:

4^{n+1} Triangle



Then we would have $4(4^n - 1)$ or $4^n - 4$ triangles covered, in which case we can place one more shape such that $4 \cdot 4^n - 1$ triangles have been covered.

Conclusion: Therefore by induction, we have proved that the remaining $4^n - 1$ triangles in a 4^n equilateral triangle with the top removed can be covered entirely by the shape for $n \in \mathbb{N}$.

4) Proof: We proceed by induction.

Base Case: The base case is when $n = 2, 3$.

For $n=2$ cents, we can only use 1 2cent stamp, so $1 \le \frac{2}{2}$ or $1 \le 1$.

For $n=3$ cents, we can only use 1 3cent stamp, so $1 \le \frac{3}{2}$.

Thus, all base cases are true.

Strong Inductive Hypothesis: Let $k \in \mathbb{N}$ and assume that k cents in postage can be made up using at most $\frac{k}{2}$ postage stamps for $k \ge 2$.

Induction Step: We want to prove that the result holds for $k+1$ cents $\le \frac{k+1}{2}$ stamps

From our hypothesis, we know that $k-1$ cents in postage can be made up using at most $\left(\frac{k-1}{2}\right)$ stamps, which can be represented as $k-1 \le \left(\frac{k-1}{2}\right)$

If we add one 2-cent stamp, then k would increase by 2 cents and the number of stamps would increase by 1.

This means that we would have $k+1 \le \frac{k-1}{2} + 1$ or $k+1 \le \frac{k+1}{2}$

Conclusion: Therefore by Strong induction, n cents in postage can be made up using at most $\frac{n}{2}$ postage stamps for $n \in \mathbb{N}$ and $n \ge 2$.