

# HomeWork3

Laurenz Ohnemüller

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## Question 1

The expected value is calculated as follows:

$$E(x) = \sum_{n=1}^{\infty} x_n * P(X = x_n)$$

(a)

If he walks away with the \$16,000 than the expected value is \$16,000.

$$E(X) = 16,000 * 1 = 16,000$$

(b)

He applies his “lifeline” to the 10th question, which means that he increases the probability that he correctly answers the question (by guessing) from 0.25 to 0.5. If he answers wrong, he leaves with just \$1,000. Nevertheless suppose he gets it right, then he moves on to the 11th question, where he has a 0.25 chance of guessing the question correct and then leaving with \$64,000. However, his chance of wrongly answering the question is 0.75, where he would leave with \$32,000. Now everything can be calculated together.

```
e_b = 0.5 * 1000 + 0.5 * (0.25*64000 + 0.75*32000)
e_b
```

```
## [1] 20500
```

The expected value is \$20,500.

(c)

If he applies his lifeline to the 11th question than, his chances of winning the 10th round are 0.25 and of losing and leaving with \$1,000 are 0.75. If he wins the round, then his chances of winning \$64,000 is as high as of leaving with \$32,000, both have the probability of 0.5. Now we apply the expected value formula again:

```
e_c = 0.75 * 1000 + 0.25 * (0.5*64000 + 0.5*32000)
e_c
```

```
## [1] 12750
```

The expected value is \$12,750.

Thus the highest expected value has option b, with \$20,500.

## Question 2

(a)

To find the PMF for the random variable X we have to differentiate between  $k=0$  and  $k>0$ .

Lets start with  $k=0$ , which can occur in two different ways (as explained in the task).

$$P(X = 0) = p + (1 - p) * \frac{\lambda^0}{0!} * e^{-\lambda} = p + (1 - p) * e^{-\lambda}$$

If we consider  $k > 0$ , then:

$$P(X = k) = (1 - p) * \frac{\lambda^k}{k!} * e^{-\lambda}$$

Here, we do not add the probability  $p$  that HEADS occur, since  $k > 0$  can only be achieved by flipping TAILS.

(b)

Note that we will compare our results with the results from the previous questions. First, we want to show that  $(1 - I) * Y$  (remember that  $I$  is the indicator random variable which can take on the values 0 and 1) equals 0 in two ways, either  $I = 1$  or  $Y = 0$  and  $I = 0$ . Now create a new random variable  $Z$ :

$$P(Z = 0) = P(I = 1) + P(I = 0) * P(Y = 0)$$

$$P(Z = 0) = p + (1 - p) * \frac{\lambda^0}{0!} * e^{-\lambda} = p + (1 - p) * e^{-\lambda}$$

This equals  $P(X=0)$  from the previous question.

Now for  $Z > 0$ , there is only one possible way. Let  $Y = k$  ( $k > 0$ ), and  $I = 0$

$$P(Z = k) = p + (1 - p) * P(Y = k)$$

$$P(Z = k) = (1 - p) * \frac{\lambda^k}{k!} * e^{-\lambda}$$

As we can see this corresponds with the previous findings. This proves that  $X$  and  $(1-I)Y$  have the same PMF and with that the same distribution.

(c)

Using the representation from (b):

$$E(X) = E((1 - I)Y) = E(1 - I) * E(Y) = (1 - p) * \lambda$$

(d)

$$Var(X) = E(X^2) - E(X)^2$$

From (c) we already know that  $E(X) = (1 - p) * \lambda$ ,

$$Var(X) = E(X^2) - (1 - p)^2 * \lambda^2$$

Now we still need to find  $E(X^2)$ , where we can use the fact that  $X$  has the same distribution as  $(1 - I)Y$ :

$$E(X^2) = E((1 - I)^2 Y^2) = E((1 - I)^2) * E(Y^2) = (1 - p) * (\lambda + \lambda^2)$$

To find  $E(Y^2)$ , we can just re-formulate the definition of the variance:

$$E(Y^2) = Var(Y) + E(Y)^2 = \lambda + \lambda^2$$

Finally, we can combine these:

$$Var(X) = E(X^2) - E(X)^2 = (1 - p) * (\lambda + \lambda^2) - (1 - p)^2 * \lambda^2$$

### Question 3

If we take the definition of the expectation:

$$E(X) = \sum_{k=1}^{\infty} x_k P(X = x_k)$$

so:

$$E(2^X) = \sum_{k=1}^{\infty} 2^k P(X = x_k)$$

and the PMF of the Poisson distribution:

$$P(X = k) = e^{-\lambda} * \frac{\lambda^k}{k!}$$

then we can insert the PMF into the definition of expectation.

$$E(2^X) = \sum_{k=1}^{\infty} 2^k * \frac{e^{-\lambda} * \lambda^k}{k!}$$

$$E(2^X) = e^{-\lambda} \sum_{k=1}^{\infty} 2^k * \frac{\lambda^k}{k!}$$

Now we apply the Taylor series:

$$E(2^X) = e^{-\lambda} * e^{2*\lambda} = e^{\lambda}$$

This proves that  $E(2^x)$  is finite.