

# Tensor Products of Pointed Sets

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**00C3** In this chapter we introduce, construct, and study tensor products of pointed sets. The most well-known among these is the *smash product of pointed sets*

$$\wedge : \mathbf{Sets}_* \times \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*,$$

introduced in **Section 5.1**, defined via a universal property as inducing a bijection between the following data:

- Pointed maps  $f : X \wedge Y \rightarrow Z$ .
- Maps of sets  $f : X \times Y \rightarrow Z$  satisfying

$$\begin{aligned} f(x_0, y) &= z_0, \\ f(x, y_0) &= z_0 \end{aligned}$$

for each  $x \in X$  and each  $y \in Y$ .

As it turns out, however, dropping either of the *bilinearity* conditions

$$\begin{aligned} f(x_0, y) &= z_0, \\ f(x, y_0) &= z_0 \end{aligned}$$

while retaining the other leads to two other tensor products of pointed sets,

$$\begin{aligned} \triangleleft : \mathbf{Sets}_* \times \mathbf{Sets}_* &\rightarrow \mathbf{Sets}_*, \\ \triangleright : \mathbf{Sets}_* \times \mathbf{Sets}_* &\rightarrow \mathbf{Sets}_*, \end{aligned}$$

called the *left* and *right tensor products of pointed sets*. In contrast to  $\wedge$ , which turns out to endow  $\mathbf{Sets}_*$  with a monoidal category structure (**Proposition 5.9.1.1**),

these do not admit invertible associators and unitors, but do endow  $\mathbf{Sets}_*$  with the structure of a skew monoidal category, however ([Propositions 3.8.1.1](#) and [4.8.1.1](#)). Finally, in addition to the tensor products  $\triangleleft$ ,  $\triangleright$ , and  $\wedge$ , we also have a “tensor product” of the form

$$\odot : \mathbf{Sets} \times \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*,$$

called the *tensor* of sets with pointed sets. All in all, these tensor products assemble into a family of functors of the form

$$\begin{aligned} \otimes_{k,\ell} : \mathbf{Mon}_{\mathbb{E}_k}(\mathbf{Sets}) \times \mathbf{Mon}_{\mathbb{E}_\ell}(\mathbf{Sets}) &\rightarrow \mathbf{Mon}_{\mathbb{E}_{k+\ell}}(\mathbf{Sets}), \\ \triangleleft_{i,k} : \mathbf{Mon}_{\mathbb{E}_k}(\mathbf{Sets}) \times \mathbf{Mon}_{\mathbb{E}_k}(\mathbf{Sets}) &\rightarrow \mathbf{Mon}_{\mathbb{E}_k}(\mathbf{Sets}), \\ \triangleright_{i,k} : \mathbf{Mon}_{\mathbb{E}_k}(\mathbf{Sets}) \times \mathbf{Mon}_{\mathbb{E}_k}(\mathbf{Sets}) &\rightarrow \mathbf{Mon}_{\mathbb{E}_k}(\mathbf{Sets}), \end{aligned}$$

where  $k, \ell, i \in \mathbb{N}$  with  $i \leq k - 1$ . Together with the Cartesian product  $\times$  of  $\mathbf{Sets}$ , the tensor products studied in this chapter form the cases:

- $(k, \ell) = (-1, -1)$  for the Cartesian product of  $\mathbf{Sets}$ ;
- $(k, \ell) = (0, -1)$  and  $(-1, 0)$  for the tensor of sets with pointed sets of [Definition 2.1.1.1](#);
- $(i, k) = (-1, 0)$  for the left and right tensor products of pointed sets of [Sections 3](#) and [4](#);
- $(k, \ell) = (-1, -1)$  for the smash product of pointed sets of [Section 5](#).

In this chapter, we will carefully define and study bilinearity for pointed sets, as well as all the tensor products described above. Then, in [??](#), we will extend these to tensor products involving also monoids and commutative monoids, which will end up covering all cases up to  $k, \ell \leq 2$ , and hence *all* cases since  $\mathbb{E}_k$ -monoids on  $\mathbf{Sets}$  are the same as  $\mathbb{E}_2$ -monoids on  $\mathbf{Sets}$  when  $k \geq 2$ .

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## 00C4 1 Bilinear Morphisms of Pointed Sets

### 00C5 1.1 Left Bilinear Morphisms of Pointed Sets

Let  $(X, x_0)$ ,  $(Y, y_0)$ , and  $(Z, z_0)$  be pointed sets.

**00C6 Definition 1.1.1.1.** A **left bilinear morphism of pointed sets from  $(X \times Y, (x_0, y_0))$  to  $(Z, z_0)$**  is a map of sets

$$f: X \times Y \rightarrow Z$$

satisfying the following condition:<sup>1,2</sup>

(★) *Left Unital Bilinearity.* The diagram

$$\begin{array}{ccccc}
 & & \text{pt} \times \text{pt} & & \\
 & \text{id}_{\text{pt}} \times \epsilon_Y \nearrow & & \searrow \sim & \\
 \text{pt} \times Y & & & & \text{pt} \\
 [x_0] \times \text{id}_Y \searrow & & & & \nearrow [z_0] \\
 X \times Y & \xrightarrow{f} & Z & & 
 \end{array}$$

commutes, i.e. for each  $y \in Y$ , we have

$$f(x_0, y) = z_0.$$

<sup>1</sup>*Slogan:* The map  $f$  is left bilinear if it preserves basepoints in its first argument.

<sup>2</sup>Succinctly,  $f$  is bilinear if we have

$$f(x_0, y) = z_0$$

**00C7 Definition 1.1.1.2.** The **set of left bilinear morphisms of pointed sets** from  $(X \times Y, (x_0, y_0))$  to  $(Z, z_0)$  is the set  $\text{Hom}_{\text{Sets}_*}^{\otimes, \text{L}}(X \times Y, Z)$  defined by

$$\text{Hom}_{\text{Sets}_*}^{\otimes, \text{L}}(X \times Y, Z) \stackrel{\text{def}}{=} \{f \in \text{Hom}_{\text{Sets}}(X \times Y, Z) \mid f \text{ is left bilinear}\}.$$

## 00C8 1.2 Right Bilinear Morphisms of Pointed Sets

Let  $(X, x_0)$ ,  $(Y, y_0)$ , and  $(Z, z_0)$  be pointed sets.

**00C9 Definition 1.2.1.1.** A **right bilinear morphism of pointed sets** from  $(X \times Y, (x_0, y_0))$  to  $(Z, z_0)$  is a map of sets

$$f: X \times Y \rightarrow Z$$

satisfying the following condition:<sup>3,4</sup>

(★) *Right Unital Bilinearity.* The diagram

$$\begin{array}{ccccc} & & \text{pt} \times \text{pt} & & \\ & \nearrow \epsilon_X \times \text{id}_{\text{pt}} & & \searrow \sim & \\ X \times \text{pt} & & & & \text{pt} \\ & \searrow \text{id}_X \times [y_0] & & \nearrow [z_0] & \\ X \times Y & \xrightarrow{f} & Z & & \end{array}$$

commutes, i.e. for each  $x \in X$ , we have

$$f(x, y_0) = z_0.$$

**00CA Definition 1.2.1.2.** The **set of right bilinear morphisms of pointed sets** from  $(X \times Y, (x_0, y_0))$  to  $(Z, z_0)$  is the set  $\text{Hom}_{\text{Sets}_*}^{\otimes, \text{R}}(X \times Y, Z)$  defined by

$$\text{Hom}_{\text{Sets}_*}^{\otimes, \text{R}}(X \times Y, Z) \stackrel{\text{def}}{=} \{f \in \text{Hom}_{\text{Sets}}(X \times Y, Z) \mid f \text{ is right bilinear}\}.$$

for each  $y \in Y$ .

<sup>3</sup>*Slogan:* The map  $f$  is right bilinear if it preserves basepoints in its second argument.

<sup>4</sup>Succinctly,  $f$  is bilinear if we have

$$f(x, y_0) = z_0$$

for each  $x \in X$ .

**00CB 1.3 Bilinear Morphisms of Pointed Sets**

Let  $(X, x_0)$ ,  $(Y, y_0)$ , and  $(Z, z_0)$  be pointed sets.

**00CC Definition 1.3.1.1.** A **bilinear morphism of pointed sets from**  $(X \times Y, (x_0, y_0))$  **to**  $(Z, z_0)$  is a map of sets

$$f: X \times Y \rightarrow Z$$

that is both left bilinear and right bilinear.

**00CD Remark 1.3.1.2.** In detail, a **bilinear morphism of pointed sets from**  $(X \times Y, (x_0, y_0))$  **to**  $(Z, z_0)$  is a map of sets

$$f: (X \times Y, (x_0, y_0)) \rightarrow (Z, z_0)$$

satisfying the following conditions:<sup>5,6</sup>

1. *Left Unital Bilinearity.* The diagram

$$\begin{array}{ccccc}
 & & \text{pt} \times \text{pt} & & \\
 & \text{id}_{\text{pt}} \times \epsilon_Y \nearrow & & \text{---} \sim \text{---} \searrow & \\
 \text{pt} \times Y & & & & \text{pt} \\
 & \downarrow [\text{x}_0] \times \text{id}_Y & & & \downarrow [\text{z}_0] \\
 X \times Y & \xrightarrow{f} & Z & & 
 \end{array}$$

commutes, i.e. for each  $y \in Y$ , we have

$$f(x_0, y) = z_0.$$

<sup>5</sup>Slogan: The map  $f$  is bilinear if it preserves basepoints in each argument.

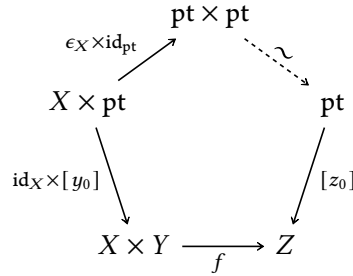
<sup>6</sup>Succinctly,  $f$  is bilinear if we have

$$f(x_0, y) = z_0,$$

$$f(x, y_0) = z_0$$

for each  $x \in X$  and each  $y \in Y$ .

2. *Right Unital Bilinearity.* The diagram



commutes, i.e. for each  $x \in X$ , we have

$$f(x, y_0) = z_0.$$

**00CE Definition 1.3.1.3.** The **set of bilinear morphisms of pointed sets from**  $(X \times Y, (x_0, y_0))$  **to**  $(Z, z_0)$  **is the set**  $\text{Hom}_{\text{Sets}_*}^{\otimes}(X \times Y, Z)$  **defined by**

$$\text{Hom}_{\text{Sets}_*}^{\otimes}(X \times Y, Z) \stackrel{\text{def}}{=} \{f \in \text{Hom}_{\text{Sets}}(X \times Y, Z) \mid f \text{ is bilinear}\}.$$

## 00CF 2 Tensors and Cotensors of Pointed Sets by Sets

### 00CG 2.1 Tensors of Pointed Sets by Sets

Let  $(X, x_0)$  be a pointed set and let  $A$  be a set.

**00CH Definition 2.1.1.1.** The **tensor of**  $(X, x_0)$  **by**  $A$ <sup>7</sup> **is the pointed set**<sup>8</sup>  $A \odot (X, x_0)$  **satisfying the following universal property:**

(UP) We have a bijection

$$\text{Sets}_*(A \odot X, K) \cong \text{Sets}(A, \text{Sets}_*(X, K)),$$

natural in  $(K, k_0) \in \text{Obj}(\text{Sets}_*)$ .

**00CJ Remark 2.1.1.2.** The universal property in **Definition 2.1.1.1** is equivalent to the following one:

<sup>7</sup>Further Terminology: Also called the **copower of**  $(X, x_0)$  **by**  $A$ .

<sup>8</sup>Further Notation: Often written  $A \odot X$  for simplicity.

(UP) We have a bijection

$$\mathbf{Sets}_*(A \odot X, K) \cong \mathbf{Sets}_{\mathbb{E}_0}^{\otimes}(A \times X, K),$$

natural in  $(K, k_0) \in \mathbf{Obj}(\mathbf{Sets}_*)$ , where  $\mathbf{Sets}_{\mathbb{E}_0}^{\otimes}(A \times X, K)$  is the set defined by

$$\mathbf{Sets}_{\mathbb{E}_0}^{\otimes}(A \times X, K) \stackrel{\text{def}}{=} \left\{ f \in \mathbf{Sets}(A \times X, K) \left| \begin{array}{l} \text{for each } a \in A, \text{ we} \\ \text{have } f(a, x_0) = k_0 \end{array} \right. \right\}.$$

*Proof.* We claim we have a bijection

$$\mathbf{Sets}(A, \mathbf{Sets}_*(X, K)) \cong \mathbf{Sets}_{\mathbb{E}_0}^{\otimes}(A \times X, K)$$

natural in  $(K, k_0) \in \mathbf{Obj}(\mathbf{Sets}_*)$ . Indeed, this bijection is a restriction of the bijection

$$\mathbf{Sets}(A, \mathbf{Sets}(X, K)) \cong \mathbf{Sets}(A \times X, K)$$

of **Constructions With Sets, Item 2** of **Proposition 1.3.1.2**:

- A map

$$\begin{aligned} \xi: A &\longrightarrow \mathbf{Sets}_*(X, K), \\ a &\longmapsto (\xi_a: X \rightarrow K), \end{aligned}$$

in  $\mathbf{Sets}(A, \mathbf{Sets}_*(X, K))$  gets sent to the map

$$\xi^\dagger: A \times X \rightarrow K$$

defined by

$$\xi^\dagger(a, x) \stackrel{\text{def}}{=} \xi_a(x)$$

for each  $(a, x) \in A \times X$ , which indeed lies in  $\mathbf{Sets}_{\mathbb{E}_0}^{\otimes}(A \times X, K)$ , as we have

$$\begin{aligned} \xi^\dagger(a, x_0) &\stackrel{\text{def}}{=} \xi_a(x_0) \\ &\stackrel{\text{def}}{=} k_0 \end{aligned}$$

for each  $a \in A$ , where we have used that  $\xi_a \in \mathbf{Sets}_*(X, K)$  is a morphism of pointed sets.



- Conversely, a map

$$\xi: A \times X \rightarrow K$$

in  $\text{Sets}_{\mathbb{E}_0}^{\otimes}(A \times X, K)$  gets sent to the map

$$\begin{aligned} \xi^\dagger: A &\longrightarrow \text{Sets}_*(X, K), \\ a &\longmapsto \left( \xi_a^\dagger: X \rightarrow K \right), \end{aligned}$$

where

$$\xi_a^\dagger: X \rightarrow K$$

is the map defined by

$$\xi_a^\dagger(x) \stackrel{\text{def}}{=} \xi(a, x)$$

for each  $x \in X$ , and indeed lies in  $\text{Sets}_*(X, K)$ , as we have

$$\begin{aligned} \xi_a^\dagger(x_0) &\stackrel{\text{def}}{=} \xi(a, x_0) \\ &\stackrel{\text{def}}{=} k_0. \end{aligned}$$

This finishes the proof.  $\square$

**00CK Construction 2.1.1.3.** Concretely, the **tensor of**  $(X, x_0)$  **by**  $A$  is the pointed set  $A \odot (X, x_0)$  consisting of:

- *The Underlying Set.* The set  $A \odot X$  given by

$$A \odot X \cong \bigvee_{a \in A} (X, x_0),$$

where  $\bigvee_{a \in A} (X, x_0)$  is the wedge product of the  $A$ -indexed family  $((X, x_0))_{a \in A}$  of **Pointed Sets, Definition 3.2.1.1.**

- *The Basepoint.* The point  $[(a, x_0)] = [(a', x_0)]$  of  $\bigvee_{a \in A} (X, x_0)$ .

*Proof.* (Proven below in a bit.)  $\square$

**00CL Notation 2.1.1.4.** We write  $a \odot x$  for the element  $[(a, x)]$  of

$$\begin{aligned} A \odot X &\cong \bigvee_{a \in A} (X, x_0) \\ &\stackrel{\text{def}}{=} \left( \bigsqcup_{i \in I} X_i \right) / \sim. \end{aligned}$$

**00CM Remark 2.1.1.5.** Taking the tensor of any element of  $A$  with the basepoint  $x_0$  of  $X$  leads to the same element in  $A \odot X$ , i.e. we have

$$a \odot x_0 = a' \odot x_0,$$

for each  $a, a' \in A$ . This is due to the equivalence relation  $\sim$  on

$$\bigvee_{a \in A} (X, x_0) \stackrel{\text{def}}{=} \bigsqcup_{a \in A} X / \sim$$

identifying  $(a, x_0)$  with  $(a', x_0)$ , so that the equivalence class  $a \odot x_0$  is independent from the choice of  $a \in A$ .

*Proof.* We claim we have a bijection

$$\text{Sets}_*(A \odot X, K) \cong \text{Sets}(A, \text{Sets}_*(X, K))$$

natural in  $(K, k_0) \in \text{Obj}(\text{Sets}_*)$ .

- *Map I.* We define a map

$$\Phi_K: \text{Sets}_*(A \odot X, K) \rightarrow \text{Sets}(A, \text{Sets}_*(X, K))$$

by sending a morphism of pointed sets

$$\xi: (A \odot X, a \odot x_0) \rightarrow (K, k_0)$$

to the map of sets

$$\begin{aligned} \xi^\dagger: A &\longrightarrow \text{Sets}_*(X, K), \\ a &\longmapsto (\xi_a: X \rightarrow K), \end{aligned}$$

where

$$\xi_a: (X, x_0) \rightarrow (K, k_0)$$

is the morphism of pointed sets defined by

$$\xi_a(x) \stackrel{\text{def}}{=} \xi(a \odot x)$$

for each  $x \in X$ . Note that we have

$$\begin{aligned} \xi_a(x_0) &\stackrel{\text{def}}{=} \xi(a \odot x_0) \\ &= k_0, \end{aligned}$$

so that  $\xi_a$  is indeed a morphism of pointed sets, where we have used that  $\xi$  is a morphism of pointed sets.

- *Map II.* We define a map

$$\Psi_K : \text{Sets}(A, \text{Sets}_*(X, K)) \rightarrow \text{Sets}_*(A \odot X, K)$$

given by sending a map

$$\begin{aligned} \xi &: A \rightarrow \text{Sets}_*(X, K), \\ a &\mapsto (\xi_a : X \rightarrow K), \end{aligned}$$

to the morphism of pointed sets

$$\xi^\dagger : (A \odot X, a \odot x_0) \rightarrow (K, k_0)$$

defined by

$$\xi^\dagger(a \odot x) \stackrel{\text{def}}{=} \xi_a(x)$$

for each  $a \odot x \in A \odot X$ . Note that  $\xi^\dagger$  is indeed a morphism of pointed sets, as we have

$$\begin{aligned} \xi^\dagger(a \odot x_0) &\stackrel{\text{def}}{=} \xi_a(x_0) \\ &= k_0, \end{aligned}$$

where we have used that  $\xi(a) \in \text{Sets}_*(X, K)$  is a morphism of pointed sets.

- *Invertibility I.* We claim that

$$\Psi_K \circ \Phi_K = \text{id}_{\text{Sets}_*(A \odot X, K)}.$$

Indeed, given a morphism of pointed sets

$$\xi : (A \odot X, a \odot x_0) \rightarrow (K, k_0),$$

we have

$$\begin{aligned} [\Psi_K \circ \Phi_K](\xi) &= \Psi_K(\Phi_K(\xi)) \\ &= \Psi_K(\llbracket a \mapsto \llbracket x \mapsto \xi(a \odot x) \rrbracket \rrbracket) \\ &= \Psi_K(\llbracket a' \mapsto \llbracket x' \mapsto \xi(a' \odot x') \rrbracket \rrbracket) \\ &= \llbracket a \odot x \mapsto \text{ev}_x(\text{ev}_a(\llbracket a' \mapsto \llbracket x' \mapsto \xi(a' \odot x') \rrbracket \rrbracket)) \rrbracket \\ &= \llbracket a \odot x \mapsto \text{ev}_x(\llbracket x' \mapsto \xi(a \odot x') \rrbracket) \rrbracket \\ &= \llbracket a \odot x \mapsto \xi(a \odot x) \rrbracket \\ &= \xi. \end{aligned}$$

- *Invertibility II.* We claim that

$$\Phi_K \circ \Psi_K = \text{id}_{\text{Sets}(A, \text{Sets}_*(X, K))}.$$

Indeed, given a morphism  $\xi: A \rightarrow \text{Sets}_*(X, K)$ , we have

$$\begin{aligned} [\Phi_K \circ \Psi_K](\xi) &= \Phi_K(\Psi_K(\xi)) \\ &= \Phi_K(\llbracket a \odot x \mapsto \xi_a(x) \rrbracket) \\ &= \llbracket a \mapsto \llbracket x \mapsto \xi_a(x) \rrbracket \rrbracket \\ &= \llbracket a \mapsto \xi(a) \rrbracket \\ &= \xi. \end{aligned}$$

- *Naturality of  $\Phi$ .* We need to show that, given a morphism of pointed sets

$$\phi: (K, k_0) \rightarrow (K', k'_0),$$

the diagram

$$\begin{array}{ccc} \text{Sets}_*(A \odot X, K) & \xrightarrow{\Phi_K} & \text{Sets}(A, \text{Sets}_*(X, K)) \\ \phi_* \downarrow & & \downarrow (\phi_*)_* \\ \text{Sets}_*(A \odot X, K') & \xrightarrow{\Phi_{K'}} & \text{Sets}(A, \text{Sets}_*(X, K')) \end{array}$$

commutes. Indeed, given a morphism of pointed sets

$$\xi: (A \odot X, a \odot x_0) \rightarrow (K, k_0),$$

we have

$$\begin{aligned} [\Phi_{K'} \circ \phi_*](\xi) &= \Phi_{K'}(\phi_*(\xi)) \\ &= \Phi_{K'}(\phi \circ \xi) \\ &= (\phi \circ \xi)^\dagger \\ &= \llbracket a \mapsto \phi \circ \xi(a \odot -) \rrbracket \\ &= \llbracket a \mapsto \phi_*(\xi(a \odot -)) \rrbracket \\ &= (\phi_*)_* (\llbracket a \mapsto \xi(a \odot -) \rrbracket) \\ &= (\phi_*)_*(\Phi_K(\xi)) \\ &= [(\phi_*)_* \circ \Phi_K](\xi). \end{aligned}$$

- *Naturality of  $\Psi$ .* Since  $\Phi$  is natural and  $\Phi$  is a componentwise inverse to  $\Psi$ , it follows from **Categories, Item 2** of **Proposition 8.6.1.2** that  $\Psi$  is also natural.

This finishes the proof.  $\square$

**00CN Proposition 2.1.1.6.** Let  $(X, x_0)$  be a pointed set and let  $A$  be a set.

**00CP** 1. *Functoriality.* The assignments  $A, (X, x_0), (A, (X, x_0))$  define functors

$$\begin{aligned} A \odot - &: \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*, \\ - \odot X &: \mathbf{Sets} \rightarrow \mathbf{Sets}_*, \\ -_1 \odot -_2 &: \mathbf{Sets} \times \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*. \end{aligned}$$

In particular, given:

- A map of sets  $f: A \rightarrow B$ ;
- A pointed map  $\phi: (X, x_0) \rightarrow (Y, y_0)$ ;

the induced map

$$f \odot \phi: A \odot X \rightarrow B \odot Y$$

is given by

$$[f \odot \phi](a \odot x) \stackrel{\text{def}}{=} f(a) \odot \phi(x)$$

for each  $a \odot x \in A \odot X$ .

**00CQ** 2. *Adjointness I.* We have an adjunction

$$(- \odot X \dashv \mathbf{Sets}_*(X, -)) : \mathbf{Sets} \begin{array}{c} \xrightarrow{- \odot X} \\ \perp \\ \xleftarrow{\mathbf{Sets}_*(X, -)} \end{array} \mathbf{Sets}_*,$$

witnessed by a bijection

$$\mathbf{Sets}_*(A \odot X, K) \cong \mathbf{Sets}(A, \mathbf{Sets}_*(X, K)),$$

natural in  $A \in \mathbf{Obj}(\mathbf{Sets})$  and  $X, Y \in \mathbf{Obj}(\mathbf{Sets}_*)$ .

00CR 3. *Adjointness II.* We have an adjunctions

$$(A \odot - \dashv A \pitchfork -): \text{Sets}_* \begin{array}{c} \xrightarrow{A \odot -} \\ \perp \\ \xleftarrow{A \pitchfork -} \end{array} \text{Sets}_*,$$

witnessed by a bijection

$$\text{Hom}_{\text{Sets}_*}(A \odot X, Y) \cong \text{Hom}_{\text{Sets}_*}(X, A \pitchfork Y),$$

natural in  $A \in \text{Obj}(\text{Sets})$  and  $X, Y \in \text{Obj}(\text{Sets}_*)$ .

00CS 4. *As a Weighted Colimit.* We have

$$A \odot X \cong \text{colim}^{[A]}(X),$$

where in the right hand side we write:

- $A$  for the functor  $A: \text{pt} \rightarrow \text{Sets}$  picking  $A \in \text{Obj}(\text{Sets})$ ;
- $X$  for the functor  $X: \text{pt} \rightarrow \text{Sets}_*$  picking  $(X, x_0) \in \text{Obj}(\text{Sets}_*)$ .

00CT 5. *Iterated Tensors.* We have an isomorphism of pointed sets

$$A \odot (B \odot X) \cong (A \times B) \odot X,$$

natural in  $A, B \in \text{Obj}(\text{Sets})$  and  $(X, x_0) \in \text{Obj}(\text{Sets}_*)$ .

00CU 6. *Interaction With Homs.* We have a natural isomorphism

$$\text{Sets}_*(A \odot X, -) \cong A \pitchfork \text{Sets}_*(X, -).$$

00CV 7. *The Tensor Evaluation Map.* For each  $X, Y \in \text{Obj}(\text{Sets}_*)$ , we have a map

$$\text{ev}_{X,Y}^\odot: \text{Sets}_*(X, Y) \odot X \rightarrow Y,$$

natural in  $X, Y \in \text{Obj}(\text{Sets}_*)$ , and given by

$$\text{ev}_{X,Y}^\odot(f \odot x) \stackrel{\text{def}}{=} f(x)$$

for each  $f \odot x \in \text{Sets}_*(X, Y) \odot X$ .

- 00CW 8. *The Tensor Coevaluation Map.* For each  $A \in \text{Obj}(\text{Sets})$  and each  $X \in \text{Obj}(\text{Sets}_*)$ , we have a map

$$\text{coev}_{A,X}^\odot: A \rightarrow \text{Sets}_*(X, A \odot X),$$

natural in  $A \in \text{Obj}(\text{Sets})$  and  $X \in \text{Obj}(\text{Sets}_*)$ , and given by

$$\text{coev}_{A,X}^\odot(a) \stackrel{\text{def}}{=} \llbracket x \mapsto a \odot x \rrbracket$$

for each  $a \in A$ .

*Proof.* **Item 1, Functoriality:** This is the special case of ??, ?? of ?? for when  $C = \text{Sets}_*$ .

**Item 2, Adjointness I:** This is simply a rephrasing of **Definition 2.1.1.1**.

**Item 3, Adjointness II:** This is the special case of ??, ?? of ?? for when  $C = \text{Sets}_*$ .

**Item 4, As a Weighted Colimit:** This is the special case of ??, ?? of ?? for when  $C = \text{Sets}_*$ .

**Item 5, Iterated Tensors:** This is the special case of ??, ?? of ?? for when  $C = \text{Sets}_*$ .

**Item 6, Interaction With Homs:** This is the special case of ??, ?? of ?? for when  $C = \text{Sets}_*$ .

**Item 7, The Tensor Evaluation Map:** This is the special case of ??, ?? of ?? for when  $C = \text{Sets}_*$ .

**Item 8, The Tensor Coevaluation Map:** This is the special case of ??, ?? of ?? for when  $C = \text{Sets}_*$ .  $\square$

## 00CX 2.2 Cotensors of Pointed Sets by Sets

Let  $(X, x_0)$  be a pointed set and let  $A$  be a set.

- 00CY **Definition 2.2.1.1.** The **cotensor of**  $(X, x_0)$  **by**  $A^9$  is the pointed set<sup>10</sup>  $A \pitchfork (X, x_0)$  satisfying the following universal property:

(UP) We have a bijection

$$\text{Sets}_*(K, A \pitchfork X) \cong \text{Sets}(A, \text{Sets}_*(K, X)),$$

natural in  $(K, k_0) \in \text{Obj}(\text{Sets}_*)$ .

<sup>9</sup>Further Terminology: Also called the **power of**  $(X, x_0)$  **by**  $A$ .

<sup>10</sup>Further Notation: Often written  $A \pitchfork X$  for simplicity.

**00CZ Remark 2.2.1.2.** The universal property of **Definition 2.2.1.1** is equivalent to the following one:

(UP) We have a bijection

$$\text{Sets}_*(K, A \pitchfork X) \cong \text{Sets}_{\mathbb{E}_0}^{\otimes}(A \times K, X),$$

natural in  $(K, k_0) \in \text{Obj}(\text{Sets}_*)$ , where  $\text{Sets}_{\mathbb{E}_0}^{\otimes}(A \times K, X)$  is the set defined by

$$\text{Sets}_{\mathbb{E}_0}^{\otimes}(A \times K, X) \stackrel{\text{def}}{=} \left\{ f \in \text{Sets}(A \times K, X) \left| \begin{array}{l} \text{for each } a \in A, \text{ we} \\ \text{have } f(a, k_0) = x_0 \end{array} \right. \right\}.$$

*Proof.* This follows from the bijection

$$\text{Sets}(A, \text{Sets}_*(K, X)) \cong \text{Sets}_{\mathbb{E}_0}^{\otimes}(A \times K, X),$$

natural in  $(K, k_0) \in \text{Obj}(\text{Sets}_*)$  constructed in the proof of **Remark 2.1.1.2**.  $\square$

**00D0 Construction 2.2.1.3.** Concretely, the **cotensor of  $(X, x_0)$  by  $A$**  is the pointed set  $A \pitchfork (X, x_0)$  consisting of:

- *The Underlying Set.* The set  $A \pitchfork X$  given by

$$A \pitchfork X \cong \bigwedge_{a \in A} (X, x_0),$$

where  $\bigwedge_{a \in A} (X, x_0)$  is the smash product of the  $A$ -indexed family  $((X, x_0))_{a \in A}$  of **Definition 6.1.1.1**.

- *The Basepoint.* The point  $[(x_0)_{a \in A}] = [(x_0, x_0, x_0, \dots)]$  of  $\bigwedge_{a \in A} (X, x_0)$ .

*Proof.* We claim we have a bijection

$$\text{Sets}_*(K, A \pitchfork X) \cong \text{Sets}(A, \text{Sets}_*(K, X)),$$

natural in  $(K, k_0) \in \text{Obj}(\text{Sets}_*)$ .

- *Map  $I$ .* We define a map

$$\Phi_K: \text{Sets}_*(K, A \pitchfork X) \rightarrow \text{Sets}(A, \text{Sets}_*(K, X)),$$



by sending a morphism of pointed sets

$$\xi: (K, k_0) \rightarrow (A \pitchfork X, [(x_0)_{a \in A}])$$

to the map of sets

$$\begin{aligned} \xi^\dagger: A &\longrightarrow \text{Sets}_*(K, X), \\ a &\longmapsto (\xi_a: K \rightarrow X), \end{aligned}$$

where

$$\xi_a: (K, k_0) \rightarrow (X, x_0)$$

is the morphism of pointed sets defined by

$$\xi_a(k) = \begin{cases} x_a^k & \text{if } \xi(k) \neq [(x_0)_{a \in A}], \\ x_0 & \text{if } \xi(k) = [(x_0)_{a \in A}] \end{cases}$$

for each  $k \in K$ , where  $x_a^k$  is the  $a$ th component of  $\xi(k) = [(x_a^k)_{a \in A}]$ . Note that:

1. The definition of  $\xi_a(k)$  is independent of the choice of equivalence class. Indeed, suppose we have

$$\begin{aligned} \xi(k) &= \left[ (x_a^k)_{a \in A} \right] \\ &= \left[ (y_a^k)_{a \in A} \right] \end{aligned}$$

with  $x_a^k \neq y_a^k$  for some  $a \in A$ . Then there exist  $a_x, a_y \in A$  such that  $x_{a_x}^k = y_{a_y}^k = x_0$ . The equivalence relation  $\sim$  on  $\prod_{a \in A} X$  then forces

$$\begin{aligned} \left[ (x_a^k)_{a \in A} \right] &= [(x_0)_{a \in A}], \\ \left[ (y_a^k)_{a \in A} \right] &= [(x_0)_{a \in A}], \end{aligned}$$

however, and  $\xi_a(k)$  is defined to be  $x_0$  in this case.

2. The map  $\xi_a$  is indeed a morphism of pointed sets, as we have

$$\xi_a(k_0) = x_0$$

since  $\xi(k_0) = [(x_0)_{a \in A}]$  as  $\xi$  is a morphism of pointed sets and  $\xi_a(k_0)$ , defined to be the  $a$ th component of  $[(x_0)_{a \in A}]$ , is equal to  $x_0$ .

- *Map II.* We define a map

$$\Psi_K : \text{Sets}(A, \text{Sets}_*(K, X)) \rightarrow \text{Sets}_*(K, A \pitchfork X),$$

given by sending a map

$$\begin{aligned} \xi : A &\rightarrow \text{Sets}_*(K, X), \\ a &\mapsto (\xi_a : K \rightarrow X), \end{aligned}$$

to the morphism of pointed sets

$$\xi^\dagger : (K, k_0) \rightarrow (A \pitchfork X, [(x_0)_{a \in A}])$$

defined by

$$\xi^\dagger(k) \stackrel{\text{def}}{=} [(\xi_a(k))_{a \in A}]$$

for each  $k \in K$ . Note that  $\xi^\dagger$  is indeed a morphism of pointed sets, as we have

$$\begin{aligned} \xi^\dagger(k_0) &\stackrel{\text{def}}{=} [(\xi_a(k_0))_{a \in A}] \\ &= x_0, \end{aligned}$$

where we have used that  $\xi_a \in \text{Sets}_*(K, X)$  is a morphism of pointed sets for each  $a \in A$ .

- *Naturality of  $\Psi$ .* We need to show that, given a morphism of pointed sets

$$\phi : (K, k_0) \rightarrow (K', k'_0),$$

the diagram

$$\begin{array}{ccc} \text{Sets}(A, \text{Sets}_*(K', X)) & \xrightarrow{\Psi_{K'}} & \text{Sets}_*(K', A \pitchfork X) \\ (\phi^*)_* \downarrow & & \downarrow \phi^* \\ \text{Sets}(A, \text{Sets}_*(K, X)) & \xrightarrow{\Psi_K} & \text{Sets}_*(K, A \pitchfork X) \end{array}$$

commutes. Indeed, given a map of sets

$$\begin{aligned} \xi : A &\rightarrow \text{Sets}_*(K', X), \\ a &\mapsto (\xi_a : K' \rightarrow X), \end{aligned}$$

we have

$$\begin{aligned}
[\Psi_K \circ (\phi^*)_*](\xi) &= \Psi_K((\phi^*)_*(\xi)) \\
&= \Psi_K((\phi^*)_*(\llbracket a \mapsto \xi_a \rrbracket)) \\
&= \Psi_K(\llbracket a \mapsto \phi^*(\xi_a) \rrbracket) \\
&= \Psi_K(\llbracket a \mapsto \llbracket k \mapsto \xi_a(\phi(k)) \rrbracket \rrbracket) \\
&= \llbracket k \mapsto [(\xi_a(\phi(k)))_{a \in A}] \rrbracket \\
&= \phi^*(\llbracket k' \mapsto [(\xi_a(k'))_{a \in A}] \rrbracket) \\
&= \phi^*(\Psi_{K'}(\xi)) \\
&= [\phi^* \circ \Psi_{K'}](\xi).
\end{aligned}$$

- *Naturality of  $\Phi$ .* Since  $\Psi$  is natural and  $\Psi$  is a componentwise inverse to  $\Phi$ , it follows from [Categories, Item 2](#) of [Proposition 8.6.1.2](#) that  $\Phi$  is also natural.
- *Invertibility I.* We claim that

$$\Psi_K \circ \Phi_K = \text{id}_{\text{Sets}_*(K, A \pitchfork X)}.$$

Indeed, given a morphism of pointed sets

$$\xi: (K, k_0) \rightarrow (A \pitchfork X, [(x_0)_{a \in A}])$$

we have

$$\begin{aligned}
[\Psi_K \circ \Phi_K](\xi) &= \Psi_K(\Phi_K(\xi)) \\
&= \Psi_K(\llbracket a \mapsto \xi_a \rrbracket) \\
&= \Psi_K(\llbracket a' \mapsto \xi_{a'} \rrbracket) \\
&= \llbracket k \mapsto [(\text{ev}_a(\llbracket a' \mapsto \xi_{a'}(k) \rrbracket))_{a \in A}] \rrbracket \\
&= \llbracket k \mapsto [(\xi_a(k))_{a \in A}] \rrbracket.
\end{aligned}$$

Now, we have two cases:

1. If  $\xi(k) = [(x_0)_{a \in A}]$ , we have

$$\begin{aligned}
[\Psi_K \circ \Phi_K](\xi) &= \dots \\
&= \llbracket k \mapsto [(\xi_a(k))_{a \in A}] \rrbracket \\
&= \llbracket k \mapsto [(x_0)_{a \in A}] \rrbracket \\
&= \llbracket k \mapsto \xi(k) \rrbracket \\
&= \xi.
\end{aligned}$$

2. If  $\xi(k) \neq [(x_0)_{a \in A}]$  and  $\xi(k) = [(x_a^k)_{a \in A}]$  instead, we have

$$\begin{aligned} [\Psi_K \circ \Phi_K](\xi) &= \dots \\ &= \llbracket k \mapsto [(\xi_a(k))_{a \in A}] \rrbracket \\ &= \llbracket k \mapsto [(x_a^k)_{a \in A}] \rrbracket \\ &= \llbracket k \mapsto \xi(k) \rrbracket \\ &= \xi. \end{aligned}$$

In both cases, we have  $[\Psi_K \circ \Phi_K](\xi) = \xi$ , and thus we are done.

- *Invertibility II.* We claim that

$$\Phi_K \circ \Psi_K = \text{id}_{\text{Sets}(A, \text{Sets}_*(K, X))}.$$

Indeed, given a morphism  $\xi: A \rightarrow \text{Sets}_*(K, X)$ , we have

$$\begin{aligned} [\Phi_K \circ \Psi_K](\xi) &= \Phi_K(\Psi_K(\xi)) \\ &= \Phi_K(\llbracket k \mapsto [(\xi_a(k))_{a \in A}] \rrbracket) \\ &= \llbracket a \mapsto \llbracket k \mapsto \xi_a(k) \rrbracket \rrbracket \\ &= \xi \end{aligned}$$

This finishes the proof.  $\square$

**00D1 Proposition 2.2.1.4.** Let  $(X, x_0)$  be a pointed set and let  $A$  be a set.

**00D2** 1. *Functoriality.* The assignments  $A, (X, x_0), (A, (X, x_0))$  define functors

$$\begin{aligned} A &\curvearrowright -: \text{Sets}_* \rightarrow \text{Sets}_*, \\ - &\curvearrowright X: \text{Sets}^{\text{op}} \rightarrow \text{Sets}_*, \\ -_1 &\curvearrowright -_2: \text{Sets}^{\text{op}} \times \text{Sets}_* \rightarrow \text{Sets}_*. \end{aligned}$$

In particular, given:

- A map of sets  $f: A \rightarrow B$ ;
- A pointed map  $\phi: (X, x_0) \rightarrow (Y, y_0)$ ;

the induced map

$$f \odot \phi: A \pitchfork X \rightarrow B \pitchfork Y$$

is given by

$$[f \odot \phi]([ (x_a)_{a \in A} ]) \stackrel{\text{def}}{=} [(\phi(x_{f(a)}))_{a \in A}]$$

for each  $[ (x_a)_{a \in A} ] \in A \pitchfork X$ .

**00D3** 2. *Adjointness I.* We have an adjunction

$$(- \pitchfork X \dashv \text{Sets}_*(-, X)): \text{Sets}^{\text{op}} \begin{array}{c} \xrightarrow{- \pitchfork X} \\ \perp \\ \xleftarrow{\text{Sets}_*(-, X)} \end{array} \text{Sets}_*,$$

witnessed by a bijection

$$\text{Sets}_*^{\text{op}}(A \pitchfork X, K) \cong \text{Sets}(A, \text{Sets}_*(K, X)),$$

i.e. by a bijection

$$\text{Sets}_*(K, A \pitchfork X) \cong \text{Sets}(A, \text{Sets}_*(K, X)),$$

natural in  $A \in \text{Obj}(\text{Sets})$  and  $X, Y \in \text{Obj}(\text{Sets}_*)$ .

**00D4** 3. *Adjointness II.* We have an adjunctions

$$(A \odot - \dashv A \pitchfork -): \text{Sets}_* \begin{array}{c} \xrightarrow{A \odot -} \\ \perp \\ \xleftarrow{A \pitchfork -} \end{array} \text{Sets}_*,$$

witnessed by a bijection

$$\text{Hom}_{\text{Sets}_*}(A \odot X, Y) \cong \text{Hom}_{\text{Sets}_*}(X, A \pitchfork Y),$$

natural in  $A \in \text{Obj}(\text{Sets})$  and  $X, Y \in \text{Obj}(\text{Sets}_*)$ .

**00D5** 4. *As a Weighted Limit.* We have

$$A \pitchfork X \cong \lim^{[A]}(X),$$

where in the right hand side we write:

- $A$  for the functor  $A: \mathbf{pt} \rightarrow \mathbf{Sets}$  picking  $A \in \mathbf{Obj}(\mathbf{Sets})$ ;
- $X$  for the functor  $X: \mathbf{pt} \rightarrow \mathbf{Sets}_*$  picking  $(X, x_0) \in \mathbf{Obj}(\mathbf{Sets}_*)$ .

00D6 5. *Iterated Cotensors.* We have an isomorphism of pointed sets

$$A \pitchfork (B \pitchfork X) \cong (A \times B) \pitchfork X,$$

natural in  $A, B \in \mathbf{Obj}(\mathbf{Sets})$  and  $(X, x_0) \in \mathbf{Obj}(\mathbf{Sets}_*)$ .

00D7 6. *Commutativity With Homs.* We have natural isomorphisms

$$\begin{aligned} A \pitchfork \mathbf{Sets}_*(X, -) &\cong \mathbf{Sets}_*(A \odot X, -), \\ A \pitchfork \mathbf{Sets}_*(-, Y) &\cong \mathbf{Sets}_*(-, A \pitchfork Y). \end{aligned}$$

00D8 7. *The Cotensor Evaluation Map.* For each  $X, Y \in \mathbf{Obj}(\mathbf{Sets}_*)$ , we have a map

$$\mathrm{ev}_{X,Y}^{\pitchfork}: X \rightarrow \mathbf{Sets}_*(X, Y) \pitchfork Y,$$

natural in  $X, Y \in \mathbf{Obj}(\mathbf{Sets}_*)$ , and given by

$$\mathrm{ev}_{X,Y}^{\pitchfork}(x) \stackrel{\mathrm{def}}{=} \left[ (f(x))_{f \in \mathbf{Sets}_*(X,Y)} \right]$$

for each  $x \in X$ .

00D9 8. *The Cotensor Coevaluation Map.* For each  $X \in \mathbf{Obj}(\mathbf{Sets}_*)$  and each  $A \in \mathbf{Obj}(\mathbf{Sets})$ , we have a map

$$\mathrm{coev}_{A,X}^{\pitchfork}: A \rightarrow \mathbf{Sets}_*(A \pitchfork X, X),$$

natural in  $X \in \mathbf{Obj}(\mathbf{Sets}_*)$  and  $A \in \mathbf{Obj}(\mathbf{Sets})$ , and given by

$$\mathrm{coev}_{A,X}^{\pitchfork}(a) \stackrel{\mathrm{def}}{=} \llbracket [(x_b)_{b \in A}] \mapsto x_a \rrbracket$$

for each  $a \in A$ .

*Proof.* **Item 1, Functoriality:** This is the special case of ??, ?? of ?? for when  $C = \mathbf{Sets}_*$ .

**Item 2, Adjointness I:** This is simply a rephrasing of **Definition 2.2.1.1**.

**Item 3, : Adjointness II:** This is the special case of ??, ?? of ?? for when  $C = \mathbf{Sets}_*$ .

**Item 4, As a Weighted Limit:** This is the special case of ??, ?? of ?? for when  $C = \mathbf{Sets}_*$ .

*Item 5, Iterated Cotensors:* This is the special case of ??, ?? of ?? for when  $C = \mathbf{Sets}_*$ .

*Item 6, Commutativity With Homs:* This is the special case of ??, ?? of ?? for when  $C = \mathbf{Sets}_*$ .

*Item 7, The Cotensor Evaluation Map:* This is the special case of ??, ?? of ?? for when  $C = \mathbf{Sets}_*$ .

*Item 8, The Cotensor Coevaluation Map:* This is the special case of ??, ?? of ?? for when  $C = \mathbf{Sets}_*$ .  $\square$

### 00DA 3 The Left Tensor Product of Pointed Sets

#### 00DB 3.1 Foundations

Let  $(X, x_0)$  and  $(Y, y_0)$  be pointed sets.

00DC **Definition 3.1.1.1.** The **left tensor product of pointed sets** is the functor<sup>11</sup>

$$\triangleleft : \mathbf{Sets}_* \times \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*$$

defined as the composition

$$\mathbf{Sets}_* \times \mathbf{Sets}_* \xrightarrow{\text{id} \times \overline{\omega}} \mathbf{Sets}_* \times \mathbf{Sets} \xrightarrow{\beta_{\mathbf{Sets}_*, \mathbf{Sets}}^{\mathbf{Cats}_2}} \mathbf{Sets} \times \mathbf{Sets}_* \xrightarrow{\odot} \mathbf{Sets}_*,$$

where:

- $\overline{\omega} : \mathbf{Sets}_* \rightarrow \mathbf{Sets}$  is the forgetful functor from pointed sets to sets.
- $\beta_{\mathbf{Sets}_*, \mathbf{Sets}}^{\mathbf{Cats}_2} : \mathbf{Sets}_* \times \mathbf{Sets} \xrightarrow{\cong} \mathbf{Sets} \times \mathbf{Sets}_*$  is the braiding of  $\mathbf{Cats}_2$ , i.e. the functor witnessing the isomorphism

$$\mathbf{Sets}_* \times \mathbf{Sets} \cong \mathbf{Sets} \times \mathbf{Sets}_*.$$

- $\odot : \mathbf{Sets} \times \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*$  is the tensor functor of **Item 1** of **Proposition 2.1.1.6**.

00DD **Remark 3.1.1.2.** The left tensor product of pointed sets satisfies the following natural bijection:

$$\mathbf{Sets}_*(X \triangleleft Y, Z) \cong \text{Hom}_{\mathbf{Sets}_*}^{\otimes, L}(X \times Y, Z).$$

That is to say, the following data are in natural bijection:

<sup>11</sup>Further Notation: Also written  $\triangleleft_{\mathbf{Sets}_*}$ .

1. Pointed maps  $f: X \triangleleft Y \rightarrow Z$ .
2. Maps of sets  $f: X \times Y \rightarrow Z$  satisfying  $f(x_0, y) = z_0$  for each  $y \in Y$ .

**00DE Remark 3.1.1.3.** The left tensor product of pointed sets may be described as follows:

- The left tensor product of  $(X, x_0)$  and  $(Y, y_0)$  is the pair  $((X \triangleleft Y, x_0 \triangleleft y_0), \iota)$  consisting of
  - A pointed set  $(X \triangleleft Y, x_0 \triangleleft y_0)$ ;
  - A left bilinear morphism of pointed sets  $\iota: (X \times Y, (x_0, y_0)) \rightarrow X \triangleleft Y$ ;

satisfying the following universal property:

(UP) Given another such pair  $((Z, z_0), f)$  consisting of

- \* A pointed set  $(Z, z_0)$ ;
- \* A left bilinear morphism of pointed sets  $f: (X \times Y, (x_0, y_0)) \rightarrow Z$ ;

there exists a unique morphism of pointed sets  $X \triangleleft Y \xrightarrow{\exists!} Z$  making the diagram

$$\begin{array}{ccc}
 & & X \triangleleft Y \\
 & \nearrow \iota & \downarrow \exists! \\
 X \times Y & \xrightarrow{f} & Z
 \end{array}$$

commute.

**00DF Construction 3.1.1.4.** In detail, the **left tensor product of  $(X, x_0)$  and  $(Y, y_0)$**  is the pointed set  $(X \triangleleft Y, [x_0])$  consisting of

- *The Underlying Set.* The set  $X \triangleleft Y$  defined by

$$\begin{aligned}
 X \triangleleft Y &\stackrel{\text{def}}{=} |Y| \odot X \\
 &\cong \bigvee_{y \in Y} (X, x_0),
 \end{aligned}$$

where  $|Y|$  denotes the underlying set of  $(Y, y_0)$ ;



- *The Underlying Basepoint.* The point  $[(y_0, x_0)]$  of  $\bigvee_{y \in Y} (X, x_0)$ , which is equal to  $[(y, x_0)]$  for any  $y \in Y$ .

**00DG** **Notation 3.1.1.5.** We write<sup>12</sup>  $x \triangleleft y$  for the element  $[(y, x)]$  of

$$X \triangleleft Y \cong |Y| \odot X.$$

**00DH** **Remark 3.1.1.6.** Employing the notation introduced in **Notation 3.1.1.5**, we have

$$x_0 \triangleleft y_0 = x_0 \triangleleft y$$

for each  $y \in Y$ , and

$$x_0 \triangleleft y = x_0 \triangleleft y'$$

for each  $y, y' \in Y$ .

**00DJ** **Proposition 3.1.1.7.** Let  $(X, x_0)$  and  $(Y, y_0)$  be pointed sets.

**00DK** 1. *Functoriality.* The assignments  $X, Y, (X, Y) \mapsto X \triangleleft Y$  define functors

$$\begin{aligned} X \triangleleft - &: \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*, \\ - \triangleleft Y &: \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*, \\ -_1 \triangleleft -_2 &: \mathbf{Sets}_* \times \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*. \end{aligned}$$

In particular, given pointed maps

$$\begin{aligned} f &: (X, x_0) \rightarrow (A, a_0), \\ g &: (Y, y_0) \rightarrow (B, b_0), \end{aligned}$$

the induced map

$$f \triangleleft g: X \triangleleft Y \rightarrow A \triangleleft B$$

is given by

$$[f \triangleleft g](x \triangleleft y) \stackrel{\text{def}}{=} f(x) \triangleleft g(y)$$

for each  $x \triangleleft y \in X \triangleleft Y$ .

---

<sup>12</sup>*Further Notation:* Also written  $x \triangleleft_{\mathbf{Sets}_*} y$ .

00DL 2. *Adjointness I.* We have an adjunction

$$\left( - \triangleleft Y \dashv [Y, -]_{\mathbf{Sets}_*}^{\triangleleft} \right): \mathbf{Sets}_* \begin{array}{c} \xrightarrow{- \triangleleft Y} \\ \perp \\ \xleftarrow{[Y, -]_{\mathbf{Sets}_*}^{\triangleleft}} \end{array} \mathbf{Sets}_*,$$

witnessed by a bijection of sets

$$\mathrm{Hom}_{\mathbf{Sets}_*}(X \triangleleft Y, Z) \cong \mathrm{Hom}_{\mathbf{Sets}_*}\left(X, [Y, Z]_{\mathbf{Sets}_*}^{\triangleleft}\right)$$

natural in  $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathbf{Sets}_*)$ , where  $[X, Y]_{\mathbf{Sets}_*}^{\triangleleft}$  is the pointed set of [Definition 3.2.1.1](#).

00DM 3. *Adjointness II.* The functor

$$X \triangleleft - : \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*$$

does not admit a right adjoint.

00DN 4. *Adjointness III.* We have a bijection of sets

$$\mathrm{Hom}_{\mathbf{Sets}_*}(X \triangleleft Y, Z) \cong \mathrm{Hom}_{\mathbf{Sets}}(|Y|, \mathbf{Sets}_*(X, Z))$$

natural in  $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathbf{Sets}_*)$ .

*Proof. Item 1, Functoriality:* Clear.

*Item 2, Adjointness I:* This follows from [Item 3](#) of [Proposition 2.1.1.6](#).

*Item 3, Adjointness II:* For  $X \triangleleft -$  to admit a right adjoint would require it to preserve colimits by [??](#), [??](#) of [??](#). However, we have

$$\begin{aligned} X \triangleleft \mathrm{pt} &\stackrel{\mathrm{def}}{=} |\mathrm{pt}| \odot X \\ &\cong X \\ &\not\cong \mathrm{pt}, \end{aligned}$$

and thus we see that  $X \triangleleft -$  does not have a right adjoint.

*Item 4, Adjointness III:* This follows from [Item 2](#) of [Proposition 2.1.1.6](#).  $\square$

00DP **Remark 3.1.1.8.** Here is some intuition on why  $X \triangleleft -$  fails to be a left adjoint.

[Item 4](#) of [Proposition 3.1.1.7](#) states that we have a natural bijection

$$\mathrm{Hom}_{\mathbf{Sets}_*}(X \triangleleft Y, Z) \cong \mathrm{Hom}_{\mathbf{Sets}}(|Y|, \mathbf{Sets}_*(X, Z)),$$

so it would be reasonable to wonder whether a natural bijection of the form

$$\mathrm{Hom}_{\mathbf{Sets}_*}(X \triangleleft Y, Z) \cong \mathrm{Hom}_{\mathbf{Sets}_*}(Y, \mathbf{Sets}_*(X, Z)),$$

also holds, which would give  $X \triangleleft - \dashv \mathbf{Sets}_*(X, -)$ . However, such a bijection would require every map

$$f: X \triangleleft Y \rightarrow Z$$

to satisfy

$$f(x \triangleleft y_0) = z_0$$

for each  $x \in X$ , whereas we are imposing such a basepoint preservation condition only for elements of the form  $x_0 \triangleleft y$ . Thus  $\mathbf{Sets}_*(X, -)$  can't be a right adjoint for  $X \triangleleft -$ , and as shown by **Item 3** of **Proposition 3.1.1.7**, no functor can.<sup>13</sup>

### 00DQ 3.2 The Left Internal Hom of Pointed Sets

Let  $(X, x_0)$  and  $(Y, y_0)$  be pointed sets.

00DR **Definition 3.2.1.1.** The **left internal Hom of pointed sets** is the functor

$$[-, -]_{\mathbf{Sets}_*}^{\triangleleft} : \mathbf{Sets}_*^{\mathrm{op}} \times \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*$$

defined as the composition

$$\mathbf{Sets}_*^{\mathrm{op}} \times \mathbf{Sets}_* \xrightarrow{\omega \times \mathrm{id}} \mathbf{Sets}^{\mathrm{op}} \times \mathbf{Sets}_* \xrightarrow{\pitchfork} \mathbf{Sets}_*,$$

where:

- $\omega: \mathbf{Sets}_* \rightarrow \mathbf{Sets}$  is the forgetful functor from pointed sets to sets.
- $\pitchfork: \mathbf{Sets}^{\mathrm{op}} \times \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*$  is the cotensor functor of **Item 1** of **Proposition 2.2.1.4**.

*Proof.* For a proof that  $[-, -]_{\mathbf{Sets}_*}^{\triangleleft}$  is indeed the left internal Hom of  $\mathbf{Sets}_*$  with respect to the left tensor product of pointed sets, see **Item 2** of **Proposition 3.1.1.7**.  $\square$

<sup>13</sup>The functor  $\mathbf{Sets}_*(X, -)$  is instead right adjoint to  $X \wedge -$ , the smash product of pointed sets of **Definition 5.1.1.1**. See **Item 2** of **Proposition 5.1.1.9**.

**00DS Remark 3.2.1.2.** The left internal Hom of pointed sets satisfies the following universal property:

$$\mathbf{Sets}_*(X \triangleleft Y, Z) \cong \mathbf{Sets}_*\left(X, [Y, Z]_{\mathbf{Sets}_*}^{\triangleleft}\right)$$

That is to say, the following data are in bijection:

1. Pointed maps  $f: X \triangleleft Y \rightarrow Z$ .
2. Pointed maps  $f: X \rightarrow [Y, Z]_{\mathbf{Sets}_*}^{\triangleleft}$ .

**00DT Remark 3.2.1.3.** In detail, the **left internal Hom of**  $(X, x_0)$  **and**  $(Y, y_0)$  is the pointed set  $\left([X, Y]_{\mathbf{Sets}_*}^{\triangleleft}, [(y_0)_{x \in X}]\right)$  consisting of

- *The Underlying Set.* The set  $[X, Y]_{\mathbf{Sets}_*}^{\triangleleft}$  defined by

$$\begin{aligned} [X, Y]_{\mathbf{Sets}_*}^{\triangleleft} &\stackrel{\text{def}}{=} |X| \pitchfork Y \\ &\cong \bigwedge_{x \in X} (Y, y_0), \end{aligned}$$

where  $|X|$  denotes the underlying set of  $(X, x_0)$ ;

- *The Underlying Basepoint.* The point  $[(y_0)_{x \in X}]$  of  $\bigwedge_{x \in X} (Y, y_0)$ .

**00DU Proposition 3.2.1.4.** Let  $(X, x_0)$  and  $(Y, y_0)$  be pointed sets.

**00DV** 1. *Functoriality.* The assignments  $X, Y, (X, Y) \mapsto [X, Y]_{\mathbf{Sets}_*}^{\triangleleft}$  define functors

$$\begin{aligned} [X, -]_{\mathbf{Sets}_*}^{\triangleleft} &: \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*, \\ [-, Y]_{\mathbf{Sets}_*}^{\triangleleft} &: \mathbf{Sets}_*^{\text{op}} \rightarrow \mathbf{Sets}_*, \\ [-_1, -_2]_{\mathbf{Sets}_*}^{\triangleleft} &: \mathbf{Sets}_*^{\text{op}} \times \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*. \end{aligned}$$

In particular, given pointed maps

$$\begin{aligned} f &: (X, x_0) \rightarrow (A, a_0), \\ g &: (Y, y_0) \rightarrow (B, b_0), \end{aligned}$$

the induced map

$$[f, g]_{\mathbf{Sets}_*}^{\triangleleft} : [A, Y]_{\mathbf{Sets}_*}^{\triangleleft} \rightarrow [X, B]_{\mathbf{Sets}_*}^{\triangleleft}$$

is given by

$$[f, g]_{\mathbf{Sets}_*}^{\triangleleft} \left( [(y_a)_{a \in A}] \right) \stackrel{\text{def}}{=} [(g(y_{f(x)}))_{x \in X}]$$

for each  $[(y_a)_{a \in A}] \in [A, Y]_{\mathbf{Sets}_*}^{\triangleleft}$ .

00DW 2. *Adjointness I.* We have an adjunction

$$\left( - \triangleleft Y \dashv [Y, -]_{\mathbf{Sets}_*}^{\triangleleft} \right): \mathbf{Sets}_* \begin{array}{c} \xrightarrow{- \triangleleft Y} \\ \perp \\ \xleftarrow{[Y, -]_{\mathbf{Sets}_*}^{\triangleleft}} \end{array} \mathbf{Sets}_*$$

witnessed by a bijection of sets

$$\text{Hom}_{\mathbf{Sets}_*}(X \triangleleft Y, Z) \cong \text{Hom}_{\mathbf{Sets}_*}\left(X, [Y, Z]_{\mathbf{Sets}_*}^{\triangleleft}\right)$$

natural in  $(X, x_0), (Y, y_0), (Z, z_0) \in \text{Obj}(\mathbf{Sets}_*)$

00DX 3. *Adjointness II.* The functor

$$X \triangleleft -: \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*$$

does not admit a right adjoint.

*Proof.* **Item 1, Functoriality:** Clear.

**Item 2, Adjointness I:** This is a repetition of **Item 2** of **Proposition 3.1.1.7**, and is proved there.

**Item 3, Adjointness II:** This is a repetition of **Item 3** of **Proposition 3.1.1.7**, and is proved there.  $\square$

### 00DY 3.3 The Left Skew Unit

00DZ **Definition 3.3.1.1.** The left skew unit of the left tensor product of pointed sets is the functor

$$\mathbb{1}_{\mathbf{Sets}_*, \triangleleft}: \text{pt} \rightarrow \mathbf{Sets}_*$$

defined by

$$\mathbb{1}_{\mathbf{Sets}_*}^{\triangleleft} \stackrel{\text{def}}{=} S^0.$$

### 00E0 3.4 The Left Skew Associator

00E1 **Definition 3.4.1.1.** The **skew associator of the left tensor product of pointed sets** is the natural transformation

$$\alpha^{\text{Sets}_*, \triangleleft} : \triangleleft \circ (\triangleleft \times \text{id}_{\text{Sets}_*}) \Longrightarrow \triangleleft \circ (\text{id}_{\text{Sets}_*} \times \triangleleft) \circ \alpha^{\text{Cats}}_{\text{Sets}_*, \text{Sets}_*, \text{Sets}_*}$$

as in the diagram

$$\begin{array}{ccc}
 & \text{Sets}_* \times (\text{Sets}_* \times \text{Sets}_*) & \\
 \alpha^{\text{Cats}}_{\text{Sets}_*, \text{Sets}_*, \text{Sets}_*} \nearrow & & \searrow \text{id} \times \triangleleft \\
 (\text{Sets}_* \times \text{Sets}_*) \times \text{Sets}_* & & \text{Sets}_* \times \text{Sets}_* \\
 \triangleleft \times \text{id} \searrow & \alpha^{\text{Sets}_*, \triangleleft} \nearrow & \searrow \triangleleft \\
 \text{Sets}_* \times \text{Sets}_* & \xrightarrow{\triangleleft} & \text{Sets}_*
 \end{array}$$

whose component

$$\alpha^{\text{Sets}_*, \triangleleft}_{X,Y,Z} : (X \triangleleft Y) \triangleleft Z \rightarrow X \triangleleft (Y \triangleleft Z)$$

at  $(X, x_0), (Y, y_0), (Z, z_0) \in \text{Obj}(\text{Sets}_*)$  is given by

$$\begin{aligned}
 (X \triangleleft Y) \triangleleft Z &\stackrel{\text{def}}{=} |Z| \odot (X \triangleleft Y) \\
 &\stackrel{\text{def}}{=} |Z| \odot (|Y| \odot X) \\
 &\cong \bigvee_{z \in Z} |Y| \odot X \\
 &\cong \bigvee_{z \in Z} \left( \bigvee_{y \in Y} X \right) \\
 &\rightarrow \bigvee_{[(z,y)] \in \bigvee_{z \in Z} Y} X \\
 &\cong \bigvee_{[(z,y)] \in |Z| \odot Y} X \\
 &\cong ||Z| \odot Y| \odot X \\
 &\stackrel{\text{def}}{=} |Y \triangleleft Z| \odot X \\
 &\stackrel{\text{def}}{=} X \triangleleft (Y \triangleleft Z),
 \end{aligned}$$

where the map

$$\bigvee_{z \in Z} \left( \bigvee_{y \in Y} X \right) \rightarrow \bigvee_{(z,y) \in \bigvee_{z \in Z} Y} X$$

is given by  $[(z, [(y, x)])] \mapsto [([(z, y)], x)]$ .

*Proof.* (Proven below in a bit.) □

**00E2 Remark 3.4.1.2.** Unwinding the notation for elements, we have

$$\begin{aligned} [(z, [(y, x)])] &\stackrel{\text{def}}{=} [(z, x \triangleleft y)] \\ &\stackrel{\text{def}}{=} (x \triangleleft y) \triangleleft z \end{aligned}$$

and

$$\begin{aligned} [([(z, y)], x)] &\stackrel{\text{def}}{=} [(y \triangleleft z, x)] \\ &\stackrel{\text{def}}{=} x \triangleleft (y \triangleleft z). \end{aligned}$$

So, in other words,  $\alpha_{X,Y,Z}^{\text{Sets}_*, \triangleleft}$  acts on elements via

$$\alpha_{X,Y,Z}^{\text{Sets}_*, \triangleleft} ((x \triangleleft y) \triangleleft z) \stackrel{\text{def}}{=} x \triangleleft (y \triangleleft z)$$

for each  $(x \triangleleft y) \triangleleft z \in (X \triangleleft Y) \triangleleft Z$ .

**00E3 Remark 3.4.1.3.** Taking  $y = y_0$ , we see that the morphism  $\alpha_{X,Y,Z}^{\text{Sets}_*, \triangleleft}$  acts on elements as

$$\alpha_{X,Y,Z}^{\text{Sets}_*, \triangleleft} ((x \triangleleft y_0) \triangleleft z) \stackrel{\text{def}}{=} x \triangleleft (y_0 \triangleleft z).$$

However, by the definition of  $\triangleleft$ , we have  $y_0 \triangleleft z = y_0 \triangleleft z'$  for all  $z, z' \in Z$ , preventing  $\alpha_{X,Y,Z}^{\text{Sets}_*, \triangleleft}$  from being non-invertible.

*Proof.* Firstly, note that, given  $(X, x_0), (Y, y_0), (Z, z_0) \in \text{Obj}(\text{Sets}_*)$ , the map

$$\alpha_{X,Y,Z}^{\text{Sets}_*, \triangleleft} : (X \triangleleft Y) \triangleleft Z \rightarrow X \triangleleft (Y \triangleleft Z)$$

is indeed a morphism of pointed sets, as we have

$$\alpha_{X,Y,Z}^{\text{Sets}_*, \triangleleft} ((x_0 \triangleleft y_0) \triangleleft z_0) = x_0 \triangleleft (y_0 \triangleleft z_0).$$

Next, we claim that  $\alpha^{\text{Sets}_*, \triangleleft}$  is a natural transformation. We need to show that,

given morphisms of pointed sets

$$\begin{aligned} f &: (X, x_0) \rightarrow (X', x'_0), \\ g &: (Y, y_0) \rightarrow (Y', y'_0), \\ h &: (Z, z_0) \rightarrow (Z', z'_0) \end{aligned}$$

the diagram

$$\begin{array}{ccc} (X \triangleleft Y) \triangleleft Z & \xrightarrow{(f \triangleleft g) \triangleleft h} & (X' \triangleleft Y') \triangleleft Z' \\ \alpha_{X,Y,Z}^{\text{Sets}_*, \triangleleft} \downarrow & & \downarrow \alpha_{X',Y',Z'}^{\text{Sets}_*, \triangleleft} \\ X \triangleleft (Y \triangleleft Z) & \xrightarrow{f \triangleleft (g \triangleleft h)} & X' \triangleleft (Y' \triangleleft Z') \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc} (x \triangleleft y) \triangleleft z & \longmapsto & (f(x) \triangleleft g(y)) \triangleleft h(z) \\ \downarrow & & \downarrow \\ x \triangleleft (y \triangleleft z) & \longmapsto & f(x) \triangleleft (g(y) \triangleleft h(z)) \end{array}$$

and hence indeed commutes, showing  $\alpha^{\text{Sets}_*, \triangleleft}$  to be a natural transformation. This finishes the proof.  $\square$

### 00E4 3.5 The Left Skew Left Unitor

00E5 **Definition 3.5.1.1.** The **skew left unitor of the left tensor product of pointed sets** is the natural transformation

$$\lambda^{\text{Sets}_*, \triangleleft} : \triangleleft \circ (\mathbb{1}_{\text{Sets}_*} \times \text{id}_{\text{Sets}_*}) \xrightarrow{\sim} \lambda_{\text{Sets}_*}^{\text{Cats}_2}$$

whose component

$$\lambda_X^{\text{Sets}_*, \triangleleft} : S^0 \triangleleft X \rightarrow X$$



at  $(X, x_0) \in \text{Obj}(\text{Sets}_*)$  is given by the composition

$$\begin{aligned} S^0 \triangleleft X &\cong |X| \odot S^0 \\ &\cong \bigvee_{x \in X} S^0 \\ &\rightarrow X, \end{aligned}$$

where  $\bigvee_{x \in X} S^0 \rightarrow X$  is the map given by

$$\begin{aligned} [(x, 0)] &\mapsto x_0, \\ [(x, 1)] &\mapsto x. \end{aligned}$$

*Proof.* (Proven below in a bit.) □

**00E6 Remark 3.5.1.2.** In other words,  $\lambda_X^{\text{Sets}_*, \triangleleft}$  acts on elements as

$$\begin{aligned} \lambda_X^{\text{Sets}_*, \triangleleft}(0 \triangleleft x) &\stackrel{\text{def}}{=} x_0, \\ \lambda_X^{\text{Sets}_*, \triangleleft}(1 \triangleleft x) &\stackrel{\text{def}}{=} x \end{aligned}$$

for each  $1 \triangleleft x \in S^0 \triangleleft X$ .

**00E7 Remark 3.5.1.3.** The morphism  $\lambda_X^{\text{Sets}_*, \triangleleft}$  is almost invertible, with its would-be-inverse

$$\phi_X: X \rightarrow S^0 \triangleleft X$$

given by

$$\phi_X(x) \stackrel{\text{def}}{=} 1 \triangleleft x$$

for each  $x \in X$ . Indeed, we have

$$\begin{aligned} \left[ \lambda_X^{\text{Sets}_*, \triangleleft} \circ \phi \right](x) &= \lambda_X^{\text{Sets}_*, \triangleleft}(\phi(x)) \\ &= \lambda_X^{\text{Sets}_*, \triangleleft}(1 \triangleleft x) \\ &= x \\ &= [\text{id}_X](x) \end{aligned}$$

so that

$$\lambda_X^{\text{Sets}_*, \triangleleft} \circ \phi = \text{id}_X$$

and

$$\begin{aligned}
 \left[ \phi \circ \lambda_X^{\text{Sets}_*, \triangleleft} \right] (1 \triangleleft x) &= \phi \left( \lambda_X^{\text{Sets}_*, \triangleleft} (1 \triangleleft x) \right) \\
 &= \phi(x) \\
 &= 1 \triangleleft x \\
 &= [\text{id}_{S^0 \triangleleft X}] (1 \triangleleft x),
 \end{aligned}$$

but

$$\begin{aligned}
 \left[ \phi \circ \lambda_X^{\text{Sets}_*, \triangleleft} \right] (0 \triangleleft x) &= \phi \left( \lambda_X^{\text{Sets}_*, \triangleleft} (0 \triangleleft x) \right) \\
 &= \phi(x_0) \\
 &= 1 \triangleleft x_0,
 \end{aligned}$$

where  $0 \triangleleft x \neq 1 \triangleleft x_0$ . Thus

$$\phi \circ \lambda_X^{\text{Sets}_*, \triangleleft} \stackrel{?}{=} \text{id}_{S^0 \triangleleft X}$$

holds for all elements in  $S^0 \triangleleft X$  except one.

*Proof.* Firstly, note that, given  $(X, x_0) \in \text{Obj}(\text{Sets}_*)$ , the map

$$\lambda_X^{\text{Sets}_*, \triangleleft} : S^0 \triangleleft X \rightarrow X$$

is indeed a morphism of pointed sets, as we have

$$\lambda_X^{\text{Sets}_*, \triangleleft} (0 \triangleleft x_0) = x_0.$$

Next, we claim that  $\lambda^{\text{Sets}_*, \triangleleft}$  is a natural transformation. We need to show that, given a morphism of pointed sets

$$f : (X, x_0) \rightarrow (Y, y_0),$$

the diagram

$$\begin{array}{ccc}
 S^0 \triangleleft X & \xrightarrow{\text{id}_{S^0} \triangleleft f} & S^0 \triangleleft Y \\
 \lambda_X^{\text{Sets}_*, \triangleleft} \downarrow & & \downarrow \lambda_Y^{\text{Sets}_*, \triangleleft} \\
 X & \xrightarrow{f} & Y
 \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc} 0 \triangleleft x & & 0 \triangleleft x \mapsto 0 \triangleleft f(x) \\ \downarrow & & \downarrow \\ x_0 \mapsto f(x_0) & & y_0 \end{array}$$

and

$$\begin{array}{ccc} 1 \triangleleft x \mapsto 1 \triangleleft f(x) \\ \downarrow \quad \quad \downarrow \\ x \mapsto f(x) \end{array}$$

and hence indeed commutes, showing  $\lambda^{\text{Sets}_*, \triangleleft}$  to be a natural transformation. This finishes the proof.  $\square$

### 00E8 3.6 The Left Skew Right Unitor

00E9 **Definition 3.6.1.1.** The **skew right unitor of the left tensor product of pointed sets** is the natural transformation

$$\rho^{\text{Sets}_*, \triangleleft} : \rho_{\text{Sets}_*}^{\text{Cats}_2} \xrightarrow{\sim} \triangleleft \circ (\text{id} \times \mathbb{1}^{\text{Sets}_*}),$$

whose component

$$\rho_X^{\text{Sets}_*, \triangleleft} : X \rightarrow X \triangleleft S^0$$

at  $(X, x_0) \in \text{Obj}(\text{Sets}_*)$  is given by the composition

$$\begin{aligned} X &\rightarrow X \vee X \\ &\cong |S^0| \odot X \\ &\cong X \triangleleft S^0, \end{aligned}$$

where  $X \rightarrow X \vee X$  is the map sending  $X$  to the second factor of  $X$  in  $X \vee X$ .

*Proof.* (Proven below in a bit.) □

**00EA Remark 3.6.1.2.** In other words,  $\rho_X^{\text{Sets}_*, \triangleleft}$  acts on elements as

$$\rho_X^{\text{Sets}_*, \triangleleft}(x) \stackrel{\text{def}}{=} [(1, x)]$$

i.e. by

$$\rho_X^{\text{Sets}_*, \triangleleft}(x) \stackrel{\text{def}}{=} x \triangleleft 1$$

for each  $x \in X$ .

**00EB Remark 3.6.1.3.** The morphism  $\rho_X^{\text{Sets}_*, \triangleleft}$  is non-invertible, as it is non-surjective when viewed as a map of sets, since the elements  $x \triangleleft 0$  of  $X \triangleleft S^0$  with  $x \neq x_0$  are outside the image of  $\rho_X^{\text{Sets}_*, \triangleleft}$ , which sends  $x$  to  $x \triangleleft 1$ .

*Proof.* Firstly, note that, given  $(X, x_0) \in \text{Obj}(\text{Sets}_*)$ , the map

$$\rho_X^{\text{Sets}_*, \triangleleft} : X \rightarrow X \triangleleft S^0$$

is indeed a morphism of pointed sets as we have

$$\begin{aligned} \rho_X^{\text{Sets}_*, \triangleleft}(x_0) &= x_0 \triangleleft 1 \\ &= x_0 \triangleleft 0. \end{aligned}$$

Next, we claim that  $\rho^{\text{Sets}_*, \triangleleft}$  is a natural transformation. We need to show that, given a morphism of pointed sets

$$f : (X, x_0) \rightarrow (Y, y_0),$$

the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \rho_X^{\text{Sets}_*, \triangleleft} \downarrow & & \downarrow \rho_Y^{\text{Sets}_*, \triangleleft} \\ X \triangleleft S^0 & \xrightarrow{f \triangleleft \text{id}_{S^0}} & Y \triangleleft S^0 \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc} x & \longmapsto & f(x) \\ \downarrow & & \downarrow \\ x \triangleleft 0 & \longmapsto & f(x) \triangleleft 0 \end{array}$$

and hence indeed commutes, showing  $\rho^{\text{Sets}_*, \triangleleft}$  to be a natural transformation. This finishes the proof. □

### 00EC 3.7 The Diagonal

00ED **Definition 3.7.1.1.** The **diagonal of the left tensor product of pointed sets** is the natural transformation

$$\Delta^\triangleleft : \text{id}_{\text{Sets}_*} \Rightarrow \triangleleft \circ \Delta_{\text{Sets}_*}^{\text{Cats}_2},$$

whose component

$$\Delta_X^\triangleleft : (X, x_0) \rightarrow (X \triangleleft X, x_0 \triangleleft x_0)$$

at  $(X, x_0) \in \text{Obj}(\text{Sets}_*)$  is given by

$$\Delta_X^\triangleleft(x) \stackrel{\text{def}}{=} x \triangleleft x$$

for each  $x \in X$ .

*Proof. Being a Morphism of Pointed Sets:* We have

$$\Delta_X^\triangleleft(x_0) \stackrel{\text{def}}{=} x_0 \triangleleft x_0,$$

and thus  $\Delta_X^\triangleleft$  is a morphism of pointed sets.

*Naturality:* We need to show that, given a morphism of pointed sets

$$f : (X, x_0) \rightarrow (Y, y_0),$$

the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \Delta_X^\triangleleft \downarrow & & \downarrow \Delta_Y^\triangleleft \\ X \triangleleft X & \xrightarrow{f \triangleleft f} & Y \triangleleft Y \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc} x & \longmapsto & f(x) \\ \downarrow & & \downarrow \\ x \triangleleft x & \longmapsto & f(x) \triangleleft f(x) \end{array}$$

and hence indeed commutes, showing  $\Delta^\triangleleft$  to be natural.  $\square$

### 3.8 The Left Skew Monoidal Structure on Pointed Sets Associated to

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$\triangleleft$

00EF

**Proposition 3.8.1.1.** The category  $\mathbf{Sets}_*$  admits a left-closed left skew monoidal category structure consisting of

- *The Underlying Category.* The category  $\mathbf{Sets}_*$  of pointed sets;
- *The Left Skew Monoidal Product.* The left tensor product functor

$$\triangleleft : \mathbf{Sets}_* \times \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*$$

of **Definition 3.1.1.1**;

- *The Left Internal Skew Hom.* The left internal Hom functor

$$[-, -]_{\mathbf{Sets}_*}^{\triangleleft} : \mathbf{Sets}_*^{\text{op}} \times \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*$$

of **Definition 3.2.1.1**;

- *The Left Skew Monoidal Unit.* The functor

$$\mathbb{1}^{\mathbf{Sets}_*, \triangleleft} : \text{pt} \rightarrow \mathbf{Sets}_*$$

of **Definition 3.3.1.1**;

- *The Left Skew Associators.* The natural transformation

$$\alpha^{\mathbf{Sets}_*, \triangleleft} : \triangleleft \circ (\triangleleft \times \text{id}_{\mathbf{Sets}_*}) \Longrightarrow \triangleleft \circ (\text{id}_{\mathbf{Sets}_*} \times \triangleleft) \circ \alpha_{\mathbf{Sets}_*, \mathbf{Sets}_*, \mathbf{Sets}_*}^{\text{Cats}}$$

of **Definition 3.4.1.1**;

- *The Left Skew Left Unitors.* The natural transformation

$$\lambda^{\mathbf{Sets}_*, \triangleleft} : \triangleleft \circ (\mathbb{1}^{\mathbf{Sets}_*} \times \text{id}_{\mathbf{Sets}_*}) \xrightarrow{\sim} \lambda_{\mathbf{Sets}_*}^{\text{Cats}_2}$$

of **Definition 3.5.1.1**;

- *The Left Skew Right Unitors.* The natural transformation

$$\rho^{\mathbf{Sets}_*, \triangleleft} : \rho_{\mathbf{Sets}_*}^{\text{Cats}_2} \xrightarrow{\sim} \triangleleft \circ (\text{id} \times \mathbb{1}^{\mathbf{Sets}_*})$$

of **Definition 3.6.1.1**.

*Proof. The Pentagon Identity:* Let  $(W, w_0)$ ,  $(X, x_0)$ ,  $(Y, y_0)$  and  $(Z, z_0)$  be pointed sets. We have to show that the diagram

$$\begin{array}{ccc}
 & (W \triangleleft (X \triangleleft Y)) \triangleleft Z & \\
 \alpha_{W,X,Y}^{\text{Sets}_*, \triangleleft} \triangleleft \text{id}_Z \nearrow & & \searrow \alpha_{W,X \triangleleft Y,Z}^{\text{Sets}_*, \triangleleft} \\
 ((W \triangleleft X) \triangleleft Y) \triangleleft Z & & W \triangleleft ((X \triangleleft Y) \triangleleft Z) \\
 \alpha_{W \triangleleft X,Y,Z}^{\text{Sets}_*, \triangleleft} \searrow & & \swarrow \text{id}_W \triangleleft \alpha_{X,Y,Z}^{\text{Sets}_*, \triangleleft} \\
 (W \triangleleft X) \triangleleft (Y \triangleleft Z) & \xrightarrow{\alpha_{W,X,Y \triangleleft Z}^{\text{Sets}_*, \triangleleft}} & W \triangleleft (X \triangleleft (Y \triangleleft Z))
 \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc}
 & (w \triangleleft (x \triangleleft y)) \triangleleft z & \\
 \swarrow & & \searrow \\
 ((w \triangleleft x) \triangleleft y) \triangleleft z & & w \triangleleft ((x \triangleleft y) \triangleleft z) \\
 \searrow & & \swarrow \\
 (w \triangleleft x) \triangleleft (y \triangleleft z) & \longmapsto & w \triangleleft (x \triangleleft (y \triangleleft z))
 \end{array}$$

and thus we see that the pentagon identity is satisfied.

*The Left Skew Left Triangle Identity:* Let  $(X, x_0)$  and  $(Y, y_0)$  be pointed sets. We have to show that the diagram

$$\begin{array}{ccc}
 (S^0 \triangleleft X) \triangleleft Y & \xrightarrow{\alpha_{S^0, X, Y}^{\text{Sets}_*, \triangleleft}} & S^0 \triangleleft (X \triangleleft Y) \\
 \searrow \lambda_X^{\text{Sets}_*, \triangleleft} \triangleleft \text{id}_Y & & \downarrow \lambda_{X \triangleleft Y}^{\text{Sets}_*, \triangleleft} \\
 & & X \triangleleft Y
 \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc}
 (0 \triangleleft x) \triangleleft y & \longmapsto & 0 \triangleleft (x \triangleleft y) \\
 \searrow & & \downarrow \\
 & & x_0 \triangleleft y = x_0 \triangleleft y_0
 \end{array}$$

and

$$\begin{array}{ccc}
 (1 \triangleleft x) \triangleleft y & \longmapsto & 1 \triangleleft (x \triangleleft y) \\
 \searrow & & \downarrow \\
 & & x \triangleleft y
 \end{array}$$

and hence indeed commutes. Thus the left skew triangle identity is satisfied.

*The Left Skew Right Triangle Identity:* Let  $(X, x_0)$  and  $(Y, y_0)$  be pointed sets. We have to show that the diagram

$$\begin{array}{ccc}
 X \triangleleft Y & & \\
 \downarrow \rho_{X \triangleleft Y}^{\text{Sets}_*, \triangleleft} & \searrow \text{id}_X \triangleleft \rho_Y^{\text{Sets}_*, \triangleleft} & \\
 (X \triangleleft Y) \triangleleft S^0 & \xrightarrow{\alpha_{X, Y, S^0}^{\text{Sets}_*, \triangleleft}} & X \triangleleft (Y \triangleleft S^0)
 \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc}
 x \triangleleft y & & \\
 \downarrow & \searrow & \\
 (x \triangleleft y) \triangleleft 1 & \longmapsto & x \triangleleft (y \triangleleft 1)
 \end{array}$$



and hence indeed commutes. Thus the right skew triangle identity is satisfied.

*The Left Skew Middle Triangle Identity:* Let  $(X, x_0)$  and  $(Y, y_0)$  be pointed sets. We have to show that the diagram

$$\begin{array}{ccc}
 X \triangleleft Y & \xlongequal{\quad} & X \triangleleft Y \\
 \rho_X^{\text{Sets}_*, \triangleleft} \triangleleft \text{id}_Y \downarrow & & \uparrow \text{id}_A \triangleleft \lambda_Y^{\text{Sets}_*, \triangleleft} \\
 (X \triangleleft S^0) \triangleleft Y & \xrightarrow[\alpha_{X, S^0, Y}^{\text{Sets}_*, \triangleleft}]{} & X \triangleleft (S^0 \triangleleft Y)
 \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc}
 x \triangleleft y & \xrightarrow{\quad} & x \triangleleft y \\
 \downarrow & & \uparrow \\
 (x \triangleleft 1) \triangleleft y & \xrightarrow{\quad} & x \triangleleft (1 \triangleleft y)
 \end{array}$$

and hence indeed commutes. Thus the right skew triangle identity is satisfied.

*The Zig-Zag Identity:* We have to show that the diagram

$$\begin{array}{ccc}
 S^0 & \xrightarrow{\rho_{S^0}^{\text{Sets}_*, \triangleleft}} & S^0 \triangleleft S^0 \\
 \searrow & & \downarrow \lambda_{S^0}^{\text{Sets}_*, \triangleleft} \\
 & & S^0
 \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc}
 0 & \xrightarrow{\quad} & 0 \triangleleft 1 \\
 \swarrow & & \downarrow \\
 & & 0
 \end{array}$$

and

$$\begin{array}{ccc}
 1 & \xrightarrow{\quad} & 1 \triangleleft 1 \\
 \swarrow & & \downarrow \\
 & & 1
 \end{array}$$

and hence indeed commutes. Thus the zig-zag identity is satisfied.

*Left Skew Monoidal Left-Closedness:* This follows from **Item 2** of **Proposition 3.1.1.7**.

□

### 00EG 3.9 Monoids With Respect to the Left Tensor Product of Pointed Sets

00EH **Proposition 3.9.1.1.** The category of monoids on  $(\text{Sets}_*, \triangleleft, S^0)$  is isomorphic to the category of “monoids with left zero”<sup>14</sup> and morphisms between them.

*Proof.* Monoids on  $(\text{Sets}_*, \triangleleft, S^0)$ : A monoid on  $(\text{Sets}_*, \triangleleft, S^0)$  consists of:

- *The Underlying Object.* A pointed set  $(A, 0_A)$ .
- *The Multiplication Morphism.* A morphism of pointed sets

$$\mu_A: A \triangleleft A \rightarrow A,$$

determining a left bilinear morphism of pointed sets

$$\begin{aligned} A \times A &\longrightarrow A \\ (a, b) &\longmapsto ab. \end{aligned}$$

- *The Unit Morphism.* A morphism of pointed sets

$$\eta_A: S^0 \rightarrow A$$

picking an element  $1_A$  of  $A$ .

satisfying the following conditions:

1. *Associativity.* The diagram

$$\begin{array}{ccccc} & & A \triangleleft (A \triangleleft A) & & \\ & \nearrow \alpha_{A,A,A}^{\text{Sets}_*, \triangleleft} & & \searrow \text{id}_A \triangleleft \mu_A & \\ (A \triangleleft A) \triangleleft A & & & & A \triangleleft A \\ & \searrow \mu_A \triangleleft \text{id}_A & & \nearrow \mu_A & \\ & A \triangleleft A & \xrightarrow{\mu_A} & A & \end{array}$$

<sup>14</sup>A monoid with left zero is defined similarly as the monoids with zero of ???. Succinctly, they

2. *Left Unitality.* The diagram

$$\begin{array}{ccc}
 S^0 \triangleleft A & \xrightarrow{\eta_A \times \text{id}_A} & A \triangleleft A \\
 & \searrow \lambda_A^{\text{Sets}_*, \triangleleft} & \downarrow \mu_A \\
 & & A
 \end{array}$$

commutes.

3. *Right Unitality.* The diagram

$$\begin{array}{ccc}
 A & \xrightarrow{\rho_A^{\text{Sets}_*, \triangleleft}} & A \triangleleft S^0 \\
 \parallel & & \downarrow \text{id}_A \times \eta_A \\
 A & \xleftarrow{\mu_A} & A \triangleleft A
 \end{array}$$

commutes.

Being a left-bilinear morphism of pointed sets, the multiplication map satisfies

$$0_A a = 0_A$$

for each  $a \in A$ . Now, the associativity, left unitality, and right unitality conditions act on elements as follows:

1. *Associativity.* The associativity condition acts as

$$\begin{array}{ccc}
 & & a \triangleleft (b \triangleleft c) \\
 & \swarrow & \searrow \\
 (a \triangleleft b) \triangleleft c & & (a \triangleleft b) \triangleleft c \quad a \triangleleft bc \\
 \searrow & & \swarrow \\
 ab \triangleleft c & \xrightarrow{\quad} & (ab)c \\
 & & \downarrow \\
 & & a(bc)
 \end{array}$$

---

are monoids  $(A, \mu_A, \eta_A)$  with a special element  $0_A$  satisfying

$$0_A a = 0_A$$

for each  $a \in A$ .

This gives

$$(ab)c = a(bc)$$

for each  $a, b, c \in A$ .

2. *Left Unitality*. The left unitality condition acts:

(a) On  $0 \triangleleft a$  as

$$\begin{array}{ccc} 0 \triangleleft a & & 0 \triangleleft a \mapsto 0_A \triangleleft a \\ & \searrow & \downarrow \\ & 0_A & 0_A a. \end{array}$$

(b) On  $1 \triangleleft a$  as

$$\begin{array}{ccc} 1 \triangleleft a & & 1 \triangleleft a \mapsto 1_A \triangleleft a \\ & \searrow & \downarrow \\ & a & 1_A a. \end{array}$$

This gives

$$\begin{aligned} 1_A a &= a, \\ 0_A a &= 0_A \end{aligned}$$

for each  $a \in A$ .

3. *Right Unitality*. The right unitality condition acts as

$$\begin{array}{ccc} a & \xrightarrow{\quad} & a \triangleleft 1 \\ \downarrow & & \downarrow \\ a & \xleftarrow{\quad} & a \triangleleft 1_A \end{array}$$

This gives

$$a 1_A = a$$

for each  $a \in A$ .

Thus we see that monoids with respect to  $\triangleleft$  are exactly monoids with left zero. *Morphisms of Monoids on  $(\text{Sets}_*, \triangleleft, S^0)$* : A morphism of monoids on  $(\text{Sets}_*, \triangleleft, S^0)$  from  $(A, \mu_A, \eta_A, 0_A)$  to  $(B, \mu_B, \eta_B, 0_B)$  is a morphism of pointed sets

$$f: (A, 0_A) \rightarrow (B, 0_B)$$

satisfying the following conditions:

1. *Compatibility With the Multiplication Morphisms.* The diagram

$$\begin{array}{ccc} A \triangleleft A & \xrightarrow{f \triangleleft f} & B \triangleleft B \\ \mu_A \downarrow & & \downarrow \mu_B \\ A & \xrightarrow{f} & B \end{array}$$

commutes.

2. *Compatibility With the Unit Morphisms.* The diagram

$$\begin{array}{ccc} S^0 & \xrightarrow{\eta_A} & A \\ & \searrow \eta_B & \downarrow f \\ & & B \end{array}$$

commutes.

These act on elements as

$$\begin{array}{ccc} a \triangleleft b & & a \triangleleft b \mapsto f(a) \triangleleft f(b) \\ \downarrow & & \downarrow \\ ab \mapsto f(ab) & & f(a)f(b) \end{array}$$

and

$$\begin{array}{ccc} 0 & \mapsto & 0_A \\ & \searrow & \downarrow \\ & & f(0_A) \end{array}$$

and

$$\begin{array}{ccc}
 1 & & 1 \xrightarrow{\quad} 1_A \\
 \searrow & & \downarrow \\
 & 1_B & f(1_A)
 \end{array}$$

giving

$$f(ab) = f(a)f(b),$$

$$f(0_A) = 0_B,$$

$$f(1_A) = 1_B,$$

for each  $a, b \in A$ , which is exactly a morphism of monoids with left zero.

*Identities and Composition:* Similarly, the identities and composition of  $\text{Mon}(\text{Sets}_*, \triangleleft, S^0)$  can be easily seen to agree with those of monoids with left zero, which finishes the proof.  $\square$

## 00EJ 4 The Right Tensor Product of Pointed Sets

### 00EK 4.1 Foundations

Let  $(X, x_0)$  and  $(Y, y_0)$  be pointed sets.

00EL **Definition 4.1.1.1.** The **right tensor product of pointed sets** is the functor<sup>15</sup>

$$\triangleright : \text{Sets}_* \times \text{Sets}_* \rightarrow \text{Sets}_*$$

defined as the composition

$$\text{Sets}_* \times \text{Sets}_* \xrightarrow{\text{忘} \times \text{id}} \text{Sets} \times \text{Sets}_* \xrightarrow{\odot} \text{Sets}_*,$$

where:

- $\text{忘} : \text{Sets}_* \rightarrow \text{Sets}$  is the forgetful functor from pointed sets to sets.
- $\odot : \text{Sets} \times \text{Sets}_* \rightarrow \text{Sets}_*$  is the tensor functor of **Item 1** of **Proposition 2.1.1.6**.

<sup>15</sup>*Further Notation:* Also written  $\triangleright_{\text{Sets}_*}$ .

**00EM Remark 4.1.1.2.** The right tensor product of pointed sets satisfies the following natural bijection:

$$\text{Sets}_*(X \triangleright Y, Z) \cong \text{Hom}_{\text{Sets}_*}^{\otimes, R}(X \times Y, Z).$$

That is to say, the following data are in natural bijection:

1. Pointed maps  $f: X \triangleright Y \rightarrow Z$ .
2. Maps of sets  $f: X \times Y \rightarrow Z$  satisfying  $f(x, y_0) = z_0$  for each  $x \in X$ .

**00EN Remark 4.1.1.3.** The right tensor product of pointed sets may be described as follows:

- The right tensor product of  $(X, x_0)$  and  $(Y, y_0)$  is the pair  $((X \triangleright Y, x_0 \triangleright y_0), \iota)$  consisting of
  - A pointed set  $(X \triangleright Y, x_0 \triangleright y_0)$ ;
  - A right bilinear morphism of pointed sets  $\iota: (X \times Y, (x_0, y_0)) \rightarrow X \triangleright Y$ ;

satisfying the following universal property:

**(UP)** Given another such pair  $((Z, z_0), f)$  consisting of

- \* A pointed set  $(Z, z_0)$ ;
- \* A right bilinear morphism of pointed sets  $f: (X \times Y, (x_0, y_0)) \rightarrow X \triangleright Y$ ;

there exists a unique morphism of pointed sets  $X \triangleright Y \xrightarrow{\exists!} Z$  making the diagram

$$\begin{array}{ccc} & X \triangleright Y & \\ \iota \nearrow & & \downarrow \exists! \\ X \times Y & \xrightarrow{f} & Z \end{array}$$

commute.

**00EP Construction 4.1.1.4.** In detail, the **right tensor product of  $(X, x_0)$  and  $(Y, y_0)$**  is the pointed set  $(X \triangleright Y, [y_0])$  consisting of:

- *The Underlying Set.* The set  $X \triangleright Y$  defined by

$$\begin{aligned} X \triangleright Y &\stackrel{\text{def}}{=} |X| \odot Y \\ &\cong \bigvee_{x \in X} (Y, y_0), \end{aligned}$$

where  $|X|$  denotes the underlying set of  $(X, x_0)$ .

- *The Underlying Basepoint.* The point  $[(x_0, y_0)]$  of  $\bigvee_{x \in X} (Y, y_0)$ , which is equal to  $[(x, y_0)]$  for any  $x \in X$ .

**00EQ** **Notation 4.1.1.5.** We write<sup>16</sup>  $x \triangleright y$  for the element  $[(x, y)]$  of

$$X \triangleright Y \cong |X| \odot Y.$$

**00ER** **Remark 4.1.1.6.** Employing the notation introduced in **Notation 4.1.1.5**, we have

$$x_0 \triangleright y_0 = x \triangleright y_0$$

for each  $x \in X$ , and

$$x \triangleright y_0 = x' \triangleright y_0$$

for each  $x, x' \in X$ .

**00ES** **Proposition 4.1.1.7.** Let  $(X, x_0)$  and  $(Y, y_0)$  be pointed sets.

**00ET** 1. *Functoriality.* The assignments  $X, Y, (X, Y) \mapsto X \triangleright Y$  define functors

$$\begin{aligned} X \triangleright - &: \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*, \\ - \triangleright Y &: \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*, \\ -_1 \triangleright -_2 &: \mathbf{Sets}_* \times \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*. \end{aligned}$$

In particular, given pointed maps

$$\begin{aligned} f &: (X, x_0) \rightarrow (A, a_0), \\ g &: (Y, y_0) \rightarrow (B, b_0), \end{aligned}$$

the induced map

$$f \triangleright g: X \triangleright Y \rightarrow A \triangleright B$$

is given by

$$[f \triangleright g](x \triangleright y) \stackrel{\text{def}}{=} f(x) \triangleright g(y)$$

for each  $x \triangleright y \in X \triangleright Y$ .

---

<sup>16</sup>*Further Notation:* Also written  $x \triangleright_{\mathbf{Sets}_*} y$ .



00EU 2. *Adjointness I.* We have an adjunction

$$\left( X \triangleright - \dashv [X, -]_{\text{Sets}_*}^{\triangleright} \right): \text{Sets}_* \begin{array}{c} \xrightarrow{X \triangleright -} \\ \perp \\ \xleftarrow{[X, -]_{\text{Sets}_*}^{\triangleright}} \end{array} \text{Sets}_*$$

witnessed by a bijection of sets

$$\text{Hom}_{\text{Sets}_*}(X \triangleright Y, Z) \cong \text{Hom}_{\text{Sets}_*}\left(Y, [X, Z]_{\text{Sets}_*}^{\triangleright}\right)$$

natural in  $(X, x_0), (Y, y_0), (Z, z_0) \in \text{Obj}(\text{Sets}_*)$ , where  $[X, Y]_{\text{Sets}_*}^{\triangleright}$  is the pointed set of [Definition 4.2.1.1](#).

00EV 3. *Adjointness II.* The functor

$$- \triangleright Y: \text{Sets}_* \rightarrow \text{Sets}_*$$

does not admit a right adjoint.

00EW 4. *Adjointness III.* We have a bijection of sets

$$\text{Hom}_{\text{Sets}_*}(X \triangleright Y, Z) \cong \text{Hom}_{\text{Sets}}(|X|, \text{Sets}_*(Y, Z))$$

natural in  $(X, x_0), (Y, y_0), (Z, z_0) \in \text{Obj}(\text{Sets}_*)$ .

*Proof. Item 1, Functoriality:* Clear.

*Item 2, Adjointness I:* This follows from [Item 3](#) of [Proposition 2.1.1.6](#).

*Item 3, Adjointness II:* For  $- \triangleright Y$  to admit a right adjoint would require it to preserve colimits by [??](#), [??](#) of [??](#). However, we have

$$\begin{aligned} \text{pt} \triangleright X &\stackrel{\text{def}}{=} |\text{pt}| \odot X \\ &\cong X \\ &\neq \text{pt}, \end{aligned}$$

and thus we see that  $- \triangleright Y$  does not have a right adjoint.

*Item 4, Adjointness III:* This follows from [Item 2](#) of [Proposition 2.1.1.6](#).  $\square$

00EX **Remark 4.1.1.8.** Here is some intuition on why  $- \triangleright Y$  fails to be a left adjoint.

[Item 4](#) of [Proposition 3.1.1.7](#) states that we have a natural bijection

$$\text{Hom}_{\text{Sets}_*}(X \triangleright Y, Z) \cong \text{Hom}_{\text{Sets}}(|X|, \text{Sets}_*(Y, Z)),$$

so it would be reasonable to wonder whether a natural bijection of the form

$$\mathrm{Hom}_{\mathbf{Sets}_*}(X \triangleright Y, Z) \cong \mathrm{Hom}_{\mathbf{Sets}_*}(X, \mathbf{Sets}_*(Y, Z)),$$

also holds, which would give  $- \triangleright Y \dashv \mathbf{Sets}_*(Y, -)$ . However, such a bijection would require every map

$$f: X \triangleright Y \rightarrow Z$$

to satisfy

$$f(x_0 \triangleright y) = z_0$$

for each  $x \in X$ , whereas we are imposing such a basepoint preservation condition only for elements of the form  $x \triangleright y_0$ . Thus  $\mathbf{Sets}_*(Y, -)$  can't be a right adjoint for  $- \triangleright Y$ , and as shown by **Item 3** of **Proposition 4.1.1.7**, no functor can.<sup>17</sup>

## 00EY 4.2 The Right Internal Hom of Pointed Sets

Let  $(X, x_0)$  and  $(Y, y_0)$  be pointed sets.

00EZ **Definition 4.2.1.1.** The **right internal Hom of pointed sets** is the functor

$$[-, -]_{\mathbf{Sets}_*}^{\triangleright} : \mathbf{Sets}_*^{\mathrm{op}} \times \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*$$

defined as the composition

$$\mathbf{Sets}_*^{\mathrm{op}} \times \mathbf{Sets}_* \xrightarrow{\omega \times \mathrm{id}} \mathbf{Sets}^{\mathrm{op}} \times \mathbf{Sets}_* \xrightarrow{\pitchfork} \mathbf{Sets}_*,$$

where:

- $\omega: \mathbf{Sets}_* \rightarrow \mathbf{Sets}$  is the forgetful functor from pointed sets to sets.
- $\pitchfork: \mathbf{Sets}^{\mathrm{op}} \times \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*$  is the cotensor functor of **Item 1** of **Proposition 2.2.1.4**.

*Proof.* For a proof that  $[-, -]_{\mathbf{Sets}_*}^{\triangleright}$  is indeed the right internal Hom of  $\mathbf{Sets}_*$  with respect to the right tensor product of pointed sets, see **Item 2** of **Proposition 4.1.1.7**.  $\square$

00F0 **Remark 4.2.1.2.** We have

$$[-, -]_{\mathbf{Sets}_*}^{\triangleleft} = [-, -]_{\mathbf{Sets}_*}^{\triangleright}.$$

<sup>17</sup>The functor  $\mathbf{Sets}_*(Y, -)$  is instead right adjoint to  $- \wedge Y$ , the smash product of pointed sets

**00F1 Remark 4.2.1.3.** The right internal Hom of pointed sets satisfies the following universal property:

$$\mathbf{Sets}_*(X \triangleright Y, Z) \cong \mathbf{Sets}_*\left(Y, [X, Z]_{\mathbf{Sets}_*}^{\triangleright}\right)$$

That is to say, the following data are in bijection:

1. Pointed maps  $f: X \triangleright Y \rightarrow Z$ .
2. Pointed maps  $f: Y \rightarrow [X, Z]_{\mathbf{Sets}_*}^{\triangleright}$ .

**00F2 Remark 4.2.1.4.** In detail, the **right internal Hom of**  $(X, x_0)$  **and**  $(Y, y_0)$  is the pointed set  $\left([X, Y]_{\mathbf{Sets}_*}^{\triangleright}, [(y_0)_{x \in X}]\right)$  consisting of

- *The Underlying Set.* The set  $[X, Y]_{\mathbf{Sets}_*}^{\triangleright}$  defined by

$$\begin{aligned} [X, Y]_{\mathbf{Sets}_*}^{\triangleright} &\stackrel{\text{def}}{=} |X| \pitchfork Y \\ &\cong \bigwedge_{x \in X} (Y, y_0), \end{aligned}$$

where  $|X|$  denotes the underlying set of  $(X, x_0)$ ;

- *The Underlying Basepoint.* The point  $[(y_0)_{x \in X}]$  of  $\bigwedge_{x \in X} (Y, y_0)$ .

**00F3 Proposition 4.2.1.5.** Let  $(X, x_0)$  and  $(Y, y_0)$  be pointed sets.

**00F4** 1. *Functoriality.* The assignments  $X, Y, (X, Y) \mapsto [X, Y]_{\mathbf{Sets}_*}^{\triangleright}$  define functors

$$\begin{aligned} [X, -]_{\mathbf{Sets}_*}^{\triangleright} &: \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*, \\ [-, Y]_{\mathbf{Sets}_*}^{\triangleright} &: \mathbf{Sets}_*^{\text{op}} \rightarrow \mathbf{Sets}_*, \\ [-_1, -_2]_{\mathbf{Sets}_*}^{\triangleright} &: \mathbf{Sets}_*^{\text{op}} \times \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*. \end{aligned}$$

In particular, given pointed maps

$$\begin{aligned} f &: (X, x_0) \rightarrow (A, a_0), \\ g &: (Y, y_0) \rightarrow (B, b_0), \end{aligned}$$

---

of [Definition 5.1.1.1](#). See [Item 2](#) of [Proposition 5.1.1.9](#).

the induced map

$$[f, g]_{\text{Sets}_*}^{\triangleright} : [A, Y]_{\text{Sets}_*}^{\triangleright} \rightarrow [X, B]_{\text{Sets}_*}^{\triangleright}$$

is given by

$$[f, g]_{\text{Sets}_*}^{\triangleright} \left( [(y_a)_{a \in A}] \right) \stackrel{\text{def}}{=} [(g(y_{f(x)}))_{x \in X}]$$

for each  $[(y_a)_{a \in A}] \in [A, Y]_{\text{Sets}_*}^{\triangleright}$ .

00F5 2. *Adjointness I.* We have an adjunction

$$\left( X \triangleright - \dashv [X, -]_{\text{Sets}_*}^{\triangleright} \right) : \text{Sets}_* \begin{array}{c} \xrightarrow{X \triangleright -} \\ \perp \\ \xleftarrow{[X, -]_{\text{Sets}_*}^{\triangleright}} \end{array} \text{Sets}_*$$

witnessed by a bijection of sets

$$\text{Hom}_{\text{Sets}_*}(X \triangleright Y, Z) \cong \text{Hom}_{\text{Sets}_*}(Y, [X, Z]_{\text{Sets}_*}^{\triangleright})$$

natural in  $(X, x_0), (Y, y_0), (Z, z_0) \in \text{Obj}(\text{Sets}_*)$ , where  $[X, Y]_{\text{Sets}_*}^{\triangleright}$  is the pointed set of [Definition 4.2.1.1](#).

00F6 3. *Adjointness II.* The functor

$$- \triangleright Y : \text{Sets}_* \rightarrow \text{Sets}_*$$

does not admit a right adjoint.

*Proof.* [Item 1, Functoriality](#): Clear.

[Item 2, Adjointness I](#): This is a repetition of [Item 2](#) of [Proposition 4.1.1.7](#), and is proved there.

[Item 3, Adjointness II](#): This is a repetition of [Item 3](#) of [Proposition 4.1.1.7](#), and is proved there.  $\square$

### 00F7 4.3 The Right Skew Unit

00F8 **Definition 4.3.1.1.** The **right skew unit of the right tensor product of pointed sets** is the functor

$$\mathbb{1}^{\text{Sets}_*, \triangleright} : \text{pt} \rightarrow \text{Sets}_*$$

defined by

$$\mathbb{1}_{\text{Sets}_*}^{\triangleright} \stackrel{\text{def}}{=} S^0.$$

00F9 **4.4 The Right Skew Associator**

00FA **Definition 4.4.1.1.** The **skew associator of the right tensor product of pointed sets** is the natural transformation

$$\alpha^{\text{Sets}_*, \triangleright} : \triangleright \circ (\text{id}_{\text{Sets}_*} \times \triangleright) \Longrightarrow \triangleright \circ (\triangleright \times \text{id}_{\text{Sets}_*}) \circ \alpha_{\text{Sets}_*, \text{Sets}_*, \text{Sets}_*}^{\text{Cats}, -1}$$

as in the diagram

$$\begin{array}{ccccc}
 & & (\text{Sets}_* \times \text{Sets}_*) \times \text{Sets}_* & & \\
 & \nearrow \alpha_{\text{Sets}_*, \text{Sets}_*, \text{Sets}_*}^{\text{Cats}, -1} & & \searrow \triangleright \times \text{id} & \\
 \text{Sets}_* \times (\text{Sets}_* \times \text{Sets}_*) & & & & \text{Sets}_* \times \text{Sets}_* \\
 \downarrow \text{id} \times \triangleright & \nearrow \alpha^{\text{Sets}_*, \triangleright} & & \searrow \triangleright & \\
 \text{Sets}_* \times \text{Sets}_* & \xrightarrow{\triangleright} & \text{Sets}_* & & 
 \end{array}$$

whose component

$$\alpha_{X,Y,Z}^{\text{Sets}_*, \triangleright} : X \triangleright (Y \triangleright Z) \rightarrow (X \triangleright Y) \triangleright Z$$

at  $(X, x_0), (Y, y_0), (Z, z_0) \in \text{Obj}(\text{Sets}_*)$  is given by

$$\begin{aligned}
 X \triangleright (Y \triangleright Z) &\stackrel{\text{def}}{=} |X| \odot (Y \triangleright Z) \\
 &\stackrel{\text{def}}{=} |X| \odot (|Y| \odot Z) \\
 &\cong \bigvee_{x \in X} (|Y| \odot Z) \\
 &\cong \bigvee_{x \in X} \left( \bigvee_{y \in Y} Z \right) \\
 &\rightarrow \bigvee_{[(x,y)] \in \bigvee_{x \in X} Y} Z \\
 &\cong \bigvee_{[(x,y)] \in |X| \odot Y} Z \\
 &\cong ||X| \odot Y| \odot Z \\
 &\stackrel{\text{def}}{=} |X \triangleright Y| \odot Z \\
 &\stackrel{\text{def}}{=} (X \triangleright Y) \triangleright Z,
 \end{aligned}$$

where the map

$$\bigvee_{x \in X} \left( \bigvee_{y \in Y} Z \right) \rightarrow \bigvee_{[(x,y)] \in \bigvee_{x \in X} Y} Z$$

is given by  $[(x, [(y, z)])] \mapsto [([(x, y)], z)]$ .

*Proof.* (Proven below in a bit.) □

**00FB Remark 4.4.1.2.** Unwinding the notation for elements, we have

$$\begin{aligned} [(x, [(y, z)])] &\stackrel{\text{def}}{=} [(x, y \triangleright z)] \\ &\stackrel{\text{def}}{=} x \triangleright (y \triangleright z) \end{aligned}$$

and

$$\begin{aligned} [([(x, y)], z)] &\stackrel{\text{def}}{=} [(x \triangleright y, z)] \\ &\stackrel{\text{def}}{=} (x \triangleright y) \triangleright z. \end{aligned}$$

So, in other words,  $\alpha_{X,Y,Z}^{\text{Sets}_*, \triangleright}$  acts on elements via

$$\alpha_{X,Y,Z}^{\text{Sets}_*, \triangleright} (x \triangleright (y \triangleright z)) \stackrel{\text{def}}{=} (x \triangleright y) \triangleright z$$

for each  $x \triangleright (y \triangleright z) \in X \triangleright (Y \triangleright Z)$ .

**00FC Remark 4.4.1.3.** Taking  $y = y_0$ , we see that the morphism  $\alpha_{X,Y,Z}^{\text{Sets}_*, \triangleright}$  acts on elements as

$$\alpha_{X,Y,Z}^{\text{Sets}_*, \triangleright} (x \triangleright (y_0 \triangleright z)) \stackrel{\text{def}}{=} (x \triangleright y_0) \triangleright z.$$

However, by the definition of  $\triangleright$ , we have  $x \triangleright y_0 = x' \triangleright y_0$  for all  $x, x' \in X$ , preventing  $\alpha_{X,Y,Z}^{\text{Sets}_*, \triangleright}$  from being non-invertible.

*Proof.* Firstly, note that, given  $(X, x_0), (Y, y_0), (Z, z_0) \in \text{Obj}(\text{Sets}_*)$ , the map

$$\alpha_{X,Y,Z}^{\text{Sets}_*, \triangleright} : X \triangleright (Y \triangleright Z) \rightarrow (X \triangleright Y) \triangleright Z$$

is indeed a morphism of pointed sets, as we have

$$\alpha_{X,Y,Z}^{\text{Sets}_*, \triangleright} (x_0 \triangleright (y_0 \triangleright z_0)) = (x_0 \triangleright y_0) \triangleright z_0.$$

Next, we claim that  $\alpha^{\text{Sets}_*, \triangleright}$  is a natural transformation. We need to show that,

given morphisms of pointed sets

$$\begin{aligned} f &: (X, x_0) \rightarrow (X', x'_0), \\ g &: (Y, y_0) \rightarrow (Y', y'_0), \\ h &: (Z, z_0) \rightarrow (Z', z'_0) \end{aligned}$$

the diagram

$$\begin{array}{ccc} X \triangleright (Y \triangleright Z) & \xrightarrow{f \triangleright (g \triangleright h)} & X' \triangleright (Y' \triangleright Z') \\ \downarrow \alpha_{X,Y,Z}^{\text{Sets}_*, \triangleright} & & \downarrow \alpha_{X',Y',Z'}^{\text{Sets}_*, \triangleright} \\ (X \triangleright Y) \triangleright Z & \xrightarrow{(f \triangleright g) \triangleright h} & (X' \triangleright Y') \triangleright Z' \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc} x \triangleright (y \triangleright z) & \longmapsto & f(x) \triangleright (g(y) \triangleright h(z)) \\ \downarrow & & \downarrow \\ (x \triangleright y) \triangleright z & \longmapsto & (f(x) \triangleright g(y)) \triangleright h(z) \end{array}$$

and hence indeed commutes, showing  $\alpha^{\text{Sets}_*, \triangleright}$  to be a natural transformation. This finishes the proof.  $\square$

#### 00FD 4.5 The Right Skew Left Unitor

00FE **Definition 4.5.1.1.** The **skew left unitor of the right tensor product of pointed sets** is the natural transformation

$$\lambda^{\text{Sets}_*, \triangleright} : \lambda_{\text{Sets}_*}^{\text{Cats}_2} \xrightarrow{\sim} \triangleright \circ (\mathbb{1}^{\text{Sets}_*} \times \text{id}_{\text{Sets}_*})$$

whose component

$$\lambda_X^{\text{Sets}_*, \triangleright} : X \rightarrow S^0 \triangleright X$$

at  $(X, x_0) \in \text{Obj}(\text{Sets}_*)$  is given by the composition

$$\begin{aligned} X &\rightarrow X \vee X \\ &\cong |S^0| \odot X \\ &\cong S^0 \triangleright X, \end{aligned}$$

where  $X \rightarrow X \vee X$  is the map sending  $X$  to the second factor of  $X$  in  $X \vee X$ .

*Proof.* (Proven below in a bit.)  $\square$

**00FF Remark 4.5.1.2.** In other words,  $\lambda_X^{\text{Sets}_*, \triangleright}$  acts on elements as

$$\lambda_X^{\text{Sets}_*, \triangleright}(x) \stackrel{\text{def}}{=} [(1, x)]$$

i.e. by

$$\lambda_X^{\text{Sets}_*, \triangleright}(x) \stackrel{\text{def}}{=} 1 \triangleright x$$

for each  $x \in X$ .

**00FG Remark 4.5.1.3.** The morphism  $\lambda_X^{\text{Sets}_*, \triangleright}$  is non-invertible, as it is non-surjective when viewed as a map of sets, since the elements  $0 \triangleright x$  of  $S^0 \triangleright X$  with  $x \neq x_0$  are outside the image of  $\lambda_X^{\text{Sets}_*, \triangleright}$ , which sends  $x$  to  $1 \triangleright x$ .

*Proof.* Firstly, note that, given  $(X, x_0) \in \text{Obj}(\text{Sets}_*)$ , the map

$$\lambda_X^{\text{Sets}_*, \triangleright} : X \rightarrow S^0 \triangleright X$$

is indeed a morphism of pointed sets, as we have

$$\begin{aligned} \lambda_X^{\text{Sets}_*, \triangleright}(x_0) &= 1 \triangleright x_0 \\ &= 0 \triangleright x_0. \end{aligned}$$

Next, we claim that  $\lambda^{\text{Sets}_*, \triangleright}$  is a natural transformation. We need to show that, given a morphism of pointed sets

$$f : (X, x_0) \rightarrow (Y, y_0),$$

the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \lambda_X^{\text{Sets}_*, \triangleright} \downarrow & & \downarrow \lambda_Y^{\text{Sets}_*, \triangleright} \\ S^0 \triangleright X & \xrightarrow{\text{id}_{S^0} \triangleright f} & S^0 \triangleright Y \end{array}$$



commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc} x & \longmapsto & f(x) \\ \downarrow & & \downarrow \\ 1 \triangleright x & \longmapsto & 1 \triangleright f(x) \end{array}$$

and hence indeed commutes, showing  $\lambda^{\text{Sets}_*, \triangleright}$  to be a natural transformation. This finishes the proof.  $\square$

#### 00FH 4.6 The Right Skew Right Unitor

00FJ **Definition 4.6.1.1.** The **skew right unitor of the right tensor product of pointed sets** is the natural transformation

$$\rho^{\text{Sets}_*, \triangleright} : \triangleright \circ (\text{id} \times \mathbb{1}^{\text{Sets}_*}) \xrightarrow{\sim} \rho_{\text{Sets}_*}^{\text{Cats}_2},$$

whose component

$$\rho_X^{\text{Sets}_*, \triangleright} : X \triangleright S^0 \rightarrow X$$

at  $(X, x_0) \in \text{Obj}(\text{Sets}_*)$  is given by the composition

$$\begin{aligned} X \triangleright S^0 &\cong |X| \odot S^0 \\ &\cong \bigvee_{x \in X} S^0 \\ &\rightarrow X, \end{aligned}$$

where  $\bigvee_{x \in X} S^0 \rightarrow X$  is the map given by

$$\begin{aligned} [(x, 0)] &\mapsto x_0, \\ [(x, 1)] &\mapsto x. \end{aligned}$$

*Proof.* (Proven below in a bit.) □

**00FK Remark 4.6.1.2.** In other words,  $\rho_X^{\text{Sets}_*, \triangleright}$  acts on elements as

$$\begin{aligned}\rho_X^{\text{Sets}_*, \triangleright}(x \triangleright 0) &\stackrel{\text{def}}{=} x_0, \\ \rho_X^{\text{Sets}_*, \triangleright}(x \triangleright 1) &\stackrel{\text{def}}{=} x\end{aligned}$$

for each  $x \triangleright 1 \in X \triangleright S^0$ .

**00FL Remark 4.6.1.3.** The morphism  $\rho_X^{\text{Sets}_*, \triangleright}$  is almost invertible, with its would-be-inverse

$$\phi_X: X \rightarrow X \triangleright S^0$$

given by

$$\phi_X(x) \stackrel{\text{def}}{=} x \triangleright 1$$

for each  $x \in X$ . Indeed, we have

$$\begin{aligned}\left[\rho_X^{\text{Sets}_*, \triangleright} \circ \phi\right](x) &= \rho_X^{\text{Sets}_*, \triangleright}(\phi(x)) \\ &= \rho_X^{\text{Sets}_*, \triangleright}(x \triangleright 1) \\ &= x \\ &= [\text{id}_X](x)\end{aligned}$$

so that

$$\rho_X^{\text{Sets}_*, \triangleright} \circ \phi = \text{id}_X$$

and

$$\begin{aligned}\left[\phi \circ \rho_X^{\text{Sets}_*, \triangleright}\right](x \triangleright 1) &= \phi\left(\rho_X^{\text{Sets}_*, \triangleright}(x \triangleright 1)\right) \\ &= \phi(x) \\ &= x \triangleright 1 \\ &= [\text{id}_{X \triangleright S^0}](x \triangleright 1),\end{aligned}$$

but

$$\begin{aligned}\left[\phi \circ \rho_X^{\text{Sets}_*, \triangleright}\right](x \triangleright 0) &= \phi\left(\rho_X^{\text{Sets}_*, \triangleright}(x \triangleright 0)\right) \\ &= \phi(x_0) \\ &= 1 \triangleright x_0,\end{aligned}$$

where  $x \triangleright 0 \neq 1 \triangleright x_0$ . Thus

$$\phi \circ \rho_X^{\text{Sets}_*, \triangleright} \stackrel{?}{=} \text{id}_{X \triangleright S^0}$$

holds for all elements in  $X \triangleright S^0$  except one.

*Proof.* Firstly, note that, given  $(X, x_0) \in \text{Obj}(\text{Sets}_*)$ , the map

$$\rho_X^{\text{Sets}_*, \triangleright} : X \triangleright S^0 \rightarrow X$$

is indeed a morphism of pointed sets as we have

$$\rho_X^{\text{Sets}_*, \triangleright}(x_0 \triangleright 0) = x_0.$$

Next, we claim that  $\rho^{\text{Sets}_*, \triangleright}$  is a natural transformation. We need to show that, given a morphism of pointed sets

$$f : (X, x_0) \rightarrow (Y, y_0),$$

the diagram

$$\begin{array}{ccc} X \triangleright S^0 & \xrightarrow{f \triangleright \text{id}_{S^0}} & Y \triangleright S^0 \\ \rho_X^{\text{Sets}_*, \triangleright} \downarrow & & \downarrow \rho_Y^{\text{Sets}_*, \triangleright} \\ X & \xrightarrow{f} & Y \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc} x \triangleright 0 & & x \triangleright 0 \mapsto f(x) \triangleright 0 \\ \downarrow & & \downarrow \\ x_0 & \mapsto & f(x_0) \end{array} \qquad \begin{array}{ccc} & & \\ & & y_0 \end{array}$$

and

$$\begin{array}{ccc} x \triangleright 1 & \mapsto & f(x) \triangleright 1 \\ \downarrow & & \downarrow \\ x & \mapsto & f(x) \end{array}$$

and hence indeed commutes, showing  $\rho^{\text{Sets}_*, \triangleright}$  to be a natural transformation. This finishes the proof.  $\square$

**00FM 4.7 The Diagonal**

**00FN Definition 4.7.1.1.** The **diagonal of the right tensor product of pointed sets** is the natural transformation

$$\Delta^\triangleright : \text{id}_{\text{Sets}_*} \Rightarrow \triangleright \circ \Delta_{\text{Sets}_*}^{\text{Cats}_2},$$

whose component

$$\Delta_X^\triangleright : (X, x_0) \rightarrow (X \triangleright X, x_0 \triangleright x_0)$$

at  $(X, x_0) \in \text{Obj}(\text{Sets}_*)$  is given by

$$\Delta_X^\triangleright(x) \stackrel{\text{def}}{=} x \triangleright x$$

for each  $x \in X$ .

*Proof. Being a Morphism of Pointed Sets:* We have

$$\Delta_X^\triangleright(x_0) \stackrel{\text{def}}{=} x_0 \triangleright x_0,$$

and thus  $\Delta_X^\triangleright$  is a morphism of pointed sets.

*Naturality:* We need to show that, given a morphism of pointed sets

$$f : (X, x_0) \rightarrow (Y, y_0),$$

the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \Delta_X^\triangleright \downarrow & & \downarrow \Delta_Y^\triangleright \\ X \triangleright X & \xrightarrow{f \triangleright f} & Y \triangleright Y \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc} x & \longmapsto & f(x) \\ \downarrow & & \downarrow \\ x \triangleright x & \longmapsto & f(x) \triangleright f(x) \end{array}$$

and hence indeed commutes, showing  $\Delta^\triangleright$  to be natural.  $\square$

#### 4.8 The Right Skew Monoidal Structure on Pointed Sets Associated to $\triangleright$

00FP

00FQ **Proposition 4.8.1.1.** The category  $\mathbf{Sets}_*$  admits a right-closed right skew monoidal category structure consisting of

- *The Underlying Category.* The category  $\mathbf{Sets}_*$  of pointed sets;
- *The Right Skew Monoidal Product.* The right tensor product functor

$$\triangleright : \mathbf{Sets}_* \times \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*$$

of **Definition 4.1.1.1**;

- *The Right Internal Skew Hom.* The right internal Hom functor

$$[-, -]_{\mathbf{Sets}_*}^{\triangleright} : \mathbf{Sets}_*^{\text{op}} \times \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*$$

of **Definition 4.2.1.1**;

- *The Right Skew Monoidal Unit.* The functor

$$\mathbb{1}^{\mathbf{Sets}_*, \triangleright} : \text{pt} \rightarrow \mathbf{Sets}_*$$

of **Definition 4.3.1.1**;

- *The Right Skew Associators.* The natural transformation

$$\alpha^{\mathbf{Sets}_*, \triangleright} : \triangleright \circ (\text{id}_{\mathbf{Sets}_*} \times \triangleright) \Longrightarrow \triangleright \circ (\triangleright \times \text{id}_{\mathbf{Sets}_*}) \circ \alpha_{\mathbf{Sets}_*, \mathbf{Sets}_*, \mathbf{Sets}_*}^{\mathbf{Cats}, -1}$$

of **Definition 4.4.1.1**;

- *The Right Skew Left Unitors.* The natural transformation

$$\lambda^{\mathbf{Sets}_*, \triangleright} : \lambda_{\mathbf{Sets}_*}^{\mathbf{Cats}_2} \xrightarrow{\sim} \triangleright \circ (\mathbb{1}^{\mathbf{Sets}_*} \times \text{id}_{\mathbf{Sets}_*})$$

of **Definition 4.5.1.1**;

- *The Right Skew Right Unitors.* The natural transformation

$$\rho^{\mathbf{Sets}_*, \triangleright} : \triangleright \circ (\text{id} \times \mathbb{1}^{\mathbf{Sets}_*}) \xrightarrow{\sim} \rho_{\mathbf{Sets}_*}^{\mathbf{Cats}_2}$$

of **Definition 4.6.1.1**.

*Proof. The Pentagon Identity:* Let  $(W, w_0)$ ,  $(X, x_0)$ ,  $(Y, y_0)$  and  $(Z, z_0)$  be pointed sets. We have to show that the diagram

$$\begin{array}{ccccc}
 & & W \triangleright ((X \triangleright Y) \triangleright Z) & & \\
 & \nearrow \alpha_{W,X,Y}^{\text{Sets}_*, \triangleright} \text{id}_Z & & \nwarrow \alpha_{W,X \triangleright Y,Z}^{\text{Sets}_*, \triangleright} & \\
 W \triangleright (X \triangleright (Y \triangleright Z)) & & & & (W \triangleright (X \triangleright Y)) \triangleright Z \\
 \searrow \alpha_{W \triangleright X,Y,Z}^{\text{Sets}_*, \triangleright} & & & & \searrow \text{id}_W \triangleright \alpha_{X,Y,Z}^{\text{Sets}_*, \triangleright} \\
 (W \triangleright X) \triangleright (Y \triangleright Z) & \xrightarrow{\alpha_{W,X,Y \triangleright Z}^{\text{Sets}_*, \triangleright}} & & & ((W \triangleright X) \triangleright Y) \triangleright Z
 \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccccc}
 & & w \triangleright ((x \triangleright y) \triangleright z) & & \\
 & \nearrow & & \nwarrow & \\
 w \triangleright (x \triangleright (y \triangleright z)) & & & & (w \triangleright (x \triangleright y)) \triangleright z \\
 \searrow & & & & \searrow \\
 (w \triangleright x) \triangleright (y \triangleright z) & \longmapsto & & & ((w \triangleright x) \triangleright y) \triangleright z
 \end{array}$$

and thus we see that the pentagon identity is satisfied.

*The Right Skew Left Triangle Identity:* Let  $(X, x_0)$  and  $(Y, y_0)$  be pointed sets. We have to show that the diagram

$$\begin{array}{ccc}
 X \triangleright Y & & \\
 \lambda_{X \triangleright Y}^{\text{Sets}_*, \triangleright} \downarrow & \searrow \lambda_X^{\text{Sets}_*, \triangleright} \triangleright \text{id}_Y & \\
 S^0 \triangleright (X \triangleright Y) & \xrightarrow{\alpha_{S^0, X, Y}^{\text{Sets}_*, \triangleright}} & (S^0 \triangleright X) \triangleright Y
 \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc}
 x \triangleright y & & \\
 \downarrow & \searrow & \\
 1 \triangleright (x \triangleright y) & \longmapsto & (1 \triangleright x) \triangleright y
 \end{array}$$

and hence indeed commutes. Thus the left skew triangle identity is satisfied.

*The Right Skew Right Triangle Identity:* Let  $(X, x_0)$  and  $(Y, y_0)$  be pointed sets. We have to show that the diagram

$$\begin{array}{ccc}
 X \triangleright (Y \triangleright S^0) & \xrightarrow{\text{id}_X \triangleright \rho_Y^{\text{Sets}_*, \triangleright}} & (X \triangleright Y) \triangleright S^0 \\
 \searrow \alpha_{S^0, X, Y}^{\text{Sets}_*, \triangleright} & & \downarrow \rho_{X \triangleright Y}^{\text{Sets}_*, \triangleright} \\
 & & X \triangleright Y
 \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc}
 x \triangleright (y \triangleright 0) & \longmapsto & (x \triangleright y) \triangleright 0 \\
 \searrow & & \downarrow \\
 & & x \triangleright y_0 = x_0 \triangleright y_0
 \end{array}$$

and

$$\begin{array}{ccc}
 x \triangleright (y \triangleright 1) & \longmapsto & (x \triangleright y) \triangleright 1 \\
 \searrow & & \downarrow \\
 & & x \triangleright y
 \end{array}$$

and hence indeed commutes. Thus the right skew triangle identity is satisfied.

*The Right Skew Middle Triangle Identity:* Let  $(X, x_0)$  and  $(Y, y_0)$  be pointed sets. We have to show that the diagram

$$\begin{array}{ccc}
 X \triangleright Y & \xlongequal{\quad} & X \triangleright Y \\
 \text{id}_X \triangleright \lambda_Y^{\text{Sets}_*, \triangleright} \downarrow & & \uparrow \rho_X^{\text{Sets}_*, \triangleright} \triangleright \text{id}_Y \\
 X \triangleright (S^0 \triangleright Y) & \xrightarrow[\alpha_{X, S^0, Y}^{\text{Sets}_*, \triangleright}]{} & (X \triangleright S^0) \triangleright Y
 \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc}
 x \triangleright y & \xrightarrow{\quad} & x \triangleright y \\
 \downarrow & & \uparrow \\
 x \triangleright (1 \triangleright y) & \xrightarrow{\quad} & (x \triangleright 1) \triangleright y
 \end{array}$$

and hence indeed commutes. Thus the right skew triangle identity is satisfied.

*The Zig-Zag Identity:* We have to show that the diagram

$$\begin{array}{ccc}
 S^0 & \xrightarrow{\lambda_{S^0}^{\text{Sets}_*, \triangleright}} & S^0 \triangleright S^0 \\
 \searrow & & \downarrow \rho_{S^0}^{\text{Sets}_*, \triangleright} \\
 & & S^0
 \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc}
 0 & \xrightarrow{\quad} & 1 \triangleright 0 \\
 \searrow & & \downarrow \\
 & & 0
 \end{array}$$

and

$$\begin{array}{ccc}
 1 & \xrightarrow{\quad} & 1 \triangleright 1 \\
 \searrow & & \downarrow \\
 & & 1
 \end{array}$$



and hence indeed commutes. Thus the zig-zag identity is satisfied.

*Right Skew Monoidal Right-Closedness:* This follows from **Item 2** of **Proposition 4.1.1.7**.  $\square$

## 4.9 Monoids With Respect to the Right Tensor Product of Pointed Sets

00FR

**Proposition 4.9.1.1.** The category of monoids on  $(\text{Sets}_*, \triangleright, S^0)$  is isomorphic to the category of “monoids with right zero”<sup>18</sup> and morphisms between them.

00FS

*Proof.* Monoids on  $(\text{Sets}_*, \triangleright, S^0)$ : A monoid on  $(\text{Sets}_*, \triangleright, S^0)$  consists of:

- *The Underlying Object.* A pointed set  $(A, 0_A)$ .
- *The Multiplication Morphism.* A morphism of pointed sets

$$\mu_A: A \triangleright A \rightarrow A,$$

determining a right bilinear morphism of pointed sets

$$\begin{aligned} A \times A &\longrightarrow A \\ (a, b) &\longmapsto ab. \end{aligned}$$

- *The Unit Morphism.* A morphism of pointed sets

$$\eta_A: S^0 \rightarrow A$$

picking an element  $1_A$  of  $A$ .

satisfying the following conditions:

---

<sup>18</sup>A monoid with right zero is defined similarly as the monoids with zero of ???. Succinctly, they are monoids  $(A, \mu_A, \eta_A)$  with a special element  $0_A$  satisfying

$$0_A a = 0_A$$

for each  $a \in A$ .

1. *Associativity.* The diagram

$$\begin{array}{ccccc}
 & & A \triangleright (A \triangleright A) & & \\
 \alpha_{A,A,A}^{\text{Sets}_*, \triangleright} \nearrow & & & \searrow \text{id}_A \triangleright \mu_A & \\
 (A \triangleright A) \triangleright A & & & & A \triangleright A \\
 \mu_A \triangleright \text{id}_A \searrow & & & \nearrow \mu_A & \\
 & A \triangleright A & \xrightarrow{\mu_A} & A &
 \end{array}$$

2. *Left Unitality.* The diagram

$$\begin{array}{ccccc}
 A & \xrightarrow{\lambda_A^{\text{Sets}_*, \triangleright}} & S^0 \triangleright A & & \\
 \parallel & & \downarrow \eta_A \times \text{id}_A & & \\
 A & \xleftarrow{\mu_A} & A \triangleright A & &
 \end{array}$$

commutes.

3. *Right Unitality.* The diagram

$$\begin{array}{ccc}
 A \triangleright S^0 & \xrightarrow{\text{id}_A \times \eta_A} & A \triangleright A \\
 \searrow \rho_A^{\text{Sets}_*, \triangleright} & & \downarrow \mu_A \\
 & & A
 \end{array}$$

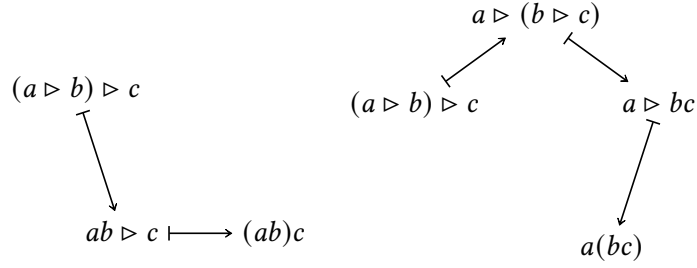
commutes.

Being a right-bilinear morphism of pointed sets, the multiplication map satisfies

$$0_A a = 0_A$$

for each  $a \in A$ . Now, the associativity, left unitality, and right unitality conditions act on elements as follows:

1. *Associativity.* The associativity condition acts as

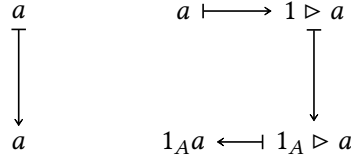


This gives

$$(ab)c = a(bc)$$

for each  $a, b, c \in A$ .

2. *Left Unitality.* The left unitality condition acts as



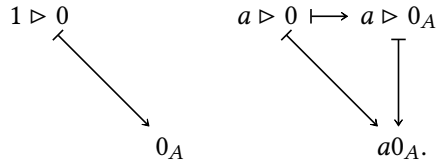
This gives

$$1_A a = a$$

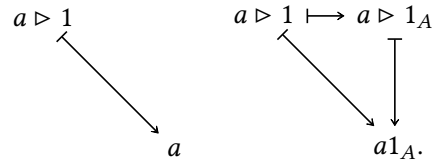
for each  $a \in A$ .

3. *Right Unitality.* The right unitality condition acts:

(a) On  $1 \triangleright 0$  as



(b) On  $a \triangleright 1$  as



This gives

$$\begin{aligned} a1_A &= a, \\ a0_A &= 0_A \end{aligned}$$

for each  $a \in A$ .

Thus we see that monoids with respect to  $\triangleright$  are exactly monoids with right zero. *Morphisms of Monoids on  $(\text{Sets}_*, \triangleright, S^0)$* : A morphism of monoids on  $(\text{Sets}_*, \triangleright, S^0)$  from  $(A, \mu_A, \eta_A, 0_A)$  to  $(B, \mu_B, \eta_B, 0_B)$  is a morphism of pointed sets

$$f: (A, 0_A) \rightarrow (B, 0_B)$$

satisfying the following conditions:

1. *Compatibility With the Multiplication Morphisms.* The diagram

$$\begin{array}{ccc} A \triangleright A & \xrightarrow{f \triangleright f} & B \triangleright B \\ \mu_A \downarrow & & \downarrow \mu_B \\ A & \xrightarrow{f} & B \end{array}$$

commutes.

2. *Compatibility With the Unit Morphisms.* The diagram

$$\begin{array}{ccc} S^0 & \xrightarrow{\eta_A} & A \\ & \searrow \eta_B & \downarrow f \\ & & B \end{array}$$

commutes.

These act on elements as

$$\begin{array}{ccc} a \triangleright b & & a \triangleright b \longmapsto f(a) \triangleright f(b) \\ \downarrow & & \downarrow \\ ab \longmapsto f(ab) & & f(a)f(b) \end{array}$$

and

$$\begin{array}{ccc} 0 & & 0 \mapsto 0_A \\ & \searrow & \downarrow \\ & & f(0_A) \\ & & \downarrow \\ & & 0_B \end{array}$$

and

$$\begin{array}{ccc} 1 & & 1 \mapsto 1_A \\ & \searrow & \downarrow \\ & & f(1_A) \\ & & \downarrow \\ & & 1_B \end{array}$$

giving

$$\begin{aligned} f(ab) &= f(a)f(b), \\ f(0_A) &= 0_B, \\ f(1_A) &= 1_B, \end{aligned}$$

for each  $a, b \in A$ , which is exactly a morphism of monoids with right zero.

*Identities and Composition:* Similarly, the identities and composition of  $\text{Mon}(\text{Sets}_*, \triangleright, S^0)$  can be easily seen to agree with those of monoids with right zero, which finishes the proof.  $\square$

## 00FT 5 The Smash Product of Pointed Sets

### 00FU 5.1 Foundations

Let  $(X, x_0)$  and  $(Y, y_0)$  be pointed sets.

**00FV Definition 5.1.1.1.** The **smash product** of  $(X, x_0)$  and  $(Y, y_0)$ <sup>19</sup> is the pointed set  $X \wedge Y$ <sup>20</sup> satisfying the bijection

$$\text{Sets}_*(X \wedge Y, Z) \cong \text{Hom}_{\text{Sets}_*}^{\otimes}(X \times Y, Z),$$

naturally in  $(X, x_0), (Y, y_0), (Z, z_0) \in \text{Obj}(\text{Sets}_*)$ .

<sup>19</sup>*Further Terminology:* In the context of monoids with zero as models for  $\mathbb{F}_1$ -algebras, the smash product  $X \wedge Y$  is also called the **tensor product of  $\mathbb{F}_1$ -modules of  $(X, x_0)$  and  $(Y, y_0)$**  or the **tensor product of  $(X, x_0)$  and  $(Y, y_0)$  over  $\mathbb{F}_1$** .

<sup>20</sup>*Further Notation:* In the context of monoids with zero as models for  $\mathbb{F}_1$ -algebras, the smash

**00FW Remark 5.1.1.2.** That is to say, the smash product of pointed sets is defined so as to induce a bijection between the following data:

- Pointed maps  $f: X \wedge Y \rightarrow Z$ .
- Maps of sets  $f: X \times Y \rightarrow Z$  satisfying

$$\begin{aligned} f(x_0, y) &= z_0, \\ f(x, y_0) &= z_0 \end{aligned}$$

for each  $x \in X$  and each  $y \in Y$ .

**00FX Remark 5.1.1.3.** The smash product of pointed sets may be described as follows:

- The smash product of  $(X, x_0)$  and  $(Y, y_0)$  is the pair  $((X \wedge Y, x_0 \wedge y_0), \iota)$  consisting of
  - A pointed set  $(X \wedge Y, x_0 \wedge y_0)$ ;
  - A bilinear morphism of pointed sets  $\iota: (X \times Y, (x_0, y_0)) \rightarrow X \wedge Y$ ;

satisfying the following universal property:

(UP) Given another such pair  $((Z, z_0), f)$  consisting of

- \* A pointed set  $(Z, z_0)$ ;
- \* A bilinear morphism of pointed sets  $f: (X \times Y, (x_0, y_0)) \rightarrow Z$ ;

there exists a unique morphism of pointed sets  $X \wedge Y \xrightarrow{\exists!} Z$  making the diagram

$$\begin{array}{ccc} & X \wedge Y & \\ \iota \nearrow & & \downarrow \exists! \\ X \times Y & \xrightarrow{f} & Z \end{array}$$

commute.

**00FY Construction 5.1.1.4.** Concretely, the **smash product** of  $(X, x_0)$  and  $(Y, y_0)$  is the pointed set  $(X \wedge Y, x_0 \wedge y_0)$  consisting of

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- *The Underlying Set.* The set  $X \wedge Y$  defined by

$$X \wedge Y \cong (X \times Y) / \sim_R,$$

where  $\sim_R$  is the equivalence relation on  $X \times Y$  obtained by declaring

$$\begin{aligned} (x_0, y) &\sim_R (x_0, y'), \\ (x, y_0) &\sim_R (x', y_0) \end{aligned}$$

for each  $x, x' \in X$  and each  $y, y' \in Y$ ;

- *The Basepoint.* The element  $[(x_0, y_0)]$  of  $X \wedge Y$  given by the equivalence class of  $(x_0, y_0)$  under the equivalence relation  $\sim$  on  $X \times Y$ .

*Proof.* By **Equivalence Relations and Apartness Relations, Item 6 of Proposition 5.2.1.3**, we have a natural bijection

$$\text{Sets}_*(X \wedge Y, Z) \cong \text{Hom}_{\text{Sets}}^R(X \times Y, Z).$$

Now, by definition,  $\text{Hom}_{\text{Sets}}^R(X \times Y, Z)$  is the set

$$\text{Hom}_{\text{Sets}}^R(X \times Y, Z) \stackrel{\text{def}}{=} \left\{ f \in \text{Hom}_{\text{Sets}}(X \times Y, Z) \left| \begin{array}{l} \text{for each } x, y \in X, \text{ if} \\ (x, y) \sim_R (x', y'), \text{ then} \\ f(x, y) = f(x', y') \end{array} \right. \right\}.$$

However, the condition  $(x, y) \sim_R (x', y')$  only holds when:

1. We have  $x = x'$  and  $y = y'$ .
2. The following conditions are satisfied:
  - (a) We have  $x = x_0$  or  $y = y_0$ .
  - (b) We have  $x' = x_0$  or  $y' = y_0$ .

So, given  $f \in \text{Hom}_{\text{Sets}}(X \times Y, Z)$  with a corresponding  $\bar{f}: X \wedge Y \rightarrow Z$ , the latter case above implies

$$\begin{aligned} f(x_0, y) &= f(x, y_0) \\ &= f(x_0, y_0), \end{aligned}$$

---

product  $X \wedge Y$  is also denoted  $X \otimes_{\mathbb{R}_1} Y$ .

and since  $\bar{f}: X \wedge Y \rightarrow Z$  is a pointed map, we have

$$\begin{aligned} f(x_0, y_0) &= \bar{f}(x_0, y_0) \\ &= z_0. \end{aligned}$$

Thus the elements  $f$  in  $\text{Hom}_{\text{Sets}}(X \times Y, Z)$  are precisely those functions  $f: X \times Y \rightarrow Z$  satisfying the equalities

$$\begin{aligned} f(x_0, y) &= z_0, \\ f(x, y_0) &= z_0 \end{aligned}$$

for each  $x \in X$  and each  $y \in Y$ , giving an equality

$$\text{Hom}_{\text{Sets}}^R(X \times Y, Z) = \text{Hom}_{\text{Sets}_*}^{\otimes}(X \times Y, Z)$$

of sets, which when composed with our earlier isomorphism

$$\text{Sets}_*(X \wedge Y, Z) \cong \text{Hom}_{\text{Sets}}^R(X \times Y, Z)$$

gives our desired natural bijection, finishing the proof.  $\square$

**00FZ Remark 5.1.1.5.** It is also somewhat common to write

$$X \wedge Y \stackrel{\text{def}}{=} \frac{X \times Y}{X \vee Y},$$

identifying  $X \vee Y$  with the subspace  $(\{x_0\} \times Y) \cup (X \times \{y_0\})$  of  $X \times Y$ , and having the quotient be defined by declaring  $(x, y) \sim (x', y')$  iff we have  $(x, y), (x', y') \in X \vee Y$ .

**00G0 Notation 5.1.1.6.** We write  $x \wedge y$  for the element  $[(x, y)]$  of

$$X \wedge Y \cong X \times Y / \sim.$$

**00G1 Remark 5.1.1.7.** Employing the notation introduced in **Notation 5.1.1.6**, we have

$$\begin{aligned} x_0 \wedge y_0 &= x \wedge y_0, \\ &= x_0 \wedge y \end{aligned}$$

for each  $x \in X$  and each  $y \in Y$ , and

$$\begin{aligned} x \wedge y_0 &= x' \wedge y_0, \\ x_0 \wedge y &= x_0 \wedge y' \end{aligned}$$

for each  $x, x' \in X$  and each  $y, y' \in Y$ .



**00G2 Example 5.1.1.8.** Here are some examples of smash products of pointed sets.

**00G3** 1. *Smashing With pt.* For any pointed set  $X$ , we have isomorphisms of pointed sets

$$\begin{aligned} \text{pt} \wedge X &\cong \text{pt}, \\ X \wedge \text{pt} &\cong \text{pt}. \end{aligned}$$

**00G4** 2. *Smashing With  $S^0$ .* For any pointed set  $X$ , we have isomorphisms of pointed sets

$$\begin{aligned} S^0 \wedge X &\cong X, \\ X \wedge S^0 &\cong X. \end{aligned}$$

**00G5 Proposition 5.1.1.9.** Let  $(X, x_0)$  and  $(Y, y_0)$  be pointed sets.

**00G6** 1. *Functoriality.* The assignments  $X, Y, (X, Y) \mapsto X \wedge Y$  define functors

$$\begin{aligned} X \wedge - &: \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*, \\ - \wedge Y &: \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*, \\ -_1 \wedge -_2 &: \mathbf{Sets}_* \times \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*. \end{aligned}$$

In particular, given pointed maps

$$\begin{aligned} f &: (X, x_0) \rightarrow (A, a_0), \\ g &: (Y, y_0) \rightarrow (B, b_0), \end{aligned}$$

the induced map

$$f \wedge g: X \wedge Y \rightarrow A \wedge B$$

is given by

$$[f \wedge g](x \wedge y) \stackrel{\text{def}}{=} f(x) \wedge g(y)$$

for each  $x \wedge y \in X \wedge Y$ .

**00G7** 2. *Adjointness.* We have adjunctions

$$\begin{aligned} (X \wedge - \dashv \mathbf{Sets}_*(X, -)) &: \mathbf{Sets}_* \begin{array}{c} \xrightarrow{X \wedge -} \\ \perp \\ \xleftarrow{\mathbf{Sets}_*(X, -)} \end{array} \mathbf{Sets}_*, \\ (- \wedge Y \dashv \mathbf{Sets}_*(Y, -)) &: \mathbf{Sets}_* \begin{array}{c} \xrightarrow{- \wedge Y} \\ \perp \\ \xleftarrow{\mathbf{Sets}_*(Y, -)} \end{array} \mathbf{Sets}_*. \end{aligned}$$



*Proof. Item 1, Functoriality:* The map  $f \wedge g$  comes from **Equivalence Relations and Apartness Relations, Item 4** of **Proposition 5.2.1.3** via the map

$$f \wedge g: X \times Y \rightarrow A \wedge B$$

sending  $(x, y)$  to  $f(x) \wedge g(y)$ , which we need to show satisfies

$$[f \wedge g](x, y) = [f \wedge g](x', y')$$

for each  $(x, y), (x', y') \in X \times Y$  with  $(x, y) \sim_R (x', y')$ , where  $\sim_R$  is the relation constructing  $X \wedge Y$  as

$$X \wedge Y \cong (X \times Y) / \sim_R$$

in **Construction 5.1.1.4**. The condition defining  $\sim$  is that at least one of the following conditions is satisfied:

1. We have  $x = x'$  and  $y = y'$ ;
2. Both of the following conditions are satisfied:
  - (a) We have  $x = x_0$  or  $y = y_0$ .
  - (b) We have  $x' = x_0$  or  $y' = y_0$ .

We have five cases:

1. In the first case, we clearly have

$$[f \wedge g](x, y) = [f \wedge g](x', y')$$

since  $x = x'$  and  $y = y'$ .

2. If  $x = x_0$  and  $x' = x_0$ , we have

$$\begin{aligned} [f \wedge g](x_0, y) &\stackrel{\text{def}}{=} f(x_0) \wedge g(y) \\ &= a_0 \wedge g(y) \\ &= a_0 \wedge g(y') \\ &= f(x_0) \wedge g(y') \\ &\stackrel{\text{def}}{=} [f \wedge g](x_0, y'). \end{aligned}$$

3. If  $x = x_0$  and  $y' = y_0$ , we have

$$\begin{aligned}
 [f \wedge g](x_0, y) &\stackrel{\text{def}}{=} f(x_0) \wedge g(y) \\
 &= a_0 \wedge g(y) \\
 &= a_0 \wedge b_0 \\
 &= f(x') \wedge b_0 \\
 &= f(x') \wedge g(y_0) \\
 &\stackrel{\text{def}}{=} [f \wedge g](x', y_0).
 \end{aligned}$$

4. If  $y = y_0$  and  $x' = x_0$ , we have

$$\begin{aligned}
 [f \wedge g](x, y_0) &\stackrel{\text{def}}{=} f(x) \wedge g(y_0) \\
 &= f(x) \wedge b_0 \\
 &= a_0 \wedge b_0 \\
 &= a_0 \wedge g(y') \\
 &= f(x_0) \wedge g(y') \\
 &\stackrel{\text{def}}{=} [f \wedge g](x_0, y').
 \end{aligned}$$

5. If  $y = y_0$  and  $y' = y_0$ , we have

$$\begin{aligned}
 [f \wedge g](x, y_0) &\stackrel{\text{def}}{=} f(x) \wedge g(y_0) \\
 &= f(x) \wedge b_0 \\
 &= f(x') \wedge b_0 \\
 &= f(x) \wedge g(y_0) \\
 &\stackrel{\text{def}}{=} [f \wedge g](x', y_0).
 \end{aligned}$$

Thus  $f \wedge g$  is well-defined. Next, we claim that  $\wedge$  preserves identities and composition:

- *Preservation of Identities.* We have

$$\begin{aligned}
 [\text{id}_X \wedge \text{id}_Y](x \wedge y) &\stackrel{\text{def}}{=} \text{id}_X(x) \wedge \text{id}_Y(y) \\
 &= x \wedge y \\
 &= [\text{id}_{X \wedge Y}](x \wedge y)
 \end{aligned}$$

for each  $x \wedge y \in X \wedge Y$ , and thus

$$\text{id}_X \wedge \text{id}_Y = \text{id}_{X \wedge Y}.$$

- *Preservation of Composition.* Given pointed maps

$$\begin{aligned} f &: (X, x_0) \rightarrow (X', x'_0), \\ h &: (X', x'_0) \rightarrow (X'', x''_0), \\ g &: (Y, y_0) \rightarrow (Y', y'_0), \\ k &: (Y', y'_0) \rightarrow (Y'', y''_0), \end{aligned}$$

we have

$$\begin{aligned} [(h \circ f) \wedge (k \circ g)](x \wedge y) &\stackrel{\text{def}}{=} h(f(x)) \wedge k(g(y)) \\ &\stackrel{\text{def}}{=} [h \wedge k](f(x) \wedge g(y)) \\ &\stackrel{\text{def}}{=} [h \wedge k]([f \wedge g](x \wedge y)) \\ &\stackrel{\text{def}}{=} [(h \wedge k) \circ (f \wedge g)](x \wedge y) \end{aligned}$$

for each  $x \wedge y \in X \wedge Y$ , and thus

$$(h \circ f) \wedge (k \circ g) = (h \wedge k) \circ (f \wedge g).$$

This finishes the proof.

**Item 2, Adjointness:** We prove only the adjunction  $- \wedge Y \dashv \mathbf{Sets}_*(Y, -)$ , witnessed by a natural bijection

$$\text{Hom}_{\mathbf{Sets}_*}(X \wedge Y, Z) \cong \text{Hom}_{\mathbf{Sets}_*}(X, \mathbf{Sets}_*(Y, Z)),$$

as the proof of the adjunction  $X \wedge - \dashv \mathbf{Sets}_*(X, -)$  is similar. We claim we have a bijection

$$\text{Hom}_{\mathbf{Sets}_*}^{\otimes}(X \times Y, Z) \cong \text{Hom}_{\mathbf{Sets}_*}(X, \mathbf{Sets}_*(Y, Z))$$

natural in  $(X, x_0), (Y, y_0), (Z, z_0) \in \text{Obj}(\mathbf{Sets}_*)$ , implying the desired adjunction. Indeed, this bijection is a restriction of the bijection

$$\mathbf{Sets}(X \times Y, Z) \cong \mathbf{Sets}(X, \mathbf{Sets}(Y, Z))$$

of **Constructions With Sets**, **Item 2** of **Proposition 1.3.1.2**:

- A map

$$\xi: X \times Y \rightarrow Z$$

in  $\text{Hom}_{\text{Sets}_*}^{\otimes}(X \times Y, Z)$  gets sent to the pointed map

$$\begin{aligned} \xi^{\dagger}: (X, x_0) &\rightarrow (\mathbf{Sets}_*(Y, Z), \Delta_{z_0}), \\ x &\longmapsto (\xi_x^{\dagger}: Y \rightarrow Z), \end{aligned}$$

where  $\xi_x^{\dagger}: Y \rightarrow Z$  is the map defined by

$$\xi_x^{\dagger}(y) \stackrel{\text{def}}{=} \xi(x, y)$$

for each  $y \in Y$ , where:

- The map  $\xi^{\dagger}$  is indeed pointed, as we have

$$\begin{aligned} \xi_{x_0}^{\dagger}(y) &\stackrel{\text{def}}{=} \xi(x_0, y) \\ &\stackrel{\text{def}}{=} z_0 \end{aligned}$$

for each  $y \in Y$ . Thus  $\xi_{x_0}^{\dagger} = \Delta_{z_0}$  and  $\xi^{\dagger}$  is pointed.

- The map  $\xi_x^{\dagger}$  indeed lies in  $\mathbf{Sets}_*(Y, Z)$ , as we have

$$\begin{aligned} \xi_x^{\dagger}(y_0) &\stackrel{\text{def}}{=} \xi(x, y_0) \\ &\stackrel{\text{def}}{=} z_0. \end{aligned}$$

- Conversely, a map

$$\begin{aligned} \xi: (X, x_0) &\rightarrow (\mathbf{Sets}_*(Y, Z), \Delta_{z_0}), \\ x &\longmapsto (\xi_x: Y \rightarrow Z), \end{aligned}$$

in  $\text{Hom}_{\text{Sets}_*}(X, \mathbf{Sets}_*(Y, Z))$  gets sent to the map

$$\xi^{\dagger}: X \times Y \rightarrow Z$$

defined by

$$\xi^{\dagger}(x, y) \stackrel{\text{def}}{=} \xi_x(y)$$

for each  $(x, y) \in X \times Y$ , which indeed lies in  $\text{Hom}_{\text{Sets}_*}^{\otimes}(X \times Y, Z)$ , as:

– *Left Bilinearity.* We have

$$\begin{aligned}\xi^\dagger(x_0, y) &\stackrel{\text{def}}{=} \xi_{x_0}(y) \\ &\stackrel{\text{def}}{=} \Delta_{z_0}(y) \\ &\stackrel{\text{def}}{=} z_0\end{aligned}$$

for each  $y \in Y$ , since  $\xi_{x_0} = \Delta_{z_0}$  as  $\xi$  is assumed to be a pointed map.

– *Right Bilinearity.* We have

$$\begin{aligned}\xi^\dagger(x, y_0) &\stackrel{\text{def}}{=} \xi_x(y_0) \\ &\stackrel{\text{def}}{=} z_0\end{aligned}$$

for each  $x \in X$ , since  $\xi_x \in \mathbf{Sets}_*(Y, Z)$  is a morphism of pointed sets.

This finishes the proof.

*Item 3, Enriched Adjointness:* This follows from [Item 2](#) and [??, ?? of ??](#).

*Item 4, As a Pushout:* Following the description of [Constructions With Sets](#), [Remark 2.4.1.2](#), we have

$$\text{pt} \coprod_{X \vee Y} (X \times Y) \cong (\text{pt} \times (X \times Y)) / \sim,$$

where  $\sim$  identifies the element  $\star$  in  $\text{pt}$  with all elements of the form  $(x_0, y)$  and  $(x, y_0)$  in  $X \times Y$ . Thus [Equivalence Relations and Apartness Relations](#), [Item 4](#) of [Proposition 5.2.1.3](#) coupled with [Remark 5.1.1.7](#) then gives us a well-defined map

$$\text{pt} \coprod_{X \vee Y} (X \times Y) \rightarrow X \wedge Y$$

via  $[(\star, (x, y))] \mapsto x \wedge y$ , with inverse

$$X \wedge Y \rightarrow \text{pt} \coprod_{X \vee Y} (X \times Y)$$

given by  $x \wedge y \mapsto [(\star, (x, y))]$ .

*Item 5, Distributivity Over Wedge Sums:* This follows from [Proposition 5.9.1.1](#), [??, ?? of ??](#), and the fact that  $\vee$  is the coproduct in  $\mathbf{Sets}_*$  ([Pointed Sets](#), [Definition 3.3.1.1](#)).  $\square$

## 00GB 5.2 The Internal Hom of Pointed Sets

Let  $(X, x_0)$  and  $(Y, y_0)$  be pointed sets.

00GC **Definition 5.2.1.1.** The **internal Hom**<sup>21</sup> of pointed sets from  $(X, x_0)$  to  $(Y, y_0)$  is the pointed set  $\mathbf{Sets}_*((X, x_0), (Y, y_0))$ <sup>22</sup> consisting of:

- *The Underlying Set.* The set  $\mathbf{Sets}_*((X, x_0), (Y, y_0))$  of morphisms of pointed sets from  $(X, x_0)$  to  $(Y, y_0)$ .
- *The Basepoint.* The element

$$\Delta_{y_0} : (X, x_0) \rightarrow (Y, y_0)$$

of  $\mathbf{Sets}_*((X, x_0), (Y, y_0))$  given by

$$\Delta_{y_0}(x) \stackrel{\text{def}}{=} y_0$$

for each  $x \in X$ .

*Proof.* For a proof that  $\mathbf{Sets}_*$  is indeed the internal Hom of  $\mathbf{Sets}_*$  with respect to the smash product of pointed sets, see **Item 2** of **Proposition 5.1.1.9**.  $\square$

00GD **Proposition 5.2.1.2.** Let  $(X, x_0)$  and  $(Y, y_0)$  be pointed sets.

00GE 1. *Functoriality.* The assignments  $X, Y, (X, Y) \mapsto \mathbf{Sets}_*(X, Y)$  define functors

$$\begin{aligned} \mathbf{Sets}_*(X, -) &: \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*, \\ \mathbf{Sets}_*(-, Y) &: \mathbf{Sets}_*^{\text{op}} \rightarrow \mathbf{Sets}_*, \\ \mathbf{Sets}_*(-_1, -_2) &: \mathbf{Sets}_*^{\text{op}} \times \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*. \end{aligned}$$

In particular, given pointed maps

$$\begin{aligned} f &: (X, x_0) \rightarrow (A, a_0), \\ g &: (Y, y_0) \rightarrow (B, b_0), \end{aligned}$$

<sup>21</sup>The pointed set  $\mathbf{Sets}_*(X, Y)$  is the internal **Hom** of  $\mathbf{Sets}_*$  with respect to the smash product of **Tensor Products of Pointed Sets**, **Definition 5.1.1.1**; see **Tensor Products of Pointed Sets**, **Item 2** of **Proposition 5.1.1.9**.

<sup>22</sup>*Further Notation:* Also written  $\mathbf{Hom}_{\mathbf{Sets}_*}(X, Y)$ .



the induced map

$$\mathbf{Sets}_*(f, g): \mathbf{Sets}_*(A, Y) \rightarrow \mathbf{Sets}_*(X, B)$$

is given by

$$[\mathbf{Sets}_*(f, g)](\phi) \stackrel{\text{def}}{=} g \circ \phi \circ f$$

for each  $\phi \in \mathbf{Sets}_*(A, Y)$ .

00GF

2. *Adjointness.* We have adjunctions

$$\begin{aligned} (X \wedge - \dashv \mathbf{Sets}_*(X, -)): \quad & \mathbf{Sets}_* \begin{array}{c} \xrightarrow{X \wedge -} \\ \perp \\ \xleftarrow{\mathbf{Sets}_*(X, -)} \end{array} \mathbf{Sets}_*, \\ (- \wedge Y \dashv \mathbf{Sets}_*(Y, -)): \quad & \mathbf{Sets}_* \begin{array}{c} \xrightarrow{- \wedge Y} \\ \perp \\ \xleftarrow{\mathbf{Sets}_*(Y, -)} \end{array} \mathbf{Sets}_*, \end{aligned}$$

witnessed by bijections

$$\begin{aligned} \text{Hom}_{\mathbf{Sets}_*}(X \wedge Y, Z) &\cong \text{Hom}_{\mathbf{Sets}_*}(X, \mathbf{Sets}_*(Y, Z)), \\ \text{Hom}_{\mathbf{Sets}_*}(X \wedge Y, Z) &\cong \text{Hom}_{\mathbf{Sets}_*}(X, \mathbf{Sets}_*(A, Z)), \end{aligned}$$

natural in  $(X, x_0), (Y, y_0), (Z, z_0) \in \text{Obj}(\mathbf{Sets}_*)$ .

00GG

3. *Enriched Adjointness.* We have  $\mathbf{Sets}_*$ -enriched adjunctions

$$\begin{aligned} (X \wedge - \dashv \mathbf{Sets}_*(X, -)): \quad & \mathbf{Sets}_* \begin{array}{c} \xrightarrow{X \wedge -} \\ \perp \\ \xleftarrow{\mathbf{Sets}_*(X, -)} \end{array} \mathbf{Sets}_*, \\ (- \wedge Y \dashv \mathbf{Sets}_*(Y, -)): \quad & \mathbf{Sets}_* \begin{array}{c} \xrightarrow{- \wedge Y} \\ \perp \\ \xleftarrow{\mathbf{Sets}_*(Y, -)} \end{array} \mathbf{Sets}_*, \end{aligned}$$

witnessed by isomorphisms of pointed sets

$$\begin{aligned} \mathbf{Sets}_*(X \wedge Y, Z) &\cong \mathbf{Sets}_*(X, \mathbf{Sets}_*(Y, Z)), \\ \mathbf{Sets}_*(X \wedge Y, Z) &\cong \mathbf{Sets}_*(X, \mathbf{Sets}_*(A, Z)), \end{aligned}$$

natural in  $(X, x_0), (Y, y_0), (Z, z_0) \in \text{Obj}(\mathbf{Sets}_*)$ .

*Proof. Item 1, Functoriality:* This follows from **Constructions With Sets, Item 1** of **Proposition 3.5.1.2** and from the equalities

$$\begin{aligned} g \circ \Delta_{y_0} &= \Delta_{z_0}, \\ \Delta_{y_0} \circ f &= \Delta_{y_0} \end{aligned}$$

for morphisms  $f: (K, k_0) \rightarrow (X, x_0)$  and  $g: (Y, y_0) \rightarrow (Z, z_0)$ , which guarantee pre- and postcomposition by morphisms of pointed sets to also be morphisms of pointed sets.

*Item 2, Adjointness:* This is a repetition of **Item 2** of **Proposition 5.1.1.9**, and is proved there.

*Item 3, Enriched Adjointness:* This is a repetition of **Item 3** of **Proposition 5.1.1.9**, and is proved there.  $\square$

### 00GH 5.3 The Monoidal Unit

**00GJ Definition 5.3.1.1.** The **monoidal unit of the smash product of pointed sets** is the functor

$$\mathbb{1}^{\text{Sets}_*} : \text{pt} \rightarrow \text{Sets}_*$$

defined by

$$\mathbb{1}_{\text{Sets}_*} \stackrel{\text{def}}{=} S^0.$$

### 00GK 5.4 The Associator

**00GL Definition 5.4.1.1.** The **associator of the smash product of pointed sets** is the natural isomorphism

$$\alpha^{\text{Sets}_*} : \wedge \circ (\wedge \times \text{id}_{\text{Sets}_*}) \xrightarrow{\sim} \wedge \circ (\text{id}_{\text{Sets}_*} \times \wedge) \circ \alpha_{\text{Sets}_*, \text{Sets}_*, \text{Sets}_*}^{\text{Cats}},$$

as in the diagram

$$\begin{array}{ccccc} & & \text{Sets}_* \times (\text{Sets}_* \times \text{Sets}_*) & & \\ & \nearrow \alpha_{\text{Sets}_*, \text{Sets}_*, \text{Sets}_*}^{\text{Cats}} & & \searrow \text{id} \times \wedge & \\ (\text{Sets}_* \times \text{Sets}_*) \times \text{Sets}_* & & & & \text{Sets}_* \times \text{Sets}_* \\ \downarrow \wedge \times \text{id} & \nearrow \alpha_{\text{Sets}_*} & & \downarrow \wedge & \\ \text{Sets}_* \times \text{Sets}_* & \xrightarrow{\wedge} & \text{Sets}_* & & \end{array}$$

whose component

$$\alpha_{X,Y,Z}^{\text{Sets}_*}: (X \wedge Y) \wedge Z \xrightarrow{\cong} X \wedge (Y \wedge Z)$$

at  $(X, x_0), (Y, y_0), (Z, z_0) \in \text{Obj}(\text{Sets}_*)$  is given by

$$\alpha_{X,Y,Z}^{\text{Sets}_*}((x \wedge y) \wedge z) \stackrel{\text{def}}{=} x \wedge (y \wedge z)$$

for each  $(x \wedge y) \wedge z \in (X \wedge Y) \wedge Z$ .

*Proof. Well-Definedness:* Let  $[((x, y), z)] = [((x', y'), z')]$  be an element in  $(X \wedge Y) \wedge Z$ . Then either:

1. We have  $x = x', y = y',$  and  $z = z'.$
2. Both of the following conditions are satisfied:
  - (a) We have  $x = x_0$  or  $y = y_0$  or  $z = z_0.$
  - (b) We have  $x' = x_0$  or  $y' = y_0$  or  $z' = z_0.$

In the first case,  $\alpha_{X,Y,Z}^{\text{Sets}_*}$  clearly sends both elements to the same element in  $X \wedge (Y \wedge Z)$ . Meanwhile, in the latter case both elements are equal to the basepoint  $(x_0 \wedge y_0) \wedge z_0$  of  $(X \wedge Y) \wedge Z$ , which gets sent to the basepoint  $x_0 \wedge (y_0 \wedge z_0)$  of  $X \wedge (Y \wedge Z)$ .

*Being a Morphism of Pointed Sets:* As just mentioned, we have

$$\alpha_{X,Y,Z}^{\text{Sets}_*}((x_0 \wedge y_0) \wedge z_0) \stackrel{\text{def}}{=} x_0 \wedge (y_0 \wedge z_0),$$

and thus  $\alpha_{X,Y,Z}^{\text{Sets}_*}$  is a morphism of pointed sets.

*Invertibility:* Clearly, the inverse of  $\alpha_{X,Y,Z}^{\text{Sets}_*}$  is given by the morphism

$$\alpha_{X,Y,Z}^{\text{Sets}_*, -1}: X \wedge (Y \wedge Z) \xrightarrow{\cong} (X \wedge Y) \wedge Z$$

defined by

$$\alpha_{X,Y,Z}^{\text{Sets}_*, -1}(x \wedge (y \wedge z)) \stackrel{\text{def}}{=} (x \wedge y) \wedge z$$

for each  $x \wedge (y \wedge z) \in X \wedge (Y \wedge Z)$ .

*Naturality:* We need to show that, given morphisms of pointed sets

$$\begin{aligned} f: (X, x_0) &\rightarrow (X', x'_0), \\ g: (Y, y_0) &\rightarrow (Y', y'_0), \\ h: (Z, z_0) &\rightarrow (Z', z'_0) \end{aligned}$$

the diagram

$$\begin{array}{ccc}
 (X \wedge Y) \wedge Z & \xrightarrow{(f \wedge g) \wedge h} & (X' \wedge Y') \wedge Z' \\
 \alpha_{X,Y,Z}^{\text{Sets}_*} \downarrow & & \downarrow \alpha_{X',Y',Z'}^{\text{Sets}_*} \\
 X \wedge (Y \wedge Z) & \xrightarrow{f \wedge (g \wedge h)} & X' \wedge (Y' \wedge Z')
 \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc}
 (x \wedge y) \wedge z & \longmapsto & (f(x) \wedge g(y)) \wedge h(z) \\
 \downarrow & & \downarrow \\
 x \wedge (y \wedge z) & \longmapsto & f(x) \wedge (g(y) \wedge h(z))
 \end{array}$$

and hence indeed commutes, showing  $\alpha^{\text{Sets}_*}$  to be a natural transformation.

*Being a Natural Isomorphism:* Since  $\alpha^{\text{Sets}_*}$  is natural and  $\alpha^{\text{Sets}_*, -1}$  is a component-wise inverse to  $\alpha^{\text{Sets}_*}$ , it follows from **Categories, Item 2** of **Proposition 8.6.1.2** that  $\alpha^{\text{Sets}_*, -1}$  is also natural. Thus  $\alpha^{\text{Sets}_*}$  is a natural isomorphism.  $\square$

## 00GM 5.5 The Left Unitor

00GN **Definition 5.5.1.1.** The **left unitor of the smash product of pointed sets** is the natural isomorphism

$$\begin{array}{ccc}
 \text{pt} \times \text{Sets}_* & \xrightarrow{\mathbb{1}_{\text{Sets}_*} \times \text{id}} & \text{Sets}_* \times \text{Sets}_* \\
 \downarrow \lambda_{\text{Sets}_*}^{\text{Cats}_2} & \swarrow \lambda_{\text{Sets}_*} & \downarrow \wedge \\
 & & \text{Sets}_*
 \end{array}$$

$\lambda^{\text{Sets}_*} : \wedge \circ (\mathbb{1}^{\text{Sets}_*} \times \text{id}_{\text{Sets}_*}) \xRightarrow{\sim} \lambda_{\text{Sets}_*}^{\text{Cats}_2}$

whose component

$$\lambda_X^{\text{Sets}_*} : S^0 \wedge X \xrightarrow{\cong} X$$

at  $X \in \text{Obj}(\text{Sets}_*)$  is given by

$$\begin{aligned}
 0 \wedge x &\mapsto x_0, \\
 1 \wedge x &\mapsto x.
 \end{aligned}$$

*Proof. Well-Definedness:* Let  $[(x, y)] = [(x', y')]$  be an element in  $S^0 \wedge X$ . Then either:

1. We have  $x = x'$  and  $y = y'$ .
2. Both of the following conditions are satisfied:
  - (a) We have  $x = 0$  or  $y = x_0$ .
  - (b) We have  $x' = 0$  or  $y' = x_0$ .

In the first case,  $\lambda_X^{\text{Sets}_*}$  clearly sends both elements to the same element in  $X$ . Meanwhile, in the latter case both elements are equal to the basepoint  $0 \wedge x_0$  of  $S^0 \wedge X$ , which gets sent to the basepoint  $x_0$  of  $X$ .

*Being a Morphism of Pointed Sets:* As just mentioned, we have

$$\lambda_X^{\text{Sets}_*}(0 \wedge x_0) \stackrel{\text{def}}{=} x_0,$$

and thus  $\lambda_X^{\text{Sets}_*}$  is a morphism of pointed sets.

*Invertibility:* The inverse of  $\lambda_X^{\text{Sets}_*}$  is the morphism

$$\lambda_X^{\text{Sets}_*, -1} : X \xrightarrow{\cong} S^0 \wedge X$$

defined by

$$\lambda_X^{\text{Sets}_*, -1}(x) \stackrel{\text{def}}{=} 1 \wedge x$$

for each  $x \in X$ . Indeed:

- *Invertibility I.* We have

$$\begin{aligned} \left[ \lambda_X^{\text{Sets}_*, -1} \circ \lambda_X^{\text{Sets}_*} \right] (0 \wedge x) &= \lambda_X^{\text{Sets}_*, -1} \left( \lambda_X^{\text{Sets}_*}(0 \wedge x) \right) \\ &= \lambda_X^{\text{Sets}_*, -1}(x_0) \\ &= 1 \wedge x_0 \\ &= 0 \wedge x, \end{aligned}$$

and

$$\begin{aligned} \left[ \lambda_X^{\text{Sets}_*, -1} \circ \lambda_X^{\text{Sets}_*} \right] (1 \wedge x) &= \lambda_X^{\text{Sets}_*, -1} \left( \lambda_X^{\text{Sets}_*}(1 \wedge x) \right) \\ &= \lambda_X^{\text{Sets}_*, -1}(x) \\ &= 1 \wedge x \end{aligned}$$

for each  $x \in X$ , and thus we have

$$\lambda_X^{\text{Sets}_*, -1} \circ \lambda_X^{\text{Sets}_*} = \text{id}_{S^0 \wedge X}.$$

- *Invertibility II.* We have

$$\begin{aligned} \left[ \lambda_X^{\text{Sets}_*} \circ \lambda_X^{\text{Sets}_*, -1} \right] (x) &= \lambda_X^{\text{Sets}_*} \left( \lambda_X^{\text{Sets}_*, -1} (x) \right) \\ &= \lambda_X^{\text{Sets}_*, -1} (1 \wedge x) \\ &= x \end{aligned}$$

for each  $x \in X$ , and thus we have

$$\lambda_X^{\text{Sets}_*} \circ \lambda_X^{\text{Sets}_*, -1} = \text{id}_X.$$

This shows  $\lambda_X^{\text{Sets}_*}$  to be invertible.

*Naturality:* We need to show that, given a morphism of pointed sets

$$f: (X, x_0) \rightarrow (Y, y_0),$$

the diagram

$$\begin{array}{ccc} S^0 \wedge X & \xrightarrow{\text{id}_{S^0} \wedge f} & S^0 \wedge Y \\ \lambda_X^{\text{Sets}_*} \downarrow & & \downarrow \lambda_Y^{\text{Sets}_*} \\ X & \xrightarrow{f} & Y \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc} 0 \wedge x & & 0 \wedge x \mapsto 0 \wedge f(x) \\ \downarrow & & \downarrow \\ x_0 \mapsto & f(x_0) & y_0 \end{array}$$

and

$$\begin{array}{ccc} 1 \wedge x \mapsto & 1 \wedge f(x) & \\ \downarrow & & \downarrow \\ x \mapsto & f(x) & \end{array}$$

and hence indeed commutes, showing  $\lambda^{\text{Sets}_*}$  to be a natural transformation.

*Being a Natural Isomorphism:* Since  $\lambda^{\text{Sets}_*}$  is natural and  $\lambda^{\text{Sets}_*, -1}$  is a componentwise inverse to  $\lambda^{\text{Sets}_*}$ , it follows from **Categories, Item 2 of Proposition 8.6.1.2** that  $\lambda^{\text{Sets}_*, -1}$  is also natural. Thus  $\lambda^{\text{Sets}_*}$  is a natural isomorphism.  $\square$

00GP **5.6 The Right Unitor**

00GQ **Definition 5.6.1.1.** The **right unitor of the smash product of pointed sets** is the natural isomorphism

$$\rho^{\text{Sets}_*} : \wedge \circ (\text{id} \times \mathbb{1}^{\text{Sets}_*}) \xrightarrow{\sim} \rho_{\text{Sets}_*}^{\text{Cats}_2},$$

whose component

$$\rho_X^{\text{Sets}_*} : X \wedge S^0 \xrightarrow{\cong} X$$

at  $X \in \text{Obj}(\text{Sets}_*)$  is given by

$$\begin{aligned} x \wedge 0 &\mapsto x_0, \\ x \wedge 1 &\mapsto x. \end{aligned}$$

*Proof. Well-Definedness:* Let  $[(x, y)] = [(x', y')]$  be an element in  $X \wedge S^0$ . Then either:

1. We have  $x = x'$  and  $y = y'$ .
2. Both of the following conditions are satisfied:
  - (a) We have  $x = x_0$  or  $y = 0$ .
  - (b) We have  $x' = x_0$  or  $y' = 0$ .

In the first case,  $\rho_X^{\text{Sets}_*}$  clearly sends both elements to the same element in  $X$ . Meanwhile, in the latter case both elements are equal to the basepoint  $x_0 \wedge 0$  of  $X \wedge S^0$ , which gets sent to the basepoint  $x_0$  of  $X$ .

*Being a Morphism of Pointed Sets:* As just mentioned, we have

$$\rho_X^{\text{Sets}_*}(x_0 \wedge 0) \stackrel{\text{def}}{=} x_0,$$

and thus  $\rho_X^{\text{Sets}_*}$  is a morphism of pointed sets.

*Invertibility:* The inverse of  $\rho_X^{\text{Sets}_*}$  is the morphism

$$\rho_X^{\text{Sets}_*, -1} : X \xrightarrow{\cong} X \wedge S^0$$

defined by

$$\rho_X^{\text{Sets}_*, -1}(x) \stackrel{\text{def}}{=} x \wedge 1$$

for each  $x \in X$ . Indeed:

- *Invertibility I.* We have

$$\begin{aligned} \left[ \rho_X^{\text{Sets}_*, -1} \circ \rho_X^{\text{Sets}_*} \right] (x \wedge 0) &= \rho_X^{\text{Sets}_*, -1} \left( \rho_X^{\text{Sets}_*} (x \wedge 0) \right) \\ &= \rho_X^{\text{Sets}_*, -1} (x_0) \\ &= x_0 \wedge 1 \\ &= x \wedge 0, \end{aligned}$$

and

$$\begin{aligned} \left[ \rho_X^{\text{Sets}_*, -1} \circ \rho_X^{\text{Sets}_*} \right] (x \wedge 1) &= \rho_X^{\text{Sets}_*, -1} \left( \rho_X^{\text{Sets}_*} (x \wedge 1) \right) \\ &= \rho_X^{\text{Sets}_*, -1} (x) \\ &= x \wedge 1 \end{aligned}$$

for each  $x \in X$ , and thus we have

$$\rho_X^{\text{Sets}_*, -1} \circ \rho_X^{\text{Sets}_*} = \text{id}_{X \wedge S^0}.$$

- *Invertibility II.* We have

$$\begin{aligned} \left[ \rho_X^{\text{Sets}_*} \circ \rho_X^{\text{Sets}_*, -1} \right] (x) &= \rho_X^{\text{Sets}_*} \left( \rho_X^{\text{Sets}_*, -1} (x) \right) \\ &= \rho_X^{\text{Sets}_*, -1} (x \wedge 1) \\ &= x \end{aligned}$$

for each  $x \in X$ , and thus we have

$$\rho_X^{\text{Sets}_*} \circ \rho_X^{\text{Sets}_*, -1} = \text{id}_X.$$



This shows  $\rho_X^{\text{Sets}_*}$  to be invertible.

*Naturality:* We need to show that, given a morphism of pointed sets

$$f: (X, x_0) \rightarrow (Y, y_0),$$

the diagram

$$\begin{array}{ccc} X \wedge S^0 & \xrightarrow{f \wedge \text{id}_{S^0}} & Y \wedge S^0 \\ \rho_X^{\text{Sets}_*} \downarrow & & \downarrow \rho_Y^{\text{Sets}_*} \\ X & \xrightarrow{f} & Y \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc} x \wedge 0 & & x \wedge 0 \mapsto f(x) \wedge 0 \\ \downarrow & & \downarrow \\ x_0 \mapsto f(x_0) & & y_0 \end{array}$$

and

$$\begin{array}{ccc} x \wedge 1 \mapsto f(x) \wedge 1 & & \\ \downarrow & & \downarrow \\ x \mapsto f(x) & & \end{array}$$

and hence indeed commutes, showing  $\rho^{\text{Sets}_*}$  to be a natural transformation.

*Being a Natural Isomorphism:* Since  $\rho^{\text{Sets}_*}$  is natural and  $\rho^{\text{Sets}_*, -1}$  is a component-wise inverse to  $\rho^{\text{Sets}_*}$ , it follows from **Categories, Item 2** of **Proposition 8.6.1.2** that  $\rho^{\text{Sets}_*, -1}$  is also natural. Thus  $\rho^{\text{Sets}_*}$  is a natural isomorphism.  $\square$

## 00GR 5.7 The Symmetry

**00GS Definition 5.7.1.1.** The **symmetry of the smash product of pointed sets** is the natural isomorphism

$$\sigma^{\text{Sets}_*}: \wedge \xrightarrow{\sim} \wedge \circ \sigma_{\text{Sets}_*, \text{Sets}_*}^{\text{Cats}_2}, \quad \begin{array}{ccc} \text{Sets}_* \times \text{Sets}_* & \xrightarrow{\wedge} & \text{Sets}_* \\ \sigma_{\text{Sets}_*, \text{Sets}_*}^{\text{Cats}_2} \searrow & \sigma_{\text{Sets}_*}^{\text{Sets}_*} \Downarrow & \nearrow \wedge \\ & \text{Sets}_* \times \text{Sets}_* & \end{array}$$

whose component

$$\sigma_{X,Y}^{\text{Sets}_*} : X \wedge Y \xrightarrow{\cong} Y \wedge X$$

at  $X, Y \in \text{Obj}(\text{Sets}_*)$  is defined by

$$\sigma_{X,Y}^{\text{Sets}_*}(x \wedge y) \stackrel{\text{def}}{=} y \wedge x$$

for each  $x \wedge y \in X \wedge Y$ .

*Proof. Well-Definedness:* Let  $[(x, y)] = [(x', y')]$  be an element in  $X \wedge Y$ . Then either:

1. We have  $x = x'$  and  $y = y'$ .
2. Both of the following conditions are satisfied:
  - (a) We have  $x = x_0$  or  $y = y_0$ .
  - (b) We have  $x' = x_0$  or  $y' = y_0$ .

In the first case,  $\sigma_X^{\text{Sets}_*}$  clearly sends both elements to the same element in  $X$ . Meanwhile, in the latter case both elements are equal to the basepoint  $x_0 \wedge y_0$  of  $X \wedge Y$ , which gets sent to the basepoint  $y_0 \wedge x_0$  of  $Y \wedge X$ .

*Being a Morphism of Pointed Sets:* As just mentioned, we have

$$\sigma_X^{\text{Sets}_*}(x_0 \wedge y_0) \stackrel{\text{def}}{=} y_0 \wedge x_0,$$

and thus  $\sigma_X^{\text{Sets}_*}$  is a morphism of pointed sets.

*Invertibility:* Clearly, the inverse of  $\sigma_{X,Y}^{\text{Sets}_*}$  is given by the morphism

$$\sigma_{X,Y}^{\text{Sets}_*, -1} : Y \wedge X \xrightarrow{\cong} X \wedge Y$$

defined by

$$\sigma_{X,Y}^{\text{Sets}_*, -1}(y \wedge x) \stackrel{\text{def}}{=} x \wedge y$$

for each  $y \wedge x \in Y \wedge X$ .

*Naturality:* We need to show that, given morphisms of pointed sets

$$\begin{aligned} f &: (X, x_0) \rightarrow (A, a_0), \\ g &: (Y, y_0) \rightarrow (B, b_0) \end{aligned}$$

the diagram

$$\begin{array}{ccc}
 X \wedge Y & \xrightarrow{f \wedge g} & A \wedge B \\
 \sigma_{X,Y}^{\text{Sets}_*} \downarrow & & \downarrow \sigma_{A,B}^{\text{Sets}_*} \\
 Y \wedge X & \xrightarrow{g \wedge f} & B \wedge A
 \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc}
 x \wedge y & \longmapsto & f(x) \wedge g(y) \\
 \downarrow & & \downarrow \\
 y \wedge x & \longmapsto & g(y) \wedge f(x)
 \end{array}$$

and hence indeed commutes, showing  $\sigma^{\text{Sets}_*}$  to be a natural transformation.

*Being a Natural Isomorphism:* Since  $\sigma^{\text{Sets}_*}$  is natural and  $\sigma^{\text{Sets}_*, -1}$  is a component-wise inverse to  $\sigma^{\text{Sets}_*}$ , it follows from **Categories, Item 2 of Proposition 8.6.1.2** that  $\sigma^{\text{Sets}_*, -1}$  is also natural. Thus  $\sigma^{\text{Sets}_*}$  is a natural isomorphism.  $\square$

## 00GT 5.8 The Diagonal

00GU **Definition 5.8.1.1.** The **diagonal of the smash product of pointed sets** is the natural transformation

$$\Delta^\wedge : \text{id}_{\text{Sets}_*} \Rightarrow \wedge \circ \Delta_{\text{Sets}_*}^{\text{Cats}_2},$$

whose component

$$\Delta_X^\wedge : (X, x_0) \rightarrow (X \wedge X, x_0 \wedge x_0)$$

at  $(X, x_0) \in \text{Obj}(\text{Sets}_*)$  is given by the composition

$$\begin{aligned}
 (X, x_0) &\xrightarrow{\Delta_X^\wedge} (X \times X, (x_0, x_0)) \\
 &\longrightarrow ((X \times X)/\sim, [(x_0, x_0)]) \\
 &\stackrel{\text{def}}{=} (X \wedge X, x_0 \wedge x_0)
 \end{aligned}$$

in  $\mathbf{Sets}_*$ , and thus by

$$\Delta_X^\wedge(x) \stackrel{\text{def}}{=} x \wedge x$$

for each  $x \in X$ .

*Proof. Being a Morphism of Pointed Sets:* We have

$$\Delta_X^\wedge(x_0) \stackrel{\text{def}}{=} x_0 \wedge x_0,$$

and thus  $\Delta_X^\wedge$  is a morphism of pointed sets.

*Naturality:* We need to show that, given a morphism of pointed sets

$$f: (X, x_0) \rightarrow (Y, y_0),$$

the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \Delta_X^\wedge \downarrow & & \downarrow \Delta_Y^\wedge \\ X \wedge X & \xrightarrow{f \wedge f} & Y \wedge Y \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc} x & \longmapsto & f(x) \\ \downarrow & & \downarrow \\ x \wedge x & \longmapsto & f(x) \wedge f(x) \end{array}$$

and hence indeed commutes, showing  $\Delta^\wedge$  to be natural.  $\square$

**00GV Proposition 5.8.1.2.** Let  $(X, x_0) \in \mathbf{Obj}(\mathbf{Sets}_*)$ .

**00GW** 1. *Monoidality.* The diagonal

$$\Delta^\wedge: \mathbf{id}_{\mathbf{Sets}_*} \Longrightarrow \wedge \circ \Delta_{\mathbf{Sets}_*}^{\mathbf{Cats}_2},$$

of the smash product of pointed sets is a monoidal natural transformation:

**00GX** (a) *Compatibility With Strong Monoidality Constraints.* For each  $(X, x_0), (Y, y_0) \in$

$\text{Obj}(\text{Sets}_*)$ , the diagram

$$\begin{array}{ccc} X \wedge Y & \xrightarrow{\Delta_X^\wedge \wedge \Delta_Y^\wedge} & (X \wedge X) \wedge (Y \wedge Y) \\ & \searrow \Delta_{X \wedge Y}^\wedge & \downarrow \wr \\ & & (X \wedge Y) \wedge (X \wedge Y) \end{array}$$

commutes.

00GY

(b) *Compatibility With Strong Unitality Constraints.* The diagrams

$$\begin{array}{ccc} S^0 & \xrightarrow{\Delta_{S^0}^\wedge} & S^0 \wedge S^0 \\ \parallel & \downarrow \lambda_{S^0}^{\text{Sets}_*} & \\ S^0 & & S^0 \end{array} \quad \begin{array}{ccc} S^0 & \xrightarrow{\Delta_{S^0}^\wedge} & S^0 \wedge S^0 \\ \parallel & \downarrow \rho_{S^0}^{\text{Sets}_*} & \\ S^0 & & S^0 \end{array}$$

commute, i.e. we have

$$\begin{aligned} \Delta_{S^0}^\wedge &= \lambda_{S^0}^{\text{Sets}_*, -1} \\ &= \rho_{S^0}^{\text{Sets}_*, -1}, \end{aligned}$$

where we recall that the equalities

$$\begin{aligned} \lambda_{S^0}^{\text{Sets}_*} &= \rho_{S^0}^{\text{Sets}_*}, \\ \lambda_{S^0}^{\text{Sets}_*, -1} &= \rho_{S^0}^{\text{Sets}_*, -1} \end{aligned}$$

are always true in any monoidal category by ??, ?? of ??.

00GZ

2. *The Diagonal of the Unit.* The component

$$\Delta_{S^0}^\wedge : S^0 \xrightarrow{\cong} S^0 \wedge S^0$$

of  $\Delta^\wedge$  at  $S^0$  is an isomorphism.

*Proof.* **Item 1, Monoidality:** We claim that  $\Delta^\wedge$  is indeed monoidal:

1. **Item 1a: Compatibility With Strong Monoidality Constraints:** We need to

show that the diagram

$$\begin{array}{ccc}
 X \wedge Y & \xrightarrow{\Delta_X^\wedge \wedge \Delta_Y^\wedge} & (X \wedge X) \wedge (Y \wedge Y) \\
 & \searrow \Delta_{X \wedge Y}^\wedge & \downarrow \wr \\
 & & (X \wedge Y) \wedge (X \wedge Y)
 \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc}
 x \wedge y & \xrightarrow{\quad} & (x \wedge x) \wedge (y \wedge y) \\
 & \searrow & \downarrow \\
 & & (x \wedge y) \wedge (x \wedge y)
 \end{array}$$

and hence indeed commutes.

2. *Item 1b: Compatibility With Strong Unitality Constraints:* As shown in the proof of **Definition 5.5.1.1**, the inverse of the left unitor of  $\mathbf{Sets}_*$  with respect to the smash product of pointed sets at  $(X, x_0) \in \mathbf{Obj}(\mathbf{Sets}_*)$  is given by

$$\lambda_X^{\mathbf{Sets}_*, -1}(x) \stackrel{\text{def}}{=} 1 \wedge x$$

for each  $x \in X$ , so when  $X = S^0$ , we have

$$\begin{aligned}
 \lambda_{S^0}^{\mathbf{Sets}_*, -1}(0) &\stackrel{\text{def}}{=} 1 \wedge 0, \\
 \lambda_{S^0}^{\mathbf{Sets}_*, -1}(1) &\stackrel{\text{def}}{=} 1 \wedge 1.
 \end{aligned}$$

But since  $1 \wedge 0 = 0 \wedge 0$  and

$$\begin{aligned}
 \Delta_{S^0}^\wedge(0) &\stackrel{\text{def}}{=} 0 \wedge 0, \\
 \Delta_{S^0}^\wedge(1) &\stackrel{\text{def}}{=} 1 \wedge 1,
 \end{aligned}$$

it follows that we indeed have  $\Delta_{S^0}^\wedge = \lambda_{S^0}^{\mathbf{Sets}_*, -1}$ .

This finishes the proof.

*Item 2, The Diagonal of the Unit:* This follows from **Item 1** and the invertibility of the left/right unitor of  $\mathbf{Sets}_*$  with respect to  $\wedge$ , proved in the proof of **Definition 5.5.1.1** for the left unitor or the proof of **Definition 5.6.1.1** for the right unitor.  $\square$

**00H0 5.9 The Monoidal Structure on Pointed Sets Associated to  $\wedge$** **00H1 Proposition 5.9.1.1.** The category  $\mathbf{Sets}_*$  admits a closed monoidal category with diagonals structure consisting of

- *The Underlying Category.* The category  $\mathbf{Sets}_*$  of pointed sets;
- *The Monoidal Product.* The smash product functor

$$\wedge : \mathbf{Sets}_* \times \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*$$

of **Item 1** of **Proposition 5.1.1.9**;

- *The Internal Hom.* The internal Hom functor

$$\mathbf{Sets}_* : \mathbf{Sets}_*^{\text{op}} \times \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*$$

of **Item 1** of **Proposition 5.2.1.2**;

- *The Monoidal Unit.* The functor

$$\mathbb{1}^{\mathbf{Sets}_*} : \text{pt} \rightarrow \mathbf{Sets}_*$$

of **Definition 5.3.1.1**;

- *The Associators.* The natural isomorphism

$$\alpha^{\mathbf{Sets}_*} : \wedge \circ (\wedge \times \text{id}_{\mathbf{Sets}_*}) \xrightarrow{\sim} \wedge \circ (\text{id}_{\mathbf{Sets}_*} \times \wedge) \circ \alpha_{\mathbf{Sets}_*, \mathbf{Sets}_*, \mathbf{Sets}_*}^{\mathbf{Cats}}$$

of **Definition 5.4.1.1**;

- *The Left Unitors.* The natural isomorphism

$$\lambda^{\mathbf{Sets}_*} : \wedge \circ (\mathbb{1}^{\mathbf{Sets}_*} \times \text{id}_{\mathbf{Sets}_*}) \xrightarrow{\sim} \lambda_{\mathbf{Sets}_*}^{\mathbf{Cats}_2}$$

of **Definition 5.5.1.1**;

- *The Right Unitors.* The natural isomorphism

$$\rho^{\mathbf{Sets}_*} : \wedge \circ (\text{id} \times \mathbb{1}^{\mathbf{Sets}_*}) \xrightarrow{\sim} \rho_{\mathbf{Sets}_*}^{\mathbf{Cats}_2}$$

of **Definition 5.6.1.1**;

- *The Symmetry.* The natural isomorphism

$$\sigma^{\text{Sets}_*} : \wedge \xrightarrow{\sim} \wedge \circ \sigma_{\text{Sets}_*, \text{Sets}_*}^{\text{Cats}_2}$$

of [Definition 5.7.1.1](#);

- *The Diagonals.* The monoidal natural transformation

$$\Delta^\wedge : \text{id}_{\text{Sets}_*} \Longrightarrow \wedge \circ \Delta_{\text{Sets}_*}^{\text{Cats}_2}$$

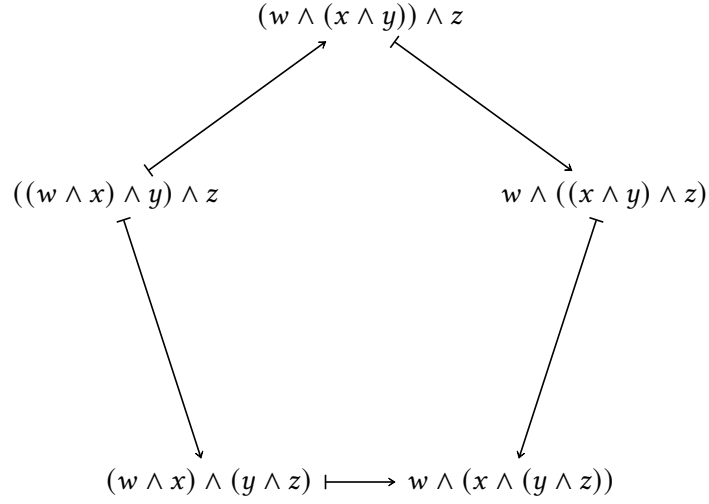
of [Definition 5.8.1.1](#).

*Proof. The Pentagon Identity:* Let  $(W, w_0)$ ,  $(X, x_0)$ ,  $(Y, y_0)$  and  $(Z, z_0)$  be pointed sets. We have to show that the diagram

$$\begin{array}{ccccc}
 & & (W \wedge (X \wedge Y)) \wedge Z & & \\
 & \nearrow \alpha_{W, X, Y}^{\text{Sets}_*} \wedge \text{id}_Z & & \searrow \alpha_{W, X \wedge Y, Z}^{\text{Sets}_*} & \\
 ((W \wedge X) \wedge Y) \wedge Z & & & & W \wedge ((X \wedge Y) \wedge Z) \\
 \searrow \alpha_{W \wedge X, Y, Z}^{\text{Sets}_*} & & & & \searrow \text{id}_W \wedge \alpha_{X, Y, Z}^{\text{Sets}_*} \\
 (W \wedge X) \wedge (Y \wedge Z) & \xrightarrow[\alpha_{W, X, Y \wedge Z}^{\text{Sets}_*}]{} & W \wedge (X \wedge (Y \wedge Z)) & & 
 \end{array}$$



commutes. Indeed, this diagram acts on elements as

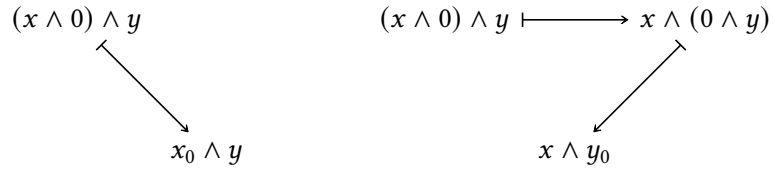


and thus we see that the pentagon identity is satisfied.

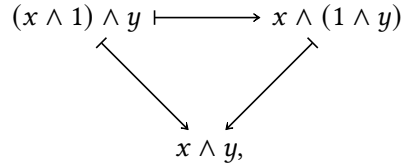
*The Triangle Identity:* Let  $(X, x_0)$  and  $(Y, y_0)$  be pointed sets. We have to show that the diagram

$$\begin{array}{ccc}
 (X \wedge S^0) \wedge Y & \xrightarrow{\alpha_{X, S^0, Y}^{\text{Sets}_*}} & X \wedge (S^0 \wedge Y) \\
 \searrow \rho_X^{\text{Sets}_*} \wedge \text{id}_Y & & \swarrow \text{id}_X \wedge \lambda_Y^{\text{Sets}_*} \\
 & X \wedge Y &
 \end{array}$$

commutes. Indeed, this diagram acts on elements as



and



and thus we see that the triangle identity is satisfied.

*The Left Hexagon Identity:* Let  $(X, x_0)$ ,  $(Y, y_0)$ , and  $(Z, z_0)$  be pointed sets. We have to show that the diagram

$$\begin{array}{ccc}
 & (X \wedge Y) \wedge Z & \\
 \alpha_{X,Y,Z}^{\text{Sets}*} \swarrow & & \searrow \beta_{X,Y}^{\text{Sets}*} \wedge \text{id}_Z \\
 X \wedge (Y \wedge Z) & & (Y \wedge X) \wedge Z \\
 \downarrow \beta_{X,Y \wedge Z}^{\text{Sets}*} & & \downarrow \alpha_{Y,X,Z}^{\text{Sets}*} \\
 (Y \wedge Z) \wedge X & & Y \wedge (X \wedge Z) \\
 \searrow \alpha_{Y,Z,X}^{\text{Sets}*} & & \swarrow \text{id}_Y \wedge \beta_{X,Z}^{\text{Sets}*} \\
 & Y \wedge (Z \wedge X) &
 \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc}
 & (x \wedge y) \wedge z & \\
 \swarrow & & \searrow \\
 x \wedge (y \wedge z) & & (y \wedge x) \wedge z \\
 \downarrow & & \downarrow \\
 (y \wedge z) \wedge x & & y \wedge (x \wedge z) \\
 \swarrow & & \searrow \\
 & y \wedge (z \wedge x) &
 \end{array}$$

and thus we see that the left hexagon identity is satisfied.

*The Right Hexagon Identity:* Let  $(X, x_0)$ ,  $(Y, y_0)$ , and  $(Z, z_0)$  be pointed sets. We

have to show that the diagram

$$\begin{array}{ccc}
 & X \wedge (Y \wedge Z) & \\
 \swarrow (\alpha_{X,Y,Z}^{\text{Sets}_*})^{-1} & & \searrow \text{id}_X \wedge \beta_{Y,Z}^{\text{Sets}_*} \\
 (X \wedge Y) \wedge Z & & X \wedge (Z \wedge Y) \\
 \downarrow \beta_{X \wedge Y, Z}^{\text{Sets}_*} & & \downarrow (\alpha_{X,Z,Y}^{\text{Sets}_*})^{-1} \\
 Z \wedge (X \wedge Y) & & (X \wedge Z) \wedge Y \\
 \swarrow (\alpha_{Z,X,Y}^{\text{Sets}_*})^{-1} & & \searrow \beta_{X,Z}^{\text{Sets}_*} \wedge \text{id}_Y \\
 & (Z \wedge X) \wedge Y &
 \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc}
 & x \wedge (y \wedge z) & \\
 \swarrow & & \searrow \\
 (x \wedge y) \wedge z & & x \wedge (z \wedge y) \\
 \downarrow & & \downarrow \\
 z \wedge (x \wedge y) & & (x \wedge z) \wedge y \\
 \swarrow & & \searrow \\
 & (z \wedge x) \wedge y &
 \end{array}$$

and thus we see that the right hexagon identity is satisfied.

*Monoidal Closedness:* This follows from **Item 2** of **Proposition 5.1.1.9**.

*Existence of Monoidal Diagonals:* This follows from **Items 1** and **2** of **Proposition 5.8.1.2**.  $\square$

## 00H2 5.10 Universal Properties of the Smash Product of Pointed Sets I

**00H3 Theorem 5.10.1.1.** The symmetric monoidal structure on the category  $\text{Sets}_*$  is uniquely determined by the following requirements:

1. *Two-Sided Preservation of Colimits.* The smash product

$$\wedge : \mathbf{Sets}_* \times \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*$$

of  $\mathbf{Sets}_*$  preserves colimits separately in each variable.

2. *The Unit Object Is  $S^0$ .* We have  $\mathbb{1}_{\mathbf{Sets}_*} = S^0$ .

*Proof.* Omitted. □

#### 00H4 5.11 Universal Properties of the Smash Product of Pointed Sets II

00H5 **Theorem 5.11.1.1.** The symmetric monoidal structure on the category  $\mathbf{Sets}_*$  is the unique symmetric monoidal structure on  $\mathbf{Sets}_*$  such that the free pointed set functor

$$(-)^+ : \mathbf{Sets} \rightarrow \mathbf{Sets}_*$$

admits a symmetric monoidal structure.

*Proof.* See [GGN15, Theorem 5.1]. □

#### 00H6 5.12 Monoids With Respect to the Smash Product of Pointed Sets

00H7 **Proposition 5.12.1.1.** The category of monoids on  $(\mathbf{Sets}_*, \wedge, S^0)$  is isomorphic to the category of monoids with zero and morphisms between them.

*Proof.* See ??, in particular ??, ??, ??, and ??. □

#### 00H8 5.13 Comonoids With Respect to the Smash Product of Pointed Sets

00H9 **Proposition 5.13.1.1.** The symmetric monoidal functor

$$((-)^+, (-)^{+, \times}, (-)_{\mathbb{1}}^{+, \times}) : (\mathbf{Sets}, \times, \text{pt}) \rightarrow (\mathbf{Sets}_*, \wedge, S^0),$$

of **Pointed Sets**, Item 4 of **Proposition 4.1.1.2** lifts to an equivalence of categories

$$\begin{aligned} \text{CoMon}(\mathbf{Sets}_*, \wedge, S^0) &\stackrel{\text{eq.}}{\cong} \text{CoMon}(\mathbf{Sets}, \times, \text{pt}) \\ &\cong \mathbf{Sets}. \end{aligned}$$

*Proof.* See [PS19, Lemma 2.4]. □

## 00HA 6 Miscellany

### 00HB 6.1 The Smash Product of a Family of Pointed Sets

Let  $\{(X_i, x_0^i)\}_{i \in I}$  be a family of pointed sets.

00HC **Definition 6.1.1.1.** The **smash product of the family**  $\{(X_i, x_0^i)\}_{i \in I}$  is the pointed set  $\bigwedge_{i \in I} X_i$  consisting of:

- *The Underlying Set.* The set  $\bigwedge_{i \in I} X_i$  defined by

$$\bigwedge_{i \in I} X_i \stackrel{\text{def}}{=} \left( \prod_{i \in I} X_i \right) / \sim,$$

where  $\sim$  is the equivalence relation on  $\prod_{i \in I} X_i$  obtained by declaring

$$(x_i)_{i \in I} \sim (y_i)_{i \in I}$$

if there exist  $i_0 \in I$  such that  $x_{i_0} = x_0$  and  $y_{i_0} = y_0$ , for each  $(x_i)_{i \in I}, (y_i)_{i \in I} \in \prod_{i \in I} X_i$ .

- *The Basepoint.* The element  $[(x_0)_{i \in I}]$  of  $\bigwedge_{i \in I} X_i$ .

# Appendices

## A Other Chapters

### Sets

1. Sets
2. Constructions With Sets
3. Pointed Sets
4. Tensor Products of Pointed Sets

### Relations

5. Relations

### 6. Constructions With Relations

7. Equivalence Relations and Apartness Relations

### Category Theory

8. Categories

### Bicategories

9. Types of Morphisms in Bicat-  
egories

## References

- [GGN15] David Gepner, Moritz Groth, and Thomas Nikolaus. “Universality of Multiplicative Infinite Loop Space Machines”. In: *Algebr. Geom. Topol.* 15.6 (2015), pp. 3107–3153. ISSN: 1472-2747. DOI: [10.2140/agt.2015.15.3107](https://doi.org/10.2140/agt.2015.15.3107). URL: <https://doi.org/10.2140/agt.2015.15.3107> (cit. on p. 100).
- [PS19] Maximilien Péroux and Brooke Shipley. “Coalgebras in Symmetric Monoidal Categories of Spectra”. In: *Homology Homotopy Appl.* 21.1 (2019), pp. 1–18. ISSN: 1532-0073. DOI: [10.4310/HHA.2019.v21.n1.a1](https://doi.org/10.4310/HHA.2019.v21.n1.a1). URL: <https://doi.org/10.4310/HHA.2019.v21.n1.a1> (cit. on p. 100).