

# Flocking of Multi-Agents With a Virtual Leader

Housheng Su, Xiaofan Wang, *Senior Member, IEEE*, and Zongli Lin, *Fellow, IEEE*

**Abstract**—All agents being informed and the virtual leader traveling at a constant velocity are the two critical assumptions seen in the recent literature on flocking in multi-agent systems. Under these assumptions, Olfati-Saber in a recent IEEE TRANSACTIONS ON AUTOMATIC CONTROL paper proposed a flocking algorithm which by incorporating a navigational feedback enables a group of agents to track a virtual leader. This paper revisits the problem of multi-agent flocking in the absence of the above two assumptions. We first show that, even when only a fraction of agents are informed, the Olfati-Saber flocking algorithm still enables all the informed agents to move with the desired constant velocity, and an uninformed agent to also move with the same desired velocity if it can be influenced by the informed agents from time to time during the evolution. Numerical simulation demonstrates that a very small group of the informed agents can cause most of the agents to move with the desired velocity and the larger the informed group is the bigger portion of agents will move with the desired velocity. In the situation where the virtual leader travels with a varying velocity, we propose modification to the Olfati-Saber algorithm and show that the resulting algorithm enables the asymptotic tracking of the virtual leader. That is, the position and velocity of the center of mass of all agents will converge exponentially to those of the virtual leader. The convergent rate is also given.

**Index Terms**—Distributed control, flocking, informed agents, nonlinear systems, virtual leader.

## I. INTRODUCTION

**F**LOCKING is the phenomenon in which a large number of agents, using only limited environmental information and simple rules, organize into a coordinated motion. Flocking behavior exists in nature in the form of flocking of birds, schooling of fish, swarming of bacteria, and so on [1], [2]. Flocking problems have attracted much attention among researchers in biology, physics and computer science for decades [3]–[10]. Recently, there has also been a surge of interest among control theorists in flocking problems, partly due to the wide applications of flocking in many control areas including cooperative control of mobile robots and design of mobile sensor networks [11]–[13].

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H. Su and X. Wang are with the Department of Automation, Shanghai Jiao Tong University, Shanghai 200240, China (e-mail: wwwshs@sjtu.edu.cn; xfwang@sjtu.edu.cn).

Z. Lin is with the Charles L. Brown Department of Electrical and Computer Engineering, University of Virginia, Charlottesville, VA 22904-4743 USA (e-mail: zl5y@virginia.edu).

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The classical flocking model proposed by Reynolds [3] in 1980s consists of three heuristic rules: 1) Separation: avoid collision with nearby flockmates; 2) Alignment: attempt to match velocity with nearby flockmates; 3) Cohesion: attempt to stay close to nearby flockmates. Over the years, many variants of these three rules and additional rules have been suggested, including obstacle avoidance and goal seeking. Many algorithms to realize these rules have also been proposed [14]–[26]. For example, graph theory [14], [15] and Lyapunov-based approach [16] have been used to investigate consensus (alignment) problem, local attractive/repulsive potential between neighboring agents to deal with separation and cohesion problem [17], [18], and artificial potentials [19]–[21] and gyroscopic forces [22] for obstacle avoidance. Leader (virtual leader)/follower approach to goal seeking (tracking) problem has also been investigated [23]–[26].

Recently, flocking algorithms based on Reynolds flocking model were proposed by a combination of a velocity consensus component with a local artificial potential field [27]–[29]. In particular, Olfati-Saber [29] provided a theoretical and computational framework for design and analysis of scalable flocking algorithms. This framework includes three flocking algorithms. The first algorithm embodies the three rules of Reynolds and the third algorithm has obstacle avoidance capabilities. The second algorithm is the main flocking algorithm for a group of agents moving in a free space and is aimed at causing the group to track the path of a virtual leader by including a navigational feedback mechanism to every agent. It is assumed in the second algorithm that every agent in the group is an informed agent which has the information of the virtual leader. The benefit of this assumption is that it guarantees that all agents remain cohesive and eventually move with the same velocity. However, such an assumption is in contrast with some phenomena in the nature and may be difficult to implement in engineering applications.

In some nature examples, few individuals have the pertinent information, such as knowledge of the location of a food source, or of a migration route. For example, a few informed individuals within a fish school are known to be able to guide the school to navigate towards a target, and only about 5% of the bees within a honeybee swarm can lead the group to a new nest site [30]. Recently, using a simple discrete-time model of group formation, Couzin *et al.* [30] revealed through simulation that only a very small proportion of informed individuals is required to achieve a great accuracy of group motion in a preferred motion.

The first objective of this paper is to show that most agents will move with the desired velocity even when only a very small fraction of agents are informed agents in a large group of agents. In particular, under the assumption that the desired velocity is a constant vector, we will prove that the Olfati-Saber algorithm enables all the informed agents to move with the desired velocity, and an uninformed agent to also move with the same desired velocity if it can be influenced by the informed agents

from time to time during the evolution. These results will not only help to understand flocking behaviors in the nature, but also provide a framework for guiding the design of engineering multi-agent systems with a small fraction of agents being informed.

Showing that some uninformed agents can also approach the same desired velocity even when they do not have constant access to one informed agent turns out to be rather involved. Due to the existence of a nonlinear gradient-based term in the control protocol, the graph theory and related stochastic matrix [15], [23] and Lyapunov-based approaches [16], which have been used to study the consensus problems under the assumption of frequent enough network connectivity of the underlying topology, cannot be directly applied to the flocking problems studied in this paper.

A key step in establishing these results is the application of the LaSalle Invariance Principle, which establishes convergence properties once the boundedness of the trajectories of all agents is guaranteed. When all agents are informed agents, such boundedness property of the trajectories is relatively straightforward consequence of the negative semi-definiteness of the derivative of a certain positive semi-definite function  $Q$ , which is the total artificial potential energy and the total relative kinetic energy between all agents and the virtual leader. However, when not all agents are informed agents, the boundedness of trajectories of those uninformed agents no longer follow directly from the negative semi-definiteness of  $\dot{Q}$ . A careful examination of the intricate behaviors of these uninformed agents is required. A coordinate system for the collective dynamics of all agents is also adopted which makes the virtual leader as a reference. Such a coordinate system also facilitates the proof that the velocities of the agents eventually match that of the virtual leader as the time goes by.

In the situation when the velocity of the virtual leader is not a constant vector, Olfati-Saber [29] has shown that his second algorithm is able to cause all the agents in the group to eventually achieve the same velocity, which, however, is in general not equal to the velocity of the virtual leader. This motivates us to propose modification to the Olfati-Saber second algorithm so that the resulting algorithm will enable the asymptotic tracking of the virtual leader. That is, the position and velocity of the center of mass (COM) of all agents will converge exponentially to those of the virtual leader. In establishing this result, we will also obtain the rate of convergence.

The remainder of the paper is organized as follows. Section II recalls some background materials and motivates the problems to be solved in this paper. Section III establishes flocking results when only a fraction of agents are informed. Section IV deals with the situation when the virtual leader does not move with a constant velocity. Section V includes the results from the simulation study. Finally, Section VI draws a conclusion to the paper.

## II. BACKGROUNDS AND PROBLEM STATEMENT

We consider  $N$  agents moving in an  $n$  dimensional Euclidean space. The motion of each agent is described by two integrators as

$$\begin{cases} \dot{q}_i = p_i \\ \dot{p}_i = u_i, \quad i = 1, 2, \dots, N \end{cases} \quad (1)$$

where  $q_i, p_i \in \mathbf{R}^n$  are respectively the position and velocity vectors of agent  $i$ , and  $u_i \in \mathbf{R}^n$  is the (acceleration) control input acting on agent  $i$ . For notational convenience, we also define

$$q = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_N \end{bmatrix}, \quad p = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_N \end{bmatrix}. \quad (2)$$

Each agent has a limited communication capability and it can only communicate with agents within its neighborhood region. The neighboring set of agent  $i$  at time  $t$  is denoted as:

$$\mathcal{N}_i(t) = \{j : \|q_i - q_j\| < r, j = 1, 2, \dots, N, j \neq i\} \quad (3)$$

where  $\|\cdot\|$  is the Euclidean norm. In the above definition, we have assumed that all agents have an identical influencing/sensing radius  $r$ . During the course of motion, the relative distances between agents may vary with time, so the neighbors of each agent may change. We define the neighboring graph  $\mathcal{G}(t) = \{\mathcal{V}, \mathcal{E}(t)\}$  to be an undirected graph consisting of a set of vertices  $\mathcal{V} = \{1, 2, \dots, N\}$ , whose elements represent agents in the group, and a set of edges  $\mathcal{E}(t) = \{(i, j) \in \mathcal{V} \times \mathcal{V} : i \sim j\}$ , containing unordered pairs of vertices that represent neighboring relations at time  $t$ . Vertices  $i$  and  $j$  are said to be adjacent at time  $t$  if  $(i, j) \in \mathcal{E}(t)$ . A path of length  $l$  between vertices  $i$  and  $j$  is a sequence of  $l + 1$  distinct vertices starting with  $i$  and ending with  $j$  such that consecutive vertices in the sequence are adjacent.

In his second algorithm, Olfati-Saber [29] proposes a distributed controller for each agent which only utilizes the information of other agents in its neighborhood and that of the virtual leader. Under his control scheme, agent  $i$  is steered via its acceleration input  $u_i$  that consists of three components

$$u_i = f_i^g + f_i^d + f_i^\gamma. \quad (4)$$

The first component of (4),  $f_i^g$ , is used to regulate position between agent  $i$  and its flockmates. This term is responsible for collision avoidance and cohesion in the group and is derived from the field produced by a collective potential function  $V_i(q)$ , which depends on the relative distance between agent  $i$  and its flockmates, and is defined as

$$\begin{aligned} V_i(q) &= \sum_{j \in \mathcal{V} \setminus \{i\}} \psi_\alpha(\|q_i - q_j\|_\sigma) \\ &= \sum_{j \notin \mathcal{N}_i(t), j \neq i} \psi_\alpha(\|r\|_\sigma) + \sum_{j \in \mathcal{N}_i(t)} \psi_\alpha(\|q_i - q_j\|_\sigma) \end{aligned} \quad (5)$$

where the  $\sigma$ -norm  $\|\cdot\|_\sigma$  of a vector is a map  $\mathbf{R}^n \rightarrow \mathbf{R}_+$  defined as

$$\|z\|_\sigma = \frac{1}{\varepsilon} \left[ \sqrt{1 + \varepsilon \|z\|^2} - 1 \right]$$

with a parameter  $\varepsilon > 0$ . Unlike the norm  $\|z\|$ , which is not differentiable at  $z = 0$ , the map  $\|z\|_\sigma$  is differentiable everywhere. This property of  $\sigma$ -norm enables the construction of smooth collective potential functions for agents. The function  $\psi_\alpha : \mathbf{R}_+ \rightarrow \mathbf{R}_+$  (see Fig. 1) is a nonnegative smooth pairwise potential function of the distance  $\|q_i - q_j\|_\sigma$  between agents  $i$

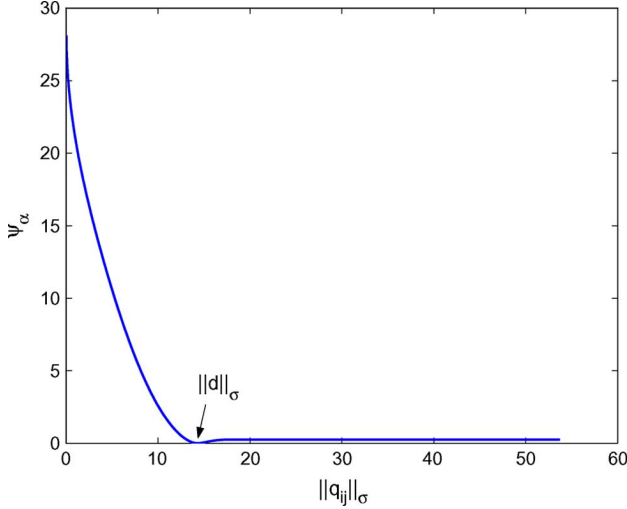


Fig. 1. Smooth pairwise potential function  $\psi_\alpha$ .

and  $j$ , such that  $\psi_\alpha$  reaches its maximum as  $\|q_i - q_j\|_\sigma \rightarrow 0$ , attains its unique minimum when agents  $i$  and  $j$  are located at a desired distance  $\|d\|_\sigma$ , and is constant for  $\|q_i - q_j\|_\sigma \geq \|r\|_\sigma$ .

The second component of (4),  $f_i^d$  regulates the velocity of agent  $i$  to the weighted average velocity of its flockmates. This term is responsible for velocity alignment. The third component of (4),  $f_i^\gamma$ , is a navigational feedback term, which drives agent  $i$  to track the virtual leader with the following model of motion,

$$\begin{cases} \dot{q}_\gamma = p_\gamma \\ \dot{p}_\gamma = f_\gamma(q_\gamma, p_\gamma) \end{cases} \quad (6)$$

where  $q_\gamma, p_\gamma, f_\gamma \in \mathbf{R}^n$  are position, velocity and control input (acceleration) of the virtual leader with  $(q_\gamma(0), p_\gamma(0)) = (q_d, p_d)$ .

More specifically, the control protocol in the second Olfati-Saber flocking algorithm of [29] is given by,

$$u_i = - \sum_{j \in \mathcal{N}_i(t)} \nabla_{q_i} \psi_\alpha(\|q_j - q_i\|_\sigma) + \sum_{j \in \mathcal{N}_i(t)} a_{ij}(t)(p_j - p_i) - c_1(q_i - q_\gamma) - c_2(p_i - p_\gamma), \quad c_1, c_2 > 0 \quad (7)$$

where  $A(t) = (a_{ij}(t))$  is the adjacent matrix which is defined as

$$a_{ij}(t) = \begin{cases} 0, & \text{if } j = i \\ \rho_h(\|q_j - q_i\|_\sigma / \|r\|_\sigma), & \text{if } j \neq i \end{cases} \quad (8)$$

with the bump function  $\rho_h(z)$ ,  $h \in (0, 1)$ , being

$$\rho_h(z) = \begin{cases} 1, & \text{if } z \in [0, h) \\ 0.5 \left[ 1 + \cos \left( \pi \frac{z-h}{1-h} \right) \right], & \text{if } z \in [h, 1] \\ 0, & \text{otherwise.} \end{cases} \quad (9)$$

It is then clear that  $a_{ij}(t) = a_{ji}(t) > 0$  if  $(j, i) \in \mathcal{E}(t)$  and  $a_{ij}(t) = a_{ji}(t) = 0$  if  $j \neq i$  but  $(j, i) \notin \mathcal{E}(t)$ . The two positive constants  $c_1$  and  $c_2$  represent weights of the navigational feedback. We observe that the control protocol (7) requires all agents to have the knowledge of the virtual leader,  $(q_\gamma, p_\gamma)$ .

Under the reasonable assumption that the initial velocity mismatch and inertia are finite, Olfati-Saber proved that three rules of Reynolds hold for a group of agents under the influence of

the control protocol (7). In particular, it was established in [29] that, the control protocol (7) enables a group of agents to track a virtual leader moving at a constant velocity.

Motivated by these results of [29], we will examine in this paper to what extent the control protocol (7) can still perform when not all agents are informed with the knowledge of the virtual leader and how to strengthen this control protocol when the virtual leader moves at a varying velocity.

### III. FLOCKING WITH A MINORITY OF INFORMED AGENTS

#### A. Algorithm Description and Main Results

In the flocking control protocol (7), it is assumed that each agent is an informed agent who has the information about the virtual leader (i.e., its position and velocity). In this section, we assume that only some, but not all, of the agents are informed agents who are given the information about the virtual leader. Consequently, the control input for agent  $i$ , (7), is modified as

$$u_i = - \sum_{j \in \mathcal{N}_i(t)} \nabla_{q_i} \psi_\alpha(\|q_j - q_i\|_\sigma) + \sum_{j \in \mathcal{N}_i(t)} a_{ij}(t)(p_j - p_i) - h_i [c_1(q_i - q_\gamma) + c_2(p_i - p_\gamma)], \quad c_1, c_2 > 0 \quad (10)$$

where  $h_i = 1$  if agent  $i$  is informed and  $h_i = 0$  otherwise. Without loss of generality, we assume that the first  $M_0$  ( $1 \leq M_0 \leq N$ ) agents are informed agents, that is,  $h_i = 1$  for  $i = 1, 2, \dots, M_0$ , and  $h_i = 0$  for  $i = M_0 + 1, M_0 + 2, \dots, N$ . We further assume that the virtual leader moves along a fixed direction with a constant velocity  $p_d$ . Therefore, the dynamic (6) for the virtual leader can be simplified to

$$\dot{q}_\gamma = p_d, \quad q_\gamma(0) = q_d. \quad (11)$$

We are interested in the behavior of the whole group of agents in the situation when only a small fraction of the agents are informed agents (i.e.,  $1 \leq M_0 \ll N$ ). As an informed agent not only is influenced by the virtual leader but might also be influenced by some uninformed agents, it is not obvious that an informed agent will definitely track the virtual leader, not to mention those uninformed agents. A careful analysis will however show that not only all the informed agents but also some uninformed agents will track the virtual leader. Furthermore, we will carry out extensive simulation to reveal that the majority of the uninformed agents will indeed track the virtual leader even when only a small fraction of a large group of agents are informed agents.

Before stating the main results of this section, we need the following definition. Denote the union of all neighboring graphs across a nonempty finite time interval  $[t_i, t_{i+1})$ ,  $t_{i+1} > t_i$  as  $\hat{\mathcal{G}}(t_i, t_{i+1})$ , whose edges are the union of the edges of those neighboring graphs. For an uninformed agent, if there is a path between this agent and one informed agent in the union  $\hat{\mathcal{G}}(t_i, t_{i+1})$ , then we say that there exists a joint path between the uninformed agent and the informed agent during the finite time interval  $[t_i, t_{i+1})$ . We divide all the uninformed agents into two types. An uninformed agent is called a type I uninformed agent if there exists an infinite sequence of contiguous, nonempty and uniformly bounded time-intervals  $[t_i, t_{i+1})$ ,  $i = 0, 1, 2, \dots$ , such that across each time interval there exists a joint path

between this agent and one informed agent. Otherwise, it is called a type II uninformed agent.

It is important to note that a type I uninformed agent is not required to stay in touch with an informed agent all the time. It only requires to be linked with an informed agent through a joint path during each time interval  $[t_i, t_{i+1})$ . Such a joint link is established as long as this type I uninformed agent is connected with an informed agent at any given time instant during the time interval  $[t_i, t_{i+1})$ , or it is connected at a time instant during  $[t_i, t_{i+1})$  with another type I uninformed agent which is connected with an informed agent at a different time instant during  $[t_i, t_{i+1})$ . In other words, a type I uninformed agent is only required to get in touch, directly or indirectly, with an informed agent from time to time.

As the intervals  $[t_i, t_{i+1})$ s are uniformly bounded, a type I uninformed agent will not stay out of touch with an informed agent for a period longer than the bound on these intervals. Thus, there exists a sufficiently large  $T > 0$  such that, for all  $t \geq T$ , there does not exist any joint path between any type II uninformed agent and any informed agent. This assumption implies that an uninformed agent that is disconnected from all informed agents for a long enough period of time will stay disconnected from them forever. Under this assumption, all the informed agents and type I uninformed agents cannot be influenced by type II uninformed agents directly or indirectly for all  $t \geq T$ . This assumption is reasonable in that it appears to represent a real situation. For example, the probability of a bird that has gone astray for a week coming back to the flock is very low.

To state the main results of this section, we define the sum of the total artificial potential energy and the total relative kinetic energy between all agents and the virtual leader as follows,

$$Q(q, p) = \frac{1}{2} \sum_{i=1}^N [U_i(q) + (p_i - p_\gamma)^T (p_i - p_\gamma)] \quad (12)$$

where

$$U_i(q) = \sum_{j=1, j \neq i}^N \psi_\alpha(\|q_i - q_j\|_\sigma) + h_i c_1 (q_i - q_\gamma)^T (q_i - q_\gamma). \quad (13)$$

We note that  $Q$  is a nonnegative valued function.

Our analysis results can then be summarized in the following theorem.

**Theorem 1:** Consider a system of  $N$  mobile agents, each with dynamics (1) and steered by the control protocol (10). Suppose that the initial energy  $Q_0 := Q(q(0), p(0))$  is finite. Then the following statements hold:

- i) The distance between each informed agent and the virtual leader is not greater than  $\sqrt{2Q_0}/c_1$  for all  $t \geq 0$ ;
- ii) The velocities of all informed agents and type I uninformed agents approach the desired velocity  $p_\gamma$  asymptotically.
- iii) If the initial energy  $Q_0$  of the group is less than  $(\bar{k} + 1)c^*$ ,  $c^* = \psi_\alpha(0)$ , for some  $\bar{k} \in \mathbf{Z}_+$ , then at most  $\bar{k}$  distinct pairs of agents could possibly collide ( $\bar{k} = 0$  guarantees a collision free motion).

The proof is carried out next in Sections III-B–III-D. Before we carry out this proof, we define the Laplacian of graph  $\mathcal{G}(t)$  with adjacent matrix  $A(t)$  as  $L(t) = \Delta(A(t)) - A(t)$ , where the degree matrix  $\Delta(A(t))$  is a diagonal matrix with the  $i$ th diagonal element being  $\sum_{j=1}^N a_{ij}(t)$ . Denote the eigenvalues of  $L(t)$  as  $\lambda_1(L(t)) \leq \lambda_2(L(t)) \leq \dots \leq \lambda_N(L(t))$ . Then,  $\lambda_1(L(t)) = 0$  and  $(1, 1, \dots, 1)^T \in \mathbf{R}^N$  is the corresponding eigenvector. Moreover, if  $\mathcal{G}(t)$  is a connected graph, then  $\lambda_2(L(t)) > 0$  [31]. The corresponding  $n$  dimensional graph Laplacian is defined as  $\hat{L}(t) = L(t) \otimes I_n$ , where  $I_n$  is the identity matrix of order  $n$  and  $\otimes$  stands for the Kronecker product. This multi-dimensional Laplacian satisfies the following sum of squares property [29],

$$z^T \hat{L}(t) z = \frac{1}{2} \sum_{(i,j) \in \mathcal{E}} a_{ij}(t) \|z_j - z_i\|^2 \quad (14)$$

where

$$z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_N \end{bmatrix} \in \mathbf{R}^{Nn}$$

and  $z_i \in \mathbf{R}^n$  for  $i = 1, 2, \dots, N$ .

### B. Cohesive Analysis

We first carry out the cohesiveness analysis, which contains the proof of Part i) of Theorem 1.

Denote the position difference vector and the velocity difference vector between agent  $i$  and the virtual leader as  $\tilde{q}_i = q_i - q_\gamma$  and  $\tilde{p}_i = p_i - p_\gamma$ , respectively. We have

$$\begin{cases} \dot{\tilde{q}}_i = \tilde{p}_i \\ \dot{\tilde{p}}_i = u_i, \end{cases} \quad i = 1, 2, \dots, N. \quad (15)$$

Let  $q_{ij} = q_i - q_j$  and  $\tilde{q}_{ij} = \tilde{q}_i - \tilde{q}_j$ . Clearly,  $\tilde{q}_{ij} = q_{ij}$ , and hence the collective potential function (5) can be rewritten as

$$\begin{aligned} \tilde{V}_i(\tilde{q}_{ij}) &= V_i(q) \\ &= \sum_{j \in \mathcal{V} \setminus \{i\}} \psi_\alpha(\|\tilde{q}_{ij}\|_\sigma) \\ &= \sum_{j \notin \mathcal{N}_i(t), j \neq i} \psi_\alpha(\|r\|_\sigma) + \sum_{j \in \mathcal{N}_i(t)} \psi_\alpha(\|\tilde{q}_{ij}\|_\sigma). \end{aligned} \quad (16)$$

Similarly, the control protocol for agent  $i$  is rewritten as

$$\begin{aligned} u_i &= - \sum_{j \in \mathcal{N}_i(t)} \nabla_{\tilde{q}_i} \psi_\alpha(\|\tilde{q}_{ij}\|_\sigma) \\ &\quad + \sum_{j \in \mathcal{N}_i(t)} a_{ij}(t) (\tilde{p}_j - \tilde{p}_i) - h_i [c_1 \tilde{q}_i + c_2 \tilde{p}_i] \end{aligned} \quad (17)$$

and the nonnegative valued energy function (12) can be rewritten as

$$Q(\tilde{q}, \tilde{p}) = \frac{1}{2} \sum_{i=1}^N (U_i(\tilde{q}) + \tilde{p}_i^T \tilde{p}_i) \quad (18)$$

where

$$\begin{aligned} U_i(\tilde{q}) &= \sum_{j=1, j \neq i}^N \psi_\alpha(\|\tilde{q}_{ij}\|_\sigma) + h_i c_1 \tilde{q}_i^T \tilde{q}_i \\ &= \tilde{V}_i(\tilde{q}_{ij}) + h_i c_1 \tilde{q}_i^T \tilde{q}_i \end{aligned} \quad (19)$$

and

$$\tilde{q} = \begin{bmatrix} \tilde{q}_1 \\ \tilde{q}_2 \\ \vdots \\ \tilde{q}_N \end{bmatrix}, \quad \tilde{p} = \begin{bmatrix} \tilde{p}_1 \\ \tilde{p}_2 \\ \vdots \\ \tilde{p}_N \end{bmatrix}.$$

Clearly,  $Q(\tilde{q}, \tilde{p})$  is a positive semi-definite function of  $(\tilde{q}, \tilde{p})$ .

Due to the symmetry of the pairwise potential function  $\psi_\alpha$  and the adjacent matrix  $A(t)$ , we have

$$\frac{\partial \psi_\alpha(\|\tilde{q}_{ij}\|_\sigma)}{\partial \tilde{q}_{ij}} = \frac{\partial \psi_\alpha(\|\tilde{q}_{ij}\|_\sigma)}{\partial \tilde{q}_i} = -\frac{\partial \psi_\alpha(\|\tilde{q}_{ij}\|_\sigma)}{\partial \tilde{q}_j} \quad (20)$$

and by (15),

$$\frac{1}{2} \sum_{i=1}^N \dot{U}_i = \sum_{i=1}^N \left( \tilde{p}_i^T \nabla_{\tilde{q}_i} \tilde{V}_i(\tilde{q}_{ij}) + h_i c_1 \tilde{p}_i^T \tilde{q}_i \right) \quad (21)$$

where  $\dot{U}_i = dU_i/dt$ . Therefore, the derivative of  $Q$  along the trajectories of the agents and the virtual leader is given by,

$$\begin{aligned} \dot{Q} &= \frac{1}{2} \sum_{i=1}^N \dot{U}_i + \sum_{i=1}^N \tilde{p}_i^T \dot{\tilde{p}}_i \\ &= -\tilde{p}^T [(L(t) + c_2 H(t)) \otimes I_n] \tilde{p} \end{aligned} \quad (22)$$

where

$$H(t) = \begin{bmatrix} h_1 & 0 & \cdots & 0 \\ 0 & h_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & h_N \end{bmatrix}.$$

Since  $L(t)$  and  $H(t)$  are both positive semi-definite matrices [32],  $L(t) + c_2 H(t)$  is also a positive semi-definite matrix. Therefore,  $\dot{Q} \leq 0$ , which implies that  $Q(t)$  is a non-increasing function of time  $t$  and thus  $Q(t) \leq Q_0$  for all  $t \geq 0$ . It then follows from (18) and (19) that  $c_1 \tilde{q}_i^T \tilde{q}_i \leq 2Q_0$  for any informed agent  $i$ . Hence, the distance between an informed agent  $i$  and the virtual leader is not greater than  $\sqrt{2Q_0/c_1}$  for all  $t \geq 0$ .

### C. Velocity Matching Analysis

We now continue with the asymptotic velocity matching analysis, which contains the proof of Part ii) of Theorem 1.

From Part i) of Theorem 1, the distance between any informed agent and the virtual leader is finite for all  $t \geq 0$ . It follows from (18) and (19) that  $\tilde{p}_i^T \tilde{p}_i \leq 2Q_0$ ,  $i = 1, 2, \dots, N$ , or equivalently,  $\|\tilde{p}_i\| \leq \sqrt{2Q_0}$ ,  $i = 1, 2, \dots, N$ . That is, the velocity difference between any agent and the virtual leader is not greater than  $\sqrt{2Q_0}$  for all  $t \geq 0$ . Since there exists a joint path between a type I uninformed agent and one informed agent in each finite time-interval  $[t_i, t_{i+1})$ ,  $i = 0, 1, 2, \dots$ , the distance between any type I uninformed agent and virtual leader is also finite for all  $t \geq 0$ .

Let  $M$ ,  $M_0 \leq M \leq N$ , be the total number of informed agents and Type I uninformed agents in the group. Without loss of generality, we label type I uninformed agents from  $M_0 + 1$  to  $M$ . By definition, all the informed agents and the type I uninformed agents cannot be influenced by a type II uninformed agents directly or indirectly for all  $t \geq T$ . For any  $t \geq T$ , we consider the positive semi-definite function

$$\check{Q}(\check{q}, \check{p}) = \frac{1}{2} \sum_{i=1}^M \left( \check{U}_i(\check{q}) + \check{p}_i^T \check{p}_i \right) \quad (23)$$

where

$$\check{U}_i(\check{q}) = \sum_{j=1, j \neq i}^M \psi_\alpha(\|\check{q}_{ij}\|_\sigma) + h_i c_1 \check{q}_i^T \check{q}_i \quad (24)$$

and

$$\check{q} = \begin{bmatrix} \check{q}_1 \\ \check{q}_2 \\ \vdots \\ \check{q}_M \end{bmatrix}, \quad \check{p} = \begin{bmatrix} \check{p}_1 \\ \check{p}_2 \\ \vdots \\ \check{p}_M \end{bmatrix}.$$

From (18) and (23), it is clear that  $\check{Q} \leq Q$ . The derivative of  $\check{Q}$  along the trajectories of the agents and the virtual leader is given by,

$$\begin{aligned} \dot{\check{Q}} &= \frac{1}{2} \sum_{i=1}^M \dot{\check{U}}_i + \sum_{i=1}^M \check{p}_i^T \dot{\check{p}}_i \\ &= -\check{p}^T \left[ (\check{L}(t) + c_2 \check{H}(t)) \otimes I_n \right] \check{p} \end{aligned} \quad (25)$$

where  $\check{L}(t) = [\check{l}_{ij}(t)]$  with

$$\check{l}_{ij}(t) = \begin{cases} -a_{ij}(t), & \text{if } i \neq j \\ \sum_{j=1}^M a_{ij}(t), & \text{if } i = j \end{cases}$$

and

$$\check{H}(t) = \begin{bmatrix} h_1 & 0 & \cdots & 0 \\ 0 & h_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & h_M \end{bmatrix}.$$

Following the similar analysis that led to  $\dot{Q} \leq 0$ , we can arrive at that  $\dot{\check{Q}} \leq 0$  and consequently,  $\check{Q} \leq \check{Q}_T \leq Q_T \leq Q_0$  for all  $t \geq T$ , where  $\check{Q}_T = \check{Q}(\check{q}(T), \check{p}(T))$ . Clearly, the set of all  $(\check{q}, \check{p})$  such that  $\check{Q} \leq Q_0$ , that is

$$\Omega = \left\{ [\check{q}^T, \check{p}^T]^T \in \mathbf{R}^{2Mn} : \check{Q}(\check{q}, \check{p}) \leq Q_0 \right\}$$

is an invariant set. We next show that it is also a compact set. By (23) and (24),  $\tilde{p}_i^T \tilde{p}_i \leq 2Q_0$ , for all  $i = 1, 2, \dots, M$ , and hence all  $\tilde{p}_i$ ,  $i = 1, 2, \dots, M$ , are bounded. Similarly, all  $\tilde{q}_i$ ,  $i = 1, 2, \dots, M_0$ , are bounded, that is, the positions of all informed agents are bounded. We next consider  $\tilde{q}_i$ ,  $i = M_0 + 1, M_0 + 2, \dots, M$ . Each of these  $\tilde{q}_i$ 's is the position of a type I uninformed agent, whose distance to an informed agent is, by the definition of the type I uninformed agents, bounded, due to the facts that the velocity of each agent is bounded and that the time intervals  $[t_i, t_{i+1})$  are uniformly bounded. Consequently,

the positions of all type I uninformed agents are also bounded. This shows that the set  $\Omega$  is compact.

It then follows from the LaSalle Invariance Principle [33] that all trajectories of the agents that start from  $\Omega$  will converge to the largest invariant set inside the region

$$S = \left\{ \left[ \check{q}^T, \check{p}^T \right]^T \in \mathbf{R}^{2Mn} : \dot{Q} = 0 \right\}.$$

Denote the set of all the informed agents and type I uninformed agents as  $\check{V} = \{1, 2, \dots, M\}$ . During the evolution, the graph  $\check{G}(t) = \{\check{V}, \check{E}(t)\}$ ,  $\check{E}(t) = \{(i, j) \in \check{V} \times \check{V} : i \sim j\}$ , may not always be connected. Suppose that  $\check{G}(t)$  consists of  $m(t)$ ,  $1 \leq m(t) \leq M$ , connected subgraphs for any  $t \geq T$  and there are  $M_k(t)$  agents in the  $k$ th,  $1 \leq k \leq m(t)$ , connected subgraph at time  $t$ . Clearly,  $M = M_1(t) + M_2(t) + \dots + M_{m(t)}(t)$ . A connected subgraph is called an informed connected subgraph if it includes at least one informed agent. We assume that the graph  $\check{G}(t)$  consists of  $m_1(t)$ ,  $1 \leq m_1(t) \leq M_0$ , informed connected subgraphs.

As in the earlier analysis, we have that both  $\check{L}(t) \otimes I_n$  and  $\check{H}(t) \otimes I_n$  are positive semi-definite. Thus, it follows from (25) that  $\dot{Q} = 0$  if and only if  $\check{p}^T (\check{L}(t) \otimes I_n) \check{p} = 0$  and  $\check{p}^T (\check{H}(t) \otimes I_n) \check{p} = 0$ .

Without loss of generality, we assume that the informed connected subgraphs are numbered from 1 to  $m_1(t)$ . For any time  $t \geq T$ , there exists an orthogonal permutation matrix  $P(t) \in \mathbf{R}^{M \times M}$  such that  $\check{L}(t)$  can be transformed into a block diagonal matrix of the form

$$\begin{aligned} \hat{L}(t) &= P(t) \check{L}(t) P^T(t) \\ &= \begin{bmatrix} \check{L}_1(t) & \dots & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & \check{L}_{m_1(t)}(t) & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \dots & \check{L}_{m(t)}(t) \end{bmatrix} \end{aligned}$$

where  $\check{L}_k(t) \in \mathbf{R}^{M_k(t) \times M_k(t)}$  is the Laplacian matrix associated with the  $k$ th connected subgraph of the graph  $\check{G}(t)$ . The indices of the state vector can be rearranged such that

$$\hat{p} = \begin{bmatrix} \check{p}_1^1 \\ \check{p}_2^1 \\ \vdots \\ \check{p}_{m(t)}^1 \end{bmatrix} = (P(t) \otimes I_n) \check{p}, \quad \check{p}^k = \begin{bmatrix} \check{p}_1^k \\ \check{p}_2^k \\ \vdots \\ \check{p}_{M_k(t)}^k \end{bmatrix}.$$

where  $\check{p}^k$  is the velocity difference vector of the  $M_k(t)$  agents within the  $k$ th connected subgraph. We have

$$\begin{aligned} &\hat{p}^T (\hat{L}(t) \otimes I_n) \hat{p} \\ &= \left( (P(t) \otimes I_n) \check{p} \right)^T \left( (P(t) \check{L}(t) P^T(t)) \otimes I_n \right) (P(t) \otimes I_n) \check{p} \\ &= \check{p}^T \left( (P^T(t) \otimes I_n) \left( (P(t) \check{L}(t) P^T(t)) \otimes I_n \right) (P(t) \otimes I_n) \check{p} \right) \\ &= \check{p}^T \left( (P^T(t) P(t) \check{L}(t) P^T(t)) \otimes I_n \right) (P(t) \otimes I_n) \check{p} \\ &= \check{p}^T \left( (\check{L}(t) P^T(t) P(t)) \otimes I_n \right) \check{p} \\ &= \check{p}^T (\check{L}(t) \otimes I_n) \check{p}. \end{aligned} \quad (26)$$

Therefore

$$\begin{aligned} -\check{p}^T (\check{L}(t) \otimes I_n) \check{p} &= - \begin{bmatrix} (\check{p}^1)^T & \dots & (\check{p}^{m_1(t)})^T & \dots & (\check{p}^{m(t)})^T \end{bmatrix} \\ &\times \begin{bmatrix} \check{L}_1(t) & \dots & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & \check{L}_{m_1(t)}(t) & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \dots & \check{L}_{m(t)}(t) \end{bmatrix} \otimes I_n \begin{bmatrix} \check{p}^1 \\ \check{p}^{m_1(t)} \\ \vdots \\ \check{p}^{m(t)} \end{bmatrix}. \end{aligned} \quad (27)$$

Clearly,  $-\check{p}^T (\check{L}(t) \otimes I_n) \check{p} = 0$  if and only if

$$(\check{p}^k)^T (\check{L}_k(t) \otimes I_n) \check{p}^k = 0, \quad k = 1, 2, \dots, m(t). \quad (28)$$

By the sum of squares property [29],

$$(\check{p}^k)^T (\check{L}_k(t) \otimes I_n) \check{p}^k = \frac{1}{2} \sum_{(i,j) \in \mathcal{E}(t)} a_{ij}(t) \left\| \check{p}_i^k - \check{p}_j^k \right\|^2.$$

Therefore, (28) is equivalent to

$$\check{p}_1^k = \check{p}_2^k = \dots = \check{p}_{M_k(t)}^k, \quad k = 1, 2, \dots, m(t). \quad (29)$$

Similarly, for any  $t \geq T$ , we have

$$\begin{aligned} -\check{p}^T (\check{H}(t) \otimes I_n) \check{p} &= - \begin{bmatrix} (\check{p}^1)^T & \dots & (\check{p}^{m_1(t)})^T & \dots & (\check{p}^{m(t)})^T \end{bmatrix} \\ &\times \begin{bmatrix} \check{H}_1(t) & \dots & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & \check{H}_{m_1(t)}(t) & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \dots & \check{H}_{m(t)}(t) \end{bmatrix} \otimes I_n \begin{bmatrix} \check{p}^1 \\ \check{p}^{m_1(t)} \\ \vdots \\ \check{p}^{m(t)} \end{bmatrix}. \end{aligned} \quad (30)$$

where  $\check{H}_k(t) \in \mathbf{R}^{M_k(t) \times M_k(t)}$  is the diagonal matrix associated with the  $k$ th connected subgraph with the  $i$ th diagonal element  $h_i^k$ ,  $i = 1, 2, \dots, M_k(t)$ , which is equal to one for an informed agent and zero for an uninformed agent.

Clearly,  $-\check{p}^T (\check{H}(t) \otimes I_n) \check{p} = 0$  if and only if

$$(\check{p}^k)^T (\check{H}_k(t) \otimes I_n) \check{p}^k = 0, \quad k = 1, 2, \dots, m(t) \quad (31)$$

which implies that the velocity difference between all informed agents and the virtual leader is zero. Since there is at least one informed agent in each informed connected subgraph, without loss of generality, we assume that the first agent in each informed connected subgraph is an informed agent. We have

$$\check{p}_1^k = 0, \quad k = 1, 2, \dots, m_1(t). \quad (32)$$

In view of (29), (31) and (32),

$$-\check{p}^T (\check{L}(t) \otimes I_n) \check{p} - \check{p}^T (\check{H}(t) \otimes I_n) \check{p} = 0.$$

if and only if

$$\check{p}_1^k = \check{p}_2^k = \dots = \check{p}_{M_k(t)}^k = 0, \quad k = 1, 2, \dots, m_1(t)$$

and

$$\check{p}_1^k = \check{p}_2^k = \dots = \check{p}_{M_k(t)}^k, \quad k = m_1(t)+1, m_1(t)+2, \dots, m(t).$$

We next show that, for  $t \geq T$ , all connected subgraphs are informed connected subgraphs, i.e.,  $m(t) = m_1(t)$  for  $t$  large

enough. This would imply that the velocities of all informed agents and type I uninformed agents approach the desired velocity  $p_\gamma$  asymptotically. To this end, let us assume that the interaction network switches at time  $t' \in [T, \infty)$ , i.e., at time  $t'$ , either an agent  $i$  is disconnected from another agent  $j$  or an agent  $i$  is connected with another agent  $j$ . In either case, we have  $\tilde{p}_i(t') \neq \tilde{p}_j(t')$ . It then follows from (25), (27) and (30) that

$$\dot{Q}(t') < 0$$

which contradicts with the fact that  $\dot{Q}(t) = 0$ . Thus, we conclude that the interaction network will not switch for all  $t \geq T$ . By their definition, all type I uninformed agents must have a path to one informed agent in the fixed interaction network for all  $t \geq T$ . Consequently,  $m(t) = m_1(t)$  for all  $t \geq T$ .

#### D. Collision Avoidance Analysis

Finally, we proceed to carry out the collision-avoidance analysis, which contains the proof of Part iii) of Theorem 1.

In view of (22),  $Q$  is a non-increasing function of time  $t$  and thus  $Q(t) \leq Q_0$  for all  $t \geq 0$ . The rest of the proof of this part is similar to that of part (iv) of Theorem 2 in [29]. Suppose that the initial energy  $Q_0$  is less than  $(\bar{k} + 1)c^*$  and more than  $\bar{k}$  distinct pairs of agents collide at a given time  $t^* > 0$ , i.e., at least  $\bar{k} + 1$  distinct pairs of agents collide at time  $t^* > 0$ . Then, at time  $t^* > 0$ , the collective potential of the system is at least  $(\bar{k} + 1)\psi_\alpha(0)$ . Recalling that  $Q_0 \geq Q(t)$  for all  $t \geq 0$ , we have

$$Q_0 \geq Q(t^*) \geq (\bar{k} + 1)\psi_\alpha(0)$$

which contradicts the assumption that  $Q_0 < (\bar{k} + 1)c^*$ . This shows that no more than  $\bar{k}$  distinct pairs of agents can possibly collide at any time  $t \geq 0$ . Consequently, if  $\bar{k} = 0$ , no collision will occur.

### IV. FLOCKING WITH A VIRTUAL LEADER OF VARYING VELOCITY

#### A. Algorithm Description and Main Result

In the situation where all agents are informed agents and the virtual leader move with a constant velocity  $p_\gamma$ , Theorem 1 guarantees that all agents will finally move with the desired velocity  $p_\gamma$ . In the situation where the velocity  $p_\gamma(t)$  of the virtual leader varies with time, even though it has been established in Theorem 2 of [29] that all agents in the group can finally achieve a common velocity, this common velocity in general does not match  $p_\gamma(t)$ . In this section, we focus on the situation where the virtual leader move with a varying velocity. In order for the common velocity of all the agents to match that of a virtual leader of varying velocity,  $p_\gamma(t)$ , we propose to incorporate the information of acceleration of the virtual leader into the protocol (7) as follows,

$$\begin{aligned} u_i = & - \sum_{j \in \mathcal{N}_i(t)} \nabla_{q_i} \psi_\alpha(\|q_j - q_i\|_\sigma) \\ & + \sum_{j \in \mathcal{N}_i(t)} a_{ij}(t)(p_j - p_i) + f_\gamma(q_\gamma, p_\gamma) - c_1(q_i - q_\gamma) \\ & - c_2(p_i - p_\gamma), \quad c_1, c_2 > 0. \end{aligned} \quad (33)$$

Denote the position and velocity of the center of mass (COM) of all agents in the group as

$$\bar{q} = \frac{\sum_{i=1}^N q_i}{N}, \quad \bar{p} = \frac{\sum_{i=1}^N p_i}{N}.$$

The sum of the total artificial potential energy and the total relative kinetic energy between all agents and the virtual leader in this case is defined as follows,

$$Q(q, p) = \frac{1}{2} \sum_{i=1}^N (U_i(q) + (p_i - p_\gamma)^T (p_i - p_\gamma)) \quad (34)$$

where

$$U_i(q) = \sum_{j=1, j \neq i}^N \psi_\alpha(\|q_{ij}\|_\sigma) + c_1(q_i - q_\gamma)^T (q_i - q_\gamma). \quad (35)$$

Note that  $Q$  is a nonnegative valued function. Our main results on the tracking of the virtual leader can then be stated in the following theorem, whose proof will be carried out in Section IV-B below.

**Theorem 2:** Consider a system of  $N$  mobile agents, each with dynamics (1) and steered by the control protocol (33). Suppose that the initial energy  $Q_0 = Q(q(0), p(0))$  is finite. Then the following statements hold:

- i) The distance between any agent and the virtual leader is not greater than  $\sqrt{2Q_0/c_1}$  for all  $t \geq 0$ ;
- ii) The velocity of each agent approaches the desired velocity  $p_\gamma$  asymptotically;
- iii) If both the initial velocity and position of the COM are equal to the initial velocity and position of the virtual leader, then the velocity (or position) of the COM will be equal to the velocity (or position) of the virtual leader for all  $t \geq 0$ ; otherwise the velocity (or position) of the COM will converge exponentially to the velocity (or position) of virtual leader with a time constant  $\tau$ , where

$$\tau = \begin{cases} \frac{c_2}{2}, & \text{if } c_2^2 \leq 4c_1 \\ \frac{c_2 - \sqrt{c_2^2 - 4c_1}}{2}, & \text{if } c_2^2 > 4c_1; \end{cases}$$

- iv) Almost every configuration the system approaches is a local minimum of the total artificial potential energy  $(1/2) \sum_{i=1}^N U_i(q)$ ;
- v) If the initial energy  $Q_0$  of the group of agents is less than  $(\bar{k} + 1)c^*$ ,  $c^* = \psi_\alpha(0)$ , for some  $\bar{k} \in \mathbb{Z}_+$ , then at most  $\bar{k}$  distinct pairs of agents could possibly collide ( $\bar{k} = 0$  guarantees a collision-free motion).

#### B. Theoretical Analysis

We first prove Part i) of Theorem 2. Denote the position difference vector and the velocity difference vector between agent  $i$  and the virtual leader as  $\tilde{q}_i = q_i - q_\gamma$  and  $\tilde{p}_i = p_i - p_\gamma$ , respectively. Then,

$$\begin{cases} \dot{\tilde{q}}_i = \tilde{p}_i \\ \dot{\tilde{p}}_i = u_i - f_\gamma(q_\gamma, p_\gamma), \quad i = 1, 2, \dots, N. \end{cases} \quad (36)$$

We will also denote

$$\tilde{q} = \begin{bmatrix} \tilde{q}_1 \\ \tilde{q}_2 \\ \vdots \\ \tilde{q}_N \end{bmatrix}, \quad \tilde{p} = \begin{bmatrix} \tilde{p}_1 \\ \tilde{p}_2 \\ \vdots \\ \tilde{p}_N \end{bmatrix}.$$

Thus, the control protocol for agent  $i$ , (33), can be rewritten as

$$u_i = - \sum_{j \in \mathcal{N}_i(t)} \nabla_{\tilde{q}_i} \psi_\alpha(\|\tilde{q}_{ij}\|_\sigma) + \sum_{j \in \mathcal{N}_i(t)} a_{ij}(t)(\tilde{p}_j - \tilde{p}_i) + f_\gamma(q_\gamma, p_\gamma) - c_1 \tilde{q}_i - c_2 \tilde{p}_i. \quad (37)$$

From the previous analysis, we recall that the collective potential function (5) can be rewritten as

$$\begin{aligned} \tilde{V}_i(\tilde{q}_{ij}) &= V_i(q) \\ &= \sum_{j \in \mathcal{V} \setminus \{i\}} \psi_\alpha(\|\tilde{q}_{ij}\|_\sigma) \\ &= \sum_{j \notin \mathcal{N}_i(t), j \neq i} \psi_\alpha(\|r\|_\sigma) + \sum_{j \in \mathcal{N}_i(t)} \psi_\alpha(\|\tilde{q}_{ij}\|_\sigma). \end{aligned} \quad (38)$$

Also, the nonnegative valued function (34) can be rewritten as

$$Q(\tilde{q}, \tilde{p}) = Q(q, p) = \frac{1}{2} \sum_{i=1}^N (U_i(\tilde{q}) + \tilde{p}_i^T \tilde{p}_i) \quad (39)$$

where

$$U_i(\tilde{q}) = \sum_{j=1, j \neq i}^N \psi_\alpha(\|\tilde{q}_{ij}\|_\sigma) + c_1 \tilde{q}_i^T \tilde{q}_i = \tilde{V}_i(\tilde{q}_{ij}) + c_1 \tilde{q}_i^T \tilde{q}_i. \quad (40)$$

Clearly,  $Q(\tilde{q}, \tilde{p})$  is a positive definite function of  $(\tilde{q}, \tilde{p})$ .

Due to the symmetry of the pairwise potential function  $\psi_\alpha$  and the adjacent matrix  $A(t)$ ,

$$\frac{1}{2} \sum_{i=1}^N \dot{U}_i = \sum_{i=1}^N \left( \tilde{p}_i^T \nabla_{\tilde{q}_i} \tilde{V}_i(\tilde{q}_{ij}) + c_1 \tilde{p}_i^T \dot{\tilde{q}}_i \right). \quad (41)$$

Therefore, the derivative of  $Q$  along the trajectories of the agents and the virtual leader is given by,

$$\dot{Q} = \frac{1}{2} \sum_{i=1}^N \dot{U}_i + \sum_{i=1}^N \tilde{p}_i^T \dot{\tilde{p}}_i = -\tilde{p}^T [(L(t) + c_2 I_N) \otimes I_n] \tilde{p}. \quad (42)$$

Recalling that  $L(t)$  is a positive semi-definite matrix [32], we have  $\dot{Q} \leq 0$ , which implies that  $Q(t) := Q(\tilde{q}, \tilde{p})$  is a non-increasing function of time  $t$  and thus  $Q(t) \leq Q_0$  for all  $t \geq 0$ . From (39) and (40), we have  $c_1 \tilde{q}_i^T \tilde{q}_i \leq 2Q_0$  for any agent  $i$ . Therefore, the distance between any agent and the virtual leader is not greater than  $\sqrt{2Q_0/c_1}$  for all  $t \geq 0$ .

We now proceed to prove Parts ii) and iii) of Theorem 2. Since  $Q$  is positive definite and  $\dot{Q} \leq 0$ , the set

$$\Omega = \{[\tilde{q}^T, \tilde{p}^T]^T \in \mathbf{R}^{2Nn} : Q \leq Q_0\}$$

is a compact invariant set. It then follows from the LaSalle Invariance Principle that all trajectories of the agents that start from  $\Omega$  converge to the largest invariant set inside the region

$$S = \{[\tilde{q}^T, \tilde{p}^T]^T \in \mathbf{R}^{2Nn} : \dot{Q} = 0\}.$$

From (42), we have

$$\dot{Q} = -\tilde{p}^T [(L(t) + c_2 I_N) \otimes I_n] \tilde{p} = -\tilde{p}^T (L(t) \otimes I_n) \tilde{p} - c_2 \tilde{p}^T \tilde{p}. \quad (43)$$

Hence  $\dot{Q} = 0$  is equivalent to

$$\tilde{p}_1 \equiv \tilde{p}_2 \equiv \cdots \equiv \tilde{p}_N = 0$$

which occurs only when

$$p_1 \equiv p_2 \equiv \cdots \equiv p_N \equiv p_\gamma. \quad (44)$$

This proves Part ii).

It follows from the control protocol (33) and the symmetry of  $\psi_\alpha$  and  $A(t)$  that

$$\begin{aligned} \bar{u} &= \frac{\sum_{i=1}^N u_i}{N} \\ &= f_\gamma(q_\gamma, p_\gamma) \\ &\quad - \frac{1}{N} \sum_{i=1}^N \left( \sum_{j \in \mathcal{N}_i(t)} \nabla_{q_i} \psi_\alpha(\|q_j - q_i\|_\sigma) \right. \\ &\quad \left. + \sum_{j \in \mathcal{N}_i(t)} a_{ij}(t)(p_i - p_j) + c_1(q_i - q_\gamma) \right. \\ &\quad \left. + c_2(p_i - p_\gamma) \right) \\ &= f_\gamma(q_\gamma, p_\gamma) - c_1(\bar{q} - q_\gamma) - c_2(\bar{p} - p_\gamma) \end{aligned}$$

which in turn implies that

$$\begin{cases} \dot{\bar{q}} = \bar{p} \\ \dot{\bar{p}} = \bar{p}_\gamma - c_1(\bar{q} - q_\gamma) - c_2(\bar{p} - p_\gamma). \end{cases} \quad (45)$$

The solution of (45) can be obtained in three separate cases. For  $c_2^2 > 4c_1$ ,

$$\begin{cases} \bar{q}(t) = q_\gamma(t) - \frac{1}{\sqrt{c_2^2 - 4c_1}} \\ \quad \times \left( \bar{p}(0) - p_\gamma(0) + \frac{c_2 - \sqrt{c_2^2 - 4c_1}}{2} (\bar{q}(0) - q_\gamma(0)) \right) \\ \quad \times e^{\frac{-c_2 - \sqrt{c_2^2 - 4c_1}}{2} t} + \frac{1}{\sqrt{c_2^2 - 4c_1}} \\ \quad \times \left( \bar{p}(0) - p_\gamma(0) + \frac{c_2 + \sqrt{c_2^2 - 4c_1}}{2} (\bar{q}(0) - q_\gamma(0)) \right) \\ \quad \times e^{\frac{-c_2 + \sqrt{c_2^2 - 4c_1}}{2} t} \\ \bar{p}(t) = p_\gamma(t) + \frac{c_2 + \sqrt{c_2^2 - 4c_1}}{2\sqrt{c_2^2 - 4c_1}} \\ \quad \times \left( \bar{p}(0) - p_\gamma(0) + \frac{c_2 - \sqrt{c_2^2 - 4c_1}}{2} (\bar{q}(0) - q_\gamma(0)) \right) \\ \quad \times e^{\frac{-c_2 - \sqrt{c_2^2 - 4c_1}}{2} t} + \frac{-c_2 + \sqrt{c_2^2 - 4c_1}}{2\sqrt{c_2^2 - 4c_1}} \\ \quad \times \left( \bar{p}(0) - p_\gamma(0) + \frac{c_2 + \sqrt{c_2^2 - 4c_1}}{2} (\bar{q}(0) - q_\gamma(0)) \right) \\ \quad \times e^{\frac{-c_2 + \sqrt{c_2^2 - 4c_1}}{2} t} \end{cases} \quad (46)$$



for  $c_2^2 = 4c_1$ ,

$$\begin{cases} \bar{q}(t) = q_\gamma(t) + (\bar{q}(0) - q_\gamma(0) \\ + (\bar{p}(0) - p_\gamma(0) + \frac{2c_1}{c_2} (\bar{q}(0) - q_\gamma(0))) t) e^{-\frac{c_2}{2}t} \\ \bar{p}(t) = p_\gamma(t) + (\bar{p}(0) - p_\gamma(0) \\ + (-\frac{c_2}{2} (\bar{p}(0) - p_\gamma(0)) - c_1 (\bar{q}(0) - q_\gamma(0))) t) e^{-\frac{c_2}{2}t} \end{cases} \quad (47)$$

and for  $c_2^2 < 4c_1$ , see (48), shown at the bottom of the page, where  $\bar{q}(0)$  and  $\bar{p}(0)$  are initial position and velocity of the COM of the group, and  $q_\gamma(0)$  and  $p_\gamma(0)$  are the initial position and velocity of the virtual leader. The solution (46)–(48) indicates that, if  $\bar{q}(0) = q_\gamma(0)$  and  $\bar{p}(0) = p_\gamma(0)$ , then the position and velocity of the COM will be equal to those of the virtual leader for all  $t \geq 0$ , otherwise they will converge exponentially to those of the virtual leader with a time constant of  $c_2/2$  when  $c_2^2 \leq 4c_1$ , or  $(c_2 - \sqrt{4c_1 - c_2^2})/2$  when  $c_2^2 > 4c_1$ . This completes the proof of Part iii).

We now prove Part iv) of Theorem 2. From (44), we see that, in steady state,

$$\dot{p}_1 = \dot{p}_2 = \dots = \dot{p}_N = \dot{p}_\gamma = f_\gamma(q_\gamma, p_\gamma)$$

which implies that  $u_i = f_\gamma(q_\gamma, p_\gamma)$ . It thus follows from (37) and (40) that

$$\nabla_q \left( \frac{1}{2} \sum_{i=1}^N U_i(q) \right) = 0. \quad (49)$$

Thus, the configuration converges asymptotically to a fixed configuration that is an extremum of  $(1/2) \sum_{i=1}^N U_i(q)$ . If the solution of (49) starts at the an equilibrium, which is local maxima or saddle points, the solution will invariant at all time. Then, not all solutions of (49) converge to local minima. However, any point but a local minimum is an unstable equilibrium [29]. Thus, almost every configuration the system approaches is a local minimum of the total artificial potential energy  $(1/2) \sum_{i=1}^N U_i(q)$ .

Finally, we prove Part v) of Theorem 2. From the earlier analysis, we recall that  $Q \leq Q_0$ . The results then follow the same arguments as used in the proof of Part iii) of Theorem 1. This completes the proof of Theorem 2.

By comparing the control protocol (7) with the control protocol (33), we observe that the information of both the acceleration and velocity of the virtual leader is sufficient to guarantee every agent to track a virtual leader of varying velocity.

The information of velocity alone cannot guarantee the group to achieve desired velocity.

We recall that Shi *et al.* [24] proposed another algorithm to track a virtual leader, in which every agent in the group only knows the information of velocity of the virtual leader. With the assumption of the neighboring graph is always connected, all the agents can achieve stable flocking motion, but can only track a desired constant velocity. This assumption is also difficult to implement in practice. Without the information of position of the virtual leader, convergence cannot be guaranteed in arbitrary switching information exchange topology.

## V. SIMULATION STUDY

### A. Flocking With a Fraction of Agents Informed

Simulation was performed on 100 agents moving in a 2-dimensional space under the influence of the control protocol (10). Initial positions and initial velocities of the 100 agents were chosen randomly from the boxes  $[0,55] \times [0,55]$  and  $[0,0.01] \times [0,0.01]$ , respectively, and the initial position and velocity of the virtual leader were set at  $q_\gamma(0) = [10, 10]^T$  and  $p_\gamma(0) = [1, 1]^T$ . The influencing/sensing radius was chosen as  $r = 4$ , the desired distance  $d = 3.3$ ,  $\varepsilon = 0.1$  for the  $\sigma$ -norm,  $h = 0.9$  for the bump function  $\rho_h(\cdot)$ , and  $c_1 = c_2 = 0.5$ .

Some simulation results are shown in Fig. 2. In the figure, the solid lines represent the neighboring relations, the solid lines with arrows represent the velocities, and the hexagram represents the desired position, which is where the virtual leader is. There are ten informed agents which are chosen randomly from the group and marked with circles. Fig. 2(a) shows the group initial state which is highly disconnected. As time evolves, the size of the largest cluster increases and more and more agents move with the same velocity. Eventually, 86 of the 100 agents move with the desired velocity (see plot (f)).

To further substantiate the results shown in Fig. 2, we ran extensive simulations on groups of different sizes. In order for the comparison to be reasonable, initial positions of the agents were chosen randomly from a  $[0, L] \times [0, L]$  box so that the density is fixed at  $\rho = N/L^2 = 0.05$ . Initial velocities of the  $N$  agents were chosen randomly from  $[0, 4] \times [0, 4]$ , and the initial position and velocity of the virtual leader were set at  $q_\gamma(0) = [10, 10]^T$  and  $p_\gamma(0) = [3, 3]^T$ . The influencing/sensing radius was set at  $r = 3$ , the desired distance  $d = 2.5$ ,  $\varepsilon = 0.1$  for the

$$\begin{cases} \bar{q}(t) = q_\gamma(t) + \left( (\bar{q}(0) - q_\gamma(0)) \cos \frac{\sqrt{4c_1 - c_2^2}}{2} t \right. \\ \quad \left. + \frac{2}{\sqrt{4c_1 - c_2^2}} \left( \frac{c_2}{2} (\bar{q}(0) - q_\gamma(0)) + \bar{p}(0) - p_\gamma(0) \right) \sin \frac{\sqrt{4c_1 - c_2^2}}{2} t \right) e^{-\frac{c_2}{2}t} \\ \bar{p}(t) = p_\gamma(t) + \left( (\bar{p}(0) - p_\gamma(0)) \cos \frac{\sqrt{4c_1 - c_2^2}}{2} t \right. \\ \quad \left. + \left( -\frac{\sqrt{4c_1 - c_2^2}}{2} (\bar{q}(0) - q_\gamma(0)) - \frac{2}{\sqrt{4c_1 - c_2^2}} \left( \frac{c_2}{2} (\bar{q}(0) - q_\gamma(0)) + \bar{p}(0) - p_\gamma(0) \right) \right) \sin \frac{\sqrt{4c_1 - c_2^2}}{2} t \right) e^{-\frac{c_2}{2}t} \end{cases} \quad (48)$$

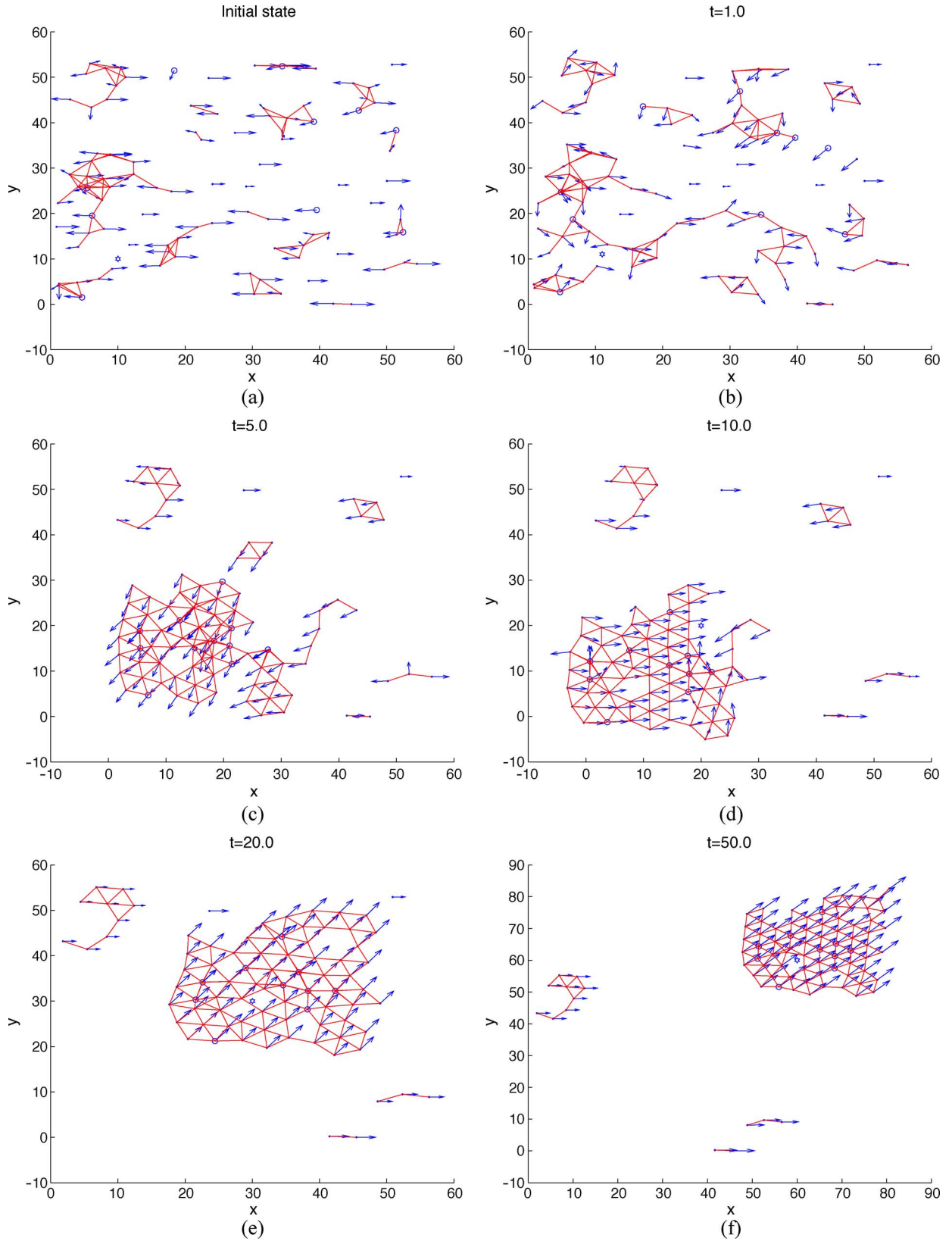


Fig. 2. 2-D flocking for 100 agents applying algorithm (10) with 10 informed agents.

$\sigma$ -norm,  $h = 0.2$  for the bump function  $\rho_h(\cdot)$ , and  $c_1 = c_2 = 0.5$ .

We randomly select a fraction  $\delta$  of the  $N$  agents as the informed agents. Fig. 3 shows the relationship between the frac-

tion  $\eta$  of the agents that eventually move with the desired velocity and the fraction  $\delta$  of the informed agents with  $N = 100, 300, 500$  and  $1000$ , respectively. All estimates are the results of averaging over 50 realizations. Obviously, for any given group

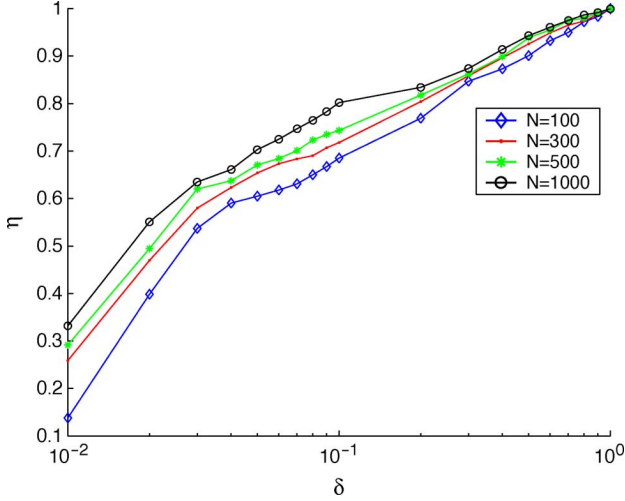


Fig. 3. Fraction of agents with the desired velocity as a function of the fraction of informed agents. All estimates are the results of averaging over 50 realizations.

size  $N$ , the fraction  $\eta$  of agents that move with the desired velocity is an increasing function of the fraction  $\delta$  of informed agents. Furthermore, the larger the group, the smaller the fraction  $\delta$  of informed agents is needed to guide the group with a given fraction  $\eta$ . For example, in order for 80% of the agents to move with the same desired velocity, about 27% of the agents should be informed agents when the group size is  $N = 100$ , but only about 10% of the agents need to be the informed agents when the group has  $N = 1000$  agents. Thus, for sufficiently large groups, only a very small fraction of informed agents will guide most agents in the group.

#### B. Flocking With a Virtual Leader of Varying Velocity

Simulations for protocol (7) (algorithm 2 in [29]) and protocol (33) were performed with ten agents moving in a 3-dimensional space, and a virtual leader whose acceleration function  $f_\gamma(q_\gamma, p_\gamma)$  satisfies the Lorenz equation (see Fig. 4),

$$\begin{cases} \dot{p}_{\gamma x} = 10(p_{\gamma y} - p_{\gamma x}) \\ \dot{p}_{\gamma y} = 28p_{\gamma x} - p_{\gamma x}p_{\gamma z} - p_{\gamma y} \\ \dot{p}_{\gamma z} = p_{\gamma x}p_{\gamma y} - \frac{8}{3}p_{\gamma z} \end{cases}$$

Simulation results are shown in Figs. 5 and 6. In the simulation, initial positions and velocities of the ten agents were chosen randomly from the same cube  $[0, 15] \times [0, 15] \times [0, 15]$ , and the initial position and velocity of the virtual leader were set at  $q_\gamma(0) = [10, 10, 10]^T$  and  $p_\gamma(0) = [3, 3, 3]^T$ . The influencing/sensing radius was chosen at  $r = 4.8$ , the desired distance  $d = 4$ ,  $\varepsilon = 0.1$  for the  $\sigma$ -norm,  $h = 0.2$  for the bump function  $\rho_h(\cdot)$ , and  $c_1 = c_2 = 0.5$ . The solid lines in the figures represent the neighboring relations, the dotted lines with arrows represent the agent velocities, and the hexagram represents the desired position. In both Figs. 5 and 6, plot (a) shows the initial states of the agents; plot (b) presents the configuration and velocities of the group of agents at  $t = 30$  seconds; plot (c) depicts the motion trajectories of all agents from  $t = 0$  to 30 seconds; plot (d) shows the convergence of velocity, from which we can

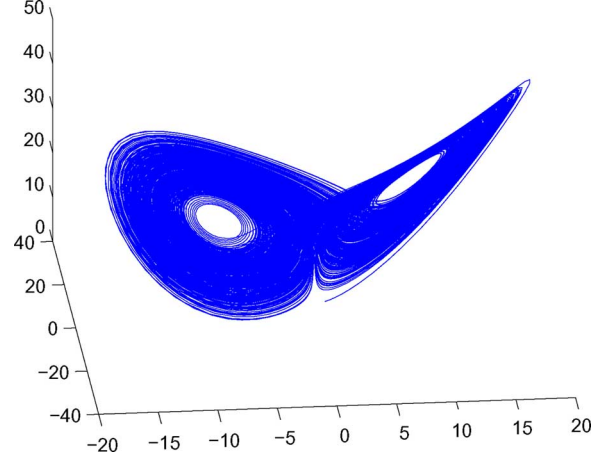


Fig. 4. The attractor of Lorenz system.

see that all the agents eventually achieve the same velocity; plots (e) and (f) depict the position and velocity difference between the COM and the desired position  $q_\gamma(t)$  and velocity  $p_\gamma(t)$ , respectively.

It is obvious from Figs. 5 and 6 that both control protocols (7) and (33) are capable of achieving stable flocking motion. Under the protocol (7), the position and velocity of the COM fluctuate around those of the virtual leader (see Fig. 5, plots (e) and (f)). However, under the protocol (33), the position and velocity of the COM converge exponentially to those of the virtual leader [see Fig. 6, plots (e) and (f)].

#### C. Flocking With a Fraction of Agents Informed and a Virtual Leader of Varying Velocity

This section presents simulation results for the situation when only a fraction of agents are informed agents and in the same time, the velocity of the virtual leader is varying with time. The control protocol for agent  $i$  is naturally chosen as,

$$u_i = - \sum_{j \in \mathcal{N}_i(t)} \nabla_{q_i} \psi_\alpha(\|q_j - q_i\|_\sigma) - \sum_{j \in \mathcal{N}_i(t)} a_{ij}(p_i - p_j) + h_i [f_\gamma(q_\gamma, p_\gamma) + c_1(q_\gamma - q_i) + c_2(p_\gamma - p_i)]$$

where  $h_i = 1$  if agent  $i$  is an informed agent, and  $h_i = 0$  otherwise. Simulation results are shown in Fig. 7. In the simulation, the acceleration vector of the virtual leader was selected as  $f_\gamma = [-0.3, 0.3]^T$  and all other parameters were chosen as those in the simulation shown in Fig. 2. Plot (a) shows the group initial state, which is highly disconnected. As time evolves, the size of the largest cluster increases and more and more agents move with the same velocity. Eventually, 89 of the 100 agents move with the desired velocity [see plot (f)].

## VI. CONCLUSION

In this paper, a flocking algorithm proposed by Olfati-Saber for a group of agents to track a virtual leader is extended from two directions. First, the flocking algorithm is generalized to the case that only a fraction of agents are informed agents. We

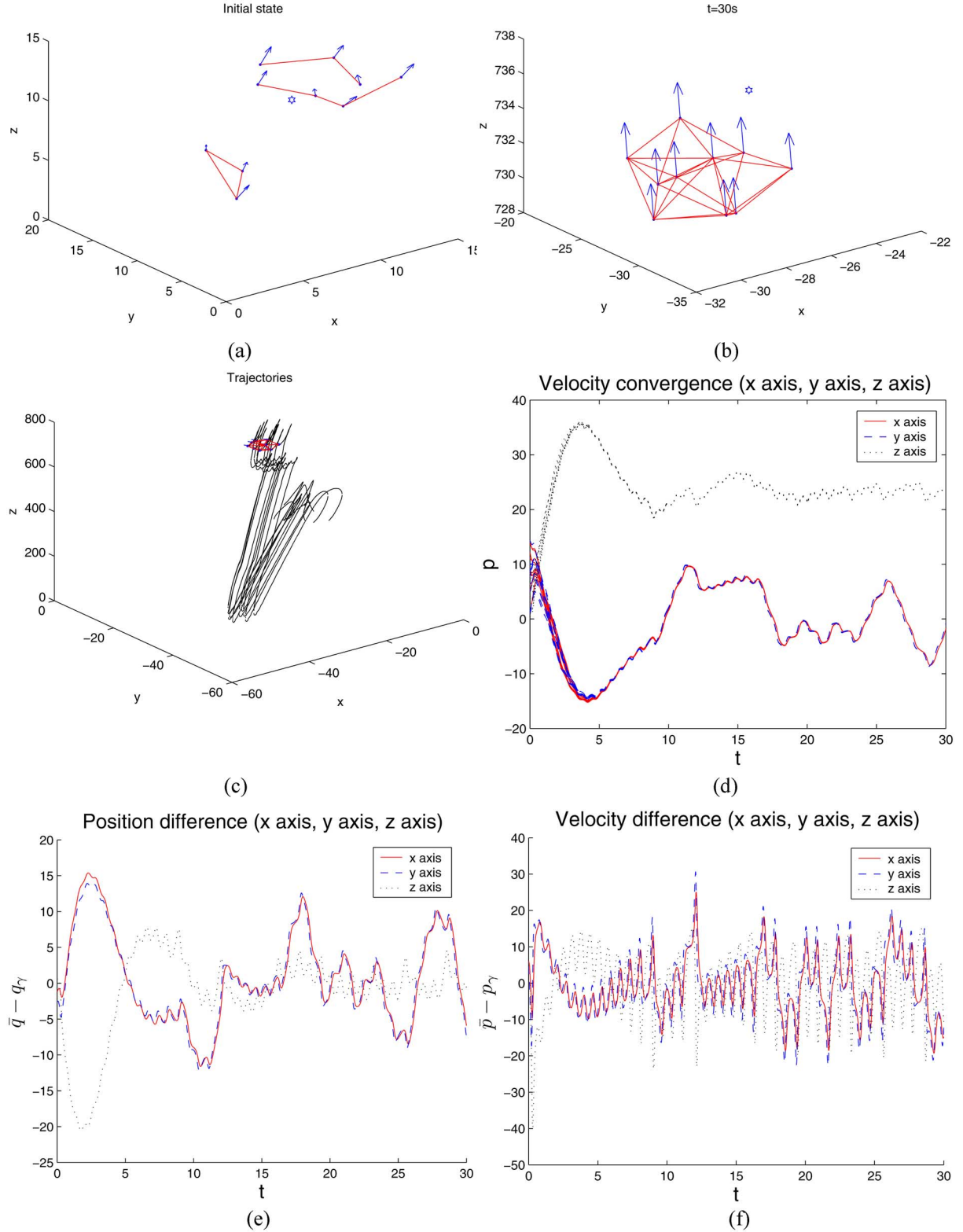


Fig. 5. 3-D flocking for 10 agents applying Algorithm 2 in [29]. (a) Initial state; (b) configuration and velocities at  $t = 30$  s; (c) trajectories of all agents; (d) velocity convergence; (e) position difference between the COM and  $q_\gamma(t)$ ; (f) velocity difference between the COM and  $p_\gamma(t)$ .

established that all the informed agents will move with the desired velocity. Furthermore, if an uninformed agent can be influenced by the informed agents from time to time during the evolution, this uninformed agent will also move with the desired velocity. We demonstrate numerically that a very small

proportion of informed agents can make most of agents move with the desired velocity, and the larger the group the higher is the proportion of agents that will eventually move with the desired velocity. Second, we modified the algorithm for a group of agents in order to track a virtual leader of varying velocity.

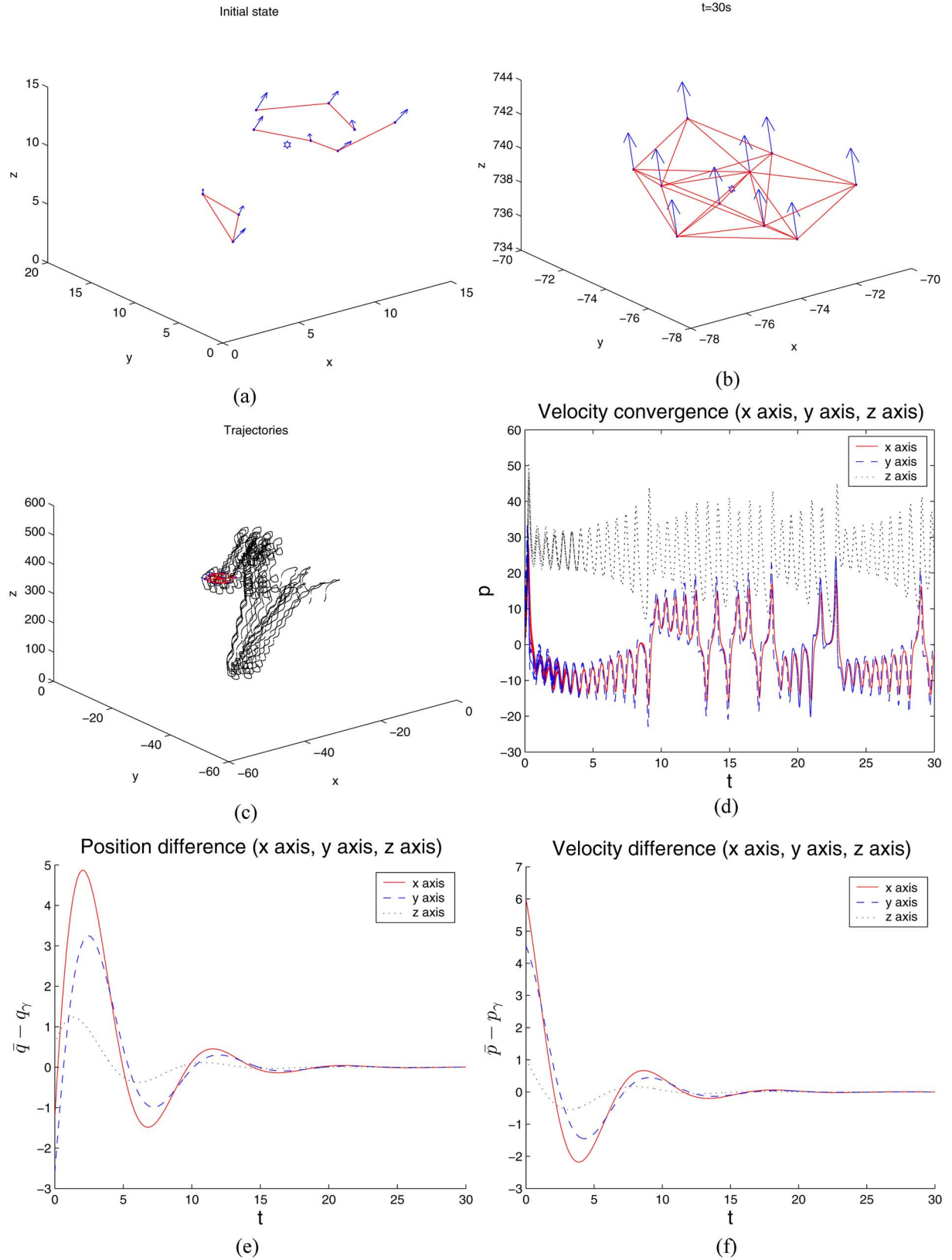


Fig. 6. 3-D flocking for 10 agents applying Algorithm (33). (a) Initial state; (b) configuration and velocities at  $t = 30$  s; (c) trajectories of all agents; (d) velocity convergence; (e) position difference between the COM and  $q_\gamma(t)$ ; (f) velocity difference between the COM and  $p_\gamma(t)$ .



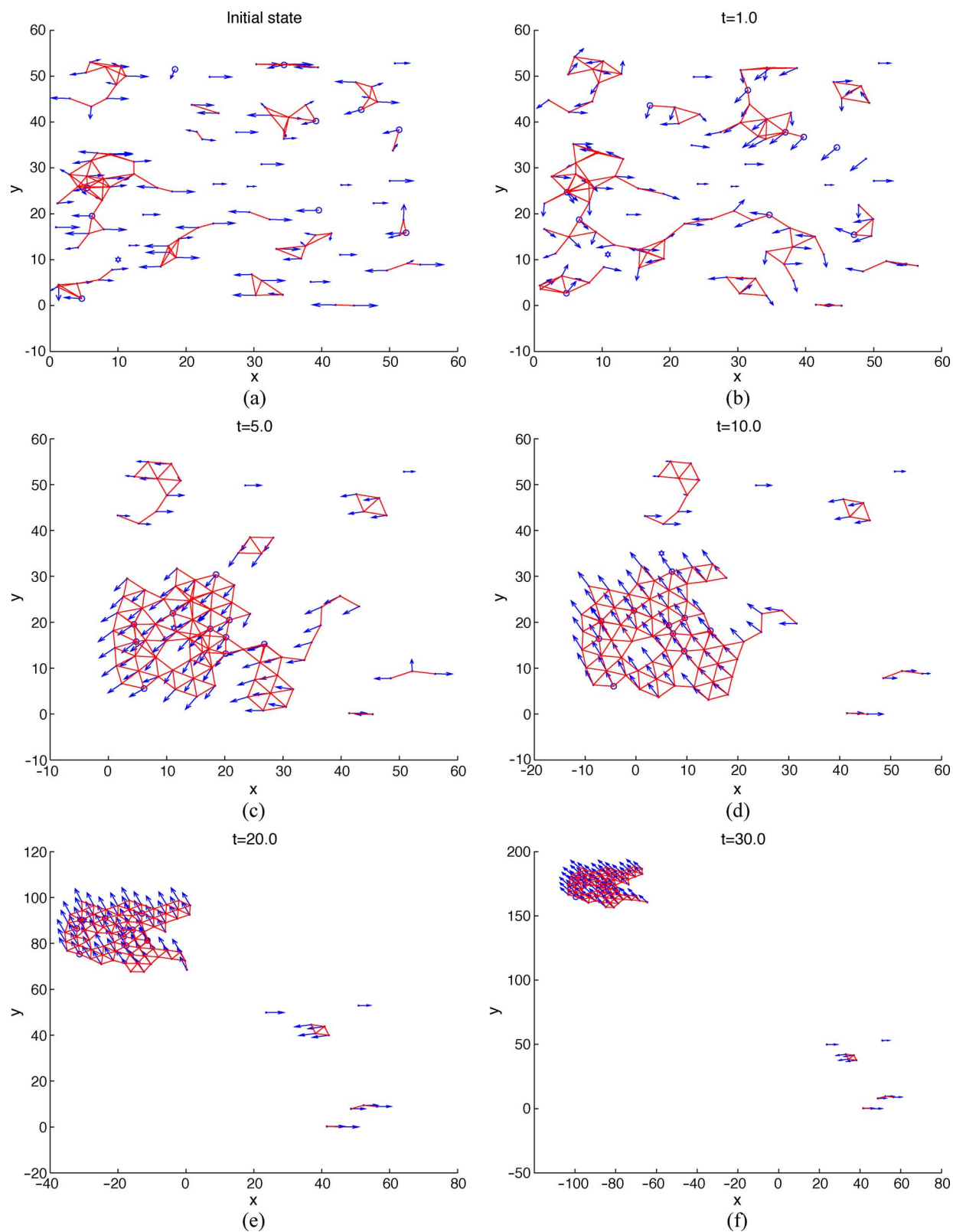


Fig. 7. 2-D flocking for 100 agents with 10 informed agents and a virtual leader of varying velocity.

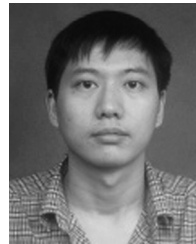
Moreover, we established that the position and velocity of the COM will converge exponentially to those of virtual leader. The convergent rate was also given.

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**Housheng Su** was born in Wuhan, China, in 1979. He received the B.S. degree in automatic control and the M.S. degree in control theory and control engineering from Wuhan University of Technology, Wuhan, China, in 2002 and 2005, respectively, and is currently pursuing the Ph.D. degree in control theory and control engineering at Shanghai Jiao Tong University, Shanghai, China.

His research interests lie in the areas of multi-agent coordination control theory and its applications to autonomous robotics and mobile sensor networks.



**Xiaofan Wang** (M'01–SM'05) received the B.Sc. degree in mathematics from Suzhou University, Jiangsu, China, in 1986, the M.Sc. degree in computational mathematics from Nanjing Normal University, Nanjing, China, in 1991, and the Ph.D. degree in control theory and applications from the Southeast University, Nanjing, in 1996.

Currently, he is a Professor with the Department of Automation, Shanghai Jiao Tong University, Shanghai, China. His research interests include analysis and control of complex dynamical networks.

Dr. Wang received the National Science Fund for Distinguished Young Scholars of China Award in 2002 and the Best Transactions Paper Award from the IEEE Circuits and Systems Society in 2005.



**Zongli Lin** (S'90–M'94–SM'98–F'07) received the B.S. degree in mathematics and computer science from Xiamen University, Xiamen, China, in 1983, the M.Eng. degree in automatic control from the Chinese Academy of Space Technology, Beijing, in 1989, and the Ph.D. degree in electrical and computer engineering from Washington State University, Pullman, in 1994.

He is a Professor of electrical and computer engineering at the University of Virginia, Charlottesville. His current research interests include nonlinear control, robust control, and control applications.

Dr. Lin was an Associate Editor of the IEEE TRANSACTIONS ON AUTOMATIC CONTROL (2001–2003) and the Corresponding Editor for Conference Activities of the IEEE CONTROL SYSTEMS MAGAZINE (2005–2007). He currently serves on the editorial boards of several journals, including *Automatica*, *Systems & Control Letters*, *IEEE/ASME TRANSACTIONS ON MECHATRONICS* and *IEEE CONTROL SYSTEMS MAGAZINE*. He is an elected member of the Board of Governors of the IEEE Control Systems Society.