# On Pinning Synchronization of Directed and Undirected Complex Dynamical Networks

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***Abstract*—This paper presents some low-dimensional pinning criteria for global synchronization of both directed and undirected complex networks, and proposes specifically pinning schemes to select pinned nodes by investigating the relationship among pinning synchronization, network topology, and the coupling strength. The paper answers the challenging questions in pinning control of complex networks: 1) what sufficient conditions can guarantee global asymptotic stability of the pinning process; 2) what nodes should be chosen as pinned candidates; and 3) how many nodes are needed to be pinned for a fixed coupling strength? Furthermore, an adaptive pinning control scheme is developed to achieve synchronization of general complex networks. Numerical examples are given to verify our theoretical analysis.**

***Index Terms*—Adaptive tuning, directed and undirected complex networks, low-dimensional pinning criteria, pinning control, synchronization.**

## I. Introduction

In the past few decades, much effort has been devoted to the control and synchronization of complex dynamical networks. However, a typical real-world complex network usually consists of large number of nodes and links, and thus, it is practically impossible to apply control actions to all nodes. Fortunately, recent studies show that a complex network can be synchronized onto some desired trajectory by placing local feedback injections on a small fraction of network nodes, which is now known as pinning control.

There has been much literature on pinning control and synchronization of undirected complex dynamical networks. In [1], a pinning control scheme was developed for spatially extended chaotic systems. Wang and Chen [2] deeply investigated both specifically and randomly pinning schemes for scale-free networks and pointed out that better synchronization performance can be achieved by specifically pinning the most highly connected nodes. In [3], the concept of virtual control was proposed to show that the control actions applied to the pinned nodes can be propagated to the rest of network nodes through the couplings in the network, and eventually, result in the synchronization of the whole network. In [4], the pinning synchronization of complex networks with time-delay couplings was studied. Sorrentino *et al.*[5] defined the concept of pinning controllability, and discussed the roles of the coupling and control gains, as well as of the number of pinned nodes. In [6], the global pinning controllability of complex networks was studied and some sufficient pinning conditions were established. Lu [7] developed adaptive pinning controllers using neighborhood information of each node. In [2], [8], and [9], some low-dimensional pinning conditions based on uncontrolled part of the coupling matrix were derived for undirected complex networks, which inspired us to extend these results to both directed and undirected complex networks.

In contrast with the intensive studies on undirected complex networks, there are only few studies on pinning synchronization of directed complex networks mainly due to the asymmetry of the coupling configuration matrix. Based on master stability function (MSF) [10], some stability criteria were derived to pin continuous-time and discrete-time complex networks with general topology to their equilibrium points [11]. Zhou *et al.* [12], [13] proposed some local and global criteria for pinning adaptive synchronization of general complex networks. In [14], it was proved that it is possible to pin a complex network by a single controller under certain conditions. In [15], the concept of ControlRank (CR) was defined and a directed scale-free network was pinned to its equilibrium. In [16], it was shown that the pinning control actions should be applied to roots of trees in a spanning forest of the underlying interaction graph. The relationship between the pinning control effectiveness and the network topology was investigated [16], [17].

To the best of our knowledge, how to choose the pinned nodes for general complex dynamical networks still remains to be a challenging problem. This paper aims at developing effective pinned-node selection schemes and practical pinning control schemes for the synchronization of complex networks with general topology. Recent papers show that the synchronizability is determined by the network structure and the coupling strength [5]–[9], [11]–[20]. Thus, this paper deeply investigates the relationship among pinning synchronization, the network topology, the coupling strength, and pinning feedback gains.

The most important two contributions of this paper are to establish some low-dimensional pinning criteria and propose an effective pinned-node selection scheme for directed complex networks. Under some mild assumptions, we show that the

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nodes whose out-degrees are bigger than their in-degrees of a directed network should be chosen as pinned candidates, and point out that the randomly pinning scheme may not guarantee the synchronization of directed complex networks.

The rest of the paper is organized as follows. In Section II, some preliminaries are given. Section III presents a general complex network model and some assumptions. Section IV proposes some low-dimensional pinning conditions for general complex dynamical networks. Section V presents pinned-node selection schemes for directed and undirected complex networks. In Section VI, an adaptive pinning control scheme is developed. Section VII provides two numerical simulation examples to validate theoretical results. Conclusions are drawn in Section VIII.

## II. Preliminaries

### *Notations*

Here, we list some mathematical notations used throughout the paper. Let $I_N$ be an $N$-dimensional identity matrix. Denote the transpose of the matrix $A \in R^{n\times n}$ and the vector $x \in R^n$ as $A^T$ and $x^T$, respectively. The inverse of the matrix $A$ is indicated with $A^{-1}$. Denote the positive (semi) definiteness of the matrix $A$ by $A > 0 (A \geq 0)$, and write $A > B (A \geq B)$ if $A - B > 0 (A - B \geq 0)$. For the matrix $G \in R^{N\times N}$, the $i$th row and the $i$th column of $G$ is called the $i$th row–column pair of $G$ [2]. We denote $G_l \in R^{(N-l)\times(N-l)}$ as a minor matrix of $G \in R^{N\times N}$ by removing arbitrary $l (1 \leq l < N)$ row–column pairs of $G$ [2]. Denote $\lambda_{\max}(A)$ as the maximum eigenvalue of the matrix $A$. Let $A \otimes B$ be the Kronecker product of two matrices $A$ and $B$ [21].

### *Lemmas*

Some lemmas are needed to derive the main results.

*Lemma 1:* (Schur complement [22]). The following linear matrix inequality (LMI)

$$\begin{bmatrix} Q(x) & S(x) \\ S(x)^T & R(x) \end{bmatrix} < 0$$

where $Q(x) = Q(x)^T$ and $R(x) = R(x)^T$, is equivalent to any one of the following conditions:

1) $Q(x) < 0, R(x) - S(x)^T Q(x)^{-1} S(x) < 0$.
2) $R(x) < 0, Q(x) - S(x) R(x)^{-1} S(x)^T < 0$.

*Lemma 2:* Assume that $M = (m_{ij})_{N\times N}$ is symmetric. Let $D = \mathrm{diag}(d_1, \ldots, d_l, \underbrace{0, \ldots, 0}_{N-l})$ in which $d_i > 0, i = 1, \ldots, l$. When $d_i > 0, i = 1, \ldots, l$ are sufficiently large, $M - D < 0$ is equivalent to $M_l < 0$, where $M_l$ is the minor matrix of $M$ by removing its first $l$ row–column pairs.

*Proof:* Let $\tilde{D} = \mathrm{diag}(d_1, \ldots, d_l)$.

Using matrix decomposition [9], we have

$$M - D = \begin{bmatrix} A - \tilde{D} & B \\ B^T & M_l \end{bmatrix}$$

where $A = (a_{ij})_{l\times l}$, $a_{ij} = a_{ji} = m_{ij}$, $i, j = 1, \ldots, l$, $B = (b_{ij})_{l\times(N-l)}$, $b_{ij} = m_{ij}$, $i = 1, \ldots, l$, $j = l+1, \ldots, N$, $M_l = (m_{l_{ij}})_{(N-l)\times(N-l)}$, $m_{l_{ij}} = m_{l_{ji}} = m_{i+l,j+l}$, and $i, j = 1, \ldots, N - l$.

With Lemma 1, we know that if $M - D < 0$, then $M_l < 0$. We only need to prove that if $M_l < 0$, then $M - D < 0$. When $d_i > 0$, $i = 1, \ldots, l$ are sufficiently large such that $\tilde{D} > \lambda_{\max}\left(A - BM_l^{-1}B^T\right) I_l$ holds [9], it is obvious to see that $A - \tilde{D} - BM_l^{-1}B^T < 0$. Then by Lemma 1, we can conclude that $M - D < 0$.

So, we finish the proof.

*Remark 1:* Let $d = \min_{1\leq i\leq l}\{d_i\}$. From the proof of Lemma 2, we know that $M - D < 0$ is equivalent to $M_l < 0$ when $d > \lambda_{\max}\left(A - BM_l^{-1}B^T\right)$, which gives an estimated lower bound of $d_i > 0$, $i = 1, \ldots, l$.

*Lemma 3:* Assume that $A, B$ are $N$ by $N$ Hermitian matrices. Let $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_N$, $\beta_1 \geq \beta_2 \geq \ldots \geq \beta_N$, and $\gamma_1 \geq \gamma_2 \ldots \geq \gamma_N$ be eigenvalues of $A$, $B$, and $A + B$, respectively. Then, one has $\alpha_i + \beta_N \leq \gamma_i \leq \alpha_i + \beta_1$, $i = 1, \ldots, N$.

## III. Problem Statement

In this section, we consider a general complex dynamical network consisting of $N$ identical nodes with linearly diffusive couplings, and each node is an $n$-dimensional dynamic system. The network is described by [15]

$$\dot{x}_i = f(x_i) + c\sum_{j=1}^{N} g_{ij}\Gamma x_j, \quad i = 1, \ldots, N \tag{1}$$

where $x_i = (x_{i1}, \ldots, x_{in})^T$ is the state variable of the $i$th node, $f : R^n \to R^n$ is a nonlinear vector function, the positive constant $c > 0$ is the coupling strength, the positive definite matrix $\Gamma = \mathrm{diag}(r_1, \ldots, r_n) > 0$ is the inner coupling matrix, and $G = (g_{ij})_{N\times N}$ is called the coupling configuration matrix whose elements are defined as follows: if there is a directed link from node $j$ to node $i (i \neq j)$, then $g_{ij} = 1$, otherwise $g_{ij} = 0$ and $g_{ii} = -\sum_{j=1, j\neq i}^{N} g_{ij}$, $i = 1, \ldots, N$. Thus, we have $\sum_{j=1}^{N} g_{ij} = 0$.

*Remark 2:* The coupling configuration matrix $G$ represents the topological structure of network (1). In this paper, the matrix $G$ is not assumed to be symmetric or irreducible, and generally, it has complex eigenvalues. Let $\mathrm{DegIn}(i)$ and $\mathrm{DegOut}(i)$ denote the in-degree and out-degree [24] of the $i$th node, respectively. According to the definition of $G$ in (1), it is easy to verify that

$$-g_{ii} = \sum_{j=1, j\neq i}^{N} g_{ij} = \mathrm{DegIn}(i)$$
$$\sum_{j=1, j\neq i}^{N} g_{ji} = \mathrm{DegOut}(i), \quad i = 1, \ldots, N. \tag{2}$$

In later section, we will use the degree information in (2) to determine what nodes should be pinned first.

Suppose that we want to control network (1) onto the solution of the isolated node described by [8], [9], [12]–[14].

$$\dot{s} = f(s). \tag{3}$$

The objective of synchronization is to design controllers for network (1) to achieve $\lim_{t\to\infty}(x_i(t) - s(t)) = 0$, $i = 1, \ldots, N$. We can apply control actions to a small fraction $\delta (0 < \delta \ll 1)$ of the total nodes in network (1)[2]–[9], [11]–[17]. Suppose that nodes $i_1, \ldots, i_l$ in network (1) are selected to be pinned, where $l = \lfloor \delta N \rfloor$ is the smaller, but nearest integer to $\delta N$ [2]. Without loss of generality, assume that the first $l$ nodes are selected to be pinned. Otherwise, we can rearrange the order of the nodes [9]. In later section, we will provide schemes to choose pinned nodes. Apply local control actions to network (1) and the pinning controlled network is written as follows[9]:

$$\begin{aligned} \dot{x}_i &= f(x_i) + c\sum_{j=1}^{N} g_{ij}\Gamma x_j + u_i, \quad i = 1,\ldots,l \\ \dot{x}_i &= f(x_i) + c\sum_{j=1}^{N} g_{ij}\Gamma x_j \quad i = l+1,\ldots,N \end{aligned} \tag{4}$$

where $u_i = -d_i\Gamma(x_i - s)$, $d_i > 0$, $i = 1,\ldots,l$.

To derive the stability criteria for the pinning controlled network (4), we need to make the following assumptions.

*Assumption 1:* (A1) [9]. There exists a constant $\theta > 0$, such that the nonlinear function $f$ in (1) satisfies

$$(x-y)^T(f(x) - f(y)) \le \theta(x-y)^T\Gamma(x-y) \quad \forall\ x, y \in R^n. \tag{5}$$

*Assumption 2:* (A2) [7]. Define two sets $V = \{i_1,\ldots,i_N\}$ and $V_{\mathrm{pin}} = \{i_1,\ldots,i_l\}$ as the sets of total nodes and the selected pinned nodes for the controlled network (4), respectively. All nodes in $V\backslash V_{\mathrm{pin}}$ are accessible from the pinned node set $V_{\mathrm{pin}}$, i.e., for any node $i \in V\backslash V_{\mathrm{pin}}$, we can always find a node $j \in V_{\mathrm{pin}}$, such that there is a directed path from node $j$ to node $i$.

## IV. Pinning Criteria for Synchronization of Directed and Undirected Complex Networks

In this section, we investigate the stability criteria for the pinning controlled general complex network (4), and present some low-dimensional conditions to guarantee global asymptotic stability of the pinning process.

Assume that the pinning-controlled complex network (4) satisfies assumptions (A1) and (A2). Construct the following symmetric matrix $H = (h_{ij})_{N\times N}$:

$$H = \theta I_N + c\frac{G+G^T}{2} \tag{6}$$

where $\theta > 0$ satisfies (A1).

Let $D = \mathrm{diag}(d_1,\ldots,d_l,\underbrace{0,\ldots,0}_{N-l})$ with $d_i > 0$, $i = 1,\ldots,l$. Using matrix decomposition, we have

$$H - D = \begin{bmatrix} A - \tilde{D} & B \\ B^T & C \end{bmatrix} \tag{7}$$

where $A = (a_{ij})_{l\times l}$, $a_{ij} = a_{ji} = h_{ij}$, $i,j = 1,\ldots,l$, $\tilde{D} = \mathrm{diag}(d_1,\ldots,d_l)$, $B = (b_{ij})_{l\times(N-l)}$, $b_{ij} = h_{ij}$, $i = 1,\ldots,l$, $j = l+1,\ldots,N$, and $C = \left(\theta I_N + c((G+G^T)/2)\right)_l$ is the minor matrix of $H$ by removing its first $l$ row--column pairs.

*Theorem 1:* Under the assumptions (A1) and (A2), the pinning controlled network (4) is globally synchronized when the following two conditions are satisfied:

$$d_i > \lambda_{\max}(A - BC^{-1}B^T), \qquad i = 1,\ldots,l \tag{8}$$

$$\lambda_{\max}\left(\left(\frac{(G+G^T)}{2}\right)_l\right) < -\frac{\theta}{c} \tag{9}$$

where $d_i$ is the pinning feedback gain, $A$, $B$, $C$ are defined in (7), and $((G+G^T)/2)_l$ is the minor matrix of $((G+G^T)/2)$ by removing its first $l$ row-column pairs, $\theta > 0$ satisfies (A1).

*Proof:* Let $e_i = x_i - s$, $i = 1,\ldots,N$. From (3) and (4), we have the following error system:

$$\begin{aligned} \dot{e}_i = f(e_i+s) - f(s) + c\sum_{j=1}^{N} g_{ij}\Gamma e_j - d_i\Gamma e_i, \\ i = 1,\ldots,N \end{aligned} \tag{10}$$

where $d_i > 0$, $i = 1,\ldots,l$ and $d_i = 0$, $i = l+1,\ldots,N$.

Construct the Lyapunov functional candidate [9]

$$V(t) = \frac{1}{2}\sum_{i=1}^{N} e_i^T(t)e_i(t). \tag{11}$$

The derivative of $V(t)$ along the trajectory (10) is as follows:

$$\begin{aligned} \dot{V} &= \sum_{i=1}^{N} e_i^T\left[f(e_i+s) - f(s) + c\sum_{j=1}^{N} g_{ij}\Gamma e_j - d_i\Gamma e_i\right] \\ &\le \sum_{i=1}^{N} e_i^T\left[\theta\Gamma e_i + c\sum_{j=1}^{N} g_{ij}\Gamma e_j - d_i\Gamma e_i\right] \\ &= e^T[(\theta I_N + cG - D)\otimes\Gamma]e \\ &= e^T\left[(\theta I_N - D)\otimes\Gamma + \frac{c}{2}G\otimes\Gamma + \frac{c}{2}(G\otimes\Gamma)^T\right]e \end{aligned} \tag{12}$$

where $e = \left(e_1^T,\ldots,e_N^T\right)^T$ and $D = \mathrm{diag}(d_1,\ldots,d_l,\underbrace{0,\ldots,0}_{N-l})$.

Recall that $\Gamma = \mathrm{diag}(r_1,\ldots,r_n)$. In review of (6) and (12), we have

$$\begin{aligned} \dot{V} &\le e^T\left[\left(\theta I_N + c\frac{G+G^T}{2} - D\right)\otimes\Gamma\right]e \\ &= e^T[(H-D)\otimes\Gamma]e. \end{aligned} \tag{13}$$

By Lemma 2 and Remark 1, from (7) and the given condition (8), we know that $H - D < 0$ is equivalent to $C = \left(\theta I_N + c((G+G^T)/2)\right)_l < 0$ when the pinning feedback gains satisfy $d_i > \lambda_{\max}(A - BC^{-1}B^T), i = 1,\ldots,l$. Therefore, we only need to show that $C = \left(\theta I_N + c((G+G^T)/2)\right)_l < 0$.

By Lemma 3, we have

$$\lambda_{\max}\left(\left(\theta I_N + c\frac{G+G^T}{2}\right)_l\right) \le \theta + c\lambda_{\max}\left(\left(\frac{G+G^T}{2}\right)_l\right). \tag{14}$$

From the given condition (9) of the theorem, it is easy to have $\theta + c\lambda_{\max}\left(((G+G^T)/2)_l\right) < 0$. Then, in view of (14), we know that $\lambda_{\max}\left(\left(\theta I_N + c((G+G^T)/2)\right)_l\right) < 0$, which indicates that $C = \left(\theta I_N + c((G+G^T)/2)\right)_l < 0$. Therefore, we can conclude that $H - D = \theta I_N + c((G+G^T)/2) - D < 0$. Considering that $\Gamma > 0$, from (13) we have $\dot{V} < 0$. Thus, the pinning controlled network (4) is globally synchronized.

By now the proof is completed.

*Remark 3:* An undirected network can be treated as a directed network with symmetric coupling matrix, i.e., $G = G^T$. Therefore, Theorem 1 really establishes sufficient conditions for pinning synchronization of both directed and undirected complex networks. Pinning condition (8) gives a low bound of pinning feedback gains, which provides some guidance to design pinning feedback gains. For a given pinned-node selection scheme and a constant coupling strength, we can use pinning condition (9) to determine the least number $l_0$ of pinned nodes, which satisfies $\lambda_{\max}\left(((G+G^T)/2)_{l_0-1}\right) \geq -(\theta/c)$ and $\lambda_{\max}\left(((G+G^T)/2)_{l_0}\right) < -(\theta/c)$ to reach network synchronization.

*Remark 4:* Suppose that we have selected $l$ pinned nodes to satisfy pinning condition (9). Then, the coupling strength is expected to satisfy $c > \left(-\theta/\lambda_{\max}(((G+G^T)/2)_l)\right)$. Usually, the theoretical value of $c$ is much larger than that needed in practice [7], [9], [14]. When $c$ is small, it is not guaranteed that we can find a small fraction of network nodes such that pinning condition (9) holds. To achieve synchronization, we can use global or local approaches to adaptively tune the overall or local coupling strengths [7], [9], [14], [25]–[29]. In this paper, we focus on the pinning synchronization of a general complex network with a fixed coupling strength.

## V. Pinned-Node Selection Schemes for Directed and Undirected Complex Dynamical Networks

Up to this point, one may raise two questions: is it possible to find some pinned nodes such that pinning condition (9) is satisfied and how to locate them in the network? In this section, two pinned-node selection schemes are presented for directed and undirected complex networks to guide what nodes can be chosen to be pinned to reach network synchronization.

### A. Pinned-Node Selection Scheme for Directed Complex Dynamical Networks

To satisfy pinning condition (9), at least we need to choose some pinned candidates such that $\lambda_{\max}\left(((G+G^T)/2)_l\right) \leq 0$. To achieve this goal, we give the following theorem.

*Theorem 2:* Suppose that the controlled directed complex dynamical network (4) satisfies assumption (A2). Denote $\mathrm{DegIn}(i)$ and $\mathrm{DegOut}(i)$ as in-degree and out-degree of the $i$th node, respectively. Assume that the first $l$ nodes satisfy $\mathrm{DegOut}(i) > \mathrm{DegIn}(i)$, $i = 1, \ldots, l$, and other $N-l$ nodes satisfy $\mathrm{DegOut}(i) \leq \mathrm{DegIn}(i)$, $i = l+1, \ldots, N$. Then, we always have $\lambda_{\max}\left(((G+G^T)/2)_l\right) \leq 0$ in which $(G+G^T/2)_l$ is the minor matrix of $(G+G^T/2)$ by removing its first $l$ row–column pairs.

*Proof:* Let

$$\overline{G} = (\overline{g}_{ij})_{N\times N} = \frac{G+G^T}{2} \tag{15}$$

where $\overline{g}_{ij} = \overline{g}_{ji} = (g_{ij}+g_{ji})/2, i, j = 1, \ldots, N$.

According to (2) stated in Remark 2 and (15), we have

$$\begin{aligned} \overline{g}_{ii} &= -\mathrm{DegIn}(i) \\ \sum_{j=1, j\neq i}^{N} \overline{g}_{ij} &= \frac{\mathrm{DegIn}(i)+\mathrm{DegOut}(i)}{2}, \qquad i = 1, \ldots, N. \end{aligned} \tag{16}$$

Decompose matrix $\overline{G}$ in (15) as follows:

$$\overline{G} = \begin{bmatrix} \overline{A} & \overline{B} \\ \overline{B}^T & \overline{G}_l \end{bmatrix} \tag{17}$$

where $\overline{A} = (\overline{a}_{ij})_{l\times l}$ and $\overline{G}_l = (\overline{g}_{l_{ij}})_{(N-l)\times(N-l)}$ are symmetric, $\overline{a}_{ij} = \overline{a}_{ji} = \overline{g}_{ij}, i, j = 1, \ldots, l$, $\overline{B} = (\overline{b}_{ij})_{l\times(N-l)}$, $\overline{b}_{ij} = \overline{g}_{ij} \geq 0, i = 1, \ldots, l, j = l+1, \ldots, N$, and $\overline{G}_l = \left((G+G^T)/2\right)_l$ with $\overline{g}_{l_{ij}} = \overline{g}_{l_{ji}} = \overline{g}_{i+l,j+l}, i, j = 1, \ldots, N-l$.

Denote $S(i)$ and $S_l(i)$ as the row-sums of the $i$th row of $\overline{G}$ and $\overline{G}_l$, respectively. From (15)–(17), we have

$$\begin{aligned} S(i) &= \frac{\mathrm{DegOut}(i)-\mathrm{DegIn}(i)}{2}, \quad i = 1, \ldots, N \\ S_l(i) &= \overline{g}_{l_{ii}} + \sum_{j=1, j\neq i}^{N-l} \overline{g}_{l_{ij}}, \quad i = 1, \ldots, N-l. \end{aligned} \tag{18}$$

Based on the definitions of $G$ in (1) and $\overline{G}$ in (15) and (16), we see that $\overline{g}_{ij} \geq 0 (i \neq j)$. By the definitions of $\overline{B}$ and $\overline{B}^T$ in (17), we know that all of their elements are nonnegative. Furthermore, matrices $\overline{B}$ and $\overline{B}^T$ cannot be zero matrices. Otherwise, assumption (A2) will not hold. Then, we have

$$S_l(i) \leq S(i+l), \quad i = 1, \ldots, N-l. \tag{19}$$

Let $\lambda$ be any eigenvalue of $\overline{G}_l$. From (18), (19), and Gerschgorin's Circle Theorem [23], we can conclude that

$$\begin{aligned} \lambda &\leq \overline{g}_{l_{ii}} + \sum_{j=1, j\neq i}^{N-l} \overline{g}_{l_{ij}} \\ &= S_l(i) \leq S(i+l) = \frac{\mathrm{DegOut}(i+l)-\mathrm{DegIn}(i+l)}{2}, \\ &\qquad i \in \{1, \ldots, N-l\}. \end{aligned} \tag{20}$$

By the conditions of the theorem, we know that $\mathrm{DegOut}(i) \leq \mathrm{DegIn}(i), i = l+1, \ldots, N$. From (20), we have $\lambda \leq 0$. Therefore, the maximum eigenvalue of $\overline{G}_l$ satisfies $\lambda_{\max}(\overline{G}_l) \leq 0$, i.e., $\lambda_{\max}\left(((G+G^T)/2)_l\right) \leq 0$.

By now we finish the proof of Theorem 2.

*Remark 5:* Theorem 2 provides some useful guidance to select pinned nodes to achieve network synchronization. According to Theorem 2 and (20), the nodes whose out-degrees are bigger than their in-degrees should be specifically selected as pinned candidates. If the randomly pinning scheme is used, we may have $\lambda_{\max}\left(((G+G^T)/2)_l\right) > 0$ if the out-degrees of some unpinned nodes are bigger than their in-degrees. Furthermore, assumption (A2) may not hold when the pinned nodes are randomly chosen. Therefore, the randomly pinning scheme may not guarantee the synchronization of a directed network even if it has a very large coupling strength. For a given small fraction $\delta (0 < \delta \ll 1)$, we present the following specifically pinning scheme to choose pinned nodes for directed networks based on Theorem 2.

Step 1) Define a degree-difference vector: $\mathrm{DegDif}(i) = \mathrm{DegOut}(i) - \mathrm{DegIn}(i), i = 1, \ldots, N$.

Step 2) Rearrange the network nodes: the nodes with zero in-degrees, followed by other nodes in descending order based on their degree-difference. For the nodes with same degree difference, we sort them in descending order according to their out-degrees. Let $l = 1$.

Step 3) Select the first $l$ network nodes as pinned candidates. Evaluate $\lambda_{\max}\left(((G+G^T)/2)_l\right)$ and check if pinning condition (9) is satisfied.

Step 4) If pinning condition (9) and assumption (A2) are not satisfied and $l < \lfloor \delta N \rfloor$, let $l = l+1$, go to step 3. Otherwise, end.

When the coupling strength $c$ is large enough, pinning condition (9) is easily to be satisfied, and the network can be synchronized by pinning a small fraction of network nodes. However, for a small coupling strength, it is not guaranteed that we can always find a small fraction of network nodes as pinned nodes to achieve synchronization. To overcome this difficulty, one can consider adaptively adjusting the coupling strength [7], [9], [14], [25]–[29].

### *B. Pinned-Node Selection Scheme for Undirected Complex Dynamical Networks*

An undirected complex dynamical network can be treated as a special kind of directed network. For an undirected network, all nodes have same out-degrees and in-degrees. Therefore, we have $\mathrm{DegOut}(i) = \mathrm{DegIn}(i)$, $i = 1,\dots,N$ and $G = G^T$. From Theorem 2, we have the following corollary.

*Corollary:* Suppose that the controlled undirected complex network (4) satisfies assumption (A2). Choosing arbitrary $l(1 \le l < N)$ nodes, we always have $\lambda_{\max}(G_l) \le 0$, where $G_l$ is the minor matrix of $G$ by removing its first $l$ row--column pairs.

*Remark 6:* According to this corollary, we know that both randomly and specifically pinning schemes are applicable for the synchronization of undirected networks. This confirms the work in [2] and [14]. Wang and Chen [2] investigated randomly and specifically pinning schemes for scale-free networks and pointed out that it is better to successively pin the most highly connected nodes. Usually, we can rearrange network nodes in descending order based on their degrees and choose the first $l$ network nodes as pinned candidates to satisfy pinning condition (9).

## VI. Adaptive Pinning Control Scheme for Directed and Undirected Complex Dynamical Networks

With the proposed pinning conditions and pinned-node selection schemes, this section further investigates how to achieve the synchronization of a general complex network. For the pinning controlled network (4), one may choose pinning feedback gains according to pinning condition (8). However, this condition involves matrix decomposition, and thus, it is not quite practical. Furthermore, it usually gives much larger feedback gains than those needed in practice. A better way is to use adaptive approach to tune the pinning feedback gains [7]–[9], [12], [13].

The following theorem develops an adaptive pinning control scheme for the controlled complex network (4).

*Theorem 3:* Suppose that $l$ pinned nodes have been selected such that pinning condition (9) holds. Under the assumptions (A1) and (A2), the pinning-controlled complex dynamical network (4) is globally synchronized when the pinning feedback gains are tuned by

$$\dot{d}_i = q_i(x_i - s)^T\Gamma(x_i - s), \quad i = 1,\dots,l \tag{21}$$

where $q_i > 0$, $i = 1,\dots,l$ are small positive constants.

*Proof:* Let $e_i = x_i - s$, $i = 1,\dots,N$.

It follows from (3), (4), and (21) that we have the following error system:

$$\dot{e}_i = f(e_i+s) - f(s) + c\sum_{j=1}^{N} g_{ij}\Gamma e_j - d_i\Gamma e_i, \quad i = 1,\dots,N \tag{22}$$

where $d_i > 0$, $i = 1,\dots,l$; $d_i = 0$, $i = l+1,\dots,N$, and $\dot{d}_i = q_i(x_i - s)^T\Gamma(x_i - s)$, $i = 1,\dots,l$ in which $q_i > 0$ is an appropriate positive constant.

Construct the Lyapunov functional candidate [9]

$$V(t) = \frac{1}{2}\sum_{i=1}^{N} e_i^T(t)e_i(t) + \sum_{i=1}^{l}\frac{1}{2q_i}(d_i - \tilde{d})^2 \tag{23}$$

where $\tilde{d} > 0$ is a constant to be determined next.

The derivative of $V(t)$ along the trajectory (22) is as follows:

$$\begin{aligned}\dot{V} &= \sum_{i=1}^{N} e_i^T\left[f(e_i+s) - f(s) + c\sum_{j=1}^{N} g_{ij}\Gamma e_j - d_i\Gamma e_i\right] \\ &\quad + \sum_{i=1}^{l}(d_i - \tilde{d})e_i^T\Gamma e_i \\ &\le \sum_{i=1}^{N} e_i^T\left(\theta\Gamma e_i + c\sum_{j=1}^{N} g_{ij}\Gamma e_j\right) - \tilde{d}\sum_{i=1}^{l} e_i^T\Gamma e_i \\ &= e^T\left[\left(\theta I_N + c\frac{G+G^T}{2} - D) \otimes \Gamma\right)\right]e \end{aligned} \tag{24}$$

where $e = \left(e_1^T,\dots,e_N^T\right)^T$, $D = \mathrm{diag}(\underbrace{\tilde{d},\dots,\tilde{d}}_{l},\underbrace{0,\dots,0}_{N-l})$.

The assumptions (A1) and (A2) hold and pinning condition (9) is supposed to be satisfied. Letting $\tilde{d} > 0$ be sufficiently large, by Lemma 2, Theorem 1, and its proof, we have $\dot{V} < 0$. Therefore, we conclude that the pinning-controlled complex network (4) is globally synchronized when the pinning feedback gains are tuned by (21).

The proof is thus completed.

## VII. Numerical Results

This section provides two numerical examples to verify the effectiveness of the proposed techniques. We first give an example of a small directed network, followed by an example of a large-scale directed scale-free network. The quantity $E(t) = \sqrt{\sum_{i=1}^{N}(x_i(t) - s(t))^T(x_i(t) - s(t))/N}$ is used to measure the quality of the pinning process [14].

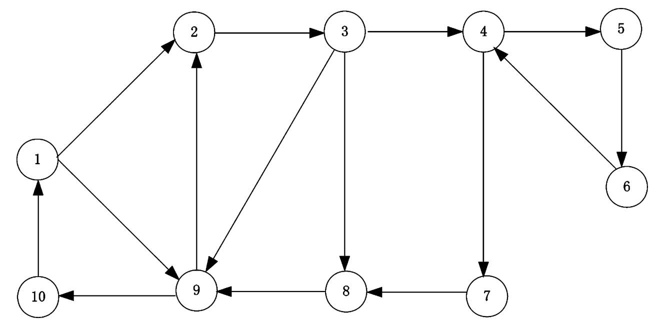


Fig. 1. Simple directed complex network with ten nodes.

### A. Simulation Example I

Consider a simple directed complex dynamical network shown in Fig. 1.

The directed complex network in Fig. 1 consisting of ten identical Chua's systems [30] is described by

$$\dot{x}_i = f(x_i) + c\sum_{j=1}^{10} g_{ij}\Gamma x_j, \quad i = 1, \dots, 10 \tag{25}$$

where $x_i = (x_{i1}, x_{i2}, x_{i3})^T$ is the state variable of the $i$th node

$$f(x_i) = \begin{cases} \alpha(x_{i2} - x_{i1} - F(x_{i1})) \\ x_{i1} - x_{i2} + x_{i3} \\ -\beta x_{i2} \end{cases} \tag{26}$$

where $\alpha = 10$ and $\beta = 14.87$

$$F(x_{i1}) = bx_{i1} + \frac{a-b}{2}(|x_{i1}+1| - |x_{i1}-1|)$$

in which $a = -1.27$ and $b = -0.68$, where

$$G = (g_{ij})_{10\times 10} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -2 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

$c = 60$, $\Gamma = \mathrm{diag}(1, 1, 1)$.

Now study if a parameter $\theta$ can be found to satisfy assumption (A1) in (5). Rewrite (26) as follows:

$$f(x_i) = Ax_i + \phi(x_i) \tag{27}$$

where $A = \begin{bmatrix} -\alpha & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & 0 \end{bmatrix}$ and $\phi(x_i) = \begin{bmatrix} -\alpha F(x_{i1}) \\ 0 \\ 0 \end{bmatrix}$.

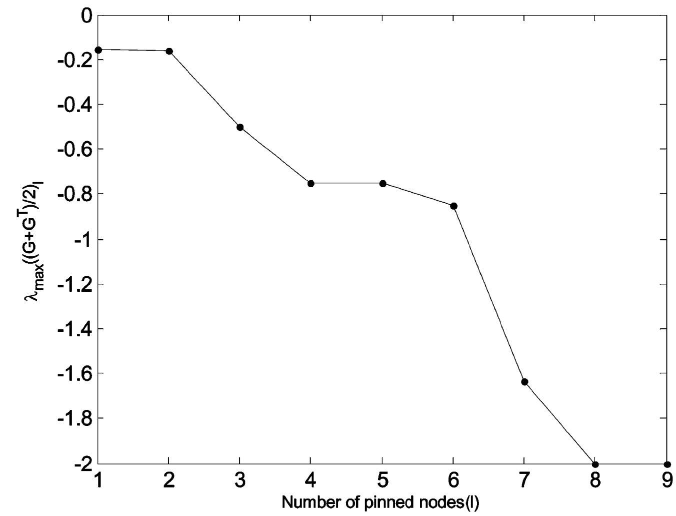


Fig. 2. $\lambda_{\max}\left(((G+G^T)/2)_l\right)$ versus the number of pinned nodes for a simple directed network.

Let $e_i = x_i - s$, where $s$ is the solution of the isolated node. Then, we have [30]

$$\begin{aligned} e_i^T(f(x_i) - f(s)) &= \frac{1}{2}e_i^T(A + A^T)e_i + e_i^T(\phi(x_i) - \phi(s)) \\ &\leq \frac{1}{2}e_i^T(A + A^T)e_i + |\alpha a|\, e_{i1}^T e_{i1} \\ &= \frac{1}{2}e_i^T(\tilde{A} + \tilde{A}^T)e_i \\ &\leq \frac{1}{2}\lambda_{\max}(\tilde{A} + \tilde{A}^T)e_i^T e_i \end{aligned} \tag{28}$$

where $\tilde{A} = A + \mathrm{diag}(|\alpha a|, 0, 0)$.

Note that $\Gamma = \mathrm{diag}(1, 1, 1)$. Thus, assumption (A1) is satisfied if we choose the parameter $\theta$ as follows:

$$\theta = \frac{1}{2}\lambda_{\max}(\tilde{A} + \tilde{A}^T) = 9.0620. \tag{29}$$

According to pinning condition (9), we have

$$\lambda_{\max}\left(\left(\frac{G+G^T}{2}\right)_l\right) < -\frac{\theta}{c} = -\frac{9.0620}{60} = -0.1510. \tag{30}$$

Examining Fig. 1, we note that the out-degrees of nodes 3 and 1 are bigger than their in-degrees. According to the pinned-node selection scheme for directed complex networks in Remark 5, rearrange the network nodes and their new order is: 3, 1, 4, 5, 6, 7, 10, 9, 2, 8. We choose $l$ from 1 to 9 and depict $\lambda_{\max}\left((G+G^T/2)_l\right)$, which decreases with the increase of $l$ in Fig. 2.

When $l = 1$, we choose node 3 as the pinned candidate and have $\lambda_{\max}\left(((G+G^T)/2)_l\right) = -0.1534$, which satisfies pinning condition (30). Also, it is easy to verify that assumption (A2) holds when we pin node 3. Therefore, the network with ten nodes in Fig. 1 can be globally synchronized by *only* pinning a single node, i.e., node 3.

With pinning condition (8), we estimate the lower bound of the pinning feedback gain, which is shown in Fig. 3. We note that the estimated feedback gain decreases dramatically when

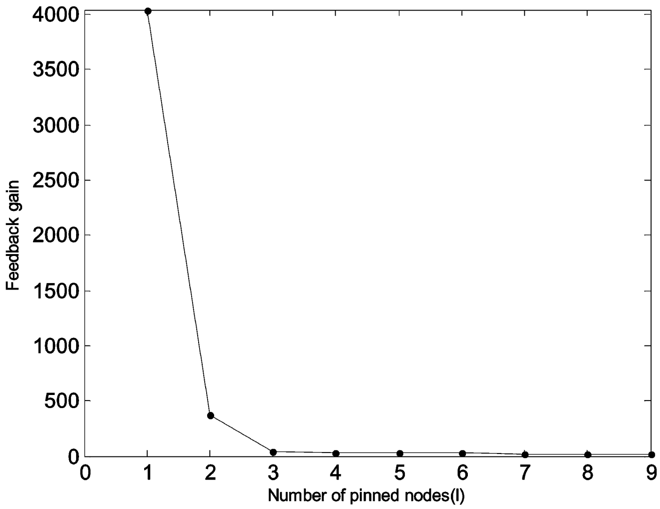


Fig. 3. Estimated feedback gain versus the number of pinned nodes for a simple directed network.

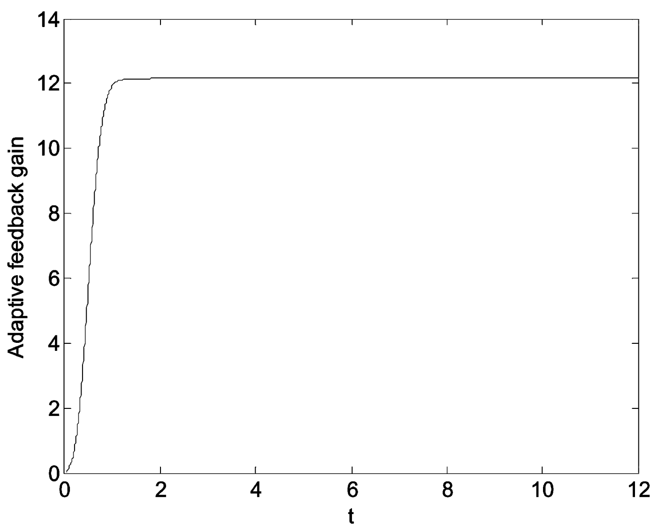


Fig. 4. Adaptive pinning feedback gain for a simple pinning-controlled directed network.

$l = 2$ and 3. When node 3 is pinned, the estimated lower bound of the feedback gain is $d = 4039.8$.

Based on Theorem 3, we apply pinning control action to node 3 and tune the feedback gain with (21). The evolutions of the pinning feedback gain and the synchronization error are illustrated in Figs. 4 and 5, respectively. The pinning feedback gain converges to a constant value of 11.9733, which is much smaller than its estimated lower bound. Fig. 5 shows that the network with ten nodes in Fig. 1 is synchronized by *only* pinning a single node.

### B. Simulation Example 2

Now we investigate the pining synchronization of a large-scale directed scale-free network. In [15] and [31], some techniques are proposed to establish directed networks. We adopt the method in [15] to generate a directed scale-free network, which is strongly connected and the in-degree and out-degree of each

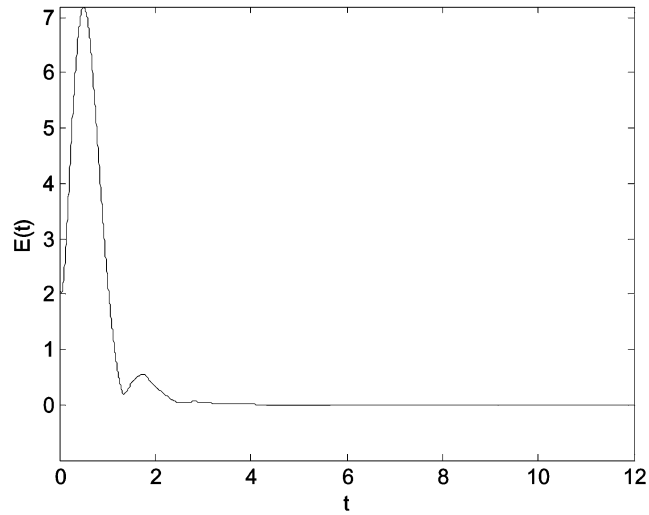


Fig. 5. Time evolution of synchronization error for a simple pinning-controlled directed network.

individual node both follow power law distribution. Thus, assumption (A2) holds. The generation of an undirected scale-free network is advised (see [32]).

A large-scale directed scale-free network with 500 identical nodes of Lü's systems [12] is described by

$$\dot{x}_i = f(x_i) + c\sum_{j=1}^{N} g_{ij}\Gamma x_j, \quad i = 1, \ldots, N \tag{31}$$

where $N = 500$, $c = 50$, $G = (g_{ij})_{N\times N}$ is an asymmetric diffusive coupling matrix, $\Gamma = \mathrm{diag}(0.1, 2, 0.5)$, $x_i = (x_{i1}, x_{i2}, x_{i3})^T$ is the state variable of the $i$th node, and

$$f(x_i) = \begin{cases} 36(x_{i2} - x_{i1}) \\ 20x_{i2} - x_{i1}x_{i3} \\ x_{i1}x_{i2} - 3x_{i3}. \end{cases} \tag{32}$$

The isolated node is Lü's system described by $\dot{s} = f(s)$, where $s = (s_1, s_2, s_3)^T$. It is known that Lü attractor is bounded by $|s_j| \le M_j$, $j = 1, 2, 3$. According to [12], we can choose $M_1 = 25$, $M_2 = 30$, and $M_3 = 45$.

Let $e_i = x_i - s$. Using the method in [9], we have

$$\begin{aligned} &(x_i - s)^T(f(x_i) - f(s)) \\ &\quad = \begin{bmatrix} e_{i1} & e_{i2} & e_{i3} \end{bmatrix} \begin{bmatrix} 36(e_{i2} - e_{i1}) \\ 20e_{i2} - s_1e_{i3} - s_3e_{i1} - e_{i1}e_{i3} \\ -3e_{i3} + s_1e_{i2} + s_2e_{i1} + e_{i1}e_{i2} \end{bmatrix} \\ &\quad = -36e_{i1}^2 + 20e_{i2}^2 - 3e_{i3}^2 + (36 - s_3)e_{i1}e_{i2} + s_2e_{i1}e_{i3} \\ &\quad \le -36e_{i1}^2 + 20e_{i2}^2 - 3e_{i3}^2 + (36 + M_3)\,|e_{i1}e_{i2}| \\ &\qquad + M_2\,|e_{i1}e_{i3}| \\ &\quad \le \left(-36 + \rho\frac{36 + M_3}{2} + \eta\frac{M_2}{2}\right)e_{i1}^2 \\ &\qquad + \left(20 + \frac{36 + M_3}{2\rho}\right)e_{i2}^2 + \left(-3 + \frac{M_2}{2\eta}\right)e_{i3}^2. \end{aligned} \tag{33}$$

Choosing $\rho = 0.7285$ and $\eta = 0.6850$, we have

$$\begin{aligned} (x_i - s)^T(f(x_i) - f(s)) &\le 37.7963(0.1e_{i1}^2 + 2e_{i2}^2 + 0.5e_{i3}^2) \\ &= 37.7963e_i^T\Gamma e_i. \end{aligned} \tag{34}$$

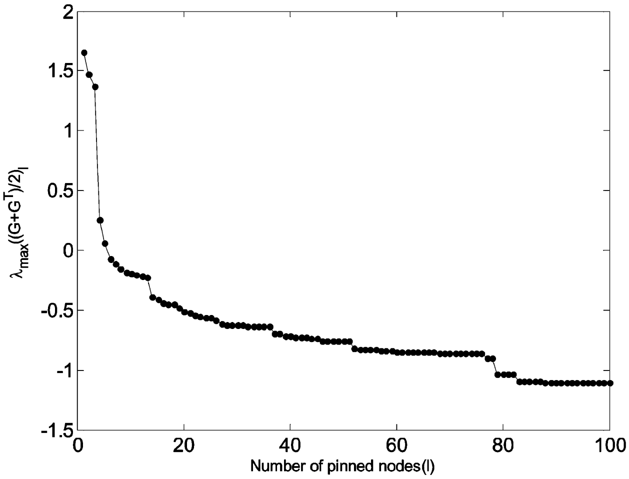


Fig. 6. $\lambda_{\max}\left(((G+G^T)/2)_l\right)$ versus the number of pinned nodes for a directed scale-free network.

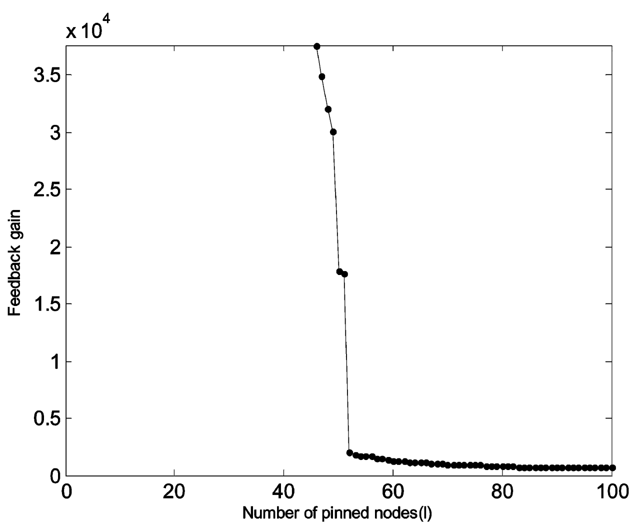


Fig. 7. Estimated feedback gain versus the number of pinned nodes for a directed scale-free network.

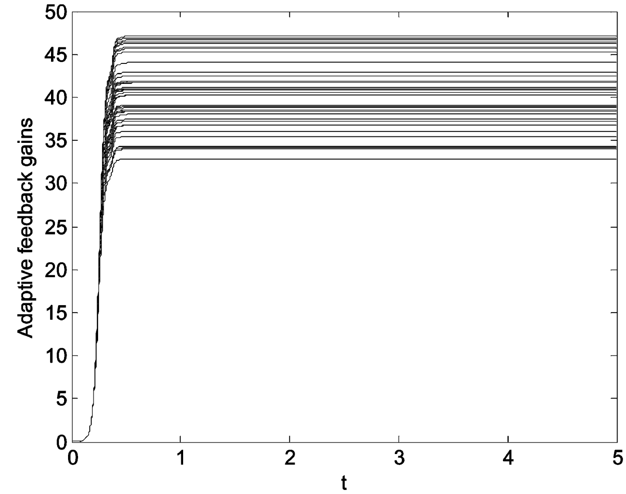


Fig. 8. Adaptive pinning feedback gains for a pinning-controlled directed scale-free network.

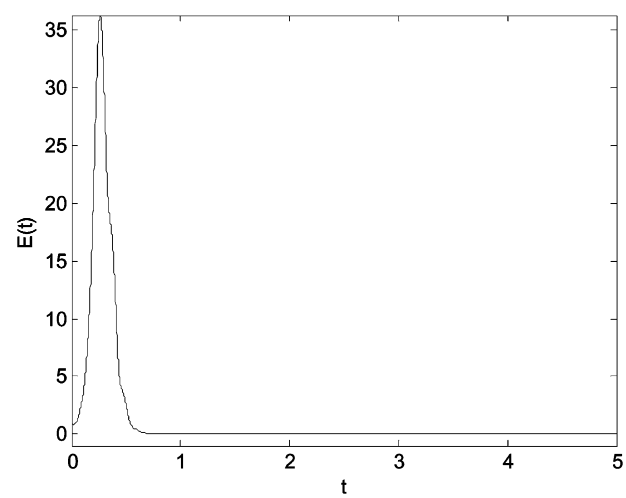


Fig. 9. Time evolution of synchronization error for a pinning-controlled directed scale-free network.

Therefore, we can choose $\theta = 37.7963$ to satisfy assumption (A1). According to pinning condition (9), we have

$$\lambda_{\max}\left(\left(\frac{G+G^T}{2}\right)_l\right) < -\frac{\theta}{c} = -\frac{37.7963}{50} = -0.7559. \tag{35}$$

Using the pinned-node selection scheme for directed complex networks in Remark 5, we rearrange network nodes. Select $\delta = 0.02$ as the pinning fraction. Choose $l$ from 1 to 100 and depict $\lambda_{\max}\left(((G+G^T)/2)_l\right)$, which decreases with the number of pinned nodes in Fig. 6. Especially, when $l = 45$ and 46, we have $\lambda_{\max}\left(((G+G^T)/2)_{45}\right) = -0.7327$ and $\lambda_{\max}\left(((G+G^T)/2)_{46}\right) = -0.7569$.

Then, we can determine that the least number of pinned nodes is 46 by pinning condition (35) and Remark 3. Therefore, we choose the first 46 rearranged nodes of network (31) as pinned nodes.

We calculate the lower bound of the pinning feedback gain shown in Fig. 7 with pinning condition (8). It is noted that the estimated feedback gain decreases with the increase of $l$. When the first 46 rearranged network nodes are chosen to be pinned, the lower bound of the feedback gain is $d = 37622$, which is quite large.

Based on Theorem 3, we apply pinning control actions to the selected 46 nodes of the directed scale-free network (31) and tune feedback gains with (21). The evolutions of the pinning feedback gains and the synchronization error are illustrated in Figs. 8 and 9, respectively. We can see that the directed scale-free network with 500 nodes is globally synchronized by *only* pinning 46 network nodes.

## VIII. Conclusion

In this paper, the pinning synchronization of directed and undirected complex dynamical networks has been intensively studied. Without assuming symmetry and irreducibility of the couplings, we have established low-dimensional pinning criteria for general complex dynamical networks. Then, we have proposed effective pinned-node selection schemes for complex networks with general topology. For directed networks, we have shown that those nodes whose out-degrees are bigger than their in-degrees should be chosen as pinned candidates. Especially, we point out that the randomly pinning scheme may not ensure the synchronization of directed networks. To achieve synchronization of general complex networks, we have developed a simple adaptive pinning control scheme. Detailed and satisfactory numerical results have validated the feasibility and the correctness of the proposed techniques. We hope this study will be helpful for the research and applications on pinning control and synchronization of general complex networks.

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