Flocking in Fixed and Switching Networks

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Abstract

The work of this paper is inspired by the flocking phenomenon observed in [3]. We introduce a class of local control laws for a group of mobile agents that result in: (i) global alignment of their velocity vectors, (ii) convergence of their speeds to a common one, (iii) collision avoidance, and (iv) local minimization of the agents artificial potential energy. These are made possible through local control action by exploiting the algebraic graph theoretic properties of the underlying interconnection graph. Algebraic connectivity affects the performance and robustness properties of the overall closed loop system. We show how the stability of the flocking motion of the group is directly associated with the connectivity properties of the interconnection network and is robust to arbitrary switching of the network topology.

Index Terms

Multi-agent systems, cooperative control, nonsmooth systems, algebraic graph theory.

I. Introduction

Over the last years, the problem of coordinating the motion of multiple autonomous agents has attracted significant attention. Research is motivated by recent advances in communication and computation, as well as inspiring links to problems in biology, social behavior, statistical physics, and computer graphics. Efforts have been directed in trying to understand how a group of autonomous moving creatures such as flocks of birds, schools of fish, crowds of people [4], [5], or man-made mobile autonomous agents, can cluster in formations without centralized coordination.

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Such problems have also been studied in ecology and theoretical biology, in the context of animal aggregation and social cohesion in animal groups (see for example [6], [7]). A computer model mimicking animal aggregation was proposed by [3]. Following the work of [3] several other computer models have appeared in the literature and led to creation of a new area in computer graphics known as *artificial life* [3], [8]. At the same time, several researchers in the area of statistical physics and complexity theory have addressed flocking and schooling behavior in the context of non-equilibrium phenomena in many-degree-of-freedom dynamical systems and self organization in systems of self-propelled particles [9]–[11]. Similar problems have become a major thrust in systems and control theory, in the context of cooperative control, distributed control of multiple vehicles and formation control; see for example [12]–[25]. The main goal of the above papers is to develop a decentralized control strategy so that a global objective, such as a tight formation with desired inter-vehicle distances, is achieved.

Reynolds [3] aimed at generating a computer animation model of the motion of bird flocks and fish schools. The author called the generic simulated flocking creatures "boids". The basic flocking model consists of three simple steering behaviors which describe how an individual agent maneuvers based on the positions and velocities its nearby flockmates:

- **Separation:** steer to avoid crowding local flockmates.
- **Alignment:** steer towards the average heading of local flockmates.
- **Cohesion:** steer to move toward the average position of local flockmates.

In Reynolds' model, each agent has direct access to the whole scene's geometric description, but flocking requires that it reacts only to flockmates within a certain small neighborhood around itself. The neighborhood is characterized by a distance and an angle, measured from the agent's direction of flight. Flockmates outside this local neighborhood are ignored. The neighborhood could be considered a model of limited perception (as by fish in murky water), or just the the region in which flockmates influence an agent's steering. The superposition of these three rules results in all agents moving as a flock while avoiding collisions.

Vicsek et al. [9] proposed a model which, although developed independently, turns out to be a special case of [3] where all agents move with the same speed (no dynamics), and only follow an alignment rule. In [9], each agent heading is updated as the average of the headings of the agent and its nearest neighbors, plus some additive noise. Numerical simulations in [9] indicate a coherent collective motion, in which the headings of all agents converge to a common value, a surprising result in the physics community that was followed by a series of papers. The first rigorous proof of convergence for Vicsek's model (in the noise-free case) was recently given by [20]. Generalizations of this model include a leader follower strategy, in which one agent acts as a group leader and the other agents would just follow the aforementioned cohesion/separation/alignment rules, resulting in leader following.

Motivation for this work comes primarily from the need to theoretically explain the flocking phenomenon of [3]. Inspired by Reynolds flocking model, we construct local control laws that allow a group of mobile agents with double integrator dynamics to align their velocities, move with a common speed and achieve desired inter-agent distances while avoiding collisions with each other. We believe that these control laws capture the essence of Reynolds model, both in terms of the nature of local interactions and with respect to the overall objective.

We theoretically establish the stability properties of the interconnected closed loop system by combining results from classical and nonsmooth control theory, robot navigation, mechanics and algebraic graph theory. Stability is shown to rely on the connectivity properties of the graph that represents agent interconnections, in terms of not only asymptotic convergence but also convergence speed and robustness with respect to arbitrary changes in the interconnection topology. Exploiting modern results from algebraic graph theory, these properties are directly related to the topology of the network through the eigenvalues of the Laplacian of the graph. Collision avoidance and pairwise distance convergence is ensured through the application of a set of local artificial potential fields [26], [27]. Similar results regarding collective flocking behavior have been independently produced by [28], although the analysis techniques, both in terms of collision avoidance and velocity stabilization, are fundamentally different.

We first investigate the case where the topology of the control interactions between the agents is fixed. Each agent regulates its position and orientation based on a fixed set of "neighbors". In this case, the control inputs for the agent are smooth and the stability analysis is based on the classic version of LaSalle's invariant principle, facilitated by the algebraic properties of the interconnection graph that allow the connectivity properties of the network to be reflected on the convergence estimate. Then we turn our attention to the case where agent interactions are local, limited within a certain neighborhood around each agent. The time varying nature of the interconnection topology introduces discontinuities in the control inputs, which in turns give rise to a set of discontinuous differential equations describing the system dynamics. System stability is analyzed using nonsmooth Lyapunov stability,[1] which is reviewed briefly in the Appendix. As in the smooth case, the connectivity properties of the switching graph are instrumental in establishing global asymptotic stability. This paper extends further our earlier work, by lifting the assumptions on the regularity of the potential functions and on the uniqueness of the solutions. Not having to check whether the (nonsmooth) potential function is regular reduces significantly the complexity of the design problem.

II. Problem Formulation

The group we wish to coordinate consists of N mobile agents. Each mobile agent is a dynamical system moving on the plane. Generalizations to three dimensions and more complex dynamics are possible, but for simplicity we let each agent be described by a double integrator:

$$\dot{r}_i = v_i \tag{1a}$$

$$\dot{v}_i = u_i \quad i = 1, \ldots, N\ , \tag{1b}$$

where $r_i = (x_i, y_i)^T$ is the position of agent i, $v_i = (\dot{x}_i, \dot{y}_i)^T$ its velocity and $u_i = (u_x, u_y)^T$ its acceleration inputs. Let the relative position vector between agents i and j be denoted $r_{ij} = r_i - r_j$. Agent i is steered via its acceleration input u_i which consists of two components (Figure 1):

$$u_i = \alpha_i + a_i\ . \tag{2}$$

The first component of (2), α_i aims at aligning the velocity vectors of all the agents and to make them move with a common speed and direction. Component a_i is thought to be a vector in the direction of the negated gradient of an artificial potential function, V_i. In this way a_i will contribute to collision avoidance and cohesion in the group.

In our interpretation of Reynold's notion of flocking, a group of mobile agents is said to (asymptotically) flock, when all agents attain the same velocity vector, distances between the

[1] The nonsmooth approach is not the only way to deal with the problem of switching interconnections but is, in our opinion, technically more direct and intuitive than any alternatives. Discontinuities can arise both from nonsmooth inter-agent potential functions and from changes in the neighboring relations between agents. In both cases, one can try to "smoothen" the potential functions at the edge of the neighboring regions, and introduce differentiable "transition functions" for agent velocity interactions. Designing those differentiable potential functions requires ingenuity, introduces generalized Laplacians and defeats the purpose of "smoothening for simplification". It fact, it steers analysis into an area where few results are known.

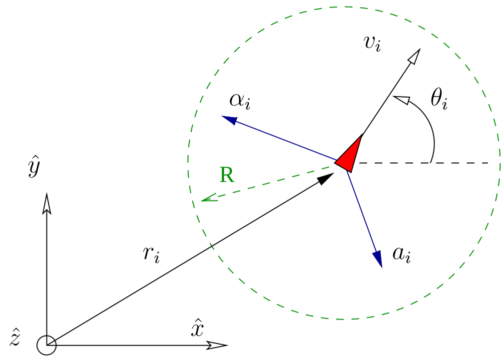

Fig. 1. Control inputs applied to agent i.

agents are stabilized, and no collisions between them occur. The problem here is to design the control input (2) so that in the group of mobile agents, velocities are synchronized and pair-wise distances stabilized, giving rise to an emergent cooperative behavior that resembles flocking.

A. Algebraic Graph Theory

Stability analysis of the group of agents builds around several results on algebraic graph theory. This necessitates a brief introduction of related graph theoretic notation and terminology. The interested reader is referred to [29] for details.

An (undirected) *graph* $\mathcal{G}$ consists of a *vertex set*, $\mathcal{V}$, and an *edge set* $\mathcal{E}$, where an edge is an unordered pair of distinct vertices in $\mathcal{G}$. If $x, y \in \mathcal{V}$, and $(x, y) \in \mathcal{E}$, then x and y are said to be *adjacent*, or neighbors and we denote this by writing $x \sim y$. A graph is called *complete* if any two vertices are neighbors. The number of neighbors of each vertex is its *valency* or *degree*. A *path* of length r from vertex x to vertex y is a sequence of $r + 1$ distinct vertices starting with x and ending with y such that consecutive vertices are adjacent. If there is a path between any two vertices of a graph $\mathcal{G}$, then $\mathcal{G}$ is said to be *connected*.

The *valency matrix* $\Delta(\mathcal{G})$ of a graph $\mathcal{G}$ is a diagonal matrix with rows and columns indexed by $\mathcal{V}$, in which the (i, i)-entry is the valency of vertex i. An *orientation* of a graph $\mathcal{G}$ is the assignment of a direction to each edge, so that the edge (i, j) is now an arc from vertex i to vertex j. We denote by $\mathcal{G}^\sigma$ the graph $\mathcal{G}$ with orientation σ. The *incidence matrix* $B(\mathcal{G}^\sigma)$ of an oriented graph $\mathcal{G}^\sigma$ is the matrix whose rows and columns are indexed by the vertices and edges of $\mathcal{G}$ respectively, such that the i, j entry of $B(\mathcal{G}^\sigma)$ is equal to 1 if edge j is incoming to vertex

i, -1 if edge j is outcoming from vertex i, and 0 otherwise. The symmetric matrix defined as:

$$L(\mathcal{G}) = B(\mathcal{G}^\sigma)B(\mathcal{G}^\sigma)^T$$

is called the *Laplacian* of $\mathcal{G}$ and is independent of the choice of orientation σ. It is known that the Laplacian captures many interesting properties of the graph. Among those, is the fact that L is always symmetric and positive semidefinite, and the algebraic multiplicity of its zero eigenvalue is equal to the number of connected components in the graph. For a connected graph, the n-dimensional eigenvector associated with the single zero eigenvalue is the vector of ones, $\mathbf{1}_n$. The second smallest eigenvalue, λ_2 is positive and is known as the algebraic connectivity of the graph, because it is directly related to how the nodes are interconnected.

In what follows, we will use graph theoretic terminology to represent the interconnections between the agents in the group. The connectivity properties of the induced graph will prove crucial for establishing the stability of the flocking motion of the group.

III. Coordination Strategy

In this section we introduce local control laws of the form of (2), which cause the group of mobile agents to flock asymptotically. The control laws are uniform for all agents and can accommodate a large class of artificial potential functions. The controller of agent i requires state information from a subset of the agent's flockmates, called the *neighbor set*, $\mathcal{N}_i$ of agent i. Neighboring relations may reflect physical proximity between two agents, or the existence of a communication channel. A neighboring relation induces a control interconnection between the two neighbors. The network of such interconnections is represented by means of a graph:

Definition 1 (Neighboring graph) *The neighboring graph, $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$, is an undirected graph consisting of:*

- *a set of vertices (nodes), $\mathcal{V} = \{n_1, \ldots, n_N\}$, indexed by the agents in the group, and*
- *a set of edges, $\mathcal{E} = \{(n_i, n_j) \in \mathcal{V} \times \mathcal{V} \mid n_i \sim n_j\}$, containing unordered pairs of nodes that represent neighboring relations.*

In the sequel, we will not distinguish between the edge (n_i, n_j) and the respective pair of indices, (i, j). The control law for agent i should not require state information from all its groupmates, but rather from a subset which we call neighbors:

$$\mathcal{N}_i \triangleq \{j \mid i \sim j\} \subseteq \{1, \ldots, N\} \setminus \{i\}.$$

The neighboring set of agent i, $\mathcal{N}_i$, can represent the set of agents with which i is allowed to communicate (giving rise to a fixed, logical interconnection network), or the set of agents which i can sense, transit or receive information. In the latter case, the neighboring set may express physical proximity, since sensing and communication capabilities can also be spatially-related, giving rise to a dynamic, distance-dependent interconnection network. These two cases motivate the stability analysis of Sections IV and V, respectively.

The control input for agent i is defined as:

$$u_i = \underbrace{-\sum_{j\in\mathcal{N}_i}(v_i - v_j)}_{\alpha_i} \underbrace{-\sum_{j\in\mathcal{N}_i}\nabla_{r_i}V_{ij}}_{a_i} . \tag{3}$$

Function V_{ij} depends on the distance between the neighbors and defined as follows,

Definition 2 (Potential function) *Potential V_{ij} is a differentiable, nonnegative, radially unbounded function of the distance $\|r_{ij}\|$ between agents i and j, such that*

1) $V_{ij}(\|r_{ij}\|) \to \infty$ *as* $\|r_{ij}\| \to 0$,
2) V_{ij} *attains its unique minimum when agents i and j are located at a desired distance.*

This definition ensures that minimization of the inter-agent potential functions implies cohesion and separation in the group. Having defined V_{ij} we can now express agent i total potential as (Figure 2),

$$V_i = \sum_{j\in\mathcal{N}_i} V_{ij}(\|r_{ij}\|), \tag{4}$$

Fig. 2. Example of inter-agent artificial potential function: $V(\|r_{ij}\|) = \log^2(\|r_{ij}\|) + \frac{1}{\|r_{ij}\|}$

IV. Fixed Interconnection Topology

If the interconnection topology of the group is represented by a time invariant but connected graph, then control laws (3) create an asymptotically stable equilibrium manifold on which the group satisfies the conditions for flocking as described above. Each agent maintains the same set of neighbors, implying that the neighboring graph is constant. The main consequence of time invariance is that the mechanical energy of the group is differentiable, the agent control laws are smooth and classic Lyapunov theory can be applied.

Consider the following nonnegative function:

$$W(r,v) = \frac{1}{2}\sum_{i=1}^{N}(V_i + v_i^T v_i), \tag{5}$$

where r is the stack vector of all agent positions r_i, and v is the stack vector of all agent velocities v_i.

Using LaSalle's invariant principle we can show that the closed loop system of agents (1)-(3) flocks, provided that the neighboring graph is connected:

Theorem 1 (Flocking in a fixed network) *Consider a system of N mobile agents with dynamics* (1)*, each steered by control law* (3) *and assume that the neighboring graph is connected. Then all agent velocity vectors become asymptotically the same, collisions between interconnected agents are avoided and the system approaches a configuration that locally minimizes all agent potentials.*

Proof: The level sets of W define compact sets in the space of agent velocities and relative distances. The set $\{r_{ij}, v_i\}$ such that $W \leq c$, for $c > 0$ is closed by continuity. Boundedness follows from connectivity: from $W \leq c$ we have that $V_{ij} \leq c$. Connectivity ensures that a path connecting nodes i and j has length at most $N-1$. Thus $\|r_{ij}\| \leq V_{ij}^{-1}\big(c(N-1)\big)$. Similarly, $v_i^T v_i \leq c$ yielding $\|v\|_i \leq \sqrt{c}$. Thus, the set

$$\Omega = \{(v_i, r_{ij}) \mid W \leq c\} \tag{6}$$

is compact. The derivative of W defined in (5) is:

$$\dot{W} = \frac{1}{2}\sum_{i=1}^{N}\dot{V}_i - \sum_{i=1}^{N} v_i^T \left(\sum_{j\sim i}(v_i - v_j) + \nabla_{r_i} V_i\right). \tag{7}$$

Note however that due to the symmetric nature of V_{ij},

$$\frac{1}{2}\sum_{i=1}^{N}\dot{V}_i=\sum_{j\sim i}\dot{r}_{ij}^T\nabla_{r_{ij}}V_{ij}=\sum_{j\sim i}(\dot{r}_i^T\nabla_{r_{ij}}V_{ij}-\dot{r}_j^T\nabla_{r_{ij}}V_{ij})=\sum_{j\sim i}(\dot{r}_i^T\nabla_{r_i}V_{ij}+\dot{r}_j^T\nabla_{r_j}V_{ij})=\sum_{i=1}^{N}\dot{r}_i^T\nabla_{r_i}V_i.$$

Thus, (7) simplifies to

$$\dot{W}=\sum_{i=1}^{N}v_i^T\nabla_{r_i}V_i-\sum_{i=1}^{N}v_i^T\left(\sum_{j\sim i}(v_i-v_j)+\nabla_{r_i}V_i\right)=-\sum_{i=1}^{N}v_i^T\sum_{j\sim i}(v_i-v_j)=-v^T(L\otimes I_2)v,$$

where v is the stack vector of all agent (three dimensional) velocity vectors, L is the Laplacian of the neighboring graph and $\otimes$ denotes the Kronecker matrix product. Expanding this matrix product, it is straightforward to see that $\dot{W}$ can be written

$$\dot{W}=-v_x^TLv_x-v_y^TLv_y. \tag{8}$$

where v_x and v_y are the stack vectors of the components of the agent velocities along $\hat{x}$ and $\hat{y}$ directions (Figure 1), respectively.

For a connected neighboring graph, L is positive semidefinite and the eigenvector associated with the single zero eigenvalue is the N-dimensional vector of ones. This means that $\dot{W}$ will only be zero whenever both v_x and v_y belong to $\mathrm{span}\{\mathbf{1}\}$, implying that all agent velocities have the same components and are therefore equal. It follows immediately that $\dot{\bar{r}}=0$, where $\bar{r}$ is the stack vector of all pairwise distances $r_{ij}\ \forall\ (i,j)\in N\times N$.

Applying LaSalle's invariant principle to the system described by the vector field $(\dot{\bar{r}},\dot{v})$, it follows that if the initial conditions of the system lie in Ω, its trajectories will converge to the largest invariant set inside the region $S=\{v\mid\dot{W}=0\}$. Note that Ω can be made arbitrarily large, ensuring semi-global asymptotic stability of the invariant set. In S, the agent velocity dynamics are

$$\dot{v}=-\begin{bmatrix}\nabla_{r_1}V_1\\ \vdots\\ \nabla r_N V_N\end{bmatrix}=-(B\otimes I_2)\begin{bmatrix}\vdots\\ \nabla_{r_{ij}}V_{ij}\\ \vdots\end{bmatrix}$$

which, by a slight abuse of notation, can be expanded to

$$\dot{v}_x=-B[\nabla_{r_{ij}}V_{ij}]_x, \qquad \dot{v}_y=-B[\nabla_{r_{ij}}V_{ij}]_y.$$

Thus, both $\dot{v}_x$ and $\dot{v}_y$ belong in the range of the incidence matrix B. For a connected graph, $\mathrm{range}(B)=\mathrm{span}\{\mathbf{1}\}^\perp$ and therefore

$$\dot{v}_x,\ \dot{v}_y\in\mathrm{span}\{\mathbf{1}\}^\perp.$$

On the other hand, in the invariant set within S

$$v_x, v_y \in \operatorname{span}\{\mathbf{1}\} \Rightarrow \dot{v}_x,\ \dot{v}_y \in \operatorname{span}\{\mathbf{1}\},$$

which leads to contradiction unless

$$\dot{v}_x, \dot{v}_y \in \operatorname{span}\{\mathbf{1}\} \cap \operatorname{span}\{\mathbf{1}\}^{\perp} \equiv \{0\}.$$

This means that the agents velocities do not change in steady state and that the potential V_i of each agent i is (locally) minimized. ■

Corollary 1 (Distance setpoint stabilization) *If the neighboring graph is a tree, then inter-agent distances can be stabilized to desired setpoints.*

Proof: For a tree, the number of edges is $N-1$ and thus B is full rank. In this case,

$$(B \otimes I_2)\begin{bmatrix} \vdots \\ \nabla_{r_{ij}} V_{ij} \\ \vdots \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} \vdots \\ \nabla_{r_{ij}} V_{ij} \\ \vdots \end{bmatrix} = 0,$$

Let r_d be the configuration where V_{ij} attains its unique minimum. Then $\frac{\partial V_{ij}}{\partial \|r_{ij}\|} = 0$ implies that $\|r_{ij}\| = r_d$. ■

Corollary 2 (Convergence speed) *Velocity synchronization is accelerated as as the algebraic connectivity of the neighboring graph increases.*

Proof: Let us decompose the velocities v_x and v_y into two components

$$v_x = v_{x_p} \oplus v_{x_n},\ \text{ where } v_{x_p} \in \operatorname{span}\{\mathbf{1}\}, v_{x_n} \in \operatorname{span}\{\mathbf{1}\}^{\perp},$$

$$v_y = v_{y_p} \oplus v_{y_n},\ \text{ where } v_{y_p} \in \operatorname{span}\{\mathbf{1}\}, v_{y_n} \in \operatorname{span}\{\mathbf{1}\}^{\perp}.$$

Then from (8), since $L = BB^T$ we have that

$$\dot{W} = -v_{x_n}^T L v_{x_n} - v_{y_n}^T L v_{y_n} + v^T f_d,$$

where f_d is the stack vector of all disturbances. For a connected graph, B^T is full rank in $\operatorname{span}\{\mathbf{1}\}^{\perp}$ and therefore,

$$\dot{W} \leq -\lambda_2(\|v_{x_p}\|^2 + \|v_{y_p}\|^2) + \|v\| = -\lambda_2 \|v_p\|^2$$

where λ_2 is the second smallest eigenvalue of the Laplacian, and $\|v_p\|$, expresses the magnitude of velocity misalignments. It is known that the addition of a new edge in a graph generally increases the eigenvalues of the Laplacian [30]. Hence, increasing the connectivity of the neighboring graph results to faster convergence. ■

Note that unless two agents are interconnected under the fixed neighboring graph topology, they cannot be aware of each other presence in their close vicinity and collision avoidance between them cannot be ensured. In order to ensure collision avoidance between all agents, we need the neighboring graph to be complete.

Corollary 3 (Collision avoidance in fixed networks) *If the fixed interconnection topology corresponds to a complete graph, then collision avoidance is ensured.*

Proof: A complete graph contains all possible edges and therefore every agent is interconnected to every other agent. Since Ω is positively invariant, all inter-agent potential functions have to remain bounded. However, $V_{ij} \to \infty$ whenever $\|r_{ij}\| \to 0$, and by continuity $\|r_{ij}\| > 0$, $\forall\ t > 0$ and $\forall\ (i, j) \in N \times N$. ■

Collisions can also be avoided in the case where neighboring interconnections depend dynamically on the distance between the agents. Group motion under such conditions is discussed in the following section.

V. SWITCHING INTERCONNECTION TOPOLOGY

One of the most interesting characteristics of the control scheme is that its stability is not affected by changes in the neighboring graph. This robusteness property with respect to interconnnection topology variations can be particularly useful in cases where the agents are subject to sensing and communication constraints. In this section we relax the assumption that interconnection topology is fixed and we let neighboring relations depend on physical proximity between the agents. Thus, two agents are considered interconnected when their distance is below a certain threshold, R:

$$i \sim j \Leftrightarrow \|r_{ij}\| < R.$$

The control law for agent i is still expressed by (3), with the only difference being that $\mathcal{N}_i$ now changes dynamically as a function of the distance between the agents. Such changes introduce discontinuities in (3) which in turn give rise to a set of discontinuous differential equations for the dynamics of agent i.

The stability of the discontinuous dynamics will be analyzed using differential inclusions [31] and nonsmooth analysis [32]. A brief review of nonsmooth analysis and stability is given in Appendix . In a switching interconnection topology, the agent dynamics can be expressed by means of a differential inclusion:

$$\dot{r}_i = v_i \tag{9a}$$

$$\dot{v}_i \overset{\text{a.e}}{\in} K[u_i] \quad i = 1, \dots, N\,, \tag{9b}$$

where $K[\cdot]$ is a differential inclusion (see Appendix) and $\mathrm{a.e}$ stands for "almost everywhere". Existence and uniqueness of the solutions of (9) is guaranteed by the boundedness of u_i.

The stability analysis is based on a nonsmooth version of LaSalle's invariant principle [33], using the nonnegative Lyapunov-like function

$$Q(r,v) = \frac{1}{2}\sum_{i=1}^{N}(V_i + v_i^T v_i) + \sum_{(i,j)\notin\mathcal{E}} V_{ij}(R).$$

Equivalently, one express Q in terms of a nonsmooth inter-agent potential function, which is constant for two agents that are not neighbors:

Definition 3 (Nonsmooth potential function) *Potential function U_{ij} is a nonsmooth, nonnegative, radially bounded function of the distance $\|r_{ij}\|$ between agents i and j, such that*

1) $U_{ij}(\|r_{ij}\|) \to \infty$ *as* $\|r_{ij}\| \to 0$,
2) U_{ij} *attains its unique minimum when agents i and j are located at a desired distance.*
3) U_{ij} *is constant for* $\|r_{ij}\| \geq R \in \mathbb{R}_+$.

Based on the above Definition, one can define a "saturated version" of V_{ij}:

$$U_{ij}(\|r_{ij}\|) = \begin{cases} V_{ij}(\|r_{ij}\|), & \|r_{ij}\| < R, \\ V_{ij}(R), & \|r_{ij}\| \geq R \end{cases}, \tag{10}$$

as follows:

$$Q = \frac{1}{2}\sum_{i=1}^{N}\Big(\sum_{j=1}^{N} U_{ij} + v_i^T v_i\Big). \tag{11}$$

Function Q is continuous everywhere but nonsmooth whenever $\|r_{ij}\| = R$ for some $(i,j) \in N \times N$ (Figure 3).

We can now generalize Theorem 1 to the case where the interconnection topology switches arbitrarily between connected neighboring graphs:

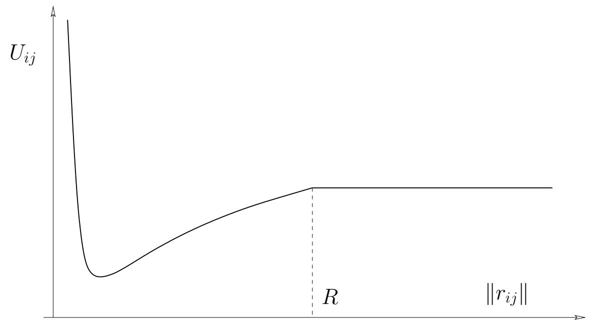

Fig. 3. A saturated nonsmooth artificial potential function.

Theorem 2 *[Flocking in switching networks] Consider a system of N mobile agents with dynamics* (9)*, each steered by control law* (3) *and assume that the neighboring graph is connected. Then all pairwise velocity differences converge asymptotically to zero, collisions between the agents are avoided, and the system approaches a configuration that locally minimizes all agent potentials.*

Proof: We differentiate function Q as it is expressed in (11). For all points where $\|r_{ij}\| \neq R$, for any $(i,j) \in N \times N$, the time derivative of Q is calculated as in (7) yielding:

$$\dot{Q} = -v^T (L \otimes I_2) v, \quad \|r_{ij}\| \neq R \quad \forall (i,j) \in N \times N.$$

If for some $(i,j) \in N \times N$ we have $\|r_{ij}\| = R$, then we need to consider the generalized time derivative of Q.

Lemma 1 *The (partial) generalized gradient of U_{ij} with respect to r_i at R is empty:*

$$\partial_{r_i} U_{ij}(R) = \emptyset. \tag{12}$$

Proof: of Lemma 1 The generalized derivative of U_{ij} at R along w, namely $U_{ij}^{\circ}(R)$, is determined by the expression:

$$U_{ij}^{\circ}(R; w) \triangleq \max\{\langle \zeta, w \rangle \mid \zeta \in \partial U_{ij}(R)\}.$$

Depending on the sign of w we distinguish the two cases:

1) if $w > 0$ then $0 \geq \zeta w$, which means that all $\zeta \in \partial U_{ij}(R)$ have to be nonpositive;
2) if $w < 0$ then $\zeta w \leq c < 0$ which means that all $\zeta \in \partial U_{ij}(R)$ have to be positive.

Since the direction of w is arbitrary, $\partial U_{ij}(R) = \emptyset$.

Function U_{ij} is a composition of a continuous function $U_{ij}(s)$ from the positive reals to the positive reals with $\|r_{ij}\|$. The norm $\|r_{ij}\|$ is a smooth (hence strictly differentiable) function of both position vectors r_i, r_j when $r_i \neq r_j$. Note that $r_i = r_j$ corresponds to collision configurations in the exterior of Ω, which are naturally excluded. Function, $U_{ij}(s)$ is locally Lipschitz and regular for all $s > 0$. Therefore [32]:

$$\partial_{r_i} U_{ij}(\|r_{ij}\|) = \partial_{r_{ij}} U_{ij}(\|r_{ij}\|) \cdot \frac{\partial \|r_{ij}\|}{\partial r_i}$$

At R where U_{ij} is not differentiable, $\partial_{r_{ij}} U_{ij}(R) = \emptyset$, and thus, $\partial_{r_i} U_{ij}(d) = \emptyset$. ∎

Remark 1 *The fact that the generalized gradient of U_{ij} is empty can be geometrically justified considering this set to be the polar of the contingent cone of U_{ij} (the tangent cone in the case where U_{ij} is regular).*

Using the property of finite sums of generalized gradients [32] we obtain

$$\partial Q \subset \sum_{i=1}^{N} \Big[\sum_{j=1}^{N} \partial_{r_i} U_{ij}^T + v_i^T \Big].$$

Then the worst case for the rate of change of Q, captured by the term (c.f. Appendix, Theorem 4)

$$m(r, v) = \max \left\{ Q^{\circ}(r, v; \phi) \mid \phi \in \left(\begin{smallmatrix} v \\ K[u] \end{smallmatrix} \right) \right\},$$

will be:

$$m = \max \left\{ \sum_{i=1}^{N} \left(\bigcap_{\xi_i} \xi_i^T v_i \right) - v^T K \left[(L_t \otimes I_2) v + \begin{pmatrix} \vdots \\ \nabla_{r_i} V_i \\ \vdots \end{pmatrix} \right] \right\},$$

where $\xi_i \in \sum_{j=1}^{N} \partial_{r_i} U_{ij}$, and L_t and $\nabla_{r_i} V_i = \sum_{j \in \mathcal{N}_i} \nabla_{r_i} V_{ij}$ are switching over time depending on the neighboring set $\mathcal{N}_i$ of each agent i. Recalling that $\partial U_{ij}(R) = \emptyset$ (Lemma 1) and using some differential inclusion algebra for sums, (finite) cartesian products and multiplications with continuous matrices [34], we obtain

$$\begin{aligned} m &\in \sum_{i=1}^{N} (\nabla_{r_i} V_i)^T v_i - v^T K [(L_t \otimes I_2) v] - \sum_{i=1}^{N} v_i^T \nabla_{r_i} V_i \\ &= -\, \overline{\mathrm{co}} \{ v_x^T L_t v_x + v_y^T L_t v_y \}. \end{aligned} \tag{13}$$

For any graph, the right hand of (13) will be an interval of the form $[e, 0]$, with $e < 0$. Therefore it is always $q \leq 0$, for all $q \in \dot{\tilde{Q}}$. If the graph is connected, then this interval contains 0 *only* when $v_x, v_y \in \mathrm{span}\{\mathbf{1}\}$.

Applying Theorem 4 (a nonsmooth version of the invariant principle) proposed by [35] to the system described by the vector field $(\dot{\bar{r}}, \dot{v})$, it follows that for initial conditions in Ω, the Filippov trajectories of the system converge to a subset of $\{v \mid v_x, v_y \in \operatorname{span}\{\mathbf{1}\}\}$ in which $m \geq 0$, and since L_t is positive semi-definite, the (weakly) invariant set will consist of configurations where

$$m = 0 \Rightarrow \dot{r}_{ij} = v_i - v_j = 0, \quad \forall (i,j) \in N \times N.$$

In this set, the system dynamics reduces to

$$\dot{v} \in -K \left[(B_t \otimes I_2) \left[\cdots \ (\nabla_{r_{ij}} V_{ij})^T \ \cdots \right]^T \right]$$

which implies that both $\dot{v}_x$ and $\dot{v}_y$ belong in the range of the switching incidence matrix B_t. For a connected graph, $\operatorname{range}(B_t) = \operatorname{span}\{\mathbf{1}\}^\perp$ and therefore

$$\dot{v}_x, \dot{v}_y \in \operatorname{span}\{\mathbf{1}\} \cap \operatorname{span}\{\mathbf{1}\}^\perp \equiv \{0\}. \tag{14}$$

From the above we conclude that

1) v does not change in steady state (and thus switching eventually stops), and
2) the potential V_i of each agent is minimized.

■

The issue of maintaining connectivity in the group while the network topology is switching based on the distance between the agents is a major one. In the present analysis, this assumption is instrumental in showing the stability of the flocking motion of the group. The nonsmooth invariance theorem of Ryan [35], Theorem 4, does not require Ω to be compact, but it does require that trajectories remain in Ω for all time. If connectivity is lost, one cannot guarantee that $r_{ij} \in \Omega$ and thus stability cannot be established.

Remark 2 (Differentiable Inter-Agent Potentials) *An alternative to the nonsmooth inter-agent potential function* (10) *would be to define a differentiable function of the distance, which satisfies the requirements of Definition 3. Satisfing the requirements of Definition 3 with a function that is differentiable at R is not trivial, especially if the distance with the minimum potential, r_{min},* and *the neighborhood boundary, R, is fixed. One candidate can be the following function:*

$$U_{ij} = \begin{cases} a_1\|r_{ij}\| - \log(\|r_{ij}\|) - \frac{a_2}{\|r_{ij}\|}, & \text{if } \|r_{ij}\| < R \\ a_1 R - \log(R) - \frac{a_2}{R}, & \text{if } \|r_{ij}\| \geq R \end{cases},$$

with

$$a_1 = \frac{1}{r_{min} + R} \qquad\qquad a_2 = \frac{R\, r_{min}}{r_{min} + R}$$

Then, the Lyapunov-like function will be regular and one can use the invariance principle of [36] to establish stability.

VI. Numerical Simulations

This section presents the results of a numerical implementation of the proposed control scheme on a group of ten mobile agents. The number of agents in the group was kept that small for clarity of presentation. We investigate both the case of fixed topology and the case of dynamic, distance-dependent interconnections. Convergence is verified in cases, and case related characteristics are identified.

The case of fixed topology is investigated first. A group of ten mobile agents with dynamics (1) is initialized with random initial (x, y) positions in the range of $(-2.5, 2.5) \times (-2.5, 2.5)$m. Velocities were also randomly selected with magnitudes in the $(0, 1)$m/s range and arbitrary directions. A randomly generated $0-1$ adjacency matrix defined a connected neighboring graph. Then the group motion evolves according to the closed loop system (1)-(3), and successive snapshots of this evolution are captured in Figure 4, for a time period of 100 simulation seconds. The particular time instant where the snapshot was taken is recorded below each frame. In Figure 4 the position of the agents is depicted by black dots and interconnections are represented by line segments connecting the agent locations. The path of each agent is shown by a dotted line and agent velocities are given as small arrows, which are scaled up at steady state to show how the vectors have been synchronized.

It is worth noting that a fixed interconnection topology cannot ensure collision avoidance between two agents, unless these are interconnected; otherwise there is no way an agent is aware of the presence of a flockmate in its vicinity. This is demonstrated in Figure 4 by the proximity of the two agents near the center of the frame corresponding to steady state. Although the agents do not collide, they may come dangerously close.

This scenario is fortunately avoided in the case where the interconnection topology is distance-dependent. In this case, whenever two agents are found close to each other, an interconnection is established which guarantees collision free motion via the action of the potential forces. Figure 5

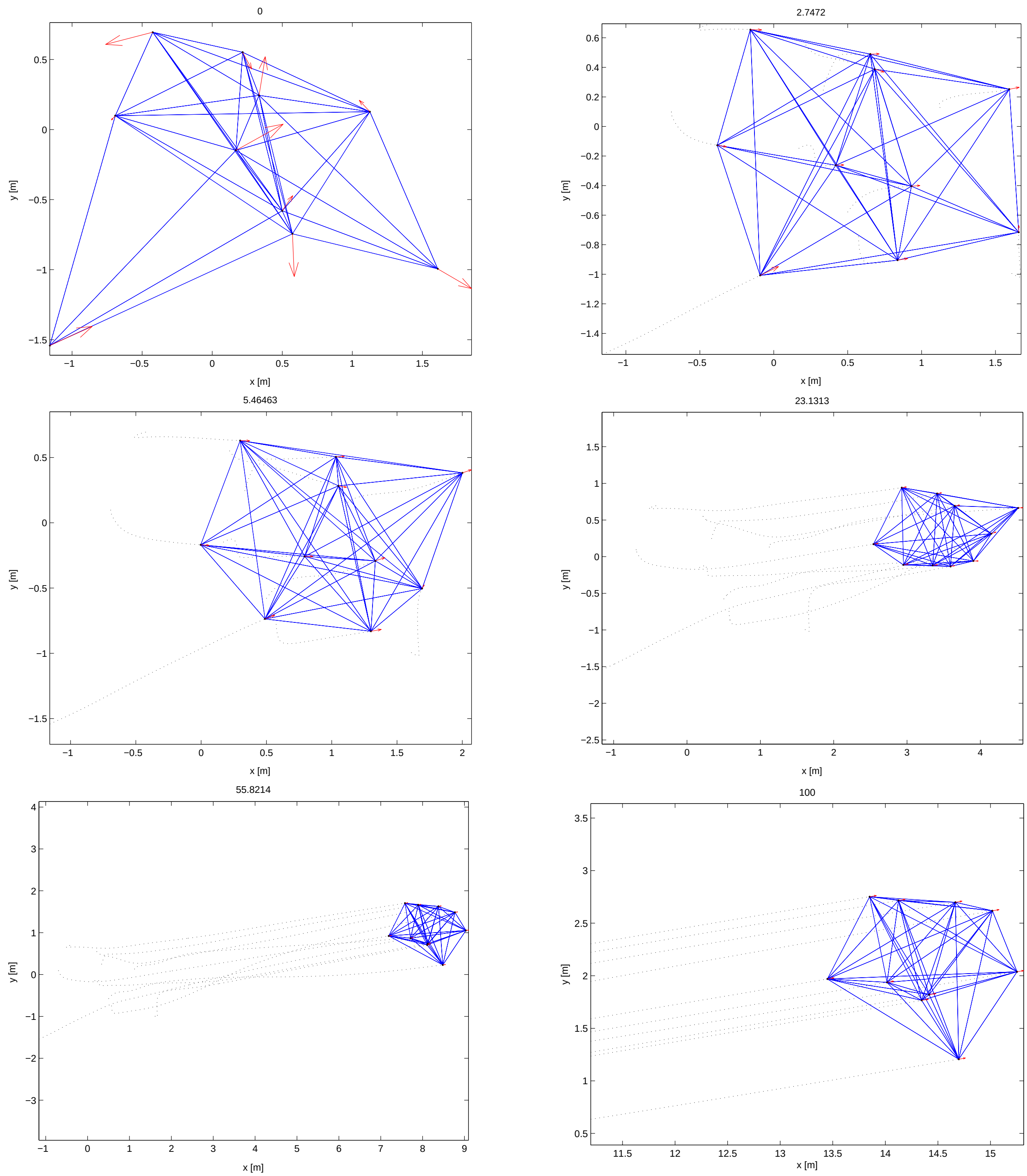

Fig. 4. Successive simulation time snapshots of flocking with fixed interconnection topology.

describes the evolution of a group of ten agents in which the topology is switching according to inter-agent distance. The agents are randomly initialized within the same range of positions and velocities. The neighborhood radius below which two agents can sense the state of each other was set to 2m. They reach a steady state where all velocities are synchronized after 100 simulation seconds. In steady state, connectivity is high because the inter-agent potential fields pull the agents close enough to establish a control interconnection, however the neighboring graph is not necessarily complete.

The dynamic nature of the interconnection topology in the case where the neighboring links are distance dependent can be depicted in Figure 6, where the valencies of all nodes in the interconnection graph are plotted versus time. It can be seen that the number of neighbors for each agent generally increases, up to a point where switching ends, and node valencies stabilize to some steady state values.

Velocity synchronization in both cases is demonstrated in Figures 7-8. While Figures 4-5 have shown that all agents eventually move in the same direction, Figures 7-8 establish the convergence of agent speeds as well.

VII. Conclusions

In this paper we introduce a local control law for a group of mobile agents that allows them to stabilize their pairwise distances, avoid collisions and move as a coherent group having a common velocity vector. We show that this behavior is robust to arbitrary changes in the interconnections between the agents. The control law is based on a combination of a velocity alignment component with a local artificial potential field. The potential field for each agent is a superposition of individual fields associated with its neighbors and is minimized in a distributed way. Both the minimization of the agents potentials and the stability of the flocking motion is established by exploiting the algebraic connectivity of the underlying interconnection graph.

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References

[1] H. G. Tanner, A. Jadbabaie, and G. J. Pappas, "Stable flocking of mobile agents, Part I: Fixed topology," in *Proceedings of the IEEE Conference on Decision and Control*, Maui, Hawaii, December 9-12 2003, pp. 2010–2015.

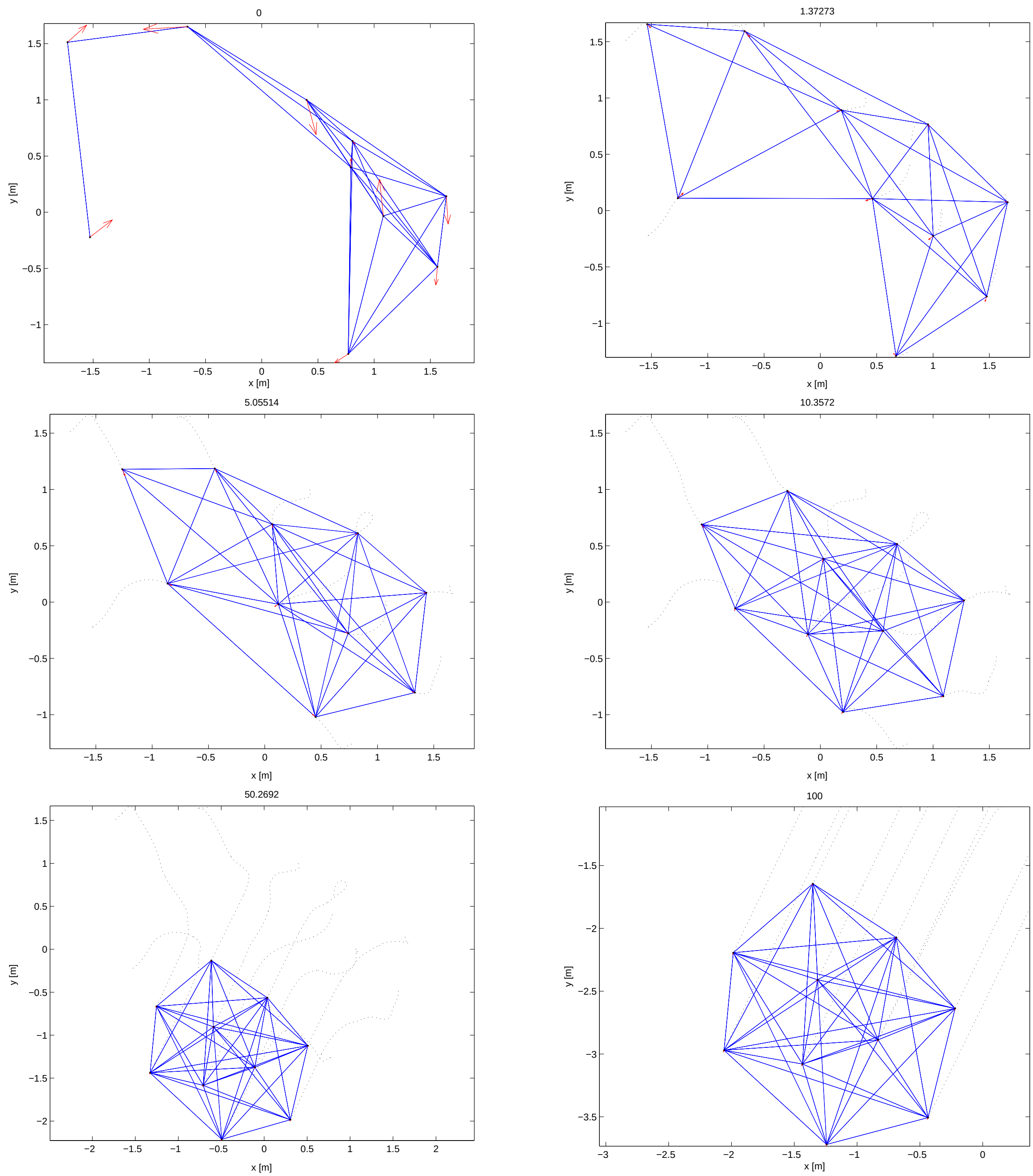

Fig. 5. Successive simulation time snapshots of flocking with dynamic interconnection topology.

[2] ——, "Stable flocking of mobile agents, Part II: Dynamic topology," in *Proceedings of the IEEE Conference on Decision and Control*, Maui, Hawaii, December 9-12 2003, pp. 2016–2021.

[3] C. Reynolds, "Flocks, birds, and schools: a distributed behavioral model," *Computer Graphics*, vol. 21, pp. 25–34, 1987.

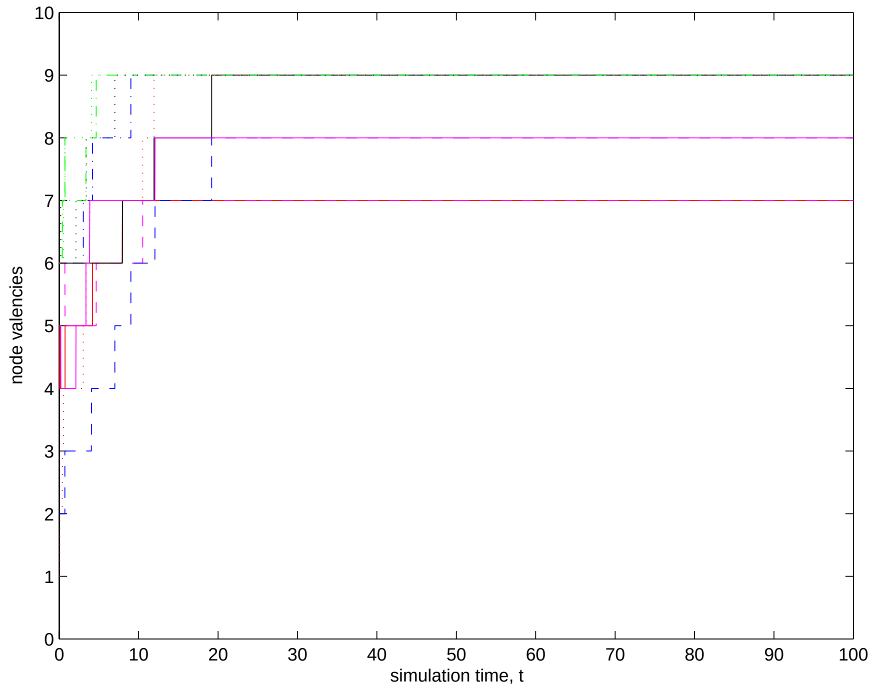

Fig. 6. Time history of the valencies of the nodes in the interconnection graph, for the case of switching topology.

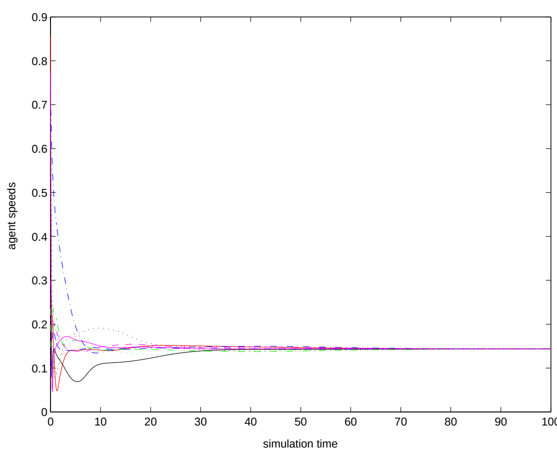

Fig. 7. Convergence of speeds with fixed topology.

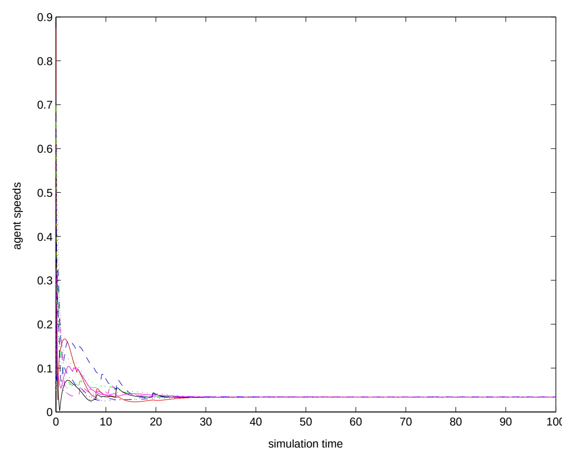

Fig. 8. Convergence of speeds with dynamic topology.

[4] T. Vicsek, “A question of scale,” *Nature*, vol. 411, p. 421, May 24 2001.

[5] D. J. Low, “Following the crowd,” *Nature*, vol. 407, pp. 465–466, 2000.

[6] D. Grunbaum and A. Okubo, “Modeling social animal aggregations,” *Frontiers in Theoretical Biology*, vol. 100 of Lecture Notes in Biomathematics, pp. 296–325, 1994.

[7] G. Flierl, D. Grunbaum, S. Levin, and D. Olson, “From individuals to aggregations: the interplay between behavior and

physics," *Journal of Theoretical Biology*, vol. 196, pp. 397–454, 1999.

[8] D. Terzopoulos, "Artificial life for computer graphics," *Communications of the ACM*, vol. 42, no. 8, pp. 32–42, 1999.

[9] T. Vicsek, A. Czirok, E. B. Jacob, I. Cohen, and O. Schochet, "Novel type of phase transitions in a system of self-driven particles," *Physical Review Letters*, vol. 75, pp. 1226–1229, 1995.

[10] J. Toner and Y. Tu, "Flocks, herds, and schools: A quantitative theory of flocking," *Physical Review E.*, vol. 58, pp. 4828–4858, 1998.

[11] H. Levine and W. J. Rappel, "Self organization in systems of self-propelled particles," *Physical Review E*, vol. 63, pp. 208–211, 2001.

[12] N. Leonard and E. Friorelli, "Virtual leaders, artificial potentials and coordinated control of groups," in *IEEE Conference on Decision and Control*, Orlando, FL, 2001.

[13] J. Cortés, S. Martinez, T. Karatas, and F. Bullo, "Coverage control for mobile sensing networks," *IEEE Transactions on Robotics and Automation*, vol. 20, no. 2, pp. 243–255, April 2004.

[14] R. Olfati and R. M. Murray, "Distributed structural stabilization and tracking for formations of dynamic multi-agents," in *IEEE Conference on Decision and Control*, Las Vegas, NV, 2002, pp. 209–215.

[15] J. H. Reif and H. Wang, "Social potential fields: A distributed behavioral control for autonomous robots," *Robotics and Autonomous Systems*, vol. 27, pp. 171–194, 1999.

[16] J. A. Fax and R. M. Murray, "Graph Laplacians and stabilization of vehicle formations," *15th IFAC Congress, Barcelona, Spain*, 2002.

[17] Y. Liu, K. M. Passino, and M. M. Polycarpou, "Stability analysis of m-dimensional asynchronous swarms with a fixed communication topology," *IEEE Transactions on Automatic Control*, vol. 48, no. 1, pp. 76–95, February 2003.

[18] V. Gazi and K. M. Passino, "Stability analysis of swarms," *IEEE Transactions on Automatic Control*, vol. 48, no. 4, pp. 692–696, April 2003.

[19] P. Tabuada, G. J. Pappas, and P. Lima, "Feasible formations of multi-agent systems," in *Proceedings of the American Control Conference*, Arlington, VA, 2001.

[20] A. Jadbabaie, J. Lin, and A. S. Morse, "Coordination of groups of mobile autonomous agents using nearest neighbor rules," *IEEE Transactions on Automatic Control*, vol. 48, no. 6, pp. 988–1001, July 2002.

[21] P. Ögren, M. Egerstedt, and X. Hu, "A control Lyapunov function approach to multi-agent coordination," *IEEE Transactions on Robotics and Automation*, vol. 18, no. 5, October 2002.

[22] J. P. Desai, J. P. Ostrowski, and V. Kumar, "Modeling and control of formations of nonholonomic mobile robots," *IEEE Transactions on Robotics and Automation*, vol. 17, no. 6, pp. 905–908, 2001.

[23] E. Justh and P. Krishnaprasad, "A simple control law for UAV formation flying," Institute for Systems Research, Tech. Rep. 2002-38, 2002.

[24] R. Vidal, O. Shakernia, and S. Sastry, "Distributed formation control with omnidirectional vision based motion segmentation and visual servoing," *Robotics and Automation Magazine*, 2003, (to appear).

[25] A. Pant, P. Seiler, T. J. Koo, and K. Hedrick, "Mesh stability of unmanned aerial vehicle clusters," in *Proceedings of the American Control Conference*, Arlington, VA, June 2001, pp. 62–68.

[26] O. Khatib, "Real-time obstacle avoidance for manipulators and mobile robots," *The International Journal of Robotics Research*, vol. 5, no. 1, pp. 396–404, Spring 1986.

[27] E. Rimon and D. Koditschek, "Exact robot navigation using artificial potential functions," *IEEE Transactions on Robotics and Automation*, vol. 8, no. 5, pp. 501–518, October 1992.

[28] R. Olfati-Saber and R. M. Murray, “Consensus problems in networks of agents with switching topology and time-delays,” *IEEE Transactions on Automatic Control*, vol. 49, no. 9, pp. 1520–1533, September 2004.

[29] C. Godsil and G. Royle, *Algebraic Graph Theory*. New York: Springer Graduate Texts in Mathematics # 207, 2001.

[30] R. Merris, “Laplacian matrices of graphs: A survey,” *Linear Algebra and its Applications*, vol. 197, pp. 143–176, 1994.

[31] A. F. Filippov, *Differential equations with discontinuous right-hand side*, ser. Mathematics and Its Applications (Soviet Series), F. M. Arscott, Ed. The Netherlands: Kluwer Academic Publishers, 1988.

[32] F. H. Clarke, *Optimization and Nonsmooth Analysis*, ser. Classics in applied mathematics; 5. Philadelphia: SIAM, 1990.

[33] D. Shevitz and B. Paden, “Lyapunov stability theory of nonsmooth systems,” *IEEE Transactions on Automatic Control*, vol. 39, no. 9, pp. 1910–1914, 1994.

[34] B. Paden and S. Sastry, “A calculus for computing filipov’s differential inclusion with application to the variable structure control of robot manipulators,” *IEEE Transactions on Circuits and Systems*, vol. CAS-34, no. 1, pp. 73–82, 1987.

[35] E. P. Ryan, “An integral invariance principle for differential inclusions with applications in adaptive control,” *SIAM Journal on Control and Optimization*, vol. 36, no. 3, pp. 960–980, May 1998.

[36] A. Bacciotti and F. Ceragioli, “Stability and stabilization of discontinuous systems and nonsmooth lyapunov functions,” *ESAIM: Control Optimisation and Calculus of Variations*, vol. 4, pp. 361–376, 1999.

[37] F. Clarke, Y. Ledyaev, R. Stern, and P. Wolenski, *Nonsmooth Analysis and Control Theory*, ser. Graduate Texts in Mathematics; 178. New York: Springer, 1998.

APPENDIX

The purpose of this section is to briefly introduce the mathematical machinery related to nonsmooth stability analysis. We begin with a definition of our notion of solutions of differential equations with discontinuous right hand sides:

Definition 4 ([34]) *Consider the following differential equation in which the right hand side can be discontinuous:*

$$\dot{x} = f(x) \tag{15}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ *is measurable and essentially locally bounded and* n *is finite. A vector function* $x(\cdot)$ *is called a solution of* (15) *on* $[t_0, t_1]$*, where if* $x(\cdot)$ *is absolutely continuous on* $[t_0, t_1]$ *and for almost all* $t \in [t_0, t_1]$

$$\dot{x} = K[f](x)$$

where

$$K[f](x) \triangleq \overline{co}\{\lim_{x_i \to x} f(x_i) \mid x_i \notin M_f \cup M\}$$

where $M_f \subset \mathbb{R}^n$*,* $\mu(M_f) = 0$ *and* $M \subset \mathbb{R}^n$*,* $\mu(M) = 0$*.*

According to this definition, a trajectory $x(t)$ is considered a solution of the discontinuous differential equation (15) if its tangent vector, where defined, belongs in the convex closure of the limit of the vector fields defined by (15) in a decreasingly small neighborhood of the solution point. Being able to exclude a set of measure zero, is critical since one can thus define solutions even at points where the vector field in (15) is not defined.

A sligthtly more general definition of (maximal) solutions can be found in [35]:

Definition 5 ([35]) *Consider the autonomous initial-value problem*

$$\dot{x}(t) \in X\big(x(t)\big), \quad x(t) \in G, \quad x(t_0) = x^0, \tag{16}$$

where $G \neq \emptyset$ is an open subset of $\mathbb{R}^N$. The set-values map $(x) \mapsto X(x) \subset \mathbb{R}^N$ in (16) *is assumed to be upper semicontinuous on $\mathbb{R} \times G$, with nonempty, convex, and compact values.*

This definition is sufficient to ensure that the solution is absolutely continuous on compact subintervals $I \in \mathbb{R}$ ($x(t) \in AC(I;G)$). Ryan defines solutions to be maximal if they cannot be extended any further in time:

Definition 6 ([35]) *A solution of* (16) *is said to be maximal if it does not have a proper right extension which is also a solution of* (16)*.*

Then, it can be shown that all solutions of (16) can be thought to be maximal:

Proposition 1 ([35]) *Every solution of* (16) *can be extended to a maximal solution.*

A maximal solution is called precompact if it always stays in the closure of G:

Definition 7 ([35]) *A solution $x \in AC([t_0, \omega); G)$ of* (16) *is precompact if it is maximal and the closure $\mathrm{cl}\big(x([t_0, \omega))\big)$ of its trajectory is a compact subset of G.*

Lyapunov stability has been extended to nonsmooth systems [33], [36]. Establishing stability results in this framework requires working with generalized derivatives [32], whenever classical derivatives are not defined.

Definition 8 ([32]) *Let f be Lipschitz near a given point x and let w be any vector in a Banach space X. The generalized directional derivative of f at x in the direction w, denoted $f^\circ(x; w)$*

is defined as follows:

$$f^\circ(x;w) \triangleq \limsup_{\substack{y\to x\\ t\downarrow 0}} \frac{f(y+tw)-f(y)}{t}$$

The generalized gradient, on the other hand, is generally a set of vectors, which reduces to the single classical gradient in the case where the function is differentiable:

Definition 9 ([32]) *The generalized gradient of f at x, denoted $\partial f(x)$, is the subset of X^* given by:*

$$\partial f(x) \triangleq \{\zeta \in X^* \mid f^\circ(x;w) \geq \langle \zeta, w\rangle,\ \forall w \in X\}$$

In the special case where X is finite dimensional, we have the following convenient characterization of the generalized gradient:

Theorem 3 ([37]) *Let $x \in \mathbb{R}^n$ and let $f : \mathbb{R}^n \to \mathbb{R}$ be Lipschitz near x. Let Ω be any subset of zero measure in $\mathbb{R}^n$, and let Ω_f be the set of points in $\mathbb{R}^n$ at which f fails to be differentiable. Then*

$$\partial f(x) \triangleq \mathrm{co}\{\lim_{x_i\to x} \nabla f(x_i) \mid x_i \notin \Omega, x_i \notin \Omega_f\}$$

A *weakly invariant set* is defined to be the set where at least one of the (possibly multiple) maximal solutions of (16) stays forever in the set:

Definition 10 ([35]) *Relative to* (16)*, $S \subset \mathbb{R}^N$ is said to be weakly invariant set if, for each $x^0 \in S \cap G$, there exists at least one maximal solution $x \in AC([0,\omega);G)$ of* (16) *with $\omega = \infty$ and with trajectory $x([0,\omega))$ in S.*

Shevitz and Paden [33] proposed a nonsmooth version of LaSalle's invariant principle, and Bacciotti and Ceragioli [36] have given an alternative nonsmooth characterization of the invariant principle which also applies to the case where uniqueness of solutions cannot be guaranteed. Here, we choose to apply the invariant principle introduced by [35] since, not only does it not require uniqueness of solutions, but also lifts the regularity requirement from the Lyapunov-like function. The price to pay is that one can only guarantee the invariance of at least a single solution. This is unavoidable, if one is to relax the requirement on uniqueness of solutions [36].

Theorem 4 ([35]) *Let $V : G \to \mathbb{R}$ be locally Lipschitz. Define*

$$m : G \to \mathbb{R}, z \mapsto m(z) \triangleq \max\{V^\circ(z;\phi) \mid \phi \in X(z)\}.$$

Suppose that $U \subset G$ *is non-empty and that* $m(z) \leq 0$ *for all* $z \in U$*. If* x *is a precompact solution of* (16) *with trajectory in* U*, then for some constant* $c \in V(\mathrm{cl}(U) \cap G)$*,* x *approaches the largest weakly invariant set in* $\Sigma \cap V^{-1}(c)$*, where*

$$\Sigma = \{z \in \mathrm{cl}(U) \cap G \mid m(z) \geq 0\}.$$

In the above theorem, the invariant set is defined more generally, but this does not restrict our analysis. In essense, it allows the generalized time derivative of V to be positive on the boundary of U. From Definitions 9 and 4, one can see that $m(z)$ is nothing but the maximal element in $\dot{\tilde{V}}$ of [33], if V turns out to be regular.