

Constructions With Relations

The Clowder Project Authors

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This chapter contains some material about constructions with relations. Notably, we discuss and explore:

1. The existence or non-existence of Kan extensions and Kan lifts in the 2-category **Rel** (Section 2).
2. The various kinds of constructions involving relations, such as graphs, domains, ranges, unions, intersections, products, inverse relations, composition of relations, and collages (Section 3).
3. The adjoint pairs

$$\begin{aligned} R_* \dashv R_{-1} : \mathcal{P}(A) &\rightleftarrows \mathcal{P}(B), \\ R^{-1} \dashv R_! : \mathcal{P}(B) &\rightleftarrows \mathcal{P}(A) \end{aligned}$$

of functors (morphisms of posets) between $\mathcal{P}(A)$ and $\mathcal{P}(B)$ induced by a relation $R : A \rightarrowtail B$, as well as the properties of R_* , R_{-1} , R^{-1} , and $R_!$ (Section 4).

Of particular note are the following points:

- (a) These two pairs of adjoint functors are the counterpart for relations of the adjoint triple $f_* \dashv f^{-1} \dashv f_!$ induced by a function $f : A \rightarrow B$ studied in **Constructions With Sets, Section 4**.
- (b) We have $R_{-1} = R^{-1}$ iff R is total and functional (Item 8 of Proposition 4.2.1.3).
- (c) As a consequence of the previous item, when R comes from a function f , the pair of adjunctions

$$R_* \dashv R_{-1} = R^{-1} \dashv R_!$$

reduces to the triple adjunction

$$f_* \dashv f^{-1} \dashv f_!$$

from [Constructions With Sets, Section 4](#).

- (d) The pairs $R_* \dashv R_{-1}$ and $R^{-1} \dashv R_!$ turn out to be rather important later on, as they appear in the definition and study of continuous, open, and closed relations between topological spaces (??, ??).

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1 Co/Limits in the Category of Relations

This section is currently just a stub, and will be properly developed later on.

2 Kan Extensions and Kan Lifts in the 2-Category of Relations

2.1 Left Kan Extensions in Rel

Proposition 2.1.1.1. Let $R : A \rightarrow B$ be a relation.

1. *Non-Existence of All Left Kan Extensions in Rel.* Not all relations in **Rel** admit left Kan extensions.
2. *Characterisation of Relations Admitting Left Kan Extensions Along Them.* The following conditions are equivalent:
 - (a) The left Kan extension

$$\text{Lan}_R : \mathbf{Rel}(A, X) \rightarrow \mathbf{Rel}(B, X)$$

along R exists.

- (b) The relation R admits a left adjoint in **Rel**.
- (c) The relation R is of the form f^{-1} (as in [Definition 3.2.1.1](#)) for some function f .

Proof. **Item 1, Non-Existence of All Left Kan Extensions in Rel:** Omitted, but will eventually follow [Fosco Loregian's comment](#) on [\[MO 460656\]](#).

Item 2, Characterisation of Relations Admitting Left Kan Extensions Along Them: Omitted, but will eventually follow [Tim Campion's answer to](#) [\[MO 460656\]](#). \square

Question 2.1.1.2. Given relations $S: A \rightarrowtail X$ and $R: A \rightarrowtail B$, is there a characterisation of when the left Kan extension

$$\mathrm{Lan}_S(R): B \rightarrowtail X$$

exists in terms of properties of R and S ?

This question also appears as [MO 461592].

Question 2.1.1.3. As shown in [Item 2](#) of [Proposition 2.1.1.1](#), the left Kan extension

$$\mathrm{Lan}_R: \mathbf{Rel}(A, X) \rightarrow \mathbf{Rel}(B, X)$$

along a relation of the form $R = f^{-1}$ exists. Is there an explicit description of it, similarly to the explicit description of right Kan extensions given in [Proposition 2.3.1.1](#)?

This question also appears as [MO 461592].

2.2 Left Kan Lifts in **Rel**

Proposition 2.2.1.1. Let $R: A \rightarrowtail B$ be a relation.

1. *Non-Existence of All Left Kan Lifts in **Rel**.* Not all relations in **Rel** admit left Kan lifts.
2. *Characterisation of Relations Admitting Left Kan Lifts Along Them.* The following conditions are equivalent:
 - (a) The left Kan lift

$$\mathrm{Lift}_R: \mathbf{Rel}(X, B) \rightarrow \mathbf{Rel}(X, A)$$

along R exists.

- (b) The relation R admits a right adjoint in **Rel**.
- (c) The relation R is of the form $\mathrm{Gr}(f)$ (as in [Definition 3.1.1.1](#)) for some function f .

Proof. [Item 1](#), *Non-Existence of All Left Kan Lifts in **Rel***: Omitted, but will eventually follow (the dual of) [Fosco Loregian's comment](#) on [MO 460656].

[Item 2](#), *Characterisation of Relations Admitting Left Kan Lifts Along Them*: Omitted, but will eventually follow [Tim Campion's answer to](#) [MO 460656]. \square

Question 2.2.1.2. Given relations $S: A \rightarrowtail X$ and $R: A \rightarrowtail B$, is there a characterisation of when the left Kan lift

$$\mathrm{Lift}_S(R): X \rightarrowtail A$$

exists in terms of properties of R and S ?

This question also appears as [MO 461592].

Question 2.2.1.3. As shown in Item 2 of Proposition 2.2.1.1, the left Kan lift

$$\mathrm{Lift}_R: \mathbf{Rel}(X, B) \rightarrow \mathbf{Rel}(X, A)$$

along a relation of the form $R = \mathrm{Gr}(f)$ exists. Is there an explicit description of it, similarly to the explicit description of right Kan lifts given in Proposition 2.4.1.1? This question also appears as [MO 461592].

2.3 Right Kan Extensions in **Rel**

Let $R: A \rightarrowtail B$ be a relation.

Proposition 2.3.1.1. The right Kan extension

$$\mathrm{Ran}_R: \mathbf{Rel}(A, X) \rightarrow \mathbf{Rel}(B, X)$$

along R in **Rel** exists and is given by

$$\mathrm{Ran}_R(S) \stackrel{\mathrm{def}}{=} \int_{a \in A} \mathbf{Hom}_{\{t, f\}}(R_a^{-2}, S_a^{-1})$$

for each $S \in \mathbf{Rel}(A, X)$, so that the following conditions are equivalent:

1. We have $b \sim_{\mathrm{Ran}_R(S)} x$.
2. For each $a \in A$, if $a \sim_R b$, then $a \sim_S x$.

Proof. We have

$$\begin{aligned}
 \text{Hom}_{\mathbf{Rel}(A,X)}(S \diamond R, T) &\cong \int_{a \in A} \int_{x \in X} \mathbf{Hom}_{\{t,f\}}((S \diamond R)_a^x, T_a^x) \\
 &\cong \int_{a \in A} \int_{x \in X} \mathbf{Hom}_{\{t,f\}}((\int_{b \in B} S_b^x \times R_a^b), T_a^x) \\
 &\cong \int_{a \in A} \int_{x \in X} \int_{b \in B} \mathbf{Hom}_{\{t,f\}}(S_b^x \times R_a^b, T_a^x) \\
 &\cong \int_{a \in A} \int_{x \in X} \int_{b \in B} \mathbf{Hom}_{\{t,f\}}(S_b^x, \mathbf{Hom}_{\{t,f\}}(R_a^b, T_a^x)) \\
 &\cong \int_{b \in B} \int_{x \in X} \int_{a \in A} \mathbf{Hom}_{\{t,f\}}(S_b^x, \mathbf{Hom}_{\{t,f\}}(R_a^b, T_a^x)) \\
 &\cong \int_{b \in B} \int_{x \in X} \mathbf{Hom}_{\{t,f\}}(S_b^x, \int_{a \in A} \mathbf{Hom}_{\{t,f\}}(R_a^b, T_a^x)) \\
 &\cong \text{Hom}_{\mathbf{Rel}(B,X)}(S, \int_{a \in A} \mathbf{Hom}_{\{t,f\}}(R_a^{-2}, T_a^{-1}))
 \end{aligned}$$

naturally in each $S \in \mathbf{Rel}(B, X)$ and each $T \in \mathbf{Rel}(A, X)$, showing that

$$\int_{a \in A} \mathbf{Hom}_{\{t,f\}}(R_a^{-2}, T_a^{-1})$$

is right adjoint to the precomposition functor $- \diamond R$, being thus the right Kan extension along R . Here we have used the following results, respectively (i.e. for each \cong sign):

1. **Relations, Item 1 of Proposition 1.1.1.5.**
2. **Definition 3.12.1.1.**
3. **??, ?? of ??.**
4. **Sets, Proposition 2.2.1.5.**
5. **??, ?? of ??.**
6. **??, ?? of ??.**
7. **Relations, Item 1 of Proposition 1.1.1.5.**

This finishes the proof. □

2.4 Right Kan Lifts in **Rel**

Let $R: A \rightarrowtail B$ be a relation.

Proposition 2.4.1.1. The right Kan lift

$$\text{Rift}_R: \mathbf{Rel}(X, B) \rightarrow \mathbf{Rel}(X, A)$$

along R in **Rel** exists and is given by

$$\text{Rift}_R(S) \stackrel{\text{def}}{=} \int_{b \in B} \mathbf{Hom}_{\{t, f\}}(R_{-1}^b, S_{-2}^b)$$

for each $S \in \mathbf{Rel}(X, B)$, so that the following conditions are equivalent:

1. We have $x \sim_{\text{Rift}_R(S)} a$.
2. For each $b \in B$, if $a \sim_R b$, then $x \sim_S b$.

Proof. We have

$$\begin{aligned} \mathbf{Hom}_{\mathbf{Rel}(X, B)}(R \diamond S, T) &\cong \int_{x \in X} \int_{b \in B} \mathbf{Hom}_{\{t, f\}}((R \diamond S)_x^b, T_x^b) \\ &\cong \int_{x \in X} \int_{b \in B} \mathbf{Hom}_{\{t, f\}}((\int_{a \in A} R_a^b \times S_x^a), T_x^b) \\ &\cong \int_{x \in X} \int_{b \in B} \int_{a \in A} \mathbf{Hom}_{\{t, f\}}(R_a^b \times S_x^a, T_x^b) \\ &\cong \int_{x \in X} \int_{b \in B} \int_{a \in A} \mathbf{Hom}_{\{t, f\}}(S_x^a, \mathbf{Hom}_{\{t, f\}}(R_a^b, T_x^b)) \\ &\cong \int_{x \in X} \int_{a \in A} \int_{b \in B} \mathbf{Hom}_{\{t, f\}}(S_x^a, \mathbf{Hom}_{\{t, f\}}(R_a^b, T_x^b)) \\ &\cong \int_{x \in X} \int_{a \in A} \mathbf{Hom}_{\{t, f\}}(S_x^a, \int_{b \in B} \mathbf{Hom}_{\{t, f\}}(R_a^b, T_x^b)) \\ &\cong \mathbf{Hom}_{\mathbf{Rel}(X, A)}(S, \int_{b \in B} \mathbf{Hom}_{\{t, f\}}(R_{-1}^b, T_{-2}^b)) \end{aligned}$$

naturally in each $S \in \mathbf{Rel}(X, A)$ and each $T \in \mathbf{Rel}(X, B)$, showing that

$$\int_{b \in B} \mathbf{Hom}_{\{t, f\}}(R_{-1}^b, S_{-2}^b)$$

is right adjoint to the postcomposition functor $R \diamond -$, being thus the right Kan lift along R . Here we have used the following results, respectively (i.e. for each \cong sign):

1. Relations, Item 1 of Proposition 1.1.1.5.
2. Definition 3.12.1.1.
3. ??, ?? of ??.
4. Sets, Proposition 2.2.1.5.
5. ??, ?? of ??.
6. ??, ?? of ??.
7. Relations, Item 1 of Proposition 1.1.1.5.

This finishes the proof. \square

3 More Constructions With Relations

3.1 The Graph of a Function

Let $f: A \rightarrow B$ be a function.

Definition 3.1.1.1. The **graph of f** is the relation $\text{Gr}(f): A \rightarrow B$ defined as follows:¹

- Viewing relations from A to B as subsets of $A \times B$, we define

$$\text{Gr}(f) \stackrel{\text{def}}{=} \{(a, f(a)) \in A \times B \mid a \in A\}.$$

- Viewing relations from A to B as functions $A \times B \rightarrow \{\text{true}, \text{false}\}$, we define

$$[\text{Gr}(f)](a, b) \stackrel{\text{def}}{=} \begin{cases} \text{true} & \text{if } b = f(a), \\ \text{false} & \text{otherwise} \end{cases}$$

for each $(a, b) \in A \times B$.

- Viewing relations from A to B as functions $A \rightarrow \mathcal{P}(B)$, we define

$$[\text{Gr}(f)](a) \stackrel{\text{def}}{=} \{f(a)\}$$

for each $a \in A$, i.e. we define $\text{Gr}(f)$ as the composition

$$A \xrightarrow{f} B \xrightarrow{\chi_B} \mathcal{P}(B).$$

¹Further Notation: We write $\text{Gr}(A)$ for $\text{Gr}(\text{id}_A)$, and call it the **graph** of A .

Proposition 3.1.1.2. Let $f: A \rightarrow B$ be a function.

1. *Functoriality.* The assignment $A \mapsto \text{Gr}(A)$ defines a functor

$$\text{Gr}: \text{Sets} \rightarrow \text{Rel}$$

where

- *Action on Objects.* For each $A \in \text{Obj}(\text{Sets})$, we have

$$\text{Gr}(A) \stackrel{\text{def}}{=} A.$$

- *Action on Morphisms.* For each $A, B \in \text{Obj}(\text{Sets})$, the action on Hom-sets

$$\text{Gr}_{A,B}: \text{Sets}(A, B) \rightarrow \underbrace{\text{Rel}(\text{Gr}(A), \text{Gr}(B))}_{\stackrel{\text{def}}{=} \text{Rel}(A, B)}$$

of Gr at (A, B) is defined by

$$\text{Gr}_{A,B}(f) \stackrel{\text{def}}{=} \text{Gr}(f),$$

where $\text{Gr}(f)$ is the graph of f as in [Definition 3.1.1.1](#).

In particular:

- *Preservation of Identities.* We have

$$\text{Gr}(\text{id}_A) = \chi_A$$

for each $A \in \text{Obj}(\text{Sets})$.

- *Preservation of Composition.* We have

$$\text{Gr}(g \circ f) = \text{Gr}(g) \diamond \text{Gr}(f)$$

for each pair of functions $f: A \rightarrow B$ and $g: B \rightarrow C$.

2. *Adjointness Inside **Rel**.* We have an adjunction

$$\left(\text{Gr}(f) \dashv f^{-1} \right): A \begin{array}{c} \xrightarrow{\text{Gr}(f)} \\ \dashv \\ \xleftarrow{f^{-1}} \end{array} B$$

in **Rel**, where f^{-1} is the inverse of f of [Definition 3.2.1.1](#).

3. *Adjointness.* We have an adjunction

$$(\text{Gr} \dashv \mathcal{P}_*) : \text{Sets} \begin{array}{c} \xrightarrow{\text{Gr}} \\ \perp \\ \xleftarrow{\mathcal{P}_*} \end{array} \text{Rel},$$

witnessed by a bijection of sets

$$\text{Rel}(\text{Gr}(A), B) \cong \text{Sets}(A, \mathcal{P}(B))$$

natural in $A \in \text{Obj}(\text{Sets})$ and $B \in \text{Obj}(\text{Rel})$.

4. *Interaction With Inverses.* We have

$$\begin{aligned} \text{Gr}(f)^\dagger &= f^{-1}, \\ (f^{-1})^\dagger &= \text{Gr}(f). \end{aligned}$$

5. *Cocontinuity.* The functor $\text{Gr} : \text{Sets} \rightarrow \text{Rel}$ of **Item 1** preserves colimits.

6. *Characterisations.* Let $R : A \rightarrowtail B$ be a relation. The following conditions are equivalent:

- (a) There exists a function $f : A \rightarrow B$ such that $R = \text{Gr}(f)$.
- (b) The relation R is total and functional.
- (c) The weak and strong inverse images of R agree, i.e. we have $R^{-1} = R_{-1}$.
- (d) The relation R has a right adjoint R^\dagger in Rel .

Proof. **Item 1**, *Functoriality*: Clear.

Item 2, *Adjointness Inside Rel*: We need to check that there are inclusions

$$\begin{aligned} \chi_A &\subset f^{-1} \diamond \text{Gr}(f), \\ \text{Gr}(f) \diamond f^{-1} &\subset \chi_B. \end{aligned}$$

These correspond respectively to the following conditions:

- 1. For each $a \in A$, there exists some $b \in B$ such that $a \sim_{\text{Gr}(f)} b$ and $b \sim_{f^{-1}} a$.
- 2. For each $a, b \in A$, if $a \sim_{\text{Gr}(f)} b$ and $b \sim_{f^{-1}} a$, then $a = b$.

In other words, the first condition states that the image of any $a \in A$ by f is nonempty, whereas the second condition states that f is not multivalued. As f is a function, both of these statements are true, and we are done.

Item 3, Adjointness: The stated bijection follows from [Relations, Remark 1.1.1.4](#), with naturality being clear.

Item 4, Interaction With Inverses: Clear.

Item 5, Cocontinuity: Omitted.

Item 6, Characterisations: We claim that [Items 6a to 6d](#) are indeed equivalent:

- [Item 6a](#) \iff [Item 6b](#). This is shown in the proof of ?? of ??.
- [Item 6b](#) \implies [Item 6c](#). If R is total and functional, then, for each $a \in A$, the set $R(a)$ is a singleton, implying that

$$\begin{aligned} R^{-1}(V) &\stackrel{\text{def}}{=} \{a \in A \mid R(a) \cap V \neq \emptyset\}, \\ R_{-1}(V) &\stackrel{\text{def}}{=} \{a \in A \mid R(a) \subset V\} \end{aligned}$$

are equal for all $V \in \mathcal{P}(B)$, as the conditions $R(a) \cap V \neq \emptyset$ and $R(a) \subset V$ are equivalent when $R(a)$ is a singleton.

- [Item 6c](#) \implies [Item 6b](#). We claim that R is indeed total and functional:
 - *Totality.* If we had $R(a) = \emptyset$ for some $a \in A$, then we would have $a \in R_{-1}(\emptyset)$, so that $R_{-1}(\emptyset) \neq \emptyset$. But since $R^{-1}(\emptyset) = \emptyset$, this would imply $R_{-1}(\emptyset) \neq R^{-1}(\emptyset)$, a contradiction. Thus $R(a) \neq \emptyset$ for all $a \in A$ and R is total.
 - *Functionality.* If $R^{-1} = R_{-1}$, then we have

$$\begin{aligned} \{a\} &= R^{-1}(\{b\}) \\ &= R_{-1}(\{b\}) \end{aligned}$$

for each $b \in R(a)$ and each $a \in A$, and thus $R(a) \subset \{b\}$. But since R is total, we must have $R(a) = \{b\}$, and thus we see that R is functional.

- [Item 6a](#) \iff [Item 6d](#). This follows from [Relations, Proposition 3.3.1.1](#).

This finishes the proof. □

3.2 The Inverse of a Function

Let $f: A \rightarrow B$ be a function.

Definition 3.2.1.1. The **inverse of f** is the relation $f^{-1}: B \rightarrowtail A$ defined as follows:

- Viewing relations from B to A as subsets of $B \times A$, we define

$$f^{-1} \stackrel{\text{def}}{=} \{(b, f^{-1}(b)) \in B \times A \mid a \in A\},$$

where

$$f^{-1}(b) \stackrel{\text{def}}{=} \{a \in A \mid f(a) = b\}$$

for each $b \in B$.

- Viewing relations from B to A as functions $B \times A \rightarrow \{\text{true}, \text{false}\}$, we define

$$f^{-1}(b, a) \stackrel{\text{def}}{=} \begin{cases} \text{true} & \text{if there exists } a \in A \text{ with } f(a) = b, \\ \text{false} & \text{otherwise} \end{cases}$$

for each $(b, a) \in B \times A$.

- Viewing relations from B to A as functions $B \rightarrow \mathcal{P}(A)$, we define

$$f^{-1}(b) \stackrel{\text{def}}{=} \{a \in A \mid f(a) = b\}$$

for each $b \in B$.

Proposition 3.2.1.2. Let $f: A \rightarrow B$ be a function.

1. *Functoriality.* The assignment $A \mapsto A, f \mapsto f^{-1}$ defines a functor

$$(-)^{-1}: \text{Sets} \rightarrow \text{Rel}$$

where

- *Action on Objects.* For each $A \in \text{Obj}(\text{Sets})$, we have

$$[(-)^{-1}](A) \stackrel{\text{def}}{=} A.$$

- *Action on Morphisms.* For each $A, B \in \text{Obj}(\text{Sets})$, the action on Hom-

sets

$$(-)^{-1}_{A,B} : \mathbf{Sets}(A, B) \rightarrow \mathbf{Rel}(A, B)$$

of $(-)^{-1}$ at (A, B) is defined by

$$(-)^{-1}_{A,B}(f) \stackrel{\text{def}}{=} [(-)^{-1}](f),$$

where f^{-1} is the inverse of f as in **Definition 3.2.1.1**.

In particular:

- *Preservation of Identities*. We have

$$\text{id}_A^{-1} = \chi_A$$

for each $A \in \mathbf{Obj}(\mathbf{Sets})$.

- *Preservation of Composition*. We have

$$(g \circ f)^{-1} = g^{-1} \diamond f^{-1}$$

for pair of functions $f : A \rightarrow B$ and $g : B \rightarrow C$.

2. *Adjointness Inside **Rel***. We have an adjunction

$$\left(\text{Gr}(f) \dashv f^{-1} \right): A \begin{array}{c} \xrightarrow{\text{Gr}(f)} \\ \perp \\ \xleftarrow{f^{-1}} \end{array} B$$

in **Rel**.

3. *Interaction With Inverses of Relations*. We have

$$(f^{-1})^\dagger = \text{Gr}(f),$$

$$\text{Gr}(f)^\dagger = f^{-1}.$$

Proof. **Item 1**, *Functoriality*: Clear.

Item 2, *Adjointness Inside **Rel***: This is proved in **Item 2** of **Proposition 3.1.1.2**.

Item 3, *Interaction With Inverses of Relations*: Clear. □

3.3 Representable Relations

Let A and B be sets.

Definition 3.3.1.1. Let $f: A \rightarrow B$ and $g: B \rightarrow A$ be functions.²

1. The **representable relation associated to f** is the relation $\chi_f: A \dashv\vdash B$ defined as the composition

$$A \times B \xrightarrow{f \times \text{id}_B} B \times B \xrightarrow{\chi_B} \{\text{true}, \text{false}\},$$

i.e. given by declaring $a \sim_{\chi_f} b$ iff $f(a) = b$.

2. The **corepresentable relation associated to g** is the relation $\chi^g: B \dashv\vdash A$ defined as the composition

$$B \times A \xrightarrow{g \times \text{id}_A} A \times A \xrightarrow{\chi_A} \{\text{true}, \text{false}\},$$

i.e. given by declaring $b \sim_{\chi^g} a$ iff $g(b) = a$.

3.4 The Domain and Range of a Relation

Let A and B be sets.

²More generally, given functions

$$\begin{aligned} f: A &\rightarrow C, \\ g: B &\rightarrow D \end{aligned}$$

and a relation $B \dashv\vdash D$, we may consider the composite relation

$$A \times B \xrightarrow{f \times g} C \times D \xrightarrow{R} \{\text{true}, \text{false}\},$$

for which we have $a \sim_{R \circ (f \times g)} b$ iff $f(a) \sim_R g(b)$.

Definition 3.4.1.1. Let $R \subset A \times B$ be a relation.^{3,4}

1. The **domain of R** is the subset $\text{dom}(R)$ of A defined by

$$\text{dom}(R) \stackrel{\text{def}}{=} \left\{ a \in A \left| \begin{array}{l} \text{there exists some } b \in B \\ \text{such that } a \sim_R b \end{array} \right. \right\}.$$

2. The **range of R** is the subset $\text{range}(R)$ of B defined by

$$\text{range}(R) \stackrel{\text{def}}{=} \left\{ b \in B \left| \begin{array}{l} \text{there exists some } a \in A \\ \text{such that } a \sim_R b \end{array} \right. \right\}.$$

3.5 Binary Unions of Relations

Let A and B be sets and let R and S be relations from A to B .

Definition 3.5.1.1. The **union of R and S** ⁵ is the relation $R \cup S$ from A to B defined as follows:

³Following ??, ??, we may compute the (characteristic functions associated to the) domain and range of a relation using the following colimit formulas:

$$\begin{aligned} \chi_{\text{dom}(R)}(a) &\cong \text{colim}_{b \in B} (R_a^b) & (a \in A) \\ &\cong \bigvee_{b \in B} R_a^b, \\ \chi_{\text{range}(R)}(b) &\cong \text{colim}_{a \in A} (R_a^b) & (b \in B) \\ &\cong \bigvee_{a \in A} R_a^b, \end{aligned}$$

where the join \bigvee is taken in the poset $(\{\text{true}, \text{false}\}, \preceq)$ of [Constructions With Sets, Definition 2.2.1.3](#).

⁴Viewing R as a function $R: A \rightarrow \mathcal{P}(B)$, we have

$$\begin{aligned} \text{dom}(R) &\cong \text{colim}_{y \in Y} (R(y)) \\ &\cong \bigcup_{y \in Y} R(y), \\ \text{range}(R) &\cong \text{colim}_{x \in X} (R(x)) \\ &\cong \bigcup_{x \in X} R(x), \end{aligned}$$

⁵*Further Terminology:* Also called the **binary union of R and S** , for emphasis.

- Viewing relations from A to B as subsets of $A \times B$, we define⁶

$$R \cup S \stackrel{\text{def}}{=} \{(a, b) \in B \times A \mid \text{we have } a \sim_R b \text{ or } a \sim_S b\}.$$

- Viewing relations from A to B as functions $A \rightarrow \mathcal{P}(B)$, we define

$$[R \cup S](a) \stackrel{\text{def}}{=} R(a) \cup S(a)$$

for each $a \in A$.

Proposition 3.5.1.2. Let R, S, R_1 , and R_2 be relations from A to B , and let S_1 and S_2 be relations from B to C .

1. *Interaction With Inverses.* We have

$$(R \cup S)^\dagger = R^\dagger \cup S^\dagger.$$

2. *Interaction With Composition.* We have

$$(S_1 \diamond R_1) \cup (S_2 \diamond R_2) \stackrel{\text{poss.}}{\neq} (S_1 \cup S_2) \diamond (R_1 \cup R_2).$$

Proof. **Item 1, Interaction With Inverses:** Clear.

Item 2, Interaction With Composition: Unwinding the definitions, we see that:

1. The condition for $(S_1 \diamond R_1) \cup (S_2 \diamond R_2)$ is:

- (a) There exists some $b \in B$ such that:

- i. $a \sim_{R_1} b$ and $b \sim_{S_1} c$;

or

- i. $a \sim_{R_2} b$ and $b \sim_{S_2} c$;

3. The condition for $(S_1 \cup S_2) \diamond (R_1 \cup R_2)$ is:

- (a) There exists some $b \in B$ such that:

- i. $a \sim_{R_1} b$ or $a \sim_{R_2} b$;

and

- i. $b \sim_{S_1} c$ or $b \sim_{S_2} c$.

These two conditions may fail to agree (counterexample omitted), and thus the two resulting relations on $A \times C$ may differ. \square

⁶This is the same as the union of R and S as subsets of $A \times B$.

3.6 Unions of Families of Relations

Let A and B be sets and let $\{R_i\}_{i \in I}$ be a family of relations from A to B .

Definition 3.6.1.1. The **union of the family** $\{R_i\}_{i \in I}$ is the relation $\bigcup_{i \in I} R_i$ from A to B defined as follows:

- Viewing relations from A to B as subsets of $A \times B$, we define⁷

$$\bigcup_{i \in I} R_i \stackrel{\text{def}}{=} \left\{ (a, b) \in (A \times B)^{\times I} \mid \begin{array}{l} \text{there exists some } i \in I \\ \text{such that } a \sim_{R_i} b \end{array} \right\}.$$

- Viewing relations from A to B as functions $A \rightarrow \mathcal{P}(B)$, we define

$$\left[\bigcup_{i \in I} R_i \right] (a) \stackrel{\text{def}}{=} \bigcup_{i \in I} R_i(a)$$

for each $a \in A$.

Proposition 3.6.1.2. Let A and B be sets and let $\{R_i\}_{i \in I}$ be a family of relations from A to B .

1. *Interaction With Inverses.* We have

$$\left(\bigcup_{i \in I} R_i \right)^\dagger = \bigcup_{i \in I} R_i^\dagger.$$

Proof. **Item 1**, *Interaction With Inverses*: Clear. □

3.7 Binary Intersections of Relations

Let A and B be sets and let R and S be relations from A to B .

Definition 3.7.1.1. The **intersection of R and S** ⁸ is the relation $R \cap S$ from A to B defined as follows:

- Viewing relations from A to B as subsets of $A \times B$, we define⁹

$$R \cap S \stackrel{\text{def}}{=} \{(a, b) \in B \times A \mid \text{we have } a \sim_R b \text{ and } a \sim_S b\}.$$

⁷This is the same as the union of $\{R_i\}_{i \in I}$ as a collection of subsets of $A \times B$.

⁸*Further Terminology:* Also called the **binary intersection of R and S** , for emphasis.

⁹This is the same as the intersection of R and S as subsets of $A \times B$.

- Viewing relations from A to B as functions $A \rightarrow \mathcal{P}(B)$, we define

$$[R \cap S](a) \stackrel{\text{def}}{=} R(a) \cap S(a)$$

for each $a \in A$.

Proposition 3.7.1.2. Let R, S, R_1 , and R_2 be relations from A to B , and let S_1 and S_2 be relations from B to C .

1. *Interaction With Inverses.* We have

$$(R \cap S)^\dagger = R^\dagger \cap S^\dagger.$$

2. *Interaction With Composition.* We have

$$(S_1 \diamond R_1) \cap (S_2 \diamond R_2) = (S_1 \cap S_2) \diamond (R_1 \cap R_2).$$

Proof. **Item 1**, *Interaction With Inverses*: Clear.

Item 2, *Interaction With Composition*: Unwinding the definitions, we see that:

1. The condition for $(S_1 \diamond R_1) \cap (S_2 \diamond R_2)$ is:

- (a) There exists some $b \in B$ such that:

- i. $a \sim_{R_1} b$ and $b \sim_{S_1} c$;

and

- i. $a \sim_{R_2} b$ and $b \sim_{S_2} c$;

3. The condition for $(S_1 \cap S_2) \diamond (R_1 \cap R_2)$ is:

- (a) There exists some $b \in B$ such that:

- i. $a \sim_{R_1} b$ and $a \sim_{R_2} b$;

and

- i. $b \sim_{S_1} c$ and $b \sim_{S_2} c$.

These two conditions agree, and thus so do the two resulting relations on $A \times C$. \square

3.8 Intersections of Families of Relations

Let A and B be sets and let $\{R_i\}_{i \in I}$ be a family of relations from A to B .

Definition 3.8.1.1. The **intersection of the family** $\{R_i\}_{i \in I}$ is the relation $\bigcap_{i \in I} R_i$ defined as follows:

- Viewing relations from A to B as subsets of $A \times B$, we define¹⁰

$$\bigcap_{i \in I} R_i \stackrel{\text{def}}{=} \left\{ (a, b) \in (A \times B)^{\times I} \mid \begin{array}{l} \text{for each } i \in I, \\ \text{we have } a \sim_{R_i} b \end{array} \right\}.$$

- Viewing relations from A to B as functions $A \rightarrow \mathcal{P}(B)$, we define

$$\left[\bigcap_{i \in I} R_i \right] (a) \stackrel{\text{def}}{=} \bigcap_{i \in I} R_i(a)$$

for each $a \in A$.

Proposition 3.8.1.2. Let A and B be sets and let $\{R_i\}_{i \in I}$ be a family of relations from A to B .

1. *Interaction With Inverses.* We have

$$\left(\bigcap_{i \in I} R_i \right)^\dagger = \bigcap_{i \in I} R_i^\dagger.$$

Proof. **Item 1**, *Interaction With Inverses*: Clear. □

3.9 Binary Products of Relations

Let A , B , X , and Y be sets, let $R: A \rightarrow B$ be a relation from A to B , and let $S: X \rightarrow Y$ be a relation from X to Y .

Definition 3.9.1.1. The **product of R and S** ¹¹ is the relation $R \times S$ from $A \times X$ to $B \times Y$ defined as follows:

- Viewing relations from $A \times X$ to $B \times Y$ as subsets of $(A \times X) \times (B \times Y)$,

¹⁰This is the same as the intersection of $\{R_i\}_{i \in I}$ as a collection of subsets of $A \times B$.

¹¹*Further Terminology:* Also called the **binary product of R and S** , for emphasis.

we define $R \times S$ as the Cartesian product of R and S as subsets of $A \times X$ and $B \times Y$.¹²

- Viewing relations from $A \times X$ to $B \times Y$ as functions $A \times X \rightarrow \mathcal{P}(B \times Y)$, we define $R \times S$ as the composition

$$A \times X \xrightarrow{R \times S} \mathcal{P}(B) \times \mathcal{P}(Y) \xrightarrow{\mathcal{P}_{B,Y}^\otimes} \mathcal{P}(B \times Y)$$

in Sets, i.e. by

$$[R \times S](a, x) \stackrel{\text{def}}{=} R(a) \times S(x)$$

for each $(a, x) \in A \times X$.

Proposition 3.9.1.2. Let A, B, X , and Y be sets.

1. *Interaction With Inverses.* Let

$$\begin{aligned} R &: A \rightarrowtail A, \\ S &: X \rightarrowtail X \end{aligned}$$

We have

$$(R \times S)^\dagger = R^\dagger \times S^\dagger.$$

2. *Interaction With Composition.* Let

$$\begin{aligned} R_1 &: A \rightarrowtail B, \\ S_1 &: B \rightarrowtail C, \\ R_2 &: X \rightarrowtail Y, \\ S_2 &: Y \rightarrowtail Z \end{aligned}$$

be relations. We have

$$(S_1 \diamond R_1) \times (S_2 \diamond R_2) = (S_1 \times S_2) \diamond (R_1 \times R_2).$$

Proof. **Item 1**, *Interaction With Inverses*: Unwinding the definitions, we see that:

1. We have $(a, x) \sim_{(R \times S)^\dagger} (b, y)$ iff:

¹²That is, $R \times S$ is the relation given by declaring $(a, x) \sim_{R \times S} (b, y)$ iff $a \sim_R b$ and $x \sim_S y$.

- We have $(b, y) \sim_{R \times S} (a, x)$, i.e. iff:
 - We have $b \sim_R a$;
 - We have $y \sim_S x$;
- 2. We have $(a, x) \sim_{R^\dagger \times S^\dagger} (b, y)$ iff:
 - We have $a \sim_{R^\dagger} b$ and $x \sim_{S^\dagger} y$, i.e. iff:
 - We have $b \sim_R a$;
 - We have $y \sim_S x$.

These two conditions agree, and thus the two resulting relations on $A \times X$ are equal.

Item 2, Interaction With Composition: Unwinding the definitions, we see that:

1. We have $(a, x) \sim_{(S_1 \diamond R_1) \times (S_2 \diamond R_2)} (c, z)$ iff:
 - (a) We have $a \sim_{S_1 \diamond R_1} c$ and $x \sim_{S_2 \diamond R_2} z$, i.e. iff:
 - i. There exists some $b \in B$ such that $a \sim_{R_1} b$ and $b \sim_{S_1} c$;
 - ii. There exists some $y \in Y$ such that $x \sim_{R_2} y$ and $y \sim_{S_2} z$;
2. We have $(a, x) \sim_{(S_1 \times S_2) \diamond (R_1 \times R_2)} (c, z)$ iff:
 - (a) There exists some $(b, y) \in B \times Y$ such that $(a, x) \sim_{R_1 \times R_2} (b, y)$ and $(b, y) \sim_{S_1 \times S_2} (c, z)$, i.e. such that:
 - i. We have $a \sim_{R_1} b$ and $x \sim_{R_2} y$;
 - ii. We have $b \sim_{S_1} c$ and $y \sim_{S_2} z$.

These two conditions agree, and thus the two resulting relations from $A \times X$ to $C \times Z$ are equal. \square

3.10 Products of Families of Relations

Let $\{A_i\}_{i \in I}$ and $\{B_i\}_{i \in I}$ be families of sets, and let $\{R_i : A_i \rightarrow B_i\}_{i \in I}$ be a family of relations.

Definition 3.10.1.1. The **product of the family** $\{R_i\}_{i \in I}$ is the relation $\prod_{i \in I} R_i$ from $\prod_{i \in I} A_i$ to $\prod_{i \in I} B_i$ defined as follows:

- Viewing relations as subsets, we define $\prod_{i \in I} R_i$ as its product as a family

of sets, i.e. we have

$$\prod_{i \in I} R_i \stackrel{\text{def}}{=} \left\{ (a_i, b_i)_{i \in I} \in \prod_{i \in I} (A_i \times B_i) \mid \begin{array}{l} \text{for each } i \in I, \\ \text{we have } a_i \sim_{R_i} b_i \end{array} \right\}.$$

- Viewing relations as functions to powersets, we define

$$\left[\prod_{i \in I} R_i \right] ((a_i)_{i \in I}) \stackrel{\text{def}}{=} \prod_{i \in I} R_i(a_i)$$

for each $(a_i)_{i \in I} \in \prod_{i \in I} R_i$.

3.11 The Inverse of a Relation

Let A , B , and C be sets and let $R \subset A \times B$ be a relation.

Definition 3.11.1.1. The **inverse of R** ¹³ is the relation R^\dagger defined as follows:

- Viewing relations as subsets, we define

$$R^\dagger \stackrel{\text{def}}{=} \{(b, a) \in B \times A \mid \text{we have } b \sim_R a\}.$$

- Viewing relations as functions $A \times B \rightarrow \{\text{true}, \text{false}\}$, we define

$$[R^\dagger]_b^a \stackrel{\text{def}}{=} R_a^b$$

for each $(b, a) \in B \times A$.

- Viewing relations as functions $A \rightarrow \mathcal{P}(B)$, we define

$$\begin{aligned} [R^\dagger](b) &\stackrel{\text{def}}{=} R^\dagger(\{b\}) \\ &\stackrel{\text{def}}{=} \{a \in A \mid b \in R(a)\} \end{aligned}$$

for each $b \in B$, where $R^\dagger(\{b\})$ is the fibre of R over $\{b\}$.

Example 3.11.1.2. Here are some examples of inverses of relations.

1. *Less Than Equal Signs.* We have $(\leq)^\dagger = \geq$.

¹³*Further Terminology:* Also called the **opposite of R** , the **transpose of R** , or the **converse of R** .

2. *Greater Than Equal Signs.* Dually to **Item 1**, we have $(\geq)^\dagger = \leq$.
3. *Functions.* Let $f: A \rightarrow B$ be a function. We have

$$\begin{aligned}\text{Gr}(f)^\dagger &= f^{-1}, \\ (f^{-1})^\dagger &= \text{Gr}(f).\end{aligned}$$

Proposition 3.11.1.3. Let $R: A \rightarrowtail B$ and $S: B \rightarrowtail C$ be relations.

1. *Functoriality.* The assignment $R \mapsto R^\dagger$ defines a functor (i.e. morphism of posets)

$$(-)^\dagger: \mathbf{Rel}(A, B) \rightarrow \mathbf{Rel}(B, A).$$

In particular, given relations $R, S: A \rightarrowtail B$, we have:

$$(\star) \text{ If } R \subset S, \text{ then } R^\dagger \subset S^\dagger.$$

2. *Interaction With Ranges and Domains.* We have

$$\begin{aligned}\text{dom}(R^\dagger) &= \text{range}(R), \\ \text{range}(R^\dagger) &= \text{dom}(R).\end{aligned}$$

3. *Interaction With Composition I.* We have

$$(S \diamond R)^\dagger = R^\dagger \diamond S^\dagger.$$

4. *Interaction With Composition II.* We have

$$\begin{aligned}\chi_B &\subset R \diamond R^\dagger, \\ \chi_A &\subset R^\dagger \diamond R.\end{aligned}$$

5. *Invertibility.* We have

$$(R^\dagger)^\dagger = R.$$

6. *Identity.* We have

$$\chi_A^\dagger = \chi_A.$$

Proof. **Item 1**, Functoriality: Clear.

Item 2, Interaction With Ranges and Domains: Clear.

Item 3, Interaction With Composition I: Clear.

Item 4, Interaction With Composition II: Clear.

Item 5, Invertibility: Clear.

Item 6, Identity: Clear. □

3.12 Composition of Relations

Let A, B , and C be sets and let $R: A \rightarrowtail B$ and $S: B \rightarrowtail C$ be relations.

Definition 3.12.1.1. The **composition of R and S** is the relation $S \diamond R$ defined as follows:

- Viewing relations from A to C as subsets of $A \times C$, we define

$$S \diamond R \stackrel{\text{def}}{=} \left\{ (a, c) \in A \times C \left| \begin{array}{l} \text{there exists some } b \in B \text{ such} \\ \text{that } a \sim_R b \text{ and } b \sim_S c \end{array} \right. \right\}.$$

- Viewing relations as functions $A \times B \rightarrow \{\text{true}, \text{false}\}$, we define

$$\begin{aligned} (S \diamond R)^{-1}_{-2} &\stackrel{\text{def}}{=} \int^{b \in B} S_b^{-1} \times R_{-2}^b \\ &= \bigvee_{b \in B} S_b^{-1} \times R_{-2}^b, \end{aligned}$$

where the join \bigvee is taken in the poset $(\{\text{true}, \text{false}\}, \preceq)$ of [Sets, Definition 2.2.1.3](#).

- Viewing relations as functions $A \rightarrow \mathcal{P}(B)$, we define

$$S \diamond R \stackrel{\text{def}}{=} \text{Lan}_{\chi_B}(S) \circ R, \quad \begin{array}{ccc} B & \xrightarrow{S} & \mathcal{P}(C), \\ \chi_B \downarrow & \swarrow & \nearrow \text{Lan}_{\chi_B}(S) \\ A & \xrightarrow{R} & \mathcal{P}(B) \end{array}$$

where $\text{Lan}_{\chi_B}(S)$ is computed by the formula

$$\begin{aligned} [\text{Lan}_{\chi_B}(S)](V) &\cong \int^{y \in B} \chi_{\mathcal{P}(B)}(\chi_y, V) \odot S_y \\ &\cong \int^{y \in B} \chi_V(y) \odot S_y \\ &\cong \bigcup_{y \in B} \chi_V(y) \odot S_y \\ &\cong \bigcup_{y \in V} S_y \end{aligned}$$

for each $V \in \mathcal{P}(B)$. In other words, $S \diamond R$ is defined by¹⁴

$$\begin{aligned} [S \diamond R](a) &\stackrel{\text{def}}{=} S(R(a)) \\ &\stackrel{\text{def}}{=} \bigcup_{x \in R(a)} S(x). \end{aligned}$$

for each $a \in A$.

Example 3.12.1.2. Here are some examples of composition of relations.

1. *Composing Less/Greater Than Equal With Greater/Less Than Equal Signs.* We have

$$\begin{aligned} \leq \diamond \geq &= \sim_{\text{triv}}, \\ \geq \diamond \leq &= \sim_{\text{triv}}. \end{aligned}$$

2. *Composing Less/Greater Than Equal Signs With Less/Greater Than Equal Signs.* We have

$$\begin{aligned} \leq \diamond \leq &= \leq, \\ \geq \diamond \geq &= \geq. \end{aligned}$$

Proposition 3.12.1.3. Let $R: A \rightarrowtail B$, $S: B \rightarrowtail C$, and $T: C \rightarrowtail D$ be relations.

1. *Interaction With Ranges and Domains.* We have

$$\begin{aligned} \text{dom}(S \diamond R) &\subset \text{dom}(R), \\ \text{range}(S \diamond R) &\subset \text{range}(S). \end{aligned}$$

2. *Associativity.* We have

$$(T \diamond S) \diamond R = T \diamond (S \diamond R).$$

3. *Unitality.* We have

$$\begin{aligned} \chi_B \diamond R &= R, \\ R \diamond \chi_A &= R. \end{aligned}$$

¹⁴That is: the relation R may send $a \in A$ to a number of elements $\{b_i\}_{i \in I}$ in B , and then

4. *Interaction With Inverses.* We have

$$(S \diamond R)^\dagger = R^\dagger \diamond S^\dagger.$$

5. *Interaction With Composition.* We have

$$\begin{aligned}\chi_B &\subset R \diamond R^\dagger, \\ \chi_A &\subset R^\dagger \diamond R.\end{aligned}$$

Proof. **Item 1**, *Interaction With Ranges and Domains:* Clear.

Item 2, *Associativity:* Indeed, we have

$$\begin{aligned}(T \diamond S) \diamond R &\stackrel{\text{def}}{=} \left(\int^{c \in C} T_c^{-1} \times S_{-2}^c \right) \diamond R \\ &\stackrel{\text{def}}{=} \int^{b \in B} \left(\int^{c \in C} T_c^{-1} \times S_b^c \right) \diamond R_{-2}^b \\ &= \int^{b \in B} \int^{c \in C} (T_c^{-1} \times S_b^c) \diamond R_{-2}^b \\ &= \int^{c \in C} \int^{b \in B} (T_c^{-1} \times S_b^c) \diamond R_{-2}^b \\ &= \int^{c \in C} \int^{b \in B} T_c^{-1} \times (S_b^c \diamond R_{-2}^b) \\ &= \int^{c \in C} T_c^{-1} \times \left(\int^{b \in B} S_b^c \diamond R_{-2}^b \right) \\ &\stackrel{\text{def}}{=} \int^{c \in C} T_c^{-1} \times (S \diamond R)_{-2}^c \\ &\stackrel{\text{def}}{=} T \diamond (S \diamond R).\end{aligned}$$

In the language of relations, given $a \in A$ and $d \in D$, the stated equality witnesses the equivalence of the following two statements:

1. We have $a \sim_{(T \diamond S) \diamond R} d$, i.e. there exists some $b \in B$ such that:

(a) We have $a \sim_R b$;

the relation S may send the image of each of the b_i 's to a number of elements $\{S(b_i)\}_{i \in I} = \left\{ \{c_{j_i}\}_{j_i \in J_i} \right\}_{i \in I}$ in C .

(b) We have $b \sim_{T \diamond S} d$, i.e. there exists some $c \in C$ such that:

- i. We have $b \sim_S c$;
- ii. We have $c \sim_T d$;

2. We have $a \sim_{T \diamond (S \diamond R)} d$, i.e. there exists some $c \in C$ such that:

(a) We have $a \sim_{S \diamond R} c$, i.e. there exists some $b \in B$ such that:

- i. We have $a \sim_R b$;
- ii. We have $b \sim_S c$;

(b) We have $c \sim_T d$;

both of which are equivalent to the statement

· There exist $b \in B$ and $c \in C$ such that $a \sim_R b \sim_S c \sim_T d$.

Item 3, Unitality: Indeed, we have

$$\begin{aligned}
 \chi_B \diamond R &\stackrel{\text{def}}{=} \int^{x \in B} (\chi_B)_x^{-1} \times R_{-2}^x \\
 &= \bigvee_{x \in B} (\chi_B)_x^{-1} \times R_{-2}^x \\
 &= \bigvee_{\substack{x \in B \\ x = -1}} R_{-2}^x \\
 &= R_{-2}^{-1},
 \end{aligned}$$

and

$$\begin{aligned}
 R \diamond \chi_A &\stackrel{\text{def}}{=} \int^{x \in A} R_x^{-1} \times (\chi_A)_{-2}^x \\
 &= \bigvee_{x \in B} R_x^{-1} \times (\chi_A)_{-2}^x \\
 &= \bigvee_{\substack{x \in B \\ x = -2}} R_x^{-1} \\
 &= R_{-2}^{-1}.
 \end{aligned}$$

In the language of relations, given $a \in A$ and $b \in B$:

- The equality

$$\chi_B \diamond R = R$$

witnesses the equivalence of the following two statements:

1. We have $a \sim_b B$.
2. There exists some $b' \in B$ such that:
 - (a) We have $a \sim_R b'$
 - (b) We have $b' \sim_{\chi_B} b$, i.e. $b' = b$.

- The equality

$$R \diamond \chi_A = R$$

witnesses the equivalence of the following two statements:

1. There exists some $a' \in A$ such that:
 - (a) We have $a \sim_{\chi_B} a'$, i.e. $a = a'$.
 - (b) We have $a' \sim_R b$
2. We have $a \sim_b B$.

Item 4, Interaction With Inverses: Clear.

Item 5, Interaction With Composition: Clear. □

3.13 The Collage of a Relation

Let A and B be sets and let $R: A \rightarrow B$ be a relation from A to B .

Definition 3.13.1.1. The **collage of R** ¹⁵ is the poset $\mathbf{Coll}(R) \stackrel{\text{def}}{=} (\text{Coll}(R), \preceq_{\mathbf{Coll}(R)})$ consisting of:

- *The Underlying Set.* The set $\text{Coll}(R)$ defined by

$$\text{Coll}(R) \stackrel{\text{def}}{=} A \coprod B.$$

- *The Partial Order.* The partial order

$$\preceq_{\mathbf{Coll}(R)}: \text{Coll}(R) \times \text{Coll}(R) \rightarrow \{\text{true}, \text{false}\}$$

on $\text{Coll}(R)$ defined by

$$\preceq(a, b) \stackrel{\text{def}}{=} \begin{cases} \text{true} & \text{if } a = b \text{ or } a \sim_R b, \\ \text{false} & \text{otherwise.} \end{cases}$$

¹⁵*Further Terminology:* Also called the **cograph of R** .

Proposition 3.13.1.2. Let A and B be sets and let $R: A \rightarrowtail B$ be a relation from A to B .

1. *Functoriality I.* The assignment $R \mapsto \mathbf{Coll}(R)$ defines a functor¹⁶

$$\mathbf{Coll}: \mathbf{Rel}(A, B) \rightarrow \mathbf{Pos}_{/\Delta^1}(A, B),$$

where

- *Action on Objects.* For each $R \in \mathbf{Obj}(\mathbf{Rel}(A, B))$, we have

$$[\mathbf{Coll}](R) \stackrel{\text{def}}{=} (\mathbf{Coll}(R), \phi_R)$$

for each $R \in \mathbf{Rel}(A, B)$, where

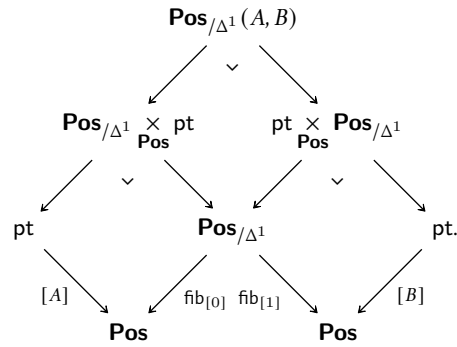
- The poset $\mathbf{Coll}(R)$ is the collage of R of [Definition 3.13.1.1](#).
- The morphism $\phi_R: \mathbf{Coll}(R) \rightarrow \Delta^1$ is given by

$$\phi_R(x) \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if } x \in A, \\ 1 & \text{if } x \in B \end{cases}$$

¹⁶Here $\mathbf{Pos}_{/\Delta^1}(A, B)$ is the category defined as the pullback

$$\mathbf{Pos}_{/\Delta^1}(A, B) \stackrel{\text{def}}{=} \mathbf{pt}_{[A], \mathbf{Pos}, \text{fib}_0} \times_{\mathbf{Pos}_{/\Delta^1} \times_{\text{fib}_1, \mathbf{Pos}, [B]} \mathbf{pt}},$$

as in the diagram



Explicitly, an object of $\mathbf{Pos}_{/\Delta^1}(A, B)$ is a pair (X, ϕ_X) consisting of

- A poset X ;
- A morphism $\phi_X: X \rightarrow \Delta^1$;

such that $\phi_X^{-1}(0) = A$ and $\phi_X^{-1}(1) = B$, with morphisms between such objects being morphisms of posets over Δ^1 .

for each $x \in \mathbf{Coll}(R)$.

- *Action on Morphisms.* For each $R, S \in \mathbf{Obj}(\mathbf{Rel}(A, B))$, the action on Hom-sets

$$\mathbf{Coll}_{R,S}: \mathbf{Hom}_{\mathbf{Rel}(A,B)}(R, S) \rightarrow \mathbf{Pos}(\mathbf{Coll}(R), \mathbf{Coll}(S))$$

of **Coll** at (R, S) is given by sending an inclusion

$$\iota: R \subset S$$

to the morphism

$$\mathbf{Coll}(\iota): \mathbf{Coll}(R) \rightarrow \mathbf{Coll}(S)$$

of posets over Δ^1 defined by

$$[\mathbf{Coll}(\iota)](x) \stackrel{\text{def}}{=} x$$

for each $x \in \mathbf{Coll}(R)$.¹⁷

2. *Equivalence.* The functor of **Item 1** is an equivalence of categories.

Proof. **Item 1**, *Functoriality*: Clear.

Item 2, *Equivalence*: Omitted. □

4 Functoriality of Powersets

4.1 Direct Images

Let A and B be sets and let $R: A \rightarrow B$ be a relation.

Definition 4.1.1.1. The **direct image function associated to R** is the function

$$R_*: \mathcal{P}(A) \rightarrow \mathcal{P}(B)$$

¹⁷Note that this is indeed a morphism of posets: if $x \preceq_{\mathbf{Coll}(R)} y$, then $x = y$ or $x \sim_R y$, so we have either $x = y$ or $x \sim_S y$ (as $R \subset S$), and thus $x \preceq_{\mathbf{Coll}(S)} y$.

defined by^{18,19}

$$\begin{aligned} R_*(U) &\stackrel{\text{def}}{=} R(U) \\ &\stackrel{\text{def}}{=} \bigcup_{a \in U} R(a) \\ &= \left\{ b \in B \mid \begin{array}{l} \text{there exists some } a \in U \\ \text{such that } b \in R(a) \end{array} \right\} \end{aligned}$$

for each $U \in \mathcal{P}(A)$.

Remark 4.1.1.2. Identifying subsets of A with relations from pt to A via **Constructions With Sets, Item 3** of **Proposition 4.3.1.6**, we see that the direct image function associated to R is equivalently the function

$$R_*: \underbrace{\mathcal{P}(A)}_{\cong \text{Rel}(\text{pt}, A)} \rightarrow \underbrace{\mathcal{P}(B)}_{\cong \text{Rel}(\text{pt}, B)}$$

defined by

$$R_*(U) \stackrel{\text{def}}{=} R \diamond U$$

for each $U \in \mathcal{P}(A)$, where $R \diamond U$ is the composition

$$\text{pt} \xrightarrow{U} A \xrightarrow{R} B.$$

Proposition 4.1.1.3. Let $R: A \rightarrowtail B$ be a relation.

1. *Functoriality.* The assignment $U \mapsto R_*(U)$ defines a functor

$$R_*: (\mathcal{P}(A), \subset) \rightarrow (\mathcal{P}(B), \subset)$$

where

- *Action on Objects.* For each $U \in \mathcal{P}(A)$, we have

$$[R_*](U) \stackrel{\text{def}}{=} R_*(U).$$

¹⁸*Further Terminology:* The set $R(U)$ is called the **direct image of U by R** .

¹⁹We also have

$$R_*(U) = B \setminus R_!(A \setminus U);$$

see **Item 7** of **Proposition 4.1.1.3**.

· *Action on Morphisms.* For each $U, V \in \mathcal{P}(A)$:

- If $U \subset V$, then $R_*(U) \subset R_*(V)$.

2. *Adjointness.* We have an adjunction

$$(R_* \dashv R_{-1}): \mathcal{P}(A) \begin{matrix} \xrightarrow{R_*} \\ \perp \\ \xleftarrow{R_{-1}} \end{matrix} \mathcal{P}(B),$$

witnessed by a bijections of sets

$$\mathrm{Hom}_{\mathcal{P}(A)}(R_*(U), V) \cong \mathrm{Hom}_{\mathcal{P}(A)}(U, R_{-1}(V)),$$

natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$, i.e. such that:

(★) The following conditions are equivalent:

- We have $R_*(U) \subset V$.
- We have $U \subset R_{-1}(V)$.

3. *Preservation of Colimits.* We have an equality of sets

$$R_*\left(\bigcup_{i \in I} U_i\right) = \bigcup_{i \in I} R_*(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have equalities

$$\begin{aligned} R_*(U) \cup R_*(V) &= R_*(U \cup V), \\ R_*(\emptyset) &= \emptyset, \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

4. *Oplax Preservation of Limits.* We have an inclusion of sets

$$R_*\left(\bigcap_{i \in I} U_i\right) \subset \bigcap_{i \in I} R_*(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have inclusions

$$\begin{aligned} R_*(U \cap V) &\subset R_*(U) \cap R_*(V), \\ R_*(A) &\subset B, \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

5. *Symmetric Strict Monoidality With Respect to Unions.* The direct image function of **Item 1** has a symmetric strict monoidal structure

$$(R_*, R_*^\otimes, R_{*|\mathbb{1}}^\otimes) : (\mathcal{P}(A), \cup, \emptyset) \rightarrow (\mathcal{P}(B), \cup, \emptyset),$$

being equipped with equalities

$$\begin{aligned} R_{*|U,V}^\otimes : R_*(U) \cup R_*(V) &\xrightarrow{=} R_*(U \cup V), \\ R_{*|\mathbb{1}}^\otimes : \emptyset &\xrightarrow{=} \emptyset, \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

6. *Symmetric Oplax Monoidality With Respect to Intersections.* The direct image function of **Item 1** has a symmetric oplax monoidal structure

$$(R_*, R_*^\otimes, R_{*|\mathbb{1}}^\otimes) : (\mathcal{P}(A), \cap, A) \rightarrow (\mathcal{P}(B), \cap, B),$$

being equipped with inclusions

$$\begin{aligned} R_{*|U,V}^\otimes : R_*(U \cap V) &\subset R_*(U) \cap R_*(V), \\ R_{*|\mathbb{1}}^\otimes : R_*(A) &\subset B, \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

7. *Relation to Direct Images With Compact Support.* We have

$$R_*(U) = B \setminus R_!(A \setminus U)$$

for each $U \in \mathcal{P}(A)$.

Proof. **Item 1**, *Functoriality*: Clear.

Item 2, *Adjointness*: This follows from ??, ?? of ??.

Item 3, *Preservation of Colimits*: This follows from **Item 2** and ??, ?? of ??.

Item 4, *Oplax Preservation of Limits*: Omitted.

Item 5, *Symmetric Strict Monoidality With Respect to Unions*: This follows from **Item 3**.

Item 6, *Symmetric Oplax Monoidality With Respect to Intersections*: This follows from **Item 4**.

Item 7, *Relation to Direct Images With Compact Support*: The proof proceeds in the same way as in the case of functions (**Constructions With Sets**, **Item 9** of **Proposition 4.4.1.4**): applying **Item 7** of **Proposition 4.4.1.3** to $A \setminus U$, we have

$$\begin{aligned} R_!(A \setminus U) &= B \setminus R_*(A \setminus (A \setminus U)) \\ &= B \setminus R_*(U). \end{aligned}$$

Taking complements, we then obtain

$$\begin{aligned} R_*(U) &= B \setminus (B \setminus R_*(U)), \\ &= B \setminus R_!(A \setminus U), \end{aligned}$$

which finishes the proof. \square

Proposition 4.1.1.4. Let $R: A \rightarrowtail B$ be a relation.

1. *Functionality I.* The assignment $R \mapsto R_*$ defines a function

$$(-)_*: \text{Rel}(A, B) \rightarrow \text{Sets}(\mathcal{P}(A), \mathcal{P}(B)).$$

2. *Functionality II.* The assignment $R \mapsto R_*$ defines a function

$$(-)_*: \text{Rel}(A, B) \rightarrow \text{Pos}((\mathcal{P}(A), \subset), (\mathcal{P}(B), \subset)).$$

3. *Interaction With Identities.* For each $A \in \text{Obj}(\text{Sets})$, we have²⁰

$$(\chi_A)_* = \text{id}_{\mathcal{P}(A)}.$$

4. *Interaction With Composition.* For each pair of composable relations $R: A \rightarrowtail B$ and $S: B \rightarrowtail C$, we have²¹

$$(S \diamond R)_* = S_* \circ R_*,$$

$$\begin{array}{ccc} \mathcal{P}(A) & \xrightarrow{R_*} & \mathcal{P}(B) \\ & \searrow (S \diamond R)_* & \downarrow S_* \\ & & \mathcal{P}(C). \end{array}$$

²⁰That is, the postcomposition function

$$(\chi_A)_*: \text{Rel}(\text{pt}, A) \rightarrow \text{Rel}(\text{pt}, A)$$

is equal to $\text{id}_{\text{Rel}(\text{pt}, A)}$.

²¹That is, we have

$$(S \diamond R)_* = S_* \circ R_*,$$

$$\begin{array}{ccc} \text{Rel}(\text{pt}, A) & \xrightarrow{R_*} & \text{Rel}(\text{pt}, B) \\ & \searrow (S \diamond R)_* & \downarrow S_* \\ & & \text{Rel}(\text{pt}, C). \end{array}$$

Proof. **Item 1, Functionality I:** Clear.

Item 2, Functionality II: Clear.

Item 3, Interaction With Identities: Indeed, we have

$$\begin{aligned}
 (\chi_A)_*(U) &\stackrel{\text{def}}{=} \bigcup_{a \in U} \chi_A(a) \\
 &\stackrel{\text{def}}{=} \bigcup_{a \in U} \{a\} \\
 &= U \\
 &\stackrel{\text{def}}{=} \text{id}_{\mathcal{P}(A)}(U)
 \end{aligned}$$

for each $U \in \mathcal{P}(A)$. Thus $(\chi_A)_* = \text{id}_{\mathcal{P}(A)}$.

Item 4, Interaction With Composition: Indeed, we have

$$\begin{aligned}
 (S \diamond R)_*(U) &\stackrel{\text{def}}{=} \bigcup_{a \in U} [S \diamond R](a) \\
 &\stackrel{\text{def}}{=} \bigcup_{a \in U} S(R(a)) \\
 &\stackrel{\text{def}}{=} \bigcup_{a \in U} S_*(R(a)) \\
 &= S_*\left(\bigcup_{a \in U} R(a)\right) \\
 &\stackrel{\text{def}}{=} S_*(R_*(U)) \\
 &\stackrel{\text{def}}{=} [S_* \circ R_*](U)
 \end{aligned}$$

for each $U \in \mathcal{P}(A)$, where we used **Item 3** of **Proposition 4.1.1.3**. Thus $(S \diamond R)_* = S_* \circ R_*$. \square

4.2 Strong Inverse Images

Let A and B be sets and let $R: A \rightarrowtail B$ be a relation.

Definition 4.2.1.1. The **strong inverse image function associated to R** is the function

$$R_{-1}: \mathcal{P}(B) \rightarrow \mathcal{P}(A)$$

defined by²²

$$R_{-1}(V) \stackrel{\text{def}}{=} \{a \in A \mid R(a) \subset V\}$$

for each $V \in \mathcal{P}(B)$.

Remark 4.2.1.2. Identifying subsets of B with relations from pt to B via **Constructions With Sets, Item 3** of **Proposition 4.3.1.6**, we see that the inverse image function associated to R is equivalently the function

$$R_{-1}: \underbrace{\mathcal{P}(B)}_{\cong \text{Rel}(\text{pt}, B)} \rightarrow \underbrace{\mathcal{P}(A)}_{\cong \text{Rel}(\text{pt}, A)}$$

defined by

$$R_{-1}(V) \stackrel{\text{def}}{=} \text{Rift}_R(V),$$

and being explicitly computed by

$$\begin{aligned} R_{-1}(V) &\stackrel{\text{def}}{=} \text{Rift}_R(V) \\ &\cong \int_{b \in B} \text{Hom}_{\{t, f\}}(R_{-1}^b, V_{-2}^b), \end{aligned}$$

where we have used **Proposition 2.4.1.1**.

²²*Further Terminology:* The set $R_{-1}(V)$ is called the **strong inverse image of V by R** .

Proof. We have

$$\begin{aligned}
 \text{Rift}_R(V) &\cong \int_{b \in B} \text{Hom}_{\{t,f\}}(R_{-1}^b, V_{-2}^b) \\
 &= \left\{ a \in A \mid \int_{b \in B} \text{Hom}_{\{t,f\}}(R_a^b, V_{\star}^b) = \text{true} \right\} \\
 &= \left\{ a \in A \mid \begin{array}{l} \text{for each } b \in B, \text{ at least one of the} \\ \text{following conditions hold:} \\ \begin{array}{l} 1. \text{ We have } R_a^b = \text{false} \\ 2. \text{ The following conditions hold:} \\ \begin{array}{l} (a) \text{ We have } R_a^b = \text{true} \\ (b) \text{ We have } V_{\star}^b = \text{true} \end{array} \end{array} \end{array} \right\} \\
 &= \left\{ a \in A \mid \begin{array}{l} \text{for each } b \in B, \text{ at least one of the} \\ \text{following conditions hold:} \\ \begin{array}{l} 1. \text{ We have } b \notin R(a) \\ 2. \text{ The following conditions hold:} \\ \begin{array}{l} (a) \text{ We have } b \in R(a) \\ (b) \text{ We have } b \in V \end{array} \end{array} \end{array} \right\} \\
 &= \{ a \in A \mid \text{for each } b \in R(a), \text{ we have } b \in V \} \\
 &= \{ a \in A \mid R(a) \subset V \} \\
 &\stackrel{\text{def}}{=} R_{-1}(V).
 \end{aligned}$$

This finishes the proof. \square

Proposition 4.2.1.3. Let $R: A \dashv B$ be a relation.

1. *Functoriality.* The assignment $V \mapsto R_{-1}(V)$ defines a functor

$$R_{-1}: (\mathcal{P}(B), \subset) \rightarrow (\mathcal{P}(A), \subset)$$

where

- *Action on Objects.* For each $V \in \mathcal{P}(B)$, we have

$$[R_{-1}](V) \stackrel{\text{def}}{=} R_{-1}(V).$$

· *Action on Morphisms.* For each $U, V \in \mathcal{P}(B)$:

- If $U \subset V$, then $R_{-1}(U) \subset R_{-1}(V)$.

2. *Adjointness.* We have an adjunction

$$(R_* \dashv R_{-1}): \mathcal{P}(A) \begin{array}{c} \xrightarrow{R_*} \\ \perp \\ \xleftarrow{R_{-1}} \end{array} \mathcal{P}(B),$$

witnessed by a bijections of sets

$$\mathrm{Hom}_{\mathcal{P}(A)}(R_*(U), V) \cong \mathrm{Hom}_{\mathcal{P}(A)}(U, R_{-1}(V)),$$

natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$, i.e. such that:

(★) The following conditions are equivalent:

- We have $R_*(U) \subset V$.
- We have $U \subset R_{-1}(V)$.

3. *Lax Preservation of Colimits.* We have an inclusion of sets

$$\bigcup_{i \in I} R_{-1}(U_i) \subset R_{-1}\left(\bigcup_{i \in I} U_i\right),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(B)^{\times I}$. In particular, we have inclusions

$$\begin{aligned} R_{-1}(U) \cup R_{-1}(V) &\subset R_{-1}(U \cup V), \\ \emptyset &\subset R_{-1}(\emptyset), \end{aligned}$$

natural in $U, V \in \mathcal{P}(B)$.

4. *Preservation of Limits.* We have an equality of sets

$$R_{-1}\left(\bigcap_{i \in I} U_i\right) = \bigcap_{i \in I} R_{-1}(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(B)^{\times I}$. In particular, we have equalities

$$\begin{aligned} R_{-1}(U \cap V) &= R_{-1}(U) \cap R_{-1}(V), \\ R_{-1}(B) &= B, \end{aligned}$$

natural in $U, V \in \mathcal{P}(B)$.

5. *Symmetric Lax Monoidality With Respect to Unions.* The direct image with compact support function of **Item 1** has a symmetric lax monoidal structure

$$(R_{-1}, R_{-1}^{\otimes}, R_{-1|\mathbb{1}}): (\mathcal{P}(A), \cup, \emptyset) \rightarrow (\mathcal{P}(B), \cup, \emptyset),$$

being equipped with inclusions

$$\begin{aligned} R_{-1|U,V}^{\otimes}: R_{-1}(U) \cup R_{-1}(V) &\subset R_{-1}(U \cup V), \\ R_{-1|\mathbb{1}}^{\otimes}: \emptyset &\subset R_{-1}(\emptyset), \end{aligned}$$

natural in $U, V \in \mathcal{P}(B)$.

6. *Symmetric Strict Monoidality With Respect to Intersections.* The direct image function of **Item 1** has a symmetric strict monoidal structure

$$(R_{-1}, R_{-1}^{\otimes}, R_{-1|\mathbb{1}}): (\mathcal{P}(A), \cap, A) \rightarrow (\mathcal{P}(B), \cap, B),$$

being equipped with equalities

$$\begin{aligned} R_{-1|U,V}^{\otimes}: R_{-1}(U \cap V) &\xrightarrow{=} R_{-1}(U) \cap R_{-1}(V), \\ R_{-1|\mathbb{1}}^{\otimes}: R_{-1}(A) &\xrightarrow{=} B, \end{aligned}$$

natural in $U, V \in \mathcal{P}(B)$.

7. *Interaction With Weak Inverse Images I.* We have

$$R_{-1}(V) = A \setminus R^{-1}(B \setminus V)$$

for each $V \in \mathcal{P}(B)$.

8. *Interaction With Weak Inverse Images II.* Let $R: A \dashv B$ be a relation from A to B .

- (a) If R is a total relation, then we have an inclusion of sets

$$R_{-1}(V) \subset R^{-1}(V)$$

natural in $V \in \mathcal{P}(B)$.

- (b) If R is total and functional, then the above inclusion is in fact an equality.

(c) Conversely, if we have $R_{-1} = R^{-1}$, then R is total and functional.

Proof. **Item 1**, *Functoriality*: Clear.

Item 2, *Adjointness*: This follows from ??, ?? of ??.

Item 3, *Lax Preservation of Colimits*: Omitted.

Item 4, *Preservation of Limits*: This follows from **Item 2** and ??, ?? of ??.

Item 5, *Symmetric Lax Monoidality With Respect to Unions*: This follows from **Item 3**.

Item 6, *Symmetric Strict Monoidality With Respect to Intersections*: This follows from **Item 4**.

Item 7, *Interaction With Weak Inverse Images I*: We claim we have an equality

$$R_{-1}(B \setminus V) = A \setminus R^{-1}(V).$$

Indeed, we have

$$R_{-1}(B \setminus V) = \{a \in A \mid R(a) \subset B \setminus V\},$$

$$A \setminus R^{-1}(V) = \{a \in A \mid R(a) \cap V = \emptyset\}.$$

Taking $V = B \setminus V$ then implies the original statement.

Item 8, *Interaction With Weak Inverse Images II*: **Item 8a** is clear, while **Items 8b** and **8c** follow from **Item 6** of **Proposition 3.1.1.2**. \square

Proposition 4.2.1.4. Let $R: A \rightarrowtail B$ be a relation.

1. *Functionality I*. The assignment $R \mapsto R_{-1}$ defines a function

$$(-)_{-1}: \text{Sets}(A, B) \rightarrow \text{Sets}(\mathcal{P}(A), \mathcal{P}(B)).$$

2. *Functionality II*. The assignment $R \mapsto R_{-1}$ defines a function

$$(-)_{-1}: \text{Sets}(A, B) \rightarrow \text{Pos}((\mathcal{P}(A), \subset), (\mathcal{P}(B), \subset)).$$

3. *Interaction With Identities*. For each $A \in \text{Obj}(\text{Sets})$, we have

$$(\text{id}_A)_{-1} = \text{id}_{\mathcal{P}(A)}.$$

4. *Interaction With Composition*. For each pair of composable relations $R: A \rightarrowtail B$ and $S: B \rightarrowtail C$, we have

$$(S \diamond R)_{-1} = R_{-1} \circ S_{-1},$$

$$\begin{array}{ccc} \mathcal{P}(C) & \xrightarrow{S_{-1}} & \mathcal{P}(B) \\ & \searrow (S \diamond R)_{-1} & \downarrow R_{-1} \\ & & \mathcal{P}(A). \end{array}$$

Proof. **Item 1, Functionality I:** Clear.

Item 2, Functionality II: Clear.

Item 3, Interaction With Identities: Indeed, we have

$$\begin{aligned} (\chi_A)_{-1}(U) &\stackrel{\text{def}}{=} \{a \in A \mid \chi_A(a) \subset U\} \\ &\stackrel{\text{def}}{=} \{a \in A \mid \{a\} \subset U\} \\ &= U \end{aligned}$$

for each $U \in \mathcal{P}(A)$. Thus $(\chi_A)_{-1} = \text{id}_{\mathcal{P}(A)}$.

Item 4, Interaction With Composition: Indeed, we have

$$\begin{aligned} (S \diamond R)_{-1}(U) &\stackrel{\text{def}}{=} \{a \in A \mid [S \diamond R](a) \subset U\} \\ &\stackrel{\text{def}}{=} \{a \in A \mid S(R(a)) \subset U\} \\ &\stackrel{\text{def}}{=} \{a \in A \mid S_*(R(a)) \subset U\} \\ &= \{a \in A \mid R(a) \subset S_{-1}(U)\} \\ &\stackrel{\text{def}}{=} R_{-1}(S_{-1}(U)) \\ &\stackrel{\text{def}}{=} [R_{-1} \circ S_{-1}](U) \end{aligned}$$

for each $U \in \mathcal{P}(C)$, where we used **Item 2** of **Proposition 4.2.1.3**, which implies that the conditions

- We have $S_*(R(a)) \subset U$.
- We have $R(a) \subset S_{-1}(U)$.

are equivalent. Thus $(S \diamond R)_{-1} = R_{-1} \circ S_{-1}$. □

4.3 Weak Inverse Images

Let A and B be sets and let $R: A \rightarrowtail B$ be a relation.

Definition 4.3.1.1. The **weak inverse image function associated to R** ²³ is the function

$$R^{-1}: \mathcal{P}(B) \rightarrow \mathcal{P}(A)$$

defined by²⁴

$$R^{-1}(V) \stackrel{\text{def}}{=} \{a \in A \mid R(a) \cap V \neq \emptyset\}$$

for each $V \in \mathcal{P}(B)$.

²³*Further Terminology:* Also called simply the **inverse image function associated to R** .

²⁴*Further Terminology:* The set $R^{-1}(V)$ is called the **weak inverse image of V by R** or simply the

Remark 4.3.1.2. Identifying subsets of B with relations from B to pt via [Constructions With Sets, Item 3](#) of [Proposition 4.3.1.6](#), we see that the weak inverse image function associated to R is equivalently the function

$$R^{-1}: \underbrace{\mathcal{P}(B)}_{\cong \text{Rel}(B, \text{pt})} \rightarrow \underbrace{\mathcal{P}(A)}_{\cong \text{Rel}(A, \text{pt})}$$

defined by

$$R^{-1}(V) \stackrel{\text{def}}{=} V \diamond R$$

for each $V \in \mathcal{P}(A)$, where $R \diamond V$ is the composition

$$A \xrightarrow{R} B \xrightarrow{V} \text{pt}.$$

Explicitly, we have

$$\begin{aligned} R^{-1}(V) &\stackrel{\text{def}}{=} V \diamond R \\ &\stackrel{\text{def}}{=} \int^{b \in B} V_b^{-1} \times R_{-2}^b. \end{aligned}$$

inverse image of V by R .

Proof. We have

$$\begin{aligned}
 V \diamond R &\stackrel{\text{def}}{=} \int^{b \in B} V_b^{-1} \times R_{-2}^b \\
 &= \left\{ a \in A \mid \int^{b \in B} V_b^{\star} \times R_a^b = \text{true} \right\} \\
 &= \left\{ a \in A \mid \begin{array}{l} \text{there exists } b \in B \text{ such that the} \\ \text{following conditions hold:} \\ \quad 1. \text{ We have } V_b^{\star} = \text{true} \\ \quad 2. \text{ We have } R_a^b = \text{true} \end{array} \right\} \\
 &= \left\{ a \in A \mid \begin{array}{l} \text{there exists } b \in B \text{ such that the} \\ \text{following conditions hold:} \\ \quad 1. \text{ We have } b \in V \\ \quad 2. \text{ We have } b \in R(a) \end{array} \right\} \\
 &= \{a \in A \mid \text{there exists } b \in V \text{ such that } b \in R(a)\} \\
 &= \{a \in A \mid R(a) \cap V \neq \emptyset\} \\
 &\stackrel{\text{def}}{=} R^{-1}(V)
 \end{aligned}$$

This finishes the proof. \square

Proposition 4.3.1.3. Let $R: A \dashv B$ be a relation.

1. *Functoriality.* The assignment $V \mapsto R^{-1}(V)$ defines a functor

$$R^{-1}: (\mathcal{P}(B), \subset) \rightarrow (\mathcal{P}(A), \subset)$$

where

- *Action on Objects.* For each $V \in \mathcal{P}(B)$, we have

$$[R^{-1}](V) \stackrel{\text{def}}{=} R^{-1}(V).$$

- *Action on Morphisms.* For each $U, V \in \mathcal{P}(B)$:
 - If $U \subset V$, then $R^{-1}(U) \subset R^{-1}(V)$.

2. *Adjointness.* We have an adjunction

$$\left(R^{-1} \dashv R_!\right): \mathcal{P}(B) \begin{matrix} \xrightarrow{R^{-1}} \\ \perp \\ \xleftarrow{R_!} \end{matrix} \mathcal{P}(A),$$

witnessed by a bijections of sets

$$\mathrm{Hom}_{\mathcal{P}(A)}(R^{-1}(U), V) \cong \mathrm{Hom}_{\mathcal{P}(A)}(U, R_!(V)),$$

natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$, i.e. such that:

(★) The following conditions are equivalent:

- We have $R^{-1}(U) \subset V$.
- We have $U \subset R_!(V)$.

3. *Preservation of Colimits.* We have an equality of sets

$$R^{-1}\left(\bigcup_{i \in I} U_i\right) = \bigcup_{i \in I} R^{-1}(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(B)^{\times I}$. In particular, we have equalities

$$\begin{aligned} R^{-1}(U) \cup R^{-1}(V) &= R^{-1}(U \cup V), \\ R^{-1}(\emptyset) &= \emptyset, \end{aligned}$$

natural in $U, V \in \mathcal{P}(B)$.

4. *Oplax Preservation of Limits.* We have an inclusion of sets

$$R^{-1}\left(\bigcap_{i \in I} U_i\right) \subset \bigcap_{i \in I} R^{-1}(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(B)^{\times I}$. In particular, we have inclusions

$$\begin{aligned} R^{-1}(U \cap V) &\subset R^{-1}(U) \cap R^{-1}(V), \\ R^{-1}(A) &\subset B, \end{aligned}$$

natural in $U, V \in \mathcal{P}(B)$.

5. *Symmetric Strict Monoidality With Respect to Unions.* The direct image function of **Item 1** has a symmetric strict monoidal structure

$$(R^{-1}, R^{-1, \otimes}, R_{\perp}^{-1, \otimes}): (\mathcal{P}(A), \cup, \emptyset) \rightarrow (\mathcal{P}(B), \cup, \emptyset),$$

being equipped with equalities

$$\begin{aligned} R_{U,V}^{-1, \otimes}: R^{-1}(U) \cup R^{-1}(V) &\xrightarrow{=} R^{-1}(U \cup V), \\ R_{\perp}^{-1, \otimes}: \emptyset &\xrightarrow{=} \emptyset, \end{aligned}$$

natural in $U, V \in \mathcal{P}(B)$.

6. *Symmetric Oplax Monoidality With Respect to Intersections.* The direct image function of **Item 1** has a symmetric oplax monoidal structure

$$(R^{-1}, R^{-1, \otimes}, R_{\perp}^{-1, \otimes}): (\mathcal{P}(A), \cap, A) \rightarrow (\mathcal{P}(B), \cap, B),$$

being equipped with inclusions

$$\begin{aligned} R_{U,V}^{-1, \otimes}: R^{-1}(U \cap V) &\subset R^{-1}(U) \cap R^{-1}(V), \\ R_{\perp}^{-1, \otimes}: R^{-1}(A) &\subset B, \end{aligned}$$

natural in $U, V \in \mathcal{P}(B)$.

7. *Interaction With Strong Inverse Images I.* We have

$$R^{-1}(V) = A \setminus R_{-1}(B \setminus V)$$

for each $V \in \mathcal{P}(B)$.

8. *Interaction With Strong Inverse Images II.* Let $R: A \rightarrowtail B$ be a relation from A to B .

- (a) If R is a total relation, then we have an inclusion of sets

$$R_{-1}(V) \subset R^{-1}(V)$$

natural in $V \in \mathcal{P}(B)$.

- (b) If R is total and functional, then the above inclusion is in fact an equality.

(c) Conversely, if we have $R_{-1} = R^{-1}$, then R is total and functional.

Proof. **Item 1, Functoriality:** Clear.

Item 2, Adjointness: This follows from ??, ?? of ??.

Item 3, Preservation of Colimits: This follows from **Item 2** and ??, ?? of ??.

Item 4, Oplax Preservation of Limits: Omitted.

Item 5, Symmetric Strict Monoidality With Respect to Unions: This follows from **Item 3**.

Item 6, Symmetric Oplax Monoidality With Respect to Intersections: This follows from **Item 4**.

Item 7, Interaction With Strong Inverse Images I: This follows from **Item 7** of **Proposition 4.2.1.3**.

Item 8, Interaction With Strong Inverse Images II: This was proved in **Item 8** of **Proposition 4.2.1.3**. \square

Proposition 4.3.1.4. Let $R: A \dashrightarrow B$ be a relation.

1. *Functionality I.* The assignment $R \mapsto R^{-1}$ defines a function

$$(-)^{-1}: \text{Rel}(A, B) \rightarrow \text{Sets}(\mathcal{P}(A), \mathcal{P}(B)).$$

2. *Functionality II.* The assignment $R \mapsto R^{-1}$ defines a function

$$(-)^{-1}: \text{Rel}(A, B) \rightarrow \text{Pos}((\mathcal{P}(A), \subset), (\mathcal{P}(B), \subset)).$$

3. *Interaction With Identities.* For each $A \in \text{Obj}(\text{Sets})$, we have²⁵

$$(\chi_A)^{-1} = \text{id}_{\mathcal{P}(A)}.$$

4. *Interaction With Composition.* For each pair of composable relations $R: A \dashrightarrow$

²⁵That is, the postcomposition

$$(\chi_A)^{-1}: \text{Rel}(\text{pt}, A) \rightarrow \text{Rel}(\text{pt}, A)$$

is equal to $\text{id}_{\text{Rel}(\text{pt}, A)}$.

B and $S: B \rightarrow C$, we have²⁶

$$(S \diamond R)^{-1} = R^{-1} \circ S^{-1},$$

$$\begin{array}{ccc} \mathcal{P}(C) & \xrightarrow{S^{-1}} & \mathcal{P}(B) \\ & \searrow (S \diamond R)^{-1} & \downarrow R^{-1} \\ & & \mathcal{P}(A). \end{array}$$

Proof. **Item 1, Functionality I:** Clear.

Item 2, Functionality II: Clear.

Item 3, Interaction With Identities: This follows from **Categories, Item 5** of **Proposition 1.6.1.2**.

Item 4, Interaction With Composition: This follows from **Categories, Item 2** of **Proposition 1.6.1.2**. \square

4.4 Direct Images With Compact Support

Let A and B be sets and let $R: A \rightarrow B$ be a relation.

Definition 4.4.1.1. The **direct image with compact support function** associated to R is the function

$$R_! : \mathcal{P}(A) \rightarrow \mathcal{P}(B)$$

defined by^{27,28}

$$\begin{aligned} R_!(U) &\stackrel{\text{def}}{=} \left\{ b \in B \mid \begin{array}{l} \text{for each } a \in A, \text{ if we have} \\ b \in R(a), \text{ then } a \in U \end{array} \right\} \\ &= \{ b \in B \mid R^{-1}(b) \subset U \} \end{aligned}$$

²⁶That is, we have

$$(S \diamond R)^{-1} = R^{-1} \circ S^{-1},$$

$$\begin{array}{ccc} \text{Rel}(\text{pt}, C) & \xrightarrow{R^{-1}} & \text{Rel}(\text{pt}, B) \\ & \searrow (S \diamond R)^{-1} & \downarrow S^{-1} \\ & & \text{Rel}(\text{pt}, A). \end{array}$$

²⁷*Further Terminology:* The set $R_!(U)$ is called the **direct image with compact support of U by R** .

²⁸We also have

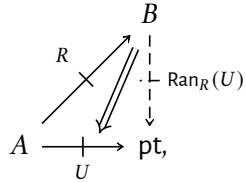
$$R_!(U) = B \setminus R_*(A \setminus U);$$

for each $U \in \mathcal{P}(A)$.

Remark 4.4.1.2. Identifying subsets of B with relations from pt to B via **Constructions With Sets, Item 3** of **Proposition 4.3.1.6**, we see that the direct image with compact support function associated to R is equivalently the function

$$R_! : \underbrace{\mathcal{P}(A)}_{\cong \text{Rel}(A, \text{pt})} \rightarrow \underbrace{\mathcal{P}(B)}_{\cong \text{Rel}(B, \text{pt})}$$

defined by

$$R_!(U) \stackrel{\text{def}}{=} \text{Ran}_R(U),$$


being explicitly computed by

$$\begin{aligned} R^*(U) &\stackrel{\text{def}}{=} \text{Ran}_R(U) \\ &\cong \int_{a \in A} \text{Hom}_{\{\text{t}, \text{f}\}}(R_a^{-2}, U_a^{-1}), \end{aligned}$$

where we have used **Proposition 2.3.1.1**.

see **Item 7** of **Proposition 4.4.1.3**.

Proof. We have

$$\begin{aligned}
 \text{Ran}_R(V) &\cong \int_{a \in A} \text{Hom}_{\{t,f\}}(R_a^{-2}, U_a^{-1}) \\
 &= \left\{ b \in B \mid \int_{a \in A} \text{Hom}_{\{t,f\}}(R_a^b, U_a^\star) = \text{true} \right\} \\
 &= \left\{ b \in B \mid \begin{array}{l} \text{for each } a \in A, \text{ at least one of the} \\ \text{following conditions hold:} \\ \begin{array}{l} 1. \text{ We have } R_a^b = \text{false} \\ 2. \text{ The following conditions hold:} \\ \begin{array}{l} (a) \text{ We have } R_a^b = \text{true} \\ (b) \text{ We have } U_a^\star = \text{true} \end{array} \end{array} \end{array} \right\} \\
 &= \left\{ b \in B \mid \begin{array}{l} \text{for each } a \in A, \text{ at least one of the} \\ \text{following conditions hold:} \\ \begin{array}{l} 1. \text{ We have } b \notin R(a) \\ 2. \text{ The following conditions hold:} \\ \begin{array}{l} (a) \text{ We have } b \in R(a) \\ (b) \text{ We have } a \in U \end{array} \end{array} \end{array} \right\} \\
 &= \left\{ b \in B \mid \begin{array}{l} \text{for each } a \in A, \text{ if we have} \\ b \in R(a), \text{ then } a \in U \end{array} \right\} \\
 &= \{ b \in B \mid R^{-1}(b) \subset U \} \\
 &\stackrel{\text{def}}{=} R^{-1}(U).
 \end{aligned}$$

This finishes the proof. \square

Proposition 4.4.1.3. Let $R: A \rightarrow B$ be a relation.

1. *Functoriality.* The assignment $U \mapsto R_!(U)$ defines a functor

$$R_!: (\mathcal{P}(A), \subset) \rightarrow (\mathcal{P}(B), \subset)$$

where

- *Action on Objects.* For each $U \in \mathcal{P}(A)$, we have

$$[R_!](U) \stackrel{\text{def}}{=} R_!(U).$$

- *Action on Morphisms.* For each $U, V \in \mathcal{P}(A)$:
 - If $U \subset V$, then $R_!(U) \subset R_!(V)$.

2. *Adjointness.* We have an adjunction

$$\left(R^{-1} \dashv R_!\right): \mathcal{P}(B) \begin{array}{c} \xrightarrow{R^{-1}} \\ \perp \\ \xleftarrow{R_!} \end{array} \mathcal{P}(A),$$

witnessed by a bijections of sets

$$\mathrm{Hom}_{\mathcal{P}(A)}(R^{-1}(U), V) \cong \mathrm{Hom}_{\mathcal{P}(A)}(U, R_!(V)),$$

natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$, i.e. such that:

(★) The following conditions are equivalent:

- We have $R^{-1}(U) \subset V$.
- We have $U \subset R_!(V)$.

3. *Lax Preservation of Colimits.* We have an inclusion of sets

$$\bigcup_{i \in I} R_!(U_i) \subset R_!\left(\bigcup_{i \in I} U_i\right),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have inclusions

$$\begin{aligned} R_!(U) \cup R_!(V) &\subset R_!(U \cup V), \\ \emptyset &\subset R_!(\emptyset), \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

4. *Preservation of Limits.* We have an equality of sets

$$R_!\left(\bigcap_{i \in I} U_i\right) = \bigcap_{i \in I} R_!(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have equalities

$$\begin{aligned} R_!(U \cap V) &= R_!(U) \cap R_!(V), \\ R_!(A) &= B, \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

5. *Symmetric Lax Monoidality With Respect to Unions.* The direct image with compact support function of **Item 1** has a symmetric lax monoidal structure

$$(R_!, R_!^\otimes, R_{!|\mathbb{1}}^\otimes): (\mathcal{P}(A), \cup, \emptyset) \rightarrow (\mathcal{P}(B), \cup, \emptyset),$$

being equipped with inclusions

$$\begin{aligned} R_{!|U,V}^\otimes: R_!(U) \cup R_!(V) &\subset R_!(U \cup V), \\ R_{!|\mathbb{1}}^\otimes: \emptyset &\subset R_!(\emptyset), \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

6. *Symmetric Strict Monoidality With Respect to Intersections.* The direct image function of **Item 1** has a symmetric strict monoidal structure

$$(R_!, R_!^\otimes, R_{!|\mathbb{1}}^\otimes): (\mathcal{P}(A), \cap, A) \rightarrow (\mathcal{P}(B), \cap, B),$$

being equipped with equalities

$$\begin{aligned} R_{!|U,V}^\otimes: R_!(U \cap V) &\xrightarrow{=} R_!(U) \cap R_!(V), \\ R_{!|\mathbb{1}}^\otimes: R_!(A) &\xrightarrow{=} B, \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

7. *Relation to Direct Images.* We have

$$R_!(U) = B \setminus R_*(A \setminus U)$$

for each $U \in \mathcal{P}(A)$.

Proof. **Item 1**, *Functoriality*: Clear.

Item 2, *Adjointness*: This follows from ??, ?? of ??.

Item 3, *Lax Preservation of Colimits*: Omitted.

Item 4, *Preservation of Limits*: This follows from **Item 2** and ??, ?? of ??.

Item 5, *Symmetric Lax Monoidality With Respect to Unions*: This follows from **Item 3**.

Item 6, *Symmetric Strict Monoidality With Respect to Intersections*: This follows from **Item 4**.

Item 7, *Relation to Direct Images*: This follows from **Item 7** of **Proposition 4.1.1.3**.

Alternatively, we may prove it directly as follows, with the proof proceeding in the same way as in the case of functions (**Constructions With Sets**, **Item 9** of **Proposition 4.6.1.6**).

We claim that $R_!(U) = B \setminus R_*(A \setminus U)$:

- *The First Implication.* We claim that

$$R_!(U) \subset B \setminus R_*(A \setminus U).$$

Let $b \in R_!(U)$. We need to show that $b \notin R_*(A \setminus U)$, i.e. that there is no $a \in A \setminus U$ such that $b \in R(a)$.

This is indeed the case, as otherwise we would have $a \in R^{-1}(b)$ and $a \notin U$, contradicting $R^{-1}(b) \subset U$ (which holds since $b \in R_!(U)$).

Thus $b \in B \setminus R_*(A \setminus U)$.

- *The Second Implication.* We claim that

$$B \setminus R_*(A \setminus U) \subset R_!(U).$$

Let $b \in B \setminus R_*(A \setminus U)$. We need to show that $b \in R_!(U)$, i.e. that $R^{-1}(b) \subset U$.

Since $b \notin R_*(A \setminus U)$, there exists no $a \in A \setminus U$ such that $b \in R(a)$, and hence $R^{-1}(b) \subset U$.

Thus $b \in R_!(U)$.

This finishes the proof. \square

Proposition 4.4.1.4. Let $R: A \rightarrowtail B$ be a relation.

1. *Functionality I.* The assignment $R \mapsto R_!$ defines a function

$$(-)_!: \text{Sets}(A, B) \rightarrow \text{Sets}(\mathcal{P}(A), \mathcal{P}(B)).$$

2. *Functionality II.* The assignment $R \mapsto R_!$ defines a function

$$(-)_!: \text{Sets}(A, B) \rightarrow \text{Hom}_{\text{Pos}}((\mathcal{P}(A), \subset), (\mathcal{P}(B), \subset)).$$

3. *Interaction With Identities.* For each $A \in \text{Obj}(\text{Sets})$, we have

$$(\text{id}_A)_! = \text{id}_{\mathcal{P}(A)}.$$

4. *Interaction With Composition.* For each pair of composable relations $R: A \rightarrowtail B$ and $S: B \rightarrowtail C$, we have

$$(S \diamond R)_! = S_! \circ R_!,$$

$$\begin{array}{ccc} \mathcal{P}(A) & \xrightarrow{R_!} & \mathcal{P}(B) \\ & \searrow (S \diamond R)_! & \downarrow S_! \\ & & \mathcal{P}(C). \end{array}$$

Proof. **Item 1, Functionality I:** Clear.

Item 2, Functionality II: Clear.

Item 3, Interaction With Identities: Indeed, we have

$$\begin{aligned} (\chi_A)_!(U) &\stackrel{\text{def}}{=} \{a \in A \mid \chi_A^{-1}(a) \subset U\} \\ &\stackrel{\text{def}}{=} \{a \in A \mid \{a\} \subset U\} \\ &= U \end{aligned}$$

for each $U \in \mathcal{P}(A)$. Thus $(\chi_A)_! = \text{id}_{\mathcal{P}(A)}$.

Item 4, Interaction With Composition: Indeed, we have

$$\begin{aligned} (S \diamond R)_!(U) &\stackrel{\text{def}}{=} \{c \in C \mid [S \diamond R]^{-1}(c) \subset U\} \\ &\stackrel{\text{def}}{=} \{c \in C \mid S^{-1}(R^{-1}(c)) \subset U\} \\ &= \{c \in C \mid R^{-1}(c) \subset S_!(U)\} \\ &\stackrel{\text{def}}{=} R_!(S_!(U)) \\ &\stackrel{\text{def}}{=} [R_! \circ S_!](U) \end{aligned}$$

for each $U \in \mathcal{P}(C)$, where we used **Item 2** of **Proposition 4.4.1.3**, which implies that the conditions

- We have $S^{-1}(R^{-1}(c)) \subset U$.
- We have $R^{-1}(c) \subset S_!(U)$.

are equivalent. Thus $(S \diamond R)_! = S_! \circ R_!$. □

4.5 Functoriality of Powersets

Proposition 4.5.1.1. The assignment $X \mapsto \mathcal{P}(X)$ defines functors²⁹

$$\begin{aligned} \mathcal{P}_* &: \text{Rel} \rightarrow \text{Sets}, \\ \mathcal{P}_{-1} &: \text{Rel}^{\text{op}} \rightarrow \text{Sets}, \\ \mathcal{P}^{-1} &: \text{Rel}^{\text{op}} \rightarrow \text{Sets}, \\ \mathcal{P}_! &: \text{Rel} \rightarrow \text{Sets} \end{aligned}$$

where

²⁹The functor $\mathcal{P}_* : \text{Rel} \rightarrow \text{Sets}$ admits a left adjoint; see **Item 3** of **Proposition 3.1.1.2**.

- *Action on Objects.* For each $A \in \text{Obj}(\text{Rel})$, we have

$$\begin{aligned}\mathcal{P}_*(A) &\stackrel{\text{def}}{=} \mathcal{P}(A), \\ \mathcal{P}_{-1}(A) &\stackrel{\text{def}}{=} \mathcal{P}(A), \\ \mathcal{P}^{-1}(A) &\stackrel{\text{def}}{=} \mathcal{P}(A), \\ \mathcal{P}_!(A) &\stackrel{\text{def}}{=} \mathcal{P}(A).\end{aligned}$$

- *Action on Morphisms.* For each morphism $R: A \rightarrowtail B$ of Rel , the images

$$\begin{aligned}\mathcal{P}_*(R): \mathcal{P}(A) &\rightarrow \mathcal{P}(B), \\ \mathcal{P}_{-1}(R): \mathcal{P}(B) &\rightarrow \mathcal{P}(A), \\ \mathcal{P}^{-1}(R): \mathcal{P}(B) &\rightarrow \mathcal{P}(A), \\ \mathcal{P}_!(R): \mathcal{P}(A) &\rightarrow \mathcal{P}(B)\end{aligned}$$

of R by \mathcal{P}_* , \mathcal{P}_{-1} , \mathcal{P}^{-1} , and $\mathcal{P}_!$ are defined by

$$\begin{aligned}\mathcal{P}_*(R) &\stackrel{\text{def}}{=} R_*, \\ \mathcal{P}_{-1}(R) &\stackrel{\text{def}}{=} R_{-1}, \\ \mathcal{P}^{-1}(R) &\stackrel{\text{def}}{=} R^{-1}, \\ \mathcal{P}_!(R) &\stackrel{\text{def}}{=} R_!,\end{aligned}$$

as in [Definitions 4.1.1.1, 4.2.1.1, 4.3.1.1](#) and [4.4.1.1](#).

Proof. This follows from [Items 3 and 4 of Proposition 4.1.1.4](#), [Items 3 and 4 of Proposition 4.2.1.4](#), [Items 3 and 4 of Proposition 4.3.1.4](#), and [Items 3 and 4 of Proposition 4.4.1.4](#). \square

4.6 Functoriality of Powersets: Relations on Powersets

Let A and B be sets and let $R: A \rightarrowtail B$ be a relation.

Definition 4.6.1.1. The **relation on powersets associated to R** is the relation

$$\mathcal{P}(R): \mathcal{P}(A) \rightarrowtail \mathcal{P}(B)$$

defined by³⁰

$$\mathcal{P}(R)_U^V \stackrel{\text{def}}{=} \mathbf{Rel}(\chi_{\text{pt}}, V \diamond R \diamond U)$$

for each $U \in \mathcal{P}(A)$ and each $V \in \mathcal{P}(B)$.

Remark 4.6.1.2. In detail, we have $U \sim_{\mathcal{P}(R)} V$ iff the following equivalent conditions hold:

- We have $\chi_{\text{pt}} \subset V \diamond R \diamond U$.
- We have $(V \diamond R \diamond U)_{\star}^{\star} = \text{true}$, i.e. we have

$$\int^{a \in A} \int^{b \in B} V_b^{\star} \times R_a^b \times U_{\star}^a = \text{true}.$$

- There exists some $a \in A$ and some $b \in B$ such that:
 - We have $U_{\star}^a = \text{true}$.
 - We have $R_a^b = \text{true}$.
 - We have $V_b^{\star} = \text{true}$.
- There exists some $a \in A$ and some $b \in B$ such that:
 - We have $a \in U$.
 - We have $a \sim_R b$.
 - We have $b \in V$.

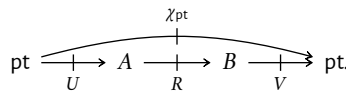
Proposition 4.6.1.3. The assignment $R \mapsto \mathcal{P}(R)$ defines a functor

$$\mathcal{P}: \text{Rel} \rightarrow \text{Rel}.$$

Proof. Omitted. □

Appendices

³⁰Illustration:



A Other Chapters

Sets

1. [Sets](#)
2. [Constructions With Sets](#)
3. [Pointed Sets](#)
4. [Tensor Products of Pointed Sets](#)

Relations

5. [Relations](#)

6. [Constructions With Relations](#)

7. [Equivalence Relations and Apartness Relations](#)

Category Theory

8. [Categories](#)

Bicategories

9. [Types of Morphisms in Bicategories](#)

References

- [MO 460656] [Emily de Oliveira Santos](#). *Existence and characterisations of left Kan extensions and liftings in the bicategory of relations I*. MathOverflow. URL: <https://mathoverflow.net/q/460656> (cit. on pp. 3, 4).
- [MO 461592] [Emily de Oliveira Santos](#). *Existence and characterisations of left Kan extensions and liftings in the bicategory of relations II*. MathOverflow. URL: <https://mathoverflow.net/q/461592> (cit. on pp. 4, 5).