Linear algebra review

Euclidean vector space

- $\mathbb{R}^d$: d-dimensional Euclidean space (a vector space)
- A vector $v \in \mathbb{R}^d$ is represented as d-tuple of real numbers

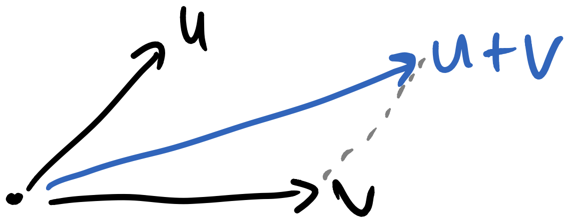
$$v = (v_1, v_2, \dots, v_d)$$

- Matrix $A \in \mathbb{R}^{k \times d}$ is a linear map from $\mathbb{R}^d$ to $\mathbb{R}^k$

$$v \mapsto u = Av \text{ where } u_i = \sum_{j=1}^{d} A_{i,j} v_j$$

- Can think of a vector $v \in \mathbb{R}^d$ as a matrix in $\mathbb{R}^{d \times 1}$ (column vector)

Vector addition : for $u, v \in \mathbb{R}^d$



$\boxed{\text{If } w = u + v, \text{ then } w_i = u_i + v_i}$

Vector scaling : for $c \in \mathbb{R}$, $v \in \mathbb{R}^d$

$(c > 1)$

$\boxed{\text{If } w = cv \text{ then } w_i = c v_i}$

$(-1 < c < 0)$

Linear independence

- Vectors $v^{(1)}, \dots, v^{(k)} \in \mathbb{R}^d$ are **linearly dependent** if there are scalars $c_1, \dots, c_k \in \mathbb{R}$, not all equal to zero, such that
$$c_1 v^{(1)} + \cdots + c_k v^{(k)} = 0$$
- If there are no such scalars $c_1, \dots, c_k$, then the vectors are **linearly independent**.

Subspaces

- A collection of k linearly independent vectors $v^{(1)}, \ldots, v^{(k)} \in \mathbb{R}^d$ defines a ***k*-dimensional subspace**

$$\operatorname{span}\left(v^{(1)}, \ldots, v^{(k)}\right)$$
$$= \left\{x \in \mathbb{R}^d : c_1 v^{(1)} + \cdots + c_k v^{(k)} \text{ for some } c_1, \ldots, c_k \in \mathbb{R}\right\}$$

- The vectors $v^{(1)}, \ldots, v^{(k)}$ are a **basis** for the subspace.

Inner products and norms

- **Standard inner product** on $\mathbb{R}^d$:

$$\langle u, v\rangle = \sum_{i=1}^{d} u_i v_i$$

- The **Euclidean norm** is induced by this standard inner product:

$$\|u\|_2 = \sqrt{\langle u, u\rangle}$$

- Two vectors u, v are **orthogonal** if $\langle u, v\rangle = 0$.

Orthonormal basis

- Let W be a k-dimensional subspace of $\mathbb{R}^d$.
- There are $q^{(1)}, \dots, q^{(k)} \in W$ such that

$$\langle q^{(i)}, q^{(j)} \rangle = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$$

- Such a collection of vectors is called an **orthonormal basis** (ONB).

- Using Gram-Schmidt, we can obtain an ONB from any basis for W.

Representation w.r.t. an ONB

- Let W be a k-dimensional subspace with ONB $q^{(1)}, \dots, q^{(k)}$.
- Take $u = c_1 q^{(1)} + \cdots c_k q^{(k)} \in W$.
- **Claim**: $c_i = \langle u, q^{(i)} \rangle$.

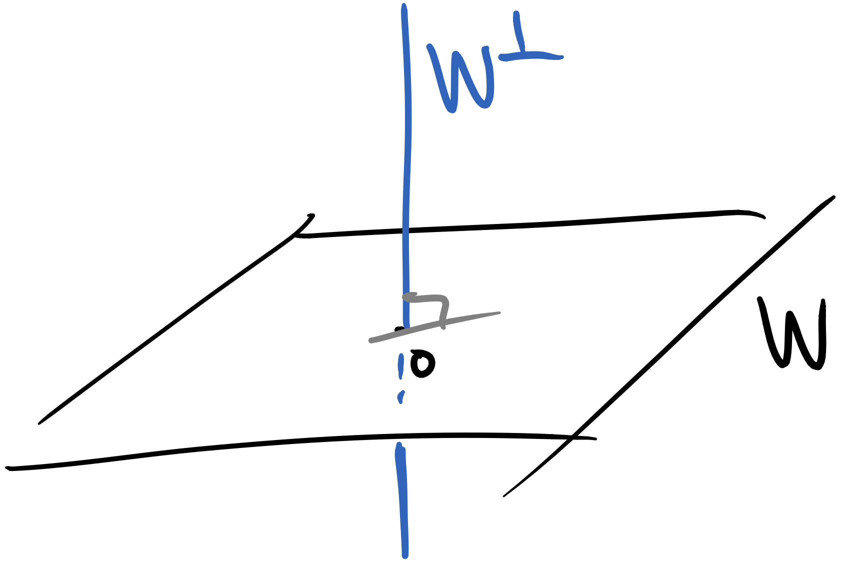
if i != j, <q(i), q(j)> = 0

Let W be a subspace.

Define $W^{\perp} := \{u \in \mathbb{R}^d : \langle u, w \rangle = 0 \text{ for } \underline{\underline{\text{all}}}\ w \in W\}$,

the <u>orthogonal complement of W</u>.

a subspace of dimension <u>$d - \dim(W)$</u>.

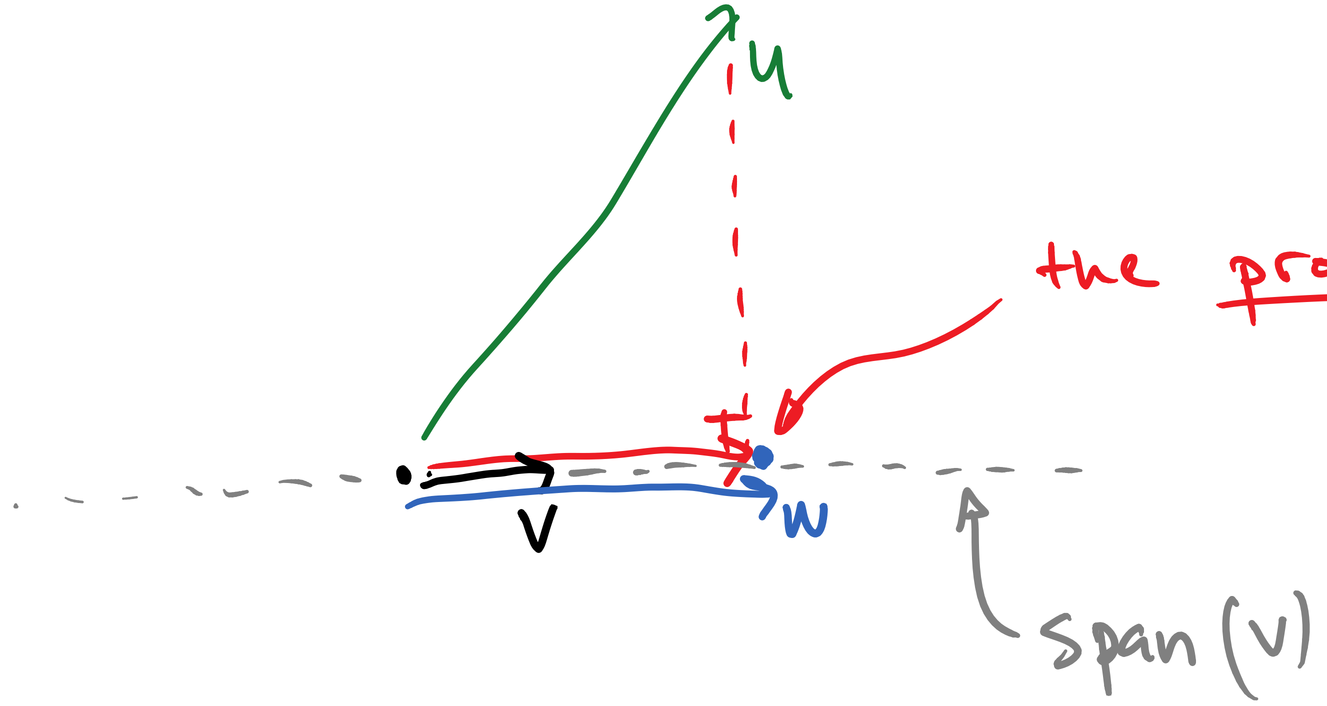


<u>Note</u>: union of an ONB for W and an ONB for $W^{\perp}$ is an ONB for $\mathbb{R}^d$!

Projection

- The (**orthogonal**) **projection** of $u \in \mathbb{R}^d$ onto a subspace W is the point $w \in \mathbb{R}^d$ that minimizes $\|u - w\|_2$.
- If $q^{(1)}, \dots, q^{(k)}$ is an ONB for W, then the projection is a linear map (i.e., a matrix) given by

$$[q^{(1)} \quad \cdots \quad q^{(k)}] \begin{bmatrix} \left(q^{(1)}\right)^T \\ \vdots \\ \left(q^{(k)}\right)^T \end{bmatrix} = \sum_{i=1}^{k} q^{(i)} \left(q^{(i)}\right)^T$$



the projection of u onto $\underbrace{\text{span}\{v\}}_{\text{one-dim subspace}}$.

Claim: $w = \dfrac{\langle u, v \rangle}{\|v\|_2^2} v$

Cauchy-Schwarz inequality

$\langle u, v \rangle \leq \|u\|_2 \|v\|_2$ for any $u, v \in \mathbb{R}^d$.

[Equality holds if $u = cv$ for some $c > 0$.]