Weeks 6&7 - CS230

1) a) The relation A on $\mathbb{R}$: $\{(x,y) \in \mathbb{R}^2,\ y \geq x\}$

b) The relation B on $\mathbb{R}$: $\{(x,y) \in \mathbb{R}^2,\ x \neq y\}$

c) The relation C on $\mathbb{R}$: $\{(x,y) \in \mathbb{R}^2,\ x \neq -y\}$

d) The relation D on $\mathbb{Z}$: $\{(x,y) \in \mathbb{Z}^2,\ x > y\}$

e) The relation E on $\mathbb{Z}$: $\{(x,y) \in \mathbb{Z}^2,\ 3 \mid (x-y)\}$

f) The relation F on $\mathbb{Z}$: $\{(x,y) \in \mathbb{Z}^2,\ |x+y| < 3\}$

2) a) $\boxed{\text{65536 different binary relations on set A}}$

$2^{4^2} = 2^{16} = 2^{|A| * |A|}$

$\boxed{\text{15 different equivalence relations}}$

All Partitions of A: $\{\{1\},\{2\},\{3\},\{4\}\}$,

$\{\{1,2\},\{3\},\{4\}\}$, $\{\{1,2\},\{3,4\}\}$, $\{\{1,2,3,4\}\}$,

$\{\{1,2,3\},\{4\}\}$, $\{\{1\},\{2,3,4\}\}$, $\{\{1\},\{2,3\},\{4\}\}$,

$\{\{1,3\},\{2,4\}\}$, $\{\{1,3\},\{2\},\{4\}\}$, $\{\{1,4\},\{2\},\{3\}\}$

$\{\{1,4\},\{2,3\}\}$, $\{\{1,2,4\},\{3\}\}$, $\{\{1\},\{2\},\{3,4\}\}$

$\{\{1\},\{2,4\},\{3\}\}$, $\{\{1,3,4\},\{2\}\}$

2b) Proof : We can express the relation R on $\mathbb{Z}$ as $\{(x,y) \in \mathbb{Z}^2, \ 2 \mid (3x-5y)\}$

In order to prove that R is an equivalence relation, we must show that R is reflexive, Symmetric, and transitive.

Reflexive : $R(x,x) \rightarrow 3x - 5x = -2x$

We know $-2x$ is even by the definition of an even number where $2a, a \in \mathbb{Z}$ will represent an even number. Thus we know R is reflexive

Symmetric : $R(x,y) \rightarrow 3x - 5y$

$R(y,x) \rightarrow 3y - 5x$

We can rearrange $R(y,x)$ such that

$$3y - 5x = 3y - 8y - 5x + 8x$$
$$= 3x - 5y + 2(4x - 4y)$$

Using this expression, if we know $3x - 5y$ is even and $2(4x-4y)$ will always be even since it resembles the definition of an even number where $2b, b \in \mathbb{Z}$, then then we know $R(y,x)$ will also be even since the addition of two even numbers will produce an even number. Thus, R must be Symmetric.

Transitive : $R(x,y)$ and $R(y,z)$

We can write $3x - 5y = 2k$ and $3y - 5z = 2m$, where $k, m \in \mathbb{Z}$, assuming that $3x-5y$ and $3y-5z$ are even. We can then simplify these equations:

$$3x - 5y = 2k \qquad 3y - 5z = 2m$$
$$3x = 2k + 5y \qquad -5z = 2m - 3y$$

$$3x - 5z = 2k + 5y + 2m - 3y = 2(k + m + y)$$

even by definition of an even number

Thus we can observe that $R(x,z)$ is transitive since when $3x-5y$ and $3y - 5z$ are even, then $3x - 5z$ would also be even.

Since we have proved that R is ~~R~~reflexive, Symmetric, and transitive, then R is an equivalent relation. □

R has two equivalence classes : the set of all odd integers and the set of all even integers.

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c) Relation A on $\mathbb{N}$: $\{(x,y) \in \mathbb{N}^2$: x and y end in the same number of 0's in digits$\}$

There are an infinite number of equivalence classes, where each class represents numbers that end in the same number of 0's in digits and each class has an infinite number of elements in their sets,

We can show that A would be an equivalence Relation:

Reflexive: The same number would end in the same number of 0's as itself.

Symmetric: If we switch x and y, they would both still end in the same number of 0's since flipping the numbers won't change how many 0's the numbers will end in.

Transitive: We know that the relation is transitive since if x and y both end in the same number of 0's and y and z both end in the same number of 0's, then x and z would have the same number of 0's. Assuming that there exists relations between x and y and y and z, then we can represent the relation's transitive property as:

$A(x,y)$ and $A(y,z)$ implies $A(x,z)$ where x and z are both natural numbers that end in the same number of 0's

Thus, we have justified that A is an equivalence relation by showing that A is reflexive, symmetric and transitive, and that A has an infinite number of equivalence classes where each class has an infinite number of elements in their sets.

3a) We can prove that the set $A = \{(n,a) \in \mathbb{N} \times \mathbb{R} : a = \pi n\}$ is countably infinite by showing that the function $f: \mathbb{N} \to A$ where $f(n) = (n, \pi n)$ is bijective, or both injective and surjective.

injective: We can observe that $f(x)$ and $f(y)$ would represent:

$$f(x) = f(y) \longrightarrow (x, \pi x) = (y, \pi y)$$

$x = y$ and $\pi x = \pi y$ would imply $x = y$, so we know that f is injective.

Surjective: If we consider the point $(z, \pi z)$, where $z \in \mathbb{N}$, we can observe that this output happens when the input is z in $f(z)$, such that $f(z) = (z, \pi z)$, so f is surjective.

Thus, we have proven that f is a bijection, which means that by definition, A must be countably infinite since $\mathbb{N}$ was our domain in bijective f and we reviewed in class that $\mathbb{N}$ is also countably infinite, so $|A| = |\mathbb{N}|$. $\square$

~~3) a) The set $A = \{(n,a) \in \mathbb{N} \times \mathbb{R} : a = \pi n\}$ resembles the set $\{(1,\pi), (2,2\pi), (3,3\pi) \ldots\}$. With this information, we can observe that $|A| = |\mathbb{N}|$ since the x-coordinate for each ordered pair resembles an element in $\mathbb{N}$ and the y-coordinate is just multiplying the x coordinate value by π. Since we know that $|\mathbb{N}|$ is countably infinite as we reviewed in class, and $|A| = |\mathbb{N}|$ as previously explained, then we know that A must also be countably infinite.~~

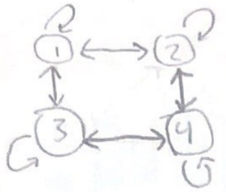
3 b) Proof: To prove that the set $\mathbb{R} \setminus \mathbb{Q}$ of irrational numbers is uncountable, we can utilize a proof by contradiction, where the set $\mathbb{R} \setminus \mathbb{Q}$ of irrational numbers is countable. We can observe that the set of irrational numbers combined with the set of rational numbers must equal the set of real numbers or

$$(\mathbb{R} \setminus \mathbb{Q}) \cup \mathbb{Q} = \mathbb{R}.$$

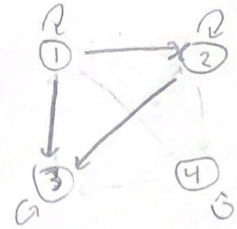
We can also observe that the set $\mathbb{Q}$ of rational numbers is countable as stated in the proofs textbook and so the union of two countable sets must also be countable, which is a theorem we reviewed in class. However, this is a contradiction since we know $\mathbb{R}$ is an uncountable set, which we went over in class, and we are saying that the union of two countable sets is also countable.

This, we can conclude by a proof by contradiction that the set $\mathbb{R} \setminus \mathbb{Q}$ of irrational numbers must be uncountable. $\square$

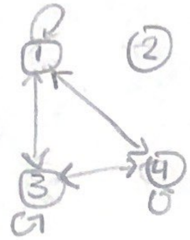
4) a)



b)



c)



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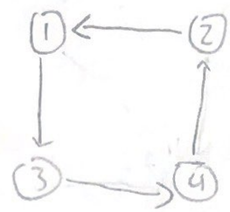

5) Proof: In order to prove that R is reflexive, we must prove that there is an element $a \in A$ such that aRx for every $x \in A$, where $x = a$ or aRa. Since we know the relation is symmetric, we know that aRx implies xRa for all $x \in A$.

Using the definition of a transitive relation, we can observe that if

aRx implies xRa from the symmetric relation, then

aRa, which by definition of a reflexive relation, means

that we can conclude R is reflexive.

$\square$