Categories

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This chapter contains some elementary material about categories, functors, and natural transformations. Notably, we discuss and explore:

1. Categories (Section 1).
2. The quadruple adjunction $\pi_0 \dashv (-)_{\mathsf{disc}} \dashv \mathsf{Obj} \dashv (-)_{\mathsf{indisc}}$ between the category of categories and the category of sets (Section 2).
3. Groupoids, categories in which all morphisms admit inverses (Section 3).
4. Functors (Section 4).
5. The conditions one may impose on functors in decreasing order of importance:
    (a) Section 5 introduces the foundationally important conditions one may impose on functors, such as faithfulness, conservativity, essential surjectivity, etc.
    (b) Section 6 introduces more conditions one may impose on functors that are still important but less omni-present than those of Section 5, such as being dominant, being a monomorphism, being pseudomonic, etc.
    (c) Section 7 introduces some rather rare or uncommon conditions one may impose on functors that are nevertheless still useful to explicit record in this chapter.
6. Natural transformations (Section 8).
7. The various categorical and 2-categorical structures formed by categories, functors, and natural transformations (Section 9).

Contents

1 Categories

1.1 Foundations

Definition 1.1.1.1. A **category** $(\mathcal{C}, \circ^{\mathcal{C}}, \mathbb{1}^{\mathcal{C}})$ consists of:

- *Objects.* A class $\mathrm{Obj}(\mathcal{C})$ of **objects**.
- *Morphisms.* For each $A, B \in \mathrm{Obj}(\mathcal{C})$, a class $\mathrm{Hom}_{\mathcal{C}}(A, B)$, called the **class of morphisms of $\mathcal{C}$ from A to B**.
- *Identities.* For each $A \in \mathrm{Obj}(\mathcal{C})$, a map of sets

  $$\mathbb{1}^{\mathcal{C}}_A \colon \mathrm{pt} \to \mathrm{Hom}_{\mathcal{C}}(A, A),$$

  called the **unit map of $\mathcal{C}$ at A**, determining a morphism

  $$\mathrm{id}_A \colon A \to A$$

  of $\mathcal{C}$, called the **identity morphism of A**.
- *Composition.* For each $A, B, C \in \mathrm{Obj}(\mathcal{C})$, a map of sets

  $$\circ^{\mathcal{C}}_{A,B,C} \colon \mathrm{Hom}_{\mathcal{C}}(B, C) \times \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{C}}(A, C),$$

  called the **composition map of $\mathcal{C}$ at (A, B, C)**.

such that the following conditions are satisfied:

1. *Associativity.* The diagram

$$\begin{tikzcd}
 & \mathrm{Hom}_{\mathcal{C}}(C,D) \times (\mathrm{Hom}_{\mathcal{C}}(B,C) \times \mathrm{Hom}_{\mathcal{C}}(A,B)) \arrow[dr, "\mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(C,D)} \times \circ^{\mathcal{C}}_{A,B,C}"] & \\
(\mathrm{Hom}_{\mathcal{C}}(C,D) \times \mathrm{Hom}_{\mathcal{C}}(B,C)) \times \mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[ur, dashed, "\alpha^{\mathsf{Sets}}_{\mathrm{Hom}_{\mathcal{C}}(C,D),\mathrm{Hom}_{\mathcal{C}}(B,C),\mathrm{Hom}_{\mathcal{C}}(A,B)}", "\sim"'] \arrow[d, "\circ^{\mathcal{C}}_{B,C,D} \times \mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(A,B)}"'] & & \mathrm{Hom}_{\mathcal{C}}(C,D) \times \mathrm{Hom}_{\mathcal{C}}(A,C) \arrow[d, "\circ^{\mathcal{C}}_{A,C,D}"] \\
\mathrm{Hom}_{\mathcal{C}}(B,D) \times \mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[rr, "\circ^{\mathcal{C}}_{A,B,D}"'] & & \mathrm{Hom}_{\mathcal{C}}(A,D)
\end{tikzcd}$$

commutes, i.e. for each composable triple (f, g, h) of morphisms of C, we have

$$(f \circ g) \circ h = f \circ (g \circ h).$$

2. *Left Unitality.* The diagram

$$\begin{tikzcd}
\mathrm{pt} \times \mathrm{Hom}_C(A, B) \arrow[d, "\mathbb{1}^C_B \times \mathrm{id}_{\mathrm{Hom}_C(A,B)}"'] \arrow[drr, dashed, "\lambda^{\mathsf{Sets}}_{\mathrm{Hom}_C(A,B)}", "\sim"'] & & \\
\mathrm{Hom}_C(B, B) \times \mathrm{Hom}_C(A, B) \arrow[rr, "\circ^C_{A,B,B}"'] & & \mathrm{Hom}_C(A, B)
\end{tikzcd}$$

commutes, i.e. for each morphism $f : A \longrightarrow B$ of C, we have

$$\mathrm{id}_B \circ f = f.$$

3. *Right Unitality.* The diagram

$$\begin{tikzcd}
\mathrm{Hom}_C(A, B) \times \mathrm{pt} \arrow[d, "\mathrm{id}_{\mathrm{Hom}_C(A,B)} \times \mathbb{1}^C_A"'] \arrow[drr, dashed, "\rho^{\mathsf{Sets}}_{\mathrm{Hom}_C(A,B)}", "\sim"'] & & \\
\mathrm{Hom}_C(A, B) \times \mathrm{Hom}_C(A, A) \arrow[rr, "\circ^C_{A,A,B}"'] & & \mathrm{Hom}_C(A, B)
\end{tikzcd}$$

commutes, i.e. for each morphism $f : A \longrightarrow B$ of C, we have

$$f \circ \mathrm{id}_A = f.$$

Notation 1.1.1.2. Let C be a category.

1. We also write $C(A, B)$ for $\mathrm{Hom}_C(A, B)$.
2. We write $\mathrm{Mor}(C)$ for the class of all morphisms of C.

Definition 1.1.1.3. Let κ be a regular cardinal. A category C is

1. **Locally small** if, for each $A, B \in \mathrm{Obj}(C)$, the class $\mathrm{Hom}_C(A, B)$ is a set.
2. **Locally essentially small** if, for each $A, B \in \mathrm{Obj}(C)$, the class

$$\mathrm{Hom}_C(A, B)/\{\text{isomorphisms}\}$$

is a set.

3. **Small** if C is locally small and $\mathrm{Obj}(C)$ is a set.
4. κ-**Small** if C is locally small, $\mathrm{Obj}(C)$ is a set, and we have $\#\mathrm{Obj}(C) < \kappa$.

1.2 Examples of Categories

Example 1.2.1.1. The **punctual category**[1] is the category pt where

- *Objects.* We have
$$\mathrm{Obj}(\mathrm{pt}) \stackrel{\mathrm{def}}{=} \{\star\}.$$
- *Morphisms.* The unique Hom-set of pt is defined by
$$\mathrm{Hom}_{\mathsf{pt}}(\star, \star) \stackrel{\mathrm{def}}{=} \{\mathrm{id}_\star\}.$$
- *Identities.* The unit map
$$\mathbb{1}^{\mathsf{pt}}_\star \colon \mathrm{pt} \to \mathrm{Hom}_{\mathsf{pt}}(\star, \star)$$
of pt at $\star$ is defined by
$$\mathrm{id}^{\mathsf{pt}}_\star \stackrel{\mathrm{def}}{=} \mathrm{id}_\star.$$
- *Composition.* The composition map
$$\circ^{\mathsf{pt}}_{\star,\star,\star} \colon \mathrm{Hom}_{\mathsf{pt}}(\star, \star) \times \mathrm{Hom}_{\mathsf{pt}}(\star, \star) \to \mathrm{Hom}_{\mathsf{pt}}(\star, \star)$$
of pt at $(\star, \star, \star)$ is given by the bijection $\mathrm{pt} \times \mathrm{pt} \cong \mathrm{pt}$.

Example 1.2.1.2. We have an isomorphism of categories[2]

$$\mathsf{Mon} \cong \mathsf{pt} \underset{\mathsf{Sets}}{\times} \mathsf{Cats}, \qquad \begin{tikzcd} \mathsf{Mon} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Cats} \arrow[d, "\mathrm{Obj}"] \\ \mathsf{pt} \arrow[r, "{[\mathrm{pt}]}"'] & \mathsf{Sets} \end{tikzcd}$$

[1] *Further Terminology:* Also called the **singleton category**.

[2] This can be enhanced to an isomorphism of 2-categories

$$\mathsf{Mon}_{\mathsf{2disc}} \cong \mathsf{pt}_{\mathsf{bi}} \underset{\mathsf{Sets}_{\mathsf{2disc}}}{\times} \mathsf{Cats}_{\mathsf{2},*}, \qquad \begin{tikzcd} \mathsf{Mon}_{\mathsf{2disc}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Cats}_{\mathsf{2},*} \arrow[d, "\mathrm{Obj}"] \\ \mathsf{pt}_{\mathsf{bi}} \arrow[r, "{[\mathrm{pt}]}"'] & \mathsf{Sets}_{\mathsf{2disc}} \end{tikzcd}$$

via the delooping functor $\mathsf{B}\colon \mathsf{Mon} \to \mathsf{Cats}$ of **??** of **??**, exhibiting monoids as exactly those categories having a single object.

Proof. Omitted. □

Example 1.2.1.3. The **empty category** is the category $\emptyset_{\mathsf{cat}}$ where

- *Objects.* We have
$$\mathrm{Obj}(\emptyset_{\mathsf{cat}}) \stackrel{\text{def}}{=} \emptyset.$$
- *Morphisms.* We have
$$\mathrm{Mor}(\emptyset_{\mathsf{cat}}) \stackrel{\text{def}}{=} \emptyset.$$
- *Identities and Composition.* Having no objects, $\emptyset_{\mathsf{cat}}$ has no unit nor composition maps.

Example 1.2.1.4. The n**th ordinal category** is the category $\mathbb{n}$ where[3]

- *Objects.* We have
$$\mathrm{Obj}(\mathbb{n}) \stackrel{\text{def}}{=} \{[0], \ldots, [n]\}.$$

between the discrete 2-category $\mathsf{Mon}_{\mathsf{2disc}}$ on Mon and the 2-category of pointed categories with one object.

[3]In other words, $\mathbb{n}$ is the category associated to the poset
$$[0] \to [1] \to \cdots \to [n-1] \to [n].$$
The category $\mathbb{n}$ for $n \geq 2$ may also be defined in terms of $\mathbb{0}$ and joins (**??**, **??**): we have isomorphisms of categories
$$\begin{aligned}
\mathbb{1} &\cong \mathbb{0} \star \mathbb{0},\\
\mathbb{2} &\cong \mathbb{1} \star \mathbb{0}\\
&\cong (\mathbb{0} \star \mathbb{0}) \star \mathbb{0},\\
\mathbb{3} &\cong \mathbb{2} \star \mathbb{0}\\
&\cong (\mathbb{1} \star \mathbb{0}) \star \mathbb{0}\\
&\cong ((\mathbb{0} \star \mathbb{0}) \star \mathbb{0}) \star \mathbb{0},\\
\mathbb{4} &\cong \mathbb{3} \star \mathbb{0}\\
&\cong (\mathbb{2} \star \mathbb{0}) \star \mathbb{0}\\
&\cong ((\mathbb{1} \star \mathbb{0}) \star \mathbb{0}) \star \mathbb{0}\\
&\cong (((\mathbb{0} \star \mathbb{0}) \star \mathbb{0}) \star \mathbb{0}) \star \mathbb{0},
\end{aligned}$$
and so on.

- *Morphisms.* For each $[i], [j] \in \mathrm{Obj}(\mathbb{n})$, we have

$$\mathrm{Hom}_{\mathbb{n}}([i],[j]) \stackrel{\text{def}}{=} \begin{cases} \{\mathrm{id}_{[i]}\} & \text{if } [i]=[j], \\ \{[i] \to [j]\} & \text{if } [j] < [i], \\ \emptyset & \text{if } [j] > [i]. \end{cases}$$

- *Identities.* For each $[i] \in \mathrm{Obj}(\mathbb{n})$, the unit map

$$\mathbb{1}^{\mathbb{n}}_{[i]} \colon \mathrm{pt} \to \mathrm{Hom}_{\mathbb{n}}([i],[i])$$

of $\mathbb{n}$ at $[i]$ is defined by

$$\mathrm{id}^{\mathbb{n}}_{[i]} \stackrel{\text{def}}{=} \mathrm{id}_{[i]}.$$

- *Composition.* For each $[i], [j], [k] \in \mathrm{Obj}(\mathbb{n})$, the composition map

$$\circ^{\mathbb{n}}_{[i],[j],[k]} \colon \mathrm{Hom}_{\mathbb{n}}([j],[k]) \times \mathrm{Hom}_{\mathbb{n}}([i],[j]) \to \mathrm{Hom}_{\mathbb{n}}([i],[k])$$

of $\mathbb{n}$ at $([i],[j],[k])$ is defined by

$$\begin{aligned} \mathrm{id}_{[i]} \circ \mathrm{id}_{[i]} &= \mathrm{id}_{[i]}, \\ ([j] \to [k]) \circ ([i] \to [j]) &= ([i] \to [k]). \end{aligned}$$

Example 1.2.1.5. Here we list some of the other categories appearing throughout this work.

1. The category Sets_* of pointed sets of Pointed Sets, Definition 1.3.1.1.
2. The category Rel of sets and relations of Relations, Definition 2.1.1.1.
3. The category $\mathsf{Span}(A, B)$ of spans from a set A to a set B of **??**, **??**.
4. The category $\mathsf{ISets}(K)$ of K-indexed sets of **??**, **??**.
5. The category ISets of indexed sets of **??**, **??**.
6. The category $\mathsf{FibSets}(K)$ of K-fibred sets of **??**, **??**.
7. The category $\mathsf{FibSets}$ of fibred sets of **??**, **??**.
8. Categories of functors $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$ as in Definition 9.1.1.1.
9. The category of categories Cats of Definition 9.2.1.1.
10. The category of groupoids Grpd of Definition 9.4.1.1.

1.3 Posetal Categories

Definition 1.3.1.1. Let $(X, \preceq_X)$ be a poset.

1. The **posetal category associated to** $(X, \preceq_X)$ is the category X_{pos} where
   - *Objects.* We have
     $$\mathrm{Obj}(X_{\mathsf{pos}}) \overset{\text{def}}{=} X.$$
   - *Morphisms.* For each $a, b \in \mathrm{Obj}(X_{\mathsf{pos}})$, we have
     $$\mathrm{Hom}_{X_{\mathsf{pos}}}(a, b) \overset{\text{def}}{=} \begin{cases} \mathrm{pt} & \text{if } a \preceq_X b, \\ \emptyset & \text{otherwise.} \end{cases}$$
   - *Identities.* For each $a \in \mathrm{Obj}(X_{\mathsf{pos}})$, the unit map
     $$\mathbb{1}^{X_{\mathsf{pos}}}_a \colon \mathrm{pt} \to \mathrm{Hom}_{X_{\mathsf{pos}}}(a, a)$$
     of X_{pos} at a is given by the identity map.
   - *Composition.* For each $a, b, c \in \mathrm{Obj}(X_{\mathsf{pos}})$, the composition map
     $$\circ^{X_{\mathsf{pos}}}_{a,b,c} \colon \mathrm{Hom}_{X_{\mathsf{pos}}}(b, c) \times \mathrm{Hom}_{X_{\mathsf{pos}}}(a, b) \to \mathrm{Hom}_{X_{\mathsf{pos}}}(a, c)$$
     of X_{pos} at (a, b, c) is defined as either the inclusion $\emptyset \hookrightarrow \mathrm{pt}$ or the identity map of pt, depending on whether we have $a \preceq_X b$, $b \preceq_X c$, and $a \preceq_X c$.
2. A category C is **posetal**[4] if C is equivalent to X_{pos} for some poset $(X, \preceq_X)$.

Proposition 1.3.1.2. Let $(X, \preceq_X)$ be a poset and let C be a category.

1. *Functoriality.* The assignment $(X, \preceq_X) \mapsto X_{\mathsf{pos}}$ defines a functor
   $$(-)_{\mathsf{pos}} \colon \mathsf{Pos} \to \mathsf{Cats}.$$
2. *Fully Faithfulness.* The functor $(-)_{\mathsf{pos}}$ of Item 1 is fully faithful.
3. *Characterisations.* The following conditions are equivalent:

[4] *Further Terminology:* Also called a **thin** category or a $(0, 1)$-**category**.

(a) The category $\mathcal{C}$ is posetal.

(b) For each $A, B \in \mathrm{Obj}(\mathcal{C})$ and each $f, g \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, we have $f = g$.

Proof. *Item 1, Functoriality*: Omitted.
Item 2, Fully Faithfulness: Omitted.
Item 3, Characterisations: Clear. □

1.4 Subcategories

Let $\mathcal{C}$ be a category.

Definition 1.4.1.1. A **subcategory** of $\mathcal{C}$ is a category $\mathcal{A}$ satisfying the following conditions:

1. *Objects.* We have $\mathrm{Obj}(\mathcal{A}) \subset \mathrm{Obj}(\mathcal{C})$.

2. *Morphisms.* For each $A, B \in \mathrm{Obj}(\mathcal{A})$, we have
$$\mathrm{Hom}_{\mathcal{A}}(A, B) \subset \mathrm{Hom}_{\mathcal{C}}(A, B).$$

3. *Identities.* For each $A \in \mathrm{Obj}(\mathcal{A})$, we have
$$\mathbb{1}^{\mathcal{A}}_{A} = \mathbb{1}^{\mathcal{C}}_{A}.$$

4. *Composition.* For each $A, B, C \in \mathrm{Obj}(\mathcal{A})$, we have
$$\circ^{\mathcal{A}}_{A,B,C} = \circ^{\mathcal{C}}_{A,B,C}.$$

Definition 1.4.1.2. A subcategory $\mathcal{A}$ of $\mathcal{C}$ is **full** if the canonical inclusion functor $\mathcal{A} \to \mathcal{C}$ is full, i.e. if, for each $A, B \in \mathrm{Obj}(\mathcal{A})$, the inclusion
$$\iota_{A,B} \colon \mathrm{Hom}_{\mathcal{A}}(A, B) \hookrightarrow \mathrm{Hom}_{\mathcal{C}}(A, B)$$
is surjective (and thus bijective).

Definition 1.4.1.3. A subcategory $\mathcal{A}$ of a category $\mathcal{C}$ is **strictly full** if it satisfies the following conditions:

1. *Fullness.* The subcategory $\mathcal{A}$ is full.

2. *Closedness Under Isomorphisms.* The class $\mathrm{Obj}(\mathcal{A})$ is closed under isomorphisms.[5]

Definition 1.4.1.4. A subcategory $\mathcal{A}$ of C is **wide**[6] if $\mathrm{Obj}(\mathcal{A}) = \mathrm{Obj}(C)$.

1.5 Skeletons of Categories

Definition 1.5.1.1. A[7] **skeleton** of a category C is a full subcategory $\mathsf{Sk}(C)$ with one object from each isomorphism class of objects of C.

Definition 1.5.1.2. A category C is **skeletal** if $C \cong \mathsf{Sk}(C)$.[8]

Proposition 1.5.1.3. Let C be a category.

1. *Existence.* Assuming the axiom of choice, $\mathsf{Sk}(C)$ always exists.
2. *Pseudofunctoriality.* The assignment $C \mapsto \mathsf{Sk}(C)$ defines a pseudofunctor
$$\mathsf{Sk}\colon \mathsf{Cats}_2 \to \mathsf{Cats}_2.$$
3. *Uniqueness Up to Equivalence.* Any two skeletons of C are equivalent.
4. *Inclusions of Skeletons Are Equivalences.* The inclusion
$$\iota_C\colon \mathsf{Sk}(C) \hookrightarrow C$$
of a skeleton of C into C is an equivalence of categories.

Proof. *Item 1, Existence*: See [nLab23, Section "Existence of Skeletons of Categories"].
Item 2, Pseudofunctoriality: See [nLab23, Section "Skeletons as an Endo-Pseudofunctor on $\mathfrak{Cat}$"].
Item 3, Uniqueness Up to Equivalence: Clear.
Item 4, Inclusions of Skeletons Are Equivalences: Clear. □

[5] That is, given $A \in \mathrm{Obj}(\mathcal{A})$ and $C \in \mathrm{Obj}(C)$, if $C \cong A$, then $C \in \mathrm{Obj}(\mathcal{A})$.
[6] *Further Terminology:* Also called **lluf**.
[7] Due to Item 3 of Proposition 1.5.1.3, we often refer to any such full subcategory $\mathsf{Sk}(C)$ of C as *the* skeleton of C.
[8] That is, C is **skeletal** if isomorphic objects of C are equal.

1.6 Precomposition and Postcomposition

Let $\mathcal{C}$ be a category and let $A, B, C \in \mathrm{Obj}(\mathcal{C})$.

Definition 1.6.1.1. Let $f : A \to B$ and $g : B \to C$ be morphisms of $\mathcal{C}$.

1. The **precomposition function associated to** f is the function

   $$f^* : \mathrm{Hom}_{\mathcal{C}}(B, C) \to \mathrm{Hom}_{\mathcal{C}}(A, C)$$

   defined by

   $$f^*(\phi) \stackrel{\mathrm{def}}{=} \phi \circ f$$

   for each $\phi \in \mathrm{Hom}_{\mathcal{C}}(B, C)$.

2. The **postcomposition function associated to** g is the function

   $$g_* : \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{C}}(A, C)$$

   defined by

   $$g_*(\phi) \stackrel{\mathrm{def}}{=} g \circ \phi$$

   for each $\phi \in \mathrm{Hom}_{\mathcal{C}}(A, B)$.

Proposition 1.6.1.2. Let $A, B, C, D \in \mathrm{Obj}(\mathcal{C})$ and let $f : A \to B$ and $g : B \to C$ be morphisms of $\mathcal{C}$.

1. *Interaction Between Precomposition and Postcomposition.* We have

   $$g_* \circ f^* = f^* \circ g_*, \qquad \begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(B, C) \arrow[r, "g_*"] \arrow[d, "f^*"'] & \mathrm{Hom}_{\mathcal{C}}(B, D) \arrow[d, "f^*"] \\ \mathrm{Hom}_{\mathcal{C}}(A, C) \arrow[r, "g_*"'] & \mathrm{Hom}_{\mathcal{C}}(A, D). \end{tikzcd}$$

2. *Interaction With Composition I.* We have

$$(g \circ f)^* = f^* \circ g^*, \qquad \begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(X,A) \arrow[r, "f_*"] \arrow[dr, "(g\circ f)_*"'] & \mathrm{Hom}_{\mathcal{C}}(X,B) \arrow[d, "g_*"] \\ & \mathrm{Hom}_{\mathcal{C}}(X,C), \end{tikzcd}$$

$$(g \circ f)_* = g_* \circ f_*, \qquad \begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(C,X) \arrow[r, "g^*"] \arrow[dr, "(g\circ f)^*"'] & \mathrm{Hom}_{\mathcal{C}}(B,X) \arrow[d, "f^*"] \\ & \mathrm{Hom}_{\mathcal{C}}(A,X). \end{tikzcd}$$

3. *Interaction With Composition II.* We have

$$\begin{tikzcd} \mathrm{pt} \arrow[r, "{[f]}"] \arrow[dr, "{[g\circ f]}"'] & \mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[d, "g_*"] \\ & \mathrm{Hom}_{\mathcal{C}}(A,C) \end{tikzcd} \qquad \begin{aligned} [g \circ f] &= g_* \circ [f], \\ [g \circ f] &= f^* \circ [g], \end{aligned} \qquad \begin{tikzcd} \mathrm{pt} \arrow[r, "{[g]}"] \arrow[dr, "{[g\circ f]}"'] & \mathrm{Hom}_{\mathcal{C}}(B,C) \arrow[d, "f^*"] \\ & \mathrm{Hom}_{\mathcal{C}}(A,C). \end{tikzcd}$$

4. *Interaction With Composition III.* We have

$$f^* \circ \circ^{\mathcal{C}}_{A,B,C} = \circ^{\mathcal{C}}_{X,B,C} \circ (f^* \times \mathrm{id}), \qquad \begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(B,C) \times \mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[r, "\circ^{\mathcal{C}}_{A,B,C}"] \arrow[d, "\mathrm{id}\times f^*"'] & \mathrm{Hom}_{\mathcal{C}}(A,C) \arrow[d, "f^*"] \\ \mathrm{Hom}_{\mathcal{C}}(B,C) \times \mathrm{Hom}_{\mathcal{C}}(X,B) \arrow[r, "\circ^{\mathcal{C}}_{X,B,C}"'] & \mathrm{Hom}_{\mathcal{C}}(X,C), \end{tikzcd}$$

$$g_* \circ \circ^{\mathcal{C}}_{A,B,C} = \circ^{\mathcal{C}}_{A,B,D} \circ (\mathrm{id} \times g_*), \qquad \begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(B,C) \times \mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[r, "\circ^{\mathcal{C}}_{A,B,C}"] \arrow[d, "g_*\times\mathrm{id}"'] & \mathrm{Hom}_{\mathcal{C}}(A,C) \arrow[d, "g^*"] \\ \mathrm{Hom}_{\mathcal{C}}(B,D) \times \mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[r, "\circ^{\mathcal{C}}_{A,B,D}"'] & \mathrm{Hom}_{\mathcal{C}}(A,D). \end{tikzcd}$$

5. *Interaction With Identities.* We have

$$\begin{aligned} (\mathrm{id}_A)^* &= \mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(A,B)}, \\ (\mathrm{id}_B)_* &= \mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(A,B)}. \end{aligned}$$

Proof. *Item 1, Interaction Between Precomposition and Postcomposition*: Clear.
Item 2, Interaction With Composition I: Clear.
Item 3, Interaction With Composition II: Clear.
Item 4, Interaction With Composition III: Clear.
Item 5, Interaction With Identities: Clear. □

2 The Quadruple Adjunction With Sets

2.1 Statement

Let $\mathcal{C}$ be a category.

Proposition 2.1.1.1. We have a quadruple adjunction

$$(\pi_0 \dashv (-)_{\mathsf{disc}} \dashv \mathrm{Obj} \dashv (-)_{\mathsf{indisc}}) \colon \begin{tikzcd} \mathsf{Sets} \arrow[r, "(-)_{\mathsf{disc}}" description, bend left=25] \arrow[r, "(-)_{\mathsf{indisc}}"', bend right=60] & \mathsf{Cats}, \arrow[l, "\pi_0"', bend right=60] \arrow[l, "\mathrm{Obj}" description, bend left=25] \end{tikzcd}$$

witnessed by bijections of sets

$$\begin{aligned} \mathrm{Hom}_{\mathsf{Sets}}(\pi_0(\mathcal{C}), X) &\cong \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, X_{\mathsf{disc}}), \\ \mathrm{Hom}_{\mathsf{Cats}}(X_{\mathsf{disc}}, \mathcal{C}) &\cong \mathrm{Hom}_{\mathsf{Sets}}(X, \mathrm{Obj}(\mathcal{C})), \\ \mathrm{Hom}_{\mathsf{Sets}}(\mathrm{Obj}(\mathcal{C}), X) &\cong \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, X_{\mathsf{indisc}}), \end{aligned}$$

natural in $\mathcal{C} \in \mathrm{Obj}(\mathsf{Cats})$ and $X \in \mathrm{Obj}(\mathsf{Sets})$, where

- The functor
$$\pi_0 \colon \mathsf{Cats} \to \mathsf{Sets},$$
the **connected components functor**, is the functor sending a category to its set of connected components of Definition 2.2.2.1.

- The functor
$$(-)_{\mathsf{disc}} \colon \mathsf{Sets} \to \mathsf{Cats},$$
the **discrete category functor**, is the functor sending a set to its associated discrete category of Item 1.

- The functor
$$\mathrm{Obj}\colon \mathsf{Cats} \to \mathsf{Sets},$$
the **object functor**, is the functor sending a category to its set of objects.

- The functor
$$(-)_{\mathsf{indisc}}\colon \mathsf{Sets} \to \mathsf{Cats},$$
the **indiscrete category functor**, is the functor sending a set to its associated indiscrete category of Item 1.

Proof. Omitted. □

2.2 Connected Components and Connected Categories

2.2.1 Connected Components of Categories

Let $\mathcal{C}$ be a category.

Definition 2.2.1.1. A **connected component** of $\mathcal{C}$ is a full subcategory $\mathcal{I}$ of $\mathcal{C}$ satisfying the following conditions:[9]

1. *Non-Emptiness.* We have $\mathrm{Obj}(\mathcal{I}) \neq \emptyset$.

2. *Connectedness.* There exists a zigzag of arrows between any two objects of $\mathcal{I}$.

2.2.2 Sets of Connected Components of Categories

Let $\mathcal{C}$ be a category.

Definition 2.2.2.1. The **set of connected components of** $\mathcal{C}$ is the set $\pi_0(\mathcal{C})$ whose elements are the connected components of $\mathcal{C}$.

Proposition 2.2.2.2. Let $\mathcal{C}$ be a category.

1. *Functoriality.* The assignment $\mathcal{C} \mapsto \pi_0(\mathcal{C})$ defines a functor
$$\pi_0\colon \mathsf{Cats} \to \mathsf{Sets}.$$

[9] In other words, a **connected component** of $\mathcal{C}$ is an element of the set $\mathrm{Obj}(\mathcal{C})/\sim$ with $\sim$ the equivalence relation generated by the relation $\sim'$ obtained by declaring $A \sim' B$ iff there exists a morphism of $\mathcal{C}$ from A to B.

2. *Adjointness.* We have a quadruple adjunction

$$\left(\pi_0 \dashv (-)_{\mathsf{disc}} \dashv \mathsf{Obj} \dashv (-)_{\mathsf{indisc}}\right)\colon \begin{tikzcd} \mathsf{Sets} \arrow[r, shift left=1, "(-)_{\mathsf{disc}}" description] \arrow[r, bend right=60, "(-)_{\mathsf{indisc}}"'] \arrow[r, phantom, shift left=6, "\perp"] \arrow[r, phantom, "\perp"] \arrow[r, phantom, shift right=6, "\perp"] & \mathsf{Cats}. \arrow[l, bend right=60, "\pi_0"'] \arrow[l, shift left=1, "\mathsf{Obj}" description] \end{tikzcd}$$

3. *Interaction With Groupoids.* If $\mathcal{C}$ is a groupoid, then we have an isomorphism of categories

$$\pi_0(\mathcal{C}) \cong \mathrm{K}(\mathcal{C}),$$

where $\mathrm{K}(\mathcal{C})$ is the set of isomorphism classes of $\mathcal{C}$ of **??**.

4. *Preservation of Colimits.* The functor π_0 of Item 1 preserves colimits. In particular, we have bijections of sets

$$\begin{aligned} \pi_0(\mathcal{C} \textstyle\coprod \mathcal{D}) &\cong \pi_0(\mathcal{C}) \textstyle\coprod \pi_0(\mathcal{D}), \\ \pi_0(\mathcal{C} \textstyle\coprod_{\mathcal{E}} \mathcal{D}) &\cong \pi_0(\mathcal{C}) \textstyle\coprod_{\pi_0(\mathcal{E})} \pi_0(\mathcal{D}), \\ \pi_0(\mathrm{CoEq}(\mathcal{C} \underset{G}{\overset{F}{\rightrightarrows}} \mathcal{D})) &\cong \mathrm{CoEq}(\pi_0(\mathcal{C}) \underset{\pi_0(G)}{\overset{\pi_0(F)}{\rightrightarrows}} \pi_0(\mathcal{D})), \end{aligned}$$

natural in $\mathcal{C}, \mathcal{D}, \mathcal{E} \in \mathsf{Obj}(\mathsf{Cats})$.

5. *Symmetric Strong Monoidality With Respect to Coproducts.* The connected components functor of Item 1 has a symmetric strong monoidal structure

$$\left(\pi_0, \pi_0^{\coprod}, \pi_{0|\mathbb{1}}^{\coprod}\right)\colon (\mathsf{Cats}, \textstyle\coprod, \emptyset_{\mathsf{cat}}) \longrightarrow (\mathsf{Sets}, \textstyle\coprod, \emptyset),$$

being equipped with isomorphisms

$$\begin{aligned} \pi_{0|\mathcal{C},\mathcal{D}}^{\coprod}\colon \pi_0(\mathcal{C}) \textstyle\coprod \pi_0(\mathcal{D}) &\xrightarrow{\cong} \pi_0(\mathcal{C} \textstyle\coprod \mathcal{D}), \\ \pi_{0|\mathbb{1}}^{\coprod}\colon \emptyset &\xrightarrow{\cong} \pi_0(\emptyset_{\mathsf{cat}}), \end{aligned}$$

natural in $\mathcal{C}, \mathcal{D} \in \mathsf{Obj}(\mathsf{Cats})$.

6. *Symmetric Strong Monoidality With Respect to Products.* The connected components functor of Item 1 has a symmetric strong monoidal structure

$$\left(\pi_0, \pi_0^{\times}, \pi_{0|\mathbb{1}}^{\times}\right)\colon (\mathsf{Cats}, \times, \mathrm{pt}) \longrightarrow (\mathsf{Sets}, \times, \mathrm{pt}),$$

being equipped with isomorphisms

$$\pi^{\times}_{0|C,\mathcal{D}} \colon \pi_0(C) \times \pi_0(\mathcal{D}) \xrightarrow{\cong} \pi_0(C \times \mathcal{D}),$$
$$\pi^{\times}_{0|\mathbb{1}} \colon \mathrm{pt} \xrightarrow{\cong} \pi_0(\mathrm{pt}),$$

natural in $C, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$.

Proof. *Item 1, Functoriality*: Clear.
Item 2, Adjointness: This is proved in Proposition 2.1.1.1.
Item 3, Interaction With Groupoids: Clear.
Item 4, Preservation of Colimits: This follows from Item 2 and **??** of **??**.
Item 5, Symmetric Strong Monoidality With Respect to Coproducts: Clear.
Item 6, Symmetric Strong Monoidality With Respect to Products: Clear. □

2.2.3 Connected Categories

Definition 2.2.3.1. A category C is **connected** if $\pi_0(C) \cong \mathrm{pt}$.[10,11]

2.3 Discrete Categories

Definition 2.3.1.1. Let X be a set.

1. The **discrete category on** X is the category X_{disc} where
   - *Objects.* We have
   $$\mathrm{Obj}(X_{\mathsf{disc}}) \overset{\mathrm{def}}{=} X.$$
   - *Morphisms.* For each $A, B \in \mathrm{Obj}(X_{\mathsf{disc}})$, we have
   $$\mathrm{Hom}_{X_{\mathsf{disc}}}(A, B) \overset{\mathrm{def}}{=} \begin{cases} \mathrm{id}_A & \text{if } A = B, \\ \emptyset & \text{if } A \neq B. \end{cases}$$
   - *Identities.* For each $A \in \mathrm{Obj}(X_{\mathsf{disc}})$, the unit map
   $$\mathbb{1}^{X_{\mathsf{disc}}}_A \colon \mathrm{pt} \to \mathrm{Hom}_{X_{\mathsf{disc}}}(A, A)$$
   of X_{disc} at A is defined by
   $$\mathrm{id}^{X_{\mathsf{disc}}}_A \overset{\mathrm{def}}{=} \mathrm{id}_A.$$

[10] *Further Terminology:* A category is **disconnected** if it is not connected.
[11] *Example:* A groupoid is connected iff any two of its objects are isomorphic.

- *Composition.* For each $A, B, C \in \mathrm{Obj}(X_{\mathsf{disc}})$, the composition map

$$\circ^{X_{\mathsf{disc}}}_{A,B,C} \colon \mathrm{Hom}_{X_{\mathsf{disc}}}(B, C) \times \mathrm{Hom}_{X_{\mathsf{disc}}}(A, B) \to \mathrm{Hom}_{X_{\mathsf{disc}}}(A, C)$$

  of X_{disc} at (A, B, C) is defined by

$$\mathrm{id}_A \circ \mathrm{id}_A \stackrel{\text{def}}{=} \mathrm{id}_A.$$

2. A category C is **discrete** if it is equivalent to X_{disc} for some set X.

Proposition 2.3.1.2. Let X be a set.

1. *Functoriality.* The assignment $X \mapsto X_{\mathsf{disc}}$ defines a functor

$$(-)_{\mathsf{disc}} \colon \mathsf{Sets} \to \mathsf{Cats}.$$

2. *Adjointness.* We have a quadruple adjunction

$$\begin{tikzcd}(\pi_0 \dashv (-)_{\mathsf{disc}} \dashv \mathsf{Obj} \dashv (-)_{\mathsf{indisc}}) \colon \mathsf{Sets} \arrow[r, bend left=15, "(-)_{\mathsf{disc}}"] \arrow[r, bend right=50, "(-)_{\mathsf{indisc}}"'] & \mathsf{Cats}. \arrow[l, bend right=50, "\pi_0"'] \arrow[l, bend left=15, "\mathsf{Obj}"] \end{tikzcd}$$

3. *Symmetric Strong Monoidality With Respect to Coproducts.* The functor of Item 1 has a symmetric strong monoidal structure

$$\left((-)_{\mathsf{disc}}, (-)^{\coprod}_{\mathsf{disc}}, (-)^{\coprod}_{\mathsf{disc}|\mathbb{1}}\right) \colon (\mathsf{Sets}, \textstyle\coprod, \emptyset) \to (\mathsf{Cats}, \textstyle\coprod, \emptyset_{\mathsf{cat}}),$$

   being equipped with isomorphisms

$$(-)^{\coprod}_{\mathsf{disc}|X,Y} \colon X_{\mathsf{disc}} \textstyle\coprod Y_{\mathsf{disc}} \xrightarrow{\cong} (X \textstyle\coprod Y)_{\mathsf{disc}},$$
$$(-)^{\coprod}_{\mathsf{disc}|\mathbb{1}} \colon \emptyset_{\mathsf{cat}} \xrightarrow{\cong} \emptyset_{\mathsf{disc}},$$

   natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

4. *Symmetric Strong Monoidality With Respect to Products.* The functor of Item 1 has a symmetric strong monoidal structure

$$\left((-)_{\mathsf{disc}}, (-)^{\times}_{\mathsf{disc}}, (-)^{\times}_{\mathsf{disc}|\mathbb{1}}\right) \colon (\mathsf{Sets}, \times, \mathrm{pt}) \to (\mathsf{Cats}, \times, \mathrm{pt}),$$

being equipped with isomorphisms

$$(-)^{\times}_{\mathsf{disc}|X,Y} \colon X_{\mathsf{disc}} \times Y_{\mathsf{disc}} \xrightarrow{\cong} (X \times Y)_{\mathsf{disc}},$$
$$(-)^{\times}_{\mathsf{disc}|\mathbb{1}} \colon \mathrm{pt} \xrightarrow{\cong} \mathrm{pt}_{\mathsf{disc}},$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

Proof. *Item 1, Functoriality*: Clear.
Item 2, Adjointness: This is proved in Proposition 2.1.1.1.
Item 3, Symmetric Strong Monoidality With Respect to Coproducts: Clear.
Item 4, Symmetric Strong Monoidality With Respect to Products: Clear. □

2.4 Indiscrete Categories

Definition 2.4.1.1. Let X be a set.

1. The **indiscrete category on** X[12] is the category X_{indisc} where
   - *Objects.* We have
     $$\mathrm{Obj}(X_{\mathsf{indisc}}) \stackrel{\text{def}}{=} X.$$
   - *Morphisms.* For each $A, B \in \mathrm{Obj}(X_{\mathsf{indisc}})$, we have
     $$\begin{aligned}\mathrm{Hom}_{X_{\mathsf{disc}}}(A, B) &\stackrel{\text{def}}{=} \{[A] \to [B]\}\\ &\cong \mathrm{pt}.\end{aligned}$$
   - *Identities.* For each $A \in \mathrm{Obj}(X_{\mathsf{indisc}})$, the unit map
     $$\mathbb{1}^{X_{\mathsf{indisc}}}_{A} \colon \mathrm{pt} \to \mathrm{Hom}_{X_{\mathsf{indisc}}}(A, A)$$
     of X_{indisc} at A is defined by
     $$\mathrm{id}^{X_{\mathsf{indisc}}}_{A} \stackrel{\text{def}}{=} \{[A] \to [A]\}.$$
   - *Composition.* For each $A, B, C \in \mathrm{Obj}(X_{\mathsf{indisc}})$, the composition map
     $$\circ^{X_{\mathsf{indisc}}}_{A,B,C} \colon \mathrm{Hom}_{X_{\mathsf{indisc}}}(B, C) \times \mathrm{Hom}_{X_{\mathsf{indisc}}}(A, B) \to \mathrm{Hom}_{X_{\mathsf{indisc}}}(A, C)$$
     of X_{disc} at (A, B, C) is defined by
     $$([B] \to [C]) \circ ([A] \to [B]) \stackrel{\text{def}}{=} ([A] \to [C]).$$

[12] *Further Terminology:* Sometimes called the **chaotic category on** X.

2. A category $\mathcal{C}$ is **indiscrete** if it is equivalent to X_{indisc} for some set X.

Proposition 2.4.1.2. Let X be a set.

1. *Functoriality.* The assignment $X \mapsto X_{\mathsf{indisc}}$ defines a functor

$$(-)_{\mathsf{indisc}} \colon \mathsf{Sets} \to \mathsf{Cats}.$$

2. *Adjointness.* We have a quadruple adjunction

$$\left(\pi_0 \dashv (-)_{\mathsf{disc}} \dashv \mathsf{Obj} \dashv (-)_{\mathsf{indisc}}\right)\colon \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left=25, "(-)_{\mathsf{disc}}" description] \arrow[r, bend right=60, "(-)_{\mathsf{indisc}}"'] \arrow[r, phantom, shift left=11, "\perp"] \arrow[r, phantom, "\perp"] \arrow[r, phantom, shift right=11, "\perp"] & \mathsf{Cats}. \arrow[l, bend right=60, "\pi_0"'] \arrow[l, bend left=25, "\mathsf{Obj}" description] \end{tikzcd}$$

3. *Symmetric Strong Monoidality With Respect to Products.* The functor of Item 1 has a symmetric strong monoidal structure

$$\left((-)_{\mathsf{indisc}}, (-)^{\times}_{\mathsf{indisc}}, (-)^{\times}_{\mathsf{indisc}|\mathbb{1}}\right) \colon (\mathsf{Sets}, \times, \mathsf{pt}) \to (\mathsf{Cats}, \times, \mathsf{pt}),$$

being equipped with isomorphisms

$$(-)^{\times}_{\mathsf{indisc}|X,Y} \colon X_{\mathsf{indisc}} \times Y_{\mathsf{indisc}} \xrightarrow{\cong} (X \times Y)_{\mathsf{indisc}},$$
$$(-)^{\times}_{\mathsf{indisc}|\mathbb{1}} \colon \mathsf{pt} \xrightarrow{\cong} \mathsf{pt}_{\mathsf{indisc}},$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

Proof. *Item 1, Functoriality*: Clear.
Item 2, Adjointness: This is proved in Proposition 2.1.1.1.
Item 3, Symmetric Strong Monoidality With Respect to Products: Clear. □

3 Groupoids

3.1 Foundations

Let $\mathcal{C}$ be a category.

Definition 3.1.1.1. A morphism $f: A \to B$ of C is an **isomorphism** if there exists a morphism $f^{-1}: B \to A$ of C such that

$$\begin{aligned} f \circ f^{-1} &= \mathrm{id}_B, \\ f^{-1} \circ f &= \mathrm{id}_A. \end{aligned}$$

Notation 3.1.1.2. We write $\mathrm{Iso}_C(A, B)$ for the set of all isomorphisms in C from A to B.

Definition 3.1.1.3. A **groupoid** is a category in which every morphism is an isomorphism.

3.2 The Groupoid Completion of a Category

Let C be a category.

Definition 3.2.1.1. The **groupoid completion of** C[13] is the pair $(\mathrm{K}_0(C), \iota_C)$ consisting of

- A groupoid $\mathrm{K}_0(C)$;
- A functor $\iota_C : C \to \mathrm{K}_0(C)$;

satisfying the following universal property:[14]

(UP) Given another such pair $(\mathcal{G}, i)$, there exists a unique functor $\mathrm{K}_0(C) \xrightarrow{\exists!} \mathcal{G}$ making the diagram

$$\begin{tikzcd} & \mathrm{K}_0(C) \arrow[d, dashed, "\exists!"] \\ C \arrow[ur, "\iota_C"] \arrow[r, "i"'] & \mathcal{G} \end{tikzcd}$$

commute.

Construction 3.2.1.2. Concretely, the groupoid completion of C is the Gabriel–Zisman localisation $\mathrm{Mor}(C)^{-1}C$ of C at the set $\mathrm{Mor}(C)$ of all morphisms of C; see **??**, **??**.
(To be expanded upon later on.)

[13] *Further Terminology:* Also called the **Grothendieck groupoid of** C or the **Grothendieck groupoid completion of** C.

[14] See Item 5 of Proposition 3.2.1.3 for an explicit construction.

Proof. Omitted. □

Proposition 3.2.1.3. Let $\mathcal{C}$ be a category.

1. *Functoriality.* The assignment $\mathcal{C} \mapsto \mathrm{K}_0(\mathcal{C})$ defines a functor

$$\mathrm{K}_0 \colon \mathsf{Cats} \to \mathsf{Grpd}.$$

2. *2-Functoriality.* The assignment $\mathcal{C} \mapsto \mathrm{K}_0(\mathcal{C})$ defines a 2-functor

$$\mathrm{K}_0 \colon \mathsf{Cats}_2 \to \mathsf{Grpd}_2.$$

3. *Adjointness.* We have an adjunction

$$(\mathrm{K}_0 \dashv \iota)\colon \quad \begin{tikzcd} \mathsf{Cats} \arrow[r, shift left, "\mathrm{K}_0"] \arrow[r, phantom, "\perp"] & \mathsf{Grpd}, \arrow[l, hook', shift left, "\iota"] \end{tikzcd}$$

witnessed by a bijection of sets

$$\mathrm{Hom}_{\mathsf{Grpd}}(\mathrm{K}_0(\mathcal{C}), \mathcal{G}) \cong \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, \mathcal{G}),$$

natural in $\mathcal{C} \in \mathrm{Obj}(\mathsf{Cats})$ and $\mathcal{G} \in \mathrm{Obj}(\mathsf{Grpd})$, forming, together with the functor Core of Item 1 of Proposition 3.3.1.4, a triple adjunction

$$(\mathrm{K}_0 \dashv \iota \dashv \mathsf{Core})\colon \quad \begin{tikzcd} \mathsf{Cats} \arrow[r, bend left, "\mathrm{K}_0"] \arrow[r, bend right, "\mathsf{Core}"'] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, phantom, shift right=2, "\perp"] & \mathsf{Grpd}, \arrow[l, hook', "\iota"] \end{tikzcd}$$

witnessed by bijections of sets

$$\begin{aligned} \mathrm{Hom}_{\mathsf{Grpd}}(\mathrm{K}_0(\mathcal{C}), \mathcal{G}) &\cong \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, \mathcal{G}), \\ \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{G}, \mathcal{D}) &\cong \mathrm{Hom}_{\mathsf{Grpd}}(\mathcal{G}, \mathsf{Core}(\mathcal{D})), \end{aligned}$$

natural in $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$ and $\mathcal{G} \in \mathrm{Obj}(\mathsf{Grpd})$.

4. *2-Adjointness.* We have a 2-adjunction

$$(\mathrm{K}_0 \dashv \iota)\colon \quad \begin{tikzcd} \mathsf{Cats} \arrow[r, shift left, "\mathrm{K}_0"] \arrow[r, phantom, "\perp_2"] & \mathsf{Grpd}, \arrow[l, hook', shift left, "\iota"] \end{tikzcd}$$

witnessed by an isomorphism of categories

$$\mathsf{Fun}(\mathrm{K}_0(\mathcal{C}), \mathcal{G}) \cong \mathsf{Fun}(\mathcal{C}, \mathcal{G}),$$

natural in $\mathcal{C} \in \mathsf{Obj}(\mathsf{Cats})$ and $\mathcal{G} \in \mathsf{Obj}(\mathsf{Grpd})$, forming, together with the 2-functor Core of Item 2 of Proposition 3.3.1.4, a triple 2-adjunction

$$(\mathrm{K}_0 \dashv \iota \dashv \mathsf{Core})\colon \quad \begin{tikzcd} \mathsf{Cats} \arrow[r, bend left=45, "\mathrm{K}_0"] \arrow[r, bend right=45, "\mathsf{Core}"'] \arrow[r, phantom, shift left=3, "\perp_2"] \arrow[r, phantom, shift right=3, "\perp_2"] & \mathsf{Grpd}, \arrow[l, hook', "\iota"'] \end{tikzcd}$$

witnessed by isomorphisms of categories

$$\begin{aligned} \mathsf{Fun}(\mathrm{K}_0(\mathcal{C}), \mathcal{G}) &\cong \mathsf{Fun}(\mathcal{C}, \mathcal{G}), \\ \mathsf{Fun}(\mathcal{G}, \mathcal{D}) &\cong \mathsf{Fun}(\mathcal{G}, \mathsf{Core}(\mathcal{D})), \end{aligned}$$

natural in $\mathcal{C}, \mathcal{D} \in \mathsf{Obj}(\mathsf{Cats})$ and $\mathcal{G} \in \mathsf{Obj}(\mathsf{Grpd})$.

5. *Interaction With Classifying Spaces.* We have an isomorphism of groupoids

$$\mathrm{K}_0(\mathcal{C}) \cong \Pi_{\leq 1}(|\mathrm{N}_\bullet(\mathcal{C})|),$$

natural in $\mathcal{C} \in \mathsf{Obj}(\mathsf{Cats})$; i.e. the diagram

$$\begin{tikzcd} \mathsf{Cats} \arrow[r, "\mathrm{K}_0"] \arrow[d, "\mathrm{N}_\bullet"'] & \mathsf{Grp} \\ \mathsf{sSets} \arrow[r, "|-|"'] & \mathsf{Top} \arrow[u, "\Pi_{\leq 1}"'] \end{tikzcd}$$

commutes up to natural isomorphism.

6. *Symmetric Strong Monoidality With Respect to Coproducts.* The groupoid completion functor of Item 1 has a symmetric strong monoidal structure

$$(\mathrm{K}_0, \mathrm{K}_0^{\coprod}, \mathrm{K}_{0|\mathbb{1}}^{\coprod})\colon (\mathsf{Cats}, \textstyle\coprod, \emptyset_{\mathsf{cat}}) \longrightarrow (\mathsf{Grpd}, \textstyle\coprod, \emptyset_{\mathsf{cat}})$$

being equipped with isomorphisms

$$\begin{aligned} \mathrm{K}_{0|\mathcal{C},\mathcal{D}}^{\coprod}&\colon \mathrm{K}_0(\mathcal{C}) \textstyle\coprod \mathrm{K}_0(\mathcal{D}) \xrightarrow{\cong} \mathrm{K}_0(\mathcal{C} \textstyle\coprod \mathcal{D}), \\ \mathrm{K}_{0|\mathbb{1}}^{\coprod}&\colon \emptyset_{\mathsf{cat}} \xrightarrow{\cong} \mathrm{K}_0(\emptyset_{\mathsf{cat}}), \end{aligned}$$

natural in $\mathcal{C}, \mathcal{D} \in \mathsf{Obj}(\mathsf{Cats})$.

7. *Symmetric Strong Monoidality With Respect to Products.* The groupoid completion functor of Item 1 has a symmetric strong monoidal structure

$$(\mathrm{K}_0, \mathrm{K}_0^{\times}, \mathrm{K}_{0|\mathbb{1}}^{\times}) \colon (\mathsf{Cats}, \times, \mathrm{pt}) \to (\mathsf{Grpd}, \times, \mathrm{pt})$$

being equipped with isomorphisms

$$\mathrm{K}_{0|\mathcal{C},\mathcal{D}}^{\times} \colon \mathrm{K}_0(\mathcal{C}) \times \mathrm{K}_0(\mathcal{D}) \xrightarrow{\cong} \mathrm{K}_0(\mathcal{C} \times \mathcal{D}),$$
$$\mathrm{K}_{0|\mathbb{1}}^{\times} \colon \mathrm{pt} \xrightarrow{\cong} \mathrm{K}_0(\mathrm{pt}),$$

natural in $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$.

Proof. *Item 1, Functoriality*: Omitted.
Item 2, 2-Functoriality: Omitted.
Item 3, Adjointness: Omitted.
Item 4, 2-Adjointness: Omitted.
Item 5, Interaction With Classifying Spaces: See Corollary 18.33 of `https://web.ma.utexas.edu/users/dafr/M392C-2012/Notes/lecture18.pdf`.
Item 6, Symmetric Strong Monoidality With Respect to Coproducts: Omitted.
Item 7, Symmetric Strong Monoidality With Respect to Products: Omitted. □

3.3 The Core of a Category

Let $\mathcal{C}$ be a category.

Definition 3.3.1.1. The **core** of $\mathcal{C}$ is the pair $(\mathsf{Core}(\mathcal{C}), \iota_{\mathcal{C}})$ consisting of

- A groupoid $\mathsf{Core}(\mathcal{C})$;
- A functor $\iota_{\mathcal{C}} \colon \mathsf{Core}(\mathcal{C}) \hookrightarrow \mathcal{C}$;

satisfying the following universal property:

(UP) Given another such pair $(\mathcal{G}, i)$, there exists a unique functor $\mathcal{G} \xrightarrow{\exists!} \mathsf{Core}(\mathcal{C})$ making the diagram

$$\begin{tikzcd} & \mathsf{Core}(\mathcal{C}) \arrow[d, hook, "\iota_{\mathcal{C}}"] \\ \mathcal{G} \arrow[ur, dashed, "\exists!"] \arrow[r, "i"'] & \mathcal{C} \end{tikzcd}$$

commute.

Notation 3.3.1.2. We also write $\mathcal{C}^{\simeq}$ for $\mathsf{Core}(\mathcal{C})$.

Construction 3.3.1.3. The core of $\mathcal{C}$ is the wide subcategory of $\mathcal{C}$ spanned by the isomorphisms of $\mathcal{C}$, i.e. the category $\mathsf{Core}(\mathcal{C})$ where[15]

1. *Objects.* We have
$$\mathsf{Obj}(\mathsf{Core}(\mathcal{C})) \overset{\text{def}}{=} \mathsf{Obj}(\mathcal{C}).$$
2. *Morphisms.* The morphisms of $\mathsf{Core}(\mathcal{C})$ are the isomorphisms of $\mathcal{C}$.

Proof. This follows from the fact that functors preserve isomorphisms (Item 1 of Proposition 4.1.1.6). □

Proposition 3.3.1.4. Let $\mathcal{C}$ be a category.

1. *Functoriality.* The assignment $\mathcal{C} \mapsto \mathsf{Core}(\mathcal{C})$ defines a functor
$$\mathsf{Core}\colon \mathsf{Cats} \longrightarrow \mathsf{Grpd}.$$
2. *2-Functoriality.* The assignment $\mathcal{C} \mapsto \mathsf{Core}(\mathcal{C})$ defines a 2-functor
$$\mathsf{Core}\colon \mathsf{Cats}_2 \longrightarrow \mathsf{Grpd}_2.$$
3. *Adjointness.* We have an adjunction
$$(\iota \dashv \mathsf{Core})\colon \quad \begin{tikzcd} \mathsf{Grpd} \arrow[r, bend left, "\iota"] \arrow[r, phantom, "\perp"] & \mathsf{Cats}, \arrow[l, bend left, "\mathsf{Core}"] \end{tikzcd}$$
witnessed by a bijection of sets
$$\mathrm{Hom}_{\mathsf{Cats}}(\mathcal{G}, \mathcal{D}) \cong \mathrm{Hom}_{\mathsf{Grpd}}(\mathcal{G}, \mathsf{Core}(\mathcal{D})),$$
natural in $\mathcal{G} \in \mathsf{Obj}(\mathsf{Grpd})$ and $\mathcal{D} \in \mathsf{Obj}(\mathsf{Cats})$, forming, together with the functor K_0 of Item 1 of Proposition 3.2.1.3, a triple adjunction
$$(\mathrm{K}_0 \dashv \iota \dashv \mathsf{Core})\colon \quad \begin{tikzcd} \mathsf{Cats} \arrow[r, bend left, "\mathrm{K}_0"] \arrow[r, bend right, "\mathsf{Core}"'] & \mathsf{Grpd}, \arrow[l, "\iota"'] \end{tikzcd}$$

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[15] *Slogan:* The groupoid $\mathsf{Core}(\mathcal{C})$ is the maximal subgroupoid of $\mathcal{C}$.

witnessed by bijections of sets

$$\begin{aligned}\mathrm{Hom}_{\mathsf{Grpd}}(\mathrm{K}_0(\mathcal{C}),\mathcal{G}) &\cong \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C},\mathcal{G}),\\ \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{G},\mathcal{D}) &\cong \mathrm{Hom}_{\mathsf{Grpd}}(\mathcal{G},\mathsf{Core}(\mathcal{D})),\end{aligned}$$

natural in $\mathcal{C},\mathcal{D}\in\mathrm{Obj}(\mathsf{Cats})$ and $\mathcal{G}\in\mathrm{Obj}(\mathsf{Grpd})$.

4. *2-Adjointness.* We have an adjunction

$$(\iota\dashv\mathsf{Core})\colon\quad \begin{tikzcd}\mathsf{Grpd} \arrow[r, bend left, "\iota"] \arrow[r, phantom, "\perp_2"] & \mathsf{Cats}, \arrow[l, bend left, "\mathsf{Core}"]\end{tikzcd}$$

witnessed by an isomorphism of categories

$$\mathsf{Fun}(\mathcal{G},\mathcal{D})\cong\mathsf{Fun}(\mathcal{G},\mathsf{Core}(\mathcal{D})),$$

natural in $\mathcal{G}\in\mathrm{Obj}(\mathsf{Grpd})$ and $\mathcal{D}\in\mathrm{Obj}(\mathsf{Cats})$, forming, together with the 2-functor K_0 of Item 2 of Proposition 3.2.1.3, a triple 2-adjunction

$$(\mathrm{K}_0\dashv\iota\dashv\mathsf{Core})\colon\quad \begin{tikzcd}\mathsf{Cats} \arrow[r, bend left=50, "\mathrm{K}_0"] \arrow[r, bend right=50, "\mathsf{Core}"'] & \mathsf{Grpd}, \arrow[l, "\iota" description] \arrow[l, phantom, bend right=25, "\perp_2"] \arrow[l, phantom, bend left=25, "\perp_2"]\end{tikzcd}$$

witnessed by isomorphisms of categories

$$\begin{aligned}\mathsf{Fun}(\mathrm{K}_0(\mathcal{C}),\mathcal{G}) &\cong \mathsf{Fun}(\mathcal{C},\mathcal{G}),\\ \mathsf{Fun}(\mathcal{G},\mathcal{D}) &\cong \mathsf{Fun}(\mathcal{G},\mathsf{Core}(\mathcal{D})),\end{aligned}$$

natural in $\mathcal{C},\mathcal{D}\in\mathrm{Obj}(\mathsf{Cats})$ and $\mathcal{G}\in\mathrm{Obj}(\mathsf{Grpd})$.

5. *Symmetric Strong Monoidality With Respect to Products.* The core functor of Item 1 has a symmetric strong monoidal structure

$$(\mathsf{Core},\mathsf{Core}^{\times},\mathsf{Core}^{\times}_{\mathbb{1}})\colon(\mathsf{Cats},\times,\mathrm{pt})\longrightarrow(\mathsf{Grpd},\times,\mathrm{pt})$$

being equipped with isomorphisms

$$\begin{aligned}\mathsf{Core}^{\times}_{\mathcal{C},\mathcal{D}}\colon\mathsf{Core}(\mathcal{C})\times\mathsf{Core}(\mathcal{D}) &\xrightarrow{\cong}\mathsf{Core}(\mathcal{C}\times\mathcal{D}),\\ \mathsf{Core}^{\times}_{\mathbb{1}}\colon\mathrm{pt}&\xrightarrow{\cong}\mathsf{Core}(\mathrm{pt}),\end{aligned}$$

natural in $\mathcal{C},\mathcal{D}\in\mathrm{Obj}(\mathsf{Cats})$.

6. *Symmetric Strong Monoidality With Respect to Coproducts.* The core functor of Item 1 has a symmetric strong monoidal structure

$$(\mathsf{Core}, \mathsf{Core}^{\coprod}, \mathsf{Core}^{\coprod}_{\mathbb{1}}) \colon (\mathsf{Cats}, \coprod, \emptyset_{\mathsf{cat}}) \to (\mathsf{Grpd}, \coprod, \emptyset_{\mathsf{cat}})$$

being equipped with isomorphisms

$$\mathsf{Core}^{\coprod}_{\mathcal{C},\mathcal{D}} \colon \mathsf{Core}(\mathcal{C}) \coprod \mathsf{Core}(\mathcal{D}) \xrightarrow{\cong} \mathsf{Core}(\mathcal{C} \coprod \mathcal{D}),$$
$$\mathsf{Core}^{\coprod}_{\mathbb{1}} \colon \emptyset_{\mathsf{cat}} \xrightarrow{\cong} \mathsf{Core}(\emptyset_{\mathsf{cat}}),$$

natural in $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$.

Proof. *Item 1, Functoriality*: Omitted.
Item 2, 2-Functoriality: Omitted.
Item 3, Adjointness: Omitted.
Item 4, 2-Adjointness: Omitted.
Item 5, Symmetric Strong Monoidality With Respect to Products: Omitted.
Item 6, Symmetric Strong Monoidality With Respect to Coproducts: Omitted. □

4 Functors

4.1 Foundations

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 4.1.1.1. A **functor** $F \colon \mathcal{C} \to \mathcal{D}$ **from** $\mathcal{C}$ **to** $\mathcal{D}$[16] consists of:

1. *Action on Objects.* A map of sets

$$F \colon \mathrm{Obj}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{D}),$$

called the **action on objects of** F.

2. *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathcal{C})$, a map

$$F_{A,B} \colon \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F(A), F(B)),$$

called the **action on morphisms of** F **at** (A, B)[17].

[16] *Further Terminology:* Also called a **covariant functor**.
[17] *Further Terminology:* Also called **action on Hom-sets of** F **at** (A, B).

satisfying the following conditions:

1. *Preservation of Identities.* For each $A \in \mathrm{Obj}(\mathcal{C})$, the diagram

$$\begin{tikzcd}
\mathrm{pt} \arrow[d, "\mathbb{1}^{\mathcal{C}}_{A}"'] \arrow[dr, "\mathbb{1}^{\mathcal{D}}_{F(A)}"] & \\
\mathrm{Hom}_{\mathcal{C}}(A,A) \arrow[r, "F_{A,A}"'] & \mathrm{Hom}_{\mathcal{D}}(F(A),F(A))
\end{tikzcd}$$

   commutes, i.e. we have

$$F(\mathrm{id}_A) = \mathrm{id}_{F(A)}.$$

2. *Preservation of Composition.* For each $A, B, C \in \mathrm{Obj}(\mathcal{C})$, the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(B,C) \times \mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[r, "\circ^{\mathcal{C}}_{A,B,C}"] \arrow[d, "F_{B,C} \times F_{A,B}"'] & \mathrm{Hom}_{\mathcal{C}}(A,C) \arrow[d, "F_{A,C}"] \\
\mathrm{Hom}_{\mathcal{D}}(F(B),F(C)) \times \mathrm{Hom}_{\mathcal{D}}(F(A),F(B)) \arrow[r, "\circ^{\mathcal{D}}_{F(A),F(B),F(C)}"'] & \mathrm{Hom}_{\mathcal{D}}(F(A),F(C))
\end{tikzcd}$$

   commutes, i.e. for each composable pair (g, f) of morphisms of $\mathcal{C}$, we have

$$F(g \circ f) = F(g) \circ F(f).$$

Notation 4.1.1.2. Let $\mathcal{C}$ and $\mathcal{D}$ be categories, and write $\mathcal{C}^{\mathsf{op}}$ for the opposite category of $\mathcal{C}$ of **??**, **??**.

1. Given a functor

$$F: \mathcal{C} \to \mathcal{D},$$

   we also write F_A for $F(A)$.

2. Given a functor

$$F: \mathcal{C}^{\mathsf{op}} \to \mathcal{D},$$

   we also write F^A for $F(A)$.

3. Given a functor

$$F: \mathcal{C} \times \mathcal{C} \to \mathcal{D},$$

   we also write $F_{A,B}$ for $F(A, B)$.

4. Given a functor
$$F\colon \mathcal{C}^{\mathsf{op}}\times\mathcal{C}\to\mathcal{D},$$
we also write F^A_B for $F(A,B)$.

We employ a similar notation for morphisms, writing e.g. F_f for $F(f)$ given a functor $F\colon\mathcal{C}\to\mathcal{D}$.

Notation 4.1.1.3. Following the notation $[\![x\mapsto f(x)]\!]$ for a function $f\colon X\to Y$ introduced in Sets, Notation 1.1.1.2, we will sometimes denote a functor $F\colon\mathcal{C}\to\mathcal{D}$ by
$$F\stackrel{\mathrm{def}}{=}[\![A\mapsto F(A)]\!],$$
specially when the action on morphisms of F is clear from its action on objects.

Example 4.1.1.4. The **identity functor** of a category $\mathcal{C}$ is the functor $\mathrm{id}_{\mathcal{C}}\colon\mathcal{C}\to\mathcal{C}$ where

1. *Action on Objects.* For each $A\in\mathrm{Obj}(\mathcal{C})$, we have
$$\mathrm{id}_{\mathcal{C}}(A)\stackrel{\mathrm{def}}{=}A.$$

2. *Action on Morphisms.* For each $A,B\in\mathrm{Obj}(\mathcal{C})$, the action on morphisms
$$(\mathrm{id}_{\mathcal{C}})_{A,B}\colon\mathrm{Hom}_{\mathcal{C}}(A,B)\to\underbrace{\mathrm{Hom}_{\mathcal{C}}(\mathrm{id}_{\mathcal{C}}(A),\mathrm{id}_{\mathcal{C}}(B))}_{\stackrel{\mathrm{def}}{=}\mathrm{Hom}_{\mathcal{C}}(A,B)}$$
of $\mathrm{id}_{\mathcal{C}}$ at (A,B) is defined by
$$(\mathrm{id}_{\mathcal{C}})_{A,B}\stackrel{\mathrm{def}}{=}\mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(A,B)}.$$

Proof. *Preservation of Identities*: We have $\mathrm{id}_{\mathcal{C}}(\mathrm{id}_A)\stackrel{\mathrm{def}}{=}\mathrm{id}_A$ for each $A\in\mathrm{Obj}(\mathcal{C})$ by definition.

Preservation of Compositions: For each composable pair $A\xrightarrow{f}B\xrightarrow{g}B$ of morphisms of $\mathcal{C}$, we have
$$\begin{aligned}\mathrm{id}_{\mathcal{C}}(g\circ f)&\stackrel{\mathrm{def}}{=}g\circ f\\&\stackrel{\mathrm{def}}{=}\mathrm{id}_{\mathcal{C}}(g)\circ\mathrm{id}_{\mathcal{C}}(f).\end{aligned}$$

This finishes the proof. □

Definition 4.1.1.5. The **composition** of two functors $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{E}$ is the functor $G \circ F$ where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathcal{C})$, we have

$$[G \circ F](A) \stackrel{\text{def}}{=} G(F(A)).$$

- *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathcal{C})$, the action on morphisms

$$(G \circ F)_{A,B}\colon \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{E}}(G_{F_A}, G_{F_B})$$

  of $G \circ F$ at (A, B) is defined by

$$[G \circ F](f) \stackrel{\text{def}}{=} G(F(f)).$$

Proof. *Preservation of Identities*: For each $A \in \mathrm{Obj}(\mathcal{C})$, we have

$$\begin{aligned} G_{F_{\mathrm{id}_A}} &= G_{\mathrm{id}_{F_A}} && \text{(functoriality of } F) \\ &= \mathrm{id}_{G_{F_A}}. && \text{(functoriality of } G) \end{aligned}$$

Preservation of Composition: For each composable pair (g, f) of morphisms of $\mathcal{C}$, we have

$$\begin{aligned} G_{F_{g \circ f}} &= G_{F_g \circ F_f} && \text{(functoriality of } F) \\ &= G_{F_g} \circ G_{F_f}. && \text{(functoriality of } G) \end{aligned}$$

This finishes the proof. □

Proposition 4.1.1.6. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor.

1. *Preservation of Isomorphisms.* If f is an isomorphism in $\mathcal{C}$, then $F(f)$ is an isomorphism in $\mathcal{D}$.[18]

Proof. *Item 1, Preservation of Isomorphisms*: Indeed, we have

$$\begin{aligned} F(f)^{-1} \circ F(f) &= F(f^{-1} \circ f) \\ &= F(\mathrm{id}_A) \\ &= \mathrm{id}_{F(A)} \end{aligned}$$

[18] When the converse holds, we call F *conservative*, see Definition 5.4.1.1.

and

$$\begin{aligned} F(f) \circ F(f)^{-1} &= F(f \circ f^{-1}) \\ &= F(\mathrm{id}_B) \\ &= \mathrm{id}_{F(B)}, \end{aligned}$$

showing $F(f)$ to be an isomorphism. □

4.2 Contravariant Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories, and let $\mathcal{C}^{\mathsf{op}}$ denote the opposite category of $\mathcal{C}$ of **??**, **??**.

Definition 4.2.1.1. A **contravariant functor** from $\mathcal{C}$ to $\mathcal{D}$ is a functor from $\mathcal{C}^{\mathsf{op}}$ to $\mathcal{D}$.

Remark 4.2.1.2. In detail, a **contravariant functor** from $\mathcal{C}$ to $\mathcal{D}$ consists of:

1. *Action on Objects.* A map of sets

   $$F\colon \mathrm{Obj}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{D}),$$

   called the **action on objects of** F.

2. *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathcal{C})$, a map

   $$F_{A,B}\colon \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F(B), F(A)),$$

   called the **action on morphisms of** F **at** (A, B).

satisfying the following conditions:

1. *Preservation of Identities.* For each $A \in \mathrm{Obj}(\mathcal{C})$, the diagram

   $$\begin{tikzcd} \mathrm{pt} \arrow[d, "\mathbb{1}^{\mathcal{C}}_A"'] \arrow[dr, "\mathbb{1}^{\mathcal{D}}_{F(A)}"] & \\ \mathrm{Hom}_{\mathcal{C}}(A, A) \arrow[r, "F_{A,A}"'] & \mathrm{Hom}_{\mathcal{D}}(F(A), F(A)) \end{tikzcd}$$

   commutes, i.e. we have

   $$F(\mathrm{id}_A) = \mathrm{id}_{F(A)}.$$

2. *Preservation of Composition.* For each $A, B, C \in \mathrm{Obj}(\mathcal{C})$, the diagram

$$\begin{tikzcd}
 & \mathrm{Hom}_{\mathcal{D}}(F(C), F(B)) \times \mathrm{Hom}_{\mathcal{D}}(F(B), F(A)) \arrow[dr, dashed, "\sigma^{\mathsf{Sets}}_{\mathrm{Hom}_{\mathcal{D}}(F(C),F(B)),\mathrm{Hom}_{\mathcal{D}}(F(B),F(A))}", "\sim"'] & \\
\mathrm{Hom}_{\mathcal{C}}(B, C) \times \mathrm{Hom}_{\mathcal{C}}(A, B) \arrow[ur, "F_{B,C} \times F_{A,B}"] \arrow[d, "\circ^{\mathcal{C}}_{A,B,C}"'] & & \mathrm{Hom}_{\mathcal{D}}(F(B), F(A)) \times \mathrm{Hom}_{\mathcal{D}}(F(C), F(B)) \arrow[d, "\circ^{\mathcal{D}}_{F(C),F(B),F(A)}"] \\
\mathrm{Hom}_{\mathcal{C}}(A, C) \arrow[rr, "F_{A,C}"'] & & \mathrm{Hom}_{\mathcal{D}}(F(C), F(A))
\end{tikzcd}$$

commutes, i.e. for each composable pair (g, f) of morphisms of $\mathcal{C}$, we have

$$F(g \circ f) = F(f) \circ F(g).$$

Remark 4.2.1.3. Throughout this work we will not use the term "contravariant" functor, speaking instead simply of functors $F \colon \mathcal{C}^{\mathsf{op}} \to \mathcal{D}$. We will usually, however, write

$$F_{A,B} \colon \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F(B), F(A))$$

for the action on morphisms

$$F_{A,B} \colon \mathrm{Hom}_{\mathcal{C}^{\mathsf{op}}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F(A), F(B))$$

of F, as well as write $F(g \circ f) = F(f) \circ F(g)$.

4.3 Forgetful Functors

Definition 4.3.1.1. There isn't a precise definition of a **forgetful functor**.

Remark 4.3.1.2. Despite there not being a formal or precise definition of a forgetful functor, the term is often very useful in practice, similarly to the word "canonical". The idea is that a "forgetful functor" is a functor that forgets structure or properties, and is best explained through examples, such as the ones below (see Examples 4.3.1.3 and 4.3.1.4).

Example 4.3.1.3. Examples of forgetful functors that forget structure include:

1. *Forgetting Group Structures.* The functor $\mathsf{Grp} \to \mathsf{Sets}$ sending a group (G, μ_G, η_G) to its underlying set G, forgetting the multiplication and unit maps μ_G and η_G of G.

2. *Forgetting Topologies.* The functor $\mathsf{Top} \to \mathsf{Sets}$ sending a topological space $(X, \mathcal{T}_X)$ to its underlying set X, forgetting the topology $\mathcal{T}_X$.

3. *Forgetting Fibrations.* The functor $\mathsf{FibSets}(K) \to \mathsf{Sets}$ sending a K-fibred set $\phi_X \colon X \to K$ to the set X, forgetting the map ϕ_X and the base set K.

Example 4.3.1.4. Examples of forgetful functors that forget properties include:

1. *Forgetting Commutativity.* The inclusion functor $\iota \colon \mathsf{CMon} \hookrightarrow \mathsf{Mon}$ which forgets the property of being commutative.

2. *Forgetting Inverses.* The inclusion functor $\iota \colon \mathsf{Grp} \hookrightarrow \mathsf{Mon}$ which forgets the property of having inverses.

Notation 4.3.1.5. Throughout this work, we will denote forgetful functors that forget structure by 忘, e.g. as in

$$\text{忘} \colon \mathsf{Grp} \to \mathsf{Sets}.$$

The symbol 忘, pronounced *wasureru* (see Item 1 of Remark 4.3.1.6 below), means *to forget*, and is a kanji found in the following words in Japanese and Chinese:

1. 忘れる, transcribed as *wasureru*, meaning *to forget*.

2. 忘却関手, transcribed as *boukyaku kanshu*, meaning *forgetful functor*.

3. 忘记 or 忘記, transcribed as *wàngjì*, meaning *to forget*.

4. 遗忘函子 or 遺忘函子, transcribed as *yíwàng hánz☒*, meaning *forgetful functor*.

Remark 4.3.1.6. Here we collect the pronunciation of the words in Notation 4.3.1.5 for accuracy and completeness.

1. Pronunciation of 忘れる:

- Audio: see https://topological-modular-forms.github.io/the-clowder-project/static/sounds/wasureru-01.mp3
- IPA broad transcription: [wäsɯɾe̞ɾɯ].
- IPA narrow transcription: [ɰᵝäsɨᵝɾe̞ɾɯ̟ᵝ].

2. Pronunciation of 忘却関手: Pronunciation:

- Audio: see https://topological-modular-forms.github.io/the-clowder-project/static/sounds/wasureru-02.mp3
- IPA broad transcription: [bo̞ːkʲäkɯ kä̃ɰ̃ɕɯ].
- IPA narrow transcription: [bo̞ːkʲäkɯ̟ᵝ kä̃ɰ̃ɕɯᵝ].

3. Pronunciation of 忘记:

- Audio: see https://topological-modular-forms.github.io/the-clowder-project/static/sounds/wasureru-03.ogg
- Broad IPA transcription: [wɑŋtɕi].
- Sinological IPA transcription: [wɑŋ$^{51-53}$t͡ɕi^{51}].

4. Pronunciation of 遗忘函子:

- Audio: see https://topological-modular-forms.github.io/the-clowder-project/static/sounds/wasureru-04.mp3
- Broad IPA transcription: [iwɑŋ xänt͡ʂʐ̩ɨ].
- Sinological IPA transcription: [i^{35}wɑŋ^{51} xän^{35}t͡ʂʐ̩$^{214-21(4)}$].

4.4 The Natural Transformation Associated to a Functor

Definition 4.4.1.1. Every functor $F\colon \mathcal{C} \to \mathcal{D}$ defines a natural transformation[19]

$$F^{\dagger}\colon \mathrm{Hom}_{\mathcal{C}} \Longrightarrow \mathrm{Hom}_{\mathcal{D}} \circ (F^{\mathsf{op}} \times F), \qquad \begin{tikzcd} \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \arrow[rr, "F^{\mathsf{op}} \times F"] \arrow[dr, "\mathrm{Hom}_{\mathcal{C}}"'] & \arrow[d, phantom, "\Longrightarrow F^{\dagger}"] & \mathcal{D}^{\mathsf{op}} \times \mathcal{D} \arrow[dl, "\mathrm{Hom}_{\mathcal{D}}"] \\ & \mathsf{Sets}, & \end{tikzcd}$$

[19] This is the 1-categorical version of Constructions With Sets, Item 1 of Proposition 4.1.1.3.

called the **natural transformation associated to** F, consisting of the collection

$$\left\{F^{\dagger}_{A,B}\colon \mathrm{Hom}_{\mathcal{C}}(A,B) \to \mathrm{Hom}_{\mathcal{D}}(F_A,F_B)\right\}_{(A,B)\in\mathrm{Obj}(\mathcal{C}^{\mathsf{op}}\times\mathcal{C})}$$

with

$$F^{\dagger}_{A,B} \overset{\mathrm{def}}{=} F_{A,B}.$$

Proof. The naturality condition for $F^{\dagger}$ is the requirement that for each morphism

$$(\phi,\psi)\colon (X,Y) \to (A,B)$$

of $\mathcal{C}^{\mathsf{op}}\times\mathcal{C}$, the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(X,Y) \arrow[r, "\phi^*\circ\psi_*=\psi_*\circ\phi^*"] \arrow[d, "F_{X,Y}"'] & \mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[d, "F_{A,B}"] \\
\mathrm{Hom}_{\mathcal{D}}(F_X,F_Y) \arrow[r, "F(\phi)^*\circ F(\psi)_*=F(\psi)_*\circ F(\phi)^*"'] & \mathrm{Hom}_{\mathcal{D}}(F_A,F_B),
\end{tikzcd}$$

acting on elements as

$$\begin{tikzcd}
f \arrow[r, mapsto] \arrow[d, mapsto] & \psi\circ f\circ\phi \arrow[d, mapsto] \\
F(f) \arrow[r, mapsto] & F(\psi)\circ F(f)\circ F(\psi) = F(\psi\circ f\circ\phi)
\end{tikzcd}$$

commutes, which follows from the functoriality of F. ☐

Proposition 4.4.1.2. Let $F\colon \mathcal{C}\to\mathcal{D}$ and $G\colon \mathcal{D}\to\mathcal{E}$ be functors.

1. *Interaction With Natural Isomorphisms.* The following conditions are equivalent:

   (a) The natural transformation $F^{\dagger}\colon \mathrm{Hom}_{\mathcal{C}} \Longrightarrow \mathrm{Hom}_{\mathcal{D}}\circ(F^{\mathsf{op}}\times F)$ associated to F is a natural isomorphism.

   (b) The functor F is fully faithful.

2. *Interaction With Composition.* We have an equality of pasting diagrams

$$
\begin{tikzcd}
\mathcal{C}^{\mathsf{op}} \times \mathcal{C} \arrow[r, "F^{\mathsf{op}}\times F"] \arrow[dr, "\mathrm{Hom}_{\mathcal{C}}"', ""{name=A}] & \mathcal{D}^{\mathsf{op}} \times \mathcal{D} \arrow[r, "G^{\mathsf{op}}\times G"] \arrow[d, "\mathrm{Hom}_{\mathcal{D}}" description] \arrow[dl, phantom, "\overset{F^{\dagger}}{\Rightarrow}"] & \mathcal{E}^{\mathsf{op}} \times \mathcal{E} \arrow[dl, "\mathrm{Hom}_{\mathcal{E}}"] \arrow[dll, phantom, "\overset{G^{\dagger}}{\Rightarrow}"] \\
 & \mathsf{Sets} &
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{C}^{\mathsf{op}} \times \mathcal{C} \arrow[rr, "(G\circ F)^{\mathsf{op}}\times (G\circ F)"] \arrow[dr, "\mathrm{Hom}_{\mathcal{C}}"'] & \arrow[d, phantom, "\overset{(G\circ F)^{\dagger}}{\Rightarrow}"] & \mathcal{E}^{\mathsf{op}} \times \mathcal{E}, \arrow[dl, "\mathrm{Hom}_{\mathcal{E}}"] \\
 & \mathsf{Sets} &
\end{tikzcd}
$$

in Cats_2, i.e. we have

$$(G \circ F)^{\dagger} = (G^{\dagger} \star \mathrm{id}_{F^{\mathsf{op}}\times F}) \circ F^{\dagger}.$$

3. *Interaction With Identities.* We have

$$\mathrm{id}^{\dagger}_{\mathcal{C}} = \mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(-_1,-_2)},$$

i.e. the natural transformation associated to $\mathrm{id}_{\mathcal{C}}$ is the identity natural transformation of the functor $\mathrm{Hom}_{\mathcal{C}}(-_1, -_2)$.

Proof. Item 1, *Interaction With Natural Isomorphisms*: Clear.
Item 2, *Interaction With Composition*: Clear.
Item 3, *Interaction With Identities*: Clear. □

5 Conditions on Functors

5.1 Faithful Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 5.1.1.1. A functor $F: \mathcal{C} \to \mathcal{D}$ is **faithful** if, for each $A, B \in \mathrm{Obj}(\mathcal{C})$, the action on morphisms

$$F_{A,B}: \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F_A, F_B)$$

of F at (A, B) is injective.

Proposition 5.1.1.2. Let $F: \mathcal{C} \to \mathcal{D}$ be a functor.

1. *Interaction With Postcomposition.* The following conditions are equivalent:

   (a) The functor $F: \mathcal{C} \to \mathcal{D}$ is faithful.

(b) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor

$$F_* \colon \mathsf{Fun}(\mathcal{X}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{X}, \mathcal{D})$$

is faithful.

(c) The functor $F \colon \mathcal{C} \to \mathcal{D}$ is a representably faithful morphism in Cats_2 in the sense of Types of Morphisms in Bicategories, Definition 1.1.1.1.

2. *Interaction With Precomposition I.* Let $F \colon \mathcal{C} \to \mathcal{D}$ be a functor.

   (a) If F is faithful, then the precomposition functor

   $$F^* \colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

   can fail to be faithful.

   (b) Conversely, if the precomposition functor

   $$F^* \colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

   is faithful, then F *can fail* to be faithful.

3. *Interaction With Precomposition II.* If F is essentially surjective, then the precomposition functor

   $$F^* \colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

   is faithful.

4. *Interaction With Precomposition III.* The following conditions are equivalent:

   (a) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor

   $$F^* \colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

   is faithful.

   (b) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor

   $$F^* \colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

   is conservative.

(c) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor

$$F^* \colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

is monadic.

(d) The functor $F \colon \mathcal{C} \to \mathcal{D}$ is a corepresentably faithful morphism in Cats_2 in the sense of Types of Morphisms in Bicategories, Definition 2.1.1.1.

(e) The components

$$\eta_G \colon G \Longrightarrow \mathrm{Ran}_F(G \circ F)$$

of the unit

$$\eta \colon \mathrm{id}_{\mathsf{Fun}(\mathcal{D},\mathcal{X})} \Longrightarrow \mathrm{Ran}_F \circ F^*$$

of the adjunction $F^* \dashv \mathrm{Ran}_F$ are all monomorphisms.

(f) The components

$$\epsilon_G \colon \mathrm{Lan}_F(G \circ F) \Longrightarrow G$$

of the counit

$$\epsilon \colon \mathrm{Lan}_F \circ F^* \Longrightarrow \mathrm{id}_{\mathsf{Fun}(\mathcal{D},\mathcal{X})}$$

of the adjunction $\mathrm{Lan}_F \dashv F^*$ are all epimorphisms.

(g) The functor F is dominant (Definition 6.1.1.1), i.e. every object of $\mathcal{D}$ is a retract of some object in $\mathrm{Im}(F)$:

($\star$) For each $B \in \mathrm{Obj}(\mathcal{D})$, there exist:
- An object A of $\mathcal{C}$;
- A morphism $s \colon B \to F(A)$ of $\mathcal{D}$;
- A morphism $r \colon F(A) \to B$ of $\mathcal{D}$;

such that $r \circ s = \mathrm{id}_B$.

Proof. *Item 1, Interaction With Postcomposition*: Omitted.
Item 2, Interaction With Precomposition I: See [MSE 733163] for Item 2a. Item 2b follows from Item 3 and the fact that there are essentially surjective functors that are not faithful.
Item 3, Interaction With Precomposition II: Omitted, but see https://unimath.github.io/doc/UniMath/d4de26f//UniMath.CategoryTheory.precomp_fully_faithful.html for a formalised proof.
Item 4, Interaction With Precomposition III: We claim Items 4a to 4g are equivalent:

- *Items 4a and 4d Are Equivalent:* This is true by the definition of corepresentably faithful morphism; see Types of Morphisms in Bicategories, Definition 2.1.1.1.

- *Items 4a to 4c and 4g Are Equivalent:* See [Adá+01, Proposition 4.1] or alternatively [Fre09, Lemmas 3.1 and 3.2] for the equivalence between Items 4a and 4g.

- *Items 4a, 4e and 4f Are Equivalent:* See **??**, **??** of **??**.

This finishes the proof. □

5.2 Full Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 5.2.1.1. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **full** if, for each $A, B \in \mathrm{Obj}(\mathcal{C})$, the action on morphisms

$$F_{A,B}\colon \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F_A, F_B)$$

of F at (A, B) is surjective.

Proposition 5.2.1.2. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor.

1. *Interaction With Postcomposition.* The following conditions are equivalent:

   (a) The functor $F\colon \mathcal{C} \to \mathcal{D}$ is full.

   (b) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor

   $$F_*\colon \mathsf{Fun}(\mathcal{X}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{X}, \mathcal{D})$$

   is full.

   (c) The functor $F\colon \mathcal{C} \to \mathcal{D}$ is a representably full morphism in Cats_2 in the sense of Types of Morphisms in Bicategories, Definition 1.2.1.1.

2. *Interaction With Precomposition I.* If F is full, then the precomposition functor

   $$F^*\colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

   can fail to be full.

3. *Interaction With Precomposition II.* If the precomposition functor

$$F^* \colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

is full, then F *can fail* to be full.

4. *Interaction With Precomposition III.* If F is essentially surjective and full, then the precomposition functor

$$F^* \colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

is full (and also faithful by Item 3 of Proposition 5.1.1.2).

5. *Interaction With Precomposition IV.* The following conditions are equivalent:

   (a) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor

   $$F^* \colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

   is full.

   (b) The functor $F \colon \mathcal{C} \to \mathcal{D}$ is a corepresentably full morphism in Cats_2 in the sense of Types of Morphisms in Bicategories, Definition 2.1.1.1.

   (c) The components

   $$\eta_G \colon G \Longrightarrow \mathrm{Ran}_F(G \circ F)$$

   of the unit

   $$\eta \colon \mathrm{id}_{\mathsf{Fun}(\mathcal{D}, \mathcal{X})} \Longrightarrow \mathrm{Ran}_F \circ F^*$$

   of the adjunction $F^* \dashv \mathrm{Ran}_F$ are all retractions/split epimorphisms.

   (d) The components

   $$\epsilon_G \colon \mathrm{Lan}_F(G \circ F) \Longrightarrow G$$

   of the counit

   $$\epsilon \colon \mathrm{Lan}_F \circ F^* \Longrightarrow \mathrm{id}_{\mathsf{Fun}(\mathcal{D}, \mathcal{X})}$$

   of the adjunction $\mathrm{Lan}_F \dashv F^*$ are all sections/split monomorphisms.

(e) For each $B \in \text{Obj}(\mathcal{D})$, there exist:

- An object A_B of $\mathcal{C}$;
- A morphism $s_B \colon B \to F(A_B)$ of $\mathcal{D}$;
- A morphism $r_B \colon F(A_B) \to B$ of $\mathcal{D}$;

satisfying the following condition:

($\star$) For each $A \in \text{Obj}(\mathcal{C})$ and each pair of morphisms

$$\begin{aligned} r &\colon F(A) \to B, \\ s &\colon B \to F(A) \end{aligned}$$

of $\mathcal{D}$, we have

$$[(A_B, s_B, r_B)] = [(A, s, r \circ s_B \circ r_B)]$$

in $\int^{A \in \mathcal{C}} h^{B'}_{F_A} \times h^{F_A}_{B}$.

Proof. *Item 1, Interaction With Postcomposition*: Omitted.
Item 2, Interaction With Precomposition I: Omitted.
Item 3, Interaction With Precomposition II: See [BS10, p. 47].
Item 4, Interaction With Precomposition III: Omitted, but see `https://unimath.github.io/doc/UniMath/d4de26f//UniMath.CategoryTheory.precomp_fully_faithful.html` for a formalised proof.
Item 5, Interaction With Precomposition IV: We claim Items 5a to 5e are equivalent:

- *Items 5a and 5b Are Equivalent:* This is true by the definition of corepresentably full morphism; see Types of Morphisms in Bicategories, Definition 2.2.1.1.
- *Items 5a, 5c and 5d Are Equivalent:* See **??**, **??** of **??**.
- *Items 5a and 5e Are Equivalent:* See [Adá+01, Item (b) of Remark 4.3].

This finishes the proof. □

Question 5.2.1.3. Item 5 of Proposition 5.2.1.2 gives a characterisation of the functors F for which F^* is full, but the characterisations given there are really messy. Are there better ones?
This question also appears as [`MO 468121b`].

5.3 Fully Faithful Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 5.3.1.1. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **fully faithful** if F is full and faithful, i.e. if, for each $A, B \in \mathrm{Obj}(\mathcal{C})$, the action on morphisms

$$F_{A,B}\colon \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F_A, F_B)$$

of F at (A, B) is bijective.

Proposition 5.3.1.2. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor.

1. *Characterisations.* The following conditions are equivalent:
   (a) The functor F is fully faithful.
   (b) We have a pullback square

$$\mathsf{Arr}(\mathcal{C}) \cong (\mathcal{C} \times \mathcal{C}) \times_{\mathcal{D}\times\mathcal{D}} \mathsf{Arr}(\mathcal{D}), \qquad \begin{tikzcd} \mathsf{Arr}(\mathcal{C}) \arrow[r, "\mathsf{Arr}(F)"] \arrow[d, "\mathrm{src}\times\mathrm{tgt}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Arr}(\mathcal{D}) \arrow[d, "\mathrm{src}\times\mathrm{tgt}"] \\ \mathcal{C}\times\mathcal{C} \arrow[r, "F\times F"'] & \mathcal{D}\times\mathcal{D} \end{tikzcd}$$

   in Cats.

2. *Conservativity.* If F is fully faithful, then F is conservative.

3. *Essential Injectivity.* If F is fully faithful, then F is essentially injective.

4. *Interaction With Co/Limits.* If F is fully faithful, then F reflects co/limits.

5. *Interaction With Postcomposition.* The following conditions are equivalent:
   (a) The functor $F\colon \mathcal{C} \to \mathcal{D}$ is fully faithful.
   (b) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor

$$F_*\colon \mathsf{Fun}(\mathcal{X}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{X}, \mathcal{D})$$

   is fully faithful.
   (c) The functor $F\colon \mathcal{C} \to \mathcal{D}$ is a representably fully faithful morphism in Cats_2 in the sense of Types of Morphisms in Bicategories, Definition 1.3.1.1.

6. *Interaction With Precomposition I.* If F is fully faithful, then the precomposition functor

$$F^*\colon \mathsf{Fun}(\mathcal{D},\mathcal{X}) \to \mathsf{Fun}(\mathcal{C},\mathcal{X})$$

can fail to be fully faithful.

7. *Interaction With Precomposition II.* If the precomposition functor

$$F^*\colon \mathsf{Fun}(\mathcal{D},\mathcal{X}) \to \mathsf{Fun}(\mathcal{C},\mathcal{X})$$

is fully faithful, then F *can fail* to be fully faithful (and in fact it can also fail to be either full or faithful).

8. *Interaction With Precomposition III.* If F is essentially surjective and full, then the precomposition functor

$$F^*\colon \mathsf{Fun}(\mathcal{D},\mathcal{X}) \to \mathsf{Fun}(\mathcal{C},\mathcal{X})$$

is fully faithful.

9. *Interaction With Precomposition IV.* The following conditions are equivalent:

    (a) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor

    $$F^*\colon \mathsf{Fun}(\mathcal{D},\mathcal{X}) \to \mathsf{Fun}(\mathcal{C},\mathcal{X})$$

    is fully faithful.

    (b) The precomposition functor

    $$F^*\colon \mathsf{Fun}(\mathcal{D},\mathsf{Sets}) \to \mathsf{Fun}(\mathcal{C},\mathsf{Sets})$$

    is fully faithful.

    (c) The functor

    $$\mathrm{Lan}_F\colon \mathsf{Fun}(\mathcal{C},\mathsf{Sets}) \to \mathsf{Fun}(\mathcal{D},\mathsf{Sets})$$

    is fully faithful.

    (d) The functor F is a corepresentably fully faithful morphism in Cats_2 in the sense of Types of Morphisms in Bicategories, Definition 2.3.1.1.

(e) The functor F is absolutely dense.

(f) The components

$$\eta_G \colon G \Longrightarrow \mathrm{Ran}_F(G \circ F)$$

of the unit

$$\eta \colon \mathrm{id}_{\mathsf{Fun}(\mathcal{D},\mathcal{X})} \Longrightarrow \mathrm{Ran}_F \circ F^*$$

of the adjunction $F^* \dashv \mathrm{Ran}_F$ are all isomorphisms.

(g) The components

$$\epsilon_G \colon \mathrm{Lan}_F(G \circ F) \Longrightarrow G$$

of the counit

$$\epsilon \colon \mathrm{Lan}_F \circ F^* \Longrightarrow \mathrm{id}_{\mathsf{Fun}(\mathcal{D},\mathcal{X})}$$

of the adjunction $\mathrm{Lan}_F \dashv F^*$ are all isomorphisms.

(h) The natural transformation

$$\alpha \colon \mathrm{Lan}_{h_F}(h^F) \Longrightarrow h$$

with components

$$\alpha_{B',B} \colon \int^{A \in \mathcal{C}} h^{B'}_{F_A} \times h^{F_A}_{B} \to h^{B'}_{B}$$

given by

$$\alpha_{B',B}([(\phi, \psi)]) = \psi \circ \phi$$

is a natural isomorphism.

(i) For each $B \in \mathrm{Obj}(\mathcal{D})$, there exist:

- An object A_B of $\mathcal{C}$;
- A morphism $s_B \colon B \to F(A_B)$ of $\mathcal{D}$;
- A morphism $r_B \colon F(A_B) \to B$ of $\mathcal{D}$;

satisfying the following conditions:

i. The triple $(F(A_B), r_B, s_B)$ is a retract of B, i.e. we have $r_B \circ s_B = \mathrm{id}_B$.

ii. For each morphism $f\colon B' \to B$ of $\mathcal{D}$, we have

$$[(A_B, s_{B'}, f \circ r_{B'})] = [(A_B, s_B \circ f, r_B)]$$

in $\int^{A \in C} h^{B'}_{F_A} \times h^{F_A}_{B}$.

Proof. Item 1, Characterisations: Omitted.
Item 2, Conservativity: This is a repetition of Item 2 of Proposition 5.4.1.2, and is proved there.
Item 3, Essential Injectivity: Omitted.
Item 4, Interaction With Co/Limits: Omitted.
Item 5, Interaction With Postcomposition: This follows from Item 1 of Proposition 5.1.1.2 and Item 1 of Proposition 5.2.1.2.
Item 6, Interaction With Precomposition I: See [`MSE 733161`] for an example of a fully faithful functor whose precomposition with which fails to be full.
Item 7, Interaction With Precomposition II: See [`MSE 749304`, Item 3].
Item 8, Interaction With Precomposition III: Omitted, but see `https://unimath.github.io/doc/UniMath/d4de26f//UniMath.CategoryTheory.precomp_fully_faithful.html` for a formalised proof.
Item 9, Interaction With Precomposition IV: We claim Items 9a to 9i are equivalent:

- *Items 9a and 9d Are Equivalent:* This is true by the definition of corepresentably fully faithful morphism; see Types of Morphisms in Bicategories, Definition 2.3.1.1.

- *Items 9a, 9f and 9g Are Equivalent:* See **??**, **??** of **??**.

- *Items 9a to 9c Are Equivalent:* This follows from [Low15, Proposition A.1.5].

- *Items 9a, 9e, 9h and 9i Are Equivalent:* See [Fre09, Theorem 4.1] and [Adá+01, Theorem 1.1].

This finishes the proof. □

5.4 Conservative Functors

Let C and $\mathcal{D}$ be categories.

Definition 5.4.1.1. A functor $F\colon C \to \mathcal{D}$ is **conservative** if it satisfies the following condition:[20]

(★) For each $f \in \mathrm{Mor}(\mathcal{C})$, if $F(f)$ is an isomorphism in $\mathcal{D}$, then f is an isomorphism in $\mathcal{C}$.

Proposition 5.4.1.2. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor.

1. *Characterisations.* The following conditions are equivalent:

   (a) The functor F is conservative.

   (b) For each $f \in \mathrm{Mor}(\mathcal{C})$, the morphism $F(f)$ is an isomorphism in $\mathcal{D}$ iff f is an isomorphism in $\mathcal{C}$.

2. *Interaction With Fully Faithfulness.* Every fully faithful functor is conservative.

3. *Interaction With Precomposition.* The following conditions are equivalent:

   (a) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor

   $$F^*\colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

   is conservative.

   (b) The equivalent conditions of Item 4 of Proposition 5.1.1.2 are satisfied.

Proof. *Item 1, Characterisations*: This follows from Item 1 of Proposition 4.1.1.6. *Item 2, Interaction With Fully Faithfulness*: Let $F\colon \mathcal{C} \to \mathcal{D}$ be a fully faithful functor, let $f\colon A \to B$ be a morphism of $\mathcal{C}$, and suppose that F_f is an isomorphism. We have

$$\begin{aligned} F(\mathrm{id}_B) &= \mathrm{id}_{F(B)} \\ &= F(f) \circ F(f)^{-1} \\ &= F(f \circ f^{-1}). \end{aligned}$$

Similarly, $F(\mathrm{id}_A) = F(f^{-1} \circ f)$. But since F is fully faithful, we must have

$$\begin{aligned} f \circ f^{-1} &= \mathrm{id}_B, \\ f^{-1} \circ f &= \mathrm{id}_A, \end{aligned}$$

showing f to be an isomorphism. Thus F is conservative. □

[20] *Slogan:* A functor F is **conservative** if it reflects isomorphisms.

Question 5.4.1.3. Is there a characterisation of functors $F\colon \mathcal{C} \to \mathcal{D}$ satisfying the following condition:

($\star$) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor

$$F_*\colon \mathsf{Fun}(\mathcal{X}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{X}, \mathcal{D})$$

is conservative?

This question also appears as [MO 468121a].

5.5 Essentially Injective Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 5.5.1.1. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **essentially injective** if it satisfies the following condition:

($\star$) For each $A, B \in \mathrm{Obj}(\mathcal{C})$, if $F(A) \cong F(B)$, then $A \cong B$.

Question 5.5.1.2. Is there a characterisation of functors $F\colon \mathcal{C} \to \mathcal{D}$ such that:

1. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor

$$F^*\colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

   is essentially injective, i.e. if $\phi \circ F \cong \psi \circ F$, then $\phi \cong \psi$ for all functors ϕ and ψ?

2. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor

$$F_*\colon \mathsf{Fun}(\mathcal{X}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{X}, \mathcal{D})$$

   is essentially injective, i.e. if $F \circ \phi \cong F \circ \psi$, then $\phi \cong \psi$?

This question also appears as [MO 468121a].

5.6 Essentially Surjective Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 5.6.1.1. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **essentially surjective**[21] if it satisfies the following condition:

($\star$) For each $D \in \mathrm{Obj}(\mathcal{D})$, there exists some object A of $\mathcal{C}$ such that $F(A) \cong D$.

Question 5.6.1.2. Is there a characterisation of functors $F\colon \mathcal{C} \to \mathcal{D}$ such that:

1. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor

$$F^*\colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

is essentially surjective?

2. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor

$$F_*\colon \mathsf{Fun}(\mathcal{X}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{X}, \mathcal{D})$$

is essentially surjective?

This question also appears as [MO 468121a].

5.7 Equivalences of Categories

Definition 5.7.1.1. Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

1. An **equivalence of categories** between $\mathcal{C}$ and $\mathcal{D}$ consists of a pair of functors

$$\begin{aligned} F &\colon \mathcal{C} \to \mathcal{D}, \\ G &\colon \mathcal{D} \to \mathcal{C} \end{aligned}$$

together with natural isomorphisms

$$\begin{aligned} \eta &\colon \mathrm{id}_{\mathcal{C}} \overset{\sim}{\Longrightarrow} G \circ F, \\ \epsilon &\colon F \circ G \overset{\sim}{\Longrightarrow} \mathrm{id}_{\mathcal{D}}. \end{aligned}$$

2. An **adjoint equivalence of categories** between $\mathcal{C}$ and $\mathcal{D}$ is an equivalence (F, G, η, ϵ) between $\mathcal{C}$ and $\mathcal{D}$ which is also an adjunction.

[21] *Further Terminology:* Also called an **eso** functor, where the name "eso" comes from *essentially*

Proposition 5.7.1.2. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor.

1. *Characterisations.* If $\mathcal{C}$ and $\mathcal{D}$ are small[22], then the following conditions are equivalent:[23]

   (a) The functor F is an equivalence of categories.

   (b) The functor F is fully faithful and essentially surjective.

   (c) The induced functor

   $$\restriction F\mathsf{Sk}(\mathcal{C})\colon \mathsf{Sk}(\mathcal{C}) \to \mathsf{Sk}(\mathcal{D})$$

   is an *isomorphism* of categories.

   (d) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor

   $$F^*\colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

   is an equivalence of categories.

   (e) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor

   $$F_*\colon \mathsf{Fun}(\mathcal{X}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{X}, \mathcal{D})$$

   is an equivalence of categories.

2. *Two-Out-of-Three.* Let

   $$\begin{tikzcd} \mathcal{C} \arrow[rr, "G\circ F"] \arrow[dr, "F"'] & & \mathcal{E} \\ & \mathcal{D} \arrow[ur, "G"'] & \end{tikzcd}$$

   be a diagram in Cats. If two out of the three functors among F, G, and $G \circ F$ are equivalences of categories, then so is the third.

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surjective on objects.

[22] Otherwise there will be size issues. One can also work with large categories and universes, or require F to be *constructively* essentially surjective; see [MSE 1465107].

[23] In ZFC, the equivalence between Item 1a and Item 1b is equivalent to the axiom of choice; see [MO 119454].

In Univalent Foundations, this is true without requiring neither the axiom of choice nor the law

3. *Stability Under Composition.* Let

$$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D} \underset{G'}{\overset{F'}{\rightleftarrows}} \mathcal{E}$$

be a diagram in Cats. If (F, G) and (F', G') are equivalences of categories, then so is their composite $(F' \circ F, G' \circ G)$.

4. *Equivalences vs.Adjoint Equivalences.* Every equivalence of categories can be promoted to an adjoint equivalence.[24]

5. *Interaction With Groupoids.* If $\mathcal{C}$ and $\mathcal{D}$ are groupoids, then the following conditions are equivalent:

   (a) The functor F is an equivalence of groupoids.

   (b) The following conditions are satisfied:

      i. The functor F induces a bijection
      $$\pi_0(F) \colon \pi_0(\mathcal{C}) \to \pi_0(\mathcal{D})$$
      of sets.

      ii. For each $A \in \mathrm{Obj}(\mathcal{C})$, the induced map
      $$F_{x,x} \colon \mathrm{Aut}_{\mathcal{C}}(A) \to \mathrm{Aut}_{\mathcal{D}}(F_A)$$
      is an isomorphism of groups.

Proof. *Item 1, Characterisations*: We claim that Items 1a to 1e are indeed equivalent:

1. *Item 1a* $\Longrightarrow$ *Item 1b:* Clear.

2. *Item 1b* $\Longrightarrow$ *Item 1a:* Since F is essentially surjective and $\mathcal{C}$ and $\mathcal{D}$ are small, we can choose, using the axiom of choice, for each $B \in \mathrm{Obj}(\mathcal{D})$, an object j_B of $\mathcal{C}$ and an isomorphism $i_B \colon B \to F_{j_B}$ of $\mathcal{D}$.

   Since F is fully faithful, we can extend the assignment $B \mapsto j_B$ to a *unique* functor $j \colon \mathcal{D} \to \mathcal{C}$ such that the isomorphisms $i_B \colon B \to F_{j_B}$ assemble into a natural isomorphism $\eta \colon \mathrm{id}_{\mathcal{D}} \overset{\sim}{\Longrightarrow} F \circ j$, with a similar natural isomorphism $\epsilon \colon \mathrm{id}_{\mathcal{C}} \overset{\sim}{\Longrightarrow} j \circ F$. Hence F is an equivalence.

of excluded middle.

[24]More precisely, we can promote an equivalence of categories (F, G, η, ϵ) to adjoint equiva-

3. *Item 1a* $\implies$ *Item 1c*: This follows from Item 4 of Proposition 1.5.1.3.
4. *Item 1c* $\implies$ *Item 1a*: Omitted.
5. *Items 1a, 1d and 1e Are Equivalent:* This follows from **??**.

This finishes the proof of Item 1.
Item 2, Two-Out-of-Three: Omitted.
Item 3, Stability Under Composition: Clear.
Item 4, Equivalences vs.Adjoint Equivalences: See [Rie17, Proposition 4.4.5].
Item 5, Interaction With Groupoids: See [nLa24, Proposition 4.4]. □

5.8 Isomorphisms of Categories

Definition 5.8.1.1. An **isomorphism of categories** is a pair of functors

$$\begin{aligned} F &\colon \mathcal{C} \to \mathcal{D}, \\ G &\colon \mathcal{D} \to \mathcal{C} \end{aligned}$$

such that we have

$$G \circ F = \mathrm{id}_{\mathcal{C}},$$

$$F \circ G = \mathrm{id}_{\mathcal{D}}.$$

Example 5.8.1.2. Categories can be equivalent but non-isomorphic. For example, the category consisting of two isomorphic objects is equivalent to pt, but not isomorphic to it.

Proposition 5.8.1.3. Let $F \colon \mathcal{C} \to \mathcal{D}$ be a functor.

1. *Characterisations.* If $\mathcal{C}$ and $\mathcal{D}$ are small, then the following conditions are equivalent:
   (a) The functor F is an isomorphism of categories.
   (b) The functor F is fully faithful and bijective on objects.
   (c) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor
   $$F^* \colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$
   is an isomorphism of categories.

(d) For each $X \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor

$$F_* \colon \mathsf{Fun}(\mathcal{X}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{X}, \mathcal{D})$$

is an isomorphism of categories.

Proof. *Item 1, Characterisations*: We claim that Items 1a to 1d are indeed equivalent:

1. *Items 1a and 1b Are Equivalent:* Omitted, but similar to Item 1 of Proposition 5.7.1.2.

2. *Items 1a, 1c and 1d Are Equivalent:* This follows from **??**.

This finishes the proof. □

6 More Conditions on Functors

6.1 Dominant Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 6.1.1.1. A functor $F \colon \mathcal{C} \to \mathcal{D}$ is **dominant** if every object of $\mathcal{D}$ is a retract of some object in $\mathrm{Im}(F)$, i.e.:

($\star$) For each $B \in \mathrm{Obj}(\mathcal{D})$, there exist:

- An object A of $\mathcal{C}$;
- A morphism $r \colon F(A) \to B$ of $\mathcal{D}$;
- A morphism $s \colon B \to F(A)$ of $\mathcal{D}$;

such that we have

$$r \circ s = \mathrm{id}_B, \qquad \begin{tikzcd} B \arrow[r, "s"] \arrow[dr, "\mathrm{id}_B"'] & F(A) \arrow[d, "r"] \\ & B. \end{tikzcd}$$

Proposition 6.1.1.2. Let $F, G \colon \mathcal{C} \rightrightarrows \mathcal{D}$ be functors and let $I \colon \mathcal{X} \to \mathcal{C}$ be a functor.

1. *Interaction With Right Whiskering.* If I is full and dominant, then the map

$$- \star \mathrm{id}_I \colon \mathrm{Nat}(F, G) \to \mathrm{Nat}(F \circ I, G \circ I)$$

   is a bijection.

2. *Interaction With Adjunctions.* Let $(F, G) \colon \mathcal{C} \rightleftarrows \mathcal{D}$ be an adjunction.

   (a) If F is dominant, then G is faithful.

   (b) The following conditions are equivalent:

      i. The functor G is full.

      ii. The restriction

$$\upharpoonright G\mathrm{Im}_F \colon \mathrm{Im}(F) \to \mathcal{C}$$

      of G to $\mathrm{Im}(F)$ is full.

Proof. *Item 1, Interaction With Right Whiskering*: See [DFH75, Proposition 1.4]. *Item 2, Interaction With Adjunctions*: See [DFH75, Proposition 1.7]. □

Question 6.1.1.3. Is there a characterisation of functors $F \colon \mathcal{C} \to \mathcal{D}$ such that:

1. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor

$$F^* \colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

   is dominant?

2. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor

$$F_* \colon \mathsf{Fun}(\mathcal{X}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{X}, \mathcal{D})$$

   is dominant?

This question also appears as [`MO 468121a`].

6.2 Monomorphisms of Categories

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

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lences (F, G, η', ϵ) and (F, G, η, ϵ').

Definition 6.2.1.1. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is a **monomorphism of categories** if it is a monomorphism in Cats (see **??**, **??**).

Proposition 6.2.1.2. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor.

1. *Characterisations.* The following conditions are equivalent:
   (a) The functor F is a monomorphism of categories.
   (b) The functor F is injective on objects and morphisms, i.e. F is injective on objects and the map
   $$F\colon \mathrm{Mor}(\mathcal{C}) \to \mathrm{Mor}(\mathcal{D})$$
   is injective.

Proof. *Item 1*, *Characterisations*: Omitted. □

Question 6.2.1.3. Is there a characterisation of functors $F\colon \mathcal{C} \to \mathcal{D}$ such that:

1. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor
   $$F^*\colon \mathsf{Fun}(\mathcal{D},\mathcal{X}) \to \mathsf{Fun}(\mathcal{C},\mathcal{X})$$
   is a monomorphism of categories?
2. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor
   $$F_*\colon \mathsf{Fun}(\mathcal{X},\mathcal{C}) \to \mathsf{Fun}(\mathcal{X},\mathcal{D})$$
   is a monomorphism of categories?

This question also appears as [MO 468121a].

6.3 Epimorphisms of Categories

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 6.3.1.1. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is a **epimorphism of categories** if it is a epimorphism in Cats (see **??**, **??**).

Proposition 6.3.1.2. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor.

1. *Characterisations.* The following conditions are equivalent:[25]

   (a) The functor F is a epimorphism of categories.

   (b) For each morphism $f\colon A \to B$ of $\mathcal{D}$, we have a diagram

$$\begin{tikzcd}
 & & & & A \arrow[dllll, "\phi_1"'] \arrow[dll, "\phi_2"'] \arrow[d, "\phi_3"] \arrow[drr, "\phi_m"] \arrow[drrrr, dashed, "\mathrm{id}_A"] & & & & \\
X_1 \arrow[dr, dashed, "\alpha_1"] \arrow[ddrrrr, dashed, bend right, "\alpha_0"'] & & X_2 \arrow[dl, dashed, "\alpha_2"'] \arrow[dr, dashed, "\alpha_3"] & & X_3 \arrow[dl, dashed, "\alpha_4"'] \arrow[dr, dashed, "\alpha_6"] & \cdots & X_m \arrow[dl, dashed, "\alpha_{2m-2}"'] \arrow[dr, dashed, "\alpha_{2m-1}"] & & A \arrow[dl, dashed, "\alpha_{2m}"'] \\
 & Y_1 \arrow[drrr, "\psi_1"] & & Y_2 \arrow[dr, "\psi_2"] & & \cdots \arrow[dl, "\psi_3"'] & & Y_m \arrow[dlll, "\psi_m"'] & \\
 & & & & B & & & &
\end{tikzcd}$$

   in $\mathcal{D}$ satisfying the following conditions:

   i. We have $f = \alpha_0 \circ \phi_1$.
   ii. We have $f = \psi_m \circ \alpha_{2m}$.
   iii. For each $0 \leq i \leq 2m$, we have $\alpha_i \in \mathrm{Mor}(\mathrm{Im}(F))$.

2. *Surjectivity on Objects.* If F is an epimorphism of categories, then F is surjective on objects.

Proof. Item 1, Characterisations: See [Isb68].
Item 2, Surjectivity on Objects: Omitted. □

Question 6.3.1.3. Is there a characterisation of functors $F\colon \mathcal{C} \to \mathcal{D}$ such that:

1. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor

$$F^*\colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

   is an epimorphism of categories?

[25] *Further Terminology:* This statement is known as **Isbell's zigzag theorem**.

2. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor

$$F_*\colon \mathsf{Fun}(\mathcal{X}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{X}, \mathcal{D})$$

is an epimorphism of categories?

This question also appears as [MO 468121a].

6.4 Pseudomonic Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 6.4.1.1. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **pseudomonic** if it satisfies the following conditions:

1. For all diagrams of the form

$$\begin{tikzcd} \mathcal{X} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & \mathcal{C} \arrow[r, "F"] & \mathcal{D}, \end{tikzcd}$$

if we have

$$\mathrm{id}_F \star \alpha = \mathrm{id}_F \star \beta,$$

then $\alpha = \beta$.

2. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$ and each natural isomorphism

$$\beta\colon F \circ \phi \overset{\sim}{\Longrightarrow} F \circ \psi, \qquad \begin{tikzcd} \mathcal{X} \arrow[r, bend left, "F\circ\phi"] \arrow[r, bend right, "F\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{D}, \end{tikzcd}$$

there exists a natural isomorphism

$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} \mathcal{X} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{C} \end{tikzcd}$$

such that we have an equality

$$\begin{tikzcd} \mathcal{X} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{C} \arrow[r, "F"] & \mathcal{D} \end{tikzcd} \quad = \quad \begin{tikzcd} \mathcal{X} \arrow[r, bend left, "F\circ\phi"] \arrow[r, bend right, "F\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{D} \end{tikzcd}$$

of pasting diagrams, i.e. such that we have

$$\beta = \mathrm{id}_F \star \alpha.$$

Proposition 6.4.1.2. Let $F \colon \mathcal{C} \to \mathcal{D}$ be a functor.

1. *Characterisations.* The following conditions are equivalent:

   (a) The functor F is pseudomonic.

   (b) The functor F satisfies the following conditions:

      i. The functor F is faithful, i.e. for each $A, B \in \mathrm{Obj}(\mathcal{C})$, the action on morphisms

      $$F_{A,B} \colon \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F_A, F_B)$$

      of F at (A, B) is injective.

      ii. For each $A, B \in \mathrm{Obj}(\mathcal{C})$, the restriction

      $$F^{\mathrm{iso}}_{A,B} \colon \mathrm{Iso}_{\mathcal{C}}(A, B) \to \mathrm{Iso}_{\mathcal{D}}(F_A, F_B)$$

      of the action on morphisms of F at (A, B) to isomorphisms is surjective.

   (c) We have an isocomma square of the form

   $$\mathcal{C} \overset{\text{eq.}}{\cong} \mathcal{C} \overset{\leftrightarrow}{\times}_{\mathcal{D}} \mathcal{C}, \qquad \begin{tikzcd} \mathcal{C} \arrow[r, "\mathrm{id}_{\mathcal{C}}"] \arrow[d, "\mathrm{id}_{\mathcal{C}}"'] \arrow[dr, phantom, "\Leftrightarrow"] & \mathcal{C} \arrow[d, "F"] \\ \mathcal{C} \arrow[r, "F"'] & \mathcal{D} \end{tikzcd}$$

   in Cats_2 up to equivalence.

   (d) We have an isocomma square of the form

   $$\mathcal{C} \overset{\text{eq.}}{\cong} \mathcal{C} \overset{\leftrightarrow}{\times}_{\mathsf{Arr}(\mathcal{D})} \mathcal{D}, \qquad \begin{tikzcd} \mathcal{C} \arrow[r, hook] \arrow[d, "F"'] \arrow[dr, phantom, "\Leftrightarrow"] & \mathsf{Arr}(\mathcal{C}) \arrow[d, "\mathsf{Arr}(F)"] \\ \mathcal{D} \arrow[r, hook] & \mathsf{Arr}(\mathcal{D}) \end{tikzcd}$$

   in Cats_2 up to equivalence.

(e) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition[26] functor

$$F_* \colon \mathsf{Fun}(\mathcal{X}, C) \to \mathsf{Fun}(\mathcal{X}, \mathcal{D})$$

is pseudomonic.

2. *Conservativity.* If F is pseudomonic, then F is conservative.

3. *Essential Injectivity.* If F is pseudomonic, then F is essentially injective.

Proof. *Item 1*, *Characterisations*: Omitted.
Item 2, *Conservativity*: Omitted.
Item 3, *Essential Injectivity*: Omitted. □

6.5 Pseudoepic Functors

Let C and $\mathcal{D}$ be categories.

Definition 6.5.1.1. A functor $F \colon C \to \mathcal{D}$ is **pseudoepic** if it satisfies the following conditions:

1. For all diagrams of the form

$$\begin{tikzcd} C \arrow[r, "F"] & \mathcal{D} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & \mathcal{X}, \end{tikzcd}$$

if we have

$$\alpha \star \mathrm{id}_F = \beta \star \mathrm{id}_F,$$

then $\alpha = \beta$.

2. For each $X \in \mathrm{Obj}(C)$ and each 2-isomorphism

$$\beta \colon \phi \circ F \overset{\sim}{\Longrightarrow} \psi \circ F, \qquad \begin{tikzcd} C \arrow[r, bend left, "\phi\circ F"] \arrow[r, bend right, "\psi\circ F"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{X} \end{tikzcd}$$

[26] Asking the precomposition functors

$$F^* \colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(C, \mathcal{X})$$

to be pseudomonic leads to pseudoepic functors; see Item 1b of Item 1 of Proposition 6.5.1.2.

of $\mathcal{C}$, there exists a 2-isomorphism

$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{X} \end{tikzcd}$$

of $\mathcal{C}$ such that we have an equality

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] & \mathcal{D} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{X} \end{tikzcd} = \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "\phi\circ F"] \arrow[r, bend right, "\psi\circ F"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{X} \end{tikzcd}$$

of pasting diagrams in $\mathcal{C}$, i.e. such that we have

$$\beta = \alpha \star \mathrm{id}_F.$$

Proposition 6.5.1.2. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor.

1. *Characterisations.* The following conditions are equivalent:

   (a) The functor F is pseudoepic.

   (b) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the functor

   $$F^*\colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

   given by precomposition by F is pseudomonic.

   (c) We have an isococomma square of the form

   $$\mathcal{D} \overset{\text{eq.}}{\cong} \mathcal{D} \overset{\leftrightarrow}{\coprod}_{\mathcal{C}} \mathcal{D}, \qquad \begin{tikzcd} \mathcal{D} & \mathcal{D} \arrow[l, "\mathrm{id}_{\mathcal{D}}"'] \\ \mathcal{D} \arrow[u, "\mathrm{id}_{\mathcal{D}}"] \arrow[ur, phantom, "\Leftrightarrow"] & \mathcal{C} \arrow[l, "F"] \arrow[u, "F"'] \end{tikzcd}$$

   in Cats_2 up to equivalence.

2. *Dominance.* If F is pseudoepic, then F is dominant (Definition 6.1.1.1).

Proof. *Item 1, Characterisations*: Omitted.

Item 2, *Dominance*: If F is pseudoepic, then

$$F^*\colon \mathsf{Fun}(\mathcal{D},\mathcal{X}) \to \mathsf{Fun}(\mathcal{C},\mathcal{X})$$

is pseudomonic for all $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, and thus in particular faithful. By Item 4g of Item 4 of Proposition 5.1.1.2, this is equivalent to requiring F to be dominant.

□

Question 6.5.1.3. Is there a nice characterisation of the pseudoepic functors, similarly to the characterisaiton of pseudomonic functors given in Item 1b of Item 1 of Proposition 6.4.1.2?
This question also appears as [MO 321971].

Question 6.5.1.4. A pseudomonic and pseudoepic functor is dominant, faithful, essentially injective, and full on isomorphisms. Is it necessarily an equivalence of categories? If not, how bad can this fail, i.e. how far can a pseudomonic and pseudoepic functor be from an equivalence of categories?
This question also appears as [MO 468334].

Question 6.5.1.5. Is there a characterisation of functors $F\colon \mathcal{C} \to \mathcal{D}$ such that:

1. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor

$$F^*\colon \mathsf{Fun}(\mathcal{D},\mathcal{X}) \to \mathsf{Fun}(\mathcal{C},\mathcal{X})$$

   is pseudoepic?

2. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor

$$F_*\colon \mathsf{Fun}(\mathcal{X},\mathcal{C}) \to \mathsf{Fun}(\mathcal{X},\mathcal{D})$$

   is pseudoepic?

This question also appears as [MO 468121a].

7 Even More Conditions on Functors

7.1 Injective on Objects Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 7.1.1.1. A functor $F: \mathcal{C} \to \mathcal{D}$ is **injective on objects** if the action on objects

$$F: \mathrm{Obj}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{D})$$

of F is injective.

Proposition 7.1.1.2. Let $F: \mathcal{C} \to \mathcal{D}$ be a functor.

1. *Characterisations.* The following conditions are equivalent:
   (a) The functor F is injective on objects.
   (b) The functor F is an isocofibration in Cats_2.

Proof. Item 1, Characterisations: Omitted. ☐

7.2 Surjective on Objects Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 7.2.1.1. A functor $F: \mathcal{C} \to \mathcal{D}$ is **surjective on objects** if the action on objects

$$F: \mathrm{Obj}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{D})$$

of F is surjective.

7.3 Bijective on Objects Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 7.3.1.1. A functor $F: \mathcal{C} \to \mathcal{D}$ is **bijective on objects**[27] if the action on objects

$$F: \mathrm{Obj}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{D})$$

of F is a bijection.

7.4 Functors Representably Faithful on Cores

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 7.4.1.1. A functor $F: \mathcal{C} \to \mathcal{D}$ is **representably faithful on cores** if,

[27] *Further Terminology:* Also called a **bo** functor.

for each $X \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition by F functor

$$F_*\colon \mathsf{Core}(\mathsf{Fun}(\mathcal{X}, \mathcal{C})) \to \mathsf{Core}(\mathsf{Fun}(\mathcal{X}, \mathcal{D}))$$

is faithful.

Remark 7.4.1.2. In detail, a functor $F\colon \mathcal{C} \to \mathcal{D}$ is **representably faithful on cores** if, given a diagram of the form

$$\begin{tikzcd} \mathcal{X} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & \mathcal{C} \arrow[r, "F"] & \mathcal{D}, \end{tikzcd}$$

if α and β are natural isomorphisms and we have

$$\mathrm{id}_F \star \alpha = \mathrm{id}_F \star \beta,$$

then $\alpha = \beta$.

Question 7.4.1.3. Is there a characterisation of functors representably faithful on cores?

7.5 Functors Representably Full on Cores

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 7.5.1.1. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **representably full on cores** if, for each $X \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition by F functor

$$F_*\colon \mathsf{Core}(\mathsf{Fun}(\mathcal{X}, \mathcal{C})) \to \mathsf{Core}(\mathsf{Fun}(\mathcal{X}, \mathcal{D}))$$

is full.

Remark 7.5.1.2. In detail, a functor $F\colon \mathcal{C} \to \mathcal{D}$ is **representably full on cores** if, for each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$ and each natural isomorphism

$$\beta\colon F \circ \phi \overset{\sim}{\Longrightarrow} F \circ \psi, \qquad \begin{tikzcd} \mathcal{X} \arrow[r, bend left, "F\circ\phi"] \arrow[r, bend right, "F\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{D}, \end{tikzcd}$$

there exists a natural isomorphism

$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} \mathcal{X} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{C} \end{tikzcd}$$

such that we have an equality

$$\begin{tikzcd} \mathcal{X} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{C} \arrow[r, "F"] & \mathcal{D} \end{tikzcd} = \begin{tikzcd} \mathcal{X} \arrow[r, bend left, "F\circ\phi"] \arrow[r, bend right, "F\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{D} \end{tikzcd}$$

of pasting diagrams in Cats_2, i.e. such that we have

$$\beta = \mathrm{id}_F \star \alpha.$$

Question 7.5.1.3. Is there a characterisation of functors representably full on cores?
This question also appears as [MO 468121a].

7.6 Functors Representably Fully Faithful on Cores

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 7.6.1.1. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **representably fully faithful on cores** if, for each $X \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition by F functor

$$F_*\colon \mathsf{Core}(\mathsf{Fun}(\mathcal{X}, \mathcal{C})) \to \mathsf{Core}(\mathsf{Fun}(\mathcal{X}, \mathcal{D}))$$

is fully faithful.

Remark 7.6.1.2. In detail, a functor $F\colon \mathcal{C} \to \mathcal{D}$ is **representably fully faithful on cores** if it satisfies the conditions in Remarks 7.4.1.2 and 7.5.1.2, i.e.:

1. For all diagrams of the form

$$\begin{tikzcd} \mathcal{X} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & \mathcal{C} \arrow[r, "F"] & \mathcal{D}, \end{tikzcd}$$

   with α and β natural isomorphisms, if we have $\mathrm{id}_F \star \alpha = \mathrm{id}_F \star \beta$, then $\alpha = \beta$.

2. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$ and each natural isomorphism

$$\beta \colon F \circ \phi \overset{\sim}{\Longrightarrow} F \circ \psi, \qquad \begin{tikzcd} \mathcal{X} \arrow[r, bend left, "F\circ\phi"] \arrow[r, bend right, "F\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{D} \end{tikzcd}$$

of $\mathcal{C}$, there exists a natural isomorphism

$$\alpha \colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} \mathcal{X} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{C} \end{tikzcd}$$

of $\mathcal{C}$ such that we have an equality

$$\begin{tikzcd} \mathcal{X} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{C} \arrow[r, "F"] & \mathcal{D} \end{tikzcd} = \begin{tikzcd} \mathcal{X} \arrow[r, bend left, "F\circ\phi"] \arrow[r, bend right, "F\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{D} \end{tikzcd}$$

of pasting diagrams in Cats_2, i.e. such that we have

$$\beta = \mathrm{id}_F \star \alpha.$$

Question 7.6.1.3. Is there a characterisation of functors representably fully faithful on cores?

7.7 Functors Corepresentably Faithful on Cores

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 7.7.1.1. A functor $F \colon \mathcal{C} \to \mathcal{D}$ is **corepresentably faithful on cores** if, for each $X \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition by F functor

$$F_* \colon \mathsf{Core}(\mathsf{Fun}(\mathcal{X}, \mathcal{C})) \to \mathsf{Core}(\mathsf{Fun}(\mathcal{X}, \mathcal{D}))$$

is faithful.

Remark 7.7.1.2. In detail, a functor $F \colon \mathcal{C} \to \mathcal{D}$ is **corepresentably faithful on cores** if, given a diagram of the form

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] & \mathcal{D} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & \mathcal{X}, \end{tikzcd}$$

if α and β are natural isomorphisms and we have

$$\alpha \star \mathrm{id}_F = \beta \star \mathrm{id}_F,$$

then $\alpha = \beta$.

Question 7.7.1.3. Is there a characterisation of functors corepresentably faithful on cores?

7.8 Functors Corepresentably Full on Cores

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 7.8.1.1. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **corepresentably full on cores** if, for each $X \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition by F functor

$$F_*\colon \mathsf{Core}(\mathsf{Fun}(\mathcal{X}, \mathcal{C})) \to \mathsf{Core}(\mathsf{Fun}(\mathcal{X}, \mathcal{D}))$$

is full.

Remark 7.8.1.2. In detail, a functor $F\colon \mathcal{C} \to \mathcal{D}$ is **corepresentably full on cores** if, for each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$ and each natural isomorphism

$$\beta\colon \phi \circ F \overset{\sim}{\Longrightarrow} \psi \circ F, \qquad \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "\phi\circ F"] \arrow[r, bend right, "\psi\circ F"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{X}, \end{tikzcd}$$

there exists a natural isomorphism

$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{X} \end{tikzcd}$$

such that we have an equality

$$\begin{tikzcd} \mathcal{X} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{C} \arrow[r, "F"] & \mathcal{D} \end{tikzcd} = \begin{tikzcd} \mathcal{X} \arrow[r, bend left, "F\circ\phi"] \arrow[r, bend right, "F\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{D} \end{tikzcd}$$

of pasting diagrams in Cats_2, i.e. such that we have

$$\beta = \alpha \star \mathrm{id}_F.$$

Question 7.8.1.3. Is there a characterisation of functors corepresentably full on cores?
This question also appears as [MO 468121a].

7.9 Functors Corepresentably Fully Faithful on Cores

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 7.9.1.1. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **corepresentably fully faithful on cores** if, for each $X \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition by F functor

$$F_*\colon \mathsf{Core}(\mathsf{Fun}(\mathcal{X}, \mathcal{C})) \to \mathsf{Core}(\mathsf{Fun}(\mathcal{X}, \mathcal{D}))$$

is fully faithful.

Remark 7.9.1.2. In detail, a functor $F\colon \mathcal{C} \to \mathcal{D}$ is **corepresentably fully faithful on cores** if it satisfies the conditions in Remarks 7.7.1.2 and 7.8.1.2, i.e.:

1. For all diagrams of the form

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] & \mathcal{D} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & \mathcal{X}, \end{tikzcd}$$

   if α and β are natural isomorphisms and we have

$$\alpha \star \mathrm{id}_F = \beta \star \mathrm{id}_F,$$

   then $\alpha = \beta$.

2. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$ and each natural isomorphism

$$\beta\colon \phi \circ F \overset{\sim}{\Longrightarrow} \psi \circ F, \qquad \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "\phi\circ F"] \arrow[r, bend right, "\psi\circ F"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{X}, \end{tikzcd}$$

   there exists a natural isomorphism

$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{X} \end{tikzcd}$$

   such that we have an equality

$$\begin{tikzcd} \mathcal{X} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{C} \arrow[r, "F"] & \mathcal{D} \end{tikzcd} = \begin{tikzcd} \mathcal{X} \arrow[r, bend left, "F\circ\phi"] \arrow[r, bend right, "F\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{D} \end{tikzcd}$$

   of pasting diagrams in Cats_2, i.e. such that we have

$$\beta = \alpha \star \mathrm{id}_F.$$

Question 7.9.1.3. Is there a characterisation of functors corepresentably fully faithful on cores?

8 Natural Transformations

8.1 Transformations

Let $\mathcal{C}$ and $\mathcal{D}$ be categories and $F, G \colon \mathcal{C} \rightrightarrows \mathcal{D}$ be functors.

Definition 8.1.1.1. A **transformation**[28] $\alpha \colon F \Rightarrow G$ **from** F **to** G is a collection

$$\{\alpha_A \colon F(A) \to G(A)\}_{A \in \mathrm{Obj}(\mathcal{C})}$$

of morphisms of $\mathcal{D}$.

Notation 8.1.1.2. We write $\mathrm{Trans}(F, G)$ for the set of transformations from F to G.

8.2 Natural Transformations

Let $\mathcal{C}$ and $\mathcal{D}$ be categories and $F, G \colon \mathcal{C} \rightrightarrows \mathcal{D}$ be functors.

Definition 8.2.1.1. A **natural transformation** $\alpha \colon F \Longrightarrow G$ **from** F **to** G is a transformation

$$\{\alpha_A \colon F(A) \to G(A)\}_{A \in \mathrm{Obj}(\mathcal{C})}$$

from F to G such that, for each morphism $f \colon A \to B$ of $\mathcal{C}$, the diagram

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\alpha_A"'] & F(B) \arrow[d, "\alpha_B"] \\ G(A) \arrow[r, "G(f)"'] & G(B) \end{tikzcd}$$

commutes.[29]

Remark 8.2.1.2. We denote natural transformations in diagrams as

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{D}. \end{tikzcd}$$

[28] *Further Terminology:* Also called an **unnatural transformation** for emphasis.
[29] *Further Terminology:* The morphism $\alpha_A \colon F_A \to G_A$ is called the **component of** α **at** A.

Notation 8.2.1.3. We write $\mathrm{Nat}(F, G)$ for the set of natural transformations from F to G.

Example 8.2.1.4. The **identity natural transformation** $\mathrm{id}_F : F \Longrightarrow F$ **of** F is the natural transformation consisting of the collection

$$\left\{\mathrm{id}_{F(A)} : F(A) \to F(A)\right\}_{A \in \mathrm{Obj}(C)}.$$

Proof. The naturality condition for id_F is the requirement that, for each morphism $f : A \to B$ of C, the diagram

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\mathrm{id}_{F(A)}"'] & F(B) \arrow[d, "\mathrm{id}_{F(B)}"] \\ F(A) \arrow[r, "F(f)"'] & F(B) \end{tikzcd}$$

commutes, which follows from unitality of the composition of C. □

Definition 8.2.1.5. Two natural transformations $\alpha, \beta : F \Longrightarrow G$ are **equal** if we have

$$\alpha_A = \beta_A$$

for each $A \in \mathrm{Obj}(C)$.

8.3 Vertical Composition of Natural Transformations

Definition 8.3.1.1. The **vertical composition** of two natural transformations $\alpha : F \Longrightarrow G$ and $\beta : G \Longrightarrow H$ as in the diagram

$$\begin{tikzcd} C \arrow[r, bend left=60, "F"{name=F}] \arrow[r, "G"{name=G, description}] \arrow[r, bend right=60, "H"'{name=H}] & \mathcal{D} \arrow[from=F, to=G, Rightarrow, "\alpha"'] \arrow[from=G, to=H, Rightarrow, "\beta"'] \end{tikzcd}$$

is the natural transformation $\beta \circ \alpha : F \Longrightarrow H$ consisting of the collection

$$\{(\beta \circ \alpha)_A : F(A) \to H(A)\}_{A \in \mathrm{Obj}(C)}$$

with

$$(\beta \circ \alpha)_A \stackrel{\mathrm{def}}{=} \beta_A \circ \alpha_A$$

for each $A \in \mathrm{Obj}(C)$.

Proof. The naturality condition for $\beta \circ \alpha$ is the requirement that the boundary of the diagram

$$\begin{tikzcd}
F(A) \arrow[r, "F(f)"] \arrow[d, "\alpha_A"'] \arrow[dr, phantom, "(1)"] & F(B) \arrow[d, "\alpha_B"] \\
G(A) \arrow[r, "G(f)" description] \arrow[d, "\beta_A"'] \arrow[dr, phantom, "(2)"] & G(B) \arrow[d, "\beta_B"] \\
H(A) \arrow[r, "H(f)"'] & H(B)
\end{tikzcd}$$

commutes. Since

1. Subdiagram (1) commutes by the naturality of α.

2. Subdiagram (2) commutes by the naturality of β.

so does the boundary diagram. Hence $\beta \circ \alpha$ is a natural transformation. $\square$

Proposition 8.3.1.2. Let $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ be categories.

1. *Functionality.* The assignment $(\beta, \alpha) \mapsto \beta \circ \alpha$ defines a function

$$\circ_{F,G,H} \colon \mathrm{Nat}(G,H) \times \mathrm{Nat}(F,G) \to \mathrm{Nat}(F,H).$$

2. *Associativity.* Let $F, G, H, K \colon \mathcal{C} \overset{\rightrightarrows}{\rightrightarrows} \mathcal{D}$ be functors. The diagram

$$\begin{tikzcd}
 & \mathrm{Nat}(H,K) \times (\mathrm{Nat}(G,H) \times \mathrm{Nat}(F,G)) \arrow[dr, "\mathrm{id}_{\mathrm{Nat}(H,K)} \times \circ_{F,G,H}"] & \\
(\mathrm{Nat}(H,K) \times \mathrm{Nat}(G,H)) \times \mathrm{Nat}(F,G) \arrow[ur, dashed, "\alpha^{\mathsf{Sets}}_{\mathrm{Nat}(H,K),\mathrm{Nat}(G,H),\mathrm{Nat}(F,G)}", "\sim"'] \arrow[d, "\circ_{G,H,K} \times \mathrm{id}_{\mathrm{Nat}(F,G)}"'] & & \mathrm{Nat}(H,K) \times \mathrm{Nat}(F,H) \arrow[d, "\circ_{F,H,K}"] \\
\mathrm{Nat}(G,K) \times \mathrm{Nat}(F,G) \arrow[rr, "\circ_{F,G,K}"'] & & \mathrm{Nat}(F,K)
\end{tikzcd}$$

commutes, i.e. given natural transformations

$$F \overset{\alpha}{\Longrightarrow} G \overset{\beta}{\Longrightarrow} H \overset{\gamma}{\Longrightarrow} K,$$

we have

$$(\gamma \circ \beta) \circ \alpha = \gamma \circ (\beta \circ \alpha).$$

3. *Unitality.* Let $F, G \colon \mathcal{C} \rightrightarrows \mathcal{D}$ be functors.

   (a) *Left Unitality.* The diagram

$$\begin{tikzcd}
\mathrm{pt} \times \mathrm{Nat}(F,G) \arrow[d, "{[\mathrm{id}_G] \times \mathrm{id}_{\mathrm{Nat}(F,G)}}"'] \arrow[dr, dashed, "\lambda^{\mathsf{Sets}}_{\mathrm{Nat}(F,G)}", "\sim"'] & \\
\mathrm{Nat}(G,G) \times \mathrm{Nat}(F,G) \arrow[r, "\circ_{F,G,G}"'] & \mathrm{Nat}(F,G)
\end{tikzcd}$$

   commutes, i.e. given a natural transformation $\alpha \colon F \Longrightarrow G$, we have

$$\mathrm{id}_G \circ \alpha = \alpha.$$

   (b) *Right Unitality.* The diagram

$$\begin{tikzcd}
\mathrm{Nat}(F,G) \times \mathrm{pt} \arrow[d, "{\mathrm{id}_{\mathrm{Nat}(F,G)} \times [\mathrm{id}_F]}"'] \arrow[dr, dashed, "\rho^{\mathsf{Sets}}_{\mathrm{Nat}(F,G)}", "\sim"'] & \\
\mathrm{Nat}(F,G) \times \mathrm{Nat}(F,F) \arrow[r, "\circ^{\mathcal{C}}_{F,F,G}"'] & \mathrm{Nat}(F,G)
\end{tikzcd}$$

   commutes, i.e. given a natural transformation $\alpha \colon F \Longrightarrow G$, we have

$$\alpha \circ \mathrm{id}_F = \alpha.$$

4. *Middle Four Exchange.* Let $F_1, F_2, F_3 \colon \mathcal{C} \to \mathcal{D}$ and $G_1, G_2, G_3 \colon \mathcal{D} \to \mathcal{E}$ be functors. The diagram

$$\begin{tikzcd}
(\mathrm{Nat}(G_2,G_3) \times \mathrm{Nat}(G_1,G_2)) \times (\mathrm{Nat}(F_2,F_3) \times \mathrm{Nat}(F_1,F_2)) \arrow[d, "\circ_{G_1,G_2,G_3} \times \circ_{F_1,F_2,F_3}"'] & & (\mathrm{Nat}(G_2,G_3) \times \mathrm{Nat}(F_2,F_3)) \times (\mathrm{Nat}(G_1,G_2) \times \mathrm{Nat}(F_1,F_2)) \arrow[ll, dashed, "\mu_4", "\sim"'] \arrow[d, "\star_{F_2,F_3,G_2,G_3} \times \star_{F_1,F_2,G_1,G_2}"] \\
\mathrm{Nat}(G_1,G_3) \times \mathrm{Nat}(F_1,F_3) \arrow[dr, "\star_{F_1,F_3,G_1,G_3}"'] & & \mathrm{Nat}(G_2 \circ F_2, G_3 \circ F_3) \times \mathrm{Nat}(G_1 \circ F_1, G_2 \circ F_2) \arrow[dl, "\circ_{G_1 \circ F_1, G_2 \circ F_2, G_3 \circ F_3}"] \\
& \mathrm{Nat}(G_1 \circ F_1, G_3 \circ F_3) &
\end{tikzcd}$$

commutes, i.e. given a diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=60, "F_1"] \arrow[r, "F_2" description] \arrow[r, bend right=60, "F_3"'] & \mathcal{D} \arrow[r, bend left=60, "G_1"] \arrow[r, "G_2" description] \arrow[r, bend right=60, "G_3"'] & \mathcal{E} \end{tikzcd}$$

with $\alpha\Downarrow$ between F_1 and F_2, $\alpha'\Downarrow$ between F_2 and F_3, $\beta\Downarrow$ between G_1 and G_2, $\beta'\Downarrow$ between G_2 and G_3,

in Cats_2, we have

$$(\beta' \star \alpha') \circ (\beta \star \alpha) = (\beta' \circ \beta) \star (\alpha' \circ \alpha).$$

Proof. *Item 1*, *Functionality*: Clear.
Item 2, *Associativity*: Indeed, we have

$$\begin{aligned} ((\gamma \circ \beta) \circ \alpha)_A &\overset{\text{def}}{=} (\gamma \circ \beta)_A \circ \alpha_A \\ &\overset{\text{def}}{=} (\gamma_A \circ \beta_A) \circ \alpha_A \\ &= \gamma_A \circ (\beta_A \circ \alpha_A) \\ &\overset{\text{def}}{=} \gamma_A \circ (\beta \circ \alpha)_A \\ &\overset{\text{def}}{=} (\gamma \circ (\beta \circ \alpha))_A \end{aligned}$$

for each $A \in \text{Obj}(\mathcal{C})$, showing the desired equality.
Item 3, *Unitality*: We have

$$\begin{aligned} (\text{id}_G \circ \alpha)_A &= \text{id}_G \circ \alpha_A \\ &= \alpha_A, \\ (\alpha \circ \text{id}_F)_A &= \alpha_A \circ \text{id}_F \\ &= \alpha_A \end{aligned}$$

for each $A \in \text{Obj}(\mathcal{C})$, showing the desired equality.
Item 4, *Middle Four Exchange*: This is proved in Item 4 of Proposition 8.4.1.3. □

8.4 Horizontal Composition of Natural Transformations

Definition 8.4.1.1. The **horizontal composition**[30,31] of two natural transformations $\alpha\colon F \Longrightarrow G$ and $\beta\colon H \Longrightarrow K$ as in the diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{D} \arrow[r, bend left, "H"] \arrow[r, bend right, "K"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{E} \end{tikzcd}$$

of α and β is the natural transformation

$$\beta \star \alpha\colon (H \circ F) \Longrightarrow (K \circ G),$$

as in the diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "H\circ F"] \arrow[r, bend right, "K\circ G"'] \arrow[r, phantom, "\beta\star\alpha\Downarrow"] & \mathcal{E}, \end{tikzcd}$$

consisting of the collection

$$\{(\beta \star \alpha)_A\colon H(F(A)) \to K(G(A))\}_{A\in\mathrm{Obj}(\mathcal{C})},$$

of morphisms of $\mathcal{E}$ with

$$\begin{aligned}(\beta \star \alpha)_A &\overset{\text{def}}{=} \beta_{G(A)} \circ H(\alpha_A)\\ &= K(\alpha_A) \circ \beta_{F(A)},\end{aligned} \qquad \begin{tikzcd} H(F(A)) \arrow[r, "H(\alpha_A)"] \arrow[d, "\beta_{F(A)}"'] & H(G(A)) \arrow[d, "\beta_{G(A)}"] \\ K(F(A)) \arrow[r, "K(\alpha_A)"'] & K(G(A)). \end{tikzcd}$$

Proof. First, we claim that we indeed have

$$\beta_{G(A)} \circ H(\alpha_A) = K(\alpha_A) \circ \beta_{F(A)}, \qquad \begin{tikzcd} H(F(A)) \arrow[r, "H(\alpha_A)"] \arrow[d, "\beta_{F(A)}"'] & H(G(A)) \arrow[d, "\beta_{G(A)}"] \\ K(F(A)) \arrow[r, "K(\alpha_A)"'] & K(G(A)). \end{tikzcd}$$

[30] *Further Terminology:* Also called the **Godement product** of α and β.

[31] Horizontal composition forms a map

$$\star_{(F,H),(G,K)}\colon \mathrm{Nat}(H,K) \times \mathrm{Nat}(F,G) \to \mathrm{Nat}(H \circ F, K \circ G).$$

This is, however, simply the naturality square for β applied to the morphism $\alpha_A \colon F(A) \to G(A)$. Next, we check the naturality condition for $\beta \star \alpha$, which is the requirement that the boundary of the diagram

$$\begin{tikzcd}
H(F(A)) \arrow[r, "H(F(f))"] \arrow[d, "H(\alpha_A)"'] \arrow[dr, phantom, "(1)"] & H(F(B)) \arrow[d, "H(\alpha_B)"] \\
H(G(A)) \arrow[r, "H(G(f))" description] \arrow[d, "\beta_{G(A)}"'] \arrow[dr, phantom, "(2)"] & H(G(B)) \arrow[d, "\beta_{G(B)}"] \\
K(G(A)) \arrow[r, "K(G(f))"'] & K(G(B))
\end{tikzcd}$$

commutes. Since

1. Subdiagram (1) commutes by the naturality of α.

2. Subdiagram (2) commutes by the naturality of β.

so does the boundary diagram. Hence $\beta \circ \alpha$ is a natural transformation.[32] □

Definition 8.4.1.2. Let

$$\begin{tikzcd}
\mathcal{X} \arrow[r, "F"] & \mathcal{C} \arrow[r, bend left, "\phi"{name=U}] \arrow[r, bend right, "\psi"'{name=D}] & \mathcal{D} \arrow[r, "G"] & \mathcal{Y}
\arrow[from=U, to=D, Rightarrow, "\alpha"']
\end{tikzcd}$$

be a diagram in Cats_2.

1. The **left whiskering of** α **with** G is the natural transformation[33]

$$\mathrm{id}_G \star \alpha \colon G \circ \phi \Longrightarrow G \circ \psi.$$

---

[32] *Reference:* [Bor94, Proposition 1.3.4].

[33] *Further Notation:* Also written $G\alpha$ or $G \star \alpha$, although we won't use either of these notations

2. The **right whiskering of** α **with** F is the natural transformation[34]

$$\alpha \star \mathrm{id}_F \colon \phi \circ F \Longrightarrow \psi \circ F.$$

Proposition 8.4.1.3. Let $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ be categories.

1. *Functionality.* The assignment $(\beta, \alpha) \mapsto \beta \star \alpha$ defines a function

$$\star_{(F,G),(H,K)} \colon \mathrm{Nat}(H, K) \times \mathrm{Nat}(F, G) \to \mathrm{Nat}(H \circ F, K \circ G).$$

2. *Associativity.* Let

$$\mathcal{C} \underset{G_1}{\overset{F_1}{\rightrightarrows}} \mathcal{D} \underset{G_2}{\overset{F_2}{\rightrightarrows}} \mathcal{E} \underset{G_3}{\overset{F_3}{\rightrightarrows}} \mathcal{F}$$

be a diagram in Cats_2. The diagram

$$\begin{tikzcd}
\mathrm{Nat}(F_3, G_3) \times \mathrm{Nat}(F_2, G_2) \times \mathrm{Nat}(F_1, G_1) \arrow[r, "\star_{(F_2,G_2),(F_3,G_3)} \times \mathsf{id}"] \arrow[d, "\mathsf{id} \times \star_{(F_1,G_1),(F_2,G_2)}"'] & \mathrm{Nat}(F_3 \circ F_2, G_3 \circ G_2) \times \mathrm{Nat}(F_1, G_1) \arrow[d, "\star_{(F_3 \circ F_2),(G_3 \circ G_2, F_1, G_1)}"] \\
\mathrm{Nat}(F_3, G_3) \times \mathrm{Nat}(F_2 \circ F_1, G_2 \circ G_1) \arrow[r, "\star_{(F_2 \circ F_1),(G_2 \circ G_1, F_3, G_3)}"'] & \mathrm{Nat}(F_3 \circ F_2 \circ F_1, G_3 \circ G_2 \circ G_1)
\end{tikzcd}$$

commutes, i.e. given natural transformations

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "F_1"] \arrow[r, bend right, "G_1"'] \arrow[r, phantom, "\alpha \Downarrow"] & \mathcal{D} \arrow[r, bend left, "F_2"] \arrow[r, bend right, "G_2"'] \arrow[r, phantom, "\beta \Downarrow"] & \mathcal{E} \arrow[r, bend left, "F_3"] \arrow[r, bend right, "G_3"'] \arrow[r, phantom, "\gamma \Downarrow"] & \mathcal{F},
\end{tikzcd}$$

we have

$$(\gamma \star \beta) \star \alpha = \gamma \star (\beta \star \alpha).$$

3. *Interaction With Identities.* Let $F \colon \mathcal{C} \to \mathcal{D}$ and $G \colon \mathcal{D} \to \mathcal{E}$ be functors. The diagram

$$\begin{tikzcd}
\mathrm{pt} \times \mathrm{pt} \arrow[r, "{[\mathrm{id}_G] \times [\mathrm{id}_F]}"] & \mathrm{Nat}(G, G) \times \mathrm{Nat}(F, F) \arrow[d, "\star_{(F,F),(G,G)}"] \\
\mathrm{pt} \arrow[u, dashed, "\wr"] \arrow[r, "{[\mathrm{id}_{G \circ F}]}"'] & \mathrm{Nat}(G \circ F, G \circ F)
\end{tikzcd}$$

in this work.

[34] *Further Notation:* Also written αF or $\alpha \star F$, although we won't use either of these notations in

commutes, i.e. we have

$$\mathrm{id}_G \star \mathrm{id}_F = \mathrm{id}_{G \circ F}.$$

4. *Middle Four Exchange.* Let $F_1, F_2, F_3 \colon \mathcal{C} \to \mathcal{D}$ and $G_1, G_2, G_3 \colon \mathcal{D} \to \mathcal{E}$ be functors. The diagram

$$\begin{tikzcd}
(\mathrm{Nat}(G_2,G_3) \times \mathrm{Nat}(G_1,G_2)) \times (\mathrm{Nat}(F_2,F_3) \times \mathrm{Nat}(F_1,F_2)) \arrow[rr, dashed, leftrightarrow, "\mu_4", "\sim"'] \arrow[d, "\circ_{G_1,G_2,G_3} \times \circ_{F_1,F_2,F_3}"'] & & (\mathrm{Nat}(G_2,G_3) \times \mathrm{Nat}(F_2,F_3)) \times (\mathrm{Nat}(G_1,G_2) \times \mathrm{Nat}(F_1,F_2)) \arrow[d, "\star_{F_2,F_3,G_2,G_3} \times \star_{F_1,F_2,G_1,G_2}"] \\
\mathrm{Nat}(G_1,G_3) \times \mathrm{Nat}(F_1,F_3) \arrow[dr, "\star_{F_1,F_3,G_1,G_3}"'] & & \mathrm{Nat}(G_2 \circ F_2, G_3 \circ F_3) \times \mathrm{Nat}(G_1 \circ F_1, G_2 \circ F_2) \arrow[dl, "\circ_{G_1 \circ F_1, G_2 \circ F_2, G_3 \circ F_3}"] \\
& \mathrm{Nat}(G_1 \circ F_1, G_3 \circ F_3) &
\end{tikzcd}$$

commutes, i.e. given a diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=60, "F_1"{name=U}] \arrow[r, "F_2"{name=M, description}] \arrow[r, bend right=60, "F_3"'{name=L}] & \mathcal{D} \arrow[r, bend left=60, "G_1"{name=U2}] \arrow[r, "G_2"{name=M2, description}] \arrow[r, bend right=60, "G_3"'{name=L2}] & \mathcal{E}
\arrow[from=U, to=M, Rightarrow, "\alpha"']
\arrow[from=M, to=L, Rightarrow, "\alpha'"']
\arrow[from=U2, to=M2, Rightarrow, "\beta"']
\arrow[from=M2, to=L2, Rightarrow, "\beta'"']
\end{tikzcd}$$

in Cats_2, we have

$$(\beta' \star \alpha') \circ (\beta \star \alpha) = (\beta' \circ \beta) \star (\alpha' \circ \alpha).$$

Proof. *Item 1, Functionality*: Clear.
Item 2, Associativity: Omitted.
Item 3, Interaction With Identities: We have

$$\begin{aligned}
(\mathrm{id}_G \star \mathrm{id}_F)_A &\stackrel{\text{def}}{=} (\mathrm{id}_G)_{F_A} \circ G_{(\mathrm{id}_F)_A} \\
&\stackrel{\text{def}}{=} \mathrm{id}_{G_{F_A}} \circ G_{\mathrm{id}_{F_A}} \\
&= \mathrm{id}_{G_{F_A}} \circ \mathrm{id}_{G_{F_A}} \\
&= \mathrm{id}_{G_{F_A}} \\
&\stackrel{\text{def}}{=} (\mathrm{id}_{G \circ F})_A
\end{aligned}$$

for each $A \in \mathrm{Obj}(\mathcal{C})$, showing the desired equality.
Item 4, *Middle Four Exchange*: Let $A \in \mathrm{Obj}(\mathcal{C})$ and consider the diagram

$$\begin{tikzcd}
& & G_1(F_3(A)) \arrow[dr, "\beta_{F_3(A)}"] & & \\
G_1(F_1(A)) \arrow[r, "G_1(\alpha_A)"] & G_1(F_2(A)) \arrow[ur, "G_1(\alpha'_A)"] \arrow[dr, "\beta_{F_2(A)}"'] & (1) & G_2(F_3(A)) \arrow[r, "\beta'_{F_3(A)}"] & G_3(F_3(A)). \\
& & G_2(F_2(A)) \arrow[ur, "G_2(\alpha'_A)"'] & &
\end{tikzcd}$$

The top composition

$$\begin{tikzcd}
& & G_1(F_3(A)) \arrow[dr, "\beta_{F_3(A)}"] & & \\
G_1(F_1(A)) \arrow[r, "G_1(\alpha_A)"] & G_1(F_2(A)) \arrow[ur, "G_1(\alpha'_A)"] \arrow[dr, "\beta_{F_2(A)}"'] & (1) & G_2(F_3(A)) \arrow[r, "\beta'_{F_3(A)}"] & G_3(F_3(A)). \\
& & G_2(F_2(A)) \arrow[ur, "G_2(\alpha'_A)"'] & &
\end{tikzcd}$$

is given by $((\beta' \circ \beta) \star (\alpha' \circ \alpha))_A$, while the bottom composition

$$\begin{tikzcd}
& & G_1(F_3(A)) \arrow[dr, "\beta_{F_3(A)}"] & & \\
G_1(F_1(A)) \arrow[r, "G_1(\alpha_A)"] & G_1(F_2(A)) \arrow[ur, "G_1(\alpha'_A)"] \arrow[dr, "\beta_{F_2(A)}"'] & (1) & G_2(F_3(A)) \arrow[r, "\beta'_{F_3(A)}"] & G_3(F_3(A)). \\
& & G_2(F_2(A)) \arrow[ur, "G_2(\alpha'_A)"'] & &
\end{tikzcd}$$

this work.

is given by $((\beta' \star \alpha') \circ (\beta \star \alpha))_A$. Now, Subdiagram (1) corresponds to the naturality condition

$$G_2(\alpha'_A) \circ \beta_{F_2(A)} = \beta_{F_3}(A) \circ G_1(\alpha'_A), \qquad \begin{tikzcd} G_1(F_2(A)) \arrow[r, "G_1(\alpha'_A)"] \arrow[d, "\beta_{F_2(A)}"'] & G_1(F_3(A)) \arrow[d, "\beta_{F_3(A)}"] \\ G_2(F_2(A)) \arrow[r, "G_2(\alpha'_A)"'] & G_2(F_3(A)) \end{tikzcd}$$

for $\beta\colon G_1 \Longrightarrow G_2$ at $\alpha'_A\colon F_2(A) \to F_3(A)$, and thus commutes. Thus we have

$$((\beta' \circ \beta) \star (\alpha' \circ \alpha))_A = ((\beta' \star \alpha') \circ (\beta \star \alpha))_A$$

for each $A \in \mathrm{Obj}(\mathcal{C})$ and therefore

$$(\beta' \star \alpha') \circ (\beta \star \alpha) = (\beta' \circ \beta) \star (\alpha' \circ \alpha).$$

This finishes the proof. □

8.5 Properties of Natural Transformations

Proposition 8.5.1.1. Let $F, G\colon \mathcal{C} \rightrightarrows \mathcal{D}$ be functors. The following data are equivalent:[35]

1. A natural transformation $\alpha\colon F \Longrightarrow G$.
2. A functor $[\alpha]\colon \mathcal{C} \to \mathcal{D}^{\mathbb{1}}$ filling the diagram

$$\begin{tikzcd} & \mathcal{D} \\ \mathcal{C} \arrow[ur, "F"] \arrow[r, "{[\alpha]}"] \arrow[dr, "G"'] & \mathcal{D}^{\mathbb{1}}. \arrow[u, "\mathrm{ev}_0"'] \arrow[d, "\mathrm{ev}_1"] \\ & \mathcal{D} \end{tikzcd}$$

[35]Taken from [MO 64365].

3. A functor $[\alpha]\colon C \times \mathbb{1} \to \mathcal{D}$ filling the diagram

$$\begin{tikzcd}
C \arrow[dr, "F"] & \\
C \times \mathbb{1} \arrow[u, two heads, "\mathrm{ev}_0"] \arrow[d, two heads, "\mathrm{ev}_1"'] \arrow[r, dashed, "{[\alpha]}"] & \mathcal{D}. \\
C \arrow[ur, "G"'] &
\end{tikzcd}$$

Proof. From Item 1 to Item 2 and Back: We may identify $\mathcal{D}^{\mathbb{1}}$ with $\mathsf{Arr}(\mathcal{D})$. Given a natural transformation $\alpha\colon F \Longrightarrow G$, we have a functor

$$\begin{aligned}
[\alpha]\colon C &\longrightarrow \mathcal{D}^{\mathbb{1}} \\
A &\longmapsto \alpha_A \\
(f\colon A \to B) &\longmapsto \left(\begin{tikzcd} F_A \arrow[r, "F_f"] \arrow[d, "\alpha_A"'] & F_B \arrow[d, "\alpha_B"] \\ G_A \arrow[r, "G_f"'] & G_B \end{tikzcd} \right)
\end{aligned}$$

making the diagram in Item 2 commute. Conversely, every such functor gives rise to a natural transformation from F to G, and these constructions are inverse to each other.
From Item 2 to Item 3 and Back: This follows from Item 3 of Proposition 9.1.1.2. □

8.6 Natural Isomorphisms

Let C and $\mathcal{D}$ be categories and let $F, G\colon C \rightrightarrows \mathcal{D}$ be functors.

Definition 8.6.1.1. A natural transformation $\alpha\colon F \Longrightarrow G$ is a **natural isomorphism** if there exists a natural transformation $\alpha^{-1}\colon G \Longrightarrow F$ such that

$$\begin{aligned}
\alpha^{-1} \circ \alpha &= \mathrm{id}_F, \\
\alpha \circ \alpha^{-1} &= \mathrm{id}_G.
\end{aligned}$$

Proposition 8.6.1.2. Let $\alpha\colon F \Longrightarrow G$ be a natural transformation.

1. *Characterisations.* The following conditions are equivalent:

   (a) The natural transformation α is a natural isomorphism.

   (b) For each $A \in \mathrm{Obj}(C)$, the morphism $\alpha_A\colon F_A \to G_A$ is an isomorphism.

2. *Componentwise Inverses of Natural Transformations Assemble Into Natural Transformations.* Let $\alpha^{-1}\colon G \Longrightarrow F$ be a transformation such that, for each $A \in \mathrm{Obj}(C)$, we have

$$\begin{aligned}\alpha_A^{-1} \circ \alpha_A &= \mathrm{id}_{F(A)},\\ \alpha_A \circ \alpha_A^{-1} &= \mathrm{id}_{G(A)}.\end{aligned}$$

   Then α^{-1} is a natural transformation.

Proof. *Item 1, Characterisations*: The implication Item 1a $\Longrightarrow$ Item 1b is clear, whereas the implication Item 1b $\Longrightarrow$ Item 1a follows from Item 2.
Item 2, Componentwise Inverses of Natural Transformations Assemble Into Natural Transformations: The naturality condition for α^{-1} corresponds to the commutativity of the diagram

$$\begin{tikzcd} G(A) \arrow[r, "G(f)"] \arrow[d, "\alpha_A^{-1}"'] & G(B) \arrow[d, "\alpha_B^{-1}"] \\ F(A) \arrow[r, "F(f)"'] & F(B) \end{tikzcd}$$

for each $A, B \in \mathrm{Obj}(C)$ and each $f \in \mathrm{Hom}_C(A, B)$. Considering the diagram

$$\begin{tikzcd} G(A) \arrow[r, "G(f)"] \arrow[d, "\alpha_A^{-1}"'] \arrow[dr, phantom, "(1)"] & G(B) \arrow[d, "\alpha_B^{-1}"] \\ F(A) \arrow[r, "F(f)" description] \arrow[d, "\alpha_A"'] \arrow[dr, phantom, "(2)"] & F(B) \arrow[d, "\alpha_B"] \\ G(A) \arrow[r, "G(f)"'] & G(B), \end{tikzcd}$$

where the boundary diagram as well as Subdiagram (2) commute, we have

$$\begin{aligned} G(f) &= G(f) \circ \mathrm{id}_{G(A)} \\ &= G(f) \circ \alpha_A \circ \alpha_A^{-1} \\ &= \alpha_B \circ F(f) \circ \alpha_A^{-1}. \end{aligned}$$

Postcomposing both sides with α_B^{-1}, we get

$$\begin{aligned} \alpha_B^{-1} \circ G(f) &= \alpha_B^{-1} \circ \alpha_B \circ F(f) \circ \alpha_A^{-1} \\ &= \mathrm{id}_{F(B)} \circ F(f) \circ \alpha_A^{-1} \\ &= F(f) \circ \alpha_A^{-1}, \end{aligned}$$

which is the naturality condition we wanted to show. Thus α^{-1} is a natural transformation. □

9 Categories of Categories

9.1 Functor Categories

Let C be a category and $\mathcal{D}$ be a small category.

Definition 9.1.1.1. The **category of functors from C to $\mathcal{D}$**[36] is the category $\mathsf{Fun}(C, \mathcal{D})$[37] where

- *Objects.* The objects of $\mathsf{Fun}(C, \mathcal{D})$ are functors from C to $\mathcal{D}$.
- *Morphisms.* For each $F, G \in \mathrm{Obj}(\mathsf{Fun}(C, \mathcal{D}))$, we have

$$\mathrm{Hom}_{\mathsf{Fun}(C,\mathcal{D})}(F, G) \stackrel{\text{def}}{=} \mathrm{Nat}(F, G).$$

- *Identities.* For each $F \in \mathrm{Obj}(\mathsf{Fun}(C, \mathcal{D}))$, the unit map

$$\mathbb{1}_F^{\mathsf{Fun}(C,\mathcal{D})} : \mathrm{pt} \to \mathrm{Nat}(F, F)$$

  of $\mathsf{Fun}(C, \mathcal{D})$ at F is given by

$$\mathrm{id}_F^{\mathsf{Fun}(C,\mathcal{D})} \stackrel{\text{def}}{=} \mathrm{id}_F,$$

  where $\mathrm{id}_F : F \Longrightarrow F$ is the identity natural transformation of F of Example 8.2.1.4.

[36] *Further Terminology:* Also called the **functor category** $\mathsf{Fun}(C, \mathcal{D})$.
[37] *Further Notation:* Also written $\mathcal{D}^C$ and $[C, \mathcal{D}]$.

- *Composition.* For each $F, G, H \in \mathrm{Obj}(\mathsf{Fun}(\mathcal{C}, \mathcal{D}))$, the composition map

$$\circ^{\mathsf{Fun}(\mathcal{C},\mathcal{D})}_{F,G,H} \colon \mathrm{Nat}(G, H) \times \mathrm{Nat}(F, G) \to \mathrm{Nat}(F, H)$$

  of $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$ at (F, G, H) is given by

$$\beta \circ^{\mathsf{Fun}(\mathcal{C},\mathcal{D})}_{F,G,H} \alpha \overset{\text{def}}{=} \beta \circ \alpha,$$

  where $\beta \circ \alpha$ is the vertical composition of α and β of Item 1 of Proposition 8.3.1.2.

Proposition 9.1.1.2. Let $\mathcal{C}$ and $\mathcal{D}$ be categories and let $F \colon \mathcal{C} \to \mathcal{D}$ be a functor.

1. *Functoriality.* The assignments $\mathcal{C}, \mathcal{D}, (\mathcal{C}, \mathcal{D}) \mapsto \mathsf{Fun}(\mathcal{C}, \mathcal{D})$ define functors

$$\begin{aligned} \mathsf{Fun}(\mathcal{C}, -_2) &\colon \mathsf{Cats} \to \mathsf{Cats}, \\ \mathsf{Fun}(-_1, \mathcal{D}) &\colon \mathsf{Cats}^{\mathsf{op}} \to \mathsf{Cats}, \\ \mathsf{Fun}(-_1, -_2) &\colon \mathsf{Cats}^{\mathsf{op}} \times \mathsf{Cats} \to \mathsf{Cats}. \end{aligned}$$

2. *2-Functoriality.* The assignments $\mathcal{C}, \mathcal{D}, (\mathcal{C}, \mathcal{D}) \mapsto \mathsf{Fun}(\mathcal{C}, \mathcal{D})$ define 2-functors

$$\begin{aligned} \mathsf{Fun}(\mathcal{C}, -_2) &\colon \mathsf{Cats}_2 \to \mathsf{Cats}_2, \\ \mathsf{Fun}(-_1, \mathcal{D}) &\colon \mathsf{Cats}_2^{\mathsf{op}} \to \mathsf{Cats}_2, \\ \mathsf{Fun}(-_1, -_2) &\colon \mathsf{Cats}_2^{\mathsf{op}} \times \mathsf{Cats}_2 \to \mathsf{Cats}_2. \end{aligned}$$

3. *Adjointness.* We have adjunctions

$$(\mathcal{C} \times - \dashv \mathsf{Fun}(\mathcal{C}, -)) \colon \quad \begin{tikzcd} \mathsf{Cats} \arrow[r, bend left, "\mathcal{C}\times-"] \arrow[r, phantom, "\perp"] & \mathsf{Cats}, \arrow[l, bend left, "\mathsf{Fun}(\mathcal{C},-)"] \end{tikzcd}$$

$$(- \times \mathcal{D} \dashv \mathsf{Fun}(\mathcal{D}, -)) \colon \quad \begin{tikzcd} \mathsf{Cats} \arrow[r, bend left, "-\times\mathcal{D}"] \arrow[r, phantom, "\perp"] & \mathsf{Cats}, \arrow[l, bend left, "\mathsf{Fun}(\mathcal{D},-)"] \end{tikzcd}$$

   witnessed by bijections of sets

$$\begin{aligned} \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) &\cong \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{D}, \mathsf{Fun}(\mathcal{C}, \mathcal{E})), \\ \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) &\cong \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, \mathsf{Fun}(\mathcal{D}, \mathcal{E})), \end{aligned}$$

   natural in $\mathcal{C}, \mathcal{D}, \mathcal{E} \in \mathrm{Obj}(\mathsf{Cats})$.

4. *2-Adjointness.* We have 2-adjunctions

$$(\mathcal{C} \times - \dashv \mathsf{Fun}(\mathcal{C}, -)) \colon \quad \begin{tikzcd} \mathsf{Cats}_2 \arrow[r, bend left, "\mathcal{C}\times -"] \arrow[r, phantom, "\perp_2"] & \mathsf{Cats}_2, \arrow[l, bend left, "\mathsf{Fun}(\mathcal{C},-)"] \end{tikzcd}$$

$$(- \times \mathcal{D} \dashv \mathsf{Fun}(\mathcal{D}, -)) \colon \quad \begin{tikzcd} \mathsf{Cats}_2 \arrow[r, bend left, "-\times\mathcal{D}"] \arrow[r, phantom, "\perp_2"] & \mathsf{Cats}_2, \arrow[l, bend left, "\mathsf{Fun}(\mathcal{D},-)"] \end{tikzcd}$$

witnessed by isomorphisms of categories

$$\mathsf{Fun}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) \cong \mathsf{Fun}(\mathcal{D}, \mathsf{Fun}(\mathcal{C}, \mathcal{E})),$$
$$\mathsf{Fun}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) \cong \mathsf{Fun}(\mathcal{C}, \mathsf{Fun}(\mathcal{D}, \mathcal{E})),$$

natural in $\mathcal{C}, \mathcal{D}, \mathcal{E} \in \mathrm{Obj}(\mathsf{Cats}_2)$.

5. *Interaction With Punctual Categories.* We have a canonical isomorphism of categories

$$\mathsf{Fun}(\mathsf{pt}, \mathcal{C}) \cong \mathcal{C},$$

natural in $\mathcal{C} \in \mathrm{Obj}(\mathsf{Cats})$.

6. *Objectwise Computation of Co/Limits.* Let

$$D \colon \mathcal{I} \to \mathsf{Fun}(\mathcal{C}, \mathcal{D})$$

be a diagram in $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$. We have isomorphisms

$$\lim(D)_A \cong \lim_{i \in \mathcal{I}}(D_i(A)),$$
$$\operatorname{colim}(D)_A \cong \operatorname*{colim}_{i \in \mathcal{I}}(D_i(A)),$$

naturally in $A \in \mathrm{Obj}(\mathcal{C})$.

7. *Interaction With Co/Completeness.* If $\mathcal{E}$ is co/complete, then so is $\mathsf{Fun}(\mathcal{C}, \mathcal{E})$.

8. *Monomorphisms and Epimorphisms.* Let $\alpha \colon F \Longrightarrow G$ be a morphism of $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$. The following conditions are equivalent:

   (a) The natural transformation

   $$\alpha \colon F \Longrightarrow G$$

   is a monomorphism (resp. epimorphism) in $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$.

(b) For each $A \in \mathrm{Obj}(\mathcal{C})$, the morphism

$$\alpha_A \colon F_A \to G_A$$

is a monomorphism (resp. epimorphism) in $\mathcal{D}$.

Proof. Item 1, *Functoriality*: Omitted.
Item 2, *2-Functoriality*: Omitted.
Item 3, *Adjointness*: Omitted.
Item 4, *2-Adjointness*: Omitted.
Item 5, *Interaction With Punctual Categories*: Omitted.
Item 6, *Objectwise Computation of Co/Limits*: Omitted.
Item 7, *Interaction With Co/Completeness*: This follows from **??**.
Item 8, *Monomorphisms and Epimorphisms*: Omitted. □

9.2 The Category of Categories and Functors

Definition 9.2.1.1. The **category of (small) categories and functors** is the category Cats where

- *Objects.* The objects of Cats are small categories.
- *Morphisms.* For each $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$, we have

  $$\mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, \mathcal{D}) \stackrel{\text{def}}{=} \mathrm{Obj}(\mathsf{Fun}(\mathcal{C}, \mathcal{D})).$$

- *Identities.* For each $\mathcal{C} \in \mathrm{Obj}(\mathsf{Cats})$, the unit map

  $$\mathbb{1}^{\mathsf{Cats}}_{\mathcal{C}} \colon \mathrm{pt} \to \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, \mathcal{C})$$

  of Cats at $\mathcal{C}$ is defined by

  $$\mathrm{id}^{\mathsf{Cats}}_{\mathcal{C}} \stackrel{\text{def}}{=} \mathrm{id}_{\mathcal{C}},$$

  where $\mathrm{id}_{\mathcal{C}} \colon \mathcal{C} \to \mathcal{C}$ is the identity functor of $\mathcal{C}$ of Example 4.1.1.4.

- *Composition.* For each $\mathcal{C}, \mathcal{D}, \mathcal{E} \in \mathrm{Obj}(\mathsf{Cats})$, the composition map

  $$\circ^{\mathsf{Cats}}_{\mathcal{C},\mathcal{D},\mathcal{E}} \colon \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{D}, \mathcal{E}) \times \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, \mathcal{D}) \to \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, \mathcal{E})$$

  of Cats at $(\mathcal{C}, \mathcal{D}, \mathcal{E})$ is given by

  $$G \circ^{\mathsf{Cats}}_{\mathcal{C},\mathcal{D},\mathcal{E}} F \stackrel{\text{def}}{=} G \circ F,$$

  where $G \circ F \colon \mathcal{C} \to \mathcal{E}$ is the composition of F and G of Definition 4.1.1.5.

Proposition 9.2.1.2. Let $\mathcal{C}$ be a category.

1. *Co/Completeness.* The category Cats is complete and cocomplete.

2. *Cartesian Monoidal Structure.* The quadruple $(\mathsf{Cats}, \times, \mathrm{pt}, \mathsf{Fun})$ is a Cartesian closed monoidal category.

Proof. *Item 1, Co/Completeness*: Omitted.
Item 2, Cartesian Monoidal Structure: Omitted. □

9.3 The 2-Category of Categories, Functors, and Natural Transformations

Definition 9.3.1.1. The 2-**category of (small) categories, functors, and natural transformations** is the 2-category Cats_2 where

- *Objects.* The objects of Cats_2 are small categories.

- Hom-*Categories.* For each $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats}_2)$, we have

$$\mathsf{Hom}_{\mathsf{Cats}_2}(\mathcal{C}, \mathcal{D}) \overset{\text{def}}{=} \mathsf{Fun}(\mathcal{C}, \mathcal{D}).$$

- *Identities.* For each $\mathcal{C} \in \mathrm{Obj}(\mathsf{Cats}_2)$, the unit functor

$$\mathbb{1}^{\mathsf{Cats}_2}_{\mathcal{C}} \colon \mathrm{pt} \to \mathsf{Fun}(\mathcal{C}, \mathcal{C})$$

  of Cats_2 at $\mathcal{C}$ is the functor picking the identity functor $\mathrm{id}_{\mathcal{C}} \colon \mathcal{C} \to \mathcal{C}$ of $\mathcal{C}$.

- *Composition.* For each $\mathcal{C}, \mathcal{D}, \mathcal{E} \in \mathrm{Obj}(\mathsf{Cats}_2)$, the composition bifunctor

$$\circ^{\mathsf{Cats}_2}_{\mathcal{C},\mathcal{D},\mathcal{E}} \colon \mathsf{Hom}_{\mathsf{Cats}_2}(\mathcal{D}, \mathcal{E}) \times \mathsf{Hom}_{\mathsf{Cats}_2}(\mathcal{C}, \mathcal{D}) \to \mathsf{Hom}_{\mathsf{Cats}_2}(\mathcal{C}, \mathcal{E})$$

  of Cats_2 at $(\mathcal{C}, \mathcal{D}, \mathcal{E})$ is the functor where

  - *Action on Objects.* For each object $(G, F) \in \mathrm{Obj}(\mathsf{Hom}_{\mathsf{Cats}_2}(\mathcal{D}, \mathcal{E}) \times \mathsf{Hom}_{\mathsf{Cats}_2}(\mathcal{C}, \mathcal{D}))$, we have

$$\circ^{\mathsf{Cats}_2}_{\mathcal{C},\mathcal{D},\mathcal{E}}(G, F) \overset{\text{def}}{=} G \circ F.$$

  - *Action on Morphisms.* For each morphism $(\beta, \alpha) \colon (K, H) \Longrightarrow (G, F)$

of $\mathsf{Hom}_{\mathsf{Cats}_2}(\mathcal{D}, \mathcal{E}) \times \mathsf{Hom}_{\mathsf{Cats}_2}(\mathcal{C}, \mathcal{D})$, we have

$$\circ^{\mathsf{Cats}_2}_{\mathcal{C},\mathcal{D},\mathcal{E}}(\beta, \alpha) \stackrel{\text{def}}{=} \beta \star \alpha,$$

where $\beta \star \alpha$ is the horizontal composition of α and β of Definition 8.4.1.1.

Proposition 9.3.1.2. Let $\mathcal{C}$ be a category.

1. *2-Categorical Co/Completeness.* The 2-category Cats_2 is complete and cocomplete as a 2-category, having all 2-categorical and bicategorical co/limits.

Proof. *Item 1*, *Co/Completeness*: Omitted. □

9.4 The Category of Groupoids

Definition 9.4.1.1. The **category of (small) groupoids** is the full subcategory Grpd of Cats spanned by the groupoids.

9.5 The 2-Category of Groupoids

Definition 9.5.1.1. The **2-category of (small) groupoids** is the full sub-2-category Grpd_2 of Cats_2 spanned by the groupoids.

Appendices

A Other Chapters

Sets

1. Sets
2. Constructions With Sets
3. Pointed Sets
4. Tensor Products of Pointed Sets

Relations

5. Relations
6. Constructions With Relations
7. Equivalence Relations and Apartness Relations

Category Theory

8. Categories

Bicategories

9. Types of Morphisms in Bicategories

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