Constructions With Relations

The Clowder Project Authors

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This chapter contains some material about constructions with relations. Notably, we discuss and explore:

1. The existence or non-existence of Kan extensions and Kan lifts in the 2-category **Rel** (Section 2).

2. The various kinds of constructions involving relations, such as graphs, domains, ranges, unions, intersections, products, inverse relations, composition of relations, and collages (Section 3).

3. The adjoint pairs

$$\begin{aligned} R_* \dashv R_{-1} &\colon \mathcal{P}(A) \rightleftarrows \mathcal{P}(B), \\ R^{-1} \dashv R_! &\colon \mathcal{P}(B) \rightleftarrows \mathcal{P}(A) \end{aligned}$$

   of functors (morphisms of posets) between $\mathcal{P}(A)$ and $\mathcal{P}(B)$ induced by a relation $R\colon A \nrightarrow B$, as well as the properties of R_*, R_{-1}, R^{-1}, and $R_!$ (Section 4).

   Of particular note are the following points:

   (a) These two pairs of adjoint functors are the counterpart for relations of the adjoint triple $f_* \dashv f^{-1} \dashv f_!$ induced by a function $f\colon A \longrightarrow B$ studied in Constructions With Sets, Section 4.

   (b) We have $R_{-1} = R^{-1}$ iff R is total and functional (Item 8 of Proposition 4.2.4).

   (c) As a consequence of the previous item, when R comes from a function f, the pair of adjunctions

   $$R_* \dashv R_{-1} = R^{-1} \dashv R_!$$

   reduces to the triple adjunction

   $$f_* \dashv f^{-1} \dashv f_!$$

   from Constructions With Sets, Section 4.

(d) The pairs $R_* \dashv R_{-1}$ and $R^{-1} \dashv R_!$ turn out to be rather important later on, as they appear in the definition and study of continuous, open, and closed relations between topological spaces (**??**, **??**).

Contents

1 Co/Limits in the Category of Relations

This section is currently just a stub, and will be properly developed later on.

2 Kan Extensions and Kan Lifts in the 2-Category of Relations

2.1 Left Kan Extensions in Rel

Proposition 2.1.1 ► Left Kan Extensions in Rel

Let $R\colon A \nrightarrow B$ be a relation.

1. *Non-Existence of All Left Kan Extensions in* **Rel**. Not all relations in **Rel** admit left Kan extensions.

2. *Characterisation of Relations Admitting Left Kan Extensions Along Them.* The following conditions are equivalent:

   (a) The left Kan extension

   $$\mathrm{Lan}_R\colon \mathbf{Rel}(A, X) \to \mathbf{Rel}(B, X)$$

   along R exists.

   (b) The relation R admits a left adjoint in **Rel**.

   (c) The relation R is of the form f^{-1} (as in Definition 3.2.1) for some function f.

Proof 2.1.2 ► Proof of Proposition 2.1.1

Item 1: Non-Existence of All Left Kan Extensions in Rel

Omitted, but will eventually follow Fosco Loregian's comment on [MO 460656].

Item 2: Characterisation of Relations Admitting Left Kan Extensions Along Ther

Omitted, but will eventually follow Tim Campion's answer to to [MO 460656]. □

QUESTION 2.1.3 ► EXISTENCE OF SPECIFIC LEFT KAN EXTENSIONS OF RELATIONS

Given relations $S\colon A \nrightarrow X$ and $R\colon A \nrightarrow B$, is there a characterisation of when the left Kan extension

$$\mathrm{Lan}_S(R)\colon B \nrightarrow X$$

exists in terms of properties of R and S?
This question also appears as [MO 461592].

QUESTION 2.1.4 ► EXPLICIT DESCRIPTION OF LEFT KAN EXTENSIONS ALONG FUNCTIONS

As shown in Item 2 of Proposition 2.1.1, the left Kan extension

$$\mathrm{Lan}_R\colon \mathbf{Rel}(A,X) \to \mathbf{Rel}(B,X)$$

along a relation of the form $R = f^{-1}$ exists. Is there a explicit description of it, similarly to the explicit description of right Kan extensions given in Proposition 2.3.1? This question also appears as [MO 461592].

2.2 Left Kan Lifts in Rel

PROPOSITION 2.2.1 ► LEFT KAN LIFTS IN Rel

Let $R\colon A \nrightarrow B$ be a relation.

1. *Non-Existence of All Left Kan Lifts in* **Rel**. Not all relations in **Rel** admit left Kan lifts.

2. *Characterisation of Relations Admitting Left Kan Lifts Along Them.* The following conditions are equivalent:

   (a) The left Kan lift

   $$\mathrm{Lift}_R\colon \mathbf{Rel}(X,B) \to \mathbf{Rel}(X,A)$$

   along R exists.

   (b) The relation R admits a right adjoint in **Rel**.

   (c) The relation R is of the form $\mathrm{Gr}(f)$ (as in Definition 3.1.1) for some function f.

PROOF 2.2.2 ► PROOF OF PROPOSITION 2.2.1

Item 1: Non-Existence of All Left Kan Lifts in Rel

Omitted, but will eventually follow (the dual of) Fosco Loregian's comment on [MO 460656].

Item 2: Characterisation of Relations Admitting Left Kan Lifts Along Them

Omitted, but will eventually follow Tim Campion's answer to to [MO 460656]. □

QUESTION 2.2.3 ► EXISTENCE OF SPECIFIC LEFT KAN LIFTS OF RELATIONS

Given relations $S\colon A \mathrel{\rightarrow\!\!\!\!\!+} X$ and $R\colon A \mathrel{\rightarrow\!\!\!\!\!+} B$, is there a characterisation of when the left Kan lift

$$\mathrm{Lift}_S(R)\colon X \mathrel{\rightarrow\!\!\!\!\!+} A$$

exists in terms of properties of R and S?
This question also appears as [MO 461592].

QUESTION 2.2.4 ► EXPLICIT DESCRIPTION OF LEFT KAN LIFTS ALONG FUNCTIONS

As shown in Item 2 of Proposition 2.2.1, the left Kan lift

$$\mathrm{Lift}_R\colon \mathbf{Rel}(X, B) \longrightarrow \mathbf{Rel}(X, A)$$

along a relation of the form $R = \mathrm{Gr}(f)$ exists. Is there a explicit description of it, similarly to the explicit description of right Kan lifts given in Proposition 2.4.1?
This question also appears as [MO 461592].

2.3 Right Kan Extensions in Rel

Let $R\colon A \mathrel{\rightarrow\!\!\!\!\!+} B$ be a relation.

PROPOSITION 2.3.1 ► EXISTENCE OF RIGHT KAN EXTENSIONS IN Rel

The right Kan extension

$$\mathrm{Ran}_R\colon \mathrm{Rel}(A, X) \longrightarrow \mathrm{Rel}(B, X)$$

along R in **Rel** exists and is given by

$$\mathrm{Ran}_R(S) \stackrel{\text{def}}{=} \int_{a\in A} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R_a^{-2}, S_a^{-1})$$

for each $S \in \mathrm{Rel}(A, X)$, so that the following conditions are equivalent:

1. We have $b \sim_{\mathrm{Ran}_R(S)} x$.
2. For each $a \in A$, if $a \sim_R b$, then $a \sim_S x$.

Proof 2.3.2 ► Proof of Proposition 2.3.1

We have

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{Rel}(A,X)}(S \diamond R, T) &\cong \int_{a\in A}\int_{x\in X} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}((S\diamond R)_a^x, T_a^x)\\
&\cong \int_{a\in A}\int_{x\in X} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}((\int^{b\in B} S_b^x \times R_a^b), T_a^x)\\
&\cong \int_{a\in A}\int_{x\in X}\int_{b\in B} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(S_b^x \times R_a^b, T_a^x)\\
&\cong \int_{a\in A}\int_{x\in X}\int_{b\in B} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(S_b^x, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R_a^b, T_a^x))\\
&\cong \int_{b\in B}\int_{x\in X}\int_{a\in A} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(S_b^x, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R_a^b, T_a^x))\\
&\cong \int_{b\in B}\int_{x\in X} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(S_b^x, \int_{a\in A} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R_a^b, T_a^x))\\
&\cong \mathrm{Hom}_{\mathbf{Rel}(B,X)}(S, \int_{a\in A} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R_a^{-2}, T_a^{-1}))
\end{aligned}$$

naturally in each $S \in \mathbf{Rel}(B, X)$ and each $T \in \mathbf{Rel}(A, X)$, showing that

$$\int_{a\in A} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R_a^{-2}, T_a^{-1})$$

is right adjoint to the precomposition functor $- \diamond R$, being thus the right Kan extension along R. Here we have used the following results, respectively (i.e. for each $\cong$ sign):

1. Relations, Item 1 of Proposition 1.1.6.

2. Definition 3.12.1.
3. **??**, **??** of **??**.
4. Sets, Proposition 2.2.5.
5. **??**, **??** of **??**.
6. **??**, **??** of **??**.
7. Relations, Item 1 of Proposition 1.1.6.

This finishes the proof.

2.4 Right Kan Lifts in Rel

Let $R\colon A \nrightarrow B$ be a relation.

Proposition 2.4.1 ► Existence of Right Kan Lifts in Rel

The right Kan lift

$$\mathrm{Rift}_R\colon \mathrm{Rel}(X,B) \to \mathrm{Rel}(X,A)$$

along R in **Rel** exists and is given by

$$\mathrm{Rift}_R(S) \stackrel{\text{def}}{=} \int_{b\in B} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R^b_{-_1}, S^b_{-_2})$$

for each $S \in \mathrm{Rel}(X,B)$, so that the following conditions are equivalent:

1. We have $x \sim_{\mathrm{Rift}_R(S)} a$.
2. For each $b \in B$, if $a \sim_R b$, then $x \sim_S b$.

PROOF 2.4.2 ▶ PROOF OF PROPOSITION 2.4.1

We have

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{Rel}(X,B)}(R \diamond S, T) &\cong \int_{x\in X}\int_{b\in B} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}((R\diamond S)^b_x, T^b_x)\\
&\cong \int_{x\in X}\int_{b\in B} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}((\int^{a\in A} R^b_a \times S^a_x), T^b_x)\\
&\cong \int_{x\in X}\int_{b\in B}\int_{a\in A} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R^b_a \times S^a_x, T^b_x)\\
&\cong \int_{x\in X}\int_{b\in B}\int_{a\in A} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(S^a_x, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R^b_a, T^b_x))\\
&\cong \int_{x\in X}\int_{a\in A}\int_{b\in B} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(S^a_x, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R^b_a, T^b_x))\\
&\cong \int_{x\in X}\int_{a\in A} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(S^a_x, \int_{b\in B} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R^b_a, T^b_x))\\
&\cong \mathrm{Hom}_{\mathbf{Rel}(X,A)}(S, \int_{b\in B} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R^b_{-_1}, T^b_{-_2}))
\end{aligned}$$

naturally in each $S \in \mathbf{Rel}(X, A)$ and each $T \in \mathbf{Rel}(X, B)$, showing that

$$\int_{b\in B} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R^b_{-_1}, S^b_{-_2})$$

is right adjoint to the postcomposition functor $R \diamond -$, being thus the right Kan lift along R. Here we have used the following results, respectively (i.e. for each $\cong$ sign):

1. Relations, Item 1 of Proposition 1.1.6.
2. Definition 3.12.1.
3. **??**, **??** of **??**.
4. Sets, Proposition 2.2.5.
5. **??**, **??** of **??**.
6. **??**, **??** of **??**.
7. Relations, Item 1 of Proposition 1.1.6.

This finishes the proof. ■

3 More Constructions With Relations

3.1 The Graph of a Function

Let $f\colon A \longrightarrow B$ be a function.

Definition 3.1.1 ► The Graph of a Function

The **graph of** f is the relation $\mathrm{Gr}(f)\colon A \nrightarrow B$ defined as follows:[1]

- Viewing relations from A to B as subsets of $A \times B$, we define

$$\mathrm{Gr}(f) \overset{\text{def}}{=} \{(a, f(a)) \in A \times B \mid a \in A\}.$$

- Viewing relations from A to B as functions $A \times B \longrightarrow \{\mathsf{true}, \mathsf{false}\}$, we define

$$[\mathrm{Gr}(f)](a, b) \overset{\text{def}}{=} \begin{cases} \mathsf{true} & \text{if } b = f(a), \\ \mathsf{false} & \text{otherwise} \end{cases}$$

  for each $(a, b) \in A \times B$.

- Viewing relations from A to B as functions $A \longrightarrow \mathcal{P}(B)$, we define

$$[\mathrm{Gr}(f)](a) \overset{\text{def}}{=} \{f(a)\}$$

  for each $a \in A$, i.e. we define $\mathrm{Gr}(f)$ as the composition

$$A \xrightarrow{f} B \xrightarrow{\chi_B} \mathcal{P}(B).$$

[1] *Further Notation:* We write $\mathrm{Gr}(A)$ for $\mathrm{Gr}(\mathrm{id}_A)$, and call it the **graph** of A.

Proposition 3.1.2 ► Properties of Graphs of Functions

Let $f\colon A \longrightarrow B$ be a function.

1. *Functoriality.* The assignment $A \mapsto \mathrm{Gr}(A)$ defines a functor

$$\mathrm{Gr}\colon \mathsf{Sets} \longrightarrow \mathsf{Rel}$$

   where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$\mathrm{Gr}(A) \stackrel{\mathrm{def}}{=} A.$$

- *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathsf{Sets})$, the action on Hom-sets

$$\mathrm{Gr}_{A,B} \colon \mathsf{Sets}(A, B) \to \underbrace{\mathrm{Rel}(\mathrm{Gr}(A), \mathrm{Gr}(B))}_{\stackrel{\mathrm{def}}{=}\mathrm{Rel}(A,B)}$$

  of Gr at (A, B) is defined by

$$\mathrm{Gr}_{A,B}(f) \stackrel{\mathrm{def}}{=} \mathrm{Gr}(f),$$

  where $\mathrm{Gr}(f)$ is the graph of f as in Definition 3.1.1.

In particular:

- *Preservation of Identities.* We have

$$\mathrm{Gr}(\mathrm{id}_A) = \chi_A$$

  for each $A \in \mathrm{Obj}(\mathsf{Sets})$.
- *Preservation of Composition.* We have

$$\mathrm{Gr}(g \circ f) = \mathrm{Gr}(g) \diamond \mathrm{Gr}(f)$$

  for each pair of functions $f \colon A \to B$ and $g \colon B \to C$.

2. *Adjointness Inside* **Rel**. We have an adjunction

$$\left(\mathrm{Gr}(f) \dashv f^{-1}\right)\colon \quad \begin{tikzcd} A \arrow[r, bend left, "\mathrm{Gr}(f)"] \arrow[r, phantom, "\perp"] & B \arrow[l, bend left, "f^{-1}"] \end{tikzcd}$$

in **Rel**, where f^{-1} is the inverse of f of Definition 3.2.1.

3. *Adjointness.* We have an adjunction

$$(\mathrm{Gr} \dashv \mathcal{P}_*) \colon \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "\mathrm{Gr}"] \arrow[r, phantom, "\perp"] & \mathsf{Rel}, \arrow[l, bend left, "\mathcal{P}_*"] \end{tikzcd}$$

witnessed by a bijection of sets

$$\mathsf{Rel}(\mathrm{Gr}(A), B) \cong \mathsf{Sets}(A, \mathcal{P}(B))$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$ and $B \in \mathrm{Obj}(\mathsf{Rel})$.

4. *Interaction With Inverses.* We have

$$\begin{aligned} \mathrm{Gr}(f)^{\dagger} &= f^{-1}, \\ (f^{-1})^{\dagger} &= \mathrm{Gr}(f). \end{aligned}$$

5. *Cocontinuity.* The functor $\mathrm{Gr} \colon \mathsf{Sets} \to \mathsf{Rel}$ of Item 1 preserves colimits.

6. *Characterisations.* Let $R \colon A \nrightarrow B$ be a relation. The following conditions are equivalent:

   (a) There exists a function $f \colon A \to B$ such that $R = \mathrm{Gr}(f)$.

   (b) The relation R is total and functional.

   (c) The weak and strong inverse images of R agree, i.e. we have $R^{-1} = R_{-1}$.

   (d) The relation R has a right adjoint $R^{\dagger}$ in Rel.

Proof 3.1.3 ► Proof of Proposition 3.1.2

Item 1: Functoriality

Clear.

Item 2: Adjointness Inside Rel

We need to check that there are inclusions

$$\begin{aligned}\chi_A &\subset f^{-1} \diamond \mathrm{Gr}(f),\\ \mathrm{Gr}(f) \diamond f^{-1} &\subset \chi_B.\end{aligned}$$

These correspond respectively to the following conditions:

1. For each $a \in A$, there exists some $b \in B$ such that $a \sim_{\mathrm{Gr}(f)} b$ and $b \sim_{f^{-1}} a$.
2. For each $a, b \in A$, if $a \sim_{\mathrm{Gr}(f)} b$ and $b \sim_{f^{-1}} a$, then $a = b$.

In other words, the first condition states that the image of any $a \in A$ by f is nonempty, whereas the second condition states that f is not multivalued. As f is a function, both of these statements are true, and we are done.

Item 3: Adjointness

The stated bijection follows from Relations, Remark 1.1.4, with naturality being clear.

Item 4: Interaction With Inverses

Clear.

Item 5: Cocontinuity

Omitted.

Item 6: Characterisations

We claim that Items 6a to 6d are indeed equivalent:

- *Item 6a* $\iff$ *Item 6b*. This is shown in the proof of **??** of **??**.
- *Item 6b* $\implies$ *Item 6c*. If R is total and functional, then, for each $a \in A$, the set $R(a)$ is a singleton, implying that

  $$\begin{aligned}R^{-1}(V) &\overset{\text{def}}{=} \{a \in A \mid R(a) \cap V \neq \emptyset\},\\ R_{-1}(V) &\overset{\text{def}}{=} \{a \in A \mid R(a) \subset V\}\end{aligned}$$

  are equal for all $V \in \mathcal{P}(B)$, as the conditions $R(a) \cap V \neq \emptyset$ and $R(a) \subset V$ are equivalent when $R(a)$ is a singleton.
- *Item 6c* $\implies$ *Item 6b*. We claim that R is indeed total and functional:

- *Totality.* If we had $R(a) = \emptyset$ for some $a \in A$, then we would have $a \in R_{-1}(\emptyset)$, so that $R_{-1}(\emptyset) \neq \emptyset$. But since $R^{-1}(\emptyset) = \emptyset$, this would imply $R_{-1}(\emptyset) \neq R^{-1}(\emptyset)$, a contradiction. Thus $R(a) \neq \emptyset$ for all $a \in A$ and R is total.
- *Functionality.* If $R^{-1} = R_{-1}$, then we have

$$\begin{aligned}\{a\} &= R^{-1}(\{b\}) \\ &= R_{-1}(\{b\})\end{aligned}$$

for each $b \in R(a)$ and each $a \in A$, and thus $R(a) \subset \{b\}$. But since R is total, we must have $R(a) = \{b\}$, and thus we see that R is functional.

- *Item 6a* $\iff$ *Item 6d*. This follows from Relations, Proposition 3.3.1.

This finishes the proof. ∎

3.2 The Inverse of a Function

Let $f : A \to B$ be a function.

Definition 3.2.1 ► The Inverse of a Function

The **inverse of** f is the relation $f^{-1} : B \nrightarrow A$ defined as follows:

- Viewing relations from B to A as subsets of $B \times A$, we define

$$f^{-1} \stackrel{\text{def}}{=} \left\{(b, f^{-1}(b)) \in B \times A \,\middle|\, a \in A\right\},$$

where

$$f^{-1}(b) \stackrel{\text{def}}{=} \{a \in A \mid f(a) = b\}$$

for each $b \in B$.

- Viewing relations from B to A as functions $B \times A \to \{\text{true}, \text{false}\}$, we define

$$f^{-1}(b, a) \stackrel{\text{def}}{=} \begin{cases} \text{true} & \text{if there exists } a \in A \text{ with } f(a) = b, \\ \text{false} & \text{otherwise} \end{cases}$$

for each $(b, a) \in B \times A$.

- Viewing relations from B to A as functions $B \to \mathcal{P}(A)$, we define

$$f^{-1}(b) \overset{\text{def}}{=} \{a \in A \mid f(a) = b\}$$

for each $b \in B$.

Proposition 3.2.2 ▶ Properties of Inverses of Functions

Let $f\colon A \to B$ be a function.

1. *Functoriality*. The assignment $A \mapsto A, f \mapsto f^{-1}$ defines a functor

$$(-)^{-1}\colon \mathsf{Sets} \to \mathsf{Rel}$$

where

- *Action on Objects*. For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$\left[(-)^{-1}\right](A) \overset{\text{def}}{=} A.$$

- *Action on Morphisms*. For each $A, B \in \mathrm{Obj}(\mathsf{Sets})$, the action on Hom-sets

$$(-)^{-1}_{A,B}\colon \mathsf{Sets}(A, B) \to \mathsf{Rel}(A, B)$$

of $(-)^{-1}$ at (A, B) is defined by

$$(-)^{-1}_{A,B}(f) \overset{\text{def}}{=} \left[(-)^{-1}\right](f),$$

where f^{-1} is the inverse of f as in Definition 3.2.1.

In particular:

- *Preservation of Identities*. We have

$$\mathrm{id}_A^{-1} = \chi_A$$

for each $A \in \mathrm{Obj}(\mathsf{Sets})$.

- *Preservation of Composition*. We have

$$(g \circ f)^{-1} = g^{-1} \diamond f^{-1}$$

for pair of functions $f\colon A \to B$ and $g\colon B \to C$.

2. *Adjointness Inside* **Rel**. We have an adjunction

$$
(\mathsf{Gr}(f) \dashv f^{-1})\colon \quad \begin{tikzcd} A \arrow[r, bend left, "\mathsf{Gr}(f)", "\shortmid"{marking}] \arrow[r, phantom, "\perp"] & B \arrow[l, bend left, "f^{-1}", "\shortmid"{marking}] \end{tikzcd}
$$

in **Rel**.

3. *Interaction With Inverses of Relations.* We have

$$
\begin{aligned}
(f^{-1})^{\dagger} &= \mathsf{Gr}(f), \\
\mathsf{Gr}(f)^{\dagger} &= f^{-1}.
\end{aligned}
$$

Proof 3.2.3 ▶ Proof of Proposition 3.2.2

Item 1: Functoriality

Clear.

Item 2: Adjointness Inside **Rel**

This is proved in Item 2 of Proposition 3.1.2.

Item 3: Interaction With Inverses of Relations

Clear.

3.3 Representable Relations

Let A and B be sets.

Definition 3.3.1 ▶ Representable Relations

Let $f\colon A \longrightarrow B$ and $g\colon B \longrightarrow A$ be functions.[1]

1. The **representable relation associated to** f is the relation $\chi_f\colon A \nrightarrow B$ defined as the composition

$$
A \times B \xrightarrow{f \times \mathrm{id}_B} B \times B \xrightarrow{\chi_B} \{\mathsf{true}, \mathsf{false}\},
$$

i.e. given by declaring $a \sim_{\chi_f} b$ iff $f(a) = b$.

2. The **corepresentable relation associated to** g is the relation $\chi^g\colon B \nrightarrow A$ defined as the composition

$$B \times A \xrightarrow{g\times\mathrm{id}_A} A \times A \xrightarrow{\chi_A} \{\mathsf{true}, \mathsf{false}\},$$

i.e. given by declaring $b \sim_{\chi^g} a$ iff $g(b) = a$.

[1]More generally, given functions

$$\begin{aligned} f\colon A &\to C,\\ g\colon B &\to D \end{aligned}$$

and a relation $B \nrightarrow D$, we may consider the composite relation

$$A \times B \xrightarrow{f\times g} C \times D \xrightarrow{R} \{\mathsf{true}, \mathsf{false}\},$$

for which we have $a \sim_{R\circ(f\times g)} b$ iff $f(a) \sim_R g(b)$.

3.4 The Domain and Range of a Relation

Let A and B be sets.

Definition 3.4.1 ► The Domain and Range of a Relation

Let $R \subset A \times B$ be a relation.[1,2]

1. The **domain of** R is the subset $\mathrm{dom}(R)$ of A defined by

$$\mathrm{dom}(R) \overset{\mathrm{def}}{=} \left\{ a \in A \;\middle|\; \begin{array}{l}\text{there exists some } b \in B\\ \text{such that } a \sim_R b\end{array} \right\}.$$

2. The **range of** R is the subset $\mathrm{range}(R)$ of B defined by

$$\mathrm{range}(R) \overset{\mathrm{def}}{=} \left\{ b \in B \;\middle|\; \begin{array}{l}\text{there exists some } a \in A\\ \text{such that } a \sim_R b\end{array} \right\}.$$

[1] Following ??, ??, we may compute the (characteristic functions associated to the) domain and range of a relation using the following colimit formulas:

$$\begin{aligned}\chi_{\mathrm{dom}(R)}(a) &\cong \operatorname*{colim}_{b\in B}(R^b_a) \qquad (a\in A)\\ &\cong \bigvee_{b\in B} R^b_a,\\ \chi_{\mathrm{range}(R)}(b) &\cong \operatorname*{colim}_{a\in A}(R^b_a) \qquad (b\in B)\\ &\cong \bigvee_{a\in A} R^b_a,\end{aligned}$$

where the join $\bigvee$ is taken in the poset $(\{\mathsf{true},\mathsf{false}\},\preceq)$ of Constructions With Sets, Definition 2.2.3.

[2] Viewing R as a function $R\colon A\longrightarrow\mathcal{P}(B)$, we have

$$\begin{aligned}\mathrm{dom}(R) &\cong \operatorname*{colim}_{y\in Y}(R(y))\\ &\cong \bigcup_{y\in Y} R(y),\\ \mathrm{range}(R) &\cong \operatorname*{colim}_{x\in X}(R(x))\\ &\cong \bigcup_{x\in X} R(x),\end{aligned}$$

3.5 Binary Unions of Relations

Let A and B be sets and let R and S be relations from A to B.

Definition 3.5.1 ► Binary Unions of Relations

The **union of R and S**[1] is the relation $R\cup S$ from A to B defined as follows:

- Viewing relations from A to B as subsets of $A\times B$, we define[2]

$$R\cup S \overset{\mathrm{def}}{=} \{(a,b)\in B\times A \mid \text{we have } a\sim_R b \text{ or } a\sim_S b\}.$$

- Viewing relations from A to B as functions $A\longrightarrow\mathcal{P}(B)$, we define

$$[R\cup S](a) \overset{\mathrm{def}}{=} R(a)\cup S(a)$$

for each $a\in A$.

[1] *Further Terminology:* Also called the **binary union of R and S**, for emphasis.

[2] This is the same as the union of R and S as subsets of $A\times B$.

PROPOSITION 3.5.2 ► PROPERTIES OF BINARY UNIONS OF RELATIONS

Let R, S, R_1, and R_2 be relations from A to B, and let S_1 and S_2 be relations from B to C.

1. *Interaction With Inverses.* We have

$$(R \cup S)^{\dagger} = R^{\dagger} \cup S^{\dagger}.$$

2. *Interaction With Composition.* We have

$$(S_1 \diamond R_1) \cup (S_2 \diamond R_2) \overset{\text{poss.}}{\neq} (S_1 \cup S_2) \diamond (R_1 \cup R_2).$$

PROOF 3.5.3 ► PROOF OF PROPOSITION 3.5.2

Item 1: Interaction With Inverses

Clear.

Item 2: Interaction With Composition

Unwinding the definitions, we see that:

1. The condition for $(S_1 \diamond R_1) \cup (S_2 \diamond R_2)$ is:
   (a) There exists some $b \in B$ such that:
      i. $a \sim_{R_1} b$ and $b \sim_{S_1} c$;

   or

      i. $a \sim_{R_2} b$ and $b \sim_{S_2} c$;

3. The condition for $(S_1 \cup S_2) \diamond (R_1 \cup R_2)$ is:
   (a) There exists some $b \in B$ such that:
      i. $a \sim_{R_1} b$ or $a \sim_{R_2} b$;

   and

      i. $b \sim_{S_1} c$ or $b \sim_{S_2} c$.

These two conditions may fail to agree (counterexample omitted), and thus the two resulting relations on $A \times C$ may differ. ▭

3.6 Unions of Families of Relations

Let A and B be sets and let $\{R_i\}_{i \in I}$ be a family of relations from A to B.

Definition 3.6.1 ► The Union of a Family of Relations

The **union of the family** $\{R_i\}_{i\in I}$ is the relation $\bigcup_{i\in I} R_i$ from A to B defined as follows:

- Viewing relations from A to B as subsets of $A \times B$, we define[1]

$$\bigcup_{i\in I} R_i \overset{\text{def}}{=} \left\{ (a,b) \in (A\times B)^{\times I} \;\middle|\; \begin{array}{l}\text{there exists some } i \in I\\ \text{such that } a \sim_{R_i} b\end{array} \right\}.$$

- Viewing relations from A to B as functions $A \to \mathcal{P}(B)$, we define

$$\left[\bigcup_{i\in I} R_i\right](a) \overset{\text{def}}{=} \bigcup_{i\in I} R_i(a)$$

  for each $a \in A$.

[1] This is the same as the union of $\{R_i\}_{i\in I}$ as a collection of subsets of $A \times B$.

Proposition 3.6.2 ► Properties of Unions of Families of Relations

Let A and B be sets and let $\{R_i\}_{i\in I}$ be a family of relations from A to B.

1. *Interaction With Inverses.* We have

$$(\bigcup_{i\in I} R_i)^{\dagger} = \bigcup_{i\in I} R_i^{\dagger}.$$

Proof 3.6.3 ► Proof of Proposition 3.6.2

Item 1: Interaction With Inverses

Clear. ∎

3.7 Binary Intersections of Relations

Let A and B be sets and let R and S be relations from A to B.

Definition 3.7.1 ▶ Binary Intersections of Relations

The **intersection of R and S**[1] is the relation $R \cap S$ from A to B defined as follows:

- Viewing relations from A to B as subsets of $A \times B$, we define[2]

$$R \cap S \stackrel{\text{def}}{=} \{(a, b) \in B \times A \mid \text{we have } a \sim_R b \text{ and } a \sim_S b\}.$$

- Viewing relations from A to B as functions $A \to \mathcal{P}(B)$, we define

$$[R \cap S](a) \stackrel{\text{def}}{=} R(a) \cap S(a)$$

for each $a \in A$.

[1] *Further Terminology:* Also called the **binary intersection of R and S**, for emphasis.
[2] This is the same as the intersection of R and S as subsets of $A \times B$.

Proposition 3.7.2 ▶ Properties of Binary Intersections of Relations

Let R, S, R_1, and R_2 be relations from A to B, and let S_1 and S_2 be relations from B to C.

1. *Interaction With Inverses.* We have

$$(R \cap S)^{\dagger} = R^{\dagger} \cap S^{\dagger}.$$

2. *Interaction With Composition.* We have

$$(S_1 \diamond R_1) \cap (S_2 \diamond R_2) = (S_1 \cap S_2) \diamond (R_1 \cap R_2).$$

Proof 3.7.3 ▶ Proof of Proposition 3.7.2

Item 1: Interaction With Inverses

Clear.

Item 2: Interaction With Composition

Unwinding the definitions, we see that:

1. The condition for $(S_1 \diamond R_1) \cap (S_2 \diamond R_2)$ is:
   (a) There exists some $b \in B$ such that:

i. $a \sim_{R_1} b$ and $b \sim_{S_1} c$;

and

i. $a \sim_{R_2} b$ and $b \sim_{S_2} c$;

3. The condition for $(S_1 \cap S_2) \diamond (R_1 \cap R_2)$ is:

   (a) There exists some $b \in B$ such that:

   i. $a \sim_{R_1} b$ and $a \sim_{R_2} b$;

   and

   i. $b \sim_{S_1} c$ and $b \sim_{S_2} c$.

These two conditions agree, and thus so do the two resulting relations on $A \times C$.

3.8 Intersections of Families of Relations

Let A and B be sets and let $\{R_i\}_{i\in I}$ be a family of relations from A to B.

Definition 3.8.1 ► The Intersection of a Family of Relations

The **intersection of the family** $\{R_i\}_{i\in I}$ is the relation $\bigcup_{i\in I} R_i$ defined as follows:

- Viewing relations from A to B as subsets of $A \times B$, we define[1]

$$\bigcup_{i\in I} R_i \overset{\text{def}}{=} \left\{ (a,b) \in (A \times B)^{\times I} \;\middle|\; \begin{array}{l} \text{for each } i \in I, \\ \text{we have } a \sim_{R_i} b \end{array} \right\}.$$

- Viewing relations from A to B as functions $A \to \mathcal{P}(B)$, we define

$$\left[\bigcap_{i\in I} R_i\right](a) \overset{\text{def}}{=} \bigcap_{i\in I} R_i(a)$$

  for each $a \in A$.

[1] This is the same as the intersection of $\{R_i\}_{i\in I}$ as a collection of subsets of $A \times B$.

Proposition 3.8.2 ► Properties of Intersections of Families of Relations

Let A and B be sets and let $\{R_i\}_{i\in I}$ be a family of relations from A to B.

1. *Interaction With Inverses.* We have

$$\Big(\bigcap_{i\in I} R_i\Big)^{\dagger} = \bigcap_{i\in I} R_i^{\dagger}.$$

Proof 3.8.3 ► Proof of Proposition 3.8.2

Item 1: Interaction With Inverses

Clear.

3.9 Binary Products of Relations

Let A, B, X, and Y be sets, let $R\colon A \nrightarrow B$ be a relation from A to B, and let $S\colon X \nrightarrow Y$ be a relation from X to Y.

Definition 3.9.1 ► Binary Products of Relations

The **product of R and S**[1] is the relation $R\times S$ from $A\times X$ to $B\times Y$ defined as follows:

- Viewing relations from $A\times X$ to $B\times Y$ as subsets of $(A\times X)\times(B\times Y)$, we define $R\times S$ as the Cartesian product of R and S as subsets of $A\times X$ and $B\times Y$.[2]
- Viewing relations from $A\times X$ to $B\times Y$ as functions $A\times X \to \mathcal{P}(B\times Y)$, we define $R\times S$ as the composition

$$A\times X \xrightarrow{R\times S} \mathcal{P}(B)\times\mathcal{P}(Y) \overset{\mathcal{P}^{\otimes}_{B,Y}}{\hookrightarrow} \mathcal{P}(B\times Y)$$

in Sets, i.e. by

$$[R\times S](a,x) \overset{\text{def}}{=} R(a)\times S(x)$$

for each $(a,x)\in A\times X$.

[1] *Further Terminology:* Also called the **binary product of R and S**, for emphasis.

[2] That is, $R\times S$ is the relation given by declaring $(a,x)\sim_{R\times S}(b,y)$ iff $a\sim_R b$ and $x\sim_S y$.

Proposition 3.9.2 ► Properties of Binary Products of Relations

Let A, B, X, and Y be sets.

1. *Interaction With Inverses.* Let

$$R : A \nrightarrow A,$$
$$S : X \nrightarrow X$$

   We have

$$(R \times S)^{\dagger} = R^{\dagger} \times S^{\dagger}.$$

2. *Interaction With Composition.* Let

$$R_1 : A \nrightarrow B,$$
$$S_1 : B \nrightarrow C,$$
$$R_2 : X \nrightarrow Y,$$
$$S_2 : Y \nrightarrow Z$$

   be relations. We have

$$(S_1 \diamond R_1) \times (S_2 \diamond R_2) = (S_1 \times S_2) \diamond (R_1 \times R_2).$$

Proof 3.9.3 ► Proof of Proposition 3.9.2

Item 1: Interaction With Inverses

Unwinding the definitions, we see that:

1. We have $(a, x) \sim_{(R \times S)^{\dagger}} (b, y)$ iff:
   - We have $(b, y) \sim_{R \times S} (a, x)$, i.e. iff:
     - We have $b \sim_R a$;
     - We have $y \sim_S x$;
2. We have $(a, x) \sim_{R^{\dagger} \times S^{\dagger}} (b, y)$ iff:
   - We have $a \sim_{R^{\dagger}} b$ and $x \sim_{S^{\dagger}} y$, i.e. iff:
     - We have $b \sim_R a$;

– We have $y \sim_S x$.

These two conditions agree, and thus the two resulting relations on $A \times X$ are equal.

Item 2: Interaction With Composition

Unwinding the definitions, we see that:

1. We have $(a, x) \sim_{(S_1 \diamond R_1) \times (S_2 \diamond R_2)} (c, z)$ iff:
   (a) We have $a \sim_{S_1 \diamond R_1} c$ and $x \sim_{S_2 \diamond R_2} z$, i.e. iff:
      i. There exists some $b \in B$ such that $a \sim_{R_1} b$ and $b \sim_{S_1} c$;
      ii. There exists some $y \in Y$ such that $x \sim_{R_2} y$ and $y \sim_{S_2} z$;
2. We have $(a, x) \sim_{(S_1 \times S_2) \diamond (R_1 \times R_2)} (c, z)$ iff:
   (a) There exists some $(b, y) \in B \times Y$ such that $(a, x) \sim_{R_1 \times R_2} (b, y)$ and $(b, y) \sim_{S_1 \times S_2} (c, z)$, i.e. such that:
      i. We have $a \sim_{R_1} b$ and $x \sim_{R_2} y$;
      ii. We have $b \sim_{S_1} c$ and $y \sim_{S_2} z$.

These two conditions agree, and thus the two resulting relations from $A \times X$ to $C \times Z$ are equal. □

3.10 Products of Families of Relations

Let $\{A_i\}_{i \in I}$ and $\{B_i\}_{i \in I}$ be families of sets, and let $\{R_i \colon A_i \nrightarrow B_i\}_{i \in I}$ be a family of relations.

Definition 3.10.1 ▶ The Product of a Family of Relations

The **product of the family** $\{R_i\}_{i \in I}$ is the relation $\prod_{i \in I} R_i$ from $\prod_{i \in I} A_i$ to $\prod_{i \in I} B_i$ defined as follows:

- Viewing relations as subsets, we define $\prod_{i \in I} R_i$ as its product as a family of sets, i.e. we have

$$\prod_{i \in I} R_i \overset{\text{def}}{=} \left\{ (a_i, b_i)_{i \in I} \in \prod_{i \in I} (A_i \times B_i) \;\middle|\; \begin{array}{l} \text{for each } i \in I, \\ \text{we have } a_i \sim_{R_i} b_i \end{array} \right\}.$$

- Viewing relations as functions to powersets, we define

$$\left[\prod_{i\in I} R_i\right]((a_i)_{i\in I}) \overset{\text{def}}{=} \prod_{i\in I} R_i(a_i)$$

for each $(a_i)_{i\in I} \in \prod_{i\in I} R_i$.

3.11 The Inverse of a Relation

Let A, B, and C be sets and let $R \subset A \times B$ be a relation.

Definition 3.11.1 ► The Inverse of a Relation

The **inverse of** R[1] is the relation $R^\dagger$ defined as follows:

- Viewing relations as subsets, we define

$$R^\dagger \overset{\text{def}}{=} \{(b, a) \in B \times A \mid \text{we have } b \sim_R a\}.$$

- Viewing relations as functions $A \times B \longrightarrow \{\text{true}, \text{false}\}$, we define

$$[R^\dagger]^a_b \overset{\text{def}}{=} R^b_a$$

for each $(b, a) \in B \times A$.

- Viewing relations as functions $A \longrightarrow \mathcal{P}(B)$, we define

$$\begin{aligned} [R^\dagger](b) &\overset{\text{def}}{=} R^\dagger(\{b\}) \\ &\overset{\text{def}}{=} \{a \in A \mid b \in R(a)\} \end{aligned}$$

for each $b \in B$, where $R^\dagger(\{b\})$ is the fibre of R over $\{b\}$.

[1] *Further Terminology:* Also called the **opposite of** R, the **transpose of** R, or the **converse of** R.

Example 3.11.2 ► Examples of Inverses of Relations

Here are some examples of inverses of relations.

1. *Less Than Equal Signs.* We have $(\leq)^\dagger = \geq$.

2. *Greater Than Equal Signs.* Dually to Item 1, we have $(\geq)^\dagger = \leq$.

3. *Functions.* Let $f\colon A \to B$ be a function. We have

$$\begin{aligned}\mathrm{Gr}(f)^\dagger &= f^{-1},\\ (f^{-1})^\dagger &= \mathrm{Gr}(f).\end{aligned}$$

Proposition 3.11.3 ► Properties of Inverses of Relations

Let $R\colon A \nrightarrow B$ and $S\colon B \nrightarrow C$ be relations.

1. *Functoriality.* The assignment $R \mapsto R^\dagger$ defines a functor (i.e. morphism of posets)
$$(-)^\dagger\colon \mathbf{Rel}(A, B) \to \mathbf{Rel}(B, A).$$
In particular, given relations $R, S\colon A \rightrightarrows B$, we have:
   ($\star$) If $R \subset S$, then $R^\dagger \subset S^\dagger$.

2. *Interaction With Ranges and Domains.* We have
$$\begin{aligned}\mathrm{dom}(R^\dagger) &= \mathrm{range}(R),\\ \mathrm{range}(R^\dagger) &= \mathrm{dom}(R).\end{aligned}$$

3. *Interaction With Composition I.* We have
$$(S \diamond R)^\dagger = R^\dagger \diamond S^\dagger.$$

4. *Interaction With Composition II.* We have
$$\begin{aligned}\chi_B &\subset R \diamond R^\dagger,\\ \chi_A &\subset R^\dagger \diamond R.\end{aligned}$$

5. *Invertibility.* We have
$$(R^\dagger)^\dagger = R.$$

6. *Identity.* We have
$$\chi_A^\dagger = \chi_A.$$

Proof 3.11.4 ▶ Proof of Proposition 3.11.3

Item 1: Functoriality

Clear.

Item 2: Interaction With Ranges and Domains

Clear.

Item 3: Interaction With Composition I

Clear.

Item 4: Interaction With Composition II

Clear.

Item 5: Invertibility

Clear.

Item 6: Identity

Clear.

3.12 Composition of Relations

Let A, B, and C be sets and let $R\colon A \nrightarrow B$ and $S\colon B \nrightarrow C$ be relations.

Definition 3.12.1 ▶ Composition of Relations

The **composition of** R **and** S is the relation $S \diamond R$ defined as follows:

- Viewing relations from A to C as subsets of $A \times C$, we define

$$S \diamond R \overset{\text{def}}{=} \left\{ (a, c) \in A \times C \,\middle|\, \begin{array}{l}\text{there exists some } b \in B \text{ such}\\ \text{that } a \sim_R b \text{ and } b \sim_S c\end{array} \right\}.$$

- Viewing relations as functions $A \times B \to \{\text{true}, \text{false}\}$, we define

$$\begin{aligned}(S \diamond R)^{-1}_{-2} &\overset{\text{def}}{=} \int^{b \in B} S^{-1}_b \times R^b_{-2}\\ &= \bigvee_{b \in B} S^{-1}_b \times R^b_{-2},\end{aligned}$$

where the join $\bigvee$ is taken in the poset $(\{\text{true}, \text{false}\}, \preceq)$ of Sets, Definition 2.2.3.

- Viewing relations as functions $A \to \mathcal{P}(B)$, we define

$$S \diamond R \overset{\text{def}}{=} \mathrm{Lan}_{\chi_B}(S) \circ R, \qquad \begin{tikzcd} & B \arrow[r, "S"] \arrow[d, "\chi_B"'] & \mathcal{P}(C), \\ A \arrow[r, "R"'] & \mathcal{P}(B) \arrow[ur, bend right, "\mathrm{Lan}_{\chi_B}(S)"'] \arrow[ur, phantom, shift left=3, "\Leftarrow"] & \end{tikzcd}$$

where $\mathrm{Lan}_{\chi_B}(S)$ is computed by the formula

$$\begin{aligned} [\mathrm{Lan}_{\chi_B}(S)](V) &\cong \int^{y \in B} \chi_{\mathcal{P}(B)}(\chi_y, V) \odot S_y \\ &\cong \int^{y \in B} \chi_V(y) \odot S_y \\ &\cong \bigcup_{y \in B} \chi_V(y) \odot S_y \\ &\cong \bigcup_{y \in V} S_y \end{aligned}$$

for each $V \in \mathcal{P}(B)$. In other words, $S \diamond R$ is defined by[1]

$$\begin{aligned} [S \diamond R](a) &\overset{\text{def}}{=} S(R(a)) \\ &\overset{\text{def}}{=} \bigcup_{x \in R(a)} S(x). \end{aligned}$$

for each $a \in A$.

[1] That is: the relation R may send $a \in A$ to a number of elements $\{b_i\}_{i \in I}$ in B, and then the relation S may send the image of each of the b_i's to a number of elements $\{S(b_i)\}_{i \in I} = \left\{\left\{c_{j_i}\right\}_{j_i \in J_i}\right\}_{i \in I}$ in C.

Example 3.12.2 ► Examples of Composition of Relations

Here are some examples of composition of relations.

1. *Composing Less/Greater Than Equal With Greater/Less Than Equal Signs.* We

have

$$\leq \diamond \geq \; = \sim_{\text{triv}},$$
$$\geq \diamond \leq \; = \sim_{\text{triv}} .$$

2. *Composing Less/Greater Than Equal Signs With Less/Greater Than Equal Signs.* We have

$$\leq \diamond \leq \; = \; \leq,$$
$$\geq \diamond \geq \; = \; \geq.$$

Proposition 3.12.3 ▶ Properties of Composition of Relations

Let $R\colon A \nrightarrow B$, $S\colon B \nrightarrow C$, and $T\colon C \nrightarrow D$ be relations.

1. *Interaction With Ranges and Domains.* We have

$$\operatorname{dom}(S \diamond R) \subset \operatorname{dom}(R),$$
$$\operatorname{range}(S \diamond R) \subset \operatorname{range}(S).$$

2. *Associativity.* We have

$$(T \diamond S) \diamond R = T \diamond (S \diamond R).$$

3. *Unitality.* We have

$$\chi_B \diamond R = R,$$
$$R \diamond \chi_A = R.$$

4. *Interaction With Inverses.* We have

$$(S \diamond R)^\dagger = R^\dagger \diamond S^\dagger.$$

5. *Interaction With Composition.* We have

$$\chi_B \subset R \diamond R^\dagger,$$
$$\chi_A \subset R^\dagger \diamond R.$$

Proof 3.12.4 ► Proof of Proposition 3.12.3

Item 1: Interaction With Ranges and Domains

Clear.

Item 2: Associativity

Indeed, we have

$$\begin{aligned}
(T \diamond S) \diamond R &\overset{\text{def}}{=} \Big(\int^{c \in C} T_c^{-1} \times S^c_{-_2}\Big) \diamond R \\
&\overset{\text{def}}{=} \int^{b \in B} \Big(\int^{c \in C} T_c^{-1} \times S^c_b\Big) \diamond R^b_{-_2} \\
&= \int^{b \in B} \int^{c \in C} (T_c^{-1} \times S^c_b) \diamond R^b_{-_2} \\
&= \int^{c \in C} \int^{b \in B} (T_c^{-1} \times S^c_b) \diamond R^b_{-_2} \\
&= \int^{c \in C} \int^{b \in B} T_c^{-1} \times (S^c_b \diamond R^b_{-_2}) \\
&= \int^{c \in C} T_c^{-1} \times \Big(\int^{b \in B} S^c_b \diamond R^b_{-_2}\Big) \\
&\overset{\text{def}}{=} \int^{c \in C} T_c^{-1} \times (S \diamond R)^c_{-_2} \\
&\overset{\text{def}}{=} T \diamond (S \diamond R).
\end{aligned}$$

In the language of relations, given $a \in A$ and $d \in D$, the stated equality witnesses the equivalence of the following two statements:

1. We have $a \sim_{(T \diamond S) \diamond R} d$, i.e. there exists some $b \in B$ such that:
   (a) We have $a \sim_R b$;
   (b) We have $b \sim_{T \diamond S} d$, i.e. there exists some $c \in C$ such that:
      i. We have $b \sim_S c$;
      ii. We have $c \sim_T d$;
2. We have $a \sim_{T \diamond (S \diamond R)} d$, i.e. there exists some $c \in C$ such that:
   (a) We have $a \sim_{S \diamond R} c$, i.e. there exists some $b \in B$ such that:

i. We have $a \sim_R b$;

ii. We have $b \sim_S c$;

(b) We have $c \sim_T d$;

both of which are equivalent to the statement

- There exist $b \in B$ and $c \in C$ such that $a \sim_R b \sim_S c \sim_T d$.

Item 3: Unitality

Indeed, we have

$$\begin{aligned}
\chi_B \diamond R &\stackrel{\text{def}}{=} \int^{x \in B} (\chi_B)^{-1}_{x} \times R^{x}_{-2} \\
&= \bigvee_{x \in B} (\chi_B)^{-1}_{x} \times R^{x}_{-2} \\
&= \bigvee_{\substack{x \in B \\ x = -1}} R^{x}_{-2} \\
&= R^{-1}_{-2},
\end{aligned}$$

and

$$\begin{aligned}
R \diamond \chi_A &\stackrel{\text{def}}{=} \int^{x \in A} R^{-1}_{x} \times (\chi_A)^{x}_{-2} \\
&= \bigvee_{x \in B} R^{-1}_{x} \times (\chi_A)^{x}_{-2} \\
&= \bigvee_{\substack{x \in B \\ x = -2}} R^{-1}_{x} \\
&= R^{-1}_{-2}.
\end{aligned}$$

In the language of relations, given $a \in A$ and $b \in B$:

- The equality

$$\chi_B \diamond R = R$$

witnesses the equivalence of the following two statements:

1. We have $a \sim_b B$.

2. There exists some $b' \in B$ such that:
   (a) We have $a \sim_R b'$
   (b) We have $b' \sim_{\chi_B} b$, i.e. $b' = b$.

- The equality
$$R \diamond \chi_A = R$$
witnesses the equivalence of the following two statements:
  1. There exists some $a' \in A$ such that:
     (a) We have $a \sim_{\chi_B} a'$, i.e. $a = a'$.
     (b) We have $a' \sim_R b$
  2. We have $a \sim_b B$.

Item 4: Interaction With Inverses

Clear.

Item 5: Interaction With Composition

Clear.

3.13 The Collage of a Relation

Let A and B be sets and let $R\colon A \nrightarrow B$ be a relation from A to B.

Definition 3.13.1 ► The Collage of a Relation

The **collage of** R[1] is the poset $\mathbf{Coll}(R) \overset{\text{def}}{=} (\mathrm{Coll}(R), \preceq_{\mathbf{Coll}(R)})$ consisting of:

- *The Underlying Set.* The set $\mathrm{Coll}(R)$ defined by
$$\mathrm{Coll}(R) \overset{\text{def}}{=} A \coprod B.$$
- *The Partial Order.* The partial order
$$\preceq_{\mathbf{Coll}(R)} : \mathrm{Coll}(R) \times \mathrm{Coll}(R) \longrightarrow \{\mathrm{true}, \mathrm{false}\}$$
on $\mathrm{Coll}(R)$ defined by
$$\preceq(a,b) \overset{\text{def}}{=} \begin{cases} \mathrm{true} & \text{if } a = b \text{ or } a \sim_R b, \\ \mathrm{false} & \text{otherwise.} \end{cases}$$

[1] *Further Terminology:* Also called the **cograph of** R.

Proposition 3.13.2 ► Properties of Collages of Relations

Let A and B be sets and let $R\colon A \nrightarrow B$ be a relation from A to B.

1. *Functoriality I.* The assignment $R \mapsto \mathbf{Coll}(R)$ defines a functor[1]

$$\mathbf{Coll}\colon \mathbf{Rel}(A,B) \to \mathsf{Pos}_{/\Delta^1}(A,B),$$

where

- *Action on Objects.* For each $R \in \mathrm{Obj}(\mathbf{Rel}(A,B))$, we have

$$[\mathbf{Coll}](R) \stackrel{\text{def}}{=} (\mathbf{Coll}(R), \phi_R)$$

for each $R \in \mathbf{Rel}(A,B)$, where

 – The poset $\mathbf{Coll}(R)$ is the collage of R of Definition 3.13.1.
 – The morphism $\phi_R\colon \mathbf{Coll}(R) \to \Delta^1$ is given by

$$\phi_R(x) \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if } x \in A, \\ 1 & \text{if } x \in B \end{cases}$$

 for each $x \in \mathbf{Coll}(R)$.

- *Action on Morphisms.* For each $R, S \in \mathrm{Obj}(\mathbf{Rel}(A,B))$, the action on Hom-sets

$$\mathbf{Coll}_{R,S}\colon \mathrm{Hom}_{\mathbf{Rel}(A,B)}(R,S) \to \mathsf{Pos}(\mathbf{Coll}(R), \mathbf{Coll}(S))$$

of $\mathbf{Coll}$ at (R,S) is given by sending an inclusion

$$\iota\colon R \subset S$$

to the morphism

$$\mathbf{Coll}(\iota)\colon \mathbf{Coll}(R) \to \mathbf{Coll}(S)$$

of posets over Δ^1 defined by

$$[\mathbf{Coll}(\iota)](x) \stackrel{\text{def}}{=} x$$

for each $x \in \mathbf{Coll}(R)$.[2]

2. *Equivalence.* The functor of Item 1 is an equivalence of categories.

[1]Here $\mathsf{Pos}_{/\Delta^1}(A,B)$ is the category defined as the pullback

$$\mathsf{Pos}_{/\Delta^1}(A,B) \overset{\text{def}}{=} \mathrm{pt} \underset{[A],\mathsf{Pos},\mathrm{fib}_0}{\times} \mathsf{Pos}_{/\Delta^1} \underset{\mathrm{fib}_1,\mathsf{Pos},[B]}{\times} \mathrm{pt},$$

as in the diagram

$$\begin{tikzcd}
 & & \mathbf{Pos}_{/\Delta^1}(A,B) \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45] & & \\
 & \mathbf{Pos}_{/\Delta^1} \underset{\mathbf{Pos}}{\times} \mathrm{pt} \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45] & & \mathrm{pt} \underset{\mathbf{Pos}}{\times} \mathbf{Pos}_{/\Delta^1} \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45] & \\
\mathrm{pt} \arrow[dr, "{[A]}"'] & & \mathbf{Pos}_{/\Delta^1} \arrow[dl, "\mathrm{fib}_{[0]}"] \arrow[dr, "\mathrm{fib}_{[1]}"'] & & \mathrm{pt}. \arrow[dl, "{[B]}"] \\
 & \mathbf{Pos} & & \mathbf{Pos} &
\end{tikzcd}$$

Explicitly, an object of $\mathsf{Pos}_{/\Delta^1}(A,B)$ is a pair (X,ϕ_X) consisting of

- A poset X;
- A morphism $\phi_X\colon X \to \Delta^1$;

such that $\phi_X^{-1}(0) = A$ and $\phi_X^{-1}(0) = B$, with morphisms between such objects being morphisms of posets over Δ^1.

[2]Note that this is indeed a morphism of posets: if $x \preceq_{\mathbf{Coll}(R)} y$, then $x = y$ or $x \sim_R y$, so we have either $x = y$ or $x \sim_S y$ (as $R \subset S$), and thus $x \preceq_{\mathbf{Coll}(S)} y$.

Proof 3.13.3 ► Proof of Proposition 3.13.2

Item 1: Functoriality

Clear.

Item 2: Equivalence

Omitted.

4 Functoriality of Powersets

4.1 Direct Images

Let A and B be sets and let $R\colon A \nrightarrow B$ be a relation.

Definition 4.1.1 ▶ Direct Images

The **direct image function associated to** R is the function

$$R_* \colon \mathcal{P}(A) \to \mathcal{P}(B)$$

defined by[1,2]

$$\begin{aligned} R_*(U) &\stackrel{\text{def}}{=} R(U) \\ &\stackrel{\text{def}}{=} \bigcup_{a \in U} R(a) \\ &= \left\{ b \in B \;\middle|\; \begin{array}{l} \text{there exists some } a \in U \\ \text{such that } b \in R(a) \end{array} \right\} \end{aligned}$$

for each $U \in \mathcal{P}(A)$.

[1] *Further Terminology:* The set $R(U)$ is called the **direct image of** U **by** R.
[2] We also have

$$R_*(U) = B \setminus R_!(A \setminus U);$$

see Item 7 of Proposition 4.1.3.

Remark 4.1.2 ▶ Unwinding Definition 4.1.1

Identifying subsets of A with relations from pt to A via Constructions With Sets, Item 3 of Proposition 4.3.9, we see that the direct image function associated to R is equivalently the function

$$R_* \colon \underbrace{\mathcal{P}(A)}_{\cong \mathrm{Rel}(\mathrm{pt}, A)} \to \underbrace{\mathcal{P}(B)}_{\cong \mathrm{Rel}(\mathrm{pt}, B)}$$

defined by

$$R_*(U) \stackrel{\text{def}}{=} R \diamond U$$

for each $U \in \mathcal{P}(A)$, where $R \diamond U$ is the composition

$$\mathrm{pt} \overset{U}{\nrightarrow} A \overset{R}{\nrightarrow} B.$$

Proposition 4.1.3 ► Properties of Direct Image Functions

Let $R\colon A \nrightarrow B$ be a relation.

1. *Functoriality.* The assignment $U \mapsto R_*(U)$ defines a functor

$$R_*\colon (\mathcal{P}(A), \subset) \to (\mathcal{P}(B), \subset)$$

where

- *Action on Objects.* For each $U \in \mathcal{P}(A)$, we have

$$[R_*](U) \stackrel{\text{def}}{=} R_*(U).$$

- *Action on Morphisms.* For each $U, V \in \mathcal{P}(A)$:
  - If $U \subset V$, then $R_*(U) \subset R_*(V)$.

2. *Adjointness.* We have an adjunction

$$\begin{tikzcd} (R_* \dashv R_{-1})\colon \quad \mathcal{P}(A) \arrow[r, bend left, "R_*"] \arrow[r, phantom, "\perp"] & \mathcal{P}(B), \arrow[l, bend left, "R_{-1}"] \end{tikzcd}$$

witnessed by a bijections of sets

$$\mathrm{Hom}_{\mathcal{P}(A)}(R_*(U), V) \cong \mathrm{Hom}_{\mathcal{P}(A)}(U, R_{-1}(V)),$$

natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$, i.e. such that:

($\star$) The following conditions are equivalent:
  - We have $R_*(U) \subset V$.
  - We have $U \subset R_{-1}(V)$.

3. *Preservation of Colimits.* We have an equality of sets

$$R_*\Big(\bigcup_{i\in I} U_i\Big) = \bigcup_{i\in I} R_*(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have equalities

$$R_*(U) \cup R_*(V) = R_*(U \cup V),$$
$$R_*(\emptyset) = \emptyset,$$

natural in $U, V \in \mathcal{P}(A)$.

4. *Oplax Preservation of Limits.* We have an inclusion of sets

$$R_*(\bigcap_{i\in I} U_i) \subset \bigcap_{i\in I} R_*(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have inclusions

$$R_*(U \cap V) \subset R_*(U) \cap R_*(V),$$
$$R_*(A) \subset B,$$

natural in $U, V \in \mathcal{P}(A)$.

5. *Symmetric Strict Monoidality With Respect to Unions.* The direct image function of Item 1 has a symmetric strict monoidal structure

$$(R_*, R_*^{\otimes}, R_{*|\mathbb{1}}^{\otimes}) \colon (\mathcal{P}(A), \cup, \emptyset) \longrightarrow (\mathcal{P}(B), \cup, \emptyset),$$

being equipped with equalities

$$R_{*|U,V}^{\otimes} \colon R_*(U) \cup R_*(V) \xrightarrow{=} R_*(U \cup V),$$
$$R_{*|\mathbb{1}}^{\otimes} \colon \emptyset \xrightarrow{=} \emptyset,$$

natural in $U, V \in \mathcal{P}(A)$.

6. *Symmetric Oplax Monoidality With Respect to Intersections.* The direct image function of Item 1 has a symmetric oplax monoidal structure

$$(R_*, R_*^{\otimes}, R_{*|\mathbb{1}}^{\otimes}) \colon (\mathcal{P}(A), \cap, A) \longrightarrow (\mathcal{P}(B), \cap, B),$$

being equipped with inclusions

$$R_{*|U,V}^{\otimes} \colon R_*(U \cap V) \subset R_*(U) \cap R_*(V),$$
$$R_{*|\mathbb{1}}^{\otimes} \colon R_*(A) \subset B,$$

natural in $U, V \in \mathcal{P}(A)$.

7. *Relation to Direct Images With Compact Support.* We have

$$R_*(U) = B \setminus R_!(A \setminus U)$$

for each $U \in \mathcal{P}(A)$.

PROOF 4.1.4 ► PROOF OF PROPOSITION 4.1.3

Item 1: Functoriality

Clear.

Item 2: Adjointness

This follows from **??**, **??** of **??**.

Item 3: Preservation of Colimits

This follows from Item 2 and **??**, **??** of **??**.

Item 4: Oplax Preservation of Limits

Omitted.

Item 5: Symmetric Strict Monoidality With Respect to Unions

This follows from Item 3.

Item 6: Symmetric Oplax Monoidality With Respect to Intersections

This follows from Item 4.

Item 7: Relation to Direct Images With Compact Support

The proof proceeds in the same way as in the case of functions (Constructions With Sets, Item 9 of Proposition 4.4.4): applying Item 7 of Proposition 4.4.4 to $A \setminus U$, we have

$$\begin{aligned} R_!(A \setminus U) &= B \setminus R_*(A \setminus (A \setminus U)) \\ &= B \setminus R_*(U). \end{aligned}$$

Taking complements, we then obtain

$$\begin{aligned} R_*(U) &= B \setminus (B \setminus R_*(U)), \\ &= B \setminus R_!(A \setminus U), \end{aligned}$$

which finishes the proof. □

PROPOSITION 4.1.5 ► PROPERTIES OF THE DIRECT IMAGE FUNCTION OPERATION

Let $R \colon A \nrightarrow B$ be a relation.

1. *Functionality I.* The assignment $R \mapsto R_*$ defines a function

$$(-)_* \colon \mathrm{Rel}(A,B) \to \mathsf{Sets}(\mathcal{P}(A),\mathcal{P}(B)).$$

2. *Functionality II.* The assignment $R \mapsto R_*$ defines a function

$$(-)_* \colon \mathrm{Rel}(A,B) \to \mathsf{Pos}((\mathcal{P}(A),\subset),(\mathcal{P}(B),\subset)).$$

3. *Interaction With Identities.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have[1]

$$(\chi_A)_* = \mathrm{id}_{\mathcal{P}(A)}.$$

4. *Interaction With Composition.* For each pair of composable relations $R \colon A \nrightarrow B$ and $S \colon B \nrightarrow C$, we have[2]

$$(S \diamond R)_* = S_* \circ R_*, \qquad \begin{tikzcd} \mathcal{P}(A) \arrow[r, "R_*"] \arrow[dr, "(S\diamond R)_*"'] & \mathcal{P}(B) \arrow[d, "S_*"] \\ & \mathcal{P}(C). \end{tikzcd}$$

[1] That is, the postcomposition function

$$(\chi_A)_* \colon \mathrm{Rel}(\mathrm{pt},A) \to \mathrm{Rel}(\mathrm{pt},A)$$

is equal to $\mathrm{id}_{\mathrm{Rel}(\mathrm{pt},A)}$.

[2] That is, we have

$$(S \diamond R)_* = S_* \circ R_*, \qquad \begin{tikzcd} \mathrm{Rel}(\mathrm{pt},A) \arrow[r, "R_*"] \arrow[dr, "(S\diamond R)_*"'] & \mathrm{Rel}(\mathrm{pt},B) \arrow[d, "S_*"] \\ & \mathrm{Rel}(\mathrm{pt},C). \end{tikzcd}$$

Proof 4.1.6 ► Proof of Proposition 4.1.5

Item 1: Functionality I

Clear.

Item 2: Functionality II

Clear.

Item 3: Interaction With Identities

Indeed, we have

$$\begin{aligned}
(\chi_A)_*(U) &\overset{\text{def}}{=} \bigcup_{a \in U} \chi_A(a) \\
&\overset{\text{def}}{=} \bigcup_{a \in U} \{a\} \\
&= U \\
&\overset{\text{def}}{=} \mathrm{id}_{\mathcal{P}(A)}(U)
\end{aligned}$$

for each $U \in \mathcal{P}(A)$. Thus $(\chi_A)_* = \mathrm{id}_{\mathcal{P}(A)}$.

Item 4: Interaction With Composition

Indeed, we have

$$\begin{aligned}
(S \diamond R)_*(U) &\overset{\text{def}}{=} \bigcup_{a \in U} [S \diamond R](a) \\
&\overset{\text{def}}{=} \bigcup_{a \in U} S(R(a)) \\
&\overset{\text{def}}{=} \bigcup_{a \in U} S_*(R(a)) \\
&= S_*(\bigcup_{a \in U} R(a)) \\
&\overset{\text{def}}{=} S_*(R_*(U)) \\
&\overset{\text{def}}{=} [S_* \circ R_*](U)
\end{aligned}$$

for each $U \in \mathcal{P}(A)$, where we used Item 3 of Proposition 4.1.3. Thus $(S \diamond R)_* = S_* \circ R_*$.

4.2 Strong Inverse Images

Let A and B be sets and let $R \colon A \nrightarrow B$ be a relation.

Definition 4.2.1 ► Strong Inverse Images

The **strong inverse image function associated to** R is the function

$$R_{-1}\colon \mathcal{P}(B) \to \mathcal{P}(A)$$

defined by[1]

$$R_{-1}(V) \stackrel{\text{def}}{=} \{a \in A \mid R(a) \subset V\}$$

for each $V \in \mathcal{P}(B)$.

[1] *Further Terminology:* The set $R_{-1}(V)$ is called the **strong inverse image of** V **by** R.

Remark 4.2.2 ► Unwinding Definition 4.2.1

Identifying subsets of B with relations from pt to B via Constructions With Sets, Item 3 of Proposition 4.3.9, we see that the inverse image function associated to R is equivalently the function

$$R_{-1}\colon \underbrace{\mathcal{P}(B)}_{\cong \mathrm{Rel}(\mathrm{pt},B)} \to \underbrace{\mathcal{P}(A)}_{\cong \mathrm{Rel}(\mathrm{pt},A)}$$

defined by

$$R_{-1}(V) \stackrel{\text{def}}{=} \mathrm{Rift}_R(V), \qquad \begin{tikzcd} & A \arrow[d, "R", "\shortmid"{marking}] \\ \mathrm{pt} \arrow[ur, dashed, "\mathrm{Rift}_R(V)", "\shortmid"{marking}] \arrow[r, "V"', "\shortmid"{marking}] & B, \end{tikzcd}$$

and being explicitly computed by

$$\begin{aligned} R_{-1}(V) &\stackrel{\text{def}}{=} \mathrm{Rift}_R(V) \\ &\cong \int_{b\in B} \mathrm{Hom}_{\{\mathrm{t},\mathrm{f}\}}\left(R^{b}_{-_1}, V^{b}_{-_2}\right), \end{aligned}$$

where we have used Proposition 2.4.1.

PROOF 4.2.3 ▶ PROOF OF REMARK 4.2.2

We have

$$
\begin{aligned}
\mathrm{Rift}_R(V) &\cong \int_{b\in B} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^b_{-_1}, V^b_{-_2}\right)\\
&= \left\{a \in A \;\middle|\; \int_{b\in B} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^b_a, V^b_\star\right) = \mathsf{true}\right\}\\
&= \left\{ a \in A \;\middle|\; \begin{array}{l} \text{for each } b \in B\text{, at least one of the} \\ \text{following conditions hold:} \\ \quad 1.\ \text{We have } R^b_a = \mathsf{false} \\ \quad 2.\ \text{The following conditions hold:} \\ \qquad \text{(a) We have } R^b_a = \mathsf{true} \\ \qquad \text{(b) We have } V^b_\star = \mathsf{true} \end{array} \right\}\\
&= \left\{ a \in A \;\middle|\; \begin{array}{l} \text{for each } b \in B\text{, at least one of the} \\ \text{following conditions hold:} \\ \quad 1.\ \text{We have } b \notin R(a) \\ \quad 2.\ \text{The following conditions hold:} \\ \qquad \text{(a) We have } b \in R(a) \\ \qquad \text{(b) We have } b \in V \end{array} \right\}\\
&= \{a \in A \mid \text{for each } b \in R(a)\text{, we have } b \in V\}\\
&= \{a \in A \mid R(a) \subset V\}\\
&\stackrel{\text{def}}{=} R_{-1}(V).
\end{aligned}
$$

This finishes the proof.

PROPOSITION 4.2.4 ▶ PROPERTIES OF STRONG INVERSE IMAGES

Let $R\colon A \nrightarrow B$ be a relation.

1. *Functoriality.* The assignment $V \mapsto R_{-1}(V)$ defines a functor

$$R_{-1}\colon (\mathcal{P}(B), \subset) \to (\mathcal{P}(A), \subset)$$

where

- *Action on Objects.* For each $V \in \mathcal{P}(B)$, we have

$$[R_{-1}](V) \stackrel{\text{def}}{=} R_{-1}(V).$$

- *Action on Morphisms.* For each $U, V \in \mathcal{P}(B)$:
  - If $U \subset V$, then $R_{-1}(U) \subset R_{-1}(V)$.

2. *Adjointness.* We have an adjunction

$$(R_* \dashv R_{-1}) \colon \quad \begin{tikzcd} \mathcal{P}(A) \arrow[r, bend left, "R_*"] \arrow[r, phantom, "\perp"] & \mathcal{P}(B), \arrow[l, bend left, "R_{-1}"] \end{tikzcd}$$

witnessed by a bijections of sets

$$\mathrm{Hom}_{\mathcal{P}(A)}(R_*(U), V) \cong \mathrm{Hom}_{\mathcal{P}(A)}(U, R_{-1}(V)),$$

natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$, i.e. such that:

($\star$) The following conditions are equivalent:
  - We have $R_*(U) \subset V$.
  - We have $U \subset R_{-1}(V)$.

3. *Lax Preservation of Colimits.* We have an inclusion of sets

$$\bigcup_{i \in I} R_{-1}(U_i) \subset R_{-1}(\bigcup_{i \in I} U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(B)^{\times I}$. In particular, we have inclusions

$$\begin{gathered} R_{-1}(U) \cup R_{-1}(V) \subset R_{-1}(U \cup V), \\ \emptyset \subset R_{-1}(\emptyset), \end{gathered}$$

natural in $U, V \in \mathcal{P}(B)$.

4. *Preservation of Limits.* We have an equality of sets

$$R_{-1}(\bigcap_{i \in I} U_i) = \bigcap_{i \in I} R_{-1}(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(B)^{\times I}$. In particular, we have equalities

$$R_{-1}(U \cap V) = R_{-1}(U) \cap R_{-1}(V),$$
$$R_{-1}(B) = B,$$

natural in $U, V \in \mathcal{P}(B)$.

5. *Symmetric Lax Monoidality With Respect to Unions.* The direct image with compact support function of Item 1 has a symmetric lax monoidal structure

$$(R_{-1}, R^{\otimes}_{-1}, R^{\otimes}_{-1|\mathbb{1}}) \colon (\mathcal{P}(A), \cup, \emptyset) \to (\mathcal{P}(B), \cup, \emptyset),$$

being equipped with inclusions

$$R^{\otimes}_{-1|U,V} \colon R_{-1}(U) \cup R_{-1}(V) \subset R_{-1}(U \cup V),$$
$$R^{\otimes}_{-1|\mathbb{1}} \colon \emptyset \subset R_{-1}(\emptyset),$$

natural in $U, V \in \mathcal{P}(B)$.

6. *Symmetric Strict Monoidality With Respect to Intersections.* The direct image function of Item 1 has a symmetric strict monoidal structure

$$(R_{-1}, R^{\otimes}_{-1}, R^{\otimes}_{-1|\mathbb{1}}) \colon (\mathcal{P}(A), \cap, A) \to (\mathcal{P}(B), \cap, B),$$

being equipped with equalities

$$R^{\otimes}_{-1|U,V} \colon R_{-1}(U \cap V) \xrightarrow{=} R_{-1}(U) \cap R_{-1}(V),$$
$$R^{\otimes}_{-1|\mathbb{1}} \colon R_{-1}(A) \xrightarrow{=} B,$$

natural in $U, V \in \mathcal{P}(B)$.

7. *Interaction With Weak Inverse Images I.* We have

$$R_{-1}(V) = A \setminus R^{-1}(B \setminus V)$$

for each $V \in \mathcal{P}(B)$.

8. *Interaction With Weak Inverse Images II.* Let $R \colon A \nrightarrow B$ be a relation from A to B.

(a) If R is a total relation, then we have an inclusion of sets

$$R_{-1}(V) \subset R^{-1}(V)$$

natural in $V \in \mathcal{P}(B)$.

(b) If R is total and functional, then the above inclusion is in fact an equality.

(c) Conversely, if we have $R_{-1} = R^{-1}$, then R is total and functional.

Proof 4.2.5 ▶ Proof of Proposition 4.2.4

Item 1: Functoriality

Clear.

Item 2: Adjointness

This follows from **??**, **??** of **??**.

Item 3: Lax Preservation of Colimits

Omitted.

Item 4: Preservation of Limits

This follows from Item 2 and **??**, **??** of **??**.

Item 5: Symmetric Lax Monoidality With Respect to Unions

This follows from Item 3.

Item 6: Symmetric Strict Monoidality With Respect to Intersections

This follows from Item 4.

Item 7: Interaction With Weak Inverse Images I

We claim we have an equality

$$R_{-1}(B \setminus V) = A \setminus R^{-1}(V).$$

Indeed, we have

$$\begin{aligned} R_{-1}(B \setminus V) &= \{a \in A \mid R(a) \subset B \setminus V\}, \\ A \setminus R^{-1}(V) &= \{a \in A \mid R(a) \cap V = \emptyset\}. \end{aligned}$$

Taking $V = B \setminus V$ then implies the original statement.

Item 8: Interaction With Weak Inverse Images II

Item 8a is clear, while Items 8b and 8c follow from Item 6 of Proposition 3.1.2. ∎

Proposition 4.2.6 ▶ Properties of the Strong Inverse Image Function Operation

Let $R\colon A \nrightarrow B$ be a relation.

1. *Functionality I.* The assignment $R \mapsto R_{-1}$ defines a function
$$(-)_{-1}\colon \mathsf{Sets}(A,B) \to \mathsf{Sets}(\mathcal{P}(A),\mathcal{P}(B)).$$

2. *Functionality II.* The assignment $R \mapsto R_{-1}$ defines a function
$$(-)_{-1}\colon \mathsf{Sets}(A,B) \to \mathsf{Pos}((\mathcal{P}(A),\subset),(\mathcal{P}(B),\subset)).$$

3. *Interaction With Identities.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have
$$(\mathrm{id}_A)_{-1} = \mathrm{id}_{\mathcal{P}(A)}.$$

4. *Interaction With Composition.* For each pair of composable relations $R\colon A \nrightarrow B$ and $S\colon B \nrightarrow C$, we have
$$(S \diamond R)_{-1} = R_{-1} \circ S_{-1}, \qquad \begin{tikzcd} \mathcal{P}(C) \arrow[r, "S_{-1}"] \arrow[dr, "(S\diamond R)_{-1}"'] & \mathcal{P}(B) \arrow[d, "R_{-1}"] \\ & \mathcal{P}(A). \end{tikzcd}$$

Proof 4.2.7 ▶ Proof of Proposition 4.2.6

Item 1: Functionality I

Clear.

Item 2: Functionality II

Clear.

Item 3: Interaction With Identities

Indeed, we have

$$\begin{aligned}(\chi_A)_{-1}(U) &\stackrel{\text{def}}{=} \{a \in A \mid \chi_A(a) \subset U\}\\ &\stackrel{\text{def}}{=} \{a \in A \mid \{a\} \subset U\}\\ &= U\end{aligned}$$

for each $U \in \mathcal{P}(A)$. Thus $(\chi_A)_{-1} = \mathrm{id}_{\mathcal{P}(A)}$.

Item 4: Interaction With Composition

Indeed, we have

$$\begin{aligned}(S \diamond R)_{-1}(U) &\stackrel{\text{def}}{=} \{a \in A \mid [S \diamond R](a) \subset U\}\\ &\stackrel{\text{def}}{=} \{a \in A \mid S(R(a)) \subset U\}\\ &\stackrel{\text{def}}{=} \{a \in A \mid S_*(R(a)) \subset U\}\\ &= \{a \in A \mid R(a) \subset S_{-1}(U)\}\\ &\stackrel{\text{def}}{=} R_{-1}(S_{-1}(U))\\ &\stackrel{\text{def}}{=} [R_{-1} \circ S_{-1}](U)\end{aligned}$$

for each $U \in \mathcal{P}(C)$, where we used Item 2 of Proposition 4.2.4, which implies that the conditions

- We have $S_*(R(a)) \subset U$.
- We have $R(a) \subset S_{-1}(U)$.

are equivalent. Thus $(S \diamond R)_{-1} = R_{-1} \circ S_{-1}$.

4.3 Weak Inverse Images

Let A and B be sets and let $R\colon A \nrightarrow B$ be a relation.

Definition 4.3.1 ► Weak Inverse Images

The **weak inverse image function associated to** R[1] is the function

$$R^{-1}\colon \mathcal{P}(B) \to \mathcal{P}(A)$$

defined by[2]

$$R^{-1}(V) \stackrel{\text{def}}{=} \{a \in A \mid R(a) \cap V \neq \emptyset\}$$

for each $V \in \mathcal{P}(B)$.

[1] *Further Terminology:* Also called simply the **inverse image function associated to** R.

[2] *Further Terminology:* The set $R^{-1}(V)$ is called the **weak inverse image of** V **by** R or simply the **inverse image of** V **by** R.

Remark 4.3.2 ► Unwinding Definition 4.3.1

Identifying subsets of B with relations from B to pt via Constructions With Sets, Item 3 of Proposition 4.3.9, we see that the weak inverse image function associated to R is equivalently the function

$$R^{-1} \colon \underbrace{\mathcal{P}(B)}_{\cong \mathrm{Rel}(B,\mathrm{pt})} \to \underbrace{\mathcal{P}(A)}_{\cong \mathrm{Rel}(A,\mathrm{pt})}$$

defined by

$$R^{-1}(V) \stackrel{\text{def}}{=} V \diamond R$$

for each $V \in \mathcal{P}(A)$, where $R \diamond V$ is the composition

$$A \overset{R}{\nrightarrow} B \overset{V}{\nrightarrow} \mathrm{pt}.$$

Explicitly, we have

$$\begin{aligned} R^{-1}(V) &\stackrel{\text{def}}{=} V \diamond R \\ &\stackrel{\text{def}}{=} \int^{b \in B} V^{-1}_{b} \times R^{b}_{-_2}. \end{aligned}$$

Proof 4.3.3 ▶ Proof of Remark 4.3.2

We have

$$\begin{aligned}
V \diamond R &\stackrel{\text{def}}{=} \int^{b\in B} V_b^{-1} \times R^b_{-_2} \\
&= \left\{ a \in A \;\middle|\; \int^{b\in B} V_b^{\star} \times R^b_a = \text{true} \right\} \\
&= \left\{ a \in A \;\middle|\; \begin{array}{l}\text{there exists } b \in B \text{ such that the} \\ \text{following conditions hold:} \\ \quad 1.\ \text{We have } V_b^{\star} = \text{true} \\ \quad 2.\ \text{We have } R^b_a = \text{true}\end{array} \right\} \\
&= \left\{ a \in A \;\middle|\; \begin{array}{l}\text{there exists } b \in B \text{ such that the} \\ \text{following conditions hold:} \\ \quad 1.\ \text{We have } b \in V \\ \quad 2.\ \text{We have } b \in R(a)\end{array} \right\} \\
&= \{a \in A \mid \text{there exists } b \in V \text{ such that } b \in R(a)\} \\
&= \{a \in A \mid R(a) \cap V \neq \emptyset\} \\
&\stackrel{\text{def}}{=} R^{-1}(V)
\end{aligned}$$

This finishes the proof.

Proposition 4.3.4 ▶ Properties of Weak Inverse Image Functions

Let $R \colon A \nrightarrow B$ be a relation.

1. *Functoriality.* The assignment $V \mapsto R^{-1}(V)$ defines a functor

$$R^{-1} \colon (\mathcal{P}(B), \subset) \to (\mathcal{P}(A), \subset)$$

where

- *Action on Objects.* For each $V \in \mathcal{P}(B)$, we have

$$[R^{-1}](V) \stackrel{\text{def}}{=} R^{-1}(V).$$

- *Action on Morphisms.* For each $U, V \in \mathcal{P}(B)$:
  - If $U \subset V$, then $R^{-1}(U) \subset R^{-1}(V)$.

2. *Adjointness.* We have an adjunction

$$\left(R^{-1} \dashv R_!\right)\colon \quad \begin{tikzcd} \mathcal{P}(B) \arrow[r, bend left, "R^{-1}"] \arrow[r, phantom, "\perp"] & \mathcal{P}(A), \arrow[l, bend left, "R_!"] \end{tikzcd}$$

witnessed by a bijections of sets

$$\mathrm{Hom}_{\mathcal{P}(A)}(R^{-1}(U), V) \cong \mathrm{Hom}_{\mathcal{P}(A)}(U, R_!(V)),$$

natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$, i.e. such that:

($\star$) The following conditions are equivalent:
  - We have $R^{-1}(U) \subset V$.
  - We have $U \subset R_!(V)$.

3. *Preservation of Colimits.* We have an equality of sets

$$R^{-1}\Big(\bigcup_{i\in I} U_i\Big) = \bigcup_{i\in I} R^{-1}(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(B)^{\times I}$. In particular, we have equalities

$$\begin{aligned} R^{-1}(U) \cup R^{-1}(V) &= R^{-1}(U \cup V), \\ R^{-1}(\emptyset) &= \emptyset, \end{aligned}$$

natural in $U, V \in \mathcal{P}(B)$.

4. *Oplax Preservation of Limits.* We have an inclusion of sets

$$R^{-1}\Big(\bigcap_{i\in I} U_i\Big) \subset \bigcap_{i\in I} R^{-1}(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(B)^{\times I}$. In particular, we have inclusions

$$\begin{aligned} R^{-1}(U \cap V) &\subset R^{-1}(U) \cap R^{-1}(V), \\ R^{-1}(A) &\subset B, \end{aligned}$$

natural in $U, V \in \mathcal{P}(B)$.

5. *Symmetric Strict Monoidality With Respect to Unions.* The direct image function of Item 1 has a symmetric strict monoidal structure

$$(R^{-1}, R^{-1,\otimes}, R_{\mathbb{1}}^{-1,\otimes})\colon (\mathcal{P}(A), \cup, \emptyset) \to (\mathcal{P}(B), \cup, \emptyset),$$

being equipped with equalities

$$\begin{gathered} R_{U,V}^{-1,\otimes}\colon R^{-1}(U) \cup R^{-1}(V) \xrightarrow{=} R^{-1}(U \cup V), \\ R_{\mathbb{1}}^{-1,\otimes}\colon \emptyset \xrightarrow{=} \emptyset, \end{gathered}$$

natural in $U, V \in \mathcal{P}(B)$.

6. *Symmetric Oplax Monoidality With Respect to Intersections.* The direct image function of Item 1 has a symmetric oplax monoidal structure

$$(R^{-1}, R^{-1,\otimes}, R_{\mathbb{1}}^{-1,\otimes})\colon (\mathcal{P}(A), \cap, A) \to (\mathcal{P}(B), \cap, B),$$

being equipped with inclusions

$$\begin{gathered} R_{U,V}^{-1,\otimes}\colon R^{-1}(U \cap V) \subset R^{-1}(U) \cap R^{-1}(V), \\ R_{\mathbb{1}}^{-1,\otimes}\colon R^{-1}(A) \subset B, \end{gathered}$$

natural in $U, V \in \mathcal{P}(B)$.

7. *Interaction With Strong Inverse Images I.* We have

$$R^{-1}(V) = A \setminus R_{-1}(B \setminus V)$$

for each $V \in \mathcal{P}(B)$.

8. *Interaction With Strong Inverse Images II.* Let $R\colon A \nrightarrow B$ be a relation from A to B.

   (a) If R is a total relation, then we have an inclusion of sets

   $$R_{-1}(V) \subset R^{-1}(V)$$

   natural in $V \in \mathcal{P}(B)$.

   (b) If R is total and functional, then the above inclusion is in fact an equality.

   (c) Conversely, if we have $R_{-1} = R^{-1}$, then R is total and functional.

Proof 4.3.5 ► Proof of Proposition 4.3.4

Item 1: Functoriality

Clear.

Item 2: Adjointness

This follows from ??, ?? of ??.

Item 3: Preservation of Colimits

This follows from Item 2 and ??, ?? of ??.

Item 4: Oplax Preservation of Limits

Omitted.

Item 5: Symmetric Strict Monoidality With Respect to Unions

This follows from Item 3.

Item 6: Symmetric Oplax Monoidality With Respect to Intersections

This follows from Item 4.

Item 7: Interaction With Strong Inverse Images I

This follows from Item 7 of Proposition 4.2.4.

Item 8: Interaction With Strong Inverse Images II

This was proved in Item 8 of Proposition 4.2.4. □

Proposition 4.3.6 ► Properties of the Weak Inverse Image Function Operation

Let $R\colon A \nrightarrow B$ be a relation.

1. *Functionality I.* The assignment $R \mapsto R^{-1}$ defines a function

$$(-)^{-1}\colon \mathrm{Rel}(A,B) \to \mathrm{Sets}(\mathcal{P}(A),\mathcal{P}(B)).$$

2. *Functionality II.* The assignment $R \mapsto R^{-1}$ defines a function

$$(-)^{-1}\colon \mathrm{Rel}(A,B) \to \mathrm{Pos}((\mathcal{P}(A),\subset),(\mathcal{P}(B),\subset)).$$

3. *Interaction With Identities.* For each $A \in \mathrm{Obj}(\mathrm{Sets})$, we have[1]

$$(\chi_A)^{-1} = \mathrm{id}_{\mathcal{P}(A)}.$$

4. *Interaction With Composition.* For each pair of composable relations $R\colon A \nrightarrow B$ and $S\colon B \nrightarrow C$, we have[2]

$$(S \diamond R)^{-1} = R^{-1} \circ S^{-1}, \qquad \begin{tikzcd} \mathcal{P}(C) \arrow[r, "S^{-1}"] \arrow[dr, "(S\diamond R)^{-1}"'] & \mathcal{P}(B) \arrow[d, "R^{-1}"] \\ & \mathcal{P}(A). \end{tikzcd}$$

[1]That is, the postcomposition

$$(\chi_A)^{-1}\colon \mathrm{Rel}(\mathrm{pt}, A) \to \mathrm{Rel}(\mathrm{pt}, A)$$

is equal to $\mathrm{id}_{\mathrm{Rel}(\mathrm{pt},A)}$.

[2]That is, we have

$$(S \diamond R)^{-1} = R^{-1} \circ S^{-1}, \qquad \begin{tikzcd} \mathrm{Rel}(\mathrm{pt}, C) \arrow[r, "R^{-1}"] \arrow[dr, "(S\diamond R)^{-1}"'] & \mathrm{Rel}(\mathrm{pt}, B) \arrow[d, "S^{-1}"] \\ & \mathrm{Rel}(\mathrm{pt}, A). \end{tikzcd}$$

Proof 4.3.7 ► Proof of Proposition 4.3.6

Item 1: Functionality I

Clear.

Item 2: Functionality II

Clear.

Item 3: Interaction With Identities

This follows from Categories, Item 5 of Proposition 1.6.2.

Item 4: Interaction With Composition

This follows from Categories, Item 2 of Proposition 1.6.2. ∎

4.4 Direct Images With Compact Support

Let A and B be sets and let $R\colon A \nrightarrow B$ be a relation.

Definition 4.4.1 ► Direct Images With Compact Support

The **direct image with compact support function associated to** R is the function

$$R_! \colon \mathcal{P}(A) \to \mathcal{P}(B)$$

defined by[1,2]

$$\begin{aligned} R_!(U) &\stackrel{\text{def}}{=} \left\{ b \in B \,\middle|\, \begin{array}{l} \text{for each } a \in A, \text{ if we have} \\ b \in R(a), \text{ then } a \in U \end{array} \right\} \\ &= \left\{ b \in B \,\middle|\, R^{-1}(b) \subset U \right\} \end{aligned}$$

for each $U \in \mathcal{P}(A)$.

[1] *Further Terminology:* The set $R_!(U)$ is called the **direct image with compact support of** U **by** R.

[2] We also have

$$R_!(U) = B \setminus R_*(A \setminus U);$$

see Item 7 of Proposition 4.4.4.

Remark 4.4.2 ► Unwinding Definition 4.4.1

Identifying subsets of B with relations from pt to B via Constructions With Sets, Item 3 of Proposition 4.3.9, we see that the direct image with compact support function associated to R is equivalently the function

$$R_! \colon \underbrace{\mathcal{P}(A)}_{\cong \mathrm{Rel}(A,\mathrm{pt})} \to \underbrace{\mathcal{P}(B)}_{\cong \mathrm{Rel}(B,\mathrm{pt})}$$

defined by

$$R_!(U) \stackrel{\text{def}}{=} \mathrm{Ran}_R(U), \qquad \begin{tikzcd} & B \arrow[d, dashed, "\mathrm{Ran}_R(U)", "\shortmid"] \\ A \arrow[ur, "R", "\shortmid"'] \arrow[r, "U"', "\shortmid"] \arrow[ur, phantom, bend left=0, "\Leftarrow" description] & \mathrm{pt}, \end{tikzcd}$$

being explicitly computed by

$$\begin{aligned} R^*(U) &\stackrel{\text{def}}{=} \mathrm{Ran}_R(U) \\ &\cong \int_{a \in A} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R_a^{-2}, U_a^{-1}), \end{aligned}$$

where we have used Proposition 2.3.1.

Proof 4.4.3 ▶ Proof of Remark 4.4.2

We have

$$
\begin{aligned}
\mathrm{Ran}_R(V) &\cong \int_{a\in A} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R_a^{-_2}, U_a^{-_1})\\
&= \left\{ b \in B \,\middle|\, \int_{a\in A} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R_a^b, U_a^\star) = \mathsf{true} \right\}\\
&= \left\{ b \in B \,\middle|\, \begin{array}{l} \text{for each } a \in A \text{, at least one of the} \\ \text{following conditions hold:} \\ \quad 1.\ \text{We have } R_a^b = \mathsf{false} \\ \quad 2.\ \text{The following conditions hold:} \\ \qquad \text{(a) We have } R_a^b = \mathsf{true} \\ \qquad \text{(b) We have } U_a^\star = \mathsf{true} \end{array} \right\}\\
&= \left\{ b \in B \,\middle|\, \begin{array}{l} \text{for each } a \in A \text{, at least one of the} \\ \text{following conditions hold:} \\ \quad 1.\ \text{We have } b \notin R(A) \\ \quad 2.\ \text{The following conditions hold:} \\ \qquad \text{(a) We have } b \in R(a) \\ \qquad \text{(b) We have } a \in U \end{array} \right\}\\
&= \left\{ b \in B \,\middle|\, \begin{array}{l} \text{for each } a \in A \text{, if we have} \\ b \in R(a) \text{, then } a \in U \end{array} \right\}\\
&= \left\{ b \in B \,\middle|\, R^{-1}(b) \subset U \right\}\\
&\overset{\text{def}}{=} R^{-1}(U).
\end{aligned}
$$

This finishes the proof. ∎

Proposition 4.4.4 ► Properties of Direct Images With Compact Support

Let $R\colon A \nrightarrow B$ be a relation.

1. *Functoriality.* The assignment $U \mapsto R_!(U)$ defines a functor

$$R_!\colon (\mathcal{P}(A), \subset) \to (\mathcal{P}(B), \subset)$$

where

- *Action on Objects.* For each $U \in \mathcal{P}(A)$, we have

$$[R_!](U) \stackrel{\text{def}}{=} R_!(U).$$

- *Action on Morphisms.* For each $U, V \in \mathcal{P}(A)$:
  - If $U \subset V$, then $R_!(U) \subset R_!(V)$.

2. *Adjointness.* We have an adjunction

$$\big(R^{-1} \dashv R_!\big)\colon \quad \begin{tikzcd} \mathcal{P}(B) \arrow[r, bend left, "R^{-1}"] \arrow[r, phantom, "\perp"] & \mathcal{P}(A), \arrow[l, bend left, "R_!"] \end{tikzcd}$$

witnessed by a bijections of sets

$$\mathrm{Hom}_{\mathcal{P}(A)}(R^{-1}(U), V) \cong \mathrm{Hom}_{\mathcal{P}(A)}(U, R_!(V)),$$

natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$, i.e. such that:

($\star$) The following conditions are equivalent:
  - We have $R^{-1}(U) \subset V$.
  - We have $U \subset R_!(V)$.

3. *Lax Preservation of Colimits.* We have an inclusion of sets

$$\bigcup_{i \in I} R_!(U_i) \subset R_!\Big(\bigcup_{i \in I} U_i\Big),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have inclusions

$$R_!(U) \cup R_!(V) \subset R_!(U \cup V),$$
$$\emptyset \subset R_!(\emptyset),$$

natural in $U, V \in \mathcal{P}(A)$.

4. *Preservation of Limits.* We have an equality of sets

$$R_!\Bigl(\bigcap_{i\in I} U_i\Bigr) = \bigcap_{i\in I} R_!(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have equalities

$$\begin{aligned} R_!(U \cap V) &= R_!(U) \cap R_!(V), \\ R_!(A) &= B, \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

5. *Symmetric Lax Monoidality With Respect to Unions.* The direct image with compact support function of Item 1 has a symmetric lax monoidal structure

$$(R_!, R_!^{\otimes}, R_{!|\mathbb{1}}^{\otimes}) \colon (\mathcal{P}(A), \cup, \emptyset) \longrightarrow (\mathcal{P}(B), \cup, \emptyset),$$

being equipped with inclusions

$$\begin{aligned} R_{!|U,V}^{\otimes} &\colon R_!(U) \cup R_!(V) \subset R_!(U \cup V), \\ R_{!|\mathbb{1}}^{\otimes} &\colon \emptyset \subset R_!(\emptyset), \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

6. *Symmetric Strict Monoidality With Respect to Intersections.* The direct image function of Item 1 has a symmetric strict monoidal structure

$$(R_!, R_!^{\otimes}, R_{!|\mathbb{1}}^{\otimes}) \colon (\mathcal{P}(A), \cap, A) \longrightarrow (\mathcal{P}(B), \cap, B),$$

being equipped with equalities

$$\begin{aligned} R_{!|U,V}^{\otimes} &\colon R_!(U \cap V) \xrightarrow{=} R_!(U) \cap R_!(V), \\ R_{!|\mathbb{1}}^{\otimes} &\colon R_!(A) \xrightarrow{=} B, \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

7. *Relation to Direct Images.* We have

$$R_!(U) = B \setminus R_*(A \setminus U)$$

for each $U \in \mathcal{P}(A)$.

Proof 4.4.5 ▶ Proof of Proposition 4.4.4

Item 1: Functoriality

Clear.

Item 2: Adjointness

This follows from **??**, **??** of **??**.

Item 3: Lax Preservation of Colimits

Omitted.

Item 4: Preservation of Limits

This follows from Item 2 and **??**, **??** of **??**.

Item 5: Symmetric Lax Monoidality With Respect to Unions

This follows from Item 3.

Item 6: Symmetric Strict Monoidality With Respect to Intersections

This follows from Item 4.

Item 7: Relation to Direct Images

This follows from Item 7 of Proposition 4.1.3. Alternatively, we may prove it directly as follows, with the proof proceeding in the same way as in the case of functions (Constructions With Sets, Item 9 of Proposition 4.6.6).

We claim that $R_!(U) = B \setminus R_*(A \setminus U)$:

- *The First Implication.* We claim that

  $$R_!(U) \subset B \setminus R_*(A \setminus U).$$

  Let $b \in R_!(U)$. We need to show that $b \notin R_*(A \setminus U)$, i.e. that there is no $a \in A \setminus U$ such that $b \in R(a)$.

  This is indeed the case, as otherwise we would have $a \in R^{-1}(b)$ and $a \notin U$, contradicting $R^{-1}(b) \subset U$ (which holds since $b \in R_!(U)$).

  Thus $b \in B \setminus R_*(A \setminus U)$.

- *The Second Implication.* We claim that

  $$B \setminus R_*(A \setminus U) \subset R_!(U).$$

  Let $b \in B \setminus R_*(A \setminus U)$. We need to show that $b \in R_!(U)$, i.e. that $R^{-1}(b) \subset U$.

Since $b \notin R_*(A \setminus U)$, there exists no $a \in A \setminus U$ such that $b \in R(a)$, and hence $R^{-1}(b) \subset U$.

Thus $b \in R_!(U)$.

This finishes the proof. ∎

Proposition 4.4.6 ► Properties of the Direct Image With Compact Support Function Operation

Let $R\colon A \nrightarrow B$ be a relation.

1. *Functionality I.* The assignment $R \mapsto R_!$ defines a function

$$(-)_!\colon \mathsf{Sets}(A,B) \to \mathsf{Sets}(\mathcal{P}(A),\mathcal{P}(B)).$$

2. *Functionality II.* The assignment $R \mapsto R_!$ defines a function

$$(-)_!\colon \mathsf{Sets}(A,B) \to \mathsf{Hom}_{\mathsf{Pos}}((\mathcal{P}(A),\subset),(\mathcal{P}(B),\subset)).$$

3. *Interaction With Identities.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$(\mathrm{id}_A)_! = \mathrm{id}_{\mathcal{P}(A)}.$$

4. *Interaction With Composition.* For each pair of composable relations $R\colon A \nrightarrow B$ and $S\colon B \nrightarrow C$, we have

$$(S \diamond R)_! = S_! \circ R_!, \qquad \begin{tikzcd} \mathcal{P}(A) \arrow[r, "R_!"] \arrow[dr, "(S\diamond R)_!"'] & \mathcal{P}(B) \arrow[d, "S_!"] \\ & \mathcal{P}(C). \end{tikzcd}$$

Proof 4.4.7 ► Proof of Proposition 4.4.6

Item 1: Functionality I

Clear.

Item 2: Functionality II

Clear.

Item 3: Interaction With Identities

Indeed, we have

$$\begin{aligned}(\chi_A)_!(U) &\overset{\text{def}}{=} \left\{a \in A \,\middle|\, \chi_A^{-1}(a) \subset U\right\}\\ &\overset{\text{def}}{=} \{a \in A \mid \{a\} \subset U\}\\ &= U\end{aligned}$$

for each $U \in \mathcal{P}(A)$. Thus $(\chi_A)_! = \mathrm{id}_{\mathcal{P}(A)}$.

Item 4: Interaction With Composition

Indeed, we have

$$\begin{aligned}(S \diamond R)_!(U) &\overset{\text{def}}{=} \left\{c \in C \,\middle|\, [S \diamond R]^{-1}(c) \subset U\right\}\\ &\overset{\text{def}}{=} \left\{c \in C \,\middle|\, S^{-1}(R^{-1}(c)) \subset U\right\}\\ &= \left\{c \in C \,\middle|\, R^{-1}(c) \subset S_!(U)\right\}\\ &\overset{\text{def}}{=} R_!(S_!(U))\\ &\overset{\text{def}}{=} [R_! \circ S_!](U)\end{aligned}$$

for each $U \in \mathcal{P}(C)$, where we used Item 2 of Proposition 4.4.4, which implies that the conditions

- We have $S^{-1}(R^{-1}(c)) \subset U$.
- We have $R^{-1}(c) \subset S_!(U)$.

are equivalent. Thus $(S \diamond R)_! = S_! \circ R_!$. ∎

4.5 Functoriality of Powersets

Proposition 4.5.1 ▶ Functoriality of Powersets I

The assignment $X \mapsto \mathcal{P}(X)$ defines functors[1]

$$\begin{aligned} \mathcal{P}_* \colon \mathsf{Rel} &\to \mathsf{Sets}, \\ \mathcal{P}_{-1} \colon \mathsf{Rel}^{\mathsf{op}} &\to \mathsf{Sets}, \\ \mathcal{P}^{-1} \colon \mathsf{Rel}^{\mathsf{op}} &\to \mathsf{Sets}, \\ \mathcal{P}_! \colon \mathsf{Rel} &\to \mathsf{Sets} \end{aligned}$$

where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathsf{Rel})$, we have

$$\begin{aligned} \mathcal{P}_*(A) &\overset{\text{def}}{=} \mathcal{P}(A), \\ \mathcal{P}_{-1}(A) &\overset{\text{def}}{=} \mathcal{P}(A), \\ \mathcal{P}^{-1}(A) &\overset{\text{def}}{=} \mathcal{P}(A), \\ \mathcal{P}_!(A) &\overset{\text{def}}{=} \mathcal{P}(A). \end{aligned}$$

- *Action on Morphisms.* For each morphism $R \colon A \nrightarrow B$ of Rel, the images

$$\begin{aligned} \mathcal{P}_*(R) \colon \mathcal{P}(A) &\to \mathcal{P}(B), \\ \mathcal{P}_{-1}(R) \colon \mathcal{P}(B) &\to \mathcal{P}(A), \\ \mathcal{P}^{-1}(R) \colon \mathcal{P}(B) &\to \mathcal{P}(A), \\ \mathcal{P}_!(R) \colon \mathcal{P}(A) &\to \mathcal{P}(B) \end{aligned}$$

  of R by $\mathcal{P}_*$, $\mathcal{P}_{-1}$, $\mathcal{P}^{-1}$, and $\mathcal{P}_!$ are defined by

$$\begin{aligned} \mathcal{P}_*(R) &\overset{\text{def}}{=} R_*, \\ \mathcal{P}_{-1}(R) &\overset{\text{def}}{=} R_{-1}, \\ \mathcal{P}^{-1}(R) &\overset{\text{def}}{=} R^{-1}, \\ \mathcal{P}_!(R) &\overset{\text{def}}{=} R_!, \end{aligned}$$

  as in Definitions 4.1.1, 4.2.1, 4.3.1 and 4.4.1.

[1] The functor $\mathcal{P}_* \colon \mathsf{Rel} \to \mathsf{Sets}$ admits a left adjoint; see Item 3 of Proposition 3.1.2.

Proof 4.5.2 ▶ Proof of Proposition 4.5.1

This follows from Items 3 and 4 of Proposition 4.1.5, Items 3 and 4 of Proposition 4.2.6, Items 3 and 4 of Proposition 4.3.6, and Items 3 and 4 of Proposition 4.4.6.

4.6 Functoriality of Powersets: Relations on Powersets

Let A and B be sets and let $R\colon A \nrightarrow B$ be a relation.

Definition 4.6.1 ▶ The Relation on Powersets Associated to a Relation

The **relation on powersets associated to** R is the relation

$$\mathcal{P}(R)\colon \mathcal{P}(A) \nrightarrow \mathcal{P}(B)$$

defined by[1]

$$\mathcal{P}(R)^{V}_{U} \overset{\text{def}}{=} \mathbf{Rel}(\chi_{\mathrm{pt}}, V \diamond R \diamond U)$$

for each $U \in \mathcal{P}(A)$ and each $V \in \mathcal{P}(B)$.

[1] Illustration:

$$\begin{tikzcd} \mathrm{pt} \arrow[r, "U"'] \arrow[rrr, bend left, "\chi_{\mathrm{pt}}"] & A \arrow[r, "R"'] & B \arrow[r, "V"'] & \mathrm{pt}. \end{tikzcd}$$

Remark 4.6.2 ▶ Unwinding Definition 4.6.1

In detail, we have $U \sim_{\mathcal{P}(R)} V$ iff the following equivalent conditions hold:

- We have $\chi_{\mathrm{pt}} \subset V \diamond R \diamond U$.
- We have $(V \diamond R \diamond U)^{\star}_{\star} = \mathrm{true}$, i.e. we have

$$\int^{a\in A}\int^{b\in B} V^{\star}_{b} \times R^{b}_{a} \times U^{a}_{\star} = \mathrm{true}.$$

- There exists some $a \in A$ and some $b \in B$ such that:
  - We have $U^{a}_{\star} = \mathrm{true}$.
  - We have $R^{b}_{a} = \mathrm{true}$.
  - We have $V^{\star}_{b} = \mathrm{true}$.

- There exists some $a \in A$ and some $b \in B$ such that:
  - We have $a \in U$.
  - We have $a \sim_R b$.
  - We have $b \in V$.

Proposition 4.6.3 ▶ Functoriality of Powersets II

The assignment $R \mapsto \mathcal{P}(R)$ defines a functor

$$\mathcal{P}\colon \mathsf{Rel} \longrightarrow \mathsf{Rel}.$$

Proof 4.6.4 ▶ Proof of Proposition 4.6.3

Omitted.

Appendices

A Other Chapters

References

[MO 460656] Emily de Oliveira Santos. *Existence and characterisations of left Kan extensions and liftings in the bicategory of relations I*. MathOverflow. URL: https://mathoverflow.net/q/460656 (cit. on pp. 3, 5).

[MO 461592] Emily de Oliveira Santos. *Existence and characterisations of left Kan extensions and liftings in the bicategory of relations II*. MathOverflow. URL: https://mathoverflow.net/q/461592 (cit. on pp. 4, 5).