Types of Morphisms in Bicategories

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In this chapter, we study special kinds of morphisms in bicategories:

1. *Monomorphisms and Epimorphisms in Bicategories (Sections 1 and 2).* There is a large number of different notions capturing the idea of a "monomorphism" or of an "epimorphism" in a bicategory.

   Arguably, the notion that best captures these concepts is that of a *pseudomonic morphism* (Definition 1.10.1.1) and of a *pseudoepic morphism* (Definition 2.10.1.1), although the other notions introduced in Sections 1 and 2 are also interesting on their own.

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1 Monomorphisms in Bicategories

1.1 Representably Faithful Morphisms

Let C be a bicategory.

Definition 1.1.1.1. A 1-morphism $f\colon A \to B$ of C is **representably faithful**[1] if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f_*\colon \mathsf{Hom}_C(X, A) \to \mathsf{Hom}_C(X, B)$$

given by postcomposition by f is faithful.

Remark 1.1.1.2. In detail, f is representably faithful if, for all diagrams in C of the form

$$\begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & A \arrow[r, "f"] & B, \end{tikzcd}$$

if we have

$$\mathrm{id}_f \star \alpha = \mathrm{id}_f \star \beta,$$

then $\alpha = \beta$.

Example 1.1.1.3. Here are some examples of representably faithful morphisms.

1. *Representably Faithful Morphisms in* Cats_2*.* The representably faithful morphisms in Cats_2 are precisely the faithful functors; see Categories, Item 1 of Proposition 5.1.1.2.

2. *Representably Faithful Morphisms in* **Rel***.* Every morphism of **Rel** is representably faithful; see Relations, Item 1 of Proposition 3.8.1.1.

[1] *Further Terminology:* Also called simply a **faithful morphism**, based on Item 1 of

1.2 Representably Full Morphisms

Let $\mathcal{C}$ be a bicategory.

Definition 1.2.1.1. A 1-morphism $f\colon A \to B$ of $\mathcal{C}$ is **representably full**[2] if, for each $X \in \mathrm{Obj}(\mathcal{C})$, the functor

$$f_*\colon \mathsf{Hom}_{\mathcal{C}}(X, A) \to \mathsf{Hom}_{\mathcal{C}}(X, B)$$

given by postcomposition by f is full.

Remark 1.2.1.2. In detail, f is representably full if, for each $X \in \mathrm{Obj}(\mathcal{C})$ and each 2-morphism

$$\beta\colon f \circ \phi \Longrightarrow f \circ \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"] \arrow[r, bend right, "f\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & B \end{tikzcd}$$

of $\mathcal{C}$, there exists a 2-morphism

$$\alpha\colon \phi \Longrightarrow \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & A \end{tikzcd}$$

of $\mathcal{C}$ such that we have an equality

$$\begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & A \arrow[r, "f"] & B \end{tikzcd} \quad = \quad \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"] \arrow[r, bend right, "f\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & B \end{tikzcd}$$

of pasting diagrams in $\mathcal{C}$, i.e. such that we have

$$\beta = \mathrm{id}_f \star \alpha.$$

Example 1.2.1.3. Here are some examples of representably full morphisms.

1. *Representably Full Morphisms in* Cats_2. The representably full morphisms in Cats_2 are precisely the full functors; see Categories, Item 1 of Proposition 5.2.1.2.

2. *Representably Full Morphisms in* $\mathbf{Rel}$. The representably full morphisms in $\mathbf{Rel}$ are characterised in Relations, Item 2 of Proposition 3.8.1.1.

Example 1.1.1.3.

[2] *Further Terminology:* Also called simply a **full morphism**, based on Item 1 of

1.3 Representably Fully Faithful Morphisms

Let $\mathcal{C}$ be a bicategory.

Definition 1.3.1.1. A 1-morphism $f\colon A \to B$ of $\mathcal{C}$ is **representably fully faithful**[3] if the following equivalent conditions are satisfied:

1. The 1-morphism f is representably faithful (Definition 1.1.1.1) and representably full (Definition 1.2.1.1).
2. For each $X \in \mathrm{Obj}(\mathcal{C})$, the functor
$$f_*\colon \mathsf{Hom}_{\mathcal{C}}(X, A) \to \mathsf{Hom}_{\mathcal{C}}(X, B)$$
given by postcomposition by f is fully faithful.

Remark 1.3.1.2. In detail, f is representably fully faithful if the conditions in Remark 1.1.1.2 and Remark 1.2.1.2 hold:

1. For all diagrams in $\mathcal{C}$ of the form
$$\begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & A \arrow[r, "f"] & B, \end{tikzcd}$$
if we have
$$\mathrm{id}_f \star \alpha = \mathrm{id}_f \star \beta,$$
then $\alpha = \beta$.
2. For each $X \in \mathrm{Obj}(\mathcal{C})$ and each 2-morphism
$$\beta\colon f \circ \phi \Longrightarrow f \circ \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"] \arrow[r, bend right, "f\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & B \end{tikzcd}$$
of $\mathcal{C}$, there exists a 2-morphism
$$\alpha\colon \phi \Longrightarrow \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & A \end{tikzcd}$$

Example 1.2.1.3.

[3] *Further Terminology:* Also called simply a **fully faithful morphism**, based on Item 1

of C such that we have an equality

$$X \overset{\phi}{\underset{\psi}{\overset{\alpha\Downarrow}{\rightrightarrows}}} A \xrightarrow{f} B \quad = \quad X \overset{f\circ\phi}{\underset{f\circ\psi}{\overset{\beta\Downarrow}{\rightrightarrows}}} B$$

of pasting diagrams in C, i.e. such that we have

$$\beta = \mathrm{id}_f \star \alpha.$$

Example 1.3.1.3. Here are some examples of representably fully faithful morphisms.

1. *Representably Fully Faithful Morphisms in* Cats_2*.* The representably fully faithful morphisms in Cats_2 are precisely the fully faithful functors; see Categories, Item 5 of Proposition 5.3.1.2.

2. *Representably Fully Faithful Morphisms in* $\mathbf{Rel}$*.* The representably fully faithful morphisms of $\mathbf{Rel}$ coincide (Relations, Item 3 of Proposition 3.8.1.1) with the representably full morphisms in $\mathbf{Rel}$, which are characterised in Relations, Item 2 of Proposition 3.8.1.1.

1.4 Morphisms Representably Faithful on Cores

Let C be a bicategory.

Definition 1.4.1.1. A 1-morphism $f\colon A \to B$ of C is **representably faithful on cores** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f_*\colon \mathsf{Core}(\mathsf{Hom}_C(X, A)) \to \mathsf{Core}(\mathsf{Hom}_C(X, B))$$

given by postcomposition by f is faithful.

Remark 1.4.1.2. In detail, f is representably faithful on cores if, for all diagrams in C of the form

$$X \overset{\phi}{\underset{\psi}{\overset{\alpha\Downarrow\Downarrow\beta}{\rightrightarrows}}} A \xrightarrow{f} B,$$

if α and β are 2-isomorphisms and we have

$$\mathrm{id}_f \star \alpha = \mathrm{id}_f \star \beta,$$

then $\alpha = \beta$.

1.5 Morphisms Representably Full on Cores

Let $\mathcal{C}$ be a bicategory.

Definition 1.5.1.1. A 1-morphism $f\colon A \to B$ of $\mathcal{C}$ is **representably full on cores** if, for each $X \in \mathrm{Obj}(\mathcal{C})$, the functor

$$f_*\colon \mathsf{Core}(\mathsf{Hom}_{\mathcal{C}}(X, A)) \to \mathsf{Core}(\mathsf{Hom}_{\mathcal{C}}(X, B))$$

given by postcomposition by f is full.

Remark 1.5.1.2. In detail, f is representably full on cores if, for each $X \in \mathrm{Obj}(\mathcal{C})$ and each 2-isomorphism

$$\beta\colon f \circ \phi \overset{\sim}{\Longrightarrow} f \circ \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"{name=U}] \arrow[r, bend right, "f\circ\psi"'{name=D}] & B \arrow[from=U, to=D, phantom, "\beta\Downarrow"] \end{tikzcd}$$

of $\mathcal{C}$, there exists a 2-isomorphism

$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "\phi"{name=U}] \arrow[r, bend right, "\psi"'{name=D}] & A \arrow[from=U, to=D, phantom, "\alpha\Downarrow"] \end{tikzcd}$$

of $\mathcal{C}$ such that we have an equality

$$\begin{tikzcd} X \arrow[r, bend left, "\phi"{name=U}] \arrow[r, bend right, "\psi"'{name=D}] & A \arrow[r, "f"] & B \arrow[from=U, to=D, phantom, "\alpha\Downarrow"] \end{tikzcd} = \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"{name=U}] \arrow[r, bend right, "f\circ\psi"'{name=D}] & B \arrow[from=U, to=D, phantom, "\beta\Downarrow"] \end{tikzcd}$$

of pasting diagrams in $\mathcal{C}$, i.e. such that we have

$$\beta = \mathrm{id}_f \star \alpha.$$

1.6 Morphisms Representably Fully Faithful on Cores

Let $\mathcal{C}$ be a bicategory.

Definition 1.6.1.1. A 1-morphism $f\colon A \to B$ of $\mathcal{C}$ is **representably fully faithful on cores** if the following equivalent conditions are satisfied:

1. The 1-morphism f is representably faithful on cores (Definition 1.5.1.1) and representably full on cores (Definition 1.4.1.1).

2. For each $X \in \mathrm{Obj}(\mathcal{C})$, the functor

$$f_* \colon \mathsf{Core}(\mathsf{Hom}_{\mathcal{C}}(X, A)) \to \mathsf{Core}(\mathsf{Hom}_{\mathcal{C}}(X, B))$$

given by postcomposition by f is fully faithful.

Remark 1.6.1.2. In detail, f is representably fully faithful on cores if the conditions in Remark 1.4.1.2 and Remark 1.5.1.2 hold:

1. For all diagrams in $\mathcal{C}$ of the form

$$\begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & A \arrow[r, "f"] & B, \end{tikzcd}$$

if α and β are 2-isomorphisms and we have

$$\mathrm{id}_f \star \alpha = \mathrm{id}_f \star \beta,$$

then $\alpha = \beta$.

2. For each $X \in \mathrm{Obj}(\mathcal{C})$ and each 2-isomorphism

$$\beta \colon f \circ \phi \overset{\sim}{\Longrightarrow} f \circ \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"] \arrow[r, bend right, "f\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & B \end{tikzcd}$$

of $\mathcal{C}$, there exists a 2-isomorphism

$$\alpha \colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & A \end{tikzcd}$$

of $\mathcal{C}$ such that we have an equality

$$\begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & A \arrow[r, "f"] & B \end{tikzcd} = \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"] \arrow[r, bend right, "f\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & B \end{tikzcd}$$

of pasting diagrams in $\mathcal{C}$, i.e. such that we have

$$\beta = \mathrm{id}_f \star \alpha.$$

1.7 Representably Essentially Injective Morphisms

Let $\mathcal{C}$ be a bicategory.

Definition 1.7.1.1. A 1-morphism $f\colon A \to B$ of $\mathcal{C}$ is **representably essentially injective** if, for each $X \in \mathrm{Obj}(\mathcal{C})$, the functor

$$f_*\colon \mathsf{Hom}_{\mathcal{C}}(X, A) \to \mathsf{Hom}_{\mathcal{C}}(X, B)$$

given by postcomposition by f is essentially injective.

Remark 1.7.1.2. In detail, f is representably essentially injective if, for each pair of morphisms $\phi, \psi\colon X \rightrightarrows A$ of $\mathcal{C}$, the following condition is satisfied:

($\star$) If $f \circ \phi \cong f \circ \psi$, then $\phi \cong \psi$.

1.8 Representably Conservative Morphisms

Let $\mathcal{C}$ be a bicategory.

Definition 1.8.1.1. A 1-morphism $f\colon A \to B$ of $\mathcal{C}$ is **representably conservative** if, for each $X \in \mathrm{Obj}(\mathcal{C})$, the functor

$$f_*\colon \mathsf{Hom}_{\mathcal{C}}(X, A) \to \mathsf{Hom}_{\mathcal{C}}(X, B)$$

given by postcomposition by f is conservative.

Remark 1.8.1.2. In detail, f is representably conservative if, for each pair of morphisms $\phi, \psi\colon X \rightrightarrows A$ and each 2-morphism

$$\alpha\colon \phi \Longrightarrow \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "\phi"{name=U}] \arrow[r, bend right, "\psi"'{name=D}] & A \arrow[Rightarrow, from=U, to=D, "\alpha"'] \end{tikzcd}$$

of $\mathcal{C}$, if the 2-morphism

$$\mathrm{id}_f \star \alpha\colon f \circ \phi \Longrightarrow f \circ \psi, \qquad \begin{tikzcd} X \arrow[rr, bend left, "f\circ\phi"{name=U}] \arrow[rr, bend right, "f\circ\psi"'{name=D}] & & B \arrow[Rightarrow, from=U, to=D, "\mathrm{id}_f\star\alpha"'] \end{tikzcd}$$

is a 2-isomorphism, then so is α.

of Example 1.3.1.3.

1.9 Strict Monomorphisms

Let $\mathcal{C}$ be a bicategory.

Definition 1.9.1.1. A 1-morphism $f\colon A \to B$ of $\mathcal{C}$ is a **strict monomorphism** if, for each $X \in \mathrm{Obj}(\mathcal{C})$, the functor

$$f_*\colon \mathsf{Hom}_{\mathcal{C}}(X, A) \to \mathsf{Hom}_{\mathcal{C}}(X, B)$$

given by postcomposition by f is injective on objects, i.e. its action on objects

$$f_*\colon \mathrm{Obj}(\mathsf{Hom}_{\mathcal{C}}(X, A)) \to \mathrm{Obj}(\mathsf{Hom}_{\mathcal{C}}(X, B))$$

is injective.

Remark 1.9.1.2. In detail, f is a strict monomorphism in $\mathcal{C}$ if, for each diagram in $\mathcal{C}$ of the form

$$X \underset{\psi}{\overset{\phi}{\rightrightarrows}} A \xrightarrow{f} B,$$

if $f \circ \phi = f \circ \psi$, then $\phi = \psi$.

Example 1.9.1.3. Here are some examples of strict monomorphisms.

1. *Strict Monomorphisms in* Cats_2. The strict monomorphisms in Cats_2 are precisely the functors which are injective on objects and injective on morphisms; see Categories, Item 1 of Proposition 6.2.1.2.

2. *Strict Monomorphisms in* Rel. The strict monomorphisms in Rel are characterised in Relations, Proposition 3.7.1.1.

1.10 Pseudomonic Morphisms

Let $\mathcal{C}$ be a bicategory.

Definition 1.10.1.1. A 1-morphism $f\colon A \to B$ of $\mathcal{C}$ is **pseudomonic** if, for each $X \in \mathrm{Obj}(\mathcal{C})$, the functor

$$f_*\colon \mathsf{Hom}_{\mathcal{C}}(X, A) \to \mathsf{Hom}_{\mathcal{C}}(X, B)$$

given by postcomposition by f is pseudomonic.

Remark 1.10.1.2. In detail, a 1-morphism $f\colon A \to B$ of $\mathcal{C}$ is pseudomonic if it satisfies the following conditions:

1. For all diagrams in $\mathcal{C}$ of the form

$$\begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & A \arrow[r, "f"] & B, \end{tikzcd}$$

if we have

$$\mathrm{id}_f \star \alpha = \mathrm{id}_f \star \beta,$$

then $\alpha = \beta$.

2. For each $X \in \mathrm{Obj}(\mathcal{C})$ and each 2-isomorphism

$$\beta\colon f \circ \phi \overset{\sim}{\Longrightarrow} f \circ \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"] \arrow[r, bend right, "f\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & B \end{tikzcd}$$

of $\mathcal{C}$, there exists a 2-isomorphism

$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & A \end{tikzcd}$$

of $\mathcal{C}$ such that we have an equality

$$\begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & A \arrow[r, "f"] & B \end{tikzcd} \quad = \quad \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"] \arrow[r, bend right, "f\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & B \end{tikzcd}$$

of pasting diagrams in $\mathcal{C}$, i.e. such that we have

$$\beta = \mathrm{id}_f \star \alpha.$$

Proposition 1.10.1.3. Let $f\colon A \to B$ be a 1-morphism of $\mathcal{C}$.

1. *Characterisations.* The following conditions are equivalent:

   (a) The morphism f is pseudomonic.

   (b) The morphism f is representably full on cores and representably faithful.

(c) We have an isocomma square of the form

$$A \overset{\text{eq.}}{\cong} A \overset{\leftrightarrow}{\times}_B A, \qquad \begin{tikzcd} A \arrow[r, "\mathrm{id}_A"] \arrow[d, "\mathrm{id}_A"'] & A \arrow[d, "F"] \\ A \arrow[r, "F"'] & B \arrow[ul, phantom, "\Leftrightarrow"] \end{tikzcd}$$

in $\mathcal{C}$ up to equivalence.

2. *Interaction With Cotensors.* If $\mathcal{C}$ has cotensors with $\mathbb{1}$, then the following conditions are equivalent:

(a) The morphism f is pseudomonic.

(b) We have an isocomma square of the form

$$A \overset{\text{eq.}}{\cong} A \overset{\leftrightarrow}{\times}_{\mathbb{1} \pitchfork F} B, \qquad \begin{tikzcd} A \arrow[r, hook] \arrow[d, "F"'] & \mathbb{1} \pitchfork A \arrow[d, "\mathbb{1} \pitchfork F"] \\ B \arrow[r, hook] & \mathbb{1} \pitchfork B \arrow[ul, phantom, "\Leftrightarrow"] \end{tikzcd}$$

in $\mathcal{C}$ up to equivalence.

Proof. *Item 1, Characterisations*: Omitted.
Item 2, Interaction With Cotensors: Omitted. □

2 Epimorphisms in Bicategories

2.1 Corepresentably Faithful Morphisms

Let $\mathcal{C}$ be a bicategory.

Definition 2.1.1.1. A 1-morphism $f\colon A \to B$ of $\mathcal{C}$ is **corepresentably faithful** if, for each $X \in \mathrm{Obj}(\mathcal{C})$, the functor

$$f^*\colon \mathsf{Hom}_{\mathcal{C}}(B, X) \to \mathsf{Hom}_{\mathcal{C}}(A, X)$$

given by precomposition by f is faithful.

Remark 2.1.1.2. In detail, f is corepresentably faithful if, for all diagrams in $\mathcal{C}$ of the form

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & X, \end{tikzcd}$$

if we have

$$\alpha \star \mathrm{id}_f = \beta \star \mathrm{id}_f,$$

then $\alpha = \beta$.

Example 2.1.1.3. Here are some examples of corepresentably faithful morphisms.

1. *Corepresentably Faithful Morphisms in* Cats_2*.* The corepresentably faithful morphisms in Cats_2 are characterised in Categories, Item 4 of Proposition 5.1.1.2.

2. *Corepresentably Faithful Morphisms in* **Rel***.* Every morphism of **Rel** is corepresentably faithful; see Relations, Item 1 of Proposition 3.10.1.1.

2.2 Corepresentably Full Morphisms

Let $\mathcal{C}$ be a bicategory.

Definition 2.2.1.1. A 1-morphism $f \colon A \to B$ of $\mathcal{C}$ is **corepresentably full** if, for each $X \in \mathrm{Obj}(\mathcal{C})$, the functor

$$f^* \colon \mathsf{Hom}_{\mathcal{C}}(B, X) \to \mathsf{Hom}_{\mathcal{C}}(A, X)$$

given by precomposition by f is full.

Remark 2.2.1.2. In detail, f is corepresentably full if, for each $X \in \mathrm{Obj}(\mathcal{C})$ and each 2-morphism

$$\beta \colon \phi \circ f \Longrightarrow \psi \circ f, \qquad \begin{tikzcd} A \arrow[r, bend left, "\phi\circ f"{name=U}] \arrow[r, bend right, "\psi\circ f"'{name=D}] & X \arrow[from=U, to=D, Rightarrow, "\beta"'] \end{tikzcd}$$

of $\mathcal{C}$, there exists a 2-morphism

$$\alpha \colon \phi \Longrightarrow \psi, \qquad \begin{tikzcd} B \arrow[r, bend left, "\phi"{name=U}] \arrow[r, bend right, "\psi"'{name=D}] & X \arrow[from=U, to=D, Rightarrow, "\alpha"'] \end{tikzcd}$$

of $\mathcal{C}$ such that we have an equality

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "\phi"{name=U}] \arrow[r, bend right, "\psi"'{name=D}] & X \arrow[from=U, to=D, Rightarrow, "\alpha"'] \end{tikzcd} = \begin{tikzcd} A \arrow[r, bend left, "\phi\circ f"{name=U}] \arrow[r, bend right, "\psi\circ f"'{name=D}] & X \arrow[from=U, to=D, Rightarrow, "\beta"'] \end{tikzcd}$$

of pasting diagrams in $\mathcal{C}$, i.e. such that we have

$$\beta = \alpha \star \mathrm{id}_f.$$

Example 2.2.1.3. Here are some examples of corepresentably full morphisms.

1. *Corepresentably Full Morphisms in* Cats_2. The corepresentably full morphisms in Cats_2 are characterised in Categories, Item 5 of Proposition 5.2.1.2.

2. *Corepresentably Full Morphisms in* $\mathbf{Rel}$. The corepresentably full morphisms in $\mathbf{Rel}$ are characterised in Relations, Item 2 of Proposition 3.10.1.1.

2.3 Corepresentably Fully Faithful Morphisms

Let $\mathcal{C}$ be a bicategory.

Definition 2.3.1.1. A 1-morphism $f\colon A \to B$ of $\mathcal{C}$ is **corepresentably fully faithful**[4] if the following equivalent conditions are satisfied:

1. The 1-morphism f is corepresentably full (Definition 2.2.1.1) and corepresentably faithful (Definition 2.1.1.1).

2. For each $X \in \mathrm{Obj}(\mathcal{C})$, the functor

$$f^*\colon \mathsf{Hom}_{\mathcal{C}}(B, X) \to \mathsf{Hom}_{\mathcal{C}}(A, X)$$

given by precomposition by f is fully faithful.

Remark 2.3.1.2. In detail, f is corepresentably fully faithful if the conditions in Remark 2.1.1.2 and Remark 2.2.1.2 hold:

1. For all diagrams in $\mathcal{C}$ of the form

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\,\Downarrow\beta"] & X, \end{tikzcd}$$

if we have

$$\alpha \star \mathrm{id}_f = \beta \star \mathrm{id}_f,$$

then $\alpha = \beta$.

[4] *Further Terminology:* Corepresentably fully faithful morphisms have also been called **lax epimorphisms** in the literature (e.g. in [Adá+01]), though we will always use the name "corepresentably fully faithful morphism" instead in this work.

2. For each $X \in \mathrm{Obj}(\mathcal{C})$ and each 2-morphism

$$\beta\colon \phi \circ f \Longrightarrow \psi \circ f, \qquad \begin{tikzcd} A \arrow[r, bend left, "\phi\circ f"] \arrow[r, bend right, "\psi\circ f"'] \arrow[r, phantom, "\beta\Downarrow"] & X \end{tikzcd}$$

of $\mathcal{C}$, there exists a 2-morphism

$$\alpha\colon \phi \Longrightarrow \psi, \qquad \begin{tikzcd} B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & X \end{tikzcd}$$

of $\mathcal{C}$ such that we have an equality

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & X \end{tikzcd} = \begin{tikzcd} A \arrow[r, bend left, "\phi\circ f"] \arrow[r, bend right, "\psi\circ f"'] \arrow[r, phantom, "\beta\Downarrow"] & X \end{tikzcd}$$

of pasting diagrams in $\mathcal{C}$, i.e. such that we have

$$\beta = \alpha \star \mathrm{id}_f.$$

Example 2.3.1.3. Here are some examples of corepresentably fully faithful morphisms.

1. *Corepresentably Fully Faithful Morphisms in* Cats_2. The fully faithful epimorphisms in Cats_2 are characterised in Categories, Item 9 of Proposition 5.3.1.2.

2. *Corepresentably Fully Faithful Morphisms in* $\mathbf{Rel}$. The corepresentably fully faithful morphisms of $\mathbf{Rel}$ coincide (Relations, Item 3 of Proposition 3.10.1.1) with the corepresentably full morphisms in $\mathbf{Rel}$, which are characterised in Relations, Item 2 of Proposition 3.10.1.1.

2.4 Morphisms Corepresentably Faithful on Cores

Let $\mathcal{C}$ be a bicategory.

Definition 2.4.1.1. A 1-morphism $f\colon A \to B$ of $\mathcal{C}$ is **corepresentably faithful on cores** if, for each $X \in \mathrm{Obj}(\mathcal{C})$, the functor

$$f^*\colon \mathsf{Core}(\mathsf{Hom}_{\mathcal{C}}(B, X)) \to \mathsf{Core}(\mathsf{Hom}_{\mathcal{C}}(A, X))$$

given by precomposition by f is faithful.

Remark 2.4.1.2. In detail, f is corepresentably faithful on cores if, for all diagrams in $\mathcal{C}$ of the form

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & X, \end{tikzcd}$$

if α and β are 2-isomorphisms and we have

$$\alpha \star \mathrm{id}_f = \beta \star \mathrm{id}_f,$$

then $\alpha = \beta$.

2.5 Morphisms Corepresentably Full on Cores

Let $\mathcal{C}$ be a bicategory.

Definition 2.5.1.1. A 1-morphism $f\colon A \to B$ of $\mathcal{C}$ is **corepresentably full on cores** if, for each $X \in \mathrm{Obj}(\mathcal{C})$, the functor

$$f^*\colon \mathsf{Core}(\mathsf{Hom}_{\mathcal{C}}(B, X)) \to \mathsf{Core}(\mathsf{Hom}_{\mathcal{C}}(A, X))$$

given by precomposition by f is full.

Remark 2.5.1.2. In detail, f is corepresentably full on cores if, for each $X \in \mathrm{Obj}(\mathcal{C})$ and each 2-isomorphism

$$\beta\colon \phi \circ f \overset{\sim}{\Longrightarrow} \psi \circ f, \qquad \begin{tikzcd} A \arrow[r, bend left, "\phi\circ f"] \arrow[r, bend right, "\psi\circ f"'] \arrow[r, phantom, "\beta\Downarrow"] & X \end{tikzcd}$$

of $\mathcal{C}$, there exists a 2-isomorphism

$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & X \end{tikzcd}$$

of $\mathcal{C}$ such that we have an equality

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & X \end{tikzcd} = \begin{tikzcd} A \arrow[r, bend left, "\phi\circ f"] \arrow[r, bend right, "\psi\circ f"'] \arrow[r, phantom, "\beta\Downarrow"] & X \end{tikzcd}$$

of pasting diagrams in $\mathcal{C}$, i.e. such that we have

$$\beta = \alpha \star \mathrm{id}_f.$$

2.6 Morphisms Corepresentably Fully Faithful on Cores

Let C be a bicategory.

Definition 2.6.1.1. A 1-morphism $f\colon A \to B$ of C is **corepresentably fully faithful on cores** if the following equivalent conditions are satisfied:

1. The 1-morphism f is corepresentably full on cores (Definition 2.5.1.1) and corepresentably faithful on cores (Definition 2.1.1.1).

2. For each $X \in \mathrm{Obj}(C)$, the functor

$$f^*\colon \mathsf{Core}(\mathsf{Hom}_C(B, X)) \to \mathsf{Core}(\mathsf{Hom}_C(A, X))$$

given by precomposition by f is fully faithful.

Remark 2.6.1.2. In detail, f is corepresentably fully faithful on cores if the conditions in Remark 2.4.1.2 and Remark 2.5.1.2 hold:

1. For all diagrams in C of the form

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & X, \end{tikzcd}$$

if α and β are 2-isomorphisms and we have

$$\alpha \star \mathrm{id}_f = \beta \star \mathrm{id}_f,$$

then $\alpha = \beta$.

2. For each $X \in \mathrm{Obj}(C)$ and each 2-isomorphism

$$\beta\colon \phi \circ f \overset{\sim}{\Longrightarrow} \psi \circ f, \qquad \begin{tikzcd} A \arrow[r, bend left, "\phi\circ f"] \arrow[r, bend right, "\psi\circ f"'] \arrow[r, phantom, "\beta\Downarrow"] & X \end{tikzcd}$$

of C, there exists a 2-isomorphism

$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & X \end{tikzcd}$$

of $\mathcal{C}$ such that we have an equality

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & X \end{tikzcd} = \begin{tikzcd} A \arrow[r, bend left, "\phi\circ f"] \arrow[r, bend right, "\psi\circ f"'] \arrow[r, phantom, "\beta\Downarrow"] & X \end{tikzcd}$$

of pasting diagrams in $\mathcal{C}$, i.e. such that we have

$$\beta = \alpha \star \mathrm{id}_f.$$

2.7 Corepresentably Essentially Injective Morphisms

Let $\mathcal{C}$ be a bicategory.

Definition 2.7.1.1. A 1-morphism $f\colon A \to B$ of $\mathcal{C}$ is **corepresentably essentially injective** if, for each $X \in \mathrm{Obj}(\mathcal{C})$, the functor

$$f^*\colon \mathsf{Hom}_{\mathcal{C}}(B, X) \to \mathsf{Hom}_{\mathcal{C}}(A, X)$$

given by precomposition by f is essentially injective.

Remark 2.7.1.2. In detail, f is corepresentably essentially injective if, for each pair of morphisms $\phi, \psi\colon B \rightrightarrows X$ of $\mathcal{C}$, the following condition is satisfied:

($\star$) If $\phi \circ f \cong \psi \circ f$, then $\phi \cong \psi$.

2.8 Corepresentably Conservative Morphisms

Let $\mathcal{C}$ be a bicategory.

Definition 2.8.1.1. A 1-morphism $f\colon A \to B$ of $\mathcal{C}$ is **corepresentably conservative** if, for each $X \in \mathrm{Obj}(\mathcal{C})$, the functor

$$f^*\colon \mathsf{Hom}_{\mathcal{C}}(B, X) \to \mathsf{Hom}_{\mathcal{C}}(A, X)$$

given by precomposition by f is conservative.

Remark 2.8.1.2. In detail, f is corepresentably conservative if, for each pair of morphisms $\phi, \psi\colon B \rightrightarrows X$ and each 2-morphism

$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & X \end{tikzcd}$$

of C, if the 2-morphism

$$\alpha \star \mathrm{id}_f \colon \phi \circ f \Longrightarrow \psi \circ f, \qquad \begin{tikzcd} A \arrow[r, bend left, "\phi\circ f"{name=U}] \arrow[r, bend right, "\psi\circ f"'{name=D}] & X \arrow[from=U, to=D, Rightarrow, "\alpha\star\mathrm{id}_f"] \end{tikzcd}$$

is a 2-isomorphism, then so is α.

2.9 Strict Epimorphisms

Let C be a bicategory.

Definition 2.9.1.1. A 1-morphism $f \colon A \to B$ is a **strict epimorphism in** C if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f^* \colon \mathsf{Hom}_C(B, X) \to \mathsf{Hom}_C(A, X)$$

given by precomposition by f is injective on objects, i.e. its action on objects

$$f_* \colon \mathrm{Obj}(\mathsf{Hom}_C(B, X)) \to \mathrm{Obj}(\mathsf{Hom}_C(A, X))$$

is injective.

Remark 2.9.1.2. In detail, f is a strict epimorphism if, for each diagram in C of the form

$$A \xrightarrow{f} B \underset{\psi}{\overset{\phi}{\rightrightarrows}} X,$$

if $\phi \circ f = \psi \circ f$, then $\phi = \psi$.

Example 2.9.1.3. Here are some examples of strict epimorphisms.

1. *Strict Epimorphisms in* Cats_2. The strict epimorphisms in Cats_2 are characterised in Categories, Item 1 of Proposition 6.3.1.2.

2. *Strict Epimorphisms in* **Rel**. The strict epimorphisms in **Rel** are characterised in Relations, Proposition 3.9.1.1.

2.10 Pseudoepic Morphisms

Let C be a bicategory.

Definition 2.10.1.1. A 1-morphism $f \colon A \to B$ of C is **pseudoepic** if, for each $X \in \mathrm{Obj}(\mathcal{C})$, the functor

$$f^* \colon \mathsf{Hom}_{\mathcal{C}}(B, X) \to \mathsf{Hom}_{\mathcal{C}}(A, X)$$

given by precomposition by f is pseudomonic.

Remark 2.10.1.2. In detail, a 1-morphism $f \colon A \to B$ of $\mathcal{C}$ is pseudoepic if it satisfies the following conditions:

1. For all diagrams in $\mathcal{C}$ of the form

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & X, \end{tikzcd}$$

   if we have

$$\alpha \star \mathrm{id}_f = \beta \star \mathrm{id}_f,$$

   then $\alpha = \beta$.

2. For each $X \in \mathrm{Obj}(\mathcal{C})$ and each 2-isomorphism

$$\beta \colon \phi \circ f \overset{\sim}{\Longrightarrow} \psi \circ f, \qquad \begin{tikzcd} A \arrow[r, bend left, "\phi\circ f"] \arrow[r, bend right, "\psi\circ f"'] \arrow[r, phantom, "\beta\Downarrow"] & X \end{tikzcd}$$

   of $\mathcal{C}$, there exists a 2-isomorphism

$$\alpha \colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & X \end{tikzcd}$$

   of $\mathcal{C}$ such that we have an equality

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & X \end{tikzcd} = \begin{tikzcd} A \arrow[r, bend left, "\phi\circ f"] \arrow[r, bend right, "\psi\circ f"'] \arrow[r, phantom, "\beta\Downarrow"] & X \end{tikzcd}$$

   of pasting diagrams in $\mathcal{C}$, i.e. such that we have

$$\beta = \alpha \star \mathrm{id}_f.$$

Proposition 2.10.1.3. Let $f \colon A \to B$ be a 1-morphism of $\mathcal{C}$.

1. *Characterisations.* The following conditions are equivalent:

   (a) The morphism f is pseudoepic.

   (b) The morphism f is corepresentably full on cores and corepresentably faithful.

   (c) We have an isocхаракт square of the form

$$B \overset{\text{eq.}}{\cong} B \overset{\leftrightarrow}{\coprod}_A B, \qquad \begin{tikzcd} B & B \arrow[l, "\mathrm{id}_B"'] \\ B \arrow[u, "\mathrm{id}_B"] & A \arrow[l, "F"] \arrow[u, "F"'] \arrow[ul, phantom, "\Leftrightarrow"] \end{tikzcd}$$

   in $\mathcal{C}$ up to equivalence.

Proof. *Item 1*, *Characterisations*: Omitted. □

Appendices

A Other Chapters

Sets

1. Sets
2. Constructions With Sets
3. Pointed Sets
4. Tensor Products of Pointed Sets

Relations

5. Relations
6. Constructions With Relations
7. Equivalence Relations and Apartness Relations

Category Theory

8. Categories

Bicategories

9. Types of Morphisms in Bicategories

References

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