Partial Controllability of A Leader-Follower Dynamic Network

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Abstract: Generally speaking, controllability of multi-agent systems refer to transferring the remaining agents of such system from any arbitrary initial state to any final state by controlling dynamics of a small amount of agents under exchanged information between each other, which makes the system reflect the effect of a whole, that is the complete controllability. Comparing with the complete controllability of the general multi-agent systems, in reality, it may not be necessary to actuate all nodes to the desired configuration. This paper addresses the partial controllability of the discrete-time multi-agent system with a leader under fixed topology, proposes the concept of controllable node group. Some necessary and sufficient conditions are given for partial controllability of the discrete-time multi-agent system with a single leader. Moreover, the partial controllability criteria of some special cases are given. Finally, an numerical example and simulations are given to illustrate the correctness of the theoretical results.

Key Words: Multi-Agent Systems, Complete Controllability, Partial Controllability

1 INTRODUCTION

In recent years, the multi-agent system has become a new branch involving engineering, ecology, biology, sociology, computer science, communication and sensing technology, which attracts lots of researchers' interests and concerns [1]-[9]. Studies in this field have been greatly inspired by the ubiquitous cooperative behavior in real world, such as bird flocks, fish schools, ant swarms and bacteria colonies. In the context of multi-agent networks, however, the issue of the controllability of the multi-agent systems presents new features and difficulties [10]-[13].

The controllability problem of multi-agent systems has been first put forward by Tanner in [14], where one of the agents was regarded as a leader, necessary and sufficient condition of the controllability was presented through the Laplacian matrix under a fixed time-invariant nearest-neighbor topology. Based on those, some important sufficient/necessary conditions and related results for the controllability of multi-agent system are given. From [15] the system can be divided into fixed topology and switched topology. On the other hand, the system can be divided into the first-order integrator, the second-order integrator and high-order integrator such as [16]-[18]. Corresponding to the continuous-time multi-agent systems, the discrete-time multi-agent systems with leaders via switching topology can have more special results in [19]. In [16], sufficient conditions for the controllability of a switching network of multi-agent systems with a leader were presented. In [20], the controllability of a leader-follower discrete-time dynamic network with switching topology was shown. [13] investigated the switching controllability of discrete-time multi-agent systems with a single time-delay and multiple time-delays, in which the switching controllability of discrete-time multi-agent systems is determined by the information from the leaders to the followers and also the discrete-time multi-agent systems are controllable even if each of its subsystems is uncontrollable. [21] established a theoretical framework for the controllability of higher-order systems and it proved the strongly accessible and small time locally controllability. [22] studies the group controllability problems of discrete-time multi-agent systems with a single time-delay on undirected networks in which both switching topology and fixed topology are addressed. [23] shown the results of some tree structure systems.

At present, the existing results merely concern the complete controllability of the whole system. However, the uncontrollability of multi-agent systems is rarely concerned [24]. In other words, the states of specific agents are drawn to be paid attention. Comparing with the complete controllability of the general multi-agent systems, in reality, it may not be necessary to actuate all nodes to the desired configuration, that is, it is not essential that all agents in the system being controllable. This paper addresses the partial controllability of the discrete-time multi-agent system with a leader under fixed topology, proposes the concept of controllable node group. Some algebraic properties of the controllability matrix and restrictions on dynamic behaviors of

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the whole system will be investigated. The reminder of this paper is organized as follows. Section 2 gives the model and some preliminaries. Section 3 presents the main results, and some simulations are given in Section 4. Finally, Section 5 gives the conclusion.

2 PRELIMINARIES AND PROBLEM FORMULATION

2.1 Graphs Preliminaries [9]

A *weighted directed graph* $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ consists of a *vertex set* $\mathcal{V} = \{v_1, v_2, \cdots, v_N\}$ and an *edge set* $\mathcal{E} = \{(v_i, v_j) : v_i, v_j \in \mathcal{V}\}$, where an *edge* is an ordered pair of distinct vertices of $\mathcal{V}$, and the nonsymmetric weighted adjacency matrix $\mathcal{A} = [a_{ij}]$, with $a_{ij} > 0$ if and only if $e_{ij} \in \mathcal{E}$ and $a_{ij} = 0$ if not. If all the elements of $\mathcal{V}$ are unordered pairs, then the graph is called an *undirected graph*. If $v_i, v_j \in \mathcal{V}$, and $(v_i, v_j) \in \mathcal{E}$, then we say that v_i and v_j are adjacent or v_j is a neighbor of v_i. The neighborhood set of node v_i is denoted by $\mathcal{N}_i = \{v_j \in \mathcal{V} : (v_i, v_j) \in \mathcal{E}\}$. The number of neighbors of each vertex is its *degree*. A graph is called *complete* if every pair of vertices are adjacent. A *path* of length r from v_i to v_j in a graph is a sequence of $r+1$ distinct vertices starting with v_i and ending with v_j such that consecutive vertices are adjacent. If there is a path between any two vertices of $\mathcal{G}$, then $\mathcal{G}$ is *connected*. The *degree matrix* $\triangle(\mathcal{G})$ of $\mathcal{G}$ is a diagonal matrix with rows and columns indexed by $\mathcal{V}$, in which the (v_i, v_i)-entry is the degree of vertex v_i. The symmetric matrix defined as

$$L(\mathcal{G}) = \triangle(\mathcal{G}) - A(\mathcal{G})$$

is the *Laplacian* of $\mathcal{G}$. The Laplacian is always symmetric and positive semi-definite, and the algebraic multiplicity of its zero eigenvalue is equal to the number of connected components in the graph. Define a network $(\mathcal{G}, x)$ with state $x \in \mathcal{R}^k$ and topology graph $\mathcal{G}$, where the network has N agents and the $i-th$ agent $v_i \in \mathcal{R}$ is its state. Each agent updates its state based on the information available from its neighbors.

2.2 Problem Formulation

Consider a multi-agent system with n followers (labeled from 1 to n) and a leader (labeled 0) described by

$$x_i(k+1) = x_i(k) - \sum_{j \in \mathcal{N}_{i_j}} a_{ij}(x_i(k) - x_j(k)) - a_{i0}(x_i(k) - x_0(k)), \tag{1}$$

where $k \in J_k$, $x_i \in \mathcal{R}$ is the state of agent i; $A = [a_{ij}] \in \mathcal{R}^{n \times n}$ with each $a_{ij} \geq 0$ is the interactive matrix; and if there exists information from leader 0 to agent i, then $a_{i0} = 1$, otherwise $a_{i0} = 0$; J_k is a discrete time index set. The dimension of x_i could be arbitrary, due to it is the same for all agents. For convenience, we will consider the one-dimensional case here. Let $x = (x_1, \ldots, x_n)^T$ be the vector of all the followers, then the system (1) can be rewritten as:

$$\begin{aligned} x(k+1) &= (I - L - R)x(k) + rx_0(k) \\ &\triangleq Fx(k) + rx_0(k), \end{aligned} \tag{2}$$

where $F \triangleq I - L - R$, matrix $R = diag(a_{10}, a_{20}, a_{30}, \ldots, a_{n0})$, I is the identity matrix, the vector $r = (a_{10}, a_{20}, a_{30}, \ldots, a_{n0})^T$ and L is Laplacian matrix with

$$l_{ij} = \begin{cases} -a_{ij}, & i \neq j \text{ and } j \in \mathcal{N}_{i_j} \\ \sum\limits_{j \in \mathcal{N}_{i_j}} a_{ij}, & i = j \\ 0, & \text{otherwise} \end{cases} .$$

Moreover, the Laplacian matrix L has the following properties:
i) the off-diagonal elements are all negative or zero;
ii) the row sums are equal to the column sums and all take the value of zero.

Remark 1 *If x is m-dimensional, then system (2) can turn into the form $x(k+1) = (F \otimes I_m)x(k) + (r \otimes I_m)x_0(k)$, where I_m is the m-dimensional identity matrix.*

This paper focuses on the partial controllability of discrete-time multi-agent systems under fixed topology. First, we propose the concept of partial controllability of discrete-time multi-agent systems.

Definition 1 *(Controllable node group)*
Agents $x_{i1}, x_{i2}, \ldots, x_{im}$ in system (2) is said to be controllable if for any initial states $x_{i1}(h), x_{i2}(h), \ldots, x_{im}(h)$ and target states $x^*_{i1}, x^*_{i2}, \ldots, x^*_{im}$, there exist external inputs $x_0(k)$ on the leaders and a finite $l > h$, such that$x_{i1}(l) = x^*_{i1}, x_{i2}(l) = x^*_{i2}, \ldots, x_{im}(l) = x^*_{im}$, regardless of states of the other agents. Agents $x_{i1}, x_{i2}, \ldots, x_{im}$ are said to form a controllable node group .

Definition 2 *(Partial controllability [24])*
For multi-agent system (2), check whether the part is controllable, regardless of states of the other agents. Partial controllability embodies the unique characteristic of multi-agent systems. In system (2), x_i represents the state of agent i, therefore in the vector x, each agent corresponds to a specific entry and this correspondence is fixed.

3 PARTIAL CONTROLLABILITY OF DISCRETE-TIME MULTI-AGENT SYSTEMS

In this section, some necessary and sufficient criteria about the partial controllability of discrete-time multi-agent systems are given in the following.

Theorem 1 *(Partial Gram Criterion) In system (2), nodes in the same group $\tilde{V} = \{\mathcal{V}_{i1}, \mathcal{V}_{i2}, \mathcal{V}_{i3}, \ldots, \mathcal{V}_{im}\}$ is controllable if and only if the principal minor formulated by the $i_1, i_2, i_3, \ldots, i_m - th$ columns and rows in the Grammian matrix $W_c[h, l]$ (h is the initial time and l is the target time), denoted as $\tilde{W}$ is invertible, where*

$$W_c[h, l] = \sum_{k=h}^{l-1} F^{l-k-1} r r^T (F^T)^{l-k-1}.$$

Proof: Without loss of generality, we suppose that node group contains $1, 2, \cdots, m$ nodes.
(Sufficiency): Let $\tilde{x} = (x_1(h), x_2(h), \ldots, x_m(h))^T \in \mathbb{R}^m$ and

$\tilde{x}^* = (x^*_1, x^*_2, \ldots, x^*_m)^T \in \mathbb{R}^m$ be the initial states and target states of nodes in that part respectively. Denote $x(l) = ((\tilde{x}^*)^T, x^*_{m+1}, x^*_{m+2}, \ldots, x^*_n)^T \in \mathbb{R}^n$, and the initial states of the whole system is $x(h) = (x_1(h), x_2(h), \ldots, x_n(h))^T \in \mathbb{R}^n$. In the following, we will show exists $\tilde{z} \in \mathbb{R}^m, z = (\tilde{z}^T, 0, 0, \ldots, 0) \in \mathbb{R}^n$ such that when $x_0 = -r^T(F^T)^{l-h-1}z$,

$$\begin{aligned} x(l) &= F^{l-h}x(h) + \sum_{k=h}^{l-1} F^{l-k-1} r x_0(k) \\ &= F^{l-h}x(h) + \sum_{k=h}^{l-1} F^{l-k-1} r(-r^T)(F^T)^{l-k-1} z \\ &= F^{l-h}x(h) - W_c[h,l]z. \end{aligned}$$

Denote $\tilde{y}$ as the first m entries of $F^{l-h}x(h)$, let $\tilde{z} = \tilde{W}^{-1}(\tilde{y} - \tilde{x}^*)$, the first r entries of $W_c[h,l]z$ will be $\tilde{y} - \tilde{x}^*$. This ensures that $x_i(l) = x^*_i, i = 1, 2, \ldots, m$, which mean that the partial group which contains agents $i_1 = 1, i_2 = 2, \ldots, i_m = m$ is controllable.

(Necessity): If $\tilde{W}$ is not invertible, there exists $\tilde{\alpha} \in R^m, \tilde{\alpha} \neq 0, \alpha = (\tilde{\alpha}, 0, 0, \ldots, 0)^T \in R^n$ such that $\alpha^T W_c[h,l]\alpha = 0$. Since the node group which contains nodes $x_1, x_2, \ldots, x_m$ are controllable, there exists $x_0(k)$ and $l > h$ such that

$$x(l) = F^{l-h}x(h) + \sum_{k=h}^{l-1} F^{l-k-1} r x_0(k),$$

for any $\tilde{x}(h)$ and any $\tilde{x}^*$, regardless of the states of the other agents. As $\alpha \neq 0$, select $x(h)$ and $x(l)$ such that $\alpha^T(x(l) - F^{l-h}x(h)) \neq 0$, which yields

$$\alpha^T \sum_{k=h}^{l-1} F^{l-k-1} r x_0(k) \neq 0. \tag{3}$$

However,

$$\begin{aligned} \alpha^T W_c[h,l]\alpha &= \sum_{k=h}^{l-1} \alpha^T F^{l-k-1} r r^T (F^T)^{l-k-1} \alpha \\ &= \sum_{k=h}^{l-1} \| \alpha^T F^{l-k-1} r \|^2 \\ &= 0, \end{aligned}$$

for

$$\alpha^T F^{l-k-1} r = 0, h < k < l-1.$$

Therefore

$$\sum_{k=h}^{l-1} \alpha^T F^{l-k-1} r x_0(k) = 0,$$

which makes a contradiction with system (3). Therefore, $\tilde{W}$ is invertible.

Theorem 2 *(Partial Rank Criterion) In system (2), the partial group which contains agents $\tilde{V} = \{\mathscr{V}_{i1}, \mathscr{V}_{i2}, \mathscr{V}_{i3}, \ldots, \mathscr{V}_{im}\}$ is controllable if and only if the $i_1, i_2, i_3, \ldots, i_m - th$ rows in the controllability matrix $C = [r, Fr, F^2r, \ldots, F^{n-1}r]$ are linearly independent.*

Proof: Without loss of generality, we suppose that part contains $1, 2, \cdots, m$ nodes.

(Sufficiency): If the part $\tilde{V}$ is uncontrollable, according to Theorem 1, $\tilde{W}$ is not invertible for all $l > h$, so there exists $\tilde{\alpha} \neq 0, \tilde{\alpha} \in R^m$such that $\tilde{\alpha}^T \tilde{W} \tilde{\alpha} = 0$. Denote $\alpha = (\tilde{\alpha}^T, 0, 0, \ldots, 0)^T \in R^n$, we can have

$$\begin{aligned} \alpha^T W_c[h,l]\alpha &= \sum_{k=h}^{l-1} \alpha^T F^{l-k-1} r r^T (F^T)^{l-k-1} \alpha \\ &= \sum_{k=h}^{l-1} \| \alpha^T F^{l-k-1} r \|^2 \\ &= 0. \end{aligned}$$

Therefore, $\alpha^T F^{l-k-1} r = 0$ for all $h < k < l-1$. For $k = l-1$, then $\alpha^T r = 0$; for $k = l-2$, then $\alpha^T F r = 0$; for $k = l-3$, then $\alpha^T F^2 r = 0$;...; for $l-h = n$, then $\alpha^T F^{n-1} r = 0$, therefore $\alpha^T [r, Fr, F^2r, \ldots, F^{n-1}r] = 0$. Since $\alpha = (\tilde{\alpha}, 0, 0, \ldots, 0), \tilde{\alpha} \neq 0$, the first m rows of C must be linearly relevant, which makes a contradiction.

(Necessity): If the first m rows of C are linearly relevant, there must exists $\tilde{\alpha} \neq 0 \in R^m, \alpha = (\tilde{\alpha}, 0, 0, \ldots, 0) \in R^n$ such that $\alpha C = 0$. Since $C = [r, Fr, F^2r, \ldots, F^{n-1}r]$, then $\alpha r = 0, \alpha F r = 0, \alpha F^2 r = 0, \ldots, \alpha F^{n-1} r = 0$, and then $\alpha r r^T \alpha = 0, \alpha F r r^T F^T \alpha = 0, \alpha F^2 r r^T (F^T)^2 \alpha = 0, \ldots, \alpha F^{n-1} r r^T (F^T)^{n-1} \alpha = 0$, thus

$$W_c[h,l] = \sum_{k=h}^{l-1} F^{l-k-1} r r^T (F^T)^{l-k-1},$$

then we can have

$$\sum_{k=h}^{l-1} \alpha F^{l-k-1} r r^T (F^T)^{l-k-1} \alpha^T = 0,$$

and

$$\alpha^T W_c[h,l]\alpha = 0.$$

Because of the first m entry of α are not all zero, we declare that $\tilde{W}$ is not invertible, and the part which contains agents $i_1 = 1, i_2 = 2, \ldots, i_m = m$ is not controllable. This is a contradiction.

Remark 2 *If $\tilde{V} = \{\mathscr{V}_{i1}, \mathscr{V}_{i2}, \mathscr{V}_{i3}, \ldots, \mathscr{V}_{im}\}$ in system (2) is uncontrollable, then the rows in the controllability matrix corresponding to those nodes are linearly dependent.*

Remark 3 *If there is a part in system (2) which contains m nodes is controllable, then the rank of the controllability matrix C at least is m.*

Corollary 1 *For system (2), if $\tilde{V} = \{\mathscr{V}_{i_1}, \mathscr{V}_{i_2}, \mathscr{V}_{i_3}, \ldots, \mathscr{V}_{i_m}\}$ is controllable, then there exist $i_1, i_2, , i_m - th$ rows in $(\lambda I + F; r)$ are linearly independent for all λ_L(or equivalently $\lambda \in C$).*

Proof: Without loss of generality, we suppose that part contains $1, 2, \cdots, m$ nodes. Assume that there exists an eigenvalue λ_0 of F, such that the first m rows of $(\lambda_0 I + F; r)$ are linearly dependent, then there exists $\tilde{\alpha} \neq 0, \tilde{\alpha} \in R^m$, $\alpha = (\tilde{\alpha}^T, 0, 0, \ldots, 0)^T \in R^n$, such that $\alpha(\lambda_0 I +$

$F;r) = 0$, and thus we can get $\alpha[r, Fr, F^2 r, \ldots, F^{n-1}r] = \alpha[r, -\lambda_0 r, \lambda_0^2 r, \ldots, (-\lambda)^{n-1} r] = 0$, since the first m entries of α are not all 0, the first m rows of $[r, Fr, F^2 r, \ldots, F^{n-1}r]$ are linearly dependent, so the partial group which contains agents $1, 2, \ldots, m$ is uncontrollable. This is a contradiction.

Definition 3 *[19] An isolated agent is the one who has no information with the other agents (i.e., its degree is zero).*

Corollary 2 *If there is an isolated agent in the partial group, then that part in system (2) is uncontrollable.*

Proof: Without loss of generality, we assume that part contains the agents $\tilde{V} = \{\mathscr{V}_{i1}, \mathscr{V}_{i2}, \mathscr{V}_{i3}, \ldots, \mathscr{V}_{im}\} = \{1, 2, \ldots, m\}$, and the agent labeled 1 is the isolated one, then

$$r = \begin{bmatrix} 0 \\ * \\ \vdots \\ * \end{bmatrix}, \quad F = I - L - R = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & & & \\ \vdots & & * & \\ 0 & & & \end{bmatrix}.$$

Easily, we can have

$$C = [r, Fr, \cdots, F^{n-1}r] = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ & & & & \\ & & * & & \\ & & & & \end{bmatrix}.$$

The first row of matrix C are all zeros, then the part which contains $1, 2, \cdots, m$ nodes is uncontrollable.

Corollary 3 *System (2) with complete graph is uncontrollable.*

Proof: If the graph is complete, then

$$r = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}, F = \begin{bmatrix} 1-n & 1 & \cdots & 1 & 1 \\ 1 & 1-n & \cdots & 1 & 1 \\ & & \ddots & & \\ 1 & 1 & \cdots & 1 & 1-n \end{bmatrix}.$$

By computing, we have $\text{rank}(C) = 1$, then system (2) is uncontrollable, obviously, arbitrary part is uncontrollable.

Corollary 4 *Each part in the path is controllable.*

Proof:Without loss of generality, assume that the paths nodes labelled $1, 2, \ldots, n$ and the nodes in the part are $1, 2, \ldots, m$, we suggest that x_0 is the neighbor of the first agent, then

$$r = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \; F = \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 \\ 1 & -1 & 1 & \cdots & 0 \\ & & \ddots & & \\ 0 & \cdots & 1 & -1 & 1 \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

By computing, we can get

$$rank(C) = rank \left\{ \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} * \\ 1 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \cdots \begin{bmatrix} * \\ \vdots \\ * \\ * \\ 1 \end{bmatrix} \right\} = n,$$

so the the first m rows in C are linearly independent, then each part in the path is partial controllable.

Corollary 5 *If the partial group has followers which receive information from the same agent, whether that agent is leader or follower, that partial group is uncontrollable.*

Proof: Without loss of generality, we assume the part contains agents $\tilde{V} = \{\mathscr{V}_{i1}, \mathscr{V}_{i2}, \mathscr{V}_{i3}, \ldots, \mathscr{V}_{im}\} = \{1, 2, \ldots, m\}$, and the agents labeled 1 and 2 receive information from the same agent. If that agent is the leader x_0, we can easy get

$$r = \begin{bmatrix} 1 \\ 1 \\ * \\ \vdots \\ * \end{bmatrix}, L = \begin{bmatrix} 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 \\ & & * & & \\ & & & & \end{bmatrix}, F = \begin{bmatrix} 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 \\ & & * & & \\ & & & & \end{bmatrix}.$$

If that agent is follower, we can get

$$L = \begin{bmatrix} 1 & 0 & \ldots & 0 & -1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 & -1 & 0 & \ldots & 0 \\ & & & & & & & \\ & & & & * & & & \\ & & & & & & & \end{bmatrix}, r = \begin{bmatrix} 0 \\ 0 \\ * \\ \vdots \\ * \end{bmatrix},$$

then

$$F = \begin{bmatrix} 0 & 0 & \ldots & 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & 1 & 0 & \ldots & 0 \\ & & & & & & & \\ & & & & * & & & \\ & & & & & & & \end{bmatrix}.$$

So $1, 2, \cdots, m$ rows of matrix C are linearly dependent, and the part which contains agents $1, 2, \cdots, m$ is uncontrollable.

4 EXAMPLE AND SIMULATIONS

In this section, we will give an example and simulation results to illustrate the effective of the theoretical results.

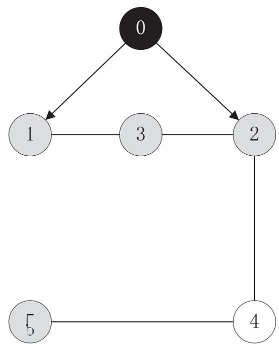

Figure 1: The Topology.

For convenience, all weights are assumed to be equal to 1. We consider a six-agent network with agent 0 as the leader

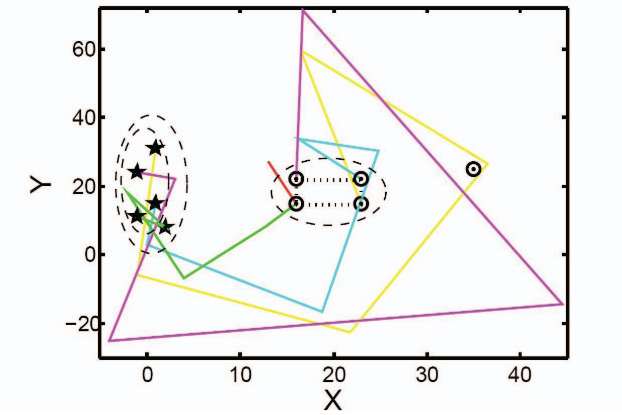

Figure 2: A rectangle.

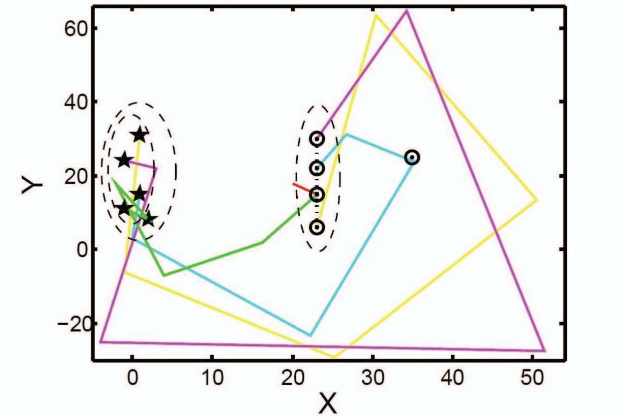

Figure 3: A straight line.

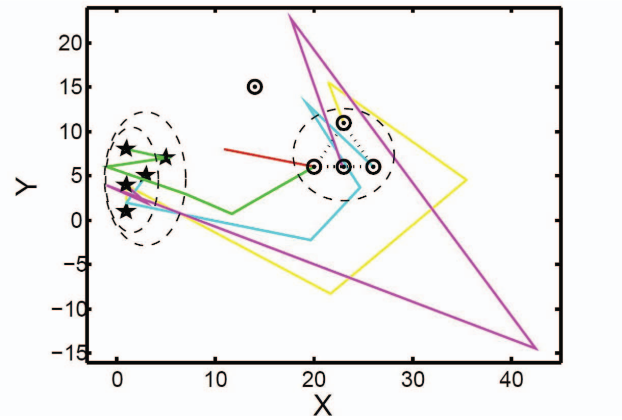

Figure 4: A triangle.

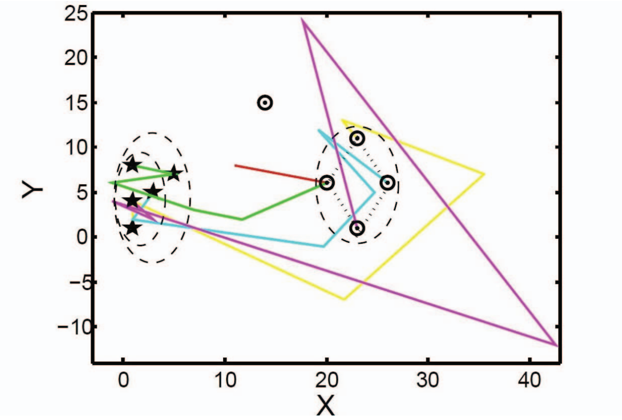

Figure 5: A diamond.

and the remaining agents as followers under fixed topology described by Fig. 1, we can get

$$L = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 2 & -1 & -1 & 0 \\ -1 & -1 & 2 & 0 & 0 \\ 0 & -1 & 0 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}, \quad r = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Then

$$F = I - L - R = \begin{bmatrix} -1 & 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

We suppose that $h = 0, l = 5$, then

$$\begin{aligned} W_c[h,l] &= \sum_{k=h}^{l-1} F^{l-k-1} r r^T (F^T)^{l-k-1} \\ &= \sum_{k=0}^{4} F^{4-k} r r^T (F^T)^{4-k}. \end{aligned}$$

By calculation, we can get

$$W_c[h,l] = \begin{bmatrix} 604 & 1810 & -1172 & -926 & 280 \\ 1810 & 5483 & -3519 & -2785 & 843 \\ -1172 & -3519 & 2279 & 1802 & -545 \\ -926 & -2785 & 1802 & 1427 & -432 \\ 280 & 843 & -545 & -432 & 131 \end{bmatrix}.$$

So

$$\tilde{W} = \begin{bmatrix} 604 & 1810 & -1172 & 280 \\ 1810 & 5483 & -3519 & 843 \\ -1172 & -3519 & 2279 & -545 \\ 280 & 843 & -545 & 131 \end{bmatrix},$$

then rank($\tilde{W}$)=4, and then $\tilde{W}$ is invertible. From Theorem 1, the group which contains agents $1,2,3,5$ is controllable. Figs. 2-5 show the simulation results of partial controllability in system (2). The vertices (the black star dots) begin from random initial positions. Interconnections are depicted as lines (yellow, green, cyan, mauve and red, respectively) which connect the corresponding vertices. Finally, the vertices are controlled to a rectangle configuration, a straight line configuration, a triangle configuration, a diamond configuration, respectively.

5 CONCLUSION

This paper has analyzed the partial controllability of discrete-time multi-agent systems under the fixed topology. The concept of partial controllability problem is proposed. The necessary and sufficient criterion partial controllability Gram Criterion and partial controllability rank criterion are proved, and taking those criterion as examples, the partial group is controllable if and only if the principal minor formulated by the corresponding columns and rows in the Grammian matrix is invertible and the partial group is controllable if and only if the corresponding rows in the controllability matrix are linearly independent.

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