Equivalence Relations and Apartness Relations

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This chapter contains some material about reflexive, symmetric, transitive, equivalence, and apartness relations.

Contents

1 Reflexive Relations

1.1 Foundations

Let A be a set.

Definition 1.1.1 ▶ Reflexive Relations

A **reflexive relation** is equivalently:[1]

- An $\mathbb{E}_0$-monoid in $(\mathrm{N}_\bullet(\mathbf{Rel}(A, A)), \chi_A)$.
- A pointed object in $(\mathbf{Rel}(A, A), \chi_A)$.

[1] Note that since $\mathbf{Rel}(A, A)$ is posetal, reflexivity is a property of a relation, rather than extra structure.

Remark 1.1.2 ▶ Unwinding Definition 1.1.1

In detail, a relation R on A is **reflexive** if we have an inclusion

$$\eta_R \colon \chi_A \subset R$$

of relations in $\mathbf{Rel}(A, A)$, i.e. if, for each $a \in A$, we have $a \sim_R a$.

Definition 1.1.3 ▶ The Po/Set of Reflexive Relations on a Set

Let A be a set.

1. The **set of reflexive relations on** A is the subset $\mathrm{Rel}^{\mathrm{refl}}(A, A)$ of $\mathrm{Rel}(A, A)$ spanned by the reflexive relations.
2. The **poset of relations on** A is is the subposet $\mathbf{Rel}^{\mathrm{refl}}(A, A)$ of $\mathbf{Rel}(A, A)$ spanned by the reflexive relations.

Proposition 1.1.4 ▶ Properties of Reflexive Relations

Let R and S be relations on A.

1. *Interaction With Inverses.* If R is reflexive, then so is $R^\dagger$.
2. *Interaction With Composition.* If R and S are reflexive, then so is $S \diamond R$.

Proof 1.1.5 ▶ Proof of Proposition 1.1.4

Item 1: Interaction With Inverses

Clear.

Item 2: Interaction With Composition

Clear. □

1.2 The Reflexive Closure of a Relation

Let R be a relation on A.

Definition 1.2.1 ▶ The Reflexive Closure of a Relation

The **reflexive closure** of $\sim_R$ is the relation $\sim_R^{\mathrm{refl}}$[1] satisfying the following universal property:[2]

(★) Given another reflexive relation $\sim_S$ on A such that $R \subset S$, there exists an inclusion $\sim_R^{\mathrm{refl}} \subset \sim_S$.

[1] *Further Notation:* Also written R^{refl}.

[2] *Slogan:* The reflexive closure of R is the smallest reflexive relation containing R.

Construction 1.2.2 ▶ The Reflexive Closure of a Relation

Concretely, $\sim_R^{\mathrm{refl}}$ is the free pointed object on R in $(\mathbf{Rel}(A, A), \chi_A)$[1], being given by

$$\begin{aligned} R^{\mathrm{refl}} &\overset{\mathrm{def}}{=} R \coprod^{\mathbf{Rel}(A,A)} \Delta_A \\ &= R \cup \Delta_A \\ &= \{(a, b) \in A \times A \mid \text{we have } a \sim_R b \text{ or } a = b\}. \end{aligned}$$

[1] Or, equivalently, the free $\mathbb{E}_0$-monoid on R in $(\mathsf{N}_\bullet(\mathbf{Rel}(A, A)), \chi_A)$.

Proof 1.2.3 ▶ Proof of Construction 1.2.2

Clear. □

Proposition 1.2.4 ► Properties of the Reflexive Closure of a Relation

Let R be a relation on A.

1. *Adjointness.* We have an adjunction
$$\left((-)^{\mathrm{refl}} \dashv 忘\right)\colon \quad \begin{tikzcd} \mathbf{Rel}(A,A) \arrow[r, bend left, "(-)^{\mathrm{refl}}"] \arrow[r, phantom, "\perp"] & \mathbf{Rel}^{\mathrm{refl}}(A,A), \arrow[l, bend left, "忘"] \end{tikzcd}$$
witnessed by a bijection of sets
$$\mathbf{Rel}^{\mathrm{refl}}(R^{\mathrm{refl}}, S) \cong \mathbf{Rel}(R, S),$$
natural in $R \in \mathrm{Obj}(\mathbf{Rel}^{\mathrm{refl}}(A,A))$ and $S \in \mathrm{Obj}(\mathbf{Rel}(A,A))$.
2. *The Reflexive Closure of a Reflexive Relation.* If R is reflexive, then $R^{\mathrm{refl}} = R$.
3. *Idempotency.* We have
$$(R^{\mathrm{refl}})^{\mathrm{refl}} = R^{\mathrm{refl}}.$$
4. *Interaction With Inverses.* We have
$$\left(R^{\dagger}\right)^{\mathrm{refl}} = \left(R^{\mathrm{refl}}\right)^{\dagger}, \qquad \begin{tikzcd} \mathrm{Rel}(A,A) \arrow[r, "(-)^{\mathrm{refl}}"] \arrow[d, "(-)^{\dagger}"'] & \mathrm{Rel}(A,A) \arrow[d, "(-)^{\dagger}"] \\ \mathrm{Rel}(A,A) \arrow[r, "(-)^{\mathrm{refl}}"'] & \mathrm{Rel}(A,A). \end{tikzcd}$$
5. *Interaction With Composition.* We have
$$(S \diamond R)^{\mathrm{refl}} = S^{\mathrm{refl}} \diamond R^{\mathrm{refl}}, \qquad \begin{tikzcd} \mathrm{Rel}(A,A) \times \mathrm{Rel}(A,A) \arrow[r, "\diamond"] \arrow[d, "(-)^{\mathrm{refl}} \times (-)^{\mathrm{refl}}"'] & \mathrm{Rel}(A,A) \arrow[d, "(-)^{\mathrm{refl}}"] \\ \mathrm{Rel}(A,A) \times \mathrm{Rel}(A,A) \arrow[r, "\diamond"'] & \mathrm{Rel}(A,A). \end{tikzcd}$$

Proof 1.2.5 ▶ Proof of Proposition 1.2.4

Item 1: Adjointness

This is a rephrasing of the universal property of the reflexive closure of a relation, stated in Definition 1.2.1.

Item 2: The Reflexive Closure of a Reflexive Relation

Clear.

Item 3: Idempotency

This follows from Item 2.

Item 4: Interaction With Inverses

Clear.

Item 5: Interaction With Composition

This follows from Item 2 of Proposition 1.1.4.

2 Symmetric Relations

2.1 Foundations

Let A be a set.

Definition 2.1.1 ▶ Symmetric Relations

A relation R on A is **symmetric** if we have $R^{\dagger} = R$.

Remark 2.1.2 ▶ Unwinding Definition 2.1.1

In detail, a relation R is symmetric if it satisfies the following condition:

($\star$) For each $a, b \in A$, if $a \sim_R b$, then $b \sim_R a$.

Definition 2.1.3 ▶ The Po/Set of Symmetric Relations on a Set

Let A be a set.

1. The **set of symmetric relations on** A is the subset $\mathrm{Rel}^{\mathrm{symm}}(A, A)$ of $\mathrm{Rel}(A, A)$ spanned by the symmetric relations.

2. The **poset of relations on** A is is the subposet $\mathbf{Rel}^{\mathrm{symm}}(A, A)$ of $\mathbf{Rel}(A, A)$ spanned by the symmetric relations.

Proposition 2.1.4 ► Properties of Symmetric Relations

Let R and S be relations on A.

1. *Interaction With Inverses.* If R is symmetric, then so is $R^{\dagger}$.
2. *Interaction With Composition.* If R and S are symmetric, then so is $S \diamond R$.

Proof 2.1.5 ► Proof of Proposition 2.1.4

Item 1: Interaction With Inverses

Clear.

Item 2: Interaction With Composition

Clear.

2.2 The Symmetric Closure of a Relation

Let R be a relation on A.

Definition 2.2.1 ► The Symmetric Closure of a Relation

The **symmetric closure** of $\sim_R$ is the relation $\sim_R^{\mathrm{symm}}$[1] satisfying the following universal property:[2]

(★) Given another symmetric relation $\sim_S$ on A such that $R \subset S$, there exists an inclusion $\sim_R^{\mathrm{symm}} \subset \sim_S$.

[1] *Further Notation:* Also written R^{symm}.
[2] *Slogan:* The symmetric closure of R is the smallest symmetric relation containing R.

Construction 2.2.2 ► The Symmetric Closure of a Relation

Concretely, $\sim_R^{\mathrm{symm}}$ is the symmetric relation on A defined by

$$\begin{aligned} R^{\mathrm{symm}} &\overset{\mathrm{def}}{=} R \cup R^{\dagger} \\ &= \{(a, b) \in A \times A \mid \text{we have } a \sim_R b \text{ or } b \sim_R a\}. \end{aligned}$$

PROOF 2.2.3 ▶ PROOF OF CONSTRUCTION 2.2.2

Clear. □

PROPOSITION 2.2.4 ▶ PROPERTIES OF THE SYMMETRIC CLOSURE OF A RELATION

Let R be a relation on A.

1. *Adjointness.* We have an adjunction

$$((-)^{\mathsf{symm}} \dashv 忘) \colon \quad \begin{tikzcd} \mathbf{Rel}(A,A) \arrow[r, bend left, "(-)^{\mathsf{symm}}"] \arrow[r, phantom, "\perp"] & \mathbf{Rel}^{\mathsf{symm}}(A,A), \arrow[l, bend left, "忘"] \end{tikzcd}$$

   witnessed by a bijection of sets

$$\mathbf{Rel}^{\mathsf{symm}}(R^{\mathsf{symm}}, S) \cong \mathbf{Rel}(R, S),$$

   natural in $R \in \mathsf{Obj}(\mathbf{Rel}^{\mathsf{symm}}(A,A))$ and $S \in \mathsf{Obj}(\mathbf{Rel}(A,A))$.

2. *The Symmetric Closure of a Symmetric Relation.* If R is symmetric, then $R^{\mathsf{symm}} = R$.

3. *Idempotency.* We have

$$(R^{\mathsf{symm}})^{\mathsf{symm}} = R^{\mathsf{symm}}.$$

4. *Interaction With Inverses.* We have

$$\left(R^{\dagger}\right)^{\mathsf{symm}} = \left(R^{\mathsf{symm}}\right)^{\dagger}, \qquad \begin{tikzcd} \mathsf{Rel}(A,A) \arrow[r, "(-)^{\mathsf{symm}}"] \arrow[d, "(-)^{\dagger}"'] & \mathsf{Rel}(A,A) \arrow[d, "(-)^{\dagger}"] \\ \mathsf{Rel}(A,A) \arrow[r, "(-)^{\mathsf{symm}}"'] & \mathsf{Rel}(A,A). \end{tikzcd}$$

5. *Interaction With Composition.* We have

$$(S \diamond R)^{\mathsf{symm}} = S^{\mathsf{symm}} \diamond R^{\mathsf{symm}}, \qquad \begin{tikzcd} \mathsf{Rel}(A,A) \times \mathsf{Rel}(A,A) \arrow[r, "\diamond"] \arrow[d, "(-)^{\mathsf{symm}} \times (-)^{\mathsf{symm}}"'] & \mathsf{Rel}(A,A) \arrow[d, "(-)^{\mathsf{symm}}"] \\ \mathsf{Rel}(A,A) \times \mathsf{Rel}(A,A) \arrow[r, "\diamond"'] & \mathsf{Rel}(A,A). \end{tikzcd}$$

Proof 2.2.5 ▶ Proof of Proposition 2.2.4

Item 1: Adjointness

This is a rephrasing of the universal property of the symmetric closure of a relation, stated in Definition 2.2.1.

Item 2: The Symmetric Closure of a Symmetric Relation

Clear.

Item 3: Idempotency

This follows from Item 2.

Item 4: Interaction With Inverses

Clear.

Item 5: Interaction With Composition

This follows from Item 2 of Proposition 2.1.4.

3 Transitive Relations

3.1 Foundations

Let A be a set.

Definition 3.1.1 ▶ Transitive Relations

A **transitive relation** is equivalently:[1]

- A non-unital $\mathbb{E}_1$-monoid in $(\mathrm{N}_\bullet(\mathbf{Rel}(A, A)), \diamond)$.
- A non-unital monoid in $(\mathbf{Rel}(A, A), \diamond)$.

[1] Note that since $\mathbf{Rel}(A, A)$ is posetal, transitivity is a property of a relation, rather than extra structure.

Remark 3.1.2 ▶ Unwinding Definition 3.1.1

In detail, a relation R on A is **transitive** if we have an inclusion

$$\mu_R : R \diamond R \subset R$$

of relations in $\mathbf{Rel}(A, A)$, i.e. if, for each $a, c \in A$, the following condition is satisfied:

($\star$) If there exists some $b \in A$ such that $a \sim_R b$ and $b \sim_R c$, then $a \sim_R c$.

Definition 3.1.3 ► The Po/Set of Transitive Relations on a Set

Let A be a set.

1. The **set of transitive relations from** A **to** B is the subset $\mathrm{Rel}^{\mathrm{trans}}(A)$ of $\mathrm{Rel}(A, A)$ spanned by the transitive relations.

2. The **poset of relations from** A **to** B is is the subposet $\mathbf{Rel}^{\mathrm{trans}}(A)$ of $\mathbf{Rel}(A, A)$ spanned by the transitive relations.

Proposition 3.1.4 ► Properties of Transitive Relations

Let R and S be relations on A.

1. *Interaction With Inverses.* If R is transitive, then so is $R^\dagger$.

2. *Interaction With Composition.* If R and S are transitive, then $S \diamond R$ ***may fail to be transitive***.

Proof 3.1.5 ► Proof of Proposition 3.1.4

Item 1: Interaction With Inverses

Clear.

Item 2: Interaction With Composition

See [MSE 2096272].[1]

[1] *Intuition:* Transitivity for R and S fails to imply that of $S \diamond R$ because the composition operation for relations intertwines R and S in an incompatible way:

1. If $a \sim_{S \diamond R} c$ and $c \sim_{S \diamond r} e$, then:
   (a) There is some $b \in A$ such that:
      i. $a \sim_R b$;
      ii. $b \sim_S c$;
   (b) There is some $d \in A$ such that:

i. $c \sim_R d$;
ii. $d \sim_S e$.

3.2 The Transitive Closure of a Relation

Let R be a relation on A.

Definition 3.2.1 ▶ The Transitive Closure of a Relation

The **transitive closure** of $\sim_R$ is the relation $\sim_R^{\mathrm{trans}}$[1] satisfying the following universal property:[2]

(★) Given another transitive relation $\sim_S$ on A such that $R \subset S$, there exists an inclusion $\sim_R^{\mathrm{trans}} \subset \sim_S$.

[1] *Further Notation:* Also written R^{trans}.
[2] *Slogan:* The transitive closure of R is the smallest transitive relation containing R.

Construction 3.2.2 ▶ The Transitive Closure of a Relation

Concretely, $\sim_R^{\mathrm{trans}}$ is the free non-unital monoid on R in $(\mathbf{Rel}(A, A), \diamond)$[1], being given by

$$\begin{aligned} R^{\mathrm{trans}} &\stackrel{\mathrm{def}}{=} \coprod_{n=1}^{\infty} R^{\diamond n} \\ &\stackrel{\mathrm{def}}{=} \bigcup_{n=1}^{\infty} R^{\diamond n} \\ &\stackrel{\mathrm{def}}{=} \left\{ (a, b) \in A \times B \;\middle|\; \begin{array}{l} \text{there exists some } (x_1, \ldots, x_n) \in R^{\times n} \\ \text{such that } a \sim_R x_1 \sim_R \cdots \sim_R x_n \sim_R b \end{array} \right\}. \end{aligned}$$

[1] Or, equivalently, the free non-unital $\mathbb{E}_1$-monoid on R in $(\mathsf{N}_\bullet(\mathbf{Rel}(A, A)), \diamond)$.

Proof 3.2.3 ▶ Proof of Construction 3.2.2

Clear. □

Proposition 3.2.4 ▶ Properties of the Transitive Closure of a Relation

Let R be a relation on A.

1. *Adjointness.* We have an adjunction

$$\left((-)^{\mathrm{trans}} \dashv 忘\right) : \quad \begin{tikzcd} \mathbf{Rel}(A,A) \arrow[r, bend left, "(-)^{\mathrm{trans}}"] \arrow[r, phantom, "\perp"] & \mathbf{Rel}^{\mathrm{trans}}(A,A), \arrow[l, bend left, "忘"] \end{tikzcd}$$

   witnessed by a bijection of sets

$$\mathbf{Rel}^{\mathrm{trans}}(R^{\mathrm{trans}}, S) \cong \mathbf{Rel}(R, S),$$

   natural in $R \in \mathrm{Obj}(\mathbf{Rel}^{\mathrm{trans}}(A,A))$ and $S \in \mathrm{Obj}(\mathbf{Rel}(A,B))$.

2. *The Transitive Closure of a Transitive Relation.* If R is transitive, then $R^{\mathrm{trans}} = R$.

3. *Idempotency.* We have

$$\left(R^{\mathrm{trans}}\right)^{\mathrm{trans}} = R^{\mathrm{trans}}.$$

4. *Interaction With Inverses.* We have

$$\left(R^{\dagger}\right)^{\mathrm{trans}} = \left(R^{\mathrm{trans}}\right)^{\dagger}, \qquad \begin{tikzcd} \mathrm{Rel}(A,A) \arrow[r, "(-)^{\mathrm{trans}}"] \arrow[d, "(-)^{\dagger}"'] & \mathrm{Rel}(A,A) \arrow[d, "(-)^{\dagger}"] \\ \mathrm{Rel}(A,A) \arrow[r, "(-)^{\mathrm{trans}}"'] & \mathrm{Rel}(A,A). \end{tikzcd}$$

5. *Interaction With Composition.* We have

$$(S \diamond R)^{\mathrm{trans}} \overset{\text{poss.}}{\neq} S^{\mathrm{trans}} \diamond R^{\mathrm{trans}}, \qquad \begin{tikzcd} \mathrm{Rel}(A,A) \times \mathrm{Rel}(A,A) \arrow[r, "\diamond"] \arrow[d, "(-)^{\mathrm{trans}} \times (-)^{\mathrm{trans}}"'] \arrow[dr, phantom, "\times"] & \mathrm{Rel}(A,A) \arrow[d, "(-)^{\mathrm{trans}}"] \\ \mathrm{Rel}(A,A) \times \mathrm{Rel}(A,A) \arrow[r, "\diamond"'] & \mathrm{Rel}(A,A). \end{tikzcd}$$

Proof 3.2.5 ▶ Proof of Proposition 3.2.4

Item 1: Adjointness

This is a rephrasing of the universal property of the transitive closure of a relation, stated in Definition 3.2.1.

Item 2: The Transitive Closure of a Transitive Relation

Clear.

Item 3: Idempotency

This follows from Item 2.

Item 4: Interaction With Inverses

We have

$$\begin{aligned}(R^{\dagger})^{\mathrm{trans}} &= \bigcup_{n=1}^{\infty}(R^{\dagger})^{\diamond n}\\ &= \bigcup_{n=1}^{\infty}(R^{\diamond n})^{\dagger}\\ &= \left(\bigcup_{n=1}^{\infty}R^{\diamond n}\right)^{\dagger}\\ &= (R^{\mathrm{trans}})^{\dagger},\end{aligned}$$

where we have used, respectively:

1. Construction 3.2.2.
2. Constructions With Relations, Item 4 of Proposition 3.12.3.
3. Constructions With Relations, Item 1 of Proposition 3.6.2.
4. Construction 3.2.2.

Item 5: Interaction With Composition

This follows from Item 2 of Proposition 3.1.4. □

4 Equivalence Relations

4.1 Foundations

Let A be a set.

Definition 4.1.1 ▶ Equivalence Relations

A relation R is an **equivalence relation** if it is reflexive, symmetric, and transitive.[1]

[1] *Further Terminology:* If instead R is just symmetric and transitive, then it is called a **partial equivalence relation**.

Example 4.1.2 ▶ The Kernel of a Function

The **kernel of a function** $f\colon A \longrightarrow B$ is the equivalence relation $\sim_{\mathrm{Ker}(f)}$ on A obtained by declaring $a \sim_{\mathrm{Ker}(f)} b$ iff $f(a) = f(b)$.[1]

[1] The kernel $\mathrm{Ker}(f)\colon A \nrightarrow A$ of f is the underlying functor of the monad induced by the adjunction $\mathrm{Gr}(f) \dashv f^{-1}\colon A \rightleftarrows B$ in **Rel** of Constructions With Relations, Item 2 of Proposition 3.1.2.

Definition 4.1.3 ▶ The Po/Set of Equivalence Relations on a Set

Let A and B be sets.

1. The **set of equivalence relations from A to B** is the subset $\mathrm{Rel}^{\mathrm{eq}}(A, B)$ of $\mathrm{Rel}(A, B)$ spanned by the equivalence relations.
2. The **poset of relations from A to B** is is the subposet $\mathbf{Rel}^{\mathrm{eq}}(A, B)$ of $\mathbf{Rel}(A, B)$ spanned by the equivalence relations.

4.2 The Equivalence Closure of a Relation

Let R be a relation on A.

Definition 4.2.1 ▶ The Equivalence Closure of a Relation

The **equivalence closure**[1] of $\sim_R$ is the relation $\sim_R^{\mathrm{eq}}$[2] satisfying the following universal property:[3]

($\star$) Given another equivalence relation $\sim_S$ on A such that $R \subset S$, there exists an inclusion $\sim_R^{\mathrm{eq}} \subset \sim_S$.

[1] *Further Terminology:* Also called the **equivalence relation associated to** $\sim_R$.
[2] *Further Notation:* Also written R^{eq}.
[3] *Slogan:* The equivalence closure of R is the smallest equivalence relation containing R.

Construction 4.2.2 ▶ The Equivalence Closure of a Relation

Concretely, $\sim_R^{\mathrm{eq}}$ is the equivalence relation on A defined by

$$\begin{aligned} R^{\mathrm{eq}} &\stackrel{\mathrm{def}}{=} ((R^{\mathrm{refl}})^{\mathrm{symm}})^{\mathrm{trans}} \\ &= ((R^{\mathrm{symm}})^{\mathrm{trans}})^{\mathrm{refl}} \\ &= \left\{ (a,b) \in A \times B \;\middle|\; \begin{array}{l} \text{there exists } (x_1, \ldots, x_n) \in R^{\times n} \text{ satisfying at} \\ \text{least one of the following conditions:} \\ \quad 1. \text{ The following conditions are satisfied:} \\ \qquad \text{(a) We have } a \sim_R x_1 \text{ or } x_1 \sim_R a; \\ \qquad \text{(b) We have } x_i \sim_R x_{i+1} \text{ or } x_{i+1} \sim_R x_i \\ \qquad \quad \text{for each } 1 \leq i \leq n-1; \\ \qquad \text{(c) We have } b \sim_R x_n \text{ or } x_n \sim_R b; \\ \quad 2. \text{ We have } a = b. \end{array} \right\}. \end{aligned}$$

Proof 4.2.3 ▶ Proof of Construction 4.2.2

From the universal properties of the reflexive, symmetric, and transitive closures of a relation (Definitions 1.2.1, 2.2.1 and 3.2.1), we see that it suffices to prove that:

1. The symmetric closure of a reflexive relation is still reflexive.
2. The transitive closure of a symmetric relation is still symmetric.

which are both clear. □

Proposition 4.2.4 ▶ Properties of Equivalence Relations

Let R be a relation on A.

1. *Adjointness.* We have an adjunction

$$\left((-)^{\mathrm{eq}} \dashv 忘\right) \colon \quad \mathbf{Rel}(A,B) \underset{忘}{\overset{(-)^{\mathrm{eq}}}{\rightleftarrows}} \mathbf{Rel}^{\mathrm{eq}}(A,B),$$

witnessed by a bijection of sets

$$\mathbf{Rel}^{\mathrm{eq}}(R^{\mathrm{eq}}, S) \cong \mathbf{Rel}(R, S),$$

natural in $R \in \mathrm{Obj}(\mathbf{Rel}^{\mathrm{eq}}(A, B))$ and $S \in \mathrm{Obj}(\mathbf{Rel}(A, B))$.

2. *The Equivalence Closure of an Equivalence Relation.* If R is an equivalence relation, then $R^{\mathrm{eq}} = R$.

3. *Idempotency.* We have

$$(R^{\mathrm{eq}})^{\mathrm{eq}} = R^{\mathrm{eq}}.$$

Proof 4.2.5 ► Proof of Proposition 4.2.4

Item 1: Adjointness

This is a rephrasing of the universal property of the equivalence closure of a relation, stated in Definition 4.2.1.

Item 2: The Equivalence Closure of an Equivalence Relation

Clear.

Item 3: Idempotency

This follows from Item 2. □

5 Quotients by Equivalence Relations

5.1 Equivalence Classes

Let A be a set, let R be a relation on A, and let $a \in A$.

Definition 5.1.1 ► Equivalence Classes

The **equivalence class associated to** a is the set $[a]$ defined by

$$\begin{aligned} [a] &\stackrel{\text{def}}{=} \{x \in X \mid x \sim_R a\} \\ &= \{x \in X \mid a \sim_R x\}. && \text{(since } R \text{ is symmetric)} \end{aligned}$$

5.2 Quotients of Sets by Equivalence Relations

Let A be a set and let R be a relation on A.

The **quotient of X by R** is the set $X/\sim_R$ defined by

$$X/\sim_R \overset{\text{def}}{=} \{[a] \in \mathcal{P}(X) \mid a \in X\}.$$

Remark 5.2.2 ▶ Why Use "Equivalence" Relations for Quotient Sets

The reason we define quotient sets for equivalence relations only is that each of the properties of being an equivalence relation—reflexivity, symmetry, and transitivity—ensures that the equivalences classes $[a]$ of X under R are well-behaved:

- *Reflexivity.* If R is reflexive, then, for each $a \in X$, we have $a \in [a]$.
- *Symmetry.* The equivalence class $[a]$ of an element a of X is defined by

  $$[a] \overset{\text{def}}{=} \{x \in X \mid x \sim_R a\},$$

  but we could equally well define

  $$[a]' \overset{\text{def}}{=} \{x \in X \mid a \sim_R x\}$$

  instead. This is not a problem when R is symmetric, as we then have $[a] = [a]'$.[1]
- *Transitivity.* If R is transitive, then $[a]$ and $[b]$ are disjoint iff $a \nsim_R b$, and equal otherwise.

[1] When categorifying equivalence relations, one finds that $[a]$ and $[a]'$ correspond to presheaves and copresheaves; see **??**, **??**.

Proposition 5.2.3 ▶ Properties of Quotient Sets

Let $f\colon X \to Y$ be a function and let R be a relation on X.

1. *As a Coequaliser.* We have an isomorphism of sets

   $$X/\sim_R^{\text{eq}} \cong \operatorname{CoEq}\left(R \hookrightarrow X \times X \underset{\mathrm{pr}_2}{\overset{\mathrm{pr}_1}{\rightrightarrows}} X\right),$$

   where $\sim_R^{\text{eq}}$ is the equivalence relation generated by $\sim_R$.

2. *As a Pushout.* We have an isomorphism of sets[1]

$$X/{\sim_R^{\mathrm{eq}}} \cong X \coprod_{\mathrm{Eq}(\mathrm{pr}_1,\mathrm{pr}_2)} X, \qquad \begin{tikzcd} X/{\sim_R^{\mathrm{eq}}} \arrow[dr, phantom, very near start, "\ulcorner"] & X \arrow[l] \\ X \arrow[u] & \mathrm{Eq}(\mathrm{pr}_1, \mathrm{pr}_2). \arrow[l] \arrow[u] \end{tikzcd}$$

   where $\sim_R^{\mathrm{eq}}$ is the equivalence relation generated by $\sim_R$.

3. *The First Isomorphism Theorem for Sets.* We have an isomorphism of sets[2,3]

$$X/{\sim_{\mathrm{Ker}(f)}} \cong \mathrm{Im}(f).$$

4. *Descending Functions to Quotient Sets, I.* Let R be an equivalence relation on X. The following conditions are equivalent:

   (a) There exists a map

$$\overline{f}\colon X/{\sim_R} \to Y$$

   making the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "q"'] & Y \\ X/{\sim_R} \arrow[ur, dotted, "\exists", "\overline{f}"'] \end{tikzcd}$$

   commute.

   (b) We have $R \subset \mathrm{Ker}(f)$.

   (c) For each $x, y \in X$, if $x \sim_R y$, then $f(x) = f(y)$.

5. *Descending Functions to Quotient Sets, II.* Let R be an equivalence relation on X. If the conditions of Item 4 hold, then $\overline{f}$ is the *unique* map making the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "q"'] & Y \\ X/{\sim_R} \arrow[ur, dashed, "\exists!", "\overline{f}"'] \end{tikzcd}$$

   commute.

6. *Descending Functions to Quotient Sets, III.* Let R be an equivalence relation on X. We have a bijection

$$\mathrm{Hom}_{\mathsf{Sets}}(X/{\sim_R}, Y) \cong \mathrm{Hom}^R_{\mathsf{Sets}}(X, Y),$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$, given by the assignment $f \mapsto \overline{f}$ of Items 4 and 5, where $\mathrm{Hom}^R_{\mathsf{Sets}}(X, Y)$ is the set defined by

$$\mathrm{Hom}^R_{\mathsf{Sets}}(X, Y) \overset{\mathrm{def}}{=} \left\{ f \in \mathrm{Hom}_{\mathsf{Sets}}(X, Y) \;\middle|\; \begin{array}{l} \text{for each } x, y \in X, \\ \text{if } x \sim_R y, \text{ then} \\ f(x) = f(y) \end{array} \right\}.$$

7. *Descending Functions to Quotient Sets, IV.* Let R be an equivalence relation on X. If the conditions of Item 4 hold, then the following conditions are equivalent:

   (a) The map $\overline{f}$ is an injection.

   (b) We have $R = \mathrm{Ker}(f)$.

   (c) For each $x, y \in X$, we have $x \sim_R y$ iff $f(x) = f(y)$.

8. *Descending Functions to Quotient Sets, V.* Let R be an equivalence relation on X. If the conditions of Item 4 hold, then the following conditions are equivalent:

   (a) The map $f \colon X \longrightarrow Y$ is surjective.

   (b) The map $\overline{f} \colon X/{\sim_R} \longrightarrow Y$ is surjective.

9. *Descending Functions to Quotient Sets, VI.* Let R be a relation on X and let $\sim^{\mathrm{eq}}_R$ be the equivalence relation associated to R. The following conditions are equivalent:

   (a) The map f satisfies the equivalent conditions of Item 4:

      - There exists a map

$$\overline{f} \colon X/{\sim^{\mathrm{eq}}_R} \longrightarrow Y$$

making the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, two heads, "q"'] & Y \\ X/\sim_R^{\text{eq}} \arrow[ur, dotted, "\exists", "\overline{f}"'] & \end{tikzcd}$$

commute.

- For each $x, y \in X$, if $x \sim_R^{\text{eq}} y$, then $f(x) = f(y)$.

(b) For each $x, y \in X$, if $x \sim_R y$, then $f(x) = f(y)$.

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[1]Dually, we also have an isomorphism of sets

$$\mathrm{Eq}(\mathrm{pr}_1, \mathrm{pr}_2) \cong X \times_{X/\sim_R^{\text{eq}}} X, \qquad \begin{tikzcd} \mathrm{Eq}(\mathrm{pr}_1, \mathrm{pr}_2) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d] \\ X \arrow[r] & X/\sim_R^{\text{eq}}. \end{tikzcd}$$

[2]*Further Terminology*: The set $X/\sim_{\mathrm{Ker}(f)}$ is often called the **coimage of** f, and denoted by $\mathrm{Coim}(f)$.

[3]In a sense this is a result relating the monad in **Rel** induced by f with the comonad in **Rel** induced by f, as the kernel and image

$$\begin{aligned} \mathrm{Ker}(f) &\colon X \nrightarrow X, \\ \mathrm{Im}(f) &\subset Y \end{aligned}$$

of f are the underlying functors of (respectively) the induced monad and comonad of the adjunction

$$\left(\mathrm{Gr}(f) \dashv f^{-1}\right)\colon \quad \begin{tikzcd} A \arrow[r, bend left, "\mathrm{Gr}(f)", "\shortmid"{marking}] \arrow[r, phantom, "\perp"] & B \arrow[l, bend left, "f^{-1}", "\shortmid"{marking}] \end{tikzcd}$$

of Constructions With Relations, Item 2 of Proposition 3.1.2.

Proof 5.2.4 ► Proof of Proposition 5.2.3

Item 1: As a Coequaliser

Omitted.

Item 2: As a Pushout

Omitted.

Item 3: The First Isomorphism Theorem for Sets

Clear.

Item 4: Descending Functions to Quotient Sets, I

See [Pro24c].

Item 5: Descending Functions to Quotient Sets, II

See [Pro24d].

Item 6: Descending Functions to Quotient Sets, III

This follows from Items 5 and 6.

Item 7: Descending Functions to Quotient Sets, IV

See [Pro24b].

Item 8: Descending Functions to Quotient Sets, V

See [Pro24a].

Item 9: Descending Functions to Quotient Sets, VI

The implication Item 9a $\implies$ Item 9b is clear.
Conversely, suppose that, for each $x, y \in X$, if $x \sim_R y$, then $f(x) = f(y)$. Spelling out the definition of the equivalence closure of R, we see that the condition $x \sim_R^{\text{eq}} y$ unwinds to the following:

($\star$) There exist $(x_1, \ldots, x_n) \in R^{\times n}$ satisfying at least one of the following conditions:

1. The following conditions are satisfied:
   (a) We have $x \sim_R x_1$ or $x_1 \sim_R x$;
   (b) We have $x_i \sim_R x_{i+1}$ or $x_{i+1} \sim_R x_i$ for each $1 \leq i \leq n-1$;
   (c) We have $y \sim_R x_n$ or $x_n \sim_R y$;
2. We have $x = y$.

Now, if $x = y$, then $f(x) = f(y)$ trivially; otherwise, we have

$$\begin{aligned} f(x) &= f(x_1), \\ f(x_1) &= f(x_2), \\ &\vdots \\ f(x_{n-1}) &= f(x_n), \\ f(x_n) &= f(y), \end{aligned}$$

and $f(x) = f(y)$, as we wanted to show.

Appendices

A Other Chapters

References

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