# Topology and Geometry Seminar - IIIT Delhi

(Organizers: Aritra Bhowmick, Sachchidanand Prasad, Sandip Samanta)

Existence of Higher Extremal Kähler Metrics on a Minimal Ruled Surface - Talk 2

**Overview of Some Concepts Related to Holomorphic Line Bundles (Prerequisites)**


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# Holomorphic Fibre Bundle

Let $B$, $E$, $F$ be complex manifolds and $\pi : E \to B$ be a surjective holomorphic mapping. Let $\{U_\alpha\}_{\alpha \in \Lambda}$ be an open covering of $B$ such that for each $\alpha \in \Lambda$ there exists a biholomorphism $\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F \subseteq B \times F$ further satisfying $\pi_B|_{U_\alpha \times F} \circ \varphi_\alpha = \pi|_{\pi^{-1}(U_\alpha)}$ where $\pi_B : B \times F \to B$ is the standard projection. Furthermore for any $\alpha, \beta \in \Lambda$ if $U_\alpha \cap U_\beta \neq \emptyset$ then let $\varphi_\beta \circ \varphi_\alpha^{-1} : (U_\alpha \cap U_\beta) \times F \to (U_\alpha \cap U_\beta) \times F$ be a holomorphic function which then at each point $p \in U_\alpha \cap U_\beta$ will induce a biholomorphism $\varphi_{\beta\alpha}(p) : F \to F$ given by $\varphi_\beta \circ \varphi_\alpha^{-1}(p, x) = (p, \varphi_{\beta\alpha}(p)(x))$ for all $x \in F$ such that the $F$-automorphism-valued functions $\varphi_{\beta\alpha}$ will be holomorphic and will satisfy the compatibility condition $\varphi_{\gamma\beta} \circ \varphi_{\beta\alpha} = \varphi_{\gamma\alpha}$ for all $\alpha, \beta, \gamma \in \Lambda$ whenever $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$. Then the entire data mentioned above is called as a *holomorphic fibre bundle*.

# Holomorphic Vector Bundle

A *holomorphic vector bundle* is a holomorphic fibre bundle in which the typical fibre $F$ is the complex $r$-space $\mathbb{C}^r$, and additionally the induced functions $\varphi_{\beta\alpha}(p) : \mathbb{C}^r \to \mathbb{C}^r$ are $\mathbb{C}$-linear isomorphisms, and hence $\varphi_{\beta\alpha}$ are matrix-valued functions with the compatibility condition being given in terms of matrix multiplication. The non-negative integer $r$ is called as the *rank* of the holomorphic vector bundle. A holomorphic vector bundle of rank 1 is called as a *holomorphic line bundle*.

**Examples:** The holomorphic tangent and cotangent bundles of a complex manifold, the canonical and anticanonical bundles of a complex manifold.

# Complex Vector Bundle

A *complex vector bundle* is on the contrary a smooth vector bundle with the same data as that of a holomorphic vector bundle, except that the local trivializations and the transition maps are just smooth functions (instead of being holomorphic), but the induced functions $\varphi_{\beta\alpha}(p) : \mathbb{C}^r \to \mathbb{C}^r$ are still required to be $\mathbb{C}$-linear isomorphisms.

**Examples:** The complexified tangent and cotangent bundles of a complex manifold.

# Hermitian Metric on a Holomorphic Vector Bundle

A *Hermitian metric* on a holomorphic vector bundle is a smoothly varying family of Hermitian (complex sesquilinear) inner products on each fibre of the bundle.

There exists a unique torsion-free affine connection $\nabla$ on a Hermitian holomorphic vector bundle which is compatible with the complex structure as well as with the Hermitian metric, called as *the Chern connection*. This is a canonical connection on a Hermitian holomorphic vector bundle just like the Levi-Civita connection on a Riemannian manifold.

The *curvature* of a Hermitian holomorphic vector bundle is then defined to be the extent to which the covariant derivatives given in terms of the Chern connection fail to commute, i.e. the *curvature tensor* is given by:

$$F_{k\bar{l}} = \nabla_k \nabla_{\bar{l}} - \nabla_{\bar{l}} \nabla_k$$

# Hermitian Metric on a Complex Manifold

A Hermitian metric $g$ on a complex manifold $M$ by definition provides for a smoothly varying family of Hermitian inner products on each holomorphic tangent space of the manifold, i.e. it naturally endows the holomorphic tangent bundle of the manifold with a Hermitian metric.

So associated to $g$ we have the unique Chern connection on the holomorphic tangent bundle $T^{(1,0)}M$. But since $g$ is a Riemannian metric on the underlying smooth manifold of $M$, there exists the unique Levi-Civita connection on the real tangent bundle $TM$ afforded by Riemannian geometry. The question that arises is: When do the two canonical connections coincide?

**Answer:** The Chern connection and the Levi-Civita connection coincide if and only if the Hermitian metric $g$ satisfies the Kähler condition.

# Ricci Form, First Chern Form and First Chern Class

Let $M$ be a compact Kähler $n$-manifold and $\omega$ be a Kähler metric on $M$.
The *Ricci form* of $\omega$ is defined as:

$$\mathrm{Ric}\,(\omega) = -\sqrt{-1}\partial\bar{\partial}\,\ln\det\,(\omega)$$

where $\det\,(\omega) = \det H\,(\omega)$, $H\,(\omega)$ being the Hermitian matrix of $\omega$. $\mathrm{Ric}\,(\omega)$ is a closed real $(1,1)$-form on $M$. If $\eta$ is any other Kähler metric on $M$ then it can be checked that:

$$[\mathrm{Ric}\,(\eta)] = [\mathrm{Ric}\,(\omega)] \in H^{(1,1)}\,(M,\mathbb{R}) \subseteq H^2\,(M,\mathbb{R})$$

The *first Chern form* of $\omega$ and the *first Chern class* of $M$ (which is as a result independent of the choice of $\omega$) are defined as:

$$c_1\,(\omega) = \frac{1}{2\pi}\,\mathrm{Ric}\,(\omega)\,, \quad c_1\,(M) = \frac{1}{2\pi}\,[\mathrm{Ric}\,(\omega)]$$

# Kähler-Einstein and cscK Metrics

## Definition (Kähler-Einstein Metric)

The Kähler metric $\omega$ is said to be a *Kähler-Einstein metric* on $M$ if $\mathrm{Ric}\,(\omega) = \lambda\omega$ for some constant $\lambda \in \mathbb{R}$.

The constant $\lambda = \lambda\,(\omega)$ which appears above is called as the *Ricci curvature* of the Kähler-Einstein metric $\omega$.
The *scalar curvature* of $\omega$, denoted by $S\,(\omega) : M \to \mathbb{R}$, is a smooth function given by the following formula:

$$n\,\mathrm{Ric}\,(\omega) \wedge \omega^{n-1} = S\,(\omega)\,\omega^{n}$$

## Definition (cscK Metric)

The Kähler metric $\omega$ is said to be a *constant scalar curvature Kähler (cscK) metric* on $M$ if $S\,(\omega)$ is a constant function on $M$.

We clearly have the following implication:

$$\omega \text{ is Kähler-Einstein} \implies \omega \text{ is cscK}$$

# Extremal Kähler Metric

## Definition (Calabi Functional; Calabi)

Let $\Omega \in H^{(1,1)}(M, \mathbb{R})$ be a Kähler class and $\Omega^+$ denote the set of all Kähler metrics in $\Omega$. The Calabi functional on $\Omega^+$ is defined as:

$$\mathrm{Cal} : \Omega^+ \to \mathbb{R}, \ \ \mathrm{Cal}(\omega) = \int_M S(\omega)^2 \omega^n, \ \ \omega \in \Omega^+$$

## Definition (Extremal Kähler Metric; Calabi)

$\omega \in \Omega^+$ is said to be an *extremal Kähler metric* if $\omega$ is a critical point of Cal on $\Omega^+$.

## Theorem (The Euler-Lagrange Equation for an Extremal Kähler Metric; Calabi)

*$\omega$ is an extremal Kähler metric on $M$ if and only if $\nabla^{1,0} S(\omega) = \left(\bar{\partial} S(\omega)\right)^{\sharp}$ is a real holomorphic vector field on $M$.*

We clearly have the following implication:

$$\omega \text{ is cscK} \implies \omega \text{ is extremal Kähler}$$

# Higher Chern Forms and Higher Chern Classes

Consider the invariant homogeneous polynomials $P_k$ of degree $k$ with $1 \leq k \leq n$ in the following expansion:

$$\det(I + A) = 1 + \sum_{k=1}^{n} P_k(A)$$

Let $\omega$ be a Kähler metric on $M$, $H(\omega)$ be the Hermitian matrix of $\omega$ and $\Theta(\omega) = \bar{\partial}\left(H^{-1}\partial H\right)(\omega)$ be the *curvature form matrix* of $\omega$. The $k^{th}$ *Chern form* of $\omega$ is defined as:

$$c_k(\omega) = P_k\left(\frac{\sqrt{-1}}{2\pi}\Theta(\omega)\right)$$

$c_k(\omega)$ is a closed real $(k,k)$-form on $M$. The $k^{th}$ *Chern class* of $M$ (which can again be verified to be independent of the choice of $\omega$) is defined as:

$$c_k(M) = [c_k(\omega)] \in H^{(k,k)}(M, \mathbb{R}) \subseteq H^{2k}(M, \mathbb{R})$$

# Higher cscK and Higher Extremal Kähler Metrics

### Definition (Higher cscK Metric; Pingali)

A Kähler metric $\omega$ on $M$ is said to be *higher constant scalar curvature Kähler (higher cscK)* if $c_n(\omega) = \frac{\lambda}{n!(2\pi)^n}\omega^n$ for some constant $\lambda \in \mathbb{R}$.

### Definition (Higher Extremal Kähler Metric; Pingali)

A Kähler metric $\omega$ on $M$ is said to be *higher extremal Kähler* if $c_n(\omega) = \frac{\lambda}{n!(2\pi)^n}\omega^n$ for some smooth function $\lambda : M \to \mathbb{R}$ such that $\nabla^{1,0}\lambda = \left(\bar{\partial}\lambda\right)^\sharp$ is a real holomorphic vector field on $M$.

We again have the following implication:

$$\omega \text{ is higher cscK} \implies \omega \text{ is higher extremal Kähler}$$

The smooth real-valued function $\lambda = \lambda(\omega)$ which appears in the above 2 definitions can be dubbed by analogy as the *"higher scalar curvature"* of $\omega$.

# "Canonical" Kähler Metrics on Compact Kähler Manifolds

Let $M$ be a compact Kähler manifold. We have the following 3 well-known and well-studied notions of "canonical" Kähler metrics in a given fixed Kähler class $\Omega$ on $M$:

$$\{\text{Kähler-Einstein Metrics}\} \subseteq \{\text{cscK Metrics}\}$$
$$\{\text{cscK Metrics}\} \subseteq \{\text{Extremal Kähler Metrics}\}$$

The definitions of these 3 notions are related to the first Chern class $c_1(M) \in H^{(1,1)}(M, \mathbb{R}) \subseteq H^2(M, \mathbb{R})$.
Taking the analogy of these to the level of the top Chern class $c_n(M) \in H^{(n,n)}(M, \mathbb{R}) = H^{2n}(M, \mathbb{R})$, Pingali introduced the following 2 new notions of canonical Kähler metrics in the Kähler class $\Omega$:

$$\{\text{Higher cscK Metrics}\} \subseteq \{\text{Higher Extremal Kähler Metrics}\}$$

# Reference Books for Complex Differential Geometry and Kähler Geometry

- Complex Geometry: An Introduction - Daniel Huybrechts
- An Introduction to Extremal Kähler Metrics - Gábor Székelyhidi
- Complex Analytic and Differential Geometry - Jean-Pierre Demailly

Thank You For
Your Kind Attention!