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Finite-Time Consensus of Switched Multiagent Systems

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***Abstract*—This paper focuses on the finite-time consensus (FTC) problem of switched multiagent system (MAS) which is composed of continuous-time and discrete-time subsystems. Different from the existing results, each agent of this system is controlled by switching control method. To achieve consensus in finite time for the switched MAS, two types of consensus protocols (the FTC protocol and the fixed-time consensus ($\mathbf{F}_d$TC) protocol) are proposed. By using algebraic graph theory, Lyapunov theory and matrix theory, it is proved that the FTC problem in strongly connected network and leader-following network can be solved, respectively. When the initial states of agents are not available, the $\mathbf{F}_d$TC protocol is applied to solve the FTC problem. Simulations are provided to illustrate the effectiveness of our theoretical results.**

***Index Terms*—Consensus, finite-time, fixed-time, switched multiagent systems (MASs).**

I. Introduction

OVER the past few decades, various control problems for different systems have been investigated, for instance, neutral-type neural networks [1], [2], nonlinear systems [3]–[6], Markovian jump systems [7]. In recent years, the distributed coordination control of multiagent systems (MASs) has attracted much attention in control field. This is mainly due to its superiority in engineering field, such as the flexibility of controller design, strong robustness, and so on. Take large scale systems as an example, centralized controller is not easy to achieve the goal because of the system's complexity. However, large scale systems can be divided into multiple subsystems, then distributed control is applied for the system. For solving different problems motivated by various objectives, lots of research results have been provided, for instance, consensus problem [8], [9], topology selection [10], controllability [11]–[14], optimal control [15], event-time driven control [16], etc.

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Consensus as a basic problem, aims to design appropriate protocol to guarantee agents to achieve agreement on certain quantity. From the viewpoint of dynamical behaviors of agents, the research results of MASs consist of single-integrator dynamics [17], double-integrator dynamics [18]–[21], nonlinear dynamics [22], [23], etc. On the analysis of consensus problems, the convergence rate plays a significant role, which reflects the performance of consensus protocol. In order to accelerate the convergence speed, researchers established and proposed some methods [24]. However, these conclusions focus on asymptotic convergence, namely, agents cannot achieve consensus in finite time. Actually, requiring an arbitrary long time to reach consensus is often unacceptable in some practical situations. Therefore, some researchers were attracted to explore the finite-time consensus (FTC) problem. For continuous-time MASs, a variety of FTC protocols have been presented [25]–[28]. In [29], two protocols were established to solve the FTC problem under time-invariant undirected topology. By employing finite-time semistability theory, the finite-time rendezvous problem was developed in [30]. Whereas the protocols mentioned in [29] and [30] involve discontinuous dynamics, which may lead to a variety of negative effects [25]. Therefore, continuous-time finite-time protocols were presented. Wang and Xiao [25] proposed a continuous-time protocol to solve the FTC problem under both the bidirectional and unidirectional interaction cases. The FTC problem of MASs with respect to a monotonic function and heterogeneous MASs were investigated in [31] and [32], respectively. In [33], FTC problem of second-order MASs without velocity measurements was investigated under undirected connected graph. Liu *et al.* [34] proved that FTC of MASs with a switching protocol can be achieved under a strongly connected and detail-balanced topology. It is noteworthy that the estimated bound of convergence time under the aforementioned FTC protocols is associate with initial states of agents, which may limit some applications. The fixed-time consensus (F_dTC) protocol was presented so as to deal with this problem. For solving the F_dTC problem, Parsegov *et al.* [35] proposed an F_dTC protocol and obtained that the estimated bound of settling time is independent of the initial states of agents. Leader–follower F_dTC of MAS was investigated under undirected connected network in [36]. Fu and Wang [37] further considered the tracking problem for second-order MASs with bounded input uncertainties and proved that the fixed-time tracking problem can be solved under interaction network among the followers is undirected connected. In addition, there exist some other research topics concerning finite-time problems, such as finite-time formation control [27], finite-time containment control [28], and so on. For discrete-time MASs, the finite-time distributed averaging algorithm with fully-connected topology was first briefly discussed in [38]. By using a matrix factorization approach, the finite-time average consensus was solved under distance regular graph in [39].

A switched system is composed of a finite number of subsystems, and a switching law managing the switching among these subsystems [40]–[43]. For MASs, when switching behaviors occur in the topologies, it can be found that the system matrix changes with topology switches [17], [44]. Therefore, the MAS with switching topologies is regarded as a kind of switched system which consists of only continuous-time or discrete-time subsystems. Zheng and Wang [45] considered the switched MAS consisting of continuous-time and discrete-time subsystems, and investigated the consensus problem for different topologies. Since realistic communication among agents may change, the consensus problem of the switched MAS under random networks was also solved in [46]. Moreover, containment control of such switched MAS was considered in [47]. In reality, such switched MAS exists widely in applications. For instance, an MAS is controlled either by a physically implemented regulator or by a digitally implemented one with a switching rule between them synchronously, i.e., the switched MAS is composed of continuous-time and discrete-time subsystems. However, so far, results concerning FTC were concerned with the MAS consisting of only continuous-time subsystem or discrete-time subsystem.

This paper aims at designing appropriate consensus protocols to solve the FTC problem of the switched MAS proposed in [45]. Zheng and Wang [45] studied the asymptotic consensus problem of switched MAS. Different from [45], we mainly study the FTC problem, namely, the consensus of agents can be achieved in finite time. The main problem we need to solve is how to design an effective consensus protocol for the switched MAS to solve the FTC problem. Difficulty comes partly from how to cope with the different dynamical behaviors and switching behaviors. It is difficult to apply the idea of FTC of continuous-time MASs (discrete-time MASs) to the present case. Motivated by Zheng and Wang [45], two types of distributed consensus protocols are designed for the switched MAS in this paper. The main contribution of this paper includes the following.

1) An FTC protocol is designed for the switched MAS. By using this consensus protocol, we show that if the sum of time intervals T_0, over which the continuous-time subsystem is activated, is larger than a finite constant, the FTC problem can be solved under strongly connected network.
2) When the initial states of agents are not available, we propose an F_dTC protocol to solve the F_dTC problem. We obtain that the sum of time intervals T_0 is uncorrelated with the initial states of agents.
3) By using these two protocols, the FTC problem and the F_dTC problem under leader-following network are also investigated, respectively.

Notation: Let R be the set of real numbers, R^n denotes the n-dimensional real vector space and I_n be identity matrix. $\mathbf{1} = [1, \ldots, 1]^T \in R^n$, $\omega = [\omega_1, \omega_2, \ldots, \omega_n]^T \in R^n$ and $\omega > 0$, $\mathcal{I}_n = \{1, 2, \ldots, n\}$, and $\bar{d} = \max_{i \in \mathcal{I}_n}\{d_{ii}\}$ denotes maximum degree of agent. A^T denotes the transpose of matrix A or vector A. $R^{n\times n}$ and $C^{n\times n}$ denote $n \times n$ real matrix and $n \times n$ complex matrix, respectively. $B = [b_{ij}] \in R^{n\times n}$, $B \geq 0$ if all $b_{ij} \geq 0$. We say that B is a non-negative matrix if $B \geq 0$. Both $W \in R^{n\times n}$ and $\bar{W} \in R^{(n-1)\times(n-1)}$ are the diagonal matrices with $w_i > 0$ as the (i, i) entry. $\bar{b} = [b_1, b_2, \ldots, b_{n-1}]^T = [a_{1n}, a_{2n}, \ldots, a_{(n-1)n}]^T \geq 0$ where there is at least one $b_i > 0$, and $\tilde{b} = \mathrm{diag}(\bar{b}) = \mathrm{diag}\{b_1, b_2, \ldots, b_{n-1}\}$. $J\{a_1, \ldots, a_n\}$ represents Jordan matrix with diagonal entry a_i and $\mathrm{sign}(\cdot)$ is the sign function. $|\cdot|$ represents the absolute value. $\lfloor x \rfloor$ is the largest integer not greater than x. $\bar{x}_k(t) = \max_{i\in V} x_i(t)$, $\underline{x}_k(t) = \min_{i\in V} x_i(t)$ and $\alpha_0 = \max_{ij} \alpha_{ij}$. $\mathrm{span}(\mathbf{1}) = \{\xi \in R^n : \xi = r\mathbf{1}, r \in R\}$.

II. Preliminaries and Problem Formulation

A. Preliminaries

The communication relationship between agents is described by graph $\mathcal{G}(A) = (V, E, A)$ with vertex set $V = \{\nu_1, \nu_2, \ldots, \nu_n\}$, edge set $E = \{e_{ij}\} \subseteq V \times V$ and non-negative matrix $A = [a_{ij}]_{n\times n}$. If $(\nu_j, \nu_i) \in E$, agents i and j are adjacent and $a_{ij} > 0$. The degree matrix $D = [d_{ij}]_{n\times n}$ is a diagonal matrix with $d_{ii} = \sum_{j\in N_i} a_{ij}$ and the Laplacian matrix of the graph is defined as $L = [l_{ij}]_{n\times n} = D - A$. For Laplacian matrix L, $\lambda_i(L)$ denotes the ith eigenvalue of L and $0 = \lambda_1(L) \leq \lambda_2(L) \leq \ldots \leq \lambda_n(L)$. The directed graph $\mathcal{G}(A)$ is said to satisfy the detail-balanced condition if there exist some scalars $\omega_i > 0$ such that $\omega_i a_{ij} = \omega_j a_{ji}$ for all $i, j \in \mathcal{I}_n$. For more details, please refer to [32]. For MASs, agent n is a leader, i.e., $a_{n1} = a_{n2} = \cdots = a_{n(n-1)} = 0$ and $\bar{a} = [a_{1n}, a_{2n}, \ldots, a_{(n-1)n}]^T \geq 0$. $L = \begin{bmatrix} L_{FF} & -\bar{b} \\ 0_{1\times(n-1)} & 0_{1\times 1} \end{bmatrix}$ where $L_{FF} \in R^{(n-1)\times(n-1)}$, and $\bar{b} \in R^{(n-1)\times 1}$. Followers' interaction subgraph is $\mathcal{G}(\bar{A})$.

B. Problem Formulation

We consider an MAS which consists of n agents, and each agent is controlled by switching control method (continuous-time control and sampled-data control). Thus, the agent takes the switched dynamics, it switches between continuous-time dynamics and discrete-time dynamics.

The continuous-time dynamics is

$$\dot{x}_i(t) = u_i(t), \quad i \in \mathcal{I}_n \tag{1a}$$

and discrete-time dynamics is

$$x_i(t+1) = x_i(t) + hu_i(t), \quad i \in \mathcal{I}_n \tag{1b}$$

where $x_i(t) \in R$ and $u_i(t) \in R$ are the state and control input of agent i, respectively. Sampling period $h > 0$ and initial value $x_0 = [x_1(0), \ldots, x_n(0)]^T$. Assume that the synchronous switch is applied for each agent.

Throughout this paper, there is a sequence of time instants $0 \leq t_1 < \bar{t}_1 < t_2 < \bar{t}_2 < \cdots < \bar{t}_{k-1} < t_k < \bar{t}_k < \cdots$ for system (1) that satisfies the following assumption.

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Assumption 1:

1) Continuous-time subsystem (1a) is activated when $t \in (t_k, \bar{t}_k]$.
2) Discrete-time subsystem (1b) is activated when $t \in (\bar{t}_{k-1}, t_k]$.
3) $\bar{t}_k - t_k \geq \tau^*$ where τ^* is constant.

Note that Assumption 1 guarantees system (1) to be always composed of only the continuous-time subsystem (or continuous-time and discrete-time subsystems). There does not exist the case that system (1) is composed of only the discrete-time subsystem. Moreover, system (1) can be viewed as a system consisting of continuous-time subsystem and discrete-time subsystem at different time interval.

Definition 1 [32]: System (1) reaches FTC if for any initial state $x_0 \in R^n$ and $i, j \in \mathcal{I}_n$, there is a settling time T such that $\lim_{t\to T}(x_j(t) - x_i(t)) = 0$ and $x_j(t) = x_i(t)$, $\forall t \geq T$.

Lemma 1 [48]: A complex matrix $B \in C^{n\times n}$ is nonsingular if B is irreducibly diagonally dominant.

Lemma 2 [25]: Let $\pi_1, \pi_2, \ldots, \pi_n \geq 0$ and $0 < \kappa \leq 1$, then

$$\sum_{i=1}^{n} \pi_i^{\kappa} \geq \left(\sum_{i=1}^{n} \pi_i\right)^{\kappa}.$$

Lemma 3 (Comparison Theorem [49]): Let $U(t, u)$ be continuous on an open (t, u)-set E and $u = u^0(t)$ the maximal solution of $(du/dt) = U(t, u)$, $u(t_0) = u_0$. Let $v(t)$ be a continuous function on $[t_0, t_0 + a]$ satisfying the conditions $v(t_0) \leq u_0$, $(t, v(t)) \in E$, and $v(t)$ has a right derivative $D_R(v(t))$ on $t_0 \leq t < t_0 + a$ such that $D_R(v(t)) \leq U(t, v(t))$. Then, on a common interval of existence of $u^0(t)$ and $v(t)$, $v(t) \leq u^0(t)$.

Lemma 4: Suppose that graph $\mathcal{G}(A)$ is strongly connected and satisfies the detailed balance condition, and $0 < h < (1/\bar{d})$. Then, for all $i \in \mathcal{I}_n$, it holds that $\lambda_i \leq 0$ and $\lambda_i = 0$ is algebraically simple, where λ_i is the ith eigenvalue of matrix $(I - hL)^T W(I - hL) - W$.

Proof: From the definition of L and $0 < h < (1/\bar{d})$, we have that

$$I - hL = I - h(D - A) = (I - hD) + hA$$

is a non-negative matrix and its row sums are 1. Hence, matrix $(I-hL)^T W(I-hL)$ is also non-negative. Because graph $\mathcal{G}(A)$ satisfies the detailed balance condition, there exist some scalars $\omega_i > 0$ such that $\omega_i a_{ij} = \omega_j a_{ji}$ for all $i, j \in \mathcal{I}_n$. Thus, we get $WA = A^T W$ and $WL = L^T W$. It follows from $WL = L^T W$ and $L\mathbf{1} = 0$ that $(I - hL)^T W(I - hL)\mathbf{1} = W(I - hL)\mathbf{1} = \omega$. Consequently, there must exist a zero eigenvalue corresponding to a right eigenvector $\mathbf{1}$ for matrix $(I - hL)^T W(I - hL) - W$.

Let $(I - hL)^T W(I - hL) = \tilde{W}$, then

$$\tilde{W} - W = \begin{bmatrix} \tilde{\omega}_{11} - \omega_1 & \ldots & \tilde{\omega}_{1n} \\ \vdots & \ddots & \vdots \\ \tilde{\omega}_{n1} & \ldots & \tilde{\omega}_{nn} - \omega_n \end{bmatrix}$$

it is obvious that $\tilde{\omega}_{ii} - \omega_i \leq 0$ and $\sum_{j=1}^{n} \tilde{\omega}_{ij} - \omega_i = 0$. By Gersgorin Disk theorem [48], all the eigenvalues of $\tilde{W} - W$ are located in the following region:

$$\bigcup_{i=1}^{n} \left\{ z \in C : |z - (\tilde{\omega}_{ii} - \omega_i)| \leq \sum_{\substack{j=1 \\ i\neq j}}^{n} \left|\tilde{\omega}_{ij}\right| = |\tilde{\omega}_{ii} - \omega_i| \right\}$$

it is easy to see that all the eigenvalues $\lambda_i \leq 0$.

Next, we prove that zero eigenvalue is algebraically simple. Let $\lambda_1(L), \ldots, \lambda_n(L)$ denote the eigenvalues of L, we have $P^{-1}LP = J\{\lambda_1(L), \ldots, \lambda_n(L)\}$. Then, we have $P^{-1}(-2hL + h^2L^2)P = J\{-2h\lambda_1(L) + h^2\lambda_1^2(L), \ldots, -2h\lambda_n(L) + h^2\lambda_n^2(L)\}$. From Gersgorin Disk theorem, it is easy to obtain $0 < h < (1/\bar{d}) \leq (2/\lambda_n(L))$. Since $\mathcal{G}(A)$ is strongly connected, we obtain $\mathrm{rank}(-2hL + h^2L^2) = n - 1$. By Sylvester inequality [48], $\mathrm{rank}((I - hL)^T W(I - hL) - W) = \mathrm{rank}(W(-2hL + h^2L^2)) = n - 1$. Hence, $\lambda_i < 0$, $i = 2, \ldots, n$ and $\lambda_1 = 0$ is algebraically simple. ■

Lemma 5: Suppose that followers' interaction subgraph $\mathcal{G}(\bar{A})$ is strongly connected and satisfies the detailed balance condition, and $0 < h < (1/\bar{d} + b_i)$, $i \in \mathcal{I}_{n-1}$. Then, for all $i \in \mathcal{I}_{n-1}$, it holds that $\bar{\lambda}_i < 0$ where $\bar{\lambda}_i$ is the ith eigenvalue of matrix $(I - hL_{FF})^T \bar{W}(I - hL_{FF}) - \bar{W}$.

Proof: Since $0 < h < (1/\bar{d} + b_i)$ and matrix $\bar{A}$ is a non-negative matrix, we have that

$$I - hL_{FF} = I - h\left(\left(\bar{D} + \tilde{b}\right) - \bar{A}\right) = \left(I - h\left(\bar{D} + \tilde{b}\right)\right) + h\bar{A}$$

is a non-negative matrix. Similar to Lemma 4, we have $\bar{W}\bar{A} = \bar{A}^T\bar{W}$ and $\bar{W}L_{FF} = L_{FF}^T\bar{W}$. Due to $L_{FF}\mathbf{1} = \bar{b}$, we have $(I - hL_{FF})^T\bar{W}(I - hL_{FF})\mathbf{1} = \bar{W}(\mathbf{1} - 2h\bar{b} + h^2\bar{l})$, $\bar{l} = [l_{11}b_1 + l_{12}b_2 + \cdots + l_{1n-1}b_{n-1}, \ldots, l_{n-11}b_1 + l_{n-22}b_2 + \cdots + l_{n-1n-1}b_{n-1}]^T$ and $l_{ji} \leq 0$, $j = 1, \ldots, n-1$, $j \neq i$. Owing to $0 < h < [1/(\bar{d} + b_i)]$, we have $(I - hL_{FF})^T\bar{W}(I - hL_{FF})\mathbf{1} \leq \bar{W}\mathbf{1}$. By Gersgorin Disk theorem [48], all eigenvalues of $(I - hL_{FF})^T\bar{W}(I - hL_{FF}) - \bar{W}$ are nonpositive.

Because graph $\mathcal{G}(\bar{A})$ is strongly connected and there is at least one agent connected to the leader, there is at least one l_{ii} such that $|l_{ii}| > \sum_{j\neq i}|l_{ij}|$, $i, j \in \mathcal{I}_{n-1}$. By Lemma 1, it follows that L_{FF} is nonsingular, i.e., there is no zero eigenvalue for matrix L_{FF}. By Gersgorin Disk theorem, we have $0 < h < [1/(\bar{d} + b_i)] \leq (2/\lambda_{n-1}(L_{FF}))$. According to Lemma 4, we can get $\mathrm{rank}(-2hL_{FF} + h^2L_{FF}^2) = n - 1$. Therefore, $(I - hL_{FF})^T\bar{W}(I - hL_{FF}) - \bar{W} = \bar{W}(-2hL_{FF} + h^2L_{FF}^2)$ is full rank. Consequently, there is no zero eigenvalue and $\bar{\lambda}_i < 0$, $i = 1, \ldots, n - 1$. ■

The asymptotic consensus problem for switched MAS has been investigated in [45]. In this paper, two types of consensus protocols which guarantee the switched MAS to achieve consensus in finite time are designed.

III. FTC Protocol for the Switched MAS

This section proposed FTC protocol for system (1) to solve FTC problem. For continuous-time and discrete-time subsystems, we design different consensus protocols, respectively. It is presented as follows:

$$u_i(t) = \begin{cases} \sum_{j=1}^{n} a_{ij}\mathrm{sig}\left(x_j(t) - x_i(t)\right)^{\alpha_{ij}}, t \in \left(t_k, \bar{t}_k\right] \\ \sum_{j=1}^{n} a_{ij}\left(x_j(t) - x_i(t)\right), t \in \left(\bar{t}_{k-1}, t_k\right] \end{cases} \quad (2)$$

where $0 < \alpha_{ij} < 1$ and $\mathrm{sig}(\mu)^{\alpha_{ij}} = \mathrm{sign}(\mu)|\mu|^{\alpha_{ij}}$. Note that $\mathrm{sig}(\mu)^{\alpha_{ij}}$ is a continuous function. When $\alpha_{ij} = 1$, it will become the consensus protocol proposed in [45] which solves the asymptotical consensus problem. In this paper, we assume that $\alpha_{ij} = \alpha_{ji}$.

Next, we will show that system (1) with protocol (2) achieves FTC under certain mathematical conditions. To solve this problem, we introduce $U = \{\xi \in R^n : \omega^T\xi = 0$ and $\|\xi\| = 1\}$. Since $\omega > 0$, it is clear that $\xi \neq 0$ and $\xi \notin \mathrm{span}(\mathbf{1})$. L is Laplacian matrix of undirected connected graph, then $\xi^T L\xi > 0$.

Theorem 1: Suppose that the interaction network $\mathscr{G}(A)$ is strongly connected and satisfies the detailed balance condition, and $0 < h < (1/\bar{d})$. Then, system (1) with protocol (2) reaches FTC if there exist finite k and $\bar{t}^* \in [t_k, \bar{t}_k)$ such that

$$T_0 \geq \frac{4}{(1-\alpha_0)K} V^{\frac{1-\alpha_0}{2}}(\gamma(0)) \tag{3}$$

where

$$T_0 = \bar{t}^* - t_k + \sum_{i=1}^{k-1}\left(\bar{t}_i - t_i\right)$$

$$V(\gamma(0)) = \frac{1}{2}\sum_{i=1}^{n}\omega_i\left(x_i(0) - \frac{1}{\sum_{j=1}^n \omega_j}\sum_{j=1}^{n}\omega_j x_j(0)\right)^2$$

and $K = (K_1K_2)^{[(1+\alpha_0)/2]}$ with

$$K_1 = \frac{1}{\sum_{i,j=1}^n \left(\omega_i a_{ij}\right)^{\frac{2}{\alpha_0+1}}} \min_{\substack{i,j\in\mathcal{I}_n \\ a_{ij}\neq 0}} \left(\omega_i a_{ij}\right)^{\frac{2}{\alpha_0+1}} \times \left(\bar{x}_k(0) - \underline{x}_k(0)\right)^{2\left(\frac{\alpha_{ij}+1}{\alpha_0+1}-1\right)} > 0$$

and

$$K_2 = \frac{4c_0}{\omega_{\max}},\ c_0 = \min_{\xi\in U} \xi^T L(B)\xi > 0$$
$$U = \left\{\xi \in R^n : \omega^T\xi = 0 \text{ and } \|\xi\| = 1\right\}$$
$$B = \left[\left(\omega_i a_{ij}\right)^{\frac{2}{\alpha_0+1}}\right] \in R^{n\times n}.$$

Proof: First, we show that there exist finite k satisfying (3). From Assumption 1, we know that $T_0 \geq (k-1)\tau^*$. Then for any finite $k \geq \lfloor (4V^{[(1-\alpha_0)/2]}(0)/(1-\alpha_0)K\tau^*)\rfloor + 1$, (3) holds.

The interaction network $\mathscr{G}(A)$ is strongly connected and satisfies the detailed balance condition, i.e., there exists a positive column vector $\omega = [\omega_1, \omega_2, \ldots, \omega_n]^T$ such that $\omega_i a_{ij} = \omega_j a_{ji}$ for all $i, j \in \mathcal{I}_n$.

Let $\upsilon(t) = (1/\sum_{i=1}^n \omega_i)\sum_{i=1}^n \omega_i x_i(t)$. Due to $\alpha_{ij} = \alpha_{ji}$, $\omega_i a_{ij} = \omega_j a_{ji}$ for all $i, j \in \mathcal{I}_n$, we have

$$\begin{aligned}\dot{\upsilon}(t) &= \frac{1}{\sum_{i=1}^n \omega_i}\sum_{i=1}^{n}\omega_i\dot{x}_i(t)\\ &= \frac{1}{\sum_{i=1}^n \omega_i}\sum_{i=1}^{n}\omega_i\sum_{j=1}^{n}a_{ij}\mathrm{sig}\left(x_j(t) - x_i(t)\right)^{\alpha_{ij}}\\ &= 0\end{aligned}$$

and

$$\begin{aligned}\upsilon(t+1) &= \frac{1}{\sum_{i=1}^n \omega_i}\sum_{i=1}^{n}\omega_i x_i(t+1)\\ &= \frac{1}{\sum_{i=1}^n \omega_i}\sum_{i=1}^{n}\omega_i\left(x_i(t) + h\sum_{j=1}^{n}a_{ij}\left(x_j(t) - x_i(t)\right)\right)\\ &= \frac{1}{\sum_{i=1}^n \omega_i}\sum_{i=1}^{n}\omega_i x_i(t)\\ &\quad + h\frac{1}{\sum_{i=1}^n \omega_i}\sum_{i=1}^{n}\omega_i\sum_{j=1}^{n}a_{ij}\left(x_j(t) - x_i(t)\right)\\ &= \upsilon(t).\end{aligned}$$

Therefore, $\upsilon(t)$ is time-invariant. Let $\gamma_i(t) = x_i(t) - \upsilon(t)$, we have $\omega^T\gamma(t) = 0$ and $\gamma(t+1) = (I - hL)x(t) - \mathbf{1}\upsilon(t) = (I - hL)(x(t) - \mathbf{1}\upsilon(t)) = (I - hL)\gamma(t)$. Take Lyapunov function $V(\gamma(t)) = (1/2)\sum_{i=1}^n \omega_i\gamma_i^2(t) = (1/2)\gamma^T(t)W\gamma(t)$ for continuous-time subsystem (1a) and discrete-time subsystem (1b).

When $t \in (\bar{t}_{k-1}, t_k]$, discrete-time subsystem (1b) is activated. Thus, we have

$$\begin{aligned}&V(\gamma(t+1)) - V(\gamma(t))\\ &\quad= \frac{1}{2}\left(\gamma^T(t+1)W\gamma(t+1) - \gamma^T(t)W\gamma(t)\right)\\ &\quad= \frac{1}{2}\gamma^T(t)\left((I-hL)^T W(I-hL) - W\right)\gamma(t)\\ &\quad\leq \frac{1}{2}\lambda_{\max}\left((I-hL)^T W(I-hL) - W\right)\gamma^T(t)\gamma(t)\\ &\quad\leq 0\end{aligned} \tag{4}$$

where the last inequality follows from Lemma 4.

When $t \in (t_k, \bar{t}_k]$, continuous-time subsystem (1a) is activated. Similar to the proof technique of Wang and Xiao [25], we have

$$\begin{aligned}\dot{V}(\gamma(t)) &= \sum_{i=1}^{n}\omega_i\gamma_i(t)\sum_{j=1}^{n}a_{ij}\mathrm{sig}\left(\gamma_j(t) - \gamma_i(t)\right)^{\alpha_{ij}}\\ &= \frac{1}{2}\sum_{i,j=1}^{n}\Big(\omega_i a_{ij}\gamma_i(t)\mathrm{sig}\left(\gamma_j(t) - \gamma_i(t)\right)^{\alpha_{ij}}\\ &\qquad + \omega_j a_{ji}\gamma_j(t)\mathrm{sig}(\gamma_i(t) - \gamma_j(t))^{\alpha_{ji}}\Big)\\ &= -\frac{1}{2}\sum_{i,j=1}^{n}\omega_i a_{ij}\left|\gamma_j(t) - \gamma_i(t)\right|^{\alpha_{ij}+1}\\ &= -\frac{1}{2}\sum_{i,j=1}^{n}\left(\left(\omega_i a_{ij}\right)^{\frac{2}{\alpha_0+1}}\left|\gamma_j(t) - \gamma_i(t)\right|^{\frac{2(\alpha_{ij}+1)}{\alpha_0+1}}\right)^{\frac{\alpha_0+1}{2}}\\ &\leq -\frac{1}{2}\left(\frac{\sum_{i,j=1}^n\left(\omega_i a_{ij}\right)^{\frac{2}{\alpha_0+1}}\left|\gamma_j(t) - \gamma_i(t)\right|^{\frac{2(\alpha_{ij}+1)}{\alpha_0+1}}}{2\gamma^T(t)L(B)\gamma(t)}\right.\\ &\qquad\left.\times V(\gamma(t))\frac{2\gamma^T(t)L(B)\gamma(t)}{V(\gamma(t))}\right)^{\frac{\alpha_0+1}{2}}\end{aligned} \tag{5}$$

where the last inequality follows from Lemma 2.

We know that $\bar{x}_k(t) = \max_{i\in V} x_i(t)$ and $\underline{x}_k(t) = \min_{i\in V} x_i(t)$. When $t \in (\bar{t}_{k-1}, t_k]$, by Lemma 4, matrix $I - hL = [\bar{l}_{ij}]_{n\times n}$ is a row stochastic matrix, i.e., $\sum_{j=1}^n \bar{l}_{ij} = 1$. Obviously, $\bar{x}_k(t+1) = \max_{i\in V} \sum_{j=1}^n \bar{l}_{ij}x_j(t) \le \max_{i\in V} \sum_{j=1}^n \bar{l}_{ij}\bar{x}_j(t) = \bar{x}_k(t)$, i.e., $\bar{x}_k(t+1) \le \bar{x}_k(t)$. We show that $\underline{x}_k(t+1) \ge \underline{x}_k(t)$ in precisely the same way. When $t \in (t_k, \bar{t}_k]$, it is obvious that $\dot{\bar{x}}_k(t) \le 0$ and $\dot{\underline{x}}_k(t) \ge 0$. Thus, $\bar{x}_k(t) - \underline{x}_k(t)$ is a nonincreasing for all $t > 0$. Hence, $|\gamma_j(t) - \gamma_i(t)| \le \bar{x}_k(t) - \underline{x}_k(t) \le \bar{x}_k(0) - \underline{x}_k(0)$. From [25], we have

$$\frac{\sum_{i,j=1}^n \left(\omega_i a_{ij}\right)^{\frac{2}{\alpha_0+1}} \left|\gamma_j(t) - \gamma_i(t)\right|^{\frac{2(\alpha_{ij}+1)}{\alpha_0+1}}}{2\gamma^T(t)L(B)\gamma(t)} \ge K_1 > 0$$

where K_1 is defined as in (3).

Since $U = \{\xi : \omega^T\xi = 0 \text{ and } \|\xi\| = 1\}$ and $\omega > 0$, we have $\xi \ne 0$ and $\xi \notin \text{span}\{\mathbf{1}\}$. By properties of Laplacian matrix under undirected connected graph, we have $\xi^T L(B)\xi > 0$. It follows from $\omega^T\gamma(t) = 0$ that

$$\frac{2\gamma^T(t)L(B)\gamma(t)}{V(\gamma(t))} \ge \frac{4}{\omega_{\max}} \frac{\gamma^T(t)}{\sqrt{\gamma^T(t)\gamma(t)}} L(B) \frac{\gamma(t)}{\sqrt{\gamma^T(t)\gamma(t)}} \ge K_2$$

where $L(B)$ and K_2 are defined as in (3).

Hence, it is easy to obtain $\dot{V}(\gamma(t)) \le -(1/2)KV^{[(\alpha_0+1)/2]}(\gamma(t))$ for $t \in (t_k, \bar{t}_k]$. By the comparison theorem

$$V^{\frac{1-\alpha_0}{2}}(\gamma(t)) \le -\frac{1-\alpha_0}{4}K(t - t_k) + V^{\frac{1-\alpha_0}{2}}(\gamma(t_k)). \quad (6)$$

From (4) and (5), it is easy to know that

$$\begin{aligned} V(\gamma(0)) &\ge V(\gamma(t_1)) \ge V\big(\gamma(\bar{t}_1)\big) \ge V(\gamma(t_2)) \\ &\ge V\big(\gamma(\bar{t}_2)\big) \ge \cdots \ge V(\gamma(t_k)) \ge V\big(\gamma(\bar{t}_k)\big) \ge \cdots \end{aligned} \quad (7)$$

Next, we show that there exists $\bar{t}^* \in (t_k, \bar{t}_k]$ such that $-[(1-\alpha_0)/4]K(\bar{t}^* - t_k) + V^{[(1-\alpha_0)/2]}(\gamma(t_k)) \le 0$. Based on (6), (7), and $1 - \alpha_0 > 0$

$$\begin{aligned} &-\frac{1-\alpha_0}{4}K\big(\bar{t}^* - t_k\big) + V^{\frac{1-\alpha_0}{2}}(\gamma(t_k)) \\ &\quad\le -\frac{1-\alpha_0}{4}K\big(\bar{t}^* - t_k\big) + V^{\frac{1-\alpha_0}{2}}\big(\gamma\big(\bar{t}_{k-1}\big)\big) \\ &\quad\le -\frac{1-\alpha_0}{4}K\big(\bar{t}^* - t_k + \bar{t}_{k-1} - t_{k-1}\big) + V^{\frac{1-\alpha_0}{2}}(\gamma(t_{k-1})) \\ &\quad\;\;\vdots \\ &\quad\le -\frac{1-\alpha_0}{4}K\big(\bar{t}^* - t_k + \bar{t}_{k-1} - t_{k-1} + \cdots + \bar{t}_1 - t_1\big) \\ &\qquad + V^{\frac{1-\alpha_0}{2}}(\gamma(0)). \end{aligned}$$

Hence, it follows from (3) that there exists $\bar{t}^* \in (t_k, \bar{t}_k]$ such that $-[(1-\alpha_0)/4]K(\bar{t}^* - t_k) + V^{[(1-\alpha_0)/2]}(\gamma(t_k)) \le 0$. By virtue of $V(\gamma(t)) \ge 0$, $\dot{V}(\gamma(t)) \le 0$ and (6), we get $V(\gamma(t)) = 0$ for $t \ge \bar{t}^*$. Thus system (1) reaches consensus in finite time. ■

Remark 1: If the interaction network is a undirected connected network, the analogous theoretical result can be established for system (1) with protocol (2) to solve the finite-time average consensus problem. It is clear that the convergence speed of system (1) is influenced by $\lambda_2(L(B))$ where $B = [(a_{ij})^{[2/(\alpha_0+1)]}] \in R^{n\times n}$. Furthermore, the larger algebraic connectivity of $\mathscr{G}(B)$ is, the shorter the convergence time is.

Theorem 2: Suppose that system (1) has a leader (labeled as n) and $n-1$ followers (labeled as $1, \ldots, n-1$), and the interaction network $\mathscr{G}(\bar{A})$ among the followers is strongly connected and satisfies the detailed balance condition, and $0 < h < [1/(\bar{d} + b_i)]$, $i \in \mathcal{I}_{n-1}$. Then, system (1) with protocol (2) reaches FTC if there exist finite k and $\bar{t}^* \in (t_k, \bar{t}_k]$ such that

$$T_0 \ge \frac{4}{(1-\alpha_0)\bar{K}} \bar{V}^{\frac{1-\alpha_0}{2}}(\gamma(0)) \quad (8)$$

where

$$T_0 = \bar{t}^* - t_k + \sum_{i=1}^{k-1}\big(\bar{t}_i - t_i\big), \;\; \bar{V}(\gamma(0)) = \frac{1}{2}\sum_{i=1}^{n-1} \omega_i(x_i(0) - x_n)^2$$

and $\bar{K} = (\bar{K}_1\bar{K}_2)^{[(1+\alpha_0)/2]}$ with

$$\begin{aligned} \bar{K}_1 = &\frac{1}{\sum_{i,j=1}^n \left(\omega_i a_{ij}\right)^{\frac{2}{\alpha_0+1}} + 2\sum_{i=1}^n b_i^{\frac{2}{\alpha_0+1}}} \min_{\substack{i,j\in\mathcal{I}_n \\ a_{ij}\ne 0}} \left(\omega_i a_{ij}\right)^{\frac{2}{\alpha_0+1}} \\ &\times \left(\bar{x}_k(0) - \underline{x}_k(0)\right)^{2\left(\frac{\alpha_{ij}+1}{\alpha_0+1} - 1\right)} > 0 \end{aligned}$$

and

$$\bar{K}_2 = \frac{4\lambda_1(L(\bar{B}) + \tilde{b})}{\omega_{\max}}, \;\; \bar{B} = \left[\left(\omega_i a_{ij}\right)^{\frac{2}{\alpha_0+1}}\right] \in R^{(n-1)\times(n-1)}.$$

Proof: The interaction network among the followers is strongly connected and satisfies the detailed balance condition, i.e., there exists a positive column vector $\bar{\omega} = [\omega_1, \omega_2, \ldots, \omega_{n-1}]^T$ such that $\omega_i a_{ij} = \omega_j a_{ji}$ for all $i, j \in \mathcal{I}_{n-1}$. We rewrite protocol (2) as follows:

$$u_i(t) = \begin{cases} \sum_{j=1}^{n-1} a_{ij}\text{sig}\big(x_j(t) - x_i(t)\big)^{\alpha_{ij}} + a_{in}\text{sig}(x_n - x_i)^{\alpha_{in}} \\ t \in \big(t_k, \bar{t}_k\big] \\ \sum_{j=1}^{n-1} a_{ij}\big(x_j(t) - x_i(t)\big) + a_{in}(x_n - x_i) \\ t \in \big(\bar{t}_{k-1}, t_k\big] \end{cases} \quad (9)$$

where

$$a_{in} = \begin{cases} b_i > 0 & \text{if agent } i \text{ is connected to agent } n \\ 0 & \text{otherwise} \end{cases}$$

denotes whether the follower i is connected to the leader n and $i \in \mathcal{I}_{n-1}$.

First, we show that there exist finite k satisfying (8). The proof is similar to that of theorem 1 and it is omitted.

Let $\gamma_i(t) = x_i(t) - x_n$, $i \in \mathcal{I}_{n-1}$, we have $\bar{\gamma}(t+1) = x(t+1) - \mathbf{1}x_n = (I - hL_{FF})x(t) + (h\bar{b} - \mathbf{1})x_n = (I - hL_{FF})\bar{\gamma}(t)$, $i \in \mathcal{I}_{n-1}$, where $L_{FF}\mathbf{1} = \bar{b}$. Take Lyapunov function $\bar{V}(\gamma(t)) = (1/2)\sum_{i=1}^{n-1}\omega_i\gamma_i^2(t) = (1/2)\bar{\gamma}^T(t)\bar{W}\bar{\gamma}(t)$ for continuous-time subsystem (1a) and discrete-time subsystem (1b).

When $t \in (\bar{t}_{k-1}, t_k]$, discrete-time subsystem (1b) is activated. Thus, we have

$$\begin{aligned}
&\bar{V}(\gamma(t+1)) - \bar{V}(\gamma(t)) \\
&\quad= \frac{1}{2}\left(\bar{\gamma}^T(t+1)\bar{W}\bar{\gamma}(t+1) - \bar{\gamma}^T(t)\bar{W}\bar{\gamma}(t)\right) \\
&\quad= \frac{1}{2}\bar{\gamma}^T(t)\left((I-hL_{FF})^T\bar{W}(I-hL_{FF}) - \bar{W}\right)\bar{\gamma}(t) \\
&\quad\le \frac{1}{2}\lambda_{\max}\left((I-hL_{FF})^T\bar{W}(I-hL_{FF}) - \bar{W}\right)\bar{\gamma}^T(t)\bar{\gamma}(t) \\
&\quad\le 0
\end{aligned} \tag{10}$$

where the last inequality follows from Lemma 5.

When $t \in (t_k, \bar{t}_k]$, continuous-time subsystem (1a) is activated. Therefore

$$\begin{aligned}
&\dot{\bar{V}}(\gamma(t)) \\
&\quad= \sum_{i=1}^{n-1} \omega_i\gamma_i(t)\left(\sum_{j=1}^{n-1} a_{ij}\mathrm{sig}\left(\gamma_j(t)-\gamma_i(t)\right)^{\alpha_{ij}} - b_i\mathrm{sig}(\gamma_i(t))^{\alpha_{in}}\right) \\
&\quad= -\frac{1}{2}\sum_{i,j=1}^{n-1} \omega_i a_{ij}\left|\gamma_j(t)-\gamma_i(t)\right|^{\alpha_{ij}+1} - \sum_{i=1}^{n-1}\omega_i b_i|\gamma_i(t)|^{\alpha_{in}+1} \\
&\quad\le -\frac{1}{2}\left(\frac{\Pi_1}{\Pi_2}\frac{\Pi_2}{\bar{V}(\gamma(t))}\bar{V}(\gamma(t))\right)^{\frac{\alpha_0+1}{2}}
\end{aligned} \tag{11}$$

where

$$\frac{\Pi_1}{\Pi_2} = \frac{\sum_{i,j=1}^{n}\left(\omega_i a_{ij}\right)^{\frac{2}{\alpha_0+1}}\left|\gamma_j(t)-\gamma_i(t)\right|^{\frac{2(\alpha_{ij}+1)}{\alpha_0+1}}}{2\bar{\gamma}^T(t)\left(L(\bar{B})+\tilde{b}\right)\bar{\gamma}(t)} + \frac{\sum_{i=1}^{n-1}(\omega_i b_i)^{\frac{2}{\alpha_0+1}}|\gamma_i(t)|^{\frac{2(\alpha_{ij}+1)}{\alpha_0+1}}}{2\bar{\gamma}^T(t)\left(L(\bar{B})+\tilde{b}\right)\bar{\gamma}(t)}.$$

By using the proof method in Theorem 1, it is easy to obtain $(\Pi_1/\Pi_2) \ge \bar{K}_1 > 0$.

From [25], we know that $L(\bar{B}) + \tilde{b}$ is a positive definite matrix and $\lambda_1(L(\bar{B}) + \tilde{b}) > 0$. Thus

$$\begin{aligned}
\frac{\Pi_2}{\bar{V}(\gamma(t))} &= \frac{2\bar{\gamma}^T(t)(L(\bar{B})+\tilde{b})\bar{\gamma}(t)}{\frac{1}{2}\bar{\gamma}^T(t)\bar{W}\bar{\gamma}(t)} \\
&\ge 4\frac{\lambda_1(L(\bar{B})+\tilde{b})}{\omega_{\max}} = \bar{K}_2 > 0.
\end{aligned}$$

Similar to Theorem 1, we get $\bar{V}(\gamma(t)) = 0$ for $t \ge \bar{t}^*$. Therefore, system (1) reaches consensus in finite time. ■

Remark 2: It is noteworthy that the time interval τ^* is not arbitrarily small if we want to guarantee the FTC of system (1). From the proof of Theorems 1 and 2, we can find that system (1) can reach consensus asymptotically if τ^* is arbitrarily small.

Note that T_0 is dependent on the initial states of agents when protocol (2) is applied. The initial states of agents must be available if we want to get T_0. However, there exists the case that the initial states of agents are unavailable in some applications. To calculate T_0 without the information of initial states of agents, we will propose an $\mathrm{F_dTC}$ protocol for the switched MAS to solve this problem in the following part.

IV. F_dTC Protocol for the Switched MAS

In this section, the F_dTC protocol is proposed for system (1). It is presented as follows:

$$u_i(t) = \begin{cases} \sum_{j=1}^{n} a_{ij}\left(x_j(t)-x_i(t)\right)^{\frac{m}{r}} + \sum_{j=1}^{n} a_{ij}\left(x_j(t)-x_i(t)\right)^{\frac{p}{q}} \\ t \in \left(t_k, \bar{t}_k\right] \\ \sum_{j=1}^{n} a_{ij}\left(x_j(t)-x_i(t)\right), \ \ t \in \left(\bar{t}_{k-1}, t_k\right] \end{cases} \tag{12}$$

where m, r, p, and q are positive odd integers such that $m > r$ and $p < q$. When $(m/r) = 1$ and $(p/q) = 1$, it will become the consensus protocol proposed in [45] which solves the asymptotical consensus problem.

Note that $f_1(x) = x^{(m/r)}$ and $f_2(x) = x^{(p/q)}$ are odd functions. It is quite clear that function $f(x) = f_1(x) + f_2(x)$ is a odd function. Moreover, it is easy to obtain that $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = 0$, i.e., $f(x)$ is continuous function. Therefore, the consensus protocol for continuous-time subsystem is continuous.

Theorem 3: Suppose that interaction network $\mathscr{G}(A)$ is strongly connected and satisfies the detailed balance condition, and $0 < h < (1/\bar{d})$. Then, system (1) with protocol (12) reaches FTC if there exist finite k and $\bar{t}^* \in (t_k, \bar{t}_k]$ such that

$$T_0 \ge \frac{1}{\tilde{K}_1(k_1-1)} + \frac{1}{\tilde{K}_2(1-k_2)} \tag{13}$$

where

$$\begin{aligned}
T_0 &= \bar{t}^* - t_k + \sum_{i=1}^{k-1}\left(\bar{t}_i - t_i\right) \\
\tilde{K}_1 &= n^{\frac{r-m}{2r}}\left(\frac{c_1}{\omega_{\max}}\right)^{\frac{m+r}{2r}} \\
\tilde{K}_2 &= 2^{\frac{p}{q}}\left(\frac{c_2}{\omega_{\max}}\right)^{\frac{p+q}{2q}} \\
k_1 &= \frac{m+r}{2r} > 1
\end{aligned}$$

and $k_2 = [(p+q)/2q] < 1$. $c_1 = \min_{\xi\in U}\xi^T L(\tilde{B}_1)\xi > 0$ and $c_2 = \min_{\xi\in U}\xi^T L(\tilde{B}_2)\xi > 0$, $U = \{\xi \in R^n : \omega^T\xi = 0 \text{ and } \|\xi\| = 1\}$, $\tilde{B}_1 = [(\omega_i a_{ij})^{[2r(m+r)]}] \in R^{n\times n}$ and $\tilde{B}_2 = [(\omega_i a_{ij})^{[2q/(p+q)]}] \in R^{n\times n}$.

Proof: First, we show that there exist finite k satisfying (13). By Assumption 1, we get $T_0 \ge (k-1)\tau^*$. It is clear that condition (13) is satisfied for any finite $k \ge \lfloor(1/\tau^*\tilde{K}_1(k_1-1)) + (1/\tau^*\tilde{K}_2(k_2-1))\rfloor + 1$. Moreover, there must exist a $t_{k^*}^+ \in (t_k, \bar{t}_k]$ such that $t_{k^*}^+ - t_{k^*} + \cdots + \bar{t}_1 - t_1 \ge (1/\tilde{K}_1(k_1-1))$ and $\bar{t}^* - t_k + \cdots + \bar{t}_{k^*} - t_{k^*}^+ \ge (1/\tilde{K}_2(1-k_2))$.

The interaction network is strongly connected and satisfies the detailed balance condition, i.e., there exists a positive column vector $\omega = [\omega_1, \omega_2, \ldots, \omega_n]^T$ such that $\omega_i a_{ij} = \omega_j a_{ji}$ for all $i, j \in \mathcal{I}_n$. Let $\upsilon(t) = (1/\sum_{i=1}^n \omega_i)\sum_{i=1}^n \omega_i x_i(t)$ and $\gamma_i(t) = x_i(t) - \upsilon(t)$. Similar to Theorem 1, $\upsilon(t)$ is time-invariant and $\gamma(t+1) = (I - hL)\gamma(t)$. Take Lyapunov function $V(\gamma(t)) = (1/2)\sum_{i=1}^n \omega_i\gamma_i^2(t) = (1/2)\gamma^T(t)W\gamma(t)$ for continuous-time subsystem (1a) and discrete-time subsystem (1b).

When $t \in (\bar{t}_{k-1}, t_k]$, discrete-time subsystem (1b) is activated. Similar to Theorem 1, we have

$$V(\gamma(t+1)) \le V(\gamma(t)). \tag{14}$$

When $t \in (t_k, \bar{t}_k]$, continuous-time subsystem (1a) is activated. Therefore

$$\begin{aligned}\dot{V}(\gamma(t)) &= \sum_{i=1}^{n}\omega_i\gamma_i(t)\left(\sum_{j=1}^{n}a_{ij}\big(\gamma_j(t)-\gamma_i(t)\big)^{\frac{m}{r}} + \sum_{j=1}^{n}a_{ij}\big(\gamma_j(t)-\gamma_i(t)\big)^{\frac{p}{q}}\right)\\ &= -\frac{1}{2}\sum_{i,j=1}^{n}\left(\big(\omega_i a_{ij}\big)^{\frac{2r}{m+r}}\big(\gamma_j(t)-\gamma_i(t)\big)^2\right)^{\frac{m+r}{2r}}\\ &\quad -\frac{1}{2}\sum_{i,j=1}^{n}\left(\big(\omega_i a_{ij}\big)^{\frac{2q}{p+q}}\big(\gamma_j(t)-\gamma_i(t)\big)^2\right)^{\frac{p+q}{2q}}\\ &\le -\frac{1}{2}(2n)^{\frac{r-m}{2r}}\left(\sum_{i,j=1}^{n}\big(\omega_i a_{ij}\big)^{\frac{2r}{m+r}}\big(\gamma_j(t)-\gamma_i(t)\big)^2\right)^{\frac{m+r}{2r}}\\ &\quad -\frac{1}{2}\left(\sum_{i,j=1}^{n}\big(\omega_i a_{ij}\big)^{\frac{2q}{p+q}}\big(\gamma_j(t)-\gamma_i(t)\big)^2)\right)^{\frac{p+q}{2q}}\end{aligned} \tag{15}$$

where the last inequality follows from Lemma 2 and Holder inequality.

Similar to Theorem 1, we have

$$\frac{\sum_{i,j=1}^{n}\big(\omega_i a_{ij}\big)^{\frac{2r}{m+r}}\big(\gamma_j(t)-\gamma_i(t)\big)^2}{V(\gamma(t))} \ge \frac{4c_1}{\omega_{\max}} \tag{16}$$

$$\frac{\sum_{i,j=1}^{n}\big(\omega_i a_{ij}\big)^{\frac{2q}{p+q}}\big(\gamma_j(t)-\gamma_i(t)\big)^2}{V(\gamma(t))} \ge \frac{4c_2}{\omega_{\max}}. \tag{17}$$

Using $m > r$, $p < q$, (16) and (17) leads to

$$\dot{V}(\gamma(t)) \le -\tilde{K}_1 V^{\frac{m+r}{2r}}(\gamma(t)) - \tilde{K}_2 V^{\frac{p+q}{2q}}(\gamma(t)) \tag{18}$$

where $\tilde{K}_1 = n^{[(r-m)/2r)}((c_1/\omega_{\max}))^{[(m+r)/2r]} > 0$ and $\tilde{K}_2 = 2^{(p/q)}((c_2/\omega_{\max}))^{[(p+q)/2q]} > 0$.

Obviously, for $t \in (t_k, \bar{t}_k]$, we have $\dot{V}(\gamma(t)) \le -\tilde{K}_1 V^{k_1}(\gamma(t))$ and $\dot{V}(\gamma(t)) \le -\tilde{K}_2 V^{k_2}(\gamma(t))$. By the Comparison theorem, we have

$$V^{1-k_1}(\gamma(t)) \ge -\tilde{K}_1(1-k_1)(t-t_k) + V^{1-k_1}(\gamma(t_k)) \tag{19}$$

and

$$V^{1-k_2}(\gamma(t)) \le -\tilde{K}_2(1-k_2)(t-t_k) + V^{1-k_2}(\gamma(t_k)). \tag{20}$$

We know from (14) and (18) that

$$\begin{aligned}&V(\gamma(0)) \ge V(\gamma(t_1)) \ge V\big(\gamma\big(\bar{t}_1\big)\big) \ge V(\gamma(t_2))\\ &\quad \ge V\big(\gamma\big(\bar{t}_2\big)\big) \ge \cdots \ge V(\gamma(t_k)) \ge V\big(\gamma\big(\bar{t}_k\big)\big) \ge \cdots\end{aligned} \tag{21}$$

When $V(\gamma(0)) < 1$, we show that there must exist $\bar{t}^* \in (t_k, \bar{t}_k]$ such that $-\tilde{K}_2(1-k_2)(\bar{t}^*-t_k) + V^{1-k_2}(\gamma(t_k)) \le 0$. Based on (20) and (21), we get

$$\begin{aligned}&-\tilde{K}_2(1-k_2)\big(\bar{t}^*-t_k\big) + V^{1-k_2}(\gamma(t_k))\\ &\quad\le -\tilde{K}_2(1-k_2)\big(\bar{t}^*-t_k\big) + V^{1-k_2}\big(\gamma\big(\bar{t}_{k-1}\big)\big)\\ &\quad\le -\tilde{K}_2(1-k_2)\big(\bar{t}^*-t_k+\bar{t}_{k-1}-t_{k-1}\big) + V^{1-k_2}(\gamma(t_{k-1}))\\ &\quad\ \ \vdots\\ &\quad\le -\tilde{K}_2(1-k_2)\big(\bar{t}^*-t_k+\bar{t}_{k-1}-t_{k-1}+\cdots+\bar{t}_1-t_1\big)+1.\end{aligned} \tag{22}$$

Hence, it follows from (13) that there exists $\bar{t}^* \in (t_k, \bar{t}_k]$ such that $-\tilde{K}_2(1-k_2)(\bar{t}^*-t_k) + V^{1-k_2}(\gamma(t_k)) \le 0$.

When $V(\gamma(0)) \ge 1$, we show that there must exist $t_{k^*}^+ \in (t_k, \bar{t}_k]$ such that $V(\gamma(t_{k^*}^+)) < 1$. Based on (19) and (21), we get

$$\begin{aligned}&V^{1-k_1}(\gamma(t_1)) \ge V^{1-k_1}(\gamma(0))\\ &V^{1-k_1}\big(\gamma\big(\bar{t}_1\big)\big) \ge -\tilde{K}_1(1-k_1)\big(\bar{t}_1-t_1\big) + V^{1-k_1}(\gamma(t_1))\\ &V^{1-k_1}(\gamma(t_2)) \ge V^{1-k_1}\big(\gamma\big(\bar{t}_1\big)\big)\\ &V^{1-k_1}\big(\gamma\big(\bar{t}_2\big)\big) \ge -\tilde{K}_1(1-k_1)\big(\bar{t}_2-t_2\big) + V^{1-k_1}(\gamma(t_2))\\ &\qquad\vdots\\ &V^{1-k_1}\big(\gamma\big(t_{k^*}^+\big)\big) \ge -\tilde{K}_1(1-k_1)\big(t_{k^*}^+-t_k\big) + V^{1-k_1}(\gamma(t_k)).\end{aligned} \tag{23}$$

Since there exists $t_{k^*}^+ - t_{k^*} + \cdots + \bar{t}_1 - t_1 \ge (1/\tilde{K}_1(k_1-1))$, we easily obtain

$$\begin{aligned}V^{1-k_1}\big(\gamma\big(t_{k^*}^+\big)\big) &\ge -\tilde{K}_1(1-k_1)\big(t_{k^*}^+-t_k+\cdots+\bar{t}_1-t_1\big)\\ &\quad + V^{1-k_1}(\gamma(0)) > 1\end{aligned}$$

it follows from $1-k_1 < 0$ that $V(\gamma(t_{k^*}^+)) < 1$. Hence, there must exist $t_{k^*}^+ \in (t_k, \bar{t}_k]$ such that $V(\gamma(t_{k^*}^+)) < 1$. By virtue of (22) and $\bar{t}^* - t_k + \cdots + \bar{t}_{k^*} - t_{k^*}^+ \ge (1/\tilde{K}_2(1-k_2))$, it follows that $-\tilde{K}_2(1-k_2)(\bar{t}^*-t_k) + V^{1-k_2}(\gamma(t_k)) \le 0$. Owing to $V(\gamma(t)) \ge 0$, $\dot{V}(\gamma(t)) \le 0$ and (20), we get $V(\gamma(t)) = 0$ for $t \ge \bar{t}^*$. Therefore, system (1) reaches consensus in finite time. ■

Theorem 4: Suppose that system (1) has a leader (labeled as n) and $n-1$ followers (labeled as $1, \ldots, n-1$), and the interaction network $\mathscr{G}(\bar{A})$ among the followers is strongly connected and satisfies the detailed balance condition, and $0 < h < [1/(\bar{d}+b_i)]$, $i \in \mathcal{I}_{n-1}$. Then, system (1) with protocol (12) reaches FTC if there exist finite k and $\bar{t}^* \in (t_k, \bar{t}_k]$ such that

$$T_0 \ge \frac{1}{\vec{K}_1(k_1-1)} + \frac{1}{\vec{K}_2(1-k_2)} \tag{24}$$

where

$$T_0 = \bar{t}^* - t_k + \sum_{i=1}^{k-1}\big(\bar{t}_i - t_i\big)$$

$$\vec{K}_1 = 2(n-1)^{\frac{r-m}{2r}}\left(\frac{\lambda_1\big(L\big(\vec{B}_1\big)+\vec{b}_1\big)}{\omega_{\max}}\right)^{\frac{m+r}{2r}} > 0$$

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and

$$\vec{K}_2 = -2^{\frac{p}{q}} \left(\frac{\lambda_1 \left(L(\vec{B}_2) + \vec{b}_2 \right)}{\omega_{\max}} \right)^{\frac{p+q}{2q}} > 0, \quad k_1 = \frac{m+r}{2r} > 1$$

and $k_2 = [(p+q)/2q] < 1$

$$\vec{B}_1 = \left[\left(\omega_i a_{ij} \right)^{\frac{2r}{m+r}} \right] \in R^{(n-1)\times(n-1)}$$

$$\vec{B}_2 = \left[\left(\omega_i a_{ij} \right)^{\frac{2q}{p+q}} \right] \in R^{(n-1)\times(n-1)}$$

$$\vec{b}_1 = \frac{\left(\omega_i \tilde{b} \right)^{\frac{2r}{m+r}}}{2} \text{ and } \vec{b}_2 = \frac{\left(\omega_i \tilde{b} \right)^{\frac{2q}{p+q}}}{2}.$$

Proof: The interaction network among the followers is strongly connected and satisfies the detailed balance condition, i.e., there exists a positive column vector $\bar{\omega} = [\omega_1, \omega_2, \ldots, \omega_{n-1}]^T$ such that $\omega_i a_{ij} = \omega_j a_{ji}$ for all $i, j \in \mathcal{I}_{n-1}$. We rewrite protocol (12) as follows:

$$u_i(t) = \begin{cases} \sum_{j=1}^{n-1} a_{ij} \left(x_j(t) - x_i(t) \right)^{\frac{m}{r}} + a_{in}(x_n - x_i)^{\frac{m}{r}} \\ \quad + \sum_{j=1}^{n-1} a_{ij} \left(x_j(t) - x_i(t) \right)^{\frac{p}{q}} + a_{in}(x_n - x_i)^{\frac{p}{q}} \\ t \in \left(t_k, \bar{t}_k \right] \\ \sum_{j=1}^{n-1} a_{ij} \left(x_j(t) - x_i(t) \right) + a_{in}(x_n - x_i) \\ t \in \left(\bar{t}_{k-1}, t_k \right] \end{cases} \tag{25}$$

where a_{in} is defined as in Theorem 2 and $i \in \mathcal{I}_{n-1}$.

First, we show that there exist finite k satisfying (24). The proof is similar to that of Theorem 3 and it is omitted. Moreover, there must exist $t_{k^*}^+ \in [t_k, \bar{t}_k)$ such that $t_{k^*}^+ - t_{k^*} + \cdots + \bar{t}_1 - t_1 \geq (1/\vec{K}_1(k_1 - 1))$ and $\bar{t}^* - t_k + \cdots + \bar{t}_{k^*} - t_{k^*}^+ \geq (1/\vec{K}_2(1 - k_2))$.

Let $\gamma_i(t) = x_i(t) - x_n$, $i \in \mathcal{I}_{n-1}$. Take Lyapunov function $\bar{V}(\gamma(t)) = (1/2)\sum_{i=1}^{n-1} \omega_i \gamma_i^2(t) = (1/2)\bar{\gamma}^T(t)\bar{W}\bar{\gamma}(t)$ for continuous-time subsystem (1a) and discrete-time subsystem (1b).

When $t \in (\bar{t}_{k-1}, t_k]$, discrete-time subsystem (1b) is activated. Similar to Theorem 2, we have

$$\bar{V}(\gamma(t+1)) \leq \bar{V}(\gamma(t)). \tag{26}$$

When $t \in (t_k, \bar{t}_k]$, continuous-time subsystem (1a) is activated. Therefore

$$\begin{aligned} &\dot{\bar{V}}(\gamma(t)) \\ &= \sum_{i=1}^{n-1} \omega_i \gamma_i(t) \left(\sum_{j=1}^{n-1} a_{ij} \left(\gamma_j(t) - \gamma_i(t) \right)^{\frac{m}{r}} - b_i \mathrm{sig}(\gamma_i(t))^{\frac{m}{r}} \right) \\ &\quad + \sum_{i=1}^{n-1} \omega_i \gamma_i(t) \left(\sum_{j=1}^{n-1} a_{ij} \left(\gamma_j(t) - \gamma_i(t) \right)^{\frac{p}{q}} - b_i \mathrm{sig}(\gamma_i(t))^{\frac{p}{q}} \right) \\ &\leq -\frac{1}{2}(2(n-1))^{\frac{r-m}{2r}} \left(\sum_{i,j=1}^{n-1} \left(\omega_i a_{ij} \right)^{\frac{2r}{m+r}} \left(\gamma_j(t) - \gamma_i(t) \right)^2 \right)^{\frac{m+r}{2r}} \\ &\quad - (n-1)^{\frac{r-m}{2r}} \left(\sum_{i=1}^{n-1} (\omega_i b_i)^{\frac{2r}{m+r}} (\gamma_i(t))^2 \right)^{\frac{m+r}{2r}} \\ &\quad - \frac{1}{2} \left(\sum_{i,j=1}^{n-1} \left(\omega_i a_{ij} \right)^{\frac{2q}{p+q}} \left(\gamma_j(t) - \gamma_i(t) \right)^2 \right)^{\frac{p+q}{2q}} \\ &\quad - \left(\sum_{i=1}^{n-1} (\omega_i b_i)^{\frac{2q}{p+q}} (\gamma_i(t))^2 \right)^{\frac{p+q}{2q}}. \end{aligned} \tag{27}$$

By Lemma 2, Holder inequality and $[(r-m)/r] < 0$, we have

$$\begin{aligned} &\dot{\bar{V}}(\gamma(t)) \\ &\leq -2^{\frac{-m}{r}} (n-1)^{\frac{r-m}{2r}} \left(\sum_{i,j=1}^{n-1} \left(\omega_i a_{ij} \right)^{\frac{2r}{m+r}} \left(\gamma_j(t) - \gamma_i(t) \right)^2 \right. \\ &\qquad \left. + \sum_{i=1}^{n-1} (\omega_i b_i)^{\frac{2r}{m+r}} (\gamma_i(t))^2 \right)^{\frac{m+r}{2r}} \\ &\quad - \frac{1}{2} \left(\sum_{i,j=1}^{n-1} \left(\omega_i a_{ij} \right)^{\frac{2q}{p+q}} \left(\gamma_j(t) - \gamma_i(t) \right)^2 \right. \\ &\qquad \left. + \sum_{i=1}^{n-1} (\omega_i b_i)^{\frac{2q}{p+q}} (\gamma_i(t))^2 \right)^{\frac{p+q}{2q}} \\ &\leq -\vec{K}_1 V^{\frac{m+r}{2r}}(\gamma(t)) - \vec{K}_2 V^{\frac{p+q}{2q}}(\gamma(t)) \end{aligned} \tag{28}$$

where

$$\vec{K}_1 = 2(n-1)^{\frac{r-m}{2r}} \left(\frac{\lambda_1 \left(L(\vec{B}) + \vec{b} \right)}{\omega_{\max}} \right)^{\frac{m+r}{2r}} > 0$$

and

$$\vec{K}_2 = -2^{\frac{p}{q}} \left(\frac{\lambda_1 \left(L(\vec{B}) + \vec{b} \right)}{\omega_{\max}} \right)^{\frac{p+q}{2q}} > 0.$$

Similar to Theorem 3, we obtain $\bar{V}(\gamma(t)) = 0$ for $t \geq \bar{t}^*$. Therefore, system (1) reaches consensus in finite time. ∎

Remark 3: It is clear that T_0 in Theorems 3 or 4 is irrelevant to the initial states of agents. It is closely related to the algebraic connectivity, order n of system (1) and design parameters m, r, p, and q of protocol (12).

When $0 < h < (1/\lambda_{\max}(L))$ and the interaction network is undirected connected, it can be found that system (1) which is composed of only discrete-time subsystem (1b) can not reach consensus in finite time. Assume that system $x(k+1) = (I - hL)x(k)$ reaches consensus in finite time, i.e., $x(k+1) = c\mathbf{1}$, therefore we have $x(k) = (I - hL)^{-1}x(k+1)$. Due to $(I - hL)\mathbf{1} = \mathbf{1}$, it is easy to obtain $(I - hL)^{-1}\mathbf{1} = \mathbf{1}$, i.e., row sums of $(I - hL)^{-1}$ are 1. Thus, we have $x(k) = c\mathbf{1} = x(k+1)$. By using the similar method, we get $x(0) = c\mathbf{1}$ which leads to a contradiction. In this paper, by using the FTC protocol, we can find that continuous-time subsystem (1a) and discrete-time subsystem (1b) can solve FTC under proper switching law even if discrete-time subsystem (1b) cannot reach consensus in finite time.

Moreover, in this paper, we can point out that discrete-time MASs can achieve consensus in finite time if there exists

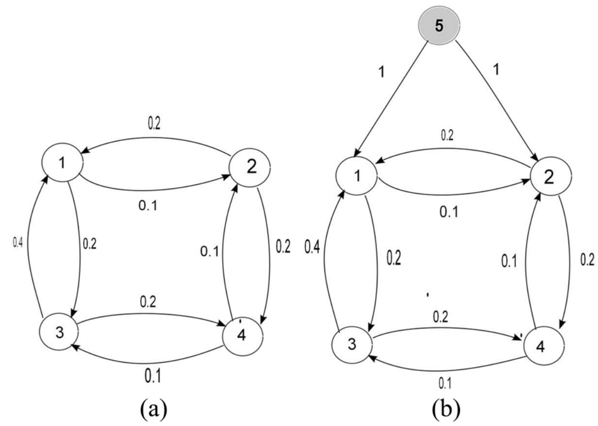

Fig. 1. (a) Strongly connected network. (b) Leader-following network.

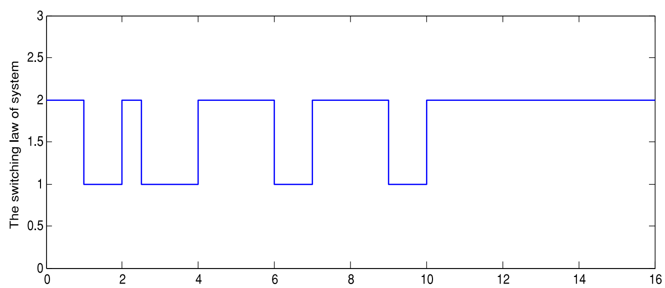

Fig. 2. Switching law of system (1) with FTC protocol.

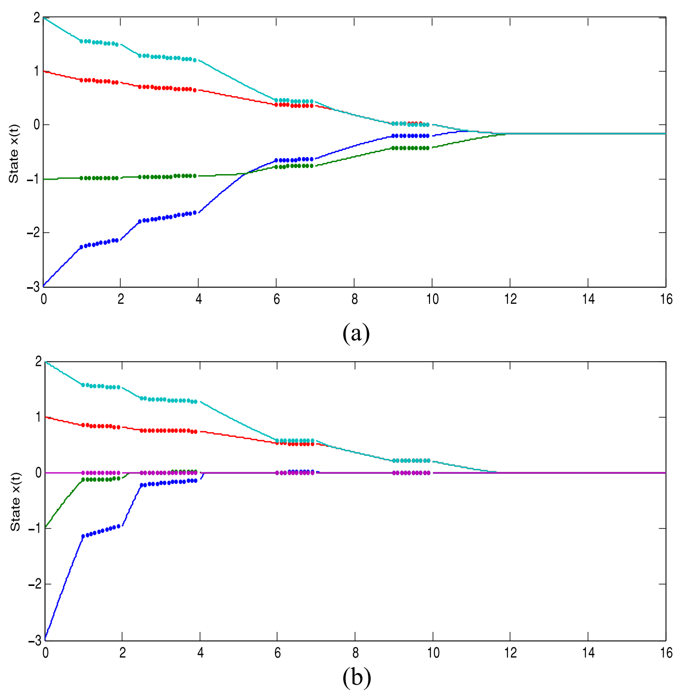

Fig. 3. (a) System (1) with protocol (2). (b) System (1) with protocol (9).

the dynamical switching behavior for this system. It can be found that the network only needs to be strongly connected and satisfy the detailed balance condition.

V. Simulations

In this section, simulations are provided to illustrate the effectiveness of our theoretical results. We consider two

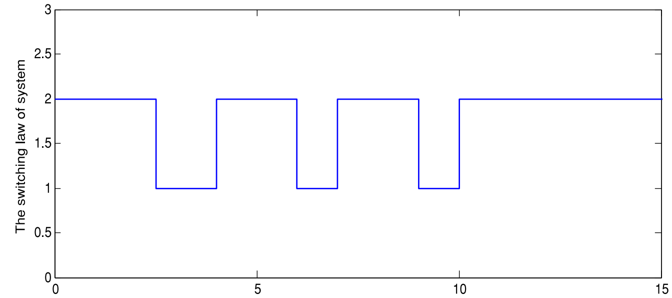

Fig. 4. Switching law of system (1) with F_dTC protocol.

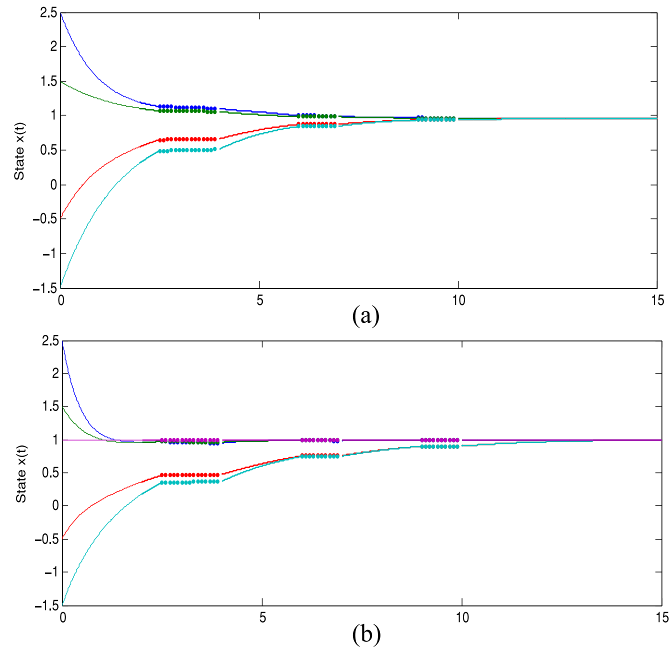

Fig. 5. (a) System (1) with protocol (12). (b) System (1) with protocol (25).

classes of graphs in Fig. 1 to describe the communication relationship of agents. Fig. 1(a) describes a strongly connected network which satisfies the detail-balanced condition. Fig. 1(b) describes a leader-following network. Suppose the vertices 1—4 denote 4 followers and the vertice 5 denotes the leader.

Example 1: The switching law of system (1) under FTC protocol is shown in Fig. 2. We assume $\omega = [1, 2, 2, 1]^T$ and $\alpha_{ij} = \alpha_{ji} = 0.2$ for all $i, j \in \mathcal{I}_n$. The initial states of agents are $x_0 = [-3, -1, 1, 2]^T$ and $x_0 = [-3, -1, 1, 2, 0]^T$ for strongly connected network Fig. 1(a) and leader-following network Fig. 1(b), respectively. By calculation, we can get $0 < h \leq (5/3)$ and $0 < h \leq (5/8)$ for different networks, respectively. For convenience, we choose $h = 0.01$. The state trajectories of system (1) under FTC protocol (2) are shown in Fig. 3(a). The state trajectories of system (1) under FTC protocol (9) are shown in Fig. 3(b).

Example 2: The switching law of system (1) under F_dTC protocol is shown in Fig. 4. The directed communication topology, ω and h are the same as Example 1. Parameters $m = 9$, $r = 7$, $p = 5$, and $q = 7$. The initial values of all the agents are $x_0 = [2.5, 1.5, -0.5, -1.5]^T$

and $x_0 = [2.5, 1.5, -0.5, -1.5, 1]^T$ for strongly connected network Fig. 1(a) and leader-following network Fig. 1(b), respectively. The state trajectories of system (1) under $\mathrm{F_dTC}$ protocol (12) are shown in Fig. 5(a). The state trajectories of system (1) under $\mathrm{F_dTC}$ protocol (25) are shown in Fig. 5(b).

VI. Conclusion

This paper discussed the FTC of the switched MAS. Two effective consensus protocols (FTC protocol and $\mathrm{F_dTC}$ protocol) were proposed to solve the FTC problem and the $\mathrm{F_dTC}$ problem, respectively. First, by using the FTC protocol for the switched MAS, we proved that the FTC problem under two classes of special directed networks can be solved, respectively. Then, we considered the $\mathrm{F_dTC}$ of the switched MAS. By applying the $\mathrm{F_dTC}$ protocol, we showed that the FTC problem can be solved and T_0 is independent of the initial states of agents. Future work will focus on FTC problem of the switched MAS under general directed network, FTC of the switched MAS with switching topology and finite-time containment control of the switched MAS, etc.

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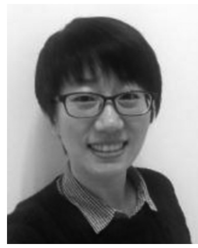

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