Exam 1

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1 Problem 1

$y = \frac{1}{3}x^2$, $y' = \frac{2}{3}x$. The tangent line equation for $x = 1$,

$$y = y'(1)(x-1) + y(1)$$

$$y = \frac{2}{3}(x-1) + \frac{1}{3} = \frac{2}{3}x - \frac{1}{3}$$

2 Problem 2

a. $\frac{x}{\sqrt{1-x}}$

$$\frac{d}{dx}\left(\frac{x}{\sqrt{1-x}}\right) = \frac{d}{dx}\left(x \cdot (1-x)^{-\frac{1}{2}}\right)$$

$$= \frac{1}{\sqrt{1-x}} + \frac{x}{2(1-x)^{\frac{3}{2}}}$$

b. $\frac{\cos(2x)}{x}$

$$\frac{d}{dx}\left(\frac{\cos(2x)}{x}\right) = \frac{d}{dx}\left(\cos(2x) \cdot x^{-1}\right)$$

$$= \frac{-2\sin(2x)}{x} - \frac{\cos(2x)}{x^2} = -\frac{2x\sin(2x) + \cos(2x)}{x^2}$$

c. $e^{2f(x)} = g(x)$

$$g'(x) = e^{2f(x)} \cdot 2f'(x)$$

d. $\ln(\sin x)$

$$\frac{d}{dx}(\ln(\sin x)) = \frac{\cos x}{\sin x} = \cot x$$

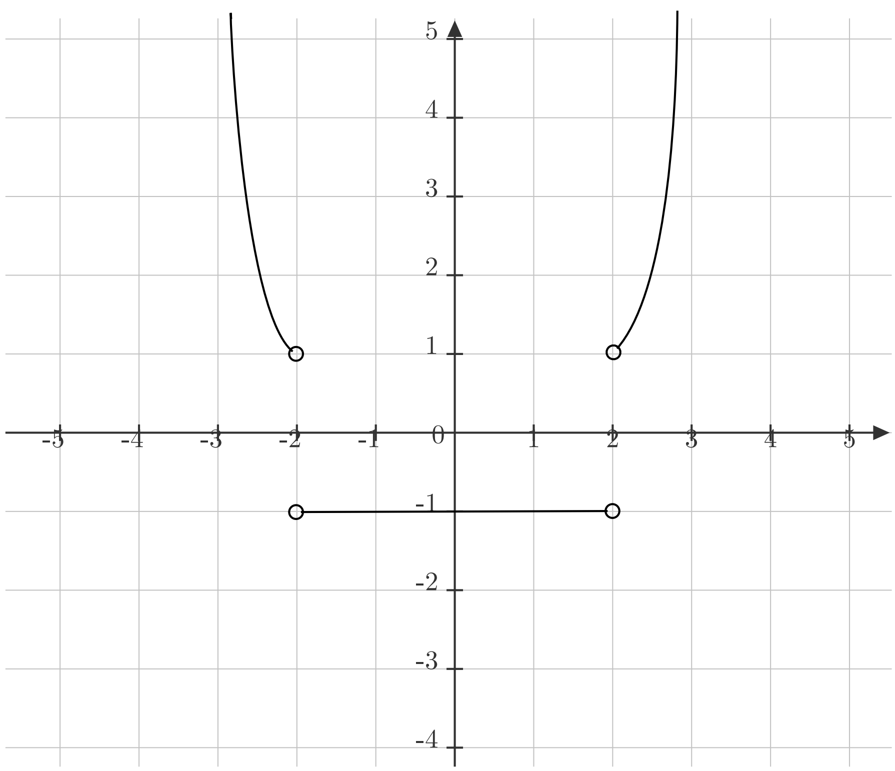
3 Problem 3

$y^4 + xy = 4$

$$\frac{d}{dx}\left(y^4 + xy\right) = \frac{d}{dx}4$$
$$4y^3\frac{dy}{dx} + \left(y + x\frac{dy}{dx}\right) = 0$$
$$\frac{dy}{dx} = -\frac{y}{4y^3 + x}$$

So, $\frac{dy}{dx}$ at $x = 3$ and $y = 1$ is $-\frac{1}{7}$

4 Problem 4



5 Problem 5

$$f(x) = \begin{cases} ax + b, & x < 1 \\ x^4 + x + 1, & x \geq 1 \end{cases}$$

For $f(x)$, to be differentiable its limit when $x \to 1^-$ must be equal to $f(1)$ and the slopes of the functions at point $x = 1$ must coincide.

$$\lim_{x \to 1^-} ax + b = f(1)$$

$$a + b = 3$$

The derivative of function $f(x)$,

$$f'(x) = \begin{cases} a, & x < 1 \\ 4x^3 + 1, & x \geq 1 \end{cases}$$

Then,

$$f'(1) = a$$

$$a = 5$$

So, $a = 5$ and $b = -2$.

6 Problem 6

a. $f(x) = (1 + 2x)^{10} \Rightarrow f'(x) = 20(1 + 2x)^9 \Rightarrow f'(0) = 20$

b. $f(x) = \sqrt{\cos x} \Rightarrow f'(x) = -\frac{\sin x}{2\sqrt{\cos x}} \Rightarrow f'(0) = 0$

7 Problem 7

$$\frac{d}{dx}a^x = \lim_{\Delta x \to 0} \frac{a^{x+\Delta x} - a^x}{\Delta x} = \lim_{\Delta x \to 0} \frac{a^x \left(a^{\Delta x} - 1\right)}{\Delta x} = a^x \lim_{\Delta x \to 0} \frac{a^{\Delta x} - 1}{\Delta x}$$

Let $M(a) = \lim_{\Delta x \to 0} \frac{a^{\Delta x} - 1}{\Delta x}$. Then,

$$\frac{d}{dx}a^x = M(a)a^x$$