Solving Parabolic PDE

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1 Introduction

Original Proposed equations: (McGarry et.al, 2016)

$$\frac{\partial A}{\partial t} + \frac{\partial}{\partial x}(Au) = 0 \tag{1}$$

$$\frac{\partial u}{\partial t} + \frac{\partial P}{\partial x} + k_R u = 0 \tag{2}$$

$$A = A_0 + k_p P \tag{3}$$

I propose a new system of pdes to better illustrate the model
Propose Hagen–Poiseuille equation to develop a new mathematical model.

$$Q = Au \tag{4}$$

Conservation of Mass

$$\frac{\partial A}{\partial t} + \frac{\partial}{\partial x} Q = 0 \tag{5}$$

Conservation of Momentum Fluid

$$Q = Au = -\frac{1}{8\pi\mu} A^2 \frac{\partial P}{\partial x} = -k_0 A^2 \frac{\partial P}{\partial x} \tag{6}$$

$$k_0 = \frac{1}{8\pi\mu} \tag{7}$$

Elastic statics thin-walled elastic cylinder stress linear to independence
ε_n is strain (dimensionless), σ_n is Hooke stress

$$E\varepsilon_n = \sigma_H \tag{8}$$

$$\varepsilon_n \sim \frac{\Delta r}{r} \tag{9}$$

$$\sigma_H = \frac{(P - P_0)r}{l} \tag{10}$$

l is thickness, r is the tube's radius

make substitution

$$\frac{\Delta r}{r} = \frac{(P - P_0)r}{lE} \tag{11}$$

Because change of radius is proportional to change of Area

$$\frac{\Delta r}{r} \sim \frac{(A - A_0)}{A} \tag{12}$$

$$\frac{dA}{A} = \frac{d\pi r^2}{\pi r^2} = \frac{2\pi r}{\pi r^2} = \frac{2}{r}dr \tag{13}$$

Substitution

$$\tag{14}$$

$$\frac{El}{2r}\frac{(A - A_0)}{A_0} = P - P_0 \tag{15}$$

$$E^* = \frac{El}{2r} \tag{16}$$

$$E^* \left(\frac{A - A_0}{A_0}\right) = (P - P_0) \tag{17}$$

$$P = P_0 + E^* \frac{(A - A_0)}{A_0} \tag{18}$$

$$\frac{\partial P}{\partial x} = \frac{\partial P_0}{\partial x} + \frac{E^*}{A_0}\frac{\partial A}{\partial x} \tag{19}$$

Substitution to equation 1

$$\frac{dA}{dt} + \frac{\partial}{\partial x} - k_0 A^2 \left(\frac{\partial P_0}{\partial x} + \frac{E^*}{A_0}\frac{\partial A}{\partial x}\right) \tag{20}$$

$$\frac{\partial A}{\partial t} + \frac{\partial}{\partial x}(-k_0 A^2 \frac{\partial P}{\partial x}) - \frac{\partial}{\partial x}(\frac{k_0 E^*}{A_0} A^2 \frac{\partial A}{\partial x}) = 0 \tag{21}$$

$$Q_0 = A_0 u_0 = -k_0 A_0^2 \frac{\partial P_0}{\partial x} \tag{22}$$

$$u_0 = -k_0 A_0 \frac{\partial P_0}{\partial x} \tag{23}$$

$$\frac{\partial A}{\partial t} + \frac{\partial}{\partial x}\frac{A^2}{A_0}u_0 - \frac{\partial}{\partial x}\frac{K_0 E^*}{A_0} A^2 \frac{\partial A}{\partial x} = 0 \tag{24}$$

Nondimensionalization

$$A = A_0 A' \tag{25}$$

$$x = Lx' \tag{26}$$

$$t = t_0 t' = (\frac{L}{u_0})t' \tag{27}$$

$$E^* = E_0 E^{*\prime} \tag{28}$$

A': dimensionless scaler factor x': (0,1) t': (0,1)

$$\frac{\partial A}{\partial t} = \left(\frac{A_0}{\frac{L}{u_0}}\right)\frac{\partial A'}{\partial t'} = \left(\frac{A_0 u_0}{L}\right)\frac{\partial A'}{\partial t'} \tag{29}$$

$$\frac{\partial}{\partial x}\left(\frac{A^2}{A_0}u_0\right) = \frac{\partial\left(\frac{A_0^2 \cdot {A'}^2}{A_0}u_0\right)}{L\partial(x')} = \frac{\partial(A_0 {A'}^2 u_0)}{L\partial x'} = \frac{A_0 u_0}{L}(\frac{\partial {A'}^2}{\partial x'}) \tag{30}$$

$$\frac{\partial}{\partial x}\frac{K_0 E^*}{A_0}A^2\frac{\partial A}{\partial x} = \frac{K_0}{A_0}\frac{\partial}{\partial x}(E^*(A_0)^2 {A'}^2\frac{A_0}{L}\frac{\partial A'}{\partial x\prime}) = \frac{K_0 E_0}{A_0 L}\frac{\partial}{\partial x\prime}(E^{*\prime}\frac{(A_0)^3}{L}{A'}^2\frac{\partial A'}{\partial x\prime}) \tag{31}$$

simplify

$$\frac{K_0 E_0}{A_0 L}\frac{\partial}{\partial x\prime}(E^{*\prime}\frac{(A_0)^3}{L}{A'}^2\frac{\partial A'}{\partial x\prime}) = \frac{A_0^2 K_0 E_0}{L^2}\frac{\partial}{\partial x\prime}(E^{*\prime}{A'}^2\frac{\partial A'}{\partial x\prime}) \tag{32}$$

Combine all the terms:

$$\left(\frac{A_0 u_0}{L}\right)\frac{\partial A'}{\partial t'} + \frac{A_0 u_0}{L}(\frac{\partial {A'}^2}{\partial x'}) + \frac{A_0^2 K_0 E_0}{L^2}\frac{\partial}{\partial x\prime}(E^{*\prime}{A'}^2\frac{\partial A'}{\partial x\prime}) = 0 \tag{33}$$

divide $\frac{A_0 u_0}{L}$ and get rid of primes

$$\frac{\partial A}{\partial t} + \frac{\partial}{\partial x}A^2 - \frac{k_0 E_0 A_0}{L u_0}\frac{\partial}{\partial x}\left(E^* A^2\frac{\partial A}{\partial x}\right) = 0 \tag{34}$$

$$\frac{k_0 E_0 A_0}{L u_0} = \alpha \tag{35}$$

Therefore, the parabolic equations are:

$$\frac{\partial A}{\partial t} + \frac{\partial}{\partial x}\left(A^2\right) - \alpha\frac{\partial}{\partial x}\left(E^* A^2\frac{\partial A}{\partial x}\right) = 0 \tag{36}$$