Problem Set 3

Aleksandr Efremov

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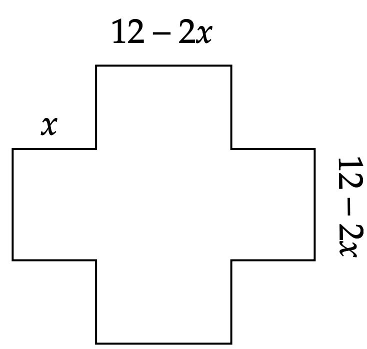
1 Part I

1.1 Lecture 11

1.1.1 2C-1



The volume of the box is,

$$V = (12 - 2x)^2 x$$

We need to maximize V with constraint $0 < x < 6$, since $V = 0$ at the endpoints $x = 0$ and $x = 6$. The derivative of V is,

$$\frac{dV}{dx} = 12x^2 - 96x + 144$$

The critical points are at $x = 2$ and $x = 6$. Hence, the maximum V is at $x = 2$.

1.1.2 2C-2

The length of the fence is given by the formula $L = 2x + y$, where x and y are sides of the rectangular area. The area is given by the equation $xy = 20000$. We need to minimize L.

First rewrite formula for L in terms of x,

$$L = 2x + \frac{20000}{x}$$

The minimum should be at some critical point between 0 and ∞, since at the endpoints $x = 0$ and $x = \infty$ L goes to ∞. The derivative of the function L is,

$$\frac{dL}{dx} = 2 - \frac{20000}{x^2}$$

The only critical point (greater than 0) is at $x = 100$. So the shortest length of the fence needed is $L = 400$.

1.1.3 2C-5

Let r be the radius of the cylinder and h be its height. The area of the cylinder is given by the formula $A = \pi r^2 + p\pi rh$. The volume of the cylinder is given by the formula $V = \pi r^2 h$. We need to maximize V. First we find h in terms of r,

$$h = \frac{A - \pi r^2}{2\pi r}$$

Then plug h into the formula for V,

$$V = \pi r^2 \left(\frac{A - \pi r^2}{2\pi r} \right) = \frac{1}{2} \left(Ar - \pi r^3 \right)$$

We have two endpoints $r = 0$ and $r = \infty$. At $r = 0$ the volume $V = 0$, and at $r = \infty$ the volume $V = -\infty$. Hence, the maximum must be at critical points.

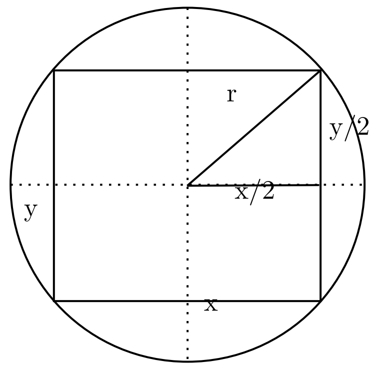
$$\frac{dV}{dr} = \frac{1}{2} \left(A - 3\pi r^2 \right)$$

$\frac{dV}{dr} = 0$ at $r = \sqrt{\frac{A}{3\pi}}$, then,

$$h = \frac{A - \pi \frac{A}{3\pi}}{2\pi \sqrt{\frac{A}{3\pi}}} = \frac{A}{3\pi \sqrt{\frac{A}{3\pi}}} = \frac{A}{\sqrt{3\pi A}} = \sqrt{\frac{A}{3\pi}}$$

Hence, the maximum volume is when $r = h$.

1.1.4 2C-11



We need to maximize $S = cxy^3$, where c is a constant. At the endpoints $x = 0$ or $y = 0$ the strength S is also zero. Hence, the maximum must be at critical points when $S' = 0$. We can express S as a function of x and treat y as an implicit function of x. So,

$$\frac{dS}{dx} = c\left(y^3 + 3xy^2\frac{dy}{dx}\right)$$

From the picture above we got the equation,

$$\frac{1}{4}\left(x^2 + y^2\right) = r^2$$

$$x^2 + y^2 = 4r^2$$

Differentiating both sides we get,

$$\frac{dy}{dx} = -\frac{x}{y}$$

Hence,

$$\frac{dS}{dx} = 0 \Rightarrow y^3 + 3xy^2\left(-\frac{x}{y}\right) = 0 \Rightarrow \frac{y}{x} = \sqrt{3}$$

1.1.5 2C-13a

Let R be the total revenue the company gets and x denotes some increase or decrease in the fare and passengers. The total revenue equals to the product of the number of passengers and the fare per ticket. Then,

$$R = (100 - 2x)(200 + 5x)$$

The function goes to $-\infty$ at the ends. So, the maximums must be in between at the critical point. $\frac{dR}{dx} = -20x + 100$. So, the critical point is at $x = 5$, and therefore the price that maximizes the revenue is 225\$

1.1.6 2C-13b

Let P be the profit the company gets. The profit equals to the product of the amount of kilowatt hours consumed and the price per kilowatt hour. Then,

$$P = \left(10 - \frac{x}{10^5}\right)\left(10^5\left(10 - \frac{p}{2}\right)\right)$$

Notice that the price per kilowatt hour is dependent on the amount of total kilowatt hours consumed. So,

$$x = 10^5\left(10 - \frac{p}{2}\right)$$

Plugging in this formula for x in P we get,

$$P = \left(10 - \frac{10^5\left(10 - \frac{p}{2}\right)}{10^5}\right)\left(10^5\left(10 - \frac{p}{2}\right)\right) = \frac{10^6 p}{2} - \frac{10^5 p^2}{4}$$

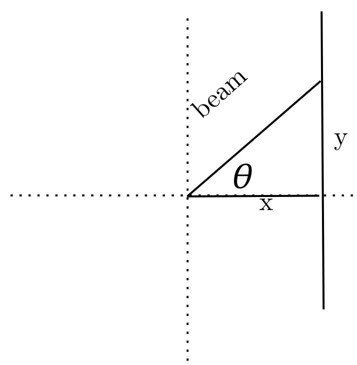
Now we need to maximize P. The minimum price is $p = 4$, because then $x = 8 \times 10^5$ which is the maximum amount the company can produce. The maximum price is $p = 20$, because then $x = 0$, which is the minimum amount the company can produce. If $p = 4$, then $P = 1600000$ cents $= 16000\$$. If $p = 20$, then $P = 0\$$. We also need to check the critical point.

$$\frac{dP}{dp} = \frac{10^6}{2} - \frac{10^5 p}{2}$$

The critical point is at $p = 10$. And when $p = 10$, then $P = 2500000$ cents $= 25000\$$. So the maximum profit is when price is equal to 10 cents.

1.2 Lecture 12

1.2.1 2E-2



Let x be the distance from the beacon to the shoreline, y be a position of the spot of light of the beam on the shoreline and θ be an angle between perpendicular line from the beacon to the shoreline and the beam. Notice, that x is constant and equals to 4 miles, y and θ are changing with respect to time t measured in minutes. So, we have to find $\frac{dy}{dt}$ measured in miles per minute when the angle between the shoreline and the beam is $\frac{\pi}{3}$. If the angle between the shoreline and the beam is $\frac{\pi}{3}$, then $\theta = \frac{\pi}{6}$. Also, we are given $\frac{d\theta}{dt} = 3 = 6\pi$

$$\tan\theta = \frac{y}{4}$$

Differentiating both sides we get,

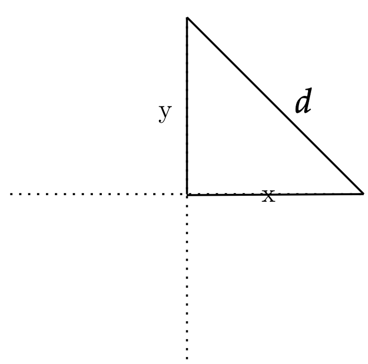
$$\sec^2\theta \frac{d\theta}{dt} = \frac{1}{4}\frac{dy}{dt}$$

$$\frac{dy}{dt} = 4\sec^2\theta \frac{d\theta}{dt}$$

So, when $\theta = \frac{\pi}{6}$,

$$\left.\frac{dy}{dt}\right|_{\theta=\frac{\pi}{6}} = 4 \cdot \frac{4}{3} \cdot 6\pi = 32\pi$$

1.2.2 2E-3



Let y be the distance from the cross point to the current position of the first boat, and x is the distance from the cross point to the current position of the second boat. Then D is the distance between them and $D^2 = x^2 + y^2$.

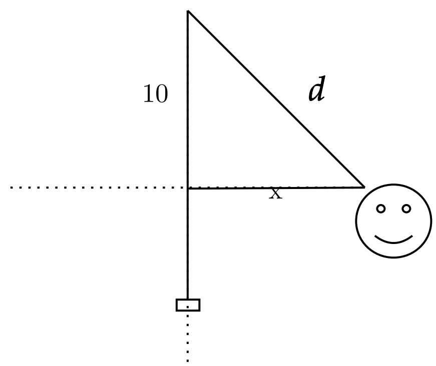
$$\frac{d}{dt}D^2 = \frac{d}{dt}\left(x^2 + y^2\right)$$

$$\frac{dD}{dt} = \frac{x\frac{dx}{dt} + y\frac{dy}{dt}}{D}$$

We have to find $\frac{dD}{dt}$ when $x = 10$, $y = 15$, $\frac{dy}{dt} = 30$, $\frac{dx}{dt} = 30$ and $D = 5\sqrt{13}$,

$$\frac{dD}{dt} = \frac{10 \cdot 30 + 15 \cdot 30}{5\sqrt{13}} = \frac{150}{\sqrt{13}}$$

1.2.3 2E-5



Let x be the distance from the person to the point directly under the pulley, and D is a length of the rope from the pulley to the person. We need to find $\frac{dx}{dt}$, given that $\frac{dD}{dt} = 4$, when $x = 20$ and $D = \sqrt{10^2 + 20^2} = 10\sqrt{5}$. Then,

$$D^2 = 100 + x^2$$

Differentiating both sides we get,

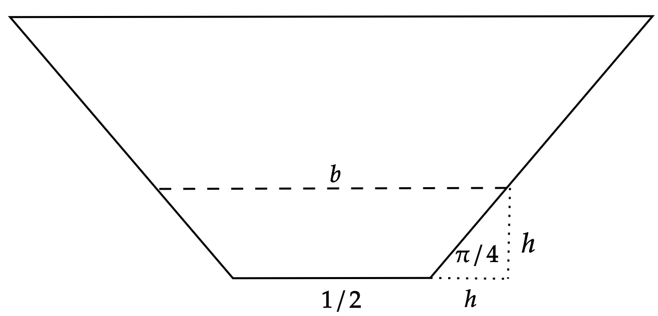
$$\frac{d}{dt}D^2 = \frac{d}{dt}\left(100 + x^2\right)$$

$$\frac{dx}{dt} = \frac{D\frac{dD}{dt}}{x}$$

So evaluating at,

$$\frac{dx}{dt} = 2\sqrt{5}$$

1.2.4 2E-7



$$V = \frac{1}{2}\left(\frac{1}{2} + \frac{1}{2} + 2h\right)4h = 2h + 4h^2$$

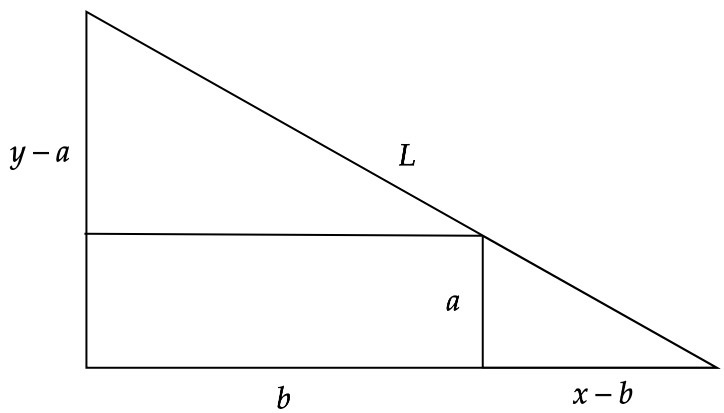
$$V' = 2h' + 8hh'$$
$$h' = \frac{V'}{2+8h}$$

So $h' = \frac{1}{6}$ when $V' = 1$ and $h = \frac{1}{2}$

2 Part II

2.1 Problem 1

2.1.1 Problem 1-a



$$\frac{y-a}{b} = \frac{a}{x-b} \Rightarrow y = \frac{ax}{x-b}$$
$$L = \sqrt{x^2+y^2} = \sqrt{x^2 + \frac{a^2x^2}{(x-b)^2}} = \frac{x}{x-b}\sqrt{(x-b)^2+a^2}$$
$$\frac{dL}{dx} = \frac{x}{\sqrt{(x-b)^2+a^2}} - \frac{b\sqrt{(x-b)^2+a^2}}{(x-b)^2}$$

At the endpoints $x = b$ and $x = \infty$ the function L goes to ∞, so the maximum must be in between at a critical point.

$$\frac{dL}{dx} = 0 \Rightarrow \frac{x}{\sqrt{(x-b)^2+a^2}} = \frac{b\sqrt{(x-b)^2+a^2}}{(x-b)^2} \Rightarrow x(x-b)^2 - b(x-b)^2 = a^2b \Rightarrow x = a^{2/3}b^{1/3} + b$$

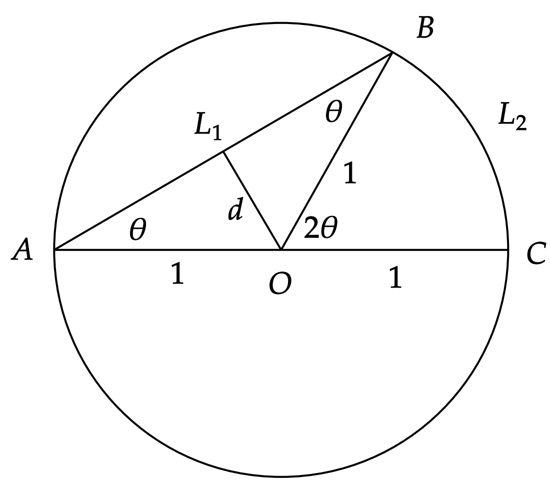
Evaluating L we get,

$$L = \frac{x}{x-b}\sqrt{(x-b)^2+a^2} = x\sqrt{1+\frac{a^2}{(x-b)^2}} = x\sqrt{1+\frac{a^2}{a^{4/3}b^{2/3}}} = x\sqrt{1+\frac{a^{2/3}}{b^{2/3}}}$$

Notice that $\frac{x}{b} = 1 + \frac{a^{2/3}}{b^{2/3}}$,

$$L = x\sqrt{1 + \frac{a^{2/3}}{b^{2/3}}} = x\sqrt{\frac{x}{b}} = \frac{x^{3/2}}{\sqrt{b}} = \frac{(a^{2/3}b^{1/3} + b)^{3/2}}{\sqrt{b}} = \frac{(b^{1/3}(a^{2/3} + b^{2/3}))^{3/2}}{\sqrt{b}} = (a^{2/3}+b^{2/3})^{3/2}$$

2.1.2 Problem 1-b



The time for the man to make to the point C is given by the formula,

$$L = \frac{L_1}{3} + \frac{L_2}{6}$$

The length L_1 is,

$$L_1 = 2\sqrt{r^2 - d^2} = 2\sqrt{1 - d^2} = 2\sqrt{1 - \sin^2\theta} = 2\cos\theta$$

The length L_2 is,

$$L_2 = 2\theta \cdot r = 2\theta$$

Then,

$$L = \frac{2\cos\theta}{3} + \frac{\theta}{3}$$

Check endpoints. If $\theta = 0$, then $L = \frac{2}{3}$. If $\theta = \frac{\pi}{2}$, then $L = \frac{\pi}{6}$. Find the critical point of the function,

$$\frac{dL}{d\theta} = \frac{1}{3} - \frac{2\sin\theta}{3}$$

$$\frac{1}{3} - \frac{2\sin\theta}{3} = 0 \Rightarrow \theta = \frac{\pi}{6}$$

If $\theta = \frac{\pi}{6}$, then $L = \frac{\sqrt{3}}{2}$. So the minimum is $L = \frac{2}{3}$ when $\theta = 0$.

2.2 Problem 2

$$x^{2/3} + y^{2/3} = 1$$

Let's rewrite the equation as a function y in terms of x in the first quadrant.

$$y = (1 - x^{2/3})^{3/2}$$

The $h(x) = y'(a)(x - a) + y(a)$ is a function of the tangent line to the curve y at an arbitrary point a.

To find the derivative y' we can use implicit differentiation,

$$\frac{d}{dx}(x^{2/3} + y^{2/3}) = \frac{d}{dx}1$$

$$\frac{dy}{dx} = -\frac{y^{1/3}}{x^{1/3}} = -\frac{(1 - x^{2/3})^{1/2}}{x^{1/3}}$$

So $h(x)$ is,

$$h(x) = -\frac{(1 - a^{2/3})^{1/2}}{a^{1/3}}(x - a) + (1 - a^{2/3})^{3/2}$$

The length of the portion of the tangent line is given by the formula,

$$L = \sqrt{h(0)^2 + x_1^2}$$

For such x_1 that $h(x_1) = 0$.

$$h(0) = a^{2/3}(1-a^{2/3})^{1/2}+(1-a^{2/3})^{3/2} = (1-a^{2/3})^{1/2}(a^{2/3}+(1-a^{2/3})) = (1-a^{2/3})^{1/2}$$

So $h(0) = (1 - a^{2/3})^{1/2}$. Solve the equation $h(x) = 0$.

$$-\frac{(1 - a^{2/3})^{1/2}}{a^{1/3}}(x - a) + (1 - a^{2/3})^{3/2} = 0$$

$$x = \frac{-a^{1/3}(1 - a^{2/3})^{3/2}}{-(1 - a^{2/3})^{1/2}} + a$$

$$x = a^{1/3}(1 - a^{2/3}) + a$$

$$x = a^{1/3}$$

So, we can compute the length L,

$$L = \sqrt{\left((1 - a^{2/3})^{1/2}\right)^2 + a^{2/3}} = \sqrt{(1 - a^{2/3}) + a^{2/3}} = 1$$

2.3 Sensitivity of measurement, revisited.

Recall that in problem 2, PS1/Part II, $L^2 + 20,000^2 = h^2$. Use implicit differentiation to calculate dL/dh

$$\frac{d}{dh}\left(L^2 + 20,000^2\right) = \frac{d}{dh}h^2$$

$$2L \cdot \frac{dL}{dh} = 2h$$

$$\frac{dL}{dh} = \frac{h}{L}$$

Compare the linear approximation dL/dh to the error $\Delta L/\Delta h$ computed in examples on PS1.

h	Δh	$\Delta L/\Delta h$	dL/dh	$\Delta L/\Delta h <= dL/dh$
25000	1	1.666607414	1.666666667	True
25000	0.1	1.666660741	1.666666667	True
25000	0.01	1.666666074	1.666666667	True
25000	-0.01	1.666667259	1.666666667	False
25000	-0.1	1.666672593	1.666666667	False
25000	-1	1.666725933	1.666666667	False
20001	1	82.84728347	100.00375	True
20001	0.1	97.62153854	100.00375	True
20001	0.01	99.75500156	100.00375	True
20001	-0.01	100.2549984	100.00375	False
20001	-0.1	102.6370585	100.00375	False
20001	-1	200.0025	100.00375	False

Explain why $\Delta L/\Delta h \leq dL/dh$ if the derivative is evaluated at the right endpoint of the interval of uncertainty (or, in other words, $\Delta h > 0$).

Let's find $\frac{d^2L}{dh^2}$.

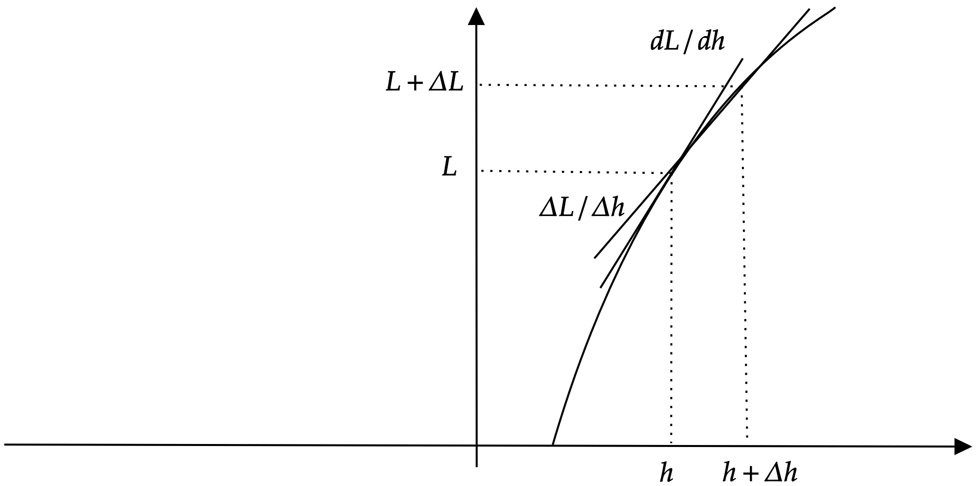
$$\frac{d^2L}{dh^2} = \frac{L - \frac{h^2}{L}}{L^2}$$

Since $\frac{d^2L}{dh^2} < 0$, the graph is concave down. To show that we compute the limit of $\frac{d^2L}{dh^2}$ as $x \to 20000^+$ and $x \to \infty$ and show that $\frac{d^2L}{dh^2} \neq 0$.

$$\lim_{x \to 20000^+} \frac{L - \frac{h^2}{L}}{L^2} = \frac{0^+ - \infty}{0^+} = -\infty$$

$$\lim_{x\to\infty} \frac{L - \frac{h^2}{L}}{L^2} = \lim_{x\to\infty} \frac{1 - \frac{h^2}{L^2}}{L} = \frac{1-1}{\infty} = 0$$

And in fact $\frac{d^2L}{dh^2} \neq 0$ because in that case h^2 would be equal to L^2 which is impossible. Below is a sketch of the concave down graph.



So the graph shows that when $\Delta h > 0$, then $dL/dh \geq \Delta L/\Delta h$ (i.e. the slope of the tangent line can't be smaller than the slope of the secant line).

In what range of values of h is it true that $|\Delta L| \leq 2|\Delta h|$.

$|\Delta L| \leq 2|\Delta h|$ implies that $\frac{|\Delta L|}{|\Delta h|} \leq 2$. From the previous part we know that $\frac{\Delta L}{\Delta h} \leq \frac{dL}{dh}$ when $\Delta h > 0$. So, if $\frac{dL}{dh} \leq 2$, then $\frac{\Delta L}{\Delta h} \leq 2$. It means that we need to solve an equation $\frac{dL}{dh} \leq 2$.

$$\frac{h}{L} \leq 2$$
$$h^2 \leq 4(h^2 - 20,000^2)$$
$$h^2 \geq \frac{4 \cdot 20,000^2}{3}$$
$$h \geq \frac{40,000}{\sqrt{3}}$$

If $h = \frac{40000}{\sqrt{3}}$ then $\frac{dL}{dh} = 2$, and that means that Δh must be greater than zero at these point, because otherwise $\frac{\Delta L}{\Delta h} < 2$. At other points Δh can be negative but $h + \Delta h \geq \frac{40000}{\sqrt{3}}$ for the same reason.

Therefore $|\Delta L| \leq 2|\Delta h|$ is true if h in range $[23094, \infty]$.

Suppose that the Planet Quirk is a not only flat, but one-dimensional (a straight line). There are several satellites at height $20,000$ kilometers and you get readings saying that satellite 1 is directly above the point $x_1 \pm 10^{-10}$ and is at a distance $h_1 = 21,000 \pm 10^{-2}$ from you, satellite 2 is directly above $x_2 \pm 10^{-10}$ and at a distance $h_2 = 52,000 \pm 10^{-2}$. Where are you and to what accuracy? Hint: Consider separately the cases $x_1 < x_2$ and $x_2 < x_1$.

Let p be a point at which the person is located on the planet. Then L_1 is a distance from p to x_1 and L_2 is a distance from p to x_2.

$$L_1 = \sqrt{h_1^2 - 20,000^2} \approx 6403$$

$$L_2 = \sqrt{h_2^2 - 20,000^2} = 48000$$

The error in our measurements is ΔL, which is $\Delta L \approx \frac{dL}{dh}\Delta h$. So,

$$\Delta L_1 = \frac{h_1}{L_1}\Delta h_1 = \frac{21000}{6403} \cdot \pm 0.01 \approx \pm 0.03$$

$$\Delta L_2 = \frac{h_2}{L_2}\Delta h_2 = \frac{52000}{48000} \cdot \pm 0.01 \approx \pm 0.01$$

Let's consider the case $x_1 < x_2$. We are closer to x_1, so there are two subcases $p < x_1 < x_2$ and $x_1 < p < x_2$. In the former case

$$p = x_1 - L_1 \pm (10^{-10} + 0.03)$$

$$p = x_2 - L_2 \pm (10^{-10} + 0.01)$$

In the latter case

$$p = x_1 + L_1 \pm (10^{-10} + 0.03)$$

$$p = x_2 - L_2 \pm (10^{-10} + 0.01)$$

Let's consider the case $x_2 < x_1$. We are closer to x_1, so there are to subcases $x_2 < p < x_1$ and $x_2 < x_1 < p$. In the former case

$$p = x_1 - L_1 \pm (10^{-10} + 0.03)$$

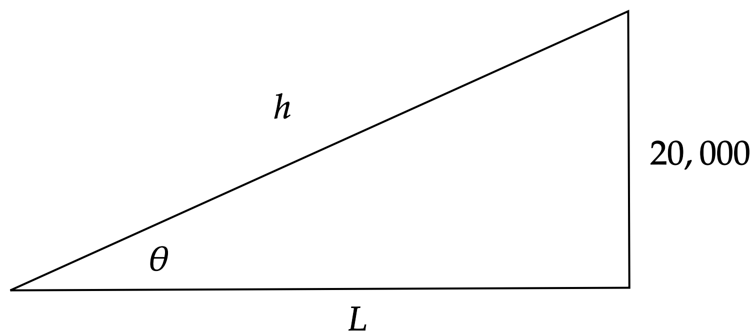
$$p = x_2 + L_2 \pm (10^{-10} + 0.01)$$

In the latter case

$$p = x_1 + L_1 \pm (10^{-10} + 0.03)$$

$$p = x_2 + L_2 \pm (10^{-10} + 0.01)$$

Express dL/dh in terms of the angle between the line of sight to the satellite and the horizontal from the person on the ground. (When expressed using the line-of-sight angle, the formula also works for a curved planet like Earth.)

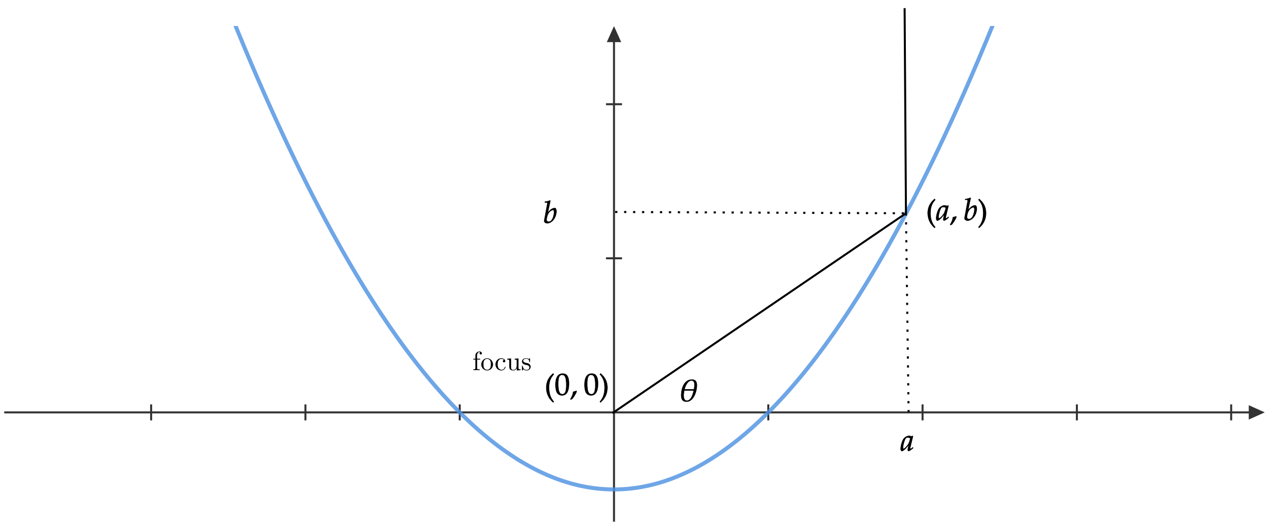


Since we found $dL/dh = h/L$ using implicit differentiation, then dL/dh in terms of θ should be $\sec\theta$.

2.4 More sensitivity of measurement.

Consider a parabolic mirror with equation $y = -1/4 + x^2$ and focus at the origin. (See Problem Set 1.) A ray of light traveling down vertically along the line $x = a$ hits the mirror at the point (a, b) where $b = -1/4+a^2$ and goes to the origin along a ray at angle θ measured from the positive x-axis.

a) Find the formula for $\tan\theta$ in terms of a and b, and calculate $d\theta/da$ using implicit differentiation. (Express your answer in terms of a and θ.)



$$\tan\theta = \frac{b}{a}$$

$$\frac{d}{da}(\tan\theta) = \frac{d}{da}\left(\frac{b}{a}\right)$$

$$\sec^2\theta \frac{d\theta}{da} = \frac{\frac{db}{da}a - b}{a^2}$$

$$\frac{d\theta}{da} = \frac{2a^2 + 1/4 - a^2}{a^2\sec^2\theta} = \frac{a^2 + 1/4}{a^2\sec^2\theta}$$

b) If the telescope records a star at $\theta = -\pi/6$ and the measurement is accurate to 10^{-3} radians, use part (a) to give an estimate as to the location of the star in the variable a.

If $\theta = -\pi/6$, then

$$\frac{b}{a} = -\frac{\sqrt{3}}{3} \Rightarrow 3a^2 + \sqrt{3}a - \frac{3}{4} = 0$$

Solving the equation above we get $a = \frac{1}{2\sqrt{3}}$ (ignore negative a). If θ is a function in terms of a, then error in θ is approximately $\Delta\theta \approx \frac{d\theta}{da}\Delta a \Rightarrow \Delta a \approx \Delta\theta / \frac{d\theta}{da}$. Hence,

$$\frac{d\theta}{da} = \frac{(1/2\sqrt{3})^2 + 1/4}{(1/2\sqrt{3})^2 \sec^2(-\pi/6)} = 3$$

So, if $\Delta\theta = 10^{-3}$, then $\Delta a \approx 10^{-3}/3$. Hence, the location of the start in the variable a is approximately $\frac{1}{2\sqrt{3}} \pm 10^{-3}/3$

(optional; no credit) Solve for a as a function of θ alone and doublecheck your answers to parts (a) and (b).

Let h be *hypotenuse*, then from Pythagorean theorem

$$h^2 = a^2 + \left(-1/4 + a^2\right)^2 \Rightarrow h = a^2 + 1/4$$

Then

$$\sin\theta = \frac{a^2 - 1/4}{a^2 + 1/4}$$

$$a = \frac{1}{2}\sqrt{\frac{1+\sin\theta}{1-\sin\theta}}$$

$$\frac{da}{d\theta} = \frac{\cos\theta}{2\sqrt{\frac{1+\sin\theta}{1-\sin\theta}}(1-\sin\theta)^2}$$

If $\theta = -\pi/6$, then $a = \frac{1}{2\sqrt{3}}$, $\frac{da}{d\theta} = 1/3$, $\Delta a \approx \frac{da}{d\theta}\Delta\theta = 10^{-3}/3$. Hence, $a \approx \frac{1}{2\sqrt{3}} \pm 10^{-3}/3$.

2.5 Newton's method.

a) Compute the cube root of 9 to 6 significant figures using Newton's method. Give the general formula, and list numerical values, starting with $x_0 = 2$. At what iteration k does the method surpass the accuracy of your calculator or computer? (Display your answers to the accuracy of your calculator or computer.)
b) For each step x_k, $k = 0, 1, \dots$, say whether the value is i) larger or smaller than $9^{1/3}$; ii) larger or smaller than the preceding value x_{k-1}. Illustrate on the graph of $x^3 - 9$ why this is so.

If $x = \sqrt[3]{9}$, then $f(x) = x^3 - 9$ and $f'(x) = 3x^2$. So,

$$x_k = x_{k-1} - \frac{f(x_{k-1})}{f'(x_{k-1})}$$

According to my calculator $x = \sqrt[3]{9} \approx 2.080084$.

k	x_k	(i)	(ii)
0	2	smaller	-
1	2.083333	larger	larger
2	2.080089	larger	smaller
3	2.080084	same	smaller

So using the Newton method the accuracy of the answer is the same as the calculator's accuracy at $k = 3$.

c) Find a quadratic approximation to $9^{1/3}$, and estimate the difference between the quadratic approximation and the exact answer. (Hint: To get a reasonable quadratic approximation, use $9 = 8(1 + 1/8)$.)

To find a quadratic approximation to $9^{1/3}$ we could approximate function $f(x) = 2(1 + x)^{1/3}$ at $x \approx 0$. So,

$$f'(x) = \frac{2}{3}(1 + x)^{-1/6}$$

$$f''(x) = -\frac{1}{9}(1 + x)^{-2/3}$$

$$f(x) \approx f(0) + f'(0)x + \frac{f''(0)}{2}x^2 = 2 + \frac{2}{3}x - \frac{1}{18}x^2$$

$$f(1/18) \approx 2 + \frac{1}{12} - \frac{1}{1152} = 2.082465$$

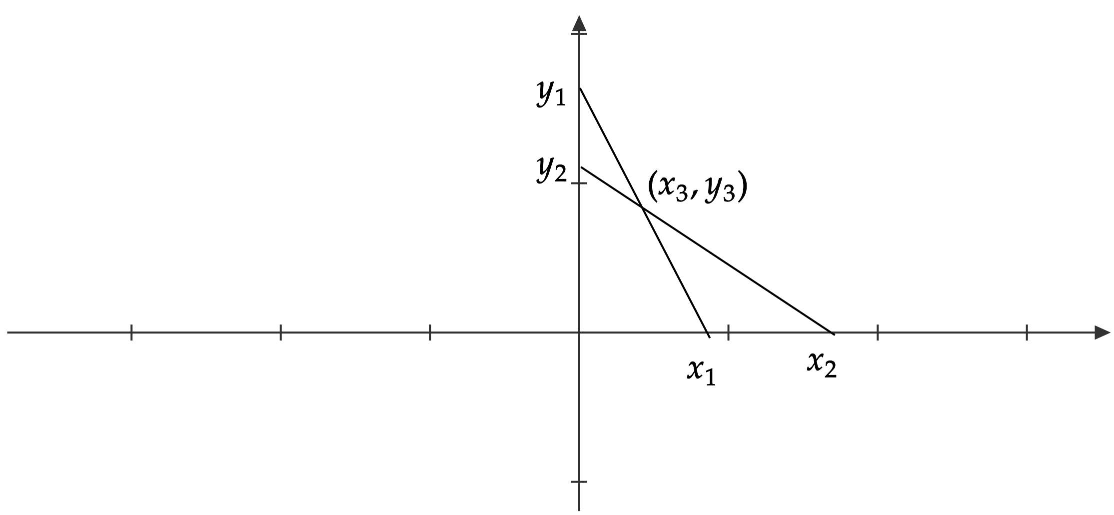
So, the difference between quadratic approximation and the exact number is $|2.082465 - 2.080084| \approx 3 \times 10^{-3}$.

2.6 Hypocycloid, again.

Here we derive the equation for the hypocycloid of Problem 2 from the sweeping out property directly. This takes quite a bit longer. We will look at the hypocycloid from yet another (easier) point of view later on.

Think of the first quadrant of the xy-plane as representing the region to the right of a wall with the ground as the positive x-axis and the wall as the positive y-axis. A unit length ladder is placed vertically against the wall. The bottom of the ladder is at $x = 0$ and slides to the right along the x-axis until the ladder is horizontal. At the same time, the top of the ladder is dragged down the y-axis ending at the origin $(0, 0)$. We are going to describe the region swept out by this motion, in other words, the blurry region formed in a photograph of the motion if the eye of the camera is open the whole time.

a) Suppose that L_1 is the line segment from $(0, y_1)$ to $(x_1, 0)$ and L_2 is the line segment from $(0, y_2)$ to $(x_2, 0)$. Find the formula for the point of intersection (x_3, y_3) of the two line segments. Don't expect the formula to be simple: It must involve all four parameters x_1, x_2, y_1, and y_2. But simplify as much as possible!



We have a system of linear equations.

$$y_3 = -\frac{y_1}{x_1}x_3 + y_1$$

$$y_3 = -\frac{y_2}{x_2}x_3 + y_2$$

Solving for y_3 and x_3 we get

$$x_3 = \frac{y_2 - y_1}{y_2/x_2 - y_1/x_1}$$

$$y_3 = \frac{x_1 - x_2}{x_1/y_1 - x_2/y_2}$$

b) Write the equation involving x_2 and y_2 that expresses the property that ladder L_2 has length one. We will suppose that L_1 represents the ladder at a fixed position, and L_2 tends to L_1. Thus

$$x_2 = x_1 + \Delta x; \quad y_2 = y_1 + \Delta y$$

Use implicit differentiation (related rates) to find

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

(Express the limit as a function of the fixed values x_1 and y_1.)

If L_2 has length one, then

$$x_2^2 + y_2^2 = 1$$

Thus, the general equation expressed in x and y is

$$x^2 + y^2 = 1$$

By using implicit differentiation we can find dy/dx

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = -\frac{x}{y}$$

If we need the derivative at point x_1 we have

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = -\frac{x_1}{y_1}$$

c) Substitute $x_2 = x_1 + \Delta x$ and $y_2 = y_1 + \Delta y$ into the formula in part (a) for x_3 and use part (b) to compute

$$X = \lim_{x_2 \to x_1} x_3 = \lim_{\Delta x \to 0} x_3$$

Simplify as much as possible. Deduce, by symmetry alone, the formula for

$$Y = \lim_{x_2 \to x_1} y_3$$

$$x_3 = \frac{y_2 - y_1}{y_2/x_2 - y_1/x_1} = \frac{x_1 x_2 (y_2 - y_1)}{x_1 y_2 - x_2 y_1} = \frac{\Delta y \cdot x_1 \cdot (x_1 + \Delta x)}{x_1 \Delta y - y_1 \Delta x} = \frac{\Delta y}{\Delta x} \cdot \left(\frac{x_1 (x_1 + \Delta x)}{x_1 \frac{\Delta y}{\Delta x} - y_1} \right)$$

$$X = \lim_{\Delta x \to 0} \left[\frac{\Delta y}{\Delta x} \cdot \left(\frac{x_1 (x_1 + \Delta x)}{x_1 \frac{\Delta y}{\Delta x} - y_1} \right) \right] = -\frac{x_1}{y_1} \cdot \frac{x_1^2}{-\frac{x_1^2}{y_1} - y_1} = \frac{-x_1^3}{-x_1^2 - y_1^2} = \frac{x_1^3}{x_1^2 + y_1^2}$$

Since $x^2 + y^2 = 1$, then $x_1^2 + y_1^2 = 1$. Hence, $X = x_1^3$. Similarly, by symmetry $Y = y_1^3$.

d) Show that $X^{2/3} + Y^{2/3} = 1$. (The limit point (X, Y) that you found in part (c) is expressed as a function of x_1 and y_1. This is the unique point of the ladder L_1 that is also part of the boundary curve of the region swept out by the family of ladders.)

$$X^{2/3} + Y^{2/3} = \left(x_1^2\right)^{2/3} + \left(y_1^2\right)^{2/3} = x_1^2 + y_1^2 = 1$$