Hamiltonian dynamics on Poisson manifolds
- Generalities on Poisson structures
- Hamiltonian systems
- A first approach to the numerical analysis on Poisson manifolds

Symplectic groupoids
- Basic notions on groupoids
- Basics on symplectic groupoids
- Lagrangian bisections
- Birealisations
- Hamilton-Jacobi equation

Hamiltonian Poisson integrators
- Butcher series
- Explicit construction through birealisations
- Numerical tests around a singularity: a Lotka-Volterra system
- Numerical tests of symmetry preservation: the rigid body

Perspectives
- Poisson integrators in solid mechanics (with L. Le Marrec and V. Carlier)
- Deformation theory of symplectic groupoids and plasma physics

Groupoid

Definition

A groupoid over a set M is a set $\mathcal{G}$ with

1. Two maps α and $\beta \colon \mathcal{G} \to M$ (*source* and *target*). We write $\mathcal{G}^{(k)} = \{(g_i)_{1 \le i \le k} \in \mathcal{G}^k, \quad \forall 1 \le i \le k-1, \alpha(g_{i+1}) = \beta(g_i)\}$
2. A *product* $\cdot \colon \mathcal{G}^{(2)} \to \mathcal{G}$, such that
   2.1 $\alpha(g \cdot h) = \alpha(g)$ and $\beta(g \cdot h) = \beta(h)$ (compatibility with source and target)
   2.2 $\forall (g, h, k) \in \mathcal{G}^{(3)}, \quad (g \cdot h) \cdot k = g \cdot (h \cdot k)$ (associativity)
3. An *identity section* $i \colon M \to \mathcal{G}$ such that $g \cdot i(\alpha(g)) = i(\beta(g)) \cdot g = g$.
4. An *inverse* $(.)^{-1} \colon \mathcal{G} \to \mathcal{G}$ such that $\alpha(g^{-1}) = \beta(g)$, $\beta(g^{-1}) = \alpha(g)$ and

$$g^{-1} \cdot g = i(\alpha(g)), \quad g \cdot g^{-1} = i(\beta(g)).$$

M is said to be the *objects* and $\mathcal{G}$ is said to be the *arrows*.

Lie groupoid

Definition

A Lie groupoid is a groupoid $\mathcal{G} \rightrightarrows M$ such that

- $\mathcal{G}$ and M are smooth manifolds
- the product $\cdot\colon \mathcal{G}^{(2)} \to \mathcal{G}$ and the identity section $i\colon M \to \mathcal{G}$ are smooth maps
- $\alpha, \beta\colon \mathcal{G} \to M$ are smooth submersions.

Remark

In the literature, it is common not to assume the differentiable topology of $\mathcal{G}$ to be Hausdorff separable. In this lecture, all the manifolds are assumed to be separable except if they are explicitly introduced as Lie groupoids.

More generally, the topology of groupoids is an intricate topic[1].

[1] Lie Groupoids and Lie Algebroids in Differential Geometry, Mackenzie, 1987

Examples of Lie groupoids

The pair groupoid of a manifold M
$\mathcal{G} = M \times M$
$\alpha(x, y) = x$, $\beta(x, y) = y$
$(x, y) \cdot (y, z) = (x, z)$
$i(x) = (x, x)$

The action groupoid $\mathcal{G} = G \times M$ of an action of G on M
$\alpha(g, x) = x$, $\beta(g, x) = g.x$
$(g, x) \cdot (h, g.x) = (hg, x)$
$i(x) = (1, x)$

$G \times \mathfrak{g}^*$
$\alpha(g, x) = x$, $\beta(g, x) = \mathrm{Ad}_g^* x$
$(g, x) \cdot (h, \mathrm{Ad}_g^* x) = (gh, x)$
$i(x) = (1, x)$

Induced singular foliation

Any Lie groupoid $\mathcal{G} \rightrightarrows M$ induces a partition of the base M:

$$x \sim y \Longleftrightarrow \exists p \in \mathcal{G}, \quad \alpha(p) = x \quad \& \quad \beta(p) = y$$

Theorem

This partition is a singular foliation. In particular, each leaf is an immersed submanifold.

Example

- The foliation of a pair groupoid has only one leaf.
- The foliation of an action groupoid is made of the orbits of the action.
- The foliation of $G \times \mathfrak{g}^* \rightrightarrows \mathfrak{g}^*$ is made of the coadjoint orbits.

Bisections

Definition (Bisection)

A bisection is a submanifold L of $\mathcal{G}$ such that $\alpha_{|L}\colon L \to M$ and $\beta_{|L}\colon L \to M$ are diffeomorphisms.

Proposition

The set of bisections is a group: for any bisections L_1 and L_2,

$$L_1 \cdot L_2 = \{l_1 \cdot l_2 \in \mathcal{G}, l_1 \in L_1, l_2 \in L_2, \beta(l_1) = \alpha(l_2)\}.$$

Proposition

Any bisection $L \subset \mathcal{G}$ induces a diffeomorphism

$$\Phi^L\colon \begin{array}{ccc} M & \to & M \\ x & \mapsto & \beta \circ (\alpha_{|L})^{-1}(x) \end{array}$$

such that $x \in M$ and $\Phi^L(x) \in M$ remain on the same leaf.

Symplectic groupoids

Definition (Symplectic groupoid)

A symplectic groupoid is a Lie groupoid $\mathcal{G} \rightrightarrows M$ such that $\mathcal{G}$ is equipped with a symplectic form ω such that

$$m^*\omega = \mathrm{pr}_1^*\omega + \mathrm{pr}_2^*\omega$$

where $m \colon \mathcal{G}^{(2)} \to \mathcal{G}$ is the product and $\mathrm{pr}_i \colon \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ is the projection on the i-th factor.

Terminology

ω is said to be multiplicative.
Roughly speaking, "$\omega(g \cdot h) = \omega(g) + \omega(h)$".

Remark

The graph of the product is isotropic –even Lagrangian– in $\mathcal{G} \times \mathcal{G} \times \bar{\mathcal{G}}$, where $\bar{\mathcal{G}}$ is equipped with $-\omega$.

Some properties of symplectic groupoids

Proposition

1. *We identify M with $i(M) \subset \mathcal{G}$. Then, M is Lagrangian in $\mathcal{G}$.*
2. *The inverse map is an anti-symplectomorphism:*

$$i^*\omega = -\omega$$

3. *The fibers of the source are symplectically orthogonal to the fibers of the target:*

$$\forall f, g \in \mathcal{C}^\infty(M), \{\alpha^* f, \beta^* g\} = 0.$$

4. *There is a unique Poisson structure π on M such that the source map α is a Poisson map and the target map β is an anti-Poisson map. $\mathcal{G} \rightrightarrows M$ is said to be integrating (M, π).*

Remark (Informal)

A symplectic groupoid desingularizes the symplectic foliation on the base.

Examples

The cotangent bundle T^*M of any manifold M – equipped with the canonical symplectic form on T^*M and source and target equal the cotangent projection – integrates the zero Poisson structure on M.

(M, Ω) is integrated by the pair groupoid $M \times \overline{M} \rightrightarrows M$.

The action groupoid $T^*G \simeq G \times \mathfrak{g}^* \rightrightarrows \mathfrak{g}^*$ – equipped with the canonical symplectic form on T^*G – integrates $\mathfrak{g}^*$.

A symplectic groupoid for any cluster Poisson structure has been computed in *Symplectic groupoids for cluster manifolds*, Li Rupel, 2020.

Lagrangian Bisections

Let $(\mathcal{G}, \omega) \rightrightarrows M$ be a symplectic groupoid.

Definition (Lagrangian bisection)

A Lagrangian bisection is a bisection which is Lagrangian as a submanifold of $(\mathcal{G}, \omega)$.

Lagrangian bisections are a subgroup of the group of bisections.

Lemma

The group of bisections acts on $\mathcal{G}$.

Lemma

The group of Lagrangian bisections acts on $\mathcal{G}$ by symplectomorphisms.

Induced diffeomorphisms

Let us assume that the fibers of α are source-connected.

Theorem

Let $L \subset \mathcal{G}$ be a Lagrangian bisection of $(\mathcal{G}, \omega)$. The induced diffeomorphism

$$\Phi^L = \beta \circ (\alpha_{|L})^{-1} \colon M \to M$$

1. *is a Poisson diffeomorphism on the base*
2. *and stays on a leaf.*

Question

Let $H \in \mathcal{C}^\infty(M)$ a Hamiltonian and $t \in \mathbb{R}$ some time. Which Lagrangian bisection L recovers the Hamiltonian flow ϕ_t^H ? Namely, how to chose L such that

$$\Phi^L = \phi_t^H ?$$

Weinstein tubular neighborhood theorem

Let $(\mathcal{G}, \omega) \rightrightarrows M$ be a symplectic groupoid.

Theorem (Weinstein tubular neighborhood theorem)

There exists a symplectomorphism from an open neighborhood of $M \subset \mathcal{G}$ onto an open neighborhood of the zero section of T^*M.

One can transport any local symplectic groupoid towards a neighborhood of the identity section inside the cotangent bundle of the base. There, the symplectic form is known and provides many Lagrangian bisections:

Remark

1. *Any closed 1-form on M provides a Lagrangian submanifold of T^*M.*
2. *Locally, any image of a sufficiently small closed 1-form is a Lagrangian bisection.*

Birealisation

The notion of birealisation axiomatises this embedding.

Definition (Birealisation)

A birealisation of a Poisson manifold (M, π) is a triple $(\mathbb{G}, \alpha, \beta)$ such that

- $\mathbb{G} \subset T^*M$ is a neighborhood of the zero section
- $\alpha \colon \mathbb{G} \twoheadrightarrow M$ – *the source* – and $\beta \colon \mathbb{G} \twoheadrightarrow M$ – *the target* – are smooth submersions such that
  1. $\alpha \circ 0 = \beta \circ 0 = \mathrm{Id}_M$
  2. α is a Poisson and β an anti-Poisson morphism for the canonical Poisson bracket on $\mathbb{G} \subset (T^*M, \omega)$
  3. $\forall f, g \in \mathcal{C}^\infty(M)$, $\{\alpha^* f, \beta^* g\}_\omega = 0$.

Remark

A birealisation is a local symplectic groupoid transformed by Weinstein tubular neighborhood theorem.

Examples of birealisations

for the canonical Poisson structure $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ of $T^*\mathbb{R} \simeq \mathbb{R}^2$

$$\left\{ \begin{array}{l} \alpha\colon T^*T^*\mathbb{R} \to T^*\mathbb{R}\colon (q,p,\xi_q,\xi_p) \mapsto (q - \frac{1}{2}\xi_p, p + \frac{1}{2}\xi_q) \\ \beta\colon T^*T^*\mathbb{R} \to T^*\mathbb{R}\colon (q,p,\xi_q,\xi_p) \mapsto (q + \frac{1}{2}\xi_p, p - \frac{1}{2}\xi_q) \end{array} \right. .$$

---

for the canonical Poisson structure on $so(n)^* \simeq so(n)$

$$\left\{ \begin{array}{l} \alpha\colon so(n) \times so(n) \to so(n) : (A,x) \mapsto (1 + \frac{A}{2}).x.(1 - \frac{A}{2}) \\ \beta\colon so(n) \times so(n) \to so(n)\colon (A,x) \mapsto (1 - \frac{A}{2}).x.(1 + \frac{A}{2}) \end{array} \right. .$$

---

for the cluster Poisson brackets

$$\left\{ \begin{array}{l} \alpha\colon T^*\mathbb{R}^n \to \mathbb{R}^n\colon (x,p) \mapsto \left(e^{-\frac{1}{2}\sum_i a_{ij}x_i p_i}.x_j\right)_{j=1,\dots,n} \\ \beta\colon T^*\mathbb{R}^n \to \mathbb{R}^n\colon (x,p) \mapsto \left(e^{\frac{1}{2}\sum_i a_{ij}x_i p_i}.x_j\right)_{j=1,\dots,n} \end{array} \right. .$$

Remark

The computation of a birealisation for a generic Poisson structure is a hard task ! We will talk about it later.

Birealisations ease symplectic groupoid computations

- Cotangent lifts of Poisson automorphisms are morphisms of local symplectic groupoids
- Lagrangian bisections are images of closed 1-forms
- Smooth families of Lagrangian bisections are smooth families of closed 1-forms

Hamilton-Jacobi equation

Let $H \in \mathcal{C}^\infty(M)$. Which Lagrangian bisection induces the flow of H as diffeomorphism on the base ?

Theorem (C., 2022)

1. *Hamilton-Jacobi equation*

$$\begin{cases} \partial_t S_t(x) & = H(\alpha(d_x S_t)) \\ S_0 & = 0 \end{cases} \tag{2}$$

   admits a unique solution $(S_t)_t$ for t in a neighborhood of 0. $(S_t)_t$ is the Hamilton-Jacobi transform of H.

2. *Such a solution allows to compute the flow of H by:*

$$\begin{cases} \phi_t^H(x) = \beta(d_{\bar{x}} S_t) \\ \\ \textit{where } \bar{x} \textit{ is the unique solution of } \alpha(d_{\bar{x}} S_t) = x \end{cases}$$

Hamilton-Jacobi equation at finite order

Theorem (C., 2022)

Let $S^i_t = \sum_{j=1}^{i} \frac{t^j}{j!} S_j$ be a solution of Hamilton-Jacobi equation (2) *at order i. Then,*

$$\begin{cases} \alpha(d_{\bar{x}_n} S^i_{\Delta t}) & = & x_n \\ x_{n+1} & = & \beta(d_{\bar{x}_n} S^i_{\Delta t}) \end{cases} \tag{3}$$

provides a Hamiltonian Poisson integrator at order i and time-step Δt for the Hamiltonian flow of H.

Remark

The resulting numerical methods are implicit: the non-linear equation (3) *is to be solved at each time-step.*

Remark

The truncated Lagrangian bisection induces a time-dependent Hamiltonian flow. A ***Hamiltonian Poisson integrator*** *is an integrator that follows the flow of a time-dependent Hamiltonian.*

Exercises for next week

We use the Frobenius metric $< A, B >= \mathrm{Tr}(AB)$ to identify $so(d)^* \simeq so(d)$. $so(d)$ becomes then equipped with the Lie-Poisson structure coming from $so(d)^*$.

1. Use the content of the previous chapter to prove that the isospectral method

$$\begin{aligned} W_n &= (I + \tfrac{\Delta t}{2}\overline{W}_n)\overline{W}_n(I - \tfrac{\Delta t}{2}\overline{W}_n) \\ W_{n+1} &= (I - \tfrac{\Delta t}{2}\overline{W}_n)\overline{W}_n(I + \tfrac{\Delta t}{2}\overline{W}_n) \end{aligned}$$

   preserves the Poisson structure of $so(d)$.

2. Find the Hamiltonian for which the map $W_n \mapsto W_{n+1}$ is a Hamiltonian Poisson integrator at order 1.

Bibliography for the second lecture

- Chapter 7 of *Poisson Structures and their Normal Forms*, Jean-Paul Dufour and Nguyen Tien Zung, Birkhäuser, 2005
  $\rightarrow$ symplectic groupoids

- *Geometric Models for Noncommutative Algebras*, Ana Cannas da Silva, Alan Weinstein, American Mathematical Society, 1999
  $\rightarrow$ groupoids

- *Symplectic groupoids for Poisson integrators*, O. C., Journal of Geometry and Physics, 2023