

# Pointed Sets

The Clowder Project Authors

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This chapter contains some foundational material on pointed sets.

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# 1 Pointed Sets

## 1.1 Foundations

### DEFINITION 1.1.1 ► POINTED SETS

A **pointed set**<sup>1</sup> is equivalently:

- An  $\mathbb{E}_0$ -monoid in  $(\mathbf{N}_\bullet(\mathbf{Sets}), \text{pt})$ .
- A pointed object in  $(\mathbf{Sets}, \text{pt})$ .

<sup>1</sup>*Further Terminology:* In the context of monoids with zero as models for  $\mathbb{F}_1$ -algebras, pointed sets are viewed as  $\mathbb{F}_1$ -**modules**.

### REMARK 1.1.2 ► UNWINDING DEFINITION 1.1.1

In detail, a **pointed set** is a pair  $(X, x_0)$  consisting of:

- *The Underlying Set.* A set  $X$ , called the **underlying set of**  $(X, x_0)$ .
- *The Basepoint.* A morphism

$$[x_0] : \text{pt} \rightarrow X$$

in  $\mathbf{Sets}$ , determining an element  $x_0 \in X$ , called the **basepoint of**  $X$ .

### EXAMPLE 1.1.3 ► THE ZERO SPHERE

The **0-sphere**<sup>1</sup> is the pointed set  $(S^0, 0)$ <sup>2</sup> consisting of:

- *The Underlying Set.* The set  $S^0$  defined by

$$S^0 \stackrel{\text{def}}{=} \{0, 1\}.$$

- *The Basepoint.* The element 0 of  $S^0$ .

<sup>1</sup>*Further Terminology:* In the context of monoids with zero as models for  $\mathbb{F}_1$ -algebras, the 0-sphere is viewed as the **underlying pointed set of the field with one element**.

<sup>2</sup>*Further Notation:* In the context of monoids with zero as models for  $\mathbb{F}_1$ -algebras,  $S^0$  is also denoted  $(\mathbb{F}_1, 0)$ .

**EXAMPLE 1.1.4 ► THE TRIVIAL POINTED SET**

The **trivial pointed set** is the pointed set  $(\text{pt}, \star)$  consisting of:

- *The Underlying Set.* The punctual set  $\text{pt} \stackrel{\text{def}}{=} \{\star\}$ .
- *The Basepoint.* The element  $\star$  of  $\text{pt}$ .

**EXAMPLE 1.1.5 ► THE UNDERLYING POINTED SET OF A SEMIMODULE**

The **underlying pointed set** of a semimodule  $(M, \alpha_M)$  is the pointed set  $(M, 0_M)$ .

**EXAMPLE 1.1.6 ► THE UNDERLYING POINTED SET OF A MODULE**

The **underlying pointed set** of a module  $(M, \alpha_M)$  is the pointed set  $(M, 0_M)$ .

**1.2 Morphisms of Pointed Sets****DEFINITION 1.2.1 ► MORPHISMS OF POINTED SETS**

A **morphism of pointed sets**<sup>1,2</sup> is equivalently:

- A morphism of  $\mathbb{B}_0$ -monoids in  $(N_\bullet(\text{Sets}), \text{pt})$ .
- A morphism of pointed objects in  $(\text{Sets}, \text{pt})$ .

<sup>1</sup>Further Terminology: Also called a **pointed function**.

<sup>2</sup>Further Terminology: In the context of monoids with zero as models for  $\mathbb{F}_1$ -algebras, morphisms of pointed sets are also called **morphism of  $\mathbb{F}_1$ -modules**.

**REMARK 1.2.2 ► UNWINDING DEFINITION 1.2.1**

In detail, a **morphism of pointed sets**  $f: (X, x_0) \rightarrow (Y, y_0)$  is a morphism of sets  $f: X \rightarrow Y$  such that the diagram

$$\begin{array}{ccc} & \text{pt} & \\ [x_0] \swarrow & & \searrow [y_0] \\ X & \xrightarrow{f} & Y \end{array}$$

commutes, i.e. such that

$$f(x_0) = y_0.$$

### 1.3 The Category of Pointed Sets

#### DEFINITION 1.3.1 ► THE CATEGORY OF POINTED SETS

The **category of pointed sets** is the category  $\mathbf{Sets}_*$  defined equivalently as

- The homotopy category of the  $\infty$ -category  $\mathbf{Mon}_{\mathbb{B}_0}(\mathbf{N}_\bullet(\mathbf{Sets}), \text{pt})$  of ??, ??;
- The category  $\mathbf{Sets}_*$  of ??, ??.

#### REMARK 1.3.2 ► UNWINDING DEFINITION 1.3.1

In detail, the **category of pointed sets** is the category  $\mathbf{Sets}_*$  where

- *Objects.* The objects of  $\mathbf{Sets}_*$  are pointed sets;
- *Morphisms.* The morphisms of  $\mathbf{Sets}_*$  are morphisms of pointed sets;
- *Identities.* For each  $(X, x_0) \in \text{Obj}(\mathbf{Sets}_*)$ , the unit map

$$\mathbb{1}_{(X, x_0)}^{\mathbf{Sets}_*} : \text{pt} \rightarrow \mathbf{Sets}_*((X, x_0), (X, x_0))$$

of  $\mathbf{Sets}_*$  at  $(X, x_0)$  is defined by<sup>1</sup>

$$\text{id}_{(X, x_0)}^{\mathbf{Sets}_*} \stackrel{\text{def}}{=} \text{id}_X;$$

- *Composition.* For each  $(X, x_0), (Y, y_0), (Z, z_0) \in \text{Obj}(\mathbf{Sets}_*)$ , the composition map

$$\circ_{(X, x_0), (Y, y_0), (Z, z_0)}^{\mathbf{Sets}_*} : \mathbf{Sets}_*((Y, y_0), (Z, z_0)) \times \mathbf{Sets}_*((X, x_0), (Y, y_0)) \rightarrow \mathbf{Sets}_*((X, x_0), (Z, z_0))$$

of  $\mathbf{Sets}_*$  at  $((X, x_0), (Y, y_0), (Z, z_0))$  is defined by<sup>2</sup>

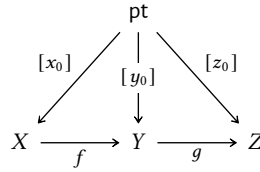
$$g \circ_{(X, x_0), (Y, y_0), (Z, z_0)}^{\mathbf{Sets}_*} f \stackrel{\text{def}}{=} g \circ f.$$

<sup>1</sup>Note that  $\text{id}_X$  is indeed a morphism of pointed sets, as we have  $\text{id}_X(x_0) = x_0$ .

<sup>2</sup>Note that the composition of two morphisms of pointed sets is indeed a morphism of pointed sets, as we have

$$\begin{aligned} g(f(x_0)) &= g(y_0) \\ &= z_0, \end{aligned}$$

or



in terms of diagrams.

## 1.4 Elementary Properties of Pointed Sets

### PROPOSITION 1.4.1 ► ELEMENTARY PROPERTIES OF POINTED SETS

Let  $(X, x_0)$  be a pointed set.

1. *Completeness.* The category  $\mathbf{Sets}_*$  of pointed sets and morphisms between them is complete, having in particular:
  - (a) Products, described as in [Definition 2.3.1](#);
  - (b) Pullbacks, described as in [Definition 2.4.1](#);
  - (c) Equalisers, described as in [Definition 2.5.1](#).
2. *Cocompleteness.* The category  $\mathbf{Sets}_*$  of pointed sets and morphisms between them is cocomplete, having in particular:
  - (a) Coproducts, described as in [Definition 3.3.1](#);
  - (b) Pushouts, described as in [Definition 3.4.1](#);
  - (c) Coequalisers, described as in [Definition 3.5.1](#).
3. *Failure To Be Cartesian Closed.* The category  $\mathbf{Sets}_*$  is not Cartesian closed.<sup>1</sup>
4. *Morphisms From the Monoidal Unit.* We have a bijection of sets<sup>2</sup>

$$\mathbf{Sets}_*(S^0, X) \cong X,$$

natural in  $(X, x_0) \in \text{Obj}(\text{Sets}_*)$ , internalising also to an isomorphism of pointed sets

$$\text{Sets}_*(S^0, X) \cong (X, x_0),$$

again natural in  $(X, x_0) \in \text{Obj}(\text{Sets}_*)$ .

5. *Relation to Partial Functions.* We have an equivalence of categories<sup>3</sup>

$$\text{Sets}_* \xrightarrow{\cong} \text{Sets}^{\text{part.}}$$

between the category of pointed sets and pointed functions between them and the category of sets and partial functions between them, where:

(a) *From Pointed Sets to Sets With Partial Functions.* The equivalence

$$\xi: \text{Sets}_* \xrightarrow{\cong} \text{Sets}^{\text{part.}}$$

sends:

- i. A pointed set  $(X, x_0)$  to  $X$ .
- ii. A pointed function

$$f: (X, x_0) \rightarrow (Y, y_0)$$

to the partial function

$$\xi_f: X \rightarrow Y$$

defined on  $f^{-1}(Y \setminus y_0)$  and given by

$$\xi_f(x) \stackrel{\text{def}}{=} f(x)$$

for each  $x \in f^{-1}(Y \setminus y_0)$ .

(b) *From Sets With Partial Functions to Pointed Sets.* The equivalence

$$\xi^{-1}: \text{Sets}^{\text{part.}} \xrightarrow{\cong} \text{Sets}_*$$

sends:

- i. A set  $X$  is to the pointed set  $(X, \star)$  with  $\star$  an element that is not in  $X$ .

ii. A partial function

$$f: X \rightarrow Y$$

defined on  $U \subset X$  to the pointed function

$$\xi_f^{-1}: (X, x_0) \rightarrow (Y, y_0)$$

defined by

$$\xi_f(x) \stackrel{\text{def}}{=} \begin{cases} f(x) & \text{if } x \in U, \\ y_0 & \text{otherwise.} \end{cases}$$


for each  $x \in X$ .

<sup>1</sup>The category  $\mathbf{Sets}_*$  does admit monoidal closed structures however; see [Tensor Products of Pointed Sets](#).

<sup>2</sup>In other words, the forgetful functor

$$\omega: \mathbf{Sets}_* \rightarrow \mathbf{Sets}$$

defined on objects by sending a pointed set to its underlying set is corepresentable by  $S^0$ .

<sup>3</sup> **Warning:** This is not an isomorphism of categories, only an equivalence.

#### PROOF 1.4.2 ► PROOF OF PROPOSITION 1.4.1

##### Item 1: Completeness

This follows from (the proofs) of [Definitions 2.3.1](#), [2.4.1](#) and [2.5.1](#) and [??](#), [??](#).

##### Item 2: Cocompleteness

This follows from (the proofs) of [Definitions 3.3.1](#), [3.4.1](#) and [3.5.1](#) and [??](#), [??](#).

##### Item 3: Failure To Be Cartesian Closed

See [\[MSE 2855868\]](#).

##### Item 4: Morphisms From the Monoidal Unit

Since a morphism from  $S^0$  to a pointed set  $(X, x_0)$  sends  $0 \in S^0$  to  $x_0$  and then can send  $1 \in S^0$  to any element of  $X$ , we obtain a bijection between pointed maps  $S^0 \rightarrow X$  and the elements of  $X$ .

The isomorphism then

$$\mathbf{Sets}_*(S^0, X) \cong (X, x_0)$$

follows by noting that  $\Delta_{x_0} : S^0 \rightarrow X$ , the basepoint of  $\mathbf{Sets}_*(S^0, X)$ , corresponds to the pointed map  $S^0 \rightarrow X$  picking the element  $x_0$  of  $X$ , and thus we see that the bijection between pointed maps  $S^0 \rightarrow X$  and elements of  $X$  is compatible with basepoints, lifting to an isomorphism of pointed sets.

Item 5: Relation to Partial Functions

See [MSE 884460].



## 2 Limits of Pointed Sets

### 2.1 The Terminal Pointed Set

#### DEFINITION 2.1.1 ► THE TERMINAL POINTED SET

The **terminal pointed set** is the pair  $((\text{pt}, \star), \{!_X\}_{(X, x_0) \in \text{Obj}(\mathbf{Sets}_*)})$  consisting of:

- *The Limit.* The pointed set  $(\text{pt}, \star)$ .
- *The Cone.* The collection of morphisms of pointed sets

$$\{!_X : (X, x_0) \rightarrow (\text{pt}, \star)\}_{(X, x_0) \in \text{Obj}(\mathbf{Sets})}$$

defined by

$$!_X(x) \stackrel{\text{def}}{=} \star$$

for each  $x \in X$  and each  $(X, x_0) \in \text{Obj}(\mathbf{Sets})$ .

#### PROOF 2.1.2 ► PROOF OF DEFINITION 2.1.1

We claim that  $(\text{pt}, \star)$  is the terminal object of  $\mathbf{Sets}_*$ . Indeed, suppose we have a diagram of the form

$$(X, x_0) \quad (\text{pt}, \star)$$

in  $\mathbf{Sets}_*$ . Then there exists a unique morphism of pointed sets

$$\phi : (X, x_0) \rightarrow (\text{pt}, \star)$$

making the diagram

$$(X, x_0) \xrightarrow[\exists!]{\phi} (\text{pt}, \star)$$



commute, namely  $!_X$ .



## 2.2 Products of Families of Pointed Sets

Let  $\{(X_i, x_0^i)\}_{i \in I}$  be a family of pointed sets.

### DEFINITION 2.2.1 ► THE PRODUCT OF A FAMILY OF POINTED SETS

The **product** of  $\{(X_i, x_0^i)\}_{i \in I}$  is the pair  $((\prod_{i \in I} X_i, (x_0^i)_{i \in I}), \{\text{pr}_i\}_{i \in I})$  consisting of:

- *The Limit.* The pointed set  $(\prod_{i \in I} X_i, (x_0^i)_{i \in I})$ .
- *The Cone.* The collection

$$\left\{ \text{pr}_i : \left( \prod_{i \in I} X_i, (x_0^i)_{i \in I} \right) \rightarrow (X_i, x_0^i) \right\}_{i \in I}$$

of maps given by

$$\text{pr}_i((x_j)_{j \in I}) \stackrel{\text{def}}{=} x_i$$

for each  $(x_j)_{j \in I} \in \prod_{i \in I} X_i$  and each  $i \in I$ .

### PROOF 2.2.2 ► PROOF OF DEFINITION 2.2.1

We claim that  $(\prod_{i \in I} X_i, (x_0^i)_{i \in I})$  is the categorical product of  $\{(X_i, x_0^i)\}_{i \in I}$  in  $\text{Sets}_*$ . Indeed, suppose we have, for each  $i \in I$ , a diagram of the form

$$\begin{array}{ccc} (P, *) & & \\ & \searrow p_i & \\ (\prod_{i \in I} X_i, (x_0^i)_{i \in I}) & \xrightarrow{\text{pr}_i} & (X_i, x_0^i) \end{array}$$

in  $\text{Sets}_*$ . Then there exists a unique morphism of pointed sets

$$\phi : (P, *) \rightarrow \left( \prod_{i \in I} X_i, (x_0^i)_{i \in I} \right)$$

making the diagram

$$\begin{array}{ccc}
 (P, *) & & \\
 \downarrow \phi \exists! & \searrow p_i & \\
 (\prod_{i \in I} X_i, (x_0^i)_{i \in I}) & \xrightarrow{\text{pr}_i} & (X_i, x_0^i)
 \end{array}$$

commute, being uniquely determined by the condition  $\text{pr}_i \circ \phi = p_i$  for each  $i \in I$  via

$$\phi(x) = (p_i(x))_{i \in I}$$

for each  $x \in P$ . Note that this is indeed a morphism of pointed sets, as we have

$$\begin{aligned}
 \phi(*) &= (p_i(*))_{i \in I} \\
 &= (x_0^i)_{i \in I},
 \end{aligned}$$

where we have used that  $p_i$  is a morphism of pointed sets for each  $i \in I$ . 

#### PROPOSITION 2.2.3 ► PROPERTIES OF PRODUCTS OF FAMILIES OF POINTED SETS

Let  $\{(X_i, x_0^i)\}_{i \in I}$  be a family of pointed sets.

1. *Functoriality.* The assignment  $\{(X_i, x_0^i)\}_{i \in I} \mapsto (\prod_{i \in I} X_i, (x_0^i)_{i \in I})$  defines a functor

$$\prod_{i \in I}: \text{Fun}(I_{\text{disc}}, \text{Sets}_*) \rightarrow \text{Sets}_*.$$

#### PROOF 2.2.4 ► PROOF OF PROPOSITION 2.2.3

Item 1: Functoriality

This follows from ??, ?? of ??.



## 2.3 Products

Let  $(X, x_0)$  and  $(Y, y_0)$  be pointed sets.

**DEFINITION 2.3.1** ► PRODUCTS OF POINTED SETS

The **product** of  $(X, x_0)$  and  $(Y, y_0)$  is the pair consisting of:

- *The Limit.* The pointed set  $(X \times Y, (x_0, y_0))$ .
- *The Cone.* The morphisms of pointed sets

$$\begin{aligned} \text{pr}_1 &: (X \times Y, (x_0, y_0)) \rightarrow (X, x_0), \\ \text{pr}_2 &: (X \times Y, (x_0, y_0)) \rightarrow (Y, y_0) \end{aligned}$$

defined by

$$\begin{aligned} \text{pr}_1(x, y) &\stackrel{\text{def}}{=} x, \\ \text{pr}_2(x, y) &\stackrel{\text{def}}{=} y \end{aligned}$$

for each  $(x, y) \in X \times Y$ .

**PROOF 2.3.2** ► PROOF OF DEFINITION 2.3.1

We claim that  $(X \times Y, (x_0, y_0))$  is the categorical product of  $(X, x_0)$  and  $(Y, y_0)$  in  $\mathbf{Sets}_*$ . Indeed, suppose we have a diagram of the form

$$\begin{array}{ccccc} & & (P, *) & & \\ & \swarrow p_1 & & \searrow p_2 & \\ (X, x_0) & \xleftarrow{\text{pr}_1} & (X \times Y, (x_0, y_0)) & \xrightarrow{\text{pr}_2} & (Y, y_0) \end{array}$$

in  $\mathbf{Sets}_*$ . Then there exists a unique morphism of pointed sets

$$\phi: (P, *) \rightarrow (X \times Y, (x_0, y_0))$$

making the diagram

$$\begin{array}{ccccc} & & (P, *) & & \\ & \swarrow p_1 & \downarrow \phi \mid \exists! & \searrow p_2 & \\ (X, x_0) & \xleftarrow{\text{pr}_1} & (X \times Y, (x_0, y_0)) & \xrightarrow{\text{pr}_2} & (Y, y_0) \end{array}$$

commute, being uniquely determined by the conditions

$$\text{pr}_1 \circ \phi = p_1,$$


$$\text{pr}_2 \circ \phi = p_2$$

via

$$\phi(x) = (p_1(x), p_2(x))$$

for each  $x \in P$ . Note that this is indeed a morphism of pointed sets, as we have

$$\begin{aligned} \phi(*) &= (p_1(*), p_2(*)) \\ &= (x_0, y_0), \end{aligned}$$

where we have used that  $p_1$  and  $p_2$  are morphisms of pointed sets. 

### PROPOSITION 2.3.3 ► PROPERTIES OF PRODUCTS OF POINTED SETS

Let  $(X, x_0)$ ,  $(Y, y_0)$ , and  $(Z, z_0)$  be pointed sets.

1. *Functoriality.* The assignments

$$(X, x_0), (Y, y_0), ((X, x_0), (Y, y_0)) \mapsto (X \times Y, (x_0, y_0))$$

define functors

$$X \times -: \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*,$$

$$- \times Y: \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*,$$

$$-_1 \times -_2: \mathbf{Sets}_* \times \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*,$$

defined in the same way as the functors of [Constructions With Sets, Item 1](#) of [Proposition 1.3.3](#).

2. *Associativity.* We have an isomorphism of pointed sets

$$((X \times Y) \times Z, ((x_0, y_0), z_0)) \cong (X \times (Y \times Z), (x_0, (y_0, z_0)))$$

natural in  $(X, x_0), (Y, y_0), (Z, z_0) \in \text{Obj}(\mathbf{Sets}_*)$ .

3. *Unitality*. We have isomorphisms of pointed sets

$$(\text{pt}, \star) \times (X, x_0) \cong (X, x_0),$$

$$(X, x_0) \times (\text{pt}, \star) \cong (X, x_0),$$

natural in  $(X, x_0) \in \text{Obj}(\text{Sets}_*)$ .

4. *Commutativity*. We have an isomorphism of pointed sets

$$(X \times Y, (x_0, y_0)) \cong (Y \times X, (y_0, x_0)),$$

natural in  $(X, x_0), (Y, y_0) \in \text{Obj}(\text{Sets}_*)$ .

5. *Symmetric Monoidality*. The triple  $(\text{Sets}_*, \times, (\text{pt}, \star))$  is a symmetric monoidal category.

#### PROOF 2.3.4 ► PROOF OF PROPOSITION 2.3.3

##### Item 1: Functoriality

This is a special case of functoriality of limits, ??, ?? of ??.

##### Item 2: Associativity

This follows from **Constructions With Sets**, Item 3 of Proposition 1.3.3.

##### Item 3: Unitality

This follows from **Constructions With Sets**, Item 4 of Proposition 1.3.3.

##### Item 4: Commutativity

This follows from **Constructions With Sets**, Item 5 of Proposition 1.3.3.

##### Item 5: Symmetric Monoidality

This follows from **Constructions With Sets**, Item 12 of Proposition 1.3.3. 

## 2.4 Pullbacks

Let  $(X, x_0)$ ,  $(Y, y_0)$ , and  $(Z, z_0)$  be pointed sets and let  $f: (X, x_0) \rightarrow (Z, z_0)$  and  $g: (Y, y_0) \rightarrow (Z, z_0)$  be morphisms of pointed sets.

**DEFINITION 2.4.1 ► PULLBACKS OF POINTED SETS**

The **pullback of**  $(X, x_0)$  **and**  $(Y, y_0)$  **over**  $(Z, z_0)$  **along**  $(f, g)$  is the pair consisting of:

- *The Limit.* The pointed set  $(X \times_Z Y, (x_0, y_0))$ .
- *The Cone.* The morphisms of pointed sets

$$\text{pr}_1: (X \times_Z Y, (x_0, y_0)) \rightarrow (X, x_0),$$

$$\text{pr}_2: (X \times_Z Y, (x_0, y_0)) \rightarrow (Y, y_0)$$

defined by

$$\text{pr}_1(x, y) \stackrel{\text{def}}{=} x,$$

$$\text{pr}_2(x, y) \stackrel{\text{def}}{=} y$$

for each  $(x, y) \in X \times_Z Y$ .

**PROOF 2.4.2 ► PROOF OF DEFINITION 2.4.1**

We claim that  $X \times_Z Y$  is the categorical pullback of  $(X, x_0)$  and  $(Y, y_0)$  over  $(Z, z_0)$  with respect to  $(f, g)$  in  $\text{Sets}_*$ . First we need to check that the relevant pullback diagram commutes, i.e. that we have

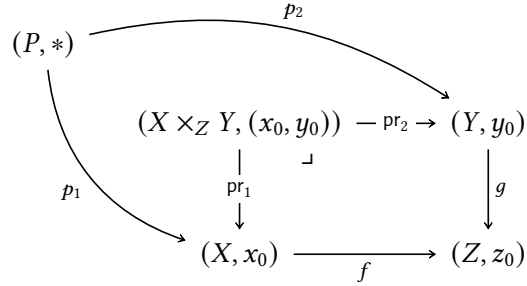
$$\begin{array}{ccc} (X \times_Z Y, (x_0, y_0)) & \xrightarrow{\text{pr}_2} & (Y, y_0) \\ \text{pr}_1 \downarrow & & \downarrow g \\ (X, x_0) & \xrightarrow{f} & (Z, z_0). \end{array}$$

$f \circ \text{pr}_1 = g \circ \text{pr}_2,$

Indeed, given  $(x, y) \in X \times_Z Y$ , we have

$$\begin{aligned} [f \circ \text{pr}_1](x, y) &= f(\text{pr}_1(x, y)) \\ &= f(x) \\ &= g(y) \\ &= g(\text{pr}_2(x, y)) \\ &= [g \circ \text{pr}_2](x, y), \end{aligned}$$

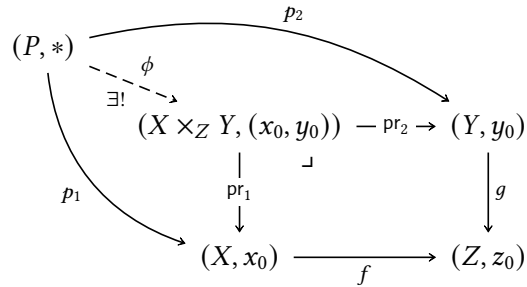
where  $f(x) = g(y)$  since  $(x, y) \in X \times_Z Y$ . Next, we prove that  $X \times_Z Y$  satisfies the universal property of the pullback. Suppose we have a diagram of the form



in  $\mathbf{Sets}_*$ . Then there exists a unique morphism of pointed sets

$$\phi: (P, *) \rightarrow (X \times_Z Y, (x_0, y_0))$$

making the diagram



commute, being uniquely determined by the conditions

$$\text{pr}_1 \circ \phi = p_1,$$

$$\text{pr}_2 \circ \phi = p_2$$

via

$$\phi(x) = (p_1(x), p_2(x))$$

for each  $x \in P$ , where we note that  $(p_1(x), p_2(x)) \in X \times Y$  indeed lies in  $X \times_Z Y$  by the condition


$$f \circ p_1 = g \circ p_2,$$

which gives

$$f(p_1(x)) = g(p_2(x))$$

for each  $x \in P$ , so that  $(p_1(x), p_2(x)) \in X \times_Z Y$ . Lastly, we note that  $\phi$  is indeed a morphism of pointed sets, as we have

$$\begin{aligned}\phi(*) &= (p_1(*), p_2(*)) \\ &= (x_0, y_0),\end{aligned}$$

where we have used that  $p_1$  and  $p_2$  are morphisms of pointed sets. 

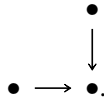
### PROPOSITION 2.4.3 ► PROPERTIES OF PULLBACKS OF POINTED SETS

Let  $(X, x_0)$ ,  $(Y, y_0)$ ,  $(Z, z_0)$ , and  $(A, a_0)$  be pointed sets.

1. *Functoriality.* The assignment  $(X, Y, Z, f, g) \mapsto X \times_{f, Z, g} Y$  defines a functor

$$-_1 \times_{-3} -_1 : \text{Fun}(\mathcal{P}, \text{Sets}_*) \rightarrow \text{Sets}_*,$$

where  $\mathcal{P}$  is the category that looks like this:



In particular, the action on morphisms of  $-_1 \times_{-3} -_1$  is given by sending a morphism

$$\begin{array}{ccccc} X \times_Z Y & \xrightarrow{\quad} & Y & & \\ \downarrow & \lrcorner & \downarrow g & \searrow \psi & \\ & X' \times_{Z'} Y' & \xrightarrow{\quad} & Y' & \\ \downarrow & \lrcorner & \downarrow & & \downarrow g' \\ X & \xrightarrow{f} & Z & \searrow \chi & \\ \downarrow \phi & & \downarrow & & \downarrow \\ & X' & \xrightarrow{f'} & Z' & \end{array}$$



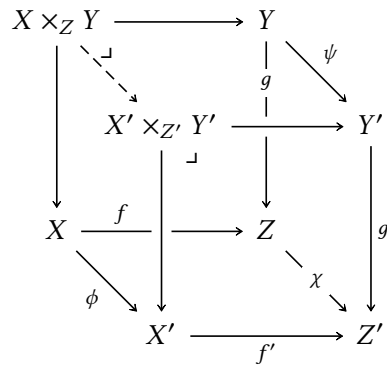
in  $\text{Fun}(\mathcal{P}, \text{Sets}_*)$  to the morphism of pointed sets

$$\xi: (X \times_Z Y, (x_0, y_0)) \xrightarrow{\exists!} (X' \times_{Z'} Y', (x'_0, y'_0))$$

given by

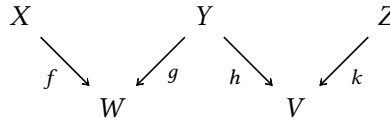
$$\xi(x, y) \stackrel{\text{def}}{=} (\phi(x), \psi(y))$$

for each  $(x, y) \in X \times_Z Y$ , which is the unique morphism of pointed sets making the diagram



commute.

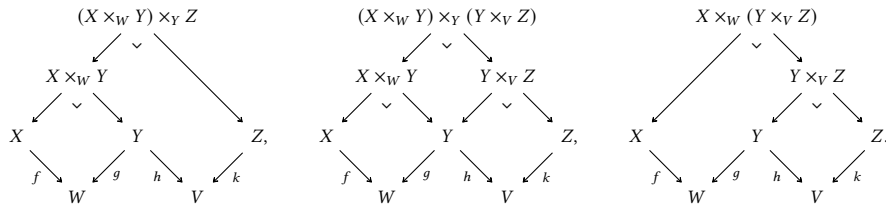
2. *Associativity.* Given a diagram



in  $\text{Sets}_*$ , we have isomorphisms of pointed sets

$$(X \times_W Y) \times_V Z \cong (X \times_W Y) \times_Y (Y \times_V Z) \cong X \times_W (Y \times_V Z),$$

where these pullbacks are built as in the diagrams



3. *Unitality.* We have isomorphisms of pointed sets

$$\begin{array}{ccc}
 A & \xlongequal{\quad} & A \\
 f \downarrow & \lrcorner & \downarrow f \\
 X & \xlongequal{\quad} & X
 \end{array}
 \quad
 \begin{array}{l}
 X \times_X A \cong A, \\
 A \times_X X \cong A,
 \end{array}
 \quad
 \begin{array}{ccc}
 A & \xrightarrow{f} & X \\
 \parallel \downarrow & \lrcorner & \downarrow \parallel \\
 X & \xrightarrow{f} & X.
 \end{array}$$

4. *Commutativity.* We have an isomorphism of pointed sets

$$\begin{array}{ccc}
 A \times_X B & \longrightarrow & B \\
 \downarrow & \lrcorner & \downarrow g \\
 A & \xrightarrow{f} & X,
 \end{array}
 \quad
 A \times_X B \cong B \times_X A
 \quad
 \begin{array}{ccc}
 B \times_X A & \longrightarrow & A \\
 \downarrow & \lrcorner & \downarrow f \\
 B & \xrightarrow{g} & X.
 \end{array}$$

5. *Interaction With Products.* We have an isomorphism of pointed sets

$$\begin{array}{ccc}
 X \times Y & \longrightarrow & Y \\
 \downarrow & \lrcorner & \downarrow !_Y \\
 X & \xrightarrow{!_X} & \text{pt.}
 \end{array}
 \quad
 X \times_{\text{pt}} Y \cong X \times Y,$$

6. *Symmetric Monoidality.* The triple  $(\text{Sets}_*, \times_X, X)$  is a symmetric monoidal category.

#### PROOF 2.4.4 ► PROOF OF PROPOSITION 2.4.3

##### Item 1: Functoriality

This is a special case of functoriality of co/limits, ??, ?? of ??, with the explicit expression for  $\xi$  following from the commutativity of the cube pullback diagram.

##### Item 2: Associativity

This follows from [Constructions With Sets](#), [Item 2 of Proposition 1.4.5](#).

## Item 3: Unitality

This follows from [Constructions With Sets, Item 3 of Proposition 1.4.5](#).

## Item 4: Commutativity

This follows from [Constructions With Sets, Item 4 of Proposition 1.4.5](#).

## Item 5: Interaction With Products

This follows from [Constructions With Sets, Item 6 of Proposition 1.4.5](#).

## Item 6: Symmetric Monoidality

This follows from [Constructions With Sets, Item 7 of Proposition 1.4.5](#). 

## 2.5 Equalisers

Let  $f, g: (X, x_0) \rightrightarrows (Y, y_0)$  be morphisms of pointed sets.

### DEFINITION 2.5.1 ► EQUALISERS OF POINTED SETS

The **equaliser** of  $(f, g)$  is the pair consisting of:

- *The Limit.* The pointed set  $(\text{Eq}(f, g), x_0)$ .
- *The Cone.* The morphism of pointed sets

$$\text{eq}(f, g): (\text{Eq}(f, g), x_0) \hookrightarrow (X, x_0)$$

given by the canonical inclusion  $\text{eq}(f, g) \hookrightarrow \text{Eq}(f, g) \hookrightarrow X$ .

### PROOF 2.5.2 ► PROOF OF DEFINITION 2.5.1

We claim that  $(\text{Eq}(f, g), x_0)$  is the categorical equaliser of  $f$  and  $g$  in  $\text{Sets}_*$ . First we need to check that the relevant equaliser diagram commutes, i.e. that we have

$$f \circ \text{eq}(f, g) = g \circ \text{eq}(f, g),$$

which indeed holds by the definition of the set  $\text{Eq}(f, g)$ . Next, we prove that  $\text{Eq}(f, g)$  satisfies the universal property of the equaliser. Suppose we have a dia-

gram of the form

$$\begin{array}{ccccc} (\text{Eq}(f, g), x_0) & \xrightarrow{\text{eq}(f, g)} & (X, x_0) & \xrightleftharpoons[g]{f} & (Y, y_0) \\ & \nearrow e & & & \\ (E, *) & & & & \end{array}$$

in  $\text{Sets}_*$ . Then there exists a unique morphism of pointed sets

$$\phi: (E, *) \rightarrow (\text{Eq}(f, g), x_0)$$

making the diagram

$$\begin{array}{ccccc} (\text{Eq}(f, g), x_0) & \xrightarrow{\text{eq}(f, g)} & (X, x_0) & \xrightleftharpoons[g]{f} & (Y, y_0) \\ \uparrow \phi \exists! & \nearrow e & & & \\ (E, *) & & & & \end{array}$$

commute, being uniquely determined by the condition

$$\text{eq}(f, g) \circ \phi = e$$

via

$$\phi(x) = e(x)$$

for each  $x \in E$ , where we note that  $e(x) \in \text{Eq}(f, g)$  indeed lies in  $\text{Eq}(f, g)$  by the condition


$$f \circ e = g \circ e,$$

which gives

$$f(e(x)) = g(e(x))$$

for each  $x \in E$ , so that  $e(x) \in \text{Eq}(f, g)$ . Lastly, we note that  $\phi$  is indeed a morphism of pointed sets, as we have

$$\begin{aligned} \phi(*) &= e(*) \\ &= x_0, \end{aligned}$$

where we have used that  $e$  is a morphism of pointed sets. 

### PROPOSITION 2.5.3 ► PROPERTIES OF EQUALISERS OF POINTED SETS

Let  $(X, x_0)$  and  $(Y, y_0)$  be pointed sets and let  $f, g, h: (X, x_0) \rightarrow (Y, y_0)$  be morphisms of pointed sets.

1. *Associativity.* We have isomorphisms of pointed sets

$$\underbrace{\text{Eq}(f \circ \text{eq}(g, h), g \circ \text{eq}(g, h))}_{=\text{Eq}(f \circ \text{eq}(g, h), h \circ \text{eq}(g, h))} \cong \text{Eq}(f, g, h) \cong \underbrace{\text{Eq}(f \circ \text{eq}(f, g), h \circ \text{eq}(f, g))}_{=\text{Eq}(g \circ \text{eq}(f, g), h \circ \text{eq}(f, g))},$$

where  $\text{Eq}(f, g, h)$  is the limit of the diagram

$$(X, x_0) \begin{array}{c} \xrightarrow{f} \\ \xrightarrow[g]{h} \end{array} (Y, y_0)$$

in  $\text{Sets}_*$ , being explicitly given by

$$\text{Eq}(f, g, h) \cong \{a \in A \mid f(a) = g(a) = h(a)\}.$$

2. *Unitality.* We have an isomorphism of pointed sets

$$\text{Eq}(f, f) \cong X.$$

3. *Commutativity.* We have an isomorphism of pointed sets

$$\text{Eq}(f, g) \cong \text{Eq}(g, f).$$

### PROOF 2.5.4 ► PROOF OF PROPOSITION 2.5.3

#### Item 1: Associativity

This follows from **Constructions With Sets, Item 1** of **Proposition 1.5.3**.

#### Item 2: Unitality

This follows from **Constructions With Sets, Item 2** of **Proposition 1.5.3**.

#### Item 3: Commutativity

This follows from **Constructions With Sets, Item 3** of **Proposition 1.5.3**. 

## 3 Colimits of Pointed Sets

### 3.1 The Initial Pointed Set

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|

The **initial pointed set** is the pair  $((\text{pt}, \star), \{\iota_X\}_{(X, x_0) \in \text{Obj}(\text{Sets}_*)})$  consisting of:

- *The Limit.* The pointed set  $(\text{pt}, \star)$ .
- *The Cone.* The collection of morphisms of pointed sets

$$\{\iota_X : (\text{pt}, \star) \rightarrow (X, x_0)\}_{(X, x_0) \in \text{Obj}(\text{Sets})}$$

defined by

$$\iota_X(\star) \stackrel{\text{def}}{=} x_0.$$

#### PROOF 3.1.2 ► PROOF OF DEFINITION 3.1.1

We claim that  $(\text{pt}, \star)$  is the initial object of  $\text{Sets}_*$ . Indeed, suppose we have a diagram of the form

$$(\text{pt}, \star) \quad (X, x_0)$$

in  $\text{Sets}_*$ . Then there exists a unique morphism of pointed sets

$$\phi : (\text{pt}, \star) \rightarrow (X, x_0)$$

making the diagram

$$(\text{pt}, \star) \xrightarrow[\exists!]{\phi} (X, x_0)$$

commute, namely  $\iota_X$ .



### 3.2 Coproducts of Families of Pointed Sets

Let  $\{(X_i, x_0^i)\}_{i \in I}$  be a family of pointed sets.

#### DEFINITION 3.2.1 ► COPRODUCTS OF FAMILIES OF POINTED SETS

The **coproduct of the family**  $\{(X_i, x_0^i)\}_{i \in I}$ , also called their **wedge sum**, is the pair consisting of:

- *The Colimit.* The pointed set  $(\bigvee_{i \in I} X_i, p_0)$  consisting of:

– *The Underlying Set.* The set  $\bigvee_{i \in I} X_i$  defined by

$$\bigvee_{i \in I} X_i \stackrel{\text{def}}{=} \left( \coprod_{i \in I} X_i \right) / \sim,$$

where  $\sim$  is the equivalence relation on  $\coprod_{i \in I} X_i$  given by declaring

$$(i, x_0^i) \sim (j, x_0^j)$$

for each  $i, j \in I$ .

– *The Basepoint.* The element  $p_0$  of  $\bigvee_{i \in I} X_i$  defined by

$$\begin{aligned} p_0 &\stackrel{\text{def}}{=} [(i, x_0^i)] \\ &= [(j, x_0^j)] \end{aligned}$$

for any  $i, j \in I$ .

• *The Cocone.* The collection

$$\left\{ \text{inj}_i : (X_i, x_0^i) \rightarrow \left( \bigvee_{i \in I} X_i, p_0 \right) \right\}_{i \in I}$$

of morphism of pointed sets given by

$$\text{inj}_i(x) \stackrel{\text{def}}{=} (i, x)$$

for each  $x \in X_i$  and each  $i \in I$ .



**PROOF 3.2.2 ► PROOF OF DEFINITION 3.2.1**

We claim that  $(\bigvee_{i \in I} X_i, p_0)$  is the categorical coproduct of  $\{(X_i, x_0^i)\}_{i \in I}$  in  $\mathbf{Sets}_*$ . Indeed, suppose we have, for each  $i \in I$ , a diagram of the form

$$\begin{array}{ccc} & & (C, *) \\ & \nearrow \iota_i & \\ (X_i, x_0^i) & \xrightarrow{\text{inj}_i} & (\bigvee_{i \in I} X_i, p_0) \end{array}$$

in  $\mathbf{Sets}_*$ . Then there exists a unique morphism of pointed sets

$$\phi: (\bigvee_{i \in I} X_i, p_0) \rightarrow (C, *)$$

making the diagram


$$\begin{array}{ccc} & & (C, *) \\ & \nearrow \iota_i & \uparrow \exists! \phi \\ (X_i, x_0^i) & \xrightarrow{\text{inj}_i} & (\bigvee_{i \in I} X_i, p_0) \end{array}$$

commute, being uniquely determined by the condition  $\phi \circ \text{inj}_i = \iota_i$  for each  $i \in I$  via

$$\phi([(i, x)]) = \iota_i(x)$$

for each  $[(i, x)] \in \bigvee_{i \in I} X_i$ , where we note that  $\phi$  is indeed a morphism of pointed sets, as we have

$$\begin{aligned} \phi(p_0) &= \iota_i([(i, x_0^i)]) \\ &= *, \end{aligned}$$

as  $\iota_i$  is a morphism of pointed sets. 

**PROPOSITION 3.2.3 ► PROPERTIES OF COPRODUCTS OF FAMILIES OF POINTED SETS**

Let  $\{(X_i, x_0^i)\}_{i \in I}$  be a family of pointed sets.

1. *Functoriality.* The assignment  $\{(X_i, x_0^i)\}_{i \in I} \mapsto (\bigvee_{i \in I} X_i, p_0)$  defines a functor

$$\bigvee_{i \in I} : \text{Fun}(I_{\text{disc}}, \text{Sets}_*) \rightarrow \text{Sets}_*.$$

**PROOF 3.2.4 ► PROOF OF PROPOSITION 3.2.3**

Item 1: Functoriality

This follows from ??, ?? of ??.

**3.3 Coproducts**

Let  $(X, x_0)$  and  $(Y, y_0)$  be pointed sets.

**DEFINITION 3.3.1 ► COPRODUCTS OF POINTED SETS**

The **coproduct** of  $(X, x_0)$  and  $(Y, y_0)$ , also called their **wedge sum**, is the pair consisting of:

- *The Colimit.* The pointed set  $(X \vee Y, p_0)$  consisting of:

- *The Underlying Set.* The set  $X \vee Y$  defined by

$$\begin{aligned} (X \vee Y, p_0) &\stackrel{\text{def}}{=} (X, x_0) \amalg (Y, y_0) \\ &\cong (X \amalg_{\text{pt}} Y, p_0) \\ &\cong (X \amalg Y / \sim, p_0), \end{aligned} \quad \begin{array}{ccc} X & \vee & Y \\ \uparrow & \ulcorner & \uparrow \\ X & & Y \\ \downarrow & \llcorner & \downarrow \\ X & \xleftarrow{[x_0]} & \text{pt} \end{array}$$

where  $\sim$  is the equivalence relation on  $X \amalg Y$  obtained by declaring  $(0, x_0) \sim (1, y_0)$ .

- *The Basepoint.* The element  $p_0$  of  $X \vee Y$  defined by

$$\begin{aligned} p_0 &\stackrel{\text{def}}{=} [(0, x_0)] \\ &= [(1, y_0)]. \end{aligned}$$

· *The Cocone.* The morphisms of pointed sets

$$\text{inj}_1 : (X, x_0) \rightarrow (X \vee Y, p_0),$$

$$\text{inj}_2 : (Y, y_0) \rightarrow (X \vee Y, p_0),$$

given by

$$\text{inj}_1(x) \stackrel{\text{def}}{=} [(0, x)],$$

$$\text{inj}_2(y) \stackrel{\text{def}}{=} [(1, y)],$$

for each  $x \in X$  and each  $y \in Y$ .

#### PROOF 3.3.2 ► PROOF OF DEFINITION 3.3.1

We claim that  $(X \vee Y, p_0)$  is the categorical coproduct of  $(X, x_0)$  and  $(Y, y_0)$  in  $\text{Sets}_*$ . Indeed, suppose we have a diagram of the form

$$\begin{array}{ccccc} & & (C, *) & & \\ & \nearrow \iota_X & & \nwarrow \iota_Y & \\ (X, x_0) & \xrightarrow{\text{inj}_X} & (X \vee Y, p_0) & \xleftarrow{\text{inj}_Y} & (Y, y_0) \end{array}$$

in  $\text{Sets}$ . Then there exists a unique morphism of pointed sets

$$\phi : (X \vee Y, p_0) \rightarrow (C, *)$$

making the diagram

$$\begin{array}{ccccc} & & (C, *) & & \\ & \nearrow \iota_X & \uparrow \phi \mid \exists! & \nwarrow \iota_Y & \\ (X, x_0) & \xrightarrow{\text{inj}_X} & (X \vee Y, p_0) & \xleftarrow{\text{inj}_Y} & (Y, y_0) \end{array}$$

commute, being uniquely determined by the conditions

$$\phi \circ \text{inj}_X = \iota_X,$$


$$\phi \circ \text{inj}_Y = \iota_Y$$

via

$$\phi(z) = \begin{cases} \iota_X(x) & \text{if } z = [(0, x)] \text{ with } x \in X, \\ \iota_Y(y) & \text{if } z = [(1, y)] \text{ with } y \in Y \end{cases}$$

for each  $z \in X \vee Y$ , where we note that  $\phi$  is indeed a morphism of pointed sets, as we have

$$\begin{aligned} \phi(p_0) &= \iota_X([(0, x_0)]) \\ &= \iota_Y([(1, y_0)]) \\ &= *, \end{aligned}$$

as  $\iota_X$  and  $\iota_Y$  are morphisms of pointed sets. 

### PROPOSITION 3.3.3 ► PROPERTIES OF WEDGE SUMS OF POINTED SETS

Let  $(X, x_0)$  and  $(Y, y_0)$  be pointed sets.

1. *Functoriality.* The assignments

$$(X, x_0), (Y, y_0), ((X, x_0), (Y, y_0)) \mapsto (X \vee Y, p_0)$$

define functors

$$\begin{aligned} X \vee - &: \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*, \\ - \vee Y &: \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*, \\ -_1 \vee -_2 &: \mathbf{Sets}_* \times \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*. \end{aligned}$$

2. *Associativity.* We have an isomorphism of pointed sets

$$(X \vee Y) \vee Z \cong X \vee (Y \vee Z),$$

natural in  $(X, x_0), (Y, y_0), (Z, z_0) \in \mathbf{Sets}_*$ .

3. *Unitality.* We have isomorphisms of pointed sets

$$\begin{aligned} (\mathbf{pt}, *) \vee (X, x_0) &\cong (X, x_0), \\ (X, x_0) \vee (\mathbf{pt}, *) &\cong (X, x_0), \end{aligned}$$

natural in  $(X, x_0) \in \mathbf{Sets}_*$ .

4. *Commutativity.* We have an isomorphism of pointed sets

$$X \vee Y \cong Y \vee X,$$

natural in  $(X, x_0), (Y, y_0) \in \mathbf{Sets}_*$ .

5. *Symmetric Monoidality.* The triple  $(\mathbf{Sets}_*, \vee, \text{pt})$  is a symmetric monoidal category.

6. *The Fold Map.* We have a natural transformation

$$\nabla: \vee \circ \Delta_{\mathbf{Sets}_*}^{\mathbf{Cats}} \Rightarrow \text{id}_{\mathbf{Sets}_*},$$

called the **fold map**, whose component

$$\nabla_X: X \vee X \rightarrow X$$

at  $X$  is given by

$$\nabla_X(p) \stackrel{\text{def}}{=} \begin{cases} x & \text{if } p = [(0, x)], \\ x & \text{if } p = [(1, x)] \end{cases}$$

for each  $p \in X \vee X$ .

#### PROOF 3.3.4 ► PROOF OF PROPOSITION 3.3.3

Item 1: Functoriality

This follows from ??, ?? of ??.

Item 2: Associativity

Clear.

Item 3: Unitality

Clear.

## Item 4: Commutativity

Clear.

## Item 5: Symmetric Monoidality

Omitted.

## Item 6: The Fold Map

Naturality for the transformation  $\nabla$  is the statement that, given a morphism of pointed sets  $f: (X, x_0) \rightarrow (Y, y_0)$ , we have

$$\nabla_Y \circ (f \vee f) = f \circ \nabla_X,$$

$$\begin{array}{ccc} X \vee X & \xrightarrow{\nabla_X} & X \\ f \vee f \downarrow & & \downarrow f \\ Y \vee Y & \xrightarrow{\nabla_Y} & Y. \end{array}$$

Indeed, we have

$$\begin{aligned} [\nabla_Y \circ (f \vee f)]([i, x]) &= \nabla_Y([i, f(x)]) \\ &= f(x) \\ &= f(\nabla_X([i, x])) \\ &= [f \circ \nabla_X]([i, x]) \end{aligned}$$

for each  $[i, x] \in X \vee X$ , and thus  $\nabla$  is indeed a natural transformation. 

### 3.4 Pushouts

Let  $(X, x_0)$ ,  $(Y, y_0)$ , and  $(Z, z_0)$  be pointed sets and let  $f: (Z, z_0) \rightarrow (X, x_0)$  and  $g: (Z, z_0) \rightarrow (Y, y_0)$  be morphisms of pointed sets.

#### DEFINITION 3.4.1 ► PUSHOUTS OF POINTED SETS

The **pushout of  $(X, x_0)$  and  $(Y, y_0)$  over  $(Z, z_0)$  along  $(f, g)$**  is the pair consisting of:

- *The Colimit.* The pointed set  $(X \coprod_{f, Z, g} Y, p_0)$ , where:
  - The set  $X \coprod_{f, Z, g} Y$  is the pushout (of unpointed sets) of  $X$  and  $Y$  over

$Z$  with respect to  $f$  and  $g$ ;

– We have  $p_0 = [x_0] = [y_0]$ .

• *The Cocone.* The morphisms of pointed sets

$$\text{inj}_1: (X, x_0) \rightarrow (X \amalg_Z Y, p_0),$$

$$\text{inj}_2: (Y, y_0) \rightarrow (X \amalg_Z Y, p_0)$$

given by

$$\text{inj}_1(x) \stackrel{\text{def}}{=} [(0, x)]$$

$$\text{inj}_2(y) \stackrel{\text{def}}{=} [(1, y)]$$

for each  $x \in X$  and each  $y \in Y$ .

#### PROOF 3.4.2 ► PROOF OF DEFINITION 3.4.1

Firstly, we note that indeed  $[x_0] = [y_0]$ , as we have

$$x_0 = f(z_0),$$

$$y_0 = g(z_0)$$

since  $f$  and  $g$  are morphisms of pointed sets, with the relation  $\sim$  on  $X \amalg_Z Y$  then identifying  $x_0 = f(z_0) \sim g(z_0) = y_0$ .

We now claim that  $(X \amalg_Z Y, p_0)$  is the categorical pushout of  $(X, x_0)$  and  $(Y, y_0)$  over  $(Z, z_0)$  with respect to  $(f, g)$  in  $\mathbf{Sets}_*$ . First we need to check that the relevant pushout diagram commutes, i.e. that we have

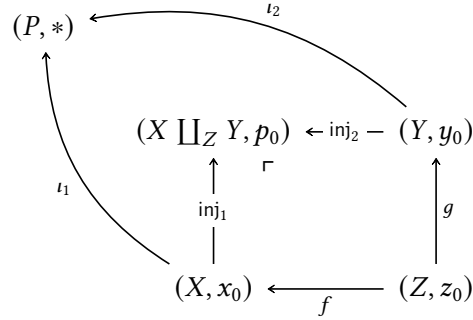
$$\text{inj}_1 \circ f = \text{inj}_2 \circ g,$$

$$\begin{array}{ccc} (X \amalg_Z Y, p_0) & \xleftarrow{\text{inj}_2} & (Y, y_0) \\ \text{inj}_1 \uparrow & & \uparrow g \\ (X, x_0) & \xleftarrow{f} & (Z, z_0). \end{array}$$

Indeed, given  $z \in Z$ , we have

$$\begin{aligned}
 [\text{inj}_1 \circ f](z) &= \text{inj}_1(f(z)) \\
 &= [(0, f(z))] \\
 &= [(1, g(z))] \\
 &= \text{inj}_2(g(z)) \\
 &= [\text{inj}_2 \circ g](z),
 \end{aligned}$$

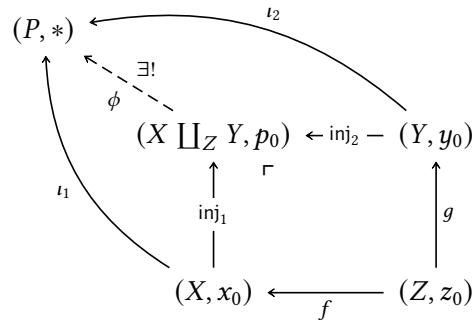
where  $[(0, f(z))] = [(1, g(z))]$  by the definition of the relation  $\sim$  on  $X \amalg Y$  (the coproduct of unpointed sets of  $X$  and  $Y$ ). Next, we prove that  $X \amalg_Z Y$  satisfies the universal property of the pushout. Suppose we have a diagram of the form



in  $\mathbf{Sets}_*$ . Then there exists a unique morphism of pointed sets

$$\phi: (X \amalg_Z Y, p_0) \rightarrow (P, *)$$

making the diagram





commute, being uniquely determined by the conditions

$$\begin{aligned}\phi \circ \text{inj}_1 &= \iota_1, \\ \phi \circ \text{inj}_2 &= \iota_2\end{aligned}$$

via


$$\phi(p) = \begin{cases} \iota_1(x) & \text{if } x = [(0, x)], \\ \iota_2(y) & \text{if } x = [(1, y)] \end{cases}$$

for each  $p \in X \amalg_Z Y$ , where the well-definedness of  $\phi$  is proven in the same way as in the proof of **Constructions With Sets, Definition 2.4.1**. Finally, we show that  $\phi$  is indeed a morphism of pointed sets, as we have

$$\begin{aligned}\phi(p_0) &= \phi([(0, x_0)]) \\ &= \iota_1(x_0) \\ &= *,\end{aligned}$$

or alternatively

$$\begin{aligned}\phi(p_0) &= \phi([(1, y_0)]) \\ &= \iota_2(y_0) \\ &= *,\end{aligned}$$

where we use that  $\iota_1$  (resp.  $\iota_2$ ) is a morphism of pointed sets. 

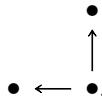
#### PROPOSITION 3.4.3 ► PROPERTIES OF PUSHOUTS OF POINTED SETS

Let  $(X, x_0)$ ,  $(Y, y_0)$ ,  $(Z, z_0)$ , and  $(A, a_0)$  be pointed sets.

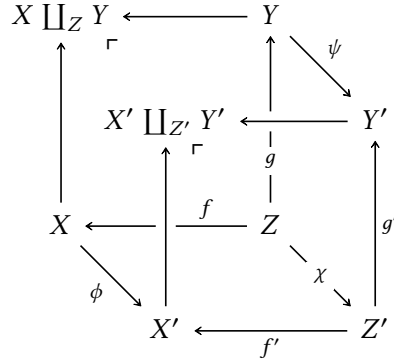
1. *Functoriality.* The assignment  $(X, Y, Z, f, g) \mapsto X \amalg_{f, Z, g} Y$  defines a functor

$$-_1 \amalg_{-3} -_1: \text{Fun}(\mathcal{P}, \text{Sets}) \rightarrow \text{Sets}_*,$$

where  $\mathcal{P}$  is the category that looks like this:



In particular, the action on morphisms of  $-1 \amalg_{-3} -1$  is given by sending a morphism



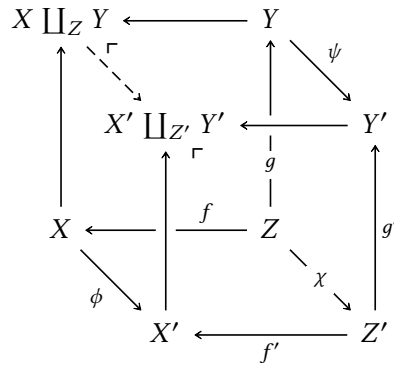
in  $\text{Fun}(\mathcal{P}, \text{Sets}_*)$  to the morphism of pointed sets

$$\xi: (X \amalg_Z Y, p_0) \xrightarrow{\exists!} (X' \amalg_{Z'} Y', p'_0)$$

given by

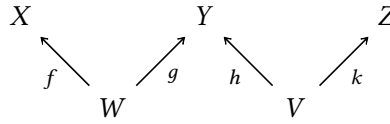
$$\xi(p) \stackrel{\text{def}}{=} \begin{cases} \phi(x) & \text{if } p = [(0, x)], \\ \psi(y) & \text{if } p = [(1, y)] \end{cases}$$

for each  $p \in X \amalg_Z Y$ , which is the unique morphism of pointed sets making the diagram



commute.

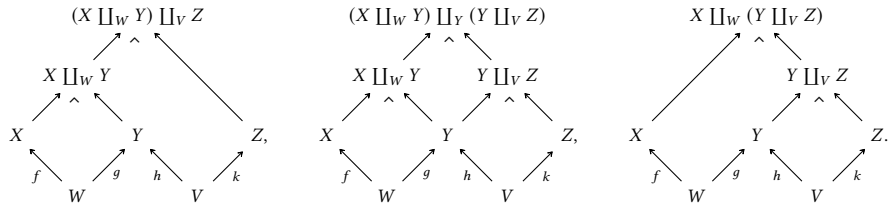
2. *Associativity.* Given a diagram



in Sets, we have isomorphisms of pointed sets

$$(X \amalg_W Y) \amalg_V Z \cong (X \amalg_W Y) \amalg_Y (Y \amalg_V Z) \cong X \amalg_W (Y \amalg_V Z),$$

where these pullbacks are built as in the diagrams



3. *Unitality.* We have isomorphisms of sets

$$\begin{array}{ccc}
 A & \xlongequal{\quad} & A \\
 \uparrow f & \lrcorner & \uparrow f \\
 X & \xlongequal{\quad} & X
 \end{array}
 \quad
 \begin{array}{l}
 X \amalg_X A \cong A, \\
 A \amalg_X X \cong A,
 \end{array}
 \quad
 \begin{array}{ccc}
 A & \xleftarrow{f} & X \\
 \parallel & \lrcorner & \parallel \\
 X & \xleftarrow{f} & X.
 \end{array}$$

4. *Commutativity.* We have an isomorphism of sets

$$\begin{array}{ccc}
 X \amalg_Z Y & \xleftarrow{\quad} & Y \\
 \uparrow \lrcorner & & \uparrow g \\
 X & \xleftarrow{f} & Z,
 \end{array}
 \quad
 X \amalg_Z Y \cong Y \amalg_Z X
 \quad
 \begin{array}{ccc}
 Y \amalg_Z X & \xleftarrow{\quad} & X \\
 \uparrow \lrcorner & & \uparrow f \\
 Y & \xleftarrow{g} & Z.
 \end{array}$$

5. *Interaction With Coproducts.* We have

$$X \amalg_{\text{pt}} Y \cong X \vee Y,$$

$$\begin{array}{ccc} X \vee Y & \longleftarrow & Y \\ \uparrow \ulcorner & & \uparrow [y_0] \\ X & \xleftarrow{[x_0]} & \text{pt.} \end{array}$$

6. *Symmetric Monoidality.* The triple  $(\text{Sets}_*, \amalg_X, (X, x_0))$  is a symmetric monoidal category.

#### PROOF 3.4.4 ► PROOF OF PROPOSITION 3.4.3

##### Item 1: Functoriality

This is a special case of functoriality of co/limits, ??, ?? of ??, with the explicit expression for  $\xi$  following from the commutativity of the cube pushout diagram.

##### Item 2: Associativity

This follows from [Constructions With Sets, Item 2 of Proposition 2.4.6](#).

##### Item 3: Unitality

This follows from [Constructions With Sets, Item 3 of Proposition 2.4.6](#).


##### Item 4: Commutativity

This follows from [Constructions With Sets, Item 4 of Proposition 2.4.6](#).

##### Item 5: Interaction With Coproducts

Clear.

##### Item 6: Symmetric Monoidality

Omitted. 

### 3.5 Coequalisers

Let  $f, g: (X, x_0) \rightrightarrows (Y, y_0)$  be morphisms of pointed sets.

#### DEFINITION 3.5.1 ► COEQUALISERS OF POINTED SETS

The **coequaliser of**  $(f, g)$  is the pointed set  $(\text{CoEq}(f, g), [y_0])$ .

## PROOF 3.5.2 ► PROOF OF DEFINITION 3.5.1

We claim that  $(\text{CoEq}(f, g), [y_0])$  is the categorical coequaliser of  $f$  and  $g$  in  $\text{Sets}_*$ . First we need to check that the relevant coequaliser diagram commutes, i.e. that we have

$$\text{coeq}(f, g) \circ f = \text{coeq}(f, g) \circ g.$$

Indeed, we have

$$\begin{aligned} [\text{coeq}(f, g) \circ f](x) &\stackrel{\text{def}}{=} [\text{coeq}(f, g)](f(x)) \\ &\stackrel{\text{def}}{=} [f(x)] \\ &= [g(x)] \\ &\stackrel{\text{def}}{=} [\text{coeq}(f, g)](g(x)) \\ &\stackrel{\text{def}}{=} [\text{coeq}(f, g) \circ g](x) \end{aligned}$$

for each  $x \in X$ . Next, we prove that  $\text{CoEq}(f, g)$  satisfies the universal property of the coequaliser. Suppose we have a diagram of the form


$$\begin{array}{ccc} (X, x_0) & \xrightarrow[g]{f} (Y, y_0) & \xrightarrow{\text{coeq}(f, g)} (\text{CoEq}(f, g), [y_0]) \\ & \searrow c & \\ & & (C, *) \end{array}$$

in  $\text{Sets}$ . Then, since  $c(f(a)) = c(g(a))$  for each  $a \in A$ , it follows from [Equivalence Relations and Apartness Relations, Items 4 and 5 of Proposition 5.2.3](#) that there exists a unique map  $\phi: \text{CoEq}(f, g) \xrightarrow{\exists!} C$  making the diagram

$$\begin{array}{ccc} (X, x_0) & \xrightarrow[g]{f} (Y, y_0) & \xrightarrow{\text{coeq}(f, g)} (\text{CoEq}(f, g), [y_0]) \\ & \searrow c & \downarrow \phi \mid \exists! \\ & & (C, *) \end{array}$$

commute, where we note that  $\phi$  is indeed a morphism of pointed sets since

$$\begin{aligned} \phi([y_0]) &= [\phi \circ \text{coeq}(f, g)]([y_0]) \\ &= c([y_0]) \\ &= *, \end{aligned}$$

where we have used that  $c$  is a morphism of pointed sets. 

### PROPOSITION 3.5.3 ► PROPERTIES OF COEQUALISERS OF POINTED SETS

Let  $(X, x_0)$  and  $(Y, y_0)$  be pointed sets and let  $f, g, h: (X, x_0) \rightarrow (Y, y_0)$  be morphisms of pointed sets.

1. *Associativity.* We have isomorphisms of pointed sets

$$\underbrace{\text{CoEq}(\text{coeq}(f, g) \circ f, \text{coeq}(f, g) \circ h)}_{=\text{CoEq}(\text{coeq}(f, g) \circ g, \text{coeq}(f, g) \circ h)} \cong \text{CoEq}(f, g, h) \cong \underbrace{\text{CoEq}(\text{coeq}(g, h) \circ f, \text{coeq}(g, h) \circ g)}_{=\text{CoEq}(\text{coeq}(g, h) \circ f, \text{coeq}(g, h) \circ h)},$$

where  $\text{CoEq}(f, g, h)$  is the colimit of the diagram

$$(X, x_0) \begin{array}{c} \xrightarrow{f} \\ \xrightarrow[g]{h} \end{array} (Y, y_0)$$

in  $\text{Sets}_*$ .

2. *Unitality.* We have an isomorphism of pointed sets

$$\text{CoEq}(f, f) \cong B.$$

3. *Commutativity.* We have an isomorphism of pointed sets

$$\text{CoEq}(f, g) \cong \text{CoEq}(g, f).$$

### PROOF 3.5.4 ► PROOF OF PROPOSITION 3.5.3

Item 1: Associativity

This follows from [Constructions With Sets](#), [Item 1 of Proposition 2.5.6](#).

Item 2: Unitality

This follows from [Constructions With Sets](#), [Item 2 of Proposition 2.5.6](#).

Item 3: Commutativity

This follows from [Constructions With Sets](#), [Item 3 of Proposition 2.5.6](#). 

## 4 Constructions With Pointed Sets

### 4.1 Free Pointed Sets

Let  $X$  be a set.

#### DEFINITION 4.1.1 ► FREE POINTED SETS

The **free pointed set on  $X$**  is the pointed set  $X^+$  consisting of:

- *The Underlying Set.* The set  $X^+$  defined by<sup>1</sup>

$$\begin{aligned} X^+ &\stackrel{\text{def}}{=} X \amalg \text{pt} \\ &\stackrel{\text{def}}{=} X \amalg \{\star\}. \end{aligned}$$

- *The Basepoint.* The element  $\star$  of  $X^+$ .

<sup>1</sup>*Further Notation:* We sometimes write  $\star_X$  for the basepoint of  $X^+$  for clarity when there are multiple free pointed sets involved in the current discussion.

#### PROPOSITION 4.1.2 ► PROPERTIES OF FREE POINTED SETS

Let  $X$  be a set.

1. *Functoriality.* The assignment  $X \mapsto X^+$  defines a functor

$$(-)^+ : \text{Sets} \rightarrow \text{Sets}_*,$$

where

- *Action on Objects.* For each  $X \in \text{Obj}(\text{Sets})$ , we have

$$[(-)^+](X) \stackrel{\text{def}}{=} X^+,$$

where  $X^+$  is the pointed set of [Definition 4.1.1](#);

- *Action on Morphisms.* For each morphism  $f : X \rightarrow Y$  of  $\text{Sets}$ , the image

$$f^+ : X^+ \rightarrow Y^+$$

of  $f$  by  $(-)^+$  is the map of pointed sets defined by

$$f^+(x) \stackrel{\text{def}}{=} \begin{cases} f(x) & \text{if } x \in X, \\ \star_Y & \text{if } x = \star_X. \end{cases}$$

2. *Adjointness.* We have an adjunction

$$((-)^+ \dashv \text{忘}): \text{Sets} \begin{array}{c} \xrightarrow{(-)^+} \\ \perp \\ \xleftarrow{\text{忘}} \end{array} \text{Sets}_*,$$

witnessed by a bijection of sets

$$\text{Sets}_*((X^+, \star_X), (Y, y_0)) \cong \text{Sets}(X, Y),$$

natural in  $X \in \text{Obj}(\text{Sets})$  and  $(Y, y_0) \in \text{Obj}(\text{Sets}_*)$ .

3. *Symmetric Strong Monoidality With Respect to Wedge Sums.* The free pointed set functor of **Item 1** has a symmetric strong monoidal structure

$$((-)^+, (-)^+, \amalg, (-)^+, \amalg): (\text{Sets}, \amalg, \emptyset) \rightarrow (\text{Sets}_*, \vee, \text{pt}),$$

being equipped with isomorphisms of pointed sets

$$\begin{aligned} (-)^+_{X,Y} \amalg: X^+ \vee Y^+ &\xrightarrow{\cong} (X \amalg Y)^+, \\ (-)^+_{\mathbb{1}} \amalg: \text{pt} &\xrightarrow{\cong} \emptyset^+, \end{aligned}$$

natural in  $X, Y \in \text{Obj}(\text{Sets})$ .

4. *Symmetric Strong Monoidality With Respect to Smash Products.* The free pointed set functor of **Item 1** has a symmetric strong monoidal structure

$$((-)^+, (-)^{+, \times}, (-)^{+, \times}): (\text{Sets}, \times, \text{pt}) \rightarrow (\text{Sets}_*, \wedge, S^0),$$

being equipped with isomorphisms of pointed sets

$$\begin{aligned} (-)^{+, \times}_{X,Y} \times: X^+ \wedge Y^+ &\xrightarrow{\cong} (X \times Y)^+, \\ (-)^{+, \times}_{\mathbb{1}} \times: S^0 &\xrightarrow{\cong} \text{pt}^+, \end{aligned}$$

natural in  $X, Y \in \text{Obj}(\text{Sets})$ .



## PROOF 4.1.3 ► PROOF OF PROPOSITION 4.1.2

## Item 1: Functoriality

Clear.

## Item 2: Adjointness

We claim there's an adjunction  $(-)^+ \dashv \mathbf{Sets}_*$ , witnessed by a bijection of sets

$$\mathbf{Sets}_*((X^+, \star_X), (Y, y_0)) \cong \mathbf{Sets}(X, Y),$$

natural in  $X \in \mathbf{Obj}(\mathbf{Sets})$  and  $(Y, y_0) \in \mathbf{Obj}(\mathbf{Sets}_*)$ .

• *Map I.* We define a map

$$\Phi_{X,Y}: \mathbf{Sets}_*((X^+, \star_X), (Y, y_0)) \rightarrow \mathbf{Sets}(X, Y)$$

by sending a pointed function

$$\xi: (X^+, \star_X) \rightarrow (Y, y_0)$$

to the function

$$\xi^\dagger: X \rightarrow Y$$

given by

$$\xi^\dagger(x) \stackrel{\text{def}}{=} \xi(x)$$

for each  $x \in X$ .

• *Map II.* We define a map

$$\Psi_{X,Y}: \mathbf{Sets}(X, Y) \rightarrow \mathbf{Sets}_*((X^+, \star_X), (Y, y_0))$$

given by sending a function  $\xi: X \rightarrow Y$  to the pointed function

$$\xi^\dagger: (X^+, \star_X) \rightarrow (Y, y_0)$$

defined by

$$\xi^\dagger(x) \stackrel{\text{def}}{=} \begin{cases} \xi(x) & \text{if } x \in X, \\ y_0 & \text{if } x = \star_X \end{cases}$$

for each  $x \in X^+$ .

- *Invertibility I.* We claim that

$$\Psi_{X,Y} \circ \Phi_{X,Y} = \text{id}_{\text{Sets}_*((X^+, \star_X), (Y, y_0))},$$

which is clear.

- *Invertibility II.* We claim that

$$\Phi_{X,Y} \circ \Psi_{X,Y} = \text{id}_{\text{Sets}(X,Y)},$$

which is clear.

- *Naturality for  $\Phi$ , Part I.* We need to show that, given a pointed function  $g: (Y, y_0) \rightarrow (Y', y'_0)$ , the diagram

$$\begin{array}{ccc} \text{Sets}_*((X^+, \star_X), (Y, y_0)) & \xrightarrow{\Phi_{X,Y}} & \text{Sets}(X, Y) \\ g_* \downarrow & & \downarrow g_* \\ \text{Sets}_*((X^+, \star_X), (Y', y'_0)) & \xrightarrow{\Phi_{X,Y'}} & \text{Sets}(X, Y') \end{array}$$

commutes. Indeed, given a pointed function

$$\xi^\dagger: (X^+, \star_X) \rightarrow (Y, y_0)$$

we have

$$\begin{aligned} [\Phi_{X,Y'} \circ g_*](\xi) &= \Phi_{X,Y'}(g_*(\xi)) \\ &= \Phi_{X,Y'}(g \circ \xi) \\ &= g \circ \xi \\ &= g \circ \Phi_{X,Y'}(\xi) \\ &= g_*(\Phi_{X,Y'}(\xi)) \\ &= [g_* \circ \Phi_{X,Y'}](\xi). \end{aligned}$$

- *Naturality for  $\Phi$ , Part II.* We need to show that, given a pointed function

$f: (X, x_0) \rightarrow (X', x'_0)$ , the diagram

$$\begin{array}{ccc} \text{Sets}_*((X'^+, \star_X), (Y, y_0)) & \xrightarrow{\Phi_{X', Y}} & \text{Sets}(X', Y) \\ f^* \downarrow & & \downarrow f^* \\ \text{Sets}_*((X^+, \star_X), (Y, y_0)) & \xrightarrow{\Phi_{X, Y}} & \text{Sets}(X, Y) \end{array}$$

commutes. Indeed, given a function

$$\xi: X' \rightarrow Y,$$

we have

$$\begin{aligned} [\Phi_{X, Y} \circ f^*](\xi) &= \Phi_{X, Y}(f^*(\xi)) \\ &= \Phi_{X, Y}(\xi \circ f) \\ &= \xi \circ f \\ &= \Phi_{X', Y}(\xi) \circ f \\ &= f^*(\Phi_{X', Y}(\xi)) \\ &= f^*(\Phi_{X', Y}(\xi)) \\ &= [f^* \circ \Phi_{X', Y}](\xi). \end{aligned}$$

- *Naturality for  $\Psi$ .* Since  $\Phi$  is natural in each argument and  $\Phi$  is a componentwise inverse to  $\Psi$  in each argument, it follows from [Categories, Item 2 of Proposition 8.6.2](#) that  $\Psi$  is also natural in each argument.

#### Item 3: Symmetric Strong Monoidality With Respect to Wedge Sums

The isomorphism

$$\phi: X^+ \vee Y^+ \xrightarrow{\cong} (X \amalg Y)^+$$

is given by

$$\phi(z) = \begin{cases} x & \text{if } z = [(0, x)] \text{ with } x \in X, \\ y & \text{if } z = [(1, y)] \text{ with } y \in Y, \\ \star_X \amalg_Y & \text{if } z = [(0, \star_X)], \\ \star_X \amalg_Y & \text{if } z = [(1, \star_Y)] \end{cases}$$

for each  $z \in X^+ \vee Y^+$ , with inverse

$$\phi^{-1}: (X \amalg Y)^+ \xrightarrow{\cong} X^+ \vee Y^+$$

given by

$$\phi^{-1}(z) \stackrel{\text{def}}{=} \begin{cases} [(0, x)] & \text{if } z = [(0, x)], \\ [(0, y)] & \text{if } z = [(1, y)], \\ p_0 & \text{if } z = \star_X \amalg_Y \end{cases}$$

for each  $z \in (X \amalg Y)^+$ .

Meanwhile, the isomorphism  $\text{pt} \cong \emptyset^+$  is given by sending  $\star_X$  to  $\star_\emptyset$ .

That these isomorphisms satisfy the coherence conditions making the functor  $(-)^+$  symmetric strong monoidal can be directly checked element by element.

#### Item 4: Symmetric Strong Monoidality With Respect to Smash Products

The isomorphism

$$\phi: X^+ \wedge Y^+ \xrightarrow{\cong} (X \times Y)^+$$

is given by

$$\phi(x \wedge y) = \begin{cases} (x, y) & \text{if } x \neq \star_X \text{ and } y \neq \star_Y \\ \star_{X \times Y} & \text{otherwise} \end{cases}$$

for each  $x \wedge y \in X^+ \wedge Y^+$ , with inverse


$$\phi^{-1}: (X \times Y)^+ \xrightarrow{\cong} X^+ \wedge Y^+$$

given by

$$\phi^{-1}(z) \stackrel{\text{def}}{=} \begin{cases} x \wedge y & \text{if } z = (x, y) \text{ with } (x, y) \in X \times Y, \\ \star_X \wedge \star_Y & \text{if } z = \star_{X \times Y}, \end{cases}$$

for each  $z \in (X \times Y)^+$ .

Meanwhile, the isomorphism  $S^0 \cong \text{pt}^+$  is given by sending  $\star$  to  $1 \in S^0 = \{0, 1\}$  and  $\star_{\text{pt}}$  to  $0 \in S^0$ .

That these isomorphisms satisfy the coherence conditions making the functor  $(-)^+$  symmetric strong monoidal can be directly checked element by element. 

## Appendices

### A Other Chapters

#### Sets

- |  |  |
|--|--|
| 1. <a href="#">Sets</a>                            | 7. <a href="#">Equivalence Relations and Apartness Relations</a> |
| 2. <a href="#">Constructions With Sets</a>         |  |
| 3. <a href="#">Pointed Sets</a>                    | <b>Category Theory</b>   |
| 4. <a href="#">Tensor Products of Pointed Sets</a> | 8. <a href="#">Categories</a>                                    |
| <b>Relations</b>                                   | <b>Bicategories</b>  |
| 5. <a href="#">Relations</a>                       |  |
| 6. <a href="#">Constructions With Relations</a>    | 9. <a href="#">Types of Morphisms in Bicategories</a>            |

## References

- [MSE 2855868] [Qiaochu Yuan](#). *Is the category of pointed sets Cartesian closed?* Mathematics Stack Exchange. URL: <https://math.stackexchange.com/q/2855868> (cit. on p. 7).
- [MSE 884460] [Martin Brandenburg](#). *Why are the category of pointed sets and the category of sets and partial functions “essentially the same”?* Mathematics Stack Exchange. URL: <https://math.stackexchange.com/q/884460> (cit. on p. 8).