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Consensusability of Multi-agent Systems with Delay and Packet Dropout Under Predictor-like Protocols

Juanjuan Xu, Huanshui Zhang and Lihua Xie

Abstract—**This paper considers the consensusability of multi-agent systems with delay and packet dropout. By proposing a kind of predictor-like protocol, sufficient and necessary conditions are given for the mean-square consensusability in terms of system matrices, time delay, communication graph and the packet drop probability. Moreover, sufficient and necessary conditions are also obtained for the formationability of multi-agent systems.**

Index Terms—**Consensus, Delay, Packet Dropout, Predictor-like protocol, Formationable, Multi-agent system.**

I. Introduction

Multi-agent systems have attracted much attention in various scientific communities due to their broad applications in many areas including distributed computation [16], formation control [6], distributed sensor networks [3], etc. Consensus is the most fundamental control problem in multi-agent systems. Due to the fact that each individual agent lacks of global knowledge of the whole system and can only interact with its neighbors, one key issue of consensus is to study conditions under which the consensus can be achieved under a given protocol and the other is the design of a consensus protocol. Numerous results have been reported in the literature on the design of distributed consensus protocols for multi-agent systems. See [14], [20] and references therein. For the consensusability problem, [17] and [19] gave a necessary and sufficient condition for continuous-time and discrete-time multi-agent systems in the deterministic case, respectively. [31] studied the multi-agent systems with multiplicative noise and time delay.

Time delays are unavoidable in information acquisition and transmission of practical multi-agent systems and should be taken into account in designing the consensus protocol. An initial study was given in [20] which provides a necessary and sufficient condition on the upper bound of time delays under the assumption that all the delays are equal and time-invariant. Sufficient conditions have been given in [1] for average consensus with constant, time varying and nonuniform time delays. [18] sutdied the output consensus for multi-agent systems with different types of time delays including communication delay, homogeneous and heterogeneous input delays. [2] considered discrete-time multi-agent systems with dynamically changing topologies and time-varying communication delays.

In addition, random link failures or transmission noises exist widely in networked multi-agent systems, which motivates the study of the stochastic consensus problem. In the literature, [4] provided two kinds of average consensus protocols, namely biased compensation method and balanced compensation method to handle random link failures. It was shown in [24] that in the presence of noises the consensus value will diverge when the traditional consensus algorithms are applied. Under a fixed topology, necessary and sufficient conditions were given in [13] for mean square average consensus. [10] derived a sufficient condition for the switching topology case. For multi-agent systems with multiplicative-noise, [12] revealed that multiplicative noises may enhance the almost sure consensus, but may have damaging effect on the mean square consensus. [11] studied the mean square consensus for linear discrete-time systems by solving a modified algebraic Riccati equation. [30] investigated the stochastic consensus problem. [31] gave the stochastic consentability analysis of linear multi-agent systems with time delays and multiplicative noises. Though many research achievements have been made for multi-agent systems with either time delay or multiplicative noise, there is little progress for discrete-time multi-agent systems with both input delay and packet dropout. The consensus problem for the latter remains challenging. More recently, substantial progress for the optimal LQ control of single agent system has been made by the approach of the forward and backward difference equations. See [27] and [29] for details.

Inspired by the work [27] and [29], we will study the consensusability problem of multi-agent systems with delay and packet dropout. Different from the consensus protocols in the literature where the protocol is mostly in the feedback form of the current state or delayed state, a new kind of predictor-like consensus protocol is proposed in this paper to deal with the delay. Sufficient and necessary conditions are given for the mean-square consensusability in terms of system matrix, time delay, communication graph and the packet dropout probability under the predictor-like protocol. It will be shown that the derived results can be reduced to the deterministic case in the literature. Moreover, sufficient and necessary conditions are obtained for the formationability of multi-agent systems.

The remainder of the paper is organized as follows. Section II presents some preliminary knowledge about algebraic graph theory. Problem formulation is given in Section III. Section IV shows preliminaries on modified Riccati equation. Main results

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are stated in Section V. Some concluding remarks are given in the last section. Related theorems and proofs are given in Appendix.

The following notations will be used throughout this paper: R^n denotes the set of n-dimensional vectors; x' denotes the transpose of x; a symmetric matrix $M > 0$ (≥ 0) means that M is strictly positive-definite (positive semi-definite). $\hat{x}(k|t) \triangleq E[x(k)|\mathcal{F}_{t-1}]$ denotes the conditional expectation with respect to the filtration $\mathcal{F}_{t-1}$. $\lambda_i(A)$ means the ith eigenvalue of matrix A.

II. Algebraic Graph Theory

In this paper, the information exchange among agents is modeled by an undirected graph. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ be a diagraph with the set of vertices $\mathcal{E} = \{1, \ldots, N\}$, the set of edges $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$, and the weighted adjacency matrix $\mathcal{A} = [a_{ij}] \in \mathbf{R}^{N\times N}$ is symmetric. In $\mathcal{G}$, the i-th vertex represents the i-th agent. Let $a_{ij} > 0$ if and only if $(i, j) \in \mathcal{E}$, i.e., there is a communication link between agents i and j. Undirected graph $\mathcal{G}$ is connected if any two distinct agents of $\mathcal{G}$ can be connected along some edges of $\mathcal{G}$. For agent i, the degree is defined as $d_i \triangleq \sum_{j=1}^{N} a_{ij}$. Diagonal matrix $\mathcal{D} \triangleq diag\{d_1, \ldots, d_N\}$ is used to denote the degree matrix of diagraph $\mathcal{G}$. Denote the Laplacian matrix by $L_G = \mathcal{D} - \mathcal{A}$. The eigenvalues of L_G are denoted by $\lambda_i(L_G) \in R, i = 1, \cdots, N$, and an ascending order in magnitude is written as $0 = \lambda_1(L_G) \leq \cdots \leq \lambda_N(L_G)$, that is, the Laplacian matrix L_G of an undirected graph has at least one zero eigenvalue and all the nonzero eigenvalues are in the open right half plane. Furthermore, L_G has exactly one zero eigenvalue if and only if G is connected [7].

III. Problem Formulation

Consider a multi-agent system as depicted in Fig. 1 where the dynamic for $i = 1, \cdots, N$ is given by

$$x_i(k+1) \quad = \quad Ax_i(k) + \gamma(k)Bu_i(k-d), \tag{1}$$

while $x_i \in R^n$ is the state of the ith agent, $u_i \in R^m$ is the control input of the ith agent, A, B are constant matrices with appropriate dimensions. d represents the input delay. $\gamma(k) = 1$ denotes that the data packet has been successfully delivered to the plant, and $\gamma(k) = 0$ signifies the dropout of the data packet. Without loss of generality, the random process $\{\gamma(k), k \geq 0\}$ is modeled as an independent and identically distributed (i.i.d.) Bernoulli process with probability distribution $P(\gamma(k) = 0) = p$ and $P(\gamma(k) = 1) = 1 - p$, where $p \in (0, 1)$ is said to be the packet dropout rate. The initial values are given by $x_i(0), u_i(-1), \cdots, u_i(-d)$. Note that the channel fading and time delay occur simultaneously due to the unreliable network connecting the controller i and the agent i. Moreover, the information exchange between the controllers of agent i and j happens in the controller processor.

Remark 1: The packet dropout processes in the multi-agent system under study are assumed to be identical. While this assumption is restrictive in practice, it allows us to derive some necessary and sufficient conditions for consensusability of multi-agent systems with both delay and packet dropout and gain insights into the interplay among system dynamic, delay and network topology. They could also shed some light on resolving the non-identical γ case which is interesting and is left for our future study. We note that in some situation, e.g., the multi-agent system as shown in Fig. 1 where the control signals are transmitted to the agents through a wireless network and the packet dropouts could be caused by an attacker which jams randomly the network. In this case, the packet dropout processes for all agents can be assumed to be homogeneous.

We further make the following general assumption.

Assumption 1: All the eigenvalues of A are either on or outside the unit circle and B has full column rank.

Assumption 2: System $(A, B, 0, A^dB)$ is mean-square stabilizable, that is, for the system

$$x(k+1) = Ax(k) + Bu(k) + \nu(k)A^dBu(k)$$

where $\nu(k)$ is a sequence of white noise with zero mean and unit covariance, there exists a feedback controller $u(k) = Kx(k)$ with K being a time-invariant matrix such that the closed-loop system is mean-square stable, i.e. $\lim_{k\to\infty} E\|x(k)\|^2 = 0$.

Assumption 3: The undirected graph is connected.

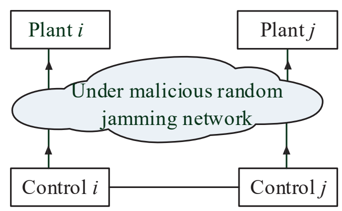

Fig. 1. Multi-agent system with unreliable networks

Denote $w(k) = \gamma(k) - E[\gamma(k)]$, then system (1) is reformulated as

$$\begin{aligned} x_i(k+1) \quad &= \quad Ax_i(k) + (1-p)Bu_i(k-d) \\ &\quad\quad +w(k)Bu_i(k-d), \quad i = 1, \cdots, N, \end{aligned} \tag{2}$$

where $\{w(k), k \in N\}$ is a sequence of random variables defined on $(\Omega, \mathcal{F}, \mathcal{P}; \mathcal{F}_k)$ with $E[w(k)] = 0$ and $E[w(k)w(s)] = p(1-p)\delta_{ks}$. For simplicity, we denote $\mu = 1 - p$ and $\sigma^2 = p(1-p)$.

In the literature [6], [20], the relative state $x_j(k) - x_i(k)$ between agents is used to design the consensus protocol like $u_i(k) = K\sum_{j\in N_i}\left[x_j(k) - x_i(k)\right]$. Different from the existing results, we firstly calculate the following predictor using each agent's own state and historical inputs for $k \geq d$ as follows:

$$\begin{aligned} &\hat{x}_i(k|k-d) = E[x_i(k)|\mathcal{F}_{k-d-1}] \\ &= \quad A^dx_i(k-d) + \mu\sum_{j=1}^{d} A^{j-1}Bu_i(k-d-j). \end{aligned} \tag{3}$$

Then the relative predictor $\hat{x}_j(k|k-d) - \hat{x}_i(k|k-d)$ is applied to design the consensus protocol. To be specific, the distributed

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protocol for $k \geq d$ is described as

$$u_i(k-d) = K\sum_{j\in N_i}\big[\hat{x}_j(k|k-d) - \hat{x}_i(k|k-d)\big]. \quad (4)$$

The aim is to find sufficient and necessary conditions for the mean-square consensusability of multi-agent system (2) under protocol (4) where the definition on the mean square consensusability is given below.

Definition 1: The discrete-time multi-agent system (2) with a fixed undirected graph is said to be mean-square consensusable under protocol (4) if for any finite initial values $x_i(0), u_i(-d), \cdots, u_i(-1)$, there exists a control gain K such that the controller (4) enforces consensus, i.e. $\lim_{k\to\infty} E\|x_j(k) - x_i(k)\|^2 = 0,\ \forall i, j = 1, \cdots, N$.

By substituting (4) into (2), the closed-loop multi-agent system becomes

$$\begin{aligned} x_i(k+1) &= Ax_i(k) + \mu BK\sum_{j\in N_i}\big[\hat{x}_j(k|k-d) \\ &\quad -\hat{x}_i(k|k-d)\big] + BK\sum_{j\in N_i}\big[\hat{x}_j(k|k-d) \\ &\quad -\hat{x}_i(k|k-d)\big]w(k), \quad k \geq d. \end{aligned} \quad (5)$$

Let $X(k) = \begin{bmatrix} x_1(k) & \cdots & x_N(k)\end{bmatrix}'$, $\hat{X}(k|k-d) = \begin{bmatrix} \hat{x}_1(k|k-d) & \cdots & \hat{x}_N(k|k-d)\end{bmatrix}'$, then (5) can be reformulated as

$$\begin{aligned} X(k+1) &= (I_N\otimes A)X(k) - \mu(L_G\otimes BK)\hat{X}(k|k-d) \\ &\quad -w(k)(L_G\otimes BK)\hat{X}(k|k-d), \quad k\geq d. \end{aligned} \quad (6)$$

Denote $\bar{X}(k) = \frac{1}{N}\sum_{i=1}^N x_i(k)$, then

$$\begin{aligned} \bar{X}(k+1) &= \frac{1}{N}(1_N\otimes I_n)'X(k+1) \\ &= A\bar{X}(k) - (\mu/N)(1_N' L_G\otimes BK)\hat{X}(k|k-d) \\ &\quad -(w(k)/N)(1_N' L_G\otimes BK)\hat{X}(k|k-d) \\ &= A\bar{X}(k), \end{aligned} \quad (7)$$

where $1_N' L_G = 0$ has been used in the derivation of the last equality. Given the initial condition $\bar{X}(0) = \frac{1}{N}\sum_{i=1}^N x_i(0)$ and equation (7), it yields that $\bar{X}(k)$ is deterministic. This further implies that $\hat{\bar{X}}(k|s) = E[\bar{X}(k)|\mathcal{F}_{s-1}] = \bar{X}(k)$ for any positive integer s. We now present the dynamic equation of $\delta(k+1) = X(k+1) - (1_N\otimes I_n)\bar{X}(k+1)$ with $\hat{\delta}(k|k-d) = \hat{X}(k|k-d) - (1_N\otimes I_n)\hat{\bar{X}}(k|k-d) = \hat{X}(k|k-d) - (1_N\otimes I_n)\bar{X}(k)$. It is obtained by subtracting (7) from (6) that

$$\begin{aligned} \delta(k+1) &= (I_N\otimes A)X(k) - \mu(L_G\otimes BK)\hat{X}(k|k-d) \\ &\quad -w(k)(L_G\otimes BK)\hat{X}(k|k-d) - (I_N\otimes A) \\ &\quad \times(1_N\otimes I_n)\bar{X}(k) \\ &= (I_N\otimes A)X(k) - \mu(L_G\otimes BK)\hat{X}(k|k-d) \\ &\quad -w(k)(L_G\otimes BK)\hat{X}(k|k-d) - (I_N\otimes A) \\ &\quad \times(1_N\otimes I_n)\bar{X}(k) + \mu(L_G\otimes BK)(1_N\otimes I_n) \\ &\quad \times\hat{\bar{X}}(k|k-d) + w(k)(L_G\otimes BK)(1_N\otimes I_n) \\ &\quad \times\hat{\bar{X}}(k|k-d) \\ &= (I_N\otimes A)\delta(k) - \mu(L_G\otimes BK)\hat{\delta}(k|k-d) \\ &\quad -w(k)(L_G\otimes BK)\hat{\delta}(k|k-d), \quad k\geq d. \end{aligned} \quad (8)$$

Select $\phi_i \in R^N$ such that $\phi_i' L_G = \lambda_i(L_G)\phi_i'$ and form a unitary matrix $\Phi = \begin{bmatrix} \frac{1}{\sqrt{N}} & \phi_2 & \cdots & \phi_N\end{bmatrix}$ to transform L_G into a diagonal form $diag\{0, \lambda_2(L_G), \cdots, \lambda_N(L_G)\} = \Phi' L_G \Phi$. Let $\tilde{\delta}(k) = (\Phi'\otimes I_n)\delta(k) = \begin{bmatrix}\tilde{\delta}_1(k) & \cdots & \tilde{\delta}_N(k)\end{bmatrix}'$. Together with the property of Kronecker product, it holds that $\tilde{\delta}_1(k) = 0$ and for $i = 2, \ldots, N$,

$$\begin{aligned} \tilde{\delta}_i(k+1) &= A\tilde{\delta}_i(k) - \mu\lambda_i(L_G)BK\hat{\tilde{\delta}}_i(k|k-d) \\ &\quad -\lambda_i(L_G)w(k)BK\hat{\tilde{\delta}}_i(k|k-d), k\geq d. \end{aligned} \quad (9)$$

Theorem 1: The multi-agent system (2) achieves mean-square consensus under protocol (4) if and only if the systems in (9) are mean-square stable simultaneously.

Proof. "Necessity" The simultaneous mean-square stability of (9) follows from the derivation of (5)-(9).

"Sufficiency" Since $\lim_{k\to\infty} E\|\tilde{\delta}_i(k)\|^2 = 0$, then $\lim_{k\to\infty} E\|\delta_i(k)\|^2 = 0$. This implies that $\lim_{k\to\infty} E\|x_i(k) - \bar{X}(k)\|^2 = 0$ for $i = 1, \cdots, N$. Thus,

$$\begin{aligned} & \lim_{k\to\infty} E\|x_j(k) - x_i(k)\|^2 \\ \leq\ & \lim_{k\to\infty} E\|x_j(k) - \bar{X}(k)\|^2 + \lim_{k\to\infty} E\|x_i(k) - \bar{X}(k)\|^2 \\ =\ & 0. \end{aligned}$$

The proof is now completed. ■

IV. Preliminaries on Modified Riccati Equation

Based on Theorem 1, the simultaneous stabilizability of the systems in (9) is necessary for consensusability. To this end, we shall present some results with respect to the stabilizability criterion and further investigate a corresponding modified algebraic Riccati equation. Firstly, the following equivalent conditions have been given in [23].

Lemma 1: The following statements are equivalent.

1) System

$$\begin{aligned} x(k+1) &= Ax(k) + \mu Bu(k-d) \\ &\quad +w(k)Bu(k-d) \end{aligned} \quad (10)$$

is mean-square stable under the controller $u(k-d) = K\hat{x}(k|k-d)$.

2) System

$$x(k+1) = Ax(k) + \mu Bu(k) + w(k)A^d Bu(k) \quad (11)$$

is mean-square stabilizable under the controller $u(k) = Kx(k)$.

3) For any $Q > 0$, there exist matrices K and $P > 0$ satisfying the following equation:

$$\begin{aligned} P &= Q + (A+\mu BK)'P(A+\mu BK) \\ &\quad +\sigma^2 K'B'(A')^d PA^d BK. \end{aligned} \quad (12)$$

4) There exist matrices K and $P > 0$ satisfying the following equation:

$$\begin{aligned} P &> (A+\mu BK)'P(A+\mu BK) \\ &\quad +\sigma^2 K'B'(A')^d PA^d BK. \end{aligned} \quad (13)$$

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In particular, it has also been shown in [23] that for some $Q > 0$, the existence of a unique positive definite solution to the algebraic Riccati equation

$$\begin{aligned} P &= A'PA + Q - \mu^2 A'PB\Big[R + \mu^2 B'PB \\ &\quad + \sigma^2 B'(A')^d PA^d B\Big]^{-1} B'PA \end{aligned}$$

is necessary and sufficient for the mean-square stabilizability of system (10). Motivated by the results in [23], we define the parameterized algebraic Riccati equation (PARE)

$$\begin{aligned} P &= A'PA + Q - \gamma A'PB\Big[R + B'PB \\ &\quad + B'(A')^d PA^d B\Big]^{-1} B'PA \end{aligned} \tag{14}$$

and denote

$$\begin{aligned} g_\gamma(P) &= A'PA + Q - \gamma A'PB\Big[R + B'PB \\ &\quad + B'(A')^d PA^d B\Big]^{-1} B'PA, \qquad (15) \\ \Phi(K,P) &= (1-\gamma)(A'PA + Q) + \gamma(F_1'PF_1 \\ &\quad + F_2'PF_2 + K'RK + Q), \qquad (16) \\ \Psi(K,P) &= F_1'PF_1 + F_2'PF_2 + K'RK + Q, \qquad (17) \end{aligned}$$

where $F_1 = A + BK, F_2 = A^d BK$.

Theorem 2: Consider the PARE (14). Let A be unstable, $(A, B, 0, A^d B)$ is mean-square stabilizable and $Q > 0, R > 0$. Then the following hold.

1) The PARE has a unique strictly positive definite solution if and only if $\gamma > \gamma_c$, where γ_c is the critical value defined as
$$\gamma_c = \inf\{\gamma \in [0,1] | P = g_\gamma(P), P > 0\}.$$
2) The critical value γ_c satisfies the following analytical bounds: $\underline{\gamma} \le \gamma_c \le \overline{\gamma}$ where $\underline{\gamma}$ and $\overline{\gamma}$ are defined by
$$\begin{aligned} \underline{\gamma} &= arginf_\gamma\{\exists S | (1-\gamma)A'SA + Q = S, S \ge 0\} \\ \overline{\gamma} &= arginf_\gamma\{\exists (K,P) | P > \Phi(K,P)\} \end{aligned}$$
3) The critical value can be numerically computed by the solution of the following quasiconvex LMI optimization problem
$$\gamma_c = argmin_\gamma \Delta_\gamma(Y,Z) > 0, 0 \le Y \le I$$
$$\Delta_\gamma(Y,Z) =$$
$$\begin{bmatrix} Y & Y & \sqrt{\gamma} Z R^{\frac{1}{2}} & \sqrt{\gamma}(AY + BZ')' & \sqrt{\gamma}(A^d BZ')' & \sqrt{1-\gamma} YA' \\ Y & Q^{-1} & 0 & 0 & 0 & 0 \\ \sqrt{\gamma} R^{\frac{1}{2}} Z' & 0 & I & 0 & 0 & 0 \\ \sqrt{\gamma}(AY + BZ') & 0 & 0 & Y & 0 & 0 \\ \sqrt{\gamma} A^d BZ' & 0 & 0 & 0 & Y & 0 \\ \sqrt{1-\gamma} AY & 0 & 0 & 0 & 0 & Y \end{bmatrix}$$

Proof. Based on Theorem 6, 7, 8 and 9 in Appendix, the results follow by using a similar proof to that of Lemma 5.4 in [21].

V. Mean-square Consensusability

Denote for $i = 2, \cdots, N$,

$$\gamma_i = \frac{\mu^2}{\mu^2 + \sigma^2} \frac{4\Big(\lambda_i(L_G)[\lambda_2(L_G) + \lambda_N(L_G)] - \lambda_i^2(L_G)\Big)}{[\lambda_N(L_G) + \lambda_2(L_G)]^2}.$$

It is noted that

$$\begin{aligned} \gamma_2 &= \frac{\mu^2}{\mu^2 + \sigma^2} \frac{4\lambda_2(L_G)\lambda_N(L_G)}{[\lambda_N(L_G) + \lambda_2(L_G)]^2} \\ &= \frac{\mu^2}{\mu^2 + \sigma^2}\Big[1 - \Big(\frac{\lambda_N(L_G) - \lambda_2(L_G)}{\lambda_N(L_G) + \lambda_2(L_G)}\Big)^2\Big]. \end{aligned}$$

We now present the main result of the mean-square consensusability for multi-agent system (2).

Theorem 3: Let Assumption 1-3 hold. If $\gamma_2 > \gamma_c$ where γ_c is given in Theorem 2, then the multi-agent system (2) is mean-square consensusable under protocol (4).

Proof. Consider the Riccati equation

$$\begin{aligned} P &= A'PA + Q - \gamma_i A'PB\Big[R + B'PB \\ &\quad + B'(A')^d PA^d B\Big]^{-1} B'PA. \end{aligned} \tag{18}$$

Since

$$\begin{aligned} &\frac{4\Big(\lambda_i(L_G)[\lambda_2(L_G) + \lambda_N(L_G)] - \lambda_i^2(L_G)\Big)}{[\lambda_N(L_G) + \lambda_2(L_G)]^2} \\ &- \frac{4\lambda_2(L_G)\lambda_N(L_G)}{[\lambda_N(L_G) + \lambda_2(L_G)]^2} \\ = &\frac{4\Big(\lambda_2(L_G) - \lambda_i(L_G)\Big)\Big(\lambda_i(L_G) - \lambda_N(L_G)\Big)}{[\lambda_N(L_G) + \lambda_2(L_G)]^2} \ge 0, \end{aligned}$$

then it follows that $\gamma_i \ge \gamma_2 > \gamma_c$ for $i > 2$. Using Theorem 2, the Riccati equation (18) admits a solution $P > 0$. Since B has a full column rank, then $B'PB + B'(A')^d PA^d B > 0$. Using the fact that $M^{-1} < N^{-1}$ when $M > N > 0$ and $R > 0, Q > 0$, we have

$$\begin{aligned} P &> A'PA - \gamma_i A'PB\Big[B'PB \\ &\quad + B'(A')^d PA^d B\Big]^{-1} B'PA. \end{aligned} \tag{19}$$

From $p \in (0,1)$, one has $\mu > 0$ and $\sigma^2 > 0$ which yields that $\mu^2 B'PB > 0$. Thus (19) further implies that

$$\begin{aligned} P &> A'PA - \bar{\gamma}_i(L_G) A'PB\Big[\mu^2 B'PB \\ &\quad + \sigma^2 B'(A')^d PA^d B\Big]^{-1} B'PA, \end{aligned} \tag{20}$$

where $\bar{\gamma}_i = \mu^2 \frac{4\Big(\lambda_i(L_G)[\lambda_2(L_G) + \lambda_N(L_G)] - \lambda_i^2(L_G)\Big)}{[\lambda_N(L_G) + \lambda_2(L_G)]^2}$. By letting the feedback gain matrix

$$\begin{aligned} K &= \frac{2\mu}{\lambda_2(L_G) + \lambda_N(L_G)}\Big[\mu^2 B'PB \\ &\quad + \sigma^2 B'(A')^d PA^d B\Big]^{-1} B'PA, \end{aligned} \tag{21}$$

the Riccati equation (20) is equivalently rewritten as

$$\begin{aligned} P &> [A - \lambda_i(L_G)\mu BK]' P [A - \lambda_i(L_G)\mu BK] \\ &\quad + \sigma^2 \lambda_i^2(L_G) K'B'(A')^d PA^d BK. \end{aligned} \tag{22}$$

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Combining with Lemma 1, system (9) is mean-square stabilizable. This yields that the multi-agent system (2) is mean-square consensusable. The proof is now completed. ■

Remark 2: Noting that $\mu = 1-p$ and $\sigma^2 = p(1-p)$, the condition $\gamma_2 > \gamma_c$ in Theorem 3 becomes $(1-p)\Big[1-\Big(\frac{\lambda_N(L_G)-\lambda_2(L_G)}{\lambda_N(L_G)+\lambda_2(L_G)}\Big)^2\Big] > \gamma_c$.

Remark 3: When time delay is 0, the above sufficient condition is reduced to $\frac{\mu^2}{\mu^2+\sigma^2}\Big[1-\Big(\frac{\lambda_N(L_G)-\lambda_2(L_G)}{\lambda_N(L_G)+\lambda_2(L_G)}\Big)^2\Big] > \gamma_c$ which is consistent with the result obtained in [15] for the consensusability of discrete-time linear multi-agent systems over analog fading networks where μ and σ^2 are corresponding to the expectation and the covariance of identical channel fading.

We next give a necessary condition for the mean-square consensusability of multi-agent system (2).

Theorem 4: Under Assumption 1, 3 and $Rank(B) = 1$, the multi-agent system (2) is mean-square consensusable under protocol (4) only if

$$\Pi_i|\lambda_i^u(A)|^2 < \Big(\frac{1+\lambda_2(L_G)/\lambda_N(L_G)}{1-\lambda_2(L_G)/\lambda_N(L_G)}\Big)^2, \tag{23}$$

where $\lambda_i^u(A)$ denotes the unstable eigenvalue of matrix A.

Proof. Using Theorem 1, systems (9) are mean-square stable simultaneously for all $i = 2,\ldots,N$. By applying Lemma 1, the following systems

$$\tilde\delta_i(k+1) \;=\; A\tilde\delta_i(k) - \lambda_i\mu BK\tilde\delta_i(k) - w(k)\lambda_i A^d BK\tilde\delta_i(k)$$

are mean-square stable for all $i = 2,\ldots,N$. Combining with the fact that $\lim_{k\to\infty} E\|\tilde\delta_i(k)\|^2 = 0$ implies that $\lim_{k\to\infty} E\big[\tilde\delta_i(k)\big] = 0$, it yields that $A-\lambda_i\mu BK$ is Schur stable, i.e. all the eigenvalues of $A-\lambda_i\mu BK$ are within the unit disk. The result then follows from [19]. ■

Remark 4: Consider the case of $Rank(B) = 1$. When the communication is delay free and packets can be perfectly delivered, that is, $d = 0$ and $p = 0$, $\gamma_c = 1-\frac{1}{\Pi_i|\lambda_i(A)|^2}$ which has been obtained in [22]. From Theorem 3, $\gamma_2 > \gamma_c$ is reduced to (23). Together with Theorem 4, (23) is necessary and sufficient for the consensusability of multi-agent systems (2) under protocol (4). This is consistent with Theorem 3.1 in [19] for the deterministic linear multi-agent systems under $u_i(k) = K\sum_{j\in N_i}\big[x_j(k)-x_i(k)\big]$.

We then study the scalar multi-agent systems. A necessary and sufficient condition will be shown below for the consensusability.

Theorem 5: Let $A = a \geq 1, B = b > 0$ be constants, the multi-agent system (2) is mean-square consensusable by the control protocol (4) if and only if

$$\frac{\mu^2}{(\mu^2+a^{2d}\sigma^2)}\Big[1-\frac{\Big(\lambda_N(L_G)-\lambda_2(L_G)\Big)^2}{\Big(\lambda_2(L_G)+\lambda_N(L_G)\Big)^2}\Big] > 1-\frac{1}{a^2}. \tag{24}$$

Proof. The equivalent condition (22) for the consensusability is reduced to

$$\begin{aligned} &a^2 - 2\lambda_i(L_G)\mu abk + \lambda_i^2(L_G)\mu^2b^2k^2 + a^{2d}\sigma^2\lambda_i^2(L_G)b^2k^2 \\ &< \;1, \end{aligned} \tag{25}$$

that is,

$$\lambda_i(L_G)^2(\mu^2+a^{2d}\sigma^2)b^2k^2 - 2\lambda_i(L_G)\mu abk + a^2 < 1. \tag{26}$$

"Necessity" Since $b>0$, one has from (26) that

$$\begin{aligned} &\frac{\mu a - \sqrt{(\mu a)^2-(\mu^2+a^{2d}\sigma^2)(a^2-1)}}{\lambda_i(L_G)(\mu^2+a^{2d}\sigma^2)b} \leq k \\ \leq\;\; &\frac{\mu a + \sqrt{(\mu a)^2-(\mu^2+a^{2d}\sigma^2)(a^2-1)}}{\lambda_i(L_G)(\mu^2+a^{2d}\sigma^2)b} \end{aligned}$$

Thus, we obtain that

$$\begin{aligned} \bigcap_{i=2}^{N}\Big(&\frac{\mu a - \sqrt{(\mu a)^2-(\mu^2+a^{2d}\sigma^2)(a^2-1)}}{\lambda_i(L_G)(\mu^2+a^{2d}\sigma^2)b}, \\ &\frac{\mu a - \sqrt{(\mu a)^2-(\mu^2+a^{2d}\sigma^2)(a^2-1)}}{\lambda_i(L_G)(\mu^2+a^{2d}\sigma^2)b}\Big) \neq \emptyset. \end{aligned}$$

Using $\lambda_2(L_G) < \lambda_i(L_G) < \lambda_N(L_G)$, it is further derived that

$$\begin{aligned} &\frac{\mu - \sqrt{(\mu)^2-(\mu^2+a^{2d}\sigma^2)(1-\frac{1}{a^2})}}{\lambda_2(L_G)} \\ \leq\;\; &\frac{\mu + \sqrt{(\mu)^2-(\mu^2+a^{2d}\sigma^2)(1-\frac{1}{a^2})}}{\lambda_N(L_G)}. \end{aligned}$$

By applying some algebraic transformations, we have

$$\Big[\frac{\Big(\lambda_N(L_G)-\lambda_2(L_G)\Big)^2}{\Big(\lambda_2(L_G)+\lambda_N(L_G)\Big)^2}-1\Big]\mu^2 \leq -(\mu^2+a^{2d}\sigma^2)(1-\frac{1}{a^2}).$$

Thus, (24) follows.

"Sufficiency" From (24), it yields that

$$\begin{aligned} &\frac{\mu^2}{(\mu^2+a^{2d}\sigma^2)}\frac{4\Big[\lambda_i(L_G)\Big(\lambda_N(L_G)+\lambda_2(L_G)\Big)-\lambda_i^2(L_G)\Big]}{(\lambda_2(L_G)+\lambda_N(L_G))^2} \\ &> \;1-\frac{1}{a^2}. \end{aligned}$$

By selecting the feedback gain in the form of (21), that is, $k = \frac{2\mu}{(\mu^2+\sigma^2a^{2d})\Big(\lambda_2(L_G)+\lambda_N(L_G)\Big)}\frac{a}{b}$, we have the inequality (25). Thus, system (2) is mean-square consensusable. The proof is now completed. ■

Remark 5: For system (2) with delay and $p = 0$, the advantage of using the predictor-like protocol (4) is that the allowable delay for consensus can be arbitrarily large. However, when using the protocol without delay compensation, there exists a maximum delay margin within which consensus can be achieved [9]. Take the case of $Rank(B) = 1$ for example, by combining Theorem 3, Theorem 4 with Lemma 5.4 in [21], the equivalent condition for consensus of system (2) is $\Pi_i|\lambda_i^u(A)|^2 < \Big(\frac{1+\lambda_2(L_G)/\lambda_N(L_G)}{1-\lambda_2(L_G)/\lambda_N(L_G)}\Big)^2$. This is exactly the necessary and sufficient condition to ensure the consensus for system (2) without delay obtained in [19]. This indicates that system (2) is consensusable for any large delay under the basic assumption. Furthermore, recalling Theorem 3 in [25], for scalar system with input delay, when $1+\frac{\lambda_2(L_G)}{\lambda_N(L_G)} \leq A < \frac{1+\lambda_2(L_G)/\lambda_N(L_G)}{1-\lambda_2(L_G)/\lambda_N(L_G)}$ or $-\frac{1+\lambda_2(L_G)/\lambda_N(L_G)}{1-\lambda_2(L_G)/\lambda_N(L_G)} \leq A \leq -1$, no delay is allowed for consensusability via relative state feedback

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protocols. This illustrates the advantage of using predictor-like protocol (4) which can tolerate any large delay.

As an important application, the result on consensusability is extended to the formationability of the discrete-time multi-agent systems (2). In particular, given a formation vector $H = \begin{bmatrix} H_1' & \cdots & H_N' \end{bmatrix}'$, the following control protocol is adopted to study the formation problem of the discrete-time multi-agent system:

$$\begin{aligned} u_i(k-d) &= K \sum_{j\in N_i} \Big(\big[\hat{x}_j(k|k-d) - H_j\big] \\ &\quad - \big[\hat{x}_i(k|k-d) - H_i\big] \Big), \end{aligned} \tag{27}$$

where $H_i - H_j$ is the desired formuation vector between agent i and agent j. Noting that the common knowledge of the directions of reference axes is required for all the agents, the protocol $u_i(k) = K\sum_{j\in N_i}\Big(\big[x_j(k) - H_j\big] - \big[x_i(k) - H_i\big]\Big)$ has been widely adopted in formation control [19] and references therein, we now apply the predictor-like protocol (4) to the formationable problem.

Definition 2: The discrete-time multi-agent system (2) is said to be formationable under protocol (27) if for any finite $x_i(0), u_i(-d), \cdots, u_i(-1)$, there exists a control gain K in (27) such that $\lim_{k\to\infty} E\|\big[x_j(k) - H_j\big] - \big[x_i(k) - H_i\big]\|^2 = 0, \forall i, j = 1, \cdots, N$.

Based on Theorem 3, sufficient and necessary conditions on formationability of the discrete-time multi-agent systems is stated as follows.

Corollary 1: Assume that Assumption 1-3 hold and $A(H_i - H_j) = (H_i - H_j)$, $\forall i, j = 1, \cdots, N$. The following statements hold:

1) If $\gamma_2 > \gamma_c$ where γ_c is given in Theorem 2, then the multi-agent system (2) is mean-square formationable under protocol (27).
2) Let $Rank(B) = 1$, the multi-agent system (2) is mean-square consensusable under protocol (27) only if (23) holds.
3) Let $A = a \geq 1, B = b > 0$, the multi-agent system (7) is mean-square formationable under protocol (27) if $\frac{\mu^2}{(\mu^2 + a^{2d}\sigma^2)}\Big[1 - \frac{\Big(\lambda_N(L_G) - \lambda_2(L_G)\Big)^2}{\Big(\lambda_2(L_G) + \lambda_N(L_G)\Big)^2}\Big] > 1 - \frac{1}{a^2}$.

Proof. Denote $\delta_i(k) = \big[x_i(k) - H_i\big] - \big[\bar{X}(k) - \bar{H}\big]$ where $\bar{X}(k) = \frac{1}{N}\sum_{i=1}^N x_i(k)$, $\bar{H} = \frac{1}{N}\sum_{i=1}^N H_i$. Then mean-square formationability is equivalent to that $\lim_{k\to\infty} E\|\delta_i(k)\|^2 = 0$. By stacking δ_i into a column vector $\delta(k) = \begin{bmatrix} \delta_1'(k) & \cdots & \delta_N'(k) \end{bmatrix}$, the following dynamical equation is in force:

$$\begin{aligned} \delta(k+1) &= (I_N \otimes A)\delta(k) - \mu(L_G \otimes BK)\hat{\delta}(k|k-d) \\ &\quad - w(k)(L_G \otimes BK)\hat{\delta}(k|k-d) \\ &\quad + \big[I_N \otimes (A - I_n)\big] \begin{bmatrix} H_1 - \bar{H} \\ \vdots \\ H_N - \bar{H} \end{bmatrix}. \end{aligned}$$

Together with $A(H_i - H_j) = (H_i - H_j)$, it follows that $(A - I_n)\big](H_i - \bar{H}) = 0$. The above equation is reformulated as

$$\begin{aligned} \delta(k+1) &= (I_N \otimes A)\delta(k) - \mu(L_G \otimes BK)\hat{\delta}(k|k-d) \\ &\quad - w(k)(L_G \otimes BK)\hat{\delta}(k|k-d). \end{aligned}$$

The remainder of the proof follows from Theorem 1, 3, 4 and 5. The proof is now completed. ■

VI. CONCLUSIONS

In this paper, we studied the consensusability of multi-agent systems with delay and packet dropout. By proposing a kind of predictor-like protocol, sufficient and necessary conditions have been given for the mean-square consensusability in terms of system matrices, time delay, communication graph and the packetdrop probability. It has been shown that the derived results are exactly the necessary and sufficient condition obtained in [19] when there are no delay and packet dropouts. Moreover, sufficient and necessary conditions have been obtained for the formationability of multi-agent systems.

APPENDIX

The following results can be obtained by similar discussions as in [22]. We give some brief proofs for the completion of the work.

Lemma 2: Assume that $P \in \{S \in R^{n\times n}, S \geq 0\}, R > 0, Q > 0$. Then the following statements hold.

1) With $K_P = -\Big[R + B'PB + B'(A')^d PA^d B\Big]^{-1} B'PA$, $g_\gamma(P) = \Phi(K_P, P)$.
2) $g_\gamma(P) = \min_K \Phi(K, P) \leq \Phi(K, P)$.
3) If $P_1 \leq P_2$, then $g_\gamma(P_1) \leq g_\gamma(P_2)$.
4) If $\gamma_1 \leq \gamma_2$, then $g_{\gamma_1}(P) \geq g_{\gamma_2}(P)$.
5) If $\alpha \in [0, 1]$, then $g_\gamma\Big(\alpha P_1 + (1-\alpha)P_2\Big) \geq \alpha g_\gamma(P_1) + (1-\alpha)g_\gamma(P_2)$.
6) $g_\gamma(P) \geq (1-\gamma)A'PA + Q$.
7) Provided that the equation $(1-\gamma)A'XA + Q = X$ has a solution $X > 0$. If $\bar{P} \geq g_\gamma(\bar{P})$, then $\bar{P} > 0$.

Proof.

1) Using the definition of K_P, we have
$$\begin{aligned} \Phi(K_P, P) &= A'PA + Q - \gamma A'PB\Big[R + B'PB \\ &\quad + B'(A')^d PA^d B\Big]^{-1} B'PA = g_\gamma(P). \end{aligned}$$
2) By using the definitions of $\Phi(K, P)$ and $\Psi(K, P)$, it holds that $\min_K \Phi(K, P) = \min_K \Psi(K, P)$. Combining with the fact that $P \geq 0, R > 0$, the minimizer K can be found by using $\frac{\partial \Psi(K,P)}{\partial K} = 0$, that is $0 = B'P(A + BK) + B'(A^d)'RA^d BK + RK$. This implies that $K = -\Big[R + B'PB + B'(A')^d PA^d B\Big]^{-1} B'PA$. Together with the fact 1), the result follows.
3) If $P_1 \leq P_2$, we have by using the above two facts
$$\begin{aligned} g_\gamma(P_1) &= \Phi(K_{P_1}, P_1) \leq \Phi(K_{P_2}, P_1) \\ &\leq \Phi(K_{P_2}, P_2) = g_\gamma(P_2). \end{aligned}$$

4) Noting that $A'PB\Big[R + B'PB + B'(A')^d PA^d B\Big]^{-1} B'PA \geq 0$, the fact follows directly.
5) Let $Z = \alpha P_1 + (1-\alpha)P_2$, then
$$g_\gamma(Z) = (1-\gamma)(A'ZA + Q) + \gamma\Psi(K_Z, Z).$$
Further rewriting $\Psi(K_Z, Z)$ yields that
$$\begin{aligned}\Psi(K_Z, Z) &= \alpha\Psi(K_Z, P_1) + (1-\alpha)\Psi(K_Z, P_2)\\ &\geq \alpha\Psi(K_{P_1}, P_1) + (1-\alpha)\Psi(K_{P_2}, P_2).\end{aligned}$$
Thus
$$\begin{aligned}g_\gamma(Z) &\geq (1-\gamma)(A'ZA + Q) + \gamma\alpha\Psi(K_{P_1}, P_1)\\ &\quad + \gamma(1-\alpha)\Psi(K_{P_2}, P_2)\\ &= \alpha g_\gamma(P_1) + (1-\alpha) g_\gamma(P_2).\end{aligned}$$
6) By using the facts that $F_1'PF_1 \geq 0, F_2'PF_2 \geq 0, K'RK \geq 0$, the result is straightforward.
7) Using the above fact, it follows that $\bar{P} \geq g_\gamma(\bar{P}) \geq (1-\gamma)A'\bar{P}A + Q$. Combining with $(1-\gamma)A'XA + Q = X$, there holds that $\bar{P} - X \geq (1-\gamma)A'(\bar{P} - X)A$, which gives $\bar{P} - X \geq 0$. Since $X > 0$, it is thus obtained that $\bar{P} > 0$.

Theorem 6: Suppose there exists a matrix $\tilde{K}$ and a positive-definite matrix $\tilde{P}$ such that $\tilde{P} > \Phi(\tilde{K}, \tilde{P})$. Then

1) for any initial condition P_0, the MARE converges, and the limit is independent of the initial condition $\lim_{t\to\infty} P_t = \lim_{t\to\infty} g_\gamma^t(P_0) = \bar{P}$.
2) $\bar{P}$ is the unique positive-semidefinite fixed point of the MARE.

Proof.

1) We first let the initial condition be $Q_0 = 0$. Let $Q_k = g_\lambda^k(0)$. Since $0 = Q_0 \leq Q_1 = Q$. From 3) of Lemma 2, it follows that $Q_1 = g_\gamma(Q_0) \leq g_\gamma(Q_1) = Q_2$. By induction, it is obtained that $Q_t \leq Q_{t+1}$ for $t \geq 0$. We show the sequence has an upper bound. Define the linear operator $\mathcal{L}(Y) = (1-\gamma)A'YA + \gamma(F_1'YF_1 + F_2'YF_2)$. Noting that $\tilde{P} > \Phi(\tilde{K}, \tilde{P}) = \mathcal{L}(\tilde{P}) + Q + \gamma K'RK \geq \mathcal{L}(\tilde{P})$. On the other hand, we have $Q_{t+1} = g_\gamma(Q_t) \leq \Phi(K_{\tilde{P}}, Q_t) = \mathcal{L}(\tilde{P}) + Q + \gamma K_{\tilde{P}}' RK_{\tilde{P}}$. In view of $Q + \gamma K_{\tilde{P}}' RK_{\tilde{P}} \geq 0$ and using Lemma 3 in [22], we conclude that there exists M_{Q_0} such that $Q_t \leq M_{Q_0}$ for $t \geq 0$. Accordingly, the sequence converges, i.e. $\lim_{t\to\infty} Q_t = \bar{P}$ and $\bar{P} = g_\gamma(\bar{P})$.
We next consider the case that the initial condition is selected as $R_0 \geq \bar{P}$. First, define $\bar{K} = -\Big[R + B'\bar{P}B + B'(A')^d \bar{P} A^d B\Big]^{-1} B'\bar{P}A$, $\bar{F}_1 = A + B\bar{K}$, $\bar{F}_2 = A^d B\bar{K}$ and $\hat{\mathcal{L}}(Y) = (1-\gamma)A'YA + \gamma(\bar{F}_1'Y\bar{F}_1 + \bar{F}_2'Y\bar{F}_2)$. It is noted that $\bar{P} = g_\gamma(\bar{P}) = \hat{\mathcal{L}}(\bar{P}) + Q + \gamma\bar{K}'R\bar{K} > \hat{\mathcal{L}}(\bar{P})$ where $Q > 0$ has been used in the derivation of last inequality. Using again Lemma 3 in [22], we have that $\lim_{t\to\infty} \hat{\mathcal{L}}^t(Y) = 0$ for all $Y \geq 0$. Since $R_0 \geq \bar{P}$, then $R_1 = g_\gamma(R_0) \geq g_\gamma(\bar{P}) = \bar{P}$. By induction, it follows that $R_t \geq \bar{P}$ for $t \geq 0$. Noting that
$$\begin{aligned}0 &\leq R_{t+1} - \bar{P} = g_\gamma(R_t) - g_\gamma(\bar{P})\\ &= \Phi(K_{R_t}, R_t) - \Phi(K_{\bar{P}}, \bar{P})\\ &\leq \Phi(K_{\bar{P}}, R_t) - \Phi(K_{\bar{P}}, \bar{P})\\ &= (1-\gamma)A'(R_t - \bar{P})A + \gamma\bar{F}_1'(R_t - \bar{P})\bar{F}_1\\ &\quad + \gamma\bar{F}_2'(R_t - \bar{P})\bar{F}_2 = \hat{\mathcal{L}}(R_t - \bar{P}) \to 0, t \to \infty,\end{aligned}$$
which gives that $\lim_{t\to\infty} R_{t+1} = \bar{P}$.
We now prove that the Riccati iteration converges to $\bar{P}$ for all initial values $P_0 \geq 0$. Let $Q_0 = 0$ and $R_0 = P_0 + \bar{P}$, it is obvious that $Q_0 \leq P_0 \leq R_0$. Consider the Riccati iterations initialized at Q_0, P_0 and R_0. It then follows that $Q_t \leq P_t \leq R_t, \forall t \geq 0$. Based on the above discussions, it has already been obtained that $\lim_{t\to\infty} Q_t = \lim_{t\to\infty} R_t = \bar{P}$. This implies that $\lim_{t\to\infty} P_t = \bar{P}$.
2) It is now claimed that the solution is unique. Otherwise, let $\hat{P}$ be another solution, i.e., $\hat{P} = g_\gamma(\hat{P})$ and let the initial value be $\hat{P}$. Thus we have a constant sequence with $\hat{P}$. Using the above prove, we have that the constant sequence also converges to $\bar{P}$. Thus $\hat{P} = \bar{P}$. The proof is now completed.

Theorem 7: If $(A, B, 0, A^d B)$ is mean-square stabilizable and A is unstable. Then there exists a $\gamma_c \in [0, 1)$ such that
$$\begin{aligned}&\lim_{t\to\infty} P_t = +\infty, \text{for } 0 \leq \gamma \leq \lambda_c \text{ and } \exists P_0 \geq 0\\ &P_t \leq M_{P_0}\ \forall t, \text{for } \lambda_c < \gamma \leq 1 \text{ and } \forall P_0 \geq 0\end{aligned}$$
where $M_{P_0} > 0$ depends on the initial condition $P_0 \geq 0$.

Proof. If $\gamma = 1$, the Riccati difference equation becomes the delay-dependent Riccati equation in [23] and [27] which has been shown to converge to a unique positive definite solution under the mean-square stabilizability of $(A, B, 0, A^d B)$ for the zero initial value. Based on similar discussions in Theorem 6, the Riccati iteration converges to a fixed point for any initial values $P_0 \geq 0$. Hence, P_t is always bounded for any initial values $P_0 \geq 0$. If $\gamma = 0$, the equation is reduced to $P_{t+1} = A'P_tA + Q$. If A is unstable, there always exists one initial value $P_0 \geq 0$ such that P_t is unbounded. Accordingly, the critical value $\gamma_c \in [0, 1)$ exists. We now prove there exists a single critical value. In fact, for any $\gamma > \gamma_c$, it is obtained that $P_{t+1} = g_\gamma(P_t) \leq g_{\gamma_c}(P_t)$ which is bounded. This completes the proof.

Theorem 8: If $(A, B, 0, A^d B)$ is mean-square stabilizable and A is unstable. Then the critical value satisfies $\underline{\gamma} \leq \gamma_c \leq \overline{\gamma}$ where
$$\begin{aligned}\underline{\gamma} &= arginf_\gamma\{\exists S | (1-\lambda)A'SA + Q = S, S \geq 0\}\\ \overline{\gamma} &= arginf_\gamma\{\exists (K, P) | P > \Phi(K, P)\}\end{aligned}$$

Proof. Consider $S_{t+1} = (1-\gamma)A'S_tA + Q$ with $S_0 = 0$, it is obtained that $\lim_{t\to\infty} S_t = \infty$ for $\gamma < \underline{\gamma}$ in the proof of Theorem 3 in [22]. Noting that the initial value $P_0 \geq 0$, i.e. $P_0 \geq S_0$. Assume that $P_t \geq S_t$. From 6) of Lemma 2, it holds that $P_{t+1} \geq (1-\gamma)A'P_tA + Q \geq (1-\gamma)A'S_tA + Q = S_{t+1}$. By induction, we have that $P_t \geq S_t, \forall t \geq 0, \forall P_0 \geq 0$. This implies that $\lim_{t\to\infty} P_t \geq \lim_{t\to\infty} S_t = \infty$. That is, P_t is unbounded for any $\gamma < \underline{\gamma}$ and any initial values $P_0 \geq 0$.

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Therefore, $\gamma_c \geq \underline{\gamma}$. On the other hand, when $\gamma > \overline{\gamma}$, there exists X such that $X > \Phi(K,X) \geq g_\gamma(X)$. Using 7) of Lemma 2, it yields that $X > 0$. Using Lemma 3 of [22], P_t is bounded. That is, $\gamma_c \leq \underline{\gamma}$.

Theorem 9: If $(A, B, 0, A^d B)$ is mean-square stabilizable, then the following statements are equivalent.

1) $\exists X$ such that $X > g_\gamma(X)$.
2) $\exists K, X > 0$ such that $X > \Phi(K,X)$.
3) $\exists Z$ and $0 \leq Y \leq I$ such that

$$\Gamma_\gamma(Y,Z) = \begin{bmatrix} Y & \sqrt{\gamma}(AY+BZ)' & \sqrt{\gamma}(A^dBZ)' & \sqrt{1-\gamma}(AY)' \\ \sqrt{\gamma}(AY+BZ) & Y & 0 & 0 \\ \sqrt{\gamma}A^dBZ & 0 & Y & 0 \\ \sqrt{1-\gamma}AY & 0 & 0 & Y \end{bmatrix} > 0.$$

Proof. Using facts 1) and 2) in Lemma 2, the equivalence between 1) and 2) follows. We now establish the equivalence between 2) and 3). Let $F = A+BK$, then $X > \Phi(K,X)$ is in fact $X > (1-\gamma)A'XA + \gamma F'XF + \gamma K'B'(A')^dXA^dBK + \gamma K'RK + Q$. By using Schur complement, the inequality is equivalent to

$$\begin{bmatrix} X - (1-\gamma)A'XA - \gamma K'B'(A')^dXA^dBK & \sqrt{\gamma}F' \\ \sqrt{\gamma}F & X^{-1} \end{bmatrix} > 0.$$

By applying similar procedures to Theorem 5 in [22], the result can be obtained. So we omit the details.

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