Pointed Sets

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This chapter contains some foundational material on pointed sets.

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1 Pointed Sets

1.1 Foundations

Definition 1.1.1.1. A **pointed set**¹ is equivalently:

- An \mathbb{E}_0 -monoid in $(N_{\bullet}(\mathsf{Sets}), \mathsf{pt})$.
- A pointed object in (Sets, pt).

Remark 1.1.1.2. In detail, a **pointed set** is a pair (X, x_0) consisting of:

- The Underlying Set. A set X, called the underlying set of (X, x_0) .
- The Basepoint. A morphism

$$[x_0] \colon \mathrm{pt} \to X$$

in Sets, determining an element $x_0 \in X$, called the basepoint of X.

Example 1.1.1.3. The 0-sphere² is the pointed set $(S^0,0)^3$ consisting of:

• The Underlying Set. The set S^0 defined by

$$S^0 \stackrel{\text{\tiny def}}{=} \{0,1\}.$$

• The Basepoint. The element 0 of S^0 .

Example 1.1.1.4. The trivial pointed set is the pointed set (pt, \star) consisting of:

- The Underlying Set. The punctual set pt $\stackrel{\text{def}}{=} \{ \star \}$.
- The Basepoint. The element \star of pt.

Example 1.1.1.5. The underlying pointed set of a semimodule (M, α_M) is the pointed set $(M, 0_M)$.

Example 1.1.1.6. The underlying pointed set of a module (M, α_M) is the pointed set $(M, 0_M)$.

¹Further Terminology: In the context of monoids with zero as models for \mathbb{F}_1 -algebras, pointed sets are viewed as \mathbb{F}_1 -modules.

² Further Terminology: In the context of monoids with zero as models for \mathbb{F}_1 -algebras, the 0-sphere is viewed as the underlying pointed set of the field with one element.

³ Further Notation: In the context of monoids with zero as models for \mathbb{F}_1 -algebras, S^0

1.2 Morphisms of Pointed Sets

Definition 1.2.1.1. A morphism of pointed sets^{4,5} is equivalently:

- A morphism of \mathbb{E}_0 -monoids in $(N_{\bullet}(\mathsf{Sets}), \mathsf{pt})$.
- A morphism of pointed objects in (Sets, pt).

Remark 1.2.1.2. In detail, a morphism of pointed sets $f:(X,x_0) \to (Y,y_0)$ is a morphism of sets $f:X\to Y$ such that the diagram

$$\begin{array}{ccc}
& \text{pt} \\
[x_0] & & [y_0] \\
X & \xrightarrow{f} Y
\end{array}$$

commutes, i.e. such that

$$f(x_0) = y_0.$$

1.3 The Category of Pointed Sets

Definition 1.3.1.1. The category of pointed sets is the category Sets_{*} defined equivalently as

- The homotopy category of the ∞-category Mon_{E0}(N_•(Sets), pt) of ??,
 ??;
- The category Sets* of ??, ??.

Remark 1.3.1.2. In detail, the category of pointed sets is the category Sets_* where

- *Objects*. The objects of Sets* are pointed sets;
- Morphisms. The morphisms of Sets* are morphisms of pointed sets;
- Identities. For each $(X, x_0) \in \text{Obj}(\mathsf{Sets}_*)$, the unit map

$$\mathbb{1}^{\mathsf{Sets}_*}_{(X,x_0)} \colon \mathrm{pt} \to \mathsf{Sets}_*((X,x_0),(X,x_0))$$

of Sets_{*} at (X, x_0) is defined by⁶

$$\operatorname{id}_{(X,x_0)}^{\mathsf{Sets}_*} \stackrel{\text{def}}{=} \operatorname{id}_X;$$

is also denoted $(\mathbb{F}_1, 0)$.

⁴Further Terminology: Also called a **pointed function**.

⁵ Further Terminology: In the context of monoids with zero as models for \mathbb{F}_1 -algebras, morphisms of pointed sets are also called **morphism of** \mathbb{F}_1 -modules.

⁶Note that id_X is indeed a morphism of pointed sets, as we have $id_X(x_0) = x_0$.

• Composition. For each $(X, x_0), (Y, y_0), (Z, z_0) \in \text{Obj}(\mathsf{Sets}_*)$, the composition map

$$\circ^{\mathsf{Sets}_*}_{(X,x_0),(Y,y_0),(Z,z_0)} \colon \mathsf{Sets}_*((Y,y_0),(Z,z_0)) \times \mathsf{Sets}_*((X,x_0),(Y,y_0)) \to \mathsf{Sets}_*((X,x_0),(Z,z_0))$$

of Sets_{*} at $((X, x_0), (Y, y_0), (Z, z_0))$ is defined by⁷

$$g \circ_{(X,x_0),(Y,y_0),(Z,z_0)}^{\mathsf{Sets}_*} f \stackrel{\mathrm{def}}{=} g \circ f.$$

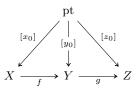
1.4 Elementary Properties of Pointed Sets

Proposition 1.4.1.1. Let (X, x_0) be a pointed set.

- 1. Completeness. The category Sets_{*} of pointed sets and morphisms between them is complete, having in particular:
 - (a) Products, described as in Definition 2.3.1.1;
 - (b) Pullbacks, described as in Definition 2.4.1.1;
 - (c) Equalisers, described as in Definition 2.5.1.1.
- 2. Cocompleteness. The category Sets_{*} of pointed sets and morphisms between them is cocomplete, having in particular:
 - (a) Coproducts, described as in Definition 3.3.1.1;
 - (b) Pushouts, described as in Definition 3.4.1.1;
 - (c) Coequalisers, described as in Definition 3.5.1.1.
- 3. Failure To Be Cartesian Closed. The category Sets_* is not Cartesian closed. 8

$$g(f(x_0)) = g(y_0)$$
$$= z_0,$$

or



in terms of diagrams.

 $^{^7}$ Note that the composition of two morphisms of pointed sets is indeed a morphism of pointed sets, as we have

 $^{^8}$ The category Sets_* does admit monoidal closed structures however; see $\overline{\mathsf{Tensor}}$ $\underline{\mathsf{Products}}$

4. Morphisms From the Monoidal Unit. We have a bijection of sets⁹

$$\mathsf{Sets}_*(S^0, X) \cong X,$$

natural in $(X, x_0) \in \text{Obj}(\mathsf{Sets}_*)$, internalising also to an isomorphism of pointed sets

$$\mathbf{Sets}_*(S^0, X) \cong (X, x_0),$$

again natural in $(X, x_0) \in \text{Obj}(\mathsf{Sets}_*)$.

5. Relation to Partial Functions. We have an equivalence of categories 10

$$\mathsf{Sets}_* \stackrel{\mathrm{eq.}}{\cong} \mathsf{Sets}^{\mathrm{part.}}$$

between the category of pointed sets and pointed functions between them and the category of sets and partial functions between them, where:

(a) From Pointed Sets to Sets With Partial Functions. The equivalence

$$\xi \colon \mathsf{Sets}_* \stackrel{\cong}{\to} \mathsf{Sets}^{\mathrm{part.}}$$

sends:

- i. A pointed set (X, x_0) to X.
- ii. A pointed function

$$f: (X, x_0) \to (Y, y_0)$$

to the partial function

$$\xi_f \colon X \to Y$$

defined on $f^{-1}(Y \setminus y_0)$ and given by

$$\xi_f(x) \stackrel{\text{def}}{=} f(x)$$

for each
$$x \in f^{-1}(Y \setminus y_0)$$
.

忘:
$$\mathsf{Sets}_* \to \mathsf{Sets}$$

defined on objects by sending a pointed set to its underlying set is corepresentable by S^0 .

 10 Warning: This is not an isomorphism of categories, only an equivalence.

of Pointed Sets.

⁹In other words, the forgetful functor

(b) From Sets With Partial Functions to Pointed Sets. The equivalence

$$\xi^{-1} \colon \mathsf{Sets}^{\mathsf{part.}} \stackrel{\cong}{\to} \mathsf{Sets}_*$$

sends:

- i. A set X is to the pointed set (X, \star) with \star an element that is not in X.
- ii. A partial function

$$f \colon X \to Y$$

defined on $U \subset X$ to the pointed function

$$\xi_f^{-1} \colon (X, x_0) \to (Y, y_0)$$

defined by

$$\xi_f(x) \stackrel{\text{def}}{=} \begin{cases} f(x) & \text{if } x \in U, \\ y_0 & \text{otherwise.} \end{cases}$$

for each $x \in X$.

Proof. Item 1, Completeness: This follows from (the proofs) of Definitions 2.3.1.1, 2.4.1.1 and 2.5.1.1 and ??, ??.

Item 2, Cocompleteness: This follows from (the proofs) of Definitions 3.3.1.1, 3.4.1.1 and 3.5.1.1 and ??, ??.

Item 3, Failure To Be Cartesian Closed: See [MSE 2855868].

Item 4, Morphisms From the Monoidal Unit: Since a morphism from S^0 to a pointed set (X, x_0) sends $0 \in S^0$ to x_0 and then can send $1 \in S^0$ to any element of X, we obtain a bijection between pointed maps $S^0 \to X$ and the elements of X.

The isomorphism then

$$\mathbf{Sets}_*(S^0,X)\cong (X,x_0)$$

follows by noting that $\Delta_{x_0} \colon S^0 \to X$, the basepoint of $\mathbf{Sets}_*(S^0, X)$, corresponds to the pointed map $S^0 \to X$ picking the element x_0 of X, and thus we see that the bijection between pointed maps $S^0 \to X$ and elements of X is compatible with basepoints, lifting to an isomorphism of pointed sets.

Item 5, Relation to Partial Functions: See [MSE 884460].

2 Limits of Pointed Sets

2.1 The Terminal Pointed Set

Definition 2.1.1.1. The **terminal pointed set** is the pair $((pt, \star), \{!_X\}_{(X,x_0) \in Obj(Sets_*)})$ consisting of:

- The Limit. The pointed set (pt, \star) .
- The Cone. The collection of morphisms of pointed sets

$$\{!_X \colon (X, x_0) \to (\operatorname{pt}, \star)\}_{(X, x_0) \in \operatorname{Obj}(\mathsf{Sets})}$$

defined by

$$!_X(x) \stackrel{\text{def}}{=} \star$$

for each $x \in X$ and each $(X, x_0) \in \text{Obj}(\mathsf{Sets})$.

Proof. We claim that (pt, \star) is the terminal object of Sets_* . Indeed, suppose we have a diagram of the form

$$(X, x_0)$$
 (pt, \star)

in Sets_{*}. Then there exists a unique morphism of pointed sets

$$\phi \colon (X, x_0) \to (\operatorname{pt}, \star)$$

making the diagram

$$(X, x_0) \xrightarrow{-\phi} (\operatorname{pt}, \star)$$

commute, namely $!_X$.

2.2 Products of Families of Pointed Sets

Let $\{(X_i, x_0^i)\}_{i \in I}$ be a family of pointed sets.

Definition 2.2.1.1. The **product of** $\{(X_i, x_0^i)\}_{i \in I}$ is the pair $((\prod_{i \in I} X_i, (x_0^i)_{i \in I}), \{\operatorname{pr}_i\}_{i \in I})$ consisting of:

- The Limit. The pointed set $(\prod_{i \in I} X_i, (x_0^i)_{i \in I})$.
- The Cone. The collection

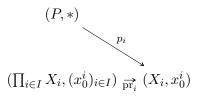
$$\left\{\operatorname{pr}_i \colon (\prod_{i \in I} X_i, (x_0^i)_{i \in I}) \to (X_i, x_0^i)\right\}_{i \in I}$$

of maps given by

$$\operatorname{pr}_i((x_j)_{j\in I}) \stackrel{\text{def}}{=} x_i$$

for each $(x_j)_{j\in I} \in \prod_{i\in I} X_i$ and each $i\in I$.

Proof. We claim that $(\prod_{i \in I} X_i, (x_0^i)_{i \in I})$ is the categorical product of $\{(X_i, x_0^i)\}_{i \in I}$ in Sets_{*}. Indeed, suppose we have, for each $i \in I$, a diagram of the form



in Sets_{*}. Then there exists a unique morphism of pointed sets

$$\phi \colon (P, *) \to (\prod_{i \in I} X_i, (x_0^i)_{i \in I})$$

making the diagram

$$(P,*)$$

$$\phi \mid \exists !$$

$$(\prod_{i \in I} X_i, (x_0^i)_{i \in I}) \underset{\overrightarrow{\operatorname{pr}_i}}{\rightarrow} (X_i, x_0^i)$$

commute, being uniquely determined by the condition $\operatorname{pr}_i \circ \phi = p_i$ for each $i \in I$ via

$$\phi(x) = (p_i(x))_{i \in I}$$

for each $x \in P$. Note that this is indeed a morphism of pointed sets, as we have

$$\phi(*) = (p_i(*))_{i \in I}$$

= $(x_0^i)_{i \in I}$,

where we have used that p_i is a morphism of pointed sets for each $i \in I$. \square

Proposition 2.2.1.2. Let $\{(X_i, x_0^i)\}_{i \in I}$ be a family of pointed sets.

1. Functoriality. The assignment $\{(X_i, x_0^i)\}_{i \in I} \mapsto (\prod_{i \in I} X_i, (x_0^i)_{i \in I})$ defines a functor

$$\prod_{i \in I} : \mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}_*) o \mathsf{Sets}_*.$$

Proof. Item 1, Functoriality: This follows from ??, ?? of ??.

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2.3 Products

Let (X, x_0) and (Y, y_0) be pointed sets.

Definition 2.3.1.1. The **product of** (X, x_0) **and** (Y, y_0) is the pair consisting of:

- The Limit. The pointed set $(X \times Y, (x_0, y_0))$.
- The Cone. The morphisms of pointed sets

$$\operatorname{pr}_1: (X \times Y, (x_0, y_0)) \to (X, x_0),$$

 $\operatorname{pr}_2: (X \times Y, (x_0, y_0)) \to (Y, y_0)$

defined by

$$\operatorname{pr}_1(x,y) \stackrel{\text{def}}{=} x,$$

 $\operatorname{pr}_2(x,y) \stackrel{\text{def}}{=} y$

for each $(x,y) \in X \times Y$.

Proof. We claim that $(X \times Y, (x_0, y_0))$ is the categorical product of (X, x_0) and (Y, y_0) in Sets_* . Indeed, suppose we have a diagram of the form

$$(X, x_0) \underset{\text{pr}_1}{\longleftarrow} (X \times Y, (x_0, y_0)) \xrightarrow{p_2} (Y, y_0)$$

in Sets_{*}. Then there exists a unique morphism of pointed sets

$$\phi \colon (P, *) \to (X \times Y, (x_0, y_0))$$

making the diagram

$$(P,*) \xrightarrow{p_1} (P,*)$$

$$\downarrow^{\downarrow} \exists !$$

$$(X,x_0) \xleftarrow{\text{Dr.}} (X \times Y,(x_0,y_0)) \xrightarrow{\text{Dr.}} (Y,y_0)$$

commute, being uniquely determined by the conditions

$$\operatorname{pr}_1 \circ \phi = p_1,$$
$$\operatorname{pr}_2 \circ \phi = p_2$$

2.3 Products 10

via

$$\phi(x) = (p_1(x), p_2(x))$$

for each $x \in P$. Note that this is indeed a morphism of pointed sets, as we have

$$\phi(*) = (p_1(*), p_2(*))$$

= $(x_0, y_0),$

where we have used that p_1 and p_2 are morphisms of pointed sets. \Box

Proposition 2.3.1.2. Let (X, x_0) , (Y, y_0) , and (Z, z_0) be pointed sets.

1. Functoriality. The assignments

$$(X, x_0), (Y, y_0), ((X, x_0), (Y, y_0)) \mapsto (X \times Y, (x_0, y_0))$$

define functors

$$\begin{split} X\times -\colon \mathsf{Sets}_* \to \mathsf{Sets}_*, \\ -\times Y\colon \mathsf{Sets}_* \to \mathsf{Sets}_*, \\ -_1\times -_2\colon \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*, \end{split}$$

defined in the same way as the functors of Constructions With Sets, Item 1 of Proposition 1.3.1.2.

2. Associativity. We have an isomorphism of pointed sets

$$((X \times Y) \times Z, ((x_0, y_0), z_0)) \cong (X \times (Y \times Z), (x_0, (y_0, z_0)))$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \text{Obj}(\mathsf{Sets}_*).$

3. Unitality. We have isomorphisms of pointed sets

$$(\operatorname{pt}, \star) \times (X, x_0) \cong (X, x_0),$$

$$(X, x_0) \times (\operatorname{pt}, \star) \cong (X, x_0),$$

natural in $(X, x_0) \in \text{Obj}(\mathsf{Sets}_*)$.

4. Commutativity. We have an isomorphism of pointed sets

$$(X \times Y, (x_0, y_0)) \cong (Y \times X, (y_0, x_0)),$$

natural in $(X, x_0), (Y, y_0) \in \text{Obj}(\mathsf{Sets}_*).$

5. Symmetric Monoidality. The triple $(\mathsf{Sets}_*, \times, (\mathsf{pt}, \star))$ is a symmetric monoidal category.

Proof. Item 1, Functoriality: This is a special case of functoriality of limits, ??, ?? of ??.

Item 2, Associativity: This follows from Constructions With Sets, Item 3 of Proposition 1.3.1.2.

Item 3, Unitality: This follows from Constructions With Sets, Item 4 of Proposition 1.3.1.2.

Item 4, Commutativity: This follows from Constructions With Sets, Item 5 of Proposition 1.3.1.2.

Item 5, Symmetric Monoidality: This follows from Constructions With Sets, Item 12 of Proposition 1.3.1.2.

2.4 Pullbacks

Let (X, x_0) , (Y, y_0) , and (Z, z_0) be pointed sets and let $f: (X, x_0) \to (Z, z_0)$ and $g: (Y, y_0) \to (Z, z_0)$ be morphisms of pointed sets.

Definition 2.4.1.1. The pullback of (X, x_0) and (Y, y_0) over (Z, z_0) along (f, g) is the pair consisting of:

- The Limit. The pointed set $(X \times_Z Y, (x_0, y_0))$.
- The Cone. The morphisms of pointed sets

$$\operatorname{pr}_1: (X \times_Z Y, (x_0, y_0)) \to (X, x_0),$$

 $\operatorname{pr}_2: (X \times_Z Y, (x_0, y_0)) \to (Y, y_0)$

defined by

$$\operatorname{pr}_1(x,y) \stackrel{\text{def}}{=} x,$$

 $\operatorname{pr}_2(x,y) \stackrel{\text{def}}{=} y$

for each $(x,y) \in X \times_Z Y$.

Proof. We claim that $X \times_Z Y$ is the categorical pullback of (X, x_0) and (Y, y_0) over (Z, z_0) with respect to (f, g) in Sets_* . First we need to check that the relevant pullback diagram commutes, i.e. that we have

$$f \circ \operatorname{pr}_{1} = g \circ \operatorname{pr}_{2}, \qquad \left(X \times_{Z} Y, (x_{0}, y_{0}) \right) \xrightarrow{\operatorname{pr}_{2}} (Y, y_{0})$$
$$\downarrow^{g} \qquad \left(X, x_{0} \right) \xrightarrow{f} (Z, z_{0}).$$

Indeed, given $(x, y) \in X \times_Z Y$, we have

$$[f \circ pr_1](x, y) = f(pr_1(x, y))$$

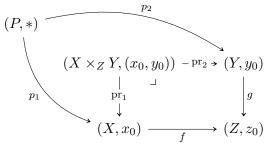
$$= f(x)$$

$$= g(y)$$

$$= g(pr_2(x, y))$$

$$= [g \circ pr_2](x, y),$$

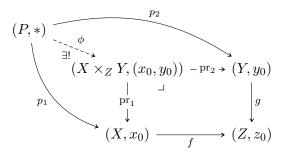
where f(x) = g(y) since $(x, y) \in X \times_Z Y$. Next, we prove that $X \times_Z Y$ satisfies the universal property of the pullback. Suppose we have a diagram of the form



in Sets_{*}. Then there exists a unique morphism of pointed sets

$$\phi \colon (P, *) \to (X \times_Z Y, (x_0, y_0))$$

making the diagram



commute, being uniquely determined by the conditions

$$\operatorname{pr}_1 \circ \phi = p_1,$$
$$\operatorname{pr}_2 \circ \phi = p_2$$

via

$$\phi(x) = (p_1(x), p_2(x))$$

for each $x \in P$, where we note that $(p_1(x), p_2(x)) \in X \times Y$ indeed lies in $X \times_Z Y$ by the condition

$$f \circ p_1 = g \circ p_2$$
,

which gives

$$f(p_1(x)) = g(p_2(x))$$

for each $x \in P$, so that $(p_1(x), p_2(x)) \in X \times_Z Y$. Lastly, we note that ϕ is indeed a morphism of pointed sets, as we have

$$\phi(*) = (p_1(*), p_2(*))$$

= $(x_0, y_0),$

where we have used that p_1 and p_2 are morphisms of pointed sets. \Box

Proposition 2.4.1.2. Let (X, x_0) , (Y, y_0) , (Z, z_0) , and (A, a_0) be pointed sets.

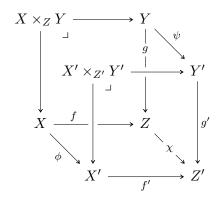
1. Functoriality. The assignment $(X,Y,Z,f,g)\mapsto X\times_{f,Z,g}Y$ defines a functor

$$-1 \times_{-3} -1$$
: $\operatorname{\mathsf{Fun}}(\mathcal{P}, \operatorname{\mathsf{Sets}}_*) \to \operatorname{\mathsf{Sets}}_*,$

where \mathcal{P} is the category that looks like this:



In particular, the action on morphisms of $-1 \times_{-3} -1$ is given by sending a morphism



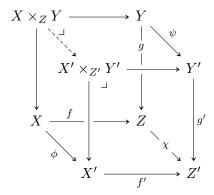
in $Fun(\mathcal{P}, \mathsf{Sets}_*)$ to the morphism of pointed sets

$$\xi \colon (X \times_Z Y, (x_0, y_0)) \xrightarrow{\exists !} (X' \times_{Z'} Y', (x'_0, y'_0))$$

given by

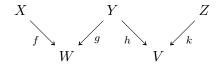
$$\xi(x,y) \stackrel{\text{\tiny def}}{=} (\phi(x), \psi(y))$$

for each $(x, y) \in X \times_Z Y$, which is the unique morphism of pointed sets making the diagram



commute.

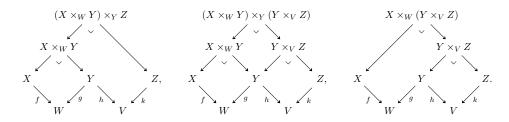
2. Associativity. Given a diagram



in Sets_{*}, we have isomorphisms of pointed sets

$$(X \times_W Y) \times_V Z \cong (X \times_W Y) \times_Y (Y \times_V Z) \cong X \times_W (Y \times_V Z),$$

where these pullbacks are built as in the diagrams



3. Unitality. We have isomorphisms of pointed sets

4. Commutativity. We have an isomorphism of pointed sets

$$A \times_X B \longrightarrow B$$

$$\downarrow \qquad \qquad \downarrow g \qquad A \times_X B \cong B \times_X A \qquad \qquad \downarrow \qquad \downarrow f$$

$$A \xrightarrow{f} X, \qquad \qquad B \xrightarrow{g} X.$$

5. Interaction With Products. We have an isomorphism of pointed sets

$$X \times_{\mathrm{pt}} Y \cong X \times Y, \qquad \begin{array}{c} X \times Y \longrightarrow Y \\ & \downarrow !_{Y} \\ X \xrightarrow{\quad \mid !_{X} \quad } \mathrm{pt}. \end{array}$$

6. Symmetric Monoidality. The triple $(\mathsf{Sets}_*, \times_X, X)$ is a symmetric monoidal category.

Proof. Item 1, Functoriality: This is a special case of functoriality of co/limits, ??, ?? of ??, with the explicit expression for ξ following from the commutativity of the cube pullback diagram.

Item 2, Associativity: This follows from Constructions With Sets, Item 2 of Proposition 1.4.1.3.

Item 3, Unitality: This follows from Constructions With Sets, Item 3 of Proposition 1.4.1.3.

Item 4, Commutativity: This follows from Constructions With Sets, Item 4 of Proposition 1.4.1.3.

Item 5, Interaction With Products: This follows from Constructions With Sets, Item 6 of Proposition 1.4.1.3.

Item 6, Symmetric Monoidality: This follows from Constructions With Sets, Item 7 of Proposition 1.4.1.3.

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2.5 Equalisers

Let $f, g: (X, x_0) \rightrightarrows (Y, y_0)$ be morphisms of pointed sets.

Definition 2.5.1.1. The equaliser of (f,g) is the pair consisting of:

- The Limit. The pointed set $(\text{Eq}(f,g),x_0)$.
- The Cone. The morphism of pointed sets

$$eq(f,g): (Eq(f,g),x_0) \hookrightarrow (X,x_0)$$

given by the canonical inclusion $eq(f,g) \hookrightarrow Eq(f,g) \hookrightarrow X$.

Proof. We claim that $(\text{Eq}(f,g),x_0)$ is the categorical equaliser of f and g in Sets_* . First we need to check that the relevant equaliser diagram commutes, i.e. that we have

$$f \circ eq(f,g) = g \circ eq(f,g),$$

which indeed holds by the definition of the set Eq(f,g). Next, we prove that Eq(f,g) satisfies the universal property of the equaliser. Suppose we have a diagram of the form

$$(\operatorname{Eq}(f,g),x_0) \xrightarrow{\operatorname{eq}(f,g)} (X,x_0) \xrightarrow{f} (Y,y_0)$$

$$(E,*)$$

in Sets_{*}. Then there exists a unique morphism of pointed sets

$$\phi \colon (E, *) \to (\text{Eq}(f, g), x_0)$$

making the diagram

$$(\operatorname{Eq}(f,g),x_0) \xrightarrow{\operatorname{eq}(f,g)} (X,x_0) \xrightarrow{f} (Y,y_0)$$

$$\downarrow \phi \mid \exists! \qquad e$$

$$(E,*)$$

commute, being uniquely determined by the condition

$$eq(f,g) \circ \phi = e$$

via

$$\phi(x) = e(x)$$

for each $x \in E$, where we note that $e(x) \in A$ indeed lies in Eq(f,g) by the condition

$$f \circ e = q \circ e$$
,

which gives

$$f(e(x)) = g(e(x))$$

for each $x \in E$, so that $e(x) \in \text{Eq}(f, g)$. Lastly, we note that ϕ is indeed a morphism of pointed sets, as we have

$$\phi(*) = e(*)$$
$$= x_0,$$

where we have used that e is a morphism of pointed sets.

Proposition 2.5.1.2. Let (X, x_0) and (Y, y_0) be pointed sets and let $f, g, h: (X, x_0) \to (Y, y_0)$ be morphisms of pointed sets.

1. Associativity. We have isomorphisms of pointed sets

$$\underbrace{\mathrm{Eq}(f \circ \mathrm{eq}(g,h), g \circ \mathrm{eq}(g,h))}_{=\mathrm{Eq}(f \circ \mathrm{eq}(g,h), h \circ \mathrm{eq}(g,h))} \cong \underbrace{\mathrm{Eq}(f,g,h)}_{=\mathrm{Eq}(g \circ \mathrm{eq}(f,g), h \circ \mathrm{eq}(f,g))} \underbrace{\underbrace{\mathrm{Eq}(f \circ \mathrm{eq}(f,g), h \circ \mathrm{eq}(f,g))}_{=\mathrm{Eq}(g \circ \mathrm{eq}(f,g), h \circ \mathrm{eq}(f,g))}}_{=\mathrm{Eq}(g \circ \mathrm{eq}(f,g), h \circ \mathrm{eq}(f,g))}$$

where Eq(f, g, h) is the limit of the diagram

$$(X, x_0) \xrightarrow{f} (Y, y_0)$$

in Sets_{*}, being explicitly given by

$$\operatorname{Eq}(f, g, h) \cong \{ a \in A \mid f(a) = g(a) = h(a) \}.$$

2. Unitality. We have an isomorphism of pointed sets

$$\operatorname{Eq}(f, f) \cong X$$
.

3. Commutativity. We have an isomorphism of pointed sets

$$\operatorname{Eq}(f,g) \cong \operatorname{Eq}(g,f).$$

Proof. Item 1, Associativity: This follows from Constructions With Sets, Item 1 of Proposition 1.5.1.2.

Item 2, Unitality: This follows from Constructions With Sets, Item 4 of Proposition 1.5.1.2.

Item 3, Commutativity: This follows from Constructions With Sets, Item 5 of Proposition 1.5.1.2. \Box

3 Colimits of Pointed Sets

3.1 The Initial Pointed Set

Definition 3.1.1.1. The **initial pointed set** is the pair $((pt, \star), \{\iota_X\}_{(X,x_0) \in \text{Obj}(\mathsf{Sets}_*)})$ consisting of:

- The Limit. The pointed set (pt, \star) .
- The Cone. The collection of morphisms of pointed sets

$$\{\iota_X \colon (\mathrm{pt},\star) \to (X,x_0)\}_{(X,x_0) \in \mathrm{Obj}(\mathsf{Sets})}$$

defined by

$$\iota_X(\star) \stackrel{\text{def}}{=} x_0.$$

Proof. We claim that (pt, \star) is the initial object of Sets_* . Indeed, suppose we have a diagram of the form

$$(pt, \star)$$
 (X, x_0)

in Sets_{*}. Then there exists a unique morphism of pointed sets

$$\phi \colon (\mathrm{pt}, \star) \to (X, x_0)$$

making the diagram

$$(\mathrm{pt},\star) \xrightarrow{-\frac{\phi}{\exists 1}} (X,x_0)$$

commute, namely ι_X .

3.2 Coproducts of Families of Pointed Sets

Let $\{(X_i, x_0^i)\}_{i \in I}$ be a family of pointed sets.

Definition 3.2.1.1. The **coproduct of the family** $\{(X_i, x_0^i)\}_{i \in I}$, also called their **wedge sum**, is the pair consisting of:

- The Colimit. The pointed set $(\bigvee_{i\in I} X_i, p_0)$ consisting of:
 - The Underlying Set. The set $\bigvee_{i \in I} X_i$ defined by

$$\bigvee_{i \in I} X_i \stackrel{\text{\tiny def}}{=} (\coprod_{i \in I} X_i) / {\sim},$$

where \sim is the equivalence relation on $\coprod_{i \in I} X_i$ given by declaring

$$(i, x_0^i) \sim (j, x_0^j)$$

for each $i, j \in I$.

- The Basepoint. The element p_0 of $\bigvee_{i \in I} X_i$ defined by

$$p_0 \stackrel{\text{def}}{=} [(i, x_0^i)]$$
$$= [(j, x_0^j)]$$

for any $i, j \in I$.

• The Cocone. The collection

$$\left\{ \operatorname{inj}_i \colon (X_i, x_0^i) \to (\bigvee_{i \in I} X_i, p_0) \right\}_{i \in I}$$

of morphism of pointed sets given by

$$\operatorname{inj}_i(x) \stackrel{\text{def}}{=} (i, x)$$

for each $x \in X_i$ and each $i \in I$.

Proof. We claim that $(\bigvee_{i \in I} X_i, p_0)$ is the categorical coproduct of $\{(X_i, x_0^i)\}_{i \in I}$ in Sets_{*}. Indeed, suppose we have, for each $i \in I$, a diagram of the form

$$(X_i, x_0^i) \xrightarrow[\inf]{i \text{inj}_i} (\bigvee_{i \in I} X_i, p_0)$$

in Sets_{*}. Then there exists a unique morphism of pointed sets

$$\phi \colon (\bigvee_{i \in I} X_i, p_0) \to (C, *)$$

making the diagram

$$(C,*)$$

$$\downarrow_{i}$$

commute, being uniquely determined by the condition $\phi \circ \operatorname{inj}_i = \iota_i$ for each $i \in I$ via

$$\phi([(i,x)]) = \iota_i(x)$$

for each $[(i,x)] \in \bigvee_{i \in I} X_i$, where we note that ϕ is indeed a morphism of pointed sets, as we have

$$\phi(p_0) = \iota_i([(i, x_0^i)])$$

= *,

as ι_i is a morphism of pointed sets.

Proposition 3.2.1.2. Let $\{(X_i, x_0^i)\}_{i \in I}$ be a family of pointed sets.

1. Functoriality. The assignment $\{(X_i, x_0^i)\}_{i \in I} \mapsto (\bigvee_{i \in I} X_i, p_0)$ defines a functor

$$\bigvee_{i \in I} \colon \mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}_*) \to \mathsf{Sets}_*.$$

Proof. Item 1, Functoriality: This follows from ??, ?? of ??. □

3.3 Coproducts

Let (X, x_0) and (Y, y_0) be pointed sets.

Definition 3.3.1.1. The **coproduct of** (X, x_0) **and** (Y, y_0) , also called their **wedge sum**, is the pair consisting of:

- The Colimit. The pointed set $(X \vee Y, p_0)$ consisting of:
 - The Underlying Set. The set $X \vee Y$ defined by

$$(X \lor Y, p_0) \stackrel{\text{def}}{=} (X, x_0) \coprod (Y, y_0) \qquad X \lor Y \longleftarrow Y$$

$$\cong (X \coprod_{\text{pt}} Y, p_0) \qquad \qquad \uparrow \qquad \qquad \uparrow \qquad \qquad \downarrow [y_0]$$

$$\cong (X \coprod Y/\sim, p_0), \qquad X \longleftarrow_{[x_0]} \text{pt},$$

where \sim is the equivalence relation on $X \coprod Y$ obtained by declaring $(0, x_0) \sim (1, y_0)$.

- The Basepoint. The element p_0 of $X \vee Y$ defined by

$$p_0 \stackrel{\text{def}}{=} [(0, x_0)]$$

= $[(1, y_0)].$

• The Cocone. The morphisms of pointed sets

$$\operatorname{inj}_1: (X, x_0) \to (X \vee Y, p_0),$$

 $\operatorname{inj}_2: (Y, y_0) \to (X \vee Y, p_0),$

given by

$$\begin{aligned} & \operatorname{inj}_1(x) \stackrel{\text{def}}{=} [(0,x)], \\ & \operatorname{inj}_2(y) \stackrel{\text{def}}{=} [(1,y)], \end{aligned}$$

for each $x \in X$ and each $y \in Y$.

Proof. We claim that $(X \vee Y, p_0)$ is the categorical coproduct of (X, x_0) and (Y, y_0) in Sets_{*}. Indeed, suppose we have a diagram of the form

$$(X, x_0) \xrightarrow[\operatorname{inj}_X]{(C, *)} \leftarrow \iota_Y$$

$$(X, x_0) \xrightarrow[\operatorname{inj}_X]{(X \vee Y, p_0)} \leftarrow \iota_Y$$

in Sets. Then there exists a unique morphism of pointed sets

$$\phi \colon (X \vee Y, p_0) \to (C, *)$$

making the diagram

$$(X, x_0) \xrightarrow[\operatorname{inj}_X]{(C, *)} \leftarrow \iota_Y$$

$$\downarrow^{\bullet}_{[\exists !}$$

$$(X, x_0) \xrightarrow[\operatorname{inj}_X]{(X \vee Y, p_0)} \xleftarrow[\operatorname{inj}_Y]{(Y, y_0)}$$

commute, being uniquely determined by the conditions

$$\phi \circ \operatorname{inj}_X = \iota_X,$$
$$\phi \circ \operatorname{inj}_V = \iota_Y$$

via

$$\phi(z) = \begin{cases} \iota_X(x) & \text{if } z = [(0, x)] \text{ with } x \in X, \\ \iota_Y(y) & \text{if } z = [(1, y)] \text{ with } y \in Y \end{cases}$$

for each $z \in X \vee Y$, where we note that ϕ is indeed a morphism of pointed

sets, as we have

$$\phi(p_0) = \iota_X([(0, x_0)])$$

= $\iota_Y([(1, y_0)])$
= *,

as ι_X and ι_Y are morphisms of pointed sets.

Proposition 3.3.1.2. Let (X, x_0) and (Y, y_0) be pointed sets.

1. Functoriality. The assignments

$$(X, x_0), (Y, y_0), ((X, x_0), (Y, y_0)) \mapsto (X \vee Y, p_0)$$

define functors

$$X \lor -: \mathsf{Sets}_* \to \mathsf{Sets}_*,$$
 $- \lor Y : \mathsf{Sets}_* \to \mathsf{Sets}_*,$
 $-_1 \lor -_2 : \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*.$

2. Associativity. We have an isomorphism of pointed sets

$$(X \vee Y) \vee Z \cong X \vee (Y \vee Z),$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathsf{Sets}_*$.

3. Unitality. We have isomorphisms of pointed sets

$$(pt, *) \lor (X, x_0) \cong (X, x_0),$$

 $(X, x_0) \lor (pt, *) \cong (X, x_0),$

natural in $(X, x_0) \in \mathsf{Sets}_*$.

4. Commutativity. We have an isomorphism of pointed sets

$$X \vee Y \cong Y \vee X$$
,

natural in $(X, x_0), (Y, y_0) \in \mathsf{Sets}_*$.

5. Symmetric Monoidality. The triple $(\mathsf{Sets}_*, \vee, \mathsf{pt})$ is a symmetric monoidal category.

6. The Fold Map. We have a natural transformation

$$\begin{array}{c} \text{Sets}_* \times \text{Sets}_* \\ \nabla \colon \vee \circ \Delta^{\mathsf{Cats}}_{\mathsf{Sets}_*} \Longrightarrow \mathrm{id}_{\mathsf{Sets}_*}, \qquad \begin{array}{c} \Delta^{\mathsf{Cats}}_{\mathsf{Sets}_*} \\ \nabla & \nabla \\ \mathsf{Sets}_* \end{array} \begin{array}{c} \nabla & \nabla \\ \mathsf{Sets}_* \end{array} \begin{array}{c} \nabla & \nabla \\ \mathsf{Sets}_* \end{array}$$

called the **fold map**, whose component

$$\nabla_X \colon X \vee X \to X$$

at X is given by

$$\nabla_X(p) \stackrel{\text{def}}{=} \begin{cases} x & \text{if } p = [(0, x)], \\ x & \text{if } p = [(1, x)] \end{cases}$$

for each $p \in X \vee X$.

Proof. Item 1, Functoriality: This follows from ??, ?? of ??.

Item 2, Associativity: Clear.

Item 3, Unitality: Clear.

Item 4, Commutativity: Clear.

Item 5, Symmetric Monoidality: Omitted.

Item 6, The Fold Map: Naturality for the transformation ∇ is the statement that, given a morphism of pointed sets $f:(X,x_0)\to (Y,y_0)$, we have

$$\nabla_{Y} \circ (f \vee f) = f \circ \nabla_{X}, \quad \begin{array}{c} X \vee X \xrightarrow{\nabla_{X}} X \\ \downarrow^{f} \\ Y \vee Y \xrightarrow{\nabla_{Y}} Y. \end{array}$$

Indeed, we have

$$\begin{split} [\nabla_Y \circ (f \vee f)]([(i,x)]) &= \nabla_Y([(i,f(x))]) \\ &= f(x) \\ &= f(\nabla_X([(i,x)])) \\ &= [f \circ \nabla_X]([(i,x)]) \end{split}$$

for each $[(i,x)] \in X \vee X$, and thus ∇ is indeed a natural transformation. \square

3.4 Pushouts

Let (X, x_0) , (Y, y_0) , and (Z, z_0) be pointed sets and let $f: (Z, z_0) \to (X, x_0)$ and $g: (Z, z_0) \to (Y, y_0)$ be morphisms of pointed sets.

Definition 3.4.1.1. The pushout of (X, x_0) and (Y, y_0) over (Z, z_0) along (f, g) is the pair consisting of:

- The Colimit. The pointed set $(X \coprod_{f,Z,g} Y, p_0)$, where:
 - The set $X \coprod_{f,Z,g} Y$ is the pushout (of unpointed sets) of X and Y over Z with respect to f and g;
 - We have $p_0 = [x_0] = [y_0]$.
- The Cocone. The morphisms of pointed sets

$$\operatorname{inj}_1 \colon (X, x_0) \to (X \coprod_Z Y, p_0),$$

 $\operatorname{inj}_2 \colon (Y, y_0) \to (X \coprod_Z Y, p_0)$

given by

$$\operatorname{inj}_{1}(x) \stackrel{\text{def}}{=} [(0, x)]
 \operatorname{inj}_{2}(y) \stackrel{\text{def}}{=} [(1, y)]$$

for each $x \in X$ and each $y \in Y$.

Proof. Firstly, we note that indeed $[x_0] = [y_0]$, as we have

$$x_0 = f(z_0),$$

$$y_0 = g(z_0)$$

since f and g are morphisms of pointed sets, with the relation \sim on $X \coprod_Z Y$ then identifying $x_0 = f(z_0) \sim g(z_0) = y_0$.

We now claim that $(X \coprod_Z Y, p_0)$ is the categorical pushout of (X, x_0) and (Y, y_0) over (Z, z_0) with respect to (f, g) in Sets_* . First we need to check that the relevant pushout diagram commutes, i.e. that we have

$$(X \coprod_{Z} Y, p_{0}) \stackrel{\text{inj}_{2}}{\longleftarrow} (Y, y_{0})$$

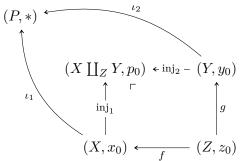
$$\text{inj}_{1} \circ f = \text{inj}_{2} \circ g, \qquad \text{inj}_{1} \qquad \qquad \uparrow g$$

$$(X, x_{0}) \stackrel{f}{\longleftarrow} (Z, z_{0}).$$

Indeed, given $z \in \mathbb{Z}$, we have

$$\begin{split} [\inf_1 \circ f](z) &= \inf_1(f(z)) \\ &= [(0, f(z))] \\ &= [(1, g(z))] \\ &= \inf_2(g(z)) \\ &= [\inf_2 \circ g](z), \end{split}$$

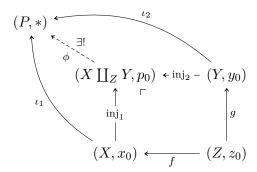
where [(0, f(z))] = [(1, g(z))] by the definition of the relation \sim on $X \coprod Y$ (the coproduct of unpointed sets of X and Y). Next, we prove that $X \coprod_Z Y$ satisfies the universal property of the pushout. Suppose we have a diagram of the form



in Sets_{*}. Then there exists a unique morphism of pointed sets

$$\phi \colon (X \coprod_Z Y, p_0) \to (P, *)$$

making the diagram



commute, being uniquely determined by the conditions

$$\phi \circ \operatorname{inj}_1 = \iota_1,$$

$$\phi \circ \operatorname{inj}_2 = \iota_2$$

via

$$\phi(p) = \begin{cases} \iota_1(x) & \text{if } x = [(0, x)], \\ \iota_2(y) & \text{if } x = [(1, y)] \end{cases}$$

for each $p \in X \coprod_Z Y$, where the well-definedness of ϕ is proven in the same way as in the proof of Constructions With Sets, Definition 2.4.1.1. Finally, we show that ϕ is indeed a morphism of pointed sets, as we have

$$\phi(p_0) = \phi([(0, x_0)])$$
= $\iota_1(x_0)$
= *.

or alternatively

$$\phi(p_0) = \phi([(1, y_0)]) = \iota_2(y_0) = *,$$

where we use that ι_1 (resp. ι_2) is a morphism of pointed sets.

Proposition 3.4.1.2. Let (X, x_0) , (Y, y_0) , (Z, z_0) , and (A, a_0) be pointed sets

1. Functoriality. The assignment $(X,Y,Z,f,g)\mapsto X\coprod_{f,Z,g}Y$ defines a functor

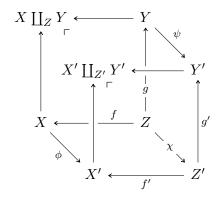
$$-_1 \coprod_{-_3} -_1 \colon \mathsf{Fun}(\mathcal{P},\mathsf{Sets}) o \mathsf{Sets}_*,$$

where \mathcal{P} is the category that looks like this:



In particular, the action on morphisms of $-1 \coprod_{-3} -1$ is given by sending

a morphism



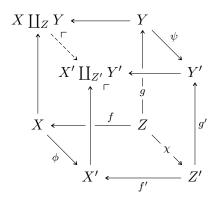
in $\operatorname{\mathsf{Fun}}(\mathcal{P},\operatorname{\mathsf{Sets}}_*)$ to the morphism of pointed sets

$$\xi \colon (X \coprod_Z Y, p_0) \xrightarrow{\exists !} (X' \coprod_{Z'} Y', p'_0)$$

given by

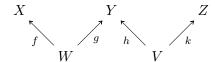
$$\xi(p) \stackrel{\text{def}}{=} \begin{cases} \phi(x) & \text{if } p = [(0, x)], \\ \psi(y) & \text{if } p = [(1, y)] \end{cases}$$

for each $p \in X \coprod_Z Y$, which is the unique morphism of pointed sets making the diagram



commute.

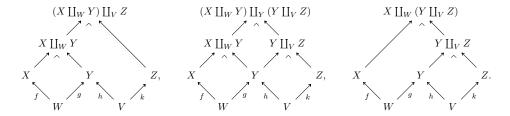
2. Associativity. Given a diagram



in Sets, we have isomorphisms of pointed sets

$$(X \coprod_W Y) \coprod_V Z \cong (X \coprod_W Y) \coprod_Y (Y \coprod_V Z) \cong X \coprod_W (Y \coprod_V Z),$$

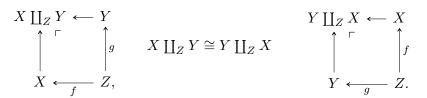
where these pullbacks are built as in the diagrams



3. Unitality. We have isomorphisms of sets



4. Commutativity. We have an isomorphism of sets



5. Interaction With Coproducts. We have

$$X \coprod_{\mathrm{pt}} Y \cong X \vee Y,$$

$$X \bigvee_{\Gamma} Y \longleftarrow Y$$

$$X \swarrow_{[x_0]} pt.$$

6. Symmetric Monoidality. The triple (Sets_{*}, \coprod_X , (X, x_0)) is a symmetric monoidal category.

Proof. Item 1, Functoriality: This is a special case of functoriality of co/limits, ??, ?? of ??, with the explicit expression for ξ following from the commutativity of the cube pushout diagram.

Item 2, Associativity: This follows from Constructions With Sets, Item 2 of Proposition 2.4.1.4.

Item 3, Unitality: This follows from Constructions With Sets, Item 3 of Proposition 2.4.1.4.

Item 4, Commutativity: This follows from Constructions With Sets, Item 4 of Proposition 2.4.1.4.

Item 5, Interaction With Coproducts: Clear.

Item 6, Symmetric Monoidality: Omitted.

3.5 Coequalisers

Let $f, g: (X, x_0) \rightrightarrows (Y, y_0)$ be morphisms of pointed sets.

Definition 3.5.1.1. The **coequaliser of** (f, g) is the pointed set $(CoEq(f, g), [y_0])$.

Proof. We claim that $(CoEq(f, g), [y_0])$ is the categorical coequaliser of f and g in $Sets_*$. First we need to check that the relevant coequaliser diagram commutes, i.e. that we have

$$coeq(f,g) \circ f = coeq(f,g) \circ g.$$

Indeed, we have

$$[\operatorname{coeq}(f,g) \circ f](x) \stackrel{\text{def}}{=} [\operatorname{coeq}(f,g)](f(x))$$

$$\stackrel{\text{def}}{=} [f(x)]$$

$$= [g(x)]$$

$$\stackrel{\text{def}}{=} [\operatorname{coeq}(f,g)](g(x))$$

$$\stackrel{\text{def}}{=} [\operatorname{coeq}(f,g) \circ g](x)$$

for each $x \in X$. Next, we prove that CoEq(f,g) satisfies the universal property of the coequaliser. Suppose we have a diagram of the form

$$(X, x_0) \xrightarrow{f} (Y, y_0) \xrightarrow{\operatorname{coeq}(f,g)} (\operatorname{CoEq}(f,g), [y_0])$$

$$(C, *)$$

in Sets. Then, since c(f(a)) = c(g(a)) for each $a \in A$, it follows from Equivalence Relations and Apartness Relations, Items 4 and 5 of Proposition 5.2.1.3 that there exists a unique map $\phi \colon \mathrm{CoEq}(f,g) \xrightarrow{\exists !} C$ making the diagram

commute, where we note that ϕ is indeed a morphism of pointed sets since

$$\phi([y_0]) = [\phi \circ \text{coeq}(f, g)]([y_0])$$

= $c([y_0])$
= *,

where we have used that c is a morphism of pointed sets.

Proposition 3.5.1.2. Let (X, x_0) and (Y, y_0) be pointed sets and let $f, g, h: (X, x_0) \to (Y, y_0)$ be morphisms of pointed sets.

1. Associativity. We have isomorphisms of pointed sets

$$\underbrace{\mathrm{CoEq}(\mathrm{coeq}(f,g) \circ f, \mathrm{coeq}(f,g) \circ h)}_{=\mathrm{CoEq}(\mathrm{coeq}(f,g) \circ g, \mathrm{coeq}(f,g) \circ h)} \cong \underbrace{\mathrm{CoEq}(\mathrm{coeq}(g,h) \circ f, \mathrm{coeq}(g,h) \circ g)}_{=\mathrm{CoEq}(\mathrm{coeq}(g,h) \circ f, \mathrm{coeq}(g,h) \circ h)}$$

where CoEq(f, g, h) is the colimit of the diagram

$$(X, x_0) \xrightarrow{f \atop g \atop h} (Y, y_0)$$

in $Sets_*$.

2. Unitality. We have an isomorphism of pointed sets

$$CoEq(f, f) \cong B$$
.

3. Commutativity. We have an isomorphism of pointed sets

$$CoEq(f,g) \cong CoEq(g,f).$$

Proof. Item 1, Associativity: This follows from Constructions With Sets, Item 1 of Proposition 2.5.1.4.

Item 2, Unitality: This follows from Constructions With Sets, Item 4 of Proposition 2.5.1.4.

Item 3, Commutativity: This follows from Constructions With Sets, Item 5 of Proposition 2.5.1.4. \Box

4 Constructions With Pointed Sets

4.1 Free Pointed Sets

Let X be a set.

Definition 4.1.1.1. The **free pointed set on** X is the pointed set X^+ consisting of:

• The Underlying Set. The set X^+ defined by 11

$$X^{+} \stackrel{\mathrm{def}}{=} X \coprod \mathrm{pt}$$
$$\stackrel{\mathrm{def}}{=} X \coprod \{\star\}.$$

• The Basepoint. The element \star of X^+ .

Proposition 4.1.1.2. Let X be a set.

1. Functoriality. The assignment $X \mapsto X^+$ defines a functor

$$(-)^+ \colon \mathsf{Sets} \to \mathsf{Sets}_*,$$

where

• Action on Objects. For each $X \in \text{Obj}(\mathsf{Sets})$, we have

$$[(-)^+](X) \stackrel{\text{def}}{=} X^+,$$

where X^+ is the pointed set of Definition 4.1.1.1;

• Action on Morphisms. For each morphism $f: X \to Y$ of Sets, the image

$$f^+\colon X^+ \to Y^+$$

of f by $(-)^+$ is the map of pointed sets defined by

$$f^+(x) \stackrel{\text{def}}{=} \begin{cases} f(x) & \text{if } x \in X, \\ \star_Y & \text{if } x = \star_X. \end{cases}$$

¹¹Further Notation: We sometimes write \star_X for the basepoint of X^+ for clarity when there are multiple free pointed sets involved in the current discussion.

2. Adjointness. We have an adjunction

$$((-)^+ \dashv \overline{z})$$
: Sets $\underbrace{(-)^+}_{\overline{z}}$ Sets_{*},

witnessed by a bijection of sets

$$\mathsf{Sets}_*((X^+, \star_X), (Y, y_0)) \cong \mathsf{Sets}(X, Y),$$

natural in $X \in \text{Obj}(\mathsf{Sets})$ and $(Y, y_0) \in \text{Obj}(\mathsf{Sets}_*)$.

3. Symmetric Strong Monoidality With Respect to Wedge Sums. The free pointed set functor of Item 1 has a symmetric strong monoidal structure

$$((-)^+, (-)^{+,\coprod}, (-)^{+,\coprod}_1) \colon (\mathsf{Sets}, \coprod, \emptyset) \to (\mathsf{Sets}_*, \vee, \mathsf{pt}),$$

being equipped with isomorphisms of pointed sets

$$(-)_{X,Y}^{+,\coprod} \colon X^{+} \vee Y^{+} \xrightarrow{\cong} (X \coprod Y)^{+},$$
$$(-)_{1}^{+,\coprod} \colon \operatorname{pt} \xrightarrow{\cong} \emptyset^{+},$$

natural in $X, Y \in \text{Obj}(\mathsf{Sets})$.

4. Symmetric Strong Monoidality With Respect to Smash Products. The free pointed set functor of Item 1 has a symmetric strong monoidal structure

$$((-)^+,(-)^{+,\times},(-)^{+,\times}_{1})\colon (\mathsf{Sets},\times,\mathrm{pt})\to (\mathsf{Sets}_*,\wedge,S^0),$$

being equipped with isomorphisms of pointed sets

$$(-)_{X,Y}^{+,\times} \colon X^+ \wedge Y^+ \xrightarrow{\cong} (X \times Y)^+,$$
$$(-)_{\mathbb{1}}^{+,\times} \colon S^0 \xrightarrow{\cong} \mathrm{pt}^+,$$

natural in $X, Y \in \text{Obj}(\mathsf{Sets})$.

Proof. Item 1, Functoriality: Clear.

Item 2, Adjointness: We claim there's an adjunction $(-)^+$ \dashv 忘, witnessed by a bijection of sets

$$\mathsf{Sets}_*((X^+, \star_X), (Y, y_0)) \cong \mathsf{Sets}(X, Y),$$

natural in $X \in \text{Obj}(\mathsf{Sets})$ and $(Y, y_0) \in \text{Obj}(\mathsf{Sets}_*)$.

• Map I. We define a map

$$\Phi_{X,Y} \colon \mathsf{Sets}_*((X^+, \star_X), (Y, y_0)) \to \mathsf{Sets}(X, Y)$$

by sending a pointed function

$$\xi \colon (X^+, \star_X) \to (Y, y_0)$$

to the function

$$\xi^{\dagger} \colon X \to Y$$

given by

$$\xi^{\dagger}(x) \stackrel{\text{def}}{=} \xi(x)$$

for each $x \in X$.

• Map II. We define a map

$$\Psi_{X,Y} \colon \mathsf{Sets}(X,Y) \to \mathsf{Sets}_*((X^+,\star_X),(Y,y_0))$$

given by sending a function $\xi \colon X \to Y$ to the pointed function

$$\xi^{\dagger} \colon (X^+, \star_X) \to (Y, y_0)$$

defined by

$$\xi^{\dagger}(x) \stackrel{\text{def}}{=} \begin{cases} \xi(x) & \text{if } x \in X, \\ y_0 & \text{if } x = \star_X \end{cases}$$

for each $x \in X^+$.

• Invertibility I. We claim that

$$\Psi_{X,Y} \circ \Phi_{X,Y} = \mathrm{id}_{\mathsf{Sets}_*((X^+,\star_X),(Y,y_0))},$$

which is clear.

• Invertibility II. We claim that

$$\Phi_{X,Y} \circ \Psi_{X,Y} = \mathrm{id}_{\mathsf{Sets}(X,Y)},$$

which is clear.

• Naturality for Φ , Part I. We need to show that, given a pointed function

$$g\colon (Y,y_0)\to (Y',y_0')$$
, the diagram

$$\begin{split} \mathsf{Sets}_*((X^+, \star_X), (Y, y_0)) & \stackrel{\Phi_{X,Y}}{\longrightarrow} \mathsf{Sets}(X, Y) \\ g_* & & & \downarrow g_* \\ \mathsf{Sets}_*((X^+, \star_X), (Y', y_0')),_{\Phi_{X,Y'}} \mathsf{Sets}(X, Y') \end{split}$$

commutes. Indeed, given a pointed function

$$\xi^{\dagger} \colon (X^+, \star_X) \to (Y, y_0)$$

we have

$$\begin{split} [\Phi_{X,Y'} \circ g_*](\xi) &= \Phi_{X,Y'}(g_*(\xi)) \\ &= \Phi_{X,Y'}(g \circ \xi) \\ &= g \circ \xi \\ &= g \circ \Phi_{X,Y'}(\xi) \\ &= g_*(\Phi_{X,Y'}(\xi)) \\ &= [g_* \circ \Phi_{X,Y'}](\xi). \end{split}$$

• Naturality for Φ , Part II. We need to show that, given a pointed function $f:(X,x_0)\to (X',x_0')$, the diagram

$$\begin{split} \mathsf{Sets}_*((X^{',+}, \star_X), (Y, y_0)) & \stackrel{\Phi_{X',Y}}{\longrightarrow} \mathsf{Sets}(X', Y) \\ f^* & & \downarrow f^* \\ \mathsf{Sets}_*((X^+, \star_X), (Y, y_0)) & \xrightarrow{\Phi_{X,Y}} \mathsf{Sets}(X, Y) \end{split}$$

commutes. Indeed, given a function

$$\xi \colon X' \to Y$$
,

we have

$$[\Phi_{X,Y} \circ f^*](\xi) = \Phi_{X,Y}(f^*(\xi))$$

$$= \Phi_{X,Y}(\xi \circ f)$$

$$= \xi \circ f$$

$$= \Phi_{X',Y}(\xi) \circ f$$

$$= f^*(\Phi_{X',Y}(\xi))$$

$$= f^*(\Phi_{X',Y}(\xi))$$

$$= [f^* \circ \Phi_{X',Y}](\xi).$$

• Naturality for Ψ . Since Φ is natural in each argument and Φ is a componentwise inverse to Ψ in each argument, it follows from Categories, Item 2 of Proposition 8.6.1.2 that Ψ is also natural in each argument.

Item 3, Symmetric Strong Monoidality With Respect to Wedge Sums: The isomorphism

$$\phi \colon X^+ \vee Y^+ \xrightarrow{\cong} (X \coprod Y)^+$$

is given by

$$\phi(z) = \begin{cases} x & \text{if } z = [(0, x)] \text{ with } x \in X, \\ y & \text{if } z = [(1, y)] \text{ with } y \in Y, \\ \star_{X \coprod Y} & \text{if } z = [(0, \star_X)], \\ \star_{X \coprod Y} & \text{if } z = [(1, \star_Y)] \end{cases}$$

for each $z \in X^+ \vee Y^+$, with inverse

$$\phi^{-1} \colon (X \coprod Y)^+ \xrightarrow{\cong} X^+ \lor Y^+$$

given by

$$\phi^{-1}(z) \stackrel{\text{def}}{=} \begin{cases} [(0, x)] & \text{if } z = [(0, x)], \\ [(0, y)] & \text{if } z = [(1, y)], \\ p_0 & \text{if } z = \star_{X \coprod Y} \end{cases}$$

for each $z \in (X \coprod Y)^+$.

Meanwhile, the isomorphism pt $\cong \emptyset^+$ is given by sending \star_X to \star_\emptyset . That these isomorphisms satisfy the coherence conditions making the functor $(-)^+$ symmetric strong monoidal can be directly checked element by element. Item 4, Symmetric Strong Monoidality With Respect to Smash Products:

The isomorphism $\phi \colon X^+ \wedge Y^+ \stackrel{\cong}{\longrightarrow} (X \times Y)^+$

is given by

$$\phi(x \land y) = \begin{cases} (x, y) & \text{if } x \neq \star_X \text{ and } y \neq \star_Y \\ \star_{X \times Y} & \text{otherwise} \end{cases}$$

for each $x \wedge y \in X^+ \wedge Y^+$, with inverse

$$\phi^{-1} \colon (X \times Y)^+ \xrightarrow{\cong} X^+ \wedge Y^+$$

given by

$$\phi^{-1}(z) \stackrel{\text{def}}{=} \begin{cases} x \wedge y & \text{if } z = (x, y) \text{ with } (x, y) \in X \times Y, \\ \star_X \wedge \star_Y & \text{if } z = \star_{X \times Y}, \end{cases}$$

for each $z \in (X \coprod Y)^+$.

Meanwhile, the isomorphism $S^0 \cong \operatorname{pt}^+$ is given by sending \star to $1 \in S^0 = \{0,1\}$ and $\star_{\operatorname{pt}}$ to $0 \in S^0$.

That these isomorphisms satisfy the coherence conditions making the functor $(-)^+$ symmetric strong monoidal can be directly checked element by element.

Appendices

A Other Chapters

Sets

- 1. Sets
- 2. Constructions With Sets
- 3. Pointed Sets
- 4. Tensor Products of Pointed Sets

Relations

5. Relations

- 6. Constructions With Relations
- 7. Equivalence Relations and Apartness Relations

Category Theory

8. Categories

Bicategories

9. Types of Morphisms in Bicategories

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References

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[MSE 884460] Martin Brandenburg. Why are the category of pointed sets and the category of sets and partial functions "essentially the same"? Mathematics Stack Exchange. URL: https://math.stackexchange.com/q/884460 (cit. on p. 6).