COMS 4771 Lecture 2

1. Classification problems (review of some probability)
2. Classifiers via generative models
3. Evaluating classifiers

Classification problems

Terminology and notation

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- **Note**: possible to see both $(x, 1)$ and $(x, 2)$ for same input x.
  (Why is this realistic?)

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Statistical setting

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where $(X, Y) \sim P$.

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  $$\operatorname{err}(f) := \Pr[f(X) \neq Y].$$

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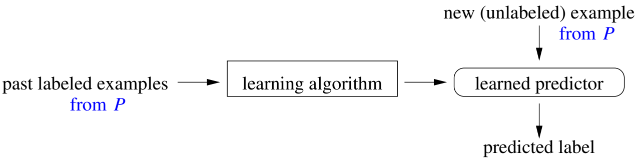
- **Key assumption**: Data $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$ are i.i.d. random labeled examples with distribution P—i.e., an i.i.d. sample from P.



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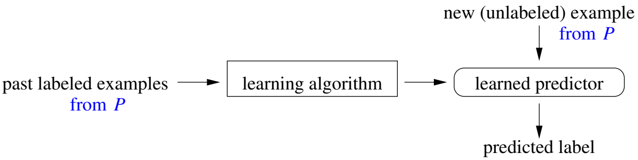

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What's next:

- What does an accurate classifier look like?
- How do we exploit the key assumption to construct an accurate classifier?

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- Therefore:

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The above quantity ($\ddagger$) is minimized (for this $x \in \mathcal{X}$) when

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The classifier f with property $(\star)$ for all $x \in \mathcal{X}$ is called the **Bayes classifier**, and it has the smallest prediction error $(\dagger)$ among *all classifiers*.

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$$f^\star(x) := \arg\max_{y \in \mathcal{Y}} \Pr\left[Y = y \,\middle|\, X = x\right]$$

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Question: What can these decision boundaries look like?

Structure of the Bayes classifier

By Bayes' rule:

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If X has a probability density (rather than a probability mass function), replace $\Pr[X = \cdot \mid Y = y]$ with **class conditional density** $p_y(\cdot)$.

Example: Gaussian class conditional densities

Suppose $\mathcal{X} = \mathbb{R}$, $\mathcal{Y} = \{0, 1\}$, and the distribution P of (X, Y) is as follows.

- **Class prior**:

$$\Pr[Y = y] = \pi_y, \quad y \in \{0, 1\}$$

  for some real numbers $\pi_0, \pi_1 \in [0, 1]$ satisfying $\pi_0 + \pi_1 = 1$.

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- **Class conditional density** for class $y \in \{0, 1\}$:

$$p_y(x) = \frac{1}{\sqrt{2\pi\sigma_y^2}} \exp\left(-\frac{(x - \mu_y)^2}{2\sigma_y^2}\right)$$

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- **Bayes classifier**:

$$\begin{aligned} f^\star(x) &= \operatorname*{arg\,max}_{y \in \{0,1\}} \Pr\left[Y = y \,\middle|\, X = x\right] \\ &= \begin{cases} 1 & \text{if } \dfrac{\pi_1}{\sigma_1} \exp\left(-\dfrac{(x-\mu_1)^2}{2\sigma_1^2}\right) > \dfrac{\pi_0}{\sigma_0} \exp\left(-\dfrac{(x-\mu_0)^2}{2\sigma_0^2}\right); \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

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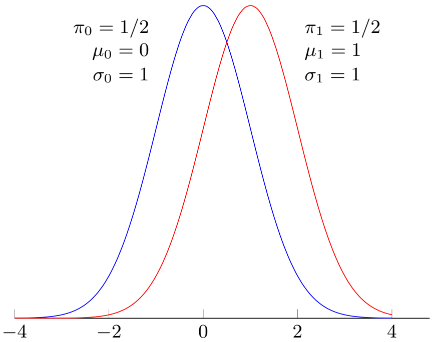

$1/2$ of x's from $\mathcal{N}(0, 1)$ (w/ $y = 0$)
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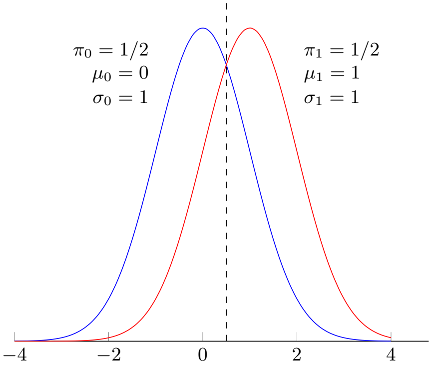

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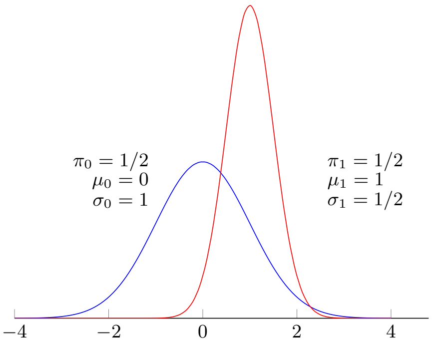

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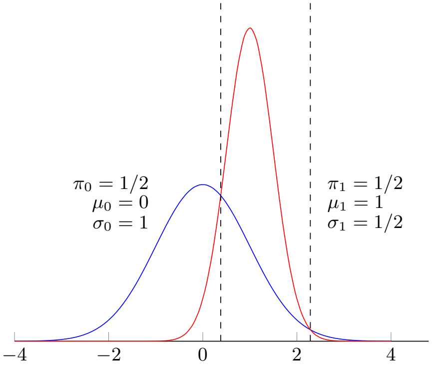

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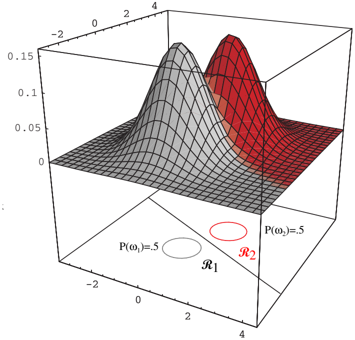
Bayes classifier:

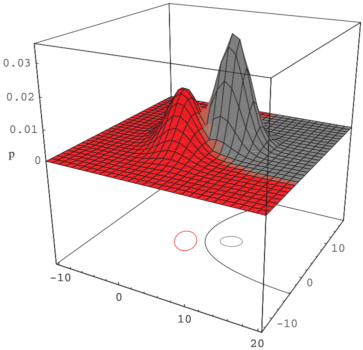
$$f^\star(x) = \begin{cases} 1 & \text{if } x \in [0.38, 2.29]; \\ 0 & \text{otherwise.} \end{cases}$$

$\mathcal{X} = \mathbb{R}^d$, $\mathcal{Y} = \{0, 1\}$, class conditional densities are Gaussians in $\mathbb{R}^d$ $(d = 2)$.



$\boldsymbol{\Sigma}_0 = \boldsymbol{\Sigma}_1$

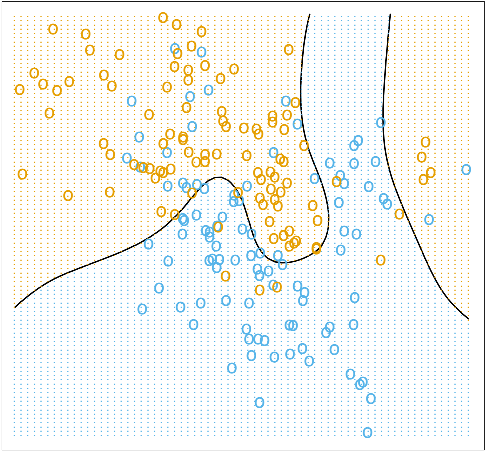
Bayes classifier:
linear separator



$\boldsymbol{\Sigma}_0 \neq \boldsymbol{\Sigma}_1$

Bayes classifier:
quadratic separator

Bayes classifier in general



In general, Bayes classifier may be rather complicated!

Classifiers via generative models

Plug-in classifiers

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We'll use **"generative" statistical models** to estimate P, and then form approximation to $\Pr[Y = y|X = x]$.

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- Estimating class priors is easy (e.g., Bernoulli MLE for binary classes).

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     (i.e., class priors and class conditional distributions).
  2. **Plug-in** approximations to formula to form classifier $\hat{f}$.
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- Estimating class conditional distributions hard in general.

Plug-in classifiers using generative models

- **Plug-in classifiers using "generative" statistical models**:
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  Usually just use *simple* parametric models.

Example: Gaussian class conditional densities

$\mathcal{X} = \mathbb{R}^d$, $\mathcal{Y} = \{1, 2, \ldots, K\}$

- **Class priors**: MLE estimate of π_y is

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Class conditional density $\mathcal{N}(\boldsymbol{\mu}_y, \boldsymbol{\Sigma}_y)$: MLE estimate of $(\boldsymbol{\mu}_y, \boldsymbol{\Sigma}_y)$ is

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Caveat: $\widehat{\boldsymbol{\Sigma}}_y$ could be singular!

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Next time: methods for modeling decision boundary directly.

Evaluating classifiers

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  This generally **under-estimates** the true error $\operatorname{err}(\hat{f})$.

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General methodology
Given your pile of labeled examples, (randomly) split into two disjoint groups:

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Assuming T is an i.i.d. sample from P:

- The test error is an unbiased estimate of $\mathrm{err}(\hat{f})$: $\mathbb{E}[\mathrm{err}(\hat{f}, T) \mid S] = \mathrm{err}(\hat{f})$.
  [Expectation is over random draw of T.]
- The standard deviation of $\mathrm{err}(\hat{f}, T)$ is $\sqrt{\dfrac{\mathrm{err}(\hat{f})(1 - \mathrm{err}(\hat{f}))}{|T|}}$.

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  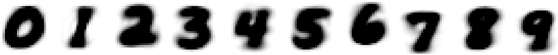

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  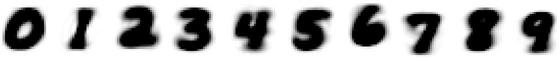

- **Training error**: $\mathrm{err}(\hat{f}, S) = 0.0346$

  Test error: $\mathrm{err}(\hat{f}, T) = 0.0415$
- **True error**? Unknown, but perhaps $0.039 \leq \mathrm{err}(\hat{f}) \leq 0.044$ or so.