# Calculating the order book of a Constant Function Market Maker

AMM technical note

v1.1

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**Abstract**

In this note we discuss the different types of order books that exist in financial markets (notably, cumulative by marginal price and by effective price; marginal and marginal bucketed) and how to compute them in for a Constant Function Market Makers. We give explicit functions for the CPMM case x*y=k.



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# 1 Introduction

This is a short technical note, laying out the mathematics of how to get from the invariant function of a constant function market maker ("CFMM") to the various types of order book views that we are interested in. We also run the specific calculations in the case of a standard $x \cdot y = k$ constant product market maker ("CPMM").

# 2 AMMs and order books

In a traditional exchange, the *order book* describes the trade intentions registered with the exchange, ie how much tokens are there to sell in a specific pair at a specific price. The order book proper displays the *marginal* view, ie for every price (range) it displays how much liquidity is available at this price (range). The *cumulative order book* – or *liquidity depth chart* as it is also known – aggregates the liquidity from the current market price (or any other reasonable proxy) to the price indicated on the x-axis of the chart.

Order book display presents us with four independent choices

1. which order book to display (cumulative-effective-price, cumulative-marginal-price, marginal, marginal bucketed)
2. the direction in which prices are displayed (eg USDC per ETH, or ETH per USDC),
3. the unit in which the liquidity is accounted for (eg ETH, or USDC), and
4. the bid book (AMM *buys* base token; the green book on the left) or the ask book (AMM *sells* base token; the red book on the right)

The choice (2) above is addressed by the choice of base and quote token, and reversing the roles of $x, y$ will address it. We assume throughout this note that the base token $x$ is ETH and that the quote token $y$ is USDC. Therefore all prices are in $y$ *per* $x$, ie USDC per ETH.

We discuss here both the formulas for general CFMMs, as well specifically for the constant product case $k = x \cdot y$. We assume that we already have the AMM's invariant function in the form $y = y(x)$. In this case the AMM's *marginal price* is the derivative $P = -dy/dx = -y'(x)$. The effective price $\pi$ is

$$\pi(\Delta x) = -\frac{\Delta y}{\Delta x} = \frac{k}{x(x + \Delta x)} \tag{2.1}$$

In all the order book calculations that follow we assume that the the AMM's current base token (ETH) liquidity is $x_0$ and its quote token (USDC) liquidity is $y_0$, giving a current price of $P_0 = y_0/x_0$, and of course we also have $k = x_0 \cdot y_0$ in case of the constant product market maker, or more generally $y_0 = y(x_0)$.

# 3 Cumulative order book by marginal price

The cumulative order book by marginal price shows how much tokens hit the market / are absorbed from the market. This is not a bona-fide order book in the sense that this is less interesting for traders and therefore unlikely to be displayed in the user interface of an exchange. Nevertheless it is an ecomomically important quantity as it shows how the AMM reacts to changes in market conditions.

## 3.1 Base token view

We have seen that the marginal price as a function of $x$ is $P = -y'(x)$. Reversing the role of $P$ and $x$, and substituting $x = x_0 + \Delta x$ yields the cumulative order book, or liquidity depth plot, where the liquidity is expressed in the base token (ETH). The bid side is $\Delta x < 0$, and the ask side is $\Delta x > 0$. For display purposes, we always use the positive number $|\Delta x|$, leading to a cusp at $P = P_0$, $\Delta x = 0$.

Please refer to the chart *Cumulative order book by marginal price (ETH)* for an example in the case of a standard CPMM. In this case it is easy to calculate the liquidity depth manually as

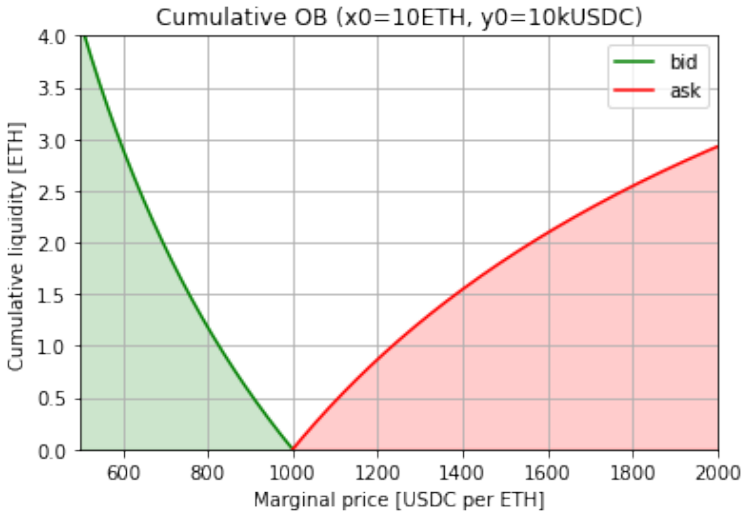


Figure 1: Cumulative order book by marginal price (ETH)

$$\Delta x(P) = \left| \sqrt{\frac{k}{P}} - \sqrt{\frac{k}{P_0}} \right| = x_0 \left| \sqrt{\frac{P_0}{P}} - 1 \right| \tag{3.1}$$

where the inner term is positive on the base token (ETH) ask side, and negative on its bid side.

### 3.2 Quote token view

To obtain the cumulative order book with liquidity expressed in the quote token (USDC), we first convert $x$ into $y$ using the invariant function $y(x)$. Then we express $y$ as $y = y_0 + \Delta y$ where $y_0 = y(x_0)$. As before, we display $|\Delta y|$ as positive number. Bid and ask are now reversed as those terms now refer to the quote token (USDC) rather than the base token. To ensure that bid remains on the left, the x-axis is inverted.

Please refer to the chart *Cumulative order book by marginal price (USDC)* for the corresponding example. The manual calculation in the constant product case yields

$$\Delta y(P) = \sqrt{k} \left| \sqrt{P} - \sqrt{P_0} \right| = y_0 \left| \sqrt{\frac{P}{P_0}} - 1 \right| \tag{3.2}$$

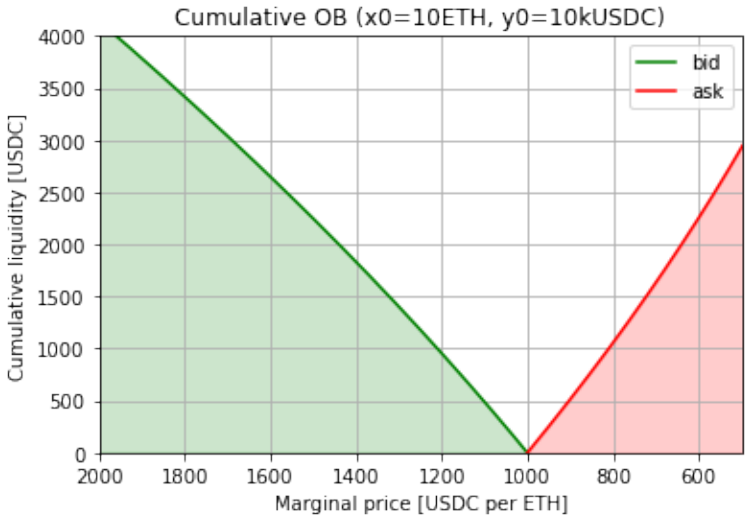


Figure 2: Cumulative order book by marginal price (USDC)

where the inner term is positive on the quote token (USDC) bid side, and negative on its ask side.

# 4 Cumulative order book by effective price (liquidity depth)

The cumulative order book by effective price, also referred to as *liquidity depth*, is the more interesting cumulative chart for traders, as it shows the amount of tokens available to buy and sell at a given transaction price.

## 4.1 Base token view

For the base token view, we start with the general part of the equation 2.1 that defines the effective price $\pi(\Delta x)$ and revert it, yielding

$$\Delta x(\pi) = [\pi(\Delta x)]^{-1} \tag{4.1}$$

where $[.]^{-1}$ denotes the reverse of a function. For the CPMM, we revert the specific part of the equation 2.1, giving the formula below that is also plotted in the chart *Liquidity*

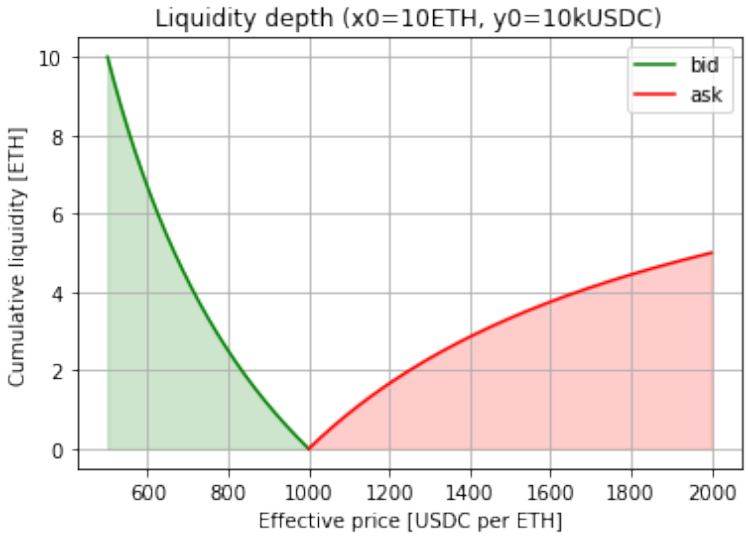


Figure 3: Liquidity depth (ETH)

*depth (ETH)*:

$$\Delta y(\pi) = \left| \frac{k}{\pi x} - x \right| = \left| \frac{y}{\pi} - x \right| \tag{4.2}$$

## 4.2 Quote token view

We know that $\pi = -\Delta y / \Delta x$, hence

$$\Delta y(\pi) = \pi \, \Delta x(\pi) \tag{4.3}$$

In the constant product case, we get the formula below that is also plotted in the chart *Liquidity depth (USDC)*. As before, we inverted the x-axis to ensure that the (USDC) bid book is on the left.

$$\Delta y(\pi) = \left| -\frac{k}{x} + \pi x \right| = |\pi x - y| \tag{4.4}$$

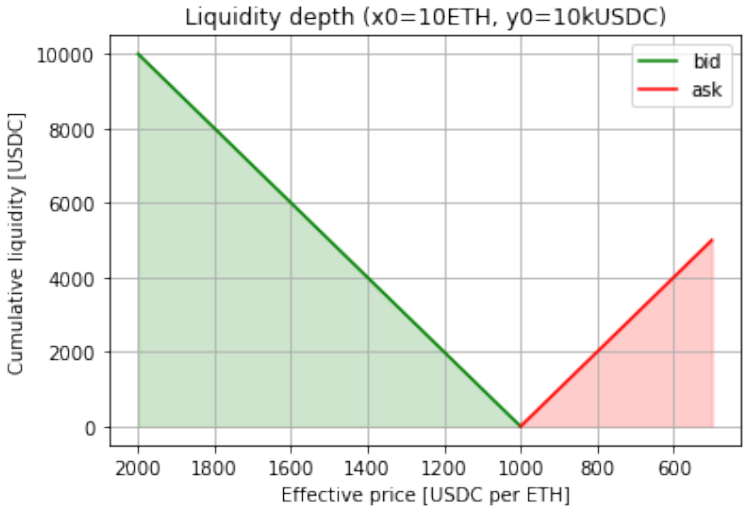


Figure 4: Liquidity depth (USDC)

# 5 Marginal order book

## 5.1 Base token view

The marginal order book is the derivative of the cumulative order book – the one expressed as function of the marginal price not that off the effective price – with respect to marginal price. It is thus obtained as

$$\Delta x'(P) = \frac{1}{y''(x(P))} \tag{5.1}$$

The units are somewhat awkward – when multiplied with a price range $dP$ we get units of the quote token (ETH), so in our example it would be *ETH per (USDC per ETH)* which formally but not particularly sensibly can be simplified to USD.

Please refer to the chart *Order book (ETH)* for the corresponding example. The formula in the constant product case is

$$\Delta x'(P) = \frac{\sqrt{k}}{2\sqrt{P^3}} \tag{5.2}$$

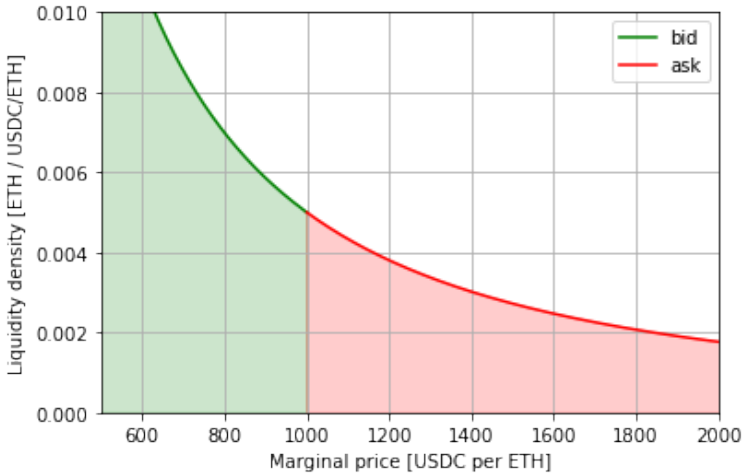


Figure 5: Order book (ETH)

### 5.2 Quote token view

The marginal order book expressed in the quote token (USDC) is easily obtained from the one expressed in the base token (ETH) by multiplying $\Delta x'(P)$ with $P$, yielding

$$\Delta y'(P) = P\,\Delta x'(P) = \frac{P}{y''(x(P))} \tag{5.3}$$

Please refer to the chart *Order book (USDC)* for the corresponding example, again with inverted x-axis. The associated formula is

$$\Delta y'(P) = \frac{\sqrt{k}}{2\sqrt{P}} \tag{5.4}$$

## 6 Bucketed marginal order book

As noted above, the marginal order book of an AMM has the somewhat peculiar units of *Token per Price* where *Token* can be for example ETH or USDC, and *Price* in this case would be either *USDC/ETH* or *ETH/USDC*. This gives not only units that are somewhat hard to understand (eg, *ETH / USDC/ETH = USD* for the order book

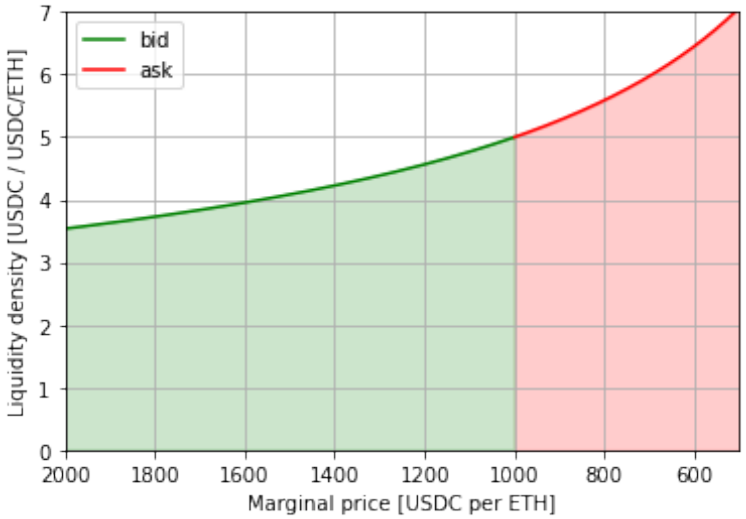


Figure 6: Order book (USDC)

denoted in ETH) but also the numbers are hard to understand.

In order to make numbers more intelligible for the end user in this case it makes sense to *bucket* the order book, ie to split the price range into price buckets, ie intervals $I_P$ with

$$I_i = I_{P_i} = \left[P_i - \frac{1}{2}\Delta P,\ P_i + \frac{1}{2}\Delta P\right] \tag{6.1}$$

and $P_{i+1} - P_i = \Delta P$ with some suitable $P_0, \Delta P$.

For the *base token view*, we can either integrate equation 5.1 over the interval $I_i$ (numerically, if need be), or evaluate equation 3.1 at the boundaries of the interval and take the difference. For the *quote token view*, we do the same, except the equation to integrate is 5.3 and the one to evaluate at the boundaries is 3.2. We have shown an example for a bucketed liquidity plot in the chart *Bucketed marginal order book (ETH)*. Apart from the segment shown on the x-axis it is equivalent to the chart *Order book (ETH)*, but now the liquidity is a proper ETH number. For example, 0.5 at 625 means that there is 0.5ETH of liquidity in the price range between 600 and 650 USDC per ETH.

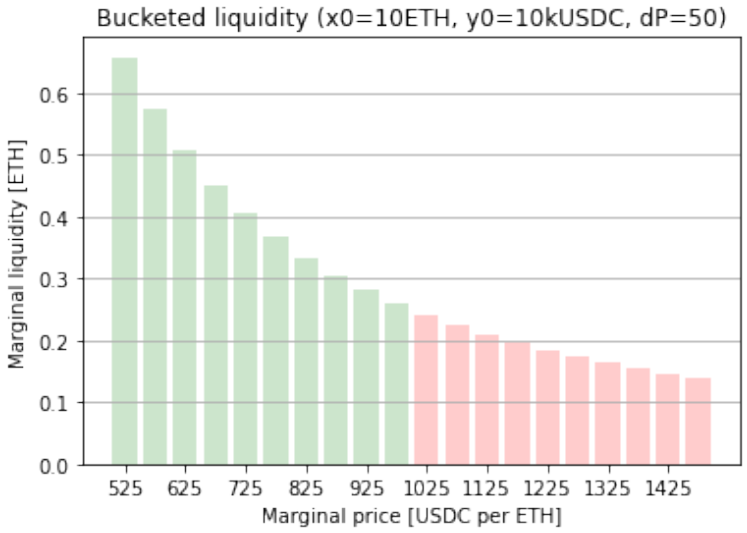


Figure 7: Bucketed marginal order book (ETH)

Assuming the buckets are small enough we can also approximate the corresponding base token (USDC) book by multiplying the value with the central price of the range rather than going through the detailed calculations.

## 7 Conclusion

In this short note we looked at how to construct the order books from the invariant functions of CFMMs, and gave specific formulas for CPMMs. **The associated Jupyter notebook producing the charts is available here**.