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A new characteristic of switching topology and synchronization of linear multi-agent systems

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Abstract—**The concept of synchronizability exponent is proposed to define a quantitative characteristic of switching topology. Meanwhile, the Lyapunov exponent is introduced as a quantitative characteristic of agent dynamics. In terms of the two exponents, we present a quantitative synchronization condition that the Lyapunov exponent of agent dynamics is less than the synchronizability exponent of switching topology. This condition describes the relationship between agent dynamics and switching topology, which does not seem to have been explicitly considered so far. Under the new quantitative condition, we design state feedback and output feedback protocols to achieve synchronization of linear multi-agent systems. We further show that the output feedback protocol can be constructed by separately designing the state feedback protocol and the state observers, namely the separation principle holds.**

Index Terms—**Linear multi-agent systems, synchronization, synchronizability exponent, Lyapunov exponent, infinite matrix product**

I. Introduction

In a multi-agent system, all agents interact over a network and the topology of the network is described using graph theory. In the early works on multi-agent systems, only fixed topology is considered. In [1] switching topology was first explicitly introduced to describe the time-varying interaction relationship in the Vicsek model. Switching topology may be more practical, but it makes difficult the study of multi-agent systems.

In a multi-agent system with switching topology, we can see two opposing tendencies as the topology varies with time. One tendency is that the agents that are linked together interact and coordinate with each other for some collective goal. The other is that the agents that are separated from the others evolve independently of and may deviate from the others. To achieve synchronization of a multi-agent system with switching topology, conditions are needed to ensure the first tendency to dominate the second.

First of all, joint connectivity condition is necessary for switching topology. Under this condition, all agents can be "linked together" across the time intervals of some given length, though they may not be linked together at any instant. Joint connectivity condition (together with the dwell time assumption in the continuous-time case) is sufficient for consensus of single-integrator multi-agent systems [1], [2], [3], [4], [5], [6], [7]. While for synchronization of general linear multi-agent systems, these conditions are no longer sufficient. Further conditions are necessary for network topology or agent dynamics.

When an agent is isolated from the others for a period of time under a switching topology, it will evolve without interacting with the others. The speeds of the isolated agents have a strong effect on synchronization. Intuitively, if an isolated agent moves too fast, it may deviate significantly from the others during the time when isolated and destroy synchronization. Under a switching topology, at any time there may be some agents isolated. In order to achieve synchronization under switching topology, it is natural to impose conditions to restrict the speed of isolated agents. In [8], [9], [10], [11], [12], the dynamic matrix of agents is, or is assumed to be, neutrally stable. A less restrictive condition is given in [13], [14], [15] where the eigenvalues of the dynamic matrix of agents are assumed to be in the closed left half plane. In effect, these conditions restrict the isolated agents to move at most at a polynomial rate.

When agents fail to satisfy the above eigenvalue conditions, the conditions stronger than the joint connectivity are imposed on network topology in the literature. In [16], [17], [18], [19], [20], [21], [22], [23], [24], the network topologies are assumed to be both fixed and connected, where all agents coordinate with their fixed neighbors constantly. In [25] the network topology is allowed to be switching but required to be frequently connected. That is, although it may not always be connected, it must be connected at intervals of some length. If the network topology is allowed to be switching and is just jointly connected, it seems difficult to synchronize the agents which move at an exponential rate when isolated.

It should be noted that in the cited papers, the synchronization conditions are discussed separately for network topology and agent dynamics, and no conditions are concerned with the relationship between them. However, an interesting tradeoff can be observed by examining these conditions: when the network topologies are jointly connected, the agent dynamics need to satisfy some eigenvalue conditions; when the network topologies are connected or frequently connected, the eigenvalue conditions are not necessary. That is, the conditions on the network topology are not irrelevant to the conditions on the agent dynamics, though they are stated separately. There must be some relationship between network topology and agent dynamics.

Because network topology and agent dynamics belong to different areas of mathematics and their mathematical natures

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are totally different, it is difficult to combine them together and describe their relationship. In this paper, we will give a quantitative description of such a relationship. We associate each switching topology with a sequence of orthogonal projectors and look at the switching topology from the viewpoint of infinite product of the sequence. A convergence characteristic of the infinite product is named as the synchronizability exponent of the switching topology. It defines a quantitative characteristic of switching topology that differs from those topological properties such as joint connectivity. Besides, the Lyapunov exponent is introduced to quantify the speeds of isolated agents. The synchronizability exponent and Lyapunov exponent can be viewed respectively as measures of the synchronizing tendency and the desynchronizing tendency in a multi-agent system. Thus, the dominance of synchronizing tendency over desynchronizing can be described by the quantitative relation that the synchronizability exponent is greater than the Lyapunov exponent. Under this quantitative relation and the standard joint connectivity assumption, state feedback protocols are designed to synchronize the agents. If agents are in addition observable, output feedback protocols can achieve synchronization as well. Furthermore, it can be shown that the separation principle holds for synchronization of linear multi-agent systems. Namely, the output feedback protocol can be constructed by separately designing the state feedback protocol and the state observers.

The rest of the paper is organized as follows. In Section II, we define synchronizability exponent for switching topology and introduce the Lyapunov exponent for agent dynamics. In terms of the two exponents, Theorem 1 and 2 are stated. Section III is devoted to the proof of Theorem 1. We relate synchronization of linear multi-agent systems to the convergence of an infinite matrix product, then prove the theorem using infinite matrix product method. In Section IV, we give the proof of Theorem 2. A rigorous proof of the separation principle for output feedback protocol design is given as well. The further discussions are presented in Section V. In Section VI, a simulation example is presented to illustrate the main results. Section VII concludes the paper.

II. Preliminaries and main results

A. Algebraic graph theory

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be an undirected graph of order N, where $\mathcal{V} = \{1, 2, \ldots, N\}$ is the node set, $\mathcal{E} \subset \{\{i, j\} : i \neq j, i, j \in \mathcal{V}\}$ is the edge set. A path joining nodes i and j is an edge sequence of the form $\{i, i_1\}, \{i_1, i_2\}, \ldots, \{i_{d-1}, i_d\}, \{i_d, j\}$. Graph $\mathcal{G}$ is said to be connected if any two distinct nodes can be joined by a path. The adjacency matrix $\mathcal{A}(\mathcal{G}) = (a_{ij})$ of graph $\mathcal{G}$ is an $N \times N$ matrix, where $a_{ij} = 1$ if $\{i, j\} \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. The Laplacian $\mathcal{L}(\mathcal{G}) = (\ell_{ij}) \in \mathbb{R}^{N \times N}$ is defined as $\ell_{ii} = \sum_{i=1}^{N} a_{ij}$ and $\ell_{ij} = -a_{ij}$ for $i \neq j$.

The Laplacian is important in algebraic graph theory, which provides us a powerful tool in the study of graphs. We consider the null space of the Laplacian and introduce the orthogonal projector onto it. For a graph $\mathcal{G}$, we let $\mathcal{N}_{\mathcal{G}}$ be the null space of the Laplacian $\mathcal{L}(\mathcal{G})$, namely

$$\mathcal{N}_{\mathcal{G}} = \{x \in \mathbb{R}^N : \mathcal{L}(\mathcal{G})x = 0\}$$

By $P_{\mathcal{G}}$ we denote the orthogonal projector onto $\mathcal{N}_{\mathcal{G}}$. If $\mathcal{N}_{\mathcal{G}}$ has an orthonormal basis Q, then $P_{\mathcal{G}} = QQ^{\mathrm{T}}$. When $\mathcal{G}$ is connected, then $\mathcal{N}_{\mathcal{G}} = \mathrm{span}\{\mathbf{1}\}$. In this case, $P_{\mathcal{G}}$ is abbreviated as P, namely, $P = \frac{1}{n}\mathbf{1}\mathbf{1}^{\mathrm{T}}$.

From the definition of the Laplacian, we have $\mathcal{L}(\mathcal{G})\mathbf{1} = 0$ or $\mathrm{span}\{\mathbf{1}\} \subset \mathcal{N}_{\mathcal{G}}$. So, it is easy to verify that $P_{\mathcal{G}} - P$ is the orthogonal projector onto the orthogonal complement of $\mathbf{1}$ in $\mathcal{N}_{\mathcal{G}}$. In our study, the graph associated orthogonal projector $P_{\mathcal{G}} - P$ plays an important role. Our analysis of switching topology and the proofs of the main results are performed based on the related orthogonal projectors.

The union of a set of graphs $\{\mathcal{G}_\lambda = (\mathcal{V}, \mathcal{E}_\lambda) : \lambda \in \Lambda\}$ is defined and denoted by $\bigcup_{\lambda \in \Lambda} \mathcal{G}_\lambda = \left(\mathcal{E}, \bigcup_{\lambda \in \Lambda} \mathcal{G}_\lambda\right)$. A set of graphs is said to be jointly connected if its union is connected. A lemma in [1] says that if a graph set $\{\mathcal{G}_1, \ldots, \mathcal{G}_r\}$ is jointly connected, then

$$\bigcap_{i=1}^{r} \mathcal{N}_{\mathcal{G}_i} = \mathrm{span}\{\mathbf{1}\}$$

In terms of the associated orthogonal projectors, we give a new property of a jointly connected set of graphs.

Lemma 1. *For any positive integers r and N there exists a number $0 < \delta(N, r) < 1$ for which if a set $\{\mathcal{G}_1, \ldots, \mathcal{G}_r\}$ of r graphs of order N is jointly connected, then*

$$\|(P_{\mathcal{G}_r} - P) \cdots (P_{\mathcal{G}_1} - P)\| \leq \delta(N, r).$$

The proof is given in Appendix A.

B. Synchronizability exponent and Lyapunov exponent

Assume that $\{\mathcal{G}_p : p \in \mathcal{P}\}$ is the collection of all possible undirected graphs of order N and is parameterized by index set $\mathcal{P}$. A switching topology is defined as $\mathcal{G}(t) = \mathcal{G}_{\sigma(t)}$, where $\sigma : [0, \infty) \to \mathcal{P}$ is a right continuous piecewise constant function. The discontinuity points of σ, $t_0 = 0 < t_1 < t_2 < \ldots$, are named the switching instants of $\mathcal{G}(t)$. Letting $\tau_k = t_{k+1} - t_k$, we call $\tau = \inf\{\tau_k : k = 0, 1, 2, \ldots\}$ the dwell time of $\mathcal{G}(t)$.

Consider a group of N identical agents, labeled 1 through N, described by the following dynamics

$$\begin{aligned} \dot{x}_i &= Ax_i + Bu_i \\ y_i &= Cx_i, i = 1, \ldots, N \end{aligned} \tag{1}$$

where $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$, and $y_i \in \mathbb{R}^r$ are the ith agent's state, input, and output, respectively, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, and $C \in \mathbb{R}^{r \times n}$ are constant matrices. A switching topology $\mathcal{G}(t) = \mathcal{G}_{\sigma(t)}$ describes the network topology of the N agents: only when $\{i, j\}$ is an edge of $\mathcal{G}(t)$ can agent i and j interact with each other at time t .

In the literature, the description of switching topology is mainly concerned with the joint connectivity property. In this paper, we will present the concept of synchronizability exponent to describe a new characteristic of switching topology.

As described in the previous subsection, at each switching instant t_k the switching topology $\mathcal{G}(t)$ determines an orthogonal projector $P_{\mathcal{G}(t_k)} - P$, which is abbreviated as $P(t_k) - P$ hereafter. Therefore, we get a sequence of orthogonal projectors $\{P(t_k) - P\}_{k=0}^{\infty}$ and then an infinite product of the sequence. Infinite matrix product is an important topic in

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matrix theory and many convergence results can be found in the literature (see, for example, [26], [27], [28]). By a theorem in [26], the infinite product of $\{P(t_k)-P\}_{k=0}^{\infty}$ can be easily shown to converge to an orthogonal projector. However, what we are interested here is not such common convergence property. We view the projector sequence as a matrix-valued function of switching instant t_k and consider a quantitative characteristic of the infinite product—its convergence rate with respect to t_k. We call it the synchronizability exponent of $\mathcal{G}(t)$.

Definition 1. *Assume that switching topology $\mathcal{G}(t)$ has switching instants $\{t_k\}_{k=0}^{\infty}$. For any positive integer q, define*

$$\rho(q) = -\varlimsup_{k\to\infty} \frac{\ln\|(P(t_{k+q-1})-P)\cdots(P(t_{k+1})-P)(P(t_k)-P)\|}{t_{k+q}-t_k} \tag{2}$$

where, by convention, $\ln 0=-\infty$. We call

$$\rho = \sup\{\rho(q): q=1,2,\ldots\}$$

the synchronizability exponent of $\mathcal{G}(t)$.

A switching topology $\mathcal{G}(t)$, as a piecewise constant graph-valued function, is determined by two distinct sequences, i.e., the graph sequence $\{\mathcal{G}(t_k)\}_{k=0}^{\infty}$ and the switching instant sequence $\{t_k\}_{k=0}^{\infty}$. In the literature, only the graph sequence $\{\mathcal{G}(t_k)\}_{k=0}^{\infty}$ is considered in the descriptions of switching topology. In contrast, the switching instant sequence $\{t_k\}_{k=0}^{\infty}$ is also taken into account in the definition of synchronizability exponent.

Proposition 1.
1) *If there exists a positive integer M such that among any consecutive M switching instants there exists at least one at which $\mathcal{G}(t)$ is connected, then the synchronization exponent $\rho=+\infty$. In particular, $\rho=+\infty$ if $\mathcal{G}(t)$ is fixed and connected.*
2) *If there exist $T>0, \tau>0$ such that the dwell time of $\mathcal{G}(t)$ is not less than τ and $\mathcal{G}[t,t+T)=\bigcup_{t\le s<t+T}\mathcal{G}(s)$ is connected for any $t\ge 0$, then $\rho>0$.*
3) *Let $\tilde{\rho}$ denote the synchronizability exponent of $\tilde{\mathcal{G}}(t)$. If $\tilde{\mathcal{G}}(t)=\mathcal{G}(\kappa t)$ for some $\kappa>0$, then $\tilde{\rho}=\kappa\rho$.*

The second item of the proposition shows that the synchronizability exponent is positive under the joint connectivity condition. The last item shows that the synchronizability exponent of $\mathcal{G}(t)$ varies with the switching frequency of $\mathcal{G}(t)$. The proof of Proposition 1 is given in Appendix B.

Remark 1. *Assume that there exists a positive integer R for which the union of any R consecutive graphs $\mathcal{G}(t_k)$ is connected, and assume that there exists $\delta>0$ for which if $\mathcal{G}(t_k)$ is disconnected then $t_{k+1}-t_k\le\delta$. Then*

$$\rho \ge -\frac{\ln\bar{\varepsilon}}{\delta R} > 0$$

where $\bar{\varepsilon}=\sup\{\|(P(t_{k+R-1})-P)\cdots(P(t_k)-P)\|: k\ge 0\}$ and $\bar{\varepsilon}<1$ form Lemma 1.

Remark 2. *The joint spectral radius of a finite set Σ of matrices is defined as [27]*

$$\rho(\Sigma) = \varlimsup_{t\to\infty}\big(\max_{M_1,\ldots,M_t\in\Sigma}\|M_t\cdots M_1\|^{\frac{1}{t}}\big)$$

It is often used in the study of infinite products of the form $\prod_{i=1}^{\infty}A_i$, where the factors A_i are arbitrarily drawn from Σ. In $\prod_{i=1}^{\infty}(P(t_i)-P)$, the factors $P(t_i)-P$ are not arbitrarily drawn from the orthogonal projector set—they are subject to the restriction that $\mathcal{G}(t)$ is jointly connected, hence the joint spectral radius is not a suitable tool here.

Consider a continuous-time linear system

$$\dot{x} = A(t)x$$

and let $x(t,x_0)$ be the solution with initial state x_0. The Lyapunov exponent of solution $x(t,x_0)$ is defined and denoted by

$$\bar{\chi}(x_0) = \varlimsup_{t\to\infty}\frac{\ln\|x(t,x_0)\|}{t}$$

If $A(t)$ is continuous and bounded, there are a finite number of possible Lyapunov exponents and in the case when $A(t)=A$ is time-invariant, the Lyapunov exponents are actually the real parts of A's eigenvalues [29].

When an agent of (1) is isolated, it will evolve according to its free dynamics

$$\dot{x}_i = Ax_i \tag{3}$$

The Lyapunov exponent of an agent is defined as the largest Lyapunov exponent of its free dynamics, which specifies the largest exponential rate at which the isolated agents move. As mentioned above, the Lyapunov exponent of agents is equal to the largest real part of the eigenvalues of A.

C. Statement of main results

We will consider synchronization of multi-agent system (1). The objective is to give new synchronization conditions and to design state feedback and output feedback protocols to exponentially synchronize the N agents. We will prove the following two theorems in the present paper.

Theorem 1. *Assume that*
1) *the pair (A,B) is controllable;*
2) *there exist $\tau>0, T>0$ for which the dwell time of $\mathcal{G}(t)$ is not less than τ and $\mathcal{G}[t,t+T)$ is connected for any $t\ge 0$.*

If the agents have Lyapunov exponent χ and the switching topology $\mathcal{G}(t)$ has synchronizability exponent ρ and they satisfy

$$\chi < \rho \tag{4}$$

then state feedback protocols can be designed to synchronize all the solutions of (1). In detail, for any β satisfying $0<\beta<\rho-\chi$, we can design state feedback protocols of the form

$$u_i = -\sum_{j=1}^{N} a_{ij}(t)\,K(x_i-x_j)$$

where $(a_{ij}(t))$ are the adjacency matrix of $\mathcal{G}(t)$, under which the solutions of (1) satisfy

$$\|x_i(t)-x_j(t)\| \le Ce^{-\beta t}\|x(0)\|,\ i,j=1,\ldots,N \tag{5}$$

where $x(0)=(x_1^{\mathrm{T}}(0),\ldots,x_N^{\mathrm{T}}(0))^{\mathrm{T}}$ and $x_i(0), i=1,\ldots,N$ are the initial states, $C>0$ is independent of t.

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Theorem 2. *In addition to the assumptions and condition of Theorem 1, if (A,C) is observable, for any β satisfying $0<\beta<\rho-\chi$ we can design observer-based output feedback protocols of the form*

$$\dot{\hat{x}}_i = A\hat{x}_i + Bu_i + L(y_i - C\hat{x}_i)$$
$$u_i = -\sum_{j=1}^{N} a_{ij}(t)K(\hat{x}_i - \hat{x}_j), i=1,\dots,N$$

to synchronize the solution of (1) in the sense of (5).

We can further show that if β in (5) is not specified in advance, the output synchronization protocols can be designed using the separation principle. Namely, the output feedback protocols can be constructed by separately designing the state feedback protocols and the state observers of agents.

Remark 3. *The quantitative condition (4) reveals a fundamental relation between agent dynamics and underlying topology. Moreover, it is also an essential eigenvalue condition. In the literature, linear agents are not allowed to have exponentially unstable eigenvalues when switching topology is jointly connected. By contrast, condition (4) allows linear agents to have exponentially unstable eigenvalues as long as their real parts are less than $\rho>0$. This makes Theorem 1 and 2 significantly different from the existing results. Another point worth stressing is that in the two theorems the exponential synchronization rate is explicitly given, which relies on the gap between the two exponents.*

III. Proof of Theorem 1

In the literature on linear multi-agent systems, two methods, namely the Lyapunov function method and infinite matrix product method, are widely used. Infinite matrix product method is a powerful tool for the consensus of single integrator multi-agent systems. This is because the states of coupled integrators can be expressed in the form of the nonnegative matrix product. However, this method is seldom used in general linear multi-agent systems. Most of the results for general multi-agents systems are obtained by the Lyapunov function method, which often involves complicated non-smooth analysis. In this section, we will develop a new technique for using infinite matrix product method to prove Theorem 1. For this purpose, we need to establish the relationship between synchronization and infinite matrix product.

A. Synchronization and infinite matrix product

We consider a state feedback protocol of the general form

$$u_i = -\sum_{j=1}^{N} a_{ij}(t)\,K(x_i - x_j) \tag{6}$$

where $\mathcal{A}(\mathcal{G}(t)) = (a_{ij}(t))$ is the adjacency matrix of $\mathcal{G}(t)$, K is the control gain matrix. Applying the protocol to (1) and denoting $x=(x_1^{\mathrm{T}},\dots,x_N^{\mathrm{T}})^{\mathrm{T}}$, we obtain

$$\dot{x} = \big[I_N\otimes A - \mathcal{L}(t)\otimes BK\big]x \tag{7}$$

where $\mathcal{L}(t)=\mathcal{L}(\mathcal{G}(t))$ is the Laplacian of $\mathcal{G}(t)$. Assume that $0=t_0<t_1\ldots<t_i<\ldots$ are the switching instants of $\mathcal{G}(t)$. The coupled system (7) can be further rewritten in the form

$$\dot{x} = \big[I_N\otimes A - \mathcal{L}(t_i)\otimes BK\big]x,\ t_i\le t<t_{i+1}, i=0,1,\ldots \tag{8}$$

Assume that the Laplacian $\mathcal{L}(t_i)$ has rank r_i. We denote the eigenvalues of $\mathcal{L}(t_i)$ by

$$\lambda_1^{(i)}\ge\ldots\ge\lambda_{r_i}^{(i)}>\lambda_{r_i+1}^{(i)}=\ldots=\lambda_N^{(i)}=0$$

and the corresponding orthonormal eigenvectors by

$$\xi_1^{(i)},\ldots,\xi_{r_i}^{(i)},\xi_{r_i+1}^{(i)},\ldots,\xi_N^{(i)}$$

Let

$$G_i=(\xi_1^{(i)},\ldots,\xi_{r_i}^{(i)}), H_i=(\xi_{r_i+1}^{(i)},\ldots,\xi_N^{(i)})$$

and define an orthogonal transformation for $t_i\le t<t_{i+1}$,

$$x=(G_i,H_i)\otimes I_n\cdot\begin{pmatrix} z_1\\ z_2\end{pmatrix} \tag{9}$$

where $z_1\in\mathbb{R}^{r_i n}, z_2\in\mathbb{R}^{(N-r_i)n}$. In the new coordinate, system (8) reads as

$$\begin{aligned}\dot{z}_1 &= \big[I_{r_i}\otimes A - \Lambda_i\otimes(BK)\big]z_1\\ \dot{z}_2 &= \big[I_{N-r_i}\otimes A\big]z_2,\ t_i\le t<t_{i+1}, i=0,1,\ldots\end{aligned} \tag{10}$$

where $\Lambda_i=\mathrm{diag}\big(\lambda_1^{(i)},\ldots,\lambda_{r_i}^{(i)}\big)$. This means that through the graph-related state transformation, system (8) is decomposed into two components in the interval $[t_i,t_{i+1})$: z_1-component evolves in the range space of $\mathcal{L}(\mathcal{G}(t_i))\times I$ and z_2-component evolves in the null space of $\mathcal{L}(\mathcal{G}(t_i))\times I$. Only z_1-component is affected by the gain matrix K.

The solution of (10) for $t_i\le t<t_{i+1}$ is

$$\begin{pmatrix} z_1(t)\\ z_2(t)\end{pmatrix}=\begin{pmatrix} e^{M_i(K)(t-t_i)} & \\ & I_{N-r_i}\otimes e^{A(t-t_i)}\end{pmatrix}\begin{pmatrix} z_1(t_i)\\ z_2(t_i)\end{pmatrix} \tag{11}$$

where

$$M_i(K)=I_{r_i}\otimes A-\Lambda_i\otimes(BK) \tag{12}$$

From (9), we obtain the solution of (8) for $t_i\le t<t_{i+1}$

$$x(t)=\big[\widetilde{G}_i\,e^{M_i(K)(t-t_i)}\widetilde{G}_i^{\mathrm{T}}+P(t_i)\otimes e^{A(t-t_i)}\big]x(t_i) \tag{13}$$

where

$$\widetilde{G}_i=G_i\otimes I_n, P(t_i)=H_iH_i^{\mathrm{T}}$$

Letting $t\to t_{i+1}$ and by the continuity of the state trajectories, we then get

$$x(t_{i+1})=\big[\widetilde{G}_i\,e^{M_i(K)\tau_i}\widetilde{G}_i^{\mathrm{T}}+P(t_i)\otimes e^{A\tau_i}\big]x(t_i) \tag{14}$$

Since $H_i=(\xi_{r_i+1}^{(i)},\ldots,\xi_N^{(i)})$ constitutes an orthonormal basis of the null space of $\mathcal{L}(\mathcal{G}(t_i))$, so $P(t_i)=H_iH_i^{\mathrm{T}}$ is exactly the orthogonal projector onto $\mathcal{N}_{\mathcal{G}(t_i)}$. Moreover, from $\mathcal{L}(\mathcal{G}(t_i))\mathbf{1}=0$ or $\mathbf{1}\in\mathcal{N}_{\mathcal{G}(t_i)}=\mathrm{span}(\xi_{r_i+1}^{(i)},\ldots,\xi_N^{(i)})$, it turns out that

$$P(t_i)P=PP(t_i)=P, PG_i=0 \tag{15}$$

From (13), (14), and (15), we can directly verify the following two equalities by computing the both sides

$$(I-P\otimes I)x(t_{i+1})=\Xi_i(\tau_i)(I-P\otimes I_n)x(t_i) \tag{16}$$

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and

$$x(t) - P \otimes e^{A(t-t_i)} x(t_i) = \Xi_i(t-t_i)(I - P \otimes I_n)x(t_i) \quad (17)$$

where $t_i \leq t < t_{i+1}$ and

$$\Xi_i(s) = \widetilde{G}_i \, e^{M_i(K)s} \, \widetilde{G}_i^{\mathrm{T}} + (P(t_i) - P) \otimes e^{As}$$

Regarding (16) as a recursive formula and combining it with (17), we conclude that

$$x(t) - P \otimes e^{A(t-t_k)} x(t_k) = \Xi_k(t-t_k) \prod_{i=0}^{k-1} \Xi_i(\tau_i) \, x(0) \quad (18)$$

where $k = \max\{i : t_i < t\}$. Noticing that

$$P \otimes e^{A(t-t_k)} \cdot x(t_k) = \mathbf{1} \otimes \Big(\frac{1}{N} \sum_{j=1}^{N} e^{A(t-t_k)} x_j(t_k) \Big)$$

we thus express the deviation of the individual agents' states from their average in the form of matrix product. If the product can be made to converge to 0 as $t \to \infty$, the agents' states will converge to their average and the desired synchronization result follows. Therefore, we successfully relate synchronization of (7) to the convergence of an infinite matrix product. This relationship is ubiquitous to general linear multi-agent systems, which provides us a new way to deal with synchronization of linear multi-agent systems.

In the following two subsections, we will define a parametrized gain matrix $K = K(\alpha)$ and choose α such that the matrix product converges to 0 as $t \to \infty$ at the specified rate β.

B. Technical lemmas

Let

$$\Phi_i(s) = \widetilde{G}_i e^{M_i(K)s} \widetilde{G}_i^{\mathrm{T}}, \Psi_i(s) = (P(t_i) - P) \otimes e^{As}, \quad (19)$$

and express $\Xi_i(s)$ as

$$\Xi_i(s) = \Phi_i(s) + \Psi_i(s) \quad (20)$$

where $\Phi_i(s)$ is dependent on K and $\Psi_i(s)$ is free of K. Since

$$\begin{aligned} \Xi_i^{\mathrm{T}}(s)\Xi_i(s) &= \Phi_i^{\mathrm{T}}(s)\Phi_i(s) + \Psi_i^{\mathrm{T}}(s)\Psi_i(s) \\ &\geq \Psi_i^{\mathrm{T}}(s)\Psi_i(s) = (P(t_i) - P) \otimes (e^{As})^{\mathrm{T}} \cdot e^{As} \end{aligned}$$

we have $\|\Xi_i(\tau_i)\| \geq \|e^{A\tau_i}\|$ if $\mathcal{G}(t_i)$ is disconnected. Thus, it may happen that $\|\Xi_i(\tau_i)\| \geq 1$ and even $\|\Xi_i(\tau_i)\| > 1$. So we cannot make $\|\Xi_i(\tau_i)\| < 1$ to render the infinite product of $\Xi_i(\tau_i)$ convergent. On the other hand, $\Xi_i(\tau_i)$ are not nonnegative matrices in general, so the nonnegative matrix theory is not applicable here. In what follows we will deal with the infinite product of $\Xi_i(\tau_i)$ in a new way. We first present some technical lemmas.

The second smallest eigenvalue of the Laplacian of a graph is called the algebraic connectivity of the graph [30]. A result in [31] (Theorem 4.2) shows that if an undirected graph $\mathcal{G}$ of order N is connected, its algebraic connectivity is not less than $\frac{4}{Nd}$, where d is the diameter of $\mathcal{G}$ and $d \leq N-1$. From this result, if $\mathcal{G}$ is disconnected and has components $\mathcal{G}_1, \ldots, \mathcal{G}_r$, the nonzero eigenvalues of $\mathcal{L}(\mathcal{G}_i)$, $i = 1, \ldots, r$, are not less than $\frac{4}{N(N-1)}$. Using the fact that the spectra of a disconnected graph are the union of the spectra of its components, we have:

Lemma 2. *If undirected graph $\mathcal{G}$ is of order N and λ is any nonzero eigenvalue of $\mathcal{L}(\mathcal{G})$, then*

$$\lambda \geq \frac{4}{N(N-1)} \quad (21)$$

For $\alpha \geq 0$, we define the weighted controllability Gramian

$$W(\alpha) = \int_0^{\frac{\tau}{3}} e^{-\alpha t} \cdot e^{-\frac{A}{2}t} B B^{\mathrm{T}} e^{-\frac{A^{\mathrm{T}}}{2}t} dt \quad (22)$$

where $\tau > 0$ is specified in assumption 2) of Theorem 1. From the controllability of (A, B), we have $W(\alpha) > 0$. Let

$$K(\alpha) = \frac{N(N-1)}{4} B^{\mathrm{T}} W^{-1}(\alpha) \quad (23)$$

Lemma 3. *Assume that undirected graph $\mathcal{G}$ is of order N and (A, B) is a controllable pair. Then, for any $\alpha \geq 0$,*

$$\left\| e^{(A - \lambda B K(\alpha))\, t} \right\| \leq C_0 e^{\frac{\tau}{6}\alpha} e^{-\alpha t}, \; t \geq 0 \quad (24)$$

holds uniformly for all nonzero eigenvalues λ of $\mathcal{L}(\mathcal{G})$, where $C_0 = (\lambda_1 \lambda_2^{-1})^{\frac{1}{2}}$, λ_1 and λ_2 are the maximum and minimum eigenvalue of $W(0)$ respectively.

The lemma can be viewed as a sharp version of the Squashing Lemma of [32]. The proof is given in Appendix C.

In view of Lemma 3, we define the gain matrix K of (6) as

$$K = K(\alpha) \quad (25)$$

where $\alpha \geq 0$ serves as a design parameter. Observing that the $M_i(K)$ in (12) becomes

$$M_i(K(\alpha)) = \mathrm{diag}(A - \lambda_1^{(i)} B K(\alpha), \ldots, A - \lambda_{r_i}^{(i)} B K(\alpha))$$

we have by Lemma 3

$$\begin{aligned} &\|\Phi_i(s)\| = \|e^{M_i(K(\alpha))s}\| \\ &= \max\left\{ \|e^{(A - \lambda_1^{(i)} B K(\alpha))s}\|, \ldots, \|e^{(A - \lambda_{r_i}^{(i)} B K(\alpha))s}\| \right\} \\ &\leq C_0 e^{\frac{1}{6}\alpha\tau} e^{-\alpha s}, s \geq 0 \end{aligned} \quad (26)$$

From $\tau_i \geq \tau$ it follows that for any $i \geq 0$

$$\|\Phi_i(\tau_i)\| \leq C_0 e^{-\frac{5}{6}\alpha\tau} \quad (27)$$

Furthermore, we have

Lemma 4. *Under assumption 1) and 2) of Theorem 1, for any $\gamma > 0$ and any positive integer m there is a positive number $\kappa(\gamma, m)$ such that when $\alpha \geq \kappa(\gamma, m)$,*

$$\left\| \prod_{i=0}^{m-1} \Xi_i(\tau_i) - \prod_{i=0}^{m-1} \Psi_i(\tau_i) \right\| \leq \frac{1}{2} e^{-\gamma(\tau_0 + \ldots \tau_{m-1})} \quad (28)$$

The lemma says that for any positive integer m, we can choose a suitable α for which the product of any m consecutive $\Xi_i(\tau_i)$ is arbitrarily close to that of m consecutive $\Psi_i(\tau_i)$. However, this cannot immediately lead to the proof of Theorem 1, because the chosen α is dependent on m.

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Remark 4. *It should be pointed out that if $\mathcal{G}(t)$ is restricted to vary in a set S of some specified graphs, the factor $\frac{1}{4}N(N-1)$ in (23) can be replaced by $\frac{1}{\lambda_S}$, where λ_S is the smallest of nonzero eigenvalues of all $\mathcal{L}(\mathcal{G})$, $\mathcal{G} \in S$.*

C. Proof of Theorem 1

Now we are in a position to prove Theorem 1. Let

$$\Omega(t) = \Xi_k(t-t_k)\prod_{i=0}^{k-1}\Xi_i(\tau_i) \tag{29}$$

From (18), we only need to prove that for any β satisfying $0<\beta<\rho-\chi$, there exist $C_1>0, \alpha_0>0$ such that when $\alpha \geq \alpha_0$,

$$\|\Omega(t)\| \leq C_1 e^{-\beta t},\ t\geq 0 \tag{30}$$

Proof of Theorem 1: Choose $\varepsilon>0$ satisfying $\chi+\beta+\varepsilon<\rho$. By the definition of synchronizability exponent, for real number $\chi+\beta+\varepsilon<\rho$, there exist q such that $\rho(q)>\chi+\beta+\varepsilon$. So by (2), there is an integer $\ell\geq 0$ such that for all $k\geq \ell$,

$$\frac{\ln\|(P(t_{k+q-1})-P)\cdots(P(t_k)-P)\|}{t_{k+q}-t_k} < -(\chi+\beta+\varepsilon) \tag{31}$$

or equivalently,

$$\|(P(t_{k+q-1})-P)\cdots(P(t_k)-P)\| < e^{-(\chi+\beta+\varepsilon)(t_{k+q}-t_k)} \tag{32}$$

Because adding or removing a finite number of factors does not affect the convergence of an infinite product, we assume $\ell=0$ for the sake of conciseness.

Since χ is equal to the largest real part of eigenvalues of A, we have as $t\to\infty$, $e^{-(\chi+\varepsilon)t}\|e^{At}\|\to 0$. Hence, there is a positive number T_1 such that when $t\geq T_1$,

$$\|e^{At}\| < \tfrac{1}{2}e^{(\chi+\varepsilon)t} \tag{33}$$

Let $h=[\frac{T_1}{q\tau}]+1$, $R=qh$, and let $p=[\frac{k-1}{R}]$. We decompose $\Omega(t)$ into $p+2$ blocks and express it in the form

$$\Omega(t) = \big(\Xi_k(t-t_k)\big)\Big(\prod_{i=pR}^{k-1}\Xi_i(t_i)\Big)\prod_{j=0}^{p-1}\Big(\prod_{i=jR}^{(j+1)R-1}\Xi_i(\tau_i)\Big) \tag{34}$$

We first make an estimation for $\Xi_i(s)$ for $0\leq s\leq\tau_i$, and then for each block of (34) separately.

Estimate $\Xi_i(s)$ for $0\leq s\leq \tau_i$. If $\mathcal{G}(t_i)$ is disconnected, then $\tau_i<T$ by assumption 2). Thus, for $0\leq s\leq\tau_i$ we have

$$\begin{aligned}\|\Xi_i(s)\| &\leq \|\Phi_i(s)\|+\|\Psi_i(s)\| \\ &\leq C_0 e^{\frac{\tau}{6}\alpha}e^{-\alpha s}+\|P(t_i)-P\|\|e^{As}\| \\ &\leq \big(C_0 e^{\frac{\tau}{6}\alpha}+e^{\|A\|T}e^{\alpha T}\big)e^{-\alpha s}\end{aligned}$$

If $\mathcal{G}(t_i)$ is connected, then $P(t_i)=P$ or $\Psi_i(s)=0$, and

$$\|\Xi_i(s)\| = \|\Phi_i(s)\| \leq C_0 e^{\frac{\tau}{6}\alpha}e^{-\alpha s}$$

Thus for any $i\geq 0$, it always holds that

$$\|\Xi_i(s)\| \leq \big(C_0 e^{\frac{\tau}{6}\alpha}+e^{\|A\|T}e^{\alpha T}\big)e^{-\alpha s},\ 0\leq s\leq \tau_i \tag{35}$$

Estimate the block $\Xi_k(t-t_k)$. Since $t-t_k<\tau_k$, from (35) we immediately have

$$\|\Xi(t-t_k)\| \leq \big(C_0 e^{\frac{\tau}{6}\alpha}+e^{\|A\|T}e^{\alpha T}\big)e^{-\alpha(t-t_k)} \tag{36}$$

Estimate the block $\prod_{i=pR}^{k-1}\Xi_i(t_i)$. Since

$$k-pR<(p+1)R-pR=R$$

from (35) again we obtain

$$\begin{aligned}\Big\|\prod_{i=pR}^{k-1}\Xi_i(\tau_i)\Big\| &\leq \prod_{i=pR}^{k-1}\|\Xi_i(\tau_i)\| \\ &\leq \big(C_0 e^{\frac{\tau}{6}\alpha}+e^{\|A\|T}e^{\alpha T}\big)^{k-pR}e^{-\alpha(t_k-t_{pR})} \\ &\leq \big(C_0 e^{\frac{\tau}{6}\alpha}+e^{\|A\|T}e^{\alpha T}\big)^{R-1}e^{-\alpha(t_k-t_{pR})}\end{aligned} \tag{37}$$

Estimate the blocks $\prod_{i=jR}^{(j+1)R-1}\Xi_i(\tau_i)$, $j=0,\dots,p-1$. Applying Lemma 4 with $\gamma=\beta, m=R$, we infer that there exists $\kappa(\beta,R)$ such that when $\alpha\geq\kappa(\beta,R)$,

$$\begin{aligned}\Big\|\prod_{i=jR}^{(j+1)R-1}\Xi_i(\tau_i)\Big\| &\leq \tfrac{1}{2}e^{-\beta(t_{(j+1)R}-t_{jR})}+\Big\|\prod_{i=jR}^{(j+1)R-1}\Psi_i(\tau_i)\Big\| \\ &\leq \tfrac{1}{2}e^{-\beta(t_{(j+1)R}-t_{jR})} \\ &\quad+\Big\|\prod_{i=jR}^{(j+1)R-1}(P(t_i)-P)\Big\|\|e^{A(t_{(j+1)R}-t_{jR})}\|\end{aligned} \tag{38}$$

On the one hand, from (32) it follows that

$$\begin{aligned}\Big\|\prod_{i=jR}^{(j+1)R-1}(P(t_i)-P)\Big\| &\leq \prod_{s=0}^{h-1}\Big\|\prod_{i=jR+sq}^{jR+(s+1)q-1}(P(t_i)-P)\Big\| \\ &\leq e^{-(\chi+\beta+\varepsilon)(t_{(j+1)R}-t_{jR})}\end{aligned} \tag{39}$$

On the other hand, since $t_{(j+1)R}-t_{jR}\geq R\tau=qh\tau>T_1$, it follows from (33) that

$$\|e^{A(t_{(j+1)R}-t_{jR})}\| < \tfrac{1}{2}e^{(\chi+\varepsilon)(t_{(j+1)R}-t_{jR})} \tag{40}$$

Substituting (39) and (40) into (38), we obtain

$$\Big\|\prod_{i=jR}^{(j+1)R-1}\Xi_i(\tau_i)\Big\| \leq e^{-\beta(t_{(j+1)R}-t_{jR})},\ j=0,\dots,p-1 \tag{41}$$

Combining the estimations (36), (37), and (41) and choosing $\alpha_0=\max\{\beta,\kappa(\beta,R)\}$, we conclude that when $\alpha\geq\alpha_0$

$$\begin{aligned}\|\Omega(t)\| \leq& \|\Xi_k(t-t_k)\|\Big\|\prod_{i=pR}^{k-1}\Xi_i(\tau_i)\Big\|\cdot\prod_{j=0}^{p-1}\Big\|\prod_{i=jR}^{(j+1)R-1}\Xi_i(\tau_i)\Big\| \\ \leq& C_1 e^{-\beta t},\ t\geq 0\end{aligned}$$

where $C_1=\big(C_0 e^{\frac{\tau}{6}\alpha}+e^{\|A\|T}e^{\alpha T}\big)^R$. The proof is completed. ■

IV. Proof of Theorem 2

A. Synchronization and infinite matrix product

As in the state feedback protocol case, we first establish the relationship between synchronization and infinite matrix product in the output feedback protocol case.

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Consider an observer-based feedback output protocol of the general form

$$\begin{aligned}\dot{\hat{x}}_i &= A\hat{x}_i + Bu_i + L(y_i - C\hat{x}_i)\\ u_i &= -\sum_{j=1}^{N} a_{ij}(t)K(\hat{x}_i - \hat{x}_j), i = 1, \ldots, N\end{aligned} \tag{42}$$

Using the notation defined in the proof of Theorem 1 and letting

$$\hat{x} = (\hat{x}_1^{\mathrm{T}}, \ldots, \hat{x}_N^{\mathrm{T}})^{\mathrm{T}}$$

we express the coupled system in the form

$$\begin{aligned}\dot{x} =&[I_N \otimes A]x - [\mathcal{L}(t) \otimes BK]\hat{x}\\ \dot{\hat{x}} =&[I_N \otimes LC]x\\ &+ [I_N \otimes (A - LC) - \mathcal{L}(t) \otimes BK]\hat{x}\end{aligned} \tag{43}$$

Letting $\tilde{x} = \hat{x} - x$, we get

$$\begin{aligned}\dot{x} =&[I_N \otimes A - \mathcal{L}(t) \otimes BK]x + [\mathcal{L}(t) \otimes BK]\tilde{x}\\ \dot{\tilde{x}} =&[I_N \otimes (A - LC)]\tilde{x}\end{aligned} \tag{44}$$

For $t_i \le t < t_{i+1}$, applying the graph based transformation (9) on x gives

$$\begin{aligned}\dot{z}_1 =&M_i(K)z_1 + \Lambda_i G_i^{\mathrm{T}} \otimes (BK)\tilde{x}\\ \dot{z}_2 =&[I_{N-r_i} \otimes A]z_2\\ \dot{\tilde{x}} =&[I_N \otimes (A - LC)]\tilde{x},\ t_i \le t < t_{i+1}, i = 0, 1, \ldots\end{aligned} \tag{45}$$

The solution of (45) for $t_i \le t < t_{i+1}$ is

$$\begin{aligned}z_1(t) &= e^{M_i(K)(t-t_i)}z_1(t_i) + \Theta_i(t - t_i)\tilde{x}(t_i)\\ z_2(t) &= I_{N-r_i} \otimes e^{A(t-t_i)}z_2(t_i)\\ \tilde{x}(t) &= I_N \otimes e^{(A-LC)(t-t_i)}\tilde{x}(t_i)\end{aligned} \tag{46}$$

where

$$\begin{aligned}\Theta_i(s) = \int_{t_i}^{t_i+s} &\big[e^{M_i(K)(t_i+s-\lambda)}\\ &\cdot (\Lambda_i G_i^{\mathrm{T}}) \otimes (BKe^{(A-LC)(\lambda-t_i)})\big]d\lambda\end{aligned}$$

From the transformation (9), the solution of (44) for $t_i \le t < t_{i+1}$ has the form

$$\begin{pmatrix}x(t)\\ \tilde{x}(t)\end{pmatrix} = \begin{pmatrix}\widetilde{G}_i e^{M_i(K)(t-t_i)}\widetilde{G}_i^{\mathrm{T}} + P(t_i) \otimes e^{A(t-t_i)} & \widetilde{G}_i\Theta_i(t - t_i)\\ 0 & I_N \otimes e^{(A-LC)(t-t_i)}\end{pmatrix}\begin{pmatrix}x(t_i)\\ \tilde{x}(t_i)\end{pmatrix}$$

Let

$$\tilde{\Phi}_i(s) = \begin{pmatrix}\Phi_i(s) & \widetilde{G}_i\Theta_i(s)\\ 0 & I_N \otimes e^{(A-LC)s}\end{pmatrix}$$

$$\tilde{\Psi}_i(s) = \begin{pmatrix}\Psi_i(s) & 0\\ 0 & 0\end{pmatrix}$$

where $\Phi_i(s), \Psi_i(s)$ is defined in (19). Then

$$\tilde{\Xi}_i(s) = \begin{pmatrix}\widetilde{G}_i e^{M_i(K)s}\widetilde{G}_i^{\mathrm{T}} + P(t_i) \otimes e^{As} & \widetilde{G}_i\Theta_i s\\ 0 & I_N \otimes e^{(A-LC)s}\end{pmatrix} = \tilde{\Phi}_i(s) + \tilde{\Psi}_i(s)$$

As in (16) and (17), the following two equations can be directly verified

$$\begin{aligned}&\begin{pmatrix}I - P \otimes I & 0\\ 0 & I\end{pmatrix}\begin{pmatrix}x(t_{i+1})\\ \tilde{x}(t_{i+1})\end{pmatrix}\\ &\quad =\tilde{\Xi}_i(\tau_i)\begin{pmatrix}I - P \otimes I & 0\\ 0 & I\end{pmatrix}\begin{pmatrix}x(t_i)\\ \tilde{x}(t_i)\end{pmatrix}\end{aligned}$$

and

$$\begin{aligned}&\begin{pmatrix}x(t)\\ \tilde{x}(t)\end{pmatrix} - \begin{pmatrix}P \otimes e^{A(t-t_i)}x(t_i)\\ 0\end{pmatrix}\\ &\quad = \tilde{\Xi}_i(t - t_i)\begin{pmatrix}I - P \otimes I & 0\\ 0 & I\end{pmatrix}\begin{pmatrix}x(t_i)\\ \tilde{x}(t_i)\end{pmatrix}\end{aligned}$$

where $t_k \le t < t_{k+1}$. Letting $k = \max\{i, t_i < t\}$, we can obtain the relation

$$\begin{aligned}&\begin{pmatrix}x(t) - P \otimes e^{A(t-t_i)}x(t_i)\\ \tilde{x}(t)\end{pmatrix}\\ &\quad = \tilde{\Xi}_k(t - t_k)\prod_{i=0}^{k-1}\tilde{\Xi}_i(\tau_i)\begin{pmatrix}x(0)\\ \tilde{x}(0)\end{pmatrix}\end{aligned} \tag{47}$$

If the matrix product in (47) can be made to converge to 0 as $t \to \infty$, all agents' states will synchronize to their average and the output feedback synchronization result follows.

B. Proof of Theorem 2

We will define a parametrized feedback gain $K = K(\alpha)$ and a parameterized observer gain $L = L(\tilde{\alpha})$ and choose $\alpha, \tilde{\alpha}$ such that the matrix product converges to 0 as $t \to \infty$ at the specified rate β.

Proof of Theorem 2: Let

$$\tilde{W}(\tilde{\alpha}) = \int_0^{\frac{\tau}{3}} e^{-\tilde{\alpha}t}e^{-\frac{A^{\mathrm{T}}}{2}t}C^{\mathrm{T}}Ce^{-\frac{A}{2}t}dt$$

where $\tau > 0$ is specified in assumption 2) of Theorem 1, and let

$$L(\tilde{\alpha}) = \tilde{W}(\tilde{\alpha})^{-1}C^{\mathrm{T}} \tag{48}$$

The definition of $L(\tilde{\alpha})$ is meaningful due to the observability of (A, C). Repeating the argument for Lemma 3, we can prove that for any $\tilde{\alpha} \ge 0$,

$$\|e^{(A-L(\tilde{\alpha})C)t}\| \le \widetilde{C}_0 e^{\frac{1}{6}\tilde{\alpha}\tau}e^{-\tilde{\alpha}t}, t \ge 0 \tag{49}$$

where $\widetilde{C}_0 = (\tilde{\lambda}_1\tilde{\lambda}_2^{-1})^{\frac{1}{2}}$, $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ are respectively the maximum and minimum eigenvalue of $\tilde{W}(0)$.

The parameterized feedback gain and the observer gain in (42) are respectively defined as

$$K = K(\alpha),\ L = L(\tilde{\alpha}) \tag{50}$$

where $K(\alpha)$ is defined in (23)

As for the proof of Theorem 1, we will derive the inequalities for $\tilde{\Phi}(s)$, $\tilde{\Phi}(\tau_i)$, and $\prod_{i=0}^{m-1}\tilde{\Xi}(\tau_i)$ as in Subsection III-B for $\Phi(s)$, $\Phi(\tau_i)$, and $\prod_{i=0}^{m-1}\Xi(\tau_i)$.

By a norm-compression inequality [33], it follows that

$$\begin{aligned}\|\tilde{\Phi}_i(s)\| &= \left\|\begin{pmatrix}\|\widetilde{G}_i e^{M_i(K(\alpha))s}\widetilde{G}_i^{\mathrm{T}}\| & \|\widetilde{G}_i\Theta_i(s)\|\\ 0 & \|I_N \otimes e^{(A-L(\tilde{\alpha})C)s}\|\end{pmatrix}\right\|\\ &\le \|e^{M_i(K(\alpha))s}\| + \|\Theta_i(s)\| + \|e^{(A-L(\tilde{\alpha})C)s}\|\end{aligned} \tag{51}$$

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Estimating $\Theta_i(s)$ by inequalities (26) and (49) yields

$$\begin{aligned}\|\Theta_i(s)\| \leq &\int_{t_i}^{t_i+s} \left[\|e^{M_i(K(\alpha))(t_i+s-\lambda)}\|\cdot\right.\\ &\left.\|\Lambda_i\|\|BK(\alpha)\|\|e^{(A-L(\tilde{\alpha})C)(\lambda-t_i)}\|\right]d\lambda\\ \leq &\|\Lambda_i\|\|B\|\|K(\alpha)\|C_0\tilde{C}_0 e^{\frac{\tau\alpha}{6}} e^{\frac{\tau\tilde{\alpha}}{6}}\cdot\\ &\int_{t_i}^{t_i+s} e^{-\alpha(t_i+s-\lambda)-\tilde{\alpha}(\lambda-t_i)}d\lambda\end{aligned} \tag{52}$$

Since the eigenvalues of the Laplacian $\mathcal{L}(\mathcal{G})$ are not greater than $2(N-1)$, we have $\|\Lambda_i\| \leq 2(N-1)$. Furthermore, since $\|W^{-1}(\alpha)\| \leq \lambda_2^{-1} e^{\frac{1}{3}\alpha\tau}$ (see (73) in Appendix C), we obtain

$$\begin{aligned}\|K(\alpha)\| =&\left\|\tfrac{1}{4}N(N-1)B^{\mathrm{T}}W^{-1}(\alpha)\right\|\\ \leq&\tfrac{1}{4}N(N-1)\|B^{\mathrm{T}}\|\lambda_2^{-1}e^{\frac{1}{3}\alpha\tau}\end{aligned}$$

Let $\tilde{\alpha} = \alpha + 1$. A simple computation yields that the integral in the last line of (52) is less than $e^{-\alpha s}$. Therefore, we have

$$\|\Theta_i(s)\| \leq \mu e^{\frac{2}{3}\alpha\tau} e^{-\alpha s} \tag{53}$$

where $\mu = \frac{1}{2}N(N-1)^2\lambda_2^{-1}C_0\tilde{C}_0 e^{\frac{1}{6}\tau}\|B\|^2$. Substituting (26), (49), and (53) into (51), we obtain the inequality for $\tilde{\Phi}_i(s)$

$$\|\tilde{\Phi}_i(s)\| \leq \tilde{\mu} e^{\frac{2}{3}\alpha\tau} e^{-\alpha s},\ s \geq 0 \tag{54}$$

where $\tilde{\mu} > 0$ independent of α, and the inequality for $\tilde{\Phi}_i(\tau_i)$

$$\|\tilde{\Phi}_i(\tau_i)\| \leq \tilde{\mu} e^{\frac{2}{3}\alpha\tau} e^{-\alpha\tau_i} \leq \tilde{\mu} e^{-\frac{1}{3}\alpha\tau} \tag{55}$$

From (55) we can further show, as in the proof of Lemma 4, that Lemma 4 is also true for $\prod_{i=0}^{m-1}\tilde{\Xi}(\tau_i)$ and $\prod_{i=0}^{m-1}\tilde{\Psi}(\tau_i)$.

Exactly repeating the argument in the proof of Theorem 1, we can prove that for any β satisfying $0 < \beta < \rho - \chi$ there exist $\bar{\alpha}_0 > 0$ and $\bar{C}_1 > 0$ such that when $\alpha \geq \bar{\alpha}_0, \tilde{\alpha} = \alpha + 1$

$$\left\|\tilde{\Xi}_k(t-t_k)\prod_{i=0}^{k-1}\tilde{\Xi}_i(\tau_i)\right\| < \bar{C}_1 e^{-\beta t},\ t \geq 0$$

from which the claim follows. ■

C. Separation principle

In the proof of Theorem 2, the choice of $\tilde{\alpha} = \alpha + 1$ is dependent on α. This means that the design of observers is not independent of the state feedback protocol. However, if we do not care the convergence rate, the output feedback protocol can be designed by the separation principle.

We design the following state observers for each agent

$$\dot{\hat{x}}_i = A\hat{x}_i + Bu_i + L(y_i - C\hat{x}_i), i = 1, \ldots, N \tag{56}$$

where the observer gain L is any matrix such that $A - LC$ is Hurwitz. So there exist $\hat{\alpha} > 0$ and $\hat{C} > 0$ such that

$$\|e^{(A-LC)t}\| \leq \hat{C} e^{-\hat{\alpha} t} \tag{57}$$

Combining these state observers with the state feedback protocol designed in the proof of Theorem 1, we obtain the output feedback protocol

$$\begin{aligned}\dot{\hat{x}}_i &= A\hat{x}_i + Bu_i + L(y_i - C\hat{x}_i)\\ u_i &= -\sum_{j=1}^{N} a_{ij}(t)K(\alpha)(\hat{x}_i - \hat{x}_j), i = 1, \ldots, N\end{aligned} \tag{58}$$

We will show that protocol (58) is still capable of achieving synchronization of (1).

Note that

$$\tilde{\Xi}_i(s) = \tilde{\Phi}_i(s) + \tilde{\Psi}_i(s) = \begin{pmatrix} \Xi_i(s) & \tilde{G}_i\Theta_i(s) \\ 0 & I_N \otimes e^{(A-LC)s} \end{pmatrix}$$

is an upper triangular matrix. A direct computation gives

$$\tilde{\Omega}(t) = \tilde{\Xi}_k(t-t_k)\prod_{i=0}^{k-1}\tilde{\Xi}_i(\tau_i) = \begin{pmatrix} \Omega(t) & \tilde{\Theta}(t) \\ 0 & I_N \otimes e^{(A-LC)t} \end{pmatrix}$$

where $\Omega(t)$ is defined in (29) and

$$\begin{aligned}\tilde{\Theta}(t) = &\Xi_k(t-t_k)\Xi_{k-1}(\tau_{k-1})\cdots\Xi_1(\tau_1)\Theta_0(\tau_0) +\\ &\Xi_k(t-t_k)\cdots\Xi_2(\tau_2)\Theta_1(\tau_1)\cdot I_N \otimes e^{(A-LC)t_1} +\\ &\cdots +\\ &\Xi_k(t-t_k)\Theta_{k-1}(\tau_{k-1})\cdot I_N \otimes e^{(A-LC)t_{k-1}} +\\ &+ \Theta_k(t-t_k)\cdot I_N \otimes e^{(A-LC)t_k}\end{aligned}$$

To show $\tilde{\Omega}(t)$ exponentially converges to 0 as $t \to \infty$, we only need to prove that $\tilde{\Theta}(t)$ exponentially converges to 0 as $t \to \infty$.

It is easy to check that

$$\int_{t_i}^{t_i+s} e^{-\alpha(t_i+s-\lambda)}e^{-\hat{\alpha}(\lambda-t_i)}d\lambda \leq \begin{cases} (\alpha-\hat{\alpha})^{-1}e^{-\hat{\alpha}s}, & \alpha > \hat{\alpha} \\ (\hat{\alpha}-\alpha)^{-1}e^{-\alpha s}, & \hat{\alpha} > \alpha \\ s e^{-\alpha s}, & \alpha = \hat{\alpha} \end{cases}$$

and the maximum value of $s\, e^{-\frac{1}{2}\alpha s}$ is $2(\alpha e)^{-1}$. So the integral in the last line of (52) satisfies, for any $\alpha > 0$, $\hat{\alpha} > 0$,

$$\int_{t_i}^{t_i+s} e^{-\alpha(t_i+s-\lambda)}e^{-\hat{\alpha}(\lambda-t_i)}d\lambda \leq C(\alpha,\hat{\alpha})\, e^{-\frac{1}{2}\gamma s}, s \geq 0$$

where $\gamma = \min\{\alpha, \hat{\alpha}\}$ and

$$C(\alpha,\hat{\alpha}) = \begin{cases} |\alpha-\hat{\alpha}|^{-1}, & \alpha \neq \hat{\alpha} \\ 2(\alpha e)^{-1}, & \alpha = \hat{\alpha} \end{cases}$$

Substituting this inequality into (52) leads to

$$\|\Theta_i(s)\| \leq C_2 e^{-\frac{1}{2}\gamma s}, s \geq 0 \tag{59}$$

where C_2 is a suitable number independent of s.

Repeating the same argument as for $\Omega(t)$ in Subsection III-C, we have for any $k > j \geq 0$

$$\|\Xi_k(t-t_k)\Xi_{k-1}(\tau_{k-1})\cdots\Xi_j(\tau_j)\| \leq C_1 e^{-\beta(t-t_j)} \tag{60}$$

Estimating each of the $k+1$ terms in $\tilde{\Theta}(t)$ by (57), (59), and (60 gives

$$\|\tilde{\Theta}(t)\| \leq (k+1)C_3 e^{-\tilde{\gamma}t},\ t \geq 0 \tag{61}$$

where $\tilde{\gamma} = \min\{\frac{1}{2}\gamma, \hat{\alpha}, \beta\} > 0$ and

$$C_3 = \max\{C_1C_2, C_1C_2\hat{C}, C_2\hat{C}\}$$

Since $t > t_k = \tau_0 + \cdots + \tau_{k-1} \geq k\tau$, we have $k < \frac{t}{\tau}$. Substituting it into (61), we conclude that $\tilde{\Theta}(t)$ exponentially converges to 0 as $t \to \infty$. So the product $\tilde{\Omega}(t)$ is shown to exponentially converge to 0 as $t \to \infty$. That is, the separation principle holds.

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V. Further discussions

1: Recall that by Proposition 1, a switching topology satisfying assumption 2) of Theorem 1 has synchronizability exponent $\rho > 0$. So the Lyapunov exponent satisfying the condition (4) is permitted to be positive. This means that the agents are allowed to have unstable eigenvalues. However, this is not the case for the existing results [10], [11], [12], [13], [14], [15], where if switching topology is jointly connected, the eigenvalues are not allowed to have positive real parts. To the best of our knowledge, it seems that no result is available under switching topology for synchronizing the agents with exponentially unstable dynamics.

2: Output feedback synchronization of multi-agent systems has been extensively studied and a variety of dynamic output feedback protocols were proposed [14], [15], [22], [34]. However, none of them has the form of (42). In [14], [34], the proposed output feedback protocol is designed based on the main results in [35]. It has the form

$$\begin{aligned}\dot{\eta}_i &= (A+BK)\eta_i + \sum_{j=1}^{N} a_{ij}(t)(\eta_j - \eta_i + \hat{x}_i - \hat{x}_j)\\ \dot{\hat{x}}_i &= A\hat{x}_i + Bu_i + H(C\hat{x}_i - y_i)\\ u_i &= K\eta_i\end{aligned} \tag{62}$$

It can be seen that in addition to the state observers x_i, the auxiliary dynamics η_i are involved. In [22] a reduced-order observer-based consensus protocol is introduced. It has the form

$$\begin{aligned}\dot{v}_i &= Fv_i + Gy_i + TBu_i\\ u_i &= cKQ_1\sum_{j=1}^{N} a_{ij}(y_i - y_j) + cKQ_2\sum_{j=1}^{N} a_{ij}(v_i - v_j)\end{aligned} \tag{63}$$

The dynamics v_i are designed as the reduced-order observers of agents' states, but the matrices Q_1 and Q_2 in u_i are dependent of the matrix T in the observers. In short, in the designs of the output feedback protocols in [14], [15], [22], [34], the separation principle is not used as in design of output feedback stabilizing controllers for linear systems.

To the best of our knowledge, it seems that the separation principle has not been rigorously proved for output feedback synchronization of general linear multi-agent systems.

3: Below are some corollaries derived by combining Theorems 1 and 2 with Proposition 1.

A switching topology $\mathcal{G}(t)$ is called frequently connected if there exists a positive integer M such that among any consecutive M switching instants there is at least one at which $\mathcal{G}(t)$ is connected. From Proposition 1, a frequently connected topology has synchronizability exponent $\rho = +\infty$. Thus, condition (4) of Theorem 1 holds trivially. Therefore we have the following corollary—the first part is the main result of [25].

Corollary 1. *Assume that assumptions 1) and 2) of Theorem 1 hold and in addition $\mathcal{G}(t)$ is frequently connected. Then, the conclusion of Theorem 1 follows for any given $\beta > 0$. Besides, if (A, C) is observable, the conclusion of Theorem 2 follows for any given $\beta > 0$.*

By Proposition 1, the switching topology satisfying assumption 2) of Theorem 1 has synchronizability exponent $\rho > 0$. If the agents have Lyapunov exponent $\chi \le 0$, then condition (4) of Theorem 1 holds trivially. Therefore we have the following corollary—the first part is one of the main results of [36].

Corollary 2. *Assume that assumptions 1) and 2) of Theorems 1 hold and eigenvalues of A are all in the closed left half plane. Then, the conclusion of Theorem 1 follows for some $\beta > 0$. Besides, if (A, C) is observable, the conclusion of Theorem 2 follows for some $\beta > 0$.*

By Proposition 1, the synchronizability exponent ρ can be made larger by adjusting the switching frequency of switching topology (if possible). This implies that if condition (4) of Theorem 1 does not hold, it can be recovered if the adjustment is allowed. Therefore we have:

Corollary 3. *Assume that assumptions 1) and 2) of Theorem 1 hold but condition (4) fails to hold. Let $\kappa > 0$ such that $\kappa\rho > \chi$. Then the conclusion of Theorem 1 follows if $\mathcal{G}(t)$ is replaced by $\mathcal{G}(\kappa t)$. Moreover, if (A, C) is observable, the conclusion of Theorem 2 follows in the same way.*

Remark 5. *Switching topologies $\mathcal{G}(\kappa t)$ and $\mathcal{G}(t)$ differ only in their switching frequencies. So far the importance of switching frequency for multi-agent systems has not been recognized. Corollary 3 tells us that the switching frequency has a significant effect on synchronizability of multi-agent systems. In summary, when the switching topology of a linear multi-agent system is jointly connected and its switching frequency can be adjusted, the controllability of agents means their synchronizability.*

VI. Simulation example

Consider a group of 20 linear systems described by

$$\begin{aligned}\dot{x}_i &= Ax_i + Bu_i\\ y_i &= Cx_i,\ i = 1, \ldots, 20\end{aligned} \tag{64}$$

where $x_i(t) \in \mathbb{R}^2, u_i, y_i \in \mathbb{R}$, and

$$A = \begin{pmatrix} 2/3 & -4\\ 7/5 & 0\end{pmatrix},\ B = \begin{pmatrix} 0\\ 1\end{pmatrix},\ C = \begin{pmatrix} 1 & 0\end{pmatrix}$$

Two graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ on $\{1, \ldots, 20\}$ are depicted in Fig. 1. The smallest of nonzero eigenvalues of the Laplacians $\mathcal{L}(\mathcal{G}_1)$ and $\mathcal{L}(\mathcal{G}_2)$ is 1. Choose numbers $\tau_0, \tau_1, \ldots$ at random independently from the interval $[0.1, 0.2]$ and let $t_0 = 0, t_k = \tau_0 + \cdots + \tau_{k-1}$ for $k = 1, 2, \ldots$. Define switching topology

$$\mathcal{G}(t) = \begin{cases}\mathcal{G}_1, & t_{2k-1} \le t < t_{2k}\\ \mathcal{G}_2, & t_{2k} \le t < t_{2k+1}\end{cases}$$

which satisfies assumption 2) of Theorem 1 with $\tau = 0.1$ and $T = 0.4$. By Remark 1 ($R = 4, \delta = 0.2$), a lower bound of the synchronizability exponent ρ is

$$\rho \ge -\frac{5}{4}\ln\frac{8}{5\sqrt{5}} > 0.418$$

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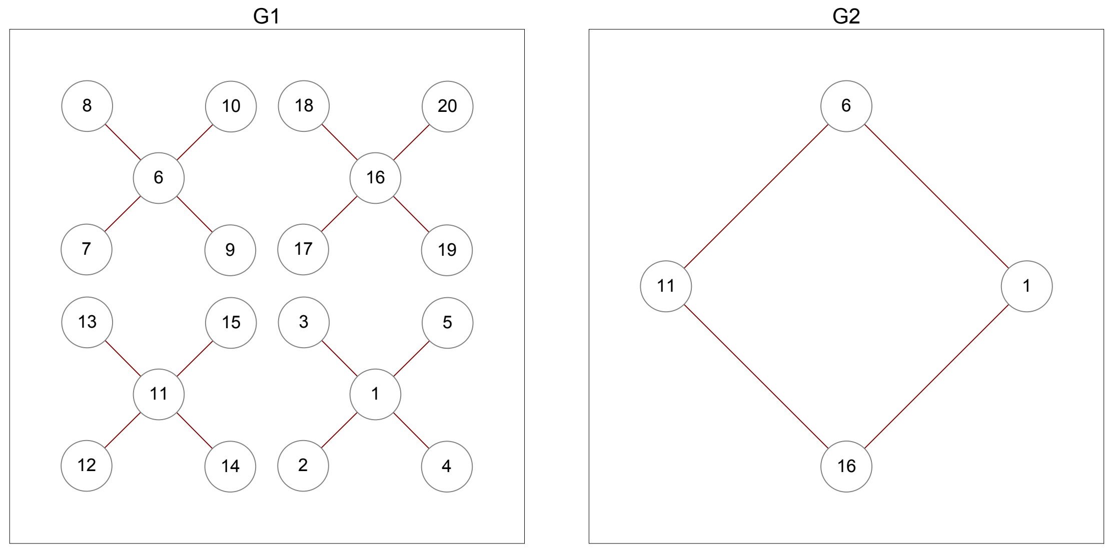

Figure 1. Two Graphs, $\mathcal{G}1$ and $\mathcal{G}2$, defined on $\{1, 2, \ldots, 20\}$. The isolated nodes are suppressed

It is easy to check that pair (A, B) is controllable and A has unstable eigenvalues

$$\frac{1}{3} + i\frac{\sqrt{1235}}{15}, \frac{1}{3} - i\frac{\sqrt{1235}}{15}$$

The 20 agents have Lyapunov exponent $\chi = \frac{1}{3}$. So $\chi < \rho$ and condition (4) of Theorem 1 holds. From Theorem 1 the state feedback protocol of the form

$$u_i = -\sum_{j=1}^{N} a_{ij}(t)\, K(\alpha)(x_i - x_j), i = 1, \ldots, 20 \tag{65}$$

synchronizes the trajectories of 20 agents to a common trajectory, where $K(\alpha)$ is defined in (23) and α is the design parameter.

A procedure for computing α is given in the proof of Lemma 4, but it is mainly for its existence. Here, we directly set $\alpha = 10$. The initial states for each agent are independently drawn from a bivariate normal distribution with mean $\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$ and covariance matrix $\left(\begin{smallmatrix}20 & 0\\0 & 20\end{smallmatrix}\right)$. The simulation is run for 65 switchings. The simulation result is shown in Fig. 2.

Moreover, pair (A, C) is observable, so the synchronization can be achieved by an output feedback protocol. We build the output feedback protocol using the separation principle. Design the state observers for each agent of the form

$$\dot{\hat{x}}_i = A\hat{x}_i + Bu_i + L(y_i - C\hat{x}_i),\, i = 1, \ldots, 20 \tag{66}$$

where the observer gain is chosen as $L = \begin{pmatrix} 6 \\ -1/3 \end{pmatrix}$, which makes $A - LC$ Hurwitz. Combining the observers with state feedback protocol (65) yields the observer-based couplings

$$u_i = -\sum_{j=1}^{N} a_{ij}(t) K(10)(\hat{x}_i - \hat{x}_j),\, i = 1, \ldots, 20$$

The initial states for 20 agents and their observers are drawn in the same way as above. The simulation is also run for 65 switchings. The trajectories of the 20 agents are shown in Fig. 3; the trajectories of the 20 observers are shown in Fig. 4.

VII. Conclusion

Network topology specifies how agents interact with each other, so its description is significant in the study of multi-agent systems. However, in the literature, the descriptions are mainly devoted to the connectivity property of network topology, while its other aspects and its relationship to agent dynamics are often overlooked. In the present paper, we propose the synchronizability exponent concept for switching topology to quantify its dynamic characteristic and, as usual, introduce the Lyapunov exponent to quantify the agents' characteristic. Using the two exponents, we describe a relationship between agent dynamics and network topology and present new conditions for synchronization of linear multi-agent systems. Since network topology and agent dynamics belong to different areas of mathematics and their mathematical natures are totally different, such a relationship is not easy to obtain.

We also develop an effective method for synchronization of general linear multi-agent systems. Infinite matrix product is a powerful tool for the consensus of the single-integrator multi-agent systems. In this paper, we adapt it to general linear multi-agent systems. This method, which differs from the Lyapunov function method, may be more effective and less conservative for linear multi-agent systems. Under the quantitative condition together with joint connectivity assumption, we prove two main synchronization results by infinite matrix product method. We further show that the separation principle is also valid for output feedback synchronization of linear multi-agent systems. Besides the quantitative condition, two significant features make our results different from the existing ones: under jointly connected topology, the synchronized agents are allowed to have exponentially unstable eigenvalues; the synchronization rate is explicitly given.

Finally, we would like to notice that we have restricted ourselves to the case of undirected network topology. Though the extension of our method to the directed network topology case would be a challenging issue, we believe that something like synchronizability exponent would be applicable to directed

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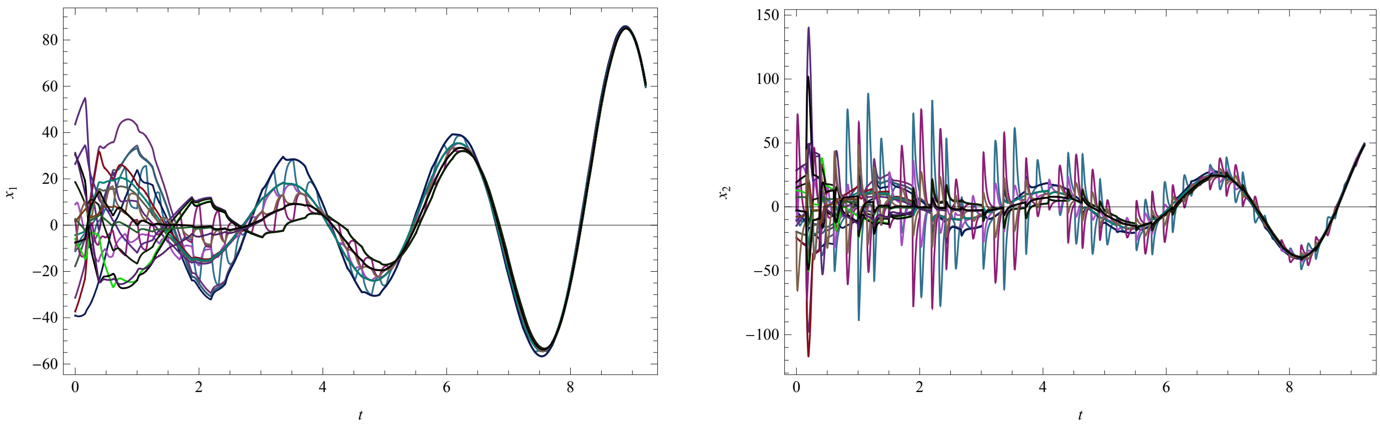

Figure 2. Trajectories of 20 agents under the state feedback protocol.

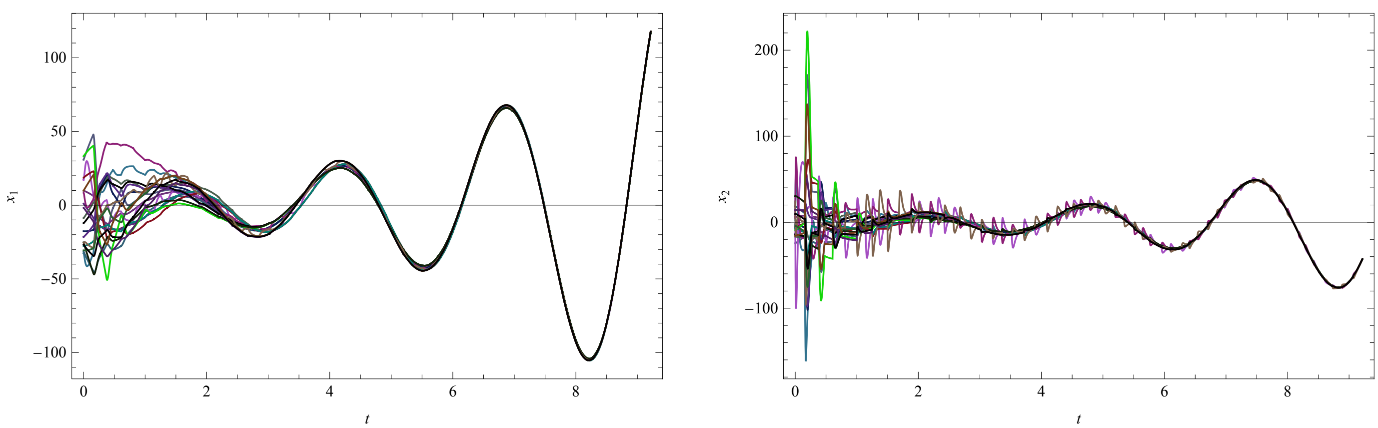

Figure 3. Trajectories of 20 agents under the output feedback protocol.

switching topology.

Appendix

A. Proof of Lemma 1

Proof: Let x be a solution of the following system of equations

$$(I-(P_{\mathcal{G}_i}-P))x=0, i=1,\dots,r. \tag{67}$$

Pre-multiplying the ith equation by $\mathcal{L}(\mathcal{G}_i)$ yields

$$\mathcal{L}(\mathcal{G}_i)x=0, i=1,\dots,r$$

Then $x\in\bigcap_{i=1}^r\{x:\mathcal{L}(\mathcal{G}_i)x=0\}$. Since $\bigcup_{i=1}^r\mathcal{G}_i$ is connected, we have (by Lemma 3 of [1])

$$\bigcap_{i=1}^r\{x:\mathcal{L}(\mathcal{G}_i)x=0\}=\operatorname{span}\{\mathbf{1}\}$$

So, x must take the form $x=a\mathbf{1}$ for some $a\in\mathbb{R}$. Substituting it into one of (67) gives

$$x=(P_{\mathcal{G}_i}-P)x=a(P_{\mathcal{G}_i}-P)\mathbf{1}=a(P_{\mathcal{G}_i}\mathbf{1}-P\mathbf{1})=0$$

That is, (67) has the unique solution $x=0$, or

$$\bigcap_{i=1}^r\{x:(P_{\mathcal{G}_i}-P)x=x\}=\{0\}$$

From a result in [1] (Lemma 4), we get

$$\|(P_{\mathcal{G}_r}-P)\cdots(P_{\mathcal{G}_1}-P)\|<1$$

Let $\delta(N,r)$ be the maximum of the norms of all such products. Because there are a finite number of such products, then $\delta(N,r)<1$. This completes the proof. ■

B. Proof of Proposition 1

Proof: (1) The condition means that among any consecutive M switching instants there is at least one instant at which $P(t_i)=P$, so for any $k\geq 0$,

$$(P(t_{k+M-1})-P)\cdots(P(t_{k+1})-P)(P(t_k)-P)=0$$

Thus, $\rho(M)=+\infty$ and $\rho(\mathcal{G}(t))=+\infty$.

(2) We let $R=\left[\frac{T}{\tau}\right]+1$ and consider, for fixed $k\geq 0$, the product $P(t_{k+R-1})-P)\cdots(P(t_{k+1})-P)(P(t_k)-P)$.

Case (i): For all $k\leq i<k+R$, $\mathcal{G}(t_i)$ are disconnected. Then we have $\tau<t_{i+1}-t_i<T$ for all $k\leq i<k+R$, and

$$T<\tau R\leq t_{k+R}-t_k\leq TR$$

Thus, $\mathcal{G}([t_k,t_{k+R}))=\bigcup_{i=k}^{k+R-1}\mathcal{G}(t_i)$ is connected for $t_{k+R}-t_k>T$. From Lemma 1 it follows that there exists $0<\varepsilon<1$ independent of k for which

$$\|(P(t_{k+R-1})-P)\cdots(P(t_k)-P)\|\leq\varepsilon<1$$

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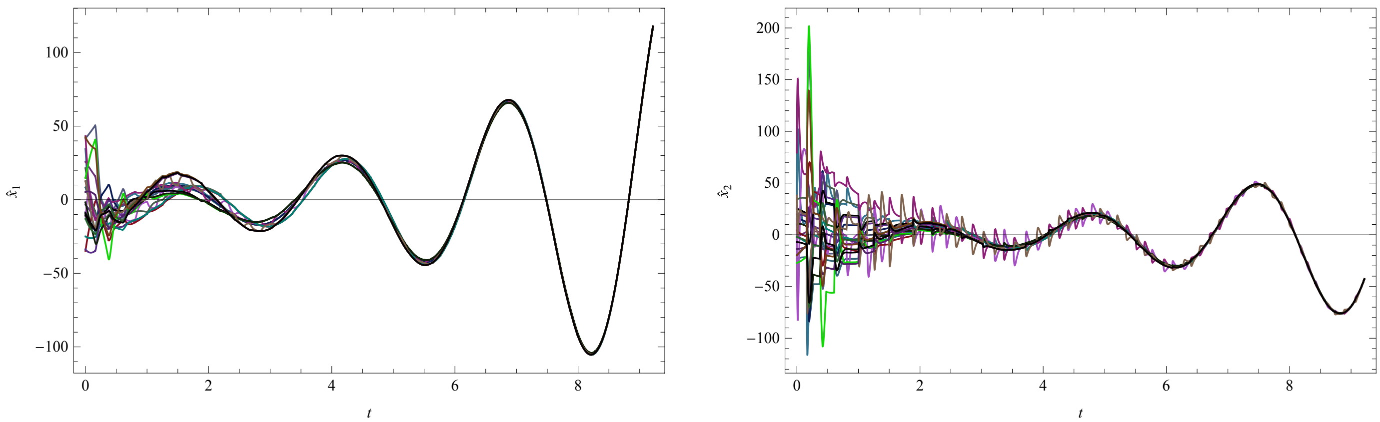

Figure 4. Trajectories of 20 agents' observers under the output feedback protocol.

Therefore

$$\frac{\ln\|(P(t_{k+R-1})-P)\cdots(P(t_k)-P)\|}{t_{k+R}-t_k}\le\frac{\ln\varepsilon}{TR}<0$$

Case (ii): For some $k\le i<k+R$, $\mathcal{G}(t_i)$ are connected. Then we have $(P(t_{k+R-1})-P)\cdots(P(t_k)-P)=0$, so that

$$\frac{\ln\|(P(t_{k+R-1})-P)\cdots(P(t_k)-P)\|}{t_{k+R}-t_k}=-\infty<\frac{\ln\varepsilon}{TR}$$

By Definition 1 it follows that

$$\rho\ge\rho(R)\ge-\frac{\ln\varepsilon}{TR}>0$$

(3) Let $\tilde{\mathcal{G}}(t)=\mathcal{G}(\kappa t)$ and $\tilde{P}(t)=P_{\tilde{\mathcal{G}}(t)}$. Then, $\tilde{\mathcal{G}}(t)$ has switching instants $\tilde{t}_i=\frac{t_k}{\kappa},i=1,2,\dots$ and $\tilde{\mathcal{G}}(\tilde{t}_i)=\mathcal{G}(t_i)$. Thus, we have $\tilde{P}(\tilde{t}_i)=P_{\tilde{\mathcal{G}}(\tilde{t}_i)}=P_{\mathcal{G}(t_i)}=P(t_i)$ and

$$\begin{aligned}&\frac{\ln\|(\tilde{P}(\tilde{t}_{k+q-1})-P)\cdots(\tilde{P}(\tilde{t}_k)-P)\|}{\tilde{t}_{k+q}-\tilde{t}_k}\\&=\kappa\,\frac{\ln\|(P(t_{k+q-1})-P)\cdots(P(t_k)-P)\|}{t_{k+q}-t_k}\end{aligned}\tag{68}$$

By Definition 1, we have $\tilde{\rho}=\kappa\rho$. ■

C. Proof of Lemma 3

Proof: Let

$$Q(t)=e^{-\alpha t}e^{-\frac{A}{2}t}BB^{\mathrm{T}}e^{-\frac{A^{\mathrm{T}}}{2}t}$$

The time derivative $Q(t)$ is

$$\dot{Q}(t)=-\alpha Q(t)-\tfrac{1}{2}AQ(t)-\tfrac{1}{2}Q(t)A^{\mathrm{T}}$$

Integrating the equation from $t=0$ to $\frac{1}{3}\tau$ yields

$$Q(\tfrac{\tau}{3})-BB^{\mathrm{T}}=-\tfrac{1}{2}(2\alpha W(\alpha)+AW(\alpha)+W(\alpha)A^{\mathrm{T}})$$

Denote $H=W^{-1}(\alpha)$. Pre- and post-multiplying this equation by H give

$$H[Q(\tfrac{\tau}{3})-BB^{\mathrm{T}}]H=-\tfrac{1}{2}(2\alpha H+HA+A^{\mathrm{T}}H)$$

or

$$HA+A^{\mathrm{T}}H=-2\alpha H+2HBB^{\mathrm{T}}H-2HQ(\tfrac{\tau}{3})H.\tag{69}$$

We now consider the initial value problem

$$\dot{x}(t)=(A-\tilde{\lambda}BB^{\mathrm{T}}H)x(t),x(0)=x_0\tag{70}$$

where $x_0\in\mathbb{R}^n$, $\tilde{\lambda}=\frac{1}{4}N(N-1)\lambda$, λ is a nonzero eigenvalue of $\mathcal{L}(\mathcal{G})$. Let $x(t)$ be the solution of (70) and define

$$V(t)=x^{\mathrm{T}}(t)Hx(t)$$

Computing $\dot{V}(t)$ gives

$$\dot{V}(t)=x^{\mathrm{T}}(t)\big[A^{\mathrm{T}}H+HA-2\tilde{\lambda}HBB^{\mathrm{T}}H\big]x(t)\tag{71}$$

Substituting (69) into (71) and noticing $\tilde{\lambda}\ge1$ from Lemma 2, we have

$$\begin{aligned}\dot{V}(t)=x^{\mathrm{T}}(t)\big[&-2\alpha H-2HQ(\tfrac{\tau}{3})H\\&+2HBB^{\mathrm{T}}H-2\tilde{\lambda}HBB^{\mathrm{T}}H\big]x(t)\le-2\alpha V(t)\end{aligned}$$

By the comparison principle [37], it follows that

$$V(t)\le e^{-2\alpha t}V(0).\tag{72}$$

Noting that $e^{-\frac{1}{3}\alpha\tau}\le e^{-\alpha t}\le1$ for $0\le t\le\frac{\tau}{4}$ and recalling the definition of $W(\alpha)$ in (22), we obtain

$$e^{-\frac{1}{3}\alpha\tau}\lambda_2I\le e^{-\frac{1}{3}\alpha\tau}W(0)\le W(\alpha)\le W(0)\le\lambda_1I$$

where λ_1 and λ_2 are respectively the maximum and minimum eigenvalue of $W(0)$. Then, it holds that

$$\lambda_1^{-1}I\le H=W^{-1}(\alpha)\le e^{\frac{1}{3}\alpha\tau}\lambda_2^{-1}I\tag{73}$$

Therefore

$$\begin{aligned}V(t)&=x^{\mathrm{T}}(t)Hx(t)\ge\lambda_1^{-1}\|x(t)\|^2\\V(0)&=x^{\mathrm{T}}(0)Hx(0)\le e^{\frac{1}{3}\alpha\tau}\lambda_2^{-1}\|x_0\|^2\end{aligned}$$

Substituting into (72) and letting $C_0=(\lambda_1\lambda_2^{-1})^{\frac{1}{2}}$ yield

$$\|x(t)\|\le C_0e^{\frac{1}{6}\tau\alpha}e^{-\alpha t}\|x_0\|\tag{74}$$

Note that the solution of the initial-value problem of (70) is

$$x(t)=e^{(A-\tilde{\lambda}BB^{\mathrm{T}}H)\,t}x_0=e^{(A-\lambda BK(\alpha))\,t}x_0$$

Substituting into (74) and using the definition of induced norm, we get

$$\left\|e^{(A-\lambda BK(\alpha))\,t}\right\|\le C_0e^{\frac{1}{6}\tau\alpha}e^{-\alpha t}$$

The proof is complete. ■

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D. Proof of Lemma 4

Proof: We choose α_1 satisfying

$$C_0 e^{-\frac{5}{6}\tau\alpha_1} < 1$$

and, for the sake of simplicity, tacitly assume $\alpha \geq \alpha_1$. Thus, for any $i \geq 0$ it turns out from (26) that

$$\|\Phi_i(\tau_i)\| \leq C_0 e^{\frac{1}{6}\alpha\tau} e^{-\alpha\tau_i} \leq C_0 e^{-\frac{5}{6}\tau\alpha_1} < 1$$

Case (i): For all $0 \leq i \leq m-1$, $\tau_i < T$. Then,

$$\|\Psi_i(\tau_i)\| = \|(P(t_i) - P) \otimes e^{A\tau_i}\| \leq \|e^{A\tau_i}\| \leq e^{\|A\|T}$$

Using inequality

$$x^m - y^m = (x-y)\sum_{i=1}^{m} x^{m-i}y^{i-1} \leq m(x-y)x^{m-1}, x > y$$

we have

$$\begin{aligned}\Gamma &= \Big\| \prod_{i=1}^{m-1}(\Phi_i(\tau_i) + \Psi_i(\tau_i)) - \prod_{i=1}^{m-1}\Psi_i(\tau_i)\Big\| \\ &\leq \prod_{i=1}^{m-1}(\|\Phi_i(\tau_i)\| + \|\Psi_i(\tau_i)\|) - \prod_{i=1}^{m-1}\|\Psi_i(\tau_i)\| \\ &\leq (C_0 e^{-\frac{5}{6}\tau\alpha} + e^{\|A\|T})^m - e^{m\|A\|T} \\ &\leq mC_0(1 + e^{\|A\|T})^{m-1} e^{-\frac{5}{6}\tau\alpha}\end{aligned}$$

Choose $\kappa_1(\gamma, m)$ such that for $\alpha \geq \kappa_1(\gamma, m)$

$$mC_0(1 + e^{\|A\|T})^{m-1} e^{-\frac{5}{6}\tau\alpha} \leq \tfrac{1}{2} e^{-\gamma m T}$$

Denoting $T_0 = \tau_0 + \cdots + \tau_{m-1}$ and noting $T_0 < mT$, we have

$$\Gamma \leq mC_0(1 + e^{\|A\|T})^{m-1} e^{-\frac{5}{6}\tau\alpha} \leq \tfrac{1}{2}e^{-\gamma m T} \leq \tfrac{1}{2}e^{-\gamma T_0} \quad (75)$$

Case (ii): There exists some $0 \leq i \leq m-1$ for which $\tau_i \geq T$. By assumption 2) of Theorem 1, $\mathcal{G}(t_i)$ must be connected for those i with $\tau_i \geq T$, and then

$$\Psi_i(\tau_i) = (P(t_i) - P) \otimes e^{A\tau_i} = 0$$

Letting $\tau_j = \max\{\tau_i, i = 0, 2, \ldots, m-1\}$, we have

$$\begin{aligned}\Gamma &= \Big\| \prod_{i=0}^{m-1}(\Phi_i(\tau_i) + \Psi_i(\tau_i))\Big\| \\ &\leq \prod_{0 \leq i \leq m-1, \tau_i \geq T} \|\Phi_i(\tau_i)\| \\ &\qquad \prod_{0 \leq i \leq m-1, \tau_i < T} (\|\Phi_i(\tau_i)\| + \|\Psi_i(\tau_i)\|) \\ &\leq \|\Phi_j(\tau_j)\|(1 + e^{\|A\|T})^{m-1} \\ &\leq C_0(1 + e^{\|A\|T})^{m-1} e^{-\frac{5}{6}\tau_j\alpha}\end{aligned}$$

Since $T_0 = \tau_0 + \cdots + \tau_{m-1} \leq m\tau_j$, we have $\tau_j \geq \frac{T_0}{m}$ and

$$\Gamma \leq C_0(1 + e^{\|A\|T})^{m-1} e^{-\frac{5T_0\alpha}{6m}}$$

Now, to prove the claim in Case (ii), we only need to choose α such that

$$C_0(1 + e^{\|A\|T})^{m-1} e^{-\frac{5T_0\alpha}{6m}} < \tfrac{1}{2}e^{-\gamma T_0} \quad (76)$$

or, equivalently, to choose α such that

$$C_0(1 + e^{\|A\|T})^{m-1} e^{-(\frac{5\alpha}{6m} - \gamma)T_0} \leq \tfrac{1}{2}$$

Noting that $m\tau < T_0$, we only need to choose $\kappa_2(\gamma, m)$ such that when $\alpha \geq \kappa_2(\gamma, m)$,

$$C_0(1 + e^{\|A\|T})^{m-1} e^{-(\frac{5\alpha}{6m} - \gamma)m\tau} \leq \tfrac{1}{2}$$

Combining the two cases and letting

$$\kappa(\gamma, m) = \max\{\alpha_1, \kappa_1(\gamma, m), \kappa_2(\gamma, m)\}$$

we complete the proof. ■

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