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Existence of Higher Extremal Kähler Metrics on a Minimal Ruled Surface - Talk 3

Existence of Higher Extremal Kähler Metrics on a Minimal Ruled Surface (Final Main Talk)

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“Canonical” Kähler Metrics on Compact Kähler Manifolds

Let M be a compact Kähler manifold. We have the following 3 well-known and well-studied notions of “canonical” Kähler metrics in a given fixed Kähler class Ω on M:

$$\{\text{Kähler-Einstein Metrics}\} \subseteq \{\text{cscK Metrics}\}$$
$$\{\text{cscK Metrics}\} \subseteq \{\text{Extremal Kähler Metrics}\}$$

The definitions of these 3 notions are related to the first Chern class $c_1(M) \in H^{(1,1)}(M, \mathbb{R}) \subseteq H^2(M, \mathbb{R})$.
Taking the analogy of these to the level of the top Chern class $c_n(M) \in H^{(n,n)}(M, \mathbb{R}) = H^{2n}(M, \mathbb{R})$, Pingali introduced the following 2 new notions of canonical Kähler metrics in the Kähler class Ω:

$$\{\text{Higher cscK Metrics}\} \subseteq \{\text{Higher Extremal Kähler Metrics}\}$$

The Minimal Ruled Surface Involved in our Problem

Let (Σ, ω_Σ) be a genus 2 compact Riemann surface equipped with a Kähler metric of constant scalar curvature -2 (and hence of surface area 2π). Let (L, h) be a degree -1 holomorphic line bundle on Σ equipped with a Hermitian metric whose curvature form is $-\omega_\Sigma$. We consider the minimal ruled complex surface $X = \mathbb{P}(L \oplus \mathcal{O})$.

Let C be a typical fibre of X, $S_\infty = \mathbb{P}(L \oplus \{0\})$ is called as the *infinity divisor* of X and $S_0 = \mathbb{P}(\{0\} \oplus \mathcal{O})$ is called as the *zero divisor* of X.

We have the following intersection formulae:

$$C^2 = 0\,, \quad S_\infty^2 = 1\,, \quad S_0^2 = -1$$
$$C \cdot S_\infty = 1\,, \quad C \cdot S_0 = 1\,, \quad S_\infty \cdot S_0 = 0$$
$$c_1(L) \cdot [\Sigma] = -1\,, \quad [\omega_\Sigma] \cdot [\Sigma] = 2\pi$$

Computing the Kähler Cone of the Minimal Ruled Surface

By the Leray-Hirsch theorem we have (upto Poincaré duality) $H^2(X, \mathbb{R}) = \mathbb{R}C \oplus \mathbb{R}S_\infty$.

Question: Starting with a general cohomology class $\alpha = aC + bS_\infty \in H^{(1,1)}(X, \mathbb{R}) \subseteq H^2(X, \mathbb{R})$ with $a, b \in \mathbb{R}$, when will α be a Kähler class on X?

Answer (Fujiki, LeBrun-Singer, Tønnesen-Friedman): By using the Nakai-Moishezon criterion (extended to the real cohomology case by Buchdahl, Lamari), α is a Kähler class on X if and only if $\alpha^2 > 0$, $\alpha \cdot C > 0$, $\alpha \cdot S_\infty > 0$, $\alpha \cdot S_0 > 0$ if and only if $a > 0$, $b > 0$.

Therefore the Kähler cone of X is precisely:

$$H^{(1,1)}(X, \mathbb{R})^+ = \{aC + bS_\infty \mid a, b > 0\}$$

Statement of our Problem

We will first consider Kähler classes only of the form $2\pi\left(C + mS_\infty\right)$ with $m > 0$, and after constructing higher extremal Kähler metrics in these Kähler classes, we can simply rescale these metrics to get higher extremal Kähler metrics in the general Kähler classes which are of the form $k\left(C + mS_\infty\right)$ with $k, m > 0$. Since being a higher extremal Kähler metric (or even a higher cscK metric) is a scale-invariant property, this simple rescaling argument works.

Problem: To find a higher extremal Kähler metric ω on X belonging to the Kähler class $2\pi\left(C + mS_\infty\right)$ by using the momentum construction method of Hwang-Singer (the Calabi ansatz procedure), i.e. satisfying the following conditions:

$$[\omega] = 2\pi\left(C + mS_\infty\right), \quad c_2\left(\omega\right) = \frac{\lambda}{2\left(2\pi\right)^2}\omega^2, \quad \nabla^{1,0}\lambda \in \mathfrak{h}\left(X\right)$$

where $\mathfrak{h}\left(X\right)$ denotes the set of all real holomorphic vector fields on X.

The Momentum Construction Method Used in our Problem

Momentum Construction Method (Hwang-Singer):

Let $p : X \to \Sigma$ be the fibre bundle projection.
Let z be a coordinate on Σ, w be a coordinate on the typical fibre of L corresponding to a local trivialization around z,
$s = \ln |(z, w)|_h^2 = \ln |w|^2 + \ln h(z)$ be the logarithm of the square of the fibrewise norm on the total space of L induced by h.
Let f be a strictly convex smooth function of $s \in \mathbb{R}$, such that $s + f(s)$ is strictly increasing.

Let ω be the Kähler metric on $X \setminus (S_0 \cup S_\infty)$ (which is supposed to extend smoothly across S_0 and S_∞) given by the following ansatz (known as the Calabi ansatz):

$$\omega = p^* \omega_\Sigma + \sqrt{-1} \partial \bar{\partial} f(s)$$

It can be checked: In order for ω to be in the Kähler class $2\pi (C + mS_\infty)$, we must have $0 \leq f'(s) \leq m$.

The Momentum Construction Method Used in our Problem (contd.)

We define the *momentum profile* of ω as $\phi(\gamma) = f''(s) \geq 0$ where $\gamma = 1 + f'(s) \in [1, m+1]$. γ is called as the *momentum variable* and $[1, m+1]$ is called as the *momentum interval*.
We then write down the expressions for ω^2 and $c_2(\omega)$ in terms of $\phi(\gamma)$:

$$\omega^2 = 2\gamma\phi p^* \omega_\Sigma \wedge \sqrt{-1}\frac{dw \wedge d\bar{w}}{|w|^2}$$

$$c_2(\omega) = \frac{1}{(2\pi)^2} p^* \omega_\Sigma \wedge \sqrt{-1}\frac{dw \wedge d\bar{w}}{|w|^2} \frac{\phi}{\gamma^2} \left(\gamma(\phi + 2\gamma)\phi'' + \phi'(\phi'\gamma - \phi)\right)$$

and then substitute them in the required equation:

$$c_2(\omega) = \frac{\lambda}{2(2\pi)^2}\omega^2$$

to obtain the expression for λ in terms of $\phi(\gamma)$:

$$\lambda = \frac{1}{\gamma^3}\left(\gamma(\phi + 2\gamma)\phi'' + \phi'(\phi'\gamma - \phi)\right)$$

where $\nabla^{1,0}\lambda$ should be a real holomorphic vector field on X, for ω to be higher extremal Kähler.

The Resultant ODE BVP

Finally ω is a higher extremal Kähler metric on X satisfying the required properties if and only if its momentum profile $\phi(\gamma)$, $\gamma \in [1, m+1]$ satisfies the following ODE for some parameters $A, B, C \in \mathbb{R}$:

$$(2\gamma + \phi)\,\phi' = A\frac{\gamma^4}{3} + B\frac{\gamma^3}{2} + C\gamma \tag{1}$$

with the following boundary conditions (which are required for ω to extend smoothly across S_0 and S_∞):

$$\phi(1) = \phi(m+1) = 0$$
$$\phi'(1) = -\phi'(m+1) = 1$$

and with $\phi > 0$ on $(1, m+1)$ (as $\phi = f'' > 0$).

Analysis of the ODE BVP

Imposing the boundary conditions on the ODE (1) gives us A, B as linear functions of C:

$$A(C) = \frac{3C}{m}\left[1 - \frac{1}{(m+1)^2}\right] - \frac{6}{m}\left[1 + \frac{1}{(m+1)^2}\right]$$
$$B(C) = -2C\left[1 + \frac{1}{m} - \frac{1}{m(m+1)^2}\right] + 4\left[1 + \frac{1}{m} + \frac{1}{m(m+1)^2}\right]$$

Defining $p(\gamma) = A(C)\frac{\gamma^3}{3} + B(C)\frac{\gamma^2}{2} + C$ and substituting $v = \frac{(2\gamma+\phi)^2}{2}$, $\gamma \in [1, m+1]$, the ODE BVP (1) reduces to the following:

$$\begin{gathered} v' = 2\sqrt{2}\sqrt{v} + p(\gamma)\gamma \quad on \quad [1, m+1] \\ v(1) = 2, \;\; v(m+1) = 2(m+1)^2 \\ v'(1) = 6, \;\; v'(m+1) = 2(m+1) \\ v > 2\gamma^2 \quad on \quad (1, m+1) \end{gathered} \tag{2}$$

Analysis of the ODE BVP (contd.)

Lemma (Pingali)

The boundary conditions $v(1) = 2$ *and* $v(m+1) = 2(m+1)^2$ *on the solution* v *of the ODE in (2) individually imply the boundary conditions* $v'(1) = 6$ *and* $v'(m+1) = 2(m+1)$ *respectively, and together imply the condition* $v > 2\gamma^2$ *on* $(1, m+1)$*.*

Thus the ODE BVP reduces to the following:

$$\begin{gathered} v' = 2\sqrt{2}\sqrt{v} + p(\gamma)\gamma \quad \text{on} \quad [1, m+1] \\ v(1) = 2, \quad v(m+1) = 2(m+1)^2 \end{gathered} \tag{3}$$

Rough Strategy of Solving the ODE BVP

We first drop the final boundary condition $v(m+1) = 2(m+1)^2$ in the ODE BVP (3) and consider the following ODE IVP:

$$\begin{gathered} v' = 2\sqrt{2}\sqrt{v} + p(\gamma)\gamma \quad on \quad [1, m+1] \\ v(1) = 2 \end{gathered} \tag{4}$$

We first get a smooth solution $v(\cdot; C)$ on $[1, m+1]$ for the above ODE IVP (4) for each value of C coming from an interval of the real line. Then we find a C_1 in that interval such that $v(m+1; C_1) > 2(m+1)^2$ and a C_2 in that interval such that $v(m+1; C_2) < 2(m+1)^2$, thereby proving that there exists a C in the interval such that $v(m+1; C) = 2(m+1)^2$.
However, as the ODE in (4) is not directly integrable, its analysis for getting the existence of a smooth solution satisfying all the required boundary conditions is quite delicate but nevertheless uses only elementary real analysis and standard ODE theory.

Main Results Obtained for our Problem

$m > 0$ is fixed determining the Kähler class under consideration.

Theorem (S.)

There exists a unique $M = M(m) > 2$*, such that for every* $C \in (-\infty, M)$ *there exists a unique smooth solution* $v(\cdot; C) : [1, m+1] \to \mathbb{R}$ *for the aforementioned ODE IVP (4) such that* $\lim_{C \to -\infty} v(m+1; C) = \infty$ *and* $\lim_{C \to M^-} v(m+1; C) = 0$*, and for every* $C \in [M, \infty)$ *there exists a unique smooth solution* $v(\cdot; C) : [1, \gamma_{\star,C}) \to \mathbb{R}$ *for a unique* $\gamma_{\star,C} \in (1, m+1]$ *such that* $v(\cdot; C)$ *cannot be continued further beyond* $\gamma_{\star,C}$*. Furthermore there exists a unique* $C = C(m) \in (-\infty, M)$ *such that* $v(m+1; C) = 2(m+1)^2$*.*

We will then have answered the question of the existence of higher extremal Kähler metrics in the Kähler classes of the form $2\pi (C + mS_\infty)$ on X affirmatively.

Main Results Obtained for our Problem (contd.)

Corollary (S.)

For each $m > 0$ there exists a higher extremal Kähler metric ω on X having Calabi symmetry and belonging to the Kähler class $2\pi\left(C + mS_\infty\right)$, i.e. satisfying the following:

$$[\omega] = 2\pi\left(C + mS_\infty\right), \quad c_2(\omega) = \frac{\lambda(\omega)}{2\left(2\pi\right)^2}\omega^2, \quad \nabla^{1,0}\lambda(\omega) \in \mathfrak{h}(X)$$

An appropriate rescaling argument gives the existence of higher extremal Kähler metrics in the Kähler classes of the form $k\left(C + mS_\infty\right)$ on X with any $k > 0$.

Corollary (S.)

For all $k, m > 0$ there exists a higher extremal Kähler metric η on X having Calabi symmetry and belonging to the Kähler class $k\left(C + mS_\infty\right)$, i.e. satisfying the following:

$$[\eta] = k\left(C + mS_\infty\right), \quad c_2(\eta) = \frac{\lambda(\eta)}{2\left(2\pi\right)^2}\eta^2, \quad \nabla^{1,0}\lambda(\eta) \in \mathfrak{h}(X)$$

Whether these Constructed Higher Extremal Kähler Metrics are Higher cscK?

Now that we have gotten the existence of higher extremal Kähler metrics in all the Kähler classes on X by the momentum construction method, we ask the next question so as to whether these momentum-constructed Kähler metrics are higher cscK. This question was answered in the negative (Pingali).

Theorem (Pingali, S.)

For all $a, b > 0$ there exists a Kähler metric η on X, which is higher extremal Kähler but not higher cscK, having Calabi symmetry and belonging to the Kähler class $aC + bS_\infty$, i.e. satisfying the following:

$$[\eta] = aC + bS_\infty, \;\; c_2(\eta) = \frac{\lambda(\eta)}{2(2\pi)^2}\eta^2, \;\; \nabla^{1,0}\lambda(\eta) \neq 0 \in \mathfrak{h}(X)$$

An Obstruction to the Existence of Higher cscK Metrics: The Top Bando-Futaki Invariant

Let M be a compact Kähler n-manifold. Given a Kähler form ω on M, by using Hodge theory we get a real $(n-1, n-1)$-form φ on M such that:

$$c_n(\omega) - Hc_n(\omega) = \sqrt{-1}\partial\bar{\partial}\varphi$$

where H denotes harmonic projection.
The n^{th} *Bando-Futaki invariant* for the Kähler class $[\omega]$ on M is defined as:

$$\mathcal{F}_n(Y, [\omega]) = \int_M \mathcal{L}_Y \varphi \wedge \omega, \quad Y \in \mathfrak{h}(M)$$

where $\mathcal{L}_Y$ denotes Lie derivative w.r.t. Y.

Theorem (Bando)

$\mathcal{F}_n$ is a real-valued function of the Kähler class $[\omega]$ alone and does not depend on the choice of the Kähler metric ω in $[\omega]$ and also on the choice of the real $(n-1, n-1)$-form φ satisfying the above equation.

A Nice Expression for the Top Bando-Futaki Invariant

The n^{th} Bando-Futaki invariant $\mathcal{F}_n(Y,[\omega])$ for any real holomorphic vector field Y and any Kähler class $[\omega]$ on M can be re-expressed as follows:

$$\mathcal{F}_n(Y,[\omega]) = -\frac{1}{n!\,(2\pi)^n}\int_M F(\lambda-\lambda_0)\,\omega^n$$

where $\lambda \in \mathcal{C}^\infty(M,\mathbb{R})$ satisfies $c_n(\omega) = \frac{\lambda}{n!(2\pi)^n}\omega^n$, $\lambda_0 \in \mathbb{R}$ satisfies $\lambda_0 = \frac{\int_M \lambda\omega^n}{\int_M \omega^n}$ and $F \in \mathcal{C}^\infty(M,\mathbb{R})$ satisfies $d(\iota_Y\omega) = \sqrt{-1}\partial\bar{\partial}F$ (where ι_Y denotes interior product or contraction w.r.t. Y).

Now if ω is a higher extremal Kähler metric on M and if we take $Y = \left(\bar{\partial}\lambda\right)^\sharp \in \mathfrak{h}(M)$, then the above expression for the n^{th} Bando-Futaki invariant further simplifies to the following:

$$\mathcal{F}_n\left(\left(\bar{\partial}\lambda\right)^\sharp,[\omega]\right) = -\frac{1}{n!\,(2\pi)^n}\int_M (\lambda-\lambda_0)^2\,\omega^n = -\|\lambda-\lambda_0\|^2_{\mathcal{L}^2(M,\omega)}$$

i.e. $F = \lambda - \lambda_0$.

Results about the Top Bando-Futaki Invariant

The n^{th} Bando-Futaki invariant $\mathcal{F}_n(\cdot, [\omega])$ for a Kähler class $[\omega]$ on M provides an obstruction to the existence of a higher cscK metric in $[\omega]$.

Theorem (Bando)

If ω is a higher cscK metric then $\mathcal{F}_n(\cdot, [\omega]) \equiv 0$ on $\mathfrak{h}(M)$.

The converse holds true if there exists a higher extremal Kähler metric in the given Kähler class.

Theorem (S.)

Let ω be a higher extremal Kähler metric on M. Then ω is higher cscK if and only if $\mathcal{F}_n(\cdot, [\omega]) \equiv 0$ on $\mathfrak{h}(M)$.

Corollary (S.)

Let ω be a higher cscK metric on M. Then every higher extremal Kähler metric in the Kähler class $[\omega]$ is higher cscK.

Non-Existence of Higher cscK Metrics on $X = \mathbb{P}(L \oplus \mathcal{O})$

Since there exists a higher extremal Kähler metric which is not higher cscK in each Kähler class on $X = \mathbb{P}(L \oplus \mathcal{O})$, and since in the Kähler class of a higher cscK metric every higher extremal Kähler representative has to be higher cscK - from these two things we can conclude the non-existence of any kind of higher cscK metrics on X.

Corollary (S.)

For all $a, b > 0$ there does not exist a higher cscK metric in the Kähler class $aC + bS_\infty$ on X, i.e. there do not exist any higher cscK metrics on X.

Reference Books for Complex Differential Geometry and Kähler Geometry

- Complex Geometry: An Introduction - Daniel Huybrechts
- An Introduction to Extremal Kähler Metrics - Gábor Székelyhidi
- Complex Analytic and Differential Geometry - Jean-Pierre Demailly
- Compact Complex Surfaces - Wolf P. Barth, Klaus Hulek, Chris A. M. Peters and Antonius Van de Ven

Relevant Research Papers

- Extremal Kähler Metrics and Extremal Kähler Metrics II - Eugenio Calabi
- Extremal Kähler Metrics on Minimal Ruled Surfaces - Christina W. Tønnesen-Friedman
- A Momentum Construction for Circle-Invariant Kähler Metrics - Andrew D. Hwang and Michael A. Singer
- A Note on Higher Extremal Metrics - Vamsi Pritham Pingali
- An Obstruction to the Existence of Einstein Kähler Metrics - Akito Futaki
- An Obstruction for Chern Class Forms to be Harmonic - Shigetoshi Bando
- Existence of Higher Extremal Kähler Metrics on a Minimal Ruled Surface - Rajas Sandeep Sompurkar

Thank You For
Your Kind Attention!