Stefan Loesch[1]

Valuing concentrated liquidity positions using option pricing techniques

DRAFT

[1] stefan@topaze.blue (topaze.blue)

Valuing concentrated liquidity positions using option pricing techniques

DRAFT

v0.5 [DRAFT]

Stefan Loesch

stefan@topaze.blue

11 Jan 2023

Abstract

In this note we apply Black-Scholes valuation techniques to concentrated liquidity positions. We start from the self-financing investment strategy associated with the k=xy market maker, and we replicate the rebalancing ("Gamma") income of a concentrated liquidity position using European digital call options. We then establish the implication of this analysis for regular and concentrated LP positions and provide a number of numerical anlysis'.

topaze.blue

Contents

1 Introduction

We have previously established a relationship between weighted constant product AMMs ("WCPMMs") and power law profiles, where the weighting factor $0 < \alpha < 1$ corresponds to the power law exponent S^α, and where $\alpha = \frac{1}{2}$ is the standard equally weighted case. An important finding is that, because the power law profile is an eigenvector of the Black Scholes operator, the power law shape is preserved indefinitely, and the Black Scholes PDE becomes an ODE with the growth factor of $\sigma^2/8$ for the symmetric case [3].

When we move to concentrated liquidiy of the virtual-token-balances type, as we implictly do in both Uniswap v3 and in Carbon for example, we cut the power law curve into vertical strips that can received liquidity independently from each other. Those strips are no longer eigenvectors of the Black Scholes equation, and therefore they'll diffuse value across the strip boundaries as time goes by. In the concentrated liquidity case, each of the strips is akin to a digital call spread, and we have hundreds of strips diffusing value into each other.

In this note we first analyze the simplest possible scenario for a such concentrated-liquidity-style split: we divide our standard constant product curve into two segments, at the price point P_0. We then extend those results to concentrated liquidity positions that are bounded at both sides, and present a number of analytical calculations.

2 AMMs, investment strategies and power law profiles

We here only recall briefly the most important identifies we will use in this paper. For a more detailed explanation, see [3,4].

Any power law profile S^α is an eigenvector of the Black Scholes equation, and the associated eigenvalue is $\frac{1}{2}\sigma^2\alpha(1-\alpha) - (r - r_f)\alpha + r$ where r, r_f are the domestic and foreign interest rates respectively. In what follows we will assume zero rates, so the eigenvalue expression simplifies to $\frac{1}{2}\sigma^2\alpha(1-\alpha)$, and to $\sigma^2/8$ for the case $\alpha = \frac{1}{2}$.

In the case of the power law profiles $\nu^\alpha(S,t) \propto S^\alpha$, the Black Scholes PDE becomes an ODE of the form $\partial_t \nu^\alpha = c\nu^\alpha$ with some constant c, which integrates to the well known exponential solution. Specifically for $\nu = \nu^{\frac{1}{2}}$

$$\nu(S,t) = \nu_0(S) \cdot \exp\left(\pm\frac{\sigma^2}{8}t\right) \tag{2.1}$$

where the sign in the exponent depends on whether we are looking backwards or forwards, and how we define the flow of time. Also, as by construction we started with a square root profile, we know that $\nu_0(S) = \mathrm{const} \cdot \sqrt{S}$.

If we compute the Cash Delta Δ_c of the square root profile ν, we find that

$$\Delta_c[\nu] = S\partial_S\, \nu = \frac{1}{2}\nu \tag{2.2}$$

In other words: in order to delta hedge ν, half of the value of the hedge portfolio is in the risk asset, and the other half is in the numeraire asset. This happens to be the exact investment strategy of a CPMM, which establishes the equivalence between this particular AMM and this particular strategy.

There is one key difference however: the portfolio of the investment strategy grows with an instanenous rate of $\sigma^2/8$, whilst the portfolio of an AMM does not grow at all. The difference between the two has been described in [3], and has more recently been christened *Loss Versus Rebalancing*, short *LVR* [5]. We refer to it as Λ:

$$\Lambda(S_t,t) = \left(e^{\frac{\sigma^2}{8}t} - 1\right)\sqrt{\frac{S_t}{S_0}} \tag{2.3}$$

where S_0 is the spot price at the reference time $t = 0$. We have drawn the growth rate $\sigma^2/8$ determining the rebalancing income as function of the volatility in the chart *Rebalancing income*. We see that for typical vol ranges of 20-80%, the annual growth rate is between 1-8%, so in line with other financial investments.

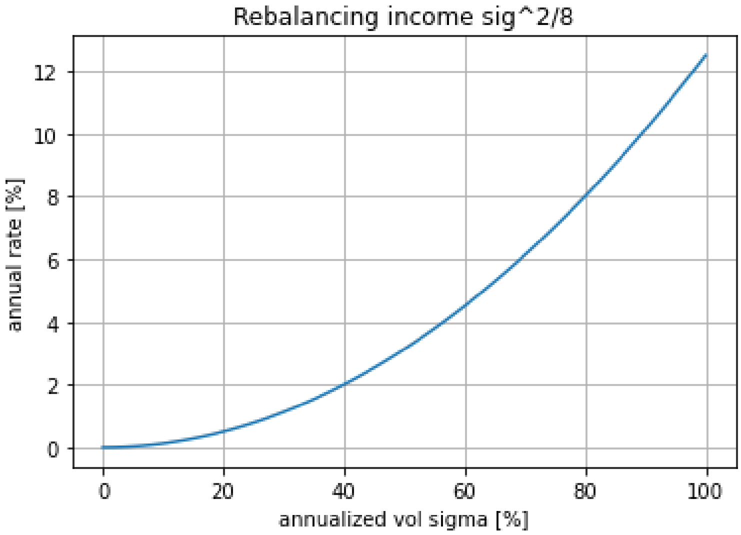

Figure 1: Rebalancing income

3 The split investment strategy

3.1 Rules

We assume our market is ETHUSDC, so prices are quoted as USDC per ETH, and the initial and reference spot price is 1,000. The *split investment strategy* we introduce has the following rules

1. At inception, both investors invest the same amount
2. Investor A, the *"downside investor"*, *"owns"* the area below 1,000; investor B, the *"upside investor"*, owns the area above 1,000
3. Each investor runs the portfolio rebalancing strategy whenever the spot is in their area
4. The investor running the strategy accrues the entire value of running it; the other investor's funds are *"parked"*
5. The downside investor A parks funds in the numeraire (USDC), the upside investor parks funds in the risk asset (ETH)

6. Either of the investors can ask for their money back, at which point the investor with parked funds gets the funds they parked, and the other one the remainder

3.2 A brief example

We now run through a short example:

- Initially, each investor invested \$10,000, so the initial balance is \$10,000 and 10 ETH.
- The price of ETH then moves up immediately and stays in the upside region; the portfolio value at time t is therefore $\$20k \cdot e^{\frac{\sigma^2}{8}t}\sqrt{S_t/S_0}$, out of which $\$10k$ belong to A and the remainder to B; note that the portfolio contains at least $\$10k$ in USDC collateral because $S_t > S_0$, and therefore the portfolio value is $> \$20k$.
- At time t_1 the spot goes back down to 1,000, and remains below 1,000 thereafter. At t_1, the portfolio value is $\$20k \cdot e^{\frac{\sigma^2}{8}t_1}$, out of which $\$10k$ belong to A and the remainder, $\$10k + \$20k \cdot (e^{\frac{\sigma^2}{8}t_1} - 1)$, belong to B. B's holdings are now parked in ETH, and we assume that it is 11 ETH.
- After spending a while below 1,000, the spot climbs to 1,000 again, hitting it at time t_2. At this point, the portfolio value is $\$20k \cdot e^{\frac{\sigma^2}{8}t_2}$. We assume that the spot spent about twice as much time down as up, and that A now has \$12,000. We recall that A has 11 ETH, so the total portfolio value is \$23,000.
- Finally, at time t_3 and at a spot value of 1,440, investor A decides to pull the plug. We assume $t_3 - t_2 = t_1$ so the growth factor from the Gamma is another 10%. On top of this there is the growth factor from spot move which is $\sqrt{1.44} = 1.2$, ie 20%. Therefore, total portfolio value is $23,000 \cdot 1.32 = 30,360$ out of which $\$12,000$ go to A, and the remainder to B.

3.3 Discussion

We now discuss in more detail why we are looking at those exact rules. The short answer is that they are a good model of what happens in a concentrated liquidity scenario. The most important point to make is the following:

> Whilst the above strategy looks like it requires two investors working toghether, actually each of the investors can run their part of the strategy on their own.

All they have to do is to "park" their assets whilst the spot is outside their range, and to resume trading when it is back inside. As for the parking, we note that rule (5) is designed to mirror the behavior of concentrated liquidiy ranges: whenever the spot is out of range, the investor is fully invested in the asset that went *down*.

The main difference between this model and a concentrated liquidity investment is that liquidity providers earn *fees* whilst the spot is in range, whilst the investors here earn the rebalancing income on the Gamma of the strategy.

4 Analysis and valuation

4.1 Valuation of the investment strategies

The reason why he have presented this strategy as a two-investor strategy is that in this case we know what the total value is: just like the value of a portfolio of mutually exclusive and completely exhaustive digital options must be equal to the value of a zero coupon bond, the combined value of the two strategies here must be the value of the square root profile, including the exponential growth term.

The portfolio value has two major parts to it, a delta compenent from the directional exposure to the spot, and an accrual component from the rebalancing income. The delta component itself is split into two major parts

- the *in-range* part, where the directional exposure is $\sqrt{S}$, and
- the *out-of-range* part where the directional exposure depends on the "parking"

position

There is also a third, and somewhat more complex part, which is the directional exposure of the accrued income. This depends on whether said income is being kept *on curve* as in the example described, or, if not, in which asset it is held.

The rebalancing income at time t is mildly stochastic, in that it depends on the spot price S_t, specifically its square root. In the absence of rates, during an interval of length dt the income is

$$\frac{\sigma^2 N_0}{8\sqrt{S_0}}\sqrt{S_t}\,dt \tag{4.1}$$

where N_0 is the dollar amount invested in the portofolio at time $t = 0$.

As we show in Appendix B, we can replicate the square root profile with a portfolio of European digital call options $C_D^{K,T}$, a finite sized one at strike K_0 to start off the series, and then a continous profile of strikes thereafter

$$\sqrt{K_0}\,C_D^{K_0,T}(S) + \int_{K_0}^{\infty} \frac{dK}{2\sqrt{K}}\,C_D^{K,T}(S) = \sqrt{S}\Bigg|_{S>=K_0} \tag{4.2}$$

We show the profile of a single digital call in the chart *Digital call profile* where we see that it starts out as a step function (blue curve) at time $t = 0$, and then diffuses the more the further we get away. Also the curve gets diminishes because of discounting.

Long dated digital options in a Black-Scholes world have a somewhat peculiar behaviour: as most paths end up around zero, all digital calls eventually go to zero, even without discounting. However, as we can see in the chart *Digital calls over time*, in the short run, out-of-the-money options will increase in value, whilst in-the-money ones decrease.

The two equations above show as (1) the income we earn at time t conditional on the spot price, and (2) how to replicate this profile with digital calls. Combining those two

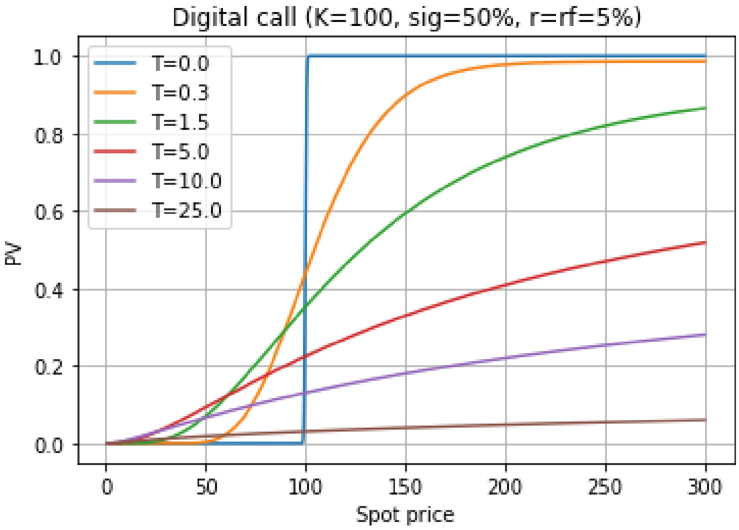

Figure 2: Digital call profile

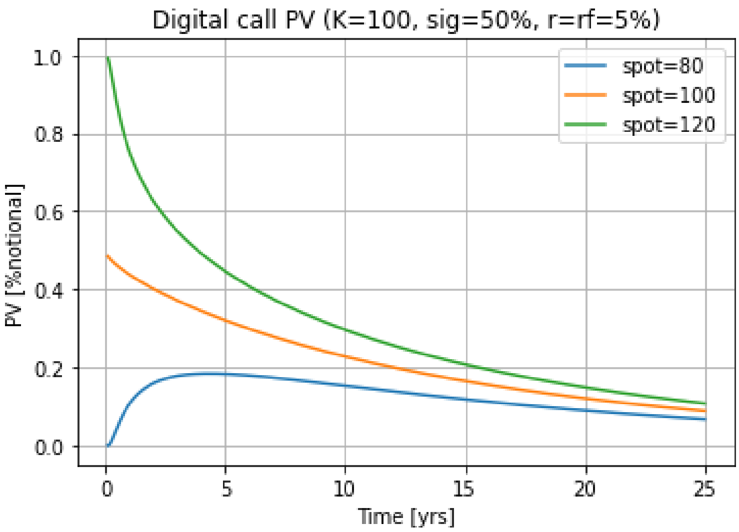

Figure 3: Digital calls over time

therefore yield the present value (PV) of the expected rebalancing income stream

$$PV_S\left[\Lambda\right] = \frac{N\sigma^2}{8\sqrt{S_0}} \left(\int_0^\infty dt \left(\sqrt{S_a}\, PV_S[D^{S_a,t}] + \int_{S_a}^{S_b} \frac{dK}{2\sqrt{K}}\, PV_S[D^{K,t}] \right) \right) \tag{4.3}$$

In the above equation, $PV_S[D^{K,t}$ denotes the present value, conditional on a spot value of S, of the digital call at strike K.

We have taken a few shortcuts here. Firstly, our digital calls contain discounting, whilst our rebalancing income assume rates are zero. This is necessary because despite there being interest rates in the cash world, the moment we hold USDC we forego this interest income. We have also introduced two new boundary terms, S_a, S_b. This is because we now moved on to more to a proper concentrated liquidity environment where the world is split into strips rather than into only two pieces. The prices S_a, S_b correspond to the lower and upper boundary of the concentrated liquidity range respectively.

In the second integral, if the range S_a, S_b is relatively narrow, we can approximate it with a single digital option, simplifying the formula above considerably, to

$$PV_S\left[\Lambda\right] = \frac{N\sigma^2}{8}\sqrt{\frac{S_{ab}}{S_0}} \int_0^\infty PV_S[\bar{D}^{S_{ab},t}]\, dt \tag{4.4}$$

where $S_{ab} = \sqrt{S_a S_b}$ or another suitable average value with $S_a < S_{ab} < S_b$, and $\bar{D}^{S_{ab}}$ is the the digial call *spread* associated with the interval. Today's value of the fee income of a segment liquidity position is the time integral over the call spread positions.

We have approximated this integral with sum over discrete time steps, as shown in in the chart *Value of lambda*. The chart shows the segment from 90..110 when the initial spot was at 100, and the notional was \$10,000, so the figures can be interpreted as basis points. The volatility is 30%, discount rates are at 5%, and the time period is restricted to 1 year. Over the first year, the PV of the income in this case is 40bp at the money, and falls of quickly beyond 80-120. If we consider a 10 year period however (chart *Value*

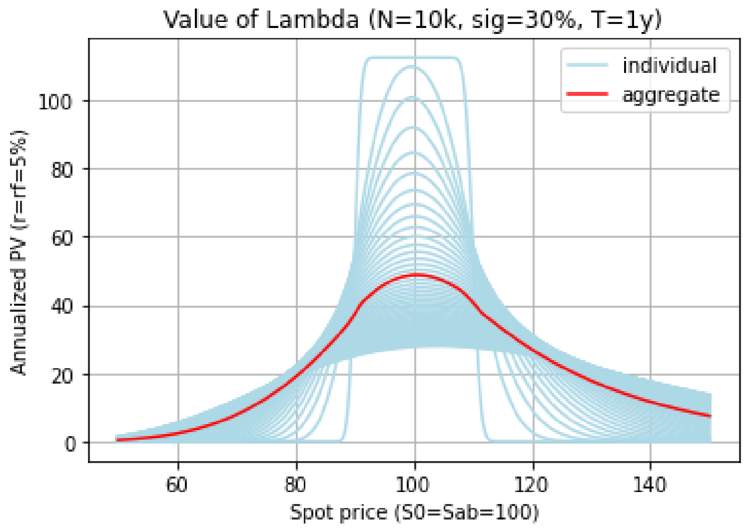

Figure 4: Value of Lambda

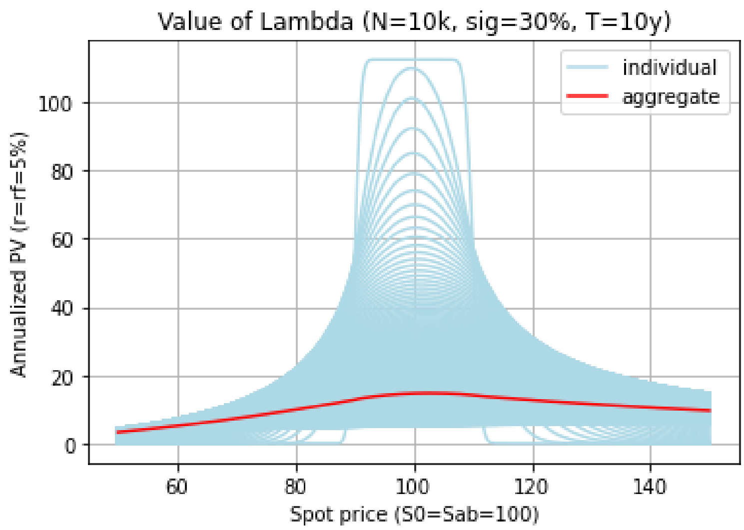

Figure 5: Value of Lambda (10y)

of Lambda (10y)) then the distribution is significantly flatter, and the annualized at the money value is about half, at 20bp.

4.2 Carry-over to concentrated liquidity

In our forthcoming paper [2] we will prove the following, very important proposition

> In efficient markets, the fees arbitrageurs pay to LPs be exactly equal to the rebalancing income Λ of the associated trading strategy.

This means that our analysis carries over directly to the case of AMM LPs: if markets are efficient – and of course we know they are not, because efficient markets do not have AMMs, which is the Achilles heel of every analysis applying continous-time financial modelling techniques to AMMs – then we know that the value captured by AMM LPs is *at least* the income generated by the rebalancing strategy. The income generated from end-customers – sometimes in some unfortunate turn of words referred to as "noise-traders" – comes on top of this.

The implications of this are somewhat complex, because of a common misconception about the model of the financial world that the Black-Scholes analysis uses: the Black-Scholes universe *is not risk neutral*; rather, in a Black-Scholes world all options can be hedged, so there is no risk, so one may as well assume that there is no risk premium. As a consequence of this, on *unhedged* positions, Black-Scholes expected payoffs are *not* real world expected payoffs because in the real world there *is*, or at least can be, as risk premium.

With this out of the way we can turn back to the analysis of *expected* AMM LP returns: in the *risk-neutral* Black Scholes world they expect to at least break even on average. Not path-by-path: just like when selling a call and pocketing the premium, some paths may lose, some may win. However, in expectation they break even from the arbitrage flow alone, and the income from end-customer flow is on top. In the real world however there is a risk premium, and a risk premium introduces a drift between the two assets. This drift will dominate the price dynamics in the long run, which means that, over the long run, AMM LPs will *expect* to suffer Divergence Loss, which is what Impermanent Loss should really be called. So, in summary:

> Whenever a risk premium is present in a trading pair, AMM LPs expect to lose money in the long run because of persistent Divergence Loss.

On a path-by-path basis, the portfolio's value will be first and foremost dominated by the delta exposure the investor is taking. We have drawn the result for different investor classes in the chart *Value profile split*:

- A generic AMM investor (grey) sees a square-root shaped curve
- The "downside" investor A (green) sees the square root curve on the downside, and a flat curve – corresponding to a numeraire position – on the upside
- The "upside" investor B (blue) sees the square root curve on the upside, and a $y = x$ cuve – corresponding to a numeraire position – on the downside
- The concentrated liquidity investor (red) sees a short-put-plus-cash position: risk asset on the downside, numeraire on the upside, and an area with non-zero Gamma in-between (that the red curve is below the other ones in the chart is of no consequence)

The above chart did not take the rebalancing income of $T\sigma^2/8$ into account, which ensures that in the center of the distribution there is a profit. We have drawn this in the *Break even points* chart that centers on said middle. The break event points are were the respective investor curves (grey, green and blue) meet the yellow curve that represents the linear portfolio.

We have also plotted the upper and lower break even points as a function of time and volatility in the chart *Break even chart*.

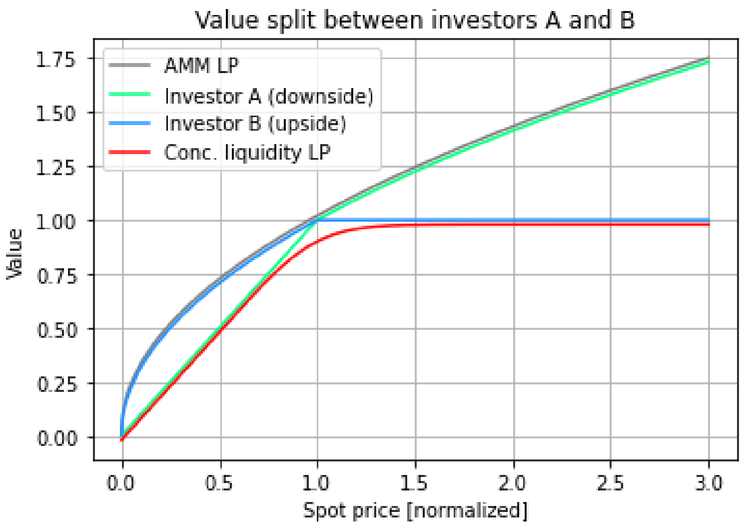

Figure 6: Value profile split

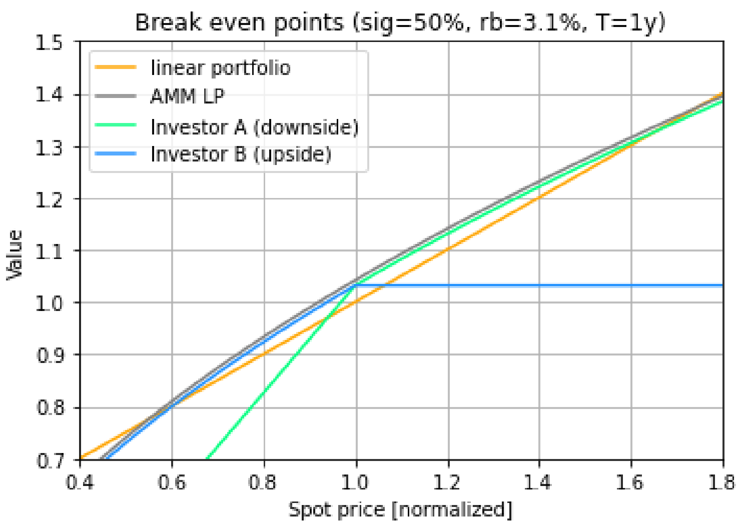

Figure 7: Break even points

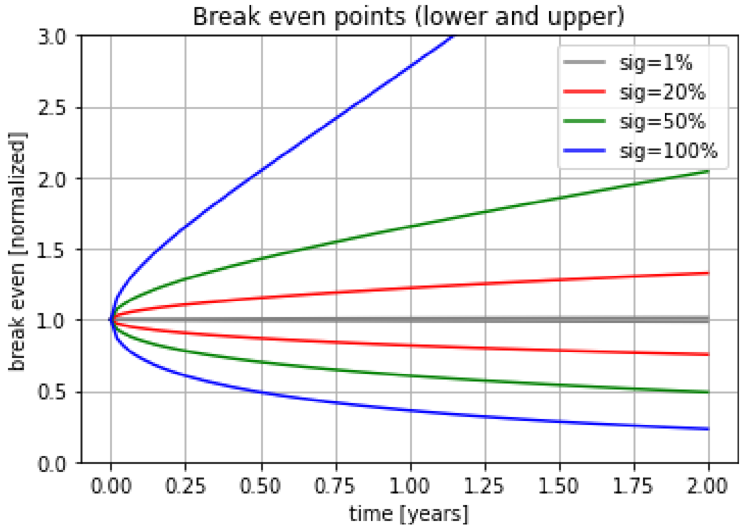

Figure 8: Break even chart

5 Conclusion

We had previously established the link between a constant proportion investment strategy and a power law profile European option strategy [3]. Power law profiles are invariant under the Black Scholes transition which simplified the analysis. Segmenting the liquidity into concentrated liquidity buckets complicates this, and in this note we draw the link between concentrated liquidity positions, their associated self-financing strategies, and the corresponding European options and their evolution over time, something that to our knowledge had not been done before.

In summary, we show that to establish the relationship we need to look at two elements in separation. The first element is the delta exposure, which is well-known to be the square root profile for the standard k=x*y AMM, close to a short put option with constant maturity for a concentrated liquidity AMM, and a chimera position which on one end is the square root profile, and on the other end the put for our investors A and B.

The second element, which we introduced in this paper, is the rebalancing gain. In the standard AMM, this term proportional to the underlying profile, with the well-known proportionality factor of $e^{T\sigma^2/8} - 1$. We established that in the other cases, the rebalancing gain can be represented by a partial square root profile that can be decomposed into a continous portfolio of digital calls. In the concentrated liquidity case we found that, in case of narrow ranges S_a, S_b, the continous could be approximated by the single digital call spread $\sqrt{\bar{S}}\, CS_D^{\bar{S},t}$ where $\bar{S}$ is a suitable average of S with $S_a < \bar{S} < S_b$.

6 References

1. *Bancor Protocol v2.1 - Economic and Quantitative Finance Analysis*, Loesch Hindman 2020 [Bancor]
2. *Impermanent Loss in Uniswap v3*, Loesch Hindman Richardson Welch 2021 [Arxiv]
3. *The quantitative finance aspects of AMMs*, Loesch 2022 [theAMMbook.org] [Arxiv]
4. *Optimal fee levels for Automated Market Makers*, Loesch 2023 (forthcoming)
5. *Automated Market Making and Loss-Versus-Rebalancing*, Milionis Moallemi Roughgarden Zhang 2022, [Arxiv]
6. Other recent profitability studies **TODO:ADD**
7. *Options, Futures, and Other Derivatives*, Hull [Pearson]

A Pricing a European digital option

A European digital call pays 1$ at *maturity* T if and only if $S > K$. Its price is

$$C_d = D\, N(d_2) \tag{A.1}$$

The price of the digital call that pays one unit of the risk asset is

$$\bar{C}_d = DF\, N(d_1) \tag{A.2}$$

The price of the respective digital put is $P_d = D - C_d$ in both cases. $D = e^{-rT}$ is the *discount factor*, and $F = S\, e^{(r-r_f)T}$ is the *forward*. The coefficients $d_{1/2}$ are

$$d_{1/2} = \frac{\ln \frac{F}{K} \pm \frac{\sigma^2}{2}\, T}{\sigma\sqrt{T}} \tag{A.3}$$

The coefficient d_1 takes the plus sign, and $d_2 = d_1 - \sigma\sqrt{T}$.

B European option profile matching

It is well known that every European option profile can be matched with call options by matching the Gamma of the calls to the second derivative of the profile [2]. Here we want to do something similar: we want to match a profile with digital call options.

A digital call at maturity is a step function. We recall that $\Theta(x)$ is the step function that is zero on the negative numbers, and unity on the positive ones. In a functional analysis sense, $\Theta = \int \delta$ or $\Theta' = \delta$, ie the derivative of the Heaviside function is the Dirac delta function.

Being given a suitably regular function $\nu(x)$ we are looking for a weight function $w(k)$ so that

$$\nu(x) = \int w(k)\, \Theta(x-k) dk \tag{B.1}$$

We can write $w = W'$ as the derivative of a function W. Integrating by parts yields

$$\nu(x) = \int W(k) \delta(x-k)\, dk = W(x) \tag{B.2}$$

plus some boundary term. In other words, we find that the weight function

$$w(x) = \nu'(x) \tag{B.3}$$

If we want to match $\nu(x) = \sqrt{x}$ we find

$$\sqrt{x} = \sqrt{\epsilon} + \int\limits_{\epsilon}^{\infty} \frac{\Theta(x-k)}{2\sqrt{k}} dk \tag{B.4}$$

where we added the $\sqrt{\epsilon} > 0$ term to avoid problematic boundary conditions at $k = 0$.