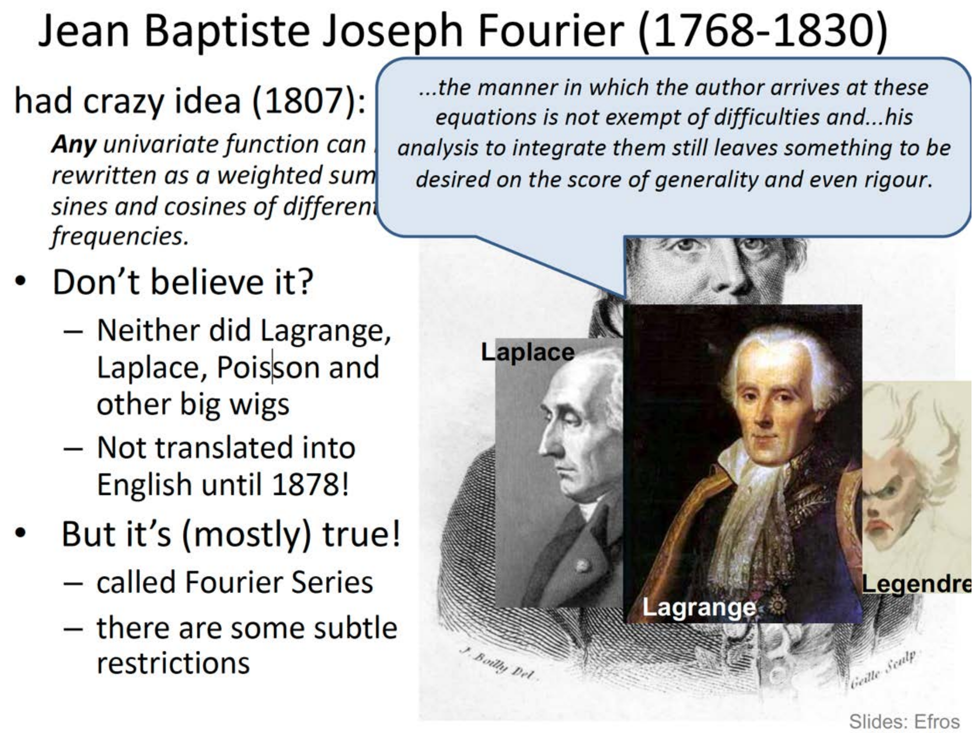
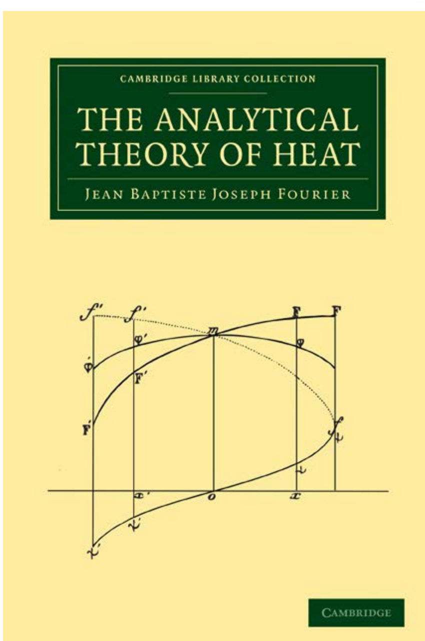
Class 5. Transformations and Filters

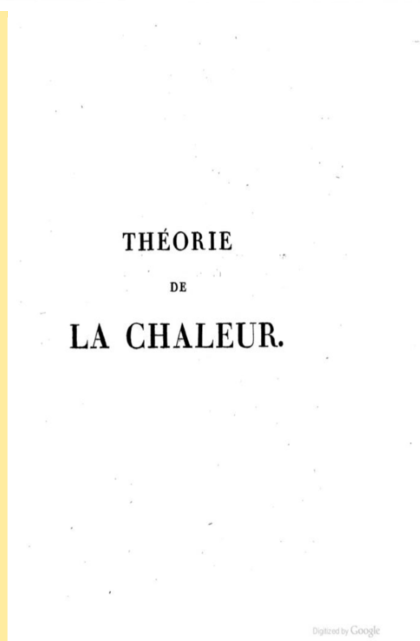
1. Fourier Transformations

- https://upload.wikimedia.org/wikipedia/commons/7/72/Fourier_transform_time_and_frequency_domains_%28small%29.gif

Example: Joseph Fourier’s analytical solution to heat conduction problem (1822)







Application: Solve 1D thermal conduction problem:
Calculate temperature waves at depth based on surface temperature.

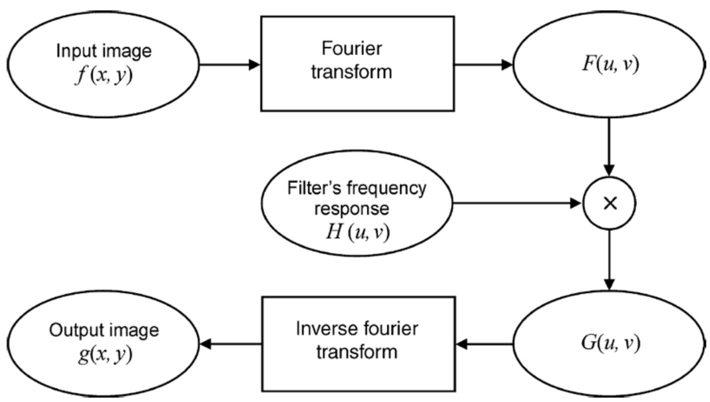
$$T(z,t) = T_0 + \sum_{s=1}^{N} T_s exp\left[-z\left(\frac{\omega_s}{2k}\right)^{1/2}\right] cos\left[\omega_s t - \phi_s - z\left(\frac{\omega_s}{2k}\right)^{1/2}\right]$$

Given surface temperature $T(z=0,t) = T_0 + \sum_{s=1}^{N} T_s cos\,(\omega_s t - \phi_s)$ and T(z=∞,0)= T_0

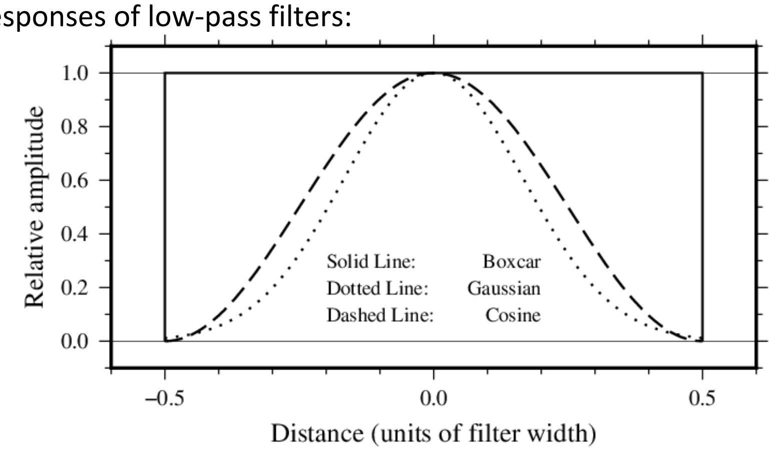
- Reference: Zhang, T., Chen, Y., Ding, M. et al (2019). Air-temperature control on diurnal variations in microseismicity at Laohugou Glacier No. 12, Qilian Mountains. Annals of Glaciology, 60(79), 125–136. https://doi.org/10.1017/aog.2018.34

2. Frequency domain filters
2.1 One-dimensional case:

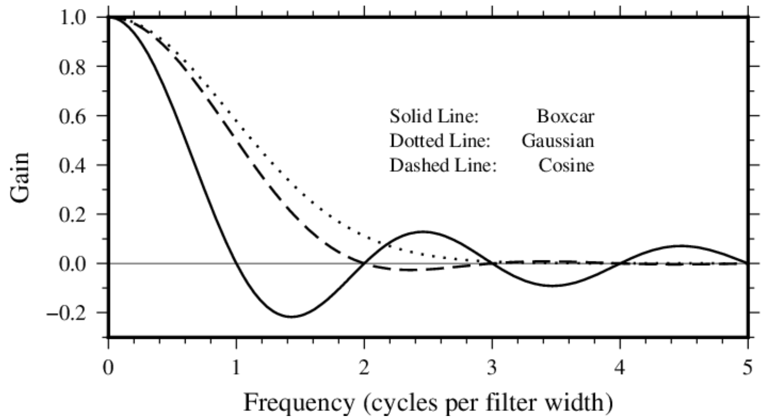
- Procedure



- Impulse responses of low-pass filters:

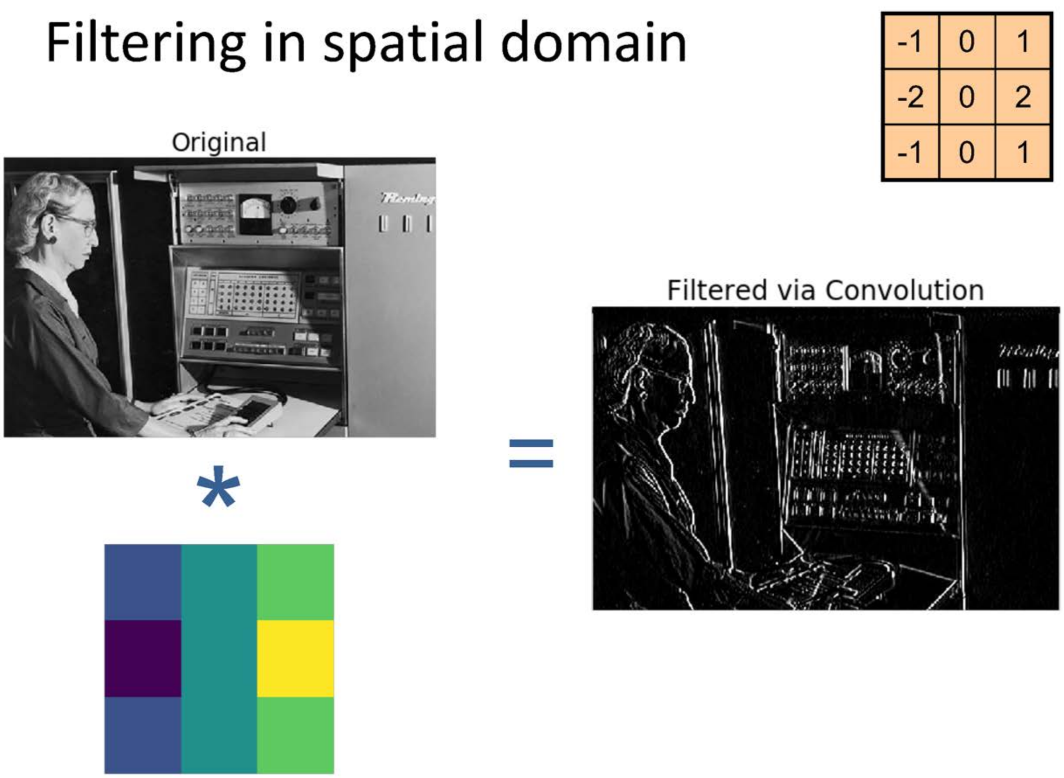


- Corresponding transfer functions:

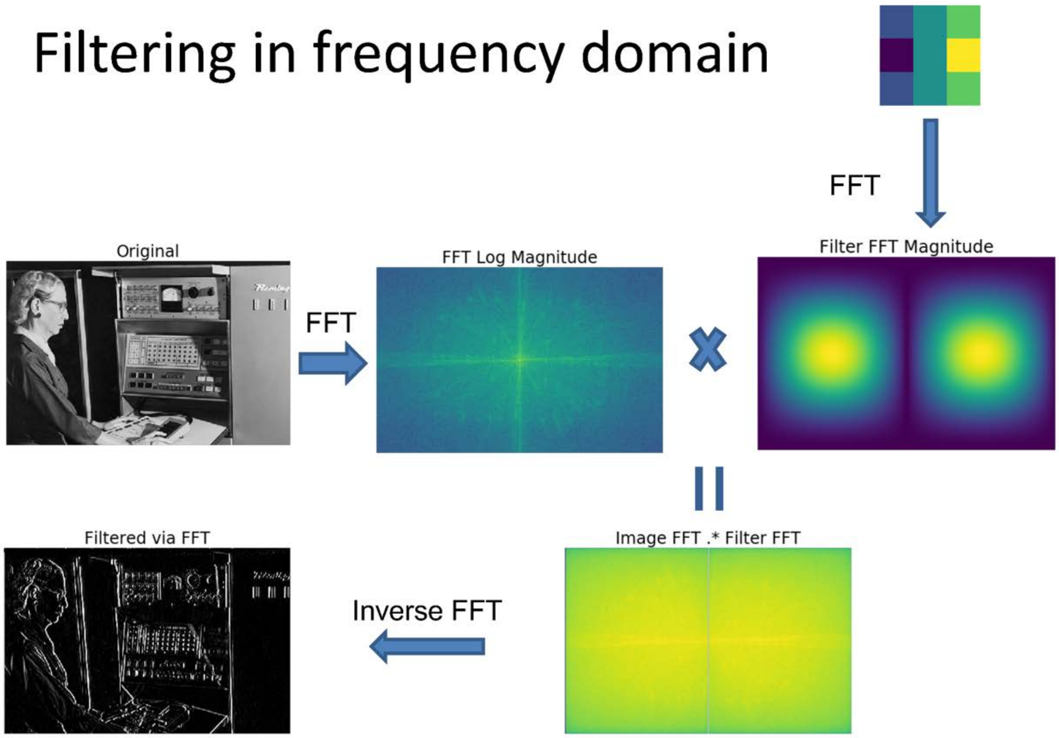


- As a general rule, the cosine and gaussian filters are “better” in the sense that they do not have the “side lobes” (large-amplitude oscillations in the transfer function) that the boxcar filter has. However, they are correspondingly “worse” in the sense that they require more work (doubling the width to achieve the same cut-off wavelength).
- High-pass filters, band-pass filters, and band-reject filters
- Edge enhancement filter (convolution filter in the spatial domain, e.g., Sobel operator)

2.2 Two-dimensional case:

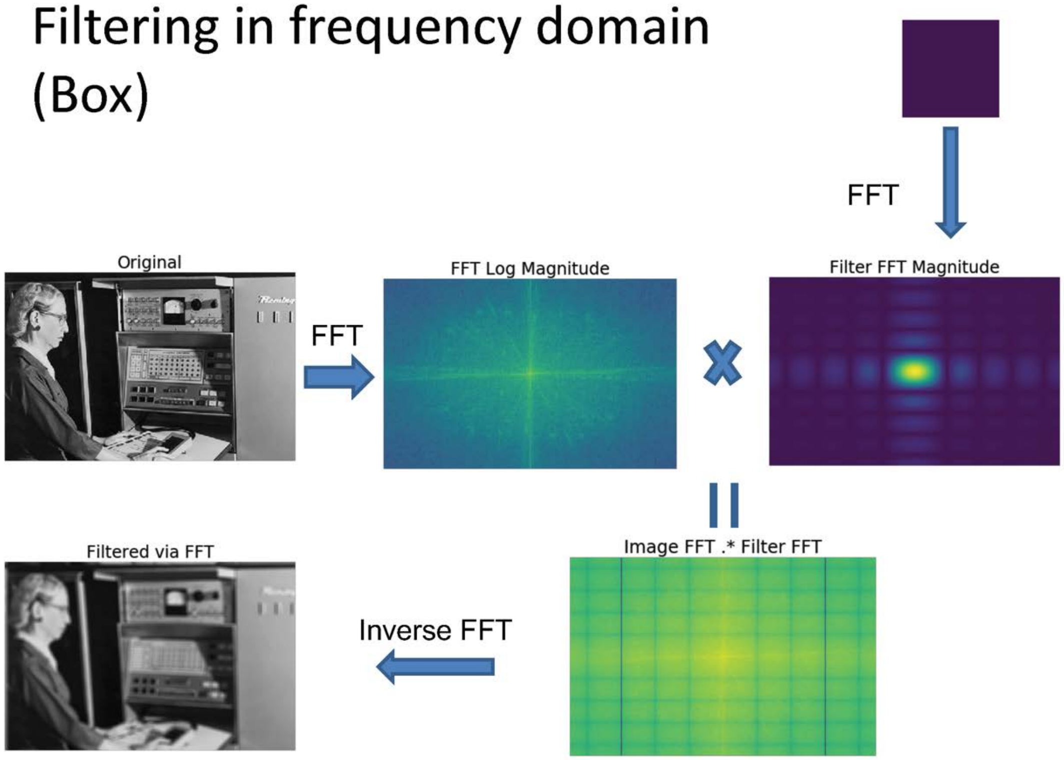


Filtering in frequency domain

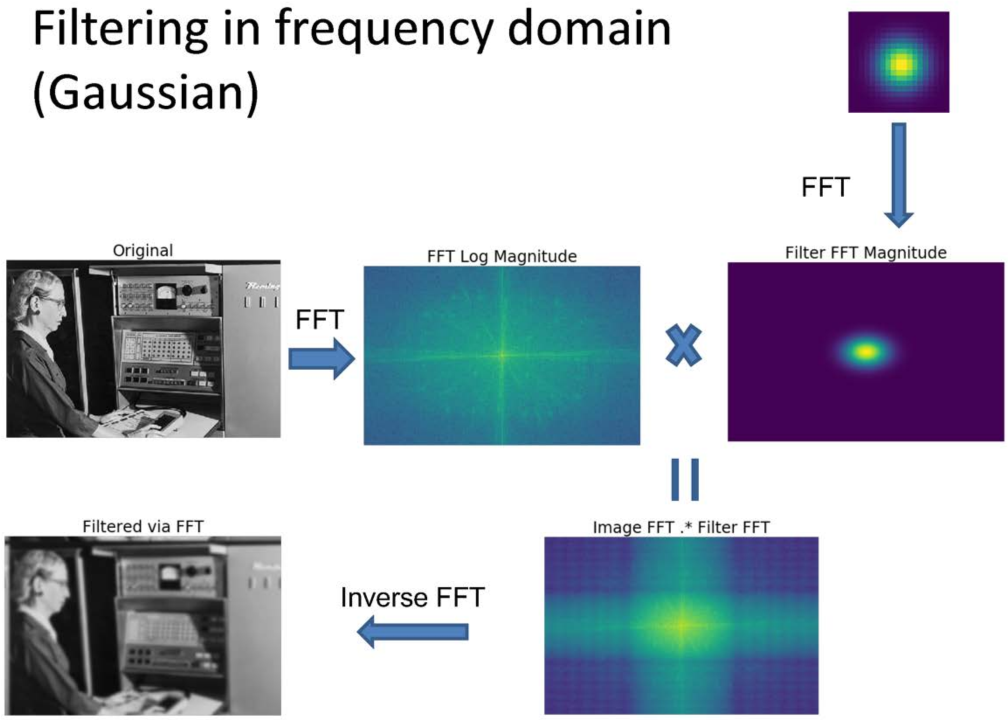


Why does the Gaussian give a nice smooth image, but the square filter give edgy artifacts?

Filtering in frequency domain (Box)



Filtering in frequency domain (Gaussian)

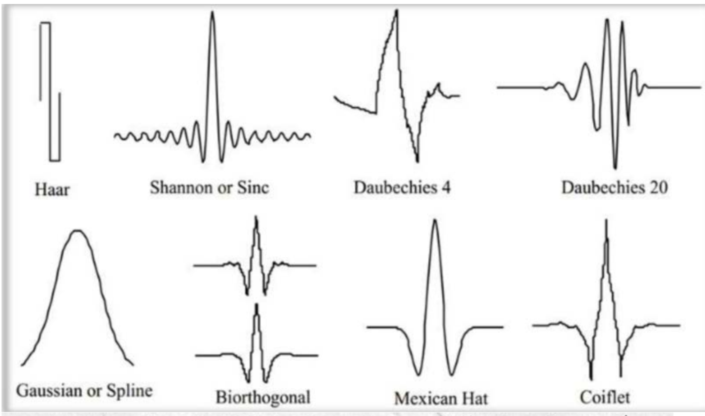


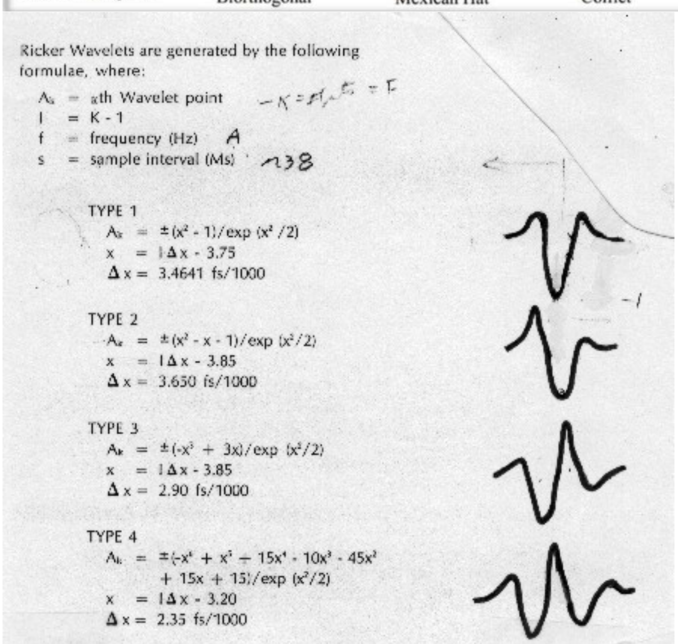
Filtering in frequency domain (Gaussian)

3. 特征函数和基函数变换

https://tracholar.github.io/math/2017/03/12/fourier-transform.html

4. Wavelet transformation





Ricker Wavelets are generated by the following formulae, where:

A_k = kth Wavelet point

l = K - 1

f = frequency (Hz) A

s = sample interval (Ms) 138

TYPE 1

$A_k = \pm(x^2 - 1)/\exp(x^2/2)$

$x = l\Delta x - 3.75$

$\Delta x = 3.4641\ fs/1000$

TYPE 2

$A_k = \pm(x^2 - x - 1)/\exp(x^2/2)$

$x = l\Delta x - 3.85$

$\Delta x = 3.650\ fs/1000$

TYPE 3

$A_k = \pm(-x^3 + 3x)/\exp(x^2/2)$

$x = l\Delta x - 3.85$

$\Delta x = 2.90\ fs/1000$

TYPE 4

$A_k = \pm(-x^6 + x^5 + 15x^4 - 10x^3 - 45x^2 + 15x + 15)/\exp(x^2/2)$

$x = l\Delta x - 3.20$

$\Delta x = 2.35\ fs/1000$

Comparison with Fourier transform and time-frequency analysis [edit]

Transform	Representation	Input
Fourier transform	$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi}\, dx$	ξ frequency
Time-frequency analysis	$X(t, f)$	t time; f frequency
Wavelet transform	$X(a, b) = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} \overline{\Psi\left(\frac{t-b}{a}\right)} x(t)\, dt$	a scaling; b time shift factor

Wavelets have some slight benefits over Fourier transforms in reducing computations when examining specific frequencies. However, they are rarely more sensitive, and indeed, the common Morlet wavelet is mathematically identical to a short-time Fourier transform using a Gaussian window function.[13] The exception is when searching for signals of a known, non-sinusoidal shape (e.g., heartbeats); in that case, using matched wavelets can outperform standard STFT/Morlet analyses.[14]

Example: 2D Discrete Wavelet Analysis: https://www.mathworks.com/help/wavelet/ug/two-dimensional-discrete-wavelet-analysis.html

5. Spherical Harmonic Transformation

Normalization Conventions for Real Associated Legendre and Spherical Harmonic Functions

4π normalized

$$\bar{P}_{lm}(\mu)=\sqrt{(2-\delta_{m0})(2l+1)\frac{(l-m)!}{(l+m)!}}P_{lm}(\mu)$$

$$\int_{-1}^{1}\bar{P}_{lm}(\mu)\,\bar{P}_{l'm}(\mu)=2(2-\delta_{0m})\,\delta_{ll'}$$

$$\int_{\Omega}Y_{lm}(\theta,\phi)\,Y_{l'm'}(\theta,\phi)\,d\Omega=4\pi\,\delta_{ll'}\,\delta_{mm'}$$

$$S_{fg}(l)=\sum_{m=-l}^{l}f_{lm}\,g_{lm}$$

Schmidt seminormalized

$$\bar{P}_{lm}(\mu)=\sqrt{(2-\delta_{m0})\frac{(l-m)!}{(l+m)!}}P_{lm}(\mu)$$

$$\int_{-1}^{1}\bar{P}_{lm}(\mu)\,\bar{P}_{l'm}(\mu)=\frac{2(2-\delta_{0m})}{(2l+1)}\,\delta_{ll'}$$

$$\int_{\Omega}Y_{lm}(\theta,\phi)\,Y_{l'm'}(\theta,\phi)\,d\Omega=\frac{4\pi}{(2l+1)}\,\delta_{ll'}\,\delta_{mm'}$$

$$S_{fg}(l)=\frac{1}{(2l+1)}\sum_{m=-l}^{l}f_{lm}\,g_{lm}$$

Orthonormalized

$$\bar{P}_{lm}(\mu)=\sqrt{\frac{(2-\delta_{0m})(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}}P_{lm}(\mu)$$

$$\int_{-1}^{1}\bar{P}_{lm}(\mu)\,\bar{P}_{l'm}(\mu)=\frac{(2-\delta_{0m})}{2\pi}\,\delta_{ll'}$$

$$\int_{\Omega}Y_{lm}(\theta,\phi)\,Y_{l'm'}(\theta,\phi)\,d\Omega=\delta_{ll'}\,\delta_{mm'}$$

$$S_{fg}(l)=\frac{1}{4\pi}\sum_{m=-l}^{l}f_{lm}\,g_{lm}$$

Unnormalized

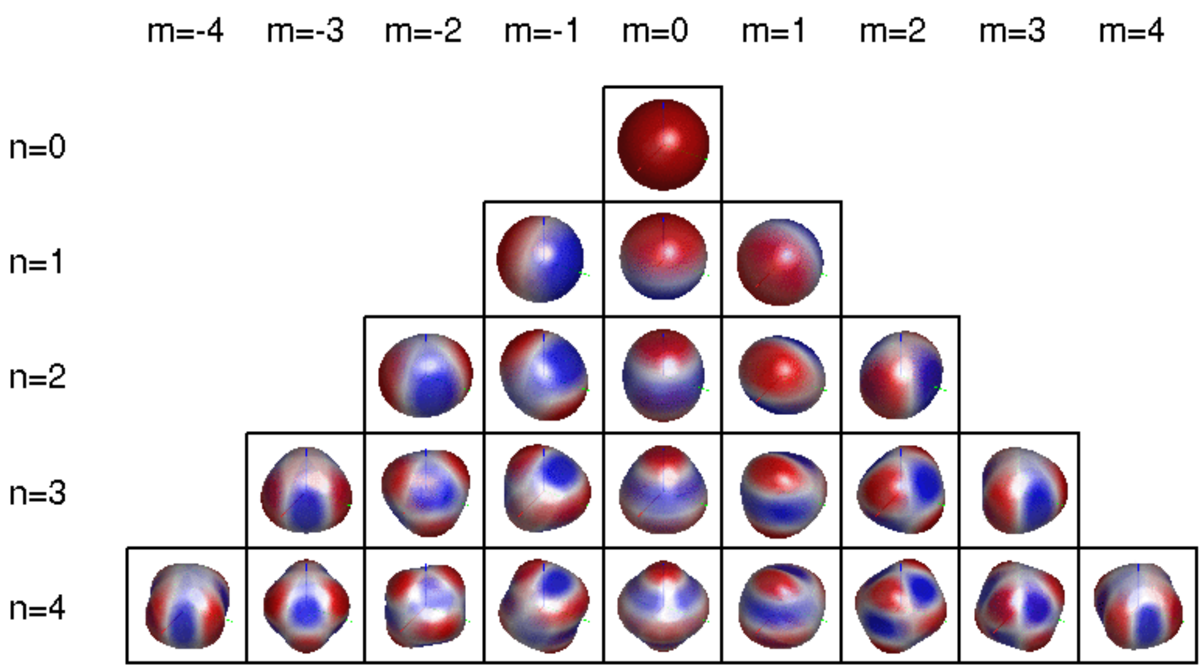
$$\bar{P}_{lm}(\mu)=P_{lm}(\mu)$$

$$\int_{-1}^{1}\bar{P}_{lm}(\mu)\,\bar{P}_{l'm}(\mu)=\frac{2}{(2l+1)}\frac{(l+m)!}{(l-m)!}\,\delta_{ll'}$$

$$\int_{\Omega}Y_{lm}(\theta,\phi)\,Y_{l'm'}(\theta,\phi)\,d\Omega=\frac{4\pi\,(l+m)!}{(2-\delta_{0m})(2l+1)(l-m)!}\,\delta_{ll'}\,\delta_{mm'}$$

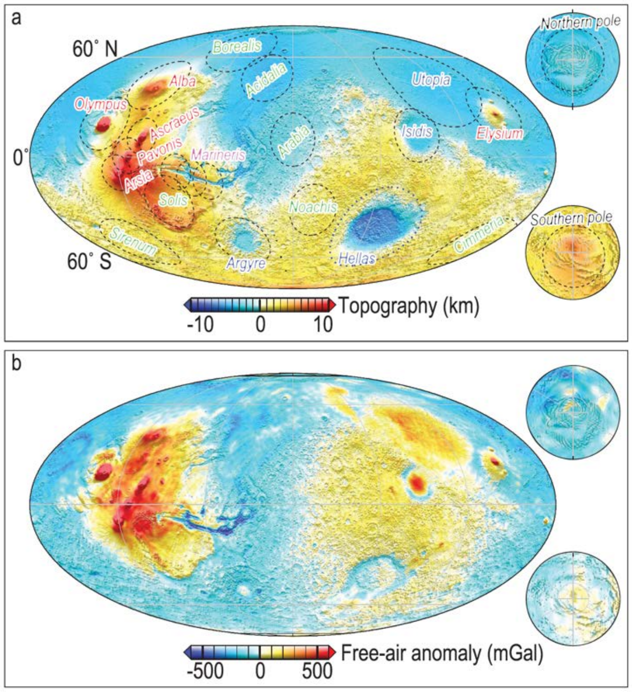
$$S_{fg}(l)=\sum_{m=-l}^{l}\frac{(l+m)!}{(2-\delta_{0m})(2l+1)(l-m)!}f_{lm}\,g_{lm}$$

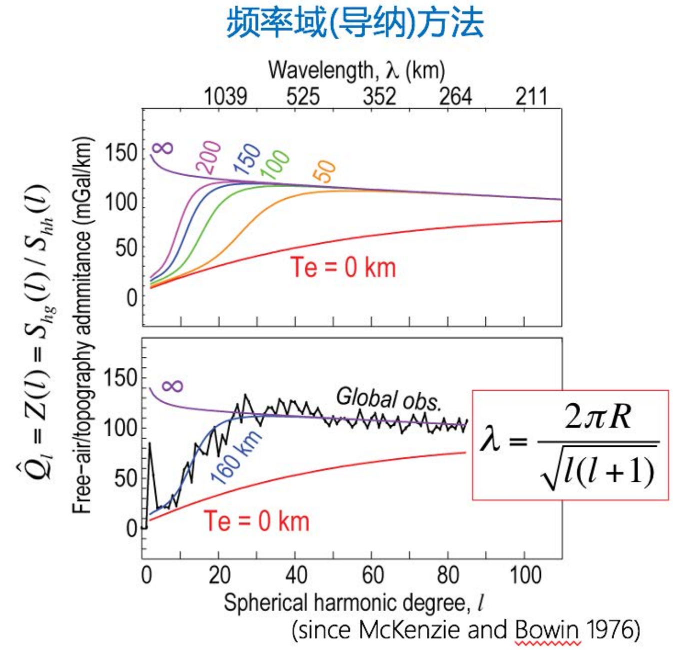
The spherical harmonics have two indices. n determines how many nodal circles the spherical harmonic modes have, and $|m|$ counts how many of them run through north and south pole. Therefore it is clear that the spherical harmonics can be seen as the spatial resolution. Another nice property: they are orthogonal; e.g. only the first one has a non-zero mean value.



From https://iaem.at/Members/zotter/spherical_soundfield/spherical-harmonics

Application 1: Spectral analysis for spherical data



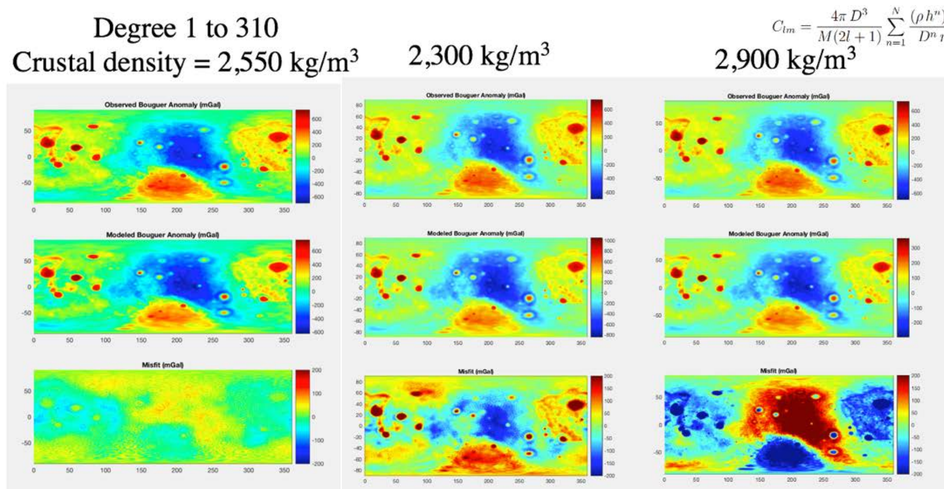


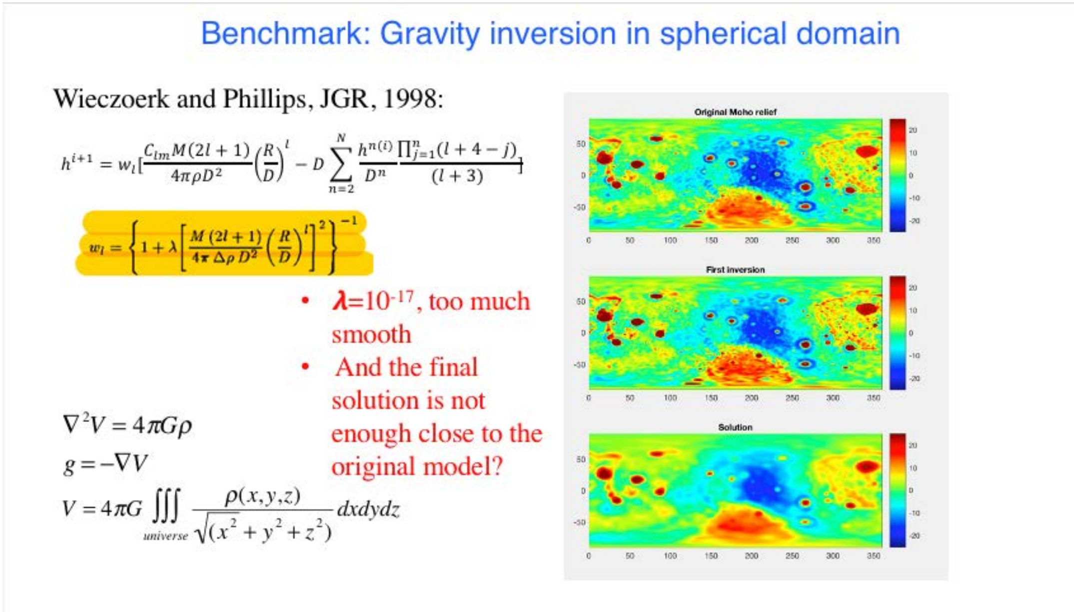
Application 2: Gravity modeling

Benchmark: Gravity modeling in spherical domain

- Forward modeling formula (Wieczorek et al., Science, 2013; Wieczoerk and Phillips, JGR, 1998): $g(\mathbf{r}) = \frac{GM}{r^2} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left(\frac{R_0}{r}\right)^l (l+1) C_{lm} Y_{lm}(\theta, \varphi)$.

$$C_{lm} = \frac{4\pi D^3}{M(2l+1)} \sum_{n=1}^{N} \frac{(\rho h^n)_{lm}}{D^n n!} \frac{\prod_{j=1}^{n}(l+4-j)}{(l+3)},$$





6. Slepian’s Concentration Problem

- Space concentration: seek those functions that are optimally concentrated within a spherical cap.
- Spectral concentration: seek those functions that are optimally concentrated within an effective spherical harmonic bandwidth.
- Ref: Wieczorek, M. A., & Simons, F. J. (2005). Localized spectral analysis on the sphere. *Geophysical Journal International*, *162*(3), 655–675.