Homework4

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9/29/2019

Question 1:

(a)

To find the CDF with a given PDF of X, we have to integrate the PDF:

$$f_x(x) = 12x^2(1-x) = 12x^2 - 12x^3$$

that can be done using this formula:

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)dt = \int_{0}^{x} 12t^2 - 12t^3 dt = 4 * t^3 - 3 * t^4 \Big|_0^x = 4x^3 - 3x^4$$

(b) Now we can easily find $P(0 < X < 0.5)$ using the above calculated CDF $P(X \leq x)$:

$$P(0 < X < 0.5) = F(0.5) - F(0) = (4 * 0.5^3 - 3 * 0.5^4) - (4 * 0^3 - 3 * 0^4) = \frac{5}{16}$$

(c) We can calculate the mean by using the general way of expressing it and plugging it in our function with our given limits:

$$E(X) = \int x f_x(x)dx = \int_0^1 x * (12x^2 - 12x^3) = \int_0^1 12x^3 - 12x^4 = 3x^4 - 12 * \frac{x^5}{5}\Big|_0^1 = \frac{3}{5}$$

To obtain the variance, we have to calculate first $E(X^2)$, which we can do similar to the previous calculation using LOTUS:

$$E(X^2) = \int_0^1 x^2 * (12x^2 - 12x^3) = \int_0^1 12x^4 - 12x^5 = \frac{2}{5}$$

Now we use the defintion of the variance:

$$Var(X) = E(X^2) - E(X)^2 = \frac{2}{5} - (\frac{3}{5})^2 = \frac{1}{25}$$

Question 2

First, we want to calculate the CDF of X, so:

$$F_x(X) = P(X \leq x)$$

We can use our definition of X and re-write this as:

$$P(max(U_1, .., U_n) \leq x)$$

and since $U_1, ..., U_n$ can be seen as an independet, indetically distributed Uniform distribution, we can observe each U_n indepently and multiply them together later. But before we want to find the CDF: the CDF F(x) = x, because the PDF of a uniform distribution is a constant $c = \frac{1}{b-a} = \frac{1}{1-0} = 1$ in the case Unif(0,1) and if we integrate our PDF/constant we end up here. As already mentioned, we can now look at them indepentendly :

$$P(U_1 \leq x), ..., P(U_n \leq x)$$

and each of them is equal to x. So the CDF from 0 to 1 is x^n. Now we can easily find the PDF through deriving:

$$F(x) = x^n; F'(x) = f_x(x) = n * x^{n-1}$$

and lastly we can find the expected value EX by computing the integral from 0 to 1 of $x * f_x(x)$:

$$E(X) = \int_0^1 x * f_x(x) = \int_0^1 x * (n * x^{n-1}) = \int_0^1 n * x^n = n * \int_0^1 x^n = n * \frac{1}{n+1} * x^{n+1}\Big|_0^1 = \frac{n}{n+1}$$

Question 3

(a)

The definition of the standard normal CDF Φ is:

$$\Phi_y(z) = \int_{-\infty}^{z} f_y(x)dx$$

We are looking for the CDF of Y $P(Y \leq z) = P(|X| \leq z)$. From this, we can simply see that the CDF equals 0 for $z \leq 0$, since the absolute value can never be negative. Furthermore, we want to cover the case $z > 0$, where we can re-write this as because X can now also take on negativ values:

$$P(|X| \leq z) = P(-z \leq X \leq z) = \int_{-z}^{z} \varphi(x)dx = \Phi_x(z) - \Phi_x(-z)$$

Now we can apply an important property of the Normal distribution, which is symmetry:

$$\Phi(-z) = 1 - \Phi_x(z);$$

and if we plug this inside our calculation, it results in:

$$\Phi_x(z) - \Phi_x(-z) = \Phi_x(z) - (1 - \Phi_x(z)) = 2\Phi_x(z) - 1$$

Lastly, we want to specify X in terms of the standard normal distribution, where we know that for

$$Z \sim N(\mu, \sigma^2)$$

,we can say $X = \mu + \sigma Z$. If we rearrange this formula we get $Z = \frac{X-\mu}{\sigma}$. From here we can derive the CDF of X: $\Phi_x(x) = \Phi_z(x)$ and plug it finally in the above standing formula:

$$2\Phi_x(z) - 1 = 2\Phi_z(\frac{X-\mu}{\sigma}) - 1$$

Now we specified the CDF for:

$$z \leq 0 : F(Y) = P(|X| \leq z) = 0$$
$$z > 0 : F(Y) = P(|X| \leq z) = 2\Phi_x(z) - 1 = 2\Phi_z(\frac{X-\mu}{\sigma}) - 1$$

(b)

Now we are supposed to find the PDF of the random variable Y, which we can do by taking the derivative of the CDF of Y.

We know for z>0:

$$F(z) = 0; F'(z) = 0 = f_y(x)$$

furthermore we know that the CDF is not differentiable at z=0.

Lastly for z < 0:

$$F(z) = 2 * \Phi(z) - 1; F'(z) = 2 * f_x(z) = f_y(z)$$

Note here that $\Phi(z)$ is the integrand of $f_x(z)$ and we apply two laws of deriving:
1. the derivative of a constant is 0
2. while deriving the constant factor is not changing.
To that note, that we use the CDF of Y not in terms of the standard normal distribution, to make it simplier and more observable.

(c)

To check if Y is continuos, we will work with limits approaching form 0^+ and 0^-. This means that we will let z walk towards 0 from both sides.

$$lim_{z\to 0^+} f_y(z) = lim_{z\to 0^+} 2 * f_x(z) = 2 * lim_{z\to 0^+} \frac{1}{\sqrt{2\pi}} * e^{-\frac{z^2}{2}}$$

We can use that fact that the PDF of Y is twice the PDF of X, which we proved above. Here the exponential function $lim_{z\to 0^+} e^{-\frac{z^2}{2}}$ walks towards 1. So we end up with:

$$lim_{z\to 0^+} f_y(z) = 2 * \frac{1}{\sqrt{2\pi}}$$

Now lets approach z from 0^-:

$$lim_{z\to 0^-} f_y(z) = lim_{z\to 0^-} 0 = 0$$

Finally, we can observe that $lim_{z\to 0^-} f_y(z) \neq lim_{z\to 0^+} f_y(z)$ holds, which proves that f_y can not be continuos at 0. However, this should not be a problem as far as using PDF to find probabilities, since we can define z=0 as an arbitrary variable.