

Pointed Sets

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0098 This chapter contains some foundational material on pointed sets.

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0099 1 Pointed Sets

009A 1.1 Foundations

009B **Definition 1.1.1.1.** A **pointed set**¹ is equivalently:

- An \mathbb{E}_0 -monoid in $(N_\bullet(\mathbf{Sets}), \text{pt})$.
- A pointed object in $(\mathbf{Sets}, \text{pt})$.

009C **Remark 1.1.1.2.** In detail, a **pointed set** is a pair (X, x_0) consisting of:

- *The Underlying Set.* A set X , called the **underlying set of** (X, x_0) .
- *The Basepoint.* A morphism

$$[x_0] : \text{pt} \rightarrow X$$

in \mathbf{Sets} , determining an element $x_0 \in X$, called the **basepoint of** X .

009D **Example 1.1.1.3.** The **0-sphere**² is the pointed set $(S^0, 0)$ ³ consisting of:

- *The Underlying Set.* The set S^0 defined by

$$S^0 \stackrel{\text{def}}{=} \{0, 1\}.$$

- *The Basepoint.* The element 0 of S^0 .

009E **Example 1.1.1.4.** The **trivial pointed set** is the pointed set (pt, \star) consisting of:

- *The Underlying Set.* The punctual set $\text{pt} \stackrel{\text{def}}{=} \{\star\}$.
- *The Basepoint.* The element \star of pt .

009F **Example 1.1.1.5.** The **underlying pointed set** of a semimodule (M, α_M) is the pointed set $(M, 0_M)$.

009G **Example 1.1.1.6.** The **underlying pointed set** of a module (M, α_M) is the pointed set $(M, 0_M)$.

¹*Further Terminology:* In the context of monoids with zero as models for \mathbb{F}_1 -algebras, pointed sets are viewed as **\mathbb{F}_1 -modules**.

²*Further Terminology:* In the context of monoids with zero as models for \mathbb{F}_1 -algebras, the 0-sphere is viewed as the **underlying pointed set of the field with one element**.

³*Further Notation:* In the context of monoids with zero as models for \mathbb{F}_1 -algebras, S^0 is also

009H 1.2 Morphisms of Pointed Sets

009J **Definition 1.2.1.1.** A **morphism of pointed sets**^{4,5} is equivalently:

- A morphism of \mathbb{E}_0 -monoids in $(\mathbf{N}_\bullet(\mathbf{Sets}), \text{pt})$.
- A morphism of pointed objects in $(\mathbf{Sets}, \text{pt})$.

009K **Remark 1.2.1.2.** In detail, a **morphism of pointed sets** $f: (X, x_0) \rightarrow (Y, y_0)$ is a morphism of sets $f: X \rightarrow Y$ such that the diagram

$$\begin{array}{ccc} & \text{pt} & \\ [x_0] \swarrow & & \searrow [y_0] \\ X & \xrightarrow{f} & Y \end{array}$$

commutes, i.e. such that

$$f(x_0) = y_0.$$

009L 1.3 The Category of Pointed Sets

009M **Definition 1.3.1.1.** The **category of pointed sets** is the category \mathbf{Sets}_* defined equivalently as

- The homotopy category of the ∞ -category $\mathbf{Mon}_{\mathbb{E}_0}(\mathbf{N}_\bullet(\mathbf{Sets}), \text{pt})$ of ??, ??;
- The category \mathbf{Sets}_* of ??, ??.

009N **Remark 1.3.1.2.** In detail, the **category of pointed sets** is the category \mathbf{Sets}_* where

- *Objects.* The objects of \mathbf{Sets}_* are pointed sets;
- *Morphisms.* The morphisms of \mathbf{Sets}_* are morphisms of pointed sets;
- *Identities.* For each $(X, x_0) \in \text{Obj}(\mathbf{Sets}_*)$, the unit map

$$\mathbb{1}_{(X, x_0)}^{\mathbf{Sets}_*} : \text{pt} \rightarrow \mathbf{Sets}_*((X, x_0), (X, x_0))$$

denoted $(\mathbb{F}_1, 0)$.

⁴*Further Terminology:* Also called a **pointed function**.

⁵*Further Terminology:* In the context of monoids with zero as models for \mathbb{F}_1 -algebras, mor-

of \mathbf{Sets}_* at (X, x_0) is defined by⁶

$$\mathrm{id}_{(X, x_0)}^{\mathbf{Sets}_*} \stackrel{\mathrm{def}}{=} \mathrm{id}_X;$$

- *Composition.* For each $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathbf{Sets}_*)$, the composition map

$$\circ_{(X, x_0), (Y, y_0), (Z, z_0)}^{\mathbf{Sets}_*} : \mathbf{Sets}_*((Y, y_0), (Z, z_0)) \times \mathbf{Sets}_*((X, x_0), (Y, y_0)) \rightarrow \mathbf{Sets}_*((X, x_0), (Z, z_0))$$

of \mathbf{Sets}_* at $((X, x_0), (Y, y_0), (Z, z_0))$ is defined by⁷

$$g \circ_{(X, x_0), (Y, y_0), (Z, z_0)}^{\mathbf{Sets}_*} f \stackrel{\mathrm{def}}{=} g \circ f.$$

009P 1.4 Elementary Properties of Pointed Sets

009Q **Proposition 1.4.1.1.** Let (X, x_0) be a pointed set.

009R 1. *Completeness.* The category \mathbf{Sets}_* of pointed sets and morphisms between them is complete, having in particular:

- 009S (a) Products, described as in [Definition 2.3.1.1](#);
- 009T (b) Pullbacks, described as in [Definition 2.4.1.1](#);
- 009U (c) Equalisers, described as in [Definition 2.5.1.1](#).

009V 2. *Cocompleteness.* The category \mathbf{Sets}_* of pointed sets and morphisms between them is cocomplete, having in particular:

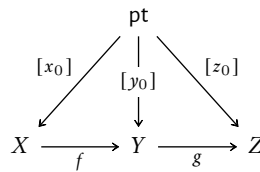
phisms of pointed sets are also called **morphism of \mathbb{F}_1 -modules**.

⁶Note that id_X is indeed a morphism of pointed sets, as we have $\mathrm{id}_X(x_0) = x_0$.

⁷Note that the composition of two morphisms of pointed sets is indeed a morphism of pointed sets, as we have

$$\begin{aligned} g(f(x_0)) &= g(y_0) \\ &= z_0, \end{aligned}$$

or



in terms of diagrams.

- 009W (a) Coproducts, described as in [Definition 3.3.1.1](#);
- 009X (b) Pushouts, described as in [Definition 3.4.1.1](#);
- 009Y (c) Coequalisers, described as in [Definition 3.5.1.1](#).
- 009Z 3. *Failure To Be Cartesian Closed.* The category \mathbf{Sets}_* is not Cartesian closed.⁸
- 00A0 4. *Morphisms From the Monoidal Unit.* We have a bijection of sets⁹

$$\mathbf{Sets}_*(S^0, X) \cong X,$$

natural in $(X, x_0) \in \mathbf{Obj}(\mathbf{Sets}_*)$, internalising also to an isomorphism of pointed sets

$$\mathbf{Sets}_*(S^0, X) \cong (X, x_0),$$

again natural in $(X, x_0) \in \mathbf{Obj}(\mathbf{Sets}_*)$.

- 00A1 5. *Relation to Partial Functions.* We have an equivalence of categories¹⁰

$$\mathbf{Sets}_* \stackrel{\text{eq.}}{\cong} \mathbf{Sets}^{\text{part.}}$$

between the category of pointed sets and pointed functions between them and the category of sets and partial functions between them, where:

- (a) *From Pointed Sets to Sets With Partial Functions.* The equivalence

$$\xi : \mathbf{Sets}_* \xrightarrow{\cong} \mathbf{Sets}^{\text{part.}}$$

sends:


- i. A pointed set (X, x_0) to X .

⁸The category \mathbf{Sets}_* does admit monoidal closed structures however; see [Tensor Products of Pointed Sets](#).

⁹In other words, the forgetful functor

$$\omega : \mathbf{Sets}_* \rightarrow \mathbf{Sets}$$

defined on objects by sending a pointed set to its underlying set is corepresentable by S^0 .

¹⁰  *Warning:* This is not an isomorphism of categories, only an equivalence.

ii. A pointed function

$$f: (X, x_0) \rightarrow (Y, y_0)$$

to the partial function

$$\xi_f: X \rightarrow Y$$

defined on $f^{-1}(Y \setminus y_0)$ and given by

$$\xi_f(x) \stackrel{\text{def}}{=} f(x)$$

for each $x \in f^{-1}(Y \setminus y_0)$.

(b) *From Sets With Partial Functions to Pointed Sets.* The equivalence

$$\xi^{-1}: \text{Sets}^{\text{part.}} \xrightarrow{\cong} \text{Sets}_*$$

sends:

- i. A set X is to the pointed set (X, \star) with \star an element that is not in X .
- ii. A partial function

$$f: X \rightarrow Y$$

defined on $U \subset X$ to the pointed function

$$\xi_f^{-1}: (X, x_0) \rightarrow (Y, y_0)$$

defined by

$$\xi_f(x) \stackrel{\text{def}}{=} \begin{cases} f(x) & \text{if } x \in U, \\ y_0 & \text{otherwise.} \end{cases}$$

for each $x \in X$.

Proof. **Item 1, Completeness:** This follows from (the proofs) of [Definitions 2.3.1.1](#), [2.4.1.1](#) and [2.5.1.1](#) and [??, ??](#).

Item 2, Cocompleteness: This follows from (the proofs) of [Definitions 3.3.1.1](#), [3.4.1.1](#) and [3.5.1.1](#) and [??, ??](#).

Item 3, Failure To Be Cartesian Closed: See [[MSE 2855868](#)].

Item 4, Morphisms From the Monoidal Unit: Since a morphism from S^0 to a pointed set (X, x_0) sends $0 \in S^0$ to x_0 and then can send $1 \in S^0$ to any element of X , we obtain a bijection between pointed maps $S^0 \rightarrow X$ and the elements of X . The isomorphism then

$$\mathbf{Sets}_*(S^0, X) \cong (X, x_0)$$

follows by noting that $\Delta_{x_0}: S^0 \rightarrow X$, the basepoint of $\mathbf{Sets}_*(S^0, X)$, corresponds to the pointed map $S^0 \rightarrow X$ picking the element x_0 of X , and thus we see that the bijection between pointed maps $S^0 \rightarrow X$ and elements of X is compatible with basepoints, lifting to an isomorphism of pointed sets.

Item 5, Relation to Partial Functions: See [MSE 884460]. \square

00A2 2 Limits of Pointed Sets

00A3 2.1 The Terminal Pointed Set

00A4 Definition 2.1.1.1. The **terminal pointed set** is the pair $((pt, \star), \{!_X\}_{(X, x_0) \in \text{Obj}(\mathbf{Sets}_*)})$ consisting of:

- *The Limit.* The pointed set (pt, \star) .
- *The Cone.* The collection of morphisms of pointed sets

$$\{!_X: (X, x_0) \rightarrow (pt, \star)\}_{(X, x_0) \in \text{Obj}(\mathbf{Sets})}$$

defined by

$$!_X(x) \stackrel{\text{def}}{=} \star$$

for each $x \in X$ and each $(X, x_0) \in \text{Obj}(\mathbf{Sets})$.

Proof. We claim that (pt, \star) is the terminal object of \mathbf{Sets}_* . Indeed, suppose we have a diagram of the form

$$(X, x_0) \quad (pt, \star)$$

in \mathbf{Sets}_* . Then there exists a unique morphism of pointed sets

$$\phi: (X, x_0) \rightarrow (pt, \star)$$

making the diagram

$$(X, x_0) \xrightarrow[\exists!]{\phi} (pt, \star)$$

commute, namely $!_X$. \square

00A5 2.2 Products of Families of Pointed Sets

Let $\{(X_i, x_0^i)\}_{i \in I}$ be a family of pointed sets.

00A6 Definition 2.2.1.1. The **product** of $\{(X_i, x_0^i)\}_{i \in I}$ is the pair $\left(\left(\prod_{i \in I} X_i, (x_0^i)_{i \in I}\right), \{\text{pr}_i\}_{i \in I}\right)$ consisting of:

- *The Limit.* The pointed set $\left(\prod_{i \in I} X_i, (x_0^i)_{i \in I}\right)$.
- *The Cone.* The collection

$$\left\{ \text{pr}_i : \left(\prod_{i \in I} X_i, (x_0^i)_{i \in I} \right) \rightarrow (X_i, x_0^i) \right\}_{i \in I}$$

of maps given by

$$\text{pr}_i \left((x_j)_{j \in I} \right) \stackrel{\text{def}}{=} x_i$$

for each $(x_j)_{j \in I} \in \prod_{i \in I} X_i$ and each $i \in I$.

Proof. We claim that $\left(\prod_{i \in I} X_i, (x_0^i)_{i \in I}\right)$ is the categorical product of $\{(X_i, x_0^i)\}_{i \in I}$ in Sets_* . Indeed, suppose we have, for each $i \in I$, a diagram of the form

$$\begin{array}{ccc} (P, *) & & \\ & \searrow p_i & \\ & \left(\prod_{i \in I} X_i, (x_0^i)_{i \in I} \right) & \xrightarrow{\text{pr}_i} (X_i, x_0^i) \end{array}$$

in Sets_* . Then there exists a unique morphism of pointed sets

$$\phi : (P, *) \rightarrow \left(\prod_{i \in I} X_i, (x_0^i)_{i \in I} \right)$$

making the diagram

$$\begin{array}{ccc} (P, *) & & \\ \downarrow \phi \quad \exists! & \searrow p_i & \\ \left(\prod_{i \in I} X_i, (x_0^i)_{i \in I} \right) & \xrightarrow{\text{pr}_i} & (X_i, x_0^i) \end{array}$$

commute, being uniquely determined by the condition $\text{pr}_i \circ \phi = p_i$ for each $i \in I$ via

$$\phi(x) = (p_i(x))_{i \in I}$$

for each $x \in P$. Note that this is indeed a morphism of pointed sets, as we have

$$\begin{aligned} \phi(*) &= (p_i(*))_{i \in I} \\ &= (x_0^i)_{i \in I}, \end{aligned}$$

where we have used that p_i is a morphism of pointed sets for each $i \in I$. \square

00A7 Proposition 2.2.1.2. Let $\{(X_i, x_0^i)\}_{i \in I}$ be a family of pointed sets.

00A8 1. *Functoriality.* The assignment $\{(X_i, x_0^i)\}_{i \in I} \mapsto \left(\prod_{i \in I} X_i, (x_0^i)_{i \in I}\right)$ defines a functor

$$\prod_{i \in I} : \text{Fun}(I_{\text{disc}}, \text{Sets}_*) \rightarrow \text{Sets}_*.$$

Proof. **Item 1, Functoriality:** This follows from ??, ?? of ??. \square

00A9 2.3 Products

Let (X, x_0) and (Y, y_0) be pointed sets.

00AA Definition 2.3.1.1. The **product of (X, x_0) and (Y, y_0)** is the pair consisting of:

- *The Limit.* The pointed set $(X \times Y, (x_0, y_0))$.
- *The Cone.* The morphisms of pointed sets

$$\begin{aligned} \text{pr}_1 : (X \times Y, (x_0, y_0)) &\rightarrow (X, x_0), \\ \text{pr}_2 : (X \times Y, (x_0, y_0)) &\rightarrow (Y, y_0) \end{aligned}$$

defined by

$$\begin{aligned} \text{pr}_1(x, y) &\stackrel{\text{def}}{=} x, \\ \text{pr}_2(x, y) &\stackrel{\text{def}}{=} y \end{aligned}$$

for each $(x, y) \in X \times Y$.

Proof. We claim that $(X \times Y, (x_0, y_0))$ is the categorical product of (X, x_0) and (Y, y_0) in \mathbf{Sets}_* . Indeed, suppose we have a diagram of the form

$$\begin{array}{ccccc} & & (P, *) & & \\ & \swarrow p_1 & & \searrow p_2 & \\ (X, x_0) & \xleftarrow{\text{pr}_1} & (X \times Y, (x_0, y_0)) & \xrightarrow{\text{pr}_2} & (Y, y_0) \end{array}$$

in \mathbf{Sets}_* . Then there exists a unique morphism of pointed sets

$$\phi: (P, *) \rightarrow (X \times Y, (x_0, y_0))$$

making the diagram

$$\begin{array}{ccccc} & & (P, *) & & \\ & \swarrow p_1 & \downarrow \phi \mid \exists! & \searrow p_2 & \\ (X, x_0) & \xleftarrow{\text{pr}_1} & (X \times Y, (x_0, y_0)) & \xrightarrow{\text{pr}_2} & (Y, y_0) \end{array}$$

commute, being uniquely determined by the conditions

$$\begin{aligned} \text{pr}_1 \circ \phi &= p_1, \\ \text{pr}_2 \circ \phi &= p_2 \end{aligned}$$

via

$$\phi(x) = (p_1(x), p_2(x))$$

for each $x \in P$. Note that this is indeed a morphism of pointed sets, as we have

$$\begin{aligned} \phi(*) &= (p_1(*), p_2(*)) \\ &= (x_0, y_0), \end{aligned}$$

where we have used that p_1 and p_2 are morphisms of pointed sets. \square

00AB Proposition 2.3.1.2. Let (X, x_0) , (Y, y_0) , and (Z, z_0) be pointed sets.

00AC 1. *Functoriality.* The assignments

$$(X, x_0), (Y, y_0), ((X, x_0), (Y, y_0)) \mapsto (X \times Y, (x_0, y_0))$$

define functors

$$\begin{aligned} X \times - &: \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*, \\ - \times Y &: \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*, \\ -_1 \times -_2 &: \mathbf{Sets}_* \times \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*, \end{aligned}$$

defined in the same way as the functors of **Constructions With Sets, Item 1** of **Proposition 1.3.1.2**.

00AD 2. *Associativity*. We have an isomorphism of pointed sets

$$((X \times Y) \times Z, ((x_0, y_0), z_0)) \cong (X \times (Y \times Z), (x_0, (y_0, z_0)))$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathbf{Obj}(\mathbf{Sets}_*)$.

00AE 3. *Unitality*. We have isomorphisms of pointed sets

$$\begin{aligned} (\mathbf{pt}, \star) \times (X, x_0) &\cong (X, x_0), \\ (X, x_0) \times (\mathbf{pt}, \star) &\cong (X, x_0), \end{aligned}$$

natural in $(X, x_0) \in \mathbf{Obj}(\mathbf{Sets}_*)$.

00AF 4. *Commutativity*. We have an isomorphism of pointed sets

$$(X \times Y, (x_0, y_0)) \cong (Y \times X, (y_0, x_0)),$$

natural in $(X, x_0), (Y, y_0) \in \mathbf{Obj}(\mathbf{Sets}_*)$.

00AG 5. *Symmetric Monoidality*. The triple $(\mathbf{Sets}_*, \times, (\mathbf{pt}, \star))$ is a symmetric monoidal category.

Proof. **Item 1, Functoriality**: This is a special case of functoriality of limits, ??, ?? of ??.

Item 2, Associativity: This follows from **Constructions With Sets, Item 3** of **Proposition 1.3.1.2**.

Item 3, Unitality: This follows from **Constructions With Sets, Item 4** of **Proposition 1.3.1.2**.

Item 4, Commutativity: This follows from **Constructions With Sets, Item 5** of **Proposition 1.3.1.2**.

Item 5, Symmetric Monoidality: This follows from **Constructions With Sets, Item 12** of **Proposition 1.3.1.2**. \square

00AH 2.4 Pullbacks

Let (X, x_0) , (Y, y_0) , and (Z, z_0) be pointed sets and let $f: (X, x_0) \rightarrow (Z, z_0)$ and $g: (Y, y_0) \rightarrow (Z, z_0)$ be morphisms of pointed sets.

00AJ Definition 2.4.1.1. The **pullback of (X, x_0) and (Y, y_0) over (Z, z_0) along (f, g)** is the pair consisting of:

- *The Limit.* The pointed set $(X \times_Z Y, (x_0, y_0))$.
- *The Cone.* The morphisms of pointed sets

$$\begin{aligned} \text{pr}_1: (X \times_Z Y, (x_0, y_0)) &\rightarrow (X, x_0), \\ \text{pr}_2: (X \times_Z Y, (x_0, y_0)) &\rightarrow (Y, y_0) \end{aligned}$$

defined by

$$\begin{aligned} \text{pr}_1(x, y) &\stackrel{\text{def}}{=} x, \\ \text{pr}_2(x, y) &\stackrel{\text{def}}{=} y \end{aligned}$$

for each $(x, y) \in X \times_Z Y$.

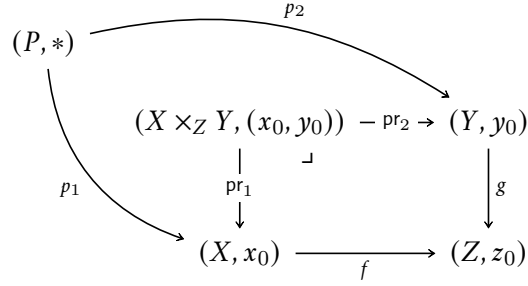
Proof. We claim that $X \times_Z Y$ is the categorical pullback of (X, x_0) and (Y, y_0) over (Z, z_0) with respect to (f, g) in \mathbf{Sets}_* . First we need to check that the relevant pullback diagram commutes, i.e. that we have

$$\begin{array}{ccc} (X \times_Z Y, (x_0, y_0)) & \xrightarrow{\text{pr}_2} & (Y, y_0) \\ \text{pr}_1 \downarrow & & \downarrow g \\ (X, x_0) & \xrightarrow{f} & (Z, z_0). \end{array} \quad f \circ \text{pr}_1 = g \circ \text{pr}_2,$$

Indeed, given $(x, y) \in X \times_Z Y$, we have

$$\begin{aligned} [f \circ \text{pr}_1](x, y) &= f(\text{pr}_1(x, y)) \\ &= f(x) \\ &= g(y) \\ &= g(\text{pr}_2(x, y)) \\ &= [g \circ \text{pr}_2](x, y), \end{aligned}$$

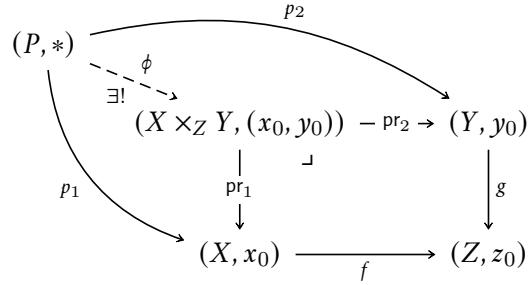
where $f(x) = g(y)$ since $(x, y) \in X \times_Z Y$. Next, we prove that $X \times_Z Y$ satisfies the universal property of the pullback. Suppose we have a diagram of the form



in \mathbf{Sets}_* . Then there exists a unique morphism of pointed sets

$$\phi: (P, *) \rightarrow (X \times_Z Y, (x_0, y_0))$$

making the diagram



commute, being uniquely determined by the conditions

$$\text{pr}_1 \circ \phi = p_1,$$

$$\text{pr}_2 \circ \phi = p_2$$

via

$$\phi(x) = (p_1(x), p_2(x))$$

for each $x \in P$, where we note that $(p_1(x), p_2(x)) \in X \times Y$ indeed lies in $X \times_Z Y$ by the condition

$$f \circ p_1 = g \circ p_2,$$

which gives

$$f(p_1(x)) = g(p_2(x))$$

for each $x \in P$, so that $(p_1(x), p_2(x)) \in X \times_Z Y$. Lastly, we note that ϕ is indeed a morphism of pointed sets, as we have

$$\begin{aligned}\phi(*) &= (p_1(*), p_2(*)) \\ &= (x_0, y_0),\end{aligned}$$

where we have used that p_1 and p_2 are morphisms of pointed sets. \square

00AK Proposition 2.4.1.2. Let (X, x_0) , (Y, y_0) , (Z, z_0) , and (A, a_0) be pointed sets.

00AL 1. *Functoriality.* The assignment $(X, Y, Z, f, g) \mapsto X \times_{f,Z,g} Y$ defines a functor

$$-_1 \times_{-3} -_1 : \text{Fun}(\mathcal{P}, \text{Sets}_*) \rightarrow \text{Sets}_*,$$

where \mathcal{P} is the category that looks like this:

$$\begin{array}{ccc} & \bullet & \\ & \downarrow & \\ \bullet & \longrightarrow & \bullet \end{array}$$

In particular, the action on morphisms of $-_1 \times_{-3} -_1$ is given by sending a morphism

$$\begin{array}{ccccc} X \times_Z Y & \xrightarrow{\quad} & Y & & \\ \downarrow & \lrcorner & \downarrow g & \searrow \psi & \\ & X' \times_{Z'} Y' & \xrightarrow{\quad} & Y' & \\ \downarrow & \lrcorner & \downarrow & & \downarrow g' \\ X & \xrightarrow{f} & Z & \searrow \chi & \\ \downarrow \phi & & \downarrow & & \downarrow \\ & X' & \xrightarrow{f'} & Z' & \end{array}$$

in $\text{Fun}(\mathcal{P}, \text{Sets}_*)$ to the morphism of pointed sets

$$\xi : (X \times_Z Y, (x_0, y_0)) \xrightarrow{\exists!} (X' \times_{Z'} Y', (x'_0, y'_0))$$

given by

$$\xi(x, y) \stackrel{\text{def}}{=} (\phi(x), \psi(y))$$

for each $(x, y) \in X \times_Z Y$, which is the unique morphism of pointed sets making the diagram

$$\begin{array}{ccccc}
 X \times_Z Y & \xrightarrow{\quad} & Y & & \\
 \downarrow & \searrow \lrcorner & \downarrow g & \searrow \psi & \\
 & X' \times_{Z'} Y' & \xrightarrow{\quad} & Y' & \\
 \downarrow & \lrcorner & \downarrow & & \downarrow g' \\
 X & \xrightarrow{f} & Z & & \\
 \downarrow \phi & & \downarrow \chi & & \\
 X' & \xrightarrow{f'} & Z' & &
 \end{array}$$

commute.

00AM

2. *Associativity.* Given a diagram

$$\begin{array}{ccccc}
 X & & Y & & Z \\
 & \searrow f & \swarrow g & \searrow h & \swarrow k \\
 & W & & V &
 \end{array}$$

in \mathbf{Sets}_* , we have isomorphisms of pointed sets

$$(X \times_W Y) \times_V Z \cong (X \times_W Y) \times_Y (Y \times_V Z) \cong X \times_W (Y \times_V Z),$$

where these pullbacks are built as in the diagrams

$$\begin{array}{ccc}
 \begin{array}{c} (X \times_W Y) \times_Y Z \\ \swarrow \quad \searrow \\ X \times_W Y \quad Y \\ \swarrow \quad \searrow \\ X \quad Y \\ \swarrow f \quad \searrow g \quad \searrow h \quad \swarrow k \\ W \quad V \end{array} &
 \begin{array}{c} (X \times_W Y) \times_Y (Y \times_V Z) \\ \swarrow \quad \searrow \\ X \times_W Y \quad Y \times_V Z \\ \swarrow \quad \searrow \\ X \quad Y \quad Z \\ \swarrow f \quad \searrow g \quad \searrow h \quad \swarrow k \\ W \quad V \end{array} &
 \begin{array}{c} X \times_W (Y \times_V Z) \\ \swarrow \quad \searrow \\ X \quad Y \times_V Z \\ \swarrow \quad \searrow \\ X \quad Y \quad Z \\ \swarrow f \quad \searrow g \quad \searrow h \quad \swarrow k \\ W \quad V \end{array}
 \end{array}$$

00AN

3. *Unitality.* We have isomorphisms of pointed sets

$$\begin{array}{ccc}
 \begin{array}{ccc} A & \xlongequal{\quad} & A \\ \downarrow f & \lrcorner & \downarrow f \\ X & \xlongequal{\quad} & X \end{array} &
 \begin{array}{l} X \times_X A \cong A, \\ A \times_X X \cong A, \end{array} &
 \begin{array}{ccc} A & \xrightarrow{f} & X \\ \parallel \lrcorner & & \parallel \\ X & \xrightarrow{f} & X. \end{array}
 \end{array}$$

00AP 4. *Commutativity.* We have an isomorphism of pointed sets

$$\begin{array}{ccc}
 A \times_X B & \longrightarrow & B \\
 \downarrow \lrcorner & & \downarrow g \\
 A & \xrightarrow{f} & X,
 \end{array}
 \quad A \times_X B \cong B \times_X A
 \quad
 \begin{array}{ccc}
 B \times_X A & \longrightarrow & A \\
 \downarrow \lrcorner & & \downarrow f \\
 B & \xrightarrow{g} & X.
 \end{array}$$

00AQ 5. *Interaction With Products.* We have an isomorphism of pointed sets

$$X \times_{\text{pt}} Y \cong X \times Y,
 \quad
 \begin{array}{ccc}
 X \times Y & \longrightarrow & Y \\
 \downarrow \lrcorner & & \downarrow !_Y \\
 X & \xrightarrow{!_X} & \text{pt.}
 \end{array}$$

00AR 6. *Symmetric Monoidality.* The triple $(\text{Sets}_*, \times_X, X)$ is a symmetric monoidal category.

Proof. **Item 1, Functoriality:** This is a special case of functoriality of co/limits, ??, ?? of ??, with the explicit expression for ξ following from the commutativity of the cube pullback diagram.

Item 2, Associativity: This follows from **Constructions With Sets, Item 2 of Proposition 1.4.1.3.**

Item 3, Unitality: This follows from **Constructions With Sets, Item 3 of Proposition 1.4.1.3.**

Item 4, Commutativity: This follows from **Constructions With Sets, Item 4 of Proposition 1.4.1.3.**

Item 5, Interaction With Products: This follows from **Constructions With Sets, Item 6 of Proposition 1.4.1.3.**

Item 6, Symmetric Monoidality: This follows from **Constructions With Sets, Item 7 of Proposition 1.4.1.3.** \square

00AS 2.5 Equalisers

Let $f, g: (X, x_0) \rightrightarrows (Y, y_0)$ be morphisms of pointed sets.

00AT **Definition 2.5.1.1.** The **equaliser** of (f, g) is the pair consisting of:

- *The Limit.* The pointed set $(\text{Eq}(f, g), x_0)$.
- *The Cone.* The morphism of pointed sets

$$\text{eq}(f, g): (\text{Eq}(f, g), x_0) \hookrightarrow (X, x_0)$$

given by the canonical inclusion $\text{eq}(f, g) \hookrightarrow \text{Eq}(f, g) \hookrightarrow X$.

Proof. We claim that $(\text{Eq}(f, g), x_0)$ is the categorical equaliser of f and g in Sets_* . First we need to check that the relevant equaliser diagram commutes, i.e. that we have

$$f \circ \text{eq}(f, g) = g \circ \text{eq}(f, g),$$

which indeed holds by the definition of the set $\text{Eq}(f, g)$. Next, we prove that $\text{Eq}(f, g)$ satisfies the universal property of the equaliser. Suppose we have a diagram of the form

$$\begin{array}{ccccc} (\text{Eq}(f, g), x_0) & \xrightarrow{\text{eq}(f, g)} & (X, x_0) & \xrightleftharpoons[g]{f} & (Y, y_0) \\ & \nearrow e & & & \\ (E, *) & & & & \end{array}$$

in Sets_* . Then there exists a unique morphism of pointed sets

$$\phi: (E, *) \rightarrow (\text{Eq}(f, g), x_0)$$

making the diagram

$$\begin{array}{ccccc} (\text{Eq}(f, g), x_0) & \xrightarrow{\text{eq}(f, g)} & (X, x_0) & \xrightleftharpoons[g]{f} & (Y, y_0) \\ \uparrow \phi \quad \exists! & \nearrow e & & & \\ (E, *) & & & & \end{array}$$

commute, being uniquely determined by the condition

$$\text{eq}(f, g) \circ \phi = e$$

via

$$\phi(x) = e(x)$$

for each $x \in E$, where we note that $e(x) \in A$ indeed lies in $\text{Eq}(f, g)$ by the condition

$$f \circ e = g \circ e,$$

which gives

$$f(e(x)) = g(e(x))$$

for each $x \in E$, so that $e(x) \in \text{Eq}(f, g)$. Lastly, we note that ϕ is indeed a morphism of pointed sets, as we have

$$\begin{aligned} \phi(*) &= e(*) \\ &= x_0, \end{aligned}$$

where we have used that e is a morphism of pointed sets. \square

00AU Proposition 2.5.1.2. Let (X, x_0) and (Y, y_0) be pointed sets and let $f, g, h: (X, x_0) \rightarrow (Y, y_0)$ be morphisms of pointed sets.

00AV 1. *Associativity.* We have isomorphisms of pointed sets

$$\underbrace{\text{Eq}(f \circ \text{eq}(g, h), g \circ \text{eq}(g, h))}_{=\text{Eq}(f \circ \text{eq}(g, h), h \circ \text{eq}(g, h))} \cong \text{Eq}(f, g, h) \cong \underbrace{\text{Eq}(f \circ \text{eq}(f, g), h \circ \text{eq}(f, g))}_{=\text{Eq}(g \circ \text{eq}(f, g), h \circ \text{eq}(f, g))},$$

where $\text{Eq}(f, g, h)$ is the limit of the diagram

$$(X, x_0) \begin{array}{c} \xrightarrow{f} \\ \xrightarrow[g]{h} \end{array} (Y, y_0)$$

in Sets_* , being explicitly given by

$$\text{Eq}(f, g, h) \cong \{a \in A \mid f(a) = g(a) = h(a)\}.$$

00AW 2. *Unitality.* We have an isomorphism of pointed sets

$$\text{Eq}(f, f) \cong X.$$

00AX 3. *Commutativity.* We have an isomorphism of pointed sets

$$\text{Eq}(f, g) \cong \text{Eq}(g, f).$$

Proof. **Item 1, Associativity:** This follows from **Constructions With Sets, Item 1 of Proposition 1.5.1.2.**

Item 2, Unitality: This follows from **Constructions With Sets, Item 4 of Proposition 1.5.1.2.**

Item 3, Commutativity: This follows from **Constructions With Sets, Item 5 of Proposition 1.5.1.2.** \square

00AY 3 Colimits of Pointed Sets

00AZ 3.1 The Initial Pointed Set

00B0 **Definition 3.1.1.1.** The **initial pointed set** is the pair $((pt, \star), \{\iota_X\}_{(X, x_0) \in \text{Obj}(\text{Sets}_*)})$ consisting of:

- *The Limit.* The pointed set (pt, \star) .
- *The Cone.* The collection of morphisms of pointed sets

$$\{\iota_X : (pt, \star) \rightarrow (X, x_0)\}_{(X, x_0) \in \text{Obj}(\text{Sets}_*)}$$

defined by

$$\iota_X(\star) \stackrel{\text{def}}{=} x_0.$$

Proof. We claim that (pt, \star) is the initial object of Sets_* . Indeed, suppose we have a diagram of the form

$$(pt, \star) \quad (X, x_0)$$

in Sets_* . Then there exists a unique morphism of pointed sets

$$\phi : (pt, \star) \rightarrow (X, x_0)$$

making the diagram

$$(pt, \star) \xrightarrow[\exists!]{\phi} (X, x_0)$$

commute, namely ι_X . □

00B1 3.2 Coproducts of Families of Pointed Sets

Let $\{(X_i, x_0^i)\}_{i \in I}$ be a family of pointed sets.

00B2 **Definition 3.2.1.1.** The **coproduct of the family** $\{(X_i, x_0^i)\}_{i \in I}$, also called their **wedge sum**, is the pair consisting of:

- *The Colimit.* The pointed set $(\bigvee_{i \in I} X_i, p_0)$ consisting of:

- *The Underlying Set.* The set $\bigvee_{i \in I} X_i$ defined by

$$\bigvee_{i \in I} X_i \stackrel{\text{def}}{=} \left(\bigsqcup_{i \in I} X_i \right) / \sim,$$

where \sim is the equivalence relation on $\bigsqcup_{i \in I} X_i$ given by declaring

$$(i, x_0^i) \sim (j, x_0^j)$$

for each $i, j \in I$.

- *The Basepoint.* The element p_0 of $\bigvee_{i \in I} X_i$ defined by

$$\begin{aligned} p_0 &\stackrel{\text{def}}{=} \left[(i, x_0^i) \right] \\ &= \left[(j, x_0^j) \right] \end{aligned}$$

for any $i, j \in I$.

- *The Cocone.* The collection

$$\left\{ \text{inj}_i : (X_i, x_0^i) \rightarrow \left(\bigvee_{i \in I} X_i, p_0 \right) \right\}_{i \in I}$$

of morphism of pointed sets given by

$$\text{inj}_i(x) \stackrel{\text{def}}{=} (i, x)$$

for each $x \in X_i$ and each $i \in I$.

Proof. We claim that $(\bigvee_{i \in I} X_i, p_0)$ is the categorical coproduct of $\{(X_i, x_0^i)\}_{i \in I}$ in \mathbf{Sets}_* . Indeed, suppose we have, for each $i \in I$, a diagram of the form

$$\begin{array}{ccc} & & (C, *) \\ & \nearrow \iota_i & \\ (X_i, x_0^i) & \xrightarrow{\text{inj}_i} & \left(\bigvee_{i \in I} X_i, p_0 \right) \end{array}$$

in \mathbf{Sets}_* . Then there exists a unique morphism of pointed sets

$$\phi: \left(\bigvee_{i \in I} X_i, p_0 \right) \rightarrow (C, *)$$

making the diagram

$$\begin{array}{ccc} & & (C, *) \\ & \nearrow \iota_i & \uparrow \phi \exists! \\ (X_i, x_0^i) & \xrightarrow{\text{inj}_i} & \left(\bigvee_{i \in I} X_i, p_0 \right) \end{array}$$

commute, being uniquely determined by the condition $\phi \circ \text{inj}_i = \iota_i$ for each $i \in I$ via

$$\phi([(i, x)]) = \iota_i(x)$$

for each $[(i, x)] \in \bigvee_{i \in I} X_i$, where we note that ϕ is indeed a morphism of pointed sets, as we have

$$\begin{aligned} \phi(p_0) &= \iota_i([(i, x_0^i)]) \\ &= *, \end{aligned}$$

as ι_i is a morphism of pointed sets. \square

00B3 Proposition 3.2.1.2. Let $\{(X_i, x_0^i)\}_{i \in I}$ be a family of pointed sets.

00B4 1. *Functoriality.* The assignment $\{(X_i, x_0^i)\}_{i \in I} \mapsto (\bigvee_{i \in I} X_i, p_0)$ defines a functor

$$\bigvee_{i \in I}: \mathbf{Fun}(I_{\text{disc}}, \mathbf{Sets}_*) \rightarrow \mathbf{Sets}_*.$$

Proof. **Item 1, Functoriality:** This follows from ??, ?? of ??. \square

00B5 3.3 Coproducts

Let (X, x_0) and (Y, y_0) be pointed sets.

00B6 Definition 3.3.1.1. The **coproduct of (X, x_0) and (Y, y_0)** , also called their **wedge sum**, is the pair consisting of:

· *The Colimit.* The pointed set $(X \vee Y, p_0)$ consisting of:

– *The Underlying Set.* The set $X \vee Y$ defined by

$$\begin{aligned} (X \vee Y, p_0) &\stackrel{\text{def}}{=} (X, x_0) \amalg (Y, y_0) \\ &\cong (X \amalg_{\text{pt}} Y, p_0) \\ &\cong (X \amalg Y / \sim, p_0), \end{aligned} \quad \begin{array}{ccc} X \vee Y & \longleftarrow & Y \\ \uparrow \scriptstyle \ulcorner & & \uparrow \scriptstyle [y_0] \\ X & \xleftarrow{\scriptstyle [x_0]} & \text{pt}, \end{array}$$

where \sim is the equivalence relation on $X \amalg Y$ obtained by declaring $(0, x_0) \sim (1, y_0)$.

– *The Basepoint.* The element p_0 of $X \vee Y$ defined by

$$\begin{aligned} p_0 &\stackrel{\text{def}}{=} [(0, x_0)] \\ &= [(1, y_0)]. \end{aligned}$$

· *The Cocone.* The morphisms of pointed sets

$$\begin{aligned} \text{inj}_1 &: (X, x_0) \rightarrow (X \vee Y, p_0), \\ \text{inj}_2 &: (Y, y_0) \rightarrow (X \vee Y, p_0), \end{aligned}$$

given by

$$\begin{aligned} \text{inj}_1(x) &\stackrel{\text{def}}{=} [(0, x)], \\ \text{inj}_2(y) &\stackrel{\text{def}}{=} [(1, y)], \end{aligned}$$

for each $x \in X$ and each $y \in Y$.

Proof. We claim that $(X \vee Y, p_0)$ is the categorical coproduct of (X, x_0) and (Y, y_0) in \mathbf{Sets}_* . Indeed, suppose we have a diagram of the form

$$\begin{array}{ccccc} & & (C, *) & & \\ & \nearrow \scriptstyle \iota_X & & \nwarrow \scriptstyle \iota_Y & \\ (X, x_0) & \xrightarrow{\scriptstyle \text{inj}_X} & (X \vee Y, p_0) & \xleftarrow{\scriptstyle \text{inj}_Y} & (Y, y_0) \end{array}$$

in \mathbf{Sets} . Then there exists a unique morphism of pointed sets

$$\phi: (X \vee Y, p_0) \rightarrow (C, *)$$

making the diagram

$$\begin{array}{ccccc}
 & & (C, *) & & \\
 & \nearrow \iota_X & \uparrow \phi \mid \exists! & \nwarrow \iota_Y & \\
 (X, x_0) & \xrightarrow{\text{inj}_X} & (X \vee Y, p_0) & \xleftarrow{\text{inj}_Y} & (Y, y_0)
 \end{array}$$

commute, being uniquely determined by the conditions

$$\begin{aligned}
 \phi \circ \text{inj}_X &= \iota_X, \\
 \phi \circ \text{inj}_Y &= \iota_Y
 \end{aligned}$$

via

$$\phi(z) = \begin{cases} \iota_X(x) & \text{if } z = [(0, x)] \text{ with } x \in X, \\ \iota_Y(y) & \text{if } z = [(1, y)] \text{ with } y \in Y \end{cases}$$

for each $z \in X \vee Y$, where we note that ϕ is indeed a morphism of pointed sets, as we have

$$\begin{aligned}
 \phi(p_0) &= \iota_X([(0, x_0)]) \\
 &= \iota_Y([(1, y_0)]) \\
 &= *,
 \end{aligned}$$

as ι_X and ι_Y are morphisms of pointed sets. □

00B7 Proposition 3.3.1.2. Let (X, x_0) and (Y, y_0) be pointed sets.

00B8 1. *Functoriality.* The assignments

$$(X, x_0), (Y, y_0), ((X, x_0), (Y, y_0)) \mapsto (X \vee Y, p_0)$$

define functors

$$\begin{aligned}
 X \vee - &: \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*, \\
 - \vee Y &: \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*, \\
 -_1 \vee -_2 &: \mathbf{Sets}_* \times \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*.
 \end{aligned}$$

00B9 2. *Associativity.* We have an isomorphism of pointed sets

$$(X \vee Y) \vee Z \cong X \vee (Y \vee Z),$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathbf{Sets}_*$.

00BA 3. *Unitality*. We have isomorphisms of pointed sets

$$\begin{aligned} (\text{pt}, *) \vee (X, x_0) &\cong (X, x_0), \\ (X, x_0) \vee (\text{pt}, *) &\cong (X, x_0), \end{aligned}$$

natural in $(X, x_0) \in \mathbf{Sets}_*$.

00BB 4. *Commutativity*. We have an isomorphism of pointed sets

$$X \vee Y \cong Y \vee X,$$

natural in $(X, x_0), (Y, y_0) \in \mathbf{Sets}_*$.

00BC 5. *Symmetric Monoidality*. The triple $(\mathbf{Sets}_*, \vee, \text{pt})$ is a symmetric monoidal category.

00BD 6. *The Fold Map*. We have a natural transformation

$$\nabla: \vee \circ \Delta_{\mathbf{Sets}_*}^{\text{Cats}} \Rightarrow \text{id}_{\mathbf{Sets}_*},$$

called the **fold map**, whose component

$$\nabla_X: X \vee X \rightarrow X$$

at X is given by

$$\nabla_X(p) \stackrel{\text{def}}{=} \begin{cases} x & \text{if } p = [(0, x)], \\ x & \text{if } p = [(1, x)] \end{cases}$$

for each $p \in X \vee X$.

Proof. **Item 1, Functoriality:** This follows from ??, ?? of ??.

Item 2, Associativity: Clear.

Item 3, Unitality: Clear.

Item 4, Commutativity: Clear.

Item 5, Symmetric Monoidality: Omitted.

Item 6, The Fold Map: Naturality for the transformation ∇ is the statement that, given a morphism of pointed sets $f: (X, x_0) \rightarrow (Y, y_0)$, we have

$$\nabla_Y \circ (f \vee f) = f \circ \nabla_X,$$

$$\begin{array}{ccc} X \vee X & \xrightarrow{\nabla_X} & X \\ f \vee f \downarrow & & \downarrow f \\ Y \vee Y & \xrightarrow{\nabla_Y} & Y. \end{array}$$

Indeed, we have

$$\begin{aligned} [\nabla_Y \circ (f \vee f)]([i, x]) &= \nabla_Y([i, f(x)]) \\ &= f(x) \\ &= f(\nabla_X([i, x])) \\ &= [f \circ \nabla_X]([i, x]) \end{aligned}$$

for each $[i, x] \in X \vee X$, and thus ∇ is indeed a natural transformation. \square

00BE 3.4 Pushouts

Let (X, x_0) , (Y, y_0) , and (Z, z_0) be pointed sets and let $f: (Z, z_0) \rightarrow (X, x_0)$ and $g: (Z, z_0) \rightarrow (Y, y_0)$ be morphisms of pointed sets.

00BF Definition 3.4.1.1. The **pushout of (X, x_0) and (Y, y_0) over (Z, z_0) along (f, g)** is the pair consisting of:

- *The Colimit.* The pointed set $(X \coprod_{f, Z, g} Y, p_0)$, where:
 - The set $X \coprod_{f, Z, g} Y$ is the pushout (of unpointed sets) of X and Y over Z with respect to f and g ;
 - We have $p_0 = [x_0] = [y_0]$.

- *The Cocone.* The morphisms of pointed sets

$$\begin{aligned} \text{inj}_1: (X, x_0) &\rightarrow (X \coprod_Z Y, p_0), \\ \text{inj}_2: (Y, y_0) &\rightarrow (X \coprod_Z Y, p_0) \end{aligned}$$

given by

$$\begin{aligned} \text{inj}_1(x) &\stackrel{\text{def}}{=} [(0, x)] \\ \text{inj}_2(y) &\stackrel{\text{def}}{=} [(1, y)] \end{aligned}$$

for each $x \in X$ and each $y \in Y$.

Proof. Firstly, we note that indeed $[x_0] = [y_0]$, as we have

$$\begin{aligned} x_0 &= f(z_0), \\ y_0 &= g(z_0) \end{aligned}$$

since f and g are morphisms of pointed sets, with the relation \sim on $X \amalg_Z Y$ then identifying $x_0 = f(z_0) \sim g(z_0) = y_0$.

We now claim that $(X \amalg_Z Y, p_0)$ is the categorical pushout of (X, x_0) and (Y, y_0) over (Z, z_0) with respect to (f, g) in \mathbf{Sets}_* . First we need to check that the relevant pushout diagram commutes, i.e. that we have

$$\begin{array}{ccc} (X \amalg_Z Y, p_0) & \xleftarrow{\text{inj}_2} & (Y, y_0) \\ \text{inj}_1 \uparrow & & \uparrow g \\ (X, x_0) & \xleftarrow{f} & (Z, z_0). \end{array}$$

$\text{inj}_1 \circ f = \text{inj}_2 \circ g,$

Indeed, given $z \in Z$, we have

$$\begin{aligned} [\text{inj}_1 \circ f](z) &= \text{inj}_1(f(z)) \\ &= [(0, f(z))] \\ &= [(1, g(z))] \\ &= \text{inj}_2(g(z)) \\ &= [\text{inj}_2 \circ g](z), \end{aligned}$$

where $[(0, f(z))] = [(1, g(z))]$ by the definition of the relation \sim on $X \amalg Y$ (the coproduct of unpointed sets of X and Y). Next, we prove that $X \amalg_Z Y$ satisfies the universal property of the pushout. Suppose we have a diagram of the form

$$\begin{array}{ccccc} & & & & (P, *) \\ & & & \nwarrow \iota_2 & \\ & & (X \amalg_Z Y, p_0) & \xleftarrow{\text{inj}_2} & (Y, y_0) \\ & & \uparrow \text{inj}_1 & \uparrow g & \\ & & (X, x_0) & \xleftarrow{f} & (Z, z_0) \\ & \nearrow \iota_1 & & & \end{array}$$

in \mathbf{Sets}_* . Then there exists a unique morphism of pointed sets

$$\phi: (X \amalg_Z Y, p_0) \rightarrow (P, *)$$

making the diagram

$$\begin{array}{ccc}
 (P, *) & \xleftarrow{\iota_2} & (X \amalg_Z Y, p_0) \xleftarrow{\text{inj}_2} (Y, y_0) \\
 \uparrow \iota_1 & \nwarrow \phi \quad \exists! & \uparrow g \\
 (X, x_0) & \xleftarrow{f} & (Z, z_0) \\
 & & \uparrow \text{inj}_1
 \end{array}$$

commute, being uniquely determined by the conditions

$$\phi \circ \text{inj}_1 = \iota_1,$$

$$\phi \circ \text{inj}_2 = \iota_2$$

via

$$\phi(p) = \begin{cases} \iota_1(x) & \text{if } x = [(0, x)], \\ \iota_2(y) & \text{if } x = [(1, y)] \end{cases}$$

for each $p \in X \amalg_Z Y$, where the well-definedness of ϕ is proven in the same way as in the proof of **Constructions With Sets, Definition 2.4.1.1**. Finally, we show that ϕ is indeed a morphism of pointed sets, as we have

$$\begin{aligned}
 \phi(p_0) &= \phi([(0, x_0)]) \\
 &= \iota_1(x_0) \\
 &= *,
 \end{aligned}$$

or alternatively

$$\begin{aligned}
 \phi(p_0) &= \phi([(1, y_0)]) \\
 &= \iota_2(y_0) \\
 &= *,
 \end{aligned}$$

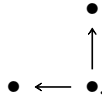
where we use that ι_1 (resp. ι_2) is a morphism of pointed sets. \square

00BG Proposition 3.4.1.2. Let (X, x_0) , (Y, y_0) , (Z, z_0) , and (A, a_0) be pointed sets.

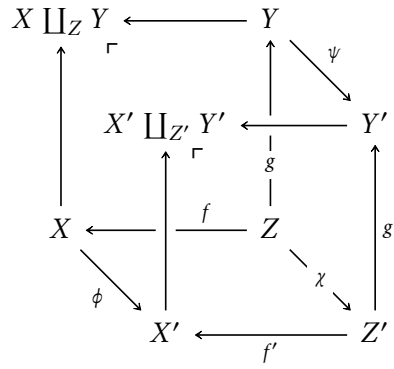
00BH 1. *Functoriality.* The assignment $(X, Y, Z, f, g) \mapsto X \amalg_{f,Z,g} Y$ defines a functor

$$-_1 \amalg_{-3} -_1 : \text{Fun}(\mathcal{P}, \text{Sets}) \rightarrow \text{Sets}_*,$$

where \mathcal{P} is the category that looks like this:



In particular, the action on morphisms of $-_1 \amalg_{-3} -_1$ is given by sending a morphism



in $\text{Fun}(\mathcal{P}, \text{Sets}_*)$ to the morphism of pointed sets

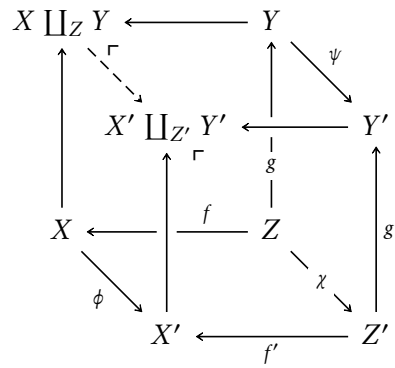
$$\xi : (X \amalg_Z Y, p_0) \xrightarrow{\exists!} (X' \amalg_{Z'} Y', p'_0)$$

given by

$$\xi(p) \stackrel{\text{def}}{=} \begin{cases} \phi(x) & \text{if } p = [(0, x)], \\ \psi(y) & \text{if } p = [(1, y)] \end{cases}$$

for each $p \in X \amalg_Z Y$, which is the unique morphism of pointed sets

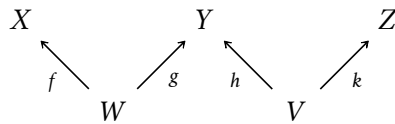
making the diagram



commute.

00BJ

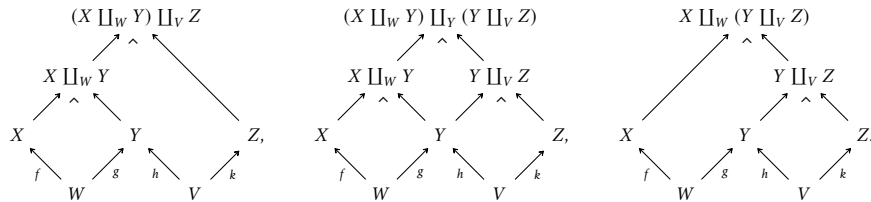
2. *Associativity.* Given a diagram



in Sets, we have isomorphisms of pointed sets

$$(X \amalg_W Y) \amalg_V Z \cong (X \amalg_W Y) \amalg_Y (Y \amalg_V Z) \cong X \amalg_W (Y \amalg_V Z),$$

where these pullbacks are built as in the diagrams



00BK

3. *Unitality.* We have isomorphisms of sets

$$\begin{array}{ccc} A & \xlongequal{\quad} & A \\ \uparrow f & \lrcorner & \uparrow f \\ X & \xlongequal{\quad} & X \end{array} \quad \begin{array}{l} X \amalg_X A \cong A, \\ A \amalg_X X \cong A, \end{array} \quad \begin{array}{ccc} A & \xleftarrow{f} & X \\ \parallel & \lrcorner & \parallel \\ X & \xleftarrow{f} & X. \end{array}$$

00BL 4. *Commutativity.* We have an isomorphism of sets

$$\begin{array}{ccc}
 X \amalg_Z Y & \longleftarrow & Y \\
 \uparrow \ulcorner & & \uparrow g \\
 X & \xleftarrow{f} & Z
 \end{array}
 \quad
 X \amalg_Z Y \cong Y \amalg_Z X
 \quad
 \begin{array}{ccc}
 Y \amalg_Z X & \longleftarrow & X \\
 \uparrow \ulcorner & & \uparrow f \\
 Y & \xleftarrow{g} & Z
 \end{array}$$

00BM 5. *Interaction With Coproducts.* We have

$$X \amalg_{\text{pt}} Y \cong X \vee Y,
 \quad
 \begin{array}{ccc}
 X \vee Y & \longleftarrow & Y \\
 \uparrow \ulcorner & & \uparrow [y_0] \\
 X & \xleftarrow{[x_0]} & \text{pt.}
 \end{array}$$

00BN 6. *Symmetric Monoidality.* The triple $(\text{Sets}_*, \amalg_X, (X, x_0))$ is a symmetric monoidal category.

Proof. **Item 1, Functoriality:** This is a special case of functoriality of co/limits, ??, ?? of ??, with the explicit expression for ξ following from the commutativity of the cube pushout diagram.

Item 2, Associativity: This follows from [Constructions With Sets, Item 2 of Proposition 2.4.1.4](#).

Item 3, Unitality: This follows from [Constructions With Sets, Item 3 of Proposition 2.4.1.4](#).

Item 4, Commutativity: This follows from [Constructions With Sets, Item 4 of Proposition 2.4.1.4](#).

Item 5, Interaction With Coproducts: Clear.

Item 6, Symmetric Monoidality: Omitted. □

00BP 3.5 Coequalisers

Let $f, g: (X, x_0) \rightrightarrows (Y, y_0)$ be morphisms of pointed sets.

00BQ **Definition 3.5.1.1.** The **coequaliser of (f, g)** is the pointed set $(\text{CoEq}(f, g), [y_0])$.

Proof. We claim that $(\text{CoEq}(f, g), [y_0])$ is the categorical coequaliser of f and g

in \mathbf{Sets}_* . First we need to check that the relevant coequaliser diagram commutes, i.e. that we have

$$\text{coeq}(f, g) \circ f = \text{coeq}(f, g) \circ g.$$

Indeed, we have

$$\begin{aligned} [\text{coeq}(f, g) \circ f](x) &\stackrel{\text{def}}{=} [\text{coeq}(f, g)](f(x)) \\ &\stackrel{\text{def}}{=} [f(x)] \\ &= [g(x)] \\ &\stackrel{\text{def}}{=} [\text{coeq}(f, g)](g(x)) \\ &\stackrel{\text{def}}{=} [\text{coeq}(f, g) \circ g](x) \end{aligned}$$

for each $x \in X$. Next, we prove that $\text{CoEq}(f, g)$ satisfies the universal property of the coequaliser. Suppose we have a diagram of the form

$$\begin{array}{ccc} (X, x_0) & \xrightarrow[g]{f} & (Y, y_0) \xrightarrow{\text{coeq}(f, g)} (\text{CoEq}(f, g), [y_0]) \\ & & \searrow c \\ & & (C, *) \end{array}$$

in \mathbf{Sets} . Then, since $c(f(a)) = c(g(a))$ for each $a \in A$, it follows from [Equivalence Relations and Apartness Relations](#), [Items 4 and 5](#) of [Proposition 5.2.1.3](#) that there exists a unique map $\phi: \text{CoEq}(f, g) \xrightarrow{\exists!} C$ making the diagram

$$\begin{array}{ccc} (X, x_0) & \xrightarrow[g]{f} & (Y, y_0) \xrightarrow{\text{coeq}(f, g)} (\text{CoEq}(f, g), [y_0]) \\ & & \searrow c \quad \downarrow \phi \exists! \\ & & (C, *) \end{array}$$

commute, where we note that ϕ is indeed a morphism of pointed sets since

$$\begin{aligned} \phi([y_0]) &= [\phi \circ \text{coeq}(f, g)]([y_0]) \\ &= c([y_0]) \\ &= *, \end{aligned}$$

where we have used that c is a morphism of pointed sets. □

00BR Proposition 3.5.1.2. Let (X, x_0) and (Y, y_0) be pointed sets and let $f, g, h: (X, x_0) \rightarrow (Y, y_0)$ be morphisms of pointed sets.

00BS 1. *Associativity.* We have isomorphisms of pointed sets

$$\underbrace{\text{CoEq}(\text{coeq}(f, g) \circ f, \text{coeq}(f, g) \circ h)}_{=\text{CoEq}(\text{coeq}(f, g) \circ g, \text{coeq}(f, g) \circ h)} \cong \text{CoEq}(f, g, h) \cong \underbrace{\text{CoEq}(\text{coeq}(g, h) \circ f, \text{coeq}(g, h) \circ g)}_{=\text{CoEq}(\text{coeq}(g, h) \circ f, \text{coeq}(g, h) \circ h)},$$

where $\text{CoEq}(f, g, h)$ is the colimit of the diagram

$$(X, x_0) \begin{array}{c} \xrightarrow{f} \\ \xrightarrow[-g]{h} \end{array} (Y, y_0)$$

in Sets_* .

00BT 2. *Unitality.* We have an isomorphism of pointed sets

$$\text{CoEq}(f, f) \cong B.$$

00BU 3. *Commutativity.* We have an isomorphism of pointed sets

$$\text{CoEq}(f, g) \cong \text{CoEq}(g, f).$$

Proof. **Item 1, Associativity:** This follows from **Constructions With Sets, Item 1** of **Proposition 2.5.1.4**.

Item 2, Unitality: This follows from **Constructions With Sets, Item 4** of **Proposition 2.5.1.4**.

Item 3, Commutativity: This follows from **Constructions With Sets, Item 5** of **Proposition 2.5.1.4**. \square

00BV 4 **Constructions With Pointed Sets**

00BW 4.1 **Free Pointed Sets**

Let X be a set.

00BX Definition 4.1.1.1. The **free pointed set on X** is the pointed set X^+ consisting of:

- *The Underlying Set.* The set X^+ defined by¹¹

$$\begin{aligned} X^+ &\stackrel{\text{def}}{=} X \amalg \text{pt} \\ &\stackrel{\text{def}}{=} X \amalg \{\star\}. \end{aligned}$$

- *The Basepoint.* The element \star of X^+ .

00BY Proposition 4.1.1.2. Let X be a set.

00BZ 1. *Functoriality.* The assignment $X \mapsto X^+$ defines a functor

$$(-)^+ : \mathbf{Sets} \rightarrow \mathbf{Sets}_*,$$

where

- *Action on Objects.* For each $X \in \mathbf{Obj}(\mathbf{Sets})$, we have

$$[(-)^+](X) \stackrel{\text{def}}{=} X^+,$$

where X^+ is the pointed set of **Definition 4.1.1.1**;

- *Action on Morphisms.* For each morphism $f: X \rightarrow Y$ of \mathbf{Sets} , the image

$$f^+ : X^+ \rightarrow Y^+$$

of f by $(-)^+$ is the map of pointed sets defined by

$$f^+(x) \stackrel{\text{def}}{=} \begin{cases} f(x) & \text{if } x \in X, \\ \star_Y & \text{if } x = \star_X. \end{cases}$$

00C0 2. *Adjointness.* We have an adjunction

$$((-)^+ \dashv \text{忘}): \mathbf{Sets} \begin{matrix} \xrightarrow{(-)^+} \\ \perp \\ \xleftarrow{\text{忘}} \end{matrix} \mathbf{Sets}_*,$$

witnessed by a bijection of sets

$$\mathbf{Sets}_*((X^+, \star_X), (Y, y_0)) \cong \mathbf{Sets}(X, Y),$$

natural in $X \in \mathbf{Obj}(\mathbf{Sets})$ and $(Y, y_0) \in \mathbf{Obj}(\mathbf{Sets}_*)$.

¹¹ *Further Notation:* We sometimes write \star_X for the basepoint of X^+ for clarity when there are

- 00C1** 3. *Symmetric Strong Monoidality With Respect to Wedge Sums.* The free pointed set functor of **Item 1** has a symmetric strong monoidal structure

$$\left((-)^+, (-)^+, \amalg, (-)^+_{\mathbb{1}} \right): (\mathbf{Sets}, \amalg, \emptyset) \rightarrow (\mathbf{Sets}_*, \vee, \text{pt}),$$

being equipped with isomorphisms of pointed sets

$$\begin{aligned} (-)^+_{X,Y} \amalg: X^+ \vee Y^+ &\xrightarrow{\cong} (X \amalg Y)^+, \\ (-)^+_{\mathbb{1}} \amalg: \text{pt} &\xrightarrow{\cong} \emptyset^+, \end{aligned}$$

natural in $X, Y \in \text{Obj}(\mathbf{Sets})$.

- 00C2** 4. *Symmetric Strong Monoidality With Respect to Smash Products.* The free pointed set functor of **Item 1** has a symmetric strong monoidal structure

$$\left((-)^+, (-)^+, \times, (-)^+_{\mathbb{1}} \right): (\mathbf{Sets}, \times, \text{pt}) \rightarrow (\mathbf{Sets}_*, \wedge, S^0),$$

being equipped with isomorphisms of pointed sets

$$\begin{aligned} (-)^+_{X,Y} \times: X^+ \wedge Y^+ &\xrightarrow{\cong} (X \times Y)^+, \\ (-)^+_{\mathbb{1}} \times: S^0 &\xrightarrow{\cong} \text{pt}^+, \end{aligned}$$

natural in $X, Y \in \text{Obj}(\mathbf{Sets})$.

Proof. **Item 1**, *Functoriality*: Clear.

Item 2, *Adjointness*: We claim there's an adjunction $(-)^+ \dashv \bar{}$, witnessed by a bijection of sets

$$\mathbf{Sets}_*((X^+, \star_X), (Y, y_0)) \cong \mathbf{Sets}(X, Y),$$

natural in $X \in \text{Obj}(\mathbf{Sets})$ and $(Y, y_0) \in \text{Obj}(\mathbf{Sets}_*)$.

• *Map I.* We define a map

$$\Phi_{X,Y}: \mathbf{Sets}_*((X^+, \star_X), (Y, y_0)) \rightarrow \mathbf{Sets}(X, Y)$$

by sending a pointed function

$$\xi: (X^+, \star_X) \rightarrow (Y, y_0)$$

to the function

$$\xi^\dagger: X \rightarrow Y$$

given by

$$\xi^\dagger(x) \stackrel{\text{def}}{=} \xi(x)$$

for each $x \in X$.

· *Map II.* We define a map

$$\Psi_{X,Y}: \text{Sets}(X, Y) \rightarrow \text{Sets}_*((X^+, \star_X), (Y, y_0))$$

given by sending a function $\xi: X \rightarrow Y$ to the pointed function

$$\xi^\dagger: (X^+, \star_X) \rightarrow (Y, y_0)$$

defined by

$$\xi^\dagger(x) \stackrel{\text{def}}{=} \begin{cases} \xi(x) & \text{if } x \in X, \\ y_0 & \text{if } x = \star_X \end{cases}$$

for each $x \in X^+$.

· *Invertibility I.* We claim that

$$\Psi_{X,Y} \circ \Phi_{X,Y} = \text{id}_{\text{Sets}_*((X^+, \star_X), (Y, y_0))},$$

which is clear.

· *Invertibility II.* We claim that

$$\Phi_{X,Y} \circ \Psi_{X,Y} = \text{id}_{\text{Sets}(X, Y)},$$

which is clear.

· *Naturality for Φ , Part I.* We need to show that, given a pointed function $g: (Y, y_0) \rightarrow (Y', y'_0)$, the diagram

$$\begin{array}{ccc} \text{Sets}_*((X^+, \star_X), (Y, y_0)) & \xrightarrow{\Phi_{X,Y}} & \text{Sets}(X, Y) \\ g_* \downarrow & & \downarrow g_* \\ \text{Sets}_*((X^+, \star_X), (Y', y'_0)) & \xrightarrow{\Phi_{X,Y'}} & \text{Sets}(X, Y') \end{array}$$

commutes. Indeed, given a pointed function

$$\xi^\dagger: (X^+, \star_X) \rightarrow (Y, y_0)$$

we have

$$\begin{aligned} [\Phi_{X,Y'} \circ g_*](\xi) &= \Phi_{X,Y'}(g_*(\xi)) \\ &= \Phi_{X,Y'}(g \circ \xi) \\ &= g \circ \xi \\ &= g \circ \Phi_{X,Y'}(\xi) \\ &= g_*(\Phi_{X,Y'}(\xi)) \\ &= [g_* \circ \Phi_{X,Y'}](\xi). \end{aligned}$$

- *Naturality for Φ , Part II.* We need to show that, given a pointed function $f: (X, x_0) \rightarrow (X', x'_0)$, the diagram

$$\begin{array}{ccc} \text{Sets}_*\left((X'^+, \star_{X'}), (Y, y_0)\right) & \xrightarrow{\Phi_{X',Y}} & \text{Sets}(X', Y) \\ f^* \downarrow & & \downarrow f^* \\ \text{Sets}_*((X^+, \star_X), (Y, y_0)) & \xrightarrow{\Phi_{X,Y}} & \text{Sets}(X, Y) \end{array}$$

commutes. Indeed, given a function

$$\xi: X' \rightarrow Y,$$

we have

$$\begin{aligned} [\Phi_{X,Y} \circ f^*](\xi) &= \Phi_{X,Y}(f^*(\xi)) \\ &= \Phi_{X,Y}(\xi \circ f) \\ &= \xi \circ f \\ &= \Phi_{X',Y}(\xi) \circ f \\ &= f^*(\Phi_{X',Y}(\xi)) \\ &= f^*(\Phi_{X',Y}(\xi)) \\ &= [f^* \circ \Phi_{X',Y}](\xi). \end{aligned}$$

- *Naturality for Ψ* . Since Φ is natural in each argument and Φ is a component-wise inverse to Ψ in each argument, it follows from [Categories, Item 2 of Proposition 8.6.1.2](#) that Ψ is also natural in each argument.

Item 3, Symmetric Strong Monoidality With Respect to Wedge Sums: The isomorphism

$$\phi: X^+ \vee Y^+ \xrightarrow{\cong} (X \amalg Y)^+$$

is given by

$$\phi(z) = \begin{cases} x & \text{if } z = [(0, x)] \text{ with } x \in X, \\ y & \text{if } z = [(1, y)] \text{ with } y \in Y, \\ \star_X \amalg_Y & \text{if } z = [(0, \star_X)], \\ \star_X \amalg_Y & \text{if } z = [(1, \star_Y)] \end{cases}$$

for each $z \in X^+ \vee Y^+$, with inverse

$$\phi^{-1}: (X \amalg Y)^+ \xrightarrow{\cong} X^+ \vee Y^+$$

given by

$$\phi^{-1}(z) \stackrel{\text{def}}{=} \begin{cases} [(0, x)] & \text{if } z = [(0, x)], \\ [(0, y)] & \text{if } z = [(1, y)], \\ p_0 & \text{if } z = \star_X \amalg_Y \end{cases}$$

for each $z \in (X \amalg Y)^+$.

Meanwhile, the isomorphism $\text{pt} \cong \emptyset^+$ is given by sending \star_X to \star_\emptyset .

That these isomorphisms satisfy the coherence conditions making the functor $(-)^+$ symmetric strong monoidal can be directly checked element by element.

Item 4, Symmetric Strong Monoidality With Respect to Smash Products: The isomorphism

$$\phi: X^+ \wedge Y^+ \xrightarrow{\cong} (X \times Y)^+$$

is given by

$$\phi(x \wedge y) = \begin{cases} (x, y) & \text{if } x \neq \star_X \text{ and } y \neq \star_Y \\ \star_{X \times Y} & \text{otherwise} \end{cases}$$

for each $x \wedge y \in X^+ \wedge Y^+$, with inverse

$$\phi^{-1}: (X \times Y)^+ \xrightarrow{\cong} X^+ \wedge Y^+$$

given by

$$\phi^{-1}(z) \stackrel{\text{def}}{=} \begin{cases} x \wedge y & \text{if } z = (x, y) \text{ with } (x, y) \in X \times Y, \\ \star_X \wedge \star_Y & \text{if } z = \star_{X \times Y}, \end{cases}$$

for each $z \in (X \amalg Y)^+$.

Meanwhile, the isomorphism $S^0 \cong \text{pt}^+$ is given by sending \star to $1 \in S^0 = \{0, 1\}$ and \star_{pt} to $0 \in S^0$.

That these isomorphisms satisfy the coherence conditions making the functor $(-)^+$ symmetric strong monoidal can be directly checked element by element.

□

Appendices

A Other Chapters

Sets

1. [Sets](#)
2. [Constructions With Sets](#)
3. [Pointed Sets](#)
4. [Tensor Products of Pointed Sets](#)

6. [Constructions With Relations](#)

7. [Equivalence Relations and Apartness Relations](#)

Category Theory

8. [Categories](#)

Bicategories

Relations

5. [Relations](#)

9. [Types of Morphisms in Bicategories](#)

References

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multiple free pointed sets involved in the current discussion.

- [MSE 884460] **Martin Brandenburg**. *Why are the category of pointed sets and the category of sets and partial functions “essentially the same”?* Mathematics Stack Exchange. URL: <https://math.stackexchange.com/q/884460> (cit. on p. 7).