INTRODUCTION TO SPIN GEOMETRY
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1. DIRAC BUNDLES AND DIRAC OPERATORS

1.1. **Clifford algebras.** As a motivating example assume that W is a (real or complex) finite dimensional vector space together with a collection of linear maps $J_1, \dots, J_n : W \to W$ satisfying the following identities:

- $J_i^2 = -\mathrm{Id}_W$ for all i,
- $J_i J_j = -J_j J_i$ for all $i \neq j$.

Let $(x^1, \dots, x^n)$ be the canonical coordinates on $\mathbb{R}^n$. We define the *Dirac operator* $\mathcal{D} : C^\infty(\mathbb{R}^n, W) \to C^\infty(\mathbb{R}^n, W)$,

$$\mathcal{D}f = \sum_{i=1}^n J_i \circ \frac{\partial f}{\partial x^i}.$$

This is a linear differential operator of first order whose square is equal to the Laplace operator:

$$\mathcal{D}^2 = \Delta = -\sum_{i=1}^n \frac{\partial^2}{(\partial x^i)^2}.$$

It was found by Paul Dirac (1928) in his description of the quantum mechanical behavior of fermions. Lorentz invariance forces the relevant differential operator to be of first order.

Spin geometry arises out of the attempt to replace $\mathbb{R}^n$ by an arbitrary Riemannian manifold (M, g).

Definition 1. Let $(V, \langle -, - \rangle)$ be a Euclidean vector space. The *Clifford algebra* $\mathrm{Cl}(V)$ is defined as a quotient of the free tensor algebra,

$$\mathrm{Cl}(V) = \bigoplus_{r=0}^\infty V^{\otimes r} \Big/ I,$$

where I is the ideal generated by all elements of the form $v \otimes v + |v|^2$ for $v \in V$. $\mathrm{Cl}(V)$ is an $\mathbb{R}$-algebra with unit $1 \in \mathbb{R} = \bigotimes^0 V$.

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For orthogonal elements $v, w \in V$ we have $v \cdot w = -w \cdot v$ in $\mathrm{Cl}(V)$ as

$$-|v|^2 - |w|^2 = -|v+w|^2 = (v+w)^2 = v^2 + vw + wv + w^2 = -|v|^2 - |w|^2 + vw + wv.$$

If V' is a second Euclidean vector space and $V \to V'$ is an isometric inclusion, we get an induced $\mathbb{R}$-algebra map $\mathrm{Cl}(V) \to \mathrm{Cl}(V')$. We write

$$\mathrm{Cl}_n := \mathrm{Cl}(\mathbb{R}^n, \langle -, - \rangle_{\mathrm{Eucl.}}).$$

Let $(e_1, \ldots, e_n)$ be the standard orthonormal basis of $\mathbb{R}^n$. There are isomorphisms of $\mathbb{R}$-algebras $\mathrm{Cl}_1 \cong \mathbb{C}$ and $\mathrm{Cl}_2 \cong \mathbb{H}$. The first one is induced by $e_1 \mapsto i$, the second one is induced by $e_1 \mapsto i$, $e_2 \mapsto j$, writing as usual $\mathbb{H} = \mathrm{span}\{1, i, j, k\}$.

The map $V \to V$, $v \mapsto -v$, is an orthogonal isomorphism and hence it induces an algebra automorphism $\alpha \colon \mathrm{Cl}(V) \to \mathrm{Cl}(V)$. Since $\alpha^2 = \mathrm{id}$, we obtain a splitting

$$\mathrm{Cl}(V) = \mathrm{Cl}^0(V) \oplus \mathrm{Cl}^1(V),$$

where $\mathrm{Cl}^0(V) = \mathrm{im}(\alpha + \mathrm{id})$ is the $+1$ eigenspace of α, and $\mathrm{Cl}^1(V) = \mathrm{im}(\alpha - \mathrm{id})$ is the -1 eigenspace of α. We call $\mathrm{Cl}^0(C)$ the *even part* of $\mathrm{Cl}(V)$, and $\mathrm{Cl}^1(V)$ the *odd part* of $\mathrm{Cl}(V)$. Note that Clifford multiplication with any element $0 \neq v \in V$ induces a linear isomorphism $\mathrm{Cl}^0(V) \cong \mathrm{Cl}^1(V)$, in particular these spaces have the same dimension. More generally, Clifford multiplication induces linear maps

$$\mathrm{Cl}^i(V) \otimes \mathrm{Cl}^j(V) \to \mathrm{Cl}^{i+j}(V)$$

where $i, j \in \mathbb{Z}/2$. In other words: $\mathrm{Cl}(V)$ is a $\mathbb{Z}/2$-graded $\mathbb{R}$-algebra. For an element $v \in \mathrm{Cl}^i(V)$, we set $|v| := i \in \mathbb{Z}/2$.

If V_1 and V_2 are Euclidean vector spaces, we define the $\mathbb{Z}/2$-graded tensor product algebra $\mathrm{Cl}(V_1) \tilde{\otimes} \mathrm{Cl}(V_2)$ as the tensor product $\mathrm{Cl}(V_1) \otimes \mathrm{Cl}(V_2)$ together with the muliplication

$$(c_1 \otimes d_1)(c_2 \otimes d_2) = (-1)^{|d_1||c_2|}(c_1 c_2) \otimes (d_1 d_2). \tag{2}$$

Proposition 3. *Let V be a Euclidean vector space and let $V_1, V_2 \subset V$ be orthogonal subspaces such that $V = V_1 \oplus V_2$. Then there exists a canonical isomorphism*

$$\mathrm{Cl}(V) \cong \mathrm{Cl}(V_1) \tilde{\otimes} \mathrm{Cl}(V_2).$$

Proof. We define two algebra homomorphisms inverse to each other. First,

$$V = V_1 \oplus V_2 \to \mathrm{Cl}(V_1) \tilde{\otimes} \mathrm{Cl}(V_2), \quad (v_1, v_2) \mapsto v_1 \otimes 1 + 1 \otimes v_2,$$

induces an algebra map $f \colon \mathrm{Cl}(V) \to \mathrm{Cl}(V_1) \tilde{\otimes} \mathrm{Cl}(V_2)$ since

$$(v_1 \otimes 1 + 1 \otimes v_2)^2 = -|v_1|^2 - |v_2|^2 = -|(v_1, v_2)|^2,$$

using the multiplication (2).

Second, using the algebra maps $\iota_1 \colon \mathrm{Cl}(V_1) \to \mathrm{Cl}(V)$ and $\iota_2 \colon \mathrm{Cl}(V_2) \to \mathrm{Cl}(V)$ induced by the inclusions $V_1 \to V$ and $V_2 \to V$, we obtain a bilinear map

$$\mathrm{Cl}(V_1) \times \mathrm{Cl}(V_2) \to \mathrm{Cl}(V), \quad (x, y) \mapsto \iota_1(x) \cdot \iota_2(y),$$

and hence a linear map $g \colon \mathrm{Cl}(V_1) \tilde{\otimes} \mathrm{Cl}(V_2) \to \mathrm{Cl}(V)$. Since $V_1 \perp V_2$, the map g is an algebra map, again using the multiplication (2).

By a straightforward calculation, one shows that f and g are inverse to each other. ∎

$\mathrm{Cl}(V_1) \tilde{\otimes} \mathrm{Cl}(V_2)$ is again a $\mathbb{Z}/2$-graded algebra with $(\mathrm{Cl}(V_1) \tilde{\otimes} \mathrm{Cl}(V_2))^\ell = \bigoplus_{i+j=\ell} \mathrm{Cl}(V_1)^i \tilde{\otimes} \mathrm{Cl}(V)^j$.

Corollary 4. *Let $n = \dim V$. Then $\dim_{\mathbb{R}} \mathrm{Cl}(V) = 2^n$.*

Proof. We decompose $V = V_1 \oplus \cdots \oplus V_n$ into pairwise orthogonal 1-dimensional subspaces. Proposition 3 implies

$$\mathrm{Cl}(V) \cong \mathrm{Cl}(V_1) \tilde{\otimes} \cdots \tilde{\otimes} \mathrm{Cl}(V_n) \cong \mathrm{Cl}_1 \tilde{\otimes} \cdots \tilde{\otimes} \mathrm{Cl}_1 .$$

As $\mathrm{Cl}_1 \cong \mathbb{C}$, the assertion follows. ∎

1.2. **Relation to the Exterior Algebra.** Recall that the exterior algebra $\Lambda^* V$ is spanned by element of the form

$$v_1 \wedge \cdots \wedge v_r \qquad v_i \in V, r \in \mathbb{N}.$$

More formally, $\Lambda^* V = TV/J$ is the quotient of the free tensor algebra $\bigoplus_{r=0}^{\infty} V^{\otimes r}$ modulo the ideal J generated by elements of the form $v \otimes v$ for $v \in V$. Consider the (obviously well defined) linear map

$$\lambda^k \colon \Lambda^k V \to \mathrm{Cl}(V), \quad v_1 \wedge \cdots \wedge v_k \mapsto \frac{1}{k!} \sum_{\sigma \in \mathrm{Sym}_n} \mathrm{sgn}(\sigma) v_{\sigma(1)} \cdots v_{\sigma(k)}. \tag{5}$$

Proposition 6. *The maps* (5) *define an isomorphism* $\lambda^* \colon \Lambda^*(V) \to \mathrm{Cl}(V)$ *of real vector spaces.*

Proof. Consider the canonical projection

$$\pi \colon \bigoplus_{r \geq 0} V^{\otimes r} \to \mathrm{Cl}(V)$$

and for $k \geq 0$, let $\mathrm{Cl}^{(k)}(V) := \pi\left(\bigoplus_{r=0}^{k} V^{\otimes r}\right) \subset \mathrm{Cl}(V)$. For each k we have a map

$$\Lambda^k(V) \to \mathrm{Cl}^{(k)}(V)/\,\mathrm{Cl}^{(k-1)}(V), \quad v_1 \wedge \ldots \wedge v_k \mapsto v_1 \cdots v_k \tag{7}$$

which is induced by (5) and is clearly surjective. Hence the map $\Lambda^*(V) \to \mathrm{Cl}(V)$ is also surjective.

Let $n = \dim V$. Then $\dim_{\mathbb{R}} \mathrm{Cl}(V) = 2^n = \dim_{\mathbb{R}} \Lambda^* V$ by Proposition 4. Hence λ^* is a linear isomorphism. □

Corollary 8. *Let* $(v_1, \ldots, v_n)$ *be a basis of* V. *Then the elements*

$$v_{i_1} \cdots v_{i_k} \qquad i_1 < \ldots < i_k, \;\; 0 \leq k \leq \dim V,$$

form a basis of the $\mathbb{R}$*-vector space* $\mathrm{Cl}(V)$. *In particular,* $V = \bigotimes^1 V$ *is contained in* $\mathrm{Cl}(V)$ *as a linear subspace.*

1.3. **Classification of complex Clifford algebras.** We set $\mathbb{Cl}(V) := \mathrm{Cl}(V) \otimes \mathbb{C}$ and $\mathbb{Cl}(n) := \mathrm{Cl}(\mathbb{R}^n) \otimes \mathbb{C}$ with the tensor product algebra structures. All algebra homomorphisms are assumed to be unital.

Lemma 9. *There are isomorphisms of* $\mathbb{C}$*-algebras*

$$\mathbb{Cl}_1 \cong \mathbb{C} \oplus \mathbb{C}\,, \quad \mathbb{Cl}_2 \cong \mathrm{Mat}(2, \mathbb{C}).$$

Proof. Consider the $\mathbb{C}$-linear map $\mathbb{C} \oplus \mathbb{C} \to \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} = \mathrm{Cl}(1) \otimes \mathbb{C}$ determined on a basis of $\mathbb{C} \oplus \mathbb{C}$ by

$$(1,0) \mapsto \tfrac{1}{2}(1 \otimes 1 + i \otimes i), \quad (0,1) \mapsto \tfrac{1}{2}(1 \otimes 1 - i \otimes i).$$

This map is clearly injective, and by a dimension count it is a $\mathbb{C}$-linear isomorphism. Since $\frac{1}{2}(1 \otimes 1 - i \otimes i)$ and $\frac{1}{2}(1 \otimes 1 - i \otimes i)$ are idempotent and their product is zero, this map is multiplicative. This shows the first claim.

For the second isomorphism we write

$$\mathrm{Mat}(2, \mathbb{C}) = \mathrm{span}\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}.$$

These matrices are closely related to the *Pauli matrices* in quantum mechanics. Since the second, third and fourth matrix squares to -1 and they anticommute with each other, this shows $\mathrm{Mat}(2, \mathbb{C}) \cong \mathbb{Cl}_2$ as $\mathbb{R}$-algebras. □

Proposition 10 (Bott periodicity)**.** *For each* $n \geq 1$ *there exists a canonical isomorphism of* $\mathbb{C}$*-algebras*

$$\mathbb{Cl}_{n+2} \cong \mathbb{Cl}_n \otimes_{\mathbb{C}} \mathbb{Cl}_2 \cong \mathbb{Cl}_n \otimes_{\mathbb{C}} \mathrm{Mat}(2, \mathbb{C}).$$

Here we are using the ungraded multiplications given by multiplications in each tensor factor.

Proof. Choose generators h_1, h_2 of $\mathbb{Cl}_2$, and $f_1, \ldots, f_n$ of $\mathbb{Cl}_n$ and $e_1, \ldots, e_{n+2}$ von $\mathbb{Cl}_{n+2}$ where these generators correspond to standard generators of $\mathbb{R}^\ell$, $\ell = 2, n, n+2$. Then the assignment

$$e_1 \mapsto 1 \otimes h_1, \quad e_2 \mapsto 1 \otimes h_2, \quad e_3 \mapsto i f_1 \otimes h_1 h_2, \; \ldots, \; e_{n+2} \mapsto i f_n \otimes h_1 h_2$$

induces a $\mathbb{C}$-algebra map $\mathbb{Cl}_{n+2} \to \mathbb{Cl}_n \otimes_{\mathbb{C}} \mathbb{Cl}_2$. One checks that this map is surjective so that the claim follows from a dimension count. □

Corollary 11. *There are* $\mathbb{C}$*-algebra isomorphisms*

$$\mathbb{Cl}_{2k} \cong \mathrm{Mat}(2^k, \mathbb{C}), \qquad \mathbb{Cl}_{2k+1} \cong \mathrm{Mat}(2^k, \mathbb{C}) \oplus \mathrm{Mat}(2^k, \mathbb{C}).$$

1.4. **Clifford modules.**

Definition 12. A (complex) *Clifford module* for $\mathrm{Cl}(V)$, or $\mathrm{Cl}(V)$-*module*, is a finite dimensional complex vector space W together with a map of $\mathbb{R}$-algebras

$$\rho : \mathrm{Cl}(V) \to \mathrm{Hom}_{\mathbb{C}}(W, W).$$

Instead of $\rho(v)(w)$ we often write $v \cdot w$.

For the sake of brevity, these notes will be restricted to complex Cl_n modules. Many constructions also work for real Cl_n-modules. In the situation at the beginning of Section 1.1, W is a Cl_n module, setting $\rho(e_i) := J_i$ for the standard orthonormal basis $(e_1, \dots, e_n)$ of $\mathbb{R}^n$ and extending this map to an algebra map $\mathrm{Cl}(\mathbb{R}^n) \to \mathrm{Hom}(W, W)$.

Example 13. Set $\Lambda^*_{\mathbb{C}}(V) := \Lambda^*(V) \otimes_{\mathbb{R}} \mathbb{C}$. Clifford multiplication of $\mathrm{Cl}(V)$ on $\mathbb{C}\mathrm{l}(V)$ defines a $\mathrm{Cl}(V)$-module structure on the $\mathbb{C}$-vector space $\mathbb{C}\mathrm{l}(V) \overset{\text{can.}}{\cong} \Lambda^*_{\mathbb{C}}(V)$. For $v \in V$ and $\omega \in \Lambda^*_{\mathbb{C}}(V)$, we have $v \cdot \omega = v \wedge \omega + \iota_v \omega$, where $\iota_v \colon \Lambda^{k+1}_{\mathbb{C}}(V) \to \Lambda^k_{\mathbb{C}}(V)$, $\iota_v\omega(v_1, \dots, v_k) = \omega(v, v_1, \dots, v_k)$, is the contraction.

If W is a $\mathrm{Cl}(V)$-module, we define the Dirac operator $\mathcal{D} : C^\infty(V, W) \to C^\infty(V, W)$ by

$$\mathcal{D}f = \sum_{i=1}^n e_i \cdot \frac{\partial}{\partial x^i} f.$$

Here $(e_1, \dots, e_n)$ is some orthonormal basis of V and $x^1, \dots, x^n \colon V \to \mathbb{R}$ are the coordinates determined by $e_1, \dots, e_n$. In other words, $\mathcal{D}$ is the composition

$$\mathcal{D}\colon C^\infty(V, W) \xrightarrow{f \mapsto \nabla f} C^\infty(V, V^* \otimes W) \xrightarrow{c} C^\infty(V, W)$$

where for $v, \xi \in V$, we define $(\nabla f(v))(\xi) := \nabla_\xi f(v)$ as the derivative of $f \colon V \to W$ in direction ξ and c is induced by the Clifford multiplication of $V^* \cong V$ on W. This description shows that $\mathcal{D}$ is independent of the choice of $(e_1, \dots, e_n)$.

In order to define Dirac operators on Riemannian manifolds (M, g), we will

(i) replace V by the tangent spaces (T_xM, g_x), $x \in M$,
(ii) replace W by a "smooth family" of Clifford modules $\mathcal{S}_x$ for $\mathrm{Cl}(T_xM)$, $x \in M$.

Part (i) is straightforward: Since (T_xM, g_x) is a Euclidean vector space, we can form its Clifford algebra $\mathrm{Cl}(T_xM)$ and define a vector bundle $\mathrm{Cl}(TM) \to M$ by $\mathrm{Cl}(TM)_x = \mathrm{Cl}(T_xM)$. More precisely, given a local trivialisation $TM|_U \cong U \times \mathbb{R}^n$ set $\mathrm{Cl}(TM)|_U := U \times \mathrm{Cl}_n$. A transition map $U \cap V \to \mathrm{O}(n)$ for two orthogonal trivialisations of TM induces a transition map $U \cap V \to \mathrm{Aut}(\mathrm{Cl}_n, \mathrm{Cl}_n)$ into the algebra automorphisms of Cl_n satisfying the cocycle condition. Hence we obtain a well defined real vector bundle $\mathrm{Cl}(TM) \to M$ where each fibre has dimension 2^n as a real vector space. This bundle is a bundle of algebras in the sense that we have a vector bundle map

$$\mathrm{Cl}(TM) \otimes_{\mathbb{R}} \mathrm{Cl}(TM) \to \mathrm{Cl}(TM)$$

which restricts to Clifford multiplication in each fibre. We call the bundle $\mathrm{Cl}(TM) \to M$ the *Clifford algebra bundle* associated to $(TM, g) \to M$.

1.5. **Dirac bundles and Dirac operators.**

Definition 14. Let M be a Riemannian manifold. A $\mathrm{Cl}(TM)$-*module bundle* is a Hermitian vector bundle $\mathcal{S} \to M$ together with a vector bundle map

$$\rho\colon \mathrm{Cl}(TM) \to \mathrm{Hom}_{\mathbb{C}}(\mathcal{S}, \mathcal{S}) \tag{15}$$

such that for each $x \in M$, the map ρ restricts to a Clifford module $\rho_x\colon \mathrm{Cl}(T_xM) \to \mathrm{Hom}_{\mathbb{C}}(\mathcal{S}_x, \mathcal{S}_x)$ and Clifford multiplication $\rho_x(v)$ with unit vectors $v \in T_xM$ defines unitary maps $\mathcal{S}_x \to \mathcal{S}_x$.

A *Dirac bundle* is a Clifford module bundle $\mathcal{S} \to M$ together with a connection $\nabla^{\mathcal{S}}$ on $\mathcal{S}$ which is compatible with the Hermitian structure and satisfies the Leibniz rule

$$\nabla^{\mathcal{S}}_X(vs) = \nabla^{TM}_X(v) \cdot s + v \cdot \nabla^{\mathcal{S}}_X(s)$$

for all $X, v \in C^\infty(M, TM)$ and $s \in C^\infty(M, \mathcal{S})$. Here, ∇^{TM} is the Levi-Civita connection of (M, g).

Definition 16. Let $\mathcal{S}$ be a Dirac bundle on a Riemannian manifold (M, g). Then the corresponding *Dirac operator* is the composition

$$\mathcal{D}\colon C^\infty(M,\mathcal{S}) \xrightarrow{\nabla^{\mathcal{S}}} C^\infty(M, T^*M\otimes\mathcal{S}) \xrightarrow{c} C^\infty(M,\mathcal{S}).$$

In terms of a local orthonormal frame $(e_1,\dots,e_n)$ of TM this may be rewritten as usual as

$$\mathcal{D}s = \sum_{i=1}^n e_i \cdot \nabla^{\mathcal{S}}_{e_i} s.$$

The Dirac operator is a first order linear differential operator.

Example 17. Let (M^n, g) be a Riemannian manifold. The exterior product $\Lambda^*_{\mathbb{C}}TM = \Lambda^*TM\otimes\mathbb{C} \to M$ is a Dirac bundle with fibrewise Clifford module structures $\mathrm{Cl}(T_xM)\otimes\Lambda^*_{\mathbb{C}}T_xM \to \Lambda^*_{\mathbb{C}}T_xM$, see Example 13. The connection is induced by the Levi-Civita connection on (M, g). We set

$$\Omega^*_{\mathbb{C}}(M) := C^\infty(M, \Lambda^*T^*M\otimes\mathbb{C}) \cong C^\infty(M, \Lambda^*_{\mathbb{C}}TM).$$

The Dirac operator $\mathcal{D} : \Omega^*_{\mathbb{C}}(M) \to \Omega^*_{\mathbb{C}}(M)$ is given by

$$\mathcal{D} \ = d + d^*$$

where d is the exterior differential and d^* is the codifferential on $\Omega^*_{\mathbb{C}}(M)$. Recall that for a k-form ω, we set $d^*(\omega) = (-1)^{nk+n+1} * d(*\omega) \in \Omega^{k-1}_{\mathbb{C}}(M)$ where $*\colon \Omega^\ell_{\mathbb{C}}(M)\to\Omega^{n-\ell}_{\mathbb{C}}(M)$ is the Hodge star operator. We have

$$\mathcal{D}^2 = dd^* + d^*d\colon \Omega^*_{\mathbb{C}}(M)\to\Omega^*_{\mathbb{C}}(M),$$

the *Hodge Laplace operator*.

1.6. **Relation to the connection Laplacian.**

Definition 18. Let (M, g) be a Riemannian manifold. Let $E\to M$ be a vector bundle with connection ∇^E. The *second covariant derivative* of $s\in C^\infty(M,E)$ in directions $X, Y \in C^\infty(M,TM)$ is defined as

$$\nabla^2_{X,Y}s := \nabla^E_X\nabla^E_Y s - \nabla^E_{\nabla^{TM}_X Y} s.$$

The second term has been inserted to make this a tensor in both X and Y. The *connection Laplacian* $\Delta\colon C^\infty(M,E)\to C^\infty(M,E)$ is defined as

$$\Delta := -\sum_{i=1}^n \nabla^2_{e_i,e_i}. \tag{19}$$

where $(e_1,\dots,e_n)$ be some local orthonormal frame of (TM, g).

This is the usual Laplace operator on $C^\infty(\mathbb{R}^n)$ when $M := \mathbb{R}^n$ is equipped with the standard Euclidean structure and smooth functions on M are regarded as sections in the trivial bundle $E = M\times\mathbb{R}\to M$ with the trivial connection. If (M, g) is a Riemannian manifold and $E = M\times\mathbb{R}\to M$ is the trivial vector bundle with the trivial connection, this is called the *Laplace Beltrami operator* of (M, g). It acts on real valued functions on M.

Theorem 20 (Weitzenböck formula)**.** *Let $\mathcal{S}\to M$ be a Dirac bundle. Then*

$$\mathcal{D}^2 = \Delta + K,$$

where $K \in C^\infty(M, \mathrm{End}(\mathcal{S}))$ is given by

$$K = \sum_{i<j} e_i\cdot e_j\cdot \mathcal{K}^{\mathcal{S}}(e_i,e_j)$$

using the curvature $\mathcal{K}^{\mathcal{S}} \in C^\infty(M, \Lambda^2TM\otimes\mathrm{End}(\mathcal{S}))$ of $(\mathcal{S},\nabla^{\mathcal{S}})$.

Proof. Choose a local orthonormal frame (e_i) of TM which is synchronous at $x\in M$, i.e., for all i, j we have

$$\nabla^{TM}_{e_i} e_j(x) = 0$$

This can be done by parallel extending an orthonormal basis of T_xM along radial geodesics starting at x. For $s \in C^\infty(M, \mathcal{S})$ we compute at x, using that $[e_i, e_j] = \nabla_{e_i} e_j - \nabla_{e_j} e_i = 0$ at x,

$$
\begin{aligned}
\mathcal{D}^2 s &= \sum_{i,j} e_j \cdot \nabla^{\mathcal{S}}_{e_j} \left(e_i \cdot \nabla^{\mathcal{S}}_{e_i} s \right) \\
&= \sum_{i,j} e_j e_i \nabla^{\mathcal{S}}_{e_j} \nabla^{\mathcal{S}}_{e_i} s \\
&= -\sum_i \nabla_i^2(s) + \sum_{j<i} e_j e_i (\nabla^{\mathcal{S}}_{e_j} \nabla^{\mathcal{S}}_{e_i} - \nabla^{\mathcal{S}}_{e_i} \nabla^{\mathcal{S}}_{e_j}) s = \Delta s + K s.
\end{aligned}
$$

□

Δ and $\mathcal{D}^2$ are differential operators of second order, while K is an endomorphism field of $\mathcal{S}$, that is, a differential operator of order 0. Our initial goal to define a square root of the Laplace operator is solved up to this zero order term. It is this zero order term that will be useful later.

2. Spin manifolds

2.1. The spin groups.

Definition 21. Let $\mathrm{Cl}_n^\times \subset \mathrm{Cl}_n$ be the multiplicative group of invertible elements. Each $0 \neq v \in \mathbb{R}^n$ lies in $\mathrm{Cl}_n^\times$. We define a subgroup $\mathrm{Spin}(n) \subset \mathrm{Cl}_n^\times$ as

$$\mathrm{Spin}(n) := \{v_1 \cdots v_{2k} \in \mathrm{Cl}_n^\times \mid v_1, \ldots, v_{2k} \in \mathbb{R}^n, |v_i| = 1, k \geq 0\}.$$

Let $v \in \mathbb{R}^n$ be of length 1 and let $x \in \mathbb{R}^n$. Computing in Cl_n, we observe

- If $x \in \mathrm{Span}\{v\}$, we have $v \cdot x \cdot v^{-1} = x$.
- If $x \perp v$, we have $v \cdot x \cdot v^{-1} = -x$.

Hence $\Phi_v \colon \mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto v \cdot x \cdot v^{-1}$, is the negative of the reflection at the hyperplane $v^\perp \subset \mathbb{R}^n$. This map extends to a group homomorphism $\Phi \colon \mathrm{Spin}(n) \to \mathrm{SO}(n)$, $v \mapsto \Phi_v$,

$$\Phi_v(x) = v \cdot x \cdot v^{-1},$$

called the *adjoint representation* of $\mathrm{Spin}(n)$ on $\mathbb{R}^n$.

Proposition 22. $\ker \Phi = \{\pm 1\}$.

Proof. Let $(e_1, \ldots, e_n)$ be the standard orthonormal basis of $\mathbb{R}^n$ and suppose that $v = v_1 \cdots v_{2k} \in \ker(\Phi)$. Write

$$v = p_0 + e_1 p_1.$$

where p_0, p_1 are polynomials in $e_2, \ldots, e_n$. Then $p_0 \in \mathrm{Cl}_n^0$ and $p_1 \in \mathrm{Cl}_n^1$. Using the assumption $v \in \ker(\Phi)$ we get $e_1 v = v e_1$ such that

$$e_1 p_0 - p_1 = p_0 e_1 + e_1 p_1 e_1 = e_1 p_0 + p_1.$$

Hence $p_1 = 0$ and v is a polynomial in $e_2, \ldots, e_n$. Proceeding by induction, we see that v does not contain any of $e_1, \ldots, e_n$ and hence $v = \pm 1$. □

Since every element of $\mathrm{SO}(n)$ is a product of $2k$ reflections where $2k \leq n$, the group homomorphism Φ is surjective, and with Proposition 22, we get

Corollary 23. $\mathrm{Spin}(n) = \{v_1 \cdots v_{2k} \mid 0 \leq 2k \leq n, |v_i| = 1\}$.

Proposition 24. $\mathrm{Spin}(n)$ *is a compact Lie group.*

Proof. The subset $\mathrm{Cl}_n^\times$ is open in $\mathrm{Cl}_n \cong \mathbb{R}^{2^n}$. Because the multiplication map $\mathrm{Cl}_n \times \mathrm{Cl}_n \to \mathrm{Cl}_n$ is bilinear, it is smooth. The same holds for the inversion map on $\mathrm{Cl}_n^\times$. It follows that $(\mathrm{Cl}_n^\times, \cdot, 1)$ is a Lie group.

We claim that $\mathrm{Spin}(n) \subset \mathrm{Cl}_n^\times$ is a compact subset. This implies that $\mathrm{Spin}(n) \subset \mathrm{Cl}_n^\times$ is a closed subgroup. By the closed subgroup theorem from Lie group theory, $\mathrm{Spin}(n) \subset \mathrm{Cl}_n^\times$ is a Lie subgroup, proving our claim.

For each $k \geq 1$ we have a continuous map

$$\lambda_k \colon \underbrace{S^{n-1} \times \cdots \times S^{n-1}}_{2k \text{ factors}} \to \mathrm{Spin}(n), \quad (v_1, \ldots, v_{2k}) \mapsto v_1 \cdots v_{2k}.$$

As $S^{n-1} \times \cdots \times S^{n-1}$ is compact, the image of λ_k is compact. Using Corollary 23, $\mathrm{Spin}(n) = \bigcup_{k=1}^{\lfloor n/2 \rfloor} \mathrm{im}(\lambda_{2k})$ is a finite union of compact sets and hence itself compact. □

The map $\Phi\colon \mathrm{Spin}(n) \to \mathrm{SO}(n)$ is smooth, being the restriction of the smooth map $\mathrm{Cl}_n^\times \to \mathrm{Aut}(\mathrm{Cl}(n))$, $v \mapsto (x \mapsto vxv^{-1})$ to the smooth submanifold $\mathrm{Spin}(n) \subset \mathrm{Cl}_n^\times$.

Proposition 25. $\Phi\colon \mathrm{Spin}(n) \to \mathrm{SO}(n)$ *is a two-fold covering map. In particular,* $\dim \mathrm{Spin}(n) = \dim \mathrm{SO}(n) = n(n-1)/2$.

Proof. It is enough to find an open neighborhood of $1 \in \mathrm{SO}(n)$ which is evenly covered by Φ. By Proposition 22 this amounts to finding an open neighborhood $U \subset \mathrm{Spin}(n)$ of $1 \in \mathrm{Spin}(n)$ so that $U \cap U^{-1} = \emptyset$. But this follows easily by the continuity of the inversion map $v \mapsto v^{-1}$ on $\mathrm{Spin}(n)$. □

Example 26.
1. $\mathrm{Spin}(1) = \mathbb{Z}/2$ and $\Phi\colon \mathrm{Spin}(1) \to \mathrm{SO}(1) = \{1\}$ is the constant map.
2. $\mathrm{Spin}(2) = \{v_1 \cdot v_2 \mid v_1, v_2 \in S^1 \subset \mathbb{R}^2\} \subset \mathrm{Cl}(2) = \mathbb{H} = \mathrm{Span}\{1, i, j, k\}$. Remember that i and j correspond to the orthonormal basis elements $e_1, e_2 \in \mathbb{R}^2$. Let $\alpha i + \beta j \in S^1 \subset \mathrm{span}\{i, j\} \subset \mathbb{H}$ and $ai + bj \in S^1 \subset \mathrm{span}\{i, j\} \subset \mathbb{H}$. Then, in $\mathbb{H} = \mathrm{Cl}_2$,
$$(\alpha i + \beta j) \cdot (ai + bj) = (-\alpha a - \beta b) + (\alpha b - \beta a)k =: \gamma + \eta k$$
where $\gamma^2 + \eta^2 = 1$. We see that $\mathrm{Spin}(2) = S^1 \subset \mathrm{span}\{1, k\}$. Let $\gamma + \eta k \in S^1 = \mathrm{Spin}(2)$ and $xi + yj \in \mathbb{R}^2$. We calculate
$$\Phi_{\gamma+\eta k}(xi + yj) = (\gamma + \eta k)(xi + yj)(\gamma - \eta k) = (\gamma + \eta k)^2(xi + yj).$$
This shows that $\Phi\colon \mathrm{Spin}(2) = S^1 \to S^1 = \mathrm{SO}(2)$ is given by $\Phi(z) = z^2$.

Proposition 27. $\mathrm{Spin}(n)$ *is connected for* $n \geq 2$ *and simply-connected for* $n \geq 3$.

Proof. We have the long exact sequence for the covering $\mathrm{Spin}(n) \to \mathrm{SO}(n)$,
$$0 \to \pi_1(\mathrm{Spin}(n)) \to \pi_1(\mathrm{SO}(n)) \to \pi_0(\mathbb{Z}/2) \to \pi_0(\mathrm{Spin}(n)) \to \pi_0(\mathrm{SO}(n)) \to 0$$
The space $\mathrm{SO}(n)$ is connected for all $n \geq 1$. Moreover $\pi_1(\mathrm{SO}(n)) = \mathbb{Z}/2$ for $n \geq 3$. For $n = 3$ this follows from $\mathrm{SO}(3) = \mathbb{R}P^3$. For higher n, we use the long exact sequence for the fibration $\mathrm{SO}(n-1) \to \mathrm{SO}(n) \to S^{n-1}$ and the fact that S^{n-1} is simply-connected for $n > 3$.

Let $n \geq 2$, The map $\pi_0(\mathbb{Z}/2) \to \pi_0(\mathrm{Spin}(n))$ is the constant map, because $+1$ and -1 in $\mathrm{Spin}(n)$ may be joined by the path $(\cos(t)e_1 + \sin(t)e_2)\, e_1$ for $t \in [0, \pi]$. Hence $\pi_0(\mathrm{Spin}(n)) \cong \pi_0(\mathrm{SO}(n))$ and $\mathrm{Spin}(n)$ is connected.

Let $n \geq 3$. From the exact sequence it follows that $\pi_1(\mathrm{SO}(n)) \to \pi_0(\mathbb{Z}/2)$ is a bijection and so the injection $\pi_1(\mathrm{Spin}(n)) \to \pi_1(\mathrm{SO}(n))$ has zero image. It follows that $\pi_1(\mathrm{Spin}(n)) = 0$. □

Corollary 28. *For* $n \geq 3$ *the map* $\Phi\colon \mathrm{Spin}(n) \to \mathrm{SO}(n)$ *is the universal covering of* $\mathrm{SO}(n)$.

2.2. **From Clifford modules to Clifford bundles.** Let W a $\mathrm{Cl}(n)$-module. By averaging some Hermitian inner product of W over the finite group in $\mathrm{Cl}(n)$ generated by $e_1, \ldots, e_n \in \mathbb{R}^n$, we can assume that the Clifford action of each unit vector in $\mathbb{R}^n$ on W is unitary. We wish to construct a $\mathrm{Cl}(TM)$-module bundle $\mathcal{S} \to M$ with typical fibre isometric to W.

Assume that M is oriented. Let $(U_i)_{i \in I}$ be a trivialising open cover for $TM \to M$. We assume without loss of generality that this cover is *good*, that is, the intersection of a finite number of the U_i is either empty or contractible. Choose orthogonal orientation preserving trivialisations
$$\psi_i\colon TM|_{U_i} \cong U_i \times \mathbb{R}^n$$
with induced transition maps $\psi_{ji} := \psi_j \circ \psi_i^{-1}\colon U_i \cap U_j \to \mathrm{SO}(n)$.

We define the bundle $\mathcal{S} \to M$ by local trivialisations $\mathcal{S}|_{U_i} \cong U_i \times W$ and transition maps $\rho_{ji} : U_i \cap U_i \to \mathrm{Aut}(W)$ satisfying

- Compatibility with Clifford multiplication: For all i, j, all $x \in U_i \cap U_j$, all $v \in \mathbb{R}^n$ and all $w \in W$, we get
$$(\psi_{ji})_x(v) \cdot (\rho_{ji})_x(w) = (\rho_{ji})_x(v \cdot w).$$
- Cocycle condition: For all i, j, k, we have
$$\rho_{ki} = \rho_{kj} \circ \rho_{ji}$$
on $U_i \cap U_j \cap U_i$.

For $x \in U_i \cap U_j$ let $\xi \in \mathrm{Spin}(n)$ be a preimage of $(\psi_{ji})_x \in \mathrm{SO}(n)$ under π. There are two choices for ξ that differ from each other by a minus sign. Setting $(\rho_{ji})_x(w) := \xi \cdot w$ for $w \in W$ we then obtain

$$(\psi_{ji})_x(v) \cdot (\rho_{ji})_x(w) = (\xi \cdot v \cdot \xi^{-1}) \cdot (\xi \cdot w) = \xi \cdot v \cdot w = (\rho_{ji})_x(v \cdot w),$$

so that the compatibility of Clifford multiplication under the transition map is satisfied.

We hence choose lifts

$$\rho_{ji} : U_i \cap U_j \to \mathrm{Spin}(n)$$

of ψ_{ji} and view each $(\rho_{ji})_x \in \mathrm{Aut}(W)$ by use of the Cl_n-multiplication on W. Since all $U_i \cap U_j$ are contractible (or empty), these lifts ρ_{ji} exist.

The transitions maps (ψ_{ji}) satisfy the cocycle condition, hence

$$\sigma_{ijk} := \rho_{ki}^{-1} \circ \rho_{kj} \circ \rho_{ji} : U_i \cap U_j \cap U_k \to \mathbb{Z}/2 = \ker \Phi$$

measuring the failure of ρ_{ji} fulfilling the cocycle condition. The family (σ_{ijk}) defines a Cech cochain in $\sigma \in C^2(M; \mathbb{Z}/2)$. It is not difficult to show that this cochain is closed, i.e. $d\sigma = 0$. It hence defines a Cech cohomology class

$$w_2(M) \in H^2(M; \mathbb{Z}/2).$$

One can show that this is equal to the *second Stiefel-Whitney class* of M. If $w_2(M) = 0$ then there is a 1-cochain $\tau \in C^1(M; \mathbb{Z}/2)$ with $d\tau = \sigma$ and τ can be used in order to repair the chosen ρ_{ji} (by multiplying them with $\pm 1 \in \ker \Phi$) so that they fulfill the cocycle condition.

Definition 29. We call M *spinable*, if $w_2(M) = 0$.

One can show that each oriented closed manifold of dimension at most 3 is a spinable. The manifold $\mathbb{C}P^2$ is not spinable.

Summarizing, we proved that for a spinable oriented Riemannian manifold (M, g) and any complex Clifford module W for $\mathrm{Cl}(n)$ we can construct a $\mathrm{Cl}(TM)$-module bundle $\mathcal{S} \to M$ with fibre W.

Definition 30. A *spin structure* on an oriented Riemannian manifold (M, g) is a collection (U_i, ψ_i, ρ_{ji}) where (U_i) is an open cover of M, the maps $\psi_i \colon TM|_{U_i} \cong U_i \times \mathbb{R}^n$ are orthogonal, orientation preserving local trivialisations of TM and $\rho_{ji} \colon U_i \cap U_j \to \mathrm{Spin}(n)$ are lifts of the transition maps $\psi_{ji} = \psi_j \circ \psi_i^{-1} \colon U_i \cap U_j \to \mathrm{SO}(n)$ such that the ρ_{ji} satisfy the cocycle condition.

Given a spin Riemannian manifold (M, g) and a Clifford module W, the bundle $\mathcal{S} \to M$ constructed above is called the *spinor bundle* associated to the $\mathrm{Cl}(n)$-module W.

Equivalence classes of spin structures (defined in an appropriate sense) on a spinable oriented Riemannian manifold (M, g) are parametrized by $H^1(M; \mathbb{Z}/2)$. For a fixed Cl_n-module W, different spin structures may lead to non-isomorphic spinor bundles $\mathcal{S} \to M$ associated to W.

In coordinate free language, the construction of the spinor bundle is as follows: The bundle of oriented orthonormal frames of TM defines an $\mathrm{SO}(n)$-principle bundle $P_{\mathrm{SO}} \to M$. If $w_2(M) = 0$, then there is a $\mathrm{Spin}(n)$-principal bundle $P_{\mathrm{Spin}} \to M$ and a Φ-equivariant map of principal bundles $P_{\mathrm{Spin}} \to P_{\mathrm{SO}}$. Equivalence classes of such Φ-equivariant maps of principal bundles $P_{\mathrm{Spin}} \to P_{\mathrm{SO}}$ correspond to spin structures on M. Now

(i) The tangent bundle $TM \to M$ is the associated bundle $P_{\mathrm{Spin}} \times_\Phi \mathbb{R}^n = P_{\mathrm{SO}} \times_{\mathrm{SO}(n)} \mathbb{R}^n$.
(ii) The spinor bundle $\mathcal{S} \to M$ for a Cl_n-module W is the associated bundle $P_{\mathrm{Spin}} \times_{\mathrm{Spin}(n)} W$.

2.3. **Connections on spinor bundles.** Let $\mathcal{S} \to M$ be the $\mathrm{Cl}(TM)$-module bundle constructed in the last subsection. It remains to construct a connection on $\mathcal{S} \to M$ which is compatible with the Hermitian structure and the $\mathrm{Cl}(TM)$-module structure. Let $TM|_U \cong U \times \mathbb{R}^n$ and $\mathcal{S}|_U \cong U \times W$ be compatible trivialisations of TM and $\mathcal{S}$ as in the last subsection. Let $\{e_1, \dots, e_n\}$ be the standard orthonormal basis of $\mathbb{R}^n$, viewed as an orthonormal frame of $TM|_U$.

Let $\mathfrak{so}(n) = \{A \in \mathbb{R}^{n \times n} | A = -A^T\}$ be the Lie algebra of $\mathrm{SO}(n)$. Let $\omega = (\omega_i^j) \in \Omega^1(U, \mathfrak{so}(n))$ be the connection form of the Levi-Civita connection on M with respect to the trivialisation $TM|_U \cong U \times \mathbb{R}^n$, that is,

$$\nabla_X^{TM} e_i = \omega(X)(e_i) = \sum_{j=1}^n \omega(X)_i^j e_j \in C^\infty(U, \mathbb{R}^n),$$

Definition 31. Let $w \in W$, viewed as a section im $C^\infty(U, W|_U)$. Then with respect to the trivialisation $\mathcal{S}|_U \cong U \times W$, the connection $\nabla^{\mathcal{S}}$ is defined by

$$\nabla^{\mathcal{S}}_X w = \frac{1}{2} \sum_{1 \le i < j \le n} \omega(X)^j_i \cdot e_i \cdot e_j \cdot w \in C^\infty(U, W).$$

One can show that this defines a global connection on $\mathcal{S}$ which is compatible with the Hermitian structure and the $\mathrm{Cl}(TM)$-module structure.

Recall that the Riemannian curvature tensor $\mathcal{R} \in \Omega^2(M, \mathrm{End}(TM)) = C^\infty(\Lambda^2 T^* M \otimes \mathrm{End}\, TM)$ of (M, g) is locally described on U by the $\mathfrak{so}(n)$-valued 2-form $\Omega = (\Omega^j_i) \in \Omega^2(U, \mathfrak{so}(n))$, where

$$\Omega = d\omega + [\omega, \omega].$$

That is, for $X, Y \in \Gamma(TM|_U)$, we have

$$\mathcal{R}(X, Y) e_i = \sum_{i=1}^n \Omega^j_i(X, Y) e_j.$$

With the connection on $\mathcal{S}$ defined in 31, the curvature of $(\mathcal{S}, \nabla^{\mathcal{S}})$ is given by

$$\mathcal{K}^{\mathcal{S}} = \frac{1}{2} \sum_{1 \le i < j \le n} \Omega^j_i e_i e_j,$$

where the right hand side acts by Clifford multiplication on sections of $E|_U = U \times W$. Using the equation $\Omega^j_i(X, Y) = \langle \mathcal{R}^{TM}(X, Y) e_i, e_j \rangle$ we therefore get

$$\mathcal{K}^{\mathcal{S}}(X, Y) = \frac{1}{2} \sum_{1 \le i < j \le n} \langle \mathcal{R}^{TM}(X, Y) e_i, e_j \rangle e_i e_j.$$

Let us compute the curvature term in the Weitzenböck formula $\mathcal{D}^2 = \Delta + K$ in Theorem 20:

$$\begin{aligned}
\sum_{j<k} e_j e_k \mathcal{K}^{\mathcal{S}}(e_j, e_k) &= \frac{1}{2} \sum_{j,k} e_j e_k \mathcal{K}^{\mathcal{S}}(e_j, e_k) = \frac{1}{8} \sum_{j,k,\alpha,\beta} e_j e_k \langle \mathcal{R}(e_j, e_k) e_\alpha, e_\beta \rangle e_\alpha e_\beta \\
&= \frac{1}{8} \sum_\beta \Bigg(\frac{1}{3} \sum_{j,k,\alpha \text{ distinct}} \langle \mathcal{R}(e_j, e_k) e_\alpha + \mathcal{R}(e_k, e_\alpha) e_j + \mathcal{R}(e_\alpha, e_j) e_k, e_\beta \rangle e_j e_k e_\alpha \\
&\quad + \sum_{j,k,(\alpha=j)} \langle \mathcal{R}(e_j, e_k) e_j, e_\beta \rangle e_j e_k e_j + \sum_{j,k,(\alpha=k)} \langle \mathcal{R}(e_j, e_k) e_k, e_\beta \rangle e_j e_k e_k \Bigg) e_\beta
\end{aligned}$$

Leaving β fixed, we have used here the anti-symmetry of $\mathcal{R}$ to reduce the three-fold sum over (j, k, α) to the case $j \neq k$. The remaining cases (j, k, α) pairwise disjoint, $j = \alpha$, and $k = \alpha$ were then gathered as individual summands. The first summand consists of three equal parts. It vanishes by the Bianchi identity. By replacing j with k in the last summand, we see that the last two summands are equal. The above expression therefore reduces to (using $e_j e_k e_j e_\beta = e_k e_\beta$)

$$\frac{1}{4} \sum_{j,k,\beta} \langle \mathcal{R}(e_j, e_k) e_j, e_\beta \rangle e_k e_\beta = -\frac{1}{4} \sum_{k,\beta} \mathrm{Ric}(e_k, e_\beta) e_k e_\beta = \frac{1}{4} \mathrm{scal}_g.$$

Theorem 32 (Schrödinger 1932, Lichnerowicz 1963)**.** *Let $\mathcal{S} \rightarrow M$ be a spinor Dirac operator for some Cl_n-representation W. Then the corresponding Dirac operator satisfies*

$$\mathcal{D}^2 = \Delta^{\mathcal{S}} + \frac{1}{4} \mathrm{scal}_g.$$

The construction of spinor bundles can be enhanced to *twisted spinor bundles*. Let (M, g) be a spin Riemannian manifold, let W be a Cl_n-module, and let $\mathcal{S} \to M$ be the associated spinor bundle. Let $E \to M$ be a Hermitian vector bundle with Hermitian connection ∇^E.

The tensor product bundle $\mathcal{S} \otimes E$ together with the tensor product Hermitian structure and the tensor product connection $\nabla^{\mathcal{S}} \otimes \mathrm{id} + \mathrm{id} \otimes \nabla^E$ is again a Dirac bundle. Here we use the $\mathrm{Cl}(TM)$-module structure on the tensor factor $\mathcal{S}$.

2.4. **The Atiyah-Singer operator.** Let $n = 2k$ be even. We have $\mathbb{C}\mathrm{l}_n \cong \mathrm{Mat}(2^k, \mathbb{C})$ and hence we obtain a complex Cl_n-module $\Sigma_n = \mathbb{C}^{2^k}$ by matrix multiplication of $\mathrm{Mat}(2^k, \mathbb{C})$. One can show that Σ_n an irreducible complex Cl_n-module and it is the only irreducible complex Cl_n-module up to isomorphism. In particular, any Cl_n-module is a direct sum of copies of Σ_n. If $n = 2k + 1$ is odd, there are two non-isomorphic irreducible Cl_n-modules $\Sigma_n^{(1)}$ and $\Sigma_n^{(2)}$, corresponding to the $\mathbb{C}$-algebra isomorphism $\mathrm{Cl}_{2k+1} \cong \mathrm{Mat}(2^k, \mathbb{C}) \oplus \mathrm{Mat}(2^k, \mathbb{C})$.

Let (M, g) be a spin Riemannian manifold of dimension n. If n is even, we define $\Sigma M \to M$ as the spinor bundle associated to the unique irreducible Cl_n-representation Σ_n. The associated Dirac operator

$$\mathcal{D}\colon C^\infty(M, \Sigma M) \to C^\infty(M, \Sigma M)$$

is called the *Atiyah-Singer operator*. If n is odd, we have two inequivalent spinor bundles $\Sigma M^{(1)} \to M$ and $\Sigma M^{(2)} \to M$ and corresponding Atiyah-Singer operators.

If $E \to M$ is a Hermitian vector bundle with compatible connection, we obtain the *twisted Atiyah-Singer operator*

$$\mathcal{D}_E\colon C^\infty(M, \Sigma M \otimes E) \to C^\infty(M, \Sigma M \otimes E).$$

Remark 33. Let n be even. Since Σ_n is the unique irreducible complex Cl_n-module, each Dirac bundle $\mathcal{S} \to M$ is a twisted spinor bundle $\Sigma M \otimes E \to M$.

2.5. **The Atiyah-Singer index theorem.** Let (M, g) be an oriented Riemannian manifold of even dimension n and let $\mathcal{S} \to M$ be a Dirac bundle. Let $(e_1, \dots, e_n)$ an oriented orthonormal basis of $\mathbb{R}^n$. The *complex volume element*

$$\omega_{\mathbb{C}} = i^k e_1 \cdot \ldots \cdot e_n \in \mathbb{C}\mathrm{l}_n$$

is invariant under orthogonal orientation preserving transformations of $\mathbb{R}^n$. Hence we obtain a well defined section of $\mathrm{Cl}(TM) \otimes \mathbb{C}$ which acts as a self adjoint endomorphism of $\mathcal{S} \to M$ by fibrewise Clifford multiplication. It satisfies $\omega_{\mathbb{C}}^2 = \mathrm{id}_{\mathcal{S}}$ and is parallel with respect to the connection $\nabla^{\mathcal{S}}$. Hence we obtain an orthogonal splitting into the ± 1 eigenspaces of $\omega_{\mathbb{C}}$,

$$\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-.$$

These subbundles are parallel. and of equal dimension. We call this the *chirality decomposition* of $\mathcal{S}$. It only exists in even dimensions. Since Clifford multiplication with elements in $\mathbb{R}^n$ anticommutes with $\omega_{\mathbb{C}}$, the Dirac operator splits as

$$\mathcal{D} = \mathcal{D}^\pm : C^\infty(M, \mathcal{S}^\pm) \to C^\infty(M, \mathcal{S}^\mp).$$

Now let (M, g) be a spin Riemannian manifold of even dimension, and let $E \to M$ be a Hermitian bundle with compatible connection. Using the chirality decomposition of the twisted spinor bundle $\Sigma M \otimes E$,

$$(\Sigma M \otimes E)^\pm = \Sigma M^\pm \otimes E,$$

we obtain the *twisted half-spin Atiyah-Singer* operator

$$\mathcal{D}_E^\pm : C^\infty(M, (\Sigma M \otimes E)^\pm) \to C^\infty(M, (\Sigma M \otimes E)^\mp).$$

If M is compact, one can show that the kernels of $\mathcal{D}_E^\pm$ are finite dimensional subspaces of $C^\infty(M, (\Sigma M \otimes E)^\pm)$. We define the *Fredholm index* of $\mathcal{D}_E^+$ as

$$\mathrm{ind}(\mathcal{D}_E^+) := \dim_{\mathbb{C}} \ker \mathcal{D}_E^+ - \dim_{\mathbb{C}} \ker \mathcal{D}_E^- \in \mathbb{Z}.$$

This number is computed by the famous Atiyan-Singer index theorem.

Theorem 34 (Atiyah-Singer)**.** *We have*

$$\mathrm{ind}(\mathcal{D}_E^+) = \int_M \widehat{\mathsf{A}}(TM) \wedge \mathsf{ch}(E).$$

Here, the $\hat{A}$-form $\widehat{\mathsf{A}}(TM) \in \Omega^{4*}(M)$ *is determined by the curvature of* (TM, ∇^{TM}) *and the* Chern form $\mathsf{ch}(E) \in \Omega^{2*}(M; \mathbb{C})$ *is determined by the curvature of* (E, ∇^E).

One can show that the right hand side of the Atiyah-Singer index theorem only depends on the diffeomorphism type of M and the isomorphism type of the complex bundle $E \to M$. In connection with the Schrödinger-Lichnerowicz formula, Theorem 32, this has strong consequences in scalar curvature geometry.

For more information on the material in these lectures, see B. Lawson, M.-L. Michelsohn: *Spin geometry*.

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