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Contents

Part I Sets 1

1 Sets **2**
- 1.1 Sets and Functions 2
- 1.2 The Enrichment of Sets in Classical Truth Values 4
- 1.A Other Chapters 10

2 Constructions With Sets **11**
- 2.1 Limits of Sets 13
- 2.2 Colimits of Sets 32
- 2.3 Operations With Sets 47
- 2.4 Powersets 63
- 2.A Other Chapters 101

3 Pointed Sets **102**
- 3.1 Pointed Sets 103
- 3.2 Limits of Pointed Sets 108
- 3.3 Colimits of Pointed Sets 118
- 3.4 Constructions With Pointed Sets 131
- 3.A Other Chapters 136

4 Tensor Products of Pointed Sets **137**
- 4.1 Bilinear Morphisms of Pointed Sets 140
- 4.2 Tensors and Cotensors of Pointed Sets by Sets 143
- 4.3 The Left Tensor Product of Pointed Sets 157
- 4.4 The Right Tensor Product of Pointed Sets 179
- 4.5 The Smash Product of Pointed Sets 200
- 4.6 Miscellany 229
- 4.A Other Chapters 230

Part II Relations **231**

5 Relations **232**
5.1 Relations 234
5.2 Categories of Relations 243
5.3 Properties of the 2-Category of Relations 256
5.4 The Left Skew Monoidal Structure on $\mathbf{Rel}(A, B)$ 276
5.5 The Right Skew Monoidal Structure on $\mathbf{Rel}(A, B)$ 281
5.A Other Chapters 286

6 Constructions With Relations **287**
6.1 Co/Limits in the Category of Relations 289
6.2 Kan Extensions and Kan Lifts in the 2-Category of Relations 289
6.3 More Constructions With Relations 293
6.4 Functoriality of Powersets 314
6.A Other Chapters 337

7 Equivalence Relations and Apartness Relations **338**
7.1 Reflexive Relations 339
7.2 Symmetric Relations 341
7.3 Transitive Relations 343
7.4 Equivalence Relations 346
7.5 Quotients by Equivalence Relations 348
7.A Other Chapters 353

Part III Category Theory **354**

8 Categories **355**
8.1 Categories 357
8.2 The Quadruple Adjunction With Sets 367
8.3 Groupoids 374
8.4 Functors 380
8.5 Conditions on Functors 389
8.6 More Conditions on Functors 403
8.7 Even More Conditions on Functors 411
8.8 Natural Transformations 417
8.9 Categories of Categories 429
8.A Other Chapters 434

Part IV Bicategories 435

9 Types of Morphisms in Bicategories **436**
9.1 Monomorphisms in Bicategories 437
9.2 Epimorphisms in Bicategories 446
9.A Other Chapters 454

Part V Extra Part 456

10 Miscellaneous Notes **457**
10.1 To Do List 457
10.A Other Chapters 464

Index of Notation **476**

Index of Set Theory **479**

Index of Category Theory **483**

Index of Higher Category Theory **485**

Contents

Part I Sets **1**

1 Sets **2**

1.1 Sets and Functions 2

1.1.1 Functions 2

1.2 The Enrichment of Sets in Classical Truth Values 4

1.2.1 (-2)-Categories 4

1.2.2 (-1)-Categories 4

1.2.3 0-Categories 8

1.2.4 Tables of Analogies Between Set Theory and Category Theory 8

1.A Other Chapters 10

2 Constructions With Sets **11**

2.1 Limits of Sets 13

2.1.1 The Terminal Set 13

2.1.2 Products of Families of Sets 13

2.1.3 Binary Products of Sets 15

2.1.4 Pullbacks 23

2.1.5 Equalisers 28

2.2 Colimits of Sets 32

2.2.1 The Initial Set 32

2.2.2 Coproducts of Families of Sets 33

2.2.3 Binary Coproducts 35

2.2.4 Pushouts 37

2.2.5 Coequalisers 44

2.3 Operations With Sets 47

2.3.1 The Empty Set 47

2.3.2 Singleton Sets 47

2.3.3 Pairings of Sets 48

2.3.4 Ordered Pairs 48
2.3.5 Sets of Maps 48
2.3.6 Unions of Families 49
2.3.7 Binary Unions 49
2.3.8 Intersections of Families 51
2.3.9 Binary Intersections 51
2.3.10 Differences 55
2.3.11 Complements 58
2.3.12 Symmetric Differences 59
2.4 Powersets 63
2.4.1 Characteristic Functions 63
2.4.2 The Yoneda Lemma for Sets 68
2.4.3 Powersets 69
2.4.4 Direct Images 84
2.4.5 Inverse Images 90
2.4.6 Direct Images With Compact Support 93
2.A Other Chapters 101

3 Pointed Sets 102
3.1 Pointed Sets 103
3.1.1 Foundations 103
3.1.2 Morphisms of Pointed Sets 104
3.1.3 The Category of Pointed Sets 104
3.1.4 Elementary Properties of Pointed Sets 105
3.2 Limits of Pointed Sets 108
3.2.1 The Terminal Pointed Set 108
3.2.2 Products of Families of Pointed Sets 108
3.2.3 Products 110
3.2.4 Pullbacks 112
3.2.5 Equalisers 116
3.3 Colimits of Pointed Sets 118
3.3.1 The Initial Pointed Set 118
3.3.2 Coproducts of Families of Pointed Sets 119
3.3.3 Coproducts 121
3.3.4 Pushouts 124
3.3.5 Coequalisers 129
3.4 Constructions With Pointed Sets 131
3.4.1 Free Pointed Sets 131
3.A Other Chapters 136

4 Tensor Products of Pointed Sets 137
4.1 Bilinear Morphisms of Pointed Sets 140
4.1.1 Left Bilinear Morphisms of Pointed Sets 140
4.1.2 Right Bilinear Morphisms of Pointed Sets 141
4.1.3 Bilinear Morphisms of Pointed Sets 141
4.2 Tensors and Cotensors of Pointed Sets by Sets 143
4.2.1 Tensors of Pointed Sets by Sets 143
4.2.2 Cotensors of Pointed Sets by Sets 150
4.3 The Left Tensor Product of Pointed Sets 157
4.3.1 Foundations 157
4.3.2 The Left Internal Hom of Pointed Sets 161
4.3.3 The Left Skew Unit 163
4.3.4 The Left Skew Associator 163
4.3.5 The Left Skew Left Unitor 166
4.3.6 The Left Skew Right Unitor 168
4.3.7 The Diagonal 170
4.3.8 The Left Skew Monoidal Structure on Pointed Sets Associated to $\lhd$ 171
4.3.9 Monoids With Respect to the Left Tensor Product of Pointed Sets 175
4.4 The Right Tensor Product of Pointed Sets 179
4.4.1 Foundations 179
4.4.2 The Right Internal Hom of Pointed Sets 182
4.4.3 The Right Skew Unit 184
4.4.4 The Right Skew Associator 185
4.4.5 The Right Skew Left Unitor 187
4.4.6 The Right Skew Right Unitor 189
4.4.7 The Diagonal 191
4.4.8 The Right Skew Monoidal Structure on Pointed Sets Associated to $\rhd$ 192
4.4.9 Monoids With Respect to the Right Tensor Product of Pointed Sets 196
4.5 The Smash Product of Pointed Sets 200
4.5.1 Foundations 200
4.5.2 The Internal Hom of Pointed Sets 210
4.5.3 The Monoidal Unit 212
4.5.4 The Associator 212
4.5.5 The Left Unitor 214
4.5.6 The Right Unitor 216
4.5.7 The Symmetry 219
4.5.8 The Diagonal 221

4.5.9 The Monoidal Structure on Pointed Sets Associated to $\wedge$ ... 224
4.5.10 Universal Properties of the Smash Product of Pointed Sets I ... 228
4.5.11 Universal Properties of the Smash Product of Pointed Sets II ... 229
4.5.12 Monoids With Respect to the Smash Product of Pointed Sets ... 229
4.5.13 Comonoids With Respect to the Smash Product of Pointed Sets ... 229
4.6 Miscellany ... 229
4.6.1 The Smash Product of a Family of Pointed Sets ... 229
4.A Other Chapters ... 230

Part II Relations **231**

5 Relations ... 232
5.1 Relations ... 234
5.1.1 Foundations ... 234
5.1.2 Relations as Decategorifications of Profunctors ... 237
5.1.3 Examples of Relations ... 239
5.1.4 Functional Relations ... 241
5.1.5 Total Relations ... 242
5.2 Categories of Relations ... 243
5.2.1 The Category of Relations ... 243
5.2.2 The Closed Symmetric Monoidal Category of Relations ... 243
5.2.3 The 2-Category of Relations ... 248
5.2.4 The Double Category of Relations ... 249
5.3 Properties of the 2-Category of Relations ... 256
5.3.1 Self-Duality ... 256
5.3.2 Isomorphisms and Equivalences in **Rel** ... 257
5.3.3 Adjunctions in **Rel** ... 258
5.3.4 Monads in **Rel** ... 261
5.3.5 Comonads in **Rel** ... 261
5.3.6 Co/Monoids in **Rel** ... 262
5.3.7 Monomorphisms in Rel ... 263
5.3.8 2-Categorical Monomorphisms in **Rel** ... 265
5.3.9 Epimorphisms in Rel ... 269
5.3.10 2-Categorical Epimorphisms in **Rel** ... 272
5.3.11 Co/Limits in Rel ... 275

5.3.12 Kan Extensions and Kan Lifts in **Rel** 276
5.3.13 Closedness of **Rel** 276
5.3.14 **Rel** as a Category of Free Algebras 276
5.4 The Left Skew Monoidal Structure on $\mathbf{Rel}(A, B)$ 276
5.4.1 The Left Skew Monoidal Product 276
5.4.2 The Left Skew Monoidal Unit........................ 277
5.4.3 The Left Skew Associators............................ 278
5.4.4 The Left Skew Left Unitors........................... 279
5.4.5 The Left Skew Right Unitors......................... 279
5.4.6 The Left Skew Monoidal Structure on $\mathbf{Rel}(A, B)$ .. 280
5.5 The Right Skew Monoidal Structure on $\mathbf{Rel}(A, B)$ 281
5.5.1 The Right Skew Monoidal Product.................. 281
5.5.2 The Right Skew Monoidal Unit...................... 282
5.5.3 The Right Skew Associators.......................... 282
5.5.4 The Right Skew Left Unitors......................... 283
5.5.5 The Right Skew Right Unitors 284
5.5.6 The Right Skew Monoidal Structure on $\mathbf{Rel}(A, B)$
285
5.A Other Chapters 286

6 Constructions With Relations 287
6.1 Co/Limits in the Category of Relations....................... 289
6.2 Kan Extensions and Kan Lifts in the 2-Category of Relations... 289
6.2.1 Left Kan Extensions in **Rel** 289
6.2.2 Left Kan Lifts in **Rel** 290
6.2.3 Right Kan Extensions in **Rel** 291
6.2.4 Right Kan Lifts in **Rel**................................ 292
6.3 More Constructions With Relations.......................... 293
6.3.1 The Graph of a Function 293
6.3.2 The Inverse of a Function 297
6.3.3 Representable Relations 298
6.3.4 The Domain and Range of a Relation 299
6.3.5 Binary Unions of Relations........................... 300
6.3.6 Unions of Families of Relations 301
6.3.7 Binary Intersections of Relations 302
6.3.8 Intersections of Families of Relations 303
6.3.9 Binary Products of Relations......................... 303
6.3.10 Products of Families of Relations 305
6.3.11 The Inverse of a Relation 306
6.3.12 Composition of Relations 307

6.3.13 The Collage of a Relation 312
6.4 Functoriality of Powersets 314
6.4.1 Direct Images 314
6.4.2 Strong Inverse Images 318
6.4.3 Weak Inverse Images 324
6.4.4 Direct Images With Compact Support 329
6.4.5 Functoriality of Powersets 335
6.4.6 Functoriality of Powersets: Relations on Powersets 336
6.A Other Chapters 337

7 Equivalence Relations and Apartness Relations 338
7.1 Reflexive Relations 339
7.1.1 Foundations 339
7.1.2 The Reflexive Closure of a Relation 339
7.2 Symmetric Relations 341
7.2.1 Foundations 341
7.2.2 The Symmetric Closure of a Relation 341
7.3 Transitive Relations 343
7.3.1 Foundations 343
7.3.2 The Transitive Closure of a Relation 344
7.4 Equivalence Relations 346
7.4.1 Foundations 346
7.4.2 The Equivalence Closure of a Relation 347
7.5 Quotients by Equivalence Relations 348
7.5.1 Equivalence Classes 348
7.5.2 Quotients of Sets by Equivalence Relations 348
7.A Other Chapters 353

Part III Category Theory 354

8 Categories 355
8.1 Categories 357
8.1.1 Foundations 357
8.1.2 Examples of Categories 360
8.1.3 Posetal Categories 362
8.1.4 Subcategories 364
8.1.5 Skeletons of Categories 364
8.1.6 Precomposition and Postcomposition 365
8.2 The Quadruple Adjunction With Sets 367
8.2.1 Statement 367

8.2.2 Connected Components and Connected Categories 368
8.2.3 Discrete Categories 370
8.2.4 Indiscrete Categories 372
8.3 Groupoids 374
8.3.1 Foundations 374
8.3.2 The Groupoid Completion of a Category 374
8.3.3 The Core of a Category 377
8.4 Functors 380
8.4.1 Foundations 380
8.4.2 Contravariant Functors 384
8.4.3 Forgetful Functors 385
8.4.4 The Natural Transformation Associated to a Functor 387
8.5 Conditions on Functors 389
8.5.1 Faithful Functors 389
8.5.2 Full Functors 391
8.5.3 Fully Faithful Functors 394
8.5.4 Conservative Functors 397
8.5.5 Essentially Injective Functors 399
8.5.6 Essentially Surjective Functors 399
8.5.7 Equivalences of Categories 400
8.5.8 Isomorphisms of Categories 402
8.6 More Conditions on Functors 403
8.6.1 Dominant Functors 403
8.6.2 Monomorphisms of Categories 404
8.6.3 Epimorphisms of Categories 405
8.6.4 Pseudomonic Functors 407
8.6.5 Pseudoepic Functors 409
8.7 Even More Conditions on Functors 411
8.7.1 Injective on Objects Functors 411
8.7.2 Surjective on Objects Functors 411
8.7.3 Bijective on Objects Functors 411
8.7.4 Functors Representably Faithful on Cores 412
8.7.5 Functors Representably Full on Cores 412
8.7.6 Functors Representably Fully Faithful on Cores 413
8.7.7 Functors Corepresentably Faithful on Cores 414
8.7.8 Functors Corepresentably Full on Cores 415
8.7.9 Functors Corepresentably Fully Faithful on Cores 415
8.8 Natural Transformations 417
8.8.1 Transformations 417

8.8.2 Natural Transformations ... 417
8.8.3 Vertical Composition of Natural Transformations . 418
8.8.4 Horizontal Composition of Natural Transformations ... 421
8.8.5 Properties of Natural Transformations ... 426
8.8.6 Natural Isomorphisms ... 428
8.9 Categories of Categories ... 429
8.9.1 Functor Categories ... 429
8.9.2 The Category of Categories and Functors ... 432
8.9.3 The 2-Category of Categories, Functors, and Natural Transformations ... 433
8.9.4 The Category of Groupoids ... 434
8.9.5 The 2-Category of Groupoids ... 434
8.A Other Chapters ... 434

Part IV Bicategories 435

9 Types of Morphisms in Bicategories ... 436
9.1 Monomorphisms in Bicategories ... 437
9.1.1 Representably Faithful Morphisms ... 437
9.1.2 Representably Full Morphisms ... 438
9.1.3 Representably Fully Faithful Morphisms ... 439
9.1.4 Morphisms Representably Faithful on Cores ... 440
9.1.5 Morphisms Representably Full on Cores ... 441
9.1.6 Morphisms Representably Fully Faithful on Cores
442
9.1.7 Representably Essentially Injective Morphisms ... 443
9.1.8 Representably Conservative Morphisms ... 443
9.1.9 Strict Monomorphisms ... 444
9.1.10 Pseudomonic Morphisms ... 444
9.2 Epimorphisms in Bicategories ... 446
9.2.1 Corepresentably Faithful Morphisms ... 446
9.2.2 Corepresentably Full Morphisms ... 447
9.2.3 Corepresentably Fully Faithful Morphisms ... 448
9.2.4 Morphisms Corepresentably Faithful on Cores ... 449
9.2.5 Morphisms Corepresentably Full on Cores ... 450
9.2.6 Morphisms Corepresentably Fully Faithful on Cores
450
9.2.7 Corepresentably Essentially Injective Morphisms .. 451
9.2.8 Corepresentably Conservative Morphisms ... 452
9.2.9 Strict Epimorphisms ... 452

9.2.10 Pseudoepic Morphisms 453
9.A Other Chapters 454

Part V Extra Part 456

10 Miscellaneous Notes 457
10.1 To Do List 457
10.1.1 Omitted Proofs To Add 457
10.1.2 Things To Explore/Add 458
10.A Other Chapters 464

Index of Notation **476**

Index of Set Theory **479**

Index of Category Theory **483**

Index of Higher Category Theory **485**

Part I

Sets

Chapter 1

Sets

This chapter (will eventually) contain material on axiomatic set theory, as well as a couple other things.

Contents

1.1 Sets and Functions **2**

1.1.1 Functions 2

1.2 The Enrichment of Sets in Classical Truth Values . **4**

1.2.1 (-2)-Categories 4

1.2.2 (-1)-Categories 4

1.2.3 0-Categories 8

1.2.4 Tables of Analogies Between Set Theory and Category Theory 8

1.A Other Chapters **10**

1.1 Sets and Functions

1.1.1 Functions

Definition 1.1.1.1.1. A **function** is a functional and total relation.

Notation 1.1.1.1.2. Throughout this work, we will sometimes denote a function $f\colon X \to Y$ by

$$f \stackrel{\text{def}}{=} [\![x \mapsto f(x)]\!].$$

1. For example, given a function

$$\Phi\colon \mathrm{Hom}_{\mathsf{Sets}}(X, Y) \to K$$

taking values on a set of functions such as $\mathrm{Hom}_{\mathsf{Sets}}(X,Y)$, we will sometimes also write

$$\Phi(f) \overset{\text{def}}{=} \Phi\big(\llbracket x \mapsto f(x) \rrbracket\big).$$

2. This notational choice is based on the lambda notation

$$f \overset{\text{def}}{=} (\lambda x.\, f(x)),$$

but uses a "$\mapsto$" symbol for better spacing and double brackets instead of either:

   (a) Square brackets $[x \mapsto f(x)]$;
   (b) Parentheses $(x \mapsto f(x))$;

hoping to improve readability when dealing with e.g.:

   (a) Equivalence classes, cf.:
      i. $\llbracket [x] \mapsto f([x]) \rrbracket$
      ii. $[[x] \mapsto f([x])]$
      iii. $(\lambda[x].\, f([x]))$
   (b) Function evaluations, cf.:
      i. $\Phi\big(\llbracket x \mapsto f(x) \rrbracket\big)$
      ii. $\Phi((x \mapsto f(x)))$
      iii. $\Phi((\lambda x.\, f(x)))$

3. We will also sometimes write $-_1, -_2$, etc. for the arguments of a function. Some examples include:

   (a) Writing $f(-_1)$ for a function $f\colon A \to B$.
   (b) Writing $f(-_1, -_2)$ for a function $f\colon A \times B \to C$.
   (c) Given a function $f\colon A \times B \to C$, writing

$$f(a, -)\colon B \to C$$

   for the function $\llbracket b \mapsto f(a,b) \rrbracket$.
   (d) Denoting a composition of the form

$$A \times B \xrightarrow{\phi \times \mathrm{id}_B} A' \times B \xrightarrow{f} C$$

   by $f(\phi(-_1), -_2)$.

4. Finally, given a function $f \colon A \to B$, we write

$$\mathrm{ev}_a(f) \overset{\text{def}}{=} f(a)$$

for the value of f at some $a \in A$.

For an example of the above notations being used in practice, see the proof of the adjunction

$$(A \times - \dashv \mathrm{Hom}_{\mathsf{Sets}}(A, -))\colon \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "A\times -"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}, \arrow[l, bend left, "\mathrm{Hom}_{\mathsf{Sets}}(A,-)"] \end{tikzcd}$$

stated in Item 2 of Proposition 2.1.3.1.2.

1.2 The Enrichment of Sets in Classical Truth Values

1.2.1 (-2)-Categories

Definition 1.2.1.1.1. A (-2)-**category** is the "necessarily true" truth value.[1,2,3]

1.2.2 (-1)-Categories

Definition 1.2.2.1.1. A (-1)-**category** is a classical truth value.

Remark 1.2.2.1.2. [4](-1)-categories should be thought of as being "categories enriched in (-2)-categories", having a collection of objects and, for each pair of objects, a Hom-object $\mathrm{Hom}(x, y)$ that is a (-2)-category (i.e. trivial).
Therefore, a (-1)-category C is either ([BS10, pp. 33–34]):

1. *Empty*, having no objects;

2. *Contractible*, having a collection of objects $\{a, b, c, \ldots\}$, but with $\mathrm{Hom}_C(a, b)$ being a (-2)-category (i.e. trivial) for all $a, b \in \mathrm{Obj}(C)$, forcing all objects of C to be uniquely isomorphic to each other.

As such, there are only two (-1)-categories, up to equivalence:

- The (-1)-category false (the empty one);

[1] Thus, there is only one (-2)-category.

[2] A $(-n)$-category for $n = 3, 4, \ldots$ is also the "necessarily true" truth value, coinciding with a (-2)-category.

[3] For motivation, see [BS10, p. 13].

[4] For more motivation, see [BS10, p. 13].

- The (-1)-category true (the contractible one).

Definition 1.2.2.1.3. The **poset of truth values**[5] is the poset $(\{\mathsf{true}, \mathsf{false}\}, \preceq)$ consisting of

- *The Underlying Set.* The set $\{\mathsf{true}, \mathsf{false}\}$ whose elements are the truth values true and false.
- *The Partial Order.* The partial order

$$\preceq\colon \{\mathsf{true}, \mathsf{false}\} \times \{\mathsf{true}, \mathsf{false}\} \longrightarrow \{\mathsf{true}, \mathsf{false}\}$$

  on $\{\mathsf{true}, \mathsf{false}\}$ defined by[6]

$$\begin{aligned} \mathsf{false} \preceq \mathsf{false} &\overset{\text{def}}{=} \mathsf{true}, \\ \mathsf{true} \preceq \mathsf{false} &\overset{\text{def}}{=} \mathsf{false}, \\ \mathsf{false} \preceq \mathsf{true} &\overset{\text{def}}{=} \mathsf{true}, \\ \mathsf{true} \preceq \mathsf{true} &\overset{\text{def}}{=} \mathsf{true}. \end{aligned}$$

Notation 1.2.2.1.4. We also write $\{\mathsf{t}, \mathsf{f}\}$ for the poset $\{\mathsf{true}, \mathsf{false}\}$.

Proposition 1.2.2.1.5. The poset of truth values $\{\mathsf{t}, \mathsf{f}\}$ is Cartesian closed with product given by[7]

$$\begin{aligned} \mathsf{t} \times \mathsf{t} &= \mathsf{t}, \\ \mathsf{t} \times \mathsf{f} &= \mathsf{f}, \\ \mathsf{f} \times \mathsf{t} &= \mathsf{f}, \\ \mathsf{f} \times \mathsf{f} &= \mathsf{f}, \end{aligned}$$

and internal Hom $\mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}$ given by the partial order of $\{\mathsf{t}, \mathsf{f}\}$, i.e. by

$$\begin{aligned} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t}, \mathsf{t}) &= \mathsf{t}, \\ \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t}, \mathsf{f}) &= \mathsf{f}, \\ \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{t}) &= \mathsf{t}, \\ \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{f}) &= \mathsf{t}. \end{aligned}$$

Proof. Existence of Products: We claim that the products $\mathsf{t} \times \mathsf{t}$, $\mathsf{t} \times \mathsf{f}$, $\mathsf{f} \times \mathsf{t}$, and $\mathsf{f} \times \mathsf{f}$ satisfy the universal property of the product in $\{\mathsf{t}, \mathsf{f}\}$. Indeed, consider

[5] *Further Terminology:* Also called the **poset of** (-1)**-categories**.

[6] This partial order coincides with logical implication.

[7] Note that $\times$ coincides with the "and" operator, while $\mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}$ coincides with the logical implication operator.

the diagrams

$$
\begin{tikzcd}
 & P_1 \arrow[dl, bend right, "p_1^1"'] \arrow[dr, bend left, "p_2^1"] \arrow[d, dashed, "\exists!"] & \\
\mathsf{t} & \underbrace{\mathsf{t} \times \mathsf{t}}_{=\mathsf{t}} \arrow[l, two heads, "\mathrm{pr}_1"] \arrow[r, two heads, "\mathrm{pr}_2"'] & \mathsf{t}
\end{tikzcd}
\quad
\begin{tikzcd}
 & P_2 \arrow[dl, bend right, "p_1^2"'] \arrow[dr, bend left, "p_2^2"] \arrow[d, dashed, "\exists!"] & \\
\mathsf{t} & \underbrace{\mathsf{t} \times \mathsf{f}}_{=\mathsf{f}} \arrow[l, two heads, "\mathrm{pr}_1"] \arrow[r, two heads, "\mathrm{pr}_2"'] & \mathsf{f}
\end{tikzcd}
\quad
\begin{tikzcd}
 & P_3 \arrow[dl, bend right, "p_1^3"'] \arrow[dr, bend left, "p_2^3"] \arrow[d, dashed, "\exists!"] & \\
\mathsf{f} & \underbrace{\mathsf{f} \times \mathsf{t}}_{=\mathsf{f}} \arrow[l, two heads, "\mathrm{pr}_1"] \arrow[r, two heads, "\mathrm{pr}_2"'] & \mathsf{t}
\end{tikzcd}
\quad
\begin{tikzcd}
 & P_4 \arrow[dl, bend right, "p_1^4"'] \arrow[dr, bend left, "p_2^4"] \arrow[d, dashed, "\exists!"] & \\
\mathsf{f} & \underbrace{\mathsf{f} \times \mathsf{f}}_{=\mathsf{f}} \arrow[l, two heads, "\mathrm{pr}_1"] \arrow[r, two heads, "\mathrm{pr}_2"'] & \mathsf{f}.
\end{tikzcd}
$$

Here:

1. If $P_1 = \mathsf{t}$, then $p_1^1 = p_2^1 = \mathrm{id}_{\mathsf{t}}$, and there's indeed a unique morphism from P_1 to t making the diagram commute, namely id_{t};
2. If $P_1 = \mathsf{f}$, then $p_1^1 = p_2^1$ are given by the unique morphism from f to t, and there's indeed a unique morphism from P_1 to t making the diagram commute, namely the unique morphism from f to t;
3. If $P_2 = \mathsf{t}$, then there is no morphism p_2^2.
4. If $P_2 = \mathsf{f}$, then p_1^2 is the unique morphism from f to t while $p_2^2 = \mathrm{id}_{\mathsf{f}}$, and there's indeed a unique morphism from P_2 to f making the diagram commute, namely id_{f};
5. The proof for P_3 is similar to the one for P_2;
6. If $P_4 = \mathsf{t}$, then there is no morphism p_1^4 or p_2^4.
7. If $P_4 = \mathsf{f}$, then $p_1^4 = p_2^4 = \mathrm{id}_{\mathsf{f}}$, and there's indeed a unique morphism from P_4 to f making the diagram commute, namely id_{f}.

Cartesian Closedness: We claim there's a bijection

$$\mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(A \times B, C) \cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(A, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(B, C)\right)$$

natural in $A, B, C \in \{\mathsf{t}, \mathsf{f}\}$. Indeed:

- For $(A, B, C) = (\mathsf{t}, \mathsf{t}, \mathsf{t})$, we have

$$\begin{aligned}
\mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t} \times \mathsf{t}, \mathsf{t}) &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t}, \mathsf{t}) \\
&= \{\mathrm{id}_{\mathsf{true}}\} \\
&\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t}, \mathsf{t}) \\
&\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(\mathsf{t}, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t}, \mathsf{t})\right).
\end{aligned}$$

- For $(A, B, C) = (\mathsf{t}, \mathsf{t}, \mathsf{f})$, we have

$$\begin{aligned}
\mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t} \times \mathsf{t}, \mathsf{f}) &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t}, \mathsf{f}) \\
&= \emptyset \\
&\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t}, \mathsf{f}) \\
&\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(\mathsf{t}, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t}, \mathsf{f})\right).
\end{aligned}$$

- For $(A, B, C) = (\mathsf{t}, \mathsf{f}, \mathsf{t})$, we have
$$\begin{aligned}\mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t} \times \mathsf{f}, \mathsf{t}) &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{t})\\ &\cong \mathrm{pt}\\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{t})\\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\big(\mathsf{f}, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{t})\big).\end{aligned}$$
- For $(A, B, C) = (\mathsf{t}, \mathsf{f}, \mathsf{f})$, we have
$$\begin{aligned}\mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t} \times \mathsf{f}, \mathsf{f}) &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{f})\\ &\cong \{\mathrm{id}_{\mathsf{false}}\}\\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{f})\\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\big(\mathsf{t}, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{f})\big).\end{aligned}$$
- For $(A, B, C) = (\mathsf{f}, \mathsf{t}, \mathsf{t})$, we have
$$\begin{aligned}\mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f} \times \mathsf{t}, \mathsf{t}) &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{t})\\ &\cong \mathrm{pt}\\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{t})\\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\big(\mathsf{f}, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t}, \mathsf{t})\big).\end{aligned}$$
- For $(A, B, C) = (\mathsf{f}, \mathsf{t}, \mathsf{f})$, we have
$$\begin{aligned}\mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f} \times \mathsf{t}, \mathsf{f}) &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{f})\\ &\cong \{\mathrm{id}_{\mathsf{false}}\}\\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{f})\\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\big(\mathsf{f}, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t}, \mathsf{f})\big).\end{aligned}$$
- For $(A, B, C) = (\mathsf{f}, \mathsf{f}, \mathsf{t})$, we have
$$\begin{aligned}\mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f} \times \mathsf{f}, \mathsf{t}) &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{t})\\ &\cong \mathrm{pt}\\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{t})\\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\big(\mathsf{f}, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{t})\big).\end{aligned}$$
- For $(A, B, C) = (\mathsf{f}, \mathsf{f}, \mathsf{f})$, we have
$$\begin{aligned}\mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f} \times \mathsf{f}, \mathsf{f}) &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{f})\\ &= \{\mathrm{id}_{\mathsf{false}}\}\\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{f})\\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\big(\mathsf{f}, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{f})\big).\end{aligned}$$

The proof of naturality is omitted. □

1.2.3 0-Categories

Definition 1.2.3.1.1. A 0-**category** is a poset.[8]

Definition 1.2.3.1.2. A 0-**groupoid** is a 0-category in which every morphism is invertible.[9]

1.2.4 Tables of Analogies Between Set Theory and Category Theory

Here we record some analogies between notions in set theory and category theory. Note that the analogies relating to presheaves relate equally well to copresheaves, as the opposite X^{op} of a set X is just X again.
Basics:

Set Theory	Category Theory
Enrichment in $\{\mathsf{true}, \mathsf{false}\}$	Enrichment in Sets
Set X	Category $\mathcal{C}$
Element $x \in X$	Object $X \in \mathsf{Obj}(\mathcal{C})$
Function	Functor
Function $X \longrightarrow \{\mathsf{true}, \mathsf{false}\}$	Functor $\mathcal{C} \longrightarrow \mathsf{Sets}$
Function $X \longrightarrow \{\mathsf{true}, \mathsf{false}\}$	Presheaf $\mathcal{C}^{\mathsf{op}} \longrightarrow \mathsf{Sets}$

Powersets and categories of presheaves:

[8] *Motivation:* A 0-category is precisely a category enriched in the poset of (-1)-categories.
[9] That is, a *set*.

Set Theory	Category Theory
Powerset $\mathcal{P}(X)$	Presheaf category $\mathsf{PSh}(C)$
Characteristic function $\chi_{\{x\}}$	Representable presheaf h_X
Characteristic embedding $\chi_{(-)}\colon X \hookrightarrow \mathcal{P}(X)$	Yoneda embedding $よ\colon C^{\mathsf{op}} \hookrightarrow \mathsf{PSh}(C)$
Characteristic relation $\chi_X(-_1, -_2)$	Hom profunctor $\mathrm{Hom}_C(-_1, -_2)$
The Yoneda lemma for sets $\mathrm{Hom}_{\mathcal{P}(X)}(\chi_x, \chi_U) = \chi_U(x)$	The Yoneda lemma for categories $\mathrm{Nat}(h_X, \mathcal{F}) \cong \mathcal{F}(X)$
The characteristic embedding is fully faithful, $\mathrm{Hom}_{\mathcal{P}(X)}\left(\chi_x, \chi_y\right) = \chi_X(x, y)$	The Yoneda embedding is fully faithful, $\mathrm{Nat}(h_X, h_Y) \cong \mathrm{Hom}_C(X, Y)$
Subsets are unions of their elements $U = \bigcup_{x \in U} \{x\}$ or $\chi_U = \operatorname*{colim}_{\chi_x \in \mathsf{Sets}(U,\{\mathsf{t},\mathsf{f}\})} (\chi_x)$	Presheaves are colimits of representables, $\mathcal{F} \cong \operatorname*{colim}_{h_X \in \int_C \mathcal{F}} (h_X)$

Categories of elements:

Set Theory	Category Theory
Assignment $U \mapsto \chi_U$	Assignment $\mathcal{F} \mapsto \int_C \mathcal{F}$ (the category of elements)
Assignment $U \mapsto \chi_U$ giving an isomorphism $\mathcal{P}(X) \cong \mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\})$	Assignment $\mathcal{F} \mapsto \int_C \mathcal{F}$ giving an equivalence $\mathsf{PSh}(C) \overset{\text{eq.}}{\cong} \mathsf{DFib}(C)$

Functions between powersets and functors between presheaf categories:

Set Theory	Category Theory
Direct image function $f_*\colon \mathcal{P}(X) \to \mathcal{P}(Y)$	Inverse image functor $f^{-1}\colon \mathsf{PSh}(C) \to \mathsf{PSh}(\mathcal{D})$
Inverse image function $f^{-1}\colon \mathcal{P}(Y) \to \mathcal{P}(X)$	Direct image functor $f_*\colon \mathsf{PSh}(\mathcal{D}) \to \mathsf{PSh}(C)$
Direct image with compact support function $f_!\colon \mathcal{P}(X) \to \mathcal{P}(Y)$	Direct image with compact support functor $f_!\colon \mathsf{PSh}(C) \to \mathsf{PSh}(\mathcal{D})$

Relations and profunctors:

Set Theory	Category Theory
Relation $R\colon X \times Y \to \{\mathsf{t}, \mathsf{f}\}$	Profunctor $\mathfrak{p}\colon \mathcal{D}^{\mathsf{op}} \times \mathcal{C} \to \mathsf{Sets}$
Relation $R\colon X \to \mathcal{P}(Y)$	Profunctor $\mathfrak{p}\colon \mathcal{C} \to \mathsf{PSh}(\mathcal{D})$
Relation as a cocontinuous morphism of posets $R\colon (\mathcal{P}(X), \subset) \to (\mathcal{P}(Y), \subset)$	Profunctor as a colimit-preserving functor $\mathfrak{p}\colon \mathsf{PSh}(\mathcal{C}) \to \mathsf{PSh}(\mathcal{D})$

Appendices

1.A Other Chapters

Sets

1. Sets
2. Constructions With Sets
3. Pointed Sets
4. Tensor Products of Pointed Sets

Relations

5. Relations
6. Constructions With Relations
7. Equivalence Relations and Apartness Relations

Category Theory

8. Categories

Bicategories

9. Types of Morphisms in Bicategories

Chapter 2

Constructions With Sets

This chapter develops some material relating to constructions with sets with an eye towards its categorical and higher-categorical counterparts to be introduced later in this work. In particular, it contains:

1. Explicit descriptions of the major types of co/limits in Sets, including in particular explicit descriptions of pushouts and coequalisers (see Definitions 2.2.4.1.1 and 2.2.5.1.1 and Remarks 2.2.4.1.2 and 2.2.5.1.2).

2. A discussion of powersets as decategorifications of categories of presheaves (Remarks 2.4.1.1.2 and 2.4.3.1.2), including a (-1)-categorical analogue of un/straightening, described in Items 1 and 2 of Proposition 2.4.3.1.6 and Remark 2.4.3.1.7.

3. A lengthy discussion of the adjoint triple

$$f_* \dashv f^{-1} \dashv f_! \colon \mathcal{P}(A) \overset{\rightleftarrows}{\to} \mathcal{P}(B)$$

   of functors (morphisms of posets) between $\mathcal{P}(A)$ and $\mathcal{P}(B)$ induced by a map of sets $f \colon A \to B$, along with a discussion of the properties of f_*, f^{-1}, and $f_!$.

   In line with the categorical viewpoint developed here, this adjoint triple may be described in terms of Kan extensions, and, as it turns out, it also shows up in some definitions and results in point-set topology, such as in e.g. notions of continuity for functions (**??**).

Contents

2.1 Limits of Sets **13**

2.1.1 The Terminal Set 13

2.1.2 Products of Families of Sets 13
2.1.3 Binary Products of Sets 15
2.1.4 Pullbacks 23
2.1.5 Equalisers 28
2.2 Colimits of Sets 32
2.2.1 The Initial Set 32
2.2.2 Coproducts of Families of Sets 33
2.2.3 Binary Coproducts 35
2.2.4 Pushouts 37
2.2.5 Coequalisers 44
2.3 Operations With Sets 47
2.3.1 The Empty Set 47
2.3.2 Singleton Sets 47
2.3.3 Pairings of Sets 48
2.3.4 Ordered Pairs 48
2.3.5 Sets of Maps 48
2.3.6 Unions of Families 49
2.3.7 Binary Unions 49
2.3.8 Intersections of Families 51
2.3.9 Binary Intersections 51
2.3.10 Differences 55
2.3.11 Complements 58
2.3.12 Symmetric Differences 59
2.4 Powersets 63
2.4.1 Characteristic Functions 63
2.4.2 The Yoneda Lemma for Sets 68
2.4.3 Powersets 69
2.4.4 Direct Images 84
2.4.5 Inverse Images 90
2.4.6 Direct Images With Compact Support 93
2.A Other Chapters 101

2.1 Limits of Sets

2.1.1 The Terminal Set

Definition 2.1.1.1.1. The **terminal set** is the pair $\left(\mathrm{pt}, \{!_A\}_{A \in \mathrm{Obj}(\mathsf{Sets})}\right)$ consisting of:

- *The Limit.* The punctual set $\mathrm{pt} \stackrel{\mathrm{def}}{=} \{\star\}$.
- *The Cone.* The collection of maps

$$\{!_A \colon A \to \mathrm{pt}\}_{A \in \mathrm{Obj}(\mathsf{Sets})}$$

  defined by

$$!_A(a) \stackrel{\mathrm{def}}{=} \star$$

  for each $a \in A$ and each $A \in \mathrm{Obj}(\mathsf{Sets})$.

Proof. We claim that pt is the terminal object of Sets. Indeed, suppose we have a diagram of the form

$$A \qquad \mathrm{pt}$$

in Sets. Then there exists a unique map $\phi \colon A \to \mathrm{pt}$ making the diagram

$$A \xrightarrow[\exists!]{\phi} \mathrm{pt}$$

commute, namely $!_A$. □

2.1.2 Products of Families of Sets

Let $\{A_i\}_{i \in I}$ be a family of sets.

Definition 2.1.2.1.1. The **product**[1] **of** $\{A_i\}_{i \in I}$ is the pair $\left(\prod_{i \in I} A_i, \{\mathrm{pr}_i\}_{i \in I}\right)$ consisting of:

- *The Limit.* The set $\prod_{i \in I} A_i$ defined by[2]

$$\prod_{i \in I} A_i \stackrel{\mathrm{def}}{=} \left\{ f \in \mathsf{Sets}\left(I, \bigcup_{i \in I} A_i\right) \;\middle|\; \begin{array}{l}\text{for each } i \in I, \text{ we} \\ \text{have } f(i) \in A_i\end{array} \right\}.$$

[1] *Further Terminology:* Also called the **Cartesian product of** $\{A_i\}_{i \in I}$.

[2] Less formally, $\prod_{i \in I} A_i$ is the set whose elements are I-indexed collections $(a_i)_{i \in I}$ with

- *The Cone.* The collection

$$\left\{\mathrm{pr}_i \colon \prod_{i\in I} A_i \to A_i\right\}_{i\in I}$$

of maps given by

$$\mathrm{pr}_i(f) \stackrel{\text{def}}{=} f(i)$$

for each $f \in \prod_{i\in I} A_i$ and each $i \in I$.

Proof. We claim that $\prod_{i\in I} A_i$ is the categorical product of $\{A_i\}_{i\in I}$ in Sets. Indeed, suppose we have, for each $i \in I$, a diagram of the form

$$\begin{tikzcd} P \arrow[dr, "p_i"] & \\ \prod_{i\in I} A_i \arrow[r, "\mathrm{pr}_i"'] & A_i \end{tikzcd}$$

in Sets. Then there exists a unique map $\phi \colon P \to \prod_{i\in I} A_i$ making the diagram

$$\begin{tikzcd} P \arrow[d, dashed, "\phi"', "\exists!"] \arrow[dr, "p_i"] & \\ \prod_{i\in I} A_i \arrow[r, "\mathrm{pr}_i"'] & A_i \end{tikzcd}$$

commute, being uniquely determined by the condition $\mathrm{pr}_i \circ \phi = p_i$ for each $i \in I$ via

$$\phi(x) = (p_i(x))_{i\in I}$$

for each $x \in P$. $\square$

Proposition 2.1.2.1.2. Let $\{A_i\}_{i\in I}$ be a family of sets.

$a_i \in A_i$ for each $i \in I$. The projection maps

$$\left\{\mathrm{pr}_i \colon \prod_{i\in I} A_i \to A_i\right\}_{i\in I}$$

are then given by

$$\mathrm{pr}_i\Big(\big(a_j\big)_{j\in I}\Big) \stackrel{\text{def}}{=} a_i$$

for each $\big(a_j\big)_{j\in I} \in \prod_{i\in I} A_i$ and each $i \in I$.

1. *Functoriality.* The assignment $\{A_i\}_{i\in I} \mapsto \prod_{i\in I} A_i$ defines a functor

$$\prod_{i\in I} : \mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}) \to \mathsf{Sets}$$

where

- *Action on Objects.* For each $(A_i)_{i\in I} \in \mathrm{Obj}(\mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}))$, we have

$$\left[\prod_{i\in I}\right]((A_i)_{i\in I}) \stackrel{\text{def}}{=} \prod_{i\in I} A_i$$

- *Action on Morphisms.* For each $(A_i)_{i\in I}, (B_i)_{i\in I} \in \mathrm{Obj}(\mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}))$, the action on Hom-sets

$$\left(\prod_{i\in I}\right)_{(A_i)_{i\in I},(B_i)_{i\in I}} : \mathrm{Nat}((A_i)_{i\in I}, (B_i)_{i\in I}) \to \mathsf{Sets}\left(\prod_{i\in I} A_i, \prod_{i\in I} B_i\right)$$

of $\prod_{i\in I}$ at $((A_i)_{i\in I}, (B_i)_{i\in I})$ is defined by sending a map

$$\{f_i \colon A_i \to B_i\}_{i\in I}$$

in $\mathrm{Nat}((A_i)_{i\in I}, (B_i)_{i\in I})$ to the map of sets

$$\prod_{i\in I} f_i \colon \prod_{i\in I} A_i \to \prod_{i\in I} B_i$$

defined by

$$\left[\prod_{i\in I} f_i\right]((a_i)_{i\in I}) \stackrel{\text{def}}{=} (f_i(a_i))_{i\in I}$$

for each $(a_i)_{i\in I} \in \prod_{i\in I} A_i$.

Proof. *Item 1*, *Functoriality*: This follows from **??** of **??**. □

2.1.3 Binary Products of Sets

Let A and B be sets.

Definition 2.1.3.1.1. The **product**[3] **of** A **and** B is the pair $(A \times B, \{\mathrm{pr}_1, \mathrm{pr}_2\})$ consisting of:

[3] *Further Terminology:* Also called the **Cartesian product of** A **and** B or the **binary Cartesian product of** A **and** B, for emphasis.

- *The Limit.* The set $A \times B$ defined by[4]

$$\begin{aligned} A \times B &\overset{\text{def}}{=} \prod_{z \in \{A,B\}} z \\ &\overset{\text{def}}{=} \{f \in \mathsf{Sets}(\{0,1\}, A \cup B) \mid \text{we have } f(0) \in A \text{ and } f(1) \in B\} \\ &\cong \{\{\{a\},\{a,b\}\} \in \mathcal{P}(\mathcal{P}(A \cup B)) \mid \text{we have } a \in A \text{ and } b \in B\}. \end{aligned}$$

- *The Cone.* The maps

$$\begin{aligned} \mathrm{pr}_1 \colon A \times B &\to A, \\ \mathrm{pr}_2 \colon A \times B &\to B \end{aligned}$$

  defined by

$$\begin{aligned} \mathrm{pr}_1(a,b) &\overset{\text{def}}{=} a, \\ \mathrm{pr}_2(a,b) &\overset{\text{def}}{=} b \end{aligned}$$

  for each $(a,b) \in A \times B$.

Proof. We claim that $A \times B$ is the categorical product of A and B in Sets. Indeed, suppose we have a diagram of the form

$$\begin{tikzcd} & P \arrow[dl, bend right, "p_1"'] \arrow[dr, bend left, "p_2"] & \\ A & A \times B \arrow[l, "\mathrm{pr}_1"] \arrow[r, "\mathrm{pr}_2"'] & B \end{tikzcd}$$

in Sets. Then there exists a unique map $\phi \colon P \to A \times B$ making the diagram

$$\begin{tikzcd} & P \arrow[dl, bend right, "p_1"'] \arrow[dr, bend left, "p_2"] \arrow[d, dashed, "\phi"', "\exists!"] & \\ A & A \times B \arrow[l, "\mathrm{pr}_1"] \arrow[r, "\mathrm{pr}_2"'] & B \end{tikzcd}$$

commute, being uniquely determined by the conditions

$$\begin{aligned} \mathrm{pr}_1 \circ \phi &= p_1, \\ \mathrm{pr}_2 \circ \phi &= p_2 \end{aligned}$$

via

$$\phi(x) = (p_1(x), p_2(x))$$

for each $x \in P$. □

This can also be thought of as the $(\mathbb{E}_{-1}, \mathbb{E}_{-1})$**-tensor product of** A **and** B.

[4]In other words, $A \times B$ is the set whose elements are ordered pairs (a,b) with $a \in A$ and

Proposition 2.1.3.1.2. Let A, B, C, and X be sets.

1. *Functoriality.* The assignments $A, B, (A, B) \mapsto A \times B$ define functors

$$\begin{aligned} A \times -&\colon \mathsf{Sets} \to \mathsf{Sets}, \\ - \times B&\colon \mathsf{Sets} \to \mathsf{Sets}, \\ -_1 \times -_2&\colon \mathsf{Sets} \times \mathsf{Sets} \to \mathsf{Sets}, \end{aligned}$$

   where $-_1 \times -_2$ is the functor where

   - *Action on Objects.* For each $(A, B) \in \mathrm{Obj}(\mathsf{Sets} \times \mathsf{Sets})$, we have

   $$[-_1 \times -_2](A, B) \stackrel{\text{def}}{=} A \times B.$$

   - *Action on Morphisms.* For each $(A, B), (X, Y) \in \mathrm{Obj}(\mathsf{Sets})$, the action on Hom-sets

   $$\times_{(A,B),(X,Y)}\colon \mathsf{Sets}(A, X) \times \mathsf{Sets}(B, Y) \to \mathsf{Sets}(A \times B, X \times Y)$$

   of $\times$ at $((A, B), (X, Y))$ is defined by sending (f, g) to the function

   $$f \times g\colon A \times B \to X \times Y$$

   defined by

   $$[f \times g](a, b) \stackrel{\text{def}}{=} (f(a), g(b))$$

   for each $(a, b) \in A \times B$.

   and where $A \times -$ and $- \times B$ are the partial functors of $-_1 \times -_2$ at $A, B \in \mathrm{Obj}(\mathsf{Sets})$.

2. *Adjointness.* We have adjunctions

$$(A \times - \dashv \mathrm{Hom}_{\mathsf{Sets}}(A, -))\colon \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "A\times-"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}, \arrow[l, bend left, "\mathrm{Hom}_{\mathsf{Sets}}(A,-)"] \end{tikzcd}$$

$$(- \times B \dashv \mathrm{Hom}_{\mathsf{Sets}}(B, -))\colon \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "-\times B"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}, \arrow[l, bend left, "\mathrm{Hom}_{\mathsf{Sets}}(B,-)"] \end{tikzcd}$$

   witnessed by bijections

$$\begin{aligned} \mathrm{Hom}_{\mathsf{Sets}}(A \times B, C) &\cong \mathrm{Hom}_{\mathsf{Sets}}(A, \mathrm{Hom}_{\mathsf{Sets}}(B, C)), \\ \mathrm{Hom}_{\mathsf{Sets}}(A \times B, C) &\cong \mathrm{Hom}_{\mathsf{Sets}}(B, \mathrm{Hom}_{\mathsf{Sets}}(A, C)), \end{aligned}$$

   natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

3. *Associativity.* We have an isomorphism of sets

$$(A \times B) \times C \cong A \times (B \times C),$$

natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

4. *Unitality.* We have isomorphisms of sets

$$\begin{aligned} \mathrm{pt} \times A &\cong A, \\ A \times \mathrm{pt} &\cong A, \end{aligned}$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$.

5. *Commutativity.* We have an isomorphism of sets

$$A \times B \cong B \times A,$$

natural in $A, B \in \mathrm{Obj}(\mathsf{Sets})$.

6. *Annihilation With the Empty Set.* We have isomorphisms of sets

$$\begin{aligned} A \times \emptyset &\cong \emptyset, \\ \emptyset \times A &\cong \emptyset, \end{aligned}$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$.

7. *Distributivity Over Unions.* We have isomorphisms of sets

$$\begin{aligned} A \times (B \cup C) &= (A \times B) \cup (A \times C), \\ (A \cup B) \times C &= (A \times C) \cup (B \times C). \end{aligned}$$

8. *Distributivity Over Intersections.* We have isomorphisms of sets

$$\begin{aligned} A \times (B \cap C) &= (A \times B) \cap (A \times C), \\ (A \cap B) \times C &= (A \times C) \cap (B \times C). \end{aligned}$$

9. *Middle-Four Exchange with Respect to Intersections.* We have an isomorphism of sets

$$(A \times B) \cap (C \times D) \cong (A \cap B) \times (C \cap D).$$

$b \in B$ as in Definition 2.3.4.1.1

10. *Distributivity Over Differences.* We have isomorphisms of sets

$$\begin{aligned} A \times (B \setminus C) &= (A \times B) \setminus (A \times C), \\ (A \setminus B) \times C &= (A \times C) \setminus (B \times C), \end{aligned}$$

natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

11. *Distributivity Over Symmetric Differences.* We have isomorphisms of sets

$$\begin{aligned} A \times (B \triangle C) &= (A \times B) \triangle (A \times C), \\ (A \triangle B) \times C &= (A \times C) \triangle (B \times C), \end{aligned}$$

natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

12. *Symmetric Monoidality.* The triple $(\mathsf{Sets}, \times, \mathrm{pt})$ is a symmetric monoidal category.

13. *Symmetric Bimonoidality.* The quintuple $(\mathsf{Sets}, \coprod, \emptyset, \times, \mathrm{pt})$ is a symmetric bimonoidal category.

Proof. Item 1, Functoriality: This follows from **??** of **??**.
Item 2, Adjointness: We prove only that there's an adjunction $- \times B \dashv \mathrm{Hom}_{\mathsf{Sets}}(B, -)$, witnessed by a bijection

$$\mathrm{Hom}_{\mathsf{Sets}}(A \times B, C) \cong \mathrm{Hom}_{\mathsf{Sets}}(A, \mathrm{Hom}_{\mathsf{Sets}}(B, C)),$$

natural in $B, C \in \mathrm{Obj}(\mathsf{Sets})$, as the proof of the existence of the adjunction $A \times - \dashv \mathrm{Hom}_{\mathsf{Sets}}(A, -)$ follows almost exactly in the same way.

- *Map I.* We define a map

$$\Phi_{B,C} \colon \mathrm{Hom}_{\mathsf{Sets}}(A \times B, C) \longrightarrow \mathrm{Hom}_{\mathsf{Sets}}(A, \mathrm{Hom}_{\mathsf{Sets}}(B, C)),$$

by sending a function

$$\xi \colon A \times B \to C$$

to the function

$$\begin{aligned} \xi^\dagger \colon A &\longrightarrow \mathrm{Hom}_{\mathsf{Sets}}(B, C), \\ a &\longmapsto \left(\xi^\dagger_a \colon B \to C\right), \end{aligned}$$

where we define

$$\xi^\dagger_a(b) \overset{\text{def}}{=} \xi(a, b)$$

for each $b \in B$. In terms of the $[\![a \mapsto f(a)]\!]$ notation of Notation 1.1.1.1.2, we have

$$\xi^\dagger \overset{\text{def}}{=} [\![a \mapsto [\![b \mapsto \xi(a, b)]\!]]\!].$$

- *Map II.* We define a map

  $$\Psi_{B,C}\colon \mathrm{Hom}_{\mathsf{Sets}}(A, \mathrm{Hom}_{\mathsf{Sets}}(B,C)), \to \mathrm{Hom}_{\mathsf{Sets}}(A\times B, C)$$

  given by sending a function

  $$\begin{aligned} \xi\colon A &\longrightarrow \mathrm{Hom}_{\mathsf{Sets}}(B,C),\\ a &\longmapsto (\xi_a\colon B\to C), \end{aligned}$$

  to the function
  $$\xi^{\dagger}\colon A\times B\to C$$
  defined by

  $$\begin{aligned} \xi^{\dagger}(a,b) &\overset{\text{def}}{=} \mathrm{ev}_b(\mathrm{ev}_a(\xi))\\ &\overset{\text{def}}{=} \mathrm{ev}_b(\xi_a)\\ &\overset{\text{def}}{=} \xi_a(b) \end{aligned}$$

  for each $(a,b)\in A\times B$.

- *Invertibility I.* We claim that

  $$\Psi_{A,B}\circ\Phi_{A,B} = \mathrm{id}_{\mathrm{Hom}_{\mathsf{Sets}}(A\times B,C)}.$$

  Indeed, given a function $\xi\colon A\times B\to C$, we have

$$\begin{aligned} \big[\Psi_{A,B}\circ\Phi_{A,B}\big](\xi) &= \Psi_{A,B}\big(\Phi_{A,B}(\xi)\big)\\ &= \Psi_{A,B}\big(\Phi_{A,B}\big(\llbracket(a,b)\mapsto\xi(a,b)\rrbracket\big)\big)\\ &= \Psi_{A,B}\big(\llbracket a\mapsto\llbracket b\mapsto\xi(a,b)\rrbracket\rrbracket\big)\\ &= \Psi_{A,B}\big(\llbracket a'\mapsto\llbracket b'\mapsto\xi(a',b')\rrbracket\rrbracket\big)\\ &= \llbracket(a,b)\mapsto\mathrm{ev}_b\big(\mathrm{ev}_a\big(\llbracket a'\mapsto\llbracket b'\mapsto\xi(a',b')\rrbracket\rrbracket\big)\big)\rrbracket\\ &= \llbracket(a,b)\mapsto\mathrm{ev}_b\big(\llbracket b'\mapsto\xi(a,b')\rrbracket\big)\rrbracket\\ &= \llbracket(a,b)\mapsto\xi(a,b)\rrbracket\\ &= \xi. \end{aligned}$$

- *Invertibility II.* We claim that

  $$\Phi_{A,B}\circ\Psi_{A,B} = \mathrm{id}_{\mathrm{Hom}_{\mathsf{Sets}}(A,\mathrm{Hom}_{\mathsf{Sets}}(B,C))}.$$

  Indeed, given a function

  $$\begin{aligned} \xi\colon A &\longrightarrow \mathrm{Hom}_{\mathsf{Sets}}(B,C),\\ a &\longmapsto (\xi_a\colon B\to C), \end{aligned}$$

we have

$$
\begin{aligned}
\left[\Phi_{A,B}\circ\Psi_{A,B}\right](\xi) &\overset{\text{def}}{=} \Phi_{A,B}\left(\Psi_{A,B}(\xi)\right)\\
&\overset{\text{def}}{=} \Phi_{A,B}\left(\llbracket (a,b)\mapsto \xi_a(b)\rrbracket\right)\\
&\overset{\text{def}}{=} \Phi_{A,B}\left(\llbracket (a',b')\mapsto \xi_{a'}(b')\rrbracket\right)\\
&\overset{\text{def}}{=} \llbracket a\mapsto \llbracket b\mapsto \mathrm{ev}_{(a,b)}\left(\llbracket (a',b')\mapsto \xi_{a'}(b')\rrbracket\right)\rrbracket\rrbracket\\
&\overset{\text{def}}{=} \llbracket a\mapsto \llbracket b\mapsto \xi_a(b)\rrbracket\rrbracket\\
&\overset{\text{def}}{=} \llbracket a\mapsto \xi_a\rrbracket\\
&\overset{\text{def}}{=} \xi.
\end{aligned}
$$

- *Naturality for Φ, Part I.* We need to show that, given a function $g\colon B\to B'$, the diagram

$$
\begin{tikzcd}
\mathrm{Hom}_{\mathsf{Sets}}(A\times B',C) \arrow[r, "\Phi_{B',C}"] \arrow[d, "\mathrm{id}_A\times g^*"'] & \mathrm{Hom}_{\mathsf{Sets}}(A,\mathrm{Hom}_{\mathsf{Sets}}(B',C)), \arrow[d, "(g^*)_*"] \\
\mathrm{Hom}_{\mathsf{Sets}}(A\times B,C) \arrow[r, "\Phi_{B,C}"'] & \mathrm{Hom}_{\mathsf{Sets}}(A,\mathrm{Hom}_{\mathsf{Sets}}(B,C))
\end{tikzcd}
$$

commutes. Indeed, given a function

$$\xi\colon A\times B'\to C,$$

we have

$$
\begin{aligned}
\left[\Phi_{B,C}\circ(\mathrm{id}_A\times g^*)\right](\xi) &= \Phi_{B,C}([\mathrm{id}_A\times g^*](\xi))\\
&= \Phi_{B,C}(\xi(-_1,g(-_2)))\\
&= [\xi(-_1,g(-_2))]^\dagger\\
&= \xi^\dagger_{-_1}(g(-_2))\\
&= (g^*)_*\left(\xi^\dagger\right)\\
&= (g^*)_*\left(\Phi_{B',C}(\xi)\right)\\
&= \left[(g^*)_*\circ\Phi_{B',C}\right](\xi).
\end{aligned}
$$

Alternatively, using the $\llbracket a\mapsto f(a)\rrbracket$ notation of Notation 1.1.1.1.2,

we have

$$\begin{aligned}\left[\Phi_{B,C}\circ(\mathrm{id}_A\times g^*)\right](\xi) &= \Phi_{B,C}([\mathrm{id}_A\times g^*](\xi))\\ &= \Phi_{B,C}\big([\mathrm{id}_A\times g^*](\llbracket(a,b')\mapsto\xi(a,b')\rrbracket)\big)\\ &= \Phi_{B,C}\big(\llbracket(a,b)\mapsto\xi(a,g(b))\rrbracket\big)\\ &= \llbracket a\mapsto\llbracket b\mapsto\xi(a,g(b))\rrbracket\rrbracket\\ &= \llbracket a\mapsto g^*\big(\llbracket b'\mapsto\xi(a,b')\rrbracket\big)\rrbracket\\ &= (g^*)_*\big(\llbracket a\mapsto\llbracket b'\mapsto\xi(a,b')\rrbracket\rrbracket\big)\\ &= (g^*)_*\big(\Phi_{B',C}\big(\llbracket(a,b')\mapsto\xi(a,b')\rrbracket\big)\big)\\ &= (g^*)_*\big(\Phi_{B',C}(\xi)\big)\\ &= \left[(g^*)_*\circ\Phi_{B',C}\right](\xi).\end{aligned}$$

- *Naturality for Φ, Part II.* We need to show that, given a function $h\colon C\to C'$, the diagram

$$\begin{tikzcd}\mathrm{Hom}_{\mathsf{Sets}}(A\times B,C) \arrow[r, "\Phi_{B,C}"] \arrow[d, "h_*"'] & \mathrm{Hom}_{\mathsf{Sets}}(A,\mathrm{Hom}_{\mathsf{Sets}}(B,C)), \arrow[d, "(h_*)_*"] \\ \mathrm{Hom}_{\mathsf{Sets}}(A\times B,C') \arrow[r, "\Phi_{B,C'}"'] & \mathrm{Hom}_{\mathsf{Sets}}(A,\mathrm{Hom}_{\mathsf{Sets}}(B,C'))\end{tikzcd}$$

  commutes. Indeed, given a function

$$\xi\colon A\times B\to C,$$

  we have

$$\begin{aligned}\left[\Phi_{B,C}\circ h_*\right](\xi) &= \Phi_{B,C}(h_*(\xi))\\ &= \Phi_{B,C}\big(h_*\big(\llbracket(a,b)\mapsto\xi(a,b)\rrbracket\big)\big)\\ &= \Phi_{B,C}\big(\llbracket(a,b)\mapsto h(\xi(a,b))\rrbracket\big)\\ &= \llbracket a\mapsto\llbracket b\mapsto h(\xi(a,b))\rrbracket\rrbracket\\ &= \llbracket a\mapsto h_*\big(\llbracket b\mapsto\xi(a,b)\rrbracket\rrbracket\big)\\ &= (h_*)_*\big(\llbracket a\mapsto\llbracket b\mapsto\xi(a,b)\rrbracket\rrbracket\big)\\ &= (h_*)_*\big(\Phi_{B,C}\big(\llbracket(a,b)\mapsto\xi(a,b)\rrbracket\big)\big)\\ &= (h_*)_*\big(\Phi_{B,C}(\xi)\big)\\ &= \left[(h_*)_*\circ\Phi_{B,C}\right](\xi).\end{aligned}$$

- *Naturality for Ψ.* Since Φ is natural in each argument and Φ is a componentwise inverse to Ψ in each argument, it follows from Item 2 of Proposition 8.8.6.1.2 that Ψ is also natural in each argument.

Item 3, *Associativity*: See [Pro24a].
Item 4, *Unitality*: Clear.
Item 5, *Commutativity*: See [Pro24b].
Item 6, *Annihilation With the Empty Set*: See [Pro24f].
Item 7, *Distributivity Over Unions*: See [Pro24e].
Item 8, *Distributivity Over Intersections*: See [Pro24g, Corollary 1].
Item 9, *Middle-Four Exchange With Respect to Intersections*: See [Pro24g, Corollary 1].
Item 10, *Distributivity Over Differences*: See [Pro24c].
Item 11, *Distributivity Over Symmetric Differences*: See [Pro24d].
Item 12, *Symmetric Monoidality*: See [`MO 382264`].
Item 13, *Symmetric Bimonoidality*: Omitted. □

2.1.4 Pullbacks

Let A, B, and C be sets and let $f\colon A \to C$ and $g\colon B \to C$ be functions.

Definition 2.1.4.1.1. The **pullback of** A **and** B **over** C **along** f **and** g[5] is the pair[6] $(A \times_C B, \{\mathrm{pr}_1, \mathrm{pr}_2\})$ consisting of:

- *The Limit.* The set $A \times_C B$ defined by
$$A \times_C B \overset{\text{def}}{=} \{(a, b) \in A \times B \mid f(a) = g(b)\}.$$
- *The Cone.* The maps
$$\begin{aligned}\mathrm{pr}_1 &\colon A \times_C B \to A,\\ \mathrm{pr}_2 &\colon A \times_C B \to B\end{aligned}$$
defined by
$$\begin{aligned}\mathrm{pr}_1(a, b) &\overset{\text{def}}{=} a,\\ \mathrm{pr}_2(a, b) &\overset{\text{def}}{=} b\end{aligned}$$
for each $(a, b) \in A \times_C B$.

Proof. We claim that $A \times_C B$ is the categorical pullback of A and B over C with respect to (f, g) in Sets. First we need to check that the relevant pullback diagram commutes, i.e. that we have

$$f \circ \mathrm{pr}_1 = g \circ \mathrm{pr}_2, \qquad \begin{tikzcd} A \times_C B \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C. \end{tikzcd}$$

[5] *Further Terminology:* Also called the **fibre product of** A **and** B **over** C **along** f **and** g.
[6] *Further Notation:* Also written $A \times_{f,C,g} B$.

Indeed, given $(a,b) \in A \times_C B$, we have

$$\begin{aligned}\left[f \circ \mathrm{pr}_1\right](a,b) &= f\big(\mathrm{pr}_1(a,b)\big)\\ &= f(a)\\ &= g(b)\\ &= g\big(\mathrm{pr}_2(a,b)\big)\\ &= \left[g \circ \mathrm{pr}_2\right](a,b),\end{aligned}$$

where $f(a) = g(b)$ since $(a,b) \in A \times_C B$. Next, we prove that $A \times_C B$ satisfies the universal property of the pullback. Suppose we have a diagram of the form

$$\begin{tikzcd} P \arrow[drr, bend left, "p_2"] \arrow[ddr, bend right, "p_1"'] & & \\ & A \times_C B \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "g"] \\ & A \arrow[r, "f"'] & C \end{tikzcd}$$

in Sets. Then there exists a unique map $\phi\colon P \to A \times_C B$ making the diagram

$$\begin{tikzcd} P \arrow[drr, bend left, "p_2"] \arrow[ddr, bend right, "p_1"'] \arrow[dr, dashed, "\phi", "\exists!"'] & & \\ & A \times_C B \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "g"] \\ & A \arrow[r, "f"'] & C \end{tikzcd}$$

commute, being uniquely determined by the conditions

$$\begin{aligned}\mathrm{pr}_1 \circ \phi &= p_1,\\ \mathrm{pr}_2 \circ \phi &= p_2\end{aligned}$$

via

$$\phi(x) = (p_1(x), p_2(x))$$

for each $x \in P$, where we note that $(p_1(x), p_2(x)) \in A \times B$ indeed lies in $A \times_C B$ by the condition

$$f \circ p_1 = g \circ p_2,$$

which gives

$$f(p_1(x)) = g(p_2(x))$$

for each $x \in P$, so that $(p_1(x), p_2(x)) \in A \times_C B$. □

Example 2.1.4.1.2. Here are some examples of pullbacks of sets.

1. *Unions via Intersections.* Let $A, B \subset X$. We have a bijection of sets

$$A \cap B \cong A \times_{A \cup B} B, \qquad \begin{tikzcd} A \cap B \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & B \arrow[d, "\iota_B"] \\ A \arrow[r, "\iota_A"'] & A \cup B. \end{tikzcd}$$

Proof. *Item 1, Unions via Intersections*: Indeed, we have

$$\begin{aligned} A \times_{A \cup B} B &\cong \{(x, y) \in A \times B \mid x = y\} \\ &\cong A \cap B. \end{aligned}$$

This finishes the proof. □

Proposition 2.1.4.1.3. Let $A, B, C,$ and X be sets.

1. *Functoriality.* The assignment $(A, B, C, f, g) \mapsto A \times_{f,C,g} B$ defines a functor

$$-_1 \times_{-_3} -_1 \colon \mathsf{Fun}(\mathcal{P}, \mathsf{Sets}) \to \mathsf{Sets},$$

where $\mathcal{P}$ is the category that looks like this:

$$\begin{tikzcd} & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet. \end{tikzcd}$$

In particular, the action on morphisms of $-_1 \times_{-_3} -_1$ is given by sending a morphism

$$\begin{tikzcd} A \times_C B \arrow[rr] \arrow[dd] \arrow[dr, phantom, very near start, "\lrcorner"] & & B \arrow[dd, "g" near start] \arrow[dr, "\psi"] & \\ & A' \times_{C'} B' \arrow[rr, crossing over] \arrow[dr, phantom, very near start, "\lrcorner"] & & B' \arrow[dd, "g'"] \\ A \arrow[rr, "f" near start] \arrow[dr, "\phi"'] & & C \arrow[dr, "\chi"] & \\ & A' \arrow[rr, "f'"'] \arrow[from=uu, crossing over] & & C' \end{tikzcd}$$

in $\mathsf{Fun}(\mathcal{P}, \mathsf{Sets})$ to the map $\xi\colon A \times_C B \xrightarrow{\exists!} A' \times_{C'} B'$ given by

$$\xi(a, b) \overset{\text{def}}{=} (\phi(a), \psi(b))$$

for each $(a, b) \in A \times_C B$, which is the unique map making the diagram

$$\begin{tikzcd}
A \times_C B \arrow[rr] \arrow[dd] \arrow[dr, dashed] \arrow[dr, phantom, very near start, "\lrcorner"] & & B \arrow[dd, "g" near start] \arrow[dr, "\psi"] & \\
& A' \times_{C'} B' \arrow[rr] \arrow[dd] \arrow[dr, phantom, very near start, "\lrcorner"] & & B' \arrow[dd, "g'"] \\
A \arrow[rr, "f" near start] \arrow[dr, "\phi"'] & & C \arrow[dr, "\chi"] & \\
& A' \arrow[rr, "f'"'] & & C'
\end{tikzcd}$$

commute.

2. *Associativity.* Given a diagram

$$\begin{tikzcd}
A \arrow[dr, "f"'] & & B \arrow[dl, "g"] \arrow[dr, "h"'] & & C \arrow[dl, "k"] \\
& X & & Y &
\end{tikzcd}$$

in Sets, we have isomorphisms of sets

$$(A \times_X B) \times_Y C \cong (A \times_X B) \times_B (B \times_Y C) \cong A \times_X (B \times_Y C),$$

where these pullbacks are built as in the diagrams

$$\begin{tikzcd}[column sep=tiny]
& & (A \times_X B) \times_Y C \arrow[dl] \arrow[ddrr] \arrow[d, phantom, "\lrcorner" rotate=-45] & & \\
& A \times_X B \arrow[dl] \arrow[dr] \arrow[d, phantom, "\lrcorner" rotate=-45] & & & \\
A \arrow[dr, "f"'] & & B \arrow[dl, "g"] \arrow[dr, "h"'] & & C, \arrow[dl, "k"] \\
& X & & Y &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=tiny]
& & (A \times_X B) \times_B (B \times_Y C) \arrow[dl] \arrow[dr] \arrow[d, phantom, "\lrcorner" rotate=-45] & & \\
& A \times_X B \arrow[dl] \arrow[dr] \arrow[d, phantom, "\lrcorner" rotate=-45] & & B \times_Y C \arrow[dl] \arrow[dr] \arrow[d, phantom, "\lrcorner" rotate=-45] & \\
A \arrow[dr, "f"'] & & B \arrow[dl, "g"] \arrow[dr, "h"'] & & C, \arrow[dl, "k"] \\
& X & & Y &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=tiny]
& & A \times_X (B \times_Y C) \arrow[ddll] \arrow[dr] \arrow[d, phantom, "\lrcorner" rotate=-45] & & \\
& & & B \times_Y C \arrow[dl] \arrow[dr] \arrow[d, phantom, "\lrcorner" rotate=-45] & \\
A \arrow[dr, "f"'] & & B \arrow[dl, "g"] \arrow[dr, "h"'] & & C. \arrow[dl, "k"] \\
& X & & Y &
\end{tikzcd}$$

3. *Unitality.* We have isomorphisms of sets

$$\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "f"'] \arrow[dr, phantom, very near start, "\lrcorner"] & A \arrow[d, "f"] \\
X \arrow[r, equal] & X
\end{tikzcd}
\qquad
\begin{aligned}
X \times_X A &\cong A, \\
A \times_X X &\cong A,
\end{aligned}
\qquad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, equal] \arrow[dr, phantom, very near start, "\lrcorner"] & X \arrow[d, equal] \\
X \arrow[r, "f"'] & X.
\end{tikzcd}$$

4. *Commutativity.* We have an isomorphism of sets

$$\begin{tikzcd} A \times_X B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & X, \end{tikzcd} \qquad A \times_X B \cong B \times_X A \qquad \begin{tikzcd} B \times_X A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\ B \arrow[r, "g"'] & X. \end{tikzcd}$$

5. *Annihilation With the Empty Set.* We have isomorphisms of sets

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \emptyset \arrow[d] \\ A \arrow[r, "f"'] & X, \end{tikzcd} \qquad \begin{aligned} A \times_X \emptyset &\cong \emptyset, \\ \emptyset \times_X A &\cong \emptyset, \end{aligned} \qquad \begin{tikzcd} \emptyset \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\ \emptyset \arrow[r] & X. \end{tikzcd}$$

6. *Interaction With Products.* We have an isomorphism of sets

$$A \times_{\mathrm{pt}} B \cong A \times B, \qquad \begin{tikzcd} A \times B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "!_B"] \\ A \arrow[r, "!_A"'] & \mathrm{pt}. \end{tikzcd}$$

7. *Symmetric Monoidality.* The triple $(\mathsf{Sets}, \times_X, X)$ is a symmetric monoidal category.

Proof. *Item 1*, *Functoriality*: This is a special case of functoriality of co/limits, **??** of **??**, with the explicit expression for ξ following from the commutativity of the cube pullback diagram.
Item 2, *Associativity*: Indeed, we have

$$\begin{aligned} (A \times_X B) \times_Y C &\cong \{((a,b),c) \in (A \times_X B) \times C \mid h(b) = k(c)\} \\ &\cong \{((a,b),c) \in (A \times B) \times C \mid f(a) = g(b) \text{ and } h(b) = k(c)\} \\ &\cong \{(a,(b,c)) \in A \times (B \times C) \mid f(a) = g(b) \text{ and } h(b) = k(c)\} \\ &\cong \{(a,(b,c)) \in A \times (B \times_Y C) \mid f(a) = g(b)\} \\ &\cong A \times_X (B \times_Y C) \end{aligned}$$

and

$$
\begin{aligned}
(A \times_X B) \times_B (B \times_Y C) &\cong \{((a,b),(b',c)) \in (A \times_X B) \times (B \times_Y C) \mid b = b'\}\\
&\cong \left\{ ((a,b),(b',c)) \in (A \times B) \times (B \times C) \;\middle|\; \begin{array}{l} f(a) = g(b), b = b', \\ \text{and } h(b') = k(c) \end{array} \right\}\\
&\cong \left\{ (a,(b,(b',c))) \in A \times (B \times (B \times C)) \;\middle|\; \begin{array}{l} f(a) = g(b), b = b', \\ \text{and } h(b') = k(c) \end{array} \right\}\\
&\cong \left\{ (a,((b,b'),c)) \in A \times ((B \times B) \times C) \;\middle|\; \begin{array}{l} f(a) = g(b), b = b', \\ \text{and } h(b') = k(c) \end{array} \right\}\\
&\cong \left\{ (a,((b,b'),c)) \in A \times ((B \times_B B) \times C) \;\middle|\; \begin{array}{l} f(a) = g(b) \text{ and} \\ h(b') = k(c) \end{array} \right\}\\
&\cong \{(a,(b,c)) \in A \times (B \times C) \mid f(a) = g(b) \text{ and } h(b) = k(c)\}\\
&\cong A \times_X (B \times_Y C),
\end{aligned}
$$

where we have used Item 3 for the isomorphism $B \times_B B \cong B$.
Item 3, Unitality: Indeed, we have

$$
\begin{aligned}
X \times_X A &\cong \{(x,a) \in X \times A \mid f(a) = x\},\\
A \times_X X &\cong \{(a,x) \in X \times A \mid f(a) = x\},
\end{aligned}
$$

which are isomorphic to A via the maps $(x,a) \mapsto a$ and $(a,x) \mapsto a$.
Item 4, Commutativity: Clear.
Item 5, Annihilation With the Empty Set: Clear.
Item 6, Interaction With Products: Clear.
Item 7, Symmetric Monoidality: Omitted. □

2.1.5 Equalisers

Let A and B be sets and let $f, g\colon A \rightrightarrows B$ be functions.

Definition 2.1.5.1.1. The **equaliser of f and g** is the pair $(\mathrm{Eq}(f,g), \mathrm{eq}(f,g))$ consisting of:

- *The Limit.* The set $\mathrm{Eq}(f,g)$ defined by

$$
\mathrm{Eq}(f,g) \overset{\text{def}}{=} \{a \in A \mid f(a) = g(a)\}.
$$

- *The Cone.* The inclusion map

$$
\mathrm{eq}(f,g)\colon \mathrm{Eq}(f,g) \hookrightarrow A.
$$

Proof. We claim that $\mathrm{Eq}(f,g)$ is the categorical equaliser of f and g in Sets.

First we need to check that the relevant equaliser diagram commutes, i.e. that we have
$$f \circ \mathrm{eq}(f, g) = g \circ \mathrm{eq}(f, g),$$
which indeed holds by the definition of the set $\mathrm{Eq}(f, g)$. Next, we prove that $\mathrm{Eq}(f, g)$ satisfies the universal property of the equaliser. Suppose we have a diagram of the form

$$\begin{tikzcd} \mathrm{Eq}(f,g) \arrow[r, hook, "\mathrm{eq}(f,g)"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\ E \arrow[ur, "e"'] & & \end{tikzcd}$$

in Sets. Then there exists a unique map $\phi \colon E \to \mathrm{Eq}(f, g)$ making the diagram

$$\begin{tikzcd} \mathrm{Eq}(f,g) \arrow[r, hook, "\mathrm{eq}(f,g)"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\ E \arrow[u, dashed, "\phi", "\exists!"'] \arrow[ur, "e"'] & & \end{tikzcd}$$

commute, being uniquely determined by the condition
$$\mathrm{eq}(f, g) \circ \phi = e$$
via
$$\phi(x) = e(x)$$
for each $x \in E$, where we note that $e(x) \in A$ indeed lies in $\mathrm{Eq}(f, g)$ by the condition
$$f \circ e = g \circ e,$$
which gives
$$f(e(x)) = g(e(x))$$
for each $x \in E$, so that $e(x) \in \mathrm{Eq}(f, g)$. □

Proposition 2.1.5.1.2. Let A, B, and C be sets.

1. *Associativity.* We have isomorphisms of sets[7]

$$\underbrace{\mathrm{Eq}\big(f \circ \mathrm{eq}(g, h), g \circ \mathrm{eq}(g, h)\big)}_{=\mathrm{Eq}(f\circ\mathrm{eq}(g,h),h\circ\mathrm{eq}(g,h))} \cong \mathrm{Eq}(f, g, h) \cong \underbrace{\mathrm{Eq}\big(f \circ \mathrm{eq}(f, g), h \circ \mathrm{eq}(f, g)\big)}_{=\mathrm{Eq}(g\circ\mathrm{eq}(f,g),h\circ\mathrm{eq}(f,g))},$$

[7]That is, the following three ways of forming "the" equaliser of (f, g, h) agree:

where $\mathrm{Eq}(f, g, h)$ is the limit of the diagram

$$A \overset{f}{\underset{h}{\xrightarrow{-g\to}}} B$$

in Sets, being explicitly given by

$$\mathrm{Eq}(f, g, h) \cong \{a \in A \mid f(a) = g(a) = h(a)\}.$$

4. *Unitality.* We have an isomorphism of sets

$$\mathrm{Eq}(f, f) \cong A.$$

---

1. Take the equaliser of (f, g, h), i.e. the limit of the diagram

$$A \overset{f}{\underset{h}{\xrightarrow{-g\to}}} B$$

in Sets.

2. First take the equaliser of f and g, forming a diagram

$$\mathrm{Eq}(f, g) \overset{\mathrm{eq}(f,g)}{\hookrightarrow} A \overset{f}{\underset{g}{\rightrightarrows}} B$$

and then take the equaliser of the composition

$$\mathrm{Eq}(f, g) \overset{\mathrm{eq}(f,g)}{\hookrightarrow} A \overset{f}{\underset{h}{\rightrightarrows}} B,$$

obtaining a subset

$$\mathrm{Eq}\big(f \circ \mathrm{eq}(f, g), h \circ \mathrm{eq}(f, g)\big) = \mathrm{Eq}\big(g \circ \mathrm{eq}(f, g), h \circ \mathrm{eq}(f, g)\big)$$

of $\mathrm{Eq}(f, g)$.

3. First take the equaliser of g and h, forming a diagram

$$\mathrm{Eq}(g, h) \overset{\mathrm{eq}(g,h)}{\hookrightarrow} A \overset{g}{\underset{h}{\rightrightarrows}} B$$

and then take the equaliser of the composition

$$\mathrm{Eq}(g, h) \overset{\mathrm{eq}(g,h)}{\hookrightarrow} A \overset{f}{\underset{g}{\rightrightarrows}} B,$$

obtaining a subset

$$\mathrm{Eq}\big(f \circ \mathrm{eq}(g, h), g \circ \mathrm{eq}(g, h)\big) = \mathrm{Eq}\big(f \circ \mathrm{eq}(g, h), h \circ \mathrm{eq}(g, h)\big)$$

of $\mathrm{Eq}(g, h)$.

5. *Commutativity.* We have an isomorphism of sets

$$\mathrm{Eq}(f, g) \cong \mathrm{Eq}(g, f).$$

6. *Interaction With Composition.* Let

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \underset{k}{\overset{h}{\rightrightarrows}} C$$

be functions. We have an inclusion of sets

$$\mathrm{Eq}\big(h \circ f \circ \mathrm{eq}(f, g), k \circ g \circ \mathrm{eq}(f, g)\big) \subset \mathrm{Eq}(h \circ f, k \circ g),$$

where $\mathrm{Eq}\big(h \circ f \circ \mathrm{eq}(f, g), k \circ g \circ \mathrm{eq}(f, g)\big)$ is the equaliser of the composition

$$\mathrm{Eq}(f, g) \overset{\mathrm{eq}(f,g)}{\hookrightarrow} A \underset{g}{\overset{f}{\rightrightarrows}} B \underset{k}{\overset{h}{\rightrightarrows}} C.$$

Proof. *Item 1, Associativity*: We first prove that $\mathrm{Eq}(f, g, h)$ is indeed given by

$$\mathrm{Eq}(f, g, h) \cong \{a \in A \mid f(a) = g(a) = h(a)\}.$$

Indeed, suppose we have a diagram of the form

$$\begin{tikzcd}
\mathrm{Eq}(f, g, h) \arrow[r, hook, "\mathrm{eq}(f,g,h)"] & A \arrow[r, shift left=2, "f"] \arrow[r, "g" description] \arrow[r, shift right=2, "h"'] & B \\
E \arrow[ur, "e"'] & &
\end{tikzcd}$$

in Sets. Then there exists a unique map $\phi \colon E \to \mathrm{Eq}(f, g, h)$, uniquely determined by the condition

$$\mathrm{eq}(f, g) \circ \phi = e$$

being necessarily given by

$$\phi(x) = e(x)$$

for each $x \in E$, where we note that $e(x) \in A$ indeed lies in $\mathrm{Eq}(f, g, h)$ by the condition

$$f \circ e = g \circ e = h \circ e,$$

which gives

$$f(e(x)) = g(e(x)) = h(e(x))$$

for each $x \in E$, so that $e(x) \in \mathrm{Eq}(f, g, h)$.

We now check the equalities

$$\mathrm{Eq}\big(f \circ \mathrm{eq}(g,h), g \circ \mathrm{eq}(g,h)\big) \cong \mathrm{Eq}(f,g,h) \cong \mathrm{Eq}\big(f \circ \mathrm{eq}(f,g), h \circ \mathrm{eq}(f,g)\big).$$

Indeed, we have

$$\begin{aligned}\mathrm{Eq}\big(f \circ \mathrm{eq}(g,h), g \circ \mathrm{eq}(g,h)\big) &\cong \big\{x \in \mathrm{Eq}(g,h) \,\big|\, \big[f \circ \mathrm{eq}(g,h)\big](a) = \big[g \circ \mathrm{eq}(g,h)\big](a)\big\}\\ &\cong \big\{x \in \mathrm{Eq}(g,h) \,\big|\, f(a) = g(a)\big\}\\ &\cong \{x \in A \mid f(a) = g(a) \text{ and } g(a) = h(a)\}\\ &\cong \{x \in A \mid f(a) = g(a) = h(a)\}\\ &\cong \mathrm{Eq}(f,g,h).\end{aligned}$$

Similarly, we have

$$\begin{aligned}\mathrm{Eq}\big(f \circ \mathrm{eq}(f,g), h \circ \mathrm{eq}(f,g)\big) &\cong \big\{x \in \mathrm{Eq}(f,g) \,\big|\, \big[f \circ \mathrm{eq}(f,g)\big](a) = \big[h \circ \mathrm{eq}(f,g)\big](a)\big\}\\ &\cong \big\{x \in \mathrm{Eq}(f,g) \,\big|\, f(a) = h(a)\big\}\\ &\cong \{x \in A \mid f(a) = h(a) \text{ and } f(a) = g(a)\}\\ &\cong \{x \in A \mid f(a) = g(a) = h(a)\}\\ &\cong \mathrm{Eq}(f,g,h).\end{aligned}$$

Item 4, *Unitality*: Clear.
Item 5, *Commutativity*: Clear.
Item 6, *Interaction With Composition*: Indeed, we have

$$\begin{aligned}\mathrm{Eq}\big(h \circ f \circ \mathrm{eq}(f,g), k \circ g \circ \mathrm{eq}(f,g)\big) &\cong \big\{a \in \mathrm{Eq}(f,g) \,\big|\, h(f(a)) = k(g(a))\big\}\\ &\cong \{a \in A \mid f(a) = g(a) \text{ and } h(f(a)) = k(g(a))\}.\end{aligned}$$

and

$$\mathrm{Eq}(h \circ f, k \circ g) \cong \{a \in A \mid h(f(a)) = k(g(a))\},$$

and thus there's an inclusion from $\mathrm{Eq}\big(h \circ f \circ \mathrm{eq}(f,g), k \circ g \circ \mathrm{eq}(f,g)\big)$ to $\mathrm{Eq}(h \circ f, k \circ g)$. □

2.2 Colimits of Sets

2.2.1 The Initial Set

Definition 2.2.1.1.1. The **initial set** is the pair $\Big(\emptyset, \{\iota_A\}_{A \in \mathrm{Obj}(\mathsf{Sets})}\Big)$ consisting of:

- *The Limit.* The empty set $\emptyset$ of Definition 2.3.1.1.1.
- *The Cone.* The collection of maps

  $$\{\iota_A : \emptyset \to A\}_{A \in \mathrm{Obj}(\mathsf{Sets})}$$

  given by the inclusion maps from $\emptyset$ to A.

Proof. We claim that $\emptyset$ is the initial object of Sets. Indeed, suppose we have a diagram of the form

$$\emptyset \qquad\qquad A$$

in Sets. Then there exists a unique map $\phi\colon \emptyset \to A$ making the diagram

$$\begin{tikzcd} \emptyset \arrow[r, dashed, "\phi", "\exists!"'] & A \end{tikzcd}$$

commute, namely the inclusion map ι_A. □

2.2.2 Coproducts of Families of Sets

Let $\{A_i\}_{i\in I}$ be a family of sets.

Definition 2.2.2.1.1. The **disjoint union of the family** $\{A_i\}_{i\in I}$ is the pair $\left(\coprod_{i\in I} A_i, \left\{\mathrm{inj}_i\right\}_{i\in I}\right)$ consisting of:

- *The Colimit.* The set $\coprod_{i\in I} A_i$ defined by

$$\coprod_{i\in I} A_i \overset{\mathrm{def}}{=} \left\{(i,x) \in I \times \left(\bigcup_{i\in I} A_i\right) \;\middle|\; x \in A_i\right\}.$$

- *The Cocone.* The collection

$$\left\{\mathrm{inj}_i\colon A_i \to \coprod_{i\in I} A_i\right\}_{i\in I}$$

  of maps given by

$$\mathrm{inj}_i(x) \overset{\mathrm{def}}{=} (i,x)$$

  for each $x \in A_i$ and each $i \in I$.

Proof. We claim that $\coprod_{i\in I} A_i$ is the categorical coproduct of $\{A_i\}_{i\in I}$ in Sets. Indeed, suppose we have, for each $i \in I$, a diagram of the form

$$\begin{tikzcd} & C \\ A_i \arrow[ur, "\iota_i"] \arrow[r, "\mathrm{inj}_i"'] & \coprod_{i\in I} A_i \end{tikzcd}$$

in Sets. Then there exists a unique map $\phi\colon \coprod_{i\in I} A_i \to C$ making the diagram

$$\begin{tikzcd} & C \\ A_i \arrow[ur, "\iota_i"] \arrow[r, "\mathrm{inj}_i"'] & \coprod_{i\in I} A_i \arrow[u, dashed, "\phi"', "\exists!"] \end{tikzcd}$$

commute, being uniquely determined by the condition $\phi \circ \mathrm{inj}_i = \iota_i$ for each $i \in I$ via

$$\phi((i,x)) = \iota_i(x)$$

for each $(i,x) \in \coprod_{i\in I} A_i$. □

Proposition 2.2.2.1.2. Let $\{A_i\}_{i\in I}$ be a family of sets.

1. *Functoriality.* The assignment $\{A_i\}_{i\in I} \mapsto \coprod_{i\in I} A_i$ defines a functor

$$\coprod_{i\in I} \colon \mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}) \to \mathsf{Sets}$$

where

- *Action on Objects.* For each $(A_i)_{i\in I} \in \mathrm{Obj}(\mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}))$, we have

$$\left[\coprod_{i\in I}\right]((A_i)_{i\in I}) \stackrel{\text{def}}{=} \coprod_{i\in I} A_i$$

- *Action on Morphisms.* For each $(A_i)_{i\in I}, (B_i)_{i\in I} \in \mathrm{Obj}(\mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}))$, the action on Hom-sets

$$\left(\coprod_{i\in I}\right)_{(A_i)_{i\in I},(B_i)_{i\in I}} \colon \mathrm{Nat}((A_i)_{i\in I}, (B_i)_{i\in I}) \to \mathsf{Sets}\left(\coprod_{i\in I} A_i, \coprod_{i\in I} B_i\right)$$

of $\coprod_{i\in I}$ at $((A_i)_{i\in I}, (B_i)_{i\in I})$ is defined by sending a map

$$\{f_i \colon A_i \to B_i\}_{i\in I}$$

in $\mathrm{Nat}((A_i)_{i\in I}, (B_i)_{i\in I})$ to the map of sets

$$\coprod_{i\in I} f_i \colon \coprod_{i\in I} A_i \to \coprod_{i\in I} B_i$$

defined by

$$\left[\coprod_{i\in I} f_i\right](i,a) \stackrel{\text{def}}{=} f_i(a)$$

for each $(i,a) \in \coprod_{i\in I} A_i$.

Proof. *Item 1*, *Functoriality*: This follows from **??** of **??**. □

2.2.3 Binary Coproducts

Let A and B be sets.

Definition 2.2.3.1.1. The **coproduct**[8] **of** A **and** B is the pair $\left(A \coprod B, \left\{\mathrm{inj}_1, \mathrm{inj}_2\right\}\right)$ consisting of:

- *The Colimit.* The set $A \coprod B$ defined by

$$\begin{aligned} A \coprod B &\stackrel{\text{def}}{=} \coprod_{z \in \{A,B\}} z \\ &\cong \{(0,a) \mid a \in A\} \cup \{(1,b) \mid b \in B\}. \end{aligned}$$

- *The Cocone.* The maps

$$\begin{aligned} \mathrm{inj}_1 &: A \to A \coprod B, \\ \mathrm{inj}_2 &: B \to A \coprod B, \end{aligned}$$

  given by

$$\begin{aligned} \mathrm{inj}_1(a) &\stackrel{\text{def}}{=} (0,a), \\ \mathrm{inj}_2(b) &\stackrel{\text{def}}{=} (1,b), \end{aligned}$$

  for each $a \in A$ and each $b \in B$.

Proof. We claim that $A \coprod B$ is the categorical coproduct of A and B in Sets. Indeed, suppose we have a diagram of the form

$$\begin{tikzcd} & C & \\ A \arrow[ur, bend left, "\iota_A"] \arrow[r, "\mathrm{inj}_A"'] & A \coprod B & B \arrow[ul, bend right, "\iota_B"'] \arrow[l, "\mathrm{inj}_B"] \end{tikzcd}$$

in Sets. Then there exists a unique map $\phi \colon A \coprod B \to C$ making the diagram

$$\begin{tikzcd} & C & \\ A \arrow[ur, bend left, "\iota_A"] \arrow[r, "\mathrm{inj}_A"'] & A \coprod B \arrow[u, dashed, "\phi", "\exists!"'] & B \arrow[ul, bend right, "\iota_B"'] \arrow[l, "\mathrm{inj}_B"] \end{tikzcd}$$

[8] *Further Terminology:* Also called the **disjoint union of** A **and** B, or the **binary disjoint union of** A **and** B, for emphasis.

commute, being uniquely determined by the conditions

$$\phi \circ \mathrm{inj}_A = \iota_A,$$
$$\phi \circ \mathrm{inj}_B = \iota_B$$

via

$$\phi(x) = \begin{cases} \iota_A(a) & \text{if } x = (0, a), \\ \iota_B(b) & \text{if } x = (1, b) \end{cases}$$

for each $x \in A \coprod B$. □

Proposition 2.2.3.1.2. Let A, B, C, and X be sets.

1. *Functoriality.* The assignment $A, B, (A, B) \mapsto A \coprod B$ defines functors

$$A \coprod - \colon \mathsf{Sets} \to \mathsf{Sets},$$
$$- \coprod B \colon \mathsf{Sets} \to \mathsf{Sets},$$
$$-_1 \coprod -_2 \colon \mathsf{Sets} \times \mathsf{Sets} \to \mathsf{Sets},$$

where $-_1 \coprod -_2$ is the functor where

- *Action on Objects.* For each $(A, B) \in \mathrm{Obj}(\mathsf{Sets} \times \mathsf{Sets})$, we have

$$[-_1 \coprod -_2](A, B) \stackrel{\text{def}}{=} A \coprod B.$$

- *Action on Morphisms.* For each $(A, B), (X, Y) \in \mathrm{Obj}(\mathsf{Sets})$, the action on Hom-sets

$$\coprod_{(A,B),(X,Y)} \colon \mathsf{Sets}(A, X) \times \mathsf{Sets}(B, Y) \to \mathsf{Sets}(A \coprod B, X \coprod Y)$$

of $\coprod$ at $((A, B), (X, Y))$ is defined by sending (f, g) to the function

$$f \coprod g \colon A \coprod B \to X \coprod Y$$

defined by

$$[f \coprod g](x) \stackrel{\text{def}}{=} \begin{cases} (0, f(a)) & \text{if } x = (0, a), \\ (1, g(b)) & \text{if } x = (1, b), \end{cases}$$

for each $x \in A \coprod B$.

and where $A \coprod -$ and $- \coprod B$ are the partial functors of $-_1 \coprod -_2$ at $A, B \in \mathrm{Obj}(\mathsf{Sets})$.

2. *Associativity.* We have an isomorphism of sets

$$(A \coprod B) \coprod C \cong A \coprod (B \coprod C),$$

natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

3. *Unitality.* We have isomorphisms of sets

$$A \coprod \emptyset \cong A,$$
$$\emptyset \coprod A \cong A,$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$.

4. *Commutativity.* We have an isomorphism of sets

$$A \coprod B \cong B \coprod A,$$

natural in $A, B \in \mathrm{Obj}(\mathsf{Sets})$.

5. *Symmetric Monoidality.* The triple $(\mathsf{Sets}, \coprod, \emptyset)$ is a symmetric monoidal category.

Proof. *Item 1, Functoriality*: This follows from **??** of **??**.
Item 2, Associativity: Clear.
Item 3, Unitality: Clear.
Item 4, Commutativity: Clear.
Item 5, Symmetric Monoidality: Omitted. □

2.2.4 Pushouts

Let A, B, and C be sets and let $f : C \to A$ and $g : C \to B$ be functions.

Definition 2.2.4.1.1. The **pushout of** A **and** B **over** C **along** f **and** g[9] is the pair[10] $\left(A \coprod_C B, \{\mathrm{inj}_1, \mathrm{inj}_2\}\right)$ consisting of:

- *The Colimit.* The set $A \coprod_C B$ defined by

$$A \coprod_C B \stackrel{\text{def}}{=} A \coprod B/{\sim_C},$$

where $\sim_C$ is the equivalence relation on $A \coprod B$ generated by $(0, f(c)) \sim_C (1, g(c))$.

[9] *Further Terminology:* Also called the **fibre coproduct of** A **and** B **over** C **along** f **and** g.
[10] *Further Notation:* Also written $A \coprod_{f,C,g} B$.

- *The Cocone.* The maps

$$\begin{aligned}\mathrm{inj}_1 &\colon A \to A \textstyle\coprod_C B,\\ \mathrm{inj}_2 &\colon B \to A \textstyle\coprod_C B\end{aligned}$$

  given by

$$\begin{aligned}\mathrm{inj}_1(a) &\stackrel{\text{def}}{=} [(0,a)]\\ \mathrm{inj}_2(b) &\stackrel{\text{def}}{=} [(1,b)]\end{aligned}$$

  for each $a \in A$ and each $b \in B$.

Proof. We claim that $A \coprod_C B$ is the categorical pushout of A and B over C with respect to (f, g) in Sets. First we need to check that the relevant pushout diagram commutes, i.e. that we have

$$\mathrm{inj}_1 \circ f = \mathrm{inj}_2 \circ g, \qquad \begin{tikzcd} A \coprod_C B & B \arrow[l, "\mathrm{inj}_2"'] \\ A \arrow[u, "\mathrm{inj}_1"] & C. \arrow[l, "f"] \arrow[u, "g"'] \end{tikzcd}$$

Indeed, given $c \in C$, we have

$$\begin{aligned}\left[\mathrm{inj}_1 \circ f\right](c) &= \mathrm{inj}_1(f(c))\\ &= [(0, f(c))]\\ &= [(1, g(c))]\\ &= \mathrm{inj}_2(g(c))\\ &= \left[\mathrm{inj}_2 \circ g\right](c),\end{aligned}$$

where $[(0, f(c))] = [(1, g(c))]$ by the definition of the relation $\sim$ on $A \coprod B$. Next, we prove that $A \coprod {}_C B$ satisfies the universal property of the pushout. Suppose we have a diagram of the form

$$\begin{tikzcd} P & & \\ & A \coprod_C B \arrow[dr, phantom, "\ulcorner", very near start] & B \arrow[l, "\mathrm{inj}_2"'] \arrow[ull, bend right, "\iota_2"'] \\ & A \arrow[u, "\mathrm{inj}_1"] \arrow[uul, bend left, "\iota_1"] & C \arrow[l, "f"] \arrow[u, "g"'] \end{tikzcd}$$

in Sets. Then there exists a unique map $\phi\colon A\coprod_C B \to P$ making the diagram

$$\begin{tikzcd}
P & & \\
& A \coprod_C B \arrow[ul, dashed, "\phi", "\exists!"'] & B \arrow[l, "\mathrm{inj}_2"'] \arrow[ull, bend right, "\iota_2"'] \\
& A \arrow[u, "\mathrm{inj}_1"] \arrow[uul, bend left, "\iota_1"] & C \arrow[l, "f"] \arrow[u, "g"'] \arrow[ul, phantom, "\ulcorner", very near end]
\end{tikzcd}$$

commute, being uniquely determined by the conditions

$$\begin{aligned}\phi \circ \mathrm{inj}_1 &= \iota_1,\\ \phi \circ \mathrm{inj}_2 &= \iota_2\end{aligned}$$

via

$$\phi(x) = \begin{cases} \iota_1(a) & \text{if } x = [(0,a)], \\ \iota_2(b) & \text{if } x = [(1,b)] \end{cases}$$

for each $x \in A \coprod_C B$, where the well-definedness of ϕ is guaranteed by the equality $\iota_1 \circ f = \iota_2 \circ g$ and the definition of the relation $\sim$ on $A \coprod B$ as follows:

1. *Case 1:* Suppose we have $x = [(0,a)] = [(0,a')]$ for some $a, a' \in A$. Then, by Remark 2.2.4.1.2, we have a sequence
$$(0,a) \sim' x_1 \sim' \cdots \sim' x_n \sim' (0,a').$$
2. *Case 2:* Suppose we have $x = [(1,b)] = [(1,b')]$ for some $b, b' \in B$. Then, by Remark 2.2.4.1.2, we have a sequence
$$(1,b) \sim' x_1 \sim' \cdots \sim' x_n \sim' (1,b').$$
3. *Case 3:* Suppose we have $x = [(0,a)] = [(1,b)]$ for some $a \in A$ and $b \in B$. Then, by Remark 2.2.4.1.2, we have a sequence
$$(0,a) \sim' x_1 \sim' \cdots \sim' x_n \sim' (1,b).$$

In all these cases, we declare $x \sim' y$ iff there exists some $c \in C$ such that $x = (0, f(c))$ and $y = (1, g(c))$ or $x = (1, g(c))$ and $y = (0, f(c))$. Then, the equality $\iota_1 \circ f = \iota_2 \circ g$ gives

$$\begin{aligned}\phi([x]) &= \phi([(0, f(c))]) \\ &\overset{\text{def}}{=} \iota_1(f(c)) \\ &= \iota_2(g(c)) \\ &\overset{\text{def}}{=} \phi([(1, g(c))]) \\ &= \phi([y]),\end{aligned}$$

with the case where $x = (1, g(c))$ and $y = (0, f(c))$ similarly giving $\phi([x]) = \phi([y])$. Thus, if $x \sim' y$, then $\phi([x]) = \phi([y])$. Applying this equality pairwise to the sequences

$$\begin{aligned}(0, a) \sim' x_1 \sim' \cdots \sim' x_n \sim' (0, a'),\\ (1, b) \sim' x_1 \sim' \cdots \sim' x_n \sim' (1, b'),\\ (0, a) \sim' x_1 \sim' \cdots \sim' x_n \sim' (1, b)\end{aligned}$$

gives

$$\begin{aligned}\phi([(0, a)]) = \phi([(0, a')]),\\ \phi([(1, b)]) = \phi([(1, b')]),\\ \phi([(0, a)]) = \phi([(1, b)]),\end{aligned}$$

showing ϕ to be well-defined. □

Remark 2.2.4.1.2. In detail, by Construction 7.4.2.1.2, the relation $\sim$ of Definition 2.2.4.1.1 is given by declaring $a \sim b$ iff one of the following conditions is satisfied:

- We have $a, b \in A$ and $a = b$;
- We have $a, b \in B$ and $a = b$;
- There exist $x_1, \ldots, x_n \in A \coprod B$ such that $a \sim' x_1 \sim' \cdots \sim' x_n \sim' b$, where we declare $x \sim' y$ if one of the following conditions is satisfied:

  1. There exists $c \in C$ such that $x = (0, f(c))$ and $y = (1, g(c))$.
  2. There exists $c \in C$ such that $x = (1, g(c))$ and $y = (0, f(c))$.

  That is: we require the following condition to be satisfied:

  (★) There exist $x_1, \ldots, x_n \in A \coprod B$ satisfying the following conditions:

1. There exists $c_0 \in C$ satisfying one of the following conditions:
   (a) We have $a = f(c_0)$ and $x_1 = g(c_0)$.
   (b) We have $a = g(c_0)$ and $x_1 = f(c_0)$.
2. For each $1 \leq i \leq n-1$, there exists $c_i \in C$ satisfying one of the following conditions:
   (a) We have $x_i = f(c_i)$ and $x_{i+1} = g(c_i)$.
   (b) We have $x_i = g(c_i)$ and $x_{i+1} = f(c_i)$.
3. There exists $c_n \in C$ satisfying one of the following conditions:
   (a) We have $x_n = f(c_n)$ and $b = g(c_n)$.
   (b) We have $x_n = g(c_n)$ and $b = f(c_n)$.

Example 2.2.4.1.3. Here are some examples of pushouts of sets.

1. *Wedge Sums of Pointed Sets.* The wedge sum of two pointed sets of Definition 3.3.3.1.1 is an example of a pushout of sets.

2. *Intersections via Unions.* Let $A, B \subset X$. We have a bijection of sets

$$A \cup B \cong A \textstyle\coprod_{A \cap B} B, \qquad \begin{tikzcd} A \cup B \arrow[dr, phantom, very near start, "\ulcorner"] & B \arrow[l, hook'] \\ A \arrow[u, hook] & A \cap B. \arrow[l, hook'] \arrow[u, hook'] \end{tikzcd}$$

Proof. *Item 1, Wedge Sums of Pointed Sets*: Follows by definition.
Item 2, Intersections via Unions: Indeed, $A \coprod_{A \cap B} B$ is the quotient of $A \coprod B$ by the equivalence relation obtained by declaring $(0, a) \sim (1, b)$ iff $a = b \in A \cap B$, which is in bijection with $A \cup B$ via the map with $[(0, a)] \mapsto a$ and $[(1, b)] \mapsto b$. $\square$

Proposition 2.2.4.1.4. Let $A, B, C,$ and X be sets.

1. *Functoriality.* The assignment $(A, B, C, f, g) \mapsto A \coprod_{f,C,g} B$ defines a functor

$$-_1 \textstyle\coprod_{-_3} -_1 \colon \mathsf{Fun}(\mathcal{P}, \mathsf{Sets}) \to \mathsf{Sets},$$

where $\mathcal{P}$ is the category that looks like this:

$$\begin{tikzcd} & \bullet \\ \bullet & \bullet. \arrow[l] \arrow[u] \end{tikzcd}$$

In particular, the action on morphisms of $-_1 \coprod_{-_3} -_1$ is given by sending a morphism

$$\begin{tikzcd}
A \coprod_C B & & B \arrow[ll] \arrow[dr, "\psi"] & \\
& A' \coprod_{C'} B' & & B' \arrow[ll] \\
A \arrow[uu] \arrow[dr, "\phi"'] & & C \arrow[ll, "f"'] \arrow[uu, "g"] \arrow[dr, "\chi"] & \\
& A' \arrow[uu] & & C' \arrow[ll, "f'"] \arrow[uu, "g'"']
\end{tikzcd}$$

in $\mathsf{Fun}(\mathcal{P}, \mathsf{Sets})$ to the map $\xi \colon A \coprod_C B \xrightarrow{\exists!} A' \coprod_{C'} B'$ given by

$$\xi(x) \stackrel{\text{def}}{=} \begin{cases} \phi(a) & \text{if } x = [(0, a)], \\ \psi(b) & \text{if } x = [(1, b)] \end{cases}$$

for each $x \in A \coprod_C B$, which is the unique map making the diagram

$$\begin{tikzcd}
A \coprod_C B \arrow[dr, dashed] & & B \arrow[ll] \arrow[dr, "\psi"] & \\
& A' \coprod_{C'} B' & & B' \arrow[ll] \\
A \arrow[uu] \arrow[dr, "\phi"'] & & C \arrow[ll, "f"'] \arrow[uu, "g"] \arrow[dr, "\chi"] & \\
& A' \arrow[uu] & & C' \arrow[ll, "f'"] \arrow[uu, "g'"']
\end{tikzcd}$$

commute.

2. *Associativity.* Given a diagram

$$\begin{tikzcd}
A & & B & & C \\
& X \arrow[ul, "f"] \arrow[ur, "g"'] & & Y \arrow[ul, "h"] \arrow[ur, "k"'] &
\end{tikzcd}$$

in Sets, we have isomorphisms of sets

$$(A \coprod_X B) \coprod_Y C \cong (A \coprod_X B) \coprod_B (B \coprod_Y C) \cong A \coprod_X (B \coprod_Y C),$$

where these pullbacks are built as in the diagrams

$$
\begin{tikzcd}[column sep=tiny, row sep=small]
 & & (A \coprod_X B) \coprod_Y C & & \\
 & A \coprod_X B \arrow[ur] & & & \\
A \arrow[ur] & & B \arrow[ul] & & C, \arrow[uull] \\
 & X \arrow[ul, "f"] \arrow[ur, "g"'] & & Y \arrow[ul, "h"] \arrow[ur, "k"'] &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=tiny, row sep=small]
 & & (A \coprod_X B) \coprod_B (B \coprod_Y C) & & \\
 & A \coprod_X B \arrow[ur] & & B \coprod_Y C \arrow[ul] & \\
A \arrow[ur] & & B \arrow[ul] \arrow[ur] & & C, \arrow[ul] \\
 & X \arrow[ul, "f"] \arrow[ur, "g"'] & & Y \arrow[ul, "h"] \arrow[ur, "k"'] &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=tiny, row sep=small]
 & & A \coprod_X (B \coprod_Y C) & & \\
 & & & B \coprod_Y C \arrow[ul] & \\
A \arrow[uurr] & & B \arrow[ur] & & C. \arrow[ul] \\
 & X \arrow[ul, "f"] \arrow[ur, "g"'] & & Y \arrow[ul, "h"] \arrow[ur, "k"'] &
\end{tikzcd}
$$

3. *Unitality.* We have isomorphisms of sets

$$
\begin{tikzcd}
A \arrow[r, equal] \arrow[dr, phantom, "\ulcorner", very near start] & A \\
X \arrow[u, "f"] \arrow[r, equal] & X \arrow[u, "f"']
\end{tikzcd}
\qquad
\begin{aligned}
X \textstyle\coprod_X A &\cong A, \\
A \textstyle\coprod_X X &\cong A,
\end{aligned}
\qquad
\begin{tikzcd}
A \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[l, "f"'] \\
X \arrow[u, equal] & X. \arrow[l, "f"] \arrow[u, equal]
\end{tikzcd}
$$

4. *Commutativity.* We have an isomorphism of sets

$$
\begin{tikzcd}
A \coprod_X B \arrow[dr, phantom, "\ulcorner", very near start] & B \arrow[l] \\
A \arrow[u] & X, \arrow[l, "f"] \arrow[u, "g"']
\end{tikzcd}
\qquad
A \textstyle\coprod_X B \cong B \textstyle\coprod_X A
\qquad
\begin{tikzcd}
B \coprod_X A \arrow[dr, phantom, "\ulcorner", very near start] & A \arrow[l] \\
B \arrow[u] & X. \arrow[l, "g"] \arrow[u, "f"']
\end{tikzcd}
$$

5. *Interaction With Coproducts.* We have

$$
A \textstyle\coprod_\emptyset B \cong A \coprod B,
\qquad
\begin{tikzcd}
A \coprod B \arrow[dr, phantom, "\ulcorner", very near start] & B \arrow[l] \\
A \arrow[u] & \emptyset. \arrow[l, hook', "\iota_A"] \arrow[u, hook, "\iota_B"']
\end{tikzcd}
$$

6. *Symmetric Monoidality.* The triple $(\mathsf{Sets}, \coprod_X, X)$ is a symmetric monoidal category.

Proof. Item 1, *Functoriality*: This is a special case of functoriality of co/limits, **??** of **??**, with the explicit expression for ξ following from the commutativity of the cube pushout diagram.
Item 2, *Associativity*: Omitted.
Item 3, *Unitality*: Omitted.
Item 4, *Commutativity*: Clear.
Item 5, *Interaction With Coproducts*: Clear.
Item 6, *Symmetric Monoidality*: Omitted. □

2.2.5 Coequalisers

Let A and B be sets and let $f, g \colon A \rightrightarrows B$ be functions.

Definition 2.2.5.1.1. The **coequaliser of** f **and** g is the pair $\left(\mathrm{CoEq}(f,g), \mathrm{coeq}(f,g)\right)$ consisting of:

- *The Colimit.* The set $\mathrm{CoEq}(f,g)$ defined by

  $$\mathrm{CoEq}(f,g) \stackrel{\text{def}}{=} B/{\sim},$$

  where $\sim$ is the equivalence relation on B generated by $f(a) \sim g(a)$.

- *The Cocone.* The map

  $$\mathrm{coeq}(f,g) \colon B \to \mathrm{CoEq}(f,g)$$

  given by the quotient map $\pi \colon B \twoheadrightarrow B/{\sim}$ with respect to the equivalence relation generated by $f(a) \sim g(a)$.

Proof. We claim that $\mathrm{CoEq}(f,g)$ is the categorical coequaliser of f and g in Sets. First we need to check that the relevant coequaliser diagram commutes, i.e. that we have

$$\mathrm{coeq}(f,g) \circ f = \mathrm{coeq}(f,g) \circ g.$$

Indeed, we have

$$\begin{aligned}
\left[\mathrm{coeq}(f,g) \circ f\right](a) &\stackrel{\text{def}}{=} \left[\mathrm{coeq}(f,g)\right](f(a)) \\
&\stackrel{\text{def}}{=} [f(a)] \\
&= [g(a)] \\
&\stackrel{\text{def}}{=} \left[\mathrm{coeq}(f,g)\right](g(a)) \\
&\stackrel{\text{def}}{=} \left[\mathrm{coeq}(f,g) \circ g\right](a)
\end{aligned}$$

for each $a \in A$. Next, we prove that $\mathrm{CoEq}(f,g)$ satisfies the universal property of the coequaliser. Suppose we have a diagram of the form

$$\begin{tikzcd}
A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, hook, "\mathrm{coeq}(f,g)"] \arrow[rd, "c"'] & \mathrm{CoEq}(f,g) \\
& & C
\end{tikzcd}$$

in Sets. Then, since $c(f(a)) = c(g(a))$ for each $a \in A$, it follows from Items 4 and 5 of Proposition 7.5.2.1.3 that there exists a unique map $\mathrm{CoEq}(f,g) \xrightarrow{\exists!}$

C making the diagram

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, hook, "\mathrm{coeq}(f,g)"] \arrow[dr, "c"'] & \mathrm{CoEq}(f,g) \arrow[d, dashed, "\exists!"] \\ & & C \end{tikzcd}$$

commute. □

Remark 2.2.5.1.2. In detail, by Construction 7.4.2.1.2, the relation $\sim$ of Definition 2.2.5.1.1 is given by declaring $a \sim b$ iff one of the following conditions is satisfied:

- We have $a = b$;
- There exist $x_1, \ldots, x_n \in B$ such that $a \sim' x_1 \sim' \cdots \sim' x_n \sim' b$, where we declare $x \sim' y$ if one of the following conditions is satisfied:
  1. There exists $z \in A$ such that $x = f(z)$ and $y = g(z)$.
  2. There exists $z \in A$ such that $x = g(z)$ and $y = f(z)$.

  That is: we require the following condition to be satisfied:

  (★) There exist $x_1, \ldots, x_n \in B$ satisfying the following conditions:
  1. There exists $z_0 \in A$ satisfying one of the following conditions:
     (a) We have $a = f(z_0)$ and $x_1 = g(z_0)$.
     (b) We have $a = g(z_0)$ and $x_1 = f(z_0)$.
  2. For each $1 \leq i \leq n - 1$, there exists $z_i \in A$ satisfying one of the following conditions:
     (a) We have $x_i = f(z_i)$ and $x_{i+1} = g(z_i)$.
     (b) We have $x_i = g(z_i)$ and $x_{i+1} = f(z_i)$.
  3. There exists $z_n \in A$ satisfying one of the following conditions:
     (a) We have $x_n = f(z_n)$ and $b = g(z_n)$.
     (b) We have $x_n = g(z_n)$ and $b = f(z_n)$.

Example 2.2.5.1.3. Here are some examples of coequalisers of sets.

1. *Quotients by Equivalence Relations.* Let R be an equivalence relation on a set X. We have a bijection of sets

$$X/\!\sim_R \,\cong \mathrm{CoEq}\left(R \hookrightarrow X \times X \underset{\mathrm{pr}_2}{\overset{\mathrm{pr}_1}{\rightrightarrows}} X\right).$$

Proof. *Item 1*, *Quotients by Equivalence Relations*: See [Pro24ad]. □

Proposition 2.2.5.1.4. Let A, B, and C be sets.

1. *Associativity.* We have isomorphisms of sets[11]

$$\underbrace{\mathrm{CoEq}(\mathrm{coeq}(f,g)\circ f,\mathrm{coeq}(f,g)\circ h)}_{=\mathrm{CoEq}(\mathrm{coeq}(f,g)\circ g,\mathrm{coeq}(f,g)\circ h)} \cong \mathrm{CoEq}(f,g,h) \cong \underbrace{\mathrm{CoEq}(\mathrm{coeq}(g,h)\circ f,\mathrm{coeq}(g,h)\circ g)}_{=\mathrm{CoEq}(\mathrm{coeq}(g,h)\circ f,\mathrm{coeq}(g,h)\circ h)},$$

where $\mathrm{CoEq}(f,g,h)$ is the colimit of the diagram

$$A \overset{f}{\underset{h}{\overset{-g\rightarrow}{\rightrightarrows}}} B$$

[11] That is, the following three ways of forming "the" coequaliser of (f,g,h) agree:

1. Take the coequaliser of (f,g,h), i.e. the colimit of the diagram

$$A \overset{f}{\underset{h}{\overset{-g\rightarrow}{\rightrightarrows}}} B$$

in Sets.

2. First take the coequaliser of f and g, forming a diagram

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \overset{\mathrm{coeq}(f,g)}{\twoheadrightarrow} \mathrm{CoEq}(f,g)$$

and then take the coequaliser of the composition

$$A \underset{h}{\overset{f}{\rightrightarrows}} B \overset{\mathrm{coeq}(f,g)}{\twoheadrightarrow} \mathrm{CoEq}(f,g),$$

obtaining a quotient

$$\mathrm{CoEq}\big(\mathrm{coeq}(f,g)\circ f,\mathrm{coeq}(f,g)\circ h\big) = \mathrm{CoEq}\big(\mathrm{coeq}(f,g)\circ g,\mathrm{coeq}(f,g)\circ h\big)$$

of $\mathrm{CoEq}(f,g)$

3. First take the coequaliser of g and h, forming a diagram

$$A \underset{h}{\overset{g}{\rightrightarrows}} B \overset{\mathrm{coeq}(g,h)}{\twoheadrightarrow} \mathrm{CoEq}(g,h)$$

and then take the coequaliser of the composition

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \overset{\mathrm{coeq}(g,h)}{\twoheadrightarrow} \mathrm{CoEq}(g,h),$$

obtaining a quotient

$$\mathrm{CoEq}\big(\mathrm{coeq}(g,h)\circ f,\mathrm{coeq}(g,h)\circ g\big) = \mathrm{CoEq}\big(\mathrm{coeq}(g,h)\circ f,\mathrm{coeq}(g,h)\circ h\big)$$

of $\mathrm{CoEq}(g,h)$.

in Sets.

4. *Unitality.* We have an isomorphism of sets
$$\mathrm{CoEq}(f, f) \cong B.$$
5. *Commutativity.* We have an isomorphism of sets
$$\mathrm{CoEq}(f, g) \cong \mathrm{CoEq}(g, f).$$
6. *Interaction With Composition.* Let
$$A \underset{g}{\overset{f}{\rightrightarrows}} B \underset{k}{\overset{h}{\rightrightarrows}} C$$
be functions. We have a surjection
$$\mathrm{CoEq}(h \circ f, k \circ g) \twoheadrightarrow \mathrm{CoEq}\big(\mathrm{coeq}(h, k) \circ h \circ f, \mathrm{coeq}(h, k) \circ k \circ g\big)$$
exhibiting $\mathrm{CoEq}\big(\mathrm{coeq}(h, k) \circ h \circ f, \mathrm{coeq}(h, k) \circ k \circ g\big)$ as a quotient of $\mathrm{CoEq}(h \circ f, k \circ g)$ by the relation generated by declaring $h(y) \sim k(y)$ for each $y \in B$.

Proof. *Item 1*, *Associativity*: Omitted.
Item 4, *Unitality*: Clear.
Item 5, *Commutativity*: Clear.
Item 6, *Interaction With Composition*: Omitted. □

2.3 Operations With Sets

2.3.1 The Empty Set

Definition 2.3.1.1.1. The **empty set** is the set $\emptyset$ defined by
$$\emptyset \overset{\text{def}}{=} \{x \in X \mid x \neq x\},$$
where A is the set in the set existence axiom, **??** of **??**.

2.3.2 Singleton Sets

Let X be a set.

Definition 2.3.2.1.1. The **singleton set containing** X is the set $\{X\}$ defined by
$$\{X\} \overset{\text{def}}{=} \{X, X\},$$
where $\{X, X\}$ is the pairing of X with itself (Definition 2.3.3.1.1).

2.3.3 Pairings of Sets

Let X and Y be sets.

Definition 2.3.3.1.1. The **pairing of** X **and** Y is the set $\{X, Y\}$ defined by

$$\{X, Y\} \overset{\text{def}}{=} \{x \in A \mid x = X \text{ or } x = Y\},$$

where A is the set in the axiom of pairing, **??** of **??**.

2.3.4 Ordered Pairs

Let A and B be sets.

Definition 2.3.4.1.1. The **ordered pair associated to** A **and** B is the set (A, B) defined by

$$(A, B) \overset{\text{def}}{=} \{\{A\}, \{A, B\}\}.$$

Proposition 2.3.4.1.2. Let A and B be sets.

1. *Uniqueness.* Let A, B, C, and D be sets. The following conditions are equivalent:

   (a) We have $(A, B) = (C, D)$.

   (b) We have $A = C$ and $B = D$.

Proof. *Item 1*, *Uniqueness*: See [Cie97, Theorem 1.2.3]. □

2.3.5 Sets of Maps

Let A and B be sets.

Definition 2.3.5.1.1. The **set of maps from** A **to** B[12] is the set $\mathrm{Hom}_{\mathsf{Sets}}(A, B)$[13] whose elements are the functions from A to B.

Proposition 2.3.5.1.2. Let A and B be sets.

1. *Functoriality.* The assignments $X, Y, (X, Y) \mapsto \mathrm{Hom}_{\mathsf{Sets}}(X, Y)$ define functors

$$\mathrm{Hom}_{\mathsf{Sets}}(X, -) \colon \mathsf{Sets} \to \mathsf{Sets},$$
$$\mathrm{Hom}_{\mathsf{Sets}}(-, Y) \colon \mathsf{Sets}^{\mathsf{op}} \to \mathsf{Sets},$$
$$\mathrm{Hom}_{\mathsf{Sets}}(-_1, -_2) \colon \mathsf{Sets}^{\mathsf{op}} \times \mathsf{Sets} \to \mathsf{Sets}.$$

Proof. *Item 1*, *Functoriality*: This follows from Items 2 and 5 of Proposition 8.1.6.1.2. □

[12] *Further Terminology:* Also called the **Hom set from** A **to** B.
[13] *Further Notation:* Also written $\mathsf{Sets}(A, B)$.

2.3.6 Unions of Families

Let $\{A_i\}_{i\in I}$ be a family of sets.

Definition 2.3.6.1.1. The **union of the family** $\{A_i\}_{i\in I}$ is the set $\bigcup_{i\in I} A_i$ defined by

$$\bigcup_{i\in I} A_i \stackrel{\mathrm{def}}{=} \{x \in F \mid \text{there exists some } i \in I \text{ such that } x \in A_i\},$$

where F is the set in the axiom of union, **??** of **??**.

2.3.7 Binary Unions

Let A and B be sets.

Definition 2.3.7.1.1. The **union**[14] **of** A **and** B is the set $A \cup B$ defined by

$$A \cup B \stackrel{\mathrm{def}}{=} \bigcup_{z\in\{A,B\}} z.$$

Proposition 2.3.7.1.2. Let X be a set.

1. *Functoriality.* The assignments $U, V, (U, V) \mapsto U \cup V$ define functors

$$\begin{aligned} U \cup - \colon (\mathcal{P}(X), \subset) &\to (\mathcal{P}(X), \subset), \\ - \cup V \colon (\mathcal{P}(X), \subset) &\to (\mathcal{P}(X), \subset), \\ -_1 \cup -_2 \colon (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \subset) &\to (\mathcal{P}(X), \subset), \end{aligned}$$

where $-_1 \cup -_2$ is the functor where

- *Action on Objects.* For each $(U, V) \in \mathcal{P}(X) \times \mathcal{P}(X)$, we have

$$[-_1 \cup -_2](U, V) \stackrel{\mathrm{def}}{=} U \cup V.$$

- *Action on Morphisms.* For each pair of morphisms

$$\begin{aligned} \iota_U &\colon U \hookrightarrow U', \\ \iota_V &\colon V \hookrightarrow V' \end{aligned}$$

of $\mathcal{P}(X) \times \mathcal{P}(X)$, the image

$$\iota_U \cup \iota_V \colon U \cup V \hookrightarrow U' \cup V'$$

of (ι_U, ι_V) by $\cup$ is the inclusion

$$U \cup V \subset U' \cup V'$$

i.e. where we have

[14]*Further Terminology:* Also called the **binary union of** A **and** B, for emphasis.

($\star$) If $U \subset U'$ and $V \subset V'$, then $U \cup V \subset U' \cup V'$.

and where $U \cup -$ and $- \cup V$ are the partial functors of $-_1 \cup -_2$ at $U, V \in \mathcal{P}(X)$.

2. *Via Intersections and Symmetric Differences.* We have an equality of sets

$$U \cup V = (U \vartriangle V) \vartriangle (U \cap V)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

3. *Associativity.* We have an equality of sets

$$(U \cup V) \cup W = U \cup (V \cup W)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

4. *Unitality.* We have equalities of sets

$$\begin{aligned} U \cup \emptyset &= U, \\ \emptyset \cup U &= U \end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

5. *Commutativity.* We have an equality of sets

$$U \cup V = V \cup U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

6. *Idempotency.* We have an equality of sets

$$U \cup U = U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

7. *Distributivity Over Intersections.* We have equalities of sets

$$\begin{aligned} U \cup (V \cap W) &= (U \cup V) \cap (U \cup W), \\ (U \cap V) \cup W &= (U \cup W) \cap (V \cup W) \end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

8. *Interaction With Characteristic Functions I.* We have

$$\chi_{U \cup V} = \max(\chi_U, \chi_V)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

9. *Interaction With Characteristic Functions II.* We have

$$\chi_{U\cup V} = \chi_U + \chi_V - \chi_{U\cap V}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

10. *Interaction With Powersets and Semirings.* The quintuple $(\mathcal{P}(X), \cup, \cap, \emptyset, X)$ is an idempotent commutative semiring.

Proof. *Item 1, Functoriality*: See [Pro24ar].
Item 2, Via Intersections and Symmetric Differences: See [Pro24bc].
Item 3, Associativity: See [Pro24be].
Item 4, Unitality: This follows from [Pro24bh] and Item 5.
Item 5, Commutativity: See [Pro24bf].
Item 6, Idempotency: See [Pro24aq].
Item 7, Distributivity Over Intersections: See [Pro24bd].
Item 8, Interaction With Characteristic Functions I: See [Pro24k].
Item 9, Interaction With Characteristic Functions II: See [Pro24k].
Item 10, Interaction With Powersets and Semirings: This follows from Items 3 to 6 and Items 3 to 5, 7 and 8 of Proposition 2.3.9.1.2. □

2.3.8 Intersections of Families

Let $\mathcal{F}$ be a family of sets.

Definition 2.3.8.1.1. The **intersection of a family $\mathcal{F}$ of sets** is the set $\bigcap_{X\in\mathcal{F}} X$ defined by

$$\bigcap_{X\in\mathcal{F}} X \overset{\mathrm{def}}{=} \left\{ z \in \bigcup_{X\in\mathcal{F}} X \;\middle|\; \text{for each } X \in \mathcal{F}, \text{ we have } z \in X \right\}.$$

2.3.9 Binary Intersections

Let X and Y be sets.

Definition 2.3.9.1.1. The **intersection**[15] **of X and Y** is the set $X \cap Y$ defined by

$$X \cap Y \overset{\mathrm{def}}{=} \bigcap_{z\in\{X,Y\}} z.$$

Proposition 2.3.9.1.2. Let X be a set.

[15] *Further Terminology:* Also called the **binary intersection of X and Y**, for emphasis.

1. *Functoriality.* The assignments $U, V, (U, V) \mapsto U \cap V$ define functors

$$\begin{aligned} U \cap - \colon (\mathcal{P}(X), \subset) &\to (\mathcal{P}(X), \subset), \\ - \cap V \colon (\mathcal{P}(X), \subset) &\to (\mathcal{P}(X), \subset), \\ -_1 \cap -_2 \colon (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \subset) &\to (\mathcal{P}(X), \subset), \end{aligned}$$

   where $-_1 \cap -_2$ is the functor where

   - *Action on Objects.* For each $(U, V) \in \mathcal{P}(X) \times \mathcal{P}(X)$, we have

     $$[-_1 \cap -_2](U, V) \overset{\text{def}}{=} U \cap V.$$

   - *Action on Morphisms.* For each pair of morphisms

     $$\begin{aligned} \iota_U &\colon U \hookrightarrow U', \\ \iota_V &\colon V \hookrightarrow V' \end{aligned}$$

     of $\mathcal{P}(X) \times \mathcal{P}(X)$, the image

     $$\iota_U \cap \iota_V \colon U \cap V \hookrightarrow U' \cap V'$$

     of (ι_U, ι_V) by $\cap$ is the inclusion

     $$U \cap V \subset U' \cap V'$$

     i.e. where we have

     ($\star$) If $U \subset U'$ and $V \subset V'$, then $U \cap V \subset U' \cap V'$.

   and where $U \cap -$ and $- \cap V$ are the partial functors of $-_1 \cap -_2$ at $U, V \in \mathcal{P}(X)$.

2. *Adjointness.* We have adjunctions

$$\big(U \cap - \dashv \mathbf{Hom}_{\mathcal{P}(X)}(U, -)\big)\colon \quad \begin{tikzcd} \mathcal{P}(X) \arrow[r, bend left, "U\cap -"] \arrow[r, phantom, "\perp"] & \mathcal{P}(X) \arrow[l, bend left, "\mathbf{Hom}_{\mathcal{P}(X)}(U,-)"] \end{tikzcd},$$

$$\big(- \cap V \dashv \mathbf{Hom}_{\mathcal{P}(X)}(V, -)\big)\colon \quad \begin{tikzcd} \mathcal{P}(X) \arrow[r, bend left, "-\cap V"] \arrow[r, phantom, "\perp"] & \mathcal{P}(X) \arrow[l, bend left, "\mathbf{Hom}_{\mathcal{P}(X)}(V,-)"] \end{tikzcd},$$

   where

$$\mathbf{Hom}_{\mathcal{P}(X)}(-_1, -_2) \colon \mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \to \mathcal{P}(X)$$

is the bifunctor defined by[16]

$$\mathbf{Hom}_{\mathcal{P}(X)}(U,V) \overset{\text{def}}{=} (X \setminus U) \cup V$$

witnessed by bijections

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{P}(X)}(U \cap V, W) &\cong \mathrm{Hom}_{\mathcal{P}(X)}\big(U, \mathbf{Hom}_{\mathcal{P}(X)}(V,W)\big),\\
\mathrm{Hom}_{\mathcal{P}(X)}(U \cap V, W) &\cong \mathrm{Hom}_{\mathcal{P}(X)}\big(V, \mathbf{Hom}_{\mathcal{P}(X)}(U,W)\big),
\end{aligned}$$

natural in $U, V, W \in \mathcal{P}(X)$, i.e. where:

(a) The following conditions are equivalent:
   i. We have $U \cap V \subset W$.
   ii. We have $U \subset \mathbf{Hom}_{\mathcal{P}(X)}(V,W)$.
   iii. We have $U \subset (X \setminus V) \cup W$.

(b) The following conditions are equivalent:
   i. We have $V \cap U \subset W$.
   ii. We have $V \subset \mathbf{Hom}_{\mathcal{P}(X)}(U,W)$.
   iii. We have $V \subset (X \setminus U) \cup W$.

3. *Associativity.* We have an equality of sets

$$(U \cap V) \cap W = U \cap (V \cap W)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

4. *Unitality.* Let X be a set and let $U \in \mathcal{P}(X)$. We have equalities of sets

$$\begin{aligned}
X \cap U &= U,\\
U \cap X &= U
\end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

5. *Commutativity.* We have an equality of sets

$$U \cap V = V \cap U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

6. *Idempotency.* We have an equality of sets

$$U \cap U = U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

[16] For intuition regarding the expression defining $\mathbf{Hom}_{\mathcal{P}(X)}(U,V)$, see Remark 2.3.9.1.3.

7. *Distributivity Over Unions.* We have equalities of sets

$$U \cap (V \cup W) = (U \cap V) \cup (U \cap W),$$
$$(U \cup V) \cap W = (U \cap W) \cup (V \cap W)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

8. *Annihilation With the Empty Set.* We have an equality of sets

$$\emptyset \cap X = \emptyset,$$
$$X \cap \emptyset = \emptyset$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

9. *Interaction With Characteristic Functions I.* We have

$$\chi_{U \cap V} = \chi_U \chi_V$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

10. *Interaction With Characteristic Functions II.* We have

$$\chi_{U \cap V} = \min(\chi_U, \chi_V)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

11. *Interaction With Powersets and Monoids With Zero.* The quadruple $((\mathcal{P}(X), \emptyset), \cap, X)$ is a commutative monoid with zero.

12. *Interaction With Powersets and Semirings.* The quintuple $(\mathcal{P}(X), \cup, \cap, \emptyset, X)$ is an idempotent commutative semiring.

Proof. *Item 1, Functoriality*: See [Pro24ap].
Item 2, Adjointness: See [MSE 267469].
Item 3, Associativity: See [Pro24v].
Item 4, Unitality: This follows from [Pro24z] and Item 5.
Item 5, Commutativity: See [Pro24w].
Item 6, Idempotency: See [Pro24ao].
Item 7, Distributivity Over Unions: See [Pro24an].
Item 8, Annihilation With the Empty Set: This follows from [Pro24x] and Item 5.
Item 9, Interaction With Characteristic Functions I: See [Pro24h].
Item 10, Interaction With Characteristic Functions II: See [Pro24h].
Item 11, Interaction With Powersets and Monoids With Zero: This follows from Items 3 to 5 and 8.
Item 12, Interaction With Powersets and Semirings: This follows from Items 3 to 6 and Items 3 to 5, 7 and 8 of Proposition 2.3.9.1.2. □

Remark 2.3.9.1.3. Since intersections are the products in $\mathcal{P}(X)$ (Item 1 of Proposition 2.4.3.1.3), the left adjoint $\mathbf{Hom}_{\mathcal{P}(X)}(U, V)$ may be thought of as a function type $[U, V]$.
Then, under the Curry–Howard correspondence, the function type $[U, V]$ corresponds to implication $U \implies V$, which is logically equivalent to the statement $\neg U \vee V$. This in turn corresponds to the set $U^{\mathsf{c}} \vee V = (X \setminus U) \cup V$.

2.3.10 Differences

Let X and Y be sets.

Definition 2.3.10.1.1. The **difference of** X **and** Y is the set $X \setminus Y$ defined by

$$X \setminus Y \stackrel{\text{def}}{=} \{a \in X \mid a \notin Y\}.$$

Proposition 2.3.10.1.2. Let X be a set.

1. *Functoriality.* The assignments $U, V, (U, V) \mapsto U \cap V$ define functors

$$\begin{aligned} U \setminus -&\colon (\mathcal{P}(X), \supset) \to (\mathcal{P}(X), \subset), \\ - \setminus V&\colon (\mathcal{P}(X), \subset) \to (\mathcal{P}(X), \subset), \\ -_1 \setminus -_2&\colon (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \supset) \to (\mathcal{P}(X), \subset), \end{aligned}$$

   where $-_1 \setminus -_2$ is the functor where

   - *Action on Objects.* For each $(U, V) \in \mathcal{P}(X) \times \mathcal{P}(X)$, we have

   $$[-_1 \setminus -_2](U, V) \stackrel{\text{def}}{=} U \setminus V.$$

   - *Action on Morphisms.* For each pair of morphisms

   $$\begin{aligned} \iota_A&\colon A \hookrightarrow B, \\ \iota_U&\colon U \hookrightarrow V \end{aligned}$$

     of $\mathcal{P}(X) \times \mathcal{P}(X)$, the image

   $$\iota_U \setminus \iota_V \colon A \setminus V \hookrightarrow B \setminus U$$

     of (ι_U, ι_V) by $\setminus$ is the inclusion

   $$A \setminus V \subset B \setminus U$$

     i.e. where we have

     ($\star$) If $A \subset B$ and $U \subset V$, then $A \setminus V \subset B \setminus U$.

and where $U \setminus -$ and $- \setminus V$ are the partial functors of $-_1 \setminus -_2$ at $U, V \in \mathcal{P}(X)$.

2. *De Morgan's Laws.* We have equalities of sets

$$\begin{aligned} X \setminus (U \cup V) &= (X \setminus U) \cap (X \setminus V), \\ X \setminus (U \cap V) &= (X \setminus U) \cup (X \setminus V) \end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

3. *Interaction With Unions I.* We have equalities of sets

$$U \setminus (V \cup W) = (U \setminus V) \cap (U \setminus W)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

4. *Interaction With Unions II.* We have equalities of sets

$$(U \setminus V) \cup W = (U \cup W) \setminus (V \setminus W)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

5. *Interaction With Unions III.* We have equalities of sets

$$\begin{aligned} U \setminus (V \cup W) &= (U \cup W) \setminus (V \cup W) \\ &= (U \setminus V) \setminus W \\ &= (U \setminus W) \setminus V \end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

6. *Interaction With Unions IV.* We have equalities of sets

$$(U \cup V) \setminus W = (U \setminus W) \cup (V \setminus W)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

7. *Interaction With Intersections.* We have equalities of sets

$$\begin{aligned} (U \setminus V) \cap W &= (U \cap W) \setminus V \\ &= U \cap (W \setminus V) \end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

8. *Interaction With Complements.* We have an equality of sets

$$U \setminus V = U \cap V^{\mathsf{c}}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

9. *Interaction With Symmetric Differences.* We have an equality of sets

$$U \setminus V = U \mathbin{\triangle} (U \cap V)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

10. *Triple Differences.* We have

$$U \setminus (V \setminus W) = (U \cap W) \cup (U \setminus V)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

11. *Left Annihilation.* We have

$$\emptyset \setminus U = \emptyset$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

12. *Right Unitality.* We have

$$U \setminus \emptyset = U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

13. *Invertibility.* We have

$$U \setminus U = \emptyset$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

14. *Interaction With Containment.* The following conditions are equivalent:

    (a) We have $V \setminus U \subset W$.

    (b) We have $V \setminus W \subset U$.

15. *Interaction With Characteristic Functions.* We have

$$\chi_{U \setminus V} = \chi_U - \chi_{U \cap V}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

Proof. *Item 1, Functoriality*: See [Pro24ah] and [Pro24al].
Item 2, De Morgan's Laws: See [Pro24p].
Item 3, Interaction With Unions I: See [Pro24q].
Item 4, Interaction With Unions II: Omitted.
Item 5, Interaction With Unions III: See [Pro24am].
Item 6, Interaction With Unions IV: See [Pro24ag].

Item 7, *Interaction With Intersections*: See [Pro24y].
Item 8, *Interaction With Complements*: See [Pro24ae].
Item 9, *Interaction With Symmetric Differences*: See [Pro24af].
Item 10, *Triple Differences*: See [Pro24ak].
Item 11, *Left Annihilation*: Clear.
Item 12, *Right Unitality*: See [Pro24ai].
Item 13, *Invertibility*: See [Pro24aj].
Item 14, *Interaction With Containment*: Omitted.
Item 15, *Interaction With Characteristic Functions*: See [Pro24i]. □

2.3.11 Complements

Let X be a set and let $U \in \mathcal{P}(X)$.

Definition 2.3.11.1.1. The **complement of** U is the set U^{c} defined by

$$\begin{aligned} U^{\mathsf{c}} &\stackrel{\text{def}}{=} X \setminus U \\ &\stackrel{\text{def}}{=} \{a \in X \mid a \notin U\}. \end{aligned}$$

Proposition 2.3.11.1.2. Let X be a set.

1. *Functoriality.* The assignment $U \mapsto U^{\mathsf{c}}$ defines a functor

$$(-)^{\mathsf{c}} \colon \mathcal{P}(X)^{\mathsf{op}} \to \mathcal{P}(X),$$

where

- *Action on Objects.* For each $U \in \mathcal{P}(X)$, we have

$$\left[(-)^{\mathsf{c}}\right](U) \stackrel{\text{def}}{=} U^{\mathsf{c}}.$$

- *Action on Morphisms.* For each morphism $\iota_U \colon U \hookrightarrow V$ of $\mathcal{P}(X)$, the image

$$\iota_U^{\mathsf{c}} \colon V^{\mathsf{c}} \hookrightarrow U^{\mathsf{c}}$$

of ι_U by $(-)^{\mathsf{c}}$ is the inclusion

$$V^{\mathsf{c}} \subset U^{\mathsf{c}}$$

i.e. where we have

(★) If $U \subset V$, then $V^{\mathsf{c}} \subset U^{\mathsf{c}}$.

2. *De Morgan's Laws.* We have equalities of sets

$$\begin{aligned} (U \cup V)^{\mathsf{c}} &= U^{\mathsf{c}} \cap V^{\mathsf{c}}, \\ (U \cap V)^{\mathsf{c}} &= U^{\mathsf{c}} \cup V^{\mathsf{c}} \end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

3. *Involutority.* We have

$$(U^{\mathsf{c}})^{\mathsf{c}} = U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

4. *Interaction With Characteristic Functions.* We have

$$\chi_{U^{\mathsf{c}}} = 1 - \chi_U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

Proof. *Item 1*, *Functoriality*: This follows from Item 1 of Proposition 2.3.10.1.2.
Item 2, *De Morgan's Laws*: See [Pro24p].
Item 3, *Involutority*: See [Pro24l].
Item 4, *Interaction With Characteristic Functions*: Clear. □

2.3.12 Symmetric Differences

Let A and B be sets.

Definition 2.3.12.1.1. The **symmetric difference of** A **and** B is the set $A \bigtriangleup B$ defined by

$$A \bigtriangleup B \overset{\text{def}}{=} (A \setminus B) \cup (B \setminus A).$$

Proposition 2.3.12.1.2. Let X be a set.

1. *Lack of Functoriality.* The assignment $(U, V) \mapsto U \bigtriangleup V$ ***need not*** define functors

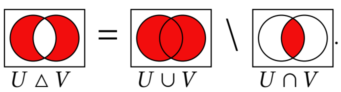
$$\begin{aligned} U \bigtriangleup -\colon (\mathcal{P}(X), \subset) &\to (\mathcal{P}(X), \subset), \\ - \bigtriangleup V\colon (\mathcal{P}(X), \subset) &\to (\mathcal{P}(X), \subset), \\ -_1 \bigtriangleup -_2\colon (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \subset) &\to (\mathcal{P}(X), \subset). \end{aligned}$$

2. *Via Unions and Intersections.* We have[17]

$$U \bigtriangleup V = (U \cup V) \setminus (U \cap V)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

[17] *Illustration:*



3. *Associativity.* We have[18]

$$(U \mathbin{\triangle} V) \mathbin{\triangle} W = U \mathbin{\triangle} (V \mathbin{\triangle} W)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

4. *Commutativity.* We have

$$U \mathbin{\triangle} V = V \mathbin{\triangle} U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

5. *Unitality.* We have

$$\begin{aligned} U \mathbin{\triangle} \emptyset &= U, \\ \emptyset \mathbin{\triangle} U &= U \end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

6. *Invertibility.* We have

$$U \mathbin{\triangle} U = \emptyset$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

7. *Interaction With Unions.* We have

$$(U \mathbin{\triangle} V) \cup (V \mathbin{\triangle} T) = (U \cup V \cup W) \setminus (U \cap V \cap W)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

8. *Interaction With Complements I.* We have

$$U \mathbin{\triangle} U^{\mathsf{c}} = X$$

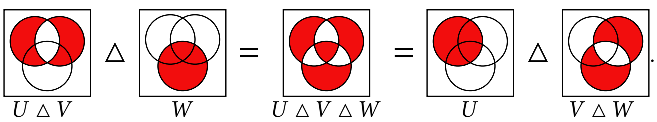
for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

9. *Interaction With Complements II.* We have

$$\begin{aligned} U \mathbin{\triangle} X &= U^{\mathsf{c}}, \\ X \mathbin{\triangle} U &= U^{\mathsf{c}} \end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

[18] *Illustration:*



10. *Interaction With Complements III.* We have

$$U^{\mathsf{c}} \mathbin{\triangle} V^{\mathsf{c}} = U \mathbin{\triangle} V$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

11. *"Transitivity"*. We have

$$(U \mathbin{\triangle} V) \mathbin{\triangle} (V \mathbin{\triangle} W) = U \mathbin{\triangle} W$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

12. *The Triangle Inequality for Symmetric Differences.* We have

$$U \mathbin{\triangle} W \subset U \mathbin{\triangle} V \cup V \mathbin{\triangle} W$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

13. *Distributivity Over Intersections.* We have

$$\begin{aligned} U \cap (V \mathbin{\triangle} W) &= (U \cap V) \mathbin{\triangle} (U \cap W), \\ (U \mathbin{\triangle} V) \cap W &= (U \cap W) \mathbin{\triangle} (V \cap W) \end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

14. *Interaction With Characteristic Functions.* We have

$$\chi_{U \triangle V} = \chi_U + \chi_V - 2\chi_{U \cap V}$$

and thus, in particular, we have

$$\chi_{U \triangle V} \equiv \chi_U + \chi_V \mod 2$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

15. *Bijectivity.* Given $A, B \subset \mathcal{P}(X)$, the maps

$$\begin{aligned} A \mathbin{\triangle} -&\colon \mathcal{P}(X) \to \mathcal{P}(X), \\ - \mathbin{\triangle} B&\colon \mathcal{P}(X) \to \mathcal{P}(X) \end{aligned}$$

are bijections with inverses given by

$$\begin{aligned} (A \mathbin{\triangle} -)^{-1} &= - \cup (A \cap -), \\ (- \mathbin{\triangle} B)^{-1} &= - \cup (B \cap -). \end{aligned}$$

Moreover, the map

$$C \mapsto C \mathbin{\triangle} (A \mathbin{\triangle} B)$$

is a bijection of $\mathcal{P}(X)$ onto itself sending A to B and B to A.

16. *Interaction With Powersets and Groups.* Let X be a set.

    (a) The quadruple $\left(\mathcal{P}(X), \triangle, \emptyset, \mathrm{id}_{\mathcal{P}(X)}\right)$ is an abelian group.[19]

    (b) Every element of $\mathcal{P}(X)$ has order 2 with respect to $\triangle$, and thus $\mathcal{P}(X)$ is a *Boolean group* (i.e. an abelian 2-group).

4. *Interaction With Powersets and Vector Spaces I.* The pair $\left(\mathcal{P}(X), \alpha_{\mathcal{P}(X)}\right)$ consisting of

    - The group $\mathcal{P}(X)$ of **??**;
    - The map $\alpha_{\mathcal{P}(X)} \colon \mathbb{F}_2 \times \mathcal{P}(X) \to \mathcal{P}(X)$ defined by

    $$\begin{aligned} 0 \cdot U &\stackrel{\text{def}}{=} \emptyset, \\ 1 \cdot U &\stackrel{\text{def}}{=} U; \end{aligned}$$

    is an $\mathbb{F}_2$-vector space.

5. *Interaction With Powersets and Vector Spaces II.* If X is finite, then:

    (a) The set of singletons sets on the elements of X forms a basis for the $\mathbb{F}_2$-vector space $\left(\mathcal{P}(X), \alpha_{\mathcal{P}(X)}\right)$ of Item 4.

    (b) We have
    $$\dim(\mathcal{P}(X)) = \#\mathcal{P}(X).$$

6. *Interaction With Powersets and Rings.* The quintuple $(\mathcal{P}(X), \triangle, \cap, \emptyset, X)$ is a commutative ring.[20]

---

[19] Here are some examples:

1. When $X = \emptyset$, we have an isomorphism of groups between $\mathcal{P}(\emptyset)$ and the trivial group:
$$\left(\mathcal{P}(\emptyset), \triangle, \emptyset, \mathrm{id}_{\mathcal{P}(\emptyset)}\right) \cong \mathrm{pt}.$$

2. When $X = \mathrm{pt}$, we have an isomorphism of groups between $\mathcal{P}(\mathrm{pt})$ and $\mathbb{Z}_{/2}$:
$$\left(\mathcal{P}(\mathrm{pt}), \triangle, \emptyset, \mathrm{id}_{\mathcal{P}(\mathrm{pt})}\right) \cong \mathbb{Z}_{/2}.$$

3. When $X = \{0, 1\}$, we have an isomorphism of groups between $\mathcal{P}(\{0,1\})$ and $\mathbb{Z}_{/2} \times \mathbb{Z}_{/2}$:
$$\left(\mathcal{P}(\{0,1\}), \triangle, \emptyset, \mathrm{id}_{\mathcal{P}(\{0,1\})}\right) \cong \mathbb{Z}_{/2} \times \mathbb{Z}_{/2}.$$

[20] *Warning:* The analogous statement replacing intersections by unions (i.e. that the quintuple $(\mathcal{P}(X), \triangle, \cup, \emptyset, X)$ is a ring) is false, however. See [Pro24ba] for a proof.
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Proof. *Item 1, Lack of Functoriality*: Omitted.
Item 2, Via Unions and Intersections: See [Pro24r].
Item 3, Associativity: See [Pro24as].
Item 4, Commutativity: See [Pro24at].
Item 5, Unitality: This follows from Item 4 and [Pro24ax].
Item 6, Invertibility: See [Pro24az].
Item 7, Interaction With Unions: See [Pro24bg].
Item 8, Interaction With Complements I: See [Pro24aw].
Item 9, Interaction With Complements II: This follows from Item 4 and [Pro24bb].
Item 10, Interaction With Complements III: See [Pro24au].
Item 11, "Transitivity": We have

$$\begin{aligned}
(U \triangle V) \triangle (V \triangle W) &= U \triangle (V \triangle (V \triangle W)) && \text{(by Item 3)}\\
&= U \triangle ((V \triangle V) \triangle W) && \text{(by Item 3)}\\
&= U \triangle (\emptyset \triangle W) && \text{(by Item 6)}\\
&= U \triangle W && \text{(by Item 5)}
\end{aligned}$$

Item 12, The Triangle Inequality for Symmetric Differences: This follows from Items 2 and 11.
Item 13, Distributivity Over Intersections: See [Pro24u].
Item 14, Interaction With Characteristic Functions: See [Pro24j].
Item 15, Bijectivity: Clear.
Item 16, Interaction With Powersets and Groups: Item 16a follows from[21] Items 3 to 6, while Item 3b follows from Item 6.
Item 4, Interaction With Powersets and Vector Spaces I: Clear.
Item 5, Interaction With Powersets and Vector Spaces II: Omitted.
Item 6, Interaction With Powersets and Rings: This follows from Items 8 and 11 of Proposition 2.3.9.1.2 and Items 13 and 16.[22] □

2.4 Powersets

2.4.1 Characteristic Functions

Let X be a set.

Definition 2.4.1.1.1. Let $U \subset X$ and let $x \in X$.

[21] Reference: [Pro24av].
[22] *Reference:* [Pro24ay].

1. The **characteristic function of** U[23] is the function[24]

$$\chi_U \colon X \to \{\mathsf{t}, \mathsf{f}\}$$

defined by

$$\chi_U(x) \overset{\text{def}}{=} \begin{cases} \mathsf{true} & \text{if } x \in U, \\ \mathsf{false} & \text{if } x \notin U \end{cases}$$

for each $x \in X$.

2. The **characteristic function of** x is the function[25]

$$\chi_x \colon X \to \{\mathsf{t}, \mathsf{f}\}$$

defined by

$$\chi_x \overset{\text{def}}{=} \chi_{\{x\}},$$

i.e. by

$$\chi_x(y) \overset{\text{def}}{=} \begin{cases} \mathsf{true} & \text{if } x = y, \\ \mathsf{false} & \text{if } x \neq y \end{cases}$$

for each $y \in X$.

3. The **characteristic relation on** X[26] is the relation[27]

$$\chi_X(-_1, -_2) \colon X \times X \to \{\mathsf{t}, \mathsf{f}\}$$

on X defined by[28]

$$\chi_X(x, y) \overset{\text{def}}{=} \begin{cases} \mathsf{true} & \text{if } x = y, \\ \mathsf{false} & \text{if } x \neq y \end{cases}$$

for each $x, y \in X$.

4. The **characteristic embedding**[29] **of** X **into** $\mathcal{P}(X)$ is the function

$$\chi_{(-)} \colon X \hookrightarrow \mathcal{P}(X)$$

[23] *Further Terminology:* Also called the **indicator function of** U.
[24] *Further Notation:* Also written $\chi_X(U, -)$ or $\chi_X(-, U)$.
[25] *Further Notation:* Also written χ^x, $\chi_X(x, -)$, or $\chi_X(-, x)$.
[26] *Further Terminology:* Also called the **identity relation on** X.
[27] *Further Notation:* Also written $\chi_{-_2}^{-_1}$, or $\sim_{\mathrm{id}}$ in the context of relations.
[28] As a subset of $X \times X$, the relation χ_X corresponds to the diagonal $\Delta_X \subset X \times X$ of X.
[29] The name "characteristic *embedding*" comes from the fact that there is an analogue of

defined by

$$\chi_{(-)}(x) \stackrel{\text{def}}{=} \chi_x$$

for each $x \in X$.

Remark 2.4.1.1.2. The definitions in Definition 2.4.1.1.1 are decategorifications of co/presheaves, representable co/presheaves, Hom profunctors, and the Yoneda embedding:[30]

1. A function

$$f : X \to \{\mathsf{t}, \mathsf{f}\}$$

is a decategorification of a presheaf

$$\mathcal{F} : \mathcal{C}^{\mathsf{op}} \to \mathsf{Sets},$$

with the characteristic functions χ_U of the subsets of X being the primordial examples (and, in fact, all examples) of these.

2. The characteristic function

$$\chi_x : X \to \{\mathsf{t}, \mathsf{f}\}$$

fully faithfulness for $\chi_{(-)}$: given a set X, we have

$$\mathrm{Hom}_{\mathcal{P}(X)}(\chi_x, \chi_y) = \chi_X(x, y),$$

for each $x, y \in X$.

[30]These statements can be made precise by using the embeddings

$$\begin{aligned}(-)_{\mathsf{disc}} &: \mathsf{Sets} \hookrightarrow \mathsf{Cats},\\ (-)_{\mathsf{disc}} &: \{\mathsf{t}, \mathsf{f}\}_{\mathsf{disc}} \hookrightarrow \mathsf{Sets}\end{aligned}$$

of sets into categories and of classical truth values into sets.

For instance, in this approach the characteristic function

$$\chi_x : X \to \{\mathsf{t}, \mathsf{f}\}$$

of an element x of X, defined by

$$\chi_x(y) \stackrel{\text{def}}{=} \begin{cases} \mathsf{true} & \text{if } x = y, \\ \mathsf{false} & \text{if } x \neq y \end{cases}$$

for each $y \in X$, is recovered as the representable presheaf

$$\mathrm{Hom}_{X_{\mathsf{disc}}}(-, x) : X_{\mathsf{disc}} \to \mathsf{Sets}$$

of the corresponding object x of X_{disc}, defined on objects by

$$\mathrm{Hom}_{X_{\mathsf{disc}}}(y, x) \stackrel{\text{def}}{=} \begin{cases} \mathrm{pt} & \text{if } x = y, \\ \emptyset & \text{if } x \neq y \end{cases}$$

for each $y \in \mathrm{Obj}(X_{\mathsf{disc}})$.

of an *element* x of X is a decategorification of the representable presheaf

$$h_X\colon C^{\mathsf{op}} \to \mathsf{Sets}$$

of an *object* x of a category C.

3. The characteristic relation

$$\chi_X(-_1,-_2)\colon X\times X \to \{\mathsf{t},\mathsf{f}\}$$

of X is a decategorification of the Hom profunctor

$$\mathrm{Hom}_C(-_1,-_2)\colon C^{\mathsf{op}}\times C \to \mathsf{Sets}$$

of a category C.

4. The characteristic embedding

$$\chi_{(-)}\colon X \hookrightarrow \mathcal{P}(X)$$

of X into $\mathcal{P}(X)$ is a decategorification of the Yoneda embedding

$$よ\colon C^{\mathsf{op}} \hookrightarrow \mathsf{PSh}(C)$$

of a category C into $\mathsf{PSh}(C)$.

5. There is also a direct parallel between unions and colimits:

   - An element of $\mathcal{P}(X)$ is a union of elements of X, viewed as one-point subsets $\{x\}\in\mathcal{P}(A)$.
   - An object of $\mathsf{PSh}(C)$ is a colimit of objects of C, viewed as representable presheaves $h_X\in\mathrm{Obj}(\mathsf{PSh}(C))$.

Proposition 2.4.1.1.3. Let X be a set.

1. *The Inclusion of Characteristic Relations Associated to a Function.* Let $f\colon A\to B$ be a function. We have an inclusion[31]

$$\chi_B\circ(f\times f)\subset\chi_A,\qquad \begin{tikzcd} A\times A \arrow[rr, "f\times f"] \arrow[dr, "\chi_A"'] & \arrow[d, phantom, "\supset"] & B\times B \arrow[dl, "\chi_B"] \\ & \{\mathsf{t},\mathsf{f}\}. & \end{tikzcd}$$

[31] This is the 0-categorical version of Definition 8.4.4.1.1.

2. *Interaction With Unions I.* We have

$$\chi_{U\cup V} = \max(\chi_U, \chi_V)$$

for each $X \in \text{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

3. *Interaction With Unions II.* We have

$$\chi_{U\cup V} = \chi_U + \chi_V - \chi_{U\cap V}$$

for each $X \in \text{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

4. *Interaction With Intersections I.* We have

$$\chi_{U\cap V} = \chi_U \chi_V$$

for each $X \in \text{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

5. *Interaction With Intersections II.* We have

$$\chi_{U\cap V} = \min(\chi_U, \chi_V)$$

for each $X \in \text{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

6. *Interaction With Differences.* We have

$$\chi_{U\setminus V} = \chi_U - \chi_{U\cap V}$$

for each $X \in \text{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

7. *Interaction With Complements.* We have

$$\chi_{U^{\mathsf{c}}} = 1 - \chi_U$$

for each $X \in \text{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

8. *Interaction With Symmetric Differences.* We have

$$\chi_{U \triangle V} = \chi_U + \chi_V - 2\chi_{U\cap V}$$

and thus, in particular, we have

$$\chi_{U \triangle V} \equiv \chi_U + \chi_V \mod 2$$

for each $X \in \text{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

9. *Interaction Between the Characteristic Embedding and Morphisms.* Let $f: X \to Y$ be a map of sets. The diagram

$$f_* \circ \chi_X = \chi_{X'} \circ f, \qquad \begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\chi_X"'] & X' \arrow[d, "\chi_{X'}"] \\ \mathcal{P}(X) \arrow[r, "f_*"'] & \mathcal{P}(X'). \end{tikzcd}$$

commutes.

Proof. *Item 1, The Inclusion of Characteristic Relations Associated to a Function*: The inclusion $\chi_B(f(a), f(b)) \subset \chi_A(a, b)$ is equivalent to the statement "if $a = b$, then $f(a) = f(b)$", which is true.
Item 2, Interaction With Unions I: This is a repetition of Item 8 of Proposition 2.3.7.1.2 and is proved there.
Item 3, Interaction With Unions II: This is a repetition of Item 9 of Proposition 2.3.7.1.2 and is proved there.
Item 4, Interaction With Intersections I: This is a repetition of Item 9 of Proposition 2.3.9.1.2 and is proved there.
Item 5, Interaction With Intersections II: This is a repetition of Item 10 of Proposition 2.3.9.1.2 and is proved there.
Item 6, Interaction With Differences: This is a repetition of Item 15 of Proposition 2.3.10.1.2 and is proved there.
Item 7, Interaction With Complements: This is a repetition of Item 4 of Proposition 2.3.11.1.2 and is proved there.
Item 8, Interaction With Symmetric Differences: This is a repetition of Item 14 of Proposition 2.3.12.1.2 and is proved there.
Item 9, Interaction Between the Characteristic Embedding and Morphisms: Indeed, we have

$$\begin{aligned} [f_* \circ \chi_X](x) &\stackrel{\text{def}}{=} f_*(\chi_X(x)) \\ &\stackrel{\text{def}}{=} f_*(\{x\}) \\ &= \{f(x)\} \\ &\stackrel{\text{def}}{=} \chi_{X'}(f(x)) \\ &\stackrel{\text{def}}{=} [\chi_{X'} \circ f](x), \end{aligned}$$

for each $x \in X$, showing the desired equality. □

2.4.2 The Yoneda Lemma for Sets

Let X be a set and let $U \subset X$ be a subset of X.

Proposition 2.4.2.1.1. We have

$$\chi_{\mathcal{P}(X)}(\chi_x, \chi_U) = \chi_U(x)$$

for each $x \in X$, giving an equality of functions

$$\chi_{\mathcal{P}(X)}\big(\chi_{(-)}, \chi_U\big) = \chi_U.$$

Proof. Clear. □

Corollary 2.4.2.1.2. The characteristic embedding is fully faithful, i.e., we have

$$\chi_{\mathcal{P}(X)}\big(\chi_x, \chi_y\big) = \chi_X(x, y)$$

for each $x, y \in X$.

Proof. This follows from Proposition 2.4.2.1.1. □

2.4.3 Powersets

Let X be a set.

Definition 2.4.3.1.1. The **powerset of** X is the set $\mathcal{P}(X)$ defined by

$$\mathcal{P}(X) \overset{\text{def}}{=} \{U \in P \mid U \subset X\},$$

where P is the set in the axiom of powerset, **??** of **??**.

Remark 2.4.3.1.2. The powerset of a set is a decategorification of the category of presheaves of a category: while[32]

- The powerset of a set X is equivalently (Items 1 and 2 of Proposition 2.4.3.1.6) the set

  $$\mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\})$$

  of functions from X to the set $\{\mathsf{t}, \mathsf{f}\}$ of classical truth values.

[32]This parallel is based on the following comparison:

- A category is enriched over the category

  $$\mathsf{Sets} \overset{\text{def}}{=} \mathsf{Cats}_0$$

  of sets (i.e. "0-categories"), with presheaves taking values on it.
- A set is enriched over the set

  $$\{\mathsf{t}, \mathsf{f}\} \overset{\text{def}}{=} \mathsf{Cats}_{-1}$$

- The category of presheaves on a category C is the category

$$\mathsf{Fun}(C^{\mathsf{op}}, \mathsf{Sets})$$

  of functors from C^{op} to the category Sets of sets.

Proposition 2.4.3.1.3. Let X be a set.

1. *Co/Completeness.* The (posetal) category (associated to) $(\mathcal{P}(X), \subset)$ is complete and cocomplete:
   (a) *Products.* The products in $\mathcal{P}(X)$ are given by intersection of subsets.
   (b) *Coproducts.* The coproducts in $\mathcal{P}(X)$ are given by union of subsets.
   (c) *Co/Equalisers.* Being a posetal category, $\mathcal{P}(X)$ only has at most one morphisms between any two objects, so co/equalisers are trivial.
2. *Cartesian Closedness.* The category $\mathcal{P}(X)$ is Cartesian closed with internal Hom

$$\mathbf{Hom}_{\mathcal{P}(X)}(-_1, -_2) : \mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \to \mathcal{P}(X)$$

   given by[33]

$$\mathbf{Hom}_{\mathcal{P}(X)}(U, V) \overset{\text{def}}{=} (X \setminus U) \cup V$$

   for each $U, V \in \mathrm{Obj}(\mathcal{P}(X))$.

Proof. *Item 1*, *Co/Completeness*: Clear.
Item 2, *Cartesian Closedness*: This follows from Item 2 of Proposition 2.3.9.1.2. □

Proposition 2.4.3.1.4. Let X be a set.

1. *Functoriality I.* The assignment $X \mapsto \mathcal{P}(X)$ defines a functor

$$\mathcal{P}_* : \mathsf{Sets} \to \mathsf{Sets},$$

   where

of classical truth values (i.e. "(-1)-categories"), with characteristic functions taking values on it.

[33]For intuition regarding the expression defining $\mathbf{Hom}_{\mathcal{P}(X)}(U, V)$, see Remark 2.3.9.1.3.

- *Action on Objects.* For each $A \in \text{Obj}(\mathsf{Sets})$, we have

$$\mathcal{P}_*(A) \stackrel{\text{def}}{=} \mathcal{P}(A).$$

- *Action on Morphisms.* For each $A, B \in \text{Obj}(\mathsf{Sets})$, the action on morphisms

$$\mathcal{P}_{*|A,B} \colon \mathsf{Sets}(A, B) \to \mathsf{Sets}(\mathcal{P}(A), \mathcal{P}(B))$$

of $\mathcal{P}_*$ at (A, B) is the map defined by by sending a map of sets $f \colon A \to B$ to the map

$$\mathcal{P}_*(f) \colon \mathcal{P}(A) \to \mathcal{P}(B)$$

defined by

$$\mathcal{P}_*(f) \stackrel{\text{def}}{=} f_*,$$

as in Definition 2.4.4.1.1.

2. *Functoriality II.* The assignment $X \mapsto \mathcal{P}(X)$ defines a functor

$$\mathcal{P}^{-1} \colon \mathsf{Sets}^{\mathsf{op}} \to \mathsf{Sets},$$

where

- *Action on Objects.* For each $A \in \text{Obj}(\mathsf{Sets})$, we have

$$\mathcal{P}^{-1}(A) \stackrel{\text{def}}{=} \mathcal{P}(A).$$

- *Action on Morphisms.* For each $A, B \in \text{Obj}(\mathsf{Sets})$, the action on morphisms

$$\mathcal{P}^{-1}_{A,B} \colon \mathsf{Sets}(A, B) \to \mathsf{Sets}(\mathcal{P}(B), \mathcal{P}(A))$$

of $\mathcal{P}^{-1}$ at (A, B) is the map defined by by sending a map of sets $f \colon A \to B$ to the map

$$\mathcal{P}^{-1}(f) \colon \mathcal{P}(B) \to \mathcal{P}(A)$$

defined by

$$\mathcal{P}^{-1}(f) \stackrel{\text{def}}{=} f^{-1},$$

as in Definition 2.4.5.1.1.

3. *Functoriality III.* The assignment $X \mapsto \mathcal{P}(X)$ defines a functor

$$\mathcal{P}_! \colon \mathsf{Sets} \to \mathsf{Sets},$$

where

- *Action on Objects.* For each $A \in \text{Obj}(\mathsf{Sets})$, we have

$$\mathcal{P}_!(A) \stackrel{\text{def}}{=} \mathcal{P}(A).$$

- *Action on Morphisms.* For each $A, B \in \text{Obj}(\mathsf{Sets})$, the action on morphisms

$$\mathcal{P}_{!|A,B} \colon \mathsf{Sets}(A, B) \to \mathsf{Sets}(\mathcal{P}(A), \mathcal{P}(B))$$

of $\mathcal{P}_!$ at (A, B) is the map defined by by sending a map of sets $f \colon A \to B$ to the map

$$\mathcal{P}_!(f) \colon \mathcal{P}(A) \to \mathcal{P}(B)$$

defined by

$$\mathcal{P}_!(f) \stackrel{\text{def}}{=} f_!,$$

as in Definition 2.4.6.1.1.

4. *Adjointness I.* We have an adjunction

$$\left(\mathcal{P}^{-1} \dashv \mathcal{P}^{-1,\mathsf{op}}\right) \colon \quad \begin{tikzcd} \mathsf{Sets}^{\mathsf{op}} \arrow[r, bend left, "\mathcal{P}^{-1}"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}, \arrow[l, bend left, "\mathcal{P}^{-1,\mathsf{op}}"] \end{tikzcd}$$

witnessed by a bijection

$$\underbrace{\mathsf{Sets}^{\mathsf{op}}(\mathcal{P}(A), B)}_{\stackrel{\text{def}}{=} \mathsf{Sets}(B, \mathcal{P}(A))} \cong \mathsf{Sets}(A, \mathcal{P}(B)),$$

natural in $A \in \text{Obj}(\mathsf{Sets})$ and $B \in \text{Obj}(\mathsf{Sets}^{\mathsf{op}})$.

5. *Adjointness II.* We have an adjunction

$$(\text{Gr} \dashv \mathcal{P}_*) \colon \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "\text{Gr}"] \arrow[r, phantom, "\perp"] & \text{Rel}, \arrow[l, bend left, "\mathcal{P}_*"] \end{tikzcd}$$

witnessed by a bijection of sets

$$\text{Rel}(\text{Gr}(A), B) \cong \mathsf{Sets}(A, \mathcal{P}(B))$$

natural in $A \in \text{Obj}(\mathsf{Sets})$ and $B \in \text{Obj}(\text{Rel})$, where Gr is the graph functor of Item 1 of Proposition 6.3.1.1.2 and $\mathcal{P}_*$ is the functor of Proposition 6.4.5.1.1.

Proof. *Item 1, Functoriality I*: This follows from Items 3 and 4 of Proposition 2.4.4.1.5.
Item 2, Functoriality II: This follows Items 3 and 4 of Proposition 2.4.5.1.4.
Item 3, Functoriality III: This follows Items 3 and 4 of Proposition 2.4.6.1.7.
Item 4, Adjointness I: We have

$$\begin{aligned}
\mathsf{Sets}^{\mathsf{op}}(\mathcal{P}(A), B) &\stackrel{\text{def}}{=} \mathsf{Sets}(B, \mathcal{P}(A)) \\
&\cong \mathsf{Sets}(B, \mathsf{Sets}(A, \{\mathsf{t}, \mathsf{f}\})) \\
&\qquad\qquad \text{(by Item 1 of Proposition 2.4.3.1.6)} \\
&\cong \mathsf{Sets}(A \times B, \{\mathsf{t}, \mathsf{f}\}) \\
&\qquad\qquad \text{(by Item 2 of Proposition 2.1.3.1.2)} \\
&\cong \mathsf{Sets}(A, \mathsf{Sets}(B, \{\mathsf{t}, \mathsf{f}\})) \\
&\qquad\qquad \text{(by Item 2 of Proposition 2.1.3.1.2)} \\
&\cong \mathsf{Sets}(A, \mathcal{P}(B)) \quad \text{(by Item 1 of Proposition 2.4.3.1.6)}
\end{aligned}$$

with all bijections natural in A and B (where we use Item 2 of Proposition 2.4.3.1.6 here).
Item 5, Adjointness II: We have

$$\begin{aligned}
\mathrm{Rel}(\mathrm{Gr}(A), B) &\cong \mathcal{P}(A \times B) \\
&\cong \mathsf{Sets}(A \times B, \{\mathsf{t}, \mathsf{f}\}) \quad \text{(by Item 1 of Proposition 2.4.3.1.6)} \\
&\cong \mathsf{Sets}(A, \mathsf{Sets}(B, \{\mathsf{t}, \mathsf{f}\})) \\
&\qquad\qquad \text{(by Item 2 of Proposition 2.1.3.1.2)} \\
&\cong \mathsf{Sets}(A, \mathcal{P}(B)) \qquad \text{(by Item 1 of Proposition 2.4.3.1.6)}
\end{aligned}$$

with all bijections natural in A (where we use Item 2 of Proposition 2.4.3.1.6 here). Explicitly, this isomorphism is given by sending a relation $R\colon \mathrm{Gr}(A) \nrightarrow B$ to the map $R^{\dagger}\colon A \to \mathcal{P}(B)$ sending a to the subset $R(a)$ of B, as in Remark 5.1.1.1.4.
Naturality in B is then the statement that given a relation $R\colon B \nrightarrow B'$, the diagram

$$\begin{tikzcd}
\mathrm{Rel}(\mathrm{Gr}(A), B) \arrow[r, "R \diamond -"] \arrow[d, dashed, "\wr"] & \mathrm{Rel}(\mathrm{Gr}(A), B') \arrow[d, dashed, "\wr"'] \\
\mathsf{Sets}(A, \mathcal{P}(B)) \arrow[r, "R_*"'] & \mathsf{Sets}(A, \mathcal{P}(B'))
\end{tikzcd}$$

commutes, which follows from Remark 6.4.1.1.2. □

Proposition 2.4.3.1.5. Let X be a set.

1. *Symmetric Strong Monoidality With Respect to Coproducts I.* The powerset functor $\mathcal{P}_*$ of Item 1 of Proposition 2.4.3.1.4 has a symmetric strong monoidal structure

$$\left(\mathcal{P}_*, \mathcal{P}_*^{\coprod}, \mathcal{P}_{*|\mathbb{1}}^{\coprod}\right) \colon (\mathsf{Sets}, \times, \mathrm{pt}) \longrightarrow (\mathsf{Sets}, \coprod, \emptyset)$$

being equipped with isomorphisms

$$\mathcal{P}_{*|X,Y}^{\coprod} \colon \mathcal{P}(X) \times \mathcal{P}(Y) \xrightarrow{\cong} \mathcal{P}(X \coprod Y),$$
$$\mathcal{P}_{*|\mathbb{1}}^{\coprod} \colon \mathrm{pt} \xrightarrow{\cong} \mathcal{P}(\emptyset),$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

2. *Symmetric Strong Monoidality With Respect to Coproducts II.* The powerset functor $\mathcal{P}^{-1}$ of Item 2 of Proposition 2.4.3.1.4 has a symmetric strong monoidal structure

$$\left(\mathcal{P}^{-1}, \mathcal{P}^{-1|\coprod}, \mathcal{P}_{\mathbb{1}}^{-1|\coprod}\right) \colon (\mathsf{Sets}^{\mathsf{op}}, \times^{\mathsf{op}}, \mathrm{pt}) \longrightarrow (\mathsf{Sets}, \coprod, \emptyset)$$

being equipped with isomorphisms

$$\mathcal{P}_{X,Y}^{-1|\coprod} \colon \mathcal{P}(X) \times \mathcal{P}(Y) \xrightarrow{\cong} \mathcal{P}(X \coprod Y),$$
$$\mathcal{P}_{\mathbb{1}}^{-1|\coprod} \colon \mathrm{pt} \xrightarrow{\cong} \mathcal{P}(\emptyset),$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

3. *Symmetric Strong Monoidality With Respect to Coproducts III.* The powerset functor $\mathcal{P}_!$ of Item 3 of Proposition 2.4.3.1.4 has a symmetric strong monoidal structure

$$\left(\mathcal{P}_!, \mathcal{P}_!^{\coprod}, \mathcal{P}_{!|\mathbb{1}}^{\coprod}\right) \colon (\mathsf{Sets}, \times, \mathrm{pt}) \longrightarrow (\mathsf{Sets}, \coprod, \emptyset)$$

being equipped with isomorphisms

$$\mathcal{P}_{!|X,Y}^{\coprod} \colon \mathcal{P}(X) \times \mathcal{P}(Y) \xrightarrow{\cong} \mathcal{P}(X \coprod Y),$$
$$\mathcal{P}_{!|\mathbb{1}}^{\coprod} \colon \mathrm{pt} \xrightarrow{\cong} \mathcal{P}(\emptyset),$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

4. *Symmetric Lax Monoidality With Respect to Products I.* The power-

set functor $\mathcal{P}_*$ of Item 1 of Proposition 2.4.3.1.4 has a symmetric lax monoidal structure

$$\left(\mathcal{P}_*, \mathcal{P}_*^{\otimes}, \mathcal{P}_{*|\mathbb{1}}^{\otimes}\right) \colon (\mathsf{Sets}, \times, \mathrm{pt}) \to (\mathsf{Sets}, \times, \mathrm{pt})$$

being equipped with morphisms

$$\mathcal{P}_{*|X,Y}^{\times} \colon \mathcal{P}(X) \times \mathcal{P}(Y) \to \mathcal{P}(X \times Y),$$
$$\mathcal{P}_{*|\mathbb{1}}^{\times} \colon \mathrm{pt} \to \mathcal{P}(\mathrm{pt}),$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$, where

- The map $\mathcal{P}_{*|X,Y}^{\times}$ is given by

$$\mathcal{P}_{*|X,Y}^{\times}(U, V) \stackrel{\text{def}}{=} U \times V$$

  for each $(U, V) \in \mathcal{P}(X) \times \mathcal{P}(Y)$,
- The map $\mathcal{P}_{*|\mathbb{1}}^{\times}$ is given by

$$\mathcal{P}_{*|\mathbb{1}}^{\times}(\star) = \mathrm{pt}.$$

5. *Symmetric Lax Monoidality With Respect to Products II.* The powerset functor $\mathcal{P}^{-1}$ of Item 2 of Proposition 2.4.3.1.4 has a symmetric lax monoidal structure

$$\left(\mathcal{P}^{-1}, \mathcal{P}^{-1|\otimes}, \mathcal{P}_{\mathbb{1}}^{-1|\otimes}\right) \colon (\mathsf{Sets}^{\mathsf{op}}, \times^{\mathsf{op}}, \mathrm{pt}) \to (\mathsf{Sets}, \times, \mathrm{pt})$$

being equipped with morphisms

$$\mathcal{P}_{X,Y}^{-1|\times} \colon \mathcal{P}(X) \times \mathcal{P}(Y) \to \mathcal{P}(X \times Y),$$
$$\mathcal{P}_{\mathbb{1}}^{\times} \colon \mathrm{pt} \to \mathcal{P}(\emptyset),$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$, defined as in Item 4.

6. *Symmetric Lax Monoidality With Respect to Products III.* The powerset functor $\mathcal{P}_!$ of Item 3 of Proposition 2.4.3.1.4 has a symmetric lax monoidal structure

$$\left(\mathcal{P}_!, \mathcal{P}_!^{\otimes}, \mathcal{P}_{!|\mathbb{1}}^{\otimes}\right) \colon (\mathsf{Sets}, \times, \mathrm{pt}) \to (\mathsf{Sets}, \times, \mathrm{pt})$$

being equipped with morphisms

$$\mathcal{P}_{!|X,Y}^{\times} \colon \mathcal{P}(X) \times \mathcal{P}(Y) \to \mathcal{P}(X \times Y),$$
$$\mathcal{P}_{!|\mathbb{1}}^{\times} \colon \mathrm{pt} \to \mathcal{P}(\emptyset),$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$, defined as in Item 4.

Proof. *Item 1, Symmetric Strong Monoidality With Respect to Coproducts I*: The isomorphism

$$\mathcal{P}^{\coprod}_{*|X,Y}\colon \mathcal{P}(X)\times\mathcal{P}(Y)\to\mathcal{P}(X\coprod Y)$$

is given by sending $(U,V)\in\mathcal{P}(X)\times\mathcal{P}(Y)$ to $U\coprod V$, with inverse given by sending a subset S of $X\coprod Y$ to the pair $(S_X,S_Y)\in\mathcal{P}(X)\times\mathcal{P}(Y)$ with

$$\begin{aligned}S_X &\stackrel{\text{def}}{=} \{x\in X \mid (0,x)\in S\}\\ S_Y &\stackrel{\text{def}}{=} \{y\in Y \mid (1,y)\in S\}.\end{aligned}$$

The isomorphism $\mathrm{pt}\cong\mathcal{P}(\emptyset)$ is given by $\star\mapsto\emptyset\in\mathcal{P}(\emptyset)$.
Naturality for the isomorphism $\mathcal{P}^{\coprod}_{*|X,Y}$ is the statement that, given maps of sets $f\colon X\to X'$ and $g\colon Y\to Y'$, the diagram

$$\begin{tikzcd}
\mathcal{P}(X)\times\mathcal{P}(Y) \arrow[r, "f_*\times g_*"] \arrow[d, dashed, "\wr"] & \mathcal{P}(X')\times\mathcal{P}(Y') \arrow[d, dashed, "\wr"'] \\
\mathcal{P}(X\coprod Y) \arrow[r, "(f\coprod g)_*"'] & \mathcal{P}(X'\coprod Y')
\end{tikzcd}$$

commutes, which is clear, as it acts on elements as

$$\begin{tikzcd}
(U,V) \arrow[r, mapsto] \arrow[d, mapsto] & (f_*(U),g_*(V)) \arrow[d, mapsto] \\
U\coprod V \arrow[r, mapsto] & (f\coprod g)_*(U\coprod V)=f_*(U)\coprod g_*(V),
\end{tikzcd}$$

where we are using Item 7 of Proposition 2.4.4.1.4.
Finally, monoidality, unity, and symmetry of $\mathcal{P}_*$ as a monoidal functor follow by checking the commutativity of the relevant diagrams on elements.
Item 2, Symmetric Strong Monoidality With Respect to Coproducts II: The proof is similar to Item 1, and is hence omitted.
Item 3, Symmetric Strong Monoidality With Respect to Coproducts III: The proof is similar to Item 1, and is hence omitted.
Item 4, Symmetric Lax Monoidality With Respect to Products I: Naturality for the morphism $\mathcal{P}^{\times}_{*|X,Y}$ is the statement that, given maps of sets $f\colon X\to X'$ and $g\colon Y\to Y'$, the diagram

$$\begin{tikzcd}
\mathcal{P}(X)\times\mathcal{P}(Y) \arrow[r, "f_*\times g_*"] \arrow[d, dashed, "\wr"] & \mathcal{P}(X')\times\mathcal{P}(Y') \arrow[d, dashed, "\wr"'] \\
\mathcal{P}(X\times Y) \arrow[r, "(f\times g)_*"'] & \mathcal{P}(X'\times Y')
\end{tikzcd}$$

commutes, which is clear, as it acts on elements as

$$\begin{tikzcd}
(U, V) \arrow[r, mapsto] \arrow[d, mapsto] & (f_*(U), g_*(V)) \arrow[d, mapsto] \\
U \times V \arrow[r, mapsto] & (f \times g)_*(U \times V) = f_*(U) \times g_*(V),
\end{tikzcd}$$

where we are using Item 8 of Proposition 2.4.4.1.4.

Finally, monoidality, unity, and symmetry of $\mathcal{P}_*$ as a monoidal functor follow by checking the commutativity of the relevant diagrams on elements.

Item 5, Symmetric Lax Monoidality With Respect to Products II: The proof is similar to Item 4, and is hence omitted.

Item 6, Symmetric Lax Monoidality With Respect to Products III: The proof is similar to Item 4, and is hence omitted. □

Proposition 2.4.3.1.6. Let X be a set.

1. *Powersets as Sets of Functions I.* The assignment $U \mapsto \chi_U$ defines a bijection
$$\chi_{(-)} : \mathcal{P}(X) \xrightarrow{\cong} \mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\}),$$
for each $X \in \mathrm{Obj}(\mathsf{Sets})$.

2. *Powersets as Sets of Functions II.* The bijection
$$\mathcal{P}(X) \cong \mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\})$$
of Item 1 is natural in $X \in \mathrm{Obj}(\mathsf{Sets})$, refining to a natural isomorphism of functors
$$\mathcal{P}^{-1} \cong \mathsf{Sets}(-, \{\mathsf{t}, \mathsf{f}\}).$$

3. *Powersets as Sets of Relations.* We have bijections
$$\begin{aligned} \mathcal{P}(X) &\cong \mathrm{Rel}(\mathrm{pt}, X), \\ \mathcal{P}(X) &\cong \mathrm{Rel}(X, \mathrm{pt}), \end{aligned}$$
natural in $X \in \mathrm{Obj}(\mathsf{Sets})$.

Proof. *Item 1, Powersets as Sets of Functions I*: Indeed, the inverse of $\chi_{(-)}$ is given by sending a function $f : X \to \{\mathsf{t}, \mathsf{f}\}$ to the subset U_f of $\mathcal{P}(X)$ defined by
$$U_f \stackrel{\text{def}}{=} \{x \in X \mid f(x) = \mathsf{true}\},$$

i.e. by $U_f = f^{-1}(\mathsf{true})$. That $\chi_{(-)}$ and $f \mapsto U_f$ are inverses is then straightforward to check.

Item 2, Powersets as Sets of Functions II: We need to check that, given a function $f\colon X \to Y$, the diagram

$$\begin{tikzcd}
\mathcal{P}(Y) \arrow[r, "f^{-1}"] \arrow[d, dashed, "\chi_{(-)}"', "\wr"] & \mathcal{P}(X) \arrow[d, dashed, "\chi_{(-)}", "\wr"'] \\
\mathsf{Sets}(Y, \{\mathsf{t}, \mathsf{f}\}) \arrow[r, "f^{*}"'] & \mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\})
\end{tikzcd}$$

commutes, i.e. that for each $V \in \mathcal{P}(Y)$, we have

$$\chi_V \circ f = \chi_{f^{-1}(V)}.$$

And indeed, we have

$$\begin{aligned}
[\chi_V \circ f](v) &\stackrel{\text{def}}{=} \chi_V(f(v)) \\
&= \begin{cases} \mathsf{true} & \text{if } f(v) \in V, \\ \mathsf{false} & \text{otherwise} \end{cases} \\
&= \begin{cases} \mathsf{true} & \text{if } v \in f^{-1}(V), \\ \mathsf{false} & \text{otherwise} \end{cases} \\
&\stackrel{\text{def}}{=} \chi_{f^{-1}(V)}(v)
\end{aligned}$$

for each $v \in V$.

Item 3, Powersets as Sets of Relations: Indeed, we have

$$\begin{aligned}
\mathrm{Rel}(\mathrm{pt}, X) &\stackrel{\text{def}}{=} \mathcal{P}(\mathrm{pt} \times X) \\
&\cong \mathcal{P}(X)
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Rel}(X, \mathrm{pt}) &\stackrel{\text{def}}{=} \mathcal{P}(X \times \mathrm{pt}) \\
&\cong \mathcal{P}(X),
\end{aligned}$$

where we have used Item 4 of Proposition 2.1.3.1.2. □

Remark 2.4.3.1.7. The bijection

$$\mathcal{P}(X) \cong \mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\})$$

of Item 1 of Proposition 2.4.3.1.6, which

- Takes a subset $U \hookrightarrow X$ of X and *straightens* it to a function $\chi_U : X \to \{\mathsf{true}, \mathsf{false}\}$;
- Takes a function $f : X \to \{\mathsf{true}, \mathsf{false}\}$ and *unstraightens* it to a subset $f^{-1}(\mathsf{true}) \hookrightarrow X$ of X;

may be viewed as the (-1)-categorical version of the un/straightening isomorphism for indexed and fibred sets

$$\underbrace{\mathsf{FibSets}(X)}_{\stackrel{\text{def}}{=}\mathsf{Sets}_{/X}} \cong \underbrace{\mathsf{ISets}(X)}_{\stackrel{\text{def}}{=}\mathsf{Fun}(X_{\mathsf{disc}},\mathsf{Sets})}$$

of **??**, where we view:

- Subsets $U \hookrightarrow X$ as analogous to X-fibred sets $\phi_X : A \to X$.
- Functions $f : X \to \{\mathsf{t}, \mathsf{f}\}$ as analogous to X-indexed sets $A : X_{\mathsf{disc}} \to \mathsf{Sets}$.

Proposition 2.4.3.1.8. Let X be a set.

1. *Universal Property.* The pair $\big(\mathcal{P}(X), \chi_{(-)}\big)$ consisting of

   - The powerset $\mathcal{P}(X)$ of X;
   - The characteristic embedding $\chi_{(-)} : X \hookrightarrow \mathcal{P}(X)$ of X into $\mathcal{P}(X)$;

   satisfies the following universal property:

   (★) Given another pair (Y, f) consisting of
   - A cocomplete poset $(Y, \preceq)$;
   - A function $f : X \to Y$;

   there exists a unique cocontinuous morphism of posets

   $$(\mathcal{P}(X), \subset) \xrightarrow{\exists!} (Y, \preceq)$$

   making the diagram

   $$\begin{tikzcd} & \mathcal{P}(X) \arrow[d, dashed, "\exists!"] \\ X \arrow[ur, "\chi_X"] \arrow[r, "f"'] & Y \end{tikzcd}$$

   commute.

2. *Adjointness.* We have an adjunction[34]

$$(\mathcal{P} \dashv 忘)\colon \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "\mathcal{P}"] \arrow[r, phantom, "\perp"] & \mathsf{Pos}^{\mathsf{cocomp.}}, \arrow[l, bend left, "忘"] \end{tikzcd}$$

witnessed by a bijection

$$\mathsf{Pos}^{\mathsf{cocomp.}}((\mathcal{P}(X), \subset), (Y, \preceq)) \cong \mathsf{Sets}(X, Y),$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets})$ and $(Y, \preceq) \in \mathrm{Obj}(\mathsf{Pos}^{\mathsf{cocomp.}})$, where the maps witnessing this bijection are given by

- The map
  $$\chi_X^*\colon \mathsf{Pos}^{\mathsf{cocomp.}}((\mathcal{P}(X), \subset), (Y, \preceq)) \to \mathsf{Sets}(X, Y)$$
  defined by
  $$\chi_X^*(f) \overset{\text{def}}{=} f \circ \chi_X,$$
  i.e. by sending a cocontinuous morphism of posets $f\colon \mathcal{P}(X) \to Y$ to the composition
  $$X \overset{\chi_X}{\hookrightarrow} \mathcal{P}(X) \xrightarrow{f} Y.$$
- The map
  $$\mathrm{Lan}_{\chi_X}\colon \mathsf{Sets}(X, Y) \to \mathsf{Pos}^{\mathsf{cocomp.}}((\mathcal{P}(X), \subset), (Y, \preceq))$$
  is given by sending a function $f\colon X \to Y$ to its left Kan extension along χ_X,
  $$\mathrm{Lan}_{\chi_X}(f)\colon \mathcal{P}(X) \to Y, \qquad \begin{tikzcd} & \mathcal{P}(X) \arrow[d, dashed, "\mathrm{Lan}_{\chi_X}(f)"] \\ X \arrow[ur, "\chi_X"] \arrow[r, "f"'] \arrow[r, phantom, bend left, "\Uparrow"] & Y. \end{tikzcd}$$
  Moreover, $\mathrm{Lan}_{\chi_X}(f)$ can be explicitly computed by
  $$\begin{aligned} \left[\mathrm{Lan}_{\chi_X}(f)\right](U) &\cong \int^{x \in X} \chi_{\mathcal{P}(X)}(\chi_x, U) \odot f(x) \\ &\cong \int^{x \in X} \chi_U(x) \odot f(x) \qquad \text{(by Proposition 2.4.2.1.1)} \\ &\cong \bigvee_{x \in X} (\chi_U(x) \odot f(x)) \end{aligned}$$
  for each $U \in \mathcal{P}(X)$, where:

[34] In this sense, $\mathcal{P}(A)$ is the free cocompletion of A. (Note that, despite its name, however,

– $\bigvee$ is the join in $(Y, \preceq)$.
– We have

$$\begin{aligned}\mathsf{true} \odot f(x) &\overset{\text{def}}{=} f(x),\\ \mathsf{false} \odot f(x) &\overset{\text{def}}{=} \varnothing_Y,\end{aligned}$$

where $\varnothing_Y$ is the minimal element of $(Y, \preceq)$.

Proof. *Item 1*, *Universal Property*: This is a rephrasing of Item 2.
Item 2, *Adjointness*: We claim we have adjunction $\mathcal{P} \dashv 忘$, witnessed by a bijection

$$\mathsf{Pos}^{\mathsf{cocomp.}}((\mathcal{P}(X), \subset), (Y, \preceq)) \cong \mathsf{Sets}(X, Y),$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets})$ and $(Y, \preceq) \in \mathrm{Obj}(\mathsf{Pos}^{\mathsf{cocomp.}})$.

- *Map I*. We define a map

  $$\Phi_{X,Y} \colon \mathsf{Pos}^{\mathsf{cocomp.}}((\mathcal{P}(X), \subset), (Y, \preceq)) \to \mathsf{Sets}(X, Y)$$

  as in the statement, by

  $$\Phi_{X,Y}(f) \overset{\text{def}}{=} f \circ \chi_X$$

  for each $f \in \mathsf{Pos}^{\mathsf{cocomp.}}((\mathcal{P}(X), \subset), (Y, \preceq))$.

- *Map II*. We define a map

  $$\Psi_{X,Y} \colon \mathsf{Sets}(X, Y) \to \mathsf{Pos}^{\mathsf{cocomp.}}((\mathcal{P}(X), \subset), (Y, \preceq))$$

  as in the statement, by

  $$\Psi_{X,Y}(f) \overset{\text{def}}{=} \mathrm{Lan}_{\chi_X}(f), \qquad \begin{tikzcd} & \mathcal{P}(X) \arrow[d, dashed, "\mathrm{Lan}_{\chi_X}(f)", "\shortmid"{marking}] \\ X \arrow[ur, "\chi_X", "\shortmid"{marking}] \arrow[r, "f"', "\shortmid"{marking}] \arrow[ur, phantom, shift right=3, "\Uparrow"] & Y, \end{tikzcd}$$

  for each $f \in \mathsf{Sets}(X, Y)$.

- *Invertibility I*. We claim that

  $$\Psi_{X,Y} \circ \Phi_{X,Y} = \mathrm{id}_{\mathsf{Pos}^{\mathsf{cocomp.}}((\mathcal{P}(X), \subset), (Y, \preceq))}.$$

this is not an idempotent operation, as we have $\mathcal{P}(\mathcal{P}(A)) \neq \mathcal{P}(A)$.)

Indeed, given a cocontinuous morphism of posets

$$\xi\colon (\mathcal{P}(X), \subset) \to (Y, \preceq),$$

we have

$$\begin{aligned}
\big[\Psi_{X,Y} \circ \Phi_{X,Y}\big](\xi) &\overset{\text{def}}{=} \Psi_{X,Y}\big(\Phi_{X,Y}(\xi)\big)\\
&\overset{\text{def}}{=} \Psi_{X,Y}(\xi \circ \chi_X)\\
&\overset{\text{def}}{=} \mathrm{Lan}_{\chi_X}(\xi \circ \chi_X)\\
&\cong \bigvee_{x\in X} \chi_{(-)}(x) \odot \xi(\chi_X(x))\\
&\overset{\text{clm}}{=} \xi,
\end{aligned}$$

where indeed

$$\begin{aligned}
\left[\bigvee_{x\in X} \chi_{(-)}(x) \odot \xi(\chi_X(x))\right](U) &\overset{\text{def}}{=} \bigvee_{x\in X} \chi_U(x) \odot \xi(\chi_X(x))\\
&= \left(\bigvee_{x\in U} \chi_U(x) \odot \xi(\chi_X(x))\right) \vee \left(\bigvee_{x\in X\setminus U} \chi_U(x) \odot \xi(\chi_X(x))\right)\\
&= \left(\bigvee_{x\in U} \xi(\chi_X(x))\right) \vee \left(\bigvee_{x\in X\setminus U} \varnothing_Y\right)\\
&= \bigvee_{x\in U} \xi(\chi_X(x))\\
&\overset{(\dagger)}{=} \xi\left(\bigvee_{x\in U} \chi_X(x)\right)\\
&= \xi(U)
\end{aligned}$$

for each $U \in \mathcal{P}(X)$, where we have used that ξ is cocontinuous for the equality $\overset{(\dagger)}{=}$.

- *Invertibility II.* We claim that

$$\Phi_{X,Y} \circ \Psi_{X,Y} = \mathrm{id}_{\mathsf{Sets}(X,Y)}.$$

Indeed, given a function $f\colon X \to Y$, we have

$$\begin{aligned}
\big[\Phi_{X,Y} \circ \Psi_{X,Y}\big](f) &\overset{\text{def}}{=} \Phi_{X,Y}\big(\Psi_{X,Y}(f)\big)\\
&\overset{\text{def}}{=} \Phi_{X,Y}\big(\mathrm{Lan}_{\chi_X}(f)\big)\\
&\overset{\text{def}}{=} \mathrm{Lan}_{\chi_X}(f) \circ \chi_X\\
&\overset{\text{clm}}{=} f,
\end{aligned}$$

where indeed

$$
\begin{aligned}
\left[\mathrm{Lan}_{\chi_X}(f)\circ\chi_X\right](x) &\overset{\text{def}}{=} \bigvee_{y\in X} \chi_{\{x\}}(y)\odot f(y)\\
&= \left(\chi_{\{x\}}(x)\odot f(x)\right)\vee\left(\bigvee_{y\in X\setminus\{x\}} \chi_{\{x\}}(y)\odot f(y)\right)\\
&= f(x)\vee\left(\bigvee_{y\in X\setminus\{x\}} \emptyset_Y\right)\\
&= f(x)\vee\emptyset_Y\\
&= f(x)
\end{aligned}
$$

for each $x\in X$.

- *Naturality for Φ, Part I.* We need to show that, given a function $f\colon X\longrightarrow X'$, the diagram

$$
\begin{tikzcd}
\mathsf{Pos}^{\mathsf{cocomp.}}((\mathcal{P}(X'),\subset),(Y,\preceq)) \arrow[r, "\Phi_{X',Y}"] \arrow[d, "\mathcal{P}_*(f)^*"'] & \mathsf{Sets}(X',Y) \arrow[d, "f^*"]\\
\mathsf{Pos}^{\mathsf{cocomp.}}((\mathcal{P}(X),\subset),(Y,\preceq)) \arrow[r, "\Phi_{X,Y}"'] & \mathsf{Sets}(X,Y)
\end{tikzcd}
$$

commutes. Indeed, given a cocontinuous morphism of posets

$$\xi\colon(\mathcal{P}(X'),\subset)\to(Y,\preceq),$$

we have

$$
\begin{aligned}
\left[\Phi_{X,Y}\circ\mathcal{P}_*(f)^*\right](\xi) &\overset{\text{def}}{=} \Phi_{X,Y}\left(\mathcal{P}_*(f)^*(\xi)\right)\\
&\overset{\text{def}}{=} \Phi_{X,Y}(\xi\circ f_*)\\
&\overset{\text{def}}{=} (\xi\circ f_*)\circ\chi_X\\
&= \xi\circ(f_*\circ\chi_X)\\
&\overset{(\dagger)}{=} \xi\circ(\chi_{X'}\circ f)\\
&= (\xi\circ\chi_{X'})\circ f\\
&\overset{\text{def}}{=} \Phi_{X',Y}(\xi)\circ f\\
&\overset{\text{def}}{=} f^*\left(\Phi_{X',Y}(\xi)\right)\\
&\overset{\text{def}}{=} \left[f^*\circ\Phi_{X',Y}\right](\xi),
\end{aligned}
$$

where we have used Item 9 of Proposition 2.4.1.1.3 for the equality $\overset{(\dagger)}{=}$ above.

- *Naturality for Φ, Part II.* We need to show that, given a cocontinuous morphism of posets

$$g\colon (Y, \preceq_Y) \to (Y', \preceq_{Y'}),$$

the diagram

$$\begin{tikzcd}
\mathsf{Pos}^{\mathsf{cocomp.}}((\mathcal{P}(X), \subset), (Y, \preceq)) \arrow[r, "\Phi_{X,Y}"] \arrow[d, "g_*"'] & \mathsf{Sets}(X, Y) \arrow[d, "g_*"] \\
\mathsf{Pos}^{\mathsf{cocomp.}}((\mathcal{P}(X), \subset), (Y', \preceq)) \arrow[r, "\Phi_{X,Y'}"'] & \mathsf{Sets}(X, Y')
\end{tikzcd}$$

commutes. Indeed, given a cocontinuous morphism of posets

$$\xi\colon (\mathcal{P}(X), \subset) \to (Y, \preceq),$$

we have

$$\begin{aligned}
\left[\Phi_{X,Y'} \circ g_*\right](\xi) &\overset{\text{def}}{=} \Phi_{X,Y'}(g_*(\xi)) \\
&\overset{\text{def}}{=} \Phi_{X,Y'}(g \circ \xi) \\
&\overset{\text{def}}{=} (g \circ \xi) \circ \chi_X \\
&= g \circ (\xi \circ \chi_X) \\
&\overset{\text{def}}{=} g \circ \left(\Phi_{X,Y}(\xi)\right) \\
&\overset{\text{def}}{=} g_*\left(\Phi_{X,Y}(\xi)\right) \\
&\overset{\text{def}}{=} \left[g_* \circ \Phi_{X,Y}\right](\xi).
\end{aligned}$$

- *Naturality for Ψ.* Since Φ is natural in each argument and Φ is a componentwise inverse to Ψ in each argument, it follows from Item 2 of Proposition 8.8.6.1.2 that Ψ is also natural in each argument.

This finishes the proof. □

2.4.4 Direct Images

Let A and B be sets and let $f: A \to B$ be a function.

Definition 2.4.4.1.1. The **direct image function associated to** f is the function

$$f_*: \mathcal{P}(A) \to \mathcal{P}(B)$$

defined by[35,36]

$$\begin{aligned} f_*(U) &\overset{\text{def}}{=} f(U) \\ &\overset{\text{def}}{=} \left\{ b \in B \;\middle|\; \begin{array}{l} \text{there exists some } a \in U \\ \text{such that } b = f(a) \end{array} \right\} \\ &= \{ f(a) \in B \mid a \in U \} \end{aligned}$$

for each $U \in \mathcal{P}(A)$.

Notation 2.4.4.1.2. Sometimes one finds the notation

$$\exists_f \colon \mathcal{P}(A) \to \mathcal{P}(B)$$

for f_*. This notation comes from the fact that the following statements are equivalent, where $b \in B$ and $U \in \mathcal{P}(A)$:

- We have $b \in \exists_f(U)$.
- There exists some $a \in U$ such that $f(a) = b$.

Remark 2.4.4.1.3. Identifying subsets of A with functions from A to $\{\mathsf{true}, \mathsf{false}\}$ via Items 1 and 2 of Proposition 2.4.3.1.6, we see that the direct image function associated to f is equivalently the function

$$f_* \colon \mathcal{P}(A) \to \mathcal{P}(B)$$

defined by

$$\begin{aligned} f_*(\chi_U) &\overset{\text{def}}{=} \operatorname{Lan}_f(\chi_U) \\ &= \operatorname{colim}\left(\left(f \overset{\rightarrow}{\times} \underline{(-_1)} \right) \overset{\text{pr}}{\twoheadrightarrow} A \xrightarrow{\chi_U} \{\mathsf{t}, \mathsf{f}\} \right) \\ &= \operatorname*{colim}_{\substack{a \in A \\ f(a) = -_1}} (\chi_U(a)) \\ &= \bigvee_{\substack{a \in A \\ f(a) = -_1}} (\chi_U(a)), \end{aligned}$$

[35] *Further Terminology:* The set $f(U)$ is called the **direct image of** U **by** f.

[36] We also have

$$f_*(U) = B \setminus f_!(A \setminus U);$$

see Item 9 of Proposition 2.4.4.1.4.

where we have used **??** for the second equality. In other words, we have

$$\begin{aligned}
[f_*(\chi_U)](b) &= \bigvee_{\substack{a\in A\\ f(a)=b}} (\chi_U(a))\\
&= \begin{cases} \mathsf{true} & \text{if there exists some } a \in A \text{ such} \\ & \text{that } f(a) = b \text{ and } a \in U, \\ \mathsf{false} & \text{otherwise} \end{cases}\\
&= \begin{cases} \mathsf{true} & \text{if there exists some } a \in U \\ & \text{such that } f(a) = b, \\ \mathsf{false} & \text{otherwise} \end{cases}
\end{aligned}$$

for each $b \in B$.

Proposition 2.4.4.1.4. Let $f : A \to B$ be a function.

1. *Functoriality.* The assignment $U \mapsto f_*(U)$ defines a functor

   $$f_* : (\mathcal{P}(A), \subset) \to (\mathcal{P}(B), \subset)$$

   where

   - *Action on Objects.* For each $U \in \mathcal{P}(A)$, we have

     $$[f_*](U) \stackrel{\text{def}}{=} f_*(U).$$

   - *Action on Morphisms.* For each $U, V \in \mathcal{P}(A)$:
     ($\star$) If $U \subset V$, then $f_*(U) \subset f_*(V)$.

2. *Triple Adjointness.* We have a triple adjunction

   $$(f_* \dashv f^{-1} \dashv f_!) \colon \quad \begin{tikzcd} \mathcal{P}(A) \arrow[r, bend left=45, "f_*"] \arrow[r, bend right=45, "f_!"'] & \mathcal{P}(B), \arrow[l, "f^{-1}" description] \end{tikzcd}$$

   witnessed by bijections of sets

   $$\begin{aligned}
   \mathrm{Hom}_{\mathcal{P}(B)}(f_*(U), V) &\cong \mathrm{Hom}_{\mathcal{P}(A)}\big(U, f^{-1}(V)\big),\\
   \mathrm{Hom}_{\mathcal{P}(A)}\big(f^{-1}(U), V\big) &\cong \mathrm{Hom}_{\mathcal{P}(A)}(U, f_!(V)),
   \end{aligned}$$

   natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$ and (respectively) $V \in \mathcal{P}(A)$ and $U \in \mathcal{P}(B)$, i.e. where:

(a) The following conditions are equivalent:
   i. We have $f_*(U) \subset V$.
   ii. We have $U \subset f^{-1}(V)$.

(b) The following conditions are equivalent:
   i. We have $f^{-1}(U) \subset V$.
   ii. We have $U \subset f_!(V)$.

3. *Preservation of Colimits.* We have an equality of sets

$$f_*\left(\bigcup_{i\in I} U_i\right) = \bigcup_{i\in I} f_*(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have equalities

$$\begin{aligned} f_*(U) \cup f_*(V) &= f_*(U \cup V), \\ f_*(\emptyset) &= \emptyset, \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

4. *Oplax Preservation of Limits.* We have an inclusion of sets

$$f_*\left(\bigcap_{i\in I} U_i\right) \subset \bigcap_{i\in I} f_*(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have inclusions

$$\begin{aligned} f_*(U \cap V) &\subset f_*(U) \cap f_*(V), \\ f_*(A) &\subset B, \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

5. *Symmetric Strict Monoidality With Respect to Unions.* The direct image function of Item 1 has a symmetric strict monoidal structure

$$\left(f_*, f_*^{\otimes}, f_{*|\mathbb{1}}^{\otimes}\right) \colon (\mathcal{P}(A), \cup, \emptyset) \to (\mathcal{P}(B), \cup, \emptyset),$$

being equipped with equalities

$$\begin{aligned} f_{*|U,V}^{\otimes} &\colon f_*(U) \cup f_*(V) \xrightarrow{=} f_*(U \cup V), \\ f_{*|\mathbb{1}}^{\otimes} &\colon \emptyset \xrightarrow{=} \emptyset, \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

6. *Symmetric Oplax Monoidality With Respect to Intersections.* The direct image function of Item 1 has a symmetric oplax monoidal structure

$$\left(f_*, f_*^{\otimes}, f_{*|\mathbb{1}}^{\otimes}\right) : (\mathcal{P}(A), \cap, A) \to (\mathcal{P}(B), \cap, B),$$

being equipped with inclusions

$$f_{*|U,V}^{\otimes} : f_*(U \cap V) \hookrightarrow f_*(U) \cap f_*(V),$$
$$f_{*|\mathbb{1}}^{\otimes} : f_*(A) \hookrightarrow B,$$

natural in $U, V \in \mathcal{P}(A)$.

7. *Interaction With Coproducts.* Let $f : A \to A'$ and $g : B \to B'$ be maps of sets. We have

$$(f \coprod g)_*(U \coprod V) = f_*(U) \coprod g_*(V)$$

for each $U \in \mathcal{P}(A)$ and each $V \in \mathcal{P}(B)$.

8. *Interaction With Products.* Let $f : A \to A'$ and $g : B \to B'$ be maps of sets. We have

$$(f \times g)_*(U \times V) = f_*(U) \times g_*(V)$$

for each $U \in \mathcal{P}(A)$ and each $V \in \mathcal{P}(B)$.

9. *Relation to Direct Images With Compact Support.* We have

$$f_*(U) = B \setminus f_!(A \setminus U)$$

for each $U \in \mathcal{P}(A)$.

Proof. *Item 1*, *Functoriality*: Clear.
Item 2, *Triple Adjointness*: This follows from Remark 2.4.4.1.3, Remark 2.4.5.1.2, Remark 2.4.6.1.3, and **??** of **??**.
Item 3, *Preservation of Colimits*: This follows from Item 2 and **??** of **??**.[37]
Item 4, *Oplax Preservation of Limits*: The inclusion $f_*(A) \subset B$ is clear. See [Pro24s] for the other inclusions.
Item 5, *Symmetric Strict Monoidality With Respect to Unions*: This follows from Item 3.
Item 6, *Symmetric Oplax Monoidality With Respect to Intersections*: This follows from Item 4.
Item 7, *Interaction With Coproducts*: Clear.

[37] See also [Pro24t].

Item 8, Interaction With Products: Clear.
Item 9, Relation to Direct Images With Compact Support: Applying Item 9 of Proposition 2.4.6.1.6 to $A \setminus U$, we have

$$\begin{aligned} f_!(A \setminus U) &= B \setminus f_*(A \setminus (A \setminus U)) \\ &= B \setminus f_*(U). \end{aligned}$$

Taking complements, we then obtain

$$\begin{aligned} f_*(U) &= B \setminus (B \setminus f_*(U)), \\ &= B \setminus f_!(A \setminus U), \end{aligned}$$

which finishes the proof. □

Proposition 2.4.4.1.5. Let $f : A \to B$ be a function.

1. *Functionality I.* The assignment $f \mapsto f_*$ defines a function
$$(-)_{*|A,B} : \mathsf{Sets}(A, B) \to \mathsf{Sets}(\mathcal{P}(A), \mathcal{P}(B)).$$

2. *Functionality II.* The assignment $f \mapsto f_*$ defines a function
$$(-)_{*|A,B} : \mathsf{Sets}(A, B) \to \mathsf{Pos}((\mathcal{P}(A), \subset), (\mathcal{P}(B), \subset)).$$

3. *Interaction With Identities.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have
$$(\mathrm{id}_A)_* = \mathrm{id}_{\mathcal{P}(A)}.$$

4. *Interaction With Composition.* For each pair of composable functions $f : A \to B$ and $g : B \to C$, we have
$$(g \circ f)_* = g_* \circ f_*, \qquad \begin{tikzcd} \mathcal{P}(A) \arrow[r, "f_*"] \arrow[dr, "(g\circ f)_*"'] & \mathcal{P}(B) \arrow[d, "g_*"] \\ & \mathcal{P}(C). \end{tikzcd}$$

Proof. *Item 1, Functionality I*: Clear.
Item 2, Functionality II: Clear.
Item 3, Interaction With Identities: This follows from Remark 2.4.4.1.3 and **??** of **??**.
Item 4, Interaction With Composition: This follows from Remark 2.4.4.1.3 and **??** of **??**. □

2.4.5 Inverse Images

Let A and B be sets and let $f\colon A \to B$ be a function.

Definition 2.4.5.1.1. The **inverse image function associated to** f is the function[38]

$$f^{-1}\colon \mathcal{P}(B) \to \mathcal{P}(A)$$

defined by[39]

$$f^{-1}(V) \overset{\text{def}}{=} \{a \in A \mid \text{we have } f(a) \in V\}$$

for each $V \in \mathcal{P}(B)$.

Remark 2.4.5.1.2. Identifying subsets of B with functions from B to $\{\mathsf{true}, \mathsf{false}\}$ via Items 1 and 2 of Proposition 2.4.3.1.6, we see that the inverse image function associated to f is equivalently the function

$$f^{*}\colon \mathcal{P}(B) \to \mathcal{P}(A)$$

defined by

$$f^{*}(\chi_V) \overset{\text{def}}{=} \chi_V \circ f$$

for each $\chi_V \in \mathcal{P}(B)$, where $\chi_V \circ f$ is the composition

$$A \xrightarrow{f} B \xrightarrow{\chi_V} \{\mathsf{true}, \mathsf{false}\}$$

in Sets.

Proposition 2.4.5.1.3. Let $f\colon A \to B$ be a function.

1. *Functoriality.* The assignment $V \mapsto f^{-1}(V)$ defines a functor

   $$f^{-1}\colon (\mathcal{P}(B), \subset) \to (\mathcal{P}(A), \subset)$$

   where

   - *Action on Objects.* For each $V \in \mathcal{P}(B)$, we have

     $$[f^{-1}](V) \overset{\text{def}}{=} f^{-1}(V).$$

   - *Action on Morphisms.* For each $U, V \in \mathcal{P}(B)$:

     ($\star$) If $U \subset V$, then $f^{-1}(U) \subset f^{-1}(V)$.

[38] *Further Notation:* Also written $f^{*}\colon \mathcal{P}(B) \to \mathcal{P}(A)$.

[39] *Further Terminology:* The set $f^{-1}(V)$ is called the **inverse image of** V **by** f.

2. *Triple Adjointness.* We have a triple adjunction

$$(f_* \dashv f^{-1} \dashv f_!)\colon \quad \begin{tikzcd} \mathcal{P}(A) \arrow[r, bend left=45, "f_*"] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, phantom, shift right=3, "\perp"] \arrow[r, bend right=45, "f_!"'] & \mathcal{P}(B) \arrow[l, "f^{-1}"'] \end{tikzcd},$$

witnessed by bijections of sets

$$\begin{aligned} \mathrm{Hom}_{\mathcal{P}(B)}(f_*(U), V) &\cong \mathrm{Hom}_{\mathcal{P}(A)}\big(U, f^{-1}(V)\big), \\ \mathrm{Hom}_{\mathcal{P}(A)}\big(f^{-1}(U), V\big) &\cong \mathrm{Hom}_{\mathcal{P}(A)}(U, f_!(V)), \end{aligned}$$

natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$ and (respectively) $V \in \mathcal{P}(A)$ and $U \in \mathcal{P}(B)$, i.e. where:

(a) The following conditions are equivalent:
   i. We have $f_*(U) \subset V$;
   ii. We have $U \subset f^{-1}(V)$;

(b) The following conditions are equivalent:
   i. We have $f^{-1}(U) \subset V$.
   ii. We have $U \subset f_!(V)$.

3. *Preservation of Colimits.* We have an equality of sets

$$f^{-1}\left(\bigcup_{i \in I} U_i\right) = \bigcup_{i \in I} f^{-1}(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(B)^{\times I}$. In particular, we have equalities

$$\begin{aligned} f^{-1}(U) \cup f^{-1}(V) &= f^{-1}(U \cup V), \\ f^{-1}(\emptyset) &= \emptyset, \end{aligned}$$

natural in $U, V \in \mathcal{P}(B)$.

4. *Preservation of Limits.* We have an equality of sets

$$f^{-1}\left(\bigcap_{i \in I} U_i\right) = \bigcap_{i \in I} f^{-1}(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(B)^{\times I}$. In particular, we have equalities

$$\begin{aligned} f^{-1}(U) \cap f^{-1}(V) &= f^{-1}(U \cap V), \\ f^{-1}(B) &= A, \end{aligned}$$

natural in $U, V \in \mathcal{P}(B)$.

5. *Symmetric Strict Monoidality With Respect to Unions.* The inverse image function of Item 1 has a symmetric strict monoidal structure

$$\left(f^{-1}, f^{-1,\otimes}, f^{-1,\otimes}_{\mathbb{1}}\right) \colon (\mathcal{P}(B), \cup, \emptyset) \longrightarrow (\mathcal{P}(A), \cup, \emptyset),$$

being equipped with equalities

$$f^{-1,\otimes}_{U,V} \colon f^{-1}(U) \cup f^{-1}(V) \xrightarrow{=} f^{-1}(U \cup V),$$
$$f^{-1,\otimes}_{\mathbb{1}} \colon \emptyset \xrightarrow{=} f^{-1}(\emptyset),$$

natural in $U, V \in \mathcal{P}(B)$.

6. *Symmetric Strict Monoidality With Respect to Intersections.* The inverse image function of Item 1 has a symmetric strict monoidal structure

$$\left(f^{-1}, f^{-1,\otimes}, f^{-1,\otimes}_{\mathbb{1}}\right) \colon (\mathcal{P}(B), \cap, B) \longrightarrow (\mathcal{P}(A), \cap, A),$$

being equipped with equalities

$$f^{-1,\otimes}_{U,V} \colon f^{-1}(U) \cap f^{-1}(V) \xrightarrow{=} f^{-1}(U \cap V),$$
$$f^{-1,\otimes}_{\mathbb{1}} \colon A \xrightarrow{=} f^{-1}(B),$$

natural in $U, V \in \mathcal{P}(B)$.

7. *Interaction With Coproducts.* Let $f \colon A \longrightarrow A'$ and $g \colon B \longrightarrow B'$ be maps of sets. We have

$$(f \textstyle\coprod g)^{-1}(U' \textstyle\coprod V') = f^{-1}(U') \textstyle\coprod g^{-1}(V')$$

for each $U' \in \mathcal{P}(A')$ and each $V' \in \mathcal{P}(B')$.

8. *Interaction With Products.* Let $f \colon A \longrightarrow A'$ and $g \colon B \longrightarrow B'$ be maps of sets. We have

$$(f \times g)^{-1}(U' \times V') = f^{-1}(U') \times g^{-1}(V')$$

for each $U' \in \mathcal{P}(A')$ and each $V' \in \mathcal{P}(B')$.

Proof. *Item 1, Functoriality*: Clear.
Item 2, Triple Adjointness: This follows from Remark 2.4.4.1.3, Remark 2.4.5.1.2, Remark 2.4.6.1.3, and **??** of **??**.
Item 3, Preservation of Colimits: This follows from Item 2 and **??** of **??**.[40]

[40] See also [Pro24ac].

Item 4, Preservation of Limits: This follows from Item 2 and **??** of **??**.[41]
Item 5, Symmetric Strict Monoidality With Respect to Unions: This follows from Item 3.
Item 6, Symmetric Strict Monoidality With Respect to Intersections: This follows from Item 4.
Item 7, Interaction With Coproducts: Clear.
Item 8, Interaction With Products: Clear. □

Proposition 2.4.5.1.4. Let $f\colon A \to B$ be a function.

1. *Functionality I.* The assignment $f \mapsto f^{-1}$ defines a function

$$(-)^{-1}_{A,B}\colon \mathsf{Sets}(A,B) \to \mathsf{Sets}(\mathcal{P}(B),\mathcal{P}(A)).$$

2. *Functionality II.* The assignment $f \mapsto f^{-1}$ defines a function

$$(-)^{-1}_{A,B}\colon \mathsf{Sets}(A,B) \to \mathsf{Pos}((\mathcal{P}(B),\subset),(\mathcal{P}(A),\subset)).$$

3. *Interaction With Identities.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$\mathrm{id}^{-1}_{A} = \mathrm{id}_{\mathcal{P}(A)}.$$

4. *Interaction With Composition.* For each pair of composable functions $f\colon A \to B$ and $g\colon B \to C$, we have

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}, \qquad \begin{tikzcd} \mathcal{P}(C) \arrow[r, "g^{-1}"] \arrow[dr, "(g\circ f)^{-1}"'] & \mathcal{P}(B) \arrow[d, "f^{-1}"] \\ & \mathcal{P}(A). \end{tikzcd}$$

Proof. *Item 1, Functionality I*: Clear.
Item 2, Functionality II: Clear.
Item 3, Interaction With Identities: This follows from Remark 2.4.5.1.2 and Item 5 of Proposition 8.1.6.1.2.
Item 4, Interaction With Composition: This follows from Remark 2.4.5.1.2 and Item 2 of Proposition 8.1.6.1.2. □

2.4.6 Direct Images With Compact Support

Let A and B be sets and let $f\colon A \to B$ be a function.

[41] See also [Pro24ab].

Definition 2.4.6.1.1. The **direct image with compact support function associated to** f is the function

$$f_!\colon \mathcal{P}(A) \to \mathcal{P}(B)$$

defined by[42,43]

$$\begin{aligned} f_!(U) &\stackrel{\text{def}}{=} \left\{ b \in B \;\middle|\; \begin{array}{l}\text{for each } a \in A\text{, if we have}\\ f(a) = b\text{, then } a \in U\end{array} \right\} \\ &= \left\{ b \in B \;\middle|\; \text{we have } f^{-1}(b) \subset U \right\} \end{aligned}$$

for each $U \in \mathcal{P}(A)$.

Notation 2.4.6.1.2. Sometimes one finds the notation

$$\forall_f\colon \mathcal{P}(A) \to \mathcal{P}(B)$$

for f_*. This notation comes from the fact that the following statements are equivalent, where $b \in B$ and $U \in \mathcal{P}(A)$:

- We have $b \in \forall_f(U)$.
- For each $a \in A$, if $b = f(a)$, then $a \in U$.

Remark 2.4.6.1.3. Identifying subsets of A with functions from A to $\{\mathsf{true}, \mathsf{false}\}$ via Items 1 and 2 of Proposition 2.4.3.1.6, we see that the direct image with compact support function associated to f is equivalently the function

$$f_!\colon \mathcal{P}(A) \to \mathcal{P}(B)$$

defined by

$$\begin{aligned} f_!(\chi_U) &\stackrel{\text{def}}{=} \mathrm{Ran}_f(\chi_U) \\ &= \lim\left(\left(\underline{(-_1)} \overset{\rightarrow}{\times} f\right) \overset{\mathrm{pr}}{\twoheadrightarrow} A \xrightarrow{\chi_U} \{\mathsf{true}, \mathsf{false}\}\right) \\ &= \lim_{\substack{a \in A \\ f(a) = -_1}} (\chi_U(a)) \\ &= \bigwedge_{\substack{a \in A \\ f(a) = -_1}} (\chi_U(a)). \end{aligned}$$

[42] *Further Terminology:* The set $f_!(U)$ is called the **direct image with compact support of** U **by** f.

[43] We also have

$$f_!(U) = B \setminus f_*(A \setminus U);$$

where we have used **??** for the second equality. In other words, we have

$$
\begin{aligned}
[f_!(\chi_U)](b) &= \bigwedge_{\substack{a\in A\\ f(a)=b}} (\chi_U(a))\\
&= \begin{cases} \mathsf{true} & \text{if, for each } a \in A \text{ such that}\\ & f(a) = b\text{, we have } a \in U,\\ \mathsf{false} & \text{otherwise}\end{cases}\\
&= \begin{cases} \mathsf{true} & \text{if } f^{-1}(b) \subset U\\ \mathsf{false} & \text{otherwise}\end{cases}
\end{aligned}
$$

for each $b \in B$.

Definition 2.4.6.1.4. Let U be a subset of A.[44,45]

1. The **image part of the direct image with compact support** $f_!(U)$ **of** U is the set $f_{!,\mathrm{im}}(U)$ defined by

$$
\begin{aligned}
f_{!,\mathrm{im}}(U) &\stackrel{\mathrm{def}}{=} f_!(U) \cap \mathrm{Im}(f)\\
&= \left\{ b \in B \,\middle|\, \begin{array}{l}\text{we have } f^{-1}(b) \subset\\ U \text{ and } f^{-1}(b) \neq \emptyset\end{array} \right\}.
\end{aligned}
$$

2. The **complement part of the direct image with compact support**

see Item 9 of Proposition 2.4.6.1.6.

[44]Note that we have

$$f_!(U) = f_{!,\mathrm{im}}(U) \cup f_{!,\mathrm{cp}}(U),$$

as

$$
\begin{aligned}
f_!(U) &= f_!(U) \cap B\\
&= f_!(U) \cap (\mathrm{Im}(f) \cup (B \setminus \mathrm{Im}(f)))\\
&= (f_!(U) \cap \mathrm{Im}(f)) \cup (f_!(U) \cap (B \setminus \mathrm{Im}(f)))\\
&\stackrel{\mathrm{def}}{=} f_{!,\mathrm{im}}(U) \cup f_{!,\mathrm{cp}}(U).
\end{aligned}
$$

[45]In terms of the meet computation of $f_!(U)$ of Remark 2.4.6.1.3, namely

$$f_!(\chi_U) = \bigwedge_{\substack{a\in A\\ f(a)=-_1}} (\chi_U(a)),$$

we see that $f_{!,\mathrm{im}}$ corresponds to meets indexed over nonempty sets, while $f_{!,\mathrm{cp}}$ corresponds to meets indexed over the empty set.

$f_!(U)$ **of** U is the set $f_{!,\mathrm{cp}}(U)$ defined by

$$\begin{aligned} f_{!,\mathrm{cp}}(U) &\overset{\mathrm{def}}{=} f_!(U) \cap (B \setminus \mathrm{Im}(f)) \\ &= B \setminus \mathrm{Im}(f) \\ &= \left\{ b \in B \;\middle|\; \begin{array}{l}\text{we have } f^{-1}(b) \subset \\ U \text{ and } f^{-1}(b) = \emptyset\end{array} \right\} \\ &= \left\{ b \in B \;\middle|\; f^{-1}(b) = \emptyset \right\}. \end{aligned}$$

Example 2.4.6.1.5. Here are some examples of direct images with compact support.

1. *The Multiplication by Two Map on the Natural Numbers.* Consider the function $f\colon \mathbb{N} \to \mathbb{N}$ given by

$$f(n) \overset{\mathrm{def}}{=} 2n$$

for each $n \in \mathbb{N}$. Since f is injective, we have

$$\begin{aligned} f_{!,\mathrm{im}}(U) &= f_*(U) \\ f_{!,\mathrm{cp}}(U) &= \{\text{odd natural numbers}\} \end{aligned}$$

for any $U \subset \mathbb{N}$.

2. *Parabolas.* Consider the function $f\colon \mathbb{R} \to \mathbb{R}$ given by

$$f(x) \overset{\mathrm{def}}{=} x^2$$

for each $x \in \mathbb{R}$. We have

$$f_{!,\mathrm{cp}}(U) = \mathbb{R}_{<0}$$

for any $U \subset \mathbb{R}$. Moreover, since $f^{-1}(x) = \left\{-\sqrt{x}, \sqrt{x}\right\}$, we have e.g.:

$$\begin{aligned} f_{!,\mathrm{im}}([0,1]) &= \{0\}, \\ f_{!,\mathrm{im}}([-1,1]) &= [0,1], \\ f_{!,\mathrm{im}}([1,2]) &= \emptyset, \\ f_{!,\mathrm{im}}([-2,-1] \cup [1,2]) &= [1,4]. \end{aligned}$$

3. *Circles.* Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x,y) \overset{\mathrm{def}}{=} x^2 + y^2$$

for each $(x,y) \in \mathbb{R}^2$. We have

$$f_{!,\mathrm{cp}}(U) = \mathbb{R}_{<0}$$

for any $U \subset \mathbb{R}^2$, and since

$$f^{-1}(r) = \begin{cases} \text{a circle of radius } r \text{ about the origin} & \text{if } r > 0, \\ \{(0,0)\} & \text{if } r = 0, \\ \emptyset & \text{if } r < 0, \end{cases}$$

we have e.g.:

$$\begin{aligned} f_{!,\mathrm{im}}([-1,1] \times [-1,1]) &= [0,1], \\ f_{!,\mathrm{im}}(([-1,1] \times [-1,1]) \setminus [-1,1] \times \{0\}) &= \emptyset. \end{aligned}$$

Proposition 2.4.6.1.6. Let $f \colon A \to B$ be a function.

1. *Functoriality.* The assignment $U \mapsto f_!(U)$ defines a functor

   $$f_! \colon (\mathcal{P}(A), \subset) \to (\mathcal{P}(B), \subset)$$

   where

   - *Action on Objects.* For each $U \in \mathcal{P}(A)$, we have

     $$[f_!](U) \overset{\text{def}}{=} f_!(U).$$

   - *Action on Morphisms.* For each $U, V \in \mathcal{P}(A)$:

     ($\star$) If $U \subset V$, then $f_!(U) \subset f_!(V)$.

2. *Triple Adjointness.* We have a triple adjunction

   $$(f_* \dashv f^{-1} \dashv f_!) \colon \quad \begin{tikzcd} \mathcal{P}(A) \arrow[r, bend left=50, "f_*"] \arrow[r, bend right=50, "f_!"'] & \mathcal{P}(B), \arrow[l, "f^{-1}"'] \end{tikzcd}$$

   witnessed by bijections of sets

   $$\begin{aligned} \mathrm{Hom}_{\mathcal{P}(B)}(f_*(U), V) &\cong \mathrm{Hom}_{\mathcal{P}(A)}(U, f^{-1}(V)), \\ \mathrm{Hom}_{\mathcal{P}(A)}(f^{-1}(U), V) &\cong \mathrm{Hom}_{\mathcal{P}(A)}(U, f_!(V)), \end{aligned}$$

   natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$ and (respectively) $V \in \mathcal{P}(A)$ and $U \in \mathcal{P}(B)$, i.e. where:

   (a) The following conditions are equivalent:

      i. We have $f_*(U) \subset V$.

ii. We have $U \subset f^{-1}(V)$.

(b) The following conditions are equivalent:

i. We have $f^{-1}(U) \subset V$.

ii. We have $U \subset f_!(V)$.

3. *Lax Preservation of Colimits.* We have an inclusion of sets

$$\bigcup_{i \in I} f_!(U_i) \subset f_!\left(\bigcup_{i \in I} U_i\right),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have inclusions

$$\begin{gathered} f_!(U) \cup f_!(V) \hookrightarrow f_!(U \cup V), \\ \emptyset \hookrightarrow f_!(\emptyset), \end{gathered}$$

natural in $U, V \in \mathcal{P}(A)$.

4. *Preservation of Limits.* We have an equality of sets

$$f_!\left(\bigcap_{i \in I} U_i\right) = \bigcap_{i \in I} f_!(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have equalities

$$\begin{gathered} f^{-1}(U \cap V) = f_!(U) \cap f^{-1}(V), \\ f_!(A) = B, \end{gathered}$$

natural in $U, V \in \mathcal{P}(A)$.

5. *Symmetric Lax Monoidality With Respect to Unions.* The direct image with compact support function of Item 1 has a symmetric lax monoidal structure

$$\left(f_!, f_!^{\otimes}, f_{!|\mathbb{1}}^{\otimes}\right) \colon (\mathcal{P}(A), \cup, \emptyset) \to (\mathcal{P}(B), \cup, \emptyset),$$

being equipped with inclusions

$$\begin{gathered} f_{!|U,V}^{\otimes} \colon f_!(U) \cup f_!(V) \hookrightarrow f_!(U \cup V), \\ f_{!|\mathbb{1}}^{\otimes} \colon \emptyset \hookrightarrow f_!(\emptyset), \end{gathered}$$

natural in $U, V \in \mathcal{P}(A)$.

6. *Symmetric Strict Monoidality With Respect to Intersections.* The direct image function of Item 1 has a symmetric strict monoidal structure

$$\left(f_!, f_!^{\otimes}, f_{!|\mathbb{1}}^{\otimes}\right) : (\mathcal{P}(A), \cap, A) \to (\mathcal{P}(B), \cap, B),$$

being equipped with equalities

$$f_{!|U,V}^{\otimes} : f_!(U \cap V) \xrightarrow{=} f_!(U) \cap f_!(V),$$
$$f_{!|\mathbb{1}}^{\otimes} : f_!(A) \xrightarrow{=} B,$$

natural in $U, V \in \mathcal{P}(A)$.

7. *Interaction With Coproducts.* Let $f : A \to A'$ and $g : B \to B'$ be maps of sets. We have

$$(f \textstyle\coprod g)_!(U \textstyle\coprod V) = f_!(U) \textstyle\coprod g_!(V)$$

for each $U \in \mathcal{P}(A)$ and each $V \in \mathcal{P}(B)$.

8. *Interaction With Products.* Let $f : A \to A'$ and $g : B \to B'$ be maps of sets. We have

$$(f \times g)_!(U \times V) = f_!(U) \times g_!(V)$$

for each $U \in \mathcal{P}(A)$ and each $V \in \mathcal{P}(B)$.

9. *Relation to Direct Images.* We have

$$f_!(U) = B \setminus f_*(A \setminus U)$$

for each $U \in \mathcal{P}(A)$.

10. *Interaction With Injections.* If f is injective, then we have

$$\begin{aligned} f_{!,\mathrm{im}}(U) &= f_*(U), \\ f_{!,\mathrm{cp}}(U) &= B \setminus \mathrm{Im}(f), \\ f_!(U) &= f_{!,\mathrm{im}}(U) \cup f_{!,\mathrm{cp}}(U) \\ &= f_*(U) \cup (B \setminus \mathrm{Im}(f)) \end{aligned}$$

for each $U \in \mathcal{P}(A)$.

11. *Interaction With Surjections.* If f is surjective, then we have

$$\begin{aligned} f_{!,\mathrm{im}}(U) &\subset f_*(U), \\ f_{!,\mathrm{cp}}(U) &= \emptyset, \\ f_!(U) &\subset f_*(U) \end{aligned}$$

for each $U \in \mathcal{P}(A)$.

Proof. *Item 1, Functoriality*: Clear.
Item 2, Triple Adjointness: This follows from Remark 2.4.4.1.3, Remark 2.4.5.1.2, Remark 2.4.6.1.3, and **??** of **??**.
Item 3, Lax Preservation of Colimits: Omitted.
Item 4, Preservation of Limits: This follows from Item 2 and **??** of **??**.
Item 5, Symmetric Lax Monoidality With Respect to Unions: This follows from Item 3.
Item 6, Symmetric Strict Monoidality With Respect to Intersections: This follows from Item 4.
Item 7, Interaction With Coproducts: Clear.
Item 8, Interaction With Products: Clear.
Item 9, Relation to Direct Images: We claim that $f_!(U) = B \setminus f_*(A \setminus U)$.

- *The First Implication.* We claim that

  $$f_!(U) \subset B \setminus f_*(A \setminus U).$$

  Let $b \in f_!(U)$. We need to show that $b \notin f_*(A \setminus U)$, i.e. that there is no $a \in A \setminus U$ such that $f(a) = b$.

  This is indeed the case, as otherwise we would have $a \in f^{-1}(b)$ and $a \notin U$, contradicting $f^{-1}(b) \subset U$ (which holds since $b \in f_!(U)$).

  Thus $b \in B \setminus f_*(A \setminus U)$.

- *The Second Implication.* We claim that

  $$B \setminus f_*(A \setminus U) \subset f_!(U).$$

  Let $b \in B \setminus f_*(A \setminus U)$. We need to show that $b \in f_!(U)$, i.e. that $f^{-1}(b) \subset U$.

  Since $b \notin f_*(A \setminus U)$, there exists no $a \in A \setminus U$ such that $b = f(a)$, and hence $f^{-1}(b) \subset U$.

  Thus $b \in f_!(U)$.

This finishes the proof of Item 9.
Item 10, Interaction With Injections: Clear.
Item 11, Interaction With Surjections: Clear. □

Proposition 2.4.6.1.7. Let $f : A \to B$ be a function.

1. *Functionality I.* The assignment $f \mapsto f_!$ defines a function

   $$(-)_{!|A,B} : \mathsf{Sets}(A, B) \to \mathsf{Sets}(\mathcal{P}(A), \mathcal{P}(B)).$$

2. *Functionality II.* The assignment $f \mapsto f_!$ defines a function

$$(-)_{!|A,B} \colon \mathsf{Sets}(A, B) \to \mathsf{Pos}((\mathcal{P}(A), \subset), (\mathcal{P}(B), \subset)).$$

3. *Interaction With Identities.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$(\mathrm{id}_A)_! = \mathrm{id}_{\mathcal{P}(A)}.$$

4. *Interaction With Composition.* For each pair of composable functions $f \colon A \to B$ and $g \colon B \to C$, we have

$$(g \circ f)_! = g_! \circ f_!, \qquad \begin{tikzcd} \mathcal{P}(A) \arrow[r, "f_!"] \arrow[dr, "(g\circ f)_!"'] & \mathcal{P}(B) \arrow[d, "g_!"] \\ & \mathcal{P}(C). \end{tikzcd}$$

Proof. *Item 1, Functionality I*: Clear.
Item 2, Functionality II: Clear.
Item 3, Interaction With Identities: This follows from Remark 2.4.6.1.3 and **??** of **??**.
Item 4, Interaction With Composition: This follows from Remark 2.4.6.1.3 and **??** of **??**. □

Appendices

2.A Other Chapters

Sets

1. Sets
2. Constructions With Sets
3. Pointed Sets
4. Tensor Products of Pointed Sets

Relations

5. Relations
6. Constructions With Relations
7. Equivalence Relations and Apartness Relations

Category Theory

8. Categories

Bicategories

9. Types of Morphisms in Bicategories

Chapter 3

Pointed Sets

This chapter contains some foundational material on pointed sets.

Contents

3.1 Pointed Sets **103**
- 3.1.1 Foundations 103
- 3.1.2 Morphisms of Pointed Sets 104
- 3.1.3 The Category of Pointed Sets 104
- 3.1.4 Elementary Properties of Pointed Sets 105

3.2 Limits of Pointed Sets **108**
- 3.2.1 The Terminal Pointed Set 108
- 3.2.2 Products of Families of Pointed Sets 108
- 3.2.3 Products 110
- 3.2.4 Pullbacks 112
- 3.2.5 Equalisers 116

3.3 Colimits of Pointed Sets **118**
- 3.3.1 The Initial Pointed Set 118
- 3.3.2 Coproducts of Families of Pointed Sets 119
- 3.3.3 Coproducts 121
- 3.3.4 Pushouts 124
- 3.3.5 Coequalisers 129

3.4 Constructions With Pointed Sets **131**
- 3.4.1 Free Pointed Sets 131

3.A Other Chapters **136**

3.1 Pointed Sets

3.1.1 Foundations

Definition 3.1.1.1.1. A **pointed set**[1] is equivalently:

- An $\mathbb{E}_0$-monoid in $(\mathsf{N}_\bullet(\mathsf{Sets}), \mathrm{pt})$.
- A pointed object in $(\mathsf{Sets}, \mathrm{pt})$.

Remark 3.1.1.1.2. In detail, a **pointed set** is a pair (X, x_0) consisting of:

- *The Underlying Set.* A set X, called the **underlying set of** (X, x_0).
- *The Basepoint.* A morphism

$$[x_0] \colon \mathrm{pt} \to X$$

  in Sets, determining an element $x_0 \in X$, called the **basepoint of** X.

Example 3.1.1.1.3. The 0-**sphere**[2] is the pointed set $\left(S^0, 0\right)$[3] consisting of:

- *The Underlying Set.* The set S^0 defined by

$$S^0 \overset{\mathrm{def}}{=} \{0, 1\}.$$

- *The Basepoint.* The element 0 of S^0.

Example 3.1.1.1.4. The **trivial pointed set** is the pointed set $(\mathrm{pt}, \star)$ consisting of:

- *The Underlying Set.* The punctual set $\mathrm{pt} \overset{\mathrm{def}}{=} \{\star\}$.
- *The Basepoint.* The element $\star$ of pt.

Example 3.1.1.1.5. The **underlying pointed set** of a semimodule (M, α_M) is the pointed set $(M, 0_M)$.

Example 3.1.1.1.6. The **underlying pointed set** of a module (M, α_M) is the pointed set $(M, 0_M)$.

[1] *Further Terminology:* In the context of monoids with zero as models for $\mathbb{F}_1$-algebras, pointed sets are viewed as $\mathbb{F}_1$-**modules**.

[2] *Further Terminology:* In the context of monoids with zero as models for $\mathbb{F}_1$-algebras, the 0-sphere is viewed as the **underlying pointed set of the field with one element**.

[3] *Further Notation:* In the context of monoids with zero as models for $\mathbb{F}_1$-algebras, S^0 is also denoted $(\mathbb{F}_1, 0)$.

3.1.2 Morphisms of Pointed Sets

Definition 3.1.2.1.1. A **morphism of pointed sets**[4,5] is equivalently:

- A morphism of $\mathbb{E}_0$-monoids in $(\mathrm{N}_\bullet(\mathsf{Sets}), \mathrm{pt})$.
- A morphism of pointed objects in $(\mathsf{Sets}, \mathrm{pt})$.

Remark 3.1.2.1.2. In detail, a **morphism of pointed sets** $f\colon (X, x_0) \to (Y, y_0)$ is a morphism of sets $f\colon X \to Y$ such that the diagram

$$\begin{tikzcd} & \mathrm{pt} \arrow[dl, "{[x_0]}"'] \arrow[dr, "{[y_0]}"] & \\ X \arrow[rr, "f"'] & & Y \end{tikzcd}$$

commutes, i.e. such that

$$f(x_0) = y_0.$$

3.1.3 The Category of Pointed Sets

Definition 3.1.3.1.1. The **category of pointed sets** is the category Sets_* defined equivalently as

- The homotopy category of the ∞-category $\mathsf{Mon}_{\mathbb{E}_0}(\mathrm{N}_\bullet(\mathsf{Sets}), \mathrm{pt})$ of **??**;
- The category Sets_* of **??**.

Remark 3.1.3.1.2. In detail, the **category of pointed sets** is the category Sets_* where

- *Objects.* The objects of Sets_* are pointed sets;
- *Morphisms.* The morphisms of Sets_* are morphisms of pointed sets;
- *Identities.* For each $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, the unit map

  $$\mathbb{1}^{\mathsf{Sets}_*}_{(X,x_0)}\colon \mathrm{pt} \to \mathsf{Sets}_*((X, x_0), (X, x_0))$$

  of Sets_* at (X, x_0) is defined by[6]

  $$\mathrm{id}^{\mathsf{Sets}_*}_{(X,x_0)} \overset{\mathrm{def}}{=} \mathrm{id}_X;$$

[4] *Further Terminology:* Also called a **pointed function**.

[5] *Further Terminology:* In the context of monoids with zero as models for $\mathbb{F}_1$-algebras, morphisms of pointed sets are also called **morphism of $\mathbb{F}_1$-modules**.

[6] Note that id_X is indeed a morphism of pointed sets, as we have $\mathrm{id}_X(x_0) = x_0$.

- *Composition.* For each $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, the composition map

$$\circ^{\mathsf{Sets}_*}_{(X,x_0),(Y,y_0),(Z,z_0)} \colon \mathsf{Sets}_*((Y, y_0), (Z, z_0)) \times \mathsf{Sets}_*((X, x_0), (Y, y_0)) \to \mathsf{Sets}_*((X, x_0), (Z, z_0))$$

  of Sets_* at $((X, x_0), (Y, y_0), (Z, z_0))$ is defined by[7]

$$g \circ^{\mathsf{Sets}_*}_{(X,x_0),(Y,y_0),(Z,z_0)} f \stackrel{\text{def}}{=} g \circ f.$$

3.1.4 Elementary Properties of Pointed Sets

Proposition 3.1.4.1.1. Let (X, x_0) be a pointed set.

1. *Completeness.* The category Sets_* of pointed sets and morphisms between them is complete, having in particular:
   (a) Products, described as in Definition 3.2.3.1.1;
   (b) Pullbacks, described as in Definition 3.2.4.1.1;
   (c) Equalisers, described as in Definition 3.2.5.1.1.
2. *Cocompleteness.* The category Sets_* of pointed sets and morphisms between them is cocomplete, having in particular:
   (a) Coproducts, described as in Definition 3.3.3.1.1;
   (b) Pushouts, described as in Definition 3.3.4.1.1;
   (c) Coequalisers, described as in Definition 3.3.5.1.1.
3. *Failure To Be Cartesian Closed.* The category Sets_* is not Cartesian closed.[8]

[7] Note that the composition of two morphisms of pointed sets is indeed a morphism of pointed sets, as we have

$$\begin{aligned} g(f(x_0)) &= g(y_0) \\ &= z_0, \end{aligned}$$

or

$$\begin{tikzcd} & \mathrm{pt} \arrow[dl, "{[x_0]}"'] \arrow[d, "{[y_0]}"] \arrow[dr, "{[z_0]}"] & \\ X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z \end{tikzcd}$$

in terms of diagrams.

[8] The category Sets_* does admit monoidal closed structures however; see Tensor Products of Pointed Sets.

4. *Morphisms From the Monoidal Unit.* We have a bijection of sets[9]

$$\mathsf{Sets}_*(S^0, X) \cong X,$$

natural in $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, internalising also to an isomorphism of pointed sets

$$\mathbf{Sets}_*(S^0, X) \cong (X, x_0),$$

again natural in $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

5. *Relation to Partial Functions.* We have an equivalence of categories[10]

$$\mathsf{Sets}_* \overset{\text{eq.}}{\cong} \mathsf{Sets}^{\text{part.}}$$

between the category of pointed sets and pointed functions between them and the category of sets and partial functions between them, where:

(a) *From Pointed Sets to Sets With Partial Functions.* The equivalence

$$\xi \colon \mathsf{Sets}_* \xrightarrow{\cong} \mathsf{Sets}^{\text{part.}}$$

sends:

i. A pointed set (X, x_0) to X.

ii. A pointed function

$$f \colon (X, x_0) \to (Y, y_0)$$

to the partial function

$$\xi_f \colon X \to Y$$

defined on $f^{-1}(Y \setminus y_0)$ and given by

$$\xi_f(x) \overset{\text{def}}{=} f(x)$$

for each $x \in f^{-1}(Y \setminus y_0)$.

[9] In other words, the forgetful functor

$$忘 \colon \mathsf{Sets}_* \to \mathsf{Sets}$$

defined on objects by sending a pointed set to its underlying set is corepresentable by S^0.

[10] *Warning:* This is not an isomorphism of categories, only an equivalence.

(b) *From Sets With Partial Functions to Pointed Sets.* The equivalence

$$\xi^{-1}\colon \mathsf{Sets}^{\mathrm{part.}} \xrightarrow{\cong} \mathsf{Sets}_*$$

sends:

i. A set X is to the pointed set $(X, \star)$ with $\star$ an element that is not in X.

ii. A partial function

$$f\colon X \to Y$$

defined on $U \subset X$ to the pointed function

$$\xi_f^{-1}\colon (X, x_0) \to (Y, y_0)$$

defined by

$$\xi_f(x) \overset{\mathrm{def}}{=} \begin{cases} f(x) & \text{if } x \in U, \\ y_0 & \text{otherwise.} \end{cases}$$

for each $x \in X$.

Proof. *Item 1*, *Completeness*: This follows from (the proofs) of Definitions 3.2.3.1.1, 3.2.4.1.1 and 3.2.5.1.1 and **??**.
Item 2, *Cocompleteness*: This follows from (the proofs) of Definitions 3.3.3.1.1, 3.3.4.1.1 and 3.3.5.1.1 and **??**.
Item 3, *Failure To Be Cartesian Closed*: See [MSE 2855868].
Item 4, *Morphisms From the Monoidal Unit*: Since a morphism from S^0 to a pointed set (X, x_0) sends $0 \in S^0$ to x_0 and then can send $1 \in S^0$ to any element of X, we obtain a bijection between pointed maps $S^0 \to X$ and the elements of X.
The isomorphism then

$$\mathsf{Sets}_*(S^0, X) \cong (X, x_0)$$

follows by noting that $\Delta_{x_0}\colon S^0 \to X$, the basepoint of $\mathsf{Sets}_*(S^0, X)$, corresponds to the pointed map $S^0 \to X$ picking the element x_0 of X, and thus we see that the bijection between pointed maps $S^0 \to X$ and elements of X is compatible with basepoints, lifting to an isomorphism of pointed sets.
Item 5, *Relation to Partial Functions*: See [MSE 884460]. □

END TEXTDBEND

3.2 Limits of Pointed Sets

3.2.1 The Terminal Pointed Set

Definition 3.2.1.1.1. The **terminal pointed set** is the pair $\left((\mathrm{pt}, \star), \{!_X\}_{(X,x_0)\in\mathrm{Obj}(\mathsf{Sets}_*)}\right)$ consisting of:

- *The Limit.* The pointed set $(\mathrm{pt}, \star)$.
- *The Cone.* The collection of morphisms of pointed sets
$$\{!_X \colon (X, x_0) \to (\mathrm{pt}, \star)\}_{(X,x_0)\in\mathrm{Obj}(\mathsf{Sets})}$$
defined by
$$!_X(x) \overset{\mathrm{def}}{=} \star$$
for each $x \in X$ and each $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets})$.

Proof. We claim that $(\mathrm{pt}, \star)$ is the terminal object of Sets_*. Indeed, suppose we have a diagram of the form
$$(X, x_0) \qquad (\mathrm{pt}, \star)$$
in Sets_*. Then there exists a unique morphism of pointed sets
$$\phi \colon (X, x_0) \to (\mathrm{pt}, \star)$$
making the diagram
$$(X, x_0) \xrightarrow[\exists!]{\phi} (\mathrm{pt}, \star)$$
commute, namely $!_X$. □

3.2.2 Products of Families of Pointed Sets

Let $\left\{(X_i, x_0^i)\right\}_{i\in I}$ be a family of pointed sets.

Definition 3.2.2.1.1. The **product of** $\left\{(X_i, x_0^i)\right\}_{i\in I}$ is the pair $\left((\prod_{i\in I} X_i, (x_0^i)_{i\in I}), \{\mathrm{pr}_i\}_{i\in I}\right)$ consisting of:

- *The Limit.* The pointed set $\left(\prod_{i\in I} X_i, (x_0^i)_{i\in I}\right)$.
- *The Cone.* The collection
$$\left\{\mathrm{pr}_i \colon \left(\prod_{i\in I} X_i, (x_0^i)_{i\in I}\right) \to (X_i, x_0^i)\right\}_{i\in I}$$
of maps given by
$$\mathrm{pr}_i\left((x_j)_{j\in I}\right) \overset{\mathrm{def}}{=} x_i$$
for each $(x_j)_{j\in I} \in \prod_{i\in I} X_i$ and each $i \in I$.

Proof. We claim that $\left(\prod_{i\in I} X_i, \left(x_0^i\right)_{i\in I}\right)$ is the categorical product of $\left\{\left(X_i, x_0^i\right)\right\}_{i\in I}$ in Sets_*. Indeed, suppose we have, for each $i \in I$, a diagram of the form

$$\begin{tikzcd} (P,*) \arrow[dr, "p_i"] & \\ \left(\prod_{i\in I} X_i, \left(x_0^i\right)_{i\in I}\right) \arrow[r, "\mathrm{pr}_i"'] & \left(X_i, x_0^i\right) \end{tikzcd}$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi\colon (P,*) \to \left(\prod_{i\in I} X_i, \left(x_0^i\right)_{i\in I}\right)$$

making the diagram

$$\begin{tikzcd} (P,*) \arrow[d, dashed, "\phi"', "\exists!"] \arrow[dr, "p_i"] & \\ \left(\prod_{i\in I} X_i, \left(x_0^i\right)_{i\in I}\right) \arrow[r, "\mathrm{pr}_i"'] & \left(X_i, x_0^i\right) \end{tikzcd}$$

commute, being uniquely determined by the condition $\mathrm{pr}_i \circ \phi = p_i$ for each $i \in I$ via

$$\phi(x) = (p_i(x))_{i\in I}$$

for each $x \in P$. Note that this is indeed a morphism of pointed sets, as we have

$$\begin{aligned} \phi(*) &= (p_i(*))_{i\in I} \\ &= \left(x_0^i\right)_{i\in I}, \end{aligned}$$

where we have used that p_i is a morphism of pointed sets for each $i \in I$. □

Proposition 3.2.2.1.2. Let $\left\{\left(X_i, x_0^i\right)\right\}_{i\in I}$ be a family of pointed sets.

1. *Functoriality.* The assignment $\left\{\left(X_i, x_0^i\right)\right\}_{i\in I} \mapsto \left(\prod_{i\in I} X_i, \left(x_0^i\right)_{i\in I}\right)$ defines a functor

$$\prod_{i\in I}\colon \mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}_*) \to \mathsf{Sets}_*.$$

Proof. *Item 1*, *Functoriality*: This follows from **??** of **??**. □

3.2.3 Products

Let (X, x_0) and (Y, y_0) be pointed sets.

Definition 3.2.3.1.1. The **product of** (X, x_0) **and** (Y, y_0) is the pair consisting of:

- *The Limit.* The pointed set $(X \times Y, (x_0, y_0))$.
- *The Cone.* The morphisms of pointed sets

  $$\begin{aligned}\mathrm{pr}_1 &: (X \times Y, (x_0, y_0)) \to (X, x_0),\\ \mathrm{pr}_2 &: (X \times Y, (x_0, y_0)) \to (Y, y_0)\end{aligned}$$

  defined by

  $$\begin{aligned}\mathrm{pr}_1(x, y) &\stackrel{\text{def}}{=} x,\\ \mathrm{pr}_2(x, y) &\stackrel{\text{def}}{=} y\end{aligned}$$

  for each $(x, y) \in X \times Y$.

Proof. We claim that $(X \times Y, (x_0, y_0))$ is the categorical product of (X, x_0) and (Y, y_0) in Sets_*. Indeed, suppose we have a diagram of the form

$$\begin{tikzcd} & (P, *) \arrow[dl, bend right, "p_1"'] \arrow[dr, bend left, "p_2"] & \\ (X, x_0) & (X \times Y, (x_0, y_0)) \arrow[l, "\mathrm{pr}_1"] \arrow[r, "\mathrm{pr}_2"'] & (Y, y_0) \end{tikzcd}$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi \colon (P, *) \to (X \times Y, (x_0, y_0))$$

making the diagram

$$\begin{tikzcd} & (P, *) \arrow[dl, bend right, "p_1"'] \arrow[dr, bend left, "p_2"] \arrow[d, dashed, "\phi"', "\exists!"] & \\ (X, x_0) & (X \times Y, (x_0, y_0)) \arrow[l, "\mathrm{pr}_1"] \arrow[r, "\mathrm{pr}_2"'] & (Y, y_0) \end{tikzcd}$$

commute, being uniquely determined by the conditions

$$\begin{aligned}\mathrm{pr}_1 \circ \phi &= p_1,\\ \mathrm{pr}_2 \circ \phi &= p_2\end{aligned}$$

via

$$\phi(x) = (p_1(x), p_2(x))$$

for each $x \in P$. Note that this is indeed a morphism of pointed sets, as we have

$$\begin{aligned}\phi(*) &= (p_1(*), p_2(*))\\ &= (x_0, y_0),\end{aligned}$$

where we have used that p_1 and p_2 are morphisms of pointed sets. □

Proposition 3.2.3.1.2. Let (X, x_0), (Y, y_0), and (Z, z_0) be pointed sets.

1. *Functoriality.* The assignments

   $$(X, x_0), (Y, y_0), ((X, x_0), (Y, y_0)) \mapsto (X \times Y, (x_0, y_0))$$

   define functors

   $$\begin{aligned}X \times -\colon \mathsf{Sets}_* &\to \mathsf{Sets}_*,\\ - \times Y\colon \mathsf{Sets}_* &\to \mathsf{Sets}_*,\\ -_1 \times -_2\colon \mathsf{Sets}_* \times \mathsf{Sets}_* &\to \mathsf{Sets}_*,\end{aligned}$$

   defined in the same way as the functors of Item 1 of Proposition 2.1.3.1.2.

2. *Associativity.* We have an isomorphism of pointed sets

   $$((X \times Y) \times Z, ((x_0, y_0), z_0)) \cong (X \times (Y \times Z), (x_0, (y_0, z_0)))$$

   natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

3. *Unitality.* We have isomorphisms of pointed sets

   $$\begin{aligned}(\mathrm{pt}, \star) \times (X, x_0) &\cong (X, x_0),\\ (X, x_0) \times (\mathrm{pt}, \star) &\cong (X, x_0),\end{aligned}$$

   natural in $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

4. *Commutativity.* We have an isomorphism of pointed sets

   $$(X \times Y, (x_0, y_0)) \cong (Y \times X, (y_0, x_0)),$$

   natural in $(X, x_0), (Y, y_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

5. *Symmetric Monoidality.* The triple $(\mathsf{Sets}_*, \times, (\mathrm{pt}, \star))$ is a symmetric monoidal category.

Proof. *Item 1*, *Functoriality*: This is a special case of functoriality of limits, **??** of **??**.
Item 2, *Associativity*: This follows from Item 3 of Proposition 2.1.3.1.2.
Item 3, *Unitality*: This follows from Item 4 of Proposition 2.1.3.1.2.
Item 4, *Commutativity*: This follows from Item 5 of Proposition 2.1.3.1.2.
Item 5, *Symmetric Monoidality*: This follows from Item 12 of Proposition 2.1.3.1.2. □

3.2.4 Pullbacks

Let (X, x_0), (Y, y_0), and (Z, z_0) be pointed sets and let $f\colon (X, x_0) \to (Z, z_0)$ and $g\colon (Y, y_0) \to (Z, z_0)$ be morphisms of pointed sets.

Definition 3.2.4.1.1. The **pullback of** (X, x_0) **and** (Y, y_0) **over** (Z, z_0) **along** (f, g) is the pair consisting of:

- *The Limit.* The pointed set $(X \times_Z Y, (x_0, y_0))$.
- *The Cone.* The morphisms of pointed sets
$$\begin{aligned} \mathrm{pr}_1 &\colon (X \times_Z Y, (x_0, y_0)) \to (X, x_0),\\ \mathrm{pr}_2 &\colon (X \times_Z Y, (x_0, y_0)) \to (Y, y_0) \end{aligned}$$
defined by
$$\begin{aligned} \mathrm{pr}_1(x, y) &\overset{\text{def}}{=} x,\\ \mathrm{pr}_2(x, y) &\overset{\text{def}}{=} y \end{aligned}$$
for each $(x, y) \in X \times_Z Y$.

Proof. We claim that $X \times_Z Y$ is the categorical pullback of (X, x_0) and (Y, y_0) over (Z, z_0) with respect to (f, g) in Sets_*. First we need to check that the relevant pullback diagram commutes, i.e. that we have

$$f \circ \mathrm{pr}_1 = g \circ \mathrm{pr}_2, \qquad \begin{tikzcd} (X \times_Z Y, (x_0, y_0)) \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] & (Y, y_0) \arrow[d, "g"] \\ (X, x_0) \arrow[r, "f"'] & (Z, z_0). \end{tikzcd}$$

Indeed, given $(x, y) \in X \times_Z Y$, we have

$$\begin{aligned} [f \circ \mathrm{pr}_1](x, y) &= f(\mathrm{pr}_1(x, y))\\ &= f(x)\\ &= g(y)\\ &= g(\mathrm{pr}_2(x, y))\\ &= [g \circ \mathrm{pr}_2](x, y), \end{aligned}$$

where $f(x) = g(y)$ since $(x, y) \in X \times_Z Y$. Next, we prove that $X \times_Z Y$ satisfies the universal property of the pullback. Suppose we have a diagram of the form

$$\begin{tikzcd}
(P,*) \arrow[drr, bend left, "p_2"] \arrow[ddr, bend right, "p_1"'] & & \\
& (X \times_Z Y, (x_0, y_0)) \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] \arrow[dr, phantom, "\lrcorner", very near start] & (Y, y_0) \arrow[d, "g"] \\
& (X, x_0) \arrow[r, "f"'] & (Z, z_0)
\end{tikzcd}$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi \colon (P, *) \to (X \times_Z Y, (x_0, y_0))$$

making the diagram

$$\begin{tikzcd}
(P,*) \arrow[drr, bend left, "p_2"] \arrow[ddr, bend right, "p_1"'] \arrow[dr, dashed, "\phi", "\exists!"'] & & \\
& (X \times_Z Y, (x_0, y_0)) \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] \arrow[dr, phantom, "\lrcorner", very near start] & (Y, y_0) \arrow[d, "g"] \\
& (X, x_0) \arrow[r, "f"'] & (Z, z_0)
\end{tikzcd}$$

commute, being uniquely determined by the conditions

$$\mathrm{pr}_1 \circ \phi = p_1,$$
$$\mathrm{pr}_2 \circ \phi = p_2$$

via

$$\phi(x) = (p_1(x), p_2(x))$$

for each $x \in P$, where we note that $(p_1(x), p_2(x)) \in X \times Y$ indeed lies in $X \times_Z Y$ by the condition

$$f \circ p_1 = g \circ p_2,$$

which gives

$$f(p_1(x)) = g(p_2(x))$$

for each $x \in P$, so that $(p_1(x), p_2(x)) \in X \times_Z Y$. Lastly, we note that ϕ is indeed a morphism of pointed sets, as we have

$$\begin{aligned} \phi(*) &= (p_1(*), p_2(*)) \\ &= (x_0, y_0), \end{aligned}$$

where we have used that p_1 and p_2 are morphisms of pointed sets. □

Proposition 3.2.4.1.2. Let (X, x_0), (Y, y_0), (Z, z_0), and (A, a_0) be pointed sets.

1. *Functoriality.* The assignment $(X, Y, Z, f, g) \mapsto X \times_{f,Z,g} Y$ defines a functor

   $$-_1 \times_{-_3} -_1 \colon \mathsf{Fun}(\mathcal{P}, \mathsf{Sets}_*) \to \mathsf{Sets}_*,$$

   where $\mathcal{P}$ is the category that looks like this:

   $$\begin{tikzcd} & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet. \end{tikzcd}$$

   In particular, the action on morphisms of $-_1 \times_{-_3} -_1$ is given by sending a morphism

   $$\begin{tikzcd} X \times_Z Y \arrow[rr] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & & Y \arrow[dd, "g" near start] \arrow[dr, "\psi"] & \\ & X' \times_{Z'} Y' \arrow[rr] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & & Y' \arrow[dd, "g'"] \\ X \arrow[rr, "f" near start] \arrow[dr, "\phi"'] & & Z \arrow[dr, "\chi"] & \\ & X' \arrow[rr, "f'"'] & & Z' \end{tikzcd}$$

   in $\mathsf{Fun}(\mathcal{P}, \mathsf{Sets}_*)$ to the morphism of pointed sets

   $$\xi \colon (X \times_Z Y, (x_0, y_0)) \xrightarrow{\exists!} \left(X' \times_{Z'} Y', \left(x'_0, y'_0\right)\right)$$

   given by

   $$\xi(x, y) \overset{\text{def}}{=} (\phi(x), \psi(y))$$

for each $(x, y) \in X \times_Z Y$, which is the unique morphism of pointed sets making the diagram

$$\begin{tikzcd}
X \times_Z Y \arrow[rr] \arrow[dr, dashed] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & & Y \arrow[dr, "\psi"] \arrow[dd, "g", near start] & \\
& X' \times_{Z'} Y' \arrow[rr] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & & Y' \arrow[dd, "g'"] \\
X \arrow[rr, "f", near start] \arrow[dr, "\phi"'] & & Z \arrow[dr, "\chi"] & \\
& X' \arrow[rr, "f'"'] & & Z'
\end{tikzcd}$$

commute.

2. *Associativity.* Given a diagram

$$\begin{tikzcd}
X \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \arrow[dr, "h"'] & & Z \arrow[dl, "k"] \\
& W & & V &
\end{tikzcd}$$

in Sets_*, we have isomorphisms of pointed sets

$$(X \times_W Y) \times_V Z \cong (X \times_W Y) \times_Y (Y \times_V Z) \cong X \times_W (Y \times_V Z),$$

where these pullbacks are built as in the diagrams

$$\begin{tikzcd}[column sep=tiny, row sep=small]
& & (X \times_W Y) \times_V Z \arrow[dl] \arrow[ddrr] \arrow[d, phantom, "\lrcorner" rotate=-45] & & \\
& X \times_W Y \arrow[dl] \arrow[dr] \arrow[d, phantom, "\lrcorner" rotate=-45] & & & \\
X \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \arrow[dr, "h"'] & & Z, \arrow[dl, "k"] \\
& W & & V &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=tiny, row sep=small]
& & (X \times_W Y) \times_Y (Y \times_V Z) \arrow[dl] \arrow[dr] \arrow[d, phantom, "\lrcorner" rotate=-45] & & \\
& X \times_W Y \arrow[dl] \arrow[dr] \arrow[d, phantom, "\lrcorner" rotate=-45] & & Y \times_V Z \arrow[dl] \arrow[dr] \arrow[d, phantom, "\lrcorner" rotate=-45] & \\
X \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \arrow[dr, "h"'] & & Z, \arrow[dl, "k"] \\
& W & & V &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=tiny, row sep=small]
& & X \times_W (Y \times_V Z) \arrow[ddll] \arrow[dr] \arrow[d, phantom, "\lrcorner" rotate=-45] & & \\
& & & Y \times_V Z \arrow[dl] \arrow[dr] \arrow[d, phantom, "\lrcorner" rotate=-45] & \\
X \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \arrow[dr, "h"'] & & Z. \arrow[dl, "k"] \\
& W & & V &
\end{tikzcd}$$

3. *Unitality.* We have isomorphisms of pointed sets

$$\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\
X \arrow[r, equal] & X
\end{tikzcd}
\qquad
\begin{aligned}
X \times_X A &\cong A, \\
A \times_X X &\cong A,
\end{aligned}
\qquad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, equal] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, equal] \\
X \arrow[r, "f"'] & X.
\end{tikzcd}$$

4. *Commutativity.* We have an isomorphism of pointed sets

$$
\begin{tikzcd}
A \times_X B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "g"] \\
A \arrow[r, "f"'] & X,
\end{tikzcd}
\qquad A \times_X B \cong B \times_X A \qquad
\begin{tikzcd}
B \times_X A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\
B \arrow[r, "g"'] & X.
\end{tikzcd}
$$

5. *Interaction With Products.* We have an isomorphism of pointed sets

$$
X \times_{\mathrm{pt}} Y \cong X \times Y, \qquad
\begin{tikzcd}
X \times Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "!_Y"] \\
X \arrow[r, "!_X"'] & \mathrm{pt}.
\end{tikzcd}
$$

6. *Symmetric Monoidality.* The triple $(\mathsf{Sets}_*, \times_X, X)$ is a symmetric monoidal category.

Proof. *Item 1, Functoriality*: This is a special case of functoriality of co/limits, **??** of **??**, with the explicit expression for ξ following from the commutativity of the cube pullback diagram.
Item 2, Associativity: This follows from Item 2 of Proposition 3.2.4.1.2.
Item 3, Unitality: This follows from Item 3 of Proposition 2.1.4.1.3.
Item 4, Commutativity: This follows from Item 4 of Proposition 2.1.4.1.3.
Item 5, Interaction With Products: This follows from Item 6 of Proposition 2.1.4.1.3.
Item 6, Symmetric Monoidality: This follows from Item 7 of Proposition 2.1.4.1.3. □

3.2.5 Equalisers

Let $f, g \colon (X, x_0) \rightrightarrows (Y, y_0)$ be morphisms of pointed sets.

Definition 3.2.5.1.1. The **equaliser of** (f, g) is the pair consisting of:

- *The Limit.* The pointed set $\left(\mathrm{Eq}(f, g), x_0\right)$.
- *The Cone.* The morphism of pointed sets

  $$\mathrm{eq}(f, g) \colon \left(\mathrm{Eq}(f, g), x_0\right) \hookrightarrow (X, x_0)$$

  given by the canonical inclusion $\mathrm{eq}(f, g) \hookrightarrow \mathrm{Eq}(f, g) \hookrightarrow X$.

Proof. We claim that $\left(\mathrm{Eq}(f, g), x_0\right)$ is the categorical equaliser of f and g in

Sets_*. First we need to check that the relevant equaliser diagram commutes, i.e. that we have

$$f \circ \mathrm{eq}(f,g) = g \circ \mathrm{eq}(f,g),$$

which indeed holds by the definition of the set $\mathrm{Eq}(f,g)$. Next, we prove that $\mathrm{Eq}(f,g)$ satisfies the universal property of the equaliser. Suppose we have a diagram of the form

$$\begin{tikzcd}
(\mathrm{Eq}(f,g), x_0) \arrow[r, hook, "\mathrm{eq}(f,g)"] & (X, x_0) \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & (Y, y_0) \\
(E, *) \arrow[ur, "e"'] & &
\end{tikzcd}$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi\colon (E,*) \to (\mathrm{Eq}(f,g), x_0)$$

making the diagram

$$\begin{tikzcd}
(\mathrm{Eq}(f,g), x_0) \arrow[r, hook, "\mathrm{eq}(f,g)"] & (X, x_0) \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & (Y, y_0) \\
(E, *) \arrow[u, dashed, "\phi", "\exists!"'] \arrow[ur, "e"'] & &
\end{tikzcd}$$

commute, being uniquely determined by the condition

$$\mathrm{eq}(f,g) \circ \phi = e$$

via

$$\phi(x) = e(x)$$

for each $x \in E$, where we note that $e(x) \in A$ indeed lies in $\mathrm{Eq}(f,g)$ by the condition

$$f \circ e = g \circ e,$$

which gives

$$f(e(x)) = g(e(x))$$

for each $x \in E$, so that $e(x) \in \mathrm{Eq}(f,g)$. Lastly, we note that ϕ is indeed a morphism of pointed sets, as we have

$$\begin{aligned} \phi(*) &= e(*) \\ &= x_0, \end{aligned}$$

where we have used that e is a morphism of pointed sets. □

Proposition 3.2.5.1.2. Let (X, x_0) and (Y, y_0) be pointed sets and let $f, g, h\colon (X, x_0) \to (Y, y_0)$ be morphisms of pointed sets.

1. *Associativity.* We have isomorphisms of pointed sets

$$\underbrace{\mathrm{Eq}(f \circ \mathrm{eq}(g,h), g \circ \mathrm{eq}(g,h))}_{=\mathrm{Eq}(f\circ\mathrm{eq}(g,h),h\circ\mathrm{eq}(g,h))} \cong \mathrm{Eq}(f,g,h) \cong \underbrace{\mathrm{Eq}(f \circ \mathrm{eq}(f,g), h \circ \mathrm{eq}(f,g))}_{=\mathrm{Eq}(g\circ\mathrm{eq}(f,g),h\circ\mathrm{eq}(f,g))},$$

   where $\mathrm{Eq}(f, g, h)$ is the limit of the diagram

$$\begin{tikzcd} (X, x_0) \arrow[r, shift left=2, "f"] \arrow[r, "g" description] \arrow[r, shift right=2, "h"'] & (Y, y_0) \end{tikzcd}$$

   in Sets_*, being explicitly given by

$$\mathrm{Eq}(f, g, h) \cong \{a \in A \mid f(a) = g(a) = h(a)\}.$$

2. *Unitality.* We have an isomorphism of pointed sets

$$\mathrm{Eq}(f, f) \cong X.$$

3. *Commutativity.* We have an isomorphism of pointed sets

$$\mathrm{Eq}(f, g) \cong \mathrm{Eq}(g, f).$$

Proof. *Item 1*, *Associativity*: This follows from Item 1 of Proposition 2.1.5.1.2.
Item 2, *Unitality*: This follows from Item 4 of Proposition 2.1.5.1.2.
Item 3, *Commutativity*: This follows from Item 5 of Proposition 2.1.5.1.2. □

3.3 Colimits of Pointed Sets

3.3.1 The Initial Pointed Set

Definition 3.3.1.1.1. The **initial pointed set** is the pair $\left((\mathrm{pt}, \star), \{\iota_X\}_{(X,x_0)\in\mathrm{Obj}(\mathsf{Sets}_*)}\right)$ consisting of:

- *The Limit.* The pointed set $(\mathrm{pt}, \star)$.
- *The Cone.* The collection of morphisms of pointed sets

$$\{\iota_X : (\mathrm{pt}, \star) \to (X, x_0)\}_{(X,x_0)\in\mathrm{Obj}(\mathsf{Sets})}$$

  defined by

$$\iota_X(\star) \stackrel{\mathrm{def}}{=} x_0.$$

Proof. We claim that $(\mathrm{pt}, \star)$ is the initial object of Sets_*. Indeed, suppose we have a diagram of the form

$$(\mathrm{pt}, \star) \qquad (X, x_0)$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi\colon (\mathrm{pt}, \star) \to (X, x_0)$$

making the diagram

$$(\mathrm{pt}, \star) \overset{\phi}{\underset{\exists!}{\dashrightarrow}} (X, x_0)$$

commute, namely ι_X. □

3.3.2 Coproducts of Families of Pointed Sets

Let $\{(X_i, x_0^i)\}_{i \in I}$ be a family of pointed sets.

Definition 3.3.2.1.1. The **coproduct of the family** $\{(X_i, x_0^i)\}_{i \in I}$, also called their **wedge sum**, is the pair consisting of:

- *The Colimit.* The pointed set $(\bigvee_{i \in I} X_i, p_0)$ consisting of:
  - *The Underlying Set.* The set $\bigvee_{i \in I} X_i$ defined by

    $$\bigvee_{i \in I} X_i \overset{\text{def}}{=} \left(\coprod_{i \in I} X_i\right)/\sim,$$

    where $\sim$ is the equivalence relation on $\coprod_{i \in I} X_i$ given by declaring

    $$\left(i, x_0^i\right) \sim \left(j, x_0^j\right)$$

    for each $i, j \in I$.
  - *The Basepoint.* The element p_0 of $\bigvee_{i \in I} X_i$ defined by

    $$\begin{aligned} p_0 &\overset{\text{def}}{=} \left[\left(i, x_0^i\right)\right] \\ &= \left[\left(j, x_0^j\right)\right] \end{aligned}$$

    for any $i, j \in I$.
- *The Cocone.* The collection

  $$\left\{\mathrm{inj}_i\colon \left(X_i, x_0^i\right) \to \left(\bigvee_{i \in I} X_i, p_0\right)\right\}_{i \in I}$$

of morphism of pointed sets given by

$$\mathrm{inj}_i(x) \stackrel{\mathrm{def}}{=} (i, x)$$

for each $x \in X_i$ and each $i \in I$.

Proof. We claim that $(\bigvee_{i \in I} X_i, p_0)$ is the categorical coproduct of $\{(X_i, x_0^i)\}_{i \in I}$ in Sets_*. Indeed, suppose we have, for each $i \in I$, a diagram of the form

$$\begin{tikzcd} & (C, *) \\ (X_i, x_0^i) \arrow[ur, "\iota_i"] \arrow[r, "\mathrm{inj}_i"'] & \left(\bigvee_{i \in I} X_i, p_0\right) \end{tikzcd}$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi \colon \left(\bigvee_{i \in I} X_i, p_0\right) \to (C, *)$$

making the diagram

$$\begin{tikzcd} & (C, *) \\ (X_i, x_0^i) \arrow[ur, "\iota_i"] \arrow[r, "\mathrm{inj}_i"'] & \left(\bigvee_{i \in I} X_i, p_0\right) \arrow[u, dashed, "\phi", "\exists!"'] \end{tikzcd}$$

commute, being uniquely determined by the condition $\phi \circ \mathrm{inj}_i = \iota_i$ for each $i \in I$ via

$$\phi([(i, x)]) = \iota_i(x)$$

for each $[(i, x)] \in \bigvee_{i \in I} X_i$, where we note that ϕ is indeed a morphism of pointed sets, as we have

$$\begin{aligned} \phi(p_0) &= \iota_i([(i, x_0^i)]) \\ &= *, \end{aligned}$$

as ι_i is a morphism of pointed sets. □

Proposition 3.3.2.1.2. Let $\{(X_i, x_0^i)\}_{i \in I}$ be a family of pointed sets.

1. *Functoriality.* The assignment $\{(X_i, x_0^i)\}_{i \in I} \mapsto (\bigvee_{i \in I} X_i, p_0)$ defines a functor

$$\bigvee_{i \in I} \colon \mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}_*) \to \mathsf{Sets}_*.$$

Proof. Item 1, *Functoriality*: This follows from **??** of **??**. □

3.3.3 Coproducts

Let (X, x_0) and (Y, y_0) be pointed sets.

Definition 3.3.3.1.1. The **coproduct of** (X, x_0) **and** (Y, y_0), also called their **wedge sum**, is the pair consisting of:

- *The Colimit.* The pointed set $(X \vee Y, p_0)$ consisting of:
  - *The Underlying Set.* The set $X \vee Y$ defined by

$$\begin{aligned}(X \vee Y, p_0) &\overset{\text{def}}{=} (X, x_0) \textstyle\coprod (Y, y_0)\\ &\cong \left(X \textstyle\coprod_{\text{pt}} Y, p_0\right)\\ &\cong (X \textstyle\coprod Y/\sim, p_0),\end{aligned}$$

$$\begin{tikzcd} X \vee Y \arrow[dr, phantom, "\ulcorner", very near start] & Y \arrow[l] \\ X \arrow[u] & \mathrm{pt}, \arrow[l, hook', "{[x_0]}"] \arrow[u, hook, "{[y_0]}"'] \end{tikzcd}$$

  where $\sim$ is the equivalence relation on $X \coprod Y$ obtained by declaring $(0, x_0) \sim (1, y_0)$.
  - *The Basepoint.* The element p_0 of $X \vee Y$ defined by

$$\begin{aligned}p_0 &\overset{\text{def}}{=} [(0, x_0)]\\ &= [(1, y_0)].\end{aligned}$$

- *The Cocone.* The morphisms of pointed sets

$$\begin{aligned}\mathrm{inj}_1 &\colon (X, x_0) \to (X \vee Y, p_0),\\ \mathrm{inj}_2 &\colon (Y, y_0) \to (X \vee Y, p_0),\end{aligned}$$

  given by

$$\begin{aligned}\mathrm{inj}_1(x) &\overset{\text{def}}{=} [(0, x)],\\ \mathrm{inj}_2(y) &\overset{\text{def}}{=} [(1, y)],\end{aligned}$$

  for each $x \in X$ and each $y \in Y$.

Proof. We claim that $(X \vee Y, p_0)$ is the categorical coproduct of (X, x_0) and (Y, y_0) in Sets_*. Indeed, suppose we have a diagram of the form

$$\begin{tikzcd} & (C, *) & \\ (X, x_0) \arrow[ur, bend left, "\iota_X"] \arrow[r, "\mathrm{inj}_X"'] & (X \vee Y, p_0) & (Y, y_0) \arrow[ul, bend right, "\iota_Y"'] \arrow[l, "\mathrm{inj}_Y"] \end{tikzcd}$$

in Sets. Then there exists a unique morphism of pointed sets

$$\phi\colon (X \vee Y, p_0) \to (C, *)$$

making the diagram

$$\begin{tikzcd}
 & (C,*) & \\
(X,x_0) \arrow[ur, bend left, "\iota_X"] \arrow[r, "\mathrm{inj}_X"'] & (X \vee Y, p_0) \arrow[u, dashed, "\phi", "\exists!"'] & (Y,y_0) \arrow[ul, bend right, "\iota_Y"'] \arrow[l, "\mathrm{inj}_Y"]
\end{tikzcd}$$

commute, being uniquely determined by the conditions

$$\begin{aligned}
\phi \circ \mathrm{inj}_X &= \iota_X,\\
\phi \circ \mathrm{inj}_Y &= \iota_Y
\end{aligned}$$

via

$$\phi(z) = \begin{cases} \iota_X(x) & \text{if } z = [(0,x)] \text{ with } x \in X, \\ \iota_Y(y) & \text{if } z = [(1,y)] \text{ with } y \in Y \end{cases}$$

for each $z \in X \vee Y$, where we note that ϕ is indeed a morphism of pointed sets, as we have

$$\begin{aligned}
\phi(p_0) &= \iota_X([(0,x_0)])\\
&= \iota_Y([(1,y_0)])\\
&= *,
\end{aligned}$$

as ι_X and ι_Y are morphisms of pointed sets. □

Proposition 3.3.3.1.2. Let (X, x_0) and (Y, y_0) be pointed sets.

1. *Functoriality.* The assignments

   $$(X,x_0),(Y,y_0),((X,x_0),(Y,y_0)) \mapsto (X \vee Y, p_0)$$

   define functors

   $$\begin{aligned}
   X \vee -\colon \mathsf{Sets}_* &\to \mathsf{Sets}_*,\\
   - \vee Y\colon \mathsf{Sets}_* &\to \mathsf{Sets}_*,\\
   -_1 \vee -_2\colon \mathsf{Sets}_* \times \mathsf{Sets}_* &\to \mathsf{Sets}_*.
   \end{aligned}$$

2. *Associativity.* We have an isomorphism of pointed sets

   $$(X \vee Y) \vee Z \cong X \vee (Y \vee Z),$$

   natural in $(X,x_0),(Y,y_0),(Z,z_0) \in \mathsf{Sets}_*$.

3. *Unitality.* We have isomorphisms of pointed sets

$$\begin{aligned}(\mathrm{pt}, *) \vee (X, x_0) &\cong (X, x_0),\\ (X, x_0) \vee (\mathrm{pt}, *) &\cong (X, x_0),\end{aligned}$$

natural in $(X, x_0) \in \mathsf{Sets}_*$.

4. *Commutativity.* We have an isomorphism of pointed sets

$$X \vee Y \cong Y \vee X,$$

natural in $(X, x_0), (Y, y_0) \in \mathsf{Sets}_*$.

5. *Symmetric Monoidality.* The triple $(\mathsf{Sets}_*, \vee, \mathrm{pt})$ is a symmetric monoidal category.

6. *The Fold Map.* We have a natural transformation

$$\nabla \colon \vee \circ \Delta^{\mathsf{Cats}}_{\mathsf{Sets}_*} \Longrightarrow \mathrm{id}_{\mathsf{Sets}_*}, \qquad \begin{tikzcd} & \mathsf{Sets}_* \times \mathsf{Sets}_* \arrow[dr, "\vee"] \arrow[d, phantom, "\Downarrow \nabla"] & \\ \mathsf{Sets}_* \arrow[ur, "\Delta^{\mathsf{Cats}}_{\mathsf{Sets}_*}"] \arrow[rr, bend right, "\mathrm{id}_{\mathsf{Sets}_*}"'] & & \mathsf{Sets}_*, \end{tikzcd}$$

called the **fold map**, whose component

$$\nabla_X \colon X \vee X \to X$$

at X is given by

$$\nabla_X(p) \stackrel{\text{def}}{=} \begin{cases} x & \text{if } p = [(0, x)], \\ x & \text{if } p = [(1, x)] \end{cases}$$

for each $p \in X \vee X$.

Proof. *Item 1*, *Functoriality*: This follows from **??** of **??**.
Item 2, *Associativity*: Clear.
Item 3, *Unitality*: Clear.
Item 4, *Commutativity*: Clear.
Item 5, *Symmetric Monoidality*: Omitted.
Item 6, *The Fold Map*: Naturality for the transformation ∇ is the statement that, given a morphism of pointed sets $f \colon (X, x_0) \to (Y, y_0)$, we have

$$\nabla_Y \circ (f \vee f) = f \circ \nabla_X, \qquad \begin{tikzcd} X \vee X \arrow[r, "\nabla_X"] \arrow[d, "f \vee f"'] & X \arrow[d, "f"] \\ Y \vee Y \arrow[r, "\nabla_Y"'] & Y. \end{tikzcd}$$

Indeed, we have

$$\begin{aligned}
[\nabla_Y \circ (f \vee f)]([(i,x)]) &= \nabla_Y([(i, f(x))]) \\
&= f(x) \\
&= f(\nabla_X([(i,x)])) \\
&= [f \circ \nabla_X]([(i,x)])
\end{aligned}$$

for each $[(i,x)] \in X \vee X$, and thus ∇ is indeed a natural transformation. □

3.3.4 Pushouts

Let (X, x_0), (Y, y_0), and (Z, z_0) be pointed sets and let $f\colon (Z, z_0) \to (X, x_0)$ and $g\colon (Z, z_0) \to (Y, y_0)$ be morphisms of pointed sets.

Definition 3.3.4.1.1. The **pushout of** (X, x_0) **and** (Y, y_0) **over** (Z, z_0) **along** (f, g) is the pair consisting of:

- *The Colimit.* The pointed set $\left(X \coprod_{f,Z,g} Y, p_0\right)$, where:
  - The set $X \coprod_{f,Z,g} Y$ is the pushout (of unpointed sets) of X and Y over Z with respect to f and g;
  - We have $p_0 = [x_0] = [y_0]$.
- *The Cocone.* The morphisms of pointed sets

$$\begin{aligned}
\mathrm{inj}_1 &\colon (X, x_0) \to (X \textstyle\coprod_Z Y, p_0), \\
\mathrm{inj}_2 &\colon (Y, y_0) \to (X \textstyle\coprod_Z Y, p_0)
\end{aligned}$$

  given by

$$\begin{aligned}
\mathrm{inj}_1(x) &\stackrel{\mathrm{def}}{=} [(0, x)] \\
\mathrm{inj}_2(y) &\stackrel{\mathrm{def}}{=} [(1, y)]
\end{aligned}$$

  for each $x \in X$ and each $y \in Y$.

Proof. Firstly, we note that indeed $[x_0] = [y_0]$, as we have

$$\begin{aligned}
x_0 &= f(z_0), \\
y_0 &= g(z_0)
\end{aligned}$$

since f and g are morphisms of pointed sets, with the relation $\sim$ on $X \coprod_Z Y$ then identifying $x_0 = f(z_0) \sim g(z_0) = y_0$.
We now claim that $(X \coprod_Z Y, p_0)$ is the categorical pushout of (X, x_0) and

(Y, y_0) over (Z, z_0) with respect to (f, g) in Sets_*. First we need to check that the relevant pushout diagram commutes, i.e. that we have

$$\mathrm{inj}_1 \circ f = \mathrm{inj}_2 \circ g, \qquad \begin{tikzcd} (X \coprod_Z Y, p_0) & (Y, y_0) \arrow[l, "\mathrm{inj}_2"'] \\ (X, x_0) \arrow[u, "\mathrm{inj}_1"] & (Z, z_0). \arrow[l, "f"] \arrow[u, "g"'] \end{tikzcd}$$

Indeed, given $z \in Z$, we have

$$\begin{aligned} \left[\mathrm{inj}_1 \circ f\right](z) &= \mathrm{inj}_1(f(z)) \\ &= [(0, f(z))] \\ &= [(1, g(z))] \\ &= \mathrm{inj}_2(g(z)) \\ &= \left[\mathrm{inj}_2 \circ g\right](z), \end{aligned}$$

where $[(0, f(z))] = [(1, g(z))]$ by the definition of the relation $\sim$ on $X \coprod Y$ (the coproduct of unpointed sets of X and Y). Next, we prove that $X \coprod_Z Y$ satisfies the universal property of the pushout. Suppose we have a diagram of the form

$$\begin{tikzcd} (P, *) & & \\ & (X \coprod_Z Y, p_0) \arrow[dr, phantom, very near start, "\ulcorner"] & (Y, y_0) \arrow[l, "\mathrm{inj}_2"'] \arrow[ull, bend right, "\iota_2"'] \\ & (X, x_0) \arrow[u, "\mathrm{inj}_1"] \arrow[uul, bend left, "\iota_1"] & (Z, z_0) \arrow[l, "f"] \arrow[u, "g"'] \end{tikzcd}$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi \colon (X \textstyle\coprod_Z Y, p_0) \to (P, *)$$

making the diagram

$$\begin{tikzcd}
(P, *) & & \\
& (X \coprod_Z Y, p_0) \arrow[ul, dashed, "\exists!", "\phi"'] \arrow[dr, phantom, "\ulcorner", very near start] & (Y, y_0) \arrow[l, "\mathrm{inj}_2"'] \arrow[ull, bend right, "\iota_2"'] \\
& (X, x_0) \arrow[u, "\mathrm{inj}_1"] \arrow[uul, bend left, "\iota_1"] & (Z, z_0) \arrow[l, "f"] \arrow[u, "g"']
\end{tikzcd}$$

commute, being uniquely determined by the conditions

$$\begin{aligned}
\phi \circ \mathrm{inj}_1 &= \iota_1, \\
\phi \circ \mathrm{inj}_2 &= \iota_2
\end{aligned}$$

via

$$\phi(p) = \begin{cases} \iota_1(x) & \text{if } x = [(0, x)], \\ \iota_2(y) & \text{if } x = [(1, y)] \end{cases}$$

for each $p \in X \coprod_Z Y$, where the well-definedness of ϕ is proven in the same way as in the proof of Definition 2.2.4.1.1. Finally, we show that ϕ is indeed a morphism of pointed sets, as we have

$$\begin{aligned}
\phi(p_0) &= \phi([(0, x_0)]) \\
&= \iota_1(x_0) \\
&= *,
\end{aligned}$$

or alternatively

$$\begin{aligned}
\phi(p_0) &= \phi([(1, y_0)]) \\
&= \iota_2(y_0) \\
&= *,
\end{aligned}$$

where we use that ι_1 (resp. ι_2) is a morphism of pointed sets. □

Proposition 3.3.4.1.2. Let (X, x_0), (Y, y_0), (Z, z_0), and (A, a_0) be pointed sets.

1. *Functoriality.* The assignment $(X, Y, Z, f, g) \mapsto X \coprod_{f,Z,g} Y$ defines a functor

$$-_1 \coprod_{-_3} -_1 \colon \mathsf{Fun}(\mathcal{P}, \mathsf{Sets}) \to \mathsf{Sets}_*,$$

where $\mathcal{P}$ is the category that looks like this:

$$
\begin{tikzcd}
 & \bullet \\
\bullet & \bullet. \arrow[l] \arrow[u]
\end{tikzcd}
$$

In particular, the action on morphisms of $-_1 \coprod_{-_3} -_1$ is given by sending a morphism

$$
\begin{tikzcd}
X \coprod_Z Y \arrow[dr, phantom, "\ulcorner", very near start] & & Y \arrow[ll] \arrow[dr, "\psi"] & \\
 & X' \coprod_{Z'} Y' \arrow[dr, phantom, "\ulcorner", very near start] & & Y' \arrow[ll] \\
X \arrow[uu] \arrow[dr, "\phi"'] & & Z \arrow[ll, "f"'] \arrow[uu, "g"] \arrow[dr, "\chi"] & \\
 & X' \arrow[uu] & & Z' \arrow[ll, "f'"] \arrow[uu, "g'"']
\end{tikzcd}
$$

in $\mathsf{Fun}(\mathcal{P}, \mathsf{Sets}_*)$ to the morphism of pointed sets

$$\xi\colon (X \coprod_Z Y, p_0) \xrightarrow{\exists!} \left(X' \coprod_{Z'} Y', p_0'\right)$$

given by

$$\xi(p) \stackrel{\text{def}}{=} \begin{cases} \phi(x) & \text{if } p = [(0, x)], \\ \psi(y) & \text{if } p = [(1, y)] \end{cases}$$

for each $p \in X \coprod_Z Y$, which is the unique morphism of pointed sets making the diagram

$$
\begin{tikzcd}
X \coprod_Z Y \arrow[dr, dashed] \arrow[dr, phantom, "\ulcorner", very near start] & & Y \arrow[ll] \arrow[dr, "\psi"] & \\
 & X' \coprod_{Z'} Y' \arrow[dr, phantom, "\ulcorner", very near start] & & Y' \arrow[ll] \\
X \arrow[uu] \arrow[dr, "\phi"'] & & Z \arrow[ll, "f"'] \arrow[uu, "g"] \arrow[dr, "\chi"] & \\
 & X' \arrow[uu] & & Z' \arrow[ll, "f'"] \arrow[uu, "g'"']
\end{tikzcd}
$$

commute.

2. *Associativity.* Given a diagram

$$\begin{tikzcd}
X & & Y & & Z \\
& W \arrow[ul, "f"] \arrow[ur, "g"'] & & V \arrow[ul, "h"] \arrow[ur, "k"'] &
\end{tikzcd}$$

in Sets, we have isomorphisms of pointed sets

$$(X \textstyle\coprod_W Y) \textstyle\coprod_V Z \cong (X \textstyle\coprod_W Y) \textstyle\coprod_Y (Y \textstyle\coprod_V Z) \cong X \textstyle\coprod_W (Y \textstyle\coprod_V Z),$$

where these pullbacks are built as in the diagrams

$$\begin{tikzcd}[column sep=tiny, row sep=small]
& & (X \coprod_W Y) \coprod_V Z & & \\
& X \coprod_W Y \arrow[ur] \arrow[ur, phantom, "\wedge", very near end] & & & \\
X \arrow[ur] & & Y \arrow[ul] \arrow[uu, phantom, "\wedge"] & & Z, \arrow[uull] \\
& W \arrow[ul, "f"] \arrow[ur, "g"'] & & V \arrow[ul, "h"] \arrow[ur, "k"'] &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=tiny, row sep=small]
& & (X \coprod_W Y) \coprod_Y (Y \coprod_V Z) & & \\
& X \coprod_W Y \arrow[ur] & & Y \coprod_V Z \arrow[ul] & \\
X \arrow[ur] & & Y \arrow[ul] \arrow[ur] & & Z, \arrow[ul] \\
& W \arrow[ul, "f"] \arrow[ur, "g"'] & & V \arrow[ul, "h"] \arrow[ur, "k"'] &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=tiny, row sep=small]
& & X \coprod_W (Y \coprod_V Z) & & \\
& & & Y \coprod_V Z \arrow[ul] & \\
X \arrow[uurr] & & Y \arrow[ur] & & Z. \arrow[ul] \\
& W \arrow[ul, "f"] \arrow[ur, "g"'] & & V \arrow[ul, "h"] \arrow[ur, "k"'] &
\end{tikzcd}$$

3. *Unitality.* We have isomorphisms of sets

$$\begin{tikzcd}
A \arrow[r, equal] & A \\
X \arrow[u, "f"] \arrow[r, equal] \arrow[ur, phantom, "\ulcorner", very near end] & X \arrow[u, "f"']
\end{tikzcd}
\qquad
\begin{aligned}
X \textstyle\coprod_X A &\cong A, \\
A \textstyle\coprod_X X &\cong A,
\end{aligned}
\qquad
\begin{tikzcd}
A & X \arrow[l, "f"'] \\
X \arrow[u, equal] \arrow[ur, phantom, "\ulcorner", very near end] & X. \arrow[l, "f"] \arrow[u, equal]
\end{tikzcd}$$

4. *Commutativity.* We have an isomorphism of sets

$$\begin{tikzcd}
X \coprod_Z Y & Y \arrow[l] \\
X \arrow[u] \arrow[ur, phantom, "\ulcorner", very near end] & Z, \arrow[l, "f"] \arrow[u, "g"']
\end{tikzcd}
\qquad
X \textstyle\coprod_Z Y \cong Y \textstyle\coprod_Z X
\qquad
\begin{tikzcd}
Y \coprod_Z X & X \arrow[l] \\
Y \arrow[u] \arrow[ur, phantom, "\ulcorner", very near end] & Z. \arrow[l, "g"] \arrow[u, "f"']
\end{tikzcd}$$

5. *Interaction With Coproducts.* We have

$$X \textstyle\coprod_{\mathrm{pt}} Y \cong X \vee Y,
\qquad
\begin{tikzcd}
X \vee Y & Y \arrow[l] \\
X \arrow[u] \arrow[ur, phantom, "\ulcorner", very near end] & \mathrm{pt}. \arrow[l, hook', "{[x_0]}"] \arrow[u, hook', "{[y_0]}"']
\end{tikzcd}$$

6. *Symmetric Monoidality.* The triple $(\mathsf{Sets}_*, \coprod_X, (X, x_0))$ is a symmetric monoidal category.

Proof. *Item 1, Functoriality*: This is a special case of functoriality of co/limits, **??** of **??**, with the explicit expression for ξ following from the commutativity of the cube pushout diagram.
Item 2, Associativity: This follows from Item 2 of Proposition 2.2.4.1.4.
Item 3, Unitality: This follows from Item 3 of Proposition 2.2.4.1.4.
Item 4, Commutativity: This follows from Item 4 of Proposition 2.2.4.1.4.
Item 5, Interaction With Coproducts: Clear.
Item 6, Symmetric Monoidality: Omitted. □

3.3.5 Coequalisers

Let $f, g \colon (X, x_0) \rightrightarrows (Y, y_0)$ be morphisms of pointed sets.

Definition 3.3.5.1.1. The **coequaliser of** (f, g) is the pointed set $\left(\mathrm{CoEq}(f, g), [y_0]\right)$.

Proof. We claim that $\left(\mathrm{CoEq}(f, g), [y_0]\right)$ is the categorical coequaliser of f and g in Sets_*. First we need to check that the relevant coequaliser diagram commutes, i.e. that we have

$$\mathrm{coeq}(f, g) \circ f = \mathrm{coeq}(f, g) \circ g.$$

Indeed, we have

$$\begin{aligned}
\left[\mathrm{coeq}(f, g) \circ f\right](x) &\stackrel{\text{def}}{=} \left[\mathrm{coeq}(f, g)\right](f(x)) \\
&\stackrel{\text{def}}{=} [f(x)] \\
&= [g(x)] \\
&\stackrel{\text{def}}{=} \left[\mathrm{coeq}(f, g)\right](g(x)) \\
&\stackrel{\text{def}}{=} \left[\mathrm{coeq}(f, g) \circ g\right](x)
\end{aligned}$$

for each $x \in X$. Next, we prove that $\mathrm{CoEq}(f, g)$ satisfies the universal property of the coequaliser. Suppose we have a diagram of the form

$$\begin{tikzcd}
(X, x_0) \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & (Y, y_0) \arrow[r, "\mathrm{coeq}(f,g)"] \arrow[dr, "c"'] & \left(\mathrm{CoEq}(f, g), [y_0]\right) \\
& & (C, *)
\end{tikzcd}$$

in Sets. Then, since $c(f(a)) = c(g(a))$ for each $a \in A$, it follows from Items 4 and 5 of Proposition 7.5.2.1.3 that there exists a unique map $\phi \colon \mathrm{CoEq}(f, g) \xrightarrow{\exists!}$

C making the diagram

$$\begin{tikzcd} (X,x_0) \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & (Y,y_0) \arrow[r, "\mathrm{coeq}(f,g)"] \arrow[dr, "c"'] & (\mathrm{CoEq}(f,g),[y_0]) \arrow[d, dashed, "\phi"', "\exists!"] \\ & & (C,*) \end{tikzcd}$$

commute, where we note that ϕ is indeed a morphism of pointed sets since

$$\begin{aligned} \phi([y_0]) &= [\phi \circ \mathrm{coeq}(f,g)]([y_0]) \\ &= c([y_0]) \\ &= *, \end{aligned}$$

where we have used that c is a morphism of pointed sets. □

Proposition 3.3.5.1.2. Let (X,x_0) and (Y,y_0) be pointed sets and let $f,g,h\colon (X,x_0) \to (Y,y_0)$ be morphisms of pointed sets.

1. *Associativity.* We have isomorphisms of pointed sets

$$\underbrace{\mathrm{CoEq}(\mathrm{coeq}(f,g)\circ f, \mathrm{coeq}(f,g)\circ h)}_{=\mathrm{CoEq}(\mathrm{coeq}(f,g)\circ g, \mathrm{coeq}(f,g)\circ h)} \cong \mathrm{CoEq}(f,g,h) \cong \underbrace{\mathrm{CoEq}(\mathrm{coeq}(g,h)\circ f, \mathrm{coeq}(g,h)\circ g)}_{=\mathrm{CoEq}(\mathrm{coeq}(g,h)\circ f, \mathrm{coeq}(g,h)\circ h)},$$

   where $\mathrm{CoEq}(f,g,h)$ is the colimit of the diagram

$$\begin{tikzcd} (X,x_0) \arrow[r, shift left=2, "f"] \arrow[r, "g" description] \arrow[r, shift right=2, "h"'] & (Y,y_0) \end{tikzcd}$$

   in Sets_*.

2. *Unitality.* We have an isomorphism of pointed sets

$$\mathrm{CoEq}(f,f) \cong B.$$

3. *Commutativity.* We have an isomorphism of pointed sets

$$\mathrm{CoEq}(f,g) \cong \mathrm{CoEq}(g,f).$$

Proof. *Item 1, Associativity*: This follows from Item 1 of Proposition 2.2.5.1.4.
Item 2, Unitality: This follows from Item 4 of Proposition 2.2.5.1.4.
Item 3, Commutativity: This follows from Item 5 of Proposition 2.2.5.1.4. □

3.4 Constructions With Pointed Sets

3.4.1 Free Pointed Sets

Let X be a set.

Definition 3.4.1.1.1. The **free pointed set on** X is the pointed set X^+ consisting of:

- *The Underlying Set.* The set X^+ defined by[11]

$$\begin{aligned} X^+ &\overset{\text{def}}{=} X \coprod \text{pt} \\ &\overset{\text{def}}{=} X \coprod \{\star\}. \end{aligned}$$

- *The Basepoint.* The element $\star$ of X^+.

Proposition 3.4.1.1.2. Let X be a set.

1. *Functoriality.* The assignment $X \mapsto X^+$ defines a functor

$$(-)^+ \colon \mathsf{Sets} \to \mathsf{Sets}_*,$$

where

- *Action on Objects.* For each $X \in \text{Obj}(\mathsf{Sets})$, we have

$$\left[(-)^+\right](X) \overset{\text{def}}{=} X^+,$$

where X^+ is the pointed set of Definition 3.4.1.1.1;

- *Action on Morphisms.* For each morphism $f \colon X \to Y$ of Sets, the image

$$f^+ \colon X^+ \to Y^+$$

of f by $(-)^+$ is the map of pointed sets defined by

$$f^+(x) \overset{\text{def}}{=} \begin{cases} f(x) & \text{if } x \in X, \\ \star_Y & \text{if } x = \star_X. \end{cases}$$

2. *Adjointness.* We have an adjunction

$$\left((-)^+ \dashv 忘\right) \colon \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "(-)^+"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_*, \arrow[l, bend left, "忘"] \end{tikzcd}$$

witnessed by a bijection of sets

$$\mathsf{Sets}_*\left((X^+, \star_X), (Y, y_0)\right) \cong \mathsf{Sets}(X, Y),$$

natural in $X \in \text{Obj}(\mathsf{Sets})$ and $(Y, y_0) \in \text{Obj}(\mathsf{Sets}_*)$.

[11] *Further Notation:* We sometimes write $\star_X$ for the basepoint of X^+ for clarity when there

3. *Symmetric Strong Monoidality With Respect to Wedge Sums.* The free pointed set functor of Item 1 has a symmetric strong monoidal structure

$$\left((-)^+, (-)^{+,\coprod}, (-)^{+,\coprod}_{\mathbb{1}}\right) \colon (\mathsf{Sets}, \coprod, \emptyset) \to (\mathsf{Sets}_*, \vee, \mathrm{pt}),$$

being equipped with isomorphisms of pointed sets

$$(-)^{+,\coprod}_{X,Y} \colon X^+ \vee Y^+ \xrightarrow{\cong} (X \coprod Y)^+,$$
$$(-)^{+,\coprod}_{\mathbb{1}} \colon \mathrm{pt} \xrightarrow{\cong} \emptyset^+,$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

4. *Symmetric Strong Monoidality With Respect to Smash Products.* The free pointed set functor of Item 1 has a symmetric strong monoidal structure

$$\left((-)^+, (-)^{+,\times}, (-)^{+,\times}_{\mathbb{1}}\right) \colon (\mathsf{Sets}, \times, \mathrm{pt}) \to \left(\mathsf{Sets}_*, \wedge, S^0\right),$$

being equipped with isomorphisms of pointed sets

$$(-)^{+,\times}_{X,Y} \colon X^+ \wedge Y^+ \xrightarrow{\cong} (X \times Y)^+,$$
$$(-)^{+,\times}_{\mathbb{1}} \colon S^0 \xrightarrow{\cong} \mathrm{pt}^+,$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

Proof. Item 1, Functoriality: Clear.
Item 2, Adjointness: We claim there's an adjunction $(-)^+ \dashv$ 忘, witnessed by a bijection of sets

$$\mathsf{Sets}_*\left((X^+, \star_X), (Y, y_0)\right) \cong \mathsf{Sets}(X, Y),$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets})$ and $(Y, y_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

- *Map I.* We define a map

$$\Phi_{X,Y} \colon \mathsf{Sets}_*\left((X^+, \star_X), (Y, y_0)\right) \to \mathsf{Sets}(X, Y)$$

by sending a pointed function

$$\xi \colon (X^+, \star_X) \to (Y, y_0)$$

are multiple free pointed sets involved in the current discussion.

to the function

$$\xi^{\dagger} \colon X \to Y$$

given by

$$\xi^{\dagger}(x) \stackrel{\text{def}}{=} \xi(x)$$

for each $x \in X$.

- *Map II.* We define a map

$$\Psi_{X,Y} \colon \mathsf{Sets}(X, Y) \to \mathsf{Sets}_*\big((X^+, \star_X), (Y, y_0)\big)$$

given by sending a function $\xi \colon X \to Y$ to the pointed function

$$\xi^{\dagger} \colon \big(X^+, \star_X\big) \to (Y, y_0)$$

defined by

$$\xi^{\dagger}(x) \stackrel{\text{def}}{=} \begin{cases} \xi(x) & \text{if } x \in X, \\ y_0 & \text{if } x = \star_X \end{cases}$$

for each $x \in X^+$.

- *Invertibility I.* We claim that

$$\Psi_{X,Y} \circ \Phi_{X,Y} = \mathrm{id}_{\mathsf{Sets}_*((X^+, \star_X),(Y, y_0))},$$

which is clear.

- *Invertibility II.* We claim that

$$\Phi_{X,Y} \circ \Psi_{X,Y} = \mathrm{id}_{\mathsf{Sets}(X,Y)},$$

which is clear.

- *Naturality for Φ, Part I.* We need to show that, given a pointed function $g \colon (Y, y_0) \to \big(Y', y_0'\big)$, the diagram

$$\begin{tikzcd} \mathsf{Sets}_*((X^+, \star_X), (Y, y_0)) \arrow[r, "\Phi_{X,Y}"] \arrow[d, "g_*"'] & \mathsf{Sets}(X, Y) \arrow[d, "g_*"] \\ \mathsf{Sets}_*\big((X^+, \star_X), \big(Y', y_0'\big)\big), \arrow[r, "\Phi_{X,Y'}"'] & \mathsf{Sets}(X, Y') \end{tikzcd}$$

commutes. Indeed, given a pointed function

$$\xi^{\dagger} \colon \big(X^+, \star_X\big) \to (Y, y_0)$$

we have

$$
\begin{aligned}
\left[\Phi_{X,Y'} \circ g_*\right](\xi) &= \Phi_{X,Y'}(g_*(\xi)) \\
&= \Phi_{X,Y'}(g \circ \xi) \\
&= g \circ \xi \\
&= g \circ \Phi_{X,Y'}(\xi) \\
&= g_*\left(\Phi_{X,Y'}(\xi)\right) \\
&= \left[g_* \circ \Phi_{X,Y'}\right](\xi).
\end{aligned}
$$

- *Naturality for Φ, Part II.* We need to show that, given a pointed function $f\colon (X, x_0) \to \left(X', x'_0\right)$, the diagram

$$
\begin{tikzcd}
\mathsf{Sets}_*\left(\left(X'^{,+}, \star_X\right), (Y, y_0)\right) \arrow[r, "\Phi_{X',Y}"] \arrow[d, "f^*"'] & \mathsf{Sets}(X', Y) \arrow[d, "f^*"] \\
\mathsf{Sets}_*((X^+, \star_X), (Y, y_0)) \arrow[r, "\Phi_{X,Y}"'] & \mathsf{Sets}(X, Y)
\end{tikzcd}
$$

  commutes. Indeed, given a function

$$\xi\colon X' \to Y,$$

  we have

$$
\begin{aligned}
\left[\Phi_{X,Y} \circ f^*\right](\xi) &= \Phi_{X,Y}(f^*(\xi)) \\
&= \Phi_{X,Y}(\xi \circ f) \\
&= \xi \circ f \\
&= \Phi_{X',Y}(\xi) \circ f \\
&= f^*\left(\Phi_{X',Y}(\xi)\right) \\
&= f^*\left(\Phi_{X',Y}(\xi)\right) \\
&= \left[f^* \circ \Phi_{X',Y}\right](\xi).
\end{aligned}
$$

- *Naturality for Ψ.* Since Φ is natural in each argument and Φ is a componentwise inverse to Ψ in each argument, it follows from Item 2 of Proposition 8.8.6.1.2 that Ψ is also natural in each argument.

Item 3, Symmetric Strong Monoidality With Respect to Wedge Sums: The isomorphism

$$\phi\colon X^+ \vee Y^+ \xrightarrow{\cong} (X \coprod Y)^+$$

is given by

$$\phi(z) = \begin{cases} x & \text{if } z = [(0,x)] \text{ with } x \in X, \\ y & \text{if } z = [(1,y)] \text{ with } y \in Y, \\ \star_{X \coprod Y} & \text{if } z = [(0,\star_X)], \\ \star_{X \coprod Y} & \text{if } z = [(1,\star_Y)] \end{cases}$$

for each $z \in X^+ \vee Y^+$, with inverse

$$\phi^{-1} \colon (X \coprod Y)^+ \xrightarrow{\cong} X^+ \vee Y^+$$

given by

$$\phi^{-1}(z) \stackrel{\text{def}}{=} \begin{cases} [(0,x)] & \text{if } z = [(0,x)], \\ [(0,y)] & \text{if } z = [(1,y)], \\ p_0 & \text{if } z = \star_{X \coprod Y} \end{cases}$$

for each $z \in (X \coprod Y)^+$.
Meanwhile, the isomorphism $\mathrm{pt} \cong \emptyset^+$ is given by sending $\star_X$ to $\star_\emptyset$.
That these isomorphisms satisfy the coherence conditions making the functor $(-)^+$ symmetric strong monoidal can be directly checked element by element.
Item 4, *Symmetric Strong Monoidality With Respect to Smash Products*: The isomorphism

$$\phi \colon X^+ \wedge Y^+ \xrightarrow{\cong} (X \times Y)^+$$

is given by

$$\phi(x \wedge y) = \begin{cases} (x,y) & \text{if } x \neq \star_X \text{ and } y \neq \star_Y \\ \star_{X \times Y} & \text{otherwise} \end{cases}$$

for each $x \wedge y \in X^+ \wedge Y^+$, with inverse

$$\phi^{-1} \colon (X \times Y)^+ \xrightarrow{\cong} X^+ \wedge Y^+$$

given by

$$\phi^{-1}(z) \stackrel{\text{def}}{=} \begin{cases} x \wedge y & \text{if } z = (x,y) \text{ with } (x,y) \in X \times Y, \\ \star_X \wedge \star_Y & \text{if } z = \star_{X \times Y}, \end{cases}$$

for each $z \in (X \coprod Y)^+$.
Meanwhile, the isomorphism $S^0 \cong \mathrm{pt}^+$ is given by sending $\star$ to $1 \in S^0 = \{0,1\}$ and $\star_{\mathrm{pt}}$ to $0 \in S^0$.
That these isomorphisms satisfy the coherence conditions making the functor $(-)^+$ symmetric strong monoidal can be directly checked element by element.

□

Appendices

3.A Other Chapters

Sets

1. Sets
2. Constructions With Sets
3. Pointed Sets
4. Tensor Products of Pointed Sets

Relations

5. Relations
6. Constructions With Relations
7. Equivalence Relations and Apartness Relations

Category Theory

8. Categories

Bicategories

9. Types of Morphisms in Bicategories

Chapter 4

Tensor Products of Pointed Sets

In this chapter we introduce, construct, and study tensor products of pointed sets. The most well-known among these is the *smash product of pointed sets*

$$\wedge \colon \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*,$$

introduced in Section 4.5.1, defined via a universal property as inducing a bijection between the following data:

- Pointed maps $f \colon X \wedge Y \to Z$.
- Maps of sets $f \colon X \times Y \to Z$ satisfying

  $$\begin{aligned} f(x_0, y) &= z_0, \\ f(x, y_0) &= z_0 \end{aligned}$$

  for each $x \in X$ and each $y \in Y$.

As it turns out, however, dropping either of the *bilinearity* conditions

$$\begin{aligned} f(x_0, y) &= z_0, \\ f(x, y_0) &= z_0 \end{aligned}$$

while retaining the other leads to two other tensor products of pointed sets,

$$\begin{aligned} \lhd \colon \mathsf{Sets}_* \times \mathsf{Sets}_* &\to \mathsf{Sets}_*, \\ \rhd \colon \mathsf{Sets}_* \times \mathsf{Sets}_* &\to \mathsf{Sets}_*, \end{aligned}$$

called the *left* and *right tensor products of pointed sets*. In contrast to $\wedge$, which turns out to endow Sets_* with a monoidal category structure (Proposition 4.5.9.1.1), these do not admit invertible associators and unitors, but do endow Sets_*

with the structure of a skew monoidal category, however (Propositions 4.3.8.1.1 and 4.4.8.1.1).

Finally, in addition to the tensor products $\lhd$, $\rhd$, and $\wedge$, we also have a "tensor product" of the form

$$\odot\colon \mathsf{Sets} \times \mathsf{Sets}_* \to \mathsf{Sets}_*,$$

called the *tensor* of sets with pointed sets. All in all, these tensor products assemble into a family of functors of the form

$$\begin{aligned}
\otimes_{k,\ell}\colon \mathsf{Mon}_{\mathbb{E}_k}(\mathsf{Sets}) \times \mathsf{Mon}_{\mathbb{E}_\ell}(\mathsf{Sets}) &\to \mathsf{Mon}_{\mathbb{E}_{k+\ell}}(\mathsf{Sets}),\\
\lhd_{i,k}\colon \mathsf{Mon}_{\mathbb{E}_k}(\mathsf{Sets}) \times \mathsf{Mon}_{\mathbb{E}_k}(\mathsf{Sets}) &\to \mathsf{Mon}_{\mathbb{E}_k}(\mathsf{Sets}),\\
\rhd_{i,k}\colon \mathsf{Mon}_{\mathbb{E}_k}(\mathsf{Sets}) \times \mathsf{Mon}_{\mathbb{E}_k}(\mathsf{Sets}) &\to \mathsf{Mon}_{\mathbb{E}_k}(\mathsf{Sets}),
\end{aligned}$$

where $k, \ell, i \in \mathbb{N}$ with $i \leq k-1$. Together with the Cartesian product $\times$ of Sets, the tensor products studied in this chapter form the cases:

- $(k, \ell) = (-1, -1)$ for the Cartesian product of Sets;
- $(k, \ell) = (0, -1)$ and $(-1, 0)$ for the tensor of sets with pointed sets of Definition 4.2.1.1.1;
- $(i, k) = (-1, 0)$ for the left and right tensor products of pointed sets of Sections 4.3 and 4.4;
- $(k, \ell) = (-1, -1)$ for the smash product of pointed sets of Section 4.5.

In this chapter, we will carefully define and study bilinearity for pointed sets, as well as all the tensor products described above. Then, in **??**, we will extend these to tensor products involving also monoids and commutative monoids, which will end up covering all cases up to $k, \ell \leq 2$, and hence *all* cases since $\mathbb{E}_k$-monoids on Sets are the same as $\mathbb{E}_2$-monoids on Sets when $k \geq 2$.

Contents

4.1 Bilinear Morphisms of Pointed Sets 140
4.1.1 Left Bilinear Morphisms of Pointed Sets 140
4.1.2 Right Bilinear Morphisms of Pointed Sets ... 141
4.1.3 Bilinear Morphisms of Pointed Sets 141
4.2 Tensors and Cotensors of Pointed Sets by Sets 143
4.2.1 Tensors of Pointed Sets by Sets 143
4.2.2 Cotensors of Pointed Sets by Sets 150
4.3 The Left Tensor Product of Pointed Sets 157
4.3.1 Foundations 157

4.3.2 The Left Internal Hom of Pointed Sets 161
4.3.3 The Left Skew Unit 163
4.3.4 The Left Skew Associator 163
4.3.5 The Left Skew Left Unitor 166
4.3.6 The Left Skew Right Unitor 168
4.3.7 The Diagonal.................................. 170
4.3.8 The Left Skew Monoidal Structure on Pointed Sets Associated to $\lhd$ 171
4.3.9 Monoids With Respect to the Left Tensor Product of Pointed Sets 175

4.4 The Right Tensor Product of Pointed Sets 179

4.4.1 Foundations.................................... 179
4.4.2 The Right Internal Hom of Pointed Sets 182
4.4.3 The Right Skew Unit 184
4.4.4 The Right Skew Associator 185
4.4.5 The Right Skew Left Unitor 187
4.4.6 The Right Skew Right Unitor................. 189
4.4.7 The Diagonal.................................. 191
4.4.8 The Right Skew Monoidal Structure on Pointed Sets Associated to $\rhd$ 192
4.4.9 Monoids With Respect to the Right Tensor Product of Pointed Sets 196

4.5 The Smash Product of Pointed Sets 200

4.5.1 Foundations.................................... 200
4.5.2 The Internal Hom of Pointed Sets 210
4.5.3 The Monoidal Unit............................ 212
4.5.4 The Associator................................. 212
4.5.5 The Left Unitor................................ 214
4.5.6 The Right Unitor 216
4.5.7 The Symmetry................................. 219
4.5.8 The Diagonal.................................. 221
4.5.9 The Monoidal Structure on Pointed Sets Associated to $\wedge$ 224
4.5.10 Universal Properties of the Smash Product of Pointed Sets I.. 228
4.5.11 Universal Properties of the Smash Product of Pointed Sets II... 229
4.5.12 Monoids With Respect to the Smash Product of Pointed Sets 229

4.5.13 Comonoids With Respect to the Smash Product of Pointed Sets 229
4.6 Miscellany **229**
4.6.1 The Smash Product of a Family of Pointed Sets 229
4.A Other Chapters **230**

4.1 Bilinear Morphisms of Pointed Sets

4.1.1 Left Bilinear Morphisms of Pointed Sets

Let (X, x_0), (Y, y_0), and (Z, z_0) be pointed sets.

Definition 4.1.1.1.1. A **left bilinear morphism of pointed sets from** $(X \times Y, (x_0, y_0))$ **to** (Z, z_0) is a map of sets

$$f : X \times Y \to Z$$

satisfying the following condition:[1,2]

(★) *Left Unital Bilinearity.* The diagram

$$\begin{tikzcd} & \mathrm{pt} \times \mathrm{pt} \arrow[dr, dashed, "\sim"] & \\ \mathrm{pt} \times Y \arrow[ur, "\mathrm{id}_{\mathrm{pt}} \times \epsilon_Y"] \arrow[d, "{[x_0] \times \mathrm{id}_Y}"'] & & \mathrm{pt} \arrow[d, "{[z_0]}"] \\ X \times Y \arrow[rr, "f"'] & & Z \end{tikzcd}$$

commutes, i.e. for each $y \in Y$, we have

$$f(x_0, y) = z_0.$$

Definition 4.1.1.1.2. The **set of left bilinear morphisms of pointed sets from** $(X \times Y, (x_0, y_0))$ **to** (Z, z_0) is the set $\mathrm{Hom}^{\otimes, \mathrm{L}}_{\mathsf{Sets}_*}(X \times Y, Z)$ defined by

$$\mathrm{Hom}^{\otimes, \mathrm{L}}_{\mathsf{Sets}_*}(X \times Y, Z) \overset{\mathrm{def}}{=} \{f \in \mathrm{Hom}_{\mathsf{Sets}}(X \times Y, Z) \mid f \text{ is left bilinear}\}.$$

[1] *Slogan:* The map f is left bilinear if it preserves basepoints in its first argument.
[2] Succinctly, f is bilinear if we have

$$f(x_0, y) = z_0$$

4.1.2 Right Bilinear Morphisms of Pointed Sets

Let (X, x_0), (Y, y_0), and (Z, z_0) be pointed sets.

Definition 4.1.2.1.1. A **right bilinear morphism of pointed sets from** $(X \times Y, (x_0, y_0))$ **to** (Z, z_0) is a map of sets

$$f : X \times Y \to Z$$

satisfying the following condition:[3,4]

($\star$) *Right Unital Bilinearity.* The diagram

$$\begin{tikzcd}
 & \mathrm{pt}\times\mathrm{pt} \arrow[dr, dashed, "\sim"] & \\
X\times\mathrm{pt} \arrow[ur, "\epsilon_X\times\mathrm{id}_{\mathrm{pt}}"] \arrow[d, "\mathrm{id}_X\times[y_0]"'] & & \mathrm{pt} \arrow[d, "{[z_0]}"] \\
X\times Y \arrow[rr, "f"'] & & Z
\end{tikzcd}$$

commutes, i.e. for each $x \in X$, we have

$$f(x, y_0) = z_0.$$

Definition 4.1.2.1.2. The **set of right bilinear morphisms of pointed sets from** $(X \times Y, (x_0, y_0))$ **to** (Z, z_0) is the set $\mathrm{Hom}^{\otimes,\mathrm{R}}_{\mathsf{Sets}_*}(X \times Y, Z)$ defined by

$$\mathrm{Hom}^{\otimes,\mathrm{R}}_{\mathsf{Sets}_*}(X \times Y, Z) \overset{\text{def}}{=} \left\{ f \in \mathrm{Hom}_{\mathsf{Sets}}(X \times Y, Z) \;\middle|\; f \text{ is right bilinear} \right\}.$$

4.1.3 Bilinear Morphisms of Pointed Sets

Let (X, x_0), (Y, y_0), and (Z, z_0) be pointed sets.

Definition 4.1.3.1.1. A **bilinear morphism of pointed sets from** $(X \times Y, (x_0, y_0))$ **to** (Z, z_0) is a map of sets

$$f : X \times Y \to Z$$

that is both left bilinear and right bilinear.

for each $y \in Y$.

[3] *Slogan:* The map f is right bilinear if it preserves basepoints in its second argument.

[4] Succinctly, f is bilinear if we have

$$f(x, y_0) = z_0$$

Remark 4.1.3.1.2. In detail, a **bilinear morphism of pointed sets from** $(X \times Y, (x_0, y_0))$ **to** (Z, z_0) is a map of sets

$$f\colon (X \times Y, (x_0, y_0)) \longrightarrow (Z, z_0)$$

satisfying the following conditions:[5,6]

1. *Left Unital Bilinearity.* The diagram

$$\begin{tikzcd} & \mathrm{pt} \times \mathrm{pt} \arrow[dr, dashed, "\sim"] & \\ \mathrm{pt} \times Y \arrow[ur, "\mathrm{id}_{\mathrm{pt}} \times \epsilon_Y"] \arrow[d, "{[x_0]} \times \mathrm{id}_Y"'] & & \mathrm{pt} \arrow[d, "{[z_0]}"] \\ X \times Y \arrow[rr, "f"'] & & Z \end{tikzcd}$$

   commutes, i.e. for each $y \in Y$, we have

$$f(x_0, y) = z_0.$$

2. *Right Unital Bilinearity.* The diagram

$$\begin{tikzcd} & \mathrm{pt} \times \mathrm{pt} \arrow[dr, dashed, "\sim"] & \\ X \times \mathrm{pt} \arrow[ur, "\epsilon_X \times \mathrm{id}_{\mathrm{pt}}"] \arrow[d, "\mathrm{id}_X \times {[y_0]}"'] & & \mathrm{pt} \arrow[d, "{[z_0]}"] \\ X \times Y \arrow[rr, "f"'] & & Z \end{tikzcd}$$

   commutes, i.e. for each $x \in X$, we have

$$f(x, y_0) = z_0.$$

Definition 4.1.3.1.3. The **set of bilinear morphisms of pointed sets from** $(X \times Y, (x_0, y_0))$ **to** (Z, z_0) is the set $\mathrm{Hom}^{\otimes}_{\mathsf{Sets}_*}(X \times Y, Z)$ defined by

$$\mathrm{Hom}^{\otimes}_{\mathsf{Sets}_*}(X \times Y, Z) \overset{\mathrm{def}}{=} \{f \in \mathrm{Hom}_{\mathsf{Sets}}(X \times Y, Z) \mid f \text{ is bilinear}\}.$$

for each $x \in X$.

[5] *Slogan:* The map f is bilinear if it preserves basepoints in each argument.

[6] Succinctly, f is bilinear if we have

$$\begin{aligned} f(x_0, y) &= z_0, \\ f(x, y_0) &= z_0 \end{aligned}$$

4.2 Tensors and Cotensors of Pointed Sets by Sets

4.2.1 Tensors of Pointed Sets by Sets

Let (X, x_0) be a pointed set and let A be a set.

Definition 4.2.1.1.1. The **tensor of** (X, x_0) **by** A[7] is the pointed set[8] $A \odot (X, x_0)$ satisfying the following universal property:

(UP) We have a bijection

$$\mathsf{Sets}_*(A \odot X, K) \cong \mathsf{Sets}(A, \mathsf{Sets}_*(X, K)),$$

natural in $(K, k_0) \in \mathsf{Obj}(\mathsf{Sets}_*)$.

Remark 4.2.1.1.2. The universal property in Definition 4.2.1.1.1 is equivalent to the following one:

(UP) We have a bijection

$$\mathsf{Sets}_*(A \odot X, K) \cong \mathsf{Sets}^{\otimes}_{\mathbb{E}_0}(A \times X, K),$$

natural in $(K, k_0) \in \mathsf{Obj}(\mathsf{Sets}_*)$, where $\mathsf{Sets}^{\otimes}_{\mathbb{E}_0}(A \times X, K)$ is the set defined by

$$\mathsf{Sets}^{\otimes}_{\mathbb{E}_0}(A \times X, K) \stackrel{\text{def}}{=} \left\{ f \in \mathsf{Sets}(A \times X, K) \;\middle|\; \begin{array}{l} \text{for each } a \in A, \text{ we} \\ \text{have } f(a, x_0) = k_0 \end{array} \right\}.$$

Proof. We claim we have a bijection

$$\mathsf{Sets}(A, \mathsf{Sets}_*(X, K)) \cong \mathsf{Sets}^{\otimes}_{\mathbb{E}_0}(A \times X, K)$$

natural in $(K, k_0) \in \mathsf{Obj}(\mathsf{Sets}_*)$. Indeed, this bijection is a restriction of the bijection

$$\mathsf{Sets}(A, \mathsf{Sets}(X, K)) \cong \mathsf{Sets}(A \times X, K)$$

of Item 2 of Proposition 2.1.3.1.2:

- A map

$$\begin{aligned} \xi \colon A &\longrightarrow \mathsf{Sets}_*(X, K), \\ a &\longmapsto (\xi_a \colon X \to K), \end{aligned}$$

for each $x \in X$ and each $y \in Y$.

[7] *Further Terminology:* Also called the **copower of** (X, x_0) **by** A.

[8] *Further Notation:* Often written $A \odot X$ for simplicity.

in $\mathsf{Sets}(A, \mathsf{Sets}_*(X, K))$ gets sent to the map

$$\xi^\dagger \colon A \times X \to K$$

defined by

$$\xi^\dagger(a, x) \stackrel{\text{def}}{=} \xi_a(x)$$

for each $(a, x) \in A \times X$, which indeed lies in $\mathsf{Sets}^{\otimes}_{\mathbb{E}_0}(A \times X, K)$, as we have

$$\begin{aligned}\xi^\dagger(a, x_0) &\stackrel{\text{def}}{=} \xi_a(x_0)\\ &\stackrel{\text{def}}{=} k_0\end{aligned}$$

for each $a \in A$, where we have used that $\xi_a \in \mathsf{Sets}_*(X, K)$ is a morphism of pointed sets.

- Conversely, a map

$$\xi \colon A \times X \to K$$

in $\mathsf{Sets}^{\otimes}_{\mathbb{E}_0}(A \times X, K)$ gets sent to the map

$$\begin{aligned}\xi^\dagger \colon A &\longrightarrow \mathsf{Sets}_*(X, K),\\ a &\longmapsto \left(\xi^\dagger_a \colon X \to K\right),\end{aligned}$$

where

$$\xi^\dagger_a \colon X \to K$$

is the map defined by

$$\xi^\dagger_a(x) \stackrel{\text{def}}{=} \xi(a, x)$$

for each $x \in X$, and indeed lies in $\mathsf{Sets}_*(X, K)$, as we have

$$\begin{aligned}\xi^\dagger_a(x_0) &\stackrel{\text{def}}{=} \xi(a, x_0)\\ &\stackrel{\text{def}}{=} k_0.\end{aligned}$$

This finishes the proof. □

Construction 4.2.1.1.3. Concretely, the **tensor of** (X, x_0) **by** A is the pointed set $A \odot (X, x_0)$ consisting of:

- *The Underlying Set.* The set $A \odot X$ given by

$$A \odot X \cong \bigvee_{a\in A} (X, x_0),$$

  where $\bigvee_{a\in A}(X, x_0)$ is the wedge product of the A-indexed family $((X, x_0))_{a\in A}$ of Definition 3.3.2.1.1.

- *The Basepoint.* The point $[(a, x_0)] = [(a', x_0)]$ of $\bigvee_{a\in A}(X, x_0)$.

Proof. (Proven below in a bit.) □

Notation 4.2.1.1.4. We write $a \odot x$ for the element $[(a, x)]$ of

$$\begin{aligned} A \odot X &\cong \bigvee_{a\in A} (X, x_0) \\ &\stackrel{\text{def}}{=} \left(\coprod_{i\in I} X_i\right)/\sim. \end{aligned}$$

Remark 4.2.1.1.5. Taking the tensor of any element of A with the basepoint x_0 of X leads to the same element in $A \odot X$, i.e. we have

$$a \odot x_0 = a' \odot x_0,$$

for each $a, a' \in A$. This is due to the equivalence relation $\sim$ on

$$\bigvee_{a\in A} (X, x_0) \stackrel{\text{def}}{=} \coprod_{a\in A} X/\sim$$

identifying (a, x_0) with (a', x_0), so that the equivalence class $a \odot x_0$ is independent from the choice of $a \in A$.

Proof. We claim we have a bijection

$$\mathsf{Sets}_*(A \odot X, K) \cong \mathsf{Sets}(A, \mathsf{Sets}_*(X, K))$$

natural in $(K, k_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

- *Map I.* We define a map

$$\Phi_K \colon \mathsf{Sets}_*(A \odot X, K) \to \mathsf{Sets}(A, \mathsf{Sets}_*(X, K))$$

  by sending a morphism of pointed sets

$$\xi \colon (A \odot X, a \odot x_0) \to (K, k_0)$$

to the map of sets

$$\begin{aligned} \xi^\dagger \colon A &\longrightarrow \mathsf{Sets}_*(X,K), \\ a &\longmapsto (\xi_a \colon X \to K), \end{aligned}$$

where

$$\xi_a \colon (X, x_0) \to (K, k_0)$$

is the morphism of pointed sets defined by

$$\xi_a(x) \stackrel{\text{def}}{=} \xi(a \odot x)$$

for each $x \in X$. Note that we have

$$\begin{aligned} \xi_a(x_0) &\stackrel{\text{def}}{=} \xi(a \odot x_0) \\ &= k_0, \end{aligned}$$

so that ξ_a is indeed a morphism of pointed sets, where we have used that ξ is a morphism of pointed sets.

- *Map II.* We define a map

$$\Psi_K \colon \mathsf{Sets}(A, \mathsf{Sets}_*(X,K)) \to \mathsf{Sets}_*(A \odot X, K)$$

given by sending a map

$$\begin{aligned} \xi \colon A &\longrightarrow \mathsf{Sets}_*(X,K), \\ a &\longmapsto (\xi_a \colon X \to K), \end{aligned}$$

to the morphism of pointed sets

$$\xi^\dagger \colon (A \odot X, a \odot x_0) \to (K, k_0)$$

defined by

$$\xi^\dagger(a \odot x) \stackrel{\text{def}}{=} \xi_a(x)$$

for each $a \odot x \in A \odot X$. Note that $\xi^\dagger$ is indeed a morphism of pointed sets, as we have

$$\begin{aligned} \xi^\dagger(a \odot x_0) &\stackrel{\text{def}}{=} \xi_a(x_0) \\ &= k_0, \end{aligned}$$

where we have used that $\xi(a) \in \mathsf{Sets}_*(X,K)$ is a morphism of pointed sets.

- *Invertibility I.* We claim that

$$\Psi_K \circ \Phi_K = \mathrm{id}_{\mathsf{Sets}_*(A \odot X, K)}.$$

Indeed, given a morphism of pointed sets

$$\xi \colon (A \odot X, a \odot x_0) \to (K, k_0),$$

we have

$$\begin{aligned}
[\Psi_K \circ \Phi_K](\xi) &= \Psi_K(\Phi_K(\xi)) \\
&= \Psi_K\big([\![a \mapsto [\![x \mapsto \xi(a \odot x)]\!]]\!]\big) \\
&= \Psi_K\big([\![a' \mapsto [\![x' \mapsto \xi(a' \odot x')]\!]]\!]\big) \\
&= [\![a \odot x \mapsto \mathrm{ev}_x\big(\mathrm{ev}_a\big([\![a' \mapsto [\![x' \mapsto \xi(a' \odot x')]\!]]\!]\big)\big)]\!] \\
&= [\![a \odot x \mapsto \mathrm{ev}_x\big([\![x' \mapsto \xi(a \odot x')]\!]\big)]\!] \\
&= [\![a \odot x \mapsto \xi(a \odot x)]\!] \\
&= \xi.
\end{aligned}$$

- *Invertibility II.* We claim that

$$\Phi_K \circ \Psi_K = \mathrm{id}_{\mathsf{Sets}(A, \mathsf{Sets}_*(X, K))}.$$

Indeed, given a morphism $\xi \colon A \to \mathsf{Sets}_*(X, K)$, we have

$$\begin{aligned}
[\Phi_K \circ \Psi_K](\xi) &= \Phi_K(\Psi_K(\xi)) \\
&= \Phi_K\big([\![a \odot x \mapsto \xi_a(x)]\!]\big) \\
&= [\![a \mapsto [\![x \mapsto \xi_a(x)]\!]]\!] \\
&= [\![a \mapsto \xi(a)]\!] \\
&= \xi.
\end{aligned}$$

- *Naturality of* Φ. We need to show that, given a morphism of pointed sets

$$\phi \colon (K, k_0) \to \big(K', k'_0\big),$$

the diagram

$$\begin{tikzcd}
\mathsf{Sets}_*(A \odot X, K) \arrow[r, "\Phi_K"] \arrow[d, "\phi_*"'] & \mathsf{Sets}(A, \mathsf{Sets}_*(X, K)) \arrow[d, "(\phi_*)_*"] \\
\mathsf{Sets}_*(A \odot X, K') \arrow[r, "\Phi_{K'}"'] & \mathsf{Sets}(A, \mathsf{Sets}_*(X, K'))
\end{tikzcd}$$

commutes. Indeed, given a morphism of pointed sets

$$\xi\colon (A \odot X, a \odot x_0) \to (K, k_0),$$

we have

$$\begin{aligned}
[\Phi_{K'} \circ \phi_*](\xi) &= \Phi_{K'}(\phi_*(\xi)) \\
&= \Phi_{K'}(\phi \circ \xi) \\
&= (\phi \circ \xi)^{\dagger} \\
&= [\![a \mapsto \phi \circ \xi(a \odot -)]\!] \\
&= [\![a \mapsto \phi_*(\xi(a \odot -))]\!] \\
&= (\phi_*)_*([\![a \mapsto \xi(a \odot -]\!])) \\
&= (\phi_*)_*(\Phi_K(\xi)) \\
&= \left[(\phi_*)_* \circ \Phi_K\right](\xi).
\end{aligned}$$

- *Naturality of* Ψ. Since Φ is natural and Φ is a componentwise inverse to Ψ, it follows from Item 2 of Proposition 8.8.6.1.2 that Ψ is also natural.

This finishes the proof. □

Proposition 4.2.1.1.6. Let (X, x_0) be a pointed set and let A be a set.

1. *Functoriality.* The assignments A, (X, x_0), $(A, (X, x_0))$ define functors

$$\begin{aligned}
A \odot - &\colon \mathsf{Sets}_* \to \mathsf{Sets}_*, \\
- \odot X &\colon \mathsf{Sets} \to \mathsf{Sets}_*, \\
-_1 \odot -_2 &\colon \mathsf{Sets} \times \mathsf{Sets}_* \to \mathsf{Sets}_*.
\end{aligned}$$

   In particular, given:

   - A map of sets $f\colon A \to B$;
   - A pointed map $\phi\colon (X, x_0) \to (Y, y_0)$;

   the induced map

$$f \odot \phi\colon A \odot X \to B \odot Y$$

   is given by

$$[f \odot \phi](a \odot x) \overset{\text{def}}{=} f(a) \odot \phi(x)$$

   for each $a \odot x \in A \odot X$.

2. *Adjointness I.* We have an adjunction

$$(-\odot X \dashv \mathsf{Sets}_*(X,-))\colon \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "-\odot X"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_*, \arrow[l, bend left, "\mathsf{Sets}_*(X,-)"] \end{tikzcd}$$

 witnessed by a bijection

$$\mathsf{Sets}_*(A\odot X, K) \cong \mathsf{Sets}(A, \mathsf{Sets}_*(X,K)),$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$ and $X, Y \in \mathrm{Obj}(\mathsf{Sets}_*)$.

3. *Adjointness II.* We have an adjunctions

$$(A\odot - \dashv A \pitchfork -)\colon \quad \begin{tikzcd} \mathsf{Sets}_* \arrow[r, bend left, "A\odot -"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_*, \arrow[l, bend left, "A\pitchfork -"] \end{tikzcd}$$

 witnessed by a bijection

$$\mathrm{Hom}_{\mathsf{Sets}_*}(A\odot X, Y) \cong \mathrm{Hom}_{\mathsf{Sets}_*}(X, A\pitchfork Y),$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$ and $X, Y \in \mathrm{Obj}(\mathsf{Sets}_*)$.

4. *As a Weighted Colimit.* We have

$$A\odot X \cong \mathrm{colim}^{[A]}(X),$$

where in the right hand side we write:

- A for the functor $A\colon \mathrm{pt} \to \mathsf{Sets}$ picking $A \in \mathrm{Obj}(\mathsf{Sets})$;
- X for the functor $X\colon \mathrm{pt} \to \mathsf{Sets}_*$ picking $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

5. *Iterated Tensors.* We have an isomorphism of pointed sets

$$A\odot(B\odot X) \cong (A\times B)\odot X,$$

natural in $A, B \in \mathrm{Obj}(\mathsf{Sets})$ and $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

6. *Interaction With Homs.* We have a natural isomorphism

$$\mathsf{Sets}_*(A\odot X, -) \cong A \pitchfork \mathsf{Sets}_*(X,-).$$

7. *The Tensor Evaluation Map.* For each $X, Y \in \mathrm{Obj}(\mathsf{Sets}_*)$, we have a map

$$\mathrm{ev}^{\odot}_{X,Y}\colon \mathsf{Sets}_*(X,Y)\odot X \to Y,$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets}_*)$, and given by

$$\mathrm{ev}^{\odot}_{X,Y}(f\odot x) \stackrel{\text{def}}{=} f(x)$$

for each $f\odot x \in \mathsf{Sets}_*(X,Y)\odot X$.

8. *The Tensor Coevaluation Map.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$ and each $X \in \mathrm{Obj}(\mathsf{Sets}_*)$, we have a map

$$\mathrm{coev}^{\odot}_{A,X} \colon A \to \mathsf{Sets}_*(X, A \odot X),$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$ and $X \in \mathrm{Obj}(\mathsf{Sets}_*)$, and given by

$$\mathrm{coev}^{\odot}_{A,X}(a) \stackrel{\text{def}}{=} [\![x \mapsto a \odot x]\!]$$

for each $a \in A$.

Proof. *Item 1*, *Functoriality*: This is the special case of **??** of **??** for when $\mathcal{C} = \mathsf{Sets}_*$.
Item 2, *Adjointness I*: This is simply a rephrasing of Definition 4.2.1.1.1.
Item 3, *: Adjointness II*: This is the special case of **??** of **??** for when $\mathcal{C} = \mathsf{Sets}_*$.
Item 4, *As a Weighted Colimit*: This is the special case of **??** of **??** for when $\mathcal{C} = \mathsf{Sets}_*$.
Item 5, *Iterated Tensors*: This is the special case of **??** of **??** for when $\mathcal{C} = \mathsf{Sets}_*$.
Item 6, *Interaction With Homs*: This is the special case of **??** of **??** for when $\mathcal{C} = \mathsf{Sets}_*$.
Item 7, *The Tensor Evaluation Map*: This is the special case of **??** of **??** for when $\mathcal{C} = \mathsf{Sets}_*$.
Item 8, *The Tensor Coevaluation Map*: This is the special case of **??** of **??** for when $\mathcal{C} = \mathsf{Sets}_*$. □

4.2.2 Cotensors of Pointed Sets by Sets

Let (X, x_0) be a pointed set and let A be a set.

Definition 4.2.2.1.1. The **cotensor of** (X, x_0) **by** A[9] is the pointed set[10] $A \pitchfork (X, x_0)$ satisfying the following universal property:

(UP) We have a bijection

$$\mathsf{Sets}_*(K, A \pitchfork X) \cong \mathsf{Sets}(A, \mathsf{Sets}_*(K, X)),$$

natural in $(K, k_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

Remark 4.2.2.1.2. The universal property of Definition 4.2.2.1.1 is equivalent to the following one:

[9] *Further Terminology:* Also called the **power of** (X, x_0) **by** A.
[10] *Further Notation:* Often written $A \pitchfork X$ for simplicity.

(UP) We have a bijection

$$\mathsf{Sets}_*(K, A \pitchfork X) \cong \mathsf{Sets}^{\otimes}_{\mathbb{E}_0}(A \times K, X),$$

natural in $(K, k_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, where $\mathsf{Sets}^{\otimes}_{\mathbb{E}_0}(A \times K, X)$ is the set defined by

$$\mathsf{Sets}^{\otimes}_{\mathbb{E}_0}(A \times K, X) \stackrel{\text{def}}{=} \left\{ f \in \mathsf{Sets}(A \times K, X) \,\middle|\, \begin{array}{l}\text{for each } a \in A, \text{ we} \\ \text{have } f(a, k_0) = x_0\end{array} \right\}.$$

Proof. This follows from the bijection

$$\mathsf{Sets}(A, \mathsf{Sets}_*(K, X)) \cong \mathsf{Sets}^{\otimes}_{\mathbb{E}_0}(A \times K, X),$$

natural in $(K, k_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$ constructed in the proof of Remark 4.2.1.1.2. □

Construction 4.2.2.1.3. Concretely, the **cotensor of** (X, x_0) **by** A is the pointed set $A \pitchfork (X, x_0)$ consisting of:

- *The Underlying Set.* The set $A \pitchfork X$ given by

$$A \pitchfork X \cong \bigwedge_{a \in A} (X, x_0),$$

  where $\bigwedge_{a \in A}(X, x_0)$ is the smash product of the A-indexed family $((X, x_0))_{a \in A}$ of Definition 4.6.1.1.1.

- *The Basepoint.* The point $[(x_0)_{a \in A}] = [(x_0, x_0, x_0, \dots)]$ of $\bigwedge_{a \in A}(X, x_0)$.

Proof. We claim we have a bijection

$$\mathsf{Sets}_*(K, A \pitchfork X) \cong \mathsf{Sets}(A, \mathsf{Sets}_*(K, X)),$$

natural in $(K, k_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

- *Map I.* We define a map

$$\Phi_K \colon \mathsf{Sets}_*(K, A \pitchfork X) \to \mathsf{Sets}(A, \mathsf{Sets}_*(K, X)),$$

  by sending a morphism of pointed sets

$$\xi \colon (K, k_0) \to (A \pitchfork X, [(x_0)_{a \in A}])$$

  to the map of sets

$$\begin{aligned} \xi^{\dagger} \colon A &\longrightarrow \mathsf{Sets}_*(K, X), \\ a &\longmapsto (\xi_a \colon K \to X), \end{aligned}$$

where
$$\xi_a\colon (K,k_0) \longrightarrow (X,x_0)$$
is the morphism of pointed sets defined by
$$\xi_a(k) = \begin{cases} x^k_a & \text{if } \xi(k) \neq [(x_0)_{a\in A}], \\ x_0 & \text{if } \xi(k) = [(x_0)_{a\in A}] \end{cases}$$
for each $k \in K$, where x^k_a is the ath component of $\xi(k) = \left[(x^k_a)_{a\in A}\right]$. Note that:

1. The definition of $\xi_a(k)$ is independent of the choice of equivalence class. Indeed, suppose we have
$$\begin{aligned} \xi(k) &= \left[\left(x^k_a\right)_{a\in A}\right] \\ &= \left[\left(y^k_a\right)_{a\in A}\right] \end{aligned}$$
with $x^k_a \neq y^k_a$ for some $a \in A$. Then there exist $a_x, a_y \in A$ such that $x^k_{a_x} = y^k_{a_y} = x_0$. The equivalence relation $\sim$ on $\prod_{a\in A} X$ then forces
$$\begin{aligned} \left[\left(x^k_a\right)_{a\in A}\right] &= [(x_0)_{a\in A}], \\ \left[\left(y^k_a\right)_{a\in A}\right] &= [(x_0)_{a\in A}], \end{aligned}$$
however, and $\xi_a(k)$ is defined to be x_0 in this case.
2. The map ξ_a is indeed a morphism of pointed sets, as we have
$$\xi_a(k_0) = x_0$$
since $\xi(k_0) = [(x_0)_{a\in A}]$ as ξ is a morphism of pointed sets and $\xi_a(k_0)$, defined to be the ath component of $[(x_0)_{a\in A}]$, is equal to x_0.

- *Map II.* We define a map
$$\Psi_K\colon \mathsf{Sets}(A, \mathsf{Sets}_*(K,X)) \to \mathsf{Sets}_*(K, A \pitchfork X),$$
given by sending a map
$$\begin{aligned} \xi\colon A &\longrightarrow \mathsf{Sets}_*(K,X), \\ a &\longmapsto (\xi_a\colon K \to X), \end{aligned}$$

to the morphism of pointed sets

$$\xi^\dagger \colon (K, k_0) \to (A \pitchfork X, [(x_0)_{a\in A}])$$

defined by

$$\xi^\dagger(k) \overset{\text{def}}{=} \left[(\xi_a(k))_{a\in A}\right]$$

for each $k \in K$. Note that $\xi^\dagger$ is indeed a morphism of pointed sets, as we have

$$\begin{aligned}\xi^\dagger(k_0) &\overset{\text{def}}{=} \left[(\xi_a(k_0))_{a\in A}\right]\\ &= x_0,\end{aligned}$$

where we have used that $\xi_a \in \mathsf{Sets}_*(K, X)$ is a morphism of pointed sets for each $a \in A$.

- *Naturality of* Ψ. We need to show that, given a morphism of pointed sets

$$\phi \colon (K, k_0) \to (K', k'_0),$$

the diagram

$$\begin{tikzcd}
\mathsf{Sets}(A, \mathsf{Sets}_*(K', X)) \arrow[r, "\Psi_{K'}"] \arrow[d, "(\phi^*)_*"'] & \mathsf{Sets}_*(K', A \pitchfork X) \arrow[d, "\phi^*"] \\
\mathsf{Sets}(A, \mathsf{Sets}_*(K, X)) \arrow[r, "\Psi_K"'] & \mathsf{Sets}_*(K, A \pitchfork X)
\end{tikzcd}$$

commutes. Indeed, given a map of sets

$$\begin{aligned}\xi \colon A &\longrightarrow \mathsf{Sets}_*(K', X),\\ a &\longmapsto (\xi_a \colon K' \to X),\end{aligned}$$

we have

$$\begin{aligned}\left[\Psi_K \circ (\phi^*)_*\right](\xi) &= \Psi_K\big((\phi^*)_*(\xi)\big)\\ &= \Psi_K\big((\phi^*)_*([\![a \mapsto \xi_a]\!])\big)\\ &= \Psi_K\big(([\![a \mapsto \phi^*(\xi_a)]\!])\big)\\ &= \Psi_K\big(([\![a \mapsto [\![k \mapsto \xi_a(\phi(k))]\!]]\!])\big)\\ &= [\![k \mapsto \left[(\xi_a(\phi(k)))_{a\in A}\right]]\!]\\ &= \phi^*\big([\![k' \mapsto \left[(\xi_a(k'))_{a\in A}\right]]\!]\big)\\ &= \phi^*(\Psi_{K'}(\xi))\\ &= [\phi^* \circ \Psi_{K'}](\xi).\end{aligned}$$

- *Naturality of* Φ. Since Ψ is natural and Ψ is a componentwise inverse to Φ, it follows from Item 2 of Proposition 8.8.6.1.2 that Φ is also natural.

- *Invertibility I.* We claim that

$$\Psi_K \circ \Phi_K = \mathrm{id}_{\mathsf{Sets}_*(K, A \pitchfork X)}.$$

Indeed, given a morphism of pointed sets

$$\xi \colon (K, k_0) \to (A \pitchfork X, [(x_0)_{a \in A}])$$

we have

$$\begin{aligned}
[\Psi_K \circ \Phi_K](\xi) &= \Psi_K(\Phi_K(\xi)) \\
&= \Psi_K\big(\llbracket a \mapsto \xi_a \rrbracket\big) \\
&= \Psi_K\big(\llbracket a' \mapsto \xi_{a'} \rrbracket\big) \\
&= \llbracket k \mapsto \big[\big(\mathrm{ev}_a\big(\llbracket a' \mapsto \xi_{a'}(k) \rrbracket\big)\big)_{a \in A}\big] \rrbracket \\
&= \llbracket k \mapsto \big[(\xi_a(k))_{a \in A}\big] \rrbracket.
\end{aligned}$$

Now, we have two cases:

1. If $\xi(k) = [(x_0)_{a \in A}]$, we have

$$\begin{aligned}
[\Psi_K \circ \Phi_K](\xi) &= \cdots \\
&= \llbracket k \mapsto \big[(\xi_a(k))_{a \in A}\big] \rrbracket \\
&= \llbracket k \mapsto [(x_0)_{a \in A}] \rrbracket \\
&= \llbracket k \mapsto \xi(k) \rrbracket \\
&= \xi.
\end{aligned}$$

2. If $\xi(k) \neq [(x_0)_{a \in A}]$ and $\xi(k) = \big[\big(x^k_a\big)_{a \in A}\big]$ instead, we have

$$\begin{aligned}
[\Psi_K \circ \Phi_K](\xi) &= \cdots \\
&= \llbracket k \mapsto \big[(\xi_a(k))_{a \in A}\big] \rrbracket \\
&= \llbracket k \mapsto \left[\left(x^k_a\right)_{a \in A}\right] \rrbracket \\
&= \llbracket k \mapsto \xi(k) \rrbracket \\
&= \xi.
\end{aligned}$$

In both cases, we have $[\Psi_K \circ \Phi_K](\xi) = \xi$, and thus we are done.

- *Invertibility II.* We claim that

$$\Phi_K \circ \Psi_K = \mathrm{id}_{\mathsf{Sets}(A,\mathsf{Sets}_*(K,X))}.$$

Indeed, given a morphism $\xi \colon A \to \mathsf{Sets}_*(K,X)$, we have

$$\begin{aligned}
[\Phi_K \circ \Psi_K](\xi) &= \Phi_K(\Psi_K(\xi)) \\
&= \Phi_K\left(\llbracket k \mapsto \left[(\xi_a(k))_{a\in A}\right]\rrbracket\right) \\
&= \llbracket a \mapsto \llbracket k \mapsto \xi_a(k)\rrbracket\rrbracket \\
&= \xi
\end{aligned}$$

This finishes the proof. □

Proposition 4.2.2.1.4. Let (X,x_0) be a pointed set and let A be a set.

1. *Functoriality.* The assignments A, (X,x_0), $(A,(X,x_0))$ define functors

$$\begin{aligned}
A \pitchfork - &\colon \mathsf{Sets}_* \to \mathsf{Sets}_*, \\
- \pitchfork X &\colon \mathsf{Sets}^{\mathsf{op}} \to \mathsf{Sets}_*, \\
-_1 \pitchfork -_2 &\colon \mathsf{Sets}^{\mathsf{op}} \times \mathsf{Sets}_* \to \mathsf{Sets}_*.
\end{aligned}$$

In particular, given:

- A map of sets $f\colon A \to B$;
- A pointed map $\phi\colon (X,x_0) \to (Y,y_0)$;

the induced map

$$f \odot \phi \colon A \pitchfork X \to B \pitchfork Y$$

is given by

$$[f \odot \phi]([(x_a)_{a\in A}]) \stackrel{\text{def}}{=} \left[\left(\phi\left(x_{f(a)}\right)\right)_{a\in A}\right]$$

for each $[(x_a)_{a\in A}] \in A \pitchfork X$.

2. *Adjointness I.* We have an adjunction

$$\left(- \pitchfork X \dashv \mathsf{Sets}_*(-,X)\right)\colon \quad \begin{tikzcd} \mathsf{Sets}^{\mathsf{op}} \arrow[r, bend left, "- \pitchfork X"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_*, \arrow[l, bend left, "\mathsf{Sets}_*(-,X)"] \end{tikzcd}$$

witnessed by a bijection

$$\mathsf{Sets}_*^{\mathsf{op}}(A \pitchfork X, K) \cong \mathsf{Sets}(A, \mathsf{Sets}_*(K,X)),$$

i.e. by a bijection

$$\mathsf{Sets}_*(K, A \pitchfork X) \cong \mathsf{Sets}(A, \mathsf{Sets}_*(K,X)),$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$ and $X, Y \in \mathrm{Obj}(\mathsf{Sets}_*)$.

3. *Adjointness II.* We have an adjunctions

$$(A \odot - \dashv A \pitchfork -)\colon \quad \begin{tikzcd} \mathsf{Sets}_* \arrow[r, bend left, "A\odot -"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_*, \arrow[l, bend left, "A\pitchfork -"] \end{tikzcd}$$

   witnessed by a bijection

$$\mathrm{Hom}_{\mathsf{Sets}_*}(A \odot X, Y) \cong \mathrm{Hom}_{\mathsf{Sets}_*}(X, A \pitchfork Y),$$

   natural in $A \in \mathrm{Obj}(\mathsf{Sets})$ and $X, Y \in \mathrm{Obj}(\mathsf{Sets}_*)$.

4. *As a Weighted Limit.* We have

$$A \pitchfork X \cong \lim^{[A]}(X),$$

   where in the right hand side we write:

   - A for the functor $A\colon \mathrm{pt} \to \mathsf{Sets}$ picking $A \in \mathrm{Obj}(\mathsf{Sets})$;
   - X for the functor $X\colon \mathrm{pt} \to \mathsf{Sets}_*$ picking $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

5. *Iterated Cotensors.* We have an isomorphism of pointed sets

$$A \pitchfork (B \pitchfork X) \cong (A \times B) \pitchfork X,$$

   natural in $A, B \in \mathrm{Obj}(\mathsf{Sets})$ and $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

6. *Commutativity With Homs.* We have natural isomorphisms

$$A \pitchfork \mathsf{Sets}_*(X, -) \cong \mathsf{Sets}_*(A \odot X, -),$$
$$A \pitchfork \mathsf{Sets}_*(-, Y) \cong \mathsf{Sets}_*(-, A \pitchfork Y).$$

7. *The Cotensor Evaluation Map.* For each $X, Y \in \mathrm{Obj}(\mathsf{Sets}_*)$, we have a map

$$\mathrm{ev}^{\pitchfork}_{X,Y}\colon X \to \mathsf{Sets}_*(X, Y) \pitchfork Y,$$

   natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets}_*)$, and given by

$$\mathrm{ev}^{\pitchfork}_{X,Y}(x) \stackrel{\text{def}}{=} \left[(f(x))_{f \in \mathsf{Sets}_*(X,Y)}\right]$$

   for each $x \in X$.

8. *The Cotensor Coevaluation Map.* For each $X \in \mathrm{Obj}(\mathsf{Sets}_*)$ and each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have a map

$$\mathrm{coev}^{\pitchfork}_{A,X}\colon A \to \mathsf{Sets}_*(A \pitchfork X, X),$$

   natural in $X \in \mathrm{Obj}(\mathsf{Sets}_*)$ and $A \in \mathrm{Obj}(\mathsf{Sets})$, and given by

$$\mathrm{coev}^{\pitchfork}_{A,X}(a) \stackrel{\text{def}}{=} [\![[(x_b)_{b \in A}] \mapsto x_a]\!]$$

   for each $a \in A$.

Proof. *Item 1*, *Functoriality*: This is the special case of **??** of **??** for when $C = \mathsf{Sets}_*$.
Item 2, *Adjointness I*: This is simply a rephrasing of Definition 4.2.2.1.1.
Item 3, *: Adjointness II*: This is the special case of **??** of **??** for when $C = \mathsf{Sets}_*$.
Item 4, *As a Weighted Limit*: This is the special case of **??** of **??** for when $C = \mathsf{Sets}_*$.
Item 5, *Iterated Cotensors*: This is the special case of **??** of **??** for when $C = \mathsf{Sets}_*$.
Item 6, *Commutativity With Homs*: This is the special case of **??** of **??** for when $C = \mathsf{Sets}_*$.
Item 7, *The Cotensor Evaluation Map*: This is the special case of **??** of **??** for when $C = \mathsf{Sets}_*$.
Item 8, *The Cotensor Coevaluation Map*: This is the special case of **??** of **??** for when $C = \mathsf{Sets}_*$. □

4.3 The Left Tensor Product of Pointed Sets

4.3.1 Foundations

Let (X, x_0) and (Y, y_0) be pointed sets.

Definition 4.3.1.1.1. The **left tensor product of pointed sets** is the functor[11]

$$\lhd\colon \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*$$

defined as the composition

$$\mathsf{Sets}_* \times \mathsf{Sets}_* \xrightarrow{\mathrm{id}\times 忘} \mathsf{Sets}_* \times \mathsf{Sets} \xrightarrow{\beta^{\mathsf{Cats}_2}_{\mathsf{Sets}_*,\mathsf{Sets}}} \mathsf{Sets} \times \mathsf{Sets}_* \xrightarrow{\odot} \mathsf{Sets}_*,$$

where:

- $忘\colon \mathsf{Sets}_* \to \mathsf{Sets}$ is the forgetful functor from pointed sets to sets.
- $\beta^{\mathsf{Cats}_2}_{\mathsf{Sets}_*,\mathsf{Sets}}\colon \mathsf{Sets}_* \times \mathsf{Sets} \xrightarrow{\cong} \mathsf{Sets} \times \mathsf{Sets}_*$ is the braiding of Cats_2, i.e. the functor witnessing the isomorphism

$$\mathsf{Sets}_* \times \mathsf{Sets} \cong \mathsf{Sets} \times \mathsf{Sets}_*.$$

- $\odot\colon \mathsf{Sets} \times \mathsf{Sets}_* \to \mathsf{Sets}_*$ is the tensor functor of Item 1 of Proposition 4.2.1.1.6.

[11] *Further Notation:* Also written $\lhd_{\mathsf{Sets}_*}$.

Remark 4.3.1.1.2. The left tensor product of pointed sets satisfies the following natural bijection:

$$\mathsf{Sets}_*(X \lhd Y, Z) \cong \mathrm{Hom}^{\otimes,\mathrm{L}}_{\mathsf{Sets}_*}(X \times Y, Z).$$

That is to say, the following data are in natural bijection:

1. Pointed maps $f \colon X \lhd Y \to Z$.
2. Maps of sets $f \colon X \times Y \to Z$ satisfying $f(x_0, y) = z_0$ for each $y \in Y$.

Remark 4.3.1.1.3. The left tensor product of pointed sets may be described as follows:

- The left tensor product of (X, x_0) and (Y, y_0) is the pair $((X \lhd Y, x_0 \lhd y_0), \iota)$ consisting of
  - A pointed set $(X \lhd Y, x_0 \lhd y_0)$;
  - A left bilinear morphism of pointed sets $\iota \colon (X \times Y, (x_0, y_0)) \to X \lhd Y$;

  satisfying the following universal property:

  (UP) Given another such pair $((Z, z_0), f)$ consisting of
    * A pointed set (Z, z_0);
    * A left bilinear morphism of pointed sets $f \colon (X \times Y, (x_0, y_0)) \to X \lhd Y$;

  there exists a unique morphism of pointed sets $X \lhd Y \xrightarrow{\exists!} Z$ making the diagram

$$\begin{tikzcd}
& X \lhd Y \arrow[d, dashed, "\exists!"] \\
X \times Y \arrow[ur, "\iota"] \arrow[r, "f"'] & Z
\end{tikzcd}$$

  commute.

Construction 4.3.1.1.4. In detail, the **left tensor product of** (X, x_0) **and** (Y, y_0) is the pointed set $(X \lhd Y, [x_0])$ consisting of

- *The Underlying Set.* The set $X \lhd Y$ defined by

$$\begin{aligned} X \lhd Y &\stackrel{\text{def}}{=} |Y| \odot X \\ &\cong \bigvee_{y \in Y} (X, x_0), \end{aligned}$$

  where $|Y|$ denotes the underlying set of (Y, y_0);

- *The Underlying Basepoint.* The point $[(y_0, x_0)]$ of $\bigvee_{y \in Y}(X, x_0)$, which is equal to $[(y, x_0)]$ for any $y \in Y$.

Notation 4.3.1.1.5. We write[12] $x \lhd y$ for the element $[(y, x)]$ of

$$X \lhd Y \cong |Y| \odot X.$$

Remark 4.3.1.1.6. Employing the notation introduced in Notation 4.3.1.1.5, we have

$$x_0 \lhd y_0 = x_0 \lhd y$$

for each $y \in Y$, and

$$x_0 \lhd y = x_0 \lhd y'$$

for each $y, y' \in Y$.

Proposition 4.3.1.1.7. Let (X, x_0) and (Y, y_0) be pointed sets.

1. *Functoriality.* The assignments $X, Y, (X, Y) \mapsto X \lhd Y$ define functors

$$\begin{aligned} X \lhd - \colon \mathsf{Sets}_* &\to \mathsf{Sets}_*, \\ - \lhd Y \colon \mathsf{Sets}_* &\to \mathsf{Sets}_*, \\ -_1 \lhd -_2 \colon \mathsf{Sets}_* \times \mathsf{Sets}_* &\to \mathsf{Sets}_*. \end{aligned}$$

   In particular, given pointed maps

$$\begin{aligned} f \colon (X, x_0) &\to (A, a_0), \\ g \colon (Y, y_0) &\to (B, b_0), \end{aligned}$$

   the induced map

$$f \lhd g \colon X \lhd Y \to A \lhd B$$

   is given by

$$[f \lhd g](x \lhd y) \overset{\text{def}}{=} f(x) \lhd g(y)$$

   for each $x \lhd y \in X \lhd Y$.

2. *Adjointness I.* We have an adjunction

$$\left(- \lhd Y \dashv [Y, -]^{\lhd}_{\mathsf{Sets}_*}\right) \colon \quad \begin{tikzcd} \mathsf{Sets}_* \arrow[r, bend left, "- \lhd Y"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_* \arrow[l, bend left, "{[Y,-]^{\lhd}_{\mathsf{Sets}_*}}"] \end{tikzcd},$$

[12] *Further Notation:* Also written $x \lhd_{\mathsf{Sets}_*} y$.

witnessed by a bijection of sets

$$\mathrm{Hom}_{\mathsf{Sets}_*}(X \lhd Y, Z) \cong \mathrm{Hom}_{\mathsf{Sets}_*}\left(X, [Y,Z]^{\lhd}_{\mathsf{Sets}_*}\right)$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, where $[X,Y]^{\lhd}_{\mathsf{Sets}_*}$ is the pointed set of Definition 4.3.2.1.1.

3. *Adjointness II*. The functor

$$X \lhd -\colon \mathsf{Sets}_* \to \mathsf{Sets}_*$$

does not admit a right adjoint.

4. *Adjointness III*. We have a bijection of sets

$$\mathrm{Hom}_{\mathsf{Sets}_*}(X \lhd Y, Z) \cong \mathrm{Hom}_{\mathsf{Sets}}(|Y|, \mathsf{Sets}_*(X, Z))$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

Proof. Item 1, Functoriality: Clear.
Item 2, Adjointness I: This follows from Item 3 of Proposition 4.2.1.1.6.
Item 3, Adjointness II: For $X \lhd -$ to admit a right adjoint would require it to preserve colimits by **??** of **??**. However, we have

$$\begin{aligned} X \lhd \mathrm{pt} &\stackrel{\text{def}}{=} |\mathrm{pt}| \odot X \\ &\cong X \\ &\ncong \mathrm{pt}, \end{aligned}$$

and thus we see that $X \lhd -$ does not have a right adjoint.
Item 4, Adjointness III: This follows from Item 2 of Proposition 4.2.1.1.6. □

Remark 4.3.1.1.8. Here is some intuition on why $X \lhd -$ fails to be a left adjoint. Item 4 of Proposition 4.3.1.1.7 states that we have a natural bijection

$$\mathrm{Hom}_{\mathsf{Sets}_*}(X \lhd Y, Z) \cong \mathrm{Hom}_{\mathsf{Sets}}(|Y|, \mathsf{Sets}_*(X, Z)),$$

so it would be reasonable to wonder whether a natural bijection of the form

$$\mathrm{Hom}_{\mathsf{Sets}_*}(X \lhd Y, Z) \cong \mathrm{Hom}_{\mathsf{Sets}_*}(Y, \mathbf{Sets}_*(X, Z)),$$

also holds, which would give $X \lhd - \dashv \mathbf{Sets}_*(X, -)$. However, such a bijection would require every map

$$f\colon X \lhd Y \to Z$$

to satisfy

$$f(x \lhd y_0) = z_0$$

for each $x \in X$, whereas we are imposing such a basepoint preservation condition only for elements of the form $x_0 \lhd y$. Thus $\mathbf{Sets}_*(X, -)$ can't be a right adjoint for $X \lhd -$, and as shown by Item 3 of Proposition 4.3.1.1.7, no functor can.[13]

4.3.2 The Left Internal Hom of Pointed Sets

Let (X, x_0) and (Y, y_0) be pointed sets.

Definition 4.3.2.1.1. The **left internal Hom of pointed sets** is the functor

$$[-,-]^{\lhd}_{\mathsf{Sets}_*} \colon \mathsf{Sets}_*^{\mathsf{op}} \times \mathsf{Sets}_* \to \mathsf{Sets}_*$$

defined as the composition

$$\mathsf{Sets}_*^{\mathsf{op}} \times \mathsf{Sets}_* \xrightarrow{忘\times\mathrm{id}} \mathsf{Sets}^{\mathsf{op}} \times \mathsf{Sets}_* \xrightarrow{\pitchfork} \mathsf{Sets}_*,$$

where:

- $忘\colon \mathsf{Sets}_* \to \mathsf{Sets}$ is the forgetful functor from pointed sets to sets.
- $\pitchfork\colon \mathsf{Sets}^{\mathsf{op}} \times \mathsf{Sets}_* \to \mathsf{Sets}_*$ is the cotensor functor of Item 1 of Proposition 4.2.2.1.4.

Proof. For a proof that $[-,-]^{\lhd}_{\mathsf{Sets}_*}$ is indeed the left internal Hom of Sets_* with respect to the left tensor product of pointed sets, see Item 2 of Proposition 4.3.1.1.7. □

Remark 4.3.2.1.2. The left internal Hom of pointed sets satisfies the following universal property:

$$\mathsf{Sets}_*(X \lhd Y, Z) \cong \mathsf{Sets}_*\left(X, [Y,Z]^{\lhd}_{\mathsf{Sets}_*}\right)$$

That is to say, the following data are in bijection:

1. Pointed maps $f\colon X \lhd Y \to Z$.
2. Pointed maps $f\colon X \to [Y,Z]^{\lhd}_{\mathsf{Sets}_*}$.

Remark 4.3.2.1.3. In detail, the **left internal Hom of** (X, x_0) **and** (Y, y_0) is the pointed set $\left([X,Y]^{\lhd}_{\mathsf{Sets}_*}, \left[(y_0)_{x\in X}\right]\right)$ consisting of

[13]The functor $\mathbf{Sets}_*(X, -)$ is instead right adjoint to $X \wedge -$, the smash product of pointed

- *The Underlying Set.* The set $[X, Y]^{\lhd}_{\mathsf{Sets}_*}$ defined by

$$\begin{aligned}[X, Y]^{\lhd}_{\mathsf{Sets}_*} &\overset{\text{def}}{=} |X| \pitchfork Y\\ &\cong \bigwedge_{x\in X}(Y, y_0),\end{aligned}$$

where $|X|$ denotes the underlying set of (X, x_0);

- *The Underlying Basepoint.* The point $\left[(y_0)_{x\in X}\right]$ of $\bigwedge_{x\in X}(Y, y_0)$.

Proposition 4.3.2.1.4. Let (X, x_0) and (Y, y_0) be pointed sets.

1. *Functoriality.* The assignments $X, Y, (X, Y) \mapsto [X, Y]^{\lhd}_{\mathsf{Sets}_*}$ define functors

$$\begin{aligned}[X, -]^{\lhd}_{\mathsf{Sets}_*} &\colon \mathsf{Sets}_* \to \mathsf{Sets}_*,\\ [-, Y]^{\lhd}_{\mathsf{Sets}_*} &\colon \mathsf{Sets}_*^{\mathsf{op}} \to \mathsf{Sets}_*,\\ [-_1, -_2]^{\lhd}_{\mathsf{Sets}_*} &\colon \mathsf{Sets}_*^{\mathsf{op}} \times \mathsf{Sets}_* \to \mathsf{Sets}_*.\end{aligned}$$

In particular, given pointed maps

$$\begin{aligned}f &\colon (X, x_0) \to (A, a_0),\\ g &\colon (Y, y_0) \to (B, b_0),\end{aligned}$$

the induced map

$$[f, g]^{\lhd}_{\mathsf{Sets}_*} \colon [A, Y]^{\lhd}_{\mathsf{Sets}_*} \to [X, B]^{\lhd}_{\mathsf{Sets}_*}$$

is given by

$$[f, g]^{\lhd}_{\mathsf{Sets}_*}\left(\left[(y_a)_{a\in A}\right]\right) \overset{\text{def}}{=} \left[\left(g\left(y_{f(x)}\right)\right)_{x\in X}\right]$$

for each $\left[(y_a)_{a\in A}\right] \in [A, Y]^{\lhd}_{\mathsf{Sets}_*}$.

2. *Adjointness I.* We have an adjunction

$$\left(- \lhd Y \dashv [Y, -]^{\lhd}_{\mathsf{Sets}_*}\right)\colon \quad \begin{tikzcd} \mathsf{Sets}_* \arrow[r, bend left, "- \lhd Y"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_*, \arrow[l, bend left, "{[Y, -]^{\lhd}_{\mathsf{Sets}_*}}"] \end{tikzcd}$$

witnessed by a bijection of sets

$$\mathrm{Hom}_{\mathsf{Sets}_*}(X \lhd Y, Z) \cong \mathrm{Hom}_{\mathsf{Sets}_*}\left(X, [Y, Z]^{\lhd}_{\mathsf{Sets}_*}\right)$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$

3. *Adjointness II.* The functor

$$X \lhd -\colon \mathsf{Sets}_* \longrightarrow \mathsf{Sets}_*$$

does not admit a right adjoint.

Proof. *Item 1*, *Functoriality*: Clear.
Item 2, *Adjointness I*: This is a repetition of Item 2 of Proposition 4.3.1.1.7, and is proved there.
Item 3, *Adjointness II*: This is a repetition of Item 3 of Proposition 4.3.1.1.7, and is proved there. □

4.3.3 The Left Skew Unit

Definition 4.3.3.1.1. The **left skew unit of the left tensor product of pointed sets** is the functor

$$\mathbb{1}^{\mathsf{Sets}_*,\lhd}\colon \mathsf{pt} \longrightarrow \mathsf{Sets}_*$$

defined by

$$\mathbb{1}^{\lhd}_{\mathsf{Sets}_*} \stackrel{\text{def}}{=} S^0.$$

4.3.4 The Left Skew Associator

Definition 4.3.4.1.1. The **skew associator of the left tensor product of pointed sets** is the natural transformation

$$\alpha^{\mathsf{Sets}_*,\lhd}\colon \lhd \circ \left(\lhd \times \mathrm{id}_{\mathsf{Sets}_*}\right) \Longrightarrow \lhd \circ \left(\mathrm{id}_{\mathsf{Sets}_*} \times \lhd\right) \circ \boldsymbol{\alpha}^{\mathsf{Cats}}_{\mathsf{Sets}_*,\mathsf{Sets}_*,\mathsf{Sets}_*}$$

as in the diagram

$$\begin{tikzcd}
 & \mathsf{Sets}_* \times (\mathsf{Sets}_* \times \mathsf{Sets}_*) \arrow[dr, "\mathrm{id}\times\lhd"] & \\
(\mathsf{Sets}_* \times \mathsf{Sets}_*) \times \mathsf{Sets}_* \arrow[ur, dashed, "\boldsymbol{\alpha}^{\mathsf{Cats}}_{\mathsf{Sets}_*,\mathsf{Sets}_*,\mathsf{Sets}_*}", "\sim"'] \arrow[dr, "\lhd\times\mathrm{id}"'] & & \mathsf{Sets}_* \times \mathsf{Sets}_* \arrow[d, "\lhd"] \\
 & \mathsf{Sets}_* \times \mathsf{Sets}_* \arrow[r, "\lhd"'] \arrow[ur, Rightarrow, "\alpha^{\mathsf{Sets}_*,\lhd}"] & \mathsf{Sets}_*,
\end{tikzcd}$$

whose component

$$\alpha^{\mathsf{Sets}_*,\lhd}_{X,Y,Z}\colon (X \lhd Y) \lhd Z \to X \lhd (Y \lhd Z)$$

at $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$ is given by

$$\begin{aligned}
(X \lhd Y) \lhd Z &\stackrel{\text{def}}{=} |Z| \odot (X \lhd Y)\\
&\stackrel{\text{def}}{=} |Z| \odot (|Y| \odot X)\\
&\cong \bigvee_{z\in Z} |Y| \odot X\\
&\cong \bigvee_{z\in Z}\left(\bigvee_{y\in Y} X\right)\\
&\to \bigvee_{[(z,y)]\in\bigvee_{z\in Z} Y} X\\
&\cong \bigvee_{[(z,y)]\in|Z|\odot Y} X\\
&\cong ||Z| \odot Y| \odot X\\
&\stackrel{\text{def}}{=} |Y \lhd Z| \odot X\\
&\stackrel{\text{def}}{=} X \lhd (Y \lhd Z),
\end{aligned}$$

where the map

$$\bigvee_{z\in Z}\left(\bigvee_{y\in Y} X\right) \to \bigvee_{(z,y)\in\bigvee_{z\in Z} Y} X$$

is given by $[(z, [(y, x)])] \mapsto [([(z, y)], x)]$.

Proof. (Proven below in a bit.) □

Remark 4.3.4.1.2. Unwinding the notation for elements, we have

$$\begin{aligned}
[(z, [(y, x)])] &\stackrel{\text{def}}{=} [(z, x \lhd y)]\\
&\stackrel{\text{def}}{=} (x \lhd y) \lhd z
\end{aligned}$$

and

$$\begin{aligned}
[([(z, y)], x)] &\stackrel{\text{def}}{=} [(y \lhd z, x)]\\
&\stackrel{\text{def}}{=} x \lhd (y \lhd z).
\end{aligned}$$

So, in other words, $\alpha^{\mathsf{Sets}_*,\lhd}_{X,Y,Z}$ acts on elements via

$$\alpha^{\mathsf{Sets}_*,\lhd}_{X,Y,Z}((x \lhd y) \lhd z) \stackrel{\text{def}}{=} x \lhd (y \lhd z)$$

for each $(x \lhd y) \lhd z \in (X \lhd Y) \lhd Z$.

Remark 4.3.4.1.3. Taking $y = y_0$, we see that the morphism $\alpha^{\mathsf{Sets}_*,\lhd}_{X,Y,Z}$ acts on elements as

$$\alpha^{\mathsf{Sets}_*,\lhd}_{X,Y,Z}((x \lhd y_0) \lhd z) \stackrel{\text{def}}{=} x \lhd (y_0 \lhd z).$$

However, by the definition of $\lhd$, we have $y_0 \lhd z = y_0 \lhd z'$ for all $z, z' \in Z$, preventing $\alpha^{\mathsf{Sets}_*,\lhd}_{X,Y,Z}$ from being non-invertible.

Proof. Firstly, note that, given $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, the map

$$\alpha^{\mathsf{Sets}_*,\lhd}_{X,Y,Z} : (X \lhd Y) \lhd Z \to X \lhd (Y \lhd Z)$$

is indeed a morphism of pointed sets, as we have

$$\alpha^{\mathsf{Sets}_*,\lhd}_{X,Y,Z}((x_0 \lhd y_0) \lhd z_0) = x_0 \lhd (y_0 \lhd z_0).$$

Next, we claim that $\alpha^{\mathsf{Sets}_*,\lhd}$ is a natural transformation. We need to show that, given morphisms of pointed sets

$$\begin{aligned} f &: (X, x_0) \to (X', x'_0), \\ g &: (Y, y_0) \to (Y', y'_0), \\ h &: (Z, z_0) \to (Z', z'_0) \end{aligned}$$

the diagram

$$\begin{tikzcd} (X \lhd Y) \lhd Z \arrow[r, "(f \lhd g) \lhd h"] \arrow[d, "\alpha^{\mathsf{Sets}_*,\lhd}_{X,Y,Z}"'] & (X' \lhd Y') \lhd Z' \arrow[d, "\alpha^{\mathsf{Sets}_*,\lhd}_{X',Y',Z'}"] \\ X \lhd (Y \lhd Z) \arrow[r, "f \lhd (g \lhd h)"'] & X' \lhd (Y' \lhd Z') \end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd} (x \lhd y) \lhd z \arrow[r, mapsto] \arrow[d, mapsto] & (f(x) \lhd g(y)) \lhd h(z) \arrow[d, mapsto] \\ x \lhd (y \lhd z) \arrow[r, mapsto] & f(x) \lhd (g(y) \lhd h(z)) \end{tikzcd}$$

and hence indeed commutes, showing $\alpha^{\mathsf{Sets}_*,\lhd}$ to be a natural transformation. This finishes the proof. □

sets of Definition 4.5.1.1.1. See Item 2 of Proposition 4.5.1.1.9.

4.3.5 The Left Skew Left Unitor

Definition 4.3.5.1.1. The **skew left unitor of the left tensor product of pointed sets** is the natural transformation

$$\lambda^{\mathsf{Sets}_*,\lhd}\colon \lhd\circ\left(\mathbb{1}^{\mathsf{Sets}_*}\times\mathrm{id}_{\mathsf{Sets}_*}\right)\xRightarrow{\sim}\boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}\qquad
\begin{tikzcd}
\mathrm{pt}\times\mathsf{Sets}_* \arrow[r, "\mathbb{1}^{\mathsf{Sets}_*}\times\mathrm{id}"] \arrow[dr, dashed, bend right, "\boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}"'] & \mathsf{Sets}_*\times\mathsf{Sets}_* \arrow[d, "\lhd"] \arrow[dl, phantom, "\Swarrow\lambda^{\mathsf{Sets}_*,\lhd}"] \\
 & \mathsf{Sets}_*,
\end{tikzcd}$$

whose component

$$\lambda^{\mathsf{Sets}_*,\lhd}_X\colon S^0\lhd X\to X$$

at $(X,x_0)\in\mathrm{Obj}(\mathsf{Sets}_*)$ is given by the composition

$$\begin{aligned}
S^0\lhd X &\cong |X|\odot S^0\\
&\cong \bigvee_{x\in X} S^0\\
&\to X,
\end{aligned}$$

where $\bigvee_{x\in X} S^0\to X$ is the map given by

$$\begin{aligned}
[(x,0)] &\mapsto x_0,\\
[(x,1)] &\mapsto x.
\end{aligned}$$

Proof. (Proven below in a bit.) □

Remark 4.3.5.1.2. In other words, $\lambda^{\mathsf{Sets}_*,\lhd}_X$ acts on elements as

$$\begin{aligned}
\lambda^{\mathsf{Sets}_*,\lhd}_X(0\lhd x) &\overset{\text{def}}{=} x_0,\\
\lambda^{\mathsf{Sets}_*,\lhd}_X(1\lhd x) &\overset{\text{def}}{=} x
\end{aligned}$$

for each $1\lhd x\in S^0\lhd X$.

Remark 4.3.5.1.3. The morphism $\lambda^{\mathsf{Sets}_*,\lhd}_X$ is almost invertible, with its would-be-inverse

$$\phi_X\colon X\to S^0\lhd X$$

given by

$$\phi_X(x)\overset{\text{def}}{=}1\lhd x$$

for each $x \in X$. Indeed, we have

$$\begin{aligned}\left[\lambda_X^{\mathsf{Sets}_*,\lhd} \circ \phi\right](x) &= \lambda_X^{\mathsf{Sets}_*,\lhd}(\phi(x)) \\ &= \lambda_X^{\mathsf{Sets}_*,\lhd}(1 \lhd x) \\ &= x \\ &= [\mathrm{id}_X](x)\end{aligned}$$

so that

$$\lambda_X^{\mathsf{Sets}_*,\lhd} \circ \phi = \mathrm{id}_X$$

and

$$\begin{aligned}\left[\phi \circ \lambda_X^{\mathsf{Sets}_*,\lhd}\right](1 \lhd x) &= \phi\Big(\lambda_X^{\mathsf{Sets}_*,\lhd}(1 \lhd x)\Big) \\ &= \phi(x) \\ &= 1 \lhd x \\ &= [\mathrm{id}_{S^0 \lhd X}](1 \lhd x),\end{aligned}$$

but

$$\begin{aligned}\left[\phi \circ \lambda_X^{\mathsf{Sets}_*,\lhd}\right](0 \lhd x) &= \phi\Big(\lambda_X^{\mathsf{Sets}_*,\lhd}(0 \lhd x)\Big) \\ &= \phi(x_0) \\ &= 1 \lhd x_0,\end{aligned}$$

where $0 \lhd x \neq 1 \lhd x_0$. Thus

$$\phi \circ \lambda_X^{\mathsf{Sets}_*,\lhd} \stackrel{?}{=} \mathrm{id}_{S^0 \lhd X}$$

holds for all elements in $S^0 \lhd X$ except one.

Proof. Firstly, note that, given $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, the map

$$\lambda_X^{\mathsf{Sets}_*,\lhd} \colon S^0 \lhd X \to X$$

is indeed a morphism of pointed sets, as we have

$$\lambda_X^{\mathsf{Sets}_*,\lhd}(0 \lhd x_0) = x_0.$$

Next, we claim that $\lambda^{\mathsf{Sets}_*,\lhd}$ is a natural transformation. We need to show that, given a morphism of pointed sets

$$f \colon (X, x_0) \to (Y, y_0),$$

the diagram

$$
\begin{tikzcd}
S^0 \lhd X \arrow[r, "\mathrm{id}_{S^0} \lhd f"] \arrow[d, "\lambda^{\mathsf{Sets}_*,\lhd}_X"'] & S^0 \lhd Y \arrow[d, "\lambda^{\mathsf{Sets}_*,\lhd}_Y"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}
$$

commutes. Indeed, this diagram acts on elements as

$$
\begin{tikzcd}
0 \lhd x \arrow[d, mapsto] & & 0 \lhd x \arrow[r, mapsto] & 0 \lhd f(x) \arrow[d, mapsto] \\
x_0 \arrow[r, mapsto] & f(x_0) & & y_0
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
1 \lhd x \arrow[r, mapsto] \arrow[d, mapsto] & 1 \lhd f(x) \arrow[d, mapsto] \\
x \arrow[r, mapsto] & f(x)
\end{tikzcd}
$$

and hence indeed commutes, showing $\lambda^{\mathsf{Sets}_*,\lhd}$ to be a natural transformation. This finishes the proof. □

4.3.6 The Left Skew Right Unitor

Definition 4.3.6.1.1. The **skew right unitor of the left tensor product of pointed sets** is the natural transformation

$$
\rho^{\mathsf{Sets}_*,\lhd}\colon \boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*} \overset{\sim}{\Longrightarrow} \lhd \circ \left(\mathrm{id} \times \mathbb{1}^{\mathsf{Sets}_*}\right),
\qquad
\begin{tikzcd}
\mathsf{Sets}_* \times \mathrm{pt} \arrow[r, "\mathrm{id}\times\mathbb{1}^{\mathsf{Sets}_*}"] \arrow[dr, dashed, bend right, "\boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}"'] \arrow[dr, phantom, bend left=10, "\Nearrow \rho^{\mathsf{Sets}_*,\lhd}"] & \mathsf{Sets}_* \times \mathsf{Sets}_* \arrow[d, "\lhd"] \\
& \mathsf{Sets}_*,
\end{tikzcd}
$$

whose component

$$
\rho^{\mathsf{Sets}_*,\lhd}_X \colon X \to X \lhd S^0
$$

at $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$ is given by the composition

$$\begin{aligned} X &\to X \vee X \\ &\cong |S^0| \odot X \\ &\cong X \lhd S^0, \end{aligned}$$

where $X \to X \vee X$ is the map sending X to the second factor of X in $X \vee X$.

Proof. (Proven below in a bit.) □

Remark 4.3.6.1.2. In other words, $\rho^{\mathsf{Sets}_*,\lhd}_X$ acts on elements as

$$\rho^{\mathsf{Sets}_*,\lhd}_X(x) \stackrel{\text{def}}{=} [(1, x)]$$

i.e. by

$$\rho^{\mathsf{Sets}_*,\lhd}_X(x) \stackrel{\text{def}}{=} x \lhd 1$$

for each $x \in X$.

Remark 4.3.6.1.3. The morphism $\rho^{\mathsf{Sets}_*,\lhd}_X$ is non-invertible, as it is non-surjective when viewed as a map of sets, since the elements $x \lhd 0$ of $X \lhd S^0$ with $x \neq x_0$ are outside the image of $\rho^{\mathsf{Sets}_*,\lhd}_X$, which sends x to $x \lhd 1$.

Proof. Firstly, note that, given $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, the map

$$\rho^{\mathsf{Sets}_*,\lhd}_X \colon X \to X \lhd S^0$$

is indeed a morphism of pointed sets as we have

$$\begin{aligned} \rho^{\mathsf{Sets}_*,\lhd}_X(x_0) &= x_0 \lhd 1 \\ &= x_0 \lhd 0. \end{aligned}$$

Next, we claim that $\rho^{\mathsf{Sets}_*,\lhd}$ is a natural transformation. We need to show that, given a morphism of pointed sets

$$f \colon (X, x_0) \to (Y, y_0),$$

the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\rho^{\mathsf{Sets}_*,\lhd}_X"'] & Y \arrow[d, "\rho^{\mathsf{Sets}_*,\lhd}_Y"] \\ X \lhd S^0 \arrow[r, "f \lhd \mathrm{id}_{S^0}"'] & Y \lhd S^0 \end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
x \arrow[r, mapsto] \arrow[d, mapsto] & f(x) \arrow[d, mapsto] \\
x \lhd 0 \arrow[r, mapsto] & f(x) \lhd 0
\end{tikzcd}$$

and hence indeed commutes, showing $\rho^{\mathsf{Sets}_*,\lhd}$ to be a natural transformation. This finishes the proof. □

4.3.7 The Diagonal

Definition 4.3.7.1.1. The **diagonal of the left tensor product of pointed sets** is the natural transformation

$$\Delta^{\lhd} \colon \mathrm{id}_{\mathsf{Sets}_*} \Longrightarrow \lhd \circ \Delta^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}, \qquad \begin{tikzcd}
\mathsf{Sets}_* \arrow[rr, "\mathrm{id}_{\mathsf{Sets}_*}"] \arrow[dr, "\Delta^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}"'] & \arrow[d, phantom, "\Downarrow\Delta^{\lhd}"] & \mathsf{Sets}_* \\
& \mathsf{Sets}_* \times \mathsf{Sets}_*, \arrow[ur, "\lhd"'] &
\end{tikzcd}$$

whose component

$$\Delta^{\lhd}_X \colon (X, x_0) \to (X \lhd X, x_0 \lhd x_0)$$

at $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$ is given by

$$\Delta^{\lhd}_X(x) \stackrel{\text{def}}{=} x \lhd x$$

for each $x \in X$.

Proof. *Being a Morphism of Pointed Sets*: We have

$$\Delta^{\lhd}_X(x_0) \stackrel{\text{def}}{=} x_0 \lhd x_0,$$

and thus $\Delta^{\lhd}_X$ is a morphism of pointed sets.
Naturality: We need to show that, given a morphism of pointed sets

$$f \colon (X, x_0) \to (Y, y_0),$$

the diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\Delta^{\lhd}_X"'] & Y \arrow[d, "\Delta^{\lhd}_Y"] \\
X \lhd X \arrow[r, "f \lhd f"'] & Y \lhd Y
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
x \arrow[r, mapsto] \arrow[d, mapsto] & f(x) \arrow[d, mapsto] \\
x \lhd x \arrow[r, mapsto] & f(x) \lhd f(x)
\end{tikzcd}$$

and hence indeed commutes, showing $\Delta^{\lhd}$ to be natural. □

4.3.8 The Left Skew Monoidal Structure on Pointed Sets Associated to $\lhd$

Proposition 4.3.8.1.1. The category Sets_* admits a left-closed left skew monoidal category structure consisting of

- *The Underlying Category.* The category Sets_* of pointed sets;
- *The Left Skew Monoidal Product.* The left tensor product functor

$$\lhd \colon \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*$$

  of Definition 4.3.1.1.1;
- *The Left Internal Skew Hom.* The left internal Hom functor

$$[-,-]^{\lhd}_{\mathsf{Sets}_*} \colon \mathsf{Sets}_*^{\mathsf{op}} \times \mathsf{Sets}_* \to \mathsf{Sets}_*$$

  of Definition 4.3.2.1.1;
- *The Left Skew Monoidal Unit.* The functor

$$\mathbb{1}^{\mathsf{Sets}_*,\lhd} \colon \mathsf{pt} \to \mathsf{Sets}_*$$

  of Definition 4.3.3.1.1;
- *The Left Skew Associators.* The natural transformation

$$\alpha^{\mathsf{Sets}_*,\lhd} \colon \lhd \circ (\lhd \times \mathrm{id}_{\mathsf{Sets}_*}) \Longrightarrow \lhd \circ (\mathrm{id}_{\mathsf{Sets}_*} \times \lhd) \circ \boldsymbol{\alpha}^{\mathsf{Cats}}_{\mathsf{Sets}_*,\mathsf{Sets}_*,\mathsf{Sets}_*}$$

  of Definition 4.3.4.1.1;
- *The Left Skew Left Unitors.* The natural transformation

$$\lambda^{\mathsf{Sets}_*,\lhd} \colon \lhd \circ \left(\mathbb{1}^{\mathsf{Sets}_*} \times \mathrm{id}_{\mathsf{Sets}_*}\right) \overset{\sim}{\Longrightarrow} \boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}$$

  of Definition 4.3.5.1.1;

- *The Left Skew Right Unitors.* The natural transformation

$$\rho^{\mathsf{Sets}_*,\lhd} \colon \boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*} \overset{\sim}{\Longrightarrow} \lhd \circ \left(\mathrm{id} \times \mathbb{1}^{\mathsf{Sets}_*}\right)$$

  of Definition 4.3.6.1.1.

Proof. The Pentagon Identity: Let (W, w_0), (X, x_0), (Y, y_0) and (Z, z_0) be pointed sets. We have to show that the diagram

$$\begin{tikzcd}
 & (W \lhd (X \lhd Y)) \lhd Z \arrow[dr, "\alpha^{\mathsf{Sets}_*,\lhd}_{W,X\lhd Y,Z}"] & \\
((W \lhd X) \lhd Y) \lhd Z \arrow[ur, "\alpha^{\mathsf{Sets}_*,\lhd}_{W,X,Y} \lhd \mathrm{id}_Z"] \arrow[d, "\alpha^{\mathsf{Sets}_*,\lhd}_{W\lhd X,Y,Z}"'] & & W \lhd ((X \lhd Y) \lhd Z) \arrow[d, "\mathrm{id}_W \lhd \alpha^{\mathsf{Sets}_*,\lhd}_{X,Y,Z}"] \\
(W \lhd X) \lhd (Y \lhd Z) \arrow[rr, "\alpha^{\mathsf{Sets}_*,\lhd}_{W,X,Y\lhd Z}"'] & & W \lhd (X \lhd (Y \lhd Z))
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
 & (w \lhd (x \lhd y)) \lhd z \arrow[dr, mapsto] & \\
((w \lhd x) \lhd y) \lhd z \arrow[ur, mapsto] \arrow[d, mapsto] & & w \lhd ((x \lhd y) \lhd z) \arrow[d, mapsto] \\
(w \lhd x) \lhd (y \lhd z) \arrow[rr, mapsto] & & w \lhd (x \lhd (y \lhd z))
\end{tikzcd}$$

and thus we see that the pentagon identity is satisfied.

The Left Skew Left Triangle Identity: Let (X, x_0) and (Y, y_0) be pointed sets. We have to show that the diagram

$$\begin{tikzcd}
\left(S^0 \lhd X\right) \lhd Y \arrow[r, "\alpha^{\mathsf{Sets}_*,\lhd}_{S^0,X,Y}"] \arrow[dr, "\lambda^{\mathsf{Sets}_*,\lhd}_X \lhd \mathrm{id}_Y"'] & S^0 \lhd (X \lhd Y) \arrow[d, "\lambda^{\mathsf{Sets}_*,\lhd}_{X \lhd Y}"] \\
 & X \lhd Y
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
(0 \lhd x) \lhd y \arrow[r, mapsto] \arrow[dr, mapsto] & 0 \lhd (x \lhd y) \arrow[d, mapsto] \\
 & x_0 \lhd y = x_0 \lhd y_0
\end{tikzcd}$$

and

$$\begin{tikzcd}
(1 \lhd x) \lhd y \arrow[r, mapsto] \arrow[dr, mapsto] & 1 \lhd (x \lhd y) \arrow[d, mapsto] \\
 & x \lhd y
\end{tikzcd}$$

and hence indeed commutes. Thus the left skew triangle identity is satisfied.

The Left Skew Right Triangle Identity: Let (X, x_0) and (Y, y_0) be pointed sets. We have to show that the diagram

$$\begin{tikzcd}
X \lhd Y \arrow[d, "\rho^{\mathsf{Sets}_*,\lhd}_{X \lhd Y}"'] \arrow[dr, "\mathrm{id}_X \lhd \rho^{\mathsf{Sets}_*,\lhd}_Y"] & \\
(X \lhd Y) \lhd S^0 \arrow[r, "\alpha^{\mathsf{Sets}_*,\lhd}_{X,Y,S^0}"'] & X \lhd \left(Y \lhd S^0\right)
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
x \lhd y \arrow[d, mapsto] \arrow[dr, mapsto] & \\
(x \lhd y) \lhd 1 \arrow[r, mapsto] & x \lhd (y \lhd 1)
\end{tikzcd}$$

and hence indeed commutes. Thus the right skew triangle identity is satisfied.

The Left Skew Middle Triangle Identity: Let (X, x_0) and (Y, y_0) be pointed sets. We have to show that the diagram

$$\begin{tikzcd}
X \lhd Y \arrow[rr, equal] \arrow[d, "\rho^{\mathsf{Sets}_*,\lhd}_X \lhd \mathrm{id}_Y"'] & & X \lhd Y \\
(X \lhd S^0) \lhd Y \arrow[rr, "\alpha^{\mathsf{Sets}_*,\lhd}_{X,S^0,Y}"'] & & X \lhd (S^0 \lhd Y) \arrow[u, "\mathrm{id}_A \lhd \lambda^{\mathsf{Sets}_*,\lhd}_Y"']
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
x \lhd y \arrow[rr, mapsto] \arrow[d, mapsto] & & x \lhd y \\
(x \lhd 1) \lhd y \arrow[rr, mapsto] & & x \lhd (1 \lhd y) \arrow[u, mapsto]
\end{tikzcd}$$

and hence indeed commutes. Thus the right skew triangle identity is satisfied.
The Zig-Zag Identity: We have to show that the diagram

$$\begin{tikzcd}
S^0 \arrow[r, "\rho^{\mathsf{Sets}_*,\lhd}_{S^0}"] \arrow[dr, equal] & S^0 \lhd S^0 \arrow[d, "\lambda^{\mathsf{Sets}_*,\lhd}_{S^0}"] \\
& S^0
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
0 \arrow[r, mapsto] \arrow[dr, mapsto] & 0 \lhd 1 \arrow[d, mapsto] \\
& 0
\end{tikzcd}$$

and

$$\begin{tikzcd}
1 \arrow[r, mapsto] \arrow[dr, mapsto] & 1 \lhd 1 \arrow[d, mapsto] \\
& 1
\end{tikzcd}$$

and hence indeed commutes. Thus the zig-zag identity is satisfied.
Left Skew Monoidal Left-Closedness: This follows from Item 2 of Proposition 4.3.1.1.7. □

4.3.9 Monoids With Respect to the Left Tensor Product of Pointed Sets

Proposition 4.3.9.1.1. The category of monoids on $(\mathsf{Sets}_*, \lhd, S^0)$ is isomorphic to the category of "monoids with left zero"[14] and morphisms between them.

Proof. Monoids on $(\mathsf{Sets}_*, \lhd, S^0)$: A monoid on $(\mathsf{Sets}_*, \lhd, S^0)$ consists of:

- *The Underlying Object.* A pointed set $(A, 0_A)$.
- *The Multiplication Morphism.* A morphism of pointed sets
$$\mu_A \colon A \lhd A \to A,$$
determining a left bilinear morphism of pointed sets
$$\begin{aligned} A \times A &\longrightarrow A \\ (a, b) &\longmapsto ab. \end{aligned}$$
- *The Unit Morphism.* A morphism of pointed sets
$$\eta_A \colon S^0 \to A$$
picking an element 1_A of A.

satisfying the following conditions:

1. *Associativity.* The diagram
$$\begin{tikzcd} & A \lhd (A \lhd A) \arrow[dr, "\mathrm{id}_A \lhd \mu_A"] & \\ (A \lhd A) \lhd A \arrow[ur, "\alpha^{\mathsf{Sets}_*, \lhd}_{A,A,A}"] \arrow[d, "\mu_A \lhd \mathrm{id}_A"'] & & A \lhd A \arrow[d, "\mu_A"] \\ A \lhd A \arrow[rr, "\mu_A"'] & & A \end{tikzcd}$$

[14] A monoid with left zero is defined similarly as the monoids with zero of **??**. Succinctly, they are monoids (A, μ_A, η_A) with a special element 0_A satisfying
$$0_A a = 0_A$$
for each $a \in A$.

2. *Left Unitality.* The diagram

$$\begin{tikzcd}
S^0 \lhd A \arrow[r, "\eta_A \times \mathrm{id}_A"] \arrow[dr, "\lambda_A^{\mathsf{Sets}_*,\lhd}"'] & A \lhd A \arrow[d, "\mu_A"] \\
& A
\end{tikzcd}$$

commutes.

3. *Right Unitality.* The diagram

$$\begin{tikzcd}
A \arrow[r, "\rho_A^{\mathsf{Sets}_*,\lhd}"] \arrow[d, equal] & A \lhd S^0 \arrow[d, "\mathrm{id}_A \times \eta_A"] \\
A & A \lhd A \arrow[l, "\mu_A"]
\end{tikzcd}$$

commutes.

Being a left-bilinear morphism of pointed sets, the multiplication map satisfies

$$0_A a = 0_A$$

for each $a \in A$. Now, the associativity, left unitality, and right unitality conditions act on elements as follows:

1. *Associativity.* The associativity condition acts as

$$\begin{tikzcd}
& & & a \lhd (b \lhd c) \arrow[dr, mapsto] & \\
(a \lhd b) \lhd c \arrow[d, mapsto] & & (a \lhd b) \lhd c \arrow[ur, mapsto] & & a \lhd bc \arrow[d, mapsto] \\
ab \lhd c \arrow[r, mapsto] & (ab)c & & & a(bc)
\end{tikzcd}$$

 This gives

$$(ab)c = a(bc)$$

 for each $a, b, c \in A$.

2. *Left Unitality.* The left unitality condition acts:

(a) On $0 \lhd a$ as

$$\begin{tikzcd} 0 \lhd a \arrow[dr, mapsto] & \\ & 0_A \end{tikzcd} \qquad \begin{tikzcd} 0 \lhd a \arrow[r, mapsto] \arrow[dr, mapsto] & 0_A \lhd a \arrow[d, mapsto] \\ & 0_A a. \end{tikzcd}$$

(b) On $1 \lhd a$ as

$$\begin{tikzcd} 1 \lhd a \arrow[dr, mapsto] & \\ & a \end{tikzcd} \qquad \begin{tikzcd} 1 \lhd a \arrow[r, mapsto] \arrow[dr, mapsto] & 1_A \lhd a \arrow[d, mapsto] \\ & 1_A a. \end{tikzcd}$$

This gives

$$1_A a = a,$$
$$0_A a = 0_A$$

for each $a \in A$.

3. *Right Unitality.* The right unitality condition acts as

$$\begin{tikzcd} a \arrow[d, mapsto] \\ a \end{tikzcd} \qquad \begin{tikzcd} a \arrow[r, mapsto] & a \lhd 1 \arrow[d, mapsto] \\ a1_A & a \lhd 1_A \arrow[l, mapsto] \end{tikzcd}$$

This gives

$$a1_A = a$$

for each $a \in A$.

Thus we see that monoids with respect to $\lhd$ are exactly monoids with left zero.

Morphisms of Monoids on $(\mathsf{Sets}_*, \lhd, S^0)$: A morphism of monoids on $(\mathsf{Sets}_*, \lhd, S^0)$ from $(A, \mu_A, \eta_A, 0_A)$ to $(B, \mu_B, \eta_B, 0_B)$ is a morphism of pointed sets

$$f\colon (A, 0_A) \to (B, 0_B)$$

satisfying the following conditions:

1. *Compatibility With the Multiplication Morphisms.* The diagram

$$\begin{tikzcd} A \lhd A \arrow[r, "f \lhd f"] \arrow[d, "\mu_A"'] & B \lhd B \arrow[d, "\mu_B"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

commutes.

2. *Compatibility With the Unit Morphisms.* The diagram

$$\begin{tikzcd} S^0 \arrow[r, "\eta_A"] \arrow[dr, "\eta_B"'] & A \arrow[d, "f"] \\ & B \end{tikzcd}$$

commutes.

These act on elements as

$$\begin{tikzcd} a \lhd b \arrow[d, mapsto] & \\ ab \arrow[r, mapsto] & f(ab) \end{tikzcd} \qquad \begin{tikzcd} a \lhd b \arrow[r, mapsto] & f(a) \lhd f(b) \arrow[d, mapsto] \\ & f(a)f(b) \end{tikzcd}$$

and

$$\begin{tikzcd} 0 \arrow[dr, mapsto] & \\ & 0_B \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[r, mapsto] & 0_A \arrow[d, mapsto] \\ & f(0_A) \end{tikzcd}$$

and

$$\begin{tikzcd} 1 \arrow[dr, mapsto] & \\ & 1_B \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[r, mapsto] & 1_A \arrow[d, mapsto] \\ & f(1_A) \end{tikzcd}$$

giving

$$\begin{aligned} f(ab) &= f(a)f(b), \\ f(0_A) &= 0_B, \\ f(1_A) &= 1_B, \end{aligned}$$

for each $a, b \in A$, which is exactly a morphism of monoids with left zero.
Identities and Composition: Similarly, the identities and composition of $\mathsf{Mon}(\mathsf{Sets}_*, \lhd, S^0)$ can be easily seen to agree with those of monoids with left zero, which finishes the proof. □

4.4 The Right Tensor Product of Pointed Sets

4.4.1 Foundations

Let (X, x_0) and (Y, y_0) be pointed sets.

Definition 4.4.1.1.1. The **right tensor product of pointed sets** is the functor[15]

$$\rhd : \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*$$

defined as the composition

$$\mathsf{Sets}_* \times \mathsf{Sets}_* \xrightarrow{忘\times\mathrm{id}} \mathsf{Sets} \times \mathsf{Sets}_* \xrightarrow{\odot} \mathsf{Sets}_*,$$

where:

- $忘 \colon \mathsf{Sets}_* \to \mathsf{Sets}$ is the forgetful functor from pointed sets to sets.
- $\odot \colon \mathsf{Sets} \times \mathsf{Sets}_* \to \mathsf{Sets}_*$ is the tensor functor of Item 1 of Proposition 4.2.1.1.6.

Remark 4.4.1.1.2. The right tensor product of pointed sets satisfies the following natural bijection:

$$\mathsf{Sets}_*(X \rhd Y, Z) \cong \mathrm{Hom}^{\otimes, \mathrm{R}}_{\mathsf{Sets}_*}(X \times Y, Z).$$

That is to say, the following data are in natural bijection:

1. Pointed maps $f \colon X \rhd Y \to Z$.
2. Maps of sets $f \colon X \times Y \to Z$ satisfying $f(x, y_0) = z_0$ for each $x \in X$.

Remark 4.4.1.1.3. The right tensor product of pointed sets may be described as follows:

- The right tensor product of (X, x_0) and (Y, y_0) is the pair $((X \rhd Y, x_0 \rhd y_0), \iota)$ consisting of
  - A pointed set $(X \rhd Y, x_0 \rhd y_0)$;

[15] *Further Notation:* Also written $\rhd_{\mathsf{Sets}_*}$.

– A right bilinear morphism of pointed sets $\iota\colon (X \times Y, (x_0, y_0)) \to X \rhd Y$;

satisfying the following universal property:

(UP) Given another such pair $((Z, z_0), f)$ consisting of

* A pointed set (Z, z_0);
* A right bilinear morphism of pointed sets $f\colon (X \times Y, (x_0, y_0)) \to X \rhd Y$;

there exists a unique morphism of pointed sets $X \rhd Y \xrightarrow{\exists!} Z$ making the diagram

$$\begin{tikzcd} & X \rhd Y \arrow[d, dashed, "\exists!"] \\ X \times Y \arrow[ur, "\iota"] \arrow[r, "f"'] & Z \end{tikzcd}$$

commute.

Construction 4.4.1.1.4. In detail, the **right tensor product of** (X, x_0) **and** (Y, y_0) is the pointed set $(X \rhd Y, [y_0])$ consisting of:

- *The Underlying Set.* The set $X \rhd Y$ defined by

$$\begin{aligned} X \rhd Y &\overset{\mathrm{def}}{=} |X| \odot Y \\ &\cong \bigvee_{x \in X} (Y, y_0), \end{aligned}$$

where $|X|$ denotes the underlying set of (X, x_0).

- *The Underlying Basepoint.* The point $[(x_0, y_0)]$ of $\bigvee_{x \in X}(Y, y_0)$, which is equal to $[(x, y_0)]$ for any $x \in X$.

Notation 4.4.1.1.5. We write[16] $x \rhd y$ for the element $[(x, y)]$ of

$$X \rhd Y \cong |X| \odot Y.$$

Remark 4.4.1.1.6. Employing the notation introduced in Notation 4.4.1.1.5, we have

$$x_0 \rhd y_0 = x \rhd y_0$$

for each $x \in X$, and

$$x \rhd y_0 = x' \rhd y_0$$

for each $x, x' \in X$.

[16] *Further Notation:* Also written $x \rhd_{\mathsf{Sets}_*} y$.

Proposition 4.4.1.1.7. Let (X, x_0) and (Y, y_0) be pointed sets.

1. *Functoriality.* The assignments $X, Y, (X, Y) \mapsto X \rhd Y$ define functors

$$
\begin{aligned}
X \rhd - \colon \mathsf{Sets}_* &\to \mathsf{Sets}_*,\\
- \rhd Y \colon \mathsf{Sets}_* &\to \mathsf{Sets}_*,\\
-_1 \rhd -_2 \colon \mathsf{Sets}_* \times \mathsf{Sets}_* &\to \mathsf{Sets}_*.
\end{aligned}
$$

   In particular, given pointed maps

$$
\begin{aligned}
f \colon (X, x_0) &\to (A, a_0),\\
g \colon (Y, y_0) &\to (B, b_0),
\end{aligned}
$$

   the induced map

$$f \rhd g \colon X \rhd Y \to A \rhd B$$

   is given by

$$[f \rhd g](x \rhd y) \overset{\text{def}}{=} f(x) \rhd g(y)$$

   for each $x \rhd y \in X \rhd Y$.

2. *Adjointness I.* We have an adjunction

$$\left(X \rhd - \dashv [X, -]^{\rhd}_{\mathsf{Sets}_*}\right)\colon \quad \begin{tikzcd} \mathsf{Sets}_* \arrow[r, bend left, "X \rhd -"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_*, \arrow[l, bend left, "{[X,-]^{\rhd}_{\mathsf{Sets}_*}}"] \end{tikzcd}$$

   witnessed by a bijection of sets

$$\mathrm{Hom}_{\mathsf{Sets}_*}(X \rhd Y, Z) \cong \mathrm{Hom}_{\mathsf{Sets}_*}\left(Y, [X, Z]^{\rhd}_{\mathsf{Sets}_*}\right)$$

   natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, where $[X, Y]^{\rhd}_{\mathsf{Sets}_*}$ is the pointed set of Definition 4.4.2.1.1.

3. *Adjointness II.* The functor

$$- \rhd Y \colon \mathsf{Sets}_* \to \mathsf{Sets}_*$$

   does not admit a right adjoint.

4. *Adjointness III.* We have a bijection of sets

$$\mathrm{Hom}_{\mathsf{Sets}_*}(X \rhd Y, Z) \cong \mathrm{Hom}_{\mathsf{Sets}}(|X|, \mathsf{Sets}_*(Y, Z))$$

   natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

Proof. *Item 1, Functoriality*: Clear.
Item 2, Adjointness I: This follows from Item 3 of Proposition 4.2.1.1.6.
Item 3, Adjointness II: For $- \rhd Y$ to admit a right adjoint would require it to preserve colimits by **??** of **??**. However, we have

$$\begin{aligned} \mathrm{pt} \rhd X &\stackrel{\text{def}}{=} |\mathrm{pt}| \odot X \\ &\cong X \\ &\ncong \mathrm{pt}, \end{aligned}$$

and thus we see that $- \rhd Y$ does not have a right adjoint.
Item 4, Adjointness III: This follows from Item 2 of Proposition 4.2.1.1.6. □

Remark 4.4.1.1.8. Here is some intuition on why $- \rhd Y$ fails to be a left adjoint. Item 4 of Proposition 4.3.1.1.7 states that we have a natural bijection

$$\mathrm{Hom}_{\mathsf{Sets}_*}(X \rhd Y, Z) \cong \mathrm{Hom}_{\mathsf{Sets}}(|X|, \mathsf{Sets}_*(Y, Z)),$$

so it would be reasonable to wonder whether a natural bijection of the form

$$\mathrm{Hom}_{\mathsf{Sets}_*}(X \rhd Y, Z) \cong \mathrm{Hom}_{\mathsf{Sets}_*}(X, \mathbf{Sets}_*(Y, Z)),$$

also holds, which would give $- \rhd Y \dashv \mathbf{Sets}_*(Y, -)$. However, such a bijection would require every map

$$f \colon X \rhd Y \to Z$$

to satisfy

$$f(x_0 \rhd y) = z_0$$

for each $x \in X$, whereas we are imposing such a basepoint preservation condition only for elements of the form $x \rhd y_0$. Thus $\mathbf{Sets}_*(Y, -)$ can't be a right adjoint for $- \rhd Y$, and as shown by Item 3 of Proposition 4.4.1.1.7, no functor can.[17]

4.4.2 The Right Internal Hom of Pointed Sets

Let (X, x_0) and (Y, y_0) be pointed sets.

Definition 4.4.2.1.1. The **right internal Hom of pointed sets** is the functor

$$[-,-]^{\rhd}_{\mathsf{Sets}_*} \colon \mathsf{Sets}_*^{\mathsf{op}} \times \mathsf{Sets}_* \to \mathsf{Sets}_*$$

defined as the composition

$$\mathsf{Sets}_*^{\mathsf{op}} \times \mathsf{Sets}_* \xrightarrow{忘 \times \mathrm{id}} \mathsf{Sets}^{\mathsf{op}} \times \mathsf{Sets}_* \xrightarrow{⋔} \mathsf{Sets}_*,$$

where:

[17] The functor $\mathbf{Sets}_*(Y, -)$ is instead right adjoint to $- \wedge Y$, the smash product of pointed

- $忘\colon \mathsf{Sets}_* \to \mathsf{Sets}$ is the forgetful functor from pointed sets to sets.
- $\pitchfork\colon \mathsf{Sets}^{\mathsf{op}} \times \mathsf{Sets}_* \to \mathsf{Sets}_*$ is the cotensor functor of Item 1 of Proposition 4.2.2.1.4.

Proof. For a proof that $[-,-]^{\rhd}_{\mathsf{Sets}_*}$ is indeed the right internal Hom of Sets_* with respect to the right tensor product of pointed sets, see Item 2 of Proposition 4.4.1.1.7. □

Remark 4.4.2.1.2. We have

$$[-,-]^{\lhd}_{\mathsf{Sets}_*} = [-,-]^{\rhd}_{\mathsf{Sets}_*}.$$

Remark 4.4.2.1.3. The right internal Hom of pointed sets satisfies the following universal property:

$$\mathsf{Sets}_*(X \rhd Y, Z) \cong \mathsf{Sets}_*\left(Y, [X,Z]^{\rhd}_{\mathsf{Sets}_*}\right)$$

That is to say, the following data are in bijection:

1. Pointed maps $f\colon X \rhd Y \to Z$.
2. Pointed maps $f\colon Y \to [X,Z]^{\rhd}_{\mathsf{Sets}_*}$.

Remark 4.4.2.1.4. In detail, the **right internal Hom of** (X,x_0) **and** (Y,y_0) is the pointed set $\left([X,Y]^{\rhd}_{\mathsf{Sets}_*}, \left[(y_0)_{x\in X}\right]\right)$ consisting of

- *The Underlying Set.* The set $[X,Y]^{\rhd}_{\mathsf{Sets}_*}$ defined by

$$\begin{aligned} [X,Y]^{\rhd}_{\mathsf{Sets}_*} &\overset{\text{def}}{=} |X| \pitchfork Y \\ &\cong \bigwedge_{x\in X} (Y,y_0), \end{aligned}$$

  where $|X|$ denotes the underlying set of (X,x_0);
- *The Underlying Basepoint.* The point $\left[(y_0)_{x\in X}\right]$ of $\bigwedge_{x\in X}(Y,y_0)$.

Proposition 4.4.2.1.5. Let (X,x_0) and (Y,y_0) be pointed sets.

1. *Functoriality.* The assignments $X, Y, (X,Y) \mapsto [X,Y]^{\rhd}_{\mathsf{Sets}_*}$ define functors

$$\begin{aligned} [X,-]^{\rhd}_{\mathsf{Sets}_*}&\colon \mathsf{Sets}_* \to \mathsf{Sets}_*, \\ [-,Y]^{\rhd}_{\mathsf{Sets}_*}&\colon \mathsf{Sets}_*^{\mathsf{op}} \to \mathsf{Sets}_*, \\ [-_1,-_2]^{\rhd}_{\mathsf{Sets}_*}&\colon \mathsf{Sets}_*^{\mathsf{op}} \times \mathsf{Sets}_* \to \mathsf{Sets}_*. \end{aligned}$$

In particular, given pointed maps

$$\begin{aligned} f\colon (X,x_0) &\longrightarrow (A,a_0),\\ g\colon (Y,y_0) &\longrightarrow (B,b_0), \end{aligned}$$

the induced map

$$[f,g]^{\triangleright}_{\mathsf{Sets}_*}\colon [A,Y]^{\triangleright}_{\mathsf{Sets}_*} \longrightarrow [X,B]^{\triangleright}_{\mathsf{Sets}_*}$$

is given by

$$[f,g]^{\triangleright}_{\mathsf{Sets}_*}\left(\left[(y_a)_{a\in A}\right]\right) \overset{\text{def}}{=} \left[\left(g\left(y_{f(x)}\right)\right)_{x\in X}\right]$$

for each $\left[(y_a)_{a\in A}\right] \in [A,Y]^{\triangleright}_{\mathsf{Sets}_*}$.

2. *Adjointness I.* We have an adjunction

$$\left(X \triangleright - \dashv [X,-]^{\triangleright}_{\mathsf{Sets}_*}\right)\colon \quad \begin{tikzcd} \mathsf{Sets}_* \arrow[r, bend left, "X\triangleright -"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_*, \arrow[l, bend left, "{[X,-]^{\triangleright}_{\mathsf{Sets}_*}}"] \end{tikzcd}$$

witnessed by a bijection of sets

$$\mathrm{Hom}_{\mathsf{Sets}_*}(X \triangleright Y, Z) \cong \mathrm{Hom}_{\mathsf{Sets}_*}\left(Y, [X,Z]^{\triangleright}_{\mathsf{Sets}_*}\right)$$

natural in $(X,x_0),(Y,y_0),(Z,z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, where $[X,Y]^{\triangleright}_{\mathsf{Sets}_*}$ is the pointed set of Definition 4.4.2.1.1.

3. *Adjointness II.* The functor

$$- \triangleright Y\colon \mathsf{Sets}_* \longrightarrow \mathsf{Sets}_*$$

does not admit a right adjoint.

Proof. *Item 1, Functoriality*: Clear.
Item 2, Adjointness I: This is a repetition of Item 2 of Proposition 4.4.1.1.7, and is proved there.
Item 3, Adjointness II: This is a repetition of Item 3 of Proposition 4.4.1.1.7, and is proved there. □

4.4.3 The Right Skew Unit

Definition 4.4.3.1.1. The **right skew unit of the right tensor product of pointed sets** is the functor

$$\mathbb{1}^{\mathsf{Sets}_*,\triangleright}\colon \mathrm{pt} \longrightarrow \mathsf{Sets}_*$$

defined by

$$\mathbb{1}^{\triangleright}_{\mathsf{Sets}_*} \overset{\text{def}}{=} S^0.$$

4.4.4 The Right Skew Associator

Definition 4.4.4.1.1. The **skew associator of the right tensor product of pointed sets** is the natural transformation

$$\alpha^{\mathsf{Sets}_*,\triangleright} \colon \triangleright \circ \left(\mathrm{id}_{\mathsf{Sets}_*} \times \triangleright\right) \Longrightarrow \triangleright \circ \left(\triangleright \times \mathrm{id}_{\mathsf{Sets}_*}\right) \circ \boldsymbol{\alpha}^{\mathsf{Cats},-1}_{\mathsf{Sets}_*,\mathsf{Sets}_*,\mathsf{Sets}_*}$$

as in the diagram

$$\begin{tikzcd}
 & (\mathsf{Sets}_* \times \mathsf{Sets}_*) \times \mathsf{Sets}_* \arrow[dr, "\triangleright\times\mathrm{id}"] & \\
\mathsf{Sets}_* \times (\mathsf{Sets}_* \times \mathsf{Sets}_*) \arrow[ur, dashed, "\boldsymbol{\alpha}^{\mathsf{Cats},-1}_{\mathsf{Sets}_*,\mathsf{Sets}_*,\mathsf{Sets}_*}", "\sim"'] \arrow[d, "\mathrm{id}\times\triangleright"'] & & \mathsf{Sets}_* \times \mathsf{Sets}_* \arrow[d, "\triangleright"] \\
\mathsf{Sets}_* \times \mathsf{Sets}_* \arrow[rr, "\triangleright"'] \arrow[from=3-1, to=2-3, phantom, "\Uparrow\alpha^{\mathsf{Sets}_*,\triangleright}"] & & \mathsf{Sets}_*,
\end{tikzcd}$$

whose component

$$\alpha^{\mathsf{Sets}_*,\triangleright}_{X,Y,Z} \colon X \triangleright (Y \triangleright Z) \to (X \triangleright Y) \triangleright Z$$

at $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$ is given by

$$\begin{aligned}
X \triangleright (Y \triangleright Z) &\overset{\text{def}}{=} |X| \odot (Y \triangleright Z) \\
&\overset{\text{def}}{=} |X| \odot (|Y| \odot Z) \\
&\cong \bigvee_{x\in X} (|Y| \odot Z) \\
&\cong \bigvee_{x\in X}\left(\bigvee_{y\in Y} Z\right) \\
&\to \bigvee_{[(x,y)]\in\bigvee_{x\in X} Y} Z \\
&\cong \bigvee_{[(x,y)]\in|X|\odot Y} Z \\
&\cong ||X| \odot Y| \odot Z \\
&\overset{\text{def}}{=} |X \triangleright Y| \odot Z \\
&\overset{\text{def}}{=} (X \triangleright Y) \triangleright Z,
\end{aligned}$$

sets of Definition 4.5.1.1.1. See Item 2 of Proposition 4.5.1.1.9.

where the map

$$\bigvee_{x\in X}\left(\bigvee_{y\in Y} Z\right) \to \bigvee_{[(x,y)]\in\bigvee_{x\in X} Y} Z$$

is given by $[(x,[(y,z)])] \mapsto [([(x,y)],z)]$.

Proof. (Proven below in a bit.) □

Remark 4.4.4.1.2. Unwinding the notation for elements, we have

$$\begin{aligned}[(x,[(y,z)])] &\stackrel{\text{def}}{=} [(x,y\rhd z)]\\ &\stackrel{\text{def}}{=} x\rhd(y\rhd z)\end{aligned}$$

and

$$\begin{aligned}[([(x,y)],z)] &\stackrel{\text{def}}{=} [(x\rhd y,z)]\\ &\stackrel{\text{def}}{=} (x\rhd y)\rhd z.\end{aligned}$$

So, in other words, $\alpha^{\mathsf{Sets}_*,\rhd}_{X,Y,Z}$ acts on elements via

$$\alpha^{\mathsf{Sets}_*,\rhd}_{X,Y,Z}(x\rhd(y\rhd z)) \stackrel{\text{def}}{=} (x\rhd y)\rhd z$$

for each $x\rhd(y\rhd z)\in X\rhd(Y\rhd Z)$.

Remark 4.4.4.1.3. Taking $y=y_0$, we see that the morphism $\alpha^{\mathsf{Sets}_*,\rhd}_{X,Y,Z}$ acts on elements as

$$\alpha^{\mathsf{Sets}_*,\rhd}_{X,Y,Z}(x\rhd(y_0\rhd z)) \stackrel{\text{def}}{=} (x\rhd y_0)\rhd z.$$

However, by the definition of $\rhd$, we have $x\rhd y_0 = x'\rhd y_0$ for all $x,x'\in X$, preventing $\alpha^{\mathsf{Sets}_*,\rhd}_{X,Y,Z}$ from being non-invertible.

Proof. Firstly, note that, given $(X,x_0),(Y,y_0),(Z,z_0)\in\mathrm{Obj}(\mathsf{Sets}_*)$, the map

$$\alpha^{\mathsf{Sets}_*,\rhd}_{X,Y,Z}\colon X\rhd(Y\rhd Z)\to(X\rhd Y)\rhd Z$$

is indeed a morphism of pointed sets, as we have

$$\alpha^{\mathsf{Sets}_*,\rhd}_{X,Y,Z}(x_0\rhd(y_0\rhd z_0)) = (x_0\rhd y_0)\rhd z_0.$$

Next, we claim that $\alpha^{\mathsf{Sets}_*,\rhd}$ is a natural transformation. We need to show that, given morphisms of pointed sets

$$\begin{aligned}f&\colon(X,x_0)\to(X',x'_0),\\ g&\colon(Y,y_0)\to(Y',y'_0),\\ h&\colon(Z,z_0)\to(Z',z'_0)\end{aligned}$$

the diagram

$$\begin{tikzcd}
X \rhd (Y \rhd Z) \arrow[r, "f \rhd (g \rhd h)"] \arrow[d, "\alpha^{\mathsf{Sets}_*,\rhd}_{X,Y,Z}"'] & X' \rhd (Y' \rhd Z') \arrow[d, "\alpha^{\mathsf{Sets}_*,\rhd}_{X',Y',Z'}"] \\
(X \rhd Y) \rhd Z \arrow[r, "(f \rhd g) \rhd h"'] & (X' \rhd Y') \rhd Z'
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
x \rhd (y \rhd z) \arrow[r, mapsto] \arrow[d, mapsto] & f(x) \rhd (g(y) \rhd h(z)) \arrow[d, mapsto] \\
(x \rhd y) \rhd z \arrow[r, mapsto] & (f(x) \rhd g(y)) \rhd h(z)
\end{tikzcd}$$

and hence indeed commutes, showing $\alpha^{\mathsf{Sets}_*,\rhd}$ to be a natural transformation. This finishes the proof. □

4.4.5 The Right Skew Left Unitor

Definition 4.4.5.1.1. The **skew left unitor of the right tensor product of pointed sets** is the natural transformation

$$\lambda^{\mathsf{Sets}_*,\rhd} \colon \boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*} \overset{\sim}{\Longrightarrow} \rhd \circ \left(\mathbb{1}^{\mathsf{Sets}_*} \times \mathrm{id}_{\mathsf{Sets}_*}\right)$$

$$\begin{tikzcd}
\mathrm{pt} \times \mathsf{Sets}_* \arrow[r, "\mathbb{1}^{\mathsf{Sets}_*} \times \mathrm{id}"] \arrow[dr, dashed, bend right, "\boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}"'] & \mathsf{Sets}_* \times \mathsf{Sets}_* \arrow[d, "\rhd"] \arrow[dl, phantom, "\Nearrow \lambda^{\mathsf{Sets}_*,\rhd}"] \\
 & \mathsf{Sets}_*,
\end{tikzcd}$$

whose component

$$\lambda^{\mathsf{Sets}_*,\rhd}_X \colon X \to S^0 \rhd X$$

at $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$ is given by the composition

$$\begin{aligned}
X &\to X \vee X \\
&\cong |S^0| \odot X \\
&\cong S^0 \rhd X,
\end{aligned}$$

where $X \to X \vee X$ is the map sending X to the second factor of X in $X \vee X$.

Proof. (Proven below in a bit.) □

Remark 4.4.5.1.2. In other words, $\lambda_X^{\mathsf{Sets}_*,\rhd}$ acts on elements as

$$\lambda_X^{\mathsf{Sets}_*,\rhd}(x) \stackrel{\text{def}}{=} [(1,x)]$$

i.e. by

$$\lambda_X^{\mathsf{Sets}_*,\rhd}(x) \stackrel{\text{def}}{=} 1 \rhd x$$

for each $x \in X$.

Remark 4.4.5.1.3. The morphism $\lambda_X^{\mathsf{Sets}_*,\rhd}$ is non-invertible, as it is non-surjective when viewed as a map of sets, since the elements $0 \rhd x$ of $S^0 \rhd X$ with $x \neq x_0$ are outside the image of $\lambda_X^{\mathsf{Sets}_*,\rhd}$, which sends x to $1 \rhd x$.

Proof. Firstly, note that, given $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, the map

$$\lambda_X^{\mathsf{Sets}_*,\rhd} : X \to S^0 \rhd X$$

is indeed a morphism of pointed sets, as we have

$$\begin{aligned} \lambda_X^{\mathsf{Sets}_*,\rhd}(x_0) &= 1 \rhd x_0 \\ &= 0 \rhd x_0. \end{aligned}$$

Next, we claim that $\lambda^{\mathsf{Sets}_*,\rhd}$ is a natural transformation. We need to show that, given a morphism of pointed sets

$$f : (X, x_0) \to (Y, y_0),$$

the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\lambda_X^{\mathsf{Sets}_*,\rhd}"'] & Y \arrow[d, "\lambda_Y^{\mathsf{Sets}_*,\rhd}"] \\ S^0 \rhd X \arrow[r, "\mathrm{id}_{S^0} \rhd f"'] & S^0 \rhd Y \end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd} x \arrow[r, mapsto] \arrow[d, mapsto] & f(x) \arrow[d, mapsto] \\ 1 \rhd x \arrow[r, mapsto] & 1 \rhd f(x) \end{tikzcd}$$

and hence indeed commutes, showing $\lambda^{\mathsf{Sets}_*,\rhd}$ to be a natural transformation. This finishes the proof. □

4.4.6 The Right Skew Right Unitor

Definition 4.4.6.1.1. The **skew right unitor of the right tensor product of pointed sets** is the natural transformation

$$\rho^{\mathsf{Sets}_*,\triangleright} \colon \triangleright \circ \left(\mathrm{id} \times \mathbb{1}^{\mathsf{Sets}_*}\right) \overset{\sim}{\Longrightarrow} \boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}, \qquad \begin{tikzcd} \mathsf{Sets}_* \times \mathrm{pt} \arrow[r, "\mathrm{id}\times\mathbb{1}^{\mathsf{Sets}_*}"] \arrow[dr, dashed, bend right, "\boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}"'] & \mathsf{Sets}_* \times \mathsf{Sets}_* \arrow[d, "\triangleright"] \arrow[dl, phantom, "\Swarrow \rho^{\mathsf{Sets}_*,\triangleright}"] \\ & \mathsf{Sets}_*, \end{tikzcd}$$

whose component

$$\rho^{\mathsf{Sets}_*,\triangleright}_X \colon X \triangleright S^0 \to X$$

at $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$ is given by the composition

$$\begin{aligned} X \triangleright S^0 &\cong |X| \odot S^0 \\ &\cong \bigvee_{x \in X} S^0 \\ &\to X, \end{aligned}$$

where $\bigvee_{x \in X} S^0 \to X$ is the map given by

$$\begin{aligned} [(x, 0)] &\mapsto x_0, \\ [(x, 1)] &\mapsto x. \end{aligned}$$

Proof. (Proven below in a bit.) □

Remark 4.4.6.1.2. In other words, $\rho^{\mathsf{Sets}_*,\triangleright}_X$ acts on elements as

$$\begin{aligned} \rho^{\mathsf{Sets}_*,\triangleright}_X (x \triangleright 0) &\overset{\text{def}}{=} x_0, \\ \rho^{\mathsf{Sets}_*,\triangleright}_X (x \triangleright 1) &\overset{\text{def}}{=} x \end{aligned}$$

for each $x \triangleright 1 \in X \triangleright S^0$.

Remark 4.4.6.1.3. The morphism $\rho^{\mathsf{Sets}_*,\triangleright}_X$ is almost invertible, with its would-be-inverse

$$\phi_X \colon X \to X \triangleright S^0$$

given by

$$\phi_X(x) \overset{\text{def}}{=} x \triangleright 1$$

for each $x \in X$. Indeed, we have

$$\begin{aligned}\left[\rho_X^{\mathsf{Sets}_*,\rhd} \circ \phi\right](x) &= \rho_X^{\mathsf{Sets}_*,\rhd}(\phi(x))\\ &= \rho_X^{\mathsf{Sets}_*,\rhd}(x \rhd 1)\\ &= x\\ &= [\mathrm{id}_X](x)\end{aligned}$$

so that

$$\rho_X^{\mathsf{Sets}_*,\rhd} \circ \phi = \mathrm{id}_X$$

and

$$\begin{aligned}\left[\phi \circ \rho_X^{\mathsf{Sets}_*,\rhd}\right](x \rhd 1) &= \phi\Big(\rho_X^{\mathsf{Sets}_*,\rhd}(x \rhd 1)\Big)\\ &= \phi(x)\\ &= x \rhd 1\\ &= [\mathrm{id}_{X \rhd S^0}](x \rhd 1),\end{aligned}$$

but

$$\begin{aligned}\left[\phi \circ \rho_X^{\mathsf{Sets}_*,\rhd}\right](x \rhd 0) &= \phi\Big(\rho_X^{\mathsf{Sets}_*,\rhd}(x \rhd 0)\Big)\\ &= \phi(x_0)\\ &= 1 \rhd x_0,\end{aligned}$$

where $x \rhd 0 \neq 1 \rhd x_0$. Thus

$$\phi \circ \rho_X^{\mathsf{Sets}_*,\rhd} \stackrel{?}{=} \mathrm{id}_{X \rhd S^0}$$

holds for all elements in $X \rhd S^0$ except one.

Proof. Firstly, note that, given $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, the map

$$\rho_X^{\mathsf{Sets}_*,\rhd} \colon X \rhd S^0 \to X$$

is indeed a morphism of pointed sets as we have

$$\rho_X^{\mathsf{Sets}_*,\rhd}(x_0 \rhd 0) = x_0.$$

Next, we claim that $\rho^{\mathsf{Sets}_*,\rhd}$ is a natural transformation. We need to show that, given a morphism of pointed sets

$$f \colon (X, x_0) \to (Y, y_0),$$

the diagram

$$\begin{tikzcd}
X \rhd S^0 \arrow[r, "f \rhd \mathrm{id}_{S^0}"] \arrow[d, "\rho^{\mathsf{Sets}_*,\rhd}_X"'] & Y \rhd S^0 \arrow[d, "\rho^{\mathsf{Sets}_*,\rhd}_Y"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
x \rhd 0 \arrow[d, mapsto] & \\
x_0 \arrow[r, mapsto] & f(x_0)
\end{tikzcd}
\qquad
\begin{tikzcd}
x \rhd 0 \arrow[r, mapsto] & f(x) \rhd 0 \arrow[d, mapsto] \\
& y_0
\end{tikzcd}$$

and

$$\begin{tikzcd}
x \rhd 1 \arrow[r, mapsto] \arrow[d, mapsto] & f(x) \rhd 1 \arrow[d, mapsto] \\
x \arrow[r, mapsto] & f(x)
\end{tikzcd}$$

and hence indeed commutes, showing $\rho^{\mathsf{Sets}_*,\rhd}$ to be a natural transformation. This finishes the proof. □

4.4.7 The Diagonal

Definition 4.4.7.1.1. The **diagonal of the right tensor product of pointed sets** is the natural transformation

$$\Delta^{\rhd} \colon \mathrm{id}_{\mathsf{Sets}_*} \Longrightarrow \rhd \circ \Delta^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}, \qquad \begin{tikzcd}
\mathsf{Sets}_* \arrow[rr, "\mathrm{id}_{\mathsf{Sets}_*}"] \arrow[dr, "\Delta^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}"'] & {} \arrow[d, phantom, "\Downarrow \Delta^{\rhd}"] & \mathsf{Sets}_* \\
& \mathsf{Sets}_* \times \mathsf{Sets}_*, \arrow[ur, "\rhd"'] &
\end{tikzcd}$$

whose component

$$\Delta^{\rhd}_X \colon (X, x_0) \to (X \rhd X, x_0 \rhd x_0)$$

at $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$ is given by

$$\Delta^{\rhd}_X(x) \overset{\text{def}}{=} x \rhd x$$

for each $x \in X$.

Proof. *Being a Morphism of Pointed Sets*: We have

$$\Delta^{\triangleright}_{X}(x_0) \stackrel{\text{def}}{=} x_0 \triangleright x_0,$$

and thus $\Delta^{\triangleright}_{X}$ is a morphism of pointed sets.
Naturality: We need to show that, given a morphism of pointed sets

$$f\colon (X, x_0) \to (Y, y_0),$$

the diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\Delta^{\triangleright}_{X}"'] & Y \arrow[d, "\Delta^{\triangleright}_{Y}"] \\
X \triangleright X \arrow[r, "f \triangleright f"'] & Y \triangleright Y
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
x \arrow[r, mapsto] \arrow[d, mapsto] & f(x) \arrow[d, mapsto] \\
x \triangleright x \arrow[r, mapsto] & f(x) \triangleright f(x)
\end{tikzcd}$$

and hence indeed commutes, showing $\Delta^{\triangleright}$ to be natural. □

4.4.8 The Right Skew Monoidal Structure on Pointed Sets Associated to $\triangleright$

Proposition 4.4.8.1.1. The category Sets_* admits a right-closed right skew monoidal category structure consisting of

- *The Underlying Category.* The category Sets_* of pointed sets;
- *The Right Skew Monoidal Product.* The right tensor product functor

  $$\triangleright\colon \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*$$

  of Definition 4.4.1.1.1;
- *The Right Internal Skew Hom.* The right internal Hom functor

  $$[-,-]^{\triangleright}_{\mathsf{Sets}_*}\colon \mathsf{Sets}_*^{\mathsf{op}} \times \mathsf{Sets}_* \to \mathsf{Sets}_*$$

  of Definition 4.4.2.1.1;

- *The Right Skew Monoidal Unit.* The functor

$$\mathbb{1}^{\mathsf{Sets}_*,\triangleright} \colon \mathrm{pt} \longrightarrow \mathsf{Sets}_*$$

  of Definition 4.4.3.1.1;

- *The Right Skew Associators.* The natural transformation

$$\alpha^{\mathsf{Sets}_*,\triangleright} \colon \triangleright \circ \left(\mathrm{id}_{\mathsf{Sets}_*} \times \triangleright\right) \Longrightarrow \triangleright \circ \left(\triangleright \times \mathrm{id}_{\mathsf{Sets}_*}\right) \circ \boldsymbol{\alpha}^{\mathsf{Cats},-1}_{\mathsf{Sets}_*,\mathsf{Sets}_*,\mathsf{Sets}_*}$$

  of Definition 4.4.4.1.1;

- *The Right Skew Left Unitors.* The natural transformation

$$\lambda^{\mathsf{Sets}_*,\triangleright} \colon \boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*} \overset{\sim}{\Longrightarrow} \triangleright \circ \left(\mathbb{1}^{\mathsf{Sets}_*} \times \mathrm{id}_{\mathsf{Sets}_*}\right)$$

  of Definition 4.4.5.1.1;

- *The Right Skew Right Unitors.* The natural transformation

$$\rho^{\mathsf{Sets}_*,\triangleright} \colon \triangleright \circ \left(\mathrm{id} \times \mathbb{1}^{\mathsf{Sets}_*}\right) \overset{\sim}{\Longrightarrow} \boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}$$

  of Definition 4.4.6.1.1.

Proof. The Pentagon Identity: Let (W, w_0), (X, x_0), (Y, y_0) and (Z, z_0) be pointed sets. We have to show that the diagram

$$\begin{tikzcd}
 & W \triangleright ((X \triangleright Y) \triangleright Z) \arrow[dr, "\alpha^{\mathsf{Sets}_*,\triangleright}_{W,X\triangleright Y,Z}"] & \\
W \triangleright (X \triangleright (Y \triangleright Z)) \arrow[ur, "\alpha^{\mathsf{Sets}_*,\triangleright}_{W,X,Y} \triangleright \mathrm{id}_Z"] \arrow[d, "\alpha^{\mathsf{Sets}_*,\triangleright}_{W\triangleright X,Y,Z}"'] & & (W \triangleright (X \triangleright Y)) \triangleright Z \arrow[d, "\mathrm{id}_W \triangleright \alpha^{\mathsf{Sets}_*,\triangleright}_{X,Y,Z}"] \\
(W \triangleright X) \triangleright (Y \triangleright Z) \arrow[rr, "\alpha^{\mathsf{Sets}_*,\triangleright}_{W,X,Y\triangleright Z}"'] & & ((W \triangleright X) \triangleright Y) \triangleright Z
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$
\begin{tikzcd}
 & w \rhd ((x \rhd y) \rhd z) \arrow[dr, mapsto] & \\
w \rhd (x \rhd (y \rhd z)) \arrow[ur, mapsto] \arrow[d, mapsto] & & (w \rhd (x \rhd y)) \rhd z \arrow[d, mapsto] \\
(w \rhd x) \rhd (y \rhd z) \arrow[rr, mapsto] & & ((w \rhd x) \rhd y) \rhd z
\end{tikzcd}
$$

and thus we see that the pentagon identity is satisfied.
The Right Skew Left Triangle Identity: Let (X, x_0) and (Y, y_0) be pointed sets. We have to show that the diagram

$$
\begin{tikzcd}
X \rhd Y \arrow[d, "\lambda^{\mathsf{Sets}_*,\rhd}_{X \rhd Y}"'] \arrow[dr, "\lambda^{\mathsf{Sets}_*,\rhd}_{X} \rhd \mathrm{id}_Y"] & \\
S^0 \rhd (X \rhd Y) \arrow[r, "\alpha^{\mathsf{Sets}_*,\rhd}_{S^0,X,Y}"'] & \left(S^0 \rhd X\right) \rhd Y
\end{tikzcd}
$$

commutes. Indeed, this diagram acts on elements as

$$
\begin{tikzcd}
x \rhd y \arrow[d, mapsto] \arrow[dr, mapsto] & \\
1 \rhd (x \rhd y) \arrow[r, mapsto] & (1 \rhd x) \rhd y
\end{tikzcd}
$$

and hence indeed commutes. Thus the left skew triangle identity is satisfied.
The Right Skew Right Triangle Identity: Let (X, x_0) and (Y, y_0) be pointed

sets. We have to show that the diagram

$$\begin{tikzcd}
X \rhd \left(Y \rhd S^0\right) \arrow[r, "\mathrm{id}_X \rhd \rho^{\mathsf{Sets}_*,\rhd}_Y"] \arrow[dr, "\alpha^{\mathsf{Sets}_*,\rhd}_{S^0,X,Y}"'] & (X \rhd Y) \rhd S^0 \arrow[d, "\rho^{\mathsf{Sets}_*,\rhd}_{X \rhd Y}"] \\
& X \rhd Y
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
x \rhd (y \rhd 0) \arrow[r, mapsto] \arrow[dr, mapsto] & (x \rhd y) \rhd 0 \arrow[d, mapsto] \\
& x \rhd y_0 = x_0 \rhd y_0
\end{tikzcd}$$

and

$$\begin{tikzcd}
x \rhd (y \rhd 1) \arrow[r, mapsto] \arrow[dr, mapsto] & (x \rhd y) \rhd 1 \arrow[d, mapsto] \\
& x \rhd y
\end{tikzcd}$$

and hence indeed commutes. Thus the right skew triangle identity is satisfied. *The Right Skew Middle Triangle Identity*: Let (X, x_0) and (Y, y_0) be pointed sets. We have to show that the diagram

$$\begin{tikzcd}
X \rhd Y \arrow[r, equal] \arrow[d, "\mathrm{id}_X \rhd \lambda^{\mathsf{Sets}_*,\rhd}_Y"'] & X \rhd Y \\
X \rhd \left(S^0 \rhd Y\right) \arrow[r, "\alpha^{\mathsf{Sets}_*,\rhd}_{X,S^0,Y}"'] & \left(X \rhd S^0\right) \rhd Y \arrow[u, "\rho^{\mathsf{Sets}_*,\rhd}_X \rhd \mathrm{id}_Y"']
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
x \rhd y \arrow[r, mapsto] \arrow[d, mapsto] & x \rhd y \\
x \rhd (1 \rhd y) \arrow[r, mapsto] & (x \rhd 1) \rhd y \arrow[u, mapsto]
\end{tikzcd}$$

and hence indeed commutes. Thus the right skew triangle identity is satisfied.

The Zig-Zag Identity: We have to show that the diagram

$$\begin{tikzcd} S^0 \arrow[r, "\lambda^{\mathsf{Sets}_*,\rhd}_{S^0}"] \arrow[dr, equal] & S^0 \rhd S^0 \arrow[d, "\rho^{\mathsf{Sets}_*,\rhd}_{S^0}"] \\ & S^0 \end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd} 0 \arrow[r, mapsto] \arrow[dr, mapsto] & 1 \rhd 0 \arrow[d, mapsto] \\ & 0 \end{tikzcd}$$

and

$$\begin{tikzcd} 1 \arrow[r, mapsto] \arrow[dr, mapsto] & 1 \rhd 1 \arrow[d, mapsto] \\ & 1 \end{tikzcd}$$

and hence indeed commutes. Thus the zig-zag identity is satisfied.
Right Skew Monoidal Right-Closedness: This follows from Item 2 of Proposition 4.4.1.1.7. □

4.4.9 Monoids With Respect to the Right Tensor Product of Pointed Sets

Proposition 4.4.9.1.1. The category of monoids on $(\mathsf{Sets}_*, \rhd, S^0)$ is isomorphic to the category of "monoids with right zero"[18] and morphisms between them.

Proof. Monoids on $(\mathsf{Sets}_*, \rhd, S^0)$: A monoid on $(\mathsf{Sets}_*, \rhd, S^0)$ consists of:

- *The Underlying Object.* A pointed set $(A, 0_A)$.
- *The Multiplication Morphism.* A morphism of pointed sets

$$\mu_A \colon A \rhd A \to A,$$

[18] A monoid with right zero is defined similarly as the monoids with zero of **??**. Succinctly, they are monoids (A, μ_A, η_A) with a special element 0_A satisfying

$$0_A a = 0_A$$

for each $a \in A$.

determining a right bilinear morphism of pointed sets

$$\begin{aligned} A \times A &\longrightarrow A \\ (a, b) &\longmapsto ab. \end{aligned}$$

- *The Unit Morphism.* A morphism of pointed sets

$$\eta_A \colon S^0 \to A$$

picking an element 1_A of A.

satisfying the following conditions:

1. *Associativity.* The diagram

$$\begin{tikzcd} & A \rhd (A \rhd A) \arrow[dr, "\mathrm{id}_A \rhd \mu_A"] & \\ (A \rhd A) \rhd A \arrow[ur, "\alpha^{\mathsf{Sets}_*, \rhd}_{A,A,A}"] \arrow[d, "\mu_A \rhd \mathrm{id}_A"'] & & A \rhd A \arrow[d, "\mu_A"] \\ A \rhd A \arrow[rr, "\mu_A"'] & & A \end{tikzcd}$$

2. *Left Unitality.* The diagram

$$\begin{tikzcd} A \arrow[r, "\lambda^{\mathsf{Sets}_*, \rhd}_A"] \arrow[d, equal] & S^0 \rhd A \arrow[d, "\eta_A \times \mathrm{id}_A"] \\ A & A \rhd A \arrow[l, "\mu_A"] \end{tikzcd}$$

commutes.

3. *Right Unitality.* The diagram

$$\begin{tikzcd} A \rhd S^0 \arrow[r, "\mathrm{id}_A \times \eta_A"] \arrow[dr, "\rho^{\mathsf{Sets}_*, \rhd}_A"'] & A \rhd A \arrow[d, "\mu_A"] \\ & A \end{tikzcd}$$

commutes.

Being a right-bilinear morphism of pointed sets, the multiplication map satisfies

$$0_A a = 0_A$$

for each $a \in A$. Now, the associativity, left unitality, and right unitality conditions act on elements as follows:

1. *Associativity.* The associativity condition acts as

$$\begin{tikzcd}
(a \rhd b) \rhd c \arrow[d, mapsto] & \\
ab \rhd c \arrow[r, mapsto] & (ab)c
\end{tikzcd}
\qquad
\begin{tikzcd}
& a \rhd (b \rhd c) \arrow[dr, mapsto] & \\
(a \rhd b) \rhd c \arrow[ur, mapsto] & & a \rhd bc \arrow[dl, mapsto] \\
& a(bc) &
\end{tikzcd}$$

 This gives

$$(ab)c = a(bc)$$

 for each $a, b, c \in A$.

2. *Left Unitality.* The left unitality condition acts as

$$\begin{tikzcd}
a \arrow[d, mapsto] \\
a
\end{tikzcd}
\qquad
\begin{tikzcd}
a \arrow[r, mapsto] & 1 \rhd a \arrow[d, mapsto] \\
1_A a & 1_A \rhd a \arrow[l, mapsto]
\end{tikzcd}$$

 This gives

$$1_A a = a$$

 for each $a \in A$.

3. *Right Unitality.* The right unitality condition acts:

 (a) On $1 \rhd 0$ as

$$\begin{tikzcd}
1 \rhd 0 \arrow[dr, mapsto] & \\
& 0_A
\end{tikzcd}
\qquad
\begin{tikzcd}
a \rhd 0 \arrow[r, mapsto] \arrow[dr, mapsto] & a \rhd 0_A \arrow[d, mapsto] \\
& a0_A.
\end{tikzcd}$$

(b) On $a \rhd 1$ as

$$
\begin{tikzcd}
a \rhd 1 \arrow[dr, mapsto] & \\
& a
\end{tikzcd}
\qquad
\begin{tikzcd}
a \rhd 1 \arrow[r, mapsto] \arrow[dr, mapsto] & a \rhd 1_A \arrow[d, mapsto] \\
& a1_A.
\end{tikzcd}
$$

This gives

$$
\begin{aligned}
a1_A &= a, \\
a0_A &= 0_A
\end{aligned}
$$

for each $a \in A$.

Thus we see that monoids with respect to $\rhd$ are exactly monoids with right zero.

Morphisms of Monoids on $(\mathsf{Sets}_*, \rhd, S^0)$: A morphism of monoids on $(\mathsf{Sets}_*, \rhd, S^0)$ from $(A, \mu_A, \eta_A, 0_A)$ to $(B, \mu_B, \eta_B, 0_B)$ is a morphism of pointed sets

$$f \colon (A, 0_A) \to (B, 0_B)$$

satisfying the following conditions:

1. *Compatibility With the Multiplication Morphisms.* The diagram

$$
\begin{tikzcd}
A \rhd A \arrow[r, "f \rhd f"] \arrow[d, "\mu_A"'] & B \rhd B \arrow[d, "\mu_B"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}
$$

commutes.

2. *Compatibility With the Unit Morphisms.* The diagram

$$
\begin{tikzcd}
S^0 \arrow[r, "\eta_A"] \arrow[dr, "\eta_B"'] & A \arrow[d, "f"] \\
& B
\end{tikzcd}
$$

commutes.

These act on elements as

$$
\begin{tikzcd}
a \triangleright b \arrow[d, mapsto] & & a \triangleright b \arrow[r, mapsto] & f(a) \triangleright f(b) \arrow[d, mapsto] \\
ab \arrow[r, mapsto] & f(ab) & & f(a)f(b)
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
0 \arrow[dr, mapsto] & & 0 \arrow[r, mapsto] & 0_A \arrow[d, mapsto] \\
 & 0_B & & f(0_A)
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
1 \arrow[dr, mapsto] & & 1 \arrow[r, mapsto] & 1_A \arrow[d, mapsto] \\
 & 1_B & & f(1_A)
\end{tikzcd}
$$

giving

$$
\begin{aligned}
f(ab) &= f(a)f(b), \\
f(0_A) &= 0_B, \\
f(1_A) &= 1_B,
\end{aligned}
$$

for each $a, b \in A$, which is exactly a morphism of monoids with right zero.
Identities and Composition: Similarly, the identities and composition of $\mathsf{Mon}(\mathsf{Sets}_*, \triangleright, S^0)$ can be easily seen to agree with those of monoids with right zero, which finishes the proof. □

4.5 The Smash Product of Pointed Sets

4.5.1 Foundations

Let (X, x_0) and (Y, y_0) be pointed sets.

Definition 4.5.1.1.1. The **smash product of** (X, x_0) **and** (Y, y_0)[19] is the

[19] *Further Terminology:* In the context of monoids with zero as models for $\mathbb{F}_1$-algebras, the smash product $X \wedge Y$ is also called the **tensor product of** $\mathbb{F}_1$**-modules of** (X, x_0) **and** (Y, y_0) or the **tensor product of** (X, x_0) **and** (Y, y_0) **over** $\mathbb{F}_1$.

pointed set $X \wedge Y$[20] satisfying the bijection

$$\mathsf{Sets}_*(X \wedge Y, Z) \cong \mathrm{Hom}^{\otimes}_{\mathsf{Sets}_*}(X \times Y, Z),$$

naturally in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

Remark 4.5.1.1.2. That is to say, the smash product of pointed sets is defined so as to induce a bijection between the following data:

- Pointed maps $f \colon X \wedge Y \to Z$.
- Maps of sets $f \colon X \times Y \to Z$ satisfying

$$f(x_0, y) = z_0,$$
$$f(x, y_0) = z_0$$

  for each $x \in X$ and each $y \in Y$.

Remark 4.5.1.1.3. The smash product of pointed sets may be described as follows:

- The smash product of (X, x_0) and (Y, y_0) is the pair $((X \wedge Y, x_0 \wedge y_0), \iota)$ consisting of
  - A pointed set $(X \wedge Y, x_0 \wedge y_0)$;
  - A bilinear morphism of pointed sets $\iota \colon (X \times Y, (x_0, y_0)) \to X \wedge Y$;

  satisfying the following universal property:

  (UP) Given another such pair $((Z, z_0), f)$ consisting of
  * A pointed set (Z, z_0);
  * A bilinear morphism of pointed sets $f \colon (X \times Y, (x_0, y_0)) \to X \wedge Y$;

  there exists a unique morphism of pointed sets $X \wedge Y \xrightarrow{\exists!} Z$ making the diagram

$$\begin{tikzcd} & X \wedge Y \arrow[d, dashed, "\exists!"] \\ X \times Y \arrow[ur, "\iota"] \arrow[r, "f"'] & Z \end{tikzcd}$$

  commute.

[20] *Further Notation:* In the context of monoids with zero as models for $\mathbb{F}_1$-algebras, the

Construction 4.5.1.1.4. Concretely, the **smash product of** (X, x_0) **and** (Y, y_0) is the pointed set $(X \wedge Y, x_0 \wedge y_0)$ consisting of

- *The Underlying Set.* The set $X \wedge Y$ defined by
$$X \wedge Y \cong (X \times Y)/\sim_R,$$
where $\sim_R$ is the equivalence relation on $X \times Y$ obtained by declaring
$$\begin{aligned}(x_0, y) &\sim_R (x_0, y'),\\ (x, y_0) &\sim_R (x', y_0)\end{aligned}$$
for each $x, x' \in X$ and each $y, y' \in Y$;
- *The Basepoint.* The element $[(x_0, y_0)]$ of $X \wedge Y$ given by the equivalence class of (x_0, y_0) under the equivalence relation $\sim$ on $X \times Y$.

Proof. By Item 6 of Proposition 7.5.2.1.3, we have a natural bijection
$$\mathsf{Sets}_*(X \wedge Y, Z) \cong \mathrm{Hom}^R_{\mathsf{Sets}}(X \times Y, Z).$$
Now, by definition, $\mathrm{Hom}^R_{\mathsf{Sets}}(X \times Y, Z)$ is the set
$$\mathrm{Hom}^R_{\mathsf{Sets}}(X \times Y, Z) \overset{\text{def}}{=} \left\{ f \in \mathrm{Hom}_{\mathsf{Sets}}(X \times Y, Z) \;\middle|\; \begin{array}{l}\text{for each } x, y \in X, \text{ if}\\ (x, y) \sim_R (x', y'), \text{ then}\\ f(x, y) = f(x', y')\end{array} \right\}.$$
However, the condition $(x, y) \sim_R (x', y')$ only holds when:

1. We have $x = x'$ and $y = y'$.
2. The following conditions are satisfied:
   (a) We have $x = x_0$ or $y = y_0$.
   (b) We have $x' = x_0$ or $y' = y_0$.

So, given $f \in \mathrm{Hom}_{\mathsf{Sets}}(X \times Y, Z)$ with a corresponding $\overline{f} \colon X \wedge Y \to Z$, the latter case above implies
$$\begin{aligned} f(x_0, y) &= f(x, y_0)\\ &= f(x_0, y_0), \end{aligned}$$

smash product $X \wedge Y$ is also denoted $X \otimes_{\mathbb{F}_1} Y$.

and since $\overline{f}\colon X \wedge Y \to Z$ is a pointed map, we have

$$\begin{aligned} f(x_0, y_0) &= \overline{f}(x_0, y_0) \\ &= z_0. \end{aligned}$$

Thus the elements f in $\mathrm{Hom}_{\mathsf{Sets}}(X \times Y, Z)$ are precisely those functions $f\colon X \times Y \to Z$ satisfying the equalities

$$\begin{aligned} f(x_0, y) &= z_0, \\ f(x, y_0) &= z_0 \end{aligned}$$

for each $x \in X$ and each $y \in Y$, giving an equality

$$\mathrm{Hom}^{R}_{\mathsf{Sets}}(X \times Y, Z) = \mathrm{Hom}^{\otimes}_{\mathsf{Sets}_*}(X \times Y, Z)$$

of sets, which when composed with our earlier isomorphism

$$\mathsf{Sets}_*(X \wedge Y, Z) \cong \mathrm{Hom}^{R}_{\mathsf{Sets}}(X \times Y, Z)$$

gives our desired natural bijection, finishing the proof. □

Remark 4.5.1.1.5. It is also somewhat common to write

$$X \wedge Y \overset{\text{def}}{=} \frac{X \times Y}{X \vee Y},$$

identifying $X \vee Y$ with the subspace $(\{x_0\} \times Y) \cup (X \times \{y_0\})$ of $X \times Y$, and having the quotient be defined by declaring $(x, y) \sim (x', y')$ iff we have $(x, y), (x', y') \in X \vee Y$.

Notation 4.5.1.1.6. We write $x \wedge y$ for the element $[(x, y)]$ of

$$X \wedge Y \cong X \times Y/{\sim}.$$

Remark 4.5.1.1.7. Employing the notation introduced in Notation 4.5.1.1.6, we have

$$\begin{aligned} x_0 \wedge y_0 &= x \wedge y_0, \\ &= x_0 \wedge y \end{aligned}$$

for each $x \in X$ and each $y \in Y$, and

$$\begin{aligned} x \wedge y_0 &= x' \wedge y_0, \\ x_0 \wedge y &= x_0 \wedge y' \end{aligned}$$

for each $x, x' \in X$ and each $y, y' \in Y$.

Example 4.5.1.1.8. Here are some examples of smash products of pointed sets.

1. *Smashing With pt.* For any pointed set X, we have isomorphisms of pointed sets
$$\begin{aligned} \mathrm{pt} \wedge X &\cong \mathrm{pt}, \\ X \wedge \mathrm{pt} &\cong \mathrm{pt}. \end{aligned}$$

2. *Smashing With S^0.* For any pointed set X, we have isomorphisms of pointed sets
$$\begin{aligned} S^0 \wedge X &\cong X, \\ X \wedge S^0 &\cong X. \end{aligned}$$

Proposition 4.5.1.1.9. Let (X, x_0) and (Y, y_0) be pointed sets.

1. *Functoriality.* The assignments $X, Y, (X, Y) \mapsto X \wedge Y$ define functors
$$\begin{aligned} X \wedge - \colon \mathsf{Sets}_* &\to \mathsf{Sets}_*, \\ - \wedge Y \colon \mathsf{Sets}_* &\to \mathsf{Sets}_*, \\ -_1 \wedge -_2 \colon \mathsf{Sets}_* \times \mathsf{Sets}_* &\to \mathsf{Sets}_*. \end{aligned}$$
   In particular, given pointed maps
$$\begin{aligned} f \colon (X, x_0) &\to (A, a_0), \\ g \colon (Y, y_0) &\to (B, b_0), \end{aligned}$$
   the induced map
$$f \wedge g \colon X \wedge Y \to A \wedge B$$
   is given by
$$[f \wedge g](x \wedge y) \overset{\text{def}}{=} f(x) \wedge g(y)$$
   for each $x \wedge y \in X \wedge Y$.

2. *Adjointness.* We have adjunctions
$$\begin{aligned} (X \wedge - \dashv \mathbf{Sets}_*(X, -)) \colon \quad & \begin{tikzcd} \mathsf{Sets}_* \arrow[r, bend left, "X\wedge -"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_* \arrow[l, bend left, "\mathbf{Sets}_*(X,-)"] \end{tikzcd}, \\ (- \wedge Y \dashv \mathbf{Sets}_*(Y, -)) \colon \quad & \begin{tikzcd} \mathsf{Sets}_* \arrow[r, bend left, "-\wedge Y"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_* \arrow[l, bend left, "\mathbf{Sets}_*(Y,-)"] \end{tikzcd}, \end{aligned}$$

witnessed by bijections

$$\begin{aligned}\mathrm{Hom}_{\mathsf{Sets}_*}(X \wedge Y, Z) &\cong \mathrm{Hom}_{\mathsf{Sets}_*}(X, \mathbf{Sets}_*(Y, Z)),\\ \mathrm{Hom}_{\mathsf{Sets}_*}(X \wedge Y, Z) &\cong \mathrm{Hom}_{\mathsf{Sets}_*}(X, \mathbf{Sets}_*(A, Z)),\end{aligned}$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

3. *Enriched Adjointness.* We have Sets_*-enriched adjunctions

$$(X \wedge - \dashv \mathbf{Sets}_*(X, -))\colon \quad \begin{tikzcd} \mathbf{Sets}_* \arrow[r, bend left, "X\wedge -"] \arrow[r, phantom, "\perp"] & \mathbf{Sets}_*, \arrow[l, bend left, "\mathbf{Sets}_*(X,-)"] \end{tikzcd}$$

$$(- \wedge Y \dashv \mathbf{Sets}_*(Y, -))\colon \quad \begin{tikzcd} \mathbf{Sets}_* \arrow[r, bend left, "-\wedge Y"] \arrow[r, phantom, "\perp"] & \mathbf{Sets}_*, \arrow[l, bend left, "\mathbf{Sets}_*(Y,-)"] \end{tikzcd}$$

witnessed by isomorphisms of pointed sets

$$\begin{aligned}\mathbf{Sets}_*(X \wedge Y, Z) &\cong \mathbf{Sets}_*(X, \mathbf{Sets}_*(Y, Z)),\\ \mathbf{Sets}_*(X \wedge Y, Z) &\cong \mathbf{Sets}_*(X, \mathbf{Sets}_*(A, Z)),\end{aligned}$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathbf{Sets}_*)$.

4. *As a Pushout.* We have an isomorphism

$$X \wedge Y \cong \mathrm{pt} \coprod_{X \vee Y} (X \times Y), \qquad \begin{tikzcd} X \wedge Y \arrow[dr, phantom, "\ulcorner", very near start] & X \times Y \arrow[l] \\ \mathrm{pt} \arrow[u] & X \vee Y, \arrow[l, "!"] \arrow[u, hook, "\iota"'] \end{tikzcd}$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets}_*)$, where the pushout is taken in Sets, and the embedding $\iota\colon X \vee Y \hookrightarrow X \times Y$ is defined following Remark 4.5.1.1.5.

5. *Distributivity Over Wedge Sums.* We have isomorphisms of pointed sets

$$\begin{aligned}X \wedge (Y \vee Z) &\cong (X \wedge Y) \vee (X \wedge Z),\\ (X \vee Y) \wedge Z &\cong (X \wedge Z) \vee (Y \wedge Z),\end{aligned}$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

Proof. Item 1, Functoriality: The map $f \wedge g$ comes from Item 4 of Proposition 7.5.2.1.3 via the map

$$f \wedge g\colon X \times Y \to A \wedge B$$

sending (x, y) to $f(x) \wedge g(y)$, which we need to show satisfies

$$[f \wedge g](x, y) = [f \wedge g](x', y')$$

for each $(x, y), (x', y') \in X \times Y$ with $(x, y) \sim_R (x', y')$, where $\sim_R$ is the relation constructing $X \wedge Y$ as

$$X \wedge Y \cong (X \times Y)/\sim_R$$

in Construction 4.5.1.1.4. The condition defining $\sim$ is that at least one of the following conditions is satisfied:

1. We have $x = x'$ and $y = y'$;
2. Both of the following conditions are satisfied:
   (a) We have $x = x_0$ or $y = y_0$.
   (b) We have $x' = x_0$ or $y' = y_0$.

We have five cases:

1. In the first case, we clearly have
   $$[f \wedge g](x, y) = [f \wedge g](x', y')$$
   since $x = x'$ and $y = y'$.
2. If $x = x_0$ and $x' = x_0$, we have
   $$\begin{aligned} [f \wedge g](x_0, y) &\overset{\text{def}}{=} f(x_0) \wedge g(y) \\ &= a_0 \wedge g(y) \\ &= a_0 \wedge g(y') \\ &= f(x_0) \wedge g(y') \\ &\overset{\text{def}}{=} [f \wedge g](x_0, y'). \end{aligned}$$
3. If $x = x_0$ and $y' = y_0$, we have
   $$\begin{aligned} [f \wedge g](x_0, y) &\overset{\text{def}}{=} f(x_0) \wedge g(y) \\ &= a_0 \wedge g(y) \\ &= a_0 \wedge b_0 \\ &= f(x') \wedge b_0 \\ &= f(x') \wedge g(y_0) \\ &\overset{\text{def}}{=} [f \wedge g](x', y_0). \end{aligned}$$

4. If $y = y_0$ and $x' = x_0$, we have

$$\begin{aligned}[f \wedge g](x, y_0) &\overset{\text{def}}{=} f(x) \wedge g(y_0)\\ &= f(x) \wedge b_0\\ &= a_0 \wedge b_0\\ &= a_0 \wedge g(y')\\ &= f(x_0) \wedge g(y')\\ &\overset{\text{def}}{=} [f \wedge g](x_0, y').\end{aligned}$$

5. If $y = y_0$ and $y' = y_0$, we have

$$\begin{aligned}[f \wedge g](x, y_0) &\overset{\text{def}}{=} f(x) \wedge g(y_0)\\ &= f(x) \wedge b_0\\ &= f(x') \wedge b_0\\ &= f(x) \wedge g(y_0)\\ &\overset{\text{def}}{=} [f \wedge g](x', y_0).\end{aligned}$$

Thus $f \wedge g$ is well-defined. Next, we claim that $\wedge$ preserves identities and composition:

- *Preservation of Identities.* We have

  $$\begin{aligned}[\mathrm{id}_X \wedge \mathrm{id}_Y](x \wedge y) &\overset{\text{def}}{=} \mathrm{id}_X(x) \wedge \mathrm{id}_Y(y)\\ &= x \wedge y\\ &= [\mathrm{id}_{X \wedge Y}](x \wedge y)\end{aligned}$$

  for each $x \wedge y \in X \wedge Y$, and thus

  $$\mathrm{id}_X \wedge \mathrm{id}_Y = \mathrm{id}_{X \wedge Y}.$$

- *Preservation of Composition.* Given pointed maps

  $$\begin{aligned}f &\colon (X, x_0) \to \left(X', x_0'\right),\\ h &\colon \left(X', x_0'\right) \to \left(X'', x_0''\right),\\ g &\colon (Y, y_0) \to \left(Y', y_0'\right),\\ k &\colon \left(Y', y_0'\right) \to \left(Y'', y_0''\right),\end{aligned}$$

  we have

  $$\begin{aligned}[(h \circ f) \wedge (k \circ g)](x \wedge y) &\overset{\text{def}}{=} h(f(x)) \wedge k(g(y))\\ &\overset{\text{def}}{=} [h \wedge k](f(x) \wedge g(y))\\ &\overset{\text{def}}{=} [h \wedge k]([f \wedge g](x \wedge y))\\ &\overset{\text{def}}{=} [(h \wedge k) \circ (f \wedge g)](x \wedge y)\end{aligned}$$

for each $x \wedge y \in X \wedge Y$, and thus

$$(h \circ f) \wedge (k \circ g) = (h \wedge k) \circ (f \wedge g).$$

This finishes the proof.

Item 2, Adjointness: We prove only the adjunction $- \wedge Y \dashv \mathbf{Sets}_*(Y, -)$, witnessed by a natural bijection

$$\mathrm{Hom}_{\mathsf{Sets}_*}(X \wedge Y, Z) \cong \mathrm{Hom}_{\mathsf{Sets}_*}(X, \mathbf{Sets}_*(Y, Z)),$$

as the proof of the adjunction $X \wedge - \dashv \mathbf{Sets}_*(X, -)$ is similar. We claim we have a bijection

$$\mathrm{Hom}^{\otimes}_{\mathsf{Sets}_*}(X \times Y, Z) \cong \mathrm{Hom}_{\mathsf{Sets}_*}(X, \mathbf{Sets}_*(Y, Z))$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, implying the desired adjunction. Indeed, this bijection is a restriction of the bijection

$$\mathsf{Sets}(X \times Y, Z) \cong \mathsf{Sets}(X, \mathsf{Sets}(Y, Z))$$

of Item 2 of Proposition 2.1.3.1.2:

- A map
$$\xi \colon X \times Y \to Z$$
in $\mathrm{Hom}^{\otimes}_{\mathsf{Sets}_*}(X \times Y, Z)$ gets sent to the pointed map
$$\begin{aligned} \xi^\dagger \colon (X, x_0) &\to \left(\mathbf{Sets}_*(Y, Z), \Delta_{z_0}\right), \\ x &\longmapsto \left(\xi^\dagger_x \colon Y \to Z\right), \end{aligned}$$
where $\xi^\dagger_x \colon Y \to Z$ is the map defined by
$$\xi^\dagger_x(y) \stackrel{\text{def}}{=} \xi(x, y)$$
for each $y \in Y$, where:
  - The map $\xi^\dagger$ is indeed pointed, as we have
  $$\begin{aligned} \xi^\dagger_{x_0}(y) &\stackrel{\text{def}}{=} \xi(x_0, y) \\ &\stackrel{\text{def}}{=} z_0 \end{aligned}$$
  for each $y \in Y$. Thus $\xi^\dagger_{x_0} = \Delta_{z_0}$ and $\xi^\dagger$ is pointed.

- The map $\xi^{\dagger}_{x}$ indeed lies in $\mathsf{Sets}_*(Y, Z)$, as we have

$$\begin{aligned}\xi^{\dagger}_{x}(y_0) &\overset{\text{def}}{=} \xi(x, y_0)\\ &\overset{\text{def}}{=} z_0.\end{aligned}$$

- Conversely, a map

$$\begin{aligned}\xi\colon (X, x_0) &\to \big(\mathsf{Sets}_*(Y, Z), \Delta_{z_0}\big),\\ x &\longmapsto (\xi_x\colon Y \to Z),\end{aligned}$$

in $\mathrm{Hom}_{\mathsf{Sets}_*}(X, \mathsf{Sets}_*(Y, Z))$ gets sent to the map

$$\xi^{\dagger}\colon X \times Y \to Z$$

defined by

$$\xi^{\dagger}(x, y) \overset{\text{def}}{=} \xi_x(y)$$

for each $(x, y) \in X \times Y$, which indeed lies in $\mathrm{Hom}^{\otimes}_{\mathsf{Sets}_*}(X \times Y, Z)$, as:

  - *Left Bilinearity*. We have

$$\begin{aligned}\xi^{\dagger}(x_0, y) &\overset{\text{def}}{=} \xi_{x_0}(y)\\ &\overset{\text{def}}{=} \Delta_{z_0}(y)\\ &\overset{\text{def}}{=} z_0\end{aligned}$$

  for each $y \in Y$, since $\xi_{x_0} = \Delta_{z_0}$ as ξ is assumed to be a pointed map.

  - *Right Bilinearity*. We have

$$\begin{aligned}\xi^{\dagger}(x, y_0) &\overset{\text{def}}{=} \xi_x(y_0)\\ &\overset{\text{def}}{=} z_0\end{aligned}$$

  for each $x \in X$, since $\xi_x \in \mathsf{Sets}_*(Y, Z)$ is a morphism of pointed sets.

This finishes the proof.

Item 3, Enriched Adjointness: This follows from Item 2 and **??** of **??**.

Item 4, As a Pushout: Following the description of Remark 2.2.4.1.2, we have

$$\mathrm{pt} \textstyle\coprod_{X \vee Y}(X \times Y) \cong (\mathrm{pt} \times (X \times Y))/{\sim},$$

where $\sim$ identifies the elemenet $\star$ in pt with all elements of the form (x_0, y) and (x, y_0) in $X \times Y$. Thus Item 4 of Proposition 7.5.2.1.3 coupled with Remark 4.5.1.1.7 then gives us a well-defined map

$$\mathrm{pt} \textstyle\coprod_{X \vee Y}(X \times Y) \to X \wedge Y$$

via $[(\star, (x, y))] \mapsto x \wedge y$, with inverse

$$X \wedge Y \to \mathrm{pt} \coprod_{X \vee Y} (X \times Y)$$

given by $x \wedge y \mapsto [(\star, (x, y))]$.
Item 5, Distributivity Over Wedge Sums: This follows from Proposition 4.5.9.1.1, **??** of **??**, and the fact that $\vee$ is the coproduct in Sets_* (Definition 3.3.3.1.1). □

4.5.2 The Internal Hom of Pointed Sets

Let (X, x_0) and (Y, y_0) be pointed sets.

Definition 4.5.2.1.1. The **internal Hom**[21] **of pointed sets from** (X, x_0) **to** (Y, y_0) is the pointed set $\mathbf{Sets}_*((X, x_0), (Y, y_0))$[22] consisting of:

- *The Underlying Set.* The set $\mathsf{Sets}_*((X, x_0), (Y, y_0))$ of morphisms of pointed sets from (X, x_0) to (Y, y_0).
- *The Basepoint.* The element

  $$\Delta_{y_0} : (X, x_0) \to (Y, y_0)$$

  of $\mathsf{Sets}_*((X, x_0), (Y, y_0))$ given by

  $$\Delta_{y_0}(x) \stackrel{\text{def}}{=} y_0$$

  for each $x \in X$.

Proof. For a proof that $\mathbf{Sets}_*$ is indeed the internal Hom of Sets_* with respect to the smash product of pointed sets, see Item 2 of Proposition 4.5.1.1.9. □

Proposition 4.5.2.1.2. Let (X, x_0) and (Y, y_0) be pointed sets.

1. *Functoriality.* The assignments $X, Y, (X, Y) \mapsto \mathbf{Sets}_*(X, Y)$ define functors

$$\mathbf{Sets}_*(X, -) : \mathsf{Sets}_* \to \mathsf{Sets}_*,$$
$$\mathbf{Sets}_*(-, Y) : \mathsf{Sets}_*^{\mathsf{op}} \to \mathsf{Sets}_*,$$
$$\mathbf{Sets}_*(-_1, -_2) : \mathsf{Sets}_*^{\mathsf{op}} \times \mathsf{Sets}_* \to \mathsf{Sets}_*.$$

[21] The pointed set $\mathbf{Sets}_*(X, Y)$ is the internal **Hom** of Sets_* with respect to the smash product of Definition 4.5.1.1.1; see Item 2 of Proposition 4.5.1.1.9.
[22] *Further Notation:* Also written $\mathbf{Hom}_{\mathbf{Sets}_*}(X, Y)$.

In particular, given pointed maps

$$f\colon (X,x_0) \to (A,a_0),$$
$$g\colon (Y,y_0) \to (B,b_0),$$

the induced map

$$\mathbf{Sets}_*(f,g)\colon \mathbf{Sets}_*(A,Y) \to \mathbf{Sets}_*(X,B)$$

is given by

$$[\mathbf{Sets}_*(f,g)](\phi) \stackrel{\text{def}}{=} g\circ\phi\circ f$$

for each $\phi \in \mathbf{Sets}_*(A,Y)$.

2. *Adjointness.* We have adjunctions

$$(X\wedge - \dashv \mathbf{Sets}_*(X,-))\colon \quad \begin{tikzcd} \mathsf{Sets}_* \arrow[r, bend left, "X\wedge-"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_*, \arrow[l, bend left, "\mathbf{Sets}_*(X,-)"] \end{tikzcd}$$

$$(-\wedge Y \dashv \mathbf{Sets}_*(Y,-))\colon \quad \begin{tikzcd} \mathsf{Sets}_* \arrow[r, bend left, "-\wedge Y"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_*, \arrow[l, bend left, "\mathbf{Sets}_*(Y,-)"] \end{tikzcd}$$

witnessed by bijections

$$\mathrm{Hom}_{\mathsf{Sets}_*}(X\wedge Y,Z) \cong \mathrm{Hom}_{\mathsf{Sets}_*}(X,\mathbf{Sets}_*(Y,Z)),$$
$$\mathrm{Hom}_{\mathsf{Sets}_*}(X\wedge Y,Z) \cong \mathrm{Hom}_{\mathsf{Sets}_*}(X,\mathbf{Sets}_*(A,Z)),$$

natural in $(X,x_0),(Y,y_0),(Z,z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

3. *Enriched Adjointness.* We have Sets_*-enriched adjunctions

$$(X\wedge - \dashv \mathbf{Sets}_*(X,-))\colon \quad \begin{tikzcd} \mathbf{Sets}_* \arrow[r, bend left, "X\wedge-"] \arrow[r, phantom, "\perp"] & \mathbf{Sets}_*, \arrow[l, bend left, "\mathbf{Sets}_*(X,-)"] \end{tikzcd}$$

$$(-\wedge Y \dashv \mathbf{Sets}_*(Y,-))\colon \quad \begin{tikzcd} \mathbf{Sets}_* \arrow[r, bend left, "-\wedge Y"] \arrow[r, phantom, "\perp"] & \mathbf{Sets}_*, \arrow[l, bend left, "\mathbf{Sets}_*(Y,-)"] \end{tikzcd}$$

witnessed by isomorphisms of pointed sets

$$\mathbf{Sets}_*(X\wedge Y,Z) \cong \mathbf{Sets}_*(X,\mathbf{Sets}_*(Y,Z)),$$
$$\mathbf{Sets}_*(X\wedge Y,Z) \cong \mathbf{Sets}_*(X,\mathbf{Sets}_*(A,Z)),$$

natural in $(X,x_0),(Y,y_0),(Z,z_0) \in \mathrm{Obj}(\mathbf{Sets}_*)$.

Proof. *Item 1, Functoriality*: This follows from Item 1 of Proposition 2.3.5.1.2 and from the equalities

$$\begin{aligned} g \circ \Delta_{y_0} &= \Delta_{z_0}, \\ \Delta_{y_0} \circ f &= \Delta_{y_0} \end{aligned}$$

for morphisms $f\colon (K, k_0) \longrightarrow (X, x_0)$ and $g\colon (Y, y_0) \longrightarrow (Z, z_0)$, which guarantee pre- and postcomposition by morphisms of pointed sets to also be morphisms of pointed sets.

Item 2, Adjointness: This is a repetition of Item 2 of Proposition 4.5.1.1.9, and is proved there.

Item 3, Enriched Adjointness: This is a repetition of Item 3 of Proposition 4.5.1.1.9, and is proved there. □

4.5.3 The Monoidal Unit

Definition 4.5.3.1.1. The **monoidal unit of the smash product of pointed sets** is the functor

$$\mathbb{1}^{\mathsf{Sets}_*}\colon \mathsf{pt} \longrightarrow \mathsf{Sets}_*$$

defined by

$$\mathbb{1}_{\mathsf{Sets}_*} \stackrel{\text{def}}{=} S^0.$$

4.5.4 The Associator

Definition 4.5.4.1.1. The **associator of the smash product of pointed sets** is the natural isomorphism

$$\alpha^{\mathsf{Sets}_*}\colon \wedge \circ \left(\wedge \times \mathrm{id}_{\mathsf{Sets}_*}\right) \stackrel{\sim}{\Longrightarrow} \wedge \circ \left(\mathrm{id}_{\mathsf{Sets}_*} \times \wedge\right) \circ \boldsymbol{\alpha}^{\mathsf{Cats}}_{\mathsf{Sets}_*,\mathsf{Sets}_*,\mathsf{Sets}_*},$$

as in the diagram

$$\begin{tikzcd}
 & \mathsf{Sets}_* \times (\mathsf{Sets}_* \times \mathsf{Sets}_*) \arrow[dr, "\mathrm{id}\times\wedge"] & \\
(\mathsf{Sets}_* \times \mathsf{Sets}_*) \times \mathsf{Sets}_* \arrow[ur, dashed, "\boldsymbol{\alpha}^{\mathsf{Cats}}_{\mathsf{Sets}_*,\mathsf{Sets}_*,\mathsf{Sets}_*}", "\sim"'] \arrow[d, "\wedge\times\mathrm{id}"'] & & \mathsf{Sets}_* \times \mathsf{Sets}_* \arrow[d, "\wedge"] \\
\mathsf{Sets}_* \times \mathsf{Sets}_* \arrow[rr, "\wedge"'] \arrow[urr, Rightarrow, "\alpha^{\mathsf{Sets}_*}"] & & \mathsf{Sets}_*,
\end{tikzcd}$$

whose component

$$\alpha^{\mathsf{Sets}_*}_{X,Y,Z}\colon (X \wedge Y) \wedge Z \xrightarrow{\cong} X \wedge (Y \wedge Z)$$

at $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$ is given by

$$\alpha^{\mathsf{Sets}_*}_{X,Y,Z}((x \wedge y) \wedge z) \overset{\text{def}}{=} x \wedge (y \wedge z)$$

for each $(x \wedge y) \wedge z \in (X \wedge Y) \wedge Z$.

Proof. Well-Definedness: Let $[((x, y), z)] = [((x', y'), z')]$ be an element in $(X \wedge Y) \wedge Z$. Then either:

1. We have $x = x'$, $y = y'$, and $z = z'$.
2. Both of the following conditions are satisfied:
   (a) We have $x = x_0$ or $y = y_0$ or $z = z_0$.
   (b) We have $x' = x_0$ or $y' = y_0$ or $z' = z_0$.

In the first case, $\alpha^{\mathsf{Sets}_*}_{X,Y,Z}$ clearly sends both elements to the same element in $X \wedge (Y \wedge Z)$. Meanwhile, in the latter case both elements are equal to the basepoint $(x_0 \wedge y_0) \wedge z_0$ of $(X \wedge Y) \wedge Z$, which gets sent to the basepoint $x_0 \wedge (y_0 \wedge z_0)$ of $X \wedge (Y \wedge Z)$.

Being a Morphism of Pointed Sets: As just mentioned, we have

$$\alpha^{\mathsf{Sets}_*}_{X,Y,Z}((x_0 \wedge y_0) \wedge z_0) \overset{\text{def}}{=} x_0 \wedge (y_0 \wedge z_0),$$

and thus $\alpha^{\mathsf{Sets}_*}_{X,Y,Z}$ is a morphism of pointed sets.

Invertibility: Clearly, the inverse of $\alpha^{\mathsf{Sets}_*}_{X,Y,Z}$ is given by the morphism

$$\alpha^{\mathsf{Sets}_*,-1}_{X,Y,Z} \colon X \wedge (Y \wedge Z) \xrightarrow{\cong} (X \wedge Y) \wedge Z$$

defined by

$$\alpha^{\mathsf{Sets}_*,-1}_{X,Y,Z}(x \wedge (y \wedge z)) \overset{\text{def}}{=} (x \wedge y) \wedge z$$

for each $x \wedge (y \wedge z) \in X \wedge (Y \wedge Z)$.

Naturality: We need to show that, given morphisms of pointed sets

$$\begin{aligned} f &\colon (X, x_0) \to (X', x'_0), \\ g &\colon (Y, y_0) \to (Y', y'_0), \\ h &\colon (Z, z_0) \to (Z', z'_0) \end{aligned}$$

the diagram

$$\begin{tikzcd} (X \wedge Y) \wedge Z \arrow[r, "(f \wedge g) \wedge h"] \arrow[d, "\alpha^{\mathsf{Sets}_*}_{X,Y,Z}"'] & (X' \wedge Y') \wedge Z' \arrow[d, "\alpha^{\mathsf{Sets}_*}_{X',Y',Z'}"] \\ X \wedge (Y \wedge Z) \arrow[r, "f \wedge (g \wedge h)"'] & X' \wedge (Y' \wedge Z') \end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
(x \wedge y) \wedge z \arrow[r, mapsto] \arrow[d, mapsto] & (f(x) \wedge g(y)) \wedge h(z) \arrow[d, mapsto] \\
x \wedge (y \wedge z) \arrow[r, mapsto] & f(x) \wedge (g(y) \wedge h(z))
\end{tikzcd}$$

and hence indeed commutes, showing α^{Sets_*} to be a natural transformation.
Being a Natural Isomorphism: Since α^{Sets_*} is natural and $\alpha^{\mathsf{Sets}_*,-1}$ is a componentwise inverse to α^{Sets_*}, it follows from Item 2 of Proposition 8.8.6.1.2 that $\alpha^{\mathsf{Sets}_*,-1}$ is also natural. Thus α^{Sets_*} is a natural isomorphism. □

4.5.5 The Left Unitor

Definition 4.5.5.1.1. The **left unitor of the smash product of pointed sets** is the natural isomorphism

$$\lambda^{\mathsf{Sets}_*} \colon \wedge \circ \left(\mathbb{1}^{\mathsf{Sets}_*} \times \mathrm{id}_{\mathsf{Sets}_*}\right) \overset{\sim}{\Longrightarrow} \lambda^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}
\qquad
\begin{tikzcd}
\mathrm{pt} \times \mathsf{Sets}_* \arrow[r, "\mathbb{1}^{\mathsf{Sets}_*} \times \mathrm{id}"] \arrow[dr, dashed, bend right, "\lambda^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}"'] & \mathsf{Sets}_* \times \mathsf{Sets}_* \arrow[d, "\wedge"] \arrow[dl, phantom, "\Swarrow \lambda^{\mathsf{Sets}_*}"] \\
& \mathsf{Sets}_*,
\end{tikzcd}$$

whose component

$$\lambda^{\mathsf{Sets}_*}_X \colon S^0 \wedge X \xrightarrow{\cong} X$$

at $X \in \mathrm{Obj}(\mathsf{Sets}_*)$ is given by

$$\begin{aligned}
0 \wedge x &\mapsto x_0, \\
1 \wedge x &\mapsto x.
\end{aligned}$$

Proof. Well-Definedness: Let $[(x, y)] = [(x', y')]$ be an element in $S^0 \wedge X$. Then either:

1. We have $x = x'$ and $y = y'$.

2. Both of the following conditions are satisfied:

   (a) We have $x = 0$ or $y = x_0$.

(b) We have $x' = 0$ or $y' = x_0$.

In the first case, $\lambda_X^{\mathsf{Sets}_*}$ clearly sends both elements to the same element in X. Meanwhile, in the latter case both elements are equal to the basepoint $0 \wedge x_0$ of $S^0 \wedge X$, which gets sent to the basepoint x_0 of X.

Being a Morphism of Pointed Sets: As just mentioned, we have

$$\lambda_X^{\mathsf{Sets}_*}(0 \wedge x_0) \stackrel{\text{def}}{=} x_0,$$

and thus $\lambda_X^{\mathsf{Sets}_*}$ is a morphism of pointed sets.

Invertibility: The inverse of $\lambda_X^{\mathsf{Sets}_*}$ is the morphism

$$\lambda_X^{\mathsf{Sets}_*,-1} : X \xrightarrow{\cong} S^0 \wedge X$$

defined by

$$\lambda_X^{\mathsf{Sets}_*,-1}(x) \stackrel{\text{def}}{=} 1 \wedge x$$

for each $x \in X$. Indeed:

- *Invertibility I.* We have

$$\begin{aligned}\left[\lambda_X^{\mathsf{Sets}_*,-1} \circ \lambda_X^{\mathsf{Sets}_*}\right](0 \wedge x) &= \lambda_X^{\mathsf{Sets}_*,-1}\left(\lambda_X^{\mathsf{Sets}_*}(0 \wedge x)\right)\\ &= \lambda_X^{\mathsf{Sets}_*,-1}(x_0)\\ &= 1 \wedge x_0\\ &= 0 \wedge x,\end{aligned}$$

  and

$$\begin{aligned}\left[\lambda_X^{\mathsf{Sets}_*,-1} \circ \lambda_X^{\mathsf{Sets}_*}\right](1 \wedge x) &= \lambda_X^{\mathsf{Sets}_*,-1}\left(\lambda_X^{\mathsf{Sets}_*}(1 \wedge x)\right)\\ &= \lambda_X^{\mathsf{Sets}_*,-1}(x)\\ &= 1 \wedge x\end{aligned}$$

  for each $x \in X$, and thus we have

$$\lambda_X^{\mathsf{Sets}_*,-1} \circ \lambda_X^{\mathsf{Sets}_*} = \mathrm{id}_{S^0 \wedge X}.$$

- *Invertibility II.* We have

$$\begin{aligned}\left[\lambda_X^{\mathsf{Sets}_*} \circ \lambda_X^{\mathsf{Sets}_*,-1}\right](x) &= \lambda_X^{\mathsf{Sets}_*}\left(\lambda_X^{\mathsf{Sets}_*,-1}(x)\right)\\ &= \lambda_X^{\mathsf{Sets}_*,-1}(1 \wedge x)\\ &= x\end{aligned}$$

  for each $x \in X$, and thus we have

$$\lambda_X^{\mathsf{Sets}_*} \circ \lambda_X^{\mathsf{Sets}_*,-1} = \mathrm{id}_X.$$

This shows $\lambda_X^{\mathsf{Sets}_*}$ to be invertible.
Naturality: We need to show that, given a morphism of pointed sets

$$f\colon (X, x_0) \to (Y, y_0),$$

the diagram

$$\begin{tikzcd}
S^0 \wedge X \arrow[r, "\mathrm{id}_{S^0} \wedge f"] \arrow[d, "\lambda_X^{\mathsf{Sets}_*}"'] & S^0 \wedge Y \arrow[d, "\lambda_Y^{\mathsf{Sets}_*}"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
0 \wedge x \arrow[d, mapsto] & \\
x_0 \arrow[r, mapsto] & f(x_0)
\end{tikzcd}
\qquad
\begin{tikzcd}
0 \wedge x \arrow[r, mapsto] & 0 \wedge f(x) \arrow[d, mapsto] \\
& y_0
\end{tikzcd}$$

and

$$\begin{tikzcd}
1 \wedge x \arrow[r, mapsto] \arrow[d, mapsto] & 1 \wedge f(x) \arrow[d, mapsto] \\
x \arrow[r, mapsto] & f(x)
\end{tikzcd}$$

and hence indeed commutes, showing $\lambda^{\mathsf{Sets}_*}$ to be a natural transformation.
Being a Natural Isomorphism: Since $\lambda^{\mathsf{Sets}_*}$ is natural and $\lambda^{\mathsf{Sets}_*,-1}$ is a componentwise inverse to $\lambda^{\mathsf{Sets}_*}$, it follows from Item 2 of Proposition 8.8.6.1.2 that $\lambda^{\mathsf{Sets}_*,-1}$ is also natural. Thus $\lambda^{\mathsf{Sets}_*}$ is a natural isomorphism. □

4.5.6 The Right Unitor

Definition 4.5.6.1.1. The **right unitor of the smash product of pointed sets** is the natural isomorphism

$$\rho^{\mathsf{Sets}_*}\colon \wedge \circ \left(\mathrm{id} \times \mathbb{1}^{\mathsf{Sets}_*}\right) \overset{\sim}{\Longrightarrow} \rho^{\mathsf{Cats}_2}_{\mathsf{Sets}_*},
\qquad
\begin{tikzcd}
\mathsf{Sets}_* \times \mathrm{pt} \arrow[r, "\mathrm{id} \times \mathbb{1}^{\mathsf{Sets}_*}"] \arrow[dr, dashed, bend right, "\rho^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}"'] & \mathsf{Sets}_* \times \mathsf{Sets}_* \arrow[d, "\wedge"] \arrow[dl, phantom, "\Swarrow \rho^{\mathsf{Sets}_*}"] \\
& \mathsf{Sets}_*,
\end{tikzcd}$$

whose component

$$\rho_X^{\mathsf{Sets}_*} : X \wedge S^0 \xrightarrow{\cong} X$$

at $X \in \mathrm{Obj}(\mathsf{Sets}_*)$ is given by

$$\begin{aligned} x \wedge 0 &\mapsto x_0, \\ x \wedge 1 &\mapsto x. \end{aligned}$$

Proof. Well-Definedness: Let $[(x, y)] = [(x', y')]$ be an element in $X \wedge S^0$. Then either:

1. We have $x = x'$ and $y = y'$.
2. Both of the following conditions are satisfied:
   (a) We have $x = x_0$ or $y = 0$.
   (b) We have $x' = x_0$ or $y' = 0$.

In the first case, $\rho_X^{\mathsf{Sets}_*}$ clearly sends both elements to the same element in X. Meanwhile, in the latter case both elements are equal to the basepoint $x_0 \wedge 0$ of $X \wedge S^0$, which gets sent to the basepoint x_0 of X.
Being a Morphism of Pointed Sets: As just mentioned, we have

$$\rho_X^{\mathsf{Sets}_*}(x_0 \wedge 0) \stackrel{\text{def}}{=} x_0,$$

and thus $\rho_X^{\mathsf{Sets}_*}$ is a morphism of pointed sets.
Invertibility: The inverse of $\rho_X^{\mathsf{Sets}_*}$ is the morphism

$$\rho_X^{\mathsf{Sets}_*,-1} : X \xrightarrow{\cong} X \wedge S^0$$

defined by

$$\rho_X^{\mathsf{Sets}_*,-1}(x) \stackrel{\text{def}}{=} x \wedge 1$$

for each $x \in X$. Indeed:

- *Invertibility I.* We have

$$\begin{aligned} \left[\rho_X^{\mathsf{Sets}_*,-1} \circ \rho_X^{\mathsf{Sets}_*}\right](x \wedge 0) &= \rho_X^{\mathsf{Sets}_*,-1}\left(\rho_X^{\mathsf{Sets}_*}(x \wedge 0)\right) \\ &= \rho_X^{\mathsf{Sets}_*,-1}(x_0) \\ &= x_0 \wedge 1 \\ &= x \wedge 0, \end{aligned}$$

and

$$\begin{aligned}\left[\rho_X^{\mathsf{Sets}_*,-1} \circ \rho_X^{\mathsf{Sets}_*}\right](x \wedge 1) &= \rho_X^{\mathsf{Sets}_*,-1}\left(\rho_X^{\mathsf{Sets}_*}(x \wedge 1)\right)\\ &= \rho_X^{\mathsf{Sets}_*,-1}(x)\\ &= x \wedge 1\end{aligned}$$

for each $x \in X$, and thus we have

$$\rho_X^{\mathsf{Sets}_*,-1} \circ \rho_X^{\mathsf{Sets}_*} = \mathrm{id}_{X \wedge S^0}.$$

- *Invertibility II.* We have

$$\begin{aligned}\left[\rho_X^{\mathsf{Sets}_*} \circ \rho_X^{\mathsf{Sets}_*,-1}\right](x) &= \rho_X^{\mathsf{Sets}_*}\left(\rho_X^{\mathsf{Sets}_*,-1}(x)\right)\\ &= \rho_X^{\mathsf{Sets}_*,-1}(x \wedge 1)\\ &= x\end{aligned}$$

for each $x \in X$, and thus we have

$$\rho_X^{\mathsf{Sets}_*} \circ \rho_X^{\mathsf{Sets}_*,-1} = \mathrm{id}_X.$$

This shows $\rho_X^{\mathsf{Sets}_*}$ to be invertible.

Naturality: We need to show that, given a morphism of pointed sets

$$f\colon (X, x_0) \to (Y, y_0),$$

the diagram

$$\begin{tikzcd} X \wedge S^0 \arrow[r, "f \wedge \mathrm{id}_{S^0}"] \arrow[d, "\rho_X^{\mathsf{Sets}_*}"'] & Y \wedge S^0 \arrow[d, "\rho_Y^{\mathsf{Sets}_*}"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd} x \wedge 0 \arrow[d, mapsto] & \\ x_0 \arrow[r, mapsto] & f(x_0) \end{tikzcd} \qquad \begin{tikzcd} x \wedge 0 \arrow[r, mapsto] & f(x) \wedge 0 \arrow[d, mapsto] \\ & y_0 \end{tikzcd}$$

and

$$\begin{tikzcd} x \wedge 1 \arrow[r, mapsto] \arrow[d, mapsto] & f(x) \wedge 1 \arrow[d, mapsto] \\ x \arrow[r, mapsto] & f(x) \end{tikzcd}$$

and hence indeed commutes, showing ρ^{Sets_*} to be a natural transformation.
Being a Natural Isomorphism: Since ρ^{Sets_*} is natural and $\rho^{\mathsf{Sets}_*,-1}$ is a componentwise inverse to ρ^{Sets_*}, it follows from Item 2 of Proposition 8.8.6.1.2 that $\rho^{\mathsf{Sets}_*,-1}$ is also natural. Thus ρ^{Sets_*} is a natural isomorphism. □

4.5.7 The Symmetry

Definition 4.5.7.1.1. The **symmetry of the smash product of pointed sets** is the natural isomorphism

$$\sigma^{\mathsf{Sets}_*} \colon \wedge \overset{\sim}{\Longrightarrow} \wedge \circ \boldsymbol{\sigma}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*,\mathsf{Sets}_*}, \qquad \begin{tikzcd} \mathsf{Sets}_* \times \mathsf{Sets}_* \arrow[rr, "\wedge"] \arrow[dr, bend right, "\boldsymbol{\sigma}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*,\mathsf{Sets}_*}"'] & \arrow[d, phantom, "\Downarrow \sigma^{\mathsf{Sets}_*}"] & \mathsf{Sets}_*, \\ & \mathsf{Sets}_* \times \mathsf{Sets}_* \arrow[ur, bend right, "\wedge"'] & \end{tikzcd}$$

whose component

$$\sigma^{\mathsf{Sets}_*}_{X,Y} \colon X \wedge Y \xrightarrow{\cong} Y \wedge X$$

at $X, Y \in \mathrm{Obj}(\mathsf{Sets}_*)$ is defined by

$$\sigma^{\mathsf{Sets}_*}_{X,Y}(x \wedge y) \overset{\mathrm{def}}{=} y \wedge x$$

for each $x \wedge y \in X \wedge Y$.

Proof. Well-Definedness: Let $[(x, y)] = [(x', y')]$ be an element in $X \wedge Y$. Then either:

1. We have $x = x'$ and $y = y'$.
2. Both of the following conditions are satisfied:
   (a) We have $x = x_0$ or $y = y_0$.
   (b) We have $x' = x_0$ or $y' = y_0$.

In the first case, $\sigma^{\mathsf{Sets}_*}_X$ clearly sends both elements to the same element in X. Meanwhile, in the latter case both elements are equal to the basepoint $x_0 \wedge y_0$ of $X \wedge Y$, which gets sent to the basepoint $y_0 \wedge x_0$ of $Y \wedge X$.

Being a Morphism of Pointed Sets: As just mentioned, we have

$$\sigma_X^{\mathsf{Sets}_*}(x_0 \wedge y_0) \stackrel{\text{def}}{=} y_0 \wedge x_0,$$

and thus $\sigma_X^{\mathsf{Sets}_*}$ is a morphism of pointed sets.
Invertibility: Clearly, the inverse of $\sigma_{X,Y}^{\mathsf{Sets}_*}$ is given by the morphism

$$\sigma_{X,Y}^{\mathsf{Sets}_*,-1} \colon Y \wedge X \xrightarrow{\cong} X \wedge Y$$

defined by

$$\sigma_{X,Y}^{\mathsf{Sets}_*,-1}(y \wedge x) \stackrel{\text{def}}{=} x \wedge y$$

for each $y \wedge x \in Y \wedge X$.
Naturality: We need to show that, given morphisms of pointed sets

$$f \colon (X, x_0) \to (A, a_0),$$
$$g \colon (Y, y_0) \to (B, b_0)$$

the diagram

$$\begin{tikzcd}
X \wedge Y \arrow[r, "f \wedge g"] \arrow[d, "\sigma_{X,Y}^{\mathsf{Sets}_*}"'] & A \wedge B \arrow[d, "\sigma_{A,B}^{\mathsf{Sets}_*}"] \\
Y \wedge X \arrow[r, "g \wedge f"'] & B \wedge A
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
x \wedge y \arrow[r, mapsto] \arrow[d, mapsto] & f(x) \wedge g(y) \arrow[d, mapsto] \\
y \wedge x \arrow[r, mapsto] & g(y) \wedge f(x)
\end{tikzcd}$$

and hence indeed commutes, showing σ^{Sets_*} to be a natural transformation.
Being a Natural Isomorphism: Since σ^{Sets_*} is natural and $\sigma^{\mathsf{Sets}_*,-1}$ is a componentwise inverse to σ^{Sets_*}, it follows from Item 2 of Proposition 8.8.6.1.2 that $\sigma^{\mathsf{Sets}_*,-1}$ is also natural. Thus σ^{Sets_*} is a natural isomorphism. □

4.5.8 The Diagonal

Definition 4.5.8.1.1. The **diagonal of the smash product of pointed sets** is the natural transformation

$$\Delta^{\wedge}\colon \mathrm{id}_{\mathsf{Sets}_*} \Longrightarrow \wedge \circ \Delta^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}, \qquad \begin{tikzcd} \mathsf{Sets}_* \arrow[rr, "\mathrm{id}_{\mathsf{Sets}_*}"] \arrow[dr, "\Delta^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}"'] & \arrow[d, phantom, "\Downarrow \Delta^{\wedge}"] & \mathsf{Sets}_* \\ & \mathsf{Sets}_* \wedge \mathsf{Sets}_*, \arrow[ur, "\wedge"'] & \end{tikzcd}$$

whose component

$$\Delta^{\wedge}_X\colon (X, x_0) \to (X \wedge X, x_0 \wedge x_0)$$

at $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$ is given by the composition

$$\begin{aligned} (X, x_0) &\xrightarrow{\Delta^{\wedge}_X} (X \times X, (x_0, x_0)) \\ &\twoheadrightarrow ((X \times X)/\sim, [(x_0, x_0)]) \\ &\overset{\text{def}}{=\!=} (X \wedge X, x_0 \wedge x_0) \end{aligned}$$

in Sets_*, and thus by

$$\Delta^{\wedge}_X(x) \overset{\text{def}}{=} x \wedge x$$

for each $x \in X$.

Proof. Being a Morphism of Pointed Sets: We have

$$\Delta^{\wedge}_X(x_0) \overset{\text{def}}{=} x_0 \wedge x_0,$$

and thus $\Delta^{\wedge}_X$ is a morphism of pointed sets.
Naturality: We need to show that, given a morphism of pointed sets

$$f\colon (X, x_0) \to (Y, y_0),$$

the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\Delta^{\wedge}_X"'] & Y \arrow[d, "\Delta^{\wedge}_Y"] \\ X \wedge X \arrow[r, "f \wedge f"'] & Y \wedge Y \end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd} x \arrow[r, mapsto] \arrow[d, mapsto] & f(x) \arrow[d, mapsto] \\ x \wedge x \arrow[r, mapsto] & f(x) \wedge f(x) \end{tikzcd}$$

and hence indeed commutes, showing $\Delta^{\wedge}$ to be natural. □

Proposition 4.5.8.1.2. Let $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

1. *Monoidality.* The diagonal

$$\Delta^{\wedge} \colon \mathrm{id}_{\mathsf{Sets}_*} \Longrightarrow \wedge \circ \Delta^{\mathsf{Cats}_2}_{\mathsf{Sets}_*},$$

   of the smash product of pointed sets is a monoidal natural transformation:

   (a) *Compatibility With Strong Monoidality Constraints.* For each $(X, x_0), (Y, y_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, the diagram

$$\begin{tikzcd} X \wedge Y \arrow[r, "\Delta^{\wedge}_X \wedge \Delta^{\wedge}_Y"] \arrow[dr, "\Delta^{\wedge}_{X \wedge Y}"'] & (X \wedge X) \wedge (Y \wedge Y) \arrow[d, dashed, "\wr"] \\ & (X \wedge Y) \wedge (X \wedge Y) \end{tikzcd}$$

   commutes.

   (b) *Compatibility With Strong Unitality Constraints.* The diagrams

$$\begin{tikzcd} S^0 \arrow[r, "\Delta^{\wedge}_{S^0}"] \arrow[dr, equals] & S^0 \wedge S^0 \arrow[d, "\lambda^{\mathsf{Sets}_*}_{S^0}"] \\ & S^0 \end{tikzcd} \qquad \begin{tikzcd} S^0 \arrow[r, "\Delta^{\wedge}_{S^0}"] \arrow[dr, equals] & S^0 \wedge S^0 \arrow[d, "\rho^{\mathsf{Sets}_*}_{S^0}"] \\ & S^0 \end{tikzcd}$$

   commute, i.e. we have

$$\begin{aligned} \Delta^{\wedge}_{S^0} &= \lambda^{\mathsf{Sets}_*, -1}_{S^0} \\ &= \rho^{\mathsf{Sets}_*, -1}_{S^0}, \end{aligned}$$

   where we recall that the equalities

$$\begin{aligned} \lambda^{\mathsf{Sets}_*}_{S^0} &= \rho^{\mathsf{Sets}_*}_{S^0}, \\ \lambda^{\mathsf{Sets}_*, -1}_{S^0} &= \rho^{\mathsf{Sets}_*, -1}_{S^0} \end{aligned}$$

   are always true in any monoidal category by **??** of **??**.

2. *The Diagonal of the Unit.* The component

$$\Delta^{\wedge}_{S^0} \colon S^0 \xrightarrow{\cong} S^0 \wedge S^0$$

of $\Delta^{\wedge}$ at S^0 is an isomorphism.

Proof. *Item 1, Monoidality*: We claim that $\Delta^{\wedge}$ is indeed monoidal:

1. *Item 1a: Compatibility With Strong Monoidality Constraints:* We need to show that the diagram

$$\begin{tikzcd} X \wedge Y \arrow[r, "\Delta^{\wedge}_X \wedge \Delta^{\wedge}_Y"] \arrow[dr, "\Delta^{\wedge}_{X \wedge Y}"'] & (X \wedge X) \wedge (Y \wedge Y) \arrow[d, dashed, "\wr"] \\ & (X \wedge Y) \wedge (X \wedge Y) \end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd} x \wedge y \arrow[r, mapsto] \arrow[dr, mapsto] & (x \wedge x) \wedge (y \wedge y) \arrow[d, mapsto] \\ & (x \wedge y) \wedge (x \wedge y) \end{tikzcd}$$

and hence indeed commutes.

2. *Item 1b: Compatibility With Strong Unitality Constraints:* As shown in the proof of Definition 4.5.5.1.1, the inverse of the left unitor of Sets_* with respect to to the smash product of pointed sets at $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$ is given by

$$\lambda^{\mathsf{Sets}_*,-1}_X(x) \stackrel{\text{def}}{=} 1 \wedge x$$

for each $x \in X$, so when $X = S^0$, we have

$$\lambda^{\mathsf{Sets}_*,-1}_{S^0}(0) \stackrel{\text{def}}{=} 1 \wedge 0,$$
$$\lambda^{\mathsf{Sets}_*,-1}_{S^0}(1) \stackrel{\text{def}}{=} 1 \wedge 1.$$

But since $1 \wedge 0 = 0 \wedge 0$ and

$$\Delta^{\wedge}_{S^0}(0) \stackrel{\text{def}}{=} 0 \wedge 0,$$
$$\Delta^{\wedge}_{S^0}(1) \stackrel{\text{def}}{=} 1 \wedge 1,$$

it follows that we indeed have $\Delta^{\wedge}_{S^0} = \lambda^{\mathsf{Sets}_*,-1}_{S^0}$.

This finishes the proof.

Item 2, The Diagonal of the Unit: This follows from Item 1 and the invertibility of the left/right unitor of Sets_* with respect to $\wedge$, proved in the proof of Definition 4.5.5.1.1 for the left unitor or the proof of Definition 4.5.6.1.1 for the right unitor. □

4.5.9 The Monoidal Structure on Pointed Sets Associated to $\wedge$

Proposition 4.5.9.1.1. The category Sets_* admits a closed monoidal category with diagonals structure consisting of

- *The Underlying Category.* The category Sets_* of pointed sets;
- *The Monoidal Product.* The smash product functor

  $$\wedge\colon \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*$$

  of Item 1 of Proposition 4.5.1.1.9;
- *The Internal Hom.* The internal Hom functor

  $$\mathbf{Sets}_*\colon \mathsf{Sets}_*^{\mathsf{op}} \times \mathsf{Sets}_* \to \mathsf{Sets}_*$$

  of Item 1 of Proposition 4.5.2.1.2;
- *The Monoidal Unit.* The functor

  $$\mathbb{1}^{\mathsf{Sets}_*}\colon \mathsf{pt} \to \mathsf{Sets}_*$$

  of Definition 4.5.3.1.1;
- *The Associators.* The natural isomorphism

  $$\alpha^{\mathsf{Sets}_*}\colon \wedge \circ (\wedge \times \mathrm{id}_{\mathsf{Sets}_*}) \overset{\sim}{\Longrightarrow} \wedge \circ (\mathrm{id}_{\mathsf{Sets}_*} \times \wedge) \circ \boldsymbol{\alpha}^{\mathsf{Cats}}_{\mathsf{Sets}_*,\mathsf{Sets}_*,\mathsf{Sets}_*}$$

  of Definition 4.5.4.1.1;
- *The Left Unitors.* The natural isomorphism

  $$\lambda^{\mathsf{Sets}_*}\colon \wedge \circ \left(\mathbb{1}^{\mathsf{Sets}_*} \times \mathrm{id}_{\mathsf{Sets}_*}\right) \overset{\sim}{\Longrightarrow} \boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}$$

  of Definition 4.5.5.1.1;
- *The Right Unitors.* The natural isomorphism

  $$\rho^{\mathsf{Sets}_*}\colon \wedge \circ \left(\mathrm{id} \times \mathbb{1}^{\mathsf{Sets}_*}\right) \overset{\sim}{\Longrightarrow} \boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}$$

  of Definition 4.5.6.1.1;

- *The Symmetry.* The natural isomorphism

$$\sigma^{\mathsf{Sets}_*} \colon \wedge \overset{\sim}{\Longrightarrow} \wedge \circ \boldsymbol{\sigma}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*,\mathsf{Sets}_*}$$

  of Definition 4.5.7.1.1;

- *The Diagonals.* The monoidal natural transformation

$$\Delta^{\wedge} \colon \mathrm{id}_{\mathsf{Sets}_*} \Longrightarrow \wedge \circ \Delta^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}$$

  of Definition 4.5.8.1.1.

Proof. The Pentagon Identity: Let (W, w_0), (X, x_0), (Y, y_0) and (Z, z_0) be pointed sets. We have to show that the diagram

$$\begin{tikzcd}
 & (W \wedge (X \wedge Y)) \wedge Z \arrow[dr, "\alpha^{\mathsf{Sets}_*}_{W,X\wedge Y,Z}"] & \\
((W \wedge X) \wedge Y) \wedge Z \arrow[ur, "\alpha^{\mathsf{Sets}_*}_{W,X,Y} \wedge \mathrm{id}_Z"] \arrow[d, "\alpha^{\mathsf{Sets}_*}_{W\wedge X,Y,Z}"'] & & W \wedge ((X \wedge Y) \wedge Z) \arrow[d, "\mathrm{id}_W \wedge \alpha^{\mathsf{Sets}_*}_{X,Y,Z}"] \\
(W \wedge X) \wedge (Y \wedge Z) \arrow[rr, "\alpha^{\mathsf{Sets}_*}_{W,X,Y\wedge Z}"'] & & W \wedge (X \wedge (Y \wedge Z))
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$
\begin{tikzcd}
 & & (w \wedge (x \wedge y)) \wedge z \arrow[drr, mapsto] & & \\
((w \wedge x) \wedge y) \wedge z \arrow[urr, mapsto] \arrow[dr, mapsto] & & & & w \wedge ((x \wedge y) \wedge z) \arrow[dl, mapsto] \\
 & (w \wedge x) \wedge (y \wedge z) \arrow[rr, mapsto] & & w \wedge (x \wedge (y \wedge z)) &
\end{tikzcd}
$$

and thus we see that the pentagon identity is satisfied.
The Triangle Identity: Let (X, x_0) and (Y, y_0) be pointed sets. We have to show that the diagram

$$
\begin{tikzcd}
(X \wedge S^0) \wedge Y \arrow[rr, "\alpha^{\mathsf{Sets}_*}_{X,S^0,Y}"] \arrow[dr, swap, "\rho^{\mathsf{Sets}_*}_X \wedge \mathrm{id}_Y"] & & X \wedge (S^0 \wedge Y) \arrow[dl, "\mathrm{id}_X \wedge \lambda^{\mathsf{Sets}_*}_Y"] \\
 & X \wedge Y &
\end{tikzcd}
$$

commutes. Indeed, this diagram acts on elements as

$$
\begin{tikzcd}
(x \wedge 0) \wedge y \arrow[dr, mapsto] & \\
 & x_0 \wedge y
\end{tikzcd}
\qquad
\begin{tikzcd}
(x \wedge 0) \wedge y \arrow[rr, mapsto] & & x \wedge (0 \wedge y) \arrow[dl, mapsto] \\
 & x \wedge y_0 &
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
(x \wedge 1) \wedge y \arrow[rr, mapsto] \arrow[dr, mapsto] & & x \wedge (1 \wedge y) \arrow[dl, mapsto] \\
 & x \wedge y, &
\end{tikzcd}
$$

and thus we see that the triangle identity is satisfied.

The Left Hexagon Identity: Let (X, x_0), (Y, y_0), and (Z, z_0) be pointed sets. We have to show that the diagram

$$
\begin{tikzcd}
 & (X \wedge Y) \wedge Z \arrow[dl, "\alpha^{\mathsf{Sets}_*}_{X,Y,Z}"'] \arrow[dr, "\beta^{\mathsf{Sets}_*}_{X,Y} \wedge \mathrm{id}_Z"] & \\
X \wedge (Y \wedge Z) \arrow[d, "\beta^{\mathsf{Sets}_*}_{X,Y \wedge Z}"'] & & (Y \wedge X) \wedge Z \arrow[d, "\alpha^{\mathsf{Sets}_*}_{Y,X,Z}"] \\
(Y \wedge Z) \wedge X \arrow[dr, "\alpha^{\mathsf{Sets}_*}_{Y,Z,X}"'] & & Y \wedge (X \wedge Z) \arrow[dl, "\mathrm{id}_Y \wedge \beta^{\mathsf{Sets}_*}_{X,Z}"] \\
 & Y \wedge (Z \wedge X) &
\end{tikzcd}
$$

commutes. Indeed, this diagram acts on elements as

$$
\begin{tikzcd}
 & (x \wedge y) \wedge z \arrow[dl, mapsto] \arrow[dr, mapsto] & \\
x \wedge (y \wedge z) \arrow[d, mapsto] & & (y \wedge x) \wedge z \arrow[d, mapsto] \\
(y \wedge z) \wedge x \arrow[dr, mapsto] & & y \wedge (x \wedge z) \arrow[dl, mapsto] \\
 & y \wedge (z \wedge x) &
\end{tikzcd}
$$

and thus we see that the left hexagon identity is satisfied.
The Right Hexagon Identity: Let (X, x_0), (Y, y_0), and (Z, z_0) be pointed sets.

We have to show that the diagram

$$\begin{tikzcd}
 & X \wedge (Y \wedge Z) \arrow[dl, "\left(\alpha^{\mathsf{Sets}_*}_{X,Y,Z}\right)^{-1}"'] \arrow[dr, "\mathrm{id}_X \wedge \beta^{\mathsf{Sets}_*}_{Y,Z}"] & \\
(X \wedge Y) \wedge Z \arrow[d, "\beta^{\mathsf{Sets}_*}_{X \wedge Y,Z}"'] & & X \wedge (Z \wedge Y) \arrow[d, "\left(\alpha^{\mathsf{Sets}_*}_{X,Z,Y}\right)^{-1}"] \\
Z \wedge (X \wedge Y) \arrow[dr, "\left(\alpha^{\mathsf{Sets}_*}_{Z,X,Y}\right)^{-1}"'] & & (X \wedge Z) \wedge Y \arrow[dl, "\beta^{\mathsf{Sets}_*}_{X,Z} \wedge \mathrm{id}_Y"] \\
 & (Z \wedge X) \wedge Y &
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
 & x \wedge (y \wedge z) \arrow[dl, mapsto] \arrow[dr, mapsto] & \\
(x \wedge y) \wedge z \arrow[d, mapsto] & & x \wedge (z \wedge y) \arrow[d, mapsto] \\
z \wedge (x \wedge y) \arrow[dr, mapsto] & & (x \wedge z) \wedge y \arrow[dl, mapsto] \\
 & (z \wedge x) \wedge y &
\end{tikzcd}$$

and thus we see that the right hexagon identity is satisfied.
Monoidal Closedness: This follows from Item 2 of Proposition 4.5.1.1.9.
Existence of Monoidal Diagonals: This follows from Items 1 and 2 of Proposition 4.5.8.1.2. □

4.5.10 Universal Properties of the Smash Product of Pointed Sets I

Theorem 4.5.10.1.1. The symmetric monoidal structure on the category Sets_* is uniquely determined by the following requirements:

1. *Two-Sided Preservation of Colimits.* The smash product

$$\wedge \colon \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*$$

of Sets_* preserves colimits separately in each variable.

2. *The Unit Object Is S^0*. We have $\mathbb{1}_{\mathsf{Sets}_*} = S^0$.

Proof. Omitted. □

4.5.11 Universal Properties of the Smash Product of Pointed Sets II

Theorem 4.5.11.1.1. The symmetric monoidal structure on the category Sets_* is the unique symmetric monoidal structure on Sets_* such that the free pointed set functor

$$(-)^+ \colon \mathsf{Sets} \to \mathsf{Sets}_*$$

admits a symmetric monoidal structure.

Proof. See [GGN15, Theorem 5.1]. □

4.5.12 Monoids With Respect to the Smash Product of Pointed Sets

Proposition 4.5.12.1.1. The category of monoids on $(\mathsf{Sets}_*, \wedge, S^0)$ is isomorphic to the category of monoids with zero and morphisms between them.

Proof. See **??**, in particular **??**, **??**, and **??**. □

4.5.13 Comonoids With Respect to the Smash Product of Pointed Sets

Proposition 4.5.13.1.1. The symmetric monoidal functor

$$\left((-)^+, (-)^{+,\times}, (-)^{+,\times}_{\mathbb{1}}\right) \colon (\mathsf{Sets}, \times, \mathrm{pt}) \to \left(\mathsf{Sets}_*, \wedge, S^0\right),$$

of Item 4 of Proposition 3.4.1.1.2 lifts to an equivalence of categories

$$\begin{aligned} \mathsf{CoMon}\left(\mathsf{Sets}_*, \wedge, S^0\right) &\overset{\text{eq.}}{\cong} \mathsf{CoMon}(\mathsf{Sets}, \times, \mathrm{pt}) \\ &\cong \mathsf{Sets}. \end{aligned}$$

Proof. See [PS19, Lemma 2.4]. □

4.6 Miscellany

4.6.1 The Smash Product of a Family of Pointed Sets

Let $\left\{\left(X_i, x_0^i\right)\right\}_{i\in I}$ be a family of pointed sets.

Definition 4.6.1.1.1. The **smash product of the family** $\left\{(X_i, x_0^i)\right\}_{i\in I}$ is the pointed set $\bigwedge_{i\in I} X_i$ consisting of:

- *The Underlying Set.* The set $\bigwedge_{i\in I} X_i$ defined by

  $$\bigwedge_{i\in I} X_i \stackrel{\text{def}}{=} \left(\prod_{i\in I} X_i\right)/\sim,$$

  where $\sim$ is the equivalence relation on $\prod_{i\in I} X_i$ obtained by declaring

  $$(x_i)_{i\in I} \sim (y_i)_{i\in I}$$

  if there exist $i_0 \in I$ such that $x_{i_0} = x_0$ and $y_{i_0} = y_0$, for each $(x_i)_{i\in I}, (y_i)_{i\in I} \in \prod_{i\in I} X_i$.

- *The Basepoint.* The element $[(x_0)_{i\in I}]$ of $\bigwedge_{i\in I} X_i$.

Appendices

4.A Other Chapters

Sets

1. Sets
2. Constructions With Sets
3. Pointed Sets
4. Tensor Products of Pointed Sets

Relations

5. Relations
6. Constructions With Relations
7. Equivalence Relations and Apartness Relations

Category Theory

8. Categories

Bicategories

9. Types of Morphisms in Bicategories

Part II

Relations

Chapter 5

Relations

This chapter contains some material about relations. Notably, we discuss and explore:

1. The definition of relations (Section 5.1.1).
2. How relations may be viewed as decategorification of profunctors (Section 5.1.2).
3. The various kind of categories that relations form, namely:
   (a) A category (Section 5.2.1).
   (b) A monoidal category (Section 5.2.2).
   (c) A 2-category (Section 5.2.3).
   (d) A double category (Section 5.2.4).
4. The various categorical properties of the 2-category of relations, including:
   (a) The self-duality of Rel and **Rel** (Proposition 5.3.1.1.1).
   (b) Identifications of equivalences and isomorphisms in **Rel** with bijections (Proposition 5.3.2.1.1).
   (c) Identifications of adjunctions in **Rel** with functions (Proposition 5.3.3.1.1).
   (d) Identifications of monads in **Rel** with preorders (Proposition 5.3.4.1.1).
   (e) Identifications of comonads in **Rel** with subsets (Proposition 5.3.5.1.1).
   (f) A description of the monoids and comonoids in **Rel** with respect to the Cartesian product (Remark 5.3.6.1.1).
   (g) Characterisations of monomorphisms in Rel (Proposition 5.3.7.1.1).

(h) Characterisations of 2-categorical notions of monomorphisms in **Rel** (Proposition 5.3.8.1.1).

(i) Characterisations of epimorphisms in Rel (Proposition 5.3.9.1.1).

(j) Characterisations of 2-categorical notions of epimorphisms in **Rel** (Proposition 5.3.10.1.1).

(k) The partial co/completeness of Rel (Proposition 5.3.11.1.1).

(l) The existence or non-existence of Kan extensions and Kan lifts in Rel (Remark 5.3.12.1.1).

(m) The closedness of **Rel** (Proposition 5.3.13.1.1).

(n) The identification of **Rel** with the category of free algebras of the powerset monad on Sets (Proposition 5.3.14.1.1).

5. A description of two notions of "skew composition" on $\mathbf{Rel}(A, B)$, giving rise to left and right skew monoidal structures analogous to the left skew monoidal structure on $\mathsf{Fun}(C, \mathcal{D})$ appearing in the definition of a relative monad (Sections 5.4 and 5.5).

Contents

5.1 Relations **234**

5.1.1 Foundations 234

5.1.2 Relations as Decategorifications of Profunctors 237

5.1.3 Examples of Relations 239

5.1.4 Functional Relations 241

5.1.5 Total Relations 242

5.2 Categories of Relations **243**

5.2.1 The Category of Relations 243

5.2.2 The Closed Symmetric Monoidal Category of Relations 243

5.2.3 The 2-Category of Relations 248

5.2.4 The Double Category of Relations 249

5.3 Properties of the 2-Category of Relations **256**

5.3.1 Self-Duality 256

5.3.2 Isomorphisms and Equivalences in **Rel** 257

5.3.3 Adjunctions in **Rel** 258

5.3.4 Monads in **Rel** 261

5.3.5 Comonads in **Rel** 261

5.3.6 Co/Monoids in **Rel** 262

5.3.7 Monomorphisms in Rel 263
5.3.8 2-Categorical Monomorphisms in **Rel** 265
5.3.9 Epimorphisms in Rel 269
5.3.10 2-Categorical Epimorphisms in **Rel** 272
5.3.11 Co/Limits in Rel 275
5.3.12 Kan Extensions and Kan Lifts in **Rel** 276
5.3.13 Closedness of **Rel** 276
5.3.14 **Rel** as a Category of Free Algebras 276
5.4 The Left Skew Monoidal Structure on Rel(A, B) ... 276
5.4.1 The Left Skew Monoidal Product 276
5.4.2 The Left Skew Monoidal Unit 277
5.4.3 The Left Skew Associators 278
5.4.4 The Left Skew Left Unitors 279
5.4.5 The Left Skew Right Unitors 279
5.4.6 The Left Skew Monoidal Structure on **Rel**(A, B)
280
5.5 The Right Skew Monoidal Structure on Rel(A, B) . 281
5.5.1 The Right Skew Monoidal Product 281
5.5.2 The Right Skew Monoidal Unit 282
5.5.3 The Right Skew Associators 282
5.5.4 The Right Skew Left Unitors 283
5.5.5 The Right Skew Right Unitors 284
5.5.6 The Right Skew Monoidal Structure on **Rel**(A, B)
285
5.A Other Chapters 286

5.1 Relations

5.1.1 Foundations

Let A and B be sets.

Definition 5.1.1.1.1. A **relation** $R\colon A \nrightarrow B$ **from** A **to** B[1,2] is a subset R of $A \times B$.

Notation 5.1.1.1.2. Let $R\colon A \nrightarrow B$ be a relation.

[1] *Further Terminology:* Also called a **multivalued function from** A **to** B, a **relation over** A **and** B, **relation on** A **and** B, a **binary relation over** A **and** B, or a **binary relation on** A **and** B.

[2] *Further Terminology:* When $A = B$, we also call $R \subset A \times A$ a **relation on** A.

1. Given elements $a \in A$ and $b \in B$ and a relation $R\colon A \nrightarrow B$, we write $a \sim_R b$ to mean $(a,b) \in R$.

2. Viewing R as a function

$$R\colon A \times B \to \{\mathsf{t}, \mathsf{f}\}$$

via Remark 5.1.1.1.4, we write R^b_a for the value of R at (a,b).[3]

Definition 5.1.1.1.3. Let A and B be sets.

1. The **set of relations from** A **to** B is the set $\mathrm{Rel}(A,B)$ defined by

$$\mathrm{Rel}(A,B) \stackrel{\text{def}}{=} \{\text{Relations from } A \text{ to } B\}.$$

2. The **poset of relations from** A **to** B is the poset

$$\mathbf{Rel}(A,B) \stackrel{\text{def}}{=} (\mathrm{Rel}(A,B), \subset)$$

consisting of:

- *The Underlying Set.* The set $\mathrm{Rel}(A,B)$ of Item 1.
- *The Partial Order.* The partial order

$$\subset\colon \mathrm{Rel}(A,B) \times \mathrm{Rel}(A,B) \to \{\mathsf{true}, \mathsf{false}\}$$

on $\mathrm{Rel}(A,B)$ given by inclusion of relations.

3. The **category of relations from** A **to** B is the posetal category $\mathbf{Rel}(A,B)$[4] associated to the poset $\mathbf{Rel}(A,B)$ of Item 2 via Definition 8.1.3.1.1.

Remark 5.1.1.1.4. A relation from A to B is equivalently:[5]

1. A subset of $A \times B$.

2. A function from $A \times B$ to $\{\mathsf{true}, \mathsf{false}\}$.

3. A function from A to $\mathcal{P}(B)$.

[3]The choice R^b_a in place of R^a_b is to keep the notation consistent with the notation we will later employ for profunctors.

[4]Here we choose to slightly abuse notation by writing $\mathbf{Rel}(A,B)$ (instead of e.g. $\mathbf{Rel}(A,B)_{\mathsf{pos}}$) for the posetal category of relations from A to B, even though the same notation is used for the poset of relations from A to B.

[5]*Intuition:* In particular, we may think of a relation $R\colon A \to \mathcal{P}(B)$ from A to B as a multivalued function from A to B (including the possibility of a given $a \in A$ having no value at all).

4. A function from B to $\mathcal{P}(A)$.

5. A cocontinuous morphism of posets from $(\mathcal{P}(A), \subset)$ to $(\mathcal{P}(B), \subset)$.

That is: we have bijections of sets

$$\begin{aligned}\mathrm{Rel}(A, B) &\stackrel{\mathrm{def}}{=} \mathcal{P}(A \times B),\\ &\cong \mathrm{Hom}_{\mathsf{Sets}}(A \times B, \{\mathsf{true}, \mathsf{false}\}),\\ &\cong \mathrm{Hom}_{\mathsf{Sets}}(A, \mathcal{P}(B)),\\ &\cong \mathrm{Hom}_{\mathsf{Sets}}(B, \mathcal{P}(A)),\\ &\cong \mathrm{Hom}^{\mathrm{cocont}}_{\mathsf{Pos}}(\mathcal{P}(A), \mathcal{P}(B)),\end{aligned}$$

natural in $A, B \in \mathrm{Obj}(\mathsf{Sets})$.

Proof. We claim that Items 1 to 5 are indeed equivalent:

- *Item 1* $\iff$ *Item 2*: This is a special case of Items 1 and 2 of Proposition 2.4.3.1.6.

- *Item 2* $\iff$ *Item 3*: This follows from the bijections

$$\begin{aligned}\mathrm{Hom}_{\mathsf{Sets}}(A \times B, \{\mathsf{true}, \mathsf{false}\}) &\cong \mathrm{Hom}_{\mathsf{Sets}}(A, \mathrm{Hom}_{\mathsf{Sets}}(B, \{\mathsf{true}, \mathsf{false}\}))\\ &\cong \mathrm{Hom}_{\mathsf{Sets}}(A, \mathcal{P}(B)),\end{aligned}$$

  where the last bijection is from Items 1 and 2 of Proposition 2.4.3.1.6.

- *Item 2* $\iff$ *Item 4*: This follows from the bijections

$$\begin{aligned}\mathrm{Hom}_{\mathsf{Sets}}(A \times B, \{\mathsf{true}, \mathsf{false}\}) &\cong \mathrm{Hom}_{\mathsf{Sets}}(B, \mathrm{Hom}_{\mathsf{Sets}}(B, \{\mathsf{true}, \mathsf{false}\}))\\ &\cong \mathrm{Hom}_{\mathsf{Sets}}(B, \mathcal{P}(A)),\end{aligned}$$

  where again the last bijection is from Items 1 and 2 of Proposition 2.4.3.1.6.

- *Item 2* $\iff$ *Item 5*: This follows from the universal property of the powerset $\mathcal{P}(X)$ of a set X as the free cocompletion of X via the characteristic embedding

$$\chi_X : X \hookrightarrow \mathcal{P}(X)$$

  of X into $\mathcal{P}(X)$, Item 2 of Proposition 2.4.3.1.8.

  In particular, the bijection

$$\mathrm{Rel}(A, B) \cong \mathrm{Hom}^{\mathrm{cocont}}_{\mathsf{Pos}}(\mathcal{P}(A), \mathcal{P}(B))$$

is given by taking a relation $R\colon A \nrightarrow B$, passing to its associated function $f\colon A \to \mathcal{P}(B)$ from A to B and then extending f from A to all of $\mathcal{P}(A)$ by taking its left Kan extension along χ_X.

This coincides with the direct image function $f_*\colon \mathcal{P}(A) \to \mathcal{P}(B)$ of Definition 2.4.4.1.1.

This finishes the proof. □

Proposition 5.1.1.1.5. Let A and B be sets and let $R, S\colon A \nrightarrow B$ be relations.

1. *End Formula for the Set of Inclusions of Relations.* We have

$$\mathrm{Hom}_{\mathbf{Rel}(A,B)}(R,S) \cong \int_{a\in A}\int_{b\in B} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^b_a, S^b_a\right).$$

Proof. *Item 1, End Formula for the Set of Inclusions of Relations*: Unwinding the expression inside the end on the right hand side, we have

$$\int_{a\in A}\int_{b\in B} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^b_a, S^b_a\right) \cong \begin{cases} \mathrm{pt} & \text{if, for each } a \in A \text{ and each } b \in B, \\ & \text{we have } \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^b_a, S^b_a\right) \cong \mathrm{pt} \\ \emptyset & \text{otherwise.} \end{cases}$$

Since we have $\mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^b_a, S^b_a\right) = \{\mathsf{true}\} \cong \mathrm{pt}$ exactly when $R^b_a = \mathsf{false}$ or $R^b_a = S^b_a = \mathsf{true}$, we get

$$\int_{a\in A}\int_{b\in B} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^b_a, S^b_a\right) \cong \begin{cases} \mathrm{pt} & \text{if, for each } a \in A \text{ and each } b \in B, \\ & \text{if } a \sim_R b, \text{ then } a \sim_S b, \\ \emptyset & \text{otherwise.} \end{cases}$$

On the left hand-side, we have

$$\mathrm{Hom}_{\mathbf{Rel}(A,B)}(R,S) \cong \begin{cases} \mathrm{pt} & \text{if } R \subset S, \\ \emptyset & \text{otherwise.} \end{cases}$$

It is then clear that the conditions for each set to evaluate to pt (up to isomorphism) are equivalent, implying that those two sets are isomorphic. □

5.1.2 Relations as Decategorifications of Profunctors

Remark 5.1.2.1.1. The notion of a relation is a decategorification of that of a profunctor:

1. A profunctor from a category $\mathcal{C}$ to a category $\mathcal{D}$ is a functor

$$\mathfrak{p}\colon \mathcal{D}^{\mathsf{op}} \times \mathcal{C} \to \mathsf{Sets}.$$

2. A relation on sets A and B is a function

$$R\colon A\times B\longrightarrow\{\mathsf{true},\mathsf{false}\}.$$

Here we notice that:

- The opposite X^{op} of a set X is itself, as $(-)^{\mathsf{op}}\colon \mathsf{Cats}\longrightarrow\mathsf{Cats}$ restricts to the identity endofunctor on Sets.
- The values that profunctors and relations take are analogous:
  - A category is enriched over the category
    $$\mathsf{Sets}\stackrel{\text{def}}{=}\mathsf{Cats}_0$$
    of sets, with profunctors taking values on it.
  - A set is enriched over the set
    $$\{\mathsf{true},\mathsf{false}\}\stackrel{\text{def}}{=}\mathsf{Cats}_{-1}$$
    of classical truth values, with relations taking values on it.

Remark 5.1.2.1.2. Extending Remark 5.1.2.1.1, the equivalent definitions of relations in Remark 5.1.1.1.4 are also related to the corresponding ones for profunctors (**??**), which state that a profunctor $\mathfrak{p}\colon \mathcal{C}\nrightarrow\mathcal{D}$ is equivalently:

1. A functor $\mathfrak{p}\colon \mathcal{D}^{\mathsf{op}}\times\mathcal{C}\longrightarrow\mathsf{Sets}$.
2. A functor $\mathfrak{p}\colon \mathcal{C}\longrightarrow\mathsf{PSh}(\mathcal{D})$.
3. A functor $\mathfrak{p}\colon \mathcal{D}^{\mathsf{op}}\longrightarrow\mathsf{Fun}(\mathcal{C},\mathsf{Sets})$.
4. A colimit-preserving functor $\mathfrak{p}\colon \mathsf{PSh}(\mathcal{C})\longrightarrow\mathsf{PSh}(\mathcal{D})$.

Indeed:

- The equivalence between Items 1 and 2 (and also that between Items 1 and 3, which is proved analogously) is an instance of currying, both for profunctors as well as for relations, using the isomorphisms

$$\begin{aligned}\mathsf{Sets}(A\times B,\{\mathsf{true},\mathsf{false}\})&\cong\mathsf{Sets}(A,\mathsf{Sets}(B,\{\mathsf{true},\mathsf{false}\}))\\&\cong\mathsf{Sets}(A,\mathcal{P}(B)),\\\mathsf{Fun}(\mathcal{D}^{\mathsf{op}}\times\mathcal{D},\mathsf{Sets})&\cong\mathsf{Fun}(\mathcal{C},\mathsf{Fun}(\mathcal{D}^{\mathsf{op}},\mathsf{Sets}))\\&\cong\mathsf{Fun}(\mathcal{C},\mathsf{PSh}(\mathcal{D})).\end{aligned}$$

- The equivalence between Items 1 and 3 follows from the universal properties of:
  - The powerset $\mathcal{P}(X)$ of a set X as the free cocompletion of X via the characteristic embedding

$$\chi_{(-)}\colon X \hookrightarrow \mathcal{P}(X)$$

    of X into $\mathcal{P}(X)$, as stated and proved in Item 2 of Proposition 2.4.3.1.8.
  - The category $\mathsf{PSh}(\mathcal{C})$ of presheaves on a category $\mathcal{C}$ as the free cocompletion of $\mathcal{C}$ via the Yoneda embedding

$$よ\colon \mathcal{C} \hookrightarrow \mathsf{PSh}(\mathcal{C})$$

    of $\mathcal{C}$ into $\mathsf{PSh}(\mathcal{C})$, as stated and proved in **??** of **??**.

5.1.3 Examples of Relations

Example 5.1.3.1.1. The **trivial relation on** A **and** B is the relation $\sim_{\text{triv}}$ defined equivalently as follows:

1. As a subset of $A \times B$, we have

$$\sim_{\text{triv}} \overset{\text{def}}{=} A \times B.$$

2. As a function from $A \times B$ to $\{\mathsf{true}, \mathsf{false}\}$, the relation $\sim_{\text{triv}}$ is the constant function

$$\Delta_{\mathsf{true}}\colon A \times B \longrightarrow \{\mathsf{true}, \mathsf{false}\}$$

   from $A \times B$ to $\{\mathsf{true}, \mathsf{false}\}$ taking the value true.

3. As a function from A to $\mathcal{P}(B)$, the relation $\sim_{\text{triv}}$ is the function

$$\Delta_{\mathsf{true}}\colon A \longrightarrow \mathcal{P}(B)$$

   defined by

$$\Delta_{\mathsf{true}}(a) \overset{\text{def}}{=} B$$

   for each $a \in A$.

4. Lastly, it is the unique relation R on A and B such that we have $a \sim_R b$ for each $a \in A$ and each $b \in B$.

Example 5.1.3.1.2. The **cotrivial relation on** A **and** B is the relation $\sim_{\text{cotriv}}$ defined equivalently as follows:

1. As a subset of $A \times B$, we have

$$\sim_{\text{cotriv}} \stackrel{\text{def}}{=} \emptyset.$$

2. As a function from $A \times B$ to $\{\mathsf{true}, \mathsf{false}\}$, the relation $\sim_{\text{cotriv}}$ is the constant function

$$\Delta_{\mathsf{false}} \colon A \times B \longrightarrow \{\mathsf{true}, \mathsf{false}\}$$

from $A \times B$ to $\{\mathsf{true}, \mathsf{false}\}$ taking the value false.

3. As a function from A to $\mathcal{P}(B)$, the relation $\sim_{\text{cotriv}}$ is the function

$$\Delta_{\mathsf{false}} \colon A \longrightarrow \mathcal{P}(B)$$

defined by

$$\Delta_{\mathsf{false}}(a) \stackrel{\text{def}}{=} \emptyset$$

for each $a \in A$.

4. Lastly, it is the unique relation R on A and B such that we have $a \nsim_R b$ for each $a \in A$ and each $b \in B$.

Example 5.1.3.1.3. The characteristic relation

$$\chi_X(-_1, -_2) \colon X \times X \longrightarrow \{\mathsf{t}, \mathsf{f}\}$$

on X of Item 3 of Definition 2.4.1.1.1, defined by

$$\chi_X(x, y) \stackrel{\text{def}}{=} \begin{cases} \mathsf{true} & \text{if } x = y, \\ \mathsf{false} & \text{if } x \neq y \end{cases}$$

for each $x, y \in X$, is another example of a relation.

Example 5.1.3.1.4. Square roots are examples of relations:

1. *Square Roots in* $\mathbb{R}$. The assignment $x \mapsto \sqrt{x}$ defines a relation

$$\sqrt{-} \colon \mathbb{R} \longrightarrow \mathcal{P}(\mathbb{R})$$

from $\mathbb{R}$ to itself, being explicitly given by

$$\sqrt{x} \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if } x = 0, \\ \left\{-\sqrt{|x|}, \sqrt{|x|}\right\} & \text{if } x \neq 0. \end{cases}$$

2. *Square Roots in* $\mathbb{Q}$. Square roots in $\mathbb{Q}$ are similar to square roots in $\mathbb{R}$, though now additionally it may also occur that $\sqrt{-}\colon \mathbb{Q} \to \mathcal{P}(\mathbb{Q})$ sends a rational number x (e.g. 2) to the empty set (since $\sqrt{2} \notin \mathbb{Q}$).

Example 5.1.3.1.5. The complex logarithm defines a relation

$$\log\colon \mathbb{C} \to \mathcal{P}(\mathbb{C})$$

from $\mathbb{C}$ to itself, where we have

$$\log(a+bi) \stackrel{\text{def}}{=} \left\{ \log\left(\sqrt{a^2+b^2}\right) + i\arg(a+bi) + (2\pi i)k \;\middle|\; k \in \mathbb{Z} \right\}$$

for each $a + bi \in \mathbb{C}$.

Example 5.1.3.1.6. See [Wik24] for more examples of relations, such as antiderivation, inverse trigonometric functions, and inverse hyperbolic functions.

5.1.4 Functional Relations

Let A and B be sets.

Definition 5.1.4.1.1. A relation $R\colon A \nrightarrow B$ is **functional** if, for each $a \in A$, the set $R(a)$ is either empty or a singleton.

Proposition 5.1.4.1.2. Let $R\colon A \nrightarrow B$ be a relation.

1. *Characterisations.* The following conditions are equivalent:

   (a) The relation R is functional.

   (b) We have $R \diamond R^{\dagger} \subset \chi_B$.

Proof. Item 1, Characterisations: We claim that Items 1a and 1b are indeed equivalent:

- *Item 1a* $\implies$ *Item 1b:* Let $(b, b') \in B \times B$. We need to show that

  $$\left[R \diamond R^{\dagger}\right](b, b') \preceq_{\{\mathsf{t},\mathsf{f}\}} \chi_B(b, b'),$$

  i.e. that if there exists some $a \in A$ such that $b \sim_{R^{\dagger}} a$ and $a \sim_R b'$, then $b = b'$. But since $b \sim_{R^{\dagger}} a$ is the same as $a \sim_R b$, we have both $a \sim_R b$ and $a \sim_R b'$ at the same time, which implies $b = b'$ since R is functional.

- *Item 1b* $\implies$ *Item 1a:* Suppose that we have $a \sim_R b$ and $a \sim_R b'$ for $b, b' \in B$. We claim that $b = b'$:

1. Since $a \sim_R b$, we have $b \sim_{R^\dagger} a$.
2. Since $R \diamond R^\dagger \subset \chi_B$, we have

   $$\left[R \diamond R^\dagger\right](b, b') \preceq_{\{\mathrm{t},\mathrm{f}\}} \chi_B(b, b'),$$

   and since $b \sim_{R^\dagger} a$ and $a \sim_R b'$, it follows that $\left[R \diamond R^\dagger\right](b, b') = \mathsf{true}$, and thus $\chi_B(b, b') = \mathsf{true}$ as well, i.e. $b = b'$.

This finishes the proof. □

5.1.5 Total Relations

Let A and B be sets.

Definition 5.1.5.1.1. A relation $R\colon A \nrightarrow B$ is **total** if, for each $a \in A$, we have $R(a) \neq \emptyset$.

Proposition 5.1.5.1.2. Let $R\colon A \nrightarrow B$ be a relation.

1. *Characterisations.* The following conditions are equivalent:
   (a) The relation R is total.
   (b) We have $\chi_A \subset R^\dagger \diamond R$.

Proof. *Item 1, Characterisations*: We claim that Items 1a and 1b are indeed equivalent:

- *Item 1a* $\implies$ *Item 1b*: We have to show that, for each $(a, a') \in A$, we have

  $$\chi_A(a, a') \preceq_{\{\mathrm{t},\mathrm{f}\}} \left[R^\dagger \diamond R\right](a, a'),$$

  i.e. that if $a = a'$, then there exists some $b \in B$ such that $a \sim_R b$ and $b \sim_{R^\dagger} a'$ (i.e. $a \sim_R b$ again), which follows from the totality of R.

- *Item 1b* $\implies$ *Item 1a*: Given $a \in A$, since $\chi_A \subset R^\dagger \diamond R$, we must have

  $$\{a\} \subset \left[R^\dagger \diamond R\right](a),$$

  implying that there must exist some $b \in B$ such that $a \sim_R b$ and $b \sim_{R^\dagger} a$ (i.e. $a \sim_R b$) and thus $R(a) \neq \emptyset$, as $b \in R(a)$.

This finishes the proof. □

5.2 Categories of Relations

5.2.1 The Category of Relations

Definition 5.2.1.1.1. The **category of relations** is the category Rel where

- *Objects.* The objects of Rel are sets.
- *Morphisms.* For each $A, B \in \mathrm{Obj}(\mathsf{Sets})$, we have
$$\mathsf{Rel}(A, B) \stackrel{\text{def}}{=} \mathrm{Rel}(A, B).$$
- *Identities.* For each $A \in \mathrm{Obj}(\mathsf{Rel})$, the unit map
$$\mathbb{1}^{\mathsf{Rel}}_{A} \colon \mathrm{pt} \to \mathrm{Rel}(A, A)$$
of Rel at A is defined by
$$\mathrm{id}^{\mathsf{Rel}}_{A} \stackrel{\text{def}}{=} \chi_A(-_1, -_2),$$
where $\chi_A(-_1, -_2)$ is the characteristic relation of A of Item 3 of Definition 2.4.1.1.1.
- *Composition.* For each $A, B, C \in \mathrm{Obj}(\mathsf{Rel})$, the composition map
$$\circ^{\mathsf{Rel}}_{A,B,C} \colon \mathrm{Rel}(B, C) \times \mathrm{Rel}(A, B) \to \mathrm{Rel}(A, C)$$
of Rel at (A, B, C) is defined by
$$S \circ^{\mathsf{Rel}}_{A,B,C} R \stackrel{\text{def}}{=} S \diamond R$$
for each $(S, R) \in \mathrm{Rel}(B, C) \times \mathrm{Rel}(A, B)$, where $S \diamond R$ is the composition of S and R of Definition 6.3.12.1.1.

5.2.2 The Closed Symmetric Monoidal Category of Relations

5.2.2.1 The Monoidal Product

Definition 5.2.2.1.1. The **monoidal product of** Rel is the functor
$$\times \colon \mathsf{Rel} \times \mathsf{Rel} \to \mathsf{Rel}$$
where

- *Action on Objects.* For each $A, B \in \mathrm{Obj}(\mathsf{Rel})$, we have
$$\times(A, B) \stackrel{\text{def}}{=} A \times B,$$
where $A \times B$ is the Cartesian product of sets of Definition 2.1.3.1.1.

- *Action on Morphisms.* For each $(A, C), (B, D) \in \mathrm{Obj}(\mathsf{Rel} \times \mathsf{Rel})$, the action on morphisms

  $$\times_{(A,C),(B,D)} : \mathrm{Rel}(A, B) \times \mathrm{Rel}(C, D) \to \mathrm{Rel}(A \times C, B \times D)$$

  of $\times$ is given by sending a pair of morphisms (R, S) of the form

  $$\begin{aligned} R &: A \nrightarrow B, \\ S &: C \nrightarrow D \end{aligned}$$

  to the relation

  $$R \times S : A \times C \nrightarrow B \times D$$

  of Definition 6.3.9.1.1.

5.2.2.2 The Monoidal Unit

Definition 5.2.2.2.1. The **monoidal unit of** Rel is the functor

$$\mathbb{1}^{\mathsf{Rel}} : \mathrm{pt} \to \mathsf{Rel}$$

picking the set

$$\mathbb{1}_{\mathsf{Rel}} \stackrel{\text{def}}{=} \mathrm{pt}$$

of Rel.

5.2.2.3 The Associator

Definition 5.2.2.3.1. The **associator of** Rel is the natural isomorphism

$$\alpha^{\mathsf{Rel}} : \times \circ ((\times) \times \mathrm{id}) \stackrel{\sim}{\Longrightarrow} \times \circ (\mathrm{id} \times (\times)) \circ \boldsymbol{\alpha}^{\mathsf{Cats}}_{\mathsf{Rel},\mathsf{Rel},\mathsf{Rel}},$$

as in the diagram

$$\begin{tikzcd} & \mathsf{Rel} \times (\mathsf{Rel} \times \mathsf{Rel}) \arrow[dr, "\mathrm{id} \times (\times)", "\shortmid" marking] & \\ (\mathsf{Rel} \times \mathsf{Rel}) \times \mathsf{Rel} \arrow[ur, dashed, "\boldsymbol{\alpha}^{\mathsf{Cats}}_{\mathsf{Rel},\mathsf{Rel},\mathsf{Rel}}", "\sim"'] \arrow[d, "(\times) \times \mathrm{id}"', "\shortmid" marking] & \arrow[phantom, "\Uparrow \alpha^{\mathsf{Rel}}"] & \mathsf{Rel} \times \mathsf{Rel} \arrow[d, "\times", "\shortmid" marking] \\ \mathsf{Rel} \times \mathsf{Rel} \arrow[rr, "\times"', "\shortmid" marking] & & \mathsf{Rel}, \end{tikzcd}$$

whose component

$$\alpha^{\mathsf{Rel}}_{A,B,C} : (A \times B) \times C \nrightarrow A \times (B \times C)$$

at $A, B, C \in \mathrm{Obj}(\mathsf{Rel})$ is the relation defined by declaring

$$((a,b),c) \sim_{\alpha^{\mathsf{Rel}}_{A,B,C}} (a',(b',c'))$$

iff $a = a'$, $b = b'$, and $c = c'$.

5.2.2.4 The Left Unitor

Definition 5.2.2.4.1. The **left unitor of** Rel is the natural isomorphism

$$\lambda^{\mathsf{Rel}} \colon \times \circ \left(\mathbb{1}^{\mathsf{Rel}} \times \mathrm{id}\right) \overset{\sim}{\Longrightarrow} \boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathsf{Rel}}, \qquad \begin{tikzcd} \mathrm{pt} \times \mathsf{Rel} \arrow[r, "\mathbb{1}^{\mathsf{Rel}} \times \mathrm{id}"] \arrow[dr, dashed, bend right, "\boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathsf{Rel}}"'] & \mathsf{Rel} \times \mathsf{Rel}, \arrow[d, "\times"] \arrow[dl, phantom, "\Swarrow \lambda^{\mathsf{Rel}}"] \\ & \mathsf{Rel} \end{tikzcd}$$

whose component

$$\lambda^{\mathsf{Rel}}_A \colon \mathbb{1}_{\mathsf{Rel}} \times A \nrightarrow A$$

at A is defined by declaring

$$(\star, a) \sim_{\lambda^{\mathsf{Rel}}_A} b$$

iff $a = b$.

5.2.2.5 The Right Unitor

Definition 5.2.2.5.1. The **right unitor of** Rel is the natural isomorphism

$$\rho^{\mathsf{Rel}} \colon \times \circ \left(\mathrm{id} \times \mathbb{1}^{\mathsf{Rel}}\right) \overset{\sim}{\Longrightarrow} \boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathsf{Rel}}, \qquad \begin{tikzcd} \mathsf{Rel} \times \mathrm{pt} \arrow[r, "\mathrm{id} \times \mathbb{1}^{\mathsf{Rel}}"] \arrow[dr, dashed, bend right, "\boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathsf{Rel}}"'] & \mathsf{Rel} \times \mathsf{Rel}, \arrow[d, "\times"] \arrow[dl, phantom, "\Swarrow \rho^{\mathsf{Rel}}"] \\ & \mathsf{Rel} \end{tikzcd}$$

whose component

$$\rho^{\mathsf{Rel}}_A \colon A \times \mathbb{1}_{\mathsf{Rel}} \nrightarrow A$$

at A is defined by declaring

$$(a, \star) \sim_{\rho^{\mathsf{Rel}}_A} b$$

iff $a = b$.

5.2.2.6 The Symmetry

Definition 5.2.2.6.1. The **symmetry of** Rel is the natural isomorphism

$$\sigma^{\mathsf{Rel}}\colon \times \Longrightarrow \times \circ \boldsymbol{\sigma}^{\mathsf{Cats}_2}_{\mathsf{Rel},\mathsf{Rel}}, \qquad \begin{tikzcd} \mathsf{Rel}\times\mathsf{Rel} \arrow[rr, "\times"] \arrow[dr, "\boldsymbol{\sigma}^{\mathsf{Cats}_2}_{\mathsf{Rel},\mathsf{Rel}}"'] & {} \arrow[d, phantom, "\overset{\sigma^{\mathsf{Rel}}}{\Downarrow}"] & \mathsf{Rel}, \\ & \mathsf{Rel}\times\mathsf{Rel} \arrow[ur, "\times"'] & \end{tikzcd}$$

whose component

$$\sigma^{\mathsf{Rel}}_{A,B}\colon A\times B \longrightarrow B\times A$$

at (A,B) is defined by declaring

$$(a,b) \sim_{\sigma^{\mathsf{Rel}}_{A,B}} (b',a')$$

iff $a = a'$ and $b = b'$.

5.2.2.7 The Internal Hom

Definition 5.2.2.7.1. The **internal Hom of** Rel is the functor

$$\mathrm{Rel}\colon \mathsf{Rel}^{\mathsf{op}}\times\mathsf{Rel} \longrightarrow \mathsf{Rel}$$

defined

- On objects by sending $A,B \in \mathrm{Obj}(\mathsf{Rel})$ to the set $\mathrm{Rel}(A,B)$ of Item 1 of Definition 5.1.1.1.3.
- On morphisms by pre/post-composition defined as in Definition 6.3.12.1.1.

Proposition 5.2.2.7.2. Let $A,B,C \in \mathrm{Obj}(\mathsf{Rel})$.

1. *Adjointness.* We have adjunctions

$$(A\times - \dashv \mathrm{Rel}(A,-))\colon \quad \begin{tikzcd} \mathsf{Rel} \arrow[r, bend left, "A\times -"] \arrow[r, phantom, "\perp"] & \mathsf{Rel}, \arrow[l, bend left, "\mathrm{Rel}(A,-)"] \end{tikzcd}$$

$$(-\times B \dashv \mathrm{Rel}(B,-))\colon \quad \begin{tikzcd} \mathsf{Rel} \arrow[r, bend left, "-\times B"] \arrow[r, phantom, "\perp"] & \mathsf{Rel}, \arrow[l, bend left, "\mathrm{Rel}(B,-)"] \end{tikzcd}$$

witnessed by bijections

$$\mathrm{Rel}(A\times B, C) \cong \mathrm{Rel}(A, \mathrm{Rel}(B,C)),$$
$$\mathrm{Rel}(A\times B, C) \cong \mathrm{Rel}(B, \mathrm{Rel}(A,C)),$$

natural in $A,B,C \in \mathrm{Obj}(\mathsf{Rel})$.

Proof. *Item 1*, *Adjointness*: Indeed, we have

$$\begin{aligned}\mathrm{Rel}(A\times B,C) &\overset{\text{def}}{=} \mathsf{Sets}(A\times B\times C,\{\mathsf{true},\mathsf{false}\})\\ &\overset{\text{def}}{=} \mathrm{Rel}(A,B\times C)\\ &\overset{\text{def}}{=} \mathrm{Rel}(A,\mathrm{Rel}(B,C)),\end{aligned}$$

and similarly for the bijection $\mathrm{Rel}(A\times B,C)\cong\mathrm{Rel}(B,\mathrm{Rel}(A,C))$. □

5.2.2.8 The Closed Symmetric Monoidal Category of Relations

Proposition 5.2.2.8.1. The category Rel admits a closed symmetric monoidal category structure consisting of[6]

- *The Underlying Category.* The category Rel of sets and relations of Definition 5.2.1.1.1.
- *The Monoidal Product.* The functor

  $$\times\colon \mathrm{Rel}\times\mathrm{Rel}\longrightarrow\mathrm{Rel}$$

  of Definition 5.2.2.1.1.
- *The Internal Hom.* The internal Hom functor

  $$\mathbf{Rel}\colon \mathrm{Rel}^{\mathsf{op}}\times\mathrm{Rel}\longrightarrow\mathrm{Rel}$$

  of Definition 5.2.2.7.1.
- *The Monoidal Unit.* The functor

  $$\mathbb{1}^{\mathrm{Rel}}\colon \mathsf{pt}\longrightarrow\mathrm{Rel}$$

  of Definition 5.2.2.2.1.
- *The Associators.* The natural isomorphism

  $$\alpha^{\mathrm{Rel}}\colon \times\circ(\times\times\mathrm{id}_{\mathrm{Rel}})\overset{\sim}{\Longrightarrow}\times\circ(\mathrm{id}_{\mathrm{Rel}}\times\times)\circ\boldsymbol{\alpha}^{\mathsf{Cats}}_{\mathrm{Rel},\mathrm{Rel},\mathrm{Rel}}$$

  of Definition 5.2.2.3.1.
- *The Left Unitors.* The natural isomorphism

  $$\lambda^{\mathrm{Rel}}\colon \times\circ\left(\mathbb{1}^{\mathrm{Rel}}\times\mathrm{id}_{\mathrm{Rel}}\right)\overset{\sim}{\Longrightarrow}\boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathrm{Rel}}$$

[6] *Warning:* This is not a Cartesian monoidal structure, as the product on Rel is in fact

of Definition 5.2.2.4.1.

- *The Right Unitors.* The natural isomorphism

$$\rho^{\mathrm{Rel}} : \times \circ \left(\mathsf{id} \times \mathbb{1}^{\mathrm{Rel}}\right) \overset{\sim}{\Longrightarrow} \boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathrm{Rel}}$$

of Definition 5.2.2.5.1.

- *The Symmetry.* The natural isomorphism

$$\sigma^{\mathrm{Rel}} : \times \overset{\sim}{\Longrightarrow} \times \circ \boldsymbol{\sigma}^{\mathsf{Cats}_2}_{\mathrm{Rel},\mathrm{Rel}}$$

of Definition 5.2.2.6.1.

Proof. Omitted. □

5.2.3 The 2-Category of Relations

Definition 5.2.3.1.1. The 2-**category of relations** is the locally posetal 2-category Rel where

- *Objects.* The objects of Rel are sets.
- ***Hom**-Objects.* For each $A, B \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$\begin{aligned}\mathrm{Hom}_{\mathsf{Rel}}(A, B) &\overset{\text{def}}{=} \mathbf{Rel}(A, B)\\ &\overset{\text{def}}{=} (\mathrm{Rel}(A, B), \subset).\end{aligned}$$

- *Identities.* For each $A \in \mathrm{Obj}(\mathsf{Rel})$, the unit map

$$\mathbb{1}^{\mathsf{Rel}}_A : \mathrm{pt} \to \mathbf{Rel}(A, A)$$

of Rel at A is defined by

$$\mathrm{id}^{\mathsf{Rel}}_A \overset{\text{def}}{=} \chi_A(-_1, -_2),$$

where $\chi_A(-_1, -_2)$ is the characteristic relation of A of Item 3 of Definition 2.4.1.1.1.

given by the disjoint union of sets; see ??.
END TEXTDBEND

- *Composition.* For each $A, B, C \in \mathrm{Obj}(\mathsf{Rel})$, the composition map[7]

$$\circ^{\mathsf{Rel}}_{A,B,C} \colon \mathbf{Rel}(B,C) \times \mathbf{Rel}(A,B) \longrightarrow \mathbf{Rel}(A,C)$$

 of Rel at (A, B, C) is defined by

$$S \circ^{\mathsf{Rel}}_{A,B,C} R \stackrel{\text{def}}{=} S \diamond R$$

 for each $(S, R) \in \mathsf{Rel}(B,C) \times \mathsf{Rel}(A,B)$, where $S \diamond R$ is the composition of S and R of Definition 6.3.12.1.1.

5.2.4 The Double Category of Relations

5.2.4.1 The Double Category of Relations

Definition 5.2.4.1.1. The **double category of relations** is the locally posetal double category $\mathsf{Rel}^{\mathsf{dbl}}$ where

- *Objects.* The objects of $\mathsf{Rel}^{\mathsf{dbl}}$ are sets.
- *Vertical Morphisms.* The vertical morphisms of $\mathsf{Rel}^{\mathsf{dbl}}$ are maps of sets $f \colon A \longrightarrow B$.
- *Horizontal Morphisms.* The horizontal morphisms of $\mathsf{Rel}^{\mathsf{dbl}}$ are relations $R \colon A \nrightarrow X$.
- 2-*Morphisms.* A 2-cell

$$\begin{tikzcd} A \arrow[r, "R", "|"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & B \arrow[d, "g"] \\ X \arrow[r, "S"', "|"{marking}] & Y \end{tikzcd}$$

 of $\mathsf{Rel}^{\mathsf{dbl}}$ is either non-existent or an inclusion of relations of the form

$$R \subset S \circ (f \times g), \qquad \begin{tikzcd} A \times B \arrow[r, "R"] \arrow[d, "f \times g"'] \arrow[dr, phantom, "\circlearrowleft"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ X \times Y \arrow[r, "S"'] & \{\mathsf{true}, \mathsf{false}\}. \end{tikzcd}$$

[7]Note that this is indeed a morphism of posets: given relations $R_1, R_2 \in \mathbf{Rel}(A, B)$ and

- *Horizontal Identities.* The horizontal unit functor of $\mathsf{Rel}^{\mathsf{dbl}}$ is the functor of Definition 5.2.4.2.1.

- *Vertical Identities.* For each $A \in \mathrm{Obj}\left(\mathsf{Rel}^{\mathsf{dbl}}\right)$, we have

$$\mathrm{id}_A^{\mathsf{Rel}^{\mathsf{dbl}}} \stackrel{\text{def}}{=} \mathrm{id}_A.$$

- *Identity 2-Morphisms.* For each horizontal morphism $R\colon A \nrightarrow B$ of $\mathsf{Rel}^{\mathsf{dbl}}$, the identity 2-morphism

$$\begin{tikzcd} A \arrow[r, "R", "\mid"{marking}] \arrow[d, "\mathrm{id}_A"'] \arrow[dr, phantom, "\substack{\| \\ \mathrm{id}_R \\ \Downarrow}"] & B \arrow[d, "\mathrm{id}_B"] \\ A \arrow[r, "R"', "\mid"{marking}] & B \end{tikzcd}$$

of R is the identity inclusion

$$R \subset R, \qquad \begin{tikzcd} B \times A \arrow[r, "R"] \arrow[d, "\mathrm{id}_B \times \mathrm{id}_A"'] \arrow[dr, phantom, "\subset"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ B \times A \arrow[r, "R"'] & \{\mathsf{true}, \mathsf{false}\}. \end{tikzcd}$$

- *Horizontal Composition.* The horizontal composition functor of $\mathsf{Rel}^{\mathsf{dbl}}$ is the functor of Definition 5.2.4.3.1.

- *Vertical Composition of 1-Morphisms.* For each composable pair $A \xrightarrow{F} B \xrightarrow{G} C$ of vertical morphisms of $\mathsf{Rel}^{\mathsf{dbl}}$, i.e. maps of sets, we have

$$g \circ^{\mathsf{Rel}^{\mathsf{dbl}}} f \stackrel{\text{def}}{=} g \circ f.$$

- *Vertical Composition of 2-Morphisms.* The vertical composition of 2-morphisms in $\mathsf{Rel}^{\mathsf{dbl}}$ is defined as in Definition 5.2.4.4.1.

- *Associators.* The associators of $\mathsf{Rel}^{\mathsf{dbl}}$ is defined as in Definition 5.2.4.5.1.

- *Left Unitors.* The left unitors of $\mathsf{Rel}^{\mathsf{dbl}}$ is defined as in Definition 5.2.4.6.1.

- *Right Unitors.* The right unitors of $\mathsf{Rel}^{\mathsf{dbl}}$ is defined as in Definition 5.2.4.7.1.

5.2.4.2 Horizontal Identities

Definition 5.2.4.2.1. The **horizontal unit functor** of $\mathsf{Rel}^{\mathsf{dbl}}$ is the functor

$$\mathbb{1}^{\mathsf{Rel}^{\mathsf{dbl}}} \colon \mathsf{Rel}^{\mathsf{dbl}}_0 \to \mathsf{Rel}^{\mathsf{dbl}}_1$$

of $\mathsf{Rel}^{\mathsf{dbl}}$ is the functor where

- *Action on Objects.* For each $A \in \mathrm{Obj}\left(\mathsf{Rel}^{\mathsf{dbl}}_0\right)$, we have

$$\mathbb{1}_A \stackrel{\text{def}}{=} \chi_A(-_1, -_2).$$

- *Action on Morphisms.* For each vertical morphism $f \colon A \to B$ of $\mathsf{Rel}^{\mathsf{dbl}}$, i.e. each map of sets f from A to B, the identity 2-morphism

$$\begin{tikzcd} A \arrow[r, "\mathbb{1}_A", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow \mathbb{1}_f"] & A \arrow[d, "f"] \\ B \arrow[r, "\mathbb{1}_B"', "\shortmid"{marking}] & B \end{tikzcd}$$

of f is the inclusion

$$\chi_B \circ (f \times f) \subset \chi_A, \qquad \begin{tikzcd} A \times A \arrow[r, "\chi_A(-_1,-_2)"] \arrow[d, "f \times f"'] \arrow[dr, phantom, "\subset"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ B \times B \arrow[r, "\chi_B(-_1,-_2)"'] & \{\mathsf{true}, \mathsf{false}\} \end{tikzcd}$$

of Item 1 of Proposition 2.4.1.1.3.

5.2.4.3 Horizontal Composition

Definition 5.2.4.3.1. The **horizontal composition functor** of $\mathsf{Rel}^{\mathsf{dbl}}$ is the functor

$$\odot^{\mathsf{Rel}^{\mathsf{dbl}}} \colon \mathsf{Rel}^{\mathsf{dbl}}_1 \underset{\mathsf{Rel}^{\mathsf{dbl}}_0}{\times} \mathsf{Rel}^{\mathsf{dbl}}_1 \to \mathsf{Rel}^{\mathsf{dbl}}_1$$

of $\mathsf{Rel}^{\mathsf{dbl}}$ is the functor where

$S_1, S_2 \in \mathbf{Rel}(B, C)$ such that

$$\begin{aligned} R_1 &\subset R_2, \\ S_1 &\subset S_2, \end{aligned}$$

- *Action on Objects.* For each composable pair $A \overset{R}{\nrightarrow} B \overset{S}{\nrightarrow} C$ of horizontal morphisms of $\mathsf{Rel}^{\mathsf{dbl}}$, we have

$$S \odot R \overset{\text{def}}{=} S \diamond R,$$

where $S \diamond R$ is the composition of R and S of Definition 6.3.12.1.1.

- *Action on Morphisms.* For each horizontally composable pair

$$\begin{tikzcd} A \arrow[r, "R", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & B \arrow[d, "g"] \\ X \arrow[r, "T"', "\shortmid"{marking}] & Y \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "S", "\shortmid"{marking}] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\beta"] & C \arrow[d, "h"] \\ Y \arrow[r, "U"', "\shortmid"{marking}] & Z \end{tikzcd}$$

of 2-morphisms of $\mathsf{Rel}^{\mathsf{dbl}}$, i.e. for each pair

$$\begin{tikzcd} A \times B \arrow[r, "R"] \arrow[d, "f \times g"'] \arrow[dr, phantom, "\subset"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ X \times Y \arrow[r, "T"'] & \{\mathsf{true}, \mathsf{false}\} \end{tikzcd} \qquad \begin{tikzcd} B \times C \arrow[r, "S"] \arrow[d, "g \times h"'] \arrow[dr, phantom, "\subset"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ Y \times Z \arrow[r, "U"'] & \{\mathsf{true}, \mathsf{false}\} \end{tikzcd}$$

of inclusions of relations, the horizontal composition

$$\begin{tikzcd} A \arrow[r, "S \odot R", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\beta\odot\alpha"] & C \arrow[d, "h"] \\ X \arrow[r, "U \odot T"', "\shortmid"{marking}] & Z \end{tikzcd}$$

of α and β is the inclusion of relations[8]

$$(U \diamond T) \circ (f \times h) \subset (S \diamond R) \qquad \begin{tikzcd} A \times C \arrow[r, "S \diamond R"] \arrow[d, "f \times h"'] \arrow[dr, phantom, "\subset"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ X \times Z \arrow[r, "U \diamond T"'] & \{\mathsf{true}, \mathsf{false}\}. \end{tikzcd}$$

we have also $S_1 \diamond R_1 \subset S_2 \diamond R_2$.

[8]This is justified by noting that, given $(a, c) \in A \times C$, the statement

- We have $a \sim_{(U \diamond T)\circ(f \times h)} c$, i.e. $f(a) \sim_{U \diamond T} h(c)$, i.e. there exists some $y \in Y$ such that:

5.2.4.4 Vertical Composition of 2-Morphisms

Definition 5.2.4.4.1. The **vertical composition** in $\mathsf{Rel}^{\mathsf{dbl}}$ is defined as follows: for each vertically composable pair

$$
\begin{tikzcd}
A \arrow[r, "R", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & X \arrow[d, "g"] \\
B \arrow[r, "S"', "\shortmid"{marking}] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r, "S", "\shortmid"{marking}] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow\beta"] & Y \arrow[d, "k"] \\
C \arrow[r, "T"', "\shortmid"{marking}] & Z
\end{tikzcd}
$$

of 2-morphisms of $\mathsf{Rel}^{\mathsf{dbl}}$, i.e. for each each pair

$$
\begin{tikzcd}
A \times X \arrow[r, "R"] \arrow[d, "f\times g"'] \arrow[dr, phantom, "\subset"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\
B \times Y \arrow[r, "S"'] & \{\mathsf{true}, \mathsf{false}\}
\end{tikzcd}
\qquad
\begin{tikzcd}
B \times Y \arrow[r, "S"] \arrow[d, "h\times k"'] \arrow[dr, phantom, "\subset"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\
C \times Z \arrow[r, "T"'] & \{\mathsf{true}, \mathsf{false}\}
\end{tikzcd}
$$

of inclusions of relations, we define the vertical composition

$$
\begin{tikzcd}
A \arrow[r, "R", "\shortmid"{marking}] \arrow[d, "h\circ f"'] \arrow[dr, phantom, "\Downarrow\beta\circ\alpha"] & X \arrow[d, "k\circ g"] \\
C \arrow[r, "T"', "\shortmid"{marking}] & Z
\end{tikzcd}
$$

---

1. We have $f(a) \sim_T y$;
2. We have $y \sim_U h(c)$;

is implied by the statement

- We have $a \sim_{S\diamond R} c$, i.e. there exists some $b \in B$ such that:
  1. We have $a \sim_R b$;
  2. We have $b \sim_S c$;

since:

- If $a \sim_R b$, then $f(a) \sim_T g(b)$, as $T \circ (f \times g) \subset R$;
- If $b \sim_S c$, then $g(b) \sim_U h(c)$, as $U \circ (g \times h) \subset S$.

of α and β as the inclusion of relations

$$T \circ [(h \circ f) \times (k \circ g)] \subset R, \qquad \begin{tikzcd} A \times X \arrow[r, "R"] \arrow[d, "(h\circ f)\times(k\circ g)"'] \arrow[dr, phantom, "\subset"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ C \times Z \arrow[r, "T"'] & \{\mathsf{true}, \mathsf{false}\} \end{tikzcd}$$

given by the pasting of inclusions[9]

$$\begin{tikzcd} A \times X \arrow[r, "R"] \arrow[d, "f\times g"'] \arrow[dr, phantom, "\subset"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ B \times Y \arrow[r, "S"] \arrow[d, "h\times k"'] \arrow[dr, phantom, "\subset"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ C \times Z \arrow[r, "T"'] & \{\mathsf{true}, \mathsf{false}\}. \end{tikzcd}$$

5.2.4.5 The Associators

Definition 5.2.4.5.1. For each composable triple

$$A \overset{R}{\nrightarrow} B \overset{S}{\nrightarrow} C \overset{T}{\nrightarrow} D$$

of horizontal morphisms of $\mathsf{Rel}^{\mathsf{dbl}}$, the component

$$\alpha^{\mathsf{Rel}^{\mathsf{dbl}}}_{T,S,R} \colon (T \odot S) \odot R \overset{\sim}{\Longrightarrow} T \odot (S \odot R), \qquad \begin{tikzcd} A \arrow[r, "R", "|"] \arrow[d, "\mathrm{id}_A"'] & B \arrow[r, "S", "|"] \arrow[d, phantom, "\Downarrow \alpha^{\mathsf{Rel}^{\mathsf{dbl}}}_{T,S,R}"] & C \arrow[r, "T", "|"] & D \arrow[d, "\mathrm{id}_D"] \\ A \arrow[r, "R"', "|"] & B \arrow[r, "S"', "|"] & C \arrow[r, "T"', "|"] & D \end{tikzcd}$$

---

[9]This is justified by noting that, given $(a, x) \in A \times X$, the statement

- We have $h(f(a)) \sim_T k(g(x))$;

is implied by the statement

- We have $a \sim_R x$;

since

- If $a \sim_R x$, then $f(a) \sim_S g(x)$, as $S \circ (f \times g) \subset R$;
- If $b \sim_S y$, then $h(b) \sim_T k(y)$, as $T \circ (h \times k) \subset S$, and thus, in particular:
  - If $f(a) \sim_S g(x)$, then $h(f(a)) \sim_T k(g(x))$.

of the associator of $\mathsf{Rel}^{\mathsf{dbl}}$ at (R, S, T) is the identity inclusion[10]

$$(T \diamond S) \diamond R = T \diamond (S \diamond R) \qquad \begin{tikzcd} A \times B \arrow[r, "(T\diamond S)\diamond R"] \arrow[d, equal] \arrow[dr, phantom, "\mathbin{/\!\!/}"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ A \times B \arrow[r, "T\diamond(S\diamond R)"'] & \{\mathsf{true}, \mathsf{false}\}. \end{tikzcd}$$

5.2.4.6 The Left Unitors

Definition 5.2.4.6.1. For each horizontal morphism $R\colon A \nrightarrow B$ of $\mathsf{Rel}^{\mathsf{dbl}}$, the component

$$\lambda^{\mathsf{Rel}^{\mathsf{dbl}}}_R \colon \mathbb{1}_B \odot R \overset{\sim}{\Longrightarrow} R, \qquad \begin{tikzcd} A \arrow[r, "R", "\shortmid"{marking}] \arrow[d, "\mathrm{id}_A"'] & B \arrow[r, "\mathbb{1}_B", "\shortmid"{marking}] \arrow[d, phantom, "\lambda^{\mathsf{Rel}^{\mathsf{dbl}}}_R \Downarrow"] & B \arrow[d, "\mathrm{id}_B"] \\ A \arrow[rr, "R"', "\shortmid"{marking}] & & B \end{tikzcd}$$

of the left unitor of $\mathsf{Rel}^{\mathsf{dbl}}$ at R is the identity inclusion[11]

$$R = \chi_B \diamond R, \qquad \begin{tikzcd} A \times B \arrow[r, "\chi_B\diamond R"] \arrow[d, equal] \arrow[dr, phantom, "\mathbin{/\!\!/}"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ A \times B \arrow[r, "R"'] & \{\mathsf{true}, \mathsf{false}\}. \end{tikzcd}$$

5.2.4.7 The Right Unitors

Definition 5.2.4.7.1. For each horizontal morphism $R\colon A \nrightarrow B$ of $\mathsf{Rel}^{\mathsf{dbl}}$, the component

$$\rho^{\mathsf{Rel}^{\mathsf{dbl}}}_R \colon R \odot \mathbb{1}_A \overset{\sim}{\Longrightarrow} R, \qquad \begin{tikzcd} A \arrow[r, "\mathbb{1}_A", "\shortmid"{marking}] \arrow[d, "\mathrm{id}_A"'] & A \arrow[r, "R", "\shortmid"{marking}] \arrow[d, phantom, "\rho^{\mathsf{Rel}^{\mathsf{dbl}}}_R \Downarrow"] & B \arrow[d, "\mathrm{id}_B"] \\ A \arrow[rr, "R"', "\shortmid"{marking}] & & B \end{tikzcd}$$

[10] This is justified by Item 2 of Proposition 6.3.12.1.3.
[11] This is justified by Item 3 of Proposition 6.3.12.1.3.

of the right unitor of $\mathsf{Rel}^{\mathsf{dbl}}$ at R is the identity inclusion[12]

$$R = R \diamond \chi_A, \qquad \begin{tikzcd} A \times B \arrow[r, "R \diamond \chi_A"] \arrow[d, equal] \arrow[dr, phantom, "\swarrow\!\!\!\!\,\,\,/"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ A \times B \arrow[r, "R"'] & \{\mathsf{true}, \mathsf{false}\}. \end{tikzcd}$$

5.3 Properties of the 2-Category of Relations

5.3.1 Self-Duality

Proposition 5.3.1.1.1. The (2-)category of relations is self-dual:

1. *Self-Duality I.* We have an isomorphism

   $$\mathrm{Rel}^{\mathsf{op}} \overset{\text{eq.}}{\cong} \mathrm{Rel}$$

   of categories.

2. *Self-Duality II.* We have a 2-isomorphism

   $$\mathbf{Rel}^{\mathsf{op}} \overset{\text{eq.}}{\cong} \mathbf{Rel}$$

   of 2-categories.

Proof. *Item 1*, *Self-Duality I*: We claim that the functor

$$F \colon \mathsf{Rel}^{\mathsf{op}} \to \mathsf{Rel}$$

given by the identity on objects and by $R \mapsto R^{\dagger}$ on morphisms is an isomorphism of categories.

By Item 1 of Proposition 8.5.8.1.3, it suffices to show that F is bijective on objects (which is clear) and fully faithful. Indeed, the map

$$(-)^{\dagger} \colon \mathrm{Rel}(A, B) \to \mathrm{Rel}(B, A)$$

defined by the assignment $R \mapsto R^{\dagger}$ is a bijection by Item 5 of Proposition 6.3.11.1.3, showing F to be fully faithful.

Item 2, *Self-Duality II*: We claim that the 2-functor

$$F \colon \mathsf{Rel}^{\mathsf{op}} \to \mathsf{Rel}$$

given by the identity on objects, by $R \mapsto R^{\dagger}$ on morphisms, and by preserving inclusions on 2-morphisms via Item 1 of Proposition 6.3.11.1.3, is an isomorphism of categories.

By **??** of **??**, it suffices to show that F is:

[12] This is justified by Item 3 of Proposition 6.3.12.1.3.

- Bijective on objects, which is clear.
- Bijective on 1-morphisms, which was shown in Item 1.
- Bijective on 2-morphisms, which follows from Item 1 of Proposition 6.3.11.1.3.

Thus F is indeed a 2-isomorphism of categories. □

5.3.2 Isomorphisms and Equivalences in Rel

Let $R\colon A \nrightarrow B$ be a relation from A to B.

Proposition 5.3.2.1.1. The following conditions are equivalent:

1. The relation $R\colon A \nrightarrow B$ is an equivalence in Rel, i.e.:

   (★) There exists a relation $R^{-1}\colon B \nrightarrow A$ from B to A together with isomorphisms

   $$\begin{aligned} R^{-1} \diamond R &\cong \chi_A, \\ R \diamond R^{-1} &\cong \chi_B. \end{aligned}$$

2. The relation $R\colon A \nrightarrow B$ is an isomorphism in Rel, i.e.:

   (★) There exists a relation $R^{-1}\colon B \nrightarrow A$ from B to A such that we have

   $$\begin{aligned} R^{-1} \diamond R &= \chi_A, \\ R \diamond R^{-1} &= \chi_B. \end{aligned}$$

3. There exists a bijection $f\colon A \xrightarrow{\cong} B$ with $R = \mathrm{Gr}(f)$.

Proof. We claim that Items 1 to 3 are indeed equivalent:

- *Item 1 $\iff$ Item 2:* This follows from the fact that Rel is locally posetal, so that natural isomorphisms and equalities of 1-morphisms in Rel coincide.
- *Item 2 $\implies$ Item 3:* The equalities in Item 2 imply $R \dashv R^{-1}$, and thus by Proposition 5.3.3.1.1, there exists a function $f_R\colon A \to B$ associated to R, where, for each $a \in A$, the image $f_R(a)$ of a by f_R is the unique element of $R(a)$, which implies $R = \mathrm{Gr}(f_R)$ in particular. Furthermore, we have $R^{-1} = f_R^{-1}$ (as in Definition 6.3.2.1.1). The conditions from

Item 2 then become the following:

$$\begin{aligned} f_R^{-1} \diamond f_R &= \chi_A, \\ f_R \diamond f_R^{-1} &= \chi_B. \end{aligned}$$

All that is left is to show then is that f_R is a bijection:

- *The Function f_R Is Injective.* Let $a, b \in A$ and suppose that $f_R(a) = f_R(b)$. Since $a \sim_R f_R(a)$ and $f_R(a) = f_R(b) \sim_{R^{-1}} b$, the condition $f_R^{-1} \diamond f_R = \chi_A$ implies that $a = b$, showing f_R to be injective.
- *The Function f_R Is Surjective.* Let $b \in B$. Applying the condition $f_R \diamond f_R^{-1} = \chi_B$ to (b, b), it follows that there exists some $a \in A$ such that $f_R^{-1}(b) = a$ and $f_R(a) = b$. This shows f_R to be surjective.

- *Item 3 $\implies$ Item 2:* By Item 2 of Proposition 6.3.1.1.2, we have an adjunction $\mathrm{Gr}(f) \dashv f^{-1}$, giving inclusions

$$\begin{aligned} \chi_A &\subset f^{-1} \diamond \mathrm{Gr}(f), \\ \mathrm{Gr}(f) \diamond f^{-1} &\subset \chi_B. \end{aligned}$$

We claim the reverse inclusions are also true:

  - *$f^{-1} \diamond Gr(f) \subset \chi_A$:* This is equivalent to the statement that if $f(a) = b$ and $f^{-1}(b) = a'$, then $a = a'$, which follows from the injectivity of f.
  - *$\chi_B \subset Gr(f) \diamond f^{-1}$:* This is equivalent to the statement that given $b \in B$ there exists some $a \in A$ such that $f^{-1}(b) = a$ and $f(a) = b$, which follows from the surjectivity of f.

This finishes the proof. □

5.3.3 Adjunctions in **Rel**

Let A and B be sets.

Proposition 5.3.3.1.1. We have a natural bijection

$$\left\{\begin{matrix}\text{Adjunctions in } \mathsf{Rel} \\ \text{from } A \text{ to } B\end{matrix}\right\} \cong \left\{\begin{matrix}\text{Functions} \\ \text{from } A \text{ to } B\end{matrix}\right\},$$

with every adjunction in **Rel** being of the form $\mathrm{Gr}(f) \dashv f^{-1}$ for some function f.

Proof. We proceed step by step:

1. *From Adjunctions in* **Rel** *to Functions.* An adjunction in **Rel** from A to B consists of a pair of relations

$$\begin{aligned} R &\colon A \nrightarrow B, \\ S &\colon B \nrightarrow A, \end{aligned}$$

together with inclusions

$$\begin{aligned} \chi_A &\subset S \diamond R, \\ R \diamond S &\subset \chi_B. \end{aligned}$$

We claim that these conditions imply that R is total and functional, i.e. that $R(a)$ is a singleton for each $a \in A$:

(a) *$R(a)$ Has an Element.* Given $a \in A$, since $\chi_A \subset S \diamond R$, we must have $\{a\} \subset S(R(a))$, implying that there exists some $b \in B$ such that $a \sim_R b$ and $b \sim_S a$, and thus $R(a) \neq \emptyset$, as $b \in R(a)$.

(b) *$R(a)$ Has No More Than One Element.* Suppose that we have $a \sim_R b$ and $a \sim_R b'$ for $b, b' \in B$. We claim that $b = b'$:

i. Since $\chi_A \subset S \diamond R$, there exists some $k \in B$ such that $a \sim_R k$ and $k \sim_S a$.

ii. Since $R \diamond S \subset \chi_B$, if $b'' \sim_S a'$ and $a' \sim_R b'''$, then $b'' = b'''$.

iii. Applying the above to $b'' = k$, $b''' = b$, and $a' = a$, since $k \sim_S a$ and $a \sim_R b'$, we have $k = b$.

iv. Similarly $k = b'$.

v. Thus $b = b'$.

Together, the above two items show $R(a)$ to be a singleton, being thus given by $\mathrm{Gr}(f)$ for some function $f \colon A \to B$, which gives a map

$$\left\{\begin{matrix}\text{Adjunctions in } \mathsf{Rel} \\ \text{from } A \text{ to } B\end{matrix}\right\} \to \left\{\begin{matrix}\text{Functions} \\ \text{from } A \text{ to } B\end{matrix}\right\}.$$

Moreover, by uniqueness of adjoints (**??** of **??**), this implies also that $S = f^{-1}$.

2. *From Functions to Adjunctions in* **Rel**. By Item 2 of Proposition 6.3.1.1.2, every function $f \colon A \to B$ gives rise to an adjunction $\mathrm{Gr}(f) \dashv f^{-1}$ in Rel, giving a map

$$\left\{\begin{matrix}\text{Functions} \\ \text{from } A \text{ to } B\end{matrix}\right\} \to \left\{\begin{matrix}\text{Adjunctions in } \mathsf{Rel} \\ \text{from } A \text{ to } B\end{matrix}\right\}.$$

3. *Invertibility: From Functions to Adjunctions Back to Functions.* We need to show that starting with a function $f\colon A \to B$, passing to $\mathrm{Gr}(f) \dashv f^{-1}$, and then passing again to a function gives f again. This is clear however, since we have $a \sim_{\mathrm{Gr}(f)} b$ iff $f(a) = b$.

4. *Invertibility: From Adjunctions to Functions Back to Adjunctions.* We need to show that, given an adjunction $R \dashv S$ in **Rel** giving rise to a function $f_{R,S}\colon A \to B$, we have

$$\begin{aligned} \mathrm{Gr}(f_{R,S}) &= R, \\ f^{-1}_{R,S} &= S. \end{aligned}$$

We check these explicitly:

- $Gr(f_{R,S}) = R$. We have

$$\begin{aligned} \mathrm{Gr}(f_{R,S}) &\overset{\text{def}}{=} \left\{(a, f_{R,S}(a)) \in A \times B \,\middle|\, a \in A\right\} \\ &\overset{\text{def}}{=} \{(a, R(a)) \in A \times B \mid a \in A\} \\ &= R. \end{aligned}$$

- $f^{-1}_{R,S} = S$. We first claim that, given $a \in A$ and $b \in B$, the following conditions are equivalent:
  - We have $a \sim_R b$.
  - We have $b \sim_S a$.

  Indeed:
  - *If $a \sim_R b$, then $b \sim_S a$:* Since $\chi_A \subset S \diamond R$, there exists $k \in B$ such that $a \sim_R k$ and $k \sim_S a$, but since $a \sim_R b$ and R is functional, we have $k = b$ and thus $b \sim_S a$.
  - *If $b \sim_S a$, then $a \sim_R b$:* First note that since R is total we have $a \sim_R b'$ for some $b' \in B$. Now, since $R \diamond S \subset \chi_B$, $b \sim_S a$, and $a \sim_R b'$, we have $b = b'$, and thus $a \sim_R b$.

  Having show this, we now have

$$\begin{aligned} f^{-1}_{R,S}(b) &\overset{\text{def}}{=} \left\{a \in A \,\middle|\, f_{R,S}(a) = b\right\} \\ &\overset{\text{def}}{=} \{a \in A \mid a \sim_R b\} \\ &= \{a \in A \mid b \sim_S a\} \\ &\overset{\text{def}}{=} S(b). \end{aligned}$$

  for each $b \in B$, showing $f^{-1}_{R,S} = S$.

This finishes the proof. □

5.3.4 Monads in Rel

Let A be a set.

Proposition 5.3.4.1.1. We have a natural identification[13]

$$\left\{\begin{matrix}\text{Monads in}\\ \mathsf{Rel}\text{ on }A\end{matrix}\right\} \cong \{\text{Preorders on }A\}.$$

Proof. A monad in Rel on A consists of a relation $R\colon A \nrightarrow A$ together with maps

$$\mu_R\colon R\diamond R \subset R,$$
$$\eta_R\colon \chi_A \subset R$$

making the diagrams

$$\begin{tikzcd}
\chi_A\diamond R \arrow[r, "\eta_R\diamond\mathrm{id}_R"] \arrow[dr, equal, "\lambda^{\mathbf{Rel}(A,B)}_R"'] & R\diamond R \arrow[d, "\mu_R"] \\
 & R
\end{tikzcd}
\qquad
\begin{tikzcd}
 & R\diamond(R\diamond R) \arrow[dl, equal, "\alpha^{\mathbf{Rel}(A,B)}_{R,R,R}"'] \arrow[dr, "\mathrm{id}_R\diamond\mu_R"] & \\
(R\diamond R)\diamond R \arrow[d, "\mu_R\diamond\mathrm{id}_R"'] & & R\diamond R \arrow[d, "\mu_R"] \\
R\diamond R \arrow[rr, "\mu_R"'] & & R
\end{tikzcd}
\qquad
\begin{tikzcd}
R\diamond\chi_A \arrow[r, "\mathrm{id}_R\diamond\eta_R"] \arrow[dr, equal, "\rho^{\mathbf{Rel}(A,B)}_R"'] & R\diamond R \arrow[d, "\mu_R"] \\
 & R
\end{tikzcd}$$

commute. However, since all morphisms involved are inclusions, the commutativity of the above diagrams is automatic, and hence all that is left is the data of the two maps μ_R and η_R, which correspond respectively to the following conditions:

1. For each $a, b, c \in A$, if $a \sim_R b$ and $b \sim_R c$, then $a \sim_R c$.
2. For each $a \in A$, we have $a \sim_R a$.

These are exactly the requirements for R to be a preorder (**??**). Conversely any preorder $\preceq$ gives rise to a pair of maps $\mu_{\preceq}$ and $\eta_{\preceq}$, forming a monad on A. □

5.3.5 Comonads in Rel

Let A be a set.

[13] See also **??** for an extension of this correspondence to "relative monads in Rel".

Proposition 5.3.5.1.1. We have a natural identification

$$\left\{\begin{matrix}\text{Comonads in}\\ \mathbf{Rel}\text{ on }A\end{matrix}\right\} \cong \{\text{Subsets of }A\}.$$

Proof. A comonad in $\mathbf{Rel}$ on A consists of a relation $R\colon A \nrightarrow A$ together with maps

$$\begin{aligned}\Delta_R\colon R &\subset R \diamond R,\\ \epsilon_R\colon R &\subset \chi_A\end{aligned}$$

making the diagrams

$$\begin{tikzcd} R \arrow[r, "\Delta_R", "\shortmid"{marking}] \arrow[dr, equal, "\lambda^{\mathbf{Rel}(A,B),-1}_R"'] & R \diamond R \arrow[d, "\epsilon_R \diamond \mathrm{id}_R", "\shortmid"{marking}] \\ & \chi_A \diamond R \end{tikzcd}
\qquad
\begin{tikzcd} & R \diamond R \arrow[dr, "\mathrm{id}_R \diamond \Delta_R", "\shortmid"{marking}] & \\ R \arrow[ur, "\Delta_R", "\shortmid"{marking}] \arrow[d, "\Delta_R"', "\shortmid"{marking}] & & R \diamond (R \diamond R) \arrow[d, equal, "\alpha^{\mathbf{Rel}(A,B),-1}_{R,R,R}"'] \\ R \diamond R \arrow[rr, "\Delta_R \diamond \mathrm{id}_R"', "\shortmid"{marking}] & & (R \diamond R) \diamond R \end{tikzcd}
\qquad
\begin{tikzcd} R \arrow[r, "\Delta_R", "\shortmid"{marking}] \arrow[dr, equal, "\rho^{\mathbf{Rel}(A,B),-1}_R"'] & R \diamond R \arrow[d, "\mathrm{id}_R \diamond \epsilon_R", "\shortmid"{marking}] \\ & R \diamond \chi_A \end{tikzcd}$$

commute. However, since all morphisms involved are inclusions, the commutativity of the above diagrams is automatic, and hence all that is left is the data of the two maps Δ_R and ϵ_R, which correspond respectively to the following conditions:

1. For each $a, b \in A$, if $a \sim_R b$, then there exists some $k \in A$ such that $a \sim_R k$ and $k \sim_R b$.

2. For each $a, b \in A$, if $a \sim_R b$, then $a = b$.

Taking $k = b$ in the first condition above shows it to be trivially satisfied, while the second condition implies $R \subset \Delta_A$, i.e. R must be a subset of A. Conversely, any subset U of A satisfies $U \subset \Delta_A$, defining a comonad as above. □

5.3.6 Co/Monoids in Rel

Remark 5.3.6.1.1. The monoids in $\mathbf{Rel}$ with respect to the Cartesian monoidal structure of Proposition 5.2.2.8.1 are called *hypermonoids*, and their theory is explored in **??**. Similarly, the comonoids in $\mathbf{Rel}$ are called *hypercomonoids*, and they are defined and studied in **??**.

5.3.7 Monomorphisms in Rel

In this section we characterise the epimorphisms in the category Rel, following ??.

Proposition 5.3.7.1.1. Let $R\colon A \nrightarrow B$ be a relation. The following conditions are equivalent:

1. The relation R is a monomorphism in Rel.
2. The direct image function
$$R_*\colon \mathcal{P}(A) \to \mathcal{P}(B)$$
associated to R is injective.
3. The direct image with compact support function
$$R_!\colon \mathcal{P}(A) \to \mathcal{P}(B)$$
associated to R is injective.

Moreover, if R is a monomorphism, then it satisfies the following condition, and the converse holds if R is total:

($\star$) For each $a, a' \in A$, if there exists some $b \in B$ such that
$$\begin{aligned} a &\sim_R b, \\ a' &\sim_R b, \end{aligned}$$
then $a = a'$.

Proof. Firstly note that Items 2 and 3 are equivalent by Item 7 of Proposition 6.4.1.1.3. We then claim that Items 1 and 2 are also equivalent:

- *Item 1* $\implies$ *Item 2:* Let $U, V \in \mathcal{P}(A)$ and consider the diagram
$$\mathrm{pt} \overset{U}{\underset{V}{\rightrightarrows}} A \xrightarrow{R} B.$$
By Remark 6.4.1.1.2, we have
$$\begin{aligned} R_*(U) &= R \diamond U, \\ R_*(V) &= R \diamond V. \end{aligned}$$
Now, if $R \diamond U = R \diamond V$, i.e. $R_*(U) = R_*(V)$, then $U = V$ since R is assumed to be a monomorphism, showing R_* to be injective.

- *Item 2* $\implies$ *Item 1*: Conversely, suppose that R_* is injective, consider the diagram

$$X \underset{T}{\overset{S}{\rightrightarrows}} A \xrightarrow{R} B,$$

and suppose that $R \diamond S = R \diamond T$. Note that, since R_* is injective, given a diagram of the form

$$\mathrm{pt} \underset{V}{\overset{U}{\rightrightarrows}} A \xrightarrow{R} B,$$

if $R_*(U) = R \diamond U = R \diamond V = R_*(V)$, then $U = V$. In particular, for each $x \in X$, we may consider the diagram

$$\mathrm{pt} \xrightarrow{[x]} X \underset{T}{\overset{S}{\rightrightarrows}} A \xrightarrow{R} B,$$

for which we have $R \diamond S \diamond [x] = R \diamond T \diamond [x]$, implying that we have

$$S(x) = S \diamond [x] = T \diamond [x] = T(x)$$

for each $x \in X$, implying $S = T$, and thus R is a monomorphism.

We can also prove this in a more abstract way, following [MSE 350788]:

- *Item 1* $\implies$ *Item 2*: Assume that R is a monomorphism.
  - We first notice that the functor $\mathrm{Rel}(\mathrm{pt}, -) \colon \mathrm{Rel} \to \mathsf{Sets}$ maps R to R_* by Remark 6.4.1.1.2.
  - Since $\mathrm{Rel}(\mathrm{pt}, -)$ preserves all limits by **??** of **??**, it follows by **??** of **??** that $\mathrm{Rel}(\mathrm{pt}, -)$ also preserves monomorphisms.
  - Since R is a monomorphism and $\mathrm{Rel}(\mathrm{pt}, -)$ maps R to R_*, it follows that R_* is also a monomorphism.
  - Since the monomorphisms in Sets are precisely the injections (**??** of **??**), it follows that R_* is injective.
- *Item 2* $\implies$ *Item 1*: Assume that R_* is injective.
  - We first notice that the functor $\mathrm{Rel}(\mathrm{pt}, -) \colon \mathrm{Rel} \to \mathsf{Sets}$ maps R to R_* by Remark 6.4.1.1.2.
  - Since the monomorphisms in Sets are precisely the injections (**??** of **??**), it follows that R_* is a monomorphism.
  - Since $\mathrm{Rel}(\mathrm{pt}, -)$ is faithful, it follows by **??** of **??** that $\mathrm{Rel}(\mathrm{pt}, -)$ reflects monomorphisms.

– Since R_* is a monomorphism and $\mathrm{Rel}(\mathrm{pt}, -)$ maps R to R_*, it follows that R is also a monomorphism.

Finally, we prove the second part of the statement. Assume that R is a monomorphism, let $a, a' \in A$ such that $a \sim_R b$ and $a' \sim_R b$ for some $b \in B$, and consider the diagram

$$\mathrm{pt} \underset{[a']}{\overset{[a]}{\rightrightarrows}} A \xrightarrow{R} B.$$

Since $\star \sim_{[a]} a$ and $a \sim_R b$, we have $\star \sim_{R \diamond [a]} b$. Similarly, $\star \sim_{R \diamond [a']} b$. Thus $R \diamond [a] = R \diamond [a']$, and since R is a monomorphism, we have $[a] = [a']$, i.e. $a = a'$.

Conversely, assume the condition

($\star$) For each $a, a' \in A$, if there exists some $b \in B$ such that

$$a \sim_R b,$$
$$a' \sim_R b,$$

then $a = a'$.

consider the diagram

$$X \underset{T}{\overset{S}{\rightrightarrows}} A \xrightarrow{R} B,$$

and let $(x, a) \in S$. Since R is total and $a \in A$, there exists some $b \in B$ such that $a \sim_R b$. In this case, we have $x \sim_{R \diamond S} b$, and since $R \diamond S = R \diamond T$, we have also $x \sim_{R \diamond T} b$. Thus there must exist some $a' \in A$ such that $x \sim_T a'$ and $a' \sim_R b$. However, since $a, a' \sim_R b$, we must have $a = a'$, and thus $(x, a) \in T$ as well.

A similar argument shows that if $(x, a) \in T$, then $(x, a) \in S$, and thus $S = T$ and it follows that R is a monomorphism. □

5.3.8 2-Categorical Monomorphisms in Rel

In this section we characterise (for now, some of) the 2-categorical monomorphisms in **Rel**, following Section 9.1.

Proposition 5.3.8.1.1. Let $R \colon A \nrightarrow B$ be a relation.

1. *Representably Faithful Morphisms in* **Rel**. Every morphism of **Rel** is a representably faithful morphism.

2. *Representably Full Morphisms in* **Rel**. The following conditions are equivalent:

    (a) The morphism $R\colon A \nrightarrow B$ is a representably full morphism.

    (b) For each pair of relations $S, T : X \rightrightarrows A$, the following condition is satisfied:

        (★) If $R \diamond S \subset R \diamond T$, then $S \subset T$.

    (c) The functor
    $$R_* : (\mathcal{P}(A), \subset) \to (\mathcal{P}(B), \subset)$$
    is full.

    (d) For each $U, V \in \mathcal{P}(A)$, if $R_*(U) \subset R_*(V)$, then $U \subset V$.

    (e) The functor
    $$R_! : (\mathcal{P}(A), \subset) \to (\mathcal{P}(B), \subset)$$
    is full.

    (f) For each $U, V \in \mathcal{P}(A)$, if $R_!(U) \subset R_!(V)$, then $U \subset V$.

3. *Representably Fully Faithful Morphisms in* **Rel**. Every representaly full morphism in **Rel** is a representably fully faithful morphism.

Proof. *Item 1, Representably Faithful Morphisms in* **Rel**: The relation R is a representably faithful morphism in **Rel** iff, for each $X \in \mathrm{Obj}(\mathbf{Rel})$, the functor
$$R_* : \mathbf{Rel}(X, A) \to \mathbf{Rel}(X, B)$$
is faithful, i.e. iff the morphism
$$R_{*|S,T} : \mathrm{Hom}_{\mathbf{Rel}(X,A)}(S, T) \to \mathrm{Hom}_{\mathbf{Rel}(X,B)}(R \diamond S, R \diamond T)$$
is injective for each $S, T \in \mathrm{Obj}(\mathbf{Rel}(X, A))$. However, $\mathrm{Hom}_{\mathbf{Rel}(X,A)}(S, T)$ is either empty or a singleton, in either case of which the map $R_{*|S,T}$ is necessarily injective.

Item 2, Representably Full Morphisms in **Rel**: We claim Items 2a to 2f are indeed equivalent:

- *Item 2a* $\iff$ *Item 2b:* This is simply a matter of unwinding definitions: The relation R is a representably full morphism in **Rel** iff, for each $X \in \mathrm{Obj}(\mathbf{Rel})$, the functor
$$R_* : \mathbf{Rel}(X, A) \to \mathbf{Rel}(X, B)$$

is full, i.e. iff the morphism

$$R_{*|S,T}\colon \mathrm{Hom}_{\mathbf{Rel}(X,A)}(S,T) \to \mathrm{Hom}_{\mathbf{Rel}(X,B)}(R \diamond S, R \diamond T)$$

is surjective for each $S, T \in \mathrm{Obj}(\mathbf{Rel}(X,A))$, i.e. iff, whenever $R \diamond S \subset R \diamond T$, we also have $S \subset T$.

- *Item 2c* $\iff$ *Item 2d*: This is also simply a matter of unwinding definitions: The functor

  $$R_*\colon (\mathcal{P}(A), \subset) \to (\mathcal{P}(B), \subset)$$

  is full iff, for each $U, V \in \mathcal{P}(A)$, the morphism

  $$R_{*|U,V}\colon \mathrm{Hom}_{\mathcal{P}(A)}(U,V) \to \mathrm{Hom}_{\mathcal{P}(B)}(R_*(U), R_*(V))$$

  is surjective, i.e. iff whenever $R_*(U) \subset R_*(V)$, we also necessarily have $U \subset V$.

- *Item 2e* $\iff$ *Item 2f*: This is once again simply a matter of unwinding definitions, and proceeds exactly in the same way as in the proof of the equivalence between Items 2c and 2d given above.

- *Item 2d* $\implies$ *Item 2f*: Suppose that the following condition is true:

  ($\star$) For each $U, V \in \mathcal{P}(A)$, if $R_*(U) \subset R_*(V)$, then $U \subset V$.

  We need to show that the condition

  ($\star$) For each $U, V \in \mathcal{P}(A)$, if $R_!(U) \subset R_!(V)$, then $U \subset V$.

  is also true. We proceed step by step:

  1. Suppose we have $U, V \in \mathcal{P}(A)$ with $R_!(U) \subset R_!(V)$.
  2. By Item 7 of Proposition 6.4.4.1.3, we have

     $$\begin{aligned} R_!(U) &= B \setminus R_*(A \setminus U), \\ R_!(V) &= B \setminus R_*(A \setminus V). \end{aligned}$$

  3. By Item 1 of Proposition 2.3.10.1.2 we have $R_*(A \setminus V) \subset R_*(A \setminus U)$.
  4. By assumption, we then have $A \setminus V \subset A \setminus U$.
  5. By Item 1 of Proposition 2.3.10.1.2 again, we have $U \subset V$.

- *Item 2f* $\implies$ *Item 2d*: Suppose that the following condition is true:

  ($\star$) For each $U, V \in \mathcal{P}(A)$, if $R_!(U) \subset R_!(V)$, then $U \subset V$.

We need to show that the condition

($\star$) For each $U, V \in \mathcal{P}(A)$, if $R_*(U) \subset R_*(V)$, then $U \subset V$.

is also true. We proceed step by step:

1. Suppose we have $U, V \in \mathcal{P}(A)$ with $R_*(U) \subset R_*(V)$.
2. By Item 7 of Proposition 6.4.1.1.3, we have

$$\begin{aligned} R_*(U) &= B \setminus R_!(A \setminus U), \\ R_*(V) &= B \setminus R_!(A \setminus V). \end{aligned}$$

3. By Item 1 of Proposition 2.3.10.1.2 we have $R_!(A \setminus V) \subset R_!(A \setminus U)$.
4. By assumption, we then have $A \setminus V \subset A \setminus U$.
5. By Item 1 of Proposition 2.3.10.1.2 again, we have $U \subset V$.

- *Item 2b* $\Longrightarrow$ *Item 2d*: Consider the diagram

$$X \overset{S}{\underset{T}{\rightrightarrows}} A \overset{R}{\longrightarrow} B,$$

and suppose that $R \diamond S \subset R \diamond T$. Note that, by assumption, given a diagram of the form

$$\mathrm{pt} \overset{U}{\underset{V}{\rightrightarrows}} A \overset{R}{\longrightarrow} B,$$

if $R_*(U) = R \diamond U \subset R \diamond V = R_*(V)$, then $U \subset V$. In particular, for each $x \in X$, we may consider the diagram

$$\mathrm{pt} \xrightarrow{[x]} X \overset{S}{\underset{T}{\rightrightarrows}} A \overset{R}{\longrightarrow} B,$$

for which we have $R \diamond S \diamond [x] \subset R \diamond T \diamond [x]$, implying that we have

$$S(x) = S \diamond [x] \subset T \diamond [x] = T(x)$$

for each $x \in X$, implying $S \subset T$.

- *Item 2d* $\Longrightarrow$ *Item 2b*: Let $U, V \in \mathcal{P}(A)$ and consider the diagram

$$\mathrm{pt} \overset{U}{\underset{V}{\rightrightarrows}} A \overset{R}{\longrightarrow} B.$$

By Remark 6.4.1.1.2, we have

$$\begin{aligned} R_*(U) &= R \diamond U, \\ R_*(V) &= R \diamond V. \end{aligned}$$

Now, if $R_*(U) \subset R_*(V)$, i.e. $R \diamond U \subset R \diamond V$, then $U \subset V$ by assumption.

??, *Fully Faithful Monomorphisms in* **Rel**: This follows from Items 1 and 2. □

Question 5.3.8.1.2. Item 2 of Proposition 5.3.8.1.1 gives a characterisation of the representably full morphisms in **Rel**.
Are there other nice characterisations of these?
This question also appears as [`MO 467527`].

5.3.9 Epimorphisms in Rel

In this section we characterise the epimorphisms in the category Rel, following **??**.

Proposition 5.3.9.1.1. Let $R\colon A \nrightarrow B$ be a relation. The following conditions are equivalent:

1. The relation R is an epimorphism in Rel.
2. The weak inverse image function

   $$R^{-1}\colon \mathcal{P}(B) \to \mathcal{P}(A)$$

   associated to R is injective.
3. The strong inverse image function

   $$R_{-1}\colon \mathcal{P}(B) \to \mathcal{P}(A)$$

   associated to R is injective.
4. The function $R\colon A \to \mathcal{P}(B)$ is "surjective on singletons":

   ($\star$) For each $b \in B$, there exists some $a \in A$ such that $R(a) = \{b\}$.

Moreover, if R is total and an epimorphism, then it satisfies the following equivalent conditions:

1. For each $b \in B$, there exists some $a \in A$ such that $a \sim_R b$.
2. We have $\mathrm{Im}(R) = B$.

Proof. Firstly note that Items 2 and 3 are equivalent by Item 7 of Proposition 6.4.2.1.3. We then claim that Items 1 and 2 are also equivalent:

- *Item 1* $\Longrightarrow$ *Item 2:* Let $U, V \in \mathcal{P}(A)$ and consider the diagram

  $$A \overset{R}{\longrightarrow\!\!\!\!\!\!\!\!\!\!+\;\;\;} B \underset{V}{\overset{U}{\rightrightarrows}} \mathrm{pt}.$$

  By Remark 6.4.1.1.2, we have

  $$\begin{aligned} R^{-1}(U) &= U \diamond R, \\ R^{-1}(V) &= V \diamond R. \end{aligned}$$

  Now, if $U \diamond R = V \diamond R$, i.e. $R^{-1}(U) = R^{-1}(V)$, then $U = V$ since R is assumed to be an epimorphism, showing R^{-1} to be injective.

- *Item 2* $\Longrightarrow$ *Item 1:* Conversely, suppose that R^{-1} is injective, consider the diagram

  $$A \overset{R}{\longrightarrow\!\!\!\!\!\!\!\!\!\!+\;\;\;} B \underset{T}{\overset{S}{\rightrightarrows}} X,$$

  and suppose that $S \diamond R = T \diamond R$. Note that, since R^{-1} is injective, given a diagram of the form

  $$A \overset{R}{\longrightarrow\!\!\!\!\!\!\!\!\!\!+\;\;\;} B \underset{V}{\overset{U}{\rightrightarrows}} \mathrm{pt},$$

  if $R^{-1}(U) = U \diamond R = V \diamond R = R^{-1}(V)$, then $U = V$. In particular, for each $x \in X$, we may consider the diagram

  $$A \overset{R}{\longrightarrow\!\!\!\!\!\!\!\!\!\!+\;\;\;} B \underset{T}{\overset{S}{\rightrightarrows}} X \overset{[x]}{\longrightarrow\!\!\!\!\!\!\!\!\!\!+\;\;\;} \mathrm{pt},$$

  for which we have $[x] \diamond S \diamond R = [x] \diamond T \diamond R$, implying that we have

  $$S^{-1}(x) = [x] \diamond S = [x] \diamond T = T^{-1}(x)$$

  for each $x \in X$, implying $S = T$, and thus R is an epimorphism.

We can also prove this in a more abstract way, following [`MSE 350788`]:

- *Item 1* $\Longrightarrow$ *Item 2:* Assume that R is an epimorphism.
  - We first notice that the functor $\mathrm{Rel}(-, \mathrm{pt}) \colon \mathrm{Rel}^{\mathsf{op}} \to \mathsf{Sets}$ maps R to R^{-1} by Remark 6.4.3.1.2.

- Since $\mathrm{Rel}(-, \mathrm{pt})$ preserves limits by **??** of **??**, it follows by **??** of **??** that $\mathrm{Rel}(-, \mathrm{pt})$ also preserves monomorphisms.
- That is: $\mathrm{Rel}(-, \mathrm{pt})$ sends monomorphisms in $\mathrm{Rel}^{\mathrm{op}}$ to monomorphisms in Sets.
- The monomorphisms $\mathrm{Rel}^{\mathrm{op}}$ are precisely the epimorphisms in Rel by **??** of **??**.
- Since R is an epimorphism and $\mathrm{Rel}(-, \mathrm{pt})$ maps R to R^{-1}, it follows that R^{-1} is a monomorphism.
- Since the monomorphisms in Sets are precisely the injections (**??** of **??**), it follows that R^{-1} is injective.

- *Item 2* $\Longrightarrow$ *Item 1:* Assume that R^{-1} is injective.
  - We first notice that the functor $\mathrm{Rel}(-, \mathrm{pt}) \colon \mathrm{Rel}^{\mathrm{op}} \to \mathsf{Sets}$ maps R to R^{-1} by Remark 6.4.3.1.2.
  - Since the monomorphisms in Sets are precisely the injections (**??** of **??**), it follows that R^{-1} is a monomorphism.
  - Since $\mathrm{Rel}(-, \mathrm{pt})$ is faithful, it follows by **??** of **??** that $\mathrm{Rel}(, \mathrm{pt})$ reflects monomorphisms.
  - That is: $\mathrm{Rel}(-, \mathrm{pt})$ reflects monomorphisms in Sets to monomorphisms in $\mathrm{Rel}^{\mathrm{op}}$.
  - The monomorphisms $\mathrm{Rel}^{\mathrm{op}}$ are precisely the epimorphisms in Rel by **??** of **??**.
  - Since R^{-1} is a monomorphism and $\mathrm{Rel}(-, \mathrm{pt})$ maps R to R^{-1}, it follows that R is an epimorphism.

We also claim that Items 2 and 4 are equivalent, following [`MO 350788`]:

- *Item 2* $\Longrightarrow$ *Item 4:* Since $B \setminus \{b\} \subset B$ and R^{-1} is injective, we have $R^{-1}(B \setminus \{b\}) \subsetneq R^{-1}(B)$. So taking some $a \in R^{-1}(B) \setminus R^{-1}(B \setminus \{b\})$ we get an element of A such that $R(a) = \{b\}$.

- *Item 4* $\Longrightarrow$ *Item 2:* Let $U, V \subset B$ with $U \neq V$. Without loss of generality, we can assume $U \setminus V \neq \emptyset$; otherwise just swap U and V. Let then $b \in U \setminus V$. By assumption, there exists an $a \in A$ with $R(a) = \{b\}$. Then $a \in R^{-1}(U)$ but $a \notin R^{-1}(V)$, and thus $R^{-1}(U) \neq R^{-1}(V)$, showing R^{-1} to be injective.

Finally, we prove the second part of the statement. So assume R is a total epimorphism in Rel and consider the diagram

$$A \overset{R}{\longrightarrow\!\!\!\!\!\!\!+\;\;} B \underset{T}{\overset{S}{\rightrightarrows}} \{0, 1\},$$

where $b \sim_S 0$ for each $b \in B$ and where we have

$$b \sim_T \begin{cases} 0 & \text{if } b \in \mathrm{Im}(R), \\ 1 & \text{otherwise} \end{cases}$$

for each $b \in B$. Since R is total, we have $a \sim_{S \diamond R} 0$ and $a \sim_{T \diamond R} 0$ for all $a \in A$, and no element of A is related to 1 by $S \diamond R$ or $T \diamond R$. Thus $S \diamond R = T \diamond R$, and since R is an epimorphism, we have $S = T$. But by the definition of T, this implies $\mathrm{Im}(R) = B$. □

5.3.10 2-Categorical Epimorphisms in **Rel**

In this section we characterise (for now, some of) the 2-categorical epimorphisms in **Rel**, following Section 9.2.

Proposition 5.3.10.1.1. Let $R \colon A \nrightarrow B$ be a relation.

1. *Corepresentably Faithful Morphisms in* **Rel**. Every morphism of **Rel** is a corepresentably faithful morphism.
2. *Corepresentably Full Morphisms in* **Rel**. The following conditions are equivalent:
   (a) The morphism $R \colon A \nrightarrow B$ is a corepresentably full morphism.
   (b) For each pair of relations $S, T \colon X \rightrightarrows A$, the following condition is satisfied:
      (★) If $S \diamond R \subset T \diamond R$, then $S \subset T$.
   (c) The functor
   $$R^{-1} \colon (\mathcal{P}(B), \subset) \to (\mathcal{P}(A), \subset)$$
   is full.
   (d) For each $U, V \in \mathcal{P}(B)$, if $R^{-1}(U) \subset R^{-1}(V)$, then $U \subset V$.
   (e) The functor
   $$R_{-1} \colon (\mathcal{P}(B), \subset) \to (\mathcal{P}(A), \subset)$$
   is full.
   (f) For each $U, V \in \mathcal{P}(B)$, if $R_{-1}(U) \subset R_{-1}(V)$, then $U \subset V$.
3. *Corepresentably Fully Faithful Morphisms in* **Rel**. Every corepresentably full morphism of **Rel** is a corepresentably fully faithful morphism.

Proof. *Item 1, Corepresentably Faithful Morphisms in* **Rel**: The relation R is a corepresentably faithful morphism in **Rel** iff, for each $X \in \mathrm{Obj}(\mathbf{Rel})$, the functor

$$R^*\colon \mathbf{Rel}(B, X) \to \mathbf{Rel}(A, X)$$

is faithful, i.e. iff the morphism

$$R^*_{S,T}\colon \mathrm{Hom}_{\mathbf{Rel}(B,X)}(S, T) \to \mathrm{Hom}_{\mathbf{Rel}(A,X)}(S \diamond R, T \diamond R)$$

is injective for each $S, T \in \mathrm{Obj}(\mathbf{Rel}(B, X))$. However, $\mathrm{Hom}_{\mathbf{Rel}(B,X)}(S, T)$ is either empty or a singleton, in either case of which the map $R^*_{S,T}$ is necessarily injective.

Item 2, Corepresentably Full Morphisms in **Rel**: We claim Items 2a to 2f are indeed equivalent:

- *Item 2a* $\iff$ *Item 2b*: This is simply a matter of unwinding definitions: The relation R is a corepresentably full morphism in **Rel** iff, for each $X \in \mathrm{Obj}(\mathbf{Rel})$, the functor

  $$R^*\colon \mathbf{Rel}(B, X) \to \mathbf{Rel}(A, X)$$

  is full, i.e. iff the morphism

  $$R^*_{S,T}\colon \mathrm{Hom}_{\mathbf{Rel}(B,X)}(S, T) \to \mathrm{Hom}_{\mathbf{Rel}(A,X)}(S \diamond R, T \diamond R)$$

  is surjective for each $S, T \in \mathrm{Obj}(\mathbf{Rel}(B, X))$, i.e. iff, whenever $S \diamond R \subset T \diamond R$, we also have $S \subset T$.

- *Item 2c* $\iff$ *Item 2d*: This is also simply a matter of unwinding definitions: The functor

  $$R^{-1}\colon (\mathcal{P}(B), \subset) \to (\mathcal{P}(A), \subset)$$

  is full iff, for each $U, V \in \mathcal{P}(A)$, the morphism

  $$R^{-1}_{U,V}\colon \mathrm{Hom}_{\mathcal{P}(B)}(U, V) \to \mathrm{Hom}_{\mathcal{P}(A)}\big(R^{-1}(U), R^{-1}(V)\big)$$

  is surjective, i.e. iff whenever $R^{-1}(U) \subset R^{-1}(V)$, we also necessarily have $U \subset V$.

- *Item 2e* $\iff$ *Item 2f*: This is once again simply a matter of unwinding definitions, and proceeds exactly in the same way as in the proof of the equivalence between Items 2c and 2d given above.

- *Item 2d* $\implies$ *Item 2f*: Suppose that the following condition is true:

($\star$) For each $U, V \in \mathcal{P}(B)$, if $R^{-1}(U) \subset R^{-1}(V)$, then $U \subset V$.

We need to show that the condition

($\star$) For each $U, V \in \mathcal{P}(B)$, if $R_{-1}(U) \subset R_{-1}(V)$, then $U \subset V$.

is also true. We proceed step by step:

1. Suppose we have $U, V \in \mathcal{P}(B)$ with $R_{-1}(U) \subset R_{-1}(V)$.
2. By Item 7 of Proposition 6.4.2.1.3, we have
$$R_{-1}(U) = B \setminus R^{-1}(A \setminus U),$$
$$R_{-1}(V) = B \setminus R^{-1}(A \setminus V).$$
3. By Item 1 of Proposition 2.3.10.1.2 we have $R^{-1}(A \setminus V) \subset R^{-1}(A \setminus U)$.
4. By assumption, we then have $A \setminus V \subset A \setminus U$.
5. By Item 1 of Proposition 2.3.10.1.2 again, we have $U \subset V$.

- *Item 2f* $\Longrightarrow$ *Item 2d:* Suppose that the following condition is true:

($\star$) For each $U, V \in \mathcal{P}(B)$, if $R_{-1}(U) \subset R_{-1}(V)$, then $U \subset V$.

We need to show that the condition

($\star$) For each $U, V \in \mathcal{P}(B)$, if $R^{-1}(U) \subset R^{-1}(V)$, then $U \subset V$.

is also true. We proceed step by step:

1. Suppose we have $U, V \in \mathcal{P}(B)$ with $R^{-1}(U) \subset R^{-1}(V)$.
2. By Item 7 of Proposition 6.4.3.1.3, we have
$$R^{-1}(U) = B \setminus R_{-1}(A \setminus U),$$
$$R^{-1}(V) = B \setminus R_{-1}(A \setminus V).$$
3. By Item 1 of Proposition 2.3.10.1.2 we have $R_{-1}(A \setminus V) \subset R_{-1}(A \setminus U)$.
4. By assumption, we then have $A \setminus V \subset A \setminus U$.
5. By Item 1 of Proposition 2.3.10.1.2 again, we have $U \subset V$.

- *Item 2b* $\Longrightarrow$ *Item 2d:* Consider the diagram

$$A \overset{R}{\relbar\!\!\!\mapsto} B \underset{T}{\overset{S}{\rightrightarrows}} X,$$

and suppose that $S \diamond R \subset T \diamond R$. Note that, by assumption, given a diagram of the form

$$A \overset{R}{\longrightarrow\!\!\!\!\!\!\!+\;\;} B \overset{U}{\underset{V}{\rightrightarrows}} \mathrm{pt},$$

if $R^{-1}(U) = R \diamond U \subset R \diamond V = R^{-1}(V)$, then $U \subset V$. In particular, for each $x \in X$, we may consider the diagram

$$A \overset{R}{\longrightarrow\!\!\!\!\!\!\!+\;\;} B \overset{S}{\underset{T}{\rightrightarrows}} X \overset{[x]}{\longrightarrow\!\!\!\!\!\!\!+\;\;} \mathrm{pt},$$

for which we have $[x] \diamond S \diamond R \subset [x] \diamond T \diamond R$, implying that we have

$$S^{-1}(x) = [x] \diamond S \subset [x] \diamond T = T^{-1}(x)$$

for each $x \in X$, implying $S \subset T$.

- *Item 2d* $\Longrightarrow$ *Item 2b*: Let $U, V \in \mathcal{P}(B)$ and consider the diagram

  $$A \overset{R}{\longrightarrow\!\!\!\!\!\!\!+\;\;} B \overset{U}{\underset{V}{\rightrightarrows}} \mathrm{pt}.$$

  By Remark 6.4.1.1.2, we have

  $$R^{-1}(U) = U \diamond R,$$
  $$R^{-1}(V) = V \diamond R.$$

  Now, if $R^{-1}(U) \subset R^{-1}(V)$, i.e. $U \diamond R \subset V \diamond R$, then $U \subset V$ by assumption.

Item 3, Corepresentably Fully Faithful Morphisms in **Rel**: This follows from Items 1 and 2. □

Question 5.3.10.1.2. Item 2 of Proposition 5.3.10.1.1 gives a characterisation of the corepresentably full morphisms in **Rel**.
Are there other nice characterisations of these?
This question also appears as [`MO 467527`].

5.3.11 Co/Limits in Rel

Proposition 5.3.11.1.1. This will be properly written later on.

Proof. Omitted. □

5.3.12 Kan Extensions and Kan Lifts in Rel

Remark 5.3.12.1.1. The 2-category Rel admits all right Kan extensions and right Kan lifts, though not all left Kan extensions and neither does it admit all left Kan lifts. See Section 6.2 for a detailed discussion of this.

5.3.13 Closedness of Rel

Proposition 5.3.13.1.1. The 2-category Rel is a closed bicategory, there being, for each $R\colon A \nrightarrow B$ and set X, a pair of adjunctions

$$
\begin{tikzcd}
(R^* \dashv \mathrm{Ran}_R)\colon & \mathrm{Rel}(B, X) \arrow[r, bend left, "R^*"] \arrow[r, phantom, "\perp"] & \mathrm{Rel}(A, X), \arrow[l, bend left, "\mathrm{Ran}_R"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
(R_* \dashv \mathrm{Rift}_R)\colon & \mathrm{Rel}(X, A) \arrow[r, bend left, "R_*"] \arrow[r, phantom, "\perp"] & \mathrm{Rel}(X, B), \arrow[l, bend left, "\mathrm{Rift}_R"]
\end{tikzcd}
$$

witnessed by bijections

$$
\begin{aligned}
\mathbf{Rel}(S \diamond R, T) &\cong \mathbf{Rel}(S, \mathrm{Ran}_R(T)),\\
\mathbf{Rel}(R \diamond U, V) &\cong \mathbf{Rel}(U, \mathrm{Rift}_R(V)),
\end{aligned}
$$

natural in $S \in \mathrm{Rel}(B, X)$, $T \in \mathrm{Rel}(A, X)$, $U \in \mathrm{Rel}(X, A)$, and $V \in \mathrm{Rel}(X, B)$.

Proof. This follows from Propositions 6.2.3.1.1 and 6.2.4.1.1. □

5.3.14 Rel as a Category of Free Algebras

Proposition 5.3.14.1.1. We have an isomorphism of categories

$$\mathrm{Rel} \cong \mathsf{FreeAlg}_{\mathcal{P}_*}(\mathsf{Sets}),$$

where $\mathcal{P}_*$ is the powerset monad of **??**.

Proof. Omitted. □

5.4 The Left Skew Monoidal Structure on $\mathbf{Rel}(A, B)$

5.4.1 The Left Skew Monoidal Product

Let A and B be sets and let $J\colon A \nrightarrow B$ be a relation.

Definition 5.4.1.1.1. The **left J-skew monoidal product of $\mathbf{Rel}(A, B)$** is the functor

$$\lhd_J\colon \mathbf{Rel}(A, B) \times \mathbf{Rel}(A, B) \to \mathbf{Rel}(A, B)$$

where

- *Action on Objects.* For each $R, S \in \mathrm{Obj}(\mathbf{Rel}(A, B))$, we have

$$S \lhd_J R \overset{\mathrm{def}}{=} S \diamond \mathrm{Rift}_J(R), \qquad \begin{tikzcd} & A \arrow[r, "S", "\shortmid"{marking}] \arrow[d, "J", "\shortmid"{marking}] \arrow[dl, phantom, "\Swarrow"] & B. \\ A \arrow[r, "R"', "\shortmid"{marking}] \arrow[ur, dashed, "\mathrm{Rift}_J(R)", "\shortmid"{marking}] & B \end{tikzcd}$$

- *Action on Morphisms.* For each $R, S, R', S' \in \mathrm{Obj}(\mathbf{Rel}(A, B))$, the action on Hom-sets

$$(\lhd_J)_{(G,F),(G',F')} \colon \mathrm{Hom}_{\mathbf{Rel}(A,B)}(S, S') \times \mathrm{Hom}_{\mathbf{Rel}(A,B)}(R, R') \to \mathrm{Hom}_{\mathbf{Rel}(A,B)}(S \lhd_J R, S' \lhd_J R')$$

of $\lhd_J$ at $((R, S), (R', S'))$ is defined by[14]

$$\beta \lhd_J \alpha \overset{\mathrm{def}}{=} \beta \diamond \mathrm{Rift}_J(\alpha), \qquad \begin{tikzcd} & A \arrow[r, bend left, "S", ""{name=S}] \arrow[r, bend right, "S'"', ""'{name=Sp}] \arrow[d, "J", "\shortmid"{marking}] & B \\ A \arrow[ur, dashed, bend left, "\mathrm{Rift}_J(R)"] \arrow[ur, dashed, "\mathrm{Rift}_J(R')"'] \arrow[r, "R", ""{name=R}] \arrow[r, bend right, "R'"', ""'{name=Rp}] & B \arrow[from=S, to=Sp, Rightarrow, "\beta"'] \arrow[from=R, to=Rp, Rightarrow, "\alpha"'] \end{tikzcd}$$

for each $\beta \in \mathrm{Hom}_{\mathbf{Rel}(A,B)}(S, S')$ and each $\alpha \in \mathrm{Hom}_{\mathbf{Rel}(A,B)}(R, R')$.

5.4.2 The Left Skew Monoidal Unit

Let A and B be sets and let $J \colon A \nrightarrow B$ be a relation.

Definition 5.4.2.1.1. The **left J-skew monoidal unit of $\mathbf{Rel}(A, B)$** is the functor

$$\mathbb{1}^{\mathbf{Rel}(A,B)}_{\lhd_J} \colon \mathrm{pt} \to \mathbf{Rel}(A, B)$$

picking the object

$$\mathbb{1}^{\lhd_J}_{\mathbf{Rel}(A,B)} \overset{\mathrm{def}}{=} J$$

of $\mathbf{Rel}(A, B)$.

[14] Since $\mathbf{Rel}(A, B)$ is posetal, this is to say that if $S \subset S'$ and $R \subset R'$, then $S \lhd_J R \subset S' \lhd_J R'$.

5.4.3 The Left Skew Associators

Let A and B be sets and let $J\colon A \nrightarrow B$ be a relation.

Definition 5.4.3.1.1. The **left J-skew associator of $\mathbf{Rel}(A, B)$** is the natural transformation

$$\alpha^{\mathbf{Rel}(A,B),\triangleleft_J}\colon \triangleleft_J \circ (\triangleleft_J \times \mathrm{id}) \Longrightarrow \triangleleft_J \circ (\mathrm{id} \times \triangleleft_J) \circ \boldsymbol{\alpha}^{\mathsf{Cats}}_{\mathbf{Rel}(A,B),\mathbf{Rel}(A,B),\mathbf{Rel}(A,B)},$$

as in the diagram

$$\begin{tikzcd}
 & \mathbf{Rel}(A,B)\times(\mathbf{Rel}(A,B)\times\mathbf{Rel}(A,B)) \arrow[dr, "\mathrm{id}\times\triangleleft_J"] & \\
(\mathbf{Rel}(A,B)\times\mathbf{Rel}(A,B))\times\mathbf{Rel}(A,B) \arrow[ur, dashed, "\boldsymbol{\alpha}^{\mathsf{Cats}}_{\mathbf{Rel}(A,B),\mathbf{Rel}(A,B),\mathbf{Rel}(A,B)}", "\sim"'] \arrow[d, "\triangleleft_J\times\mathrm{id}"'] & & \mathbf{Rel}(A,B)\times\mathbf{Rel}(A,B) \arrow[d, "\triangleleft_J"] \\
\mathbf{Rel}(A,B)\times\mathbf{Rel}(A,B) \arrow[rr, "\triangleleft_J"'] \arrow[urr, Rightarrow, "\alpha^{\mathbf{Rel}(A,B),\triangleleft_J}"] & & \mathbf{Rel}(A,B),
\end{tikzcd}$$

whose component

$$\alpha^{\mathbf{Rel}(A,B),\triangleleft_J}_{T,S,R}\colon \underbrace{(T \triangleleft_J S) \triangleleft_J R}_{\overset{\mathrm{def}}{=} T\diamond\mathrm{Rift}_J(S)\diamond\mathrm{Rift}_J(R)} \hookrightarrow \underbrace{T \triangleleft_J (S \triangleleft_J R)}_{\overset{\mathrm{def}}{=} T\diamond\mathrm{Rift}_J(S\diamond\mathrm{Rift}_J(R))}$$

at (T, S, R) is given by

$$\alpha^{\mathbf{Rel}(A,B),\triangleleft_J}_{T,S,R} \overset{\mathrm{def}}{=} \mathrm{id}_T \diamond \gamma,$$

where

$$\gamma\colon \mathrm{Rift}_J(S) \diamond \mathrm{Rift}_J(R) \hookrightarrow \mathrm{Rift}_J(S \diamond \mathrm{Rift}_J(R))$$

is the inclusion adjunct to the inclusion

$$\epsilon_S \star \mathrm{id}_{\mathrm{Rift}_J(R)}\colon \underbrace{J \diamond \mathrm{Rift}_J(S) \diamond \mathrm{Rift}_J(R)}_{\overset{\mathrm{def}}{=} J_*(\mathrm{Rift}_J(S)\diamond\mathrm{Rift}_J(R))} \hookrightarrow S \diamond \mathrm{Rift}_J(R)$$

under the adjunction $J_* \dashv \mathrm{Rift}_J$, where $\epsilon\colon J \diamond \mathrm{Rift}_J \Longrightarrow \mathrm{id}_{\mathbf{Rel}(A,B)}$ is the counit of the adjunction $J_* \dashv \mathrm{Rift}_J$.

5.4.4 The Left Skew Left Unitors

Let A and B be sets and let $J\colon A \nrightarrow B$ be a relation.

Definition 5.4.4.1.1. The **left J-skew left unitor of Rel(A, B)** is the natural transformation

$$\lambda^{\mathbf{Rel}(A,B),\lhd_J}\colon \lhd_J \circ \left(\mathbb{1}^{\mathbf{Rel}(A,B)}_{\lhd_J} \times \mathsf{id}\right) \Longrightarrow \boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathbf{Rel}(A,B)}$$

as in the diagram

$$\begin{tikzcd}
\mathsf{pt} \times \mathbf{Rel}(A, B) \arrow[r, "\mathbb{1}^{\mathbf{Rel}(A,B)}_{\lhd_J} \times \mathsf{id}"] \arrow[dr, dashed, bend right, "\boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathbf{Rel}(A,B)}"'] & \mathbf{Rel}(A, B) \times \mathbf{Rel}(A, B) \arrow[d, "\lhd_J"] \arrow[dl, phantom, "\Leftarrow\lambda^{\mathbf{Rel}(A,B),\lhd_J}"] \\
 & \mathbf{Rel}(A, B),
\end{tikzcd}$$

whose component

$$\lambda^{\mathbf{Rel}(A,B),\lhd_J}_R\colon \underbrace{J \lhd_J R}_{\overset{\text{def}}{=} J \diamond \mathrm{Rift}_J(R)} \hookrightarrow R$$

at R is given by

$$\lambda^{\mathbf{Rel}(A,B),\lhd_J}_R \overset{\text{def}}{=} \epsilon_R,$$

where $\epsilon\colon J_* \diamond \mathrm{Rift}_J \Longrightarrow \mathrm{id}_{\mathbf{Rel}(A,B)}$ is the counit of the adjunction $J_* \dashv \mathrm{Rift}_J$.

5.4.5 The Left Skew Right Unitors

Let A and B be sets and let $J\colon A \nrightarrow B$ be a relation.

Definition 5.4.5.1.1. The **left J-skew right unitor of Rel(A, B)** is the natural transformation

$$\rho^{\mathbf{Rel}(A,B),\lhd_J}\colon \boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathbf{Rel}(A,B)} \Longrightarrow \lhd_J \circ \left(\mathsf{id} \times \mathbb{1}^{\mathbf{Rel}(A,B)}_{\lhd_J}\right)$$

as in the diagram

$$\begin{tikzcd}
\mathbf{Rel}(A,B)\times\mathrm{pt} \arrow[r, "\mathrm{id}\times\mathbb{1}^{\mathbf{Rel}(A,B)}_{\lhd_J}"] \arrow[dr, dashed, bend right, "\rho^{\mathsf{Cats}_2}_{\mathbf{Rel}(A,B)}"'] \arrow[dr, phantom, "\Nearrow\rho^{\mathbf{Rel}(A,B),\lhd_J}"] & \mathbf{Rel}(A,B)\times\mathbf{Rel}(A,B), \arrow[d, "\lhd_J"] \\
& \mathbf{Rel}(A,B)
\end{tikzcd}$$

whose component

$$\rho^{\mathbf{Rel}(A,B),\lhd_J}_R\colon R\hookrightarrow \underbrace{R\lhd_J J}_{\overset{\text{def}}{=}R\diamond\mathrm{Rift}_J(J)}$$

at R is given by the composition

$$\begin{aligned}
R &\overset{\sim}{\Longrightarrow} R\diamond\chi_A\\
&\overset{\mathrm{id}_R\diamond\eta_{\chi_A}}{\Longrightarrow} R\diamond\mathrm{Rift}_J(J_*(\chi_A))\\
&\overset{\text{def}}{=} R\diamond\mathrm{Rift}_J(J\diamond\chi_A)\\
&\overset{\sim}{\Longrightarrow} R\diamond\mathrm{Rift}_J(J)\\
&\overset{\text{def}}{=} R\lhd_J J,
\end{aligned}$$

where $\eta\colon\mathrm{id}_{\mathbf{Rel}(A,A)}\Longrightarrow\mathrm{Rift}_J\circ J_*$ is the unit of the adjunction $J_*\dashv\mathrm{Rift}_J$.

5.4.6 The Left Skew Monoidal Structure on Rel(A, B)

Proposition 5.4.6.1.1. The category **Rel**(A, B) admits a left skew monoidal category structure consisting of

- *The Underlying Category.* The posetal category associated to the poset **Rel**(A, B) of relations from A to B of Item 2 of Definition 5.1.1.1.3.
- *The Left Skew Monoidal Product.* The left J-skew monoidal product

  $$\lhd_J\colon\mathbf{Rel}(A,B)\times\mathbf{Rel}(A,B)\to\mathbf{Rel}(A,B)$$

  of Definition 5.4.1.1.1.
- *The Left Skew Monoidal Unit.* The functor

  $$\mathbb{1}^{\mathbf{Rel}(A,B),\lhd_J}\colon\mathrm{pt}\to\mathbf{Rel}(A,B)$$

  of Definition 5.4.2.1.1.

- *The Left Skew Associators.* The natural transformation

$$\alpha^{\mathbf{Rel}(A,B),\triangleleft_J}\colon \triangleleft_J \circ (\triangleleft_J \times \mathrm{id}) \Longrightarrow \triangleleft_J \circ (\mathrm{id} \times \triangleleft_J) \circ \boldsymbol{\alpha}^{\mathsf{Cats}}_{\mathbf{Rel}(A,B),\mathbf{Rel}(A,B),\mathbf{Rel}(A,B)}$$

  of Definition 5.4.3.1.1.

- *The Left Skew Left Unitors.* The natural transformation

$$\lambda^{\mathbf{Rel}(A,B),\triangleleft_J}\colon \triangleleft_J \circ \left(\mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleleft_J} \times \mathrm{id}\right) \Longrightarrow \boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathbf{Rel}(A,B)}$$

  of Definition 5.4.4.1.1.

- *The Left Skew Right Unitors.* The natural transformation

$$\rho^{\mathbf{Rel}(A,B),\triangleleft_J}\colon \boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathbf{Rel}(A,B)} \Longrightarrow \triangleleft_J \circ \left(\mathrm{id} \times \mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleleft_J}\right)$$

  of Definition 5.4.5.1.1.

Proof. Since **Rel**(A, B) is posetal, the commutativity of the pentagon identity, the left skew left triangle identity, the left skew right triangle identity, the left skew middle triangle identity, and the zigzag identity is automatic, and thus **Rel**(A, B) together with the data in the statement forms a left skew monoidal category. □

5.5 The Right Skew Monoidal Structure on Rel(A, B)

Let A and B be sets and let $J\colon A \nrightarrow B$ be a relation.

5.5.1 The Right Skew Monoidal Product

Definition 5.5.1.1.1. The **right J-skew monoidal product of Rel(A, B)** is the functor

$$\triangleright_J\colon \mathbf{Rel}(A, B) \times \mathbf{Rel}(A, B) \to \mathbf{Rel}(A, B)$$

where

- *Action on Objects.* For each $R, S \in \mathrm{Obj}(\mathbf{Rel}(A, B))$, we have

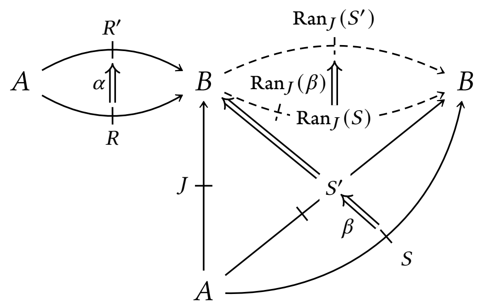
$$S \triangleright_J R \overset{\mathrm{def}}{=} \mathrm{Ran}_J(S) \diamond R, \qquad \begin{tikzcd} A \arrow[r, "R", "\mid" marking] & B \arrow[r, dashed, "\mathrm{Ran}_J(S)", "\mid" marking] & B. \\ & A \arrow[u, "J", "\mid" marking] \arrow[ur, "S"', "\mid" marking] \arrow[u, phantom, shift left=3, "\Nwarrow"] & \end{tikzcd}$$

- *Action on Morphisms.* For each $R, S, R', S' \in \mathrm{Obj}(\mathbf{Rel}(A, B))$, the action on Hom-sets

$$(\triangleright_J)_{(S,R),(S',R')} \colon \mathrm{Hom}_{\mathbf{Rel}(A,B)}(S, S') \times \mathrm{Hom}_{\mathbf{Rel}(A,B)}(R, R') \to \mathrm{Hom}_{\mathbf{Rel}(A,B)}(S \triangleright_J R, S' \triangleright_J R')$$

  of $\triangleright_J$ at $((S, R), (S', R'))$ is defined by[15]

$$\beta \triangleright_J \alpha \overset{\text{def}}{=} \mathrm{Ran}_J(\beta) \diamond \alpha,$$



  for each $\beta \in \mathrm{Hom}_{\mathbf{Rel}(A,B)}(S, S')$ and each $\alpha \in \mathrm{Hom}_{\mathbf{Rel}(A,B)}(R, R')$.

5.5.2 The Right Skew Monoidal Unit

Definition 5.5.2.1.1. The **right J-skew monoidal unit of $\mathbf{Rel}(A, B)$** is the functor

$$\mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleright_J} \colon \mathsf{pt} \to \mathbf{Rel}(A, B)$$

picking the object

$$\mathbb{1}^{\triangleright_J}_{\mathbf{Rel}(A,B)} \overset{\text{def}}{=} J$$

of $\mathbf{Rel}(A, B)$.

5.5.3 The Right Skew Associators

Definition 5.5.3.1.1. The **right J-skew associator of $\mathbf{Rel}(A, B)$** is the natural transformation

$$\alpha^{\mathbf{Rel}(A,B),\triangleright_J} \colon \triangleright_J \circ (\mathsf{id} \times \triangleright_J) \Longrightarrow \triangleright_J \circ (\triangleright_J \times \mathsf{id}) \circ \boldsymbol{\alpha}^{\mathsf{Cats},-1}_{\mathbf{Rel}(A,B),\mathbf{Rel}(A,B),\mathbf{Rel}(A,B)},$$

[15] Since $\mathbf{Rel}(A, B)$ is posetal, this is to say that if $S \subset S'$ and $R \subset R'$, then $S \triangleright_J R \subset S' \triangleright_J R'$.

as in the diagram

$$\begin{tikzcd}
 & (\mathbf{Rel}(A,B)\times\mathbf{Rel}(A,B))\times\mathbf{Rel}(A,B) \arrow[dr, "\triangleright_J\times\mathsf{id}"] & \\
\mathbf{Rel}(A,B)\times(\mathbf{Rel}(A,B)\times\mathbf{Rel}(A,B)) \arrow[ur, dashed, "\alpha^{\mathsf{Cats},-1}_{\mathbf{Rel}(A,B),\mathbf{Rel}(A,B),\mathbf{Rel}(A,B)}", "\sim"'] \arrow[d, "\mathsf{id}\times\triangleright_J"'] & \arrow[d, phantom, "\Uparrow\alpha^{\mathbf{Rel}(A,B),\triangleright_J}"] & \mathbf{Rel}(A,B)\times\mathbf{Rel}(A,B) \arrow[d, "\triangleright_J"] \\
\mathbf{Rel}(A,B)\times\mathbf{Rel}(A,B) \arrow[rr, "\triangleright_J"'] & & \mathbf{Rel}(A,B),
\end{tikzcd}$$

whose component

$$\alpha^{\mathbf{Rel}(A,B),\triangleright_J}_{T,S,R}\colon \underbrace{T\triangleright_J(S\triangleright_J R)}_{\overset{\text{def}}{=}\mathrm{Ran}_J(T)\diamond\mathrm{Ran}_J(S)\diamond R} \hookrightarrow \underbrace{(T\triangleright_J S)\triangleright_J R}_{\overset{\text{def}}{=}\mathrm{Ran}_J(\mathrm{Ran}_J(T)\diamond S)\diamond R}$$

at (T,S,R) is given by

$$\alpha^{\mathbf{Rel}(A,B),\triangleright}_{T,S,R}\overset{\text{def}}{=}\gamma\diamond\mathrm{id}_R,$$

where

$$\gamma\colon \mathrm{Ran}_J(T)\diamond\mathrm{Ran}_J(S)\hookrightarrow\mathrm{Ran}_J\big(\mathrm{Ran}_J(T)\diamond S\big)$$

is the inclusion adjunct to the inclusion

$$\mathrm{id}_{\mathrm{Ran}_J(T)}\diamond\epsilon_S\colon \underbrace{\mathrm{Ran}_J(T)\diamond\mathrm{Ran}_J(S)\diamond J}_{\overset{\text{def}}{=}J^*(\mathrm{Ran}_J(T)\diamond\mathrm{Ran}_J(S))}\hookrightarrow\mathrm{Ran}_J(T)\diamond S$$

under the adjunction $J^*\dashv\mathrm{Ran}_J$, where $\epsilon\colon \mathrm{Ran}_J\diamond J\Longrightarrow\mathrm{id}_{\mathbf{Rel}(A,B)}$ is the counit of the adjunction $J^*\dashv\mathrm{Ran}_J$.

5.5.4 The Right Skew Left Unitors

Definition 5.5.4.1.1. The **right** J**-skew left unitor of Rel**(A,B) is the natural transformation

$$\lambda^{\mathbf{Rel}(A,B),\triangleright_J}\colon \lambda^{\mathsf{Cats}_2}_{\mathbf{Rel}(A,B)}\Longrightarrow\triangleright_J\circ\left(\mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleright}\times\mathsf{id}\right),$$

as in the diagram

$$\begin{tikzcd}
\mathsf{pt} \times \mathbf{Rel}(A,B) \arrow[r, "\mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleright_J} \times \mathsf{id}"] \arrow[dr, dashed, bend right, "\lambda^{\mathsf{Cats}_2}_{\mathbf{Rel}(A,B)}"'] \arrow[dr, phantom, "\Uparrow \lambda^{\mathbf{Rel}(A,B),\triangleright_J}"] & \mathbf{Rel}(A,B) \times \mathbf{Rel}(A,B) \arrow[d, "\triangleright_J"] \\
 & \mathbf{Rel}(A,B),
\end{tikzcd}$$

whose component

$$\lambda^{\mathbf{Rel}(A,B),\triangleright_J}_R \colon R \hookrightarrow \underbrace{J \triangleright_J R}_{\overset{\text{def}}{=} \mathrm{Ran}_J(J) \diamond R}$$

at R is given by the composition

$$\begin{aligned}
R &\overset{\sim}{\Longrightarrow} \chi_B \diamond R \\
&\xrightarrow{\eta_{\chi_B} \diamond \mathrm{id}_R} \mathrm{Ran}_J(J^*(\chi_A)) \diamond R \\
&\overset{\text{def}}{=} \mathrm{Ran}_J(J^* \diamond \chi_A) \diamond R \\
&\overset{\sim}{\Longrightarrow} \mathrm{Ran}_J(J) \diamond R \\
&\overset{\text{def}}{=} R \triangleright_J J,
\end{aligned}$$

where $\eta \colon \mathrm{id}_{\mathbf{Rel}(B,B)} \Longrightarrow \mathrm{Ran}_J \circ J^*$ is the unit of the adjunction $J^* \dashv \mathrm{Ran}_J$.

5.5.5 The Right Skew Right Unitors

Definition 5.5.5.1.1. The **right J-skew right unitor of Rel(A, B)** is the natural transformation

$$\rho^{\mathbf{Rel}(A,B),\triangleright_J} \colon \triangleright_J \circ \left(\mathsf{id} \times \mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleright}\right) \Longrightarrow \rho^{\mathsf{Cats}_2}_{\mathbf{Rel}(A,B)},$$

as in the diagram

$$\begin{tikzcd}
\mathbf{Rel}(A,B) \times \mathsf{pt} \arrow[r, "\mathsf{id} \times \mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleright_J}"] \arrow[dr, dashed, bend right, "\rho^{\mathsf{Cats}_2}_{\mathbf{Rel}(A,B)}"'] \arrow[dr, phantom, "\Downarrow \rho^{\mathbf{Rel}(A,B),\triangleright_J}"] & \mathbf{Rel}(A,B) \times \mathbf{Rel}(A,B), \arrow[d, "\triangleright_J"] \\
 & \mathbf{Rel}(A,B)
\end{tikzcd}$$

whose component

$$\rho_S^{\mathbf{Rel}(A,B),\triangleright_J} \colon \underbrace{S \triangleright_J J}_{\stackrel{\text{def}}{=} \mathrm{Ran}_J(S) \diamond J} \hookrightarrow S$$

at S is given by

$$\rho_S^{\mathbf{Rel}(A,B),\triangleright_J} \stackrel{\text{def}}{=} \epsilon_R,$$

where $\epsilon \colon J^* \circ \mathrm{Ran}_J \Longrightarrow \mathrm{id}_{\mathbf{Rel}(A,B)}$ is the counit of the adjunction $J^* \dashv \mathrm{Ran}_J$.

5.5.6 The Right Skew Monoidal Structure on Rel(A, B)

Proposition 5.5.6.1.1. The category **Rel**(A, B) admits a right skew monoidal category structure consisting of

- *The Underlying Category.* The posetal category associated to the poset **Rel**(A, B) of relations from A to B of Item 2 of Definition 5.1.1.1.3.
- *The Right Skew Monoidal Product.* The right J-skew monoidal product

  $$\triangleleft_J \colon \mathbf{Rel}(A, B) \times \mathbf{Rel}(A, B) \longrightarrow \mathbf{Rel}(A, B)$$

  of Definition 5.5.1.1.1.
- *The Right Skew Monoidal Unit.* The functor

  $$\mathbb{1}^{\mathbf{Rel}(A,B),\triangleleft_J} \colon \mathsf{pt} \longrightarrow \mathbf{Rel}(A, B)$$

  of Definition 5.5.2.1.1.
- *The Right Skew Associators.* The natural transformation

  $$\alpha^{\mathbf{Rel}(A,B),\triangleright_J} \colon \triangleright_J \circ (\mathsf{id} \times \triangleright_J) \Longrightarrow \triangleright_J \circ (\triangleright_J \times \mathsf{id}) \circ \boldsymbol{\alpha}^{\mathsf{Cats},-1}_{\mathbf{Rel}(A,B),\mathbf{Rel}(A,B),\mathbf{Rel}(A,B)}$$

  of Definition 5.5.3.1.1.
- *The Right Skew Left Unitors.* The natural transformation

  $$\lambda^{\mathbf{Rel}(A,B),\triangleright_J} \colon \boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathbf{Rel}(A,B)} \Longrightarrow \triangleright_J \circ \left(\mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleright} \times \mathsf{id}\right)$$

  of Definition 5.5.4.1.1.
- *The Right Skew Right Unitors.* The natural transformation

  $$\rho^{\mathbf{Rel}(A,B),\triangleright_J} \colon \triangleright_J \circ \left(\mathsf{id} \times \mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleright}\right) \Longrightarrow \boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathbf{Rel}(A,B)}$$

  of Definition 5.5.5.1.1.

Proof. Since $\mathbf{Rel}(A, B)$ is posetal, the commutativity of the pentagon identity, the right skew left triangle identity, the right skew right triangle identity, the right skew middle triangle identity, and the zigzag identity is automatic, and thus $\mathbf{Rel}(A, B)$ together with the data in the statement forms a right skew monoidal category. □

Appendices

5.A Other Chapters

Sets

1. Sets
2. Constructions With Sets
3. Pointed Sets
4. Tensor Products of Pointed Sets

Relations

5. Relations
6. Constructions With Relations
7. Equivalence Relations and Apartness Relations

Category Theory

8. Categories

Bicategories

9. Types of Morphisms in Bicategories

Chapter 6

Constructions With Relations

This chapter contains some material about constructions with relations. Notably, we discuss and explore:

1. The existence or non-existence of Kan extensions and Kan lifts in the 2-category **Rel** (Section 6.2).

2. The various kinds of constructions involving relations, such as graphs, domains, ranges, unions, intersections, products, inverse relations, composition of relations, and collages (Section 6.3).

3. The adjoint pairs

$$\begin{aligned} R_* \dashv R_{-1} &: \mathcal{P}(A) \rightleftarrows \mathcal{P}(B), \\ R^{-1} \dashv R_! &: \mathcal{P}(B) \rightleftarrows \mathcal{P}(A) \end{aligned}$$

   of functors (morphisms of posets) between $\mathcal{P}(A)$ and $\mathcal{P}(B)$ induced by a relation $R\colon A \nrightarrow B$, as well as the properties of R_*, R_{-1}, R^{-1}, and $R_!$ (Section 6.4).

   Of particular note are the following points:

   (a) These two pairs of adjoint functors are the counterpart for relations of the adjoint triple $f_* \dashv f^{-1} \dashv f_!$ induced by a function $f\colon A \to B$ studied in Section 2.4.

   (b) We have $R_{-1} = R^{-1}$ iff R is total and functional (Item 8 of Proposition 6.4.2.1.3).

   (c) As a consequence of the previous item, when R comes from a function f, the pair of adjunctions

$$R_* \dashv R_{-1} = R^{-1} \dashv R_!$$

reduces to the triple adjunction

$$f_* \dashv f^{-1} \dashv f_!$$

from Section 2.4.

(d) The pairs $R_* \dashv R_{-1}$ and $R^{-1} \dashv R_!$ turn out to be rather important later on, as they appear in the definition and study of continuous, open, and closed relations between topological spaces (??).

Contents

6.1 Co/Limits in the Category of Relations 289

6.2 Kan Extensions and Kan Lifts in the 2-Category of Relations 289

6.2.1 Left Kan Extensions in **Rel** 289

6.2.2 Left Kan Lifts in **Rel** 290

6.2.3 Right Kan Extensions in **Rel** 291

6.2.4 Right Kan Lifts in **Rel** 292

6.3 More Constructions With Relations 293

6.3.1 The Graph of a Function 293

6.3.2 The Inverse of a Function 297

6.3.3 Representable Relations 298

6.3.4 The Domain and Range of a Relation 299

6.3.5 Binary Unions of Relations 300

6.3.6 Unions of Families of Relations 301

6.3.7 Binary Intersections of Relations 302

6.3.8 Intersections of Families of Relations 303

6.3.9 Binary Products of Relations 303

6.3.10 Products of Families of Relations 305

6.3.11 The Inverse of a Relation 306

6.3.12 Composition of Relations 307

6.3.13 The Collage of a Relation 312

6.4 Functoriality of Powersets 314

6.4.1 Direct Images 314

6.4.2 Strong Inverse Images 318

6.4.3 Weak Inverse Images 324

6.4.4 Direct Images With Compact Support 329

6.4.5 Functoriality of Powersets 335

6.4.6 Functoriality of Powersets: Relations on Powersets 336

6.A Other Chapters 337

6.1 Co/Limits in the Category of Relations

This section is currently just a stub, and will be properly developed later on.

6.2 Kan Extensions and Kan Lifts in the 2-Category of Relations

6.2.1 Left Kan Extensions in Rel

Proposition 6.2.1.1.1. Let $R\colon A \nrightarrow B$ be a relation.

1. *Non-Existence of All Left Kan Extensions in* **Rel**. Not all relations in **Rel** admit left Kan extensions.

2. *Characterisation of Relations Admitting Left Kan Extensions Along Them.* The following conditions are equivalent:

   (a) The left Kan extension

   $$\mathrm{Lan}_R\colon \mathbf{Rel}(A, X) \to \mathbf{Rel}(B, X)$$

   along R exists.

   (b) The relation R admits a left adjoint in **Rel**.

   (c) The relation R is of the form f^{-1} (as in Definition 6.3.2.1.1) for some function f.

Proof. *Item 1*, *Non-Existence of All Left Kan Extensions in* **Rel**: Omitted, but will eventually follow Fosco Loregian's comment on [MO 460656].
Item 2, *Characterisation of Relations Admitting Left Kan Extensions Along Them*: Omitted, but will eventually follow Tim Campion's answer to to [MO 460656]. □

Question 6.2.1.1.2. Given relations $S\colon A \nrightarrow X$ and $R\colon A \nrightarrow B$, is there a characterisation of when the left Kan extension

$$\mathrm{Lan}_S(R)\colon B \nrightarrow X$$

exists in terms of properties of R and S?
This question also appears as [MO 461592].

Question 6.2.1.1.3. As shown in Item 2 of Proposition 6.2.1.1.1, the left Kan extension

$$\mathrm{Lan}_R\colon \mathbf{Rel}(A, X) \to \mathbf{Rel}(B, X)$$

along a relation of the form $R = f^{-1}$ exists. Is there a explicit description of it, similarly to the explicit description of right Kan extensions given in Proposition 6.2.3.1.1?
This question also appears as [MO 461592].

6.2.2 Left Kan Lifts in Rel

Proposition 6.2.2.1.1. Let $R\colon A \nrightarrow B$ be a relation.

1. *Non-Existence of All Left Kan Lifts in* **Rel**. Not all relations in **Rel** admit left Kan lifts.

2. *Characterisation of Relations Admitting Left Kan Lifts Along Them.* The following conditions are equivalent:

   (a) The left Kan lift

   $$\mathrm{Lift}_R\colon \mathbf{Rel}(X, B) \to \mathbf{Rel}(X, A)$$

   along R exists.

   (b) The relation R admits a right adjoint in **Rel**.

   (c) The relation R is of the form $\mathrm{Gr}(f)$ (as in Definition 6.3.1.1.1) for some function f.

Proof. *Item 1, Non-Existence of All Left Kan Lifts in* **Rel**: Omitted, but will eventually follow (the dual of) Fosco Loregian's comment on [MO 460656].
Item 2, Characterisation of Relations Admitting Left Kan Lifts Along Them: Omitted, but will eventually follow Tim Campion's answer to to [MO 460656]. □

Question 6.2.2.1.2. Given relations $S\colon A \nrightarrow X$ and $R\colon A \nrightarrow B$, is there a characterisation of when the left Kan lift

$$\mathrm{Lift}_S(R)\colon X \nrightarrow A$$

exists in terms of properties of R and S?
This question also appears as [MO 461592].

Question 6.2.2.1.3. As shown in Item 2 of Proposition 6.2.2.1.1, the left Kan lift

$$\mathrm{Lift}_R\colon \mathbf{Rel}(X, B) \to \mathbf{Rel}(X, A)$$

along a relation of the form $R = \mathrm{Gr}(f)$ exists. Is there a explicit description of it, similarly to the explicit description of right Kan lifts given in Proposition 6.2.4.1.1?
This question also appears as [MO 461592].

6.2.3 Right Kan Extensions in Rel

Let $R\colon A \mathrel{-\!\!\!\!\!\mapsto} B$ be a relation.

Proposition 6.2.3.1.1. The right Kan extension

$$\mathrm{Ran}_R\colon \mathrm{Rel}(A,X) \to \mathrm{Rel}(B,X)$$

along R in **Rel** exists and is given by

$$\mathrm{Ran}_R(S) \stackrel{\mathrm{def}}{=} \int_{a\in A} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R_a^{-2}, S_a^{-1}\right)$$

for each $S \in \mathrm{Rel}(A,X)$, so that the following conditions are equivalent:

1. We have $b \sim_{\mathrm{Ran}_R(S)} x$.
2. For each $a \in A$, if $a \sim_R b$, then $a \sim_S x$.

Proof. We have

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{Rel}(A,X)}(S \diamond R, T) &\cong \int_{a\in A}\int_{x\in X} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left((S\diamond R)^x_a, T^x_a\right)\\
&\cong \int_{a\in A}\int_{x\in X} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(\left(\int^{b\in B} S^x_b \times R^b_a\right), T^x_a\right)\\
&\cong \int_{a\in A}\int_{x\in X}\int_{b\in B} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(S^x_b \times R^b_a, T^x_a\right)\\
&\cong \int_{a\in A}\int_{x\in X}\int_{b\in B} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(S^x_b, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^b_a, T^x_a\right)\right)\\
&\cong \int_{b\in B}\int_{x\in X}\int_{a\in A} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(S^x_b, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^b_a, T^x_a\right)\right)\\
&\cong \int_{b\in B}\int_{x\in X} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(S^x_b, \int_{a\in A}\mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^b_a, T^x_a\right)\right)\\
&\cong \mathrm{Hom}_{\mathbf{Rel}(B,X)}\left(S, \int_{a\in A}\mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R_a^{-2}, T_a^{-1}\right)\right)
\end{aligned}$$

naturally in each $S \in \mathbf{Rel}(B,X)$ and each $T \in \mathbf{Rel}(A,X)$, showing that

$$\int_{a\in A}\mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R_a^{-2}, T_a^{-1}\right)$$

is right adjoint to the precomposition functor $-\diamond R$, being thus the right Kan extension along R. Here we have used the following results, respectively (i.e. for each $\cong$ sign):

1. Item 1 of Proposition 5.1.1.1.5.

2. Definition 6.3.12.1.1.
3. ?? of ??.
4. Proposition 1.2.2.1.5.
5. ?? of ??.
6. ?? of ??.
7. Item 1 of Proposition 5.1.1.1.5.

This finishes the proof. □

6.2.4 Right Kan Lifts in Rel

Let $R\colon A \nrightarrow B$ be a relation.

Proposition 6.2.4.1.1. The right Kan lift

$$\mathrm{Rift}_R\colon \mathrm{Rel}(X,B) \to \mathrm{Rel}(X,A)$$

along R in **Rel** exists and is given by

$$\mathrm{Rift}_R(S) \stackrel{\text{def}}{=} \int_{b\in B} \mathbf{Hom}_{\{\mathrm{t},\mathrm{f}\}}\left(R^b_{-_1}, S^b_{-_2}\right)$$

for each $S \in \mathrm{Rel}(X,B)$, so that the following conditions are equivalent:

1. We have $x \sim_{\mathrm{Rift}_R(S)} a$.
2. For each $b \in B$, if $a \sim_R b$, then $x \sim_S b$.

Proof. We have

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{Rel}(X,B)}(R \diamond S, T) &\cong \int_{x\in X}\int_{b\in B} \mathbf{Hom}_{\{\mathrm{t},\mathrm{f}\}}\left((R\diamond S)^b_x, T^b_x\right)\\
&\cong \int_{x\in X}\int_{b\in B} \mathbf{Hom}_{\{\mathrm{t},\mathrm{f}\}}\left(\left(\int^{a\in A} R^b_a \times S^a_x\right), T^b_x\right)\\
&\cong \int_{x\in X}\int_{b\in B}\int_{a\in A} \mathbf{Hom}_{\{\mathrm{t},\mathrm{f}\}}\left(R^b_a \times S^a_x, T^b_x\right)\\
&\cong \int_{x\in X}\int_{b\in B}\int_{a\in A} \mathbf{Hom}_{\{\mathrm{t},\mathrm{f}\}}\left(S^a_x, \mathbf{Hom}_{\{\mathrm{t},\mathrm{f}\}}\left(R^b_a, T^b_x\right)\right)\\
&\cong \int_{x\in X}\int_{a\in A}\int_{b\in B} \mathbf{Hom}_{\{\mathrm{t},\mathrm{f}\}}\left(S^a_x, \mathbf{Hom}_{\{\mathrm{t},\mathrm{f}\}}\left(R^b_a, T^b_x\right)\right)\\
&\cong \int_{x\in X}\int_{a\in A} \mathbf{Hom}_{\{\mathrm{t},\mathrm{f}\}}\left(S^a_x, \int_{b\in B}\mathbf{Hom}_{\{\mathrm{t},\mathrm{f}\}}\left(R^b_a, T^b_x\right)\right)\\
&\cong \mathrm{Hom}_{\mathbf{Rel}(X,A)}\left(S, \int_{b\in B}\mathbf{Hom}_{\{\mathrm{t},\mathrm{f}\}}\left(R^b_{-_1}, T^b_{-_2}\right)\right)
\end{aligned}$$

naturally in each $S \in \mathbf{Rel}(X, A)$ and each $T \in \mathbf{Rel}(X, B)$, showing that

$$\int_{b\in B} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^b_{-_1}, S^b_{-_2}\right)$$

is right adjoint to the postcomposition functor $R \diamond -$, being thus the right Kan lift along R. Here we have used the following results, respectively (i.e. for each $\cong$ sign):

1. Item 1 of Proposition 5.1.1.1.5.
2. Definition 6.3.12.1.1.
3. **??** of **??**.
4. Proposition 1.2.2.1.5.
5. **??** of **??**.
6. **??** of **??**.
7. Item 1 of Proposition 5.1.1.1.5.

This finishes the proof. □

6.3 More Constructions With Relations

6.3.1 The Graph of a Function

Let $f : A \longrightarrow B$ be a function.

Definition 6.3.1.1.1. The **graph of** f is the relation $\mathrm{Gr}(f) : A \nrightarrow B$ defined as follows:[1]

- Viewing relations from A to B as subsets of $A \times B$, we define

$$\mathrm{Gr}(f) \overset{\text{def}}{=} \{(a, f(a)) \in A \times B \mid a \in A\}.$$

- Viewing relations from A to B as functions $A \times B \longrightarrow \{\mathsf{true}, \mathsf{false}\}$, we define

$$[\mathrm{Gr}(f)](a, b) \overset{\text{def}}{=} \begin{cases} \mathsf{true} & \text{if } b = f(a), \\ \mathsf{false} & \text{otherwise} \end{cases}$$

  for each $(a, b) \in A \times B$.

[1] *Further Notation:* We write $\mathrm{Gr}(A)$ for $\mathrm{Gr}(\mathrm{id}_A)$, and call it the **graph** of A.

- Viewing relations from A to B as functions $A \to \mathcal{P}(B)$, we define

$$[\mathrm{Gr}(f)](a) \overset{\text{def}}{=} \{f(a)\}$$

for each $a \in A$, i.e. we define $\mathrm{Gr}(f)$ as the composition

$$A \xrightarrow{f} B \xrightarrow{\chi_B} \mathcal{P}(B).$$

Proposition 6.3.1.1.2. Let $f\colon A \to B$ be a function.

1. *Functoriality.* The assignment $A \mapsto \mathrm{Gr}(A)$ defines a functor

$$\mathrm{Gr}\colon \mathsf{Sets} \to \mathrm{Rel}$$

where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$\mathrm{Gr}(A) \overset{\text{def}}{=} A.$$

- *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathsf{Sets})$, the action on Hom-sets

$$\mathrm{Gr}_{A,B}\colon \mathsf{Sets}(A, B) \to \underbrace{\mathrm{Rel}(\mathrm{Gr}(A), \mathrm{Gr}(B))}_{\overset{\text{def}}{=}\mathrm{Rel}(A,B)}$$

of Gr at (A, B) is defined by

$$\mathrm{Gr}_{A,B}(f) \overset{\text{def}}{=} \mathrm{Gr}(f),$$

where $\mathrm{Gr}(f)$ is the graph of f as in Definition 6.3.1.1.1.

In particular:

- *Preservation of Identities.* We have

$$\mathrm{Gr}(\mathrm{id}_A) = \chi_A$$

for each $A \in \mathrm{Obj}(\mathsf{Sets})$.

- *Preservation of Composition.* We have

$$\mathrm{Gr}(g \circ f) = \mathrm{Gr}(g) \diamond \mathrm{Gr}(f)$$

for each pair of functions $f\colon A \to B$ and $g\colon B \to C$.

2. *Adjointness Inside* **Rel**. We have an adjunction

$$(\mathrm{Gr}(f) \dashv f^{-1})\colon\quad \begin{tikzcd} A \arrow[r, bend left, "\mathrm{Gr}(f)", "\shortmid" marking] \arrow[r, phantom, "\perp"] & B \arrow[l, bend left, "f^{-1}", "\shortmid" marking] \end{tikzcd}$$

in **Rel**, where f^{-1} is the inverse of f of Definition 6.3.2.1.1.

3. *Adjointness.* We have an adjunction

$$(\mathrm{Gr} \dashv \mathcal{P}_*)\colon\quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "\mathrm{Gr}"] \arrow[r, phantom, "\perp"] & \mathrm{Rel}, \arrow[l, bend left, "\mathcal{P}_*"] \end{tikzcd}$$

witnessed by a bijection of sets

$$\mathrm{Rel}(\mathrm{Gr}(A), B) \cong \mathsf{Sets}(A, \mathcal{P}(B))$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$ and $B \in \mathrm{Obj}(\mathrm{Rel})$.

4. *Interaction With Inverses.* We have

$$\begin{aligned} \mathrm{Gr}(f)^{\dagger} &= f^{-1}, \\ (f^{-1})^{\dagger} &= \mathrm{Gr}(f). \end{aligned}$$

5. *Cocontinuity.* The functor $\mathrm{Gr}\colon \mathsf{Sets} \to \mathrm{Rel}$ of Item 1 preserves colimits.

6. *Characterisations.* Let $R\colon A \nrightarrow B$ be a relation. The following conditions are equivalent:

   (a) There exists a function $f\colon A \to B$ such that $R = \mathrm{Gr}(f)$.

   (b) The relation R is total and functional.

   (c) The weak and strong inverse images of R agree, i.e. we have $R^{-1} = R_{-1}$.

   (d) The relation R has a right adjoint $R^{\dagger}$ in Rel.

Proof. *Item 1*, *Functoriality*: Clear.
Item 2, *Adjointness Inside* **Rel**: We need to check that there are inclusions

$$\begin{aligned} \chi_A &\subset f^{-1} \diamond \mathrm{Gr}(f), \\ \mathrm{Gr}(f) \diamond f^{-1} &\subset \chi_B. \end{aligned}$$

These correspond respectively to the following conditions:

1. For each $a \in A$, there exists some $b \in B$ such that $a \sim_{\mathrm{Gr}(f)} b$ and $b \sim_{f^{-1}} a$.

2. For each $a, b \in A$, if $a \sim_{\mathrm{Gr}(f)} b$ and $b \sim_{f^{-1}} a$, then $a = b$.

In other words, the first condition states that the image of any $a \in A$ by f is nonempty, whereas the second condition states that f is not multivalued. As f is a function, both of these statements are true, and we are done.
Item 3, Adjointness: The stated bijection follows from Remark 5.1.1.1.4, with naturality being clear.
Item 4, Interaction With Inverses: Clear.
Item 5, Cocontinuity: Omitted.
Item 6, Characterisations: We claim that Items 6a to 6d are indeed equivalent:

- *Item 6a* $\iff$ *Item 6b*. This is shown in the proof of **??** of **??**.

- *Item 6b* $\implies$ *Item 6c*. If R is total and functional, then, for each $a \in A$, the set $R(a)$ is a singleton, implying that

$$\begin{aligned} R^{-1}(V) &\overset{\mathrm{def}}{=} \{a \in A \mid R(a) \cap V \neq \emptyset\}, \\ R_{-1}(V) &\overset{\mathrm{def}}{=} \{a \in A \mid R(a) \subset V\} \end{aligned}$$

  are equal for all $V \in \mathcal{P}(B)$, as the conditions $R(a) \cap V \neq \emptyset$ and $R(a) \subset V$ are equivalent when $R(a)$ is a singleton.

- *Item 6c* $\implies$ *Item 6b*. We claim that R is indeed total and functional:

  - *Totality.* If we had $R(a) = \emptyset$ for some $a \in A$, then we would have $a \in R_{-1}(\emptyset)$, so that $R_{-1}(\emptyset) \neq \emptyset$. But since $R^{-1}(\emptyset) = \emptyset$, this would imply $R_{-1}(\emptyset) \neq R^{-1}(\emptyset)$, a contradiction. Thus $R(a) \neq \emptyset$ for all $a \in A$ and R is total.
  - *Functionality.* If $R^{-1} = R_{-1}$, then we have

$$\begin{aligned} \{a\} &= R^{-1}(\{b\}) \\ &= R_{-1}(\{b\}) \end{aligned}$$

    for each $b \in R(a)$ and each $a \in A$, and thus $R(a) \subset \{b\}$. But since R is total, we must have $R(a) = \{b\}$, and thus we see that R is functional.

- *Item 6a* $\iff$ *Item 6d*. This follows from Proposition 5.3.3.1.1.

This finishes the proof. □

6.3.2 The Inverse of a Function

Let $f\colon A \longrightarrow B$ be a function.

Definition 6.3.2.1.1. The **inverse of** f is the relation $f^{-1}\colon B \nrightarrow A$ defined as follows:

- Viewing relations from B to A as subsets of $B \times A$, we define
$$f^{-1} \stackrel{\text{def}}{=} \left\{ \left(b, f^{-1}(b)\right) \in B \times A \,\middle|\, a \in A \right\},$$
where
$$f^{-1}(b) \stackrel{\text{def}}{=} \{ a \in A \mid f(a) = b \}$$
for each $b \in B$.

- Viewing relations from B to A as functions $B \times A \longrightarrow \{\mathsf{true}, \mathsf{false}\}$, we define
$$f^{-1}(b, a) \stackrel{\text{def}}{=} \begin{cases} \mathsf{true} & \text{if there exists } a \in A \text{ with } f(a) = b, \\ \mathsf{false} & \text{otherwise} \end{cases}$$
for each $(b, a) \in B \times A$.

- Viewing relations from B to A as functions $B \longrightarrow \mathcal{P}(A)$, we define
$$f^{-1}(b) \stackrel{\text{def}}{=} \{ a \in A \mid f(a) = b \}$$
for each $b \in B$.

Proposition 6.3.2.1.2. Let $f\colon A \longrightarrow B$ be a function.

1. *Functoriality.* The assignment $A \mapsto A, f \mapsto f^{-1}$ defines a functor
$$(-)^{-1}\colon \mathsf{Sets} \longrightarrow \mathrm{Rel}$$
where
   - *Action on Objects.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have
$$\left[(-)^{-1}\right](A) \stackrel{\text{def}}{=} A.$$
   - *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathsf{Sets})$, the action on Hom-sets
$$(-)^{-1}_{A,B}\colon \mathsf{Sets}(A, B) \longrightarrow \mathrm{Rel}(A, B)$$
of $(-)^{-1}$ at (A, B) is defined by
$$(-)^{-1}_{A,B}(f) \stackrel{\text{def}}{=} \left[(-)^{-1}\right](f),$$
where f^{-1} is the inverse of f as in Definition 6.3.2.1.1.

In particular:

- *Preservation of Identities.* We have

$$\mathrm{id}_A^{-1} = \chi_A$$

  for each $A \in \mathrm{Obj}(\mathsf{Sets})$.
- *Preservation of Composition.* We have

$$(g \circ f)^{-1} = g^{-1} \diamond f^{-1}$$

  for pair of functions $f: A \to B$ and $g: B \to C$.

2. *Adjointness Inside* **Rel**. We have an adjunction

$$\left(\mathrm{Gr}(f) \dashv f^{-1}\right): \quad \begin{tikzcd} A \arrow[r, bend left, "\mathrm{Gr}(f)", "\shortmid"{marking}] \arrow[r, phantom, "\perp"] & B \arrow[l, bend left, "f^{-1}", "\shortmid"{marking}] \end{tikzcd}$$

in **Rel**.

3. *Interaction With Inverses of Relations.* We have

$$\begin{aligned} \left(f^{-1}\right)^{\dagger} &= \mathrm{Gr}(f), \\ \mathrm{Gr}(f)^{\dagger} &= f^{-1}. \end{aligned}$$

Proof. *Item 1*, *Functoriality*: Clear.
Item 2, *Adjointness Inside* **Rel**: This is proved in Item 2 of Proposition 6.3.1.1.2.
Item 3, *Interaction With Inverses of Relations*: Clear. □

6.3.3 Representable Relations

Let A and B be sets.

Definition 6.3.3.1.1. Let $f: A \to B$ and $g: B \to A$ be functions.[2]

[2] More generally, given functions

$$\begin{aligned} f&: A \to C, \\ g&: B \to D \end{aligned}$$

and a relation $B \nrightarrow D$, we may consider the composite relation

$$A \times B \xrightarrow{f \times g} C \times D \xrightarrow{R} \{\mathsf{true}, \mathsf{false}\},$$

for which we have $a \sim_{R \circ (f \times g)} b$ iff $f(a) \sim_R g(b)$.

1. The **representable relation associated to** f is the relation $\chi_f \colon A \nrightarrow B$ defined as the composition

$$A \times B \xrightarrow{f \times \mathrm{id}_B} B \times B \xrightarrow{\chi_B} \{\mathsf{true}, \mathsf{false}\},$$

i.e. given by declaring $a \sim_{\chi_f} b$ iff $f(a) = b$.

2. The **corepresentable relation associated to** g is the relation $\chi^g \colon B \nrightarrow A$ defined as the composition

$$B \times A \xrightarrow{g \times \mathrm{id}_A} A \times A \xrightarrow{\chi_A} \{\mathsf{true}, \mathsf{false}\},$$

i.e. given by declaring $b \sim_{\chi^g} a$ iff $g(b) = a$.

6.3.4 The Domain and Range of a Relation

Let A and B be sets.

Definition 6.3.4.1.1. Let $R \subset A \times B$ be a relation.[3,4]

1. The **domain of** R is the subset $\mathrm{dom}(R)$ of A defined by

$$\mathrm{dom}(R) \overset{\mathrm{def}}{=} \left\{ a \in A \;\middle|\; \begin{array}{l} \text{there exists some } b \in B \\ \text{such that } a \sim_R b \end{array} \right\}.$$

[3] Following **??**, we may compute the (characteristic functions associated to the) domain and range of a relation using the following colimit formulas:

$$\begin{aligned} \chi_{\mathrm{dom}(R)}(a) &\cong \operatorname*{colim}_{b \in B}\left(R^b_a\right) \qquad (a \in A) \\ &\cong \bigvee_{b \in B} R^b_a, \\ \chi_{\mathrm{range}(R)}(b) &\cong \operatorname*{colim}_{a \in A}\left(R^b_a\right) \qquad (b \in B) \\ &\cong \bigvee_{a \in A} R^b_a, \end{aligned}$$

where the join $\bigvee$ is taken in the poset $(\{\mathsf{true}, \mathsf{false}\}, \preceq)$ of Definition 1.2.2.1.3.

[4] Viewing R as a function $R \colon A \to \mathcal{P}(B)$, we have

$$\begin{aligned} \mathrm{dom}(R) &\cong \operatorname*{colim}_{y \in Y}(R(y)) \\ &\cong \bigcup_{y \in Y} R(y), \\ \mathrm{range}(R) &\cong \operatorname*{colim}_{x \in X}(R(x)) \\ &\cong \bigcup_{x \in X} R(x), \end{aligned}$$

2. The **range of** R is the subset $\mathrm{range}(R)$ of B defined by

$$\mathrm{range}(R) \stackrel{\text{def}}{=} \left\{ b \in B \,\middle|\, \begin{array}{l}\text{there exists some } a \in A\\ \text{such that } a \sim_R b\end{array} \right\}.$$

6.3.5 Binary Unions of Relations

Let A and B be sets and let R and S be relations from A to B.

Definition 6.3.5.1.1. The **union of** R **and** S[5] is the relation $R \cup S$ from A to B defined as follows:

- Viewing relations from A to B as subsets of $A \times B$, we define[6]

$$R \cup S \stackrel{\text{def}}{=} \{(a, b) \in B \times A \mid \text{we have } a \sim_R b \text{ or } a \sim_S b\}.$$

- Viewing relations from A to B as functions $A \to \mathcal{P}(B)$, we define

$$[R \cup S](a) \stackrel{\text{def}}{=} R(a) \cup S(a)$$

  for each $a \in A$.

Proposition 6.3.5.1.2. Let R, S, R_1, and R_2 be relations from A to B, and let S_1 and S_2 be relations from B to C.

1. *Interaction With Inverses.* We have

$$(R \cup S)^\dagger = R^\dagger \cup S^\dagger.$$

2. *Interaction With Composition.* We have

$$(S_1 \diamond R_1) \cup (S_2 \diamond R_2) \stackrel{\text{poss.}}{\neq} (S_1 \cup S_2) \diamond (R_1 \cup R_2).$$

Proof. Item 1, Interaction With Inverses: Clear.
Item 2, Interaction With Composition: Unwinding the definitions, we see that:

1. The condition for $(S_1 \diamond R_1) \cup (S_2 \diamond R_2)$ is:
   (a) There exists some $b \in B$ such that:
      i. $a \sim_{R_1} b$ and $b \sim_{S_1} c$;
   or

[5] *Further Terminology:* Also called the **binary union of** R **and** S, for emphasis.
[6] This is the same as the union of R and S as subsets of $A \times B$.

i. $a \sim_{R_2} b$ and $b \sim_{S_2} c$;

3. The condition for $(S_1 \cup S_2) \diamond (R_1 \cup R_2)$ is:

   (a) There exists some $b \in B$ such that:

   i. $a \sim_{R_1} b$ or $a \sim_{R_2} b$;

   and

   i. $b \sim_{S_1} c$ or $b \sim_{S_2} c$.

These two conditions may fail to agree (counterexample omitted), and thus the two resulting relations on $A \times C$ may differ. □

6.3.6 Unions of Families of Relations

Let A and B be sets and let $\{R_i\}_{i \in I}$ be a family of relations from A to B.

Definition 6.3.6.1.1. The **union of the family** $\{R_i\}_{i \in I}$ is the relation $\bigcup_{i \in I} R_i$ from A to B defined as follows:

- Viewing relations from A to B as subsets of $A \times B$, we define[7]

$$\bigcup_{i \in I} R_i \overset{\text{def}}{=} \left\{ (a, b) \in (A \times B)^{\times I} \;\middle|\; \begin{array}{l} \text{there exists some } i \in I \\ \text{such that } a \sim_{R_i} b \end{array} \right\}.$$

- Viewing relations from A to B as functions $A \to \mathcal{P}(B)$, we define

$$\left[\bigcup_{i \in I} R_i\right](a) \overset{\text{def}}{=} \bigcup_{i \in I} R_i(a)$$

  for each $a \in A$.

Proposition 6.3.6.1.2. Let A and B be sets and let $\{R_i\}_{i \in I}$ be a family of relations from A to B.

1. *Interaction With Inverses.* We have

$$\left(\bigcup_{i \in I} R_i\right)^{\dagger} = \bigcup_{i \in I} R_i^{\dagger}.$$

Proof. *Item 1, Interaction With Inverses*: Clear. □

[7] This is the same as the union of $\{R_i\}_{i \in I}$ as a collection of subsets of $A \times B$.

6.3.7 Binary Intersections of Relations

Let A and B be sets and let R and S be relations from A to B.

Definition 6.3.7.1.1. The **intersection of** R **and** S[8] is the relation $R \cap S$ from A to B defined as follows:

- Viewing relations from A to B as subsets of $A \times B$, we define[9]
$$R \cap S \overset{\text{def}}{=} \{(a, b) \in B \times A \mid \text{we have } a \sim_R b \text{ and } a \sim_S b\}.$$
- Viewing relations from A to B as functions $A \to \mathcal{P}(B)$, we define
$$[R \cap S](a) \overset{\text{def}}{=} R(a) \cap S(a)$$
for each $a \in A$.

Proposition 6.3.7.1.2. Let R, S, R_1, and R_2 be relations from A to B, and let S_1 and S_2 be relations from B to C.

1. *Interaction With Inverses.* We have
$$(R \cap S)^\dagger = R^\dagger \cap S^\dagger.$$
2. *Interaction With Composition.* We have
$$(S_1 \diamond R_1) \cap (S_2 \diamond R_2) = (S_1 \cap S_2) \diamond (R_1 \cap R_2).$$

Proof. Item 1, Interaction With Inverses: Clear.
Item 2, Interaction With Composition: Unwinding the definitions, we see that:

1. The condition for $(S_1 \diamond R_1) \cap (S_2 \diamond R_2)$ is:
   (a) There exists some $b \in B$ such that:
      i. $a \sim_{R_1} b$ and $b \sim_{S_1} c$;
   and
      i. $a \sim_{R_2} b$ and $b \sim_{S_2} c$;
3. The condition for $(S_1 \cap S_2) \diamond (R_1 \cap R_2)$ is:
   (a) There exists some $b \in B$ such that:
      i. $a \sim_{R_1} b$ and $a \sim_{R_2} b$;
   and
      i. $b \sim_{S_1} c$ and $b \sim_{S_2} c$.

These two conditions agree, and thus so do the two resulting relations on $A \times C$. □

[8] *Further Terminology:* Also called the **binary intersection of** R **and** S, for emphasis.
[9] This is the same as the intersection of R and S as subsets of $A \times B$.

6.3.8 Intersections of Families of Relations

Let A and B be sets and let $\{R_i\}_{i\in I}$ be a family of relations from A to B.

Definition 6.3.8.1.1. The **intersection of the family** $\{R_i\}_{i\in I}$ is the relation $\bigcup_{i\in I} R_i$ defined as follows:

- Viewing relations from A to B as subsets of $A \times B$, we define[10]

$$\bigcup_{i\in I} R_i \overset{\text{def}}{=} \left\{ (a,b) \in (A \times B)^{\times I} \,\middle|\, \begin{array}{l} \text{for each } i \in I, \\ \text{we have } a \sim_{R_i} b \end{array} \right\}.$$

- Viewing relations from A to B as functions $A \to \mathcal{P}(B)$, we define

$$\left[\bigcap_{i\in I} R_i\right](a) \overset{\text{def}}{=} \bigcap_{i\in I} R_i(a)$$

  for each $a \in A$.

Proposition 6.3.8.1.2. Let A and B be sets and let $\{R_i\}_{i\in I}$ be a family of relations from A to B.

1. *Interaction With Inverses.* We have

$$\left(\bigcap_{i\in I} R_i\right)^{\dagger} = \bigcap_{i\in I} R_i^{\dagger}.$$

Proof. *Item 1, Interaction With Inverses*: Clear. □

6.3.9 Binary Products of Relations

Let A, B, X, and Y be sets, let $R\colon A \mathrel{\mapstochar\mkern-4mu\rightarrow} B$ be a relation from A to B, and let $S\colon X \mathrel{\mapstochar\mkern-4mu\rightarrow} Y$ be a relation from X to Y.

Definition 6.3.9.1.1. The **product of** R **and** S[11] is the relation $R \times S$ from $A \times X$ to $B \times Y$ defined as follows:

- Viewing relations from $A \times X$ to $B \times Y$ as subsets of $(A \times X) \times (B \times Y)$, we define $R \times S$ as the Cartesian product of R and S as subsets of $A \times X$ and $B \times Y$.[12]

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[10] This is the same as the intersection of $\{R_i\}_{i\in I}$ as a collection of subsets of $A \times B$.

[11] *Further Terminology:* Also called the **binary product of** R **and** S, for emphasis.

[12] That is, $R \times S$ is the relation given by declaring $(a,x) \sim_{R\times S} (b,y)$ iff $a \sim_R b$ and $x \sim_S y$.

- Viewing relations from $A \times X$ to $B \times Y$ as functions $A \times X \to \mathcal{P}(B \times Y)$, we define $R \times S$ as the composition

$$A \times X \xrightarrow{R \times S} \mathcal{P}(B) \times \mathcal{P}(Y) \overset{\mathcal{P}^{\otimes}_{B,Y}}{\hookrightarrow} \mathcal{P}(B \times Y)$$

in Sets, i.e. by

$$[R \times S](a, x) \overset{\text{def}}{=} R(a) \times S(x)$$

for each $(a, x) \in A \times X$.

Proposition 6.3.9.1.2. Let A, B, X, and Y be sets.

1. *Interaction With Inverses.* Let

$$\begin{aligned} R &: A \nrightarrow A, \\ S &: X \nrightarrow X \end{aligned}$$

We have

$$(R \times S)^{\dagger} = R^{\dagger} \times S^{\dagger}.$$

2. *Interaction With Composition.* Let

$$\begin{aligned} R_1 &: A \nrightarrow B, \\ S_1 &: B \nrightarrow C, \\ R_2 &: X \nrightarrow Y, \\ S_2 &: Y \nrightarrow Z \end{aligned}$$

be relations. We have

$$(S_1 \diamond R_1) \times (S_2 \diamond R_2) = (S_1 \times S_2) \diamond (R_1 \times R_2).$$

Proof. *Item 1, Interaction With Inverses*: Unwinding the definitions, we see that:

1. We have $(a, x) \sim_{(R \times S)^{\dagger}} (b, y)$ iff:
    - We have $(b, y) \sim_{R \times S} (a, x)$, i.e. iff:
        - We have $b \sim_R a$;
        - We have $y \sim_S x$;
2. We have $(a, x) \sim_{R^{\dagger} \times S^{\dagger}} (b, y)$ iff:
    - We have $a \sim_{R^{\dagger}} b$ and $x \sim_{S^{\dagger}} y$, i.e. iff:

- We have $b \sim_R a$;
- We have $y \sim_S x$.

These two conditions agree, and thus the two resulting relations on $A \times X$ are equal.

Item 2, Interaction With Composition: Unwinding the definitions, we see that:

1. We have $(a, x) \sim_{(S_1 \diamond R_1) \times (S_2 \diamond R_2)} (c, z)$ iff:

   (a) We have $a \sim_{S_1 \diamond R_1} c$ and $x \sim_{S_2 \diamond R_2} z$, i.e. iff:

      i. There exists some $b \in B$ such that $a \sim_{R_1} b$ and $b \sim_{S_1} c$;

      ii. There exists some $y \in Y$ such that $x \sim_{R_2} y$ and $y \sim_{S_2} z$;

2. We have $(a, x) \sim_{(S_1 \times S_2) \diamond (R_1 \times R_2)} (c, z)$ iff:

   (a) There exists some $(b, y) \in B \times Y$ such that $(a, x) \sim_{R_1 \times R_2} (b, y)$ and $(b, y) \sim_{S_1 \times S_2} (c, z)$, i.e. such that:

      i. We have $a \sim_{R_1} b$ and $x \sim_{R_2} y$;

      ii. We have $b \sim_{S_1} c$ and $y \sim_{S_2} z$.

These two conditions agree, and thus the two resulting relations from $A \times X$ to $C \times Z$ are equal. □

6.3.10 Products of Families of Relations

Let $\{A_i\}_{i \in I}$ and $\{B_i\}_{i \in I}$ be families of sets, and let $\{R_i : A_i \nrightarrow B_i\}_{i \in I}$ be a family of relations.

Definition 6.3.10.1.1. The **product of the family** $\{R_i\}_{i \in I}$ is the relation $\prod_{i \in I} R_i$ from $\prod_{i \in I} A_i$ to $\prod_{i \in I} B_i$ defined as follows:

- Viewing relations as subsets, we define $\prod_{i \in I} R_i$ as its product as a family of sets, i.e. we have

$$\prod_{i \in I} R_i \overset{\text{def}}{=} \left\{ (a_i, b_i)_{i \in I} \in \prod_{i \in I} (A_i \times B_i) \;\middle|\; \begin{array}{l} \text{for each } i \in I, \\ \text{we have } a_i \sim_{R_i} b_i \end{array} \right\}.$$

- Viewing relations as functions to powersets, we define

$$\left[\prod_{i \in I} R_i \right] ((a_i)_{i \in I}) \overset{\text{def}}{=} \prod_{i \in I} R_i(a_i)$$

  for each $(a_i)_{i \in I} \in \prod_{i \in I} R_i$.

6.3.11 The Inverse of a Relation

Let A, B, and C be sets and let $R \subset A \times B$ be a relation.

Definition 6.3.11.1.1. The **inverse of** R[13] is the relation $R^{\dagger}$ defined as follows:

- Viewing relations as subsets, we define

$$R^{\dagger} \stackrel{\text{def}}{=} \{(b, a) \in B \times A \mid \text{we have } b \sim_R a\}.$$

- Viewing relations as functions $A \times B \to \{\mathsf{true}, \mathsf{false}\}$, we define

$$\left[R^{\dagger}\right]^a_b \stackrel{\text{def}}{=} R^b_a$$

  for each $(b, a) \in B \times A$.

- Viewing relations as functions $A \to \mathcal{P}(B)$, we define

$$\begin{aligned} \left[R^{\dagger}\right](b) &\stackrel{\text{def}}{=} R^{\dagger}(\{b\}) \\ &\stackrel{\text{def}}{=} \{a \in A \mid b \in R(a)\} \end{aligned}$$

  for each $b \in B$, where $R^{\dagger}(\{b\})$ is the fibre of R over $\{b\}$.

Example 6.3.11.1.2. Here are some examples of inverses of relations.

1. *Less Than Equal Signs.* We have $(\leq)^{\dagger} = \geq$.
2. *Greater Than Equal Signs.* Dually to Item 1, we have $(\geq)^{\dagger} = \leq$.
3. *Functions.* Let $f \colon A \to B$ be a function. We have

$$\begin{aligned} \mathrm{Gr}(f)^{\dagger} &= f^{-1}, \\ \left(f^{-1}\right)^{\dagger} &= \mathrm{Gr}(f). \end{aligned}$$

Proposition 6.3.11.1.3. Let $R \colon A \nrightarrow B$ and $S \colon B \nrightarrow C$ be relations.

1. *Functoriality.* The assignment $R \mapsto R^{\dagger}$ defines a functor (i.e. morphism of posets)

$$(-)^{\dagger} \colon \mathbf{Rel}(A, B) \to \mathbf{Rel}(B, A).$$

   In particular, given relations $R, S \colon A \rightrightarrows B$, we have:

   (★) If $R \subset S$, then $R^{\dagger} \subset S^{\dagger}$.

[13] *Further Terminology:* Also called the **opposite of** R, the **transpose of** R, or the **converse**

2. *Interaction With Ranges and Domains.* We have

$$\begin{aligned}\operatorname{dom}\left(R^{\dagger}\right) &= \operatorname{range}(R),\\ \operatorname{range}\left(R^{\dagger}\right) &= \operatorname{dom}(R).\end{aligned}$$

3. *Interaction With Composition I.* We have

$$(S \diamond R)^{\dagger} = R^{\dagger} \diamond S^{\dagger}.$$

4. *Interaction With Composition II.* We have

$$\begin{aligned}\chi_B &\subset R \diamond R^{\dagger},\\ \chi_A &\subset R^{\dagger} \diamond R.\end{aligned}$$

5. *Invertibility.* We have

$$\left(R^{\dagger}\right)^{\dagger} = R.$$

6. *Identity.* We have

$$\chi_A^{\dagger} = \chi_A.$$

Proof. *Item 1*, *Functoriality*: Clear.
Item 2, *Interaction With Ranges and Domains*: Clear.
Item 3, *Interaction With Composition I*: Clear.
Item 4, *Interaction With Composition II*: Clear.
Item 5, *Invertibility*: Clear.
Item 6, *Identity*: Clear. □

6.3.12 Composition of Relations

Let A, B, and C be sets and let $R\colon A \nrightarrow B$ and $S\colon B \nrightarrow C$ be relations.

Definition 6.3.12.1.1. The **composition of** R **and** S is the relation $S \diamond R$ defined as follows:

- Viewing relations from A to C as subsets of $A \times C$, we define

$$S \diamond R \overset{\text{def}}{=} \left\{ (a, c) \in A \times C \,\middle|\, \begin{array}{l}\text{there exists some } b \in B \text{ such}\\ \text{that } a \sim_R b \text{ and } b \sim_S c\end{array} \right\}.$$

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of R.

- Viewing relations as functions $A \times B \to \{\mathsf{true}, \mathsf{false}\}$, we define

$$\begin{aligned}(S \diamond R)^{-1}_{-2} &\overset{\text{def}}{=} \int^{b \in B} S^{-1}_{b} \times R^{b}_{-2}\\ &= \bigvee_{b \in B} S^{-1}_{b} \times R^{b}_{-2},\end{aligned}$$

  where the join $\bigvee$ is taken in the poset $(\{\mathsf{true}, \mathsf{false}\}, \preceq)$ of Definition 1.2.2.1.3.

- Viewing relations as functions $A \to \mathcal{P}(B)$, we define

$$S \diamond R \overset{\text{def}}{=} \mathrm{Lan}_{\chi_B}(S) \circ R, \qquad \begin{tikzcd} & B \arrow[r, "S"] \arrow[d, hook, "\chi_B"'] & \mathcal{P}(C), \arrow[dl, phantom, "\Leftarrow"] \\ A \arrow[r, "R"'] & \mathcal{P}(B) \arrow[ur, "\mathrm{Lan}_{\chi_B}(S)"'] & \end{tikzcd}$$

  where $\mathrm{Lan}_{\chi_B}(S)$ is computed by the formula

$$\begin{aligned}\left[\mathrm{Lan}_{\chi_B}(S)\right](V) &\cong \int^{y \in B} \chi_{\mathcal{P}(B)}\left(\chi_y, V\right) \odot S_y\\ &\cong \int^{y \in B} \chi_V(y) \odot S_y\\ &\cong \bigcup_{y \in B} \chi_V(y) \odot S_y\\ &\cong \bigcup_{y \in V} S_y\end{aligned}$$

  for each $V \in \mathcal{P}(B)$. In other words, $S \diamond R$ is defined by[14]

$$\begin{aligned}[S \diamond R](a) &\overset{\text{def}}{=} S(R(a))\\ &\overset{\text{def}}{=} \bigcup_{x \in R(a)} S(x).\end{aligned}$$

  for each $a \in A$.

Example 6.3.12.1.2. Here are some examples of composition of relations.

[14]That is: the relation R may send $a \in A$ to a number of elements $\{b_i\}_{i \in I}$ in B, and then the relation S may send the image of each of the b_i's to a number of elements $\{S(b_i)\}_{i \in I} = \left\{\{c_{j_i}\}_{j_i \in J_i}\right\}_{i \in I}$ in C.

1. *Composing Less/Greater Than Equal With Greater/Less Than Equal Signs.* We have

$$\begin{aligned}\leq \diamond \geq &= \sim_{\mathrm{triv}},\\ \geq \diamond \leq &= \sim_{\mathrm{triv}}.\end{aligned}$$

2. *Composing Less/Greater Than Equal Signs With Less/Greater Than Equal Signs.* We have

$$\begin{aligned}\leq \diamond \leq &= \leq,\\ \geq \diamond \geq &= \geq.\end{aligned}$$

Proposition 6.3.12.1.3. Let $R\colon A \nrightarrow B$, $S\colon B \nrightarrow C$, and $T\colon C \nrightarrow D$ be relations.

1. *Interaction With Ranges and Domains.* We have

$$\begin{aligned}\mathrm{dom}(S \diamond R) &\subset \mathrm{dom}(R),\\ \mathrm{range}(S \diamond R) &\subset \mathrm{range}(S).\end{aligned}$$

2. *Associativity.* We have

$$(T \diamond S) \diamond R = T \diamond (S \diamond R).$$

3. *Unitality.* We have

$$\begin{aligned}\chi_B \diamond R &= R,\\ R \diamond \chi_A &= R.\end{aligned}$$

4. *Interaction With Inverses.* We have

$$(S \diamond R)^{\dagger} = R^{\dagger} \diamond S^{\dagger}.$$

5. *Interaction With Composition.* We have

$$\begin{aligned}\chi_B &\subset R \diamond R^{\dagger},\\ \chi_A &\subset R^{\dagger} \diamond R.\end{aligned}$$

Proof. *Item 1, Interaction With Ranges and Domains*: Clear.

Item 2, Associativity: Indeed, we have

$$\begin{aligned}
(T \diamond S) \diamond R &\stackrel{\text{def}}{=} \left(\int^{c\in C} T_c^{-1} \times S^c_{-_2}\right) \diamond R \\
&\stackrel{\text{def}}{=} \int^{b\in B} \left(\int^{c\in C} T_c^{-1} \times S^c_b\right) \diamond R^b_{-_2} \\
&= \int^{b\in B} \int^{c\in C} (T_c^{-1} \times S^c_b) \diamond R^b_{-_2} \\
&= \int^{c\in C} \int^{b\in B} (T_c^{-1} \times S^c_b) \diamond R^b_{-_2} \\
&= \int^{c\in C} \int^{b\in B} T_c^{-1} \times \left(S^c_b \diamond R^b_{-_2}\right) \\
&= \int^{c\in C} T_c^{-1} \times \left(\int^{b\in B} S^c_b \diamond R^b_{-_2}\right) \\
&\stackrel{\text{def}}{=} \int^{c\in C} T_c^{-1} \times (S \diamond R)^c_{-_2} \\
&\stackrel{\text{def}}{=} T \diamond (S \diamond R).
\end{aligned}$$

In the language of relations, given $a \in A$ and $d \in D$, the stated equality witnesses the equivalence of the following two statements:

1. We have $a \sim_{(T\diamond S)\diamond R} d$, i.e. there exists some $b \in B$ such that:
   (a) We have $a \sim_R b$;
   (b) We have $b \sim_{T\diamond S} d$, i.e. there exists some $c \in C$ such that:
      i. We have $b \sim_S c$;
      ii. We have $c \sim_T d$;
2. We have $a \sim_{T\diamond(S\diamond R)} d$, i.e. there exists some $c \in C$ such that:
   (a) We have $a \sim_{S\diamond R} c$, i.e. there exists some $b \in B$ such that:
      i. We have $a \sim_R b$;
      ii. We have $b \sim_S c$;
   (b) We have $c \sim_T d$;

both of which are equivalent to the statement

- There exist $b \in B$ and $c \in C$ such that $a \sim_R b \sim_S c \sim_T d$.

Item 3, Unitality: Indeed, we have

$$\begin{aligned}\chi_B \diamond R &\stackrel{\text{def}}{=} \int^{x\in B} (\chi_B)^{-1}_{x} \times R^{x}_{-_2}\\ &= \bigvee_{x\in B} (\chi_B)^{-1}_{x} \times R^{x}_{-_2}\\ &= \bigvee_{\substack{x\in B\\ x=-_1}} R^{x}_{-_2}\\ &= R^{-_1}_{-_2},\end{aligned}$$

and

$$\begin{aligned}R \diamond \chi_A &\stackrel{\text{def}}{=} \int^{x\in A} R^{-_1}_{x} \times (\chi_A)^{x}_{-_2}\\ &= \bigvee_{x\in B} R^{-_1}_{x} \times (\chi_A)^{x}_{-_2}\\ &= \bigvee_{\substack{x\in B\\ x=-_2}} R^{-_1}_{x}\\ &= R^{-_1}_{-_2}.\end{aligned}$$

In the language of relations, given $a \in A$ and $b \in B$:

- The equality
$$\chi_B \diamond R = R$$
witnesses the equivalence of the following two statements:
    1. We have $a \sim_b B$.
    2. There exists some $b' \in B$ such that:
        (a) We have $a \sim_R b'$
        (b) We have $b' \sim_{\chi_B} b$, i.e. $b' = b$.
- The equality
$$R \diamond \chi_A = R$$
witnesses the equivalence of the following two statements:
    1. There exists some $a' \in A$ such that:
        (a) We have $a \sim_{\chi_B} a'$, i.e. $a = a'$.
        (b) We have $a' \sim_R b$
    2. We have $a \sim_b B$.

Item 4, Interaction With Inverses: Clear.
Item 5, Interaction With Composition: Clear. □

6.3.13 The Collage of a Relation

Let A and B be sets and let $R\colon A \nrightarrow B$ be a relation from A to B.

Definition 6.3.13.1.1. The **collage of** R[15] is the poset $\mathbf{Coll}(R) \overset{\text{def}}{=} \left(\mathrm{Coll}(R), \preceq_{\mathbf{Coll}(R)}\right)$ consisting of:

- *The Underlying Set.* The set $\mathrm{Coll}(R)$ defined by

$$\mathrm{Coll}(R) \overset{\text{def}}{=} A \coprod B.$$

- *The Partial Order.* The partial order

$$\preceq_{\mathbf{Coll}(R)}\colon \mathrm{Coll}(R) \times \mathrm{Coll}(R) \to \{\mathsf{true}, \mathsf{false}\}$$

on $\mathrm{Coll}(R)$ defined by

$$\preceq(a, b) \overset{\text{def}}{=} \begin{cases} \mathsf{true} & \text{if } a = b \text{ or } a \sim_R b, \\ \mathsf{false} & \text{otherwise.} \end{cases}$$

Proposition 6.3.13.1.2. Let A and B be sets and let $R\colon A \nrightarrow B$ be a relation from A to B.

1. *Functoriality I.* The assignment $R \mapsto \mathbf{Coll}(R)$ defines a functor[16]

$$\mathbf{Coll}\colon \mathbf{Rel}(A, B) \to \mathsf{Pos}_{/\Delta^1}(A, B),$$

[15] *Further Terminology:* Also called the **cograph of** R.

[16] Here $\mathsf{Pos}_{/\Delta^1}(A, B)$ is the category defined as the pullback

$$\mathsf{Pos}_{/\Delta^1}(A, B) \overset{\text{def}}{=} \mathrm{pt} \underset{[A],\mathsf{Pos},\mathrm{fib}_0}{\times} \mathsf{Pos}_{/\Delta^1} \underset{\mathrm{fib}_1,\mathsf{Pos},[B]}{\times} \mathrm{pt},$$

as in the diagram

$$\begin{tikzcd}[column sep=small]
 & & & \mathsf{Pos}_{/\Delta^1}(A,B) \arrow[dl] \arrow[dr] \arrow[dd, phantom, very near start, "\lrcorner" rotate=-45] & & & \\
 & & \mathsf{Pos}_{/\Delta^1} \underset{\mathsf{Pos}}{\times} \mathrm{pt} \arrow[dl] \arrow[dr] \arrow[dd, phantom, very near start, "\lrcorner" rotate=-45] & & \mathrm{pt} \underset{\mathsf{Pos}}{\times} \mathsf{Pos}_{/\Delta^1} \arrow[dl] \arrow[dr] \arrow[dd, phantom, very near start, "\lrcorner" rotate=-45] & & \\
 & \mathrm{pt} \arrow[dr, "{[A]}"'] & & \mathsf{Pos}_{/\Delta^1} \arrow[dl, "\mathrm{fib}_{[0]}"] \arrow[dr, "\mathrm{fib}_{[1]}"'] & & \mathrm{pt}. \arrow[dl, "{[B]}"] & \\
 & & \mathsf{Pos} & & \mathsf{Pos} & &
\end{tikzcd}$$

Explicitly, an object of $\mathsf{Pos}_{/\Delta^1}(A, B)$ is a pair (X, ϕ_X) consisting of

where

- *Action on Objects.* For each $R \in \mathrm{Obj}(\mathbf{Rel}(A, B))$, we have

$$[\mathbf{Coll}](R) \stackrel{\text{def}}{=} (\mathbf{Coll}(R), \phi_R)$$

for each $R \in \mathbf{Rel}(A, B)$, where

  – The poset $\mathbf{Coll}(R)$ is the collage of R of Definition 6.3.13.1.1.
  – The morphism $\phi_R \colon \mathbf{Coll}(R) \to \Delta^1$ is given by

$$\phi_R(x) \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if } x \in A, \\ 1 & \text{if } x \in B \end{cases}$$

  for each $x \in \mathbf{Coll}(R)$.

- *Action on Morphisms.* For each $R, S \in \mathrm{Obj}(\mathbf{Rel}(A, B))$, the action on Hom-sets

$$\mathbf{Coll}_{R,S} \colon \mathrm{Hom}_{\mathbf{Rel}(A,B)}(R, S) \to \mathsf{Pos}(\mathbf{Coll}(R), \mathbf{Coll}(S))$$

of $\mathbf{Coll}$ at (R, S) is given by sending an inclusion

$$\iota \colon R \subset S$$

to the morphism

$$\mathbf{Coll}(\iota) \colon \mathbf{Coll}(R) \to \mathbf{Coll}(S)$$

of posets over Δ^1 defined by

$$[\mathbf{Coll}(\iota)](x) \stackrel{\text{def}}{=} x$$

for each $x \in \mathbf{Coll}(R)$.[17]

2. *Equivalence.* The functor of Item 1 is an equivalence of categories.

Proof. Item 1, Functoriality: Clear.
Item 2, Equivalence: Omitted. □

- A poset X;
- A morphism $\phi_X \colon X \to \Delta^1$;

such that $\phi_X^{-1}(0) = A$ and $\phi_X^{-1}(0) = B$, with morphisms between such objects being morphisms of posets over Δ^1.

[17] Note that this is indeed a morphism of posets: if $x \preceq_{\mathbf{Coll}(R)} y$, then $x = y$ or $x \sim_R y$, so we have either $x = y$ or $x \sim_S y$ (as $R \subset S$), and thus $x \preceq_{\mathbf{Coll}(S)} y$.

6.4 Functoriality of Powersets

6.4.1 Direct Images

Let A and B be sets and let $R\colon A \nrightarrow B$ be a relation.

Definition 6.4.1.1.1. The **direct image function associated to** R is the function

$$R_*\colon \mathcal{P}(A) \to \mathcal{P}(B)$$

defined by[18,19]

$$\begin{aligned} R_*(U) &\overset{\text{def}}{=} R(U) \\ &\overset{\text{def}}{=} \bigcup_{a\in U} R(a) \\ &= \left\{ b \in B \,\middle|\, \begin{array}{l} \text{there exists some } a \in U \\ \text{such that } b \in R(a) \end{array} \right\} \end{aligned}$$

for each $U \in \mathcal{P}(A)$.

Remark 6.4.1.1.2. Identifying subsets of A with relations from pt to A via Item 3 of Proposition 2.4.3.1.6, we see that the direct image function associated to R is equivalently the function

$$R_*\colon \underbrace{\mathcal{P}(A)}_{\cong \mathrm{Rel}(\mathrm{pt},A)} \to \underbrace{\mathcal{P}(B)}_{\cong \mathrm{Rel}(\mathrm{pt},B)}$$

defined by

$$R_*(U) \overset{\text{def}}{=} R \diamond U$$

for each $U \in \mathcal{P}(A)$, where $R \diamond U$ is the composition

$$\mathrm{pt} \overset{U}{\nrightarrow} A \overset{R}{\nrightarrow} B.$$

Proposition 6.4.1.1.3. Let $R\colon A \nrightarrow B$ be a relation.

1. *Functoriality.* The assignment $U \mapsto R_*(U)$ defines a functor

$$R_*\colon (\mathcal{P}(A), \subset) \to (\mathcal{P}(B), \subset)$$

where

[18] *Further Terminology:* The set $R(U)$ is called the **direct image of** U **by** R.

[19] We also have

$$R_*(U) = B \setminus R_!(A \setminus U);$$

- *Action on Objects.* For each $U \in \mathcal{P}(A)$, we have

$$[R_*](U) \stackrel{\text{def}}{=} R_*(U).$$

- *Action on Morphisms.* For each $U, V \in \mathcal{P}(A)$:
  - If $U \subset V$, then $R_*(U) \subset R_*(V)$.

2. *Adjointness.* We have an adjunction

$$(R_* \dashv R_{-1})\colon \quad \begin{tikzcd} \mathcal{P}(A) \arrow[r, bend left, "R_*"] \arrow[r, phantom, "\perp"] & \mathcal{P}(B), \arrow[l, bend left, "R_{-1}"] \end{tikzcd}$$

witnessed by a bijections of sets

$$\mathrm{Hom}_{\mathcal{P}(A)}(R_*(U), V) \cong \mathrm{Hom}_{\mathcal{P}(A)}(U, R_{-1}(V)),$$

natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$, i.e. such that:

($\star$) The following conditions are equivalent:
  - We have $R_*(U) \subset V$.
  - We have $U \subset R_{-1}(V)$.

3. *Preservation of Colimits.* We have an equality of sets

$$R_*\left(\bigcup_{i \in I} U_i\right) = \bigcup_{i \in I} R_*(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have equalities

$$R_*(U) \cup R_*(V) = R_*(U \cup V),$$
$$R_*(\emptyset) = \emptyset,$$

natural in $U, V \in \mathcal{P}(A)$.

4. *Oplax Preservation of Limits.* We have an inclusion of sets

$$R_*\left(\bigcap_{i \in I} U_i\right) \subset \bigcap_{i \in I} R_*(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have inclusions

$$R_*(U \cap V) \subset R_*(U) \cap R_*(V),$$
$$R_*(A) \subset B,$$

natural in $U, V \in \mathcal{P}(A)$.

5. *Symmetric Strict Monoidality With Respect to Unions.* The direct image function of Item 1 has a symmetric strict monoidal structure

$$\left(R_*, R_*^{\otimes}, R_{*|\mathbb{1}}^{\otimes}\right): (\mathcal{P}(A), \cup, \emptyset) \to (\mathcal{P}(B), \cup, \emptyset),$$

being equipped with equalities

$$R_{*|U,V}^{\otimes}: R_*(U) \cup R_*(V) \xrightarrow{=} R_*(U \cup V),$$
$$R_{*|\mathbb{1}}^{\otimes}: \emptyset \xrightarrow{=} \emptyset,$$

natural in $U, V \in \mathcal{P}(A)$.

6. *Symmetric Oplax Monoidality With Respect to Intersections.* The direct image function of Item 1 has a symmetric oplax monoidal structure

$$\left(R_*, R_*^{\otimes}, R_{*|\mathbb{1}}^{\otimes}\right): (\mathcal{P}(A), \cap, A) \to (\mathcal{P}(B), \cap, B),$$

being equipped with inclusions

$$R_{*|U,V}^{\otimes}: R_*(U \cap V) \subset R_*(U) \cap R_*(V),$$
$$R_{*|\mathbb{1}}^{\otimes}: R_*(A) \subset B,$$

natural in $U, V \in \mathcal{P}(A)$.

7. *Relation to Direct Images With Compact Support.* We have

$$R_*(U) = B \setminus R_!(A \setminus U)$$

for each $U \in \mathcal{P}(A)$.

Proof. *Item 1, Functoriality*: Clear.
Item 2, Adjointness: This follows from **??** of **??**.
Item 3, Preservation of Colimits: This follows from Item 2 and **??** of **??**.
Item 4, Oplax Preservation of Limits: Omitted.
Item 5, Symmetric Strict Monoidality With Respect to Unions: This follows from Item 3.
Item 6, Symmetric Oplax Monoidality With Respect to Intersections: This follows from Item 4.
Item 7, Relation to Direct Images With Compact Support: The proof proceeds in the same way as in the case of functions (**??** of Proposition 2.4.4.1.4): applying Item 7 of Proposition 6.4.4.1.3 to $A \setminus U$, we have

$$\begin{aligned} R_!(A \setminus U) &= B \setminus R_*(A \setminus (A \setminus U)) \\ &= B \setminus R_*(U). \end{aligned}$$

Taking complements, we then obtain

$$\begin{aligned} R_*(U) &= B \setminus (B \setminus R_*(U)), \\ &= B \setminus R_!(A \setminus U), \end{aligned}$$

which finishes the proof. □

Proposition 6.4.1.1.4. Let $R\colon A \nrightarrow B$ be a relation.

1. *Functionality I.* The assignment $R \mapsto R_*$ defines a function

$$(-)_*\colon \mathrm{Rel}(A, B) \to \mathsf{Sets}(\mathcal{P}(A), \mathcal{P}(B)).$$

2. *Functionality II.* The assignment $R \mapsto R_*$ defines a function

$$(-)_*\colon \mathrm{Rel}(A, B) \to \mathsf{Pos}((\mathcal{P}(A), \subset), (\mathcal{P}(B), \subset)).$$

3. *Interaction With Identities.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have[20]

$$(\chi_A)_* = \mathrm{id}_{\mathcal{P}(A)}.$$

4. *Interaction With Composition.* For each pair of composable relations $R\colon A \nrightarrow B$ and $S\colon B \nrightarrow C$, we have[21]

$$(S \diamond R)_* = S_* \circ R_*, \qquad \begin{tikzcd} \mathcal{P}(A) \arrow[r, "R_*"] \arrow[dr, "(S\diamond R)_*"'] & \mathcal{P}(B) \arrow[d, "S_*"] \\ & \mathcal{P}(C). \end{tikzcd}$$

Proof. *Item 1, Functionality I*: Clear.

see Item 7 of Proposition 6.4.1.1.3.

[20] That is, the postcomposition function

$$(\chi_A)_*\colon \mathrm{Rel}(\mathrm{pt}, A) \to \mathrm{Rel}(\mathrm{pt}, A)$$

is equal to $\mathrm{id}_{\mathrm{Rel}(\mathrm{pt},A)}$.

[21] That is, we have

$$(S \diamond R)_* = S_* \circ R_*, \qquad \begin{tikzcd} \mathrm{Rel}(\mathrm{pt}, A) \arrow[r, "R_*"] \arrow[dr, "(S\diamond R)_*"'] & \mathrm{Rel}(\mathrm{pt}, B) \arrow[d, "S_*"] \\ & \mathrm{Rel}(\mathrm{pt}, C). \end{tikzcd}$$

Item 2, Functionality II: Clear.
Item 3, Interaction With Identities: Indeed, we have

$$\begin{aligned}(\chi_A)_*(U) &\overset{\text{def}}{=} \bigcup_{a\in U} \chi_A(a)\\ &\overset{\text{def}}{=} \bigcup_{a\in U} \{a\}\\ &= U\\ &\overset{\text{def}}{=} \mathrm{id}_{\mathcal{P}(A)}(U)\end{aligned}$$

for each $U \in \mathcal{P}(A)$. Thus $(\chi_A)_* = \mathrm{id}_{\mathcal{P}(A)}$.
Item 4, Interaction With Composition: Indeed, we have

$$\begin{aligned}(S \diamond R)_*(U) &\overset{\text{def}}{=} \bigcup_{a\in U} [S \diamond R](a)\\ &\overset{\text{def}}{=} \bigcup_{a\in U} S(R(a))\\ &\overset{\text{def}}{=} \bigcup_{a\in U} S_*(R(a))\\ &= S_*\left(\bigcup_{a\in U} R(a)\right)\\ &\overset{\text{def}}{=} S_*(R_*(U))\\ &\overset{\text{def}}{=} [S_* \circ R_*](U)\end{aligned}$$

for each $U \in \mathcal{P}(A)$, where we used Item 3 of Proposition 6.4.1.1.3. Thus $(S \diamond R)_* = S_* \circ R_*$. □

6.4.2 Strong Inverse Images

Let A and B be sets and let $R \colon A \nrightarrow B$ be a relation.

Definition 6.4.2.1.1. The **strong inverse image function associated to** R is the function

$$R_{-1} \colon \mathcal{P}(B) \to \mathcal{P}(A)$$

defined by[22]

$$R_{-1}(V) \overset{\text{def}}{=} \{a \in A \mid R(a) \subset V\}$$

for each $V \in \mathcal{P}(B)$.

[22] *Further Terminology:* The set $R_{-1}(V)$ is called the **strong inverse image of** V **by** R.

Remark 6.4.2.1.2. Identifying subsets of B with relations from pt to B via Item 3 of Proposition 2.4.3.1.6, we see that the inverse image function associated to R is equivalently the function

$$R_{-1}\colon \underbrace{\mathcal{P}(B)}_{\cong\mathrm{Rel}(\mathrm{pt},B)} \to \underbrace{\mathcal{P}(A)}_{\cong\mathrm{Rel}(\mathrm{pt},A)}$$

defined by

$$R_{-1}(V) \stackrel{\mathrm{def}}{=} \mathrm{Rift}_R(V), \qquad \begin{tikzcd} & A \arrow[d, "R", "\shortmid"{marking}] \\ \mathrm{pt} \arrow[ur, dashed, "\mathrm{Rift}_R(V)", "\shortmid"{marking}] \arrow[r, "V"', "\shortmid"{marking}] & B, \arrow[ul, phantom, "\Leftarrow"] \end{tikzcd}$$

and being explicitly computed by

$$\begin{aligned} R_{-1}(V) &\stackrel{\mathrm{def}}{=} \mathrm{Rift}_R(V) \\ &\cong \int_{b\in B} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^b_{-_1}, V^b_{-_2}\right), \end{aligned}$$

where we have used Proposition 6.2.4.1.1.

Proof. We have

$$\begin{aligned}
\mathrm{Rift}_R(V) &\cong \int_{b\in B} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^b_{-_1}, V^b_{-_2}\right) \\
&= \left\{ a \in A \,\middle|\, \int_{b\in B} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^b_a, V^b_\star\right) = \mathsf{true} \right\} \\
&= \left\{ a \in A \,\middle|\, \begin{array}{l} \text{for each } b \in B \text{, at least one of the} \\ \text{following conditions hold:} \\ \quad 1.\ \text{We have } R^b_a = \mathsf{false} \\ \quad 2.\ \text{The following conditions hold:} \\ \qquad (a)\ \text{We have } R^b_a = \mathsf{true} \\ \qquad (b)\ \text{We have } V^b_\star = \mathsf{true} \end{array} \right\} \\
&= \left\{ a \in A \,\middle|\, \begin{array}{l} \text{for each } b \in B \text{, at least one of the} \\ \text{following conditions hold:} \\ \quad 1.\ \text{We have } b \notin R(a) \\ \quad 2.\ \text{The following conditions hold:} \\ \qquad (a)\ \text{We have } b \in R(a) \\ \qquad (b)\ \text{We have } b \in V \end{array} \right\} \\
&= \{a \in A \mid \text{for each } b \in R(a)\text{, we have } b \in V\} \\
&= \{a \in A \mid R(a) \subset V\} \\
&\overset{\text{def}}{=} R_{-1}(V).
\end{aligned}$$

This finishes the proof. □

Proposition 6.4.2.1.3. Let $R\colon A \nrightarrow B$ be a relation.

1. *Functoriality.* The assignment $V \mapsto R_{-1}(V)$ defines a functor

$$R_{-1}\colon (\mathcal{P}(B), \subset) \to (\mathcal{P}(A), \subset)$$

where

- *Action on Objects.* For each $V \in \mathcal{P}(B)$, we have

$$[R_{-1}](V) \overset{\text{def}}{=} R_{-1}(V).$$

- *Action on Morphisms.* For each $U, V \in \mathcal{P}(B)$:
  - If $U \subset V$, then $R_{-1}(U) \subset R_{-1}(V)$.

2. *Adjointness.* We have an adjunction

$$(R_* \dashv R_{-1})\colon \quad \begin{tikzcd} \mathcal{P}(A) \arrow[r, bend left, "R_*"] \arrow[r, phantom, "\perp"] & \mathcal{P}(B), \arrow[l, bend left, "R_{-1}"] \end{tikzcd}$$

 witnessed by a bijections of sets

$$\mathrm{Hom}_{\mathcal{P}(A)}(R_*(U), V) \cong \mathrm{Hom}_{\mathcal{P}(A)}(U, R_{-1}(V)),$$

natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$, i.e. such that:

(★) The following conditions are equivalent:
   - We have $R_*(U) \subset V$.
   - We have $U \subset R_{-1}(V)$.

3. *Lax Preservation of Colimits.* We have an inclusion of sets

$$\bigcup_{i\in I} R_{-1}(U_i) \subset R_{-1}\left(\bigcup_{i\in I} U_i\right),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(B)^{\times I}$. In particular, we have inclusions

$$R_{-1}(U) \cup R_{-1}(V) \subset R_{-1}(U \cup V),$$
$$\emptyset \subset R_{-1}(\emptyset),$$

natural in $U, V \in \mathcal{P}(B)$.

4. *Preservation of Limits.* We have an equality of sets

$$R_{-1}\left(\bigcap_{i\in I} U_i\right) = \bigcap_{i\in I} R_{-1}(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(B)^{\times I}$. In particular, we have equalities

$$R_{-1}(U \cap V) = R_{-1}(U) \cap R_{-1}(V),$$
$$R_{-1}(B) = B,$$

natural in $U, V \in \mathcal{P}(B)$.

5. *Symmetric Lax Monoidality With Respect to Unions.* The direct image with compact support function of Item 1 has a symmetric lax monoidal structure

$$\left(R_{-1}, R_{-1}^{\otimes}, R_{-1|\mathbb{1}}^{\otimes}\right)\colon (\mathcal{P}(A), \cup, \emptyset) \to (\mathcal{P}(B), \cup, \emptyset),$$

being equipped with inclusions

$$R^{\otimes}_{-1|U,V}\colon R_{-1}(U)\cup R_{-1}(V)\subset R_{-1}(U\cup V),$$
$$R^{\otimes}_{-1|\mathbb{1}}\colon \emptyset\subset R_{-1}(\emptyset),$$

natural in $U, V \in \mathcal{P}(B)$.

6. *Symmetric Strict Monoidality With Respect to Intersections.* The direct image function of Item 1 has a symmetric strict monoidal structure

$$\left(R_{-1}, R^{\otimes}_{-1}, R^{\otimes}_{-1|\mathbb{1}}\right)\colon (\mathcal{P}(A), \cap, A)\longrightarrow(\mathcal{P}(B), \cap, B),$$

being equipped with equalities

$$R^{\otimes}_{-1|U,V}\colon R_{-1}(U\cap V)\xrightarrow{=}R_{-1}(U)\cap R_{-1}(V),$$
$$R^{\otimes}_{-1|\mathbb{1}}\colon R_{-1}(A)\xrightarrow{=}B,$$

natural in $U, V \in \mathcal{P}(B)$.

7. *Interaction With Weak Inverse Images I.* We have

$$R_{-1}(V)=A\setminus R^{-1}(B\setminus V)$$

for each $V\in\mathcal{P}(B)$.

8. *Interaction With Weak Inverse Images II.* Let $R\colon A \nrightarrow B$ be a relation from A to B.

    (a) If R is a total relation, then we have an inclusion of sets

    $$R_{-1}(V)\subset R^{-1}(V)$$

    natural in $V\in\mathcal{P}(B)$.

    (b) If R is total and functional, then the above inclusion is in fact an equality.

    (c) Conversely, if we have $R_{-1}=R^{-1}$, then R is total and functional.

Proof. *Item 1*, *Functoriality*: Clear.
Item 2, *Adjointness*: This follows from **??** of **??**.
Item 3, *Lax Preservation of Colimits*: Omitted.
Item 4, *Preservation of Limits*: This follows from Item 2 and **??** of **??**.
Item 5, *Symmetric Lax Monoidality With Respect to Unions*: This follows from Item 3.

Item 6, Symmetric Strict Monoidality With Respect to Intersections: This follows from Item 4.
Item 7, Interaction With Weak Inverse Images I: We claim we have an equality

$$R_{-1}(B \setminus V) = A \setminus R^{-1}(V).$$

Indeed, we have

$$\begin{aligned} R_{-1}(B \setminus V) &= \{a \in A \mid R(a) \subset B \setminus V\}, \\ A \setminus R^{-1}(V) &= \{a \in A \mid R(a) \cap V = \emptyset\}. \end{aligned}$$

Taking $V = B \setminus V$ then implies the original statement.
Item 8, Interaction With Weak Inverse Images II: Item 8a is clear, while Items 8b and 8c follow from Item 6 of Proposition 6.3.1.1.2. □

Proposition 6.4.2.1.4. Let $R: A \nrightarrow B$ be a relation.

1. *Functoriality I.* The assignment $R \mapsto R_{-1}$ defines a function

$$(-)_{-1}: \mathsf{Sets}(A, B) \to \mathsf{Sets}(\mathcal{P}(A), \mathcal{P}(B)).$$

2. *Functoriality II.* The assignment $R \mapsto R_{-1}$ defines a function

$$(-)_{-1}: \mathsf{Sets}(A, B) \to \mathsf{Pos}((\mathcal{P}(A), \subset), (\mathcal{P}(B), \subset)).$$

3. *Interaction With Identities.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$(\mathrm{id}_A)_{-1} = \mathrm{id}_{\mathcal{P}(A)}.$$

4. *Interaction With Composition.* For each pair of composable relations $R: A \nrightarrow B$ and $S: B \nrightarrow C$, we have

$$(S \diamond R)_{-1} = R_{-1} \circ S_{-1}, \qquad \begin{tikzcd} \mathcal{P}(C) \arrow[r, "S_{-1}"] \arrow[dr, "(S\diamond R)_{-1}"'] & \mathcal{P}(B) \arrow[d, "R_{-1}"] \\ & \mathcal{P}(A). \end{tikzcd}$$

Proof. *Item 1, Functoriality I*: Clear.
Item 2, Functoriality II: Clear.
Item 3, Interaction With Identities: Indeed, we have

$$\begin{aligned} (\chi_A)_{-1}(U) &\overset{\text{def}}{=} \{a \in A \mid \chi_A(a) \subset U\} \\ &\overset{\text{def}}{=} \{a \in A \mid \{a\} \subset U\} \\ &= U \end{aligned}$$

for each $U \in \mathcal{P}(A)$. Thus $(\chi_A)_{-1} = \mathrm{id}_{\mathcal{P}(A)}$.
Item 4, Interaction With Composition: Indeed, we have

$$\begin{aligned}
(S \diamond R)_{-1}(U) &\overset{\text{def}}{=} \{a \in A \mid [S \diamond R](a) \subset U\} \\
&\overset{\text{def}}{=} \{a \in A \mid S(R(a)) \subset U\} \\
&\overset{\text{def}}{=} \{a \in A \mid S_*(R(a)) \subset U\} \\
&= \{a \in A \mid R(a) \subset S_{-1}(U)\} \\
&\overset{\text{def}}{=} R_{-1}(S_{-1}(U)) \\
&\overset{\text{def}}{=} [R_{-1} \circ S_{-1}](U)
\end{aligned}$$

for each $U \in \mathcal{P}(C)$, where we used Item 2 of Proposition 6.4.2.1.3, which implies that the conditions

- We have $S_*(R(a)) \subset U$.
- We have $R(a) \subset S_{-1}(U)$.

are equivalent. Thus $(S \diamond R)_{-1} = R_{-1} \circ S_{-1}$. □

6.4.3 Weak Inverse Images

Let A and B be sets and let $R\colon A \nrightarrow B$ be a relation.

Definition 6.4.3.1.1. The **weak inverse image function associated to** R[23] is the function

$$R^{-1}\colon \mathcal{P}(B) \to \mathcal{P}(A)$$

defined by[24]

$$R^{-1}(V) \overset{\text{def}}{=} \{a \in A \mid R(a) \cap V \neq \emptyset\}$$

for each $V \in \mathcal{P}(B)$.

Remark 6.4.3.1.2. Identifying subsets of B with relations from B to pt via Item 3 of Proposition 2.4.3.1.6, we see that the weak inverse image function associated to R is equivalently the function

$$R^{-1}\colon \underbrace{\mathcal{P}(B)}_{\cong \mathrm{Rel}(B,\mathrm{pt})} \to \underbrace{\mathcal{P}(A)}_{\cong \mathrm{Rel}(A,\mathrm{pt})}$$

defined by

$$R^{-1}(V) \overset{\text{def}}{=} V \diamond R$$

[23] *Further Terminology:* Also called simply the **inverse image function associated to** R.
[24] *Further Terminology:* The set $R^{-1}(V)$ is called the **weak inverse image of** V **by** R or

for each $V \in \mathcal{P}(A)$, where $R \diamond V$ is the composition

$$A \overset{R}{\nrightarrow} B \overset{V}{\nrightarrow} \mathrm{pt}.$$

Explicitly, we have

$$\begin{aligned} R^{-1}(V) &\stackrel{\text{def}}{=} V \diamond R \\ &\stackrel{\text{def}}{=} \int^{b \in B} V_b^{-1} \times R^b_{-_2}. \end{aligned}$$

Proof. We have

$$\begin{aligned} V \diamond R &\stackrel{\text{def}}{=} \int^{b \in B} V_b^{-1} \times R^b_{-_2} \\ &= \left\{ a \in A \,\middle|\, \int^{b \in B} V_b^{\star} \times R^b_a = \mathsf{true} \right\} \\ &= \left\{ a \in A \,\middle|\, \begin{array}{l} \text{there exists } b \in B \text{ such that the} \\ \text{following conditions hold:} \\ \quad 1. \ \text{We have } V_b^{\star} = \mathsf{true} \\ \quad 2. \ \text{We have } R^b_a = \mathsf{true} \end{array} \right\} \\ &= \left\{ a \in A \,\middle|\, \begin{array}{l} \text{there exists } b \in B \text{ such that the} \\ \text{following conditions hold:} \\ \quad 1. \ \text{We have } b \in V \\ \quad 2. \ \text{We have } b \in R(a) \end{array} \right\} \\ &= \{ a \in A \mid \text{there exists } b \in V \text{ such that } b \in R(a) \} \\ &= \{ a \in A \mid R(a) \cap V \neq \emptyset \} \\ &\stackrel{\text{def}}{=} R^{-1}(V) \end{aligned}$$

This finishes the proof. □

Proposition 6.4.3.1.3. Let $R \colon A \nrightarrow B$ be a relation.

1. *Functoriality.* The assignment $V \mapsto R^{-1}(V)$ defines a functor

$$R^{-1} \colon (\mathcal{P}(B), \subset) \to (\mathcal{P}(A), \subset)$$

   where

- *Action on Objects.* For each $V \in \mathcal{P}(B)$, we have
$$\left[R^{-1}\right](V) \stackrel{\text{def}}{=} R^{-1}(V).$$
- *Action on Morphisms.* For each $U, V \in \mathcal{P}(B)$:
  - If $U \subset V$, then $R^{-1}(U) \subset R^{-1}(V)$.

2. *Adjointness.* We have an adjunction
$$\left(R^{-1} \dashv R_!\right)\colon \quad \begin{tikzcd} \mathcal{P}(B) \arrow[r, bend left, "R^{-1}"] \arrow[r, phantom, "\perp"] & \mathcal{P}(A), \arrow[l, bend left, "R_!"] \end{tikzcd}$$
witnessed by a bijections of sets
$$\mathrm{Hom}_{\mathcal{P}(A)}\left(R^{-1}(U), V\right) \cong \mathrm{Hom}_{\mathcal{P}(A)}(U, R_!(V)),$$
natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$, i.e. such that:
   (★) The following conditions are equivalent:
   - We have $R^{-1}(U) \subset V$.
   - We have $U \subset R_!(V)$.

3. *Preservation of Colimits.* We have an equality of sets
$$R^{-1}\left(\bigcup_{i \in I} U_i\right) = \bigcup_{i \in I} R^{-1}(U_i),$$
natural in $\{U_i\}_{i \in I} \in \mathcal{P}(B)^{\times I}$. In particular, we have equalities
$$R^{-1}(U) \cup R^{-1}(V) = R^{-1}(U \cup V),$$
$$R^{-1}(\emptyset) = \emptyset,$$
natural in $U, V \in \mathcal{P}(B)$.

4. *Oplax Preservation of Limits.* We have an inclusion of sets
$$R^{-1}\left(\bigcap_{i \in I} U_i\right) \subset \bigcap_{i \in I} R^{-1}(U_i),$$
natural in $\{U_i\}_{i \in I} \in \mathcal{P}(B)^{\times I}$. In particular, we have inclusions
$$R^{-1}(U \cap V) \subset R^{-1}(U) \cap R^{-1}(V),$$
$$R^{-1}(A) \subset B,$$
natural in $U, V \in \mathcal{P}(B)$.

5. *Symmetric Strict Monoidality With Respect to Unions.* The direct image function of Item 1 has a symmetric strict monoidal structure

$$\left(R^{-1}, R^{-1,\otimes}, R^{-1,\otimes}_{\mathbb{1}}\right) \colon (\mathcal{P}(A), \cup, \emptyset) \to (\mathcal{P}(B), \cup, \emptyset),$$

being equipped with equalities

$$R^{-1,\otimes}_{U,V} \colon R^{-1}(U) \cup R^{-1}(V) \xrightarrow{=} R^{-1}(U \cup V),$$
$$R^{-1,\otimes}_{\mathbb{1}} \colon \emptyset \xrightarrow{=} \emptyset,$$

natural in $U, V \in \mathcal{P}(B)$.

6. *Symmetric Oplax Monoidality With Respect to Intersections.* The direct image function of Item 1 has a symmetric oplax monoidal structure

$$\left(R^{-1}, R^{-1,\otimes}, R^{-1,\otimes}_{\mathbb{1}}\right) \colon (\mathcal{P}(A), \cap, A) \to (\mathcal{P}(B), \cap, B),$$

being equipped with inclusions

$$R^{-1,\otimes}_{U,V} \colon R^{-1}(U \cap V) \subset R^{-1}(U) \cap R^{-1}(V),$$
$$R^{-1,\otimes}_{\mathbb{1}} \colon R^{-1}(A) \subset B,$$

natural in $U, V \in \mathcal{P}(B)$.

7. *Interaction With Strong Inverse Images I.* We have

$$R^{-1}(V) = A \setminus R_{-1}(B \setminus V)$$

for each $V \in \mathcal{P}(B)$.

8. *Interaction With Strong Inverse Images II.* Let $R \colon A \nrightarrow B$ be a relation from A to B.

   (a) If R is a total relation, then we have an inclusion of sets

   $$R_{-1}(V) \subset R^{-1}(V)$$

   natural in $V \in \mathcal{P}(B)$.

   (b) If R is total and functional, then the above inclusion is in fact an equality.

   (c) Conversely, if we have $R_{-1} = R^{-1}$, then R is total and functional.

simply the **inverse image of** V **by** R.

Proof. *Item 1, Functoriality*: Clear.
Item 2, Adjointness: This follows from **??** of **??**.
Item 3, Preservation of Colimits: This follows from Item 2 and **??** of **??**.
Item 4, Oplax Preservation of Limits: Omitted.
Item 5, Symmetric Strict Monoidality With Respect to Unions: This follows from Item 3.
Item 6, Symmetric Oplax Monoidality With Respect to Intersections: This follows from Item 4.
Item 7, Interaction With Strong Inverse Images I: This follows from Item 7 of Proposition 6.4.2.1.3.
Item 8, Interaction With Strong Inverse Images II: This was proved in Item 8 of Proposition 6.4.2.1.3. □

Proposition 6.4.3.1.4. Let $R\colon A \nrightarrow B$ be a relation.

1. *Functionality I.* The assignment $R \mapsto R^{-1}$ defines a function

$$(-)^{-1}\colon \mathrm{Rel}(A,B) \to \mathsf{Sets}(\mathcal{P}(A),\mathcal{P}(B)).$$

2. *Functionality II.* The assignment $R \mapsto R^{-1}$ defines a function

$$(-)^{-1}\colon \mathrm{Rel}(A,B) \to \mathsf{Pos}((\mathcal{P}(A),\subset),(\mathcal{P}(B),\subset)).$$

3. *Interaction With Identities.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have[25]

$$(\chi_A)^{-1} = \mathrm{id}_{\mathcal{P}(A)}.$$

4. *Interaction With Composition.* For each pair of composable relations

[25] That is, the postcomposition

$$(\chi_A)^{-1}\colon \mathrm{Rel}(\mathrm{pt},A) \to \mathrm{Rel}(\mathrm{pt},A)$$

is equal to $\mathrm{id}_{\mathrm{Rel}(\mathrm{pt},A)}$.

$R\colon A \mathrel{\mapstochar\!\!\rightarrow} B$ and $S\colon B \mathrel{\mapstochar\!\!\rightarrow} C$, we have[26]

$$(S \diamond R)^{-1} = R^{-1} \circ S^{-1}, \qquad \begin{tikzcd} \mathcal{P}(C) \arrow[r, "S^{-1}"] \arrow[dr, "(S\diamond R)^{-1}"'] & \mathcal{P}(B) \arrow[d, "R^{-1}"] \\ & \mathcal{P}(A). \end{tikzcd}$$

Proof. *Item 1, Functionality I*: Clear.
Item 2, Functionality II: Clear.
Item 3, Interaction With Identities: This follows from Item 5 of Proposition 8.1.6.1.2.
Item 4, Interaction With Composition: This follows from Item 2 of Proposition 8.1.6.1.2. □

6.4.4 Direct Images With Compact Support

Let A and B be sets and let $R\colon A \mathrel{\mapstochar\!\!\rightarrow} B$ be a relation.

Definition 6.4.4.1.1. The **direct image with compact support function associated to** R is the function

$$R_!\colon \mathcal{P}(A) \to \mathcal{P}(B)$$

defined by[27,28]

$$\begin{aligned} R_!(U) &\overset{\text{def}}{=} \left\{ b \in B \;\middle|\; \begin{array}{l} \text{for each } a \in A, \text{ if we have} \\ b \in R(a), \text{ then } a \in U \end{array} \right\} \\ &= \left\{ b \in B \;\middle|\; R^{-1}(b) \subset U \right\} \end{aligned}$$

for each $U \in \mathcal{P}(A)$.

[26] That is, we have

$$(S \diamond R)^{-1} = R^{-1} \circ S^{-1}, \qquad \begin{tikzcd} \mathrm{Rel}(\mathrm{pt}, C) \arrow[r, "R^{-1}"] \arrow[dr, "(S\diamond R)^{-1}"'] & \mathrm{Rel}(\mathrm{pt}, B) \arrow[d, "S^{-1}"] \\ & \mathrm{Rel}(\mathrm{pt}, A). \end{tikzcd}$$

[27] *Further Terminology:* The set $R_!(U)$ is called the **direct image with compact support of** U **by** R.

[28] We also have

$$R_!(U) = B \setminus R_*(A \setminus U);$$

Remark 6.4.4.1.2. Identifying subsets of B with relations from pt to B via Item 3 of Proposition 2.4.3.1.6, we see that the direct image with compact support function associated to R is equivalently the function

$$R_! \colon \underbrace{\mathcal{P}(A)}_{\cong \mathrm{Rel}(A,\mathrm{pt})} \to \underbrace{\mathcal{P}(B)}_{\cong \mathrm{Rel}(B,\mathrm{pt})}$$

defined by

$$R_!(U) \stackrel{\text{def}}{=} \mathrm{Ran}_R(U), \qquad \begin{tikzcd} & B \arrow[d, dashed, "\mathrm{Ran}_R(U)"] \\ A \arrow[ur, "R", "\times"' marking] \arrow[r, "U"', "|" marking] & \mathrm{pt}, \arrow[from=1-2, to=2-1, phantom, "\Leftarrow"] \end{tikzcd}$$

being explicitly computed by

$$\begin{aligned} R^*(U) &\stackrel{\text{def}}{=} \mathrm{Ran}_R(U) \\ &\cong \int_{a\in A} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R_a^{-2}, U_a^{-1}\right), \end{aligned}$$

where we have used Proposition 6.2.3.1.1.

see Item 7 of Proposition 6.4.4.1.3.

Proof. We have

$$
\begin{aligned}
\mathrm{Ran}_R(V) &\cong \int_{a\in A} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^{-_2}_a, U^{-_1}_a\right) \\
&= \left\{ b \in B \,\middle|\, \int_{a\in A} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^b_a, U^\star_a\right) = \mathsf{true} \right\} \\
&= \left\{ b \in B \,\middle|\, \begin{array}{l} \text{for each } a \in A\text{, at least one of the} \\ \text{following conditions hold:} \\ \quad 1.\ \text{We have } R^b_a = \mathsf{false} \\ \quad 2.\ \text{The following conditions hold:} \\ \qquad \text{(a) We have } R^b_a = \mathsf{true} \\ \qquad \text{(b) We have } U^\star_a = \mathsf{true} \end{array} \right\} \\
&= \left\{ b \in B \,\middle|\, \begin{array}{l} \text{for each } a \in A\text{, at least one of the} \\ \text{following conditions hold:} \\ \quad 1.\ \text{We have } b \notin R(A) \\ \quad 2.\ \text{The following conditions hold:} \\ \qquad \text{(a) We have } b \in R(a) \\ \qquad \text{(b) We have } a \in U \end{array} \right\} \\
&= \left\{ b \in B \,\middle|\, \begin{array}{l} \text{for each } a \in A\text{, if we have} \\ b \in R(a)\text{, then } a \in U \end{array} \right\} \\
&= \left\{ b \in B \,\middle|\, R^{-1}(b) \subset U \right\} \\
&\stackrel{\text{def}}{=} R^{-1}(U).
\end{aligned}
$$

This finishes the proof. □

Proposition 6.4.4.1.3. Let $R\colon A \nrightarrow B$ be a relation.

1. *Functoriality.* The assignment $U \mapsto R_!(U)$ defines a functor

$$R_!\colon (\mathcal{P}(A), \subset) \to (\mathcal{P}(B), \subset)$$

   where

   - *Action on Objects.* For each $U \in \mathcal{P}(A)$, we have

$$[R_!](U) \stackrel{\text{def}}{=} R_!(U).$$

   - *Action on Morphisms.* For each $U, V \in \mathcal{P}(A)$:

- If $U \subset V$, then $R_!(U) \subset R_!(V)$.

2. *Adjointness.* We have an adjunction

$$\left(R^{-1} \dashv R_!\right)\colon \quad \begin{tikzcd} \mathcal{P}(B) \arrow[r, bend left, "R^{-1}"] \arrow[r, phantom, "\perp"] & \mathcal{P}(A), \arrow[l, bend left, "R_!"] \end{tikzcd}$$

witnessed by a bijections of sets

$$\mathrm{Hom}_{\mathcal{P}(A)}\left(R^{-1}(U), V\right) \cong \mathrm{Hom}_{\mathcal{P}(A)}(U, R_!(V)),$$

natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$, i.e. such that:

($\star$) The following conditions are equivalent:
  - We have $R^{-1}(U) \subset V$.
  - We have $U \subset R_!(V)$.

3. *Lax Preservation of Colimits.* We have an inclusion of sets

$$\bigcup_{i\in I} R_!(U_i) \subset R_!\left(\bigcup_{i\in I} U_i\right),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have inclusions

$$R_!(U) \cup R_!(V) \subset R_!(U \cup V),$$
$$\emptyset \subset R_!(\emptyset),$$

natural in $U, V \in \mathcal{P}(A)$.

4. *Preservation of Limits.* We have an equality of sets

$$R_!\left(\bigcap_{i\in I} U_i\right) = \bigcap_{i\in I} R_!(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have equalities

$$R_!(U \cap V) = R_!(U) \cap R_!(V),$$
$$R_!(A) = B,$$

natural in $U, V \in \mathcal{P}(A)$.

5. *Symmetric Lax Monoidality With Respect to Unions.* The direct image

with compact support function of Item 1 has a symmetric lax monoidal structure

$$\left(R_!, R_!^{\otimes}, R_{!|\mathbb{1}}^{\otimes}\right) \colon (\mathcal{P}(A), \cup, \emptyset) \to (\mathcal{P}(B), \cup, \emptyset),$$

being equipped with inclusions

$$R_{!|U,V}^{\otimes} \colon R_!(U) \cup R_!(V) \subset R_!(U \cup V),$$
$$R_{!|\mathbb{1}}^{\otimes} \colon \emptyset \subset R_!(\emptyset),$$

natural in $U, V \in \mathcal{P}(A)$.

6. *Symmetric Strict Monoidality With Respect to Intersections.* The direct image function of Item 1 has a symmetric strict monoidal structure

$$\left(R_!, R_!^{\otimes}, R_{!|\mathbb{1}}^{\otimes}\right) \colon (\mathcal{P}(A), \cap, A) \to (\mathcal{P}(B), \cap, B),$$

being equipped with equalities

$$R_{!|U,V}^{\otimes} \colon R_!(U \cap V) \xrightarrow{=} R_!(U) \cap R_!(V),$$
$$R_{!|\mathbb{1}}^{\otimes} \colon R_!(A) \xrightarrow{=} B,$$

natural in $U, V \in \mathcal{P}(A)$.

7. *Relation to Direct Images.* We have

$$R_!(U) = B \setminus R_*(A \setminus U)$$

for each $U \in \mathcal{P}(A)$.

Proof. *Item 1, Functoriality*: Clear.
Item 2, Adjointness: This follows from **??** of **??**.
Item 3, Lax Preservation of Colimits: Omitted.
Item 4, Preservation of Limits: This follows from Item 2 and **??** of **??**.
Item 5, Symmetric Lax Monoidality With Respect to Unions: This follows from Item 3.
Item 6, Symmetric Strict Monoidality With Respect to Intersections: This follows from Item 4.
Item 7, Relation to Direct Images: This follows from Item 7 of Proposition 6.4.1.1.3. Alternatively, we may prove it directly as follows, with the proof proceeding in the same way as in the case of functions (Item 9 of Proposition 2.4.6.1.6). We claim that $R_!(U) = B \setminus R_*(A \setminus U)$:

- *The First Implication.* We claim that

$$R_!(U) \subset B \setminus R_*(A \setminus U).$$

Let $b \in R_!(U)$. We need to show that $b \notin R_*(A \setminus U)$, i.e. that there is no $a \in A \setminus U$ such that $b \in R(a)$.

This is indeed the case, as otherwise we would have $a \in R^{-1}(b)$ and $a \notin U$, contradicting $R^{-1}(b) \subset U$ (which holds since $b \in R_!(U)$).

Thus $b \in B \setminus R_*(A \setminus U)$.

- *The Second Implication.* We claim that

$$B \setminus R_*(A \setminus U) \subset R_!(U).$$

Let $b \in B \setminus R_*(A \setminus U)$. We need to show that $b \in R_!(U)$, i.e. that $R^{-1}(b) \subset U$.

Since $b \notin R_*(A \setminus U)$, there exists no $a \in A \setminus U$ such that $b \in R(a)$, and hence $R^{-1}(b) \subset U$.

Thus $b \in R_!(U)$.

This finishes the proof. □

Proposition 6.4.4.1.4. Let $R\colon A \nrightarrow B$ be a relation.

1. *Functionality I.* The assignment $R \mapsto R_!$ defines a function

$$(-)_!\colon \mathsf{Sets}(A, B) \to \mathsf{Sets}(\mathcal{P}(A), \mathcal{P}(B)).$$

2. *Functionality II.* The assignment $R \mapsto R_!$ defines a function

$$(-)_!\colon \mathsf{Sets}(A, B) \to \mathrm{Hom}_{\mathsf{Pos}}((\mathcal{P}(A), \subset), (\mathcal{P}(B), \subset)).$$

3. *Interaction With Identities.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$(\mathrm{id}_A)_! = \mathrm{id}_{\mathcal{P}(A)}.$$

4. *Interaction With Composition.* For each pair of composable relations $R\colon A \nrightarrow B$ and $S\colon B \nrightarrow C$, we have

$$(S \diamond R)_! = S_! \circ R_!, \qquad \begin{tikzcd} \mathcal{P}(A) \arrow[r, "R_!"] \arrow[dr, "(S\diamond R)_!"'] & \mathcal{P}(B) \arrow[d, "S_!"] \\ & \mathcal{P}(C). \end{tikzcd}$$

Proof. *Item 1, Functoriality I*: Clear.
Item 2, Functoriality II: Clear.
Item 3, Interaction With Identities: Indeed, we have

$$\begin{aligned}(\chi_A)_!(U) &\stackrel{\text{def}}{=} \left\{a \in A \,\middle|\, \chi_A^{-1}(a) \subset U\right\}\\ &\stackrel{\text{def}}{=} \{a \in A \mid \{a\} \subset U\}\\ &= U\end{aligned}$$

for each $U \in \mathcal{P}(A)$. Thus $(\chi_A)_! = \mathrm{id}_{\mathcal{P}(A)}$.
Item 4, Interaction With Composition: Indeed, we have

$$\begin{aligned}(S \diamond R)_!(U) &\stackrel{\text{def}}{=} \left\{c \in C \,\middle|\, [S \diamond R]^{-1}(c) \subset U\right\}\\ &\stackrel{\text{def}}{=} \left\{c \in C \,\middle|\, S^{-1}\left(R^{-1}(c)\right) \subset U\right\}\\ &= \left\{c \in C \,\middle|\, R^{-1}(c) \subset S_!(U)\right\}\\ &\stackrel{\text{def}}{=} R_!(S_!(U))\\ &\stackrel{\text{def}}{=} [R_! \circ S_!](U)\end{aligned}$$

for each $U \in \mathcal{P}(C)$, where we used Item 2 of Proposition 6.4.4.1.3, which implies that the conditions

- We have $S^{-1}\left(R^{-1}(c)\right) \subset U$.
- We have $R^{-1}(c) \subset S_!(U)$.

are equivalent. Thus $(S \diamond R)_! = S_! \circ R_!$. □

6.4.5 Functoriality of Powersets

Proposition 6.4.5.1.1. The assignment $X \mapsto \mathcal{P}(X)$ defines functors[29]

$$\begin{aligned}\mathcal{P}_* &\colon \mathsf{Rel} \to \mathsf{Sets},\\ \mathcal{P}_{-1} &\colon \mathsf{Rel}^{\mathsf{op}} \to \mathsf{Sets},\\ \mathcal{P}^{-1} &\colon \mathsf{Rel}^{\mathsf{op}} \to \mathsf{Sets},\\ \mathcal{P}_! &\colon \mathsf{Rel} \to \mathsf{Sets}\end{aligned}$$

where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathsf{Rel})$, we have

$$\begin{aligned}\mathcal{P}_*(A) &\stackrel{\text{def}}{=} \mathcal{P}(A),\\ \mathcal{P}_{-1}(A) &\stackrel{\text{def}}{=} \mathcal{P}(A),\\ \mathcal{P}^{-1}(A) &\stackrel{\text{def}}{=} \mathcal{P}(A),\\ \mathcal{P}_!(A) &\stackrel{\text{def}}{=} \mathcal{P}(A).\end{aligned}$$

[29]The functor $\mathcal{P}_* \colon \mathsf{Rel} \to \mathsf{Sets}$ admits a left adjoint; see Item 3 of Proposition 6.3.1.1.2.

- *Action on Morphisms.* For each morphism $R\colon A \nrightarrow B$ of Rel, the images

$$\begin{aligned}\mathcal{P}_*(R)\colon \mathcal{P}(A) &\to \mathcal{P}(B),\\ \mathcal{P}_{-1}(R)\colon \mathcal{P}(B) &\to \mathcal{P}(A),\\ \mathcal{P}^{-1}(R)\colon \mathcal{P}(B) &\to \mathcal{P}(A),\\ \mathcal{P}_!(R)\colon \mathcal{P}(A) &\to \mathcal{P}(B)\end{aligned}$$

  of R by $\mathcal{P}_*$, $\mathcal{P}_{-1}$, $\mathcal{P}^{-1}$, and $\mathcal{P}_!$ are defined by

$$\begin{aligned}\mathcal{P}_*(R) &\overset{\text{def}}{=} R_*,\\ \mathcal{P}_{-1}(R) &\overset{\text{def}}{=} R_{-1},\\ \mathcal{P}^{-1}(R) &\overset{\text{def}}{=} R^{-1},\\ \mathcal{P}_!(R) &\overset{\text{def}}{=} R_!,\end{aligned}$$

  as in Definitions 6.4.1.1.1, 6.4.2.1.1, 6.4.3.1.1 and 6.4.4.1.1.

Proof. This follows from Items 3 and 4 of Proposition 6.4.1.1.4, Items 3 and 4 of Proposition 6.4.2.1.4, Items 3 and 4 of Proposition 6.4.3.1.4, and Items 3 and 4 of Proposition 6.4.4.1.4. □

6.4.6 Functoriality of Powersets: Relations on Powersets

Let A and B be sets and let $R\colon A \nrightarrow B$ be a relation.

Definition 6.4.6.1.1. The **relation on powersets associated to** R is the relation

$$\mathcal{P}(R)\colon \mathcal{P}(A) \nrightarrow \mathcal{P}(B)$$

defined by[30]

$$\mathcal{P}(R)^V_U \overset{\text{def}}{=} \mathbf{Rel}\big(\chi_{\mathrm{pt}}, V \diamond R \diamond U\big)$$

for each $U \in \mathcal{P}(A)$ and each $V \in \mathcal{P}(B)$.

Remark 6.4.6.1.2. In detail, we have $U \sim_{\mathcal{P}(R)} V$ iff the following equivalent conditions hold:

- We have $\chi_{\mathrm{pt}} \subset V \diamond R \diamond U$.
- We have $(V \diamond R \diamond U)^\star_\star = \mathsf{true}$, i.e. we have

$$\int^{a\in A}\int^{b\in B} V^\star_b \times R^b_a \times U^a_\star = \mathsf{true}.$$

[30]Illustration:

$$\begin{tikzcd} \mathrm{pt} \arrow[r, "U"', "|" marking] \arrow[rrr, bend left, "\chi_{\mathrm{pt}}", "|" marking] & A \arrow[r, "R"', "|" marking] & B \arrow[r, "V"', "|" marking] & \mathrm{pt}. \end{tikzcd}$$

- There exists some $a \in A$ and some $b \in B$ such that:
  - We have $U^a_\star = \mathsf{true}$.
  - We have $R^b_a = \mathsf{true}$.
  - We have $V^\star_b = \mathsf{true}$.
- There exists some $a \in A$ and some $b \in B$ such that:
  - We have $a \in U$.
  - We have $a \sim_R b$.
  - We have $b \in V$.

Proposition 6.4.6.1.3. The assignment $R \mapsto \mathcal{P}(R)$ defines a functor

$$\mathcal{P} \colon \mathsf{Rel} \to \mathsf{Rel}.$$

Proof. Omitted. □

Appendices

6.A Other Chapters

Sets

1. Sets
2. Constructions With Sets
3. Pointed Sets
4. Tensor Products of Pointed Sets

Relations

5. Relations
6. Constructions With Relations
7. Equivalence Relations and Apartness Relations

Category Theory

8. Categories

Bicategories

9. Types of Morphisms in Bicategories

Chapter 7

Equivalence Relations and Apartness Relations

This chapter contains some material about reflexive, symmetric, transitive, equivalence, and apartness relations.

Contents

7.1 Reflexive Relations 339
- 7.1.1 Foundations 339
- 7.1.2 The Reflexive Closure of a Relation 339

7.2 Symmetric Relations 341
- 7.2.1 Foundations 341
- 7.2.2 The Symmetric Closure of a Relation 341

7.3 Transitive Relations 343
- 7.3.1 Foundations 343
- 7.3.2 The Transitive Closure of a Relation 344

7.4 Equivalence Relations 346
- 7.4.1 Foundations 346
- 7.4.2 The Equivalence Closure of a Relation 347

7.5 Quotients by Equivalence Relations 348
- 7.5.1 Equivalence Classes 348
- 7.5.2 Quotients of Sets by Equivalence Relations 348

7.A Other Chapters 353

7.1 Reflexive Relations

7.1.1 Foundations

Let A be a set.

Definition 7.1.1.1.1. A **reflexive relation** is equivalently:[1]

- An $\mathbb{E}_0$-monoid in $(\mathrm{N}_{\bullet}(\mathbf{Rel}(A, A)), \chi_A)$.
- A pointed object in $(\mathbf{Rel}(A, A), \chi_A)$.

Remark 7.1.1.1.2. In detail, a relation R on A is **reflexive** if we have an inclusion

$$\eta_R \colon \chi_A \subset R$$

of relations in $\mathbf{Rel}(A, A)$, i.e. if, for each $a \in A$, we have $a \sim_R a$.

Definition 7.1.1.1.3. Let A be a set.

1. The **set of reflexive relations on** A is the subset $\mathrm{Rel}^{\mathrm{refl}}(A, A)$ of $\mathrm{Rel}(A, A)$ spanned by the reflexive relations.
2. The **poset of relations on** A is is the subposet $\mathbf{Rel}^{\mathrm{refl}}(A, A)$ of $\mathbf{Rel}(A, A)$ spanned by the reflexive relations.

Proposition 7.1.1.1.4. Let R and S be relations on A.

1. *Interaction With Inverses.* If R is reflexive, then so is $R^{\dagger}$.
2. *Interaction With Composition.* If R and S are reflexive, then so is $S \diamond R$.

Proof. *Item 1*, *Interaction With Inverses*: Clear.
Item 2, *Interaction With Composition*: Clear. □

7.1.2 The Reflexive Closure of a Relation

Let R be a relation on A.

Definition 7.1.2.1.1. The **reflexive closure** of $\sim_R$ is the relation $\sim_R^{\mathrm{refl}}$[2] satisfying the following universal property:[3]

(★) Given another reflexive relation $\sim_S$ on A such that $R \subset S$, there exists an inclusion $\sim_R^{\mathrm{refl}} \subset \sim_S$.

[1] Note that since $\mathbf{Rel}(A, A)$ is posetal, reflexivity is a property of a relation, rather than extra structure.

[2] *Further Notation:* Also written R^{refl}.

[3] *Slogan:* The reflexive closure of R is the smallest reflexive relation containing R.

Construction 7.1.2.1.2. Concretely, $\sim_R^{\mathrm{refl}}$ is the free pointed object on R in $(\mathbf{Rel}(A,A),\chi_A)$[4], being given by

$$\begin{aligned}R^{\mathrm{refl}} &\overset{\mathrm{def}}{=} R \coprod^{\mathbf{Rel}(A,A)} \Delta_A\\ &= R \cup \Delta_A\\ &= \{(a,b)\in A\times A \mid \text{we have } a \sim_R b \text{ or } a=b\}.\end{aligned}$$

Proof. Clear. □

Proposition 7.1.2.1.3. Let R be a relation on A.

1. *Adjointness.* We have an adjunction

$$\Big((-)^{\mathrm{refl}} \dashv 忘\Big)\colon\quad \begin{tikzcd}\mathbf{Rel}(A,A) \arrow[r, bend left, "(-)^{\mathrm{refl}}"] \arrow[r, phantom, "\perp"] & \mathbf{Rel}^{\mathrm{refl}}(A,A), \arrow[l, bend left, "忘"]\end{tikzcd}$$

   witnessed by a bijection of sets

$$\mathbf{Rel}^{\mathrm{refl}}\Big(R^{\mathrm{refl}},S\Big)\cong \mathbf{Rel}(R,S),$$

   natural in $R\in\mathrm{Obj}\Big(\mathbf{Rel}^{\mathrm{refl}}(A,A)\Big)$ and $S\in\mathrm{Obj}(\mathbf{Rel}(A,A))$.

2. *The Reflexive Closure of a Reflexive Relation.* If R is reflexive, then $R^{\mathrm{refl}}=R$.

3. *Idempotency.* We have

$$\Big(R^{\mathrm{refl}}\Big)^{\mathrm{refl}} = R^{\mathrm{refl}}.$$

4. *Interaction With Inverses.* We have

$$\Big(R^{\dagger}\Big)^{\mathrm{refl}} = \Big(R^{\mathrm{refl}}\Big)^{\dagger},\qquad \begin{tikzcd}\mathrm{Rel}(A,A) \arrow[r, "(-)^{\mathrm{refl}}"] \arrow[d, "(-)^{\dagger}"'] & \mathrm{Rel}(A,A) \arrow[d, "(-)^{\dagger}"]\\ \mathrm{Rel}(A,A) \arrow[r, "(-)^{\mathrm{refl}}"'] & \mathrm{Rel}(A,A).\end{tikzcd}$$

5. *Interaction With Composition.* We have

$$(S\diamond R)^{\mathrm{refl}} = S^{\mathrm{refl}}\diamond R^{\mathrm{refl}},\qquad \begin{tikzcd}\mathrm{Rel}(A,A)\times\mathrm{Rel}(A,A) \arrow[r, "\diamond"] \arrow[d, "(-)^{\mathrm{refl}}\times(-)^{\mathrm{refl}}"'] & \mathrm{Rel}(A,A) \arrow[d, "(-)^{\mathrm{refl}}"]\\ \mathrm{Rel}(A,A)\times\mathrm{Rel}(A,A) \arrow[r, "\diamond"'] & \mathrm{Rel}(A,A).\end{tikzcd}$$

[4] Or, equivalently, the free $\mathbb{E}_0$-monoid on R in $(\mathsf{N}_{\bullet}(\mathbf{Rel}(A,A)),\chi_A)$.

Proof. *Item 1, Adjointness*: This is a rephrasing of the universal property of the reflexive closure of a relation, stated in Definition 7.1.2.1.1.
Item 2, The Reflexive Closure of a Reflexive Relation: Clear.
Item 3, Idempotency: This follows from Item 2.
Item 4, Interaction With Inverses: Clear.
Item 5, Interaction With Composition: This follows from Item 2 of Proposition 7.1.1.1.4. □

7.2 Symmetric Relations

7.2.1 Foundations

Let A be a set.

Definition 7.2.1.1.1. A relation R on A is **symmetric** if we have $R^{\dagger} = R$.

Remark 7.2.1.1.2. In detail, a relation R is symmetric if it satisfies the following condition:

(★) For each $a, b \in A$, if $a \sim_R b$, then $b \sim_R a$.

Definition 7.2.1.1.3. Let A be a set.

1. The **set of symmetric relations on** A is the subset $\mathrm{Rel}^{\mathsf{symm}}(A, A)$ of $\mathrm{Rel}(A, A)$ spanned by the symmetric relations.

2. The **poset of relations on** A is is the subposet $\mathbf{Rel}^{\mathsf{symm}}(A, A)$ of $\mathbf{Rel}(A, A)$ spanned by the symmetric relations.

Proposition 7.2.1.1.4. Let R and S be relations on A.

1. *Interaction With Inverses.* If R is symmetric, then so is $R^{\dagger}$.

2. *Interaction With Composition.* If R and S are symmetric, then so is $S \diamond R$.

Proof. *Item 1, Interaction With Inverses*: Clear.
Item 2, Interaction With Composition: Clear. □

7.2.2 The Symmetric Closure of a Relation

Let R be a relation on A.

Definition 7.2.2.1.1. The **symmetric closure** of $\sim_R$ is the relation $\sim_R^{\mathrm{symm}}$[5]

[5] *Further Notation:* Also written R^{symm}.

satisfying the following universal property:[6]

(★) Given another symmetric relation $\sim_S$ on A such that $R \subset S$, there exists an inclusion $\sim_R^{\mathrm{symm}} \subset \sim_S$.

Construction 7.2.2.1.2. Concretely, $\sim_R^{\mathrm{symm}}$ is the symmetric relation on A defined by

$$\begin{aligned} R^{\mathrm{symm}} &\overset{\mathrm{def}}{=} R \cup R^{\dagger} \\ &= \{(a,b) \in A \times A \mid \text{we have } a \sim_R b \text{ or } b \sim_R a\}. \end{aligned}$$

Proof. Clear. □

Proposition 7.2.2.1.3. Let R be a relation on A.

1. *Adjointness.* We have an adjunction

$$\left((-)^{\mathsf{symm}} \dashv 忘\right)\colon \quad \begin{tikzcd} \mathbf{Rel}(A,A) \arrow[r, bend left, "(-)^{\mathrm{symm}}"] \arrow[r, phantom, "\perp"] & \mathbf{Rel}^{\mathsf{symm}}(A,A), \arrow[l, bend left, "忘"] \end{tikzcd}$$

   witnessed by a bijection of sets

$$\mathbf{Rel}^{\mathsf{symm}}(R^{\mathrm{symm}}, S) \cong \mathbf{Rel}(R, S),$$

   natural in $R \in \mathrm{Obj}(\mathbf{Rel}^{\mathsf{symm}}(A,A))$ and $S \in \mathrm{Obj}(\mathbf{Rel}(A,A))$.

2. *The Symmetric Closure of a Symmetric Relation.* If R is symmetric, then $R^{\mathrm{symm}} = R$.

3. *Idempotency.* We have

$$(R^{\mathrm{symm}})^{\mathrm{symm}} = R^{\mathrm{symm}}.$$

4. *Interaction With Inverses.* We have

$$\left(R^{\dagger}\right)^{\mathrm{symm}} = \left(R^{\mathrm{symm}}\right)^{\dagger}, \qquad \begin{tikzcd} \mathrm{Rel}(A,A) \arrow[r, "(-)^{\mathrm{symm}}"] \arrow[d, "(-)^{\dagger}"'] & \mathrm{Rel}(A,A) \arrow[d, "(-)^{\dagger}"] \\ \mathrm{Rel}(A,A) \arrow[r, "(-)^{\mathrm{symm}}"'] & \mathrm{Rel}(A,A). \end{tikzcd}$$

[6] *Slogan:* The symmetric closure of R is the smallest symmetric relation containing R.

5. *Interaction With Composition.* We have

$$(S \diamond R)^{\mathrm{symm}} = S^{\mathrm{symm}} \diamond R^{\mathrm{symm}}, \qquad \begin{tikzcd} \mathrm{Rel}(A,A) \times \mathrm{Rel}(A,A) \arrow[r, "\diamond"] \arrow[d, "(-)^{\mathrm{symm}} \times (-)^{\mathrm{symm}}"'] & \mathrm{Rel}(A,A) \arrow[d, "(-)^{\mathrm{symm}}"] \\ \mathrm{Rel}(A,A) \times \mathrm{Rel}(A,A) \arrow[r, "\diamond"'] & \mathrm{Rel}(A,A). \end{tikzcd}$$

Proof. *Item 1, Adjointness*: This is a rephrasing of the universal property of the symmetric closure of a relation, stated in Definition 7.2.2.1.1.
Item 2, The Symmetric Closure of a Symmetric Relation: Clear.
Item 3, Idempotency: This follows from Item 2.
Item 4, Interaction With Inverses: Clear.
Item 5, Interaction With Composition: This follows from Item 2 of Proposition 7.2.1.1.4. □

7.3 Transitive Relations

7.3.1 Foundations

Let A be a set.

Definition 7.3.1.1.1. A **transitive relation** is equivalently:[7]

- A non-unital $\mathbb{E}_1$-monoid in $(\mathrm{N}_\bullet(\mathbf{Rel}(A,A)), \diamond)$.
- A non-unital monoid in $(\mathbf{Rel}(A,A), \diamond)$.

Remark 7.3.1.1.2. In detail, a relation R on A is **transitive** if we have an inclusion

$$\mu_R \colon R \diamond R \subset R$$

of relations in $\mathbf{Rel}(A,A)$, i.e. if, for each $a, c \in A$, the following condition is satisfied:

($\star$) If there exists some $b \in A$ such that $a \sim_R b$ and $b \sim_R c$, then $a \sim_R c$.

Definition 7.3.1.1.3. Let A be a set.

1. The **set of transitive relations from A to B** is the subset $\mathrm{Rel}^{\mathrm{trans}}(A)$ of $\mathrm{Rel}(A,A)$ spanned by the transitive relations.

2. The **poset of relations from A to B** is is the subposet $\mathbf{Rel}^{\mathrm{trans}}(A)$ of $\mathbf{Rel}(A,A)$ spanned by the transitive relations.

[7] Note that since $\mathbf{Rel}(A,A)$ is posetal, transitivity is a property of a relation, rather than

Proposition 7.3.1.1.4. Let R and S be relations on A.

1. *Interaction With Inverses.* If R is transitive, then so is $R^{\dagger}$.

2. *Interaction With Composition.* If R and S are transitive, then $S \diamond R$ ***may fail to be transitive***.

Proof. *Item 1*, *Interaction With Inverses*: Clear.
Item 2, *Interaction With Composition*: See [MSE 2096272].[8] □

7.3.2 The Transitive Closure of a Relation

Let R be a relation on A.

Definition 7.3.2.1.1. The **transitive closure** of $\sim_R$ is the relation $\sim_R^{\mathrm{trans}}$[9] satisfying the following universal property:[10]

(★) Given another transitive relation $\sim_S$ on A such that $R \subset S$, there exists an inclusion $\sim_R^{\mathrm{trans}} \subset \sim_S$.

Construction 7.3.2.1.2. Concretely, $\sim_R^{\mathrm{trans}}$ is the free non-unital monoid on R in $(\mathbf{Rel}(A, A), \diamond)$[11], being given by

$$\begin{aligned} R^{\mathrm{trans}} &\overset{\mathrm{def}}{=} \coprod_{n=1}^{\infty} R^{\diamond n} \\ &\overset{\mathrm{def}}{=} \bigcup_{n=1}^{\infty} R^{\diamond n} \\ &\overset{\mathrm{def}}{=} \left\{ (a, b) \in A \times B \;\middle|\; \begin{array}{l} \text{there exists some } (x_1, \ldots, x_n) \in R^{\times n} \\ \text{such that } a \sim_R x_1 \sim_R \cdots \sim_R x_n \sim_R b \end{array} \right\}. \end{aligned}$$

extra structure.

[8] *Intuition:* Transitivity for R and S fails to imply that of $S \diamond R$ because the composition operation for relations intertwines R and S in an incompatible way:

1. If $a \sim_{S \diamond R} c$ and $c \sim_{S \diamond r} e$, then:
   (a) There is some $b \in A$ such that:
      i. $a \sim_R b$;
      ii. $b \sim_S c$;
   (b) There is some $d \in A$ such that:
      i. $c \sim_R d$;
      ii. $d \sim_S e$.

[9] *Further Notation:* Also written R^{trans}.

[10] *Slogan:* The transitive closure of R is the smallest transitive relation containing R.

[11] Or, equivalently, the free non-unital $\mathbb{E}_1$-monoid on R in $(\mathrm{N}_{\bullet}(\mathbf{Rel}(A, A)), \diamond)$.

Proof. Clear. □

Proposition 7.3.2.1.3. Let R be a relation on A.

1. *Adjointness.* We have an adjunction

$$\Big((-)^{\mathrm{trans}} \dashv 忘\Big)\colon\quad \begin{tikzcd} \mathbf{Rel}(A,A) \arrow[r, bend left, "(-)^{\mathrm{trans}}"] \arrow[r, phantom, "\perp"] & \mathbf{Rel}^{\mathsf{trans}}(A,A), \arrow[l, bend left, "忘"] \end{tikzcd}$$

   witnessed by a bijection of sets

$$\mathbf{Rel}^{\mathsf{trans}}\big(R^{\mathrm{trans}}, S\big) \cong \mathbf{Rel}(R, S),$$

   natural in $R \in \mathrm{Obj}\big(\mathbf{Rel}^{\mathsf{trans}}(A,A)\big)$ and $S \in \mathrm{Obj}(\mathbf{Rel}(A,B))$.

2. *The Transitive Closure of a Transitive Relation.* If R is transitive, then $R^{\mathrm{trans}} = R$.

3. *Idempotency.* We have

$$\big(R^{\mathrm{trans}}\big)^{\mathrm{trans}} = R^{\mathrm{trans}}.$$

4. *Interaction With Inverses.* We have

$$\Big(R^{\dagger}\Big)^{\mathrm{trans}} = \Big(R^{\mathrm{trans}}\Big)^{\dagger}, \qquad \begin{tikzcd} \mathrm{Rel}(A,A) \arrow[r, "(-)^{\mathrm{trans}}"] \arrow[d, "(-)^{\dagger}"'] & \mathrm{Rel}(A,A) \arrow[d, "(-)^{\dagger}"] \\ \mathrm{Rel}(A,A) \arrow[r, "(-)^{\mathrm{trans}}"'] & \mathrm{Rel}(A,A). \end{tikzcd}$$

5. *Interaction With Composition.* We have

$$(S \diamond R)^{\mathrm{trans}} \overset{\text{poss.}}{\neq} S^{\mathrm{trans}} \diamond R^{\mathrm{trans}}, \qquad \begin{tikzcd} \mathrm{Rel}(A,A) \times \mathrm{Rel}(A,A) \arrow[r, "\diamond"] \arrow[d, "(-)^{\mathrm{trans}} \times (-)^{\mathrm{trans}}"'] \arrow[dr, phantom, "\times"] & \mathrm{Rel}(A,A) \arrow[d, "(-)^{\mathrm{trans}}"] \\ \mathrm{Rel}(A,A) \times \mathrm{Rel}(A,A) \arrow[r, "\diamond"'] & \mathrm{Rel}(A,A). \end{tikzcd}$$

Proof. *Item 1*, *Adjointness*: This is a rephrasing of the universal property of the transitive closure of a relation, stated in Definition 7.3.2.1.1.
Item 2, *The Transitive Closure of a Transitive Relation*: Clear.
Item 3, *Idempotency*: This follows from Item 2.

Item 4, Interaction With Inverses: We have

$$\begin{aligned}\left(R^{\dagger}\right)^{\mathrm{trans}} &= \bigcup_{n=1}^{\infty}\left(R^{\dagger}\right)^{\diamond n}\\ &= \bigcup_{n=1}^{\infty}\left(R^{\diamond n}\right)^{\dagger}\\ &= \left(\bigcup_{n=1}^{\infty}R^{\diamond n}\right)^{\dagger}\\ &= \left(R^{\mathrm{trans}}\right)^{\dagger},\end{aligned}$$

where we have used, respectively:

1. Construction 7.3.2.1.2.
2. Item 4 of Proposition 6.3.12.1.3.
3. Item 1 of Proposition 6.3.6.1.2.
4. Construction 7.3.2.1.2.

Item 5, Interaction With Composition: This follows from Item 2 of Proposition 7.3.1.1.4. □

7.4 Equivalence Relations

7.4.1 Foundations

Let A be a set.

Definition 7.4.1.1.1. A relation R is an **equivalence relation** if it is reflexive, symmetric, and transitive.[12]

Example 7.4.1.1.2. The **kernel of a function** $f: A \to B$ is the equivalence relation $\sim_{\mathrm{Ker}(f)}$ on A obtained by declaring $a \sim_{\mathrm{Ker}(f)} b$ iff $f(a) = f(b)$.[13]

Definition 7.4.1.1.3. Let A and B be sets.

1. The **set of equivalence relations from A to B** is the subset $\mathrm{Rel}^{\mathrm{eq}}(A, B)$ of $\mathrm{Rel}(A, B)$ spanned by the equivalence relations.
2. The **poset of relations from A to B** is is the subposet $\mathbf{Rel}^{\mathrm{eq}}(A, B)$ of $\mathbf{Rel}(A, B)$ spanned by the equivalence relations.

[12] *Further Terminology:* If instead R is just symmetric and transitive, then it is called a **partial equivalence relation**.

[13] The kernel $\mathrm{Ker}(f): A \nrightarrow A$ of f is the underlying functor of the monad induced by the

7.4.2 The Equivalence Closure of a Relation

Let R be a relation on A.

Definition 7.4.2.1.1. The **equivalence closure**[14] of $\sim_R$ is the relation $\sim_R^{\mathrm{eq}}$[15] satisfying the following universal property:[16]

(★) Given another equivalence relation $\sim_S$ on A such that $R \subset S$, there exists an inclusion $\sim_R^{\mathrm{eq}} \subset \sim_S$.

Construction 7.4.2.1.2. Concretely, $\sim_R^{\mathrm{eq}}$ is the equivalence relation on A defined by

$$
\begin{aligned}
R^{\mathrm{eq}} &\overset{\mathrm{def}}{=} \left(\left(R^{\mathrm{refl}}\right)^{\mathrm{symm}}\right)^{\mathrm{trans}} \\
&= \left(\left(R^{\mathrm{symm}}\right)^{\mathrm{trans}}\right)^{\mathrm{refl}} \\
&= \left\{ (a,b) \in A \times B \;\middle|\; \begin{array}{l} \text{there exists } (x_1, \ldots, x_n) \in R^{\times n} \text{ satisfying at} \\ \text{least one of the following conditions:} \\ \quad 1. \text{ The following conditions are satisfied:} \\ \qquad \text{(a) We have } a \sim_R x_1 \text{ or } x_1 \sim_R a; \\ \qquad \text{(b) We have } x_i \sim_R x_{i+1} \text{ or } x_{i+1} \sim_R x_i \\ \qquad \phantom{\text{(b) }} \text{for each } 1 \leq i \leq n-1; \\ \qquad \text{(c) We have } b \sim_R x_n \text{ or } x_n \sim_R b; \\ \quad 2. \text{ We have } a = b. \end{array} \right\}.
\end{aligned}
$$

Proof. From the universal properties of the reflexive, symmetric, and transitive closures of a relation (Definitions 7.1.2.1.1, 7.2.2.1.1 and 7.3.2.1.1), we see that it suffices to prove that:

1. The symmetric closure of a reflexive relation is still reflexive.
2. The transitive closure of a symmetric relation is still symmetric.

which are both clear. □

Proposition 7.4.2.1.3. Let R be a relation on A.

adjunction $\mathrm{Gr}(f) \dashv f^{-1} \colon A \rightleftarrows B$ in **Rel** of Item 2 of Proposition 6.3.1.1.2.

[14] *Further Terminology:* Also called the **equivalence relation associated to** $\sim_R$.

[15] *Further Notation:* Also written R^{eq}.

[16] *Slogan:* The equivalence closure of R is the smallest equivalence relation containing R.

1. *Adjointness.* We have an adjunction

$$\left((-)^{\mathrm{eq}} \dashv \text{忘}\right)\colon \quad \begin{tikzcd} \mathbf{Rel}(A,B) \arrow[r, bend left, "(-)^{\mathrm{eq}}"] \arrow[r, phantom, "\perp"] & \mathbf{Rel}^{\mathrm{eq}}(A,B), \arrow[l, bend left, "\text{忘}"] \end{tikzcd}$$

   witnessed by a bijection of sets

$$\mathbf{Rel}^{\mathrm{eq}}(R^{\mathrm{eq}}, S) \cong \mathbf{Rel}(R, S),$$

   natural in $R \in \mathrm{Obj}(\mathbf{Rel}^{\mathrm{eq}}(A,B))$ and $S \in \mathrm{Obj}(\mathbf{Rel}(A,B))$.

2. *The Equivalence Closure of an Equivalence Relation.* If R is an equivalence relation, then $R^{\mathrm{eq}} = R$.

3. *Idempotency.* We have

$$(R^{\mathrm{eq}})^{\mathrm{eq}} = R^{\mathrm{eq}}.$$

Proof. *Item 1, Adjointness*: This is a rephrasing of the universal property of the equivalence closure of a relation, stated in Definition 7.4.2.1.1.
Item 2, The Equivalence Closure of an Equivalence Relation: Clear.
Item 3, Idempotency: This follows from Item 2. □

7.5 Quotients by Equivalence Relations

7.5.1 Equivalence Classes

Let A be a set, let R be a relation on A, and let $a \in A$.

Definition 7.5.1.1.1. The **equivalence class associated to** a is the set $[a]$ defined by

$$\begin{aligned} [a] &\stackrel{\text{def}}{=} \{x \in X \mid x \sim_R a\} \\ &= \{x \in X \mid a \sim_R x\}. && \text{(since } R \text{ is symmetric)} \end{aligned}$$

7.5.2 Quotients of Sets by Equivalence Relations

Let A be a set and let R be a relation on A.

Definition 7.5.2.1.1. The **quotient of** X **by** R is the set $X/{\sim_R}$ defined by

$$X/{\sim_R} \stackrel{\text{def}}{=} \{[a] \in \mathcal{P}(X) \mid a \in X\}.$$

Remark 7.5.2.1.2. The reason we define quotient sets for equivalence relations only is that each of the properties of being an equivalence relation—reflexivity, symmetry, and transitivity—ensures that the equivalences classes $[a]$ of X under R are well-behaved:

- *Reflexivity.* If R is reflexive, then, for each $a \in X$, we have $a \in [a]$.
- *Symmetry.* The equivalence class $[a]$ of an element a of X is defined by
$$[a] \stackrel{\text{def}}{=} \{x \in X \mid x \sim_R a\},$$
but we could equally well define
$$[a]' \stackrel{\text{def}}{=} \{x \in X \mid a \sim_R x\}$$
instead. This is not a problem when R is symmetric, as we then have $[a] = [a]'$.[17]
- *Transitivity.* If R is transitive, then $[a]$ and $[b]$ are disjoint iff $a \nsim_R b$, and equal otherwise.

Proposition 7.5.2.1.3. Let $f\colon X \to Y$ be a function and let R be a relation on X.

1. *As a Coequaliser.* We have an isomorphism of sets
$$X/\mathord{\sim}_R^{\text{eq}} \cong \mathrm{CoEq}\left(R \hookrightarrow X \times X \underset{\mathrm{pr}_2}{\overset{\mathrm{pr}_1}{\rightrightarrows}} X\right),$$
where $\sim_R^{\text{eq}}$ is the equivalence relation generated by $\sim_R$.
2. *As a Pushout.* We have an isomorphism of sets[18]
$$X/\mathord{\sim}_R^{\text{eq}} \cong X \textstyle\coprod_{\mathrm{Eq}(\mathrm{pr}_1,\mathrm{pr}_2)} X, \qquad \begin{tikzcd} X/\mathord{\sim}_R^{\text{eq}} \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[l] \\ X \arrow[u] & \mathrm{Eq}(\mathrm{pr}_1,\mathrm{pr}_2). \arrow[l] \arrow[u] \end{tikzcd}$$

[17] When categorifying equivalence relations, one finds that $[a]$ and $[a]'$ correspond to presheaves and copresheaves; see **??**.

[18] Dually, we also have an isomorphism of sets
$$\mathrm{Eq}(\mathrm{pr}_1,\mathrm{pr}_2) \cong X \times_{X/\sim_R^{\text{eq}}} X, \qquad \begin{tikzcd} \mathrm{Eq}(\mathrm{pr}_1,\mathrm{pr}_2) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d] \\ X \arrow[r] & X/\mathord{\sim}_R^{\text{eq}}. \end{tikzcd}$$

where $\sim_R^{\mathrm{eq}}$ is the equivalence relation generated by $\sim_R$.

3. *The First Isomorphism Theorem for Sets.* We have an isomorphism of sets[19,20]

$$X/\sim_{\mathrm{Ker}(f)} \cong \mathrm{Im}(f).$$

4. *Descending Functions to Quotient Sets, I.* Let R be an equivalence relation on X. The following conditions are equivalent:

   (a) There exists a map

   $$\overline{f}\colon X/\sim_R \to Y$$

   making the diagram

   $$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "q"'] & Y \\ X/\sim_R \arrow[ur, dotted, "\overline{f}"', "\exists"] & \end{tikzcd}$$

   commute.

   (b) We have $R \subset \mathrm{Ker}(f)$.

   (c) For each $x, y \in X$, if $x \sim_R y$, then $f(x) = f(y)$.

5. *Descending Functions to Quotient Sets, II.* Let R be an equivalence relation on X. If the conditions of Item 4 hold, then $\overline{f}$ is the *unique* map making the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "q"'] & Y \\ X/\sim_R \arrow[ur, dashed, "\overline{f}"', "\exists!"] & \end{tikzcd}$$

[19]*Further Terminology:* The set $X/\sim_{\mathrm{Ker}(f)}$ is often called the **coimage of** f, and denoted by $\mathrm{Coim}(f)$.

[20]In a sense this is a result relating the monad in **Rel** induced by f with the comonad in **Rel** induced by f, as the kernel and image

$$\mathrm{Ker}(f)\colon X \nrightarrow X,$$
$$\mathrm{Im}(f) \subset Y$$

of f are the underlying functors of (respectively) the induced monad and comonad of the

commute.

6. *Descending Functions to Quotient Sets, III.* Let R be an equivalence relation on X. We have a bijection

$$\mathrm{Hom}_{\mathsf{Sets}}(X/{\sim_R}, Y) \cong \mathrm{Hom}^{R}_{\mathsf{Sets}}(X, Y),$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$, given by the assignment $f \mapsto \overline{f}$ of Items 4 and 5, where $\mathrm{Hom}^{R}_{\mathsf{Sets}}(X, Y)$ is the set defined by

$$\mathrm{Hom}^{R}_{\mathsf{Sets}}(X, Y) \overset{\mathrm{def}}{=} \left\{ f \in \mathrm{Hom}_{\mathsf{Sets}}(X, Y) \;\middle|\; \begin{array}{l} \text{for each } x, y \in X, \\ \text{if } x \sim_R y, \text{ then} \\ f(x) = f(y) \end{array} \right\}.$$

7. *Descending Functions to Quotient Sets, IV.* Let R be an equivalence relation on X. If the conditions of Item 4 hold, then the following conditions are equivalent:

   (a) The map $\overline{f}$ is an injection.

   (b) We have $R = \mathrm{Ker}(f)$.

   (c) For each $x, y \in X$, we have $x \sim_R y$ iff $f(x) = f(y)$.

8. *Descending Functions to Quotient Sets, V.* Let R be an equivalence relation on X. If the conditions of Item 4 hold, then the following conditions are equivalent:

   (a) The map $f \colon X \to Y$ is surjective.

   (b) The map $\overline{f} \colon X/{\sim_R} \to Y$ is surjective.

9. *Descending Functions to Quotient Sets, VI.* Let R be a relation on X and let $\sim^{\mathrm{eq}}_R$ be the equivalence relation associated to R. The following conditions are equivalent:

   (a) The map f satisfies the equivalent conditions of Item 4:

      - There exists a map

$$\overline{f} \colon X/{\sim^{\mathrm{eq}}_R} \to Y$$

---

adjunction

$$\left(\mathrm{Gr}(f) \dashv f^{-1}\right)\colon \begin{tikzcd} A \arrow[r, bend left, "\mathrm{Gr}(f)"] \arrow[r, phantom, "\perp"] & B \arrow[l, bend left, "f^{-1}"] \end{tikzcd}$$

making the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "q"'] & Y \\ X/{\sim_R^{\mathrm{eq}}} \arrow[ur, dotted, "\exists"', "\overline{f}"] & \end{tikzcd}$$

commute.

- For each $x, y \in X$, if $x \sim_R^{\mathrm{eq}} y$, then $f(x) = f(y)$.

(b) For each $x, y \in X$, if $x \sim_R y$, then $f(x) = f(y)$.

Proof. Item 1, As a Coequaliser: Omitted.
Item 2, As a Pushout: Omitted.
Item 3, The First Isomorphism Theorem for Sets: Clear.
Item 4, Descending Functions to Quotient Sets, I: See [Pro24o].
Item 5, Descending Functions to Quotient Sets, II: See [Pro24aa].
Item 6, Descending Functions to Quotient Sets, III: This follows from Items 5 and 6.
Item 7, Descending Functions to Quotient Sets, IV: See [Pro24n].
Item 8, Descending Functions to Quotient Sets, V: See [Pro24m].
Item 9, Descending Functions to Quotient Sets, VI: The implication Item 9a $\Longrightarrow$ Item 9b is clear.
Conversely, suppose that, for each $x, y \in X$, if $x \sim_R y$, then $f(x) = f(y)$. Spelling out the definition of the equivalence closure of R, we see that the condition $x \sim_R^{\mathrm{eq}} y$ unwinds to the following:

($\star$) There exist $(x_1, \ldots, x_n) \in R^{\times n}$ satisfying at least one of the following conditions:

1. The following conditions are satisfied:
   (a) We have $x \sim_R x_1$ or $x_1 \sim_R x$;
   (b) We have $x_i \sim_R x_{i+1}$ or $x_{i+1} \sim_R x_i$ for each $1 \leq i \leq n-1$;
   (c) We have $y \sim_R x_n$ or $x_n \sim_R y$;
2. We have $x = y$.

---

of Item 2 of Proposition 6.3.1.1.2.

Now, if $x = y$, then $f(x) = f(y)$ trivially; otherwise, we have

$$
\begin{aligned}
f(x) &= f(x_1),\\
f(x_1) &= f(x_2),\\
&\vdots\\
f(x_{n-1}) &= f(x_n),\\
f(x_n) &= f(y),
\end{aligned}
$$

and $f(x) = f(y)$, as we wanted to show. □

Appendices

7.A Other Chapters

Sets

1. Sets
2. Constructions With Sets
3. Pointed Sets
4. Tensor Products of Pointed Sets

Relations

5. Relations
6. Constructions With Relations
7. Equivalence Relations and Apartness Relations

Category Theory

8. Categories

Bicategories

9. Types of Morphisms in Bicategories

Part III

Category Theory

Chapter 8

Categories

This chapter contains some elementary material about categories, functors, and natural transformations. Notably, we discuss and explore:

1. Categories (Section 8.1).

2. The quadruple adjunction $\pi_0 \dashv (-)_{\mathsf{disc}} \dashv \mathrm{Obj} \dashv (-)_{\mathsf{indisc}}$ between the category of categories and the category of sets (Section 8.2).

3. Groupoids, categories in which all morphisms admit inverses (Section 8.3).

4. Functors (Section 8.4).

5. The conditions one may impose on functors in decreasing order of importance:

   (a) Section 8.5 introduces the foundationally important conditions one may impose on functors, such as faithfulness, conservativity, essential surjectivity, etc.

   (b) Section 8.6 introduces more conditions one may impose on functors that are still important but less omni-present than those of Section 8.5, such as being dominant, being a monomorphism, being pseudomonic, etc.

   (c) Section 8.7 introduces some rather rare or uncommon conditions one may impose on functors that are nevertheless still useful to explicit record in this chapter.

6. Natural transformations (Section 8.8).

7. The various categorical and 2-categorical structures formed by categories, functors, and natural transformations (Section 8.9).

Contents

8.1 Categories ... 357
8.1.1 Foundations ... 357
8.1.2 Examples of Categories ... 360
8.1.3 Posetal Categories ... 362
8.1.4 Subcategories ... 364
8.1.5 Skeletons of Categories ... 364
8.1.6 Precomposition and Postcomposition ... 365
8.2 The Quadruple Adjunction With Sets ... 367
8.2.1 Statement ... 367
8.2.2 Connected Components and Connected Categories ... 368
8.2.3 Discrete Categories ... 370
8.2.4 Indiscrete Categories ... 372
8.3 Groupoids ... 374
8.3.1 Foundations ... 374
8.3.2 The Groupoid Completion of a Category ... 374
8.3.3 The Core of a Category ... 377
8.4 Functors ... 380
8.4.1 Foundations ... 380
8.4.2 Contravariant Functors ... 384
8.4.3 Forgetful Functors ... 385
8.4.4 The Natural Transformation Associated to a Functor ... 387
8.5 Conditions on Functors ... 389
8.5.1 Faithful Functors ... 389
8.5.2 Full Functors ... 391
8.5.3 Fully Faithful Functors ... 394
8.5.4 Conservative Functors ... 397
8.5.5 Essentially Injective Functors ... 399
8.5.6 Essentially Surjective Functors ... 399
8.5.7 Equivalences of Categories ... 400
8.5.8 Isomorphisms of Categories ... 402
8.6 More Conditions on Functors ... 403
8.6.1 Dominant Functors ... 403
8.6.2 Monomorphisms of Categories ... 404
8.6.3 Epimorphisms of Categories ... 405

8.6.4 Pseudomonic Functors 407
8.6.5 Pseudoepic Functors 409
8.7 Even More Conditions on Functors 411
8.7.1 Injective on Objects Functors 411
8.7.2 Surjective on Objects Functors 411
8.7.3 Bijective on Objects Functors 411
8.7.4 Functors Representably Faithful on Cores ... 412
8.7.5 Functors Representably Full on Cores 412
8.7.6 Functors Representably Fully Faithful on Cores 413
8.7.7 Functors Corepresentably Faithful on Cores 414
8.7.8 Functors Corepresentably Full on Cores 415
8.7.9 Functors Corepresentably Fully Faithful on Cores 415
8.8 Natural Transformations 417
8.8.1 Transformations 417
8.8.2 Natural Transformations 417
8.8.3 Vertical Composition of Natural Transformations 418
8.8.4 Horizontal Composition of Natural Transformations 421
8.8.5 Properties of Natural Transformations 426
8.8.6 Natural Isomorphisms 428
8.9 Categories of Categories 429
8.9.1 Functor Categories 429
8.9.2 The Category of Categories and Functors 432
8.9.3 The 2-Category of Categories, Functors, and Natural Transformations 433
8.9.4 The Category of Groupoids 434
8.9.5 The 2-Category of Groupoids 434
8.A Other Chapters 434

8.1 Categories

8.1.1 Foundations

Definition 8.1.1.1.1. A **category** $(C, \circ^C, \mathbb{1}^C)$ consists of:

- *Objects.* A class $\mathrm{Obj}(\mathcal{C})$ of **objects**.
- *Morphisms.* For each $A, B \in \mathrm{Obj}(\mathcal{C})$, a class $\mathrm{Hom}_{\mathcal{C}}(A, B)$, called the **class of morphisms of** $\mathcal{C}$ **from** A **to** B.
- *Identities.* For each $A \in \mathrm{Obj}(\mathcal{C})$, a map of sets

  $$\mathbb{1}^{\mathcal{C}}_{A} \colon \mathrm{pt} \to \mathrm{Hom}_{\mathcal{C}}(A, A),$$

  called the **unit map of** $\mathcal{C}$ **at** A, determining a morphism

  $$\mathrm{id}_A \colon A \to A$$

  of $\mathcal{C}$, called the **identity morphism of** A.
- *Composition.* For each $A, B, C \in \mathrm{Obj}(\mathcal{C})$, a map of sets

  $$\circ^{\mathcal{C}}_{A,B,C} \colon \mathrm{Hom}_{\mathcal{C}}(B, C) \times \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{C}}(A, C),$$

  called the **composition map of** $\mathcal{C}$ **at** (A, B, C).

such that the following conditions are satisfied:

1. *Associativity.* The diagram

$$\begin{tikzcd}
 & \mathrm{Hom}_{\mathcal{C}}(C,D) \times (\mathrm{Hom}_{\mathcal{C}}(B,C) \times \mathrm{Hom}_{\mathcal{C}}(A,B)) \arrow[dr, "\mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(C,D)} \times \circ^{\mathcal{C}}_{A,B,C}"] & \\
(\mathrm{Hom}_{\mathcal{C}}(C,D) \times \mathrm{Hom}_{\mathcal{C}}(B,C)) \times \mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[ur, dashed, "\alpha^{\mathsf{Sets}}_{\mathrm{Hom}_{\mathcal{C}}(C,D),\mathrm{Hom}_{\mathcal{C}}(B,C),\mathrm{Hom}_{\mathcal{C}}(A,B)}", "\sim"'] \arrow[d, "\circ^{\mathcal{C}}_{B,C,D} \times \mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(A,B)}"'] & & \mathrm{Hom}_{\mathcal{C}}(C,D) \times \mathrm{Hom}_{\mathcal{C}}(A,C) \arrow[d, "\circ^{\mathcal{C}}_{A,C,D}"] \\
\mathrm{Hom}_{\mathcal{C}}(B,D) \times \mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[rr, "\circ^{\mathcal{C}}_{A,B,D}"'] & & \mathrm{Hom}_{\mathcal{C}}(A,D)
\end{tikzcd}$$

   commutes, i.e. for each composable triple (f, g, h) of morphisms of $\mathcal{C}$, we have

$$(f \circ g) \circ h = f \circ (g \circ h).$$

2. *Left Unitality.* The diagram

$$\begin{tikzcd}
\mathrm{pt} \times \mathrm{Hom}_{\mathcal{C}}(A, B) \arrow[d, "\mathbb{1}^{\mathcal{C}}_{B} \times \mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(A,B)}"'] \arrow[dr, dashed, "\lambda^{\mathsf{Sets}}_{\mathrm{Hom}_{\mathcal{C}}(A,B)}", "\sim"'] & \\
\mathrm{Hom}_{\mathcal{C}}(B, B) \times \mathrm{Hom}_{\mathcal{C}}(A, B) \arrow[r, "\circ^{\mathcal{C}}_{A,B,B}"'] & \mathrm{Hom}_{\mathcal{C}}(A, B)
\end{tikzcd}$$

commutes, i.e. for each morphism $f \colon A \longrightarrow B$ of $\mathcal{C}$, we have

$$\mathrm{id}_B \circ f = f.$$

3. *Right Unitality.* The diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(A, B) \times \mathrm{pt} \arrow[d, "\mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(A,B)} \times \mathbb{1}^{\mathcal{C}}_{A}"'] \arrow[dr, dashed, "\rho^{\mathsf{Sets}}_{\mathrm{Hom}_{\mathcal{C}}(A,B)}", "\sim"'] & \\
\mathrm{Hom}_{\mathcal{C}}(A, B) \times \mathrm{Hom}_{\mathcal{C}}(A, A) \arrow[r, "\circ^{\mathcal{C}}_{A,A,B}"'] & \mathrm{Hom}_{\mathcal{C}}(A, B)
\end{tikzcd}$$

commutes, i.e. for each morphism $f \colon A \longrightarrow B$ of $\mathcal{C}$, we have

$$f \circ \mathrm{id}_A = f.$$

Notation 8.1.1.1.2. Let $\mathcal{C}$ be a category.

1. We also write $\mathcal{C}(A, B)$ for $\mathrm{Hom}_{\mathcal{C}}(A, B)$.
2. We write $\mathrm{Mor}(\mathcal{C})$ for the class of all morphisms of $\mathcal{C}$.

Definition 8.1.1.1.3. Let κ be a regular cardinal. A category $\mathcal{C}$ is

1. **Locally small** if, for each $A, B \in \mathrm{Obj}(\mathcal{C})$, the class $\mathrm{Hom}_{\mathcal{C}}(A, B)$ is a set.
2. **Locally essentially small** if, for each $A, B \in \mathrm{Obj}(\mathcal{C})$, the class

$$\mathrm{Hom}_{\mathcal{C}}(A, B)/\{\text{isomorphisms}\}$$

is a set.

3. **Small** if $\mathcal{C}$ is locally small and $\mathrm{Obj}(\mathcal{C})$ is a set.
4. κ-**Small** if $\mathcal{C}$ is locally small, $\mathrm{Obj}(\mathcal{C})$ is a set, and we have $\#\mathrm{Obj}(\mathcal{C}) < \kappa$.

8.1.2 Examples of Categories

Example 8.1.2.1.1. The **punctual category**[1] is the category pt where

- *Objects.* We have
$$\mathrm{Obj}(\mathrm{pt}) \overset{\mathrm{def}}{=} \{\star\}.$$
- *Morphisms.* The unique Hom-set of pt is defined by
$$\mathrm{Hom}_{\mathrm{pt}}(\star, \star) \overset{\mathrm{def}}{=} \{\mathrm{id}_\star\}.$$
- *Identities.* The unit map
$$\mathbb{1}^{\mathrm{pt}}_{\star} \colon \mathrm{pt} \to \mathrm{Hom}_{\mathrm{pt}}(\star, \star)$$
of pt at $\star$ is defined by
$$\mathrm{id}^{\mathrm{pt}}_{\star} \overset{\mathrm{def}}{=} \mathrm{id}_\star.$$
- *Composition.* The composition map
$$\circ^{\mathrm{pt}}_{\star,\star,\star} \colon \mathrm{Hom}_{\mathrm{pt}}(\star, \star) \times \mathrm{Hom}_{\mathrm{pt}}(\star, \star) \to \mathrm{Hom}_{\mathrm{pt}}(\star, \star)$$
of pt at $(\star, \star, \star)$ is given by the bijection $\mathrm{pt} \times \mathrm{pt} \cong \mathrm{pt}$.

Example 8.1.2.1.2. We have an isomorphism of categories[2]

$$\mathsf{Mon} \cong \mathrm{pt} \underset{\mathsf{Sets}}{\times} \mathsf{Cats}, \qquad \begin{tikzcd} \mathsf{Mon} \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathsf{Cats} \arrow[d, "\mathrm{Obj}"] \\ \mathrm{pt} \arrow[r, "{[\mathrm{pt}]}"'] & \mathsf{Sets} \end{tikzcd}$$

via the delooping functor $\mathsf{B} \colon \mathsf{Mon} \to \mathsf{Cats}$ of **??** of **??**, exhibiting monoids as exactly those categories having a single object.

[1] *Further Terminology:* Also called the **singleton category**.

[2] This can be enhanced to an isomorphism of 2-categories

$$\mathsf{Mon}_{\mathsf{2disc}} \cong \mathrm{pt}_{\mathsf{bi}} \underset{\mathsf{Sets}_{\mathsf{2disc}}}{\times} \mathsf{Cats}_{2,*}, \qquad \begin{tikzcd} \mathsf{Mon}_{\mathsf{2disc}} \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathsf{Cats}_{2,*} \arrow[d, "\mathrm{Obj}"] \\ \mathrm{pt}_{\mathsf{bi}} \arrow[r, "{[\mathrm{pt}]}"'] & \mathsf{Sets}_{\mathsf{2disc}} \end{tikzcd}$$

Proof. Omitted. □

Example 8.1.2.1.3. The **empty category** is the category $\emptyset_{\mathsf{cat}}$ where

- *Objects.* We have
$$\mathrm{Obj}(\emptyset_{\mathsf{cat}}) \overset{\mathrm{def}}{=} \emptyset.$$
- *Morphisms.* We have
$$\mathrm{Mor}(\emptyset_{\mathsf{cat}}) \overset{\mathrm{def}}{=} \emptyset.$$
- *Identities and Composition.* Having no objects, $\emptyset_{\mathsf{cat}}$ has no unit nor composition maps.

Example 8.1.2.1.4. The ***n*th ordinal category** is the category $\mathbb{n}$ where[3]

- *Objects.* We have
$$\mathrm{Obj}(\mathbb{n}) \overset{\mathrm{def}}{=} \{[0], \ldots, [n]\}.$$
- *Morphisms.* For each $[i], [j] \in \mathrm{Obj}(\mathbb{n})$, we have
$$\mathrm{Hom}_{\mathbb{n}}([i], [j]) \overset{\mathrm{def}}{=} \begin{cases} \{\mathrm{id}_{[i]}\} & \text{if } [i] = [j], \\ \{[i] \to [j]\} & \text{if } [j] < [i], \\ \emptyset & \text{if } [j] > [i]. \end{cases}$$

---

between the discrete 2-category $\mathsf{Mon}_{\mathsf{2disc}}$ on Mon and the 2-category of pointed categories with one object.

[3] In other words, $\mathbb{n}$ is the category associated to the poset
$$[0] \to [1] \to \cdots \to [n-1] \to [n].$$
The category $\mathbb{n}$ for $n \geq 2$ may also be defined in terms of $\mathbb{0}$ and joins (**??**): we have isomorphisms of categories
$$\begin{aligned} \mathbb{1} &\cong \mathbb{0} \star \mathbb{0}, \\ \mathbb{2} &\cong \mathbb{1} \star \mathbb{0} \\ &\cong (\mathbb{0} \star \mathbb{0}) \star \mathbb{0}, \\ \mathbb{3} &\cong \mathbb{2} \star \mathbb{0} \\ &\cong (\mathbb{1} \star \mathbb{0}) \star \mathbb{0} \\ &\cong ((\mathbb{0} \star \mathbb{0}) \star \mathbb{0}) \star \mathbb{0}, \\ \mathbb{4} &\cong \mathbb{3} \star \mathbb{0} \\ &\cong (\mathbb{2} \star \mathbb{0}) \star \mathbb{0} \\ &\cong ((\mathbb{1} \star \mathbb{0}) \star \mathbb{0}) \star \mathbb{0} \\ &\cong (((\mathbb{0} \star \mathbb{0}) \star \mathbb{0}) \star \mathbb{0}) \star \mathbb{0}, \end{aligned}$$
and so on.

- *Identities.* For each $[i] \in \mathrm{Obj}(\mathbb{n})$, the unit map

$$\mathbb{1}^{\mathbb{n}}_{[i]} \colon \mathrm{pt} \to \mathrm{Hom}_{\mathbb{n}}([i],[i])$$

  of $\mathbb{n}$ at $[i]$ is defined by

$$\mathrm{id}^{\mathbb{n}}_{[i]} \overset{\text{def}}{=} \mathrm{id}_{[i]}.$$

- *Composition.* For each $[i],[j],[k] \in \mathrm{Obj}(\mathbb{n})$, the composition map

$$\circ^{\mathbb{n}}_{[i],[j],[k]} \colon \mathrm{Hom}_{\mathbb{n}}([j],[k]) \times \mathrm{Hom}_{\mathbb{n}}([i],[j]) \to \mathrm{Hom}_{\mathbb{n}}([i],[k])$$

  of $\mathbb{n}$ at $([i],[j],[k])$ is defined by

$$\begin{gathered}\mathrm{id}_{[i]} \circ \mathrm{id}_{[i]} = \mathrm{id}_{[i]},\\ ([j] \to [k]) \circ ([i] \to [j]) = ([i] \to [k]).\end{gathered}$$

Example 8.1.2.1.5. Here we list some of the other categories appearing throughout this work.

1. The category Sets_* of pointed sets of Definition 3.1.3.1.1.
2. The category Rel of sets and relations of Definition 5.2.1.1.1.
3. The category $\mathsf{Span}(A,B)$ of spans from a set A to a set B of **??**.
4. The category $\mathsf{ISets}(K)$ of K-indexed sets of **??**.
5. The category ISets of indexed sets of **??**.
6. The category $\mathsf{FibSets}(K)$ of K-fibred sets of **??**.
7. The category $\mathsf{FibSets}$ of fibred sets of **??**.
8. Categories of functors $\mathsf{Fun}(\mathcal{C},\mathcal{D})$ as in Definition 8.9.1.1.1.
9. The category of categories Cats of Definition 8.9.2.1.1.
10. The category of groupoids Grpd of Definition 8.9.4.1.1.

8.1.3 Posetal Categories

Definition 8.1.3.1.1. Let $(X,\preceq_X)$ be a poset.

1. The **posetal category associated to** $(X,\preceq_X)$ is the category X_{pos} where

- *Objects.* We have
$$\mathrm{Obj}(X_{\mathrm{pos}}) \overset{\mathrm{def}}{=} X.$$
- *Morphisms.* For each $a, b \in \mathrm{Obj}(X_{\mathrm{pos}})$, we have
$$\mathrm{Hom}_{X_{\mathrm{pos}}}(a, b) \overset{\mathrm{def}}{=} \begin{cases} \mathrm{pt} & \text{if } a \preceq_X b, \\ \emptyset & \text{otherwise.} \end{cases}$$
- *Identities.* For each $a \in \mathrm{Obj}(X_{\mathrm{pos}})$, the unit map
$$\mathbb{1}^{X_{\mathrm{pos}}}_a \colon \mathrm{pt} \to \mathrm{Hom}_{X_{\mathrm{pos}}}(a, a)$$
of X_{pos} at a is given by the identity map.
- *Composition.* For each $a, b, c \in \mathrm{Obj}(X_{\mathrm{pos}})$, the composition map
$$\circ^{X_{\mathrm{pos}}}_{a,b,c} \colon \mathrm{Hom}_{X_{\mathrm{pos}}}(b, c) \times \mathrm{Hom}_{X_{\mathrm{pos}}}(a, b) \to \mathrm{Hom}_{X_{\mathrm{pos}}}(a, c)$$
of X_{pos} at (a, b, c) is defined as either the inclusion $\emptyset \hookrightarrow \mathrm{pt}$ or the identity map of pt, depending on whether we have $a \preceq_X b$, $b \preceq_X c$, and $a \preceq_X c$.

2. A category C is **posetal**[4] if C is equivalent to X_{pos} for some poset $(X, \preceq_X)$.

Proposition 8.1.3.1.2. Let $(X, \preceq_X)$ be a poset and let C be a category.

1. *Functoriality.* The assignment $(X, \preceq_X) \mapsto X_{\mathrm{pos}}$ defines a functor
$$(-)_{\mathrm{pos}} \colon \mathsf{Pos} \to \mathsf{Cats}.$$
2. *Fully Faithfulness.* The functor $(-)_{\mathrm{pos}}$ of Item 1 is fully faithful.
3. *Characterisations.* The following conditions are equivalent:
   (a) The category C is posetal.
   (b) For each $A, B \in \mathrm{Obj}(C)$ and each $f, g \in \mathrm{Hom}_C(A, B)$, we have $f = g$.

Proof. *Item 1*, *Functoriality*: Omitted.
Item 2, *Fully Faithfulness*: Omitted.
Item 3, *Characterisations*: Clear. □

[4] *Further Terminology:* Also called a **thin** category or a $(0, 1)$-**category**.

8.1.4 Subcategories

Let $\mathcal{C}$ be a category.

Definition 8.1.4.1.1. A **subcategory** of $\mathcal{C}$ is a category $\mathcal{A}$ satisfying the following conditions:

1. *Objects.* We have $\mathrm{Obj}(\mathcal{A}) \subset \mathrm{Obj}(\mathcal{C})$.
2. *Morphisms.* For each $A, B \in \mathrm{Obj}(\mathcal{A})$, we have

$$\mathrm{Hom}_{\mathcal{A}}(A, B) \subset \mathrm{Hom}_{\mathcal{C}}(A, B).$$

3. *Identities.* For each $A \in \mathrm{Obj}(\mathcal{A})$, we have

$$\mathbb{1}^{\mathcal{A}}_{A} = \mathbb{1}^{\mathcal{C}}_{A}.$$

4. *Composition.* For each $A, B, C \in \mathrm{Obj}(\mathcal{A})$, we have

$$\circ^{\mathcal{A}}_{A,B,C} = \circ^{\mathcal{C}}_{A,B,C}.$$

Definition 8.1.4.1.2. A subcategory $\mathcal{A}$ of $\mathcal{C}$ is **full** if the canonical inclusion functor $\mathcal{A} \to \mathcal{C}$ is full, i.e. if, for each $A, B \in \mathrm{Obj}(\mathcal{A})$, the inclusion

$$\iota_{A,B} \colon \mathrm{Hom}_{\mathcal{A}}(A, B) \hookrightarrow \mathrm{Hom}_{\mathcal{C}}(A, B)$$

is surjective (and thus bijective).

Definition 8.1.4.1.3. A subcategory $\mathcal{A}$ of a category $\mathcal{C}$ is **strictly full** if it satisfies the following conditions:

1. *Fullness.* The subcategory $\mathcal{A}$ is full.
2. *Closedness Under Isomorphisms.* The class $\mathrm{Obj}(\mathcal{A})$ is closed under isomorphisms.[5]

Definition 8.1.4.1.4. A subcategory $\mathcal{A}$ of $\mathcal{C}$ is **wide**[6] if $\mathrm{Obj}(\mathcal{A}) = \mathrm{Obj}(\mathcal{C})$.

8.1.5 Skeletons of Categories

Definition 8.1.5.1.1. A[7] **skeleton** of a category $\mathcal{C}$ is a full subcategory $\mathsf{Sk}(\mathcal{C})$ with one object from each isomorphism class of objects of $\mathcal{C}$.

[5] That is, given $A \in \mathrm{Obj}(\mathcal{A})$ and $C \in \mathrm{Obj}(\mathcal{C})$, if $C \cong A$, then $C \in \mathrm{Obj}(\mathcal{A})$.
[6] *Further Terminology:* Also called **lluf**.
[7] Due to Item 3 of Proposition 8.1.5.1.3, we often refer to any such full subcategory $\mathsf{Sk}(\mathcal{C})$

Definition 8.1.5.1.2. A category $\mathcal{C}$ is **skeletal** if $\mathcal{C} \cong \mathsf{Sk}(\mathcal{C})$.[8]

Proposition 8.1.5.1.3. Let $\mathcal{C}$ be a category.

1. *Existence.* Assuming the axiom of choice, $\mathsf{Sk}(\mathcal{C})$ always exists.

2. *Pseudofunctoriality.* The assignment $\mathcal{C} \mapsto \mathsf{Sk}(\mathcal{C})$ defines a pseudofunctor
$$\mathsf{Sk}\colon \mathsf{Cats}_2 \to \mathsf{Cats}_2.$$

3. *Uniqueness Up to Equivalence.* Any two skeletons of $\mathcal{C}$ are equivalent.

4. *Inclusions of Skeletons Are Equivalences.* The inclusion
$$\iota_{\mathcal{C}}\colon \mathsf{Sk}(\mathcal{C}) \hookrightarrow \mathcal{C}$$
of a skeleton of $\mathcal{C}$ into $\mathcal{C}$ is an equivalence of categories.

Proof. *Item 1*, *Existence*: See [nLab23, Section "Existence of Skeletons of Categories"].
Item 2, *Pseudofunctoriality*: See [nLab23, Section "Skeletons as an Endo-Pseudofunctor on $\mathfrak{Cat}$"].
Item 3, *Uniqueness Up to Equivalence*: Clear.
Item 4, *Inclusions of Skeletons Are Equivalences*: Clear. □

8.1.6 Precomposition and Postcomposition

Let $\mathcal{C}$ be a category and let $A, B, C \in \mathrm{Obj}(\mathcal{C})$.

Definition 8.1.6.1.1. Let $f\colon A \to B$ and $g\colon B \to C$ be morphisms of $\mathcal{C}$.

1. The **precomposition function associated to** f is the function
$$f^*\colon \mathrm{Hom}_{\mathcal{C}}(B, C) \to \mathrm{Hom}_{\mathcal{C}}(A, C)$$
defined by
$$f^*(\phi) \stackrel{\text{def}}{=} \phi \circ f$$
for each $\phi \in \mathrm{Hom}_{\mathcal{C}}(B, C)$.

2. The **postcomposition function associated to** g is the function
$$g_*\colon \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{C}}(A, C)$$
defined by
$$g_*(\phi) \stackrel{\text{def}}{=} g \circ \phi$$
for each $\phi \in \mathrm{Hom}_{\mathcal{C}}(A, B)$.

of $\mathcal{C}$ as *the* skeleton of $\mathcal{C}$.

[8]That is, $\mathcal{C}$ is **skeletal** if isomorphic objects of $\mathcal{C}$ are equal.

Proposition 8.1.6.1.2. Let $A, B, C, D \in \mathrm{Obj}(\mathcal{C})$ and let $f\colon A \to B$ and $g\colon B \to C$ be morphisms of $\mathcal{C}$.

1. *Interaction Between Precomposition and Postcomposition.* We have

$$g_* \circ f^* = f^* \circ g_*, \qquad \begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(B,C) \arrow[r, "g_*"] \arrow[d, "f^*"'] & \mathrm{Hom}_{\mathcal{C}}(B,D) \arrow[d, "f^*"] \\ \mathrm{Hom}_{\mathcal{C}}(A,C) \arrow[r, "g_*"'] & \mathrm{Hom}_{\mathcal{C}}(A,D). \end{tikzcd}$$

2. *Interaction With Composition I.* We have

$$(g \circ f)^* = f^* \circ g^*, \qquad \begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(X,A) \arrow[r, "f_*"] \arrow[dr, "(g\circ f)_*"'] & \mathrm{Hom}_{\mathcal{C}}(X,B) \arrow[d, "g_*"] \\ & \mathrm{Hom}_{\mathcal{C}}(X,C), \end{tikzcd}$$

$$(g \circ f)_* = g_* \circ f_*, \qquad \begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(C,X) \arrow[r, "g^*"] \arrow[dr, "(g\circ f)^*"'] & \mathrm{Hom}_{\mathcal{C}}(B,X) \arrow[d, "f^*"] \\ & \mathrm{Hom}_{\mathcal{C}}(A,X). \end{tikzcd}$$

3. *Interaction With Composition II.* We have

$$\begin{tikzcd} \mathrm{pt} \arrow[r, "{[f]}"] \arrow[dr, "{[g\circ f]}"'] & \mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[d, "g_*"] \\ & \mathrm{Hom}_{\mathcal{C}}(A,C) \end{tikzcd} \qquad \begin{aligned} [g \circ f] &= g_* \circ [f], \\ [g \circ f] &= f^* \circ [g], \end{aligned} \qquad \begin{tikzcd} \mathrm{pt} \arrow[r, "{[g]}"] \arrow[dr, "{[g\circ f]}"'] & \mathrm{Hom}_{\mathcal{C}}(B,C) \arrow[d, "f^*"] \\ & \mathrm{Hom}_{\mathcal{C}}(A,C). \end{tikzcd}$$

4. *Interaction With Composition III.* We have

$$f^* \circ \circ^C_{A,B,C} = \circ^C_{X,B,C} \circ (f^* \times \mathrm{id}), \qquad \begin{tikzcd} \mathrm{Hom}_C(B,C) \times \mathrm{Hom}_C(A,B) \arrow[r, "\circ^C_{A,B,C}"] \arrow[d, "\mathrm{id}\times f^*"'] & \mathrm{Hom}_C(A,C) \arrow[d, "f^*"] \\ \mathrm{Hom}_C(B,C) \times \mathrm{Hom}_C(X,B) \arrow[r, "\circ^C_{X,B,C}"'] & \mathrm{Hom}_C(X,C), \end{tikzcd}$$

$$g_* \circ \circ^C_{A,B,C} = \circ^C_{A,B,D} \circ (\mathrm{id} \times g_*), \qquad \begin{tikzcd} \mathrm{Hom}_C(B,C) \times \mathrm{Hom}_C(A,B) \arrow[r, "\circ^C_{A,B,C}"] \arrow[d, "g_*\times\mathrm{id}"'] & \mathrm{Hom}_C(A,C) \arrow[d, "g^*"] \\ \mathrm{Hom}_C(B,D) \times \mathrm{Hom}_C(A,B) \arrow[r, "\circ^C_{A,B,D}"'] & \mathrm{Hom}_C(A,D). \end{tikzcd}$$

5. *Interaction With Identities.* We have

$$(\mathrm{id}_A)^* = \mathrm{id}_{\mathrm{Hom}_C(A,B)},$$
$$(\mathrm{id}_B)_* = \mathrm{id}_{\mathrm{Hom}_C(A,B)}.$$

Proof. *Item 1, Interaction Between Precomposition and Postcomposition*: Clear.
Item 2, Interaction With Composition I: Clear.
Item 3, Interaction With Composition II: Clear.
Item 4, Interaction With Composition III: Clear.
Item 5, Interaction With Identities: Clear. □

8.2 The Quadruple Adjunction With Sets

8.2.1 Statement

Let C be a category.

Proposition 8.2.1.1.1. We have a quadruple adjunction

$$(\pi_0 \dashv (-)_{\mathsf{disc}} \dashv \mathsf{Obj} \dashv (-)_{\mathsf{indisc}}) \colon \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left=15, "(-)_{\mathsf{disc}}"] \arrow[r, bend right=50, "(-)_{\mathsf{indisc}}"'] & \mathsf{Cats}, \arrow[l, bend right=50, "\pi_0"'] \arrow[l, bend left=15, "\mathsf{Obj}"] \end{tikzcd}$$

witnessed by bijections of sets

$$\mathrm{Hom}_{\mathsf{Sets}}(\pi_0(C), X) \cong \mathrm{Hom}_{\mathsf{Cats}}(C, X_{\mathsf{disc}}),$$
$$\mathrm{Hom}_{\mathsf{Cats}}(X_{\mathsf{disc}}, C) \cong \mathrm{Hom}_{\mathsf{Sets}}(X, \mathsf{Obj}(C)),$$
$$\mathrm{Hom}_{\mathsf{Sets}}(\mathsf{Obj}(C), X) \cong \mathrm{Hom}_{\mathsf{Cats}}(C, X_{\mathsf{indisc}}),$$

natural in $\mathcal{C} \in \mathrm{Obj}(\mathsf{Cats})$ and $X \in \mathrm{Obj}(\mathsf{Sets})$, where

- The functor
$$\pi_0 \colon \mathsf{Cats} \longrightarrow \mathsf{Sets},$$
the **connected components functor**, is the functor sending a category to its set of connected components of Definition 8.2.2.2.1.

- The functor
$$(-)_{\mathsf{disc}} \colon \mathsf{Sets} \longrightarrow \mathsf{Cats},$$
the **discrete category functor**, is the functor sending a set to its associated discrete category of Item 1.

- The functor
$$\mathrm{Obj} \colon \mathsf{Cats} \longrightarrow \mathsf{Sets},$$
the **object functor**, is the functor sending a category to its set of objects.

- The functor
$$(-)_{\mathsf{indisc}} \colon \mathsf{Sets} \longrightarrow \mathsf{Cats},$$
the **indiscrete category functor**, is the functor sending a set to its associated indiscrete category of Item 1.

Proof. Omitted. □

8.2.2 Connected Components and Connected Categories

8.2.2.1 Connected Components of Categories

Let $\mathcal{C}$ be a category.

Definition 8.2.2.1.1. A **connected component** of $\mathcal{C}$ is a full subcategory $\mathcal{I}$ of $\mathcal{C}$ satisfying the following conditions:[9]

1. *Non-Emptiness.* We have $\mathrm{Obj}(\mathcal{I}) \neq \emptyset$.

2. *Connectedness.* There exists a zigzag of arrows between any two objects of $\mathcal{I}$.

8.2.2.2 Sets of Connected Components of Categories

Let $\mathcal{C}$ be a category.

[9] In other words, a **connected component** of $\mathcal{C}$ is an element of the set $\mathrm{Obj}(\mathcal{C})/\sim$ with $\sim$

Definition 8.2.2.2.1. The **set of connected components of** $\mathcal{C}$ is the set $\pi_0(\mathcal{C})$ whose elements are the connected components of $\mathcal{C}$.

Proposition 8.2.2.2.2. Let $\mathcal{C}$ be a category.

1. *Functoriality.* The assignment $\mathcal{C} \mapsto \pi_0(\mathcal{C})$ defines a functor

$$\pi_0 \colon \mathsf{Cats} \longrightarrow \mathsf{Sets}.$$

2. *Adjointness.* We have a quadruple adjunction

$$(\pi_0 \dashv (-)_{\mathsf{disc}} \dashv \mathsf{Obj} \dashv (-)_{\mathsf{indisc}})\colon \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left=20, "(-)_{\mathsf{disc}}"] \arrow[r, bend right=70, "(-)_{\mathsf{indisc}}"'] & \mathsf{Cats}. \arrow[l, bend right=70, "\pi_0"'] \arrow[l, bend left=20, "\mathsf{Obj}"] \end{tikzcd}$$

3. *Interaction With Groupoids.* If $\mathcal{C}$ is a groupoid, then we have an isomorphism of categories

$$\pi_0(\mathcal{C}) \cong \mathrm{K}(\mathcal{C}),$$

where $\mathrm{K}(\mathcal{C})$ is the set of isomorphism classes of $\mathcal{C}$ of **??**.

4. *Preservation of Colimits.* The functor π_0 of Item 1 preserves colimits. In particular, we have bijections of sets

$$\begin{aligned} \pi_0\left(\mathcal{C} \textstyle\coprod \mathcal{D}\right) &\cong \pi_0(\mathcal{C}) \textstyle\coprod \pi_0(\mathcal{D}), \\ \pi_0\left(\mathcal{C} \textstyle\coprod_{\mathcal{E}} \mathcal{D}\right) &\cong \pi_0(\mathcal{C}) \textstyle\coprod_{\pi_0(\mathcal{E})} \pi_0(\mathcal{D}), \\ \pi_0\left(\mathrm{CoEq}\left(\mathcal{C} \overset{F}{\underset{G}{\rightrightarrows}} \mathcal{D}\right)\right) &\cong \mathrm{CoEq}\left(\pi_0(\mathcal{C}) \overset{\pi_0(F)}{\underset{\pi_0(G)}{\rightrightarrows}} \pi_0(\mathcal{D})\right), \end{aligned}$$

natural in $\mathcal{C}, \mathcal{D}, \mathcal{E} \in \mathrm{Obj}(\mathsf{Cats})$.

5. *Symmetric Strong Monoidality With Respect to Coproducts.* The connected components functor of Item 1 has a symmetric strong monoidal structure

$$\left(\pi_0, \pi_0^{\coprod}, \pi_{0|\mathbb{1}}^{\coprod}\right) \colon \left(\mathsf{Cats}, \textstyle\coprod, \emptyset_{\mathsf{cat}}\right) \longrightarrow \left(\mathsf{Sets}, \textstyle\coprod, \emptyset\right),$$

---

the equivalence relation generated by the relation $\sim'$ obtained by declaring $A \sim' B$ iff there exists a morphism of $\mathcal{C}$ from A to B.

being equipped with isomorphisms

$$\pi^{\coprod}_{0|\mathcal{C},\mathcal{D}} \colon \pi_0(\mathcal{C}) \coprod \pi_0(\mathcal{D}) \xrightarrow{\cong} \pi_0(\mathcal{C} \coprod \mathcal{D}),$$
$$\pi^{\coprod}_{0|\mathbb{1}} \colon \emptyset \xrightarrow{\cong} \pi_0(\emptyset_{\mathsf{cat}}),$$

natural in $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$.

6. *Symmetric Strong Monoidality With Respect to Products.* The connected components functor of Item 1 has a symmetric strong monoidal structure

$$\left(\pi_0, \pi_0^{\times}, \pi^{\times}_{0|\mathbb{1}}\right) \colon (\mathsf{Cats}, \times, \mathrm{pt}) \longrightarrow (\mathsf{Sets}, \times, \mathrm{pt}),$$

being equipped with isomorphisms

$$\pi^{\times}_{0|\mathcal{C},\mathcal{D}} \colon \pi_0(\mathcal{C}) \times \pi_0(\mathcal{D}) \xrightarrow{\cong} \pi_0(\mathcal{C} \times \mathcal{D}),$$
$$\pi^{\times}_{0|\mathbb{1}} \colon \mathrm{pt} \xrightarrow{\cong} \pi_0(\mathrm{pt}),$$

natural in $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$.

Proof. *Item 1*, *Functoriality*: Clear.
Item 2, *Adjointness*: This is proved in Proposition 8.2.1.1.1.
Item 3, *Interaction With Groupoids*: Clear.
Item 4, *Preservation of Colimits*: This follows from Item 2 and **??** of **??**.
Item 5, *Symmetric Strong Monoidality With Respect to Coproducts*: Clear.
Item 6, *Symmetric Strong Monoidality With Respect to Products*: Clear. □

8.2.2.3 Connected Categories

Definition 8.2.2.3.1. A category $\mathcal{C}$ is **connected** if $\pi_0(\mathcal{C}) \cong \mathrm{pt}$.[10,11]

8.2.3 Discrete Categories

Definition 8.2.3.1.1. Let X be a set.

1. The **discrete category on** X is the category X_{disc} where

   - *Objects.* We have

$$\mathrm{Obj}(X_{\mathsf{disc}}) \stackrel{\text{def}}{=} X.$$

[10] *Further Terminology:* A category is **disconnected** if it is not connected.
[11] *Example:* A groupoid is connected iff any two of its objects are isomorphic.

- *Morphisms.* For each $A, B \in \mathrm{Obj}(X_{\mathsf{disc}})$, we have

$$\mathrm{Hom}_{X_{\mathsf{disc}}}(A, B) \stackrel{\text{def}}{=} \begin{cases} \mathrm{id}_A & \text{if } A = B, \\ \emptyset & \text{if } A \neq B. \end{cases}$$

- *Identities.* For each $A \in \mathrm{Obj}(X_{\mathsf{disc}})$, the unit map

$$\mathbb{1}_A^{X_{\mathsf{disc}}} \colon \mathrm{pt} \to \mathrm{Hom}_{X_{\mathsf{disc}}}(A, A)$$

  of X_{disc} at A is defined by

$$\mathrm{id}_A^{X_{\mathsf{disc}}} \stackrel{\text{def}}{=} \mathrm{id}_A.$$

- *Composition.* For each $A, B, C \in \mathrm{Obj}(X_{\mathsf{disc}})$, the composition map

$$\circ_{A,B,C}^{X_{\mathsf{disc}}} \colon \mathrm{Hom}_{X_{\mathsf{disc}}}(B, C) \times \mathrm{Hom}_{X_{\mathsf{disc}}}(A, B) \to \mathrm{Hom}_{X_{\mathsf{disc}}}(A, C)$$

  of X_{disc} at (A, B, C) is defined by

$$\mathrm{id}_A \circ \mathrm{id}_A \stackrel{\text{def}}{=} \mathrm{id}_A.$$

2. A category $\mathcal{C}$ is **discrete** if it is equivalent to X_{disc} for some set X.

Proposition 8.2.3.1.2. Let X be a set.

1. *Functoriality.* The assignment $X \mapsto X_{\mathsf{disc}}$ defines a functor

$$(-)_{\mathsf{disc}} \colon \mathsf{Sets} \to \mathsf{Cats}.$$

2. *Adjointness.* We have a quadruple adjunction

$$(\pi_0 \dashv (-)_{\mathsf{disc}} \dashv \mathrm{Obj} \dashv (-)_{\mathsf{indisc}}) \colon \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left=20, "(-)_{\mathsf{disc}}" description] \arrow[r, bend right=60, "(-)_{\mathsf{indisc}}"'] & \mathsf{Cats} \arrow[l, bend right=60, "\pi_0"'] \arrow[l, bend left=20, "\mathrm{Obj}" description] \end{tikzcd}$$

3. *Symmetric Strong Monoidality With Respect to Coproducts.* The functor of Item 1 has a symmetric strong monoidal structure

$$\left((-)_{\mathsf{disc}}, (-)^{\coprod}_{\mathsf{disc}}, (-)^{\coprod}_{\mathsf{disc}|\mathbb{1}}\right) \colon (\mathsf{Sets}, \coprod, \emptyset) \to (\mathsf{Cats}, \coprod, \emptyset_{\mathsf{cat}}),$$

   being equipped with isomorphisms

$$(-)^{\coprod}_{\mathsf{disc}|X,Y} \colon X_{\mathsf{disc}} \coprod Y_{\mathsf{disc}} \xrightarrow{\cong} (X \coprod Y)_{\mathsf{disc}},$$

$$(-)^{\coprod}_{\mathsf{disc}|\mathbb{1}} \colon \emptyset_{\mathsf{cat}} \xrightarrow{\cong} \emptyset_{\mathsf{disc}},$$

   natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

4. *Symmetric Strong Monoidality With Respect to Products.* The functor of Item 1 has a symmetric strong monoidal structure

$$\left((-)_{\mathsf{disc}}, (-)^{\times}_{\mathsf{disc}}, (-)^{\times}_{\mathsf{disc}|\mathbb{1}}\right) \colon (\mathsf{Sets}, \times, \mathrm{pt}) \to (\mathsf{Cats}, \times, \mathrm{pt}),$$

being equipped with isomorphisms

$$(-)^{\times}_{\mathsf{disc}|X,Y} \colon X_{\mathsf{disc}} \times Y_{\mathsf{disc}} \xrightarrow{\cong} (X \times Y)_{\mathsf{disc}},$$
$$(-)^{\times}_{\mathsf{disc}|\mathbb{1}} \colon \mathrm{pt} \xrightarrow{\cong} \mathrm{pt}_{\mathsf{disc}},$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

Proof. *Item 1, Functoriality*: Clear.
Item 2, Adjointness: This is proved in Proposition 8.2.1.1.1.
Item 3, Symmetric Strong Monoidality With Respect to Coproducts: Clear.
Item 4, Symmetric Strong Monoidality With Respect to Products: Clear. □

8.2.4 Indiscrete Categories

Definition 8.2.4.1.1. Let X be a set.

1. The **indiscrete category on** X[12] is the category X_{indisc} where

   - *Objects.* We have

   $$\mathrm{Obj}(X_{\mathsf{indisc}}) \overset{\text{def}}{=} X.$$

   - *Morphisms.* For each $A, B \in \mathrm{Obj}(X_{\mathsf{indisc}})$, we have

   $$\begin{aligned}\mathrm{Hom}_{X_{\mathsf{disc}}}(A, B) &\overset{\text{def}}{=} \{[A] \to [B]\}\\ &\cong \mathrm{pt}.\end{aligned}$$

   - *Identities.* For each $A \in \mathrm{Obj}(X_{\mathsf{indisc}})$, the unit map

   $$\mathbb{1}^{X_{\mathsf{indisc}}}_A \colon \mathrm{pt} \to \mathrm{Hom}_{X_{\mathsf{indisc}}}(A, A)$$

   of X_{indisc} at A is defined by

   $$\mathrm{id}^{X_{\mathsf{indisc}}}_A \overset{\text{def}}{=} \{[A] \to [A]\}.$$

   - *Composition.* For each $A, B, C \in \mathrm{Obj}(X_{\mathsf{indisc}})$, the composition

[12] *Further Terminology:* Sometimes called the **chaotic category on** X.

map

$$\circ^{X_{\mathsf{indisc}}}_{A,B,C} \colon \mathrm{Hom}_{X_{\mathsf{indisc}}}(B,C) \times \mathrm{Hom}_{X_{\mathsf{indisc}}}(A,B) \to \mathrm{Hom}_{X_{\mathsf{indisc}}}(A,C)$$

of X_{disc} at (A,B,C) is defined by

$$([B] \to [C]) \circ ([A] \to [B]) \overset{\mathrm{def}}{=} ([A] \to [C]).$$

2. A category $\mathcal{C}$ is **indiscrete** if it is equivalent to X_{indisc} for some set X.

Proposition 8.2.4.1.2. Let X be a set.

1. *Functoriality.* The assignment $X \mapsto X_{\mathsf{indisc}}$ defines a functor

$$(-)_{\mathsf{indisc}} \colon \mathsf{Sets} \to \mathsf{Cats}.$$

2. *Adjointness.* We have a quadruple adjunction

$$(\pi_0 \dashv (-)_{\mathsf{disc}} \dashv \mathsf{Obj} \dashv (-)_{\mathsf{indisc}}) \colon \begin{tikzcd} \mathsf{Sets} \arrow[r, shift left=1, "(-)_{\mathsf{disc}}"] \arrow[r, bend right=60, "(-)_{\mathsf{indisc}}"'] & \mathsf{Cats}. \arrow[l, shift left=1, "\mathsf{Obj}"] \arrow[l, bend right=60, "\pi_0"'] \end{tikzcd}$$

3. *Symmetric Strong Monoidality With Respect to Products.* The functor of Item 1 has a symmetric strong monoidal structure

$$\left((-)_{\mathsf{indisc}}, (-)^{\times}_{\mathsf{indisc}}, (-)^{\times}_{\mathsf{indisc}|\mathbb{1}}\right) \colon (\mathsf{Sets}, \times, \mathrm{pt}) \to (\mathsf{Cats}, \times, \mathrm{pt}),$$

being equipped with isomorphisms

$$(-)^{\times}_{\mathsf{indisc}|X,Y} \colon X_{\mathsf{indisc}} \times Y_{\mathsf{indisc}} \xrightarrow{\cong} (X \times Y)_{\mathsf{indisc}},$$
$$(-)^{\times}_{\mathsf{indisc}|\mathbb{1}} \colon \mathrm{pt} \xrightarrow{\cong} \mathrm{pt}_{\mathsf{indisc}},$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

Proof. Item 1, Functoriality: Clear.
Item 2, Adjointness: This is proved in Proposition 8.2.1.1.1.
Item 3, Symmetric Strong Monoidality With Respect to Products: Clear. □

8.3 Groupoids

8.3.1 Foundations

Let $\mathcal{C}$ be a category.

Definition 8.3.1.1.1. A morphism $f: A \to B$ of $\mathcal{C}$ is an **isomorphism** if there exists a morphism $f^{-1}: B \to A$ of $\mathcal{C}$ such that

$$
\begin{aligned}
f \circ f^{-1} &= \mathrm{id}_B, \\
f^{-1} \circ f &= \mathrm{id}_A.
\end{aligned}
$$

Notation 8.3.1.1.2. We write $\mathrm{Iso}_{\mathcal{C}}(A, B)$ for the set of all isomorphisms in $\mathcal{C}$ from A to B.

Definition 8.3.1.1.3. A **groupoid** is a category in which every morphism is an isomorphism.

8.3.2 The Groupoid Completion of a Category

Let $\mathcal{C}$ be a category.

Definition 8.3.2.1.1. The **groupoid completion of** $\mathcal{C}$[13] is the pair $(\mathrm{K}_0(\mathcal{C}), \iota_{\mathcal{C}})$ consisting of

- A groupoid $\mathrm{K}_0(\mathcal{C})$;
- A functor $\iota_{\mathcal{C}}: \mathcal{C} \to \mathrm{K}_0(\mathcal{C})$;

satisfying the following universal property:[14]

(UP) Given another such pair $(\mathcal{G}, i)$, there exists a unique functor $\mathrm{K}_0(\mathcal{C}) \xrightarrow{\exists!} \mathcal{G}$ making the diagram

$$
\begin{tikzcd}
 & \mathrm{K}_0(\mathcal{C}) \arrow[d, dashed, "\exists!"] \\
\mathcal{C} \arrow[ru, "\iota_{\mathcal{C}}"] \arrow[r, "i"'] & \mathcal{G}
\end{tikzcd}
$$

commute.

[13] *Further Terminology:* Also called the **Grothendieck groupoid of** $\mathcal{C}$ or the **Grothendieck groupoid completion of** $\mathcal{C}$.

[14] See Item 5 of Proposition 8.3.2.1.3 for an explicit construction.

Construction 8.3.2.1.2. Concretely, the groupoid completion of $\mathcal{C}$ is the Gabriel–Zisman localisation $\mathrm{Mor}(\mathcal{C})^{-1}\mathcal{C}$ of $\mathcal{C}$ at the set $\mathrm{Mor}(\mathcal{C})$ of all morphisms of $\mathcal{C}$; see **??**.
(To be expanded upon later on.)

Proof. Omitted. □

Proposition 8.3.2.1.3. Let $\mathcal{C}$ be a category.

1. *Functoriality.* The assignment $\mathcal{C} \mapsto \mathrm{K}_0(\mathcal{C})$ defines a functor

$$\mathrm{K}_0 : \mathsf{Cats} \to \mathsf{Grpd}.$$

2. *2-Functoriality.* The assignment $\mathcal{C} \mapsto \mathrm{K}_0(\mathcal{C})$ defines a 2-functor

$$\mathrm{K}_0 : \mathsf{Cats}_2 \to \mathsf{Grpd}_2.$$

3. *Adjointness.* We have an adjunction

$$(\mathrm{K}_0 \dashv \iota)\colon \quad \begin{tikzcd} \mathsf{Cats} \arrow[r, shift left, "\mathrm{K}_0"] \arrow[r, phantom, "\perp"] & \mathsf{Grpd}, \arrow[l, shift left, "\iota"] \end{tikzcd}$$

   witnessed by a bijection of sets

$$\mathrm{Hom}_{\mathsf{Grpd}}(\mathrm{K}_0(\mathcal{C}), \mathcal{G}) \cong \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, \mathcal{G}),$$

   natural in $\mathcal{C} \in \mathrm{Obj}(\mathsf{Cats})$ and $\mathcal{G} \in \mathrm{Obj}(\mathsf{Grpd})$, forming, together with the functor Core of Item 1 of Proposition 8.3.3.1.4, a triple adjunction

$$(\mathrm{K}_0 \dashv \iota \dashv \mathsf{Core})\colon \quad \begin{tikzcd} \mathsf{Cats} \arrow[r, bend left, "\mathrm{K}_0"] \arrow[r, bend right, "\mathsf{Core}"'] & \mathsf{Grpd}, \arrow[l, "\iota" description] \end{tikzcd}$$

   witnessed by bijections of sets

$$\begin{aligned} \mathrm{Hom}_{\mathsf{Grpd}}(\mathrm{K}_0(\mathcal{C}), \mathcal{G}) &\cong \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, \mathcal{G}), \\ \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{G}, \mathcal{D}) &\cong \mathrm{Hom}_{\mathsf{Grpd}}(\mathcal{G}, \mathsf{Core}(\mathcal{D})), \end{aligned}$$

   natural in $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$ and $\mathcal{G} \in \mathrm{Obj}(\mathsf{Grpd})$.

4. *2-Adjointness.* We have a 2-adjunction

$$(\mathrm{K}_0 \dashv \iota)\colon\quad \begin{tikzcd} \mathsf{Cats} \arrow[r, shift left, "\mathrm{K}_0"] \arrow[r, phantom, "\perp_2"] & \mathsf{Grpd}, \arrow[l, shift left, hook', "\iota"] \end{tikzcd}$$

witnessed by an isomorphism of categories

$$\mathsf{Fun}(\mathrm{K}_0(\mathcal{C}), \mathcal{G}) \cong \mathsf{Fun}(\mathcal{C}, \mathcal{G}),$$

natural in $\mathcal{C} \in \mathrm{Obj}(\mathsf{Cats})$ and $\mathcal{G} \in \mathrm{Obj}(\mathsf{Grpd})$, forming, together with the 2-functor Core of Item 2 of Proposition 8.3.3.1.4, a triple 2-adjunction

$$(\mathrm{K}_0 \dashv \iota \dashv \mathsf{Core})\colon\quad \begin{tikzcd} \mathsf{Cats} \arrow[r, bend left=50, "\mathrm{K}_0"] \arrow[r, bend right=50, "\mathsf{Core}"'] \arrow[r, phantom, shift left=3, "\perp_2"] \arrow[r, phantom, shift right=3, "\perp_2"] & \mathsf{Grpd}, \arrow[l, hook', "\iota" description] \end{tikzcd}$$

witnessed by isomorphisms of categories

$$\begin{aligned} \mathsf{Fun}(\mathrm{K}_0(\mathcal{C}), \mathcal{G}) &\cong \mathsf{Fun}(\mathcal{C}, \mathcal{G}), \\ \mathsf{Fun}(\mathcal{G}, \mathcal{D}) &\cong \mathsf{Fun}(\mathcal{G}, \mathsf{Core}(\mathcal{D})), \end{aligned}$$

natural in $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$ and $\mathcal{G} \in \mathrm{Obj}(\mathsf{Grpd})$.

5. *Interaction With Classifying Spaces.* We have an isomorphism of groupoids

$$\mathrm{K}_0(\mathcal{C}) \cong \Pi_{\leq 1}(|\mathrm{N}_\bullet(\mathcal{C})|),$$

natural in $\mathcal{C} \in \mathrm{Obj}(\mathsf{Cats})$; i.e. the diagram

$$\begin{tikzcd} \mathsf{Cats} \arrow[r, "\mathrm{K}_0"] \arrow[d, "\mathrm{N}_\bullet"'] \arrow[dr, phantom, "\wr"] & \mathsf{Grp} \\ \mathsf{sSets} \arrow[r, "|-|"'] & \mathsf{Top} \arrow[u, "\Pi_{\leq 1}"'] \end{tikzcd}$$

commutes up to natural isomorphism.

6. *Symmetric Strong Monoidality With Respect to Coproducts.* The groupoid completion functor of Item 1 has a symmetric strong monoidal structure

$$\left(\mathrm{K}_0, \mathrm{K}_0^{\coprod}, \mathrm{K}_{0|\mathbb{1}}^{\coprod}\right)\colon (\mathsf{Cats}, \textstyle\coprod, \emptyset_{\mathsf{cat}}) \to (\mathsf{Grpd}, \textstyle\coprod, \emptyset_{\mathsf{cat}})$$

being equipped with isomorphisms

$$K^{\coprod}_{0|C,\mathcal{D}} \colon K_0(C) \coprod K_0(\mathcal{D}) \xrightarrow{\cong} K_0(C \coprod \mathcal{D}),$$
$$K^{\coprod}_{0|\mathbb{1}} \colon \emptyset_{\mathsf{cat}} \xrightarrow{\cong} K_0(\emptyset_{\mathsf{cat}}),$$

natural in $C, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$.

7. *Symmetric Strong Monoidality With Respect to Products.* The groupoid completion functor of Item 1 has a symmetric strong monoidal structure

$$\left(K_0, K_0^{\times}, K^{\times}_{0|\mathbb{1}}\right) \colon (\mathsf{Cats}, \times, \mathrm{pt}) \longrightarrow (\mathsf{Grpd}, \times, \mathrm{pt})$$

being equipped with isomorphisms

$$K^{\times}_{0|C,\mathcal{D}} \colon K_0(C) \times K_0(\mathcal{D}) \xrightarrow{\cong} K_0(C \times \mathcal{D}),$$
$$K^{\times}_{0|\mathbb{1}} \colon \mathrm{pt} \xrightarrow{\cong} K_0(\mathrm{pt}),$$

natural in $C, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$.

Proof. *Item 1*, *Functoriality*: Omitted.
Item 2, *2-Functoriality*: Omitted.
Item 3, *Adjointness*: Omitted.
Item 4, *2-Adjointness*: Omitted.
Item 5, *Interaction With Classifying Spaces*: See Corollary 18.33 of `https://web.ma.utexas.edu/users/dafr/M392C-2012/Notes/lecture18.pdf`.
Item 6, *Symmetric Strong Monoidality With Respect to Coproducts*: Omitted.
Item 7, *Symmetric Strong Monoidality With Respect to Products*: Omitted. □

8.3.3 The Core of a Category

Let C be a category.

Definition 8.3.3.1.1. The **core** of C is the pair $(\mathrm{Core}(C), \iota_C)$ consisting of

- A groupoid $\mathrm{Core}(C)$;
- A functor $\iota_C \colon \mathrm{Core}(C) \hookrightarrow C$;

satisfying the following universal property:

(UP) Given another such pair $(\mathcal{G}, i)$, there exists a unique functor $\mathcal{G} \xrightarrow{\exists!} \mathsf{Core}(\mathcal{C})$ making the diagram

$$\begin{tikzcd} & \mathsf{Core}(\mathcal{C}) \arrow[d, hook, "\iota_{\mathcal{C}}"] \\ \mathcal{G} \arrow[ur, dashed, "\exists!"] \arrow[r, "i"'] & \mathcal{C} \end{tikzcd}$$

commute.

Notation 8.3.3.1.2. We also write $\mathcal{C}^{\simeq}$ for $\mathsf{Core}(\mathcal{C})$.

Construction 8.3.3.1.3. The core of $\mathcal{C}$ is the wide subcategory of $\mathcal{C}$ spanned by the isomorphisms of $\mathcal{C}$, i.e. the category $\mathsf{Core}(\mathcal{C})$ where[15]

1. *Objects.* We have

$$\mathrm{Obj}(\mathsf{Core}(\mathcal{C})) \overset{\text{def}}{=} \mathrm{Obj}(\mathcal{C}).$$

2. *Morphisms.* The morphisms of $\mathsf{Core}(\mathcal{C})$ are the isomorphisms of $\mathcal{C}$.

Proof. This follows from the fact that functors preserve isomorphisms (Item 1 of Proposition 8.4.1.1.6). □

Proposition 8.3.3.1.4. Let $\mathcal{C}$ be a category.

1. *Functoriality.* The assignment $\mathcal{C} \mapsto \mathsf{Core}(\mathcal{C})$ defines a functor

$$\mathsf{Core}\colon \mathsf{Cats} \longrightarrow \mathsf{Grpd}.$$

2. *2-Functoriality.* The assignment $\mathcal{C} \mapsto \mathsf{Core}(\mathcal{C})$ defines a 2-functor

$$\mathsf{Core}\colon \mathsf{Cats}_2 \longrightarrow \mathsf{Grpd}_2.$$

3. *Adjointness.* We have an adjunction

$$(\iota \dashv \mathsf{Core})\colon \quad \begin{tikzcd} \mathsf{Grpd} \arrow[r, bend left, "\iota"] \arrow[r, phantom, "\perp"] & \mathsf{Cats}, \arrow[l, bend left, "\mathsf{Core}"] \end{tikzcd}$$

witnessed by a bijection of sets

$$\mathrm{Hom}_{\mathsf{Cats}}(\mathcal{G}, \mathcal{D}) \cong \mathrm{Hom}_{\mathsf{Grpd}}(\mathcal{G}, \mathsf{Core}(\mathcal{D})),$$

[15] *Slogan:* The groupoid $\mathsf{Core}(\mathcal{C})$ is the maximal subgroupoid of $\mathcal{C}$.

natural in $\mathcal{G} \in \mathrm{Obj}(\mathsf{Grpd})$ and $\mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$, forming, together with the functor K_0 of Item 1 of Proposition 8.3.2.1.3, a triple adjunction

$$
(\mathrm{K}_0 \dashv \iota \dashv \mathsf{Core}) \colon \quad \begin{tikzcd} \mathsf{Cats} \arrow[r, bend left, "\mathrm{K}_0"] \arrow[r, bend right, "\mathsf{Core}"'] & \mathsf{Grpd}, \arrow[l, hook', "\iota"'] \end{tikzcd}
$$

witnessed by bijections of sets

$$
\begin{aligned}
\mathrm{Hom}_{\mathsf{Grpd}}(\mathrm{K}_0(\mathcal{C}), \mathcal{G}) &\cong \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, \mathcal{G}),\\
\mathrm{Hom}_{\mathsf{Cats}}(\mathcal{G}, \mathcal{D}) &\cong \mathrm{Hom}_{\mathsf{Grpd}}(\mathcal{G}, \mathsf{Core}(\mathcal{D})),
\end{aligned}
$$

natural in $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$ and $\mathcal{G} \in \mathrm{Obj}(\mathsf{Grpd})$.

4. *2-Adjointness.* We have an adjunction

$$
(\iota \dashv \mathsf{Core}) \colon \quad \begin{tikzcd} \mathsf{Grpd} \arrow[r, bend left, hook, "\iota"] \arrow[r, phantom, "\perp_2"] & \mathsf{Cats}, \arrow[l, bend left, "\mathsf{Core}"] \end{tikzcd}
$$

witnessed by an isomorphism of categories

$$
\mathsf{Fun}(\mathcal{G}, \mathcal{D}) \cong \mathsf{Fun}(\mathcal{G}, \mathsf{Core}(\mathcal{D})),
$$

natural in $\mathcal{G} \in \mathrm{Obj}(\mathsf{Grpd})$ and $\mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$, forming, together with the 2-functor K_0 of Item 2 of Proposition 8.3.2.1.3, a triple 2-adjunction

$$
(\mathrm{K}_0 \dashv \iota \dashv \mathsf{Core}) \colon \quad \begin{tikzcd} \mathsf{Cats} \arrow[r, bend left, "\mathrm{K}_0"] \arrow[r, bend right, "\mathsf{Core}"'] & \mathsf{Grpd}, \arrow[l, hook', "\iota"'] \end{tikzcd}
$$

witnessed by isomorphisms of categories

$$
\begin{aligned}
\mathsf{Fun}(\mathrm{K}_0(\mathcal{C}), \mathcal{G}) &\cong \mathsf{Fun}(\mathcal{C}, \mathcal{G}),\\
\mathsf{Fun}(\mathcal{G}, \mathcal{D}) &\cong \mathsf{Fun}(\mathcal{G}, \mathsf{Core}(\mathcal{D})),
\end{aligned}
$$

natural in $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$ and $\mathcal{G} \in \mathrm{Obj}(\mathsf{Grpd})$.

5. *Symmetric Strong Monoidality With Respect to Products.* The core functor of Item 1 has a symmetric strong monoidal structure

$$
\left(\mathsf{Core}, \mathsf{Core}^{\times}, \mathsf{Core}^{\times}_{\mathbb{1}}\right) \colon (\mathsf{Cats}, \times, \mathrm{pt}) \to (\mathsf{Grpd}, \times, \mathrm{pt})
$$

being equipped with isomorphisms

$$\mathsf{Core}^{\times}_{\mathcal{C},\mathcal{D}} \colon \mathsf{Core}(\mathcal{C}) \times \mathsf{Core}(\mathcal{D}) \xrightarrow{\cong} \mathsf{Core}(\mathcal{C} \times \mathcal{D}),$$
$$\mathsf{Core}^{\times}_{\mathbb{1}} \colon \mathrm{pt} \xrightarrow{\cong} \mathsf{Core}(\mathrm{pt}),$$

natural in $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$.

6. *Symmetric Strong Monoidality With Respect to Coproducts.* The core functor of Item 1 has a symmetric strong monoidal structure

$$\left(\mathsf{Core}, \mathsf{Core}^{\coprod}, \mathsf{Core}^{\coprod}_{\mathbb{1}}\right) \colon (\mathsf{Cats}, \coprod, \emptyset_{\mathsf{cat}}) \to (\mathsf{Grpd}, \coprod, \emptyset_{\mathsf{cat}})$$

being equipped with isomorphisms

$$\mathsf{Core}^{\coprod}_{\mathcal{C},\mathcal{D}} \colon \mathsf{Core}(\mathcal{C}) \coprod \mathsf{Core}(\mathcal{D}) \xrightarrow{\cong} \mathsf{Core}(\mathcal{C} \coprod \mathcal{D}),$$
$$\mathsf{Core}^{\coprod}_{\mathbb{1}} \colon \emptyset_{\mathsf{cat}} \xrightarrow{\cong} \mathsf{Core}(\emptyset_{\mathsf{cat}}),$$

natural in $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$.

Proof. Item 1, Functoriality: Omitted.
Item 2, 2-Functoriality: Omitted.
Item 3, Adjointness: Omitted.
Item 4, 2-Adjointness: Omitted.
Item 5, Symmetric Strong Monoidality With Respect to Products: Omitted.
Item 6, Symmetric Strong Monoidality With Respect to Coproducts: Omitted.
□

8.4 Functors

8.4.1 Foundations

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 8.4.1.1.1. A **functor** $F \colon \mathcal{C} \to \mathcal{D}$ **from** $\mathcal{C}$ **to** $\mathcal{D}$[16] consists of:

1. *Action on Objects.* A map of sets

$$F \colon \mathrm{Obj}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{D}),$$

called the **action on objects of** F.

[16] *Further Terminology:* Also called a **covariant functor**.

2. *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathcal{C})$, a map

$$F_{A,B}\colon \mathrm{Hom}_{\mathcal{C}}(A,B) \to \mathrm{Hom}_{\mathcal{D}}(F(A),F(B)),$$

called the **action on morphisms of** F **at** (A,B)[17].

satisfying the following conditions:

1. *Preservation of Identities.* For each $A \in \mathrm{Obj}(\mathcal{C})$, the diagram

$$\begin{tikzcd} \mathrm{pt} \arrow[d, "\mathbb{1}^{\mathcal{C}}_{A}"'] \arrow[dr, "\mathbb{1}^{\mathcal{D}}_{F(A)}"] & \\ \mathrm{Hom}_{\mathcal{C}}(A,A) \arrow[r, "F_{A,A}"'] & \mathrm{Hom}_{\mathcal{D}}(F(A),F(A)) \end{tikzcd}$$

commutes, i.e. we have

$$F(\mathrm{id}_A) = \mathrm{id}_{F(A)}.$$

2. *Preservation of Composition.* For each $A, B, C \in \mathrm{Obj}(\mathcal{C})$, the diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(B,C) \times \mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[r, "\circ^{\mathcal{C}}_{A,B,C}"] \arrow[d, "F_{B,C}\times F_{A,B}"'] & \mathrm{Hom}_{\mathcal{C}}(A,C) \arrow[d, "F_{A,C}"] \\ \mathrm{Hom}_{\mathcal{D}}(F(B),F(C)) \times \mathrm{Hom}_{\mathcal{D}}(F(A),F(B)) \arrow[r, "\circ^{\mathcal{D}}_{F(A),F(B),F(C)}"'] & \mathrm{Hom}_{\mathcal{D}}(F(A),F(C)) \end{tikzcd}$$

commutes, i.e. for each composable pair (g, f) of morphisms of $\mathcal{C}$, we have

$$F(g \circ f) = F(g) \circ F(f).$$

Notation 8.4.1.1.2. Let $\mathcal{C}$ and $\mathcal{D}$ be categories, and write $\mathcal{C}^{\mathsf{op}}$ for the opposite category of $\mathcal{C}$ of **??**.

1. Given a functor

$$F\colon \mathcal{C} \to \mathcal{D},$$

we also write F_A for $F(A)$.

2. Given a functor

$$F\colon \mathcal{C}^{\mathsf{op}} \to \mathcal{D},$$

we also write F^A for $F(A)$.

[17] *Further Terminology:* Also called **action on Hom-sets of** F **at** (A,B).

3. Given a functor
$$F\colon \mathcal{C}\times\mathcal{C}\to\mathcal{D},$$
we also write $F_{A,B}$ for $F(A,B)$.

4. Given a functor
$$F\colon \mathcal{C}^{\mathsf{op}}\times\mathcal{C}\to\mathcal{D},$$
we also write F^A_B for $F(A,B)$.

We employ a similar notation for morphisms, writing e.g. F_f for $F(f)$ given a functor $F\colon \mathcal{C}\to\mathcal{D}$.

Notation 8.4.1.1.3. Following the notation $[\![x\mapsto f(x)]\!]$ for a function $f\colon X\to Y$ introduced in Notation 1.1.1.1.2, we will sometimes denote a functor $F\colon\mathcal{C}\to\mathcal{D}$ by
$$F\overset{\mathrm{def}}{=}[\![A\mapsto F(A)]\!],$$
specially when the action on morphisms of F is clear from its action on objects.

Example 8.4.1.1.4. The **identity functor** of a category $\mathcal{C}$ is the functor $\mathrm{id}_{\mathcal{C}}\colon\mathcal{C}\to\mathcal{C}$ where

1. *Action on Objects.* For each $A\in\mathrm{Obj}(\mathcal{C})$, we have
$$\mathrm{id}_{\mathcal{C}}(A)\overset{\mathrm{def}}{=}A.$$

2. *Action on Morphisms.* For each $A,B\in\mathrm{Obj}(\mathcal{C})$, the action on morphisms
$$(\mathrm{id}_{\mathcal{C}})_{A,B}\colon \mathrm{Hom}_{\mathcal{C}}(A,B)\to\underbrace{\mathrm{Hom}_{\mathcal{C}}(\mathrm{id}_{\mathcal{C}}(A),\mathrm{id}_{\mathcal{C}}(B))}_{\overset{\mathrm{def}}{=}\mathrm{Hom}_{\mathcal{C}}(A,B)}$$
of $\mathrm{id}_{\mathcal{C}}$ at (A,B) is defined by
$$(\mathrm{id}_{\mathcal{C}})_{A,B}\overset{\mathrm{def}}{=}\mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(A,B)}.$$

Proof. Preservation of Identities: We have $\mathrm{id}_{\mathcal{C}}(\mathrm{id}_A)\overset{\mathrm{def}}{=}\mathrm{id}_A$ for each $A\in\mathrm{Obj}(\mathcal{C})$ by definition.

Preservation of Compositions: For each composable pair $A\xrightarrow{f}B\xrightarrow{g}B$ of morphisms of $\mathcal{C}$, we have
$$\begin{aligned}\mathrm{id}_{\mathcal{C}}(g\circ f)&\overset{\mathrm{def}}{=}g\circ f\\&\overset{\mathrm{def}}{=}\mathrm{id}_{\mathcal{C}}(g)\circ\mathrm{id}_{\mathcal{C}}(f).\end{aligned}$$

This finishes the proof. □

Definition 8.4.1.1.5. The **composition** of two functors $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{E}$ is the functor $G \circ F$ where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathcal{C})$, we have
$$[G \circ F](A) \overset{\text{def}}{=} G(F(A)).$$
- *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathcal{C})$, the action on morphisms
$$(G \circ F)_{A,B}\colon \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{E}}\big(G_{F_A}, G_{F_B}\big)$$
of $G \circ F$ at (A, B) is defined by
$$[G \circ F](f) \overset{\text{def}}{=} G(F(f)).$$

Proof. *Preservation of Identities*: For each $A \in \mathrm{Obj}(\mathcal{C})$, we have

$$\begin{aligned} G_{F_{\mathrm{id}_A}} &= G_{\mathrm{id}_{F_A}} && \text{(functoriality of } F\text{)} \\ &= \mathrm{id}_{G_{F_A}}. && \text{(functoriality of } G\text{)} \end{aligned}$$

Preservation of Composition: For each composable pair (g, f) of morphisms of $\mathcal{C}$, we have

$$\begin{aligned} G_{F_{g \circ f}} &= G_{F_g \circ F_f} && \text{(functoriality of } F\text{)} \\ &= G_{F_g} \circ G_{F_f}. && \text{(functoriality of } G\text{)} \end{aligned}$$

This finishes the proof. □

Proposition 8.4.1.1.6. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor.

1. *Preservation of Isomorphisms.* If f is an isomorphism in $\mathcal{C}$, then $F(f)$ is an isomorphism in $\mathcal{D}$.[18]

Proof. *Item 1, Preservation of Isomorphisms*: Indeed, we have

$$\begin{aligned} F(f)^{-1} \circ F(f) &= F\big(f^{-1} \circ f\big) \\ &= F(\mathrm{id}_A) \\ &= \mathrm{id}_{F(A)} \end{aligned}$$

and

$$\begin{aligned} F(f) \circ F(f)^{-1} &= F\big(f \circ f^{-1}\big) \\ &= F(\mathrm{id}_B) \\ &= \mathrm{id}_{F(B)}, \end{aligned}$$

showing $F(f)$ to be an isomorphism. □

[18] When the converse holds, we call F *conservative*, see Definition 8.5.4.1.1.

8.4.2 Contravariant Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories, and let $\mathcal{C}^{\mathsf{op}}$ denote the opposite category of $\mathcal{C}$ of **??**.

Definition 8.4.2.1.1. A **contravariant functor** from $\mathcal{C}$ to $\mathcal{D}$ is a functor from $\mathcal{C}^{\mathsf{op}}$ to $\mathcal{D}$.

Remark 8.4.2.1.2. In detail, a **contravariant functor** from $\mathcal{C}$ to $\mathcal{D}$ consists of:

1. *Action on Objects.* A map of sets

$$F\colon \mathrm{Obj}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{D}),$$

called the **action on objects of** F.

2. *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathcal{C})$, a map

$$F_{A,B}\colon \mathrm{Hom}_{\mathcal{C}}(A,B) \to \mathrm{Hom}_{\mathcal{D}}(F(B),F(A)),$$

called the **action on morphisms of** F **at** (A,B).

satisfying the following conditions:

1. *Preservation of Identities.* For each $A \in \mathrm{Obj}(\mathcal{C})$, the diagram

$$\begin{tikzcd}
\mathrm{pt} \arrow[d, "\mathbb{1}^{\mathcal{C}}_{A}"'] \arrow[dr, "\mathbb{1}^{\mathcal{D}}_{F(A)}"] & \\
\mathrm{Hom}_{\mathcal{C}}(A,A) \arrow[r, "F_{A,A}"'] & \mathrm{Hom}_{\mathcal{D}}(F(A),F(A))
\end{tikzcd}$$

commutes, i.e. we have

$$F(\mathrm{id}_A) = \mathrm{id}_{F(A)}.$$

2. *Preservation of Composition.* For each $A, B, C \in \mathrm{Obj}(\mathcal{C})$, the diagram

$$\begin{tikzcd}
 & \mathrm{Hom}_{\mathcal{D}}(F(C),F(B)) \times \mathrm{Hom}_{\mathcal{D}}(F(B),F(A)) \arrow[dr, dashed, "\sigma^{\mathsf{Sets}}_{\mathrm{Hom}_{\mathcal{D}}(F(C),F(B)),\mathrm{Hom}_{\mathcal{D}}(F(B),F(A))}", "\sim"'] & \\
\mathrm{Hom}_{\mathcal{C}}(B,C) \times \mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[ur, "F_{B,C} \times F_{A,B}"] \arrow[d, "\circ^{\mathcal{C}}_{A,B,C}"'] & & \mathrm{Hom}_{\mathcal{D}}(F(B),F(A)) \times \mathrm{Hom}_{\mathcal{D}}(F(C),F(B)) \arrow[d, "\circ^{\mathcal{D}}_{F(C),F(B),F(A)}"] \\
\mathrm{Hom}_{\mathcal{C}}(A,C) \arrow[rr, "F_{A,C}"'] & & \mathrm{Hom}_{\mathcal{D}}(F(C),F(A))
\end{tikzcd}$$

commutes, i.e. for each composable pair (g, f) of morphisms of $\mathcal{C}$, we have

$$F(g \circ f) = F(f) \circ F(g).$$

Remark 8.4.2.1.3. Throughout this work we will not use the term "contravariant" functor, speaking instead simply of functors $F\colon \mathcal{C}^{\mathsf{op}} \to \mathcal{D}$. We will usually, however, write

$$F_{A,B}\colon \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F(B), F(A))$$

for the action on morphisms

$$F_{A,B}\colon \mathrm{Hom}_{\mathcal{C}^{\mathsf{op}}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F(A), F(B))$$

of F, as well as write $F(g \circ f) = F(f) \circ F(g)$.

8.4.3 Forgetful Functors

Definition 8.4.3.1.1. There isn't a precise definition of a **forgetful functor**.

Remark 8.4.3.1.2. Despite there not being a formal or precise definition of a forgetful functor, the term is often very useful in practice, similarly to the word "canonical". The idea is that a "forgetful functor" is a functor that forgets structure or properties, and is best explained through examples, such as the ones below (see Examples 8.4.3.1.3 and 8.4.3.1.4).

Example 8.4.3.1.3. Examples of forgetful functors that forget structure include:

1. *Forgetting Group Structures.* The functor $\mathsf{Grp} \to \mathsf{Sets}$ sending a group (G, μ_G, η_G) to its underlying set G, forgetting the multiplication and unit maps μ_G and η_G of G.

2. *Forgetting Topologies.* The functor $\mathsf{Top} \to \mathsf{Sets}$ sending a topological space $(X, \mathcal{T}_X)$ to its underlying set X, forgetting the topology $\mathcal{T}_X$.

3. *Forgetting Fibrations.* The functor $\mathsf{FibSets}(K) \to \mathsf{Sets}$ sending a K-fibred set $\phi_X\colon X \to K$ to the set X, forgetting the map ϕ_X and the base set K.

Example 8.4.3.1.4. Examples of forgetful functors that forget properties include:

1. *Forgetting Commutativity.* The inclusion functor $\iota\colon \mathsf{CMon} \hookrightarrow \mathsf{Mon}$ which forgets the property of being commutative.

2. *Forgetting Inverses.* The inclusion functor $\iota\colon \mathsf{Grp} \hookrightarrow \mathsf{Mon}$ which forgets the property of having inverses.

Notation 8.4.3.1.5. Throughout this work, we will denote forgetful functors that forget structure by 忘, e.g. as in

$$忘\colon \mathsf{Grp} \to \mathsf{Sets}.$$

The symbol 忘, pronounced *wasureru* (see Item 1 of Remark 8.4.3.1.6 below), means *to forget*, and is a kanji found in the following words in Japanese and Chinese:

1. 忘れる, transcribed as *wasureru*, meaning *to forget*.
2. 忘却関手, transcribed as *boukyaku kanshu*, meaning *forgetful functor*.
3. 忘记 or 忘記, transcribed as *wàngjì*, meaning *to forget*.
4. 遗忘函子 or 遺忘函子, transcribed as *yíwàng hánz*☒, meaning *forgetful functor*.

Remark 8.4.3.1.6. Here we collect the pronunciation of the words in Notation 8.4.3.1.5 for accuracy and completeness.

1. Pronunciation of 忘れる:
   - Audio: see `https://topological-modular-forms.github.io/the-clowder-project/static/sounds/wasureru-01.mp3`
   - IPA broad transcription: [wäsɯɾe̞ɾɯ].
   - IPA narrow transcription: [ɰᵝäsɨᵝɾe̞ɾɯ̟ᵝ].
2. Pronunciation of 忘却関手: Pronunciation:
   - Audio: see `https://topological-modular-forms.github.io/the-clowder-project/static/sounds/wasureru-02.mp3`
   - IPA broad transcription: [bo̞ːkʲäkɯ kä̃ɰ̃ɕɯ].
   - IPA narrow transcription: [bo̞ːkʲäkɯ̟ᵝ kä̃ɰ̃ɕɯᵝ].
3. Pronunciation of 忘记:
   - Audio: see `https://topological-modular-forms.github.io/the-clowder-project/static/sounds/wasureru-03.ogg`

- Broad IPA transcription: [waŋtɕi].
- Sinological IPA transcription: [waŋ$^{51-53}$t͡ɕi^{51}].

4. Pronunciation of 遗忘函子:

   - Audio: see https://topological-modular-forms.github.io/the-clowder-project/static/sounds/wasureru-04.mp3
   - Broad IPA transcription: [iwaŋ xänt͡sz̩ɨ].
   - Sinological IPA transcription: [i^{35}waŋ^{51} xän^{35}t͡sz̩$^{214-21(4)}$].

8.4.4 The Natural Transformation Associated to a Functor

Definition 8.4.4.1.1. Every functor $F\colon \mathcal{C} \to \mathcal{D}$ defines a natural transformation[19]

$$F^{\dagger}\colon \mathrm{Hom}_{\mathcal{C}} \Longrightarrow \mathrm{Hom}_{\mathcal{D}} \circ (F^{\mathsf{op}} \times F), \qquad \begin{tikzcd} \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \arrow[rr, "F^{\mathsf{op}}\times F"] \arrow[dr, "\mathrm{Hom}_{\mathcal{C}}"'] \arrow[rr, phantom, shift right=4ex, "\overset{F^{\dagger}}{\Longrightarrow}"] & & \mathcal{D}^{\mathsf{op}} \times \mathcal{D} \arrow[dl, "\mathrm{Hom}_{\mathcal{D}}"] \\ & \mathsf{Sets}, & \end{tikzcd}$$

called the **natural transformation associated to** F, consisting of the collection

$$\left\{F^{\dagger}_{A,B}\colon \mathrm{Hom}_{\mathcal{C}}(A,B) \to \mathrm{Hom}_{\mathcal{D}}(F_A,F_B)\right\}_{(A,B)\in\mathrm{Obj}(\mathcal{C}^{\mathsf{op}}\times\mathcal{C})}$$

with

$$F^{\dagger}_{A,B} \overset{\mathrm{def}}{=} F_{A,B}.$$

Proof. The naturality condition for $F^{\dagger}$ is the requirement that for each morphism

$$(\phi,\psi)\colon (X,Y) \to (A,B)$$

of $\mathcal{C}^{\mathsf{op}} \times \mathcal{C}$, the diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(X,Y) \arrow[r, "\phi^{*}\circ\psi_{*}=\psi_{*}\circ\phi^{*}"] \arrow[d, "F_{X,Y}"'] & \mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[d, "F_{A,B}"] \\ \mathrm{Hom}_{\mathcal{D}}(F_X,F_Y) \arrow[r, "F(\phi)^{*}\circ F(\psi)_{*}=F(\psi)_{*}\circ F(\phi)^{*}"'] & \mathrm{Hom}_{\mathcal{D}}(F_A,F_B), \end{tikzcd}$$

[19] This is the 1-categorical version of Item 1 of Proposition 2.4.1.1.3.

acting on elements as

$$\begin{tikzcd}
f \arrow[r, mapsto] \arrow[d, mapsto] & \psi \circ f \circ \phi \arrow[d, mapsto] \\
F(f) \arrow[r, mapsto] & F(\psi) \circ F(f) \circ F(\psi) = F(\psi \circ f \circ \phi)
\end{tikzcd}$$

commutes, which follows from the functoriality of F. ◻

Proposition 8.4.4.1.2. Let $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{E}$ be functors.

1. *Interaction With Natural Isomorphisms.* The following conditions are equivalent:

   (a) The natural transformation $F^{\dagger}\colon \mathrm{Hom}_{\mathcal{C}} \Longrightarrow \mathrm{Hom}_{\mathcal{D}} \circ (F^{\mathsf{op}} \times F)$ associated to F is a natural isomorphism.

   (b) The functor F is fully faithful.

2. *Interaction With Composition.* We have an equality of pasting diagrams

$$\begin{tikzcd}
\mathcal{C}^{\mathsf{op}} \times \mathcal{C} \arrow[r, "F^{\mathsf{op}} \times F"] \arrow[dr, "\mathrm{Hom}_{\mathcal{C}}"'] & \mathcal{D}^{\mathsf{op}} \times \mathcal{D} \arrow[r, "G^{\mathsf{op}} \times G"] \arrow[d, "\mathrm{Hom}_{\mathcal{D}}"] \arrow[dl, phantom, "\overset{F^{\dagger}}{\Rightarrow}"] & \mathcal{E}^{\mathsf{op}} \times \mathcal{E} \arrow[dl, "\mathrm{Hom}_{\mathcal{E}}"] \arrow[dll, phantom, "\overset{G^{\dagger}}{\Rightarrow}"] \\
 & \mathsf{Sets} &
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{C}^{\mathsf{op}} \times \mathcal{C} \arrow[rr, "(G \circ F)^{\mathsf{op}} \times (G \circ F)"] \arrow[dr, "\mathrm{Hom}_{\mathcal{C}}"'] & \arrow[d, phantom, "\overset{(G \circ F)^{\dagger}}{\Rightarrow}"] & \mathcal{E}^{\mathsf{op}} \times \mathcal{E}, \arrow[dl, "\mathrm{Hom}_{\mathcal{E}}"] \\
 & \mathsf{Sets} &
\end{tikzcd}$$

   in Cats_2, i.e. we have

$$(G \circ F)^{\dagger} = \left(G^{\dagger} \star \mathrm{id}_{F^{\mathsf{op}} \times F}\right) \circ F^{\dagger}.$$

3. *Interaction With Identities.* We have

$$\mathrm{id}_{\mathcal{C}}^{\dagger} = \mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(-_1,-_2)},$$

   i.e. the natural transformation associated to $\mathrm{id}_{\mathcal{C}}$ is the identity natural transformation of the functor $\mathrm{Hom}_{\mathcal{C}}(-_1, -_2)$.

Proof. *Item 1*, *Interaction With Natural Isomorphisms*: Clear.
Item 2, *Interaction With Composition*: Clear.
Item 3, *Interaction With Identities*: Clear. ◻

8.5 Conditions on Functors

8.5.1 Faithful Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 8.5.1.1.1. A functor $F: \mathcal{C} \to \mathcal{D}$ is **faithful** if, for each $A, B \in \mathrm{Obj}(\mathcal{C})$, the action on morphisms

$$F_{A,B}: \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F_A, F_B)$$

of F at (A, B) is injective.

Proposition 8.5.1.1.2. Let $F: \mathcal{C} \to \mathcal{D}$ be a functor.

1. *Interaction With Postcomposition.* The following conditions are equivalent:

   (a) The functor $F: \mathcal{C} \to \mathcal{D}$ is faithful.

   (b) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor

   $$F_*: \mathsf{Fun}(\mathcal{X}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{X}, \mathcal{D})$$

   is faithful.

   (c) The functor $F: \mathcal{C} \to \mathcal{D}$ is a representably faithful morphism in Cats_2 in the sense of Definition 9.1.1.1.1.

2. *Interaction With Precomposition I.* Let $F: \mathcal{C} \to \mathcal{D}$ be a functor.

   (a) If F is faithful, then the precomposition functor

   $$F^*: \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

   can fail to be faithful.

   (b) Conversely, if the precomposition functor

   $$F^*: \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

   is faithful, then F *can fail* to be faithful.

3. *Interaction With Precomposition II.* If F is essentially surjective, then the precomposition functor

   $$F^*: \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

   is faithful.

4. *Interaction With Precomposition III.* The following conditions are equivalent:

   (a) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor

   $$F^* \colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

   is faithful.

   (b) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor

   $$F^* \colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

   is conservative.

   (c) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor

   $$F^* \colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

   is monadic.

   (d) The functor $F \colon \mathcal{C} \to \mathcal{D}$ is a corepresentably faithful morphism in Cats_2 in the sense of Definition 9.2.1.1.1.

   (e) The components

   $$\eta_G \colon G \Longrightarrow \mathrm{Ran}_F(G \circ F)$$

   of the unit

   $$\eta \colon \mathrm{id}_{\mathsf{Fun}(\mathcal{D}, \mathcal{X})} \Longrightarrow \mathrm{Ran}_F \circ F^*$$

   of the adjunction $F^* \dashv \mathrm{Ran}_F$ are all monomorphisms.

   (f) The components

   $$\epsilon_G \colon \mathrm{Lan}_F(G \circ F) \Longrightarrow G$$

   of the counit

   $$\epsilon \colon \mathrm{Lan}_F \circ F^* \Longrightarrow \mathrm{id}_{\mathsf{Fun}(\mathcal{D}, \mathcal{X})}$$

   of the adjunction $\mathrm{Lan}_F \dashv F^*$ are all epimorphisms.

   (g) The functor F is dominant (Definition 8.6.1.1.1), i.e. every object of $\mathcal{D}$ is a retract of some object in $\mathrm{Im}(F)$:

   ($\star$) For each $B \in \mathrm{Obj}(\mathcal{D})$, there exist:

   - An object A of $\mathcal{C}$;
   - A morphism $s \colon B \to F(A)$ of $\mathcal{D}$;

– A morphism $r\colon F(A) \to B$ of $\mathcal{D}$;

such that $r \circ s = \mathrm{id}_B$.

Proof. *Item 1, Interaction With Postcomposition*: Omitted.
Item 2, Interaction With Precomposition I: See [MSE 733163] for Item 2a. Item 2b follows from Item 3 and the fact that there are essentially surjective functors that are not faithful.
Item 3, Interaction With Precomposition II: Omitted, but see `https://unimath.github.io/doc/UniMath/d4de26f//UniMath.CategoryTheory.precomp_fully_faithful.html` for a formalised proof.
Item 4, Interaction With Precomposition III: We claim Items 4a to 4g are equivalent:

- *Items 4a and 4d Are Equivalent:* This is true by the definition of corepresentably faithful morphism; see Definition 9.2.1.1.1.
- *Items 4a to 4c and 4g Are Equivalent:* See [Adá+01, Proposition 4.1] or alternatively [Fre09, Lemmas 3.1 and 3.2] for the equivalence between Items 4a and 4g.
- *Items 4a, 4e and 4f Are Equivalent:* See **??** of **??**.

This finishes the proof. □

8.5.2 Full Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 8.5.2.1.1. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **full** if, for each $A, B \in \mathrm{Obj}(\mathcal{C})$, the action on morphisms

$$F_{A,B}\colon \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F_A, F_B)$$

of F at (A, B) is surjective.

Proposition 8.5.2.1.2. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor.

1. *Interaction With Postcomposition.* The following conditions are equivalent:

   (a) The functor $F\colon \mathcal{C} \to \mathcal{D}$ is full.

   (b) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor

   $$F_*\colon \mathsf{Fun}(\mathcal{X}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{X}, \mathcal{D})$$

   is full.

   (c) The functor $F\colon \mathcal{C} \to \mathcal{D}$ is a representably full morphism in Cats_2 in the sense of Definition 9.1.2.1.1.

2. *Interaction With Precomposition I.* If F is full, then the precomposition functor

$$F^*\colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

   can fail to be full.

3. *Interaction With Precomposition II.* If the precomposition functor

$$F^*\colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

   is full, then F *can fail* to be full.

4. *Interaction With Precomposition III.* If F is essentially surjective and full, then the precomposition functor

$$F^*\colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

   is full (and also faithful by Item 3 of Proposition 8.5.1.1.2).

5. *Interaction With Precomposition IV.* The following conditions are equivalent:

   (a) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor

$$F^*\colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

   is full.

   (b) The functor $F\colon \mathcal{C} \to \mathcal{D}$ is a corepresentably full morphism in Cats_2 in the sense of Definition 9.2.1.1.1.

   (c) The components

$$\eta_G\colon G \Longrightarrow \mathrm{Ran}_F(G \circ F)$$

   of the unit

$$\eta\colon \mathrm{id}_{\mathsf{Fun}(\mathcal{D},\mathcal{X})} \Longrightarrow \mathrm{Ran}_F \circ F^*$$

   of the adjunction $F^* \dashv \mathrm{Ran}_F$ are all retractions/split epimorphisms.

(d) The components

$$\epsilon_G : \mathrm{Lan}_F(G \circ F) \Longrightarrow G$$

of the counit

$$\epsilon : \mathrm{Lan}_F \circ F^* \Longrightarrow \mathrm{id}_{\mathsf{Fun}(\mathcal{D},\mathcal{X})}$$

of the adjunction $\mathrm{Lan}_F \dashv F^*$ are all sections/split monomorphisms.

(e) For each $B \in \mathrm{Obj}(\mathcal{D})$, there exist:
- An object A_B of $\mathcal{C}$;
- A morphism $s_B : B \to F(A_B)$ of $\mathcal{D}$;
- A morphism $r_B : F(A_B) \to B$ of $\mathcal{D}$;

satisfying the following condition:

($\star$) For each $A \in \mathrm{Obj}(\mathcal{C})$ and each pair of morphisms

$$r : F(A) \to B,$$
$$s : B \to F(A)$$

of $\mathcal{D}$, we have

$$[(A_B, s_B, r_B)] = [(A, s, r \circ s_B \circ r_B)]$$

in $\int^{A \in \mathcal{C}} h^{B'}_{F_A} \times h^{F_A}_{B}$.

Proof. Item 1, Interaction With Postcomposition: Omitted.
Item 2, Interaction With Precomposition I: Omitted.
Item 3, Interaction With Precomposition II: See [BS10, p. 47].
Item 4, Interaction With Precomposition III: Omitted, but see `https://unimath.github.io/doc/UniMath/d4de26f//UniMath.CategoryTheory.precomp_fully_faithful.html` for a formalised proof.
Item 5, Interaction With Precomposition IV: We claim Items 5a to 5e are equivalent:

- *Items 5a and 5b Are Equivalent:* This is true by the definition of corepresentably full morphism; see Definition 9.2.2.1.1.
- *Items 5a, 5c and 5d Are Equivalent:* See **??** of **??**.
- *Items 5a and 5e Are Equivalent:* See [Adá+01, Item (b) of Remark 4.3].

This finishes the proof. □

Question 8.5.2.1.3. Item 5 of Proposition 8.5.2.1.2 gives a characterisation of the functors F for which F^* is full, but the characterisations given there are really messy. Are there better ones?
This question also appears as [`MO 468121b`].

8.5.3 Fully Faithful Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 8.5.3.1.1. A functor $F: \mathcal{C} \to \mathcal{D}$ is **fully faithful** if F is full and faithful, i.e. if, for each $A, B \in \mathrm{Obj}(\mathcal{C})$, the action on morphisms

$$F_{A,B}\colon \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F_A, F_B)$$

of F at (A, B) is bijective.

Proposition 8.5.3.1.2. Let $F: \mathcal{C} \to \mathcal{D}$ be a functor.

1. *Characterisations.* The following conditions are equivalent:

   (a) The functor F is fully faithful.

   (b) We have a pullback square

$$\mathsf{Arr}(\mathcal{C}) \cong (\mathcal{C} \times \mathcal{C}) \times_{\mathcal{D}\times\mathcal{D}} \mathsf{Arr}(\mathcal{D}), \qquad \begin{tikzcd} \mathsf{Arr}(\mathcal{C}) \arrow[r, "\mathsf{Arr}(F)"] \arrow[d, "\mathrm{src}\times\mathrm{tgt}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Arr}(\mathcal{D}) \arrow[d, "\mathrm{src}\times\mathrm{tgt}"] \\ \mathcal{C} \times \mathcal{C} \arrow[r, "F\times F"'] & \mathcal{D} \times \mathcal{D} \end{tikzcd}$$

   in Cats.

2. *Conservativity.* If F is fully faithful, then F is conservative.

3. *Essential Injectivity.* If F is fully faithful, then F is essentially injective.

4. *Interaction With Co/Limits.* If F is fully faithful, then F reflects co/limits.

5. *Interaction With Postcomposition.* The following conditions are equivalent:

   (a) The functor $F: \mathcal{C} \to \mathcal{D}$ is fully faithful.

   (b) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor

   $$F_*\colon \mathsf{Fun}(\mathcal{X}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{X}, \mathcal{D})$$

   is fully faithful.

   (c) The functor $F: \mathcal{C} \to \mathcal{D}$ is a representably fully faithful morphism in Cats_2 in the sense of Definition 9.1.3.1.1.

6. *Interaction With Precomposition I.* If F is fully faithful, then the precomposition functor

$$F^*\colon \mathsf{Fun}(\mathcal{D},\mathcal{X}) \to \mathsf{Fun}(\mathcal{C},\mathcal{X})$$

can fail to be fully faithful.

7. *Interaction With Precomposition II.* If the precomposition functor

$$F^*\colon \mathsf{Fun}(\mathcal{D},\mathcal{X}) \to \mathsf{Fun}(\mathcal{C},\mathcal{X})$$

is fully faithful, then F *can fail* to be fully faithful (and in fact it can also fail to be either full or faithful).

8. *Interaction With Precomposition III.* If F is essentially surjective and full, then the precomposition functor

$$F^*\colon \mathsf{Fun}(\mathcal{D},\mathcal{X}) \to \mathsf{Fun}(\mathcal{C},\mathcal{X})$$

is fully faithful.

9. *Interaction With Precomposition IV.* The following conditions are equivalent:

   (a) For each $\mathcal{X} \in \mathsf{Obj}(\mathsf{Cats})$, the precomposition functor

   $$F^*\colon \mathsf{Fun}(\mathcal{D},\mathcal{X}) \to \mathsf{Fun}(\mathcal{C},\mathcal{X})$$

   is fully faithful.

   (b) The precomposition functor

   $$F^*\colon \mathsf{Fun}(\mathcal{D},\mathsf{Sets}) \to \mathsf{Fun}(\mathcal{C},\mathsf{Sets})$$

   is fully faithful.

   (c) The functor

   $$\mathrm{Lan}_F\colon \mathsf{Fun}(\mathcal{C},\mathsf{Sets}) \to \mathsf{Fun}(\mathcal{D},\mathsf{Sets})$$

   is fully faithful.

   (d) The functor F is a corepresentably fully faithful morphism in Cats_2 in the sense of Definition 9.2.3.1.1.

   (e) The functor F is absolutely dense.

(f) The components

$$\eta_G\colon G \Longrightarrow \mathrm{Ran}_F(G\circ F)$$

of the unit

$$\eta\colon \mathrm{id}_{\mathsf{Fun}(\mathcal{D},\mathcal{X})} \Longrightarrow \mathrm{Ran}_F \circ F^*$$

of the adjunction $F^* \dashv \mathrm{Ran}_F$ are all isomorphisms.

(g) The components

$$\epsilon_G\colon \mathrm{Lan}_F(G\circ F) \Longrightarrow G$$

of the counit

$$\epsilon\colon \mathrm{Lan}_F \circ F^* \Longrightarrow \mathrm{id}_{\mathsf{Fun}(\mathcal{D},\mathcal{X})}$$

of the adjunction $\mathrm{Lan}_F \dashv F^*$ are all isomorphisms.

(h) The natural transformation

$$\alpha\colon \mathrm{Lan}_{h_F}\left(h^F\right) \Longrightarrow h$$

with components

$$\alpha_{B',B}\colon \int^{A\in C} h^{B'}_{F_A} \times h^{F_A}_{B} \to h^{B'}_{B}$$

given by

$$\alpha_{B',B}([(\phi,\psi)]) = \psi\circ\phi$$

is a natural isomorphism.

(i) For each $B\in \mathrm{Obj}(\mathcal{D})$, there exist:
- An object A_B of C;
- A morphism $s_B\colon B\to F(A_B)$ of $\mathcal{D}$;
- A morphism $r_B\colon F(A_B)\to B$ of $\mathcal{D}$;

satisfying the following conditions:

i. The triple $(F(A_B), r_B, s_B)$ is a retract of B, i.e. we have $r_B \circ s_B = \mathrm{id}_B$.

ii. For each morphism $f\colon B'\to B$ of $\mathcal{D}$, we have

$$[(A_B, s_{B'}, f\circ r_{B'})] = [(A_B, s_B\circ f, r_B)]$$

in $\int^{A\in C} h^{B'}_{F_A}\times h^{F_A}_{B}$.

Proof. *Item 1, Characterisations*: Omitted.
Item 2, Conservativity: This is a repetition of Item 2 of Proposition 8.5.4.1.2, and is proved there.
Item 3, Essential Injectivity: Omitted.
Item 4, Interaction With Co/Limits: Omitted.
Item 5, Interaction With Postcomposition: This follows from Item 1 of Proposition 8.5.1.1.2 and Item 1 of Proposition 8.5.2.1.2.
Item 6, Interaction With Precomposition I: See [`MSE 733161`] for an example of a fully faithful functor whose precomposition with which fails to be full.
Item 7, Interaction With Precomposition II: See [`MSE 749304`, Item 3].
Item 8, Interaction With Precomposition III: Omitted, but see `https://unimath.github.io/doc/UniMath/d4de26f//UniMath.CategoryTheory.precomp_fully_faithful.html` for a formalised proof.
Item 9, Interaction With Precomposition IV: We claim Items 9a to 9i are equivalent:

- *Items 9a and 9d Are Equivalent:* This is true by the definition of corepresentably fully faithful morphism; see Definition 9.2.3.1.1.
- *Items 9a, 9f and 9g Are Equivalent:* See **??** of **??**.
- *Items 9a to 9c Are Equivalent:* This follows from [Low15, Proposition A.1.5].
- *Items 9a, 9e, 9h and 9i Are Equivalent:* See [Fre09, Theorem 4.1] and [Adá+01, Theorem 1.1].

This finishes the proof. □

8.5.4 Conservative Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 8.5.4.1.1. A functor $F: \mathcal{C} \to \mathcal{D}$ is **conservative** if it satisfies the following condition:[20]

(★) For each $f \in \mathrm{Mor}(\mathcal{C})$, if $F(f)$ is an isomorphism in $\mathcal{D}$, then f is an isomorphism in $\mathcal{C}$.

Proposition 8.5.4.1.2. Let $F: \mathcal{C} \to \mathcal{D}$ be a functor.

1. *Characterisations.* The following conditions are equivalent:

   (a) The functor F is conservative.

[20] *Slogan:* A functor F is **conservative** if it reflects isomorphisms.

(b) For each $f \in \mathrm{Mor}(\mathcal{C})$, the morphism $F(f)$ is an isomorphism in $\mathcal{D}$ iff f is an isomorphism in $\mathcal{C}$.

2. *Interaction With Fully Faithfulness.* Every fully faithful functor is conservative.

3. *Interaction With Precomposition.* The following conditions are equivalent:

   (a) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor

   $$F^* \colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

   is conservative.

   (b) The equivalent conditions of Item 4 of Proposition 8.5.1.1.2 are satisfied.

Proof. Item 1, Characterisations: This follows from Item 1 of Proposition 8.4.1.1.6.
Item 2, Interaction With Fully Faithfulness: Let $F \colon \mathcal{C} \to \mathcal{D}$ be a fully faithful functor, let $f \colon A \to B$ be a morphism of $\mathcal{C}$, and suppose that F_f is an isomorphism. We have

$$\begin{aligned} F(\mathrm{id}_B) &= \mathrm{id}_{F(B)} \\ &= F(f) \circ F(f)^{-1} \\ &= F\left(f \circ f^{-1}\right). \end{aligned}$$

Similarly, $F(\mathrm{id}_A) = F\left(f^{-1} \circ f\right)$. But since F is fully faithful, we must have

$$\begin{aligned} f \circ f^{-1} &= \mathrm{id}_B, \\ f^{-1} \circ f &= \mathrm{id}_A, \end{aligned}$$

showing f to be an isomorphism. Thus F is conservative. □

Question 8.5.4.1.3. Is there a characterisation of functors $F \colon \mathcal{C} \to \mathcal{D}$ satisfying the following condition:

(★) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor

$$F_* \colon \mathsf{Fun}(\mathcal{X}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{X}, \mathcal{D})$$

is conservative?

This question also appears as [MO 468121a].

8.5.5 Essentially Injective Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 8.5.5.1.1. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **essentially injective** if it satisfies the following condition:

($\star$) For each $A, B \in \mathrm{Obj}(\mathcal{C})$, if $F(A) \cong F(B)$, then $A \cong B$.

Question 8.5.5.1.2. Is there a characterisation of functors $F\colon \mathcal{C} \to \mathcal{D}$ such that:

1. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor

$$F^*\colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

   is essentially injective, i.e. if $\phi \circ F \cong \psi \circ F$, then $\phi \cong \psi$ for all functors ϕ and ψ?

2. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor

$$F_*\colon \mathsf{Fun}(\mathcal{X}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{X}, \mathcal{D})$$

   is essentially injective, i.e. if $F \circ \phi \cong F \circ \psi$, then $\phi \cong \psi$?

This question also appears as [MO 468121a].

8.5.6 Essentially Surjective Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 8.5.6.1.1. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **essentially surjective**[21] if it satisfies the following condition:

($\star$) For each $D \in \mathrm{Obj}(\mathcal{D})$, there exists some object A of $\mathcal{C}$ such that $F(A) \cong D$.

Question 8.5.6.1.2. Is there a characterisation of functors $F\colon \mathcal{C} \to \mathcal{D}$ such that:

1. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor

$$F^*\colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

   is essentially surjective?

2. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor

$$F_*\colon \mathsf{Fun}(\mathcal{X}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{X}, \mathcal{D})$$

   is essentially surjective?

This question also appears as [MO 468121a].

[21] *Further Terminology:* Also called an **eso** functor, where the name "eso" comes from

8.5.7 Equivalences of Categories

Definition 8.5.7.1.1. Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

1. An **equivalence of categories** between $\mathcal{C}$ and $\mathcal{D}$ consists of a pair of functors

$$F\colon \mathcal{C} \to \mathcal{D},$$
$$G\colon \mathcal{D} \to \mathcal{C}$$

together with natural isomorphisms

$$\eta\colon \mathrm{id}_{\mathcal{C}} \overset{\sim}{\Longrightarrow} G \circ F,$$
$$\epsilon\colon F \circ G \overset{\sim}{\Longrightarrow} \mathrm{id}_{\mathcal{D}}.$$

2. An **adjoint equivalence of categories** between $\mathcal{C}$ and $\mathcal{D}$ is an equivalence (F, G, η, ϵ) between $\mathcal{C}$ and $\mathcal{D}$ which is also an adjunction.

Proposition 8.5.7.1.2. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor.

1. *Characterisations.* If $\mathcal{C}$ and $\mathcal{D}$ are small[22], then the following conditions are equivalent:[23]

   (a) The functor F is an equivalence of categories.

   (b) The functor F is fully faithful and essentially surjective.

   (c) The induced functor

   $$\restriction F\mathsf{Sk}(\mathcal{C})\colon \mathsf{Sk}(\mathcal{C}) \to \mathsf{Sk}(\mathcal{D})$$

   is an *isomorphism* of categories.

   (d) For each $\mathcal{X} \in \mathsf{Obj}(\mathsf{Cats})$, the precomposition functor

   $$F^*\colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

   is an equivalence of categories.

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essentially surjective on objects.

[22]Otherwise there will be size issues. One can also work with large categories and universes, or require F to be *constructively* essentially surjective; see [MSE 1465107].

[23]In ZFC, the equivalence between Item 1a and Item 1b is equivalent to the axiom of choice; see [MO 119454].

In Univalent Foundations, this is true without requiring neither the axiom of choice nor the law of excluded middle.

(e) For each $X \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor

$$F_*\colon \mathsf{Fun}(\mathcal{X}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{X}, \mathcal{D})$$

is an equivalence of categories.

2. *Two-Out-of-Three.* Let

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "G\circ F"] \arrow[dr, "F"'] & & \mathcal{E} \\ & \mathcal{D} \arrow[ur, "G"'] & \end{tikzcd}$$

be a diagram in Cats. If two out of the three functors among F, G, and $G \circ F$ are equivalences of categories, then so is the third.

3. *Stability Under Composition.* Let

$$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D} \underset{G'}{\overset{F'}{\rightleftarrows}} \mathcal{E}$$

be a diagram in Cats. If (F, G) and (F', G') are equivalences of categories, then so is their composite $(F' \circ F, G' \circ G)$.

4. *Equivalences vs.Adjoint Equivalences.* Every equivalence of categories can be promoted to an adjoint equivalence.[24]

5. *Interaction With Groupoids.* If $\mathcal{C}$ and $\mathcal{D}$ are groupoids, then the following conditions are equivalent:

(a) The functor F is an equivalence of groupoids.

(b) The following conditions are satisfied:

i. The functor F induces a bijection

$$\pi_0(F)\colon \pi_0(\mathcal{C}) \to \pi_0(\mathcal{D})$$

of sets.

ii. For each $A \in \mathrm{Obj}(\mathcal{C})$, the induced map

$$F_{x,x}\colon \mathrm{Aut}_{\mathcal{C}}(A) \to \mathrm{Aut}_{\mathcal{D}}(F_A)$$

is an isomorphism of groups.

[24] More precisely, we can promote an equivalence of categories (F, G, η, ϵ) to adjoint equivalences (F, G, η', ϵ) and (F, G, η, ϵ').

Proof. *Item 1, Characterisations*: We claim that Items 1a to 1e are indeed equivalent:

1. *Item 1a* $\implies$ *Item 1b:* Clear.

2. *Item 1b* $\implies$ *Item 1a:* Since F is essentially surjective and $\mathcal{C}$ and $\mathcal{D}$ are small, we can choose, using the axiom of choice, for each $B \in \mathrm{Obj}(\mathcal{D})$, an object j_B of $\mathcal{C}$ and an isomorphism $i_B \colon B \to F_{j_B}$ of $\mathcal{D}$.

   Since F is fully faithful, we can extend the assignment $B \mapsto j_B$ to a *unique* functor $j \colon \mathcal{D} \to \mathcal{C}$ such that the isomorphisms $i_B \colon B \to F_{j_B}$ assemble into a natural isomorphism $\eta \colon \mathrm{id}_{\mathcal{D}} \overset{\sim}{\Longrightarrow} F \circ j$, with a similar natural isomorphism $\epsilon \colon \mathrm{id}_{\mathcal{C}} \overset{\sim}{\Longrightarrow} j \circ F$. Hence F is an equivalence.

3. *Item 1a* $\implies$ *Item 1c:* This follows from Item 4 of Proposition 8.1.5.1.3.

4. *Item 1c* $\implies$ *Item 1a:* Omitted.

5. *Items 1a, 1d and 1e Are Equivalent:* This follows from **??**.

This finishes the proof of Item 1.
Item 2, Two-Out-of-Three: Omitted.
Item 3, Stability Under Composition: Clear.
Item 4, Equivalences vs.Adjoint Equivalences: See [Rie17, Proposition 4.4.5].
Item 5, Interaction With Groupoids: See [nLa24, Proposition 4.4]. □

8.5.8 Isomorphisms of Categories

Definition 8.5.8.1.1. An **isomorphism of categories** is a pair of functors

$$F \colon \mathcal{C} \to \mathcal{D},$$
$$G \colon \mathcal{D} \to \mathcal{C}$$

such that we have

$$G \circ F = \mathrm{id}_{\mathcal{C}},$$

$$F \circ G = \mathrm{id}_{\mathcal{D}}.$$

Example 8.5.8.1.2. Categories can be equivalent but non-isomorphic. For example, the category consisting of two isomorphic objects is equivalent to pt, but not isomorphic to it.

Proposition 8.5.8.1.3. Let $F \colon \mathcal{C} \to \mathcal{D}$ be a functor.

1. *Characterisations.* If $\mathcal{C}$ and $\mathcal{D}$ are small, then the following conditions are equivalent:

   (a) The functor F is an isomorphism of categories.

   (b) The functor F is fully faithful and bijective on objects.

   (c) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor

   $$F^*\colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

   is an isomorphism of categories.

   (d) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor

   $$F_*\colon \mathsf{Fun}(\mathcal{X}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{X}, \mathcal{D})$$

   is an isomorphism of categories.

Proof. Item 1, Characterisations: We claim that Items 1a to 1d are indeed equivalent:

1. *Items 1a and 1b Are Equivalent:* Omitted, but similar to Item 1 of Proposition 8.5.7.1.2.

2. *Items 1a, 1c and 1d Are Equivalent:* This follows from **??**.

This finishes the proof. □

8.6 More Conditions on Functors

8.6.1 Dominant Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 8.6.1.1.1. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **dominant** if every object of $\mathcal{D}$ is a retract of some object in $\mathrm{Im}(F)$, i.e.:

(★) For each $B \in \mathrm{Obj}(\mathcal{D})$, there exist:

- An object A of $\mathcal{C}$;
- A morphism $r\colon F(A) \to B$ of $\mathcal{D}$;
- A morphism $s\colon B \to F(A)$ of $\mathcal{D}$;

such that we have

$$r \circ s = \mathrm{id}_B, \qquad \begin{tikzcd} B \arrow[r, "s"] \arrow[dr, "\mathrm{id}_B"'] & F(A) \arrow[d, "r"] \\ & B. \end{tikzcd}$$

Proposition 8.6.1.1.2. Let $F, G\colon \mathcal{C} \rightrightarrows \mathcal{D}$ be functors and let $I\colon \mathcal{X} \to \mathcal{C}$ be a functor.

1. *Interaction With Right Whiskering.* If I is full and dominant, then the map
$$- \star \mathrm{id}_I \colon \mathrm{Nat}(F, G) \to \mathrm{Nat}(F \circ I, G \circ I)$$
is a bijection.

2. *Interaction With Adjunctions.* Let $(F, G)\colon \mathcal{C} \rightleftarrows \mathcal{D}$ be an adjunction.

   (a) If F is dominant, then G is faithful.

   (b) The following conditions are equivalent:

      i. The functor G is full.

      ii. The restriction
      $$\upharpoonright G\mathrm{Im}_F \colon \mathrm{Im}(F) \to \mathcal{C}$$
      of G to $\mathrm{Im}(F)$ is full.

Proof. *Item 1*, *Interaction With Right Whiskering*: See [DFH75, Proposition 1.4].
Item 2, *Interaction With Adjunctions*: See [DFH75, Proposition 1.7]. □

Question 8.6.1.1.3. Is there a characterisation of functors $F\colon \mathcal{C} \to \mathcal{D}$ such that:

1. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor
$$F^*\colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$
is dominant?

2. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor
$$F_*\colon \mathsf{Fun}(\mathcal{X}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{X}, \mathcal{D})$$
is dominant?

This question also appears as [`MO 468121a`].

8.6.2 Monomorphisms of Categories

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 8.6.2.1.1. A functor $F: C \to \mathcal{D}$ is a **monomorphism of categories** if it is a monomorphism in Cats (see **??**).

Proposition 8.6.2.1.2. Let $F: C \to \mathcal{D}$ be a functor.

1. *Characterisations.* The following conditions are equivalent:

   (a) The functor F is a monomorphism of categories.

   (b) The functor F is injective on objects and morphisms, i.e. F is injective on objects and the map

   $$F: \mathrm{Mor}(C) \to \mathrm{Mor}(\mathcal{D})$$

   is injective.

Proof. *Item 1*, *Characterisations*: Omitted. □

Question 8.6.2.1.3. Is there a characterisation of functors $F: C \to \mathcal{D}$ such that:

1. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor

   $$F^*: \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(C, \mathcal{X})$$

   is a monomorphism of categories?

2. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor

   $$F_*: \mathsf{Fun}(\mathcal{X}, C) \to \mathsf{Fun}(\mathcal{X}, \mathcal{D})$$

   is a monomorphism of categories?

This question also appears as [MO 468121a].

8.6.3 Epimorphisms of Categories

Let C and $\mathcal{D}$ be categories.

Definition 8.6.3.1.1. A functor $F: C \to \mathcal{D}$ is a **epimorphism of categories** if it is a epimorphism in Cats (see **??**).

Proposition 8.6.3.1.2. Let $F: C \to \mathcal{D}$ be a functor.

1. *Characterisations.* The following conditions are equivalent:[25]

   (a) The functor F is a epimorphism of categories.

[25] *Further Terminology:* This statement is known as **Isbell's zigzag theorem**.

(b) For each morphism $f : A \to B$ of $\mathcal{D}$, we have a diagram

$$
\begin{tikzcd}
 & & & & A \arrow[dllll, "\phi_1"'] \arrow[dll, "\phi_2"] \arrow[d, "\phi_3"] \arrow[drr, "\phi_m"] \arrow[drrrr, dashed, "\mathrm{id}_A"] & & & & \\
X_1 \arrow[dr, dashed, "\alpha_1"] \arrow[ddrrrr, dashed, bend right, "\alpha_0"'] & & X_2 \arrow[dl, dashed, "\alpha_2"'] \arrow[dr, dashed, "\alpha_3"] & & X_3 \arrow[dl, dashed, "\alpha_4"'] \arrow[dr, dashed, "\alpha_6"] & \cdots & X_m \arrow[dl, dashed, "\alpha_{2m-2}"'] \arrow[dr, dashed, "\alpha_{2m-1}"] & & A \arrow[dl, dashed, "\alpha_{2m}"'] \\
 & Y_1 \arrow[drrr, "\psi_1"] & & Y_2 \arrow[dr, "\psi_2"] & & \cdots \arrow[dl, "\psi_3"'] & & Y_m \arrow[dlll, "\psi_m"'] & \\
 & & & & B & & & &
\end{tikzcd}
$$

in $\mathcal{D}$ satisfying the following conditions:

i. We have $f = \alpha_0 \circ \phi_1$.

ii. We have $f = \psi_m \circ \alpha_{2m}$.

iii. For each $0 \leq i \leq 2m$, we have $\alpha_i \in \mathrm{Mor}(\mathrm{Im}(F))$.

2. *Surjectivity on Objects.* If F is an epimorphism of categories, then F is surjective on objects.

Proof. Item 1, Characterisations: See [Isb68].
Item 2, Surjectivity on Objects: Omitted. □

Question 8.6.3.1.3. Is there a characterisation of functors $F : \mathcal{C} \to \mathcal{D}$ such that:

1. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor

$$F^* : \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

is an epimorphism of categories?

2. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor

$$F_* : \mathsf{Fun}(\mathcal{X}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{X}, \mathcal{D})$$

is an epimorphism of categories?

This question also appears as [`MO 468121a`].

8.6.4 Pseudomonic Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 8.6.4.1.1. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **pseudomonic** if it satisfies the following conditions:

1. For all diagrams of the form

$$\begin{tikzcd} \mathcal{X} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & \mathcal{C} \arrow[r, "F"] & \mathcal{D}, \end{tikzcd}$$

if we have

$$\mathrm{id}_F \star \alpha = \mathrm{id}_F \star \beta,$$

then $\alpha = \beta$.

2. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$ and each natural isomorphism

$$\beta\colon F \circ \phi \overset{\sim}{\Longrightarrow} F \circ \psi, \qquad \begin{tikzcd} \mathcal{X} \arrow[r, bend left, "F\circ\phi"] \arrow[r, bend right, "F\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{D}, \end{tikzcd}$$

there exists a natural isomorphism

$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} \mathcal{X} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{C} \end{tikzcd}$$

such that we have an equality

$$\begin{tikzcd} \mathcal{X} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{C} \arrow[r, "F"] & \mathcal{D} \end{tikzcd} = \begin{tikzcd} \mathcal{X} \arrow[r, bend left, "F\circ\phi"] \arrow[r, bend right, "F\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{D} \end{tikzcd}$$

of pasting diagrams, i.e. such that we have

$$\beta = \mathrm{id}_F \star \alpha.$$

Proposition 8.6.4.1.2. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor.

1. *Characterisations.* The following conditions are equivalent:

   (a) The functor F is pseudomonic.

   (b) The functor F satisfies the following conditions:

i. The functor F is faithful, i.e. for each $A, B \in \mathrm{Obj}(\mathcal{C})$, the action on morphisms

$$F_{A,B} \colon \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F_A, F_B)$$

of F at (A, B) is injective.

ii. For each $A, B \in \mathrm{Obj}(\mathcal{C})$, the restriction

$$F^{\mathrm{iso}}_{A,B} \colon \mathrm{Iso}_{\mathcal{C}}(A, B) \to \mathrm{Iso}_{\mathcal{D}}(F_A, F_B)$$

of the action on morphisms of F at (A, B) to isomorphisms is surjective.

(c) We have an isocomma square of the form

$$\mathcal{C} \overset{\text{eq.}}{\cong} \mathcal{C} \overset{\leftrightarrow}{\times}_{\mathcal{D}} \mathcal{C}, \qquad \begin{tikzcd} \mathcal{C} \arrow[r, "\mathrm{id}_{\mathcal{C}}"] \arrow[d, "\mathrm{id}_{\mathcal{C}}"'] \arrow[dr, phantom, "\Leftrightarrow"] & \mathcal{C} \arrow[d, "F"] \\ \mathcal{C} \arrow[r, "F"'] & \mathcal{D} \end{tikzcd}$$

in Cats_2 up to equivalence.

(d) We have an isocomma square of the form

$$\mathcal{C} \overset{\text{eq.}}{\cong} \mathcal{C} \overset{\leftrightarrow}{\times}_{\mathsf{Arr}(\mathcal{D})} \mathcal{D}, \qquad \begin{tikzcd} \mathcal{C} \arrow[r, hook] \arrow[d, "F"'] \arrow[dr, phantom, "\Leftrightarrow"] & \mathsf{Arr}(\mathcal{C}) \arrow[d, "\mathsf{Arr}(F)"] \\ \mathcal{D} \arrow[r, hook] & \mathsf{Arr}(\mathcal{D}) \end{tikzcd}$$

in Cats_2 up to equivalence.

(e) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition[26] functor

$$F_* \colon \mathsf{Fun}(\mathcal{X}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{X}, \mathcal{D})$$

is pseudomonic.

2. *Conservativity.* If F is pseudomonic, then F is conservative.

3. *Essential Injectivity.* If F is pseudomonic, then F is essentially injective.

Proof. Item 1, Characterisations: Omitted.
Item 2, Conservativity: Omitted.
Item 3, Essential Injectivity: Omitted. □

[26] Asking the precomposition functors

$$F^* \colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

8.6.5 Pseudoepic Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 8.6.5.1.1. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **pseudoepic** if it satisfies the following conditions:

1. For all diagrams of the form

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] & \mathcal{D} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & \mathcal{X}, \end{tikzcd}$$

   if we have

$$\alpha \star \mathrm{id}_F = \beta \star \mathrm{id}_F,$$

   then $\alpha = \beta$.

2. For each $X \in \mathrm{Obj}(\mathcal{C})$ and each 2-isomorphism

$$\beta\colon \phi \circ F \overset{\sim}{\Longrightarrow} \psi \circ F, \qquad \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "\phi\circ F"] \arrow[r, bend right, "\psi\circ F"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{X} \end{tikzcd}$$

   of $\mathcal{C}$, there exists a 2-isomorphism

$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{X} \end{tikzcd}$$

   of $\mathcal{C}$ such that we have an equality

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] & \mathcal{D} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{X} \end{tikzcd} = \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "\phi\circ F"] \arrow[r, bend right, "\psi\circ F"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{X} \end{tikzcd}$$

   of pasting diagrams in $\mathcal{C}$, i.e. such that we have

$$\beta = \alpha \star \mathrm{id}_F.$$

Proposition 8.6.5.1.2. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor.

1. *Characterisations.* The following conditions are equivalent:

---

to be pseudomonic leads to pseudoepic functors; see Item 1b of Item 1 of Proposition 8.6.5.1.2.

(a) The functor F is pseudoepic.

(b) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the functor

$$F^* \colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

given by precomposition by F is pseudomonic.

(c) We have an isococomma square of the form

$$\mathcal{D} \overset{\text{eq.}}{\cong} \mathcal{D} \overset{\leftrightarrow}{\coprod}_{\mathcal{C}} \mathcal{D}, \qquad \begin{tikzcd} \mathcal{D} & \mathcal{D} \arrow[l, "\mathrm{id}_{\mathcal{D}}"'] \\ \mathcal{D} \arrow[u, "\mathrm{id}_{\mathcal{D}}"] & \mathcal{C} \arrow[l, "F"] \arrow[u, "F"'] \arrow[from=2-1, to=1-2, phantom, "\Leftrightarrow"] \end{tikzcd}$$

in Cats_2 up to equivalence.

2. *Dominance.* If F is pseudoepic, then F is dominant (Definition 8.6.1.1.1).

Proof. Item 1, Characterisations: Omitted.
Item 2, Dominance: If F is pseudoepic, then

$$F^* \colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

is pseudomonic for all $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, and thus in particular faithful. By Item 4g of Item 4 of Proposition 8.5.1.1.2, this is equivalent to requiring F to be dominant. □

Question 8.6.5.1.3. Is there a nice characterisation of the pseudoepic functors, similarly to the characterisaiton of pseudomonic functors given in Item 1b of Item 1 of Proposition 8.6.4.1.2?
This question also appears as [MO 321971].

Question 8.6.5.1.4. A pseudomonic and pseudoepic functor is dominant, faithful, essentially injective, and full on isomorphisms. Is it necessarily an equivalence of categories? If not, how bad can this fail, i.e. how far can a pseudomonic and pseudoepic functor be from an equivalence of categories?
This question also appears as [MO 468334].

Question 8.6.5.1.5. Is there a characterisation of functors $F \colon \mathcal{C} \to \mathcal{D}$ such that:

1. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor

$$F^* \colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

is pseudoepic?

2. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor

$$F_* \colon \mathsf{Fun}(\mathcal{X}, C) \to \mathsf{Fun}(\mathcal{X}, \mathcal{D})$$

is pseudoepic?

This question also appears as [MO 468121a].

8.7 Even More Conditions on Functors

8.7.1 Injective on Objects Functors

Let C and $\mathcal{D}$ be categories.

Definition 8.7.1.1.1. A functor $F \colon C \to \mathcal{D}$ is **injective on objects** if the action on objects

$$F \colon \mathrm{Obj}(C) \to \mathrm{Obj}(\mathcal{D})$$

of F is injective.

Proposition 8.7.1.1.2. Let $F \colon C \to \mathcal{D}$ be a functor.

1. *Characterisations.* The following conditions are equivalent:

   (a) The functor F is injective on objects.

   (b) The functor F is an isocofibration in Cats_2.

Proof. *Item 1*, *Characterisations*: Omitted. □

8.7.2 Surjective on Objects Functors

Let C and $\mathcal{D}$ be categories.

Definition 8.7.2.1.1. A functor $F \colon C \to \mathcal{D}$ is **surjective on objects** if the action on objects

$$F \colon \mathrm{Obj}(C) \to \mathrm{Obj}(\mathcal{D})$$

of F is surjective.

8.7.3 Bijective on Objects Functors

Let C and $\mathcal{D}$ be categories.

Definition 8.7.3.1.1. A functor $F \colon C \to \mathcal{D}$ is **bijective on objects**[27] if the action on objects

$$F \colon \mathrm{Obj}(C) \to \mathrm{Obj}(\mathcal{D})$$

of F is a bijection.

[27] *Further Terminology:* Also called a **bo** functor.

8.7.4 Functors Representably Faithful on Cores

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 8.7.4.1.1. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **representably faithful on cores** if, for each $X \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition by F functor

$$F_*\colon \mathsf{Core}(\mathsf{Fun}(\mathcal{X}, \mathcal{C})) \to \mathsf{Core}(\mathsf{Fun}(\mathcal{X}, \mathcal{D}))$$

is faithful.

Remark 8.7.4.1.2. In detail, a functor $F\colon \mathcal{C} \to \mathcal{D}$ is **representably faithful on cores** if, given a diagram of the form

$$\begin{tikzcd} \mathcal{X} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & \mathcal{C} \arrow[r, "F"] & \mathcal{D}, \end{tikzcd}$$

if α and β are natural isomorphisms and we have

$$\mathrm{id}_F \star \alpha = \mathrm{id}_F \star \beta,$$

then $\alpha = \beta$.

Question 8.7.4.1.3. Is there a characterisation of functors representably faithful on cores?

8.7.5 Functors Representably Full on Cores

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 8.7.5.1.1. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **representably full on cores** if, for each $X \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition by F functor

$$F_*\colon \mathsf{Core}(\mathsf{Fun}(\mathcal{X}, \mathcal{C})) \to \mathsf{Core}(\mathsf{Fun}(\mathcal{X}, \mathcal{D}))$$

is full.

Remark 8.7.5.1.2. In detail, a functor $F\colon \mathcal{C} \to \mathcal{D}$ is **representably full on cores** if, for each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$ and each natural isomorphism

$$\beta\colon F \circ \phi \overset{\sim}{\Longrightarrow} F \circ \psi, \qquad \begin{tikzcd} \mathcal{X} \arrow[r, bend left, "F\circ\phi"] \arrow[r, bend right, "F\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{D}, \end{tikzcd}$$

there exists a natural isomorphism

$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} \mathcal{X} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{C} \end{tikzcd}$$

such that we have an equality

$$\begin{tikzcd} \mathcal{X} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{C} \arrow[r, "F"] & \mathcal{D} \end{tikzcd} = \begin{tikzcd} \mathcal{X} \arrow[r, bend left, "F\circ\phi"] \arrow[r, bend right, "F\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{D} \end{tikzcd}$$

of pasting diagrams in Cats_2, i.e. such that we have

$$\beta = \mathrm{id}_F \star \alpha.$$

Question 8.7.5.1.3. Is there a characterisation of functors representably full on cores?
This question also appears as [MO 468121a].

8.7.6 Functors Representably Fully Faithful on Cores

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 8.7.6.1.1. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **representably fully faithful on cores** if, for each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition by F functor

$$F_*\colon \mathsf{Core}(\mathsf{Fun}(\mathcal{X}, \mathcal{C})) \to \mathsf{Core}(\mathsf{Fun}(\mathcal{X}, \mathcal{D}))$$

is fully faithful.

Remark 8.7.6.1.2. In detail, a functor $F\colon \mathcal{C} \to \mathcal{D}$ is **representably fully faithful on cores** if it satisfies the conditions in Remarks 8.7.4.1.2 and 8.7.5.1.2, i.e.:

1. For all diagrams of the form

$$\begin{tikzcd} \mathcal{X} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & \mathcal{C} \arrow[r, "F"] & \mathcal{D}, \end{tikzcd}$$

with α and β natural isomorphisms, if we have $\mathrm{id}_F \star \alpha = \mathrm{id}_F \star \beta$, then $\alpha = \beta$.

2. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$ and each natural isomorphism
$$\beta\colon F\circ\phi \overset{\sim}{\Longrightarrow} F\circ\psi, \qquad \begin{tikzcd} \mathcal{X} \arrow[r, bend left, "F\circ\phi"] \arrow[r, bend right, "F\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{D} \end{tikzcd}$$
of C, there exists a natural isomorphism
$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} \mathcal{X} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & C \end{tikzcd}$$
of C such that we have an equality
$$\begin{tikzcd} \mathcal{X} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & C \arrow[r, "F"] & \mathcal{D} \end{tikzcd} = \begin{tikzcd} \mathcal{X} \arrow[r, bend left, "F\circ\phi"] \arrow[r, bend right, "F\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{D} \end{tikzcd}$$
of pasting diagrams in Cats_2, i.e. such that we have
$$\beta = \mathrm{id}_F \star \alpha.$$

Question 8.7.6.1.3. Is there a characterisation of functors representably fully faithful on cores?

8.7.7 Functors Corepresentably Faithful on Cores

Let C and $\mathcal{D}$ be categories.

Definition 8.7.7.1.1. A functor $F\colon C \to \mathcal{D}$ is **corepresentably faithful on cores** if, for each $X \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition by F functor
$$F_*\colon \mathsf{Core}(\mathsf{Fun}(\mathcal{X}, C)) \to \mathsf{Core}(\mathsf{Fun}(\mathcal{X}, \mathcal{D}))$$
is faithful.

Remark 8.7.7.1.2. In detail, a functor $F\colon C \to \mathcal{D}$ is **corepresentably faithful on cores** if, given a diagram of the form
$$\begin{tikzcd} C \arrow[r, "F"] & \mathcal{D} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & \mathcal{X}, \end{tikzcd}$$
if α and β are natural isomorphisms and we have
$$\alpha \star \mathrm{id}_F = \beta \star \mathrm{id}_F,$$
then $\alpha = \beta$.

Question 8.7.7.1.3. Is there a characterisation of functors corepresentably faithful on cores?

8.7.8 Functors Corepresentably Full on Cores

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 8.7.8.1.1. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **corepresentably full on cores** if, for each $X \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition by F functor

$$F_*\colon \mathsf{Core}(\mathsf{Fun}(\mathcal{X},\mathcal{C})) \to \mathsf{Core}(\mathsf{Fun}(\mathcal{X},\mathcal{D}))$$

is full.

Remark 8.7.8.1.2. In detail, a functor $F\colon \mathcal{C} \to \mathcal{D}$ is **corepresentably full on cores** if, for each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$ and each natural isomorphism

$$\beta\colon \phi \circ F \overset{\sim}{\Longrightarrow} \psi \circ F, \qquad \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "\phi\circ F"] \arrow[r, bend right, "\psi\circ F"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{X}, \end{tikzcd}$$

there exists a natural isomorphism

$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{X} \end{tikzcd}$$

such that we have an equality

$$\begin{tikzcd} \mathcal{X} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{C} \arrow[r, "F"] & \mathcal{D} \end{tikzcd} \quad = \quad \begin{tikzcd} \mathcal{X} \arrow[r, bend left, "F\circ\phi"] \arrow[r, bend right, "F\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{D} \end{tikzcd}$$

of pasting diagrams in Cats_2, i.e. such that we have

$$\beta = \alpha \star \mathrm{id}_F.$$

Question 8.7.8.1.3. Is there a characterisation of functors corepresentably full on cores?
This question also appears as [MO 468121a].

8.7.9 Functors Corepresentably Fully Faithful on Cores

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 8.7.9.1.1. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **corepresentably fully faithful on cores** if, for each $X \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition by F functor

$$F_*\colon \mathsf{Core}(\mathsf{Fun}(\mathcal{X},\mathcal{C})) \to \mathsf{Core}(\mathsf{Fun}(\mathcal{X},\mathcal{D}))$$

is fully faithful.

Remark 8.7.9.1.2. In detail, a functor $F\colon \mathcal{C} \to \mathcal{D}$ is **corepresentably fully faithful on cores** if it satisfies the conditions in Remarks 8.7.7.1.2 and 8.7.8.1.2, i.e.:

1. For all diagrams of the form
$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] & \mathcal{D} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & \mathcal{X}, \end{tikzcd}$$
if α and β are natural isomorphisms and we have
$$\alpha \star \mathrm{id}_F = \beta \star \mathrm{id}_F,$$
then $\alpha = \beta$.

2. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$ and each natural isomorphism
$$\beta\colon \phi \circ F \overset{\sim}{\Longrightarrow} \psi \circ F, \qquad \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "\phi\circ F"] \arrow[r, bend right, "\psi\circ F"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{X}, \end{tikzcd}$$
there exists a natural isomorphism
$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{X} \end{tikzcd}$$
such that we have an equality
$$\begin{tikzcd} \mathcal{X} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{C} \arrow[r, "F"] & \mathcal{D} \end{tikzcd} \quad = \quad \begin{tikzcd} \mathcal{X} \arrow[r, bend left, "F\circ\phi"] \arrow[r, bend right, "F\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{D} \end{tikzcd}$$
of pasting diagrams in Cats_2, i.e. such that we have
$$\beta = \alpha \star \mathrm{id}_F.$$

Question 8.7.9.1.3. Is there a characterisation of functors corepresentably fully faithful on cores?

8.8 Natural Transformations

8.8.1 Transformations

Let $\mathcal{C}$ and $\mathcal{D}$ be categories and $F, G\colon \mathcal{C} \rightrightarrows \mathcal{D}$ be functors.

Definition 8.8.1.1.1. A **transformation**[28] $\alpha\colon F \Rightarrow G$ **from** F **to** G is a collection

$$\{\alpha_A\colon F(A) \to G(A)\}_{A\in\mathsf{Obj}(\mathcal{C})}$$

of morphisms of $\mathcal{D}$.

Notation 8.8.1.1.2. We write $\mathrm{Trans}(F, G)$ for the set of transformations from F to G.

8.8.2 Natural Transformations

Let $\mathcal{C}$ and $\mathcal{D}$ be categories and $F, G\colon \mathcal{C} \rightrightarrows \mathcal{D}$ be functors.

Definition 8.8.2.1.1. A **natural transformation** $\alpha\colon F \Longrightarrow G$ **from** F **to** G is a transformation

$$\{\alpha_A\colon F(A) \to G(A)\}_{A\in\mathsf{Obj}(\mathcal{C})}$$

from F to G such that, for each morphism $f\colon A \to B$ of $\mathcal{C}$, the diagram

$$\begin{tikzcd}
F(A) \arrow[r, "F(f)"] \arrow[d, "\alpha_A"'] & F(B) \arrow[d, "\alpha_B"] \\
G(A) \arrow[r, "G(f)"'] & G(B)
\end{tikzcd}$$

commutes.[29]

Remark 8.8.2.1.2. We denote natural transformations in diagrams as

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{D}.
\end{tikzcd}$$

Notation 8.8.2.1.3. We write $\mathrm{Nat}(F, G)$ for the set of natural transformations from F to G.

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[28] *Further Terminology:* Also called an **unnatural transformation** for emphasis.

[29] *Further Terminology:* The morphism $\alpha_A\colon F_A \to G_A$ is called the **component of** α **at** A.

Example 8.8.2.1.4. The **identity natural transformation** $\mathrm{id}_F : F \Longrightarrow F$ **of** F is the natural transformation consisting of the collection

$$\left\{\mathrm{id}_{F(A)} : F(A) \to F(A)\right\}_{A \in \mathrm{Obj}(C)}.$$

Proof. The naturality condition for id_F is the requirement that, for each morphism $f : A \to B$ of C, the diagram

$$\begin{tikzcd}
F(A) \arrow[r, "F(f)"] \arrow[d, "\mathrm{id}_{F(A)}"'] & F(B) \arrow[d, "\mathrm{id}_{F(B)}"] \\
F(A) \arrow[r, "F(f)"'] & F(B)
\end{tikzcd}$$

commutes, which follows from unitality of the composition of C. □

Definition 8.8.2.1.5. Two natural transformations $\alpha, \beta : F \Longrightarrow G$ are **equal** if we have

$$\alpha_A = \beta_A$$

for each $A \in \mathrm{Obj}(C)$.

8.8.3 Vertical Composition of Natural Transformations

Definition 8.8.3.1.1. The **vertical composition** of two natural transformations $\alpha : F \Longrightarrow G$ and $\beta : G \Longrightarrow H$ as in the diagram

$$\begin{tikzcd}
C \arrow[r, bend left=50, "F"{name=F}] \arrow[r, "G"{description, name=G}] \arrow[r, bend right=50, "H"'{name=H}] & \mathcal{D}
\arrow[from=F, to=G, Rightarrow, "\alpha"']
\arrow[from=G, to=H, Rightarrow, "\beta"']
\end{tikzcd}$$

is the natural transformation $\beta \circ \alpha : F \Longrightarrow H$ consisting of the collection

$$\left\{(\beta \circ \alpha)_A : F(A) \to H(A)\right\}_{A \in \mathrm{Obj}(C)}$$

with

$$(\beta \circ \alpha)_A \stackrel{\text{def}}{=} \beta_A \circ \alpha_A$$

for each $A \in \mathrm{Obj}(C)$.

Proof. The naturality condition for $\beta \circ \alpha$ is the requirement that the boundary of the diagram

$$\begin{tikzcd}
F(A) \arrow[r, "F(f)"] \arrow[d, "\alpha_A"'] \arrow[dr, phantom, "(1)"] & F(B) \arrow[d, "\alpha_B"] \\
G(A) \arrow[r, "G(f)" description] \arrow[d, "\beta_A"'] \arrow[dr, phantom, "(2)"] & G(B) \arrow[d, "\beta_B"] \\
H(A) \arrow[r, "H(f)"'] & H(B)
\end{tikzcd}$$

commutes. Since

1. Subdiagram (1) commutes by the naturality of α.
2. Subdiagram (2) commutes by the naturality of β.

so does the boundary diagram. Hence $\beta \circ \alpha$ is a natural transformation. □

Proposition 8.8.3.1.2. Let $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ be categories.

1. *Functionality.* The assignment $(\beta, \alpha) \mapsto \beta \circ \alpha$ defines a function
$$\circ_{F,G,H} \colon \mathrm{Nat}(G, H) \times \mathrm{Nat}(F, G) \to \mathrm{Nat}(F, H).$$
2. *Associativity.* Let $F, G, H, K \colon \mathcal{C} \overset{\rightrightarrows}{\rightrightarrows} \mathcal{D}$ be functors. The diagram
$$\begin{tikzcd}
& \mathrm{Nat}(H,K) \times (\mathrm{Nat}(G,H) \times \mathrm{Nat}(F,G)) \arrow[dr, "\mathrm{id}_{\mathrm{Nat}(H,K)} \times \circ_{F,G,H}"] & \\
(\mathrm{Nat}(H,K) \times \mathrm{Nat}(G,H)) \times \mathrm{Nat}(F,G) \arrow[ur, dashed, "\alpha^{\mathsf{Sets}}_{\mathrm{Nat}(H,K),\mathrm{Nat}(G,H),\mathrm{Nat}(F,G)}", "\sim"'] \arrow[d, "\circ_{G,H,K} \times \mathrm{id}_{\mathrm{Nat}(F,G)}"'] & & \mathrm{Nat}(H,K) \times \mathrm{Nat}(F,H) \arrow[d, "\circ_{F,H,K}"] \\
\mathrm{Nat}(G,K) \times \mathrm{Nat}(F,G) \arrow[rr, "\circ_{F,G,K}"'] & & \mathrm{Nat}(F,K)
\end{tikzcd}$$
commutes, i.e. given natural transformations
$$F \overset{\alpha}{\Longrightarrow} G \overset{\beta}{\Longrightarrow} H \overset{\gamma}{\Longrightarrow} K,$$
we have
$$(\gamma \circ \beta) \circ \alpha = \gamma \circ (\beta \circ \alpha).$$

3. *Unitality.* Let $F, G \colon \mathcal{C} \rightrightarrows \mathcal{D}$ be functors.

   (a) *Left Unitality.* The diagram

$$\begin{tikzcd}
\mathrm{pt} \times \mathrm{Nat}(F,G) \arrow[d, "{[\mathrm{id}_G] \times \mathrm{id}_{\mathrm{Nat}(F,G)}}"'] \arrow[dr, dashed, "\lambda^{\mathsf{Sets}}_{\mathrm{Nat}(F,G)}", "\sim"'] & \\
\mathrm{Nat}(G,G) \times \mathrm{Nat}(F,G) \arrow[r, "\circ_{F,G,G}"'] & \mathrm{Nat}(F,G)
\end{tikzcd}$$

   commutes, i.e. given a natural transformation $\alpha \colon F \Longrightarrow G$, we have

$$\mathrm{id}_G \circ \alpha = \alpha.$$

   (b) *Right Unitality.* The diagram

$$\begin{tikzcd}
\mathrm{Nat}(F,G) \times \mathrm{pt} \arrow[d, "{\mathrm{id}_{\mathrm{Nat}(F,G)} \times [\mathrm{id}_F]}"'] \arrow[dr, dashed, "\rho^{\mathsf{Sets}}_{\mathrm{Nat}(F,G)}", "\sim"'] & \\
\mathrm{Nat}(F,G) \times \mathrm{Nat}(F,F) \arrow[r, "\circ^{\mathcal{C}}_{F,F,G}"'] & \mathrm{Nat}(F,G)
\end{tikzcd}$$

   commutes, i.e. given a natural transformation $\alpha \colon F \Longrightarrow G$, we have

$$\alpha \circ \mathrm{id}_F = \alpha.$$

4. *Middle Four Exchange.* Let $F_1, F_2, F_3 \colon \mathcal{C} \to \mathcal{D}$ and $G_1, G_2, G_3 \colon \mathcal{D} \to \mathcal{E}$ be functors. The diagram

$$\begin{tikzcd}
(\mathrm{Nat}(G_2,G_3) \times \mathrm{Nat}(G_1,G_2)) \times (\mathrm{Nat}(F_2,F_3) \times \mathrm{Nat}(F_1,F_2)) \arrow[d, "\circ_{G_1,G_2,G_3} \times \circ_{F_1,F_2,F_3}"'] & & (\mathrm{Nat}(G_2,G_3) \times \mathrm{Nat}(F_2,F_3)) \times (\mathrm{Nat}(G_1,G_2) \times \mathrm{Nat}(F_1,F_2)) \arrow[ll, dashed, leftrightarrow, "\mu_4", "\sim"'] \arrow[d, "\star_{F_2,F_3,G_2,G_3} \times \star_{F_1,F_2,G_1,G_2}"] \\
\mathrm{Nat}(G_1,G_3) \times \mathrm{Nat}(F_1,F_3) \arrow[dr, "\star_{F_1,F_3,G_1,G_3}"'] & & \mathrm{Nat}(G_2 \circ F_2, G_3 \circ F_3) \times \mathrm{Nat}(G_1 \circ F_1, G_2 \circ F_2) \arrow[dl, "\circ_{G_1 \circ F_1, G_2 \circ F_2, G_3 \circ F_3}"] \\
& \mathrm{Nat}(G_1 \circ F_1, G_3 \circ F_3) &
\end{tikzcd}$$

   commutes, i.e. given a diagram

$$\begin{tikzcd}[column sep=huge]
\mathcal{C} \arrow[r, bend left=60, "F_1"{name=U}] \arrow[r, "F_2"{name=M, description}] \arrow[r, bend right=60, "F_3"'{name=L}] & \mathcal{D} \arrow[r, bend left=60, "G_1"{name=U2}] \arrow[r, "G_2"{name=M2, description}] \arrow[r, bend right=60, "G_3"'{name=L2}] & \mathcal{E}
\arrow[Rightarrow, from=U, to=M, "\alpha"']
\arrow[Rightarrow, from=M, to=L, "\alpha'"']
\arrow[Rightarrow, from=U2, to=M2, "\beta"']
\arrow[Rightarrow, from=M2, to=L2, "\beta'"']
\end{tikzcd}$$

in Cats_2, we have

$$(\beta' \star \alpha') \circ (\beta \star \alpha) = (\beta' \circ \beta) \star (\alpha' \circ \alpha).$$

Proof. *Item 1, Functionality*: Clear.
Item 2, Associativity: Indeed, we have

$$\begin{aligned}((\gamma \circ \beta) \circ \alpha)_A &\overset{\text{def}}{=} (\gamma \circ \beta)_A \circ \alpha_A \\ &\overset{\text{def}}{=} (\gamma_A \circ \beta_A) \circ \alpha_A \\ &= \gamma_A \circ (\beta_A \circ \alpha_A) \\ &\overset{\text{def}}{=} \gamma_A \circ (\beta \circ \alpha)_A \\ &\overset{\text{def}}{=} (\gamma \circ (\beta \circ \alpha))_A\end{aligned}$$

for each $A \in \mathrm{Obj}(\mathcal{C})$, showing the desired equality.
Item 3, Unitality: We have

$$\begin{aligned}(\mathrm{id}_G \circ \alpha)_A &= \mathrm{id}_G \circ \alpha_A \\ &= \alpha_A, \\ (\alpha \circ \mathrm{id}_F)_A &= \alpha_A \circ \mathrm{id}_F \\ &= \alpha_A\end{aligned}$$

for each $A \in \mathrm{Obj}(\mathcal{C})$, showing the desired equality.
Item 4, Middle Four Exchange: This is proved in Item 4 of Proposition 8.8.4.1.3. □

8.8.4 Horizontal Composition of Natural Transformations

Definition 8.8.4.1.1. The **horizontal composition**[30,31] of two natural transformations $\alpha\colon F \Longrightarrow G$ and $\beta\colon H \Longrightarrow K$ as in the diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{D} \arrow[r, bend left, "H"] \arrow[r, bend right, "K"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{E} \end{tikzcd}$$

of α and β is the natural transformation

$$\beta \star \alpha\colon (H \circ F) \Longrightarrow (K \circ G),$$

[30] *Further Terminology:* Also called the **Godement product** of α and β.
[31] Horizontal composition forms a map

$$\star_{(F,H),(G,K)}\colon \mathrm{Nat}(H,K) \times \mathrm{Nat}(F,G) \to \mathrm{Nat}(H \circ F, K \circ G).$$

as in the diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "H\circ F"] \arrow[r, bend right, "K\circ G"'] \arrow[r, phantom, "\Downarrow \beta\star\alpha"] & \mathcal{E}, \end{tikzcd}$$

consisting of the collection

$$\left\{(\beta \star \alpha)_A \colon H(F(A)) \to K(G(A))\right\}_{A\in\mathrm{Obj}(\mathcal{C})},$$

of morphisms of $\mathcal{E}$ with

$$\begin{aligned}(\beta \star \alpha)_A &\overset{\text{def}}{=} \beta_{G(A)} \circ H(\alpha_A)\\ &= K(\alpha_A) \circ \beta_{F(A)},\end{aligned} \qquad \begin{tikzcd} H(F(A)) \arrow[r, "H(\alpha_A)"] \arrow[d, "\beta_{F(A)}"'] & H(G(A)) \arrow[d, "\beta_{G(A)}"] \\ K(F(A)) \arrow[r, "K(\alpha_A)"'] & K(G(A)). \end{tikzcd}$$

Proof. First, we claim that we indeed have

$$\beta_{G(A)} \circ H(\alpha_A) = K(\alpha_A) \circ \beta_{F(A)}, \qquad \begin{tikzcd} H(F(A)) \arrow[r, "H(\alpha_A)"] \arrow[d, "\beta_{F(A)}"'] & H(G(A)) \arrow[d, "\beta_{G(A)}"] \\ K(F(A)) \arrow[r, "K(\alpha_A)"'] & K(G(A)). \end{tikzcd}$$

This is, however, simply the naturality square for β applied to the morphism $\alpha_A \colon F(A) \to G(A)$. Next, we check the naturality condition for $\beta \star \alpha$, which is the requirement that the boundary of the diagram

$$\begin{tikzcd} H(F(A)) \arrow[r, "H(F(f))"] \arrow[d, "H(\alpha_A)"'] \arrow[dr, phantom, "(1)"] & H(F(B)) \arrow[d, "H(\alpha_B)"] \\ H(G(A)) \arrow[r, "H(G(f))"] \arrow[d, "\beta_{G(A)}"'] \arrow[dr, phantom, "(2)"] & H(G(B)) \arrow[d, "\beta_{G(B)}"] \\ K(G(A)) \arrow[r, "K(G(f))"'] & K(G(B)) \end{tikzcd}$$

commutes. Since

1. Subdiagram (1) commutes by the naturality of α.
2. Subdiagram (2) commutes by the naturality of β.

so does the boundary diagram. Hence $\beta \circ \alpha$ is a natural transformation.[32] □

Definition 8.8.4.1.2. Let

$$\begin{tikzcd} \mathcal{X} \arrow[r, "F"] & \mathcal{C} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{D} \arrow[r, "G"] & \mathcal{Y} \end{tikzcd}$$

be a diagram in Cats_2.

1. The **left whiskering of** α **with** G is the natural transformation[33]

$$\mathrm{id}_G \star \alpha \colon G \circ \phi \Longrightarrow G \circ \psi.$$

2. The **right whiskering of** α **with** F is the natural transformation[34]

$$\alpha \star \mathrm{id}_F \colon \phi \circ F \Longrightarrow \psi \circ F.$$

Proposition 8.8.4.1.3. Let $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ be categories.

1. *Functionality.* The assignment $(\beta, \alpha) \mapsto \beta \star \alpha$ defines a function

$$\star_{(F,G),(H,K)} \colon \mathrm{Nat}(H, K) \times \mathrm{Nat}(F, G) \longrightarrow \mathrm{Nat}(H \circ F, K \circ G).$$

2. *Associativity.* Let

$$\mathcal{C} \underset{G_1}{\overset{F_1}{\rightrightarrows}} \mathcal{D} \underset{G_2}{\overset{F_2}{\rightrightarrows}} \mathcal{E} \underset{G_3}{\overset{F_3}{\rightrightarrows}} \mathcal{F}$$

be a diagram in Cats_2. The diagram

$$\begin{tikzcd} \mathrm{Nat}(F_3, G_3) \times \mathrm{Nat}(F_2, G_2) \times \mathrm{Nat}(F_1, G_1) \arrow[r, "\star_{(F_2,G_2),(F_3,G_3)} \times \mathrm{id}"] \arrow[d, "\mathrm{id} \times \star_{(F_1,G_1),(F_2,G_2)}"'] & \mathrm{Nat}(F_3 \circ F_2, G_3 \circ G_2) \times \mathrm{Nat}(F_1, G_1) \arrow[d, "\star_{(F_3 \circ F_2),(G_3 \circ G_2, F_1, G_1)}"] \\ \mathrm{Nat}(F_3, G_3) \times \mathrm{Nat}(F_2 \circ F_1, G_2 \circ G_1) \arrow[r, "\star_{(F_2 \circ F_1),(G_2 \circ G_1, F_3, G_3)}"'] & \mathrm{Nat}(F_3 \circ F_2 \circ F_1, G_3 \circ G_2 \circ G_1) \end{tikzcd}$$

[32] *Reference:* [Bor94, Proposition 1.3.4].
[33] *Further Notation:* Also written $G\alpha$ or $G \star \alpha$, although we won't use either of these notations in this work.
[34] *Further Notation:* Also written αF or $\alpha \star F$, although we won't use either of these notations

commutes, i.e. given natural transformations

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F_1"] \arrow[r, bend right, "G_1"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{D} \arrow[r, bend left, "F_2"] \arrow[r, bend right, "G_2"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{E} \arrow[r, bend left, "F_3"] \arrow[r, bend right, "G_3"'] \arrow[r, phantom, "\gamma\Downarrow"] & \mathcal{F}, \end{tikzcd}$$

we have

$$(\gamma \star \beta) \star \alpha = \gamma \star (\beta \star \alpha).$$

3. *Interaction With Identities.* Let $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{E}$ be functors. The diagram

$$\begin{tikzcd} \mathrm{pt} \times \mathrm{pt} \arrow[r, "{[\mathrm{id}_G] \times [\mathrm{id}_F]}"] & \mathrm{Nat}(G, G) \times \mathrm{Nat}(F, F) \arrow[d, "\star_{(F,F),(G,G)}"] \\ \mathrm{pt} \arrow[u, dashed, "\wr"] \arrow[r, "{[\mathrm{id}_{G\circ F}]}"'] & \mathrm{Nat}(G \circ F, G \circ F) \end{tikzcd}$$

commutes, i.e. we have

$$\mathrm{id}_G \star \mathrm{id}_F = \mathrm{id}_{G\circ F}.$$

4. *Middle Four Exchange.* Let $F_1, F_2, F_3 \colon \mathcal{C} \to \mathcal{D}$ and $G_1, G_2, G_3 \colon \mathcal{D} \to \mathcal{E}$ be functors. The diagram

$$\begin{tikzcd} (\mathrm{Nat}(G_2, G_3) \times \mathrm{Nat}(G_1, G_2)) \times (\mathrm{Nat}(F_2, F_3) \times \mathrm{Nat}(F_1, F_2)) \arrow[d, "\circ_{G_1,G_2,G_3} \times \circ_{F_1,F_2,F_3}"'] & & (\mathrm{Nat}(G_2, G_3) \times \mathrm{Nat}(F_2, F_3)) \times (\mathrm{Nat}(G_1, G_2) \times \mathrm{Nat}(F_1, F_2)) \arrow[ll, dashed, leftrightarrow, "\mu_4", "\sim"'] \arrow[d, "\star_{F_2,F_3,G_2,G_3} \times \star_{F_1,F_2,G_1,G_2}"] \\ \mathrm{Nat}(G_1, G_3) \times \mathrm{Nat}(F_1, F_3) \arrow[dr, "\star_{F_1,F_3,G_1,G_3}"'] & & \mathrm{Nat}(G_2 \circ F_2, G_3 \circ F_3) \times \mathrm{Nat}(G_1 \circ F_1, G_2 \circ F_2) \arrow[dl, "\circ_{G_1\circ F_1, G_2\circ F_2, G_3\circ F_3}"] \\ & \mathrm{Nat}(G_1 \circ F_1, G_3 \circ F_3) & \end{tikzcd}$$

commutes, i.e. given a diagram

$$\begin{tikzcd}[row sep=large] \mathcal{C} \arrow[r, bend left=60, "F_1"] \arrow[r, "F_2" description] \arrow[r, bend right=60, "F_3"'] \arrow[r, phantom, shift left=3, "\alpha\Downarrow"] \arrow[r, phantom, shift right=3, "\alpha'\Downarrow"] & \mathcal{D} \arrow[r, bend left=60, "G_1"] \arrow[r, "G_2" description] \arrow[r, bend right=60, "G_3"'] \arrow[r, phantom, shift left=3, "\beta\Downarrow"] \arrow[r, phantom, shift right=3, "\beta'\Downarrow"] & \mathcal{E} \end{tikzcd}$$

in Cats_2, we have

$$(\beta' \star \alpha') \circ (\beta \star \alpha) = (\beta' \circ \beta) \star (\alpha' \circ \alpha).$$

Proof. *Item 1, Functionality*: Clear.
Item 2, Associativity: Omitted.
Item 3, Interaction With Identities: We have

$$\begin{aligned}(\mathrm{id}_G \star \mathrm{id}_F)_A &\stackrel{\text{def}}{=} (\mathrm{id}_G)_{F_A} \circ G_{(\mathrm{id}_F)_A}\\ &\stackrel{\text{def}}{=} \mathrm{id}_{G_{F_A}} \circ G_{\mathrm{id}_{F_A}}\\ &= \mathrm{id}_{G_{F_A}} \circ \mathrm{id}_{G_{F_A}}\\ &= \mathrm{id}_{G_{F_A}}\\ &\stackrel{\text{def}}{=} (\mathrm{id}_{G\circ F})_A\end{aligned}$$

for each $A \in \mathrm{Obj}(\mathcal{C})$, showing the desired equality.
Item 4, Middle Four Exchange: Let $A \in \mathrm{Obj}(\mathcal{C})$ and consider the diagram

$$\begin{tikzcd}
 & & G_1(F_3(A)) \arrow[dr, "\beta_{F_3(A)}"] & & \\
G_1(F_1(A)) \arrow[r, "G_1(\alpha_A)"] & G_1(F_2(A)) \arrow[ur, "G_1(\alpha'_A)"] \arrow[dr, "\beta_{F_2(A)}"'] & (1) & G_2(F_3(A)) \arrow[r, "\beta'_{F_3(A)}"] & G_3(F_3(A)). \\
 & & G_2(F_2(A)) \arrow[ur, "G_2(\alpha'_A)"'] & &
\end{tikzcd}$$

The top composition

$$\begin{tikzcd}
 & & G_1(F_3(A)) \arrow[dr, "\beta_{F_3(A)}"] & & \\
G_1(F_1(A)) \arrow[r, "G_1(\alpha_A)"] & G_1(F_2(A)) \arrow[ur, "G_1(\alpha'_A)"] \arrow[dr, "\beta_{F_2(A)}"'] & (1) & G_2(F_3(A)) \arrow[r, "\beta'_{F_3(A)}"] & G_3(F_3(A)). \\
 & & G_2(F_2(A)) \arrow[ur, "G_2(\alpha'_A)"'] & &
\end{tikzcd}$$

in this work.

is given by $((\beta' \circ \beta) \star (\alpha' \circ \alpha))_A$, while the bottom composition

$$\begin{tikzcd}
 & & G_1(F_3(A)) \arrow[dr, "\beta_{F_3(A)}"] & & \\
G_1(F_1(A)) \arrow[r, "G_1(\alpha_A)"] & G_1(F_2(A)) \arrow[ur, "G_1(\alpha'_A)"] \arrow[dr, "\beta_{F_2(A)}"'] & (1) & G_2(F_3(A)) \arrow[r, "\beta'_{F_3(A)}"] & G_3(F_3(A)). \\
 & & G_2(F_2(A)) \arrow[ur, "G_2(\alpha'_A)"'] & &
\end{tikzcd}$$

is given by $((\beta' \star \alpha') \circ (\beta \star \alpha))_A$. Now, Subdiagram (1) corresponds to the naturality condition

$$G_2(\alpha'_A) \circ \beta_{F_2(A)} = \beta_{F_3}(A) \circ G_1(\alpha'_A), \qquad \begin{tikzcd}
G_1(F_2(A)) \arrow[r, "G_1(\alpha'_A)"] \arrow[d, "\beta_{F_2(A)}"'] & G_1(F_3(A)) \arrow[d, "\beta_{F_3(A)}"] \\
G_2(F_2(A)) \arrow[r, "G_2(\alpha'_A)"'] & G_2(F_3(A))
\end{tikzcd}$$

for $\beta\colon G_1 \Longrightarrow G_2$ at $\alpha'_A\colon F_2(A) \longrightarrow F_3(A)$, and thus commutes. Thus we have

$$((\beta' \circ \beta) \star (\alpha' \circ \alpha))_A = ((\beta' \star \alpha') \circ (\beta \star \alpha))_A$$

for each $A \in \mathrm{Obj}(\mathcal{C})$ and therefore

$$(\beta' \star \alpha') \circ (\beta \star \alpha) = (\beta' \circ \beta) \star (\alpha' \circ \alpha).$$

This finishes the proof. □

8.8.5 Properties of Natural Transformations

Proposition 8.8.5.1.1. Let $F, G\colon \mathcal{C} \rightrightarrows \mathcal{D}$ be functors. The following data are equivalent:[35]

1. A natural transformation $\alpha\colon F \Longrightarrow G$.

[35] Taken from [MO 64365].

2. A functor $[\alpha]\colon \mathcal{C} \to \mathcal{D}^{\mathbb{1}}$ filling the diagram

$$\begin{tikzcd} & \mathcal{D} \\ \mathcal{C} \arrow[ur, "F"] \arrow[r, "{[\alpha]}"] \arrow[dr, "G"'] & \mathcal{D}^{\mathbb{1}}. \arrow[u, "\mathrm{ev}_0"'] \arrow[d, "\mathrm{ev}_1"] \\ & \mathcal{D} \end{tikzcd}$$

3. A functor $[\alpha]\colon \mathcal{C} \times \mathbb{1} \to \mathcal{D}$ filling the diagram

$$\begin{tikzcd} \mathcal{C} \arrow[dr, "F"] & \\ \mathcal{C} \times \mathbb{1} \arrow[u, "\mathrm{ev}_0"] \arrow[d, "\mathrm{ev}_1"'] \arrow[r, "{[\alpha]}"] & \mathcal{D}. \\ \mathcal{C} \arrow[ur, "G"'] & \end{tikzcd}$$

Proof. *From Item 1 to Item 2 and Back*: We may identify $\mathcal{D}^{\mathbb{1}}$ with $\mathsf{Arr}(\mathcal{D})$. Given a natural transformation $\alpha\colon F \Longrightarrow G$, we have a functor

$$\begin{aligned} [\alpha]\colon \mathcal{C} &\longrightarrow \mathcal{D}^{\mathbb{1}} \\ A &\longmapsto \alpha_A \\ (f\colon A \to B) &\longmapsto \left(\begin{tikzcd} F_A \arrow[r, "F_f"] \arrow[d, "\alpha_A"'] & F_B \arrow[d, "\alpha_B"] \\ G_A \arrow[r, "G_f"'] & G_B \end{tikzcd} \right) \end{aligned}$$

making the diagram in Item 2 commute. Conversely, every such functor gives rise to a natural transformation from F to G, and these constructions are inverse to each other.

From Item 2 to Item 3 and Back: This follows from Item 3 of Proposition 8.9.1.1.2.

□

8.8.6 Natural Isomorphisms

Let $\mathcal{C}$ and $\mathcal{D}$ be categories and let $F, G \colon \mathcal{C} \rightrightarrows \mathcal{D}$ be functors.

Definition 8.8.6.1.1. A natural transformation $\alpha \colon F \Longrightarrow G$ is a **natural isomorphism** if there exists a natural transformation $\alpha^{-1} \colon G \Longrightarrow F$ such that

$$\begin{aligned}\alpha^{-1} \circ \alpha &= \mathrm{id}_F,\\ \alpha \circ \alpha^{-1} &= \mathrm{id}_G.\end{aligned}$$

Proposition 8.8.6.1.2. Let $\alpha \colon F \Longrightarrow G$ be a natural transformation.

1. *Characterisations.* The following conditions are equivalent:

   (a) The natural transformation α is a natural isomorphism.

   (b) For each $A \in \mathrm{Obj}(\mathcal{C})$, the morphism $\alpha_A \colon F_A \to G_A$ is an isomorphism.

2. *Componentwise Inverses of Natural Transformations Assemble Into Natural Transformations.* Let $\alpha^{-1} \colon G \Longrightarrow F$ be a transformation such that, for each $A \in \mathrm{Obj}(\mathcal{C})$, we have

$$\begin{aligned}\alpha_A^{-1} \circ \alpha_A &= \mathrm{id}_{F(A)},\\ \alpha_A \circ \alpha_A^{-1} &= \mathrm{id}_{G(A)}.\end{aligned}$$

   Then α^{-1} is a natural transformation.

Proof. *Item 1, Characterisations*: The implication Item 1a $\Longrightarrow$ Item 1b is clear, whereas the implication Item 1b $\Longrightarrow$ Item 1a follows from Item 2.
Item 2, Componentwise Inverses of Natural Transformations Assemble Into Natural Transformations: The naturality condition for α^{-1} corresponds to the commutativity of the diagram

$$\begin{tikzcd} G(A) \arrow[r, "G(f)"] \arrow[d, "\alpha_A^{-1}"'] & G(B) \arrow[d, "\alpha_B^{-1}"] \\ F(A) \arrow[r, "F(f)"'] & F(B) \end{tikzcd}$$

for each $A, B \in \mathrm{Obj}(\mathcal{C})$ and each $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$. Considering the diagram

$$\begin{tikzcd}
G(A) \arrow[r, "G(f)"] \arrow[d, "\alpha_A^{-1}"'] \arrow[dr, phantom, "(1)"] & G(B) \arrow[d, "\alpha_B^{-1}"] \\
F(A) \arrow[r, "F(f)"] \arrow[d, "\alpha_A"'] \arrow[dr, phantom, "(2)"] & F(B) \arrow[d, "\alpha_B"] \\
G(A) \arrow[r, "G(f)"'] & G(B),
\end{tikzcd}$$

where the boundary diagram as well as Subdiagram (2) commute, we have

$$\begin{aligned}
G(f) &= G(f) \circ \mathrm{id}_{G(A)} \\
&= G(f) \circ \alpha_A \circ \alpha_A^{-1} \\
&= \alpha_B \circ F(f) \circ \alpha_A^{-1}.
\end{aligned}$$

Postcomposing both sides with α_B^{-1}, we get

$$\begin{aligned}
\alpha_B^{-1} \circ G(f) &= \alpha_B^{-1} \circ \alpha_B \circ F(f) \circ \alpha_A^{-1} \\
&= \mathrm{id}_{F(B)} \circ F(f) \circ \alpha_A^{-1} \\
&= F(f) \circ \alpha_A^{-1},
\end{aligned}$$

which is the naturality condition we wanted to show. Thus α^{-1} is a natural transformation. □

8.9 Categories of Categories

8.9.1 Functor Categories

Let $\mathcal{C}$ be a category and $\mathcal{D}$ be a small category.

Definition 8.9.1.1.1. The **category of functors from $\mathcal{C}$ to $\mathcal{D}$**[36] is the category $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$[37] where

- *Objects.* The objects of $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$ are functors from $\mathcal{C}$ to $\mathcal{D}$.
- *Morphisms.* For each $F, G \in \mathrm{Obj}(\mathsf{Fun}(\mathcal{C}, \mathcal{D}))$, we have

$$\mathrm{Hom}_{\mathsf{Fun}(\mathcal{C},\mathcal{D})}(F, G) \overset{\text{def}}{=} \mathrm{Nat}(F, G).$$

[36] *Further Terminology:* Also called the **functor category** $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$.
[37] *Further Notation:* Also written $\mathcal{D}^{\mathcal{C}}$ and $[\mathcal{C}, \mathcal{D}]$.

- *Identities.* For each $F \in \mathrm{Obj}(\mathsf{Fun}(\mathcal{C}, \mathcal{D}))$, the unit map
$$\mathbb{1}_F^{\mathsf{Fun}(\mathcal{C},\mathcal{D})} \colon \mathrm{pt} \to \mathrm{Nat}(F, F)$$
of $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$ at F is given by
$$\mathrm{id}_F^{\mathsf{Fun}(\mathcal{C},\mathcal{D})} \overset{\mathrm{def}}{=} \mathrm{id}_F,$$
where $\mathrm{id}_F \colon F \Longrightarrow F$ is the identity natural transformation of F of Example 8.8.2.1.4.

- *Composition.* For each $F, G, H \in \mathrm{Obj}(\mathsf{Fun}(\mathcal{C}, \mathcal{D}))$, the composition map
$$\circ_{F,G,H}^{\mathsf{Fun}(\mathcal{C},\mathcal{D})} \colon \mathrm{Nat}(G, H) \times \mathrm{Nat}(F, G) \to \mathrm{Nat}(F, H)$$
of $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$ at (F, G, H) is given by
$$\beta \circ_{F,G,H}^{\mathsf{Fun}(\mathcal{C},\mathcal{D})} \alpha \overset{\mathrm{def}}{=} \beta \circ \alpha,$$
where $\beta \circ \alpha$ is the vertical composition of α and β of Item 1 of Proposition 8.8.3.1.2.

Proposition 8.9.1.1.2. Let $\mathcal{C}$ and $\mathcal{D}$ be categories and let $F \colon \mathcal{C} \to \mathcal{D}$ be a functor.

1. *Functoriality.* The assignments $\mathcal{C}, \mathcal{D}, (\mathcal{C}, \mathcal{D}) \mapsto \mathsf{Fun}(\mathcal{C}, \mathcal{D})$ define functors
$$\begin{aligned}\mathsf{Fun}(\mathcal{C}, -_2) &\colon \mathsf{Cats} \to \mathsf{Cats},\\ \mathsf{Fun}(-_1, \mathcal{D}) &\colon \mathsf{Cats}^{\mathsf{op}} \to \mathsf{Cats},\\ \mathsf{Fun}(-_1, -_2) &\colon \mathsf{Cats}^{\mathsf{op}} \times \mathsf{Cats} \to \mathsf{Cats}.\end{aligned}$$

2. *2-Functoriality.* The assignments $\mathcal{C}, \mathcal{D}, (\mathcal{C}, \mathcal{D}) \mapsto \mathsf{Fun}(\mathcal{C}, \mathcal{D})$ define 2-functors
$$\begin{aligned}\mathsf{Fun}(\mathcal{C}, -_2) &\colon \mathsf{Cats}_2 \to \mathsf{Cats}_2,\\ \mathsf{Fun}(-_1, \mathcal{D}) &\colon \mathsf{Cats}_2^{\mathsf{op}} \to \mathsf{Cats}_2,\\ \mathsf{Fun}(-_1, -_2) &\colon \mathsf{Cats}_2^{\mathsf{op}} \times \mathsf{Cats}_2 \to \mathsf{Cats}_2.\end{aligned}$$

3. *Adjointness.* We have adjunctions
$$(\mathcal{C} \times - \dashv \mathsf{Fun}(\mathcal{C}, -)) \colon \quad \begin{tikzcd} \mathsf{Cats} \arrow[r, bend left, "\mathcal{C}\times -"] \arrow[r, phantom, "\perp"] & \mathsf{Cats}, \arrow[l, bend left, "\mathsf{Fun}(\mathcal{C},-)"] \end{tikzcd}$$
$$(- \times \mathcal{D} \dashv \mathsf{Fun}(\mathcal{D}, -)) \colon \quad \begin{tikzcd} \mathsf{Cats} \arrow[r, bend left, "-\times\mathcal{D}"] \arrow[r, phantom, "\perp"] & \mathsf{Cats}, \arrow[l, bend left, "\mathsf{Fun}(\mathcal{D},-)"] \end{tikzcd}$$

witnessed by bijections of sets

$$\begin{aligned}\mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}\times\mathcal{D},\mathcal{E}) &\cong \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{D},\mathsf{Fun}(\mathcal{C},\mathcal{E})),\\ \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}\times\mathcal{D},\mathcal{E}) &\cong \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C},\mathsf{Fun}(\mathcal{D},\mathcal{E})),\end{aligned}$$

natural in $\mathcal{C},\mathcal{D},\mathcal{E}\in\mathrm{Obj}(\mathsf{Cats})$.

4. *2-Adjointness.* We have 2-adjunctions

$$(\mathcal{C}\times - \dashv \mathsf{Fun}(\mathcal{C},-))\colon\quad \begin{tikzcd}\mathsf{Cats}_2 \arrow[r, bend left, "\mathcal{C}\times -"] \arrow[r, phantom, "\perp_2"] & \mathsf{Cats}_2, \arrow[l, bend left, "\mathsf{Fun}(\mathcal{C},-)"]\end{tikzcd}$$

$$(-\times\mathcal{D} \dashv \mathsf{Fun}(\mathcal{D},-))\colon\quad \begin{tikzcd}\mathsf{Cats}_2 \arrow[r, bend left, "-\times\mathcal{D}"] \arrow[r, phantom, "\perp_2"] & \mathsf{Cats}_2, \arrow[l, bend left, "\mathsf{Fun}(\mathcal{D},-)"]\end{tikzcd}$$

witnessed by isomorphisms of categories

$$\begin{aligned}\mathsf{Fun}(\mathcal{C}\times\mathcal{D},\mathcal{E}) &\cong \mathsf{Fun}(\mathcal{D},\mathsf{Fun}(\mathcal{C},\mathcal{E})),\\ \mathsf{Fun}(\mathcal{C}\times\mathcal{D},\mathcal{E}) &\cong \mathsf{Fun}(\mathcal{C},\mathsf{Fun}(\mathcal{D},\mathcal{E})),\end{aligned}$$

natural in $\mathcal{C},\mathcal{D},\mathcal{E}\in\mathrm{Obj}(\mathsf{Cats}_2)$.

5. *Interaction With Punctual Categories.* We have a canonical isomorphism of categories

$$\mathsf{Fun}(\mathsf{pt},\mathcal{C})\cong\mathcal{C},$$

natural in $\mathcal{C}\in\mathrm{Obj}(\mathsf{Cats})$.

6. *Objectwise Computation of Co/Limits.* Let

$$D\colon\mathcal{I}\to\mathsf{Fun}(\mathcal{C},\mathcal{D})$$

be a diagram in $\mathsf{Fun}(\mathcal{C},\mathcal{D})$. We have isomorphisms

$$\begin{aligned}\lim(D)_A &\cong \lim_{i\in\mathcal{I}}(D_i(A)),\\ \operatorname{colim}(D)_A &\cong \operatorname*{colim}_{i\in\mathcal{I}}(D_i(A)),\end{aligned}$$

naturally in $A\in\mathrm{Obj}(\mathcal{C})$.

7. *Interaction With Co/Completeness.* If $\mathcal{E}$ is co/complete, then so is $\mathsf{Fun}(\mathcal{C},\mathcal{E})$.

8. *Monomorphisms and Epimorphisms.* Let $\alpha\colon F\Longrightarrow G$ be a morphism of $\mathsf{Fun}(\mathcal{C},\mathcal{D})$. The following conditions are equivalent:

(a) The natural transformation

$$\alpha\colon F \Longrightarrow G$$

is a monomorphism (resp. epimorphism) in $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$.

(b) For each $A \in \mathrm{Obj}(\mathcal{C})$, the morphism

$$\alpha_A\colon F_A \longrightarrow G_A$$

is a monomorphism (resp. epimorphism) in $\mathcal{D}$.

Proof. *Item 1, Functoriality*: Omitted.
Item 2, 2-Functoriality: Omitted.
Item 3, Adjointness: Omitted.
Item 4, 2-Adjointness: Omitted.
Item 5, Interaction With Punctual Categories: Omitted.
Item 6, Objectwise Computation of Co/Limits: Omitted.
Item 7, Interaction With Co/Completeness: This follows from **??**.
Item 8, Monomorphisms and Epimorphisms: Omitted. □

8.9.2 The Category of Categories and Functors

Definition 8.9.2.1.1. The **category of (small) categories and functors** is the category Cats where

- *Objects.* The objects of Cats are small categories.
- *Morphisms.* For each $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$, we have

$$\mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, \mathcal{D}) \overset{\text{def}}{=} \mathrm{Obj}(\mathsf{Fun}(\mathcal{C}, \mathcal{D})).$$

- *Identities.* For each $\mathcal{C} \in \mathrm{Obj}(\mathsf{Cats})$, the unit map

$$\mathbb{1}^{\mathsf{Cats}}_{\mathcal{C}}\colon \mathrm{pt} \longrightarrow \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, \mathcal{C})$$

of Cats at $\mathcal{C}$ is defined by

$$\mathrm{id}^{\mathsf{Cats}}_{\mathcal{C}} \overset{\text{def}}{=} \mathrm{id}_{\mathcal{C}},$$

where $\mathrm{id}_{\mathcal{C}}\colon \mathcal{C} \longrightarrow \mathcal{C}$ is the identity functor of $\mathcal{C}$ of Example 8.4.1.1.4.

- *Composition.* For each $\mathcal{C}, \mathcal{D}, \mathcal{E} \in \mathrm{Obj}(\mathsf{Cats})$, the composition map

$$\circ^{\mathsf{Cats}}_{\mathcal{C},\mathcal{D},\mathcal{E}}\colon \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{D}, \mathcal{E}) \times \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, \mathcal{D}) \longrightarrow \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, \mathcal{E})$$

of Cats at $(\mathcal{C}, \mathcal{D}, \mathcal{E})$ is given by

$$G \circ^{\mathsf{Cats}}_{\mathcal{C},\mathcal{D},\mathcal{E}} F \overset{\text{def}}{=} G \circ F,$$

where $G \circ F\colon \mathcal{C} \longrightarrow \mathcal{E}$ is the composition of F and G of Definition 8.4.1.1.5.

Proposition 8.9.2.1.2. Let $\mathcal{C}$ be a category.

1. *Co/Completeness.* The category Cats is complete and cocomplete.

2. *Cartesian Monoidal Structure.* The quadruple $(\mathsf{Cats}, \times, \mathrm{pt}, \mathsf{Fun})$ is a Cartesian closed monoidal category.

Proof. *Item 1*, *Co/Completeness*: Omitted.
Item 2, *Cartesian Monoidal Structure*: Omitted. □

8.9.3 The 2-Category of Categories, Functors, and Natural Transformations

Definition 8.9.3.1.1. The **2-category of (small) categories, functors, and natural transformations** is the 2-category Cats_2 where

- *Objects.* The objects of Cats_2 are small categories.

- Hom-*Categories.* For each $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats}_2)$, we have

$$\mathsf{Hom}_{\mathsf{Cats}_2}(\mathcal{C}, \mathcal{D}) \stackrel{\text{def}}{=} \mathsf{Fun}(\mathcal{C}, \mathcal{D}).$$

- *Identities.* For each $\mathcal{C} \in \mathrm{Obj}(\mathsf{Cats}_2)$, the unit functor

$$\mathbb{1}^{\mathsf{Cats}_2}_{\mathcal{C}} \colon \mathrm{pt} \to \mathsf{Fun}(\mathcal{C}, \mathcal{C})$$

  of Cats_2 at $\mathcal{C}$ is the functor picking the identity functor $\mathrm{id}_{\mathcal{C}} \colon \mathcal{C} \to \mathcal{C}$ of $\mathcal{C}$.

- *Composition.* For each $\mathcal{C}, \mathcal{D}, \mathcal{E} \in \mathrm{Obj}(\mathsf{Cats}_2)$, the composition bifunctor

$$\circ^{\mathsf{Cats}_2}_{\mathcal{C},\mathcal{D},\mathcal{E}} \colon \mathsf{Hom}_{\mathsf{Cats}_2}(\mathcal{D}, \mathcal{E}) \times \mathsf{Hom}_{\mathsf{Cats}_2}(\mathcal{C}, \mathcal{D}) \to \mathsf{Hom}_{\mathsf{Cats}_2}(\mathcal{C}, \mathcal{E})$$

  of Cats_2 at $(\mathcal{C}, \mathcal{D}, \mathcal{E})$ is the functor where

  - *Action on Objects.* For each object $(G, F) \in \mathrm{Obj}\big(\mathsf{Hom}_{\mathsf{Cats}_2}(\mathcal{D}, \mathcal{E}) \times \mathsf{Hom}_{\mathsf{Cats}_2}(\mathcal{C}, \mathcal{D})\big)$, we have

$$\circ^{\mathsf{Cats}_2}_{\mathcal{C},\mathcal{D},\mathcal{E}}(G, F) \stackrel{\text{def}}{=} G \circ F.$$

  - *Action on Morphisms.* For each morphism $(\beta, \alpha) \colon (K, H) \Longrightarrow (G, F)$ of $\mathsf{Hom}_{\mathsf{Cats}_2}(\mathcal{D}, \mathcal{E}) \times \mathsf{Hom}_{\mathsf{Cats}_2}(\mathcal{C}, \mathcal{D})$, we have

$$\circ^{\mathsf{Cats}_2}_{\mathcal{C},\mathcal{D},\mathcal{E}}(\beta, \alpha) \stackrel{\text{def}}{=} \beta \star \alpha,$$

    where $\beta \star \alpha$ is the horizontal composition of α and β of Definition 8.8.4.1.1.

Proposition 8.9.3.1.2. Let C be a category.

1. *2-Categorical Co/Completeness.* The 2-category Cats_2 is complete and cocomplete as a 2-category, having all 2-categorical and bicategorical co/limits.

Proof. *Item 1, Co/Completeness*: Omitted. □

8.9.4 The Category of Groupoids

Definition 8.9.4.1.1. The **category of (small) groupoids** is the full subcategory Grpd of Cats spanned by the groupoids.

8.9.5 The 2-Category of Groupoids

Definition 8.9.5.1.1. The **2-category of (small) groupoids** is the full sub-2-category Grpd_2 of Cats_2 spanned by the groupoids.

Appendices

8.A Other Chapters

Sets

1. Sets
2. Constructions With Sets
3. Pointed Sets
4. Tensor Products of Pointed Sets

Relations

5. Relations
6. Constructions With Relations
7. Equivalence Relations and Apartness Relations

Category Theory

8. Categories

Bicategories

9. Types of Morphisms in Bicategories

Part IV

Bicategories

Chapter 9

Types of Morphisms in Bicategories

In this chapter, we study special kinds of morphisms in bicategories:

1. *Monomorphisms and Epimorphisms in Bicategories (Sections 9.1 and 9.2).* There is a large number of different notions capturing the idea of a "monomorphism" or of an "epimorphism" in a bicategory.

   Arguably, the notion that best captures these concepts is that of a *pseudomonic morphism* (Definition 9.1.10.1.1) and of a *pseudoepic morphism* (Definition 9.2.10.1.1), although the other notions introduced in Sections 9.1 and 9.2 are also interesting on their own.

Contents

9.1 Monomorphisms in Bicategories 437

9.1.1 Representably Faithful Morphisms 437

9.1.2 Representably Full Morphisms 438

9.1.3 Representably Fully Faithful Morphisms 439

9.1.4 Morphisms Representably Faithful on Cores 440

9.1.5 Morphisms Representably Full on Cores 441

9.1.6 Morphisms Representably Fully Faithful on Cores 442

9.1.7 Representably Essentially Injective Morphisms 443

9.1.8 Representably Conservative Morphisms 443

9.1.9 Strict Monomorphisms 444

9.1.10 Pseudomonic Morphisms 444
9.2 Epimorphisms in Bicategories 446
9.2.1 Corepresentably Faithful Morphisms 446
9.2.2 Corepresentably Full Morphisms 447
9.2.3 Corepresentably Fully Faithful Morphisms .. 448
9.2.4 Morphisms Corepresentably Faithful on Cores 449
9.2.5 Morphisms Corepresentably Full on Cores .. 450
9.2.6 Morphisms Corepresentably Fully Faithful on Cores 450
9.2.7 Corepresentably Essentially Injective Morphisms 451
9.2.8 Corepresentably Conservative Morphisms... 452
9.2.9 Strict Epimorphisms 452
9.2.10 Pseudoepic Morphisms 453
9.A Other Chapters 454

9.1 Monomorphisms in Bicategories

9.1.1 Representably Faithful Morphisms

Let C be a bicategory.

Definition 9.1.1.1.1. A 1-morphism $f\colon A \to B$ of C is **representably faithful**[1] if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f_*\colon \mathsf{Hom}_C(X, A) \to \mathsf{Hom}_C(X, B)$$

given by postcomposition by f is faithful.

Remark 9.1.1.1.2. In detail, f is representably faithful if, for all diagrams in C of the form

$$\begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & A \arrow[r, "f"] & B, \end{tikzcd}$$

if we have

$$\mathrm{id}_f \star \alpha = \mathrm{id}_f \star \beta,$$

then $\alpha = \beta$.

[1] *Further Terminology:* Also called simply a **faithful morphism**, based on Item 1 of

Example 9.1.1.1.3. Here are some examples of representably faithful morphisms.

1. *Representably Faithful Morphisms in* Cats_2. The representably faithful morphisms in Cats_2 are precisely the faithful functors; see Item 1 of Proposition 8.5.1.1.2.

2. *Representably Faithful Morphisms in* $\mathbf{Rel}$. Every morphism of $\mathbf{Rel}$ is representably faithful; see Item 1 of Proposition 5.3.8.1.1.

9.1.2 Representably Full Morphisms

Let C be a bicategory.

Definition 9.1.2.1.1. A 1-morphism $f\colon A \to B$ of C is **representably full**[2] if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f_*\colon \mathsf{Hom}_C(X, A) \to \mathsf{Hom}_C(X, B)$$

given by postcomposition by f is full.

Remark 9.1.2.1.2. In detail, f is representably full if, for each $X \in \mathrm{Obj}(C)$ and each 2-morphism

$$\beta\colon f \circ \phi \Longrightarrow f \circ \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"] \arrow[r, bend right, "f\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & B \end{tikzcd}$$

of C, there exists a 2-morphism

$$\alpha\colon \phi \Longrightarrow \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & A \end{tikzcd}$$

of C such that we have an equality

$$\begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & A \arrow[r, "f"] & B \end{tikzcd} \quad = \quad \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"] \arrow[r, bend right, "f\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & B \end{tikzcd}$$

of pasting diagrams in C, i.e. such that we have

$$\beta = \mathrm{id}_f \star \alpha.$$

Example 9.1.1.1.3.

[2] *Further Terminology:* Also called simply a **full morphism**, based on Item 1 of

Example 9.1.2.1.3. Here are some examples of representably full morphisms.

1. *Representably Full Morphisms in* Cats_2. The representably full morphisms in Cats_2 are precisely the full functors; see Item 1 of Proposition 8.5.2.1.2.

2. *Representably Full Morphisms in* $\mathbf{Rel}$. The representably full morphisms in $\mathbf{Rel}$ are characterised in Item 2 of Proposition 5.3.8.1.1.

9.1.3 Representably Fully Faithful Morphisms

Let C be a bicategory.

Definition 9.1.3.1.1. A 1-morphism $f \colon A \to B$ of C is **representably fully faithful**[3] if the following equivalent conditions are satisfied:

1. The 1-morphism f is representably faithful (Definition 9.1.1.1.1) and representably full (Definition 9.1.2.1.1).

2. For each $X \in \mathrm{Obj}(C)$, the functor

$$f_* \colon \mathsf{Hom}_C(X, A) \to \mathsf{Hom}_C(X, B)$$

given by postcomposition by f is fully faithful.

Remark 9.1.3.1.2. In detail, f is representably fully faithful if the conditions in Remark 9.1.1.1.2 and Remark 9.1.2.1.2 hold:

1. For all diagrams in C of the form

$$\begin{tikzcd} X \arrow[r, bend left, "\phi"{name=U}] \arrow[r, bend right, "\psi"'{name=D}] \arrow[from=U, to=D, phantom, "\alpha\Downarrow\Downarrow\beta"] & A \arrow[r, "f"] & B, \end{tikzcd}$$

if we have

$$\mathrm{id}_f \star \alpha = \mathrm{id}_f \star \beta,$$

then $\alpha = \beta$.

Example 9.1.2.1.3.

[3] *Further Terminology:* Also called simply a **fully faithful morphism**, based on Item 1 of Example 9.1.3.1.3.

2. For each $X \in \mathsf{Obj}(\mathcal{C})$ and each 2-morphism

$$\beta\colon f \circ \phi \Longrightarrow f \circ \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"] \arrow[r, bend right, "f\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & B \end{tikzcd}$$

of $\mathcal{C}$, there exists a 2-morphism

$$\alpha\colon \phi \Longrightarrow \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & A \end{tikzcd}$$

of $\mathcal{C}$ such that we have an equality

$$\begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & A \arrow[r, "f"] & B \end{tikzcd} = \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"] \arrow[r, bend right, "f\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & B \end{tikzcd}$$

of pasting diagrams in $\mathcal{C}$, i.e. such that we have

$$\beta = \mathrm{id}_f \star \alpha.$$

Example 9.1.3.1.3. Here are some examples of representably fully faithful morphisms.

1. *Representably Fully Faithful Morphisms in* Cats_2. The representably fully faithful morphisms in Cats_2 are precisely the fully faithful functors; see Item 5 of Proposition 8.5.3.1.2.

2. *Representably Fully Faithful Morphisms in* **Rel**. The representably fully faithful morphisms of **Rel** coincide (Item 3 of Proposition 5.3.8.1.1) with the representably full morphisms in **Rel**, which are characterised in Item 2 of Proposition 5.3.8.1.1.

9.1.4 Morphisms Representably Faithful on Cores

Let $\mathcal{C}$ be a bicategory.

Definition 9.1.4.1.1. A 1-morphism $f\colon A \to B$ of $\mathcal{C}$ is **representably faithful on cores** if, for each $X \in \mathsf{Obj}(\mathcal{C})$, the functor

$$f_*\colon \mathsf{Core}(\mathsf{Hom}_{\mathcal{C}}(X, A)) \to \mathsf{Core}(\mathsf{Hom}_{\mathcal{C}}(X, B))$$

given by postcomposition by f is faithful.

Remark 9.1.4.1.2. In detail, f is representably faithful on cores if, for all diagrams in C of the form

$$\begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & A \arrow[r, "f"] & B, \end{tikzcd}$$

if α and β are 2-isomorphisms and we have

$$\mathrm{id}_f \star \alpha = \mathrm{id}_f \star \beta,$$

then $\alpha = \beta$.

9.1.5 Morphisms Representably Full on Cores

Let C be a bicategory.

Definition 9.1.5.1.1. A 1-morphism $f\colon A \to B$ of C is **representably full on cores** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f_*\colon \mathsf{Core}(\mathsf{Hom}_C(X, A)) \to \mathsf{Core}(\mathsf{Hom}_C(X, B))$$

given by postcomposition by f is full.

Remark 9.1.5.1.2. In detail, f is representably full on cores if, for each $X \in \mathrm{Obj}(C)$ and each 2-isomorphism

$$\beta\colon f \circ \phi \overset{\sim}{\Longrightarrow} f \circ \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"] \arrow[r, bend right, "f\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & B \end{tikzcd}$$

of C, there exists a 2-isomorphism

$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & A \end{tikzcd}$$

of C such that we have an equality

$$\begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & A \arrow[r, "f"] & B \end{tikzcd} = \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"] \arrow[r, bend right, "f\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & B \end{tikzcd}$$

of pasting diagrams in C, i.e. such that we have

$$\beta = \mathrm{id}_f \star \alpha.$$

9.1.6 Morphisms Representably Fully Faithful on Cores

Let C be a bicategory.

Definition 9.1.6.1.1. A 1-morphism $f\colon A \to B$ of C is **representably fully faithful on cores** if the following equivalent conditions are satisfied:

1. The 1-morphism f is representably faithful on cores (Definition 9.1.5.1.1) and representably full on cores (Definition 9.1.4.1.1).
2. For each $X \in \mathrm{Obj}(C)$, the functor

$$f_*\colon \mathsf{Core}(\mathsf{Hom}_C(X,A)) \to \mathsf{Core}(\mathsf{Hom}_C(X,B))$$

given by postcomposition by f is fully faithful.

Remark 9.1.6.1.2. In detail, f is representably fully faithful on cores if the conditions in Remark 9.1.4.1.2 and Remark 9.1.5.1.2 hold:

1. For all diagrams in C of the form

$$\begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & A \arrow[r, "f"] & B, \end{tikzcd}$$

if α and β are 2-isomorphisms and we have

$$\mathrm{id}_f \star \alpha = \mathrm{id}_f \star \beta,$$

then $\alpha = \beta$.

2. For each $X \in \mathrm{Obj}(C)$ and each 2-isomorphism

$$\beta\colon f\circ\phi \overset{\sim}{\Longrightarrow} f\circ\psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"] \arrow[r, bend right, "f\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & B \end{tikzcd}$$

of C, there exists a 2-isomorphism

$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & A \end{tikzcd}$$

of C such that we have an equality

$$\begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & A \arrow[r, "f"] & B \end{tikzcd} = \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"] \arrow[r, bend right, "f\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & B \end{tikzcd}$$

of pasting diagrams in C, i.e. such that we have

$$\beta = \mathrm{id}_f \star \alpha.$$

9.1.7 Representably Essentially Injective Morphisms

Let C be a bicategory.

Definition 9.1.7.1.1. A 1-morphism $f: A \to B$ of C is **representably essentially injective** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f_*: \mathsf{Hom}_C(X, A) \to \mathsf{Hom}_C(X, B)$$

given by postcomposition by f is essentially injective.

Remark 9.1.7.1.2. In detail, f is representably essentially injective if, for each pair of morphisms $\phi, \psi: X \rightrightarrows A$ of C, the following condition is satisfied:

($\star$) If $f \circ \phi \cong f \circ \psi$, then $\phi \cong \psi$.

9.1.8 Representably Conservative Morphisms

Let C be a bicategory.

Definition 9.1.8.1.1. A 1-morphism $f: A \to B$ of C is **representably conservative** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f_*: \mathsf{Hom}_C(X, A) \to \mathsf{Hom}_C(X, B)$$

given by postcomposition by f is conservative.

Remark 9.1.8.1.2. In detail, f is representably conservative if, for each pair of morphisms $\phi, \psi: X \rightrightarrows A$ and each 2-morphism

$$\alpha: \phi \Longrightarrow \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "\phi"{name=U}] \arrow[r, bend right, "\psi"'{name=D}] & A \arrow[from=U, to=D, phantom, "\alpha\Downarrow"] \end{tikzcd}$$

of C, if the 2-morphism

$$\mathrm{id}_f \star \alpha: f \circ \phi \Longrightarrow f \circ \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"{name=U}] \arrow[r, bend right, "f\circ\psi"'{name=D}] & B \arrow[from=U, to=D, phantom, "\Downarrow \mathrm{id}_f\star\alpha"] \end{tikzcd}$$

is a 2-isomorphism, then so is α.

9.1.9 Strict Monomorphisms

Let C be a bicategory.

Definition 9.1.9.1.1. A 1-morphism $f: A \to B$ of C is a **strict monomorphism** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f_*: \mathsf{Hom}_C(X, A) \to \mathsf{Hom}_C(X, B)$$

given by postcomposition by f is injective on objects, i.e. its action on objects

$$f_*: \mathrm{Obj}(\mathsf{Hom}_C(X, A)) \to \mathrm{Obj}(\mathsf{Hom}_C(X, B))$$

is injective.

Remark 9.1.9.1.2. In detail, f is a strict monomorphism in C if, for each diagram in C of the form

$$X \underset{\psi}{\overset{\phi}{\rightrightarrows}} A \xrightarrow{f} B,$$

if $f \circ \phi = f \circ \psi$, then $\phi = \psi$.

Example 9.1.9.1.3. Here are some examples of strict monomorphisms.

1. *Strict Monomorphisms in* Cats_2. The strict monomorphisms in Cats_2 are precisely the functors which are injective on objects and injective on morphisms; see Item 1 of Proposition 8.6.2.1.2.

2. *Strict Monomorphisms in* Rel. The strict monomorphisms in Rel are characterised in Proposition 5.3.7.1.1.

9.1.10 Pseudomonic Morphisms

Let C be a bicategory.

Definition 9.1.10.1.1. A 1-morphism $f: A \to B$ of C is **pseudomonic** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f_*: \mathsf{Hom}_C(X, A) \to \mathsf{Hom}_C(X, B)$$

given by postcomposition by f is pseudomonic.

Remark 9.1.10.1.2. In detail, a 1-morphism $f: A \to B$ of C is pseudomonic if it satisfies the following conditions:

1. For all diagrams in C of the form

$$\begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & A \arrow[r, "f"] & B, \end{tikzcd}$$

   if we have

$$\mathrm{id}_f \star \alpha = \mathrm{id}_f \star \beta,$$

   then $\alpha = \beta$.

2. For each $X \in \mathrm{Obj}(C)$ and each 2-isomorphism

$$\beta\colon f \circ \phi \overset{\sim}{\Longrightarrow} f \circ \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"] \arrow[r, bend right, "f\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & B \end{tikzcd}$$

   of C, there exists a 2-isomorphism

$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & A \end{tikzcd}$$

   of C such that we have an equality

$$\begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & A \arrow[r, "f"] & B \end{tikzcd} = \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"] \arrow[r, bend right, "f\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & B \end{tikzcd}$$

   of pasting diagrams in C, i.e. such that we have

$$\beta = \mathrm{id}_f \star \alpha.$$

Proposition 9.1.10.1.3. Let $f\colon A \to B$ be a 1-morphism of C.

1. *Characterisations.* The following conditions are equivalent:

   (a) The morphism f is pseudomonic.

   (b) The morphism f is representably full on cores and representably faithful.

   (c) We have an isocomma square of the form

$$A \overset{\text{eq.}}{\cong} A \overset{\leftrightarrow}{\times}_B A, \qquad \begin{tikzcd} A \arrow[r, "\mathrm{id}_A"] \arrow[d, "\mathrm{id}_A"'] & A \arrow[d, "F"] \arrow[dl, phantom, "\Leftrightarrow"] \\ A \arrow[r, "F"'] & B \end{tikzcd}$$

   in C up to equivalence.

2. *Interaction With Cotensors.* If C has cotensors with $\mathbb{1}$, then the following conditions are equivalent:

   (a) The morphism f is pseudomonic.

   (b) We have an isocomma square of the form

$$A \overset{\text{eq.}}{\cong} A \overset{\leftrightarrow}{\times}_{\mathbb{1} \pitchfork F} B, \qquad \begin{tikzcd} A \arrow[r, hook] \arrow[d, "F"'] & \mathbb{1} \pitchfork A \arrow[d, "\mathbb{1} \pitchfork F"] \arrow[dl, phantom, "\Leftrightarrow"] \\ B \arrow[r, hook] & \mathbb{1} \pitchfork B \end{tikzcd}$$

   in C up to equivalence.

Proof. *Item 1, Characterisations*: Omitted.
Item 2, Interaction With Cotensors: Omitted. □

9.2 Epimorphisms in Bicategories

9.2.1 Corepresentably Faithful Morphisms

Let C be a bicategory.

Definition 9.2.1.1.1. A 1-morphism $f: A \to B$ of C is **corepresentably faithful** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f^*: \mathsf{Hom}_C(B, X) \to \mathsf{Hom}_C(A, X)$$

given by precomposition by f is faithful.

Remark 9.2.1.1.2. In detail, f is corepresentably faithful if, for all diagrams in C of the form

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha \Downarrow \Downarrow \beta"] & X, \end{tikzcd}$$

if we have

$$\alpha \star \mathrm{id}_f = \beta \star \mathrm{id}_f,$$

then $\alpha = \beta$.

Example 9.2.1.1.3. Here are some examples of corepresentably faithful morphisms.

1. *Corepresentably Faithful Morphisms in* Cats_2. The corepresentably faithful morphisms in Cats_2 are characterised in Item 4 of Proposition 8.5.1.1.2.

2. *Corepresentably Faithful Morphisms in* $\mathbf{Rel}$. Every morphism of $\mathbf{Rel}$ is corepresentably faithful; see Item 1 of Proposition 5.3.10.1.1.

9.2.2 Corepresentably Full Morphisms

Let C be a bicategory.

Definition 9.2.2.1.1. A 1-morphism $f\colon A \to B$ of C is **corepresentably full** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f^*\colon \mathsf{Hom}_C(B, X) \to \mathsf{Hom}_C(A, X)$$

given by precomposition by f is full.

Remark 9.2.2.1.2. In detail, f is corepresentably full if, for each $X \in \mathrm{Obj}(C)$ and each 2-morphism

$$\beta\colon \phi \circ f \Longrightarrow \psi \circ f, \qquad \begin{tikzcd} A \arrow[r, bend left, "\phi\circ f"] \arrow[r, bend right, "\psi\circ f"'] \arrow[r, phantom, "\beta\Downarrow"] & X \end{tikzcd}$$

of C, there exists a 2-morphism

$$\alpha\colon \phi \Longrightarrow \psi, \qquad \begin{tikzcd} B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & X \end{tikzcd}$$

of C such that we have an equality

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & X \end{tikzcd} = \begin{tikzcd} A \arrow[r, bend left, "\phi\circ f"] \arrow[r, bend right, "\psi\circ f"'] \arrow[r, phantom, "\beta\Downarrow"] & X \end{tikzcd}$$

of pasting diagrams in C, i.e. such that we have

$$\beta = \alpha \star \mathrm{id}_f.$$

Example 9.2.2.1.3. Here are some examples of corepresentably full morphisms.

1. *Corepresentably Full Morphisms in* Cats_2. The corepresentably full morphisms in Cats_2 are characterised in Item 5 of Proposition 8.5.2.1.2.

2. *Corepresentably Full Morphisms in* Rel. The corepresentably full morphisms in Rel are characterised in **??** of Proposition 5.3.8.1.1.

9.2.3 Corepresentably Fully Faithful Morphisms

Let C be a bicategory.

Definition 9.2.3.1.1. A 1-morphism $f\colon A \to B$ of C is **corepresentably fully faithful**[4] if the following equivalent conditions are satisfied:

1. The 1-morphism f is corepresentably full (Definition 9.2.2.1.1) and corepresentably faithful (Definition 9.2.1.1.1).

2. For each $X \in \mathrm{Obj}(C)$, the functor

$$f^*\colon \mathsf{Hom}_C(B, X) \to \mathsf{Hom}_C(A, X)$$

given by precomposition by f is fully faithful.

Remark 9.2.3.1.2. In detail, f is corepresentably fully faithful if the conditions in Remark 9.2.1.1.2 and Remark 9.2.2.1.2 hold:

1. For all diagrams in C of the form

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & X, \end{tikzcd}$$

if we have

$$\alpha \star \mathrm{id}_f = \beta \star \mathrm{id}_f,$$

then $\alpha = \beta$.

2. For each $X \in \mathrm{Obj}(C)$ and each 2-morphism

$$\beta\colon \phi \circ f \Longrightarrow \psi \circ f, \qquad \begin{tikzcd} A \arrow[r, bend left, "\phi\circ f"] \arrow[r, bend right, "\psi\circ f"'] \arrow[r, phantom, "\beta\Downarrow"] & X \end{tikzcd}$$

of C, there exists a 2-morphism

$$\alpha\colon \phi \Longrightarrow \psi, \qquad \begin{tikzcd} B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & X \end{tikzcd}$$

[4] *Further Terminology:* Corepresentably fully faithful morphisms have also been called **lax epimorphisms** in the literature (e.g. in [Adá+01]), though we will always use the name "corepresentably fully faithful morphism" instead in this work.

of C such that we have an equality

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & X \end{tikzcd} = \begin{tikzcd} A \arrow[r, bend left, "\phi\circ f"] \arrow[r, bend right, "\psi\circ f"'] \arrow[r, phantom, "\beta\Downarrow"] & X \end{tikzcd}$$

of pasting diagrams in C, i.e. such that we have

$$\beta = \alpha \star \mathrm{id}_f.$$

Example 9.2.3.1.3. Here are some examples of corepresentably fully faithful morphisms.

1. *Corepresentably Fully Faithful Morphisms in* Cats_2. The fully faithful epimorphisms in Cats_2 are characterised in Item 9 of Proposition 8.5.3.1.2.

2. *Corepresentably Fully Faithful Morphisms in* $\mathbf{Rel}$. The corepresentably fully faithful morphisms of $\mathbf{Rel}$ coincide (Item 3 of Proposition 5.3.10.1.1) with the corepresentably full morphisms in $\mathbf{Rel}$, which are characterised in Item 2 of Proposition 5.3.10.1.1.

9.2.4 Morphisms Corepresentably Faithful on Cores

Let C be a bicategory.

Definition 9.2.4.1.1. A 1-morphism $f\colon A \to B$ of C is **corepresentably faithful on cores** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f^*\colon \mathsf{Core}(\mathsf{Hom}_C(B, X)) \to \mathsf{Core}(\mathsf{Hom}_C(A, X))$$

given by precomposition by f is faithful.

Remark 9.2.4.1.2. In detail, f is corepresentably faithful on cores if, for all diagrams in C of the form

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & X, \end{tikzcd}$$

if α and β are 2-isomorphisms and we have

$$\alpha \star \mathrm{id}_f = \beta \star \mathrm{id}_f,$$

then $\alpha = \beta$.

9.2.5 Morphisms Corepresentably Full on Cores

Let C be a bicategory.

Definition 9.2.5.1.1. A 1-morphism $f\colon A \to B$ of C is **corepresentably full on cores** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f^*\colon \mathsf{Core}(\mathsf{Hom}_C(B,X)) \to \mathsf{Core}(\mathsf{Hom}_C(A,X))$$

given by precomposition by f is full.

Remark 9.2.5.1.2. In detail, f is corepresentably full on cores if, for each $X \in \mathrm{Obj}(C)$ and each 2-isomorphism

$$\beta\colon \phi \circ f \overset{\sim}{\Longrightarrow} \psi \circ f, \qquad \begin{tikzcd} A \arrow[r, bend left, "\phi\circ f"] \arrow[r, bend right, "\psi\circ f"'] \arrow[r, phantom, "\beta\Downarrow"] & X \end{tikzcd}$$

of C, there exists a 2-isomorphism

$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & X \end{tikzcd}$$

of C such that we have an equality

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & X \end{tikzcd} = \begin{tikzcd} A \arrow[r, bend left, "\phi\circ f"] \arrow[r, bend right, "\psi\circ f"'] \arrow[r, phantom, "\beta\Downarrow"] & X \end{tikzcd}$$

of pasting diagrams in C, i.e. such that we have

$$\beta = \alpha \star \mathrm{id}_f.$$

9.2.6 Morphisms Corepresentably Fully Faithful on Cores

Let C be a bicategory.

Definition 9.2.6.1.1. A 1-morphism $f\colon A \to B$ of C is **corepresentably fully faithful on cores** if the following equivalent conditions are satisfied:

1. The 1-morphism f is corepresentably full on cores (Definition 9.2.5.1.1) and corepresentably faithful on cores (Definition 9.2.1.1.1).

2. For each $X \in \mathrm{Obj}(C)$, the functor

   $$f^*\colon \mathsf{Core}(\mathsf{Hom}_C(B,X)) \to \mathsf{Core}(\mathsf{Hom}_C(A,X))$$

   given by precomposition by f is fully faithful.

Remark 9.2.6.1.2. In detail, f is corepresentably fully faithful on cores if the conditions in Remark 9.2.4.1.2 and Remark 9.2.5.1.2 hold:

1. For all diagrams in C of the form

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "\phi"{name=U}] \arrow[r, bend right, "\psi"'{name=D}] & X, \arrow[from=U, to=D, phantom, "\alpha\Downarrow\Downarrow\beta"] \end{tikzcd}$$

   if α and β are 2-isomorphisms and we have

$$\alpha \star \mathrm{id}_f = \beta \star \mathrm{id}_f,$$

   then $\alpha = \beta$.

2. For each $X \in \mathrm{Obj}(C)$ and each 2-isomorphism

$$\beta \colon \phi \circ f \overset{\sim}{\Longrightarrow} \psi \circ f, \qquad \begin{tikzcd} A \arrow[r, bend left, "\phi\circ f"{name=U}] \arrow[r, bend right, "\psi\circ f"'{name=D}] & X \arrow[from=U, to=D, phantom, "\beta\Downarrow"] \end{tikzcd}$$

   of C, there exists a 2-isomorphism

$$\alpha \colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} B \arrow[r, bend left, "\phi"{name=U}] \arrow[r, bend right, "\psi"'{name=D}] & X \arrow[from=U, to=D, phantom, "\alpha\Downarrow"] \end{tikzcd}$$

   of C such that we have an equality

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "\phi"{name=U}] \arrow[r, bend right, "\psi"'{name=D}] & X \arrow[from=U, to=D, phantom, "\alpha\Downarrow"] \end{tikzcd} = \begin{tikzcd} A \arrow[r, bend left, "\phi\circ f"{name=U}] \arrow[r, bend right, "\psi\circ f"'{name=D}] & X \arrow[from=U, to=D, phantom, "\beta\Downarrow"] \end{tikzcd}$$

   of pasting diagrams in C, i.e. such that we have

$$\beta = \alpha \star \mathrm{id}_f.$$

9.2.7 Corepresentably Essentially Injective Morphisms

Let C be a bicategory.

Definition 9.2.7.1.1. A 1-morphism $f \colon A \to B$ of C is **corepresentably essentially injective** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f^* \colon \mathsf{Hom}_C(B, X) \to \mathsf{Hom}_C(A, X)$$

given by precomposition by f is essentially injective.

Remark 9.2.7.1.2. In detail, f is corepresentably essentially injective if, for each pair of morphisms $\phi, \psi \colon B \rightrightarrows X$ of C, the following condition is satisfied:

($\star$) If $\phi \circ f \cong \psi \circ f$, then $\phi \cong \psi$.

9.2.8 Corepresentably Conservative Morphisms

Let C be a bicategory.

Definition 9.2.8.1.1. A 1-morphism $f\colon A \to B$ of C is **corepresentably conservative** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f^*\colon \mathsf{Hom}_C(B,X) \to \mathsf{Hom}_C(A,X)$$

given by precomposition by f is conservative.

Remark 9.2.8.1.2. In detail, f is corepresentably conservative if, for each pair of morphisms $\phi, \psi\colon B \rightrightarrows X$ and each 2-morphism

$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} B \arrow[r, bend left, "\phi"{name=U}] \arrow[r, bend right, "\psi"'{name=D}] \arrow[from=U, to=D, Rightarrow, "\alpha"'] & X \end{tikzcd}$$

of C, if the 2-morphism

$$\alpha \star \mathrm{id}_f\colon \phi \circ f \Longrightarrow \psi \circ f, \qquad \begin{tikzcd} A \arrow[r, bend left, "\phi\circ f"{name=U}] \arrow[r, bend right, "\psi\circ f"'{name=D}] \arrow[from=U, to=D, Rightarrow, "\alpha\star\mathrm{id}_f"] & X \end{tikzcd}$$

is a 2-isomorphism, then so is α.

9.2.9 Strict Epimorphisms

Let C be a bicategory.

Definition 9.2.9.1.1. A 1-morphism $f\colon A \to B$ is a **strict epimorphism in** C if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f^*\colon \mathsf{Hom}_C(B,X) \to \mathsf{Hom}_C(A,X)$$

given by precomposition by f is injective on objects, i.e. its action on objects

$$f_*\colon \mathsf{Obj}(\mathsf{Hom}_C(B,X)) \to \mathsf{Obj}(\mathsf{Hom}_C(A,X))$$

is injective.

Remark 9.2.9.1.2. In detail, f is a strict epimorphism if, for each diagram in C of the form

$$A \xrightarrow{f} B \underset{\psi}{\overset{\phi}{\rightrightarrows}} X,$$

if $\phi \circ f = \psi \circ f$, then $\phi = \psi$.

Example 9.2.9.1.3. Here are some examples of strict epimorphisms.

1. *Strict Epimorphisms in* Cats_2. The strict epimorphisms in Cats_2 are characterised in Item 1 of Proposition 8.6.3.1.2.

2. *Strict Epimorphisms in* Rel. The strict epimorphisms in Rel are characterised in Proposition 5.3.9.1.1.

9.2.10 Pseudoepic Morphisms

Let C be a bicategory.

Definition 9.2.10.1.1. A 1-morphism $f: A \to B$ of C is **pseudoepic** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f^*: \mathsf{Hom}_C(B, X) \to \mathsf{Hom}_C(A, X)$$

given by precomposition by f is pseudomonic.

Remark 9.2.10.1.2. In detail, a 1-morphism $f: A \to B$ of C is pseudoepic if it satisfies the following conditions:

1. For all diagrams in C of the form

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & X, \end{tikzcd}$$

   if we have

$$\alpha \star \mathrm{id}_f = \beta \star \mathrm{id}_f,$$

   then $\alpha = \beta$.

2. For each $X \in \mathrm{Obj}(C)$ and each 2-isomorphism

$$\beta: \phi \circ f \overset{\sim}{\Longrightarrow} \psi \circ f, \qquad \begin{tikzcd} A \arrow[r, bend left, "\phi\circ f"] \arrow[r, bend right, "\psi\circ f"'] \arrow[r, phantom, "\beta\Downarrow"] & X \end{tikzcd}$$

   of C, there exists a 2-isomorphism

$$\alpha: \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & X \end{tikzcd}$$

of C such that we have an equality

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "\phi"{name=U}] \arrow[r, bend right, "\psi"'{name=D}] \arrow[from=U, to=D, phantom, "\alpha\Downarrow"] & X \end{tikzcd} = \begin{tikzcd} A \arrow[r, bend left, "\phi\circ f"{name=U}] \arrow[r, bend right, "\psi\circ f"'{name=D}] \arrow[from=U, to=D, phantom, "\beta\Downarrow"] & X \end{tikzcd}$$

of pasting diagrams in C, i.e. such that we have

$$\beta = \alpha \star \mathrm{id}_f.$$

Proposition 9.2.10.1.3. Let $f \colon A \to B$ be a 1-morphism of C.

1. *Characterisations.* The following conditions are equivalent:
   (a) The morphism f is pseudoepic.
   (b) The morphism f is corepresentably full on cores and corepresentably faithful.
   (c) We have an isococomma square of the form

$$B \overset{\text{eq.}}{\cong} B \overset{\leftrightarrow}{\coprod}_A B, \qquad \begin{tikzcd} B & B \arrow[l, "\mathrm{id}_B"'] \arrow[dl, phantom, "\Leftrightarrow"] \\ B \arrow[u, "\mathrm{id}_B"] & A \arrow[l, "F"] \arrow[u, "F"'] \end{tikzcd}$$

   in C up to equivalence.

Proof. *Item 1*, *Characterisations*: Omitted. □

Appendices

9.A Other Chapters

Sets

1. Sets
2. Constructions With Sets
3. Pointed Sets
4. Tensor Products of Pointed Sets

Relations

5. Relations
6. Constructions With Relations
7. Equivalence Relations and Apartness Relations

Category Theory

8. Categories

Bicategories

9. Types of Morphisms in Bicategories

Part V

Extra Part

Chapter 10

Miscellaneous Notes

Contents

10.1 To Do List **457**
10.1.1 Omitted Proofs To Add 457
10.1.2 Things To Explore/Add 458
10.A Other Chapters **464**

10.1 To Do List

10.1.1 Omitted Proofs To Add

Не так благотворна истина, как зловредна ее видимость.

Даниил Данковский

Truth does not do as much good in the world as the appearance of truth does evil.

Daniil Dankovsky

There's a very large number of omitted proofs throughout these notes. Here I list them in decreasing order of how nice it would be to add them.

Remark 10.1.1.1.1. Proofs that *need* to be added at some point:

1. **??**.

2. **??**.

3. Horizontal composition of natural transformations is associative: **??** of **??**.

4. Fully faithful functors are essentially injective: **??** of **??**.

Proofs that *would be very nice* to be added at some point:

1. Properties of pseudomonic functors: **??**.
2. Characterisation of fully faithful functors: **??** of **??**.

Proofs that *would be nice* to be added at some point:

1. Properties of posetal categories: **??**.
2. The quadruple adjunction between categories and sets: **??**.
3. Properties of groupoid completions: **??**.
4. Properties of cores: **??**.
5. F_* faithful iff F faithful: **??** of **??**.
6. F_* full iff F full: **??** of **??**.
7. Injective on objects functors are precisely the isocofibrations in Cats_2: **??** of **??**.
8. Characterisations of monomorphisms of categories: **??** of **??**.
9. Epimorphisms of categories are surjective on objects: **??** of **??**.
10. Properties of pseudoepic functors: **??**.

10.1.2 Things To Explore/Add

Here we list things to be explored/added to this work in the future.

Remark 10.1.2.1.1. Set theory through a category theory lens:

1. Isbell duality for sets.
2. Density comonads and codensity monads for sets.

Relations:

1. 2-Categorical monomorphisms and epimorphisms in $\mathbf{Rel}$.
2. Co/limits in $\mathbf{Rel}$.
3. Apartness composition, categorical properties of $\mathbf{Rel}$ with apartness, and apartness relations.

4. Apartness defines a composition for relations, but its analogue

$$\mathfrak{q} \mathbin{\square} \mathfrak{p} \overset{\text{def}}{=} \int_{A \in C} \mathfrak{p}^{-1}_{A} \coprod \mathfrak{q}^{A}_{-2}$$

fails to be unital for profunctors. Is there a less obvious analogue of apartness composition for profunctors?

5. Codensity monad $\mathrm{Ran}_J(J)$ of a relation (What about $\mathrm{Rift}_J(J)$?)

6. Relative comonads in the 2-category of relations

7. Discrete fibrations and Street fibrations in **Rel**.

8. Consider adding the sections

   - The Monoidal Bicategory of Relations
   - The Monoidal Double Category of Relations

   to Relations.

Spans:

1. Universal property of the bicategory of spans, `https://ncatlab.org/nlab/show/span`

2. Write about cospans.

Un/Straightening:

1. Write proper sections on straightening for lax functors from sets to Rel or Span (displayed sets)

Categories:

1. Expand **??** and add a proof to it.

2. Sections and retractions; retracts, `https://ncatlab.org/nlab/show/retract`.

3. Regular categories: `https://arxiv.org/pdf/2004.08964.pdf`.

4. Are pseudoepic functors those functors whose restricted Yoneda embedding is pseudomonic and Yoneda preserves absolute colimits?

5. Absolutely dense functors enriched over $\mathbb{R}^{+}$ apparently reduce to topological density

Types of Morphisms in Categories:

1. Behaviour in $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$, e.g. pointwise sections vs. sections in $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$.

2. A faithful functor from balanced category is conservative

Yoneda stuff:

1. Properties of restricted Yoneda embedding, e.g. if the restricted Yoneda embedding is full, then what can we conclude? Related: https://qchu.wordpress.com/2015/05/17/generators/

Adjunctions:

1. Adjunctions, units, counits, and fully faithfulness as in https://mathoverflow.net/questions/100808/properties-of-functors-and-their-adjoints.

2. Morphisms between adjunctions and bicategory $\mathsf{Adj}(\mathcal{C})$.

3. https://ncatlab.org/nlab/show/transformation+of+adjoints

Constructions With Categories:

1. Comparison between pseudopullbacks and isocomma categories: the "evident" functor $\mathcal{C} \times^{\mathsf{ps}}_{\mathcal{E}} \mathcal{D} \to \mathcal{C} \overset{\leftrightarrow}{\times}_{\mathcal{E}} \mathcal{D}$ is essentially surjective and full, but not faithful in general.

Co/limits:

1. Add the characterisations of absolutely dense functors given in **??** to **??**.

2. Absolutely dense functors, https://ncatlab.org/nlab/show/absolutely+dense+functor. Also theorem 1.1 here: http://www.tac.mta.ca/tac/volumes/8/n20/n20.pdf.

3. Dense functors, codense functors, and absolutely codense functors.

Co/ends:

1. Examples of co/ends: https://mathoverflow.net/a/461814

2. Cofinality for co/ends, https://mathoverflow.net/questions/353876

Fibred category theory:

1. Internal **Hom** in categories of co/Cartesian fibrations.

2. *Tensor structures on fibered categories* by Luca Terenzi: https://arxiv.org/abs/2401.13491. Check also the other papers by Luca Terenzi.

3. https://ncatlab.org/nlab/show/cartesian+natural+transformation (this is a cartesian morphism in $\mathsf{Fun}(C, \mathcal{D})$ apparently)

4. CoCartesian fibration classifying $\mathsf{Fun}(F, G)$, https://mathoverflow.net/questions/457533/cocartesian-fibration-classifying-mathrmfunf-g

Monoidal categories:

1. Free braided monoidal category with a braided monoid: https://ncatlab.org/nlab/show/vine

Skew monoidal categories:

1. Does the $\mathbb{E}_1$ tensor product of monoids admit a skew monoidal category structure?

2. Is there a (right?) skew monoidal category structure on $\mathsf{Fun}(C, \mathcal{D})$ using right Kan extensions instead of left Kan extensions?

3. Similarly, are there skew monoidal category structures on the subcategory of $\mathbf{Rel}(A, B)$ spanned by the functions using left Kan extensions and left Kan lifts?

Higher categories:

1. Internal adjunctions in Mod as in [JY21, Section 6.3]; see [JY21, Example 6.2.6].

2. Comonads in the bicategory of profunctors.

Monoids:

1. Isbell's zigzag theorem for semigroups: the following conditions are equivalent:

   (a) A morphism $f\colon A \to B$ of semigroups is an epimorphism.

   (b) For each $b \in B$, one of the following conditions is satisfied:

      - We have $f(a) = b$.

- There exist some $m \in \mathbb{N}_{\geq 1}$ and two factorisations

$$b = a_0 y_1,$$
$$b = x_m a_{2m}$$

connected by relations

$$\begin{aligned} a_0 &= x_1 a_1, \\ a_1 y_1 &= a_2 y_2, \\ x_1 a_2 &= x_2 a_3, \\ a_{2m-1} y_m &= a_{2m} \end{aligned}$$

such that, for each $1 \leq i \leq m$, we have $a_i \in \mathrm{Im}(f)$.

Wikipedia says in https://en.wikipedia.org/wiki/Isbell%27s_zigzag_theorem:

> For monoids, this theorem can be written more concisely:

Types of morphisms in bicategories:

1. Behaviour in 2-categories of pseudofunctors (or lax functors, etc.), e.g. pointwise pseudoepic morphisms in vs. pseudoepic morphisms in 2-categories of pseudofunctors.
2. Statements like "coequifiers are lax epimorphisms", Item 2 of Examples 2.4 of https://arxiv.org/abs/2109.09836, along with most of the other statements/examples there.
3. Dense, absolutely dense, etc. morphisms in bicategories

Other:

1. https://qchu.wordpress.com/
2. https://aroundtoposes.com/
3. https://ncatlab.org/nlab/show/essentially+surjective+and+full+functor
4. https://mathoverflow.net/questions/415363/objects-whose-representable-presheaf-is-a-fibration
5. https://mathoverflow.net/questions/460146/universal-property-of-isbell-duality

6. http://www.tac.mta.ca/tac/volumes/36/12/36-12abs.html (Isbell conjugacy and the reflexive completion)
7. https://ncatlab.org/nlab/show/enrichment+versus+internalisation
8. The works of Philip Saville, https://philipsaville.co.uk/
9. https://golem.ph.utexas.edu/category/2024/02/from_cartesian_to_symmetric_mo.html
10. https://mathoverflow.net/q/463855 (One-object lax transformations)
11. https://ncatlab.org/nlab/show/analytic+completion+of+a+ring
12. https://en.wikipedia.org/wiki/Quaternionic_analysis
13. https://arxiv.org/abs/2401.15051 (The Norm Functor over Schemes)
14. https://mathoverflow.net/questions/407291/ (Adjunctions with respect to profunctors)
15. https://mathoverflow.net/a/462726 (Prof is free completion of Cats under right extensions)
16. there's some cool stuff in https://arxiv.org/abs/2312.00990 (Polynomial Functors: A Mathematical Theory of Interaction), e.g. on cofunctors.
17. https://ncatlab.org/nlab/show/adjoint+lifting+theorem
18. https://ncatlab.org/nlab/show/Gabriel%E2%80%93Ulmer+duality

Appendices

10.A Other Chapters

Sets

1. Sets
2. Constructions With Sets
3. Pointed Sets
4. Tensor Products of Pointed Sets

Relations

5. Relations
6. Constructions With Relations
7. Equivalence Relations and Apartness Relations

Category Theory

8. Categories

Bicategories

9. Types of Morphisms in Bicategories

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Index of Notation

A
$[a]$, 348
(A, B), 48
$A \coprod B$, 35
$A \coprod_C B$, 37
αF, 423
$\alpha \star F$, 423
$A \odot X$, 143
$a \odot x$, 145
$A \odot (X, x_0)$, 143
$A \pitchfork X$, 150
$A \pitchfork (X, x_0)$, 150
$A \vartriangle B$, 59
$A \times B$, 15
$A \times_C B$, 23
$A \times_{f,C,g} B$, 23
$A \cup B$, 49

B
$\beta \circ \alpha$, 418
$\beta \star \alpha$, 421

C
$\mathcal{C}(A, B)$, 359
Cats, 432
Cats_2, 433
$[\mathcal{C}, \mathcal{D}]$, 429
$\chi_{(-)}$, 64
$\chi^{-1}_{-_2}$, 64
χ_U, 64
χ^x, 64
χ_x, 64
$\chi_X(-_1, -_2)$, 64
$\chi_X(-, U)$, 64
$\chi_X(-, x)$, 64
$\chi_X(U, -)$, 64
$\chi_X(x, -)$, 64
$\mathrm{CoEq}(f, g)$, 44
$\mathrm{Coim}(f)$, 350
$\mathbf{Coll}(R)$, 312
$\mathrm{Coll}(R)$, 312
$\coprod_{i \in I} A_i$, 33
$\mathsf{Core}(\mathcal{C})$, 377
$\mathcal{C}^{\simeq}$, 378

D
$\mathcal{D}^{\mathcal{C}}$, 429
$\mathrm{dom}(R)$, 299

E
$\emptyset_{\mathsf{cat}}$, 361
$\emptyset$, 32, 47
$\mathrm{Eq}(f, g)$, 28
$\mathrm{eq}(f, g)$, 28

F
$F\alpha$, 423
f^{-1}, 90

$f^{-1}(V)$, 90
$\mathbb{F}_1$, 103
$f_!$, 94
$f_{!,\mathrm{cp}}(U)$, 96
$f_{!,\mathrm{im}}(U)$, 95
$f_!(U)$, 94
f^*, 84, 365
$F \star \alpha$, 423
$\alpha \star F$, 423
$f(U)$, 85
$\mathsf{Fun}(\mathcal{C}, \mathcal{D})$, 429

G

$\mathrm{Gr}(A)$, 293
Grpd, 434
Grpd_2, 434

H

$\mathrm{Hom}^{\otimes,\mathrm{L}}_{\mathsf{Sets}_*}(X \times Y, Z)$, 140
$\mathrm{Hom}^{\otimes,\mathrm{R}}_{\mathsf{Sets}_*}(X \times Y, Z)$, 141
$\mathrm{Hom}^{\otimes}_{\mathsf{Sets}_*}(X \times Y, Z)$, 142
$\mathrm{Hom}_{\mathsf{Sets}}(A, B)$, 48
$\mathbf{Hom}_{\mathsf{Sets}_*}(X, Y)$, 210

I

id_C, 382
id_F, 418
$\mathrm{id}_F \star \alpha$, 423
$\bigcup_{i \in I} R_i$, 303
$\bigcap_{X \in \mathcal{F}} X$, 51
$\mathrm{Iso}_{\mathcal{C}}(A, B)$, 374

K

$\mathrm{K}_0(\mathcal{C})$, 374

L

$\lhd$, 157
$[-,-]^{\lhd}_{\mathsf{Sets}_*}$, 161
$\lhd_{\mathsf{Sets}_*}$, 157

M

$\mathrm{Mor}(\mathcal{C})$, 359

N

$\pitchfork$, 361
$\mathrm{Nat}(F, G)$, 417

P

$\pi_0(\mathcal{C})$, 369
$\mathcal{P}^{-1}$, 71, 335
$\star_X$, 131
$\prod_{i \in I} A_i$, 13, 108
$\mathcal{P}_!$, 71, 335
$\mathcal{P}_*$, 70, 335
pt, 13, 103, 108, 118
pt, 360
$\mathcal{P}(X)$, 69

R

$\mathrm{range}(R)$, 300
$R^\dagger$, 306
Rel, 243
$\mathbf{Rel}$, 248
$\mathrm{Rel}(A, B)$, 235
$\mathbf{Rel}(A, B)$, 235
$\mathsf{Rel}^{\mathsf{dbl}}$, 249
$\mathrm{Rel}^{\mathrm{eq}}(A, B)$, 346
$\mathbf{Rel}^{\mathrm{eq}}(A, B)$, 346
$\mathrm{Rel}^{\mathrm{refl}}(A, A)$, 339
$\mathbf{Rel}^{\mathrm{refl}}(A, A)$, 339
$\mathrm{Rel}^{\mathrm{symm}}(A, A)$, 341
$\mathbf{Rel}^{\mathrm{symm}}(A, A)$, 341
$\mathrm{Rel}^{\mathrm{trans}}(A)$, 343
$\mathbf{Rel}^{\mathrm{trans}}(A)$, 343
R^{eq}, 347

$\triangleright$, 179
$[-,-]^{\triangleright}_{\mathsf{Sets}_*}$, 182
$\triangleright_{\mathsf{Sets}_*}$, 179
$R \cap S$, 302
R^{-1}, 324
R_{-1}, 318
$R^{-1}(V)$, 324
$R_{-1}(V)$, 318
R^{refl}, 339
$R_!$, 329
$R_!(U)$, 329
R^*, 314
R^{symm}, 341
$R \times S$, 303
R^{trans}, 344
$R(U)$, 314
$R \cup S$, 300

S

$S \diamond R$, 307
$\mathsf{Sets}(A, B)$, 48
Sets_*, 104
$\mathbf{Sets}_*((X, x_0), (Y, y_0))$, 210
$\sim_{\mathrm{cotriv}}$, 239
$\sim_{\mathrm{id}}$, 64
$\sim_R^{\mathrm{refl}}$, 339
$\sim_R^{\mathrm{symm}}$, 341
$\sim_R^{\mathrm{eq}}$, 347
$\sim_R^{\mathrm{trans}}$, 344
$\sim_{\mathrm{triv}}$, 239
S^0, 103

T

$\{\mathsf{t}, \mathsf{f}\}$, 5
$\prod_{i \in I} R_i$, 305
$\mathsf{Trans}(F, G)$, 417
$\{\mathsf{true}, \mathsf{false}\}$, 5

U

U^{c}, 58
$\bigcup_{i \in I} A_i$, 49
$\bigcup_{i \in I} R_i$, 301

W

忘, 386
$\bigwedge_{i \in I} X_i$, 230

X

$\{X\}$, 47
X_{disc}, 370
X_{indisc}, 372
$X \cap Y$, 51
$x \triangleleft_{\mathsf{Sets}_*} y$, 159
$x \triangleleft y$, 159
$X \setminus Y$, 55
$X/\sim_R$, 348
$X \otimes_{\mathbb{F}_1} Y$, 202
X^+, 131
X_{pos}, 362
$x \triangleright y$, 180
$x \triangleright_{\mathsf{Sets}_*} y$, 180
$x \wedge y$, 203
$X \wedge Y$, 201
$\{X, Y\}$, 48

Index of Set Theory

B

bilinear morphism
    of pointed sets, 141
bilinear morphism of pointed sets
    left, 140
    right, 141
binary intersection, 51

C

Cartesian product, 13, 15
category of relations, 243
    associator of, 244
    internal Hom of, 246
    left unitor of, 245
    monoidal product of, 243
    monoidal unit of, 244
    right unitor of, 245
    symmetry of, 246
characteristic embedding, 64
characteristic function
    of a set, 64
    of an element, 64
characteristic relation, 64
coequaliser of sets, 44
coimage, 350
complement of a set, 58
coproduct
    of a family of pointed sets, 119

D

difference of sets, 55
disjoint union, 35
    of a family of sets, 33

E

empty set, 47
equaliser of sets, 28
equivalence class, 348
equivalence relation
    kernel, 346

F

fibre coproduct of sets, 37
fibre product of sets, 23
field with one element
    module over, 103
    module over, morphism of, 104
function, 2
    associated direct image function, 84
    associated direct image with compact support function, 94
    associated direct image with compact support function, complement part, 95

associated direct image with compact support function, image part, 95
associated inverse image function, 90
coimage of, 350
graph of, 293
inverse of, 297
kernel of, 346

I

indicator function, *see* characteristic function
initial pointed set, 118
initial set, 32
intersection of a family of sets, 51

L

left tensor product of pointed sets
diagonal, 170
left skew unit of, 163
skew associator, 163
skew left unitor, 166
skew right unitor, 168

M

(-1)-category, 4
(-2)-category, 4
module
underlying pointed set of, 103

O

ordered pairing, 48

P

pairing of two sets, 48
pointed function, *see* pointed set, morphism of
pointed set, 103
category of, 104
coequaliser of, 129
copower by a set, *see* pointed set, tensor by a set
coproduct of, 121
cotensor by a set, 150
equaliser of, 116
$\mathbb{F}_1$, 103
free, 131
internal Hom of, 210
left internal Hom of, 161
left tensor product, 157
morphism of, 104
of morphisms of pointed sets, 210
power by a set, *see* pointed set, cotensor by a set
product of, 110
pullback of, 112
pushout of, 124
right internal Hom of, 182
right tensor product, 179
tensor by a set, 143
trivial, 103
underlying a module, 103
underlying a semimodule, 103
wedge sum, 121
wedge sum of a family of pointed sets, 119
0-sphere, 103
poset
of (-1)-categories, 5
of truth values, 5
powerset, 69
product of a family of pointed sets, 108
product of a family of sets, 13

product of sets, 15
pullback of sets, 23
pushout of sets, 37

Q

quotient
  by an equivalence relation, 348

R

relation, 234
  associated direct image function, 314
  associated direct image with compact support function, 329
  associated strong inverse image function, 318
  associated weak inverse image function, 324
  category of, 235
  category of, 243
  closed symmetric monoidal category of, 247
  collage of, 312
  composition of, 307
  corepresentable, 299
  cotrivial, 239
  domain of, 299
  double category of, 249
  equivalence closure of, 347
  equivalence relation, 346
  equivalence, set of, 346
  functional, 241
  intersection of, 302
  intersection of a family of, 303
  inverse of, 306
  on powersets associated to a relation, 336
  partial equivalence relation, 346
  poset of, 235, 341, 343, 346
  product of, 303
  product of a family of, 305
  range of, 300
  reflexive, 339
  reflexive closure of, 339
  reflexive, poset of, 339
  reflexive, set of, 339
  representable, 299
  set of, 235
  symmetric, 341
  symmetric closure of, 341
  symmetric, set of, 341
  total, 242
  transitive, 343
  transitive closure of, 344
  transitive, set of, 343
  trivial, 239
  2-category of, 248
  union of, 300
  union of a family of, 301
right tensor product of pointed sets
  diagonal, 191
  right skew unit of, 184
  skew associator, 185
  skew left unitor, 187
  skew right unitor, 189

S

semimodule
  underlying pointed set of, 103
set
  Hom, 48
  of maps, 48
set of bilinear morphisms of pointed sets, 142

- left, 140
- right, 141
singleton set, 47
smash product
- of a family of pointed sets, 230
- of pointed sets, 200
smash product of pointed sets
- associator, 212
- diagonal, 221
- left unitor, 214
- monoidal unit of, 212
- right unitor, 216
- symmetry, 219
symmetric difference of sets, 59

T

terminal pointed set, 108
terminal set, 13

U

union, 49
- of a family of sets, 49

Z

0-category, 8
0-groupoid, 8

Index of Category Theory

A

adjoint equivalence of categories, 400

C

category, 357
- adjoint equivalence of, 400
- connected, 370
- connected component, 368
- connected component, set of, 369
- core of, 377
- disconnected, 370
- discrete, 371
- empty, 361
- equivalence of, 400
- groupoid completion of, 374
- indiscrete, 373
- isomorphism of, 402
- κ-small, 359
- locally essentially small, 359
- locally small, 359
- of small groupoids, 434
- skeletal, 365
- skeleton of, 364
- small, 359
- thin, 363

category of categories, 432
contravariant functor, 384

D

discrete category
- on a set, 370

E

equivalence of categories, 400

F

forgetful functor, 385
functor, 380
- bijective on objects, 411
- bo, 411
- composition of, 383
- conservative, 397
- contravariant, 384
- corepresentably faithful on cores, 414
- corepresentably full on cores, 415
- corepresentably fully faithful on cores, 415
- dominant, 403
- epimorphism, 405
- eso, 399
- essentially injective, 399
- essentially surjective, 399
- faithful, 389
- forgetful, 385
- full, 391

fully faithful, 394
identity, 382
injective on objects, 411
monomorphism, 405
pseudoepic, 409
pseudomonic, 407
representably faithful on cores, 412
representably full on cores, 412
representably fully faithful on cores, 413
surjective on objects, 411
transformation of, 417
functor category, 429

G

Godement product, *see* natural transformation, horizontal composition
Grothendieck groupoid
of a category, 374
groupoid, 374
groupoid completion, 374

I

indiscrete category
on a set, 372
Isbell's zigzag theorem, 405
isomorphism, 374
isomorphism of categories, 402

M

middle four exchange
in Cats, 420, 424

N

natural isomorphism, 428
natural transformation, 417
associated to a functor, 387
equality of, 418
horizontal composition, 421
identity natural transformation, 418
vertical composition, 418

O

ordinal category, 361

P

posetal category, 363
associated to a poset, 362
postcomposition, 365
precomposition, 365
punctual category, 360

S

singleton category, *see* punctual category
subcategory, 364
full, 364
lluf, 364
strictly full, 364
wide, 364

T

transformation between functors, 417
2-category
of small categories, 433
of small groupoids, 434

W

whiskering
left, 423
right, 423

Z

(0, 1)-category, 363

Index of Higher Category Theory

D

double category
    of relations, 249

E

epimorphism
    strict, 452

M

monomorphism
    strict, 444

O

1-morphism
    corepresentably conservative, 452
    corepresentably essentially injective, 451
    corepresentably faithful, 446
    corepresentably faithful on cores, 449
    corepresentably full, 447
    corepresentably full on cores, 450
    corepresentably fully faithful, 448
    corepresentably fully faithful on cores, 450
    pseudoepic, 453
    pseudomonic, 444
    representably conservative, 443
    representably essentially injective, 443
    representably faithful, 437
    representably faithful on cores, 440
    representably full, 438
    representably full on cores, 441
    representably fully faithful, 439
    representably fully faithful on cores, 442