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Consensus of Hybrid Multi-Agent Systems

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***Abstract*— In this brief, we consider the consensus problem of hybrid multiagent systems. First, the hybrid multiagent system is proposed, which is composed of continuous-time and discrete-time dynamic agents. Then, three kinds of consensus protocols are presented for the hybrid multiagent system. The analysis tool developed in this brief is based on the matrix theory and graph theory. With different restrictions of the sampling period, some necessary and sufficient conditions are established for solving the consensus of the hybrid multiagent system. The consensus states are also obtained under different protocols. Finally, simulation examples are provided to demonstrate the effectiveness of our theoretical results.**

***Index Terms*— Consensus, continuous time, discrete time, hybrid multiagent systems.**

I. Introduction

The investigation of multiagent coordination has regularly attracted contributions from mathematicians, physicists, and engineers over the two decades. Classic multiagent coordination of interests include consensus [1], flocking [2], containment control [3], formation control [4], coverage control [5], and distributed estimation [6]. Much interest is focusing on dynamic models of agents. Examples include the study of distributed coordination of the first-order, second-order, continuous-time, and discrete-time dynamic agents. By using the matrix theory, the graph theory, the frequency-domain analysis method, and the Lyapunov method, lots of results about multiagent coordination have been obtained [7]–[10]. A closely related problem to multiagent coordination is the synchronization of complex dynamical networks and neural networks [11]–[13].

As a fundamental problem of distributed coordination, consensus means that a group of agents reach an agreement on the consistent quantity of interest by designing appropriate control input based on local information. Vicsek *et al.* [14] proposed a discrete-time model of n agents all moving in the plane with the same speed and demonstrated by simulation that all agents move to one direction. By virtue of graph theory, Jadbabaie *et al.* [15] explained the consensus behavior of Vicsek model theoretically and showed that consensus can be achieved if the union of interaction graphs are connected frequently enough. By utilizing the preleader-follower decomposition, Wang and Xiao [16] studied the state consensus of discrete-time multiagent systems with bounded time delays. The second-order consensus of discrete-time multiagent systems was considered in [17] with nonuniform time delays. Gossip algorithms [18] and broadcast gossip algorithms [19] were also used to analyze the consensus problem, respectively. For continuous-time dynamic agents, Olfati-Saber and Murray [20] considered the consensus problem of multiagent systems with switching topologies and time delays and obtained some necessary and/or sufficient conditions for solving the average consensus. Ren and Beard [21] extended the results given in [20] and presented some more relaxable conditions for solving the consensus. Xie and Wang [22] studied the second-order consensus of multiagent systems with fixed and switching topologies. Ren [23] investigated the second-order consensus of multiagent systems in four cases. Qin and Yu [24] considered the second-order consensus problem under independent position and velocity topologies.

Hybrid systems are dynamical systems that involve the interaction of continuous and discrete dynamics. As a class of classic hybrid systems, switched systems have received much attention [25]. For multiagent systems, lots of references were concerned with consensus problem under switching topologies [20], [26]. However, Zheng and Wang [27] considered the consensus of switched multiagent system that consists of continuous-time and discrete-time subsystems. They proved that the consensus of the switched multiagent system is solvable under arbitrary switching. Zhu *et al.* [28] studied the containment control of such switched multiagent system. In the practical systems, the dynamics of the agents coupled with each others are different, i.e., the dynamics of agents are hybrid. In general, hybrid means heterogeneous in nature or composition. Different from the previous study, Zheng *et al.* [29] considered the consensus of heterogeneous multiagent system, which is composed of the first-order and second-order dynamic agents. The consensus criteria were obtained under different topologies [30], [31]. Finite-time consensus and containment control of the heterogeneous multiagent system were also studied in [32] and [33], respectively.

To the best of our knowledge, however, the existing results of consensus analysis are on homogeneous and heterogeneous multiagent systems. In the real world, natural and artificial individuals can show collective decision making. For example, autonomous robots were used to control self-organized behavioral patterns in group-living cockroaches [34]. When the continuous-time and discrete-time dynamic agents coexist and interact with each other, it is important to study the consensus problem of hybrid multiagent systems. Owing to the coexistence of continuous-time and discrete-time dynamic agents, it is difficult to design the consensus protocol and analyze the consensus problem for hybrid multiagent systems. Up to now, there does not exist any method to analyze the consensus of hybrid multi-agent systems. The main objective of this brief is to design the consensus protocols and obtain the consensus criteria of the hybrid multiagent system in different topologies. The main contribution of this brief is threefold. First, we propose the hybrid multiagent system and give the definition of consensus. Second, three kinds of consensus protocols are presented for the hybrid multiagent system. Finally, by utilizing of the graph theory, some necessary and sufficient conditions are obtained for solving the consensus of the hybrid multiagent system.

The rest of this brief is organized as follows. In Section II, we present some notions in graph theory and propose the hybrid multiagent system. In Section III, three kinds of consensus protocols are provided for solving the consensus of the hybrid multiagent

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system. In Section IV, numerical simulations are given to illustrate the effectiveness of theoretical results. Finally, some conclusions are drawn in Section V.

Notation: Throughout this brief, the following notations will be used: $\mathbb{R}$ denotes the set of real number and $\mathbb{R}^n$ denotes the $n-$dimensional real vector space. $\mathcal{I}_m = \{1, 2, \ldots, m\}$, $\mathcal{I}_n/\mathcal{I}_m = \{m+1, m+2, \ldots, n\}$. For a given vector or matrix X, X^T denotes its transpose, $\|X\|$ denotes the Euclidean norm of a vector X, and $E(X)$ denotes its mathematical expectation. A vector is nonnegative if all its elements are nonnegative. Denote by $\mathbf{1}_n$ (or $\mathbf{0}_n$) the column vector with all entries equal to one (or all zeros). I_n is an $n-$dimensional identity matrix. $\text{diag}\{a_1, a_2 \cdots, a_n\}$ defines a diagonal matrix with diagonal elements being $a_1, a_2 \cdots, a_n$. Let $\mathbf{e}_i$ denote the canonical vector with a 1 in the ith entry and 0's elsewhere. $|\lambda|$ denotes the modulus of a complex number λ.

II. Preliminaries

A. Graph Theory

A weighted directed graph $\mathscr{G}(\mathscr{A}) = (\mathscr{V}, \mathscr{E}, \mathscr{A})$ of order n consists of a vertex set $\mathscr{V} = \{s_1, s_2, \cdots, s_n\}$, an edge set $\mathscr{E} = \{e_{ij} = (s_i, s_j)\} \subset \mathscr{V} \times \mathscr{V}$ and a nonnegative matrix $\mathscr{A} = [a_{ij}]_{n\times n}$. The neighbor set of the agent i is $\mathscr{N}_i = \{j : a_{ij} > 0\}$. A directed path between two distinct vertices s_i and s_j is a finite ordered sequence of distinct edges of $\mathscr{G}$ with the form $(s_i, s_{k_1}), (s_{k_1}, s_{k_2}), \cdots, (s_{k_l}, s_j)$. A directed tree is a directed graph, where there exists a vertex called the root, such that there exists a unique directed path from this vertex to every other vertex. A directed spanning tree is a directed tree, which consists of all the nodes and some edges in $\mathscr{G}$. If a directed graph has the property that $(s_i, s_j) \in \mathscr{E} \Leftrightarrow (s_j, s_i) \in \mathscr{E}$, the directed graph is called undirected. An undirected graph is said to be connected if there exists a path between any two distinct vertices of the graph. The degree matrix $\mathscr{D} = [d_{ij}]_{n\times n}$ is a diagonal matrix with $d_{ii} = \sum_{j:s_j\in\mathscr{N}_i} a_{ij}$ and the Laplacian matrix of the graph is defined as $\mathscr{L} = [l_{ij}]_{n\times n} = \mathscr{D} - \mathscr{A}$. It is easy to see that $\mathscr{L}\mathbf{1}_n = 0$.

A nonnegative matrix is said to be a (row) stochastic matrix if all its row sums are 1. A stochastic matrix $P = [p_{ij}]_{n\times n}$ is called indecomposable and aperiodic (SIA) if $\lim_{k\to\infty} P^k = \mathbf{1}y^T$, where y is some column vector. $\mathscr{G}$ is said the graph associated with P when $(s_i, s_j) \in \mathscr{E}$ if and only if $p_{ji} > 0$. The following results propose the relationship between a stochastic matrix and its associated graph.

Lemma 1 [21]: A stochastic matrix has algebraic multiplicity equal to one for its eigenvalue $\lambda = 1$ if and only if the graph associated with the matrix has a spanning tree. Furthermore, a stochastic matrix with positive diagonal elements has the property that $|\lambda| < 1$ for every eigenvalue not equal to one.

Lemma 2 [21]: Let $A = [a_{ij}]_{n\times n}$ be a stochastic matrix. If A has an eigenvalue $\lambda = 1$ with algebraic multiplicity equal to one, and all the other eigenvalues satisfy $|\lambda| < 1$, then A is SIA, that is, $\lim_{m\to\infty} A^m = \mathbf{1}_n\nu^T$, where ν satisfies $A^T\nu = \nu$ and $\mathbf{1}_n^T\nu = 1$. Furthermore, each element of ν is nonnegative.

Based on Lemmas 1 and 2, we give the following result which will be used in the next content.

Lemma 3: Let $H = \text{diag}\{h_1, h_2, \ldots, h_n\}$ and $0 < h_i < 1/d_{ii}$, $i \in \mathcal{I}_n$. Then, $I_n - H\mathscr{L}$ is SIA, i.e., $\lim_{k\to\infty}(I_n - H\mathscr{L})^k = \mathbf{1}_n\nu^T$, if and only if graph $\mathscr{G}$ has a spanning tree. Furthermore, $(I_n - H\mathscr{L})^T\nu = \nu$, $\mathbf{1}_n^T\nu = 1$, and each element of ν is nonnegative.

Proof (Sufficiency): Let $P = I_n - H\mathscr{L}$. From the definition of $\mathscr{L}$, we have $P = K + H\mathscr{A}$, where $K = I_n - H\mathscr{D}$. It follows from $h_i \in (0, 1/d_{ii})$ that K is a positive diagonal matrix. Consequently, it is easy to obtain that P is a stochastic matrix with positive diagonal entries. Obviously, for all $i, j \in \mathcal{I}_n,\ i \neq j$, the (i, j)th entry of P is positive if and only if $a_{ij} > 0$. Then, $\mathscr{G}$ is the graph associated with P. Combining Lemmas 1 and 2, we have $\lim_{k\to\infty} P^k = \mathbf{1}_n\nu^k$ when graph $\mathscr{G}$ has a spanning tree, where ν is a nonnegative vector. Moreover, ν satisfies $P^T\nu = \nu$ and $\mathbf{1}_n^T\nu = 1$.

(Necessary): From Lemma 1, if $\mathscr{G}$ does not have a spanning tree, the algebraic multiplicity of eigenvalue $\lambda = 1$ of P is $m > 1$. Then, it is easy to prove that the rank of $\lim_{k\to\infty} P^k$ is greater than 1, which implies that $\lim_{k\to\infty} P^k \neq \mathbf{1}_n\nu^k$. ■

B. Hybrid Multiagent System

Suppose that the hybrid multiagent system consists of continuous-time and discrete-time dynamic agents. The number of agents is n, labeled 1 through n, where the number of continuous-time dynamic agents is m $(m < n)$. Without loss of generality, we assume that agent 1 through agent m are continuous-time dynamic agents. Then, each agent has the dynamics as follows:

$$\begin{cases} \dot{x}_i(t) = u_i(t), & i \in \mathcal{I}_m \\ x_i(t_{k+1}) = x_i(t_k) + u_i(t_k), t_k = kh, & k \in \mathbb{N},\ i \in \mathcal{I}_n/\mathcal{I}_m \end{cases} \tag{1}$$

where h is the sampling period, and $x_i \in \mathbb{R}$ and $u_i \in \mathbb{R}$ are the positionlike and control input of agent i, respectively. The initial conditions are $x_i(0) = x_{i0}$. Let $x(0) = [x_{10}, x_{20}, \cdots, x_{n0}]^T$.

Each agent is regarded as a vertex in a graph. Each edge $(s_i, s_j) \in \mathscr{E}$ corresponds to an available information link from agent i to agent j. Moreover, each agent updates its current state based on the information received from its neighbors. In this brief, we suppose that there exists communication behavior in hybrid multiagent system (1), i.e., there are agents i and j which make $a_{ij} > 0$.

Definition 1: Hybrid multiagent system (1) is said to reach consensus if for any initial conditions, we have

$$\lim_{t_k\to\infty} \|x_i(t_k) - x_j(t_k)\| = 0, \quad \text{for } i, j \in \mathcal{I}_n \tag{2}$$

and

$$\lim_{t\to\infty} \|x_i(t) - x_j(t)\| = 0, \quad \text{for } i, j \in \mathcal{I}_m. \tag{3}$$

III. Main Results

In this section, the consensus problem of hybrid multiagent system (1) will be considered under three kinds of control inputs (consensus protocols), respectively.

A. Case 1

In this section, we assume that all agents communicate with their neighbors and update their control inputs in the sampling time t_k. Then, the consensus protocol for hybrid multiagent system (1) is presented as follows:

$$\begin{cases} u_i(t) = \sum_{j=1}^{n} a_{ij}(x_j(t_k) - x_i(t_k)),\ t \in (t_k, t_{k+1}], & i \in \mathcal{I}_m \\ u_i(t_k) = h\sum_{j=1}^{n} a_{ij}(x_j(t_k) - x_i(t_k)), & i \in \mathcal{I}_n/\mathcal{I}_m \end{cases} \tag{4}$$

where $\mathscr{A} = [a_{ij}]_{n\times n}$ is the weighted adjacency matrix associated with the graph $\mathscr{G}$, and $h = t_{k+1} - t_k > 0$ is the sampling period.

Theorem 1: Consider a directed communication network $\mathscr{G}$ and suppose that $h < (1/(\max_{i\in\mathcal{I}_n}\{d_{ii}\}))$. Then, hybrid multiagent system (1) with protocol (4) can solve consensus if and only if the network $\mathscr{G}$ has a directed spanning tree. Moreover, the consensus state is $\nu_1^T x(0)$, where $\mathscr{L}^T\nu_1 = 0$.

Proof (Sufficiency): First, we will prove that (2) holds. From (4), we know that

$$\begin{cases} x_i(t) = x_i(t_k) + (t - t_k)\sum_{j=1}^{n} a_{ij}(x_j(t_k) - x_i(t_k)) t \in (t_k, t_{k+1}], \\ \qquad\qquad i \in \mathcal{I}_m \\ x_i(t_{k+1}) = x_i(t_k) + h\sum_{j=1}^{n} a_{ij}(x_j(t_k) - x_i(t_k)), \quad i \in \mathcal{I}_n/\mathcal{I}_m. \end{cases} \tag{5}$$

Therefore, it follows that:

$$x_i(t_{k+1}) = x_i(t_k) + h\sum_{j=1}^{n} a_{ij}(x_j(t_k) - x_i(t_k)), \quad i \in \mathcal{I}_n. \tag{6}$$

Let $x(t_k) = [x_1(t_k), x_2(t_k), \ldots, x_n(t_k)]^T$. Then, (6) can be rewritten in the matrix form as

$$x(t_{k+1}) = (I_n - h\mathscr{L})x(t_k). \tag{7}$$

According to Lemma 3, since $\mathscr{G}$ has a directed spanning tree and $h < (1/(\max_{i\in\mathcal{I}_n}\{d_{ii}\}))$, we have $\lim_{k\to\infty}(I_n - h\mathscr{L})^k = \mathbf{1}_n\nu_1^T$, where $(I_n - h\mathscr{L})^T\nu_1 = \nu_1$. Thus, it is easy to obtain

$$\lim_{t_k\to\infty} x(t_k) = \lim_{k\to\infty}(I_n - h\mathscr{L})^k x(0) = \mathbf{1}_n\nu_1^T x(0)$$

and $\mathscr{L}^T\nu_1 = 0$. As a consequence, (2) holds. Moreover

$$\lim_{t_k\to\infty} x_i(t_k) = \nu_1^T x(0), \quad \text{for } i \in \mathcal{I}_n.$$

Second, we have

$$\|x_i(t) - x_j(t)\| \le \|x_i(t) - x_i(t_k)\| + \|x_i(t_k) - x_j(t_k)\| + \|x_j(t_k) - x_j(t)\|.$$

From (5), it is easy to know that

$$\|x_i(t) - x_i(t_k)\| \le h\sum_{j=1}^{n} a_{ij}\|x_j(t_k) - x_i(t_k)\|, \quad \text{for } t \in (t_k, t_{k+1}]$$

when $t \to \infty$, we have $t_k \to \infty$. Thus

$$\lim_{t\to\infty}\|x_i(t) - x_i(t_k)\| = 0, \quad i \in \mathcal{I}_m.$$

Moreover

$$\lim_{t\to\infty} x_i(t) = \lim_{t_k\to\infty} x_i(t_k) = \nu^T x(0), \quad i \in \mathcal{I}_m$$

which implies that equation (3) holds.

Therefore, hybrid multiagent system (1) with protocol (4) can solve consensus problem with consensus state $\nu_1^T x(0)$, where $\mathscr{L}^T\nu_1 = 0$.

Necessity: From Lemma 3, we have $\lim_{k\to\infty}(I_n - h\mathscr{L})^k \neq \mathbf{1}_n\nu_1^T$ when $\mathscr{G}$ does not have a directed spanning tree. Therefore, (2) does not hold, which implies that hybrid multiagent system (1) cannot reach consensus. ■

Remark 1: In fact, the sampling period h can be chosen without knowing the structure of the communication network, since one can choose a sufficiently small $h > 0$.

B. Case 2

In this section, we still assume that the interaction among agents happens in sampling time t_k. However, different from Case 1, we assume that each continuous-time dynamic agent can observe its own state in real time. Thus, the consensus protocol for hybrid multiagent system (1) is presented as follows:

$$\begin{cases} u_i(t) = \sum_{j=1}^{n} a_{ij}(x_j(t_k) - x_i(t)), \ t \in (t_k, t_{k+1}], \quad i \in \mathcal{I}_m \\ u_i(t_k) = h\sum_{j=1}^{n} a_{ij}(x_j(t_k) - x_i(t_k)), \qquad i \in \mathcal{I}_n/\mathcal{I}_m \end{cases} \tag{8}$$

where $\mathscr{A} = [a_{ij}]_{n\times n}$ is the weighted adjacency matrix associated with the graph $\mathscr{G}$, $h = t_{k+1} - t_k > 0$ is the sampling period.

Theorem 2: Consider a directed communication network $\mathscr{G}$ and suppose that $h < (1/(\max_{i\in\mathcal{I}_n/\mathcal{I}_m}\{d_{ii}\}))$. Then, hybrid multiagent system (1) with protocol (8) can solve consensus if and only if the network $\mathscr{G}$ has a directed spanning tree. Moreover, the consensus state is $\nu^T x(0)$, where $\mathscr{L}^T H\nu = 0$

$$H = \operatorname{diag}\left\{\frac{1 - e^{-\sum_{j=1}^{n} a_{1j}h}}{\sum_{j=1}^{n} a_{1j}}, \ldots, \frac{1 - e^{-\sum_{j=1}^{n} a_{mj}h}}{\sum_{j=1}^{n} a_{mj}}, h, \ldots, h\right\}.$$

Proof (Sufficiency): First, we will prove that (2) holds. Solving hybrid multiagent system (1) with protocol (8), we have

$$\begin{cases} x_i(t) = x_i(t_k) + \dfrac{1 - e^{-\sum_{j=1}^{n} a_{ij}(t-t_k)}}{\sum_{j=1}^{n} a_{ij}} \\ \qquad \sum_{j=1}^{n} a_{ij}(x_j(t_k) - x_i(t_k)), \ t \in (t_k, t_{k+1}], \quad i \in \mathcal{I}_m \\ x_i(t_{k+1}) = x_i(t_k) + h\sum_{j=1}^{n} a_{ij}(x_j(t_k) - x_i(t_k)), \quad i \in \mathcal{I}_n/\mathcal{I}_m. \end{cases} \tag{9}$$

Accordingly, at time t_{k+1}, the states of agents are

$$\begin{cases} x_i(t_{k+1}) = x_i(t_k) + \dfrac{1 - e^{-\sum_{j=1}^{n} a_{ij}h}}{\sum_{j=1}^{n} a_{ij}} \\ \qquad \sum_{j=1}^{n} a_{ij}(x_j(t_k) - x_i(t_k)), \quad i \in \mathcal{I}_m \\ x_i(t_{k+1}) = x_i(t_k) + h\sum_{j=1}^{n} a_{ij}(x_j(t_k) - x_i(t_k)), \quad i \in \mathcal{I}_n/\mathcal{I}_m. \end{cases} \tag{10}$$

Hence, (10) can be rewritten in the compact form as

$$x(t_{k+1}) = (I_n - H\mathscr{L})x(t_k) \tag{11}$$

where

$$x(t_k) = [x_1(t_k), x_2(t_k), \ldots, x_n(t_k)]^T$$

and

$$H = \operatorname{diag}\left\{\frac{1 - e^{-\sum_{j=1}^{n} a_{1j}h}}{\sum_{j=1}^{n} a_{1j}}, \ldots, \frac{1 - e^{-\sum_{j=1}^{n} a_{mj}h}}{\sum_{j=1}^{n} a_{mj}}, h, \ldots, h\right\}.$$

It is easy to know that $(1 - e^{-\sum_{j=1}^{n} a_{ij}h}/(\sum_{j=1}^{n} a_{ij})) < 1/d_{ii}$, $i \in \mathcal{I}_m$. Owing to $h < (1/(\max_{i\in\mathcal{I}_n/\mathcal{I}_m}\{d_{ii}\}))$, we have $0 < h_i < 1/d_{ii}$ for H. From Lemma 3, because $\mathscr{G}$ has a directed spanning tree, $I_n - H\mathscr{L}$ is an SIA matrix, i.e., $\lim_{k\to\infty}(I_n - H\mathscr{L})^k = \mathbf{1}_n\nu^T$, where $(I_n - H\mathscr{L})^T\nu = \nu$. Hence

$$\lim_{t_k\to\infty} x(t_k) = \lim_{k\to\infty}(I_n - H\mathscr{L})^k x(0) = \mathbf{1}_n\nu^T x(0)$$

which means that

$$\lim_{t_k\to\infty} x_i(t_k) = \nu^T x(0), \quad \text{for } i \in \mathcal{I}_n.$$

Obviously, (2) holds. Moreover, it follows from $(I_n - H\mathscr{L})^T\nu = \nu$ that $\mathscr{L}^T H\nu = 0$.

Second, we know from (9) that

$$\|x_i(t) - x_i(t_k)\| \le \frac{1}{\sum_{i=1}^{n} a_{ij}}\sum_{j=1}^{n} a_{ij}\|x_j(t_k) - x_i(t_k)\| \quad t \in (t_k, t_{k+1}], \quad i \in \mathcal{I}_m.$$

Since $t_k \to \infty$ when $t \to \infty$

$$\lim_{t\to\infty} \|x_i(t) - x_i(t_k)\| = 0.$$

Thus, we have $\lim_{t\to\infty} x_i(t) = \nu^T x(0), \quad i \in \mathcal{I}_m$. Consequently, (3) holds.

Therefore, from Definition 1, hybrid multiagent system (1) with protocol (8) reaches consensus. Moreover, the consensus state is $\nu^T x(0)$.

Necessity: Similar to the proof of necessity in Theorem 1, we know that if the directed communication network $\mathscr{G}$ does not have a directed spanning tree, then hybrid multiagent system (1) cannot achieve consensus. ■

Remark 2: Compared with the restriction of sampling period h in Theorems 1 and 2, it is easy to find that h is only related to the out degrees of discrete-time dynamic agents if each continuous-time dynamic agent can observe the real time information.

C. Case 3

In this section, we assume that all agents interact with each other in a gossiplike manner. Some assumptions are given as follows.

(**A1**) The communication network of hybrid multiagent system (1) is undirected, i.e., $a_{ij} = a_{ji}$ for all $i, j \in \mathcal{I}_n$.

(**A2**) At time t_k, two agents i and j ($i < j$) satisfying $(s_i, s_j) \in \mathscr{E}$ are chosen with probability p_{ij}, where $p_{ij} \in (0, 1)$ and $\sum_{(s_i,s_j)\in\mathscr{E}} p_{ij} = 1$.

When agents i and j are chosen, their interplay follows the below situations, where $h = t_{k+1} - t_k > 0$ is the sampling period.

1) If i and j are continuous-time dynamic agents, i.e., $i, j \in \mathcal{I}_m$, they will communicate during $(t_k, t_{k+1}]$. The control inputs of two agents are

$$\begin{cases} u_i(t) = a_{ij}(x_j(t) - x_i(t)), t \in (t_k, t_{k+1}], & i \in \mathcal{I}_m \\ u_j(t) = a_{ji}(x_i(t) - x_j(t)), t \in (t_k, t_{k+1}], & j \in \mathcal{I}_m. \end{cases} \tag{12}$$

2) If i is continuous-time dynamic agent and j is discrete-time dynamic agent, i.e., $i \in \mathcal{I}_m$, $j \in \mathcal{I}_n/\mathcal{I}_m$, the control inputs of two agents are

$$\begin{cases} u_i(t) = a_{ij}(x_j(t_k) - x_i(t)), & i \in \mathcal{I}_m \\ u_j(t_{k+1}) = x_j(t_k) + ha_{ji}(x_i(t_k) - x_j(t_k)), & j \in \mathcal{I}_n/\mathcal{I}_m. \end{cases} \tag{13}$$

3) If i and j are discrete-time dynamic agents, i.e., $i, j \in \mathcal{I}_n/\mathcal{I}_m$, the control inputs of two agents are

$$\begin{cases} u_i(t_{k+1}) = x_i(t_k) + ha_{ij}(x_j(t) - x_i(t_k)), & i \in \mathcal{I}_n/\mathcal{I}_m \\ u_j(t_{k+1}) = x_j(t_k) + ha_{ji}(x_i(t_k) - x_j(t_k)), & j \in \mathcal{I}_n/\mathcal{I}_m. \end{cases} \tag{14}$$

For each $r \in \mathcal{I}_n/\{i, j\}$, it keeps static, that is

$$\begin{cases} u_r(t) = 0, & r \in \mathcal{I}_m \\ u_r(t_k) = x_r(t_k), & r \in \mathcal{I}_n/\mathcal{I}_m. \end{cases} \tag{15}$$

Theorem 3: Consider an undirected communication network $\mathscr{G}$. Assume that (A1) and (A2) hold and $h < (1/(\max_{i,j\in\mathcal{I}_n}\{a_{ij}\}))$. Then, hybrid multiagent system (1) with control input (12)–(15) can solve consensus in mean sense if and only if the network $\mathscr{G}$ is connected.

Proof (Sufficiency): It suffices to prove that

$$\lim_{t_k\to\infty} \mathbb{E}\|x_i(t_k) - x_j(t_k)\| = 0, \quad \text{for } i, j \in \mathcal{I}_n \tag{16}$$

and

$$\lim_{t\to\infty} \mathbb{E}\|x_i(t) - x_j(t)\| = 0, \quad \text{for } i, j \in \mathcal{I}_m \tag{17}$$

hold for any initial states.

First, we will show that (16) holds. From (12)–(15), if agents i and j are selected to interplay at time t_k, the states of all agents at time t_{k+1} are three cases.

Note that (18), as shown at the bottom of this page, can be rewritten in the following compact form as:

$$x(t_{k+1}) = \Phi_{ij} x(t_k)$$

where

$$\begin{cases} \Phi_{ij} = I_n - \dfrac{1 - e^{-2a_{ij}h}}{2}(\mathbf{e}_i - \mathbf{e}_j)(\mathbf{e}_i - \mathbf{e}_j)^T & \\ & i, j \in \mathcal{I}_m \\ \Phi_{ij} = I_n - (1 - e^{-a_{ij}h})\mathbf{e}_i(\mathbf{e}_i - \mathbf{e}_j)^T & \\ \qquad -ha_{ij}\mathbf{e}_j(\mathbf{e}_j - \mathbf{e}_i)^T, & i \in \mathcal{I}_m, j \in \mathcal{I}_n/\mathcal{I}_m \\ \Phi_{ij} = I_n - ha_{ij}(\mathbf{e}_i - \mathbf{e}_j)(\mathbf{e}_i - \mathbf{e}_j)^T, & i, j \in \mathcal{I}_n/\mathcal{I}_m. \end{cases} \tag{19}$$

According to (A2), it is not hard to know that $\{x(t_k)\}$ is a stochastic linear system

$$x(t_{k+1}) = \Phi_k x(t_k), \quad \mathbb{P}(\Phi_k = \Phi_{ij}) \xlongequal{\text{i.i.d.}} p_{ij}.$$

Therefore, it follows that:

$$\mathbb{E}(x(t_{k+1})) = \mathbb{E}(\Phi_k)\mathbb{E}(x(t_k)). \tag{20}$$

Due to $0 < h < (1/(\max_{i,j\in\mathcal{I}_n}\{a_{ij}\}))$, we have $(1 - e^{-2a_{ij}h}/(2)) \in (0, 1/2)$, $1 - e^{-a_{ij}h} \in (0, 1)$ and $ha_{ij} \in (0, 1)$. Thus, Φ_{ij} is a stochastic matrix with positive diagonal entries. Moreover, the (i, j)th and (j, i)th entries of Φ_{ij} are positive, while all other nondiagonal entries are zeros. Hence, noticing that

$$\mathbb{E}(\Phi_k) = \sum_{(s_i,s_j)\in\mathscr{E}} \Phi_{ij} p_{ij}$$

$$\text{(I)} \begin{cases} x_i(t_{k+1}) = x_i(t_k) + \dfrac{1 - e^{-2a_{ij}h}}{2}(x_j(t_k) - x_i(t_k)), & i \in \mathcal{I}_m \\ x_j(t_{k+1}) = x_j(t_k) + \dfrac{1 - e^{-2a_{ij}h}}{2}(x_i(t_k) - x_j(t_k)), & j \in \mathcal{I}_m \\ x_r(t_{k+1}) = x_r(t_k), & r \in \mathcal{I}_n/\{i, j\} \end{cases} \tag{18a}$$

$$\text{(II)} \begin{cases} x_i(t_{k+1}) = x_i(t_k) + (1 - e^{-a_{ij}h})(x_j(t_k) - x_i(t_k)), & i \in \mathcal{I}_m \\ x_j(t_{k+1}) = x_j(t_k) + ha_{ji}(x_i(t_k) - x_j(t_k)), & j \in \mathcal{I}_n/\mathcal{I}_m \\ x_r(t_{k+1}) = x_r(t_k), & r \in \mathcal{I}_n/\{i, j\} \end{cases} \tag{18b}$$

$$\text{(III)} \begin{cases} x_i(t_{k+1}) = x_i(t_k) + ha_{ij}(x_j(t) - x_i(t_k)), & j \in \mathcal{I}_n/\mathcal{I}_m \\ x_j(t_{k+1}) = x_j(t_k) + ha_{ji}(x_i(t_k) - x_j(t_k)), & j \in \mathcal{I}_n/\mathcal{I}_m \\ x_r(t_{k+1}) = x_r(t_k), & r \in \mathcal{I}_n/\{i, j\} \end{cases} \tag{18c}$$

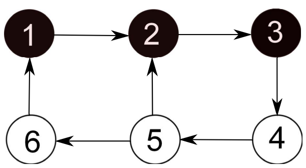

Fig. 1. Directed graph $\mathscr{G}_1$.

we know that $\mathbb{E}(\Phi_k)$ is a stochastic matrix satisfying the following:

1) All diagonal entries are positive.
2) The (i, j)th and (j, i)th entries are positive if and only if $(s_i, s_j) \in \mathscr{E}$.

Consequently, $\mathscr{G}$ is the graph associated with $\mathbb{E}(\Phi_k)$. Since $\mathscr{G}$ is connected, combining Lemmas 1 and 2, we have

$$\lim_{k\to\infty} (\mathbb{E}(\Phi_k))^k = \mathbf{1}_n \nu^{'}.$$

From (20), we have

$$\lim_{t_k\to\infty} \mathbb{E}(x(t_k)) = \lim_{k\to\infty} (\mathbb{E}(\Phi_k))^k x(0) = \mathbf{1}_n \nu^{'} x(0)$$

which implies that (16) holds.

Second, at time $t \in (t_k, t_{k+1}]$, the state of each $i \in \mathcal{I}_m$ follows the three scenarios.

1) If i is selected to communicate with its continuous-time dynamical neighbor $j \in \mathcal{I}_m$

$$x_i(t) = \frac{1+e^{-2a_{ij}(t-t_k)}}{2} x_i(t_k) + \frac{1-e^{-2a_{ij}(t-t_k)}}{2} x_j(t_k).$$

2) If i is selected to communicate with its discrete-time dynamical neighbor $j \in \mathcal{I}_n/\mathcal{I}_m$

$$x_i(t) = e^{-a_{ij}(t-t_k)} x_i(t_k) + (1 - e^{-a_{ij}(t-t_k)}) x_j(t_k).$$

3) If i is not selected, $x_i(t) = x_i(t_k)$.

Thus, we have

$$\begin{aligned}
&\mathbb{E}(x_i(t)) \\
&= \left(1 - \sum_{j\in\mathscr{N}_i} p_{ij}\right) \mathbb{E}(x_i(t_k)) \\
&\quad + \sum_{j\in\mathscr{N}_i} p_{ij}[\beta_{ij}\mathbb{E}(x_i(t_k)) + (1-\beta_{ij})\mathbb{E}(x_j(t_k))], \quad t \in (t_k, t_{k+1}]
\end{aligned} \tag{21}$$

where

$$\beta_{ij} = \begin{cases} \dfrac{1+e^{-2a_{ij}(t-t_k)}}{2}, & j \in \mathcal{I}_m \\ e^{-a_{ij}(t-t_k)}, & j \in \mathcal{I}_n/\mathcal{I}_m. \end{cases}$$

Since $t_k \to \infty$ when $t \to \infty$, it is easy to obtain from (21) that

$$\lim_{t\to\infty} \mathbb{E}(x(t)) = \lim_{t_k\to\infty} \mathbb{E}(x(t_k)) = \mathbf{1}_n \nu^{'} x(0)$$

holds for all $i \in \mathcal{I}_m$, which means that (17) holds.

Therefore, hybrid multiagent system (1) with control inputs (12)–(15) can solve consensus problem in mean sense.

Necessity: When $\mathscr{G}$ is not connected, similar to the proof of necessity in Theorem 1, we know that hybrid multiagent system (1) cannot reach consensus. ■

Remark 3: Note that hybrid multiagent system (1) presents a unified framework for both the discrete-time multiagent system and the continuous-time multiagent system. In other words, if $m = 0$, hybrid multiagent system (1) becomes a discrete-time multiagent system. And if $m = n$, hybrid multiagent system (1) becomes a continuous-time multiagent system.

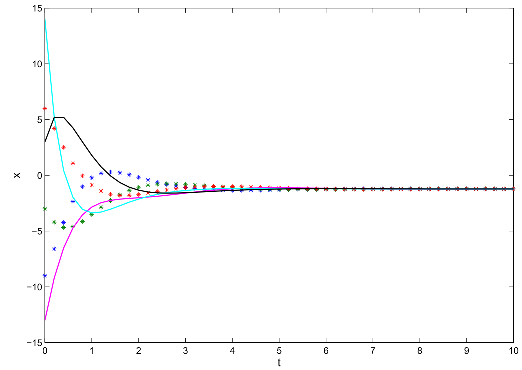

Fig. 2. State trajectories of all the agents with consensus protocol (4) and communication network $\mathscr{G}_1$, where $h = 0.2$.

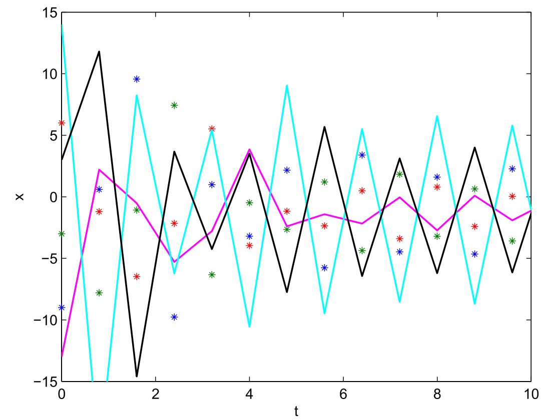

Fig. 3. State trajectories of all the agents with consensus protocol (4) and communication network $\mathscr{G}_1$, where $h = 0.8$.

IV. Simulations

In this section, we will provide some simulations to demonstrate the effectiveness of the theoretical results in this brief.

Suppose that there are six agents. The continuous-time dynamic agents and the discrete-time dynamic agents are denoted by 1–3 and 4–6, respectively. Let $x(0) = [-13, 14, 3, -9, -3, 6]^T$.

Example 1: The communication network $\mathscr{G}_1$ is shown in Fig. 1with

$$\mathscr{A}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

It can be noted that $\mathscr{G}_1$ has a directed spanning tree. It is easy to calculate that the sampling period $h = 0.2 < (1/(\max_{i=1}^{6}\{d_{ii}\}))$. By using consensus protocol (4), the state trajectories of all the agents are shown in Fig. 2, which is consistent with the sufficiency of Theorem 1. If $h = 0.8 > (1/(\max_{i=1}^{6}\{d_{ii}\}))$, the state trajectories of all the agents are shown in Fig. 3, where the consensus does not take place.

Example 2: Assume that the communication network is the same in Example 1 and $h = 0.2$. By using consensus protocol (8), the state trajectories of all the agents are shown in Fig. 4, which is consistent with the results in Theorem 2.

Example 3: Suppose that hybrid multiagent system (1) runs with control inputs (12)–(15). The communication network is shown in

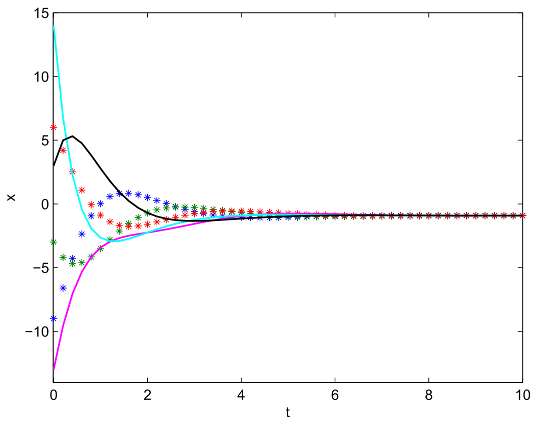

Fig. 4. State trajectories of all the agents with consensus protocol (8) and communication network $\mathscr{G}_1$, where $h = 0.2$.

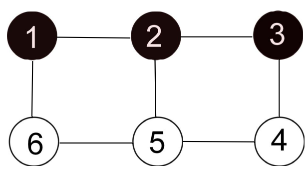

Fig. 5. Undirected graph $\mathscr{G}_2$.

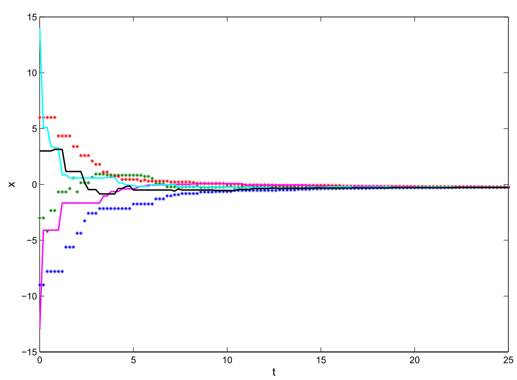

Fig. 6. State trajectories of all the agents with consensus protocol (12)–(15) and network $\mathscr{G}_2$, where $h = 0.2$.

Fig. 5 with

$$\mathscr{A}_2 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

It is shown that the sampling period $h = 0.2 < (1/(\max_{i,j=1}^{6}\{a_{ij}\}))$. At time t_k, each edge $(s_i, s_j) \in \mathscr{E}$ is chosen with probability 1/7. The state trajectories of all the agents are drawn in Fig. 6. Obviously, the simulation results are consistent with the sufficiency of Theorem 3.

V. Conclusion

In this paper, the consensus problem of a hybrid multiagent system which is composed of continuous-time and discrete-time dynamic agents was considered. First, we assumed that all agents communicate with their neighbors and update their strategies in the sampling time. When the sampling period $0 < h < (1/(\max_{i\in\mathcal{I}_n}\{d_{ii}\}))$, we proved that the hybrid multiagent system can achieve the consensus if and only if the communication network has a directed spanning tree. Then, we further assumed that each continuous-time agent can observe its own state in real time. The consensus of the hybrid multiagent system can be solved with $0 < h < (1/(\max_{i\in\mathcal{I}_n/\mathcal{I}_m}\{d_{ii}\}))$. Finally, a gossiplike consensus protocol was proposed. The necessary and sufficient condition was also given for solving the consensus problem if $0 < h < (1/(\max_{i,j\in\mathcal{I}_n}\{a_{ij}\}))$. The future work will focus on the second-order consensus of hybrid multiagent systems, and consensus of hybrid multiagent systems with time delays. ■

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