Exam 2

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1 Problem 1

Estimate the following to two decimal places (show work).

a. $\sin(\pi + 1/100)$

$$\sin(\pi + 1/100) \approx \frac{cos\pi}{100} + \sin\pi = -\frac{1}{100} + 0 = -0.01$$

b. $\sqrt{101}$

$$\sqrt{101} \approx \frac{1}{2\sqrt{100}} \cdot 1 + \sqrt{100} = \frac{201}{20} = 10.05$$

2 Problem 2

Sketch the graph of $y = \frac{4}{x} + x + 1$ on $-\infty < x < \infty$ and label all critical points and infection points with their coordinates on the graph along with the letter "C" or "I"

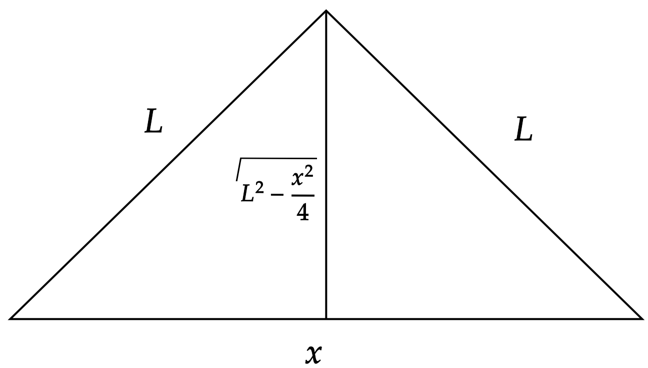
Skipped.

3 Problem 3

An architect plans to build a triangular enclosure with a fence on two sides and a wall on the third side. Each of the fence segments has fixed Length L. What is the length x of the third side if the region enclosed has the largest possible area? Show work and include an argument to show that your answer really gives the maximum area.

The area A is given by the formula

$$A = \frac{x}{2}\sqrt{L^2 - \frac{x^2}{4}} = \frac{1}{4}\sqrt{4L^2x^2 - x^4}$$



We consider only $x \geq 0$. The constraint is that $L^2 - \frac{x^2}{4} \geq 0 \Rightarrow x \leq 2L$. So if $x = 0$ or $x = 2L$, then $A = 0$. So the maximum area must be in between at critical point.

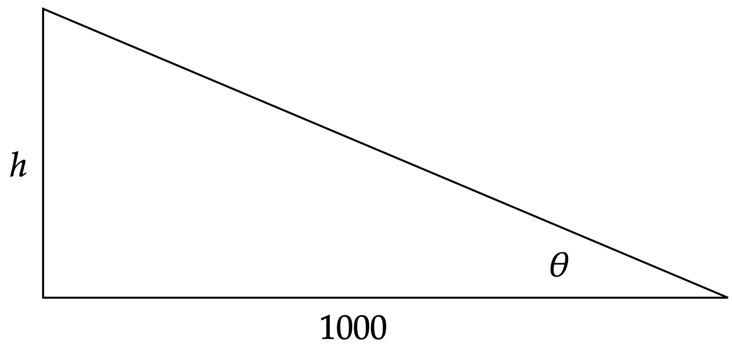
$$\frac{d}{dx}\left(\frac{1}{4}\sqrt{4L^2x^2 - x^4}\right) = 0$$

$$\frac{2L^2x - x^3}{\sqrt{L^2x^2 - x^4}} = 0$$

$$x = \sqrt{2}L$$

4 Problem 4

A rocket has launched straight up, and its altitude is $h = 10t^2$ feet after t seconds. You are on the ground 1000 feet from the launch site. The line of sight from you to the rocket makes an angle θ with the horizontal. By how many Radians per second is θ changing ten seconds after the launch?



We need to find $\frac{d\theta}{dt}$ where t is measured in seconds. We are given that $h = 10t^2$, therefore $\frac{dh}{dt}$ after 10 seconds is 200. To find $\frac{d\theta}{dt}$ we implicitly differentiate the equation $\tan\theta = \frac{h}{1000}$ with respect to t.

$$\frac{d}{dt}\tan\theta = \frac{d}{dt}\frac{h}{1000}$$

$$\sec^2\theta\frac{d\theta}{dt} = \frac{dh/dt}{1000}$$

If $t = 10$, then $h = 1000$ and $\sec^2\theta = \tan^2\theta + 1 = \frac{1000}{1000} + 1 = 2$. Then,

$$2\frac{d\theta}{dt} = \frac{200}{1000} \Rightarrow \frac{d\theta}{dt} = \frac{1}{10}$$

5 Problem 5

a. Evaluate the following indefinite integrals.

i. $\int \cos(3x)dx$

$$u = 3x; \quad du = 3dx \quad \Rightarrow \quad \frac{1}{3}\int \cos(u)du = \frac{1}{3}\sin 3x + C$$

ii. $\int xe^{x^2}dx$

$$u = x^2; \quad du = 2xdx \quad \Rightarrow \quad \frac{1}{2}\int e^u du = \frac{1}{2}e^{x^2} + C$$

b. Find $y(x)$ such that $y' = \frac{1}{y^3}$ and $y(0) = 1$.

$$\frac{dy}{dx} = \frac{1}{y^3}$$

$$y^3 dy = dx$$

Taking the integral of both sides

$$\int y^3 dy = \int dx$$

$$\frac{y^4}{4} + C = x + C$$

$$y = (4x + C)^{1/4}$$

$$y(0) = 1 \Rightarrow (4 \cdot 0 + C)^{1/4} = 1 \Rightarrow C = 1$$

$$y = (4x + 1)^{1/4}$$

6 Problem 6

Suppose that $f'(x) = e^{(x^2)}$, and $f(0) = 10$. One can conclude from the mean value theorem that

$$A < f(1) < B$$

for which numbers A and B?

The mean value theorem says

$$\frac{f(1) - f(0)}{1 - 0} = f'(c), \quad \text{for some } c \text{ s.t. } 0 < c < 1$$

Then $f(1) = e^{(c^2)} + 10$. If $0 < c < 1$, then $1 < f'(c) < e$. Hence,

$$11 < f(1) < 10 + e$$