COMS 4771 Machine Learning (Spring 2015)
Problem Set #2

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Problem 1

1.

$$
\begin{aligned}
P_{ML} &= \arg\max \prod_{i=1}^{n} P^{x_i}(1-P)^{1-x_i} \\
&= \arg\max \log \prod_{i=1}^{n} P^{x_i}(1-P)^{1-x_i} \\
&= \arg\max \sum_{i=1}^{n} \log P^{x_i}(1-P)^{1-x_i}
\end{aligned}
$$

To get the maximum value, the log-likelihood maximizer must satisfy

$$
\begin{aligned}
0 &= \sum_{i=1}^{n} \nabla_P \log P^{x_i}(1-P)^{1-x_i} \\
&= \sum_{i=1}^{n} \nabla_P \log P^{x_i}(1-P)^{1-x_i} \\
&= \sum_{i=1}^{n} (\frac{x_i}{P} + \frac{x_i - 1}{1-P}) \\
&= \frac{\sum_{i=1}^{n} x_i}{P} + \frac{\sum_{i=1}^{n} x_i - n}{1-P}
\end{aligned}
$$

Solve the equation, we could get the formula of the MLE of P.

$$P_{ML} = \frac{1}{n}\sum_{i=1}^{n} x_i \tag{1}$$

Bring P_{ML} into $\sum_{i=1}^{n} \nabla_P \log P^{x_i}(1-P)^{x_i}$, we have

$$\begin{aligned}\sum_{i=1}^{n} \nabla_P \log P^{x_i}(1-P)^{x_i} &= \sum_{i=1}^{n}(\frac{x_i}{P_{ML}} + \frac{x_i - 1}{1-P_{ML}}) \\ &= n + n(\frac{P_{ML}-1}{1-P_{ML}}) \\ &= 0\end{aligned}$$

2. The plug-in classifier based on the BNB model is

$$\underset{k\in 0,1}{arg}\{P_k \prod_{j=1}^{d} P_{j,k}^{x_j}(1-P_{j,k})^{1-x_j}\} = \underset{k\in 0,1}{arg}\{\log P_k \prod_{j=1}^{d} P_{j,k}^{x_j}(1-P_{j,k})^{1-x_j}\} \tag{2}$$

Since there were only 2 different newsgroups, we have the classifier:

$$f(x) = \begin{cases} 1 & \log P_1 \prod_{j=1}^{d} P_{j,1}^{x_j}(1-P_{j,1})^{1-x_j} > \log P_2 \prod_{j=1}^{d} P_{j,2}^{x_j}(1-P_{j,2})^{1-x_j} \\ 2 & other \end{cases}$$

Since we have

$$\begin{aligned}\log P_k \prod_{j=1}^{d} P_{j,k}^{x_j}(1-P_{j,k})^{1-x_j} &= \log P_k + \sum_{j=1}^{d} x_j \log P_{j,k} + (1-x_j)\log(1-P_{j,k}) \\ &= \log P_k + \sum_{j=1}^{d} x_j \log \frac{P_{j,k}}{1-P_{j,k}} + \log(1-P_{j,k})\end{aligned}$$

Thus,

$$\begin{aligned}&\log P_1 \prod_{j=1}^{d} P_{j,1}^{x_j}(1-P_{j,1})^{1-x_j} - \log P_2 \prod_{j=1}^{d} P_{j,2}^{x_j}(1-P_{j,2})^{1-x_j} \\ &= \log \frac{P_1}{P_2} + \sum_{j=1}^{d} x_j \log \frac{P_{j,1}(1-P_{j,2})}{P_{j,2}(1-P_{j,1})} + \log \frac{(1-P_{j,1})}{(1-P_{j,2})} \\ &= \log \frac{P_1}{P_2} + \sum_{j=1}^{d} \log \frac{(1-P_{j,1})}{(1-P_{j,2})} + \sum_{j=1}^{d} x_j \log \frac{P_{j,1}(1-P_{j,2})}{P_{j,2}(1-P_{j,1})}\end{aligned}$$

We have classifier $f^*(x)$

$$f^*(x) = \begin{cases} 1 & \log \frac{P_1}{P_2} + \sum_{j=1}^{d} \log \frac{(1-P_{j,1})}{(1-P_{j,2})} + \sum_{j=1}^{d} x_j \log \frac{P_{j,1}(1-P_{j,2})}{P_{j,2}(1-P_{j,1})} > 0 \\ 2 & other \end{cases}$$

Replace $\theta = -(\log \frac{P_1}{P_2} + \sum_{j=1}^{d} \log \frac{(1-P_{j,1})}{(1-P_{j,2})})$ and $w_j = \log \frac{P_{j,1}(1-P_{j,2})}{P_{j,2}(1-P_{j,1})}$, we could have

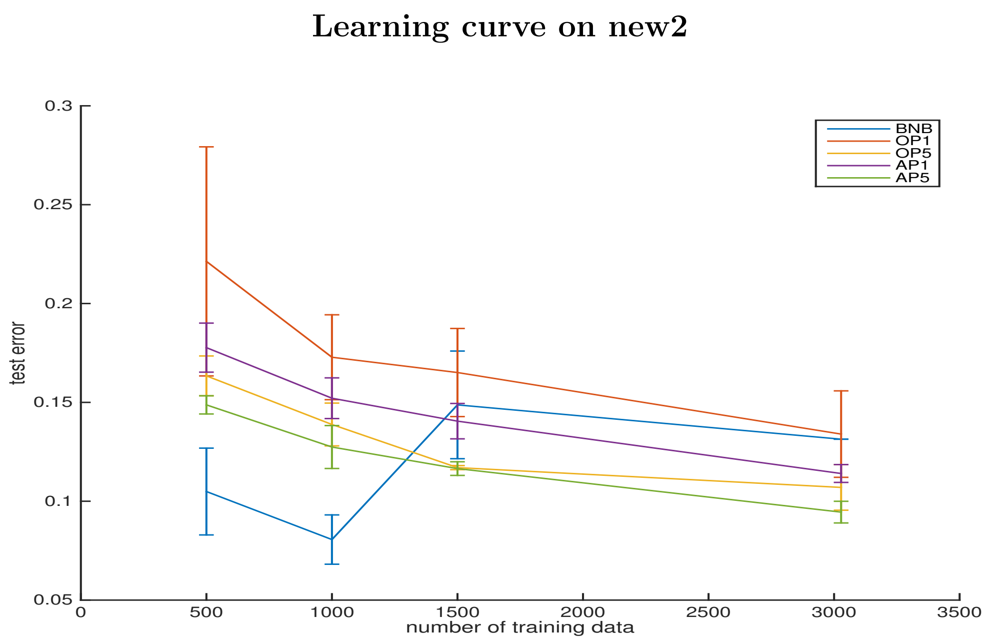
$$f^*(x) = \begin{cases} 1 & \langle \boldsymbol{x}\boldsymbol{w} \rangle - \theta > 0 \\ 2 & other \end{cases}$$

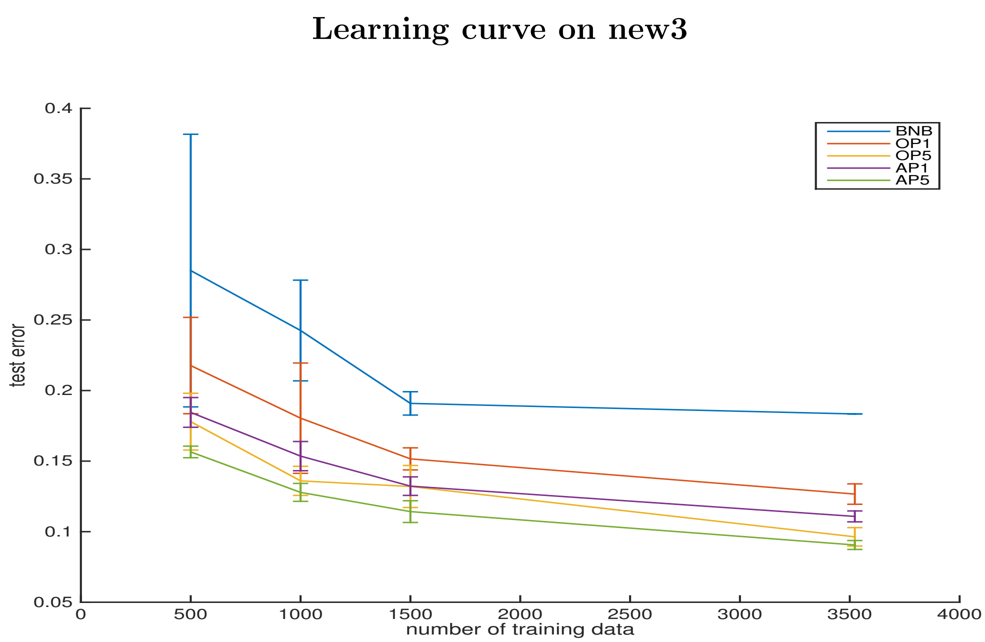
Thus, the plug-in classifier based on the BNB model here is a linear classifier.

3. The training error is 21.63 %, and the test error is 37.6%.

Problem 2

(a)
if $f_{w,\theta}(\boldsymbol{x}) \neq y$
$\boldsymbol{w} := \boldsymbol{w} + y\boldsymbol{x}; \theta := \theta - y$
end if
(b)





According to the classification result over new2 and new3. We could conclude that, if the the training set is large enough, the multi-pass online perceptron and average perceptron could have better classification performance over BNB classifier. The multi-pass perceptron could have better performance over single-pass perceptron, whether in online perceptron or average perception. If average and online perceptron have the same times of pass, the average perception could have better performance over online perceptron.

(c)

Table 1: 20 most important words

BNB_new2		BNB_new3		AP5_new2		AP5_new3	
athos	5.1816	encryption	5.5471	god	21.7711	windows	25.1524
atheism	4.7957	nsa	4.9406	jesus	18.4720	graphics	19.8811
atheists	4.7642	escrow	4.8818	christian	18.1406	window	14.4926
clh	4.7318	secret	4.8469	gun	15.4042	motif	14.2956
firearms	4.6845	pgp	4.8290	bible	15.3759	image	13.3405
occupied	4.5487	crypto	4.6750	christians	14.5415	key	12.4873
teachings	4.4191	enforcement	4.4217	keith	14.3694	space	11.9605
israelis	4.4071	government	4.3600	american	13.3989	win	11.9320
serdar	4.3944	motif	4.2610	israel	13.0465	moon	11.8423
argic	4.3944	xlib	4.2305	government	12.7633	encryption	11.1987
ohanus	4.2302	lunar	4.2082	clh	12.3217	circuit	11.1708
appressian	4.2302	font	4.1814	atheism	11.9379	format	10.6489
sahak	4.2302	voltage	4.1599	april	11.4819	file	10.5748
melkonian	4.2302	wiretap	4.1599	athos	11.4675	government	10.4239
villages	4.2154	xterm	4.1474	lord	11.2899	server	10.0757
revelation	4.2136	eff	4.1093	news	11.1809	power	10.0752
testament	4.1783	orbit	4.0987	james	10.8314	large	9.9933
livesey	4.1602	denning	4.0920	religious	10.7604	mouse	9.9714
atheist	4.1526	sternlight	4.0743	kent	10.7436	low	9.9362
solntze	4.1231	vehicle	4.0563	amendment	10.6447	version	9.8288

Problem 3

	Centering	Standardization
Plug-in Classifier (Gaussian Distribution)	no effects	have effects
1-NN Classifier (Euclidean distance)	no effects	have effects
Greedy Decision Tree (Gini index)	no effects	no effects
ERM(Linear Classifier)	no effects	no effects

(a)

a.1 Centering transformation

$$\begin{aligned}\hat{\boldsymbol{\mu}}'_y &= \frac{1}{n\hat{\pi}_y}\sum_{i=1}^{n} 1\{y_i = y\}(\boldsymbol{x}_i - \boldsymbol{\mu}) \\ &= \frac{1}{n\hat{\pi}_y}\sum_{i=1}^{n} 1\{y_i = y\}\boldsymbol{x}_i - \frac{1}{n\hat{\pi}_y}\sum_{i=1}^{n} 1\{y_i = y\}\boldsymbol{\mu} \\ &= \frac{1}{n\hat{\pi}_y}\sum_{i=1}^{n} 1\{y_i = y\}\boldsymbol{x}_i - \frac{1}{n\hat{\pi}_y}\boldsymbol{\mu}\sum_{i=1}^{n} 1\{y_i = y\} \\ &= \frac{1}{n\hat{\pi}_y}\sum_{i=1}^{n} 1\{y_i = y\}(\boldsymbol{x}_i) - \boldsymbol{\mu}\end{aligned}$$

Then we can plug-in the $\boldsymbol{\mu}'_y$ into the classifer, we have: $\boldsymbol{x}' - \hat{\boldsymbol{\mu}}'_y = \boldsymbol{x} - \boldsymbol{\mu} - \hat{\boldsymbol{\mu}}_y + \boldsymbol{\mu} = \boldsymbol{x} - \hat{\boldsymbol{\mu}}_y$ And the covariance matrix is the identity matrix $\mathbf{I}$, thus the centering transformation have no effects on the resulting learning classifier.

a.2 Standardization transformation.

$\boldsymbol{x}' - \hat{\boldsymbol{\mu}}'_y = \frac{\boldsymbol{x} - \boldsymbol{\mu} - \hat{\boldsymbol{\mu}}_y}{\boldsymbol{\sigma}} = \frac{\boldsymbol{x} - \hat{\boldsymbol{\mu}}_y}{\boldsymbol{\sigma}}$

Since the σ_i might not be equal to σ_j, some dimensions would be favored over other dimensions, thus the new classifier could produce different classifications.

(b)

Suppose we have two points: $\boldsymbol{m} \in TrainSet$ and $\boldsymbol{n} \in TestSet$.

$$||\boldsymbol{m} - \boldsymbol{n}||_2 := \sqrt{\sum_{i}^{d}(m_i - n_i)^2} \tag{3}$$

b.1 Centering transformation.

$$||\boldsymbol{m} - \boldsymbol{n}||'_2 := \sqrt{\sum_{i}^{d}(m_i - u_i - (m_i - u_i))^2} = \sqrt{\sum_{i}^{d}(m_i - n_i)^2} = ||\boldsymbol{m} - \boldsymbol{n}||_2 \tag{4}$$

Thus the centering transformation have no effects over resulting learned classifier.

b.1 Standardization transformation.

$$||\boldsymbol{m} - \boldsymbol{n}||_2' := \sqrt{\sum_i^d (\frac{m_i - u_i}{\sigma_i} - \frac{n_i - u_i}{\sigma_i})^2} = \sqrt{\sum_i^d \frac{1}{\sigma_i^2}(m_i - n_i)^2} \tag{5}$$

Since σ_i might be different for each dimension, some dimensions would be favored over other dimensions, thus the new classifier could produce different classifications.

(c)
c.1 Centering transformation.
In transformation: $\boldsymbol{x}' = \boldsymbol{x} - \boldsymbol{\mu}$, μ_i is the same for x_i. It means all training data is shifted along the same vector $\boldsymbol{\mu}$, and for any $\boldsymbol{x}_i$, it shifts the same distance along i dimension's axis. According to the definition of decision tree method, we try to find the line along i dimension to minimize uncertainty. Since we just shift whole points in the same distance along each dimension, the relative position of each points is the same, the line we want find just shifts the same distance in each dimensions respectively. Therefore, the centering transformation have no effects over resulting learned classifier.

c.2 Standardization transformation
The same as the above analysis. The decision tree is constructed by drawing a line at i dimension, which could minimize the uncertainty. What standardization transformation do at each dimension is to shift all points the distance of $\frac{\mu}{\sigma_i}$ and contract the factor of σ_i, the points' relative position at each dimension keep the same during this transformation. The transformation only affects the cutting line's position, but the uncertainty at the cutting line's two side keeps the same. Therefore, the standardization transformation have no effects over resulting learned classifier.

(d)
d.1 Centering transformation.
Linear ERM Classifier is to find the hyperplane separates the training data into two parts. In transformation: $\boldsymbol{x}' = \boldsymbol{x} - \boldsymbol{\mu}$, μ_i is the same for x_i. It means all training data is shifted along the same vector $\boldsymbol{\mu}$, and for any $\boldsymbol{x}_i$, it shifts the same distance along i dimension's axis. Thus, each points's relative position keep the same. The new classifier would have the same classifying result as the original one.

d.2 Standardization transformation
The same as the above analysis, standardization transformation shifts all points the distance of $\frac{\mu}{\sigma_i}$ and contract the factor of σ_i, the cutting position at i dimension would keep the same. Thus, the standardization transformation have no effects over resulting learned classifier.

Problem 4

(a)
$\phi : \mathbb{R}^d \to \mathbb{R}^{d+\binom{d}{2}}$
$\phi(x) := (x_1^2, x_2^2, ..., x_d^2, \sqrt{2}x_1x_2, \sqrt{2}x_1x_3, ..., \sqrt{2}x_1x_d, \sqrt{2}x_2x_3, ..., \sqrt{2}x_{d-1}x_d)$

(b)
Let's assume

$\phi_1(x)$ is the feature map for $K_1(x, x')$
$\phi_1 : \mathbb{R}^d \to \mathbb{R}^{D_1}$
$\phi_1(x) := (f_1(x), f_2(x), f_3(x), ...)$

$\phi_2(x)$ is the feature map for $K_2(x, x')$
$\phi_2 : \mathbb{R}^d \to \mathbb{R}^{D_2}$
$\phi_2(x) := (g_1(x), g_2(x), g_3(x), ...)$

We have

$$
\begin{aligned}
K_1(x, x')K_2(x, x') &= \langle \phi_1(x), \phi_1(x') \rangle \langle \phi_2(x), \phi_2(x') \rangle \\
&= \sum_{i=1}^{|D_1|} f_i(x) f_i(x') \sum_{j=1}^{|D_2|} g_j(x) g_j(x') \\
&= \sum_{i,j} f_i(x) f_i(x') g_j(x) g_j(x') \\
&= \sum_{i,j} (f_i(x) g_j(x))(f_i(x') g_j(x'))
\end{aligned}
$$

Thus, we can define a feature map $\phi_3(x)$ with a feature $c_{i,j}(x)$, and and each pair (i, j) defined as: $c_{i,j}(x) = f_i(x)g_j(x)$.
$\phi_3(x)$ is the feature map for $K_1(x, x')K_2(x, x')$
$\phi_3 : \mathbb{R}^d \to \mathbb{R}^{D_1 D_2}$
$\phi_3(x) := (c_{1,1}(x), c_{1,2}(x), ..., c_{1,D_2}(x), c_{2,1}(x), ..., c_{D_1,D_2}(x))$

Let $\phi_3(x)$ be the feature map for $K_3(x, x')$, $K_3(x, x') = K_1(x, x')K_2(x, x')$ is also a positive definite kernel function.