AE353: Design Problem 01

State Space Models

February 15, 2019

1 Goal

The code provided in `DesignProblem01` simulates the rotational motion of a spacecraft. This spacecraft has actuators that can apply torque about two different axes. This spacecraft also has sensors—an inertial measurement unit (IMU)—to measure its angular velocity. The spacecraft starts with some random angular velocity. The goal is to achieve a 1 by 3 vector angular velocity of $\vec{\omega} = \begin{bmatrix} 3 & 0 & 0 \end{bmatrix}$ rad/s.

2 Model

The rotational motion of the spacecraft, if it is modeled as a single rigid body, is governed by the ordinary differential equations

$$\tau_1 = J_1 \dot{w}_1 - (J_2 - J_3) w_2 w_3 \tag{1}$$

$$0 = J_2 \dot{w}_2 - (J_3 - J_1) w_3 w_1 \tag{2}$$

$$\tau_3 = J_3 \dot{w}_3 - (J_1 - J_2) w_1 w_2, \tag{3}$$

where w_1, w_2, w_3 are the components of angular velocity, J_1, J_2, J_3 are the principle moments of inertia, and τ_1, τ_3 are the two different torques that can be applied.

3 Linearization to State Space Model

In order to determine the linear model, we first define the State, x, Input, u, and Output, y as well as the state. Our goal is to simplify our equations of motion to the following:

$$\dot{x} = Ax + Bu, \quad y = Cx + Du$$

$$x = \begin{bmatrix} \omega_1 - \omega_e \\ \omega_2 - \omega_e \\ \omega_3 - \omega_e \end{bmatrix} \frac{rad}{s}, \quad u = \begin{bmatrix} \tau_1 - \tau_e \\ \tau_3 - \tau_e \end{bmatrix} N \cdot m, \quad y = \begin{bmatrix} \omega_1 - \omega_e \\ \omega_2 - \omega_e \\ \omega_3 - \omega_e \end{bmatrix} \frac{rad}{s}$$

Gathering all this information, we will now go to MatLab to utilize `Jacobian()` in order to find values for A, B, C and D for the non-linearized model, the code and results of this linearization are displayed below:

```
syms j1 j2 j3 w1 w2 w3 wd1 wd2 wd3 t1 t3
x = [w1; w2; w3];
u = [t1; t3];
y = x;
eqn1 = t1 == j1*wd1 - (j2 - j3)*w2*w3;
eqn2 = 0 == j2*wd2 - (j3 - j1)*w3*w1;
eqn3 = t3 == j3*wd3 - (j1 - j2)*w1*w2;
xd = [solve(eqn1, wd1); solve(eqn2, wd2); solve(eqn3, wd3)];
%Jacobian to find values for A, B, C, D
A = jacobian(xd, x)
B = jacobian(xd, u)
C = jacobian(y, x)
D = jacobian(y, u)
```

$$C_1 = \frac{J_2 - J_3}{J_1} \quad C_2 = \frac{J_1 - J_3}{J_2} \quad C_3 = \frac{J_1 - J_2}{J_3}$$

$$A = \begin{bmatrix} 0 & (\omega_3 \times C_1) & (\omega_2 \times C_1) \\ -(\omega_3 \times C_2) & 0 & -(\omega_1 \times C_2) \\ (\omega_2 \times C_3) & (\omega_1 \times C_3) & 0 \end{bmatrix}$$

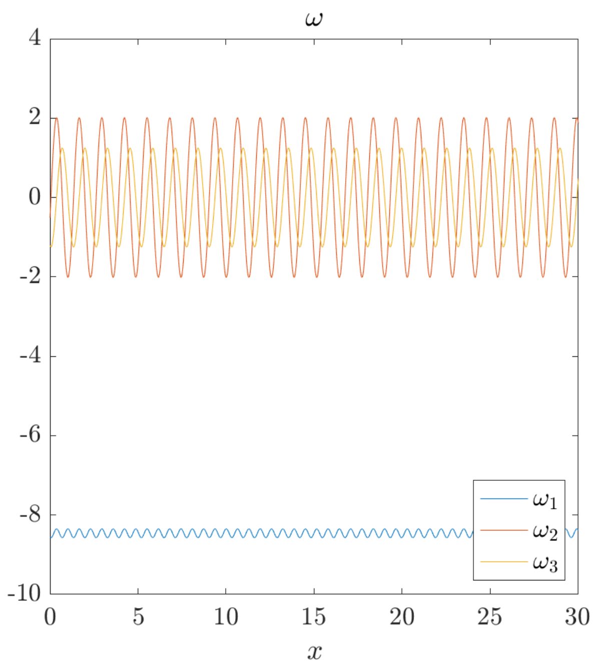
$$B = \begin{bmatrix} \frac{1}{J_1} & 0 \\ 0 & 0 \\ 0 & \frac{1}{J_3} \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

4 Test.m Configuration

In order to initialize the simulation, we created our `Test.m`, a file which will be used in future reference to call the necessary functions to run the simulation.

5 Zero Input Analysis

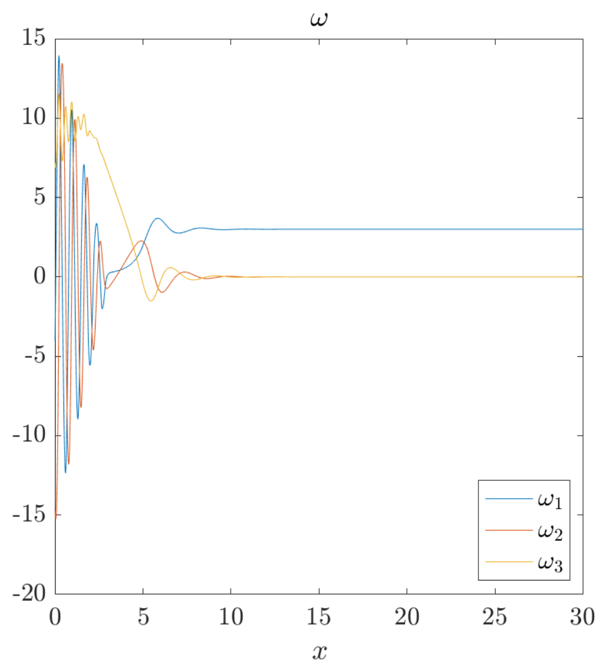
For **Zero Input**, we set `actuators.tau1` and `actuators.tau3` $= \tau_e$ in the `Controller.m` file. Finding τ_e by solving the non-linear system of motion equations provided to us in Section 2, we placed those values into '`actuators.tau1 =` ' to observe the behavior of the oscillation. Due to the fact that there are no controllers providing input to the system and producing changes in angular velocity vectors, it may be hypothesized that the satellite will not converge nor be asymptotically stable. The resulting figures for zero input are posted on the next page:



As we can see from the graphs, the satellite does not converge, is not asymptotically stable, and continues oscillating for the duration of the simulation. Additionally, we found some values of which $F = eig(A - B * K) > 0$, therefore the system is not asymptotically stable.

6 State Feedback Analysis

State feedback uses the linearized state space model to control the input, in this case τ_1 and τ_3, to obtain a desired angular velocity. If the controller is operating optimally, the initial position, $x(t_0)$, and initial velocity, $\dot{z}(t_0)$, will not change the final outcome but will affect the behavior of the process towards equilibrium. Basically, the initial values may significantly or minimally alter the initial behavior of the position and angular velocity graphs by changing the amplitude, duration to stabilize, or – depending on the K values – throwing the controller off. We modified the `Controller.m` code to take into account the linearized system by designating the input, u, to control the `actuators.tau`. Since the controller observes the angular velocity from the sensors and makes slight adjustments in the x- and z-directions through use of the gyroscopic wheels, we can hypothesize the satellite under the state feedback input will be asymptotically stable and converge. Note, the clip below will only show the additions to the Matlab code provided to us:



As we can see, the satellite converges on the desired points of angular velocity $\vec{\omega} = [3\ 0\ 0]$. As with the zero-input part of this report, we ran an eigenvalue test on the values of A_m to determine whether the system was asymptotically stable, and indeed it was asymptotically stable due to all of the real numbers existing as less than zero.

7 Conclusion

In this project, we were given a system of non-linearized equations of motion. To begin the implementation of the controller state-space model, we linearized the system into A, B, C, D matrices to configure the state, input, and output systems. We then migrated over to Matlab to designated the input values for `actuators.tau_1` and `actuators.tau3`. In our zero-input analysis section, we predicted the inability of the controller to reach our designated angular velocity values and therefore not be asymptotically stable. In our visual analysis of the produced graphs, the hypothesis was found to be valid: the system was not asymptotically stable in a zero-input condition. When we placed our state-space model into the simulation, the gyroscopes were able to bring the satellite to an asymptotically stable state. In conclusion, we can see that the implementation of state space linearization results in an asymptotically stable system and helps the satellite converge to a desired velocity. It is important to note that the `DesignProblem01.m` code randomizes the initial position and angular velocities. The initial components directly impacts the step response time, time to peak, and overshoot while – assuming the controller is active – the system maintains asymptotic stability. Ultimately, state space model linearization succeeds in changing randomized motion into an asymptotically stable system and allows us to reach our desired angular velocities.