COMS 4771 Lecture 24

1. Collaborative filtering
2. Stochastic gradient descent

Collaborative filtering

Netflix problem: 480000 users, 18000 movies.

Recommender systems

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- Get *features* for each user i and movie j (e.g., $\boldsymbol{x}_i \in \mathbb{R}^d$ and $\boldsymbol{y}_j \in \mathbb{R}^d$); goal is to predict rating as function of $(\boldsymbol{x}_i, \boldsymbol{y}_j)$.

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- **Linear function**: $\boldsymbol{x}_i \boldsymbol{y}_j^\top \mapsto \langle \boldsymbol{W}, \boldsymbol{x}_i \boldsymbol{y}_j^\top \rangle = \boldsymbol{x}_i^\top \boldsymbol{W} \boldsymbol{y}_j$ for $\boldsymbol{W} \in \mathbb{R}^{d \times d}$.
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What if you don't have any features?

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what a genius idea???
get the information from ratings
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How can we effectively formalize the intuition stated above?

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$$\mathbb{E}\left\{\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix}\right\} = \begin{bmatrix} — & \boldsymbol{u}_1^\top & — \\ — & \boldsymbol{u}_2^\top & — \\ & \vdots & \\ — & \boldsymbol{u}_m^\top & — \end{bmatrix} \begin{bmatrix} | & | & & | \\ \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_n \\ | & | & & | \end{bmatrix}$$

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great way to understand matrix!!!

We'll assume $k \leq \min\{m, n\}$ for reasons that will become clear later.

Complete ratings

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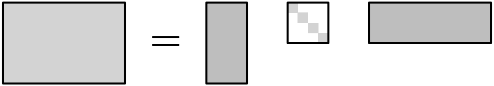

$\boldsymbol{A}$ $(m \times n)$ = $\boldsymbol{L}$ $(m \times r)$ $\boldsymbol{S}$ $(r \times r)$ $\boldsymbol{R}^\top$ $(r \times n)$

Use rows of $\boldsymbol{L}\boldsymbol{S}^{1/2}$ as the $\boldsymbol{u}_i$, and rows of $\boldsymbol{R}\boldsymbol{S}^{1/2}$ as the $\boldsymbol{v}_j$.

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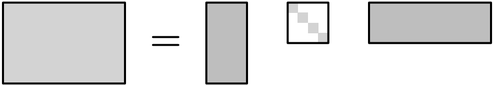

$$\underset{(m \times n)}{\boldsymbol{A}} \qquad \underset{(m \times r)}{\boldsymbol{L}} \quad \underset{(r \times r)}{\boldsymbol{S}} \quad \underset{(r \times n)}{\boldsymbol{R}^\top}$$

Use rows of $\boldsymbol{L}\boldsymbol{S}^{1/2}$ as the $\boldsymbol{u}_i$, and rows of $\boldsymbol{R}\boldsymbol{S}^{1/2}$ as the $\boldsymbol{v}_j$.

Unclear what use this is (e.g., if a new movie comes along, can we predict how users will rate it?).

Known user features

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The catch: we don't observe all of the entries of $\boldsymbol{A}$.

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Regard these $a_{i,j}$ as the training data.

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Often called (low rank) matrix completion, because this is equivalent to:

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Nevertheless, can derive an alternating minimization algorithm (similar to k-means and dictionary learning) to iteratively update $\boldsymbol{U}$ and $\boldsymbol{V}$.

Alternating minimization

Let $\Omega \subseteq [m] \times [n]$ be the entries for which $a_{i,j}$ is observed.

- Somehow initialize $\boldsymbol{u}_i \in \mathbb{R}^k$ for each $i \in [m]$ and $\boldsymbol{v}_j \in \mathbb{R}^k$ for each $j \in [n]$.
- Repeat until convergence:
  - For each user $i \in [m]$,

$$\boldsymbol{u}_i := \arg\min_{\boldsymbol{u}_i \in \mathbb{R}^k} f(\boldsymbol{U}, \boldsymbol{V})$$

  - For each movie $j \in [n]$,

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$$\boldsymbol{u}_i := \left(\sum_{j \in [n] : (i,j) \in \Omega} \boldsymbol{v}_j \boldsymbol{v}_j^\top \right)^{-1} \sum_{j \in [n] : (i,j) \in \Omega} a_{i,j} \boldsymbol{v}_j$$

  - For each movie $j \in [n]$,

$$\boldsymbol{v}_j := \left(\sum_{i \in [m] : (i,j) \in \Omega} \boldsymbol{u}_i \boldsymbol{u}_i^\top \right)^{-1} \sum_{i \in [m] : (i,j) \in \Omega} a_{i,j} \boldsymbol{u}_i .$$

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This assumes certain matrices involving $\boldsymbol{u}_i$ and $\boldsymbol{v}_j$ are invertible—but there's no guarantee of this!

Minimize regularized training error

Let $\Omega \subseteq [m] \times [n]$ be the entries for which $a_{i,j}$ is observed.

Regularized training objective:

$$f(\boldsymbol{U}, \boldsymbol{V}) \ := \ \sum_{(i,j)\in\Omega} \big(a_{i,j} - \langle \boldsymbol{u}_i, \boldsymbol{v}_j \rangle\big)^2 + \lambda \left(\sum_{i=1}^{m} \|\boldsymbol{u}_i\|_2^2 + \sum_{j=1}^{n} \|\boldsymbol{v}_j\|_2^2 \right).$$

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New updates:

- For each user $i \in [m]$,

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Prediction: For a movie j that user i has not rated, predict rating to be

$$\hat{a}_{i,j} := \langle \boldsymbol{u}_i, \boldsymbol{v}_j \rangle.$$

Other things

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- Incorporating biases

$$f(\boldsymbol{U}, \boldsymbol{V}, \boldsymbol{b}, \boldsymbol{c}, \mu) \;:=\; \sum_{(i,j)\in\Omega} \big(a_{i,j} - \langle \boldsymbol{u}_i, \boldsymbol{v}_j \rangle - b_i - c_j - \mu\big)^2 + \lambda\left(\sum_{i=1}^{m} \|\boldsymbol{u}_i\|_2^2 + \sum_{j=1}^{n} \|\boldsymbol{v}_j\|_2^2\right).$$

What about different loss functions?

Different training objective: logistic loss (each $a_{i,j} \in \{-1, +1\}$)

$$f(\boldsymbol{U}, \boldsymbol{V}) \;:=\; \sum_{(i,j)\in\Omega} \ell_{\log}(a_{i,j}\langle \boldsymbol{u}_i, \boldsymbol{v}_j\rangle).$$

No closed form minimizer of $\boldsymbol{U} \mapsto f(\boldsymbol{U}, \boldsymbol{V})$.

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$$f(\boldsymbol{U},\boldsymbol{V}) \;:=\; \sum_{(i,j)\in\Omega} \ell_{\log}(a_{i,j}\langle \boldsymbol{u}_i, \boldsymbol{v}_j\rangle).$$

No closed form minimizer of $\boldsymbol{U} \mapsto f(\boldsymbol{U},\boldsymbol{V})$.

Alternating gradient updates

For $t = 1, 2, \dots$:

- For each user $i \in [m]$,

$$\boldsymbol{u}_i^{(t+1)} \;:=\; \boldsymbol{u}_i^{(t)} - \eta \nabla_{\boldsymbol{u}_i}\left\{ f(\boldsymbol{U}^{(t)}, \boldsymbol{V}^{(t)}) \right\}.$$

- For each movie $j \in [n]$,

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(Ambiguous whether to use $\boldsymbol{u}_i^{(t)}$ or $\boldsymbol{u}_i^{(t+1)}$ when updating $\boldsymbol{v}_j^{(t)} \to \boldsymbol{v}_j^{(t+1)}$.)

Stochastic gradient descent

Unconstrained convex optimization problem

$$\min_{\boldsymbol{w} \in \mathbb{R}^d} \quad f(\boldsymbol{x})$$

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Gradient descent for differentiable objectives

- Start with some initial $\boldsymbol{w}^{(1)} \in \mathbb{R}^d$.
- For $t = 1, 2, \ldots$ until some stopping condition is satisfied.
  - Compute gradient of f at $\boldsymbol{w}^{(t)}$:

    $$\boldsymbol{\lambda}^{(t)} := \nabla f(\boldsymbol{w}^{(t)}).$$

  - Update:

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If f is "nice", GD converges to global minimizer $\boldsymbol{w}_\star$, where $\nabla f(\boldsymbol{w}_\star) = \mathbf{0}$.

Summation form objectives

In machine learning, we frequently deal with objectives of the form

$$f(\boldsymbol{w}) \ = \ \frac{1}{n}\sum_{i=1}^{n} f_i(\boldsymbol{w})$$

where $f_1, f_2, \ldots, f_n \colon \mathbb{R}^d \to \mathbb{R}$ are convex functions (e.g., loss on example i).

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Example: **Gradient descent algorithm for logistic regression**

- Start with some initial $\boldsymbol{w}^{(1)} \in \mathbb{R}^d$.
- For $t = 1, 2, \ldots$

$$\begin{aligned} \boldsymbol{w}^{(t+1)} &:= \boldsymbol{w}^{(t)} - \eta_t \nabla f(\boldsymbol{w}^{(t)}) \\ &= \boldsymbol{w}^{(t)} + \eta_t \frac{1}{|S|} \sum_{(\boldsymbol{x},y)\in S} \frac{1}{1 + e^{y\langle \boldsymbol{w}^{(t)}, \boldsymbol{x}\rangle}} y\boldsymbol{x}. \end{aligned}$$

training
loss

Summation form objectives

In fact, we often *really* care about

$$f(\boldsymbol{w}) \;=\; \mathbb{E}\big[f(\boldsymbol{w}; Z)\big]$$

where Z is a random example, and $f(\cdot; Z)$ represents the (convex) loss on this random example.

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Let us continue with this for now . . .

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Suppose at the push of a red button, we get an independent draw from the distribution P over labeled examples that we care about.

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not overall sample

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(Gradient of the loss on **a simple random example** at current weight vector.)

Can we still find a good weight vector?

Stochastic gradient descent (SGD)

randomly pick one and minimize its cost

- Start with some initial $\boldsymbol{w}^{(1)} \in \mathbb{R}^d$.
- For $t = 1, 2, \ldots$
  - Push the red button and get

$$\boldsymbol{\lambda}^{(t)} := \nabla f(\boldsymbol{w}^{(t)}; z_t).$$

just for one sample point each time

  - Update:

$$\boldsymbol{w}^{(t+1)} := \boldsymbol{w}^{(t)} - \eta_t \boldsymbol{\lambda}^{(t)}.$$

Why does this work?

We care about

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average. not a random example

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In expectation, we move in the correct direction w.r.t. the objective we actually care about!

online perceptron use this idea

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the expectation of \lambda
(rather than directly calculate)

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Implementing SGD with finite training data

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- After making a pass through the data, theory says to stop (afaik)—repeating examples cannot really be regarded as independent.
- In practice, can help to cycle through the data a few times.

there is no risk of overfitting!!!
1. very resistant to overfitting.
2. voted-perceptron

SGD with non-convex objectives

Can also apply SGD to non-convex summation objectives, e.g.,

$$f(\boldsymbol{U}, \boldsymbol{V}) \;:=\; \mathbb{E}\big[\ell_{\log}(a_{I,J}\langle \boldsymbol{u}_I, \boldsymbol{v}_J\rangle)\big]$$

(for random user I and movie J).

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Given ratings $\{a_{i,j} : (i,j) \in \Omega\}$, first randomly permute their order in Ω.

For $t = 1, 2, \dots$:

- Get next (i,j) and $a_{i,j}$.
- Updates:

$$\boldsymbol{u}_i^{(t+1)} \ := \ \boldsymbol{u}_i^{(t)} + \eta_t a_{i,j} \frac{1}{1 + \exp(a_{i,j}\langle \boldsymbol{u}_i^{(t)}, \boldsymbol{v}_j^{(t)} \rangle)} \boldsymbol{v}_j^{(t)}$$

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- (For all other (i',j'), no change: $\boldsymbol{u}_{i'}^{(t+1)} = \boldsymbol{u}_{i'}^{(t)}$ and $\boldsymbol{v}_{j'}^{(t+1)} = \boldsymbol{v}_{j'}^{(t)}$.)

Recap

- **Stochastic gradient descent** (not technically a descent method): for convex problems, converges (somewhat slowly) to the optimum of the problem we actually care about!
- Can also be applied to non-convex problems.
- **A particular non-convex problem**: matrix completion, which is often used for collaborative filtering / recommender systems.