This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TAC.2018.2867163, IEEE Transactions on Automatic Control

A NEW SECOND-ORDER SLIDING MODE AND ITS APPLICATION TO NONLINEAR CONSTRAINED SYSTEMS*

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***Abstract*—This note considers the second-order sliding mode (SOSM) control of nonlinear constrained systems. A new state-saturated-like SOSM algorithm has been constructed by using saturation technique and backstepping-like method. The finite-time stability of the closed-loop SOSM dynamics has been verified by Lyapunov analysis. The advantage of the proposed SOSM algorithm lies in that it will provide the maximum domain of attraction under the preset state constraints. An example of pendulum control system has been given to verify the effectiveness of the proposed method.**

***Index Terms*—Second-order sliding mode, Lyapunov analysis, finite-time stability, state constraints**

I. INTRODUCTION

In practical plant, the system performance is always affected by uncertainties, the influence of which should be carefully taken into account in control design [1], [2]. Many well-known methods have been developed to handle the robust control problems, such as [3]–[11] and the references therein. Among these existing robust control methods, the sliding mode control (SMC) has been considered as one of the most effective control techniques [12]–[15].

The pioneering contribution of SMC can be dated back to Emelyanov's work, while it has been popularized in the academic literature starting from the works of Utkin [16] and Itkis [17]. The basic idea of the conventional SMC can be summarized as follows. A stable sliding mode manifold in the state space is first pre-designed to guarantee that the closed-loop system will possess its desired performance. Then, a SMC law is further developed such that the sliding variables will finite-time arrive at and stay on the sliding mode manifold. As long as the system states are controlled in the sliding mode manifold, they will be insensitive to the uncertainties [18].

In most cases, the SMC laws are developed by means of satisfying the so-called reaching law, which forces the sliding variables to stay on the sliding mode manifold. In order to suppress the uncertainties, a nonlinear discontinuous term is usually injected to the sliding mode controller. Nevertheless, the discontinuous term usually brings the undesirable chattering problem, which may excite unmodeled high-frequency modes, even bring the instability for the dynamic system. To attenuate the chattering, the most widely-used technique in the literature is the boundary-layer method proposed in [19], under which the sliding variables will be confined within a region of the sliding mode manifold (i.e., boundary layer) and the disturbance rejection property is determined by the width of the boundary layer. Although the boundary-layer method can attenuate the chattering, the disturbance rejection ability of the closed-loop system degrades too because the sliding mode controller in the boundary layer actually reduces to a linear controller. Meanwhile, the width of boundary layer is also difficult to choose optimally [20].

Another method to mitigate the chattering is the SOSM method. By regarding the derivative of the actual control as the new one and designing a discontinuous SOSM controller, the chattering may be attenuated because the actual control law will be the continuous integration of its derivative. As described in [21], the second-order SMC not only keeps the main advantages of the conventional SMC, but also attenuates the chattering effect. Meanwhile, the chattering reduction is not the only possible purpose of SOSM, since it extends the relative degree of sliding variable from one to two. Consequently, the study on SOSM has been paid much attention in recent years. The frequently-used SOSM algorithms include twisting algorithm [22], super-twisting algorithm [21], [23], suboptimal algorithm [1], homogeneous algorithm [25], [26], Lyapunov-based design [24], [25], [27]–[29], etc.

One issue worth discussing is the control design problems for nonlinear systems under state constraints. It should be pointed out that the conventional SMC of nonlinear systems under state constraints has been well studied in the literature, such as [30]–[32]. The finite-time boundedness for a class of nonlinear systems has been investigates in [30] by using SMC method and a suitable SMC law has been constructed to drive the state trajectories onto the specified sliding surface during a specified finite time interval. In [31], a nearly optimal sliding mode algorithm has been proposed for the fault tolerant control problem of a nonlinear affine system with state constraints by using adaptive dynamics programming and a novel transformation technique. It can be seen from [31] that only state constraints are considered. Different from [31], the problem of both state and control input constraints in discrete-time SMC is considered in [32], where sufficient conditions for finite-time convergence of the representative point to the sliding hyperplane have been given and a new control strategy based on the reaching law approach has been proposed. Although there are many results on SMC of nonlinear constrained systems, most of them focused on the conventional first-order SMC and till now few results can be found in the literature to handle SOSM control of nonlinear constrained systems. It should be mentioned that a recent result on the design of SOSM algorithm for nonlinear systems in the presence of hard inequality constraints on both control and state variables

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* This work was supported by the NSF of China (61573170,61473080,31571571), the Jiangsu Natural Science Foundation for Distinguished Young Scholars (SBK2018010063), the PAPD of Jiangsu Higher Education Institutions and the Six Talent Peaks Project in Jiangsu Province (XNYQC-006).

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TAC.2018.2867163, IEEE Transactions on Automatic Control

has been introduced in [33], where a maximum domain of attraction is provided. To the best of the authors' knowledge, the reference [33] is the first time to consider the SOSM control of constrained nonlinear systems.

Motivated by [33], the purpose of this paper is to develop a novel SOSM method for nonlinear constrained systems with the possibility that it could provide a maximum domain of attraction when the state variables are constrained. First of all, a SOSM controller is constructed without any state constraints to ensure the finite-time stability of the sliding variables. Secondly, a saturation function will be imposed on the state variable of the derived controller yielding a state-saturated-like SOSM controller. Under the state-saturated-like controller, the sliding variables will converge to a given region determined by the saturation level, where it will reduce to the former one and ensure the local finite-time stability. Finally, it can be further proved that under the developed state-saturated-like SOSM controller, there is a maximum domain of attraction when the sliding variables satisfy the box constraint. The main contributions of the paper are two-fold. One is that a new state-saturated-like SOSM control algorithm, which could be used to obtain the maximum domain of attraction for nonlinear constrained systems, is provided. The other is that the Lyapunov analysis method has been presented to verify the finite-time stability of the sliding variables, which is stronger than finite-time convergence.

II. Preliminaries and Problem Formulation

Consider a dynamic system of the form

$$\dot{x} = f(t,x) + g(t,x)u, \quad s = s(x) \tag{1}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}$ are the state and control input, respectively; $f(t,x)$ and $g(t,x)$ are smooth functions; s is the sliding variable, which is also a smooth function. All the states of system (1) are measurable.

Assume that the relative degree of sliding variable s is 2. In other words, the control input u explicitly appears for the first time only in the second total derivative of s. Therefore, the dynamics of sliding variable s is

$$\ddot{s} = a(t,x) + b(t,x)u \tag{2}$$

where s and $\dot{s}$ are measurable since they are functions of state variable x, and the uncertainties $a(t,x) = \ddot{s}|_{u=0}$ and $b(t,x) = \frac{\partial \ddot{s}}{\partial u}$ are not exactly known smooth functions but satisfy the following assumption

Assumption 2.1: There exist positive constants C, K_m, K_M such that

$$|a(t,x)| \le C, \; K_m \le b(t,x) \le K_M.$$

Remark 2.1: As a matter of fact, the uncertainties of system (2) can also be extended to be bounded by positive functions. However, if the state variables are very large, the control effort will be large too. Note that the control saturation always exists. It may be more practical to assume that the uncertainties are bounded by positive constants. However, the uncertainties are usually linked with the state variables, which implies that the state variables should be bounded. Otherwise, Assumption 2.1 may not hold. Consequently, it is very natural to assume that the states should satisfy some constrained conditions. However, the SOSM control of constrained systems has rarely been considered.

The objective of this paper is to construct a new SOSM algorithm, which could provide the maximum domain of attraction for the SOSM dynamics with box constraint: $s \in [\underline{\delta}_1, \bar{\delta}_1], \dot{s} \in [\underline{\delta}_2, \bar{\delta}_2]$ with $\underline{\delta}_1, \underline{\delta}_2 < 0$ and $\bar{\delta}_1, \bar{\delta}_2 > 0$.

In the remainder of this section, we will list two useful lemmas which will be constantly used in the proof. For the sake of simplifying the expression, we denote $\lfloor x \rceil^{\alpha} = \mathrm{sign}(x)|x|^{\alpha}$ with any $\alpha > 0$.

Lemma 2.1: [5], [34] If $p_1 > 0$ and $0 < p_2 \le 1$, then the following inequality holds:

$$\left| \lfloor x \rceil^{p_1 p_2} - \lfloor y \rceil^{p_1 p_2} \right| \le 2^{1-p_2} \left| \lfloor x \rceil^{p_1} - \lfloor y \rceil^{p_1} \right|^{p_2}, \forall x \in \mathbb{R}, \; y \in \mathbb{R}.$$

Lemma 2.2: [35] For $\forall x_i \in \mathbb{R}, i = 1, \cdots, n$, and $0 < p \le 1$, there is $(|x_1| + \cdots + |x_n|)^p \le |x_1|^p + \cdots + |x_n|^p$.

III. A New State-saturated-like SOSM Algorithm

A SOSM controller will be constructed to finite-time stabilize system (2) without considering state constraints. By letting $s_1 = s, s_2 = \dot{s}$, the sliding dynamics (2) can be rewritten as

$$\dot{s}_1 = s_2, \; \dot{s}_2 = a(t,x) + b(t,x)u. \tag{3}$$

Then we have the following theorem.

Theorem 3.1: Under Assumption 2.1, if the SOSM controller is designed as

$$u = -\beta_2 \cdot \mathrm{sign}\left(\lfloor s_2 \rceil^{a/r_2} + \beta_1^{a/r_2} \lfloor s_1 \rceil^{a/r_1} \right) \tag{4}$$

with $a \ge r_1 = 2r_2 > 0, \; \beta_1 > 0, \; \beta_2 \ge (c_1(\beta_1) + c_2(\beta_1) + \frac{\beta_1}{12} + C)/K_m, \; c_1(\beta_1) = \frac{2^{-1-\frac{r_2}{a}} r_1}{a} \left(\frac{2^{1-\frac{r_2}{a}}(4a-r_1)}{a\beta_1} \right)^{\frac{4a-r_1}{r_1}}$ and $c_2(\beta_1) = \frac{2^{1-\frac{r_2}{a}} \beta_1^{\frac{a}{r_2}} (r_1+a)}{r_1} \left(\frac{3 \times 2^{1-\frac{r_2}{a}} (a-r_1) \beta_1^{\frac{a}{r_2}-1}}{r_1} \right)^{\frac{a-r_1}{a+r_1}} + \frac{2^{-\frac{r_2}{a}} \beta_1^{1+\frac{a}{r_2}} (r_1+2a)}{r_1} \left(\frac{3 \times 2^{-\frac{r_2}{a}} (2a-r_1) \beta_1^{\frac{a}{r_2}}}{r_1} \right)^{\frac{2a-r_1}{2a+r_1}}$, then controller (4) finite-time establishes the SOSM $s_1 = s_2 = 0$ (i.e., $s = \dot{s} = 0$) in system (1).

Proof. The proof of Theorem 3.1 is similar to that of Theorem 1 in [37]. Here we skip the proof for brevity. ■

Then, by combining controller (4) with a saturation function, a state-saturated-like SOSM controller can be given as

$$u = -\beta_2 \cdot \mathrm{sign}\left(\lfloor s_2 \rceil^{\frac{a}{r_2}} + \beta_1^{\frac{a}{r_2}} \sigma(\lfloor s_1 \rceil^{\frac{a}{r_1}}) \right) \tag{5}$$

where

$$\begin{cases} \beta_1 & > 2^{1-\frac{r_2}{a}}, \\ \beta_2 & > \max\left\{ \frac{4a}{K_m r_1}(1+\beta_1)^2 \beta_1^{\frac{a}{r_2}} + \frac{C}{K_m}, \right. \\ & \left. \frac{12c_1(\beta_1)+12c_2(\beta_1)+\beta_1+12C}{12K_m} \right\} \end{cases} \tag{6}$$

with $a, r_1, r_2, c_1(\beta_1)$, and $c_2(\beta_1)$ defined the same as in Theorem 3.1, and

$$\sigma(x) = \begin{cases} \varepsilon \cdot \mathrm{sign}(x), & \text{for } |x| > \varepsilon \\ x, & \text{for } |x| \le \varepsilon \end{cases}$$

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for any constant $\varepsilon > 0$.

Remark 3.1: It should be noted that the gains of controller (5) have to satisfy a conservative precondition (6), which implies that the parameters β_1 and β_2 may be very large. This is because the gains are overestimated during stability analysis. As a matter of fact, the precondition (6) can be partly relaxed and the gains β_1 and β_2 can be chosen to be some smaller ones, which will be reflected in the subsequent Corollary 3.1.

Next, we will show that the control design problem for system (2) is also solvable by SOSM controller (5).

We first introduce an important lemma which will be used in the proof of the main result.

Lemma 3.1: For the closed-loop system (3) and (5), the following inequality holds

$$\sup_{s_1\in\mathbb{R}}\{|\dot{\sigma}(\lfloor s_1\rceil^{a/r_1})|\} \le a_1(\beta_1)\varepsilon^{1-\frac{r_2}{a}} \tag{7}$$

provided $\left|\lfloor s_2\rceil^{\frac{a}{r_2}} + \beta_1^{\frac{a}{r_2}}\sigma(\lfloor s_1\rceil^{\frac{a}{r_1}})\right| \le \varepsilon$, where $a_1(\beta_1) = \frac{a}{r_1}(1+\beta_1)$.

Proof. To simplify the proof, we denote

$$\phi_2(s_1,s_2) = \lfloor s_2\rceil^{a/r_2} + \beta_1^{a/r_2}\sigma(\lfloor s_1\rceil^{a/r_1}).$$

By the fact $|\phi_2(s_1,s_2)| \le \varepsilon$ and Lemma 2.2, we obtain

$$|s_2| \le (\varepsilon + \beta_1^{a/r_2}\varepsilon)^{r_2/a} \le \varepsilon^{r_2/a}(1+\beta_1). \tag{8}$$

For $|\lfloor s_1\rceil^{a/r_1}| > \varepsilon$, we have $\sigma(\lfloor s_1\rceil^{a/r_1}) = \varepsilon\cdot\text{sign}(\lfloor s_1\rceil^{a/r_1})$. It is clear that

$$\sup_{s_1\in\mathbb{R}}\{|\dot{\sigma}(\lfloor s_1\rceil^{a/r_1})|\} = 0,\ \forall|\lfloor s_1\rceil^{a/r_1}| > \varepsilon. \tag{9}$$

Similarly, for $|\lfloor s_1\rceil^{a/r_1}| \le \varepsilon$, we have

$$\begin{aligned}\sup_{s_1\in\mathbb{R}}\{|\dot{\sigma}(\lfloor s_1\rceil^{a/r_1})|\} &\le \frac{a}{r_1}|s_1|^{\frac{a}{r_1}-1}|s_2| \\ &= a_1(\beta_1)\varepsilon^{1-\frac{r_2}{a}}.\end{aligned} \tag{10}$$

Combining (9) and (10) together, we have

$$\sup_{s_1\in\mathbb{R}}\{|\dot{\sigma}(\lfloor s_1\rceil^{a/r_1})|\} \le a_1(\beta_1)\varepsilon^{1-\frac{r_2}{a}}$$

provided that $|\phi_2(s_1,s_2)| \le \varepsilon$. This completes the proof. ■

With the help of Lemma 3.1, we are ready to prove the following result.

Theorem 3.2: Under Assumptions 2.1, controller (5) finite-time establishes the SOSM $s_1 = s_2 = 0$ (i.e., $s = \dot{s} = 0$) in system (1).

Proof. First of all, we will give an outline of the proof. The proof is divided into two steps. Firstly, it can be shown that the sliding variables under controller (5) will converge to a small region described by

$$Q=\left\{(s_1,s_2): \left|\lfloor s_1\rceil^{\frac{a}{r_1}}\right|<\varepsilon, \left|\lfloor s_2\rceil^{\frac{a}{r_2}}+\beta_1^{\frac{a}{r_2}}\sigma(\lfloor s_1\rceil^{\frac{a}{r_1}})\right|<\varepsilon\right\}.$$

Secondly, we will continue to show that in the region Q controller (5) will reduce to controller (4), which will further stabilize the sliding variables to zero in a finite time.

In the following, we first prove that the sliding variables s_1, s_2 will converge to the region Q in a finite time. To simplify the proof, denote $S_2 = (s_1, s_2)$.

First of all, a contradiction argument will be used to prove that there exists a time instance t_1 such that

$$|\phi_2(S_2(t_1))| = \left|\lfloor s_2(t_1)\rceil^{a/r_2} + \beta_1^{a/r_2}\sigma(\lfloor s_1(t_1)\rceil^{a/r_1})\right| \le \frac{\varepsilon}{2}.$$

Otherwise, it can be assumed that $|\phi_2(S_2(t))| > \frac{\varepsilon}{2}, \forall t \ge 0$. We first consider the case when

$$\phi_2(S_2(t))=\lfloor s_2(t)\rceil^{\frac{a}{r_2}}+\beta_1^{\frac{a}{r_2}}\sigma(\lfloor s_1(t)\rceil^{\frac{a}{r_1}})>\frac{\varepsilon}{2}, \forall t\ge 0. \tag{11}$$

With the help of (5), (11) and Assumption 2.1, we have

$$\begin{aligned}\dot{s}_2 &= a(t,x) + b(t,x)\left[-\beta_2\cdot\text{sign}\left(\phi_2(S_2)\right)\right] \\ &= a(t,x) - \beta_2 b(t,x) \\ &\le -(\beta_2 K_m - C) < 0\end{aligned} \tag{12}$$

guaranteed by (6). It implies for $t \ge 0$,

$$s_2(t) < s_2(0) - (\beta_2 K_m - C)t. \tag{13}$$

By (11) and (13), we obtain for $t \ge 0$

$$\begin{aligned}\frac{\varepsilon}{2} &< \lfloor s_2(t)\rceil^{a/r_2} + \beta_1^{a/r_2}\sigma(\lfloor s_1(t)\rceil^{a/r_1}) \\ &\le \lfloor s_2(0) - (\beta_2 K_m - C)t\rceil^{a/r_2} + \beta_1^{a/r_2}\varepsilon.\end{aligned} \tag{14}$$

As time goes to infinity, (14) leads to $\frac{\varepsilon}{2} < -\infty$, which is a contradiction. Similarly, we can show the case $\phi_2(S_2(t)) < -\frac{\varepsilon}{2},\ \forall t \ge 0$, is also impossible. In conclusion, there must exist a time instance t_1 such that

$$|\phi_2(S_2(t_1))| = \left|\lfloor s_2(t_1)\rceil^{a/r_2} + \beta_1^{a/r_2}\sigma(\lfloor s_1(t_1)\rceil^{a/r_1})\right| \le \frac{\varepsilon}{2}.$$

Next, we will prove that the following estimate holds

$$|\phi_2(S_2(t))| < \varepsilon, \quad \text{for} \quad t \ge t_1. \tag{15}$$

Suppose (15) is not true, which means there is at least one time instance t_1^* such that $|\phi_2(S_2(t_1^*))| = \varepsilon$. Specifically, there are $t_1' \in [t_1, +\infty)$ and $t_1^* \in (t_1', +\infty)$ such that either

$$\phi_2(S_2(t_1')) = \varepsilon/2 \tag{16}$$

$$\phi_2(S_2(t_1^*)) = \varepsilon \tag{17}$$

$$\frac{\varepsilon}{2} \le \phi_2(S_2(t)) \le \varepsilon,\ t \in [t_1', t_1^*] \tag{18}$$

in the positive region, or

$$\begin{aligned}&\phi_2(S_2(t_1')) = -\varepsilon/2, \\ &\phi_2(S_2(t_1^*)) = -\varepsilon, \\ &-\varepsilon \le \phi_2(S_2(t)) \le -\tfrac{\varepsilon}{2},\ t \in [t_1', t_1^*]\end{aligned}$$

as the negative case.

We first prove that the positive case (16)-(17)-(18) is impossible. By (18), we can check that (12) remains hold at time interval $[t_1', t_1^*]$, i.e.,

$$\dot{s}_2(t) \le -(\beta_2 K_m - C),\ t \in [t_1', t_1^*], \tag{19}$$

which implies

$$(\beta_2 K_m - C)(t_1^* - t_1') \le s_2(t_1') - s_2(t_1^*). \tag{20}$$

Meanwhile, by (16) and Lemma 2.2, it is not difficult to obtain

$$s_2(t_1') \le (\varepsilon/2 + \beta_1^{a/r_2}\varepsilon)^{r_2/a} \le \varepsilon^{r_2/a}(1+\beta_1). \tag{21}$$

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Similarly, by (17), we also have

$$s_2(t_1^*) \geq -\varepsilon^{r_2/a}(1+\beta_1). \tag{22}$$

Together with (21) and (22), (20) leads to the following estimate

$$t_1^* - t_1' \leq \frac{s_2(t_1') - s_2(t_1^*)}{\beta_2 K_m - C} \leq \frac{2\varepsilon^{r_2/a}(1+\beta_1)}{\beta_2 K_m - C}. \tag{23}$$

Furthermore, by (19), we have $s_2(t_1^*) \leq s_2(t_1')$, which implies

$$\begin{aligned}
&\lfloor s_2(t_1^*)\rceil^{a/r_2} + \beta_1^{a/r_2}\sigma(\lfloor s_1(t_1^*)\rceil^{a/r_1}) \\
&\leq \lfloor s_2(t_1')\rceil^{a/r_2} + \beta_1^{a/r_2}\sigma(\lfloor s_1(t_1')\rceil^{a/r_1}) - \beta_1^{a/r_2}\sigma(\lfloor s_1(t_1')\rceil^{a/r_1}) \\
&+ \beta_1^{a/r_2}\sigma(\lfloor s_1(t_1^*)\rceil^{a/r_1})
\end{aligned} \tag{24}$$

Substituting (16) and (17) into (24) obtains

$$\varepsilon/2 \leq \left|\beta_1^{\frac{a}{r_2}}\sigma(\lfloor s_1(t_1^*)\rceil^{\frac{a}{r_1}}) - \beta_1^{\frac{a}{r_2}}\sigma(\lfloor s_1(t_1')\rceil^{\frac{a}{r_1}})\right|. \tag{25}$$

Using (18), we have

$$|\phi_2(S_2)| \leq \varepsilon,\ t \in [t_1', t_1^*] \tag{26}$$

which enables us to use Lemma 3.1 to estimate (25) as

$$\begin{aligned}
\varepsilon/2 &\leq \left|\beta_1^{a/r_2}\sigma(\lfloor s_1(t_1^*)\rceil^{a/r_1}) - \beta_1^{a/r_2}\sigma(\lfloor s_1(t_1')\rceil^{a/r_1})\right| \\
&\leq \beta_1^{a/r_2} \sup_{s_1 \in \mathbb{R}}\{|\dot{\sigma}(\lfloor s_1\rceil^{a/r_1})|\}(t_1^* - t_1') \\
&\leq \beta_1^{a/r_2} a_1(\beta_1)\varepsilon^{1-\frac{r_2}{a}}(t_1^* - t_1').
\end{aligned} \tag{27}$$

Substituting (23) into (27) and by the choice of β_2 from (6), we have

$$\varepsilon/2 \leq \tfrac{2\varepsilon(1+\beta_1)}{\beta_2 K_m - C}\beta_1^{a/r_2} a_1(\beta_1) < \varepsilon/2 \tag{28}$$

which obviously is a contradiction. Therefore the case of (16)-(17)-(18) will never happen. Similarly, it can be shown that using an almost same argument as the positive case, the term $\lfloor s_2(t)\rceil^{a/r_2} + \beta_1^{a/r_2}\sigma(\lfloor s_1(t)\rceil^{a/r_1})$ will never cross $-\varepsilon$ either. Hence for $t \geq t_1$, we have

$$S_2(t) \in \Big\{S_2 : |\phi_2(S_2(t))| < \varepsilon\Big\}.$$

On the other hand, by using Lemma 2.1, for $t \geq t_1$, one obtains

$$\begin{aligned}
&\left|s_2 + \beta_1\lfloor\sigma(\lfloor s_1\rceil^{a/r_1})\rceil^{r_2/a}\right| \\
&\leq \left|\lfloor s_2\rceil^{\frac{a}{r_2}\times\frac{r_2}{a}} - \left\lfloor -\beta_1\lfloor\sigma(\lfloor s_1\rceil^{a/r_1})\rceil^{r_2/a}\right\rceil^{\frac{a}{r_2}\times\frac{r_2}{a}}\right| \\
&\leq 2^{1-\frac{r_2}{a}}\left|\lfloor s_2\rceil^{a/r_2} + \beta_1^{a/r_2}\sigma(\lfloor s_1\rceil^{a/r_1})\right|^{\frac{r_2}{a}} \\
&\leq 2^{1-\frac{r_2}{a}}\varepsilon^{\frac{r_2}{a}}.
\end{aligned} \tag{29}$$

With the help of the above relations, it can be concluded from

$$\dot{s}_1 = -\beta_1\lfloor\sigma(\lfloor s_1\rceil^{\frac{a}{r_1}})\rceil^{\frac{r_2}{a}} + (s_2 + \beta_1\lfloor\sigma(\lfloor s_1\rceil^{\frac{a}{r_1}})\rceil^{\frac{r_2}{a}}) \tag{30}$$

that for $t \geq t_1$,

$$\begin{aligned}
\dot{s}_1(t) &\leq -\mu_1\varepsilon^{\frac{r_2}{a}} < 0,\ \text{for}\ \lfloor s_1(t)\rceil^{a/r_1} \geq \varepsilon \\
\dot{s}_1(t) &\geq \mu_1\varepsilon^{\frac{r_2}{a}} > 0,\ \text{for}\ \lfloor s_1(t)\rceil^{a/r_1} \leq -\varepsilon
\end{aligned}$$

where $\mu_1 = \beta_1 - 2^{1-\frac{r_2}{a}} > 0$ with β_1 as defined in (6). It follows that there exists a $t_2 \geq t_1$ such that for $t \geq t_2$ we have $\left|\lfloor s_1\rceil^{a/r_1}\right| < \varepsilon$. Therefore, when $t \geq t_2$, $S_2(t)$ will enter and stay in the region $Q = \Big\{S_2 : \left|\lfloor s_1(t)\rceil^{a/r_1}\right| < \varepsilon, \left|\lfloor s_2(t)\rceil^{a/r_2} + \beta_1^{a/r_2}\sigma(\lfloor s_1(t)\rceil^{a/r_1})\right| < \varepsilon\Big\}$.

In the region Q , it is clear that controller (5) will reduce to controller (4). According to Theorem 3.1, one can conclude that the closed-loop system (3),(4) is finite-time stable. Thus the controller will further stabilize the sliding variables to zero in a finite time. This completes the proof of Theorem 3.2. ■

Remark 3.2: It can be clearly seen that controller (5) is constructed by combining controller (4) with a saturation function. They are both SOSM controllers which could stabilize the sliding variables to zero in a finite time. The main advantage of controller (5) lies in that it can change the phase trajectory of sliding variables by tuning the saturation level. This property allows controller (5) to provide a larger domain of attraction under state constraints than SOSM controller (4) could provide.

Corollary 3.1: Under Assumption 2.1 and controller (5) with $a = r_1 = 2r_2$, if $\beta_2 K_m - C > \frac{1}{2}\beta_1^2$, the sliding variables will first reach the terminal sliding surface $\lfloor s_2(t)\rceil^{a/r_2} + \beta_1^{a/r_2}\sigma(\lfloor s_1(t)\rceil^{a/r_1}) = 0$ and then converge to the origin along the sliding surface in a finite time.

Proof. When $a = r_1 = 2r_2$, the equation $\phi_2(S_2) = \lfloor s_2\rceil^{a/r_2} + \beta_1^{a/r_2}\sigma(\lfloor s_1\rceil^{a/r_1})$ reduces to $\phi_2(S_2) = \lfloor s_2\rceil^2 + \beta_1^2\sigma(s_1)$. Directly taking the derivative of $\phi_2(S_2)$ along system (3) yields

$$\dot{\phi}_2 = 2|s_2|\dot{s}_2 + \beta_1^2 s_2,\ \text{for}\ |s_1| \leq \varepsilon \tag{31}$$

and

$$\dot{\phi}_2 = 2|s_2|\dot{s}_2,\ \text{for}\ |s_1| > \varepsilon. \tag{32}$$

It follows from (31) and (32) that

$$\phi_2\dot{\phi}_2 = 2|s_2|\dot{s}_2\phi_2 + \beta_1^2 s_2\phi_2,\ \text{for}\ |s_1| \leq \varepsilon \tag{33}$$

and

$$\phi_2\dot{\phi}_2 = 2|s_2|\dot{s}_2\phi_2,\ \text{for}\ |s_1| > \varepsilon. \tag{34}$$

Hence, we have

$$\begin{aligned}
\phi_2\dot{\phi}_2 &\leq 2|s_2|\dot{s}_2\phi_2 + \beta_1^2|s_2||\phi_2| \\
&\leq 2|s_2|\left(a(t,x) + b(t,x)u\right)\phi_2 + \beta_1^2|s_2||\phi_2|.
\end{aligned} \tag{35}$$

With Assumption 2.1 in mind, substituting (5) into (35) yields

$$\begin{aligned}
\phi_2\dot{\phi}_2 &\leq -2\beta_2 K_m|s_2|\text{sign}(\phi_2)\phi_2 + 2C|s_2||\phi_2| + \beta_1^2|s_2||\phi_2| \\
&= -2\beta_2 K_m|\phi_2||s_2| + 2C|s_2||\phi_2| + \beta_1^2|s_2||\phi_2| \\
&= |s_2||\phi_2|\left[-2(\beta_2 K_m - C) + \beta_1^2\right].
\end{aligned} \tag{36}$$

Note that $\beta_2 K_m - C > \frac{1}{2}\beta_1^2$. This, together with (36), produces $\phi_2 \cdot \dot{\phi}_2 \leq -\left[2(\beta_2 K_m - C) - \beta_1^2\right]|s_2||\phi_2|$. According to nonsingular terminal SMC theory [38], the sliding variables will first reach the terminal sliding surface $\phi_2(S_2) = 0$ and then converge to zero in a finite time along the sliding surface, as shown in Fig. 1 (a) or (b). This completes the proof of Corollary 3.1. ■

Remark 3.3: It can be seen that when the saturation level ε is tuned to be large enough, controller (5) will reduce to controller (4). Since the saturation level ε has no direct relation

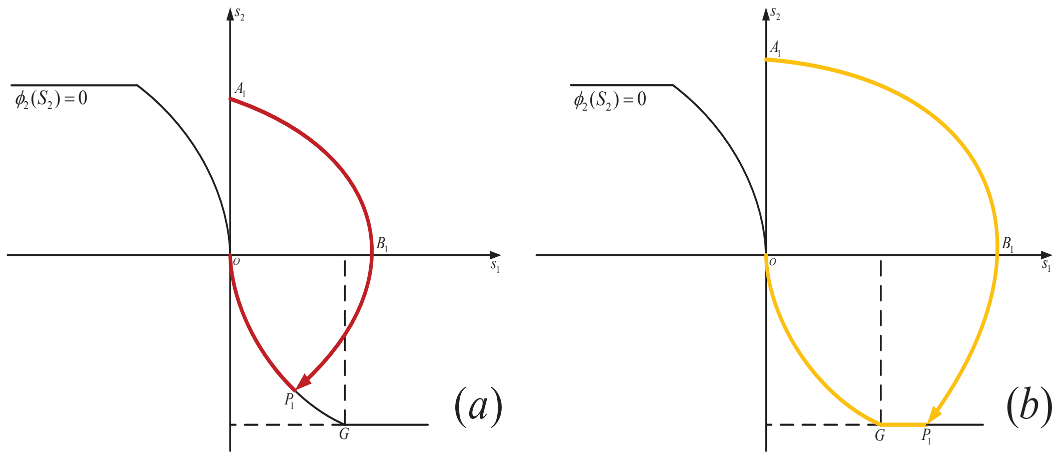

Fig. 1. Phase trajectory of system (3),(5) when $\beta_2 K_m - C > \frac{1}{2}\beta_1^2$

with the control parameters β_1 and β_2, it can be concluded from Corollary 3.1 that the conservative conditions imposed on β_2 in Theorem 3.1 may be partly relaxed. Similarly, the precondition (6) for Theorem 3.2 can also be relaxed.

IV. SOSM Control of Nonlinear Constrained Systems

In this section, we will show that controller (5) could provide the maximum domain of attraction for some nonlinear constrained systems.

For system (1), we assume that the state x satisfies a constraint, which can be described by $x \in \Phi \subset \mathbb{R}^n$ with Φ being a closed and bounded set of the origin. Consequently, the state variables are usually bounded. Under this circumstance, since the sliding variables are functions of the state x, it can be concluded that the sliding variables of system (2) will satisfy the following box constraint

$$(s, \dot{s}) \in \Pi = \left\{(s, \dot{s}) : s \in [\underline{\delta}_1, \bar{\delta}_1], \dot{s} \in [\underline{\delta}_2, \bar{\delta}_2]\right\} \tag{37}$$

with $\underline{\delta}_1, \underline{\delta}_2 < 0$ and $\bar{\delta}_1, \bar{\delta}_2 > 0$.

Remark 4.1: Note that the state variable x has been restricted to be in the bounded region Φ. Also the sliding variables s and $\dot{s}$ are functions of state variables, which implies that the sliding variables are also bounded. To this end, there exist constants $\underline{\delta}_1, \underline{\delta}_2 < 0$ and $\bar{\delta}_1, \bar{\delta}_2 > 0$ such that $(s, \dot{s}) \in \Pi$. On the other hand, according to Part B of Section III in [33], if there exists a diffeomorphism $\Omega = (\zeta, S) : R^n \to R^{n-2} \times R^2$ with $S = (s, \dot{s})$ and ζ being an internal variable, then system (1) can be transformed to system (2) plus ζ-dynamics. This also implies that there exists a new set Φ_ω such that $x \in \Phi \Leftrightarrow (S_2, \zeta) \in \Phi_\omega$. If the set Φ_ω does not depend on the internal state ζ, then the state constraint Φ may be equal to the sliding variable constraint Π.

Then we have the following result.

Theorem 4.1: When the sliding variables are constrained by (37), under controller (5) the sliding variables s_1, s_2 can be steered to the origin with the maximum domain of attraction

$$\Omega = \left\{S_2 : \underline{\delta}_1 + \frac{s_2^2(t)}{2(\beta_2 K_m - C)} < s_1(t) < \bar{\delta}_1 - \frac{s_2^2(t)}{2(\beta_2 K_m - C)}\right\} \cap \Pi.$$

Proof. To prove Theorem 4.1, we only need to show that the sliding variables will not escape from the region Ω and further show that the region Ω is the maximum domain of attraction. The region Ω can be depicted as in Fig. 2.

At first, we prove that the sliding variables will not escape from the region Ω. In other words, S_2 will not cross the segments $AB, BC, CD, DE, EF, FG, GH$ and HA (see Fig. 2).

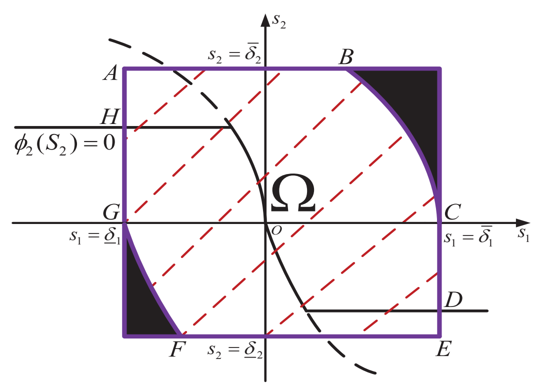

Fig. 2 Box constraint in the SOSM

Case 1 ($S_2 \in BC \cup FG$): The curve BC, which crosses the point C in the worst case, can be described by the following equation

$$\bar{\phi}_2(S_2) = s_1 + \frac{s_2^2(t)}{2(\beta_2 K_m - C)} - \bar{\delta}_1 = 0. \tag{38}$$

Taking derivative of $\bar{\phi}_2(S_2)$ along system (3) yields

$$\begin{aligned}\dot{\bar{\phi}}_2(S_2) &= s_2 + \frac{s_2(a(t,x) + b(t,x)u)}{\beta_2 K_m - C} \\ &= s_2 + \frac{s_2(a(t,x) - \beta_2 \cdot \mathrm{sign}(\phi_2(S_2)) \cdot b(t,x))}{\beta_2 K_m - C}.\end{aligned} \tag{39}$$

Note that when $S_2 \in BC$, we have $s_2 \geq 0$ and $\phi_2(S_2) = \lfloor s_2(t) \rceil^{a/r_2} + \beta_1^{a/r_2} \sigma(\lfloor s_1(t) \rceil^{a/r_1}) > 0$. It follows from (39)

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that for $\forall S_2 \in BC$,

$$\begin{aligned}\dot{\bar{\phi}}_2(S_2) &= s_2 + \frac{s_2(a(t,x) - \beta_2 b(t,x))}{\beta_2 K_m - C} \\ &\leq s_2 - \frac{(\beta_2 K_m - C)s_2}{\beta_2 K_m - C} = 0. \end{aligned} \tag{40}$$

This implies that the sliding variables will not escape from the region Ω by crossing the curve BC. Following the same lines to obtain (38), the curve FG can be described by the following equation

$$\underline{\phi}_2(S_2) = s_1 - \frac{s_2^2(t)}{2(\beta_2 K_m - C)} - \underline{\delta}_1 = 0.$$

Taking derivative of $\underline{\phi}_2(S_2)$ along system (3) yields

$$\begin{aligned}\dot{\underline{\phi}}_2(S_2) &= s_2 - \frac{s_2(a(t,x) + b(t,x)u)}{\beta_2 K_m - C} \\ &= -s_2 \left[\frac{a(t,x) - \beta_2 \cdot \mathrm{sign}(\phi_2(S_2)) \cdot b(t,x)}{\beta_2 K_m - C} - 1 \right]. \end{aligned} \tag{41}$$

Note that when $S_2 \in FG$, we have $s_2 \leq 0$ and $\phi_2(S_2) = \lfloor s_2(t) \rceil^{a/r_2} + \beta_1^{a/r_2} \sigma(\lfloor s_1(t) \rceil^{a/r_1}) < 0$. It follows from (41) that for $\forall S_2 \in FG$

$$\begin{aligned}\dot{\underline{\phi}}_2(S_2) &= -s_2 \left(\frac{a(t,x) + \beta_2 b(t,x)}{\beta_2 K_m - C} - 1 \right) \\ &\geq |s_2| \left(\frac{-C + \beta_2 K_m}{\beta_2 K_m - C} - 1 \right) \\ &\geq 0, \end{aligned} \tag{42}$$

which indicates that the phase trajectory of sliding variables will not escape from the region Ω through the segment FG. Therefore, it can be concluded that the phase trajectory will not escape the region Ω from the segments $BC \cup FG$.

Case 2 ($S_2 \in HA \cup AB$): Note that when $S_2 \in HA \cup AB$

$$\begin{aligned}\dot{s}_1 &= s_2 > 0, \\ \dot{s}_2 &= a(t,x) - \beta_2 \cdot \mathrm{sign}(\phi_2(S_2)) \cdot b(t,x) \\ &\leq -(\beta_2 K_m - C) < 0, \end{aligned}$$

which implies the phase trajectory of sliding variables is always pointing down-right. Therefore, it will not escape the region Ω from the segments $HA \cup AB$.

Case 3 ($S_2 \in DE \cup EF$): When $S_2 \in DE \cup EF$, one obtains

$$\begin{aligned}\dot{s}_1 &= s_2 < 0, \\ \dot{s}_2 &= a(t,x) - \beta_2 \cdot \mathrm{sign}(\phi_2(S_2)) \cdot b(t,x) \\ &\geq \beta_2 K_m - C > 0. \end{aligned}$$

It is immediate to check that the phase trajectory of sliding variables is always pointing up-left, which implies it will not escape the region Ω from the segments $DE \cup EF$.

Case 4 ($S_2 \in CD$): When $S_2 \in CD$, we have

$$\begin{aligned}\dot{s}_1 &= s_2 < 0, \\ \dot{s}_2 &= a(t,x) - \beta_2 \cdot \mathrm{sign}(\phi_2(S_2)) \cdot b(t,x) \\ &\leq -(\beta_2 K_m - C) < 0. \end{aligned}$$

As a consequence, the phase trajectory of sliding variables is always pointing down-left. This also indicates that the phase trajectory will not escape the region Ω from the segment CD.

Case 5 ($S_2 \in GH$): When $S_2 \in GH$, one obtains

$$\begin{aligned}\dot{s}_1 &= s_2 > 0, \\ \dot{s}_2 &= a(t,x) - \beta_2 \cdot \mathrm{sign}(\phi_2(S_2)) \cdot b(t,x) \\ &\geq \beta_2 K_m - C > 0. \end{aligned}$$

It is easy to see that the phase trajectory of sliding variables is always pointing up-right. In other words, it will not escape the region Ω from the segment GH.

As a result, it follows from **Cases 1, 2, 3, 4, 5** that the phase trajectory of the sliding variables will not escape the region Ω. Then, according to Theorem 3.2, it can be concluded that the sliding variables will be further driven to the origin in a finite time.

In the following, we will prove that the region Ω is the maximum domain of attraction. The proof is the same to that in [33]. For the completeness, we give a brief proof.

Assume that $S_2(0) \in \left\{ S_2 : s_1 + \frac{s_2^2}{2(\beta_2 K_m - C)} > \bar{\delta}_1 \right\} \cap \left\{ S_2 : s_1 \in [\underline{\delta}_1, \bar{\delta}_1], s_2 \in [\underline{\delta}_2, \bar{\delta}_2] \right\}$. Then the sliding variable s_1 and s_2 will move on a parabolic arc and the worst case of the phase trajectory of sliding variables can be described by

$$s_1 = -\frac{s_2^2}{2(\beta_2 K_m - C)} + \bar{\delta}_1 + \lambda$$

with a small $\lambda > 0$. Apparently, the above parabolic arc will intersect the line $s_1 = \bar{\delta}_1$, which implies that the given $S_2(0)$ does not belong to the domain of attraction. For the case $S_2(0) \in \left\{ S_2 : s_1 - \frac{s_2^2}{2(\beta_2 K_m - C)} < \underline{\delta}_1 \right\} \cap \left\{ S_2 : s_1 \in [\underline{\delta}_1, \bar{\delta}_1], s_2 \in [\underline{\delta}_2, \bar{\delta}_2] \right\}$, the proof is similar. Therefore, the region Ω is the maximum domain of attraction. This completes the proof of Theorem 4.1. ■

Remark 4.2: In the literature, there are several known SOSM algorithms, such as twisting algorithm, suboptimal algorithm, quasi-continuous algorithm, etc. However, when applying these algorithms, it is clear that the phase trajectory from some initial states will intersect the segment AB or EF shown in Fig. 2. In this case, it is obvious that under the same control saturation the domain of attraction of the existing SOSM algorithms is less than that given in Theorem 4.1. To fix this problem, a switching control scheme has been proposed in [33] for the same problem. The sliding mode controller proposed in [33] can be shown as follows

$$u = \begin{cases} -\beta_2 \cdot \mathrm{sign}(s_1 + \frac{s_2|s_2|}{2(\beta_2 K_m - C)}), & \text{for } S_2 \in \prod \\ -\beta_2 \cdot \mathrm{sign}(s_2), & \text{for } S_2 \notin \prod. \end{cases} \tag{43}$$

The main difference between this paper and [33] lies in that we only use one controller to obtain the maximum domain of attraction and thus the switching control can be avoided.

Remark 4.3: As a matter of fact, the SMC problems for nonlinear systems under state constraints have been widely discussed in the literatures, such as [30]–[32]. The difference between [30]–[32] and this paper is that the considered systems are different. It can be seen that the relative degree of

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sliding mode in [30]–[32] is one, while the relative degree of sliding mode in this paper is two. Consequently, the results proposed in [30]–[32] can not be directly applied to the systems considered in this paper. In addition, the maximum domain of attraction for the closed-loop system has been given in this paper, while it has never been considered in the existing results on first-order SMC of nonlinear constrained systems.

Remark 4.4: On the one hand, the saturation level ε should be large enough to ensure the rapid convergence of the sliding variables. On the other hand, it is required to be small enough to guarantee that the phase trajectory will not violate the boundary of the sliding variable constraint, because the different ε implies the different phase trajectory of the sliding variables. The above two conditions determine how to choose the saturation level ε. From Fig. 2, it can be deduced that when $CD = CE$ or $GH = GA$, the saturation level ε will achieve its maximum value. By a simple calculation, one obtains the saturation level ε can be chosen from the open interval $\left(0, \frac{\left|\min\{\bar{\delta}_2, -\underline{\delta}_2\}\right|^2}{\beta_1^2}\right)$. In order to obtain a satisfactory convergence, the value of ε should be close to $\frac{\left|\min\{\bar{\delta}_2, -\underline{\delta}_2\}\right|^2}{\beta_1^2}$.

V. Example and Simulation

In this section, we will give an example to verify the effectiveness of the proposed SOSM algorithm. Consider a pendulum system with coulomb friction and external disturbance described by the following equation [39]

$$\ddot{\theta} = \frac{1}{J}u - \frac{MgL}{2J}\sin(\theta) - \frac{V_s}{J}\dot{\theta} - \frac{P_s}{J}\text{sign}(\dot{\theta}) + d(t) \quad (44)$$

where θ is the swing angle measured from the equilibrium position, $\dot{\theta}$ is the swing speed measured of the pendulum, $M = 1.1$ Kg is the pendulum mass, $g = 9.815\ \text{m/s}^2$ is the gravitational constant, $L = 0.9$ m is the arm length, $J = ML^2 = 0.891\ \text{Kg}\cdot\text{m}^2$ is the arm inertia, $V_s = 0.18\ \text{Kg}\cdot\text{m}^2/\text{s}$ is the viscous friction coefficient, $P_s = 0.45\ \text{Kg}\cdot\text{m}^2/\text{s}$ is the coulomb friction coefficient, and $d(t)$ is the disturbance.

The control objective here is to design a controller allowing the inverted pendulum to reach the equilibrium position quickly as swing angle and swing speed satisfy the state constraint

$$\Phi = \left\{(\theta, \dot{\theta}) : \theta \in [-\frac{\pi}{4}, \frac{\pi}{3}], \dot{\theta} \in [-1.2, 1.8]\right\}. \quad (45)$$

Let $s_1 = \theta, s_2 = \dot{\theta}$. Then system (44) can be rewritten as

$$\dot{s}_1 = s_2, \quad \dot{s}_2 = a(t, x) + b(t, x)u \quad (46)$$

with $a(t, x) = -\frac{MgL}{2J}\sin(s_1) - \frac{V_s}{J}s_2 - \frac{P_s}{J}\text{sign}(s_2) + d(t), b(t, x) = \frac{1}{J}$. Additionally, it is not difficult to verify that the sliding variables also satisfy the following box constraint

$$\Pi = \left\{(s, \dot{s}) : s \in [-\frac{\pi}{4}, \frac{\pi}{3}], \dot{s} \in [-1.2, 1.8]\right\}. \quad (47)$$

Here we assume $d(t) = 0.5\sin(2t) + 0.5\cos(5t)$. When the sliding variables satisfy the box constraint (47), it is reasonable to choose $|a(t, x)| \le C = \frac{MgL}{2J} + \frac{1.8V_s}{J} + \frac{P_s}{J} + 0.5\sqrt{2} = 7.03$ and $K_m = \frac{1}{J} = 1.1$.

Select $a = 1, r_1 = 1, r_2 = \frac{1}{2}, \varepsilon = \frac{\pi}{12}$. According to Theorem 4.1, the following controller

$$u = -\beta_2 \cdot \text{sign}\left(\lfloor s_2\rceil^2 + \beta_1^2\sigma(s_1)\right). \quad (48)$$

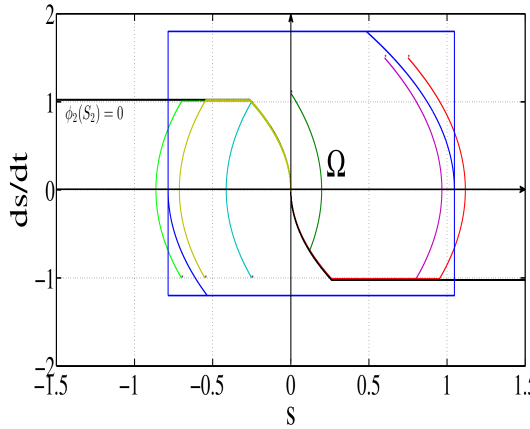

Fig. 3 Phase trajectory of $s_1 - s_2$ under controller (48) with $\beta_1 = 2, \beta_2 = 9, \varepsilon = \frac{\pi}{12}$

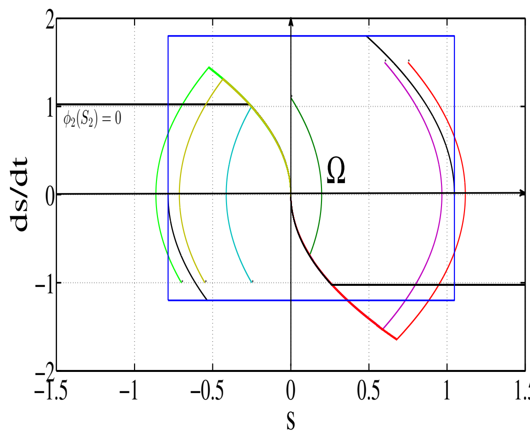

Fig. 4 Phase trajectory of $s_1 - s_2$ under controller (49) with $\beta_1 = 2, \beta_2 = 9$

will achieve the maximum domain of attraction as

$$\begin{aligned}\Omega \ &= \Big\{S_2 : -\tfrac{\pi}{4} + \tfrac{s_2^2(t)}{2\times(1.1\beta_2 - 7.03)} < s_1(t) \\ &\quad < \tfrac{\pi}{3} - \tfrac{s_2^2(t)}{2\times(1.1\beta_2 - 7.03)}\Big\} \cap \Pi.\end{aligned}$$

Let the initial states be $(0, 1.1), (0.6, 1.5), (0.75, 1.5), (-0.25, -1), (-0.55, -1), (-0.7, -1)$, respectively. We choose $\beta_1 = 2, \beta_2 = 9$, the simulation results under controller (48) are shown in Fig. 3. Without considering the state saturation, controller (48) reduces to

$$u = -\beta_2 \cdot \text{sign}\left(\lfloor s_2\rceil^2 + \beta_1^2 s_1\right). \quad (49)$$

Under the same control parameters and initial states, the simulation results under controller (49) are shown in Fig. 4.

It can be clearly seen from Figs. 4 and 5 that when the initial states of sliding variables close to the origin (e.g. $(0, 1.1)$ and $(-0.25, -1)$), the phase trajectories under controllers (48) and (49) are the same. Yet if the initial states of sliding variables close to the boundary of Ω (e.g. $(0.6, 1.5)$), the phase trajectory under controller (49) will intersect the boundary, while the

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phase trajectory under controller (48) can still be restricted in the region Ω by tuning the saturation level ε. However, if the initial states are located outside of the domain of attraction Ω (e.g. $(0.75, 1.5)$ and $(-0.7, -1)$), the phase trajectories under controllers (48) and (49) will both escape from the box constraint. This implies that controller (48) provides the larger domain of attraction by comparing with controller (49).

VI. Conclusion

In this paper, a new SOSM control scheme has been developed for the possibility of controlling nonlinear constrained systems with the maximum domain of attraction. The proposed SOSM control algorithm is based on the saturation technique and backstepping-like method. The feature of the proposed algorithm lies that the yielding controller includes a state saturation term, which could change the phase trajectory of the closed-loop sliding mode dynamics. The Lyapunov method has been used to test the finite-time stability of the sliding variables. Rigorous mathematical analysis shows that under the proposed SOSM controller a maximum domain of attraction can be obtained for the considered nonlinear systems with a preset state constraint. Further research will be focused on higher-order ($n \geq 3$) SMC design under state constraints.

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