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IFAC PapersOnLine 50-1 (2017) 8327–8332

Pinning Controllability of Complex Stochastic Networks

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Abstract: In this paper we study the pinning controllability of networks when noise affects either the node dynamics, or the communication links or the connections through which pinning control itself is being exerted. By using appropriate Lyapunov functions and the notion of almost sure exponential stability, we provide simple algebraic conditions depending on the node dynamics, networks structure, noise intensity and control parameters to guarantee that the network states converge toward the desired trajectory. Rather than observing noise to be detrimental for achieving control of the network collective behaviour, we find that under some specific conditions noise can enhance the pinning controllability of the network, making it easier to drive all network nodes towards the desired collective evolution of interest. Throughout the paper, theoretical results are illustrated via representative numerical examples showing the effectiveness of the proposed approach.

Keywords: Synchronization, stochastic systems, complex systems, networks, Brownian motion

1. INTRODUCTION

Influencing the behaviour of large-scale network systems to achieve a desired target collective evolution is a major challenge in the application of control to network science [Liu and Barabási (2016)]. Examples range from controlling natural networks like metabolic networks [Basler et al. (2016)] to engineered large-scale systems such as power grids [Cornelius et al. (2013)], and ensembles of robots performing cooperative tasks [Fax and Murray (2004)]. Pinning control is a well known strategy for network control that is often used to induce synchronization of all nodes towards the same desired trajectory. It works by, directly controlling only a small fraction of the nodes as first described in [Wang and Chen (2002)]. This makes the strategy applicable also to large networks and in those cases where only some of the nodes are accessible for control [see for example Lo Iudice et al. (2015)].

Over the past few years, pinning control strategies of different types (adaptive, optimal, nonlinear etc) and their extensions have been widely used to steer networks toward a synchronous state [see Wang and Su (2014); Turci et al. (2014) and references therein for a comprehensive recent review of pinning control strategies]. When designing a pinning control strategy, it is of great importance to appropriately select the number and location of the nodes to be controlled. This problem is known as, [*pinning controllability*] and was originally studied by Sorrentino et al. (2007) using local stability analysis and then by Porfiri and di Bernardo (2008) using global methods. Moreover, in [Wu (2008); Orouskhani et al. (2016)] optimal selection strategies of the pinned nodes are investigated. In most of the existing literature on pinning controllability, it is often assumed that the node dynamics and network links are deterministic and noise-free. However, this is not the case in real networks where noise and uncertainties are ubiquitous. For instance, in gene regulatory networks deterministic models are not able to capture the cell-to-cell fluctuations in genetic switching [Tian and Burrage (2006)]. Also, in Engineering applications the communication links coupling the nodes in the network might be subject to quantized state variables or Gaussian fading communication channels [Li et al. (2014)], while in [Griggs et al. (2016)], noise on the state measurements has been shown to be crucial when designing speed advisory systems for Intelligent Transportation systems.

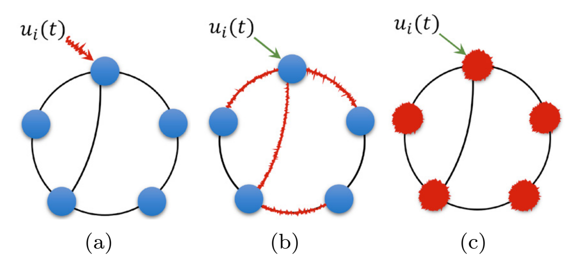

Fig. 1. Three different scenarios of networks affected by noise. (a) noise on the control action, (b) on the links, (c) on the node dynamics. Red color represents noise

In this paper we study the pinning controllability of networks when noise is present in the network. Specifically,

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10.1016/j.ifacol.2017.08.1553

we consider the cases (summarized in Fig. 1) where noise affects (i) the control input, modelling possible noisy sensors measurement and non-ideal actuators; (ii) some of the network edges (including scenarios where some links are affected by higher noise intensity than others); (iii) the node dynamics, modelling the presence of external stochastic disturbances of parameter uncertainties at the node level.

Related independent work can be found in [Mwaffo et al. (2014)] where pinning controllability is studied in networks of chaotic maps with the control signal acting in a stochastic manner. Moreover, in [Li and Zhang (2009); Li et al. (2014)] mean-square average consensus is studied for the case where node dynamics are simple integrators communicating with each other through noisy measurements (the analysis has been recently extended to generic continuous-time and discrete-time linear systems in [Zong et al. (2016)] and [Li and Chen (2016)] respectively). Pinning synchronization has been also studied in the case where node dynamics are nonlinear and noise is present on all the communication links see Tang et al. (2014). More sophisticated approaches have been proposed in [Tang and Wong (2013)], while in [Wang et al. (2014); Yang et al. (2012)] adaptive and impulsive strategies are studied; however, contrary to the spirit of the pinning strategy, control interventions are added to all nodes in the network of interest rendering these approaches unviable for large networks.

The role of noise in steering a network towards consensus in networks of dynamical agents has been discussed in the absence of pinning in [Russo and Shorten (2016)]. In this paper we go beyond the existing literature by considering generic networks of nonlinear dynamical systems affected by noise in each one of the three scenarios depicted in Fig. 1 and assuming that control is only present on a relatively small fraction of the nodes. Using Lyapunov stability theory, we are able to extend the analysis of pinning controllability to the stochastic case under investigation, providing algebraic conditions guaranteeing almost sure exponential synchronization of the closed-loop network. Most importantly, we show that, despite the fact that noise is often assumed to be detrimental for control applications, it might instead be useful for enhancing the ability of a pinning controller to drive the network towards a desired synchronous evolution.

2. PROBLEM STATEMENT

We consider a network of $N > 1$ diffusively coupled identical nonlinear systems affected by noise of the form

$$d\mathbf{x}_i = \left(\mathbf{f}(\mathbf{x}_i, t) - \sigma \sum_{j=1}^{N} \mathcal{L}_{ij}\mathbf{x}_j + \mathbf{u}_i\right) dt + \mathbf{g}(\mathbf{x}_i, \mathbf{y}_i)db$$
$$\mathbf{x}_i(0) = \mathbf{x}_{io}, \quad i, j \in \mathcal{N} := \{1, \cdots, N\} \tag{1}$$

where $\mathbf{x} \in \mathbb{R}^n$ is the state variable, $\mathbf{x}_0$ is an arbitrary initial condition, $\mathbf{f} : \Omega \subseteq \mathbb{R}^n \times \mathbb{R}^+ \cup \{0\} \mapsto \mathbb{R}^n$ belongs to the set of functions $\mathcal{C}^{2\times 1}$ (i.e. $\mathbf{f}(\mathbf{x}, t)$ is twice differentiable in $\mathbf{x}$ and differentiable in t), and it represents the node dynamics. Often, in applications, the set Ω is a closed set (for example, for a biochemical system, this is given by non-negativity constraints on the state variables). For a non-open set Ω, the fact that $\mathbf{f}(\cdot)$ belongs to $\mathcal{C}^{2\times 1}$ means that the function $\mathbf{f}(\mathbf{x}, \cdot)$ can be extended as a twice differentiable function to some open set which includes Ω, and that $\mathbf{f}(\cdot, t)$ can be extended as a differentiable function on this open set. The non-negative constant σ represents the coupling strength, $\mathbf{u}_i$ is an exogenous control input. $\mathbf{g}(\mathbf{x}_i, \mathbf{y}_i) \in \mathcal{C}$ is a nonlinear function possibly depending on the states of the neighboring agents denoted by the vector $\mathbf{y}_i$, and b is a one dimensional Brownian motion. Different choices of the function $\mathbf{g}(\mathbf{x}_i, \mathbf{y}_i)$ will be described below to model each of the three scenarios in Fig. 1.

Network (1) is represented by a graph $\mathscr{G} = (\mathcal{N}, \mathcal{E})$ where $\mathcal{N}$ is the finite set of N node indices and $\mathcal{E} \subset \mathcal{N} \times \mathcal{N}$ is the set of edges. $\mathscr{G}$ has an associated *Laplacian matrix* $\boldsymbol{\mathcal{L}} \in \mathbb{R}^{N\times N}$ whose entries $\boldsymbol{\mathcal{L}} = [\mathcal{L}_{ij}]$ are given by $\mathcal{L}_{ij} = -1$ if $i \neq j$ and $\mathcal{L}_{ii} = -\sum_{j=1}^{N} \mathcal{L}_{ij}$ otherwise. We assume $\mathscr{G} = (\mathcal{N}, \mathcal{E})$ to be undirected, unweighted and connected.

Assumption 1. There exist an arbitrary positive constant k_f such that

$$(\mathbf{x} - \mathbf{y})^T (\mathbf{f}(\mathbf{x}, t) - \mathbf{f}(\mathbf{y}, t)) \le k_f (\mathbf{x} - \mathbf{y})^T (\mathbf{x} - \mathbf{y}) \tag{2}$$

for all $\mathbf{x}, \mathbf{y} \in \Omega \subseteq \mathbb{R}^n$.

Condition (2) in Assumption 1 is known as a QUAD condition, and it has been widely used in the synchronization literature see [DeLellis et al. (2011)].

The problem we are aiming to solve is finding a control law $\mathbf{u}_i$ such that all the states of the network asymptotically converge onto a common desired state. To that aim we use the well known pinning control strategy, where the control action is injected only in a fraction of nodes [Wang and Chen (2002)].

2.1 Pinning Control

For the sake of simplicity, we consider the control action $\mathbf{u}_i(t)$ to be given by a proportional feedback of the form

$$\mathbf{u}_i(t) := -b_i\alpha (\mathbf{x}_i - \mathbf{x}_r) \tag{3}$$

where $\alpha > 0$ is the control gain, $\mathbf{x}_r$ is the reference (or target) state, and b_i is a constant value that indicates if the i-th node is controlled (or pinned), i.e. the control action is injected at the ith node ($b_i = 1$), or not ($b_i = 0$). We assume that at least one node is pinned, i.e. $b_i \neq 0$, for some $i \in \mathcal{N}$ excluding the case where no control action is applied.

Definition 2.1. The stochastic network (1) is said to achieve stochastic synchronization onto a synchronous solution $\mathbf{x}_r(t)$ if

$$\lim_{t\to+\infty} \sup \frac{1}{t} \log (|\mathbf{x}_i(t) - \mathbf{x}_r(t)|) < 0, \quad a.s.,$$

$\forall i = 1, \ldots, N$.

Definition 2.2. We say that the closed-loop network (1)-(3) is pinning controllable in a stochastic sense if there exist a value of the control gain α and a set of pinned nodes such that, all node trajectories reach stochastic synchronization onto the reference trajectory $\mathbf{x}_r(t)$.

2.2 Modeling different noise inputs

We consider three different types of scenarios as shown in Fig. 1.

We model the case where noise is present on the control signal by setting

$$\mathbf{g}(\mathbf{x}_i, \mathbf{y}_i) = -b_i \sigma^* (\mathbf{x}_i - \mathbf{x}_r) \tag{4}$$

where σ^* is a constant representing the noise intensity.

The second scenario that we consider is the case where noise is present on the communication links. In this case we set

$$\mathbf{g}(\mathbf{x}_i, \mathbf{y}_i) = \sigma^* \sum_{j=1}^{N} \mathcal{L}^*_{ij}(\mathbf{x}_j) \tag{5}$$

where $\boldsymbol{\mathcal{L}}^*$ is a Laplacian matrix possibly different from the network topology $\boldsymbol{\mathcal{L}}$ in (1), representing the structure of the network with noisy communication links. We denote such network structure by a graph $\mathscr{G}^* = (\mathcal{N}, \mathcal{E}^*, \boldsymbol{\mathcal{L}}^*)$ that we assume to be undirected and connected. We want to emphasize that different network structures may arise in the case where noise is only present on a subset of links of the network (1), this is $\mathscr{G}^* \subset \mathscr{G}$ (see Fig. 1b).

Finally, we model the third scenario in Fig. 1, where noise affects the node dynamics, by choosing

$$\mathbf{g}(\mathbf{x}_i, \mathbf{y}_i) = \mathbf{g}(\mathbf{x}_i) = \sigma^* \mathbf{A} \mathbf{x_i} \tag{6}$$

where we assumed the stochastic perturbation to the vector field of each node can be modeled by a generic linear term with $\mathbf{A} \in \mathbb{R}^{n \times n}$.

3. MAIN RESULTS

Theorem 2. Consider the controlled stochastic network (1)-(3) where noise is acting on the feedback control action, that is $\mathbf{g}(\mathbf{x}_i, \mathbf{y}_i)$ is given by (4). Assuming that (2) holds, and the reference state (or trajectory) $\mathbf{x}_r$ is a solution of the uncoupled dynamics $d\mathbf{x}_r = \mathbf{f}(\mathbf{x}_r)dt$; then, the closed-loop network (1)-(3)-(4) with $\mathscr{G}^*$ being undirected and connected, is stochastic pinning controllable if

$$\tilde{\lambda} > k_f + \frac{(\sigma^*)^2}{2} \tag{7}$$

where $\tilde{\lambda} := \lambda_{\min}(\tilde{\boldsymbol{\mathcal{L}}})$ is the minimum eigenvalue of $\tilde{\boldsymbol{\mathcal{L}}} := \alpha\boldsymbol{\mathcal{L}} + \alpha\mathbf{B}$ with $\mathbf{B} := \mathrm{diag}\{b_1, \cdots, b_N\}$ being a diagonal matrix identifying the pinned nodes.

Proof. For the sake of brevity we only give a sketch proof divided in three steps. *Step 1:* We first define the error at each node as the difference between the ith node state and the reference trajectory as $\mathbf{e}_i := \mathbf{x}_i - \mathbf{x}_r$. Next, letting $\mathbf{x}$, and $\mathbf{r}$ be the stack vectors of the node states and reference trajectory respectively, the overall error dynamics $\mathbf{e}$ can be recast in compact form as $d\mathbf{e} = \tilde{\mathbf{F}}(\mathbf{e}, t)dt + \tilde{\mathbf{G}}(\mathbf{e}, t)db$, where $\tilde{\mathbf{F}}(\mathbf{e}, t) = \mathbf{F}(\mathbf{e} + \mathbf{r}, t) - \mathbf{F}(\mathbf{r}, t) + (\mathbf{B} \otimes \mathbf{I}_n)\mathbf{r} - (\sigma(\boldsymbol{\mathcal{L}} \otimes \mathbf{I}_n) + \alpha(\mathbf{B} \otimes \mathbf{I}_n))(\mathbf{e} + \mathbf{r})$ and $\tilde{\mathbf{G}}(\mathbf{e}, t) = -\sigma^*(\mathbf{B} \otimes \mathbf{I}_n)\mathbf{e}$. Note that $\mathbf{e} = \mathbb{0}$ is the trivial solution of the closed-loop network. *Step 2:* Next, we consider the positive definite function $V(\mathbf{e}) = 1/2\mathbf{e}^T\mathbf{e}$ and we calculate $LV(\mathbf{e})$ (where L is the differential operator associated to the stochastic Lyapunov function [Mao (2008)]), yielding $LV(\mathbf{e}) \leq c_2 V(\mathbf{e})$, where $c_2 = 2k_f - 2\lambda_{min}(\tilde{\boldsymbol{\mathcal{L}}}) + (\alpha^*)^2$. *Step 3:* Finally, we estimate a lower-bound for $\mathbf{z} := \left|V_e(\mathbf{e})\tilde{\mathbf{G}}(\mathbf{e})\right|^2$ given by $\mathbf{z} \geq c_3 V^2(\mathbf{e})$. Therefore, from condition (7), we have that $c_2 < 0$, and using Theorem 3.5 in Mao (2008)(pp 123) we can conclude that $\lim_{t \to +\infty} \sup \frac{1}{t} \log(|\mathbf{e}(t)|) < 0$, *a.s.*, and the proof is complete.

Remark 3. Note that condition (7) depends on the node dynamics (via k_f), network and control structure (via $\tilde{\lambda}$) and most importantly on the intensity of the noise (via σ^*). Intuitively, this means that, for a given network structure, more control interventions are needed to overcome noise, i.e. $\tilde{\lambda}$ must be increased. We shall see that this is not the case for the other two scenarios where noise can be particularly useful under certain conditions (Proofs of the following theorems are omitted here for the sake of brevity and will be reported elsewhere.)

Theorem 4. Consider the stochastic network (1) where noise is acting on the communication links, i.e. $\mathbf{g}(\mathbf{x}_i, \mathbf{y}_i)$ is given by (5) with both $\mathscr{G}$ and $\mathscr{G}^*$ being undirected and connected, Assumption 1 holds and the control action is the proportional feedback (3). Assuming the reference state (or trajectory) $\mathbf{x}_r$ to be the solution of $d\mathbf{x}_r = \mathbf{f}(\mathbf{x}_r)dt$; then, the closed-loop network (1)-(3)-(5) is stochastic pinning controllable if

$$\tilde{\lambda} > k_f - (\sigma^*)^2 \left((\lambda_2^*)^2 - \frac{(\lambda_N^*)^2}{2} \right) \tag{8}$$

where λ_2^* and λ_N^* are the smallest non-zero and largest eigenvalues respectively of the Laplacian matrix $\boldsymbol{\mathcal{L}}^*$ associated to the graph $\mathscr{G}^*$.

Remark 5. Interestingly, the structure of the network (that is assumed to be connected) where noise is propagated plays an important role in (8). Indeed, if the eigenratio $R^* := \lambda_N^*/\lambda_2^*$ is less than $\sqrt{2}$ then for a fixed k_f, the right-hand side of inequality (8) decreases as σ^* is increased. Therefore, increasing noise levels might have a beneficial effect on synchronization allowing the network to converge for lower values of $\tilde{\lambda}$, i.e. for a smaller number of pinned nodes or for a lower value of the control gain α, or even changing the connectivity of the network structure (by rewiring $\boldsymbol{\mathcal{L}}$ or varying the coupling strength σ).

Theorem 6. Consider an undirected and connected stochastic network (1) with noisy node dynamics (6) satisfying Assumption 1. Under the control action (3) where the reference trajectory is the solution of an isolated node $d\mathbf{x}_r = \mathbf{f}(\mathbf{x}_r)dt + \sigma^*\mathbf{A}\mathbf{x}\mathrm{d}b$; the, closed-loop stochastic network (1)-(3)-(6) is pinning controllable if

$$\tilde{\lambda} > \frac{1}{2}\left(k_f - (\sigma^*)^2 \left(\lambda^2_{min}(\mathbf{A}') - \frac{\lambda^2_{max}(\mathbf{A}')}{2} \right) \right) \tag{9}$$

where $\mathbf{A}' := \mathbf{A} + \mathbf{A}^T$ and $\tilde{\lambda} := \lambda_{\min}(\tilde{\boldsymbol{\mathcal{L}}})$.

Remark 7. Contrary to the result in Theorem 2, in this case noise can be potentially useful for enhancing the stochastic pinning controllability if matrix $\mathbf{A}$ satisfies $\lambda_{max}(\mathbf{A}')/\lambda_{min}(\mathbf{A}') < \sqrt{2}$.

4. APPLICATION TO A NETWORK OF CHAOTIC OSCILLATORS

Networks of chaotic oscillators are often used as a paradigmatic benchmark, e.g. to test strategies for synchronization [Boccaletti et al. (2002)], electronic circuits [De Magistris et al. (2015)] and applications like secure communications [Lu and Chen (2006)], image processing [Perez-Munuzuri et al. (1993)] and pattern recognition [Hölzel

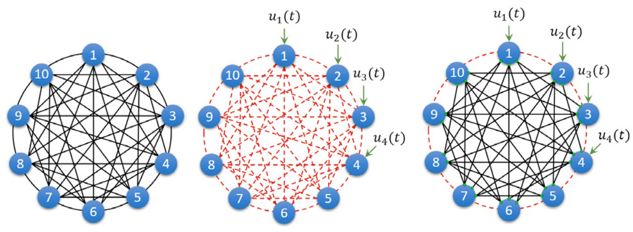


Fig. 2. Network topology and noise propagation networks: (a) free noise All-to-All network of ten nodes, (b) noise is propagated across all links, (c) noise propagates only on a ring configuration. The red-dot line represents communication links corrupted by noise.

and Krischer (2011)]. Currents efforts in this field [Pecora and Carroll (2015)] aim at using chaos synchronization as a tool for parameter estimation in dynamical systems; as well as to explore more complex collective behaviours in networks.

In this case we consider an all-to-all network (1) of $N = 10$ nodes as the one depicted in Fig. 2a. As non-linear vector-field modelling the intrinsic node dynamics of each unit we choose the well known Lorenz equation

$$\mathbf{f}(\mathbf{x}) = \begin{bmatrix} \mu(x_2 - x_1) \\ x_1(\rho - x_3) - x_2 \\ x_1x_2 - \omega x_3 \end{bmatrix} \tag{10}$$

with the parameters set as $\mu = 10, \rho = 28$ and $\omega = 2$ for which the system exhibits a chaotic solution [Huang et al. (2009)]. We further assume that only nodes 1, 2, 3 and 4 are pinned.

4.1 Scenario 1: Noise on the control input

We first consider the case where noise is present on the control input (4), the coupling strength $\sigma = 5$ and the noise intensity $\sigma^* = 0.7$. For the sake of comparison we first show the case where no noise is acting on the network as can be seen in fig. 3a. Note that the network converges to a synchronous solution different from the desired trajectory. Next, in order to use Theorem 2, we notice that as stated in Liu and Chen (2008), the Lorenz system is QUAD; therefore, a constant k_f exists such that condition (2) holds. To design the control strategy according to Theorem 2 we need an estimate of such a constant. Using the MATLAB optimization toolbox we solve a constrained nonlinear multivariate optimization problem, and we found $k_f = 14.025$. Then, from (7) we have that the stochastic network is pinning controllable if $\tilde{\lambda} > 14.51$. By setting the control gain $\alpha = 100$, and considering that the all-to-all configuration we are studying has $\lambda_2 = N$ the condition remain satisfied and the closed-loop networks converges to the desired trajectory as can be seen in Fig. 3b. We want to point out that the conditions of Theorem 2 are only sufficient; therefore, lower values of the control gain α might exist, such that the network still converges to $\mathbf{x}_r$.

4.2 Scenario 2: Noise on the communication links

Next, we consider the case where the communication links are affected by noise. Considering again the All-to-All network of Lorenz mentioned above with $\sigma = 0.7$ and $\sigma^* = 0.6$. Assume that the noise is propagated on all the network links (see Fig. 2b), so that $\lambda_2^* = \lambda_N^* = 10$ yielding $R = 1 < \sqrt{2}$. Then, using Theorem 4 we find that the network is pinning controllable if $\tilde{\lambda} > -3.97$. Then by setting $k = 10$ the condition is fulfilled and the stochastic network converges to the desired trajectory $\mathbf{x}_r$ as shown in Fig. 4a.

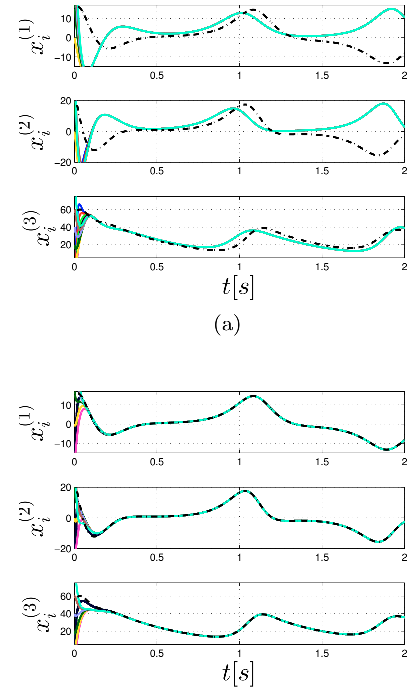


Fig. 3. Time response of the stochastic network: (a) without control, (b) Pinning the nodes 1, 2, 3 and 4. The black-dashed line represents the desired trajectory $\mathbf{x}_r$.

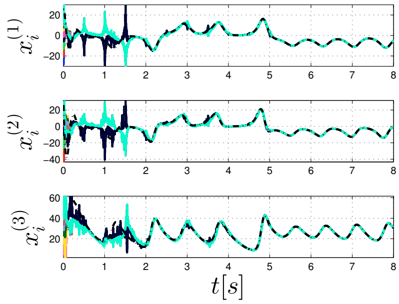
We now change the topology of the network along which noise is propagated. Particularly we choose the case where noise is only present on the links belonging to a ring configuration as can be seen in Fig. 2c. In this case the conditions of Theorem 4 are not fulfilled with the selected $\tilde{\lambda}$ since $\lambda_2^* = 0.38$ and $\lambda_N^* = 4$ so that $R = 10.47 > \sqrt{2}$, and the network does not reach the control target as can be seen in Fig. 4b. This is an important example to illustrate that in complex networks where noise acts on the diffusive links, the structure of the network topology where noise is propagated plays a crucial role on the pinning controllability of the network.

4.3 Scenario 3: Noise on the node dynamics

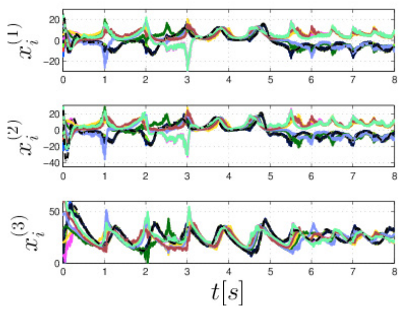
Finally we consider the case where noise is affecting the node dynamics by setting

$$\mathbf{g}(\mathbf{x}_i) = \sigma^* \mathbf{A}\mathbf{x_i}, \qquad \mathbf{A} = \begin{bmatrix} -4 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & -0.4 \end{bmatrix} \tag{11}$$

and $\sigma^* = 0.8$. Then, following theorem 6 we find that $\lambda_{min}(\mathbf{A}') = -8$ and $\lambda_{max}(\mathbf{A}') = 0.2$; therefore, the closed-loop network is stochastic pinning controllable if $\tilde{\lambda} > -13.46$. In this case any positive value for the pinning



(a)



(b)

Fig. 4. Time response of the stochastic network where noise is propagated through (a) all the network links, (b) links in a ring configuration. The black-dashed line represents the desired trajectory $\mathbf{x}_r$.

control gain α will guarantee condition (9) to be fulfilled. For instance in Fig. 5a the evolution of the network nodes is shown for $\alpha = 10$. Note that the convergence time to synchronization is approximately 10s.

For the sake of comparison we change the first entry of matrix $\mathbf{A}$

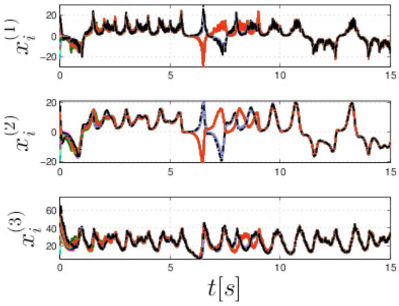
$$\mathbf{A} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & -0.4 \end{bmatrix} \qquad (12)$$

In this case, $\lambda_{min}(\mathbf{A}') = -0.8$ and $\lambda_{max}(\mathbf{A}') = 8$; so, from condition (9) one has $\tilde{\lambda} > 17.04$ in order to guarantee stochastic synchronization. When compared with the previous case we found that the threshold for ensuring synchronization has increased, so that $\alpha = 10$ does not any longer guarantee convergence to $\mathbf{x}_r$ as shown in Fig. 5b. This example illustrates how noise on the node dynamics may play an important role on the ability to control the network; since, depending on the node dynamics the pinning controllability can be enhanced so that condition (9) is fulfilled for lower values of α and even a lower number of pinned nodes (or weaker network connectivity) is required to synchronize the network.

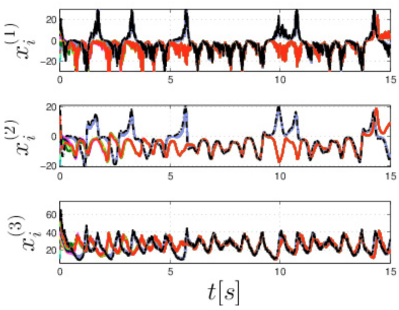
5. CONCLUSION

We discussed the analysis of pinning controllability of networks affected by noise. We considered three cases where the noise is either present in the control input, or the communication links among the nodes or affecting the node dynamics. We derived sufficient conditions to guarantee the effectiveness of the control strategy in achieving convergence to a synchronous solution. The conditions involve properties related to the node dynamics, the network structure, the intensity of the noise, the value of the control gains and the number and location of the pinned nodes. We uncovered a highly nontrivial effect of noise on the pinning controllability of the network. Specifically, we observed that noise can be beneficial allowing the network to be synchronized even in those situations when this is not possible in its absence. We validated the results on a network of chaotic oscillators that was taken as a paradigmatic benchmark case. We showed that the conditions derived in this paper can be useful for control design purposes and to select the location of the control inputs in the network.

The work presented here opens many interesting future directions implying that noise and stochastic effects can be exploited for network control and synchronization. Ongoing work is aimed at generalizing the results to the case where the coupling is nonlinear and to assess whether optimal configurations exist of the network structure over which noise is propagated that maximize the network controllability and facilitate synchronization onto some desired collective behavior.



(a)



(b)

Fig. 5. Time response of a network of 10 Lorenz affected by noise at nodes with control gain $\alpha = 10$ and matrix $\mathbf{A}$ given by: (a) (11), (b) (12). The black-dashed line represents the desired trajectory $\mathbf{x}_r$ provided by the dynamics of an uncoupled node.

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