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Stability of glassy hierarchical networks

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Abstract

The structure of interactions in most animal and human societies can be best represented by complex hierarchical networks. In order to maintain close-to-optimal function both stability and adaptability are necessary. Here we investigate the stability of hierarchical networks that emerge from the simulations of an organization type with an efficiency function reminiscent of the Hamiltonian of spin glasses. Using this quantitative approach we find a number of expected (from everyday observations) and highly non-trivial results for the obtained locally optimal networks, including, for example: (i) stability increases with growing efficiency and level of hierarchy; (ii) the same perturbation results in a larger change for more efficient states; (iii) networks with a lower level of hierarchy become more efficient after perturbation; (iv) due to the huge number of possible optimal states only a small fraction of them exhibit resilience and, finally, (v) 'attacks' targeting the nodes selectively (regarding their position in the hierarchy) can result in paradoxical outcomes.

1. Introduction

Stability is one of the most essential features of complex systems ranging from ecological [1, 2] to social [3, 4], communication [5–7] and economic networks [8], or even multi-robot systems acting as a single collective intelligent system based on the ideas and features of a high performance computing cluster [9]. The stability of a system can be investigated from several perspectives including the perhaps two most essential ones: resistance and resilience. From the resistance point of view, the main question is how resistant a system is against external perturbations. In this regard, networks that persist for longer in the presence of perturbations are considered more stable or, alternatively, they are also more stable (resistant) if a higher magnitude of perturbation is needed to deviate a metastable system from its stationary, locally optimal state. Resilience refers to how quickly a system recovers from disturbance and returns to its equilibrium or stationary state. In both of these approaches, the change of some appropriately chosen variables could be used as a tool for measuring the level of stability.

In [10] we introduced a model in order to interpret the apparently glassy behaviour of hierarchical organizations and their corresponding network of interactions. Here, glassy behaviour means that according to observations, a given organization (or, in general, most complex systems) can maintain several or many metastable states, depending on their initial structure and the perturbations they are subject to. The model [10] leads to a complex behaviour of the efficiency function associated with the performance of networked organizations, resembling the phenomena displayed by the so-called spin glass model [11–13]. Within the above approach, organizations have many local optimal states which are close to each other and, in addition, maximizing the efficiency function leads to hierarchical structure in the networks of the interactions between individual units. Here we address the question of central importance: how stable is the network structure against perturbations? What is the relation between efficiency and stability and how can stability be related to the structure of the network? Are the hierarchical structures more stable than the less hierarchical ones? To answer these questions we first need to define stability. In standard physical systems stability is defined using the second derivative of potential energy [14]: when the second derivative of potential energy is larger than zero (and the first derivative equals zero) it means that the potential energy is at a local minimum and a small perturbation

returns the system to its stable state. How can we define a somewhat analogous approach to the stability of complex networks?

The stability of complex networks has been defined in several ways. It has been studied extensively by considering the removal of random and targeted nodes and links [15–17] based in part on percolation theory. In these approaches, network connectivity is a crucial criterion for measuring stability [18]; networks are considered stable if their connectivity is unaffected by the removal of a high number of nodes and/or links. Although connectivity is an important feature of networks, their detailed structure and further global properties play fundamental roles in the interaction of individuals in social, robotic and economic systems [19], thus, the latter properties represent significant parameters when measuring stability. Different theoretical models have been developed to understand the formation of social and economic networks and, at the same time, their efficiency and stability have also been analysed [21, 22]. Changes in network variables and parameters (such as efficiency) due to changing environment or any external disturbance are typically considered as stability measurement criteria. In recent work by Gao *et al*, the resilience of multi-dimensional networked systems was measured by reducing them to one dimension [23]. In their approach, it was assumed that networks are in their steady state (fixed points), and because of a changing environment they may lose their resilience by sudden transition to other undesired fixed points. Node, weight and link removal are externally imposed perturbations to the system which has been considered an undirected network. An early, very general approach (involving an important theorem on structural stability) related to our topic was introduced by Andronov and Pontryagin [24] and later reviewed in [25] (see the discussion section).

In this paper, the stability of locally optimal states of directed complex networks is examined by adding two kinds of perturbations (noise) to the system. While after optimization the structure of the network freezes in one of its locally optimal states, the effect of noise relocates links or removes nodes in the system. Change in efficiency and global reaching centrality (level of hierarchy) [26] are studied between local and noisy states and are compared for different values of noise. Network resistance against perturbation is investigated by measuring the number of steps that have to be taken before disturbing the efficient state. Network resilience is considered by looking at the ability of the system to return to its local optimal state after turning off the noise.

2. Perturbing networks

In any complex network a general function (describing the total/global state of the system) can be defined. In our previous communication [10], we developed an efficiency function for a typical organization/system which is constructed from interacting individual units with a variety of abilities a_i (level of the potential contribution of a unit to the performance of the whole system). This function reflects the fact that the contribution resulting from the 'collaboration' of two units is proportional to their multiplied abilities and can be both positive and negative:

$$E_{\text{eff}}(p, q) = 1/N \sum_{ij} J_{ij}(p, q) a_i a_j, \tag{1}$$

where N is the number of nodes. Directed edges between individuals have signs corresponding to their harmonic ($J_{ij} = +1$) or antagonistic ($J_{ij} = -1$) relation. $J_{ij} = 0$ if the two nodes are not connected. The direction of the edges is related to the sign of the expression $a_i - a_j$ (it is pointed, in the majority of cases, from the unit with a higher ability to a unit with a lower ability). In particular, the probabilities for $J_{ij} = 1$ are $(1 - p)(1 - q)$ and pq while for $J_{ij} = -1$ they are $(1 - p)q$ or $p(1 - q)$. p and q correspond to the probabilities of an inverse direction of the edge ij (i.e., from j to i for $a_i > a_j$) and for an antagonistic collaboration, respectively. It is very important to stress at this point that we ensure that the above rules hold for the subgraphs of M edges as well!

Equation (1) has a structure similar to a spin-glass Hamiltonian. However, there is a notable 'twist' in the present interpretation: while in the case of the standard spin-glass framework it is the spins which are varied to find configurations with small free energy, in equation (1) the J_{ij}-s (i.e., the network configurations) are tuned to maximize the efficiency, E_{eff}, while the a_i-s are constant. In spite of the differences, however, as we also point out in [10], the system we consider exhibits some of the essential qualitative features of spin glasses.

The model has three parameters; the probability of antagonistic interactions (q) and the direction of an edge pointing against the larger ability node (p) and, in addition, a constraint for the maximum of the incoming plus the outgoing edges ($in + out$). The results we present are for systems of N nodes, $p = q = 0.2$ and $in + out = 10$. Before the optimization starts, a full graph of N nodes with given J_{ij}-s and edge directions is generated. Then a subgraph (within the full graph) of $M = 3N$ randomly chosen edges is created. In most cases this subgraph has a number of nodes equal to N. The efficiency function is maximized in order to find local optimal states of the networks using Monte Carlo simulation. The resulting network efficiencies and their corresponding distribution exhibit a glassy behaviour meaning that the optimization converges to many states. Maximizing the efficiency function leads to complex directed networks with hierarchical features. The

distribution of local maxima of efficiencies and their corresponding global reaching centrality (GRC), or the level of hierarchy values, indicate that optimal states fall into two categories with high and low GRC.

Global reaching centrality (GRC) is defined by the following equation [26].

$$GRC = \frac{\sum_{i \in V}[C_R^{max} - C_R(i)]}{N - 1},$$ (2)

where N is the number of nodes in the network, $C_{R(i)}$ is the local reaching centrality of the node i that is described as the number of nodes which can be reached from node i through the directed edges of the network. C_R^{max} is the maximum of $C_{R(i)}$ and the summation is over all nodes in the graph V. The question posed is: to what degree is an optimal state reached by maximizing equation (1) stable, from a resistance point of view, against external perturbation, and what is its ability to return to its optimal state after turning off the noise (resilience)?

In a mechanical system, after a long time particles tend to stay in an equilibrium state and resist any disturbance from outside. For a highly stable state, a greater external force is required to permanently perturb the system from its equilibrium or a well pronounced metastable state.

A somewhat modified version of this concept is used in this paper to define the level of stability of complex networks generated by [10]. Therefore, for an external perturbation of a given magnitude, the number of steps needed to deviate from the local optimal state and the efficiency difference caused by external perturbation or noise can be considered as quantities for checking stability. External perturbation can be a noise in a local optimal state of the system, after optimal states are achieved through Monte Carlo simulation by randomly relocating the position of the edges for temperatures close to zero. In each Monte Carlo step, the efficiency is calculated. If the efficiency is higher than the previous step it is accepted and if it is lower, it is accepted by Boltzmann probability $(\exp(-\frac{\Delta E_{eff}}{T}))$. To reach the saturated highly efficient state, temperature (T) in Boltzmann probability should be close to zero. After reaching the optimal states where efficiency saturates, we increase temperature to implement the noise which increases the Boltzmann probability. The noise is kept on until efficiency changes. Then the difference between efficiencies and GRCs in the two states (optimal and noisy states), as well as the number of steps taken to see the first change in efficiency, are calculated. The random relocation of edges in local optimal states is another way of perturbing the system; it is shown that a high value of T or noise has the same effect as the edge relocation.

Moreover, we investigate the effect of removing nodes (attacks) on the efficiency and GRC. An attack is defined as the deletion of Q number of nodes from the network in its local optimal states, with $Q = [1, 2, 3, \ldots, N_l]$, where N_l is the maximum number of nodes removed. For each of the Q attacks, we measure the efficiency and GRC after the attack. The effect of external perturbation by adding noise in local optimal states (temperature increase in Boltzmann probability in Monte Carlo simulation) and attack to the system by targeted node removal are depicted schematically in figure 1.
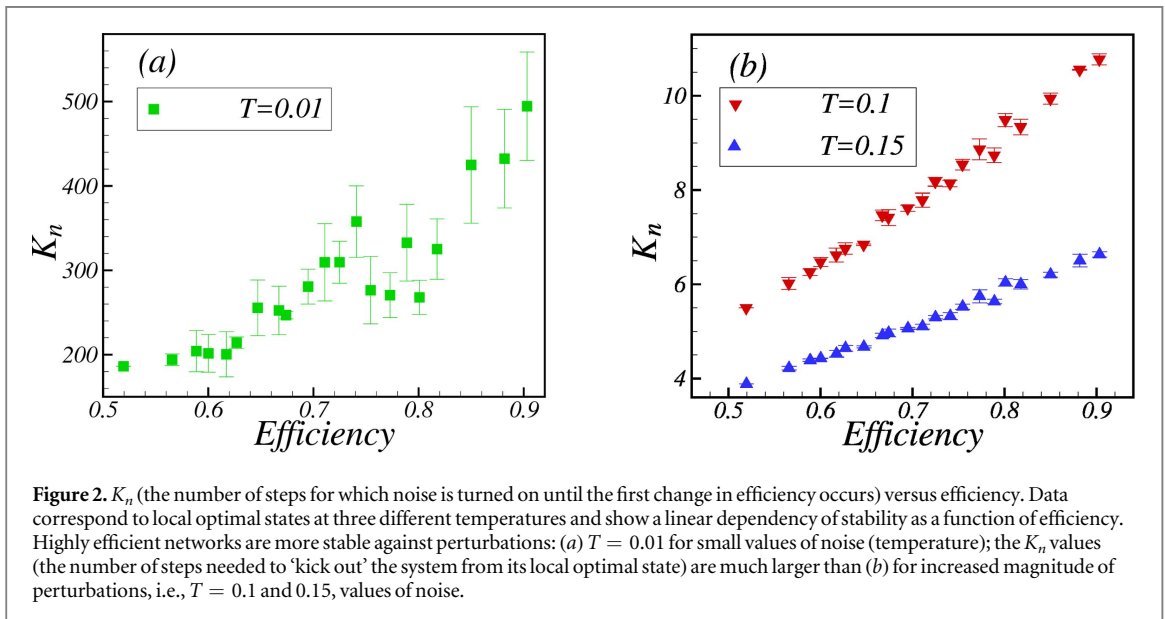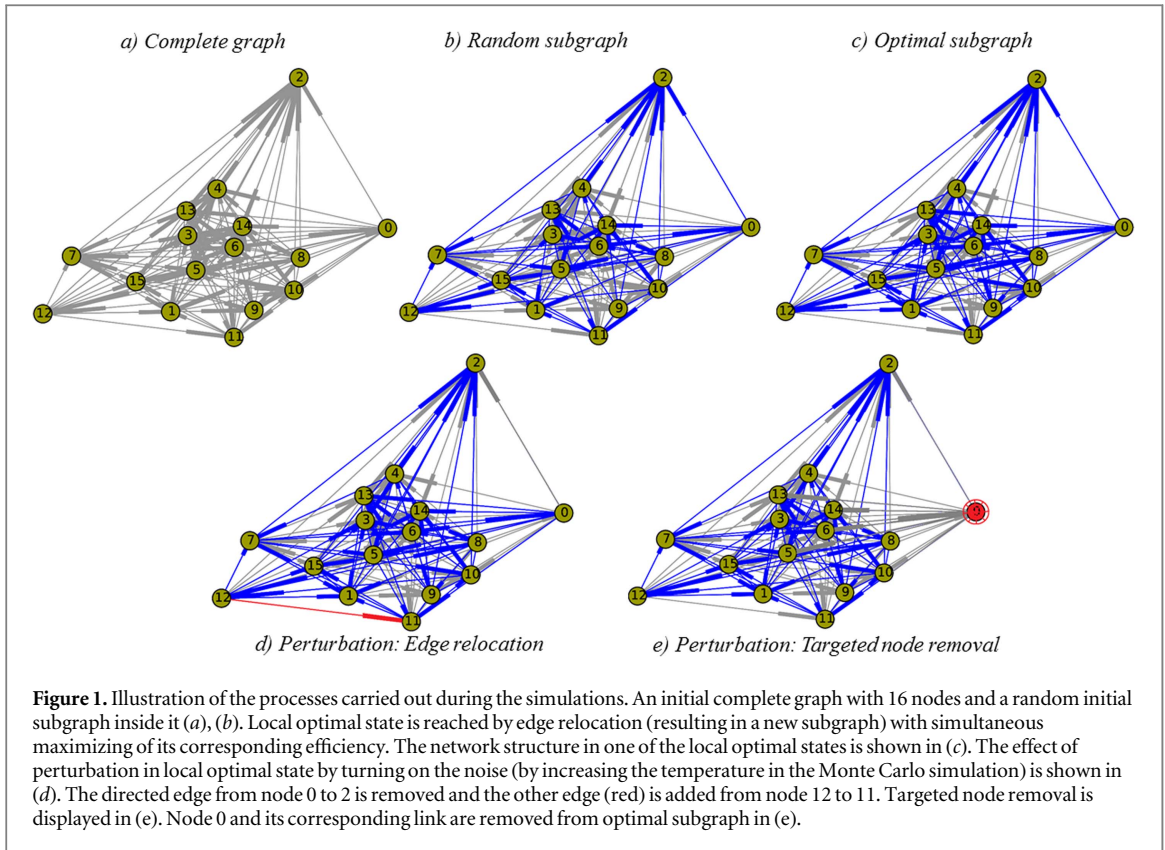
3. Results

We perform the following computational experiment to understand the relation between efficiency and stability. We start with a random graph of M edges with a certain efficiency. Then we follow the procedure in [10] to maximize efficiency as defined in equation (1). We consider $u = 100$ different optimal states chosen in a random fashion. For each optimal state, we turn on the noise and Monte Carlo simulation continues until the system abandons the optimal state in favour of an unstable state. The number of steps taken in the Monte Carlo simulation until the first change in efficiency is observed is saved. We perform this measurement $w = 100$ times per optimal state. We repeat this algorithm for all 100 different optimal states. Finally, we average over the number of Monte Carlo steps using equation (3).

$$K_n = \frac{\sum_{j=1}^{u=100}\sum_{i=1}^{w=100} k_i}{u \times w},$$ (3)

where k_i is the number of Monte Carlo steps before an optimal state is abandoned. Figure 2 displays K_n for systems with different efficiencies. Each point in figure 2 belongs to a different random initial subgraph with the same range of efficiency in their optimal states, and efficiency values are averaged over all these initial states and $u = 100$ corresponding optimal states.
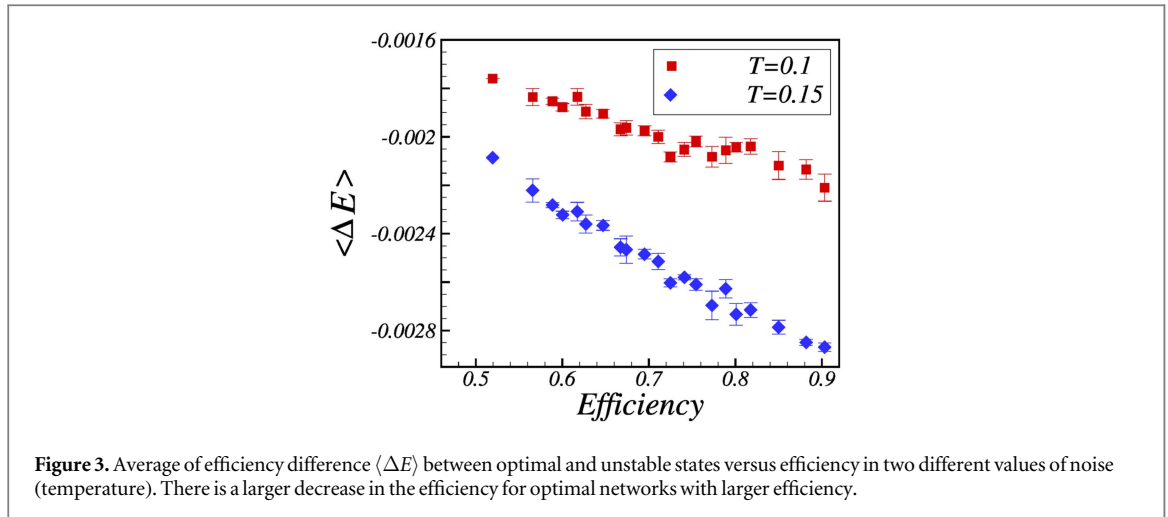
3.1. Resistance
We start our interpretation of the results in figure 2 with the observation that for a given noise (perturbation), higher values of K_n imply higher stability and vice versa. Networks with higher efficiencies need more steps to deviate from their optimal state and exhibit a higher level of resistance against external perturbation.

Figure 1. Illustration of the processes carried out during the simulations. An initial complete graph with 16 nodes and a random initial subgraph inside it (*a*), (*b*). Local optimal state is reached by edge relocation (resulting in a new subgraph) with simultaneous maximizing of its corresponding efficiency. The network structure in one of the local optimal states is shown in (*c*). The effect of perturbation in local optimal state by turning on the noise (by increasing the temperature in the Monte Carlo simulation) is shown in (*d*). The directed edge from node 0 to 2 is removed and the other edge (red) is added from node 12 to 11. Targeted node removal is displayed in (*e*). Node 0 and its corresponding link are removed from optimal subgraph in (*e*).

Figure 2. K_n (the number of steps for which noise is turned on until the first change in efficiency occurs) versus efficiency. Data correspond to local optimal states at three different temperatures and show a linear dependency of stability as a function of efficiency. Highly efficient networks are more stable against perturbations: (*a*) $T = 0.01$ for small values of noise (temperature); the K_n values (the number of steps needed to 'kick out' the system from its local optimal state) are much larger than (*b*) for increased magnitude of perturbations, i.e., $T = 0.1$ and 0.15, values of noise.

The effect of noise on our experiment is understood by comparing figures 2(a) and (b). For a given system with an averaged efficiency of 0.6, K_n is around 200 for the small noise of $T = 0.01$, whereas for larger noise of $T = 0.15$ only around $K_n = 4$ to 5 steps are enough for the system to lose stability. More importantly, all the plots in figure 2 demonstrate a linear relation between efficiency and K_n. This means that efficiency and stability have a linear dependence. In other words, highly efficient systems are more resistant to external perturbations.

Next, we study the reaction of a network to external perturbation. In particular, we would like to measure the change in efficiency upon perturbation as a function of the efficiency itself.

As in [23], it is assumed that the system is located in one of its fixed (here: metastable) points, and we are interested in the question of how a single function (here the efficiency and the GRC) behaves if external perturbations (here, increasing the temperature) are added. We consider further perturbations in the form of

Figure 3. Average of efficiency difference $\langle\Delta E\rangle$ between optimal and unstable states versus efficiency in two different values of noise (temperature). There is a larger decrease in the efficiency for optimal networks with larger efficiency.

targeted node removal. Among others, we would like to measure the change in efficiency after perturbation is applied as a function of system efficiency.

In figure 3 the average change in the efficiency denoted by $\langle\Delta E\rangle$ is shown as a function of efficiency for two values of noise (T). There is an approximately linear dependence between $\langle\Delta E\rangle$ and efficiency. For systems with higher efficiency the absolute value of $\langle\Delta E\rangle$ (reaction) of the system is larger in response to a perturbation. Since we have already established that higher efficiency translates into higher stability we can conclude, based on our findings so far, the following: systems with higher stability are less susceptible to external perturbation but once the perturbation is large enough, they undergo a more pronounced change.

Below we show that for large values of noise (perturbation), the reaction of the system ($\langle\Delta E\rangle$ and $\langle\Delta GRC\rangle$) is very similar to the random relocation of an edge in a network that operates at its optimal state. Random relocation here implies the removal of an edge and its addition between two disconnected nodes in a random way. Figure 4(a) shows the change in the efficiency for a large noise $T = 5$ (red curve) and for random relocation of an edge (green) as a function of efficiency of the corresponding optimal states. Similarly, figure 4(b) shows $\langle\Delta GRC\rangle$ as a function of GRC for two perturbation approaches, shown by the red and green data points.
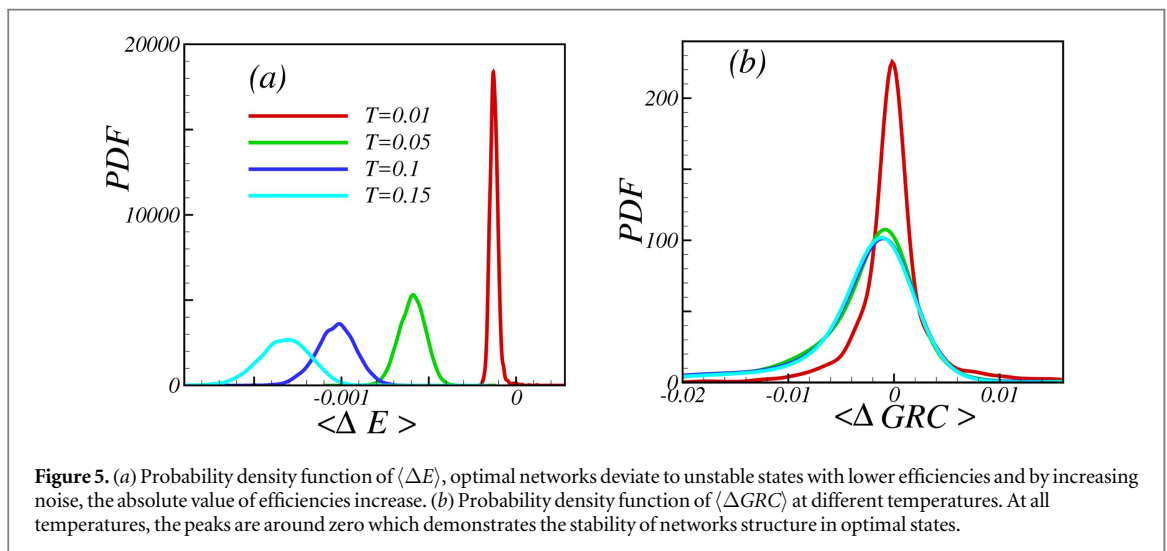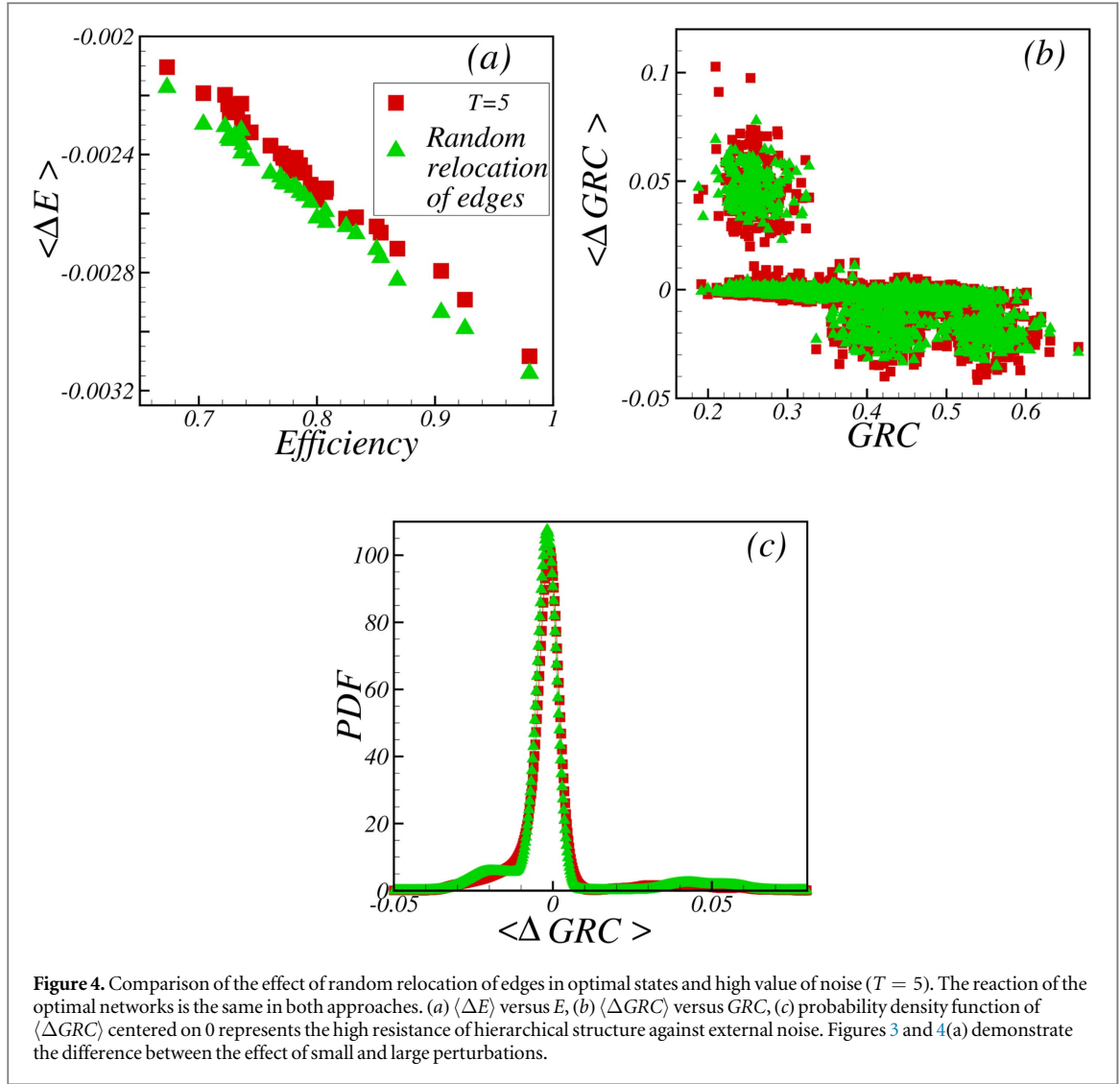
The fact that the behaviour of the efficiencies is similar when large noise or random relocation is applied as a perturbation is consistent with our expectations of what should happen for the case of large noises, since in the latter case a new edge is chosen almost randomly due to the larger value of the Boltzmann factor.

Figure 4(b) displays two regions: in less- or non-hierarchical networks with small GRC, the effect of perturbation makes the graph deviate to higher GRC states, thus $\langle\Delta GRC\rangle$ is positive. For states with large GRC the $\langle\Delta GRC\rangle$ values are negative, indicating a jump to less hierarchical states (i.e., perturbations are likely to decrease the otherwise high level of hierarchy corresponding to a high level of efficiency). Figure 4(c) demonstrates the probability density function of $\langle\Delta GRC\rangle$ with a high peak around $\langle\Delta GRC\rangle = 0$ and a side shift to the negative values which indicates that hierarchical optimal states have a higher resistance against external perturbation, preserve their structure and, in the case of noise effect, they lose their stability by jumping to the less hierarchical states. Figure 5 depicts the probability density function of $\langle\Delta E\rangle$ and $\langle\Delta GRC\rangle$ at four different temperatures.

3.2. Resilience

The network's ability to retain its basic functionality after external perturbation and return to its optimal state is studied in this section. To model this, we start with an optimal state and turn on the noise, changing T from nearly zero to $T = 0.1$. After 32 steps, the noise is switched off and the system is allowed to recover from its unstable state and converge to a local optimal state again. Figure 6 demonstrates the efficiency and GRC values over the whole process from the time that noise is turned on (step = 0) and off (step = 32) until the network saturates to its local optimal state. According to figure 6(a), when noise is turned on, the system experiences a sudden decrease in both efficiency and *GRC*. Larger values of noise impart a stronger disturbance on the system. After the noise is switched off, the efficiency and GRC increase again and converge to a higher value corresponding to one of the local optimal states.

We then calculate efficiency and GRC differences between the new state and the initial local optimal state. Figure 7(a) shows the variation of $\langle\Delta E\rangle$ versus efficiency for three different initial complete graphs. The linear trends with the negative slopes in figure 7(a) show that as the efficiency increases, the difference between two local optimal states are decreased. For larger efficiencies the network returns to the same initial optimal states

Figure 4. Comparison of the effect of random relocation of edges in optimal states and high value of noise ($T = 5$). The reaction of the optimal networks is the same in both approaches. (*a*) $\langle \Delta E \rangle$ versus E, (*b*) $\langle \Delta GRC \rangle$ versus GRC, (*c*) probability density function of $\langle \Delta GRC \rangle$ centered on 0 represents the high resistance of hierarchical structure against external noise. Figures 3 and 4(a) demonstrate the difference between the effect of small and large perturbations.

Figure 5. (*a*) Probability density function of $\langle \Delta E \rangle$, optimal networks deviate to unstable states with lower efficiencies and by increasing noise, the absolute value of efficiencies increase. (*b*) Probability density function of $\langle \Delta GRC \rangle$ at different temperatures. At all temperatures, the peaks are around zero which demonstrates the stability of networks structure in optimal states.

$\langle \Delta E \rangle = 0$ after turning off the noise. This confirms the high resilience of highly efficient networks. For less efficient states, the difference between two optimal states is larger. Figure 7(b) shows the probability density function of $\langle \Delta E \rangle$ with positive skew and a high peak close to zero.
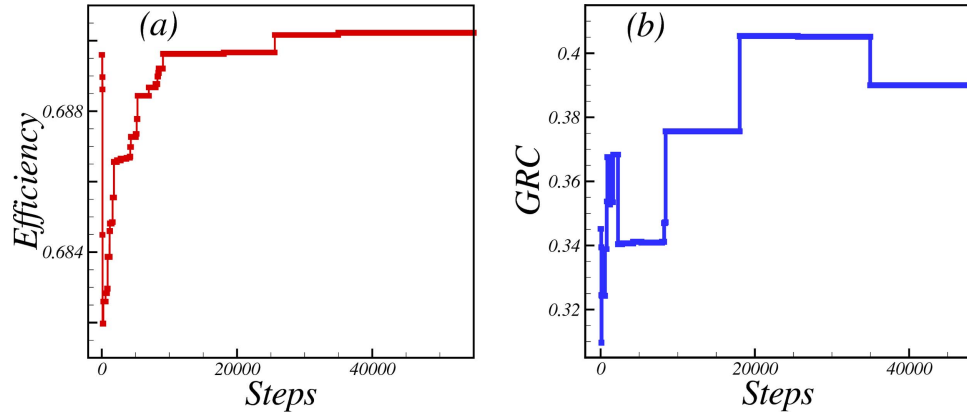
Figure 6. (*a*) Variation of efficiency during the effect of external perturbation. First the perturbation $T = 0.1$ is applied and then the noise is switched off until the efficiency saturates. (*b*) Change of GRC during the perturbation and after turning off the noise. Noise is turned off after 32 steps and the system saturates to other optimal states with higher efficiency.
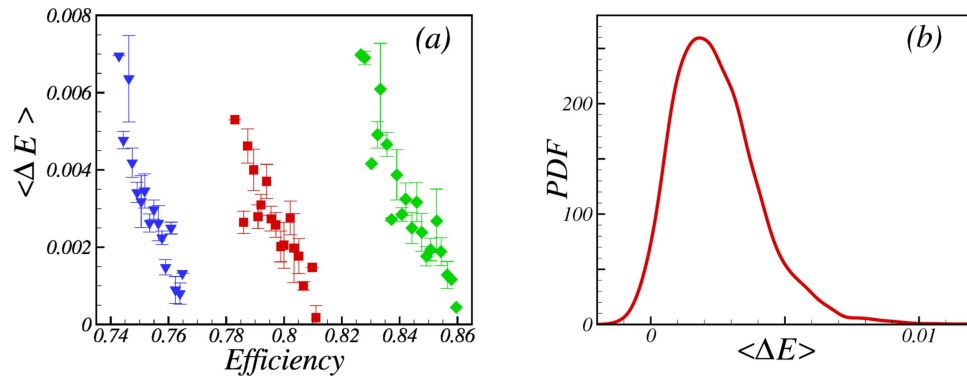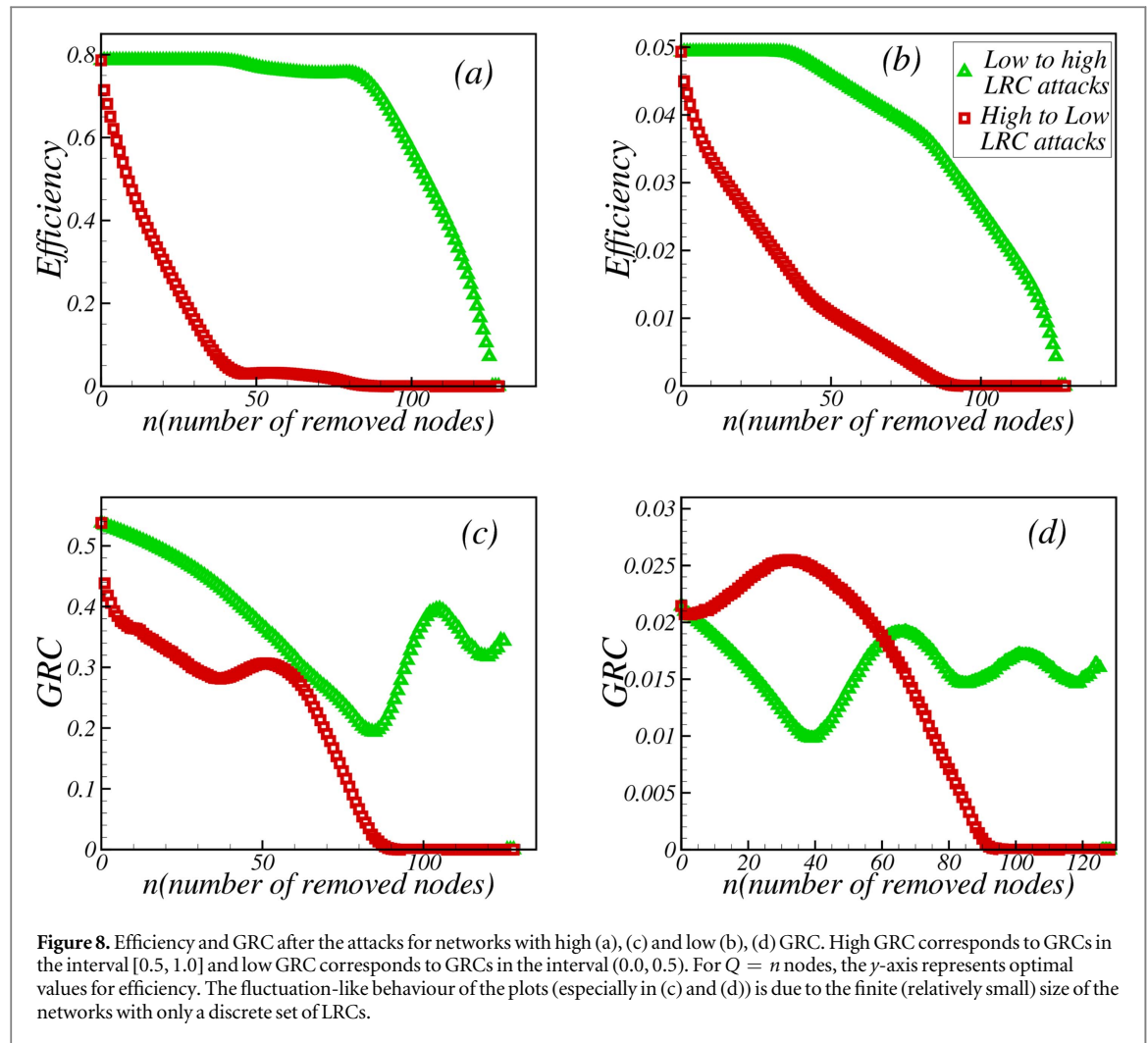

Figure 7. (a) Average efficiency difference between two local optimal states $\langle \Delta E \rangle$ before turning on the noise and after it is turned off versus efficiency (E) for three different initial conditions. After the noise is switched off the network saturates to higher efficient state, $\langle \Delta E \rangle$ is positive. For each initial state higher efficient states exhibit higher resilience: $\langle \Delta E \rangle$ converges to 0 and the network converges to the same initial optimal graph. (b) Probability density function of $\langle \Delta E \rangle$ is skewed with a peak close to zero, demonstrating the resilience of some of the efficient states and positive values, i.e., switching to a more efficient state.

3.3. Resistance against node removal

In this section, we study the resistance of optimal networks against targeted node removal. Consider, e.g., a robotic network with N robots (nodes) and edges representing existing links among robots, as proposed in [9]. Such links are intermittent during the execution of a real world mission, since the reach between two nodes, varies in time given current communications technologies. During the execution of the mission, some nodes might have their battery drained or even, in some cases, some robots might be destroyed or experiment all sorts of failures. Also, the communication among the robots is not necessarily symmetric. Thus, during a mission carried out by a network of robots of varying function the underlying structure of the signals sent within the system can indeed be interpreted as a hierarchical directed network. A possible further interpretation is that in which the nodes represent tasks to be completed by a group of robots and edges represent dependency among tasks. Another network example, in the realm of society, is a military organization, i.e. an army. Clearly, it is possible to describe an army as a directed hierarchical network. Therefore, what could happen if a general or a high-ranked military person is lost during combat? Alternatively, what happens if low-ranked soldiers are lost during a combat? We use our efficiency function to evaluate the effect of attacks on the networks we consider and suggest that the basic features we observe are likely to be applicable to other hierarchical systems as well.

In order to observe the effect on stability when nodes are lost or removed, we performed the following numerical experiments. We define an attack as the removal of Q nodes, $Q = [1, 2, 3, 4, \ldots, N]$, i.e. attack Q consists of removing Q nodes at once. We start with one local optimal state and perform Q attacks and after each attack the efficiency and GRC of the network are measured. Given the hierarchical structure of the networks studied here, a finite set of different $C_{R(i)}$ (local reaching centrality) exists for the nodes in the network i.e.

Figure 8. Efficiency and GRC after the attacks for networks with high (a), (c) and low (b), (d) GRC. High GRC corresponds to GRCs in the interval [0.5, 1.0] and low GRC corresponds to GRCs in the interval (0.0, 0.5). For $Q = n$ nodes, the y-axis represents optimal values for efficiency. The fluctuation-like behaviour of the plots (especially in (c) and (d)) is due to the finite (relatively small) size of the networks with only a discrete set of LRCs.

$C_{R(i)} = [C_R^1, \ C_R^2, \ldots, C_R^x]$ with N_1 nodes having C_R^1, N_2 nodes having C_R^2, etc. We investigate the stability of the networks in their local optimal states by removing nodes (attacks) using two approaches. First, nodes with higher $C_{R(i)}$ and their corresponding links are removed one by one, and this process is continued to lower $C_{R(i)}$ values, while the efficiency and GRC of the networks are measured after each attack. In the second approach, node removal starts from those with lower $C_{R(i)}$, and it continues to the higher ones. For a specific attack, each of the removed nodes possess a given $C_R(i)$. Hereafter, instead of $C_R(i)$, the symbol LRC is used. Thus, the LRCs are $LRC = [LRC_1, LRC_2, \ldots, LRC_x]$ with N_1 nodes having LRC_1, N_2 nodes having LRC_2, etc, $Q = [1, 2, 3, \ldots, 128]$ nodes were removed from the network in two approaches, from highest to lowest LRC and vice versa.

With the objective of analysing the stability of networks with low GRC (lower than 0.5) and high GRC (higher than 0.5), under perturbations by node removal, we performed Q attacks, with $Q = [1, 2, \ldots, 128]$ nodes, for 32 networks (18 with high GRC and 14 with low GRC), each with 16 local optimal states. The nodes were ordered by two different approaches: 1. highest to lowest LRC (red curves) and 2. lowest to highest LRC (green curves). This way, each attack Q, consisting of removing Q nodes, was carried out according to these two methods. Figure 8(a) shows that when nodes, in networks with high GRC, are removed with approach one (highest to lowest LRC), efficiency decreases faster than when nodes are removed with approach two. This is expected, according to the model presented in [10], where nodes with high ability and correspondingly high *LRC* contribute to the efficiency to a greater extent, and are at the top layer of the corresponding hierarchical network, than nodes with low ability.

A similar behaviour appears for networks with low GRC, as depicted in figure 8(b). However, within approach two (lowest to highest LRC), the removal of more than 80, out of 128 nodes, causes the efficiency to drop significantly for high GRC networks, whilst the removal of more than 40 nodes causes the efficiency to drop significantly for low GRC networks. It is clear from figure 8 that the two kinds of attacks lead to rather different outcomes for a large range of removed nodes.

These results suggest that networks with high GRC (more hierarchical) are very stable and efficient, even when losing a large quantity of nodes, in comparison with low GRC networks.

Figure [8](d) shows fluctuating behaviour for GRC, with attacks according to approach two (lowest to highest LRC). These fluctuations show that networks with low GRC are not stable against external perturbations, such as node removal. Again, there is no relevant GRC increase in approach one. While the numerical experiments performed in this work are on networks whose efficiency corresponds to that in the model presented in [10], GRC is calculated according to the general model presented in [26], which suggest that these results are applicable to general hierarchical directed networks.

4. Discussion

In [10] we introduced a rather general model for the efficiency of organizations involving both directed and collaborative or conflicting interactions among the members of the collective. The introduction of directed edges was aimed at investigating the complex structure of the actual interactions corresponding to locally optimal configurations of the system. Indeed, we found that most of the optimal states correspond to an underlying hierarchical structure—quite like the structure of organizations and other complex systems observed in nature and society.

In the present paper we considered another, equally important, aspect of our model of organizations: the stability of the states into which an optimization procedure drives the system. It has been shown (a long time ago) that the structure of a system of differential equations is closely related to the stability of its solution(s). The related notion of structural stability was introduced by Andronov and Pontryagin in 1937 [24]. The main theorem by Andronov and Pontryagin is concerned with the effect of perturbations on the trajectories corresponding to a dynamical system represented by a set of differential equations. In our paper, we do not investigate trajectories emerging as the solutions of differential equations describing the dynamics of the states of the nodes. Instead, we perturb the system in a way which results in a new network configuration which can be looked at as corresponding to the statement that we perturb the system of equations by changing the structure instead of considering the perturbations applied to the equations and thus, the behaviour without changing the structure. In fact, the notion of structural stability has become part of one of the new important directions of network theory recently. The application of the theorems related to structural stability to networks were revived in the context of controlling the dynamics of the states of the nodes [25] and the edges in a complex network.

Although in this paper we do not present a theory behind our findings, we argue that we have an understanding of the behaviour of our approach in the sense that the model we investigate behaves in many ways similarly to spin glasses. This is so, even though the process we study (optimizing for the structure of the otherwise constant values associated with the nodes) represents a qualitatively new interpretation of the spin-glass-like Hamiltonians with randomly chosen cooperative (ferromagnetic) and antagonistic (antiferromagnetic) interactions. Spin glasses can be understood on two levels, one being a qualitative understanding of the appearance of an extremely complex free energy landscape, in an analogy of our efficiency landscape. The other, truly theoretical, approach involves the notion of replica symmetry breaking (see, e.g., [11]), which is a very complex theoretical framework and works only for traditional spin glasses, being both analogous, but rather different, systems from the one we investigate. The extra complexity in our approach is due to the optimization for the network structure and the fact that we consider directed interactions (both missing from the assumptions of the systems for which the replica symmetry breaking formalism can be applied). In spite of the differences, however, as we also point out in [10], our system exhibits some of the essential qualitative features of spin glasses.

In conclusion, our two main results regarding organisations complement each other, i.e., (i) a hierarchical structure is in most cases more optimal than a non-hierarchical one and (ii) a higher level of hierarchy—again, in most cases—results in a more stable system. The above statements follow from our model, but are in good agreement with many observations regarding our everyday experience (see, e.g., [27]).

Acknowledgments

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