QCQT QE8 Quantum Control 2024 - 2025

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Problem set 4

Problem 1 | A particle of mass m and charge e is confined inside a one-dimensional infinite quantum well, with $V(x) = \begin{cases} 0, & 0<x<L \\ \infty, & x<0 \text{ or } x>L \end{cases}$

At time $t=0$ the particle is in the ground state.

For $t>0$ the particle is subject to a time dependent potential

$$V^{int}(x,t) = exE_0 e^{-\frac{t}{\tau}} \quad , \quad E_0, \tau > 0$$

• We know that for the infinite square well the eigenstates are given by

$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$$

As we said at $t=0$, the system is in the eigenstate $\psi_1(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{\pi x}{L}\right)$

• The first excited state is given by $\psi_2(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{2\pi x}{L}\right)$

• We also know that the corresponding eigenenergies are: $E_1 = \frac{\hbar^2 \pi^2}{2mL^2}$

and $E_2 = \frac{2\hbar^2 \pi^2}{mL^2}$

• We want to find the probability at time t to have a transition from the ground state to the first excited state of the system. Working in the interaction picture, as we have seen in previous lectures, this probability is given by:

$$P_{1\to 2}(t) = |\langle \psi_2 | \hat{U}_I(t,0) | \psi_1 \rangle|^2$$

Limiting ourselves in the first order, we can expand this as:

$$P_{1\to 2}(t) = \left| -\frac{i}{\hbar} \int_0^t e^{i\omega_{21} t'} V_{21}^{int}(t')\, dt' \right|^2$$

where: $\cdot\, \omega_{21} = \dfrac{E_2 - E_1}{\hbar} = \dfrac{2\hbar \pi^2}{mL^2} - \dfrac{\hbar \pi^2}{2mL^2} = \dfrac{3\hbar \pi^2}{2mL^2}$

$$\cdot\, V_{21}^{int}(t) = \langle \psi_2 | e x E_0 e^{-\frac{t}{\tau}} | \psi_1 \rangle = e E_0 \langle \psi_2 | x | \psi_1 \rangle e^{-\frac{t}{\tau}} = e E_0 \int_0^L \frac{2}{L} x \sin\left(\frac{2\pi x}{L}\right) \sin\left(\frac{\pi x}{L}\right) dx\, e^{-\frac{t}{\tau}}$$

$$= \frac{2}{L} e E_0 \int_0^L x \frac{1}{2} \left[\cos\left(\frac{\pi x}{L}\right) - \cos\left(\frac{3\pi x}{L}\right) \right] dx\, e^{-\frac{t}{\tau}}$$

Using the trig. identity $\sin A \cdot \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+$

$$= \frac{e E_0}{L} \left[\int_0^L x \cos\left(\frac{\pi x}{L}\right) dx - \int_0^L x \cos\left(\frac{3\pi x}{L}\right) \right] dx\, e^{-\frac{t}{\tau}} = \frac{e E_0}{L} \left[-\frac{2L^2}{\pi^2} + \frac{2L^2}{9\pi^2} \right] = -\frac{16 e E_0}{9\pi^2} e^{-\frac{t}{\tau}}$$

So
$$\int_0^t e^{i\omega_{21} t'} V_{21}^{int}(t')\, dt' = -\frac{16 e E_0}{9\pi^2} \int_0^t e^{i\omega_{21} t'} \cdot e^{-\frac{t'}{\tau}} dt' = -\frac{16 e E_0}{9\pi^2} \int_0^t e^{\left(i\omega_{21} - \frac{1}{\tau}\right)t'} dt'$$

$$= -\frac{16 e E_0}{9\pi^2} \cdot \frac{e^{i\omega_{21} t} e^{-\frac{t}{\tau}} - 1}{i\omega_{21} - \frac{1}{\tau}}$$

So
$$P_{1\to 2}(t) = \left| \frac{i}{\hbar} \cdot \frac{16 e E_0}{9\pi^2} \cdot \frac{e^{i\omega_{21} t} e^{-\frac{t}{\tau}} - 1}{i\omega_{21} - \frac{1}{\tau}} \right|^2$$

$$= \frac{1}{\hbar^2} \cdot \left(\frac{16 e E_0}{9\pi^2} \right)^2 \cdot \frac{1 - 2e^{-\frac{t}{\tau}} \cos(\omega_{21} t) + e^{-\frac{2t}{\tau}}}{\omega_{21}^2 + \frac{1}{\tau^2}}$$

. Then the transition probability for $t \to +\infty$ is :

$$P_{1\to 2}(+\infty) = \frac{1}{\hbar^2} \cdot \left(\frac{16eE_0}{9\pi^2}\right)^2 \cdot \frac{1 - 2e^{-\infty} \cos(+\infty) + e^{-\infty}}{w_{21}^2 + \frac{1}{\tau^2}}$$

$=0$ $=0$

$$= \frac{1}{\hbar^2} \cdot \left(\frac{16eE_0}{9\pi^2}\right)^2 \cdot \frac{1}{w_{21}^2 + \frac{1}{\tau^2}}$$

Problem 2 We have a two level quantum system being in the ground state at time $t=0$, that interacts with a weak electromagnetic field with constant detuning Δ and slowly varying Rabi Frequency $\Omega(t)$, with $\Omega(t) << |\Delta|$ and we assume that $\Omega(0)=0$

Working in the interaction picture, we have seen in the lecture notes that the interaction Hamiltonian under the Rotating Wave approximation is given by:

$$V_I^{int} = \frac{\hbar}{2} \begin{bmatrix} 0 & \Omega^*(t) e^{-i\phi} e^{-i\Delta t} \\ \Omega(t) e^{i\phi} e^{i\Delta t} & 0 \end{bmatrix}$$

We take $\phi = 0$ for simplicity, so that :

$$V_I^{int} = \frac{\hbar}{2} \begin{bmatrix} 0 & \Omega^*(t) e^{-i\Delta t} \\ \Omega(t) e^{i\Delta t} & 0 \end{bmatrix}$$

a) Using first order perturbation theory, we can say that the probability amplitude for the upper state is given by:

$$c_2^{(1)}(t) = -\frac{i}{\hbar}\int_0^t V_{21}^{int}(t')\,dt'$$

It is $V_{21}^{int}(t) = \frac{\hbar}{2}\Omega(t)e^{i\Delta t}$

So: $$c_2^{(1)}(t) = -\frac{i}{\hbar}\int_0^t \frac{\hbar}{2}\Omega(t')e^{i\Delta t'}\,dt'$$

$$= -\frac{i}{2}\int_0^t \Omega(t')e^{i\Delta t'}\,dt'$$

$$= -\frac{i}{2}\left[\frac{1}{i\Delta}\left[\Omega(t')e^{i\Delta t'}\right]_0^t - \underbrace{\frac{1}{i\Delta}\int_0^t \frac{\dot{\Omega}(t')}{2}e^{i\Delta t'}\,dt'}_{\text{We neglect this term since } \frac{\dot{\Omega}(t)}{\Omega(t)} \ll \Delta}\right]$$

$$\approx -\frac{i}{2}\left[\frac{1}{i\Delta}\left(\Omega(t)e^{i\Delta t} - \Omega(0)\right)\right]$$

$$\approx -\frac{\Omega(t)}{2\Delta}e^{i\Delta t}$$ since we know $\Omega(0)=0$

So then $P_2(t) = P_{1\to 2}(t) = |c_2^{(1)}(t)|^2 = \left|-\frac{\Omega(t)}{2\Delta}e^{i\Delta t}\right|^2 = \frac{\Omega^2(t)}{4\Delta^2}$

b) We have seen in the lecture notes that after using the adiabatic approximation for a two level system interacting with an electromagnetic field, we obtain the eigenstates in the transformed basis as: $|+\rangle = \sin\left(\frac{\theta(t)}{2}\right)|1\rangle + \cos\left(\frac{\theta(t)}{2}\right)|2\rangle$

$$|-\rangle = \cos\left(\frac{\theta(t)}{2}\right)|1\rangle - \sin\left(\frac{\theta(t)}{2}\right)|2\rangle$$

with $\cos(\theta(t)) = \frac{\Delta}{\sqrt{\Delta^2+\varrho^2(t)}}$

$$\sin(\theta(t)) = \frac{\varrho(t)}{\sqrt{\Delta^2+\varrho^2(t)}}$$

We are told that the system is initially in the ground state $|1\rangle$

Also, we recall that the Hamiltonian in the Schödinger picture is $\hat{H}(t) = \frac{\hbar}{2}\begin{bmatrix} 0 & \varrho(t) \\ \varrho(t) & 2\Delta \end{bmatrix}$

Knowing that $\varrho(0)=0$, it is $\hat{H}(0) = \begin{bmatrix} 0 & 0 \\ 0 & 2\Delta \end{bmatrix}$

It is obvious that the instanteneous eigenstates of the Hamiltonian used in the adiabatic theorem are just $|-\rangle = |1\rangle$ and $|+\rangle = |2\rangle$

Then, given that the system is in the ground state $|1\rangle$ at time 0, it will stay in this state at later times t.

Then $|\phi(t)\rangle = |-\rangle = \cos\left(\frac{\theta(t)}{2}\right)|1\rangle - \sin\left(\frac{\theta(t)}{2}\right)|2\rangle$

Then $P_2(t) = \left|-\sin\left(\frac{\theta(t)}{2}\right)\right|^2 = \sin^2\left(\frac{\theta(t)}{2}\right)$

We have shown in the lecture notes that $\theta(t) = \arctan\left(\frac{\varrho(t)}{\Delta(t)}\right)$

Then knowing that $\varrho(t) \ll |\Delta|$, we use the small angle approximation and get $\theta(t) \approx \frac{\varrho(t)}{\Delta(t)}$

So $P_2(t) \approx \sin^2\left(\frac{\Omega(t)}{2\Delta}\right)$

We use the small angle approximation and finally get $P_2(t) \approx \frac{\Omega^2(t)}{4\Delta^2}$ which is the same as the result of the previous question

Problem 3 | I

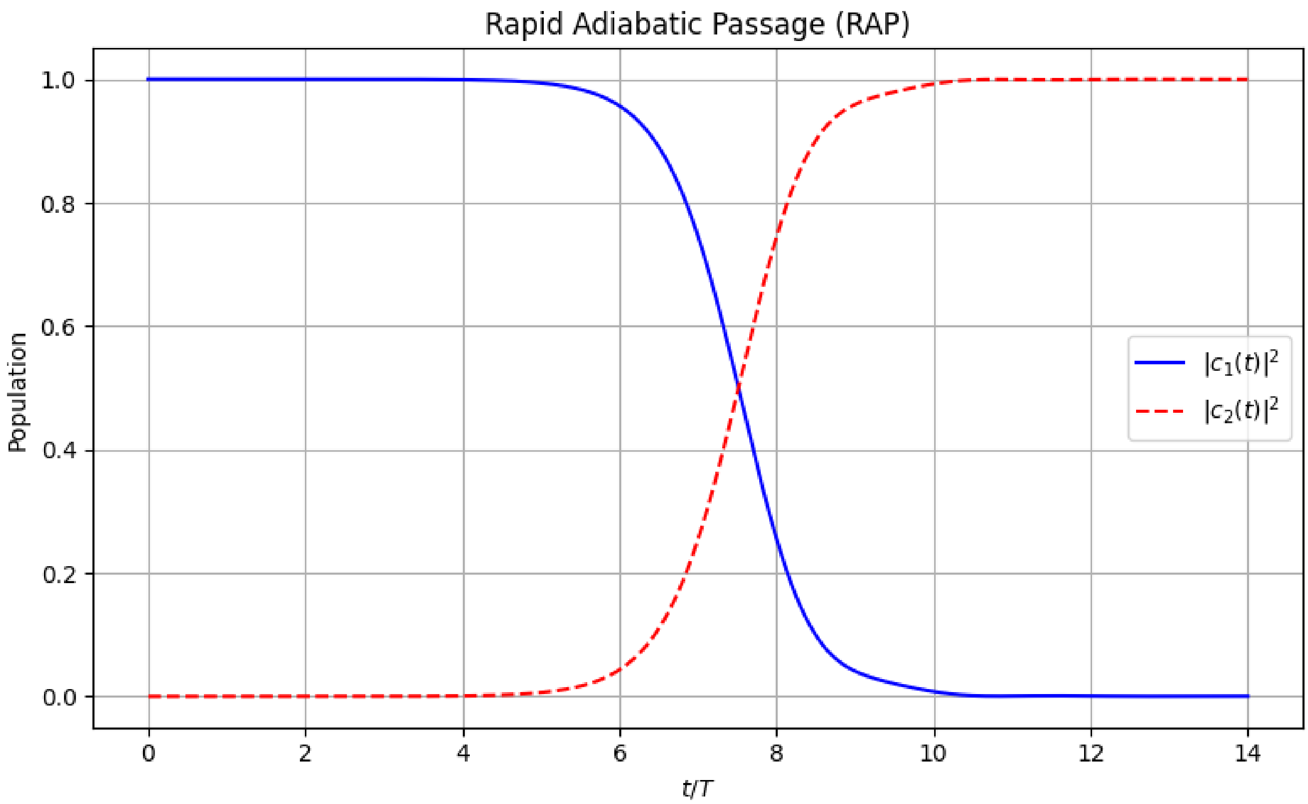
Problem set 4 - Problem 3 - I
We have a two-level quantum system, which at time $t = 0$ is in the lower state and which interacts with an electromagnetic field with time dependent detuning $\Delta(t) = \delta_0 \tanh[(t - t_0)/T]$ and time dependent Rabi frequency $\Omega(t) = \Omega_0[(t - t_0)/T]$

Given that $t_0 = 7.5T$, I had to choose parameters $\delta_0 T$ and $\Omega_0 T$ in the region between 1 and 5 such that to have efficient population transfer to the upper state using **Rapid Adiabatic Passage**.

Below you will find the plotted results after choosing:

- $\delta_0 T = 3$
- $\Omega_0 T = 3$

and also you will find the respective code.
It is obvious from the plot that we achieve efficient population transfer to the upper state.



Link for code repository: `https://github.com/topalidisd-qcqt-duth/qe8-problem-set-4/blob/main/problem-4.ipynb`

```
import numpy as np
from scipy.integrate import solve_ivp
import matplotlib.pyplot as plt

T = 1
phi = 0
omega0 = 3/ T
delta0 = 3 / T
t0 = 7.5 * T


def sech(x):
    return 1 / np.cosh(x)
```

```python
def omega(t):
    return omega0 * sech((t - t0) / T)

def delta(t):
    return delta0 * np.tanh((t - t0) / T)


def diff_eqs(t, y):
    c1, c2 = y
    dc1 = (-1j / 2) * omega(t) * np.exp(-1j * phi) * c2
    dc2 = (-1j / 2) * omega(t) * np.exp(1j * phi) * c1 - 1j * delta(t) * c2
    return [dc1, dc2]


y0 = [1.0 + 0j, 0.0 + 0j]

t_span = (0, 14 * T)
t_eval = np.linspace(*t_span, 500)

sol = solve_ivp(diff_eqs, t_span, y0, t_eval=t_eval, rtol=1e-9, atol=1e-9)


pop_c1 = np.abs(sol.y[0])**2
pop_c2 = np.abs(sol.y[1])**2

plt.figure(figsize=(8, 5))
plt.plot(sol.t / T, pop_c1, 'b', label=r'$|c_1(t)|^2$')
plt.plot(sol.t / T, pop_c2, 'r--', label=r'$|c_2(t)|^2$')
plt.xlabel(r'$t/T$')
plt.ylabel('Population')
plt.title('Rapid Adiabatic Passage (RAP)')
plt.legend()
plt.grid(True)
plt.tight_layout()
plt.show()
```

I a) We saw in the lecture notes that the transitionless quantum driving method, proposed by Berry, gives the following counter diabatic Hamiltonian:

$$\hat{H}_{CD}(t) = i\hbar \sum_n \left[|\dot{\psi}_n(t)\rangle\langle\psi_n(t)| - \langle\psi_n(t)|\dot{\psi}_n(t)\rangle\, |\psi_n(t)\rangle\langle\psi_n(t)| \right]$$

For a two level quantum system interacting with an electromagnetic field, we consider the adiabatic basis with eigenstates: $|\phi_+(t)\rangle$ and $|\phi_-(t)\rangle$

From the lecture notes we know that :

- $|\dot{\phi}_+(t)\rangle = \frac{\dot{\theta}(t)}{2} |\phi_-(t)\rangle$
- $|\dot{\phi}_-(t)\rangle = -\frac{\dot{\theta}(t)}{2} |\phi_+(t)\rangle$
- $\langle\phi_-(t)|\dot{\phi}_-(t)\rangle = 0$
- $\langle\phi_+(t)|\dot{\phi}_+(t)\rangle = 0$

We proceed to calculate the Berry's counter diabatic Hamiltonian presented above:

$$\hat{H}_{CD}(t) = i\hbar \Big[|\dot{\phi}_+(t)\rangle\langle\phi_+(t)| - \underbrace{\langle\phi_+(t)|\dot{\phi}_+(t)\rangle}_{=0} |\phi_+(t)\rangle\langle\phi_+(t)|$$
$$+ |\dot{\phi}_-(t)\rangle\langle\phi_-(t)| - \underbrace{\langle\phi_-(t)|\dot{\phi}_-(t)\rangle}_{=0} |\phi_-(t)\rangle\langle\phi_-(t)| \Big]$$

$$= i\hbar \left[|\dot{\phi}_+(t)\rangle\langle\phi_+(t)| + |\dot{\phi}_-(t)\rangle\langle\phi_-(t)| \right]$$

$$= i\hbar \left[\frac{\dot{\theta}(t)}{2} |\phi_-(t)\rangle\langle\phi_+(t)| - \frac{\dot{\theta}(t)}{2} |\phi_+(t)\rangle\langle\phi_-(t)| \right]$$

$$= \frac{\hbar}{2}\dot{\theta}(t) \underbrace{\left[i|\phi_-(t)\rangle\langle\phi_+(t)| - i|\phi_+(t)\rangle\langle\phi_-(t)| \right]}_{\sigma_y \text{ pauli matrix in the adiabatic basis}}$$

$$= \frac{\hbar}{2}\dot{\theta}(t)\sigma_y \qquad \text{or in matrix form} \quad \hat{H}_{CD}(t) = \frac{i\hbar}{2} \begin{bmatrix} 0 & -\dot{\theta}(t) \\ \dot{\theta}(t) & 0 \end{bmatrix}$$

b) We have a two level quantum system interacting with an external electromagnetic field.

The Rabi frequency is constant $\Omega(t) = \Omega_0$

and the detuning changes linearly with time $\Delta(t) = \alpha t$

The interaction time is from $-\infty$ to $+\infty$

As we have seen before the Hamiltonian is.

$$\hat{H}(t) = \frac{\hbar}{2}\begin{bmatrix} 0 & \Omega(t)e^{-i\phi} \\ \Omega(t)e^{i\phi} & 2\Delta(t) \end{bmatrix} = \frac{\hbar}{2}\begin{bmatrix} 0 & \Omega_0 e^{-i\phi} \\ \Omega_0 e^{i\phi} & 2\alpha t \end{bmatrix}$$

Now, as we showed in the previous question, the counter diabatic Hamiltonian for the two level system is:

$$\hat{H}_{CD}(t) = \frac{i\hbar}{2}\begin{bmatrix} 0 & -\dot{\theta}(t) \\ \dot{\theta}(t) & 0 \end{bmatrix}$$

We have also shown in the lecture notes that: $$\dot{\theta}(t) = \frac{\Delta(t)\overset{=0}{\dot{\Omega}(t)} - \dot{\Delta}(t)\Omega(t)}{\Omega^2(t) + \Delta^2(t)}$$

$$= -\frac{\alpha\Omega_0}{\Omega_0^2 + \alpha^2 t^2}$$

So $$\hat{H}_{CD}(t) = \frac{\hbar}{2}\begin{bmatrix} 0 & i\dfrac{\alpha\Omega_0}{\Omega_0^2 + \alpha^2 t^2} \\ -i\dfrac{\alpha\Omega_0}{\Omega_0^2 + \alpha^2 t^2} & 0 \end{bmatrix}$$

Then $\hat{H}_{total}(t) = \hat{H}(t) + \hat{H}_{CD}(t) = \frac{\hbar}{2}\begin{bmatrix} 0 & \Omega_0 e^{-i\phi} \\ \Omega_0 e^{i\phi} & 2at \end{bmatrix} + \frac{\hbar}{2}\begin{bmatrix} 0 & i\frac{a\Omega_0}{\Omega_0^2+a^2t^2} \\ -i\frac{a\Omega_0}{\Omega_0^2+a^2t^2} & 0 \end{bmatrix}$

$$= \frac{\hbar}{2}\begin{bmatrix} 0 & \Omega_0 e^{-i\phi} + i\frac{a\Omega_0}{\Omega_0^2+a^2t^2} \\ \Omega_0 e^{i\phi} - i\frac{a\Omega_0}{\Omega_0^2+a^2t^2} & 0 \end{bmatrix}$$

Problem 4 The Lindblad master equation reads as:

$$\frac{d\hat{\rho}(t)}{dt} = -\frac{i}{\hbar}\left[\hat{H}(t), \hat{\rho}(t)\right] + \sum_k \gamma_k \left(\hat{L}_k \hat{\rho}(t) \hat{L}_k^\dagger - \frac{1}{2}\left\{\hat{L}_k^\dagger \hat{L}_k, \hat{\rho}(t)\right\}\right) \tag{4.1}$$

- We have a two level system that exhibits both decay and dephasing effects.
- The density matrix for the two level system is:

$$\hat{\rho}(t) = \begin{bmatrix} \rho_{11}(t) & \rho_{12}(t) \\ \rho_{21}(t) & \rho_{22}(t) \end{bmatrix}$$

- The Lindblad operator that represents a decay transition from the excited state $|\psi_2\rangle$ to the ground state $|\psi_1\rangle$ is $L_\downarrow = |\psi_1\rangle\langle\psi_2|$

Applying $\hat{L}_\downarrow$ to $|\psi_2\rangle$ we get: $\hat{L}_\downarrow|\psi_2\rangle = |\psi_1\rangle\langle\psi_2|\psi_2\rangle = |\psi_1\rangle$

Also $\hat{L}_\downarrow$ is NOT Hermitian as $\hat{L}_\downarrow^\dagger = |\psi_2\rangle\langle\psi_1| \neq \hat{L}_\downarrow$

We also define the decay rate as $\gamma_\downarrow = \Gamma_{21}$

We will substitute the decay operator into the master equation and omit the term coming from $-\frac{i}{\hbar}[\hat{H}(t),\hat{\rho}(t)]$ as we are focusing on the decay effect. However, when fully studying the dynamics of the system we should include it.

We have: $(4.1) \Rightarrow \dfrac{d\hat{\rho}(t)}{dt} = \gamma_\downarrow\left[\hat{L}_\downarrow\hat{\rho}(t)\hat{L}_\downarrow^\dagger - \frac{1}{2}\left(\hat{L}_\downarrow^\dagger\hat{L}_\downarrow\hat{\rho}(t) + \hat{\rho}(t)\hat{L}_\downarrow^\dagger\hat{L}_\downarrow\right)\right]$ (4.2)

It is:

- $\hat{L}_\downarrow\hat{\rho}(t)\hat{L}_\downarrow^\dagger = |\psi_1\rangle\langle\psi_2|\hat{\rho}(t)|\psi_2\rangle\langle\psi_1| = \rho_{22}(t)|\psi_1\rangle\langle\psi_1| = \begin{bmatrix}\rho_{22}(t) & 0\\ 0 & 0\end{bmatrix}$

- $\hat{L}_\downarrow^\dagger\hat{L}_\downarrow = |\psi_2\rangle\langle\psi_1|\psi_1\rangle\langle\psi_2| = |\psi_2\rangle\langle\psi_2|$

- $\left\{\hat{L}_\downarrow^\dagger\hat{L}_\downarrow, \hat{\rho}(t)\right\} = |\psi_2\rangle\langle\psi_2|\hat{\rho}(t) + \hat{\rho}(t)|\psi_2\rangle\langle\psi_2| = \begin{bmatrix}0 & \rho_{12}(t)\\ \rho_{21}(t) & 2\rho_{22}(t)\end{bmatrix}$

So $(4.2) \Rightarrow \dfrac{d\hat{\rho}(t)}{dt} = \Gamma_{21}\left(\begin{bmatrix}\rho_{22}(t) & 0\\ 0 & 0\end{bmatrix} - \frac{1}{2}\begin{bmatrix}0 & \rho_{12}(t)\\ \rho_{21}(t) & 2\rho_{22}(t)\end{bmatrix}\right) = \begin{bmatrix}\Gamma_{21}\rho_{22}(t) & -\frac{\Gamma_{21}}{2}\rho_{12}(t)\\ -\frac{\Gamma_{21}}{2}\rho_{21}(t) & -\Gamma_{21}\rho_{22}(t)\end{bmatrix}$

Which gives the following terms to the equations of the density matrix elements

- $$\frac{d\rho_{11}}{dt} = \Gamma_{21}\rho_{22}$$
- $$\frac{d\rho_{12}}{dt} = -\frac{\Gamma_{21}}{2}\rho_{12}(t)$$
- $$\frac{d\rho_{21}}{dt} = -\frac{\Gamma_{21}}{2}\rho_{21}(t)$$
- $$\frac{d\rho_{22}}{dt} = -\Gamma_{21}\rho_{22}(t)$$

- The Lindblad operator that represents the dephasing for a two level system is:

$$\hat{L}_d = |\psi_2\rangle\langle\psi_2| - |\psi_1\rangle\langle\psi_1| = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

It is $\hat{L}_d^\dagger = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}^T = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \hat{L}_d$

The respective dephasing rate is $\gamma_d = \frac{\gamma_{12}^d}{2}$

We will substitute the decay operator into the master equation and omit the term coming from $-\frac{i}{\hbar}[\hat{H}(t), \hat{\rho}(t)]$ as we are focusing on the decay effect. However, when fully studying the dynamics of the system we should include it.

We have: $(4.1) \Rightarrow$

$$\frac{d\hat{\rho}(t)}{dt} = \frac{\gamma_{12}^d}{2}\left[\hat{L}_d\hat{\rho}(t)\hat{L}_d^\dagger - \frac{1}{2}\hat{L}_d^\dagger\hat{L}_d\hat{\rho}(t) + \hat{\rho}(t)\hat{L}_d^\dagger\hat{L}_d\right] \quad (4.3)$$

I_+ is:

- $\hat{L}_d \hat{\rho}(t) \hat{L}_d^\dagger = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \rho_{11}(t) & \rho_{12}(t) \\ \rho_{21}(t) & \rho_{22}(t) \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \rho_{11} & -\rho_{12} \\ -\rho_{21} & \rho_{22} \end{bmatrix}$

- $\hat{L}_\downarrow^\dagger \hat{L}_\downarrow = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \hat{I}$

- $\left\{ \hat{L}_\downarrow^\dagger \hat{L}_\downarrow, \hat{\rho}(t) \right\} = \hat{I}\hat{\rho}(t) + \hat{\rho}(t)\hat{I} = 2\hat{\rho}(t) = \begin{bmatrix} 2\rho_{11}(t) & 2\rho_{12}(t) \\ 2\rho_{21}(t) & 2\rho_{22}(t) \end{bmatrix}$

So (4.3) $\Rightarrow$ $$\frac{d\hat{\rho}(t)}{dt} = \frac{\gamma_{12}^d}{2} \left(\begin{bmatrix} \rho_{11} & -\rho_{12} \\ -\rho_{21} & \rho_{22} \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 2\rho_{11}(t) & 2\rho_{12}(t) \\ 2\rho_{21}(t) & 2\rho_{22}(t) \end{bmatrix} \right)$$

$$= \frac{\gamma_{12}^d}{2} \begin{bmatrix} 0 & -2\rho_{12}(t) \\ -2\rho_{21}(t) & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\gamma_{12}^d \rho_{12}(t) \\ -\gamma_{12}^d \rho_{21}(t) & 0 \end{bmatrix}$$

Which gives the following terms to the equations of the density matrix elements

- $\dfrac{d\rho_{12}}{dt} = -\gamma_{12}^d \rho_{12}(t)$

- $\dfrac{d\rho_{21}}{dt} = -\gamma_{12}^d \rho_{21}(t)$

Now if we combine the terms we got from the decay effect with the terms we get for the dephasing effect, we end up with the following total terms:

- $$\frac{d\rho_{11}}{dt} = \Gamma_{21}\,\rho_{22}(t)$$

- $$\frac{d\rho_{12}}{dt} = -\left(\frac{\Gamma_{21}}{2} + \gamma_{12}^{d}\right)\rho_{12}(t)$$

- $$\frac{d\rho_{22}}{dt} = -\Gamma_{21}\,\rho_{22}(t)$$

- $$\frac{d\rho_{21}}{dt} = -\left(\frac{\Gamma_{21}}{2} + \gamma_{21}^{d}\right)\rho_{21}(t)$$

- One useful property of the density matrix is that it's trace is 1, so:

$$Tr(\rho) = 1 \Rightarrow \rho_{11}(t) + \rho_{22}(t) = 1 \Rightarrow \rho_{11}(t) = 1 - \rho_{22}(t).$$

- With that in mind, the solutions to these equations are given as follows:

- $$\rho_{22}(t) = \rho_{22}(0)\,e^{-\Gamma_{21}t}$$

- $$\rho_{11}(t) = 1 - \rho_{22}(0)\,e^{-\Gamma_{21}t}$$

- $$\rho_{12}(t) = \rho_{12}(0)\,e^{-\left(\frac{\Gamma_{21}}{2} + \gamma_{21}^{d}\right)t}$$

- $$\rho_{21}(t) = \rho_{21}(0)\,e^{-\left(\frac{\Gamma_{21}}{2} + \gamma_{21}^{d}\right)t}$$

• Assuming that the initial state at time $t=0$ is: $|\psi(0)\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |2\rangle)$

Then $\rho(0) = |\psi(0)\rangle\langle\psi(0)| = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$

So $\rho_{11}(0) = \frac{1}{2}$, $\rho_{12}(0) = -\frac{1}{2}$, $\rho_{21}(0) = -\frac{1}{2}$, $\rho_{22}(0) = \frac{1}{2}$

Then

• $\rho_{22}(t) = \frac{1}{2} e^{-\Gamma_{21} t}$

• $\rho_{11}(t) = 1 - \frac{1}{2} e^{-\Gamma_{21} t}$

• $\rho_{12}(t) = -\frac{1}{2} e^{-\left(\frac{\Gamma_{21}}{2} + \gamma_{21}^{d}\right)t}$

• $\rho_{21}(t) = -\frac{1}{2} e^{-\left(\frac{\Gamma_{21}}{2} + \gamma_{21}^{d}\right)t}$

or $\hat{\rho}(t) = \begin{bmatrix} 1 - \frac{1}{2} e^{-\Gamma_{21} t} & -\frac{1}{2} e^{-\left(\frac{\Gamma_{21}}{2} + \gamma_{21}^{d}\right)t} \\ -\frac{1}{2} e^{-\left(\frac{\Gamma_{21}}{2} + \gamma_{21}^{d}\right)t} & \frac{1}{2} e^{-\Gamma_{21} t} \end{bmatrix}$

Problem 5 a) We consider the case of symmetric quantum well potentials with the centre $\Delta = 0$.

As we have seen previously in the lecture notes, the Hamiltonian in the localized basis is:

$$\hat{H} = \begin{bmatrix} E_0 & -\bar{A} \\ -\bar{A} & E_0 \end{bmatrix} \quad \text{with } \bar{A} \approx \frac{2\hbar^2 \pi^2}{ma^2} \cdot \frac{1}{k_{II} a}$$

with a the width of the quantum well and

$$k_{II} = \sqrt{\frac{2m(V_0 - E)}{\hbar}}$$

The density matrix in the localized basis is: $\rho = \begin{bmatrix} \rho_{LL} & \rho_{LR} \\ \rho_{RL} & \rho_{RR} \end{bmatrix}$

The system is initially in the left quantum well i.e. $\rho(0) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

The dynamics of the density matrix are described by:

$$\frac{d\rho}{dt} = -\frac{i}{\hbar} [\hat{H}, \hat{\rho}]$$

$$\Rightarrow \frac{d\rho}{dt} = -\frac{i}{\hbar}\left[\hat{H}\hat{\rho} - \hat{\rho}\hat{H}\right]$$

$$\Rightarrow \frac{d\rho}{dt} = -\frac{i}{\hbar}\left(\begin{bmatrix} E_0 & -\vec{A} \\ -\vec{A} & E_0 \end{bmatrix} \begin{bmatrix} \rho_{LL} & \rho_{LR} \\ \rho_{RL} & \rho_{RR} \end{bmatrix} - \begin{bmatrix} \rho_{LL} & \rho_{LR} \\ \rho_{RL} & \rho_{RR} \end{bmatrix} \begin{bmatrix} E_0 & -\vec{A} \\ -\vec{A} & E_0 \end{bmatrix} \right)$$

$$\Rightarrow \frac{d\rho}{dt} = -\frac{i}{\hbar}\left(\begin{bmatrix} E_0\rho_{LL} - \vec{A}\rho_{RL} & E_0\rho_{LR} - \vec{A}\rho_{RR} \\ -\vec{A}\rho_{LL} + E_0\rho_{RL} & -\vec{A}\rho_{LR} + E_0\rho_{RR} \end{bmatrix} - \begin{bmatrix} E_0\rho_{LL} - \vec{A}\rho_{LR} & -\vec{A}\rho_{LL} + E_0\rho_{LR} \\ E_0\rho_{RL} - \vec{A}\rho_{RR} & -\vec{A}\rho_{RL} + E_0\rho_{RR} \end{bmatrix} \right)$$

$$\Rightarrow \frac{d\rho}{dt} = -\frac{i}{\hbar}\begin{bmatrix} \vec{A}(\rho_{LR} - \rho_{RL}) & \vec{A}(\rho_{LL} - \rho_{RR}) \\ \vec{A}(\rho_{RR} - \rho_{LL}) & \vec{A}(\rho_{RL} - \rho_{LR}) \end{bmatrix}$$

$$\Rightarrow \frac{d}{dt}\begin{bmatrix} \rho_{LL} & \rho_{LR} \\ \rho_{RL} & \rho_{RR} \end{bmatrix} = -\frac{i\vec{A}}{\hbar}\begin{bmatrix} \rho_{LR} - \rho_{RL} & 1 - 2\rho_{RR} \\ 2\rho_{RR} - 1 & \rho_{RL} - \rho_{LR} \end{bmatrix}$$

It is $Tr(\rho) = 1 \Rightarrow \rho_{LL} + \rho_{RR} = 1$

$\Rightarrow \rho_{LL} = 1 - \rho_{RR}$

which gives the following differential equations:

- $\dfrac{d\rho_{LL}}{dt} = -\dfrac{i\vec{A}}{\hbar}(\rho_{LR} - \rho_{RL})$ (5.1)

- $\dfrac{d\rho_{LR}}{dt} = -\dfrac{i\vec{A}}{\hbar}(1 - 2\rho_{RR})$ (5.2)

- $\dfrac{d\rho_{RL}}{dt} = -\dfrac{i\vec{A}}{\hbar}(2\rho_{RR} - 1)$ (5.3)

- $\dfrac{d\rho_{RR}}{dt} = -\dfrac{i\vec{A}}{\hbar}(\rho_{RL} - \rho_{LR})$ (5.4)

• We differentiate (5.4) with respect to time and get:

$$(5.4) \Rightarrow \frac{d^2\rho_{RR}}{dt^2} = -\frac{i\bar{A}}{\hbar}\frac{d\rho_{RL}}{dt} + \frac{i\bar{A}}{\hbar}\frac{d\rho_{LR}}{dt}$$

We substitute from (5.2) and (5.3)

$$\Rightarrow \frac{d^2\rho_{RR}}{dt} = -\frac{i\bar{A}}{\hbar}\left[-\frac{i\bar{A}}{\hbar}(2\rho_{RR}-1)\right] + \frac{i\bar{A}}{\hbar}\left[-\frac{i\bar{A}}{\hbar}(1-2\rho_{RR})\right]$$

$$\Rightarrow \frac{d^2\rho_{RR}}{dt} = -\frac{2\bar{A}^2}{\hbar^2}\rho_{RR} + \frac{\bar{A}^2}{\hbar^2} + \frac{\bar{A}^2}{\hbar^2} - \frac{2\bar{A}^2}{\hbar^2}\rho_{RR}$$

$$\Rightarrow \frac{d^2\rho_{RR}}{dt} = -\left(\frac{2\bar{A}}{\hbar}\right)^2\rho_{RR} + 2\left(\frac{\bar{A}}{\hbar}\right)^2$$

$$\Rightarrow \frac{d^2\rho_{RR}}{dt} = -\Omega^2\rho_{RR} + \frac{\Omega^2}{2}$$ We set $\Omega = \frac{2\bar{A}}{\hbar}$

$$\Rightarrow \frac{d^2\rho_{RR}}{dt} + \Omega^2\left(\rho_{RR} - \frac{1}{2}\right) = 0$$

which corresponds to the equation of a harmonic oscillator centered at $\frac{1}{2}$.

The solution for this is known as: $\rho_{RR}(t) = A\sin(\Omega t) + B\cos(\Omega t) + \frac{1}{2}$

We know that $\rho_{RR}(0) = 0 \Rightarrow B = -\frac{1}{2}$

and $$\frac{d\rho_{RR}(0)}{dt} = -\frac{i\bar{A}}{\hbar}\left(\underbrace{\rho_{LR}(0)}_{=0} - \underbrace{\rho_{RL}(0)}_{=0}\right) \Rightarrow A\cos(0) + \frac{1}{2}\sin(0) = 0$$

$$\Rightarrow A = 0$$

So then $$\rho_{RR}(t) = \frac{1}{2} - \frac{1}{2}\cos(\Omega t) \qquad (5.5)$$

We know that $\rho_{LL}(t) = 1 - \rho_{RR}(t) \Rightarrow \rho_{LL}(t) = \frac{1}{2} + \frac{1}{2}\cos(\underline{\Omega} t)$

- Next, we substitute (5.5) into (5.2) and (5.3) and get:

$$(5.2) \Rightarrow \frac{d\rho_{LR}}{dt} = -i\frac{\underline{\Omega}}{2}\left(1 - 1 + \cos(\underline{\Omega} t)\right)$$

$$\Rightarrow \frac{d\rho_{LR}}{dt} = -i\frac{\underline{\Omega}}{2}\cos(\underline{\Omega} t)$$

$$\Rightarrow \rho_{LR} = -\frac{i}{2}\sin(\underline{\Omega} t)$$

and

$$(5.3) \Rightarrow \frac{d\rho_{RL}}{dt} = -i\frac{\underline{\Omega}}{2}\left(1 - \cos(\underline{\Omega} t) - 1\right)$$

$$\Rightarrow \frac{d\rho_{RL}}{dt} = i\frac{\underline{\Omega}}{2}\cos(\underline{\Omega} t)$$

$$\Rightarrow \rho_{RL} = \frac{i}{2}\sin(\underline{\Omega} t)$$

So
$$\hat{\rho}(t) = \begin{bmatrix} \frac{1}{2} + \frac{1}{2}\cos(\underline{\Omega} t) & -\frac{i}{2}\sin(\underline{\Omega} t) \\ \frac{i}{2}\sin(\underline{\Omega} t) & \frac{1}{2} - \frac{1}{2}\cos(\underline{\Omega} t) \end{bmatrix} \quad \text{with } \underline{\Omega} = \frac{2\tilde{A}}{\hbar}$$

b) Now we consider the effect of pure dephasing in a), with dephasing rate γ, $\Delta = 0$ and the quantum system being initially in the left quantum well ie $\rho(0) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

As we have seen in the lecture notes, the dephasing effect will give rise to an additional term in the off digonal density matrix elements' equations, such as:

- $\dfrac{d\rho_{LL}}{dt} = -\dfrac{i\bar{A}}{\hbar}\left(\rho_{LR}(t) - \rho_{RL}(t)\right)$ (5.6)

- $\dfrac{d\rho_{LR}}{dt} = -\dfrac{i\bar{A}}{\hbar}\left(\rho_{LL}(t) - \rho_{RR}(t)\right) - \gamma\rho_{LR}(t)$ (5.7)

- $\dfrac{d\rho_{RL}}{dt} = -\dfrac{i\bar{A}}{\hbar}\left(\rho_{RR}(t) - \rho_{LL}(t)\right) - \gamma\rho_{RL}(t)$ (5.8)

- $\dfrac{d\rho_{RR}}{dt} = -\dfrac{i\bar{A}}{\hbar}\left(\rho_{RL}(t) - \rho_{LR}(t)\right)$ (5.9)

We define $V(t) = \rho_{RL}(t) - \rho_{LR}(t)$

So $\dfrac{dV}{dt} = \dfrac{d\rho_{RL}}{dt} - \dfrac{d\rho_{LR}}{dt} \Rightarrow \dfrac{dV}{dt} = -\dfrac{i\bar{A}}{\hbar}\left(\rho_{RR} - \rho_{LL}\right) - \gamma\rho_{RL} + \dfrac{i\bar{A}}{\hbar}\left(\rho_{LL} - \rho_{RR}\right) + \gamma\rho_{LR}$

$\Rightarrow \dfrac{dV}{dt} = -\dfrac{2i\bar{A}}{\hbar}\left[\rho_{RR}(t) - \rho_{LL}(t)\right] - \gamma\left[\rho_{RL}(t) - \rho_{LR}(t)\right] \Rightarrow \dfrac{dV}{dt} = -\dfrac{2i\bar{A}}{\hbar}\left[\rho_{RR}(t) - \rho_{LL}(t)\right] - \gamma V(t)$ (5.10)

From (5.6) and (5.9) we have:

$$\frac{d}{dt}\left[\rho_{RR}(t)-\rho_{LL}(t)\right]=\frac{d\rho_{RR}}{dt}-\frac{d\rho_{LL}}{dt}$$

$$=-\frac{i\bar{A}}{\hbar}\left(\rho_{RL}(t)-\rho_{LR}(t)\right)+\frac{i\bar{A}}{\hbar}\left(\rho_{LR}(t)-\rho_{RL}(t)\right)$$

$$=-\frac{2i\bar{A}}{\hbar}\left[\rho_{RL}(t)-\rho_{LR}(t)\right]=-\frac{2i\bar{A}}{\hbar}V(t) \qquad (5.11)$$

We differentiate (5.10) with respect to time and using (5.11) we get:

$$(5.10) \Rightarrow \frac{d^2V}{dt^2}=-\frac{2i\bar{A}}{\hbar}\frac{d}{dt}\left[\rho_{RR}(t)-\rho_{LL}(t)\right]-\gamma\frac{dV}{dt}$$

$$\Rightarrow \frac{d^2V}{dt^2}=-\frac{2i\bar{A}}{\hbar}\cdot\left(-\frac{2i\bar{A}}{\hbar}\right)V(t)-\gamma\frac{dV}{dt}$$

$$\Rightarrow \frac{d^2V}{dt^2}=-\varrho^2V(t)-\gamma\frac{dV}{dt}$$

We set $\varrho=\frac{2\bar{A}}{\hbar}$

$$\Rightarrow \frac{d^2V}{dt^2}+\gamma\frac{dV}{dt}+\varrho^2V(t)=0 \qquad (5.12)$$

This equation is the same as in a decaying harmonic oscillator.

We are looking for solutions of the form $V(t)=Ae^{\rho t}$

We substitute into (5.12) and get: $\rho^2Ae^{\rho t}+\gamma\rho Ae^{\rho t}+\varrho^2Ae^{\rho t}=0$

$$\Rightarrow \rho^2+\gamma\rho+\varrho^2=0$$

$$\Rightarrow \rho_{1,2}=\frac{-\gamma\pm\sqrt{\gamma^2-4\varrho^2}}{2}$$

We assume $\Omega \neq \frac{\gamma}{2}$. Then the solution for $V(t)$ is:

$$V(t) = A_1 e^{-\frac{\gamma t}{2} + \sqrt{\frac{\gamma^2}{4} - \Omega^2}\, t} + A_2 e^{-\frac{\gamma t}{2} - \sqrt{\frac{\gamma^2}{4} - \Omega^2}\, t}$$

We know that our initial conditions are $\rho_{LL}(0) = 1$, $\rho_{LR}(0) = 0$, $\rho_{RL}(0) = 0$, $\rho_{RR}(0) = 0$

Then $V(0) = \rho_{RL}(0) - \rho_{LR}(0) \Rightarrow A_1 e^{\gamma/2} + A_2 e^{\gamma/2} = 0 \Rightarrow A_1 + A_2 = 0 \Rightarrow A_2 = -A_1$

So we get the simplified form: $V(t) = A_1 e^{-\frac{\gamma t}{2}} \left(e^{\sqrt{\frac{\gamma^2}{4} - \Omega^2}\, t} - e^{-\sqrt{\frac{\gamma^2}{4} - \Omega^2}\, t} \right)$

Then $$\frac{dV}{dt} = A_1 \frac{\left[\left(2\sqrt{\frac{\gamma^2}{4} - \Omega^2} - \gamma \right) e^{2\sqrt{\frac{\gamma^2}{4} - \Omega^2}\, t} + 2\sqrt{\frac{\gamma^2}{4} - \Omega^2} + \gamma \right] e^{-\sqrt{\frac{\gamma^2}{4} - \Omega^2}\, t - \frac{\gamma t}{2}}}{2}$$

Then $$\left.\frac{dV}{dt}\right|_{t=0} = \left.\frac{d\rho_{RL}}{dt}\right|_{t=0} - \left.\frac{d\rho_{LR}}{dt}\right|_{t=0}$$

$$\Rightarrow A_1 \frac{2\sqrt{\frac{\gamma^2}{4} - \Omega^2} - \gamma + 2\sqrt{\frac{\gamma^2}{4} - \Omega^2} + \gamma}{2} = -\frac{i\bar{A}}{\hbar}\left(\rho_{RR}(0) - \rho_{LL}(0)\right) - \gamma \rho_{RL}(0)$$
$$-\frac{i\bar{A}}{\hbar}\left(\rho_{LL}(0) - \rho_{RR}(0)\right) - \gamma \rho_{LR}(0)$$

$$\Rightarrow A_1 \cdot 2\sqrt{\frac{\gamma^2}{4} - \Omega^2} = i\Omega \Rightarrow A_1 = \frac{i\Omega}{2\sqrt{\frac{\gamma^2}{4} - \Omega^2}}$$

Thus it is: $V(t) = \dfrac{i\underline{\Omega}}{2\sqrt{\frac{\gamma^2}{4} - \underline{\Omega}^2}} e^{-\gamma t/2} \left(e^{\sqrt{\frac{\gamma^2}{4} - \underline{\Omega}^2}\, t} - e^{-\sqrt{\frac{\gamma^2}{4} - \underline{\Omega}^2}\, t} \right)$

Now we integrate (5.6) and obtain (as seen in the lecture notes):

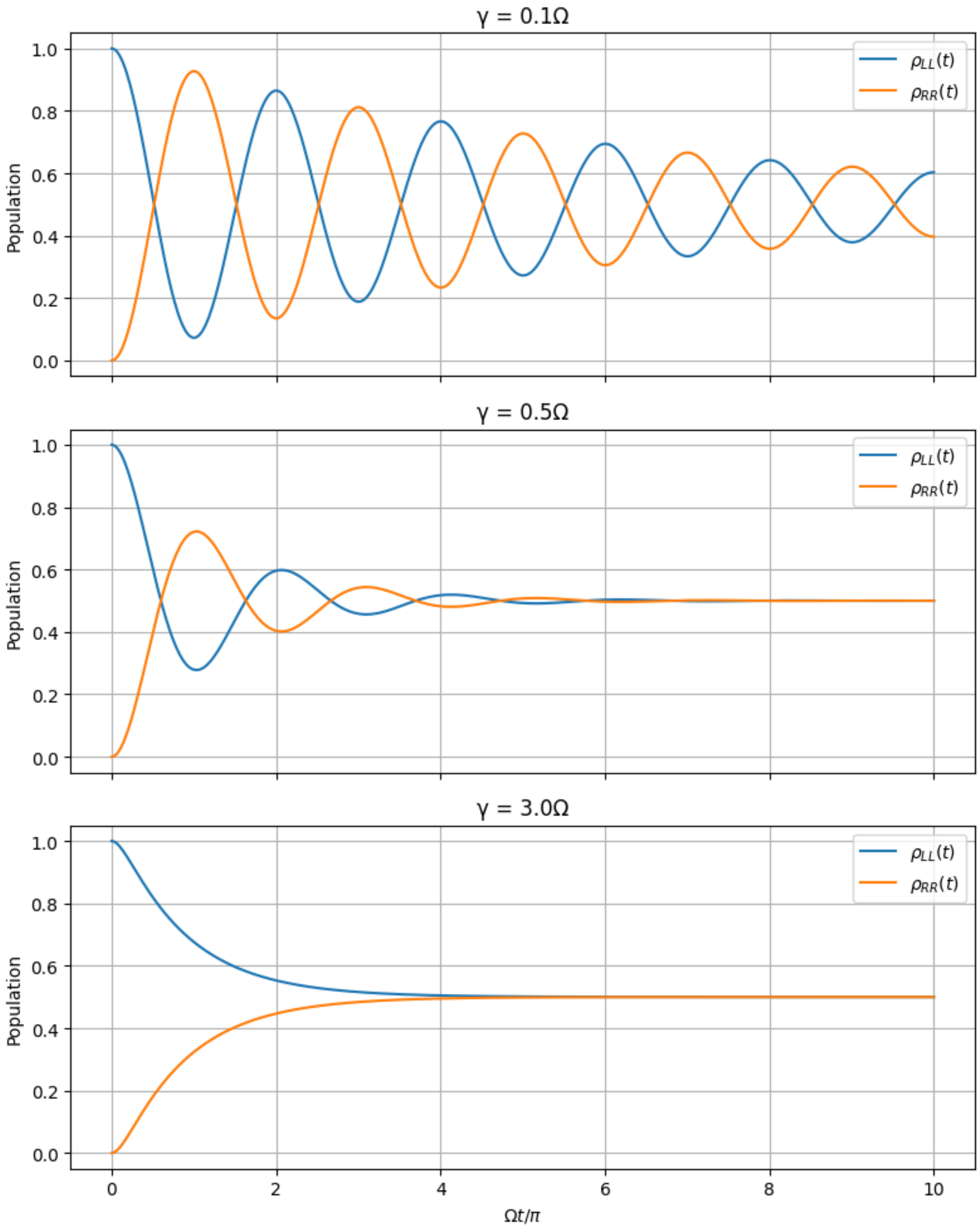
$$(5.6) \Rightarrow \rho_{LL}(t) = 1 + i\frac{\underline{\Omega}}{2}\int_0^t V(t')\,dt'$$

which gives: $$\rho_{LL}(t) = 1 - \frac{\underline{\Omega}^2}{4\sqrt{\frac{\gamma^2}{4} - \underline{\Omega}^2}} \left[\frac{e^{-\left(\frac{\gamma}{2} - \sqrt{\frac{\gamma^2}{4} - \underline{\Omega}^2}\right)t} - 1}{-\frac{\gamma}{2} + \sqrt{\frac{\gamma^2}{4} - \underline{\Omega}^2}} + \frac{e^{-\left(\frac{\gamma}{2} + \sqrt{\frac{\gamma^2}{4} - \underline{\Omega}^2}\right)t} - 1}{\frac{\gamma}{2} + \sqrt{\frac{\gamma^2}{4} - \underline{\Omega}^2}} \right]$$

$$\rho_{RR}(t) = 1 - \rho_{LL}(t)$$

with $\underline{\Omega} = \dfrac{2\bar{A}}{\hbar}$

Problem set 4 - Problem 5 b)
After studying the problem of symmetric potential wells ($\Delta = 0$), for the case that the quantum system is intially in the left quantum well, we considered the effect of pure dephasing in the results.
Below you will find the plots of $\rho_{LL}(t)$ and $\rho_{RR}(t)$ as a function of $\frac{\Omega t}{\pi}$ for $\gamma = 0.1\Omega$, $\gamma = \frac{\Omega}{2}$ and $\gamma = 3\Omega$

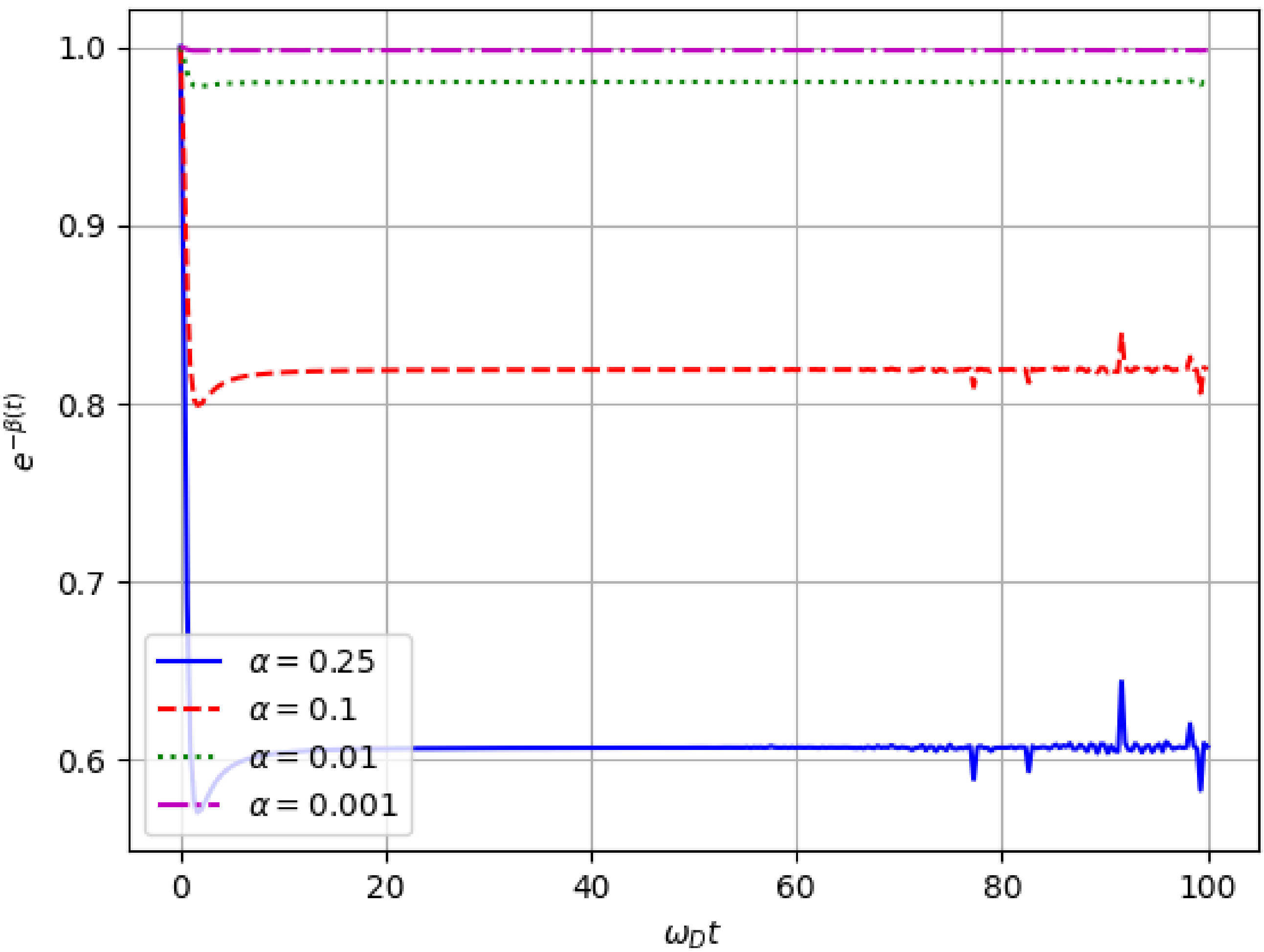


Link for code repository: `https://github.com/topalidisd-qcqt-duth/qe8-problem-set-4/blob/main/problem-5.ipynb`

```
import numpy as np
from scipy.integrate import solve_ivp
import matplotlib.pyplot as plt


Omega = 1.0
gammas = [0.1 * Omega, 0.5 * Omega, 3 * Omega]
```

```
t_vals = np.linspace(0, 10 * np.pi, 500)
rho0 =  np.array([1, 0, 0, 0], dtype=complex)

def lindblad(t, y, Omega, gamma):
    rho_LL, rho_LR, rho_RL, rho_RR = y

    d_rho_LL = - 1j * Omega/2 * (rho_LR - rho_RL)
    d_rho_RR = - 1j * Omega/2 * (rho_RL - rho_LR)
    d_rho_LR = - 1j * Omega/2 * (rho_LL - rho_RR) - gamma * rho_LR
    d_rho_RL = - 1j * Omega/2 * (rho_RR - rho_LL) - gamma * rho_RL

    return np.array([d_rho_LL, d_rho_LR, d_rho_RL, d_rho_RR], dtype=complex)


fig, axs = plt.subplots(len(gammas), 1, figsize=(8, 10), sharex=True)

for i, gamma in enumerate(gammas):
    sol = solve_ivp(lindblad, [0, 10 * np.pi], rho0, t_eval=t_vals, args=(Omega, gamma))
    t_plot = Omega * sol.t / np.pi
    rho_LL = sol.y[0]
    rho_RR = sol.y[3]

    axs[i].plot(t_plot, rho_LL, label=r'$\rho_{LL}(t)$')
    axs[i].plot(t_plot, rho_RR, label=r'$\rho_{RR}(t)$')
    axs[i].set_title(f'g = {gamma:.1f}Omega')
    axs[i].set_ylabel('Population')
    axs[i].legend()
    axs[i].grid(True)

axs[-1].set_xlabel(r'$\Omega t/\pi$')
plt.tight_layout()
plt.show()
```

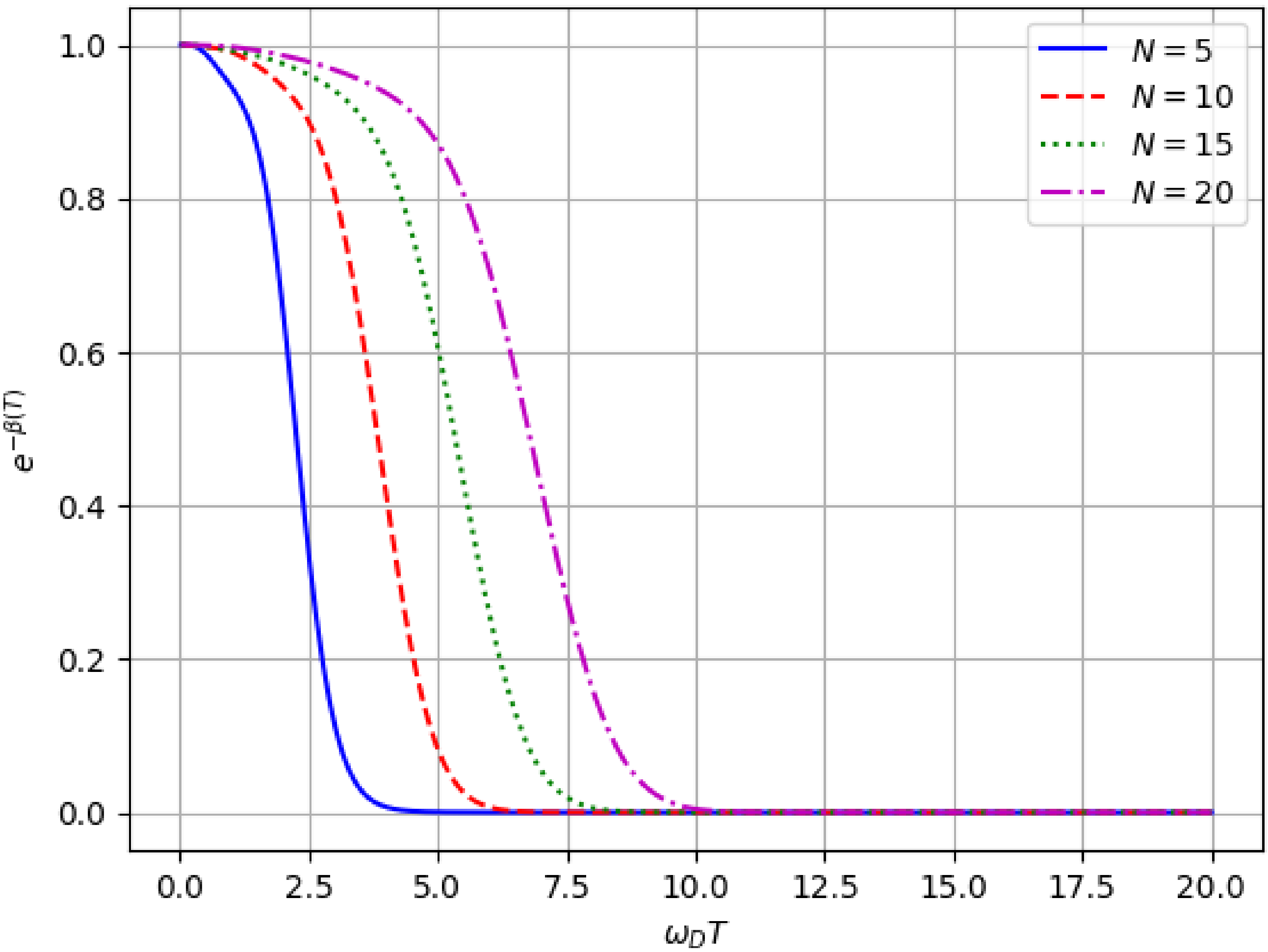
Problem set 4 - Problem 6
The time evolution of $e^{-\beta(t)}$ for the super Ohmic bath $J(\omega) = 2\alpha\omega^3 e^{-\frac{\omega}{\omega_D}}$ for $\alpha = 0.25$, $\alpha = 0.1$, $\alpha = 0.01$ and $\alpha = 0.001$




```
import numpy as np
import matplotlib.pyplot as plt
from scipy.integrate import quad

omega_D = 1
t_vals = np.linspace(0.01, 100, 300)
omegaDt = omega_D * t_vals

alphas = [0.25, 0.1, 0.01, 0.001]
styles = ['b-', 'r--', 'g:', 'm-.']
labels = [r'$\alpha=0.25$', r'$\alpha=0.1$', r'$\alpha=0.01$', r'$\alpha=0.001$']


def J(omega, t, omega_D):
    return omega * np.exp(-omega / omega_D) * np.sin(omega * t / 2)**2

for alpha, style, label in zip(alphas, styles, labels):
    beta_vals = []
    for t in t_vals:
        integral, _ = quad(J, 0, np.inf, args=(t, omega_D))
        beta_t = 4 * alpha * integral
        beta_vals.append(np.exp(-beta_t))

    plt.plot(omegaDt, beta_vals, style, label=label)
```

```
    plt.xlabel(r'$\omega_D t$')
    plt.ylabel(r'$e^{-\beta(t)}$')
    plt.grid(True)
    plt.legend()
    plt.show()
```

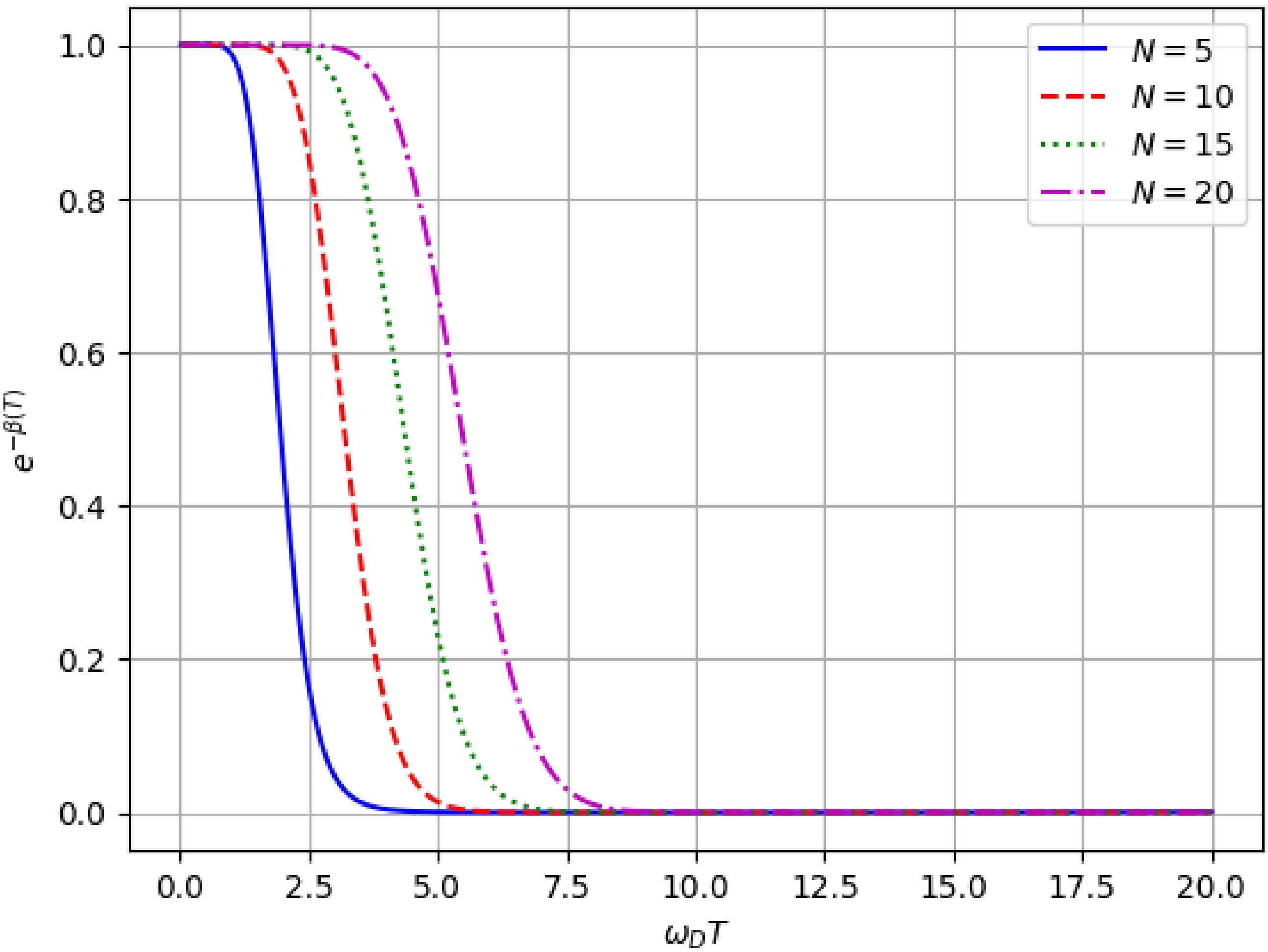
The time evolution of $e^{-\beta(t)}$ for the super Ohmic bath $J(\omega) = 2\alpha\omega^3 e^{-\frac{\omega}{\omega_D}}$ for $\alpha = 0.25$, under the application of $N = 5$, $N = 10$, $N = 15$ and $N = 20$ pulse, using the CPMG pulse sequence.




```
omega_D = 1
alpha = 0.25
T_vals = np.linspace(0.01, 20, 300)
omegaDT = omega_D * T_vals

N_vals = [5, 10, 15, 20]
styles = ['b-', 'r--', 'g:', 'm-.']
labels = [rf'$N={N}$' for N in N_vals]

def J(omega, alpha, omega_D):
    return 2 * alpha * omega**3 * np.exp(-omega / omega_D)

def cpmg(omega, T, N):
    if N % 2 == 0:
        return (4 / omega**2) * np.tan(omega * T / (2 * N + 2))**2 * np.cos(omega * T / 2)**2
    else:
        return (4 / omega**2) * np.tan(omega * T / (2 * N + 2))**2 * np.sin(omega * T / 2)**2

def integrand(omega, T, alpha, N, omega_D):
    return J(omega, alpha, omega_D) * cpmg(omega, T, N)


for N, style, label in zip(N_vals, styles, labels):
    beta_vals = []
    for T in T_vals:
        result, _ = quad(integrand, 0.01, 50, args=(T, alpha, N, omega_D))
```

```
            beta_vals.append(np.exp(-result))

        plt.plot(omegaDT, beta_vals, style, label=label)

    plt.xlabel(r'$\omega_D T$')
    plt.ylabel(r'$e^{-\beta(T)}$')
    plt.grid(True)
    plt.legend()
    plt.show()
```

The time evolution of $e^{-\beta(t)}$ for the super Ohmic bath $J(\omega) = 2\alpha\omega^3 e^{-\frac{\omega}{\omega_D}}$ for $\alpha = 0.25$, under the application of $N = 5$, $N = 10$, $N = 15$ and $N = 20$ pulse, using the UDD pulse sequence.




```
omega_D = 1
alpha = 0.25
T_vals = np.linspace(0.01, 20, 300)
omegaDT = omega_D * T_vals

N_vals = [5, 10, 15, 20]
styles = ['b-', 'r--', 'g:', 'm-.']
labels = [rf'$N={N}$' for N in N_vals]

def J(omega, alpha, omega_D):
    return 2 * alpha * omega**3 * np.exp(-omega / omega_D)

def udd(omega, T, N):
    arg = omega * T / 2
    return (16 * (N + 1)**2 / omega**2) * jv(N + 1, arg)**2

def integrand(omega, T, alpha, N, omega_D):
    return J(omega, alpha, omega_D) * udd(omega, T, N)

for N, style, label in zip(N_vals, styles, labels):
    beta_vals = []
    for T in T_vals:
        result, _ = quad(integrand, 0.01, 50, args=(T, alpha, N, omega_D))
        beta_vals.append(np.exp(-result))

    plt.plot(omegaDT, beta_vals, style, label=label)
```

```
    plt.xlabel(r'$\omega_D T$')
    plt.ylabel(r'$e^{-\beta(T)}$')
    plt.grid(True)
    plt.legend()
    plt.show()
```

Link for code repository: https://github.com/topalidisd-qcqt-duth/qe8-problem-set-4/blob/main/problem-6.ipynb