

# Categories

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This chapter contains some elementary material about categories, functors, and natural transformations. Notably, we discuss and explore:

1. Categories ([Section 1](#)).
2. The quadruple adjunction  $\pi_0 \dashv (-)_{\text{disc}} \dashv \text{Obj} \dashv (-)_{\text{indisc}}$  between the category of categories and the category of sets ([Section 2](#)).
3. Groupoids, categories in which all morphisms admit inverses ([Section 3](#)).
4. Functors ([Section 4](#)).
5. The conditions one may impose on functors in decreasing order of importance:
  - (a) [Section 5](#) introduces the foundationally important conditions one may impose on functors, such as faithfulness, conservativity, essential surjectivity, etc.
  - (b) [Section 6](#) introduces more conditions one may impose on functors that are still important but less omni-present than those of [Section 5](#), such as being dominant, being a monomorphism, being pseudomonadic, etc.
  - (c) [Section 7](#) introduces some rather rare or uncommon conditions one may impose on functors that are nevertheless still useful to explicitly record in this chapter.
6. Natural transformations ([Section 8](#)).
7. The various categorical and 2-categorical structures formed by categories, functors, and natural transformations ([Section 9](#)).

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# 1 Categories

## 1.1 Foundations

**Definition 1.1.1.1.** A **category**  $(C, \circ^C, \mathbb{1}^C)$  consists of:

- *Objects.* A class  $\text{Obj}(C)$  of **objects**.
- *Morphisms.* For each  $A, B \in \text{Obj}(C)$ , a class  $\text{Hom}_C(A, B)$ , called the **class of morphisms of  $C$  from  $A$  to  $B$** .
- *Identities.* For each  $A \in \text{Obj}(C)$ , a map of sets

$$\mathbb{1}_A^C: \text{pt} \rightarrow \text{Hom}_C(A, A),$$

called the **unit map of  $C$  at  $A$** , determining a morphism

$$\text{id}_A: A \rightarrow A$$

of  $C$ , called the **identity morphism of  $A$** .

- *Composition.* For each  $A, B, C \in \text{Obj}(C)$ , a map of sets

$$\circ_{A,B,C}^C: \text{Hom}_C(B, C) \times \text{Hom}_C(A, B) \rightarrow \text{Hom}_C(A, C),$$

called the **composition map of  $C$  at  $(A, B, C)$** .

such that the following conditions are satisfied:

1. *Associativity.* The diagram

$$\begin{array}{ccccc}
 & & \text{Hom}_C(C, D) \times (\text{Hom}_C(B, C) \times \text{Hom}_C(A, B)) & & \\
 & \nearrow \alpha_{\text{Hom}_C(C,D), \text{Hom}_C(B,C), \text{Hom}_C(A,B)}^{\text{Sets}} & & \searrow \text{id}_{\text{Hom}_C(C,D)} \times \circ_{A,B,C}^C & \\
 (\text{Hom}_C(C, D) \times \text{Hom}_C(B, C)) \times \text{Hom}_C(A, B) & & & & \text{Hom}_C(C, D) \times \text{Hom}_C(A, C) \\
 \downarrow \circ_{B,C,D}^C \times \text{id}_{\text{Hom}_C(A,B)} & & & & \downarrow \circ_{A,C,D}^C \\
 \text{Hom}_C(B, D) \times \text{Hom}_C(A, B) & \xrightarrow{\circ_{A,B,D}^C} & \text{Hom}_C(A, D) & & 
 \end{array}$$

commutes, i.e. for each composable triple  $(f, g, h)$  of morphisms of  $C$ , we have

$$(f \circ g) \circ h = f \circ (g \circ h).$$

2. *Left Unitality.* The diagram

$$\begin{array}{ccc} \text{pt} \times \text{Hom}_C(A, B) & & \\ \downarrow \mathbb{1}_B^C \times \text{id}_{\text{Hom}_C(A, B)} & \searrow \lambda_{\text{Hom}_C(A, B)}^{\text{Sets}} & \\ \text{Hom}_C(B, B) \times \text{Hom}_C(A, B) & \xrightarrow{\circ_{A, B, B}^C} & \text{Hom}_C(A, B) \end{array}$$

commutes, i.e. for each morphism  $f: A \rightarrow B$  of  $C$ , we have

$$\text{id}_B \circ f = f.$$

3. *Right Unitality.* The diagram

$$\begin{array}{ccc} \text{Hom}_C(A, B) \times \text{pt} & & \\ \downarrow \text{id}_{\text{Hom}_C(A, B)} \times \mathbb{1}_A^C & \searrow \rho_{\text{Hom}_C(A, B)}^{\text{Sets}} & \\ \text{Hom}_C(A, B) \times \text{Hom}_C(A, A) & \xrightarrow{\circ_{A, A, B}^C} & \text{Hom}_C(A, B) \end{array}$$

commutes, i.e. for each morphism  $f: A \rightarrow B$  of  $C$ , we have

$$f \circ \text{id}_A = f.$$

**Notation 1.1.1.2.** Let  $C$  be a category.

1. We also write  $C(A, B)$  for  $\text{Hom}_C(A, B)$ .
2. We write  $\text{Mor}(C)$  for the class of all morphisms of  $C$ .

**Definition 1.1.1.3.** Let  $\kappa$  be a regular cardinal. A category  $C$  is

1. **Locally small** if, for each  $A, B \in \text{Obj}(C)$ , the class  $\text{Hom}_C(A, B)$  is a set.
2. **Locally essentially small** if, for each  $A, B \in \text{Obj}(C)$ , the class

$$\text{Hom}_C(A, B) / \{\text{isomorphisms}\}$$

is a set.

3. **Small** if  $C$  is locally small and  $\text{Obj}(C)$  is a set.
4.  **$\kappa$ -Small** if  $C$  is locally small,  $\text{Obj}(C)$  is a set, and we have  $\#\text{Obj}(C) < \kappa$ .

## 1.2 Examples of Categories

**Example 1.2.1.1.** The **punctual category**<sup>1</sup> is the category  $\mathbf{pt}$  where

- *Objects.* We have

$$\mathrm{Obj}(\mathbf{pt}) \stackrel{\mathrm{def}}{=} \{\star\}.$$

- *Morphisms.* The unique Hom-set of  $\mathbf{pt}$  is defined by

$$\mathrm{Hom}_{\mathbf{pt}}(\star, \star) \stackrel{\mathrm{def}}{=} \{\mathrm{id}_{\star}\}.$$

- *Identities.* The unit map

$$\mathbb{1}_{\star}^{\mathbf{pt}} : \mathbf{pt} \rightarrow \mathrm{Hom}_{\mathbf{pt}}(\star, \star)$$

of  $\mathbf{pt}$  at  $\star$  is defined by

$$\mathrm{id}_{\star}^{\mathbf{pt}} \stackrel{\mathrm{def}}{=} \mathrm{id}_{\star}.$$

- *Composition.* The composition map

$$\circ_{\star, \star, \star}^{\mathbf{pt}} : \mathrm{Hom}_{\mathbf{pt}}(\star, \star) \times \mathrm{Hom}_{\mathbf{pt}}(\star, \star) \rightarrow \mathrm{Hom}_{\mathbf{pt}}(\star, \star)$$

of  $\mathbf{pt}$  at  $(\star, \star, \star)$  is given by the bijection  $\mathbf{pt} \times \mathbf{pt} \cong \mathbf{pt}$ .

**Example 1.2.1.2.** We have an isomorphism of categories<sup>2</sup>

$$\begin{array}{ccc} \mathbf{Mon} & \longrightarrow & \mathbf{Cats} \\ \downarrow \lrcorner & & \downarrow \mathrm{Obj} \\ \mathbf{pt} & \xrightarrow{[\mathbf{pt}]} & \mathbf{Sets} \end{array}$$

$\mathbf{Mon} \cong \mathbf{pt} \times_{\mathbf{Sets}} \mathbf{Cats},$

<sup>1</sup>Further Terminology: Also called the **singleton category**.

<sup>2</sup>This can be enhanced to an isomorphism of 2-categories

$$\begin{array}{ccc} \mathbf{Mon}_{2\mathrm{disc}} & \longrightarrow & \mathbf{Cats}_{2,*} \\ \downarrow \lrcorner & & \downarrow \mathrm{Obj} \\ \mathbf{pt}_{\mathrm{bi}} & \xrightarrow{[\mathbf{pt}]} & \mathbf{Sets}_{2\mathrm{disc}} \end{array}$$

$\mathbf{Mon}_{2\mathrm{disc}} \cong \mathbf{pt}_{\mathrm{bi}} \times_{\mathbf{Sets}_{2\mathrm{disc}}} \mathbf{Cats}_{2,*},$

via the delooping functor  $B: \mathbf{Mon} \rightarrow \mathbf{Cats}$  of ?? of ??, exhibiting monoids as exactly those categories having a single object.

*Proof.* Omitted. □

**Example 1.2.1.3.** The **empty category** is the category  $\emptyset_{\text{cat}}$  where

- *Objects.* We have

$$\text{Obj}(\emptyset_{\text{cat}}) \stackrel{\text{def}}{=} \emptyset.$$

- *Morphisms.* We have

$$\text{Mor}(\emptyset_{\text{cat}}) \stackrel{\text{def}}{=} \emptyset.$$

- *Identities and Composition.* Having no objects,  $\emptyset_{\text{cat}}$  has no unit nor composition maps.

**Example 1.2.1.4.** The  **$n$ th ordinal category** is the category  $\mathfrak{n}$  where<sup>3</sup>

- *Objects.* We have

$$\text{Obj}(\mathfrak{n}) \stackrel{\text{def}}{=} \{[0], \dots, [n]\}.$$

between the discrete 2-category  $\mathbf{Mon}_{2\text{disc}}$  on  $\mathbf{Mon}$  and the 2-category of pointed categories with one object.

<sup>3</sup>In other words,  $\mathfrak{n}$  is the category associated to the poset

$$[0] \rightarrow [1] \rightarrow \dots \rightarrow [n-1] \rightarrow [n].$$

The category  $\mathfrak{n}$  for  $n \geq 2$  may also be defined in terms of  $\emptyset$  and joins (??, ??): we have isomorphisms of categories

$$\begin{aligned} 1 &\cong \emptyset \star \emptyset, \\ 2 &\cong 1 \star \emptyset \\ &\cong (\emptyset \star \emptyset) \star \emptyset, \\ 3 &\cong 2 \star \emptyset \\ &\cong (1 \star \emptyset) \star \emptyset \\ &\cong ((\emptyset \star \emptyset) \star \emptyset) \star \emptyset, \\ 4 &\cong 3 \star \emptyset \\ &\cong (2 \star \emptyset) \star \emptyset \\ &\cong ((1 \star \emptyset) \star \emptyset) \star \emptyset \\ &\cong (((\emptyset \star \emptyset) \star \emptyset) \star \emptyset) \star \emptyset, \end{aligned}$$

and so on.

- *Morphisms.* For each  $[i], [j] \in \text{Obj}(\mathfrak{n})$ , we have

$$\text{Hom}_{\mathfrak{n}}([i], [j]) \stackrel{\text{def}}{=} \begin{cases} \{\text{id}_{[i]}\} & \text{if } [i] = [j], \\ \{[i] \rightarrow [j]\} & \text{if } [j] < [i], \\ \emptyset & \text{if } [j] > [i]. \end{cases}$$

- *Identities.* For each  $[i] \in \text{Obj}(\mathfrak{n})$ , the unit map

$$\mathbb{1}_{[i]}^{\mathfrak{n}} : \text{pt} \rightarrow \text{Hom}_{\mathfrak{n}}([i], [i])$$

of  $\mathfrak{n}$  at  $[i]$  is defined by

$$\text{id}_{[i]}^{\mathfrak{n}} \stackrel{\text{def}}{=} \text{id}_{[i]}.$$

- *Composition.* For each  $[i], [j], [k] \in \text{Obj}(\mathfrak{n})$ , the composition map

$$\circ_{[i],[j],[k]}^{\mathfrak{n}} : \text{Hom}_{\mathfrak{n}}([j], [k]) \times \text{Hom}_{\mathfrak{n}}([i], [j]) \rightarrow \text{Hom}_{\mathfrak{n}}([i], [k])$$

of  $\mathfrak{n}$  at  $([i], [j], [k])$  is defined by

$$\begin{aligned} \text{id}_{[i]} \circ \text{id}_{[i]} &= \text{id}_{[i]}, \\ ([j] \rightarrow [k]) \circ ([i] \rightarrow [j]) &= ([i] \rightarrow [k]). \end{aligned}$$

**Example 1.2.1.5.** Here we list some of the other categories appearing throughout this work.

1. The category  $\text{Sets}_*$  of pointed sets of [Pointed Sets, Definition 1.3.1.1](#).
2. The category  $\text{Rel}$  of sets and relations of [Relations, Definition 2.1.1.1](#).
3. The category  $\text{Span}(A, B)$  of spans from a set  $A$  to a set  $B$  of [??, ??](#).
4. The category  $\text{ISets}(K)$  of  $K$ -indexed sets of [??, ??](#).
5. The category  $\text{ISets}$  of indexed sets of [??, ??](#).
6. The category  $\text{FibSets}(K)$  of  $K$ -fibred sets of [??, ??](#).
7. The category  $\text{FibSets}$  of fibred sets of [??, ??](#).
8. Categories of functors  $\text{Fun}(C, \mathcal{D})$  as in [Definition 9.1.1.1](#).
9. The category of categories  $\text{Cats}$  of [Definition 9.2.1.1](#).
10. The category of groupoids  $\text{Grpd}$  of [Definition 9.4.1.1](#).



### 1.3 Posetal Categories

**Definition 1.3.1.1.** Let  $(X, \preceq_X)$  be a poset.

1. The **posetal category associated to**  $(X, \preceq_X)$  is the category  $X_{\text{pos}}$  where

- *Objects.* We have

$$\text{Obj}(X_{\text{pos}}) \stackrel{\text{def}}{=} X.$$

- *Morphisms.* For each  $a, b \in \text{Obj}(X_{\text{pos}})$ , we have

$$\text{Hom}_{X_{\text{pos}}}(a, b) \stackrel{\text{def}}{=} \begin{cases} \text{pt} & \text{if } a \preceq_X b, \\ \emptyset & \text{otherwise.} \end{cases}$$

- *Identities.* For each  $a \in \text{Obj}(X_{\text{pos}})$ , the unit map

$$\mathbb{1}_a^{X_{\text{pos}}} : \text{pt} \rightarrow \text{Hom}_{X_{\text{pos}}}(a, a)$$

of  $X_{\text{pos}}$  at  $a$  is given by the identity map.

- *Composition.* For each  $a, b, c \in \text{Obj}(X_{\text{pos}})$ , the composition map

$$\circ_{a,b,c}^{X_{\text{pos}}} : \text{Hom}_{X_{\text{pos}}}(b, c) \times \text{Hom}_{X_{\text{pos}}}(a, b) \rightarrow \text{Hom}_{X_{\text{pos}}}(a, c)$$

of  $X_{\text{pos}}$  at  $(a, b, c)$  is defined as either the inclusion  $\emptyset \hookrightarrow \text{pt}$  or the identity map of  $\text{pt}$ , depending on whether we have  $a \preceq_X b$ ,  $b \preceq_X c$ , and  $a \preceq_X c$ .

2. A category  $C$  is **posetal**<sup>4</sup> if  $C$  is equivalent to  $X_{\text{pos}}$  for some poset  $(X, \preceq_X)$ .

**Proposition 1.3.1.2.** Let  $(X, \preceq_X)$  be a poset and let  $C$  be a category.

1. *Functoriality.* The assignment  $(X, \preceq_X) \mapsto X_{\text{pos}}$  defines a functor

$$(-)_{\text{pos}} : \text{Pos} \rightarrow \text{Cats}.$$

2. *Fully Faithfulness.* The functor  $(-)_{\text{pos}}$  of **Item 1** is fully faithful.

3. *Characterisations.* The following conditions are equivalent:

---

<sup>4</sup>*Further Terminology:* Also called a **thin** category or a  $(0, 1)$ -category.

- (a) The category  $C$  is posetal.
- (b) For each  $A, B \in \text{Obj}(C)$  and each  $f, g \in \text{Hom}_C(A, B)$ , we have  $f = g$ .

*Proof.* **Item 1**, Functoriality: Omitted.

**Item 2**, Fully Faithfulness: Omitted.

**Item 3**, Characterisations: Clear. □

## 1.4 Subcategories

Let  $C$  be a category.

**Definition 1.4.1.1.** A **subcategory** of  $C$  is a category  $\mathcal{A}$  satisfying the following conditions:

1. *Objects.* We have  $\text{Obj}(\mathcal{A}) \subset \text{Obj}(C)$ .
2. *Morphisms.* For each  $A, B \in \text{Obj}(\mathcal{A})$ , we have

$$\text{Hom}_{\mathcal{A}}(A, B) \subset \text{Hom}_C(A, B).$$

3. *Identities.* For each  $A \in \text{Obj}(\mathcal{A})$ , we have

$$\mathbb{1}_A^{\mathcal{A}} = \mathbb{1}_A^C.$$

4. *Composition.* For each  $A, B, C \in \text{Obj}(\mathcal{A})$ , we have

$$\circ_{A,B,C}^{\mathcal{A}} = \circ_{A,B,C}^C.$$

**Definition 1.4.1.2.** A subcategory  $\mathcal{A}$  of  $C$  is **full** if the canonical inclusion functor  $\mathcal{A} \rightarrow C$  is full, i.e. if, for each  $A, B \in \text{Obj}(\mathcal{A})$ , the inclusion

$$\iota_{A,B}: \text{Hom}_{\mathcal{A}}(A, B) \hookrightarrow \text{Hom}_C(A, B)$$

is surjective (and thus bijective).

**Definition 1.4.1.3.** A subcategory  $\mathcal{A}$  of a category  $C$  is **strictly full** if it satisfies the following conditions:

1. *Fullness.* The subcategory  $\mathcal{A}$  is full.

2. *Closedness Under Isomorphisms.* The class  $\text{Obj}(\mathcal{A})$  is closed under isomorphisms.<sup>5</sup>

**Definition 1.4.1.4.** A subcategory  $\mathcal{A}$  of  $\mathcal{C}$  is **wide**<sup>6</sup> if  $\text{Obj}(\mathcal{A}) = \text{Obj}(\mathcal{C})$ .

## 1.5 Skeletons of Categories

**Definition 1.5.1.1.** A<sup>7</sup> **skeleton** of a category  $\mathcal{C}$  is a full subcategory  $\text{Sk}(\mathcal{C})$  with one object from each isomorphism class of objects of  $\mathcal{C}$ .

**Definition 1.5.1.2.** A category  $\mathcal{C}$  is **skeletal** if  $\mathcal{C} \cong \text{Sk}(\mathcal{C})$ .<sup>8</sup>

**Proposition 1.5.1.3.** Let  $\mathcal{C}$  be a category.

1. *Existence.* Assuming the axiom of choice,  $\text{Sk}(\mathcal{C})$  always exists.
2. *Pseudofunctoriality.* The assignment  $\mathcal{C} \mapsto \text{Sk}(\mathcal{C})$  defines a pseudofunctor

$$\text{Sk}: \text{Cats}_2 \rightarrow \text{Cats}_2.$$

3. *Uniqueness Up to Equivalence.* Any two skeletons of  $\mathcal{C}$  are equivalent.
4. *Inclusions of Skeletons Are Equivalences.* The inclusion

$$\iota_{\mathcal{C}}: \text{Sk}(\mathcal{C}) \hookrightarrow \mathcal{C}$$

of a skeleton of  $\mathcal{C}$  into  $\mathcal{C}$  is an equivalence of categories.

*Proof.* **Item 1, Existence:** See [nLab23, Section “Existence of Skeletons of Categories”].

**Item 2, Pseudofunctoriality:** See [nLab23, Section “Skeletons as an Endo-Pseudofunctor on  $\mathbf{Cat}$ ”].

**Item 3, Uniqueness Up to Equivalence:** Clear.

**Item 4, Inclusions of Skeletons Are Equivalences:** Clear. □

<sup>5</sup>That is, given  $A \in \text{Obj}(\mathcal{A})$  and  $C \in \text{Obj}(\mathcal{C})$ , if  $C \cong A$ , then  $C \in \text{Obj}(\mathcal{A})$ .

<sup>6</sup>*Further Terminology:* Also called **lluf**.

<sup>7</sup>Due to **Item 3** of **Proposition 1.5.1.3**, we often refer to any such full subcategory  $\text{Sk}(\mathcal{C})$  of  $\mathcal{C}$  as *the skeleton of  $\mathcal{C}$* .

<sup>8</sup>That is,  $\mathcal{C}$  is **skeletal** if isomorphic objects of  $\mathcal{C}$  are equal.

### 1.6 Precomposition and Postcomposition

Let  $C$  be a category and let  $A, B, C \in \text{Obj}(C)$ .

**Definition 1.6.1.1.** Let  $f: A \rightarrow B$  and  $g: B \rightarrow C$  be morphisms of  $C$ .

1. The **precomposition function associated to  $f$**  is the function

$$f^*: \text{Hom}_C(B, C) \rightarrow \text{Hom}_C(A, C)$$

defined by

$$f^*(\phi) \stackrel{\text{def}}{=} \phi \circ f$$

for each  $\phi \in \text{Hom}_C(B, C)$ .

2. The **postcomposition function associated to  $g$**  is the function

$$g_*: \text{Hom}_C(A, B) \rightarrow \text{Hom}_C(A, C)$$

defined by

$$g_*(\phi) \stackrel{\text{def}}{=} g \circ \phi$$

for each  $\phi \in \text{Hom}_C(A, B)$ .

**Proposition 1.6.1.2.** Let  $A, B, C, D \in \text{Obj}(C)$  and let  $f: A \rightarrow B$  and  $g: B \rightarrow C$  be morphisms of  $C$ .

1. *Interaction Between Precomposition and Postcomposition.* We have

$$\begin{array}{ccc}
 \text{Hom}_C(B, C) & \xrightarrow{g_*} & \text{Hom}_C(B, D) \\
 f^* \downarrow & & \downarrow f^* \\
 \text{Hom}_C(A, C) & \xrightarrow{g_*} & \text{Hom}_C(A, D).
 \end{array}$$

$g_* \circ f^* = f^* \circ g_*$

2. *Interaction With Composition I.* We have

$$\begin{array}{ccc}
 \text{Hom}_C(X, A) & \xrightarrow{f^*} & \text{Hom}_C(X, B) \\
 & \searrow (g \circ f)_* & \downarrow g_* \\
 & & \text{Hom}_C(X, C), \\
 \\ 
 \text{Hom}_C(C, X) & \xrightarrow{g^*} & \text{Hom}_C(B, X) \\
 & \searrow (g \circ f)^* & \downarrow f^* \\
 & & \text{Hom}_C(A, X).
 \end{array}$$

$(g \circ f)^* = f^* \circ g^*,$   
 $(g \circ f)_* = g_* \circ f_*,$

3. *Interaction With Composition II.* We have

$$\begin{array}{ccc}
 \text{pt} \xrightarrow{[f]} \text{Hom}_C(A, B) & & \text{pt} \xrightarrow{[g]} \text{Hom}_C(B, C) \\
 \searrow [g \circ f] \quad \downarrow g_* & [g \circ f] = g_* \circ [f], & \searrow [g \circ f] \quad \downarrow f_* \\
 \text{Hom}_C(A, C) & [g \circ f] = f_* \circ [g], & \text{Hom}_C(A, C).
 \end{array}$$

4. *Interaction With Composition III.* We have

$$\begin{array}{ccc}
 \text{Hom}_C(B, C) \times \text{Hom}_C(A, B) & \xrightarrow{\circ_{A,B,C}^C} & \text{Hom}_C(A, C) \\
 \downarrow \text{id} \times f_* & & \downarrow f_* \\
 \text{Hom}_C(B, C) \times \text{Hom}_C(X, B) & \xrightarrow{\circ_{X,B,C}^C} & \text{Hom}_C(X, C), \\
 \\ 
 \text{Hom}_C(B, C) \times \text{Hom}_C(A, B) & \xrightarrow{\circ_{A,B,C}^C} & \text{Hom}_C(A, C) \\
 \downarrow g_* \times \text{id} & & \downarrow g_* \\
 \text{Hom}_C(B, D) \times \text{Hom}_C(A, B) & \xrightarrow{\circ_{A,B,D}^C} & \text{Hom}_C(A, D).
 \end{array}$$

$f^* \circ \circ_{A,B,C}^C = \circ_{X,B,C}^C \circ (f^* \times \text{id}),$   
 $g_* \circ \circ_{A,B,C}^C = \circ_{A,B,D}^C \circ (\text{id} \times g_*),$

5. *Interaction With Identities.* We have

$$\begin{aligned}
 (\text{id}_A)^* &= \text{id}_{\text{Hom}_C(A, B)}, \\
 (\text{id}_B)_* &= \text{id}_{\text{Hom}_C(A, B)}.
 \end{aligned}$$

*Proof.* **Item 1**, Interaction Between Precomposition and Postcomposition: Clear.

**Item 2**, Interaction With Composition I: Clear.

**Item 3**, Interaction With Composition II: Clear.

**Item 4**, Interaction With Composition III: Clear.

**Item 5**, Interaction With Identities: Clear.  $\square$

## 2 The Quadruple Adjunction With Sets

### 2.1 Statement

Let  $\mathcal{C}$  be a category.

**Proposition 2.1.1.1.** We have a quadruple adjunction

$$(\pi_0 \dashv (-)_{\text{disc}} \dashv \text{Obj} \dashv (-)_{\text{indisc}}): \text{Sets} \begin{array}{c} \xrightarrow{\pi_0} \\ \perp \\ \xrightarrow{(-)_{\text{disc}}} \\ \perp \\ \xrightarrow{\text{Obj}} \\ \perp \\ \xrightarrow{(-)_{\text{indisc}}} \end{array} \text{Cats},$$

witnessed by bijections of sets

$$\begin{aligned} \text{Hom}_{\text{Sets}}(\pi_0(\mathcal{C}), X) &\cong \text{Hom}_{\text{Cats}}(\mathcal{C}, X_{\text{disc}}), \\ \text{Hom}_{\text{Cats}}(X_{\text{disc}}, \mathcal{C}) &\cong \text{Hom}_{\text{Sets}}(X, \text{Obj}(\mathcal{C})), \\ \text{Hom}_{\text{Sets}}(\text{Obj}(\mathcal{C}), X) &\cong \text{Hom}_{\text{Cats}}(\mathcal{C}, X_{\text{indisc}}), \end{aligned}$$

natural in  $\mathcal{C} \in \text{Obj}(\text{Cats})$  and  $X \in \text{Obj}(\text{Sets})$ , where

- The functor

$$\pi_0: \text{Cats} \rightarrow \text{Sets},$$

the **connected components functor**, is the functor sending a category to its set of connected components of **Definition 2.2.2.1**.

- The functor

$$(-)_{\text{disc}}: \text{Sets} \rightarrow \text{Cats},$$

the **discrete category functor**, is the functor sending a set to its associated discrete category of **Item 1**.

- The functor

$$\text{Obj} : \text{Cats} \rightarrow \text{Sets},$$

the **object functor**, is the functor sending a category to its set of objects.

- The functor

$$(-)_{\text{indisc}} : \text{Sets} \rightarrow \text{Cats},$$

the **indiscrete category functor**, is the functor sending a set to its associated indiscrete category of [Item 1](#).

*Proof.* Omitted. □

## 2.2 Connected Components and Connected Categories

### 2.2.1 Connected Components of Categories

Let  $C$  be a category.

**Definition 2.2.1.1.** A **connected component** of  $C$  is a full subcategory  $\mathcal{I}$  of  $C$  satisfying the following conditions:<sup>9</sup>

1. *Non-Emptiness.* We have  $\text{Obj}(\mathcal{I}) \neq \emptyset$ .
2. *Connectedness.* There exists a zigzag of arrows between any two objects of  $\mathcal{I}$ .

### 2.2.2 Sets of Connected Components of Categories

Let  $C$  be a category.

**Definition 2.2.2.1.** The **set of connected components of  $C$**  is the set  $\pi_0(C)$  whose elements are the connected components of  $C$ .

**Proposition 2.2.2.2.** Let  $C$  be a category.

1. *Functoriality.* The assignment  $C \mapsto \pi_0(C)$  defines a functor

$$\pi_0 : \text{Cats} \rightarrow \text{Sets}.$$

---

<sup>9</sup>In other words, a **connected component** of  $C$  is an element of the set  $\text{Obj}(C)/\sim$  with  $\sim$  the equivalence relation generated by the relation  $\sim'$  obtained by declaring  $A \sim' B$  iff there exists a morphism of  $C$  from  $A$  to  $B$ .

2. *Adjointness.* We have a quadruple adjunction

$$(\pi_0 \dashv (-)_{\text{disc}} \dashv \text{Obj} \dashv (-)_{\text{indisc}}): \text{Sets} \begin{array}{c} \xrightarrow{\pi_0} \\ \dashv \quad \perp \\ \xrightarrow{(-)_{\text{disc}}} \\ \dashv \quad \perp \\ \xrightarrow{\text{Obj}} \\ \dashv \quad \perp \\ \xrightarrow{(-)_{\text{indisc}}} \end{array} \text{Cats.}$$

3. *Interaction With Groupoids.* If  $C$  is a groupoid, then we have an isomorphism of categories

$$\pi_0(C) \cong K(C),$$

where  $K(C)$  is the set of isomorphism classes of  $C$  of ??.

4. *Preservation of Colimits.* The functor  $\pi_0$  of [Item 1](#) preserves colimits. In particular, we have bijections of sets

$$\begin{aligned} \pi_0(C \amalg \mathcal{D}) &\cong \pi_0(C) \amalg \pi_0(\mathcal{D}), \\ \pi_0(C \amalg_{\mathcal{E}} \mathcal{D}) &\cong \pi_0(C) \amalg_{\pi_0(\mathcal{E})} \pi_0(\mathcal{D}), \\ \pi_0(\text{CoEq}(C \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{D})) &\cong \text{CoEq}(\pi_0(C) \xrightarrow{\pi_0(F)} \pi_0(\mathcal{D}) \xrightarrow{\pi_0(G)} \pi_0(\mathcal{D})), \end{aligned}$$

natural in  $C, \mathcal{D}, \mathcal{E} \in \text{Obj}(\text{Cats})$ .

5. *Symmetric Strong Monoidality With Respect to Coproducts.* The connected components functor of [Item 1](#) has a symmetric strong monoidal structure

$$(\pi_0, \pi_0^{\amalg}, \pi_{0|1}): (\text{Cats}, \amalg, \emptyset_{\text{cat}}) \rightarrow (\text{Sets}, \amalg, \emptyset),$$

being equipped with isomorphisms

$$\begin{aligned} \pi_{0|C, \mathcal{D}}^{\amalg}: \pi_0(C) \amalg \pi_0(\mathcal{D}) &\xrightarrow{\cong} \pi_0(C \amalg \mathcal{D}), \\ \pi_{0|1}^{\amalg}: \emptyset &\xrightarrow{\cong} \pi_0(\emptyset_{\text{cat}}), \end{aligned}$$

natural in  $C, \mathcal{D} \in \text{Obj}(\text{Cats})$ .

6. *Symmetric Strong Monoidality With Respect to Products.* The connected components functor of [Item 1](#) has a symmetric strong monoidal structure

$$(\pi_0, \pi_0^{\times}, \pi_{0|1}^{\times}): (\text{Cats}, \times, \text{pt}) \rightarrow (\text{Sets}, \times, \text{pt}),$$



being equipped with isomorphisms

$$\begin{aligned}\pi_{0|C,\mathcal{D}}^\times : \pi_0(C) \times \pi_0(\mathcal{D}) &\xrightarrow{\cong} \pi_0(C \times \mathcal{D}), \\ \pi_{0|1}^\times : \text{pt} &\xrightarrow{\cong} \pi_0(\text{pt}),\end{aligned}$$

natural in  $C, \mathcal{D} \in \text{Obj}(\text{Cats})$ .

*Proof.* **Item 1**, Functoriality: Clear.

**Item 2**, Adjointness: This is proved in **Proposition 2.1.1.1**.

**Item 3**, Interaction With Groupoids: Clear.

**Item 4**, Preservation of Colimits: This follows from **Item 2** and ?? of ??.

**Item 5**, Symmetric Strong Monoidality With Respect to Coproducts: Clear.

**Item 6**, Symmetric Strong Monoidality With Respect to Products: Clear.  $\square$

### 2.2.3 Connected Categories

**Definition 2.2.3.1.** A category  $C$  is **connected** if  $\pi_0(C) \cong \text{pt}$ .<sup>10,11</sup>

## 2.3 Discrete Categories

**Definition 2.3.1.1.** Let  $X$  be a set.

1. The **discrete category on  $X$**  is the category  $X_{\text{disc}}$  where

- *Objects.* We have

$$\text{Obj}(X_{\text{disc}}) \stackrel{\text{def}}{=} X.$$

- *Morphisms.* For each  $A, B \in \text{Obj}(X_{\text{disc}})$ , we have

$$\text{Hom}_{X_{\text{disc}}}(A, B) \stackrel{\text{def}}{=} \begin{cases} \text{id}_A & \text{if } A = B, \\ \emptyset & \text{if } A \neq B. \end{cases}$$

- *Identities.* For each  $A \in \text{Obj}(X_{\text{disc}})$ , the unit map

$$\mathbb{1}_A^{X_{\text{disc}}} : \text{pt} \rightarrow \text{Hom}_{X_{\text{disc}}}(A, A)$$

of  $X_{\text{disc}}$  at  $A$  is defined by

$$\text{id}_A^{X_{\text{disc}}} \stackrel{\text{def}}{=} \text{id}_A.$$

<sup>10</sup>Further Terminology: A category is **disconnected** if it is not connected.

<sup>11</sup>Example: A groupoid is connected iff any two of its objects are isomorphic.

· *Composition.* For each  $A, B, C \in \text{Obj}(X_{\text{disc}})$ , the composition map

$$\circ_{A,B,C}^{X_{\text{disc}}} : \text{Hom}_{X_{\text{disc}}}(B, C) \times \text{Hom}_{X_{\text{disc}}}(A, B) \rightarrow \text{Hom}_{X_{\text{disc}}}(A, C)$$

of  $X_{\text{disc}}$  at  $(A, B, C)$  is defined by

$$\text{id}_A \circ \text{id}_B \stackrel{\text{def}}{=} \text{id}_A.$$

2. A category  $C$  is **discrete** if it is equivalent to  $X_{\text{disc}}$  for some set  $X$ .

**Proposition 2.3.1.2.** Let  $X$  be a set.

1. *Functoriality.* The assignment  $X \mapsto X_{\text{disc}}$  defines a functor

$$(-)_{\text{disc}} : \text{Sets} \rightarrow \text{Cats}.$$

2. *Adjointness.* We have a quadruple adjunction

$$(\pi_0 \dashv (-)_{\text{disc}} \dashv \text{Obj} \dashv (-)_{\text{indisc}}) : \text{Sets} \begin{array}{c} \xrightarrow{\pi_0} \\ \downarrow \perp \\ \xrightarrow{(-)_{\text{disc}}} \\ \downarrow \perp \\ \xleftarrow{\text{Obj}} \\ \downarrow \perp \\ \xleftarrow{(-)_{\text{indisc}}} \end{array} \text{Cats}.$$

3. *Symmetric Strong Monoidality With Respect to Coproducts.* The functor of **Item 1** has a symmetric strong monoidal structure

$$((-)_{\text{disc}}, (-)_{\text{disc}}^{\coprod}, (-)_{\text{disc}|\mathbb{1}}^{\coprod}) : (\text{Sets}, \coprod, \emptyset) \rightarrow (\text{Cats}, \coprod, \emptyset_{\text{cat}}),$$

being equipped with isomorphisms

$$\begin{aligned} (-)_{\text{disc}|\mathbb{1}}^{\coprod} : X_{\text{disc}} \coprod Y_{\text{disc}} &\xrightarrow{\cong} (X \coprod Y)_{\text{disc}}, \\ (-)_{\text{disc}|\mathbb{1}}^{\coprod} : \emptyset_{\text{cat}} &\xrightarrow{\cong} \emptyset_{\text{disc}}, \end{aligned}$$

natural in  $X, Y \in \text{Obj}(\text{Sets})$ .

4. *Symmetric Strong Monoidality With Respect to Products.* The functor of **Item 1** has a symmetric strong monoidal structure

$$((-)_{\text{disc}}, (-)_{\text{disc}}^{\times}, (-)_{\text{disc}|\mathbb{1}}^{\times}) : (\text{Sets}, \times, \text{pt}) \rightarrow (\text{Cats}, \times, \text{pt}),$$

being equipped with isomorphisms

$$\begin{aligned} (-)_{\text{disc}|X,Y}^{\times} : X_{\text{disc}} \times Y_{\text{disc}} &\xrightarrow{\cong} (X \times Y)_{\text{disc}}, \\ (-)_{\text{disc}|\mathbb{1}}^{\times} : \text{pt} &\xrightarrow{\cong} \text{pt}_{\text{disc}}, \end{aligned}$$

natural in  $X, Y \in \text{Obj}(\text{Sets})$ .

*Proof.* **Item 1**, Functoriality: Clear.

**Item 2**, Adjointness: This is proved in **Proposition 2.1.1.1**.

**Item 3**, Symmetric Strong Monoidality With Respect to Coproducts: Clear.

**Item 4**, Symmetric Strong Monoidality With Respect to Products: Clear.  $\square$

## 2.4 Indiscrete Categories

**Definition 2.4.1.1.** Let  $X$  be a set.

1. The **indiscrete category on  $X$** <sup>12</sup> is the category  $X_{\text{indisc}}$  where

- *Objects.* We have

$$\text{Obj}(X_{\text{indisc}}) \stackrel{\text{def}}{=} X.$$

- *Morphisms.* For each  $A, B \in \text{Obj}(X_{\text{indisc}})$ , we have

$$\begin{aligned} \text{Hom}_{X_{\text{disc}}}(A, B) &\stackrel{\text{def}}{=} \{[A] \rightarrow [B]\} \\ &\cong \text{pt}. \end{aligned}$$

- *Identities.* For each  $A \in \text{Obj}(X_{\text{indisc}})$ , the unit map

$$\mathbb{1}_A^{X_{\text{indisc}}} : \text{pt} \rightarrow \text{Hom}_{X_{\text{indisc}}}(A, A)$$

of  $X_{\text{indisc}}$  at  $A$  is defined by

$$\text{id}_A^{X_{\text{indisc}}} \stackrel{\text{def}}{=} \{[A] \rightarrow [A]\}.$$

- *Composition.* For each  $A, B, C \in \text{Obj}(X_{\text{indisc}})$ , the composition map

$$\circ_{A,B,C}^{X_{\text{indisc}}} : \text{Hom}_{X_{\text{indisc}}}(B, C) \times \text{Hom}_{X_{\text{indisc}}}(A, B) \rightarrow \text{Hom}_{X_{\text{indisc}}}(A, C)$$

of  $X_{\text{disc}}$  at  $(A, B, C)$  is defined by

$$([B] \rightarrow [C]) \circ ([A] \rightarrow [B]) \stackrel{\text{def}}{=} ([A] \rightarrow [C]).$$

---

<sup>12</sup>Further Terminology: Sometimes called the **chaotic category on  $X$** .

2. A category  $C$  is **indiscrete** if it is equivalent to  $X_{\text{indisc}}$  for some set  $X$ .

**Proposition 2.4.1.2.** Let  $X$  be a set.

1. *Functoriality.* The assignment  $X \mapsto X_{\text{indisc}}$  defines a functor

$$(-)_{\text{indisc}} : \text{Sets} \rightarrow \text{Cats}.$$

2. *Adjointness.* We have a quadruple adjunction

$$(\pi_0 \dashv (-)_{\text{disc}} \dashv \text{Obj} \dashv (-)_{\text{indisc}}) : \text{Sets} \begin{array}{c} \xrightarrow{\pi_0} \\ \downarrow \perp \\ \xrightarrow{(-)_{\text{disc}}} \\ \downarrow \perp \\ \xrightarrow{\text{Obj}} \\ \downarrow \perp \\ \xrightarrow{(-)_{\text{indisc}}} \end{array} \text{Cats}.$$

3. *Symmetric Strong Monoidality With Respect to Products.* The functor of **Item 1** has a symmetric strong monoidal structure

$$((-)_{\text{indisc}}, (-)_{\text{indisc}}^{\times}, (-)_{\text{indisc}|\mathbb{1}}^{\times}) : (\text{Sets}, \times, \text{pt}) \rightarrow (\text{Cats}, \times, \text{pt}),$$

being equipped with isomorphisms

$$\begin{aligned} (-)_{\text{indisc}|\mathbb{1}}^{\times} : X_{\text{indisc}} \times Y_{\text{indisc}} &\xrightarrow{\cong} (X \times Y)_{\text{indisc}}, \\ (-)_{\text{indisc}|\mathbb{1}}^{\times} : \text{pt} &\xrightarrow{\cong} \text{pt}_{\text{indisc}}, \end{aligned}$$

natural in  $X, Y \in \text{Obj}(\text{Sets})$ .

*Proof.* **Item 1, Functoriality:** Clear.

**Item 2, Adjointness:** This is proved in **Proposition 2.1.1.1**.

**Item 3, Symmetric Strong Monoidality With Respect to Products:** Clear. □

## 3 Groupoids

### 3.1 Foundations

Let  $C$  be a category.

**Definition 3.1.1.1.** A morphism  $f: A \rightarrow B$  of  $C$  is an **isomorphism** if there exists a morphism  $f^{-1}: B \rightarrow A$  of  $C$  such that

$$\begin{aligned} f \circ f^{-1} &= \text{id}_B, \\ f^{-1} \circ f &= \text{id}_A. \end{aligned}$$

**Notation 3.1.1.2.** We write  $\text{Iso}_C(A, B)$  for the set of all isomorphisms in  $C$  from  $A$  to  $B$ .

**Definition 3.1.1.3.** A **groupoid** is a category in which every morphism is an isomorphism.

### 3.2 The Groupoid Completion of a Category

Let  $C$  be a category.

**Definition 3.2.1.1.** The **groupoid completion of  $C$** <sup>13</sup> is the pair  $(K_0(C), \iota_C)$  consisting of

- A groupoid  $K_0(C)$ ;
- A functor  $\iota_C: C \rightarrow K_0(C)$ ;

satisfying the following universal property:<sup>14</sup>

(UP) Given another such pair  $(\mathcal{G}, i)$ , there exists a unique functor  $K_0(C) \xrightarrow{\exists!} \mathcal{G}$  making the diagram

$$\begin{array}{ccc} & & K_0(C) \\ & \nearrow \iota_C & \downarrow \exists! \\ C & \xrightarrow{i} & \mathcal{G} \end{array}$$

commute.

**Construction 3.2.1.2.** Concretely, the groupoid completion of  $C$  is the Gabriel–

<sup>13</sup>*Further Terminology:* Also called the **Grothendieck groupoid of  $C$**  or the **Grothendieck groupoid completion of  $C$** .

<sup>14</sup>See [Item 5](#) of [Proposition 3.2.1.3](#) for an explicit construction.

Zisman localisation  $\text{Mor}(C)^{-1}C$  of  $C$  at the set  $\text{Mor}(C)$  of all morphisms of  $C$ ; see ??, ??.

(To be expanded upon later on.)

*Proof.* Omitted. □

**Proposition 3.2.1.3.** Let  $C$  be a category.

1. *Functoriality.* The assignment  $C \mapsto K_0(C)$  defines a functor

$$K_0 : \text{Cats} \rightarrow \text{Grpd}.$$

2. *2-Functoriality.* The assignment  $C \mapsto K_0(C)$  defines a 2-functor

$$K_0 : \text{Cats}_2 \rightarrow \text{Grpd}_2.$$

3. *Adjointness.* We have an adjunction

$$(K_0 \dashv \iota) : \text{Cats} \begin{array}{c} \xrightarrow{K_0} \\ \perp \\ \xleftarrow{\iota} \end{array} \text{Grpd},$$

witnessed by a bijection of sets

$$\text{Hom}_{\text{Grpd}}(K_0(C), \mathcal{G}) \cong \text{Hom}_{\text{Cats}}(C, \mathcal{G}),$$

natural in  $C \in \text{Obj}(\text{Cats})$  and  $\mathcal{G} \in \text{Obj}(\text{Grpd})$ , forming, together with the functor  $\text{Core}$  of **Item 1** of **Proposition 3.3.1.4**, a triple adjunction

$$(K_0 \dashv \iota \dashv \text{Core}) : \text{Cats} \begin{array}{c} \xrightarrow{K_0} \\ \perp \\ \xleftarrow{\iota} \\ \perp \\ \xrightarrow{\text{Core}} \end{array} \text{Grpd},$$

witnessed by bijections of sets

$$\begin{aligned} \text{Hom}_{\text{Grpd}}(K_0(C), \mathcal{G}) &\cong \text{Hom}_{\text{Cats}}(C, \mathcal{G}), \\ \text{Hom}_{\text{Cats}}(\mathcal{G}, \mathcal{D}) &\cong \text{Hom}_{\text{Grpd}}(\mathcal{G}, \text{Core}(\mathcal{D})), \end{aligned}$$

natural in  $C, \mathcal{D} \in \text{Obj}(\text{Cats})$  and  $\mathcal{G} \in \text{Obj}(\text{Grpd})$ .

4. *2-Adjointness*. We have a 2-adjunction

$$(K_0 \dashv \iota): \text{Cats} \begin{array}{c} \xrightarrow{K_0} \\ \xleftarrow[\iota]{\perp_2} \end{array} \text{Grpd},$$

witnessed by an isomorphism of categories

$$\text{Fun}(K_0(C), \mathcal{G}) \cong \text{Fun}(C, \mathcal{G}),$$

natural in  $C \in \text{Obj}(\text{Cats})$  and  $\mathcal{G} \in \text{Obj}(\text{Grpd})$ , forming, together with the 2-functor  $\text{Core}$  of [Item 2](#) of [Proposition 3.3.1.4](#), a triple 2-adjunction

$$(K_0 \dashv \iota \dashv \text{Core}): \text{Cats} \begin{array}{c} \xrightarrow{K_0} \\ \xleftarrow[\iota]{\perp_2} \\ \xrightarrow[\text{Core}]{\perp_2} \end{array} \text{Grpd},$$

witnessed by isomorphisms of categories

$$\begin{aligned} \text{Fun}(K_0(C), \mathcal{G}) &\cong \text{Fun}(C, \mathcal{G}), \\ \text{Fun}(\mathcal{G}, \mathcal{D}) &\cong \text{Fun}(\mathcal{G}, \text{Core}(\mathcal{D})), \end{aligned}$$

natural in  $C, \mathcal{D} \in \text{Obj}(\text{Cats})$  and  $\mathcal{G} \in \text{Obj}(\text{Grpd})$ .

5. *Interaction With Classifying Spaces*. We have an isomorphism of groupoids

$$K_0(C) \cong \Pi_{\leq 1}(|N_\bullet(C)|),$$

natural in  $C \in \text{Obj}(\text{Cats})$ ; i.e. the diagram

$$\begin{array}{ccc} \text{Cats} & \xrightarrow{K_0} & \text{Grp} \\ \downarrow N_\bullet & \uparrow \text{isom} & \uparrow \Pi_{\leq 1} \\ \text{sSets} & \xrightarrow{|\cdot|} & \text{Top} \end{array}$$

commutes up to natural isomorphism.

6. *Symmetric Strong Monoidality With Respect to Coproducts*. The groupoid completion functor of [Item 1](#) has a symmetric strong monoidal structure

$$(K_0, K_0^{\coprod}, K_0^{\coprod}): (\text{Cats}, \coprod, \emptyset_{\text{cat}}) \rightarrow (\text{Grpd}, \coprod, \emptyset_{\text{cat}})$$

being equipped with isomorphisms

$$\begin{aligned} K_{0|C, \mathcal{D}}^{\coprod} : K_0(C) \amalg K_0(\mathcal{D}) &\xrightarrow{\cong} K_0(C \amalg \mathcal{D}), \\ K_{0|\mathbb{1}}^{\coprod} : \emptyset_{\text{cat}} &\xrightarrow{\cong} K_0(\emptyset_{\text{cat}}), \end{aligned}$$

natural in  $C, \mathcal{D} \in \text{Obj}(\text{Cats})$ .

7. *Symmetric Strong Monoidality With Respect to Products*. The groupoid completion functor of **Item 1** has a symmetric strong monoidal structure

$$(K_0, K_0^\times, K_{0|\mathbb{1}}^\times) : (\text{Cats}, \times, \text{pt}) \rightarrow (\text{Grpd}, \times, \text{pt})$$

being equipped with isomorphisms

$$\begin{aligned} K_{0|C, \mathcal{D}}^\times : K_0(C) \times K_0(\mathcal{D}) &\xrightarrow{\cong} K_0(C \times \mathcal{D}), \\ K_{0|\mathbb{1}}^\times : \text{pt} &\xrightarrow{\cong} K_0(\text{pt}), \end{aligned}$$

natural in  $C, \mathcal{D} \in \text{Obj}(\text{Cats})$ .

*Proof.* **Item 1**, *Functoriality*: Omitted.

**Item 2**, *2-Functoriality*: Omitted.

**Item 3**, *Adjointness*: Omitted.

**Item 4**, *2-Adjointness*: Omitted.

**Item 5**, *Interaction With Classifying Spaces*: See Corollary 18.33 of <https://web.ma.utexas.edu/users/dafr/M392C-2012/Notes/lecture18.pdf>.

**Item 6**, *Symmetric Strong Monoidality With Respect to Coproducts*: Omitted.

**Item 7**, *Symmetric Strong Monoidality With Respect to Products*: Omitted.  $\square$

### 3.3 The Core of a Category

Let  $C$  be a category.

**Definition 3.3.1.1.** The **core** of  $C$  is the pair  $(\text{Core}(C), \iota_C)$  consisting of

- A groupoid  $\text{Core}(C)$ ;
- A functor  $\iota_C : \text{Core}(C) \hookrightarrow C$ ;

satisfying the following universal property:



(UP) Given another such pair  $(\mathcal{G}, i)$ , there exists a unique functor  $\mathcal{G} \xrightarrow{\exists!} \text{Core}(C)$  making the diagram

$$\begin{array}{ccc} & \text{Core}(C) & \\ \nearrow \exists! & \downarrow \iota_C & \\ \mathcal{G} & \xrightarrow{i} & C \end{array}$$

commute.

**Notation 3.3.1.2.** We also write  $C^\simeq$  for  $\text{Core}(C)$ .

**Construction 3.3.1.3.** The core of  $C$  is the wide subcategory of  $C$  spanned by the isomorphisms of  $C$ , i.e. the category  $\text{Core}(C)$  where<sup>15</sup>

1. *Objects.* We have

$$\text{Obj}(\text{Core}(C)) \stackrel{\text{def}}{=} \text{Obj}(C).$$

2. *Morphisms.* The morphisms of  $\text{Core}(C)$  are the isomorphisms of  $C$ .

*Proof.* This follows from the fact that functors preserve isomorphisms (Item 1 of Proposition 4.1.1.6).  $\square$

**Proposition 3.3.1.4.** Let  $C$  be a category.

1. *Functoriality.* The assignment  $C \mapsto \text{Core}(C)$  defines a functor

$$\text{Core}: \text{Cats} \rightarrow \text{Grpd}.$$

2. *2-Functoriality.* The assignment  $C \mapsto \text{Core}(C)$  defines a 2-functor

$$\text{Core}: \text{Cats}_2 \rightarrow \text{Grpd}_2.$$

3. *Adjointness.* We have an adjunction

$$(\iota \dashv \text{Core}): \text{Grpd} \begin{array}{c} \xrightarrow{\iota} \\ \perp \\ \xleftarrow{\text{Core}} \end{array} \text{Cats},$$

<sup>15</sup>*Slogan:* The groupoid  $\text{Core}(C)$  is the maximal subgroupoid of  $C$ .

witnessed by a bijection of sets

$$\mathrm{Hom}_{\mathrm{Cats}}(\mathcal{G}, \mathcal{D}) \cong \mathrm{Hom}_{\mathrm{Grpd}}(\mathcal{G}, \mathrm{Core}(\mathcal{D})),$$

natural in  $\mathcal{G} \in \mathrm{Obj}(\mathrm{Grpd})$  and  $\mathcal{D} \in \mathrm{Obj}(\mathrm{Cats})$ , forming, together with the functor  $K_0$  of [Item 1](#) of [Proposition 3.2.1.3](#), a triple adjunction

$$(K_0 \dashv \iota \dashv \mathrm{Core}): \quad \mathrm{Cats} \begin{array}{c} \xrightarrow{K_0} \\ \perp \\ \xleftarrow{\iota} \\ \perp \\ \xrightarrow{\mathrm{Core}} \end{array} \mathrm{Grpd},$$

witnessed by bijections of sets

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Grpd}}(K_0(C), \mathcal{G}) &\cong \mathrm{Hom}_{\mathrm{Cats}}(C, \mathcal{G}), \\ \mathrm{Hom}_{\mathrm{Cats}}(\mathcal{G}, \mathcal{D}) &\cong \mathrm{Hom}_{\mathrm{Grpd}}(\mathcal{G}, \mathrm{Core}(\mathcal{D})), \end{aligned}$$

natural in  $C, \mathcal{D} \in \mathrm{Obj}(\mathrm{Cats})$  and  $\mathcal{G} \in \mathrm{Obj}(\mathrm{Grpd})$ .

4. *2-Adjointness*. We have an adjunction

$$(\iota \dashv \mathrm{Core}): \quad \mathrm{Grpd} \begin{array}{c} \xrightarrow{\iota} \\ \perp_2 \\ \xleftarrow{\mathrm{Core}} \end{array} \mathrm{Cats},$$

witnessed by an isomorphism of categories

$$\mathrm{Fun}(\mathcal{G}, \mathcal{D}) \cong \mathrm{Fun}(\mathcal{G}, \mathrm{Core}(\mathcal{D})),$$

natural in  $\mathcal{G} \in \mathrm{Obj}(\mathrm{Grpd})$  and  $\mathcal{D} \in \mathrm{Obj}(\mathrm{Cats})$ , forming, together with the 2-functor  $K_0$  of [Item 2](#) of [Proposition 3.2.1.3](#), a triple 2-adjunction

$$(K_0 \dashv \iota \dashv \mathrm{Core}): \quad \mathrm{Cats} \begin{array}{c} \xrightarrow{K_0} \\ \perp_2 \\ \xleftarrow{\iota} \\ \perp_2 \\ \xrightarrow{\mathrm{Core}} \end{array} \mathrm{Grpd},$$

witnessed by isomorphisms of categories

$$\begin{aligned} \mathrm{Fun}(K_0(C), \mathcal{G}) &\cong \mathrm{Fun}(C, \mathcal{G}), \\ \mathrm{Fun}(\mathcal{G}, \mathcal{D}) &\cong \mathrm{Fun}(\mathcal{G}, \mathrm{Core}(\mathcal{D})), \end{aligned}$$

natural in  $C, \mathcal{D} \in \mathrm{Obj}(\mathrm{Cats})$  and  $\mathcal{G} \in \mathrm{Obj}(\mathrm{Grpd})$ .

5. *Symmetric Strong Monoidality With Respect to Products.* The core functor of **Item 1** has a symmetric strong monoidal structure

$$(\text{Core}, \text{Core}^\times, \text{Core}_{\mathbb{1}}^\times) : (\text{Cats}, \times, \text{pt}) \rightarrow (\text{Grpd}, \times, \text{pt})$$

being equipped with isomorphisms

$$\begin{aligned} \text{Core}_{C, \mathcal{D}}^\times : \text{Core}(C) \times \text{Core}(\mathcal{D}) &\xrightarrow{\cong} \text{Core}(C \times \mathcal{D}), \\ \text{Core}_{\mathbb{1}}^\times : \text{pt} &\xrightarrow{\cong} \text{Core}(\text{pt}), \end{aligned}$$

natural in  $C, \mathcal{D} \in \text{Obj}(\text{Cats})$ .

6. *Symmetric Strong Monoidality With Respect to Coproducts.* The core functor of **Item 1** has a symmetric strong monoidal structure

$$(\text{Core}, \text{Core}^{\amalg}, \text{Core}_{\mathbb{1}}^{\amalg}) : (\text{Cats}, \amalg, \emptyset_{\text{cat}}) \rightarrow (\text{Grpd}, \amalg, \emptyset_{\text{cat}})$$

being equipped with isomorphisms

$$\begin{aligned} \text{Core}_{C, \mathcal{D}}^{\amalg} : \text{Core}(C) \amalg \text{Core}(\mathcal{D}) &\xrightarrow{\cong} \text{Core}(C \amalg \mathcal{D}), \\ \text{Core}_{\mathbb{1}}^{\amalg} : \emptyset_{\text{cat}} &\xrightarrow{\cong} \text{Core}(\emptyset_{\text{cat}}), \end{aligned}$$

natural in  $C, \mathcal{D} \in \text{Obj}(\text{Cats})$ .

*Proof.* **Item 1**, *Functoriality*: Omitted.

**Item 2**, *2-Functoriality*: Omitted.

**Item 3**, *Adjointness*: Omitted.

**Item 4**, *2-Adjointness*: Omitted.

**Item 5**, *Symmetric Strong Monoidality With Respect to Products*: Omitted.

**Item 6**, *Symmetric Strong Monoidality With Respect to Coproducts*: Omitted.  $\square$

## 4 Functors

### 4.1 Foundations

Let  $C$  and  $\mathcal{D}$  be categories.

**Definition 4.1.1.1.** A **functor**  $F : C \rightarrow \mathcal{D}$  **from**  $C$  **to**  $\mathcal{D}$ <sup>16</sup> consists of:

<sup>16</sup>*Further Terminology:* Also called a **covariant functor**.

1. *Action on Objects.* A map of sets

$$F: \text{Obj}(\mathcal{C}) \rightarrow \text{Obj}(\mathcal{D}),$$

called the **action on objects of  $F$** .

2. *Action on Morphisms.* For each  $A, B \in \text{Obj}(\mathcal{C})$ , a map

$$F_{A,B}: \text{Hom}_{\mathcal{C}}(A, B) \rightarrow \text{Hom}_{\mathcal{D}}(F(A), F(B)),$$

called the **action on morphisms of  $F$  at  $(A, B)$** <sup>17</sup>.

satisfying the following conditions:

1. *Preservation of Identities.* For each  $A \in \text{Obj}(\mathcal{C})$ , the diagram

$$\begin{array}{ccc} \text{pt} & & \\ \downarrow \mathbb{1}_A^{\mathcal{C}} & \searrow \mathbb{1}_{F(A)}^{\mathcal{D}} & \\ \text{Hom}_{\mathcal{C}}(A, A) & \xrightarrow{F_{A,A}} & \text{Hom}_{\mathcal{D}}(F(A), F(A)) \end{array}$$

commutes, i.e. we have

$$F(\text{id}_A) = \text{id}_{F(A)}.$$

2. *Preservation of Composition.* For each  $A, B, C \in \text{Obj}(\mathcal{C})$ , the diagram

$$\begin{array}{ccc} \text{Hom}_{\mathcal{C}}(B, C) \times \text{Hom}_{\mathcal{C}}(A, B) & \xrightarrow{\circ_{A,B,C}^{\mathcal{C}}} & \text{Hom}_{\mathcal{C}}(A, C) \\ \downarrow F_{B,C} \times F_{A,B} & & \downarrow F_{A,C} \\ \text{Hom}_{\mathcal{D}}(F(B), F(C)) \times \text{Hom}_{\mathcal{D}}(F(A), F(B)) & \xrightarrow{\circ_{F(A),F(B),F(C)}^{\mathcal{D}}} & \text{Hom}_{\mathcal{D}}(F(A), F(C)) \end{array}$$

commutes, i.e. for each composable pair  $(g, f)$  of morphisms of  $\mathcal{C}$ , we have

$$F(g \circ f) = F(g) \circ F(f).$$

---

<sup>17</sup>*Further Terminology:* Also called **action on Hom-sets of  $F$  at  $(A, B)$** .

**Notation 4.1.1.2.** Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories, and write  $\mathcal{C}^{\text{op}}$  for the opposite category of  $\mathcal{C}$  of ??, ??.

1. Given a functor

$$F: \mathcal{C} \rightarrow \mathcal{D},$$

we also write  $F_A$  for  $F(A)$ .

2. Given a functor

$$F: \mathcal{C}^{\text{op}} \rightarrow \mathcal{D},$$

we also write  $F^A$  for  $F(A)$ .

3. Given a functor

$$F: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{D},$$

we also write  $F_{A,B}$  for  $F(A, B)$ .

4. Given a functor

$$F: \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{D},$$

we also write  $F_B^A$  for  $F(A, B)$ .

We employ a similar notation for morphisms, writing e.g.  $F_f$  for  $F(f)$  given a functor  $F: \mathcal{C} \rightarrow \mathcal{D}$ .

**Notation 4.1.1.3.** Following the notation  $\llbracket x \mapsto f(x) \rrbracket$  for a function  $f: X \rightarrow Y$  introduced in [Sets](#), [Notation 1.1.1.2](#), we will sometimes denote a functor  $F: \mathcal{C} \rightarrow \mathcal{D}$  by

$$F \stackrel{\text{def}}{=} \llbracket A \mapsto F(A) \rrbracket,$$

specially when the action on morphisms of  $F$  is clear from its action on objects.

**Example 4.1.1.4.** The **identity functor** of a category  $\mathcal{C}$  is the functor  $\text{id}_{\mathcal{C}}: \mathcal{C} \rightarrow \mathcal{C}$  where

1. *Action on Objects.* For each  $A \in \text{Obj}(\mathcal{C})$ , we have

$$\text{id}_{\mathcal{C}}(A) \stackrel{\text{def}}{=} A.$$

2. *Action on Morphisms.* For each  $A, B \in \text{Obj}(\mathcal{C})$ , the action on morphisms

$$(\text{id}_{\mathcal{C}})_{A,B}: \text{Hom}_{\mathcal{C}}(A, B) \rightarrow \underbrace{\text{Hom}_{\mathcal{C}}(\text{id}_{\mathcal{C}}(A), \text{id}_{\mathcal{C}}(B))}_{\stackrel{\text{def}}{=} \text{Hom}_{\mathcal{C}}(A, B)}$$

of  $\text{id}_C$  at  $(A, B)$  is defined by

$$(\text{id}_C)_{A,B} \stackrel{\text{def}}{=} \text{id}_{\text{Hom}_C(A,B)}.$$

*Proof. Preservation of Identities:* We have  $\text{id}_C(\text{id}_A) \stackrel{\text{def}}{=} \text{id}_A$  for each  $A \in \text{Obj}(C)$  by definition.

*Preservation of Compositions:* For each composable pair  $A \xrightarrow{f} B \xrightarrow{g} C$  of morphisms of  $C$ , we have

$$\begin{aligned} \text{id}_C(g \circ f) &\stackrel{\text{def}}{=} g \circ f \\ &\stackrel{\text{def}}{=} \text{id}_C(g) \circ \text{id}_C(f). \end{aligned}$$

This finishes the proof.  $\square$

**Definition 4.1.1.5.** The **composition** of two functors  $F: C \rightarrow D$  and  $G: D \rightarrow E$  is the functor  $G \circ F$  where

- *Action on Objects.* For each  $A \in \text{Obj}(C)$ , we have

$$[G \circ F](A) \stackrel{\text{def}}{=} G(F(A)).$$

- *Action on Morphisms.* For each  $A, B \in \text{Obj}(C)$ , the action on morphisms

$$(G \circ F)_{A,B}: \text{Hom}_C(A, B) \rightarrow \text{Hom}_E(G_{F_A}, G_{F_B})$$

of  $G \circ F$  at  $(A, B)$  is defined by

$$[G \circ F](f) \stackrel{\text{def}}{=} G(F(f)).$$

*Proof. Preservation of Identities:* For each  $A \in \text{Obj}(C)$ , we have

$$\begin{aligned} G_{F_{\text{id}_A}} &= G_{\text{id}_{F_A}} && \text{(functoriality of } F) \\ &= \text{id}_{G_{F_A}}. && \text{(functoriality of } G) \end{aligned}$$

*Preservation of Composition:* For each composable pair  $(g, f)$  of morphisms of  $C$ , we have

$$\begin{aligned} G_{F_{g \circ f}} &= G_{F_g \circ F_f} && \text{(functoriality of } F) \\ &= G_{F_g} \circ G_{F_f}. && \text{(functoriality of } G) \end{aligned}$$

This finishes the proof.  $\square$

**Proposition 4.1.1.6.** Let  $F: \mathcal{C} \rightarrow \mathcal{D}$  be a functor.

1. *Preservation of Isomorphisms.* If  $f$  is an isomorphism in  $\mathcal{C}$ , then  $F(f)$  is an isomorphism in  $\mathcal{D}$ .<sup>18</sup>

*Proof.* **Item 1, Preservation of Isomorphisms:** Indeed, we have

$$\begin{aligned} F(f)^{-1} \circ F(f) &= F(f^{-1} \circ f) \\ &= F(\text{id}_A) \\ &= \text{id}_{F(A)} \end{aligned}$$

and

$$\begin{aligned} F(f) \circ F(f)^{-1} &= F(f \circ f^{-1}) \\ &= F(\text{id}_B) \\ &= \text{id}_{F(B)}, \end{aligned}$$

showing  $F(f)$  to be an isomorphism. □

## 4.2 Contravariant Functors

Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories, and let  $\mathcal{C}^{\text{op}}$  denote the opposite category of  $\mathcal{C}$  of ??, ??.

**Definition 4.2.1.1.** A **contravariant functor** from  $\mathcal{C}$  to  $\mathcal{D}$  is a functor from  $\mathcal{C}^{\text{op}}$  to  $\mathcal{D}$ .

**Remark 4.2.1.2.** In detail, a **contravariant functor** from  $\mathcal{C}$  to  $\mathcal{D}$  consists of:

1. *Action on Objects.* A map of sets

$$F: \text{Obj}(\mathcal{C}) \rightarrow \text{Obj}(\mathcal{D}),$$

called the **action on objects of  $F$** .

2. *Action on Morphisms.* For each  $A, B \in \text{Obj}(\mathcal{C})$ , a map

$$F_{A,B}: \text{Hom}_{\mathcal{C}}(A, B) \rightarrow \text{Hom}_{\mathcal{D}}(F(B), F(A)),$$

called the **action on morphisms of  $F$  at  $(A, B)$** .

---

<sup>18</sup>When the converse holds, we call  $F$  *conservative*, see [Definition 5.4.1.1](#).

satisfying the following conditions:

1. *Preservation of Identities.* For each  $A \in \text{Obj}(C)$ , the diagram

$$\begin{array}{ccc} \text{pt} & & \\ \downarrow \mathbb{1}_A^C & \searrow \mathbb{1}_{F(A)}^{\mathcal{D}} & \\ \text{Hom}_C(A, A) & \xrightarrow{F_{A,A}} & \text{Hom}_{\mathcal{D}}(F(A), F(A)) \end{array}$$

commutes, i.e. we have

$$F(\text{id}_A) = \text{id}_{F(A)}.$$

2. *Preservation of Composition.* For each  $A, B, C \in \text{Obj}(C)$ , the diagram

$$\begin{array}{ccccc} & & \text{Hom}_{\mathcal{D}}(F(C), F(B)) \times \text{Hom}_{\mathcal{D}}(F(B), F(A)) & & \\ & \nearrow F_{B,C} \times F_{A,B} & & \searrow \sigma_{\text{Hom}_{\mathcal{D}}(F(C), F(B)), \text{Hom}_{\mathcal{D}}(F(B), F(A))}^{\text{Sets}} & \\ \text{Hom}_C(B, C) \times \text{Hom}_C(A, B) & & & & \text{Hom}_{\mathcal{D}}(F(B), F(A)) \times \text{Hom}_{\mathcal{D}}(F(C), F(B)) \\ & \searrow \circ_{A,B,C}^C & & \swarrow \circ_{F(C), F(B), F(A)}^{\mathcal{D}} & \\ & \text{Hom}_C(A, C) & \xrightarrow{F_{A,C}} & \text{Hom}_{\mathcal{D}}(F(C), F(A)) & \end{array}$$

commutes, i.e. for each composable pair  $(g, f)$  of morphisms of  $C$ , we have

$$F(g \circ f) = F(f) \circ F(g).$$

**Remark 4.2.1.3.** Throughout this work we will not use the term “contravariant” functor, speaking instead simply of functors  $F: C^{\text{op}} \rightarrow \mathcal{D}$ . We will usually, however, write

$$F_{A,B}: \text{Hom}_C(A, B) \rightarrow \text{Hom}_{\mathcal{D}}(F(B), F(A))$$

for the action on morphisms

$$F_{A,B}: \text{Hom}_{C^{\text{op}}}(A, B) \rightarrow \text{Hom}_{\mathcal{D}}(F(A), F(B))$$

of  $F$ , as well as write  $F(g \circ f) = F(f) \circ F(g)$ .



### 4.3 Forgetful Functors

**Definition 4.3.1.1.** There isn't a precise definition of a **forgetful functor**.

**Remark 4.3.1.2.** Despite there not being a formal or precise definition of a forgetful functor, the term is often very useful in practice, similarly to the word “canonical”. The idea is that a “forgetful functor” is a functor that forgets structure or properties, and is best explained through examples, such as the ones below (see [Examples 4.3.1.3](#) and [4.3.1.4](#)).

**Example 4.3.1.3.** Examples of forgetful functors that forget structure include:

1. *Forgetting Group Structures.* The functor  $\text{Grp} \rightarrow \text{Sets}$  sending a group  $(G, \mu_G, \eta_G)$  to its underlying set  $G$ , forgetting the multiplication and unit maps  $\mu_G$  and  $\eta_G$  of  $G$ .
2. *Forgetting Topologies.* The functor  $\text{Top} \rightarrow \text{Sets}$  sending a topological space  $(X, \mathcal{T}_X)$  to its underlying set  $X$ , forgetting the topology  $\mathcal{T}_X$ .
3. *Forgetting Fibrations.* The functor  $\text{FibSets}(K) \rightarrow \text{Sets}$  sending a  $K$ -fibred set  $\phi_X: X \rightarrow K$  to the set  $X$ , forgetting the map  $\phi_X$  and the base set  $K$ .

**Example 4.3.1.4.** Examples of forgetful functors that forget properties include:

1. *Forgetting Commutativity.* The inclusion functor  $\iota: \text{CMon} \hookrightarrow \text{Mon}$  which forgets the property of being commutative.
2. *Forgetting Inverses.* The inclusion functor  $\iota: \text{Grp} \hookrightarrow \text{Mon}$  which forgets the property of having inverses.

**Notation 4.3.1.5.** Throughout this work, we will denote forgetful functors that forget structure by 忘, e.g. as in

$$\text{忘}: \text{Grp} \rightarrow \text{Sets}.$$

The symbol 忘, pronounced *wasureru* (see [Item 1](#) of [Remark 4.3.1.6](#) below), means *to forget*, and is a kanji found in the following words in Japanese and Chinese:

1. 忘れる, transcribed as *wasureru*, meaning *to forget*.
2. 忘却関手, transcribed as *boukyaku kanshu*, meaning *forgetful functor*.
3. 忘记 or 忘記, transcribed as *wàngjì*, meaning *to forget*.

4. 遗忘函子 or 遺忘函子, transcribed as *yíwàng hánzǐ*, meaning *forgetful functor*.

**Remark 4.3.1.6.** Here we collect the pronunciation of the words in **Notation 4.3.1.5** for accuracy and completeness.

1. Pronunciation of 忘れる:

- Audio: see <https://topological-modular-forms.github.io/the-clowder-project/static/sounds/wasureru-01.mp3>
- IPA broad transcription: [wäsurɕru].
- IPA narrow transcription: [wɿ̯⁵äsi̯⁵rɕru̯⁵].

2. Pronunciation of 忘却関手: Pronunciation:

- Audio: see <https://topological-modular-forms.github.io/the-clowder-project/static/sounds/wasureru-02.mp3>
- IPA broad transcription: [bɔ̯:käku kǎũɕu].
- IPA narrow transcription: [bɔ̯:käku̯⁵ kǎũɕu̯⁵].

3. Pronunciation of 忘记:

- Audio: see <https://topological-modular-forms.github.io/the-clowder-project/static/sounds/wasureru-03.ogg>
- Broad IPA transcription: [waŋtɕi].
- Sinological IPA transcription: [waŋ<sup>51-53</sup>tɕi<sup>51</sup>].

4. Pronunciation of 遗忘函子:

- Audio: see <https://topological-modular-forms.github.io/the-clowder-project/static/sounds/wasureru-04.mp3>
- Broad IPA transcription: [iwaŋ xǎntɕzi].
- Sinological IPA transcription: [i<sup>35</sup>waŋ<sup>51</sup> xǎn<sup>35</sup>tɕz<sup>214-21(4)</sup>].

#### 4.4 The Natural Transformation Associated to a Functor

**Definition 4.4.1.1.** Every functor  $F: \mathcal{C} \rightarrow \mathcal{D}$  defines a natural transformation<sup>19</sup>

$$F^\dagger: \text{Hom}_{\mathcal{C}} \Rightarrow \text{Hom}_{\mathcal{D}} \circ (F^{\text{op}} \times F),$$

$$\begin{array}{ccc} \mathcal{C}^{\text{op}} \times \mathcal{C} & \xrightarrow{F^{\text{op}} \times F} & \mathcal{D}^{\text{op}} \times \mathcal{D} \\ \text{Hom}_{\mathcal{C}} \searrow & \xRightarrow{F^\dagger} & \swarrow \text{Hom}_{\mathcal{D}} \\ & \text{Sets,} & \end{array}$$

called the **natural transformation associated to  $F$** , consisting of the collection

$$\left\{ F_{A,B}^\dagger: \text{Hom}_{\mathcal{C}}(A, B) \rightarrow \text{Hom}_{\mathcal{D}}(F_A, F_B) \right\}_{(A,B) \in \text{Obj}(\mathcal{C}^{\text{op}} \times \mathcal{C})}$$

with

$$F_{A,B}^\dagger \stackrel{\text{def}}{=} F_{A,B}.$$

*Proof.* The naturality condition for  $F^\dagger$  is the requirement that for each morphism

$$(\phi, \psi): (X, Y) \rightarrow (A, B)$$

of  $\mathcal{C}^{\text{op}} \times \mathcal{C}$ , the diagram

$$\begin{array}{ccc} \text{Hom}_{\mathcal{C}}(X, Y) & \xrightarrow{\phi^* \circ \psi_* = \psi_* \circ \phi^*} & \text{Hom}_{\mathcal{C}}(A, B) \\ F_{X,Y} \downarrow & & \downarrow F_{A,B} \\ \text{Hom}_{\mathcal{D}}(F_X, F_Y) & \xrightarrow{F(\phi)^* \circ F(\psi)_* = F(\psi)_* \circ F(\phi)^*} & \text{Hom}_{\mathcal{D}}(F_A, F_B), \end{array}$$

acting on elements as

$$\begin{array}{ccc} f & \longmapsto & \psi \circ f \circ \phi \\ \downarrow & & \downarrow \\ F(f) & \longmapsto & F(\psi) \circ F(f) \circ F(\psi) = F(\psi \circ f \circ \phi) \end{array}$$

commutes, which follows from the functoriality of  $F$ . □

<sup>19</sup>This is the 1-categorical version of [Constructions With Sets, Item 1](#) of [Proposition 4.1.1.3](#).

**Proposition 4.4.1.2.** Let  $F: \mathcal{C} \rightarrow \mathcal{D}$  and  $G: \mathcal{D} \rightarrow \mathcal{E}$  be functors.

1. *Interaction With Natural Isomorphisms.* The following conditions are equivalent:

- (a) The natural transformation  $F^\dagger: \text{Hom}_{\mathcal{C}} \Rightarrow \text{Hom}_{\mathcal{D}} \circ (F^{\text{op}} \times F)$  associated to  $F$  is a natural isomorphism.
- (b) The functor  $F$  is fully faithful.

2. *Interaction With Composition.* We have an equality of pasting diagrams

$$\begin{array}{ccc}
 \mathcal{C}^{\text{op}} \times \mathcal{C} & \xrightarrow{F^{\text{op}} \times F} & \mathcal{D}^{\text{op}} \times \mathcal{D} \xrightarrow{G^{\text{op}} \times G} \mathcal{E}^{\text{op}} \times \mathcal{E} \\
 \searrow \text{Hom}_{\mathcal{C}} & \nearrow F^\dagger & \downarrow \text{Hom}_{\mathcal{D}} \xrightarrow{G^\dagger} \text{Hom}_{\mathcal{E}} \\
 & \text{Hom}_{\mathcal{D}} & \text{Sets}
 \end{array}
 =
 \begin{array}{ccc}
 \mathcal{C}^{\text{op}} \times \mathcal{C} & \xrightarrow{(G \circ F)^{\text{op}} \times (G \circ F)} & \mathcal{E}^{\text{op}} \times \mathcal{E} \\
 \searrow \text{Hom}_{\mathcal{C}} & \nearrow (G \circ F)^\dagger & \downarrow \text{Hom}_{\mathcal{E}} \\
 & \text{Hom}_{\mathcal{E}} & \text{Sets}
 \end{array}$$

in  $\text{Cats}_2$ , i.e. we have

$$(G \circ F)^\dagger = (G^\dagger \star \text{id}_{F^{\text{op}} \times F}) \circ F^\dagger.$$

3. *Interaction With Identities.* We have

$$\text{id}_{\mathcal{C}}^\dagger = \text{id}_{\text{Hom}_{\mathcal{C}}(-1, -2)},$$

i.e. the natural transformation associated to  $\text{id}_{\mathcal{C}}$  is the identity natural transformation of the functor  $\text{Hom}_{\mathcal{C}}(-1, -2)$ .

*Proof.* **Item 1**, *Interaction With Natural Isomorphisms*: Clear.

**Item 2**, *Interaction With Composition*: Clear.

**Item 3**, *Interaction With Identities*: Clear. □

## 5 Conditions on Functors

### 5.1 Faithful Functors

Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories.

**Definition 5.1.1.1.** A functor  $F: \mathcal{C} \rightarrow \mathcal{D}$  is **faithful** if, for each  $A, B \in \text{Obj}(\mathcal{C})$ , the action on morphisms

$$F_{A,B}: \text{Hom}_{\mathcal{C}}(A, B) \rightarrow \text{Hom}_{\mathcal{D}}(F_A, F_B)$$

of  $F$  at  $(A, B)$  is injective.

**Proposition 5.1.1.2.** Let  $F: \mathcal{C} \rightarrow \mathcal{D}$  be a functor.

1. *Interaction With Postcomposition.* The following conditions are equivalent:

- (a) The functor  $F: \mathcal{C} \rightarrow \mathcal{D}$  is faithful.
- (b) For each  $\mathcal{X} \in \text{Obj}(\text{Cats})$ , the postcomposition functor

$$F_*: \text{Fun}(\mathcal{X}, \mathcal{C}) \rightarrow \text{Fun}(\mathcal{X}, \mathcal{D})$$

is faithful.

- (c) The functor  $F: \mathcal{C} \rightarrow \mathcal{D}$  is a representably faithful morphism in  $\text{Cats}_2$  in the sense of [Types of Morphisms in Bicategories, Definition 1.1.1.1](#).

2. *Interaction With Precomposition I.* Let  $F: \mathcal{C} \rightarrow \mathcal{D}$  be a functor.

- (a) If  $F$  is faithful, then the precomposition functor

$$F^*: \text{Fun}(\mathcal{D}, \mathcal{X}) \rightarrow \text{Fun}(\mathcal{C}, \mathcal{X})$$

can fail to be faithful.

- (b) Conversely, if the precomposition functor

$$F^*: \text{Fun}(\mathcal{D}, \mathcal{X}) \rightarrow \text{Fun}(\mathcal{C}, \mathcal{X})$$

is faithful, then  $F$  can fail to be faithful.

3. *Interaction With Precomposition II.* If  $F$  is essentially surjective, then the precomposition functor

$$F^*: \text{Fun}(\mathcal{D}, \mathcal{X}) \rightarrow \text{Fun}(\mathcal{C}, \mathcal{X})$$

is faithful.

4. *Interaction With Precomposition III.* The following conditions are equivalent:

- (a) For each  $\mathcal{X} \in \text{Obj}(\text{Cats})$ , the precomposition functor

$$F^*: \text{Fun}(\mathcal{D}, \mathcal{X}) \rightarrow \text{Fun}(\mathcal{C}, \mathcal{X})$$

is faithful.

- (b) For each  $\mathcal{X} \in \text{Obj}(\text{Cats})$ , the precomposition functor

$$F^* : \text{Fun}(\mathcal{D}, \mathcal{X}) \rightarrow \text{Fun}(\mathcal{C}, \mathcal{X})$$

is conservative.

- (c) For each  $\mathcal{X} \in \text{Obj}(\text{Cats})$ , the precomposition functor

$$F^* : \text{Fun}(\mathcal{D}, \mathcal{X}) \rightarrow \text{Fun}(\mathcal{C}, \mathcal{X})$$

is monadic.

- (d) The functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  is a corepresentably faithful morphism in  $\text{Cats}_2$  in the sense of [Types of Morphisms in Bicategories, Definition 2.1.1.1](#).

- (e) The components

$$\eta_G : G \Longrightarrow \text{Ran}_F(G \circ F)$$

of the unit

$$\eta : \text{id}_{\text{Fun}(\mathcal{D}, \mathcal{X})} \Longrightarrow \text{Ran}_F \circ F^*$$

of the adjunction  $F^* \dashv \text{Ran}_F$  are all monomorphisms.

- (f) The components

$$\epsilon_G : \text{Lan}_F(G \circ F) \Longrightarrow G$$

of the counit

$$\epsilon : \text{Lan}_F \circ F^* \Longrightarrow \text{id}_{\text{Fun}(\mathcal{D}, \mathcal{X})}$$

of the adjunction  $\text{Lan}_F \dashv F^*$  are all epimorphisms.

- (g) The functor  $F$  is dominant ([Definition 6.1.1.1](#)), i.e. every object of  $\mathcal{D}$  is a retract of some object in  $\text{Im}(F)$ :

- (★) For each  $B \in \text{Obj}(\mathcal{D})$ , there exist:

- An object  $A$  of  $\mathcal{C}$ ;
- A morphism  $s : B \rightarrow F(A)$  of  $\mathcal{D}$ ;
- A morphism  $r : F(A) \rightarrow B$  of  $\mathcal{D}$ ;

such that  $r \circ s = \text{id}_B$ .

*Proof.* **Item 1**, *Interaction With Postcomposition*: Omitted.

**Item 2**, *Interaction With Precomposition I*: See [MSE 733163] for **Item 2a**. **Item 2b** follows from **Item 3** and the fact that there are essentially surjective functors that are not faithful.

**Item 3**, *Interaction With Precomposition II*: Omitted, but see [https://unimath.github.io/doc/UniMath/d4de26f//UniMath.CategoryTheory.precomp\\_fully\\_faithful.html](https://unimath.github.io/doc/UniMath/d4de26f//UniMath.CategoryTheory.precomp_fully_faithful.html) for a formalised proof.

**Item 4**, *Interaction With Precomposition III*: We claim **Items 4a** to **4g** are equivalent:

- **Items 4a** and **4d** Are Equivalent: This is true by the definition of corepresentably faithful morphism; see **Types of Morphisms in Bicategories**, Definition 2.1.1.1.
- **Items 4a** to **4c** and **4g** Are Equivalent: See [Adá+01, Proposition 4.1] or alternatively [Fre09, Lemmas 3.1 and 3.2] for the equivalence between **Items 4a** and **4g**.
- **Items 4a**, **4e** and **4f** Are Equivalent: See ??, ?? of ??.

This finishes the proof. □

## 5.2 Full Functors

Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories.

**Definition 5.2.1.1.** A functor  $F: \mathcal{C} \rightarrow \mathcal{D}$  is **full** if, for each  $A, B \in \text{Obj}(\mathcal{C})$ , the action on morphisms

$$F_{A,B}: \text{Hom}_{\mathcal{C}}(A, B) \rightarrow \text{Hom}_{\mathcal{D}}(F_A, F_B)$$

of  $F$  at  $(A, B)$  is surjective.

**Proposition 5.2.1.2.** Let  $F: \mathcal{C} \rightarrow \mathcal{D}$  be a functor.

1. *Interaction With Postcomposition.* The following conditions are equivalent:

- (a) The functor  $F: \mathcal{C} \rightarrow \mathcal{D}$  is full.
- (b) For each  $\mathcal{X} \in \text{Obj}(\text{Cats})$ , the postcomposition functor

$$F_*: \text{Fun}(\mathcal{X}, \mathcal{C}) \rightarrow \text{Fun}(\mathcal{X}, \mathcal{D})$$

is full.

- (c) The functor  $F: \mathcal{C} \rightarrow \mathcal{D}$  is a representably full morphism in  $\mathbf{Cats}_2$  in the sense of [Types of Morphisms in Bicategories, Definition 1.2.1.1](#).

2. *Interaction With Precomposition I.* If  $F$  is full, then the precomposition functor

$$F^*: \text{Fun}(\mathcal{D}, \mathcal{X}) \rightarrow \text{Fun}(\mathcal{C}, \mathcal{X})$$

can fail to be full.

3. *Interaction With Precomposition II.* If the precomposition functor

$$F^*: \text{Fun}(\mathcal{D}, \mathcal{X}) \rightarrow \text{Fun}(\mathcal{C}, \mathcal{X})$$

is full, then  $F$  can fail to be full.

4. *Interaction With Precomposition III.* If  $F$  is essentially surjective and full, then the precomposition functor

$$F^*: \text{Fun}(\mathcal{D}, \mathcal{X}) \rightarrow \text{Fun}(\mathcal{C}, \mathcal{X})$$

is full (and also faithful by [Item 3 of Proposition 5.1.1.2](#)).

5. *Interaction With Precomposition IV.* The following conditions are equivalent:

- (a) For each  $\mathcal{X} \in \text{Obj}(\mathbf{Cats})$ , the precomposition functor

$$F^*: \text{Fun}(\mathcal{D}, \mathcal{X}) \rightarrow \text{Fun}(\mathcal{C}, \mathcal{X})$$

is full.

- (b) The functor  $F: \mathcal{C} \rightarrow \mathcal{D}$  is a corepresentably full morphism in  $\mathbf{Cats}_2$  in the sense of [Types of Morphisms in Bicategories, Definition 2.1.1.1](#).

- (c) The components

$$\eta_G: G \Longrightarrow \text{Ran}_F(G \circ F)$$

of the unit

$$\eta: \text{id}_{\text{Fun}(\mathcal{D}, \mathcal{X})} \Longrightarrow \text{Ran}_F \circ F^*$$

of the adjunction  $F^* \dashv \text{Ran}_F$  are all retractions/split epimorphisms.



(d) The components

$$\epsilon_G : \text{Lan}_F(G \circ F) \Longrightarrow G$$

of the counit

$$\epsilon : \text{Lan}_F \circ F^* \Longrightarrow \text{id}_{\text{Fun}(\mathcal{D}, \mathcal{C})}$$

of the adjunction  $\text{Lan}_F \dashv F^*$  are all sections/split monomorphisms.

(e) For each  $B \in \text{Obj}(\mathcal{D})$ , there exist:

- An object  $A_B$  of  $\mathcal{C}$ ;
- A morphism  $s_B : B \rightarrow F(A_B)$  of  $\mathcal{D}$ ;
- A morphism  $r_B : F(A_B) \rightarrow B$  of  $\mathcal{D}$ ;

satisfying the following condition:

(★) For each  $A \in \text{Obj}(\mathcal{C})$  and each pair of morphisms

$$\begin{aligned} r : F(A) &\rightarrow B, \\ s : B &\rightarrow F(A) \end{aligned}$$

of  $\mathcal{D}$ , we have

$$[(A_B, s_B, r_B)] = [(A, s, r \circ s_B \circ r_B)]$$

$$\text{in } \int^{A \in \mathcal{C}} h_{F_A}^{B'} \times h_B^{F_A}.$$

*Proof.* **Item 1**, Interaction With Postcomposition: Omitted.

**Item 2**, Interaction With Precomposition I: Omitted.

**Item 3**, Interaction With Precomposition II: See [BS10, p. 47].

**Item 4**, Interaction With Precomposition III: Omitted, but see [https://unimath.github.io/doc/UniMath/d4de26f//UniMath.CategoryTheory.precomp\\_fully\\_faithful.html](https://unimath.github.io/doc/UniMath/d4de26f//UniMath.CategoryTheory.precomp_fully_faithful.html) for a formalised proof.

**Item 5**, Interaction With Precomposition IV: We claim **Items 5a** to **5e** are equivalent:

- **Items 5a** and **5b** Are Equivalent: This is true by the definition of corepresentably full morphism; see **Types of Morphisms in Bicategories**, Definition 2.2.1.1.
- **Items 5a**, **5c** and **5d** Are Equivalent: See ??, ?? of ??.

· *Items 5a and 5e Are Equivalent:* See [Adá+01, Item (b) of Remark 4.3].

This finishes the proof.  $\square$

**Question 5.2.1.3.** Item 5 of Proposition 5.2.1.2 gives a characterisation of the functors  $F$  for which  $F^*$  is full, but the characterisations given there are really messy. Are there better ones?

This question also appears as [MO 468121b].

### 5.3 Fully Faithful Functors

Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories.

**Definition 5.3.1.1.** A functor  $F: \mathcal{C} \rightarrow \mathcal{D}$  is **fully faithful** if  $F$  is full and faithful, i.e. if, for each  $A, B \in \text{Obj}(\mathcal{C})$ , the action on morphisms

$$F_{A,B}: \text{Hom}_{\mathcal{C}}(A, B) \rightarrow \text{Hom}_{\mathcal{D}}(F_A, F_B)$$

of  $F$  at  $(A, B)$  is bijective.

**Proposition 5.3.1.2.** Let  $F: \mathcal{C} \rightarrow \mathcal{D}$  be a functor.

1. *Characterisations.* The following conditions are equivalent:

- (a) The functor  $F$  is fully faithful.
- (b) We have a pullback square

$$\begin{array}{ccc} \text{Arr}(\mathcal{C}) & \xrightarrow{\text{Arr}(F)} & \text{Arr}(\mathcal{D}) \\ \text{src} \times \text{tgt} \downarrow & \lrcorner & \downarrow \text{src} \times \text{tgt} \\ \mathcal{C} \times \mathcal{C} & \xrightarrow{F \times F} & \mathcal{D} \times \mathcal{D} \end{array}$$

$\text{Arr}(\mathcal{C}) \cong (\mathcal{C} \times \mathcal{C}) \times_{\mathcal{D} \times \mathcal{D}} \text{Arr}(\mathcal{D}),$

in  $\mathbf{Cats}$ .

- 2. *Conservativity.* If  $F$  is fully faithful, then  $F$  is conservative.
- 3. *Essential Injectivity.* If  $F$  is fully faithful, then  $F$  is essentially injective.
- 4. *Interaction With Co/Limits.* If  $F$  is fully faithful, then  $F$  reflects co/limits.
- 5. *Interaction With Postcomposition.* The following conditions are equivalent:

- (a) The functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  is fully faithful.
- (b) For each  $\mathcal{X} \in \text{Obj}(\text{Cats})$ , the postcomposition functor

$$F_* : \text{Fun}(\mathcal{X}, \mathcal{C}) \rightarrow \text{Fun}(\mathcal{X}, \mathcal{D})$$

is fully faithful.

- (c) The functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  is a representably fully faithful morphism in  $\text{Cats}_2$  in the sense of [Types of Morphisms in Bicategories, Definition 1.3.1.1](#).

6. *Interaction With Precomposition I.* If  $F$  is fully faithful, then the precomposition functor

$$F^* : \text{Fun}(\mathcal{D}, \mathcal{X}) \rightarrow \text{Fun}(\mathcal{C}, \mathcal{X})$$

can fail to be fully faithful.

7. *Interaction With Precomposition II.* If the precomposition functor

$$F^* : \text{Fun}(\mathcal{D}, \mathcal{X}) \rightarrow \text{Fun}(\mathcal{C}, \mathcal{X})$$

is fully faithful, then  $F$  can fail to be fully faithful (and in fact it can also fail to be either full or faithful).

8. *Interaction With Precomposition III.* If  $F$  is essentially surjective and full, then the precomposition functor

$$F^* : \text{Fun}(\mathcal{D}, \mathcal{X}) \rightarrow \text{Fun}(\mathcal{C}, \mathcal{X})$$

is fully faithful.

9. *Interaction With Precomposition IV.* The following conditions are equivalent:

- (a) For each  $\mathcal{X} \in \text{Obj}(\text{Cats})$ , the precomposition functor

$$F^* : \text{Fun}(\mathcal{D}, \mathcal{X}) \rightarrow \text{Fun}(\mathcal{C}, \mathcal{X})$$

is fully faithful.

- (b) The precomposition functor

$$F^* : \text{Fun}(\mathcal{D}, \text{Sets}) \rightarrow \text{Fun}(\mathcal{C}, \text{Sets})$$

is fully faithful.

(c) The functor

$$\text{Lan}_F : \text{Fun}(C, \text{Sets}) \rightarrow \text{Fun}(\mathcal{D}, \text{Sets})$$

is fully faithful.

(d) The functor  $F$  is a corepresentably fully faithful morphism in  $\text{Cats}_2$  in the sense of [Types of Morphisms in Bicategories, Definition 2.3.1.1](#).

(e) The functor  $F$  is absolutely dense.

(f) The components

$$\eta_G : G \Longrightarrow \text{Ran}_F(G \circ F)$$

of the unit

$$\eta : \text{id}_{\text{Fun}(\mathcal{D}, \mathcal{X})} \Longrightarrow \text{Ran}_F \circ F^*$$

of the adjunction  $F^* \dashv \text{Ran}_F$  are all isomorphisms.

(g) The components

$$\epsilon_G : \text{Lan}_F(G \circ F) \Longrightarrow G$$

of the counit

$$\epsilon : \text{Lan}_F \circ F^* \Longrightarrow \text{id}_{\text{Fun}(\mathcal{D}, \mathcal{X})}$$

of the adjunction  $\text{Lan}_F \dashv F^*$  are all isomorphisms.

(h) The natural transformation

$$\alpha : \text{Lan}_{h_F}(h^F) \Longrightarrow h$$

with components

$$\alpha_{B', B} : \int^{A \in C} h_{F_A}^{B'} \times h_B^{F_A} \rightarrow h_B^{B'}$$

given by

$$\alpha_{B', B}([\langle \phi, \psi \rangle]) = \psi \circ \phi$$

is a natural isomorphism.

(i) For each  $B \in \text{Obj}(\mathcal{D})$ , there exist:

- An object  $A_B$  of  $\mathcal{C}$ ;
- A morphism  $s_B: B \rightarrow F(A_B)$  of  $\mathcal{D}$ ;
- A morphism  $r_B: F(A_B) \rightarrow B$  of  $\mathcal{D}$ ;

satisfying the following conditions:

- i. The triple  $(F(A_B), r_B, s_B)$  is a retract of  $B$ , i.e. we have  $r_B \circ s_B = \text{id}_B$ .
- ii. For each morphism  $f: B' \rightarrow B$  of  $\mathcal{D}$ , we have

$$[(A_B, s_{B'}, f \circ r_{B'})] = [(A_B, s_B \circ f, r_B)]$$

$$\text{in } \int^{A \in \mathcal{C}} h_{F_A}^{B'} \times h_B^{F_A}.$$

*Proof.* **Item 1, Characterisations:** Omitted.

**Item 2, Conservativity:** This is a repetition of **Item 2** of **Proposition 5.4.1.2**, and is proved there.

**Item 3, Essential Injectivity:** Omitted.

**Item 4, Interaction With Co/Limits:** Omitted.

**Item 5, Interaction With Postcomposition:** This follows from **Item 1** of **Proposition 5.1.1.2** and **Item 1** of **Proposition 5.2.1.2**.

**Item 6, Interaction With Precomposition I:** See [MSE 733161] for an example of a fully faithful functor whose precomposition with which fails to be full.

**Item 7, Interaction With Precomposition II:** See [MSE 749304, Item 3].

**Item 8, Interaction With Precomposition III:** Omitted, but see [https://unimath.github.io/doc/UniMath/d4de26f//UniMath.CategoryTheory.precomp\\_fully\\_faithful.html](https://unimath.github.io/doc/UniMath/d4de26f//UniMath.CategoryTheory.precomp_fully_faithful.html) for a formalised proof.

**Item 9, Interaction With Precomposition IV:** We claim **Items 9a** to **9i** are equivalent:

- **Items 9a and 9d Are Equivalent:** This is true by the definition of corepresentably fully faithful morphism; see **Types of Morphisms in Bicategories, Definition 2.3.1.1**.
- **Items 9a, 9f and 9g Are Equivalent:** See ??, ?? of ??.
- **Items 9a to 9c Are Equivalent:** This follows from [Low15, Proposition A.1.5].
- **Items 9a, 9e, 9h and 9i Are Equivalent:** See [Fre09, Theorem 4.1] and [Adá+01, Theorem 1.1].

This finishes the proof. □

## 5.4 Conservative Functors

Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories.

**Definition 5.4.1.1.** A functor  $F: \mathcal{C} \rightarrow \mathcal{D}$  is **conservative** if it satisfies the following condition:<sup>20</sup>

- (★) For each  $f \in \text{Mor}(\mathcal{C})$ , if  $F(f)$  is an isomorphism in  $\mathcal{D}$ , then  $f$  is an isomorphism in  $\mathcal{C}$ .

**Proposition 5.4.1.2.** Let  $F: \mathcal{C} \rightarrow \mathcal{D}$  be a functor.

1. *Characterisations.* The following conditions are equivalent:
  - (a) The functor  $F$  is conservative.
  - (b) For each  $f \in \text{Mor}(\mathcal{C})$ , the morphism  $F(f)$  is an isomorphism in  $\mathcal{D}$  iff  $f$  is an isomorphism in  $\mathcal{C}$ .
2. *Interaction With Fully Faithfulness.* Every fully faithful functor is conservative.
3. *Interaction With Precomposition.* The following conditions are equivalent:
  - (a) For each  $\mathcal{X} \in \text{Obj}(\text{Cats})$ , the precomposition functor

$$F^*: \text{Fun}(\mathcal{D}, \mathcal{X}) \rightarrow \text{Fun}(\mathcal{C}, \mathcal{X})$$

is conservative.

- (b) The equivalent conditions of [Item 4](#) of [Proposition 5.1.1.2](#) are satisfied.

*Proof.* [Item 1, Characterisations:](#) This follows from [Item 1](#) of [Proposition 4.1.1.6](#).  
[Item 2, Interaction With Fully Faithfulness:](#) Let  $F: \mathcal{C} \rightarrow \mathcal{D}$  be a fully faithful functor, let  $f: A \rightarrow B$  be a morphism of  $\mathcal{C}$ , and suppose that  $F_f$  is an isomorphism. We have

$$\begin{aligned} F(\text{id}_B) &= \text{id}_{F(B)} \\ &= F(f) \circ F(f)^{-1} \\ &= F(f \circ f^{-1}). \end{aligned}$$

---

<sup>20</sup>*Slogan:* A functor  $F$  is **conservative** if it reflects isomorphisms.

Similarly,  $F(\text{id}_A) = F(f^{-1} \circ f)$ . But since  $F$  is fully faithful, we must have

$$\begin{aligned} f \circ f^{-1} &= \text{id}_B, \\ f^{-1} \circ f &= \text{id}_A, \end{aligned}$$

showing  $f$  to be an isomorphism. Thus  $F$  is conservative.  $\square$

**Question 5.4.1.3.** Is there a characterisation of functors  $F: C \rightarrow \mathcal{D}$  satisfying the following condition:

(★) For each  $X \in \text{Obj}(\text{Cats})$ , the postcomposition functor

$$F_*: \text{Fun}(X, C) \rightarrow \text{Fun}(X, \mathcal{D})$$

is conservative?

This question also appears as [MO 468121a].

## 5.5 Essentially Injective Functors

Let  $C$  and  $\mathcal{D}$  be categories.

**Definition 5.5.1.1.** A functor  $F: C \rightarrow \mathcal{D}$  is **essentially injective** if it satisfies the following condition:

(★) For each  $A, B \in \text{Obj}(C)$ , if  $F(A) \cong F(B)$ , then  $A \cong B$ .

**Question 5.5.1.2.** Is there a characterisation of functors  $F: C \rightarrow \mathcal{D}$  such that:

1. For each  $X \in \text{Obj}(\text{Cats})$ , the precomposition functor

$$F^*: \text{Fun}(\mathcal{D}, X) \rightarrow \text{Fun}(C, X)$$

is essentially injective, i.e. if  $\phi \circ F \cong \psi \circ F$ , then  $\phi \cong \psi$  for all functors  $\phi$  and  $\psi$ ?

2. For each  $X \in \text{Obj}(\text{Cats})$ , the postcomposition functor

$$F_*: \text{Fun}(X, C) \rightarrow \text{Fun}(X, \mathcal{D})$$

is essentially injective, i.e. if  $F \circ \phi \cong F \circ \psi$ , then  $\phi \cong \psi$ ?

This question also appears as [MO 468121a].

## 5.6 Essentially Surjective Functors

Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories.

**Definition 5.6.1.1.** A functor  $F: \mathcal{C} \rightarrow \mathcal{D}$  is **essentially surjective**<sup>21</sup> if it satisfies the following condition:

- (★) For each  $D \in \text{Obj}(\mathcal{D})$ , there exists some object  $A$  of  $\mathcal{C}$  such that  $F(A) \cong D$ .

**Question 5.6.1.2.** Is there a characterisation of functors  $F: \mathcal{C} \rightarrow \mathcal{D}$  such that:

1. For each  $\mathcal{X} \in \text{Obj}(\text{Cats})$ , the precomposition functor

$$F^*: \text{Fun}(\mathcal{D}, \mathcal{X}) \rightarrow \text{Fun}(\mathcal{C}, \mathcal{X})$$

is essentially surjective?

2. For each  $\mathcal{X} \in \text{Obj}(\text{Cats})$ , the postcomposition functor

$$F_*: \text{Fun}(\mathcal{X}, \mathcal{C}) \rightarrow \text{Fun}(\mathcal{X}, \mathcal{D})$$

is essentially surjective?

This question also appears as [MO 468121a].

## 5.7 Equivalences of Categories

**Definition 5.7.1.1.** Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories.

1. An **equivalence of categories** between  $\mathcal{C}$  and  $\mathcal{D}$  consists of a pair of functors

$$F: \mathcal{C} \rightarrow \mathcal{D},$$

$$G: \mathcal{D} \rightarrow \mathcal{C}$$

together with natural isomorphisms

$$\eta: \text{id}_{\mathcal{C}} \xrightarrow{\sim} G \circ F,$$

$$\epsilon: F \circ G \xrightarrow{\sim} \text{id}_{\mathcal{D}}.$$

<sup>21</sup> *Further Terminology:* Also called an **eso** functor, where the name “eso” comes from *essentially*



2. An **adjoint equivalence of categories** between  $\mathcal{C}$  and  $\mathcal{D}$  is an equivalence  $(F, G, \eta, \epsilon)$  between  $\mathcal{C}$  and  $\mathcal{D}$  which is also an adjunction.

**Proposition 5.7.1.2.** Let  $F: \mathcal{C} \rightarrow \mathcal{D}$  be a functor.

1. *Characterisations.* If  $\mathcal{C}$  and  $\mathcal{D}$  are small<sup>22</sup>, then the following conditions are equivalent:<sup>23</sup>

- (a) The functor  $F$  is an equivalence of categories.
- (b) The functor  $F$  is fully faithful and essentially surjective.
- (c) The induced functor

$$\uparrow F\text{Sk}(\mathcal{C}): \text{Sk}(\mathcal{C}) \rightarrow \text{Sk}(\mathcal{D})$$

is an *isomorphism* of categories.

- (d) For each  $X \in \text{Obj}(\text{Cats})$ , the precomposition functor

$$F^*: \text{Fun}(\mathcal{D}, X) \rightarrow \text{Fun}(\mathcal{C}, X)$$

is an equivalence of categories.

- (e) For each  $X \in \text{Obj}(\text{Cats})$ , the postcomposition functor

$$F_*: \text{Fun}(X, \mathcal{C}) \rightarrow \text{Fun}(X, \mathcal{D})$$

is an equivalence of categories.

2. *Two-Out-of-Three.* Let

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{G \circ F} & \mathcal{E} \\ & \searrow F \quad \nearrow G & \\ & \mathcal{D} & \end{array}$$

be a diagram in  $\text{Cats}$ . If two out of the three functors among  $F$ ,  $G$ , and  $G \circ F$  are equivalences of categories, then so is the third.

---

*surjective on objects.*

<sup>22</sup>Otherwise there will be size issues. One can also work with large categories and universes, or require  $F$  to be *constructively* essentially surjective; see [MSE 1465107].

<sup>23</sup>In ZFC, the equivalence between **Item 1a** and **Item 1b** is equivalent to the axiom of choice; see [MO 119454].

3. *Stability Under Composition.* Let

$$C \begin{array}{c} \xrightarrow{F} \\ \xleftarrow{G} \end{array} \mathcal{D} \begin{array}{c} \xrightarrow{F'} \\ \xleftarrow{G'} \end{array} \mathcal{E}$$

be a diagram in *Cats*. If  $(F, G)$  and  $(F', G')$  are equivalences of categories, then so is their composite  $(F' \circ F, G' \circ G)$ .

4. *Equivalences vs. Adjoint Equivalences.* Every equivalence of categories can be promoted to an adjoint equivalence.<sup>24</sup>

5. *Interaction With Groupoids.* If  $C$  and  $\mathcal{D}$  are groupoids, then the following conditions are equivalent:

- (a) The functor  $F$  is an equivalence of groupoids.
- (b) The following conditions are satisfied:
  - i. The functor  $F$  induces a bijection

$$\pi_0(F) : \pi_0(C) \rightarrow \pi_0(\mathcal{D})$$

of sets.

- ii. For each  $A \in \text{Obj}(C)$ , the induced map

$$F_{x,x} : \text{Aut}_C(A) \rightarrow \text{Aut}_{\mathcal{D}}(F_A)$$

is an isomorphism of groups.

*Proof.* **Item 1, Characterisations:** We claim that **Items 1a to 1e** are indeed equivalent:

1. **Item 1a**  $\implies$  **Item 1b**: Clear.
2. **Item 1b**  $\implies$  **Item 1a**: Since  $F$  is essentially surjective and  $C$  and  $\mathcal{D}$  are small, we can choose, using the axiom of choice, for each  $B \in \text{Obj}(\mathcal{D})$ , an object  $j_B$  of  $C$  and an isomorphism  $i_B : B \rightarrow F_{j_B}$  of  $\mathcal{D}$ . Since  $F$  is fully faithful, we can extend the assignment  $B \mapsto j_B$  to a *unique* functor  $j : \mathcal{D} \rightarrow C$  such that the isomorphisms  $i_B : B \rightarrow F_{j_B}$  assemble into a natural isomorphism  $\eta : \text{id}_{\mathcal{D}} \xrightarrow{\sim} F \circ j$ , with a similar natural isomorphism  $\epsilon : \text{id}_C \xrightarrow{\sim} j \circ F$ . Hence  $F$  is an equivalence.

---

In Univalent Foundations, this is true without requiring neither the axiom of choice nor the law of excluded middle.

<sup>24</sup>More precisely, we can promote an equivalence of categories  $(F, G, \eta, \epsilon)$  to adjoint equivalence

3. *Item 1a*  $\implies$  *Item 1c*: This follows from *Item 4* of *Proposition 1.5.1.3*.
4. *Item 1c*  $\implies$  *Item 1a*: Omitted.
5. *Items 1a, 1d and 1e Are Equivalent*: This follows from ??.

This finishes the proof of *Item 1*.

*Item 2, Two-Out-of-Three*: Omitted.

*Item 3, Stability Under Composition*: Clear.

*Item 4, Equivalences vs. Adjoint Equivalences*: See [Rie17, Proposition 4.4.5].

*Item 5, Interaction With Groupoids*: See [nLa24, Proposition 4.4].  $\square$

## 5.8 Isomorphisms of Categories

**Definition 5.8.1.1.** An **isomorphism of categories** is a pair of functors

$$\begin{aligned} F: \mathcal{C} &\rightarrow \mathcal{D}, \\ G: \mathcal{D} &\rightarrow \mathcal{C} \end{aligned}$$

such that we have

$$G \circ F = \text{id}_{\mathcal{C}},$$

$$F \circ G = \text{id}_{\mathcal{D}}.$$

**Example 5.8.1.2.** Categories can be equivalent but non-isomorphic. For example, the category consisting of two isomorphic objects is equivalent to  $\text{pt}$ , but not isomorphic to it.

**Proposition 5.8.1.3.** Let  $F: \mathcal{C} \rightarrow \mathcal{D}$  be a functor.

1. *Characterisations*. If  $\mathcal{C}$  and  $\mathcal{D}$  are small, then the following conditions are equivalent:
  - (a) The functor  $F$  is an isomorphism of categories.
  - (b) The functor  $F$  is fully faithful and bijective on objects.

---

lences  $(F, G, \eta', \epsilon)$  and  $(F, G, \eta, \epsilon')$ .

(c) For each  $X \in \text{Obj}(\text{Cats})$ , the precomposition functor

$$F^* : \text{Fun}(\mathcal{D}, X) \rightarrow \text{Fun}(C, X)$$

is an isomorphism of categories.

(d) For each  $X \in \text{Obj}(\text{Cats})$ , the postcomposition functor

$$F_* : \text{Fun}(X, C) \rightarrow \text{Fun}(X, \mathcal{D})$$

is an isomorphism of categories.

*Proof.* **Item 1, Characterisations:** We claim that **Items 1a to 1d** are indeed equivalent:

1. **Items 1a and 1b Are Equivalent:** Omitted, but similar to **Item 1** of **Proposition 5.7.1.2**.
2. **Items 1a, 1c and 1d Are Equivalent:** This follows from ??.

This finishes the proof.  $\square$

## 6 More Conditions on Functors

### 6.1 Dominant Functors

Let  $C$  and  $\mathcal{D}$  be categories.

**Definition 6.1.1.1.** A functor  $F : C \rightarrow \mathcal{D}$  is **dominant** if every object of  $\mathcal{D}$  is a retract of some object in  $\text{Im}(F)$ , i.e.:

(★) For each  $B \in \text{Obj}(\mathcal{D})$ , there exist:

- An object  $A$  of  $C$ ;
- A morphism  $r : F(A) \rightarrow B$  of  $\mathcal{D}$ ;
- A morphism  $s : B \rightarrow F(A)$  of  $\mathcal{D}$ ;

such that we have

$$r \circ s = \text{id}_B,$$

$$\begin{array}{ccc} B & \xrightarrow{s} & F(A) \\ & \searrow \text{id}_B & \downarrow r \\ & & B. \end{array}$$

**Proposition 6.1.1.2.** Let  $F, G: \mathcal{C} \rightrightarrows \mathcal{D}$  be functors and let  $I: \mathcal{X} \rightarrow \mathcal{C}$  be a functor.

1. *Interaction With Right Whiskering.* If  $I$  is full and dominant, then the map

$$- \star \text{id}_I: \text{Nat}(F, G) \rightarrow \text{Nat}(F \circ I, G \circ I)$$

is a bijection.

2. *Interaction With Adjunctions.* Let  $(F, G): \mathcal{C} \rightleftarrows \mathcal{D}$  be an adjunction.

- (a) If  $F$  is dominant, then  $G$  is faithful.
- (b) The following conditions are equivalent:
  - i. The functor  $G$  is full.
  - ii. The restriction

$$\upharpoonright \text{Glm}_F: \text{Im}(F) \rightarrow \mathcal{C}$$

of  $G$  to  $\text{Im}(F)$  is full.

*Proof.* **Item 1**, *Interaction With Right Whiskering*: See [DFH75, Proposition 1.4].

**Item 2**, *Interaction With Adjunctions*: See [DFH75, Proposition 1.7].  $\square$

**Question 6.1.1.3.** Is there a characterisation of functors  $F: \mathcal{C} \rightarrow \mathcal{D}$  such that:

1. For each  $\mathcal{X} \in \text{Obj}(\text{Cats})$ , the precomposition functor

$$F^*: \text{Fun}(\mathcal{D}, \mathcal{X}) \rightarrow \text{Fun}(\mathcal{C}, \mathcal{X})$$

is dominant?

2. For each  $\mathcal{X} \in \text{Obj}(\text{Cats})$ , the postcomposition functor

$$F_*: \text{Fun}(\mathcal{X}, \mathcal{C}) \rightarrow \text{Fun}(\mathcal{X}, \mathcal{D})$$

is dominant?

This question also appears as [MO 468121a].

## 6.2 Monomorphisms of Categories

Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories.

**Definition 6.2.1.1.** A functor  $F: \mathcal{C} \rightarrow \mathcal{D}$  is a **monomorphism of categories** if it is a monomorphism in  $\mathbf{Cats}$  (see ??, ??).

**Proposition 6.2.1.2.** Let  $F: \mathcal{C} \rightarrow \mathcal{D}$  be a functor.

1. *Characterisations.* The following conditions are equivalent:
  - (a) The functor  $F$  is a monomorphism of categories.
  - (b) The functor  $F$  is injective on objects and morphisms, i.e.  $F$  is injective on objects and the map

$$F: \text{Mor}(\mathcal{C}) \rightarrow \text{Mor}(\mathcal{D})$$

is injective.

*Proof.* **Item 1**, *Characterisations*: Omitted. □

**Question 6.2.1.3.** Is there a characterisation of functors  $F: \mathcal{C} \rightarrow \mathcal{D}$  such that:

1. For each  $\mathcal{X} \in \text{Obj}(\mathbf{Cats})$ , the precomposition functor

$$F^*: \text{Fun}(\mathcal{D}, \mathcal{X}) \rightarrow \text{Fun}(\mathcal{C}, \mathcal{X})$$

is a monomorphism of categories?

2. For each  $\mathcal{X} \in \text{Obj}(\mathbf{Cats})$ , the postcomposition functor

$$F_*: \text{Fun}(\mathcal{X}, \mathcal{C}) \rightarrow \text{Fun}(\mathcal{X}, \mathcal{D})$$

is a monomorphism of categories?

This question also appears as **[MO 468121a]**.

## 6.3 Epimorphisms of Categories

Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories.

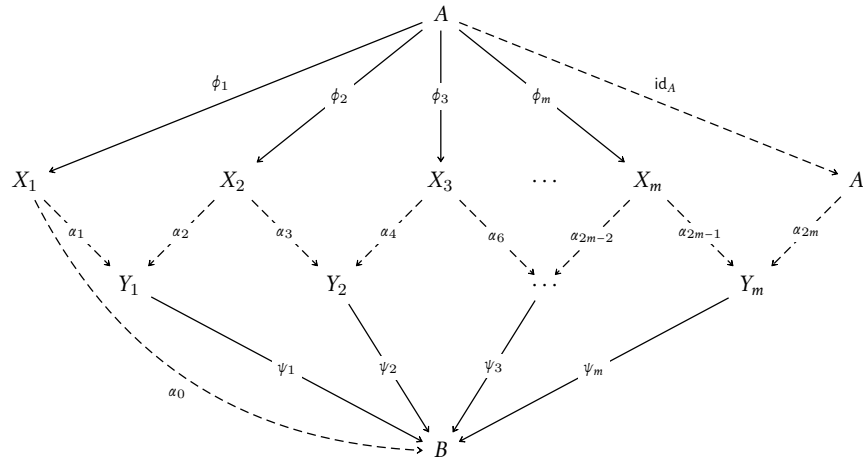
**Definition 6.3.1.1.** A functor  $F: \mathcal{C} \rightarrow \mathcal{D}$  is a **epimorphism of categories** if it is a

epimorphism in  $\mathbf{Cats}$  (see ??, ??).

**Proposition 6.3.1.2.** Let  $F: \mathcal{C} \rightarrow \mathcal{D}$  be a functor.

1. *Characterisations.* The following conditions are equivalent:<sup>25</sup>

- (a) The functor  $F$  is an epimorphism of categories.
- (b) For each morphism  $f: A \rightarrow B$  of  $\mathcal{D}$ , we have a diagram



in  $\mathcal{D}$  satisfying the following conditions:

- i. We have  $f = \alpha_0 \circ \phi_1$ .
- ii. We have  $f = \psi_m \circ \alpha_{2m}$ .
- iii. For each  $0 \leq i \leq 2m$ , we have  $\alpha_i \in \text{Mor}(\text{Im}(F))$ .

2. *Surjectivity on Objects.* If  $F$  is an epimorphism of categories, then  $F$  is surjective on objects.

*Proof.* **Item 1, Characterisations:** See [Isb68].

**Item 2, Surjectivity on Objects:** Omitted. □

**Question 6.3.1.3.** Is there a characterisation of functors  $F: \mathcal{C} \rightarrow \mathcal{D}$  such that:

- 1. For each  $\mathcal{X} \in \text{Obj}(\mathbf{Cats})$ , the precomposition functor

$$F^*: \text{Fun}(\mathcal{D}, \mathcal{X}) \rightarrow \text{Fun}(\mathcal{C}, \mathcal{X})$$

is an epimorphism of categories?

<sup>25</sup>*Further Terminology:* This statement is known as **Isbell's zigzag theorem**.

2. For each  $\mathcal{X} \in \text{Obj}(\text{Cats})$ , the postcomposition functor

$$F_*: \text{Fun}(\mathcal{X}, \mathcal{C}) \rightarrow \text{Fun}(\mathcal{X}, \mathcal{D})$$

is an epimorphism of categories?

This question also appears as [\[MO 468121a\]](#).

## 6.4 Pseudomonadic Functors

Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories.

**Definition 6.4.1.1.** A functor  $F: \mathcal{C} \rightarrow \mathcal{D}$  is **pseudomonadic** if it satisfies the following conditions:

1. For all diagrams of the form

$$\mathcal{X} \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \beta \\ \xrightarrow{\psi} \end{array} \mathcal{C} \xrightarrow{F} \mathcal{D},$$

if we have

$$\text{id}_F \star \alpha = \text{id}_F \star \beta,$$

then  $\alpha = \beta$ .

2. For each  $\mathcal{X} \in \text{Obj}(\text{Cats})$  and each natural isomorphism

$$\beta: F \circ \phi \xrightarrow{\sim} F \circ \psi, \quad \mathcal{X} \begin{array}{c} \xrightarrow{F \circ \phi} \\ \beta \Downarrow \\ \xrightarrow{F \circ \psi} \end{array} \mathcal{D},$$

there exists a natural isomorphism

$$\alpha: \phi \xrightarrow{\sim} \psi, \quad \mathcal{X} \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} \mathcal{C}$$

such that we have an equality

$$\mathcal{X} \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} \mathcal{C} \xrightarrow{F} \mathcal{D} = \mathcal{X} \begin{array}{c} \xrightarrow{F \circ \phi} \\ \beta \Downarrow \\ \xrightarrow{F \circ \psi} \end{array} \mathcal{D}$$



of pasting diagrams, i.e. such that we have

$$\beta = \text{id}_F \star \alpha.$$

**Proposition 6.4.1.2.** Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be a functor.

1. *Characterisations.* The following conditions are equivalent:

- (a) The functor  $F$  is pseudomonadic.
- (b) The functor  $F$  satisfies the following conditions:
  - i. The functor  $F$  is faithful, i.e. for each  $A, B \in \text{Obj}(\mathcal{C})$ , the action on morphisms

$$F_{A,B} : \text{Hom}_{\mathcal{C}}(A, B) \rightarrow \text{Hom}_{\mathcal{D}}(F_A, F_B)$$

of  $F$  at  $(A, B)$  is injective.

- ii. For each  $A, B \in \text{Obj}(\mathcal{C})$ , the restriction

$$F_{A,B}^{\text{iso}} : \text{Iso}_{\mathcal{C}}(A, B) \rightarrow \text{Iso}_{\mathcal{D}}(F_A, F_B)$$

of the action on morphisms of  $F$  at  $(A, B)$  to isomorphisms is surjective.

- (c) We have an isocomma square of the form

$$C \overset{\text{eq.}}{\cong} C \overset{\leftrightarrow}{\times}_{\mathcal{D}} C, \quad \begin{array}{ccc} C & \xrightarrow{\text{id}_C} & C \\ \text{id}_C \downarrow & \nearrow \text{dashed} & \downarrow F \\ C & \xrightarrow{F} & \mathcal{D} \end{array}$$

in  $\text{Cats}_2$  up to equivalence.

- (d) We have an isocomma square of the form

$$C \overset{\text{eq.}}{\cong} C \overset{\leftrightarrow}{\times}_{\text{Arr}(\mathcal{D})} \mathcal{D}, \quad \begin{array}{ccc} C & \hookrightarrow & \text{Arr}(C) \\ F \downarrow & \nearrow \text{dashed} & \downarrow \text{Arr}(F) \\ \mathcal{D} & \hookrightarrow & \text{Arr}(\mathcal{D}) \end{array}$$

in  $\text{Cats}_2$  up to equivalence.

(e) For each  $\mathcal{X} \in \text{Obj}(\text{Cats})$ , the postcomposition<sup>26</sup> functor

$$F_* : \text{Fun}(\mathcal{X}, \mathcal{C}) \rightarrow \text{Fun}(\mathcal{X}, \mathcal{D})$$

is pseudomonic.

2. *Conservativity*. If  $F$  is pseudomonic, then  $F$  is conservative.
3. *Essential Injectivity*. If  $F$  is pseudomonic, then  $F$  is essentially injective.

*Proof.* *Item 1, Characterisations*: Omitted.

*Item 2, Conservativity*: Omitted.

*Item 3, Essential Injectivity*: Omitted. □

## 6.5 Pseudoepic Functors

Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories.

**Definition 6.5.1.1.** A functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  is **pseudoepic** if it satisfies the following conditions:

1. For all diagrams of the form

$$\mathcal{C} \xrightarrow{F} \mathcal{D} \begin{array}{c} \xrightarrow{\phi} \\ \alpha \downarrow \downarrow \downarrow \beta \\ \xrightarrow{\psi} \end{array} \mathcal{X},$$

if we have

$$\alpha \star \text{id}_F = \beta \star \text{id}_F,$$

then  $\alpha = \beta$ .

2. For each  $X \in \text{Obj}(\mathcal{C})$  and each 2-isomorphism

$$\beta : \phi \circ F \xRightarrow{\sim} \psi \circ F, \quad \mathcal{C} \begin{array}{c} \xrightarrow{\phi \circ F} \\ \beta \downarrow \downarrow \downarrow \\ \xrightarrow{\psi \circ F} \end{array} \mathcal{X}$$

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<sup>26</sup>Asking the precomposition functors

$$F^* : \text{Fun}(\mathcal{D}, \mathcal{X}) \rightarrow \text{Fun}(\mathcal{C}, \mathcal{X})$$

to be pseudomonic leads to pseudoepic functors; see *Item 1b* of *Item 1* of *Proposition 6.5.1.2*.

of  $\mathcal{C}$ , there exists a 2-isomorphism

$$\alpha : \phi \xRightarrow{\sim} \psi, \quad \mathcal{D} \begin{array}{c} \xrightarrow{\phi} \\ \Downarrow \alpha \\ \xrightarrow{\psi} \end{array} \mathcal{X}$$

of  $\mathcal{C}$  such that we have an equality

$$\mathcal{C} \xrightarrow{F} \mathcal{D} \begin{array}{c} \xrightarrow{\phi} \\ \Downarrow \alpha \\ \xrightarrow{\psi} \end{array} \mathcal{X} = \mathcal{C} \begin{array}{c} \xrightarrow{\phi \circ F} \\ \Downarrow \beta \\ \xrightarrow{\psi \circ F} \end{array} \mathcal{X}$$

of pasting diagrams in  $\mathcal{C}$ , i.e. such that we have

$$\beta = \alpha \star \text{id}_F.$$

**Proposition 6.5.1.2.** Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be a functor.

1. *Characterisations.* The following conditions are equivalent:

- (a) The functor  $F$  is pseudoepic.
- (b) For each  $\mathcal{X} \in \text{Obj}(\text{Cats})$ , the functor

$$F^* : \text{Fun}(\mathcal{D}, \mathcal{X}) \rightarrow \text{Fun}(\mathcal{C}, \mathcal{X})$$

given by precomposition by  $F$  is pseudomonic.

(c) We have an isococcomma square of the form

$$\mathcal{D} \xrightarrow{\text{eq.}} \mathcal{D} \overset{\leftrightarrow}{\coprod}_C \mathcal{D}, \quad \begin{array}{ccc} \mathcal{D} & \xleftarrow{\text{id}_{\mathcal{D}}} & \mathcal{D} \\ \uparrow \text{id}_{\mathcal{D}} & \nearrow \text{dashed} & \uparrow F \\ \mathcal{D} & \xleftarrow{F} & \mathcal{C} \end{array}$$

in  $\text{Cats}_2$  up to equivalence.

2. *Dominance.* If  $F$  is pseudoepic, then  $F$  is dominant ([Definition 6.1.1.1](#)).

*Proof.* **Item 1, Characterisations:** Omitted.

*Item 2, Dominance:* If  $F$  is pseudoepic, then

$$F^* : \text{Fun}(\mathcal{D}, \mathcal{X}) \rightarrow \text{Fun}(\mathcal{C}, \mathcal{X})$$

is pseudomonic for all  $\mathcal{X} \in \text{Obj}(\text{Cats})$ , and thus in particular faithful. By [Item 4g](#) of [Item 4](#) of [Proposition 5.1.1.2](#), this is equivalent to requiring  $F$  to be dominant.  $\square$

**Question 6.5.1.3.** Is there a nice characterisation of the pseudoepic functors, similarly to the characterisation of pseudomonic functors given in [Item 1b](#) of [Item 1](#) of [Proposition 6.4.1.2](#)?

This question also appears as [\[MO 321971\]](#).

**Question 6.5.1.4.** A pseudomonic and pseudoepic functor is dominant, faithful, essentially injective, and full on isomorphisms. Is it necessarily an equivalence of categories? If not, how bad can this fail, i.e. how far can a pseudomonic and pseudoepic functor be from an equivalence of categories?

This question also appears as [\[MO 468334\]](#).

**Question 6.5.1.5.** Is there a characterisation of functors  $F : \mathcal{C} \rightarrow \mathcal{D}$  such that:

1. For each  $\mathcal{X} \in \text{Obj}(\text{Cats})$ , the precomposition functor

$$F^* : \text{Fun}(\mathcal{D}, \mathcal{X}) \rightarrow \text{Fun}(\mathcal{C}, \mathcal{X})$$

is pseudoepic?

2. For each  $\mathcal{X} \in \text{Obj}(\text{Cats})$ , the postcomposition functor

$$F_* : \text{Fun}(\mathcal{X}, \mathcal{C}) \rightarrow \text{Fun}(\mathcal{X}, \mathcal{D})$$

is pseudoepic?

This question also appears as [\[MO 468121a\]](#).

## 7 Even More Conditions on Functors

### 7.1 Injective on Objects Functors

Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories.

**Definition 7.1.1.1.** A functor  $F: C \rightarrow \mathcal{D}$  is **injective on objects** if the action on objects

$$F: \text{Obj}(C) \rightarrow \text{Obj}(\mathcal{D})$$

of  $F$  is injective.

**Proposition 7.1.1.2.** Let  $F: C \rightarrow \mathcal{D}$  be a functor.

1. *Characterisations.* The following conditions are equivalent:

- (a) The functor  $F$  is injective on objects.
- (b) The functor  $F$  is an isocofibration in  $\text{Cats}_2$ .

*Proof.* **Item 1**, *Characterisations*: Omitted. □

## 7.2 Surjective on Objects Functors

Let  $C$  and  $\mathcal{D}$  be categories.

**Definition 7.2.1.1.** A functor  $F: C \rightarrow \mathcal{D}$  is **surjective on objects** if the action on objects

$$F: \text{Obj}(C) \rightarrow \text{Obj}(\mathcal{D})$$

of  $F$  is surjective.

## 7.3 Bijective on Objects Functors

Let  $C$  and  $\mathcal{D}$  be categories.

**Definition 7.3.1.1.** A functor  $F: C \rightarrow \mathcal{D}$  is **bijective on objects**<sup>27</sup> if the action on objects

$$F: \text{Obj}(C) \rightarrow \text{Obj}(\mathcal{D})$$

of  $F$  is a bijection.

## 7.4 Functors Representably Faithful on Cores

Let  $C$  and  $\mathcal{D}$  be categories.

**Definition 7.4.1.1.** A functor  $F: C \rightarrow \mathcal{D}$  is **representably faithful on cores** if,

---

<sup>27</sup> *Further Terminology:* Also called a **bo** functor.

for each  $X \in \text{Obj}(\text{Cats})$ , the postcomposition by  $F$  functor

$$F_*: \text{Core}(\text{Fun}(X, C)) \rightarrow \text{Core}(\text{Fun}(X, D))$$

is faithful.

**Remark 7.4.1.2.** In detail, a functor  $F: C \rightarrow D$  is **representably faithful on cores** if, given a diagram of the form

$$\begin{array}{ccc} X & \xrightarrow[\psi]{\phi} & C \xrightarrow{F} D, \\ & \alpha \Downarrow \beta & \\ & & \end{array}$$

if  $\alpha$  and  $\beta$  are natural isomorphisms and we have

$$\text{id}_F \star \alpha = \text{id}_F \star \beta,$$

then  $\alpha = \beta$ .

**Question 7.4.1.3.** Is there a characterisation of functors representably faithful on cores?

## 7.5 Functors Representably Full on Cores

Let  $C$  and  $D$  be categories.

**Definition 7.5.1.1.** A functor  $F: C \rightarrow D$  is **representably full on cores** if, for each  $X \in \text{Obj}(\text{Cats})$ , the postcomposition by  $F$  functor

$$F_*: \text{Core}(\text{Fun}(X, C)) \rightarrow \text{Core}(\text{Fun}(X, D))$$

is full.

**Remark 7.5.1.2.** In detail, a functor  $F: C \rightarrow D$  is **representably full on cores** if, for each  $X \in \text{Obj}(\text{Cats})$  and each natural isomorphism

$$\beta: F \circ \phi \xrightarrow{\sim} F \circ \psi, \quad \begin{array}{ccc} X & \xrightarrow[F \circ \psi]{F \circ \phi} & D, \\ & \beta \Downarrow & \end{array}$$

there exists a natural isomorphism

$$\alpha: \phi \xRightarrow{\sim} \psi, \quad X \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} C$$

such that we have an equality

$$X \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} C \xrightarrow{F} \mathcal{D} = X \begin{array}{c} \xrightarrow{F \circ \phi} \\ \beta \Downarrow \\ \xrightarrow{F \circ \psi} \end{array} \mathcal{D}$$

of pasting diagrams in  $\mathbf{Cats}_2$ , i.e. such that we have

$$\beta = \text{id}_F \star \alpha.$$

**Question 7.5.1.3.** Is there a characterisation of functors representably full on cores?

This question also appears as [MO 468121a].

## 7.6 Functors Representably Fully Faithful on Cores

Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories.

**Definition 7.6.1.1.** A functor  $F: \mathcal{C} \rightarrow \mathcal{D}$  is **representably fully faithful on cores** if, for each  $X \in \text{Obj}(\mathbf{Cats})$ , the postcomposition by  $F$  functor

$$F_*: \text{Core}(\text{Fun}(X, \mathcal{C})) \rightarrow \text{Core}(\text{Fun}(X, \mathcal{D}))$$

is fully faithful.

**Remark 7.6.1.2.** In detail, a functor  $F: \mathcal{C} \rightarrow \mathcal{D}$  is **representably fully faithful on cores** if it satisfies the conditions in [Remarks 7.4.1.2](#) and [7.5.1.2](#), i.e.:

1. For all diagrams of the form

$$X \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \beta \\ \xrightarrow{\psi} \end{array} C \xrightarrow{F} \mathcal{D},$$

with  $\alpha$  and  $\beta$  natural isomorphisms, if we have  $\text{id}_F \star \alpha = \text{id}_F \star \beta$ , then  $\alpha = \beta$ .

2. For each  $\mathcal{X} \in \text{Obj}(\text{Cats})$  and each natural isomorphism

$$\beta: F \circ \phi \xrightarrow{\sim} F \circ \psi, \quad \mathcal{X} \begin{array}{c} \xrightarrow{F \circ \phi} \\ \beta \Downarrow \\ \xrightarrow{F \circ \psi} \end{array} \mathcal{D}$$

of  $\mathcal{C}$ , there exists a natural isomorphism

$$\alpha: \phi \xrightarrow{\sim} \psi, \quad \mathcal{X} \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} \mathcal{C}$$

of  $\mathcal{C}$  such that we have an equality

$$\mathcal{X} \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} \mathcal{C} \xrightarrow{F} \mathcal{D} = \mathcal{X} \begin{array}{c} \xrightarrow{F \circ \phi} \\ \beta \Downarrow \\ \xrightarrow{F \circ \psi} \end{array} \mathcal{D}$$

of pasting diagrams in  $\text{Cats}_2$ , i.e. such that we have

$$\beta = \text{id}_F \star \alpha.$$

**Question 7.6.1.3.** Is there a characterisation of functors representably fully faithful on cores?

## 7.7 Functors Corepresentably Faithful on Cores

Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories.

**Definition 7.7.1.1.** A functor  $F: \mathcal{C} \rightarrow \mathcal{D}$  is **corepresentably faithful on cores** if, for each  $\mathcal{X} \in \text{Obj}(\text{Cats})$ , the postcomposition by  $F$  functor

$$F_*: \text{Core}(\text{Fun}(\mathcal{X}, \mathcal{C})) \rightarrow \text{Core}(\text{Fun}(\mathcal{X}, \mathcal{D}))$$

is faithful.

**Remark 7.7.1.2.** In detail, a functor  $F: \mathcal{C} \rightarrow \mathcal{D}$  is **corepresentably faithful on cores** if, given a diagram of the form

$$\mathcal{C} \xrightarrow{F} \mathcal{D} \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \beta \\ \xrightarrow{\psi} \end{array} \mathcal{X},$$



if  $\alpha$  and  $\beta$  are natural isomorphisms and we have

$$\alpha \star \text{id}_F = \beta \star \text{id}_F,$$

then  $\alpha = \beta$ .

**Question 7.7.1.3.** Is there a characterisation of functors corepresentably faithful on cores?

## 7.8 Functors Corepresentably Full on Cores

Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories.

**Definition 7.8.1.1.** A functor  $F: \mathcal{C} \rightarrow \mathcal{D}$  is **corepresentably full on cores** if, for each  $X \in \text{Obj}(\text{Cats})$ , the postcomposition by  $F$  functor

$$F_*: \text{Core}(\text{Fun}(X, \mathcal{C})) \rightarrow \text{Core}(\text{Fun}(X, \mathcal{D}))$$

is full.

**Remark 7.8.1.2.** In detail, a functor  $F: \mathcal{C} \rightarrow \mathcal{D}$  is **corepresentably full on cores** if, for each  $X \in \text{Obj}(\text{Cats})$  and each natural isomorphism

$$\beta: \phi \circ F \xrightarrow{\sim} \psi \circ F, \quad \mathcal{C} \begin{array}{c} \xrightarrow{\phi \circ F} \\ \beta \Downarrow \\ \xrightarrow{\psi \circ F} \end{array} X,$$

there exists a natural isomorphism

$$\alpha: \phi \xrightarrow{\sim} \psi, \quad \mathcal{D} \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} X$$

such that we have an equality

$$X \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} \mathcal{C} \xrightarrow{F} \mathcal{D} = X \begin{array}{c} \xrightarrow{F \circ \phi} \\ \beta \Downarrow \\ \xrightarrow{F \circ \psi} \end{array} \mathcal{D}$$

of pasting diagrams in  $\text{Cats}_2$ , i.e. such that we have

$$\beta = \alpha \star \text{id}_F.$$

**Question 7.8.1.3.** Is there a characterisation of functors corepresentably full on cores?

This question also appears as [MO 468121a].

### 7.9 Functors Corepresentably Fully Faithful on Cores

Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories.

**Definition 7.9.1.1.** A functor  $F: \mathcal{C} \rightarrow \mathcal{D}$  is **corepresentably fully faithful on cores** if, for each  $X \in \text{Obj}(\text{Cats})$ , the postcomposition by  $F$  functor

$$F_*: \text{Core}(\text{Fun}(X, \mathcal{C})) \rightarrow \text{Core}(\text{Fun}(X, \mathcal{D}))$$

is fully faithful.

**Remark 7.9.1.2.** In detail, a functor  $F: \mathcal{C} \rightarrow \mathcal{D}$  is **corepresentably fully faithful on cores** if it satisfies the conditions in [Remarks 7.7.1.2](#) and [7.8.1.2](#), i.e.:

1. For all diagrams of the form

$$\mathcal{C} \xrightarrow{F} \mathcal{D} \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \beta \\ \xrightarrow{\psi} \end{array} X,$$

if  $\alpha$  and  $\beta$  are natural isomorphisms and we have

$$\alpha \star \text{id}_F = \beta \star \text{id}_F,$$

then  $\alpha = \beta$ .

2. For each  $X \in \text{Obj}(\text{Cats})$  and each natural isomorphism

$$\beta: \phi \circ F \xRightarrow{\sim} \psi \circ F, \quad \mathcal{C} \begin{array}{c} \xrightarrow{\phi \circ F} \\ \beta \Downarrow \\ \xrightarrow{\psi \circ F} \end{array} X,$$

there exists a natural isomorphism

$$\alpha: \phi \xRightarrow{\sim} \psi, \quad \mathcal{D} \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} X$$

such that we have an equality

$$X \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} \mathcal{C} \xrightarrow{F} \mathcal{D} = X \begin{array}{c} \xrightarrow{F \circ \phi} \\ \beta \Downarrow \\ \xrightarrow{F \circ \psi} \end{array} \mathcal{D}$$

of pasting diagrams in  $\text{Cats}_2$ , i.e. such that we have

$$\beta = \alpha \star \text{id}_F.$$

**Question 7.9.1.3.** Is there a characterisation of functors corepresentably fully faithful on cores?

## 8 Natural Transformations

### 8.1 Transformations

Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories and  $F, G : \mathcal{C} \Rightarrow \mathcal{D}$  be functors.

**Definition 8.1.1.1.** A **transformation**<sup>28</sup>  $\alpha : F \Rightarrow G$  **from**  $F$  **to**  $G$  is a collection

$$\{\alpha_A : F(A) \rightarrow G(A)\}_{A \in \text{Obj}(\mathcal{C})}$$

of morphisms of  $\mathcal{D}$ .

**Notation 8.1.1.2.** We write  $\text{Trans}(F, G)$  for the set of transformations from  $F$  to  $G$ .

### 8.2 Natural Transformations

Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories and  $F, G : \mathcal{C} \Rightarrow \mathcal{D}$  be functors.

**Definition 8.2.1.1.** A **natural transformation**  $\alpha : F \Rightarrow G$  **from**  $F$  **to**  $G$  is a transformation

$$\{\alpha_A : F(A) \rightarrow G(A)\}_{A \in \text{Obj}(\mathcal{C})}$$

from  $F$  to  $G$  such that, for each morphism  $f : A \rightarrow B$  of  $\mathcal{C}$ , the diagram

$$\begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(B) \\ \alpha_A \downarrow & & \downarrow \alpha_B \\ G(A) & \xrightarrow{G(f)} & G(B) \end{array}$$

commutes.<sup>29</sup>

<sup>28</sup> *Further Terminology:* Also called an **unnatural transformation** for emphasis.

<sup>29</sup> *Further Terminology:* The morphism  $\alpha_A : F_A \rightarrow G_A$  is called the **component of  $\alpha$  at  $A$** .

**Remark 8.2.1.2.** We denote natural transformations in diagrams as

$$C \begin{array}{c} \xrightarrow{F} \\ \alpha \Downarrow \\ \xrightarrow{G} \end{array} \mathcal{D}.$$

**Notation 8.2.1.3.** We write  $\text{Nat}(F, G)$  for the set of natural transformations from  $F$  to  $G$ .

**Example 8.2.1.4.** The **identity natural transformation**  $\text{id}_F: F \Longrightarrow F$  of  $F$  is the natural transformation consisting of the collection

$$\{\text{id}_{F(A)}: F(A) \rightarrow F(A)\}_{A \in \text{Obj}(C)}.$$

*Proof.* The naturality condition for  $\text{id}_F$  is the requirement that, for each morphism  $f: A \rightarrow B$  of  $C$ , the diagram

$$\begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(B) \\ \text{id}_{F(A)} \downarrow & & \downarrow \text{id}_{F(B)} \\ F(A) & \xrightarrow{F(f)} & F(B) \end{array}$$

commutes, which follows from unitality of the composition of  $C$ .  $\square$

**Definition 8.2.1.5.** Two natural transformations  $\alpha, \beta: F \Longrightarrow G$  are **equal** if we have

$$\alpha_A = \beta_A$$

for each  $A \in \text{Obj}(C)$ .

### 8.3 Vertical Composition of Natural Transformations

**Definition 8.3.1.1.** The **vertical composition** of two natural transformations  $\alpha: F \Longrightarrow G$  and  $\beta: G \Longrightarrow H$  as in the diagram

$$C \begin{array}{c} \xrightarrow{F} \\ \alpha \Downarrow \\ \xrightarrow{G} \\ \beta \Downarrow \\ \xrightarrow{H} \end{array} \mathcal{D}$$

is the natural transformation  $\beta \circ \alpha : F \Rightarrow H$  consisting of the collection

$$\{(\beta \circ \alpha)_A : F(A) \rightarrow H(A)\}_{A \in \text{Obj}(C)}$$

with

$$(\beta \circ \alpha)_A \stackrel{\text{def}}{=} \beta_A \circ \alpha_A$$

for each  $A \in \text{Obj}(C)$ .

*Proof.* The naturality condition for  $\beta \circ \alpha$  is the requirement that the boundary of the diagram

$$\begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(B) \\ \alpha_A \downarrow & (1) & \downarrow \alpha_B \\ G(A) & \xrightarrow{G(f)} & G(B) \\ \beta_A \downarrow & (2) & \downarrow \beta_B \\ H(A) & \xrightarrow{H(f)} & H(B) \end{array}$$

commutes. Since

1. Subdiagram (1) commutes by the naturality of  $\alpha$ .
2. Subdiagram (2) commutes by the naturality of  $\beta$ .

so does the boundary diagram. Hence  $\beta \circ \alpha$  is a natural transformation.  $\square$

**Proposition 8.3.1.2.** Let  $C$ ,  $\mathcal{D}$ , and  $\mathcal{E}$  be categories.

1. *Functionality.* The assignment  $(\beta, \alpha) \mapsto \beta \circ \alpha$  defines a function

$$\circ_{F,G,H} : \text{Nat}(G, H) \times \text{Nat}(F, G) \rightarrow \text{Nat}(F, H).$$

2. *Associativity.* Let  $F, G, H, K: \mathcal{C} \rightrightarrows \mathcal{D}$  be functors. The diagram

$$\begin{array}{ccc}
 & \text{Nat}(H, K) \times (\text{Nat}(G, H) \times \text{Nat}(F, G)) & \\
 \alpha_{\text{Nat}(H, K), \text{Nat}(G, H), \text{Nat}(F, G)}^{\text{Sets}} \nearrow \sim & & \searrow \text{id}_{\text{Nat}(H, K)} \times \circ_{F, G, H} \\
 (\text{Nat}(H, K) \times \text{Nat}(G, H)) \times \text{Nat}(F, G) & & \text{Nat}(H, K) \times \text{Nat}(F, H) \\
 \circ_{G, H, K} \times \text{id}_{\text{Nat}(F, G)} \searrow & & \searrow \circ_{F, H, K} \\
 \text{Nat}(G, K) \times \text{Nat}(F, G) & \xrightarrow{\circ_{F, G, K}} & \text{Nat}(F, K)
 \end{array}$$

commutes, i.e. given natural transformations

$$F \xRightarrow{\alpha} G \xRightarrow{\beta} H \xRightarrow{\gamma} K,$$

we have

$$(\gamma \circ \beta) \circ \alpha = \gamma \circ (\beta \circ \alpha).$$

3. *Unitality.* Let  $F, G: \mathcal{C} \rightrightarrows \mathcal{D}$  be functors.

(a) *Left Unitality.* The diagram

$$\begin{array}{ccc}
 \text{pt} \times \text{Nat}(F, G) & & \\
 [\text{id}_G] \times \text{id}_{\text{Nat}(F, G)} \downarrow & \nearrow \lambda_{\text{Nat}(F, G)}^{\text{Sets}} \sim & \\
 \text{Nat}(G, G) \times \text{Nat}(F, G) & \xrightarrow{\circ_{F, G, G}} & \text{Nat}(F, G)
 \end{array}$$

commutes, i.e. given a natural transformation  $\alpha: F \Rightarrow G$ , we have

$$\text{id}_G \circ \alpha = \alpha.$$

(b) *Right Unitality*. The diagram

$$\begin{array}{ccc}
 \text{Nat}(F, G) \times \text{pt} & & \\
 \downarrow \text{id}_{\text{Nat}(F, G)} \times [\text{id}_F] & \searrow \rho_{\text{Nat}(F, G)}^{\text{Sets}} & \\
 \text{Nat}(F, G) \times \text{Nat}(F, F) & \xrightarrow{\circ_{F, F, G}^C} & \text{Nat}(F, G)
 \end{array}$$

commutes, i.e. given a natural transformation  $\alpha : F \Rightarrow G$ , we have

$$\alpha \circ \text{id}_F = \alpha.$$

4. *Middle Four Exchange*. Let  $F_1, F_2, F_3 : \mathcal{C} \rightarrow \mathcal{D}$  and  $G_1, G_2, G_3 : \mathcal{D} \rightarrow \mathcal{E}$  be functors. The diagram

$$\begin{array}{ccc}
 (\text{Nat}(G_2, G_3) \times \text{Nat}(G_1, G_2)) \times (\text{Nat}(F_2, F_3) \times \text{Nat}(F_1, F_2)) & \xleftarrow{\mu^4} & (\text{Nat}(G_2, G_3) \times \text{Nat}(F_2, F_3)) \times (\text{Nat}(G_1, G_2) \times \text{Nat}(F_1, F_2)) \\
 \downarrow \circ_{G_1, G_2, G_3} \times \circ_{F_1, F_2, F_3} & & \downarrow \star_{F_2, F_3, G_2, G_3} \times \star_{F_1, F_2, G_1, G_2} \\
 \text{Nat}(G_1, G_3) \times \text{Nat}(F_1, F_3) & & \text{Nat}(G_2 \circ F_2, G_3 \circ F_3) \times \text{Nat}(G_1 \circ F_1, G_2 \circ F_2) \\
 \searrow \star_{F_1, F_3, G_1, G_3} & & \swarrow \circ_{G_1 \circ F_1, G_2 \circ F_2, G_3 \circ F_3} \\
 & \text{Nat}(G_1 \circ F_1, G_3 \circ F_3) &
 \end{array}$$

commutes, i.e. given a diagram

$$\begin{array}{ccccc}
 & F_1 & & G_1 & \\
 & \downarrow \alpha & & \downarrow \beta & \\
 C & \xrightarrow{F_2} & \mathcal{D} & \xrightarrow{G_2} & \mathcal{E} \\
 & \downarrow \alpha' & & \downarrow \beta' & \\
 & F_3 & & G_3 &
 \end{array}$$

in  $\text{Cats}_2$ , we have

$$(\beta' \star \alpha') \circ (\beta \star \alpha) = (\beta' \circ \beta) \star (\alpha' \circ \alpha).$$

*Proof.* **Item 1, Functionality:** Clear.

*Item 2, Associativity:* Indeed, we have

$$\begin{aligned}
 ((\gamma \circ \beta) \circ \alpha)_A &\stackrel{\text{def}}{=} (\gamma \circ \beta)_A \circ \alpha_A \\
 &\stackrel{\text{def}}{=} (\gamma_A \circ \beta_A) \circ \alpha_A \\
 &= \gamma_A \circ (\beta_A \circ \alpha_A) \\
 &\stackrel{\text{def}}{=} \gamma_A \circ (\beta \circ \alpha)_A \\
 &\stackrel{\text{def}}{=} (\gamma \circ (\beta \circ \alpha))_A
 \end{aligned}$$

for each  $A \in \text{Obj}(C)$ , showing the desired equality.

*Item 3, Unitality:* We have

$$\begin{aligned}
 (\text{id}_G \circ \alpha)_A &= \text{id}_G \circ \alpha_A \\
 &= \alpha_A, \\
 (\alpha \circ \text{id}_F)_A &= \alpha_A \circ \text{id}_F \\
 &= \alpha_A
 \end{aligned}$$

for each  $A \in \text{Obj}(C)$ , showing the desired equality.

*Item 4, Middle Four Exchange:* This is proved in *Item 4* of [Proposition 8.4.1.3](#).  $\square$

## 8.4 Horizontal Composition of Natural Transformations

**Definition 8.4.1.1.** The **horizontal composition**<sup>30,31</sup> of two natural transformations  $\alpha: F \Rightarrow G$  and  $\beta: H \Rightarrow K$  as in the diagram

$$\begin{array}{ccccc}
 C & \xrightarrow{F} & \mathcal{D} & \xrightarrow{H} & \mathcal{E} \\
 \alpha \downarrow & & \beta \downarrow & & \\
 C & \xrightarrow{G} & \mathcal{D} & \xrightarrow{K} & \mathcal{E}
 \end{array}$$

of  $\alpha$  and  $\beta$  is the natural transformation

$$\beta \star \alpha: (H \circ F) \Rightarrow (K \circ G),$$

<sup>30</sup>*Further Terminology:* Also called the **Codement product** of  $\alpha$  and  $\beta$ .

<sup>31</sup>Horizontal composition forms a map

$$\star_{(F,H),(G,K)}: \text{Nat}(H, K) \times \text{Nat}(F, G) \rightarrow \text{Nat}(H \circ F, K \circ G).$$



as in the diagram

$$\begin{array}{ccc}
 & H \circ F & \\
 C & \begin{array}{c} \xrightarrow{\quad} \\ \parallel \\ \beta \star \alpha \\ \Downarrow \\ \xrightarrow{\quad} \end{array} & \mathcal{E} \\
 & K \circ G &
 \end{array}$$

consisting of the collection

$$\{(\beta \star \alpha)_A: H(F(A)) \rightarrow K(G(A))\}_{A \in \text{Obj}(C)},$$

of morphisms of  $\mathcal{E}$  with

$$\begin{array}{ccc}
 H(F(A)) & \xrightarrow{H(\alpha_A)} & H(G(A)) \\
 \beta_{F(A)} \downarrow & & \downarrow \beta_{G(A)} \\
 K(F(A)) & \xrightarrow{K(\alpha_A)} & K(G(A)).
 \end{array}$$

$(\beta \star \alpha)_A \stackrel{\text{def}}{=} \beta_{G(A)} \circ H(\alpha_A)$   
 $= K(\alpha_A) \circ \beta_{F(A)},$

*Proof.* First, we claim that we indeed have

$$\begin{array}{ccc}
 H(F(A)) & \xrightarrow{H(\alpha_A)} & H(G(A)) \\
 \beta_{F(A)} \downarrow & & \downarrow \beta_{G(A)} \\
 K(F(A)) & \xrightarrow{K(\alpha_A)} & K(G(A)).
 \end{array}$$

$\beta_{G(A)} \circ H(\alpha_A) = K(\alpha_A) \circ \beta_{F(A)},$

This is, however, simply the naturality square for  $\beta$  applied to the morphism  $\alpha_A: F(A) \rightarrow G(A)$ . Next, we check the naturality condition for  $\beta \star \alpha$ , which is the requirement that the boundary of the diagram

$$\begin{array}{ccc}
 H(F(A)) & \xrightarrow{H(F(f))} & H(F(B)) \\
 H(\alpha_A) \downarrow & (1) & \downarrow H(\alpha_B) \\
 H(G(A)) & \xrightarrow{H(G(f))} & H(G(B)) \\
 \beta_{G(A)} \downarrow & (2) & \downarrow \beta_{G(B)} \\
 K(G(A)) & \xrightarrow{K(G(f))} & K(G(B))
 \end{array}$$

commutes. Since

1. Subdiagram (1) commutes by the naturality of  $\alpha$ .
2. Subdiagram (2) commutes by the naturality of  $\beta$ .

so does the boundary diagram. Hence  $\beta \circ \alpha$  is a natural transformation.<sup>32</sup>  $\square$

**Definition 8.4.1.2.** Let

$$\mathcal{X} \xrightarrow{F} \mathcal{C} \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} \mathcal{D} \xrightarrow{G} \mathcal{Y}$$

be a diagram in  $\mathbf{Cats}_2$ .

1. The **left whiskering of  $\alpha$  with  $G$**  is the natural transformation<sup>33</sup>

$$\mathrm{id}_G \star \alpha: G \circ \phi \Longrightarrow G \circ \psi.$$

2. The **right whiskering of  $\alpha$  with  $F$**  is the natural transformation<sup>34</sup>

$$\alpha \star \mathrm{id}_F: \phi \circ F \Longrightarrow \psi \circ F.$$

**Proposition 8.4.1.3.** Let  $\mathcal{C}$ ,  $\mathcal{D}$ , and  $\mathcal{E}$  be categories.

1. *Functionality.* The assignment  $(\beta, \alpha) \mapsto \beta \star \alpha$  defines a function

$$\star_{(F,G),(H,K)}: \mathrm{Nat}(H, K) \times \mathrm{Nat}(F, G) \rightarrow \mathrm{Nat}(H \circ F, K \circ G).$$

2. *Associativity.* Let

$$\mathcal{C} \begin{array}{c} \xrightarrow{F_1} \\ \xrightarrow{G_1} \end{array} \mathcal{D} \begin{array}{c} \xrightarrow{F_2} \\ \xrightarrow{G_2} \end{array} \mathcal{E} \begin{array}{c} \xrightarrow{F_3} \\ \xrightarrow{G_3} \end{array} \mathcal{F}$$

<sup>32</sup>Reference: [Bor94, Proposition 1.3.4].

<sup>33</sup>Further Notation: Also written  $G\alpha$  or  $G \star \alpha$ , although we won't use either of these notations in this work.

<sup>34</sup>Further Notation: Also written  $\alpha F$  or  $\alpha \star F$ , although we won't use either of these notations in this work.

be a diagram in  $\mathbf{Cats}_2$ . The diagram

$$\begin{array}{ccc}
 \text{Nat}(F_3, G_3) \times \text{Nat}(F_2, G_2) \times \text{Nat}(F_1, G_1) & \xrightarrow{\star_{(F_2, G_2), (F_3, G_3)} \times \text{id}} & \text{Nat}(F_3 \circ F_2, G_3 \circ G_2) \times \text{Nat}(F_1, G_1) \\
 \downarrow \text{id} \times \star_{(F_1, G_1), (F_2, G_2)} & & \downarrow \star_{(F_3 \circ F_2), (G_3 \circ G_2, F_1, G_1)} \\
 \text{Nat}(F_3, G_3) \times \text{Nat}(F_2 \circ F_1, G_2 \circ G_1) & \xrightarrow{\star_{(F_2 \circ F_1), (G_2 \circ G_1, F_3, G_3)}} & \text{Nat}(F_3 \circ F_2 \circ F_1, G_3 \circ G_2 \circ G_1)
 \end{array}$$

commutes, i.e. given natural transformations

$$\begin{array}{ccccc}
 C & \xrightarrow{F_1} & \mathcal{D} & \xrightarrow{F_2} & \mathcal{E} & \xrightarrow{F_3} & \mathcal{F} \\
 \alpha \Downarrow & & \beta \Downarrow & & \gamma \Downarrow & & \\
 & G_1 & & G_2 & & G_3 &
 \end{array}$$

we have

$$(\gamma \star \beta) \star \alpha = \gamma \star (\beta \star \alpha).$$

3. *Interaction With Identities.* Let  $F: C \rightarrow \mathcal{D}$  and  $G: \mathcal{D} \rightarrow \mathcal{E}$  be functors. The diagram

$$\begin{array}{ccc}
 \text{pt} \times \text{pt} & \xrightarrow{[\text{id}_G] \times [\text{id}_F]} & \text{Nat}(G, G) \times \text{Nat}(F, F) \\
 \uparrow \wr & & \downarrow \star_{(F, F), (G, G)} \\
 \text{pt} & \xrightarrow{[\text{id}_{G \circ F}]} & \text{Nat}(G \circ F, G \circ F)
 \end{array}$$

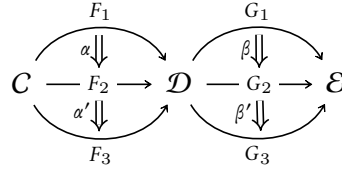
commutes, i.e. we have

$$\text{id}_G \star \text{id}_F = \text{id}_{G \circ F}.$$

4. *Middle Four Exchange.* Let  $F_1, F_2, F_3: C \rightarrow \mathcal{D}$  and  $G_1, G_2, G_3: \mathcal{D} \rightarrow \mathcal{E}$  be functors. The diagram

$$\begin{array}{ccc}
 (\text{Nat}(G_2, G_3) \times \text{Nat}(G_1, G_2)) \times (\text{Nat}(F_2, F_3) \times \text{Nat}(F_1, F_2)) & \xleftarrow{\mu_4} & (\text{Nat}(G_2, G_3) \times \text{Nat}(F_2, F_3)) \times (\text{Nat}(G_1, G_2) \times \text{Nat}(F_1, F_2)) \\
 \downarrow \circ_{G_1, G_2, G_3} \times \circ_{F_1, F_2, F_3} & & \downarrow \star_{F_2, F_3, G_2, G_3} \times \star_{F_1, F_2, G_1, G_2} \\
 \text{Nat}(G_1, G_3) \times \text{Nat}(F_1, F_3) & & \text{Nat}(G_2 \circ F_2, G_3 \circ F_3) \times \text{Nat}(G_1 \circ F_1, G_2 \circ F_2) \\
 \searrow \star_{F_1, F_3, G_1, G_3} & & \swarrow \circ_{G_1 \circ F_1, G_2 \circ F_2, G_3 \circ F_3} \\
 & \text{Nat}(G_1 \circ F_1, G_3 \circ F_3) &
 \end{array}$$

commutes, i.e. given a diagram



in  $\mathbf{Cats}_2$ , we have

$$(\beta' \star \alpha') \circ (\beta \star \alpha) = (\beta' \circ \beta) \star (\alpha' \circ \alpha).$$

*Proof.* **Item 1, Functionality:** Clear.

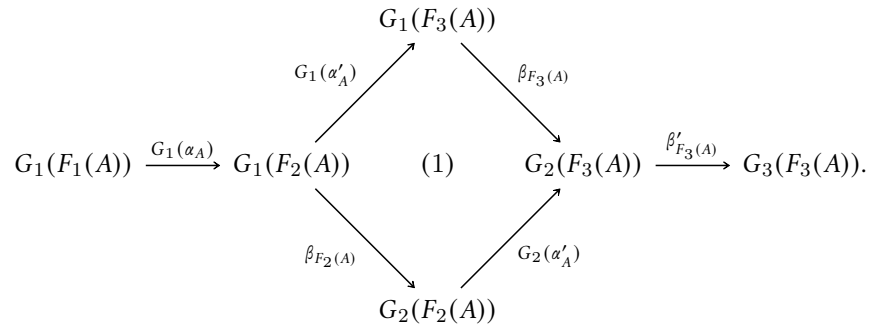
**Item 2, Associativity:** Omitted.

**Item 3, Interaction With Identities:** We have

$$\begin{aligned}
 (\mathrm{id}_G \star \mathrm{id}_F)_A &\stackrel{\mathrm{def}}{=} (\mathrm{id}_G)_{F_A} \circ G_{(\mathrm{id}_F)_A} \\
 &\stackrel{\mathrm{def}}{=} \mathrm{id}_{G_{F_A}} \circ G_{\mathrm{id}_{F_A}} \\
 &= \mathrm{id}_{G_{F_A}} \circ \mathrm{id}_{G_{F_A}} \\
 &= \mathrm{id}_{G_{F_A}} \\
 &\stackrel{\mathrm{def}}{=} (\mathrm{id}_{G \circ F})_A
 \end{aligned}$$

for each  $A \in \mathrm{Obj}(C)$ , showing the desired equality.

**Item 4, Middle Four Exchange:** Let  $A \in \mathrm{Obj}(C)$  and consider the diagram



The top composition

$$\begin{array}{ccccc}
 & & G_1(F_3(A)) & & \\
 & \nearrow^{G_1(\alpha'_A)} & & \searrow_{\beta_{F_3(A)}} & \\
 G_1(F_1(A)) & \xrightarrow{G_1(\alpha_A)} & G_1(F_2(A)) & \text{(1)} & G_2(F_3(A)) \xrightarrow{\beta'_{F_3(A)}} G_3(F_3(A)). \\
 & & \searrow_{\beta_{F_2(A)}} & & \nearrow_{G_2(\alpha'_A)} \\
 & & G_2(F_2(A)) & & 
 \end{array}$$

is given by  $((\beta' \circ \beta) \star (\alpha' \circ \alpha))_A$ , while the bottom composition

$$\begin{array}{ccccc}
 & & G_1(F_3(A)) & & \\
 & \nearrow^{G_1(\alpha'_A)} & & \searrow_{\beta_{F_3(A)}} & \\
 G_1(F_1(A)) & \xrightarrow{G_1(\alpha_A)} & G_1(F_2(A)) & \text{(1)} & G_2(F_3(A)) \xrightarrow{\beta'_{F_3(A)}} G_3(F_3(A)). \\
 & & \searrow_{\beta_{F_2(A)}} & & \nearrow_{G_2(\alpha'_A)} \\
 & & G_2(F_2(A)) & & 
 \end{array}$$

is given by  $((\beta' \star \alpha') \circ (\beta \star \alpha))_A$ . Now, Subdiagram (1) corresponds to the naturality condition

$$\begin{array}{ccc}
 G_1(F_2(A)) & \xrightarrow{G_1(\alpha'_A)} & G_1(F_3(A)) \\
 \beta_{F_2(A)} \downarrow & & \downarrow \beta_{F_3(A)} \\
 G_2(F_2(A)) & \xrightarrow{G_2(\alpha'_A)} & G_2(F_3(A))
 \end{array}$$

for  $\beta: G_1 \Rightarrow G_2$  at  $\alpha'_A: F_2(A) \rightarrow F_3(A)$ , and thus commutes. Thus we have

$$((\beta' \circ \beta) \star (\alpha' \circ \alpha))_A = ((\beta' \star \alpha') \circ (\beta \star \alpha))_A$$

for each  $A \in \text{Obj}(C)$  and therefore

$$(\beta' \star \alpha') \circ (\beta \star \alpha) = (\beta' \circ \beta) \star (\alpha' \circ \alpha).$$

This finishes the proof.  $\square$

### 8.5 Properties of Natural Transformations

**Proposition 8.5.1.1.** Let  $F, G: C \Rightarrow D$  be functors. The following data are equivalent:<sup>35</sup>

1. A natural transformation  $\alpha: F \Rightarrow G$ .
2. A functor  $[\alpha]: C \rightarrow \mathcal{D}^{\mathbb{1}}$  filling the diagram

$$\begin{array}{ccc}
 & & \mathcal{D} \\
 & \nearrow F & \uparrow \text{ev}_0 \\
 C & \xrightarrow{[\alpha]} & \mathcal{D}^{\mathbb{1}} \\
 & \searrow G & \downarrow \text{ev}_1 \\
 & & \mathcal{D}
 \end{array}$$

3. A functor  $[\alpha]: C \times \mathbb{1} \rightarrow \mathcal{D}$  filling the diagram

$$\begin{array}{ccc}
 C & & \\
 \uparrow \text{ev}_0 & \searrow F & \\
 C \times \mathbb{1} & \xrightarrow{[\alpha]} & \mathcal{D} \\
 \downarrow \text{ev}_1 & \nearrow G & \\
 C & & 
 \end{array}$$

*Proof.* From *Item 1 to Item 2 and Back*: We may identify  $\mathcal{D}^{\mathbb{1}}$  with  $\text{Arr}(\mathcal{D})$ . Given a

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<sup>35</sup>Taken from [MO 64365].

natural transformation  $\alpha: F \Rightarrow G$ , we have a functor

$$\begin{aligned} [\alpha]: C &\longrightarrow \mathcal{D}^{\mathbb{1}} \\ A &\longmapsto \alpha_A \\ (f: A \rightarrow B) &\longmapsto \left( \begin{array}{ccc} F_A & \xrightarrow{F_f} & F_B \\ \alpha_A \downarrow & & \downarrow \alpha_B \\ G_A & \xrightarrow{G_f} & G_B \end{array} \right) \end{aligned}$$

making the diagram in [Item 2](#) commute. Conversely, every such functor gives rise to a natural transformation from  $F$  to  $G$ , and these constructions are inverse to each other.

From [Item 2](#) to [Item 3](#) and Back: This follows from [Item 3](#) of [Proposition 9.1.1.2](#).  $\square$

## 8.6 Natural Isomorphisms

Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories and let  $F, G: \mathcal{C} \Rightarrow \mathcal{D}$  be functors.

**Definition 8.6.1.1.** A natural transformation  $\alpha: F \Rightarrow G$  is a **natural isomorphism** if there exists a natural transformation  $\alpha^{-1}: G \Rightarrow F$  such that

$$\begin{aligned} \alpha^{-1} \circ \alpha &= \text{id}_F, \\ \alpha \circ \alpha^{-1} &= \text{id}_G. \end{aligned}$$

**Proposition 8.6.1.2.** Let  $\alpha: F \Rightarrow G$  be a natural transformation.

1. *Characterisations.* The following conditions are equivalent:
  - (a) The natural transformation  $\alpha$  is a natural isomorphism.
  - (b) For each  $A \in \text{Obj}(\mathcal{C})$ , the morphism  $\alpha_A: F_A \rightarrow G_A$  is an isomorphism.
2. *Componentwise Inverses of Natural Transformations Assemble Into Natural Transformations.* Let  $\alpha^{-1}: G \Rightarrow F$  be a transformation such that, for each  $A \in \text{Obj}(\mathcal{C})$ , we have

$$\begin{aligned} \alpha_A^{-1} \circ \alpha_A &= \text{id}_{F(A)}, \\ \alpha_A \circ \alpha_A^{-1} &= \text{id}_{G(A)}. \end{aligned}$$

Then  $\alpha^{-1}$  is a natural transformation.

*Proof.* **Item 1, Characterisations:** The implication **Item 1a**  $\implies$  **Item 1b** is clear, whereas the implication **Item 1b**  $\implies$  **Item 1a** follows from **Item 2**.

**Item 2, Componentwise Inverses of Natural Transformations Assemble Into Natural Transformations:** The naturality condition for  $\alpha^{-1}$  corresponds to the commutativity of the diagram

$$\begin{array}{ccc} G(A) & \xrightarrow{G(f)} & G(B) \\ \alpha_A^{-1} \downarrow & & \downarrow \alpha_B^{-1} \\ F(A) & \xrightarrow{F(f)} & F(B) \end{array}$$

for each  $A, B \in \text{Obj}(C)$  and each  $f \in \text{Hom}_C(A, B)$ . Considering the diagram

$$\begin{array}{ccccc} G(A) & \xrightarrow{G(f)} & G(B) & & \\ \alpha_A^{-1} \downarrow & & (1) & & \downarrow \alpha_B^{-1} \\ F(A) & \xrightarrow{F(f)} & F(B) & & \\ \alpha_A \downarrow & & (2) & & \downarrow \alpha_B \\ G(A) & \xrightarrow{G(f)} & G(B), & & \end{array}$$

where the boundary diagram as well as Subdiagram (2) commute, we have

$$\begin{aligned} G(f) &= G(f) \circ \text{id}_{G(A)} \\ &= G(f) \circ \alpha_A \circ \alpha_A^{-1} \\ &= \alpha_B \circ F(f) \circ \alpha_A^{-1}. \end{aligned}$$

Postcomposing both sides with  $\alpha_B^{-1}$ , we get

$$\begin{aligned} \alpha_B^{-1} \circ G(f) &= \alpha_B^{-1} \circ \alpha_B \circ F(f) \circ \alpha_A^{-1} \\ &= \text{id}_{F(B)} \circ F(f) \circ \alpha_A^{-1} \\ &= F(f) \circ \alpha_A^{-1}, \end{aligned}$$

which is the naturality condition we wanted to show. Thus  $\alpha^{-1}$  is a natural transformation.  $\square$



## 9 Categories of Categories

### 9.1 Functor Categories

Let  $\mathcal{C}$  be a category and  $\mathcal{D}$  be a small category.

**Definition 9.1.1.1.** The **category of functors from  $\mathcal{C}$  to  $\mathcal{D}$** <sup>36</sup> is the category  $\text{Fun}(\mathcal{C}, \mathcal{D})$ <sup>37</sup> where

- *Objects.* The objects of  $\text{Fun}(\mathcal{C}, \mathcal{D})$  are functors from  $\mathcal{C}$  to  $\mathcal{D}$ .
- *Morphisms.* For each  $F, G \in \text{Obj}(\text{Fun}(\mathcal{C}, \mathcal{D}))$ , we have

$$\text{Hom}_{\text{Fun}(\mathcal{C}, \mathcal{D})}(F, G) \stackrel{\text{def}}{=} \text{Nat}(F, G).$$

- *Identities.* For each  $F \in \text{Obj}(\text{Fun}(\mathcal{C}, \mathcal{D}))$ , the unit map

$$\mathbb{1}_F^{\text{Fun}(\mathcal{C}, \mathcal{D})} : \text{pt} \rightarrow \text{Nat}(F, F)$$

of  $\text{Fun}(\mathcal{C}, \mathcal{D})$  at  $F$  is given by

$$\text{id}_F^{\text{Fun}(\mathcal{C}, \mathcal{D})} \stackrel{\text{def}}{=} \text{id}_F,$$

where  $\text{id}_F : F \Rightarrow F$  is the identity natural transformation of  $F$  of [Example 8.2.1.4](#).

- *Composition.* For each  $F, G, H \in \text{Obj}(\text{Fun}(\mathcal{C}, \mathcal{D}))$ , the composition map

$$\circ_{F, G, H}^{\text{Fun}(\mathcal{C}, \mathcal{D})} : \text{Nat}(G, H) \times \text{Nat}(F, G) \rightarrow \text{Nat}(F, H)$$

of  $\text{Fun}(\mathcal{C}, \mathcal{D})$  at  $(F, G, H)$  is given by

$$\beta \circ_{F, G, H}^{\text{Fun}(\mathcal{C}, \mathcal{D})} \alpha \stackrel{\text{def}}{=} \beta \circ \alpha,$$

where  $\beta \circ \alpha$  is the vertical composition of  $\alpha$  and  $\beta$  of [Item 1 of Proposition 8.3.1.2](#).

**Proposition 9.1.1.2.** Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories and let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be a functor.

<sup>36</sup> *Further Terminology:* Also called the **functor category**  $\text{Fun}(\mathcal{C}, \mathcal{D})$ .

<sup>37</sup> *Further Notation:* Also written  $\mathcal{D}^{\mathcal{C}}$  and  $[\mathcal{C}, \mathcal{D}]$ .

1. *Functoriality.* The assignments  $C, \mathcal{D}, (C, \mathcal{D}) \mapsto \text{Fun}(C, \mathcal{D})$  define functors

$$\begin{aligned}\text{Fun}(C, -_2) &: \text{Cats} \rightarrow \text{Cats}, \\ \text{Fun}(-_1, \mathcal{D}) &: \text{Cats}^{\text{op}} \rightarrow \text{Cats}, \\ \text{Fun}(-_1, -_2) &: \text{Cats}^{\text{op}} \times \text{Cats} \rightarrow \text{Cats}.\end{aligned}$$

2. *2-Functoriality.* The assignments  $C, \mathcal{D}, (C, \mathcal{D}) \mapsto \text{Fun}(C, \mathcal{D})$  define 2-functors

$$\begin{aligned}\text{Fun}(C, -_2) &: \text{Cats}_2 \rightarrow \text{Cats}_2, \\ \text{Fun}(-_1, \mathcal{D}) &: \text{Cats}_2^{\text{op}} \rightarrow \text{Cats}_2, \\ \text{Fun}(-_1, -_2) &: \text{Cats}_2^{\text{op}} \times \text{Cats}_2 \rightarrow \text{Cats}_2.\end{aligned}$$

3. *Adjointness.* We have adjunctions

$$\begin{aligned}(C \times - \dashv \text{Fun}(C, -)) &: \text{Cats} \overset{C \times -}{\underset{\text{Fun}(C, -)}{\rightleftarrows}} \text{Cats}, \\ (- \times \mathcal{D} \dashv \text{Fun}(\mathcal{D}, -)) &: \text{Cats} \overset{- \times \mathcal{D}}{\underset{\text{Fun}(\mathcal{D}, -)}{\rightleftarrows}} \text{Cats},\end{aligned}$$

witnessed by bijections of sets

$$\begin{aligned}\text{Hom}_{\text{Cats}}(C \times \mathcal{D}, \mathcal{E}) &\cong \text{Hom}_{\text{Cats}}(\mathcal{D}, \text{Fun}(C, \mathcal{E})), \\ \text{Hom}_{\text{Cats}}(C \times \mathcal{D}, \mathcal{E}) &\cong \text{Hom}_{\text{Cats}}(C, \text{Fun}(\mathcal{D}, \mathcal{E})),\end{aligned}$$

natural in  $C, \mathcal{D}, \mathcal{E} \in \text{Obj}(\text{Cats})$ .

4. *2-Adjointness.* We have 2-adjunctions

$$\begin{aligned}(C \times - \dashv \text{Fun}(C, -)) &: \text{Cats}_2 \overset{C \times -}{\underset{\text{Fun}(C, -)}{\rightleftarrows}} \text{Cats}_2, \\ (- \times \mathcal{D} \dashv \text{Fun}(\mathcal{D}, -)) &: \text{Cats}_2 \overset{- \times \mathcal{D}}{\underset{\text{Fun}(\mathcal{D}, -)}{\rightleftarrows}} \text{Cats}_2,\end{aligned}$$

witnessed by isomorphisms of categories

$$\begin{aligned}\mathrm{Fun}(C \times \mathcal{D}, \mathcal{E}) &\cong \mathrm{Fun}(\mathcal{D}, \mathrm{Fun}(C, \mathcal{E})), \\ \mathrm{Fun}(C \times \mathcal{D}, \mathcal{E}) &\cong \mathrm{Fun}(C, \mathrm{Fun}(\mathcal{D}, \mathcal{E})),\end{aligned}$$

natural in  $C, \mathcal{D}, \mathcal{E} \in \mathrm{Obj}(\mathrm{Cats}_2)$ .

5. *Interaction With Punctual Categories.* We have a canonical isomorphism of categories

$$\mathrm{Fun}(\mathrm{pt}, C) \cong C,$$

natural in  $C \in \mathrm{Obj}(\mathrm{Cats})$ .

6. *Objectwise Computation of Co/Limits.* Let

$$D: \mathcal{I} \rightarrow \mathrm{Fun}(C, \mathcal{D})$$

be a diagram in  $\mathrm{Fun}(C, \mathcal{D})$ . We have isomorphisms

$$\begin{aligned}\lim(D)_A &\cong \lim_{i \in \mathcal{I}}(D_i(A)), \\ \mathrm{colim}(D)_A &\cong \mathrm{colim}_{i \in \mathcal{I}}(D_i(A)),\end{aligned}$$

naturally in  $A \in \mathrm{Obj}(C)$ .

7. *Interaction With Co/Completeness.* If  $\mathcal{E}$  is co/complete, then so is  $\mathrm{Fun}(C, \mathcal{E})$ .  
 8. *Monomorphisms and Epimorphisms.* Let  $\alpha: F \Longrightarrow G$  be a morphism of  $\mathrm{Fun}(C, \mathcal{D})$ . The following conditions are equivalent:

- (a) The natural transformation

$$\alpha: F \Longrightarrow G$$

is a monomorphism (resp. epimorphism) in  $\mathrm{Fun}(C, \mathcal{D})$ .

- (b) For each  $A \in \mathrm{Obj}(C)$ , the morphism

$$\alpha_A: F_A \rightarrow G_A$$

is a monomorphism (resp. epimorphism) in  $\mathcal{D}$ .

*Proof.* **Item 1**, Functoriality: Omitted.

**Item 2**, 2-Functoriality: Omitted.

**Item 3**, Adjointness: Omitted.

**Item 4**, 2-Adjointness: Omitted.

**Item 5**, Interaction With Punctual Categories: Omitted.

**Item 6**, Objectwise Computation of Co/Limits: Omitted.

**Item 7**, Interaction With Co/Completeness: This follows from ??.

**Item 8**, Monomorphisms and Epimorphisms: Omitted.  $\square$

## 9.2 The Category of Categories and Functors

**Definition 9.2.1.1.** The **category of (small) categories and functors** is the category  $\mathbf{Cats}$  where

- *Objects.* The objects of  $\mathbf{Cats}$  are small categories.
- *Morphisms.* For each  $C, \mathcal{D} \in \mathbf{Obj}(\mathbf{Cats})$ , we have

$$\mathrm{Hom}_{\mathbf{Cats}}(C, \mathcal{D}) \stackrel{\mathrm{def}}{=} \mathbf{Obj}(\mathbf{Fun}(C, \mathcal{D})).$$

- *Identities.* For each  $C \in \mathbf{Obj}(\mathbf{Cats})$ , the unit map

$$\mathbb{1}_C^{\mathbf{Cats}}: \mathrm{pt} \rightarrow \mathrm{Hom}_{\mathbf{Cats}}(C, C)$$

of  $\mathbf{Cats}$  at  $C$  is defined by

$$\mathrm{id}_C^{\mathbf{Cats}} \stackrel{\mathrm{def}}{=} \mathrm{id}_C,$$

where  $\mathrm{id}_C: C \rightarrow C$  is the identity functor of  $C$  of **Example 4.1.1.4**.

- *Composition.* For each  $C, \mathcal{D}, \mathcal{E} \in \mathbf{Obj}(\mathbf{Cats})$ , the composition map

$$\circ_{C, \mathcal{D}, \mathcal{E}}^{\mathbf{Cats}}: \mathrm{Hom}_{\mathbf{Cats}}(\mathcal{D}, \mathcal{E}) \times \mathrm{Hom}_{\mathbf{Cats}}(C, \mathcal{D}) \rightarrow \mathrm{Hom}_{\mathbf{Cats}}(C, \mathcal{E})$$

of  $\mathbf{Cats}$  at  $(C, \mathcal{D}, \mathcal{E})$  is given by

$$G \circ_{C, \mathcal{D}, \mathcal{E}}^{\mathbf{Cats}} F \stackrel{\mathrm{def}}{=} G \circ F,$$

where  $G \circ F: C \rightarrow \mathcal{E}$  is the composition of  $F$  and  $G$  of **Definition 4.1.1.5**.

**Proposition 9.2.1.2.** Let  $C$  be a category.

1. *Co/Completeness*. The category  $\mathbf{Cats}$  is complete and cocomplete.
2. *Cartesian Monoidal Structure*. The quadruple  $(\mathbf{Cats}, \times, \text{pt}, \text{Fun})$  is a Cartesian closed monoidal category.

*Proof.* **Item 1**, *Co/Completeness*: Omitted.

**Item 2**, *Cartesian Monoidal Structure*: Omitted.  $\square$

### 9.3 The 2-Category of Categories, Functors, and Natural Transformations

**Definition 9.3.1.1.** The **2-category of (small) categories, functors, and natural transformations** is the 2-category  $\mathbf{Cats}_2$  where

- *Objects*. The objects of  $\mathbf{Cats}_2$  are small categories.
- *Hom-Categories*. For each  $C, \mathcal{D} \in \text{Obj}(\mathbf{Cats}_2)$ , we have

$$\text{Hom}_{\mathbf{Cats}_2}(C, \mathcal{D}) \stackrel{\text{def}}{=} \text{Fun}(C, \mathcal{D}).$$

- *Identities*. For each  $C \in \text{Obj}(\mathbf{Cats}_2)$ , the unit functor

$$\mathbb{1}_C^{\mathbf{Cats}_2} : \text{pt} \rightarrow \text{Fun}(C, C)$$

of  $\mathbf{Cats}_2$  at  $C$  is the functor picking the identity functor  $\text{id}_C : C \rightarrow C$  of  $C$ .

- *Composition*. For each  $C, \mathcal{D}, \mathcal{E} \in \text{Obj}(\mathbf{Cats}_2)$ , the composition bifunctor

$$\circ_{C, \mathcal{D}, \mathcal{E}}^{\mathbf{Cats}_2} : \text{Hom}_{\mathbf{Cats}_2}(\mathcal{D}, \mathcal{E}) \times \text{Hom}_{\mathbf{Cats}_2}(C, \mathcal{D}) \rightarrow \text{Hom}_{\mathbf{Cats}_2}(C, \mathcal{E})$$

of  $\mathbf{Cats}_2$  at  $(C, \mathcal{D}, \mathcal{E})$  is the functor where

- *Action on Objects*. For each object  $(G, F) \in \text{Obj}(\text{Hom}_{\mathbf{Cats}_2}(\mathcal{D}, \mathcal{E}) \times \text{Hom}_{\mathbf{Cats}_2}(C, \mathcal{D}))$ , we have

$$\circ_{C, \mathcal{D}, \mathcal{E}}^{\mathbf{Cats}_2}(G, F) \stackrel{\text{def}}{=} G \circ F.$$

- *Action on Morphisms*. For each morphism  $(\beta, \alpha) : (K, H) \Rightarrow (G, F)$  of  $\text{Hom}_{\mathbf{Cats}_2}(\mathcal{D}, \mathcal{E}) \times \text{Hom}_{\mathbf{Cats}_2}(C, \mathcal{D})$ , we have

$$\circ_{C, \mathcal{D}, \mathcal{E}}^{\mathbf{Cats}_2}(\beta, \alpha) \stackrel{\text{def}}{=} \beta \star \alpha,$$

where  $\beta \star \alpha$  is the horizontal composition of  $\alpha$  and  $\beta$  of **Definition 8.4.1.1**.

**Proposition 9.3.1.2.** Let  $C$  be a category.

1. *2-Categorical Co/Completeness.* The 2-category  $\text{Cats}_2$  is complete and co-complete as a 2-category, having all 2-categorical and bicategorical co/limits.

*Proof.* *Item 1, Co/Completeness:* Omitted. □

## 9.4 The Category of Groupoids

**Definition 9.4.1.1.** The **category of (small) groupoids** is the full subcategory  $\text{Grpd}$  of  $\text{Cats}$  spanned by the groupoids.

## 9.5 The 2-Category of Groupoids

**Definition 9.5.1.1.** The **2-category of (small) groupoids** is the full sub-2-category  $\text{Grpd}_2$  of  $\text{Cats}_2$  spanned by the groupoids.

# Appendices

## A Other Chapters

### Sets

1. *Sets*
2. *Constructions With Sets*
3. *Pointed Sets*
4. *Tensor Products of Pointed Sets*

### Relations

5. *Relations*

### 6. *Constructions With Relations*

7. *Equivalence Relations and Apartness Relations*

### Category Theory

8. *Categories*

### Bicategories

9. *Types of Morphisms in Bicat-  
egories*

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