Problem Set 2A

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1 Part I

1.1 Lecture 5

(1F-3) $y = x^{1/n}$

$y^n = x.$

$\frac{d}{dx}y^n = \frac{d}{dx}x \Rightarrow ny^{n-1}\frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{ny^{n-1}} \Rightarrow \frac{1}{nx^{\frac{n-1}{n}}}$

(1F-5) $\sin x + \sin y = 1/2$. We differentiate both sides to find $\frac{dy}{dx}$.

$$\frac{d}{dx}(\sin y) = \frac{d}{dx}(1/2 - \sin x) \Rightarrow \cos y\frac{dy}{dx} = -\cos x \Rightarrow \frac{dy}{dx} = -\frac{\cos x}{\cos y}$$

The horizontal tangent lines are at points where $\frac{dy}{dx} = 0 \Rightarrow \cos x = 0, \cos y \neq 0$. $\cos x = 0$ when $x = \pm\frac{\pi}{2}$.

If $x = \frac{\pi}{2}$, then,

$$\sin y = 1/2 - \sin\frac{\pi}{2} \Rightarrow \sin y = -1/2 \Rightarrow y = \frac{7\pi}{6}, \frac{11\pi}{6}$$

If $y = \frac{7\pi}{6}, \frac{11\pi}{6}$, then $\cos y \neq 0$, so points where $x = \frac{\pi}{2} + 2\pi i$ and $y = \frac{7\pi}{6} + 2\pi j, \frac{11\pi}{6} + 2\pi j$ fit.

If $x = -\frac{\pi}{2}$, then,

$$\sin y = 1/2 - \sin -\frac{\pi}{2} \Rightarrow \sin y = 3/2 \Rightarrow y = undefined$$

Then the only solutions left are points $\left(\frac{\pi}{2} + 2\pi i, \frac{7\pi}{6} + 2\pi j\right)$ and $\left(\frac{\pi}{2} + 2\pi i, \frac{11\pi}{6} + 2\pi j\right)$ for any integers i, j.

(1F-8c) $c^2 = a^2 + b^2 - 2ab\cos\theta$. $c, \cos\theta$ are constants.

$$\frac{d}{db}(c^2) = \frac{d}{db}(a^2+b^2-2ab\cos\theta) \Rightarrow 0 = 2a\frac{da}{db}+2b-2\cos\theta\left(\frac{da}{db}b + a\right) \Rightarrow \frac{da}{db} = \frac{a\cos\theta - b}{a - b\cos\theta}$$

(1A-5b) $f(x) = x^2 + 2x$

$$y = x^2 + 2x \Rightarrow y + 1 = (x+1)^2 \Rightarrow x = \pm\sqrt{y+1} - 1$$

Since this function is even it has no inverse function. But if we take the domain of function to be $[-1, \infty]$ then the inverse function $g(x) = \sqrt{x+1} - 1$. The sketch is skipped.

(5A-1a) $\tan^{-1}\sqrt{3} = \theta \Rightarrow \theta = \frac{\pi}{3}$

(5A-1b) $\sin^{-1}\frac{\sqrt{3}}{2} = \theta \Rightarrow \theta = \frac{\pi}{3}$

(5A-1c) If $\theta = \tan^{-1} 5$, then $\tan\theta = 5 \Rightarrow \tan\theta = \frac{opp}{adj} \Rightarrow hyp = \sqrt{26}$.
Then $\sin\theta = \frac{opp}{hyp} = \frac{5}{\sqrt{26}}$, $\cos\theta = \frac{adj}{hyp} = \frac{1}{\sqrt{26}}$, $\sec\theta = \frac{1}{\cos\theta} = \sqrt{26}$

(5A-3f) If $\theta = \sin^{-1}(a/x)$, then $\sin\theta = a/x \Rightarrow \frac{d}{dx}(\sin\theta) = \frac{d}{dx}(a/x)$

$= \cos\theta\frac{d\theta}{dx} = -\frac{a}{x^2} \Rightarrow \frac{d\theta}{dx} = -\frac{a}{x\sqrt{x^2-a^2}}$

(5A-3h) If $\theta = \sin^{-1}(\sqrt{1-x})$, then $\sin\theta = \sqrt{1-x} \Rightarrow \frac{d}{dx}(\sin\theta) = \frac{d}{dx}(\sqrt{1-})$

$= \cos\theta\frac{d\theta}{dx} = -\frac{1}{2\sqrt{1-x}} \Rightarrow \frac{d\theta}{dx} = -\frac{1}{2\sqrt{x(1-x)}}$

1.2 Lecture 6

(1H-1) The amount will decrease to 1/2 of the initial amount when $e^{kt} = 1/2$. Solving for t we find λ. So, $e^{kt} = 1/2 \Rightarrow t = -\frac{\ln 2}{k} \Rightarrow \lambda = -\frac{\ln 2}{k}$.

If $y(t_1) = y_1 = y_0e^{kt_1}$, then $y(t_1 + \lambda) = y_0e^{k(t_1+\lambda)} = y_0e^{kt_1} \cdot e^{k\lambda} = y_1e^{-\ln 2} = y_1/2$

(1H-2) Skipped

(1H-3a)

$$\ln(y-1) + \ln(y-1) = 2x + \ln x$$
$$(y-1)(y+1) = xe^{2x}$$
$$y^2 = xe^{2x} + 1$$

Since $y > 1$, $y = \sqrt{xe^{2x} + 1}$

(1H-5b) $y = e^x - e^{-x}$, let $u = e^x$, then $y = u + 1/u \Rightarrow u^2 - uy + 1 = 0$. Solve for u.

$$u = \frac{y \pm \sqrt{y^2 - 4}}{2}$$
$$e^x = \frac{y \pm \sqrt{y^2 - 4}}{2}$$
$$x = \ln\left(\frac{y \pm \sqrt{y^2 - 4}}{2}\right)$$

(1I-1c) $\frac{d}{dx}\left(e^{-x^2}\right) = -2xe^{-x^2}$

(1I-1d) $\frac{d}{dx}\left(x\ln x - x\right) = \ln x + 1 - 1 = \ln x$

(1I-1e) $\frac{d}{dx}\left(\ln\left(x^2\right)\right) = \frac{1}{x^2}\cdot 2x = \frac{2}{x}$

(1I-1f) $\frac{d}{dx}\left((\ln x)^2\right) = \frac{2\ln x}{x}$

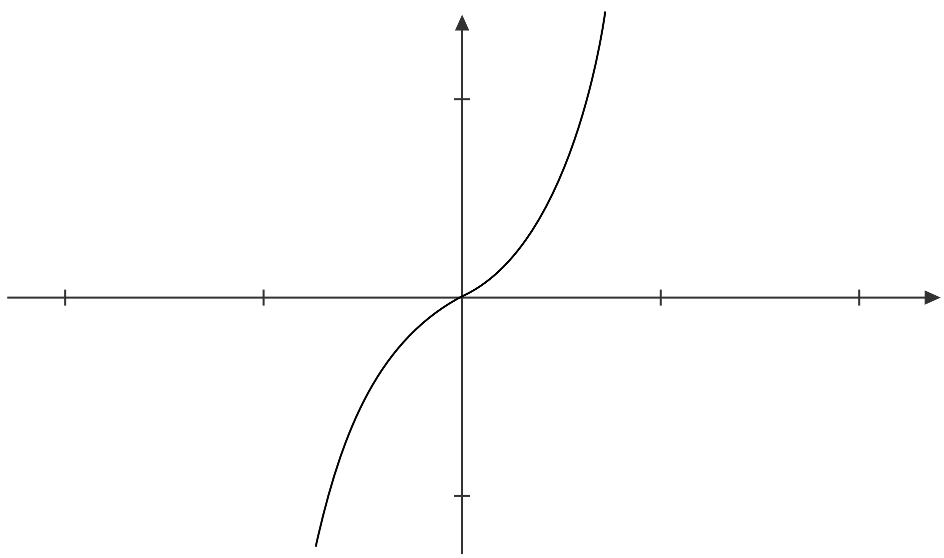
(1I-1m) $\frac{d}{dx}\left(\frac{1-e^x}{1+e^x}\right) = \frac{-e^x(1+e^x)-e^x(1-e^x)}{(1+e^x)^2} = \frac{-2e^x}{(1+e^x)^2}$

(1I-4a)

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{3n} = \lim_{n\to\infty}\left(\left(1+\frac{1}{n}\right)^{n}\right)^{3} = e^3$$

1.3 Lecture 7

(5A-5a) $y = \sinh x = \frac{1}{2}\left(e^x - e^{-x}\right)$. The function is *odd*, so it's symmetric about the origin. It has no critical points, since $\frac{d}{dx}\sinh x = \cosh x > 0$. The only point of inflection is 0, because $\frac{d^2}{dx^2}\sinh x = \sinh x$ and it equals to 0 only when $x = 0$. The function is concave up when $x > 0$ and concave down when $x < 0$. If $x \to \infty$ then $\sinh x \to \infty$, because $\lim_{x\to\infty}\frac{1}{2}\left(e^x - e^{-x}\right) = \infty - 0 = \infty$. If $x \to -\infty$ then $\sinh x \to -\infty$, because $\lim_{x\to-\infty}\frac{1}{2}\left(e^x - e^{-x}\right) = 0 - \infty = -\infty$.



(5A-5b) $\sinh^{-1} x = y$. That means $\sinh y = x$. We can solve this equation for y.

$$\frac{1}{2}\left(e^y - e^{-y}\right) = x$$

Put $u = e^y$ and solve for u first.

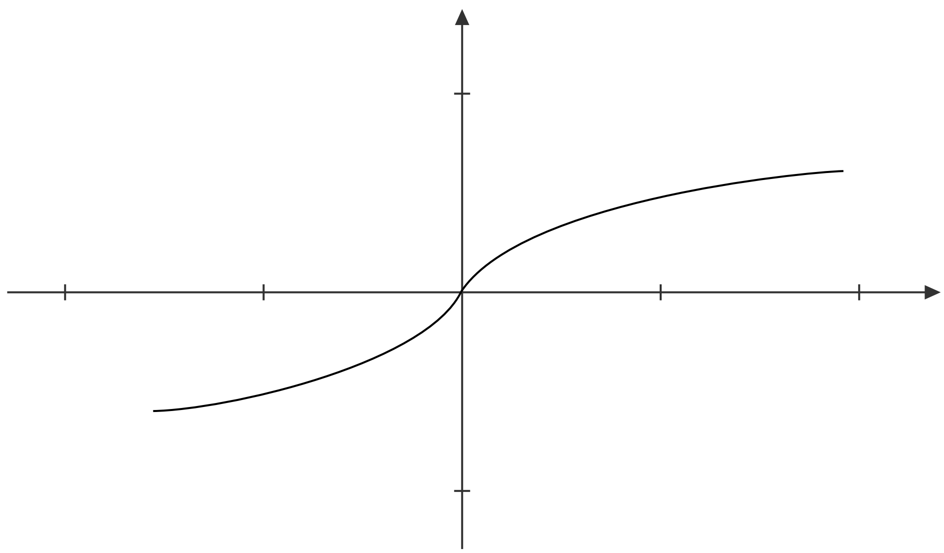
$$\left(u - \frac{1}{u}\right) = 2x$$

$$u^2 - 2xu - 1 = 0$$

$$u = \frac{2x \pm \sqrt{4x^2+4}}{2} = x \pm \sqrt{x^2+1}$$

$$e^y = x \pm \sqrt{x^2+1}$$

The only suitable definition $e^y = x + \sqrt{x^2+1}$ because $x - \sqrt{x^2+1} < 0$ and $e^y > 0$. So $y = \ln\left(x + \sqrt{x^2+1}\right)$, domain is a whole x axis.



(5A-5c)

$$\frac{d}{dx}\sinh^{-1} x = \frac{d}{dx}\left(\ln\left(x + \sqrt{x^2+1}\right)\right)$$

$$= \frac{1}{x+\sqrt{x^2+1}} \cdot \left(1 + \frac{x}{\sqrt{x^2+1}}\right) = \frac{1}{\sqrt{x^2+1}}$$

2 Part II

2.1 Problem 0

Skipped

2.2 Problem 1

Skipped

2.3 Problem 2

a)
$$\frac{d}{dx}\tan^3 x^4$$

Put $v = x^4$ and $u = \tan(v)$. Then $\frac{dv}{dx} = 4x^3$ and $\frac{du}{dx} = \sec^2(v) \cdot \frac{dv}{dx} = 4x^3\sec^2(x^4)$. Then,

$$\frac{d}{dx}u^3 = 3u^2 \cdot \frac{du}{dx} = 12x^3\tan^2(x^4)\sec^2(x^4)$$

b) $\frac{d}{dx}\left(\sin^2 y\cos^2 y\right)$. First using product rule,

$$\frac{d}{dx}\left(\sin^2 y\cos^2 y\right) = 2\sin y\cos^3 y - 2\cos y\sin^3 y = (2\sin y\cos y)(\cos^2 y - \sin^2 y)$$

$$= \sin(2y)\cos(2y) = \frac{\sin(4y)}{2}$$

Now rewrite the initial function as $f(2y)$.

$$\sin^2 y\cos^2 y = \frac{1-\cos(2y)}{2} \cdot \frac{1+\cos(2y)}{2} = \frac{1-\cos^2(2y)}{4}$$

So $f(y) = \frac{1-\cos^2 y}{4}$. Now solve $\frac{d}{dx}f(2y)$.

$$\frac{d}{dx}\left(\frac{1-\cos^2(2y)}{4}\right) = 0 - \frac{1}{4} \cdot 2\cos(2y) \cdot (-\sin(2y)) \cdot 2 = \cos(2y)\sin(2y) = \frac{\sin(4y)}{2}$$

2.4 Problem 3

Skipped

2.5 Problem 4

a) $\cos^{-1} x = y \Rightarrow \cos y = x \Rightarrow \sin y = \sqrt{1-x^2}$. Using implicit differentiation,

$$\frac{d}{dx}\cos y = \frac{d}{dx}x$$

$$-\sin y \cdot \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{-\sin y} = -\frac{1}{\sqrt{1-x^2}}$$

b) If we look at the graphs of the functions $y = \cos^{-1} x$ and $y = \sin^{-1} x$, we will see that their slopes at any point $-1 \leq x \leq 1$ have opposite values. That's why $\frac{d}{dx}\cos^{-1} x + \frac{d}{dx}\sin^{-1} x = 0$.

2.6 Problem 5

2.6.1 Section 8.2/8

a) The formula for M is $M = \frac{2}{3}\log_{10}\frac{E}{E_0}$. We need to solve for E,

$$\frac{3}{2}M = \log_{10}\frac{E}{E_0}$$

Rewrite as an exponential exponential equation,

$$10^{3/2M} = \frac{E}{E_0}$$

Then,

$$E = 10^{3/2M}E_0$$

Now let E_s be the energy of the smaller earthquake and E_l be the energy of the larger earthquake. Suppose that the magnitude of the smaller earthquake is equal to M, then the magnitude of the larger earthquake is equal to $M+1$. Then,

$$\frac{E_l}{E_s} = \frac{10^{3/2(M+1)}E_0}{10^{3/2M}E_0} = 10^{3/2}$$

c) The earthquake of magnitude 6 releases energy equal to $E = 10^{\frac{3}{2}\cdot 6}\cdot 10^{-3}\cdot 7 = 7\times 10^6$ kilowatthours. So, $\frac{7\times 10^6}{3\times 10^5} \approx 23$ days' supply could be provided by this earthquake.

2.6.2 Section 8.2/10

Proof that $\log_3 2$ is *irrational*. Assume for the purpose of contradiction that $\log_3 2$ is *rational*. Then,

$$\log_3 2 = \frac{p}{q}$$

Where p and q are positive integers and $\frac{p}{q}$ is in lowest form and $q > 0$. Then,

$$\log_3 2^q = p$$

Rewrite as an exponential equation,

$$2^q = 3^p$$

This is a contradiction. Hence, $\log_3 2$ is *irrational*.

2.6.3 Section 8.2/11

There is a flaw in multiplying by $\log\frac{1}{2}$ because it's a negative value. When multiplying by negative value the sign of inequality must be changed.

2.6.4 8.4/18

$$\ln y = \frac{1}{3}\left[\ln(x+1) + \ln(x-2) + \ln(2x+7)\right]$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{3}\left[\frac{1}{x+1} + \frac{1}{x-2} + \frac{2}{2x+7}\right]$$

$$\frac{dy}{dx} = \frac{\sqrt[3]{(x+1)(x-2)(2x+7)}}{3}\left[\frac{1}{x+1} + \frac{1}{x-2} + \frac{2}{2x+7}\right]$$

2.6.5 8.4/19a

$$y = \frac{e^x(x^2-1)}{\sqrt{6x-2}}$$

$$\ln y = x + \ln(x^2-1) - \frac{1}{2}\ln(6x-2)$$

$$\frac{1}{y}\frac{dy}{dx} = 1 + \frac{2x}{x^2-1} - \frac{3}{6x-2}$$

$$\frac{dy}{dx} = \left(1 + \frac{2x}{x^2-1} - \frac{3}{6x-2}\right)\left(\frac{e^x(x^2-1)}{\sqrt{6x-2}}\right)$$

2.7 Problem 6

Let $w = u_1u_2\dots u_n$, then we need to find w'.

$$\ln w = \ln u_1 + \ln u_2 + \dots + \ln u_n$$

$$\frac{w'}{w} = \frac{u_1'}{u_1} + \frac{u_2'}{u_2} + \dots + \frac{u_n'}{u_n}$$

$$w' = u_1'u_2\dots u_n + u_1u_2'\dots u_n + u_1u_2\dots u_n'$$