Probit MCMC

$$Y_i \mid Z_i; \beta \sim \mathbb{1}_{\{Z_i > 0\}} = \begin{cases} 1 & \text{if } Z_i > 0 \\ 0 & \text{if } Z_i \leq 0 \end{cases}$$

$$Z_i \mid \beta \sim N(x_i^T \beta^{(t)}, 1)$$

$$\beta \sim N(\beta_0, \Sigma_0)$$

Gibbs Sampler

(1) Initialize $(Z, \beta) = (Z_0^{(0)}, \beta_0^{(0)})$ for $t = 0$

(2) Sample $Z_i^{(t+1)}$ from $p(Z_i \mid \beta^{(t)}, y) \sim$ Truncated Normal $(\, ? \,)$

(3) Sample $\beta^{(t+1)}$ from $p(\beta \mid Z_i^{(t+1)}, y) \sim$ Normal $(\, ? \,)$

(4) Increment $t \mapsto t+1$ and return to (2).

$$p(z_i \mid \beta^{(t)}, y) = \frac{p(y_i, \beta^{(t)}, z_i)}{\int p(y_i, \beta^{(t)}, z)\, d\beta}$$

$$= \frac{p(y_i \mid z_i, \beta)\, p(z_i \mid \beta)}{\int p(y_i \mid z_i, \beta)\, p(z_i \mid \beta)\, d\beta}$$

$$\propto p(y_i \mid z_i, \beta)\, p(z_i \mid \beta)$$

$$\propto 1_{\{z_i > 0\}} \cdot \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(z_i - x_i^T \beta^{(t)})^2}{2}\right)$$

This is a truncated normal density

$$z_i \mid y_i = 0, \beta^{(t)} \sim TN(x_i^T \beta^{(t)}, 1, (-\infty, 0])$$

$$z_i \mid y_i = 1, \beta^{(t)} \sim TN(x_i^T \beta^{(t)}, 1, [0, \infty))$$

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$$p(\beta \mid \underline{z}^{(t+1)}, \underline{Y}) \propto p(z^{(t+1)} \mid \beta)\, p(\beta \mid \beta_0, V_0)$$

$$\propto \prod_{i=1}^{n} \left[\frac{1}{\sqrt{2\pi}} \exp\left\{ -\frac{1}{2}\left(z_i^{(t+1)} - x_i^T \beta^{(t)}\right)^2 \right] \frac{1}{\sqrt{2\pi}\, |\Sigma_0|^{-1}} \exp\left\{ \frac{-1}{2}(\beta - \beta_0)^T V_0^{-1} (\beta - \beta_0) \right\}$$

$$\propto \exp\left[-\frac{1}{2} \left\{ \sum_{i=1}^{n} \left(z_i^{(t+1)} - x_i \beta^{(t)}\right)^2 + (\beta - \beta_0)^T V_0^{-1} (\beta - \beta_0) \right\} \right]$$

Now expand and drop all constants that do not involve β

$$\propto \exp\left[-\frac{1}{2} \left\{ \sum_{i=1}^{n} \left(\cancel{z_i^{(t+1)^2}} + (x_i^T\beta)^2 - 2(x_i^T\beta) z_i \right) + \beta^T V_0^{-1} \beta - 2\beta^T V_0^{-1} \beta_0 + \cancel{\beta_0^T V_0^{-1} \beta_0} \right\} \right]$$

$$\propto \exp\left[-\frac{1}{2} \left\{ \sum_{i=1}^{n} (x_i^T\beta)^2 - 2 \sum_{i=1}^{n} (x_i^T\beta) z_i + \beta^T V_0^{-1} \beta - 2\beta^T V_0^{-1} \beta_0 \right\} \right]$$

We need to re-express this as a quadratic form in β:

$$-\frac{1}{2} (\beta - \mu_\beta)^T V_\beta^{-1} (\beta - \mu_\beta) + \text{Constant}$$

Notice $\sum_{i=1}^{n} (x_i^T\beta)^2 = \sum_{i=1}^{n} (\beta^T x_i)(x_i^T \beta) = \sum_{i=1}^{n} \beta^T (x_i x_i^T) \beta = \beta^T X^T X \beta$

Then $\sum_{i=1}^{n} (x_i^T\beta)^2 + \beta^T V_0^{-1} \beta = \beta^T (X^T X + V_0^{-1}) \beta$

$$\Rightarrow V_\beta^{-1} = (X^T X + V_0^{-1}) \Rightarrow V_\beta = (X^T X + V_0^{-1})^{-1}$$

The expanded quadratic form is proportional to the following (dropping $-1/2$) ④

$$\beta^T V_\beta^{-1} \beta \quad - 2\mu_\beta^T V_\beta^{-1} \beta \quad + \mu_\beta^T V_\beta^{-1} \mu_\beta + \text{Constant}$$

identified

$\parallel$

$\beta^T (X^T X + V_0^{-1}) \beta$

need to identify the

μ_β and the constant

The following steps help identify the quantities μ_β and the constant $= C$

Notice $\sum_{i=1}^n (x_i^T \beta) z_i = \sum_{i=1}^n \beta^T x_i z_i = \underset{(1\times p)}{\beta^T} \underset{(p \times 1)}{\sum_{i=1}^n x_i z_i}$

$$-2\beta^T \left(V_0^{-1} \beta_0 + \sum_{i=1}^n x_i z_i \right) = -2\beta^T V_0^{-1} \beta_0 - \sum_{i=1}^n (x_i^T \beta) z_i$$

Now we need to complete the square because we have a form

$$\beta^T (X^T X + V_0^{-1}) \beta \quad - \quad 2 \left(V_0^{-1} \beta_0 + \sum_{i=1}^n x_i z_i \right)^T \beta$$

and we need

$$(\beta - \mu_\beta) V_\beta^{-1} (\beta - \mu_\beta) \quad + \quad C$$

$$(\beta - \mu_\beta)^T V_\beta^{-1} (\beta - \mu_\beta) + C$$ 5

$$= \beta^T V_\beta^{-1} \beta - 2\mu_\beta^T V_\beta^{-1} \beta + \mu_\beta^T V_\beta^{-1} \mu_\beta + C = \beta^T V_\beta^{-1} \beta - 2\left(V_0^{-1}\beta_0 + \sum_{i=1}^{n} x_i z_i\right)^T \beta$$

Then we need $C = \mu_\beta^T V_\beta^{-1} \mu_\beta$

and $2\mu_\beta^T V_\beta^{-1} = 2\left(V_0^{-1}\beta_0 + \sum_{i=1}^{n} x_i z_i\right)^T$

$$\Leftrightarrow \quad \mu_\beta^T V_\beta^{-1} = \left(V_0^{-1}\beta_0 + \sum_{i=1}^{n} x_i z_i\right)^T$$

$$\Leftrightarrow \quad V_\beta^{-1} \mu_\beta = \left(V_0^{-1}\beta_0 + \sum_{i=1}^{n} x_i z_i\right)$$

$$\Leftrightarrow \quad \mu_\beta = \underset{(p\times p)}{V_\beta} \underset{(p\times 1)}{\left(V_0^{-1}\beta_0 + \sum_{i=1}^{n} x_i z_i\right)}$$

$$\Leftrightarrow \quad C = \left(V_0^{-1}\beta_0 + \sum_{i=1}^{n} x_i z_i\right)^T V_\beta V_\beta^{-1} V_\beta \left(V_0^{-1}\beta_0 + \sum_{i=1}^{n} x_i z_i\right)$$

$$C = \left(V_0^{-1}\beta_0 + \sum_{i=1}^{n} x_i z_i\right)^T V_\beta \left(V_0^{-1}\beta_0 + \sum_{i=1}^{n} (x_i z_i)\right)$$

$$\Rightarrow \beta \mid z, y \sim \mathcal{N}\left(\left(X^T X + V_0^{-1}\right) \left(V_0^{-1}\beta_0 + \sum_{i=1}^{n} x_i z_i\right), \left(X^T X + V_0^{-1}\right)^{-1} \right)$$