Derivation of the Linearized Model for Pulse Wave Propagation

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Linearized Model

The linearized model for pulse wave propagation in a flexible tube is given by the following equations:

$$\frac{\partial A}{\partial t} + A_0 \frac{\partial u}{\partial x} = 0 \quad \text{(Mass Conservation Equation)} \tag{1}$$

$$\frac{\partial u}{\partial t} + \frac{1}{\rho} \frac{\partial P}{\partial x} + K_R u = 0 \quad \text{(Momentum Conservation Equation)} \tag{2}$$

Additionally, the cross-sectional area A is related to the pressure P by:

$$A = A_0 + k_p P \tag{3}$$

where:

- A_0 is the vessel area at a reference pressure,
- k_p is the vessel compliance.

Step 1: Substitute $A = A_0 + k_p P$ into Equation (1)

Substitute $A = A_0 + k_p P$ into the continuity equation (1):

$$\frac{\partial}{\partial t}(A_0 + k_p P) + A_0 \frac{\partial u}{\partial x} = 0 \tag{4}$$

Since A_0 is a constant, its time derivative is zero:

$$k_p \frac{\partial P}{\partial t} + A_0 \frac{\partial u}{\partial x} = 0 \tag{5}$$

This is the **linearized continuity equation**.

Step 2: Solve the Linearized Continuity Equation for $\frac{\partial u}{\partial x}$

From the linearized continuity equation (5):

$$k_p \frac{\partial P}{\partial t} + A_0 \frac{\partial u}{\partial x} = 0 \tag{6}$$

Solve for $\frac{\partial u}{\partial x}$:

$$\frac{\partial u}{\partial x} = -\frac{k_p}{A_0} \frac{\partial P}{\partial t} \tag{7}$$

Step 3: Substitute $\frac{\partial u}{\partial x}$ into the Momentum Equation (2)

The momentum equation (2) is:

$$\frac{\partial u}{\partial t} + \frac{1}{\rho}\frac{\partial P}{\partial x} + K_R u = 0 \tag{8}$$

We already have $\frac{\partial u}{\partial x} = -\frac{k_p}{A_0}\frac{\partial P}{\partial t}$. To solve the system, we differentiate the momentum equation with respect to x:

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\right) + \frac{1}{\rho}\frac{\partial^2 P}{\partial x^2} + K_R\frac{\partial u}{\partial x} = 0 \tag{9}$$

Substitute $\frac{\partial u}{\partial x} = -\frac{k_p}{A_0}\frac{\partial P}{\partial t}$:

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\right) + \frac{1}{\rho}\frac{\partial^2 P}{\partial x^2} - K_R\frac{k_p}{A_0}\frac{\partial P}{\partial t} = 0 \tag{10}$$

Step 4: Express $\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\right)$ in Terms of P

From the continuity equation, we have:

$$\frac{\partial u}{\partial x} = -\frac{k_p}{A_0}\frac{\partial P}{\partial t} \tag{11}$$

Differentiate both sides with respect to time t:

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right) = -\frac{k_p}{A_0}\frac{\partial^2 P}{\partial t^2} \tag{12}$$

Now, interchange the order of differentiation (assuming smoothness):

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\right) = -\frac{k_p}{A_0}\frac{\partial^2 P}{\partial t^2} \tag{13}$$

Step 5: Substitute Back into the Momentum Equation

Substitute (13) into the momentum equation (10):

$$-\frac{k_p}{A_0}\frac{\partial^2 P}{\partial t^2} + \frac{1}{\rho}\frac{\partial^2 P}{\partial x^2} - K_R\frac{k_p}{A_0}\frac{\partial P}{\partial t} = 0 \tag{14}$$

Rearrange:

$$\frac{1}{\rho}\frac{\partial^2 P}{\partial x^2} = \frac{k_p}{A_0}\frac{\partial^2 P}{\partial t^2} + K_R\frac{k_p}{A_0}\frac{\partial P}{\partial t} \tag{15}$$

Step 6: Final Wave Equation for Pressure P

The final equation is a **damped wave equation** for the pressure P:

$$\frac{\partial^2 P}{\partial x^2} = \frac{k_p\rho}{A_0}\frac{\partial^2 P}{\partial t^2} + K_R\frac{k_p\rho}{A_0}\frac{\partial P}{\partial t} \tag{16}$$

where:

- $c^2 = \frac{k_p\rho}{A_0}$ is the wave speed squared,
- $\alpha = K_R\frac{k_p\rho}{A_0}$ is the damping coefficient.

This equation describes the propagation of pressure waves in the vessel, with damping due to viscous resistance.

Step 7: General Solution of the Damped Wave Equation

The general solution to the damped wave equation can be found using separation of variables. Assume a solution of the form:

$$P(x,t) = X(x)T(t) \tag{17}$$

Substitute into the wave equation (16):

$$X''(x)T(t) = c^2 X(x)T''(t) + \alpha X(x)T'(t) \tag{18}$$

Divide through by $X(x)T(t)$:

$$\frac{X''(x)}{X(x)} = c^2 \frac{T''(t)}{T(t)} + \alpha \frac{T'(t)}{T(t)} \tag{19}$$

Since the left-hand side depends only on x and the right-hand side depends only on t, both sides must equal a constant, say $-k^2$:

$$\frac{X''(x)}{X(x)} = -k^2 \quad \text{and} \quad c^2 \frac{T''(t)}{T(t)} + \alpha \frac{T'(t)}{T(t)} = -k^2 \tag{20}$$

Step 8: Solve the Spatial Part $X(x)$

The spatial equation is:

$$X''(x) + k^2 X(x) = 0 \tag{21}$$

The general solution is:

$$X(x) = A\cos(kx) + B\sin(kx) \tag{22}$$

where A and B are constants determined by the boundary conditions.

Step 9: Solve the Temporal Part $T(t)$

The temporal equation is:

$$c^2 T''(t) + \alpha T'(t) + k^2 T(t) = 0 \tag{23}$$

This is a second-order linear ordinary differential equation (ODE). The characteristic equation is:

$$c^2 r^2 + \alpha r + k^2 = 0 \tag{24}$$

The roots of the characteristic equation are:

$$r = \frac{-\alpha \pm \sqrt{\alpha^2 - 4c^2k^2}}{2c^2} \tag{25}$$

The nature of the roots depends on the discriminant $\Delta = \alpha^2 - 4c^2k^2$:

- **Overdamped Case** ($\Delta > 0$):
$$T(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t} \tag{26}$$

- **Critically Damped Case** ($\Delta = 0$):
$$T(t) = (C_1 + C_2 t)e^{rt} \tag{27}$$

- **Underdamped Case** ($\Delta < 0$):
$$T(t) = e^{-\frac{\alpha}{2c^2}t}\left(C_1\cos(\omega t) + C_2\sin(\omega t)\right) \tag{28}$$

  where $\omega = \frac{\sqrt{4c^2k^2 - \alpha^2}}{2c^2}$.

Step 10: Apply Initial Conditions

The initial conditions are:

- $P(x, 0) = P_0(x)$ (initial pressure distribution),
- $u(x, 0) = 0$ (initial velocity is zero).

From the linearized continuity equation:

$$\frac{\partial P}{\partial t} = -\frac{A_0}{k_p}\frac{\partial u}{\partial x} \tag{29}$$

At $t = 0$, since $u(x, 0) = 0$, its spatial derivative is also zero:

$$\frac{\partial u}{\partial x}(x, 0) = 0 \tag{30}$$

Substitute this into the equation for $\frac{\partial P}{\partial t}$:

$$\frac{\partial P}{\partial t}(x, 0) = -\frac{A_0}{k_p}\frac{\partial u}{\partial x}(x, 0) = 0 \tag{31}$$

Thus, the initial condition for $\frac{\partial P}{\partial t}$ is:

$$\frac{\partial P}{\partial t}(x, 0) = 0 \tag{32}$$

Step 11: Determine Fourier Coefficients A_n and B_n

To determine the coefficients A_n and B_n in the general solution:

$$P(x, t) = \sum_{n=1}^{\infty} \left(A_n \cos(k_n x) + B_n \sin(k_n x)\right) e^{-\frac{\alpha}{2c^2}t} \cos(\omega_n t), \tag{33}$$

we use the initial condition $P(x, 0) = P_0(x)$. At $t = 0$, the solution reduces to:

$$P(x, 0) = \sum_{n=1}^{\infty} \left(A_n \cos(k_n x) + B_n \sin(k_n x)\right). \tag{34}$$

This is a Fourier series expansion of $P_0(x)$. Using orthogonality, the coefficients A_n and B_n are given by:

$$A_n = \frac{2}{L}\int_0^L P_0(x)\cos(k_n x)\,dx, \quad B_n = \frac{2}{L}\int_0^L P_0(x)\sin(k_n x)\,dx, \tag{35}$$

where L is the length of the domain, and $k_n = \frac{n\pi}{L}$ for Dirichlet boundary conditions.

Step 12: Final Analytical Solution

Combining the spatial and temporal solutions, the general solution for $P(x, t)$ is:

$$P(x, t) = \sum_{n=1}^{\infty} \left(A_n \cos(k_n x) + B_n \sin(k_n x)\right) e^{-\frac{\alpha}{2c^2}t} \cos(\omega_n t) \tag{36}$$

where k_n are the wave numbers determined by the boundary conditions, and A_n, B_n are coefficients determined by the initial condition $P_0(x)$.

Step 13: Relationship Between $A(x,t)$ and $P(x,t)$

From the paper, the cross-sectional area $A(x,t)$ is related to the pressure $P(x,t)$ by:

$$A(x,t) = A_0 + k_p P(x,t) \tag{37}$$

where:

- A_0 is the vessel area at a reference pressure,
- k_p is the vessel compliance.

Thus, once $P(x,t)$ is known, $A(x,t)$ can be directly computed using this relationship.

Step 14: Relationship Between $u(x,t)$ and $P(x,t)$

From the linearized continuity equation:

$$k_p \frac{\partial P}{\partial t} + A_0 \frac{\partial u}{\partial x} = 0 \tag{38}$$

Solving for $\frac{\partial u}{\partial x}$:

$$\frac{\partial u}{\partial x} = -\frac{k_p}{A_0} \frac{\partial P}{\partial t} \tag{39}$$

Integrating with respect to x to find $u(x,t)$:

$$u(x,t) = -\frac{k_p}{A_0} \int_0^x \frac{\partial P}{\partial t} \, dx + u(0,t) \tag{40}$$

Here, $u(0,t)$ is the velocity at the inlet ($x = 0$), which is typically assumed to be zero (no flow at the inlet) unless otherwise specified. Thus, the velocity $u(x,t)$ is:

$$u(x,t) = -\frac{k_p}{A_0} \int_0^x \frac{\partial P}{\partial t} \, dx \tag{41}$$

Step 15: $P(x,t)$, $A(x,t)$, and $u(x,t)$

Assume the pressure $P(x,t)$ has been solved analytically or numerically. For example, if $P(x,t)$ is given by:

$$P(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos(k_n x) + B_n \sin(k_n x)\right) e^{-\frac{\alpha}{2c^2} t} \cos(\omega_n t) \tag{42}$$

then:

Cross-Sectional Area $A(x,t)$

Using the relationship $A(x,t) = A_0 + k_p P(x,t)$, substitute $P(x,t)$:

$$A(x,t) = A_0 + k_p \sum_{n=1}^{\infty} \left(A_n \cos(k_n x) + B_n \sin(k_n x)\right) e^{-\frac{\alpha}{2c^2} t} \cos(\omega_n t) \tag{43}$$

Fluid Velocity $u(x,t)$

First, compute $\frac{\partial P}{\partial t}$:

$$\frac{\partial P}{\partial t} = \sum_{n=1}^{\infty} (A_n \cos(k_n x) + B_n \sin(k_n x)) \left(-\frac{\alpha}{2c^2} e^{-\frac{\alpha}{2c^2}t} \cos(\omega_n t) - \omega_n e^{-\frac{\alpha}{2c^2}t} \sin(\omega_n t)\right) \tag{44}$$

Now, integrate $\frac{\partial P}{\partial t}$ with respect to x:

$$\int_0^x \frac{\partial P}{\partial t}\, dx = \sum_{n=1}^{\infty} \left(\frac{A_n \sin(k_n x)}{k_n} - \frac{B_n \cos(k_n x)}{k_n}\right) \left(-\frac{\alpha}{2c^2} e^{-\frac{\alpha}{2c^2}t} \cos(\omega_n t) - \omega_n e^{-\frac{\alpha}{2c^2}t} \sin(\omega_n t)\right) \tag{45}$$

Thus, the velocity $u(x,t)$ is:

$$u(x,t) = -\frac{k_p}{A_0} \sum_{n=1}^{\infty} \left(\frac{A_n \sin(k_n x)}{k_n} - \frac{B_n \cos(k_n x)}{k_n}\right) \left(-\frac{\alpha}{2c^2} e^{-\frac{\alpha}{2c^2}t} \cos(\omega_n t) - \omega_n e^{-\frac{\alpha}{2c^2}t} \sin(\omega_n t)\right) \tag{46}$$