AE353: Design Problem 01

February 16, 2019

1 Goal

The objective is to simulate the control of a spacecraft capable of setting torques about two of three axes. Implementation of a controller with state feedback and reference tracking will allow the spacecraft to achieve a steady state of $\vec{w} = \vec{0}$ rad s^{-1}. This would be useful for manned spacecraft. It also applies to docking spacecraft because zero relative rotational velocity is required for successful docking.

2 Model

The rotation of the spacecraft can be modeled by using the following system of ordinary differential equations

$$\begin{aligned}
\tau_1 &= J_1 \dot{w}_1 - (J_2 - J_3) w_2 w_3 \\
0 &= J_2 \dot{w}_2 - (J_3 - J_1) w_3 w_1 \\
\tau_3 &= J_3 \dot{w}_3 - (J_1 - J_2) w_1 w_2,
\end{aligned}$$

where w_1, w_2, w_3 are the components of angular velocity, J_1, J_2, J_3 are the principle moments of inertia, and τ_1, τ_3 are the torques about the 1 and 3 axes, respectively.

This system is nonlinear, so in order to apply a state-space model to the system, the system must be linearized. Solving the above equations for $\dot{w}_1, \dot{w}_2, \dot{w}_3$ yields

$$\dot{w}_1 = \frac{1}{J_1}\tau_1 + \frac{J_2 - J_3}{J_1} w_2 w_3 \tag{1}$$

$$\dot{w}_2 = \frac{J_3 - J_1}{J_2} w_1 w_3 \tag{2}$$

$$\dot{w}_3 = \frac{1}{J_3}\tau_3 + \frac{J_1 - J_2}{J_3} w_1 w_2 \tag{3}$$

To find a valid set of equilibrium angular velocities, set 1, 2, and 3 to 0, which yields these relations

$$0 = \frac{1}{J_1}\tau_{1e} + \frac{J_2 - J_3}{J_1} w_{2e} w_{3e} \tag{4}$$

$$0 = \frac{J_3 - J_1}{J_2} w_{1e} w_{3e} \tag{5}$$

$$0 = \frac{1}{J_3}\tau_{3e} + \frac{J_1 - J_2}{J_3} w_{1e} w_{2e} \tag{6}$$

From these three equations, $w_1, w_2, w_3, \tau_{1e}, \tau_{3e}$ can all go to 0 at equilibrium. However, 2 shows that if w_1 and w_3 both go to 0, $\dot{w}_2$ goes to 0 and w_2 can no longer change. To prevent this, the system is not linearized about the final equilibrium values, but is instead linearized about $w_{1e} = 3$, $w_{2e} = w_{3e} = \tau_{1e} = \tau_{3e} = 0$. The state x and input u are defined as

$$x = \begin{bmatrix} w_1 - w_{1e} \\ w_2 - w_{2e} \\ w_3 - w_{3e} \end{bmatrix} \qquad u = \begin{bmatrix} \tau_1 - \tau_{1e} \\ \tau_3 - \tau_{3e} \end{bmatrix} \tag{7}$$

Reference tracking will handle the movement from this intermediate equilibrium to the final equilibrium. To find the A and B matrices, take the derivative of 1,2, and 3 with respect to the equilibrium point. This process gives the relations

$$A = \begin{bmatrix} 0 & \frac{J_2-J_3}{J_1}w_3 & \frac{J_2-J_3}{J_1}w_2 \\ \frac{J_3-J_1}{J_2}w_3 & 0 & \frac{J_3-J_1}{J_2}w_1 \\ \frac{J_1-J_2}{J_3}w_2 & \frac{J_1-J_2}{J_3}w_1 & 0 \end{bmatrix}\Bigg|_{\left(\begin{bmatrix} w_{1e} \\ w_{2e} \\ w_{3e} \end{bmatrix}, \begin{bmatrix} \tau_{1e} \\ \tau_{3e} \end{bmatrix}\right)} \qquad B = \begin{bmatrix} \frac{1}{J_1} & 0 \\ 0 & 0 \\ 0 & \frac{1}{J_3} \end{bmatrix}\Bigg|_{\left(\begin{bmatrix} w_{1e} \\ w_{2e} \\ w_{3e} \end{bmatrix}, \begin{bmatrix} \tau_{1e} \\ \tau_{3e} \end{bmatrix}\right)} \tag{8}$$

Evaluating A and B at the chosen equilibrium point yields

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{3(J_3-J_1)}{J_2} \\ 0 & \frac{3(J_1-J_2)}{J_3} & 0 \end{bmatrix} \qquad B = \begin{bmatrix} \frac{1}{J_1} & 0 \\ 0 & 0 \\ 0 & \frac{1}{J_3} \end{bmatrix} \tag{9}$$

To find C and D, define y as

$$y = \begin{bmatrix} (w_1 - w_{1e}) + (w_2 - w_{2e}) + (w_3 - w_{3e}) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} (w_1 - w_{1e}) \\ (w_2 - w_{2e}) \\ (w_3 - w_{3e}) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} x \tag{10}$$

Therefore,

$$C = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 0 & 0 \end{bmatrix} \tag{11}$$

The resulting open-loop linear system can therefore be expressed as

$$\dot{x} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 3\frac{J_3-J_1}{J_2} \\ 0 & 3\frac{J_1-J_2}{J_3} & 0 \end{bmatrix} x + \begin{bmatrix} \frac{1}{J_1} & 0 \\ 0 & 0 \\ 0 & \frac{1}{J_3} \end{bmatrix} u \qquad y = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} x \tag{12}$$

3 Zero Input

Zero input occurs when u is set to zero. For this system, that means τ_1 and τ_3 are set to zero. This results in 12 simplifying to

$$\dot{x} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 3\frac{J_3-J_1}{J_2} \\ 0 & 3\frac{J_1-J_2}{J_3} & 0 \end{bmatrix} x \tag{13}$$

The eigenvalues of the resulting A are 1.7332i, -1.7332i, and 0. Since none of the real parts of the eigenvalues are negative, the system is unstable under zero input.

Figure 1 shows that when zero input is applied to the system, the system is not asymptotically stable. The sinusoidal behavior is a result of the eigenvalues of A being complex.

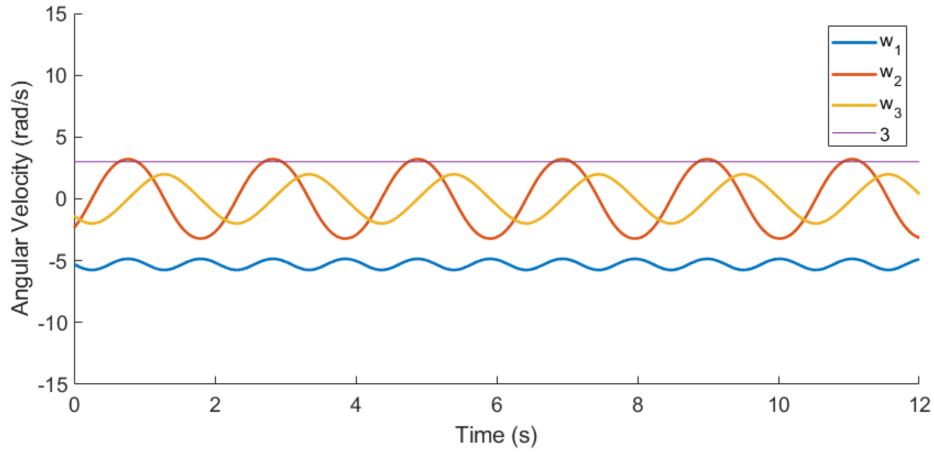

Figure 1: The angular velocities over time of a system with zero input applied.

4 State Feedback

State feedback occurs when the input u is defined as

$$u = -Kx \tag{14}$$

so the input is proportional to the state. The system therefore becomes

$$\dot{x} = Ax + Bu = Ax - BKx = (A - BK)x \tag{15}$$

Through trial-and-error, various values for K were tested. The quickest convergence to equilibrium occurs when

$$K = \begin{bmatrix} 1.4 & -0.1 & 0 \\ 0 & 6.8 & 6.3 \end{bmatrix} \tag{16}$$

The eigenvalues of the resulting $A - BK$ matrix are -12.9231, -3.9434, and -16.1096. Since all eigenvalues have a negative real part, the system is stable.

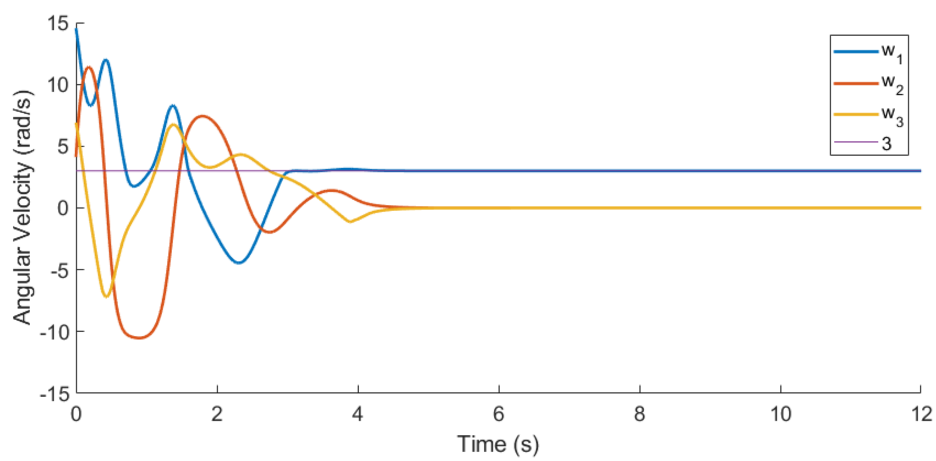

Figure 2: The angular velocities over time of a system with state feedback applied to it.

Figure 2 shows that with only state feedback, the system is able to attain asymptotic stability at the intermediate equilibrium point of $\vec{w} = 3\ \vec{e_1}$ rad s^{-1}.

5 Reference Tracking

With reference tracking, the system can be brought to an equilibrium of $\vec{w} = \vec{0}$ rad s^{-1}. In order to implement reference tracking, the input is modified to become

$$u = -Kx + k_{ref}r \tag{17}$$

In order to keep the matrix sizes consistent, k_{ref} and r are defined as

$$k_{ref} = \begin{bmatrix} k_{ref1} & 0 \\ 0 & k_{ref3} \end{bmatrix} \quad r = \begin{bmatrix} r_1 \\ r_3 \end{bmatrix} \tag{18}$$

where k_{ref1} and k_{ref3} are defined as

$$k_{ref1} = \frac{-1}{C(A-BK)^{-1}B_1} \quad k_{ref3} = \frac{-1}{C(A-BK)^{-1}B_2} \tag{19}$$

with B_1 and B_2 being the first and second columns of B respectively. r_3 is always zero; r_1 is defined as

$$r_1 = \begin{cases} 0, & \text{if } |w_2| \text{ or } |w_3| \geq 0.0001 \\ -3, & \text{if } |w_2| \text{ and } |w_3| < 0.0001 \end{cases} \tag{20}$$

in order to achieve zero rotation. This allows the controller to bring w_2 and w_3 to 0 rad s^{-1} while w_1 is sent to 3 rad s^{-1}. Then, w_1 can be brought to 0 rad s^{-1} without changing w_2 or w_3. Figure 3 shows the system achieving zero rotation when this controller is implemented. Additionally, the controller only needs a few more seconds compared to state feedback to bring the system to equilibrium.

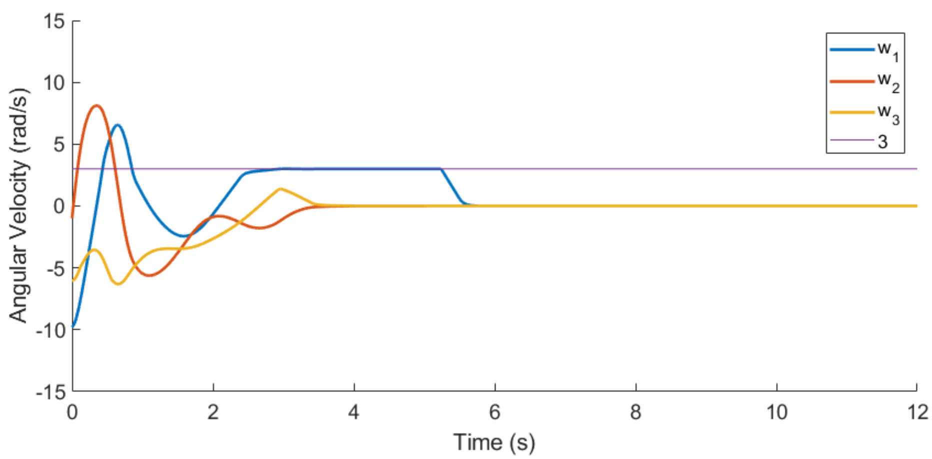

Figure 3: The angular velocities over time of a system with state feedback and reference tracking applied to it.