Introduction to Ramsey Theory

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REU Week 1 Report

June 7, 2013

Outline

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Ramsey Theorem (infinite)

Ramsey Theorem (infinite version, Ramsey 1928): An infinite complete graph with edges colored red and blue contains an infinite complete monochromatic subgraph.

Proof Idea. Choose a vertex a_0, there are an infinite number of edges of one color incident to that vertex, call this set N_0. Choose a vertex $a_1 \in N_0$, there are an infinite number of edges of one color incident to a_1. Repeat infinitely to form a chain.

$$a_0 \;\text{———}\; a_1 \;\text{———}\; a_2 \;\text{———}\; a_3 \;\text{———}\; \cdots$$

The subgraphs consisting of $a_0, a_3, \cdots$ and $a_1, a_2, \cdots$ are both monochromatic and at least one is infinite.

The theorem can be extended to any finite number of colors.

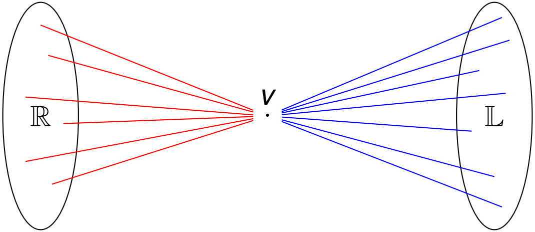
Ramsey Theorem (finite)

Ramsey Theorem (finite version, Ramsey 1928): For any pair of positive integers m and n, there exists an integer k such that any complete graph on k vertices with edges colored red and blue must contain either a monochromatic red complete subgraph on m vertices or a monochromatic blue complete subgraph on n vertices.

Proof Idea. Let $R(m, n) = k$, the particular k value required for the pair m and n. Clearly $R(m, 1) = R(1, n) = 1$. Bound $R(m, n)$ inductively from above by

$$R(m, n) \leq R(m - 1, n) + R(m, n - 1).$$

Ramsey Theorem (finite)



Case I: $|\ \mathbb{R}\ | \geq R(m-1, n)$. $\mathbb{R}$ contains a red K_{m-1} or a blue K_n, thus G contains a red K_m or a blue K_n.
Case II: $|\ \mathbb{L}\ | \geq R(m, n-1)$. $\mathbb{L}$ contains a red K_m or a blue K_{n-1}, thus G contains a red K_m or a blue K_n.
Therefore $R(m, n) \leq R(m-1, n) + R(m, n-1)$.

Outline

Ramsey Numbers

Ramsey Theory deals with finding order amongst greater chaos.

Example: if we want to find a constellation of 10 stars forming a convex polygon where no three stars are collinear, how many stars do there need to be, to guarantee its occurrence? (The Happy Ending problem). Ramsey numbers are a similar idea.

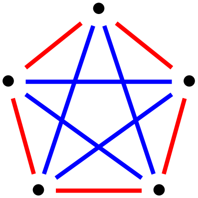
Ramsey Numbers

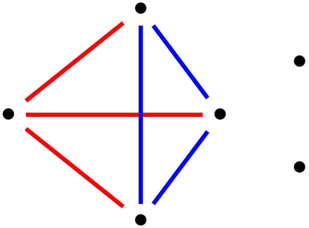
Ramsey Number: The Ramsey number $R(n_1, ..., n_c) = r$ is the least number such that if the edges of a complete graph of order r are colored with c different colors, then for some i between 1 and c, it must contain a complete subgraph of order n_i whose edges are all color i.

Motivating problem: What is the minimum number of people you must invite to a party to guarantee you will alway have 3 people who all know each other or 3 people who all do not know each other?

In other words: What is R(3,3)?

Example: $R(3,3)$

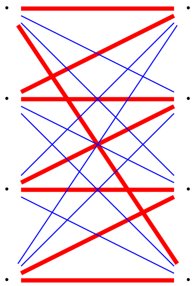




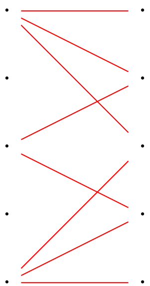
Since K_5 can be 2-colored without monochromatic triangles, $R(3,3) > 5$. Since K_6 cannot be colored without monochromatic triangles, $R(3,3) \leq 6$. Thus $R(3,3) = 6$.

Bipartite Ramsey Numbers

Two Color Bipartite Ramsey Number: With 2 colors, red and blue, the bipartite Ramsey number $b(m, n)$ is the least positive integer b such that if the edges of $K(b, b)$ are colored with red and blue, then there always exists a blue $K(m, m)$ or a red $K(n, n)$.



$b(2, 2) > 4$



≥ 13 edges in one color, degree sequence: $3, 3, 3, 2, 2$

$b(2, 2) \leq 5$

Bipartite Ramsey Numbers

Bipartite Ramsey Number: The bipartite Ramsey number $b(n_1, ..., n_k)$ is the least positive integer b such that any coloring of the edges of $K_{b,b}$, with k colors will result in a monochromatic copy of K_{n_i,n_i}, in the i-th color, for some i, $1 \leq i \leq k$.

If $n_i = m$ for all i, then we denote this number by $b_k(m)$.

Zarankiewicz Numbers

Zarankiewicz Numbers: The Zarankiewicz number $z(m, n; s, t)$ is the maximum number of edges in a subgraph of $K_{m,n}$ that does not contain $K_{s,t}$ as a subgraph.

Special Case: If $m = n$ and $s = t$, then we write $z(m, s)$ to denote the number of edges in a subgraph of $K_{m,m}$ that does not contain $K_{s,s}$ as a subgraph.

Relation to Bipartite Ramsey Numbers: For two colors, finding upper bounds for Zarankiewicz numbers help provide upper bounds on bipartite Ramsey numbers.

$$z(b, m) + z(b, n) < b^2 \quad \implies \quad b(m, n) \leq b$$

This relation can be extended to allow for more than two colors.

Zarankiewicz Numbers

Proof:

Let $z(b, m) = z_1$, and $z(b, n) = z_2$

$K_{b,b}$ has z_1 red edges that does not contain a $K_{m,m}$

$K_{b,b}$ has z_2 blue edges that does not contain a $K_{n,n}$

We also know $K_{b,b}$ has b^2 edges

If $z_1 + z_2 < b^2 \implies b(m, n) \leq b$

Since if $b(m, n) = b + 1$, $K_{b,b}$ will contain a red $K_{m,m}$ or a blue $K_{n,n}$

Outline

$b(2,5)$

It is known that $16 \leq b(2,5) \leq 19$. (Goddard et al. 2004)

Recall that $b(2,5)$ is the smallest number of vertices in each partition of a complete bipartite graph such that if the edges are colored red or blue we are guaranteed a red $K_{2,2}$ or a blue $K_{5,5}$.

We could improve the lower bound by finding edge colorings of $K_{16,16}$ which do not contain either of the forbidden subgraphs. Or we could improve the upper bound by showing that all possible colorings of $K_{18,18}$ contain one of the forbidden subgraphs. These graphs are not so large as to be computationally intractable.

$b_5(2)$

Recall that $b_5(2) = b(2,2,2,2,2)$, the number of vertices in each partite set of a complete bipartite graph such that any 5-coloring of the edges results in a monochromatic 4-cycle.

Problem of Interest: Determining values for $b_k(2)$ seems to be a difficult problem. The only known exact values are $b_2(2) = 5$, $b_3(2) = 11$, and $b_4(2) = 19$.

Some Theorems: (Dybizbański, Dzido, Radziszowski, 2013)

Theorem 1: $b_k(2) \geq k^2 + 1$

Theorem 2: $b_k(2) \leq k^2 + k - 2$

Theorem 3: $26 \leq b_5(2) \leq 28$

Possible conjecture for project: $b_5(2) = 28$

Readings

- W. Goddard, M. A. Henning and O. R. Oellermann, "Bipartite Ramsey Numbers and Zarankiewicz Numbers", *Elsevier Science* (2004).
- J. Dybizbański, T. Dzido and S. Radziszowski, "On Some Zarankiewicz Numbers and Bipartite Ramsey Numbers for Quadrilateral". Forthcoming (2013).
- R. K. Guy, "A Many-Facetted Problem of Zarankiewicz", *The Many Facets of Graph Theory* (1969).

Questions

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