Pointed Sets

The Clowder Project Authors

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0098 This chapter contains some foundational material on pointed sets.

Contents

`0099` # 1 Pointed Sets

`009A` ## 1.1 Foundations

`009B` **Definition 1.1.1.1.** A **pointed set**[1] is equivalently:

- An $\mathbb{E}_0$-monoid in $(\mathsf{N}_\bullet(\mathsf{Sets}), \mathsf{pt})$.

- A pointed object in $(\mathsf{Sets}, \mathsf{pt})$.

`009C` **Remark 1.1.1.2.** In detail, a **pointed set** is a pair (X, x_0) consisting of:

- *The Underlying Set.* A set X, called the **underlying set of** (X, x_0).

- *The Basepoint.* A morphism

$$[x_0] \colon \mathsf{pt} \to X$$

in Sets, determining an element $x_0 \in X$, called the **basepoint of** X.

`009D` **Example 1.1.1.3.** The 0-**sphere**[2] is the pointed set $(S^0, 0)$[3] consisting of:

- *The Underlying Set.* The set S^0 defined by

$$S^0 \overset{\text{def}}{=} \{0, 1\}.$$

- *The Basepoint.* The element 0 of S^0.

`009E` **Example 1.1.1.4.** The **trivial pointed set** is the pointed set $(\mathsf{pt}, \star)$ consisting of:

- *The Underlying Set.* The punctual set $\mathsf{pt} \overset{\text{def}}{=} \{\star\}$.

- *The Basepoint.* The element $\star$ of pt.

`009F` **Example 1.1.1.5.** The **underlying pointed set** of a semimodule (M, α_M) is the pointed set $(M, 0_M)$.

`009G` **Example 1.1.1.6.** The **underlying pointed set** of a module (M, α_M) is the pointed set $(M, 0_M)$.

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[1]*Further Terminology:* In the context of monoids with zero as models for $\mathbb{F}_1$-algebras, pointed sets are viewed as $\mathbb{F}_1$-**modules**.

[2]*Further Terminology:* In the context of monoids with zero as models for $\mathbb{F}_1$-algebras, the 0-sphere is viewed as the **underlying pointed set of the field with one element**.

[3]*Further Notation:* In the context of monoids with zero as models for $\mathbb{F}_1$-algebras, S^0 is also

009H   **1.2   Morphisms of Pointed Sets**

009J   **Definition 1.2.1.1.** A **morphism of pointed sets**[4,5] is equivalently:

- A morphism of $\mathbb{E}_0$-monoids in $(\mathrm{N}_\bullet(\mathsf{Sets}), \mathrm{pt})$.

- A morphism of pointed objects in $(\mathsf{Sets}, \mathrm{pt})$.

009K   **Remark 1.2.1.2.** In detail, a **morphism of pointed sets** $f\colon (X, x_0) \to (Y, y_0)$ is a morphism of sets $f\colon X \to Y$ such that the diagram

$$
\begin{array}{ccc}
 & \mathrm{pt} & \\
{\scriptstyle[x_0]}\swarrow & & \searrow{\scriptstyle[y_0]} \\
X & \xrightarrow{\ \ f\ \ } & Y
\end{array}
$$

commutes, i.e. such that

$$f(x_0) = y_0.$$

009L   **1.3   The Category of Pointed Sets**

009M   **Definition 1.3.1.1.** The **category of pointed sets** is the category Sets_* defined equivalently as

- The homotopy category of the ∞-category $\mathsf{Mon}_{\mathbb{E}_0}(\mathrm{N}_\bullet(\mathsf{Sets}), \mathrm{pt})$ of **??**, **??**;

- The category Sets_* of **??**, **??**.

009N   **Remark 1.3.1.2.** In detail, the **category of pointed sets** is the category Sets_* where

- *Objects.* The objects of Sets_* are pointed sets;

- *Morphisms.* The morphisms of Sets_* are morphisms of pointed sets;

- *Identities.* For each $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, the unit map

$$\mathbb{1}_{(X,x_0)}^{\mathsf{Sets}_*}\colon \mathrm{pt} \to \mathsf{Sets}_*((X, x_0), (X, x_0))$$

---

denoted $(\mathbb{F}_1, 0)$.

[4]*Further Terminology:* Also called a **pointed function**.

[5]*Further Terminology:* In the context of monoids with zero as models for $\mathbb{F}_1$-algebras, mor-

of Sets_* at (X, x_0) is defined by[6]

$$\mathrm{id}^{\mathsf{Sets}_*}_{(X,x_0)} \overset{\mathrm{def}}{=} \mathrm{id}_X;$$

- *Composition.* For each $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, the composition map

$$\circ^{\mathsf{Sets}_*}_{(X,x_0),(Y,y_0),(Z,z_0)} : \mathsf{Sets}_*((Y,y_0),(Z,z_0)) \times \mathsf{Sets}_*((X,x_0),(Y,y_0)) \to \mathsf{Sets}_*((X,x_0),(Z,z_0))$$

of Sets_* at $((X, x_0), (Y, y_0), (Z, z_0))$ is defined by[7]

$$g \circ^{\mathsf{Sets}_*}_{(X,x_0),(Y,y_0),(Z,z_0)} f \overset{\mathrm{def}}{=} g \circ f.$$

009P   1.4   Elementary Properties of Pointed Sets

009Q   **Proposition 1.4.1.1.** Let (X, x_0) be a pointed set.

009R   1. *Completeness.* The category Sets_* of pointed sets and morphisms between them is complete, having in particular:

009S   (a) Products, described as in Definition 2.3.1.1;

009T   (b) Pullbacks, described as in Definition 2.4.1.1;

009U   (c) Equalisers, described as in Definition 2.5.1.1.

009V   2. *Cocompleteness.* The category Sets_* of pointed sets and morphisms between them is cocomplete, having in particular:
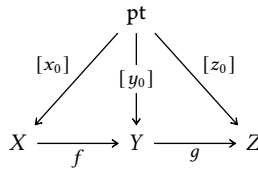
---

phisms of pointed sets are also called **morphism of $\mathbb{F}_1$-modules**.

[6]Note that id_X is indeed a morphism of pointed sets, as we have $\mathrm{id}_X(x_0) = x_0$.

[7]Note that the composition of two morphisms of pointed sets is indeed a morphism of pointed sets, as we have

$$g(f(x_0)) = g(y_0)$$
$$= z_0,$$

or

in terms of diagrams.

009W   (a) Coproducts, described as in Definition 3.3.1.1;

009X   (b) Pushouts, described as in Definition 3.4.1.1;

009Y   (c) Coequalisers, described as in Definition 3.5.1.1.

009Z   3. *Failure To Be Cartesian Closed.*   The category Sets_* is not Cartesian closed.[8]

00A0   4. *Morphisms From the Monoidal Unit.* We have a bijection of sets[9]

$$\mathsf{Sets}_*\left(S^0, X\right) \cong X,$$

natural in $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, internalising also to an isomorphism of pointed sets

$$\mathbf{Sets}_*\left(S^0, X\right) \cong (X, x_0),$$

again natural in $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

00A1   5. *Relation to Partial Functions.*   We have an equivalence of categories[10]

$$\mathsf{Sets}_* \overset{\mathrm{eq.}}{\cong} \mathsf{Sets}^{\mathrm{part.}}$$

between the category of pointed sets and pointed functions between them and the category of sets and partial functions between them, where:

(a) *From Pointed Sets to Sets With Partial Functions.* The equivalence

$$\xi \colon \mathsf{Sets}_* \overset{\cong}{\to} \mathsf{Sets}^{\mathrm{part.}}$$

sends:

i. A pointed set (X, x_0) to X.

---

[8]The category Sets_* does admit monoidal closed structures however; see Tensor Products of Pointed Sets.

[9]In other words, the forgetful functor

$$忘\colon \mathsf{Sets}_* \to \mathsf{Sets}$$

defined on objects by sending a pointed set to its underlying set is corepresentable by S^0.

[10] *Warning:* This is not an isomorphism of categories, only an equivalence.

ii. A pointed function

$$f\colon (X, x_0) \longrightarrow (Y, y_0)$$

to the partial function

$$\xi_f \colon X \rightharpoonup Y$$

defined on $f^{-1}(Y \setminus y_0)$ and given by

$$\xi_f(x) \overset{\mathrm{def}}{=} f(x)$$

for each $x \in f^{-1}(Y \setminus y_0)$.

(b) *From Sets With Partial Functions to Pointed Sets.* The equivalence

$$\xi^{-1} \colon \mathsf{Sets}^{\mathrm{part.}} \overset{\cong}{\rightarrow} \mathsf{Sets}_*$$

sends:

i. A set X is to the pointed set $(X, \star)$ with $\star$ an element that is not in X.

ii. A partial function

$$f\colon X \rightharpoonup Y$$

defined on $U \subset X$ to the pointed function

$$\xi_f^{-1} \colon (X, x_0) \longrightarrow (Y, y_0)$$

defined by

$$\xi_f(x) \overset{\mathrm{def}}{=} \begin{cases} f(x) & \text{if } x \in U, \\ y_0 & \text{otherwise.} \end{cases}$$

for each $x \in X$.

Proof. *Item 1, Completeness*: This follows from (the proofs) of Definitions 2.3.1.1, 2.4.1.1 and 2.5.1.1 and ??, ??.
Item 2, Cocompleteness: This follows from (the proofs) of Definitions 3.3.1.1, 3.4.1.1 and 3.5.1.1 and ??, ??.
Item 3, Failure To Be Cartesian Closed: See [MSE 2855868].

Item 4, Morphisms From the Monoidal Unit: Since a morphism from S^0 to a pointed set (X, x_0) sends $0 \in S^0$ to x_0 and then can send $1 \in S^0$ to any element of X, we obtain a bijection between pointed maps $S^0 \to X$ and the elements of X. The isomorphism then

$$\mathsf{Sets}_*\big(S^0, X\big) \cong (X, x_0)$$

follows by noting that $\Delta_{x_0} \colon S^0 \to X$, the basepoint of $\mathsf{Sets}_*\big(S^0, X\big)$, corresponds to the pointed map $S^0 \to X$ picking the element x_0 of X, and thus we see that the bijection between pointed maps $S^0 \to X$ and elements of X is compatible with basepoints, lifting to an isomorphism of pointed sets.

Item 5, Relation to Partial Functions: See [MSE 884460]. □

00A2   2   Limits of Pointed Sets

00A3   2.1   The Terminal Pointed Set

00A4   **Definition 2.1.1.1.** The **terminal pointed set** is the pair $\Big((\mathrm{pt}, \star), \{!_X\}_{(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)}\Big)$ consisting of:

- *The Limit.* The pointed set $(\mathrm{pt}, \star)$.

- *The Cone.* The collection of morphisms of pointed sets

$$\{!_X \colon (X, x_0) \to (\mathrm{pt}, \star)\}_{(X, x_0) \in \mathrm{Obj}(\mathsf{Sets})}$$

  defined by

$$!_X(x) \overset{\mathrm{def}}{=} \star$$

  for each $x \in X$ and each $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets})$.

Proof. We claim that $(\mathrm{pt}, \star)$ is the terminal object of Sets_*. Indeed, suppose we have a diagram of the form

$$(X, x_0) \qquad (\mathrm{pt}, \star)$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi \colon (X, x_0) \to (\mathrm{pt}, \star)$$

making the diagram

$$(X, x_0) \overset{\phi}{\underset{\exists!}{-\,{-}\!\to}} (\mathrm{pt}, \star)$$

commute, namely $!_X$. □

2.2   Products of Families of Pointed Sets

Let $\left\{\left(X_i, x_0^i\right)\right\}_{i \in I}$ be a family of pointed sets.

00A6   **Definition 2.2.1.1.** The **product of** $\left\{\left(X_i, x_0^i\right)\right\}_{i \in I}$ is the pair $\left(\left(\prod_{i \in I} X_i, \left(x_0^i\right)_{i \in I}\right), \left\{\mathrm{pr}_i\right\}_{i \in I}\right)$
consisting of:

- *The Limit.* The pointed set $\left(\prod_{i \in I} X_i, \left(x_0^i\right)_{i \in I}\right)$.

- *The Cone.* The collection

$$\left\{\mathrm{pr}_i \colon \left(\prod_{i \in I} X_i, \left(x_0^i\right)_{i \in I}\right) \to \left(X_i, x_0^i\right)\right\}_{i \in I}$$

of maps given by

$$\mathrm{pr}_i\left(\left(x_j\right)_{j \in I}\right) \overset{\mathrm{def}}{=} x_i$$

for each $\left(x_j\right)_{j \in I} \in \prod_{i \in I} X_i$ and each $i \in I$.

Proof. We claim that $\left(\prod_{i \in I} X_i, \left(x_0^i\right)_{i \in I}\right)$ is the categorical product of $\left\{\left(X_i, x_0^i\right)\right\}_{i \in I}$
in Sets_*. Indeed, suppose we have, for each $i \in I$, a diagram of the form

$$
\begin{array}{c}
(P, *) \\
\diagdown \quad p_i \\
\left(\prod_{i \in I} X_i, \left(x_0^i\right)_{i \in I}\right) \xrightarrow[\mathrm{pr}_i]{} \left(X_i, x_0^i\right)
\end{array}
$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi \colon (P, *) \to \left(\prod_{i \in I} X_i, \left(x_0^i\right)_{i \in I}\right)$$

making the diagram

$$
\begin{array}{c}
(P, *) \\
\phi \,\Big\downarrow\, \exists! \quad \diagdown \quad p_i \\
\left(\prod_{i \in I} X_i, \left(x_0^i\right)_{i \in I}\right) \xrightarrow[\mathrm{pr}_i]{} \left(X_i, x_0^i\right)
\end{array}
$$

commute, being uniquely determined by the condition $\mathrm{pr}_i \circ \phi = p_i$ for each $i \in I$ via

$$\phi(x) = (p_i(x))_{i \in I}$$

for each $x \in P$. Note that this is indeed a morphism of pointed sets, as we have

$$\phi(*) = (p_i(*))_{i \in I}$$
$$= (x_0^i)_{i \in I},$$

where we have used that p_i is a morphism of pointed sets for each $i \in I$. □

00A7 **Proposition 2.2.1.2.** Let $\left\{(X_i, x_0^i)\right\}_{i \in I}$ be a family of pointed sets.

00A8   1. *Functoriality.* The assignment $\left\{(X_i, x_0^i)\right\}_{i \in I} \mapsto \left(\prod_{i \in I} X_i, (x_0^i)_{i \in I}\right)$ defines a functor

$$\prod_{i \in I} \colon \mathsf{Fun}(I_{\mathrm{disc}}, \mathsf{Sets}_*) \to \mathsf{Sets}_*.$$

Proof. Item 1, Functoriality: This follows from **??**, **??** of **??**. □

00A9 **2.3 Products**

Let (X, x_0) and (Y, y_0) be pointed sets.

00AA **Definition 2.3.1.1.** The **product of** (X, x_0) **and** (Y, y_0) is the pair consisting of:

- *The Limit.* The pointed set $(X \times Y, (x_0, y_0))$.

- *The Cone.* The morphisms of pointed sets

$$\mathrm{pr}_1 \colon (X \times Y, (x_0, y_0)) \to (X, x_0),$$
$$\mathrm{pr}_2 \colon (X \times Y, (x_0, y_0)) \to (Y, y_0)$$

    defined by

$$\mathrm{pr}_1(x, y) \stackrel{\mathrm{def}}{=} x,$$
$$\mathrm{pr}_2(x, y) \stackrel{\mathrm{def}}{=} y$$

    for each $(x, y) \in X \times Y$.

Proof. We claim that $(X \times Y, (x_0, y_0))$ is the categorical product of (X, x_0) and (Y, y_0) in Sets_*. Indeed, suppose we have a diagram of the form

$$
\begin{array}{ccc}
& (P, *) & \\
{\scriptstyle p_1} \swarrow & & \searrow {\scriptstyle p_2} \\
(X, x_0) \xleftarrow[\mathrm{pr}_1]{} (X \times Y, (x_0, y_0)) & \xrightarrow[\mathrm{pr}_2]{} & (Y, y_0)
\end{array}
$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$
\phi \colon (P, *) \longrightarrow (X \times Y, (x_0, y_0))
$$

making the diagram

$$
\begin{array}{ccc}
& (P, *) & \\
{\scriptstyle p_1} \swarrow & {\scriptstyle \phi} \downarrow {\scriptstyle \exists!} & \searrow {\scriptstyle p_2} \\
(X, x_0) \xleftarrow[\mathrm{pr}_1]{} (X \times Y, (x_0, y_0)) & \xrightarrow[\mathrm{pr}_2]{} & (Y, y_0)
\end{array}
$$

commute, being uniquely determined by the conditions

$$
\begin{aligned}
\mathrm{pr}_1 \circ \phi &= p_1, \\
\mathrm{pr}_2 \circ \phi &= p_2
\end{aligned}
$$

via

$$
\phi(x) = (p_1(x), p_2(x))
$$

for each $x \in P$. Note that this is indeed a morphism of pointed sets, as we have

$$
\begin{aligned}
\phi(*) &= (p_1(*), p_2(*)) \\
&= (x_0, y_0),
\end{aligned}
$$

where we have used that p_1 and p_2 are morphisms of pointed sets. $\square$

00AB   **Proposition 2.3.1.2.** Let (X, x_0), (Y, y_0), and (Z, z_0) be pointed sets.

00AC   1. *Functoriality.* The assignments

$$
(X, x_0), (Y, y_0), ((X, x_0), (Y, y_0)) \longmapsto (X \times Y, (x_0, y_0))
$$

define functors

$$X \times -\colon \mathsf{Sets}_* \to \mathsf{Sets}_*,$$
$$- \times Y\colon \mathsf{Sets}_* \to \mathsf{Sets}_*,$$
$$-_1 \times -_2\colon \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*,$$

defined in the same way as the functors of Constructions With Sets, Item 1 of Proposition 1.3.1.2.

00AD   2. *Associativity.* We have an isomorphism of pointed sets

$$((X \times Y) \times Z, ((x_0, y_0), z_0)) \cong (X \times (Y \times Z), (x_0, (y_0, z_0)))$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

00AE   3. *Unitality.* We have isomorphisms of pointed sets

$$(\mathrm{pt}, \star) \times (X, x_0) \cong (X, x_0),$$
$$(X, x_0) \times (\mathrm{pt}, \star) \cong (X, x_0),$$

natural in $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

00AF   4. *Commutativity.* We have an isomorphism of pointed sets

$$(X \times Y, (x_0, y_0)) \cong (Y \times X, (y_0, x_0)),$$

natural in $(X, x_0), (Y, y_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

00AG   5. *Symmetric Monoidality.*   The triple $(\mathsf{Sets}_*, \times, (\mathrm{pt}, \star))$ is a symmetric monoidal category.

Proof. Item 1, Functoriality: This is a special case of functoriality of limits, **??**, **??** of **??**.
Item 2, Associativity: This follows from Constructions With Sets, Item 3 of Proposition 1.3.1.2.
Item 3, Unitality: This follows from Constructions With Sets, Item 4 of Proposition 1.3.1.2.
Item 4, Commutativity: This follows from Constructions With Sets, Item 5 of Proposition 1.3.1.2.
Item 5, Symmetric Monoidality: This follows from Constructions With Sets, Item 12 of Proposition 1.3.1.2.   □

2.4   Pullbacks

Let (X, x_0), (Y, y_0), and (Z, z_0) be pointed sets and let $f \colon (X, x_0) \to (Z, z_0)$ and $g \colon (Y, y_0) \to (Z, z_0)$ be morphisms of pointed sets.

Definition 2.4.1.1. The **pullback of** (X, x_0) **and** (Y, y_0) **over** (Z, z_0) **along** (f, g) is the pair consisting of:

- *The Limit.* The pointed set $(X \times_Z Y, (x_0, y_0))$.

- *The Cone.* The morphisms of pointed sets

$$
\begin{aligned}
\mathrm{pr}_1 &\colon (X \times_Z Y, (x_0, y_0)) \to (X, x_0), \\
\mathrm{pr}_2 &\colon (X \times_Z Y, (x_0, y_0)) \to (Y, y_0)
\end{aligned}
$$

defined by

$$
\begin{aligned}
\mathrm{pr}_1(x, y) &\overset{\mathrm{def}}{=} x, \\
\mathrm{pr}_2(x, y) &\overset{\mathrm{def}}{=} y
\end{aligned}
$$

for each $(x, y) \in X \times_Z Y$.

Proof. We claim that $X \times_Z Y$ is the categorical pullback of (X, x_0) and (Y, y_0) over (Z, z_0) with respect to (f, g) in Sets_*. First we need to check that the relevant pullback diagram commutes, i.e. that we have
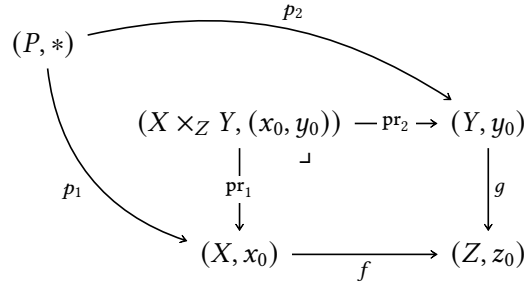
$$
f \circ \mathrm{pr}_1 = g \circ \mathrm{pr}_2,
\qquad
\begin{array}{ccc}
(X \times_Z Y, (x_0, y_0)) & \xrightarrow{\ \mathrm{pr}_2\ } & (Y, y_0) \\
{\scriptstyle \mathrm{pr}_1}\big\downarrow & & \big\downarrow{\scriptstyle g} \\
(X, x_0) & \xrightarrow[\ f\ ]{} & (Z, z_0).
\end{array}
$$

Indeed, given $(x, y) \in X \times_Z Y$, we have

$$
\begin{aligned}
\left[f \circ \mathrm{pr}_1 \right](x, y) &= f\big(\mathrm{pr}_1(x, y)\big) \\
&= f(x) \\
&= g(y) \\
&= g\big(\mathrm{pr}_2(x, y)\big) \\
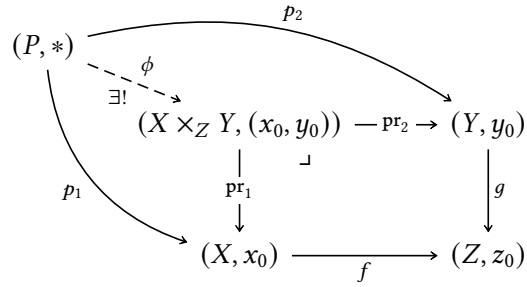&= \left[g \circ \mathrm{pr}_2 \right](x, y),
\end{aligned}
$$

where $f(x) = g(y)$ since $(x, y) \in X \times_Z Y$. Next, we prove that $X \times_Z Y$ satisfies the universal property of the pullback. Suppose we have a diagram of the form

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi \colon (P, *) \to (X \times_Z Y, (x_0, y_0))$$

making the diagram

commute, being uniquely determined by the conditions

$$\mathrm{pr}_1 \circ \phi = p_1,$$
$$\mathrm{pr}_2 \circ \phi = p_2$$

via

$$\phi(x) = (p_1(x), p_2(x))$$

for each $x \in P$, where we note that $(p_1(x), p_2(x)) \in X \times Y$ indeed lies in $X \times_Z Y$ by the condition

$$f \circ p_1 = g \circ p_2,$$

which gives

$$f(p_1(x)) = g(p_2(x))$$

for each $x \in P$, so that $(p_1(x), p_2(x)) \in X \times_Z Y$. Lastly, we note that ϕ is indeed a morphism of pointed sets, as we have

$$\phi(*) = (p_1(*), p_2(*))$$
$$= (x_0, y_0),$$

where we have used that p_1 and p_2 are morphisms of pointed sets.   □

00AK   **Proposition 2.4.1.2.** Let (X, x_0), (Y, y_0), (Z, z_0), and (A, a_0) be pointed sets.

00AL   1. *Functoriality.*   The assignment $(X, Y, Z, f, g) \mapsto X \times_{f,Z,g} Y$ defines a functor

$$-_1 \times_{-_3} -_1 \colon \mathsf{Fun}(\mathcal{P}, \mathsf{Sets}_*) \to \mathsf{Sets}_*,$$

where $\mathcal{P}$ is the category that looks like this:

In particular, the action on morphisms of $-_1 \times_{-_3} -_1$ is given by sending a morphism
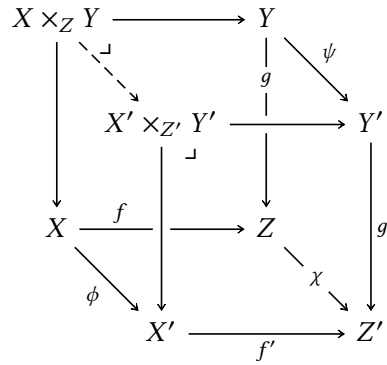

in $\mathsf{Fun}(\mathcal{P}, \mathsf{Sets}_*)$ to the morphism of pointed sets

$$\xi \colon (X \times_Z Y, (x_0, y_0)) \xrightarrow{\exists!} \left(X' \times_{Z'} Y', \left(x'_0, y'_0\right)\right)$$
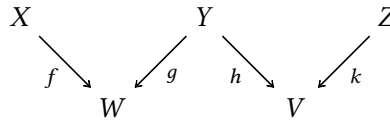
given by

$$\xi(x, y) \overset{\mathrm{def}}{=} (\phi(x), \psi(y))$$

for each $(x, y) \in X \times_Z Y$, which is the unique morphism of pointed sets making the diagram

$$
\begin{array}{c}
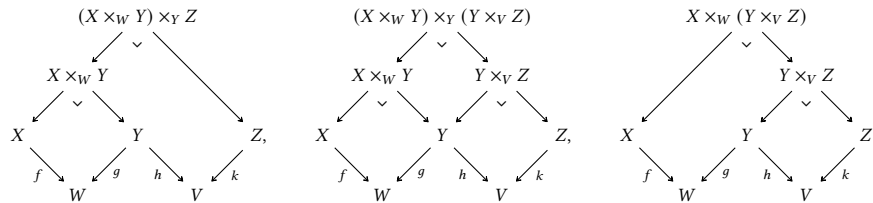\text{(commutative cube diagram with } X \times_Z Y \to Y, \\
X' \times_{Z'} Y' \to Y', \ X \xrightarrow{f} Z, \ X' \xrightarrow{f'} Z', \\
\text{maps } \psi, g, g', \phi, \chi)
\end{array}
$$

commute.

2. *Associativity.* Given a diagram

$$
X \quad\quad Y \quad\quad Z
$$
$$
f \searrow \quad \swarrow g \quad h \searrow \quad \swarrow k
$$
$$
W \quad\quad\quad V
$$

in Sets_*, we have isomorphisms of pointed sets

$$(X \times_W Y) \times_V Z \cong (X \times_W Y) \times_Y (Y \times_V Z) \cong X \times_W (Y \times_V Z),$$

where these pullbacks are built as in the diagrams

$$
\begin{array}{ccc}
(X \times_W Y) \times_Y Z & (X \times_W Y) \times_Y (Y \times_V Z) & X \times_W (Y \times_V Z) \\
X \times_W Y & X \times_W Y \quad Y \times_V Z & Y \times_V Z \\
X \quad Y \quad Z, & X \quad Y \quad Z, & X \quad Y \quad Z. \\
f \searrow \swarrow g \quad h \searrow \swarrow k & f \searrow \swarrow g \quad h \searrow \swarrow k & f \searrow \swarrow g \quad h \searrow \swarrow k \\
W \quad V & W \quad V & W \quad V
\end{array}
$$

3. *Unitality.* We have isomorphisms of pointed sets

$$
\begin{array}{ccc}
A === A & & A \xrightarrow{f} X \\
f\downarrow \quad\quad \downarrow f & X \times_X A \cong A, & \| \quad\quad \| \\
X === X & A \times_X X \cong A, & X \xrightarrow{f} X.
\end{array}
$$

00AP   4. *Commutativity.* We have an isomorphism of pointed sets

$$
\begin{array}{ccc}
A \times_X B & \longrightarrow & B \\
\downarrow & \lrcorner & \downarrow{\scriptstyle g} \\
A & \xrightarrow{\ f\ } & X,
\end{array}
\qquad
A \times_X B \cong B \times_X A
\qquad
\begin{array}{ccc}
B \times_X A & \longrightarrow & A \\
\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\
B & \xrightarrow{\ g\ } & X.
\end{array}
$$

00AQ   5. *Interaction With Products.* We have an isomorphism of pointed sets

$$
X \times_{\mathrm{pt}} Y \cong X \times Y,
\qquad
\begin{array}{ccc}
X \times Y & \longrightarrow & Y \\
\downarrow & \lrcorner & \downarrow{\scriptstyle !_Y} \\
X & \xrightarrow{\ !_X\ } & \mathrm{pt}.
\end{array}
$$

00AR   6. *Symmetric Monoidality.* The triple $(\mathsf{Sets}_*, \times_X, X)$ is a symmetric monoidal
          category.

Proof. *Item 1, Functoriality*: This is a special case of functoriality of co/limits, **??**,
?? of **??**, with the explicit expression for ξ following from the commutativity of
the cube pullback diagram.
Item 2, Associativity: This follows from Constructions With Sets, Item 2 of Propo-
sition 1.4.1.3.
Item 3, Unitality: This follows from Constructions With Sets, Item 3 of Proposi-
tion 1.4.1.3.
Item 4, Commutativity: This follows from Constructions With Sets, Item 4 of
Proposition 1.4.1.3.
Item 5, Interaction With Products: This follows from Constructions With Sets,
Item 6 of Proposition 1.4.1.3.
Item 6, Symmetric Monoidality: This follows from Constructions With Sets, Item 7
of Proposition 1.4.1.3.   □

00AS   **2.5   Equalisers**

Let $f, g\colon (X, x_0) \rightrightarrows (Y, y_0)$ be morphisms of pointed sets.

00AT   **Definition 2.5.1.1.** The **equaliser of** (f, g) is the pair consisting of:

- *The Limit.* The pointed set $\big(\mathrm{Eq}(f,g), x_0\big)$.

- *The Cone.* The morphism of pointed sets

$$\mathrm{eq}(f,g)\colon \big(\mathrm{Eq}(f,g), x_0\big) \hookrightarrow (X, x_0)$$

given by the canonical inclusion $\mathrm{eq}(f,g) \hookrightarrow \mathrm{Eq}(f,g) \hookrightarrow X$.

Proof. We claim that $\big(\mathrm{Eq}(f,g), x_0\big)$ is the categorical equaliser of f and g in Sets_*. First we need to check that the relevant equaliser diagram commutes, i.e. that we have

$$f \circ \mathrm{eq}(f,g) = g \circ \mathrm{eq}(f,g),$$

which indeed holds by the definition of the set $\mathrm{Eq}(f,g)$. Next, we prove that $\mathrm{Eq}(f,g)$ satisfies the universal property of the equaliser. Suppose we have a diagram of the form

$$\big(\mathrm{Eq}(f,g), x_0\big) \xrightarrow{\;\mathrm{eq}(f,g)\;} (X, x_0) \underset{g}{\overset{f}{\rightrightarrows}} (Y, y_0)$$

$$(E, *) \xrightarrow{\;\;e\;\;}$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi\colon (E, *) \to \big(\mathrm{Eq}(f,g), x_0\big)$$

making the diagram

$$\big(\mathrm{Eq}(f,g), x_0\big) \xrightarrow{\;\mathrm{eq}(f,g)\;} (X, x_0) \underset{g}{\overset{f}{\rightrightarrows}} (Y, y_0)$$

$$\phi \uparrow \exists! \qquad e$$

$$(E, *)$$

commute, being uniquely determined by the condition

$$\mathrm{eq}(f,g) \circ \phi = e$$

via

$$\phi(x) = e(x)$$

for each $x \in E$, where we note that $e(x) \in A$ indeed lies in $\mathrm{Eq}(f, g)$ by the condition
$$f \circ e = g \circ e,$$
which gives
$$f(e(x)) = g(e(x))$$
for each $x \in E$, so that $e(x) \in \mathrm{Eq}(f, g)$. Lastly, we note that ϕ is indeed a morphism of pointed sets, as we have
$$\phi(*) = e(*)$$
$$= x_0,$$

where we have used that e is a morphism of pointed sets. $\qquad\square$

00AU   **Proposition 2.5.1.2.** Let (X, x_0) and (Y, y_0) be pointed sets and let $f, g, h \colon (X, x_0) \to (Y, y_0)$ be morphisms of pointed sets.

00AV   1. *Associativity.* We have isomorphisms of pointed sets

$$\underbrace{\mathrm{Eq}\big(f \circ \mathrm{eq}(g, h), g \circ \mathrm{eq}(g, h)\big)}_{=\mathrm{Eq}(f\circ\mathrm{eq}(g,h),h\circ\mathrm{eq}(g,h))} \cong \mathrm{Eq}(f, g, h) \cong \underbrace{\mathrm{Eq}\big(f \circ \mathrm{eq}(f, g), h \circ \mathrm{eq}(f, g)\big)}_{=\mathrm{Eq}(g\circ\mathrm{eq}(f,g),h\circ\mathrm{eq}(f,g))},$$

where $\mathrm{Eq}(f, g, h)$ is the limit of the diagram

$$(X, x_0) \underset{h}{\overset{f}{\underset{\longrightarrow}{\overset{\longrightarrow}{\underset{g}{\longrightarrow}}}}} (Y, y_0)$$

in Sets_*, being explicitly given by

$$\mathrm{Eq}(f, g, h) \cong \{a \in A \mid f(a) = g(a) = h(a)\}.$$

00AW   2. *Unitality.* We have an isomorphism of pointed sets

$$\mathrm{Eq}(f, f) \cong X.$$

00AX   3. *Commutativity.* We have an isomorphism of pointed sets

$$\mathrm{Eq}(f, g) \cong \mathrm{Eq}(g, f).$$

Proof. *Item 1, Associativity*: This follows from Constructions With Sets, Item 1 of Proposition 1.5.1.2.

Item 2, Unitality: This follows from Constructions With Sets, Item 4 of Proposition 1.5.1.2.

Item 3, Commutativity: This follows from Constructions With Sets, Item 5 of Proposition 1.5.1.2. $\qquad\square$

00AY 3   Colimits of Pointed Sets

00AZ 3.1   The Initial Pointed Set

00B0 **Definition 3.1.1.1.** The **initial pointed set** is the pair $\left((\mathrm{pt}, \star), \{\iota_X\}_{(X,x_0)\in\mathrm{Obj}(\mathsf{Sets}_*)}\right)$ consisting of:

- *The Limit.* The pointed set $(\mathrm{pt}, \star)$.

- *The Cone.* The collection of morphisms of pointed sets

$$\{\iota_X \colon (\mathrm{pt}, \star) \to (X, x_0)\}_{(X,x_0)\in\mathrm{Obj}(\mathsf{Sets})}$$

  defined by

$$\iota_X(\star) \overset{\mathrm{def}}{=} x_0.$$

Proof. We claim that $(\mathrm{pt}, \star)$ is the initial object of Sets_*. Indeed, suppose we have a diagram of the form

$$(\mathrm{pt}, \star) \qquad (X, x_0)$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi \colon (\mathrm{pt}, \star) \to (X, x_0)$$

making the diagram

$$(\mathrm{pt}, \star) \;-\!\!\overset{\phi}{\underset{\exists!}{-}}\!\!\to\; (X, x_0)$$

commute, namely ι_X. $\qquad\square$

00B1 3.2   Coproducts of Families of Pointed Sets

Let $\left\{\left(X_i, x_0^i\right)\right\}_{i\in I}$ be a family of pointed sets.

00B2 **Definition 3.2.1.1.** The **coproduct of the family** $\left\{\left(X_i, x_0^i\right)\right\}_{i\in I}$, also called their **wedge sum**, is the pair consisting of:

- *The Colimit.* The pointed set $\left(\bigvee_{i\in I} X_i, p_0\right)$ consisting of:

– *The Underlying Set.* The set $\bigvee_{i \in I} X_i$ defined by

$$\bigvee_{i \in I} X_i \overset{\text{def}}{=} \left(\coprod_{i \in I} X_i \right) / \sim,$$

where $\sim$ is the equivalence relation on $\coprod_{i \in I} X_i$ given by declaring

$$\left(i, x_0^i \right) \sim \left(j, x_0^j \right)$$

for each $i, j \in I$.

– *The Basepoint.* The element p_0 of $\bigvee_{i \in I} X_i$ defined by

$$p_0 \overset{\text{def}}{=} \left[\left(i, x_0^i \right) \right]$$
$$= \left[\left(j, x_0^j \right) \right]$$

for any $i, j \in I$.

• *The Cocone.* The collection

$$\left\{ \mathrm{inj}_i \colon \left(X_i, x_0^i \right) \to \left(\bigvee_{i \in I} X_i, p_0 \right) \right\}_{i \in I}$$

of morphism of pointed sets given by

$$\mathrm{inj}_i(x) \overset{\text{def}}{=} (i, x)$$

for each $x \in X_i$ and each $i \in I$.

Proof. We claim that $\left(\bigvee_{i \in I} X_i, p_0 \right)$ is the categorical coproduct of $\left\{ \left(X_i, x_0^i \right) \right\}_{i \in I}$ in Sets_*. Indeed, suppose we have, for each $i \in I$, a diagram of the form

$$
\begin{array}{c}
(C, *) \\
\underset{\iota_i}{\nearrow} \\
\left(X_i, x_0^i \right) \xrightarrow[\mathrm{inj}_i]{} \left(\bigvee_{i \in I} X_i, p_0 \right)
\end{array}
$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi \colon \left(\bigvee_{i \in I} X_i, p_0 \right) \to (C, *)$$

making the diagram

$$
\begin{array}{ccc}
 & & (C, *) \\
 & {}^{\iota_i}\nearrow & {}_\phi\uparrow\vert\exists! \\
(X_i, x_0^i) & \xrightarrow[\mathrm{inj}_i]{} & \left(\displaystyle\bigvee_{i\in I} X_i, p_0\right)
\end{array}
$$

commute, being uniquely determined by the condition $\phi \circ \mathrm{inj}_i = \iota_i$ for each $i \in I$ via

$$
\phi([(i, x)]) = \iota_i(x)
$$

for each $[(i, x)] \in \bigvee_{i\in I} X_i$, where we note that ϕ is indeed a morphism of pointed sets, as we have

$$
\phi(p_0) = \iota_i\big(\big[(i, x_0^i)\big]\big)
$$
$$
= *,
$$

as ι_i is a morphism of pointed sets. $\qquad\square$

00B3 **Proposition 3.2.1.2.** Let $\big\{(X_i, x_0^i)\big\}_{i\in I}$ be a family of pointed sets.

00B4   1. *Functoriality.* The assignment $\big\{(X_i, x_0^i)\big\}_{i\in I} \mapsto (\bigvee_{i\in I} X_i, p_0)$ defines a functor

$$
\bigvee_{i\in I} : \mathsf{Fun}(I_{\mathrm{disc}}, \mathsf{Sets}_*) \to \mathsf{Sets}_*.
$$

Proof. Item 1, Functoriality: This follows from **??, ??** of **??**. $\qquad\square$

00B5 ### 3.3 Coproducts

Let (X, x_0) and (Y, y_0) be pointed sets.

00B6 **Definition 3.3.1.1.** The **coproduct of** (X, x_0) **and** (Y, y_0), also called their **wedge sum**, is the pair consisting of:

- *The Colimit.* The pointed set $(X \vee Y, p_0)$ consisting of:

– *The Underlying Set.* The set $X \vee Y$ defined by

$$(X \vee Y, p_0) \overset{\text{def}}{=} (X, x_0) \coprod (Y, y_0)$$
$$\cong \left(X \coprod_{\text{pt}} Y, p_0 \right)$$
$$\cong (X \coprod Y/\!\sim, p_0),$$

$$\begin{array}{ccc} X \vee Y & \longleftarrow & Y \\ \big\uparrow & \ulcorner & \big\uparrow {\scriptstyle [y_0]} \\ X & \underset{[x_0]}{\longleftarrow} & \text{pt,} \end{array}$$

where $\sim$ is the equivalence relation on $X \coprod Y$ obtained by declaring $(0, x_0) \sim (1, y_0)$.

– *The Basepoint.* The element p_0 of $X \vee Y$ defined by

$$p_0 \overset{\text{def}}{=} [(0, x_0)]$$
$$= [(1, y_0)].$$

• *The Cocone.* The morphisms of pointed sets

$$\text{inj}_1 \colon (X, x_0) \longrightarrow (X \vee Y, p_0),$$
$$\text{inj}_2 \colon (Y, y_0) \longrightarrow (X \vee Y, p_0),$$

given by

$$\text{inj}_1(x) \overset{\text{def}}{=} [(0, x)],$$
$$\text{inj}_2(y) \overset{\text{def}}{=} [(1, y)],$$

for each $x \in X$ and each $y \in Y$.

Proof. We claim that $(X \vee Y, p_0)$ is the categorical coproduct of (X, x_0) and (Y, y_0) in Sets_*. Indeed, suppose we have a diagram of the form

$$\begin{array}{ccc} & (C, *) & \\ {\scriptstyle \iota_X} \nearrow & & \nwarrow {\scriptstyle \iota_Y} \\ (X, x_0) \xrightarrow[\text{inj}_X]{} & (X \vee Y, p_0) & \xleftarrow[\text{inj}_Y]{} (Y, y_0) \end{array}$$

in Sets. Then there exists a unique morphism of pointed sets

$$\phi \colon (X \vee Y, p_0) \longrightarrow (C, *)$$

making the diagram

$$
\begin{array}{ccc}
 & (C, *) & \\
{}_{\iota_X}\nearrow & {}^{\phi \uparrow \exists!} & \nwarrow {}_{\iota_Y} \\
(X, x_0) \xrightarrow[\text{inj}_X]{} & (X \vee Y, p_0) & \xleftarrow[\text{inj}_Y]{} (Y, y_0)
\end{array}
$$

commute, being uniquely determined by the conditions

$$
\phi \circ \text{inj}_X = \iota_X,
$$
$$
\phi \circ \text{inj}_Y = \iota_Y
$$

via

$$
\phi(z) = \begin{cases} \iota_X(x) & \text{if } z = [(0, x)] \text{ with } x \in X, \\ \iota_Y(y) & \text{if } z = [(1, y)] \text{ with } y \in Y \end{cases}
$$

for each $z \in X \vee Y$, where we note that ϕ is indeed a morphism of pointed sets, as we have

$$
\begin{aligned}
\phi(p_0) &= \iota_X([(0, x_0)]) \\
&= \iota_Y([(1, y_0)]) \\
&= *,
\end{aligned}
$$

as ι_X and ι_Y are morphisms of pointed sets. $\square$

00B7   **Proposition 3.3.1.2.** Let (X, x_0) and (Y, y_0) be pointed sets.

00B8   1. *Functoriality.* The assignments

$$
(X, x_0), (Y, y_0), ((X, x_0), (Y, y_0)) \mapsto (X \vee Y, p_0)
$$

    define functors

$$
\begin{aligned}
X \vee -&: \mathsf{Sets}_* \to \mathsf{Sets}_*, \\
- \vee Y&: \mathsf{Sets}_* \to \mathsf{Sets}_*, \\
-_1 \vee -_2&: \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*.
\end{aligned}
$$

00B9   2. *Associativity.* We have an isomorphism of pointed sets

$$
(X \vee Y) \vee Z \cong X \vee (Y \vee Z),
$$

    natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathsf{Sets}_*$.

00BA   3. *Unitality.* We have isomorphisms of pointed sets

$$(\mathrm{pt}, *) \vee (X, x_0) \cong (X, x_0),$$
$$(X, x_0) \vee (\mathrm{pt}, *) \cong (X, x_0),$$

natural in $(X, x_0) \in \mathsf{Sets}_*$.

00BB   4. *Commutativity.* We have an isomorphism of pointed sets

$$X \vee Y \cong Y \vee X,$$

natural in $(X, x_0), (Y, y_0) \in \mathsf{Sets}_*$.

00BC   5. *Symmetric Monoidality.* The triple $(\mathsf{Sets}_*, \vee, \mathrm{pt})$ is a symmetric monoidal category.

00BD   6. *The Fold Map.* We have a natural transformation

$$\nabla \colon \vee \circ \Delta^{\mathsf{Cats}}_{\mathsf{Sets}_*} \Longrightarrow \mathrm{id}_{\mathsf{Sets}_*},$$

called the **fold map**, whose component

$$\nabla_X \colon X \vee X \to X$$

at X is given by

$$\nabla_X(p) \overset{\mathrm{def}}{=} \begin{cases} x & \text{if } p = [(0, x)], \\ x & \text{if } p = [(1, x)] \end{cases}$$

for each $p \in X \vee X$.

Proof. *Item 1, Functoriality*: This follows from **??**, **??** of **??**.
Item 2, Associativity: Clear.
Item 3, Unitality: Clear.
Item 4, Commutativity: Clear.
Item 5, Symmetric Monoidality: Omitted.

Item 6, The Fold Map: Naturality for the transformation ∇ is the statement that, given a morphism of pointed sets $f\colon (X, x_0) \to (Y, y_0)$, we have

$$\nabla_Y \circ (f \vee f) = f \circ \nabla_X,$$

$$
\begin{array}{ccc}
X \vee X & \xrightarrow{\nabla_X} & X \\
{\scriptstyle f \vee f}\downarrow & & \downarrow{\scriptstyle f} \\
Y \vee Y & \xrightarrow[\nabla_Y]{} & Y.
\end{array}
$$

Indeed, we have

$$
\begin{aligned}
[\nabla_Y \circ (f \vee f)]([(i,x)]) &= \nabla_Y([(i, f(x))]) \\
&= f(x) \\
&= f(\nabla_X([(i,x)])) \\
&= [f \circ \nabla_X]([(i,x)])
\end{aligned}
$$

for each $[(i,x)] \in X \vee X$, and thus ∇ is indeed a natural transformation. $\square$

00BE 3.4   Pushouts

Let (X, x_0), (Y, y_0), and (Z, z_0) be pointed sets and let $f\colon (Z, z_0) \to (X, x_0)$ and $g\colon (Z, z_0) \to (Y, y_0)$ be morphisms of pointed sets.

00BF **Definition 3.4.1.1.** The **pushout of** (X, x_0) **and** (Y, y_0) **over** (Z, z_0) **along** (f, g) is the pair consisting of:

- *The Colimit.* The pointed set $\left(X \coprod_{f,Z,g} Y, p_0\right)$, where:

  - The set $X \coprod_{f,Z,g} Y$ is the pushout (of unpointed sets) of X and Y over Z with respect to f and g;
  - We have $p_0 = [x_0] = [y_0]$.

- *The Cocone.* The morphisms of pointed sets

$$
\begin{aligned}
\mathrm{inj}_1 &\colon (X, x_0) \to (X \coprod_Z Y, p_0), \\
\mathrm{inj}_2 &\colon (Y, y_0) \to (X \coprod_Z Y, p_0)
\end{aligned}
$$

given by

$$
\begin{aligned}
\mathrm{inj}_1(x) &\overset{\text{def}}{=} [(0, x)] \\
\mathrm{inj}_2(y) &\overset{\text{def}}{=} [(1, y)]
\end{aligned}
$$

for each $x \in X$ and each $y \in Y$.

Proof. Firstly, we note that indeed $[x_0] = [y_0]$, as we have

$$x_0 = f(z_0),$$
$$y_0 = g(z_0)$$

since f and g are morphisms of pointed sets, with the relation $\sim$ on $X \coprod_Z Y$ then identifying $x_0 = f(z_0) \sim g(z_0) = y_0$.

We now claim that $(X \coprod_Z Y, p_0)$ is the categorical pushout of (X, x_0) and (Y, y_0) over (Z, z_0) with respect to (f, g) in Sets_*. First we need to check that the relevant pushout diagram commutes, i.e. that we have

$$\mathrm{inj}_1 \circ f = \mathrm{inj}_2 \circ g,$$

alongside the diagram:

$$
\begin{array}{ccc}
(X \coprod_Z Y, p_0) & \xleftarrow{\ \mathrm{inj}_2\ } & (Y, y_0) \\
\uparrow{\scriptstyle \mathrm{inj}_1} & & \uparrow{\scriptstyle g} \\
(X, x_0) & \xleftarrow{\ f\ } & (Z, z_0).
\end{array}
$$

Indeed, given $z \in Z$, we have

$$
\begin{aligned}
\big[\mathrm{inj}_1 \circ f\big](z) &= \mathrm{inj}_1(f(z)) \\
&= [(0, f(z))] \\
&= [(1, g(z))] \\
&= \mathrm{inj}_2(g(z)) \\
&= \big[\mathrm{inj}_2 \circ g\big](z),
\end{aligned}
$$

where $[(0, f(z))] = [(1, g(z))]$ by the definition of the relation $\sim$ on $X \coprod Y$ (the coproduct of unpointed sets of X and Y). Next, we prove that $X \coprod_Z Y$ satisfies the universal property of the pushout. Suppose we have a diagram of the form

$$
\begin{array}{ccc}
(P, *) & & \\
\uparrow{\scriptstyle \iota_1} \quad \nwarrow{\scriptstyle \iota_2} & & \\
(X \coprod_Z Y, p_0) & \xleftarrow{\ \mathrm{inj}_2\ } & (Y, y_0) \\
\uparrow{\scriptstyle \mathrm{inj}_1} & & \uparrow{\scriptstyle g} \\
(X, x_0) & \xleftarrow{\ f\ } & (Z, z_0)
\end{array}
$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi \colon (X \coprod_Z Y, p_0) \to (P, *)$$

making the diagram

commute, being uniquely determined by the conditions

$$\phi \circ \mathrm{inj}_1 = \iota_1,$$
$$\phi \circ \mathrm{inj}_2 = \iota_2$$

via

$$\phi(p) = \begin{cases} \iota_1(x) & \text{if } x = [(0, x)], \\ \iota_2(y) & \text{if } x = [(1, y)] \end{cases}$$

for each $p \in X \coprod_Z Y$, where the well-definedness of ϕ is proven in the same way as in the proof of Constructions With Sets, Definition 2.4.1.1. Finally, we show that ϕ is indeed a morphism of pointed sets, as we have

$$\begin{aligned} \phi(p_0) &= \phi([(0, x_0)]) \\ &= \iota_1(x_0) \\ &= *, \end{aligned}$$

or alternatively

$$\begin{aligned} \phi(p_0) &= \phi([(1, y_0)]) \\ &= \iota_2(y_0) \\ &= *, \end{aligned}$$
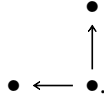
where we use that ι_1 (resp. ι_2) is a morphism of pointed sets. $\square$

00BG   **Proposition 3.4.1.2.** Let (X, x_0), (Y, y_0), (Z, z_0), and (A, a_0) be pointed sets.
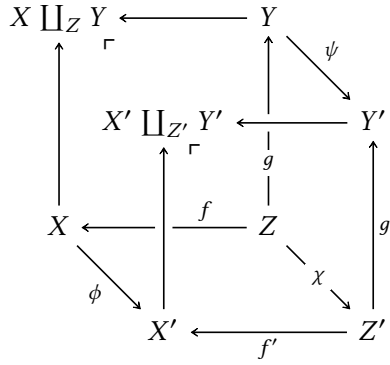
00BH   1. *Functoriality.* The assignment $(X, Y, Z, f, g) \mapsto X \coprod_{f,Z,g} Y$ defines a functor

$$-_1 \coprod\nolimits_{-_3} -_1 \colon \mathsf{Fun}(\mathcal{P}, \mathsf{Sets}) \to \mathsf{Sets}_*,$$

where $\mathcal{P}$ is the category that looks like this:

In particular, the action on morphisms of $-_1 \coprod_{-_3} -_1$ is given by sending a morphism

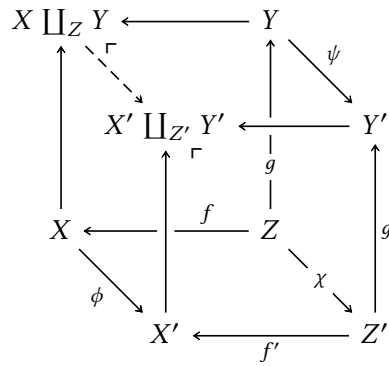in $\mathsf{Fun}(\mathcal{P}, \mathsf{Sets}_*)$ to the morphism of pointed sets

$$\xi \colon (X \coprod\nolimits_Z Y, p_0) \xrightarrow{\exists!} (X' \coprod\nolimits_{Z'} Y', p_0')$$

given by

$$\xi(p) \overset{\mathrm{def}}{=} \begin{cases} \phi(x) & \text{if } p = [(0, x)], \\ \psi(y) & \text{if } p = [(1, y)] \end{cases}$$

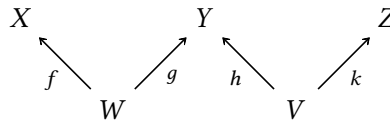for each $p \in X \coprod_Z Y$, which is the unique morphism of pointed sets
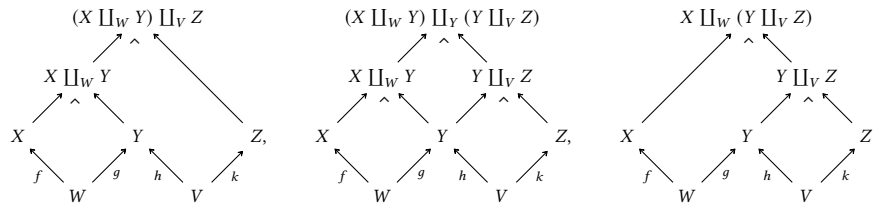
making the diagram

commute.

00BJ   2. *Associativity.* Given a diagram

in Sets, we have isomorphisms of pointed sets

$$(X \coprod_W Y) \coprod_V Z \cong (X \coprod_W Y) \coprod_Y (Y \coprod_V Z) \cong X \coprod_W (Y \coprod_V Z),$$

where these pullbacks are built as in the diagrams

00BK   3. *Unitality.* We have isomorphisms of sets

$$X \coprod_X A \cong A,$$
$$A \coprod_X X \cong A,$$

00BL 4. *Commutativity.* We have an isomorphism of sets

$$
\begin{array}{ccc}
X \coprod_Z Y & \longleftarrow & Y \\
\uparrow & \ulcorner & \uparrow g \\
X & \longleftarrow & Z, \\
& f &
\end{array}
\qquad
X \coprod_Z Y \cong Y \coprod_Z X
\qquad
\begin{array}{ccc}
Y \coprod_Z X & \longleftarrow & X \\
\uparrow & \ulcorner & \uparrow f \\
Y & \longleftarrow & Z. \\
& g &
\end{array}
$$

00BM 5. *Interaction With Coproducts.* We have

$$
X \coprod_{\mathrm{pt}} Y \cong X \vee Y,
\qquad
\begin{array}{ccc}
X \vee Y & \longleftarrow & Y \\
\uparrow & \ulcorner & \uparrow [y_0] \\
X & \longleftarrow & \mathrm{pt}. \\
& [x_0] &
\end{array}
$$

00BN 6. *Symmetric Monoidality.* The triple $(\mathsf{Sets}_*, \coprod_X, (X, x_0))$ is a symmetric monoidal category.

Proof. *Item 1, Functoriality*: This is a special case of functoriality of co/limits, **??**, **??** of **??**, with the explicit expression for ξ following from the commutativity of the cube pushout diagram.

Item 2, Associativity: This follows from Constructions With Sets, Item 2 of Proposition 2.4.1.4.

Item 3, Unitality: This follows from Constructions With Sets, Item 3 of Proposition 2.4.1.4.

Item 4, Commutativity: This follows from Constructions With Sets, Item 4 of Proposition 2.4.1.4.

Item 5, Interaction With Coproducts: Clear.

Item 6, Symmetric Monoidality: Omitted. □

00BP ## 3.5 Coequalisers

Let $f, g \colon (X, x_0) \rightrightarrows (Y, y_0)$ be morphisms of pointed sets.

00BQ **Definition 3.5.1.1.** The **coequaliser of** (f, g) is the pointed set $\big(\mathrm{CoEq}(f, g), [y_0]\big)$.

Proof. We claim that $\big(\mathrm{CoEq}(f, g), [y_0]\big)$ is the categorical coequaliser of f and g

in Sets_*. First we need to check that the relevant coequaliser diagram commutes, i.e. that we have

$$\mathrm{coeq}(f,g) \circ f = \mathrm{coeq}(f,g) \circ g.$$

Indeed, we have

$$
\begin{aligned}
\big[\mathrm{coeq}(f,g) \circ f\big](x) &\overset{\text{def}}{=} \big[\mathrm{coeq}(f,g)\big](f(x)) \\
&\overset{\text{def}}{=} [f(x)] \\
&= [g(x)] \\
&\overset{\text{def}}{=} \big[\mathrm{coeq}(f,g)\big](g(x)) \\
&\overset{\text{def}}{=} \big[\mathrm{coeq}(f,g) \circ g\big](x)
\end{aligned}
$$

for each $x \in X$. Next, we prove that $\mathrm{CoEq}(f,g)$ satisfies the universal property of the coequaliser. Suppose we have a diagram of the form

$$(X,x_0) \overset{f}{\underset{g}{\rightrightarrows}} (Y,y_0) \xrightarrow{\mathrm{coeq}(f,g)} \big(\mathrm{CoEq}(f,g), [y_0]\big)$$
$$\searrow{}^{c}$$
$$(C,*)$$

in Sets. Then, since $c(f(a)) = c(g(a))$ for each $a \in A$, it follows from Equivalence Relations and Apartness Relations, Items 4 and 5 of Proposition 5.2.1.3 that there exists a unique map $\phi \colon \mathrm{CoEq}(f,g) \xrightarrow{\exists!} C$ making the diagram

$$(X,x_0) \overset{f}{\underset{g}{\rightrightarrows}} (Y,y_0) \xrightarrow{\mathrm{coeq}(f,g)} \big(\mathrm{CoEq}(f,g), [y_0]\big)$$
$$\searrow{}^{c} \qquad \phi \downarrow \exists!$$
$$(C,*)$$

commute, where we note that ϕ is indeed a morphism of pointed sets since

$$
\begin{aligned}
\phi([y_0]) &= \big[\phi \circ \mathrm{coeq}(f,g)\big]([y_0]) \\
&= c([y_0]) \\
&= *,
\end{aligned}
$$

where we have used that c is a morphism of pointed sets.   □

00BR **Proposition 3.5.1.2.** Let (X, x_0) and (Y, y_0) be pointed sets and let $f, g, h \colon (X, x_0) \to (Y, y_0)$ be morphisms of pointed sets.

00BS   1. *Associativity.* We have isomorphisms of pointed sets

$$\underbrace{\mathrm{CoEq}(\mathrm{coeq}(f,g) \circ f, \mathrm{coeq}(f,g) \circ h)}_{=\mathrm{CoEq}(\mathrm{coeq}(f,g)\circ g,\mathrm{coeq}(f,g)\circ h)} \cong \mathrm{CoEq}(f,g,h) \cong \underbrace{\mathrm{CoEq}(\mathrm{coeq}(g,h) \circ f, \mathrm{coeq}(g,h) \circ g)}_{=\mathrm{CoEq}(\mathrm{coeq}(g,h)\circ f,\mathrm{coeq}(g,h)\circ h)},$$

where $\mathrm{CoEq}(f, g, h)$ is the colimit of the diagram

$$(X, x_0) \; \overset{f}{\underset{h}{\overset{}{\rightrightarrows}}} \; (Y, y_0)$$

in Sets_*.

00BT   2. *Unitality.* We have an isomorphism of pointed sets

$$\mathrm{CoEq}(f, f) \cong B.$$

00BU   3. *Commutativity.* We have an isomorphism of pointed sets

$$\mathrm{CoEq}(f, g) \cong \mathrm{CoEq}(g, f).$$

Proof. *Item 1, Associativity*: This follows from Constructions With Sets, Item 1 of Proposition 2.5.1.4.
Item 2, Unitality: This follows from Constructions With Sets, Item 4 of Proposition 2.5.1.4.
Item 3, Commutativity: This follows from Constructions With Sets, Item 5 of Proposition 2.5.1.4. □

00BV # 4   Constructions With Pointed Sets

00BW ## 4.1   Free Pointed Sets

Let X be a set.

00BX **Definition 4.1.1.1.** The **free pointed set on** X is the pointed set X^+ consisting of:

- *The Underlying Set.* The set X^+ defined by[11]

$$X^+ \overset{\text{def}}{=} X \coprod \mathrm{pt}$$
$$\overset{\text{def}}{=} X \coprod \{\star\}.$$

- *The Basepoint.* The element $\star$ of X^+.

00BY   **Proposition 4.1.1.2.** Let X be a set.

00BZ   1. *Functoriality.* The assignment $X \mapsto X^+$ defines a functor

$$(-)^+\colon \mathsf{Sets} \to \mathsf{Sets}_*,$$

where

- *Action on Objects.* For each $X \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$\left[(-)^+\right](X) \overset{\text{def}}{=} X^+,$$

where X^+ is the pointed set of Definition 4.1.1.1;
- *Action on Morphisms.* For each morphism $f\colon X \to Y$ of Sets, the image

$$f^+\colon X^+ \to Y^+$$

of f by $(-)^+$ is the map of pointed sets defined by

$$f^+(x) \overset{\text{def}}{=} \begin{cases} f(x) & \text{if } x \in X, \\ \star_Y & \text{if } x = \star_X. \end{cases}$$

00C0   2. *Adjointness.* We have an adjunction

$$\left((-)^+ \dashv \text{忘}\right)\colon \quad \mathsf{Sets} \underset{\text{忘}}{\overset{(-)^+}{\rightleftarrows}} \mathsf{Sets}_*,$$

witnessed by a bijection of sets

$$\mathsf{Sets}_*\left((X^+, \star_X), (Y, y_0)\right) \cong \mathsf{Sets}(X, Y),$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets})$ and $(Y, y_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

---

[11]*Further Notation:* We sometimes write $\star_X$ for the basepoint of X^+ for clarity when there are

00C1   3. *Symmetric Strong Monoidality With Respect to Wedge Sums.*   The free
pointed set functor of Item 1 has a symmetric strong monoidal structure

$$\left((-)^+, (-)^{+,\coprod}, (-)^{+,\coprod}_{\mathbb{1}}\right): (\mathsf{Sets}, \coprod, \emptyset) \to (\mathsf{Sets}_*, \vee, \mathrm{pt}),$$

being equipped with isomorphisms of pointed sets

$$(-)^{+,\coprod}_{X,Y}: X^+ \vee Y^+ \xrightarrow{\cong} (X \coprod Y)^+,$$
$$(-)^{+,\coprod}_{\mathbb{1}}: \mathrm{pt} \xrightarrow{\cong} \emptyset^+,$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

00C2   4. *Symmetric Strong Monoidality With Respect to Smash Products.* The free
pointed set functor of Item 1 has a symmetric strong monoidal structure

$$\left((-)^+, (-)^{+,\times}, (-)^{+,\times}_{\mathbb{1}}\right): (\mathsf{Sets}, \times, \mathrm{pt}) \to \left(\mathsf{Sets}_*, \wedge, S^0\right),$$

being equipped with isomorphisms of pointed sets

$$(-)^{+,\times}_{X,Y}: X^+ \wedge Y^+ \xrightarrow{\cong} (X \times Y)^+,$$
$$(-)^{+,\times}_{\mathbb{1}}: S^0 \xrightarrow{\cong} \mathrm{pt}^+,$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

Proof. Item 1, Functoriality: Clear.
Item 2, Adjointness: We claim there's an adjunction $(-)^+ \dashv \text{忘}$, witnessed by a
bijection of sets

$$\mathsf{Sets}_*\left(\left(X^+, \star_X\right), (Y, y_0)\right) \cong \mathsf{Sets}(X, Y),$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets})$ and $(Y, y_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

- *Map I.* We define a map

$$\Phi_{X,Y}: \mathsf{Sets}_*\left(\left(X^+, \star_X\right), (Y, y_0)\right) \to \mathsf{Sets}(X, Y)$$

by sending a pointed function

$$\xi: \left(X^+, \star_X\right) \to (Y, y_0)$$

to the function

$$\xi^\dagger \colon X \to Y$$

given by

$$\xi^\dagger(x) \overset{\text{def}}{=} \xi(x)$$

for each $x \in X$.

- *Map II.* We define a map

$$\Psi_{X,Y} \colon \mathsf{Sets}(X, Y) \to \mathsf{Sets}_*\big((X^+, \star_X), (Y, y_0)\big)$$

given by sending a function $\xi \colon X \to Y$ to the pointed function

$$\xi^\dagger \colon \big(X^+, \star_X\big) \to (Y, y_0)$$

defined by

$$\xi^\dagger(x) \overset{\text{def}}{=} \begin{cases} \xi(x) & \text{if } x \in X, \\ y_0 & \text{if } x = \star_X \end{cases}$$

for each $x \in X^+$.

- *Invertibility I.* We claim that

$$\Psi_{X,Y} \circ \Phi_{X,Y} = \mathrm{id}_{\mathsf{Sets}_*((X^+, \star_X), (Y, y_0))},$$

which is clear.

- *Invertibility II.* We claim that

$$\Phi_{X,Y} \circ \Psi_{X,Y} = \mathrm{id}_{\mathsf{Sets}(X,Y)},$$

which is clear.

- *Naturality for Φ, Part I.* We need to show that, given a pointed function $g \colon (Y, y_0) \to \big(Y', y_0'\big)$, the diagram

$$
\begin{array}{ccc}
\mathsf{Sets}_*((X^+, \star_X), (Y, y_0)) & \xrightarrow{\Phi_{X,Y}} & \mathsf{Sets}(X, Y) \\
\Big\downarrow{\scriptstyle g_*} & & \Big\downarrow{\scriptstyle g_*} \\
\mathsf{Sets}_*\big((X^+, \star_X), \big(Y', y_0'\big)\big), & \xrightarrow[\Phi_{X,Y'}]{} & \mathsf{Sets}(X, Y')
\end{array}
$$

commutes. Indeed, given a pointed function

$$\xi^{\dagger}\colon \left(X^{+}, \star_X\right) \to (Y, y_0)$$

we have

$$
\begin{aligned}
\left[\Phi_{X,Y'} \circ g_*\right](\xi) &= \Phi_{X,Y'}(g_*(\xi)) \\
&= \Phi_{X,Y'}(g \circ \xi) \\
&= g \circ \xi \\
&= g \circ \Phi_{X,Y'}(\xi) \\
&= g_*\big(\Phi_{X,Y'}(\xi)\big) \\
&= \left[g_* \circ \Phi_{X,Y'}\right](\xi).
\end{aligned}
$$

- *Naturality for Φ, Part II.* We need to show that, given a pointed function $f\colon (X, x_0) \to \left(X', x_0'\right)$, the diagram

$$
\begin{array}{ccc}
\mathsf{Sets}_*\!\left(\left(X'^{,+}, \star_X\right), (Y, y_0)\right) & \xrightarrow{\;\Phi_{X',Y}\;} & \mathsf{Sets}(X', Y) \\
{\scriptstyle f^*}\Big\downarrow & & \Big\downarrow{\scriptstyle f^*} \\
\mathsf{Sets}_*((X^{+}, \star_X), (Y, y_0)) & \xrightarrow[\;\Phi_{X,Y}\;]{} & \mathsf{Sets}(X, Y)
\end{array}
$$

commutes. Indeed, given a function

$$\xi\colon X' \to Y,$$

we have

$$
\begin{aligned}
\left[\Phi_{X,Y} \circ f^*\right](\xi) &= \Phi_{X,Y}(f^*(\xi)) \\
&= \Phi_{X,Y}(\xi \circ f) \\
&= \xi \circ f \\
&= \Phi_{X',Y}(\xi) \circ f \\
&= f^*\big(\Phi_{X',Y}(\xi)\big) \\
&= f^*\big(\Phi_{X',Y}(\xi)\big) \\
&= \left[f^* \circ \Phi_{X',Y}\right](\xi).
\end{aligned}
$$

- *Naturality for* Ψ. Since Φ is natural in each argument and Φ is a componen-
  twise inverse to Ψ in each argument, it follows from Categories, Item 2 of
  Proposition 8.6.1.2 that Ψ is also natural in each argument.

Item 3, Symmetric Strong Monoidality With Respect to Wedge Sums: The isomor-
phism

$$\phi \colon X^+ \vee Y^+ \xrightarrow{\cong} (X \amalg Y)^+$$

is given by

$$\phi(z) = \begin{cases} x & \text{if } z = [(0, x)] \text{ with } x \in X, \\ y & \text{if } z = [(1, y)] \text{ with } y \in Y, \\ \star_{X \amalg Y} & \text{if } z = [(0, \star_X)], \\ \star_{X \amalg Y} & \text{if } z = [(1, \star_Y)] \end{cases}$$

for each $z \in X^+ \vee Y^+$, with inverse

$$\phi^{-1} \colon (X \amalg Y)^+ \xrightarrow{\cong} X^+ \vee Y^+$$

given by

$$\phi^{-1}(z) \overset{\text{def}}{=} \begin{cases} [(0, x)] & \text{if } z = [(0, x)], \\ [(0, y)] & \text{if } z = [(1, y)], \\ p_0 & \text{if } z = \star_{X \amalg Y} \end{cases}$$

for each $z \in (X \amalg Y)^+$.

Meanwhile, the isomorphism $\text{pt} \cong \emptyset^+$ is given by sending $\star_X$ to $\star_\emptyset$.

That these isomorphisms satisfy the coherence conditions making the functor
$(-)^+$ symmetric strong monoidal can be directly checked element by element.

Item 4, Symmetric Strong Monoidality With Respect to Smash Products: The iso-
morphism

$$\phi \colon X^+ \wedge Y^+ \xrightarrow{\cong} (X \times Y)^+$$

is given by

$$\phi(x \wedge y) = \begin{cases} (x, y) & \text{if } x \neq \star_X \text{ and } y \neq \star_Y \\ \star_{X \times Y} & \text{otherwise} \end{cases}$$

for each $x \wedge y \in X^+ \wedge Y^+$, with inverse

$$\phi^{-1} \colon (X \times Y)^+ \xrightarrow{\cong} X^+ \wedge Y^+$$

given by

$$\phi^{-1}(z) \overset{\text{def}}{=} \begin{cases} x \wedge y & \text{if } z = (x, y) \text{ with } (x, y) \in X \times Y, \\ \star_X \wedge \star_Y & \text{if } z = \star_{X \times Y}, \end{cases}$$

for each $z \in (X \coprod Y)^+$.

Meanwhile, the isomorphism $S^0 \cong \text{pt}^+$ is given by sending $\star$ to $1 \in S^0 = \{0, 1\}$ and $\star_{\text{pt}}$ to $0 \in S^0$.

That these isomorphisms satisfy the coherence conditions making the functor $(-)^+$ symmetric strong monoidal can be directly checked element by element.

□

Appendices

A Other Chapters

References

[MSE 2855868]   Qiaochu Yuan. *Is the category of pointed sets Cartesian closed?* Mathematics Stack Exchange. URL: https://math.stackexchange.com/q/2855868 (cit. on p. 6).

multiple free pointed sets involved in the current discussion.

[MSE 884460]   Martin Brandenburg. *Why are the category of pointed sets and the category of sets and partial functions "essentially the same"?* Mathematics Stack Exchange. URL: https://math.stackexchange.com/q/884460 (cit. on p. 7).