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Topological and Homological Tools for the Study of Spaces from Infinitely Far Away

From Coarse Geometry to Cohomology and Stone Spaces

Coarse cohomology of a space X is an invariant that, through homological tools, captures geometric properties of X that can only be observed from infinitely afar, focusing solely on its asymptotic behaviour. In special cases [1], it can be computed using standard cohomology theories, such as singular or Alexander-Spanier cohomology.

The aim of this course is twofold: on the one hand, to motivate and provide an illustrative introduction to coarse geometry; on the other hand, to progress toward presenting some currently open research problems related to the computation of large scale invariants, particularly through the framework of topological bornological spaces.

Basic prior knowledge of homological algebra, general topology, and singular homology is assumed. The course will be held over five sessions, which will tentatively cover the following material:

1. Motivation. Groups as metric spaces

We review, through examples, the foundational philosophy of Geometric Group Theory, where each group G with a finite generating set S is associated with a metric space (G, d_S), translating algebraic properties of the group into geometric features of the space. Although this assignment initially depends on the choice of S, such dependence vanishes when viewing (G, d_S) "from infinitely far away". This motivates the introduction of coarse structures. Unlike the metric structure, the coarse structure does not depend on the choice of generating set, making the assignment

$$G \ \text{ finitely generated group } \longmapsto (G, \xi) \ \text{ coarse space}$$

a well-defined functor.

2. Coarse cohomology and the coarse category. Basic properties.

The coarse cohomology $HX^*(X)$ of a space X is introduced as a functor from the coarse category, and its coarse invariance is discussed. The main reference for this is [6]. We then particularize to metric spaces and study a few examples.

3. Ideals of contractible DGAs. The Roe product in coarse cohomology

We refine the structure of $HX^*(X)$, enhancing it to an algebra by endowing it with a product. Much like the cup product [3, §3.2] allows for a finer distinction between topological spaces, it is natural to seek products in coarse cohomology that sharpen the information it provides. A first strategy is to adapt the cup product to this context, but the resulting product in coarse cohomology turns out to be trivial. However, not all is lost: in such cases, one can define Massey triple products. In this session, we will introduce a closely related idea: the Roe product in coarse cohomology [5, §2.4]. This procedure can be abstracted to an algebraic setting, whose potential will later be exploited again.

4. Topological bornological spaces and their cohomology.

The category **TopBorn** of topological bornological spaces is defined, as in [2, §7.1.1]. We propose an invariant HB^* on these spaces, which we call bornological cohomology, and show that it can be determined from the Alexander–Spanier cohomology at infinity. We compute several examples and enrich HB^* with a product, endowing it with the structure of a differential graded algebra.

5. Low degree cohomology, extension spaces and boundaries at infinity. Open problems.

It is well known that the degree-zero singular homology of a topological space is determined by its path-connected components. Similarly, degree-zero Alexander–Spanier cohomology can be computed from its connected components. We ask whether, for a topological bornological space X, there exists a similar connection between $HB^1(X)$ and the "components at infinity of X". This question naturally leads us to give meaning to notions such as rays in X, (path-)connected components at infinity, and ultimately to consider compactifications of X. After addressing this question, we generalize the tools involved by studying certain extensions of X whose boundaries at infinity are Stone spaces [4], which can therefore be treated dually as topological spaces or as Boolean algebras (this is recent work in progress). We will conclude the course by reviewing some questions that remain open.

Referencias

[1] Arka Banerjee. *On the computation of Coarse Cohomology.* arXiv:2401.02231.

[2] Ulrich Bunke and Alexander Engel. *Homotopy Theory with Bornological Coarse Spaces.* Springer New York, NY, 1 edition, 2020.

[3] Allen Hatcher. *Algebraic Topology.* Cambridge University Press, 1 edition, 2001.

[4] Peter T. Johnstone. *Stone Spaces.* Cambridge University Press, 1982.

[5] John Roe. *Coarse Cohomology and Index Theory on Complete Riemannian Manifolds.* American Mathematical Society, 1993.

[6] John Roe. *Lectures on Coarse Geometry.* American Mathematical Society, 2003.