This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TAC.2018.2798808, IEEE Transactions on Automatic Control

Dynamic Formation Control Over Directed Networks Using Graphical Laplacian Approach

Xiuxian Li, *Member, IEEE*, and Lihua Xie, *Fellow, IEEE*

***Abstract*—This paper investigates the dynamic formation control problem for multi-agent systems over directed networks, in which desired spatial shape is time-varying instead of fixed one as usually assumed in the literature. Inspired by the fact that the existing approaches, including absolute positions based, relative positions based, inter-agent distances based, and inter-agent bearings based, for specifying a formation shape are not invariant under all three transformations: translations, rotations, and scalings, a novel specification for formation shapes is proposed that is invariant under translations, rotations, and scalings in two- and three-dimensional spaces, and thus more intrinsic. In doing so, a new notion, called matrix-valued Laplacian, for graphs is introduced in detail along with some useful properties. It is demonstrated that the matrix-valued Laplacian provides much flexibility. Subsequently, two controllers are designed for guaranteeing the achievement of a dynamic formation shape. It is proved that arbitrary dynamic geometric shape can be reached by the designed controllers. Finally, two numerical examples are provided for demonstrating the effectiveness of the theoretical results.**

***Index Terms*—Dynamic formation control, multi-agent networks, directed graphs, matrix-valued Laplacian.**

I. Introduction

Recent years have witnessed a considerable interest in cooperative control for multi-agent networks, mostly inspired by both amazing natural phenomena, such as a group of ants carrying food cooperatively, a school of fishes escaping their predator and a flock of birds migrating in V-shape, and numerous potential applications in reality, such as fulfilling missions in a cooperative manner for unmanned aerial vehicles (UAVs), unmanned underwater vehicles (UUVs), and robotic manipulators. Formation control, as a fundamental but important research direction in multi-agent networks, aims to achieve a desired spatial shape for all agents, such as flying in V-shape for multiple spacecraft, which has been extensively investigated in the literature, for example, [1]–[7] and recent surveys [8], [9], to name a few.

To date, a large variety of scenarios for the formation control problem have been taken into consideration in the literature [10]–[18], including multi-agent networks with all kinds of dynamics ranging from the simple single-integrator dynamics to nonlinear dynamics, formation control with undirected and directed sensing graphs under fixed or switching graph topologies, formation control with time-delays, formation control with the measurement of absolute positions, formation control with only the measurement of relative positions or distances between neighboring agents. For instance, formation control of second-order multi-agent networks has been studied with directed interaction graph in [10], and it has been shown that by using consensus-based strategies desired formation shape can be reached when the graph has a spanning tree. Moreover, unicycle model, describing kinematics of robots, has been addressed for formation control in [3], [11]. Also, in [13] formation control for multiple nonlinear Euler-Lagrange networks has been investigated based on the measurement of absolute position and velocity under a directed interconnection graph with a spanning tree. Besides, in the presence of communication delays, formation control has been studied for double-integrator networks, UAVs and UUVs in [14]–[17]. Most of the aforementioned research has assumed the existence of a global coordinate system, and in comparison only a common sense of direction, shared by all agents, has been used for formation control, such as, in [3], [18].

It should be pointed out that most of the results on formation control, including the aforementioned literature, are devoted to the case when the desired geometric shape is fixed, i.e., a time-invariant spatial shape. However, it is more practical to take into consideration the formation control problem in which the desired spatial shape is time-varying, called *dynamic formation control* or *time-varying formation control*, since agents perhaps need to change their spatial pattern when one desires or the environment they move in coerces them into doing so. For instance, for the purpose of entertainment performance, a collection of small spacecraft should change their flying pattern intermittently to be appealing. Taking another example, a family of UAVs may require to alter their formation shape when moving from an obstacle-free environment to a relatively narrow crevice, in order to pass the crevice safely. For this reason, in recent years dynamic formation control has been focused on in a few literature [19]–[26]. Specifically, dynamic formation control for second-order networks has been dealt with under fixed and switching directed interaction graphs in [19] and [20], respectively, and in the meantime the same problem has been solved with a leader in [24]. Additionally, dynamic formation control for linear multi-agent networks has been addressed by virtue of output feedback under fixed directed communication graph in [21], leaving the switching topology case being tackled in [22]. Also, nonlinear multi-agent networks have been taken into account for dynamic formation control in [26].

Generally speaking, the existing literature on formation control can be categorized based on 1) what kind of measurement is utilized for designing controllers, and 2) how to specify a

This work was supported by Ministry of Education of Singapore under MoE Tier 1 grant RG78/15 and Projects of Major International (Regional) Joint Research Program NSFC (Grant no. 61720106011).

X. Li and L. Xie (corresponding author) are with School of Electrical and Electronic Engineering, Nanyang Technological University, 50 Nanyang Avenue, Singapore 639798 (e-mail: xiuxianli@ntu.edu.sg; elhxie@ntu.edu.sg).

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TAC.2018.2798808, IEEE Transactions on Automatic Control

desired formation shape. For 1), the measurements that have been utilized in existing works include absolute positions, relative positions, inter-agent distances and bearings, etc. For 2), there are generally four existing methods for specifying a desired formation shape: absolute position based [27], relative position based [28], inter-agent distances based [6], [7], [29], [30], and inter-agent bearings based [31], [32]. For instance, the authors in [31], [32] have investigated formation control by virtue of only bearing measurements, where the desired formation shape is specified by inter-agent bearings. Note that the specification for formation shapes based on barycentric coordinate [33] falls into relative position based approach. It is easy to observe that all the four approaches for specifying a formation shape are not invariant under all the following three transformations: translations, rotations, and scalings. For example, the inter-agent distances based approach is invariant under only translations and rotations, but not under scalings. However, the intrinsic property of a geometric shape is that the shape is invariant when performing all the above three transformations, which naturally poses a question: How to intrinsically specify a formation shape? That is, this specification should be invariant under translations, rotations, and scalings. It is worth noting that the similar formation and affine formation in [34]–[37] are invariant under translations, rotations, and scalings, but are limited to the plane or generic configurations. Furthermore, only time-invariant formation control has been studied in [34]–[37]. In this respect, no efficient methods are reported in the literature to cope with dynamic (or time-varying) formation control problems by an intrinsic specification for a formation shape.

Motivated by the aforesaid fact, this paper aims to tackle the dynamic formation control problem for directed multi-agent networks with only knowing the structure of desired geometric shape, without employing the existing approaches for specifying a formation shape. In doing so, a novel and intrinsic specification for formation shapes is proposed which is invariant under translations, rotations, and scalings in two- and three-dimensional spaces. Along the line, a new matrix-valued Laplacian of graphs is introduced, and then two controllers are devised, showing that any desired time-varying geometric shape can be precisely achieved. The novelties of this paper are threefold. The first is to propose a new means to intrinsically specify a formation shape that is invariant under rotations, translations, and scalings, and is applicable to time-varying formation. The new specification provides more flexibility in some sense than that of the affine formation in [35]. The second is to introduce a new matrix-valued Laplacian for handling the dynamic formation control problem with the new proposed formation shape specification, which is in fact an extension of complex-valued Laplacian that has been used in [38], [39] and is proved to be a powerful tool for tackling dynamic formation control, distributed set surrounding and multi-party consensus problems. We note that a special matrix-valued Lapalician termed bearing Laplacian has been introduced in [32]. The third is to solve the dynamic formation control problem for directed multi-agent networks with the measurement of position and velocity by resorting to matrix-valued Laplacian, and it is proved that arbitrary dynamic formation shape can be precisely achieved with the entire network following a desired trajectory.

The remainder of this paper is structured as follows. Section II introduces some preliminary concepts and results. A novel specification for formation shapes is introduced in Section III and the dynamic formation control problem, by using the new matrix-valued Laplacian, is tackled when $n = 2$. Section IV provides some details for the cases in n-dimensional Euclidean space with $n \geq 3$ and Section V gives more discussions on matrix-valued Laplacian. Section VI provides several simulation examples to support the theoretical results, and the conclusion is finally drawn in Section VII.

Notations: Define $\mathcal{I}_k := \{1, 2, \ldots, k\}$ for a positive integer k. For a complex number z, denote by $Re(z), Im(z), |z|$ its real part, imaginary part and modulus, respectively. Let $diag\{a_1, a_2, \ldots, a_n\}$ and $diag\{A_1, A_2, \ldots, A_n\}$ be, respectively, a diagonal matrix with diagonal scalar entries a_k and a block-diagonal matrix with diagonal matrix entries A_k, $k \in \mathcal{I}_n$. Moreover, $\mathbb{R}^n$ and $\mathbb{C}^n$ stand for the n-dimensional real and complex Euclidean space, respectively. Given a matrix $X \in \mathbb{R}^{n\times n}$, let $\lambda_1(X), \ldots, \lambda_n(X)$ be the eigenvalues of X in the ascending order of their real parts, and $\rho(X)$ be the spectral radius of X. In addition, let I_n and $\mathbf{1}_n$ be the $n \times n$ identity matrix and the n-dimensional column vector of all entries 1, respectively, $\mathbf{0}_n$ be the n-dimensional vector of all entries 0 or $n \times n$ zero matrix determined by the context, and $\|\cdot\|$ be the standard Euclidean norm. Symbol $\otimes$ represents the Kronecker product.

II. Preliminaries

A. Graph Theory

A directed graph (or digraph) $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is composed of N nodes with the node set $\mathcal{V}$ and the edge set $\mathcal{E}$, in which an edge $(j, i) \in \mathcal{E}$ means that agent j can transmit its information to agent i, and in this case agent j is called a neighbor (or a parent) of agent i. A directed graph is called undirected if $(i, j) \in \mathcal{E}$ implies $(j, i) \in \mathcal{E}$. No self-loop is allowed. The neighbor set of agent i is denoted by $\mathcal{N}_i = \{j : (j, i) \in \mathcal{E}, j \in \mathcal{V}, j \neq i\}$. A directed path (resp. weak path) in a graph consists of a sequence of edges of the form $(i_1, i_2), (i_2, i_3), \ldots, (i_{k-1}, i_k)$, simply denoted by $i_1 i_2 \ldots i_k$, such that $(i_l, i_{l+1}) \in \mathcal{E}$ (resp. $(i_l, i_{l+1}) \in \mathcal{E}$ or $(i_{l+1}, i_l) \in \mathcal{E}$) for $l \in \mathcal{I}_{k-1}$. The directed path (resp. weak path) is called a directed cycle (resp. weak cycle) if $i_1 = i_k$. A directed graph is called strongly connected if there is at least one directed path for any pair of nodes i and j. A directed tree is a directed graph where every node has exactly one parent except one node, called the root, who has no parents. A directed spanning tree for a directed graph is a subgraph, which is exactly a directed tree, of the directed graph comprised of all its nodes and some edges.

The adjacency matrix $A = (a_{ij}) \in \mathbb{R}^{N\times N}$ is defined as: $a_{ij} > 0$ if $(j, i) \in \mathcal{E}$, $a_{ij} = 0$ $(i \neq j)$ otherwise and $a_{ii} = 0, i, j \in \mathcal{I}_N$. Assume that $a_{ij} = a_{ji}$ for undirected graphs. The Laplacian matrix $L = (l_{ij}) \in \mathbb{R}^{N\times N}$ is defined as: $l_{ii} = \sum_{j\in\mathcal{N}_i} a_{ij}$ and $l_{ij} = -a_{ij}$ for $i \neq j$. When adding a leader to the graph, we define $a_{i0} > 0$ if the leader can send information

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TAC.2018.2798808, IEEE Transactions on Automatic Control

to agent i, and $a_{i0}=0$ otherwise. Let $a_{0i}=0$ for all $i\in\mathcal{I}_N$, and define $M=L+diag\{a_{10},a_{20},\dots,a_{N0}\}$.

B. Some Useful Lemmas

This subsection lists some lemmas that will be useful in the following analysis.

Lemma 1 ([41]). *Given a directed graph with a leader, if the leader has at least one directed path to any other node, then the following statements hold: 1) M is non-singular; 2) all eigenvalues of M have positive real parts; 3) M^{-1} is nonnegative, i.e., all its entries are nonnegative; 4) there exists a positive diagonal matrix $Q=diag\{q_1,q_2,\dots,q_N\}$ such that $QM+M^TQ$ is positive definite and strictly diagonally dominant.*

To proceed, it is assumed that nonzero weights a_{ij},a_{i0} are lower and upper bounded by some positive constants $\underline{a}$ and $\bar{a}$, i.e., $\underline{a}\le a_{ij}\le\bar{a}$ and $\underline{a}\le a_{i0}\le\bar{a}$ for nonzero weights, $i,j\in\mathcal{I}_N$. For an integer $m\ge 2$, define

$$\pi_m(\underline{a},\bar{a}):=\mathcal{A}_{m-1}^{m-2}\Big(\frac{4\bar{a}}{\underline{a}}\Big)^{m-2}\Big(\frac{2\underline{a}+\bar{a}}{\underline{a}^2}\Big)+\frac{2}{\underline{a}}\sum_{i=0}^{m-3}\mathcal{A}_{m-1}^{i}\Big(\frac{4\bar{a}}{\underline{a}}\Big)^{i},\qquad(1)$$

where $\mathcal{A}_k^l:=\frac{k!}{(k-l)!}$ for integers $0\le l\le k$ with $k!$ being factorial, and the second term vanishes when $m=2$.

Lemma 2 ([42]). *Given a directed graph with a leader, if the leader has at least one directed path to any other node, then one has*

$$\begin{aligned}\frac{1}{\pi_N(\underline{a},\bar{a})}&\le Re(\lambda_1(M))\le\bar{a},\\ \rho(M)&\le(2N-1)\bar{a},\qquad N\ge 2\end{aligned}\qquad(2)$$

where $\lambda_1(M)$ and $\rho(M)$ are defined at the end of Section I.

Lemma 3 ([43]). *All roots of a polynomial $f(s)=s^2+a_1s+a_2$ with complex coefficients have negative real parts if and only if $Re(a_1)>0$ and $Re(a_1)Re(a_1a_2^*)-(Im(a_2))^2>0$, where z^* denotes the complex conjugate of $z\in\mathbb{C}$.*

III. Dynamic Formation via Matrix-Valued Laplacian

This section attempts to tackle the dynamic formation control problem by proposing a novel specification for formation shapes that is invariant under translations, rotations, and scalings. Before doing so, it is helpful to first introduce a new concept, namely, matrix-valued Laplacian, for graphs.

A. Matrix-Valued Laplacian

All concepts of a graph $\mathcal{G}=(\mathcal{V},\mathcal{E})$ with N nodes are kept the same as in Section II-A, except for the adjacency and Laplacian matrices that are defined to be matrix-valued here. Let the dimension of the agent state be $n>0$. In this setting, the adjacency matrix $W=(W_{ij})\in\mathbb{R}^{nN\times nN}$ is defined by: $W_{ij}=w_{ij}O_{ij}\in O^c(n)$ with $w_{ij}\in\mathbb{R},O_{ij}\in\mathbb{R}^{n\times n}$, and furthermore, $w_{ij}>0$, i.e., $W_{ij}\in O^{c+}(n)$, if the edge $(j,i)\in\mathcal{E}$ and $w_{ij}=0$ otherwise, where $O^c(n):=\{aQ:a\in\mathbb{R},a\ge 0,Q\in O(n)\}$, $O^{c+}(n):=\{aQ:a\in\mathbb{R},a>0,Q\in O(n)\}$, and $O(n)$ denotes the group of $n\times n$ real orthogonal matrices, i.e., $O(n)=\{Q\in\mathbb{R}^{n\times n}:Q^TQ=QQ^T=I_n\}$. Moreover, it is assumed that self-links are not allowed, i.e., $W_{ii}=\mathbf{0}_n$ for any i. Subsequently, the degree matrix $D=diag\{D_1,\dots,D_N\}$ is defined to be block-diagonal with $D_i=\sum_{j=1}^N sgn(w_{ij})I_n$ for $i\in\mathcal{I}_N$, where $sgn(\cdot)$ is the standard signum function. The Laplacian matrix $L'=(L'_{ij})$ is now defined as $L':=D-W$. Furthermore, if there exists an additional leader node that serves as only sending information to all other N nodes, not vice versa, then a block-diagonal matrix $H=diag\{H_1,\dots,H_N\},H_i\in\mathbb{R}^{n\times n}$ is defined by: $H_i=h_iO_i\in O^c(n)$ with $h_i\in\mathbb{R},O_i\in\mathbb{R}^{n\times n}$, and additionally $h_i>0$ if the node i can access the information of the leader node and $h_i=0$ otherwise. Define $M'=L'+sgn(H)$, where $sgn(H)$ is defined as follows.

Definition 1. *For a matrix $Q=(Q_{ij})\in\mathbb{R}^{nN\times nN}$ with $Q_{ij}=q_{ij}O_{ij}\in O^c(n)$ and $q_{ij}\ge 0,O_{ij}\in O(n)$, $sgn(\cdot)$ is defined to be a matrix-valued function as:*

$$\begin{aligned}&sgn(\cdot):\mathbb{R}^{nN\times nN}\to\mathbb{R}^{nN\times nN},\\ &(sgn(Q))_{ij}:=sgn(q_{ij})I_n,\qquad\forall i,j=1,\dots,N\end{aligned}\qquad(3)$$

where $(sgn(Q))_{ij}$ is the (i,j)th matrix-valued entry of $sgn(Q)$.

Remark 1. *Concerning the above matrix-valued Laplacian, it reduces to the scalar-valued Laplacian (by isomorphism $w_{ij}\cdot(\pm I_n)\mapsto\pm w_{ij}$) when $w_{ij}=0$ or 1, $O_{ij}=I_n$ in W_{ij} (the classic Laplacian with weights being 0 or 1), and when $w_{ij}=0$ or 1, $O_{ij}=\pm I_n$ in W_{ij} (the Laplacian with weights being 0 or ± 1). In fact, it is also a generalization of complex-valued Laplacian that has been empolyed in [38] and [39], which is proved to be extremely useful in handling distributed set surrounding problem and multi-party consensus problem. In [38] and [39], the weights of Laplacian $L=(l_{ij})$ are chosen to be complex numbers, i.e., $l_{ij}=-\alpha_{ij}$ if $(j,i)\in\mathcal{E}$, $l_{ij}=0$ otherwise, and $l_{ii}=\sum_{j=1}^N|\alpha_{ij}|$, where $\alpha_{ij}\in\mathbb{C}$ is a constant with $|\alpha_{ij}|=1$, and $|\cdot|$ is the modulus of a complex number. Indeed, this complex-valued Laplacian is a special case of the matrix-valued Laplacian, which is explained as follows. For a complex number α with $|\alpha|=1$, it can be written as $\alpha=e^{\iota\theta}$ for some angle $\theta\in[0,2\pi)$, where $\iota=\sqrt{-1}$ is the imaginary unit. By defining the following map*

$$\psi:\alpha=e^{\iota\theta}\mapsto\begin{pmatrix}\cos(\theta)&-\sin(\theta)\\ \sin(\theta)&\cos(\theta)\end{pmatrix},\qquad(4)$$

it is easy to see that the set $\{\alpha\in\mathbb{C}:|\alpha|=1\}$ is isomorphic to $SO(2)$, denoted by $\{\alpha\in\mathbb{C}:|\alpha|=1\}\cong SO(2)$, which means that the two sets can be identified with each other, where $SO(m):=\{Q\in\mathbb{R}^{m\times m}:Q\in O(m),det(Q)=1\}$ for an integer m. Note that $SO(m)$ is a proper subset of $O(m)$. As a consequence, the complex-valued Laplacian can be viewed as a special case of matrix-valued Laplacian when $w_{ij}=0$ or 1 and $O_{ij}\in SO(2)$ in W_{ij}. Moreover, it is noteworthy that the complex-valued Laplacian in [38] and [39] cannot cope with the case of odd-dimensional real Euclidean space, such

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TAC.2018.2798808, IEEE Transactions on Automatic Control

as $\mathbb{R}^3$ space, while the matrix-valued Laplacian can handle both even- and odd-dimensional real Euclidean space, which will be elaborated in Section V.

To proceed, one graphical notion is provided below.

Definition 2. *A weak cycle $i_1 i_2 \dots i_k i_1$ is said to be consistent if $\Pi_{m=1}^{k} W_m = I_n$, where Π means left product, i.e., $\Pi_{m=1}^{k} W_m = W_k \cdots W_2 W_1$, and $W_m = W_{i_{m+1}, i_m}$ if $(i_m, i_{m+1}) \in \mathcal{E}$ and then $W_m = W_{i_m, i_{m+1}}^{-1}$ if $(i_{m+1}, i_m) \in \mathcal{E}$. In addition, a graph is called consistent if all its weak cycles are consistent and called inconsistent otherwise.*

This definition is given in [38] and [39] for complex-valued Laplacian, which is actually in agreement with that in matrix-valued Laplacian case, as shown below.

Proposition 1. *The definition of consistency in Definition 2 is in line with that in complex-valued Laplacian case.*

Proof: The proof can be found in Appendix. ■

To this end, one fundamental result is presented as follows, which is conducive to the following analysis.

Lemma 4.
1) *For a directed graph, if it is consistent and has a spanning tree, then there exists a block-diagonal matrix $U = diag\{U_1, \dots, U_N\}$ with $U_i \in O^{c+}(n)$ such that $UWU^{-1} = sgn(W)$, and furthermore, $UL'U^{-1} = D - sgn(W)$.*
2) *For a directed graph with a leader node, if it is consistent and at least one directed path exists from the leader node to any other node, then there exists a block-diagonal matrix $V = diag\{V_1, \dots, V_N\}$ with $V_i \in O^{c+}(n)$ such that $VM'V^{-1} = D - sgn(W) + sgn(H)$.*

Proof: The proof can be found in Appendix. ■

B. Dynamic Formation Control

Equipped with the matrix-valued Laplacian, it is now ready to investigate the dynamic formation control problem for multi-agent networks without using the existing methods to specify a desired formation shape. In this section, the multi-agent network under consideration comprises N agents with the dynamics of each agent being of the form

$$\dot{x}_i(t) = v_i(t), \quad \dot{v}_i(t) = u_i(t), \qquad i \in \mathcal{I}_N \tag{5}$$

where $x_i(t), v_i(t), u_i(t) \in \mathbb{R}^n$ are the position, velocity, and control input of agent i, respectively. Throughout this section, we consider $n = 2$ and the cases when $n \geq 3$ will be discussed in next section. In order to easily generalize the results in this section to the case when $n \geq 3$, the general notation n is employed in this section unless specified otherwise.

For this problem, the desired geometric shape is time-varying, and it is expected to specify the desired spatial shape by virtue of a more intrinsical means than the existing specifications, as discussed in Section I, for geometric shapes.

To explicitly describe this problem, one additional agent, labeled as 0_1, exists in the multi-agent network with dynamics

$$\dot{x}_l(t) = v_l(t), \quad \dot{v}_l(t) = u_l(t), \tag{6}$$

where $x_l(t), v_l(t), u_l(t) \in \mathbb{R}^n$ are, respectively, the position, velocity, and control input of agent 0_1. Meanwhile, one more agent, labeled as 0_2, is required for dynamic formation control with parameter $x_b(t) \in \mathbb{R}^n$, satisfying

$$\dot{x}_b(t) = v_b(t), \quad \dot{v}_b(t) = u_b(t), \tag{7}$$

where parameters $x_b(t), v_b(t) \in \mathbb{R}^n$ are controlled by the predetermined parameter $u_b(t) \in \mathbb{R}^n$ so as to reach the goal that one desires. Note that parameters $x_b(t), v_b(t)$ are not the position and velocity of agent 0_2 generally, and agents $0_1, 0_2$ are virtual in general but can be also physical if needed. Before moving forward, the following assumption is necessary.

Assumption 1. *There is at least one directed path from both formation reference x_l and basis x_b to any node in $\mathcal{I}_N$.*

Assumption 1 is standard that has been employed in numerous literature. It should be noticed that the communication graph formed by all agents 1 to N is the same for formation reference x_l and x_b, while agents 0_1 and 0_2 can directly send information to different agents if they are not identical (they may be the same agent), as shown in Fig. 1.

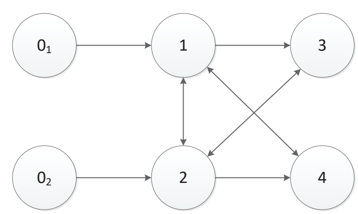

Fig. 1. One possible communication topology.

One natural question is how to describe a geometric shape if one desired shape cannot be described by agents' absolute positions, relative positions, inter-agent distances, and inter-agent bearings. The answer is as follows. The dynamic geometric shape is uniquely delineated by a parameter tuple $(x_l(t), x_b(t), r_1(t), R_1(t), \dots, r_N(t), R_N(t))$ (or simply $(x_l(t), x_b(t), C_1(t), \dots, C_N(t))$) up to translations, scalings and rotations, where parameters $0 < r_i(t) \in \mathbb{R}, R_i(t) \in SO(n)$ are prespecified, which are twice continuously differentiable (i.e., $\mathcal{C}^2$), and $C_i(t)$ is defined by

$$C_i(t) := r_i(t) R_i(t), \qquad i \in \mathcal{I}_N. \tag{8}$$

Physically, $r_i(t)$ and $R_i(t)$, for agent i, mean the adjustment of distance to $x_l(t)$ and rotation based on $x_b(t)$, respectively. Then, with $x_l(t)$ and $x_b(t)$ in hand, the dynamic formation shape is specified by the tuple $(x_l(t), x_b(t), C_1(t), \dots, C_N(t))$. Specifically, the dynamic formation control problem is defined as below.

Definition 3. *The network (5) is said to achieve dynamic formation control, specified by $(x_l(t), x_b(t), C_1(t), \dots, C_N(t))$, if for any initial states, it holds*

$$\lim_{t \to \infty} x_i(t) - x_l(t) - C_i(t) x_b(t) = 0,$$
$$i.e., \quad \lim_{t \to \infty} x_i(t) = x_l(t) + C_i(t) x_b(t), \qquad \forall i \in \mathcal{I}_N \tag{9}$$

where $x_l(t)$ and $x_b(t)$ are called formation reference and formation basis, respectively, and moreover, $r_i(t)$, $R_i(t)$ and

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TAC.2018.2798808, IEEE Transactions on Automatic Control

$C_i(t)$ are called scaling factor, rotation factor and formation factor for agent i, respectively. Furthermore, the formation is said to be static if $r_i(t), R_i(t)$, $i \in \mathcal{I}_N$, are all time-invariant.

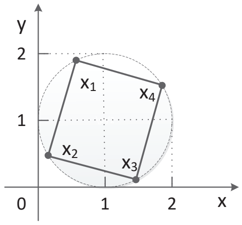

Fig. 2. The shape formed by all agents when $\theta(t) = 0$ in Example 1.

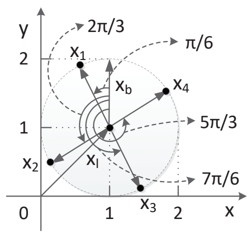

Fig. 3. The relationship among $x_l, x_b, x_i, i \in \mathcal{I}_4$ when $\theta(t) = 0$ in Example 1.

In the remainder of this paper, $x_l(t), x_b(t), C_i(t)$ (and $r_i(t), R_i(t)$) will be written as x_l, x_b, C_i (and r_i, R_i) for notational simplicity, i.e., the argument t is omitted. With reference to (9), it is straightforward to see that the formation reference x_l will determine the location of the whole network, and separately, the geometric shape is uniquely specified by formation factor C_i together with formation basis x_b up to translations, rotations, and scalings. That is, agent i adapts its position by a scaling (decided by scaling factor r_i) based on x_b and a rotation of x_b around some vector by some angle (decided by rotation factor R_i), which is more understandable when formation basis x_b is a fixed vector, such as $x_b = (0, \ldots, 0, 1)^T \in \mathbb{R}^n$. Moreover, it should be noted that the formation problem in Definition 3 is rigid formation, and formation reference x_l can be viewed as the leader for the whole network. Furthermore, the dynamic formation in Definition 3 will reduce to similar formation (cf. [34]) when fixing r_i, R_i for all $i \in \mathcal{I}_N$, where position x_l, the orientation and modulus of x_b serve as translation, rotation and scaling, respectively. Note also that the formation control problem considered here is time-varying.

Example 1. *To clearly demonstrate the meaning of Definition 3, a simple example is provided here. For a multi-agent network with $N = 4$ and $n = 2$, one possible communication graph is given in Fig. 1, which satisfies Assumption 1 by easy verification. Note that agents 0_1 and 0_2 may or may not be the same agent and they can respectively send information on x_b and x_l to different agents if they are not identical. In Definition 3, let x_l be arbitrarily time-varying, $x_b = (\sin(\theta(t)), \cos(\theta(t)))^T$, $\theta(t) \in [0, 2\pi)$, and $r_1 = \cdots = r_4 = 1$. Moreover, let us choose*

$$R_i = \begin{pmatrix} \cos(\frac{\pi}{6} + (i-1)\frac{\pi}{2}) & -\sin(\frac{\pi}{6} + (i-1)\frac{\pi}{2}) \\ \sin(\frac{\pi}{6} + (i-1)\frac{\pi}{2}) & \cos(\frac{\pi}{6} + (i-1)\frac{\pi}{2}) \end{pmatrix}, \ i \in \mathcal{I}_4$$

for which it is easy to see that $R_i \in SO(2), i \in \mathcal{I}_4$. At this point, the spatial shape specified by $(x_l, C_1, \ldots, C_4)$ is provided in Fig. 2 when $\theta(t) = 0$, which is a square. Meanwhile, the relationship among x_l, x_b, x_i is given in Fig. 3 when $\theta(t) = 0$, where the arrow is used to denote a vector, from which one can see that positions x_1, x_2, x_3, x_4 are obtained by counterclockwise rotations from x_b by angles $\pi/6, 2\pi/3, 7\pi/6, 5\pi/3$, respectively, since the matrix in (4) works as a rotation by angle θ. It is obvious to see that the formation shape in Fig. 3 is still the same square even though x_l is time-varying, which connotes that x_l does not affect the formation shape (i.e., the shape is invariant under translations), only serving as a reference for determining the location of the entire network. In addition, keeping all parameters the same as in this example, the variation of $\theta(t)$ in x_b will not have an effect on the spatial shape, still being the same square, that is, the shape is invariant under rotations. Furthermore, only changing the magnitude of x_b will not deform the geometric shape, following that the shape is invariant even under scalings. Up to now, the static formation shape has been discussed. For dynamic formation shape, it can be specified by letting the argument θ_i in R_i of the form (4) and r_i be time-varying, where, by scaling r_i, the distance between x_i and x_l can be adjusted. This is a simple example in the plane, and more discussions will be presented for $\mathbb{R}^3$ space in Section IV.

Remark 2. *In comparison with the existing shape specifications, such as absolute coordinates [19], [35] or relative coordinates [28], Definition 3 is more intrinsical for specifying spatial shapes since the specification in Definition 3 is invariant under translations, scalings, and rotations for $n = 2$, as shown in Example 1. In essence, Definition 3 decouples one spatial shape into tuning three independent parameters, r_i, R_i (based on a nonzero free parameter x_b) and x_l. In this way, it is also easier to specify complex spatial shapes, such as spiral curves and spheres in $\mathbb{R}^3$ space that can be seen from simulation examples later, than the existing shape specifications.*

To proceed, for the reference x_l and basis x_b, two block-diagonal matrices $H^k = diag\{H_1^k, \ldots, H_N^k\}$, $H_i^k \in \mathbb{R}^{n\times n}$, are defined by: $H_i^k = h_i^k O_i^k \in O^c(n)$ with $h_i^k \in \mathbb{R}, O_i^k \in \mathbb{R}^{n\times n}$, and moreover $h_i^k > 0$ if the agent i can access the information of x_k and $h_i^k = 0$ otherwise, where the index $k \in \{l, b\}$.

Now, the controller is proposed as

$$\begin{aligned} u_i = \eta_i &- k_1 \sum_{j\in\mathcal{N}_i} [v_i - \zeta_{vi} - \phi_{vi} - W_{ij}(v_j - \zeta_{vj} - \phi_{vj})] \\ &- k_1 \gamma_i^b (v_i - \zeta_{vi} - \dot{\phi}_{xi}) - k_2 \gamma_i^b (x_i - \zeta_{xi} - \phi_{xi}) \\ &- k_2 \sum_{j\in\mathcal{N}_i} [x_i - \zeta_{xi} - W_{ij}(x_j - \zeta_{xj})], \end{aligned} \quad (10)$$

where $k_1, k_2 > 0$ are constants to be specified,

$$\begin{aligned}\eta_i &:= \dot{\zeta}_{vi} + \ddot{C}_i\xi_{xi} + 2\dot{C}_i\xi_{vi} + C_i\eta_{bi},\\ \phi_{xi} &:= H_i^b\xi_{xi},\\ \phi_{vi} &:= \dot{C}_i\xi_{xi} + C_i\xi_{vi},\\ \gamma_i^b &:= sgn(h_i^b),\end{aligned}$$

and $W_{ij} = w_{ij}O_{ij} \in O^c(n)$ is the edge weight of Laplacian L' defined in last subsection. Furthermore, parameters $\zeta_{xi}, \zeta_{vi}, \xi_{xi}, \xi_{vi}, \eta_{bi}$ are the estimates of x_l, v_l, x_b, v_b, u_b for agent i, respectively, with updating laws:

$$\dot{\zeta}_{xi} = \zeta_{vi}, \tag{11a}$$

$$\begin{aligned}\dot{\zeta}_{vi} = \eta_{li} &- k_1\sum_{j\in\mathcal{N}_i} a_{ij}(\zeta_{vi}-\zeta_{vj}) - k_1\gamma_i^l(\zeta_{vi}-v_l)\\ &- k_2\sum_{j\in\mathcal{N}_i} a_{ij}(\zeta_{xi}-\zeta_{xj}) - k_2\gamma_i^l(\zeta_{xi}-x_l),\end{aligned} \tag{11b}$$

$$\begin{aligned}\dot{\eta}_{bi} = &\frac{1}{d_i^b}\Big[\sum_{j\in\mathcal{N}_i} a_{ij}\dot{\eta}_{bj} + \gamma_i^b\dot{\eta}_{b0}\Big]\\ &- \frac{k_3}{d_i^b} sig^{\alpha_1}\Big[\sum_{j\in\mathcal{N}_i} a_{ij}(\eta_{bi}-\eta_{bj}) + \gamma_i^b(\eta_{bi}-u_b)\Big],\end{aligned} \tag{12}$$

$$\begin{aligned}\dot{\eta}_{li} = &\frac{1}{d_i^l}\Big[\sum_{j\in\mathcal{N}_i} a_{ij}\dot{\eta}_{lj} + \gamma_i^l\dot{\eta}_{l0}\Big]\\ &- \frac{k_4}{d_i^l} sig^{\alpha_2}\Big[\sum_{j\in\mathcal{N}_i} a_{ij}(\eta_{li}-\eta_{lj}) + \gamma_i^l(\eta_{li}-u_l)\Big],\end{aligned} \tag{13}$$

$$\begin{aligned}\dot{\xi}_{xi} = &\frac{1}{d_i^b}\Big[\sum_{j\in\mathcal{N}_i} a_{ij}\dot{\xi}_{xj} + \gamma_i^b v_b\Big]\\ &- \frac{k_5}{d_i^b} sig^{\alpha_3}\Big[\sum_{j\in\mathcal{N}_i} a_{ij}(\xi_{xi}-\xi_{xj}) + \gamma_i^b(\xi_{xi}-x_b)\Big],\end{aligned} \tag{14}$$

and $\xi_{vi} := \dot{\xi}_{xi}$, where parameters $k_3, k_4, k_5, \alpha_l > 0$, $l = 1,2,3$, are constants to be determined, $sig^{\alpha}(z) := (sig^{\alpha}(z_1), \ldots, sig^{\alpha}(z_n))^T$ with $sig^{\alpha}(z_i) := |z_i|^{\alpha} sgn(z_i)$ for $\alpha > 0$ and a vector $z \in \mathbb{R}^n$, η_{li} is the estimate of u_l for agent i, η_{b0} and η_{l0} are respectively the estimates of u_b, u_l for formation basis x_b and reference x_l for which it is natural to assume that $\eta_{b0} = u_b$, $\eta_{l0} = u_l$ (i.e., they know their own information), and for $k \in \{l, b\}, i \in \mathcal{I}_N$

$$\begin{aligned}a_{ij} &:= sgn(w_{ij}) \quad (\text{for } j \neq i), \quad a_{ii} = 0,\\ \gamma_i^k &:= sgn(h_i^k), \qquad d_i^k := \sum_{j\in\mathcal{N}_i} a_{ij} + \gamma_i^k.\end{aligned} \tag{15}$$

Remark 3. *Regarding controller (10) for agent $i \in \mathcal{I}_N$, it has employed the information of its neighbors, i.e., x_j, v_j, and estimates $\zeta_{xj}, \zeta_{vj}, \xi_{xj}, \xi_{vj}$, besides using its own information. For dynamics (12), the update of η_{bi} for agent i requires to know the information of $\dot{\eta}_{bj}$ from its neighbor j, which is an estimate of u_b for agent j, making it difficult to implement as there will be some time delays when transmitting $\dot{\eta}_{bj}$ to agent i in practice. In this case, it can be handled by the method in [44], which also applies to dynamics (13) and (14).*

In what follows, for clarity, define

$$\begin{aligned}\bar{x}_i &= x_i - \zeta_{xi} - \phi_{xi}, & \bar{v}_i &= v_i - \zeta_{vi} - \phi_{vi},\\ \bar{\zeta}_{xi} &= \zeta_{xi} - x_l, & \bar{\zeta}_{vi} &= \zeta_{vi} - v_l,\\ \bar{\xi}_{xi} &= \xi_{xi} - x_b, & \bar{\xi}_{vi} &= \xi_{vi} - v_b,\\ \bar{\eta}_{bi} &= \eta_{bi} - u_b, & \bar{\eta}_{li} &= \eta_{li} - u_l,\end{aligned} \tag{16}$$

and let $\bar{x}, \bar{v}, \bar{\zeta}_x, \bar{\zeta}_v, \bar{\xi}_x, \bar{\xi}_v, \bar{\eta}_b, \bar{\eta}_l$ be the corresponding concatenated column vectors. Also, define $\delta_l = (\bar{\zeta}_x^T, \bar{\zeta}_v^T)^T$, $e_b = (M^b \otimes I_n)\bar{\eta}_b$, $e_l = (M^l \otimes I_n)\bar{\eta}_l$, and $e_x = (M^b \otimes I_n)\bar{\xi}_x$ with

$$\begin{aligned}&M^l := L_1 + \Gamma^l, \quad M^b := L_1 + \Gamma^b,\\ &L_1 := D_1 - A, \quad D_1 := diag\Big\{\sum_{j=1}^N a_{1j}, \ldots, \sum_{j=1}^N a_{Nj}\Big\},\\ &\Gamma^l := diag\{\gamma_1^l, \ldots, \gamma_N^l\}, \quad \Gamma^b := diag\{\gamma_1^b, \ldots, \gamma_N^b\},\end{aligned} \tag{17}$$

where $A = (a_{ij}) \in \mathbb{R}^{N\times N}$ with a_{ij} being defined in (15). Then, dynamics (11), (12), (13) and (14) can be written in a compact form:

$$\dot{\delta}_l = (F_l \otimes I_n)\delta_l + (\mathbf{0}_{nN}^T, \bar{\eta}_l^T)^T, \tag{18}$$
$$\dot{e}_b = -k_3 sig^{\alpha_1}(e_b), \tag{19}$$
$$\dot{e}_l = -k_4 sig^{\alpha_2}(e_l), \tag{20}$$
$$\dot{e}_x = -k_5 sig^{\alpha_3}(e_x), \tag{21}$$

where

$$F_l := \begin{pmatrix} \mathbf{0}_N & I_N \\ -k_2M^l & -k_1M^l \end{pmatrix}. \tag{22}$$

First, one result on dynamics (19), (20) and (21) is given below.

Theorem 1. *Under Assumption 1, the states of dynamics (19), (20) and (21) will globally converge to zero in finite time for any*

$$k_3 > 0,\ k_4 > 0,\ k_5 > 0,\ 0 < \alpha_l < 1, \qquad l = 1,2,3 \tag{23}$$

that is, there exists a real number $T_0 > 0$, such that $\bar{\eta}_b = 0$, $\bar{\eta}_l = 0$, $\bar{\xi}_x = 0$ and $\bar{\xi}_v = 0$ for all $t \geq T_0$.

Proof: For each dynamic (19), (20) or (21), by the same argument as that of Theorem 1 in [45] after simply replacing the part $tanh(sig^{\alpha}(\cdot))$ with $sig^{\alpha}(\cdot)$, it is known that there exist real numbers $T_1 > 0$, $T_2 > 0$ and $T_3 > 0$ such that $e_b = 0$ for all $t \geq T_1$, $e_l = 0$ for all $t \geq T_2$, and $e_x = 0$ for all $t \geq T_3$, and e_b, e_l, e_x are bounded. Obviously, defining $T_0 := \max\{T_1, T_2, T_3\}$ gives rise to $e_b = e_l = e_x = 0$ for all $t \geq T_0$. In addition, in view of Lemma 1 under Assumption 1, one has that M^l and M^b are both invertible, which follows that $\bar{\eta}_b = \bar{\eta}_l = \bar{\xi}_x = 0$ and further implies $\bar{\xi}_v = 0$ in light of $\bar{\xi}_v = \dot{\bar{\xi}}_x$, and they are all bounded. The proof is finished. ■

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TAC.2018.2798808, IEEE Transactions on Automatic Control

As for controller (10), it is easy to obtain that

$$\begin{aligned} u_i = \eta_i &- k_1 \sum_{j\in\mathcal{N}_i} [v_i - \zeta_{vi} - \phi_{vi} - W_{ij}(v_j - \zeta_{vj} - \phi_{vj})] \\ &- k_2 \sum_{j\in\mathcal{N}_i} [x_i - \zeta_{xi} - \phi_{xi} - W_{ij}(x_j - \zeta_{xj} - \phi_{xj})] \\ &- k_2\gamma_i^b(x_i - \zeta_{xi} - \phi_{xi}) - k_2 \sum_{j\in\mathcal{N}_i} (\phi_{xi} - W_{ij}\phi_{xj}) \\ &- k_1\gamma_i^b(v_i - \zeta_{vi} - \dot{\phi}_{xi}). \end{aligned} \tag{24}$$

At this step, let us select the edge weights W_{ij} and H_i^b as

$$W_{ij} = C_i C_j^{-1}, \quad H_i^b = C_i, \tag{25}$$

if agent i can access the information of agent j and formation basis x_b, respectively. Then, it is straightforward to verify that $\phi_{vi} = \dot{\phi}_{xi}$ and $\phi_{xi} - W_{ij}\phi_{xj} = C_i(\bar{\xi}_{xi} - \bar{\xi}_{xj})$ for any $(j,i) \in \mathcal{E}$, and consequently, substituting (24) into network (5) yields

$$\dot{\epsilon} = F_0\epsilon + (\mathbf{0}_{nN}^T, (C\bar{\eta}_b)^T)^T - (\mathbf{0}_{nN}^T, k_2(S\bar{\xi}_x)^T)^T, \tag{26}$$

where $\epsilon := (\bar{x}^T, \bar{v}^T)^T$, $C := diag\{C_1, \ldots, C_N\}$, $S := (s_{ij}) \in \mathbb{R}^{nN\times nN}$ with $s_{ij} = -C_j$ for $j \in \mathcal{N}_i$, $s_{ij} = \mathbf{0}_n$ otherwise, and $s_{ii} = \sum_{j\in\mathcal{N}_i} C_j, i \in \mathcal{I}_N$, and

$$F_0 := \begin{pmatrix} \mathbf{0}_{nN} & I_{nN} \\ -k_2M_0 & -k_1M_0 \end{pmatrix}, \quad M_0 := L' + \Gamma^b \otimes I_n, \tag{27}$$

and $L' = D - W$ with D being the degree matrix defined in last subsection. Note that F_0 is time-varying.

At this moment, it is ready to present the main result on dynamic formation control.

Theorem 2. *If Assumption 1 and condition (23) hold, then multi-agent network (5) with controller (10) can achieve dynamic formation, specified by tuple* $(x_l, x_b, C_1, \ldots, C_N)$, *by choosing weights* W_{ij}, H_i^b *as in (25), if the following conditions hold:*

$$\mu < \min_{i\in\mathcal{I}_N} \left\{ \frac{|\lambda_i^l|^2 Re(\lambda_i^l)}{Im^2(\lambda_i^l)} : Im(\lambda_i^l) \neq 0 \right\}, \tag{28}$$

$$\lim_{t\to\infty} e^{\int_0^t \lambda_{max}(s)ds} = 0, \tag{29}$$

where $\mu := k_2/k_1^2$, $\lambda_{max}(t)$ *is the largest eigenvalue of* $F_0 + F_0^T$, *and* λ_i^l*'s are the eigenvalues of* M^l *arranged in the ascending order of real parts, i.e.,* $Re(\lambda_1^l) \leq \cdots \leq Re(\lambda_N^l)$.

Proof: Consider first the dynamics (18). Invoking Theorem 1, it is known that $\bar{\eta}_l = 0$ for all $t \geq T_0$, and is bounded in time interval $[0, T_0]$, which implies that δ_l is bounded in interval $[0, T_0]$. In the following, take into account the time interval $[T_0, +\infty]$. When $t \geq T_0$, dynamics (18) reduces to

$$\dot{\delta}_l = (F_l \otimes I_n)\delta_l. \tag{30}$$

Subsequently, the eigenvalues of F_l can be calculated as

$$|\lambda I_{2N} - F_l| = |\lambda^2 I_N + (k_1\lambda + k_2)M^l| = 0, \tag{31}$$

which together with eigenvalues λ_i^l's of M^l leads to

$$\lambda^2 + k_1\lambda_i^l\lambda + k_2\lambda_i^l = 0, \quad i \in \mathcal{I}_N. \tag{32}$$

Under Assumption 1, it can be concluded that $Re(\lambda_i^l) > 0$ by Lemma 1. Therefore, in view of Lemma 3, all roots of the polynomial (32) have negative real parts if and only if condition (28) holds, that is, $\delta_l \to 0$ (i.e., $\zeta_{xi} \to x_l$ and $\zeta_{vi} \to v_l$) if and only if condition (28) holds.

Now, turning our attention to dynamics (26), along the same line as above, it can be asserted that ϵ is bounded in interval $[0, T_0]$. When $t \geq T_0$, dynamics (26) becomes

$$\dot{\epsilon} = F_0\epsilon, \tag{33}$$

which is a linear time-varying system since F_0 is time-varying. Then, invoking Theorem 8.2 in [46], it can be obtained that for all $t \geq T_0$

$$\begin{aligned} ||\epsilon(t)|| &\leq ||\epsilon(T_0)|| e^{\frac{1}{2}\int_{T_0}^t \lambda_{max}(s)ds} \\ &= ||\epsilon(T_0)|| e^{-\frac{1}{2}\int_0^{T_0} \lambda_{max}(s)ds} e^{\frac{1}{2}\int_0^t \lambda_{max}(s)ds}, \end{aligned} \tag{34}$$

which, together with the fact that $\lambda_{max}(t)$ is bounded in finite interval $[0, T_0]$, implies that $\lim_{t\to\infty} \epsilon(t) = 0$ if condition (29) holds. Combining all above analysis ends the proof. ■

Remark 4. *1) For dynamic formation control, the parameter* x_b *can be simply chosen as a fixed vector, such as* $x_b = (0, \ldots, 0, 1)^T$, *serving as the fixed axis for operating rotations* R_i*'s and scalings* r_i*'s, which is why it is called formation basis. Compared with the closely related existing results in [19]–[22], [24], [26], the result on dynamic formation control in Theorem 2 does not require to know the coordinates of a desired formation shape, that is, without using the information of* p_i*'s for a desired formation shape specified by* $p = (p_1, \ldots, p_N)^T$, $p_i \in \mathbb{R}^n$. *In this case, this paper has proposed a new type of description on the spatial shape, as defined in Definition 3, which is more intrinsical than the existing specifications based on absolute coordinate, relative coordinate, inter-agent distances, and inter-agent bearings, because the intrinsic property is that one spatial shape is invariant under translations, scalings, and rotations, implying that one spatial shape is not intrinsically dependent on nodes' positions, relative positions, inter-agent distances, and inter-agent bearings. In addition, condition (28) only builds on* λ_i^l *with nonzero imaginary part, and so the eigenvalues* λ_i^l*'s with zero imaginary parts will not impose any constraints on* μ.

2) It is noteworthy that by matrix-valued Laplacian approach, the controller (10) is designed based on the measurement of positions and velocities in a distributed manner. As a special case, when all agents can access the information of agent 0_1 *(i.e.,* x_l, v_l, u_l*) and agent* 0_2 *with fixed* x_b, *the controller (10) will simply reduce to*

$$\begin{aligned} u_i = &-k_1 \sum_{j\in\mathcal{N}_i} [v_i - v_l - \dot{C}_i x_b - W_{ij}(v_j - v_l - \dot{C}_j x_b)] \\ &- k_1(v_i - v_l - \dot{H}_i^b x_b) - k_2(x_i - x_l - H_i^b x_b) \\ &- k_2 \sum_{j\in\mathcal{N}_i} [x_i - x_l - W_{ij}(x_j - x_l)] + u_l + \ddot{C}_i x_b. \end{aligned} \tag{35}$$

Moreover, it should be noted that in Theorems 2 and 3 the formation reference x_l *and basis* x_b *are independent of each other. However, it is possible that* x_b *is known by agent* 0_1, *and in this case, the value of* x_b *will be disseminated along the transmission links of agent* 0_1, *i.e.,* $H^b = H^l$ *in this case.*

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TAC.2018.2798808, IEEE Transactions on Automatic Control

As a special case, consider the scenario when the desired formation shape is static, i.e., r_i, R_i's are independent of time t, $i \in \mathcal{I}_N$, for which we have the following result.

Theorem 3. *If Assumption 1 and condition (23) hold, then multi-agent network (5) with controller (10) can achieve static formation, specified by tuple $(x_l, x_b, C_1, \ldots, C_N)$, by choosing weights W_{ij}, H_i^b as in (25), if and only if it holds:*

$$\mu < \min_{i\in\mathcal{I}_N, k\in\{l,b\}} \left\{ \frac{|\lambda_i^k|^2 Re(\lambda_i^k)}{Im^2(\lambda_i^k)} : Im(\lambda_i^k) \neq 0 \right\}, \tag{36}$$

where $\mu = k_2/k_1^2$.

Proof: Along the same argument as the proof of Theorem 2, dynamics (26) reduces to (33), where F_0 is time-invariant here for static formation control. Before studying the property of eigenvalues of dynamics (33), it is necessary to first confirm that the interaction graph associated with W is consistent. To prove this, given an arbitrary weak cycle $i_1 i_2 \cdots i_k i_1$, and bearing in mind that $W_{ij} = C_i C_j^{-1}$ if $(j,i) \in \mathcal{E}$, it follows

$$\Pi_{m=1}^k W_m = C_1 C_k^{-1} \cdot C_k C_{k-1}^{-1} \cdots C_3 C_2^{-1} \cdot C_2 C_1^{-1} = I_n,$$

whenever $(i_l, i_{l+1}) \in \mathcal{E}$ or $(i_{l+1}, i_l) \in \mathcal{E}$ for some l's, where W_m is defined in Definition 2. Hence, the interaction graph associated with W is consistent, which together with Assumption 1, invoking Lemma 4, implies that there exists a block-diagonal matrix $V \in O^{c+}(n)$ such that $VM_0V^{-1} = D - sgn(W) + \Gamma^b \otimes I_n$. After careful observation, it is easy to see that $D - sgn(W) = L_1 \otimes I_n$ and $D - sgn(W) + \Gamma^b \otimes I_n = M^b \otimes I_n$, where L_1, M^b are defined in (17). As a result, in light of Lemma 1 under Assumption 1, all eigenvalues λ_i^b's of M^b have positive real parts, and thereby the eigenvalues of M_0, which equal λ_i^b's with algebraic multiplicity n due to $VM_0V^{-1} = M^b \otimes I_n$, have positive real parts.

At this step, similar to (32) it can be obtained that eigenvalues of F_0 in (33) satisfy

$$\lambda^2 + k_1 \lambda_i^b \lambda + k_2 \lambda_i^b = 0, \qquad i \in \mathcal{I}_N. \tag{37}$$

By virtue of Lemma 3 and $Re(\lambda_i^b) > 0$, all roots of (37) have negative real parts if and only if

$$\mu < \min_{k\in\mathcal{I}_N} \left\{ \frac{|\lambda_i^b|^2 Re(\lambda_i^b)}{Im^2(\lambda_i^b)} : Im(\lambda_i^b) \neq 0 \right\}, \tag{38}$$

that is, $\epsilon \to 0$ (i.e., $x_i - \zeta_{xi} - \phi_{xi} \to 0$ and $v_i - \zeta_{vi} - \phi_{vi} \to 0$) if and only if (38) holds, which together with the argument of Theorem 2 completes the proof. ∎

Turning back to dynamic formation control, it is worth pointing out that condition (29) is somewhat difficult to be verified especially for extremely large networks, i.e., N is pretty large. To alleviate the difficulty, another controller is now proposed as

$$u_i = \eta_i - k_1(v_i - \zeta_{vi} - \phi_{vi}) - k_2(x_i - \zeta_{xi} - \phi_{xi}), \tag{39}$$

along with updating laws (11)-(13), where all parameters are the same as in controller (10). Thus, by controller (39), network (5) can be written in a succinct form

$$\dot{\epsilon} = F_0' \epsilon + (\mathbf{0}_{nN}^T, (C\bar{\eta}_b)^T)^T, \tag{40}$$

where $\epsilon = (\bar{x}^T, \bar{v}^T)^T$, $C = diag\{C_1, \ldots, C_N\}$, and

$$F_0' := \begin{pmatrix} \mathbf{0}_{nN} & I_{nN} \\ -k_2 I_{nN} & -k_1 I_{nN} \end{pmatrix}. \tag{41}$$

At present, the following result for dynamic formation control can be provided.

Theorem 4. *If Assumption 1 and condition (23) hold, then multi-agent network (5) with controller (39) can achieve dynamic formation, specified by tuple $(x_l, x_b, C_1, \ldots, C_N)$, by choosing weights H_i^b as in (25), if and only if condition (28) holds.*

Proof: The proof is the same as that of Theorems 2 and 3. It only remains to show that all eigenvalues of F_0' has negative real parts for any $k_1, k_2 > 0$, which in fact can be proved by the same line as that of Theorem 2. This ends the proof. ∎

Note that in Theorem 4 the condition (28) is enough for guaranteeing dynamic formation control without the requirement of condition (29), which attributes to the following fact: the controller (39) for agent i is designed based on only its own information $\zeta_{xi}, \zeta_{vi}, \phi_{xi}, \phi_{vi}$ while the controller (10) builds on both its own information $\zeta_{xi}, \zeta_{vi}, \phi_{xi}, \phi_{vi}$ and its neighbors' information ζ_{vj}, ϕ_{vj}, $j \in \mathcal{N}_i$. In this way, matrix F_0' in (41) is time-invariant instead of time-varying one as F_0 in (27). Meanwhile, it should be noticed that all parameters $\zeta_{xi}, \zeta_{vi}, \phi_{xi}, \phi_{vi}$ rely on the information exchange of its neighbors, as seen from (11) and (14). Additionally, it is worth mentioning that for controller (39) the dynamics of ξ_{xi}, ξ_{vi} in (14) can also be replaced by

$$\begin{aligned} \dot{\xi}_{xi} &= \xi_{vi}, \\ \dot{\xi}_{vi} &= \eta_{bi} - k_1 \sum_{j\in\mathcal{N}_i} a_{ij}(\xi_{vi} - \xi_{vj}) - k_1 \gamma_i^b (\xi_{vi} - v_b) \\ &\quad - k_2 \sum_{j\in\mathcal{N}_i} a_{ij}(\xi_{xi} - \xi_{xj}) - k_2 \gamma_i^b (\xi_{xi} - x_b), \end{aligned}$$

by which condition (28) will be replaced by condition (36) for guaranteeing Theorem 4.

Remark 5. *Regarding controller (39), it should be noticed that it is difficult to determine the convergence rate of formation for multi-agent network (5), since in condition (36), λ_i^l and λ_i^b are complex numbers in general. However, when the interaction graph is undirected, λ_i^l and λ_i^b are both real numbers. Then, the convergence rate is given by*

$$C_{rate} := \begin{cases} e^{c_{rate,1}}, & \text{when } \mu < \frac{\lambda_i^k}{4} \text{ for some } \lambda_i^k\text{'s}; \\ e^{c_{rate,2}}, & \text{otherwise} \end{cases} \tag{42}$$

where

$$c_{rate,1} := \max\left\{ \frac{k_1(\sqrt{(\lambda_i^k)^2 - 4\mu\lambda_i^k} - \lambda_i^k)}{2} : \mu < \frac{\lambda_i^k}{4}, \; i \in \mathcal{I}_N, k \in \{l,b\} \right\},$$

$$c_{rate,2} := \max\left\{ \frac{-k_1\lambda_1^l}{2}, \frac{-k_1\lambda_1^b}{2} \right\}. \tag{43}$$

In this case, once selecting μ to satisfy condition (36), increasing the values of k_1 (by $\kappa_a k_1$) and k_2 (by $\kappa_a^2 k_2$) for a constant

$\kappa_a > 1$ (μ is invariant in this case) will drive eigenvalues $\lambda_i^l, \lambda_i^b, i \in \mathcal{I}_N$ farther away from the imaginary axis, and thus will speed up the convergence rate of formation for multi-agent network (5), which is also applicable for directed interaction graphs due to the invariance of μ in (43). Note that k_1, k_2 should not be too large when fixing μ since it will incur pretty high velocity in practice.

Although controllers (10) and (39) are given in a distributed manner, condition (28) or (36) in fact requires global information because in order to compute λ_i^l and $\lambda_i^b, i \in \mathcal{I}_N$, it is imperative to know M^l and M^b, amounting to knowing the whole communication graph spanned by all agents, formation reference x_l and formation basis x_b. To reduce this conservatism, one sufficient condition is provided in the following.

Theorem 5. *Condition (36) can be guaranteed if it holds:*

$$\mu \leq \frac{1}{\pi_N(1,1)}, \tag{44}$$

where $\pi_N(1,1)$, only dependent on N, is defined in (1), and μ is the same defined as in Theorem 2.

Proof: To analyze this condition, it is easy to see that

$$\frac{|\lambda_i^k|^2 Re(\lambda_i^k)}{Im^2(\lambda_i^k)} > Re(\lambda_i^k) \geq Re(\lambda_1^k), \qquad k \in \{l, b\} \tag{45}$$

where λ_i^l, λ_i^b are the same as defined in Theorem 2.

In the meantime, it should be noted that $D - sgn(W) = L_1 \otimes I_n$, which can be seen from the proof of Theorem 3. Observing that L_1 is the Laplacian matrix with nonzero edge weights being 1 and simultaneously, Γ^l and Γ^b, as weight matrix serving as sending information to agents in $\mathcal{I}_N$, have nonzero weights being 1 as well, which is equivalent to that $\underline{a} = \bar{a} = 1$, where $\underline{a}, \bar{a}$ (defined in Section II-B) are lower and upper bounds on edge weights, respectively. Also, note that Assumption 1 holds for the two graphs associated with $L_1 + \Gamma^l$ and $L_1 + \Gamma^b$. Therefore, invoking Lemma 2, one has

$$Re(\lambda_1^k) > \frac{1}{\pi_N(1,1)}, \qquad k \in \{l, b\} \tag{46}$$

which together with (45) results in

$$\frac{|\lambda_i^k|^2 Re(\lambda_i^k)}{Im^2(\lambda_i^k)} > \frac{1}{\pi_N(1,1)}, \qquad k \in \{l, b\} \tag{47}$$

which ends the proof. ■

It is worth pointing out that condition (44) reduces the conservatism without using global information, however, it is at the cost of making μ smaller since the right-hand side of (44) is strictly smaller than the right-hand sides of (36) that can be easily seen from the proof of Theorem 5. Finally, although this section studies double-integrator networks, high-order and even a class of nonlinear networks can also be solved for dynamic formation control as done in [35].

IV. The Case in $\mathbb{R}^n$ Space with $n \geq 3$

This section aims to further anatomize dynamic formation control in n-dimensional Euclidean space with $n \geq 3$.

For the cases when $n \geq 3$, the specification of a formation shape given in last section should be modified as follows: the orientation of u_b is fixed and thereby x_b has a fixed orientation while keeping all others unchanged. In this case, a formation shape specified by $(x_l, x_b, C_1, \dots, C_N)$ will be invariant under translations and scalings, but no longer under rotations. With this scenario, all the results in last section still hold.

It is known that $R_i \in SO(n)$ is a rotation matrix, and the question is what is the specific physical meaning for a given rotation matrix $R \in SO(n)$? That is, how to find a rotation matrix that performs a desired physical rotation? For example, in $\mathbb{R}^3$ space, if one wants to perform a rotation around z-axis by angle $\pi/4$, then how to find a rotation matrix $R \in SO(3)$ to implement it? Indeed, it can be done in $\mathbb{R}^3$ space as detailed in the sequel.

To start, it is necessary to define the *hat* map $\wedge : \mathbb{R}^3 \to \mathfrak{so}(3)$ as

$$\hat{z} = \begin{pmatrix} 0 & -z_3 & z_2 \\ z_3 & 0 & -z_1 \\ -z_2 & z_1 & 0 \end{pmatrix} \tag{48}$$

for $z = (z_1, z_2, z_3)^T \in \mathbb{R}^3$, where $\mathfrak{so}(3) := \{Q \in \mathbb{R}^{3\times 3} : Q^T = -Q\}$ [47]. It is easy to verify that $\hat{z}y = z \times y$ for any $z, y \in \mathbb{R}^3$, where $\times$ means the cross product for two 3-dimensional column vectors. By virtue of Rodrigues' rotation formula [48], it is known that for a given vector $\varsigma \in \mathbb{R}^3$ and an angle θ, exponential $e^{\theta\hat{\varsigma}}$ is a matrix for a rotation around ς by the angle θ. As a matter of fact, $SO(3)$ is a Lie group and $\mathfrak{so}(3)$ is its Lie algebra, and since $SO(3)$ is compact, every matrix $Q \in SO(3)$ can be written in the form $e^{\hat{\varsigma}}$ (or $e^{\theta\hat{\varsigma}}$) for some vector $\varsigma \in \mathbb{R}^3$ (and some angle θ) [49].

With this in hand, dynamic formation shape can be physically implemented in $\mathbb{R}^3$ space. In doing so, let us fix the formation basis as a unit vector, for example, $x_b = (0, 0, 1)^T$, and then the rotation factor $R_i, i \in \mathcal{I}_N$ can be written in the form $e^{\theta_i \hat{\varsigma}_i}$ for some angle θ_i and unit vector $\varsigma_i \in \mathbb{R}^3$, such as one that is vertical to x_b and hence can be written as $\varsigma_i = (\cos(\vartheta_i), \sin(\vartheta_i), 0)^T$ for some angle ϑ_i. Additionally, the scaling factor r_i in formation factor C_i is used to expand or shrink the distance between agent i and x_l.

Next, to make the shape specified by tuple $(x_l, x_b, C_1, \dots, C_N)$ invariant under rotations in $\mathbb{R}^3$ space, one can replace C_i in controllers (10) and (39) by $R_{bi}(t)C_i(t)$, where $R_{bi}(t)$ is an estimate of $R_b(t)$ by agent i with $R_b(t) = e^{\theta_b(t)\hat{\varsigma}_b(t)} \in SO(3)$ known by agent 0_2 for some unit vector $\varsigma_b(t) \in \mathbb{R}^3$ and angle $\theta_b(t)$. Note that $R_b(t)$ is exploited for rotations of the entire formation shape formed by all agents, rendering the formation shape $(x_l, x_b, C_1, \dots, C_N)$ invariant under rotations. In this case, each agent can estimate the values of $\varsigma_b(t), \dot{\varsigma}_b(t), \ddot{\varsigma}_b(t)$ and $\theta_b(t), \dot{\theta}_b(t), \ddot{\theta}_b(t)$ in finite time using similar algorithms to (12) and (14). As a result, the values of $R_b(t), \dot{R}_b(t), \ddot{R}_b(t)$ can be exactly estimated by all agents in finite time, and then all the results in Section III-B can be similarly obtained.

V. More Discussions on Matrix-Valued Laplacian

In this section, more applications of matrix-valued Laplacian proposed in Section III-A are discussed in addition to the application to the dynamic formation control problem, which

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TAC.2018.2798808, IEEE Transactions on Automatic Control

will be given specially for distributed set surrounding problem [38] and multi-party consensus problem [39] in the following.

First, consider the distributed set surrounding problem in [38], where a multi-agent network consisting of N agents with first-order integrator dynamics ($x_i, u_i \in \mathbb{C}$) is investigated, and the distributed set surrounding problem is defined as follows.

Definition 4 ([38])**.** *The multi-agent network $\dot{x}_i = u_i$ with $x_i, u_i \in \mathbb{C}$ being the position and input of agent $i \in \mathcal{I}_N$ is said to achieve distributed set surrounding if for any initial states*

$$\lim_{t\to\infty}[x_i - P_X(x_i)] - \alpha_{ij}[x_j - P_X(x_j)] = 0, \tag{49}$$

for any $(i,j) \in \mathcal{E}$, where the basic interaction graph is denoted by $\mathcal{G} = (\mathcal{V}, \mathcal{E})$.

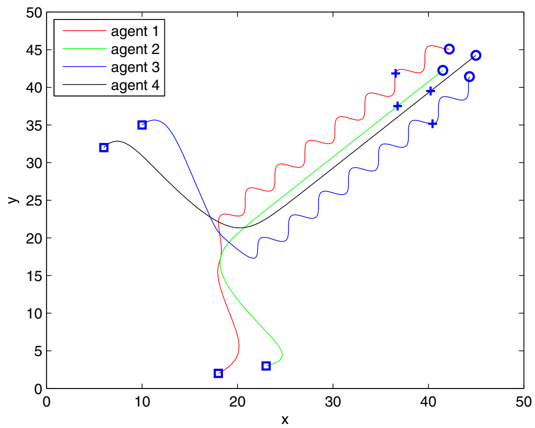

Fig. 4. Trajectories of agents in Example 2 under controller (39), where squares, pluses and hollow circles represent respectively all agents' initial positions, and two positions at two time instants.

In this definition, $X \subset \mathbb{R}^2$ is a given bounded closed convex set, $P_X(\cdot)$ denotes the projection onto X, $\alpha_{ij} \in \mathbb{C}$ is the complex-valued edge weight satisfying $|\alpha_{ij}| = 1$ if $(j,i) \in \mathcal{E}$ and $\alpha_{ij} = 0$ otherwise. As usual, assume $\alpha_{ii} = 0$ for $i \in \mathcal{I}_N$. Furthermore, given that the edge transmission may fail due to energy saving or else, the real interaction graph is denoted by $\mathcal{G}_{\sigma(t)} = (\mathcal{V}, \mathcal{E}_{\sigma(t)})$, where $\sigma : [0,\infty) \to \mathcal{S}$ is a piecewise constant function with a finite index set $\mathcal{S}$, satisfying $\mathcal{E}_{\sigma(t)} \subset \mathcal{E}$. Assume that $\mathcal{G}_{\sigma(t)}$ is uniformly jointly strongly connected (UJSC), that is, there exists $\tau > 0$ such that the union of graphs in interval $[t, t+\tau)$ is strongly connected for any $t \geq 0$. To solve this problem, the controller is proposed by

$$u_i = \sum_{j\in\mathcal{N}_i(\sigma(t))} [\alpha_{ij}(x_j - P_X(x_j)) - (x_i - P_X(x_i))], \tag{50}$$

and it has been shown that distributed set surrounding can be achieved with controller (50) if the graph $\mathcal{G}_{\sigma(t)}$ is UJSC and the basic graph $\mathcal{G}$ is consistent.

However, it should be noted that each state x_i and input u_i belong to $\mathbb{C}$. Although the above result can be straightforwardly extended to the case when $x_i \in \mathbb{C}^m, m \geq 1, i \in \mathcal{I}_N$, it can only deal with the case when x_i is in real space with even dimension since $\mathbb{C}^m \cong \mathbb{R}^{2m}$. In comparison, by using matrix-valued Laplacian with edge weight $W_{ij} = w_{ij}O_{ij} \in O^c(n)$ and setting $w_{ij} \equiv 1$, the same result in [38] can be simply generalized to the general case when state $x_i \in \mathbb{R}^n, i \in \mathcal{I}_N$, without any other modifications.

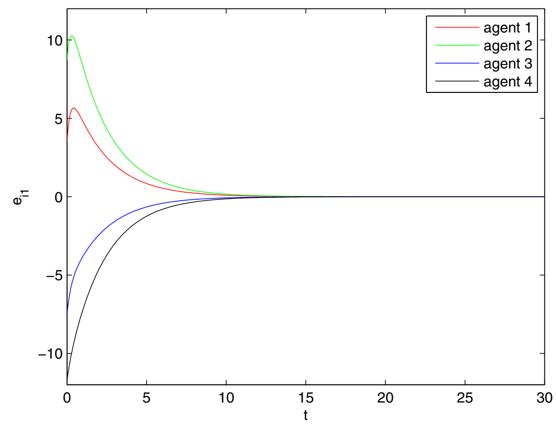

Fig. 5. Trajectories of $e_i = x_i - x_l - C_i x_b$ in the first coordinate.

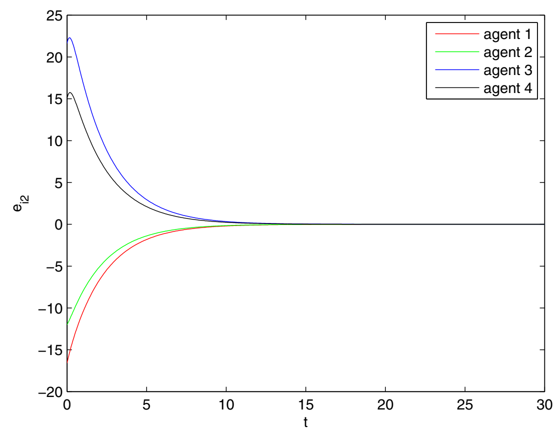

Fig. 6. Trajectories of $e_i = x_i - x_l - C_i x_b$ in the second coordinate.

As another application, multi-party consensus proposed in [39] is now discussed. Consider a multi-agent network composed of N agents with a leader, labeled as 0 with state $x_0 \in \mathbb{R}^n$, and the dynamics of each agent and the leader are governed by double-integrators as shown in (5). All follower agents are partitioned into κ parties: $\mathcal{P}_1, \ldots, \mathcal{P}_\kappa$, meaning that agents in the same party communicate cooperatively and agents in different parties interact antagonistically. Then multi-party consensus is defined as follows.

Definition 5 ([39])**.** *For a multi-agent network (5) with the leader 0, it is said to achieve multi-party consensus if given any constant $C_i = r_i R_i \in O^{c+}(n)$ satisfying $C_i = C_j$ if i, j in the same party and $C_i \neq C_j$ otherwise, and for any initial states one has*

$$\lim_{t\to\infty} ||x_i - C_i x_0|| = 0, \quad i \in \mathcal{I}_N. \tag{51}$$

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TAC.2018.2798808, IEEE Transactions on Automatic Control

To tackle the multi-party consensus problem, let $x_l \equiv 0$ and $x_b = x_0$ being the leader, then it is easy to obtain that the network (5) can achieve multi-party consensus with the controller

$$\begin{aligned} u_i = H_i^b \eta_{0i} - k_1 \sum_{j \in \mathcal{N}_i} (v_i - W_{ij} v_j) - k_1 \gamma_i^b (v_i - H_i^b v_0) \\ - k_2 \sum_{j \in \mathcal{N}_i} (x_i - W_{ij} x_j) - k_2 \gamma_i^b (x_i - H_i^b x_0), \end{aligned} \tag{52}$$

following the same line as in Theorems 3 and 5, where η_{0i} is the estimate of u_0 for agent i with the same updating law as η_{li} in (13). Furthermore, if the network (5) is replaced by linear heterogeneous multi-agent networks as studied in [39] with complex-valued Laplacian, then by following the same argument in [39], multi-party consensus can still be achieved by using matrix-valued Laplacian approach for $x_i \in \mathbb{R}^n, i \in \mathcal{I}_N$. In comparison to [39], two improvements are made by using matrix-valued Laplacian: 1) the method in [39] can only cope with the case when states $x_i \in \mathbb{C}^n$, i.e., even-dimensional real Euclidean space due to $\mathbb{C}^n \cong \mathbb{R}^{2n}$, while all real spaces can be tackled here; and 2) parameters C_i's in [39] are in a special form $SO(2)$ (note that $\{\alpha \in \mathbb{C} : |\alpha| = 1\} \cong SO(2)$), which is a special case of $C_i \in O^{c+}(n)$ here.

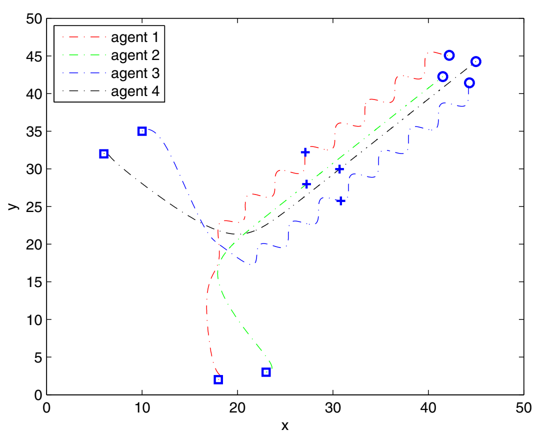

Fig. 7. Trajectories of agents in Example 2 under controller (10), where squares, pluses and hollow circles represent respectively all agents' initial positions, and two positions at two time instants.

In summary, two problems, namely, distributed set surrounding problem and multi-party consensus problem, have been presented above in order to show that some intrinsic improvements can be made by using matrix-valued Laplacian proposed in Section III-A.

VI. Simulations

This section provides two simulation examples for demonstrating the effectiveness of the theoretical results.

Example 2. *Consider multi-agent network (5) with $N = 4$ agents in $\mathbb{R}^2$ space. The desired dynamic formation shape is specified by tuple $(x_l, x_b, C_1, \ldots, C_4)$ with $x_l = (16 + t/2, 16 + t/2)^T$, $x_b = (0,1)^T$, $r_i = 2 + \mod(i,2) \cdot (1 + \sin(t))$ with $\mod(i,k)$ denoting the remainder after i is divided by an integer $k > 0$, and*

$$R_i = \begin{pmatrix} \cos(\theta_i) & -\sin(\theta_i) \\ \sin(\theta_i) & \cos(\theta_i) \end{pmatrix} \tag{53}$$

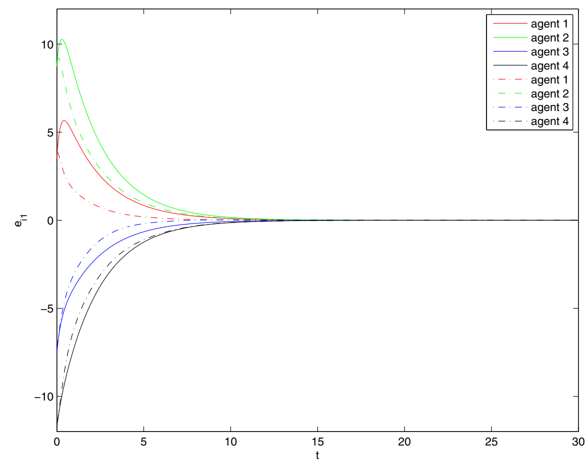

Fig. 8. Trajectories of e_i in first coordinate under controller (39) (solid lines) and controller (10) (dashed lines) in Example 2.

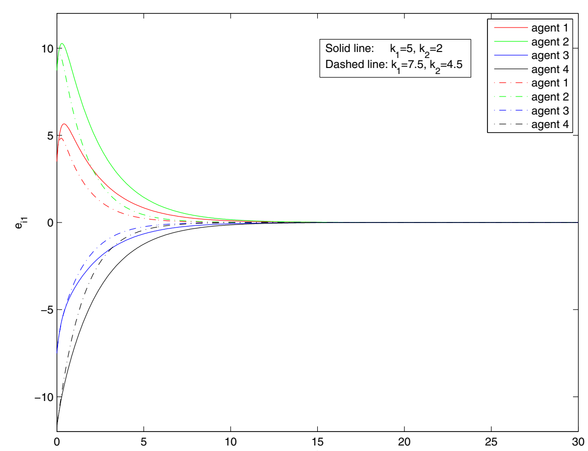

Fig. 9. Trajectories of e_i in first coordinate with different values of k_1, k_2 in Example 2.

with $\theta_1 = \pi/6, \theta_2 = 2\pi/3, \theta_3 = 7\pi/6, \theta_4 = 5\pi/3$. Setting $k_1 = 5, k_2 = 2$, $k_3 = k_4 = k_5 = 1$, $\alpha_1 = \alpha_2 = \alpha_3 = 1/2$ along with the interaction graph given in Fig. 1. After randomly selecting the initial states, all agents' trajectories are shown in Fig. 4 under controller (39), where squares denote the initial states, pluses and hollow circles mean two geometric formation shapes formed by all agents at two time instants. Furthermore, the trajectories of errors $e_i := x_i - x_l - C_i x_b$ are presented in Figs. 5 and 6 for the first and second coordinates, respectively. In Fig. 4, the geometric shape denoted by pluses at some time instant is a parallelogram while the spatial

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TAC.2018.2798808, IEEE Transactions on Automatic Control

shape outlined by hollow circles is almost a square at another time instant, which together with Figs. 5 and 6 indicates that dynamic formation is accurately reached. In the sequel, two more scenarios are discussed.

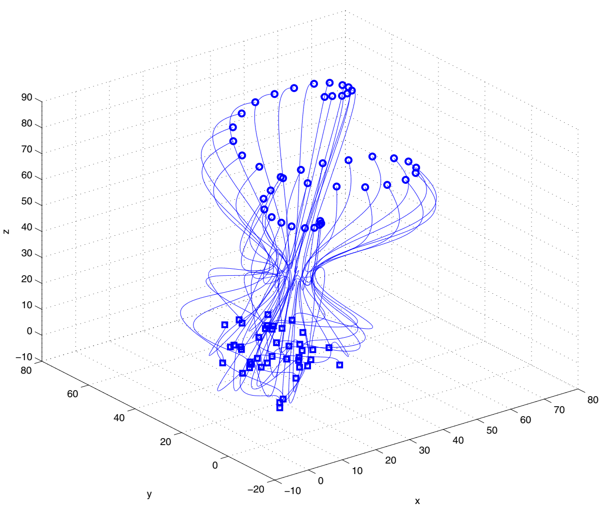

Fig. 10. Trajectories of all agents in Example 3.

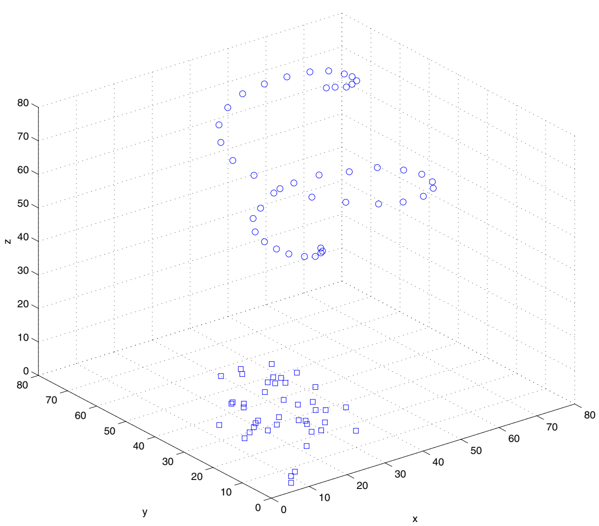

Fig. 11. Geometric shape formed by all agents, denoted by hollow circles, for Example 3, where squares represent the initial positions of all agents.

1) *Similarly, the dynamic formation specified by tuple $(x_l, x_b, C_1, \ldots, C_4)$ can also be achieved precisely by controller (10) with the same parameters as above, for which the trajectories of states are shown in Fig. 7. For the purpose of clear comparison, all errors' trajectories e_i are pictured in Fig. 8 for both controller (39) (solid lines) and controller (10) (dashed lines), where only trajectories in the first coordinate is provided for the sake of saving space. It can be easily seen from Fig. 8 that the convergence speed under controller (10) is higher than that under controller (39), which is consistent with the intuition since controller (10) has employed more information of neighbors than controller (39). However, it is hard to theoretically anatomize the convergence rate for controller (10).*
2) *Now, let $k_1 = 7.5, k_2 = 4.5$ while keeping all other parameters unchanged. By controller (39), the dynamic formation specified by tuple $(x_l, x_b, C_1, \ldots, C_4)$ can still be exactly reached. In this case, all errors' trajectories e_i are plotted in Fig. 9. Note that the trajectories of all states and all errors (in second coordinate) are omitted for saving space. By observing Fig. 9, it can be found that the trajectories, denoted by dashed lines, have faster convergence speed than that denoted by solid lines, meaning that the convergence rate can be expedited for controller (39) by increasing k_1 by a multiple $\kappa_a > 1$ ($\kappa_a = 1.5$ here) and simultaneously increasing k_2 by a multiple κ_a^2, which is in accordance with the discussion in Remark 5.*

Example 3. *Consider multi-agent network (5) with controller (10), comprised of $N = 43$ agents with states in $\mathbb{R}^3$ space. The desired dynamic formation shape is a spiral curve in the spatial space, which is specified by tuple $(x_l, x_b, C_1, \ldots, C_{43})$ (note that $C_i = r_i R_i$) with the formation reference $x_l = (1+6t, 1+6t, 6t)^T$, formation basis $x_b = (0,0,1)^T$ and $r_i = 20 + 12\sin(t)$, $R_i = exp(\theta_i \hat{\varsigma}_i)$, $\varsigma_i = (cos(\vartheta_i), sin(\vartheta_i), 0)^T$, where $\vartheta_i = 4i\pi/43$, $\theta_i = i\pi/43$, $i \in \mathcal{I}_{43}$. Consider an arbitrary interaction graph satisfying Assumption 1 and set $k_1 = 10, k_2 = 1$, $k_3 = k_4 = k_5 = 1$, $\alpha_1 = \alpha_2 = \alpha_3 = 1/2$. After randomly choosing the initial states, all agents' trajectories are shown in Fig. 10, where squares denote the initial states, and hollow circle indicates final geometric formation shape formed by all agents, which can be clearly seen from Fig. 11. Obviously, Fig. 11 shows that the spiral curve is formed by all agents, and meanwhile it can be observed from Fig. 10 that the constituted spiral curve is shrinking as time progresses at last few seconds, which connotes that dynamic formation is achieved, supporting the theoretical result.*

VII. Conclusion

The dynamic formation control problem for directed networks has been investigated in this paper. Provided that the existing specifications (i.e., based on absolute positions, relative positions, inter-agent distances, inter-agent bearings, and barycentric coordinates) for formation shapes are not invariant under all three transformations: translations, rotations, and scalings, a novel specification for geometric shapes has been proposed that can intrinsically describe a geometric shape since the proposed specification is invariant under translations, rotations, and scalings in $\mathbb{R}^2, \mathbb{R}^3$ spaces. To handle dynamic formation control, a new concept, called matrix-valued Laplacian, for graphs has been introduced and some properties have been presented. Subsequently, by matrix-valued Laplacian, the dynamic formation control problem for second-order multi-agent networks has been solved by designing two controllers. Moreover, this approach can ensure the precise achievement of arbitrary geometric shape. Finally, the theoretical results are supported by simulation examples. In addition, future work can be placed on dynamic formation control in the presence of time delays, the case without position measurements and the global coordinate system, the case with collision avoidance, and the case with heterogeneous agents as well as unreliable communication links, etc.

Appendix

Proof of Proposition 1:
Note first that in complex-valued Laplacian case, a weak cycle $i_1 i_2 \dots i_k i_1$ is said to be consistent if $\Pi_{m=1}^k w_m = 1$, where $w_m = \alpha_{i_{m+1}, i_m}$ if $(i_m, i_{m+1}) \in \mathcal{E}$ and then $w_m = \alpha_{i_m, i_{m+1}}^{-1}$ if $(i_{m+1}, i_m) \in \mathcal{E}$, and complex numbers $\alpha_{ij} = e^{\iota\theta}$'s are edge weights defined in the paragraph above (4). In this case, there exist θ_m's such that $\Pi_{m=1}^k w_m = e^{\iota(\theta_1+\theta_2+\cdots+\theta_k)} = 1$ regardless of $(i_m, i_{m+1}) \in \mathcal{E}$ or $(i_{m+1}, i_m) \in \mathcal{E}$, which implies

$$\sum_{m=1}^{k} \theta_m = 2l\pi, \qquad l = \dots, -1, 0, 1, \dots. \tag{54}$$

Meanwhile, as stated in the sentence below (4), it is known that $\{\alpha \in \mathbb{C} : |\alpha| = 1\} \cong SO(2)$ by the map (4). Therefore, it what follows it suffices to verify that

$$\prod_{m=1}^{k} \begin{pmatrix} \cos(\theta_m) & -\sin(\theta_m) \\ \sin(\theta_m) & \cos(\theta_m) \end{pmatrix} = I_2 \tag{55}$$

if and only if (54) holds. Actually, simple calculation follows that

$$\begin{aligned} &\begin{pmatrix} \cos(\theta_k) & -\sin(\theta_k) \\ \sin(\theta_k) & \cos(\theta_k) \end{pmatrix} \times \cdots \times \begin{pmatrix} \cos(\theta_1) & -\sin(\theta_1) \\ \sin(\theta_1) & \cos(\theta_1) \end{pmatrix} \\ &= \begin{pmatrix} \cos(\sum_{m=1}^k \theta_m) & -\sin(\sum_{m=1}^k \theta_m) \\ \sin(\sum_{m=1}^k \theta_m) & \cos(\sum_{m=1}^k \theta_m) \end{pmatrix}, \end{aligned} \tag{56}$$

which equals I_2 if and only if (54) holds. Combining that $SO(2)$ is a subset of $O^c(n) = \{aQ : a \geq 0, Q \in O(n)\}$ with $a = 1, n = 2$ completes the proof. ∎

Proof of Lemma 4:
The first part can be similarly proved along the same line of the proof of Theorem 1 in [39], which is thus omitted for saving space. In the following is the proof of the second part of this lemma.

For the graph $\mathcal{G}$ consisting of N nodes with a leader node 0, it can be viewed as a graph $\mathcal{G}_{N+1}$ composed of $N+1$ nodes with the node 0 as the root, and then its adjacency matrix, Laplacian matrix and degree matrix can be written as

$$\begin{aligned} &W_{N+1} = \begin{pmatrix} \mathbf{0}_n & \mathbf{0}_{nN}^T \\ H_0 & W \end{pmatrix}, \; L_{N+1} = \begin{pmatrix} \mathbf{0}_n & \mathbf{0}_{nN}^T \\ -H_0 & M' \end{pmatrix}, \\ &D_{N+1} = diag\{\mathbf{0}_n, D + sgn(H)\}, \end{aligned} \tag{57}$$

where $H_0 := (h_1 I_n, \dots, h_N I_n)^T$, and h_i, H, D, W, M' are defined in the first paragraph in Section III-A. The graph $\mathcal{G}_{N+1}$ has a spanning tree with the root being the node 0 since it is assumed that there exists at least one directed path from node 0 to any other nodes. Hence, invoking the first part of this lemma, there exists a block-diagonal matrix $V_{N+1} = diag\{v_0, V\}$ with $V = diag\{V_1, \dots, V_N\}$ and $V_i \in O^{c+}(n)$ such that $V_{N+1} L_{N+1} V^{-1} = D_{N+1} - sgn(W_{N+1})$, which by applying (57) can be written in the form

$$\begin{aligned} &\begin{pmatrix} \mathbf{0}_n & \mathbf{0}_{nN}^T \\ -VH_0V_0^{-1} & VM'V^{-1} \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{0}_n & \mathbf{0}_{nN}^T \\ -H_0 & D + sgn(H) - sgn(W) \end{pmatrix}, \end{aligned} \tag{58}$$

which obviously implies $VM'V^{-1} = D - sgn(W) + sgn(H)$. This ends the proof. ∎

Acknowledgment

The authors are grateful to the Editor, the Associate Editor and the anonymous reviewers for their insightful suggestions, which certainly improves the quality of this work.

References

[1] R. W. Beard, J. Lawton, and F. Y. Hadaegh, "A coordination architecture for spacecraft formation control," *IEEE Transactions on Control Systems Technology*, vol. 9, no. 6, pp. 777–790, 2001.
[2] J. A. Fax and R. M. Murray, "Information flow and cooperative control of vehicle formations," *IEEE Transactions on Automatic Control*, vol. 49, no. 9, pp. 1465–1476, 2004.
[3] Z. Lin, B. Francis, and M. Maggiore, "Necessary and sufficient graphical conditions for formation control of unicycles," *IEEE Transactions on Automatic Control*, vol. 50, no. 1, pp. 121–127, 2005.
[4] F. Xiao, L. Wang, J. Chen, and Y. Gao, "Finite-time formation control for multi-agent systems," *Automatica*, vol. 45, no. 11, pp. 2605–2611, 2009.
[5] D. Meng, Y. Jia, J. Du, and J. Zhang, "On iterative learning algorithms for the formation control of nonlinear multi-agent systems," *Automatica*, vol. 50, no. 1, pp. 291–295, 2014.
[6] M. Cao, C. Yu, and B. D. O. Anderson, "Formation control using range-only measurements," *Automatica*, vol. 47, no. 4, pp. 776–781, 2011.
[7] M. Deghat, B. Anderson, and Z. Lin, "Combined flocking and distance-based shape control of multi-agent formations," *IEEE Transactions on Automatic Control*, vol. 61, no. 7, pp. 1824–1837, 2016.
[8] B. Zhu, L. Xie, and D. Han, "Recent developments in control and optimization of swarm systems: A brief survey," in *Proceedings of 12th IEEE International Conference on Control and Automation*, Kathmandu, Nepal, 2016, pp. 19–24.
[9] K. K. Oh, M. C. Park, and H. S. Ahn, "A survey of multi-agent formation control," *Automatica*, vol. 53, pp. 424–440, 2015.
[10] W. Ren, "Consensus strategies for cooperative control of vehicle formations," *IET Control Theory & Applications*, vol. 1, no. 2, pp. 505–512, 2007.
[11] A. K. Das, R. Fierro, V. Kumar, J. P. Ostrowski, J. Spletzer, and C. J. Taylor, "A vision-based formation control framework," *IEEE Transactions on Robotics and Automation*, vol. 18, no. 5, pp. 813–825, 2002.
[12] H. Su, X. Wang, and Z. Lin, "Flocking of multi-agents with a virtual leader," *IEEE Transactions on Automatic Control*, vol. 54, no. 2, pp. 293–307, 2009.
[13] H. Wang, "Flocking of networked uncertain Euler-Lagrange systems on directed graphs," *Automatica*, vol. 49, no. 9, pp. 2774–2779, 2013.
[14] R. Cepeda-Gomez and L. F. Perico, "Formation control of nonholonomic vehicles under time delayed communications," *IEEE Transactions on Automation Science and Engineering*, vol. 12, no. 3, pp. 819–826, 2015.
[15] P. Millán, L. Orihuela, I. Jurado, and F. R. Rubio, "Formation control of autonomous underwater vehicles subject to communication delays," *IEEE Transactions on Control Systems Technology*, vol. 22, no. 2, pp. 770–777, 2014.
[16] P. Deshpande, P. P. Menon, and C. Edwards, "Delayed static output feedback control of a network of double integrator agents," *Automatica*, vol. 49, no. 11, pp. 3498–3501, 2013.
[17] A. Abdessameud and A. Tayebi, "Formation control of VTOL unmanned aerial vehicles with communication delays," *Automatica*, vol. 47, no. 11, pp. 2383–2394, 2011.

[18] M. Turpin, N. Michael, and V. Kumar, "Decentralized formation control with variable shapes for aerial robots," in *Proceedings of IEEE International Conference on Robotics and Automation*, 2012, pp. 23–30.

[19] X. Dong, B. Yu, Z. Shi, and Y. Zhong, "Time-varying formation control for unmanned aerial vehicles: Theories and applications," *IEEE Transactions on Control Systems Technology*, vol. 23, no. 1, pp. 340–348, 2015.

[20] X. Dong, Q. Li, R. Wang, and Z. Ren, "Time-varying formation control for second-order swarm systems with switching directed topologies," *Information Sciences*, vol. 369, pp. 1–13, 2016.

[21] X. Dong and G. Hu, "Time-varying output formation for linear multi-agent systems via dynamic output feedback control," *IEEE Transactions on Control of Network Systems*, vol. 4, no. 2, pp. 236–245, 2017.

[22] ——, "Time-varying formation control for general linear multi-agent systems with switching directed topologies," *Automatica*, vol. 73, pp. 47–55, 2016.

[23] ——, "Time-varying formation tracking for linear multi-agent systems with multiple leaders," *IEEE Transactions on Automatic Control*, vol. 62, no. 7, pp. 3658–3664, 2017.

[24] X. Dong, J. Xiang, L. Han, Q. Li, and Z. Ren, "Distributed time-varying formation tracking analysis and design for second-order multi-agent systems," *Journal of Intelligent & Robotic Systems*, vol. 86, no. 2, pp. 277–289, 2017.

[25] Z. Han, L. Wang, Z. Lin, and R. Zheng, "Formation control with size scaling via a complex Laplacian-based approach," *IEEE Transactions on Cybernetics*, vol. 46, no. 10, pp. 2348–2359, 2016.

[26] L. He, X. Sun, and Y. Lin, "Distributed adaptive control for time-varying formation tracking of a class of networked nonlinear systems," *International Journal of Control*, vol. 90, no. 7, pp. 1319–1326, 2017.

[27] W. Ren and E. Atkins, "Distributed multi-vehicle coordinated control via local information exchange," *International Journal of Robust and Nonlinear Control*, vol. 17, no. 10-11, pp. 1002–1033, 2007.

[28] K. You and L. Xie, "Network topology and communication data rate for consensusability of discrete-time multi-agent systems," *IEEE Transactions on Automatic Control*, vol. 56, no. 10, pp. 2262–2275, 2011.

[29] F. Dorfler and B. Francis, "Geometric analysis of the formation problem for autonomous robots," *IEEE Transactions on Automatic Control*, vol. 55, no. 10, pp. 2379–2384, 2010.

[30] K. K. Oh and H. S. Ahn, "Distance-based undirected formations of single-integrator and double-integrator modeled agents in n-dimensional space," *International Journal of Robust and Nonlinear Control*, vol. 24, no. 12, pp. 1809–1820, 2014.

[31] S. Zhao and D. Zelazo, "Bearing rigidity and almost global bearing-only formation stabilization," *IEEE Transactions on Automatic Control*, vol. 61, no. 5, pp. 1255–1268, 2016.

[32] S. Zhao and D. Zelazo, "Translational and scaling formation maneuver control via a bearing-based approach," *IEEE Transactions on Control of Network Systems*, vol. 4, no. 3. pp. 429–438, 2017.

[33] T. Han, Z. Lin, R. Zheng, and M. Fu, "A barycentric coordinate-based approach to formation control under directed and switching sensing graphs," *IEEE Transactions on Cybernetics*, in press, doi: 10.1109/TCYB.2017.2684461, 2017.

[34] Z. Lin, L. Wang, Z. Han, and M. Fu, "A graph Laplacian approach to coordinate-free formation stabilization for directed networks," *IEEE Transactions on Automatic Control*, vol. 61, no. 5, pp. 1269–1280, 2016.

[35] Z. Lin, L. Wang, Z. Chen, M. Fu, and Z. Han, "Necessary and sufficient graphical conditions for affine formation control," *IEEE Transactions on Automatic Control*, vol. 61, no. 10, pp. 2877–2891, 2016.

[36] Z. Lin, L. Wang, Z. Han, and M. Fu, "Distributed formation control of multi-agent systems using complex Laplacian," *IEEE Transactions on Automatic Control*, vol. 59, no. 7, pp. 1765–1777, 2014.

[37] Z. Lin, W. Ding, G. Yan, C. Yu, and A. Giua, "Leader-follower formation via complex Laplacian," *Automatica*, vol. 49, no. 6, pp. 1900–1906, 2013.

[38] Y. Lou and Y. Hong, "Distributed surrounding design of target region with complex adjacency matrices," *IEEE Transactions on Automatic Control*, vol. 60, no. 1, pp. 283–288, 2015.

[39] F. A. Yaghmaie, R. Su, F. L. Lewis, and L. Xie, "Multi-party consensus of linear heterogeneous multi-agent systems," *IEEE Transactions on Automatic Control*, vol. 62, no. 11, pp. 5578–5589, 2017.

[40] X. Li and L. Xie, "A novel approach to time-varying formation control," in *Proceedings of 11th Asian Control Conference*, Australia, 2017, pp. 1865–1870.

[41] Z. Meng, Z. Lin, and W. Ren, "Robust cooperative tracking for multiple non-identical second-order nonlinear systems," *Automatica*, vol. 49, no. 8, pp. 2363–2372, 2013.

[42] X. Li, M. Z. Q. Chen, H. Su, and C. Li, "Distributed bounds on the algebraic connectivity of graphs with application to agent networks," *IEEE Transactions on Cybernetics*, vol. 47, no. 8, pp. 2121–2131, 2017.

[43] Z. Zahreddine and E. F. Elshehawey, "On the stability of a system of differential equations with complex coefficients," *Indian Journal of Pure and Applied Mathematics*, vol. 19, no. 10, pp. 963–972, 1988.

[44] S. Liu, L. Xie, and F. L. Lewis, "Synchronization of multi-agent systems with delayed control input information from neighbors," *Automatica*, vol. 47, no. 10, pp. 2152–2164, 2011.

[45] X. Li, M. Z. Q. Chen, and H. Su, "Finite-time consensus of second-order multi-agent systems via a structural approach," *Journal of the Franklin Institute*, vol. 353, no. 15, pp. 3876–3896, 2016.

[46] W. J. Rugh, *Linear System Theory, 2nd ed*. Upper Saddle River, NJ: Prentice-Hall, 1996.

[47] T. Lee, "Global exponential attitude tracking controls on $SO(3)$," *IEEE Transactions on Automatic Control*, vol. 60, no. 10, pp. 2837–2842, 2015.

[48] R. M. Murray, Z. Li, and S. S. Sastry, *A Mathematical Introduction to Robotic Manipulation*. Boca Raton, FL: CRC Press, 1994.

[49] A. A. Kirillov, *An Introduction to Lie Groups and Lie Algebras*. New York, NY: Cambridge University Press, 2008.

[50] D. B. West, *Introduction to Graph Theory*. Upper Saddle River, NJ: Prentice-Hall, 2001.

Xiuxian Li received the B.S. degree in mathematics and applied mathematics and the M.S. degree in pure mathematics from Shandong University, Jinan, Shandong, China, in 2009 and 2012, respectively, and the Ph.D. degree in mechanical engineering from the University of Hong Kong, Hong Kong, in 2016. Since 2016, he has been a research fellow with the School of Electrical and Electronic Engineering, Nanyang Technological University, Singapore.

His research interests include formation control, multi-agent networks, cooperative and distributed optimization control, and unmanned systems.

Lihua Xie received the B.E. and M.E. degrees in electrical engineering from Nanjing University of Science and Technology in 1983 and 1986, respectively, and the Ph.D. degree in electrical engineering from the University of Newcastle, Australia, in 1992. Since 1992, he has been with the School of Electrical and Electronic Engineering, Nanyang Technological University, Singapore, where he is currently a professor and Director, Delta-NTU Corporate Laboratory for Cyber-Physical Systems. He served as the Head of Division of Control and Instrumentation from July 2011 to June 2014. He held teaching appointments in the Department of Automatic Control, Nanjing University of Science and Technology from 1986 to 1989 and Changjiang Visiting Professorship with South China University of Technology from 2006 to 2011.

Dr. Xie's research interests include robust control and estimation, networked control systems, multi-agent networks, localization and unmanned systems. He is an Editor-in-Chief for Unmanned Systems and an Associate Editor for IEEE Transactions on Network Control Systems. He has served as an editor of IET Book Series in Control and an Associate Editor of a number of journals including IEEE Transactions on Automatic Control, Automatica, IEEE Transactions on Control Systems Technology, and IEEE Transactions on Circuits and Systems-II. He is an elected member of Board of Governors, IEEE Control System Society (Jan 2016-Dec 2018). Dr. Xie is a Fellow of IEEE and Fellow of IFAC.