

Pointed Sets

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May 3, 2024

0098 This chapter contains some foundational material on pointed sets.

Contents

1	Pointed Sets	2
1.1	Foundations	2
1.2	Morphisms of Pointed Sets	3
1.3	The Category of Pointed Sets	4
1.4	Elementary Properties of Pointed Sets	5
2	Limits of Pointed Sets	8
2.1	The Terminal Pointed Set	8
2.2	Products of Families of Pointed Sets	9
2.3	Products	10
2.4	Pullbacks	13
2.5	Equalisers	19
3	Colimits of Pointed Sets	22
3.1	The Initial Pointed Set	22
3.2	Coproducts of Families of Pointed Sets	22
3.3	Coproducts	25
3.4	Pushouts	29
3.5	Coequalisers	35
4	Constructions With Pointed Sets	38
4.1	Free Pointed Sets	38
A	Other Chapters	43

0099 1 Pointed Sets

009A 1.1 Foundations

009B DEFINITION 1.1.1 ► POINTED SETS

A **pointed set**¹ is equivalently:

- An \mathbb{E}_0 -monoid in $(\mathbf{N}_\bullet(\mathbf{Sets}), \text{pt})$.
- A pointed object in $(\mathbf{Sets}, \text{pt})$.

¹*Further Terminology:* In the context of monoids with zero as models for \mathbb{F}_1 -algebras, pointed sets are viewed as \mathbb{F}_1 -**modules**.

009C REMARK 1.1.2 ► UNWINDING DEFINITION 1.1.1

In detail, a **pointed set** is a pair (X, x_0) consisting of:

- *The Underlying Set.* A set X , called the **underlying set of** (X, x_0) .
- *The Basepoint.* A morphism

$$[x_0] : \text{pt} \rightarrow X$$

in \mathbf{Sets} , determining an element $x_0 \in X$, called the **basepoint of** X .

009D EXAMPLE 1.1.3 ► THE ZERO SPHERE

The **0-sphere**¹ is the pointed set $(S^0, 0)$ ² consisting of:

- *The Underlying Set.* The set S^0 defined by

$$S^0 \stackrel{\text{def}}{=} \{0, 1\}.$$

- *The Basepoint.* The element 0 of S^0 .

¹*Further Terminology:* In the context of monoids with zero as models for \mathbb{F}_1 -algebras, the 0-sphere is viewed as the **underlying pointed set of the field with one element**.

²*Further Notation:* In the context of monoids with zero as models for \mathbb{F}_1 -algebras, S^0 is also denoted $(\mathbb{F}_1, 0)$.

009E EXAMPLE 1.1.4 ► THE TRIVIAL POINTED SET

The **trivial pointed set** is the pointed set (pt, \star) consisting of:

- *The Underlying Set.* The punctual set $\text{pt} \stackrel{\text{def}}{=} \{\star\}$.
- *The Basepoint.* The element \star of pt .

009F EXAMPLE 1.1.5 ► THE UNDERLYING POINTED SET OF A SEMIMODULE

The **underlying pointed set** of a semimodule (M, α_M) is the pointed set $(M, 0_M)$.

009G EXAMPLE 1.1.6 ► THE UNDERLYING POINTED SET OF A MODULE

The **underlying pointed set** of a module (M, α_M) is the pointed set $(M, 0_M)$.

009H 1.2 Morphisms of Pointed Sets

009J DEFINITION 1.2.1 ► MORPHISMS OF POINTED SETS

A **morphism of pointed sets**^{1,2} is equivalently:

- A morphism of \mathbb{E}_0 -monoids in $(\mathbf{N}_\bullet(\mathbf{Sets}), \text{pt})$.
- A morphism of pointed objects in $(\mathbf{Sets}, \text{pt})$.

¹Further Terminology: Also called a **pointed function**.

²Further Terminology: In the context of monoids with zero as models for \mathbb{F}_1 -algebras, morphisms of pointed sets are also called **morphism of \mathbb{F}_1 -modules**.

009K REMARK 1.2.2 ► UNWINDING DEFINITION 1.2.1

In detail, a **morphism of pointed sets** $f: (X, x_0) \rightarrow (Y, y_0)$ is a morphism of sets $f: X \rightarrow Y$ such that the diagram

$$\begin{array}{ccc} & \text{pt} & \\ [x_0] \swarrow & & \searrow [y_0] \\ X & \xrightarrow{f} & Y \end{array}$$

commutes, i.e. such that

$$f(x_0) = y_0.$$

009L 1.3 The Category of Pointed Sets

009M DEFINITION 1.3.1 ► THE CATEGORY OF POINTED SETS

The **category of pointed sets** is the category \mathbf{Sets}_* defined equivalently as

- The homotopy category of the ∞ -category $\mathbf{Mon}_{\mathbb{E}_0}(\mathbf{N}_\bullet(\mathbf{Sets}), \text{pt})$ of ??, ??;
- The category \mathbf{Sets}_* of ??, ??.

009N REMARK 1.3.2 ► UNWINDING DEFINITION 1.3.1

In detail, the **category of pointed sets** is the category \mathbf{Sets}_* where

- *Objects.* The objects of \mathbf{Sets}_* are pointed sets;
- *Morphisms.* The morphisms of \mathbf{Sets}_* are morphisms of pointed sets;
- *Identities.* For each $(X, x_0) \in \text{Obj}(\mathbf{Sets}_*)$, the unit map

$$\mathbb{1}_{(X, x_0)}^{\mathbf{Sets}_*} : \text{pt} \rightarrow \mathbf{Sets}_*((X, x_0), (X, x_0))$$

of \mathbf{Sets}_* at (X, x_0) is defined by¹

$$\text{id}_{(X, x_0)}^{\mathbf{Sets}_*} \stackrel{\text{def}}{=} \text{id}_X;$$

- *Composition.* For each $(X, x_0), (Y, y_0), (Z, z_0) \in \text{Obj}(\mathbf{Sets}_*)$, the composition map

$$\circ_{(X, x_0), (Y, y_0), (Z, z_0)}^{\mathbf{Sets}_*} : \mathbf{Sets}_*((Y, y_0), (Z, z_0)) \times \mathbf{Sets}_*((X, x_0), (Y, y_0)) \rightarrow \mathbf{Sets}_*((X, x_0), (Z, z_0))$$

of \mathbf{Sets}_* at $((X, x_0), (Y, y_0), (Z, z_0))$ is defined by²

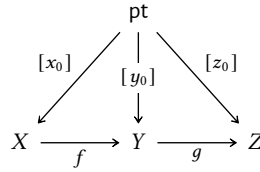
$$g \circ_{(X, x_0), (Y, y_0), (Z, z_0)}^{\mathbf{Sets}_*} f \stackrel{\text{def}}{=} g \circ f.$$

¹Note that id_X is indeed a morphism of pointed sets, as we have $\text{id}_X(x_0) = x_0$.

²Note that the composition of two morphisms of pointed sets is indeed a morphism of pointed sets, as we have

$$\begin{aligned} g(f(x_0)) &= g(y_0) \\ &= z_0, \end{aligned}$$

or



in terms of diagrams.

009P 1.4 Elementary Properties of Pointed Sets

009Q PROPOSITION 1.4.1 ► ELEMENTARY PROPERTIES OF POINTED SETS

Let (X, x_0) be a pointed set.

- 009R 1. *Completeness.* The category \mathbf{Sets}_* of pointed sets and morphisms between them is complete, having in particular:
 - 009S (a) Products, described as in [Definition 2.3.1](#);
 - 009T (b) Pullbacks, described as in [Definition 2.4.1](#);
 - 009U (c) Equalisers, described as in [Definition 2.5.1](#).
- 009V 2. *Cocompleteness.* The category \mathbf{Sets}_* of pointed sets and morphisms between them is cocomplete, having in particular:
 - 009W (a) Coproducts, described as in [Definition 3.3.1](#);
 - 009X (b) Pushouts, described as in [Definition 3.4.1](#);
 - 009Y (c) Coequalisers, described as in [Definition 3.5.1](#).
- 009Z 3. *Failure To Be Cartesian Closed.* The category \mathbf{Sets}_* is not Cartesian closed.¹
- 00A0 4. *Morphisms From the Monoidal Unit.* We have a bijection of sets²

$$\mathbf{Sets}_*(S^0, X) \cong X,$$

natural in $(X, x_0) \in \text{Obj}(\text{Sets}_*)$, internalising also to an isomorphism of pointed sets

$$\mathbf{Sets}_*(S^0, X) \cong (X, x_0),$$

again natural in $(X, x_0) \in \text{Obj}(\text{Sets}_*)$.

00A1

5. *Relation to Partial Functions.* We have an equivalence of categories³

$$\text{Sets}_* \xrightarrow{\text{eq.}} \text{Sets}^{\text{part.}}$$

between the category of pointed sets and pointed functions between them and the category of sets and partial functions between them, where:

(a) *From Pointed Sets to Sets With Partial Functions.* The equivalence

$$\xi: \text{Sets}_* \xrightarrow{\cong} \text{Sets}^{\text{part.}}$$

sends:

- i. A pointed set (X, x_0) to X .
- ii. A pointed function

$$f: (X, x_0) \rightarrow (Y, y_0)$$

to the partial function

$$\xi_f: X \rightarrow Y$$

defined on $f^{-1}(Y \setminus y_0)$ and given by

$$\xi_f(x) \stackrel{\text{def}}{=} f(x)$$

for each $x \in f^{-1}(Y \setminus y_0)$.

(b) *From Sets With Partial Functions to Pointed Sets.* The equivalence

$$\xi^{-1}: \text{Sets}^{\text{part.}} \xrightarrow{\cong} \text{Sets}_*$$

sends:

- i. A set X is to the pointed set (X, \star) with \star an element that is not in X .

ii. A partial function

$$f: X \rightarrow Y$$

defined on $U \subset X$ to the pointed function

$$\xi_f^{-1}: (X, x_0) \rightarrow (Y, y_0)$$

defined by

$$\xi_f(x) \stackrel{\text{def}}{=} \begin{cases} f(x) & \text{if } x \in U, \\ y_0 & \text{otherwise.} \end{cases}$$


for each $x \in X$.

¹The category \mathbf{Sets}_* does admit monoidal closed structures however; see [Tensor Products of Pointed Sets](#).

²In other words, the forgetful functor

$$\omega: \mathbf{Sets}_* \rightarrow \mathbf{Sets}$$

defined on objects by sending a pointed set to its underlying set is corepresentable by S^0 .

³ **Warning:** This is not an isomorphism of categories, only an equivalence.

PROOF 1.4.2 ► PROOF OF PROPOSITION 1.4.1

Item 1: Completeness

This follows from (the proofs) of [Definitions 2.3.1](#), [2.4.1](#) and [2.5.1](#) and [??](#), [??](#).

Item 2: Cocompleteness

This follows from (the proofs) of [Definitions 3.3.1](#), [3.4.1](#) and [3.5.1](#) and [??](#), [??](#).

Item 3: Failure To Be Cartesian Closed

See [\[MSE 2855868\]](#).

Item 4: Morphisms From the Monoidal Unit

Since a morphism from S^0 to a pointed set (X, x_0) sends $0 \in S^0$ to x_0 and then can send $1 \in S^0$ to any element of X , we obtain a bijection between pointed maps $S^0 \rightarrow X$ and the elements of X .

The isomorphism then

$$\mathbf{Sets}_*(S^0, X) \cong (X, x_0)$$

follows by noting that $\Delta_{x_0} : S^0 \rightarrow X$, the basepoint of $\mathbf{Sets}_*(S^0, X)$, corresponds to the pointed map $S^0 \rightarrow X$ picking the element x_0 of X , and thus we see that the bijection between pointed maps $S^0 \rightarrow X$ and elements of X is compatible with basepoints, lifting to an isomorphism of pointed sets.

Item 5: Relation to Partial Functions

See [MSE 884460].



00A2 2 Limits of Pointed Sets

00A3 2.1 The Terminal Pointed Set

00A4 DEFINITION 2.1.1 ► THE TERMINAL POINTED SET

The **terminal pointed set** is the pair $((\text{pt}, \star), \{!_X\}_{(X, x_0) \in \text{Obj}(\mathbf{Sets}_*)})$ consisting of:

- *The Limit.* The pointed set (pt, \star) .
- *The Cone.* The collection of morphisms of pointed sets

$$\{!_X : (X, x_0) \rightarrow (\text{pt}, \star)\}_{(X, x_0) \in \text{Obj}(\mathbf{Sets})}$$

defined by

$$!_X(x) \stackrel{\text{def}}{=} \star$$

for each $x \in X$ and each $(X, x_0) \in \text{Obj}(\mathbf{Sets})$.

PROOF 2.1.2 ► PROOF OF DEFINITION 2.1.1

We claim that (pt, \star) is the terminal object of \mathbf{Sets}_* . Indeed, suppose we have a diagram of the form

$$(X, x_0) \quad (\text{pt}, \star)$$

in \mathbf{Sets}_* . Then there exists a unique morphism of pointed sets

$$\phi : (X, x_0) \rightarrow (\text{pt}, \star)$$

making the diagram

$$(X, x_0) \xrightarrow[\exists!]{\phi} (\text{pt}, \star)$$

commute, namely $!_X$.



00A5 2.2 Products of Families of Pointed Sets

Let $\{(X_i, x_0^i)\}_{i \in I}$ be a family of pointed sets.

00A6

DEFINITION 2.2.1 ► THE PRODUCT OF A FAMILY OF POINTED SETS

The **product** of $\{(X_i, x_0^i)\}_{i \in I}$ is the pair $((\prod_{i \in I} X_i, (x_0^i)_{i \in I}), \{pr_i\}_{i \in I})$ consisting of:

- *The Limit.* The pointed set $(\prod_{i \in I} X_i, (x_0^i)_{i \in I})$.
- *The Cone.* The collection

$$\left\{ pr_i : \left(\prod_{i \in I} X_i, (x_0^i)_{i \in I} \right) \rightarrow (X_i, x_0^i) \right\}_{i \in I}$$

of maps given by

$$pr_i((x_j)_{j \in I}) \stackrel{\text{def}}{=} x_i$$

for each $(x_j)_{j \in I} \in \prod_{i \in I} X_i$ and each $i \in I$.

PROOF 2.2.2 ► PROOF OF DEFINITION 2.2.1

We claim that $(\prod_{i \in I} X_i, (x_0^i)_{i \in I})$ is the categorical product of $\{(X_i, x_0^i)\}_{i \in I}$ in Sets_* . Indeed, suppose we have, for each $i \in I$, a diagram of the form

$$\begin{array}{ccc} (P, *) & & \\ & \searrow p_i & \\ (\prod_{i \in I} X_i, (x_0^i)_{i \in I}) & \xrightarrow{pr_i} & (X_i, x_0^i) \end{array}$$

in Sets_* . Then there exists a unique morphism of pointed sets

$$\phi : (P, *) \rightarrow \left(\prod_{i \in I} X_i, (x_0^i)_{i \in I} \right)$$

making the diagram

$$\begin{array}{ccc}
 (P, *) & & \\
 \downarrow \phi \exists! & \searrow p_i & \\
 (\prod_{i \in I} X_i, (x_0^i)_{i \in I}) & \xrightarrow{\text{pr}_i} & (X_i, x_0^i)
 \end{array}$$

commute, being uniquely determined by the condition $\text{pr}_i \circ \phi = p_i$ for each $i \in I$ via

$$\phi(x) = (p_i(x))_{i \in I}$$

for each $x \in P$. Note that this is indeed a morphism of pointed sets, as we have

$$\begin{aligned}
 \phi(*) &= (p_i(*))_{i \in I} \\
 &= (x_0^i)_{i \in I},
 \end{aligned}$$

where we have used that p_i is a morphism of pointed sets for each $i \in I$. 

00A7

PROPOSITION 2.2.3 ► PROPERTIES OF PRODUCTS OF FAMILIES OF POINTED SETS

Let $\{(X_i, x_0^i)\}_{i \in I}$ be a family of pointed sets.

00A8

1. *Functoriality.* The assignment $\{(X_i, x_0^i)\}_{i \in I} \mapsto (\prod_{i \in I} X_i, (x_0^i)_{i \in I})$ defines a functor

$$\prod_{i \in I}: \text{Fun}(I_{\text{disc}}, \text{Sets}_*) \rightarrow \text{Sets}_*.$$

PROOF 2.2.4 ► PROOF OF PROPOSITION 2.2.3

Item 1: Functoriality

This follows from ??, ?? of ??.



00A9 2.3 Products

Let (X, x_0) and (Y, y_0) be pointed sets.

00AA

DEFINITION 2.3.1 ► PRODUCTS OF POINTED SETS

The **product** of (X, x_0) and (Y, y_0) is the pair consisting of:

- *The Limit.* The pointed set $(X \times Y, (x_0, y_0))$.
- *The Cone.* The morphisms of pointed sets

$$\text{pr}_1: (X \times Y, (x_0, y_0)) \rightarrow (X, x_0),$$

$$\text{pr}_2: (X \times Y, (x_0, y_0)) \rightarrow (Y, y_0)$$

defined by

$$\text{pr}_1(x, y) \stackrel{\text{def}}{=} x,$$

$$\text{pr}_2(x, y) \stackrel{\text{def}}{=} y$$

for each $(x, y) \in X \times Y$.

PROOF 2.3.2 ► PROOF OF DEFINITION 2.3.1

We claim that $(X \times Y, (x_0, y_0))$ is the categorical product of (X, x_0) and (Y, y_0) in Sets_* . Indeed, suppose we have a diagram of the form

$$\begin{array}{ccccc} & & (P, *) & & \\ & \swarrow p_1 & & \searrow p_2 & \\ (X, x_0) & \xleftarrow{\text{pr}_1} & (X \times Y, (x_0, y_0)) & \xrightarrow{\text{pr}_2} & (Y, y_0) \end{array}$$

in Sets_* . Then there exists a unique morphism of pointed sets

$$\phi: (P, *) \rightarrow (X \times Y, (x_0, y_0))$$

making the diagram

$$\begin{array}{ccccc} & & (P, *) & & \\ & \swarrow p_1 & \downarrow \phi \mid \exists! & \searrow p_2 & \\ (X, x_0) & \xleftarrow{\text{pr}_1} & (X \times Y, (x_0, y_0)) & \xrightarrow{\text{pr}_2} & (Y, y_0) \end{array}$$

commute, being uniquely determined by the conditions

$$\text{pr}_1 \circ \phi = p_1,$$


$$\text{pr}_2 \circ \phi = p_2$$

via

$$\phi(x) = (p_1(x), p_2(x))$$

for each $x \in P$. Note that this is indeed a morphism of pointed sets, as we have

$$\begin{aligned}\phi(*) &= (p_1(*), p_2(*)) \\ &= (x_0, y_0),\end{aligned}$$

where we have used that p_1 and p_2 are morphisms of pointed sets. 

00AB

PROPOSITION 2.3.3 ► PROPERTIES OF PRODUCTS OF POINTED SETS

Let (X, x_0) , (Y, y_0) , and (Z, z_0) be pointed sets.

00AC

1. *Functoriality.* The assignments

$$(X, x_0), (Y, y_0), ((X, x_0), (Y, y_0)) \mapsto (X \times Y, (x_0, y_0))$$

define functors

$$X \times -: \text{Sets}_* \rightarrow \text{Sets}_*,$$

$$- \times Y: \text{Sets}_* \rightarrow \text{Sets}_*,$$

$$-_1 \times -_2: \text{Sets}_* \times \text{Sets}_* \rightarrow \text{Sets}_*,$$

defined in the same way as the functors of [Constructions With Sets, Item 1](#) of [Proposition 1.3.3](#).

00AD

2. *Associativity.* We have an isomorphism of pointed sets

$$((X \times Y) \times Z, ((x_0, y_0), z_0)) \cong (X \times (Y \times Z), (x_0, (y_0, z_0)))$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \text{Obj}(\text{Sets}_*)$.

00AE

3. *Unitality.* We have isomorphisms of pointed sets

$$(\text{pt}, \star) \times (X, x_0) \cong (X, x_0),$$

$$(X, x_0) \times (\text{pt}, \star) \cong (X, x_0),$$

natural in $(X, x_0) \in \text{Obj}(\text{Sets}_*)$.

00AF

4. *Commutativity.* We have an isomorphism of pointed sets

$$(X \times Y, (x_0, y_0)) \cong (Y \times X, (y_0, x_0)),$$

natural in $(X, x_0), (Y, y_0) \in \text{Obj}(\text{Sets}_*)$.

00AG

5. *Symmetric Monoidality.* The triple $(\text{Sets}_*, \times, (\text{pt}, \star))$ is a symmetric monoidal category.

PROOF 2.3.4 ► PROOF OF PROPOSITION 2.3.3

Item 1: Functoriality

This is a special case of functoriality of limits, ??, ?? of ??.

Item 2: Associativity

This follows from [Constructions With Sets](#), [Item 3](#) of [Proposition 1.3.3](#).

Item 3: Unitality

This follows from [Constructions With Sets](#), [Item 4](#) of [Proposition 1.3.3](#).

Item 4: Commutativity

This follows from [Constructions With Sets](#), [Item 5](#) of [Proposition 1.3.3](#).

Item 5: Symmetric Monoidality

This follows from [Constructions With Sets](#), [Item 12](#) of [Proposition 1.3.3](#). 

00AH 2.4 Pullbacks

Let (X, x_0) , (Y, y_0) , and (Z, z_0) be pointed sets and let $f: (X, x_0) \rightarrow (Z, z_0)$ and $g: (Y, y_0) \rightarrow (Z, z_0)$ be morphisms of pointed sets.

00AJ

DEFINITION 2.4.1 ► PULLBACKS OF POINTED SETS

The **pullback of** (X, x_0) **and** (Y, y_0) **over** (Z, z_0) **along** (f, g) is the pair consisting of:

- *The Limit.* The pointed set $(X \times_Z Y, (x_0, y_0))$.
- *The Cone.* The morphisms of pointed sets

$$\begin{aligned} \text{pr}_1 &: (X \times_Z Y, (x_0, y_0)) \rightarrow (X, x_0), \\ \text{pr}_2 &: (X \times_Z Y, (x_0, y_0)) \rightarrow (Y, y_0) \end{aligned}$$

defined by

$$\begin{aligned} \text{pr}_1(x, y) &\stackrel{\text{def}}{=} x, \\ \text{pr}_2(x, y) &\stackrel{\text{def}}{=} y \end{aligned}$$

for each $(x, y) \in X \times_Z Y$.

PROOF 2.4.2 ► PROOF OF DEFINITION 2.4.1

We claim that $X \times_Z Y$ is the categorical pullback of (X, x_0) and (Y, y_0) over (Z, z_0) with respect to (f, g) in Sets_* . First we need to check that the relevant pullback diagram commutes, i.e. that we have

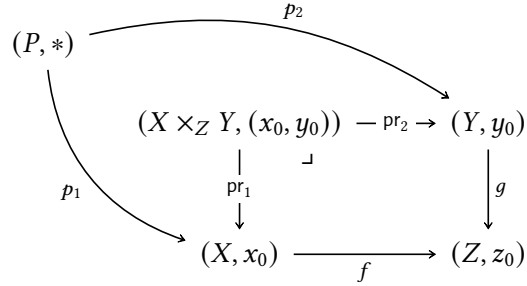
$$\begin{array}{ccc} (X \times_Z Y, (x_0, y_0)) & \xrightarrow{\text{pr}_2} & (Y, y_0) \\ \text{pr}_1 \downarrow & & \downarrow g \\ (X, x_0) & \xrightarrow{f} & (Z, z_0). \end{array}$$

$f \circ \text{pr}_1 = g \circ \text{pr}_2,$

Indeed, given $(x, y) \in X \times_Z Y$, we have

$$\begin{aligned} [f \circ \text{pr}_1](x, y) &= f(\text{pr}_1(x, y)) \\ &= f(x) \\ &= g(y) \\ &= g(\text{pr}_2(x, y)) \\ &= [g \circ \text{pr}_2](x, y), \end{aligned}$$

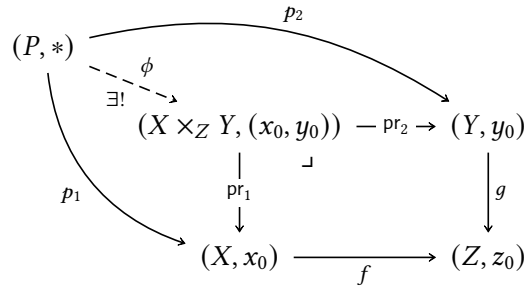
where $f(x) = g(y)$ since $(x, y) \in X \times_Z Y$. Next, we prove that $X \times_Z Y$ satisfies the universal property of the pullback. Suppose we have a diagram of the form



in \mathbf{Sets}_* . Then there exists a unique morphism of pointed sets

$$\phi: (P, *) \rightarrow (X \times_Z Y, (x_0, y_0))$$

making the diagram



commute, being uniquely determined by the conditions

$$\text{pr}_1 \circ \phi = p_1,$$

$$\text{pr}_2 \circ \phi = p_2$$

via

$$\phi(x) = (p_1(x), p_2(x))$$

for each $x \in P$, where we note that $(p_1(x), p_2(x)) \in X \times Y$ indeed lies in $X \times_Z Y$ by the condition


$$f \circ p_1 = g \circ p_2,$$

which gives

$$f(p_1(x)) = g(p_2(x))$$

for each $x \in P$, so that $(p_1(x), p_2(x)) \in X \times_Z Y$. Lastly, we note that ϕ is indeed a morphism of pointed sets, as we have

$$\begin{aligned}\phi(*) &= (p_1(*), p_2(*)) \\ &= (x_0, y_0),\end{aligned}$$

where we have used that p_1 and p_2 are morphisms of pointed sets. 

00AK

PROPOSITION 2.4.3 ► PROPERTIES OF PULLBACKS OF POINTED SETS

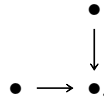
Let (X, x_0) , (Y, y_0) , (Z, z_0) , and (A, a_0) be pointed sets.

00AL

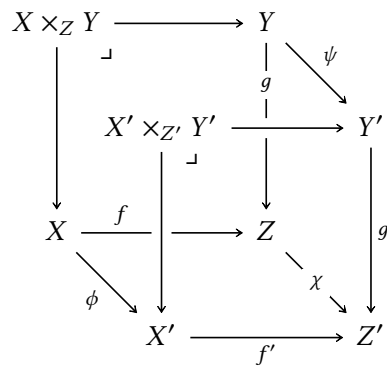
1. *Functoriality.* The assignment $(X, Y, Z, f, g) \mapsto X \times_{f, Z, g} Y$ defines a functor

$$-_1 \times_{-3} -_1 : \text{Fun}(\mathcal{P}, \text{Sets}_*) \rightarrow \text{Sets}_*,$$

where \mathcal{P} is the category that looks like this:



In particular, the action on morphisms of $-_1 \times_{-3} -_1$ is given by sending a morphism



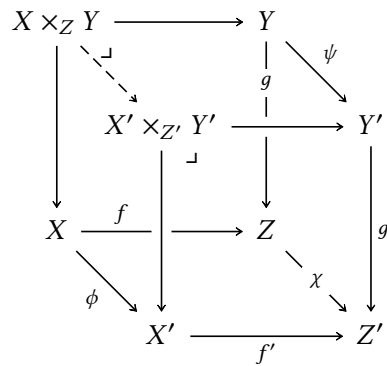
in $\text{Fun}(\mathcal{P}, \text{Sets}_*)$ to the morphism of pointed sets

$$\xi: (X \times_Z Y, (x_0, y_0)) \xrightarrow{\exists!} (X' \times_{Z'} Y', (x'_0, y'_0))$$

given by

$$\xi(x, y) \stackrel{\text{def}}{=} (\phi(x), \psi(y))$$

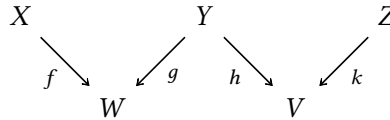
for each $(x, y) \in X \times_Z Y$, which is the unique morphism of pointed sets making the diagram



commute.

00AM

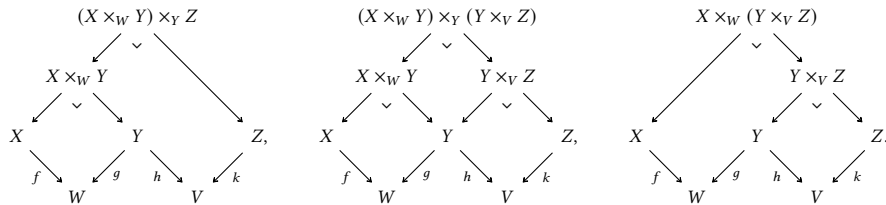
2. *Associativity.* Given a diagram



in Sets_* , we have isomorphisms of pointed sets

$$(X \times_W Y) \times_V Z \cong (X \times_W Y) \times_Y (Y \times_V Z) \cong X \times_W (Y \times_V Z),$$

where these pullbacks are built as in the diagrams



00AN

3. *Unitality.* We have isomorphisms of pointed sets

$$\begin{array}{ccc}
 A & \xlongequal{\quad} & A \\
 \downarrow f & \lrcorner & \downarrow f \\
 X & \xlongequal{\quad} & X
 \end{array}
 \quad
 \begin{array}{l}
 X \times_X A \cong A, \\
 A \times_X X \cong A,
 \end{array}
 \quad
 \begin{array}{ccc}
 A & \xrightarrow{f} & X \\
 \parallel & \lrcorner & \parallel \\
 X & \xrightarrow{f} & X.
 \end{array}$$

00AP

4. *Commutativity.* We have an isomorphism of pointed sets

$$\begin{array}{ccc}
 A \times_X B & \longrightarrow & B \\
 \downarrow & \lrcorner & \downarrow g \\
 A & \xrightarrow{f} & X,
 \end{array}
 \quad
 A \times_X B \cong B \times_X A
 \quad
 \begin{array}{ccc}
 B \times_X A & \longrightarrow & A \\
 \downarrow & \lrcorner & \downarrow f \\
 B & \xrightarrow{g} & X.
 \end{array}$$

00AQ

5. *Interaction With Products.* We have an isomorphism of pointed sets

$$\begin{array}{ccc}
 X \times Y & \longrightarrow & Y \\
 \downarrow & \lrcorner & \downarrow !_Y \\
 X & \xrightarrow{!_X} & \text{pt.}
 \end{array}
 \quad
 X \times_{\text{pt}} Y \cong X \times Y,$$

00AR

6. *Symmetric Monoidality.* The triple $(\text{Sets}_*, \times_X, X)$ is a symmetric monoidal category.

PROOF 2.4.4 ► PROOF OF PROPOSITION 2.4.3

Item 1: Functoriality

This is a special case of functoriality of co/limits, ??, ?? of ??, with the explicit expression for ξ following from the commutativity of the cube pullback diagram.

Item 2: Associativity

This follows from [Constructions With Sets](#), [Item 2](#) of [Proposition 1.4.5](#).

Item 3: Unitality

This follows from [Constructions With Sets, Item 3 of Proposition 1.4.5](#).

Item 4: Commutativity

This follows from [Constructions With Sets, Item 4 of Proposition 1.4.5](#).

Item 5: Interaction With Products

This follows from [Constructions With Sets, Item 6 of Proposition 1.4.5](#).

Item 6: Symmetric Monoidality

This follows from [Constructions With Sets, Item 7 of Proposition 1.4.5](#). 

00AS 2.5 Equalisers

Let $f, g: (X, x_0) \rightrightarrows (Y, y_0)$ be morphisms of pointed sets.

00AT DEFINITION 2.5.1 ► EQUALISERS OF POINTED SETS

The **equaliser of** (f, g) is the pair consisting of:

- *The Limit.* The pointed set $(\text{Eq}(f, g), x_0)$.
- *The Cone.* The morphism of pointed sets

$$\text{eq}(f, g): (\text{Eq}(f, g), x_0) \hookrightarrow (X, x_0)$$

given by the canonical inclusion $\text{eq}(f, g) \hookrightarrow \text{Eq}(f, g) \hookrightarrow X$.

PROOF 2.5.2 ► PROOF OF DEFINITION 2.5.1

We claim that $(\text{Eq}(f, g), x_0)$ is the categorical equaliser of f and g in Sets_* . First we need to check that the relevant equaliser diagram commutes, i.e. that we have

$$f \circ \text{eq}(f, g) = g \circ \text{eq}(f, g),$$

which indeed holds by the definition of the set $\text{Eq}(f, g)$. Next, we prove that $\text{Eq}(f, g)$ satisfies the universal property of the equaliser. Suppose we have a dia-

gram of the form

$$\begin{array}{ccccc} (\text{Eq}(f, g), x_0) & \xrightarrow{\text{eq}(f, g)} & (X, x_0) & \xrightleftharpoons[g]{f} & (Y, y_0) \\ & \nearrow e & & & \\ (E, *) & & & & \end{array}$$

in Sets_* . Then there exists a unique morphism of pointed sets

$$\phi: (E, *) \rightarrow (\text{Eq}(f, g), x_0)$$

making the diagram

$$\begin{array}{ccccc} (\text{Eq}(f, g), x_0) & \xrightarrow{\text{eq}(f, g)} & (X, x_0) & \xrightleftharpoons[g]{f} & (Y, y_0) \\ \uparrow \phi \exists! & \nearrow e & & & \\ (E, *) & & & & \end{array}$$

commute, being uniquely determined by the condition

$$\text{eq}(f, g) \circ \phi = e$$

via

$$\phi(x) = e(x)$$

for each $x \in E$, where we note that $e(x) \in \text{Eq}(f, g)$ indeed lies in $\text{Eq}(f, g)$ by the condition


$$f \circ e = g \circ e,$$

which gives

$$f(e(x)) = g(e(x))$$

for each $x \in E$, so that $e(x) \in \text{Eq}(f, g)$. Lastly, we note that ϕ is indeed a morphism of pointed sets, as we have

$$\begin{aligned} \phi(*) &= e(*) \\ &= x_0, \end{aligned}$$

where we have used that e is a morphism of pointed sets. 

00AU

PROPOSITION 2.5.3 ► PROPERTIES OF EQUALISERS OF POINTED SETS

Let (X, x_0) and (Y, y_0) be pointed sets and let $f, g, h: (X, x_0) \rightarrow (Y, y_0)$ be morphisms of pointed sets.

00AV

1. *Associativity.* We have isomorphisms of pointed sets

$$\underbrace{\text{Eq}(f \circ \text{eq}(g, h), g \circ \text{eq}(g, h))}_{=\text{Eq}(f \circ \text{eq}(g, h), h \circ \text{eq}(g, h))} \cong \text{Eq}(f, g, h) \cong \underbrace{\text{Eq}(f \circ \text{eq}(f, g), h \circ \text{eq}(f, g))}_{=\text{Eq}(g \circ \text{eq}(f, g), h \circ \text{eq}(f, g))},$$

where $\text{Eq}(f, g, h)$ is the limit of the diagram

$$(X, x_0) \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \rightarrow \\ \xrightarrow{h} \end{array} (Y, y_0)$$

in Sets_* , being explicitly given by

$$\text{Eq}(f, g, h) \cong \{a \in A \mid f(a) = g(a) = h(a)\}.$$

00AW

2. *Unitality.* We have an isomorphism of pointed sets

$$\text{Eq}(f, f) \cong X.$$

00AX

3. *Commutativity.* We have an isomorphism of pointed sets

$$\text{Eq}(f, g) \cong \text{Eq}(g, f).$$

PROOF 2.5.4 ► PROOF OF PROPOSITION 2.5.3

Item 1: Associativity

This follows from **Constructions With Sets, Item 1 of Proposition 1.5.3.**

Item 2: Unitality

This follows from **Constructions With Sets, Item 2 of Proposition 1.5.3.**

Item 3: Commutativity

This follows from **Constructions With Sets, Item 3 of Proposition 1.5.3.**



00AY 3 Colimits of Pointed Sets

00AZ 3.1 The Initial Pointed Set

00B0 DEFINITION 3.1.1 ► THE INITIAL POINTED SET

The **initial pointed set** is the pair $((pt, \star), \{\iota_X\}_{(X, x_0) \in \text{Obj}(\text{Sets}_*)})$ consisting of:

- *The Limit.* The pointed set (pt, \star) .
- *The Cone.* The collection of morphisms of pointed sets

$$\{\iota_X : (pt, \star) \rightarrow (X, x_0)\}_{(X, x_0) \in \text{Obj}(\text{Sets})}$$

defined by

$$\iota_X(\star) \stackrel{\text{def}}{=} x_0.$$

PROOF 3.1.2 ► PROOF OF DEFINITION 3.1.1

We claim that (pt, \star) is the initial object of Sets_* . Indeed, suppose we have a diagram of the form

$$(pt, \star) \quad (X, x_0)$$

in Sets_* . Then there exists a unique morphism of pointed sets

$$\phi : (pt, \star) \rightarrow (X, x_0)$$

making the diagram

$$(pt, \star) \xrightarrow[\exists!]{\phi} (X, x_0)$$

commute, namely ι_X .



00B1 3.2 Coproducts of Families of Pointed Sets

Let $\{(X_i, x_0^i)\}_{i \in I}$ be a family of pointed sets.

00B2

DEFINITION 3.2.1 ► COPRODUCTS OF FAMILIES OF POINTED SETS

The **coproduct of the family** $\{(X_i, x_0^i)\}_{i \in I}$, also called their **wedge sum**, is the pair consisting of:

- *The Colimit.* The pointed set $(\bigvee_{i \in I} X_i, p_0)$ consisting of:

- *The Underlying Set.* The set $\bigvee_{i \in I} X_i$ defined by

$$\bigvee_{i \in I} X_i \stackrel{\text{def}}{=} \left(\coprod_{i \in I} X_i \right) / \sim,$$

where \sim is the equivalence relation on $\coprod_{i \in I} X_i$ given by declaring

$$(i, x_0^i) \sim (j, x_0^j)$$

for each $i, j \in I$.

- *The Basepoint.* The element p_0 of $\bigvee_{i \in I} X_i$ defined by

$$\begin{aligned} p_0 &\stackrel{\text{def}}{=} [(i, x_0^i)] \\ &= [(j, x_0^j)] \end{aligned}$$

for any $i, j \in I$.

- *The Cocone.* The collection

$$\left\{ \text{inj}_i: (X_i, x_0^i) \rightarrow \left(\bigvee_{i \in I} X_i, p_0 \right) \right\}_{i \in I}$$

of morphism of pointed sets given by

$$\text{inj}_i(x) \stackrel{\text{def}}{=} (i, x)$$

for each $x \in X_i$ and each $i \in I$.

PROOF 3.2.2 ► PROOF OF DEFINITION 3.2.1

We claim that $(\bigvee_{i \in I} X_i, p_0)$ is the categorical coproduct of $\{(X_i, x_0^i)\}_{i \in I}$ in \mathbf{Sets}_* . Indeed, suppose we have, for each $i \in I$, a diagram of the form

$$\begin{array}{ccc} & & (C, *) \\ & \nearrow \iota_i & \\ (X_i, x_0^i) & \xrightarrow{\text{inj}_i} & \left(\bigvee_{i \in I} X_i, p_0 \right) \end{array}$$

in \mathbf{Sets}_* . Then there exists a unique morphism of pointed sets

$$\phi: \left(\bigvee_{i \in I} X_i, p_0 \right) \rightarrow (C, *)$$

making the diagram


$$\begin{array}{ccc} & & (C, *) \\ & \nearrow \iota_i & \uparrow \phi \exists! \\ (X_i, x_0^i) & \xrightarrow{\text{inj}_i} & \left(\bigvee_{i \in I} X_i, p_0 \right) \end{array}$$

commute, being uniquely determined by the condition $\phi \circ \text{inj}_i = \iota_i$ for each $i \in I$ via

$$\phi([(i, x)]) = \iota_i(x)$$

for each $[(i, x)] \in \bigvee_{i \in I} X_i$, where we note that ϕ is indeed a morphism of pointed sets, as we have

$$\begin{aligned} \phi(p_0) &= \iota_i([(i, x_0^i)]) \\ &= *, \end{aligned}$$

as ι_i is a morphism of pointed sets. 

00B3 **PROPOSITION 3.2.3 ► PROPERTIES OF COPRODUCTS OF FAMILIES OF POINTED SETS**

Let $\{(X_i, x_0^i)\}_{i \in I}$ be a family of pointed sets.

- 00B4 1. *Functoriality.* The assignment $\{(X_i, x_0^i)\}_{i \in I} \mapsto (\bigvee_{i \in I} X_i, p_0)$ defines a functor

$$\bigvee_{i \in I} : \text{Fun}(I_{\text{disc}}, \text{Sets}_*) \rightarrow \text{Sets}_*.$$

PROOF 3.2.4 ► PROOF OF PROPOSITION 3.2.3

Item 1: Functoriality

This follows from ??, ?? of ??.



00B5 **3.3 Coproducts**

Let (X, x_0) and (Y, y_0) be pointed sets.

00B6 **DEFINITION 3.3.1 ► COPRODUCTS OF POINTED SETS**

The **coproduct** of (X, x_0) and (Y, y_0) , also called their **wedge sum**, is the pair consisting of:

- *The Colimit.* The pointed set $(X \vee Y, p_0)$ consisting of:

- *The Underlying Set.* The set $X \vee Y$ defined by

$$\begin{aligned} (X \vee Y, p_0) &\stackrel{\text{def}}{=} (X, x_0) \amalg (Y, y_0) \\ &\cong (X \amalg_{\text{pt}} Y, p_0) \\ &\cong (X \amalg Y / \sim, p_0), \end{aligned} \quad \begin{array}{ccc} X \vee Y & \longleftarrow & Y \\ \uparrow \ulcorner & & \uparrow [y_0] \\ X & \xleftarrow{[x_0]} & \text{pt}, \end{array}$$

where \sim is the equivalence relation on $X \amalg Y$ obtained by declaring $(0, x_0) \sim (1, y_0)$.

- *The Basepoint.* The element p_0 of $X \vee Y$ defined by

$$\begin{aligned} p_0 &\stackrel{\text{def}}{=} [(0, x_0)] \\ &= [(1, y_0)]. \end{aligned}$$

· *The Cocone.* The morphisms of pointed sets

$$\text{inj}_1 : (X, x_0) \rightarrow (X \vee Y, p_0),$$

$$\text{inj}_2 : (Y, y_0) \rightarrow (X \vee Y, p_0),$$

given by

$$\text{inj}_1(x) \stackrel{\text{def}}{=} [(0, x)],$$

$$\text{inj}_2(y) \stackrel{\text{def}}{=} [(1, y)],$$

for each $x \in X$ and each $y \in Y$.

PROOF 3.3.2 ► PROOF OF DEFINITION 3.3.1

We claim that $(X \vee Y, p_0)$ is the categorical coproduct of (X, x_0) and (Y, y_0) in Sets_* . Indeed, suppose we have a diagram of the form

$$\begin{array}{ccccc} & & (C, *) & & \\ & \nearrow \iota_X & & \nwarrow \iota_Y & \\ (X, x_0) & \xrightarrow{\text{inj}_X} & (X \vee Y, p_0) & \xleftarrow{\text{inj}_Y} & (Y, y_0) \end{array}$$

in Sets . Then there exists a unique morphism of pointed sets

$$\phi : (X \vee Y, p_0) \rightarrow (C, *)$$

making the diagram

$$\begin{array}{ccccc} & & (C, *) & & \\ & \nearrow \iota_X & \uparrow \phi \mid \exists! & \nwarrow \iota_Y & \\ (X, x_0) & \xrightarrow{\text{inj}_X} & (X \vee Y, p_0) & \xleftarrow{\text{inj}_Y} & (Y, y_0) \end{array}$$

commute, being uniquely determined by the conditions

$$\phi \circ \text{inj}_X = \iota_X,$$


$$\phi \circ \text{inj}_Y = \iota_Y$$

via

$$\phi(z) = \begin{cases} \iota_X(x) & \text{if } z = [(0, x)] \text{ with } x \in X, \\ \iota_Y(y) & \text{if } z = [(1, y)] \text{ with } y \in Y \end{cases}$$

for each $z \in X \vee Y$, where we note that ϕ is indeed a morphism of pointed sets, as we have

$$\begin{aligned} \phi(p_0) &= \iota_X([(0, x_0)]) \\ &= \iota_Y([(1, y_0)]) \\ &= *, \end{aligned}$$

as ι_X and ι_Y are morphisms of pointed sets. 

00B7

PROPOSITION 3.3.3 ► PROPERTIES OF WEDGE SUMS OF POINTED SETS

Let (X, x_0) and (Y, y_0) be pointed sets.

00B8

1. *Functoriality.* The assignments

$$(X, x_0), (Y, y_0), ((X, x_0), (Y, y_0)) \mapsto (X \vee Y, p_0)$$

define functors

$$\begin{aligned} X \vee - &: \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*, \\ - \vee Y &: \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*, \\ -_1 \vee -_2 &: \mathbf{Sets}_* \times \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*. \end{aligned}$$

00B9

2. *Associativity.* We have an isomorphism of pointed sets

$$(X \vee Y) \vee Z \cong X \vee (Y \vee Z),$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathbf{Sets}_*$.

00BA

3. *Unitality.* We have isomorphisms of pointed sets

$$\begin{aligned} (\mathbf{pt}, *) \vee (X, x_0) &\cong (X, x_0), \\ (X, x_0) \vee (\mathbf{pt}, *) &\cong (X, x_0), \end{aligned}$$

natural in $(X, x_0) \in \mathbf{Sets}_*$.

00BB

4. *Commutativity.* We have an isomorphism of pointed sets

$$X \vee Y \cong Y \vee X,$$

natural in $(X, x_0), (Y, y_0) \in \mathbf{Sets}_*$.

00BC

5. *Symmetric Monoidality.* The triple $(\mathbf{Sets}_*, \vee, \text{pt})$ is a symmetric monoidal category.

00BD

6. *The Fold Map.* We have a natural transformation

$$\nabla: \vee \circ \Delta_{\mathbf{Sets}_*}^{\text{Cats}} \Rightarrow \text{id}_{\mathbf{Sets}_*},$$

called the **fold map**, whose component

$$\nabla_X: X \vee X \rightarrow X$$

at X is given by

$$\nabla_X(p) \stackrel{\text{def}}{=} \begin{cases} x & \text{if } p = [(0, x)], \\ x & \text{if } p = [(1, x)] \end{cases}$$

for each $p \in X \vee X$.

PROOF 3.3.4 ► PROOF OF PROPOSITION 3.3.3

Item 1: Functoriality

This follows from ??, ?? of ??.

Item 2: Associativity

Clear.

Item 3: Unitality

Clear.

Item 4: Commutativity

Clear.

Item 5: Symmetric Monoidality

Omitted.

Item 6: The Fold Map

Naturality for the transformation ∇ is the statement that, given a morphism of pointed sets $f: (X, x_0) \rightarrow (Y, y_0)$, we have

$$\nabla_Y \circ (f \vee f) = f \circ \nabla_X,$$

$$\begin{array}{ccc} X \vee X & \xrightarrow{\nabla_X} & X \\ f \vee f \downarrow & & \downarrow f \\ Y \vee Y & \xrightarrow{\nabla_Y} & Y. \end{array}$$

Indeed, we have

$$\begin{aligned} [\nabla_Y \circ (f \vee f)]([i, x]) &= \nabla_Y([i, f(x)]) \\ &= f(x) \\ &= f(\nabla_X([i, x])) \\ &= [f \circ \nabla_X]([i, x]) \end{aligned}$$

for each $[i, x] \in X \vee X$, and thus ∇ is indeed a natural transformation. 

00BE 3.4 Pushouts

Let (X, x_0) , (Y, y_0) , and (Z, z_0) be pointed sets and let $f: (Z, z_0) \rightarrow (X, x_0)$ and $g: (Z, z_0) \rightarrow (Y, y_0)$ be morphisms of pointed sets.

00BF DEFINITION 3.4.1 ► PUSHOUTS OF POINTED SETS

The **pushout of (X, x_0) and (Y, y_0) over (Z, z_0) along (f, g)** is the pair consisting of:

- *The Colimit.* The pointed set $(X \amalg_{f, Z, g} Y, p_0)$, where:

- The set $X \coprod_{f,Z,g} Y$ is the pushout (of unpointed sets) of X and Y over Z with respect to f and g ;
 - We have $p_0 = [x_0] = [y_0]$.
- *The Cocone.* The morphisms of pointed sets

$$\begin{aligned} \text{inj}_1 &: (X, x_0) \rightarrow (X \coprod_Z Y, p_0), \\ \text{inj}_2 &: (Y, y_0) \rightarrow (X \coprod_Z Y, p_0) \end{aligned}$$

given by

$$\begin{aligned} \text{inj}_1(x) &\stackrel{\text{def}}{=} [(0, x)] \\ \text{inj}_2(y) &\stackrel{\text{def}}{=} [(1, y)] \end{aligned}$$

for each $x \in X$ and each $y \in Y$.

PROOF 3.4.2 ► PROOF OF DEFINITION 3.4.1

Firstly, we note that indeed $[x_0] = [y_0]$, as we have

$$\begin{aligned} x_0 &= f(z_0), \\ y_0 &= g(z_0) \end{aligned}$$

since f and g are morphisms of pointed sets, with the relation \sim on $X \coprod_Z Y$ then identifying $x_0 = f(z_0) \sim g(z_0) = y_0$.

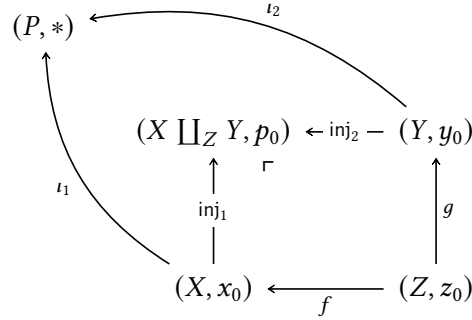
We now claim that $(X \coprod_Z Y, p_0)$ is the categorical pushout of (X, x_0) and (Y, y_0) over (Z, z_0) with respect to (f, g) in \mathbf{Sets}_* . First we need to check that the relevant pushout diagram commutes, i.e. that we have

$$\begin{array}{ccc} & (X \coprod_Z Y, p_0) & \xleftarrow{\text{inj}_2} (Y, y_0) \\ \text{inj}_1 \circ f = \text{inj}_2 \circ g, & \uparrow \text{inj}_1 & \uparrow g \\ (X, x_0) & \xleftarrow{f} & (Z, z_0). \end{array}$$

Indeed, given $z \in Z$, we have

$$\begin{aligned}
 [\text{inj}_1 \circ f](z) &= \text{inj}_1(f(z)) \\
 &= [(0, f(z))] \\
 &= [(1, g(z))] \\
 &= \text{inj}_2(g(z)) \\
 &= [\text{inj}_2 \circ g](z),
 \end{aligned}$$

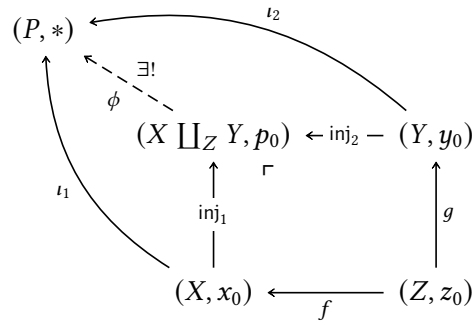
where $[(0, f(z))] = [(1, g(z))]$ by the definition of the relation \sim on $X \amalg Y$ (the coproduct of unpointed sets of X and Y). Next, we prove that $X \amalg_Z Y$ satisfies the universal property of the pushout. Suppose we have a diagram of the form



in \mathbf{Sets}_* . Then there exists a unique morphism of pointed sets

$$\phi: (X \amalg_Z Y, p_0) \rightarrow (P, *)$$

making the diagram



commute, being uniquely determined by the conditions

$$\phi \circ \text{inj}_1 = \iota_1,$$

$$\phi \circ \text{inj}_2 = \iota_2$$

via


$$\phi(p) = \begin{cases} \iota_1(x) & \text{if } x = [(0, x)], \\ \iota_2(y) & \text{if } x = [(1, y)] \end{cases}$$

for each $p \in X \amalg_Z Y$, where the well-definedness of ϕ is proven in the same way as in the proof of **Constructions With Sets, Definition 2.4.1**. Finally, we show that ϕ is indeed a morphism of pointed sets, as we have

$$\begin{aligned} \phi(p_0) &= \phi([(0, x_0)]) \\ &= \iota_1(x_0) \\ &= *, \end{aligned}$$

or alternatively

$$\begin{aligned} \phi(p_0) &= \phi([(1, y_0)]) \\ &= \iota_2(y_0) \\ &= *, \end{aligned}$$

where we use that ι_1 (resp. ι_2) is a morphism of pointed sets. 

00BG

PROPOSITION 3.4.3 ► PROPERTIES OF PUSHOUTS OF POINTED SETS

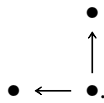
Let (X, x_0) , (Y, y_0) , (Z, z_0) , and (A, a_0) be pointed sets.

00BH

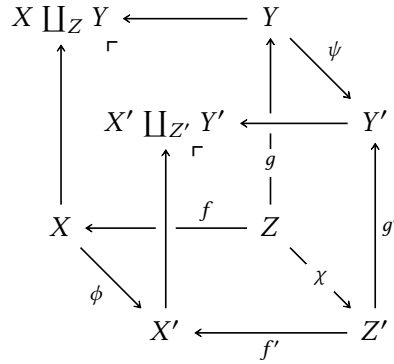
1. *Functoriality.* The assignment $(X, Y, Z, f, g) \mapsto X \amalg_{f, Z, g} Y$ defines a functor

$$-_1 \amalg_{-3} -_1 : \text{Fun}(\mathcal{P}, \text{Sets}) \rightarrow \text{Sets}_*,$$

where \mathcal{P} is the category that looks like this:



In particular, the action on morphisms of $-1 \coprod_{-3} -1$ is given by sending a morphism



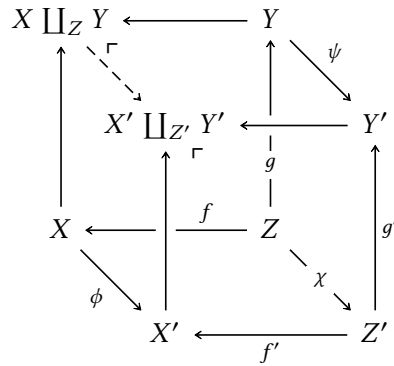
in $\text{Fun}(\mathcal{P}, \text{Sets}_*)$ to the morphism of pointed sets

$$\xi: (X \coprod_Z Y, p_0) \xrightarrow{\exists!} (X' \coprod_{Z'} Y', p'_0)$$

given by

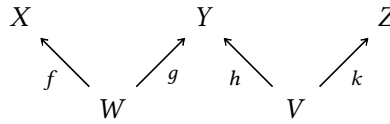
$$\xi(p) \stackrel{\text{def}}{=} \begin{cases} \phi(x) & \text{if } p = [(0, x)], \\ \psi(y) & \text{if } p = [(1, y)] \end{cases}$$

for each $p \in X \coprod_Z Y$, which is the unique morphism of pointed sets making the diagram



commute.

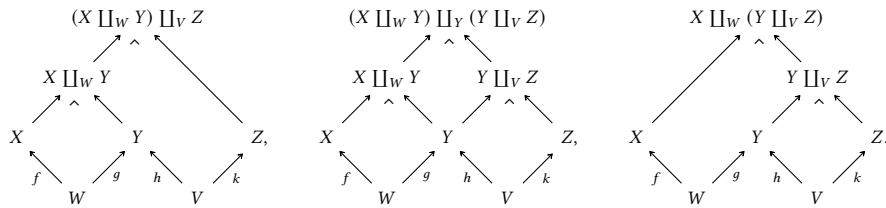
00BJ

2. *Associativity.* Given a diagram

in Sets, we have isomorphisms of pointed sets

$$(X \amalg_W Y) \amalg_V Z \cong (X \amalg_W Y) \amalg_Y (Y \amalg_V Z) \cong X \amalg_W (Y \amalg_V Z),$$

where these pullbacks are built as in the diagrams



00BK

3. *Unitality.* We have isomorphisms of sets

$$\begin{array}{ccc}
 A & \xlongequal{\quad} & A \\
 \uparrow f & \lrcorner & \uparrow f \\
 X & \xlongequal{\quad} & X
 \end{array}
 \quad
 \begin{array}{l}
 X \amalg_X A \cong A, \\
 A \amalg_X X \cong A,
 \end{array}
 \quad
 \begin{array}{ccc}
 A & \xleftarrow{f} & X \\
 \parallel & \lrcorner & \parallel \\
 X & \xleftarrow{f} & X.
 \end{array}$$

00BL

4. *Commutativity.* We have an isomorphism of sets

$$\begin{array}{ccc}
 X \amalg_Z Y & \xleftarrow{\quad} & Y \\
 \uparrow \lrcorner & & \uparrow g \\
 X & \xleftarrow{f} & Z,
 \end{array}
 \quad
 X \amalg_Z Y \cong Y \amalg_Z X
 \quad
 \begin{array}{ccc}
 Y \amalg_Z X & \xleftarrow{\quad} & X \\
 \uparrow \lrcorner & & \uparrow f \\
 Y & \xleftarrow{g} & Z.
 \end{array}$$

00BM

5. *Interaction With Coproducts.* We have

$$X \amalg_{\text{pt}} Y \cong X \vee Y,$$

$$\begin{array}{ccc} X \vee Y & \longleftarrow & Y \\ \uparrow \ulcorner & & \uparrow [y_0] \\ X & \xleftarrow{[x_0]} & \text{pt.} \end{array}$$

00BN

6. *Symmetric Monoidality.* The triple $(\text{Sets}_*, \amalg_X, (X, x_0))$ is a symmetric monoidal category.

PROOF 3.4.4 ► PROOF OF PROPOSITION 3.4.3

Item 1: Functoriality

This is a special case of functoriality of co/limits, ??, ?? of ??, with the explicit expression for ξ following from the commutativity of the cube pushout diagram.

Item 2: Associativity

This follows from [Constructions With Sets](#), [Item 2 of Proposition 2.4.6](#).

Item 3: Unitality

This follows from [Constructions With Sets](#), [Item 3 of Proposition 2.4.6](#).


Item 4: Commutativity

This follows from [Constructions With Sets](#), [Item 4 of Proposition 2.4.6](#).

Item 5: Interaction With Coproducts

Clear.

Item 6: Symmetric Monoidality

Omitted. 

00BP 3.5 Coequalisers

Let $f, g: (X, x_0) \rightrightarrows (Y, y_0)$ be morphisms of pointed sets.

00BQ

DEFINITION 3.5.1 ► COEQUALISERS OF POINTED SETS

The **coequaliser** of (f, g) is the pointed set $(\text{CoEq}(f, g), [y_0])$.

PROOF 3.5.2 ► PROOF OF DEFINITION 3.5.1

We claim that $(\text{CoEq}(f, g), [y_0])$ is the categorical coequaliser of f and g in Sets_* . First we need to check that the relevant coequaliser diagram commutes, i.e. that we have

$$\text{coeq}(f, g) \circ f = \text{coeq}(f, g) \circ g.$$

Indeed, we have

$$\begin{aligned} [\text{coeq}(f, g) \circ f](x) &\stackrel{\text{def}}{=} [\text{coeq}(f, g)](f(x)) \\ &\stackrel{\text{def}}{=} [f(x)] \\ &= [g(x)] \\ &\stackrel{\text{def}}{=} [\text{coeq}(f, g)](g(x)) \\ &\stackrel{\text{def}}{=} [\text{coeq}(f, g) \circ g](x) \end{aligned}$$

for each $x \in X$. Next, we prove that $\text{CoEq}(f, g)$ satisfies the universal property of the coequaliser. Suppose we have a diagram of the form


$$\begin{array}{ccc} (X, x_0) & \xrightarrow[g]{f} & (Y, y_0) & \xrightarrow{\text{coeq}(f, g)} & (\text{CoEq}(f, g), [y_0]) \\ & & \searrow c & & \\ & & & & (C, *) \end{array}$$

in Sets . Then, since $c(f(a)) = c(g(a))$ for each $a \in A$, it follows from [Equivalence Relations and Apartness Relations](#), [Items 4 and 5](#) of [Proposition 5.2.3](#) that there exists a unique map $\phi: \text{CoEq}(f, g) \xrightarrow{\exists!} C$ making the diagram

$$\begin{array}{ccc} (X, x_0) & \xrightarrow[g]{f} & (Y, y_0) & \xrightarrow{\text{coeq}(f, g)} & (\text{CoEq}(f, g), [y_0]) \\ & & \searrow c & & \downarrow \phi \mid \exists! \\ & & & & (C, *) \end{array}$$

commute, where we note that ϕ is indeed a morphism of pointed sets since

$$\begin{aligned}\phi([y_0]) &= [\phi \circ \text{coeq}(f, g)]([y_0]) \\ &= c([y_0]) \\ &= *,\end{aligned}$$

where we have used that c is a morphism of pointed sets. 

00BR

PROPOSITION 3.5.3 ► PROPERTIES OF COEQUALISERS OF POINTED SETS

Let (X, x_0) and (Y, y_0) be pointed sets and let $f, g, h: (X, x_0) \rightarrow (Y, y_0)$ be morphisms of pointed sets.

00BS

1. *Associativity.* We have isomorphisms of pointed sets

$$\underbrace{\text{CoEq}(\text{coeq}(f, g) \circ f, \text{coeq}(f, g) \circ h)}_{=\text{CoEq}(\text{coeq}(f, g) \circ g, \text{coeq}(f, g) \circ h)} \cong \text{CoEq}(f, g, h) \cong \underbrace{\text{CoEq}(\text{coeq}(g, h) \circ f, \text{coeq}(g, h) \circ g)}_{=\text{CoEq}(\text{coeq}(g, h) \circ f, \text{coeq}(g, h) \circ h)},$$

where $\text{CoEq}(f, g, h)$ is the colimit of the diagram

$$(X, x_0) \begin{array}{c} \xrightarrow{f} \\ \xrightarrow[-g]{} \\ \xrightarrow{h} \end{array} (Y, y_0)$$

in Sets_* .

00BT

2. *Unitality.* We have an isomorphism of pointed sets

$$\text{CoEq}(f, f) \cong B.$$

00BU

3. *Commutativity.* We have an isomorphism of pointed sets

$$\text{CoEq}(f, g) \cong \text{CoEq}(g, f).$$

PROOF 3.5.4 ► PROOF OF PROPOSITION 3.5.3

Item 1: Associativity

This follows from **Constructions With Sets**, Item 1 of **Proposition 2.5.6**.

Item 2: Unitality

This follows from **Constructions With Sets**, Item 2 of Proposition 2.5.6.

Item 3: Commutativity

This follows from **Constructions With Sets**, Item 3 of Proposition 2.5.6. 

00BV 4 Constructions With Pointed Sets

00BW 4.1 Free Pointed Sets

Let X be a set.

00BX DEFINITION 4.1.1 ► FREE POINTED SETS

The **free pointed set on X** is the pointed set X^+ consisting of:

- *The Underlying Set.* The set X^+ defined by¹

$$\begin{aligned} X^+ &\stackrel{\text{def}}{=} X \amalg \text{pt} \\ &\stackrel{\text{def}}{=} X \amalg \{\star\}. \end{aligned}$$

- *The Basepoint.* The element \star of X^+ .

¹*Further Notation:* We sometimes write \star_X for the basepoint of X^+ for clarity when there are multiple free pointed sets involved in the current discussion.

00BY PROPOSITION 4.1.2 ► PROPERTIES OF FREE POINTED SETS

Let X be a set.

- 00BZ 1. *Functoriality.* The assignment $X \mapsto X^+$ defines a functor

$$(-)^+: \mathbf{Sets} \rightarrow \mathbf{Sets}_*,$$

where

- *Action on Objects.* For each $X \in \mathbf{Obj}(\mathbf{Sets})$, we have

$$[(-)^+](X) \stackrel{\text{def}}{=} X^+,$$

where X^+ is the pointed set of **Definition 4.1.1**;

· *Action on Morphisms.* For each morphism $f: X \rightarrow Y$ of **Sets**, the image

$$f^+: X^+ \rightarrow Y^+$$

of f by $(-)^+$ is the map of pointed sets defined by

$$f^+(x) \stackrel{\text{def}}{=} \begin{cases} f(x) & \text{if } x \in X, \\ \star_Y & \text{if } x = \star_X. \end{cases}$$

00C0

2. *Adjointness.* We have an adjunction

$$((-)^+ \dashv \text{忘}): \text{Sets} \begin{matrix} \xrightarrow{(-)^+} \\ \perp \\ \xleftarrow{\text{忘}} \end{matrix} \text{Sets}_*,$$

witnessed by a bijection of sets

$$\text{Sets}_*((X^+, \star_X), (Y, y_0)) \cong \text{Sets}(X, Y),$$

natural in $X \in \text{Obj}(\text{Sets})$ and $(Y, y_0) \in \text{Obj}(\text{Sets}_*)$.

00C1

3. *Symmetric Strong Monoidality With Respect to Wedge Sums.* The free pointed set functor of **Item 1** has a symmetric strong monoidal structure

$$((-)^+, (-)^+, \amalg, (-)^+, \amalg): (\text{Sets}, \amalg, \emptyset) \rightarrow (\text{Sets}_*, \vee, \text{pt}),$$

being equipped with isomorphisms of pointed sets

$$(-)^+, \amalg_{X,Y}: X^+ \vee Y^+ \xrightarrow{\cong} (X \amalg Y)^+,$$

$$(-)^+, \amalg_{\mathbb{1}}: \text{pt} \xrightarrow{\cong} \emptyset^+,$$

natural in $X, Y \in \text{Obj}(\text{Sets})$.

00C2

4. *Symmetric Strong Monoidality With Respect to Smash Products.* The free pointed set functor of **Item 1** has a symmetric strong monoidal structure

$$((-)^+, (-)^+, \times, (-)^+, \times): (\text{Sets}, \times, \text{pt}) \rightarrow (\text{Sets}_*, \wedge, S^0),$$

being equipped with isomorphisms of pointed sets

$$(-)^+, \times_{X,Y}: X^+ \wedge Y^+ \xrightarrow{\cong} (X \times Y)^+,$$

$$(-)^+, \times_{\mathbb{1}}: S^0 \xrightarrow{\cong} \text{pt}^+,$$

natural in $X, Y \in \text{Obj}(\text{Sets})$.

PROOF 4.1.3 ► PROOF OF PROPOSITION 4.1.2

Item 1: Functoriality

Clear.

Item 2: Adjointness

We claim there's an adjunction $(-)^+ \dashv \mathbf{Sets}_*$, witnessed by a bijection of sets

$$\mathbf{Sets}_*((X^+, \star_X), (Y, y_0)) \cong \mathbf{Sets}(X, Y),$$

natural in $X \in \mathbf{Obj}(\mathbf{Sets})$ and $(Y, y_0) \in \mathbf{Obj}(\mathbf{Sets}_*)$.

• *Map I.* We define a map

$$\Phi_{X,Y}: \mathbf{Sets}_*((X^+, \star_X), (Y, y_0)) \rightarrow \mathbf{Sets}(X, Y)$$

by sending a pointed function

$$\xi: (X^+, \star_X) \rightarrow (Y, y_0)$$

to the function

$$\xi^\dagger: X \rightarrow Y$$

given by

$$\xi^\dagger(x) \stackrel{\text{def}}{=} \xi(x)$$

for each $x \in X$.

• *Map II.* We define a map

$$\Psi_{X,Y}: \mathbf{Sets}(X, Y) \rightarrow \mathbf{Sets}_*((X^+, \star_X), (Y, y_0))$$

given by sending a function $\xi: X \rightarrow Y$ to the pointed function

$$\xi^\dagger: (X^+, \star_X) \rightarrow (Y, y_0)$$

defined by

$$\xi^\dagger(x) \stackrel{\text{def}}{=} \begin{cases} \xi(x) & \text{if } x \in X, \\ y_0 & \text{if } x = \star_X \end{cases}$$

for each $x \in X^+$.

- *Invertibility I.* We claim that

$$\Psi_{X,Y} \circ \Phi_{X,Y} = \text{id}_{\text{Sets}_*((X^+, \star_X), (Y, y_0))},$$

which is clear.

- *Invertibility II.* We claim that

$$\Phi_{X,Y} \circ \Psi_{X,Y} = \text{id}_{\text{Sets}(X,Y)},$$

which is clear.

- *Naturality for Φ , Part I.* We need to show that, given a pointed function $g: (Y, y_0) \rightarrow (Y', y'_0)$, the diagram

$$\begin{array}{ccc} \text{Sets}_*((X^+, \star_X), (Y, y_0)) & \xrightarrow{\Phi_{X,Y}} & \text{Sets}(X, Y) \\ g_* \downarrow & & \downarrow g_* \\ \text{Sets}_*((X^+, \star_X), (Y', y'_0)) & \xrightarrow{\Phi_{X,Y'}} & \text{Sets}(X, Y') \end{array}$$

commutes. Indeed, given a pointed function

$$\xi^\dagger: (X^+, \star_X) \rightarrow (Y, y_0)$$

we have

$$\begin{aligned} [\Phi_{X,Y'} \circ g_*](\xi) &= \Phi_{X,Y'}(g_*(\xi)) \\ &= \Phi_{X,Y'}(g \circ \xi) \\ &= g \circ \xi \\ &= g \circ \Phi_{X,Y'}(\xi) \\ &= g_*(\Phi_{X,Y'}(\xi)) \\ &= [g_* \circ \Phi_{X,Y'}](\xi). \end{aligned}$$

- *Naturality for Φ , Part II.* We need to show that, given a pointed function

$f: (X, x_0) \rightarrow (X', x'_0)$, the diagram

$$\begin{array}{ccc} \text{Sets}_*\left(\left(X', \star_{X'}\right), (Y, y_0)\right) & \xrightarrow{\Phi_{X', Y}} & \text{Sets}(X', Y) \\ f^* \downarrow & & \downarrow f^* \\ \text{Sets}_*\left(\left(X, \star_X\right), (Y, y_0)\right) & \xrightarrow{\Phi_{X, Y}} & \text{Sets}(X, Y) \end{array}$$

commutes. Indeed, given a function

$$\xi: X' \rightarrow Y,$$

we have

$$\begin{aligned} [\Phi_{X, Y} \circ f^*](\xi) &= \Phi_{X, Y}(f^*(\xi)) \\ &= \Phi_{X, Y}(\xi \circ f) \\ &= \xi \circ f \\ &= \Phi_{X', Y}(\xi) \circ f \\ &= f^*(\Phi_{X', Y}(\xi)) \\ &= f^*(\Phi_{X', Y}(\xi)) \\ &= [f^* \circ \Phi_{X', Y}](\xi). \end{aligned}$$

- *Naturality for Ψ .* Since Φ is natural in each argument and Φ is a componentwise inverse to Ψ in each argument, it follows from [Categories, Item 2 of Proposition 8.6.2](#) that Ψ is also natural in each argument.

Item 3: Symmetric Strong Monoidality With Respect to Wedge Sums

The isomorphism

$$\phi: X^+ \vee Y^+ \xrightarrow{\cong} (X \amalg Y)^+$$

is given by

$$\phi(z) = \begin{cases} x & \text{if } z = [(0, x)] \text{ with } x \in X, \\ y & \text{if } z = [(1, y)] \text{ with } y \in Y, \\ \star_X \amalg_Y & \text{if } z = [(0, \star_X)], \\ \star_X \amalg_Y & \text{if } z = [(1, \star_Y)] \end{cases}$$

for each $z \in X^+ \vee Y^+$, with inverse

$$\phi^{-1}: (X \amalg Y)^+ \xrightarrow{\cong} X^+ \vee Y^+$$

given by

$$\phi^{-1}(z) \stackrel{\text{def}}{=} \begin{cases} [(0, x)] & \text{if } z = [(0, x)], \\ [(0, y)] & \text{if } z = [(1, y)], \\ p_0 & \text{if } z = \star_X \amalg Y \end{cases}$$

for each $z \in (X \amalg Y)^+$.

Meanwhile, the isomorphism $\text{pt} \cong \emptyset^+$ is given by sending \star_X to \star_\emptyset .

That these isomorphisms satisfy the coherence conditions making the functor $(-)^+$ symmetric strong monoidal can be directly checked element by element.

Item 4: Symmetric Strong Monoidality With Respect to Smash Products

The isomorphism

$$\phi: X^+ \wedge Y^+ \xrightarrow{\cong} (X \times Y)^+$$

is given by

$$\phi(x \wedge y) = \begin{cases} (x, y) & \text{if } x \neq \star_X \text{ and } y \neq \star_Y \\ \star_{X \times Y} & \text{otherwise} \end{cases}$$

for each $x \wedge y \in X^+ \wedge Y^+$, with inverse


$$\phi^{-1}: (X \times Y)^+ \xrightarrow{\cong} X^+ \wedge Y^+$$

given by

$$\phi^{-1}(z) \stackrel{\text{def}}{=} \begin{cases} x \wedge y & \text{if } z = (x, y) \text{ with } (x, y) \in X \times Y, \\ \star_X \wedge \star_Y & \text{if } z = \star_{X \times Y}, \end{cases}$$

for each $z \in (X \times Y)^+$.

Meanwhile, the isomorphism $S^0 \cong \text{pt}^+$ is given by sending \star to $1 \in S^0 = \{0, 1\}$ and \star_{pt} to $0 \in S^0$.

That these isomorphisms satisfy the coherence conditions making the functor $(-)^+$ symmetric strong monoidal can be directly checked element by element. 

Appendices

A Other Chapters

Sets

- | | |
|--|--|
| 1. Sets | 7. Equivalence Relations and Apartness Relations |
| 2. Constructions With Sets | |
| 3. Pointed Sets | Category Theory |
| 4. Tensor Products of Pointed Sets | 8. Categories |
| Relations | Bicategories |
| 5. Relations | |
| 6. Constructions With Relations | 9. Types of Morphisms in Bicategories |

References

- [MSE 2855868] [Qjaochu Yuan](#). *Is the category of pointed sets Cartesian closed?* Mathematics Stack Exchange. URL: <https://math.stackexchange.com/q/2855868> (cit. on p. 7).
- [MSE 884460] [Martin Brandenburg](#). *Why are the category of pointed sets and the category of sets and partial functions “essentially the same”?* Mathematics Stack Exchange. URL: <https://math.stackexchange.com/q/884460> (cit. on p. 8).