第三回レポート問題

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2022 年 5 月 21 日

目 次

3.1　ばねで結合した振り子の連成振動を議論しよう

3.1.1　1

$$\begin{cases} M\dfrac{d^2x_n}{dt^2} = -k(x_n - x_{n-1}) + k(x_{n+1} - x_n) - T\sin\theta_n \\ M\dfrac{d^2y_n}{dt^2} = T\cos\theta_n - Mg = 0 \end{cases} \tag{3.1}$$

$$M\frac{d^2x_n}{dt^2} = -k(2x_n - x_{n-1} - x_{n+1}) - \frac{Mgx_n}{l} \tag{3.2}$$

3.1.2　2

代入して、整理すると次のように定まる:

$$A_n = \frac{k}{2k - M(\omega^2 - g/l)}(A_{n-1} + A_{n+1}) \tag{3.3}$$

3.1.3　3

仮定の解を先程の等式の右辺に代入し、三角函数の和積の公式を適用すると次の式が得られる:

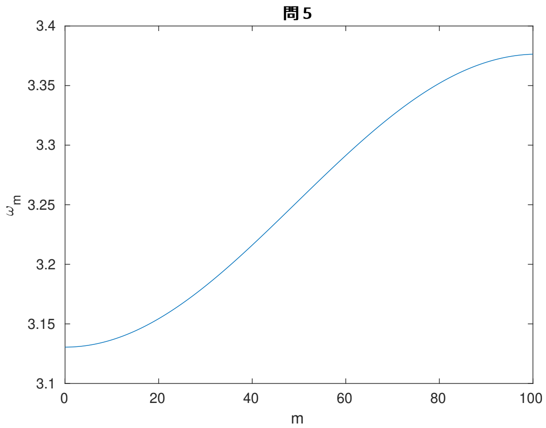
$$\begin{aligned} A_n &= \frac{k}{2k - M(\omega^2 - g/l)}(A_{n-1} + A_{n+1}) \\ &= \frac{k}{2k - M(\omega^2 - g/l)}(\sin(p(n-1)) + \sin(p(n+1))) \\ &= \frac{2k}{2k - M(\omega^2 - g/l)}A_n\cos p \\ \omega &= \sqrt{\frac{4k\sin^2(\frac{p}{2})}{M} + g/l} \end{aligned} \tag{3.4}$$

3.1.4　4

仮定の解に $x_0 = 0$ の初期条件から情報を得られず、$x_{N+1} = 0$ より、

$$\begin{aligned} A_{N+1} = \sin(p(N+1)) = 0 \Rightarrow p = \frac{m\pi}{N+1}(m \in \mathbb{N}) \\ \omega_m = \sqrt{\frac{4k\sin^2(\frac{m\pi}{2(N+1)})}{M} + g/l} \end{aligned} \tag{3.5}$$

3.1.5 5



3.1.6 6

$$x_n(t) = \sum_{i=1}^{m} \sin(\frac{in\pi}{N+1}) \sin(\omega_i t - \delta) \tag{3.6}$$

3.1.7 7

$$\begin{aligned} M\frac{d^2x_n}{dt^2} &= -k(2x_n - x_{n-1} - x_{n+1}) - \frac{Mgx_n}{l} \\ \frac{\partial^2 u(x,t)}{\partial t^2} &= \frac{kc^2}{M} \cdot \frac{\partial^2 u(x,t)}{\partial x^2} - \frac{g}{l} \cdot u(x,t) \\ \frac{\partial^2 u(x,t)}{dt^2} &= -\omega_0^2 u(x,t) + v^2 \frac{\partial^2 u(x,t)}{\partial x^2} \end{aligned} \tag{3.7}$$

3.1.8 8

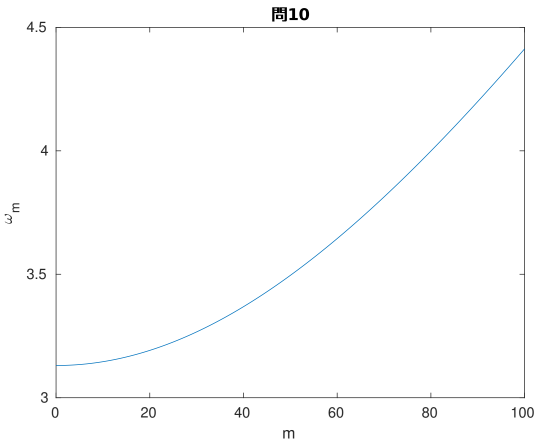
$$\begin{aligned} -\omega^2 u(x,t) &= -\omega_0^2 u(x,t) - k^2 v^2 u(x,t) \\ -\omega^2 &= -\omega_0^2 - k^2 v^2 \\ w &= \sqrt{\omega_0^2 + k^2 v^2} \end{aligned} \tag{3.8}$$

3.1.9 9

$$\begin{cases} kx + \alpha = 0 \\ kc(N+1) = m\pi \end{cases} \tag{3.9}$$

$$
\begin{aligned}
kc &= \frac{m\pi}{N+1} \\
\omega &= \sqrt{\omega_0^2 + \frac{k(\frac{m\pi}{N+1})^2}{M}}
\end{aligned}
\tag{3.10}
$$

3.1.10 10



3.1.11 11

$$
\sum_{i=0}^{m} A\sin(kx)\cdot\sin\left(\sqrt{\omega_0^2 + \frac{k(\frac{i\pi}{N+1})^2}{M}}t - \delta\right)
\tag{3.11}
$$

3.2 問題 2

3.2.1 1

$$
\begin{aligned}
\sum_{m=1}^{\infty} \sin(m\pi\frac{x}{l})\frac{d^2Q_m(t)}{dt^2} &= -\sum_{m=1}^{\infty}(\frac{m\pi v}{l})^2 \sin(m\pi\frac{x}{l})Q_m(t) \\
\sum_{m=1}^{\infty}\int_0^l \sin(n\pi\frac{x}{l})\sin(m\pi\frac{x}{l})dx \cdot \frac{d^2Q_m(t)}{dt^2} &= -\sum_{m=1}^{\infty}\int_0^l (\frac{m\pi v}{l})^2 \sin(n\pi\frac{x}{l})\sin(m\pi\frac{x}{l})dx \cdot Q_m(t) \\
\frac{d^2Q_n(t)}{dt^2} &= -\omega_n^2 Q_n(t)
\end{aligned}
\tag{3.12}
$$

ここで、n を m とおけばよい。

3.2.2 2

$$
\begin{aligned}
Q_m(t) &= A_m \sin(\omega_m t - \delta_m) \\
u(x,t) &= \sum_{m=1}^{\infty} A_m \sin(m\pi\frac{x}{l})\sin(\omega_m t - \delta_m)
\end{aligned}
\tag{3.13}
$$

3.2.3 3

$$
\begin{aligned}
K &= \sum_n \frac{M}{2}\left(\frac{du_n}{dt}\right)^2 \\
&= \frac{\rho}{2}\int_0^l \left(\frac{\partial u(x,t)}{\partial t}\right)^2 dx
\end{aligned}
\tag{3.14}
$$

3.2.4 4

$$
\begin{aligned}
U &= \sum_n Tc\left(1 + \frac{1}{2}\left(\frac{u_{n+1} - u_n}{c}\right)^2 - 1\right) \\
&= \frac{1}{2}\int_0^l \rho v^2 \left(\frac{\partial u(x,t)}{\partial x}\right)^2 dx
\end{aligned}
\tag{3.15}
$$

3.2.5　5

$$
\begin{aligned}
E &= \frac{\rho}{2}\int_0^l \left(\frac{\partial u(x,t)}{\partial t}\right)^2 + \left(v\frac{\partial u(x,t)}{\partial x}\right)^2 dx \\
&= \frac{\rho}{2}\int_0^l \left(\sum_{m=1}^{\infty} \omega_m \sin(m\pi\frac{x}{l})\frac{dQ_n(t)}{dt}\right)^2 + \left(\sum_{m=1}^{\infty} \omega_m \cos(m\pi\frac{x}{l})Q_m(t)\right)^2 dx \\
&= \frac{\rho l}{4}\sum_{m=1}^{\infty} \omega_m \left[\left(\frac{dQ_m(t)}{dt}\right)^2 + Q_m(t)^2\right]
\end{aligned} \tag{3.16}
$$

実は、$Q_m(t)$ が三角関数の形をしているから、最終的な形は前問の答えに参照していただければ、$(\rho l)/4\sum_{m=1}^{\infty} A_m\omega_m$ であることを確認できます。

3.3 描画

フーリエ級数:

$$A_m = \frac{2}{l}\int_0^l f(x)\sin(m\pi x)dx = \frac{8}{(m\pi)^2}\sin(\frac{m\pi}{2})$$