This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TAC.2018.2868997, IEEE Transactions on Automatic Control

Output Consensus of Heterogeneous Linear Multi-Agent Systems with Adaptive Event-Triggered Control

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***Abstract*—This paper investigates the output consensus problem for heterogeneous linear multi-agent systems via event-triggered control. By introducing a dynamic compensator for each agent, a fully distributed event-triggered control strategy with an adaptive event-triggering mechanism is proposed. It is shown that under the proposed control strategy, all agents asymptotically achieve output consensus with intermittent communication in a fully distributed manner. Moreover, with the proposed event-triggering mechanism, Zeno behavior is strictly excluded for each agent. Compared with existing mechanisms, the proposed event-triggering mechanism is independent of any global information and avoids the continuous monitoring issue. Finally, a numerical example is provided to illustrate the effectiveness of the proposed event-triggered control strategy.**

***Index Terms*—Multi-agent systems, output consensus, event-triggered control, distributed control, adaptive control.**

I. Introduction

Consensus of multi-agent systems (MASs) has been widely investigated in the past two decades, motivated by many practical applications such as mobile robot networks, satellite networks, and sensor networks. In the consensus problem, the objective is to design a control strategy such that a group of agents reach agreement on a quantity of particular interest. Many existing early works focus on the state consensus problem for a group of identical agents (see, e.g., [1]–[4]). However, the agents' dynamics may be different in practice. Such agents can constitute the so-called heterogeneous MASs. In this case, the output consensus problem is often studied. In recent years, some effort has been devoted to output consensus of heterogeneous MASs with general linear dynamics, referred to as heterogeneous linear MASs (see, e.g., [5]–[8]).

In the aforementioned works, the control strategies were developed under the assumption that continuous communication exists between neighboring agents. In digital implementation of these control strategies, conventional periodic sampling techniques are usually used. However, in many practical MASs, each individual agent may be equipped with a simple hardware with constrained resources such as computation and power. To reduce resource consumption, an alternative sampling control technique, known as event-triggered control, has been receiving considerable attention. In this control approach, a control task is executed only when a certain condition is satisfied. Some earlier works on event-triggered control can be found in [9]–[12] for single systems and in [13]–[17] for multi-agent systems with single- or double-integrator agent dynamics. It should be pointed out that the event-triggering mechanism proposed in [15] cannot ensure strict avoidance of the so-called Zeno behavior, which refers to the phenomenon that there exists an infinite number of events in a finite time interval. It should also be noted that the problem is addressed by the improved event-triggering mechanism in [16].

More recently, distributed event-triggered control strategies have been developed for consensus of linear MASs. For example, by utilizing the combined measurement approach, a distributed event-triggered control strategy was proposed in [18]. The event-triggering mechanism in [18] was further improved in [19] by incorporating a fixed timer such that a positive minimum inter-event time can be guaranteed. One advantage of the control strategies in [18], [19] is that each agent updates its controller only at its own triggering times. However, there exists a continuous monitoring issue, that is, to check the triggering condition, each agent needs to continuously monitor its neighbors' states. Another research line for event design is considered in [20]–[23], in which the continuous monitoring issue can be avoided. It is observed that their event-triggering mechanisms were designed in the state-dependent form [20], [21] or the time-dependent form [22], [23], where a positive constant was incorporated in the threshold to provide a positive lower bound on the inter-event times so that Zeno behavior is avoided. Nevertheless, as a tradeoff, the consensus errors cannot converge to zero, but to a neighborhood of zero.

It is noticed that the event-triggered control strategies in [18]–[23] have to rely on some global information, for instance, the eigenvalues of the Laplacian matrix and the number of the agents. This means that these control strategies are not fully distributed. Despite the importance of designing fully distributed event-triggered control strategies for consensus of linear MASs, there are few results on this topic, except for [24], in which homogeneous linear MASs were considered. This fact motivates the current study.

In this paper, a fully distributed event-triggered control strategy is proposed to solve the output consensus problem for heterogeneous linear MASs. The main advantages of the proposed control strategy are listed as follows. First, compared with the existing event-triggered control strategies on output

This work was supported by the Research Grants Council of the Hong Kong Special Administrative Region of China under Project CityU/11274916.

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This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TAC.2018.2868997, IEEE Transactions on Automatic Control

consensus in [19], [23], the proposed control strategy is fully distributed, while guaranteeing that output consensus is asymptotically achieved. Second, under the proposed event-triggering mechanism, each agent does not need to continuously monitor its neighboring information, as opposed to [18], [19], [24]. Third, the proposed event-triggering mechanism is independent of any global information, in comparison with the mechanisms in [18]–[23].

The rest of this paper is organized as follows. In Section II, the problem formulation is given. The main results are presented in Section III. An example is provided in Section IV and conclusions are drawn in Section V.

Notation: $\mathbb{N}$ denotes the set of all nonnegative integers. $\|\cdot\|$ represents the Euclidean norm for vectors or induced 2-norm for matrices. For a matrix A, let A^{T} be its transpose. $A>0$ (≥ 0) means that A is positive definite (semi-definite). For a square matrix A, its spectrum is denoted by $\sigma(A)$. $\bar{\mathbb{C}}_+$ represents the closed right-half complex plane. The symbol $\otimes$ denotes the Kronecker product. The $n\times n$ identity matrix is denoted as I_n. $\mathbf{1}_n$ stands for a column vector with all elements being 1. $\mathrm{col}(x_1,\cdots,x_N)$ represents a column vector that stacks all column vectors x_i, $i=1,\cdots,N$ together. $\mathrm{diag}(a_1,\cdots,a_N)$ denotes a diagonal matrix with scalar a_i, $i=1,\cdots,N$ as the diagonal elements. $\{a_k : k\in\mathbb{N}\}$ is a strictly increasing sequence of real numbers. For a function $f:[0,+\infty)\to\mathbb{R}$, denote the limit from the above and the below at time t by $f(t^+)=\lim_{t\to t^+} f(t)$ and $f(t^-)=\lim_{t\to t^-} f(t)$, respectively. Let the upper Dini derivative of $f(t)$ be $\mathrm{D}^+f(t)$, which is defined by $\mathrm{D}^+f(t)=\limsup_{h\to 0^+}\frac{f(t+h)-f(t)}{h}$.

II. Problem Formulation

Consider a heterogeneous linear MAS with N non-identical agents, labeled with $1,\cdots,N$. For agent i, $i\in\mathcal{N}=\{1,\cdots,N\}$, the dynamics can be described by

$$\begin{cases}\dot{x}_i(t)=A_ix_i(t)+B_iu_i(t)\\ y_i(t)=C_ix_i(t)\end{cases},\tag{1}$$

where $x_i(t)\in\mathbb{R}^{n_i}$, $u_i(t)\in\mathbb{R}^{p_i}$ and $y_i(t)\in\mathbb{R}^q$ represent the system state, control input and measured output, respectively; $A_i\in\mathbb{R}^{n_i\times n_i}$, $B_i\in\mathbb{R}^{n_i\times p_i}$ and $C_i\in\mathbb{R}^{q\times n_i}$ are constant matrices. The dynamics of system (1) are heterogeneous, in the sense that n_i, p_i, A_i, B_i and C_i, $i=1,\cdots,N$ can be different. When they are the same, system (1) becomes a homogeneous linear MAS. It is assumed that the communication topology among the agents is modeled by a fixed undirected graph $\mathcal{G}=(\mathcal{N},\mathcal{E})$, with the node set $\mathcal{N}$ and the edge set $\mathcal{E}\subseteq\mathcal{N}\times\mathcal{N}$. The adjacency matrix of $\mathcal{G}$ is defined as $\mathcal{A}=[a_{ij}]\in\mathbb{R}^{N\times N}$, for $i,j\in\mathcal{N}$, where $a_{ij}=1$, if $(j,i)\in\mathcal{E}$; otherwise $a_{ij}=0$. Note that there is no self-loop, i.e., $a_{ii}=0$. The Laplacian matrix is defined as $L=\mathcal{D}-\mathcal{A}$, where $\mathcal{D}=\mathrm{diag}\{d_1,\cdots,d_N\}$ is the degree matrix. Let the eigenvalues of L be λ_i, $i=1,\cdots,N$, satisfying $0=\lambda_1\leq\lambda_2\leq\cdots\leq\lambda_N$. For more details on graph theory, one can refer to [25]. In this paper, the objective is to design a fully distributed control strategy with intermittent communication to solve the output consensus problem, as formally defined as follows.

Definition 1 (Output Consensus Problem): Consider the heterogeneous linear MAS (1) with a communication graph $\mathcal{G}$. The output consensus problem is to find a control strategy such that for any initial conditions $x_i(0)$, $i\in\mathcal{N}$, all agents asymptotically achieve output consensus in the sense that

$$\lim_{t\to\infty}\|y_i(t)-y_j(t)\|=0,\ \forall i,\ j\in\mathcal{N}.\tag{2}$$

Furthermore, if (2) is satisfied with $\lim_{t\to\infty}y_i(t)=0$, for any $i\in\mathcal{N}$, then it is said that the output consensus problem is trivially solved, otherwise it is nontrivially solved.

The trivial case is not considered in this paper as the case can be easily addressed by stabilizing each individual agent directly. The nontrivial case of the output consensus problem is investigated under the following assumptions:

Assumption 1: The pairs (A_i,B_i), $i\in\mathcal{N}$ are stabilizable.

Assumption 2: The pairs (A_i,C_i), $i\in\mathcal{N}$ are detectable.

Assumption 3: The communication graph $\mathcal{G}$ is connected.

Before proceeding, a useful lemma is introduced.

Lemma 1: ([26]) Assume that D, E and Q are matrices with appropriate dimensions and $Q>0$. Then, for any vectors $x,y\in\mathbb{R}^n$, $2x^{\mathrm{T}}DEy\leq x^{\mathrm{T}}DQD^{\mathrm{T}}x+y^{\mathrm{T}}E^{\mathrm{T}}Q^{-1}Ey$.

III. Main Results

In this section, a fully distributed event-triggered control strategy is designed to solve the output consensus problem for the heterogeneous linear MAS (1).

A. Control Law Design

For each agent, a fully distributed event-triggered output feedback control law is proposed as follows:

$$u_i(t)=K_{1i}\xi_i(t)+K_{2i}\eta_i(t),\tag{3a}$$

$$\dot{\xi}_i(t)=A_i\xi_i(t)+B_iu_i(t)+F_i\left(C_i\xi_i(t)-y_i(t)\right),\tag{3b}$$

$$\mathrm{D}^+\eta_i(t)=A\eta_i(t)+\bar{c}_i(t)BK\sum_{j=1}^N a_{ij}(\hat{\eta}_j(t)-\hat{\eta}_i(t)),\tag{3c}$$

$$\mathrm{D}^+c_i(t)=\mathrm{e}^{2\lambda t}\left[\sum_{j=1}^N a_{ij}(\hat{\eta}_j(t)-\hat{\eta}_i(t))\right]^{\mathrm{T}}\cdot\Phi\left[\sum_{j=1}^N a_{ij}(\hat{\eta}_j(t)-\hat{\eta}_i(t))\right],\tag{3d}$$

$$t\in[t_k^i,\ t_{k+1}^i),$$

where $\xi_i(t)\in\mathbb{R}^{n_i}$ is the observer state, $\eta_i(t)\in\mathbb{R}^m$ is the compensator state, $\bar{c}_i(t)\in\mathbb{R}$ is a piecewise constant coupling gain, which is updated to be $c_i(t)$ and broadcast to agent i's neighbors whenever $c_i(t)$ reaches previous $\bar{c}_i(t)+\kappa_i$ with $\kappa_i>0$ and $\bar{c}_i(0)=c_i(0)>0$, $\lambda>0$ is a pre-specified design parameter, $\hat{\eta}_i(t)\in\mathbb{R}^m$ is the open-loop estimate of $\eta_i(t)$, t_k^i is the k-th triggering time instant of agent i, for $k\in\mathbb{N}$, and K_{1i}, K_{2i}, F_i, A, B, K and Φ are constant matrices to be designed. The dynamics of the open-loop estimate $\hat{\eta}_i(t)$ are described by

$$\begin{cases}\dot{\hat{\eta}}_i(t)=A\hat{\eta}_i(t),\ t\in[t_k^i,\ t_{k+1}^i)\\ \hat{\eta}_i(t)=\eta_i(t),\quad t=t_k^i\end{cases}.\tag{4}$$

Obviously, the estimate $\hat{\eta}_i(t)$ is reset to its true value $\eta_i(t)$ at the triggering time instants t_k^i, $k\in\mathbb{N}$. It should be emphasized

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that with the distributed control law (3a)-(3d), agent i is not required to continuously monitor any neighboring information. Instead, agent i needs to continuously estimate the compensator states of itself and its neighbors, i.e., $\eta_i(t)$ and $\eta_j(t)$, $j \in \mathcal{N}_i$, with $\mathcal{N}_i$ being the neighbor set of agent i.

Remark 1: It is worth mentioning that the parameter $\bar{c}_i(t)$ is incorporated in (3c), instead of a positive constant in the control law in [20]. In this way, the global information λ_2 is no longer used. This idea is partly motivated by the continuous-time adaptive control laws on consensus in [4]. However, different from the adaptive laws in [4], the exponential function $\mathrm{e}^{2\lambda t}$ is incorporated in the adaptive law (3d) to achieve exponential regulation, which is similar to the idea in [27].

B. Triggering Mechanism Design

In this subsection, an event-triggering mechanism is designed such that the states of all compensators can reach consensus with intermittent communication. For agent i, $i \in \mathcal{N}$, the corresponding compensator is called compensator i, which is governed by (3c)-(3d). Then, all compensators can constitute a homogeneous linear MAS with the communication graph $\mathcal{G}$, which is regarded as a virtual network layer. By exploiting this homogeneous linear MAS, an event-triggering mechanism is developed as follows.

For each agent i, $i \in \mathcal{N}$, define the measurement error as

$$e_i(t) = \hat{\eta}_i(t) - \eta_i(t),\ t \in [t_k^i,\ t_{k+1}^i), \tag{5}$$

and the relative state estimate as

$$\hat{q}_i(t) = \sum_{j=1}^{N} a_{ij}(\hat{\eta}_j(t) - \hat{\eta}_i(t)). \tag{6}$$

Let $\{t_k^i : k \in \mathbb{N}\}$ be the triggering time sequence of agent i. Then, the inter-event times are defined as $t_{k+1}^i - t_k^i$, $k \in \mathbb{N}$. To determine t_k^i, $k \in \mathbb{N}$, define a triggering function as

$$f_i(e_i(t), \hat{q}_i(t), t) = \|e_i(t)\| - \gamma_i(t)\, \|\hat{q}_i(t)\| - \delta_i(t)\mathrm{e}^{-\alpha_i t}, \tag{7}$$

where $\alpha_i > \lambda$ is a design parameter, and $\gamma_i(t)$ and $\delta_i(t)$ are two time-varying variables to be determined, which satisfy $\gamma_i(t) > 0$ and $\delta_i(t) > 0$ for all $t \geq 0$. Then, a novel event-triggering mechanism is proposed as follows:

$$t_{k+1}^i = \inf\left\{t > t_k^i \mid f_i(e_i(t), \hat{q}_i(t), t) \geq 0\right\}. \tag{8}$$

Note that the exponential convergent term $\delta_i(t)\mathrm{e}^{-\alpha_i t}$ is introduced so that Zeno behavior is excluded for each agent.

Remark 2: The event-triggering mechanism (8) can be implemented in the following way. According to (7) and (8), agent i needs to continuously monitor the compensator state $\eta_i(t)$ and the estimates $\hat{\eta}_i(t)$ and $\hat{\eta}_j(t)$, $j \in \mathcal{N}_i$, regardless of any true neighboring information. Thus, to check the triggering condition, continuous communication between neighboring agents is not required. When an event is triggered for agent i, it updates the estimate $\hat{\eta}_i(t)$ with the current state $\eta_i(t)$, and sends the current state $\eta_i(t)$ to all its neighbors in the meantime. When agent i receives any state information from its neighbors, namely, $\eta_j(t)$, $j \in \mathcal{N}_i$, it immediately updates the corresponding estimate $\hat{\eta}_j(t)$. It should be pointed out that only the compensator state and the coupling gain of each agent are exchanged through the communication network.

C. Solvability of the Output Consensus Problem

Before presenting the main result of this paper, an intermediate result on the compensator described in (3c)-(3d) and the event-triggering mechanism described in (8) will be first given. In particular, state consensus of the compensator system will be shown and Zeno behavior will be explicitly excluded for the system.

Proposition 1: Consider the homogeneous linear MAS (3c) with adaptive law (3d). Suppose that Assumption 3 holds and the pair (A, B) is controllable. Let $K = B^{\mathrm{T}}P$ and $\Phi = PBB^{\mathrm{T}}P$, where P is the unique positive definite solution of the following parametric algebraic Riccati equation (ARE)

$$PA + A^{\mathrm{T}}P - PBB^{\mathrm{T}}P + 2\lambda P + \alpha I_m = 0, \tag{9}$$

with $\lambda > 0$ and $\alpha > 0$. Let $\gamma_i(t) = \sqrt{\frac{\sigma_i}{v_i(t)(1+\theta_i(t))}}$ and $\delta_i(t) = \sqrt{\frac{\beta_i}{v_i(t)(1+\theta_i(t))}}$, where $0 < \sigma_i < 1$, $\beta_i > 0$, $\theta_i(t) = \bar{c}_i(t) + \sum_{j=1}^{N} a_{ij}\bar{c}_j(t)$, and $v_i(t)$ satisfies the following dynamics:

$$\mathrm{D}^+ v_i(t) = \mathrm{e}^{2\lambda t}\left(1 + \theta_i(t)\right)\|e_i(t)\|^2, \tag{10}$$

with $v_i(0) > 0$. If the event-triggering mechanism (8) is executed, then

1) State consensus can be exponentially achieved for all agents with the rate of $\mathrm{e}^{-\lambda t}$.
2) The adaptive parameters $c_i(t)$ and $v_i(t)$ converge to some finite positive steady-state values, respectively.
3) Zeno behavior can be strictly excluded for each agent.

Proof: See the Appendix. ■

Remark 3: If $\lambda = 0$ is chosen, the homogeneous linear MAS (3c) can asymptotically achieve state consensus under the same conditions as in Proposition 1, except that the pair (A, B) only needs to be stabilizable, rather than controllable.

Remark 4: It is observed that the event-triggering mechanisms in [19] and [20] depend on some global information, namely, λ_N and N. To avoid using these information, the adaptive parameter $v_i(t)$ is introduced into the proposed event-triggering mechanism. As a result, the variables $\gamma_i(t)$ and $\delta_i(t)$ are determined in terms of the adaptive parameter $v_i(t)$. The mechanism (8) is thus called an *adaptive event-triggering mechanism*. Compared with most existing triggering mechanisms [18]–[23], the adaptive event-triggering mechanism (8) has an advantage that no global information is required.

Now, one is ready to present the main result on the solution to the output consensus problem.

Theorem 1: Consider the heterogeneous linear MAS (1) with the distributed event-triggered control law (3a)-(3d) under Assumptions 1-3. Let $A = S$, $B = I_m$, $K = P$ and $\Phi = P^2$, where $P > 0$ is the solution of (9). For each agent i, $i \in \mathcal{N}$, choose the matrices K_{1i} and F_i such that $A_i + B_iK_{1i}$ and $A_i + F_iC_i$ are Hurwitz, respectively, and let $K_{2i} = \Gamma_i - K_{1i}\Pi_i$, where $\Pi_i \in \mathbb{R}^{n_i \times m}$ and $\Gamma_i \in \mathbb{R}^{p_i \times m}$ satisfy

$$A_i\Pi_i + B_i\Gamma_i = \Pi_i S, \tag{11}$$

$$C_i\Pi_i = R. \tag{12}$$

Then, under the adaptive event-triggering mechanism (8) with $\gamma_i(t)$ and $\delta_i(t)$ being designed as in Proposition 1, all agents asymptotically achieve output consensus, if and only if there exist an integer $m > 0$, and matrices $S \in \mathbb{R}^{m\times m}$ and $R \in \mathbb{R}^{q\times m}$, with $\sigma(S) \subset \bar{\mathbb{C}}_+$ and $R \neq 0$, such that the equations (11)-(12) have a solution pair (Π_i, Γ_i), for $i \in \mathcal{N}$. Moreover, Zeno behavior can be strictly excluded for each agent.

Proof: Consider the network consisting of all dynamic compensators. Note that (S, I_m) is always controllable. Then, the conditions in Proposition 1 are satisfied. According to the proof of Proposition 1, one has $\lim_{t\to\infty}(\eta_i(t) - \eta_j(t)) = 0$, $\lim_{t\to\infty} c_i(t) = c_\infty$, and $\lim_{t\to\infty} \hat{q}_i(t) = 0$ exponentially, for all $i,\ j \in \mathcal{N}$. Moreover, by Proposition 1, Zeno behavior is strictly excluded for each agent.

Sufficiency: Define the tracking error as $\zeta_i(t) = x_i(t) - \Pi_i\eta_i(t)$, and the observer error as $\epsilon_i(t) = x_i(t) - \xi_i(t)$, for $i \in \mathcal{N}$. Then, by (1), (3a)-(3c) and (11), the error dynamics can be written as

$$\begin{aligned} \dot{\epsilon}_i(t) &= (A_i + F_iC_i)\,\epsilon_i(t), && (13)\\ \mathrm{D}^+\zeta_i(t) &= (A_i + B_iK_{1i})\,\zeta_i(t) - B_iK_{1i}\epsilon_i(t) \\ &\quad - \bar{c}_i(t)\Pi_iP\hat{q}_i(t). && (14)\end{aligned}$$

Under Assumption 2, one can choose a proper F_i such that $A_i + F_iC_i$ is Hurwitz. Hence, it follows from (13) that $\lim_{t\to\infty}\epsilon_i(t) = 0$. Under Assumption 1, one can choose a proper K_{1i}, such that $A_i + B_iK_{1i}$ is Hurwitz. Thus, by [28, Lemma 5], it follows from (14) that $\lim_{t\to\infty}\zeta_i(t) = 0$. Furthermore, by (12), one has

$$\lim_{t\to\infty} x_i(t) = \lim_{t\to\infty}(\zeta_i(t) + \Pi_i\eta_i(t)) = \lim_{t\to\infty}\Pi_i\eta_i(t), \quad (15)$$
$$\lim_{t\to\infty}(y_i(t) - y_j(t)) = \lim_{t\to\infty} R\,(\eta_i(t) - \eta_j(t)) = 0. \quad (16)$$

Therefore, it is concluded that output consensus can be asymptotically achieved for the heterogeneous linear MAS (1).

Necessity: The proof of this part is similar to that in [19, Theorem 2], and thus omitted here. ■

Remark 5: The output consensus problem is solved in Theorem 1 using a two-step approach described as follows. In the first step, a dynamic compensator of the form (3c)-(3d) is constructed for each agent such that all the compensators constitute a homogeneous linear MAS, and then state consensus is achieved for the compensators with intermittent communication among the agents, as shown in Proposition 1. In the second step, a continuous output feedback controller of the form (3a)-(3b) is designed such that each agent can track the reference signal provided by its associated compensator.

Remark 6: It is noted that with the proposed control law in Theorem 1, each agent is required to know the matrix S, like the control laws in [5], [19], [23]. To remove this requirement, one possible approach is to use some adaptive control technique to estimate the matrix S for each agent. A similar idea can be found in [29] for solving the cooperative output regulation problem.

Remark 7: The solvability of the equations (11)-(12) is necessary and sufficient for the solvability of the output consensus problem. In [8], a sufficient condition for the solvability of the equations (11)-(12) is provided as follows. For any nonzero matrix $R \in \mathbb{R}^{q\times m}$, there exists a solution pair (Π_i, Γ_i) to the equations (11)-(12), if there exists a matrix $S \in \mathbb{R}^{m\times m}$, such that

$$\operatorname{rank}\begin{pmatrix} A_i - \rho I & B_i \\ C_i & 0 \end{pmatrix} = n_i + q, \forall\rho \in \sigma(S), i \in \mathcal{N}. \quad (17)$$

Besides, a necessary condition for the solvability of the equations (11)-(12) is provided in [8]. However, it is still an open problem to obtain a necessary and sufficient condition to guarantee that the equations (11)-(12) are solvable.

In what follows, the design procedure of the proposed event-triggered control strategy is summarized.

Step 1: Introduce the dynamic compensator (3c)-(3d) for each agent. Choose the matrices S and R, with $\sigma(S) \subset \bar{\mathbb{C}}_+$ and $R \neq 0$, such that the equations (11)-(12) have the solution pair (Π_i, Γ_i). Let $K = P$ and $\Phi = P^2$, where $P > 0$ satisfies the ARE (9).

Step 2: Introduce the state observer (3b) for each agent. Choose the matrix F_i such that $A_i + F_iC_i$ is Hurwitz.

Step 3: Design the controller (3a) for each agent. Choose the matrix K_{1i} such that $A_i+B_iK_{1i}$ is Hurwitz, and let $K_{2i} = \Gamma_i - K_{1i}\Pi_i$.

Step 4: Develop the adaptive event-triggering mechanism (8), where $\gamma_i(t)$ and $\delta_i(t)$ are designed in Proposition 1, with $v_i(t)$ being updated by the adaptive law (10).

IV. A Simulation Example

Consider the heterogenous linear MAS borrowed from [5], where the dynamics of the i-th agent are modeled by

$$\dot{x}_i = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & s_i \\ 0 & -r_i & a_i \end{pmatrix} x_i + \begin{pmatrix} 0 \\ 0 \\ b_i \end{pmatrix} u_i,$$
$$y_i = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} x_i,\ i = 1,\ \cdots,\ 4.$$

The parameters $\{a_i, b_i, s_i, r_i\}$, $i = 1, \cdots, 4$ are set as {1, 6, 2, 10}, {2, 5, 1, 8}, {1, 8, 1, 6}, and {1, 5, 1, 8}, respectively. It can be verified that the system matrix of each agent has one eigenvalue at the origin and the other two eigenvalues with positive real parts, and hence is unstable. For agent i, denote $x_i = [x_{i1},\ x_{i2},\ x_{i3}]^{\mathrm{T}}$. Then the components x_{i1}, x_{i2} and x_{i3} can be regarded as the position, velocity and actuator state of agent i, respectively. In this case, the output consensus problem is equivalent to synchronizing the positions of the agents. The communication graph among four agents is given in Fig. 1.

It is verified that Assumptions 1-3 are satisfied. Thus, the proposed control strategy in Theorem 1 can be applied to solve the output consensus problem. According to the design procedure, the parameters are chosen as follows: 1) $S = [0, 1; 0, 0]$, $R = [1, 0]$, $\Pi_i = [1, 0; 0, 1; 0, 0]$, $\Gamma_i = [0, d_i/b_i]$, $\lambda = 0.01$, $\alpha = 1$, $K = P = [0.9185, 0.4180, 0.4180, 1.2989]$, $\Phi = P^2$, $\bar{c}_i(0) = c_i(0) = 1.0$, $\kappa_i = 0.6$; 2) $F_i = [-5, -10, 15]^{\mathrm{T}}$; 3) $K_{1i} = [-15, -20, -15]$, $K_{2i} = \Gamma_i - K_{1i}\Pi_i = [15, 20 + d_i/b_i]$; 4) $\sigma_i = 0.9$, $\alpha_i = 0.5$, $\beta_i = 1$, $v_i(0) = 1.5$. The initial condition $x_i(0)$, $\xi_i(0)$ and $\eta_i(0)$, $i = 1, \cdots, 4$ can be randomly chosen.

Define the state errors between any two compensators as $e_{ij} = \eta_i - \eta_j = [e_{ij}^1,\ e_{ij}^2]^{\mathrm{T}}$, and the output errors between any two agents as $e_{ij}^y = y_i - y_j$, for $i,\ j = 1,\ \cdots,\ 4$. In simulations, state consensus is defined to be achieved for all

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TAC.2018.2868997, IEEE Transactions on Automatic Control

compensators, if $\left(\sum_{i=1}^{4}\|\sum_{j=1}^{4}a_{ij}e_{ij}\|^2\right)^{1/2} \leq 1\times 10^{-4}$. In a similar way, output consensus is also defined. Simulations have been conducted for various initial conditions. The results corresponding to one initial condition are presented in Figs. 2-5. Fig. 2 shows the state errors of the compensators. It can be seen that the state errors converge to zero. Figs. 3 and 4 depict the adaptive parameters $c_i(t)$ and $v_i(t)$, respectively, which converge to some positive values. The output errors of the agents are shown in Fig. 5. It can be observed that the output errors converge to zero. The simulation results show that the consensus times for the compensators and agents are 12.5365 and 24.3629 (s), respectively. Moreover, in the first 25 s, the minimum inter-event times of agent i, $i=1,\cdots,4$ are 0.0692, 0.1152, 0.0969 and 0.0443 (s), respectively.

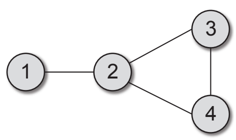

Fig. 1: Communication graph.

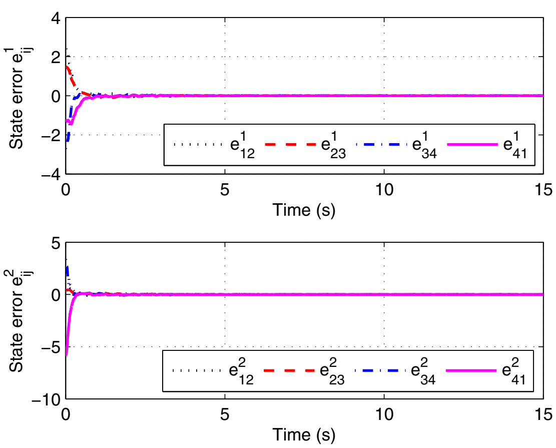

Fig. 2: State errors of the compensators.

V. Conclusion

In this paper, the output consensus problem has been solved for heterogeneous linear MASs with fixed and undirected graphs by event-triggered control. For each agent, a fully distributed event-triggered control strategy has been proposed. It has been shown that under the proposed control strategy, output consensus can be asymptotically achieved for all agents with intermittent communication in a fully distributed manner. Moreover, under the proposed adaptive event-triggering mechanism, it has been shown that Zeno behavior can be strictly excluded for each agent. Future work can be directed to the case when heterogeneous linear MASs are subject to external disturbances or the communication topologies are directed or switching.

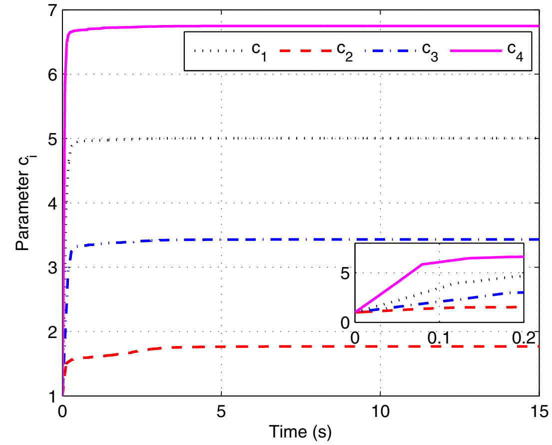

Fig. 3: Adaptive parameters c_i, $i=1,\cdots,4$.

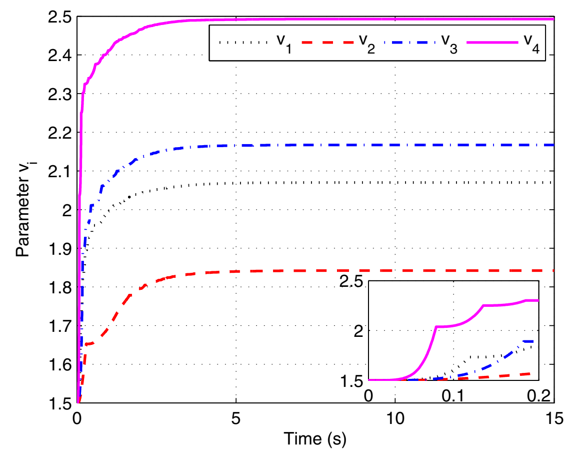

Fig. 4: Adaptive parameters v_i, $i=1,\cdots,4$.

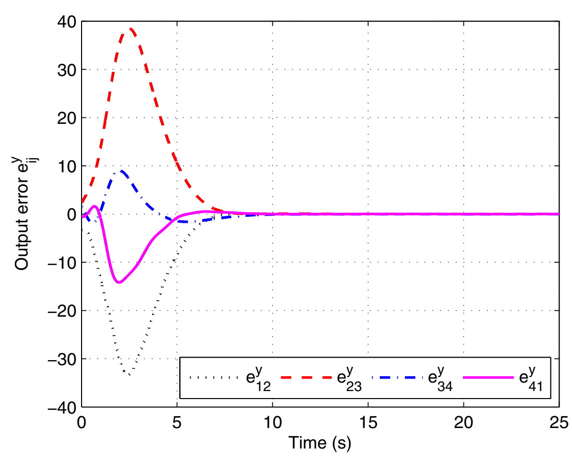

Fig. 5: Output errors of the agents.

Appendix
Proof of Proposition 1

Let $\eta(t)=\mathrm{col}(\eta_1(t),\cdots,\eta_N(t))$, $\hat{\eta}(t)=\mathrm{col}(\hat{\eta}_1(t),\cdots,\hat{\eta}_N(t))$, $e(t)=\mathrm{col}(e_1(t),\cdots,e_N(t))$, $\hat{q}(t)=\mathrm{col}(\hat{q}_1(t),$

$\cdots,\ \hat{q}_N(t))$, $\bar{c}(t) = \operatorname{diag}\{\bar{c}_1(t), \cdots, \bar{c}_N(t)\}$, and $c(t) = \operatorname{diag}\{c_1(t), \cdots, c_N(t)\}$. Then, $\hat{\eta}(t) = \eta(t) + e(t)$ and $\hat{q}(t) = -(L \otimes I_m)\hat{\eta}(t)$. By (3c), (4) and (5), the dynamics of $\eta(t)$ and $e(t)$ can be expressed as

$$\mathrm{D}^+\eta(t) = (I_N \otimes A)\,\eta(t) - (\bar{c}(t)L \otimes BK)\,\hat{\eta}(t), \tag{18}$$

$$\mathrm{D}^+e(t) = (I_N \otimes A)\,e(t) + (\bar{c}(t)L \otimes BK)\,\hat{\eta}(t). \tag{19}$$

Define the following exponential transformations: $z_i(t) = \mathrm{e}^{\lambda t}\eta_i(t)$, $\hat{z}_i(t) = \mathrm{e}^{\lambda t}\hat{\eta}_i(t)$, $\varsigma_i(t) = \mathrm{e}^{\lambda t}e_i(t)$, and $\omega_i(t) = \mathrm{e}^{\lambda t}\hat{q}_i(t) = \sum_{i=1}^N a_{ij}(\hat{z}_j(t) - \hat{z}_i(t))$. It is seen that $\lim_{t\to\infty}(\eta_i(t) - \eta_j(t)) = 0$ exponentially with the rate of $\mathrm{e}^{-\lambda t}$, if $\lim_{t\to\infty}(z_i(t) - z_j(t)) = 0$, for any $i,\ j \in \mathcal{N}$. Let $z(t) = \operatorname{col}(z_1(t), \cdots, z_N(t))$, $\hat{z}(t) = \operatorname{col}(\hat{z}_1(t), \cdots, \hat{z}_N(t))$, $\varsigma(t) = \operatorname{col}(\varsigma_1(t), \cdots, \varsigma_N(t))$, and $\omega(t) = \operatorname{col}(\omega_1(t), \cdots, \omega_N(t))$. Then, $\hat{z}(t) = z(t) + \varsigma(t)$ and $\omega(t) = -(L \otimes I_m)\hat{z}(t)$. By virtue of (18) and (19), one has

$$\mathrm{D}^+z(t) = [I_N \otimes (A + \lambda I_m)]\,z(t) - (\bar{c}(t)L \otimes BK)\,\hat{z}(t), \tag{20}$$

$$\mathrm{D}^+\varsigma(t) = [I_N \otimes (A + \lambda I_m)]\,\varsigma(t) + (\bar{c}(t)L \otimes BK)\,\hat{z}(t). \tag{21}$$

In what follows, consider the closed-loop system composed of (3d), (10), (20) and (21).

Since (A, B) is controllable, it directly follows from [30, Theorem 12.3] that $(A + \lambda I_m, B)$ is also controllable. Thus, for any given $\lambda > 0$ and an arbitrary $\alpha > 0$, the parametric ARE (9) has the unique solution P, which is positive definite and symmetric. Choose a Lyapunov-like function of the form

$$V(t) = z^{\mathrm{T}}(t)\,(L \otimes P)\,z(t) + \varsigma^{\mathrm{T}}(t)\,(I_N \otimes P)\,\varsigma(t) + \frac{1}{2}\sum_{i=1}^N \tilde{c}_i^2(t) + \frac{\alpha}{8\lambda_N}\sum_{i=1}^N \tilde{v}_i^2(t), \tag{22}$$

where $\tilde{c}_i(t) \triangleq c_i(t) - c_0$ and $\tilde{v}_i(t) \triangleq v_i(t) - v_0$, with c_0 and v_0 being two positive constants to be specified later. Let $\{t_l : l \in \mathbb{N}\}$ be a strictly increasing sequence of unique elements, which consists of the elements in $\{t_k^i : k \in \mathbb{N}\}$, for all $i \in \mathcal{N}$. It can be seen that $V(t)$ is piecewise continuous on the interval $[0, \infty)$, and differentiable on each interval $[t_l, t_{l+1})$, $l \in \mathbb{N}$.

Denote $\hat{A} = PA + A^{\mathrm{T}}P + 2\lambda P$ and $\hat{B} = PBB^{\mathrm{T}}P$. Then, it follows from (9) that $\hat{A} - \hat{B} = -\alpha I_m$. Since $K = B^{\mathrm{T}}P$ and $\Phi = PBB^{\mathrm{T}}P$, the upper Dini derivative of $V(t)$ along the trajectory of the closed-loop system on each interval $[t_l, t_{l+1})$, $l \in \mathbb{N}$ can be obtained as

$$\begin{aligned}
&\mathrm{D}^+V(t) \\
&= \hat{z}^{\mathrm{T}}(t)\left(L \otimes \hat{A} - 2L\bar{c}(t)L \otimes \hat{B}\right)\hat{z}(t) \\
&\quad + \varsigma^{\mathrm{T}}(t)\left(L \otimes \hat{A} + I_N \otimes \hat{A}\right)\varsigma(t) \\
&\quad + 2\hat{z}^{\mathrm{T}}(t)\left(-L \otimes \hat{A} + L\bar{c}(t)L \otimes \hat{B} + L\bar{c}(t) \otimes \hat{B}\right)\varsigma(t) \\
&\quad + \hat{z}^{\mathrm{T}}(t)\left[L\,(c(t) - c_0 I_N)\,L \otimes \hat{B}\right]\hat{z}(t) \\
&\quad + \frac{\alpha}{4\lambda_N}\sum_{i=1}^N (v_i(t) - v_0)\,(1 + \theta_i(t))\,\|\varsigma_i(t)\|^2.
\end{aligned} \tag{23}$$

By [25, Chapter 4], the Laplacian matrix L can be decomposed into $L = ZZ^{\mathrm{T}}$, where Z denotes the incidence matrix of graph $\mathcal{G}$. Since $c_i(0) > 0$ and $\mathrm{D}^+c_i(t) \geq 0$, $c_i(t) \geq \bar{c}_i(t) > 0$ for all $t \geq 0$ and $i \in \mathcal{N}$. Then, by Lemma 1 and the property (23) in [20], one has

$$\begin{aligned}
&-2\hat{z}^{\mathrm{T}}(t)\left(L \otimes \hat{B}\right)\varsigma(t) \leq \hat{z}^{\mathrm{T}}(t)\left(L \otimes \hat{B}\right)\hat{z}(t) \\
&\qquad + \varsigma^{\mathrm{T}}(t)\left(L \otimes \hat{B}\right)\varsigma(t), \\
&2\hat{z}^{\mathrm{T}}(t)\,(L \otimes \alpha I_m)\,\varsigma(t) \leq \frac{1}{2}\hat{z}^{\mathrm{T}}(t)\,(L \otimes \alpha I_m)\,\hat{z}(t) \\
&\qquad + 2\varsigma^{\mathrm{T}}(t)\,(L \otimes \alpha I_m)\,\varsigma(t), \\
&2\hat{z}^{\mathrm{T}}(t)\left(L\bar{c}(t)L \otimes \hat{B}\right)\varsigma(t) \leq \frac{1}{2}\hat{z}^{\mathrm{T}}(t)\left(L\bar{c}(t)L \otimes \hat{B}\right)\hat{z}(t) \\
&\qquad + 4N^2\varsigma^{\mathrm{T}}(t)\left(\theta(t) \otimes \hat{B}\right)\varsigma(t), \\
&2\hat{z}^{\mathrm{T}}(t)\left(L\bar{c}(t) \otimes \hat{B}\right)\varsigma(t) \leq \frac{1}{2}\hat{z}^{\mathrm{T}}(t)\left(L\bar{c}(t)L \otimes \hat{B}\right)\hat{z}(t) \\
&\qquad + 2\varsigma^{\mathrm{T}}(t)\left(\bar{c}(t) \otimes \hat{B}\right)\varsigma(t),
\end{aligned}$$

where $\theta(t) \triangleq \operatorname{diag}\{\theta_1(t), \cdots, \theta_N(t)\}$. Using $\hat{A} - \hat{B} = -\alpha I_m$ and substituting the preceding inequalities into (23) yield

$$\begin{aligned}
&\mathrm{D}^+V(t) \\
&\leq \hat{z}^{\mathrm{T}}(t)\Big[-\frac{1}{2}L \otimes \alpha I_m + 2L \otimes \hat{B} \\
&\quad + L\,(c(t) - \bar{c}(t) - c_0 I_N)\,L \otimes \hat{B}\Big]\hat{z}(t) \\
&\quad + \varsigma^{\mathrm{T}}(t)\Big[L \otimes \alpha I_m + (2L + 4N^2\theta(t) + 2\bar{c}(t) + I_N) \otimes \hat{B}\Big]\varsigma(t) \\
&\quad + \frac{\alpha}{4\lambda_N}\sum_{i=1}^N (v_i(t) - v_0)\,(1 + \theta_i(t))\,\|\varsigma_i(t)\|^2.
\end{aligned} \tag{24}$$

Under Assumption 3, the Laplacian matrix L has a simple zero eigenvalue while all other eigenvalues λ_i, $i = 2, \cdots, N$ are positive. Since L is symmetric, there exists an orthogonal matrix U such that $U^{\mathrm{T}}LU = \Lambda = \operatorname{diag}\{0, \lambda_2, \cdots, \lambda_N\}$, with $U^{\mathrm{T}}U = UU^{\mathrm{T}} = I_N$. Let $\bar{z}(t) = (U^{\mathrm{T}} \otimes I_m)\hat{z}(t) = \operatorname{col}(\bar{z}_1(t), \cdots, \bar{z}_N(t))$. Then, it follows from (24) that

$$\begin{aligned}
&\mathrm{D}^+V(t) \\
&\leq \sum_{i=1}^N \bar{z}_i^{\mathrm{T}}(t)\left[-\frac{1}{2}\alpha\lambda_i I_m + 2\lambda_i\hat{B} + (\kappa_i - c_0)\,\lambda_i^2\hat{B}\right]\bar{z}_i(t) \\
&\quad + \sum_{i=1}^N \left[\alpha\lambda_N + (2\lambda_N + 4N^2\theta_i(t) + 2\theta_i(t) + 1)\|\hat{B}\|\right]\|\varsigma_i(t)\|^2 \\
&\quad + \frac{\alpha}{4\lambda_N}\sum_{i=1}^N \left[v_i(t)\,(1 + \theta_i(t)) - v_0\,(1 + \theta_i(t))\right]\|\varsigma_i(t)\|^2 \\
&\leq \frac{\alpha}{2\lambda_N}\sum_{i=1}^N \left[-\lambda_i^2\,\|\bar{z}_i(t)\|^2 + \frac{1}{2}v_i(t)(1 + \theta_i(t))\,\|\varsigma_i(t)\|^2\right],
\end{aligned} \tag{25}$$

where c_0 and v_0 are chosen to be sufficiently large, such that $c_0 \geq \frac{2}{\lambda_2} + \max_{i\in\mathcal{N}}\kappa_i$ and $v_0 \geq \frac{4\lambda_N}{\alpha} \cdot \max\{\alpha\lambda_N + (2\lambda_N + 1)\|\hat{B}\|, (4N^2 + 2)\|\hat{B}\|\}$, respectively. Note that

$$\sum_{i=1}^N \lambda_i^2\,\|\bar{z}_i(t)\|^2 = \hat{z}^{\mathrm{T}}(t)(L^2 \otimes I_m)\hat{z}(t) = \sum_{i=1}^N \|\omega_i(t)\|^2. \tag{26}$$

Then, substituting (26) into (25) yields

$$\mathrm{D}^{+}V(t) \leq \frac{\alpha}{2\lambda_N}\sum_{i=1}^{N}\left[-\left\|\omega_i(t)\right\|^2+\frac{1}{2}v_i(t)(1+\theta_i(t))\left\|\varsigma_i(t)\right\|^2\right]. \quad (27)$$

If the event-triggering mechanism (8) is executed, then

$$\left\|\varsigma_i(t)\right\|^2 \leq \frac{2}{v_i(t)(1+\theta_i(t))}\left[\sigma_i\left\|\omega_i(t)\right\|^2+\beta_i \mathrm{e}^{-2(\alpha_i-\lambda)t}\right], \quad (28)$$

with $\alpha_i > \lambda$. Thus, the upper Dini derivative of $V(t)$ is further bounded by

$$\mathrm{D}^{+}V(t) \leq \frac{\alpha}{2\lambda_N}\sum_{i=1}^{N}\left[-(1-\sigma_i)\left\|\omega_i(t)\right\|^2+\beta_i \mathrm{e}^{-2(\alpha_i-\lambda)t}\right]. \quad (29)$$

Consider the following function:

$$W(t) = V(t) + \frac{\alpha}{2\lambda_N}\sum_{i=1}^{N}\frac{\beta_i}{2(\alpha_i-\lambda)}\mathrm{e}^{-2(\alpha_i-\lambda)t}, \quad (30)$$

which is piecewise continuous on the interval $[0, \infty)$. On one hand, it follows from (29) that the upper Dini derivative of $W(t)$ on each interval $[t_l,\, t_{l+1})$, $l \in \mathbb{N}$ satisfies

$$\mathrm{D}^{+}W(t) \leq -\frac{\alpha}{2\lambda_N}(1-\bar{\sigma})\left\|\omega(t)\right\|^2, \quad (31)$$

where $\bar{\sigma} \triangleq \max_{i\in\mathcal{N}}\sigma_i < 1$. On the other hand, when $t = t_l$, $l \in \mathbb{N}$,

$$\begin{aligned} W(t_l^{+}) - W(t_l^{-}) = {} & \varsigma^{\mathrm{T}}(t_l^{+})\,(I_N \otimes P)\,\varsigma(t_l^{+}) \\ & - \varsigma^{\mathrm{T}}(t_l^{-})\,(I_N \otimes P)\,\varsigma(t_l^{-}) \leq 0. \end{aligned} \quad (32)$$

Therefore, it is concluded from (31) and (32) that $W(t)$ is non-increasing on the interval $[0, \infty)$. Since $W(t) \geq 0$, $W(t)$ is bounded. Thus, $\lim_{t\to\infty} W(t)$ exists. From the definition of $W(t)$, it can be seen that $z_i(t) - z_j(t)$, $\varsigma_i(t)$, $\tilde{c}_i(t)$ and $\tilde{v}_i(t)$ are bounded over $[0, \infty)$, for any $i, j \in \mathcal{N}$.

Based on the above preliminaries, the results in Proposition 1 can be obtained by the following three steps.

Step 1: Show that $c_i(t)$ and $v_i(t)$ are convergent.

Since $c_i(0) > 0$ and $\mathrm{D}^{+}c_i(t) \geq 0$ for $t \geq 0$, then $c_i(t)$ is positive and monotonically increasing. Similarly, $v_i(t)$ is also positive and monotonically increasing. Since $\tilde{c}_i(t)$ is bounded and the value of c_0 is finite, $c_i(t)$ is bounded. Hence, $c_i(t)$ converges to a finite positive steady-state value. By the choice of v_0, the value of v_0 is finite. Since $\tilde{v}_i(t)$ is bounded, $v_i(t)$ is bounded. Hence, $v_i(t)$ converges to a finite positive steady-state value.

Step 2: Show that Zeno behavior is avoided for each agent.

Inspired by [17], this claim is proven by contradiction. Suppose that Zeno behavior happens for some agents. Without loss of generality, assume that Zeno behavior happens for agent i, $i \in \mathcal{N}$. This indicates that there exists a finite positive constant t^* such that $\lim_{k\to\infty} t_k^i = t^*$. Then, by the definition of limits of sequences, for every $\varepsilon_0 > 0$, there is an integer N_0 such that for $n \geq N_0$, $t^* - \varepsilon_0 < t_n^i \leq t^*$. Next, the evolution of $\|\varsigma_i(t)\|$ over $[t_k^i,\, t_{k+1}^i)$ is computed. By (21), the dynamics of $\varsigma_i(t)$ are given by

$$\mathrm{D}^{+}\varsigma_i(t) = (A+\lambda I_m)\,\varsigma_i(t) - \bar{c}_i(t)BK\omega_i(t). \quad (33)$$

Then, the upper Dini derivative of $\|\varsigma_i(t)\|$ on each interval $[t_k^i,\, t_{k+1}^i)$, $k \in \mathbb{N}$ satisfies

$$\mathrm{D}^{+}\|\varsigma_i(t)\| \leq \|A+\lambda I_m\|\,\|\varsigma_i(t)\| + \bar{c}_i(t)\,\|BK\|\,\|\omega_i(t)\|. \quad (34)$$

Since $\hat{z}_i(t) = z_i(t) + \varsigma_i(t)$, it is easy to show that $\hat{z}_i(t) - \hat{z}_j(t)$ is bounded over $[0, \infty)$, for any $i, j \in \mathcal{N}$. This implies that $\omega_i(t)$ is also bounded over $[0, \infty)$. From Step 1, it is seen that $c_i(t)$ is bounded over $[0, \infty)$, and so is $\bar{c}_i(t)$. Thus, there exists a finite positive number ϖ_i, such that $\varpi_i = \max_{t\in[0,\infty)}\bar{c}_i(t)\,\|BK\|\,\|\omega_i(t)\|$. It follows from (34) that

$$\mathrm{D}^{+}\|\varsigma_i(t)\| \leq \|A+\lambda I_m\|\,\|\varsigma_i(t)\| + \varpi_i, \quad (35)$$

for $t \in [t_k^i,\, t_{k+1}^i)$. Note that $\varsigma_i(t_k^i) = 0$. By the comparison lemma ([31, Lemma 3.4]), an upper bound of $\|\varsigma_i(t)\|$ over $[t_k^i,\, t_{k+1}^i)$ is given by

$$\|\varsigma_i(t)\| \leq \frac{\varpi_i}{\|A+\lambda I_m\|}\left[\mathrm{e}^{\|A+\lambda I_m\|(t-t_k^i)} - 1\right]. \quad (36)$$

Thus, under the event-triggering mechanism (8), the inter-event time $t_{k+1}^i - t_k^i$ is lower bounded by the solution τ of the following equation:

$$\frac{\varpi_i}{\|A+\lambda I_m\|}\left(\mathrm{e}^{\|A+\lambda I_m\|\tau} - 1\right) = \sqrt{\frac{\beta_i}{\bar{v}_i\left(1+\bar{\theta}_i\right)}}\,\mathrm{e}^{-(\alpha_i-\lambda)(\tau+t_k^i)}, \quad (37)$$

where $\bar{\theta}_i$ and $\bar{v}_i$ are the positive steady-state values of $\theta_i(t)$ and $v_i(t)$, respectively. Consider the case when $k \geq N_0$, and let

$$\begin{aligned} \varepsilon_0 = {} & \frac{1}{2\,\|A+\lambda I_m\|} \\ & \cdot \ln\left[\frac{\|A+\lambda I_m\|}{\varpi_i}\sqrt{\frac{\beta_i}{\bar{v}_i\left(1+\bar{\theta}_i\right)}}\,\mathrm{e}^{-(\alpha_i-\lambda)t^*} + 1\right]. \end{aligned}$$

Then, it follows from (37) that $\tau \geq 2\varepsilon_0$. Recalling that $t_k^i > t^* - \varepsilon_0$, $k \geq N_0$, one has $t_{k+1}^i \geq t_k^i + \tau > t^* + \varepsilon_0$. This contradicts the fact that $t_{k+1}^i \leq t^*$, for $k \geq N_0$. Consequently, Zeno behavior is strictly excluded for each agent. Moreover, there exists a positive number $\tilde{\tau}$ such that $t_{l+1} - t_l \geq \tilde{\tau} > 0$, for $l \in \mathbb{N}$, and $\lim_{l\to\infty} t_l = \infty$, provided that the number of the agents is finite.

Step 3: Show that $\lim_{t\to\infty}(\eta_i(t) - \eta_j(t)) = 0$ exponentially, for all $i, j \in \mathcal{N}$, with the rate of $\mathrm{e}^{-\lambda t}$.

Since $\lim_{t\to\infty} W(t)$ exists, by the Cauchy criterion for convergence, for any $\varepsilon > 0$, there exists a positive number T_0, such that for any $T_2 > T_1 > T_0$, $W(T_1^{+}) - W(T_2^{-}) < \varepsilon$. Suppose that the triggering time instants for all agents over $(T_1,\, T_2)$ are given by $t_{l_1} < t_{l_2} < \cdots < t_{l_r}$. Then, it follows from (31) and (32) that

$$\begin{aligned} & \frac{\alpha}{2\lambda_N}(1-\bar{\sigma})\int_{T_1}^{T_2}\|\omega(t)\|^2\,dt \\ = {} & \frac{\alpha}{2\lambda_N}(1-\bar{\sigma})\left(\int_{T_1}^{t_{l_1}}\|\omega(t)\|^2\,dt \right. \\ & \left. + \int_{t_{l_1}}^{t_{l_2}}\|\omega(t)\|^2\,dt + \cdots + \int_{t_{l_r}}^{T_2}\|\omega(t)\|^2\,\mathrm{d}t\right) \\ \leq {} & -\left(\int_{T_1}^{t_{l_1}}\mathrm{D}^{+}W(t)dt + \int_{t_{l_1}}^{t_{l_2}}\mathrm{D}^{+}W(t)dt + \cdots\right. \end{aligned}$$

$$
\begin{aligned}
&+\int_{t_{l_r}}^{T_2} \mathrm{D}^+ W(t)dt\Big) \\
&= W(T_1^+) - W(t_{l_1}^-) + W(t_{l_1}^+) - W(t_{l_2}^-) + \cdots \\
&\quad + W(t_{l_r}^+) - W(T_2^-) \\
&\leq W(T_1^+) - W(T_2^-) < \varepsilon.
\end{aligned}
$$

By the Cauchy criterion, $\lim_{t\to\infty} \frac{\alpha}{2\lambda_N}(1-\bar{\sigma})\int_0^t \|\omega(s)\|^2 ds$ exists, and hence $\lim_{t\to\infty}\int_0^t \|\omega(s)\|^2 ds$ exists. For any $t \in [t_l, t_{l+1})$, $l \in \mathbb{N}$, one has $\dot{\omega}(t) = (A + \lambda I_m)\omega(t)$. Thus, $\int_0^t \|\omega(s)\|^2 ds$ is twice differentiable on each interval $[t_l, t_{l+1})$, $l \in \mathbb{N}$. Note that the boundedness of $\omega(t)$ implies that $\dot{\omega}(t)$ is bounded over $[t_l, t_{l+1})$, $l \in \mathbb{N}$. Therefore, there exists a positive constant M, such that

$$
\sup_{t\in[t_l,t_{l+1}),l\in\mathbb{N}} \left|\omega^{\mathrm{T}}(t)\dot{\omega}(t)\right| \leq M.
$$

From Step 2, it is seen that $t_{l+1} - t_l \geq \tilde{\tau} > 0$, for $l \in \mathbb{N}$. Then, by the generalized Barbalat's lemma ([32, Lemma 1]), $\lim_{t\to\infty}\|\omega(t)\|^2 = 0$, which means that $\lim_{t\to\infty}\omega_i(t) = 0$, $i \in \mathcal{N}$. Furthermore, by the event-triggering mechanism (8), $\lim_{t\to\infty}\varsigma_i(t) = 0$, $i \in \mathcal{N}$. Hence, $\lim_{t\to\infty}(z_i(t) - z_j(t)) = 0$, $i, j \in \mathcal{N}$. Recalling the exponential transformations, it is concluded that $\lim_{t\to\infty}\hat{q}_i(t) = 0$, $\lim_{t\to\infty} e_i(t) = 0$, and $\lim_{t\to\infty}(\eta_i(t) - \eta_j(t)) = 0$ exponentially, for all $i, j \in \mathcal{N}$, with the rate of $\mathrm{e}^{-\lambda t}$.

The proof is thus completed.

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