Problem Set 1

Aleksandr Efremov

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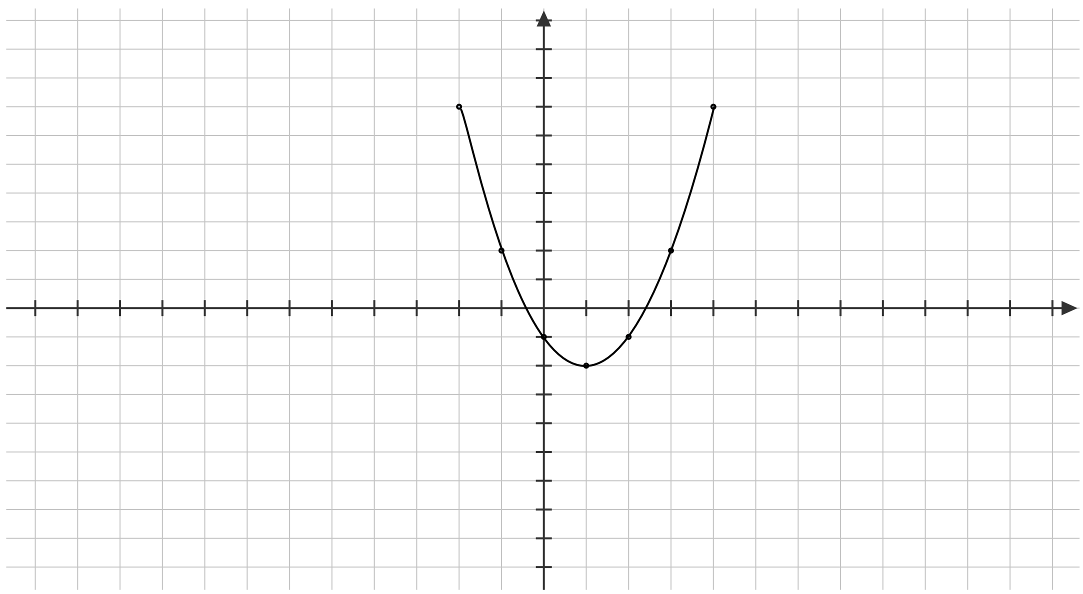
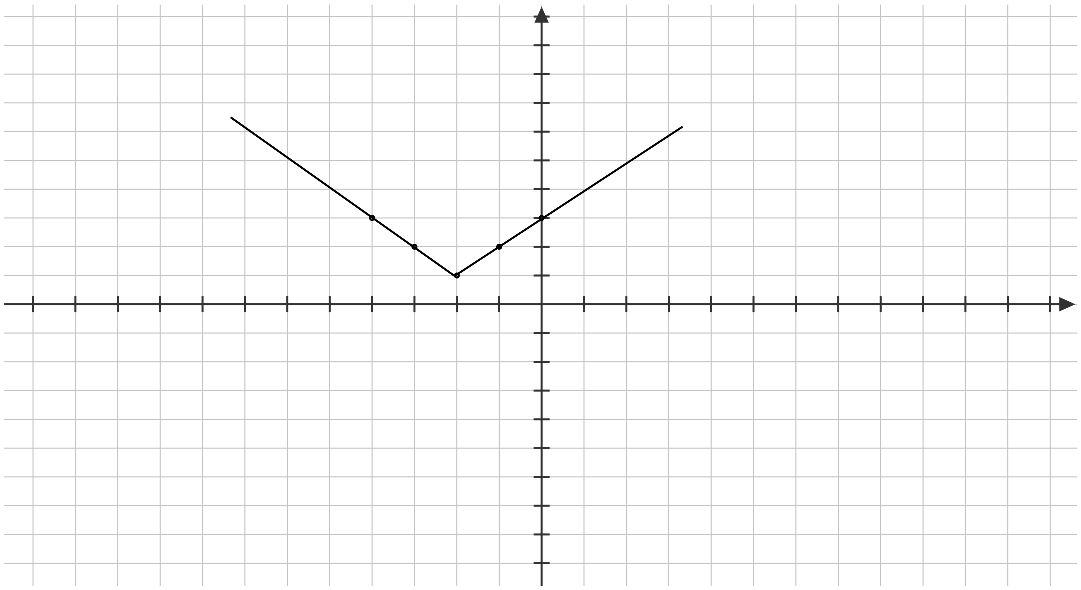
1 Part I

1.1 Recitation 0

(1A-1a) $y = x^2 - 2x - 1 = (x-1)^2 - 2$



(1A-2a) $y = 1 + |x+2|$



(1A-3a) $f(x) = \frac{x^3+3x}{1-x^4}$

$f(x)$ is *odd.*

$f(-x) = \frac{(-x)^3+3(-x)}{1-(-x)^4} = \frac{-x^3-3x}{1-x^4} = -\frac{x^3+3x}{1-x^4} = -f(x)$

(1A-3b) $f(x) = \sin^2 x$

$f(x)$ is *even.*

$f(-x) = \sin^2(-x) = \sin(-x)\sin(-x) = (-\sin x)(-\sin x) = \sin^2 x = f(x)$

(1A-3e) $f(x) = J_0(x^2)$

$f(x)$ is *even.*

$f(-x) = J_0((-x)^2) = J_0(x^2) = f(x)$

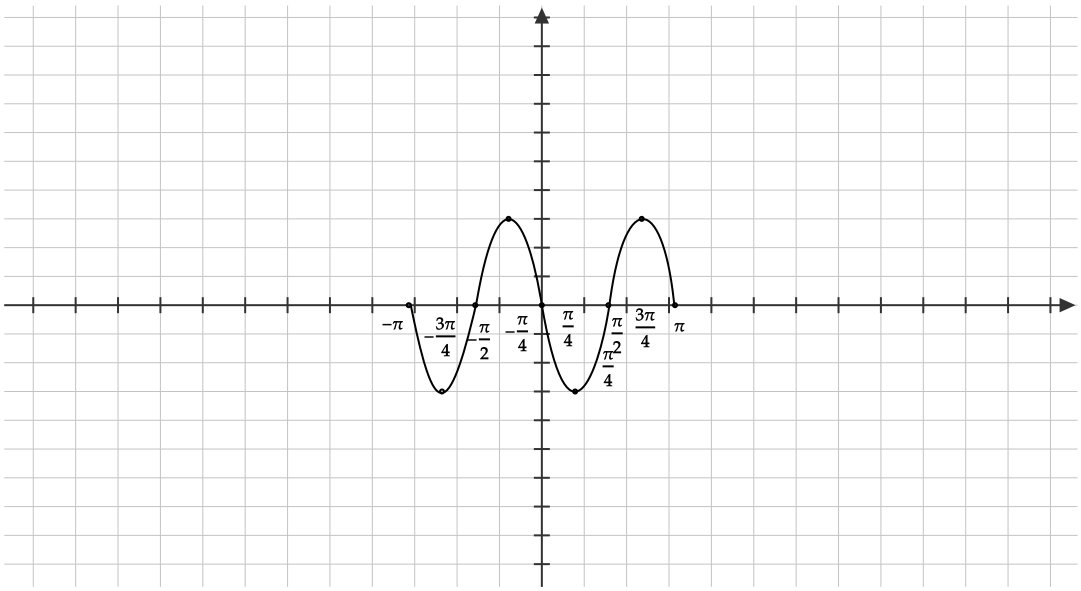
(1A-6a) $A\sin(x+c) = A\sin x\cos c + A\cos x\sin c$

$A = 2,\ c = \frac{\pi}{3}$

$2\sin(x+\frac{\pi}{3}) = 2\sin x\cos\frac{\pi}{3} + 2\cos x\sin\frac{\pi}{3} = \sin x + \sqrt{3}\cos x$

(1A-7a) $y = 3\sin(2x - \pi)$

Rewrite function as $y = 3\sin 2(x - \frac{\pi}{2})$. The period is π, amplitude is 3 and phase angle is $\frac{\pi}{2}$.



1.2 Lecture 1

(1C-3a) $f(x) = \frac{1}{2x+1}$

$$
\begin{aligned}
f'(x) &= \lim_{\Delta x \to 0} \frac{1/(2(x + \Delta x) + 1) - 1/(2x + 1)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{1/(2x + 2\Delta x + 1) - 1/(2x + 1)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{2x + 1 - 2x - 2\Delta x - 1}{(2x + 2\Delta x + 1)(2x + 1)(\Delta x)} \\
&= \lim_{\Delta x \to 0} \frac{-2\Delta x}{(2x + 2\Delta x + 1)(2x + 1)(\Delta x)} \\
&= \lim_{\Delta x \to 0} \frac{-2}{(2x + 2\Delta x + 1)(2x + 1)} \\
&= \frac{-2}{(2x + 0 + 1)(2x + 1)} = \frac{-2}{(2x + 1)^2}
\end{aligned}
$$

(1C-3b) $f(x) = 2x^2 + 5x + 4$

$$f'(x) = \lim_{\Delta x \to 0} \frac{2(x+\Delta x)^2 + 5(x+\Delta x) + 4 - 2x^2 - 5x - 4}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{4x\Delta x + 2\Delta x^2 + 5\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} 4x + 2\Delta x + 5 = 4x + 5$$

(1C-3e) For part $(1C-3a)$ the slope can't be 0 and 1 as the derivative is negative. For slope -1:

$$\frac{-2}{(2x+1)^2} = -1 \Rightarrow (2x+1)^2 = 2 \Rightarrow 2x + 1 = \pm\sqrt{2}$$

$$\Rightarrow x = -\frac{1}{2} \pm \frac{\sqrt{2}}{2}$$

So for part $(1C-3a)$ the slope is -1 at $x = -\frac{1}{2} \pm \frac{\sqrt{2}}{2}$.

For part $(1C-3b)$ the slope is 1 when $4x + 5 = 1 \Rightarrow x = -1$. The slope is -1 when $4x + 5 = -1 \Rightarrow x = -\frac{3}{2}$. The slope is 0 when $4x + 5 = 0 \Rightarrow x = -\frac{5}{4}$

(1C-4a) $f(x) = \frac{1}{2x+1}$, $f'(x) = \frac{-2}{(2x+1)^2}$

$f'(1) = \frac{-2}{9}$, $f(1) = \frac{1}{3}$

$y - \frac{1}{3} = \frac{-2}{9}(x-1) \Rightarrow y = \frac{-2}{9}x + \frac{2}{9} + \frac{1}{3} \Rightarrow y = \frac{-2x+5}{9}$

(1C-4b) $f(x) = 2x^2 + 5x + 4$, $f'(x) = 4x + 5$

$f'(a) = 4a + 5$, $f(a) = 2a^2 + 5a + 4$

$y - 2a^2 - 5a - 4 = (4a+5)(x-a) \Rightarrow y = 4ax - 4a^2 + 5x - 5a + 2a^2 + 5a + 4 \Rightarrow$
$y = 4ax + 5x - 2a^2 + 4 \Rightarrow y = (4a+5)x - 2(a^2 - 2)$

(1C-5) $y = 1 + (x-1)^2$, $y' = 2(x-1)$.
Let's find a tangent line through the point a.

$y(a) = 1 + (a-1)^2$, $y'(a) = 2(a-1)$.

The line has equation $y = y'(a)(x-a) + y(a)$. So,

$y = 2(a-1)(x-a) + 1 + (a-1)^2$.

Let's plug $x = 0$ and $y = 0$ to find points on the graph at which the tangent line goes through the origin.

$0 = 2(a-1)(0-a) + 1 + (a-1)^2 = -2a^2 + 2a + 1 + a^2 - 2a + 1 = -a^2 + 2 \Rightarrow 0 = -a^2 + 2 \Rightarrow a = \pm\sqrt{2}$

Plug a in the equation of tangent line with $\pm\sqrt{2}$ to get the tangent lines through the origin.

$y = 2(\sqrt{2}-1)(x-\sqrt{2}) + 1 + (\sqrt{2}-1)^2 = (\sqrt{2}-1)2x$

$y = 2(-\sqrt{2}-1)(x+\sqrt{2}) + 1 + (-\sqrt{2}-1)^2 = (-\sqrt{2}-1)2x$

(1C-6) Skipped

(1B-2) $s = bt - 16t^2$

a) $v = \frac{ds}{dt} = b - 32t$

b) The maximum height when $v = 0$. So,

$$b - 32t = 0$$

$$t = \frac{b}{32}$$

c) The maximum height is $s(\frac{b}{32}) = \frac{b^2}{32} - \frac{b^2}{64} = \frac{b^2}{64}$

d) The graphs of v and s

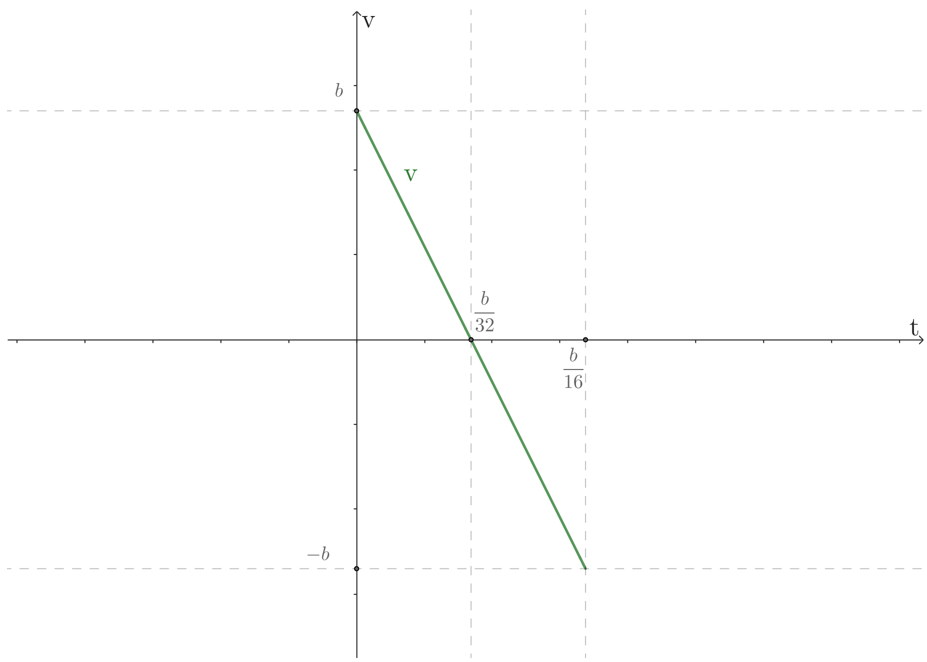


Figure 1: The graph of v (one bounce)

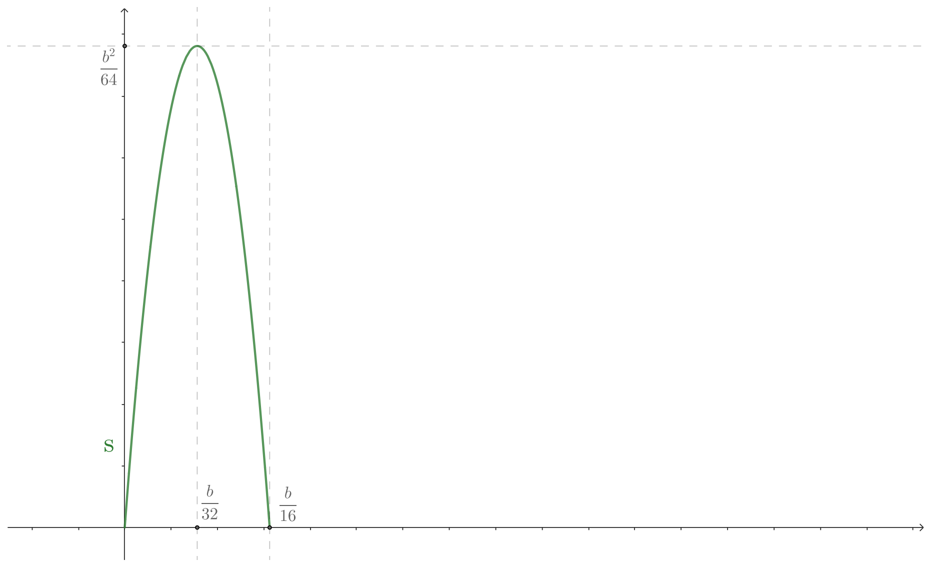


Figure 2: The graph of s (one bounce)

e) Let's take an initial velocity of the second bounce as c. Then $s = ct + 16t^2 \Rightarrow v = c + 32t$. So the maximum height of the second bounce is $\frac{c^2}{64}$. But we know that the maximum height of the second bounce is half of the first bounce, so $\frac{c^2}{64} = \frac{b^2}{64} \cdot \frac{1}{2} \Rightarrow c = \frac{b}{\sqrt{2}}$. That means that $v = \frac{b}{\sqrt{2} - 32t}$ and $s = \frac{b}{\sqrt{2}}t - 16t^2$.

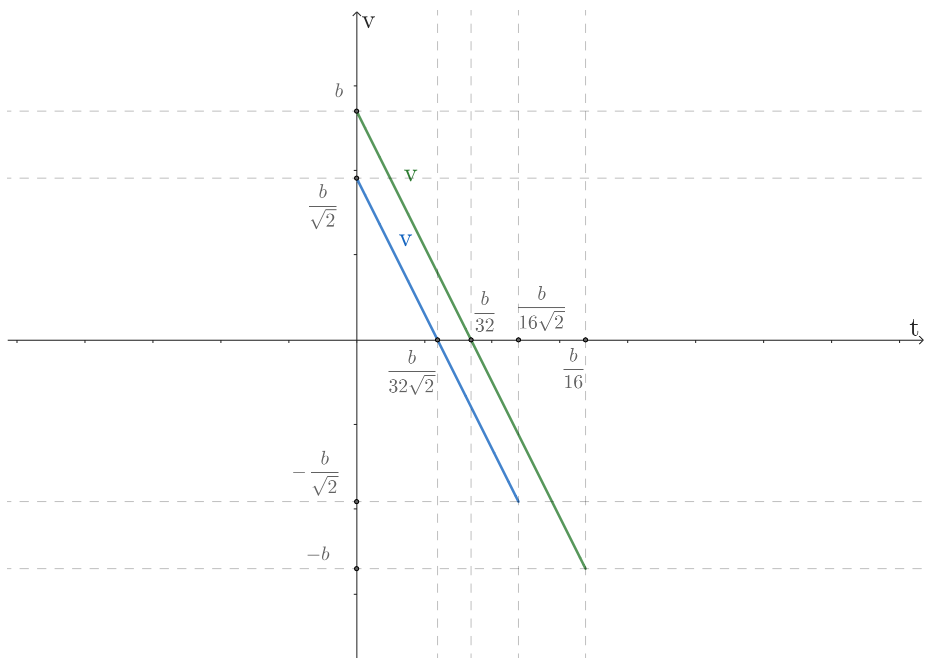


Figure 3: The graph of v (two bounces)

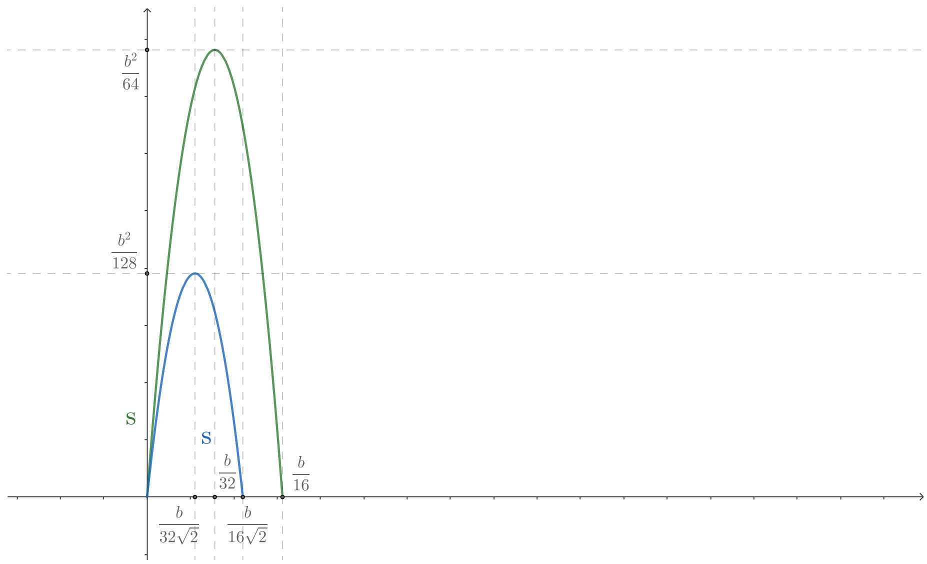


Figure 4: The graph of s (two bounces)

f) Let's examine the third bounce. If d is initial velocity then it's maximum height is at time $t = \frac{d}{32}$ (starting from 0) and its value is $\frac{d^2}{64}$, which is half of the second bounce $\frac{d^2}{64} = \frac{b^2}{128} \cdot \frac{1}{2} \Rightarrow d = \frac{b}{2}$. So the maximum height of third bounce is at time $t = \frac{b}{64}$ (starting from zero) and its value is $\frac{b^2}{256}$. The first bounce lasts $\frac{b}{16}$, the second bounce lasts $\frac{b}{16\sqrt{2}}$, the third lasts $\frac{b}{32}$ and so on. To get the time of the final landing of the ball we have to sum up all the times of individual landings of each bounce. This summation is geometric series as shown below.

$$S = (\frac{b}{16} + \frac{b}{16\sqrt{2}} + \frac{b}{32} + \dots)$$

$$\frac{b}{16}\sum_{i=0}^{\infty}\left(\frac{1}{\sqrt{2}}\right)^i = \frac{b}{16} \cdot \frac{1}{1 - \frac{1}{\sqrt{2}}}$$

(1C-1a) The area of a disk is given by function $A(r) = \pi r^2$.

$$\lim_{\Delta r \to 0} \frac{dA}{dr} = \frac{A(r + \Delta r) - A(r)}{\Delta r} = \lim_{\Delta r \to 0} \frac{\pi(r + \Delta r)^2 - \pi r^2}{\Delta r}$$

$$= \lim_{\Delta r \to 0} \frac{\pi r^2 + 2\pi r \Delta r + \pi \Delta r^2 - \pi r^2}{\Delta r} = \lim_{\Delta r \to 0} (2\pi r + \pi \Delta r) = 2\pi r$$

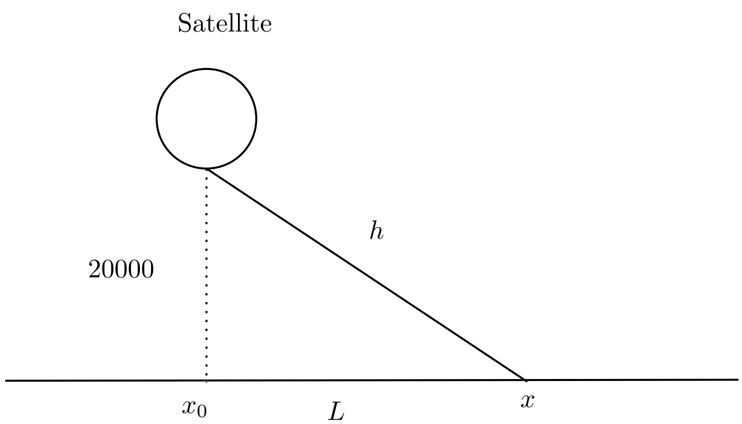
2 Part II

2.1 Problem 1

$$\frac{x-1}{x+1} = \frac{(x-1)(x-1)}{(x+1)(x-1)} = \frac{(x-1)^2}{x^2-1} = \frac{x^2-2x+1}{x^2-1} = \frac{x^2+1}{x^2-1} + \frac{-2x}{x^2-1}$$

where $\frac{x^2+1}{x^2-1}$ is even and $\frac{-2x}{x^2-1}$ is odd

2.2 Problem 2



$$L = \sqrt{h^2 - 20000^2}$$

a)

$$h_0 = 25000, L_0 = \sqrt{25000^2 - 20000^2} = 15000$$

$$\Delta h = 1, h = 25001, L = \sqrt{25001^2 - 20000^2} \approx 15001.667, \Delta L = 1.667, \frac{\Delta L}{\Delta h} = \frac{1.667}{1} = 1.667$$

$$\Delta h = 10^{-1}, h = 25000.1, L = \sqrt{25000.1^2 - 20000^2} \approx 15000.167, \Delta L = 0.167, \frac{\Delta L}{\Delta h} = \frac{0.167}{0.1} = 1.67$$

$$\Delta h = 10^{-2}, h = 25000.01, L = \sqrt{25000.01^2 - 20000^2} \approx 15000.017, \Delta L = 0.017, \frac{\Delta L}{\Delta h} = \frac{0.017}{0.01} = 1.7$$

An estimate for L is $|L - L_0| = |\Delta L| \leq 1.7|\Delta h|$

b)

$$h_0 = 20001, L_0 = \sqrt{20001^2 - 20000^2} \approx 200.002$$

$$\Delta h = 1, h = 20002, L = \sqrt{20002^2 - 20000^2} \approx 282.85, \Delta L = 82.848, \frac{\Delta L}{\Delta h} = \frac{82.848}{1} = 82.848$$

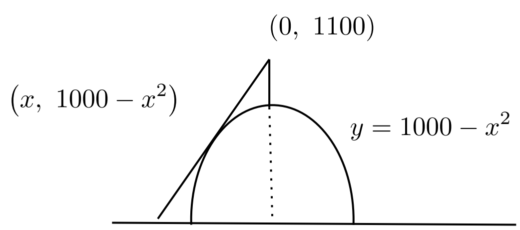
$$\Delta h = 10^{-1}, h = 20001.1, L = \sqrt{20001.1^2 - 20000^2} \approx 209.765, \Delta L = 9.763, \frac{\Delta L}{\Delta h} = \frac{9.763}{0.1} = 97.63$$

$$\Delta h = 10^{-2}, h = 20001.01, L = \sqrt{20001.01^2 - 20000^2} \approx 201, \Delta L = 0.998, \frac{\Delta L}{\Delta h} = \frac{0.998}{0.01} = 99.8$$

An estimate for L is $|L - L_0| = |\Delta L| \leq 100|\Delta h|$. The value is estimated less accurately that in part (a).

2.3 Problem 3

We have to find a point on the graph $y = 1000 - x^2$ at which the slope to the graph is going through the top of the pole. The answer would be the height y of that point. The diagram is presented below.



The slope at point x on the graph is $\frac{dy}{dx} = -2x$. At the same time the slope of the line that goes through the points $(x, 1000-x^2$ and $(0, 1000)$ is $\frac{1100-(1000-x^2)}{0-x}$. So we have to solve an equation for x.

$$\frac{1100 - (1000 - x^2)}{0 - x} = -2x$$

$$100 + x^2 = 2x^2$$

$$x^2 = 100$$

The ant begins to see the tower at height $y = 1000 - 100 = 900$.

2.4 Problem 4

(a) $y = \frac{x^2}{4p}$, the slope at point x is equal to $\frac{dy}{dx} = \frac{x}{2p}$. The tangent line has form $y = \frac{dy}{dx}x + b$. The y-intercept is equal to b when $x = 0$. So at point (x_0, y_0) we have an equation:

$$y_0 = \frac{x_0}{2p}x_0 + b$$

$$b = y0 - \frac{x_0^2}{2p}$$

$$\frac{x_0^2}{2p} = 2y_0$$

$$b = y_0 - 2y_0 = -y_0$$

(b) Let $A = (0, p)$, $B = (x_0, y_0)$ and $C = (0, -y_0)$.

The distance formula $D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

$D_{AB} = \sqrt{x_0^2 + (y_0 - p)^2} = \sqrt{x_0^2 + y_0^2 - 2y_0 p + p^2} = \sqrt{x_0^2 + \frac{x_0^4}{16p^2} - \frac{2x_0^2 p}{4p} + p^2} =$

$\sqrt{\frac{16x_0^2p^2 + x^4 - 8x_0^2p^2 + 16p^4}{16p^2}} = \sqrt{\frac{x_0^4 + 8x_0^2p^2 + 16p^4}{16p^2}} = \sqrt{\frac{(x_0^2 + 4p^2)^2}{16p^2}} = \frac{x_0^2 + 4p^2}{4p} =$

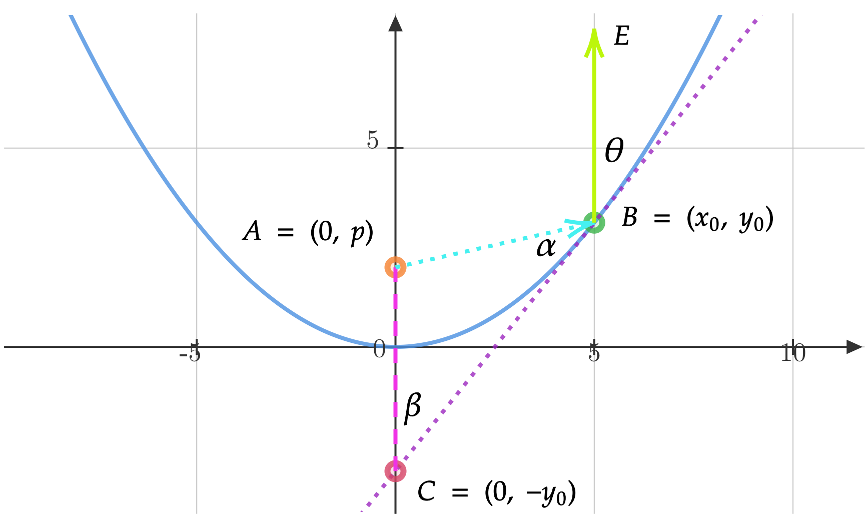
$\frac{x_0^2}{4p} + p$

$D_{AC} = \sqrt{(-y_0 - p)^2} = \sqrt{y_0^2 + 2y_0 p + p^2} = \sqrt{\frac{x_0^4}{16p^2} + \frac{2x_0^2 p}{4p} + p^2} =$

$\sqrt{\frac{x_0^4 + 8x_0^2p^2 + 16p^4}{16p^2}} = \sqrt{\frac{(x_0^2 + 4p^2)^2}{16p^2}} = \frac{x_0^2 + 4p^2}{4p} = \frac{x_0^2}{4p} + p$

The triangle is isosceles as $D_{AB} = D_{AC}$.

c) Let's take reflection point B. Without loss of generality we show that ray BE is parallel to the axis. As shown in (b) the triangle ABC is isosceles. The base of triangle is BC and angles α and β are equal. Since angles α and θ are equal by definition, it implies that $\beta = \theta$. Ray CA is parallel to the axis as shown in the picture. That implies that ray BE is also parallel to the axis.



2.5 Problem 5

a) $V = \frac{(10-t)^2}{5}$

$\frac{\Delta V}{\Delta t} = \frac{V(0)-V(5)}{0-5} = \frac{20-5}{-5} = -3\ l/m.$

b) $\frac{dV}{dt} = -\frac{2}{5}(10-t)$

When $t = 5$, then $\frac{dV}{dt} = -2\ l/m$

2.6 Problem 6

(19d)

$$\lim_{x\to\infty} x \sin\ \frac{1}{x}$$

$$x = \frac{1}{u}$$

$$\lim_{u\to 0} \frac{\sin u}{u} = 1$$

(19f)

$$\lim_{x\to 0} \frac{\sin^2 x}{3x^2} = \lim_{x\to 0} \left[\frac{1}{3}\frac{\sin x}{x}\frac{\sin x}{x}\right] = \frac{1}{3}$$

(19g)

$$\lim_{x\to 0} \frac{\sin 2x}{\sin 3x}$$

$$\begin{aligned}\frac{\sin 2x}{\sin 3x} = \frac{2\sin x\cos x}{\sin(x+2x)} &= \frac{2\sin x\cos x}{\sin x\cos 2x + \cos x\sin 2x} \\ &= \frac{2\sin x\cos x}{\sin x\cos 2x + \cos x(2\sin x\cos x)} \\ &= \frac{2\cos x}{\cos 2x + 2\cos^2 x}\end{aligned}$$

$$\lim_{x\to 0} \frac{2\cos x}{\cos 2x + 2\cos^2 x} = \frac{2}{3}$$

(20c)

$$\lim_{x\to 0} \frac{x^2}{1-\cos^2 x} = \lim_{x\to 0} \frac{x^2}{\sin^2 x} = 1$$

(20g)

$$\lim_{x\to 0}\frac{3x^2+4x}{\sin 2x}=\lim_{x\to 0}\frac{x(3x+4)}{2\sin x\cos x}=\lim_{x\to 0}\frac{x}{2\sin x}\cdot\frac{3x+4}{\cos x}=\frac{1}{2}\cdot 4=2$$

(22a)

$$L=\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta^2}$$

(a)

θ	L
0.1	0.49958
0.01	0.49999
0.001	0.49999
0.0001	0.49999

We see from the results of calculations that $L=\frac{1}{2}$.

(b)

$$\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta^2}=\lim_{\theta\to 0}\left(\frac{1-\cos\theta}{\theta^2}\cdot\frac{1+\cos\theta}{1+\cos\theta}\right)$$

$$=\lim_{\theta\to 0}\frac{1-\cos^2\theta}{\theta^2(1+\cos\theta)}=\lim_{\theta\to 0}\frac{\sin^2\theta}{\theta^2(1+\cos\theta)}$$

$$=\lim_{\theta\to 0}\left(\frac{\sin^2\theta}{\theta^2}\cdot\frac{1}{1+\cos\theta}\right)$$

$$=1\cdot\frac{1}{1+1}=\frac{1}{2}$$

2.7 Problem 7

a) $D(uvw)=(uv)'w+uvw'=(u'v+uv')w+uvw'=u'vw+uv'w+uvw'$

b) $D(u_1u_2\dots\ u_n)=u_1'u_2\dots un+u_1u_2'\dots u_n+u_1u_2\dots u_n'$

Let's prove that statement by induction. The base case is presented in (a). Suppose that the guessed formula for $D(u_1u_2\dots\ u_n)$ is true. We need to show that it's also true for $D(u_1u_2\dots\ u_nu_{n+1})$.
Let $w=u_1u_2\dots\ u_n$.
Now $D(wu_{n+1})=w'u_{n+1}+wu_{n+1}'=(u_1'u_2\dots un+u_1u_2'\dots u_n+u_1u_2\dots u_n')u_{n+1}+(u_1u_2\dots\ u_n)u_{n+1}'=u_1'u_2\dots unu_{n+1}+u_1u_2'\dots u_nu_{n+1}+u_1u_2\dots u_n'u_{n+1}+u_1u_2\dots\ u_nu_{n+1}'$.