Problem Set 2B

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1 Part I

1.1 Lecture 9

1.1.1 2A-2

$f(x) = \frac{1}{a+bx}$, $f'(x) = -\frac{b}{(a+bx)^2}$. So, by using (2) linear approximation for $f(x)$ at $x \approx 0$,

$$f(x) \approx f(0) + f'(0)(x-0) = \frac{1}{a} - \frac{bx}{a^2}$$

or, by using the basic approximation formula (5),

$$f(x) = \frac{1}{a+bx} = (a+bx)^{-1} = \frac{1}{a}\left(1+\frac{bx}{a}\right)^{-1} \approx \frac{1}{a}\left(1-\frac{bx}{a}\right) = \frac{1}{a} - \frac{bx}{a^2}$$

1.1.2 2A-3

$f(x) = \frac{(1+x)^{3/2}}{1+2x}$. By using basic approximation formulas (5) and then (4),

$$\frac{(1+x)^{3/2}}{1+2x} \approx \frac{1+3x/2}{1+2x} \approx (1+3x/2)(1-2x) \approx 1-2x+3x/2 = \frac{2-x}{2}$$

Or, by using (2),

$$f'(x) = \frac{3/2(1+2x)^{3/2}(1+x)^{1/2} - 2(1+x)^{3/2}}{(1+2x)^2}$$

$$f(x) \approx f(0) + f'(0)x = 1 - \frac{x}{2} = \frac{2-x}{2}$$

1.1.3 2A-7

$$f(x) = \frac{\sec x}{\sqrt{1-x^2}} = \sec x \cdot (1-x^2)^{-1/2} \approx \left(1+\frac{x^2}{2}\right) \cdot (1-x^2)^{-1/2} \approx \left(1+\frac{x^2}{2}\right) \cdot \left(1+\frac{x^2}{2}\right) \approx 1+x^2$$

1.1.4 2A-11

$$p = \frac{C}{v^k} = C \cdot v^{-k} = C \cdot (v_0 + \Delta v)^{-k} = \frac{C}{v_0^k}\left(1 + \frac{\Delta v}{v_0}\right)^{-k}$$
$$\approx \frac{C}{v_0^k}\left(1 - k\frac{\Delta v}{v_0} + \frac{k(k+1)}{2}\left(\frac{\Delta v}{v_0}\right)^2\right)$$

1.1.5 2A-12a

$$\frac{e^x}{1-x} \approx (1 + x + x^2/2)(1 + x + x^2) \approx 1 + 2x + \frac{5x^2}{2}$$

1.1.6 2A-12d

$$\ln(\cos x) \approx \ln(\cos 0) - \tan 0 \cdot x - \frac{\sec^2 0}{2} \cdot x^2 = -\frac{x^2}{2}$$

1.1.7 2A-12e

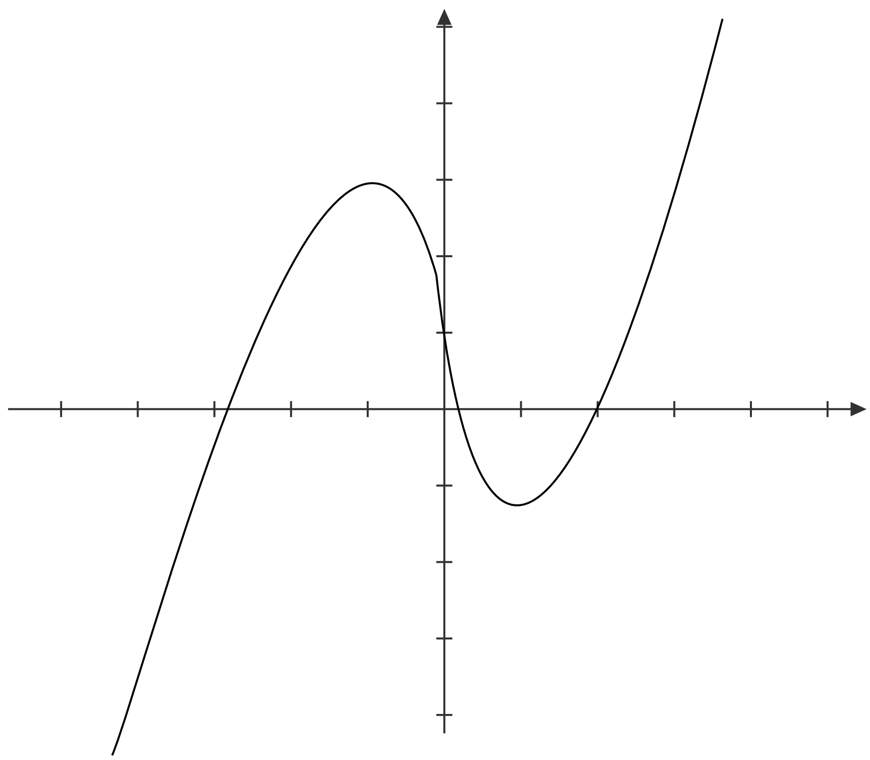
Put $x = 1 + h$.

$$(1+h)\ln(1+h) \approx (1+h)\left(h - \frac{h^2}{2}\right) \approx h + \frac{h^2}{2} = (x-1) + \frac{(x-1)^2}{2}$$

1.2 Lecture 10

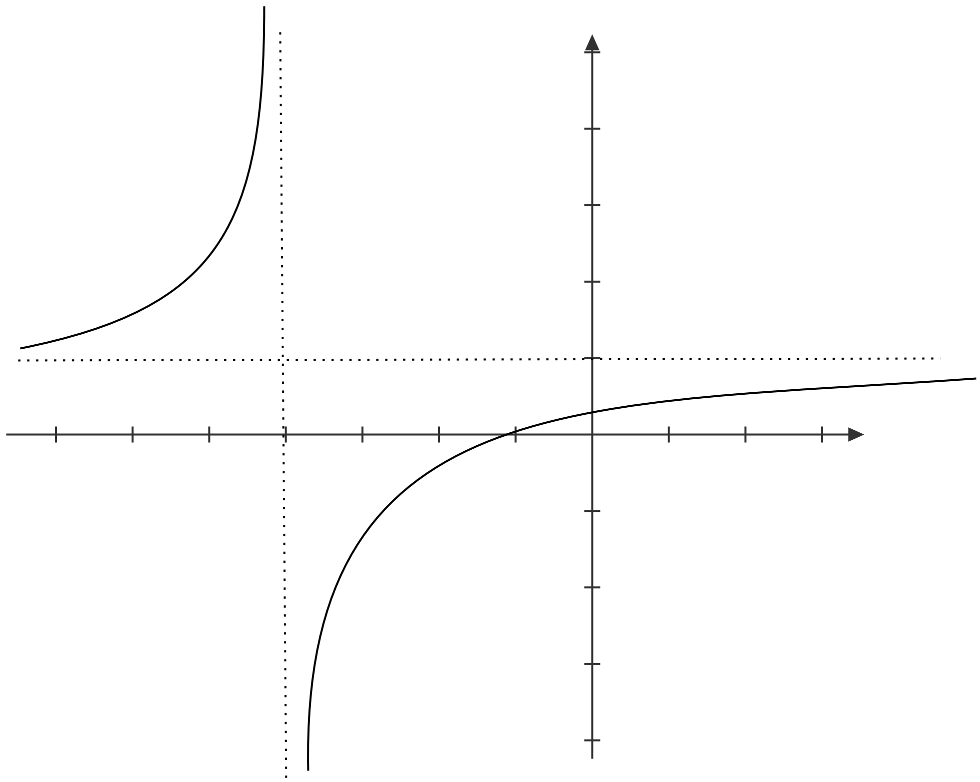
1.2.1 2B-1a

$y = x^3 - 3x + 1$, $y' = 3x^2 - 3$, the critical points at $x = \pm 1$. The *y-intercept* is 1. The function goes to ∞ as $x \to \infty$, and to $-\infty$ as $x \to -\infty$.



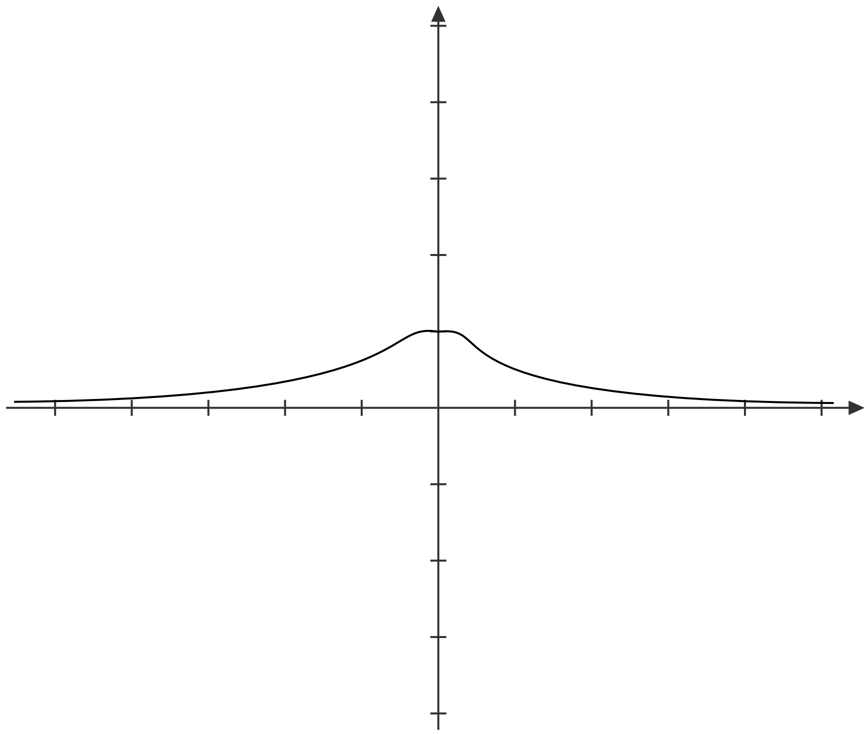
1.2.2 2B-1e

$y = \frac{x}{x+4}$, $y' = \frac{4}{(x+4)^2}$, the function has no critical points. The vertical asymptote at $x = -4$ since the function is undefined at that point. The function goes to ∞ as $x \to 4^-$, to $-\infty$ as $x \to 4^+$ and to 1 as $x \to \pm\infty$.



1.2.3 2B-1h

$y = e^{-x^2}$, $y' = -2xe^{-x^2}$. The only critical point is at $x = 0$. The *y-intercept* is 1. The function goes to 0 as $x \to \pm\infty$.



1.2.4 2B -2a

The inflection point is at $x = 0$.

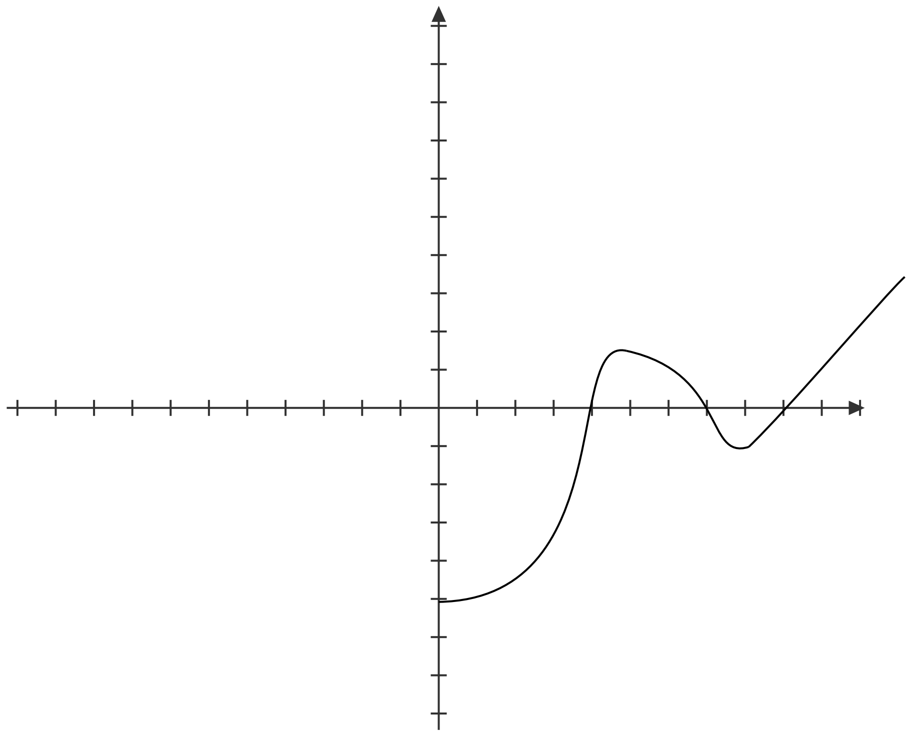
1.2.5 2B-2e

No inflection points

1.2.6 2B-2h

The critical points is at $x = \pm\frac{1}{\sqrt{2}}$

1.2.7 2B-4

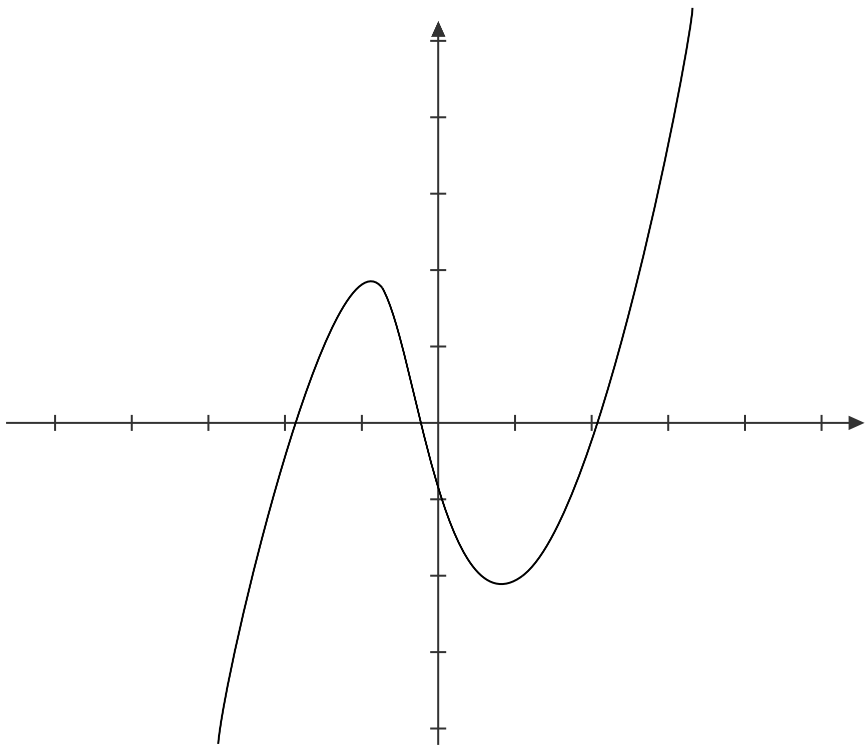


We can't say exact values of the maximum and minimum, but we know that the maximum is attained at point $x = 5$ or $x = 10$, and the minimum is attained at point $x = 0$ or $x = 8$.

1.2.8 2B-6a

$y = x^3 + 3x$.

1.2.9 2B-6b



1.2.10 2B-7a

$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$. Let $\Delta y = f(x) - f(a)$ and $\Delta x = x - a$. Then $f'(a) = \lim_{\Delta x\to 0} \frac{\Delta y}{\Delta x}$. If $f(x)$ is increasing, then,

$$\Delta x > 0 \Rightarrow \Delta y > 0 \Rightarrow \frac{\Delta y}{\Delta x} > 0 \Rightarrow f'(a) \geq 0$$

$$\Delta x < 0 \Rightarrow \Delta y < 0 \Rightarrow \frac{\Delta y}{\Delta x} > 0 \Rightarrow f'(a) \geq 0$$

In both cases $f'(a) \geq 0$.

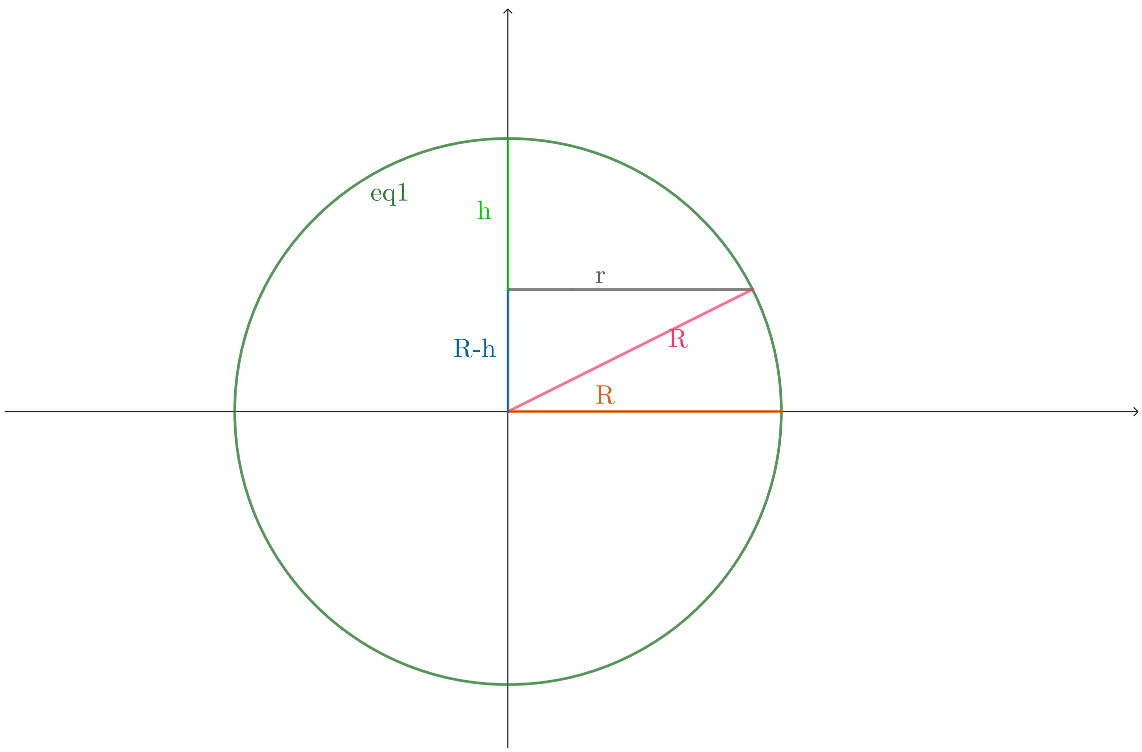
1.2.11 2B-7b

The flaw is in statement $\frac{\Delta y}{\Delta x} \Rightarrow f'(a) > 0$, because any positive function has a limit ≥ 0, not > 0. As a counterexample consider $f(x) = x^3$. This function is increasing for all x, but $f'(0) = 0$.

2 Part II

2.1 Problem 1

a) Consider a section of the sphere shown below.



So, $h = R - \sqrt{R^2 - r^2}$. Hence, the formula for the area of the cap is $2\pi R\left(R - \sqrt{R^2 - r^2}\right)$

b) Area of the cap $= 2\pi R\left(R - \sqrt{R^2 - r^2}\right) = 2\pi R^2 \left(1 - \left(1 - \left(\frac{r}{R}\right)^2\right)^{\frac{1}{2}}\right)$.

Using linear approximation to the function $(1+x)^{\frac{1}{2}}$, where $x = -\left(\frac{r}{R}\right)^2$

$$2\pi R^2 \left(1 - (1+x)^{\frac{1}{2}}\right) \approx 2\pi R^2 \left(1 - \left(1 + \frac{1}{2}x\right)\right) = 2\pi R^2 \left(\frac{r^2}{2R^2}\right) = \pi r^2$$

Using quadratic approximation to the function $(1+x)^{\frac{1}{2}}$, where $x = -\left(\frac{r}{R}\right)^2$

$$2\pi R^2 \left(1 - (1+x)^{\frac{1}{2}}\right) \approx 2\pi R^2 \left(1 - \left(1 + \frac{1}{2}x - \frac{1}{8}x^2\right)\right) = 2\pi R^2 \left(-\frac{1}{2}x + \frac{1}{8}x^2\right)$$

$$= 2\pi R^2 \left(\frac{r^2}{2R^2} + \frac{r^4}{8R^4} \right) = \pi r^2 + \frac{\pi r^4}{4R^2}$$

c) Let R be the radius of the ball and r be the radius of a dimple. The area of the ball is $4\pi R^2$. When a dimple is inserted some area is removed from the golf ball and an area equal to $2\pi r^2$ is added.

   i) If we consider the removed surface to be flat than the area removed would be equal to πr^2. Inserting 100 dimples into the ball we get the area of the ball equal to $4\pi R^2 - 100\pi r^2 + 200\pi r^2$. The area of the removed surface is equal to the linear approximation from part (b).

   ii) According to the quadratic approximation from part (b) the area of the removed surface is $\pi r^2 + \frac{\pi r^4}{4R^2}$. Inserting 100 dimples into the ball we get the area of the ball equal to $4\pi R^2 - 100\pi r^2 - \frac{25\pi r^4}{R^2} + 200\pi r^2$.

   iii) The exact formula for the removed surface is $2\pi R \left(R - \sqrt{R^2 - r^2}\right)$. Inserting 100 dimples into the ball we get the area of the ball equal to $4\pi R^2 - 100 \left(2\pi R \left(R - \sqrt{R^2 - r^2}\right)\right) + 200\pi r^2$.

   Calculate using $\pi \approx 3.14$. For part (i) the area of the ball is ≈ 35.33. For part (ii) the area is ≈ 35.31. For part (iii) the area is ≈ 35.31. Thus, the quadratic approximation is more accurate and the consideration that the removed area is flat gives wrong answer according to the rules.

2.2 Problem 2

Skipped