Week 2 Homework

1) Let a palindrome number represent an integer $p \in \mathbb{Z}$ such that p contains $2a$ digits for some natural number $a \in \mathbb{N}$, p can further be represented by each corresponding individual digit with $x_a \cdot 10^a$ where $x_a \in \mathbb{Z}$, $0 \leq x_a \leq 9$.

This means p can be expanded into the following:

$$x_1 + x_2 \cdot 10 + x_3 \cdot 10^2 + \ldots + x_a \cdot 10^a + x_a \cdot 10^{a+1} + \ldots + x_2 \cdot 10^{2a} + x_1 \cdot 10^{2a+1}$$

To simply p further, we can note that $10^n \equiv -1 \bmod 11$ for non negative integers n since we want to prove that p is divisible by 11. Using this information, we can substitute -1 for 10 in the following manner:

$$x_1 + x_2 \cdot (-1) + x_3 (-1)^2 + \ldots + x_a (-1)^a + x_a (-1)^{a+1} + \ldots + x_2 (-1)^{2a} + x_1 (-1)^{2a+1}$$

This can be rewritten as:

$$x_1 - x_2 + x_3 + \ldots + x_n - x_n + \ldots + x_3 + x_2 - x_1 = 0$$

Thus we can conclude that since $p \equiv 0 \pmod{11}$, then p will be divisible by 11 when p contains an even number of digits. $\square$

2) Scratch: $a = qb + r$ $\quad \gcd(a,b) = \gcd(b,r)$ $\quad 10 \quad 30,10$

$a = 10 \quad b = 30$

$10 = 30 \cdot q + r \quad q=0 \quad r=10$

Step 1: Proof: Let $d = \gcd(a,b)$ where $a,b,d \in \mathbb{Z}$. By the definition of greatest common divisors, the greatest common divisor of a and b is the largest integer d such that $d \mid a$ and $d \mid b$. Since we're given $a = qb + r$ by the Division Algorithm from the lemma, we can substitute $qb+r$ for a such that $d \mid qb + r$. We know that $d \mid qb$ is true due to the division definition since $d \mid b$ is true and q is an integer multiplied to b. Thus, because we know $d \mid qb+r$ must be true and $d \mid qb$ is also true, we can conclude that $d \mid r$ must also be true in order to match the validity of $d \mid qb + r$.

Step 2: Proof:
Let $e = \gcd(b,r)$ where $e,b,r \in \mathbb{Z}$. By the definition of greatest common divisors, the greatest common divisor of b and r is the largest integer e such that $e \mid b$ and $e \mid r$. Using logic that is similar to step 1 of proving the lemma true, we can substitute the given $a = qb + r$ equation, which can be rearranged to $r = a - qb$, such that $e \mid a - qb$. We know $e \mid -(qb)$ by the division definition since $e \mid b$ is true and q is an integer multiplied to b. Thus, because we know $e \mid a - qb$ must be true and $e \mid -qb$ is also true, we can conclude that $e \mid a$ must be true.

Step 3: Proof: Since we know $d \mid a$, $d \mid b$, $d \mid r$, $e \mid a$, $e \mid b$, and $e \mid r$ when we look at $\gcd(a,b)$ and $\gcd(b,r)$, then d must be equal to e. Thus because $d = \gcd(a,b)$, $e = \gcd(b,r)$, and $d = e$, then $\gcd(a,b) = \gcd(b,r)$, which completes the proof. □

□

3) Proof: Let $|A| = c + m$ and $|B| = d + m$ where $c \in \mathbb{Z}, c \geq 0$ represents the number of elements that are in set A but not in set B, $d \in \mathbb{Z}, d \geq 0$ is the number of elements that are in set B but not in set A, and $m \in \mathbb{Z}, m \geq 0$ is the number of elements that are in both set A and set B.

This means that $|A \cap B| = m$ and $|A \cup B| = m + d + c$ by definition. Thus, we can use this information to express and simplify the following:

$$|A \cap B| + |A \cup B| = m + m + d + c = c + m + d + m$$
$$|A \cap B| + |A \cup B| = |A| + |B|$$

Thus, we have shown that $|A \cup B| + |A \cap B| = |A| + |B|$ for all sets A and B. $\square$

4) Proof: Assume A, B and C are subsets of U. We will prove that $A - (B \cup C) \subseteq (A-B) \cap (A-C)$ and $(A-B) \cap (A-C) \subseteq A - (B \cup C)$. Together, this will prove that $A - (B \cup C) = (A-B) \cap (A-C)$ for all sets A, B, and C. Firstly, we need to prove $A - (B \cup C) \subseteq (A-B) \cap (A-C)$. To this end, we can assume $x \in A - (B \cup C)$.

$A - (B \cup C) = A \cap (B \cup C)^c \rightarrow$ definition of subtraction

$A \cap (B \cup C)^c = A \cap (B^c \cap C^c)$ DeMorgan's Law $\quad A \quad C \quad A^c \cap B^c$

$A \cap (B^c \cap C^c) = (A \cap B^c) \cap (A \cap C^c)$ Distributive Property

$(A \cap B^c) \cap (A \cap C^c) =$ Def of Subtraction

$(A-B) \cap (A-C)$

4] Proposition : $A-(B\cup C) = (A-B)\cap(A-C)$ for all sets A, B, and C

Proof : Assume A, B, and C are subsets of U and all complements are taken inside U. We can show that each side will be equal to the other side using different definitions. First we can show with the left hand side:

$$A-(B\cup C) = A\cap(B\cup C)^c \quad \text{Definition of Subtraction}$$
$$A\cap(B\cup C)^c = A\cap(B^c\cap C^c) \quad \text{DeMorgan's Law}$$
$$A\cap(B^c\cap C^c) = (A\cap B^c)\cap(A\cap C^c) \quad \text{Distributive Property}$$
$$(A\cap B^c)\cap(A\cap C^c) = (A-B)\cap(A-C) \quad \text{Definition of Subtraction}$$

Thus we have shown that $A-(B\cup C) = (A-B)\cap(A-C)$ for all sets A, B, and C starting with the left hand side. We can also show with the right hand side the same logic but reversed:

$$(A-B)\cap(A-C) = (A\cap B^c)\cap(A\cap C^c) \quad \text{Definition of Subtraction}$$
$$(A\cap B^c)\cap(A\cap C^c) = A\cap(B^c\cap C^c) \quad \text{Distributive Property}$$
$$A\cap(B^c\cap C^c) = A\cap(B\cup C)^c \quad \text{De Morgan's Law}$$
$$A\cap(B\cup C)^c = A-(B\cup C) \quad \text{Definition of Subtraction}$$

Thus, we have also shown that $A-(B\cup C) = (A-B)\cap(A-C)$ for all sets A, B, and C with the right hand side. Since both sides come to the same conclusion, then this concludes the proof that $A-(B\cup C) = (A-B)\cap(A-C)$ for all sets A, B, and C. $\square$

5) 1) $A \cap B = \{\{1,2\}\}$ $|A \cap B| = 1$

2) $B - A = \{\phi, \{1,2,\phi\}\}$ $|B-A| = 2$

3) A^2 or $A \times A = \{(1,1), (1,\{1,2\}), (\{1,2\},1), (\{1,2\},\{1,2\})\}$

$|A^2| = 4$

4) $\{(a,b) \in A \times B : a \in b\} = \{(1, \{1,2\}), (1, \{1,2,\phi\}), (\{1,2\},\{1,2\})\}$

cardinality $= 3$

5) $\{b \in B : |b| = 2\} = \{\{1,2\}\}$

cardinality $= 1$

6) $\{\{a\} : a \in A\} = \{\{1,2\}, \{1\}\}$ cardinality $= 2$

7) $P(B) = \{\{\phi\}, \{\{1,2\}\}, \{\{1,2,\phi\}\}, \{\phi, \{1,2\}\}, \{\phi, \{1,2,\phi\}\},$
$\{\{1,2\}, \{1,2,\phi\}\}, \{\phi, \{1,2\}, \{1,2,\phi\}\}, \phi\}$ cardinality $= 8$

8) $P(A) - B$ $P(A) = \{\phi, \{1\} \{1,2\}, \{1, \{1,2\}\}\}$

$P(A) - B = \{\{1\}, \{1, \{1,2\}\}\}$

$|P(A) - B| = 2$

9) $\bigcup_{b \in B} b$ → union of all members of B

$\bigcup_{b \in B} b = \{1, 2, \phi\}$ $\left|\bigcup_{b \in B} b\right| = 3$

10) $\bigcap_{b \in B} b = \{\}$ or ϕ $\left|\bigcap_{b \in B} b\right| = 0$