

Equivalence Relations and Apartness Relations

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00TJ This chapter contains some material about reflexive, symmetric, transitive, equivalence, and apartness relations.

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00TK 1 Reflexive Relations

00TL 1.1 Foundations

Let A be a set.

00TM **Definition 1.1.1.1.** A **reflexive relation** is equivalently:¹

- An \mathbb{E}_0 -monoid in $(\mathbf{N}_\bullet(\mathbf{Rel}(A, A)), \chi_A)$.
- A pointed object in $(\mathbf{Rel}(A, A), \chi_A)$.

00TN **Remark 1.1.1.2.** In detail, a relation R on A is **reflexive** if we have an inclusion

$$\eta_R: \chi_A \subset R$$

of relations in $\mathbf{Rel}(A, A)$, i.e. if, for each $a \in A$, we have $a \sim_R a$.

00TP **Definition 1.1.1.3.** Let A be a set.

1. The **set of reflexive relations on A** is the subset $\mathbf{Rel}^{\text{refl}}(A, A)$ of $\mathbf{Rel}(A, A)$ spanned by the reflexive relations.
2. The **poset of relations on A** is the subposet $\mathbf{Rel}^{\text{refl}}(A, A)$ of $\mathbf{Rel}(A, A)$ spanned by the reflexive relations.

00TS **Proposition 1.1.1.4.** Let R and S be relations on A .

- 00TT 1. *Interaction With Inverses.* If R is reflexive, then so is R^\dagger .
- 00TU 2. *Interaction With Composition.* If R and S are reflexive, then so is $S \diamond R$.

Proof. **Item 1, Interaction With Inverses:** Clear.

Item 2, Interaction With Composition: Clear. □

00TV 1.2 The Reflexive Closure of a Relation

Let R be a relation on A .

¹Note that since $\mathbf{Rel}(A, A)$ is posetal, reflexivity is a property of a relation, rather than extra structure.

00TW **Definition 1.2.1.1.** The **reflexive closure** of \sim_R is the relation \sim_R^{refl} ² satisfying the following universal property:³

(★) Given another reflexive relation \sim_S on A such that $R \subset S$, there exists an inclusion $\sim_R^{\text{refl}} \subset \sim_S$.

00TX **Construction 1.2.1.2.** Concretely, \sim_R^{refl} is the free pointed object on R in $(\mathbf{Rel}(A, A), \chi_A)$ ⁴, being given by

$$\begin{aligned} R^{\text{refl}} &\stackrel{\text{def}}{=} R \coprod^{\mathbf{Rel}(A, A)} \Delta_A \\ &= R \cup \Delta_A \\ &= \{(a, b) \in A \times A \mid \text{we have } a \sim_R b \text{ or } a = b\}. \end{aligned}$$

Proof. Clear. □

00TY **Proposition 1.2.1.3.** Let R be a relation on A .

00TZ 1. *Adjointness.* We have an adjunction

$$\left((-)^{\text{refl}} \dashv \overset{\circlearrowleft}{\text{忘}} \right): \mathbf{Rel}(A, A) \overset{(-)^{\text{refl}}}{\underset{\overset{\circlearrowright}{\text{忘}}}{\rightleftarrows}} \mathbf{Rel}^{\text{refl}}(A, A),$$

witnessed by a bijection of sets

$$\mathbf{Rel}^{\text{refl}}(R^{\text{refl}}, S) \cong \mathbf{Rel}(R, S),$$

natural in $R \in \text{Obj}(\mathbf{Rel}^{\text{refl}}(A, A))$ and $S \in \text{Obj}(\mathbf{Rel}(A, A))$.

00U0 2. *The Reflexive Closure of a Reflexive Relation.* If R is reflexive, then $R^{\text{refl}} = R$.

00U1 3. *Idempotency.* We have

$$(R^{\text{refl}})^{\text{refl}} = R^{\text{refl}}.$$

²*Further Notation:* Also written R^{refl} .

³*Slogan:* The reflexive closure of R is the smallest reflexive relation containing R .

⁴Or, equivalently, the free \mathbb{E}_0 -monoid on R in $(\mathbf{N}_\bullet(\mathbf{Rel}(A, A)), \chi_A)$.

00U2 4. *Interaction With Inverses.* We have

$$\begin{array}{ccc} & \text{Rel}(A, A) & \xrightarrow{(-)^{\text{refl}}} \text{Rel}(A, A) \\ (R^\dagger)^{\text{refl}} = (R^{\text{refl}})^\dagger, & \downarrow (-)^\dagger & \downarrow (-)^\dagger \\ & \text{Rel}(A, A) & \xrightarrow{(-)^{\text{refl}}} \text{Rel}(A, A). \end{array}$$

00U3 5. *Interaction With Composition.* We have

$$\begin{array}{ccc} & \text{Rel}(A, A) \times \text{Rel}(A, A) & \xrightarrow{\diamond} \text{Rel}(A, A) \\ (S \diamond R)^{\text{refl}} = S^{\text{refl}} \diamond R^{\text{refl}}, & \downarrow (-)^{\text{refl}} \times (-)^{\text{refl}} & \downarrow (-)^{\text{refl}} \\ & \text{Rel}(A, A) \times \text{Rel}(A, A) & \xrightarrow{\diamond} \text{Rel}(A, A). \end{array}$$

Proof. **Item 1, Adjointness:** This is a rephrasing of the universal property of the reflexive closure of a relation, stated in **Definition 1.2.1.1**.

Item 2, The Reflexive Closure of a Reflexive Relation: Clear.

Item 3, Idempotency: This follows from **Item 2**.

Item 4, Interaction With Inverses: Clear.

Item 5, Interaction With Composition: This follows from **Item 2** of **Proposition 1.1.1.4**.

□

00U4 2 Symmetric Relations

00U5 2.1 Foundations

Let A be a set.

00U6 **Definition 2.1.1.1.** A relation R on A is **symmetric** if we have $R^\dagger = R$.

00U7 **Remark 2.1.1.2.** In detail, a relation R is symmetric if it satisfies the following condition:

(★) For each $a, b \in A$, if $a \sim_R b$, then $b \sim_R a$.

00U8 **Definition 2.1.1.3.** Let A be a set.

00U9 1. The **set of symmetric relations on A** is the subset $\mathbf{Rel}^{\text{symm}}(A, A)$ of $\mathbf{Rel}(A, A)$ spanned by the symmetric relations.

00UA 2. The **poset of relations on A** is the subposet $\mathbf{Rel}^{\text{symm}}(A, A)$ of $\mathbf{Rel}(A, A)$ spanned by the symmetric relations.

00UB **Proposition 2.1.1.4.** Let R and S be relations on A .

00UC 1. *Interaction With Inverses.* If R is symmetric, then so is R^\dagger .

00UD 2. *Interaction With Composition.* If R and S are symmetric, then so is $S \diamond R$.

Proof. **Item 1, Interaction With Inverses:** Clear.

Item 2, Interaction With Composition: Clear. \square

00UE 2.2 The Symmetric Closure of a Relation

Let R be a relation on A .

00UF **Definition 2.2.1.1.** The **symmetric closure** of \sim_R is the relation \sim_R^{symm} ⁵ satisfying the following universal property:⁶

(★) Given another symmetric relation \sim_S on A such that $R \subset S$, there exists an inclusion $\sim_R^{\text{symm}} \subset \sim_S$.

00UG **Construction 2.2.1.2.** Concretely, \sim_R^{symm} is the symmetric relation on A defined by

$$\begin{aligned} R^{\text{symm}} &\stackrel{\text{def}}{=} R \cup R^\dagger \\ &= \{(a, b) \in A \times A \mid \text{we have } a \sim_R b \text{ or } b \sim_R a\}. \end{aligned}$$

Proof. Clear. \square

00UH **Proposition 2.2.1.3.** Let R be a relation on A .

00UJ 1. *Adjointness.* We have an adjunction

$$((-)^{\text{symm}} \dashv \text{忘}): \mathbf{Rel}(A, A) \begin{array}{c} \xrightarrow{(-)^{\text{symm}}} \\ \perp \\ \xleftarrow{\text{忘}} \end{array} \mathbf{Rel}^{\text{symm}}(A, A),$$

⁵Further Notation: Also written R^{symm} .

⁶Slogan: The symmetric closure of R is the smallest symmetric relation containing R .

witnessed by a bijection of sets

$$\mathbf{Rel}^{\text{symm}}(R^{\text{symm}}, S) \cong \mathbf{Rel}(R, S),$$

natural in $R \in \text{Obj}(\mathbf{Rel}^{\text{symm}}(A, A))$ and $S \in \text{Obj}(\mathbf{Rel}(A, A))$.

00UK 2. *The Symmetric Closure of a Symmetric Relation.* If R is symmetric, then $R^{\text{symm}} = R$.

00UL 3. *Idempotency.* We have

$$(R^{\text{symm}})^{\text{symm}} = R^{\text{symm}}.$$

00UM 4. *Interaction With Inverses.* We have

$$\begin{array}{ccc} \mathbf{Rel}(A, A) & \xrightarrow{(-)^{\text{symm}}} & \mathbf{Rel}(A, A) \\ (-)^{\dagger} \downarrow & & \downarrow (-)^{\dagger} \\ \mathbf{Rel}(A, A) & \xrightarrow{(-)^{\text{symm}}} & \mathbf{Rel}(A, A) \end{array}$$

$$(R^{\dagger})^{\text{symm}} = (R^{\text{symm}})^{\dagger},$$

00UN 5. *Interaction With Composition.* We have

$$\begin{array}{ccc} \mathbf{Rel}(A, A) \times \mathbf{Rel}(A, A) & \xrightarrow{\diamond} & \mathbf{Rel}(A, A) \\ (-)^{\text{symm}} \times (-)^{\text{symm}} \downarrow & & \downarrow (-)^{\text{symm}} \\ \mathbf{Rel}(A, A) \times \mathbf{Rel}(A, A) & \xrightarrow{\diamond} & \mathbf{Rel}(A, A) \end{array}$$

$$(S \diamond R)^{\text{symm}} = S^{\text{symm}} \diamond R^{\text{symm}},$$

Proof. **Item 1, Adjointness:** This is a rephrasing of the universal property of the symmetric closure of a relation, stated in **Definition 2.2.1.1**.

Item 2, The Symmetric Closure of a Symmetric Relation: Clear.

Item 3, Idempotency: This follows from **Item 2**.

Item 4, Interaction With Inverses: Clear.

Item 5, Interaction With Composition: This follows from **Item 2** of **Proposition 2.1.1.4**.

□

00UP 3 Transitive Relations

00UQ 3.1 Foundations

Let A be a set.

00UR **Definition 3.1.1.1.** A **transitive relation** is equivalently:⁷

- A non-unital \mathbb{E}_1 -monoid in $(\mathbf{N}_\bullet(\mathbf{Rel}(A, A)), \diamond)$.
- A non-unital monoid in $(\mathbf{Rel}(A, A), \diamond)$.

00US **Remark 3.1.1.2.** In detail, a relation R on A is **transitive** if we have an inclusion

$$\mu_R: R \diamond R \subset R$$

of relations in $\mathbf{Rel}(A, A)$, i.e. if, for each $a, c \in A$, the following condition is satisfied:

(★) If there exists some $b \in A$ such that $a \sim_R b$ and $b \sim_R c$, then $a \sim_R c$.

00UT **Definition 3.1.1.3.** Let A be a set.

00UU 1. The **set of transitive relations from A to B** is the subset $\mathbf{Rel}^{\text{trans}}(A)$ of $\mathbf{Rel}(A, A)$ spanned by the transitive relations.

00UV 2. The **poset of relations from A to B** is the subposet $\mathbf{Rel}^{\text{trans}}(A)$ of $\mathbf{Rel}(A, A)$ spanned by the transitive relations.

00UW **Proposition 3.1.1.4.** Let R and S be relations on A .

00UX 1. *Interaction With Inverses.* If R is transitive, then so is R^\dagger .

00UY 2. *Interaction With Composition.* If R and S are transitive, then $S \diamond R$ **may fail to be transitive**.

Proof. **Item 1, Interaction With Inverses:** Clear.

Item 2, Interaction With Composition: See [MSE 2096272].⁸ □

⁷Note that since $\mathbf{Rel}(A, A)$ is posetal, transitivity is a property of a relation, rather than extra structure.

⁸*Intuition:* Transitivity for R and S fails to imply that of $S \diamond R$ because the composition operation for relations intertwines R and S in an incompatible way:

00U2 3.2 The Transitive Closure of a Relation

Let R be a relation on A .

00V0 Definition 3.2.1.1. The **transitive closure** of \sim_R is the relation \sim_R^{trans} ⁹ satisfying the following universal property:¹⁰

(★) Given another transitive relation \sim_S on A such that $R \subset S$, there exists an inclusion $\sim_R^{\text{trans}} \subset \sim_S$.

00V1 Construction 3.2.1.2. Concretely, \sim_R^{trans} is the free non-unital monoid on R in $(\mathbf{Rel}(A, A), \diamond)$ ¹¹, being given by

$$\begin{aligned} R^{\text{trans}} &\stackrel{\text{def}}{=} \coprod_{n=1}^{\infty} R^{\diamond n} \\ &\stackrel{\text{def}}{=} \bigcup_{n=1}^{\infty} R^{\diamond n} \\ &\stackrel{\text{def}}{=} \left\{ (a, b) \in A \times B \mid \begin{array}{l} \text{there exists some } (x_1, \dots, x_n) \in R^{\times n} \\ \text{such that } a \sim_R x_1 \sim_R \dots \sim_R x_n \sim_R b \end{array} \right\}. \end{aligned}$$

Proof. Clear. □

00V2 Proposition 3.2.1.3. Let R be a relation on A .

00V3 1. *Adjointness.* We have an adjunction

$$((-)^{\text{trans}} \dashv \overline{}): \mathbf{Rel}(A, A) \begin{array}{c} \xrightarrow{(-)^{\text{trans}}} \\ \perp \\ \xleftarrow{\overline{}} \end{array} \mathbf{Rel}^{\text{trans}}(A, A),$$

1. If $a \sim_{S \diamond R} c$ and $c \sim_{S \diamond R} e$, then:

(a) There is some $b \in A$ such that:

- i. $a \sim_R b$;
- ii. $b \sim_S c$;

(b) There is some $d \in A$ such that:

- i. $c \sim_R d$;
- ii. $d \sim_S e$.

⁹*Further Notation:* Also written R^{trans} .

¹⁰*Slogan:* The transitive closure of R is the smallest transitive relation containing R .

¹¹Or, equivalently, the free non-unital \mathbb{E}_1 -monoid on R in $(\mathbf{N}_\bullet(\mathbf{Rel}(A, A)), \diamond)$.

witnessed by a bijection of sets

$$\mathbf{Rel}^{\text{trans}}(R^{\text{trans}}, S) \cong \mathbf{Rel}(R, S),$$

natural in $R \in \text{Obj}(\mathbf{Rel}^{\text{trans}}(A, A))$ and $S \in \text{Obj}(\mathbf{Rel}(A, B))$.

00V4 2. *The Transitive Closure of a Transitive Relation.* If R is transitive, then $R^{\text{trans}} = R$.

00V5 3. *Idempotency.* We have

$$(R^{\text{trans}})^{\text{trans}} = R^{\text{trans}}.$$

00V6 4. *Interaction With Inverses.* We have

$$\begin{array}{ccc} \mathbf{Rel}(A, A) & \xrightarrow{(-)^{\text{trans}}} & \mathbf{Rel}(A, A) \\ (-)^{\dagger} \downarrow & & \downarrow (-)^{\dagger} \\ \mathbf{Rel}(A, A) & \xrightarrow{(-)^{\text{trans}}} & \mathbf{Rel}(A, A). \end{array}$$

$$(R^{\dagger})^{\text{trans}} = (R^{\text{trans}})^{\dagger},$$

00V7 5. *Interaction With Composition.* We have

$$\begin{array}{ccc} \mathbf{Rel}(A, A) \times \mathbf{Rel}(A, A) & \xrightarrow{\diamond} & \mathbf{Rel}(A, A) \\ (S \diamond R)^{\text{trans}} \stackrel{\text{poss.}}{\neq} S^{\text{trans}} \diamond R^{\text{trans}}, & (-)^{\text{trans}} \times (-)^{\text{trans}} \downarrow & \downarrow (-)^{\text{trans}} \\ \mathbf{Rel}(A, A) \times \mathbf{Rel}(A, A) & \xrightarrow{\diamond} & \mathbf{Rel}(A, A). \end{array}$$

X

Proof. Item 1, Adjointness: This is a rephrasing of the universal property of the transitive closure of a relation, stated in **Definition 3.2.1.1**.

Item 2, The Transitive Closure of a Transitive Relation: Clear.

Item 3, Idempotency: This follows from **Item 2**.

Item 4, Interaction With Inverses: We have

$$\begin{aligned}
 (R^\dagger)^{\text{trans}} &= \bigcup_{n=1}^{\infty} (R^\dagger)^{\diamond n} \\
 &= \bigcup_{n=1}^{\infty} (R^{\diamond n})^\dagger \\
 &= \left(\bigcup_{n=1}^{\infty} R^{\diamond n} \right)^\dagger \\
 &= (R^{\text{trans}})^\dagger,
 \end{aligned}$$

where we have used, respectively:

1. [Construction 3.2.1.2.](#)
2. [Constructions With Relations, Item 4 of Proposition 3.12.1.3.](#)
3. [Constructions With Relations, Item 1 of Proposition 3.6.1.2.](#)
4. [Construction 3.2.1.2.](#)

Item 5, Interaction With Composition: This follows from [Item 2 of Proposition 3.1.1.4.](#)

□

00V8 4 Equivalence Relations

00V9 4.1 Foundations

Let A be a set.

00VA Definition 4.1.1.1. A relation R is an **equivalence relation** if it is reflexive, symmetric, and transitive.¹²

00VB Example 4.1.1.2. The **kernel of a function** $f: A \rightarrow B$ is the equivalence relation $\sim_{\text{Ker}(f)}$ on A obtained by declaring $a \sim_{\text{Ker}(f)} b$ iff $f(a) = f(b)$.¹³

¹²*Further Terminology:* If instead R is just symmetric and transitive, then it is called a **partial equivalence relation**.

¹³The kernel $\text{Ker}(f): A \dashv A$ of f is the underlying functor of the monad induced by the adjunction $\text{Gr}(f) \dashv f^{-1}: A \rightleftarrows B$ in **Rel** of [Constructions With Relations, Item 2 of Proposition 3.1.1.2.](#)

00VC **Definition 4.1.1.3.** Let A and B be sets.

- 00VD 1. The **set of equivalence relations from A to B** is the subset $\text{Rel}^{\text{eq}}(A, B)$ of $\text{Rel}(A, B)$ spanned by the equivalence relations.
- 00VE 2. The **poset of relations from A to B** is the subposet $\mathbf{Rel}^{\text{eq}}(A, B)$ of $\mathbf{Rel}(A, B)$ spanned by the equivalence relations.

00VF 4.2 The Equivalence Closure of a Relation

Let R be a relation on A .

00VG **Definition 4.2.1.1.** The **equivalence closure**¹⁴ of \sim_R is the relation \sim_R^{eq} ¹⁵ satisfying the following universal property:¹⁶

- (★) Given another equivalence relation \sim_S on A such that $R \subset S$, there exists an inclusion $\sim_R^{\text{eq}} \subset \sim_S$.

00VH **Construction 4.2.1.2.** Concretely, \sim_R^{eq} is the equivalence relation on A defined by

$$\begin{aligned}
 R^{\text{eq}} &\stackrel{\text{def}}{=} \left(\left(R^{\text{refl}} \right)^{\text{symm}} \right)^{\text{trans}} \\
 &= \left(\left(R^{\text{symm}} \right)^{\text{trans}} \right)^{\text{refl}} \\
 &= \left\{ (a, b) \in A \times B \mid \begin{array}{l} \text{there exists } (x_1, \dots, x_n) \in R^{\times n} \text{ satisfying at} \\ \text{least one of the following conditions:} \\ \\ 1. \text{ The following conditions are satisfied:} \\ \\ \quad (a) \text{ We have } a \sim_R x_1 \text{ or } x_1 \sim_R a; \\ \quad (b) \text{ We have } x_i \sim_R x_{i+1} \text{ or } x_{i+1} \sim_R x_i \\ \quad \quad \text{for each } 1 \leq i \leq n-1; \\ \quad (c) \text{ We have } b \sim_R x_n \text{ or } x_n \sim_R b; \\ \\ 2. \text{ We have } a = b. \end{array} \right\}.
 \end{aligned}$$

Proof. From the universal properties of the reflexive, symmetric, and transitive

¹⁴Further Terminology: Also called the **equivalence relation associated to \sim_R** .

¹⁵Further Notation: Also written R^{eq} .

¹⁶Slogan: The equivalence closure of R is the smallest equivalence relation containing R .

closures of a relation (**Definitions 1.2.1.1, 2.2.1.1 and 3.2.1.1**), we see that it suffices to prove that:

- 00VJ 1. The symmetric closure of a reflexive relation is still reflexive.
- 00VK 2. The transitive closure of a symmetric relation is still symmetric.

which are both clear. \square

00VL **Proposition 4.2.1.3.** Let R be a relation on A .

- 00VM 1. *Adjointness.* We have an adjunction

$$((-)^{\text{eq}} \dashv \overline{}): \mathbf{Rel}(A, B) \begin{matrix} \xrightarrow{(-)^{\text{eq}}} \\ \perp \\ \xleftarrow{\overline{}} \end{matrix} \mathbf{Rel}^{\text{eq}}(A, B),$$

witnessed by a bijection of sets

$$\mathbf{Rel}^{\text{eq}}(R^{\text{eq}}, S) \cong \mathbf{Rel}(R, S),$$

natural in $R \in \text{Obj}(\mathbf{Rel}^{\text{eq}}(A, B))$ and $S \in \text{Obj}(\mathbf{Rel}(A, B))$.

- 00VN 2. *The Equivalence Closure of an Equivalence Relation.* If R is an equivalence relation, then $R^{\text{eq}} = R$.
- 00VP 3. *Idempotency.* We have

$$(R^{\text{eq}})^{\text{eq}} = R^{\text{eq}}.$$

Proof. **Item 1, Adjointness:** This is a rephrasing of the universal property of the equivalence closure of a relation, stated in **Definition 4.2.1.1**.

Item 2, The Equivalence Closure of an Equivalence Relation: Clear.

Item 3, Idempotency: This follows from **Item 2**. \square

00VQ 5 Quotients by Equivalence Relations

00VR 5.1 Equivalence Classes

Let A be a set, let R be a relation on A , and let $a \in A$.

00VS **Definition 5.1.1.1.** The **equivalence class associated to a** is the set $[a]$ defined by

$$\begin{aligned} [a] &\stackrel{\text{def}}{=} \{x \in X \mid x \sim_R a\} \\ &= \{x \in X \mid a \sim_R x\}. \end{aligned} \quad (\text{since } R \text{ is symmetric})$$

00VT 5.2 Quotients of Sets by Equivalence Relations

Let A be a set and let R be a relation on A .

00VU Definition 5.2.1.1. The **quotient of X by R** is the set X/\sim_R defined by

$$X/\sim_R \stackrel{\text{def}}{=} \{[a] \in \mathcal{P}(X) \mid a \in X\}.$$

00VV Remark 5.2.1.2. The reason we define quotient sets for equivalence relations only is that each of the properties of being an equivalence relation—reflexivity, symmetry, and transitivity—ensures that the equivalence classes $[a]$ of X under R are well-behaved:

- *Reflexivity.* If R is reflexive, then, for each $a \in X$, we have $a \in [a]$.
- *Symmetry.* The equivalence class $[a]$ of an element a of X is defined by

$$[a] \stackrel{\text{def}}{=} \{x \in X \mid x \sim_R a\},$$

but we could equally well define

$$[a]' \stackrel{\text{def}}{=} \{x \in X \mid a \sim_R x\}$$

instead. This is not a problem when R is symmetric, as we then have $[a] = [a]'$.¹⁷

- *Transitivity.* If R is transitive, then $[a]$ and $[b]$ are disjoint iff $a \not\sim_R b$, and equal otherwise.

00VW Proposition 5.2.1.3. Let $f: X \rightarrow Y$ be a function and let R be a relation on X .

00VX 1. As a Coequaliser. We have an isomorphism of sets

$$X/\sim_R^{\text{eq}} \cong \text{CoEq} \left(R \hookrightarrow X \times X \begin{array}{c} \xrightarrow{\text{pr}_1} \\ \xrightarrow{\text{pr}_2} \end{array} X \right),$$

where \sim_R^{eq} is the equivalence relation generated by \sim_R .

¹⁷When categorifying equivalence relations, one finds that $[a]$ and $[a]'$ correspond to presheaves and copresheaves; see ??, ??.

00VY 2. *As a Pushout.* We have an isomorphism of sets¹⁸

$$X/\sim_R^{\text{eq}} \cong X \amalg_{\text{Eq}(\text{pr}_1, \text{pr}_2)} X,$$

$$\begin{array}{ccc} X/\sim_R^{\text{eq}} & \longleftarrow & X \\ \uparrow \ulcorner & & \uparrow \\ X & \longleftarrow & \text{Eq}(\text{pr}_1, \text{pr}_2). \end{array}$$

where \sim_R^{eq} is the equivalence relation generated by \sim_R .

00VZ 3. *The First Isomorphism Theorem for Sets.* We have an isomorphism of sets^{19,20}

$$X/\sim_{\text{Ker}(f)} \cong \text{Im}(f).$$

00W0 4. *Descending Functions to Quotient Sets, I.* Let R be an equivalence relation on X . The following conditions are equivalent:

(a) There exists a map

$$\bar{f}: X/\sim_R \rightarrow Y$$

¹⁸Dually, we also have an isomorphism of sets

$$\text{Eq}(\text{pr}_1, \text{pr}_2) \cong X \times_{X/\sim_R^{\text{eq}}} X,$$

$$\begin{array}{ccc} \text{Eq}(\text{pr}_1, \text{pr}_2) & \longrightarrow & X \\ \downarrow \lrcorner & & \downarrow \\ X & \longrightarrow & X/\sim_R^{\text{eq}}. \end{array}$$

¹⁹*Further Terminology:* The set $X/\sim_{\text{Ker}(f)}$ is often called the **coimage** of f , and denoted by $\text{Coim}(f)$.

²⁰In a sense this is a result relating the monad in **Rel** induced by f with the comonad in **Rel** induced by f , as the kernel and image

$$\begin{aligned} \text{Ker}(f) &: X \rightrightarrows X, \\ \text{Im}(f) &\subset Y \end{aligned}$$

of f are the underlying functors of (respectively) the induced monad and comonad of the adjunction

$$\left(\text{Gr}(f) \dashv f^{-1} \right): \begin{array}{ccc} & \text{Gr}(f) & \\ \uparrow & \text{+} & \downarrow \\ A & \xleftarrow{f^{-1}} & B \end{array}$$

of **Constructions With Relations**, Item 2 of Proposition 3.1.1.2.

making the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ q \downarrow & \nearrow \bar{f} & \\ X/\sim_R & & \end{array}$$

commute.

(b) We have $R \subset \text{Ker}(f)$.

(c) For each $x, y \in X$, if $x \sim_R y$, then $f(x) = f(y)$.

00W1

5. *Descending Functions to Quotient Sets, II.* Let R be an equivalence relation on X . If the conditions of **Item 4** hold, then \bar{f} is the *unique* map making the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ q \downarrow & \nearrow \bar{f} & \\ X/\sim_R & & \end{array}$$

commute.

00W2

6. *Descending Functions to Quotient Sets, III.* Let R be an equivalence relation on X . We have a bijection

$$\text{Hom}_{\text{Sets}}(X/\sim_R, Y) \cong \text{Hom}_{\text{Sets}}^R(X, Y),$$

natural in $X, Y \in \text{Obj}(\text{Sets})$, given by the assignment $f \mapsto \bar{f}$ of **Items 4** and **5**, where $\text{Hom}_{\text{Sets}}^R(X, Y)$ is the set defined by

$$\text{Hom}_{\text{Sets}}^R(X, Y) \stackrel{\text{def}}{=} \left\{ f \in \text{Hom}_{\text{Sets}}(X, Y) \left| \begin{array}{l} \text{for each } x, y \in X, \\ \text{if } x \sim_R y, \text{ then} \\ f(x) = f(y) \end{array} \right. \right\}.$$

00W3

7. *Descending Functions to Quotient Sets, IV.* Let R be an equivalence relation on X . If the conditions of **Item 4** hold, then the following conditions are equivalent:

- (a) The map \bar{f} is an injection.
- (b) We have $R = \text{Ker}(f)$.
- (c) For each $x, y \in X$, we have $x \sim_R y$ iff $f(x) = f(y)$.

00W4 8. *Descending Functions to Quotient Sets, V.* Let R be an equivalence relation on X . If the conditions of **Item 4** hold, then the following conditions are equivalent:

- (a) The map $f: X \rightarrow Y$ is surjective.
- (b) The map $\bar{f}: X/\sim_R \rightarrow Y$ is surjective.

00W5 9. *Descending Functions to Quotient Sets, VI.* Let R be a relation on X and let \sim_R^{eq} be the equivalence relation associated to R . The following conditions are equivalent:

- 00W6 (a) The map f satisfies the equivalent conditions of **Item 4**:
- There exists a map

$$\bar{f}: X/\sim_R^{\text{eq}} \rightarrow Y$$

making the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ q \downarrow & \exists \nearrow \bar{f} & \\ X/\sim_R^{\text{eq}} & & \end{array}$$

commute.

- For each $x, y \in X$, if $x \sim_R^{\text{eq}} y$, then $f(x) = f(y)$.
- 00W7 (b) For each $x, y \in X$, if $x \sim_R y$, then $f(x) = f(y)$.

Proof. **Item 1**, As a Coequaliser: Omitted.

Item 2, As a Pushout: Omitted.

Item 3, The First Isomorphism Theorem for Sets: Clear.

Item 4, Descending Functions to Quotient Sets, I: See [Pro24c].

Item 5, Descending Functions to Quotient Sets, II: See [Pro24d].

Item 6, Descending Functions to Quotient Sets, III: This follows from **Items 5** and **6**.

Item 7, Descending Functions to Quotient Sets, IV: See [Pro24b].

Item 8, Descending Functions to Quotient Sets, V: See [Pro24a].

Item 9, Descending Functions to Quotient Sets, VI: The implication **Item 9a** \implies **Item 9b** is clear.

Conversely, suppose that, for each $x, y \in X$, if $x \sim_R y$, then $f(x) = f(y)$. Spelling out the definition of the equivalence closure of R , we see that the condition $x \sim_R^{\text{eq}} y$ unwinds to the following:

(★) There exist $(x_1, \dots, x_n) \in R^{\times n}$ satisfying at least one of the following conditions:

1. The following conditions are satisfied:

- (a) We have $x \sim_R x_1$ or $x_1 \sim_R x$;
- (b) We have $x_i \sim_R x_{i+1}$ or $x_{i+1} \sim_R x_i$ for each $1 \leq i \leq n-1$;
- (c) We have $y \sim_R x_n$ or $x_n \sim_R y$;

2. We have $x = y$.

Now, if $x = y$, then $f(x) = f(y)$ trivially; otherwise, we have

$$\begin{aligned} f(x) &= f(x_1), \\ f(x_1) &= f(x_2), \\ &\vdots \\ f(x_{n-1}) &= f(x_n), \\ f(x_n) &= f(y), \end{aligned}$$

and $f(x) = f(y)$, as we wanted to show. \square

Appendices

A Other Chapters

Sets

1. [Sets](#)
2. [Constructions With Sets](#)
3. [Pointed Sets](#)
4. [Tensor Products of Pointed Sets](#)

Relations

5. [Relations](#)

6. [Constructions With Relations](#)

7. [Equivalence Relations and Apartness Relations](#)

Category Theory

8. [Categories](#)

Bicategories

9. [Types of Morphisms in Bicategories](#)

References

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