CS230 - Week 11 Tony Lui

Proof:

Q1] In order to prove $\sim$ is an equivalence relation, we must show that the relation is reflexive, symmetric, and transitive.

Reflexive: $f(n) \sim f(n)$ $\lim_{n\to\infty} \frac{f(n)}{f(n)} = 1$, since every function grows the same rate as itself, so $\sim$ is reflexive.

Symmetric: $f(n) \sim g(n) \longleftrightarrow g(n) \sim f(n)$

We know $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$, so (reciprocal) $\lim_{n\to\infty} \frac{g(n)}{f(n)} = 1$, since we already know that both functions grow at the same rate so $\sim$ is symmetric.

Transitive: $f(n) \sim g(n)$ and $g(n) \sim h(n)$ implies $f(n) \sim h(n)$

If we know $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$ and $\lim_{n\to\infty} \frac{g(n)}{h(n)} = 1$, then we can multiply the limits using limit properties from Theorem 21.2 in the LZ Textbook where $\lim_{n\to\infty} f(n) * \lim_{n\to\infty} g(n) = \lim_{n\to\infty} (f(n) * g(n))$; $\lim_{x\to\infty} \frac{f(x)}{g(x)} \cdot \frac{g(x)}{h(x)} = \lim_{x\to\infty} \frac{f(x)}{h(x)} = 1$, so $\sim$ is transitive.

Thus, $\sim$ is an equivalence relation since the relation is reflexive, symmetric, and transitive. $\square$

Q2) Algorithm 1: $O(n)$, where $t_1(n) = n$

↳ The 3rd operation happens linearly → max of n run-time and all other operations are constant so $O(n)$

Algorithm 2: $O(\log(n))$, where $t_2(n) = \log(n)$

↳ The 3rd operations happens in $\log_2(n)$ time and all other operations are constant so $O(\log(n))$

Q3) $(1+\frac{1}{a})(1+\frac{1}{a^2})(1+\frac{1}{a^4})\ldots(1+\frac{1}{a^{2^{100}}})$

$(1+\frac{1}{a})(1+\frac{1}{a^2})(1+\frac{1}{a^4})\ldots(1+\frac{1}{a^{2^{100}}}) \cdot \left(\frac{1-\frac{1}{a}}{1-\frac{1}{a}}\right)$ → We can multiply our original equation by $\left|\frac{(1-\frac{1}{a})}{(1-\frac{1}{a})}\right|$ since our numerator would then be a difference of squares.

$(1-\frac{1}{a})(1+\frac{1}{a})(1+\frac{1}{a^2}) =$

$(1-\frac{1}{a^2})(1+\frac{1}{a^2}) = (1-\frac{1}{a^4})$ → When multiplying by the numerator of $(1-\frac{1}{a})$, we can expand the first couple of terms, and see that this pattern would continue and the negative exponent for the term a will be equal to the next power of 2 in the sequence. Thus, we would have $1-\frac{1}{a^{2^{101}}}$ in the numerator. We then need to divide this by $1-\frac{1}{a}$ such that:

rewritten version of $1-\frac{1}{a^{2^{101}}}$ rewritten version of $1-\frac{1}{a}$

$\frac{a^{2^{101}}-1}{a^{2^{101}}} \cdot \frac{a}{a-1} =$

$$\boxed{\frac{a^{2^{101}}-1}{a^{2^{101}-1}(a-1)}}$$ → most simplified

Q4] $\boxed{\text{Recurrence Relation: } T_n = 3T_{n-1} - T_{n-3} \text{ for } n \geq 4}$

Explanation: Our first digit can be 0, 1, or 2 initially, which would mean that the remaining $n-1$ digits in the ternary sequence would result in a_{n-1} number of ternary sequences. This means that the total number of ternary sequences with digits 0,1,2 would be $3T_{n-1}$ since we have 3 possible starting terms for our ternary sequence. However, there is one sequence of numbers "012" that can not occur within our ternary sequence. In other words, the number 0 can not be followed by 1 and 2 (3 digits total, so $n-3$ digits remaining in each sequence), which means that we must exclude the associated $a_n - 3$ number of ternary sequences. If we subtract the a_{n-3} number of invalid ternary sequences with our total number of $3T_{n-1}$ ternary sequences, we would thus get: $T_n = 3T_{n-1} - T_{n-3}$

Q5] a) $\boxed{T_n = (n-1)(T_{n-1} + T_{n-2}) \text{ for } n \geq 2}$

Explanation: Assume that our first check is placed in any n^{th} persons envelope that is not the correct person's envelope. This leaves $n-1$ envelopes left remaining to be filled. We then have two seperate cases we can represent in our recurrence relation:

Case 1: If two of our checks occupy the other person's corresponding envelope, then we know that the two checks are not in their respective envelope. However, this still leaves $n-2$ envelopes to be filled, which we could then repeat our process and can represent in the recurrence relation: T_{n-2}

Case 2: With our initial check being placed in an incorrect envelope, we still need to assign the remaining checks to incorrect checks one-by-one until we have succeeded. In the event where Case 1 does not occur, we can then represent the one-by-one incorrect assignment of checks to envelopes as: T_{n-1}

Thus, our recurrence relation is $T_n = (n-1)(T_{n-1} + T_{n-2})$ for $n \geq 3$

b) $T_6 = (6-1)(T_5 + T_4)$

$T_6 = 5(44 + 9)$

$\boxed{T_6 = 265}$

$T_0 = 1$

$T_1 = 0$ $\quad T_3 = 2$

$T_2 = 1$ $\quad T_4 = 9$

$T_5 = (5-1)(T_4 + T_3)$

$= 4(9+2) = 44$