

Lecție 10

Aplicații

Un K -spatiu vectorial este format dintr-un grup abelian $(V, +)$ în care respectă următoarele axiome:

$$\textcircled{1} \cdot \alpha \cdot (x+y) = \alpha x + \alpha y$$

$$\textcircled{2} \cdot (\alpha + \beta)x = \alpha x + \beta x$$

$$\textcircled{3} \cdot \alpha(\beta x) = (\alpha\beta)x$$

$$\textcircled{4} \cdot 1 \cdot x = x$$

pt. orice $x, y \in V$ și $\alpha, \beta \in K$

\textcircled{1} Să se arate că $\mathbb{R}_+^* = (0, \infty)$ este un \mathbb{R} -spatiu vectorial în raport cu adunarea vectorilor:

$$+ : R_+^* \times R_+^* \rightarrow R_+^*$$

$$x \boxplus y = x \cdot y, \quad \forall x, y \in R_+^*$$

nă
- cu înmulțirea cu scalară:

$$\cdot : R \times R_+^* \rightarrow R_+^*$$

$$\alpha \cdot x = x^\alpha, \quad \forall x \in R_+^*, \quad \alpha \in \mathbb{R}$$

Soluție: I. $(R_+^*, +)$ grup abelian?

1. P. s.:

$$\forall x, y \in R_+^* \implies x \boxplus y \in R_+^*$$

$$x \boxplus y = \frac{x}{\in R_+^*} \cdot \frac{y}{\in R_+^*}$$

2. asociativitate:

$$\forall x, y, z \in R_+^*, (x \boxplus y) \boxplus z = x \boxplus (y \boxplus z)$$

$$(x \boxplus y) \boxplus z = (xy) \boxplus z = xyz$$

$$= x(yz) = x \boxplus (yz) = x \boxplus (y \boxplus z)$$

3. comutativitate:

$$\forall x, y \in R_+^*, x \boxplus y = y \boxplus x$$

$$x \boxplus y = x \cdot y = y \cdot x = y \boxplus x$$

④ element neutru:

$$\exists \ell \in R_+^*, \forall x \in R_+^*$$

$$x \boxplus \ell = \ell \boxplus x = x$$

com.

$$x \boxplus \ell = x \Rightarrow$$

$$x \cdot \ell = x \quad | : x \neq 0$$

$$\ell = 1 \in R_+^*$$

⑤ element simetricabil

$$\forall x \in R_+^*, \exists x' \in R_+^* \text{ a.t.}$$

$$x \boxplus x' = x' \boxplus x = \ell$$

com.

$$\Rightarrow x \boxplus x' = 1 \iff x \cdot x' = 1$$

$$x' = \frac{1}{x},$$

$$\forall x \in R_+^*$$

①, .., ⑤ $\Rightarrow (R_+^*, \boxplus)$

grup abelian

II) 1.) $d \square (x \boxplus y) = (d \square x) \boxplus (d \square y)$

$$d \square (x \boxplus y) = d \square (xy) = (xy)^d =$$

$$= x^d y^d = x^d \boxplus y^d =$$

$$= (d \square x) \boxplus (d \square y),$$

$\forall x, y \in \mathbb{R}_+^*, \lambda \in \mathbb{R}$

2) $(\lambda + \beta) \square x = (\lambda \square x) \oplus (\beta \square x)$

$$\begin{aligned}(\lambda + \beta) \square x &= x^\lambda + \beta \\&= (\lambda \square x) \oplus (\beta \square x), \quad \forall x \in \mathbb{R}_+^*, \lambda, \beta \in \mathbb{R}\end{aligned}$$

3) $(\lambda \cdot \beta) \square x = \lambda \square (\beta \square x)$

$$\begin{aligned}(\lambda \cdot \beta) \square x &= x^{\lambda \cdot \beta} = (x^\beta)^\lambda = \\&= (\beta \square x)^\lambda = \lambda \square (\beta \square x)\end{aligned}$$

4) $1 \square x = x \quad \forall x \in \mathbb{R}_+^*$

$$1 \square x = x^1 = x$$

I, II $\Rightarrow \mathbb{R}_+^*$ este un \mathbb{R} -spatiu vectorial impreună cu \oplus și op. externă \square

2. să se verifice că operațiile

$$\oplus : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$$

$$x \oplus y = \sqrt[5]{x^5 + y^5}, \quad \forall x, y \in \mathbb{R}$$

$$\square : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}, \quad \lambda \square x = \sqrt[5]{\lambda} \cdot x$$

$\forall \lambda, x \in \mathbb{R}$

definim o structură de
 \mathbb{R} - spațiu vectorial.
Soluție: I. (\mathbb{R}, \oplus) gr. abelian

1) Proprietații:

$$\forall x, y \in \mathbb{R} \Rightarrow x \oplus y \in \mathbb{R}$$

$$x \oplus y = \sqrt[5]{x^5 + y^5} \in \mathbb{R}$$

2) asociativitatea:

$$\forall x, y, z \in \mathbb{R} \Rightarrow (x \oplus y) \oplus z = x \oplus (y \oplus z)$$

$$(x \oplus y) \oplus z = \sqrt[5]{x^5 + y^5} \oplus z = \\ = \sqrt[5]{(\sqrt[5]{x^5 + y^5})^5 + z^5} = \sqrt[5]{x^5 + y^5 + z^5}$$

$$x \oplus (y \oplus z) = x \oplus \sqrt[5]{y^5 + z^5} = \\ = \sqrt[5]{x^5 + y^5 + z^5}$$

3) comutativitatea:

$$\forall x, y \in \mathbb{R}, x \oplus y = y \oplus x$$

$$\underline{x \boxplus y} = \sqrt[5]{x^5 + y^5} = \sqrt[5]{y^5 + x^5}$$

$$= y \boxplus x$$

4.) element neutru

$$\exists e \in \mathbb{R}, \forall x \in \mathbb{R} \text{ a.i. } x \boxplus e =$$

$$= e \boxplus x = x \quad \text{com.}$$

$$x \boxplus e = x \iff \sqrt[5]{x^5 + e^5} = x \quad |^5$$

$$x^5 + e^5 = x^5$$

$$e^5 = 0 \Rightarrow e = 0 \in \mathbb{R}$$

5.) elemente simetrizabile

$$\forall x \in \mathbb{R}, \exists x' \in \mathbb{R} \text{ a.i. } x \boxplus x' =$$

$$= x' \boxplus x = e \quad \text{com.}$$

$$x \boxplus x' = 0 \iff \sqrt[5]{x^5 + x'^5} = 0 \quad |^5$$

$$x^5 + x'^5 = 0$$

$$x'^5 = -x^5$$

$$x' = -x, \forall x \in \mathbb{R}$$

Din 1., .., 5.) $\Rightarrow (\mathbb{R}, \boxplus)$ este grup abelian

$$\text{II. 1.) } d \boxminus (x \boxplus y) = (d \boxminus x) \boxplus (d \boxminus y)$$

$$\forall x, y, d \in \mathbb{R}$$

$$\alpha \square x = \sqrt[5]{\alpha} \cdot x$$

$$x \oplus y = \sqrt[5]{x^5 + y^5}$$

$$\begin{aligned}\alpha \square \sqrt[5]{x^5 + y^5} &= \sqrt[5]{\alpha} \cdot \sqrt[5]{x^5 + y^5} \\ &= \sqrt[5]{\alpha(x^5 + y^5)}\end{aligned}$$

$$\begin{aligned}(\alpha \square x) \oplus (\alpha \square y) &= (\sqrt[5]{\alpha} \cdot x) \oplus \\ &\quad (\sqrt[5]{\alpha} \cdot y) = \sqrt[5]{\alpha \cdot x^5 + \alpha \cdot y^5} \\ &= \sqrt[5]{\alpha(x^5 + y^5)}\end{aligned}$$

$$2.) (\alpha + \beta) \square x = (\alpha \square x) \oplus (\beta \square x)$$

$$(\alpha + \beta) \square x = \sqrt[5]{\alpha + \beta} \cdot x$$

$$\begin{aligned}(\alpha \square x) \oplus (\beta \square x) &= \\ &= (\sqrt[5]{\alpha} \cdot x) \oplus (\sqrt[5]{\beta} \cdot x) \\ &= \sqrt[5]{(\sqrt[5]{\alpha} \cdot x)^5 + (\sqrt[5]{\beta} \cdot x)^5} \\ &= \sqrt[5]{\alpha \cdot x^5 + \beta \cdot x^5} = x \cdot \sqrt[5]{\alpha + \beta}\end{aligned}$$

$$3.) \alpha \square (\beta \square x) = (\alpha \cdot \beta) \square x$$

$$\alpha \square (\beta \square x) = \alpha \square (\sqrt[5]{\beta} \cdot x) =$$

$$= \sqrt[5]{\alpha} \cdot \sqrt[5]{\beta} \cdot x$$

$$(\alpha \cdot \beta) \square x = \sqrt[5]{\alpha \cdot \beta} \cdot x = \sqrt[5]{\alpha} \cdot \sqrt[5]{\beta} \cdot x$$

4.) $1 \square x = x$

$$1 \square x = \sqrt{1} \cdot x = x$$

$\xrightarrow{\text{I}, \text{II}}$

R este un R -spatiu vectorial impreună cu op. internă \oplus și ea externă \square

Fie V un K -sp. vectorial,

$U \subseteq V$. U.A.S.E:

(1): $U \leq_K V$

(2): a.) $0 \in U$

b.) $x, y \in U \Rightarrow x + y \in U$

c.) $x \in U, \alpha \in K \Rightarrow \alpha x \in U$

(3): a.) $0 \in U$

b.) $x, y \in U \quad \left. \begin{array}{l} \\ \alpha, \beta \in K \end{array} \right\} \Rightarrow \alpha x + \beta y \in U$

3) Care dintre următoarele submultimi ale multimii \mathbb{R}^3 sunt \mathbb{R} -subspații:

a) $A = \{x = [x_1, x_2, x_3] \in \mathbb{R}^3 \mid 2x_1 + x_2 - \underline{\underline{x_3}} = 0\}$

Soluție: $0 = [0, 0, 0] \in \mathbb{R}^3$

$$2 \cdot 0 + 0 - 0 = 0 \Rightarrow 0 \in A$$

$$x, y \in A$$

$$x = [x_1, \underline{\underline{x_2}}, \underline{\underline{x_3}}], 2x_1 + x_2 - x_3 = 0$$

$$y = [y_1, \underline{\underline{y_2}}, \underline{\underline{y_3}}], 2y_1 + y_2 - y_3 = 0$$

$$\underline{x + y} = [x_1 + y_1, x_2 + y_2, x_3 + y_3]$$

$$2(x_1 + y_1) + x_2 + y_2 - x_3 - y_3 =$$

$$= 2x_1 + 2y_1 + x_2 + y_2 - x_3 - y_3$$

$$= \underline{2x_1 + x_2 - x_3} + \underline{2y_1 + y_2 - y_3}$$

$$= 0 + 0 = 0 \Rightarrow x + y \in A$$

$$\lambda \in \mathbb{R}$$

$$x = [x_1, x_2, x_3] \in \mathbb{R}^3, x \in A$$

$$\lambda x = [\lambda x_1, \lambda x_2, \lambda x_3]$$

$$2x_1 + x_2 - 2x_3 = 2(x_1 + x_2 - x_3)$$

$$= 2 \cdot 0 = 0 \Rightarrow 2x \in A$$

$$\Rightarrow A \subseteq_{\mathbb{R}} \mathbb{R}^3$$

b) $B = \{ [x_1, x_2, x_3] \in \mathbb{R}^3 \mid 2x_1 + x_2 - x_3 = 1 \}$

$$0 = [0, 0, 0] \in \mathbb{R}^3$$

$$2 \cdot 0 + 0 - 0 = 0 \neq 1$$

$$\Rightarrow B \not\subseteq_{\mathbb{R}} \mathbb{R}^3$$

c) $C = \{ [x_1, x_2, x_3] \in \mathbb{R}^3 \mid x_1 = x_2 = x_3 \}$

$$[0, 0, 0] \in \mathbb{R}^3$$

$$0 = 0 = 0 \Rightarrow 0 \in C$$

$$x, y \in C \stackrel{?}{\Rightarrow} x + y \in C$$

$$x = [x_1, x_2, x_3], \underline{x_1 = x_2 = x_3}$$

$$y = [y_1, y_2, y_3], \underline{y_1 = y_2 = y_3}$$

$$x + y = [x_1 + y_1, x_2 + y_2, x_3 + y_3]$$

$$x_1 + y_1 = x_2 + y_2 = x_3 + y_3$$

$$\Rightarrow x + y \in C$$

$x \in C$, $\alpha \in \mathbb{R}$ a.i. $\alpha x \stackrel{?}{\in} C$

$$x = [x_1, x_2, x_3], \quad x_1 = x_2 = x_3$$

$$\alpha x = [\alpha x_1, \alpha x_2, \alpha x_3]$$

$$\alpha x_1 = \alpha x_2 = \alpha x_3$$

$$\Rightarrow \alpha x \in C$$

$$\Rightarrow C \subseteq \mathbb{R}^3$$

d) $D = \{[x_1, x_2, x_3] \in \mathbb{R}^3 \mid x_1^2 + x_2^2 = 0\}$
 $[0, 0, 0] \in \mathbb{R}^3$

$$0^2 + 0 = 0 \Rightarrow 0 \in D$$

$$x \in D, \quad x = [x_1, x_2, x_3] \in \mathbb{R}^3$$

$$\text{zu } x_1^2 + x_2^2 = 0$$

$$y \in D, \quad y = [y_1, y_2, y_3] \in \mathbb{R}^3$$

$$\text{zu } y_1^2 + y_2^2 = 0$$

$$x + y = [x_1 + y_1, x_2 + y_2, x_3 + y_3]$$

$$(x_1 + y_1)^2 + (x_2 + y_2)^2 =$$

$$= x_1^2 + 2x_1 y_1 + y_1^2 + x_2^2 + 2x_2 y_2 + y_2^2$$

$$= x_1^2 + x_2^2 + y_1^2 + y_2^2 + 2x_1 y_1 + 2x_2 y_2$$

$$= 0 + 0 + 2x_1 y_1 = 2x_1 y_1,$$

$$\Rightarrow \nabla \notin \mathbb{R}^3$$

e) $E = \mathbb{R}^3 \setminus A$

$$A = \{ [x_1, x_2, x_3] \in \mathbb{R}^3 \mid 2x_1 + x_2 - x_3 = 0 \}$$

$$0 \in A$$

$$\Rightarrow 0 \notin \mathbb{R}^3 \setminus A$$

$$\Rightarrow 0 \in E$$

$$\Rightarrow E \neq \mathbb{R}^3$$

f-) (temă)

$$F = (\mathbb{R}^3 \setminus A) \cup \{0\}$$

$$F = E \cup \{0\}$$

V un K -sp. vectorial

O aplicație f este liniară

dacă $f(\alpha \underline{x} + \beta \underline{y}) = \underline{\alpha f(x) + \beta f(y)}$

$$\underline{x}, \underline{y} \in V, \alpha, \beta \in K$$

4) Care dintre următoarele aplicații sunt liniare:

a) $f: \mathbb{R}^3 \rightarrow \mathbb{R}^3, f[x_1, x_2, x_3] =$

$$= [x_1 - x_2, x_2 - x_3, x_3 - x_1]$$

$$\alpha, \beta \in \mathbb{R}, x, y \in \mathbb{R}^3$$

$$x = [x_1, x_2, x_3], \quad y = [y_1, y_2, y_3]$$

$$\alpha x + \beta y = [\underbrace{\alpha x_1 + \beta y_1}, \underbrace{\alpha x_2 + \beta y_2}, \underbrace{\alpha x_3 + \beta y_3}]$$

$$\begin{aligned} f(\alpha x + \beta y) &= [\alpha x_1 + \beta y_1 - \alpha x_2 - \beta y_2, \\ &\quad \alpha x_2 + \beta y_2 - \alpha x_3 - \beta y_3, \alpha x_3 + \beta y_3 - \\ &\quad - \alpha x_1 - \beta y_1] = \\ &= [\alpha(x_1 - x_2), \alpha(x_2 - x_3), \alpha(x_3 - x_1)] \\ &\quad + [\beta(y_1 - y_2), \beta(y_2 - y_3), \beta(y_3 - y_1)] \\ &= \alpha[x_1 - x_2, x_2 - x_3, x_3 - x_1] + \\ &\quad + \beta[y_1 - y_2, y_2 - y_3, y_3 - y_1] \\ &= \underline{\alpha f(x) + \beta f(y)} \quad \Rightarrow \text{f este aplicație} \\ &\quad \text{liniară} \end{aligned}$$

$$\text{Ker } f = \{x = [x_1, x_2, x_3] \in \mathbb{R}^3 \mid f(x) = 0\}$$

$$\begin{aligned} f(x) = 0 &\Rightarrow [x_1 - x_2, x_2 - x_3, x_3 - x_1] \\ &= [0, 0, 0] \\ &\Rightarrow \begin{cases} x_1 - x_2 = 0 \\ x_2 - x_3 = 0 \Leftrightarrow x_1 = x_2 = x_3 \\ x_3 - x_1 = 0 \end{cases} \end{aligned}$$

$$\Rightarrow \text{Ker} f = \left\{ x = [x_1, x_2, x_3] \in \mathbb{R}^3 \mid x_1 = x_2 = x_3 \right\}$$

$$b) f: \mathbb{R}^3 \rightarrow \mathbb{R}^3, f[x_1, x_2, x_3] = [x_1 - 1, x_2 + 2, x_3 + 1]$$

$$x = [x_1, x_2, x_3] \in \mathbb{R}^3$$

$$y = [y_1, y_2, y_3] \in \mathbb{R}^3$$

$$\alpha, \beta \in \mathbb{R}$$

$$\alpha x + \beta y = [\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2, \alpha x_3 + \beta y_3]$$

$$f(\alpha x + \beta y) = [\alpha x_1 + \beta y_1 - 1, \alpha x_2 + \beta y_2 + 2, \alpha x_3 + \beta y_3 + 1]$$

$$= [\alpha x_1, \alpha x_2, \alpha x_3] + [\beta y_1, \beta y_2, \beta y_3] \\ + [-1, 2, 1]$$

$$= [\alpha x_1 - \alpha, \alpha x_2 + 2\alpha, \alpha x_3 + \alpha] + \\ + [\beta y_1 - \beta, \beta y_2 + 2\beta, \beta y_3 + \beta] \\ + [\alpha, -2\alpha, -\alpha] + [\beta, -2\beta, -\beta]$$

$$+ [-1, 2, 1] \\ = \alpha f(x) + \beta f(y) + [\alpha + \beta - 1, \\ -2(\alpha + \beta - 1), -\alpha - \beta + 1]$$

$$\neq \alpha f(x) + \beta f(y)$$

$\Rightarrow f$ nu este aplicație liniară

c) $f: \mathbb{R}^2 \rightarrow \mathbb{R}^3$

$$f[x_1, x_2] = [x_1 + x_2, x_1 - x_2, 2x_1 + x_2] \\ x, y \in \mathbb{R}^2$$

$$x = [x_1, x_2], y = [y_1, y_2]$$

$$\alpha x + \beta y = [\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2]$$

$$f(\alpha x + \beta y) = [\underline{\alpha x_1 + \beta y_1} + \underline{\alpha x_2 + \beta y_2}]$$

$$[\underline{\alpha x_1 + \beta y_1} - \underline{\alpha x_2 - \beta y_2}, \underline{2\alpha x_1 + 2\beta y_1} \\ + \underline{\alpha x_2 + \beta y_2}]$$

$$= [\alpha(x_1 + x_2), \alpha x_1 - \alpha x_2, 2\alpha x_1 + \alpha x_2]$$

$$+ [\beta y_1 + \beta y_2, \beta y_1 - \beta y_2, 2\beta y_1 + \beta y_2]$$

$$= \alpha [x_1 + x_2, x_1 - x_2, 2x_1 + x_2]$$

$$+ \beta [y_1 + y_2, y_1 - y_2, 2y_1 + y_2]$$

$$= \alpha f(x) + \beta f(y) = f_{\text{liniară}}^{\text{apl.}}$$