Momentum theory for coaxial rotors

Derivations based upon:

Leishman, J. Gordon, and Monica Syal. "Figure of merit definition for coaxial rotors." *Journal of the American Helicopter Society* 53.3 (2008): 290-300.

Table 1. Summary of minimum interference-induced power factors for coaxial rotors operating under different conditions

Case No.	Interference-Induced Power Factor, κ_{int}
1	1.4142
2	1.4142
3	1.2808
4a	1.2810
4b	1.2657

Coaxial rotor model: Equal upper and lower inflows

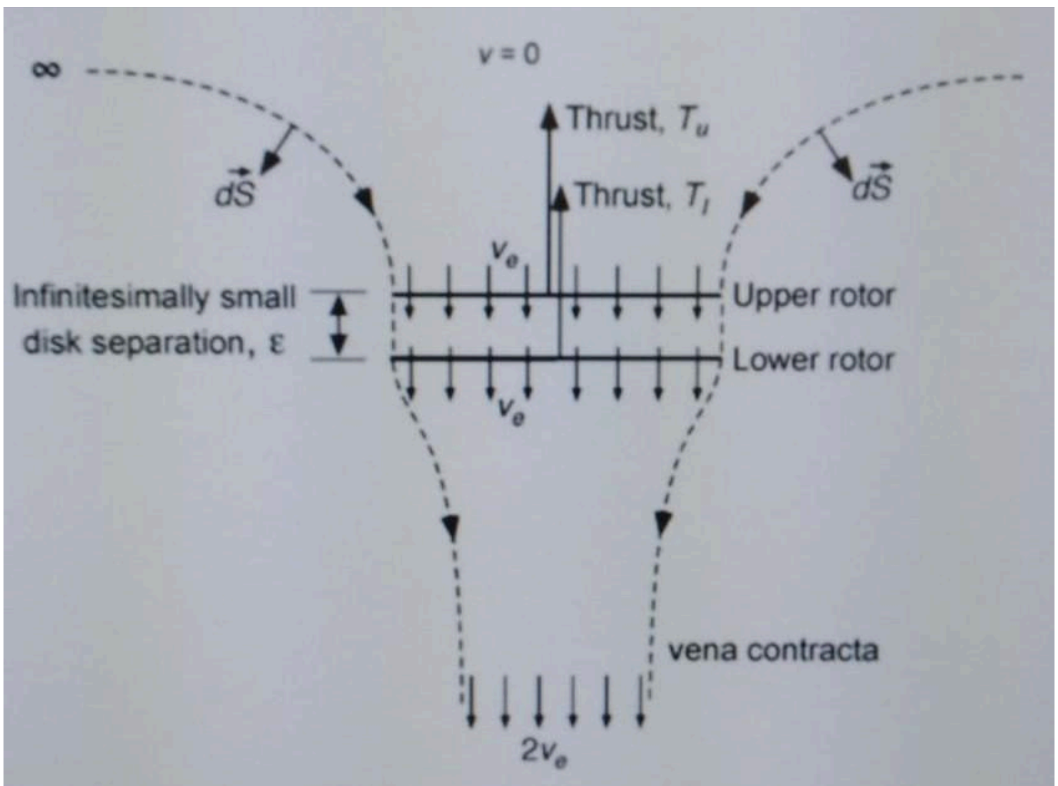


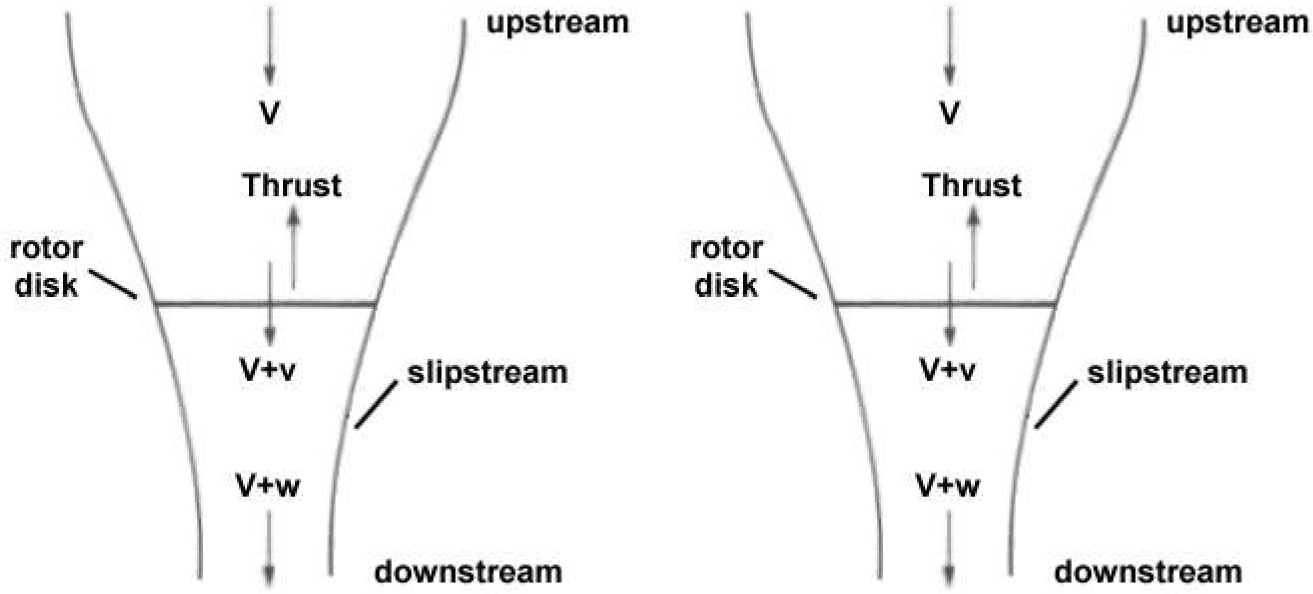
Fig. 1. Flow model of a coaxial rotor system with both rotors corotating in the same plane and sharing the same value of induced velocity.

Source: Leishman, J. Gordon, and Monica Syal. "Figure of merit definition for coaxial rotors." *Journal of the American Helicopter Society* 53.3 (2008): 290-300.

This condition is only possible at a Thrust-sharing ratio of 1. Because in MT, equal induced velocity results in equal Thrust.

This condition produces a kint = coaxEqual_P / coaxIsola_P = 1.4142

Coaxial rotor model: Isolated upper and lower inflows



Min Power as function of Thrust

```
clear all
close all
format compact

syms rho A T_0 lambda
syms T_1 T_2
assume(A>0);
assume(rho>0);
assume(T_0>0);
assume(T_1>0);
assume(T_2>0);

P_1 = sqrt(T_1^3/(2*rho*A));
P_2 = sqrt(T_2^3/(2*rho*A));
P_ind = simplify( P_1 + P_2 )
```

P_ind =

$$\frac{\sqrt{2}\,\left({T_1}^{3/2}+{T_2}^{3/2}\right)}{2\,\sqrt{A}\,\sqrt{\rho}}$$

```

% Constraint
f = T_0 - T_1 - T_2;

x = [T_1; T_2];
H = P_ind + lambda*f
```

H =

$$\frac{\sqrt{2}\,\left({T_1}^{3/2}+{T_2}^{3/2}\right)}{2\,\sqrt{A}\,\sqrt{\rho}}-\lambda\,\left(T_1-T_0+T_2\right)$$

```
% Hx = Lx + dfdx*lambda
Hx = jacobian(H, x);
Hx = simplify(Hx);
Hx = transpose(Hx)
```

Hx =

$$\begin{pmatrix}\frac{3\,\sqrt{2}\,\sqrt{T_1}}{4\,\sqrt{A}\,\sqrt{\rho}}-\lambda\\ \frac{3\,\sqrt{2}\,\sqrt{T_2}}{4\,\sqrt{A}\,\sqrt{\rho}}-\lambda\end{pmatrix}$$

```
% Hlambda = f;
% Hlambda = jacobian(H, lambda);
% Hlambda = simplify(Hlambda);

[solx, params, conditions] = solve(Hx(1) == Hx(2), T_1, 'ReturnConditions', true)
```

solx = T_2

params =
Empty sym: 1-by-0

conditions = TRUE

Min Power as function of Omega

```
clear all
close all

syms k_T omega_1 omega_2 rho A T_0 lambda c_1 c_2
assume(A>0);
assume(rho>0);
assume(T_0>0);
assume(omega_1>0);
assume(omega_2>0);
assume(c_1>0);
assume(c_2>0);

% T_1 = k_T * omega_1^2;
% T_2 = k_T * omega_2^2;
% P_1 = sqrt(T_1^3/(2*rho*A));
% P_2 = sqrt(T_2^3/(2*rho*A));
P_1 = c_1 * sqrt(omega_1^6);     % c_1 = sqrt(k_T^3/(2*rho*A))
P_2 = c_1 * sqrt(omega_2^6);     % c_1 = sqrt(k_T^3/(2*rho*A))
P_ind = simplify( P_1 + P_2 )
```

P_ind = $c_1\,\left({\omega_1}^3+{\omega_2}^3\right)$

```
% Constraint
%f = T_0/k_T - omega_1^2 - omega_2^2;
```

```
f = c_2 - omega_1^2 - omega_2^2;     % c_2 = T_0/k_T

x = [omega_1; omega_2];
H = P_ind + lambda*f
```

H = $c_1 \left(\omega_1{}^3 + \omega_2{}^3\right) - \lambda \left(\omega_1{}^2 + \omega_2{}^2 - c_2\right)$

```
% Hx = Lx + dfdx*lambda
Hx = jacobian(H, x);
Hx = simplify(Hx);
Hx = transpose(Hx)
```

Hx =

$$\begin{pmatrix} 3\, c_1\, \omega_1{}^2 - 2\, \lambda\, \omega_1 \\ 3\, c_1\, \omega_2{}^2 - 2\, \lambda\, \omega_2 \end{pmatrix}$$

```
% Hlambda = f;
% Hlambda = jacobian(H, lambda);
% Hlambda = simplify(Hlambda);

% Solution is omega_1 = omega_2 = sqrt(c_2/2)
[solx, params, conditions] = solve(Hx(1) == 0, omega_1, 'ReturnConditions', true)
```

solx =

$$\frac{2\, \lambda}{3\, c_1}$$

params =
Empty sym: 1-by-0

conditions = $0 < \lambda$

```
[solx, params, conditions] = solve(Hx(2) == 0, omega_2, 'ReturnConditions', true)
```

solx =

$$\frac{2\, \lambda}{3\, c_1}$$

params =
Empty sym: 1-by-0

conditions = $0 < \lambda$

Coaxial rotor model: Lower inflow dependence on upper rotor

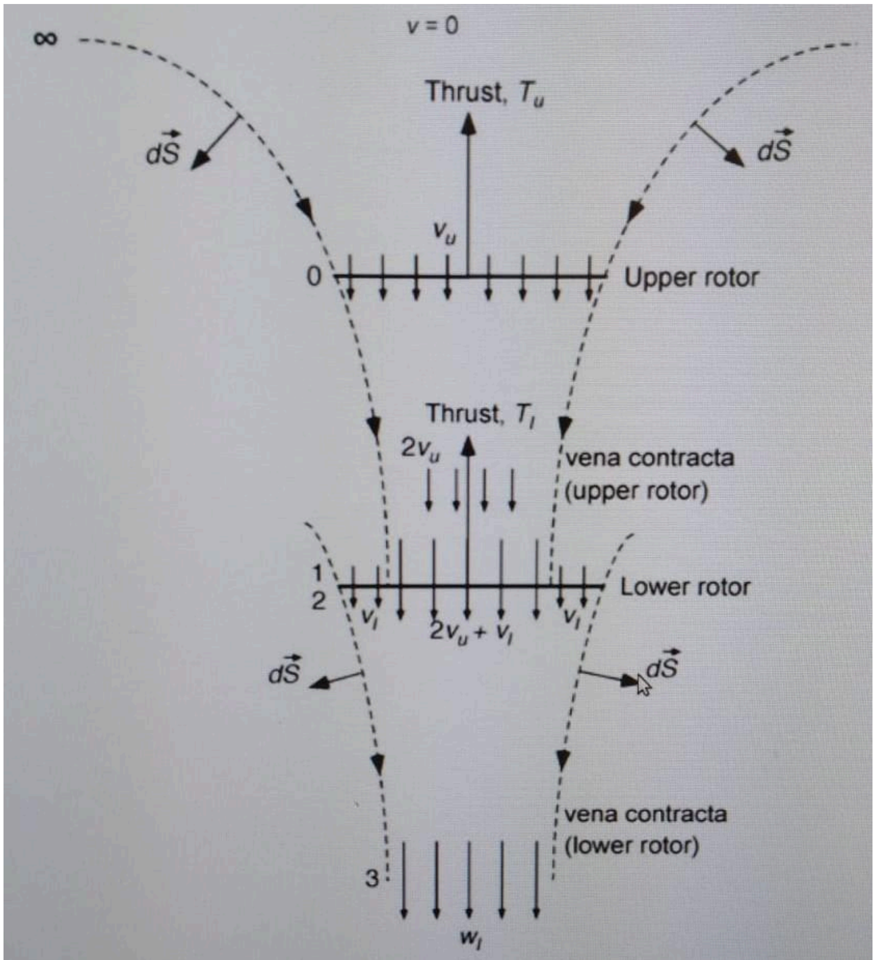
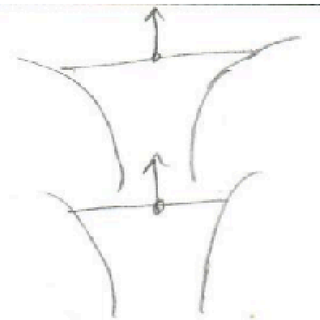


Fig. 2. Flow model of a coaxial rotor system with the lower rotor operating in the fully developed slipstream of the upper rotor.

Source: Leishman, J. Gordon, and Monica Syal. "Figure of merit definition for coaxial rotors." *Journal of the American Helicopter Society* 53.3 (2008): 290-300.

Case 3



eq 1) $T_u = 2\rho A v_u^2$

eq 2) $T_u v_u = \frac{1}{2}\rho A v_u w_u^2 \quad = P_u$

eq 3) $T_l = \rho A(v_u + v_l) w_l - 2\rho A v_u^2$

eq 4) $T_l(v_u + v_l) = \frac{1}{2}\rho A(v_u + v_l) w_l^2 - 2\rho A v_u^3 \quad = P_l$

eq 5) $\frac{T_u}{T_l} = \alpha$ eq 6) $T_u + T_l = W$ eq 7) $P = P_u + P_l$

eq 1–4) $\Longrightarrow$ 4 unknowns (v_u, w_u, v_l, w_l)
4 equations
T_u, T_l known values

$$\min \; P = \left(\frac{2\rho A v_u^2}{\alpha}\right)(v_u + v_l) + \left(2\rho A v_u^3\right)$$

st: $0 = \text{quadratic}(v_l \,;\, v_u, \alpha)$

where: $v_u = \frac{T_u^{3/2}}{\sqrt{2\rho A}}$ = upper induced velocity

$\frac{T_u}{T_l} = \alpha$ = thrust-sharing ratio

$$T_l = \rho A(v_u + v_l) w_l - T_u \Rightarrow W = \rho A(v_u + v_l) w_l$$

$$T_l(v_u + v_l) = \frac{1}{2}\rho A(v_u + v_l) w_l^2 - T_u v_u = P_l$$

$$T_l(v_u + v_l) = \frac{1}{2} W w_l - T_u v_u \Rightarrow T_u v_u + T_l(v_u + v_l) = \frac{1}{2} W w_l$$

$$w_l = \frac{2}{W}\left(T_u v_u + T_l(v_u + v_l)\right)$$

$$W = \rho A(v_u + v_l)\frac{2}{W}\left(T_u v_u + T_l(v_u + v_l)\right)$$

$$\frac{W^2}{2\rho A} = (v_u + v_l)\left(T_u v_u + T_l(v_u + v_l)\right)$$

$$\frac{W^2}{2\rho A} = (v_u + v_l)\left(v_u(T_u + T_l) + v_l(T_l)\right)$$

$$= (T_u + T_l)v_u^2 + (2T_l + T_u)v_u v_l + (T_l)v_l^2$$

$$\frac{W^2}{2\rho A} = (W)v_u^2 + [(T_l + W)v_u]v_l + (T_l)v_l^2$$

$$\frac{W^2}{2\rho A} = (W - T_u)v_l^2 + [(2W - T_u)v_u]v_l + (W v_u^2)$$

Given $\frac{T_u}{T_l} = \alpha \Rightarrow W = T_u + \frac{1}{\alpha}T_u$ and $T_u = 2\rho A v_u^2$

$$0 = \left(\frac{1}{\alpha}\right)v_l^2 + \left[\left(\frac{2+\alpha}{\alpha}\right)v_u\right]v_l - \left(\frac{1+\alpha}{\alpha^2}\right)v_u^2$$

quadratic eq on v_u

If $T = \frac{W}{2} = T_u = T_l$

$$\frac{W^2}{2\rho A} = (v_u + v_l)(2T v_u + T v_l)$$

$$\frac{(2T)^2}{2\rho A} = T(v_u + v_l)(2v_u + v_l)$$

$$\frac{2}{\rho A}T = (v_u + v_l)(2v_u + v_l)$$

$$\frac{2}{\rho A}(2\rho A v_u^2) = 2v_u^2 + v_u v_l + 2v_u v_l + v_l^2$$

$$4v_u^2 = v_l^2 + (3v_u)v_l + (2v_u^2)$$

$$0 = v_l^2 + (3v_u)v_l - (2v_u^2)$$

$$v_l = \frac{-3v_u + \left(9v_u^2 + 4\cdot 2v_u^2\right)^{1/2}}{2}$$

$$v_l = \left(\frac{-3 + \sqrt{17}}{2}\right)v_u$$

```
clear all
close all

syms rho A W lambda alpha
syms v_u v_l w_l
assume(v_u>0);
assume(v_l>0);
assume(w_l>0);

% [solx, params, conditions] = solve(P_l == T_l*(v_u + v_l), v_l, 'ReturnConditions', t
eq_v_l = (1/alpha)*v_l^2 + ((2+alpha)/alpha)*v_u*v_l - ((1+alpha)/(alpha^2))*v_u^2;
[solx, params, conditions] = solve(eq_v_l == 0,v_l, 'ReturnConditions', true)
```

solx =

$$\begin{pmatrix} -v_u - \frac{\alpha\, v_u}{2} - \frac{\alpha\, v_u \sqrt{\frac{\alpha^3 + 4\,\alpha^2 + 8\,\alpha + 4}{\alpha^3}}}{2} \\ \frac{\alpha\, v_u \sqrt{\frac{\alpha^3 + 4\,\alpha^2 + 8\,\alpha + 4}{\alpha^3}}}{2} - \frac{\alpha\, v_u}{2} - v_u \end{pmatrix}$$

params =
Empty sym: 1-by-0
conditions =

$$\begin{pmatrix} v_u < -\frac{\alpha\, v_u \left(\sqrt{\frac{\alpha^3 + 4\,\alpha^2 + 8\,\alpha + 4}{\alpha^3}} + 1\right)}{2} \\ v_u < \alpha\, v_u \left(\frac{\sqrt{\frac{\alpha^3 + 4\,\alpha^2 + 8\,\alpha + 4}{\alpha^3}}}{2} - \frac{1}{2}\right) \end{pmatrix}$$

```
%v_l =(sqrt( alpha^2 + 4*alpha + 8 + 4/alpha ) -alpha -2 )/2
v_l = solx(2)
```

v_l =

$$\frac{\alpha\, v_u \sqrt{\frac{\alpha^3 + 4\,\alpha^2 + 8\,\alpha + 4}{\alpha^3}}}{2} - \frac{\alpha\, v_u}{2} - v_u$$

$$v_l = \frac{\alpha\left(\sqrt{\dfrac{\alpha^3+4\alpha^2+8\alpha+4}{\alpha^2 \quad \alpha}}\right) - \alpha - 1}{2} v_{ul}$$

$$= \frac{\left(\sqrt{\dfrac{\alpha^3+4\alpha^2+8\alpha+4}{\alpha}}\right) - \alpha - 1}{2} v_u = \beta(\alpha)\, v_u$$

But we have

$$P_u = T_u v_u = 2\rho A v_u^3 \quad ; \quad \frac{T_u}{T_l} = \alpha \Rightarrow T_l = \frac{1}{\alpha} T_u = \frac{2\rho A v_u^2}{\alpha}$$

$$P_l = T_l (v_u + v_l) = \left(\frac{2\rho A v_u^2}{\alpha}\right)\left(v_u + v_u \beta(\alpha)\right)$$

therefore

$$P = P_u + P_l = (2\rho A v_u^3) + \left(\frac{2\rho A v_u^3}{\alpha}\right)(1+\beta(\alpha))$$

$$P(\alpha) = 2\rho A v_u^3 \left(\frac{\alpha + 1 + \beta(\alpha)}{\alpha}\right)$$

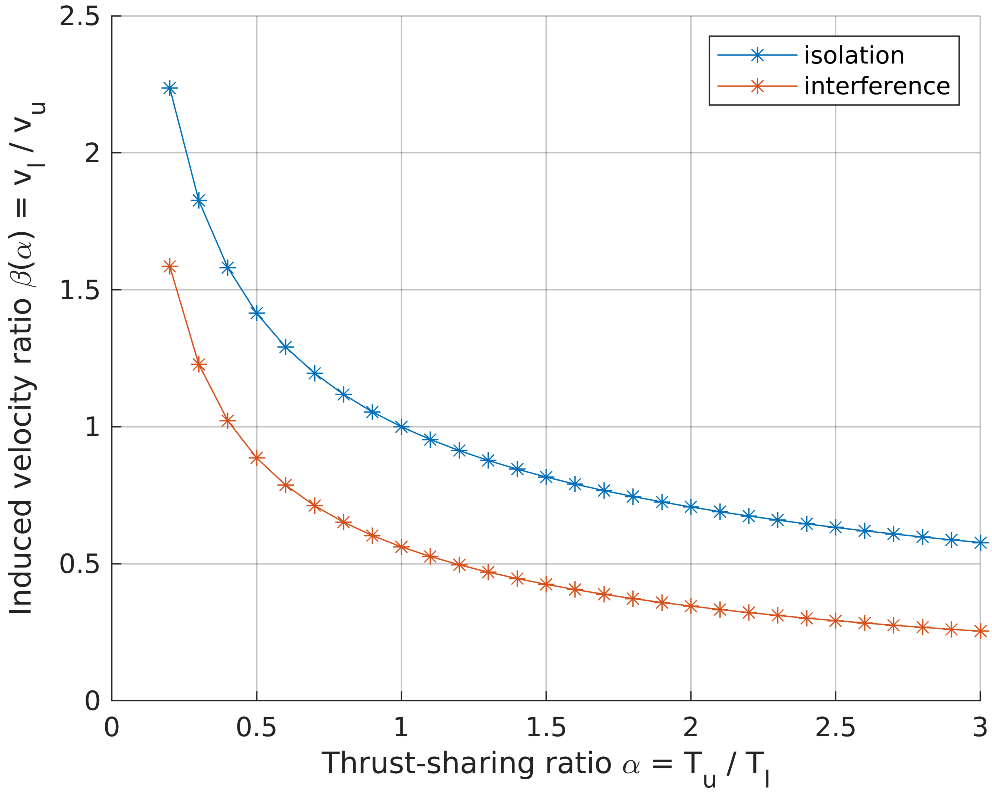
But $T_u = 2\rho A v_u^2$; $v_u = \sqrt{\dfrac{T_u}{2\rho A}}$; $T_u + \dfrac{1}{\alpha} T_u = W \Rightarrow T_u = W\left(\dfrac{\alpha}{\alpha+1}\right)$

$$\Rightarrow 2\rho A v_u^3 = T_u \sqrt{\frac{T_u}{2\rho A}} = \frac{1}{\sqrt{2\rho A}} W^{3/2} \left(\frac{\alpha}{\alpha+1}\right)^{3/2}$$

$$P(\alpha) = \frac{W^{3/2}}{\sqrt{2\rho A}} \left(\frac{\alpha}{\alpha+1}\right)^{3/2} \left(\frac{\alpha + 1 + \beta(\alpha)}{\alpha}\right)$$

Coaxial rotor model comparison

```
clear all
close all

alpha_arr = 0.2:0.1:3;
ns = length(alpha_arr);

coaxIsola_beta_arr = zeros(ns, 1);
coaxIsola_P_arr = zeros(ns, 1);
coaxInter_beta_arr = zeros(ns, 1);
coaxInter_P_arr = zeros(ns, 1);
for i=1:ns
    alpha = alpha_arr(i);

    % T_u = W*((alpha)/(alpha+1))
    % Pind = Tu*v_u = ( W*((alpha)/(alpha+1)) )*( (T_u/(2*rho*A))^0.5 )
    % Pind = ( W*normf )*( (W*((alpha)/(alpha+1))/(2*rho*A))^0.5 )
    % Pind = W^(3/2) * ((2*rhp*A)^-0.5) * ( (alpha)/(alpha+1) )^(3/2)
    normf = ( (alpha)/(alpha+1) )^(3/2);

    % isolation
    beta = sqrt(1/alpha);                  % alpha = T_u / T_l = v_u^2 / v_l^2
    coaxIsola_beta_arr(i) = beta;          % beta = sqrt(1/alpha)
    Pind = (1 + beta^3);                   % Pind = T_u*v_u*fnct(beta)
    coaxIsola_P_arr(i) = normf * Pind;     % Normalize to same total weight

    % Interference
    beta = (sqrt( alpha^2 + 4*alpha + 8 + 4/alpha ) -alpha -2 )/2;
    coaxInter_beta_arr(i) = beta;
    Pind = (alpha + 1 + beta)/(alpha);     % Pind = T_u*v_u*fnct(beta)
    coaxInter_P_arr(i) = normf * Pind;     % Normalize to same total weight
end

figure;
hold on;
grid on;
plot(alpha_arr, coaxIsola_beta_arr, '-*')
plot(alpha_arr, coaxInter_beta_arr, '-*')
xlabel('Thrust-sharing ratio \alpha = T_u / T_l')
ylabel('Induced velocity ratio \beta(\alpha) = v_l / v_u')
legend('isolation', 'interference')
```



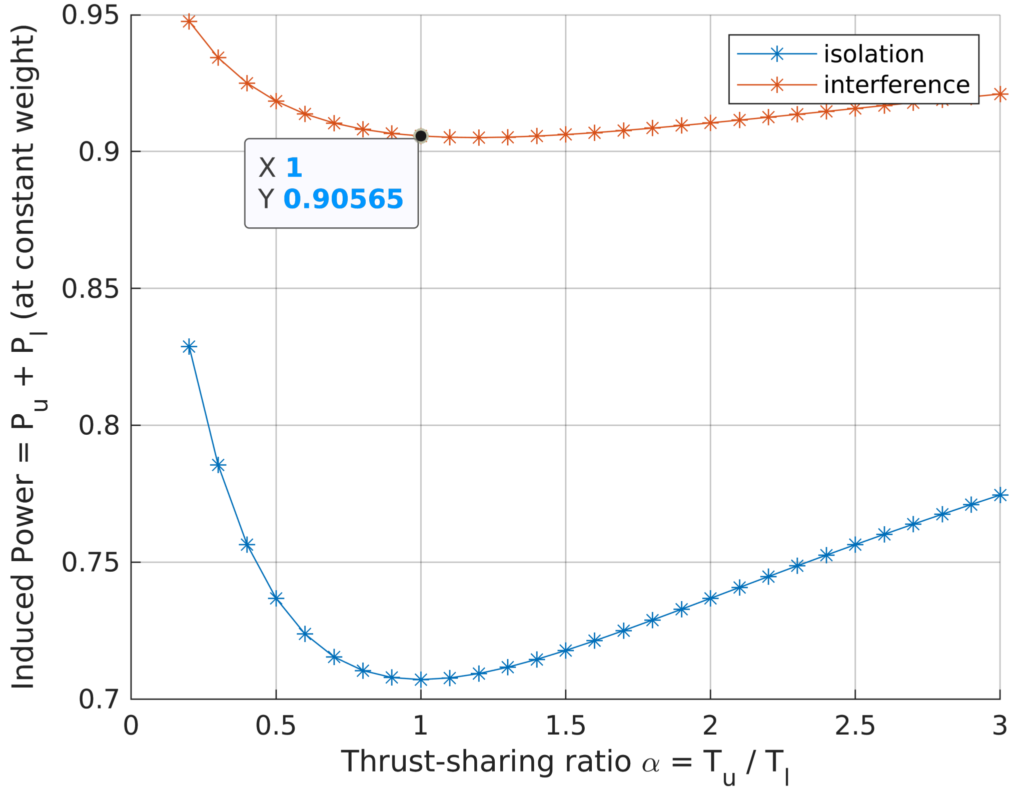

```
% Check if alpha=1 gives us back the result on Leishman's paper
% for isolation and equal thrust: v_l = 1.0000*v_u
coaxIsola_beta_arr(find(alpha_arr==1))
```

```
ans = 1
```

```
% for interference and equal thrust: v_l = 0.5616*v_u
coaxInter_beta_arr(find(alpha_arr==1))
```

```
ans = 0.5616
```

```
figure;
hold on;
grid on;
plot(alpha_arr, coaxIsola_P_arr, '-*')
plot(alpha_arr, coaxInter_P_arr, '-*')
xlabel('Thrust-sharing ratio \alpha = T_u / T_l')
ylabel('Induced Power = P_u + P_l (at constant weight)')
legend('isolation', 'interference')
```




```
% Check if alpha=1 gives us back the result on Leishman's paper
% for isolation and equal torque: P_u = P_l, v_l = 1.0000*v_u and alpha = 1.0000
alpha = 1.0000;
beta = sqrt( 1/alpha )
```

```
beta = 1
```

```
coaxIsola_P = (1 + beta)
```

```
coaxIsola_P = 2
```

```
% for interference and equal torque: P_u = P_l, v_l = 0.4375*v_u and alpha = 1.4375
alpha = 1.4375;
beta = (sqrt( alpha^2 + 4*alpha + 8 + 4/alpha ) -alpha -2 )/2
```

```
beta = 0.4376
```

```
coaxInter_P = (alpha+1+beta)/(alpha)
```

```
coaxInter_P = 2.0001
```

```
% Reproduce kint
alpha = 1.0000;
beta = sqrt( 1/alpha );
coaxIsola_P = (1 + beta);

alpha = 1.0000;
beta = (sqrt( alpha^2 + 4*alpha + 8 + 4/alpha ) -alpha -2 )/2;
coaxInter_P = (alpha+1+beta)/(alpha);
```

```
kint = coaxInter_P / coaxIsola_P
```

```
kint = 1.2808
```