week 8 - CS230 Tony Li

Q1] a) diameter : 3 b) longest cycle length : 6

c) Eulerian Circuit : $\underbrace{A-B-F-G-A}_{\text{given}}$ - C - F - E - D - C - E - A

ends where we start

Observation:

Removing all the edges from the given circuit A-B-F-G-A leaves us with a connected subgraph where every vertice has an even number of degrees, so we can observe that the remaining connected subgraph is a Eulerian circuit by LZ Theorem 7.62

Recursive Steps: uses remaining subgraph w/o A-B-F-G-A as reference

1) Start at vertex A and choose the edge A-C or A-E (doesn't matter which)

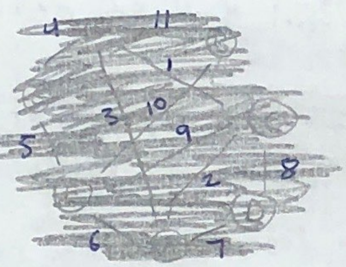
2) Next, choose an edge that has not already been traversed and without returning to vertex A.

IE if you're at vertex C, then you can choose C-E, C-D or C-F but not C-A

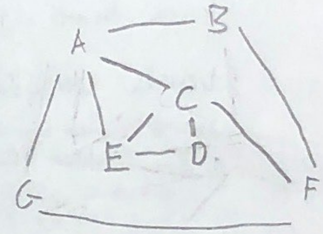
3) Repeat step 2 until there are no edges left we can traverse from step 2, and you will see that our last remaining edge we need to traverse will always lead us to our original starting vertex (A), which results in a Eulerian Circuit.

We can then observe that A-B-F-G-A is a circuit that can be joined by our recursive construction, which retains the structure of the Eulerian circuit

Q 2]



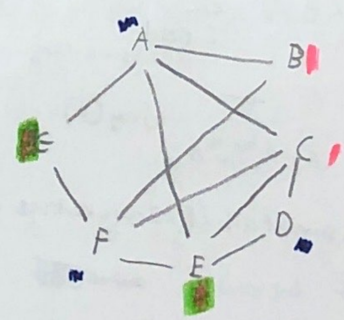
a)



b) Red Set: {B, C}

Blue Set: {A, D, F}

Green Set: {E, G}



Q3) Proof : We can prove the proposition true by proving that every completed graph with only vertices of odd degrees, must have an even number of total vertices.

Let $G = (V, E)$, where $|E| = m$, $m \in \mathbb{Z}$ and $m = 2k$, $k \in \mathbb{Z}$.

From lecture, we stated that the sum of the degrees of all vertices must equal two times the total number of edges or $2x$, $x \in \mathbb{Z}$. Knowing this, we know that the sum of ~~the~~ the degrees of all vertices must be even.

Additionally, we know that a vertex must only have either an odd number of vertices or an even number of vertices. To clarify, all vertices ~~have~~ have either an odd or even degree, and no vertice has both an odd and even degree.

Since we know that all vertices are either even ~~degree~~ or odd degree, then we know that the sum of all vertices with even degree ~~~~ plus the sum of ~~a~~ all vertices with odd degree is equal ~~to~~ the sum of all vertices, which we stated was even.

In other words:

$$\sum_{v \in \text{even degree vertices}} \deg(v) + \sum_{v \in \text{odd degree vertices}} \deg(v) = \sum_{v \in \text{all vertices}} \deg(v) = 2x,$$

We know that the sum of all even degrees must be even because an even number plus an even number sums to an even number, or $2p$, $p \in \mathbb{Z}$. Thus we can rewrite our equation such that:

$$2p + \sum_{v \in \text{odd degree vertices}} \deg(v) = 2m$$

$$\sum_{v \in \text{odd degree vertices}} \deg(v) = 2(m - p)$$

, where the sum of all odd degree vertices is even.

The only way the sum of all odd degree vertices can be even, is if there is ~~~~ an even number of odd degree vertices because ~~an odd~~ the sum of an odd number of odd numbers will be odd and the sum of an even number of odd numbers will be even. Thus, because we know the sum of all even vertices is even and the sum of all odd vertices is even, then the total number of vertices will be even.

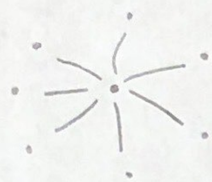
This proves that there is no connected graph with 37 vertices (odd number) in which every vertex has a degree of 3, 5, and 11 (odd), since a connected graph with only vertices of odd degrees must have an even number of total vertices. □

Q4)

a)

b)

c)



d)

e)

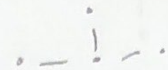
Q5) We can use graph coloring by having the chromatic number of the graph be the fewest class time periods that the college will need. To ensure there are no schedule conflicts, each vertex or course with the same color can not be connected, which represents that any courses / vertices of the same color must be scheduled at the same time and can not have any students in common. By using each color to represent a different time slot, we can create a colored graph with no schedule conflicts in the fewest time periods possible.

Q6) We can look at these three cases since $n<4$ will result in no such trees with diameter $n-2$

Case 1) $n = 4$

$n=4$: Only 1 4-vertex tree exists with a diameter of 2 exists

it looks like the following:



where all edges are connected to one vertex so one vertex has degree 3 and 3 vertices have degree 1.

When n is greater than 4, we can use case 2 and 3.

Case 2: n is odd and ≥ 5

$\boxed{\frac{(n-3)}{2} \text{ trees}}$ exist with n-vertex trees that have a diameter of $n-2$

Description:

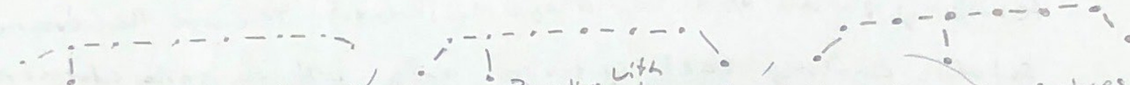
$n=5$:

$= 1$ tree, $\left(\frac{5-3}{2}\right) = 1$

$n=7$:

$=$ 2 trees, $\left(\frac{7-3}{2}\right) = 2$

$n=9$:



$= 3$ trees, $\left(\frac{9-3}{2}\right)$

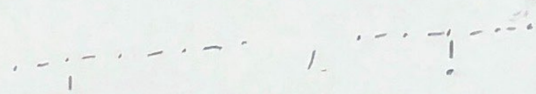
Each distinct tree will have one vertex with degree 3, 3 vertices with degree 1, and the rest with degree 2. The vertex of degree 3 for each distinct tree will be between (but not include) the two vertices that produce the diameter, up to isomorphism.

Case 3: n is even and ≥ 6

$\boxed{\frac{(n-2)}{2} \text{ trees}}$ exist with n-vertex trees that have a diameter of $n-2$

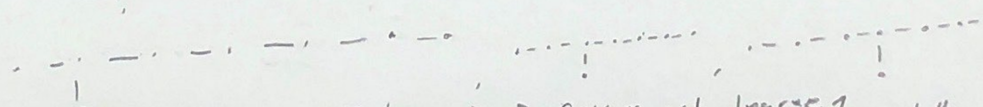
Description:

$n = 6$



$=$ 2 trees, $\left(\frac{6-2}{2}\right) = 2$

$n = 8$



$=$ 3 trees, $\left(\frac{8-2}{2}\right) = 3$

Each distinct tree will have one vertex with degree 3, 3 vertices with degree 1, and the rest with degree 2. The vertex of degree 3 for each distinct tree will be between (but not include) the two vertices that produce the diameter, up to isomorphism.

As a function we can write

$$\boxed{f(n) = \begin{cases} \frac{(n-2)}{2} & \text{when } n \text{ is even} \\ \frac{(n-3)}{2} & \text{when } n \text{ is odd} \end{cases}, \text{ when } n \geq 4}$$