Types of Morphisms in Bicategories

The Clowder Project Authors

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In this chapter, we study special kinds of morphisms in bicategories:

1. *Monomorphisms and Epimorphisms in Bicategories (Sections 1 and 2).* There is a large number of different notions capturing the idea of a "monomorphism" or of an "epimorphism" in a bicategory.

   Arguably, the notion that best captures these concepts is that of a *pseudomonic morphism* (Definition 1.10.1.1) and of a *pseudoepic morphism* (Definition 2.10.1.1), although the other notions introduced in Sections 1 and 2 are also interesting on their own.

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1.1   Representably Faithful Morphisms

Let C be a bicategory.

Definition 1.1.1.1. A 1-morphism $f : A \to B$ of C is **representably faithful**[1] if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f_* : \mathrm{Hom}_C(X, A) \to \mathrm{Hom}_C(X, B)$$

given by postcomposition by f is faithful.

Remark 1.1.1.2. In detail, f is representably faithful if, for all diagrams in C of the form

$$X \underset{\psi}{\overset{\phi}{\underset{\Longrightarrow}{\alpha \Downarrow \Downarrow \beta}}} A \xrightarrow{\ f\ } B,$$

if we have

$$\mathrm{id}_f \star \alpha = \mathrm{id}_f \star \beta,$$

then $\alpha = \beta$.

Example 1.1.1.3. Here are some examples of representably faithful morphisms.

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[1] *Further Terminology:* Also called simply a **faithful morphism**, based on Item 1 of

1. *Representably Faithful Morphisms in* Cats$_2$. The representably faithful morphisms in Cats$_2$ are precisely the faithful functors; see Categories, Item 1 of Proposition 5.1.1.2.

2. *Representably Faithful Morphisms in* **Rel**. Every morphism of **Rel** is representably faithful; see Relations, Item 1 of Proposition 3.8.1.1.

1.2   Representably Full Morphisms

Let C be a bicategory.

Definition 1.2.1.1. A 1-morphism $f \colon A \to B$ of C is **representably full**[2] if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f_* \colon \mathrm{Hom}_C(X, A) \to \mathrm{Hom}_C(X, B)$$

given by postcomposition by f is full.

Remark 1.2.1.2. In detail, f is representably full if, for each $X \in \mathrm{Obj}(C)$ and each 2-morphism

$$\beta \colon f \circ \phi \Longrightarrow f \circ \psi, \quad X \overset{f \circ \phi}{\underset{f \circ \psi}{\Longrightarrow}} B$$

of C, there exists a 2-morphism

$$\alpha \colon \phi \Longrightarrow \psi, \quad X \overset{\phi}{\underset{\psi}{\Longrightarrow}} A$$

of C such that we have an equality

$$X \overset{\phi}{\underset{\psi}{\Longrightarrow}} A \overset{f}{\longrightarrow} B \quad = \quad X \overset{f \circ \phi}{\underset{f \circ \psi}{\Longrightarrow}} B$$

of pasting diagrams in C, i.e. such that we have

$$\beta = \mathrm{id}_f \star \alpha.$$

Example 1.1.1.3.

[2]*Further Terminology:* Also called simply a **full morphism**, based on Item 1 of Example 1.2.1.3.

Example 1.2.1.3. Here are some examples of representably full morphisms.

1. *Representably Full Morphisms in* Cats_2*.* The representably full morphisms in Cats_2 are precisely the full functors; see Categories, Item 1 of Proposition 5.2.1.2.

2. *Representably Full Morphisms in* **Rel***.* The representably full morphisms in **Rel** are characterised in Relations, Item 2 of Proposition 3.8.1.1.

1.3   Representably Fully Faithful Morphisms

Let C be a bicategory.

Definition 1.3.1.1. A 1-morphism $f\colon A \to B$ of C is **representably fully faithful**[3] if the following equivalent conditions are satisfied:

1. The 1-morphism f is representably faithful (Definition 1.1.1.1) and representably full (Definition 1.2.1.1).

2. For each $X \in \mathrm{Obj}(C)$, the functor

$$f_*\colon \mathsf{Hom}_C(X, A) \to \mathsf{Hom}_C(X, B)$$

given by postcomposition by f is fully faithful.

Remark 1.3.1.2. In detail, f is representably fully faithful if the conditions in Remark 1.1.1.2 and Remark 1.2.1.2 hold:

1. For all diagrams in C of the form

$$X \underset{\psi}{\overset{\phi}{\underset{\alpha \Downarrow \Downarrow \beta}{\rightrightarrows}}} A \xrightarrow{f} B,$$

   if we have

$$\mathrm{id}_f \star \alpha = \mathrm{id}_f \star \beta,$$

   then $\alpha = \beta$.

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[3]*Further Terminology:* Also called simply a **fully faithful morphism**, based on Item 1 of

2. For each $X \in \mathrm{Obj}(C)$ and each 2-morphism

$$\beta \colon f \circ \phi \Longrightarrow f \circ \psi, \quad X \underset{f \circ \psi}{\overset{f \circ \phi}{\Longrightarrow \beta \Downarrow}} B$$

of C, there exists a 2-morphism

$$\alpha \colon \phi \Longrightarrow \psi, \quad X \underset{\psi}{\overset{\phi}{\Longrightarrow \alpha \Downarrow}} A$$

of C such that we have an equality

$$X \underset{\psi}{\overset{\phi}{\Rightarrow \alpha \Downarrow}} A \xrightarrow{\ f\ } B \quad = \quad X \underset{f \circ \psi}{\overset{f \circ \phi}{\Rightarrow \beta \Downarrow}} B$$

of pasting diagrams in C, i.e. such that we have

$$\beta = \mathrm{id}_f \star \alpha.$$

Example 1.3.1.3. Here are some examples of representably fully faithful morphisms.

1. *Representably Fully Faithful Morphisms in* Cats_2. The representably fully faithful morphisms in Cats_2 are precisely the fully faithful functors; see Categories, Item 5 of Proposition 5.3.1.2.

2. *Representably Fully Faithful Morphisms in* **Rel**. The representably fully faithful morphisms of **Rel** coincide (Relations, Item 3 of Proposition 3.8.1.1) with the representably full morphisms in **Rel**, which are characterised in Relations, Item 2 of Proposition 3.8.1.1.

1.4   Morphisms Representably Faithful on Cores

Let C be a bicategory.

Example 1.3.1.3.

Definition 1.4.1.1. A 1-morphism $f\colon A \to B$ of C is **representably faithful on cores** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f_*\colon \mathsf{Core}(\mathsf{Hom}_C(X, A)) \to \mathsf{Core}(\mathsf{Hom}_C(X, B))$$

given by postcomposition by f is faithful.

Remark 1.4.1.2. In detail, f is representably faithful on cores if, for all diagrams in C of the form

$$X \underset{\psi}{\overset{\phi}{\Longrightarrow}} {}_{\alpha\|\,\|\beta} A \xrightarrow{\ f\ } B,$$

if α and β are 2-isomorphisms and we have

$$\mathrm{id}_f \star \alpha = \mathrm{id}_f \star \beta,$$

then $\alpha = \beta$.

1.5   Morphisms Representably Full on Cores

Let C be a bicategory.

Definition 1.5.1.1. A 1-morphism $f\colon A \to B$ of C is **representably full on cores** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f_*\colon \mathsf{Core}(\mathsf{Hom}_C(X, A)) \to \mathsf{Core}(\mathsf{Hom}_C(X, B))$$

given by postcomposition by f is full.

Remark 1.5.1.2. In detail, f is representably full on cores if, for each $X \in \mathrm{Obj}(C)$ and each 2-isomorphism

$$\beta\colon f \circ \phi \overset{\sim}{\Longrightarrow} f \circ \psi, \qquad X \underset{f\circ\psi}{\overset{f\circ\phi}{\Longrightarrow}} {}_{\beta\|} B$$

of C, there exists a 2-isomorphism

$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad X \underset{\psi}{\overset{\phi}{\Longrightarrow}} {}_{\alpha\|} A$$

of C such that we have an equality

$$
X \xRightarrow[\psi]{\phi}{}^{\alpha\Downarrow} A \xrightarrow{f} B \quad = \quad X \xRightarrow[f\circ\psi]{f\circ\phi}{}^{\beta\Downarrow} B
$$

of pasting diagrams in C, i.e. such that we have

$$
\beta = \mathrm{id}_f \star \alpha.
$$

1.6 Morphisms Representably Fully Faithful on Cores

Let C be a bicategory.

Definition 1.6.1.1. A 1-morphism $f\colon A \to B$ of C is **representably fully faithful on cores** if the following equivalent conditions are satisfied:

1. The 1-morphism f is representably faithful on cores (Definition 1.5.1.1) and representably full on cores (Definition 1.4.1.1).

2. For each $X \in \mathrm{Obj}(C)$, the functor

$$
f_*\colon \mathsf{Core}(\mathsf{Hom}_C(X, A)) \to \mathsf{Core}(\mathsf{Hom}_C(X, B))
$$

given by postcomposition by f is fully faithful.

Remark 1.6.1.2. In detail, f is representably fully faithful on cores if the conditions in Remark 1.4.1.2 and Remark 1.5.1.2 hold:

1. For all diagrams in C of the form

$$
X \xRightarrow[\psi]{\phi}{}^{\alpha\Downarrow\;\Downarrow\beta} A \xrightarrow{f} B,
$$

if α and β are 2-isomorphisms and we have

$$
\mathrm{id}_f \star \alpha = \mathrm{id}_f \star \beta,
$$

then $\alpha = \beta$.

2. For each $X \in \mathrm{Obj}(C)$ and each 2-isomorphism

$$\beta\colon f \circ \phi \overset{\sim}{\Longrightarrow} f \circ \psi, \qquad X \underset{f \circ \psi}{\overset{f \circ \phi}{\Longrightarrow}} B$$

of C, there exists a 2-isomorphism

$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad X \underset{\psi}{\overset{\phi}{\Longrightarrow}} A$$

of C such that we have an equality

$$X \underset{\psi}{\overset{\phi}{\Longrightarrow}} A \overset{f}{\longrightarrow} B \quad = \quad X \underset{f \circ \psi}{\overset{f \circ \phi}{\Longrightarrow}} B$$

of pasting diagrams in C, i.e. such that we have

$$\beta = \mathrm{id}_f \star \alpha.$$

1.7 Representably Essentially Injective Morphisms

Let C be a bicategory.

Definition 1.7.1.1. A 1-morphism $f\colon A \to B$ of C is **representably essentially injective** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f_*\colon \mathrm{Hom}_C(X, A) \to \mathrm{Hom}_C(X, B)$$

given by postcomposition by f is essentially injective.

Remark 1.7.1.2. In detail, f is representably essentially injective if, for each pair of morphisms $\phi, \psi\colon X \rightrightarrows A$ of C, the following condition is satisfied:

($\star$) If $f \circ \phi \cong f \circ \psi$, then $\phi \cong \psi$.

1.8   Representably Conservative Morphisms

Let C be a bicategory.

Definition 1.8.1.1. A 1-morphism $f\colon A \to B$ of C is **representably conservative** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f_*\colon \mathrm{Hom}_C(X, A) \to \mathrm{Hom}_C(X, B)$$

given by postcomposition by f is conservative.

Remark 1.8.1.2. In detail, f is representably conservative if, for each pair of morphisms $\phi, \psi\colon X \rightrightarrows A$ and each 2-morphism

$$\alpha\colon \phi \Longrightarrow \psi, \qquad X \underset{\psi}{\overset{\phi}{\Longrightarrow}}{}_{\alpha\|} A$$

of C, if the 2-morphism

$$\mathrm{id}_f \star \alpha\colon f \circ \phi \Longrightarrow f \circ \psi, \qquad X \underset{f\circ\psi}{\overset{f\circ\phi}{\Longrightarrow}}{}_{\mathrm{id}_f\star\alpha} B$$

is a 2-isomorphism, then so is α.

1.9   Strict Monomorphisms

Let C be a bicategory.

Definition 1.9.1.1. A 1-morphism $f\colon A \to B$ of C is a **strict monomorphism** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f_*\colon \mathrm{Hom}_C(X, A) \to \mathrm{Hom}_C(X, B)$$

given by postcomposition by f is injective on objects, i.e. its action on objects

$$f_*\colon \mathrm{Obj}(\mathrm{Hom}_C(X, A)) \to \mathrm{Obj}(\mathrm{Hom}_C(X, B))$$

is injective.

Remark 1.9.1.2. In detail, f is a strict monomorphism in C if, for each diagram in C of the form

$$X \overset{\phi}{\underset{\psi}{\rightrightarrows}} A \overset{f}{\longrightarrow} B,$$

if $f \circ \phi = f \circ \psi$, then $\phi = \psi$.

Example 1.9.1.3. Here are some examples of strict monomorphisms.

1. *Strict Monomorphisms in* Cats_2. The strict monomorphisms in Cats_2 are precisely the functors which are injective on objects and injective on morphisms; see Categories, Item 1 of Proposition 6.2.1.2.

2. *Strict Monomorphisms in* **Rel**. The strict monomorphisms in **Rel** are characterised in Relations, Proposition 3.7.1.1.

1.10   Pseudomonic Morphisms

Let C be a bicategory.

Definition 1.10.1.1. A 1-morphism $f \colon A \to B$ of C is **pseudomonic** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f_* \colon \mathrm{Hom}_C(X, A) \to \mathrm{Hom}_C(X, B)$$

given by postcomposition by f is pseudomonic.

Remark 1.10.1.2. In detail, a 1-morphism $f \colon A \to B$ of C is pseudomonic if it satisfies the following conditions:

1. For all diagrams in C of the form

$$X \underset{\psi}{\overset{\phi}{\underset{\alpha \Downarrow \Downarrow \beta}{\rightrightarrows}}} A \overset{f}{\longrightarrow} B,$$

   if we have

$$\mathrm{id}_f \star \alpha = \mathrm{id}_f \star \beta,$$

   then $\alpha = \beta$.

2. For each $X \in \mathrm{Obj}(C)$ and each 2-isomorphism

$$\beta\colon f \circ \phi \overset{\sim}{\Longrightarrow} f \circ \psi, \quad X \underset{f \circ \psi}{\overset{f \circ \phi}{\Rightarrow \beta\Downarrow \Rightarrow}} B$$

of C, there exists a 2-isomorphism

$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \quad X \underset{\psi}{\overset{\phi}{\Rightarrow \alpha\Downarrow \Rightarrow}} A$$

of C such that we have an equality

$$X \underset{\psi}{\overset{\phi}{\Rightarrow \alpha\Downarrow \Rightarrow}} A \overset{f}{\longrightarrow} B \quad = \quad X \underset{f \circ \psi}{\overset{f \circ \phi}{\Rightarrow \beta\Downarrow \Rightarrow}} B$$

of pasting diagrams in C, i.e. such that we have

$$\beta = \mathrm{id}_f \star \alpha.$$

Proposition 1.10.1.3. Let $f\colon A \to B$ be a 1-morphism of C.

1. *Characterisations.* The following conditions are equivalent:

   (a) The morphism f is pseudomonic.

   (b) The morphism f is representably full on cores and representably faithful.

   (c) We have an isocomma square of the form

$$A \overset{\mathrm{eq.}}{\cong} A \overset{\leftrightarrow}{\times}_B A,$$

$$\begin{array}{ccc} A & \overset{\mathrm{id}_A}{\longrightarrow} & A \\ {\scriptstyle \mathrm{id}_A}\downarrow & \nearrow \kern-1.2em \swarrow & \downarrow{\scriptstyle F} \\ A & \underset{F}{\longrightarrow} & B \end{array}$$

   in C up to equivalence.

2. *Interaction With Cotensors.* If C has cotensors with $\mathbb{1}$, then the following conditions are equivalent:

(a) The morphism f is pseudomonic.

(b) We have an isocomma square of the form

$$A \overset{\text{eq.}}{\cong} A \overset{\leftrightarrow}{\times}_{\mathbb{1} \pitchfork F} B,$$

$$
\begin{array}{ccc}
A & \longrightarrow & \mathbb{1} \pitchfork A \\
\Big\downarrow{\scriptstyle F} & \nearrow & \Big\downarrow{\scriptstyle \mathbb{1} \pitchfork F} \\
B & \longrightarrow & \mathbb{1} \pitchfork B
\end{array}
$$

in C up to equivalence.

Proof. *Item 1, Characterisations*: Omitted.
Item 2, Interaction With Cotensors: Omitted. $\qquad\square$

2 Epimorphisms in Bicategories

2.1 Corepresentably Faithful Morphisms

Let C be a bicategory.

Definition 2.1.1.1. A 1-morphism $f \colon A \to B$ of C is **corepresentably faithful** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f^* \colon \mathsf{Hom}_C(B, X) \to \mathsf{Hom}_C(A, X)$$

given by precomposition by f is faithful.

Remark 2.1.1.2. In detail, f is corepresentably faithful if, for all diagrams in C of the form

$$A \overset{f}{\longrightarrow} B \underset{\psi}{\overset{\phi}{\underset{\alpha \Downarrow \;\; \Downarrow \beta}{\rightrightarrows}}} X,$$

if we have

$$\alpha \star \mathrm{id}_f = \beta \star \mathrm{id}_f,$$

then $\alpha = \beta$.

Example 2.1.1.3. Here are some examples of corepresentably faithful morphisms.

1. *Corepresentably Faithful Morphisms in* Cats_2. The corepresentably faithful morphisms in Cats_2 are characterised in Categories, Item 4 of Proposition 5.1.1.2.

2. *Corepresentably Faithful Morphisms in* **Rel**. Every morphism of **Rel** is corepresentably faithful; see Relations, Item 1 of Proposition 3.10.1.1.

2.2   Corepresentably Full Morphisms

Let C be a bicategory.

Definition 2.2.1.1. A 1-morphism $f \colon A \to B$ of C is **corepresentably full** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f^* \colon \mathsf{Hom}_C(B, X) \to \mathsf{Hom}_C(A, X)$$

given by precomposition by f is full.

Remark 2.2.1.2. In detail, f is corepresentably full if, for each $X \in \mathrm{Obj}(C)$ and each 2-morphism

$$\beta \colon \phi \circ f \Longrightarrow \psi \circ f, \qquad A \underset{\psi \circ f}{\overset{\phi \circ f}{\Longrightarrow}} X \quad \beta \big\Downarrow$$

of C, there exists a 2-morphism

$$\alpha \colon \phi \Longrightarrow \psi, \qquad B \underset{\psi}{\overset{\phi}{\Longrightarrow}} X \quad \alpha \big\Downarrow$$

of C such that we have an equality

$$A \overset{f}{\longrightarrow} B \underset{\psi}{\overset{\phi}{\Longrightarrow}} X \quad = \quad A \underset{\psi \circ f}{\overset{\phi \circ f}{\Longrightarrow}} X$$

of pasting diagrams in C, i.e. such that we have

$$\beta = \alpha \star \mathrm{id}_f.$$

Example 2.2.1.3. Here are some examples of corepresentably full morphisms.

1. *Corepresentably Full Morphisms in* Cats_2. The corepresentably full morphisms in Cats_2 are characterised in Categories, Item 5 of Proposition 5.2.1.2.

2. *Corepresentably Full Morphisms in* **Rel**. The corepresentably full morphisms in **Rel** are characterised in Relations, Item 2 of Proposition 3.10.1.1.

2.3   Corepresentably Fully Faithful Morphisms

Let C be a bicategory.

Definition 2.3.1.1. A 1-morphism $f\colon A \to B$ of C is **corepresentably fully faithful**[4] if the following equivalent conditions are satisfied:

1. The 1-morphism f is corepresentably full (Definition 2.2.1.1) and corepresentably faithful (Definition 2.1.1.1).

2. For each $X \in \mathrm{Obj}(C)$, the functor

$$f^*\colon \mathrm{Hom}_C(B,X) \to \mathrm{Hom}_C(A,X)$$

given by precomposition by f is fully faithful.

Remark 2.3.1.2. In detail, f is corepresentably fully faithful if the conditions in Remark 2.1.1.2 and Remark 2.2.1.2 hold:

1. For all diagrams in C of the form

$$A \xrightarrow{\ f\ } B \underset{\psi}{\overset{\phi}{\Rightarrow}}\ \alpha {\Downarrow}{\Downarrow} \beta\ X,$$

    if we have

$$\alpha \star \mathrm{id}_f = \beta \star \mathrm{id}_f,$$

    then $\alpha = \beta$.

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[4]*Further Terminology:* Corepresentably fully faithful morphisms have also been called **lax epimorphisms** in the literature (e.g. in [Adá+01]), though we will always use the name "corepresentably fully faithful morphism" instead in this work.

2. For each $X \in \mathrm{Obj}(C)$ and each 2-morphism

$$\beta\colon \phi \circ f \Longrightarrow \psi \circ f, \qquad A \underset{\psi \circ f}{\overset{\phi \circ f}{\Longrightarrow}} X$$

of C, there exists a 2-morphism

$$\alpha\colon \phi \Longrightarrow \psi, \qquad B \underset{\psi}{\overset{\phi}{\Longrightarrow}} X$$

of C such that we have an equality

$$A \xrightarrow{\ f\ } B \underset{\psi}{\overset{\phi}{\Longrightarrow}} X \quad = \quad A \underset{\psi \circ f}{\overset{\phi \circ f}{\Longrightarrow}} X$$

of pasting diagrams in C, i.e. such that we have

$$\beta = \alpha \star \mathrm{id}_f.$$

Example 2.3.1.3. Here are some examples of corepresentably fully faithful morphisms.

1. *Corepresentably Fully Faithful Morphisms in* Cats_2. The fully faithful epimorphisms in Cats_2 are characterised in Categories, Item 9 of Proposition 5.3.1.2.

2. *Corepresentably Fully Faithful Morphisms in* **Rel**. The corepresentably fully faithful morphisms of **Rel** coincide (Relations, Item 3 of Proposition 3.10.1.1) with the corepresentably full morphisms in **Rel**, which are characterised in Relations, Item 2 of Proposition 3.10.1.1.

2.4   Morphisms Corepresentably Faithful on Cores

Let C be a bicategory.

Definition 2.4.1.1. A 1-morphism $f\colon A \to B$ of C is **corepresentably faithful on cores** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f^*\colon \mathsf{Core}(\mathrm{Hom}_C(B, X)) \to \mathsf{Core}(\mathrm{Hom}_C(A, X))$$

given by precomposition by f is faithful.

Remark 2.4.1.2. In detail, f is corepresentably faithful on cores if, for all diagrams in C of the form

$$A \xrightarrow{\ f\ } B \overset{\phi}{\underset{\psi}{\Longleftrightarrow}}_{\alpha \| \| \beta} X,$$

if α and β are 2-isomorphisms and we have

$$\alpha \star \mathrm{id}_f = \beta \star \mathrm{id}_f,$$

then $\alpha = \beta$.

2.5 Morphisms Corepresentably Full on Cores

Let C be a bicategory.

Definition 2.5.1.1. A 1-morphism $f \colon A \to B$ of C is **corepresentably full on cores** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f^* \colon \mathsf{Core}(\mathsf{Hom}_C(B, X)) \to \mathsf{Core}(\mathsf{Hom}_C(A, X))$$

given by precomposition by f is full.

Remark 2.5.1.2. In detail, f is corepresentably full on cores if, for each $X \in \mathrm{Obj}(C)$ and each 2-isomorphism

$$\beta \colon \phi \circ f \overset{\sim}{\Longrightarrow} \psi \circ f, \quad A \overset{\phi \circ f}{\underset{\psi \circ f}{\Longrightarrow}}_{\beta \|} X$$

of C, there exists a 2-isomorphism

$$\alpha \colon \phi \overset{\sim}{\Longrightarrow} \psi, \quad B \overset{\phi}{\underset{\psi}{\Longrightarrow}}_{\alpha \|} X$$

of C such that we have an equality

$$A \xrightarrow{\ f\ } B \overset{\phi}{\underset{\psi}{\Longrightarrow}}_{\alpha \|} X \quad = \quad A \overset{\phi \circ f}{\underset{\psi \circ f}{\Longrightarrow}}_{\beta \|} X$$

of pasting diagrams in C, i.e. such that we have

$$\beta = \alpha \star \mathrm{id}_f.$$

2.6   Morphisms Corepresentably Fully Faithful on Cores

Let C be a bicategory.

Definition 2.6.1.1. A 1-morphism $f \colon A \to B$ of C is **corepresentably fully faithful on cores** if the following equivalent conditions are satisfied:

1. The 1-morphism f is corepresentably full on cores (Definition 2.5.1.1) and corepresentably faithful on cores (Definition 2.1.1.1).

2. For each $X \in \mathrm{Obj}(C)$, the functor

$$f^* \colon \mathsf{Core}(\mathrm{Hom}_C(B, X)) \to \mathsf{Core}(\mathrm{Hom}_C(A, X))$$

   given by precomposition by f is fully faithful.

Remark 2.6.1.2. In detail, f is corepresentably fully faithful on cores if the conditions in Remark 2.4.1.2 and Remark 2.5.1.2 hold:

1. For all diagrams in C of the form

$$A \xrightarrow{\ f\ } B \underset{\psi}{\overset{\phi}{\underset{\Longrightarrow}{\rightrightarrows}}}{}_{\alpha \Downarrow \Downarrow \beta}\, X,$$

   if α and β are 2-isomorphisms and we have

$$\alpha \star \mathrm{id}_f = \beta \star \mathrm{id}_f,$$

   then $\alpha = \beta$.

2. For each $X \in \mathrm{Obj}(C)$ and each 2-isomorphism

$$\beta \colon \phi \circ f \overset{\sim}{\Longrightarrow} \psi \circ f, \qquad A \underset{\psi \circ f}{\overset{\phi \circ f}{\underset{\beta \Downarrow}{\rightrightarrows}}} X$$

   of C, there exists a 2-isomorphism

$$\alpha \colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad B \underset{\psi}{\overset{\phi}{\underset{\alpha \Downarrow}{\rightrightarrows}}} X$$

of C such that we have an equality

$$A \xrightarrow{f} B \underset{\psi}{\overset{\phi}{\Rightarrow}}_{\alpha\Downarrow} X \quad = \quad A \underset{\psi \circ f}{\overset{\phi \circ f}{\Rightarrow}}_{\beta\Downarrow} X$$

of pasting diagrams in C, i.e. such that we have

$$\beta = \alpha \star \mathrm{id}_f.$$

2.7   Corepresentably Essentially Injective Morphisms

Let C be a bicategory.

Definition 2.7.1.1. A 1-morphism $f : A \to B$ of C is **corepresentably essentially injective** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f^* : \mathsf{Hom}_C(B, X) \to \mathsf{Hom}_C(A, X)$$

given by precomposition by f is essentially injective.

Remark 2.7.1.2. In detail, f is corepresentably essentially injective if, for each pair of morphisms $\phi, \psi : B \rightrightarrows X$ of C, the following condition is satisfied:

($\star$) If $\phi \circ f \cong \psi \circ f$, then $\phi \cong \psi$.

2.8   Corepresentably Conservative Morphisms

Let C be a bicategory.

Definition 2.8.1.1. A 1-morphism $f : A \to B$ of C is **corepresentably conservative** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f^* : \mathsf{Hom}_C(B, X) \to \mathsf{Hom}_C(A, X)$$

given by precomposition by f is conservative.

Remark 2.8.1.2. In detail, f is corepresentably conservative if, for each pair of morphisms $\phi, \psi : B \rightrightarrows X$ and each 2-morphism

$$\alpha : \phi \overset{\sim}{\Longrightarrow} \psi, \quad B \underset{\psi}{\overset{\phi}{\Rightarrow}}_{\alpha\Downarrow} X$$

of C, if the 2-morphism

$$\alpha \star \mathrm{id}_f \colon \phi \circ f \Longrightarrow \psi \circ f, \qquad A \underset{\psi \circ f}{\overset{\phi \circ f}{\Longrightarrow}} X$$

is a 2-isomorphism, then so is α.

2.9   Strict Epimorphisms

Let C be a bicategory.

Definition 2.9.1.1. A 1-morphism $f \colon A \to B$ is a **strict epimorphism in** C if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f^* \colon \mathsf{Hom}_C(B, X) \to \mathsf{Hom}_C(A, X)$$

given by precomposition by f is injective on objects, i.e. its action on objects

$$f_* \colon \mathrm{Obj}(\mathsf{Hom}_C(B, X)) \to \mathrm{Obj}(\mathsf{Hom}_C(A, X))$$

is injective.

Remark 2.9.1.2. In detail, f is a strict epimorphism if, for each diagram in C of the form

$$A \xrightarrow{\ f\ } B \underset{\psi}{\overset{\phi}{\Longrightarrow}} X,$$

if $\phi \circ f = \psi \circ f$, then $\phi = \psi$.

Example 2.9.1.3. Here are some examples of strict epimorphisms.

1. *Strict Epimorphisms in* Cats_2. The strict epimorphisms in Cats_2 are characterised in Categories, Item 1 of Proposition 6.3.1.2.

2. *Strict Epimorphisms in* **Rel**. The strict epimorphisms in **Rel** are characterised in Relations, Proposition 3.9.1.1.

2.10   Pseudoepic Morphisms

Let C be a bicategory.

Definition 2.10.1.1. A 1-morphism $f \colon A \to B$ of C is **pseudoepic** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f^* \colon \mathrm{Hom}_C(B, X) \to \mathrm{Hom}_C(A, X)$$

given by precomposition by f is pseudomonic.

Remark 2.10.1.2. In detail, a 1-morphism $f \colon A \to B$ of C is pseudoepic if it satisfies the following conditions:

1. For all diagrams in C of the form

   $$A \xrightarrow{\;f\;} B \underset{\psi}{\overset{\phi}{\Longrightarrow}} X,$$

   if we have

   $$\alpha \star \mathrm{id}_f = \beta \star \mathrm{id}_f,$$

   then $\alpha = \beta$.

2. For each $X \in \mathrm{Obj}(C)$ and each 2-isomorphism

   $$\beta \colon \phi \circ f \overset{\sim}{\Longrightarrow} \psi \circ f, \qquad A \underset{\psi \circ f}{\overset{\phi \circ f}{\Longrightarrow}} X$$

   of C, there exists a 2-isomorphism

   $$\alpha \colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad B \underset{\psi}{\overset{\phi}{\Longrightarrow}} X$$

   of C such that we have an equality

   $$A \xrightarrow{\;f\;} B \underset{\psi}{\overset{\phi}{\Longrightarrow}} X \quad = \quad A \underset{\psi \circ f}{\overset{\phi \circ f}{\Longrightarrow}} X$$

   of pasting diagrams in C, i.e. such that we have

   $$\beta = \alpha \star \mathrm{id}_f.$$

Proposition 2.10.1.3. Let $f\colon A \to B$ be a 1-morphism of C.

1. *Characterisations.* The following conditions are equivalent:

   (a) The morphism f is pseudoepic.

   (b) The morphism f is corepresentably full on cores and corepresentably faithful.

   (c) We have an isococomma square of the form

   $$B \overset{\text{eq.}}{\cong} B \overset{\leftrightarrow}{\coprod}_A B,$$

   in C up to equivalence.

Proof. *Item 1*, *Characterisations*: Omitted. $\square$

Appendices

A Other Chapters

Sets

References

[Adá+01]   Jiří Adámek, Robert El Bashir, Manuela Sobral, and Jiří Velebil. "On Functors Which Are Lax Epimorphisms". In: *Theory Appl. Categ.* 8 (2001), pp. 509–521. ISSN: 1201-561X (cit. on p. 14).