

Pointed Sets

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This chapter contains some foundational material on pointed sets.

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1 Pointed Sets

1.1 Foundations

Definition 1.1.1.1. A **pointed set**¹ is equivalently:

- An \mathbb{E}_0 -monoid in $(N_\bullet(\text{Sets}), \text{pt})$.
- A pointed object in (Sets, pt) .

Remark 1.1.1.2. In detail, a **pointed set** is a pair (X, x_0) consisting of:

- *The Underlying Set.* A set X , called the **underlying set of** (X, x_0) .
- *The Basepoint.* A morphism

$$[x_0] : \text{pt} \rightarrow X$$

in Sets , determining an element $x_0 \in X$, called the **basepoint of** X .

Example 1.1.1.3. The **0-sphere**² is the pointed set $(S^0, 0)$ ³ consisting of:

- *The Underlying Set.* The set S^0 defined by

$$S^0 \stackrel{\text{def}}{=} \{0, 1\}.$$

- *The Basepoint.* The element 0 of S^0 .

Example 1.1.1.4. The **trivial pointed set** is the pointed set (pt, \star) consisting of:

- *The Underlying Set.* The punctual set $\text{pt} \stackrel{\text{def}}{=} \{\star\}$.
- *The Basepoint.* The element \star of pt .

Example 1.1.1.5. The **underlying pointed set** of a semimodule (M, α_M) is the pointed set $(M, 0_M)$.

Example 1.1.1.6. The **underlying pointed set** of a module (M, α_M) is the pointed set $(M, 0_M)$.

¹*Further Terminology:* In the context of monoids with zero as models for \mathbb{F}_1 -algebras, pointed sets are viewed as **\mathbb{F}_1 -modules**.

²*Further Terminology:* In the context of monoids with zero as models for \mathbb{F}_1 -algebras, the 0-sphere is viewed as the **underlying pointed set of the field with one element**.

³*Further Notation:* In the context of monoids with zero as models for \mathbb{F}_1 -algebras, S^0 is also

1.2 Morphisms of Pointed Sets

Definition 1.2.1.1. A **morphism of pointed sets**^{4,5} is equivalently:

- A morphism of \mathbb{E}_0 -monoids in $(\mathbf{N}_\bullet(\mathbf{Sets}), \text{pt})$.
- A morphism of pointed objects in $(\mathbf{Sets}, \text{pt})$.

Remark 1.2.1.2. In detail, a **morphism of pointed sets** $f: (X, x_0) \rightarrow (Y, y_0)$ is a morphism of sets $f: X \rightarrow Y$ such that the diagram

$$\begin{array}{ccc} & \text{pt} & \\ [x_0] \swarrow & & \searrow [y_0] \\ X & \xrightarrow{f} & Y \end{array}$$

commutes, i.e. such that

$$f(x_0) = y_0.$$

1.3 The Category of Pointed Sets

Definition 1.3.1.1. The **category of pointed sets** is the category \mathbf{Sets}_* defined equivalently as

- The homotopy category of the ∞ -category $\mathbf{Mon}_{\mathbb{E}_0}(\mathbf{N}_\bullet(\mathbf{Sets}), \text{pt})$ of ??, ??;
- The category \mathbf{Sets}_* of ??, ??.

Remark 1.3.1.2. In detail, the **category of pointed sets** is the category \mathbf{Sets}_* where

- *Objects.* The objects of \mathbf{Sets}_* are pointed sets;
- *Morphisms.* The morphisms of \mathbf{Sets}_* are morphisms of pointed sets;
- *Identities.* For each $(X, x_0) \in \text{Obj}(\mathbf{Sets}_*)$, the unit map

$$\mathbb{1}_{(X, x_0)}^{\mathbf{Sets}_*} : \text{pt} \rightarrow \mathbf{Sets}_*((X, x_0), (X, x_0))$$

denoted $(\mathbb{F}_1, 0)$.

⁴*Further Terminology:* Also called a **pointed function**.

⁵*Further Terminology:* In the context of monoids with zero as models for \mathbb{F}_1 -algebras, mor-

of \mathbf{Sets}_* at (X, x_0) is defined by⁶

$$\mathrm{id}_{(X, x_0)}^{\mathbf{Sets}_*} \stackrel{\mathrm{def}}{=} \mathrm{id}_X;$$

- *Composition.* For each $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathbf{Sets}_*)$, the composition map

$$\circ_{(X, x_0), (Y, y_0), (Z, z_0)}^{\mathbf{Sets}_*} : \mathbf{Sets}_*((Y, y_0), (Z, z_0)) \times \mathbf{Sets}_*((X, x_0), (Y, y_0)) \rightarrow \mathbf{Sets}_*((X, x_0), (Z, z_0))$$

of \mathbf{Sets}_* at $((X, x_0), (Y, y_0), (Z, z_0))$ is defined by⁷

$$g \circ_{(X, x_0), (Y, y_0), (Z, z_0)}^{\mathbf{Sets}_*} f \stackrel{\mathrm{def}}{=} g \circ f.$$

1.4 Elementary Properties of Pointed Sets

Proposition 1.4.1.1. Let (X, x_0) be a pointed set.

1. *Completeness.* The category \mathbf{Sets}_* of pointed sets and morphisms between them is complete, having in particular:
 - (a) Products, described as in [Definition 2.3.1.1](#);
 - (b) Pullbacks, described as in [Definition 2.4.1.1](#);
 - (c) Equalisers, described as in [Definition 2.5.1.1](#).
2. *Cocompleteness.* The category \mathbf{Sets}_* of pointed sets and morphisms between them is cocomplete, having in particular:

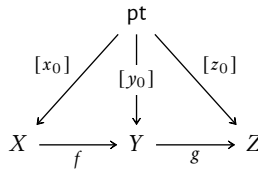
phisms of pointed sets are also called **morphism of \mathbb{F}_1 -modules**.

⁶Note that id_X is indeed a morphism of pointed sets, as we have $\mathrm{id}_X(x_0) = x_0$.

⁷Note that the composition of two morphisms of pointed sets is indeed a morphism of pointed sets, as we have

$$\begin{aligned} g(f(x_0)) &= g(y_0) \\ &= z_0, \end{aligned}$$

or



in terms of diagrams.

- (a) Coproducts, described as in [Definition 3.3.1.1](#);
- (b) Pushouts, described as in [Definition 3.4.1.1](#);
- (c) Coequalisers, described as in [Definition 3.5.1.1](#).

3. *Failure To Be Cartesian Closed.* The category \mathbf{Sets}_* is not Cartesian closed.⁸

4. *Morphisms From the Monoidal Unit.* We have a bijection of sets⁹

$$\mathbf{Sets}_*(S^0, X) \cong X,$$

natural in $(X, x_0) \in \mathbf{Obj}(\mathbf{Sets}_*)$, internalising also to an isomorphism of pointed sets

$$\mathbf{Sets}_*(S^0, X) \cong (X, x_0),$$

again natural in $(X, x_0) \in \mathbf{Obj}(\mathbf{Sets}_*)$.

5. *Relation to Partial Functions.* We have an equivalence of categories¹⁰

$$\mathbf{Sets}_* \stackrel{\text{eq.}}{\cong} \mathbf{Sets}^{\text{part.}}$$

between the category of pointed sets and pointed functions between them and the category of sets and partial functions between them, where:

(a) *From Pointed Sets to Sets With Partial Functions.* The equivalence

$$\xi: \mathbf{Sets}_* \xrightarrow{\cong} \mathbf{Sets}^{\text{part.}}$$

sends:

- i. A pointed set (X, x_0) to X .
- ii. A pointed function


$$f: (X, x_0) \rightarrow (Y, y_0)$$

⁸The category \mathbf{Sets}_* does admit monoidal closed structures however; see [Tensor Products of Pointed Sets](#).

⁹In other words, the forgetful functor

$$\omega: \mathbf{Sets}_* \rightarrow \mathbf{Sets}$$

defined on objects by sending a pointed set to its underlying set is corepresentable by S^0 .

¹⁰ *Warning:* This is not an isomorphism of categories, only an equivalence.

to the partial function

$$\xi_f : X \rightarrow Y$$

defined on $f^{-1}(Y \setminus y_0)$ and given by

$$\xi_f(x) \stackrel{\text{def}}{=} f(x)$$

for each $x \in f^{-1}(Y \setminus y_0)$.

(b) *From Sets With Partial Functions to Pointed Sets.* The equivalence

$$\xi^{-1} : \mathbf{Sets}^{\text{part.}} \xrightarrow{\cong} \mathbf{Sets}_*$$

sends:

- i. A set X is to the pointed set (X, \star) with \star an element that is not in X .
- ii. A partial function

$$f : X \rightarrow Y$$

defined on $U \subset X$ to the pointed function

$$\xi_f^{-1} : (X, x_0) \rightarrow (Y, y_0)$$

defined by

$$\xi_f(x) \stackrel{\text{def}}{=} \begin{cases} f(x) & \text{if } x \in U, \\ y_0 & \text{otherwise.} \end{cases}$$

for each $x \in X$.

Proof. **Item 1, Completeness:** This follows from (the proofs) of **Definitions 2.3.1.1, 2.4.1.1 and 2.5.1.1** and **??, ??**.

Item 2, Cocompleteness: This follows from (the proofs) of **Definitions 3.3.1.1, 3.4.1.1 and 3.5.1.1** and **??, ??**.

Item 3, Failure To Be Cartesian Closed: See **[MSE 2855868]**.

Item 4, Morphisms From the Monoidal Unit: Since a morphism from S^0 to a pointed set (X, x_0) sends $0 \in S^0$ to x_0 and then can send $1 \in S^0$ to any element of X , we obtain a bijection between pointed maps $S^0 \rightarrow X$ and the elements of X .

The isomorphism then

$$\mathbf{Sets}_*(S^0, X) \cong (X, x_0)$$

follows by noting that $\Delta_{x_0} : S^0 \rightarrow X$, the basepoint of $\mathbf{Sets}_*(S^0, X)$, corresponds to the pointed map $S^0 \rightarrow X$ picking the element x_0 of X , and thus we see that the bijection between pointed maps $S^0 \rightarrow X$ and elements of X is compatible with basepoints, lifting to an isomorphism of pointed sets.

Item 5, Relation to Partial Functions: See [MSE 884460]. \square

2 Limits of Pointed Sets

2.1 The Terminal Pointed Set

Definition 2.1.1.1. The **terminal pointed set** is the pair $((\text{pt}, \star), \{!_X\}_{(X, x_0) \in \text{Obj}(\mathbf{Sets}_*)})$ consisting of:

- *The Limit.* The pointed set (pt, \star) .
- *The Cone.* The collection of morphisms of pointed sets

$$\{!_X : (X, x_0) \rightarrow (\text{pt}, \star)\}_{(X, x_0) \in \text{Obj}(\mathbf{Sets})}$$

defined by

$$!_X(x) \stackrel{\text{def}}{=} \star$$

for each $x \in X$ and each $(X, x_0) \in \text{Obj}(\mathbf{Sets})$.

Proof. We claim that (pt, \star) is the terminal object of \mathbf{Sets}_* . Indeed, suppose we have a diagram of the form

$$(X, x_0) \quad (\text{pt}, \star)$$

in \mathbf{Sets}_* . Then there exists a unique morphism of pointed sets

$$\phi : (X, x_0) \rightarrow (\text{pt}, \star)$$

making the diagram

$$(X, x_0) \xrightarrow[\exists!]{\phi} (\text{pt}, \star)$$

commute, namely $!_X$. \square

2.2 Products of Families of Pointed Sets

Let $\{(X_i, x_0^i)\}_{i \in I}$ be a family of pointed sets.

Definition 2.2.1.1. The **product** of $\{(X_i, x_0^i)\}_{i \in I}$ is the pair $((\prod_{i \in I} X_i, (x_0^i)_{i \in I}), \{\text{pr}_i\}_{i \in I})$ consisting of:

- *The Limit.* The pointed set $(\prod_{i \in I} X_i, (x_0^i)_{i \in I})$.
- *The Cone.* The collection

$$\left\{ \text{pr}_i : \left(\prod_{i \in I} X_i, (x_0^i)_{i \in I} \right) \rightarrow (X_i, x_0^i) \right\}_{i \in I}$$

of maps given by

$$\text{pr}_i((x_j)_{j \in I}) \stackrel{\text{def}}{=} x_i$$

for each $(x_j)_{j \in I} \in \prod_{i \in I} X_i$ and each $i \in I$.

Proof. We claim that $(\prod_{i \in I} X_i, (x_0^i)_{i \in I})$ is the categorical product of $\{(X_i, x_0^i)\}_{i \in I}$ in Sets_* . Indeed, suppose we have, for each $i \in I$, a diagram of the form

$$\begin{array}{ccc} (P, *) & & \\ & \searrow p_i & \\ (\prod_{i \in I} X_i, (x_0^i)_{i \in I}) & \xrightarrow{\text{pr}_i} & (X_i, x_0^i) \end{array}$$

in Sets_* . Then there exists a unique morphism of pointed sets

$$\phi : (P, *) \rightarrow \left(\prod_{i \in I} X_i, (x_0^i)_{i \in I} \right)$$

making the diagram

$$\begin{array}{ccc} (P, *) & & \\ \downarrow \phi \quad \exists! & \searrow p_i & \\ (\prod_{i \in I} X_i, (x_0^i)_{i \in I}) & \xrightarrow{\text{pr}_i} & (X_i, x_0^i) \end{array}$$

commute, being uniquely determined by the condition $\text{pr}_i \circ \phi = p_i$ for each $i \in I$ via

$$\phi(x) = (p_i(x))_{i \in I}$$

for each $x \in P$. Note that this is indeed a morphism of pointed sets, as we have

$$\begin{aligned} \phi(*) &= (p_i(*))_{i \in I} \\ &= (x_0^i)_{i \in I}, \end{aligned}$$

where we have used that p_i is a morphism of pointed sets for each $i \in I$. \square

Proposition 2.2.1.2. Let $\{(X_i, x_0^i)\}_{i \in I}$ be a family of pointed sets.

1. *Functoriality.* The assignment $\{(X_i, x_0^i)\}_{i \in I} \mapsto (\prod_{i \in I} X_i, (x_0^i)_{i \in I})$ defines a functor

$$\prod_{i \in I}: \text{Fun}(I_{\text{disc}}, \text{Sets}_*) \rightarrow \text{Sets}_*.$$

Proof. **Item 1, Functoriality:** This follows from ??, ?? of ??. \square

2.3 Products

Let (X, x_0) and (Y, y_0) be pointed sets.

Definition 2.3.1.1. The **product of** (X, x_0) **and** (Y, y_0) is the pair consisting of:

- *The Limit.* The pointed set $(X \times Y, (x_0, y_0))$.
- *The Cone.* The morphisms of pointed sets

$$\begin{aligned} \text{pr}_1: (X \times Y, (x_0, y_0)) &\rightarrow (X, x_0), \\ \text{pr}_2: (X \times Y, (x_0, y_0)) &\rightarrow (Y, y_0) \end{aligned}$$

defined by

$$\begin{aligned} \text{pr}_1(x, y) &\stackrel{\text{def}}{=} x, \\ \text{pr}_2(x, y) &\stackrel{\text{def}}{=} y \end{aligned}$$

for each $(x, y) \in X \times Y$.

Proof. We claim that $(X \times Y, (x_0, y_0))$ is the categorical product of (X, x_0) and (Y, y_0) in \mathbf{Sets}_* . Indeed, suppose we have a diagram of the form

$$\begin{array}{ccccc} & & (P, *) & & \\ & \swarrow p_1 & & \searrow p_2 & \\ (X, x_0) & \xleftarrow{\text{pr}_1} & (X \times Y, (x_0, y_0)) & \xrightarrow{\text{pr}_2} & (Y, y_0) \end{array}$$

in \mathbf{Sets}_* . Then there exists a unique morphism of pointed sets

$$\phi: (P, *) \rightarrow (X \times Y, (x_0, y_0))$$

making the diagram

$$\begin{array}{ccccc} & & (P, *) & & \\ & \swarrow p_1 & \downarrow \phi \mid \exists! & \searrow p_2 & \\ (X, x_0) & \xleftarrow{\text{pr}_1} & (X \times Y, (x_0, y_0)) & \xrightarrow{\text{pr}_2} & (Y, y_0) \end{array}$$

commute, being uniquely determined by the conditions

$$\begin{aligned} \text{pr}_1 \circ \phi &= p_1, \\ \text{pr}_2 \circ \phi &= p_2 \end{aligned}$$

via

$$\phi(x) = (p_1(x), p_2(x))$$

for each $x \in P$. Note that this is indeed a morphism of pointed sets, as we have

$$\begin{aligned} \phi(*) &= (p_1(*), p_2(*)) \\ &= (x_0, y_0), \end{aligned}$$

where we have used that p_1 and p_2 are morphisms of pointed sets. \square

Proposition 2.3.1.2. Let (X, x_0) , (Y, y_0) , and (Z, z_0) be pointed sets.

1. *Functoriality.* The assignments

$$(X, x_0), (Y, y_0), ((X, x_0), (Y, y_0)) \mapsto (X \times Y, (x_0, y_0))$$

define functors

$$\begin{aligned} X \times - &: \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*, \\ - \times Y &: \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*, \\ -_1 \times -_2 &: \mathbf{Sets}_* \times \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*, \end{aligned}$$

defined in the same way as the functors of **Constructions With Sets, Item 1** of **Proposition 1.3.1.2**.

2. *Associativity*. We have an isomorphism of pointed sets

$$((X \times Y) \times Z, ((x_0, y_0), z_0)) \cong (X \times (Y \times Z), (x_0, (y_0, z_0)))$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathbf{Obj}(\mathbf{Sets}_*)$.

3. *Unitality*. We have isomorphisms of pointed sets

$$\begin{aligned} (\mathbf{pt}, \star) \times (X, x_0) &\cong (X, x_0), \\ (X, x_0) \times (\mathbf{pt}, \star) &\cong (X, x_0), \end{aligned}$$

natural in $(X, x_0) \in \mathbf{Obj}(\mathbf{Sets}_*)$.

4. *Commutativity*. We have an isomorphism of pointed sets

$$(X \times Y, (x_0, y_0)) \cong (Y \times X, (y_0, x_0)),$$

natural in $(X, x_0), (Y, y_0) \in \mathbf{Obj}(\mathbf{Sets}_*)$.

5. *Symmetric Monoidality*. The triple $(\mathbf{Sets}_*, \times, (\mathbf{pt}, \star))$ is a symmetric monoidal category.

Proof. **Item 1, Functoriality**: This is a special case of functoriality of limits, ??, ?? of ??.

Item 2, Associativity: This follows from **Constructions With Sets, Item 3** of **Proposition 1.3.1.2**.

Item 3, Unitality: This follows from **Constructions With Sets, Item 4** of **Proposition 1.3.1.2**.

Item 4, Commutativity: This follows from **Constructions With Sets, Item 5** of **Proposition 1.3.1.2**.

Item 5, Symmetric Monoidality: This follows from **Constructions With Sets, Item 12** of **Proposition 1.3.1.2**. \square

2.4 Pullbacks

Let (X, x_0) , (Y, y_0) , and (Z, z_0) be pointed sets and let $f: (X, x_0) \rightarrow (Z, z_0)$ and $g: (Y, y_0) \rightarrow (Z, z_0)$ be morphisms of pointed sets.

Definition 2.4.1.1. The **pullback of (X, x_0) and (Y, y_0) over (Z, z_0) along (f, g)** is the pair consisting of:

- *The Limit.* The pointed set $(X \times_Z Y, (x_0, y_0))$.
- *The Cone.* The morphisms of pointed sets

$$\begin{aligned} \text{pr}_1: (X \times_Z Y, (x_0, y_0)) &\rightarrow (X, x_0), \\ \text{pr}_2: (X \times_Z Y, (x_0, y_0)) &\rightarrow (Y, y_0) \end{aligned}$$

defined by

$$\begin{aligned} \text{pr}_1(x, y) &\stackrel{\text{def}}{=} x, \\ \text{pr}_2(x, y) &\stackrel{\text{def}}{=} y \end{aligned}$$

for each $(x, y) \in X \times_Z Y$.

Proof. We claim that $X \times_Z Y$ is the categorical pullback of (X, x_0) and (Y, y_0) over (Z, z_0) with respect to (f, g) in Sets_* . First we need to check that the relevant pullback diagram commutes, i.e. that we have

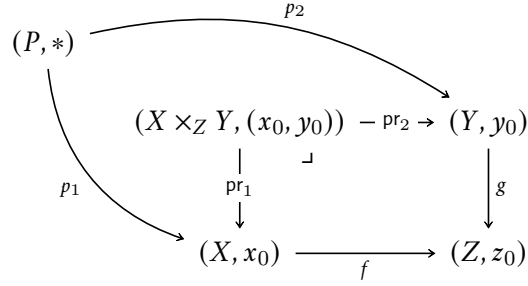
$$\begin{array}{ccc} (X \times_Z Y, (x_0, y_0)) & \xrightarrow{\text{pr}_2} & (Y, y_0) \\ \text{pr}_1 \downarrow & & \downarrow g \\ (X, x_0) & \xrightarrow{f} & (Z, z_0). \end{array}$$

$f \circ \text{pr}_1 = g \circ \text{pr}_2,$

Indeed, given $(x, y) \in X \times_Z Y$, we have

$$\begin{aligned} [f \circ \text{pr}_1](x, y) &= f(\text{pr}_1(x, y)) \\ &= f(x) \\ &= g(y) \\ &= g(\text{pr}_2(x, y)) \\ &= [g \circ \text{pr}_2](x, y), \end{aligned}$$

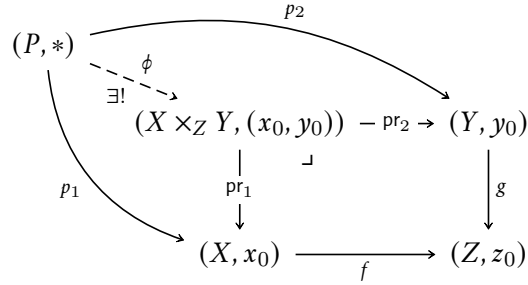
where $f(x) = g(y)$ since $(x, y) \in X \times_Z Y$. Next, we prove that $X \times_Z Y$ satisfies the universal property of the pullback. Suppose we have a diagram of the form



in \mathbf{Sets}_* . Then there exists a unique morphism of pointed sets

$$\phi: (P, *) \rightarrow (X \times_Z Y, (x_0, y_0))$$

making the diagram



commute, being uniquely determined by the conditions

$$\text{pr}_1 \circ \phi = p_1,$$

$$\text{pr}_2 \circ \phi = p_2$$

via

$$\phi(x) = (p_1(x), p_2(x))$$

for each $x \in P$, where we note that $(p_1(x), p_2(x)) \in X \times Y$ indeed lies in $X \times_Z Y$ by the condition

$$f \circ p_1 = g \circ p_2,$$

which gives

$$f(p_1(x)) = g(p_2(x))$$

for each $x \in P$, so that $(p_1(x), p_2(x)) \in X \times_Z Y$. Lastly, we note that ϕ is indeed a morphism of pointed sets, as we have

$$\begin{aligned}\phi(*) &= (p_1(*), p_2(*)) \\ &= (x_0, y_0),\end{aligned}$$

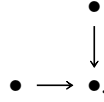
where we have used that p_1 and p_2 are morphisms of pointed sets. \square

Proposition 2.4.1.2. Let (X, x_0) , (Y, y_0) , (Z, z_0) , and (A, a_0) be pointed sets.

1. *Functoriality.* The assignment $(X, Y, Z, f, g) \mapsto X \times_{f,Z,g} Y$ defines a functor

$$-_1 \times_{-3} -_1 : \text{Fun}(\mathcal{P}, \text{Sets}_*) \rightarrow \text{Sets}_*,$$

where \mathcal{P} is the category that looks like this:



In particular, the action on morphisms of $-_1 \times_{-3} -_1$ is given by sending a morphism

$$\begin{array}{ccccc} X \times_Z Y & \xrightarrow{\quad} & Y & & \\ \downarrow & \lrcorner & \downarrow g & \searrow \psi & \\ & X' \times_{Z'} Y' & \xrightarrow{\quad} & Y' & \\ \downarrow & \lrcorner & \downarrow & & \downarrow g' \\ X & \xrightarrow{f} & Z & \searrow \chi & \\ \downarrow \phi & & \downarrow & & \downarrow \\ & X' & \xrightarrow{f'} & Z' & \end{array}$$

in $\text{Fun}(\mathcal{P}, \text{Sets}_*)$ to the morphism of pointed sets

$$\xi : (X \times_Z Y, (x_0, y_0)) \xrightarrow{\exists!} (X' \times_{Z'} Y', (x'_0, y'_0))$$

given by

$$\xi(x, y) \stackrel{\text{def}}{=} (\phi(x), \psi(y))$$

for each $(x, y) \in X \times_Z Y$, which is the unique morphism of pointed sets making the diagram

$$\begin{array}{ccccc}
 X \times_Z Y & \xrightarrow{\quad} & Y & & \\
 \downarrow & \searrow \lrcorner & \downarrow g & \searrow \psi & \\
 & X' \times_{Z'} Y' & \xrightarrow{\quad} & Y' & \\
 \downarrow & \lrcorner & \downarrow & & \downarrow g' \\
 X & \xrightarrow{f} & Z & & \\
 \searrow \phi & & \searrow \chi & & \\
 & X' & \xrightarrow{f'} & Z' &
 \end{array}$$

commute.

2. *Associativity.* Given a diagram

$$\begin{array}{ccccc}
 X & & Y & & Z \\
 & \searrow f & & \searrow g & \\
 & & W & & \\
 & & & & V \\
 & & & \searrow h & \\
 & & & & Z
 \end{array}$$

in \mathbf{Sets}_* , we have isomorphisms of pointed sets

$$(X \times_W Y) \times_V Z \cong (X \times_W Y) \times_Y (Y \times_V Z) \cong X \times_W (Y \times_V Z),$$

where these pullbacks are built as in the diagrams

$$\begin{array}{ccc}
 \begin{array}{c} (X \times_W Y) \times_Y Z \\ \swarrow \quad \searrow \\ X \times_W Y \quad Y \\ \swarrow \quad \searrow \quad \swarrow \quad \searrow \\ X \quad W \quad Y \quad V \quad Z \\ \downarrow f \quad \downarrow g \quad \downarrow h \quad \downarrow k \\ W \quad V \end{array} &
 \begin{array}{c} (X \times_W Y) \times_Y (Y \times_V Z) \\ \swarrow \quad \searrow \\ X \times_W Y \quad Y \times_V Z \\ \swarrow \quad \searrow \quad \swarrow \quad \searrow \\ X \quad W \quad Y \quad V \quad Z \\ \downarrow f \quad \downarrow g \quad \downarrow h \quad \downarrow k \\ W \quad V \end{array} &
 \begin{array}{c} X \times_W (Y \times_V Z) \\ \swarrow \quad \searrow \\ X \quad Y \times_V Z \\ \swarrow \quad \searrow \quad \swarrow \quad \searrow \\ X \quad W \quad Y \quad V \quad Z \\ \downarrow f \quad \downarrow g \quad \downarrow h \quad \downarrow k \\ W \quad V \end{array}
 \end{array}$$

3. *Unitality.* We have isomorphisms of pointed sets

$$\begin{array}{ccc}
 \begin{array}{ccc} A & \xlongequal{\quad} & A \\ \downarrow f & \lrcorner & \downarrow f \\ X & \xlongequal{\quad} & X \end{array} &
 \begin{array}{c} X \times_X A \cong A, \\ A \times_X X \cong A, \end{array} &
 \begin{array}{ccc} A & \xrightarrow{f} & X \\ \parallel \lrcorner & & \parallel \\ X & \xrightarrow{f} & X. \end{array}
 \end{array}$$

4. *Commutativity.* We have an isomorphism of pointed sets

$$\begin{array}{ccc}
 A \times_X B & \longrightarrow & B \\
 \downarrow \lrcorner & & \downarrow g \\
 A & \xrightarrow{f} & X,
 \end{array}
 \quad A \times_X B \cong B \times_X A
 \quad
 \begin{array}{ccc}
 B \times_X A & \longrightarrow & A \\
 \downarrow \lrcorner & & \downarrow f \\
 B & \xrightarrow{g} & X.
 \end{array}$$

5. *Interaction With Products.* We have an isomorphism of pointed sets

$$\begin{array}{ccc}
 X \times Y & \longrightarrow & Y \\
 \downarrow \lrcorner & & \downarrow !_Y \\
 X & \xrightarrow{!_X} & \text{pt.}
 \end{array}
 \quad X \times_{\text{pt}} Y \cong X \times Y,$$

6. *Symmetric Monoidality.* The triple $(\text{Sets}_*, \times_X, X)$ is a symmetric monoidal category.

Proof. **Item 1, Functoriality:** This is a special case of functoriality of co/limits, ??, ?? of ??, with the explicit expression for ξ following from the commutativity of the cube pullback diagram.

Item 2, Associativity: This follows from **Constructions With Sets**, Item 2 of Proposition 1.4.1.3.

Item 3, Unitality: This follows from **Constructions With Sets**, Item 3 of Proposition 1.4.1.3.

Item 4, Commutativity: This follows from **Constructions With Sets**, Item 4 of Proposition 1.4.1.3.

Item 5, Interaction With Products: This follows from **Constructions With Sets**, Item 6 of Proposition 1.4.1.3.

Item 6, Symmetric Monoidality: This follows from **Constructions With Sets**, Item 7 of Proposition 1.4.1.3. \square

2.5 Equalisers

Let $f, g: (X, x_0) \rightrightarrows (Y, y_0)$ be morphisms of pointed sets.

Definition 2.5.1.1. The **equaliser** of (f, g) is the pair consisting of:

- *The Limit.* The pointed set $(\text{Eq}(f, g), x_0)$.
- *The Cone.* The morphism of pointed sets

$$\text{eq}(f, g): (\text{Eq}(f, g), x_0) \hookrightarrow (X, x_0)$$

given by the canonical inclusion $\text{eq}(f, g) \hookrightarrow \text{Eq}(f, g) \hookrightarrow X$.

Proof. We claim that $(\text{Eq}(f, g), x_0)$ is the categorical equaliser of f and g in Sets_* . First we need to check that the relevant equaliser diagram commutes, i.e. that we have

$$f \circ \text{eq}(f, g) = g \circ \text{eq}(f, g),$$

which indeed holds by the definition of the set $\text{Eq}(f, g)$. Next, we prove that $\text{Eq}(f, g)$ satisfies the universal property of the equaliser. Suppose we have a diagram of the form

$$\begin{array}{ccc} (\text{Eq}(f, g), x_0) & \xrightarrow{\text{eq}(f, g)} & (X, x_0) \xrightleftharpoons[g]{f} (Y, y_0) \\ & \nearrow e & \\ (E, *) & & \end{array}$$

in Sets_* . Then there exists a unique morphism of pointed sets

$$\phi: (E, *) \rightarrow (\text{Eq}(f, g), x_0)$$

making the diagram

$$\begin{array}{ccc} (\text{Eq}(f, g), x_0) & \xrightarrow{\text{eq}(f, g)} & (X, x_0) \xrightleftharpoons[g]{f} (Y, y_0) \\ \uparrow \phi \exists! & \nearrow e & \\ (E, *) & & \end{array}$$

commute, being uniquely determined by the condition

$$\text{eq}(f, g) \circ \phi = e$$

via

$$\phi(x) = e(x)$$

for each $x \in E$, where we note that $e(x) \in A$ indeed lies in $\text{Eq}(f, g)$ by the condition

$$f \circ e = g \circ e,$$

which gives

$$f(e(x)) = g(e(x))$$

for each $x \in E$, so that $e(x) \in \text{Eq}(f, g)$. Lastly, we note that ϕ is indeed a morphism of pointed sets, as we have

$$\begin{aligned} \phi(*) &= e(*) \\ &= x_0, \end{aligned}$$

where we have used that e is a morphism of pointed sets. \square

Proposition 2.5.1.2. Let (X, x_0) and (Y, y_0) be pointed sets and let $f, g, h: (X, x_0) \rightarrow (Y, y_0)$ be morphisms of pointed sets.

1. *Associativity.* We have isomorphisms of pointed sets

$$\underbrace{\text{Eq}(f \circ \text{eq}(g, h), g \circ \text{eq}(g, h))}_{=\text{Eq}(f \circ \text{eq}(g, h), h \circ \text{eq}(g, h))} \cong \text{Eq}(f, g, h) \cong \underbrace{\text{Eq}(f \circ \text{eq}(f, g), h \circ \text{eq}(f, g))}_{=\text{Eq}(g \circ \text{eq}(f, g), h \circ \text{eq}(f, g))},$$

where $\text{Eq}(f, g, h)$ is the limit of the diagram

$$(X, x_0) \begin{array}{c} \xrightarrow{f} \\ \xrightarrow[g]{h} \end{array} (Y, y_0)$$

in Sets_* , being explicitly given by

$$\text{Eq}(f, g, h) \cong \{a \in A \mid f(a) = g(a) = h(a)\}.$$

2. *Unitality.* We have an isomorphism of pointed sets

$$\text{Eq}(f, f) \cong X.$$

3. *Commutativity.* We have an isomorphism of pointed sets

$$\text{Eq}(f, g) \cong \text{Eq}(g, f).$$

Proof. **Item 1, Associativity:** This follows from [Constructions With Sets, Item 1 of Proposition 1.5.1.2](#).

Item 2, Unitality: This follows from [Constructions With Sets, Item 4 of Proposition 1.5.1.2](#).

Item 3, Commutativity: This follows from [Constructions With Sets, Item 5 of Proposition 1.5.1.2](#). \square

3 Colimits of Pointed Sets

3.1 The Initial Pointed Set

Definition 3.1.1.1. The **initial pointed set** is the pair $((pt, \star), \{\iota_X\}_{(X, x_0) \in \text{Obj}(\text{Sets}_*)})$ consisting of:

- *The Limit.* The pointed set (pt, \star) .
- *The Cone.* The collection of morphisms of pointed sets

$$\{\iota_X : (pt, \star) \rightarrow (X, x_0)\}_{(X, x_0) \in \text{Obj}(\text{Sets})}$$

defined by

$$\iota_X(\star) \stackrel{\text{def}}{=} x_0.$$

Proof. We claim that (pt, \star) is the initial object of Sets_* . Indeed, suppose we have a diagram of the form

$$(pt, \star) \quad (X, x_0)$$

in Sets_* . Then there exists a unique morphism of pointed sets

$$\phi : (pt, \star) \rightarrow (X, x_0)$$

making the diagram

$$(pt, \star) \xrightarrow[\exists!]{\phi} (X, x_0)$$

commute, namely ι_X . □

3.2 Coproducts of Families of Pointed Sets

Let $\{(X_i, x_0^i)\}_{i \in I}$ be a family of pointed sets.

Definition 3.2.1.1. The **coproduct of the family** $\{(X_i, x_0^i)\}_{i \in I}$, also called their **wedge sum**, is the pair consisting of:

- *The Colimit.* The pointed set $(\bigvee_{i \in I} X_i, p_0)$ consisting of:

- *The Underlying Set.* The set $\bigvee_{i \in I} X_i$ defined by

$$\bigvee_{i \in I} X_i \stackrel{\text{def}}{=} (\coprod_{i \in I} X_i) / \sim,$$

where \sim is the equivalence relation on $\coprod_{i \in I} X_i$ given by declaring

$$(i, x_0^i) \sim (j, x_0^j)$$

for each $i, j \in I$.

- *The Basepoint.* The element p_0 of $\bigvee_{i \in I} X_i$ defined by

$$\begin{aligned} p_0 &\stackrel{\text{def}}{=} [(i, x_0^i)] \\ &= [(j, x_0^j)] \end{aligned}$$

for any $i, j \in I$.

- *The Cocone.* The collection

$$\left\{ \text{inj}_i : (X_i, x_0^i) \rightarrow \left(\bigvee_{i \in I} X_i, p_0 \right) \right\}_{i \in I}$$

of morphism of pointed sets given by

$$\text{inj}_i(x) \stackrel{\text{def}}{=} (i, x)$$

for each $x \in X_i$ and each $i \in I$.

Proof. We claim that $(\bigvee_{i \in I} X_i, p_0)$ is the categorical coproduct of $\{(X_i, x_0^i)\}_{i \in I}$ in Sets_* . Indeed, suppose we have, for each $i \in I$, a diagram of the form

$$\begin{array}{ccc} & & (C, *) \\ & \nearrow \iota_i & \\ (X_i, x_0^i) & \xrightarrow{\text{inj}_i} & \left(\bigvee_{i \in I} X_i, p_0 \right) \end{array}$$

in Sets_* . Then there exists a unique morphism of pointed sets

$$\phi : \left(\bigvee_{i \in I} X_i, p_0 \right) \rightarrow (C, *)$$

making the diagram

$$\begin{array}{ccc}
 & & (C, *) \\
 & \nearrow \iota_i & \uparrow \exists! \phi \\
 (X_i, x_0^i) & \xrightarrow{\text{inj}_i} & (\bigvee_{i \in I} X_i, p_0)
 \end{array}$$

commute, being uniquely determined by the condition $\phi \circ \text{inj}_i = \iota_i$ for each $i \in I$ via

$$\phi([(i, x)]) = \iota_i(x)$$

for each $[(i, x)] \in \bigvee_{i \in I} X_i$, where we note that ϕ is indeed a morphism of pointed sets, as we have

$$\begin{aligned}
 \phi(p_0) &= \iota_i([(i, x_0^i)]) \\
 &= *,
 \end{aligned}$$

as ι_i is a morphism of pointed sets. □

Proposition 3.2.1.2. Let $\{(X_i, x_0^i)\}_{i \in I}$ be a family of pointed sets.

1. *Functoriality.* The assignment $\{(X_i, x_0^i)\}_{i \in I} \mapsto (\bigvee_{i \in I} X_i, p_0)$ defines a functor

$$\bigvee_{i \in I} : \text{Fun}(I_{\text{disc}}, \text{Sets}_*) \rightarrow \text{Sets}_*.$$

Proof. **Item 1, Functoriality:** This follows from ??, ?? of??. □

3.3 Coproducts

Let (X, x_0) and (Y, y_0) be pointed sets.

Definition 3.3.1.1. The **coproduct of (X, x_0) and (Y, y_0)** , also called their **wedge sum**, is the pair consisting of:

- *The Colimit.* The pointed set $(X \vee Y, p_0)$ consisting of:

- *The Underlying Set.* The set $X \vee Y$ defined by

$$\begin{aligned}
 (X \vee Y, p_0) &\stackrel{\text{def}}{=} (X, x_0) \amalg (Y, y_0) \\
 &\cong (X \amalg_{\text{pt}} Y, p_0) \\
 &\cong (X \amalg Y / \sim, p_0),
 \end{aligned}
 \quad
 \begin{array}{ccc}
 X \vee Y & \longleftarrow & Y \\
 \uparrow \ulcorner & & \uparrow [y_0] \\
 X & \xleftarrow{[x_0]} & \text{pt}
 \end{array}$$

where \sim is the equivalence relation on $X \amalg Y$ obtained by declaring $(0, x_0) \sim (1, y_0)$.

- *The Basepoint.* The element p_0 of $X \vee Y$ defined by

$$\begin{aligned}
 p_0 &\stackrel{\text{def}}{=} [(0, x_0)] \\
 &= [(1, y_0)].
 \end{aligned}$$

- *The Cocone.* The morphisms of pointed sets

$$\begin{aligned}
 \text{inj}_1 &: (X, x_0) \rightarrow (X \vee Y, p_0), \\
 \text{inj}_2 &: (Y, y_0) \rightarrow (X \vee Y, p_0),
 \end{aligned}$$

given by

$$\begin{aligned}
 \text{inj}_1(x) &\stackrel{\text{def}}{=} [(0, x)], \\
 \text{inj}_2(y) &\stackrel{\text{def}}{=} [(1, y)],
 \end{aligned}$$

for each $x \in X$ and each $y \in Y$.

Proof. We claim that $(X \vee Y, p_0)$ is the categorical coproduct of (X, x_0) and (Y, y_0) in \mathbf{Sets}_* . Indeed, suppose we have a diagram of the form

$$\begin{array}{ccccc}
 & & (C, *) & & \\
 & \nearrow \iota_X & & \nwarrow \iota_Y & \\
 (X, x_0) & \xrightarrow{\text{inj}_X} & (X \vee Y, p_0) & \xleftarrow{\text{inj}_Y} & (Y, y_0)
 \end{array}$$

in \mathbf{Sets} . Then there exists a unique morphism of pointed sets

$$\phi: (X \vee Y, p_0) \rightarrow (C, *)$$

making the diagram

$$\begin{array}{ccccc}
 & & (C, *) & & \\
 & \nearrow \iota_X & \uparrow \phi \uparrow \exists! & \nwarrow \iota_Y & \\
 (X, x_0) & \xrightarrow{\text{inj}_X} & (X \vee Y, p_0) & \xleftarrow{\text{inj}_Y} & (Y, y_0)
 \end{array}$$

commute, being uniquely determined by the conditions

$$\begin{aligned}
 \phi \circ \text{inj}_X &= \iota_X, \\
 \phi \circ \text{inj}_Y &= \iota_Y
 \end{aligned}$$

via

$$\phi(z) = \begin{cases} \iota_X(x) & \text{if } z = [(0, x)] \text{ with } x \in X, \\ \iota_Y(y) & \text{if } z = [(1, y)] \text{ with } y \in Y \end{cases}$$

for each $z \in X \vee Y$, where we note that ϕ is indeed a morphism of pointed sets, as we have

$$\begin{aligned}
 \phi(p_0) &= \iota_X([(0, x_0)]) \\
 &= \iota_Y([(1, y_0)]) \\
 &= *,
 \end{aligned}$$

as ι_X and ι_Y are morphisms of pointed sets. □

Proposition 3.3.1.2. Let (X, x_0) and (Y, y_0) be pointed sets.

1. *Functoriality.* The assignments

$$(X, x_0), (Y, y_0), ((X, x_0), (Y, y_0)) \mapsto (X \vee Y, p_0)$$

define functors

$$\begin{aligned}
 X \vee - &: \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*, \\
 - \vee Y &: \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*, \\
 -_1 \vee -_2 &: \mathbf{Sets}_* \times \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*.
 \end{aligned}$$

2. *Associativity.* We have an isomorphism of pointed sets

$$(X \vee Y) \vee Z \cong X \vee (Y \vee Z),$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathbf{Sets}_*$.

3. *Unitality.* We have isomorphisms of pointed sets

$$\begin{aligned} (\text{pt}, *) \vee (X, x_0) &\cong (X, x_0), \\ (X, x_0) \vee (\text{pt}, *) &\cong (X, x_0), \end{aligned}$$

natural in $(X, x_0) \in \mathbf{Sets}_*$.

4. *Commutativity.* We have an isomorphism of pointed sets

$$X \vee Y \cong Y \vee X,$$

natural in $(X, x_0), (Y, y_0) \in \mathbf{Sets}_*$.

5. *Symmetric Monoidality.* The triple $(\mathbf{Sets}_*, \vee, \text{pt})$ is a symmetric monoidal category.

6. *The Fold Map.* We have a natural transformation

$$\nabla : \vee \circ \Delta_{\mathbf{Sets}_*}^{\text{Cats}} \Rightarrow \text{id}_{\mathbf{Sets}_*},$$

called the **fold map**, whose component

$$\nabla_X : X \vee X \rightarrow X$$

at X is given by

$$\nabla_X(p) \stackrel{\text{def}}{=} \begin{cases} x & \text{if } p = [(0, x)], \\ x & \text{if } p = [(1, x)] \end{cases}$$

for each $p \in X \vee X$.

Proof. **Item 1, Functoriality:** This follows from ??, ?? of ??.

Item 2, Associativity: Clear.

Item 3, Unitality: Clear.

Item 4, Commutativity: Clear.

Item 5, Symmetric Monoidality: Omitted.

Item 6, The Fold Map: Naturality for the transformation ∇ is the statement that, given a morphism of pointed sets $f: (X, x_0) \rightarrow (Y, y_0)$, we have

$$\nabla_Y \circ (f \vee f) = f \circ \nabla_X,$$

$$\begin{array}{ccc} X \vee X & \xrightarrow{\nabla_X} & X \\ f \vee f \downarrow & & \downarrow f \\ Y \vee Y & \xrightarrow{\nabla_Y} & Y. \end{array}$$

Indeed, we have

$$\begin{aligned} [\nabla_Y \circ (f \vee f)]([(i, x)]) &= \nabla_Y([(i, f(x))]) \\ &= f(x) \\ &= f(\nabla_X([(i, x)])) \\ &= [f \circ \nabla_X]([(i, x)]) \end{aligned}$$

for each $[(i, x)] \in X \vee X$, and thus ∇ is indeed a natural transformation. \square

3.4 Pushouts

Let (X, x_0) , (Y, y_0) , and (Z, z_0) be pointed sets and let $f: (Z, z_0) \rightarrow (X, x_0)$ and $g: (Z, z_0) \rightarrow (Y, y_0)$ be morphisms of pointed sets.

Definition 3.4.1.1. The **pushout of (X, x_0) and (Y, y_0) over (Z, z_0) along (f, g)** is the pair consisting of:

- *The Colimit.* The pointed set $(X \coprod_{f,Z,g} Y, p_0)$, where:
 - The set $X \coprod_{f,Z,g} Y$ is the pushout (of unpointed sets) of X and Y over Z with respect to f and g ;
 - We have $p_0 = [x_0] = [y_0]$.
- *The Cocone.* The morphisms of pointed sets

$$\begin{aligned} \text{inj}_1: (X, x_0) &\rightarrow (X \coprod_Z Y, p_0), \\ \text{inj}_2: (Y, y_0) &\rightarrow (X \coprod_Z Y, p_0) \end{aligned}$$

given by

$$\begin{aligned} \text{inj}_1(x) &\stackrel{\text{def}}{=} [(0, x)] \\ \text{inj}_2(y) &\stackrel{\text{def}}{=} [(1, y)] \end{aligned}$$

for each $x \in X$ and each $y \in Y$.

Proof. Firstly, we note that indeed $[x_0] = [y_0]$, as we have

$$\begin{aligned} x_0 &= f(z_0), \\ y_0 &= g(z_0) \end{aligned}$$

since f and g are morphisms of pointed sets, with the relation \sim on $X \amalg_Z Y$ then identifying $x_0 = f(z_0) \sim g(z_0) = y_0$.

We now claim that $(X \amalg_Z Y, p_0)$ is the categorical pushout of (X, x_0) and (Y, y_0) over (Z, z_0) with respect to (f, g) in \mathbf{Sets}_* . First we need to check that the relevant pushout diagram commutes, i.e. that we have

$$\begin{array}{ccc} (X \amalg_Z Y, p_0) & \xleftarrow{\text{inj}_2} & (Y, y_0) \\ \text{inj}_1 \uparrow & & \uparrow g \\ (X, x_0) & \xleftarrow{f} & (Z, z_0). \end{array}$$

$\text{inj}_1 \circ f = \text{inj}_2 \circ g,$

Indeed, given $z \in Z$, we have

$$\begin{aligned} [\text{inj}_1 \circ f](z) &= \text{inj}_1(f(z)) \\ &= [(0, f(z))] \\ &= [(1, g(z))] \\ &= \text{inj}_2(g(z)) \\ &= [\text{inj}_2 \circ g](z), \end{aligned}$$

where $[(0, f(z))] = [(1, g(z))]$ by the definition of the relation \sim on $X \amalg Y$ (the coproduct of unpointed sets of X and Y). Next, we prove that $X \amalg_Z Y$ satisfies the universal property of the pushout. Suppose we have a diagram of the form

$$\begin{array}{ccccc} & & & & (P, *) \\ & & & \nwarrow \iota_2 & \\ & & (X \amalg_Z Y, p_0) & \xleftarrow{\text{inj}_2} & (Y, y_0) \\ & & \uparrow \text{inj}_1 & \uparrow g & \\ & & (X, x_0) & \xleftarrow{f} & (Z, z_0) \\ & \nearrow \iota_1 & & & \end{array}$$

in \mathbf{Sets}_* . Then there exists a unique morphism of pointed sets

$$\phi: (X \amalg_Z Y, p_0) \rightarrow (P, *)$$

making the diagram

$$\begin{array}{ccc}
 (P, *) & \xleftarrow{\iota_2} & (X \amalg_Z Y, p_0) \xleftarrow{\text{inj}_2} (Y, y_0) \\
 \uparrow \iota_1 & \nwarrow \phi \quad \exists! & \uparrow g \\
 (X, x_0) & \xleftarrow{f} & (Z, z_0) \\
 & & \uparrow \text{inj}_1
 \end{array}$$

commute, being uniquely determined by the conditions

$$\phi \circ \text{inj}_1 = \iota_1,$$

$$\phi \circ \text{inj}_2 = \iota_2$$

via

$$\phi(p) = \begin{cases} \iota_1(x) & \text{if } x = [(0, x)], \\ \iota_2(y) & \text{if } x = [(1, y)] \end{cases}$$

for each $p \in X \amalg_Z Y$, where the well-definedness of ϕ is proven in the same way as in the proof of **Constructions With Sets, Definition 2.4.1.1**. Finally, we show that ϕ is indeed a morphism of pointed sets, as we have

$$\begin{aligned}
 \phi(p_0) &= \phi([(0, x_0)]) \\
 &= \iota_1(x_0) \\
 &= *,
 \end{aligned}$$

or alternatively

$$\begin{aligned}
 \phi(p_0) &= \phi([(1, y_0)]) \\
 &= \iota_2(y_0) \\
 &= *,
 \end{aligned}$$

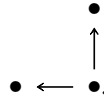
where we use that ι_1 (resp. ι_2) is a morphism of pointed sets. \square

Proposition 3.4.1.2. Let (X, x_0) , (Y, y_0) , (Z, z_0) , and (A, a_0) be pointed sets.

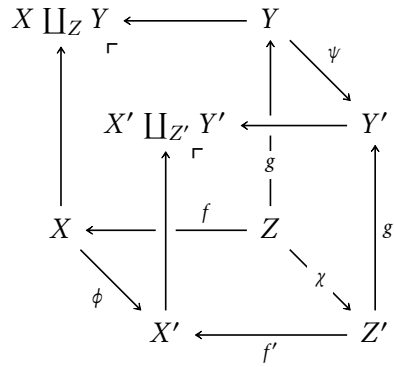
1. *Functoriality.* The assignment $(X, Y, Z, f, g) \mapsto X \amalg_{f,Z,g} Y$ defines a functor

$$-_1 \amalg_{-3} -_1 : \text{Fun}(\mathcal{P}, \text{Sets}) \rightarrow \text{Sets}_*,$$

where \mathcal{P} is the category that looks like this:



In particular, the action on morphisms of $-_1 \amalg_{-3} -_1$ is given by sending a morphism



in $\text{Fun}(\mathcal{P}, \text{Sets}_*)$ to the morphism of pointed sets

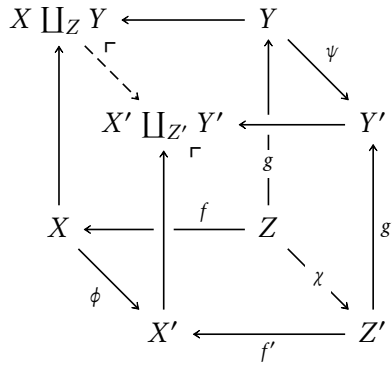
$$\xi : (X \amalg_Z Y, p_0) \xrightarrow{\exists!} (X' \amalg_{Z'} Y', p'_0)$$

given by

$$\xi(p) \stackrel{\text{def}}{=} \begin{cases} \phi(x) & \text{if } p = [(0, x)], \\ \psi(y) & \text{if } p = [(1, y)] \end{cases}$$

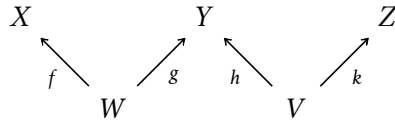
for each $p \in X \amalg_Z Y$, which is the unique morphism of pointed sets

making the diagram



commute.

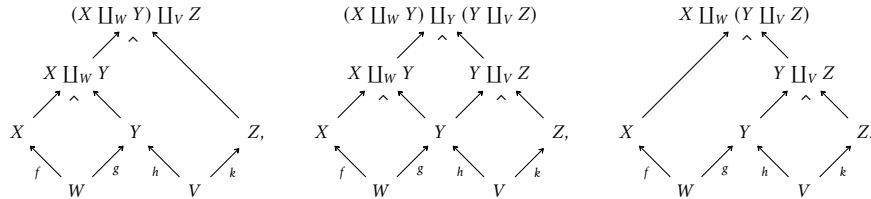
2. *Associativity.* Given a diagram



in Sets, we have isomorphisms of pointed sets

$$(X \amalg_W Y) \amalg_V Z \cong (X \amalg_W Y) \amalg_Y (Y \amalg_V Z) \cong X \amalg_W (Y \amalg_V Z),$$

where these pushbacks are built as in the diagrams



3. *Unitality.* We have isomorphisms of sets

$$\begin{array}{ccc} A & \xlongequal{\quad} & A \\ \uparrow f & \lrcorner & \uparrow f \\ X & \xlongequal{\quad} & X \end{array} \quad \begin{array}{l} X \amalg_X A \cong A, \\ A \amalg_X X \cong A, \end{array} \quad \begin{array}{ccc} A & \xleftarrow{f} & X \\ \parallel \lrcorner & & \parallel \\ X & \xleftarrow{f} & X. \end{array}$$

4. *Commutativity.* We have an isomorphism of sets

$$\begin{array}{ccc}
 X \amalg_Z Y & \xleftarrow{\quad} & Y \\
 \uparrow \scriptstyle \ulcorner & & \uparrow \scriptstyle g \\
 X & \xleftarrow{\scriptstyle f} & Z,
 \end{array}
 \quad X \amalg_Z Y \cong Y \amalg_Z X
 \quad
 \begin{array}{ccc}
 Y \amalg_Z X & \xleftarrow{\quad} & X \\
 \uparrow \scriptstyle \ulcorner & & \uparrow \scriptstyle f \\
 Y & \xleftarrow{\scriptstyle g} & Z.
 \end{array}$$

5. *Interaction With Coproducts.* We have

$$X \amalg_{\text{pt}} Y \cong X \vee Y,
 \quad
 \begin{array}{ccc}
 X \vee Y & \xleftarrow{\quad} & Y \\
 \uparrow \scriptstyle \ulcorner & & \uparrow \scriptstyle [y_0] \\
 X & \xleftarrow{\scriptstyle [x_0]} & \text{pt.}
 \end{array}$$

6. *Symmetric Monoidality.* The triple $(\text{Sets}_*, \amalg_X, (X, x_0))$ is a symmetric monoidal category.

Proof. **Item 1, Functoriality:** This is a special case of functoriality of co/limits, ??, ?? of ??, with the explicit expression for ξ following from the commutativity of the cube pushout diagram.

Item 2, Associativity: This follows from **Constructions With Sets, Item 2 of Proposition 2.4.1.4.**

Item 3, Unitality: This follows from **Constructions With Sets, Item 3 of Proposition 2.4.1.4.**

Item 4, Commutativity: This follows from **Constructions With Sets, Item 4 of Proposition 2.4.1.4.**

Item 5, Interaction With Coproducts: Clear.

Item 6, Symmetric Monoidality: Omitted. □

3.5 Coequalisers

Let $f, g: (X, x_0) \rightrightarrows (Y, y_0)$ be morphisms of pointed sets.

Definition 3.5.1.1. The **coequaliser of (f, g)** is the pointed set $(\text{CoEq}(f, g), [y_0])$.

Proof. We claim that $(\text{CoEq}(f, g), [y_0])$ is the categorical coequaliser of f and g in Sets_* . First we need to check that the relevant coequaliser diagram commutes, i.e. that we have

$$\text{coeq}(f, g) \circ f = \text{coeq}(f, g) \circ g.$$

Indeed, we have

$$\begin{aligned}
 [\text{coeq}(f, g) \circ f](x) &\stackrel{\text{def}}{=} [\text{coeq}(f, g)](f(x)) \\
 &\stackrel{\text{def}}{=} [f(x)] \\
 &= [g(x)] \\
 &\stackrel{\text{def}}{=} [\text{coeq}(f, g)](g(x)) \\
 &\stackrel{\text{def}}{=} [\text{coeq}(f, g) \circ g](x)
 \end{aligned}$$

for each $x \in X$. Next, we prove that $\text{CoEq}(f, g)$ satisfies the universal property of the coequaliser. Suppose we have a diagram of the form

$$\begin{array}{ccc}
 (X, x_0) & \xrightleftharpoons[g]{f} & (Y, y_0) \xrightarrow{\text{coeq}(f, g)} (\text{CoEq}(f, g), [y_0]) \\
 & & \searrow c \\
 & & (C, *)
 \end{array}$$

in Sets. Then, since $c(f(a)) = c(g(a))$ for each $a \in A$, it follows from [Equivalence Relations and Apartness Relations](#), [Items 4 and 5 of Proposition 5.2.1.3](#) that there exists a unique map $\phi: \text{CoEq}(f, g) \xrightarrow{\exists!} C$ making the diagram

$$\begin{array}{ccc}
 (X, x_0) & \xrightleftharpoons[g]{f} & (Y, y_0) \xrightarrow{\text{coeq}(f, g)} (\text{CoEq}(f, g), [y_0]) \\
 & & \searrow c \quad \downarrow \phi \exists! \\
 & & (C, *)
 \end{array}$$

commute, where we note that ϕ is indeed a morphism of pointed sets since

$$\begin{aligned}
 \phi([y_0]) &= [\phi \circ \text{coeq}(f, g)]([y_0]) \\
 &= c([y_0]) \\
 &= *,
 \end{aligned}$$

where we have used that c is a morphism of pointed sets. \square

Proposition 3.5.1.2. Let (X, x_0) and (Y, y_0) be pointed sets and let $f, g, h: (X, x_0) \rightarrow (Y, y_0)$ be morphisms of pointed sets.

1. *Associativity.* We have isomorphisms of pointed sets

$$\underbrace{\text{CoEq}(\text{coeq}(f, g) \circ f, \text{coeq}(f, g) \circ h)}_{=\text{CoEq}(\text{coeq}(f, g) \circ g, \text{coeq}(f, g) \circ h)} \cong \text{CoEq}(f, g, h) \cong \underbrace{\text{CoEq}(\text{coeq}(g, h) \circ f, \text{coeq}(g, h) \circ g)}_{=\text{CoEq}(\text{coeq}(g, h) \circ f, \text{coeq}(g, h) \circ h)},$$

where $\text{CoEq}(f, g, h)$ is the colimit of the diagram

$$(X, x_0) \begin{array}{c} \xrightarrow{f} \\ \xrightarrow[-g]{} \\ \xrightarrow{h} \end{array} (Y, y_0)$$

in Sets_* .

2. *Unitality.* We have an isomorphism of pointed sets

$$\text{CoEq}(f, f) \cong B.$$

3. *Commutativity.* We have an isomorphism of pointed sets

$$\text{CoEq}(f, g) \cong \text{CoEq}(g, f).$$

Proof. *Item 1, Associativity:* This follows from [Constructions With Sets, Item 1 of Proposition 2.5.1.4](#).

Item 2, Unitality: This follows from [Constructions With Sets, Item 4 of Proposition 2.5.1.4](#).

Item 3, Commutativity: This follows from [Constructions With Sets, Item 5 of Proposition 2.5.1.4](#). \square

4 Constructions With Pointed Sets

4.1 Free Pointed Sets

Let X be a set.

Definition 4.1.1.1. The **free pointed set on X** is the pointed set X^+ consisting of:

- *The Underlying Set.* The set X^+ defined by¹¹

$$\begin{aligned} X^+ &\stackrel{\text{def}}{=} X \amalg \text{pt} \\ &\stackrel{\text{def}}{=} X \amalg \{\star\}. \end{aligned}$$

¹¹ *Further Notation:* We sometimes write \star_X for the basepoint of X^+ for clarity when there are

- *The Basepoint.* The element \star of X^+ .

Proposition 4.1.1.2. Let X be a set.

1. *Functoriality.* The assignment $X \mapsto X^+$ defines a functor

$$(-)^+: \mathbf{Sets} \rightarrow \mathbf{Sets}_*,$$

where

- *Action on Objects.* For each $X \in \mathbf{Obj}(\mathbf{Sets})$, we have

$$[(-)^+](X) \stackrel{\text{def}}{=} X^+,$$

where X^+ is the pointed set of [Definition 4.1.1.1](#);

- *Action on Morphisms.* For each morphism $f: X \rightarrow Y$ of \mathbf{Sets} , the image

$$f^+: X^+ \rightarrow Y^+$$

of f by $(-)^+$ is the map of pointed sets defined by

$$f^+(x) \stackrel{\text{def}}{=} \begin{cases} f(x) & \text{if } x \in X, \\ \star_Y & \text{if } x = \star_X. \end{cases}$$

2. *Adjointness.* We have an adjunction

$$((-)^+ \dashv \text{忘}): \mathbf{Sets} \begin{matrix} \xrightarrow{(-)^+} \\ \perp \\ \xleftarrow{\text{忘}} \end{matrix} \mathbf{Sets}_*,$$

witnessed by a bijection of sets

$$\mathbf{Sets}_*((X^+, \star_X), (Y, y_0)) \cong \mathbf{Sets}(X, Y),$$

natural in $X \in \mathbf{Obj}(\mathbf{Sets})$ and $(Y, y_0) \in \mathbf{Obj}(\mathbf{Sets}_*)$.

3. *Symmetric Strong Monoidality With Respect to Wedge Sums.* The free pointed set functor of [Item 1](#) has a symmetric strong monoidal structure

$$((-)^+, (-)^+, \coprod, (-)^+_{\perp}): (\mathbf{Sets}, \coprod, \emptyset) \rightarrow (\mathbf{Sets}_*, \vee, \text{pt}),$$

being equipped with isomorphisms of pointed sets

$$\begin{aligned} (-)_{X,Y}^+ \amalg : X^+ \vee Y^+ &\xrightarrow{\cong} (X \amalg Y)^+, \\ (-)_{\mathbb{1}}^+ \amalg : \text{pt} &\xrightarrow{\cong} \emptyset^+, \end{aligned}$$

natural in $X, Y \in \text{Obj}(\text{Sets})$.

4. *Symmetric Strong Monoidality With Respect to Smash Products.* The free pointed set functor of **Item 1** has a symmetric strong monoidal structure

$$((-)^+, (-)^{+, \times}, (-)_{\mathbb{1}}^{+, \times}) : (\text{Sets}, \times, \text{pt}) \rightarrow (\text{Sets}_*, \wedge, S^0),$$

being equipped with isomorphisms of pointed sets

$$\begin{aligned} (-)_{X,Y}^{+, \times} : X^+ \wedge Y^+ &\xrightarrow{\cong} (X \times Y)^+, \\ (-)_{\mathbb{1}}^{+, \times} : S^0 &\xrightarrow{\cong} \text{pt}^+, \end{aligned}$$

natural in $X, Y \in \text{Obj}(\text{Sets})$.

Proof. **Item 1, Functoriality:** Clear.

Item 2, Adjointness: We claim there's an adjunction $(-)^+ \dashv \text{忘}$, witnessed by a bijection of sets

$$\text{Sets}_*((X^+, \star_X), (Y, y_0)) \cong \text{Sets}(X, Y),$$

natural in $X \in \text{Obj}(\text{Sets})$ and $(Y, y_0) \in \text{Obj}(\text{Sets}_*)$.

· *Map I.* We define a map

$$\Phi_{X,Y} : \text{Sets}_*((X^+, \star_X), (Y, y_0)) \rightarrow \text{Sets}(X, Y)$$

by sending a pointed function

$$\xi : (X^+, \star_X) \rightarrow (Y, y_0)$$

to the function

$$\xi^\dagger : X \rightarrow Y$$

given by

$$\xi^\dagger(x) \stackrel{\text{def}}{=} \xi(x)$$

for each $x \in X$.

- *Map II.* We define a map

$$\Psi_{X,Y} : \text{Sets}(X, Y) \rightarrow \text{Sets}_*((X^+, \star_X), (Y, y_0))$$

given by sending a function $\xi : X \rightarrow Y$ to the pointed function

$$\xi^\dagger : (X^+, \star_X) \rightarrow (Y, y_0)$$

defined by

$$\xi^\dagger(x) \stackrel{\text{def}}{=} \begin{cases} \xi(x) & \text{if } x \in X, \\ y_0 & \text{if } x = \star_X \end{cases}$$

for each $x \in X^+$.

- *Invertibility I.* We claim that

$$\Psi_{X,Y} \circ \Phi_{X,Y} = \text{id}_{\text{Sets}_*((X^+, \star_X), (Y, y_0))},$$

which is clear.

- *Invertibility II.* We claim that

$$\Phi_{X,Y} \circ \Psi_{X,Y} = \text{id}_{\text{Sets}(X, Y)},$$

which is clear.

- *Naturality for Φ , Part I.* We need to show that, given a pointed function $g : (Y, y_0) \rightarrow (Y', y'_0)$, the diagram

$$\begin{array}{ccc} \text{Sets}_*((X^+, \star_X), (Y, y_0)) & \xrightarrow{\Phi_{X,Y}} & \text{Sets}(X, Y) \\ g_* \downarrow & & \downarrow g_* \\ \text{Sets}_*((X^+, \star_X), (Y', y'_0)) & \xrightarrow{\Phi_{X,Y'}} & \text{Sets}(X, Y') \end{array}$$

commutes. Indeed, given a pointed function

$$\xi^\dagger : (X^+, \star_X) \rightarrow (Y, y_0)$$

multiple free pointed sets involved in the current discussion.

we have

$$\begin{aligned}
 [\Phi_{X,Y'} \circ g_*](\xi) &= \Phi_{X,Y'}(g_*(\xi)) \\
 &= \Phi_{X,Y'}(g \circ \xi) \\
 &= g \circ \xi \\
 &= g \circ \Phi_{X,Y'}(\xi) \\
 &= g_*(\Phi_{X,Y'}(\xi)) \\
 &= [g_* \circ \Phi_{X,Y'}](\xi).
 \end{aligned}$$

- *Naturality for Φ , Part II.* We need to show that, given a pointed function $f: (X, x_0) \rightarrow (X', x'_0)$, the diagram

$$\begin{array}{ccc}
 \text{Sets}_*((X'^+, \star_X), (Y, y_0)) & \xrightarrow{\Phi_{X',Y}} & \text{Sets}(X', Y) \\
 \downarrow f^* & & \downarrow f^* \\
 \text{Sets}_*((X^+, \star_X), (Y, y_0)) & \xrightarrow{\Phi_{X,Y}} & \text{Sets}(X, Y)
 \end{array}$$

commutes. Indeed, given a function

$$\xi: X' \rightarrow Y,$$

we have

$$\begin{aligned}
 [\Phi_{X,Y} \circ f^*](\xi) &= \Phi_{X,Y}(f^*(\xi)) \\
 &= \Phi_{X,Y}(\xi \circ f) \\
 &= \xi \circ f \\
 &= \Phi_{X',Y}(\xi) \circ f \\
 &= f^*(\Phi_{X',Y}(\xi)) \\
 &= f^*(\Phi_{X',Y}(\xi)) \\
 &= [f^* \circ \Phi_{X',Y}](\xi).
 \end{aligned}$$

- *Naturality for Ψ .* Since Φ is natural in each argument and Φ is a componentwise inverse to Ψ in each argument, it follows from [Categories, Item 2 of Proposition 8.6.1.2](#) that Ψ is also natural in each argument.

Item 3, Symmetric Strong Monoidality With Respect to Wedge Sums: The isomorphism

$$\phi: X^+ \vee Y^+ \xrightarrow{\cong} (X \amalg Y)^+$$

is given by

$$\phi(z) = \begin{cases} x & \text{if } z = [(0, x)] \text{ with } x \in X, \\ y & \text{if } z = [(1, y)] \text{ with } y \in Y, \\ \star_X \amalg_Y & \text{if } z = [(0, \star_X)], \\ \star_X \amalg_Y & \text{if } z = [(1, \star_Y)] \end{cases}$$

for each $z \in X^+ \vee Y^+$, with inverse

$$\phi^{-1}: (X \amalg Y)^+ \xrightarrow{\cong} X^+ \vee Y^+$$

given by

$$\phi^{-1}(z) \stackrel{\text{def}}{=} \begin{cases} [(0, x)] & \text{if } z = [(0, x)], \\ [(0, y)] & \text{if } z = [(1, y)], \\ p_0 & \text{if } z = \star_X \amalg_Y \end{cases}$$

for each $z \in (X \amalg Y)^+$.

Meanwhile, the isomorphism $\text{pt} \cong \emptyset^+$ is given by sending \star_X to \star_\emptyset .

That these isomorphisms satisfy the coherence conditions making the functor $(-)^+$ symmetric strong monoidal can be directly checked element by element.

Item 4, Symmetric Strong Monoidality With Respect to Smash Products: The isomorphism

$$\phi: X^+ \wedge Y^+ \xrightarrow{\cong} (X \times Y)^+$$

is given by

$$\phi(x \wedge y) = \begin{cases} (x, y) & \text{if } x \neq \star_X \text{ and } y \neq \star_Y \\ \star_{X \times Y} & \text{otherwise} \end{cases}$$

for each $x \wedge y \in X^+ \wedge Y^+$, with inverse

$$\phi^{-1}: (X \times Y)^+ \xrightarrow{\cong} X^+ \wedge Y^+$$

given by

$$\phi^{-1}(z) \stackrel{\text{def}}{=} \begin{cases} x \wedge y & \text{if } z = (x, y) \text{ with } (x, y) \in X \times Y, \\ \star_X \wedge \star_Y & \text{if } z = \star_{X \times Y}, \end{cases}$$

for each $z \in (X \amalg Y)^+$.

Meanwhile, the isomorphism $S^0 \cong \text{pt}^+$ is given by sending \star to $1 \in S^0 = \{0, 1\}$ and \star_{pt} to $0 \in S^0$.

That these isomorphisms satisfy the coherence conditions making the functor $(-)^+$ symmetric strong monoidal can be directly checked element by element.

□

Appendices

A Other Chapters

Sets

1. Sets
2. Constructions With Sets
3. Pointed Sets
4. Tensor Products of Pointed Sets

6. Constructions With Relations

7. Equivalence Relations and Apartness Relations

Category Theory

8. Categories

Bicategories

9. Types of Morphisms in Bicategories

Relations

5. Relations

References

- [MSE 2855868] [Qiaochu Yuan](https://math.stackexchange.com/q/2855868). *Is the category of pointed sets Cartesian closed?* Mathematics Stack Exchange. URL: <https://math.stackexchange.com/q/2855868> (cit. on p. 6).
- [MSE 884460] [Martin Brandenburg](https://math.stackexchange.com/q/884460). *Why are the category of pointed sets and the category of sets and partial functions “essentially the same”?* Mathematics Stack Exchange. URL: <https://math.stackexchange.com/q/884460> (cit. on p. 7).