

# Miscellaneous Notes

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## 1 To Do List

### 1.1 Omitted Proofs To Add

Не так благотворна истина, как  
зловредна ее видимость.

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*Даниил Данковский*

Truth does not do as much good in the  
world as the appearance of truth does  
evil.

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*Daniil Dankovsky*

There's a very large number of omitted proofs throughout these notes. Here I list them in decreasing order of how nice it would be to add them.

**Remark 1.1.1.1.** Proofs that *need* to be added at some point:

1. **Tensor Products of Pointed Sets, Theorem 5.10.1.1.**
2. **Tensor Products of Pointed Sets, Theorem 5.11.1.1.**

3. Horizontal composition of natural transformations is associative: [Categories](#), [Item 2 of Proposition 8.4.1.3](#).
4. Fully faithful functors are essentially injective: [Categories](#), [Item 3 of Proposition 5.3.1.2](#).

Proofs that *would be very nice* to be added at some point:

1. Properties of pseudomonadic functors: [Categories](#), [Proposition 6.4.1.2](#).
2. Characterisation of fully faithful functors: [Categories](#), [Item 1 of Proposition 5.3.1.2](#).

Proofs that *would be nice* to be added at some point:

1. Properties of posetal categories: [Categories](#), [Proposition 1.3.1.2](#).
2. The quadruple adjunction between categories and sets: [Categories](#), [Proposition 2.1.1.1](#).
3. Properties of groupoid completions: [Categories](#), [Proposition 3.2.1.3](#).
4. Properties of cores: [Categories](#), [Proposition 3.3.1.4](#).
5.  $F_*$  faithful iff  $F$  faithful: [Categories](#), [Item 1 of Proposition 5.1.1.2](#).
6.  $F_*$  full iff  $F$  full: [Categories](#), [Item 1 of Proposition 5.2.1.2](#).
7. Injective on objects functors are precisely the isofibrations in  $\mathbf{Cats}_2$ : [Categories](#), [Item 1 of Proposition 7.1.1.2](#).
8. Characterisations of monomorphisms of categories: [Categories](#), [Item 1 of Proposition 6.2.1.2](#).
9. Epimorphisms of categories are surjective on objects: [Categories](#), [Item 2 of Proposition 6.3.1.2](#).
10. Properties of pseudoepic functors: [Categories](#), [Proposition 6.5.1.2](#).

## 1.2 Things To Explore/Add

Here we list things to be explored/added to this work in the future.

**Remark 1.2.1.1.** Set theory through a category theory lens:

1. Isbell duality for sets.
2. Density comonads and codensity monads for sets.

Relations:

1. 2-Categorical monomorphisms and epimorphisms in **Rel**.
2. Co/limits in **Rel**.
3. Apartness composition, categorical properties of **Rel** with apartness, and apartness relations.
4. Apartness defines a composition for relations, but its analogue

$$q \square p \stackrel{\text{def}}{=} \int_{A \in C} p_A^{-1} \amalg q_{-2}^A$$

fails to be unital for profunctors. Is there a less obvious analogue of apartness composition for profunctors?

5. Codensity monad  $\text{Ran}_J(J)$  of a relation (What about  $\text{Rift}_J(J)$ ?)
6. Relative comonads in the 2-category of relations
7. Discrete fibrations and Street fibrations in **Rel**.
8. Consider adding the sections
  - The Monoidal Bicategory of Relations
  - The Monoidal Double Category of Relations

to **Relations**.

Spans:

1. Universal property of the bicategory of spans, <https://ncatlab.org/nlab/show/span>
2. Write about cospans.

Un/Straightening:

1. Write proper sections on straightening for lax functors from sets to Rel or Span (displayed sets)

Categories:

1. Expand ?? and add a proof to it.
2. Sections and retractions; retracts, <https://ncatlab.org/nlab/show/retract>.
3. Regular categories: <https://arxiv.org/pdf/2004.08964.pdf>.
4. Are pseudoeptic functors those functors whose restricted Yoneda embedding is pseudomonadic and Yoneda preserves absolute colimits?
5. Absolutely dense functors enriched over  $\mathbb{R}^+$  apparently reduce to topological density

Types of Morphisms in Categories:

1. Behaviour in  $\text{Fun}(C, \mathcal{D})$ , e.g. pointwise sections vs. sections in  $\text{Fun}(C, \mathcal{D})$ .
2. A faithful functor from balanced category is conservative

Yoneda stuff:

1. Properties of restricted Yoneda embedding, e.g. if the restricted Yoneda embedding is full, then what can we conclude? Related: <https://qchu.wordpress.com/2015/05/17/generators/>

Adjunctions:

1. Adjunctions, units, counits, and fully faithfulness as in <https://mathoverflow.net/questions/100808/properties-of-functors-and-their-adjoints>.
2. Morphisms between adjunctions and bicategory  $\text{Adj}(C)$ .
3. <https://ncatlab.org/nlab/show/transformation+of+adjoints>

Constructions With Categories:

1. Comparison between pseudopullbacks and isocomma categories: the “evident” functor  $C \times_{\mathcal{E}}^{\text{ps}} \mathcal{D} \rightarrow C \overset{\leftrightarrow}{\times}_{\mathcal{E}} \mathcal{D}$  is essentially surjective and full, but not faithful in general.

Co/limits:

1. Add the characterisations of absolutely dense functors given in ?? to ??.
2. Absolutely dense functors, <https://ncatlab.org/nlab/show/absolutely+dense+functor>. Also theorem 1.1 here: <http://www.tac.mta.ca/tac/volumes/8/n20/n20.pdf>.
3. Dense functors, codense functors, and absolutely codense functors.

Co/ends:

1. Examples of co/ends: <https://mathoverflow.net/a/461814>
2. Cofinality for co/ends, <https://mathoverflow.net/questions/353876>

Fibred category theory:

1. Internal **Hom** in categories of co/Cartesian fibrations.
2. *Tensor structures on fibered categories* by Luca Terenzi: <https://arxiv.org/abs/2401.13491>. Check also the other papers by Luca Terenzi.
3. <https://ncatlab.org/nlab/show/cartesian+natural+transformation> (this is a cartesian morphism in  $\text{Fun}(C, \mathcal{D})$  apparently)
4. CoCartesian fibration classifying  $\text{Fun}(F, G)$ , <https://mathoverflow.net/questions/457533/cocartesian-fibration-classifying-mathrmfun-f-g>

Monoidal categories:

1. Free braided monoidal category with a braided monoid: <https://ncatlab.org/nlab/show/vine>

Skew monoidal categories:

1. Does the  $\mathbb{E}_1$  tensor product of monoids admit a skew monoidal category structure?

2. Is there a (right?) skew monoidal category structure on  $\text{Fun}(C, \mathcal{D})$  using right Kan extensions instead of left Kan extensions?
3. Similarly, are there skew monoidal category structures on the subcategory of  $\mathbf{Rel}(A, B)$  spanned by the functions using left Kan extensions and left Kan lifts?

Higher categories:

1. Internal adjunctions in  $\text{Mod}$  as in [JY21, Section 6.3]; see [JY21, Example 6.2.6].
2. Comonads in the bicategory of profunctors.

Monoids:

1. Isbell's zigzag theorem for semigroups: the following conditions are equivalent:
  - (a) A morphism  $f: A \rightarrow B$  of semigroups is an epimorphism.
  - (b) For each  $b \in B$ , one of the following conditions is satisfied:
    - We have  $f(a) = b$ .
    - There exist some  $m \in \mathbb{N}_{\geq 1}$  and two factorisations

$$b = a_0 y_1,$$

$$b = x_m a_{2m}$$

connected by relations

$$a_0 = x_1 a_1,$$

$$a_1 y_1 = a_2 y_2,$$

$$x_1 a_2 = x_2 a_3,$$

$$a_{2m-1} y_m = a_{2m}$$

such that, for each  $1 \leq i \leq m$ , we have  $a_i \in \text{Im}(f)$ .

Wikipedia says in [https://en.wikipedia.org/wiki/Isbell%27s\\_zigzag\\_theorem](https://en.wikipedia.org/wiki/Isbell%27s_zigzag_theorem):

For monoids, this theorem can be written more concisely:

Types of morphisms in bicategories:

1. Behaviour in 2-categories of pseudofunctors (or lax functors, etc.), e.g. pointwise pseudoepic morphisms in vs. pseudoepic morphisms in 2-categories of pseudofunctors.
2. Statements like “coequifiers are lax epimorphisms”, Item 2 of Examples 2.4 of <https://arxiv.org/abs/2109.09836>, along with most of the other statements/examples there.
3. Dense, absolutely dense, etc. morphisms in bicategories

Other:

1. <https://qchu.wordpress.com/>
2. <https://aroundtoposes.com/>
3. <https://ncatlab.org/nlab/show/essentially+surjective+and+full+functor>
4. <https://mathoverflow.net/questions/415363/objects-whose-representable-presheaf-is-a-fibration>
5. <https://mathoverflow.net/questions/460146/universal-property-of-isbell-duality>
6. <http://www.tac.mta.ca/tac/volumes/36/12/36-12abs.html> ( Isbell conjugacy and the reflexive completion )
7. <https://ncatlab.org/nlab/show/enrichment+versus+internalisation>
8. The works of Philip Saville, <https://philipsaville.co.uk/>
9. [https://golem.ph.utexas.edu/category/2024/02/from\\_cartesian\\_to\\_symmetric\\_mo.html](https://golem.ph.utexas.edu/category/2024/02/from_cartesian_to_symmetric_mo.html)
10. <https://mathoverflow.net/q/463855> (One-object lax transformations)
11. <https://ncatlab.org/nlab/show/analytic+completion+of+a+ring>

12. [https://en.wikipedia.org/wiki/Quaternionic\\_analysis](https://en.wikipedia.org/wiki/Quaternionic_analysis)
13. <https://arxiv.org/abs/2401.15051> (The Norm Functor over Schemes)
14. <https://mathoverflow.net/questions/407291/> (Adjunctions with respect to profunctors)
15. <https://mathoverflow.net/a/462726> (Prof is free completion of Cats under right extensions)
16. there's some cool stuff in <https://arxiv.org/abs/2312.00990> (Polynomial Functors: A Mathematical Theory of Interaction), e.g. on cofunctors.
17. <https://ncatlab.org/nlab/show/adjoint+lifting+theorem>
18. <https://ncatlab.org/nlab/show/Gabriel%E2%80%93Ulmer+du+ality>

## Appendices

### A Other Chapters

#### Sets

1. [Sets](#)
2. [Constructions With Sets](#)
3. [Pointed Sets](#)
4. [Tensor Products of Pointed Sets](#)

#### Relations

5. [Relations](#)

6. [Constructions With Relations](#)

7. [Equivalence Relations and Apartness Relations](#)

#### Category Theory

8. [Categories](#)

#### Bicategories

9. [Types of Morphisms in Bicategories](#)



## References

- [JY21] Niles Johnson and Donald Yau. *2-Dimensional Categories*. Oxford University Press, Oxford, 2021, pp. xix+615. ISBN: 978-0-19-887138-5; 978-0-19-887137-8. DOI: [10 . 1093 / oso / 9780198871378 . 001 . 0001](https://doi.org/10.1093/oso/9780198871378.001.0001). URL: <https://doi.org/10.1093/oso/9780198871378.001.0001> (cit. on p. 6).