Miscellaneous Notes

The Clowder Project Authors

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1 To Do List

1.1 Omitted Proofs To Add

Не так благотворна истина, как зловредна ее видимость.

Даниил Данковский

Truth does not do as much good in the world as the appearance of truth does evil.

Daniil Dankovsky

There's a very large number of omitted proofs throughout these notes. Here I list them in decreasing order of how nice it would be to add them.

Remark 1.1.1 ► Omitted Proofs To Add

Proofs that *need* to be added at some point:

1. Tensor Products of Pointed Sets, Theorem 5.10.1.
2. Tensor Products of Pointed Sets, Theorem 5.11.1.

3. Horizontal composition of natural transformations is associative: Categories, Item 2 of Proposition 8.4.4.
4. Fully faithful functors are essentially injective: Categories, Item 3 of Proposition 5.3.2.

Proofs that *would be very nice* to be added at some point:

1. Properties of pseudomonic functors: Categories, Proposition 6.4.2.
2. Characterisation of fully faithful functors: Categories, Item 1 of Proposition 5.3.2.

Proofs that *would be nice* to be added at some point:

1. Properties of posetal categories: Categories, Proposition 1.3.2.
2. The quadruple adjunction between categories and sets: Categories, Proposition 2.1.1.
3. Properties of groupoid completions: Categories, Proposition 3.2.4.
4. Properties of cores: Categories, Proposition 3.3.5.
5. F_* faithful iff F faithful: Categories, Item 1 of Proposition 5.1.2.
6. F_* full iff F full: Categories, Item 1 of Proposition 5.2.2.
7. Injective on objects functors are precisely the isocofibrations in Cats_2: Categories, Item 1 of Proposition 7.1.2.
8. Characterisations of monomorphisms of categories: Categories, Item 1 of Proposition 6.2.2.
9. Epimorphisms of categories are surjective on objects: Categories, Item 2 of Proposition 6.3.2.
10. Properties of pseudoepic functors: Categories, Proposition 6.5.2.

1.2 Things To Explore/Add

Here we list things to be explored/added to this work in the future.

Remark 1.2.1 ► Things To Explore/Add

Set theory through a category theory lens:

1. Isbell duality for sets.
2. Density comonads and codensity monads for sets.

Relations:

1. 2-Categorical monomorphisms and epimorphisms in **Rel**.
2. Co/limits in **Rel**.
3. Apartness composition, categorical properties of **Rel** with apartness, and apartness relations.
4. Apartness defines a composition for relations, but its analogue

$$\mathfrak{q} \mathbin{\square} \mathfrak{p} \overset{\text{def}}{=} \int_{A \in C} \mathfrak{p}^{-1}_{A} \coprod \mathfrak{q}^{A}_{-_2}$$

fails to be unital for profunctors. Is there a less obvious analogue of apartness composition for profunctors?

5. Codensity monad $\mathrm{Ran}_J(J)$ of a relation (What about $\mathrm{Rift}_J(J)$?)
6. Relative comonads in the 2-category of relations
7. Discrete fibrations and Street fibrations in **Rel**.
8. Consider adding the sections
   - The Monoidal Bicategory of Relations
   - The Monoidal Double Category of Relations

   to Relations.

Spans:

1. Universal property of the bicategory of spans, `https://ncatlab.org/nlab/show/span`

2. Write about cospans.

Un/Straightening:

1. Write proper sections on straightening for lax functors from sets to Rel or Span (displayed sets)

Categories:

1. Expand **??** and add a proof to it.

2. Sections and retractions; retracts, https://ncatlab.org/nlab/show/retract.

3. Regular categories: https://arxiv.org/pdf/2004.08964.pdf.

4. Are pseudoepic functors those functors whose restricted Yoneda embedding is pseudomonic and Yoneda preserves absolute colimits?

5. Absolutely dense functors enriched over $\mathbb{R}^{+}$ apparently reduce to topological density

Types of Morphisms in Categories:

1. Behaviour in $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$, e.g. pointwise sections vs. sections in $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$.

2. A faithful functor from balanced category is conservative

Yoneda stuff:

1. Properties of restricted Yoneda embedding, e.g. if the restricted Yoneda embedding is full, then what can we conclude? Related: https://qchu.wordpress.com/2015/05/17/generators/

Adjunctions:

1. Adjunctions, units, counits, and fully faithfulness as in https://mathoverflow.net/questions/100808/properties-of-functors-and-their-adjoints.

2. Morphisms between adjunctions and bicategory $\mathsf{Adj}(\mathcal{C})$.

3. https://ncatlab.org/nlab/show/transformation+of+adjoints

Constructions With Categories:

1. Comparison between pseudopullbacks and isocomma categories: the "evident" functor $\mathcal{C} \times^{\mathsf{ps}}_{\mathcal{E}} \mathcal{D} \to \mathcal{C} \overset{\leftrightarrow}{\times}_{\mathcal{E}} \mathcal{D}$ is essentially surjective and full, but not faithful in general.

Co/limits:

1. Add the characterisations of absolutely dense functors given in **??** to **??**.
2. Absolutely dense functors, https://ncatlab.org/nlab/show/absolutely+dense+functor. Also theorem 1.1 here: http://www.tac.mta.ca/tac/volumes/8/n20/n20.pdf.
3. Dense functors, codense functors, and absolutely codense functors.

Co/ends:

1. Examples of co/ends: https://mathoverflow.net/a/461814
2. Cofinality for co/ends, https://mathoverflow.net/questions/353876

Fibred category theory:

1. Internal **Hom** in categories of co/Cartesian fibrations.
2. *Tensor structures on fibered categories* by Luca Terenzi: https://arxiv.org/abs/2401.13491. Check also the other papers by Luca Terenzi.
3. https://ncatlab.org/nlab/show/cartesian+natural+transformation (this is a cartesian morphism in $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$ apparently)
4. CoCartesian fibration classifying $\mathsf{Fun}(F, G)$, https://mathoverflow.net/questions/457533/cocartesian-fibration-classifying-mathrmfunf-g

Monoidal categories:

1. Free braided monoidal category with a braided monoid: https://ncatlab.org/nlab/show/vine

Skew monoidal categories:

1. Does the $\mathbb{E}_1$ tensor product of monoids admit a skew monoidal category structure?
2. Is there a (right?) skew monoidal category structure on $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$ using right Kan extensions instead of left Kan extensions?
3. Similarly, are there skew monoidal category structures on the subcategory of $\mathbf{Rel}(A, B)$ spanned by the functions using left Kan extensions and left Kan lifts?

Higher categories:

1. Internal adjunctions in Mod as in [JY21, Section 6.3]; see [JY21, Example 6.2.6].
2. Comonads in the bicategory of profunctors.

Monoids:

1. Isbell's zigzag theorem for semigroups: the following conditions are equivalent:
   (a) A morphism $f\colon A \longrightarrow B$ of semigroups is an epimorphism.
   (b) For each $b \in B$, one of the following conditions is satisfied:
      - We have $f(a) = b$.
      - There exist some $m \in \mathbb{N}_{\geq 1}$ and two factorisations
        $$\begin{aligned} b &= a_0 y_1, \\ b &= x_m a_{2m} \end{aligned}$$
        connected by relations
        $$\begin{aligned} a_0 &= x_1 a_1, \\ a_1 y_1 &= a_2 y_2, \\ x_1 a_2 &= x_2 a_3, \\ a_{2m-1} y_m &= a_{2m} \end{aligned}$$
        such that, for each $1 \leq i \leq m$, we have $a_i \in \mathrm{Im}(f)$.

Wikipedia says in https://en.wikipedia.org/wiki/Isbell%27s_zigzag_theorem:

> For monoids, this theorem can be written more concisely:

Types of morphisms in bicategories:

1. Behaviour in 2-categories of pseudofunctors (or lax functors, etc.), e.g. pointwise pseudoepic morphisms in vs. pseudoepic morphisms in 2-categories of pseudofunctors.
2. Statements like "coequifiers are lax epimorphisms", Item 2 of Examples 2.4 of https://arxiv.org/abs/2109.09836, along with most of the other statements/examples there.
3. Dense, absolutely dense, etc. morphisms in bicategories

Other:

1. https://qchu.wordpress.com/
2. https://aroundtoposes.com/
3. https://ncatlab.org/nlab/show/essentially+surjective+and+full+functor
4. https://mathoverflow.net/questions/415363/objects-whose-representable-presheaf-is-a-fibration
5. https://mathoverflow.net/questions/460146/universal-property-of-isbell-duality
6. http://www.tac.mta.ca/tac/volumes/36/12/36-12abs.html (Isbell conjugacy and the reflexive completion)
7. https://ncatlab.org/nlab/show/enrichment+versus+internalisation
8. The works of Philip Saville, https://philipsaville.co.uk/
9. https://golem.ph.utexas.edu/category/2024/02/from_cartesian_to_symmetric_mo.html

10. https://mathoverflow.net/q/463855 (One-object lax transformations)
11. https://ncatlab.org/nlab/show/analytic+completion+of+a+ring
12. https://en.wikipedia.org/wiki/Quaternionic_analysis
13. https://arxiv.org/abs/2401.15051 (The Norm Functor over Schemes)
14. https://mathoverflow.net/questions/407291/ (Adjunctions with respect to profunctors)
15. https://mathoverflow.net/a/462726 (Prof is free completion of Cats under right extensions)
16. there's some cool stuff in https://arxiv.org/abs/2312.00990 (Polynomial Functors: A Mathematical Theory of Interaction), e.g. on cofunctors.
17. https://ncatlab.org/nlab/show/adjoint+lifting+theorem
18. https://ncatlab.org/nlab/show/Gabriel%E2%80%93Ulmer+duality

Appendices

A Other Chapters

References

[JY21] Niles Johnson and Donald Yau. *2-Dimensional Categories*. Oxford University Press, Oxford, 2021, pp. xix+615. ISBN: 978-0-19-887138-5; 978-0-19-887137-8. DOI: 10.1093/oso/9780198871378.001.0001. URL: https://doi.org/10.1093/oso/9780198871378.001.0001.