COMS 4771 Lecture 20

1. Maximum entropy

Probabilistic modeling

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- Is there a general approach for
  (i) picking a probability model, and
  (ii) parameter estimation?
- How do familiar models (as above) fit into this approach?

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Say you observe

$$\frac{1}{n}\sum_{i=1}^{n} T_1(x_i) = 0.22, \quad \frac{1}{n}\sum_{i=1}^{n} T_2(x_i) = 0.32, \quad \ldots$$

Now what distribution should you pick?

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How do we measure how "random" a distribution is?

Measuring randomness

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- **Fair 32-sided die**?

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- **Two independent fair coin tosses**? Two units of randomness.
  - **Two dependent fair coin tosses**? Less randomness.
- **Fair 32-sided die**? This is equivalent to five independent fair coin tosses, so five units of randomness.

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5. (**Expansibility**) If $X \sim (p_1, p_2, \ldots, p_d)$ and $Y \sim (p_1, p_2, \ldots, p_d, 0)$, then
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6. (**Symmetry**) If $X \sim (p_1, p_2, \ldots, p_d)$ and $Y \sim (p_{\sigma(1)}, p_{\sigma(2)}, \ldots, p_{\sigma(d)})$ for some permutation σ on $\{1, 2, \ldots, d\}$, then
$$H(X) = H(Y).$$

Measuring randomness

The **only measure of randomness that satisfies the desiderata** is

$$H(X) = -\sum_{x \in \mathcal{X}} \Pr(X = x) \log_2 \Pr(X = x)$$

which is called (Shannon) **entropy**. (Note: $0 \log 0 = 0$ by convention.)

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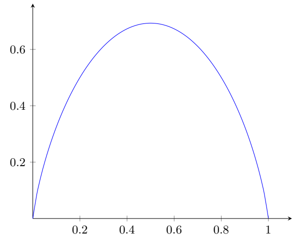
Also may write this as

$$H(P) \;=\; \mathbb{E}_{X \sim P}\Big[-\log_2(P(X))\Big] \;=\; \mathbb{E}_{X \sim P}\left[\log_2 \frac{1}{P(X)}\right].$$

"Bits" $=$ units of entropy with $\log_2$. "Nats" $=$ units of entropy with $\ln$.
Different logarithm bases just change things by constant factors.

Entropy



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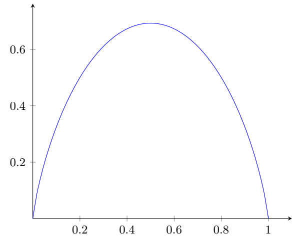

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Distribution over $\mathcal{X}$ with highest entropy: **uniform distribution**

$$H(\text{uniform}) \;=\; -\sum_{x\in\mathcal{X}} \frac{1}{|\mathcal{X}|}\log\frac{1}{|\mathcal{X}|} \;=\; \log|\mathcal{X}|.$$

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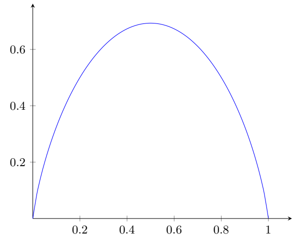

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Distribution over $\mathcal{X}$ with least entropy: **point mass at any** $x^* \in \mathcal{X}$

$$H(\delta_{x^*}) \;=\; -\sum_{x \in \mathcal{X}} \mathbb{1}\{x = x^*\} \log \mathbb{1}\{x = x^*\} \;=\; 0.$$

Interpretations of entropy

Surprise: $\log \frac{1}{P(x)}$ measures the amount of *surprise* you should feel if you observe $x \in \mathcal{X}$, according to the distribution P.

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Upshot: Entropy measures the average information content of a RV.

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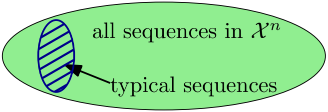

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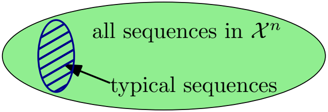

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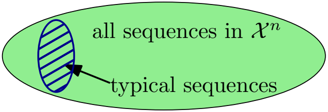

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- **Number of typical sequences**:

$$\text{Between } (1-\varepsilon) 2^{n(H(P)-\varepsilon)} \text{ and } 2^{n(H(P)+\varepsilon)}.$$

Far fewer than $|\mathcal{X}|^n$ when $H(P) \ll \log_2 |\mathcal{X}|$.

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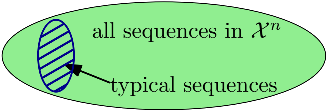

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the larger the H(P), the larger the set of typical sequences

extract typical seqeunce

Upshot: $H(P)$ characterizes the number of typical i.i.d. sequences from P.

Entropy: recap

- Entropy is a fundamental measure of the
    - randomness
    - uncertainty
    - information content

  in a probability distribution.
- Quantifies achievable rates for data compression.
- Quantifies number of typical i.i.d. sequences.
- ...

Maximum entropy principle

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Observe a random sample $x_1, x_2, \ldots, x_n$ **of words from** $\mathcal{X}$,
and record some features $T_1, T_2, \ldots, T_k \colon \mathcal{X} \to \mathbb{R}$: e.g.,

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more general way in depict whether a sample meets certain condition

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Maximum entropy principle (Jaynes, 1957):

Pick the distribution that agrees with the empirical observations, but is otherwise as "random" as possible.

Our empirical observations from sample $x_1, x_2, \ldots, x_n \in \mathcal{X}$:

Text

$$b_i \ := \ \frac{1}{n} \sum_{j=1}^{n} T_i(x_j) \ = \ \widehat{\mathbb{E}}[T_i(X)] \quad \text{for } i = 1, 2, \ldots, k$$

where $\widehat{\mathbb{E}}[\,\cdot\,]$ is expectation w.r.t. *empirical distribution based on the sample*.

Maximum entropy optimization problem

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$$\begin{aligned} \max_{P \in \Delta(\mathcal{X})} \quad & H(P) \\ \text{s.t.} \quad & \mathbb{E}_{X \sim P}[T_i(X)] \ = \ b_i \quad \text{for all } i = 1, 2, \ldots, k \end{aligned}$$

(where $\Delta(\mathcal{X})$ is the space of probability distributions over $\mathcal{X}$).

Maximum entropy optimization problem

just for constrain purpose, has nothing to do with entropy,
which is the target!!

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the expectation
of Ti

X is random variable

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get the right P
through all possible
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must meet
expirical
distribution

(where $\Delta(\mathcal{X})$ is the space of probability distributions over $\mathcal{X}$).

Without the constraints (i.e., before observations are made), $\max_{P \in \Delta(\mathcal{X})} H(P)$ is achieved by the *uniform distribution* over $\mathcal{X}$.

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Generalization of maximum entropy principle:

> *Pick the distribution that agrees with the empirical observations, but is otherwise as close to π as possible.*

Want $\pi =$ uniform $\Longrightarrow$ maximum entropy.

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Want $\pi = \text{uniform} \implies$ maximum entropy.

How do we measure how close two probability distributions are?

Relative entropy

Entropy: expected information content measured by P, where expectation is w.r.t. random draw from P.

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Relative entropy: expected information content measured by Q, where expectation is w.r.t. random draw from P

$$\mathrm{RE}(P\|Q) \;:=\; \mathbb{E}_{X\sim P}\left[\ln \frac{1}{Q(X)}\right] - H(P).$$

(and we subtract off $H(P)$ so it is zero when $P = Q$).

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use Q to measure(compare) with P. all random draw from P

(and we subtract off $H(P)$ so it is zero when $P = Q$).

More typical form:

$$\mathrm{RE}(P\|Q) \;=\; \sum_{x\in\mathcal{X}} P(x)\ln\frac{P(x)}{Q(x)}.$$

note the expansion

Properties of relative entropy

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the larger the less randomness.
we don't care about constrain at this time

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- Also called "Kullback-Leibler divergence".

Maximum entropy optimization problem

Maximum entropy optimization problem with base distribution π:

$$\begin{aligned} \min_{P \in \Delta(\mathcal{X})} \quad & \mathrm{RE}(P \| \pi) \\ \text{s.t.} \quad & \mathbb{E}_{X \sim P}[T_i(X)] \; = \; b_i \quad \text{for all } i = 1, 2, \ldots, k. \end{aligned}$$

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both P(x) and Q(x) are kinds of probability arragement(distribution)

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for given x, each of its T(x) is meet with related b. (a list of b, with imposed feature requirement)

Convex objective function, with **linear** (in)equality constraints.

Entropy projection

Note that *any* feasible solution P must satisfy

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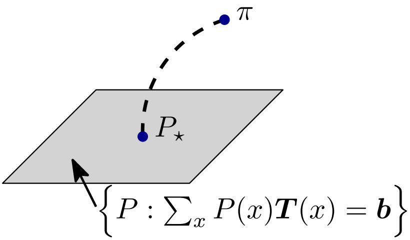

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project the base into the constrain plane

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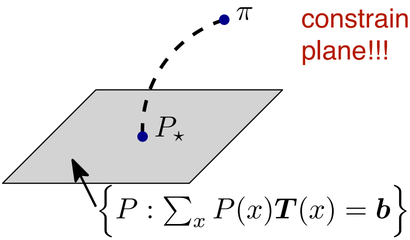

constrain plane!!!

Similar to the Euclidean projection of π onto an affine hyperplane, except we **use relative entropy instead of Euclidean distance**: an **entropy projection**.

Solution form

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Claim: A solution $P_\star$ to the optimization problem must have the form

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for some $\boldsymbol{\eta} \in \mathbb{R}^k$, where

$$Z(\boldsymbol{\eta}) \; = \; \sum_{x \in \mathcal{X}} \exp\Big\{ \langle \boldsymbol{\eta}, \boldsymbol{T}(x) \rangle \Big\} \cdot \pi(x)$$

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Consider any other feasible solution P—i.e., P is a valid probability distribution and (like $P_\star$) satisfies

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\end{aligned}$$

the minimization is achieved when p = p*

$\square$

Interpretation of the solution form

From our earlier example:

- $T_1(x) = \mathbb{1}\{x \text{ ends with an 'e'}\}$
- $T_2(x) = \mathbb{1}\{x \text{ has more than five characters}\}$
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Maximum entropy solution is of the form

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How do we get these $\boldsymbol{\eta}$ parameters?

Exponential families

The $\boldsymbol{\eta}$ parameters for distributions of the form

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This relationship is revealed through the study of these distribution families, called **exponential families**.

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Zn is the normalizer!

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This relationship is revealed through the study of these distribution families, called **exponential families**.

Many familiar probability models are exponential families:

> Bernoulli, binomial, Poisson, exponential, Gaussian, gamma, categorical, multinomial, Dirichlet, . . .

Recap

- Maximum entropy approach to probabilistic modeling: choose the most non-committal distribution that agrees with the empirical observation.
- Solution must have the form

$$P_{\boldsymbol{\eta}}(x) = \frac{1}{Z(\boldsymbol{\eta})} \cdot \exp\Big\{\langle \boldsymbol{\eta}, \boldsymbol{T}(x) \rangle\Big\} \cdot \pi(x),$$

  corresponds to the entropy projection of the base distribution π onto an affine hyperplane.
- Extracting the $\boldsymbol{\eta}$ parameters: next time, via study of exponential families.