Rapid Communications

Controllability of complex networks: Choosing the best driver set

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Identifying the best driver set in a complex network is an unsolved problem in the application of pinning control methods. We choose the eigenratio of the augmented Laplacian matrix, and the best driver set is the subset of nodes providing the most effective pinning control strategy, i.e., the one for which synchronization of the whole network to the reference state is attained over the widest range of the coupling parameter. In this Rapid Communication, we propose a centrality measure based on a sensitivity analysis of the Laplacian matrix of the connection graph to find an approximate solution to this problem. The proposed metric is computationally efficient as it requires only a single eigendecomposition of the Laplacian matrix. Numerical results on a number of sample networks show that the proposed metric has a significantly better accuracy than the currently used heuristics, and in most cases can correctly identify the true optimal set, which is obtainable through a combinatorial search.

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Networks are ubiquitous in nature and many natural and man-made systems can be modeled as networked systems. Dynamical systems interacting through a network may exhibit collective behaviors such as synchronization, consensus, opinion formation, flocking, and unusual phase transitions [1,2]. These behaviors are highly dependent on the structure of the interaction network as well as the dynamics of agents [3–7]. Although coupled dynamical systems can develop spontaneous synchronous patterns if their coupling strength lies in an appropriate range, in some applications one needs to control a fraction of nodes, known as driver nodes, in order to facilitate the synchrony [8,9].

Synchronization of the whole network into a reference state, which is called pinning control, has been an attractive research area in the control of large-scale systems in many disciplines including biology [10] and engineering [11]. It has significant applications such as maintaining synchrony in power grids [12], averaging through consensus in sensor networks [2], and control of flocking [13]. It has been shown that one can efficiently synchronize a network by controlling only a small fraction of nodes as drivers [14]. However, not all nodes have the same influence when selected as the driver. Pinning controllability can be evaluated in terms of the required control action, the number of nodes to be pinned, and the feedback topology [15]. An important unsolved challenge in this context is to find the set of best drivers, leading to the best pinning control performance. Depending on the choice of pinning controllability metric, one can interpret what "optimality" means. Here, we choose the eigenratio of the augmented Laplacian matrix that is the most widely used metric for the pinning controllability in the literature [15–18]. Based on this metric, having two sets of driver nodes S_1 and S_2 with the same size, S_1 is argued to provide more effective pinning control than S_2, if the network synchronizes to the reference state for a larger range of coupling parameter when the nodes in S_1 are chosen as drivers.

Currently, there is no solution to find the set of optimal N_d drivers. One can find the global optimal solution through a brute force search on all possible combinations of nodes in sets of size N_d, and choosing the set with the best performance. However, as it requires computing a synchronization criterion for all possible combinations, this combinatorial process is not practical for many cases, especially for large-scale networks [19]. An obvious trial choice for the optimal set is to select vital nodes (e.g., those with the highest degree, betweenness, or closeness centrality) as drivers. These heuristic methods, although computationally cost effective, often result in noneffective pinning controllability [14,16,20]. Recently, evolutionary optimization algorithms have also been applied to find a set of the most influential nodes in pinning control, resulting in a better performance than the above heuristic methods [3,16]. However, evolutionary optimization algorithms also require computing an objective function, i.e., the synchronization criteria, at every step of the optimization process, and they often converge only after many steps, where the number of steps grows exponentially with network size. Therefore, such methods can only be applied to relatively small networks.

This Rapid Communication proposes an analytical approach to finding the set of N_d optimal drivers. The eigenratio of the augmented Laplacian matrix is considered as the pinning controllability index. This index has been frequently used in the past when implementing the pinning control of dynamical networks [15–18]. Our approach to obtaining the set of optimal drivers is based on a sensitivity analysis of the eigenratio [21,22]. It requires a single computation of the eigenvectors, indicating that it is appropriate for large-scale networks. Let us consider a dynamical network including N

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identical individual dynamical nodes and an undirected and unweighted connection network. The equations of motion of the network read

$$\frac{d\boldsymbol{x}_i}{dt} = F(\boldsymbol{x}_i) - \sigma \sum_{j=1}^{N} l_{ij} H \boldsymbol{x}_j, \quad i = 1, 2, \ldots, N, \tag{1}$$

where $\boldsymbol{x}_i \in R^n$ is the n-dimensional state vector, $F : R^n \to R^n$ defines the individual dynamical systems' state equation, and σ represents the unified coupling strength. $\boldsymbol{L} = [l_{ij}]$ is the Laplacian matrix, that is, a zero-row sum matrix with off-diagonal elements equal to -1, if there is a link, and 0 otherwise [23]. The diagonal elements of $\boldsymbol{L}$ are the corresponding degrees of the nodes. H is the projection matrix determining the coupled elements of the oscillators. The objective of pinning control is to pin all nodes to the following desired state [24],

$$\frac{d[\boldsymbol{s}(t)]}{dt} = F[\boldsymbol{s}(t)]. \tag{2}$$

Indeed, the reference state is considered to have identical dynamics as the individual dynamical systems. In order to pin the dynamical network to $\boldsymbol{s}$ (t), linear state feedback controllers are applied only to driver nodes, and the equations of motion read as

$$\frac{d\boldsymbol{x}_i}{dt} = F(\boldsymbol{x}_i) - \sigma \sum_{j=1}^{N} l_{ij} H \boldsymbol{x}_j - \sigma \beta_i k_i (\boldsymbol{s} - \boldsymbol{x}_i),$$
$$i = 1, 2, \ldots, N, \tag{3}$$

where k_i is the feedback control gain, $\beta_i = 1$ when node i is a driver node, and otherwise $\beta_i = 0$. When $\boldsymbol{x}_1(t) = \boldsymbol{x}_2(t) = \cdots = \boldsymbol{x}_N(t) = \boldsymbol{s}(t)$, one can state that the network has been synchronized to the reference state. Inspired by the master stability function approach [25], the local stability of the synchronized system can be evaluated in terms of the following N decoupled blocks [16,26],

$$\frac{d\boldsymbol{\zeta}_i}{dt} = [DF(\boldsymbol{s}) - a_i DH(\boldsymbol{s})]\boldsymbol{\zeta}_i, \quad i = 1, 2, \ldots, N, \tag{4}$$

where D represents the Jacobian, $\boldsymbol{\zeta}_i = \boldsymbol{x}_i - \boldsymbol{s}$, and $a_i = \sigma \lambda_{ci}$, with λ_{ci} being the ith eigenvalue of the augmented symmetric Laplacian matrix,

$$\boldsymbol{C} = \begin{bmatrix} l_{11} + k_1\beta_1 & l_{12} & \cdots & l_{1N} \\ l_{12} & l_{22} + k_2\beta_2 & \cdots & l_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ l_{1N} & l_{2N} & \cdots & l_{NN} + k_N\beta_N \end{bmatrix}. \tag{5}$$

Here, it is assumed that the adjacency matrix for $\boldsymbol{C}$ is connected. The eigenvalues of the symmetric matrix $\boldsymbol{C}$ are real and may be ordered as $0 < \lambda_{c1} \leqslant \lambda_{c2} \leqslant \cdots \leqslant \lambda_{cN}$. It has been shown that the eigenratio $\eta = \lambda_{c1}/\lambda_{cN}$ accounts for pinning controllability, and larger values indicate better controllability [15–18]. Interestingly, this technique decouples the dynamical properties of the open-loop network from spectral properties of the network as well as from the set of the drivers [15]. In other words, the pinning controllability of the networks can be evaluated through spectral properties of matrix $\boldsymbol{C}$ regardless of the dynamics of the individual nodes. According to Eq. (5), selecting node i as a driver can be represented as a perturbation on the related diagonal element l_{ii} in the original Laplacian matrix $\boldsymbol{L}$, i.e., by setting $\beta_i = 1$. Here, we assume that perturbations, caused by control gains k_i, on all diagonal elements of the matrix $\boldsymbol{C}$ are uniform and $k_i > 0$. In order to obtain a theory resulting in the optimal set, the perturbations need to be much smaller than the diagonal entries of the original Laplacian matrix $\boldsymbol{L}$, i.e., much smaller than the smallest degree of the network. For larger perturbations, the accuracy of the method might decline. From the eigenvalue perturbation theory [27,28], changes in the eigenvalue λ_m of the matrix $\boldsymbol{L}$ caused by perturbation in the parameter p is

$$\frac{d\lambda_m}{dp} = \boldsymbol{y}_m^T \frac{d\boldsymbol{L}(p)}{dp} \boldsymbol{x}_m, \quad m = 1, 2, \ldots, N, \tag{6}$$

where $\boldsymbol{y}_m^T$ and $\boldsymbol{x}_m$ are the left and right eigenvectors of $\boldsymbol{L}$ corresponding to λ_m, respectively, and $\boldsymbol{y}_m^T \boldsymbol{x}_m = 1$. Selecting node i as the driver affects only a diagonal element of $\boldsymbol{L}$ which results in

$$\frac{d\eta}{dl_{ii}} = \frac{(\boldsymbol{y}_1^i \boldsymbol{x}_1^i)\lambda_N - (\boldsymbol{y}_N^i \boldsymbol{x}_N^i)\lambda_1}{(\lambda_N)^2}, \tag{7}$$

where the superscript i indicates the ith element of the vector. The vector $\boldsymbol{x}_1 = \boldsymbol{1}_N$ is the eigenvector of $\boldsymbol{L}$ corresponding to λ_1, where $\boldsymbol{1}_N$ is a vector with N elements all equal to 1. For undirected graphs we have $\boldsymbol{y}_n = \boldsymbol{x}_n$; therefore Eq. (7) can be written as

$$\frac{d\eta}{dl_{ii}} = \frac{1}{\lambda_N}\left[1 - \eta\left(\boldsymbol{x}_N^i\right)^2\right]. \tag{8}$$

It shows that node i with a maximum value of $(\boldsymbol{x}_N^i)^2$ results in smaller $d\eta/dl_{ii}$, meaning that η is nearer to the maximum value. That is, the node which is associated with the largest component $(\boldsymbol{x}_N^i)^2$ will have the largest value of $\eta = \lambda_{c1}/\lambda_{cN}$, resulting from a perturbation to the diagonal l_{ii}. This leads us to define the "controllability centrality" $\Psi(i)$ for node i as [22]

$$\Psi(i) = \left(\boldsymbol{x}_N^i\right)^2, \quad i = 1, 2, \ldots, N, \tag{9}$$

where, as before, $\boldsymbol{x}_N$ is the eigenvector corresponding to the largest eigenvalue of the Laplacian matrix of the graph. Thus, the node with the maximum value of controllability centrality is the best driver node for the network. However, considering practical constraints, more than one driver node is normally needed to control the whole network. In order to obtain a set of μ nodes (S_μ) that if selected as the driver will have the maximal influence on the pinning controllability, we look for the subset which issues the strongest effect on the eigenratio $\eta(l_{ii})$ where $i \in S_\mu$. We write the amplitude of the gradient of η as

$$|\nabla\eta|^2 = \nabla\eta^T \cdot \nabla\eta = \sum_{i \in S_\mu} \left(\frac{\partial\eta}{\partial l_{ii}}\right)^2. \tag{10}$$

Considering $\eta \ll 1$ and using Eq. (8), this can be simplified to

$$|\nabla\eta|^2 = \sum_{i \in S_\mu} \frac{1}{\lambda_N^2}\left[1 - \eta\left(\boldsymbol{x}_N^i\right)^2\right]^2 \approx \frac{1}{\lambda_N^2}\left[\mu - 2\eta \sum_{i \in S_\mu} \left(\boldsymbol{x}_N^i\right)^2\right]. \tag{11}$$

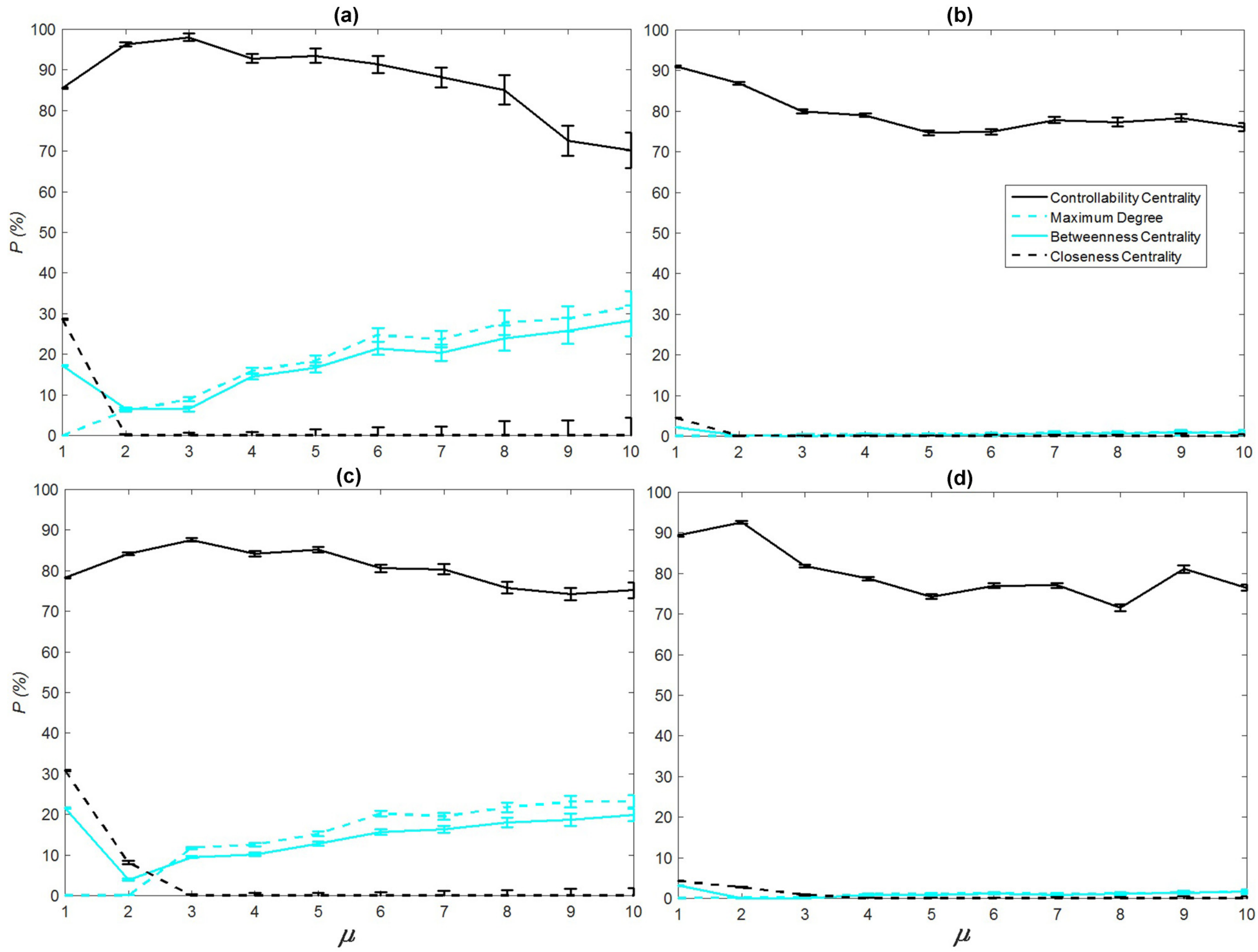

FIG. 1. Precision P of the proposed controllability centrality (solid black line), and heuristic methods including considering hub nodes with maximum degree [dashed cyan (light gray) line], maximum betweenness [solid cyan (light gray) line], and maximum closeness (dashed black line). Graphs show P for finding a driver set of μ nodes ($\mu = 1, 2, \ldots, 10$). Networks have a scale-free structure with $N = 1000$ and (a) $m = 2$ and $B = 0$, (b) $m = 10$ and $B = 0$, (c) $m = 2$ and $B = 5$, and (d) $m = 10$ and $B = 5$ (see text for a description of these parameters). Data show mean values with error bars corresponding to standard error over 100 realizations.

Therefore, controllability centrality can be defined for the subset S_μ in the same way as it is defined in Eq. (9) for a single node,

$$\Psi(S_\mu) = \sum_{i \in S_\mu} \left(\boldsymbol{x}_N^i\right)^2. \tag{12}$$

The optimization problem to find the subset of μ nodes with maximum influence on η reads as

$$\underset{S_\mu}{\text{Maximize}}\ \ \Psi(S_\mu). \tag{13}$$

In other words, among the subsets of μ nodes, the one maximizing $\Psi(S_\mu)$, i.e., minimizing Eq. (11), is the most influential subset to be considered as the driver for pinning controllability. Interestingly, the controllability centrality proposed in (12) is independent of the control gain k_i. This is valid only for uniform control gains among driver nodes and when the conditions of the perturbation theory hold, that is, control gains are sufficiently small. It also shows the submodularity feature, as it is a monotone increasing function, as

$$\Psi(S_{\mu+1}) = \left(\boldsymbol{x}_N^{\mu+1}\right)^2 + \Psi(S_\mu), \tag{14}$$

meaning that all interesting properties of submodular functions can be applied to this case as well. Based on the submodular property, in order to add a new driver node to the subset previously controlling the network, the best candidate is the node with the highest controllability centrality. In other words, once the eigendecomposition of the Laplacian $\boldsymbol{L}$ (i.e., the case without any drivers) is done, the best driver set of μ nodes is the set of top-μ nodes ranked by the controllability centrality measure. This is a significant result as it only requires a single eigendecomposition and has complexity comparable to many standard heuristic methods.

In order to assess the performance of the proposed metric, we apply the proposed controllability centrality measure on synthetic scale-free and small-world networks and compare its performance with a number of heuristic methods. We use the algorithm proposed in Refs. [29,30] to generate scale-free networks. Starting with a fully connected graph of small size,

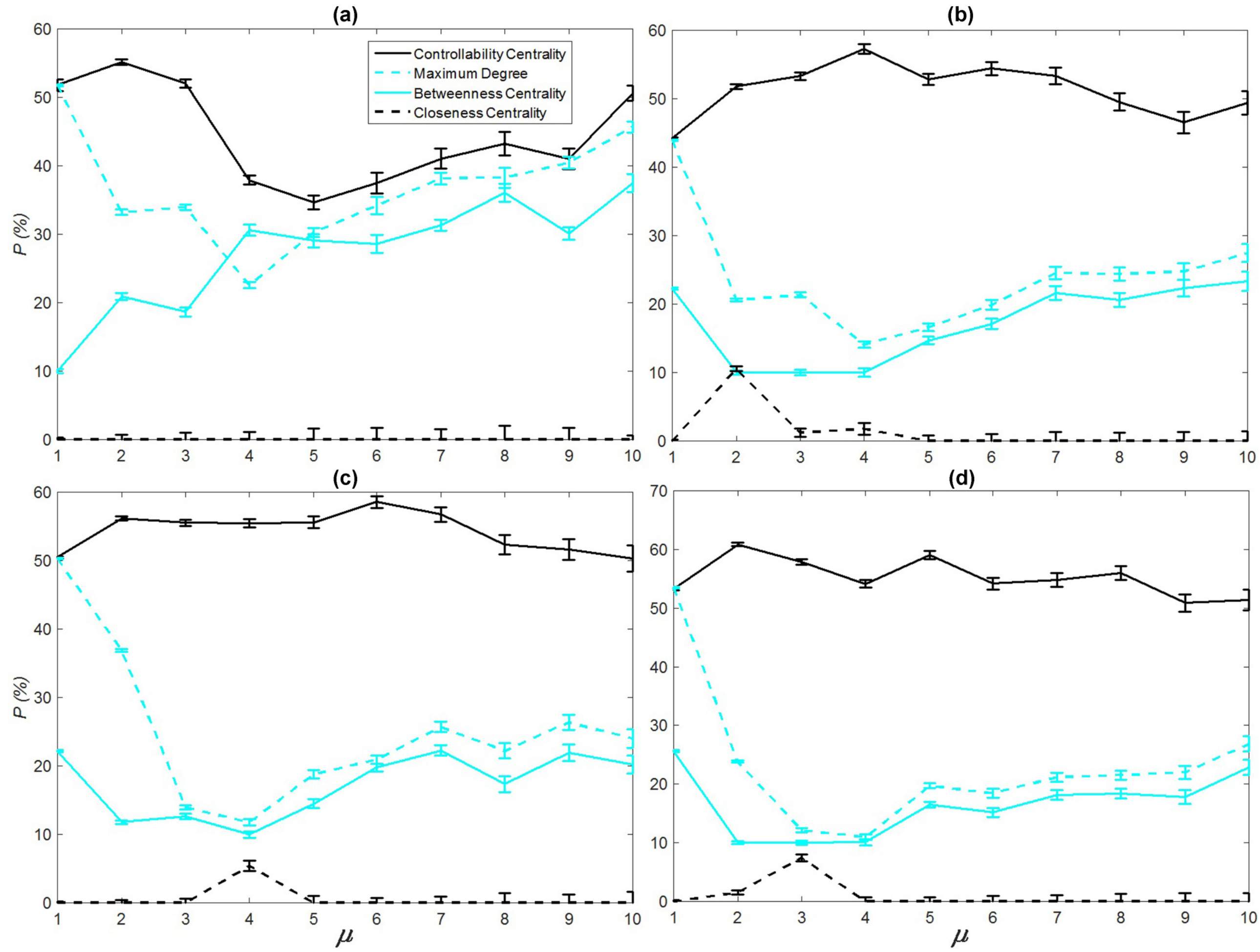

FIG. 2. Precision P of the proposed controllability centrality (solid black line), and heuristic methods including considering hub nodes with maximum degree [dashed cyan (light gray) line], maximum betweenness [cyan (light gray) line], and maximum closeness (dashed black line). Graphs show P for finding a driver set of μ nodes ($\mu = 1, 2, \ldots, 10$). Networks have a Watts-Strogatz structure with $N = 1000$ nodes and $m = 2$ and rewiring probability of (a) $p = 0.2$, (b) $p = 0.5$, (c) $p = 0.7$, and (d) $p = 0.9$. Data show mean values with error bars corresponding to standard error over 100 realizations.

at each step a new node is added to the network and creates m links with the already existing nodes. The probability of creating an edge between the newly added nodes and an existing node i is $(d_i + B)/\sum_j(d_j + B)$, where d_i is the degree of node i and B is a constant controlling the heterogeneity of the network; as B increases, the heterogeneity of the network decreases [29,30]. To construct small-world networks, the original algorithm proposed by Watts and Strogatz is used [31]. First, a ring graph is considered in which each node is connected to its m-nearest neighbors. Then, the links are rewired with probability p avoiding self-loops and multiple connections. We compare the performance of the proposed metric with a number of heuristics including selecting the set of top-μ hub nodes with high degree, betweenness, or closeness centrality as the drivers. Such heuristics have been previously used to form the set of drivers.

The synthetic networks are constructed with size $N = 1000$ and different average degree and heterogeneity levels. We first obtain the true optimal driver set, which we refer to as the ground truth and use this to compare the precision of different methods. The ground-truth optimal set is obtained through examining all possible combinations. Let us consider obtaining the optimal driver set with μ nodes. First, all μ combinations of N nodes are selected one by one and the variation of η by perturbing any of them is obtained. The perturbations considered on the diagonal elements of the original Laplacian matrix take into account uniform and nonzero control gains for the drivers. The μ combinations are then sorted in descending order based on the amount variation of η; the one at the top is selected as the best set. We run extensive numerical simulations to obtain the ground-truth optimal sets for the networks for $\mu = 1, 2, \ldots, 10$. The precision P of each method for a given μ is obtained as follows. One hundred realizations of each network type are considered and the set with μ nodes predicted by that particular method is obtained. Then the precision is obtained as $P = (n_p/n)100\%$, where n is the total number of runs for each case ($n = 100$ here) and n_p is the number of times that all nodes in the set predicted by the method match those in the ground-truth optimal set. For example, $P = 90\%$ for an algorithm indicates that the algorithm correctly predicts all nodes of the ground-truth optimal set in 90% of the runs.

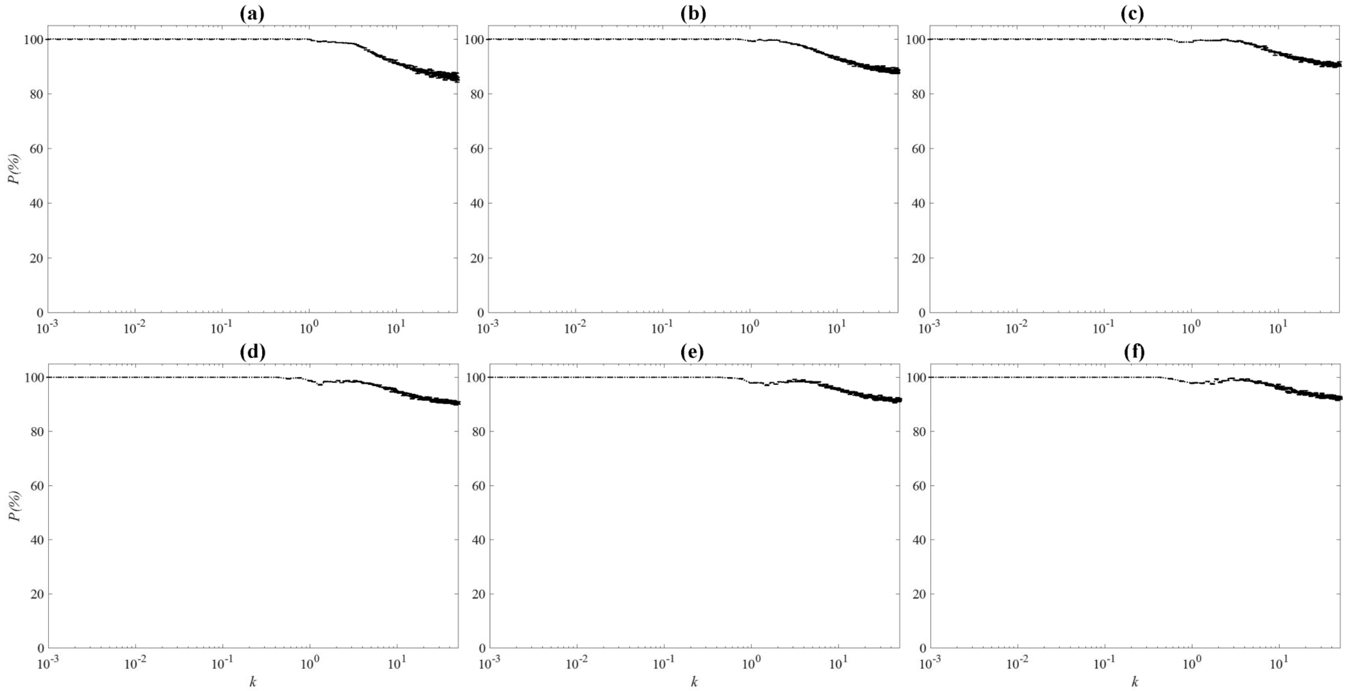

FIG. 3. Precision P of the proposed controllability centrality metric. Graphs show P as a function of control gain k for different values of μ. The networks have a scale-free structure with $N = 100$, $m = 2.5$, and $B = 0$. The driver sets have (a) $\mu = 1$, (b) $\mu = 2$, (c) $\mu = 3$, (d) $\mu = 4$, (e) $\mu = 5$, and (f) $\mu = 6$ nodes. The control gain k is applied uniformly to all drivers and varies from $k = 0.001$ to $k = 50$. Data show mean values with error bars corresponding to standard error over 300 realizations.

Figure 1 shows the precision P as a function of μ for scale-free networks with different average degrees and B values. The proposed controllability centrality metric has significantly better precision than the other heuristic methods. It always has a precision of higher than 70% and even in some cases close to 100%. This is a significant performance, indicating the proposed methods, although being computationally efficient, can correctly predict all nodes in the ground-truth optimal sets in most cases. The heuristic methods have a close to zero precision for $\mu>2$ in less heterogeneous networks, i.e., $B = 5$, while the controllability centrality still has high precision in such cases. The closeness-based method has the poorest performance among the heuristics. Figure 2 shows P as a function of rewiring probability p in small-world networks with $N = 1000$ and $m = 2$. The controllability centrality can correctly predict the ground truth in almost half of the cases in these networks. Except for $p = 0.2$ for which the controllability centrality is slightly better than degree- and betweenness-based methods, its precision is significantly higher than these heuristics in other cases. Similar to scale-free networks, the method based on closeness centrality has the worst precision. The proposed approximate method indeed works much better in heterogeneous networks than degree-homogeneous ones, as the nodes and subset of nodes are more distinguishable from each other in such networks. Precision of the proposed metric is acceptable although it is based on the first-order approximation of Eq. (6). It is worth noting that the approximation provided by this first-order equation is closer to real when perturbation on the diagonal elements of Eq. (5) is smaller, i.e., when the perturbed node has a higher degree. Therefore, we expect less precision for this approximation in networks with homogeneous and/or low-degree nodes. This is also evident from our simulations, as the precision for scale-free networks is much higher than that of Watts-Strogatz networks.

One may be surprised by such a poor quality of the heuristic methods. Indeed, the criterion used here is rather strong and to be counted as a precise set of size μ, all μ nodes suggested by the algorithm must be included in the optimal set. Although the heuristic methods may correctly find many of the optimal nodes (e.g., in many cases, they can correctly find 7–9 nodes for $\mu = 10$), they fail to recover the ground-truth optimal set in the majority of the cases. However, the proposed metric can correctly find all nodes within the optimal set with a much higher precision, especially in scale-free networks for which its precision is above 80% in the majority of cases. In other words, we do not show how effective a pinning control scheme can be in improving the synchrony as enough evidence has been already provided in the literature. Our problem is to find the best driver(s).

In conclusion, we introduced a pinning controllability centrality measure to obtain a driver set of μ nodes. The method is based on a single eigendecomposition of the original Laplacian matrix (without information of the drivers), indicating its computational efficiency. Although being simple to compute, its precision in correctly identifying the ground-truth optimal set is significantly better than the heuristic methods.

The proposed perturbation-based metric assumes uniform control gain for all drivers. Although the diagonal elements of matrix $\boldsymbol{C}$ have both k_i and β_i (the gain and location of drivers), our perturbation technique does not distinguish k_i and β_i, as only one parameter can be considered in the perturbation. Such an approach may raise an argument about the accuracy of the proposed metric when the control gains vary. Indeed, according to Eq. (12), the optimal driver set is k_i independent when the perturbations are uniform and much smaller than the smallest degree of the network. To test this, we study how the accuracy of the proposed metric changes by varying the control gains. We run simulations for networks with scale-free structures with $N = 100$ nodes and average degree $m = 5$ with a minimum degree of 1. The control gain k_i is changed from 0.001 to 50 having driver sets of different cardinality $\mu = 1, 2, \dots, 6$. Deriving the ground truth and calculating the accuracy is performed in the same way as it is done above. Our numerical results (Fig. 3) show that for control gains sufficiently smaller than the minimum degree of the graph, the proposed perturbation theory is highly accurate and the theory perfectly matches the simulation, i.e., accuracy is 100%. Even after increasing the control gain by 50 times of the minimum degree of the graph, the accuracy of the proposed metric is still well above 85%. Indeed, when the control gain increases, the accuracy of the proposed perturbation-based method slightly declines, as the first-order approximation used here becomes less accurate. Future directions of our research could be to further consider optimizing the control gains together with the location of the driver nodes, i.e., designing pinning control with optimal control cost.

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