ON FLOATING BODIES.

BOOK I.

Postulate 1.

"Let it be supposed that a fluid is of such a character that, its parts lying evenly and being continuous, that part which is thrust the less is driven along by that which is thrust the more; and that each of its parts is thrust by the fluid which is above it in a perpendicular direction if the fluid be sunk in anything and compressed by anything else."

Proposition 1.

If a surface be cut by a plane always passing through a certain point, and if the section be always a circumference [of a circle] whose centre is the aforesaid point, the surface is that of a sphere.

For, if not, there will be some two lines drawn from the point to the surface which are not equal.

Suppose O to be the fixed point, and A, B to be two points on the surface such that OA, OB are unequal. Let the surface be cut by a plane passing through OA, OB. Then the section is, by hypothesis, a circle whose centre is O.

Thus $OA = OB$; which is contrary to the assumption. Therefore the surface cannot but be a sphere.

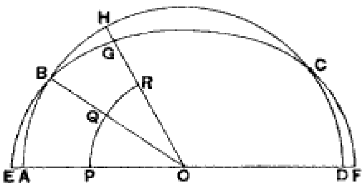
Proposition 2.

The surface of any fluid at rest is the surface of a sphere whose centre is the same as that of the earth.

Suppose the surface of the fluid cut by a plane through O, the centre of the earth, in the curve $ABCD$.

$ABCD$ shall be the circumference of a circle.

For, if not, some of the lines drawn from O to the curve will be unequal. Take one of them, OB, such that OB is greater than some of the lines from O to the curve and less than others. Draw a circle with OB as radius. Let it be EBF, which will therefore fall partly within and partly without the surface of the fluid.



Draw OGH making with OB an angle equal to the angle EOB, and meeting the surface in H and the circle in G. Draw also in the plane an arc of a circle PQR with centre O and within the fluid.

Then the parts of the fluid along PQR are uniform and continuous, and the part PQ is compressed by the part between it and AB, while the part QR is compressed by the part between QR and BH. Therefore the parts along PQ, QR will be unequally compressed, and the part which is compressed the less will be set in motion by that which is compressed the more.

Therefore there will not be rest; which is contrary to the hypothesis.

Hence the section of the surface will be the circumference of a circle whose centre is O; and so will all other sections by planes through O.

Therefore the surface is that of a sphere with centre O.

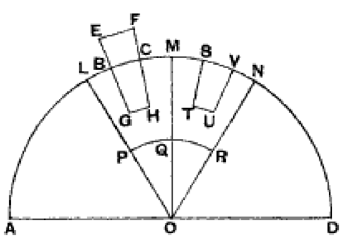
Proposition 3.

Of solids those which, size for size, are of equal weight with a fluid will, if let down into the fluid, be immersed so that they do not project above the surface but do not sink lower.

If possible, let a certain solid $EFHG$ of equal weight, volume for volume, with the fluid remain immersed in it so that part of it, $EBCF$, projects above the surface.

Draw through O, the centre of the earth, and through the solid a plane cutting the surface of the fluid in the circle $ABCD$.

Conceive a pyramid with vertex O and base a parallelogram at the surface of the fluid, such that it includes the immersed portion of the solid. Let this pyramid be cut by the plane of $ABCD$ in OL, OM. Also let a sphere within the fluid and below GH be described with centre O, and let the plane of $ABCD$ cut this sphere in PQR.



Conceive also another pyramid in the fluid with vertex O, continuous with the former pyramid and equal and similar to it. Let the pyramid so described be cut in OM, ON by the plane of $ABCD$.

Lastly, let $STUV$ be a part of the fluid within the second pyramid equal and similar to the part $BGHC$ of the solid, and let SV be at the surface of the fluid.

Then the pressures on PQ, QR are unequal, that on PQ being the greater. Hence the part at QR will be set in motion by that at PQ, and the fluid will not be at rest; which is contrary to the hypothesis.

Therefore the solid will not stand out above the surface.

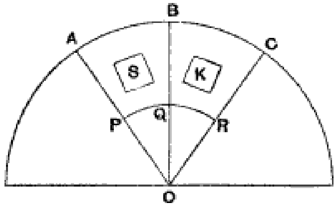
Nor will it sink further, because all the parts of the fluid will be under the same pressure.

Proposition 4.

A solid lighter than a fluid will, if immersed in it, not be completely submerged, but part of it will project above the surface.

In this case, after the manner of the previous proposition, we assume the solid, if possible, to be completely submerged and the fluid to be at rest in that position, and we conceive (1) a pyramid with its vertex at O, the centre of the earth, including the solid, (2) another pyramid continuous with the former and equal and similar to it, with the same vertex O, (3) a portion of the fluid within this latter pyramid equal to the immersed solid in the other pyramid, (4) a sphere with centre O whose surface is below the immersed solid and the part of the fluid in the second pyramid corresponding thereto. We suppose a plane to be drawn through the centre O cutting the surface of the fluid in the circle ABC, the solid in S, the first pyramid in OA, OB, the second pyramid in OB, OC, the portion of the fluid in the second pyramid in K, and the inner sphere in PQR.

Then the pressures on the parts of the fluid at PQ, QR are unequal, since S is lighter than K. Hence there will not be rest; which is contrary to the hypothesis.

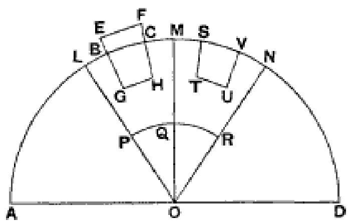


Therefore the solid S cannot, in a condition of rest, be completely submerged.

Proposition 5.

Any solid lighter than a fluid will, if placed in the fluid, be so far immersed that the weight of the solid will be equal to the weight of the fluid displaced.

For let the solid be $EGHF$, and let $BGHC$ be the portion of it immersed when the fluid is at rest. As in Prop. 3, conceive a pyramid with vertex O including the solid, and another pyramid with the same vertex continuous with the former and equal and similar to it. Suppose a portion of the fluid $STUV$ at the base of the second pyramid to be equal and similar to the immersed portion of the solid; and let the construction be the same as in Prop. 3.



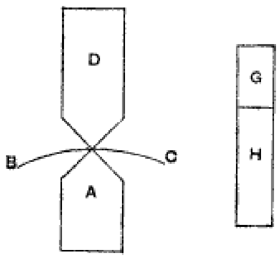
Then, since the pressure on the parts of the fluid at PQ, QR must be equal in order that the fluid may be at rest, it follows that the weight of the portion $STUV$ of the fluid must be equal to the weight of the solid $EGHF$. And the former is equal to the weight of the fluid displaced by the immersed portion of the solid $BGHC$.

Proposition 6.

If a solid lighter than a fluid be forcibly immersed in it, the solid will be driven upwards by a force equal to the difference between its weight and the weight of the fluid displaced.

For let A be completely immersed in the fluid, and let G represent the weight of A, and $(G + H)$ the weight of an equal volume of the fluid. Take a solid D, whose weight is H

and add it to A. Then the weight of $(A+D)$ is less than that of an equal volume of the fluid; and, if $(A+D)$ is immersed in the fluid, it will project so that its weight will be equal to the weight of the fluid displaced. But its weight is $(G+H)$.



Therefore the weight of the fluid displaced is $(G+H)$, and hence the volume of the fluid displaced is the volume of the solid A. There will accordingly be rest with A immersed and D projecting.

Thus the weight of D balances the upward force exerted by the fluid on A, and therefore the latter force is equal to H, which is the difference between the weight of A and the weight of the fluid which A displaces.

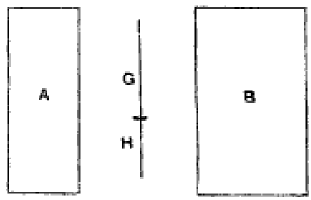
Proposition 7.

A solid heavier than a fluid will, if placed in it, descend to the bottom of the fluid, and the solid will, when weighed in the fluid, be lighter than its true weight by the weight of the fluid displaced.

(1) The first part of the proposition is obvious, since the part of the fluid under the solid will be under greater pressure, and therefore the other parts will give way until the solid reaches the bottom.

(2) Let A be a solid heavier than the same volume of the fluid, and let $(G+H)$ represent its weight, while G represents the weight of the same volume of the fluid.

Take a solid B lighter than the same volume of the fluid, and such that the weight of B is G, while the weight of the same volume of the fluid is $(G+H)$.



Let A and B be now combined into one solid and immersed. Then, since $(A+B)$ will be of the same weight as the same volume of fluid, both weights being equal to $(G+H)+G$, it follows that $(A+B)$ will remain stationary in the fluid.

Therefore the force which causes A by itself to sink must be equal to the upward force exerted by the fluid on B by itself. This latter is equal to the difference between $(G+H)$ and G [Prop. 6]. Hence A is depressed by a force equal to H, i.e. its weight in the fluid is H, or the difference between $(G+H)$ and G.

[This proposition may, I think, safely be regarded as decisive of the question how Archimedes determined the proportions of gold and silver contained in the famous crown (cf. Introduction, Chapter I.). The proposition suggests in fact the following method.

Let W represent the weight of the crown, w_1 and w_2 the weights of the gold and silver in it respectively, so that $W=w_1+w_2$.

(1) Take a weight W of pure gold and weigh it in a fluid. The apparent loss of weight is then equal to the weight of the fluid displaced. If F_1 denote this weight, F_1 is thus known as the result of the operation of weighing.

It follows that the weight of fluid displaced by a weight w_1 of gold is $\frac{w_1}{W} \cdot F_1$.

(2) Take a weight W of pure silver and perform the same operation. If F_2 be the loss of weight when the silver is weighed in the fluid, we find in like manner that the weight of fluid displaced by w_2 is $\frac{w_2}{W} \cdot F_2$.

(3) Lastly, weigh the crown itself in the fluid, and let F be the loss of weight. Therefore the weight of fluid displaced by the crown is F.

It follows that $$\frac{w_1}{W} \cdot F_1 + \frac{w_2}{W} \cdot F_2 = F,$$

or $$w_1 F_1 + w_2 F_2 = (w_1 + w_2) F,$$

whence $$\frac{w_1}{w_2} = \frac{F_2 - F}{F - F_1}.$$

This procedure corresponds pretty closely to that described in the poem *de ponderibus et mensuris* (written probably about 500 A.D.)* purporting to explain Archimedes' method. According to the author of this poem, we first take two equal weights of pure gold and pure silver respectively and weigh them against each other when both immersed in water; this gives the relation between their weights in water and therefore between their loss of weight in water. Next we take the mixture of gold and silver and an equal weight of pure silver and weigh them against each other in water in the same manner.

The other version of the method used by Archimedes is that given by Vitruvius†, according to which he measured successively the *volumes* of fluid displaced by three equal weights, (1) the crown, (2) the same weight of gold, (3) the same weight of silver, respectively. Thus, if as before the weight of the crown is W, and it contains weights w_1 and w_2 of gold and silver respectively,

(1) the crown displaces a certain quantity of fluid, V say.

(2) the weight W of gold displaces a certain volume of

* Torelli's *Archimedes*, p. 364; Hultsch, *Metrol. Script.* II. 95 sq., and Prolegomena § 118.

† *De architect.* IX. 3.

fluid, V_1 say; therefore a weight w_1 of gold displaces a volume $\frac{w_1}{W} . V_1$ of fluid.

(3) the weight W of silver displaces a certain volume of fluid, say V_2; therefore a weight w_2 of silver displaces a volume $\frac{w_2}{W} . V_2$ of fluid.

It follows that $$V = \frac{w_1}{W} . V_1 + \frac{w_2}{W} . V_2,$$

whence, since $$W = w_1 + w_2,$$
$$\frac{w_1}{w_2} = \frac{V_2 - V}{V - V_1};$$

and this ratio is obviously equal to that before obtained, viz. $\frac{F_2 - F}{F - F_1}$.]

Postulate 2.

"Let it be granted that bodies which are forced upwards in a fluid are forced upwards along the perpendicular [to the surface] which passes through their centre of gravity."

Proposition 8.

If a solid in the form of a segment of a sphere, and of a substance lighter than a fluid, be immersed in it so that its base does not touch the surface, the solid will rest in such a position that its axis is perpendicular to the surface; and, if the solid be forced into such a position that its base touches the fluid on one side and be then set free, it will not remain in that position but will return to the symmetrical position.

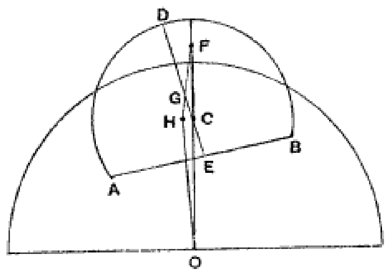
[The proof of this proposition is wanting in the Latin version of Tartaglia. Commandinus supplied a proof of his own in his edition.]

Proposition 9.

If a solid in the form of a segment of a sphere, and of a substance lighter than a fluid, be immersed in it so that its base is completely below the surface, the solid will rest in such a position that its axis is perpendicular to the surface.

[The proof of this proposition has only survived in a mutilated form. It deals moreover with only one case out of three which are distinguished at the beginning, viz. that in which the segment is greater than a hemisphere, while figures only are given for the cases where the segment is equal to, or less than, a hemisphere.]

Suppose, first, that the segment is greater than a hemisphere. Let it be cut by a plane through its axis and the centre of the earth; and, if possible, let it be at rest in the position shown in the figure, where AB is the intersection of the plane with the base of the segment, DE its axis, C the centre of the sphere of which the segment is a part, O the centre of the earth.



The centre of gravity of the portion of the segment outside the fluid, as F, lies on OC produced, its axis passing through C.

Let G be the centre of gravity of the segment. Join FG, and produce it to H so that

$FG : GH =$ (volume of immersed portion) : (rest of solid).

Join OH.

Then the weight of the portion of the solid outside the fluid acts along FO, and the pressure of the fluid on the immersed portion along OH, while the weight of the immersed portion acts along HO and is by hypothesis less than the pressure of the fluid acting along OH.

Hence there will not be equilibrium, but the part of the segment towards A will ascend and the part towards B descend, until DE assumes a position perpendicular to the surface of the fluid.

ON FLOATING BODIES.

BOOK II.

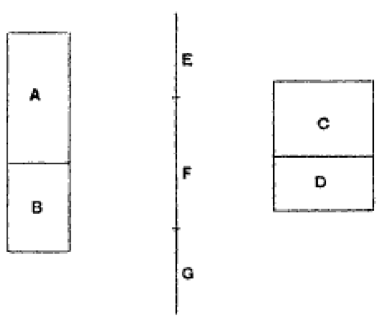
Proposition 1.

If a solid lighter than a fluid be at rest in it, the weight of the solid will be to that of the same volume of the fluid as the immersed portion of the solid is to the whole.

Let $(A+B)$ be the solid, B the portion immersed in the fluid.

Let $(C+D)$ be an equal volume of the fluid, C being equal in volume to A and B to D.

Further suppose the line E to represent the weight of the solid $(A+B)$, $(F+G)$ to represent the weight of $(C+D)$, and G that of D.



Then

$$\text{weight of } (A+B) : \text{weight of } (C+D) = E : (F+G) \ldots (1).$$

And the weight of $(A+B)$ is equal to the weight of a volume B of the fluid [I. 5], i.e. to the weight of D.

That is to say, $E=G$.

Hence, by (1),

$$\begin{aligned}\text{weight of } (A+B) : \text{weight of } (C+D) &= G : F+G\\ &= D : C+D\\ &= B : A+B.\end{aligned}$$

Proposition 2.

If a right segment of a paraboloid of revolution whose axis is not greater than $\frac{3}{4}p$ (where p is the principal parameter of the generating parabola), and whose specific gravity is less than that of a fluid, be placed in the fluid with its axis inclined to the vertical at any angle, but so that the base of the segment does not touch the surface of the fluid, the segment of the paraboloid will not remain in that position but will return to the position in which its axis is vertical.

Let the axis of the segment of the paraboloid be AN, and through AN draw a plane perpendicular to the surface of the fluid. Let the plane intersect the paraboloid in the parabola BAB', the base of the segment of the paraboloid in BB', and the plane of the surface of the fluid in the chord QQ' of the parabola.

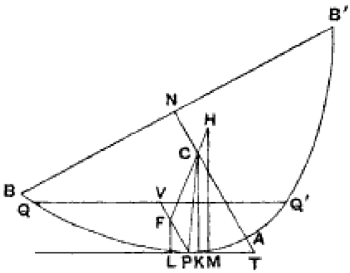
Then, since the axis AN is placed in a position not perpendicular to QQ', BB' will not be parallel to QQ'.

Draw the tangent PT to the parabola which is parallel to QQ', and let P be the point of contact*.

[From P draw PV parallel to AN meeting QQ' in V. Then PV will be a diameter of the parabola, and also the axis of the portion of the paraboloid immersed in the fluid.

* The rest of the proof is wanting in the version of Tartaglia, but is given in brackets as supplied by Commandinus.

Let C be the centre of gravity of the paraboloid BAB', and F that of the portion immersed in the fluid. Join FC and produce it to H so that H is the centre of gravity of the remaining portion of the paraboloid above the surface.



Then, since $AN = \frac{3}{2} AC$*,

and $AN \not> \frac{3}{4} p$,

it follows that $AC \not> \dfrac{p}{2}$.

Therefore, if CP be joined, the angle CPT is acute†. Hence, if CK be drawn perpendicular to PT, K will fall between P and T. And, if FL, HM be drawn parallel to CK to meet PT, they will each be perpendicular to the surface of the fluid.

Now the force acting on the immersed portion of the segment of the paraboloid will act upwards along LF, while the weight of the portion outside the fluid will act downwards along HM.

Therefore there will not be equilibrium, but the segment

* As the determination of the centre of gravity of a segment of a paraboloid which is here assumed does not appear in any extant work of Archimedes, or in any known work by any other Greek mathematician, it appears probable that it was investigated by Archimedes himself in some treatise now lost.

† The truth of this statement is easily proved from the property of the subnormal. For, if the normal at P meet the axis in G, AG is greater than $\dfrac{p}{2}$ except in the case where the normal is the normal at the vertex A itself. But the latter case is excluded here because, by hypothesis, AN is not placed vertically. Hence, P being a different point from A, AG is always greater than AC; and, since the angle TPG is right, the angle TPC must be acute.

will turn so that B will rise and B' will fall, until AN takes the vertical position.]

[For purposes of comparison the trigonometrical equivalent of this and other propositions will be appended.

Suppose that the angle NTP, at which in the above figure the axis AN is inclined to the surface of the fluid, is denoted by θ.

Then the coordinates of P referred to AN and the tangent at A as axes are

$$\frac{p}{4}\cot^2\theta, \quad \frac{p}{2}\cot\theta,$$

where p is the principal parameter.

Suppose that $AN = h$, $PV = k$.

If now x' be the distance from T of the orthogonal projection of F on TP, and x the corresponding distance for the point C, we have

$$x' = \frac{p}{2}\cot^2\theta \,.\, \cos\theta + \frac{p}{2}\cot\theta \,.\, \sin\theta + \frac{2}{3}k\cos\theta,$$

$$x = \frac{p}{4}\cot^2\theta \,.\, \cos\theta + \frac{2}{3}h\cos\theta,$$

whence $$x' - x = \cos\theta\left\{\frac{p}{4}(\cot^2\theta + 2) - \frac{2}{3}(h-k)\right\}.$$

In order that the segment of the paraboloid may turn in the direction of increasing the angle PTN, x' must be greater than x, or the expression just found must be positive.

This will always be the case, whatever be the value of θ, if

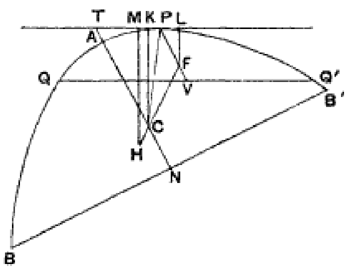
$$\frac{p}{2} \not< \frac{2h}{3},$$

or $$h \not> \tfrac{3}{4}p.]$$

Proposition 3.

If a right segment of a paraboloid of revolution whose axis is not greater than $\frac{3}{4}p$ (where p is the parameter), and whose specific gravity is less than that of a fluid, be placed in the fluid with its axis inclined at any angle to the vertical, but so that its *base is entirely submerged, the solid will not remain in that position but will return to the position in which the axis is vertical.*

Let the axis of the paraboloid be AN, and through AN draw a plane perpendicular to the surface of the fluid intersecting the paraboloid in the parabola BAB', the base of the segment in BNB', and the plane of the surface of the fluid in the chord QQ' of the parabola.



Then, since AN, as placed, is not perpendicular to the surface of the fluid, QQ' and BB' will not be parallel.

Draw PT parallel to QQ' and touching the parabola at P. Let PT meet NA produced in T. Draw the diameter PV bisecting QQ' in V. PV is then the axis of the portion of the paraboloid above the surface of the fluid.

Let C be the centre of gravity of the whole segment of the paraboloid, F that of the portion above the surface. Join FC and produce it to H so that H is the centre of gravity of the immersed portion.

Then, since $AC \ngtr \frac{p}{2}$, the angle CPT is an acute angle, as in the last proposition.

Hence, if CK be drawn perpendicular to PT, K will fall between P and T. Also, if HM, FL be drawn parallel to CK, they will be perpendicular to the surface of the fluid.

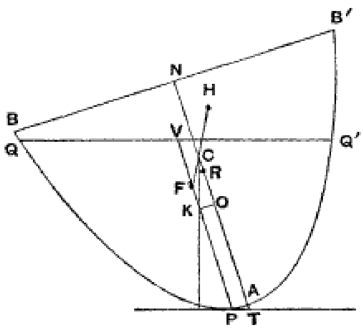
And the force acting on the submerged portion will act upwards along HM, while the weight of the rest will act downwards along LF produced.

Thus the paraboloid will turn until it takes the position in which AN is vertical.

Proposition 4.

Given a right segment of a paraboloid of revolution whose axis AN is greater than $\frac{3}{4}p$ (where p is the parameter), and whose specific gravity is less than that of a fluid but bears to it a ratio not less than $(AN - \frac{3}{4}p)^2 : AN^2$, if the segment of the paraboloid be placed in the fluid with its axis at any inclination to the vertical, but so that its base does not touch the surface of the fluid, it will not remain in that position but will return to the position in which its axis is vertical.

Let the axis of the segment of the paraboloid be AN, and let a plane be drawn through AN perpendicular to the surface of the fluid and intersecting the segment in the parabola BAB', the base of the segment in BB', and the surface of the fluid in the chord QQ' of the parabola.



Then AN, as placed, will not be perpendicular to QQ'.

Draw PT parallel to QQ' and touching the parabola at P. Draw the diameter PV bisecting QQ' in V. Thus PV will be the axis of the submerged portion of the solid.

Let C be the centre of gravity of the whole solid, F that of the immersed portion. Join FC and produce it to H so that H is the centre of gravity of the remaining portion.

Now, since $\quad AN = \frac{3}{2}AC,$

and $\quad AN > \frac{3}{4}p,$

it follows that $\quad AC > \dfrac{p}{2}.$

Measure CO along CA equal to $\frac{p}{2}$, and OR along OC equal to $\frac{1}{2}AO$.

Then, since $\quad AN = \frac{3}{2}AC,$

and $\quad AR = \frac{3}{2}AO,$

we have, by subtraction,

$$NR = \frac{3}{2}OC.$$

That is, $\quad AN - AR = \frac{3}{2}OC$

$$= \tfrac{3}{4}p,$$

or $\quad AR = (AN - \frac{3}{4}p).$

Thus $\quad (AN - \frac{3}{4}p)^2 : AN^2 = AR^2 : AN^2,$

and therefore the ratio of the specific gravity of the solid to that of the fluid is, by the enunciation, not less than the ratio $AR^2 : AN^2$.

But, by Prop. 1, the former ratio is equal to the ratio of the immersed portion to the whole solid, i.e. to the ratio $PV^2 : AN^2$ [*On Conoids and Spheroids*, Prop. 24].

Hence $\quad PV^2 : AN^2 \nless AR^2 : AN^2,$

or $\quad PV \nless AR.$

It follows that

$$PF \left(= \tfrac{2}{3}PV\right) \nless \tfrac{2}{3}AR$$
$$\nless AO.$$

If, therefore, OK be drawn from O perpendicular to OA, it will meet PF between P and F.

Also, if CK be joined, the triangle KCO is equal and similar to the triangle formed by the normal, the subnormal and the ordinate at P (since $CO = \frac{1}{2}p$ or the subnormal, and KO is equal to the ordinate).

Therefore CK is parallel to the normal at P, and therefore perpendicular to the tangent at P and to the surface of the fluid.

Hence, if parallels to CK be drawn through F, H, they will be perpendicular to the surface of the fluid, and the force acting on the submerged portion of the solid will act upwards along the former, while the weight of the other portion will act downwards along the latter.

Therefore the solid will not remain in its position but will turn until AN assumes a vertical position.

[Using the same notation as before (note following Prop. 2), we have

$$x' - x = \cos\theta \left\{\frac{p}{4}(\cot^2\theta + 2) - \frac{2}{3}(h-k)\right\},$$

and the *minimum* value of the expression within the bracket, for different values of θ, is

$$\frac{p}{2} - \frac{2}{3}(h-k),$$

corresponding to the position in which AM is vertical, or $\theta = \frac{\pi}{2}$.

Therefore there will be stable equilibrium in that position only, provided that

$$k \nless (h - \tfrac{3}{4}p),$$

or, if s be the ratio of the specific gravity of the solid to that of the fluid ($= k^2/h^2$ in this case),

$$s \nless (h - \tfrac{3}{4}p)^2/h^2.]$$

Proposition 5.

Given a right segment of a paraboloid of revolution such that its axis AN is greater than $\frac{3}{4}p$ (where p is the parameter), and its specific gravity is less than that of a fluid but in a ratio to it not greater than the ratio $\{AN^2 - (AN - \frac{3}{4}p)^2\} : AN^2$, if the segment be placed in the fluid with its axis inclined at any angle to the vertical, but so that its base is completely submerged, it will not remain in that position but will return to the position in which AN is vertical.

Let a plane be drawn through AN, as placed, perpendicular to the surface of the fluid and cutting the segment of the paraboloid in the parabola BAB', the base of the segment in BB', and the plane of the surface of the fluid in the chord QQ' of the parabola.

Draw the tangent PT parallel to QQ', and the diameter PV, bisecting QQ', will accordingly be the axis of the portion of the paraboloid above the surface of the fluid.

Let F be the centre of gravity of the portion above the surface, C that of the whole solid, and produce FC to H, the centre of gravity of the immersed portion.

As in the last proposition, $AC > \frac{p}{2}$, and we measure CO along CA equal to $\frac{p}{2}$, and OR along OC equal to $\frac{1}{2}AO$.

Then $AN = \frac{3}{2}AC$, and $AR = \frac{3}{2}AO$;

and we derive, as before,

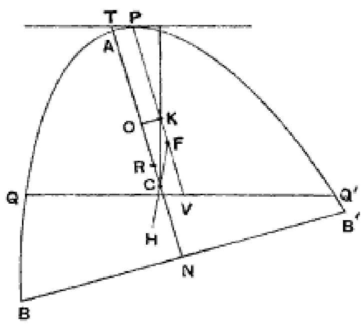
$$AR = (AN - \tfrac{3}{4}p).$$

Now, by hypothesis,

(spec. gravity of solid) : (spec. gravity of fluid)

$$\ngtr \{AN^2 - (AN - \tfrac{3}{4}p)^2\} : AN^2$$

$$\ngtr (AN^2 - AR^2) : AN^2.$$



Therefore

(portion submerged) : (whole solid)

$$\ngtr (AN^2 - AR^2) : AN^2,$$

and (whole solid) : (portion above surface)

$$\ngtr AN^2 : AR^2.$$

Thus $$AN^2 : PV^2 \ngtr AN^2 : AR^2,$$

whence $$PV \nless AR,$$

and $$PF \nless \tfrac{2}{3}AR$$

$$\nless AO.$$

Therefore, if a perpendicular to AC be drawn from O, it will meet PF in some point K between P and F.

And, since $CO = \frac{1}{2}p$, CK will be perpendicular to PT, as in the last proposition.

Now the force acting on the submerged portion of the solid will act upwards through H, and the weight of the other portion downwards through F, in directions parallel in both cases to CK; whence the proposition follows.

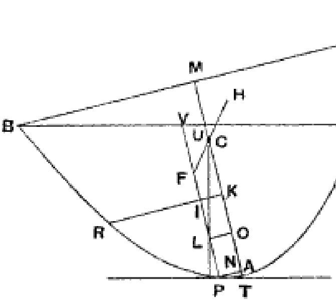
Proposition 6.

If a right segment of a paraboloid lighter than a fluid be such that its axis AM is greater than $\frac{3}{4}p$, but $AM : \frac{1}{2}p < 15 : 4$, and if the segment be placed in the fluid with its axis so inclined to the vertical that its base touches the fluid, it will never remain in such a position that the base touches the surface in one point only.

Suppose the segment of the paraboloid to be placed in the position described, and let the plane through the axis AM perpendicular to the surface of the fluid intersect the segment of the paraboloid in the parabolic segment BAB' and the plane of the surface of the fluid in BQ.

Take C on AM such that $AC = 2CM$ (or so that C is the centre of gravity of the segment of the paraboloid), and measure CK along CA such that

$$AM : CK = 15 : 4.$$



Thus $AM : CK > AM : \frac{1}{2}p$, by hypothesis; therefore $CK < \frac{1}{2}p$.

Measure CO along CA equal to $\frac{1}{2}p$. Also draw KR perpendicular to AC meeting the parabola in R.

Draw the tangent PT parallel to BQ, and through P draw the diameter PV bisecting BQ in V and meeting KR in I.

Then $$PV : PI \underset{\text{or} >}{=} KM : AK,$$
"*for this is proved.*"*

And $$CK = \tfrac{4}{15}AM = \tfrac{2}{5}AC;$$
whence $$AK = AC - CK = \tfrac{3}{5}AC = \tfrac{2}{5}AM.$$

Thus $$KM = \tfrac{3}{5}AM.$$

Therefore $$KM = \tfrac{3}{2}AK.$$

It follows that
$$PV \underset{\text{or} >}{=} \tfrac{3}{2}PI,$$
so that $$PI \underset{\text{or} <}{=} 2IV.$$

Let F be the centre of gravity of the immersed portion of the paraboloid, so that $PF = 2FV$. Produce FC to H, the centre of gravity of the portion above the surface.

Draw OL perpendicular to PV.

* We have no hint as to the work in which the proof of this proposition was contained. The following proof is shorter than Robertson's (in the Appendix to Torelli's edition).

Let BQ meet AM in U, and let PN be the ordinate from P to AM.

We have to prove that $PV \cdot AK \underset{\text{or} >}{=} PI \cdot KM$, or in other words that
$(PV \cdot AK - PI \cdot KM)$ is *positive or zero*.

Now $$PV \cdot AK - PI \cdot KM = AK \cdot PV - (AK - AN)(AM - AK)$$
$$= AK^2 - AK(AM + AN - PV) + AM \cdot AN$$
$$= AK^2 - AK \cdot UM + AM \cdot AN,$$
(since $AN = AT$).

Now $$UM : BM = NT : PN.$$

Therefore $$UM^2 : p \cdot AM = 4AN^2 : p \cdot AN,$$
whence $$UM^2 = 4AM \cdot AN,$$
or $$AM \cdot AN = \frac{UM^2}{4}.$$

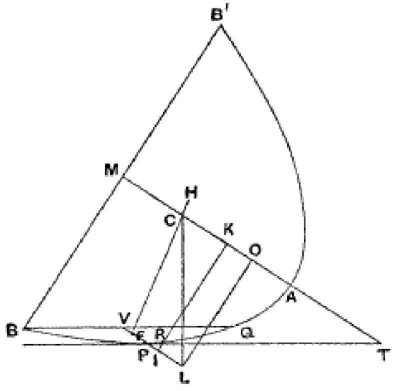
Therefore $$PV \cdot AK - PI \cdot KM = AK^2 - AK \cdot UM + \frac{UM^2}{4}$$
$$= \left(AK - \frac{UM}{2}\right)^2,$$
and accordingly $(PV \cdot AK - PI \cdot KM)$ cannot be negative.

Then, since $CO = \frac{1}{2}p$, CL must be perpendicular to PT and therefore to the surface of the fluid.

And the forces acting on the immersed portion of the paraboloid and the portion above the surface act respectively upwards and downwards along lines through F and H parallel to CL.

Hence the paraboloid cannot remain in the position in which B just touches the surface, but must turn in the direction of increasing the angle PTM.

The proof is the same in the case where the point I is not on VP but on VP produced, as in the second figure*.



[With the notation used on p. 266, if the base BB' touch the surface of the fluid at B, we have

$$BM = BV \sin\theta + PN,$$

and, by the property of the parabola,

$$BV^2 = (p + 4AN)\,PV$$
$$= pk\,(1 + \cot^2\theta).$$

Therefore $$\sqrt{ph} = \sqrt{pk} + \frac{p}{2}\cot\theta.$$

To obtain the result of the proposition, we have to eliminate k between this equation and

$$x' - x = \cos\theta\left\{\frac{p}{4}(\cot^2\theta + 2) - \frac{2}{3}(h - k)\right\}.$$

* It is curious that the figures given by Torelli, Nizze and Heiberg are all incorrect, as they all make the point which I have called I lie on BQ instead of VP produced.

We have, from the first equation,

$$k = h - \sqrt{ph}\cot\theta + \frac{p}{4}\cot^2\theta,$$

or $$h - k = \sqrt{ph}\cot\theta - \frac{p}{4}\cot^2\theta.$$

Therefore

$$x' - x = \cos\theta\left\{\frac{p}{4}(\cot^2\theta + 2) - \frac{2}{3}\left(\sqrt{ph}\cot\theta - \frac{p}{4}\cot^2\theta\right)\right\}$$

$$= \cos\theta\left\{\frac{p}{4}\left(\tfrac{5}{3}\cot^2\theta + 2\right) - \tfrac{2}{3}\sqrt{ph}\cot\theta\right\}.$$

If then the solid can never rest in the position described, but must turn in the direction of increasing the angle PTM, the expression within the bracket must be positive whatever be the value of θ.

Therefore $$\left(\tfrac{2}{3}\right)^2 ph < \tfrac{5}{6}p^2,$$

or $$h < \tfrac{15}{8}p.]$$

Proposition 7.

Given a right segment of a paraboloid of revolution lighter than a fluid and such that its axis AM is greater than $\frac{3}{4}p$, but $AM : \frac{1}{2}p < 15 : 4$, if the segment be placed in the fluid so that its base is entirely submerged, it will never rest in such a position that the base touches the surface of the fluid at one point only.

Suppose the solid so placed that one point of the base only (B) touches the surface of the fluid. Let the plane through B and the axis AM cut the solid in the parabolic segment BAB' and the plane of the surface of the fluid in the chord BQ of the parabola.

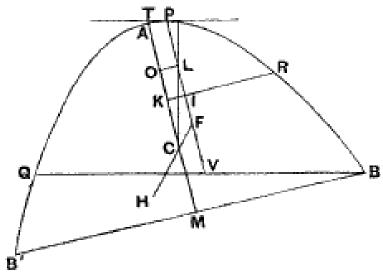
Let C be the centre of gravity of the segment, so that $AC = 2CM$; and measure CK along CA such that

$$AM : CK = 15 : 4.$$

It follows that $$CK < \tfrac{1}{2}p.$$

Measure CO along CA equal to $\frac{1}{2}p$. Draw KR perpendicular to AM meeting the parabola in R.

Let PT, touching at P, be the tangent to the parabola which is parallel to BQ, and PV the diameter bisecting BQ, i.e. the axis of the portion of the paraboloid above the surface.



Then, as in the last proposition, we prove that

$$PV \underset{\text{or} >}{=} \tfrac{3}{2} PI,$$

and $$PI \underset{\text{or} <}{=} 2IV.$$

Let F be the centre of gravity of the portion of the solid above the surface; join FC and produce it to H, the centre of gravity of the portion submerged.

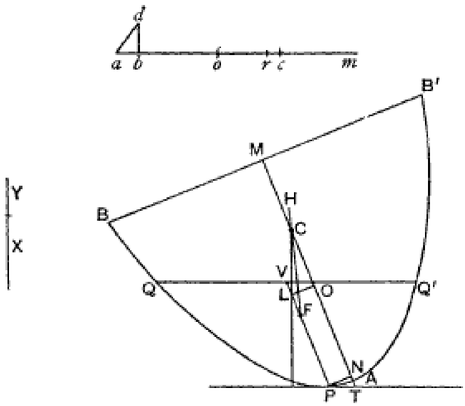
Draw OL perpendicular to PV; and, as before, since $CO = \frac{1}{2} p$, CL is perpendicular to the tangent PT. And the lines through H, F parallel to CL are perpendicular to the surface of the fluid; thus the proposition is established as before.

The proof is the same if the point I is not on VP but on VP produced.

Proposition 8.

Given a solid in the form of a right segment of a paraboloid of revolution whose axis AM is greater than $\frac{3}{4} p$, but such that $AM : \frac{1}{2} p < 15 : 4$, and whose specific gravity bears to that of a fluid a ratio less than $(AM - \frac{3}{4} p)^2 : AM^2$, then, if the solid be placed in the fluid so that its base does not touch the fluid and its axis is inclined at an angle to the vertical, the solid will not return to the position in which its axis is vertical and will not *remain in any position except that in which its axis makes with the surface of the fluid a certain angle to be described.*

Let am be taken equal to the axis AM, and let c be a point on am such that $ac = 2cm$. Measure co along ca equal to $\frac{1}{2}p$, and or along oc equal to $\frac{1}{2}ao$.



Let $X + Y$ be a straight line such that

(spec. gr. of solid) : (spec. gr. of fluid) $= (X + Y)^2 : am^2 \ldots\ldots(\alpha)$,

and suppose $X = 2Y$.

Now $$ar = \tfrac{3}{2}\,ao = \tfrac{3}{2}\left(\tfrac{2}{3}\,am - \tfrac{1}{2}p\right)$$
$$= am - \tfrac{3}{4}\,p$$
$$= AM - \tfrac{3}{4}\,p.$$

Therefore, by hypothesis,

$$(X + Y)^2 : am^2 < ar^2 : am^2,$$

whence $(X + Y) < ar$, and therefore $X < ao$.

Measure ob along oa equal to X, and draw bd perpendicular to ab and of such length that

$$bd^2 = \tfrac{1}{2}\,co\,.\,ab \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\beta).$$

Join ad.

Now let the solid be placed in the fluid with its axis AM inclined at an angle to the vertical. Through AM draw a plane perpendicular to the surface of the fluid, and let this plane cut the paraboloid in the parabola BAB' and the plane of the surface of the fluid in the chord QQ' of the parabola.

Draw the tangent PT parallel to QQ', touching at P, and let PV be the diameter bisecting QQ' in V (or the axis of the immersed portion of the solid), and PN the ordinate from P.

Measure AO along AM equal to ao, and OC along OM equal to oc, and draw OL perpendicular to PV.

I. Suppose the angle OTP greater than the angle dab.

Thus $$PN^2 : NT^2 > db^2 : ba^2.$$

But $$PN^2 : NT^2 = p : 4AN$$
$$= co : NT,$$

and $$db^2 : ba^2 = \tfrac{1}{2}co : ab, \text{ by } (\beta).$$

Therefore $$NT < 2ab,$$

or $$AN < ab,$$

whence $$NO > bo \quad (\text{since } ao = AO)$$
$$> X.$$

Now $$(X+Y)^2 : am^2 = (\text{spec. gr. of solid}) : (\text{spec. gr. of fluid})$$
$$= (\text{portion immersed}) : (\text{rest of solid})$$
$$= PV^2 : AM^2,$$

so that $$X + Y = PV.$$

But $$PL\ (= NO) > X$$
$$> \tfrac{2}{3}(X+Y), \quad \text{since } X = 2Y,$$
$$> \tfrac{2}{3}PV,$$

or $$PV < \tfrac{3}{2}PL,$$

and therefore $$PL > 2LV.$$

Take a point F on PV so that $PF = 2FV$, i.e. so that F is the centre of gravity of the immersed portion of the solid.

Also $AC = ac = \frac{2}{3}am = \frac{2}{3}AM$, and therefore C is the centre of gravity of the whole solid.

Join FC and produce it to H, the centre of gravity of the portion of the solid above the surface.

Now, since $CO = \frac{1}{2}p$, CL is perpendicular to the surface of the fluid; therefore so are the parallels to CL through F and H. But the force on the immersed portion acts upwards through F and that on the rest of the solid downwards through H.

Therefore the solid will not rest but turn in the direction of diminishing the angle MTP.

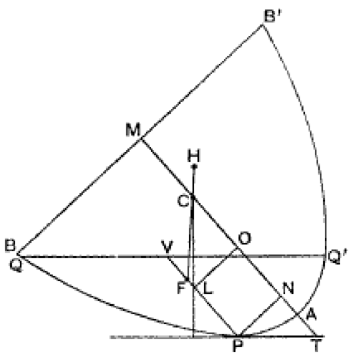
II. Suppose the angle OTP less than the angle dab. In this case, we shall have, instead of the above results, the following,

$$AN > ab,$$

$$NO < X.$$

Also $$PV > \tfrac{3}{2}PL,$$

and therefore $$PL < 2LV.$$



Make PF equal to $2FV$, so that F is the centre of gravity of the immersed portion.

And, proceeding as before, we prove in this case that the solid will turn in the direction of *increasing* the angle MTP.

III. When the angle MTP is equal to the angle dab, equalities replace inequalities in the results obtained, and L is itself the centre of gravity of the immersed portion. Thus all the forces act in one straight line, the perpendicular CL; therefore there is equilibrium, and the solid will rest in the position described.

[With the notation before used

$$x' - x = \cos\theta \left\{\frac{p}{4}(\cot^2\theta + 2) - \frac{2}{3}(h-k)\right\},$$

and a position of equilibrium is obtained by equating to zero the expression within the bracket. We have then

$$\frac{p}{4}\cot^2\theta = \frac{2}{3}(h-k) - \frac{p}{2}.$$

It is easy to verify that the angle θ satisfying this equation is the identical angle determined by Archimedes. For, in the above proposition,

$$\frac{3X}{2} = PV = k,$$

whence $$ab = \frac{2}{3}h - \frac{p}{2} - \frac{2}{3}k = \frac{2}{3}(h-k) - \frac{p}{2}.$$

Also $$bd^2 = \frac{p}{4}\,.\,ab.$$

It follows that

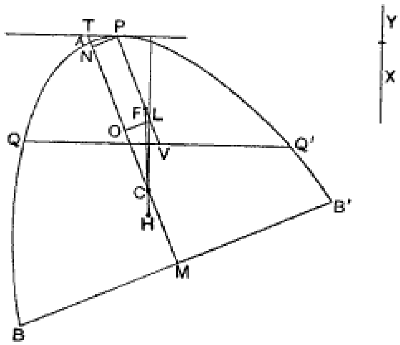
$$\cot^2 dab = ab^2/bd^2 = \frac{4}{p}\left\{\frac{2}{3}(h-k) - \frac{p}{2}\right\}.]$$

Proposition 9.

Given a solid in the form of a right segment of a paraboloid of revolution whose axis AM is greater than $\frac{3}{4}p$, but such that $AM : \frac{1}{2}p < 15 : 4$, and whose specific gravity bears to that of a fluid a ratio greater than $\{AM^2 - (AM - \frac{3}{4}p)^2\} : AM^2$, then, if the solid be placed in the fluid with its axis inclined at an angle to the vertical but so that its base is entirely below the surface, the solid will not return to the position in which its axis is vertical and will not remain in any position except that in which its axis makes with the surface of the fluid an angle equal to that described in the last proposition.

Take am equal to AM, and take c on am such that $ac = 2cm$. Measure co along ca equal to $\frac{1}{2}p$, and ar along ac such that $ar = \frac{3}{2}ao$.

Let $X+Y$ be such a line that
(spec. gr. of solid) : (spec. gr. of fluid) $= \{am^2 - (X+Y)^2\} : am^2$,
and suppose $X = 2Y$.



Now $$ar = \tfrac{3}{2}ao$$
$$= \tfrac{3}{2}(\tfrac{2}{3}am - \tfrac{1}{2}p)$$
$$= AM - \tfrac{3}{4}p.$$

Therefore, by hypothesis,
$$am^2 - ar^2 : am^2 < \{am^2 - (X+Y)^2\} : am^2,$$
whence $$X + Y < ar,$$
and therefore $$X < ao.$$

Make ob (measured along oa) equal to X, and draw bd perpendicular to ba and of such length that
$$bd^2 = \tfrac{1}{2}co \cdot ab.$$
Join ad.

Now suppose the solid placed as in the figure with its axis AM inclined to the vertical. Let the plane through AM perpendicular to the surface of the fluid cut the solid in the parabola BAB' and the surface of the fluid in QQ'.

Let PT be the tangent parallel to QQ', PV the diameter bisecting QQ' (or the axis of the portion of the paraboloid above the surface), PN the ordinate from P.

I. Suppose the angle MTP greater than the angle dab. Let AM be cut as before in C and O so that $AC = 2CM$, $OC = \frac{1}{2}p$, and accordingly AM, am are equally divided. Draw OL perpendicular to PV.

Then, we have, as in the last proposition,

$$PN^2 : NT^2 > db^2 : ba^2,$$

whence $\quad co : NT > \frac{1}{2}co : ab,$

and therefore $\quad AN < ab.$

It follows that $\quad NO > bo$

$$> X.$$

Again, since the specific gravity of the solid is to that of the fluid as the immersed portion of the solid to the whole,

$$AM^2 - (X+Y)^2 : AM^2 = AM^2 - PV^2 : AM^2,$$

or $\quad (X+Y)^2 : AM^2 = PV^2 : AM^2.$

That is, $\quad X + Y = PV.$

And $\quad PL \text{ (or } NO) > X$

$$> \tfrac{2}{3}PV,$$

so that $\quad PL > 2LV.$

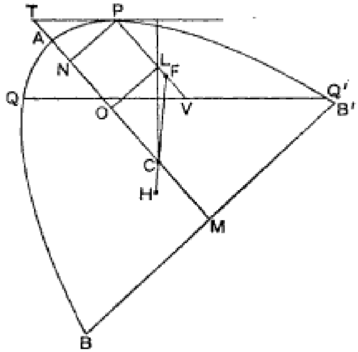
Take F on PV so that $PF = 2FV$. Then F is the centre of gravity of the portion of the solid above the surface.

Also C is the centre of gravity of the whole solid. Join FC and produce it to H, the centre of gravity of the immersed portion.

Then, since $CO = \frac{1}{2}p$, CL is perpendicular to PT and to the surface of the fluid; and the force acting on the immersed portion of the solid acts upwards along the parallel to CL through H, while the weight of the rest of the solid acts downwards along the parallel to CL through F.

Hence the solid will not rest but turn in the direction of diminishing the angle MTP.

II. Exactly as in the last proposition, we prove that, if the angle MTP be less than the angle dab, the solid will not remain in its position but will turn in the direction of increasing the angle MTP.



III. If the angle MTP is equal to the angle dab, the solid will rest in that position, because L and F will coincide, and all the forces will act along the one line CL.

Proposition 10.

Given a solid in the form of a right segment of a paraboloid of revolution in which the axis AM is of a length such that $AM : \frac{1}{2}p > 15 : 4$, and supposing the solid placed in a fluid of greater specific gravity so that its base is entirely above the surface of the fluid, to investigate the positions of rest.

(**Preliminary.**)

Suppose the segment of the paraboloid to be cut by a plane through its axis AM in the parabolic segment BAB_1 of which BB_1 is the base.

Divide AM at C so that $AC = 2CM$, and measure CK along CA so that

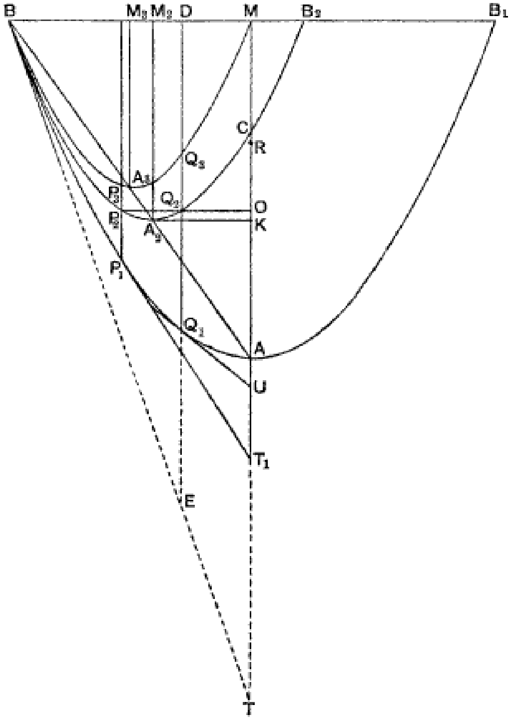
$$AM : CK = 15 : 4 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\alpha),$$

whence, by the hypothesis, $CK > \frac{1}{2}p$.

Suppose CO measured along CA equal to $\frac{1}{2}p$, and take a point R on AM such that $MR = \frac{3}{2}CO$.

Thus
$$\begin{aligned} AR &= AM - MR \\ &= \tfrac{3}{2}(AC - CO) \\ &= \tfrac{3}{2}AO. \end{aligned}$$

Join BA, draw KA_2 perpendicular to AM meeting BA in A_2 bisect BA in A_3, and draw A_2M_2, A_3M_3 parallel to AM meeting BM in M_2, M_3 respectively.



On A_2M_2, A_3M_3 as axes describe parabolic segments similar to the segment BAB_1. (It follows, by similar triangles, that BM will be the base of the segment whose axis is A_3M_3 and BB_2 the base of that whose axis is A_2M_2, where $BB_2 = 2BM_2$.)

The parabola BA_2B_2 will then pass through C.

[For
$$\begin{aligned} BM_2 : M_2M &= BM_2 : A_2K \\ &= KM : AK \\ &= CM + CK : AC - CK \\ &= (\tfrac{1}{3} + \tfrac{4}{15})\, AM : (\tfrac{2}{3} - \tfrac{4}{15})\, AM \\ &= 9 : 6 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\beta) \\ &= MA : AC. \end{aligned}$$

Thus C is seen to be on the parabola BA_2B_2 by the converse of Prop. 4 of the *Quadrature of the Parabola.*]

Also, if a perpendicular to AM be drawn from O, it will meet the parabola BA_2B_2 in two points, as Q_2, P_2. Let $Q_1Q_2Q_3D$ be drawn through Q_2 parallel to AM meeting the parabolas BAB_1, BA_3M respectively in Q_1, Q_3 and BM in D; and let $P_1P_2P_3$ be the corresponding parallel to AM through P_2. Let the tangents to the outer parabola at P_1, Q_1 meet MA produced in T_1, U respectively.

Then, since the three parabolic segments are similar and similarly situated, with their bases in the same straight line and having one common extremity, and since $Q_1Q_2Q_3D$ is a diameter common to all three segments, it follows that

$$Q_1Q_2 : Q_2Q_3 = (B_2B_1 : B_1B) . (BM : MB_2)^*.$$

Now $\qquad B_2B_1 : B_1B = MM_2 : BM \qquad$ (dividing by 2)

$\qquad = 2 : 5, \qquad$ by means of (β) above.

And $\qquad BM : MB_2 = BM : (2BM_2 - BM)$

$\qquad = 5 : (6 - 5), \qquad$ by means of (β),

$\qquad = 5 : 1.$

* This result is assumed without proof, no doubt as being an easy deduction from Prop. 5 of the *Quadrature of the Parabola*. It may be established as follows.

First, since AA_2A_3B is a straight line, and $AN = AT$ with the ordinary notation (where PT is the tangent at P and PN the ordinate), it follows, by similar triangles, that the tangent at B to the outer parabola is a tangent to each of the other two parabolas at the same point B.

Now, by the proposition quoted, if $DQ_3Q_2Q_1$ produced meet the tangent BT in E,

$$EQ_3 : Q_3D = BD : DM,$$

whence $\qquad EQ_3 : ED = BD : BM.$

Similarly $\qquad EQ_2 : ED = BD : BB_2,$

and $\qquad EQ_1 : ED = BD : BB_1.$

The first two proportions are equivalent to

$$EQ_3 : ED = BD . BB_2 : BM . BB_2,$$

and $\qquad EQ_2 : ED = BD . BM : BM . BB_2.$

By subtraction,

$$Q_2Q_3 : ED = BD . MB_2 : BM . BB_2.$$

Similarly $\qquad Q_1Q_2 : ED = BD . B_2B_1 : BB_2 . BB_1.$

It follows that

$$Q_1Q_2 : Q_2Q_3 = (B_2B_1 : B_1B) . (BM : MB_2).$$

It follows that

$$Q_1Q_2 : Q_2Q_3 = 2 : 1,$$

or

$$\left.\begin{aligned} Q_1Q_2 &= 2Q_2Q_3. \\ \text{Similarly}\qquad P_1P_2 &= 2P_2P_3. \end{aligned}\right\}$$

Also, since

$$MR = \tfrac{3}{2}CO = \tfrac{3}{4}p,$$

$$\begin{aligned} AR &= AM - MR \\ &= AM - \tfrac{3}{4}p. \end{aligned}$$

(Enunciation.)

If the segment of the paraboloid be placed in the fluid with its base entirely above the surface, then

(I.) *if*

$$(\textit{spec. gr. of solid}) : (\textit{spec. gr. of fluid}) \not< AR^2 : AM^2$$
$$[\not< (AM - \tfrac{3}{4}p)^2 : AM^2],$$

the solid will rest in the position in which its axis AM is vertical;

(II.) *if*

$$(\textit{spec. gr. of solid}) : (\textit{spec. gr. of fluid}) < AR^2 : AM^2$$
$$\textit{but} > Q_1Q_3^2 : AM^2,$$

the solid will not rest with its base touching the surface of the fluid in one point only, but in such a position that its base does not touch the surface at any point and its axis makes with the surface an angle greater than U;

(III. *a*) *if*

$$(\textit{spec. gr. of solid}) : (\textit{spec. gr. of fluid}) = Q_1Q_3^2 : AM^2,$$

the solid will rest and remain in the position in which the base touches the surface of the fluid at one point only and the axis makes with the surface an angle equal to U;

(III. *b*) *if*

$$(\textit{spec. gr. of solid}) : (\textit{spec. gr. of fluid}) = P_1P_3^2 : AM^2,$$

the solid will rest with its base touching the surface of the fluid at one point only and with its axis inclined to the surface at an angle equal to T_1;

(IV.) *if*

$$(\textit{spec. gr. of solid}) : (\textit{spec. gr. of fluid}) > P_1P_3^2 : AM^2$$
$$\textit{but} < Q_1Q_3^2 : AM^2,$$

the solid will rest and remain in a position with its base more submerged;

(V.) *if*

$$(\textit{spec. gr. of solid}) : (\textit{spec. gr. of fluid}) < P_1P_3^2 : AM^2,$$

the solid will rest in a position in which its axis is inclined to the surface of the fluid at an angle less than T_1, but so that the base does not even touch the surface at one point.

(**Proof.**)

(I.) Since $AM > \frac{3}{4}p$, and

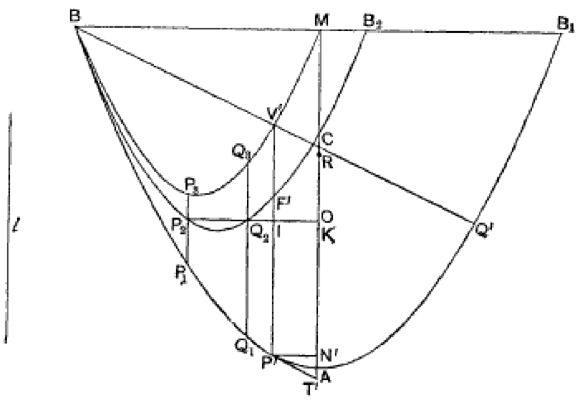
$$(\text{spec. gr. of solid}) : (\text{spec. gr. of fluid}) \nless (AM - \tfrac{3}{4}p)^2 : AM^2,$$

it follows, by Prop. 4, that the solid will be in stable equilibrium with its axis vertical.

(II.) In this case

$$(\text{spec. gr. of solid}) : (\text{spec. gr. of fluid}) < AR^2 : AM^2$$

$$\text{but} > Q_1Q_3^2 : AM^2.$$



Suppose the ratio of the specific gravities to be equal to

$$l^2 : AM^2,$$

so that $l < AR$ but $> Q_1Q_2$.

Place $P'V'$ between the two parabolas BAB_1, BP_3Q_3M equal to l and parallel to AM*; and let $P'V'$ meet the intermediate parabola in F'.

Then, by the same proof as before, we obtain

$$P'F' = 2F'V'.$$

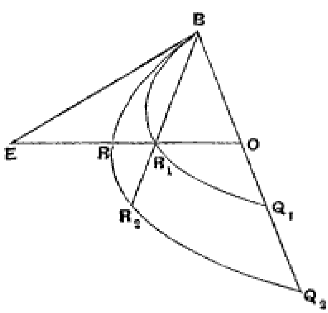
Let $P'T'$, the tangent at P' to the outer parabola, meet MA in T', and let $P'N'$ be the ordinate at P'.

Join BV' and produce it to meet the outer parabola in Q'. Let OQ_2P_2 meet $P'V'$ in I.

Now, since, in two similar and similarly situated parabolic

* Archimedes does not give the solution of this problem, but it can be supplied as follows.

Let BR_1Q_1, BRQ_2 be two similar and similarly situated parabolic segments with their bases in the same straight line, and let BE be the common tangent at B.



Suppose the problem solved, and let ERR_1O, parallel to the axes, meet the parabolas in R, R_1 and BQ_2 in O, making the intercept RR_1 equal to l.

Then, we have, as usual,

$$ER_1 : EO = BO : BQ_1$$
$$= BO \cdot BQ_2 : BQ_1 \cdot BQ_2,$$

and
$$ER : EO = BO : BQ_2$$
$$= BO \cdot BQ_1 : BQ_1 \cdot BQ_2.$$

By subtraction,
$$RR_1 : EO = BO \cdot Q_1Q_2 : BQ_1 \cdot BQ_2,$$

or
$$BO \cdot OE = l \cdot \frac{BQ_1 \cdot BQ_2}{Q_1Q_2}, \text{ which is known.}$$

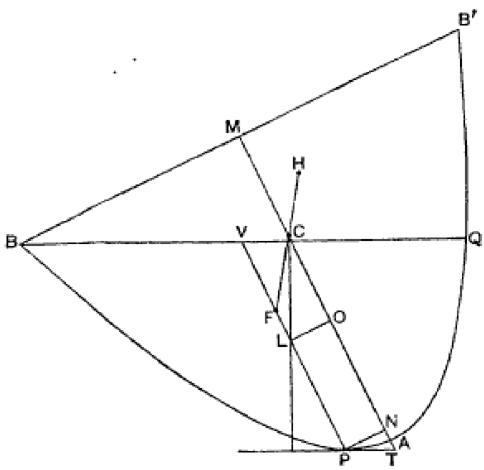
And the ratio $BO : OE$ is known. Therefore BO^2, or OE^2, can be found, and therefore O.

segments with bases BM, BB_1 in the same straight line, BV', BQ' are drawn making the same angle with the bases,

$$BV' : BQ' = BM : BB_1{}^*$$
$$= 1 : 2,$$

so that $$BV' = V'Q'.$$

Suppose the segment of the paraboloid placed in the fluid, as described, with its axis inclined at an angle to the vertical, and with its base touching the surface at one point B only. Let the solid be cut by a plane through the axis and per-



pendicular to the surface of the fluid, and let the plane intersect the solid in the parabolic segment BAB' and the plane of the surface of the fluid in BQ.

Take the points C, O on AM as before described. Draw

* To prove this, suppose that, in the figure on the opposite page, BR_1 is produced to meet the outer parabola in R_2.

We have, as before,

$$ER_1 : EO = BO : BQ_1,$$
$$ER : EO = BO : BQ_2,$$

whence $$ER_1 : ER = BQ_2 : BQ_1.$$

And, since R_1 is a point within the outer parabola,

$$ER : ER_1 = BR_1 : BR_2, \text{ in like manner.}$$

Hence $$BQ_1 : BQ_2 = BR_1 : BR_2.$$

the tangent parallel to BQ touching the parabola in P and meeting AM in T; and let PV be the diameter bisecting BQ (i.e. the axis of the immersed portion of the solid).

Then

$$l^2 : AM^2 = (\text{spec. gr. of solid}) : (\text{spec. gr. of fluid})$$
$$= (\text{portion immersed}) : (\text{whole solid})$$
$$= PV^2 : AM^2,$$

whence $$P'V' = l = PV.$$

Thus the segments in the two figures, namely $BP'Q'$, BPQ, are equal and similar.

Therefore $$\angle PTN = \angle P'T'N'.$$

Also $$AT = AT',\ AN = AN',\ PN = P'N'.$$

Now, in the first figure, $P'I < 2IV'$.

Therefore, if OL be perpendicular to PV in the second figure,

$$PL < 2LV.$$

Take F on LV so that $PF = 2FV$, i.e. so that F is the centre of gravity of the immersed portion of the solid. And C is the centre of gravity of the whole solid. Join FC and produce it to H, the centre of gravity of the portion above the surface.

Now, since $CO = \frac{1}{2}p$, CL is perpendicular to the tangent at P and to the surface of the fluid. Thus, as before, we prove that the solid will not rest with B touching the surface, but will turn in the direction of increasing the angle PTN.

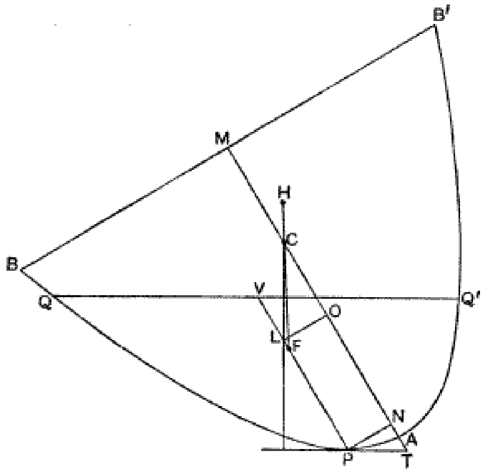
Hence, in the position of rest, the axis AM must make with the surface of the fluid an angle greater than the angle U which the tangent at Q_1 makes with AM.

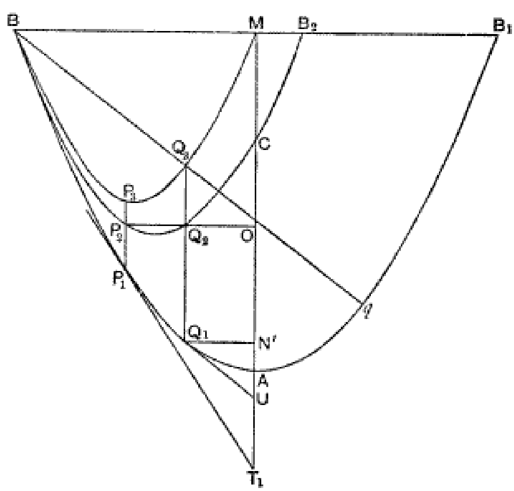
(III. a) In this case

$$(\text{spec. gr. of solid}) : (\text{spec. gr. of fluid}) = Q_1Q_3^2 : AM^2.$$

Let the segment of the paraboloid be placed in the fluid so that its base nowhere touches the surface of the fluid, and its axis is inclined at an angle to the vertical.

Let the plane through AM perpendicular to the surface of the fluid cut the paraboloid in the parabola BAB' and the





plane of the surface of the fluid in QQ'. Let PT be the tangent parallel to QQ', PV the diameter bisecting QQ', PN the ordinate at P.

Divide AM as before at C, O.

In the other figure let Q_1N' be the ordinate at Q_1. Join BQ_3 and produce it to meet the outer parabola in q. Then $BQ_3 = Q_3q$, and the tangent Q_1U is parallel to Bq. Now

$$\begin{aligned} Q_1Q_3^2 : AM^2 &= (\text{spec. gr. of solid}) : (\text{spec. gr. of fluid}) \\ &= (\text{portion immersed}) : (\text{whole solid}) \\ &= PV^2 : AM^2. \end{aligned}$$

Therefore $Q_1Q_3 = PV$; and the segments QPQ', BQ_1q of the paraboloid are equal in volume. And the base of one passes through B, while the base of the other passes through Q, a point nearer to A than B is.

It follows that the angle between QQ' and BB' is less than the angle B_1Bq.

Therefore $\angle U < \angle PTN$,

whence $AN' > AN$,

and therefore $N'O$ (or Q_1Q_2) $< PL$,

where OL is perpendicular to PV.

It follows, since $Q_1Q_2 = 2Q_2Q_3$, that

$$PL > 2LV.$$

Therefore F, the centre of gravity of the immersed portion of the solid, is between P and L, while, as before, CL is perpendicular to the surface of the fluid.

Producing FC to H, the centre of gravity of the portion of the solid above the surface, we see that the solid must turn in the direction of diminishing the angle PTN until one point B of the base just touches the surface of the fluid.

When this is the case, we shall have a segment BPQ equal and similar to the segment BQ_1q, the angle PTN will be equal to the angle U, and AN will be equal to AN'.

Hence in this case $PL = 2LV$, and F, L coincide, so that F, C, H are all in one vertical straight line.

Thus the paraboloid will remain in the position in which one point B of the base touches the surface of the fluid, and the axis makes with the surface an angle equal to U.

(III. b) In the case where

$$(\text{spec. gr. of solid}) : (\text{spec. gr. of fluid}) = P_1P_3^2 : AM^2,$$

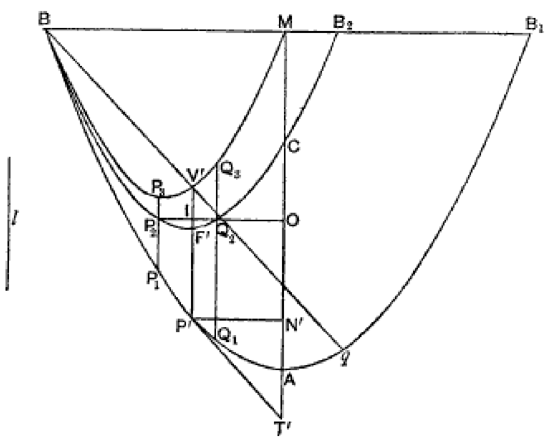
we can prove in the same way that, if the solid be placed in the fluid so that its axis is inclined to the vertical and its base does not anywhere touch the surface of the fluid, the solid will take up and rest in the position in which one point only of the base touches the surface, and the axis is inclined to it at an angle equal to T_1 (in the figure on p. 284).

(IV.) In this case

$$(\text{spec. gr. of solid}) : (\text{spec. gr. of fluid}) > P_1P_3^2 : AM^2$$
$$\text{but} < Q_1Q_3^2 : AM^2.$$

Suppose the ratio to be equal to $l^2 : AM^2$, so that l is greater than P_1P_3 but less than Q_1Q_3.

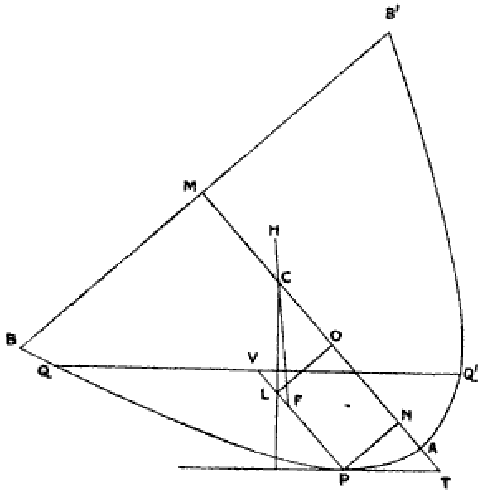
Place $P'V'$ between the parabolas BP_1Q_1, BP_3Q_3 so that $P'V'$ is equal to l and parallel to AM, and let $P'V'$ meet the intermediate parabola in F' and OQ_2P_2 in I.



Join BV' and produce it to meet the outer parabola in q.

Then, as before, $BV' = V'q$, and accordingly the tangent $P'T'$ at P' is parallel to Bq. Let $P'N'$ be the ordinate of P'.

1. Now let the segment be placed in the fluid, *first*, with its axis so inclined to the vertical that its base does not anywhere touch the surface of the fluid.



Let the plane through AM perpendicular to the surface of the fluid cut the paraboloid in the parabola BAB' and the plane of the surface of the fluid in QQ'. Let PT be the tangent parallel to QQ', PV the diameter bisecting QQ'. Divide AM at C, O as before, and draw OL perpendicular to PV.

Then, as before, we have $PV = l = P'V'$.

Thus the segments $BP'q$, QPQ' of the paraboloid are equal in volume; and it follows that the angle between QQ' and BB' is less than the angle B_1Bq.

Therefore $\qquad \angle P'T'N' < \angle PTN$,

and hence $\qquad AN' > AN$,

so that $\qquad NO > N'O$,

i.e. $\qquad PL > P'I$

$\qquad > P'F'$, *a fortiori*.

Thus $PL > 2LV$, so that F, the centre of gravity of the immersed portion of the solid, is between L and P, while CL is perpendicular to the surface of the fluid.

If then we produce FC to H, the centre of gravity of the portion of the solid above the surface, we prove that the solid will not rest but turn in the direction of diminishing the angle PTN.

2. Next let the paraboloid be so placed in the fluid that its base touches the surface of the fluid at one point B only, and let the construction proceed as before.

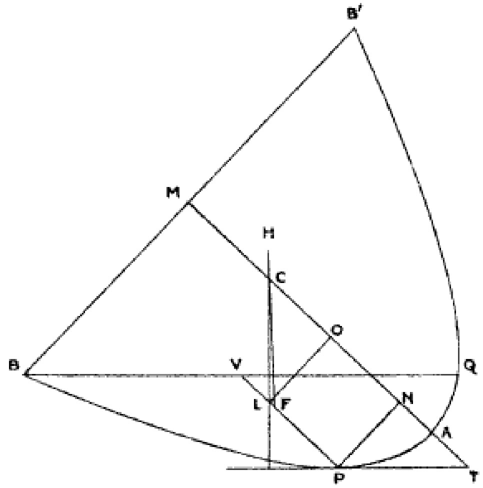
Then $PV = P'V'$, and the segments BPQ, $BP'q$ are equal and similar, so that

$$\angle PTN = \angle P'T'N'.$$

It follows that $AN = AN'$, $NO = N'O$,

and therefore $P'I = PL$,

whence $PL > 2LV$.



Thus F again lies between P and L, and, as before, the paraboloid will turn in the direction of diminishing the angle PTN, i.e. so that the base will be more submerged.

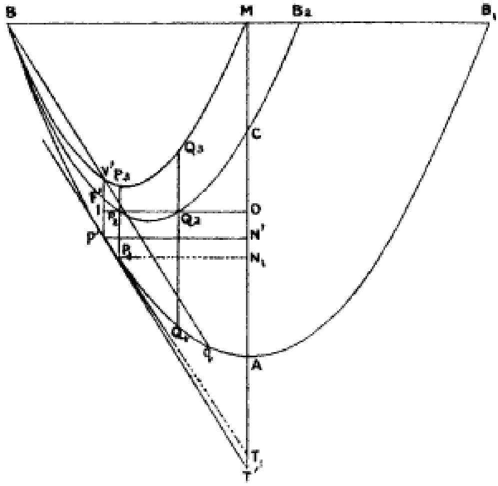
(V.) In this case

$$(\text{spec. gr. of solid}) : (\text{spec. gr. of fluid}) < P_1P_3{}^2 : AM^2.$$

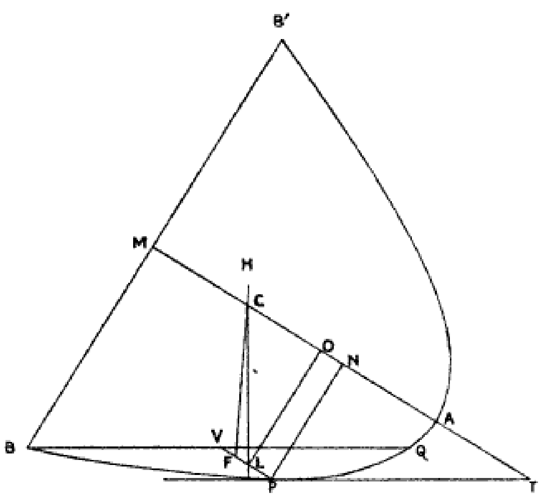
If then the ratio is equal to $l^2 : AM^2$, $l < P_1P_3$. Place $P'V'$ between the parabolas BP_1Q_1 and BP_3Q_3 equal in length to l

and parallel to AM. Let $P'V'$ meet the intermediate parabola in F' and OP_2 in I.

Join BV' and produce it to meet the outer parabola in q. Then, as before, $BV' = V'q$, and the tangent $P'T'$ is parallel to Bq.



1. Let the paraboloid be so placed in the fluid that its base touches the surface at one point only.



Let the plane through AM perpendicular to the surface of the fluid cut the paraboloid in the parabolic section BAB' and the plane of the surface of the fluid in BQ.

Making the usual construction, we find

$$PV = l = P'V',$$

and the segments BPQ, BP_1q are equal and similar.

Therefore $\qquad \angle PTN = \angle P'T'N',$

and $\qquad AN = AN',\ N'O = NO.$

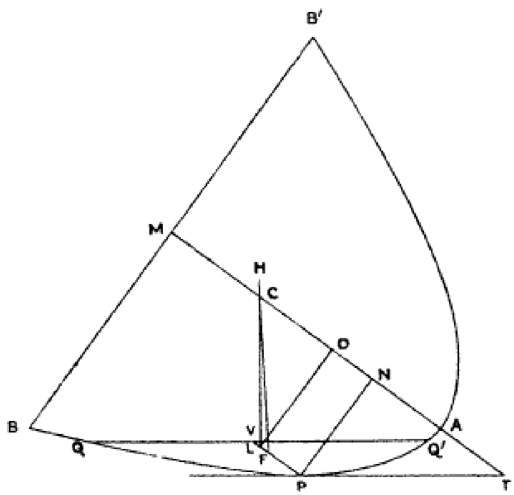
Therefore $\qquad PL = P'I,$

whence it follows that $\quad PL < 2LV.$

Thus F, the centre of gravity of the immersed portion of the solid, lies between L and V, while CL is perpendicular to the surface of the fluid.

Producing FC to H, the centre of gravity of the portion above the surface, we prove, as usual, that there will not be rest, but the solid will turn in the direction of increasing the angle PTN, so that the base will not anywhere touch the surface.

2. The solid will however rest in a position where its axis makes with the surface of the fluid an angle less than T_1.



For let it be placed so that the angle PTN is not less than T_1.

Then, with the same construction as before, $PV = l = P'V'$.

And, since $\angle T \nless \angle T_1$,

$$AN \ngtr AN_1,$$

and therefore $NO \nless N_1O$, where P_1N_1 is the ordinate of P_1.

Hence $PL \nless P_1P_2$.

But $P_1P_2 > P'F'$.

Therefore $PL > \frac{2}{3}PV$,

so that F, the centre of gravity of the immersed portion of the solid, lies between P and L.

Thus the solid will turn in the direction of diminishing the angle PTN until that angle becomes less than T_1.

[As before, if x, x' be the distances from T of the orthogonal projections of C, F respectively on TP, we have

$$x' - x = \cos\theta \left\{\frac{p}{4}(\cot^2\theta + 2) - \frac{2}{3}(h - k)\right\} \quad\ldots\ldots\ldots(1),$$

where $h = AM$, $k = PV$.

Also, if the base BB' touch the surface of the fluid at one point B, we have further, as in the note following Prop. 6,

$$\sqrt{ph} = \sqrt{pk} + \frac{p}{2}\cot\theta \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

and

$$h - k = \sqrt{ph}\cot\theta - \frac{p}{4}\cot^2\theta \quad\ldots\ldots\ldots\ldots(3).$$

Therefore, to find the relation between h and the angle θ at which the axis of the paraboloid is inclined to the surface of the fluid in a position of equilibrium with B just touching the surface, we eliminate k and equate the expression in (1) to zero; thus

$$\frac{p}{4}(\cot^2\theta + 2) - \frac{2}{3}\left(\sqrt{ph}\cot\theta - \frac{p}{4}\cot^2\theta\right) = 0,$$

or

$$5p\cot^2\theta - 8\sqrt{ph}\cot\theta + 6p = 0 \quad\ldots\ldots\ldots\ldots(4).$$

The two values of θ are given by the equations

$$5\sqrt{p}\cot\theta = 4\sqrt{h} \pm \sqrt{16h - 30p} \quad\text{............ (5).}$$

The lower sign corresponds to the angle U, and the upper sign to the angle T_1, in the proposition of Archimedes, as can be verified thus.

In the first figure of Archimedes (p. 284 above) we have

$$AK = \tfrac{2}{5}h,$$

$$M_2D^2 = \tfrac{3}{5}p \,.\, OK = \tfrac{3}{5}p\,(\tfrac{2}{3}h - \tfrac{2}{5}h - \tfrac{1}{2}p)$$

$$= \frac{3p}{5}\left(\frac{4h}{15} - \frac{p}{2}\right).$$

If $P_1P_2P_3$ meet BM in D', it follows that

$$\left.\begin{matrix} M_3D \\ M_3D' \end{matrix}\right\} = M_2D \pm M_3M_2$$

$$= \sqrt{\frac{3p}{5}\left(\frac{4h}{15} - \frac{p}{2}\right)} \pm \frac{1}{10}\sqrt{ph},$$

and
$$\left.\begin{matrix} MD \\ MD' \end{matrix}\right\} = MM_2 \mp M_2D$$

$$= \frac{2}{5}\sqrt{ph} \mp \sqrt{\frac{3p}{5}\left(\frac{4h}{15} - \frac{p}{2}\right)}.$$

Now, from the property of the parabola,

$$\cot U = 2MD/p,$$

$$\cot T_1 = 2MD'/p,$$

so that
$$\frac{p}{2}\cot\begin{Bmatrix} U \\ T_1 \end{Bmatrix} = \frac{2}{5}\sqrt{ph} \mp \sqrt{\frac{3p}{5}\left(\frac{4h}{15} - \frac{p}{2}\right)},$$

or
$$5\sqrt{p}\cot\begin{Bmatrix} U \\ T_1 \end{Bmatrix} = 4\sqrt{h} \mp \sqrt{16h - 30p},$$

which agrees with the result (5) above.

To find the corresponding ratio of the specific gravities, or k^2/h^2, we have to use equations (2) and (5) and to express k in terms of h and p.

Equation (2) gives, on the substitution in it of the value of $\cot\theta$ contained in (5),

$$\sqrt{k} = \sqrt{h} - \tfrac{1}{10}(4\sqrt{h} \pm \sqrt{16h - 30p})$$
$$= \tfrac{3}{5}\sqrt{h} \mp \tfrac{1}{10}\sqrt{16h - 30p},$$

whence we obtain, by squaring,

$$k = \tfrac{13}{25}h - \tfrac{3}{10}p \mp \tfrac{3}{25}\sqrt{h(16h - 30p)} \quad \text{.........} \ (6).$$

The lower sign corresponds to the angle U and the upper to the angle T_1, and, in order to verify the results of Archimedes, we have simply to show that the two values of k are equal to Q_1Q_3, P_1P_3 respectively.

Now it is easily seen that

$$Q_1Q_3 = h/2 - MD^2/p + 2M_3D^2/p,$$
$$P_1P_3 = h/2 - MD'^2/p + 2M_3D'^2/p.$$

Therefore, using the values of MD, MD', M_3D, M_3D' above found, we have

$$\left.\begin{matrix} Q_1Q_3 \\ P_1P_3 \end{matrix}\right\} = \frac{h}{2} + \frac{3}{5}\left(\frac{4h}{15} - \frac{p}{2}\right) - \frac{7h}{50} \pm \frac{6}{5}\sqrt{\frac{3h}{5}\left(\frac{4h}{15} - \frac{p}{2}\right)}$$
$$= \tfrac{13}{25}h - \tfrac{3}{10}p \pm \tfrac{3}{25}\sqrt{h(16h - 30p)},$$

which are the values of k given in (6) above.]