Topology and Geometry Seminar - IIIT Delhi

(Organizers: Aritra Bhowmick, Sachchidanand Prasad, Sandip Samanta)

Existence of Higher Extremal Kähler Metrics on a Minimal Ruled Surface - Talk 1

Overview of Some Concepts Related to Compact Kähler Manifolds (Prerequisites)

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Background Material Assumed for the Series of Talks

Differential Geometry:-

Familiarity with smooth manifolds and the related notions of tangent and cotangent bundles, vector fields and differential forms, exterior differential and de Rham cohomology, orientability of smooth manifolds, integration on oriented smooth manifolds; Basic familiarity with smooth vector bundles and more general smooth fibre bundles over smooth manifolds; Basic familiarity with Lie groups and Lie group actions on smooth manifolds

Riemannian Geometry:-

Familiarity with the notion of a Riemannian metric on a smooth manifold and the resultant notions of the Levi-Civita connection and curvature, covariant differentiation of vector fields, the Riemannian volume form and integration on oriented Riemannian manifolds, the canonical isomorphisms of the tangent and cotangent bundles induced by the Riemannian metric; Basic familiarity with the notion of a Riemannian metric on a smooth vector bundle
vector bundle

Background Material Assumed for the Series of Talks (contd.)

One Variable Complex Analysis: The theory of holomorphic functions of one complex variable

Several Complex Variables (SCV): Partial derivatives of a (real- or complex-valued) function defined on a domain in $\mathbb{C}^n$ with respect to the complex coordinates $z_1, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_n$; Dolbeault differential operators - ∂ and $\bar{\partial}$ on $\mathbb{C}^n$ providing the total differential with respect to the holomorphic and the antiholomorphic coordinates respectively

Holomorphic Function of Several Complex Variables: A complex-valued function of several complex variables is said to be *holomorphic* (or *complex analytic*) if it is holomorphic in each variable separately.

The Notion of a Complex Manifold - A Smooth Manifold Endowed with a Compatible Holomorphic Atlas

Definition (Complex Manifold)

Let M be a connected, second countable, metrizable topological space and let n be a fixed non-negative integer. Let $\{U_\alpha\}_{\alpha\in\Lambda}$ be an open covering of M, and for each $\alpha \in \Lambda$ let $\varphi_\alpha : U_\alpha \to V_\alpha \subseteq \mathbb{C}^n$ be a homeomorphism onto an open subset of $\mathbb{C}^n$. Further for all $\alpha, \beta \in \Lambda$ if $U_\alpha \cap U_\beta \neq \emptyset$ then let the *transition map* $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \subseteq V_\alpha \to \varphi_\beta(U_\alpha \cap U_\beta) \subseteq V_\beta$ be a holomorphic function. Then $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in\Lambda}$ is called as a *compatible holomorphic atlas* or equivalently a *complex structure* on M, and M endowed with $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in\Lambda}$ is then called as a *complex manifold* of (complex) dimension n.

Definition (Holomorphic Function on a Complex Manifold)

Let M be a complex manifold of dimension n. A function $f : M \to \mathbb{C}$ is said to be a *holomorphic function* on M if for each $\alpha \in \Lambda$ the function $f \circ \varphi_\alpha^{-1} : V_\alpha \to \mathbb{C}$ is a holomorphic function.

Some Standard Examples of Complex Manifolds

- The complex Cartesian n-space $\mathbb{C}^n$, and more generally any domain (non-empty, open, connected set) in $\mathbb{C}^n$, and in particular the complex plane $\mathbb{C}$, the punctured complex plane $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ and the open standard unit disk $\mathbb{D}$
- The Riemann sphere $\mathbb{S}^2$ is a *Riemann surface* (complex manifold of complex dimension 1)
- The complex matrix groups $\mathrm{M}_n(\mathbb{C})$, $\mathrm{GL}_n(\mathbb{C})$ and $\mathrm{SL}_n(\mathbb{C})$ are *complex Lie groups* of complex dimensions n^2, n^2 and $n^2 - 1$ respectively

Standard Fact: Every complex manifold is orientable.

Some Standard Examples of Complex Manifolds (contd.)

Constructing New Examples from Older Ones

Result (Open Complex Submanifold): A non-empty, open, connected subset of a complex manifold is also a complex manifold of the same dimension.

Result (Closed Complex Hypersurface): If $f : M \to \mathbb{C}$ is a holomorphic function on a complex manifold M of dimension n, and if f is a submersion (i.e. in this case f has got constant rank 1 over $\mathbb{C}$) and if $0 \in f(M)$, then the level set $f^{-1}(0) \subseteq M$ is a complex manifold of dimension $n-1$.

More General Result (Constant Rank Theorem - Closed Complex Submanifold): If $f : M \to N$ is a holomorphic function from a complex n-manifold M to a complex m-manifold N, and if f has got constant rank r over $\mathbb{C}$, then for any $q \in f(M)$ the level set $f^{-1}(q) \subseteq M$ is a complex manifold of dimension $n-r$.

Some Standard Examples of Complex Manifolds (contd.)

Some Well-Known Quotient Spaces of Complex Manifolds

Theorem (The Orbit Manifold of a Holomorphic Lie Group Action)

Let G be a complex Lie group acting holomorphically on a complex manifold M such that the action is free and proper. Then the orbit space of the action denoted by M/G is a complex manifold such that $\dim_{\mathbb{C}}(M/G) = \dim_{\mathbb{C}}(M) - \dim_{\mathbb{C}}(G)$.

Corollary (The Quotient Lie Group by a Closed, Normal Subgroup)

Let G be a complex Lie group and H be a closed, normal complex Lie subgroup of G. Then the quotient group G/H is a complex Lie group such that $\dim_{\mathbb{C}}(G/H) = \dim_{\mathbb{C}}(G) - \dim_{\mathbb{C}}(H)$.

Some Standard Examples of Complex Manifolds (contd.)

The Complex Projective Spaces and the Complex Tori

- **The Complex Projective n-Space:** $\mathbb{C}^*$ acts on $\mathbb{C}^{n+1} \setminus \{0\}$ as $\lambda \cdot (z_0, \ldots, z_n) = (\lambda z_0, \ldots, \lambda z_n)$. The orbit manifold of the group action is defined as $\mathbb{CP}^n$ which is a compact complex manifold of dimension n. A generic point on $\mathbb{CP}^n$ is denoted by the *homogeneous coordinates* $[z_0 : \ldots : z_n]$ which represents the unique complex line in $\mathbb{C}^{n+1}$ passing through the point $(z_0, \ldots, z_n)$ and the origin.
- **The Complex n-Torus:** $\left(\mathbb{Z}\left[\sqrt{-1}\right]\right)^n$ is a discrete additive subgroup of $\mathbb{C}^n$. The quotient group is defined as $\mathbb{T}^{2n}$ which is a compact complex Lie group of dimension n. It can be verified that $\mathbb{T}^{2n} = \left(\mathbb{S}^1\right)^{2n}$.

Complex Projective Manifolds

Definition (Complex Projective Manifold)

Let $f_1, \ldots, f_k$ be homogeneous polynomials in the complex variables $z_0, \ldots, z_n$. Define M to be the set of all points of $\mathbb{CP}^n$ whose homogeneous coordinates correspond to points in the common zero set of $f_1, \ldots, f_k$, i.e.:

$$M = \left\{ [z_0 : \ldots : z_n] \in \mathbb{CP}^n \;\middle|\; f_j\left(z_0, \ldots, z_n\right) = 0, \forall j = 1, \ldots, k \right\}$$

If M is a smooth submanifold of $\mathbb{CP}^n$ then M will be a complex submanifold of $\mathbb{CP}^n$ which is then called as a *complex projective manifold*.

Almost Complex Structure on a Real Vector Space

Let V be an n-dimensional real vector space. A linear operator $J : V \to V$ satisfying $J^2 = -I$ is called as an *almost complex structure* on V.

If there exists an almost complex structure on V then the dimension of V over $\mathbb{R}$ must be even i.e. $n = 2k$.

In that case define scalar multiplication on V by $\sqrt{-1}$ as $\sqrt{-1}v = J(v)$. This makes V into a complex vector space of dimension $\frac{n}{2} = k$ over $\mathbb{C}$.

Conversely on any even-dimensional real vector space there exists an almost complex structure making it into a complex vector space of half the real dimension: Let $\{v_1, \ldots, v_k, w_1, \ldots, w_k\}$ be an ordered basis of V, then define $J : V \to V$ by $J(v_i) = w_i$ and $J(w_i) = -v_i$ for $i = 1, \ldots, k$.

Almost Complex Structure on a Smooth Manifold

An *almost complex structure* on a smooth manifold is a smooth assignment of an almost complex structure to every real tangent space of the manifold. In other words it is a smooth vector bundle endomorphism $J : TM \to TM$ of the real tangent bundle TM such that $J^2 = -I$.

If there exists an almost complex structure on a smooth manifold M then the dimension of M over $\mathbb{R}$ must be even. The smooth manifold M equipped with the almost complex structure J is called as an *almost complex manifold*.

Result: Every almost complex manifold is orientable.

Note: An even-dimensional smooth manifold need not admit an almost complex structure.

Integrable Almost Complex Structures

A complex structure on a smooth manifold naturally induces an almost complex structure on it called as the *induced almost complex structure*.

An almost complex structure on a smooth manifold which arises from a complex structure is called as an *integrable almost complex structure*.

Theorem (Criteria for the Integrability of an Almost Complex Structure - Newlander-Nirenberg Theorem)

An almost complex structure J on a smooth manifold M is integrable if and only if any one of the following equivalent conditions holds true:

1. $\bar{\partial}^2 = 0$
2. $\partial^2 = 0$
3. $d = \partial + \bar{\partial}$

where d is the smooth exterior differential on M, while ∂ and $\bar{\partial}$ are the Dolbeault exterior differentials associated with J.

Review of Hermitian and Kähler Metrics

Let M be a complex manifold and J be the induced almost complex structure on M.

Definition (Hermitian Metric)

Let g be a Riemannian metric on M. Then g is said to be a *Hermitian metric* if $g_p(J_pX, J_pY) = g_p(X, Y)$ for all $X, Y \in T_pM$ for each $p \in M$.

Definition (Kähler Metric)

Let g be a Hermitian metric on M. Define $\omega_p(X, Y) = g_p(J_pX, Y)$ for all $X, Y \in T_pM$ for each $p \in M$. Then ω is a real smooth $(1,1)$-form on M. Then g is said to be a *Kähler metric* if ω is closed i.e. $d\omega = 0$, in which case ω is said to be the associated *Kähler form* of g.

Since g and ω are in one-one correspondence, we often simply say that ω is a Kähler metric on M.

Some Basic Definitions Involving Kähler Metrics

- If ω is a Kähler metric on M then (M, ω) is called as a *Kähler manifold*.
- Since ω is a closed $(1,1)$-form, we get the cohomology class $[\omega] \in H^{(1,1)}(M, \mathbb{R})$ which is called as the *Kähler class* of ω.
- In general a cohomology class $\mathcal{K} \in H^{(1,1)}(M, \mathbb{R})$ is called as a *Kähler class* on M if there exists a Kähler metric η on M such that $\mathcal{K} = [\eta]$.
- The set of all Kähler classes on M is called as the *Kähler cone* of M and is denoted by $H^{(1,1)}(M, \mathbb{R})^+$.

$\partial\bar{\partial}$-Lemma

Lemma ($\partial\bar{\partial}$-Lemma)

Let (M, ω) be a compact Kähler manifold. If ξ, η are closed real smooth $(1, 1)$-forms belonging to the same de Rham cohomology class on M then there exists a smooth function $f : M \to \mathbb{R}$ such that $\xi = \eta + \sqrt{-1}\partial\bar{\partial}f$.

Thus Kähler metrics in a given Kähler class on M can be parametrized by smooth real-valued functions on M.

Riemann, Ricci and Scalar Curvatures of a Kähler Metric

Let M be an n-dimensional complex manifold and g be a Kähler metric on M with associated Kähler form ω.
In terms of local holomorphic coordinates $\left(z^1, ..., z^n\right)$ on M, the metric g is completely determined by the $n \times n$ positive definite Hermitian matrix $g_{i,\bar{j}}$ given by:

$$g_{i,\bar{j}} = g\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^j}\right)$$

Let $g^{i,\bar{j}}$ denote the matrix inverse of $g_{i,\bar{j}}$.
The *Riemann curvature tensor* of g is given by:

$$R_{i,\bar{j},k,\bar{l}} = -\frac{\partial}{\partial z^k}\frac{\partial}{\partial \bar{z}^l} g_{i,\bar{j}} + g^{p,\bar{q}}\left(\frac{\partial}{\partial z^k} g_{i,\bar{q}}\right)\left(\frac{\partial}{\partial \bar{z}^l} g_{p,\bar{j}}\right)$$

Riemann, Ricci and Scalar Curvatures of a Kähler Metric (contd.)

The *Ricci curvature tensor* of g is the metric contraction of its Riemann curvature tensor:

$$R_{i,\bar{j}} = g^{k,\bar{l}} R_{i,\bar{j},k,\bar{l}}$$

The *scalar curvature* of g is the trace of the Ricci curvature tensor:

$$R = g^{i,\bar{j}} R_{i,\bar{j}}$$

Ricci Form, First Chern Form and First Chern Class

Let M be a compact Kähler n-manifold and ω be a Kähler metric on M.
The *Ricci form* of ω is defined as:

$$\mathrm{Ric}\,(\omega) = -\sqrt{-1}\partial\bar{\partial}\ln\det(\omega)$$

where $\det(\omega) = \det H(\omega)$, $H(\omega)$ being the Hermitian matrix of ω.
$\mathrm{Ric}\,(\omega)$ is a closed real $(1,1)$-form on M. If η is any other Kähler metric on M then it can be checked that:

$$[\mathrm{Ric}\,(\eta)] = [\mathrm{Ric}\,(\omega)] \in H^{(1,1)}(M, \mathbb{R})$$

The *first Chern form* of ω and the *first Chern class* of M (which is as a result independent of the choice of ω) are defined as:

$$c_1(\omega) = \frac{1}{2\pi}\mathrm{Ric}\,(\omega)\,, \quad c_1(M) = \frac{1}{2\pi}[\mathrm{Ric}\,(\omega)]$$

Kähler-Einstein and cscK Metrics

Definition (Kähler-Einstein Metric)

ω is said to be a Kähler-Einstein metric on M if $\mathrm{Ric}(\omega) = \lambda\omega$ for some constant $\lambda \in \mathbb{R}$.

The constant $\lambda = \lambda(\omega)$ which appears above is called as the *Ricci curvature* of ω.
The *scalar curvature* of ω, denoted by $S(\omega) : M \to \mathbb{R}$, is a smooth function given by the following formula:

$$n\,\mathrm{Ric}(\omega) \wedge \omega^{n-1} = S(\omega)\,\omega^n$$

Definition (cscK Metric)

ω is said to be a cscK metric on M if $S(\omega) \in \mathbb{R}$ is a constant.

We clearly have the following implication:

$$\omega \text{ is Kähler-Einstein} \implies \omega \text{ is cscK}$$

Extremal Kähler Metric

Definition (Calabi Functional; Calabi)

Let $\Omega \in H^{(1,1)}(M, \mathbb{R})$ be a Kähler class and Ω^+ denote the set of all Kähler metrics in Ω. The Calabi functional on Ω^+ is defined as:

$$\mathrm{Cal} : \Omega^+ \to \mathbb{R}, \ \ \mathrm{Cal}(\omega) = \int_M S(\omega)^2 \, \omega^n, \ \ \omega \in \Omega^+$$

Definition (Extremal Kähler Metric; Calabi)

$\omega \in \Omega^+$ is said to be an extremal Kähler metric if ω is a critical point of Cal on Ω^+.

Theorem (The Euler-Lagrange Equation for an Extremal Kähler Metric; Calabi)

ω is an extremal Kähler metric on M if and only if $\nabla^{1,0} S(\omega) = \left(\bar{\partial} S(\omega)\right)^{\sharp}$ is a real holomorphic vector field on M.

We clearly have the following implication:

$$\omega \text{ is cscK} \implies \omega \text{ is extremal Kähler}$$

Reference Books for Complex Differential Geometry and Kähler Geometry

- Complex Geometry: An Introduction - Daniel Huybrechts
- An Introduction to Extremal Kähler Metrics - Gábor Székelyhidi
- Complex Analytic and Differential Geometry - Jean-Pierre Demailly

Thank You For
Your Kind Attention!