CS230 - Week 9-10 Assignment Tony Cai

Q1] $\boxed{3! \cdot 6! \cdot 4! \cdot 3! = 622080}$

→ 3 subjects exist, and we can rearrange the 3 subjects in a maximum of 3 ways

3 potential objects for first slot * 2 subjects for 2nd slot * 1 subject for 1st slot

→ There are 6! ways to arrange the math books, 4! ways to arrange physics books, and 3! ways to arrange chemistry books.

Q2] a) all 4 cards are red: $\binom{26}{4} = \frac{26!}{22! \cdot 4!} = 14950$ (26 red cards, 4 cards chosen)

all 4 cards are of same suit: $4({}_{13}C_4) = 4\left(\frac{13!}{4!(9!)}\right) = 2860$

4 different suits; 13 cards for each suit, you choose 4

$14950 + 2860 - 2({}_{13}C_4)$ → $2\left(\frac{13!}{4!(9!)}\right) = 1430$

$|A| + |B| - |A \cap B|$

there are 2 different suits that are red

$14950 + 2860 - 1430 = \boxed{16380 \text{ hands}}$

b) all 4 cards are red: $\binom{26}{4} = \frac{26!}{22! \cdot 4!} = 14950$ like in 2a)

all 4 cards are from a diff. suit: $13^4 =$

13 cards from each suit, 4 suits $(13 \cdot 13 \cdot 13 \cdot 13)$

hearts, diamonds, clubs, spades

$14950 + 28561 - 0 = \boxed{43511 \text{ hands}}$

$|A| + |B| - |A \cap B| = |A \cup B|$

→ there exists no hand that is all red and ~~of each suit~~ all different suits

Q3] The following statement is <u>false.</u> This is because $A \cap B \cap C$ does <u>not</u> mean all sets are disjoint from each other, but rather one set is disjoint from either the two other sets combined or both sets individually. For example: Let $x \in A$, $x \in B$, $x \notin C$

$x \notin A \cap B \cap C$ but,

$x \in A \cap B$

If this is true, then $|A \cup B \cup C|$ does not always equal $|A| + |B| + |C|$. This is because $|A \cup B \cup C|$ does not take into account the fact that $|A|, |B|, |C|$ could have an intersection of elements, which means $|A| + |B| + |C|$ could lead to double counted elements.

If $A \cap B \cap C = \emptyset$, then $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C|$

would be a better conditional statement.

Q4]

1000 - 9999 :

$9 * 9 * 8 * 7 = \boxed{4536}$

digits 1-9 we can choose from

we have 10 digits to choose from (0 - 9), minus 1 because we can't repeat

8 digits left we can choose

7 digits left we can choose without repeating

1 - 10 = 9 digits

10 - 99 = 9 * 9 = 81 digits

100 - 999 = 9 * 9 * 8 = 648 digits

1-10 digits	10-99 digits	100-999 digits	1000-9999 digits
9	+ 81	+ 648	+ 4536

$9 + 81 + 648 + 4536 = \boxed{5274 \text{ digits}}$

Q5]

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

a) $$(3x-2y)^9 = \sum_{k=0}^{9} \binom{9}{k} (-2y)^k (3x)^{9-k} = \sum_{k=0}^{9} \binom{9}{k} (-2)^k \cdot y^k \cdot 3^{9-k} \cdot x^{9-k}$$

$x = -2y$
$y = 3x$

compare to $x^6 y^3$,

$k = 3$

$$\sum_{k=0}^{9} \binom{9}{3} (-2)^3 \cdot y^3 \cdot 3^6 \cdot x^6$$

$$= {}_9C_3 \cdot -8 \cdot 3^6 \cdot x^6 \cdot y^3$$

$$= 84 \cdot -8 \cdot 3^6 \cdot x^6 \cdot y^3$$

$$= \boxed{-489888}\, x^6 y^3$$

b)

proof] From our equation $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \dots (-1)^n \binom{n}{n} = 0$, we can set the variables x,y such that x = -1 and y = 1. Plugging these into the binomial theorem, we would get:

$$(-1+1)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k 1^{n-k}$$

Simplified and expanded form $\longrightarrow 0^n = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \dots (-1)^n \binom{n}{n}$

$$0^n = 0$$

Thus we have proved with the binomial theorem that the equation is true. $\square$

Q6) a) $w + x + y + z = 100$

100 stars/objects, 4 variables/sections, 3 dividing lines/bars

+4 since the numbers can be 0

$104 - 1 = 103$

$\binom{103}{3}$

So we can put the 3 bars in a total of 103 places

$$\frac{103!}{3!\,(100)!} = \boxed{176851}$$

b) $0 < w < x < y < z$. We can set this inequality equal to 100 such that $w + x + y + z = 100$. We can also let $a_1 = w$, $a_2 = x - 1$, $a_3 = y - 2$, $a_4 = z - 3$

we would get: $a_1 + a_2 + a_3 + a_4 = w + x + y + z - 6 = 94$ and

our inequality would be $1 \leq a_1 \leq a_2 \leq a_3 \leq a_4$.

Using what we simplified, we would have $\boxed{P_4(94) \text{ solutions}}$ since the new problem is a standard partition and isomorphic to 6 a).

7) 1) $\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n$

The right side of the equation, 3^n, represents the different number of ways of painting n ordered objects one of three colors: RED, BLUE, or GREEN.

The left side of the equation, $\sum_{k=0}^{n} 2^k \binom{n}{k}$, can represent the k objects we chose that are NOT RED, in which there are $\binom{n}{k}$ ways to do this. This means that there would be 2^k ways to color the k objects either BLUE or GREEN since RED is no longer an option. The remaining $n-k$ objects we would have left after k objects have been colored BLUE or GREEN, must therefore be RED. This means that the total number of NOT RED objects can be $0 \leq k \leq n$, so the LHS is $\sum_{k=0}^{n} 2^k \binom{n}{k}$. Thus, we have shown that the LHS and RHS both count the same set and are equal. $\square$

7) a) $\binom{n}{2}\binom{n-2}{k-2} = \binom{n}{k}\binom{k}{2}$ algebraic proof: $\binom{n}{m}\binom{n-m}{k-m} = \binom{n}{k}\binom{k}{m}$

looks like $m=2$

expanded: $\frac{n!}{k!(n-k)!}\frac{k!}{m!(k-m)!} = \frac{n!}{m!(n-m)!}\frac{(n-m)!}{(k-m)!(n-k)!}$

$$\frac{n!}{m!(k-m)!(n-k)!} = \frac{n!}{m!(k-m)!(n-k)!}$$

Written Proof:

We can prove that the LHS and RHS of the combinatorial identity are equal to the same set if we first consider a situation where a group of size n people need to to choose a committee of size k people as well as 2 coleaders who are included members of the committee. We can show that the LHS and RHS both represent the number of ways that the committee and coleaders can be chosen. With the $\boxed{\text{RHS,}}$ $\binom{n}{k}\binom{k}{2}$, the k Committee members are first chosen from the n group of people, $\binom{n}{k}$, and then the 2 coleaders are chosen from the k commitee members, $\binom{k}{2}$

On the $\boxed{\text{LHS}}$ $\binom{n}{2}\binom{n-2}{k-2}$, the 2 coleaders are originally chosen from the group of n people $\binom{n}{2}$. Since the 2 coleaders are part of the committee, then we can choose the remaining members for the committee or $k-2$ members out of the $n-2$ group of people. $\binom{n-2}{k-2}$. Thus, we can observe that the LHS and RHS both count the different number of ways that the committee and coleaders can be chosen since they count the same set and are equal. □