

Constructions With Relations

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This chapter contains some material about constructions with relations. Notably, we discuss and explore:

1. The existence or non-existence of Kan extensions and Kan lifts in the 2-category **Rel** ([Section 2](#)).
2. The various kinds of constructions involving relations, such as graphs, domains, ranges, unions, intersections, products, inverse relations, composition of relations, and collages ([Section 3](#)).
3. The adjoint pairs

$$\begin{aligned} R_* \dashv R_{-1} &: \mathcal{P}(A) \rightleftarrows \mathcal{P}(B), \\ R^{-1} \dashv R_! &: \mathcal{P}(B) \rightleftarrows \mathcal{P}(A) \end{aligned}$$

of functors (morphisms of posets) between $\mathcal{P}(A)$ and $\mathcal{P}(B)$ induced by a relation $R: A \rightarrowtail B$, as well as the properties of R_* , R_{-1} , R^{-1} , and $R_!$ ([Section 4](#)).

Of particular note are the following points:

- (a) These two pairs of adjoint functors are the counterpart for relations of the adjoint triple $f_* \dashv f^{-1} \dashv f_!$ induced by a function $f: A \rightarrow B$ studied in [Constructions With Sets, Section 4](#).
- (b) We have $R_{-1} = R^{-1}$ iff R is total and functional ([Item 8 of Proposition 4.2.1.3](#)).
- (c) As a consequence of the previous item, when R comes from a function f , the pair of adjunctions

$$R_* \dashv R_{-1} = R^{-1} \dashv R_!$$

reduces to the triple adjunction

$$f_* \dashv f^{-1} \dashv f^!$$

from **Constructions With Sets, Section 4**.

- (d) The pairs $R_* \dashv R_{-1}$ and $R^{-1} \dashv R_!$ turn out to be rather important later on, as they appear in the definition and study of continuous, open, and closed relations between topological spaces (??, ??).

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1 Co/Limits in the Category of Relations

This section is currently just a stub, and will be properly developed later on.

2 Kan Extensions and Kan Lifts in the 2-Category of Relations

2.1 Left Kan Extensions in Rel

Proposition 2.1.1.1. Let $R: A \rightarrow B$ be a relation.

1. *Non-Existence of All Left Kan Extensions in Rel.* Not all relations in **Rel** admit left Kan extensions.
2. *Characterisation of Relations Admitting Left Kan Extensions Along Them.* The following conditions are equivalent:

- (a) The left Kan extension

$$\mathrm{Lan}_R: \mathbf{Rel}(A, X) \rightarrow \mathbf{Rel}(B, X)$$

along R exists.

- (b) The relation R admits a left adjoint in **Rel**.
- (c) The relation R is of the form f^{-1} (as in **Definition 3.2.1.1**) for some function f .

Proof. **Item 1, Non-Existence of All Left Kan Extensions in Rel:** Omitted, but will eventually follow **Fosco Loregian's comment** on [MO 460656].

Item 2, Characterisation of Relations Admitting Left Kan Extensions Along Them: Omitted, but will eventually follow **Tim Champion's answer to** to [MO 460656].

□

Question 2.1.1.2. Given relations $S: A \rightarrowtail X$ and $R: A \rightarrowtail B$, is there a characterisation of when the left Kan extension

$$\text{Lan}_S(R): B \rightarrowtail X$$

exists in terms of properties of R and S ?

This question also appears as [MO 461592].

Question 2.1.1.3. As shown in **Item 2** of **Proposition 2.1.1.1**, the left Kan extension

$$\text{Lan}_R: \mathbf{Rel}(A, X) \rightarrow \mathbf{Rel}(B, X)$$

along a relation of the form $R = f^{-1}$ exists. Is there an explicit description of it, similarly to the explicit description of right Kan extensions given in **Proposition 2.3.1.1**?

This question also appears as [MO 461592].

2.2 Left Kan Lifts in **Rel**

Proposition 2.2.1.1. Let $R: A \rightarrowtail B$ be a relation.

1. *Non-Existence of All Left Kan Lifts in **Rel**.* Not all relations in **Rel** admit left Kan lifts.
2. *Characterisation of Relations Admitting Left Kan Lifts Along Them.* The following conditions are equivalent:

- (a) The left Kan lift

$$\text{Lift}_R: \mathbf{Rel}(X, B) \rightarrow \mathbf{Rel}(X, A)$$

along R exists.

- (b) The relation R admits a right adjoint in **Rel**.
- (c) The relation R is of the form $\text{Gr}(f)$ (as in **Definition 3.1.1.1**) for some function f .

Proof. **Item 1, Non-Existence of All Left Kan Lifts in **Rel**:** Omitted, but will eventually follow (the dual of) **Fosco Loregian's comment** on [MO 460656].

Item 2, Characterisation of Relations Admitting Left Kan Lifts Along Them: Omitted, but will eventually follow **Tim Champion's answer** to [MO 460656]. \square

Question 2.2.1.2. Given relations $S: A \rightarrowtail X$ and $R: A \rightarrowtail B$, is there a characterisation of when the left Kan lift

$$\mathrm{Lift}_S(R): X \rightarrowtail A$$

exists in terms of properties of R and S ?

This question also appears as [MO 461592].

Question 2.2.1.3. As shown in Item 2 of Proposition 2.2.1.1, the left Kan lift

$$\mathrm{Lift}_R: \mathbf{Rel}(X, B) \rightarrow \mathbf{Rel}(X, A)$$

along a relation of the form $R = \mathrm{Gr}(f)$ exists. Is there an explicit description of it, similarly to the explicit description of right Kan lifts given in Proposition 2.4.1.1? This question also appears as [MO 461592].

2.3 Right Kan Extensions in **Rel**

Let $R: A \rightarrowtail B$ be a relation.

Proposition 2.3.1.1. The right Kan extension

$$\mathrm{Ran}_R: \mathbf{Rel}(A, X) \rightarrow \mathbf{Rel}(B, X)$$

along R in **Rel** exists and is given by

$$\mathrm{Ran}_R(S) \stackrel{\mathrm{def}}{=} \int_{a \in A} \mathbf{Hom}_{\{t, f\}}(R_a^{-2}, S_a^{-1})$$

for each $S \in \mathbf{Rel}(A, X)$, so that the following conditions are equivalent:

1. We have $b \sim_{\mathrm{Ran}_R(S)} x$.
2. For each $a \in A$, if $a \sim_R b$, then $a \sim_S x$.

Proof. We have

$$\begin{aligned}
\mathbf{Hom}_{\mathbf{Rel}(A,X)}(S \diamond R, T) &\cong \int_{a \in A} \int_{x \in X} \mathbf{Hom}_{\{t,f\}}((S \diamond R)_a^x, T_a^x) \\
&\cong \int_{a \in A} \int_{x \in X} \mathbf{Hom}_{\{t,f\}}((\int_{b \in B} S_b^x \times R_a^b), T_a^x) \\
&\cong \int_{a \in A} \int_{x \in X} \int_{b \in B} \mathbf{Hom}_{\{t,f\}}(S_b^x \times R_a^b, T_a^x) \\
&\cong \int_{a \in A} \int_{x \in X} \int_{b \in B} \mathbf{Hom}_{\{t,f\}}(S_b^x, \mathbf{Hom}_{\{t,f\}}(R_a^b, T_a^x)) \\
&\cong \int_{b \in B} \int_{x \in X} \int_{a \in A} \mathbf{Hom}_{\{t,f\}}(S_b^x, \mathbf{Hom}_{\{t,f\}}(R_a^b, T_a^x)) \\
&\cong \int_{b \in B} \int_{x \in X} \mathbf{Hom}_{\{t,f\}}(S_b^x, \int_{a \in A} \mathbf{Hom}_{\{t,f\}}(R_a^b, T_a^x)) \\
&\cong \mathbf{Hom}_{\mathbf{Rel}(B,X)}(S, \int_{a \in A} \mathbf{Hom}_{\{t,f\}}(R_a^{-2}, T_a^{-1}))
\end{aligned}$$

naturally in each $S \in \mathbf{Rel}(B, X)$ and each $T \in \mathbf{Rel}(A, X)$, showing that

$$\int_{a \in A} \mathbf{Hom}_{\{t,f\}}(R_a^{-2}, T_a^{-1})$$

is right adjoint to the precomposition functor $- \diamond R$, being thus the right Kan extension along R . Here we have used the following results, respectively (i.e. for each \cong sign):

1. **Relations, Item 1 of Proposition 1.1.1.5.**
2. **Definition 3.12.1.1.**
3. **??, ?? of ??.**
4. **Sets, Proposition 2.2.1.5.**
5. **??, ?? of ??.**
6. **??, ?? of ??.**
7. **Relations, Item 1 of Proposition 1.1.1.5.**

This finishes the proof. □

2.4 Right Kan Lifts in **Rel**

Let $R: A \rightarrowtail B$ be a relation.

Proposition 2.4.1.1. The right Kan lift

$$\text{Rift}_R: \mathbf{Rel}(X, B) \rightarrow \mathbf{Rel}(X, A)$$

along R in **Rel** exists and is given by

$$\text{Rift}_R(S) \stackrel{\text{def}}{=} \int_{b \in B} \mathbf{Hom}_{\{t, f\}}(R_{-1}^b, S_{-2}^b)$$

for each $S \in \mathbf{Rel}(X, B)$, so that the following conditions are equivalent:

1. We have $x \sim_{\text{Rift}_R(S)} a$.
2. For each $b \in B$, if $a \sim_R b$, then $x \sim_S b$.

Proof. We have

$$\begin{aligned} \mathbf{Hom}_{\mathbf{Rel}(X, B)}(R \diamond S, T) &\cong \int_{x \in X} \int_{b \in B} \mathbf{Hom}_{\{t, f\}}((R \diamond S)_x^b, T_x^b) \\ &\cong \int_{x \in X} \int_{b \in B} \mathbf{Hom}_{\{t, f\}}((\int_{a \in A} R_a^b \times S_x^a), T_x^b) \\ &\cong \int_{x \in X} \int_{b \in B} \int_{a \in A} \mathbf{Hom}_{\{t, f\}}(R_a^b \times S_x^a, T_x^b) \\ &\cong \int_{x \in X} \int_{b \in B} \int_{a \in A} \mathbf{Hom}_{\{t, f\}}(S_x^a, \mathbf{Hom}_{\{t, f\}}(R_a^b, T_x^b)) \\ &\cong \int_{x \in X} \int_{a \in A} \int_{b \in B} \mathbf{Hom}_{\{t, f\}}(S_x^a, \mathbf{Hom}_{\{t, f\}}(R_a^b, T_x^b)) \\ &\cong \int_{x \in X} \int_{a \in A} \mathbf{Hom}_{\{t, f\}}(S_x^a, \int_{b \in B} \mathbf{Hom}_{\{t, f\}}(R_a^b, T_x^b)) \\ &\cong \mathbf{Hom}_{\mathbf{Rel}(X, A)}(S, \int_{b \in B} \mathbf{Hom}_{\{t, f\}}(R_{-1}^b, T_{-2}^b)) \end{aligned}$$

naturally in each $S \in \mathbf{Rel}(X, A)$ and each $T \in \mathbf{Rel}(X, B)$, showing that

$$\int_{b \in B} \mathbf{Hom}_{\{t, f\}}(R_{-1}^b, S_{-2}^b)$$

is right adjoint to the postcomposition functor $R \diamond -$, being thus the right Kan lift along R . Here we have used the following results, respectively (i.e. for each \cong sign):

1. Relations, Item 1 of Proposition 1.1.1.5.
2. Definition 3.12.1.1.
3. ??, ?? of ??.
4. Sets, Proposition 2.2.1.5.
5. ??, ?? of ??.
6. ??, ?? of ??.
7. Relations, Item 1 of Proposition 1.1.1.5.

This finishes the proof. \square

3 More Constructions With Relations

3.1 The Graph of a Function

Let $f: A \rightarrow B$ be a function.

Definition 3.1.1.1. The **graph of f** is the relation $\text{Gr}(f): A \rightarrowtail B$ defined as follows:¹

- Viewing relations from A to B as subsets of $A \times B$, we define

$$\text{Gr}(f) \stackrel{\text{def}}{=} \{(a, f(a)) \in A \times B \mid a \in A\}.$$

- Viewing relations from A to B as functions $A \times B \rightarrow \{\text{true}, \text{false}\}$, we define

$$[\text{Gr}(f)](a, b) \stackrel{\text{def}}{=} \begin{cases} \text{true} & \text{if } b = f(a), \\ \text{false} & \text{otherwise} \end{cases}$$

for each $(a, b) \in A \times B$.

- Viewing relations from A to B as functions $A \rightarrow \mathcal{P}(B)$, we define

$$[\text{Gr}(f)](a) \stackrel{\text{def}}{=} \{f(a)\}$$

for each $a \in A$, i.e. we define $\text{Gr}(f)$ as the composition

$$A \xrightarrow{f} B \xrightarrow{\chi_B} \mathcal{P}(B).$$

¹*Further Notation:* We write $\text{Gr}(A)$ for $\text{Gr}(\text{id}_A)$, and call it the **graph** of A .

Proposition 3.1.1.2. Let $f: A \rightarrow B$ be a function.

1. *Functoriality.* The assignment $A \mapsto \text{Gr}(A)$ defines a functor

$$\text{Gr}: \text{Sets} \rightarrow \text{Rel}$$

where

- *Action on Objects.* For each $A \in \text{Obj}(\text{Sets})$, we have

$$\text{Gr}(A) \stackrel{\text{def}}{=} A.$$

- *Action on Morphisms.* For each $A, B \in \text{Obj}(\text{Sets})$, the action on Hom-sets

$$\text{Gr}_{A,B}: \text{Sets}(A, B) \rightarrow \underbrace{\text{Rel}(\text{Gr}(A), \text{Gr}(B))}_{\stackrel{\text{def}}{=} \text{Rel}(A, B)}$$

of Gr at (A, B) is defined by

$$\text{Gr}_{A,B}(f) \stackrel{\text{def}}{=} \text{Gr}(f),$$

where $\text{Gr}(f)$ is the graph of f as in **Definition 3.1.1.1**.

In particular:

- *Preservation of Identities.* We have

$$\text{Gr}(\text{id}_A) = \chi_A$$

for each $A \in \text{Obj}(\text{Sets})$.

- *Preservation of Composition.* We have

$$\text{Gr}(g \circ f) = \text{Gr}(g) \diamond \text{Gr}(f)$$

for each pair of functions $f: A \rightarrow B$ and $g: B \rightarrow C$.

2. *Adjointness Inside Rel.* We have an adjunction

$$(\text{Gr}(f) \dashv f^{-1}): A \begin{array}{c} \xrightarrow{\text{Gr}(f)} \\ \perp \\ \xleftarrow{f^{-1}} \end{array} B$$

in **Rel**, where f^{-1} is the inverse of f of **Definition 3.2.1.1**.

3. *Adjointness.* We have an adjunction

$$(\text{Gr} \dashv \mathcal{P}_*) : \text{Sets} \begin{array}{c} \xrightarrow{\text{Gr}} \\ \perp \\ \xleftarrow{\mathcal{P}_*} \end{array} \text{Rel},$$

witnessed by a bijection of sets

$$\text{Rel}(\text{Gr}(A), B) \cong \text{Sets}(A, \mathcal{P}(B))$$

natural in $A \in \text{Obj}(\text{Sets})$ and $B \in \text{Obj}(\text{Rel})$.

4. *Interaction With Inverses.* We have

$$\begin{aligned} \text{Gr}(f)^\dagger &= f^{-1}, \\ (f^{-1})^\dagger &= \text{Gr}(f). \end{aligned}$$

5. *Cocontinuity.* The functor $\text{Gr} : \text{Sets} \rightarrow \text{Rel}$ of **Item 1** preserves colimits.

6. *Characterisations.* Let $R : A \rightarrowtail B$ be a relation. The following conditions are equivalent:

- (a) There exists a function $f : A \rightarrow B$ such that $R = \text{Gr}(f)$.
- (b) The relation R is total and functional.
- (c) The weak and strong inverse images of R agree, i.e. we have $R^{-1} = R_{-1}$.
- (d) The relation R has a right adjoint R^\dagger in Rel .

Proof. **Item 1, Functoriality:** Clear.

Item 2, Adjointness Inside Rel: We need to check that there are inclusions

$$\begin{aligned} \chi_A &\subset f^{-1} \diamond \text{Gr}(f), \\ \text{Gr}(f) \diamond f^{-1} &\subset \chi_B. \end{aligned}$$

These correspond respectively to the following conditions:

- 1. For each $a \in A$, there exists some $b \in B$ such that $a \sim_{\text{Gr}(f)} b$ and $b \sim_{f^{-1}} a$.
- 2. For each $a, b \in A$, if $a \sim_{\text{Gr}(f)} b$ and $b \sim_{f^{-1}} a$, then $a = b$.

In other words, the first condition states that the image of any $a \in A$ by f is nonempty, whereas the second condition states that f is not multivalued. As f is a function, both of these statements are true, and we are done.

Item 3, Adjointness: The stated bijection follows from [Relations, Remark 1.1.1.4](#), with naturality being clear.

Item 4, Interaction With Inverses: Clear.

Item 5, Cocontinuity: Omitted.

Item 6, Characterisations: We claim that [Items 6a](#) to [6d](#) are indeed equivalent:

- [Item 6a](#) \iff [Item 6b](#). This is shown in the proof of ?? of ??.
- [Item 6b](#) \implies [Item 6c](#). If R is total and functional, then, for each $a \in A$, the set $R(a)$ is a singleton, implying that

$$\begin{aligned} R^{-1}(V) &\stackrel{\text{def}}{=} \{a \in A \mid R(a) \cap V \neq \emptyset\}, \\ R_{-1}(V) &\stackrel{\text{def}}{=} \{a \in A \mid R(a) \subset V\} \end{aligned}$$

are equal for all $V \in \mathcal{P}(B)$, as the conditions $R(a) \cap V \neq \emptyset$ and $R(a) \subset V$ are equivalent when $R(a)$ is a singleton.

- [Item 6c](#) \implies [Item 6b](#). We claim that R is indeed total and functional:
 - *Totality.* If we had $R(a) = \emptyset$ for some $a \in A$, then we would have $a \in R_{-1}(\emptyset)$, so that $R_{-1}(\emptyset) \neq \emptyset$. But since $R^{-1}(\emptyset) = \emptyset$, this would imply $R_{-1}(\emptyset) \neq R^{-1}(\emptyset)$, a contradiction. Thus $R(a) \neq \emptyset$ for all $a \in A$ and R is total.
 - *Functionality.* If $R^{-1} = R_{-1}$, then we have

$$\begin{aligned} \{a\} &= R^{-1}(\{b\}) \\ &= R_{-1}(\{b\}) \end{aligned}$$

for each $b \in R(a)$ and each $a \in A$, and thus $R(a) \subset \{b\}$. But since R is total, we must have $R(a) = \{b\}$, and thus we see that R is functional.

- [Item 6a](#) \iff [Item 6d](#). This follows from [Relations, Proposition 3.3.1.1](#).

This finishes the proof. \square

3.2 The Inverse of a Function

Let $f: A \rightarrow B$ be a function.

Definition 3.2.1.1. The **inverse of f** is the relation $f^{-1}: B \rightarrowtail A$ defined as follows:

- Viewing relations from B to A as subsets of $B \times A$, we define

$$f^{-1} \stackrel{\text{def}}{=} \{(b, f^{-1}(b)) \in B \times A \mid a \in A\},$$

where

$$f^{-1}(b) \stackrel{\text{def}}{=} \{a \in A \mid f(a) = b\}$$

for each $b \in B$.

- Viewing relations from B to A as functions $B \times A \rightarrow \{\text{true}, \text{false}\}$, we define

$$f^{-1}(b, a) \stackrel{\text{def}}{=} \begin{cases} \text{true} & \text{if there exists } a \in A \text{ with } f(a) = b, \\ \text{false} & \text{otherwise} \end{cases}$$

for each $(b, a) \in B \times A$.

- Viewing relations from B to A as functions $B \rightarrow \mathcal{P}(A)$, we define

$$f^{-1}(b) \stackrel{\text{def}}{=} \{a \in A \mid f(a) = b\}$$

for each $b \in B$.

Proposition 3.2.1.2. Let $f: A \rightarrow B$ be a function.

1. *Functoriality.* The assignment $A \mapsto A, f \mapsto f^{-1}$ defines a functor

$$(-)^{-1}: \text{Sets} \rightarrow \text{Rel}$$

where

- *Action on Objects.* For each $A \in \text{Obj}(\text{Sets})$, we have

$$[(-)^{-1}](A) \stackrel{\text{def}}{=} A.$$

- *Action on Morphisms.* For each $A, B \in \text{Obj}(\text{Sets})$, the action on Hom-sets

$$(-)^{-1}_{A,B}: \text{Sets}(A, B) \rightarrow \text{Rel}(A, B)$$

of $(-)^{-1}$ at (A, B) is defined by

$$(-)^{-1}_{A,B}(f) \stackrel{\text{def}}{=} [(-)^{-1}](f),$$

where f^{-1} is the inverse of f as in [Definition 3.2.1.1](#).

In particular:

- *Preservation of Identities.* We have

$$\text{id}_A^{-1} = \chi_A$$

for each $A \in \text{Obj}(\text{Sets})$.

- *Preservation of Composition.* We have

$$(g \circ f)^{-1} = g^{-1} \diamond f^{-1}$$

for pair of functions $f: A \rightarrow B$ and $g: B \rightarrow C$.

2. *Adjointness Inside Rel.* We have an adjunction

$$(\text{Gr}(f) \dashv f^{-1}): A \begin{array}{c} \xrightarrow{\text{Gr}(f)} \\ \perp \\ \xleftarrow{f^{-1}} \end{array} B$$

in **Rel**.

3. *Interaction With Inverses of Relations.* We have

$$\begin{aligned} (f^{-1})^\dagger &= \text{Gr}(f), \\ \text{Gr}(f)^\dagger &= f^{-1}. \end{aligned}$$

Proof. [Item 1](#), *Functoriality*: Clear.

[Item 2](#), *Adjointness Inside Rel*: This is proved in [Item 2](#) of [Proposition 3.1.1.2](#).

[Item 3](#), *Interaction With Inverses of Relations*: Clear. \square

3.3 Representable Relations

Let A and B be sets.

Definition 3.3.1.1. Let $f: A \rightarrow B$ and $g: B \rightarrow A$ be functions.²

1. The **representable relation associated to f** is the relation $\chi_f: A \dashv\vdash B$ defined as the composition

$$A \times B \xrightarrow{f \times \text{id}_B} B \times B \xrightarrow{\chi_B} \{\text{true}, \text{false}\},$$

i.e. given by declaring $a \sim_{\chi_f} b$ iff $f(a) = b$.

2. The **corepresentable relation associated to g** is the relation $\chi^g: B \dashv\vdash A$ defined as the composition

$$B \times A \xrightarrow{g \times \text{id}_A} A \times A \xrightarrow{\chi_A} \{\text{true}, \text{false}\},$$

i.e. given by declaring $b \sim_{\chi^g} a$ iff $g(b) = a$.

3.4 The Domain and Range of a Relation

Let A and B be sets.

²More generally, given functions

$$f: A \rightarrow C,$$

$$g: B \rightarrow D$$

and a relation $B \dashv\vdash D$, we may consider the composite relation

$$A \times B \xrightarrow{f \times g} C \times D \xrightarrow{R} \{\text{true}, \text{false}\},$$

for which we have $a \sim_{R \circ (f \times g)} b$ iff $f(a) \sim_R g(b)$.

Definition 3.4.1.1. Let $R \subset A \times B$ be a relation.^{3,4}

1. The **domain of R** is the subset $\text{dom}(R)$ of A defined by

$$\text{dom}(R) \stackrel{\text{def}}{=} \left\{ a \in A \left| \begin{array}{l} \text{there exists some } b \in B \\ \text{such that } a \sim_R b \end{array} \right. \right\}.$$

2. The **range of R** is the subset $\text{range}(R)$ of B defined by

$$\text{range}(R) \stackrel{\text{def}}{=} \left\{ b \in B \left| \begin{array}{l} \text{there exists some } a \in A \\ \text{such that } a \sim_R b \end{array} \right. \right\}.$$

3.5 Binary Unions of Relations

Let A and B be sets and let R and S be relations from A to B .

Definition 3.5.1.1. The **union of R and S** ⁵ is the relation $R \cup S$ from A to B defined as follows:

³Following ??, ??, we may compute the (characteristic functions associated to the) domain and range of a relation using the following colimit formulas:

$$\begin{aligned} \chi_{\text{dom}(R)}(a) &\cong \text{colim}_{b \in B} (R_a^b) & (a \in A) \\ &\cong \bigvee_{b \in B} R_a^b, \\ \chi_{\text{range}(R)}(b) &\cong \text{colim}_{a \in A} (R_a^b) & (b \in B) \\ &\cong \bigvee_{a \in A} R_a^b, \end{aligned}$$

where the join \bigvee is taken in the poset $(\{\text{true}, \text{false}\}, \preceq)$ of **Constructions With Sets**, **Definition 2.2.1.3**.

⁴Viewing R as a function $R: A \rightarrow \mathcal{P}(B)$, we have

$$\begin{aligned} \text{dom}(R) &\cong \text{colim}_{y \in Y} (R(y)) \\ &\cong \bigcup_{y \in Y} R(y), \\ \text{range}(R) &\cong \text{colim}_{x \in X} (R(x)) \\ &\cong \bigcup_{x \in X} R(x), \end{aligned}$$

⁵*Further Terminology:* Also called the **binary union of R and S** , for emphasis.

- Viewing relations from A to B as subsets of $A \times B$, we define⁶

$$R \cup S \stackrel{\text{def}}{=} \{(a, b) \in B \times A \mid \text{we have } a \sim_R b \text{ or } a \sim_S b\}.$$

- Viewing relations from A to B as functions $A \rightarrow \mathcal{P}(B)$, we define

$$[R \cup S](a) \stackrel{\text{def}}{=} R(a) \cup S(a)$$

for each $a \in A$.

Proposition 3.5.1.2. Let R, S, R_1 , and R_2 be relations from A to B , and let S_1 and S_2 be relations from B to C .

1. *Interaction With Inverses.* We have

$$(R \cup S)^\dagger = R^\dagger \cup S^\dagger.$$

2. *Interaction With Composition.* We have

$$(S_1 \diamond R_1) \cup (S_2 \diamond R_2) \stackrel{\text{poss.}}{\neq} (S_1 \cup S_2) \diamond (R_1 \cup R_2).$$

Proof. **Item 1, Interaction With Inverses:** Clear.

Item 2, Interaction With Composition: Unwinding the definitions, we see that:

1. The condition for $(S_1 \diamond R_1) \cup (S_2 \diamond R_2)$ is:

- (a) There exists some $b \in B$ such that:

- i. $a \sim_{R_1} b$ and $b \sim_{S_1} c$;

or

- i. $a \sim_{R_2} b$ and $b \sim_{S_2} c$;

3. The condition for $(S_1 \cup S_2) \diamond (R_1 \cup R_2)$ is:

- (a) There exists some $b \in B$ such that:

- i. $a \sim_{R_1} b$ or $a \sim_{R_2} b$;

and

- i. $b \sim_{S_1} c$ or $b \sim_{S_2} c$.

These two conditions may fail to agree (counterexample omitted), and thus the two resulting relations on $A \times C$ may differ. \square

⁶This is the same as the union of R and S as subsets of $A \times B$.

3.6 Unions of Families of Relations

Let A and B be sets and let $\{R_i\}_{i \in I}$ be a family of relations from A to B .

Definition 3.6.1.1. The **union of the family** $\{R_i\}_{i \in I}$ is the relation $\bigcup_{i \in I} R_i$ from A to B defined as follows:

- Viewing relations from A to B as subsets of $A \times B$, we define⁷

$$\bigcup_{i \in I} R_i \stackrel{\text{def}}{=} \left\{ (a, b) \in (A \times B)^{\times I} \mid \begin{array}{l} \text{there exists some } i \in I \\ \text{such that } a \sim_{R_i} b \end{array} \right\}.$$

- Viewing relations from A to B as functions $A \rightarrow \mathcal{P}(B)$, we define

$$\left[\bigcup_{i \in I} R_i \right] (a) \stackrel{\text{def}}{=} \bigcup_{i \in I} R_i(a)$$

for each $a \in A$.

Proposition 3.6.1.2. Let A and B be sets and let $\{R_i\}_{i \in I}$ be a family of relations from A to B .

1. *Interaction With Inverses.* We have

$$\left(\bigcup_{i \in I} R_i \right)^\dagger = \bigcup_{i \in I} R_i^\dagger.$$

Proof. **Item 1, Interaction With Inverses:** Clear. □

3.7 Binary Intersections of Relations

Let A and B be sets and let R and S be relations from A to B .

Definition 3.7.1.1. The **intersection of R and S** ⁸ is the relation $R \cap S$ from A to B defined as follows:

- Viewing relations from A to B as subsets of $A \times B$, we define⁹

$$R \cap S \stackrel{\text{def}}{=} \{(a, b) \in B \times A \mid \text{we have } a \sim_R b \text{ and } a \sim_S b\}.$$

⁷This is the same as the union of $\{R_i\}_{i \in I}$ as a collection of subsets of $A \times B$.

⁸*Further Terminology:* Also called the **binary intersection of R and S** , for emphasis.

⁹This is the same as the intersection of R and S as subsets of $A \times B$.

- Viewing relations from A to B as functions $A \rightarrow \mathcal{P}(B)$, we define

$$[R \cap S](a) \stackrel{\text{def}}{=} R(a) \cap S(a)$$

for each $a \in A$.

Proposition 3.7.1.2. Let R, S, R_1 , and R_2 be relations from A to B , and let S_1 and S_2 be relations from B to C .

1. *Interaction With Inverses.* We have

$$(R \cap S)^\dagger = R^\dagger \cap S^\dagger.$$

2. *Interaction With Composition.* We have

$$(S_1 \diamond R_1) \cap (S_2 \diamond R_2) = (S_1 \cap S_2) \diamond (R_1 \cap R_2).$$

Proof. **Item 1, Interaction With Inverses:** Clear.

Item 2, Interaction With Composition: Unwinding the definitions, we see that:

1. The condition for $(S_1 \diamond R_1) \cap (S_2 \diamond R_2)$ is:

- (a) There exists some $b \in B$ such that:

- i. $a \sim_{R_1} b$ and $b \sim_{S_1} c$;

and

- i. $a \sim_{R_2} b$ and $b \sim_{S_2} c$;

3. The condition for $(S_1 \cap S_2) \diamond (R_1 \cap R_2)$ is:

- (a) There exists some $b \in B$ such that:

- i. $a \sim_{R_1} b$ and $a \sim_{R_2} b$;

and

- i. $b \sim_{S_1} c$ and $b \sim_{S_2} c$.

These two conditions agree, and thus so do the two resulting relations on $A \times C$. \square

3.8 Intersections of Families of Relations

Let A and B be sets and let $\{R_i\}_{i \in I}$ be a family of relations from A to B .

Definition 3.8.1.1. The **intersection of the family** $\{R_i\}_{i \in I}$ is the relation $\bigcup_{i \in I} R_i$ defined as follows:

- Viewing relations from A to B as subsets of $A \times B$, we define¹⁰

$$\bigcup_{i \in I} R_i \stackrel{\text{def}}{=} \left\{ (a, b) \in (A \times B)^{\times I} \left| \begin{array}{l} \text{for each } i \in I, \\ \text{we have } a \sim_{R_i} b \end{array} \right. \right\}.$$

- Viewing relations from A to B as functions $A \rightarrow \mathcal{P}(B)$, we define

$$\left[\bigcap_{i \in I} R_i \right] (a) \stackrel{\text{def}}{=} \bigcap_{i \in I} R_i(a)$$

for each $a \in A$.

Proposition 3.8.1.2. Let A and B be sets and let $\{R_i\}_{i \in I}$ be a family of relations from A to B .

1. *Interaction With Inverses.* We have

$$\left(\bigcap_{i \in I} R_i \right)^\dagger = \bigcap_{i \in I} R_i^\dagger.$$

Proof. **Item 1, Interaction With Inverses:** Clear. □

3.9 Binary Products of Relations

Let A , B , X , and Y be sets, let $R: A \rightarrowtail B$ be a relation from A to B , and let $S: X \rightarrowtail Y$ be a relation from X to Y .

Definition 3.9.1.1. The **product of R and S** ¹¹ is the relation $R \times S$ from $A \times X$ to $B \times Y$ defined as follows:

- Viewing relations from $A \times X$ to $B \times Y$ as subsets of $(A \times X) \times (B \times Y)$,

¹⁰This is the same as the intersection of $\{R_i\}_{i \in I}$ as a collection of subsets of $A \times B$.

¹¹*Further Terminology:* Also called the **binary product of R and S** , for emphasis.

we define $R \times S$ as the Cartesian product of R and S as subsets of $A \times X$ and $B \times Y$.¹²

- Viewing relations from $A \times X$ to $B \times Y$ as functions $A \times X \rightarrow \mathcal{P}(B \times Y)$, we define $R \times S$ as the composition

$$A \times X \xrightarrow{R \times S} \mathcal{P}(B) \times \mathcal{P}(Y) \xrightarrow{\mathcal{P}_{B,Y}^\otimes} \mathcal{P}(B \times Y)$$

in Sets, i.e. by

$$[R \times S](a, x) \stackrel{\text{def}}{=} R(a) \times S(x)$$

for each $(a, x) \in A \times X$.

Proposition 3.9.1.2. Let A, B, X , and Y be sets.

1. *Interaction With Inverses.* Let

$$R: A \rightarrowtail A,$$

$$S: X \rightarrowtail X$$

We have

$$(R \times S)^\dagger = R^\dagger \times S^\dagger.$$

2. *Interaction With Composition.* Let

$$R_1: A \rightarrowtail B,$$

$$S_1: B \rightarrowtail C,$$

$$R_2: X \rightarrowtail Y,$$

$$S_2: Y \rightarrowtail Z$$

be relations. We have

$$(S_1 \diamond R_1) \times (S_2 \diamond R_2) = (S_1 \times S_2) \diamond (R_1 \times R_2).$$

Proof. **Item 1**, *Interaction With Inverses*: Unwinding the definitions, we see that:

1. We have $(a, x) \sim_{(R \times S)^\dagger} (b, y)$ iff:

¹²That is, $R \times S$ is the relation given by declaring $(a, x) \sim_{R \times S} (b, y)$ iff $a \sim_R b$ and $x \sim_S y$.

- We have $(b, y) \sim_{R \times S} (a, x)$, i.e. iff:
 - We have $b \sim_R a$;
 - We have $y \sim_S x$;
- 2. We have $(a, x) \sim_{R^\dagger \times S^\dagger} (b, y)$ iff:
 - We have $a \sim_{R^\dagger} b$ and $x \sim_{S^\dagger} y$, i.e. iff:
 - We have $b \sim_R a$;
 - We have $y \sim_S x$.

These two conditions agree, and thus the two resulting relations on $A \times X$ are equal.

Item 2, Interaction With Composition: Unwinding the definitions, we see that:

1. We have $(a, x) \sim_{(S_1 \diamond R_1) \times (S_2 \diamond R_2)} (c, z)$ iff:
 - (a) We have $a \sim_{S_1 \diamond R_1} c$ and $x \sim_{S_2 \diamond R_2} z$, i.e. iff:
 - i. There exists some $b \in B$ such that $a \sim_{R_1} b$ and $b \sim_{S_1} c$;
 - ii. There exists some $y \in Y$ such that $x \sim_{R_2} y$ and $y \sim_{S_2} z$;
2. We have $(a, x) \sim_{(S_1 \times S_2) \diamond (R_1 \times R_2)} (c, z)$ iff:
 - (a) There exists some $(b, y) \in B \times Y$ such that $(a, x) \sim_{R_1 \times R_2} (b, y)$ and $(b, y) \sim_{S_1 \times S_2} (c, z)$, i.e. such that:
 - i. We have $a \sim_{R_1} b$ and $x \sim_{R_2} y$;
 - ii. We have $b \sim_{S_1} c$ and $y \sim_{S_2} z$.

These two conditions agree, and thus the two resulting relations from $A \times X$ to $C \times Z$ are equal. \square

3.10 Products of Families of Relations

Let $\{A_i\}_{i \in I}$ and $\{B_i\}_{i \in I}$ be families of sets, and let $\{R_i : A_i \rightarrow B_i\}_{i \in I}$ be a family of relations.

Definition 3.10.1.1. The **product of the family** $\{R_i\}_{i \in I}$ is the relation $\prod_{i \in I} R_i$ from $\prod_{i \in I} A_i$ to $\prod_{i \in I} B_i$ defined as follows:

- Viewing relations as subsets, we define $\prod_{i \in I} R_i$ as its product as a family

of sets, i.e. we have

$$\prod_{i \in I} R_i \stackrel{\text{def}}{=} \left\{ (a_i, b_i)_{i \in I} \in \prod_{i \in I} (A_i \times B_i) \left| \begin{array}{l} \text{for each } i \in I, \\ \text{we have } a_i \sim_{R_i} b_i \end{array} \right. \right\}.$$

- Viewing relations as functions to powersets, we define

$$\left[\prod_{i \in I} R_i \right] ((a_i)_{i \in I}) \stackrel{\text{def}}{=} \prod_{i \in I} R_i(a_i)$$

for each $(a_i)_{i \in I} \in \prod_{i \in I} R_i$.

3.11 The Inverse of a Relation

Let A , B , and C be sets and let $R \subset A \times B$ be a relation.

Definition 3.11.1.1. The **inverse of R** ¹³ is the relation R^\dagger defined as follows:

- Viewing relations as subsets, we define

$$R^\dagger \stackrel{\text{def}}{=} \{(b, a) \in B \times A \mid \text{we have } b \sim_R a\}.$$

- Viewing relations as functions $A \times B \rightarrow \{\text{true}, \text{false}\}$, we define

$$[R^\dagger]_b^a \stackrel{\text{def}}{=} R_a^b$$

for each $(b, a) \in B \times A$.

- Viewing relations as functions $A \rightarrow \mathcal{P}(B)$, we define

$$\begin{aligned} [R^\dagger](b) &\stackrel{\text{def}}{=} R^\dagger(\{b\}) \\ &\stackrel{\text{def}}{=} \{a \in A \mid b \in R(a)\} \end{aligned}$$

for each $b \in B$, where $R^\dagger(\{b\})$ is the fibre of R over $\{b\}$.

Example 3.11.1.2. Here are some examples of inverses of relations.

1. *Less Than Equal Signs.* We have $(\leq)^\dagger = \geq$.
2. *Greater Than Equal Signs.* Dually to **Item 1**, we have $(\geq)^\dagger = \leq$.

¹³*Further Terminology:* Also called the **opposite of R** , the **transpose of R** , or the **converse of R** .

3. *Functions.* Let $f: A \rightarrow B$ be a function. We have

$$\begin{aligned}\text{Gr}(f)^\dagger &= f^{-1}, \\ (f^{-1})^\dagger &= \text{Gr}(f).\end{aligned}$$

Proposition 3.11.1.3. Let $R: A \rightarrowtail B$ and $S: B \rightarrowtail C$ be relations.

1. *Functoriality.* The assignment $R \mapsto R^\dagger$ defines a functor (i.e. morphism of posets)

$$(-)^\dagger: \mathbf{Rel}(A, B) \rightarrow \mathbf{Rel}(B, A).$$

In particular, given relations $R, S: A \rightarrowtail B$, we have:

$$(\star) \text{ If } R \subset S, \text{ then } R^\dagger \subset S^\dagger.$$

2. *Interaction With Ranges and Domains.* We have

$$\begin{aligned}\text{dom}(R^\dagger) &= \text{range}(R), \\ \text{range}(R^\dagger) &= \text{dom}(R).\end{aligned}$$

3. *Interaction With Composition I.* We have

$$(S \diamond R)^\dagger = R^\dagger \diamond S^\dagger.$$

4. *Interaction With Composition II.* We have

$$\begin{aligned}\chi_B &\subset R \diamond R^\dagger, \\ \chi_A &\subset R^\dagger \diamond R.\end{aligned}$$

5. *Invertibility.* We have

$$(R^\dagger)^\dagger = R.$$

6. *Identity.* We have

$$\chi_A^\dagger = \chi_A.$$

Proof. **Item 1, Functoriality:** Clear.

Item 2, Interaction With Ranges and Domains: Clear.

Item 3, Interaction With Composition I: Clear.

Item 4, Interaction With Composition II: Clear.

Item 5, Invertibility: Clear.

Item 6, Identity: Clear. □

3.12 Composition of Relations

Let A , B , and C be sets and let $R: A \rightarrowtail B$ and $S: B \rightarrowtail C$ be relations.

Definition 3.12.1.1. The **composition of R and S** is the relation $S \diamond R$ defined as follows:

- Viewing relations from A to C as subsets of $A \times C$, we define

$$S \diamond R \stackrel{\text{def}}{=} \left\{ (a, c) \in A \times C \left| \begin{array}{l} \text{there exists some } b \in B \text{ such} \\ \text{that } a \sim_R b \text{ and } b \sim_S c \end{array} \right. \right\}.$$

- Viewing relations as functions $A \times B \rightarrow \{\text{true}, \text{false}\}$, we define

$$\begin{aligned} (S \diamond R)^{-1}_{-2} &\stackrel{\text{def}}{=} \int^{b \in B} S_b^{-1} \times R_{-2}^b \\ &= \bigvee_{b \in B} S_b^{-1} \times R_{-2}^b, \end{aligned}$$

where the join \bigvee is taken in the poset $(\{\text{true}, \text{false}\}, \preceq)$ of **Sets, Definition 2.2.1.3**.

- Viewing relations as functions $A \rightarrow \mathcal{P}(B)$, we define

$$S \diamond R \stackrel{\text{def}}{=} \text{Lan}_{\chi_B}(S) \circ R,$$

where $\text{Lan}_{\chi_B}(S)$ is computed by the formula

$$\begin{aligned} [\text{Lan}_{\chi_B}(S)](V) &\cong \int^{y \in B} \chi_{\mathcal{P}(B)}(\chi_y, V) \odot S_y \\ &\cong \int^{y \in B} \chi_V(y) \odot S_y \\ &\cong \bigcup_{y \in B} \chi_V(y) \odot S_y \\ &\cong \bigcup_{y \in V} S_y \end{aligned}$$

for each $V \in \mathcal{P}(B)$. In other words, $S \diamond R$ is defined by¹⁴

$$\begin{aligned} [S \diamond R](a) &\stackrel{\text{def}}{=} S(R(a)) \\ &\stackrel{\text{def}}{=} \bigcup_{x \in R(a)} S(x). \end{aligned}$$

for each $a \in A$.

Example 3.12.1.2. Here are some examples of composition of relations.

1. *Composing Less/Greater Than Equal With Greater/Less Than Equal Signs.* We have

$$\begin{aligned} \leq \diamond \geq &= \sim_{\text{triv}}, \\ \geq \diamond \leq &= \sim_{\text{triv}}. \end{aligned}$$

2. *Composing Less/Greater Than Equal Signs With Less/Greater Than Equal Signs.* We have

$$\begin{aligned} \leq \diamond \leq &= \leq, \\ \geq \diamond \geq &= \geq. \end{aligned}$$

Proposition 3.12.1.3. Let $R: A \rightarrowtail B$, $S: B \rightarrowtail C$, and $T: C \rightarrowtail D$ be relations.

1. *Interaction With Ranges and Domains.* We have

$$\begin{aligned} \text{dom}(S \diamond R) &\subset \text{dom}(R), \\ \text{range}(S \diamond R) &\subset \text{range}(S). \end{aligned}$$

2. *Associativity.* We have

$$(T \diamond S) \diamond R = T \diamond (S \diamond R).$$

3. *Unitality.* We have

$$\begin{aligned} \chi_B \diamond R &= R, \\ R \diamond \chi_A &= R. \end{aligned}$$

¹⁴That is: the relation R may send $a \in A$ to a number of elements $\{b_i\}_{i \in I}$ in B , and then

4. *Interaction With Inverses.* We have

$$(S \diamond R)^\dagger = R^\dagger \diamond S^\dagger.$$

5. *Interaction With Composition.* We have

$$\begin{aligned}\chi_B &\subset R \diamond R^\dagger, \\ \chi_A &\subset R^\dagger \diamond R.\end{aligned}$$

Proof. **Item 1, Interaction With Ranges and Domains:** Clear.

Item 2, Associativity: Indeed, we have

$$\begin{aligned}(T \diamond S) \diamond R &\stackrel{\text{def}}{=} \left(\int^{c \in C} T_c^{-1} \times S_{-2}^c \right) \diamond R \\ &\stackrel{\text{def}}{=} \int^{b \in B} \left(\int^{c \in C} T_c^{-1} \times S_b^c \right) \diamond R_{-2}^b \\ &= \int^{b \in B} \int^{c \in C} (T_c^{-1} \times S_b^c) \diamond R_{-2}^b \\ &= \int^{c \in C} \int^{b \in B} (T_c^{-1} \times S_b^c) \diamond R_{-2}^b \\ &= \int^{c \in C} \int^{b \in B} T_c^{-1} \times (S_b^c \diamond R_{-2}^b) \\ &= \int^{c \in C} T_c^{-1} \times \left(\int^{b \in B} S_b^c \diamond R_{-2}^b \right) \\ &\stackrel{\text{def}}{=} \int^{c \in C} T_c^{-1} \times (S \diamond R)_{-2}^c \\ &\stackrel{\text{def}}{=} T \diamond (S \diamond R).\end{aligned}$$

In the language of relations, given $a \in A$ and $d \in D$, the stated equality witnesses the equivalence of the following two statements:

1. We have $a \sim_{(T \diamond S) \diamond R} d$, i.e. there exists some $b \in B$ such that:

(a) We have $a \sim_R b$;

the relation S may send the image of each of the b_i 's to a number of elements $\{S(b_i)\}_{i \in I} = \left\{ \{c_{j_i}\}_{j_i \in J_i} \right\}_{i \in I}$ in C .

(b) We have $b \sim_{T \diamond S} d$, i.e. there exists some $c \in C$ such that:

- i. We have $b \sim_S c$;
- ii. We have $c \sim_T d$;

2. We have $a \sim_{T \diamond (S \diamond R)} d$, i.e. there exists some $c \in C$ such that:

(a) We have $a \sim_{S \diamond R} c$, i.e. there exists some $b \in B$ such that:

- i. We have $a \sim_R b$;
- ii. We have $b \sim_S c$;

(b) We have $c \sim_T d$;

both of which are equivalent to the statement

- There exist $b \in B$ and $c \in C$ such that $a \sim_R b \sim_S c \sim_T d$.

Item 3, Unitality: Indeed, we have

$$\begin{aligned}
 \chi_B \diamond R &\stackrel{\text{def}}{=} \int^{x \in B} (\chi_B)_x^{-1} \times R_{-2}^x \\
 &= \bigvee_{x \in B} (\chi_B)_x^{-1} \times R_{-2}^x \\
 &= \bigvee_{\substack{x \in B \\ x = -1}} R_{-2}^x \\
 &= R_{-2}^{-1},
 \end{aligned}$$

and

$$\begin{aligned}
 R \diamond \chi_A &\stackrel{\text{def}}{=} \int^{x \in A} R_x^{-1} \times (\chi_A)_{-2}^x \\
 &= \bigvee_{x \in B} R_x^{-1} \times (\chi_A)_{-2}^x \\
 &= \bigvee_{\substack{x \in B \\ x = -2}} R_x^{-1} \\
 &= R_{-2}^{-1}.
 \end{aligned}$$

In the language of relations, given $a \in A$ and $b \in B$:

- The equality

$$\chi_B \diamond R = R$$

witnesses the equivalence of the following two statements:

1. We have $a \sim_b B$.
2. There exists some $b' \in B$ such that:
 - (a) We have $a \sim_R b'$
 - (b) We have $b' \sim_{\chi_B} b$, i.e. $b' = b$.

- The equality

$$R \diamond \chi_A = R$$

witnesses the equivalence of the following two statements:

1. There exists some $a' \in A$ such that:
 - (a) We have $a \sim_{\chi_B} a'$, i.e. $a = a'$.
 - (b) We have $a' \sim_R b$
2. We have $a \sim_b B$.

Item 4, Interaction With Inverses: Clear.

Item 5, Interaction With Composition: Clear. □

3.13 The Collage of a Relation

Let A and B be sets and let $R: A \rightarrow B$ be a relation from A to B .

Definition 3.13.1.1. The **collage of R** ¹⁵ is the poset $\mathbf{Coll}(R) \stackrel{\text{def}}{=} (\text{Coll}(R), \preceq_{\mathbf{Coll}(R)})$ consisting of:

- *The Underlying Set.* The set $\text{Coll}(R)$ defined by

$$\text{Coll}(R) \stackrel{\text{def}}{=} A \coprod B.$$

- *The Partial Order.* The partial order

$$\preceq_{\mathbf{Coll}(R)}: \text{Coll}(R) \times \text{Coll}(R) \rightarrow \{\text{true}, \text{false}\}$$

on $\text{Coll}(R)$ defined by

$$\preceq(a, b) \stackrel{\text{def}}{=} \begin{cases} \text{true} & \text{if } a = b \text{ or } a \sim_R b, \\ \text{false} & \text{otherwise.} \end{cases}$$

¹⁵*Further Terminology:* Also called the **cograph of R** .

Proposition 3.13.1.2. Let A and B be sets and let $R: A \rightarrowtail B$ be a relation from A to B .

1. *Functoriality I.* The assignment $R \mapsto \mathbf{Coll}(R)$ defines a functor¹⁶

$$\mathbf{Coll}: \mathbf{Rel}(A, B) \rightarrow \mathbf{Pos}_{/\Delta^1}(A, B),$$

where

- *Action on Objects.* For each $R \in \mathbf{Obj}(\mathbf{Rel}(A, B))$, we have

$$[\mathbf{Coll}](R) \stackrel{\text{def}}{=} (\mathbf{Coll}(R), \phi_R)$$

for each $R \in \mathbf{Rel}(A, B)$, where

- The poset $\mathbf{Coll}(R)$ is the collage of R of [Definition 3.13.1.1](#).
- The morphism $\phi_R: \mathbf{Coll}(R) \rightarrow \Delta^1$ is given by

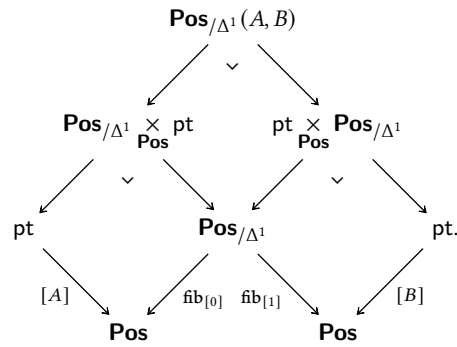
$$\phi_R(x) \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if } x \in A, \\ 1 & \text{if } x \in B \end{cases}$$

for each $x \in \mathbf{Coll}(R)$.

¹⁶Here $\mathbf{Pos}_{/\Delta^1}(A, B)$ is the category defined as the pullback

$$\mathbf{Pos}_{/\Delta^1}(A, B) \stackrel{\text{def}}{=} \text{pt}_{[A], \mathbf{Pos}, \text{fib}_0} \times_{\mathbf{Pos}_{/\Delta^1} \text{fib}_1, \mathbf{Pos}, [B]} \text{pt},$$

as in the diagram



Explicitly, an object of $\mathbf{Pos}_{/\Delta^1}(A, B)$ is a pair (X, ϕ_X) consisting of

- A poset X ;
- A morphism $\phi_X: X \rightarrow \Delta^1$;

- *Action on Morphisms.* For each $R, S \in \text{Obj}(\mathbf{Rel}(A, B))$, the action on Hom-sets

$$\mathbf{Coll}_{R,S}: \text{Hom}_{\mathbf{Rel}(A,B)}(R, S) \rightarrow \text{Pos}(\mathbf{Coll}(R), \mathbf{Coll}(S))$$

of \mathbf{Coll} at (R, S) is given by sending an inclusion

$$\iota: R \subset S$$

to the morphism

$$\mathbf{Coll}(\iota): \mathbf{Coll}(R) \rightarrow \mathbf{Coll}(S)$$

of posets over Δ^1 defined by

$$[\mathbf{Coll}(\iota)](x) \stackrel{\text{def}}{=} x$$

for each $x \in \mathbf{Coll}(R)$.¹⁷

2. *Equivalence.* The functor of **Item 1** is an equivalence of categories.

Proof. **Item 1, Functoriality:** Clear.

Item 2, Equivalence: Omitted. □

4 Functoriality of Powersets

4.1 Direct Images

Let A and B be sets and let $R: A \rightarrowtail B$ be a relation.

Definition 4.1.1.1. The **direct image function associated to R** is the function

$$R_*: \mathcal{P}(A) \rightarrow \mathcal{P}(B)$$

such that $\phi_X^{-1}(0) = A$ and $\phi_Y^{-1}(0) = B$, with morphisms between such objects being morphisms of posets over Δ^1 .

¹⁷Note that this is indeed a morphism of posets: if $x \preceq_{\mathbf{Coll}(R)} y$, then $x = y$ or $x \sim_R y$, so we have either $x = y$ or $x \sim_S y$ (as $R \subset S$), and thus $x \preceq_{\mathbf{Coll}(S)} y$.

defined by^{18,19}

$$\begin{aligned} R_*(U) &\stackrel{\text{def}}{=} R(U) \\ &\stackrel{\text{def}}{=} \bigcup_{a \in U} R(a) \\ &= \left\{ b \in B \left| \begin{array}{l} \text{there exists some } a \in U \\ \text{such that } b \in R(a) \end{array} \right. \right\} \end{aligned}$$

for each $U \in \mathcal{P}(A)$.

Remark 4.1.1.2. Identifying subsets of A with relations from pt to A via **Constructions With Sets, Item 3** of **Proposition 4.3.1.6**, we see that the direct image function associated to R is equivalently the function

$$R_*: \underbrace{\mathcal{P}(A)}_{\cong \text{Rel}(\text{pt}, A)} \rightarrow \underbrace{\mathcal{P}(B)}_{\cong \text{Rel}(\text{pt}, B)}$$

defined by

$$R_*(U) \stackrel{\text{def}}{=} R \diamond U$$

for each $U \in \mathcal{P}(A)$, where $R \diamond U$ is the composition

$$\text{pt} \xrightarrow{U} A \xrightarrow{R} B.$$

Proposition 4.1.1.3. Let $R: A \rightarrowtail B$ be a relation.

1. *Functoriality.* The assignment $U \mapsto R_*(U)$ defines a functor

$$R_*: (\mathcal{P}(A), \subset) \rightarrow (\mathcal{P}(B), \subset)$$

where

- *Action on Objects.* For each $U \in \mathcal{P}(A)$, we have

$$[R_*](U) \stackrel{\text{def}}{=} R_*(U).$$

¹⁸*Further Terminology:* The set $R(U)$ is called the **direct image of U by R** .

¹⁹We also have

$$R_*(U) = B \setminus R_!(A \setminus U);$$

see **Item 7** of **Proposition 4.1.1.3**.

- *Action on Morphisms.* For each $U, V \in \mathcal{P}(A)$:
 - If $U \subset V$, then $R_*(U) \subset R_*(V)$.

2. *Adjointness.* We have an adjunction

$$(R_* \dashv R_{-1}): \mathcal{P}(A) \begin{matrix} \xrightarrow{R_*} \\ \perp \\ \xleftarrow{R_{-1}} \end{matrix} \mathcal{P}(B),$$

witnessed by a bijections of sets

$$\mathrm{Hom}_{\mathcal{P}(A)}(R_*(U), V) \cong \mathrm{Hom}_{\mathcal{P}(A)}(U, R_{-1}(V)),$$

natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$, i.e. such that:

- (★) The following conditions are equivalent:
- We have $R_*(U) \subset V$.
 - We have $U \subset R_{-1}(V)$.

3. *Preservation of Colimits.* We have an equality of sets

$$R_*\left(\bigcup_{i \in I} U_i\right) = \bigcup_{i \in I} R_*(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have equalities

$$\begin{aligned} R_*(U) \cup R_*(V) &= R_*(U \cup V), \\ R_*(\emptyset) &= \emptyset, \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

4. *Oplax Preservation of Limits.* We have an inclusion of sets

$$R_*\left(\bigcap_{i \in I} U_i\right) \subset \bigcap_{i \in I} R_*(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have inclusions

$$\begin{aligned} R_*(U \cap V) &\subset R_*(U) \cap R_*(V), \\ R_*(A) &\subset B, \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

5. *Symmetric Strict Monoidality With Respect to Unions.* The direct image function of **Item 1** has a symmetric strict monoidal structure

$$(R_*, R_*^\otimes, R_{*|\mathbb{1}}^\otimes): (\mathcal{P}(A), \cup, \emptyset) \rightarrow (\mathcal{P}(B), \cup, \emptyset),$$

being equipped with equalities

$$\begin{aligned} R_{*|U,V}^\otimes: R_*(U) \cup R_*(V) &\xrightarrow{=} R_*(U \cup V), \\ R_{*|\mathbb{1}}^\otimes: \emptyset &\xrightarrow{=} \emptyset, \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

6. *Symmetric Oplax Monoidality With Respect to Intersections.* The direct image function of **Item 1** has a symmetric oplax monoidal structure

$$(R_*, R_*^\otimes, R_{*|\mathbb{1}}^\otimes): (\mathcal{P}(A), \cap, A) \rightarrow (\mathcal{P}(B), \cap, B),$$

being equipped with inclusions

$$\begin{aligned} R_{*|U,V}^\otimes: R_*(U \cap V) &\subset R_*(U) \cap R_*(V), \\ R_{*|\mathbb{1}}^\otimes: R_*(A) &\subset B, \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

7. *Relation to Direct Images With Compact Support.* We have

$$R_*(U) = B \setminus R_!(A \setminus U)$$

for each $U \in \mathcal{P}(A)$.

Proof. **Item 1**, *Functoriality*: Clear.

Item 2, *Adjointness*: This follows from ??, ?? of ??.

Item 3, *Preservation of Colimits*: This follows from **Item 2** and ??, ?? of ??.

Item 4, *Oplax Preservation of Limits*: Omitted.

Item 5, *Symmetric Strict Monoidality With Respect to Unions*: This follows from **Item 3**.

Item 6, *Symmetric Oplax Monoidality With Respect to Intersections*: This follows from **Item 4**.

Item 7, *Relation to Direct Images With Compact Support*: The proof proceeds in

the same way as in the case of functions ([Constructions With Sets, Item 9 of Proposition 4.4.1.4](#)): applying [Item 7 of Proposition 4.4.1.3](#) to $A \setminus U$, we have

$$\begin{aligned} R_!(A \setminus U) &= B \setminus R_*(A \setminus (A \setminus U)) \\ &= B \setminus R_*(U). \end{aligned}$$

Taking complements, we then obtain

$$\begin{aligned} R_*(U) &= B \setminus (B \setminus R_*(U)), \\ &= B \setminus R_!(A \setminus U), \end{aligned}$$

which finishes the proof. \square

Proposition 4.1.1.4. Let $R: A \rightarrowtail B$ be a relation.

1. *Functionality I.* The assignment $R \mapsto R_*$ defines a function

$$(-)_*: \text{Rel}(A, B) \rightarrow \text{Sets}(\mathcal{P}(A), \mathcal{P}(B)).$$

2. *Functionality II.* The assignment $R \mapsto R_*$ defines a function

$$(-)_*: \text{Rel}(A, B) \rightarrow \text{Pos}((\mathcal{P}(A), \subset), (\mathcal{P}(B), \subset)).$$

3. *Interaction With Identities.* For each $A \in \text{Obj}(\text{Sets})$, we have²⁰

$$(\chi_A)_* = \text{id}_{\mathcal{P}(A)}.$$

4. *Interaction With Composition.* For each pair of composable relations $R: A \rightarrowtail$

²⁰That is, the postcomposition function

$$(\chi_A)_*: \text{Rel}(\text{pt}, A) \rightarrow \text{Rel}(\text{pt}, A)$$

is equal to $\text{id}_{\text{Rel}(\text{pt}, A)}$.

B and $S: B \rightarrow C$, we have²¹

$$(S \diamond R)_* = S_* \circ R_*,$$

$$\begin{array}{ccc} \mathcal{P}(A) & \xrightarrow{R_*} & \mathcal{P}(B) \\ & \searrow (S \diamond R)_* & \downarrow S_* \\ & & \mathcal{P}(C). \end{array}$$

Proof. **Item 1, Functionality I:** Clear.

Item 2, Functionality II: Clear.

Item 3, Interaction With Identities: Indeed, we have

$$\begin{aligned} (\chi_A)_*(U) &\stackrel{\text{def}}{=} \bigcup_{a \in U} \chi_A(a) \\ &\stackrel{\text{def}}{=} \bigcup_{a \in U} \{a\} \\ &= U \\ &\stackrel{\text{def}}{=} \text{id}_{\mathcal{P}(A)}(U) \end{aligned}$$

for each $U \in \mathcal{P}(A)$. Thus $(\chi_A)_* = \text{id}_{\mathcal{P}(A)}$.

²¹That is, we have

$$(S \diamond R)_* = S_* \circ R_*,$$

$$\begin{array}{ccc} \text{Rel}(\text{pt}, A) & \xrightarrow{R_*} & \text{Rel}(\text{pt}, B) \\ & \searrow (S \diamond R)_* & \downarrow S_* \\ & & \text{Rel}(\text{pt}, C). \end{array}$$

Item 4, Interaction With Composition: Indeed, we have

$$\begin{aligned}
 (S \diamond R)_*(U) &\stackrel{\text{def}}{=} \bigcup_{a \in U} [S \diamond R](a) \\
 &\stackrel{\text{def}}{=} \bigcup_{a \in U} S(R(a)) \\
 &\stackrel{\text{def}}{=} \bigcup_{a \in U} S_*(R(a)) \\
 &= S_*\left(\bigcup_{a \in U} R(a)\right) \\
 &\stackrel{\text{def}}{=} S_*(R_*(U)) \\
 &\stackrel{\text{def}}{=} [S_* \circ R_*](U)
 \end{aligned}$$

for each $U \in \mathcal{P}(A)$, where we used *Item 3* of *Proposition 4.1.1.3*. Thus $(S \diamond R)_* = S_* \circ R_*$. \square

4.2 Strong Inverse Images

Let A and B be sets and let $R: A \rightarrow B$ be a relation.

Definition 4.2.1.1. The **strong inverse image function associated to R** is the function

$$R_{-1}: \mathcal{P}(B) \rightarrow \mathcal{P}(A)$$

defined by²²

$$R_{-1}(V) \stackrel{\text{def}}{=} \{a \in A \mid R(a) \subset V\}$$

for each $V \in \mathcal{P}(B)$.

Remark 4.2.1.2. Identifying subsets of B with relations from pt to B via *Constructions With Sets*, *Item 3* of *Proposition 4.3.1.6*, we see that the inverse image function associated to R is equivalently the function

$$\begin{array}{ccc}
 R_{-1}: & \underbrace{\mathcal{P}(B)}_{\cong \text{Rel}(\text{pt}, B)} & \rightarrow & \underbrace{\mathcal{P}(A)}_{\cong \text{Rel}(\text{pt}, A)}
 \end{array}$$

²²*Further Terminology:* The set $R_{-1}(V)$ is called the **strong inverse image of V by R** .

defined by

$$R_{-1}(V) \stackrel{\text{def}}{=} \text{Rift}_R(V),$$

and being explicitly computed by

$$\begin{aligned} R_{-1}(V) &\stackrel{\text{def}}{=} \text{Rift}_R(V) \\ &\cong \int_{b \in B} \text{Hom}_{\{t, f\}}(R_{-1}^b, V_{-2}^b), \end{aligned}$$

where we have used [Proposition 2.4.1.1](#).

Proof. We have

$$\begin{aligned}
\text{Rift}_R(V) &\cong \int_{b \in B} \text{Hom}_{\{t,f\}}(R_{-1}^b, V_{-2}^b) \\
&= \left\{ a \in A \mid \int_{b \in B} \text{Hom}_{\{t,f\}}(R_a^b, V_{\star}^b) = \text{true} \right\} \\
&= \left\{ a \in A \mid \begin{array}{l} \text{for each } b \in B, \text{ at least one of the} \\ \text{following conditions hold:} \\ \\ 1. \text{ We have } R_a^b = \text{false} \\ 2. \text{ The following conditions hold:} \\ \\ \quad (a) \text{ We have } R_a^b = \text{true} \\ \quad (b) \text{ We have } V_{\star}^b = \text{true} \end{array} \right\} \\
&= \left\{ a \in A \mid \begin{array}{l} \text{for each } b \in B, \text{ at least one of the} \\ \text{following conditions hold:} \\ \\ 1. \text{ We have } b \notin R(a) \\ 2. \text{ The following conditions hold:} \\ \\ \quad (a) \text{ We have } b \in R(a) \\ \quad (b) \text{ We have } b \in V \end{array} \right\} \\
&= \{ a \in A \mid \text{for each } b \in R(a), \text{ we have } b \in V \} \\
&= \{ a \in A \mid R(a) \subset V \} \\
&\stackrel{\text{def}}{=} R_{-1}(V).
\end{aligned}$$

This finishes the proof. \square

Proposition 4.2.1.3. Let $R: A \rightarrowtail B$ be a relation.

1. *Functoriality.* The assignment $V \mapsto R_{-1}(V)$ defines a functor

$$R_{-1}: (\mathcal{P}(B), \subset) \rightarrow (\mathcal{P}(A), \subset)$$

where

- *Action on Objects.* For each $V \in \mathcal{P}(B)$, we have

$$[R_{-1}](V) \stackrel{\text{def}}{=} R_{-1}(V).$$

- *Action on Morphisms.* For each $U, V \in \mathcal{P}(B)$:
 - If $U \subset V$, then $R_{-1}(U) \subset R_{-1}(V)$.

2. *Adjointness.* We have an adjunction

$$(R_* \dashv R_{-1}): \mathcal{P}(A) \begin{array}{c} \xrightarrow{R_*} \\ \perp \\ \xleftarrow{R_{-1}} \end{array} \mathcal{P}(B),$$

witnessed by a bijections of sets

$$\mathrm{Hom}_{\mathcal{P}(A)}(R_*(U), V) \cong \mathrm{Hom}_{\mathcal{P}(A)}(U, R_{-1}(V)),$$

natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$, i.e. such that:

- (★) The following conditions are equivalent:
- We have $R_*(U) \subset V$.
 - We have $U \subset R_{-1}(V)$.

3. *Lax Preservation of Colimits.* We have an inclusion of sets

$$\bigcup_{i \in I} R_{-1}(U_i) \subset R_{-1}\left(\bigcup_{i \in I} U_i\right),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(B)^{\times I}$. In particular, we have inclusions

$$\begin{aligned} R_{-1}(U) \cup R_{-1}(V) &\subset R_{-1}(U \cup V), \\ \emptyset &\subset R_{-1}(\emptyset), \end{aligned}$$

natural in $U, V \in \mathcal{P}(B)$.

4. *Preservation of Limits.* We have an equality of sets

$$R_{-1}\left(\bigcap_{i \in I} U_i\right) = \bigcap_{i \in I} R_{-1}(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(B)^{\times I}$. In particular, we have equalities

$$\begin{aligned} R_{-1}(U \cap V) &= R_{-1}(U) \cap R_{-1}(V), \\ R_{-1}(B) &= B, \end{aligned}$$

natural in $U, V \in \mathcal{P}(B)$.

5. *Symmetric Lax Monoidality With Respect to Unions.* The direct image with compact support function of **Item 1** has a symmetric lax monoidal structure

$$(R_{-1}, R_{-1}^{\otimes}, R_{-1|1}^{\otimes}) : (\mathcal{P}(A), \cup, \emptyset) \rightarrow (\mathcal{P}(B), \cup, \emptyset),$$

being equipped with inclusions

$$\begin{aligned} R_{-1|U,V}^{\otimes} : R_{-1}(U) \cup R_{-1}(V) &\subset R_{-1}(U \cup V), \\ R_{-1|1}^{\otimes} : \emptyset &\subset R_{-1}(\emptyset), \end{aligned}$$

natural in $U, V \in \mathcal{P}(B)$.

6. *Symmetric Strict Monoidality With Respect to Intersections.* The direct image function of **Item 1** has a symmetric strict monoidal structure

$$(R_{-1}, R_{-1}^{\otimes}, R_{-1|1}^{\otimes}) : (\mathcal{P}(A), \cap, A) \rightarrow (\mathcal{P}(B), \cap, B),$$

being equipped with equalities

$$\begin{aligned} R_{-1|U,V}^{\otimes} : R_{-1}(U \cap V) &\xrightarrow{=} R_{-1}(U) \cap R_{-1}(V), \\ R_{-1|1}^{\otimes} : R_{-1}(A) &\xrightarrow{=} B, \end{aligned}$$

natural in $U, V \in \mathcal{P}(B)$.

7. *Interaction With Weak Inverse Images I.* We have

$$R_{-1}(V) = A \setminus R^{-1}(B \setminus V)$$

for each $V \in \mathcal{P}(B)$.

8. *Interaction With Weak Inverse Images II.* Let $R : A \dashrightarrow B$ be a relation from A to B .

- (a) If R is a total relation, then we have an inclusion of sets

$$R_{-1}(V) \subset R^{-1}(V)$$

natural in $V \in \mathcal{P}(B)$.

- (b) If R is total and functional, then the above inclusion is in fact an equality.

(c) Conversely, if we have $R_{-1} = R^{-1}$, then R is total and functional.

Proof. **Item 1**, *Functoriality*: Clear.

Item 2, *Adjointness*: This follows from ??, ?? of ??.

Item 3, *Lax Preservation of Colimits*: Omitted.

Item 4, *Preservation of Limits*: This follows from **Item 2** and ??, ?? of ??.

Item 5, *Symmetric Lax Monoidality With Respect to Unions*: This follows from **Item 3**.

Item 6, *Symmetric Strict Monoidality With Respect to Intersections*: This follows from **Item 4**.

Item 7, *Interaction With Weak Inverse Images I*: We claim we have an equality

$$R_{-1}(B \setminus V) = A \setminus R^{-1}(V).$$

Indeed, we have

$$\begin{aligned} R_{-1}(B \setminus V) &= \{a \in A \mid R(a) \subset B \setminus V\}, \\ A \setminus R^{-1}(V) &= \{a \in A \mid R(a) \cap V = \emptyset\}. \end{aligned}$$

Taking $V = B \setminus V$ then implies the original statement.

Item 8, *Interaction With Weak Inverse Images II*: **Item 8a** is clear, while **Items 8b** and **8c** follow from **Item 6** of **Proposition 3.1.1.2**. \square

Proposition 4.2.1.4. Let $R: A \multimap B$ be a relation.

1. *Functionality I*. The assignment $R \mapsto R_{-1}$ defines a function

$$(-)_{-1}: \text{Sets}(A, B) \rightarrow \text{Sets}(\mathcal{P}(A), \mathcal{P}(B)).$$

2. *Functionality II*. The assignment $R \mapsto R_{-1}$ defines a function

$$(-)_{-1}: \text{Sets}(A, B) \rightarrow \text{Pos}((\mathcal{P}(A), \subset), (\mathcal{P}(B), \subset)).$$

3. *Interaction With Identities*. For each $A \in \text{Obj}(\text{Sets})$, we have

$$(\text{id}_A)_{-1} = \text{id}_{\mathcal{P}(A)}.$$

4. *Interaction With Composition*. For each pair of composable relations $R: A \multimap$

B and $S: B \rightarrowtail C$, we have

$$(S \diamond R)_{-1} = R_{-1} \circ S_{-1},$$

$$\begin{array}{ccc} \mathcal{P}(C) & \xrightarrow{S_{-1}} & \mathcal{P}(B) \\ & \searrow (S \diamond R)_{-1} & \downarrow R_{-1} \\ & & \mathcal{P}(A). \end{array}$$

Proof. **Item 1, Functionality I:** Clear.

Item 2, Functionality II: Clear.

Item 3, Interaction With Identities: Indeed, we have

$$\begin{aligned} (\chi_A)_{-1}(U) &\stackrel{\text{def}}{=} \{a \in A \mid \chi_A(a) \subset U\} \\ &\stackrel{\text{def}}{=} \{a \in A \mid \{a\} \subset U\} \\ &= U \end{aligned}$$

for each $U \in \mathcal{P}(A)$. Thus $(\chi_A)_{-1} = \text{id}_{\mathcal{P}(A)}$.

Item 4, Interaction With Composition: Indeed, we have

$$\begin{aligned} (S \diamond R)_{-1}(U) &\stackrel{\text{def}}{=} \{a \in A \mid [S \diamond R](a) \subset U\} \\ &\stackrel{\text{def}}{=} \{a \in A \mid S(R(a)) \subset U\} \\ &\stackrel{\text{def}}{=} \{a \in A \mid S_*(R(a)) \subset U\} \\ &= \{a \in A \mid R(a) \subset S_{-1}(U)\} \\ &\stackrel{\text{def}}{=} R_{-1}(S_{-1}(U)) \\ &\stackrel{\text{def}}{=} [R_{-1} \circ S_{-1}](U) \end{aligned}$$

for each $U \in \mathcal{P}(C)$, where we used **Item 2 of Proposition 4.2.1.3**, which implies that the conditions

- We have $S_*(R(a)) \subset U$.
- We have $R(a) \subset S_{-1}(U)$.

are equivalent. Thus $(S \diamond R)_{-1} = R_{-1} \circ S_{-1}$. □

4.3 Weak Inverse Images

Let A and B be sets and let $R: A \rightarrowtail B$ be a relation.

Definition 4.3.1.1. The **weak inverse image function associated to** R^{23} is the function

$$R^{-1}: \mathcal{P}(B) \rightarrow \mathcal{P}(A)$$

defined by²⁴

$$R^{-1}(V) \stackrel{\text{def}}{=} \{a \in A \mid R(a) \cap V \neq \emptyset\}$$

for each $V \in \mathcal{P}(B)$.

Remark 4.3.1.2. Identifying subsets of B with relations from B to pt via **Constructions With Sets, Item 3** of **Proposition 4.3.1.6**, we see that the weak inverse image function associated to R is equivalently the function

$$R^{-1}: \underbrace{\mathcal{P}(B)}_{\cong \text{Rel}(B, \text{pt})} \rightarrow \underbrace{\mathcal{P}(A)}_{\cong \text{Rel}(A, \text{pt})}$$

defined by

$$R^{-1}(V) \stackrel{\text{def}}{=} V \diamond R$$

for each $V \in \mathcal{P}(A)$, where $R \diamond V$ is the composition

$$A \xrightarrow{R} B \xrightarrow{V} \text{pt}.$$

Explicitly, we have

$$\begin{aligned} R^{-1}(V) &\stackrel{\text{def}}{=} V \diamond R \\ &\stackrel{\text{def}}{=} \int^{b \in B} V_b^{-1} \times R_{-2}^b. \end{aligned}$$

²³*Further Terminology:* Also called simply the **inverse image function associated to** R .

²⁴*Further Terminology:* The set $R^{-1}(V)$ is called the **weak inverse image of** V **by** R or simply the **inverse image of** V **by** R .

Proof. We have

$$\begin{aligned}
 V \diamond R &\stackrel{\text{def}}{=} \int^{b \in B} V_b^{-1} \times R_{-2}^b \\
 &= \left\{ a \in A \mid \int^{b \in B} V_b^* \times R_a^b = \text{true} \right\} \\
 &= \left\{ a \in A \mid \begin{array}{l} \text{there exists } b \in B \text{ such that the} \\ \text{following conditions hold:} \\ 1. \text{ We have } V_b^* = \text{true} \\ 2. \text{ We have } R_a^b = \text{true} \end{array} \right\} \\
 &= \left\{ a \in A \mid \begin{array}{l} \text{there exists } b \in B \text{ such that the} \\ \text{following conditions hold:} \\ 1. \text{ We have } b \in V \\ 2. \text{ We have } b \in R(a) \end{array} \right\} \\
 &= \{ a \in A \mid \text{there exists } b \in V \text{ such that } b \in R(a) \} \\
 &= \{ a \in A \mid R(a) \cap V \neq \emptyset \} \\
 &\stackrel{\text{def}}{=} R^{-1}(V)
 \end{aligned}$$

This finishes the proof. \square

Proposition 4.3.1.3. Let $R: A \rightarrowtail B$ be a relation.

1. *Functoriality.* The assignment $V \mapsto R^{-1}(V)$ defines a functor

$$R^{-1}: (\mathcal{P}(B), \subset) \rightarrow (\mathcal{P}(A), \subset)$$

where

- *Action on Objects.* For each $V \in \mathcal{P}(B)$, we have

$$[R^{-1}](V) \stackrel{\text{def}}{=} R^{-1}(V).$$

- *Action on Morphisms.* For each $U, V \in \mathcal{P}(B)$:
 - If $U \subset V$, then $R^{-1}(U) \subset R^{-1}(V)$.

2. *Adjointness.* We have an adjunction

$$(R^{-1} \dashv R_!) : \mathcal{P}(B) \begin{matrix} \xrightarrow{R^{-1}} \\ \perp \\ \xleftarrow{R_!} \end{matrix} \mathcal{P}(A),$$

witnessed by a bijections of sets

$$\mathrm{Hom}_{\mathcal{P}(A)}(R^{-1}(U), V) \cong \mathrm{Hom}_{\mathcal{P}(A)}(U, R_!(V)),$$

natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$, i.e. such that:

(★) The following conditions are equivalent:

- We have $R^{-1}(U) \subset V$.
- We have $U \subset R_!(V)$.

3. *Preservation of Colimits.* We have an equality of sets

$$R^{-1}\left(\bigcup_{i \in I} U_i\right) = \bigcup_{i \in I} R^{-1}(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(B)^{\times I}$. In particular, we have equalities

$$\begin{aligned} R^{-1}(U) \cup R^{-1}(V) &= R^{-1}(U \cup V), \\ R^{-1}(\emptyset) &= \emptyset, \end{aligned}$$

natural in $U, V \in \mathcal{P}(B)$.

4. *Oplax Preservation of Limits.* We have an inclusion of sets

$$R^{-1}\left(\bigcap_{i \in I} U_i\right) \subset \bigcap_{i \in I} R^{-1}(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(B)^{\times I}$. In particular, we have inclusions

$$\begin{aligned} R^{-1}(U \cap V) &\subset R^{-1}(U) \cap R^{-1}(V), \\ R^{-1}(A) &\subset B, \end{aligned}$$

natural in $U, V \in \mathcal{P}(B)$.

5. *Symmetric Strict Monoidality With Respect to Unions.* The direct image

function of **Item 1** has a symmetric strict monoidal structure

$$(R^{-1}, R^{-1, \otimes}, R_{\perp}^{-1, \otimes}) : (\mathcal{P}(A), \cup, \emptyset) \rightarrow (\mathcal{P}(B), \cup, \emptyset),$$

being equipped with equalities

$$\begin{aligned} R_{U,V}^{-1, \otimes} : R^{-1}(U) \cup R^{-1}(V) &\xrightarrow{=} R^{-1}(U \cup V), \\ R_{\perp}^{-1, \otimes} : \emptyset &\xrightarrow{=} \emptyset, \end{aligned}$$

natural in $U, V \in \mathcal{P}(B)$.

6. *Symmetric Oplax Monoidality With Respect to Intersections.* The direct image function of **Item 1** has a symmetric oplax monoidal structure

$$(R^{-1}, R^{-1, \otimes}, R_{\perp}^{-1, \otimes}) : (\mathcal{P}(A), \cap, A) \rightarrow (\mathcal{P}(B), \cap, B),$$

being equipped with inclusions

$$\begin{aligned} R_{U,V}^{-1, \otimes} : R^{-1}(U \cap V) &\subset R^{-1}(U) \cap R^{-1}(V), \\ R_{\perp}^{-1, \otimes} : R^{-1}(A) &\subset B, \end{aligned}$$

natural in $U, V \in \mathcal{P}(B)$.

7. *Interaction With Strong Inverse Images I.* We have

$$R^{-1}(V) = A \setminus R_{-1}(B \setminus V)$$

for each $V \in \mathcal{P}(B)$.

8. *Interaction With Strong Inverse Images II.* Let $R : A \dashrightarrow B$ be a relation from A to B .

- (a) If R is a total relation, then we have an inclusion of sets

$$R_{-1}(V) \subset R^{-1}(V)$$

natural in $V \in \mathcal{P}(B)$.

- (b) If R is total and functional, then the above inclusion is in fact an equality.

- (c) Conversely, if we have $R_{-1} = R^{-1}$, then R is total and functional.

Proof. **Item 1, Functoriality:** Clear.

Item 2, Adjointness: This follows from ??, ?? of ??.

Item 3, Preservation of Colimits: This follows from **Item 2** and ??, ?? of ??.

Item 4, Oplax Preservation of Limits: Omitted.

Item 5, Symmetric Strict Monoidality With Respect to Unions: This follows from **Item 3**.

Item 6, Symmetric Oplax Monoidality With Respect to Intersections: This follows from **Item 4**.

Item 7, Interaction With Strong Inverse Images I: This follows from **Item 7** of **Proposition 4.2.1.3**.

Item 8, Interaction With Strong Inverse Images II: This was proved in **Item 8** of **Proposition 4.2.1.3**. \square

Proposition 4.3.1.4. Let $R: A \rightarrowtail B$ be a relation.

1. *Functionality I.* The assignment $R \mapsto R^{-1}$ defines a function

$$(-)^{-1}: \text{Rel}(A, B) \rightarrow \text{Sets}(\mathcal{P}(A), \mathcal{P}(B)).$$

2. *Functionality II.* The assignment $R \mapsto R^{-1}$ defines a function

$$(-)^{-1}: \text{Rel}(A, B) \rightarrow \text{Pos}((\mathcal{P}(A), \subset), (\mathcal{P}(B), \subset)).$$

3. *Interaction With Identities.* For each $A \in \text{Obj}(\text{Sets})$, we have²⁵

$$(\chi_A)^{-1} = \text{id}_{\mathcal{P}(A)}.$$

4. *Interaction With Composition.* For each pair of composable relations $R: A \rightarrowtail$

²⁵That is, the postcomposition

$$(\chi_A)^{-1}: \text{Rel}(\text{pt}, A) \rightarrow \text{Rel}(\text{pt}, A)$$

is equal to $\text{id}_{\text{Rel}(\text{pt}, A)}$.

B and $S: B \rightarrowtail C$, we have²⁶

$$(S \diamond R)^{-1} = R^{-1} \circ S^{-1},$$

$$\begin{array}{ccc} \mathcal{P}(C) & \xrightarrow{S^{-1}} & \mathcal{P}(B) \\ & \searrow (S \diamond R)^{-1} & \downarrow R^{-1} \\ & & \mathcal{P}(A). \end{array}$$

Proof. **Item 1, Functionality I:** Clear.

Item 2, Functionality II: Clear.

Item 3, Interaction With Identities: This follows from **Categories, Item 5** of **Proposition 1.6.1.2**.

Item 4, Interaction With Composition: This follows from **Categories, Item 2** of **Proposition 1.6.1.2**. \square

4.4 Direct Images With Compact Support

Let A and B be sets and let $R: A \rightarrowtail B$ be a relation.

Definition 4.4.1.1. The **direct image with compact support function associated to R** is the function

$$R_! : \mathcal{P}(A) \rightarrow \mathcal{P}(B)$$

defined by^{27,28}

$$\begin{aligned} R_!(U) &\stackrel{\text{def}}{=} \left\{ b \in B \mid \begin{array}{l} \text{for each } a \in A, \text{ if we have} \\ b \in R(a), \text{ then } a \in U \end{array} \right\} \\ &= \{ b \in B \mid R^{-1}(b) \subset U \} \end{aligned}$$

²⁶That is, we have

$$(S \diamond R)^{-1} = R^{-1} \circ S^{-1},$$

$$\begin{array}{ccc} \text{Rel}(\text{pt}, C) & \xrightarrow{R^{-1}} & \text{Rel}(\text{pt}, B) \\ & \searrow (S \diamond R)^{-1} & \downarrow S^{-1} \\ & & \text{Rel}(\text{pt}, A). \end{array}$$

²⁷*Further Terminology:* The set $R_!(U)$ is called the **direct image with compact support of U by R** .

²⁸We also have

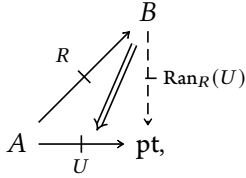
$$R_!(U) = B \setminus R_*(A \setminus U);$$

for each $U \in \mathcal{P}(A)$.

Remark 4.4.1.2. Identifying subsets of B with relations from pt to B via **Constructions With Sets, Item 3** of **Proposition 4.3.1.6**, we see that the direct image with compact support function associated to R is equivalently the function

$$R_! : \underbrace{\mathcal{P}(A)}_{\cong \text{Rel}(A, \text{pt})} \rightarrow \underbrace{\mathcal{P}(B)}_{\cong \text{Rel}(B, \text{pt})}$$

defined by

$$R_!(U) \stackrel{\text{def}}{=} \text{Ran}_R(U),$$


being explicitly computed by

$$\begin{aligned} R^*(U) &\stackrel{\text{def}}{=} \text{Ran}_R(U) \\ &\cong \int_{a \in A} \text{Hom}_{\{\text{t}, \text{f}\}}(R_a^{-2}, U_a^{-1}), \end{aligned}$$

where we have used **Proposition 2.3.1.1**.

see **Item 7** of **Proposition 4.4.1.3**.

Proof. We have

$$\begin{aligned}
 \text{Ran}_R(V) &\cong \int_{a \in A} \text{Hom}_{\{\text{t}, \text{f}\}}(R_a^{-2}, U_a^{-1}) \\
 &= \left\{ b \in B \mid \int_{a \in A} \text{Hom}_{\{\text{t}, \text{f}\}}(R_a^b, U_a^\star) = \text{true} \right\} \\
 &= \left\{ b \in B \mid \begin{array}{l} \text{for each } a \in A, \text{ at least one of the} \\ \text{following conditions hold:} \\ \\ 1. \text{ We have } R_a^b = \text{false} \\ 2. \text{ The following conditions hold:} \\ \\ \quad (a) \text{ We have } R_a^b = \text{true} \\ \quad (b) \text{ We have } U_a^\star = \text{true} \end{array} \right\} \\
 &= \left\{ b \in B \mid \begin{array}{l} \text{for each } a \in A, \text{ at least one of the} \\ \text{following conditions hold:} \\ \\ 1. \text{ We have } b \notin R(a) \\ 2. \text{ The following conditions hold:} \\ \\ \quad (a) \text{ We have } b \in R(a) \\ \quad (b) \text{ We have } a \in U \end{array} \right\} \\
 &= \left\{ b \in B \mid \begin{array}{l} \text{for each } a \in A, \text{ if we have} \\ b \in R(a), \text{ then } a \in U \end{array} \right\} \\
 &= \{ b \in B \mid R^{-1}(b) \subset U \} \\
 &\stackrel{\text{def}}{=} R^{-1}(U).
 \end{aligned}$$

This finishes the proof. \square

Proposition 4.4.1.3. Let $R: A \rightarrowtail B$ be a relation.

1. *Functoriality.* The assignment $U \mapsto R_!(U)$ defines a functor

$$R_!: (\mathcal{P}(A), \subset) \rightarrow (\mathcal{P}(B), \subset)$$

where

- *Action on Objects.* For each $U \in \mathcal{P}(A)$, we have

$$[R_!](U) \stackrel{\text{def}}{=} R_!(U).$$

- *Action on Morphisms.* For each $U, V \in \mathcal{P}(A)$:
 - If $U \subset V$, then $R_!(U) \subset R_!(V)$.

2. *Adjointness.* We have an adjunction

$$(R^{-1} \dashv R_!): \mathcal{P}(B) \begin{matrix} \xrightarrow{R^{-1}} \\ \perp \\ \xleftarrow{R_!} \end{matrix} \mathcal{P}(A),$$

witnessed by a bijections of sets

$$\mathrm{Hom}_{\mathcal{P}(A)}(R^{-1}(U), V) \cong \mathrm{Hom}_{\mathcal{P}(A)}(U, R_!(V)),$$

natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$, i.e. such that:

- (★) The following conditions are equivalent:
- We have $R^{-1}(U) \subset V$.
 - We have $U \subset R_!(V)$.

3. *Lax Preservation of Colimits.* We have an inclusion of sets

$$\bigcup_{i \in I} R_!(U_i) \subset R_!\left(\bigcup_{i \in I} U_i\right),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have inclusions

$$\begin{aligned} R_!(U) \cup R_!(V) &\subset R_!(U \cup V), \\ \emptyset &\subset R_!(\emptyset), \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

4. *Preservation of Limits.* We have an equality of sets

$$R_!\left(\bigcap_{i \in I} U_i\right) = \bigcap_{i \in I} R_!(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have equalities

$$\begin{aligned} R_!(U \cap V) &= R_!(U) \cap R_!(V), \\ R_!(A) &= B, \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

5. *Symmetric Lax Monoidality With Respect to Unions.* The direct image with compact support function of **Item 1** has a symmetric lax monoidal structure

$$(R_!, R_!^\otimes, R_{!|\mathbb{1}}^\otimes) : (\mathcal{P}(A), \cup, \emptyset) \rightarrow (\mathcal{P}(B), \cup, \emptyset),$$

being equipped with inclusions

$$\begin{aligned} R_{!|U,V}^\otimes : R_!(U) \cup R_!(V) &\subset R_!(U \cup V), \\ R_{!|\mathbb{1}}^\otimes : \emptyset &\subset R_!(\emptyset), \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

6. *Symmetric Strict Monoidality With Respect to Intersections.* The direct image function of **Item 1** has a symmetric strict monoidal structure

$$(R_!, R_!^\otimes, R_{!|\mathbb{1}}^\otimes) : (\mathcal{P}(A), \cap, A) \rightarrow (\mathcal{P}(B), \cap, B),$$

being equipped with equalities

$$\begin{aligned} R_{!|U,V}^\otimes : R_!(U \cap V) &\xrightarrow{=} R_!(U) \cap R_!(V), \\ R_{!|\mathbb{1}}^\otimes : R_!(A) &\xrightarrow{=} B, \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

7. *Relation to Direct Images.* We have

$$R_!(U) = B \setminus R_*(A \setminus U)$$

for each $U \in \mathcal{P}(A)$.

Proof. **Item 1**, *Functoriality*: Clear.

Item 2, *Adjointness*: This follows from ??, ?? of ??.

Item 3, *Lax Preservation of Colimits*: Omitted.

Item 4, *Preservation of Limits*: This follows from **Item 2** and ??, ?? of ??.

Item 5, *Symmetric Lax Monoidality With Respect to Unions*: This follows from **Item 3**.

Item 6, *Symmetric Strict Monoidality With Respect to Intersections*: This follows from **Item 4**.

Item 7, *Relation to Direct Images*: This follows from **Item 7** of **Proposition 4.1.1.3**.

Alternatively, we may prove it directly as follows, with the proof proceeding in the same way as in the case of functions (**Constructions With Sets**, **Item 9** of **Proposition 4.6.1.6**).

We claim that $R_!(U) = B \setminus R_*(A \setminus U)$:

- *The First Implication.* We claim that

$$R_!(U) \subset B \setminus R_*(A \setminus U).$$

Let $b \in R_!(U)$. We need to show that $b \notin R_*(A \setminus U)$, i.e. that there is no $a \in A \setminus U$ such that $b \in R(a)$.

This is indeed the case, as otherwise we would have $a \in R^{-1}(b)$ and $a \notin U$, contradicting $R^{-1}(b) \subset U$ (which holds since $b \in R_!(U)$).

Thus $b \in B \setminus R_*(A \setminus U)$.

- *The Second Implication.* We claim that

$$B \setminus R_*(A \setminus U) \subset R_!(U).$$

Let $b \in B \setminus R_*(A \setminus U)$. We need to show that $b \in R_!(U)$, i.e. that $R^{-1}(b) \subset U$.

Since $b \notin R_*(A \setminus U)$, there exists no $a \in A \setminus U$ such that $b \in R(a)$, and hence $R^{-1}(b) \subset U$.

Thus $b \in R_!(U)$.

This finishes the proof. \square

Proposition 4.4.1.4. Let $R: A \rightarrowtail B$ be a relation.

1. *Functionality I.* The assignment $R \mapsto R_!$ defines a function

$$(-)_!: \text{Sets}(A, B) \rightarrow \text{Sets}(\mathcal{P}(A), \mathcal{P}(B)).$$

2. *Functionality II.* The assignment $R \mapsto R_!$ defines a function

$$(-)_!: \text{Sets}(A, B) \rightarrow \text{Hom}_{\text{Pos}}((\mathcal{P}(A), \subset), (\mathcal{P}(B), \subset)).$$

3. *Interaction With Identities.* For each $A \in \text{Obj}(\text{Sets})$, we have

$$(\text{id}_A)_! = \text{id}_{\mathcal{P}(A)}.$$

4. *Interaction With Composition.* For each pair of composable relations $R: A \rightarrowtail B$ and $S: B \rightarrowtail C$, we have

$$(S \diamond R)_! = S_! \circ R_!, \quad \begin{array}{ccc} \mathcal{P}(A) & \xrightarrow{R_!} & \mathcal{P}(B) \\ & \searrow (S \diamond R)_! & \downarrow S_! \\ & & \mathcal{P}(C). \end{array}$$

Proof. **Item 1, Functionality I:** Clear.

Item 2, Functionality II: Clear.

Item 3, Interaction With Identities: Indeed, we have

$$\begin{aligned} (\chi_A)_!(U) &\stackrel{\text{def}}{=} \{a \in A \mid \chi_A^{-1}(a) \subset U\} \\ &\stackrel{\text{def}}{=} \{a \in A \mid \{a\} \subset U\} \\ &= U \end{aligned}$$

for each $U \in \mathcal{P}(A)$. Thus $(\chi_A)_! = \text{id}_{\mathcal{P}(A)}$.

Item 4, Interaction With Composition: Indeed, we have

$$\begin{aligned} (S \diamond R)_!(U) &\stackrel{\text{def}}{=} \{c \in C \mid [S \diamond R]^{-1}(c) \subset U\} \\ &\stackrel{\text{def}}{=} \{c \in C \mid S^{-1}(R^{-1}(c)) \subset U\} \\ &= \{c \in C \mid R^{-1}(c) \subset S_!(U)\} \\ &\stackrel{\text{def}}{=} R_!(S_!(U)) \\ &\stackrel{\text{def}}{=} [R_! \circ S_!](U) \end{aligned}$$

for each $U \in \mathcal{P}(C)$, where we used **Item 2** of **Proposition 4.4.1.3**, which implies that the conditions

- We have $S^{-1}(R^{-1}(c)) \subset U$.
- We have $R^{-1}(c) \subset S_!(U)$.

are equivalent. Thus $(S \diamond R)_! = S_! \circ R_!$. □

4.5 Functoriality of Powersets

Proposition 4.5.1.1. The assignment $X \mapsto \mathcal{P}(X)$ defines functors²⁹

$$\begin{aligned} \mathcal{P}_* &: \text{Rel} \rightarrow \text{Sets}, \\ \mathcal{P}_{-1} &: \text{Rel}^{\text{op}} \rightarrow \text{Sets}, \\ \mathcal{P}^{-1} &: \text{Rel}^{\text{op}} \rightarrow \text{Sets}, \\ \mathcal{P}_! &: \text{Rel} \rightarrow \text{Sets} \end{aligned}$$

where

²⁹The functor $\mathcal{P}_* : \text{Rel} \rightarrow \text{Sets}$ admits a left adjoint; see **Item 3** of **Proposition 3.1.1.2**.

- *Action on Objects.* For each $A \in \text{Obj}(\text{Rel})$, we have

$$\begin{aligned}\mathcal{P}_*(A) &\stackrel{\text{def}}{=} \mathcal{P}(A), \\ \mathcal{P}_{-1}(A) &\stackrel{\text{def}}{=} \mathcal{P}(A), \\ \mathcal{P}^{-1}(A) &\stackrel{\text{def}}{=} \mathcal{P}(A), \\ \mathcal{P}_!(A) &\stackrel{\text{def}}{=} \mathcal{P}(A).\end{aligned}$$

- *Action on Morphisms.* For each morphism $R: A \rightarrowtail B$ of Rel , the images

$$\begin{aligned}\mathcal{P}_*(R) &: \mathcal{P}(A) \rightarrow \mathcal{P}(B), \\ \mathcal{P}_{-1}(R) &: \mathcal{P}(B) \rightarrow \mathcal{P}(A), \\ \mathcal{P}^{-1}(R) &: \mathcal{P}(B) \rightarrow \mathcal{P}(A), \\ \mathcal{P}_!(R) &: \mathcal{P}(A) \rightarrow \mathcal{P}(B)\end{aligned}$$

of R by \mathcal{P}_* , \mathcal{P}_{-1} , \mathcal{P}^{-1} , and $\mathcal{P}_!$ are defined by

$$\begin{aligned}\mathcal{P}_*(R) &\stackrel{\text{def}}{=} R_*, \\ \mathcal{P}_{-1}(R) &\stackrel{\text{def}}{=} R_{-1}, \\ \mathcal{P}^{-1}(R) &\stackrel{\text{def}}{=} R^{-1}, \\ \mathcal{P}_!(R) &\stackrel{\text{def}}{=} R_!,\end{aligned}$$

as in [Definitions 4.1.1.1, 4.2.1.1, 4.3.1.1](#) and [4.4.1.1](#).

Proof. This follows from [Items 3 and 4 of Proposition 4.1.1.4](#), [Items 3 and 4 of Proposition 4.2.1.4](#), [Items 3 and 4 of Proposition 4.3.1.4](#), and [Items 3 and 4 of Proposition 4.4.1.4](#). \square

4.6 Functoriality of Powersets: Relations on Powersets

Let A and B be sets and let $R: A \rightarrowtail B$ be a relation.

Definition 4.6.1.1. The **relation on powersets associated to R** is the relation

$$\mathcal{P}(R): \mathcal{P}(A) \rightarrowtail \mathcal{P}(B)$$

defined by³⁰

$$\mathcal{P}(R)_U^V \stackrel{\text{def}}{=} \mathbf{Rel}(\chi_{\text{pt}}, V \diamond R \diamond U)$$

for each $U \in \mathcal{P}(A)$ and each $V \in \mathcal{P}(B)$.

Remark 4.6.1.2. In detail, we have $U \sim_{\mathcal{P}(R)} V$ iff the following equivalent conditions hold:

- We have $\chi_{\text{pt}} \subset V \diamond R \diamond U$.
- We have $(V \diamond R \diamond U)_{\star}^{\star} = \text{true}$, i.e. we have

$$\int^{a \in A} \int^{b \in B} V_b^{\star} \times R_a^b \times U_{\star}^a = \text{true}.$$

- There exists some $a \in A$ and some $b \in B$ such that:
 - We have $U_{\star}^a = \text{true}$.
 - We have $R_a^b = \text{true}$.
 - We have $V_b^{\star} = \text{true}$.
- There exists some $a \in A$ and some $b \in B$ such that:
 - We have $a \in U$.
 - We have $a \sim_R b$.
 - We have $b \in V$.

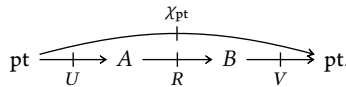
Proposition 4.6.1.3. The assignment $R \mapsto \mathcal{P}(R)$ defines a functor

$$\mathcal{P}: \mathbf{Rel} \rightarrow \mathbf{Rel}.$$

Proof. Omitted. □

Appendices

³⁰Illustration:



A Other Chapters

Sets

1. [Sets](#)
2. [Constructions With Sets](#)
3. [Pointed Sets](#)
4. [Tensor Products of Pointed Sets](#)

Relations

5. [Relations](#)

6. [Constructions With Relations](#)

7. [Equivalence Relations and Apartness Relations](#)

Category Theory

8. [Categories](#)

Bicategories

9. [Types of Morphisms in Bicategories](#)

References

- [MO 460656] [Emily de Oliveira Santos](#). *Existence and characterisations of left Kan extensions and liftings in the bicategory of relations I*. Math-Overflow. URL: <https://mathoverflow.net/q/460656> (cit. on pp. 3, 4).
- [MO 461592] [Emily de Oliveira Santos](#). *Existence and characterisations of left Kan extensions and liftings in the bicategory of relations II*. Math-Overflow. URL: <https://mathoverflow.net/q/461592> (cit. on pp. 4, 5).