Tensor Products of Pointed Sets

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In this chapter we introduce, construct, and study tensor products of pointed sets. The most well-known among these is the *smash product of pointed sets*

$$\wedge \colon \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*,$$

introduced in Section 5.1, defined via a universal property as inducing a bijection between the following data:

- Pointed maps $f \colon X \wedge Y \to Z$.
- Maps of sets $f \colon X \times Y \to Z$ satisfying

  $$\begin{aligned} f(x_0, y) &= z_0, \\ f(x, y_0) &= z_0 \end{aligned}$$

  for each $x \in X$ and each $y \in Y$.

As it turns out, however, dropping either of the *bilinearity* conditions

$$\begin{aligned} f(x_0, y) &= z_0, \\ f(x, y_0) &= z_0 \end{aligned}$$

while retaining the other leads to two other tensor products of pointed sets,

$$\begin{aligned} \lhd \colon \mathsf{Sets}_* \times \mathsf{Sets}_* &\to \mathsf{Sets}_*, \\ \rhd \colon \mathsf{Sets}_* \times \mathsf{Sets}_* &\to \mathsf{Sets}_*, \end{aligned}$$

called the *left* and *right tensor products of pointed sets*. In contrast to $\wedge$, which turns out to endow Sets_* with a monoidal category structure (Proposition 5.9.1.1), these do not admit invertible associators and unitors, but do endow Sets_* with the structure of a skew monoidal category, however (Propositions 3.8.1.1 and 4.8.1.1). Finally, in addition to the tensor products $\lhd$, $\rhd$, and $\wedge$, we also have a "tensor product" of the form

$$\odot\colon \mathsf{Sets} \times \mathsf{Sets}_* \to \mathsf{Sets}_*,$$

called the *tensor* of sets with pointed sets. All in all, these tensor products assemble into a family of functors of the form

$$\begin{aligned}
\otimes_{k,\ell}\colon \mathsf{Mon}_{\mathbb{E}_k}(\mathsf{Sets}) \times \mathsf{Mon}_{\mathbb{E}_\ell}(\mathsf{Sets}) &\to \mathsf{Mon}_{\mathbb{E}_{k+\ell}}(\mathsf{Sets}),\\
\lhd_{i,k}\colon \mathsf{Mon}_{\mathbb{E}_k}(\mathsf{Sets}) \times \mathsf{Mon}_{\mathbb{E}_k}(\mathsf{Sets}) &\to \mathsf{Mon}_{\mathbb{E}_k}(\mathsf{Sets}),\\
\rhd_{i,k}\colon \mathsf{Mon}_{\mathbb{E}_k}(\mathsf{Sets}) \times \mathsf{Mon}_{\mathbb{E}_k}(\mathsf{Sets}) &\to \mathsf{Mon}_{\mathbb{E}_k}(\mathsf{Sets}),
\end{aligned}$$

where $k, \ell, i \in \mathbb{N}$ with $i \leq k-1$. Together with the Cartesian product $\times$ of Sets, the tensor products studied in this chapter form the cases:

- $(k,\ell) = (-1,-1)$ for the Cartesian product of Sets;
- $(k,\ell) = (0,-1)$ and $(-1,0)$ for the tensor of sets with pointed sets of Definition 2.1.1.1;
- $(i,k) = (-1,0)$ for the left and right tensor products of pointed sets of Sections 3 and 4;
- $(k,\ell) = (-1,-1)$ for the smash product of pointed sets of Section 5.

In this chapter, we will carefully define and study bilinearity for pointed sets, as well as all the tensor products described above. Then, in **??**, we will extend these to tensor products involving also monoids and commutative monoids, which will end up covering all cases up to $k, \ell \leq 2$, and hence *all* cases since $\mathbb{E}_k$-monoids on Sets are the same as $\mathbb{E}_2$-monoids on Sets when $k \geq 2$.

Contents

1 Bilinear Morphisms of Pointed Sets

1.1 Left Bilinear Morphisms of Pointed Sets

Let (X, x_0), (Y, y_0), and (Z, z_0) be pointed sets.

Definition 1.1.1.1. A **left bilinear morphism of pointed sets from** $(X \times Y, (x_0, y_0))$ **to** (Z, z_0) is a map of sets

$$f\colon X \times Y \to Z$$

satisfying the following condition:[1,2]

($\star$) *Left Unital Bilinearity.* The diagram

$$\begin{tikzcd}
& \mathrm{pt} \times \mathrm{pt} \arrow[dr, dashed, "\sim"] & \\
\mathrm{pt} \times Y \arrow[ur, "\mathrm{id}_{\mathrm{pt}} \times \epsilon_Y"] \arrow[d, "{[x_0] \times \mathrm{id}_Y}"'] & & \mathrm{pt} \arrow[d, "{[z_0]}"] \\
X \times Y \arrow[rr, "f"'] & & Z
\end{tikzcd}$$

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[1] *Slogan:* The map f is left bilinear if it preserves basepoints in its first argument.

[2] Succinctly, f is bilinear if we have

$$f(x_0, y) = z_0$$

for each $y \in Y$.

commutes, i.e. for each $y \in Y$, we have

$$f(x_0, y) = z_0.$$

Definition 1.1.1.2. The **set of left bilinear morphisms of pointed sets from** $(X \times Y, (x_0, y_0))$ **to** (Z, z_0) is the set $\mathrm{Hom}^{\otimes,\mathrm{L}}_{\mathsf{Sets}_*}(X \times Y, Z)$ defined by

$$\mathrm{Hom}^{\otimes,\mathrm{L}}_{\mathsf{Sets}_*}(X \times Y, Z) \overset{\mathrm{def}}{=} \{f \in \mathrm{Hom}_{\mathsf{Sets}}(X \times Y, Z) \mid f \text{ is left bilinear}\}.$$

1.2 Right Bilinear Morphisms of Pointed Sets

Let (X, x_0), (Y, y_0), and (Z, z_0) be pointed sets.

Definition 1.2.1.1. A **right bilinear morphism of pointed sets from** $(X \times Y, (x_0, y_0))$ **to** (Z, z_0) is a map of sets

$$f : X \times Y \to Z$$

satisfying the following condition:[3,4]

(★) *Right Unital Bilinearity.* The diagram

$$\begin{tikzcd}
 & \mathrm{pt} \times \mathrm{pt} \arrow[dr, dashed, "\sim"] & \\
X \times \mathrm{pt} \arrow[ur, "\epsilon_X \times \mathrm{id}_{\mathrm{pt}}"] \arrow[d, "\mathrm{id}_X \times [y_0]"'] & & \mathrm{pt} \arrow[d, "{[z_0]}"] \\
X \times Y \arrow[rr, "f"'] & & Z
\end{tikzcd}$$

commutes, i.e. for each $x \in X$, we have

$$f(x, y_0) = z_0.$$

Definition 1.2.1.2. The **set of right bilinear morphisms of pointed sets from** $(X \times Y, (x_0, y_0))$ **to** (Z, z_0) is the set $\mathrm{Hom}^{\otimes,\mathrm{R}}_{\mathsf{Sets}_*}(X \times Y, Z)$ defined by

$$\mathrm{Hom}^{\otimes,\mathrm{R}}_{\mathsf{Sets}_*}(X \times Y, Z) \overset{\mathrm{def}}{=} \left\{f \in \mathrm{Hom}_{\mathsf{Sets}}(X \times Y, Z) \,\middle|\, f \text{ is right bilinear}\right\}.$$

[3] *Slogan:* The map f is right bilinear if it preserves basepoints in its second argument.

[4] Succinctly, f is bilinear if we have

$$f(x, y_0) = z_0$$

1.3 Bilinear Morphisms of Pointed Sets

Let (X, x_0), (Y, y_0), and (Z, z_0) be pointed sets.

Definition 1.3.1.1. A **bilinear morphism of pointed sets from** $(X \times Y, (x_0, y_0))$ **to** (Z, z_0) is a map of sets

$$f : X \times Y \to Z$$

that is both left bilinear and right bilinear.

Remark 1.3.1.2. In detail, a **bilinear morphism of pointed sets from** $(X \times Y, (x_0, y_0))$ **to** (Z, z_0) is a map of sets

$$f : (X \times Y, (x_0, y_0)) \to (Z, z_0)$$

satisfying the following conditions:[5,6]

1. *Left Unital Bilinearity.* The diagram

$$\begin{tikzcd}
 & \mathrm{pt} \times \mathrm{pt} \arrow[dr, dashed, "\sim"] & \\
\mathrm{pt} \times Y \arrow[ur, "\mathrm{id}_{\mathrm{pt}} \times \epsilon_Y"] \arrow[d, "{[x_0] \times \mathrm{id}_Y}"'] & & \mathrm{pt} \arrow[d, "{[z_0]}"] \\
X \times Y \arrow[rr, "f"'] & & Z
\end{tikzcd}$$

commutes, i.e. for each $y \in Y$, we have

$$f(x_0, y) = z_0.$$

for each $x \in X$.

[5] *Slogan:* The map f is bilinear if it preserves basepoints in each argument.

[6] Succinctly, f is bilinear if we have

$$\begin{aligned} f(x_0, y) &= z_0, \\ f(x, y_0) &= z_0 \end{aligned}$$

for each $x \in X$ and each $y \in Y$.

2. *Right Unital Bilinearity.* The diagram

$$
\begin{tikzcd}
 & \mathrm{pt}\times\mathrm{pt} \arrow[dr, dashed, "\sim"] & \\
X\times\mathrm{pt} \arrow[ur, "\epsilon_X\times\mathrm{id}_{\mathrm{pt}}"] \arrow[d, "\mathrm{id}_X\times{[y_0]}"'] & & \mathrm{pt} \arrow[d, "{[z_0]}"] \\
X\times Y \arrow[rr, "f"'] & & Z
\end{tikzcd}
$$

commutes, i.e. for each $x \in X$, we have

$$f(x, y_0) = z_0.$$

Definition 1.3.1.3. The **set of bilinear morphisms of pointed sets from** $(X \times Y, (x_0, y_0))$ **to** (Z, z_0) is the set $\mathrm{Hom}^{\otimes}_{\mathsf{Sets}_*}(X \times Y, Z)$ defined by

$$\mathrm{Hom}^{\otimes}_{\mathsf{Sets}_*}(X \times Y, Z) \stackrel{\text{def}}{=} \{f \in \mathrm{Hom}_{\mathsf{Sets}}(X \times Y, Z) \mid f \text{ is bilinear}\}.$$

2 Tensors and Cotensors of Pointed Sets by Sets

2.1 Tensors of Pointed Sets by Sets

Let (X, x_0) be a pointed set and let A be a set.

Definition 2.1.1.1. The **tensor of** (X, x_0) **by** A[7] is the pointed set[8] $A \odot (X, x_0)$ satisfying the following universal property:

(UP) We have a bijection

$$\mathsf{Sets}_*(A \odot X, K) \cong \mathsf{Sets}(A, \mathsf{Sets}_*(X, K)),$$

natural in $(K, k_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

Remark 2.1.1.2. The universal property in Definition 2.1.1.1 is equivalent to the following one:

[7] *Further Terminology:* Also called the **copower of** (X, x_0) **by** A.
[8] *Further Notation:* Often written $A \odot X$ for simplicity.

(UP) We have a bijection

$$\mathsf{Sets}_*(A \odot X, K) \cong \mathsf{Sets}^{\otimes}_{\mathbb{E}_0}(A \times X, K),$$

natural in $(K, k_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, where $\mathsf{Sets}^{\otimes}_{\mathbb{E}_0}(A \times X, K)$ is the set defined by

$$\mathsf{Sets}^{\otimes}_{\mathbb{E}_0}(A \times X, K) \overset{\text{def}}{=} \left\{ f \in \mathsf{Sets}(A \times X, K) \,\middle|\, \begin{array}{l}\text{for each } a \in A, \text{ we}\\ \text{have } f(a, x_0) = k_0\end{array} \right\}.$$

Proof. We claim we have a bijection

$$\mathsf{Sets}(A, \mathsf{Sets}_*(X, K)) \cong \mathsf{Sets}^{\otimes}_{\mathbb{E}_0}(A \times X, K)$$

natural in $(K, k_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$. Indeed, this bijection is a restriction of the bijection

$$\mathsf{Sets}(A, \mathsf{Sets}(X, K)) \cong \mathsf{Sets}(A \times X, K)$$

of Constructions With Sets, Item 2 of Proposition 1.3.1.2:

- A map
$$\begin{aligned} \xi \colon A &\longrightarrow \mathsf{Sets}_*(X, K), \\ a &\longmapsto (\xi_a \colon X \to K), \end{aligned}$$
in $\mathsf{Sets}(A, \mathsf{Sets}_*(X, K))$ gets sent to the map
$$\xi^\dagger \colon A \times X \to K$$
defined by
$$\xi^\dagger(a, x) \overset{\text{def}}{=} \xi_a(x)$$
for each $(a, x) \in A \times X$, which indeed lies in $\mathsf{Sets}^{\otimes}_{\mathbb{E}_0}(A \times X, K)$, as we have
$$\begin{aligned} \xi^\dagger(a, x_0) &\overset{\text{def}}{=} \xi_a(x_0) \\ &\overset{\text{def}}{=} k_0 \end{aligned}$$
for each $a \in A$, where we have used that $\xi_a \in \mathsf{Sets}_*(X, K)$ is a morphism of pointed sets.

- Conversely, a map
$$\xi\colon A\times X\to K$$
in $\mathsf{Sets}^{\otimes}_{\mathbb{E}_0}(A\times X,K)$ gets sent to the map
$$\begin{aligned}\xi^{\dagger}\colon A &\longrightarrow \mathsf{Sets}_*(X,K),\\ a &\longmapsto (\xi^{\dagger}_a\colon X\to K),\end{aligned}$$
where
$$\xi^{\dagger}_a\colon X\to K$$
is the map defined by
$$\xi^{\dagger}_a(x)\overset{\text{def}}{=}\xi(a,x)$$
for each $x\in X$, and indeed lies in $\mathsf{Sets}_*(X,K)$, as we have
$$\begin{aligned}\xi^{\dagger}_a(x_0) &\overset{\text{def}}{=}\xi(a,x_0)\\ &\overset{\text{def}}{=}k_0.\end{aligned}$$

This finishes the proof. □

Construction 2.1.1.3. Concretely, the **tensor of** (X,x_0) **by** A is the pointed set $A\odot(X,x_0)$ consisting of:

- *The Underlying Set.* The set $A\odot X$ given by
$$A\odot X\cong\bigvee_{a\in A}(X,x_0),$$
where $\bigvee_{a\in A}(X,x_0)$ is the wedge product of the A-indexed family $((X,x_0))_{a\in A}$ of Pointed Sets, Definition 3.2.1.1.

- *The Basepoint.* The point $[(a,x_0)]=[(a',x_0)]$ of $\bigvee_{a\in A}(X,x_0)$.

Proof. (Proven below in a bit.) □

Notation 2.1.1.4. We write $a\odot x$ for the element $[(a,x)]$ of
$$\begin{aligned}A\odot X &\cong\bigvee_{a\in A}(X,x_0)\\ &\overset{\text{def}}{=}\left(\coprod_{i\in I}X_i\right)/\sim.\end{aligned}$$

Remark 2.1.1.5. Taking the tensor of any element of A with the basepoint x_0 of X leads to the same element in $A \odot X$, i.e. we have

$$a \odot x_0 = a' \odot x_0,$$

for each $a, a' \in A$. This is due to the equivalence relation $\sim$ on

$$\bigvee_{a\in A} (X, x_0) \stackrel{\text{def}}{=} \coprod_{a\in A} X/\sim$$

identifying (a, x_0) with (a', x_0), so that the equivalence class $a \odot x_0$ is independent from the choice of $a \in A$.

Proof. We claim we have a bijection

$$\mathsf{Sets}_*(A \odot X, K) \cong \mathsf{Sets}(A, \mathsf{Sets}_*(X, K))$$

natural in $(K, k_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

- *Map I.* We define a map

  $$\Phi_K \colon \mathsf{Sets}_*(A \odot X, K) \to \mathsf{Sets}(A, \mathsf{Sets}_*(X, K))$$

  by sending a morphism of pointed sets

  $$\xi \colon (A \odot X, a \odot x_0) \to (K, k_0)$$

  to the map of sets

  $$\begin{aligned} \xi^\dagger \colon A &\longrightarrow \mathsf{Sets}_*(X, K), \\ a &\longmapsto (\xi_a \colon X \to K), \end{aligned}$$

  where

  $$\xi_a \colon (X, x_0) \to (K, k_0)$$

  is the morphism of pointed sets defined by

  $$\xi_a(x) \stackrel{\text{def}}{=} \xi(a \odot x)$$

  for each $x \in X$. Note that we have

  $$\begin{aligned} \xi_a(x_0) &\stackrel{\text{def}}{=} \xi(a \odot x_0) \\ &= k_0, \end{aligned}$$

  so that ξ_a is indeed a morphism of pointed sets, where we have used that ξ is a morphism of pointed sets.

- *Map II.* We define a map

$$\Psi_K \colon \mathsf{Sets}(A, \mathsf{Sets}_*(X, K)) \to \mathsf{Sets}_*(A \odot X, K)$$

given by sending a map

$$\begin{aligned} \xi \colon A &\longrightarrow \mathsf{Sets}_*(X, K), \\ a &\longmapsto (\xi_a \colon X \to K), \end{aligned}$$

to the morphism of pointed sets

$$\xi^\dagger \colon (A \odot X, a \odot x_0) \to (K, k_0)$$

defined by

$$\xi^\dagger(a \odot x) \stackrel{\text{def}}{=} \xi_a(x)$$

for each $a \odot x \in A \odot X$. Note that $\xi^\dagger$ is indeed a morphism of pointed sets, as we have

$$\begin{aligned} \xi^\dagger(a \odot x_0) &\stackrel{\text{def}}{=} \xi_a(x_0) \\ &= k_0, \end{aligned}$$

where we have used that $\xi(a) \in \mathsf{Sets}_*(X, K)$ is a morphism of pointed sets.

- *Invertibility I.* We claim that

$$\Psi_K \circ \Phi_K = \mathrm{id}_{\mathsf{Sets}_*(A \odot X, K)}.$$

Indeed, given a morphism of pointed sets

$$\xi \colon (A \odot X, a \odot x_0) \to (K, k_0),$$

we have

$$\begin{aligned} [\Psi_K \circ \Phi_K](\xi) &= \Psi_K(\Phi_K(\xi)) \\ &= \Psi_K(\llbracket a \mapsto \llbracket x \mapsto \xi(a \odot x) \rrbracket \rrbracket) \\ &= \Psi_K(\llbracket a' \mapsto \llbracket x' \mapsto \xi(a' \odot x') \rrbracket \rrbracket) \\ &= \llbracket a \odot x \mapsto \mathrm{ev}_x(\mathrm{ev}_a(\llbracket a' \mapsto \llbracket x' \mapsto \xi(a' \odot x') \rrbracket \rrbracket)) \rrbracket \\ &= \llbracket a \odot x \mapsto \mathrm{ev}_x(\llbracket x' \mapsto \xi(a \odot x') \rrbracket) \rrbracket \\ &= \llbracket a \odot x \mapsto \xi(a \odot x) \rrbracket \\ &= \xi. \end{aligned}$$

- *Invertibility II.* We claim that

$$\Phi_K \circ \Psi_K = \mathrm{id}_{\mathsf{Sets}(A,\mathsf{Sets}_*(X,K))}.$$

Indeed, given a morphism $\xi \colon A \to \mathsf{Sets}_*(X, K)$, we have

$$\begin{aligned}[\Phi_K \circ \Psi_K](\xi) &= \Phi_K(\Psi_K(\xi)) \\ &= \Phi_K([\![a \odot x \mapsto \xi_a(x)]\!]) \\ &= [\![a \mapsto [\![x \mapsto \xi_a(x)]\!]]\!] \\ &= [\![a \mapsto \xi(a)]\!] \\ &= \xi.\end{aligned}$$

- *Naturality of* Φ. We need to show that, given a morphism of pointed sets

$$\phi \colon (K, k_0) \to (K', k_0'),$$

the diagram

$$\begin{tikzcd} \mathsf{Sets}_*(A \odot X, K) \arrow[r, "\Phi_K"] \arrow[d, "\phi_*"'] & \mathsf{Sets}(A, \mathsf{Sets}_*(X, K)) \arrow[d, "(\phi_*)_*"] \\ \mathsf{Sets}_*(A \odot X, K') \arrow[r, "\Phi_{K'}"'] & \mathsf{Sets}(A, \mathsf{Sets}_*(X, K')) \end{tikzcd}$$

commutes. Indeed, given a morphism of pointed sets

$$\xi \colon (A \odot X, a \odot x_0) \to (K, k_0),$$

we have

$$\begin{aligned}[\Phi_{K'} \circ \phi_*](\xi) &= \Phi_{K'}(\phi_*(\xi)) \\ &= \Phi_{K'}(\phi \circ \xi) \\ &= (\phi \circ \xi)^\dagger \\ &= [\![a \mapsto \phi \circ \xi(a \odot -)]\!] \\ &= [\![a \mapsto \phi_*(\xi(a \odot -))]\!] \\ &= (\phi_*)_*([\![a \mapsto \xi(a \odot -]\!])) \\ &= (\phi_*)_*(\Phi_K(\xi)) \\ &= [(\phi_*)_* \circ \Phi_K](\xi).\end{aligned}$$

- *Naturality of* Ψ. Since Φ is natural and Φ is a componentwise inverse to Ψ, it follows from Categories, Item 2 of Proposition 8.6.1.2 that Ψ is also natural.

This finishes the proof. □

Proposition 2.1.1.6. Let (X, x_0) be a pointed set and let A be a set.

1. *Functoriality.* The assignments A, (X, x_0), $(A, (X, x_0))$ define functors

$$\begin{aligned} A \odot - &\colon \mathsf{Sets}_* \longrightarrow \mathsf{Sets}_*, \\ - \odot X &\colon \mathsf{Sets} \longrightarrow \mathsf{Sets}_*, \\ -_1 \odot -_2 &\colon \mathsf{Sets} \times \mathsf{Sets}_* \longrightarrow \mathsf{Sets}_*. \end{aligned}$$

   In particular, given:

   - A map of sets $f \colon A \longrightarrow B$;
   - A pointed map $\phi \colon (X, x_0) \longrightarrow (Y, y_0)$;

   the induced map

$$f \odot \phi \colon A \odot X \longrightarrow B \odot Y$$

   is given by

$$[f \odot \phi](a \odot x) \stackrel{\text{def}}{=} f(a) \odot \phi(x)$$

   for each $a \odot x \in A \odot X$.

2. *Adjointness I.* We have an adjunction

$$\left(- \odot X \dashv \mathsf{Sets}_*(X, -)\right) \colon \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "- \odot X"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_*, \arrow[l, bend left, "\mathsf{Sets}_*(X, -)"] \end{tikzcd}$$

   witnessed by a bijection

$$\mathsf{Sets}_*(A \odot X, K) \cong \mathsf{Sets}(A, \mathsf{Sets}_*(X, K)),$$

   natural in $A \in \operatorname{Obj}(\mathsf{Sets})$ and $X, Y \in \operatorname{Obj}(\mathsf{Sets}_*)$.

3. *Adjointness II.* We have an adjunctions

$$(A \odot - \dashv A \pitchfork -)\colon \quad \begin{tikzcd} \mathsf{Sets}_* \arrow[r, bend left, "A\odot -"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_*, \arrow[l, bend left, "A\pitchfork -"] \end{tikzcd}$$

witnessed by a bijection

$$\mathrm{Hom}_{\mathsf{Sets}_*}(A \odot X, Y) \cong \mathrm{Hom}_{\mathsf{Sets}_*}(X, A \pitchfork Y),$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$ and $X, Y \in \mathrm{Obj}(\mathsf{Sets}_*)$.

4. *As a Weighted Colimit.* We have

$$A \odot X \cong \mathrm{colim}^{[A]}(X),$$

where in the right hand side we write:

- A for the functor $A\colon \mathrm{pt} \to \mathsf{Sets}$ picking $A \in \mathrm{Obj}(\mathsf{Sets})$;
- X for the functor $X\colon \mathrm{pt} \to \mathsf{Sets}_*$ picking $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

5. *Iterated Tensors.* We have an isomorphism of pointed sets

$$A \odot (B \odot X) \cong (A \times B) \odot X,$$

natural in $A, B \in \mathrm{Obj}(\mathsf{Sets})$ and $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

6. *Interaction With Homs.* We have a natural isomorphism

$$\mathsf{Sets}_*(A \odot X, -) \cong A \pitchfork \mathsf{Sets}_*(X, -).$$

7. *The Tensor Evaluation Map.* For each $X, Y \in \mathrm{Obj}(\mathsf{Sets}_*)$, we have a map

$$\mathrm{ev}^{\odot}_{X,Y}\colon \mathsf{Sets}_*(X, Y) \odot X \to Y,$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets}_*)$, and given by

$$\mathrm{ev}^{\odot}_{X,Y}(f \odot x) \overset{\text{def}}{=} f(x)$$

for each $f \odot x \in \mathsf{Sets}_*(X, Y) \odot X$.

8. *The Tensor Coevaluation Map.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$ and each $X \in \mathrm{Obj}(\mathsf{Sets}_*)$, we have a map

$$\mathrm{coev}^{\odot}_{A,X} \colon A \longrightarrow \mathsf{Sets}_*(X, A \odot X),$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$ and $X \in \mathrm{Obj}(\mathsf{Sets}_*)$, and given by

$$\mathrm{coev}^{\odot}_{A,X}(a) \overset{\text{def}}{=} [\![x \mapsto a \odot x]\!]$$

for each $a \in A$.

Proof. *Item 1*, *Functoriality*: This is the special case of **??**, **??** of **??** for when $\mathcal{C} = \mathsf{Sets}_*$.
Item 2, *Adjointness I*: This is simply a rephrasing of Definition 2.1.1.1.
Item 3, *: Adjointness II*: This is the special case of **??**, **??** of **??** for when $\mathcal{C} = \mathsf{Sets}_*$.
Item 4, *As a Weighted Colimit*: This is the special case of **??**, **??** of **??** for when $\mathcal{C} = \mathsf{Sets}_*$.
Item 5, *Iterated Tensors*: This is the special case of **??**, **??** of **??** for when $\mathcal{C} = \mathsf{Sets}_*$.
Item 6, *Interaction With Homs*: This is the special case of **??**, **??** of **??** for when $\mathcal{C} = \mathsf{Sets}_*$.
Item 7, *The Tensor Evaluation Map*: This is the special case of **??**, **??** of **??** for when $\mathcal{C} = \mathsf{Sets}_*$.
Item 8, *The Tensor Coevaluation Map*: This is the special case of **??**, **??** of **??** for when $\mathcal{C} = \mathsf{Sets}_*$. □

2.2 Cotensors of Pointed Sets by Sets

Let (X, x_0) be a pointed set and let A be a set.

Definition 2.2.1.1. The **cotensor of** (X, x_0) **by** A[9] is the pointed set[10] $A \pitchfork (X, x_0)$ satisfying the following universal property:

(UP) We have a bijection

$$\mathsf{Sets}_*(K, A \pitchfork X) \cong \mathsf{Sets}(A, \mathsf{Sets}_*(K, X)),$$

natural in $(K, k_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

[9] *Further Terminology:* Also called the **power of** (X, x_0) **by** A.
[10] *Further Notation:* Often written $A \pitchfork X$ for simplicity.

Remark 2.2.1.2. The universal property of Definition 2.2.1.1 is equivalent to the following one:

(UP) We have a bijection

$$\mathsf{Sets}_*(K, A \pitchfork X) \cong \mathsf{Sets}^{\otimes}_{\mathbb{E}_0}(A \times K, X),$$

natural in $(K, k_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, where $\mathsf{Sets}^{\otimes}_{\mathbb{E}_0}(A \times K, X)$ is the set defined by

$$\mathsf{Sets}^{\otimes}_{\mathbb{E}_0}(A \times K, X) \stackrel{\text{def}}{=} \left\{ f \in \mathsf{Sets}(A \times K, X) \;\middle|\; \begin{array}{l} \text{for each } a \in A \text{, we} \\ \text{have } f(a, k_0) = x_0 \end{array} \right\}.$$

Proof. This follows from the bijection

$$\mathsf{Sets}(A, \mathsf{Sets}_*(K, X)) \cong \mathsf{Sets}^{\otimes}_{\mathbb{E}_0}(A \times K, X),$$

natural in $(K, k_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$ constructed in the proof of Remark 2.1.1.2. □

Construction 2.2.1.3. Concretely, the **cotensor of** (X, x_0) **by** A is the pointed set $A \pitchfork (X, x_0)$ consisting of:

- *The Underlying Set.* The set $A \pitchfork X$ given by

$$A \pitchfork X \cong \bigwedge_{a \in A} (X, x_0),$$

  where $\bigwedge_{a \in A}(X, x_0)$ is the smash product of the A-indexed family $((X, x_0))_{a \in A}$ of Definition 6.1.1.1.

- *The Basepoint.* The point $[(x_0)_{a \in A}] = [(x_0, x_0, x_0, \ldots)]$ of $\bigwedge_{a \in A}(X, x_0)$.

Proof. We claim we have a bijection

$$\mathsf{Sets}_*(K, A \pitchfork X) \cong \mathsf{Sets}(A, \mathsf{Sets}_*(K, X)),$$

natural in $(K, k_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

- *Map I.* We define a map

$$\Phi_K : \mathsf{Sets}_*(K, A \pitchfork X) \to \mathsf{Sets}(A, \mathsf{Sets}_*(K, X)),$$

by sending a morphism of pointed sets

$$\xi\colon (K, k_0) \to (A \pitchfork X, [(x_0)_{a\in A}])$$

to the map of sets

$$\begin{aligned} \xi^\dagger\colon A &\longrightarrow \mathsf{Sets}_*(K, X), \\ a &\longmapsto (\xi_a\colon K \to X), \end{aligned}$$

where

$$\xi_a\colon (K, k_0) \to (X, x_0)$$

is the morphism of pointed sets defined by

$$\xi_a(k) = \begin{cases} x^k_a & \text{if } \xi(k) \neq [(x_0)_{a\in A}], \\ x_0 & \text{if } \xi(k) = [(x_0)_{a\in A}] \end{cases}$$

for each $k \in K$, where x^k_a is the ath component of $\xi(k) = [(x^k_a)_{a\in A}]$. Note that:

1. The definition of $\xi_a(k)$ is independent of the choice of equivalence class. Indeed, suppose we have

$$\begin{aligned} \xi(k) &= [(x^k_a)_{a\in A}] \\ &= [(y^k_a)_{a\in A}] \end{aligned}$$

with $x^k_a \neq y^k_a$ for some $a \in A$. Then there exist $a_x, a_y \in A$ such that $x^k_{a_x} = y^k_{a_y} = x_0$. The equivalence relation $\sim$ on $\prod_{a\in A} X$ then forces

$$\begin{aligned} [(x^k_a)_{a\in A}] &= [(x_0)_{a\in A}], \\ [(y^k_a)_{a\in A}] &= [(x_0)_{a\in A}], \end{aligned}$$

however, and $\xi_a(k)$ is defined to be x_0 in this case.

2. The map ξ_a is indeed a morphism of pointed sets, as we have

$$\xi_a(k_0) = x_0$$

since $\xi(k_0) = [(x_0)_{a\in A}]$ as ξ is a morphism of pointed sets and $\xi_a(k_0)$, defined to be the ath component of $[(x_0)_{a\in A}]$, is equal to x_0.

- *Map II.* We define a map

$$\Psi_K \colon \mathsf{Sets}(A, \mathsf{Sets}_*(K, X)) \to \mathsf{Sets}_*(K, A \pitchfork X),$$

given by sending a map

$$\begin{aligned} \xi \colon A &\longrightarrow \mathsf{Sets}_*(K, X), \\ a &\longmapsto (\xi_a \colon K \to X), \end{aligned}$$

to the morphism of pointed sets

$$\xi^\dagger \colon (K, k_0) \to (A \pitchfork X, [(x_0)_{a \in A}])$$

defined by

$$\xi^\dagger(k) \stackrel{\text{def}}{=} [(\xi_a(k))_{a \in A}]$$

for each $k \in K$. Note that $\xi^\dagger$ is indeed a morphism of pointed sets, as we have

$$\begin{aligned} \xi^\dagger(k_0) &\stackrel{\text{def}}{=} [(\xi_a(k_0))_{a \in A}] \\ &= x_0, \end{aligned}$$

where we have used that $\xi_a \in \mathsf{Sets}_*(K, X)$ is a morphism of pointed sets for each $a \in A$.

- *Naturality of* Ψ. We need to show that, given a morphism of pointed sets

$$\phi \colon (K, k_0) \to (K', k_0'),$$

the diagram

$$\begin{tikzcd} \mathsf{Sets}(A, \mathsf{Sets}_*(K', X)) \arrow[r, "\Psi_{K'}"] \arrow[d, "(\phi^*)_*"'] & \mathsf{Sets}_*(K', A \pitchfork X) \arrow[d, "\phi^*"] \\ \mathsf{Sets}(A, \mathsf{Sets}_*(K, X)) \arrow[r, "\Psi_K"'] & \mathsf{Sets}_*(K, A \pitchfork X) \end{tikzcd}$$

commutes. Indeed, given a map of sets

$$\begin{aligned} \xi \colon A &\longrightarrow \mathsf{Sets}_*(K', X), \\ a &\longmapsto (\xi_a \colon K' \to X), \end{aligned}$$

we have

$$\begin{aligned}[\Psi_K \circ (\phi^*)_*](\xi) &= \Psi_K((\phi^*)_*(\xi))\\ &= \Psi_K((\phi^*)_*([\![a \mapsto \xi_a]\!]))\\ &= \Psi_K(([\![a \mapsto \phi^*(\xi_a)]\!]))\\ &= \Psi_K(([\![a \mapsto [\![k \mapsto \xi_a(\phi(k))]\!]]\!]))\\ &= [\![k \mapsto [(\xi_a(\phi(k)))_{a\in A}]]\!]\\ &= \phi^*([\![k' \mapsto [(\xi_a(k'))_{a\in A}]]\!])\\ &= \phi^*(\Psi_{K'}(\xi))\\ &= [\phi^* \circ \Psi_{K'}](\xi).\end{aligned}$$

- *Naturality of* Φ. Since Ψ is natural and Ψ is a componentwise inverse to Φ, it follows from Categories, Item 2 of Proposition 8.6.1.2 that Φ is also natural.

- *Invertibility I.* We claim that

  $$\Psi_K \circ \Phi_K = \mathrm{id}_{\mathsf{Sets}_*(K, A \pitchfork X)}.$$

  Indeed, given a morphism of pointed sets

  $$\xi\colon (K, k_0) \to (A \pitchfork X, [(x_0)_{a\in A}])$$

  we have

  $$\begin{aligned}[\Psi_K \circ \Phi_K](\xi) &= \Psi_K(\Phi_K(\xi))\\ &= \Psi_K([\![a \mapsto \xi_a]\!])\\ &= \Psi_K([\![a' \mapsto \xi_{a'}]\!])\\ &= [\![k \mapsto [(\mathrm{ev}_a([\![a' \mapsto \xi_{a'}(k)]\!]))_{a\in A}]]\!]\\ &= [\![k \mapsto [(\xi_a(k))_{a\in A}]]\!].\end{aligned}$$

  Now, we have two cases:

  1. If $\xi(k) = [(x_0)_{a\in A}]$, we have

     $$\begin{aligned}[\Psi_K \circ \Phi_K](\xi) &= \cdots\\ &= [\![k \mapsto [(\xi_a(k))_{a\in A}]]\!]\\ &= [\![k \mapsto [(x_0)_{a\in A}]]\!]\\ &= [\![k \mapsto \xi(k)]\!]\\ &= \xi.\end{aligned}$$

2. If $\xi(k) \neq [(x_0)_{a\in A}]$ and $\xi(k) = [(x^k_a)_{a\in A}]$ instead, we have

$$\begin{aligned}[\Psi_K \circ \Phi_K](\xi) &= \cdots \\ &= [\![k \mapsto [(\xi_a(k))_{a\in A}]]\!] \\ &= [\![k \mapsto [(x^k_a)_{a\in A}]]\!] \\ &= [\![k \mapsto \xi(k)]\!] \\ &= \xi.\end{aligned}$$

In both cases, we have $[\Psi_K \circ \Phi_K](\xi) = \xi$, and thus we are done.

- *Invertibility II.* We claim that

$$\Phi_K \circ \Psi_K = \mathrm{id}_{\mathsf{Sets}(A,\mathsf{Sets}_*(K,X))}.$$

Indeed, given a morphism $\xi\colon A \to \mathsf{Sets}_*(K,X)$, we have

$$\begin{aligned}[\Phi_K \circ \Psi_K](\xi) &= \Phi_K(\Psi_K(\xi)) \\ &= \Phi_K([\![k \mapsto [(\xi_a(k))_{a\in A}]]\!]) \\ &= [\![a \mapsto [\![k \mapsto \xi_a(k)]\!]]\!] \\ &= \xi\end{aligned}$$

This finishes the proof. □

Proposition 2.2.1.4. Let (X, x_0) be a pointed set and let A be a set.

1. *Functoriality.* The assignments A, (X, x_0), $(A, (X, x_0))$ define functors

$$\begin{gathered}A \pitchfork -\colon \mathsf{Sets}_* \to \mathsf{Sets}_*, \\ - \pitchfork X\colon \mathsf{Sets}^{\mathsf{op}} \to \mathsf{Sets}_*, \\ -_1 \pitchfork -_2\colon \mathsf{Sets}^{\mathsf{op}} \times \mathsf{Sets}_* \to \mathsf{Sets}_*.\end{gathered}$$

In particular, given:

- A map of sets $f\colon A \to B$;
- A pointed map $\phi\colon (X, x_0) \to (Y, y_0)$;

the induced map

$$f \odot \phi \colon A \pitchfork X \to B \pitchfork Y$$

is given by

$$[f \odot \phi]([(x_a)_{a \in A}]) \stackrel{\text{def}}{=} [(\phi(x_{f(a)}))_{a \in A}]$$

for each $[(x_a)_{a \in A}] \in A \pitchfork X$.

2. *Adjointness I.* We have an adjunction

$$(- \pitchfork X \dashv \mathsf{Sets}_*(-, X)) \colon \quad \begin{tikzcd} \mathsf{Sets}^{\mathsf{op}} \arrow[r, bend left, "- \pitchfork X"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_*, \arrow[l, bend left, "\mathsf{Sets}_*(-, X)"] \end{tikzcd}$$

witnessed by a bijection

$$\mathsf{Sets}_*^{\mathsf{op}}(A \pitchfork X, K) \cong \mathsf{Sets}(A, \mathsf{Sets}_*(K, X)),$$

i.e. by a bijection

$$\mathsf{Sets}_*(K, A \pitchfork X) \cong \mathsf{Sets}(A, \mathsf{Sets}_*(K, X)),$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$ and $X, Y \in \mathrm{Obj}(\mathsf{Sets}_*)$.

3. *Adjointness II.* We have an adjunctions

$$(A \odot - \dashv A \pitchfork -) \colon \quad \begin{tikzcd} \mathsf{Sets}_* \arrow[r, bend left, "A \odot -"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_*, \arrow[l, bend left, "A \pitchfork -"] \end{tikzcd}$$

witnessed by a bijection

$$\mathrm{Hom}_{\mathsf{Sets}_*}(A \odot X, Y) \cong \mathrm{Hom}_{\mathsf{Sets}_*}(X, A \pitchfork Y),$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$ and $X, Y \in \mathrm{Obj}(\mathsf{Sets}_*)$.

4. *As a Weighted Limit.* We have

$$A \pitchfork X \cong \lim^{[A]}(X),$$

where in the right hand side we write:

- A for the functor $A\colon \mathrm{pt} \to \mathsf{Sets}$ picking $A \in \mathrm{Obj}(\mathsf{Sets})$;
- X for the functor $X\colon \mathrm{pt} \to \mathsf{Sets}_*$ picking $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

5. *Iterated Cotensors.* We have an isomorphism of pointed sets

$$A \pitchfork (B \pitchfork X) \cong (A \times B) \pitchfork X,$$

natural in $A, B \in \mathrm{Obj}(\mathsf{Sets})$ and $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

6. *Commutativity With Homs.* We have natural isomorphisms

$$\begin{aligned} A \pitchfork \mathsf{Sets}_*(X, -) &\cong \mathsf{Sets}_*(A \odot X, -), \\ A \pitchfork \mathsf{Sets}_*(-, Y) &\cong \mathsf{Sets}_*(-, A \pitchfork Y). \end{aligned}$$

7. *The Cotensor Evaluation Map.* For each $X, Y \in \mathrm{Obj}(\mathsf{Sets}_*)$, we have a map

$$\mathrm{ev}^{\pitchfork}_{X,Y}\colon X \to \mathsf{Sets}_*(X, Y) \pitchfork Y,$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets}_*)$, and given by

$$\mathrm{ev}^{\pitchfork}_{X,Y}(x) \stackrel{\text{def}}{=} [(f(x))_{f \in \mathsf{Sets}_*(X,Y)}]$$

for each $x \in X$.

8. *The Cotensor Coevaluation Map.* For each $X \in \mathrm{Obj}(\mathsf{Sets}_*)$ and each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have a map

$$\mathrm{coev}^{\pitchfork}_{A,X}\colon A \to \mathsf{Sets}_*(A \pitchfork X, X),$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets}_*)$ and $A \in \mathrm{Obj}(\mathsf{Sets})$, and given by

$$\mathrm{coev}^{\pitchfork}_{A,X}(a) \stackrel{\text{def}}{=} [\![[(x_b)_{b \in A}] \mapsto x_a]\!]$$

for each $a \in A$.

Proof. *Item 1*, *Functoriality*: This is the special case of **??**, **??** of **??** for when $\mathcal{C} = \mathsf{Sets}_*$.
Item 2, *Adjointness I*: This is simply a rephrasing of Definition 2.2.1.1.
Item 3, *: Adjointness II*: This is the special case of **??**, **??** of **??** for when $\mathcal{C} = \mathsf{Sets}_*$.
Item 4, *As a Weighted Limit*: This is the special case of **??**, **??** of **??** for when $\mathcal{C} = \mathsf{Sets}_*$.

Item 5, Iterated Cotensors: This is the special case of **??**, **??** of **??** for when $\mathcal{C} = \mathsf{Sets}_*$.
Item 6, Commutativity With Homs: This is the special case of **??**, **??** of **??** for when $\mathcal{C} = \mathsf{Sets}_*$.
Item 7, The Cotensor Evaluation Map: This is the special case of **??**, **??** of **??** for when $\mathcal{C} = \mathsf{Sets}_*$.
Item 8, The Cotensor Coevaluation Map: This is the special case of **??**, **??** of **??** for when $\mathcal{C} = \mathsf{Sets}_*$. □

3 The Left Tensor Product of Pointed Sets

3.1 Foundations

Let (X, x_0) and (Y, y_0) be pointed sets.

Definition 3.1.1.1. The **left tensor product of pointed sets** is the functor[11]

$$\lhd \colon \mathsf{Sets}_* \times \mathsf{Sets}_* \longrightarrow \mathsf{Sets}_*$$

defined as the composition

$$\mathsf{Sets}_* \times \mathsf{Sets}_* \xrightarrow{\mathrm{id}\times 忘} \mathsf{Sets}_* \times \mathsf{Sets} \xrightarrow{\beta^{\mathsf{Cats}_2}_{\mathsf{Sets}_*,\mathsf{Sets}}} \mathsf{Sets} \times \mathsf{Sets}_* \xrightarrow{\odot} \mathsf{Sets}_*,$$

where:

- $忘 \colon \mathsf{Sets}_* \to \mathsf{Sets}$ is the forgetful functor from pointed sets to sets.
- $\beta^{\mathsf{Cats}_2}_{\mathsf{Sets}_*,\mathsf{Sets}} \colon \mathsf{Sets}_* \times \mathsf{Sets} \xrightarrow{\cong} \mathsf{Sets} \times \mathsf{Sets}_*$ is the braiding of Cats_2, i.e. the functor witnessing the isomorphism

$$\mathsf{Sets}_* \times \mathsf{Sets} \cong \mathsf{Sets} \times \mathsf{Sets}_*.$$

- $\odot \colon \mathsf{Sets} \times \mathsf{Sets}_* \longrightarrow \mathsf{Sets}_*$ is the tensor functor of Item 1 of Proposition 2.1.1.6.

Remark 3.1.1.2. The left tensor product of pointed sets satisfies the following natural bijection:

$$\mathsf{Sets}_*(X \lhd Y, Z) \cong \mathrm{Hom}^{\otimes,\mathrm{L}}_{\mathsf{Sets}_*}(X \times Y, Z).$$

That is to say, the following data are in natural bijection:

[11] *Further Notation:* Also written $\lhd_{\mathsf{Sets}_*}$.

1. Pointed maps $f\colon X \lhd Y \to Z$.
2. Maps of sets $f\colon X \times Y \to Z$ satisfying $f(x_0, y) = z_0$ for each $y \in Y$.

Remark 3.1.1.3. The left tensor product of pointed sets may be described as follows:

- The left tensor product of (X, x_0) and (Y, y_0) is the pair $((X \lhd Y, x_0 \lhd y_0), \iota)$ consisting of
  - A pointed set $(X \lhd Y, x_0 \lhd y_0)$;
  - A left bilinear morphism of pointed sets $\iota\colon (X \times Y, (x_0, y_0)) \to X \lhd Y$;

  satisfying the following universal property:

  (UP) Given another such pair $((Z, z_0), f)$ consisting of
    * A pointed set (Z, z_0);
    * A left bilinear morphism of pointed sets $f\colon (X \times Y, (x_0, y_0)) \to X \lhd Y$;

    there exists a unique morphism of pointed sets $X \lhd Y \xrightarrow{\exists!} Z$ making the diagram

$$\begin{tikzcd} & X \lhd Y \arrow[d, dashed, "\exists!"] \\ X \times Y \arrow[ur, "\iota"] \arrow[r, "f"'] & Z \end{tikzcd}$$

    commute.

Construction 3.1.1.4. In detail, the **left tensor product of** (X, x_0) **and** (Y, y_0) is the pointed set $(X \lhd Y, [x_0])$ consisting of

- *The Underlying Set.* The set $X \lhd Y$ defined by

$$\begin{aligned} X \lhd Y &\stackrel{\text{def}}{=} |Y| \odot X \\ &\cong \bigvee_{y \in Y} (X, x_0), \end{aligned}$$

  where $|Y|$ denotes the underlying set of (Y, y_0);

- *The Underlying Basepoint.* The point $[(y_0, x_0)]$ of $\bigvee_{y \in Y}(X, x_0)$, which is equal to $[(y, x_0)]$ for any $y \in Y$.

Notation 3.1.1.5. We write[12] $x \lhd y$ for the element $[(y, x)]$ of

$$X \lhd Y \cong |Y| \odot X.$$

Remark 3.1.1.6. Employing the notation introduced in Notation 3.1.1.5, we have

$$x_0 \lhd y_0 = x_0 \lhd y$$

for each $y \in Y$, and

$$x_0 \lhd y = x_0 \lhd y'$$

for each $y, y' \in Y$.

Proposition 3.1.1.7. Let (X, x_0) and (Y, y_0) be pointed sets.

1. *Functoriality.* The assignments $X, Y, (X, Y) \mapsto X \lhd Y$ define functors

$$\begin{aligned} X \lhd - &\colon \mathsf{Sets}_* \to \mathsf{Sets}_*, \\ - \lhd Y &\colon \mathsf{Sets}_* \to \mathsf{Sets}_*, \\ -_1 \lhd -_2 &\colon \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*. \end{aligned}$$

   In particular, given pointed maps

$$\begin{aligned} f &\colon (X, x_0) \to (A, a_0), \\ g &\colon (Y, y_0) \to (B, b_0), \end{aligned}$$

   the induced map

$$f \lhd g \colon X \lhd Y \to A \lhd B$$

   is given by

$$[f \lhd g](x \lhd y) \stackrel{\text{def}}{=} f(x) \lhd g(y)$$

   for each $x \lhd y \in X \lhd Y$.

[12] *Further Notation:* Also written $x \lhd_{\mathsf{Sets}_*} y$.

2. *Adjointness I.* We have an adjunction

$$\left(- \lhd Y \dashv [Y,-]^{\lhd}_{\mathsf{Sets}_*}\right)\colon \quad \begin{tikzcd} \mathsf{Sets}_* \arrow[r, bend left, "- \lhd Y"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_*, \arrow[l, bend left, "{[Y,-]^{\lhd}_{\mathsf{Sets}_*}}"] \end{tikzcd}$$

   witnessed by a bijection of sets

$$\mathrm{Hom}_{\mathsf{Sets}_*}(X \lhd Y, Z) \cong \mathrm{Hom}_{\mathsf{Sets}_*}(X, [Y, Z]^{\lhd}_{\mathsf{Sets}_*})$$

   natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, where $[X, Y]^{\lhd}_{\mathsf{Sets}_*}$ is the pointed set of Definition 3.2.1.1.

3. *Adjointness II.* The functor

$$X \lhd - \colon \mathsf{Sets}_* \longrightarrow \mathsf{Sets}_*$$

   does not admit a right adjoint.

4. *Adjointness III.* We have a bijection of sets

$$\mathrm{Hom}_{\mathsf{Sets}_*}(X \lhd Y, Z) \cong \mathrm{Hom}_{\mathsf{Sets}}(|Y|, \mathsf{Sets}_*(X, Z))$$

   natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

Proof. *Item 1, Functoriality*: Clear.
Item 2, Adjointness I: This follows from Item 3 of Proposition 2.1.1.6.
Item 3, Adjointness II: For $X \lhd -$ to admit a right adjoint would require it to preserve colimits by **??**, **??** of **??**. However, we have

$$\begin{aligned} X \lhd \mathrm{pt} &\stackrel{\text{def}}{=} |\mathrm{pt}| \odot X \\ &\cong X \\ &\not\cong \mathrm{pt}, \end{aligned}$$

and thus we see that $X \lhd -$ does not have a right adjoint.
Item 4, Adjointness III: This follows from Item 2 of Proposition 2.1.1.6. □

Remark 3.1.1.8. Here is some intuition on why $X \lhd -$ fails to be a left adjoint. Item 4 of Proposition 3.1.1.7 states that we have a natural bijection

$$\mathrm{Hom}_{\mathsf{Sets}_*}(X \lhd Y, Z) \cong \mathrm{Hom}_{\mathsf{Sets}}(|Y|, \mathsf{Sets}_*(X, Z)),$$

so it would be reasonable to wonder whether a natural bijection of the form

$$\mathrm{Hom}_{\mathsf{Sets}_*}(X \lhd Y, Z) \cong \mathrm{Hom}_{\mathsf{Sets}_*}(Y, \mathbf{Sets}_*(X, Z)),$$

also holds, which would give $X \lhd - \dashv \mathbf{Sets}_*(X, -)$. However, such a bijection would require every map

$$f : X \lhd Y \to Z$$

to satisfy

$$f(x \lhd y_0) = z_0$$

for each $x \in X$, whereas we are imposing such a basepoint preservation condition only for elements of the form $x_0 \lhd y$. Thus $\mathbf{Sets}_*(X, -)$ can't be a right adjoint for $X \lhd -$, and as shown by Item 3 of Proposition 3.1.1.7, no functor can.[13]

3.2 The Left Internal Hom of Pointed Sets

Let (X, x_0) and (Y, y_0) be pointed sets.

Definition 3.2.1.1. The **left internal Hom of pointed sets** is the functor

$$[-,-]^{\lhd}_{\mathsf{Sets}_*} : \mathsf{Sets}_*^{\mathsf{op}} \times \mathsf{Sets}_* \to \mathsf{Sets}_*$$

defined as the composition

$$\mathsf{Sets}_*^{\mathsf{op}} \times \mathsf{Sets}_* \xrightarrow{忘 \times \mathrm{id}} \mathsf{Sets}^{\mathsf{op}} \times \mathsf{Sets}_* \xrightarrow{\pitchfork} \mathsf{Sets}_*,$$

where:

- $忘 : \mathsf{Sets}_* \to \mathsf{Sets}$ is the forgetful functor from pointed sets to sets.
- $\pitchfork : \mathsf{Sets}^{\mathsf{op}} \times \mathsf{Sets}_* \to \mathsf{Sets}_*$ is the cotensor functor of Item 1 of Proposition 2.2.1.4.

Proof. For a proof that $[-,-]^{\lhd}_{\mathsf{Sets}_*}$ is indeed the left internal Hom of Sets_* with respect to the left tensor product of pointed sets, see Item 2 of Proposition 3.1.1.7. □

Remark 3.2.1.2. The left internal Hom of pointed sets satisfies the following

[13] The functor $\mathbf{Sets}_*(X, -)$ is instead right adjoint to $X \wedge -$, the smash product of pointed sets of Definition 5.1.1.1. See Item 2 of Proposition 5.1.1.9.

universal property:

$$\mathsf{Sets}_*(X \lhd Y, Z) \cong \mathsf{Sets}_*(X, [Y, Z]^{\lhd}_{\mathsf{Sets}_*})$$

That is to say, the following data are in bijection:

1. Pointed maps $f\colon X \lhd Y \to Z$.
2. Pointed maps $f\colon X \to [Y, Z]^{\lhd}_{\mathsf{Sets}_*}$.

Remark 3.2.1.3. In detail, the **left internal Hom of** (X, x_0) **and** (Y, y_0) is the pointed set $([X, Y]^{\lhd}_{\mathsf{Sets}_*}, [(y_0)_{x\in X}])$ consisting of

- *The Underlying Set.* The set $[X, Y]^{\lhd}_{\mathsf{Sets}_*}$ defined by

$$\begin{aligned}[X, Y]^{\lhd}_{\mathsf{Sets}_*} &\overset{\text{def}}{=} |X| \pitchfork Y\\ &\cong \bigwedge_{x\in X}(Y, y_0),\end{aligned}$$

  where $|X|$ denotes the underlying set of (X, x_0);

- *The Underlying Basepoint.* The point $[(y_0)_{x\in X}]$ of $\bigwedge_{x\in X}(Y, y_0)$.

Proposition 3.2.1.4. Let (X, x_0) and (Y, y_0) be pointed sets.

1. *Functoriality.* The assignments $X, Y, (X, Y) \mapsto [X, Y]^{\lhd}_{\mathsf{Sets}_*}$ define functors

$$\begin{aligned}[X, -]^{\lhd}_{\mathsf{Sets}_*} &: \mathsf{Sets}_* \to \mathsf{Sets}_*,\\ [-, Y]^{\lhd}_{\mathsf{Sets}_*} &: \mathsf{Sets}_*^{\mathsf{op}} \to \mathsf{Sets}_*,\\ [-_1, -_2]^{\lhd}_{\mathsf{Sets}_*} &: \mathsf{Sets}_*^{\mathsf{op}} \times \mathsf{Sets}_* \to \mathsf{Sets}_*.\end{aligned}$$

   In particular, given pointed maps

$$\begin{aligned}f &\colon (X, x_0) \to (A, a_0),\\ g &\colon (Y, y_0) \to (B, b_0),\end{aligned}$$

   the induced map

$$[f, g]^{\lhd}_{\mathsf{Sets}_*}\colon [A, Y]^{\lhd}_{\mathsf{Sets}_*} \to [X, B]^{\lhd}_{\mathsf{Sets}_*}$$

   is given by

$$[f, g]^{\lhd}_{\mathsf{Sets}_*}([(y_a)_{a\in A}]) \overset{\text{def}}{=} [(g(y_{f(x)}))_{x\in X}]$$

   for each $[(y_a)_{a\in A}] \in [A, Y]^{\lhd}_{\mathsf{Sets}_*}$.

2. *Adjointness I*. We have an adjunction

$$\Big(-\lhd Y \dashv [Y,-]^{\lhd}_{\mathsf{Sets}_*}\Big)\colon \quad \begin{tikzcd} \mathsf{Sets}_* \arrow[r, bend left, "-\lhd Y"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_*, \arrow[l, bend left, "{[Y,-]^{\lhd}_{\mathsf{Sets}_*}}"] \end{tikzcd}$$

witnessed by a bijection of sets

$$\mathrm{Hom}_{\mathsf{Sets}_*}(X\lhd Y, Z) \cong \mathrm{Hom}_{\mathsf{Sets}_*}(X, [Y,Z]^{\lhd}_{\mathsf{Sets}_*})$$

natural in $(X,x_0), (Y,y_0), (Z,z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$

3. *Adjointness II*. The functor

$$X \lhd -\colon \mathsf{Sets}_* \longrightarrow \mathsf{Sets}_*$$

does not admit a right adjoint.

Proof. *Item 1, Functoriality*: Clear.
Item 2, Adjointness I: This is a repetition of Item 2 of Proposition 3.1.1.7, and is proved there.
Item 3, Adjointness II: This is a repetition of Item 3 of Proposition 3.1.1.7, and is proved there. □

3.3 The Left Skew Unit

Definition 3.3.1.1. The **left skew unit of the left tensor product of pointed sets** is the functor

$$\mathbb{1}^{\mathsf{Sets}_*,\lhd}\colon \mathrm{pt} \longrightarrow \mathsf{Sets}_*$$

defined by

$$\mathbb{1}^{\lhd}_{\mathsf{Sets}_*} \stackrel{\text{def}}{=} S^0.$$

3.4 The Left Skew Associator

Definition 3.4.1.1. The **skew associator of the left tensor product of pointed sets** is the natural transformation

$$\alpha^{\mathsf{Sets}_*,\lhd}\colon \lhd\circ(\lhd\times \mathrm{id}_{\mathsf{Sets}_*}) \Longrightarrow \lhd\circ(\mathrm{id}_{\mathsf{Sets}_*}\times\lhd)\circ\boldsymbol{\alpha}^{\mathsf{Cats}}_{\mathsf{Sets}_*,\mathsf{Sets}_*,\mathsf{Sets}_*}$$

as in the diagram

$$
\begin{tikzcd}
 & & \mathsf{Sets}_* \times (\mathsf{Sets}_* \times \mathsf{Sets}_*) \arrow[drr, "\mathrm{id}\times\triangleleft"] & & \\
(\mathsf{Sets}_* \times \mathsf{Sets}_*) \times \mathsf{Sets}_* \arrow[urr, dashed, "\boldsymbol{\alpha}^{\mathsf{Cats}}_{\mathsf{Sets}_*,\mathsf{Sets}_*,\mathsf{Sets}_*}", "\sim"'] \arrow[dr, "\triangleleft\times\mathrm{id}"'] & & & & \mathsf{Sets}_* \times \mathsf{Sets}_* \arrow[dl, "\triangleleft"] \\
 & \mathsf{Sets}_* \times \mathsf{Sets}_* \arrow[rr, "\triangleleft"'] \arrow[uur, phantom, "\Uparrow \alpha^{\mathsf{Sets}_*,\triangleleft}"] & & \mathsf{Sets}_*, &
\end{tikzcd}
$$

whose component

$$\alpha^{\mathsf{Sets}_*,\triangleleft}_{X,Y,Z} \colon (X \triangleleft Y) \triangleleft Z \to X \triangleleft (Y \triangleleft Z)$$

at $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$ is given by

$$
\begin{aligned}
(X \triangleleft Y) \triangleleft Z &\stackrel{\mathrm{def}}{=} |Z| \odot (X \triangleleft Y) \\
&\stackrel{\mathrm{def}}{=} |Z| \odot (|Y| \odot X) \\
&\cong \bigvee_{z\in Z} |Y| \odot X \\
&\cong \bigvee_{z\in Z} \Big(\bigvee_{y\in Y} X\Big) \\
&\to \bigvee_{[(z,y)]\in \bigvee_{z\in Z} Y} X \\
&\cong \bigvee_{[(z,y)]\in |Z|\odot Y} X \\
&\cong \big||Z| \odot Y\big| \odot X \\
&\stackrel{\mathrm{def}}{=} |Y \triangleleft Z| \odot X \\
&\stackrel{\mathrm{def}}{=} X \triangleleft (Y \triangleleft Z),
\end{aligned}
$$

where the map

$$\bigvee_{z\in Z} \Big(\bigvee_{y\in Y} X\Big) \to \bigvee_{(z,y)\in \bigvee_{z\in Z} Y} X$$

is given by $[(z, [(y, x)])] \mapsto [([(z, y)], x)]$.

Proof. (Proven below in a bit.) □

Remark 3.4.1.2. Unwinding the notation for elements, we have

$$\begin{aligned}[(z, [(y, x)])] &\stackrel{\text{def}}{=} [(z, x \lhd y)] \\ &\stackrel{\text{def}}{=} (x \lhd y) \lhd z\end{aligned}$$

and

$$\begin{aligned}[([(z, y)], x)] &\stackrel{\text{def}}{=} [(y \lhd z, x)] \\ &\stackrel{\text{def}}{=} x \lhd (y \lhd z).\end{aligned}$$

So, in other words, $\alpha^{\mathsf{Sets}_*,\lhd}_{X,Y,Z}$ acts on elements via

$$\alpha^{\mathsf{Sets}_*,\lhd}_{X,Y,Z}((x \lhd y) \lhd z) \stackrel{\text{def}}{=} x \lhd (y \lhd z)$$

for each $(x \lhd y) \lhd z \in (X \lhd Y) \lhd Z$.

Remark 3.4.1.3. Taking $y = y_0$, we see that the morphism $\alpha^{\mathsf{Sets}_*,\lhd}_{X,Y,Z}$ acts on elements as

$$\alpha^{\mathsf{Sets}_*,\lhd}_{X,Y,Z}((x \lhd y_0) \lhd z) \stackrel{\text{def}}{=} x \lhd (y_0 \lhd z).$$

However, by the definition of $\lhd$, we have $y_0 \lhd z = y_0 \lhd z'$ for all $z, z' \in Z$, preventing $\alpha^{\mathsf{Sets}_*,\lhd}_{X,Y,Z}$ from being non-invertible.

Proof. Firstly, note that, given $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, the map

$$\alpha^{\mathsf{Sets}_*,\lhd}_{X,Y,Z} \colon (X \lhd Y) \lhd Z \longrightarrow X \lhd (Y \lhd Z)$$

is indeed a morphism of pointed sets, as we have

$$\alpha^{\mathsf{Sets}_*,\lhd}_{X,Y,Z}((x_0 \lhd y_0) \lhd z_0) = x_0 \lhd (y_0 \lhd z_0).$$

Next, we claim that $\alpha^{\mathsf{Sets}_*,\lhd}$ is a natural transformation. We need to show that, given morphisms of pointed sets

$$\begin{aligned}f &\colon (X, x_0) \longrightarrow (X', x_0'), \\ g &\colon (Y, y_0) \longrightarrow (Y', y_0'), \\ h &\colon (Z, z_0) \longrightarrow (Z', z_0')\end{aligned}$$

the diagram

$$\begin{tikzcd}
(X \lhd Y) \lhd Z \arrow[r, "(f \lhd g) \lhd h"] \arrow[d, "\alpha^{\mathsf{Sets}_*, \lhd}_{X,Y,Z}"'] & (X' \lhd Y') \lhd Z' \arrow[d, "\alpha^{\mathsf{Sets}_*, \lhd}_{X',Y',Z'}"] \\
X \lhd (Y \lhd Z) \arrow[r, "f \lhd (g \lhd h)"'] & X' \lhd (Y' \lhd Z')
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
(x \lhd y) \lhd z \arrow[r, mapsto] \arrow[d, mapsto] & (f(x) \lhd g(y)) \lhd h(z) \arrow[d, mapsto] \\
x \lhd (y \lhd z) \arrow[r, mapsto] & f(x) \lhd (g(y) \lhd h(z))
\end{tikzcd}$$

and hence indeed commutes, showing $\alpha^{\mathsf{Sets}_*, \lhd}$ to be a natural transformation. This finishes the proof. □

3.5 The Left Skew Left Unitor

Definition 3.5.1.1. The **skew left unitor of the left tensor product of pointed sets** is the natural transformation

$$\lambda^{\mathsf{Sets}_*, \lhd} \colon \lhd \circ (\mathbb{1}^{\mathsf{Sets}_*} \times \mathrm{id}_{\mathsf{Sets}_*}) \overset{\sim}{\Longrightarrow} \boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*} \qquad \begin{tikzcd}
\mathsf{pt} \times \mathsf{Sets}_* \arrow[r, "\mathbb{1}^{\mathsf{Sets}_*} \times \mathrm{id}"] \arrow[dr, dashed, bend right, "\boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}"'] & \mathsf{Sets}_* \times \mathsf{Sets}_* \arrow[d, "\lhd"] \arrow[dl, phantom, "\Swarrow \lambda^{\mathsf{Sets}_*, \lhd}"] \\
 & \mathsf{Sets}_*,
\end{tikzcd}$$

whose component

$$\lambda^{\mathsf{Sets}_*, \lhd}_X \colon S^0 \lhd X \to X$$

at $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$ is given by the composition

$$\begin{aligned}
S^0 \lhd X &\cong |X| \odot S^0 \\
&\cong \bigvee_{x \in X} S^0 \\
&\to X,
\end{aligned}$$

where $\bigvee_{x\in X} S^0 \to X$ is the map given by

$$\begin{aligned}[(x,0)] &\mapsto x_0,\\ [(x,1)] &\mapsto x.\end{aligned}$$

Proof. (Proven below in a bit.) □

Remark 3.5.1.2. In other words, $\lambda^{\mathsf{Sets}_*,\triangleleft}_X$ acts on elements as

$$\begin{aligned}\lambda^{\mathsf{Sets}_*,\triangleleft}_X(0\triangleleft x) &\stackrel{\text{def}}{=} x_0,\\ \lambda^{\mathsf{Sets}_*,\triangleleft}_X(1\triangleleft x) &\stackrel{\text{def}}{=} x\end{aligned}$$

for each $1\triangleleft x\in S^0\triangleleft X$.

Remark 3.5.1.3. The morphism $\lambda^{\mathsf{Sets}_*,\triangleleft}_X$ is almost invertible, with its would-be-inverse

$$\phi_X\colon X\to S^0\triangleleft X$$

given by

$$\phi_X(x)\stackrel{\text{def}}{=}1\triangleleft x$$

for each $x\in X$. Indeed, we have

$$\begin{aligned}[\lambda^{\mathsf{Sets}_*,\triangleleft}_X\circ\phi](x) &= \lambda^{\mathsf{Sets}_*,\triangleleft}_X(\phi(x))\\ &= \lambda^{\mathsf{Sets}_*,\triangleleft}_X(1\triangleleft x)\\ &= x\\ &= [\mathrm{id}_X](x)\end{aligned}$$

so that

$$\lambda^{\mathsf{Sets}_*,\triangleleft}_X\circ\phi=\mathrm{id}_X$$

and

$$\begin{aligned}[\phi\circ\lambda^{\mathsf{Sets}_*,\triangleleft}_X](1\triangleleft x) &= \phi(\lambda^{\mathsf{Sets}_*,\triangleleft}_X(1\triangleleft x))\\ &= \phi(x)\\ &= 1\triangleleft x\\ &= [\mathrm{id}_{S^0\triangleleft X}](1\triangleleft x),\end{aligned}$$

but

$$\begin{aligned}[\phi \circ \lambda^{\mathsf{Sets}_*,\lhd}_X](0 \lhd x) &= \phi(\lambda^{\mathsf{Sets}_*,\lhd}_X(0 \lhd x)) \\ &= \phi(x_0) \\ &= 1 \lhd x_0,\end{aligned}$$

where $0 \lhd x \neq 1 \lhd x_0$. Thus

$$\phi \circ \lambda^{\mathsf{Sets}_*,\lhd}_X \stackrel{?}{=} \mathrm{id}_{S^0 \lhd X}$$

holds for all elements in $S^0 \lhd X$ except one.

Proof. Firstly, note that, given $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, the map

$$\lambda^{\mathsf{Sets}_*,\lhd}_X \colon S^0 \lhd X \longrightarrow X$$

is indeed a morphism of pointed sets, as we have

$$\lambda^{\mathsf{Sets}_*,\lhd}_X(0 \lhd x_0) = x_0.$$

Next, we claim that $\lambda^{\mathsf{Sets}_*,\lhd}$ is a natural transformation. We need to show that, given a morphism of pointed sets

$$f \colon (X, x_0) \longrightarrow (Y, y_0),$$

the diagram

$$\begin{tikzcd} S^0 \lhd X \arrow[r, "\mathrm{id}_{S^0} \lhd f"] \arrow[d, "\lambda^{\mathsf{Sets}_*,\lhd}_X"'] & S^0 \lhd Y \arrow[d, "\lambda^{\mathsf{Sets}_*,\lhd}_Y"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd} 0 \lhd x \arrow[d, mapsto] & & 0 \lhd x \arrow[r, mapsto] & 0 \lhd f(x) \arrow[d, mapsto] \\ x_0 \arrow[r, mapsto] & f(x_0) & & y_0 \end{tikzcd}$$

and

$$\begin{tikzcd} 1 \lhd x \arrow[r, mapsto] \arrow[d, mapsto] & 1 \lhd f(x) \arrow[d, mapsto] \\ x \arrow[r, mapsto] & f(x) \end{tikzcd}$$

and hence indeed commutes, showing $\lambda^{\mathsf{Sets}_*,\lhd}$ to be a natural transformation. This finishes the proof. □

3.6 The Left Skew Right Unitor

Definition 3.6.1.1. The **skew right unitor of the left tensor product of pointed sets** is the natural transformation

$$\rho^{\mathsf{Sets}_*,\lhd} \colon \boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*} \overset{\sim}{\Longrightarrow} \lhd \circ (\mathrm{id} \times \mathbb{1}^{\mathsf{Sets}_*}), \qquad \begin{tikzcd} \mathsf{Sets}_* \times \mathrm{pt} \arrow[r, "\mathrm{id}\times\mathbb{1}^{\mathsf{Sets}_*}"] \arrow[dr, dashed, bend right, "\boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}"'] & \mathsf{Sets}_* \times \mathsf{Sets}_* \arrow[d, "\lhd"] \arrow[dl, phantom, "\Uparrow \rho^{\mathsf{Sets}_*,\lhd}"] \\ & \mathsf{Sets}_*, \end{tikzcd}$$

whose component

$$\rho^{\mathsf{Sets}_*,\lhd}_X \colon X \to X \lhd S^0$$

at $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$ is given by the composition

$$\begin{aligned} X &\to X \vee X \\ &\cong |S^0| \odot X \\ &\cong X \lhd S^0, \end{aligned}$$

where $X \to X \vee X$ is the map sending X to the second factor of X in $X \vee X$.

Proof. (Proven below in a bit.) □

Remark 3.6.1.2. In other words, $\rho^{\mathsf{Sets}_*,\lhd}_X$ acts on elements as

$$\rho^{\mathsf{Sets}_*,\lhd}_X(x) \overset{\text{def}}{=} [(1, x)]$$

i.e. by

$$\rho_X^{\mathsf{Sets}_*,\lhd}(x) \stackrel{\text{def}}{=} x \lhd 1$$

for each $x \in X$.

Remark 3.6.1.3. The morphism $\rho_X^{\mathsf{Sets}_*,\lhd}$ is non-invertible, as it is non-surjective when viewed as a map of sets, since the elements $x \lhd 0$ of $X \lhd S^0$ with $x \neq x_0$ are outside the image of $\rho_X^{\mathsf{Sets}_*,\lhd}$, which sends x to $x \lhd 1$.

Proof. Firstly, note that, given $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, the map

$$\rho_X^{\mathsf{Sets}_*,\lhd} \colon X \to X \lhd S^0$$

is indeed a morphism of pointed sets as we have

$$\begin{aligned} \rho_X^{\mathsf{Sets}_*,\lhd}(x_0) &= x_0 \lhd 1 \\ &= x_0 \lhd 0. \end{aligned}$$

Next, we claim that $\rho^{\mathsf{Sets}_*,\lhd}$ is a natural transformation. We need to show that, given a morphism of pointed sets

$$f \colon (X, x_0) \to (Y, y_0),$$

the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\rho_X^{\mathsf{Sets}_*,\lhd}"'] & Y \arrow[d, "\rho_Y^{\mathsf{Sets}_*,\lhd}"] \\ X \lhd S^0 \arrow[r, "f \lhd \mathrm{id}_{S^0}"'] & Y \lhd S^0 \end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd} x \arrow[r, mapsto] \arrow[d, mapsto] & f(x) \arrow[d, mapsto] \\ x \lhd 0 \arrow[r, mapsto] & f(x) \lhd 0 \end{tikzcd}$$

and hence indeed commutes, showing $\rho^{\mathsf{Sets}_*,\lhd}$ to be a natural transformation. This finishes the proof. □

3.7 The Diagonal

Definition 3.7.1.1. The **diagonal of the left tensor product of pointed sets** is the natural transformation

$$\Delta^{\triangleleft}\colon \mathrm{id}_{\mathsf{Sets}_*} \Longrightarrow \triangleleft \circ \Delta^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}, \qquad \begin{tikzcd} \mathsf{Sets}_* \arrow[rr, bend left, "\mathrm{id}_{\mathsf{Sets}_*}"] \arrow[dr, swap, "\Delta^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}"] & \arrow[d, phantom, "\Downarrow \Delta^{\triangleleft}"] & \mathsf{Sets}_* \\ & \mathsf{Sets}_* \times \mathsf{Sets}_*, \arrow[ur, swap, "\triangleleft"] & \end{tikzcd}$$

whose component

$$\Delta^{\triangleleft}_X\colon (X, x_0) \longrightarrow (X \triangleleft X, x_0 \triangleleft x_0)$$

at $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$ is given by

$$\Delta^{\triangleleft}_X(x) \stackrel{\text{def}}{=} x \triangleleft x$$

for each $x \in X$.

Proof. Being a Morphism of Pointed Sets: We have

$$\Delta^{\triangleleft}_X(x_0) \stackrel{\text{def}}{=} x_0 \triangleleft x_0,$$

and thus $\Delta^{\triangleleft}_X$ is a morphism of pointed sets.
Naturality: We need to show that, given a morphism of pointed sets

$$f\colon (X, x_0) \longrightarrow (Y, y_0),$$

the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, swap, "\Delta^{\triangleleft}_X"] & Y \arrow[d, "\Delta^{\triangleleft}_Y"] \\ X \triangleleft X \arrow[r, swap, "f \triangleleft f"] & Y \triangleleft Y \end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd} x \arrow[r, mapsto] \arrow[d, mapsto] & f(x) \arrow[d, mapsto] \\ x \triangleleft x \arrow[r, mapsto] & f(x) \triangleleft f(x) \end{tikzcd}$$

and hence indeed commutes, showing $\Delta^{\triangleleft}$ to be natural. □

3.8 The Left Skew Monoidal Structure on Pointed Sets Associated to $\lhd$

Proposition 3.8.1.1. The category Sets_* admits a left-closed left skew monoidal category structure consisting of

- *The Underlying Category.* The category Sets_* of pointed sets;
- *The Left Skew Monoidal Product.* The left tensor product functor

  $$\lhd \colon \mathsf{Sets}_* \times \mathsf{Sets}_* \longrightarrow \mathsf{Sets}_*$$

  of Definition 3.1.1.1;
- *The Left Internal Skew Hom.* The left internal Hom functor

  $$[-,-]^{\lhd}_{\mathsf{Sets}_*} \colon \mathsf{Sets}^{\mathsf{op}}_* \times \mathsf{Sets}_* \longrightarrow \mathsf{Sets}_*$$

  of Definition 3.2.1.1;
- *The Left Skew Monoidal Unit.* The functor

  $$\mathbb{1}^{\mathsf{Sets}_*,\lhd} \colon \mathsf{pt} \longrightarrow \mathsf{Sets}_*$$

  of Definition 3.3.1.1;
- *The Left Skew Associators.* The natural transformation

  $$\alpha^{\mathsf{Sets}_*,\lhd} \colon \lhd \circ (\lhd \times \mathrm{id}_{\mathsf{Sets}_*}) \Longrightarrow \lhd \circ (\mathrm{id}_{\mathsf{Sets}_*} \times \lhd) \circ \boldsymbol{\alpha}^{\mathsf{Cats}}_{\mathsf{Sets}_*,\mathsf{Sets}_*,\mathsf{Sets}_*}$$

  of Definition 3.4.1.1;
- *The Left Skew Left Unitors.* The natural transformation

  $$\lambda^{\mathsf{Sets}_*,\lhd} \colon \lhd \circ (\mathbb{1}^{\mathsf{Sets}_*} \times \mathrm{id}_{\mathsf{Sets}_*}) \overset{\sim}{\Longrightarrow} \boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}$$

  of Definition 3.5.1.1;
- *The Left Skew Right Unitors.* The natural transformation

  $$\rho^{\mathsf{Sets}_*,\lhd} \colon \boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*} \overset{\sim}{\Longrightarrow} \lhd \circ (\mathsf{id} \times \mathbb{1}^{\mathsf{Sets}_*})$$

  of Definition 3.6.1.1.

Proof. The Pentagon Identity: Let (W, w_0), (X, x_0), (Y, y_0) and (Z, z_0) be pointed sets. We have to show that the diagram

$$\begin{tikzcd}
 & & (W \triangleleft (X \triangleleft Y)) \triangleleft Z \arrow[drr, "\alpha^{\mathsf{Sets}_*,\triangleleft}_{W,X\triangleleft Y,Z}"] & & \\
((W \triangleleft X) \triangleleft Y) \triangleleft Z \arrow[urr, "\alpha^{\mathsf{Sets}_*,\triangleleft}_{W,X,Y} \triangleleft \mathrm{id}_Z"] \arrow[dr, "\alpha^{\mathsf{Sets}_*,\triangleleft}_{W\triangleleft X,Y,Z}"'] & & & & W \triangleleft ((X \triangleleft Y) \triangleleft Z) \arrow[dl, "\mathrm{id}_W \triangleleft \alpha^{\mathsf{Sets}_*,\triangleleft}_{X,Y,Z}"] \\
 & (W \triangleleft X) \triangleleft (Y \triangleleft Z) \arrow[rr, "\alpha^{\mathsf{Sets}_*,\triangleleft}_{W,X,Y\triangleleft Z}"'] & & W \triangleleft (X \triangleleft (Y \triangleleft Z)) &
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
 & & (w \triangleleft (x \triangleleft y)) \triangleleft z \arrow[drr, mapsto] & & \\
((w \triangleleft x) \triangleleft y) \triangleleft z \arrow[urr, mapsto] \arrow[dr, mapsto] & & & & w \triangleleft ((x \triangleleft y) \triangleleft z) \arrow[dl, mapsto] \\
 & (w \triangleleft x) \triangleleft (y \triangleleft z) \arrow[rr, mapsto] & & w \triangleleft (x \triangleleft (y \triangleleft z)) &
\end{tikzcd}$$

and thus we see that the pentagon identity is satisfied.

The Left Skew Left Triangle Identity: Let (X, x_0) and (Y, y_0) be pointed sets. We have to show that the diagram

$$\begin{tikzcd}
(S^0 \lhd X) \lhd Y \arrow[r, "\alpha^{\mathsf{Sets}_*,\lhd}_{S^0,X,Y}"] \arrow[dr, "\lambda^{\mathsf{Sets}_*,\lhd}_X \lhd \mathrm{id}_Y"'] & S^0 \lhd (X \lhd Y) \arrow[d, "\lambda^{\mathsf{Sets}_*,\lhd}_{X \lhd Y}"] \\
& X \lhd Y
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
(0 \lhd x) \lhd y \arrow[r, mapsto] \arrow[dr, mapsto] & 0 \lhd (x \lhd y) \arrow[d, mapsto] \\
& x_0 \lhd y = x_0 \lhd y_0
\end{tikzcd}$$

and

$$\begin{tikzcd}
(1 \lhd x) \lhd y \arrow[r, mapsto] \arrow[dr, mapsto] & 1 \lhd (x \lhd y) \arrow[d, mapsto] \\
& x \lhd y
\end{tikzcd}$$

and hence indeed commutes. Thus the left skew triangle identity is satisfied.
The Left Skew Right Triangle Identity: Let (X, x_0) and (Y, y_0) be pointed sets. We have to show that the diagram

$$\begin{tikzcd}
X \lhd Y \arrow[d, "\rho^{\mathsf{Sets}_*,\lhd}_{X \lhd Y}"'] \arrow[dr, "\mathrm{id}_X \lhd \rho^{\mathsf{Sets}_*,\lhd}_Y"] & \\
(X \lhd Y) \lhd S^0 \arrow[r, "\alpha^{\mathsf{Sets}_*,\lhd}_{X,Y,S^0}"'] & X \lhd (Y \lhd S^0)
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
x \lhd y \arrow[d, mapsto] \arrow[dr, mapsto] & \\
(x \lhd y) \lhd 1 \arrow[r, mapsto] & x \lhd (y \lhd 1)
\end{tikzcd}$$

and hence indeed commutes. Thus the right skew triangle identity is satisfied.
The Left Skew Middle Triangle Identity: Let (X, x_0) and (Y, y_0) be pointed sets. We have to show that the diagram

$$\begin{tikzcd}
X \lhd Y \arrow[rr, equal] \arrow[d, "\rho_X^{\mathsf{Sets}_*,\lhd} \lhd \mathrm{id}_Y"'] & & X \lhd Y \\
(X \lhd S^0) \lhd Y \arrow[rr, "\alpha_{X,S^0,Y}^{\mathsf{Sets}_*,\lhd}"'] & & X \lhd (S^0 \lhd Y) \arrow[u, "\mathrm{id}_A \lhd \lambda_Y^{\mathsf{Sets}_*,\lhd}"']
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
x \lhd y \arrow[r, mapsto] \arrow[d, mapsto] & x \lhd y \\
(x \lhd 1) \lhd y \arrow[r, mapsto] & x \lhd (1 \lhd y) \arrow[u, mapsto]
\end{tikzcd}$$

and hence indeed commutes. Thus the right skew triangle identity is satisfied.
The Zig-Zag Identity: We have to show that the diagram

$$\begin{tikzcd}
S^0 \arrow[r, "\rho_{S^0}^{\mathsf{Sets}_*,\lhd}"] \arrow[dr, equal] & S^0 \lhd S^0 \arrow[d, "\lambda_{S^0}^{\mathsf{Sets}_*,\lhd}"] \\
& S^0
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
0 \arrow[r, mapsto] \arrow[dr, mapsto] & 0 \lhd 1 \arrow[d, mapsto] \\
& 0
\end{tikzcd}$$

and

$$\begin{tikzcd}
1 \arrow[r, mapsto] \arrow[dr, mapsto] & 1 \lhd 1 \arrow[d, mapsto] \\
& 1
\end{tikzcd}$$

and hence indeed commutes. Thus the zig-zag identity is satisfied.
Left Skew Monoidal Left-Closedness: This follows from Item 2 of Proposition 3.1.1.7.

□

3.9 Monoids With Respect to the Left Tensor Product of Pointed Sets

Proposition 3.9.1.1. The category of monoids on $(\mathsf{Sets}_*, \lhd, S^0)$ is isomorphic to the category of "monoids with left zero"[14] and morphisms between them.

Proof. Monoids on $(\mathsf{Sets}_*, \lhd, S^0)$: A monoid on $(\mathsf{Sets}_*, \lhd, S^0)$ consists of:

- *The Underlying Object.* A pointed set $(A, 0_A)$.
- *The Multiplication Morphism.* A morphism of pointed sets

  $$\mu_A \colon A \lhd A \to A,$$

  determining a left bilinear morphism of pointed sets

  $$\begin{aligned} A \times A &\longrightarrow A \\ (a, b) &\longmapsto ab. \end{aligned}$$

- *The Unit Morphism.* A morphism of pointed sets

  $$\eta_A \colon S^0 \to A$$

  picking an element 1_A of A.

satisfying the following conditions:

1. *Associativity.* The diagram

$$\begin{tikzcd} & A \lhd (A \lhd A) \arrow[dr, "\mathrm{id}_A \lhd \mu_A"] & \\ (A \lhd A) \lhd A \arrow[ur, "\alpha^{\mathsf{Sets}_*, \lhd}_{A,A,A}"] \arrow[d, "\mu_A \lhd \mathrm{id}_A"'] & & A \lhd A \arrow[d, "\mu_A"] \\ A \lhd A \arrow[rr, "\mu_A"'] & & A \end{tikzcd}$$

[14] A monoid with left zero is defined similarly as the monoids with zero of **??**. Succinctly, they

2. *Left Unitality.* The diagram

$$\begin{tikzcd} S^0 \lhd A \arrow[r, "\eta_A \times \mathrm{id}_A"] \arrow[dr, "\lambda_A^{\mathsf{Sets}_*, \lhd}"'] & A \lhd A \arrow[d, "\mu_A"] \\ & A \end{tikzcd}$$

commutes.

3. *Right Unitality.* The diagram

$$\begin{tikzcd} A \arrow[r, "\rho_A^{\mathsf{Sets}_*, \lhd}"] \arrow[d, equal] & A \lhd S^0 \arrow[d, "\mathrm{id}_A \times \eta_A"] \\ A & A \lhd A \arrow[l, "\mu_A"] \end{tikzcd}$$

commutes.

Being a left-bilinear morphism of pointed sets, the multiplication map satisfies

$$0_A a = 0_A$$

for each $a \in A$. Now, the associativity, left unitality, and right unitality conditions act on elements as follows:

1. *Associativity.* The associativity condition acts as

$$\begin{tikzcd} & & & a \lhd (b \lhd c) \arrow[dl, mapsto] \arrow[dr, mapsto] & \\ (a \lhd b) \lhd c \arrow[d, mapsto] & & (a \lhd b) \lhd c & & a \lhd bc \arrow[d, mapsto] \\ ab \lhd c \arrow[r, mapsto] & (ab)c & & & a(bc) \end{tikzcd}$$

---

are monoids (A, μ_A, η_A) with a special element 0_A satisfying

$$0_A a = 0_A$$

for each $a \in A$.

This gives

$$(ab)c = a(bc)$$

for each $a, b, c \in A$.

2. *Left Unitality.* The left unitality condition acts:

   (a) On $0 \lhd a$ as

$$\begin{tikzcd} 0 \lhd a \arrow[dr, mapsto] & \\ & 0_A \end{tikzcd} \qquad \begin{tikzcd} 0 \lhd a \arrow[r, mapsto] \arrow[dr, mapsto] & 0_A \lhd a \arrow[d, mapsto] \\ & 0_A a. \end{tikzcd}$$

   (b) On $1 \lhd a$ as

$$\begin{tikzcd} 1 \lhd a \arrow[dr, mapsto] & \\ & a \end{tikzcd} \qquad \begin{tikzcd} 1 \lhd a \arrow[r, mapsto] \arrow[dr, mapsto] & 1_A \lhd a \arrow[d, mapsto] \\ & 1_A a. \end{tikzcd}$$

   This gives

$$\begin{aligned} 1_A a &= a, \\ 0_A a &= 0_A \end{aligned}$$

   for each $a \in A$.

3. *Right Unitality.* The right unitality condition acts as

$$\begin{tikzcd} a \arrow[d, mapsto] \\ a \end{tikzcd} \qquad \begin{tikzcd} a \arrow[r, mapsto] & a \lhd 1 \arrow[d, mapsto] \\ a1_A & a \lhd 1_A \arrow[l, mapsto] \end{tikzcd}$$

   This gives

$$a1_A = a$$

   for each $a \in A$.

Thus we see that monoids with respect to $\lhd$ are exactly monoids with left zero.

Morphisms of Monoids on $(\mathsf{Sets}_*, \lhd, S^0)$: A morphism of monoids on $(\mathsf{Sets}_*, \lhd, S^0)$ from $(A, \mu_A, \eta_A, 0_A)$ to $(B, \mu_B, \eta_B, 0_B)$ is a morphism of pointed sets

$$f\colon (A, 0_A) \to (B, 0_B)$$

satisfying the following conditions:

1. *Compatibility With the Multiplication Morphisms.* The diagram

$$\begin{tikzcd} A \lhd A \arrow[r, "f \lhd f"] \arrow[d, "\mu_A"'] & B \lhd B \arrow[d, "\mu_B"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

   commutes.

2. *Compatibility With the Unit Morphisms.* The diagram

$$\begin{tikzcd} S^0 \arrow[r, "\eta_A"] \arrow[dr, "\eta_B"'] & A \arrow[d, "f"] \\ & B \end{tikzcd}$$

   commutes.

These act on elements as

$$\begin{tikzcd} a \lhd b \arrow[d, mapsto] & & a \lhd b \arrow[r, mapsto] & f(a) \lhd f(b) \arrow[d, mapsto] \\ ab \arrow[r, mapsto] & f(ab) & & f(a)f(b) \end{tikzcd}$$

and

$$\begin{tikzcd} 0 \arrow[dr, mapsto] & & 0 \arrow[r, mapsto] & 0_A \arrow[d, mapsto] \\ & 0_B & & f(0_A) \end{tikzcd}$$

and

$$
\begin{tikzcd}
1 \arrow[dr, mapsto] & & 1 \arrow[r, mapsto] & 1_A \arrow[d, mapsto] \\
& 1_B & & f(1_A)
\end{tikzcd}
$$

giving

$$
\begin{aligned}
f(ab) &= f(a)f(b), \\
f(0_A) &= 0_B, \\
f(1_A) &= 1_B,
\end{aligned}
$$

for each $a, b \in A$, which is exactly a morphism of monoids with left zero.

Identities and Composition: Similarly, the identities and composition of $\mathsf{Mon}(\mathsf{Sets}_*, \lhd, S^0)$ can be easily seen to agree with those of monoids with left zero, which finishes the proof. □

4 The Right Tensor Product of Pointed Sets

4.1 Foundations

Let (X, x_0) and (Y, y_0) be pointed sets.

Definition 4.1.1.1. The **right tensor product of pointed sets** is the functor[15]

$$\rhd : \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*$$

defined as the composition

$$\mathsf{Sets}_* \times \mathsf{Sets}_* \xrightarrow{忘 \times \mathrm{id}} \mathsf{Sets} \times \mathsf{Sets}_* \xrightarrow{\odot} \mathsf{Sets}_*,$$

where:

- $忘 : \mathsf{Sets}_* \to \mathsf{Sets}$ is the forgetful functor from pointed sets to sets.
- $\odot : \mathsf{Sets} \times \mathsf{Sets}_* \to \mathsf{Sets}_*$ is the tensor functor of Item 1 of Proposition 2.1.1.6.

[15] *Further Notation:* Also written $\rhd_{\mathsf{Sets}_*}$.

Remark 4.1.1.2. The right tensor product of pointed sets satisfies the following natural bijection:

$$\mathsf{Sets}_*(X \rhd Y, Z) \cong \mathrm{Hom}^{\otimes,\mathrm{R}}_{\mathsf{Sets}_*}(X \times Y, Z).$$

That is to say, the following data are in natural bijection:

1. Pointed maps $f\colon X \rhd Y \to Z$.
2. Maps of sets $f\colon X \times Y \to Z$ satisfying $f(x, y_0) = z_0$ for each $x \in X$.

Remark 4.1.1.3. The right tensor product of pointed sets may be described as follows:

- The right tensor product of (X, x_0) and (Y, y_0) is the pair $((X \rhd Y, x_0 \rhd y_0), \iota)$ consisting of
  - A pointed set $(X \rhd Y, x_0 \rhd y_0)$;
  - A right bilinear morphism of pointed sets $\iota\colon (X \times Y, (x_0, y_0)) \to X \rhd Y$;

  satisfying the following universal property:

  (UP) Given another such pair $((Z, z_0), f)$ consisting of
  - $*$ A pointed set (Z, z_0);
  - $*$ A right bilinear morphism of pointed sets $f\colon (X \times Y, (x_0, y_0)) \to X \rhd Y$;

  there exists a unique morphism of pointed sets $X \rhd Y \xrightarrow{\exists!} Z$ making the diagram

$$\begin{tikzcd} & X \rhd Y \arrow[d, dashed, "\exists!"] \\ X \times Y \arrow[ur, "\iota"] \arrow[r, "f"'] & Z \end{tikzcd}$$

  commute.

Construction 4.1.1.4. In detail, the **right tensor product of** (X, x_0) **and** (Y, y_0) is the pointed set $(X \rhd Y, [y_0])$ consisting of:

- *The Underlying Set.* The set $X \triangleright Y$ defined by

$$\begin{aligned} X \triangleright Y &\overset{\text{def}}{=} |X| \odot Y \\ &\cong \bigvee_{x \in X} (Y, y_0), \end{aligned}$$

  where $|X|$ denotes the underlying set of (X, x_0).

- *The Underlying Basepoint.* The point $[(x_0, y_0)]$ of $\bigvee_{x \in X}(Y, y_0)$, which is equal to $[(x, y_0)]$ for any $x \in X$.

Notation 4.1.1.5. We write[16] $x \triangleright y$ for the element $[(x, y)]$ of

$$X \triangleright Y \cong |X| \odot Y.$$

Remark 4.1.1.6. Employing the notation introduced in Notation 4.1.1.5, we have

$$x_0 \triangleright y_0 = x \triangleright y_0$$

for each $x \in X$, and

$$x \triangleright y_0 = x' \triangleright y_0$$

for each $x, x' \in X$.

Proposition 4.1.1.7. Let (X, x_0) and (Y, y_0) be pointed sets.

1. *Functoriality.* The assignments $X, Y, (X, Y) \mapsto X \triangleright Y$ define functors

$$\begin{aligned} X \triangleright - &: \mathsf{Sets}_* \longrightarrow \mathsf{Sets}_*, \\ - \triangleright Y &: \mathsf{Sets}_* \longrightarrow \mathsf{Sets}_*, \\ -_1 \triangleright -_2 &: \mathsf{Sets}_* \times \mathsf{Sets}_* \longrightarrow \mathsf{Sets}_*. \end{aligned}$$

   In particular, given pointed maps

$$\begin{aligned} f &: (X, x_0) \longrightarrow (A, a_0), \\ g &: (Y, y_0) \longrightarrow (B, b_0), \end{aligned}$$

   the induced map

$$f \triangleright g : X \triangleright Y \longrightarrow A \triangleright B$$

   is given by

$$[f \triangleright g](x \triangleright y) \overset{\text{def}}{=} f(x) \triangleright g(y)$$

   for each $x \triangleright y \in X \triangleright Y$.

[16] *Further Notation:* Also written $x \triangleright_{\mathsf{Sets}_*} y$.

2. *Adjointness I.* We have an adjunction

$$\left(X \rhd - \dashv [X,-]^{\rhd}_{\mathsf{Sets}_*}\right)\colon \quad \begin{tikzcd} \mathsf{Sets}_* \arrow[r, bend left, "X\rhd -"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_*, \arrow[l, bend left, "{[X,-]^{\rhd}_{\mathsf{Sets}_*}}"] \end{tikzcd}$$

witnessed by a bijection of sets

$$\mathrm{Hom}_{\mathsf{Sets}_*}(X \rhd Y, Z) \cong \mathrm{Hom}_{\mathsf{Sets}_*}(Y, [X,Z]^{\rhd}_{\mathsf{Sets}_*})$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, where $[X,Y]^{\rhd}_{\mathsf{Sets}_*}$ is the pointed set of Definition 4.2.1.1.

3. *Adjointness II.* The functor

$$- \rhd Y\colon \mathsf{Sets}_* \longrightarrow \mathsf{Sets}_*$$

does not admit a right adjoint.

4. *Adjointness III.* We have a bijection of sets

$$\mathrm{Hom}_{\mathsf{Sets}_*}(X \rhd Y, Z) \cong \mathrm{Hom}_{\mathsf{Sets}}(|X|, \mathsf{Sets}_*(Y,Z))$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

Proof. *Item 1, Functoriality*: Clear.
Item 2, Adjointness I: This follows from Item 3 of Proposition 2.1.1.6.
Item 3, Adjointness II: For $- \rhd Y$ to admit a right adjoint would require it to preserve colimits by **??**, **??** of **??**. However, we have

$$\begin{aligned} \mathrm{pt} \rhd X &\stackrel{\text{def}}{=} |\mathrm{pt}| \odot X \\ &\cong X \\ &\ncong \mathrm{pt}, \end{aligned}$$

and thus we see that $- \rhd Y$ does not have a right adjoint.
Item 4, Adjointness III: This follows from Item 2 of Proposition 2.1.1.6. □

Remark 4.1.1.8. Here is some intuition on why $- \rhd Y$ fails to be a left adjoint. Item 4 of Proposition 3.1.1.7 states that we have a natural bijection

$$\mathrm{Hom}_{\mathsf{Sets}_*}(X \rhd Y, Z) \cong \mathrm{Hom}_{\mathsf{Sets}}(|X|, \mathsf{Sets}_*(Y,Z)),$$

so it would be reasonable to wonder whether a natural bijection of the form

$$\mathrm{Hom}_{\mathsf{Sets}_*}(X \rhd Y, Z) \cong \mathrm{Hom}_{\mathsf{Sets}_*}(X, \mathbf{Sets}_*(Y, Z)),$$

also holds, which would give $- \rhd Y \dashv \mathbf{Sets}_*(Y, -)$. However, such a bijection would require every map

$$f \colon X \rhd Y \to Z$$

to satisfy

$$f(x_0 \rhd y) = z_0$$

for each $x \in X$, whereas we are imposing such a basepoint preservation condition only for elements of the form $x \rhd y_0$. Thus $\mathbf{Sets}_*(Y, -)$ can't be a right adjoint for $- \rhd Y$, and as shown by Item 3 of Proposition 4.1.1.7, no functor can.[17]

4.2 The Right Internal Hom of Pointed Sets

Let (X, x_0) and (Y, y_0) be pointed sets.

Definition 4.2.1.1. The **right internal Hom of pointed sets** is the functor

$$[-,-]^{\rhd}_{\mathsf{Sets}_*} \colon \mathsf{Sets}_*^{\mathsf{op}} \times \mathsf{Sets}_* \to \mathsf{Sets}_*$$

defined as the composition

$$\mathsf{Sets}_*^{\mathsf{op}} \times \mathsf{Sets}_* \xrightarrow{忘 \times \mathrm{id}} \mathsf{Sets}^{\mathsf{op}} \times \mathsf{Sets}_* \xrightarrow{\pitchfork} \mathsf{Sets}_*,$$

where:

- $忘 \colon \mathsf{Sets}_* \to \mathsf{Sets}$ is the forgetful functor from pointed sets to sets.
- $\pitchfork \colon \mathsf{Sets}^{\mathsf{op}} \times \mathsf{Sets}_* \to \mathsf{Sets}_*$ is the cotensor functor of Item 1 of Proposition 2.2.1.4.

Proof. For a proof that $[-,-]^{\rhd}_{\mathsf{Sets}_*}$ is indeed the right internal Hom of Sets_* with respect to the right tensor product of pointed sets, see Item 2 of Proposition 4.1.1.7. □

Remark 4.2.1.2. We have

$$[-,-]^{\lhd}_{\mathsf{Sets}_*} = [-,-]^{\rhd}_{\mathsf{Sets}_*}.$$

[17] The functor $\mathbf{Sets}_*(Y, -)$ is instead right adjoint to $- \wedge Y$, the smash product of pointed sets

Remark 4.2.1.3. The right internal Hom of pointed sets satisfies the following universal property:

$$\mathsf{Sets}_*(X \rhd Y, Z) \cong \mathsf{Sets}_*(Y, [X, Z]^{\rhd}_{\mathsf{Sets}_*})$$

That is to say, the following data are in bijection:

1. Pointed maps $f \colon X \rhd Y \to Z$.
2. Pointed maps $f \colon Y \to [X, Z]^{\rhd}_{\mathsf{Sets}_*}$.

Remark 4.2.1.4. In detail, the **right internal Hom of** (X, x_0) **and** (Y, y_0) is the pointed set $([X, Y]^{\rhd}_{\mathsf{Sets}_*}, [(y_0)_{x\in X}])$ consisting of

- *The Underlying Set.* The set $[X, Y]^{\rhd}_{\mathsf{Sets}_*}$ defined by
$$\begin{aligned}[X, Y]^{\rhd}_{\mathsf{Sets}_*} &\overset{\text{def}}{=} |X| \pitchfork Y\\ &\cong \bigwedge_{x\in X} (Y, y_0),\end{aligned}$$
where $|X|$ denotes the underlying set of (X, x_0);
- *The Underlying Basepoint.* The point $[(y_0)_{x\in X}]$ of $\bigwedge_{x\in X}(Y, y_0)$.

Proposition 4.2.1.5. Let (X, x_0) and (Y, y_0) be pointed sets.

1. *Functoriality.* The assignments $X, Y, (X, Y) \mapsto [X, Y]^{\rhd}_{\mathsf{Sets}_*}$ define functors
$$\begin{aligned}[X, -]^{\rhd}_{\mathsf{Sets}_*} &\colon \mathsf{Sets}_* \to \mathsf{Sets}_*,\\ [-, Y]^{\rhd}_{\mathsf{Sets}_*} &\colon \mathsf{Sets}_*^{\mathsf{op}} \to \mathsf{Sets}_*,\\ [-_1, -_2]^{\rhd}_{\mathsf{Sets}_*} &\colon \mathsf{Sets}_*^{\mathsf{op}} \times \mathsf{Sets}_* \to \mathsf{Sets}_*.\end{aligned}$$
In particular, given pointed maps
$$\begin{aligned}f &\colon (X, x_0) \to (A, a_0),\\ g &\colon (Y, y_0) \to (B, b_0),\end{aligned}$$
the induced map
$$[f, g]^{\rhd}_{\mathsf{Sets}_*} \colon [A, Y]^{\rhd}_{\mathsf{Sets}_*} \to [X, B]^{\rhd}_{\mathsf{Sets}_*}$$

is given by

$$[f, g]^{\rhd}_{\mathsf{Sets}_*}([(y_a)_{a \in A}]) \overset{\text{def}}{=} [(g(y_{f(x)}))_{x \in X}]$$

for each $[(y_a)_{a \in A}] \in [A, Y]^{\rhd}_{\mathsf{Sets}_*}$.

2. *Adjointness I.* We have an adjunction

$$\left(X \rhd - \dashv [X, -]^{\rhd}_{\mathsf{Sets}_*}\right) \colon \quad \begin{tikzcd} \mathsf{Sets}_* \arrow[r, bend left, "X \rhd -"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_*, \arrow[l, bend left, "{[X,-]^{\rhd}_{\mathsf{Sets}_*}}"] \end{tikzcd}$$

witnessed by a bijection of sets

$$\mathrm{Hom}_{\mathsf{Sets}_*}(X \rhd Y, Z) \cong \mathrm{Hom}_{\mathsf{Sets}_*}(Y, [X, Z]^{\rhd}_{\mathsf{Sets}_*})$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, where $[X, Y]^{\rhd}_{\mathsf{Sets}_*}$ is the pointed set of Definition 4.2.1.1.

3. *Adjointness II.* The functor

$$- \rhd Y \colon \mathsf{Sets}_* \longrightarrow \mathsf{Sets}_*$$

does not admit a right adjoint.

Proof. Item 1, Functoriality: Clear.
Item 2, Adjointness I: This is a repetition of Item 2 of Proposition 4.1.1.7, and is proved there.
Item 3, Adjointness II: This is a repetition of Item 3 of Proposition 4.1.1.7, and is proved there. □

4.3 The Right Skew Unit

Definition 4.3.1.1. The **right skew unit of the right tensor product of pointed sets** is the functor

$$\mathbb{1}^{\mathsf{Sets}_*, \rhd} \colon \mathrm{pt} \longrightarrow \mathsf{Sets}_*$$

defined by

$$\mathbb{1}^{\rhd}_{\mathsf{Sets}_*} \overset{\text{def}}{=} S^0.$$

of Definition 5.1.1.1. See Item 2 of Proposition 5.1.1.9.

4.4 The Right Skew Associator

Definition 4.4.1.1. The **skew associator of the right tensor product of pointed sets** is the natural transformation

$$\alpha^{\mathsf{Sets}_*,\triangleright} : \triangleright \circ (\mathrm{id}_{\mathsf{Sets}_*} \times \triangleright) \Longrightarrow \triangleright \circ (\triangleright \times \mathrm{id}_{\mathsf{Sets}_*}) \circ \boldsymbol{\alpha}^{\mathsf{Cats},-1}_{\mathsf{Sets}_*,\mathsf{Sets}_*,\mathsf{Sets}_*}$$

as in the diagram

$$\begin{tikzcd}
 & (\mathsf{Sets}_* \times \mathsf{Sets}_*) \times \mathsf{Sets}_* \arrow[dr, "\triangleright \times \mathrm{id}"] & \\
\mathsf{Sets}_* \times (\mathsf{Sets}_* \times \mathsf{Sets}_*) \arrow[ur, dashed, "\boldsymbol{\alpha}^{\mathsf{Cats},-1}_{\mathsf{Sets}_*,\mathsf{Sets}_*,\mathsf{Sets}_*}", "\sim"'] \arrow[d, "\mathrm{id} \times \triangleright"'] & & \mathsf{Sets}_* \times \mathsf{Sets}_* \arrow[d, "\triangleright"] \\
\mathsf{Sets}_* \times \mathsf{Sets}_* \arrow[rr, "\triangleright"'] \arrow[urr, phantom, "\Uparrow \alpha^{\mathsf{Sets}_*,\triangleright}"] & & \mathsf{Sets}_*,
\end{tikzcd}$$

whose component

$$\alpha^{\mathsf{Sets}_*,\triangleright}_{X,Y,Z} : X \triangleright (Y \triangleright Z) \to (X \triangleright Y) \triangleright Z$$

at $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$ is given by

$$\begin{aligned}
X \triangleright (Y \triangleright Z) &\overset{\text{def}}{=} |X| \odot (Y \triangleright Z) \\
&\overset{\text{def}}{=} |X| \odot (|Y| \odot Z) \\
&\cong \bigvee_{x \in X} (|Y| \odot Z) \\
&\cong \bigvee_{x \in X} \Big(\bigvee_{y \in Y} Z\Big) \\
&\to \bigvee_{[(x,y)] \in \bigvee_{x \in X} Y} Z \\
&\cong \bigvee_{[(x,y)] \in |X| \odot Y} Z \\
&\cong ||X| \odot Y| \odot Z \\
&\overset{\text{def}}{=} |X \triangleright Y| \odot Z \\
&\overset{\text{def}}{=} (X \triangleright Y) \triangleright Z,
\end{aligned}$$

where the map

$$\bigvee_{x\in X}\Big(\bigvee_{y\in Y} Z\Big) \to \bigvee_{[(x,y)]\in\bigvee_{x\in X} Y} Z$$

is given by $[(x,[(y,z)])] \mapsto [([(x,y)],z)]$.

Proof. (Proven below in a bit.) □

Remark 4.4.1.2. Unwinding the notation for elements, we have

$$\begin{aligned}[(x,[(y,z)])] &\stackrel{\text{def}}{=} [(x, y \rhd z)]\\ &\stackrel{\text{def}}{=} x \rhd (y \rhd z)\end{aligned}$$

and

$$\begin{aligned}[([(x,y)],z)] &\stackrel{\text{def}}{=} [(x \rhd y, z)]\\ &\stackrel{\text{def}}{=} (x \rhd y) \rhd z.\end{aligned}$$

So, in other words, $\alpha^{\mathsf{Sets}_*,\rhd}_{X,Y,Z}$ acts on elements via

$$\alpha^{\mathsf{Sets}_*,\rhd}_{X,Y,Z}(x \rhd (y \rhd z)) \stackrel{\text{def}}{=} (x \rhd y) \rhd z$$

for each $x \rhd (y \rhd z) \in X \rhd (Y \rhd Z)$.

Remark 4.4.1.3. Taking $y = y_0$, we see that the morphism $\alpha^{\mathsf{Sets}_*,\rhd}_{X,Y,Z}$ acts on elements as

$$\alpha^{\mathsf{Sets}_*,\rhd}_{X,Y,Z}(x \rhd (y_0 \rhd z)) \stackrel{\text{def}}{=} (x \rhd y_0) \rhd z.$$

However, by the definition of $\rhd$, we have $x \rhd y_0 = x' \rhd y_0$ for all $x, x' \in X$, preventing $\alpha^{\mathsf{Sets}_*,\rhd}_{X,Y,Z}$ from being non-invertible.

Proof. Firstly, note that, given $(X,x_0), (Y,y_0), (Z,z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, the map

$$\alpha^{\mathsf{Sets}_*,\rhd}_{X,Y,Z} \colon X \rhd (Y \rhd Z) \to (X \rhd Y) \rhd Z$$

is indeed a morphism of pointed sets, as we have

$$\alpha^{\mathsf{Sets}_*,\rhd}_{X,Y,Z}(x_0 \rhd (y_0 \rhd z_0)) = (x_0 \rhd y_0) \rhd z_0.$$

Next, we claim that $\alpha^{\mathsf{Sets}_*,\rhd}$ is a natural transformation. We need to show that, given morphisms of pointed sets

$$\begin{aligned} f &\colon (X,x_0) \to (X',x'_0),\\ g &\colon (Y,y_0) \to (Y',y'_0),\\ h &\colon (Z,z_0) \to (Z',z'_0)\end{aligned}$$

the diagram

$$\begin{tikzcd}
X \rhd (Y \rhd Z) \arrow[r, "f \rhd (g \rhd h)"] \arrow[d, "\alpha^{\mathsf{Sets}_*,\rhd}_{X,Y,Z}"'] & X' \rhd (Y' \rhd Z') \arrow[d, "\alpha^{\mathsf{Sets}_*,\rhd}_{X',Y',Z'}"] \\
(X \rhd Y) \rhd Z \arrow[r, "(f \rhd g) \rhd h"'] & (X' \rhd Y') \rhd Z'
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
x \rhd (y \rhd z) \arrow[r, mapsto] \arrow[d, mapsto] & f(x) \rhd (g(y) \rhd h(z)) \arrow[d, mapsto] \\
(x \rhd y) \rhd z \arrow[r, mapsto] & (f(x) \rhd g(y)) \rhd h(z)
\end{tikzcd}$$

and hence indeed commutes, showing $\alpha^{\mathsf{Sets}_*,\rhd}$ to be a natural transformation. This finishes the proof. □

4.5 The Right Skew Left Unitor

Definition 4.5.1.1. The **skew left unitor of the right tensor product of pointed sets** is the natural transformation

$$\lambda^{\mathsf{Sets}_*,\rhd} \colon \boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*} \overset{\sim}{\Longrightarrow} \rhd \circ (\mathbb{1}^{\mathsf{Sets}_*} \times \mathrm{id}_{\mathsf{Sets}_*}) \qquad \begin{tikzcd}
\mathsf{pt} \times \mathsf{Sets}_* \arrow[r, "\mathbb{1}^{\mathsf{Sets}_*} \times \mathsf{id}"] \arrow[dr, dashed, bend right, "\boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}"'] & \mathsf{Sets}_* \times \mathsf{Sets}_* \arrow[d, "\rhd"] \arrow[dl, phantom, "\Nearrow \lambda^{\mathsf{Sets}_*,\rhd}"] \\
 & \mathsf{Sets}_*,
\end{tikzcd}$$

whose component

$$\lambda^{\mathsf{Sets}_*,\rhd}_X \colon X \to S^0 \rhd X$$

at $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$ is given by the composition

$$\begin{aligned} X &\to X \vee X \\ &\cong |S^0| \odot X \\ &\cong S^0 \rhd X, \end{aligned}$$

where $X \to X \vee X$ is the map sending X to the second factor of X in $X \vee X$.

Proof. (Proven below in a bit.) $\square$

Remark 4.5.1.2. In other words, $\lambda_X^{\mathsf{Sets}_*,\triangleright}$ acts on elements as

$$\lambda_X^{\mathsf{Sets}_*,\triangleright}(x) \overset{\text{def}}{=} [(1, x)]$$

i.e. by

$$\lambda_X^{\mathsf{Sets}_*,\triangleright}(x) \overset{\text{def}}{=} 1 \triangleright x$$

for each $x \in X$.

Remark 4.5.1.3. The morphism $\lambda_X^{\mathsf{Sets}_*,\triangleright}$ is non-invertible, as it is non-surjective when viewed as a map of sets, since the elements $0 \triangleright x$ of $S^0 \triangleright X$ with $x \neq x_0$ are outside the image of $\lambda_X^{\mathsf{Sets}_*,\triangleright}$, which sends x to $1 \triangleright x$.

Proof. Firstly, note that, given $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, the map

$$\lambda_X^{\mathsf{Sets}_*,\triangleright} : X \to S^0 \triangleright X$$

is indeed a morphism of pointed sets, as we have

$$\begin{aligned} \lambda_X^{\mathsf{Sets}_*,\triangleright}(x_0) &= 1 \triangleright x_0 \\ &= 0 \triangleright x_0. \end{aligned}$$

Next, we claim that $\lambda^{\mathsf{Sets}_*,\triangleright}$ is a natural transformation. We need to show that, given a morphism of pointed sets

$$f : (X, x_0) \to (Y, y_0),$$

the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\lambda_X^{\mathsf{Sets}_*,\triangleright}"'] & Y \arrow[d, "\lambda_Y^{\mathsf{Sets}_*,\triangleright}"] \\ S^0 \triangleright X \arrow[r, "\mathrm{id}_{S^0} \triangleright f"'] & S^0 \triangleright Y \end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd} x \arrow[r, mapsto] \arrow[d, mapsto] & f(x) \arrow[d, mapsto] \\ 1 \triangleright x \arrow[r, mapsto] & 1 \triangleright f(x) \end{tikzcd}$$

and hence indeed commutes, showing $\lambda^{\mathsf{Sets}_*,\rhd}$ to be a natural transformation. This finishes the proof. □

4.6 The Right Skew Right Unitor

Definition 4.6.1.1. The **skew right unitor of the right tensor product of pointed sets** is the natural transformation

$$\rho^{\mathsf{Sets}_*,\rhd} \colon \rhd \circ (\mathrm{id} \times \mathbb{1}^{\mathsf{Sets}_*}) \overset{\sim}{\Longrightarrow} \boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}, \qquad \begin{tikzcd} \mathsf{Sets}_* \times \mathrm{pt} \arrow[r, "\mathrm{id}\times\mathbb{1}^{\mathsf{Sets}_*}"] \arrow[dr, dashed, bend right, "\boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}"'] & \mathsf{Sets}_* \times \mathsf{Sets}_* \arrow[d, "\rhd"] \arrow[dl, phantom, "\Swarrow\rho^{\mathsf{Sets}_*,\rhd}"] \\ & \mathsf{Sets}_*, \end{tikzcd}$$

whose component

$$\rho^{\mathsf{Sets}_*,\rhd}_X \colon X \rhd S^0 \to X$$

at $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$ is given by the composition

$$\begin{aligned} X \rhd S^0 &\cong |X| \odot S^0 \\ &\cong \bigvee_{x\in X} S^0 \\ &\to X, \end{aligned}$$

where $\bigvee_{x\in X} S^0 \to X$ is the map given by

$$\begin{aligned} [(x,0)] &\mapsto x_0, \\ [(x,1)] &\mapsto x. \end{aligned}$$

Proof. (Proven below in a bit.) □

Remark 4.6.1.2. In other words, $\rho^{\mathsf{Sets}_*,\rhd}_X$ acts on elements as

$$\begin{aligned} \rho^{\mathsf{Sets}_*,\rhd}_X(x \rhd 0) &\overset{\text{def}}{=} x_0, \\ \rho^{\mathsf{Sets}_*,\rhd}_X(x \rhd 1) &\overset{\text{def}}{=} x \end{aligned}$$

for each $x \rhd 1 \in X \rhd S^0$.

Remark 4.6.1.3. The morphism $\rho_X^{\mathsf{Sets}_*,\rhd}$ is almost invertible, with its would-be-inverse

$$\phi_X \colon X \to X \rhd S^0$$

given by

$$\phi_X(x) \stackrel{\text{def}}{=} x \rhd 1$$

for each $x \in X$. Indeed, we have

$$\begin{aligned}[\rho_X^{\mathsf{Sets}_*,\rhd} \circ \phi](x) &= \rho_X^{\mathsf{Sets}_*,\rhd}(\phi(x)) \\ &= \rho_X^{\mathsf{Sets}_*,\rhd}(x \rhd 1) \\ &= x \\ &= [\mathrm{id}_X](x)\end{aligned}$$

so that

$$\rho_X^{\mathsf{Sets}_*,\rhd} \circ \phi = \mathrm{id}_X$$

and

$$\begin{aligned}[\phi \circ \rho_X^{\mathsf{Sets}_*,\rhd}](x \rhd 1) &= \phi(\rho_X^{\mathsf{Sets}_*,\rhd}(x \rhd 1)) \\ &= \phi(x) \\ &= x \rhd 1 \\ &= [\mathrm{id}_{X \rhd S^0}](x \rhd 1),\end{aligned}$$

but

$$\begin{aligned}[\phi \circ \rho_X^{\mathsf{Sets}_*,\rhd}](x \rhd 0) &= \phi(\rho_X^{\mathsf{Sets}_*,\rhd}(x \rhd 0)) \\ &= \phi(x_0) \\ &= 1 \rhd x_0,\end{aligned}$$

where $x \rhd 0 \neq 1 \rhd x_0$. Thus

$$\phi \circ \rho_X^{\mathsf{Sets}_*,\rhd} \stackrel{?}{=} \mathrm{id}_{X \rhd S^0}$$

holds for all elements in $X \rhd S^0$ except one.

Proof. Firstly, note that, given $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, the map

$$\rho_X^{\mathsf{Sets}_*,\rhd} : X \rhd S^0 \to X$$

is indeed a morphism of pointed sets as we have

$$\rho_X^{\mathsf{Sets}_{*},\triangleright}(x_0 \triangleright 0) = x_0.$$

Next, we claim that $\rho^{\mathsf{Sets}_{*},\triangleright}$ is a natural transformation. We need to show that, given a morphism of pointed sets

$$f\colon (X, x_0) \to (Y, y_0),$$

the diagram

$$\begin{tikzcd}
X \triangleright S^0 \arrow[r, "f \triangleright \mathrm{id}_{S^0}"] \arrow[d, "\rho_X^{\mathsf{Sets}_{*},\triangleright}"'] & Y \triangleright S^0 \arrow[d, "\rho_Y^{\mathsf{Sets}_{*},\triangleright}"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
x \triangleright 0 \arrow[d, mapsto] & \\
x_0 \arrow[r, mapsto] & f(x_0)
\end{tikzcd}
\qquad
\begin{tikzcd}
x \triangleright 0 \arrow[r, mapsto] & f(x) \triangleright 0 \arrow[d, mapsto] \\
& y_0
\end{tikzcd}$$

and

$$\begin{tikzcd}
x \triangleright 1 \arrow[r, mapsto] \arrow[d, mapsto] & f(x) \triangleright 1 \arrow[d, mapsto] \\
x \arrow[r, mapsto] & f(x)
\end{tikzcd}$$

and hence indeed commutes, showing $\rho^{\mathsf{Sets}_{*},\triangleright}$ to be a natural transformation. This finishes the proof. □

4.7 The Diagonal

Definition 4.7.1.1. The **diagonal of the right tensor product of pointed sets** is the natural transformation

$$\Delta^{\triangleright} \colon \mathrm{id}_{\mathsf{Sets}_*} \Longrightarrow \triangleright \circ \Delta^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}, \qquad \begin{tikzcd} \mathsf{Sets}_* \arrow[rr, bend left, "\mathrm{id}_{\mathsf{Sets}_*}"] \arrow[dr, "\Delta^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}"'] \arrow[rr, phantom, shift right=3ex, "\Downarrow \Delta^{\triangleright}"] & & \mathsf{Sets}_* \\ & \mathsf{Sets}_* \times \mathsf{Sets}_*, \arrow[ur, "\triangleright"'] & \end{tikzcd}$$

whose component

$$\Delta^{\triangleright}_X \colon (X, x_0) \longrightarrow (X \triangleright X, x_0 \triangleright x_0)$$

at $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$ is given by

$$\Delta^{\triangleright}_X(x) \stackrel{\mathrm{def}}{=} x \triangleright x$$

for each $x \in X$.

Proof. Being a Morphism of Pointed Sets: We have

$$\Delta^{\triangleright}_X(x_0) \stackrel{\mathrm{def}}{=} x_0 \triangleright x_0,$$

and thus $\Delta^{\triangleright}_X$ is a morphism of pointed sets.
Naturality: We need to show that, given a morphism of pointed sets

$$f \colon (X, x_0) \to (Y, y_0),$$

the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\Delta^{\triangleright}_X"'] & Y \arrow[d, "\Delta^{\triangleright}_Y"] \\ X \triangleright X \arrow[r, "f \triangleright f"'] & Y \triangleright Y \end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd} x \arrow[r, mapsto] \arrow[d, mapsto] & f(x) \arrow[d, mapsto] \\ x \triangleright x \arrow[r, mapsto] & f(x) \triangleright f(x) \end{tikzcd}$$

and hence indeed commutes, showing $\Delta^{\triangleright}$ to be natural. □

4.8 The Right Skew Monoidal Structure on Pointed Sets Associated to $\triangleright$

Proposition 4.8.1.1. The category Sets_* admits a right-closed right skew monoidal category structure consisting of

- *The Underlying Category.* The category Sets_* of pointed sets;
- *The Right Skew Monoidal Product.* The right tensor product functor

$$\triangleright\colon \mathsf{Sets}_* \times \mathsf{Sets}_* \longrightarrow \mathsf{Sets}_*$$

  of Definition 4.1.1.1;
- *The Right Internal Skew Hom.* The right internal Hom functor

$$[-,-]^{\triangleright}_{\mathsf{Sets}_*}\colon \mathsf{Sets}^{\mathsf{op}}_* \times \mathsf{Sets}_* \longrightarrow \mathsf{Sets}_*$$

  of Definition 4.2.1.1;
- *The Right Skew Monoidal Unit.* The functor

$$\mathbb{1}^{\mathsf{Sets}_*,\triangleright}\colon \mathsf{pt} \longrightarrow \mathsf{Sets}_*$$

  of Definition 4.3.1.1;
- *The Right Skew Associators.* The natural transformation

$$\alpha^{\mathsf{Sets}_*,\triangleright}\colon \triangleright \circ (\mathrm{id}_{\mathsf{Sets}_*} \times \triangleright) \Longrightarrow \triangleright \circ (\triangleright \times \mathrm{id}_{\mathsf{Sets}_*}) \circ \boldsymbol{\alpha}^{\mathsf{Cats},-1}_{\mathsf{Sets}_*,\mathsf{Sets}_*,\mathsf{Sets}_*}$$

  of Definition 4.4.1.1;
- *The Right Skew Left Unitors.* The natural transformation

$$\lambda^{\mathsf{Sets}_*,\triangleright}\colon \boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*} \overset{\sim}{\Longrightarrow} \triangleright \circ (\mathbb{1}^{\mathsf{Sets}_*} \times \mathrm{id}_{\mathsf{Sets}_*})$$

  of Definition 4.5.1.1;
- *The Right Skew Right Unitors.* The natural transformation

$$\rho^{\mathsf{Sets}_*,\triangleright}\colon \triangleright \circ (\mathrm{id} \times \mathbb{1}^{\mathsf{Sets}_*}) \overset{\sim}{\Longrightarrow} \boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}$$

  of Definition 4.6.1.1.

Proof. The Pentagon Identity: Let (W, w_0), (X, x_0), (Y, y_0) and (Z, z_0) be pointed sets. We have to show that the diagram

$$\begin{tikzcd}
 & & W \triangleright ((X \triangleright Y) \triangleright Z) \arrow[drr, "\alpha^{\mathsf{Sets}_*,\triangleright}_{W,X\triangleright Y,Z}"] & & \\
W \triangleright (X \triangleright (Y \triangleright Z)) \arrow[urr, "\alpha^{\mathsf{Sets}_*,\triangleright}_{W,X,Y} \triangleright \mathrm{id}_Z"] \arrow[dr, "\alpha^{\mathsf{Sets}_*,\triangleright}_{W\triangleright X,Y,Z}"'] & & & & (W \triangleright (X \triangleright Y)) \triangleright Z \arrow[dl, "\mathrm{id}_W \triangleright \alpha^{\mathsf{Sets}_*,\triangleright}_{X,Y,Z}"] \\
 & (W \triangleright X) \triangleright (Y \triangleright Z) \arrow[rr, "\alpha^{\mathsf{Sets}_*,\triangleright}_{W,X,Y\triangleright Z}"'] & & ((W \triangleright X) \triangleright Y) \triangleright Z &
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
 & & w \triangleright ((x \triangleright y) \triangleright z) \arrow[drr, mapsto] & & \\
w \triangleright (x \triangleright (y \triangleright z)) \arrow[urr, mapsto] \arrow[dr, mapsto] & & & & (w \triangleright (x \triangleright y)) \triangleright z \arrow[dl, mapsto] \\
 & (w \triangleright x) \triangleright (y \triangleright z) \arrow[rr, mapsto] & & ((w \triangleright x) \triangleright y) \triangleright z &
\end{tikzcd}$$

and thus we see that the pentagon identity is satisfied.
The Right Skew Left Triangle Identity: Let (X, x_0) and (Y, y_0) be pointed sets. We have to show that the diagram

$$\begin{tikzcd} X \rhd Y \arrow[d, "\lambda^{\mathsf{Sets}_*,\rhd}_{X \rhd Y}"'] \arrow[dr, "\lambda^{\mathsf{Sets}_*,\rhd}_{X} \rhd \mathrm{id}_Y"] & \\ S^0 \rhd (X \rhd Y) \arrow[r, "\alpha^{\mathsf{Sets}_*,\rhd}_{S^0,X,Y}"'] & (S^0 \rhd X) \rhd Y \end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd} x \rhd y \arrow[d, mapsto] \arrow[dr, mapsto] & \\ 1 \rhd (x \rhd y) \arrow[r, mapsto] & (1 \rhd x) \rhd y \end{tikzcd}$$

and hence indeed commutes. Thus the left skew triangle identity is satisfied.
The Right Skew Right Triangle Identity: Let (X, x_0) and (Y, y_0) be pointed sets. We have to show that the diagram

$$\begin{tikzcd} X \rhd (Y \rhd S^0) \arrow[r, "\mathrm{id}_X \rhd \rho^{\mathsf{Sets}_*,\rhd}_{Y}"] \arrow[dr, "\alpha^{\mathsf{Sets}_*,\rhd}_{S^0,X,Y}"'] & (X \rhd Y) \rhd S^0 \arrow[d, "\rho^{\mathsf{Sets}_*,\rhd}_{X \rhd Y}"] \\ & X \rhd Y \end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd} x \rhd (y \rhd 0) \arrow[r, mapsto] \arrow[dr, mapsto] & (x \rhd y) \rhd 0 \arrow[d, mapsto] \\ & x \rhd y_0 = x_0 \rhd y_0 \end{tikzcd}$$

and

$$\begin{tikzcd} x \rhd (y \rhd 1) \arrow[r, mapsto] \arrow[dr, mapsto] & (x \rhd y) \rhd 1 \arrow[d, mapsto] \\ & x \rhd y \end{tikzcd}$$

and hence indeed commutes. Thus the right skew triangle identity is satisfied.
The Right Skew Middle Triangle Identity: Let (X, x_0) and (Y, y_0) be pointed sets. We have to show that the diagram

$$\begin{tikzcd}
X \rhd Y \arrow[r, equal] \arrow[d, "\mathrm{id}_X \rhd \lambda^{\mathsf{Sets}_*,\rhd}_Y"'] & X \rhd Y \\
X \rhd (S^0 \rhd Y) \arrow[r, "\alpha^{\mathsf{Sets}_*,\rhd}_{X,S^0,Y}"'] & (X \rhd S^0) \rhd Y \arrow[u, "\rho^{\mathsf{Sets}_*,\rhd}_X \rhd \mathrm{id}_Y"']
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
x \rhd y \arrow[r, mapsto] \arrow[d, mapsto] & x \rhd y \\
x \rhd (1 \rhd y) \arrow[r, mapsto] & (x \rhd 1) \rhd y \arrow[u, mapsto]
\end{tikzcd}$$

and hence indeed commutes. Thus the right skew triangle identity is satisfied.
The Zig-Zag Identity: We have to show that the diagram

$$\begin{tikzcd}
S^0 \arrow[r, "\lambda^{\mathsf{Sets}_*,\rhd}_{S^0}"] \arrow[dr, equal] & S^0 \rhd S^0 \arrow[d, "\rho^{\mathsf{Sets}_*,\rhd}_{S^0}"] \\
 & S^0
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
0 \arrow[r, mapsto] \arrow[dr, mapsto] & 1 \rhd 0 \arrow[d, mapsto] \\
 & 0
\end{tikzcd}$$

and

$$\begin{tikzcd}
1 \arrow[r, mapsto] \arrow[dr, mapsto] & 1 \rhd 1 \arrow[d, mapsto] \\
 & 1
\end{tikzcd}$$

and hence indeed commutes. Thus the zig-zag identity is satisfied.
Right Skew Monoidal Right-Closedness: This follows from Item 2 of Proposition 4.1.1.7. □

4.9 Monoids With Respect to the Right Tensor Product of Pointed Sets

Proposition 4.9.1.1. The category of monoids on $(\mathsf{Sets}_*, \triangleright, S^0)$ is isomorphic to the category of "monoids with right zero"[18] and morphisms between them.

Proof. Monoids on $(\mathsf{Sets}_*, \triangleright, S^0)$: A monoid on $(\mathsf{Sets}_*, \triangleright, S^0)$ consists of:

- *The Underlying Object.* A pointed set $(A, 0_A)$.
- *The Multiplication Morphism.* A morphism of pointed sets

  $$\mu_A \colon A \triangleright A \to A,$$

  determining a right bilinear morphism of pointed sets

  $$\begin{aligned} A \times A &\longrightarrow A \\ (a, b) &\longmapsto ab. \end{aligned}$$

- *The Unit Morphism.* A morphism of pointed sets

  $$\eta_A \colon S^0 \to A$$

  picking an element 1_A of A.

satisfying the following conditions:

[18] A monoid with right zero is defined similarly as the monoids with zero of **??**. Succinctly, they are monoids (A, μ_A, η_A) with a special element 0_A satisfying

$$0_A a = 0_A$$

for each $a \in A$.

1. *Associativity.* The diagram

$$
\begin{tikzcd}
 & A \rhd (A \rhd A) \arrow[dr, "\mathrm{id}_A \rhd \mu_A"] & \\
(A \rhd A) \rhd A \arrow[ur, "\alpha^{\mathsf{Sets}_*,\rhd}_{A,A,A}"] \arrow[d, "\mu_A \rhd \mathrm{id}_A"'] & & A \rhd A \arrow[d, "\mu_A"] \\
A \rhd A \arrow[rr, "\mu_A"'] & & A
\end{tikzcd}
$$

2. *Left Unitality.* The diagram

$$
\begin{tikzcd}
A \arrow[r, "\lambda^{\mathsf{Sets}_*,\rhd}_A"] \arrow[d, equal] & S^0 \rhd A \arrow[d, "\eta_A \times \mathrm{id}_A"] \\
A & A \rhd A \arrow[l, "\mu_A"]
\end{tikzcd}
$$

commutes.

3. *Right Unitality.* The diagram

$$
\begin{tikzcd}
A \rhd S^0 \arrow[r, "\mathrm{id}_A \times \eta_A"] \arrow[dr, "\rho^{\mathsf{Sets}_*,\rhd}_A"'] & A \rhd A \arrow[d, "\mu_A"] \\
 & A
\end{tikzcd}
$$

commutes.

Being a right-bilinear morphism of pointed sets, the multiplication map satisfies

$$0_A a = 0_A$$

for each $a \in A$. Now, the associativity, left unitality, and right unitality conditions act on elements as follows:

1. *Associativity.* The associativity condition acts as

$$\begin{tikzcd} (a \rhd b) \rhd c \arrow[d, mapsto] & & \\ ab \rhd c \arrow[r, mapsto] & (ab)c \end{tikzcd} \qquad \begin{tikzcd} & a \rhd (b \rhd c) \arrow[dr, mapsto] & \\ (a \rhd b) \rhd c \arrow[ur, mapsto] & & a \rhd bc \arrow[d, mapsto] \\ & & a(bc) \end{tikzcd}$$

This gives

$$(ab)c = a(bc)$$

for each $a, b, c \in A$.

2. *Left Unitality.* The left unitality condition acts as

$$\begin{tikzcd} a \arrow[d, mapsto] \\ a \end{tikzcd} \qquad \begin{tikzcd} a \arrow[r, mapsto] & 1 \rhd a \arrow[d, mapsto] \\ 1_A a & 1_A \rhd a \arrow[l, mapsto] \end{tikzcd}$$

This gives

$$1_A a = a$$

for each $a \in A$.

3. *Right Unitality.* The right unitality condition acts:

   (a) On $1 \rhd 0$ as

$$\begin{tikzcd} 1 \rhd 0 \arrow[dr, mapsto] & \\ & 0_A \end{tikzcd} \qquad \begin{tikzcd} a \rhd 0 \arrow[r, mapsto] \arrow[dr, mapsto] & a \rhd 0_A \arrow[d, mapsto] \\ & a0_A. \end{tikzcd}$$

   (b) On $a \rhd 1$ as

$$\begin{tikzcd} a \rhd 1 \arrow[dr, mapsto] & \\ & a \end{tikzcd} \qquad \begin{tikzcd} a \rhd 1 \arrow[r, mapsto] \arrow[dr, mapsto] & a \rhd 1_A \arrow[d, mapsto] \\ & a1_A. \end{tikzcd}$$

This gives

$$\begin{aligned} a1_A &= a, \\ a0_A &= 0_A \end{aligned}$$

for each $a \in A$.

Thus we see that monoids with respect to $\rhd$ are exactly monoids with right zero. *Morphisms of Monoids on* $(\mathsf{Sets}_*, \rhd, S^0)$: A morphism of monoids on $(\mathsf{Sets}_*, \rhd, S^0)$ from $(A, \mu_A, \eta_A, 0_A)$ to $(B, \mu_B, \eta_B, 0_B)$ is a morphism of pointed sets

$$f\colon (A, 0_A) \to (B, 0_B)$$

satisfying the following conditions:

1. *Compatibility With the Multiplication Morphisms.* The diagram

$$\begin{tikzcd} A \rhd A \arrow[r, "f \rhd f"] \arrow[d, "\mu_A"'] & B \rhd B \arrow[d, "\mu_B"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

commutes.

2. *Compatibility With the Unit Morphisms.* The diagram

$$\begin{tikzcd} S^0 \arrow[r, "\eta_A"] \arrow[dr, "\eta_B"'] & A \arrow[d, "f"] \\ & B \end{tikzcd}$$

commutes.

These act on elements as

$$\begin{tikzcd} a \rhd b \arrow[d, mapsto] & \\ ab \arrow[r, mapsto] & f(ab) \end{tikzcd} \qquad \begin{tikzcd} a \rhd b \arrow[r, mapsto] & f(a) \rhd f(b) \arrow[d, mapsto] \\ & f(a)f(b) \end{tikzcd}$$

and

$$\begin{tikzcd} 0 \arrow[dr, mapsto] & & 0 \arrow[r, mapsto] & 0_A \arrow[d, mapsto] \\ & 0_B & & f(0_A) \end{tikzcd}$$

and

$$\begin{tikzcd} 1 \arrow[dr, mapsto] & & 1 \arrow[r, mapsto] & 1_A \arrow[d, mapsto] \\ & 1_B & & f(1_A) \end{tikzcd}$$

giving

$$\begin{aligned} f(ab) &= f(a)f(b), \\ f(0_A) &= 0_B, \\ f(1_A) &= 1_B, \end{aligned}$$

for each $a, b \in A$, which is exactly a morphism of monoids with right zero.
Identities and Composition: Similarly, the identities and composition of $\mathsf{Mon}(\mathsf{Sets}_*, \rhd, S^0)$ can be easily seen to agree with those of monoids with right zero, which finishes the proof. □

5 The Smash Product of Pointed Sets

5.1 Foundations

Let (X, x_0) and (Y, y_0) be pointed sets.

Definition 5.1.1.1. The **smash product of** (X, x_0) **and** (Y, y_0)[19] is the pointed set $X \wedge Y$[20] satisfying the bijection

$$\mathsf{Sets}_*(X \wedge Y, Z) \cong \mathrm{Hom}^{\otimes}_{\mathsf{Sets}_*}(X \times Y, Z),$$

naturally in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

[19] *Further Terminology:* In the context of monoids with zero as models for $\mathbb{F}_1$-algebras, the smash product $X \wedge Y$ is also called the **tensor product of** $\mathbb{F}_1$**-modules of** (X, x_0) **and** (Y, y_0) or the **tensor product of** (X, x_0) **and** (Y, y_0) **over** $\mathbb{F}_1$.

[20] *Further Notation:* In the context of monoids with zero as models for $\mathbb{F}_1$-algebras, the smash

Remark 5.1.1.2. That is to say, the smash product of pointed sets is defined so as to induce a bijection between the following data:

- Pointed maps $f : X \wedge Y \to Z$.
- Maps of sets $f : X \times Y \to Z$ satisfying

$$\begin{aligned} f(x_0, y) &= z_0, \\ f(x, y_0) &= z_0 \end{aligned}$$

  for each $x \in X$ and each $y \in Y$.

Remark 5.1.1.3. The smash product of pointed sets may be described as follows:

- The smash product of (X, x_0) and (Y, y_0) is the pair $((X \wedge Y, x_0 \wedge y_0), \iota)$ consisting of
  - A pointed set $(X \wedge Y, x_0 \wedge y_0)$;
  - A bilinear morphism of pointed sets $\iota\colon (X \times Y, (x_0, y_0)) \to X \wedge Y$;

  satisfying the following universal property:

  (UP) Given another such pair $((Z, z_0), f)$ consisting of
  * A pointed set (Z, z_0);
  * A bilinear morphism of pointed sets $f\colon (X \times Y, (x_0, y_0)) \to X \wedge Y$;

  there exists a unique morphism of pointed sets $X \wedge Y \xrightarrow{\exists!} Z$ making the diagram

$$\begin{tikzcd} & X \wedge Y \arrow[d, dashed, "\exists!"] \\ X \times Y \arrow[ur, "\iota"] \arrow[r, "f"'] & Z \end{tikzcd}$$

  commute.

Construction 5.1.1.4. Concretely, the **smash product of** (X, x_0) **and** (Y, y_0) is the pointed set $(X \wedge Y, x_0 \wedge y_0)$ consisting of

- *The Underlying Set.* The set $X \wedge Y$ defined by

$$X \wedge Y \cong (X \times Y)/{\sim_R},$$

where $\sim_R$ is the equivalence relation on $X \times Y$ obtained by declaring

$$\begin{aligned}(x_0, y) &\sim_R (x_0, y'),\\ (x, y_0) &\sim_R (x', y_0)\end{aligned}$$

for each $x, x' \in X$ and each $y, y' \in Y$;

- *The Basepoint.* The element $[(x_0, y_0)]$ of $X \wedge Y$ given by the equivalence class of (x_0, y_0) under the equivalence relation $\sim$ on $X \times Y$.

Proof. By Equivalence Relations and Apartness Relations, Item 6 of Proposition 5.2.1.3, we have a natural bijection

$$\mathsf{Sets}_*(X \wedge Y, Z) \cong \mathrm{Hom}^R_{\mathsf{Sets}}(X \times Y, Z).$$

Now, by definition, $\mathrm{Hom}^R_{\mathsf{Sets}}(X \times Y, Z)$ is the set

$$\mathrm{Hom}^R_{\mathsf{Sets}}(X \times Y, Z) \stackrel{\text{def}}{=} \left\{ f \in \mathrm{Hom}_{\mathsf{Sets}}(X \times Y, Z) \;\middle|\; \begin{array}{l}\text{for each } x, y \in X, \text{ if}\\ (x, y) \sim_R (x', y'), \text{ then}\\ f(x, y) = f(x', y')\end{array} \right\}.$$

However, the condition $(x, y) \sim_R (x', y')$ only holds when:

1. We have $x = x'$ and $y = y'$.
2. The following conditions are satisfied:
   (a) We have $x = x_0$ or $y = y_0$.
   (b) We have $x' = x_0$ or $y' = y_0$.

So, given $f \in \mathrm{Hom}_{\mathsf{Sets}}(X \times Y, Z)$ with a corresponding $\overline{f} \colon X \wedge Y \to Z$, the latter case above implies

$$\begin{aligned} f(x_0, y) &= f(x, y_0)\\ &= f(x_0, y_0), \end{aligned}$$

product $X \wedge Y$ is also denoted $X \otimes_{\mathbb{F}_1} Y$.

and since $\overline{f}\colon X \wedge Y \to Z$ is a pointed map, we have

$$\begin{aligned} f(x_0, y_0) &= \overline{f}(x_0, y_0) \\ &= z_0. \end{aligned}$$

Thus the elements f in $\mathrm{Hom}_{\mathsf{Sets}}(X \times Y, Z)$ are precisely those functions $f\colon X \times Y \to Z$ satisfying the equalities

$$\begin{aligned} f(x_0, y) &= z_0, \\ f(x, y_0) &= z_0 \end{aligned}$$

for each $x \in X$ and each $y \in Y$, giving an equality

$$\mathrm{Hom}^{R}_{\mathsf{Sets}}(X \times Y, Z) = \mathrm{Hom}^{\otimes}_{\mathsf{Sets}_*}(X \times Y, Z)$$

of sets, which when composed with our earlier isomorphism

$$\mathsf{Sets}_*(X \wedge Y, Z) \cong \mathrm{Hom}^{R}_{\mathsf{Sets}}(X \times Y, Z)$$

gives our desired natural bijection, finishing the proof. □

Remark 5.1.1.5. It is also somewhat common to write

$$X \wedge Y \overset{\text{def}}{=} \frac{X \times Y}{X \vee Y},$$

identifying $X \vee Y$ with the subspace $(\{x_0\} \times Y) \cup (X \times \{y_0\})$ of $X \times Y$, and having the quotient be defined by declaring $(x, y) \sim (x', y')$ iff we have $(x, y), (x', y') \in X \vee Y$.

Notation 5.1.1.6. We write $x \wedge y$ for the element $[(x, y)]$ of

$$X \wedge Y \cong X \times Y/{\sim}.$$

Remark 5.1.1.7. Employing the notation introduced in Notation 5.1.1.6, we have

$$\begin{aligned} x_0 \wedge y_0 &= x \wedge y_0, \\ &= x_0 \wedge y \end{aligned}$$

for each $x \in X$ and each $y \in Y$, and

$$\begin{aligned} x \wedge y_0 &= x' \wedge y_0, \\ x_0 \wedge y &= x_0 \wedge y' \end{aligned}$$

for each $x, x' \in X$ and each $y, y' \in Y$.

Example 5.1.1.8. Here are some examples of smash products of pointed sets.

1. *Smashing With pt.* For any pointed set X, we have isomorphisms of pointed sets
$$\begin{aligned} \mathrm{pt} \wedge X &\cong \mathrm{pt}, \\ X \wedge \mathrm{pt} &\cong \mathrm{pt}. \end{aligned}$$

2. *Smashing With S^0.* For any pointed set X, we have isomorphisms of pointed sets
$$\begin{aligned} S^0 \wedge X &\cong X, \\ X \wedge S^0 &\cong X. \end{aligned}$$

Proposition 5.1.1.9. Let (X, x_0) and (Y, y_0) be pointed sets.

1. *Functoriality.* The assignments $X, Y, (X, Y) \mapsto X \wedge Y$ define functors
$$\begin{aligned} X \wedge -\colon \mathsf{Sets}_* &\to \mathsf{Sets}_*, \\ - \wedge Y\colon \mathsf{Sets}_* &\to \mathsf{Sets}_*, \\ -_1 \wedge -_2\colon \mathsf{Sets}_* \times \mathsf{Sets}_* &\to \mathsf{Sets}_*. \end{aligned}$$
In particular, given pointed maps
$$\begin{aligned} f\colon (X, x_0) &\to (A, a_0), \\ g\colon (Y, y_0) &\to (B, b_0), \end{aligned}$$
the induced map
$$f \wedge g\colon X \wedge Y \to A \wedge B$$
is given by
$$[f \wedge g](x \wedge y) \stackrel{\text{def}}{=} f(x) \wedge g(y)$$
for each $x \wedge y \in X \wedge Y$.

2. *Adjointness.* We have adjunctions
$$(X \wedge - \dashv \mathbf{Sets}_*(X, -))\colon \quad \begin{tikzcd} \mathsf{Sets}_* \arrow[r, bend left, "X \wedge -"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_*, \arrow[l, bend left, "\mathbf{Sets}_*(X,-)"] \end{tikzcd}$$
$$(- \wedge Y \dashv \mathbf{Sets}_*(Y, -))\colon \quad \begin{tikzcd} \mathsf{Sets}_* \arrow[r, bend left, "- \wedge Y"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_*, \arrow[l, bend left, "\mathbf{Sets}_*(Y,-)"] \end{tikzcd}$$

witnessed by bijections

$$\begin{aligned}\mathrm{Hom}_{\mathsf{Sets}_*}(X \wedge Y, Z) &\cong \mathrm{Hom}_{\mathsf{Sets}_*}(X, \mathbf{Sets}_*(Y, Z)),\\ \mathrm{Hom}_{\mathsf{Sets}_*}(X \wedge Y, Z) &\cong \mathrm{Hom}_{\mathsf{Sets}_*}(X, \mathbf{Sets}_*(A, Z)),\end{aligned}$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

3. *Enriched Adjointness.* We have Sets_*-enriched adjunctions

$$(X \wedge - \dashv \mathbf{Sets}_*(X, -))\colon\quad \begin{tikzcd} \mathbf{Sets}_* \arrow[r, bend left, "X\wedge -"] \arrow[r, phantom, "\perp"] & \mathbf{Sets}_*, \arrow[l, bend left, "\mathbf{Sets}_*(X,-)"] \end{tikzcd}$$

$$(- \wedge Y \dashv \mathbf{Sets}_*(Y, -))\colon\quad \begin{tikzcd} \mathbf{Sets}_* \arrow[r, bend left, "-\wedge Y"] \arrow[r, phantom, "\perp"] & \mathbf{Sets}_*, \arrow[l, bend left, "\mathbf{Sets}_*(Y,-)"] \end{tikzcd}$$

witnessed by isomorphisms of pointed sets

$$\begin{aligned}\mathbf{Sets}_*(X \wedge Y, Z) &\cong \mathbf{Sets}_*(X, \mathbf{Sets}_*(Y, Z)),\\ \mathbf{Sets}_*(X \wedge Y, Z) &\cong \mathbf{Sets}_*(X, \mathbf{Sets}_*(A, Z)),\end{aligned}$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathbf{Sets}_*)$.

4. *As a Pushout.* We have an isomorphism

$$X \wedge Y \cong \mathrm{pt} \coprod_{X \vee Y} (X \times Y), \qquad \begin{tikzcd} X \wedge Y \arrow[dr, phantom, "\ulcorner", very near start] & X \times Y \arrow[l] \\ \mathrm{pt} \arrow[u] & X \vee Y, \arrow[l, "!"] \arrow[u, hook, "\iota"'] \end{tikzcd}$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets}_*)$, where the pushout is taken in Sets, and the embedding $\iota\colon X \vee Y \hookrightarrow X \times Y$ is defined following Remark 5.1.1.5.

5. *Distributivity Over Wedge Sums.* We have isomorphisms of pointed sets

$$\begin{aligned}X \wedge (Y \vee Z) &\cong (X \wedge Y) \vee (X \wedge Z),\\ (X \vee Y) \wedge Z &\cong (X \wedge Z) \vee (Y \wedge Z),\end{aligned}$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

Proof. *Item 1*, *Functoriality*: The map $f \wedge g$ comes from Equivalence Relations and Apartness Relations, Item 4 of Proposition 5.2.1.3 via the map

$$f \wedge g \colon X \times Y \longrightarrow A \wedge B$$

sending (x, y) to $f(x) \wedge g(y)$, which we need to show satisfies

$$[f \wedge g](x, y) = [f \wedge g](x', y')$$

for each $(x, y), (x', y') \in X \times Y$ with $(x, y) \sim_R (x', y')$, where $\sim_R$ is the relation constructing $X \wedge Y$ as

$$X \wedge Y \cong (X \times Y)/\sim_R$$

in Construction 5.1.1.4. The condition defining $\sim$ is that at least one of the following conditions is satisfied:

1. We have $x = x'$ and $y = y'$;
2. Both of the following conditions are satisfied:
   (a) We have $x = x_0$ or $y = y_0$.
   (b) We have $x' = x_0$ or $y' = y_0$.

We have five cases:

1. In the first case, we clearly have

   $$[f \wedge g](x, y) = [f \wedge g](x', y')$$

   since $x = x'$ and $y = y'$.

2. If $x = x_0$ and $x' = x_0$, we have

   $$\begin{aligned} [f \wedge g](x_0, y) &\overset{\text{def}}{=} f(x_0) \wedge g(y) \\ &= a_0 \wedge g(y) \\ &= a_0 \wedge g(y') \\ &= f(x_0) \wedge g(y') \\ &\overset{\text{def}}{=} [f \wedge g](x_0, y'). \end{aligned}$$

3. If $x = x_0$ and $y' = y_0$, we have

$$\begin{aligned}[f \wedge g](x_0, y) &\overset{\text{def}}{=} f(x_0) \wedge g(y) \\ &= a_0 \wedge g(y) \\ &= a_0 \wedge b_0 \\ &= f(x') \wedge b_0 \\ &= f(x') \wedge g(y_0) \\ &\overset{\text{def}}{=} [f \wedge g](x', y_0).\end{aligned}$$

4. If $y = y_0$ and $x' = x_0$, we have

$$\begin{aligned}[f \wedge g](x, y_0) &\overset{\text{def}}{=} f(x) \wedge g(y_0) \\ &= f(x) \wedge b_0 \\ &= a_0 \wedge b_0 \\ &= a_0 \wedge g(y') \\ &= f(x_0) \wedge g(y') \\ &\overset{\text{def}}{=} [f \wedge g](x_0, y').\end{aligned}$$

5. If $y = y_0$ and $y' = y_0$, we have

$$\begin{aligned}[f \wedge g](x, y_0) &\overset{\text{def}}{=} f(x) \wedge g(y_0) \\ &= f(x) \wedge b_0 \\ &= f(x') \wedge b_0 \\ &= f(x) \wedge g(y_0) \\ &\overset{\text{def}}{=} [f \wedge g](x', y_0).\end{aligned}$$

Thus $f \wedge g$ is well-defined. Next, we claim that $\wedge$ preserves identities and composition:

- *Preservation of Identities.* We have

$$\begin{aligned}[\mathrm{id}_X \wedge \mathrm{id}_Y](x \wedge y) &\overset{\text{def}}{=} \mathrm{id}_X(x) \wedge \mathrm{id}_Y(y) \\ &= x \wedge y \\ &= [\mathrm{id}_{X \wedge Y}](x \wedge y)\end{aligned}$$

  for each $x \wedge y \in X \wedge Y$, and thus

$$\mathrm{id}_X \wedge \mathrm{id}_Y = \mathrm{id}_{X \wedge Y}.$$

- *Preservation of Composition.* Given pointed maps

$$\begin{aligned} f &\colon (X, x_0) \to (X', x'_0),\\ h &\colon (X', x'_0) \to (X'', x''_0),\\ g &\colon (Y, y_0) \to (Y', y'_0),\\ k &\colon (Y', y'_0) \to (Y'', y''_0), \end{aligned}$$

  we have

$$\begin{aligned} [(h \circ f) \wedge (k \circ g)](x \wedge y) &\stackrel{\text{def}}{=} h(f(x)) \wedge k(g(y))\\ &\stackrel{\text{def}}{=} [h \wedge k](f(x) \wedge g(y))\\ &\stackrel{\text{def}}{=} [h \wedge k]([f \wedge g](x \wedge y))\\ &\stackrel{\text{def}}{=} [(h \wedge k) \circ (f \wedge g)](x \wedge y) \end{aligned}$$

  for each $x \wedge y \in X \wedge Y$, and thus

$$(h \circ f) \wedge (k \circ g) = (h \wedge k) \circ (f \wedge g).$$

This finishes the proof.
Item 2, Adjointness: We prove only the adjunction $- \wedge Y \dashv \mathbf{Sets}_*(Y, -)$, witnessed by a natural bijection

$$\mathrm{Hom}_{\mathsf{Sets}_*}(X \wedge Y, Z) \cong \mathrm{Hom}_{\mathsf{Sets}_*}(X, \mathbf{Sets}_*(Y, Z)),$$

as the proof of the adjunction $X \wedge - \dashv \mathbf{Sets}_*(X, -)$ is similar. We claim we have a bijection

$$\mathrm{Hom}^{\otimes}_{\mathsf{Sets}_*}(X \times Y, Z) \cong \mathrm{Hom}_{\mathsf{Sets}_*}(X, \mathbf{Sets}_*(Y, Z))$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, impliying the desired adjunction. Indeed, this bijection is a restriction of the bijection

$$\mathsf{Sets}(X \times Y, Z) \cong \mathsf{Sets}(X, \mathsf{Sets}(Y, Z))$$

of Constructions With Sets, Item 2 of Proposition 1.3.1.2:

- A map

$$\xi \colon X \times Y \to Z$$

in $\mathrm{Hom}^{\otimes}_{\mathsf{Sets}_*}(X \times Y, Z)$ gets sent to the pointed map

$$\begin{aligned} \xi^\dagger \colon (X, x_0) &\to (\mathbf{Sets}_*(Y, Z), \Delta_{z_0}), \\ x &\longmapsto (\xi^\dagger_x \colon Y \to Z), \end{aligned}$$

where $\xi^\dagger_x \colon Y \to Z$ is the map defined by

$$\xi^\dagger_x(y) \overset{\text{def}}{=} \xi(x, y)$$

for each $y \in Y$, where:

- The map $\xi^\dagger$ is indeed pointed, as we have

$$\begin{aligned} \xi^\dagger_{x_0}(y) &\overset{\text{def}}{=} \xi(x_0, y) \\ &\overset{\text{def}}{=} z_0 \end{aligned}$$

  for each $y \in Y$. Thus $\xi^\dagger_{x_0} = \Delta_{z_0}$ and $\xi^\dagger$ is pointed.
- The map $\xi^\dagger_x$ indeed lies in $\mathbf{Sets}_*(Y, Z)$, as we have

$$\begin{aligned} \xi^\dagger_x(y_0) &\overset{\text{def}}{=} \xi(x, y_0) \\ &\overset{\text{def}}{=} z_0. \end{aligned}$$

- Conversely, a map

$$\begin{aligned} \xi \colon (X, x_0) &\to (\mathbf{Sets}_*(Y, Z), \Delta_{z_0}), \\ x &\longmapsto (\xi_x \colon Y \to Z), \end{aligned}$$

in $\mathrm{Hom}_{\mathsf{Sets}_*}(X, \mathbf{Sets}_*(Y, Z))$ gets sent to the map

$$\xi^\dagger \colon X \times Y \to Z$$

defined by

$$\xi^\dagger(x, y) \overset{\text{def}}{=} \xi_x(y)$$

for each $(x, y) \in X \times Y$, which indeed lies in $\mathrm{Hom}^{\otimes}_{\mathsf{Sets}_*}(X \times Y, Z)$, as:

  - *Left Bilinearity.* We have

$$\begin{aligned} \xi^\dagger(x_0, y) &\overset{\text{def}}{=} \xi_{x_0}(y) \\ &\overset{\text{def}}{=} \Delta_{z_0}(y) \\ &\overset{\text{def}}{=} z_0 \end{aligned}$$

    for each $y \in Y$, since $\xi_{x_0} = \Delta_{z_0}$ as ξ is assumed to be a pointed map.

– *Right Bilinearity.* We have

$$\begin{aligned}\xi^{\dagger}(x, y_0) &\stackrel{\text{def}}{=} \xi_x(y_0)\\ &\stackrel{\text{def}}{=} z_0\end{aligned}$$

for each $x \in X$, since $\xi_x \in \mathbf{Sets}_*(Y, Z)$ is a morphism of pointed sets.

This finishes the proof.
Item 3, Enriched Adjointness: This follows from Item 2 and **??**, **??** of **??**.
Item 4, As a Pushout: Following the description of Constructions With Sets, Remark 2.4.1.2, we have

$$\mathrm{pt} \coprod_{X \vee Y}(X \times Y) \cong (\mathrm{pt} \times (X \times Y))/\sim,$$

where $\sim$ identifies the elemenet $\star$ in pt with all elements of the form (x_0, y) and (x, y_0) in $X \times Y$. Thus Equivalence Relations and Apartness Relations, Item 4 of Proposition 5.2.1.3 coupled with Remark 5.1.1.7 then gives us a well-defined map

$$\mathrm{pt} \coprod_{X \vee Y}(X \times Y) \to X \wedge Y$$

via $[(\star, (x, y))] \mapsto x \wedge y$, with inverse

$$X \wedge Y \to \mathrm{pt} \coprod_{X \vee Y}(X \times Y)$$

given by $x \wedge y \mapsto [(\star, (x, y))]$.
Item 5, Distributivity Over Wedge Sums: This follows from Proposition 5.9.1.1, **??**, **??** of **??**, and the fact that $\vee$ is the coproduct in Sets_* (Pointed Sets, Definition 3.3.1.1). □

5.2 The Internal Hom of Pointed Sets

Let (X, x_0) and (Y, y_0) be pointed sets.

Definition 5.2.1.1. The **internal Hom**[21] **of pointed sets from** (X, x_0) **to** (Y, y_0) is the pointed set $\mathbf{Sets}_*((X, x_0), (Y, y_0))$[22] consisting of:

[21] The pointed set $\mathbf{Sets}_*(X, Y)$ is the internal **Hom** of Sets_* with respect to the smash product of Tensor Products of Pointed Sets, Definition 5.1.1.1; see Tensor Products of Pointed Sets, Item 2 of Proposition 5.1.1.9.

[22] *Further Notation:* Also written $\mathbf{Hom}_{\mathbf{Sets}_*}(X, Y)$.

- *The Underlying Set.* The set $\mathsf{Sets}_*((X, x_0), (Y, y_0))$ of morphisms of pointed sets from (X, x_0) to (Y, y_0).
- *The Basepoint.* The element

$$\Delta_{y_0} \colon (X, x_0) \to (Y, y_0)$$

of $\mathsf{Sets}_*((X, x_0), (Y, y_0))$ given by

$$\Delta_{y_0}(x) \overset{\text{def}}{=} y_0$$

for each $x \in X$.

Proof. For a proof that $\textbf{Sets}_*$ is indeed the internal Hom of Sets_* with respect to the smash product of pointed sets, see Item 2 of Proposition 5.1.1.9. □

Proposition 5.2.1.2. Let (X, x_0) and (Y, y_0) be pointed sets.

1. *Functoriality.* The assignments $X, Y, (X, Y) \mapsto \textbf{Sets}_*(X, Y)$ define functors

$$\begin{aligned} \textbf{Sets}_*(X, -) \colon \mathsf{Sets}_* &\to \mathsf{Sets}_*, \\ \textbf{Sets}_*(-, Y) \colon \mathsf{Sets}_*^{\mathsf{op}} &\to \mathsf{Sets}_*, \\ \textbf{Sets}_*(-_1, -_2) \colon \mathsf{Sets}_*^{\mathsf{op}} \times \mathsf{Sets}_* &\to \mathsf{Sets}_*. \end{aligned}$$

In particular, given pointed maps

$$\begin{aligned} f &\colon (X, x_0) \to (A, a_0), \\ g &\colon (Y, y_0) \to (B, b_0), \end{aligned}$$

the induced map

$$\textbf{Sets}_*(f, g) \colon \textbf{Sets}_*(A, Y) \to \textbf{Sets}_*(X, B)$$

is given by

$$[\textbf{Sets}_*(f, g)](\phi) \overset{\text{def}}{=} g \circ \phi \circ f$$

for each $\phi \in \textbf{Sets}_*(A, Y)$.

2. *Adjointness.* We have adjunctions

$$
\begin{tikzcd}
(X \wedge - \dashv \mathbf{Sets}_*(X,-))\colon \quad \mathsf{Sets}_* \arrow[r, bend left, "X\wedge-"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_*, \arrow[l, bend left, "\mathbf{Sets}_*(X{,}-)"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
(- \wedge Y \dashv \mathbf{Sets}_*(Y,-))\colon \quad \mathsf{Sets}_* \arrow[r, bend left, "-\wedge Y"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_*, \arrow[l, bend left, "\mathbf{Sets}_*(Y{,}-)"]
\end{tikzcd}
$$

witnessed by bijections

$$
\begin{aligned}
\mathrm{Hom}_{\mathsf{Sets}_*}(X \wedge Y, Z) &\cong \mathrm{Hom}_{\mathsf{Sets}_*}(X, \mathbf{Sets}_*(Y, Z)),\\
\mathrm{Hom}_{\mathsf{Sets}_*}(X \wedge Y, Z) &\cong \mathrm{Hom}_{\mathsf{Sets}_*}(X, \mathbf{Sets}_*(A, Z)),
\end{aligned}
$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

3. *Enriched Adjointness.* We have Sets_*-enriched adjunctions

$$
\begin{tikzcd}
(X \wedge - \dashv \mathbf{Sets}_*(X,-))\colon \quad \mathbf{Sets}_* \arrow[r, bend left, "X\wedge-"] \arrow[r, phantom, "\perp"] & \mathbf{Sets}_*, \arrow[l, bend left, "\mathbf{Sets}_*(X{,}-)"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
(- \wedge Y \dashv \mathbf{Sets}_*(Y,-))\colon \quad \mathbf{Sets}_* \arrow[r, bend left, "-\wedge Y"] \arrow[r, phantom, "\perp"] & \mathbf{Sets}_*, \arrow[l, bend left, "\mathbf{Sets}_*(Y{,}-)"]
\end{tikzcd}
$$

witnessed by isomorphisms of pointed sets

$$
\begin{aligned}
\mathbf{Sets}_*(X \wedge Y, Z) &\cong \mathbf{Sets}_*(X, \mathbf{Sets}_*(Y, Z)),\\
\mathbf{Sets}_*(X \wedge Y, Z) &\cong \mathbf{Sets}_*(X, \mathbf{Sets}_*(A, Z)),
\end{aligned}
$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathbf{Sets}_*)$.

Proof. Item 1, Functoriality: This follows from Constructions With Sets, Item 1 of Proposition 3.5.1.2 and from the equalities

$$
\begin{aligned}
g \circ \Delta_{y_0} &= \Delta_{z_0},\\
\Delta_{y_0} \circ f &= \Delta_{y_0}
\end{aligned}
$$

for morphisms $f\colon (K, k_0) \to (X, x_0)$ and $g\colon (Y, y_0) \to (Z, z_0)$, which guarantee pre- and postcomposition by morphisms of pointed sets to also be morphisms of pointed sets.

Item 2, Adjointness: This is a repetition of Item 2 of Proposition 5.1.1.9, and is proved there.

Item 3, Enriched Adjointness: This is a repetition of Item 3 of Proposition 5.1.1.9, and is proved there. □

5.3 The Monoidal Unit

Definition 5.3.1.1. The **monoidal unit of the smash product of pointed sets** is the functor

$$\mathbb{1}^{\mathsf{Sets}_*}\colon \mathsf{pt} \longrightarrow \mathsf{Sets}_*$$

defined by

$$\mathbb{1}_{\mathsf{Sets}_*} \overset{\text{def}}{=} S^0.$$

5.4 The Associator

Definition 5.4.1.1. The **associator of the smash product of pointed sets** is the natural isomorphism

$$\alpha^{\mathsf{Sets}_*}\colon \wedge \circ (\wedge \times \mathrm{id}_{\mathsf{Sets}_*}) \overset{\sim}{\Longrightarrow} \wedge \circ (\mathrm{id}_{\mathsf{Sets}_*} \times \wedge) \circ \boldsymbol{\alpha}^{\mathsf{Cats}}_{\mathsf{Sets}_*,\mathsf{Sets}_*,\mathsf{Sets}_*},$$

as in the diagram

$$\begin{tikzcd}
 & & \mathsf{Sets}_* \times (\mathsf{Sets}_* \times \mathsf{Sets}_*) \arrow[drr, "\mathrm{id}\times\wedge"] & & \\
(\mathsf{Sets}_* \times \mathsf{Sets}_*) \times \mathsf{Sets}_* \arrow[urr, dashed, "\boldsymbol{\alpha}^{\mathsf{Cats}}_{\mathsf{Sets}_*,\mathsf{Sets}_*,\mathsf{Sets}_*}", "\sim"'] \arrow[dr, "\wedge\times\mathrm{id}"'] & & & & \mathsf{Sets}_* \times \mathsf{Sets}_* \arrow[dl, "\wedge"] \\
 & \mathsf{Sets}_* \times \mathsf{Sets}_* \arrow[rr, "\wedge"'] \arrow[urrr, phantom, "\Nearrow\alpha^{\mathsf{Sets}_*}"] & & \mathsf{Sets}_*, &
\end{tikzcd}$$

whose component

$$\alpha^{\mathsf{Sets}_*}_{X,Y,Z}\colon (X \wedge Y) \wedge Z \xrightarrow{\cong} X \wedge (Y \wedge Z)$$

at $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$ is given by

$$\alpha^{\mathsf{Sets}_*}_{X,Y,Z}((x \wedge y) \wedge z) \overset{\text{def}}{=} x \wedge (y \wedge z)$$

for each $(x \wedge y) \wedge z \in (X \wedge Y) \wedge Z$.

Proof. Well-Definedness: Let $[((x, y), z)] = [((x', y'), z')]$ be an element in $(X \wedge Y) \wedge Z$. Then either:

1. We have $x = x'$, $y = y'$, and $z = z'$.
2. Both of the following conditions are satisfied:
   (a) We have $x = x_0$ or $y = y_0$ or $z = z_0$.
   (b) We have $x' = x_0$ or $y' = y_0$ or $z' = z_0$.

In the first case, $\alpha^{\mathsf{Sets}_*}_{X,Y,Z}$ clearly sends both elements to the same element in $X \wedge (Y \wedge Z)$. Meanwhile, in the latter case both elements are equal to the basepoint $(x_0 \wedge y_0) \wedge z_0$ of $(X \wedge Y) \wedge Z$, which gets sent to the basepoint $x_0 \wedge (y_0 \wedge z_0)$ of $X \wedge (Y \wedge Z)$.
Being a Morphism of Pointed Sets: As just mentioned, we have

$$\alpha^{\mathsf{Sets}_*}_{X,Y,Z}((x_0 \wedge y_0) \wedge z_0) \stackrel{\text{def}}{=} x_0 \wedge (y_0 \wedge z_0),$$

and thus $\alpha^{\mathsf{Sets}_*}_{X,Y,Z}$ is a morphism of pointed sets.
Invertibility: Clearly, the inverse of $\alpha^{\mathsf{Sets}_*}_{X,Y,Z}$ is given by the morphism

$$\alpha^{\mathsf{Sets}_*,-1}_{X,Y,Z} \colon X \wedge (Y \wedge Z) \xrightarrow{\cong} (X \wedge Y) \wedge Z$$

defined by

$$\alpha^{\mathsf{Sets}_*,-1}_{X,Y,Z}(x \wedge (y \wedge z)) \stackrel{\text{def}}{=} (x \wedge y) \wedge z$$

for each $x \wedge (y \wedge z) \in X \wedge (Y \wedge Z)$.
Naturality: We need to show that, given morphisms of pointed sets

$$\begin{aligned} f &\colon (X, x_0) \to (X', x_0'), \\ g &\colon (Y, y_0) \to (Y', y_0'), \\ h &\colon (Z, z_0) \to (Z', z_0') \end{aligned}$$

the diagram

$$\begin{tikzcd} (X \wedge Y) \wedge Z \arrow[r, "(f \wedge g) \wedge h"] \arrow[d, "\alpha^{\mathsf{Sets}_*}_{X,Y,Z}"'] & (X' \wedge Y') \wedge Z' \arrow[d, "\alpha^{\mathsf{Sets}_*}_{X',Y',Z'}"] \\ X \wedge (Y \wedge Z) \arrow[r, "f \wedge (g \wedge h)"'] & X' \wedge (Y' \wedge Z') \end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
(x \wedge y) \wedge z \arrow[r, mapsto] \arrow[d, mapsto] & (f(x) \wedge g(y)) \wedge h(z) \arrow[d, mapsto] \\
x \wedge (y \wedge z) \arrow[r, mapsto] & f(x) \wedge (g(y) \wedge h(z))
\end{tikzcd}$$

and hence indeed commutes, showing α^{Sets_*} to be a natural transformation.
Being a Natural Isomorphism: Since α^{Sets_*} is natural and $\alpha^{\mathsf{Sets}_*,-1}$ is a componentwise inverse to α^{Sets_*}, it follows from Categories, Item 2 of Proposition 8.6.1.2 that $\alpha^{\mathsf{Sets}_*,-1}$ is also natural. Thus α^{Sets_*} is a natural isomorphism. □

5.5 The Left Unitor

Definition 5.5.1.1. The **left unitor of the smash product of pointed sets** is the natural isomorphism

$$\lambda^{\mathsf{Sets}_*} \colon \wedge \circ (\mathbb{1}^{\mathsf{Sets}_*} \times \mathrm{id}_{\mathsf{Sets}_*}) \xRightarrow{\sim} \boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*} \qquad \begin{tikzcd}
\mathrm{pt} \times \mathsf{Sets}_* \arrow[r, "\mathbb{1}^{\mathsf{Sets}_*} \times \mathrm{id}"] \arrow[rd, dashed, bend right, "\boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}"'] & \mathsf{Sets}_* \times \mathsf{Sets}_* \arrow[d, "\wedge"] \arrow[ld, phantom, "\Swarrow \lambda^{\mathsf{Sets}_*}"] \\
 & \mathsf{Sets}_*,
\end{tikzcd}$$

whose component

$$\lambda^{\mathsf{Sets}_*}_X \colon S^0 \wedge X \xrightarrow{\cong} X$$

at $X \in \mathrm{Obj}(\mathsf{Sets}_*)$ is given by

$$\begin{aligned}
0 \wedge x &\mapsto x_0, \\
1 \wedge x &\mapsto x.
\end{aligned}$$

Proof. Well-Definedness: Let $[(x, y)] = [(x', y')]$ be an element in $S^0 \wedge X$. Then either:

1. We have $x = x'$ and $y = y'$.

2. Both of the following conditions are satisfied:

   (a) We have $x = 0$ or $y = x_0$.

   (b) We have $x' = 0$ or $y' = x_0$.

In the first case, $\lambda_X^{\mathsf{Sets}_*}$ clearly sends both elements to the same element in X. Meanwhile, in the latter case both elements are equal to the basepoint $0 \wedge x_0$ of $S^0 \wedge X$, which gets sent to the basepoint x_0 of X.
Being a Morphism of Pointed Sets: As just mentioned, we have

$$\lambda_X^{\mathsf{Sets}_*}(0 \wedge x_0) \stackrel{\text{def}}{=} x_0,$$

and thus $\lambda_X^{\mathsf{Sets}_*}$ is a morphism of pointed sets.
Invertibility: The inverse of $\lambda_X^{\mathsf{Sets}_*}$ is the morphism

$$\lambda_X^{\mathsf{Sets}_*,-1}\colon X \xrightarrow{\cong} S^0 \wedge X$$

defined by

$$\lambda_X^{\mathsf{Sets}_*,-1}(x) \stackrel{\text{def}}{=} 1 \wedge x$$

for each $x \in X$. Indeed:

- *Invertibility I.* We have

$$\begin{aligned}
[\lambda_X^{\mathsf{Sets}_*,-1} \circ \lambda_X^{\mathsf{Sets}_*}](0 \wedge x) &= \lambda_X^{\mathsf{Sets}_*,-1}(\lambda_X^{\mathsf{Sets}_*}(0 \wedge x)) \\
&= \lambda_X^{\mathsf{Sets}_*,-1}(x_0) \\
&= 1 \wedge x_0 \\
&= 0 \wedge x,
\end{aligned}$$

  and

$$\begin{aligned}
[\lambda_X^{\mathsf{Sets}_*,-1} \circ \lambda_X^{\mathsf{Sets}_*}](1 \wedge x) &= \lambda_X^{\mathsf{Sets}_*,-1}(\lambda_X^{\mathsf{Sets}_*}(1 \wedge x)) \\
&= \lambda_X^{\mathsf{Sets}_*,-1}(x) \\
&= 1 \wedge x
\end{aligned}$$

  for each $x \in X$, and thus we have

$$\lambda_X^{\mathsf{Sets}_*,-1} \circ \lambda_X^{\mathsf{Sets}_*} = \mathrm{id}_{S^0 \wedge X}.$$

- *Invertibility II.* We have

$$\begin{aligned}[\lambda_X^{\mathsf{Sets}_*} \circ \lambda_X^{\mathsf{Sets}_*,-1}](x) &= \lambda_X^{\mathsf{Sets}_*}(\lambda_X^{\mathsf{Sets}_*,-1}(x)) \\ &= \lambda_X^{\mathsf{Sets}_*,-1}(1 \wedge x) \\ &= x\end{aligned}$$

  for each $x \in X$, and thus we have

$$\lambda_X^{\mathsf{Sets}_*} \circ \lambda_X^{\mathsf{Sets}_*,-1} = \mathrm{id}_X.$$

This shows $\lambda_X^{\mathsf{Sets}_*}$ to be invertible.

Naturality: We need to show that, given a morphism of pointed sets

$$f\colon (X, x_0) \longrightarrow (Y, y_0),$$

the diagram

$$\begin{tikzcd} S^0 \wedge X \arrow[r, "\mathrm{id}_{S^0} \wedge f"] \arrow[d, "\lambda_X^{\mathsf{Sets}_*}"'] & S^0 \wedge Y \arrow[d, "\lambda_Y^{\mathsf{Sets}_*}"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd} 0 \wedge x \arrow[d, mapsto] & & 0 \wedge x \arrow[r, mapsto] & 0 \wedge f(x) \arrow[d, mapsto] \\ x_0 \arrow[r, mapsto] & f(x_0) & & y_0 \end{tikzcd}$$

and

$$\begin{tikzcd} 1 \wedge x \arrow[r, mapsto] \arrow[d, mapsto] & 1 \wedge f(x) \arrow[d, mapsto] \\ x \arrow[r, mapsto] & f(x) \end{tikzcd}$$

and hence indeed commutes, showing $\lambda^{\mathsf{Sets}_*}$ to be a natural transformation.

Being a Natural Isomorphism: Since $\lambda^{\mathsf{Sets}_*}$ is natural and $\lambda^{\mathsf{Sets}_*,-1}$ is a componentwise inverse to $\lambda^{\mathsf{Sets}_*}$, it follows from Categories, Item 2 of Proposition 8.6.1.2 that $\lambda^{\mathsf{Sets}_*,-1}$ is also natural. Thus $\lambda^{\mathsf{Sets}_*}$ is a natural isomorphism. □

5.6 The Right Unitor

Definition 5.6.1.1. The **right unitor of the smash product of pointed sets** is the natural isomorphism

$$\rho^{\mathsf{Sets}_*}\colon \wedge\circ(\mathrm{id}\times\mathbb{1}^{\mathsf{Sets}_*}) \overset{\sim}{\Longrightarrow} \boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}, \qquad \begin{tikzcd} \mathsf{Sets}_*\times\mathsf{pt} \arrow[r, "\mathrm{id}\times\mathbb{1}^{\mathsf{Sets}_*}"] \arrow[dr, dashed, bend right, "\boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}"'] & \mathsf{Sets}_*\times\mathsf{Sets}_* \arrow[d, "\wedge"] \arrow[dl, phantom, "\Swarrow\rho^{\mathsf{Sets}_*}"] \\ & \mathsf{Sets}_*, \end{tikzcd}$$

whose component

$$\rho^{\mathsf{Sets}_*}_X\colon X\wedge S^0 \xrightarrow{\cong} X$$

at $X\in\mathrm{Obj}(\mathsf{Sets}_*)$ is given by

$$\begin{aligned} x\wedge 0 &\mapsto x_0,\\ x\wedge 1 &\mapsto x. \end{aligned}$$

Proof. Well-Definedness: Let $[(x,y)] = [(x',y')]$ be an element in $X\wedge S^0$. Then either:

1. We have $x = x'$ and $y = y'$.
2. Both of the following conditions are satisfied:
    (a) We have $x = x_0$ or $y = 0$.
    (b) We have $x' = x_0$ or $y' = 0$.

In the first case, $\rho^{\mathsf{Sets}_*}_X$ clearly sends both elements to the same element in X. Meanwhile, in the latter case both elements are equal to the basepoint $x_0\wedge 0$ of $X\wedge S^0$, which gets sent to the basepoint x_0 of X.
Being a Morphism of Pointed Sets: As just mentioned, we have

$$\rho^{\mathsf{Sets}_*}_X(x_0\wedge 0) \overset{\mathrm{def}}{=} x_0,$$

and thus $\rho^{\mathsf{Sets}_*}_X$ is a morphism of pointed sets.

Invertibility: The inverse of $\rho^{\mathsf{Sets}_*}_X$ is the morphism

$$\rho^{\mathsf{Sets}_*,-1}_X \colon X \xrightarrow{\cong} X \wedge S^0$$

defined by

$$\rho^{\mathsf{Sets}_*,-1}_X(x) \overset{\text{def}}{=} x \wedge 1$$

for each $x \in X$. Indeed:

- *Invertibility I*. We have

$$\begin{aligned}[\rho^{\mathsf{Sets}_*,-1}_X \circ \rho^{\mathsf{Sets}_*}_X](x \wedge 0) &= \rho^{\mathsf{Sets}_*,-1}_X(\rho^{\mathsf{Sets}_*}_X(x \wedge 0))\\ &= \rho^{\mathsf{Sets}_*,-1}_X(x_0)\\ &= x_0 \wedge 1\\ &= x \wedge 0,\end{aligned}$$

  and

$$\begin{aligned}[\rho^{\mathsf{Sets}_*,-1}_X \circ \rho^{\mathsf{Sets}_*}_X](x \wedge 1) &= \rho^{\mathsf{Sets}_*,-1}_X(\rho^{\mathsf{Sets}_*}_X(x \wedge 1))\\ &= \rho^{\mathsf{Sets}_*,-1}_X(x)\\ &= x \wedge 1\end{aligned}$$

  for each $x \in X$, and thus we have

$$\rho^{\mathsf{Sets}_*,-1}_X \circ \rho^{\mathsf{Sets}_*}_X = \mathrm{id}_{X \wedge S^0}.$$

- *Invertibility II*. We have

$$\begin{aligned}[\rho^{\mathsf{Sets}_*}_X \circ \rho^{\mathsf{Sets}_*,-1}_X](x) &= \rho^{\mathsf{Sets}_*}_X(\rho^{\mathsf{Sets}_*,-1}_X(x))\\ &= \rho^{\mathsf{Sets}_*,-1}_X(x \wedge 1)\\ &= x\end{aligned}$$

  for each $x \in X$, and thus we have

$$\rho^{\mathsf{Sets}_*}_X \circ \rho^{\mathsf{Sets}_*,-1}_X = \mathrm{id}_X.$$

This shows $\rho^{\mathsf{Sets}_*}_X$ to be invertible.

Naturality: We need to show that, given a morphism of pointed sets

$$f\colon (X,x_0) \to (Y,y_0),$$

the diagram

$$\begin{tikzcd} X \wedge S^0 \arrow[r, "f\wedge \mathrm{id}_{S^0}"] \arrow[d, "\rho^{\mathsf{Sets}_*}_X"'] & Y \wedge S^0 \arrow[d, "\rho^{\mathsf{Sets}_*}_Y"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd} x \wedge 0 \arrow[d, mapsto] & \\ x_0 \arrow[r, mapsto] & f(x_0) \end{tikzcd} \qquad \begin{tikzcd} x \wedge 0 \arrow[r, mapsto] & f(x) \wedge 0 \arrow[d, mapsto] \\ & y_0 \end{tikzcd}$$

and

$$\begin{tikzcd} x \wedge 1 \arrow[r, mapsto] \arrow[d, mapsto] & f(x) \wedge 1 \arrow[d, mapsto] \\ x \arrow[r, mapsto] & f(x) \end{tikzcd}$$

and hence indeed commutes, showing ρ^{Sets_*} to be a natural transformation.
Being a Natural Isomorphism: Since ρ^{Sets_*} is natural and $\rho^{\mathsf{Sets}_*,-1}$ is a componentwise inverse to ρ^{Sets_*}, it follows from Categories, Item 2 of Proposition 8.6.1.2 that $\rho^{\mathsf{Sets}_*,-1}$ is also natural. Thus ρ^{Sets_*} is a natural isomorphism. □

5.7 The Symmetry

Definition 5.7.1.1. The **symmetry of the smash product of pointed sets** is the natural isomorphism

$$\sigma^{\mathsf{Sets}_*}\colon \wedge \overset{\sim}{\Longrightarrow} \wedge \circ \boldsymbol{\sigma}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*,\mathsf{Sets}_*}, \qquad \begin{tikzcd} \mathsf{Sets}_* \times \mathsf{Sets}_* \arrow[rr, "\wedge"] \arrow[dr, bend right, "\sigma^{\mathsf{Cats}_2}_{\mathsf{Sets}_*,\mathsf{Sets}_*}"'] & \arrow[d, phantom, "\overset{\Vert}{\underset{\Downarrow}{\sigma^{\mathsf{Sets}_*}}}"] & \mathsf{Sets}_*, \\ & \mathsf{Sets}_* \times \mathsf{Sets}_* \arrow[ur, bend right, "\wedge"'] & \end{tikzcd}$$

whose component

$$\sigma^{\mathsf{Sets}_*}_{X,Y}\colon X \wedge Y \xrightarrow{\cong} Y \wedge X$$

at $X, Y \in \mathrm{Obj}(\mathsf{Sets}_*)$ is defined by

$$\sigma^{\mathsf{Sets}_*}_{X,Y}(x \wedge y) \overset{\text{def}}{=} y \wedge x$$

for each $x \wedge y \in X \wedge Y$.

Proof. Well-Definedness: Let $[(x, y)] = [(x', y')]$ be an element in $X \wedge Y$. Then either:

1. We have $x = x'$ and $y = y'$.
2. Both of the following conditions are satisfied:
   (a) We have $x = x_0$ or $y = y_0$.
   (b) We have $x' = x_0$ or $y' = y_0$.

In the first case, $\sigma^{\mathsf{Sets}_*}_{X}$ clearly sends both elements to the same element in X. Meanwhile, in the latter case both elements are equal to the basepoint $x_0 \wedge y_0$ of $X \wedge Y$, which gets sent to the basepoint $y_0 \wedge x_0$ of $Y \wedge X$.
Being a Morphism of Pointed Sets: As just mentioned, we have

$$\sigma^{\mathsf{Sets}_*}_{X}(x_0 \wedge y_0) \overset{\text{def}}{=} y_0 \wedge x_0,$$

and thus $\sigma^{\mathsf{Sets}_*}_{X}$ is a morphism of pointed sets.
Invertibility: Clearly, the inverse of $\sigma^{\mathsf{Sets}_*}_{X,Y}$ is given by the morphism

$$\sigma^{\mathsf{Sets}_*,-1}_{X,Y}\colon Y \wedge X \xrightarrow{\cong} X \wedge Y$$

defined by

$$\sigma^{\mathsf{Sets}_*,-1}_{X,Y}(y \wedge x) \overset{\text{def}}{=} x \wedge y$$

for each $y \wedge x \in Y \wedge X$.
Naturality: We need to show that, given morphisms of pointed sets

$$\begin{aligned} f&\colon (X, x_0) \to (A, a_0),\\ g&\colon (Y, y_0) \to (B, b_0) \end{aligned}$$

the diagram

$$\begin{tikzcd}
X \wedge Y \arrow[r, "f \wedge g"] \arrow[d, "\sigma^{\mathsf{Sets}_*}_{X,Y}"'] & A \wedge B \arrow[d, "\sigma^{\mathsf{Sets}_*}_{A,B}"] \\
Y \wedge X \arrow[r, "g \wedge f"'] & B \wedge A
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
x \wedge y \arrow[r, mapsto] \arrow[d, mapsto] & f(x) \wedge g(y) \arrow[d, mapsto] \\
y \wedge x \arrow[r, mapsto] & g(y) \wedge f(x)
\end{tikzcd}$$

and hence indeed commutes, showing σ^{Sets_*} to be a natural transformation.

Being a Natural Isomorphism: Since σ^{Sets_*} is natural and $\sigma^{\mathsf{Sets}_*,-1}$ is a componentwise inverse to σ^{Sets_*}, it follows from Categories, Item 2 of Proposition 8.6.1.2 that $\sigma^{\mathsf{Sets}_*,-1}$ is also natural. Thus σ^{Sets_*} is a natural isomorphism. □

5.8 The Diagonal

Definition 5.8.1.1. The **diagonal of the smash product of pointed sets** is the natural transformation

$$\Delta^{\wedge} \colon \mathrm{id}_{\mathsf{Sets}_*} \Longrightarrow \wedge \circ \Delta^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}, \qquad \begin{tikzcd}
\mathsf{Sets}_* \arrow[rr, bend left, "\mathrm{id}_{\mathsf{Sets}_*}"] \arrow[dr, "\Delta^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}"'] & \arrow[d, phantom, "\Downarrow \Delta^{\wedge}"] & \mathsf{Sets}_* \\
& \mathsf{Sets}_* \wedge \mathsf{Sets}_*, \arrow[ur, "\wedge"'] &
\end{tikzcd}$$

whose component

$$\Delta^{\wedge}_X \colon (X, x_0) \to (X \wedge X, x_0 \wedge x_0)$$

at $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$ is given by the composition

$$\begin{aligned}
(X, x_0) &\xrightarrow{\Delta^{\wedge}_X} (X \times X, (x_0, x_0)) \\
&\twoheadrightarrow ((X \times X)/\sim, [(x_0, x_0)]) \\
&\overset{\text{def}}{=\!=} (X \wedge X, x_0 \wedge x_0)
\end{aligned}$$

in Sets_*, and thus by

$$\Delta^{\wedge}_X(x) \stackrel{\text{def}}{=} x \wedge x$$

for each $x \in X$.

Proof. *Being a Morphism of Pointed Sets*: We have

$$\Delta^{\wedge}_X(x_0) \stackrel{\text{def}}{=} x_0 \wedge x_0,$$

and thus $\Delta^{\wedge}_X$ is a morphism of pointed sets.
Naturality: We need to show that, given a morphism of pointed sets

$$f \colon (X, x_0) \to (Y, y_0),$$

the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\Delta^{\wedge}_X"'] & Y \arrow[d, "\Delta^{\wedge}_Y"] \\ X \wedge X \arrow[r, "f \wedge f"'] & Y \wedge Y \end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd} x \arrow[r, mapsto] \arrow[d, mapsto] & f(x) \arrow[d, mapsto] \\ x \wedge x \arrow[r, mapsto] & f(x) \wedge f(x) \end{tikzcd}$$

and hence indeed commutes, showing $\Delta^{\wedge}$ to be natural. □

Proposition 5.8.1.2. Let $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

1. *Monoidality.* The diagonal

$$\Delta^{\wedge} \colon \mathrm{id}_{\mathsf{Sets}_*} \Longrightarrow \wedge \circ \Delta^{\mathsf{Cats}_2}_{\mathsf{Sets}_*},$$

of the smash product of pointed sets is a monoidal natural transformation:

(a) *Compatibility With Strong Monoidality Constraints.* For each $(X, x_0), (Y, y_0) \in$

$\mathrm{Obj}(\mathsf{Sets}_*)$, the diagram

$$\begin{tikzcd} X \wedge Y \arrow[r, "\Delta^{\wedge}_X \wedge \Delta^{\wedge}_Y"] \arrow[dr, "\Delta^{\wedge}_{X \wedge Y}"'] & (X \wedge X) \wedge (Y \wedge Y) \arrow[d, dashed, "\wr"] \\ & (X \wedge Y) \wedge (X \wedge Y) \end{tikzcd}$$

commutes.

(b) *Compatibility With Strong Unitality Constraints.* The diagrams

$$\begin{tikzcd} S^0 \arrow[r, "\Delta^{\wedge}_{S^0}"] \arrow[dr, equal] & S^0 \wedge S^0 \arrow[d, "\lambda^{\mathsf{Sets}_*}_{S^0}"] \\ & S^0 \end{tikzcd} \qquad \begin{tikzcd} S^0 \arrow[r, "\Delta^{\wedge}_{S^0}"] \arrow[dr, equal] & S^0 \wedge S^0 \arrow[d, "\rho^{\mathsf{Sets}_*}_{S^0}"] \\ & S^0 \end{tikzcd}$$

commute, i.e. we have

$$\begin{aligned} \Delta^{\wedge}_{S^0} &= \lambda^{\mathsf{Sets}_*,-1}_{S^0} \\ &= \rho^{\mathsf{Sets}_*,-1}_{S^0}, \end{aligned}$$

where we recall that the equalities

$$\begin{aligned} \lambda^{\mathsf{Sets}_*}_{S^0} &= \rho^{\mathsf{Sets}_*}_{S^0}, \\ \lambda^{\mathsf{Sets}_*,-1}_{S^0} &= \rho^{\mathsf{Sets}_*,-1}_{S^0} \end{aligned}$$

are always true in any monoidal category by **??**, **??** of **??**.

2. *The Diagonal of the Unit.* The component

$$\Delta^{\wedge}_{S^0} \colon S^0 \xrightarrow{\cong} S^0 \wedge S^0$$

of $\Delta^{\wedge}$ at S^0 is an isomorphism.

Proof. *Item 1*, *Monoidality*: We claim that $\Delta^{\wedge}$ is indeed monoidal:

1. *Item 1a*: *Compatibility With Strong Monoidality Constraints:* We need to

show that the diagram

$$\begin{tikzcd}
X \wedge Y \arrow[r, "\Delta^{\wedge}_{X} \wedge \Delta^{\wedge}_{Y}"] \arrow[dr, "\Delta^{\wedge}_{X \wedge Y}"'] & (X \wedge X) \wedge (Y \wedge Y) \arrow[d, dashed, "\wr"] \\
 & (X \wedge Y) \wedge (X \wedge Y)
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
x \wedge y \arrow[r, mapsto] \arrow[dr, mapsto] & (x \wedge x) \wedge (y \wedge y) \arrow[d, mapsto] \\
 & (x \wedge y) \wedge (x \wedge y)
\end{tikzcd}$$

and hence indeed commutes.

2. *Item 1b: Compatibility With Strong Unitality Constraints:* As shown in the proof of Definition 5.5.1.1, the inverse of the left unitor of Sets_* with respect to to the smash product of pointed sets at $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$ is given by

$$\lambda^{\mathsf{Sets}_*,-1}_{X}(x) \stackrel{\text{def}}{=} 1 \wedge x$$

for each $x \in X$, so when $X = S^0$, we have

$$\begin{aligned}
\lambda^{\mathsf{Sets}_*,-1}_{S^0}(0) &\stackrel{\text{def}}{=} 1 \wedge 0, \\
\lambda^{\mathsf{Sets}_*,-1}_{S^0}(1) &\stackrel{\text{def}}{=} 1 \wedge 1.
\end{aligned}$$

But since $1 \wedge 0 = 0 \wedge 0$ and

$$\begin{aligned}
\Delta^{\wedge}_{S^0}(0) &\stackrel{\text{def}}{=} 0 \wedge 0, \\
\Delta^{\wedge}_{S^0}(1) &\stackrel{\text{def}}{=} 1 \wedge 1,
\end{aligned}$$

it follows that we indeed have $\Delta^{\wedge}_{S^0} = \lambda^{\mathsf{Sets}_*,-1}_{S^0}$.

This finishes the proof.

Item 2, The Diagonal of the Unit: This follows from Item 1 and the invertibility of the left/right unitor of Sets_* with respect to $\wedge$, proved in the proof of Definition 5.5.1.1 for the left unitor or the proof of Definition 5.6.1.1 for the right unitor. □

5.9 The Monoidal Structure on Pointed Sets Associated to $\wedge$

Proposition 5.9.1.1. The category Sets_* admits a closed monoidal category with diagonals structure consisting of

- *The Underlying Category.* The category Sets_* of pointed sets;
- *The Monoidal Product.* The smash product functor

  $$\wedge \colon \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*$$

  of Item 1 of Proposition 5.1.1.9;
- *The Internal Hom.* The internal Hom functor

  $$\mathbf{Sets}_* \colon \mathsf{Sets}_*^{\mathsf{op}} \times \mathsf{Sets}_* \to \mathsf{Sets}_*$$

  of Item 1 of Proposition 5.2.1.2;
- *The Monoidal Unit.* The functor

  $$\mathbb{1}^{\mathsf{Sets}_*} \colon \mathsf{pt} \to \mathsf{Sets}_*$$

  of Definition 5.3.1.1;
- *The Associators.* The natural isomorphism

  $$\alpha^{\mathsf{Sets}_*} \colon \wedge \circ (\wedge \times \mathrm{id}_{\mathsf{Sets}_*}) \overset{\sim}{\Longrightarrow} \wedge \circ (\mathrm{id}_{\mathsf{Sets}_*} \times \wedge) \circ \boldsymbol{\alpha}^{\mathsf{Cats}}_{\mathsf{Sets}_*,\mathsf{Sets}_*,\mathsf{Sets}_*}$$

  of Definition 5.4.1.1;
- *The Left Unitors.* The natural isomorphism

  $$\lambda^{\mathsf{Sets}_*} \colon \wedge \circ (\mathbb{1}^{\mathsf{Sets}_*} \times \mathrm{id}_{\mathsf{Sets}_*}) \overset{\sim}{\Longrightarrow} \boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}$$

  of Definition 5.5.1.1;
- *The Right Unitors.* The natural isomorphism

  $$\rho^{\mathsf{Sets}_*} \colon \wedge \circ (\mathsf{id} \times \mathbb{1}^{\mathsf{Sets}_*}) \overset{\sim}{\Longrightarrow} \boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}$$

  of Definition 5.6.1.1;

- *The Symmetry.* The natural isomorphism
$$\sigma^{\mathsf{Sets}_*}\colon \wedge \overset{\sim}{\Longrightarrow} \wedge \circ \boldsymbol{\sigma}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*,\mathsf{Sets}_*}$$
of Definition 5.7.1.1;
- *The Diagonals.* The monoidal natural transformation
$$\Delta^{\wedge}\colon \mathrm{id}_{\mathsf{Sets}_*} \Longrightarrow \wedge \circ \Delta^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}$$
of Definition 5.8.1.1.

Proof. The Pentagon Identity: Let (W, w_0), (X, x_0), (Y, y_0) and (Z, z_0) be pointed sets. We have to show that the diagram

$$\begin{tikzcd}
 & (W \wedge (X \wedge Y)) \wedge Z \arrow[dr, "\alpha^{\mathsf{Sets}_*}_{W,X\wedge Y,Z}"] & \\
((W \wedge X) \wedge Y) \wedge Z \arrow[ur, "\alpha^{\mathsf{Sets}_*}_{W,X,Y}\wedge \mathrm{id}_Z"] \arrow[d, "\alpha^{\mathsf{Sets}_*}_{W\wedge X,Y,Z}"'] & & W \wedge ((X \wedge Y) \wedge Z) \arrow[d, "\mathrm{id}_W \wedge \alpha^{\mathsf{Sets}_*}_{X,Y,Z}"] \\
(W \wedge X) \wedge (Y \wedge Z) \arrow[rr, "\alpha^{\mathsf{Sets}_*}_{W,X,Y\wedge Z}"'] & & W \wedge (X \wedge (Y \wedge Z))
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$
\begin{tikzcd}
& & (w \wedge (x \wedge y)) \wedge z \arrow[drr, mapsto] & & \\
((w \wedge x) \wedge y) \wedge z \arrow[urr, mapsto] \arrow[dr, mapsto] & & & & w \wedge ((x \wedge y) \wedge z) \arrow[dl, mapsto] \\
& (w \wedge x) \wedge (y \wedge z) \arrow[rr, mapsto] & & w \wedge (x \wedge (y \wedge z)) &
\end{tikzcd}
$$

and thus we see that the pentagon identity is satisfied.

The Triangle Identity: Let (X, x_0) and (Y, y_0) be pointed sets. We have to show that the diagram

$$
\begin{tikzcd}
(X \wedge S^0) \wedge Y \arrow[rr, "\alpha^{\mathsf{Sets}_*}_{X,S^0,Y}"] \arrow[dr, "\rho^{\mathsf{Sets}_*}_X \wedge \mathrm{id}_Y"'] & & X \wedge (S^0 \wedge Y) \arrow[dl, "\mathrm{id}_X \wedge \lambda^{\mathsf{Sets}_*}_Y"] \\
& X \wedge Y &
\end{tikzcd}
$$

commutes. Indeed, this diagram acts on elements as

$$
\begin{tikzcd}
(x \wedge 0) \wedge y \arrow[dr, mapsto] & \\
& x_0 \wedge y
\end{tikzcd}
\qquad
\begin{tikzcd}
(x \wedge 0) \wedge y \arrow[rr, mapsto] & & x \wedge (0 \wedge y) \arrow[dl, mapsto] \\
& x \wedge y_0 &
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
(x \wedge 1) \wedge y \arrow[rr, mapsto] \arrow[dr, mapsto] & & x \wedge (1 \wedge y) \arrow[dl, mapsto] \\
& x \wedge y, &
\end{tikzcd}
$$

and thus we see that the triangle identity is satisfied.
The Left Hexagon Identity: Let (X, x_0), (Y, y_0), and (Z, z_0) be pointed sets. We have to show that the diagram

$$
\begin{tikzcd}
 & (X \wedge Y) \wedge Z \arrow[dl, "\alpha^{\mathsf{Sets}_*}_{X,Y,Z}"'] \arrow[dr, "\beta^{\mathsf{Sets}_*}_{X,Y} \wedge \mathrm{id}_Z"] & \\
X \wedge (Y \wedge Z) \arrow[d, "\beta^{\mathsf{Sets}_*}_{X,Y \wedge Z}"'] & & (Y \wedge X) \wedge Z \arrow[d, "\alpha^{\mathsf{Sets}_*}_{Y,X,Z}"] \\
(Y \wedge Z) \wedge X \arrow[dr, "\alpha^{\mathsf{Sets}_*}_{Y,Z,X}"'] & & Y \wedge (X \wedge Z) \arrow[dl, "\mathrm{id}_Y \wedge \beta^{\mathsf{Sets}_*}_{X,Z}"] \\
 & Y \wedge (Z \wedge X) &
\end{tikzcd}
$$

commutes. Indeed, this diagram acts on elements as

$$
\begin{tikzcd}
 & (x \wedge y) \wedge z \arrow[dl, mapsto] \arrow[dr, mapsto] & \\
x \wedge (y \wedge z) \arrow[d, mapsto] & & (y \wedge x) \wedge z \arrow[d, mapsto] \\
(y \wedge z) \wedge x \arrow[dr, mapsto] & & y \wedge (x \wedge z) \arrow[dl, mapsto] \\
 & y \wedge (z \wedge x) &
\end{tikzcd}
$$

and thus we see that the left hexagon identity is satisfied.
The Right Hexagon Identity: Let (X, x_0), (Y, y_0), and (Z, z_0) be pointed sets. We

have to show that the diagram

$$\begin{tikzcd}
 & X \wedge (Y \wedge Z) \arrow[dl, "(\alpha^{\mathsf{Sets}_*}_{X,Y,Z})^{-1}"'] \arrow[dr, "\mathrm{id}_X \wedge \beta^{\mathsf{Sets}_*}_{Y,Z}"] & \\
(X \wedge Y) \wedge Z \arrow[d, "\beta^{\mathsf{Sets}_*}_{X \wedge Y, Z}"'] & & X \wedge (Z \wedge Y) \arrow[d, "(\alpha^{\mathsf{Sets}_*}_{X,Z,Y})^{-1}"] \\
Z \wedge (X \wedge Y) \arrow[dr, "(\alpha^{\mathsf{Sets}_*}_{Z,X,Y})^{-1}"'] & & (X \wedge Z) \wedge Y \arrow[dl, "\beta^{\mathsf{Sets}_*}_{X,Z} \wedge \mathrm{id}_Y"] \\
 & (Z \wedge X) \wedge Y &
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
 & x \wedge (y \wedge z) \arrow[dl, mapsto] \arrow[dr, mapsto] & \\
(x \wedge y) \wedge z \arrow[d, mapsto] & & x \wedge (z \wedge y) \arrow[d, mapsto] \\
z \wedge (x \wedge y) \arrow[dr, mapsto] & & (x \wedge z) \wedge y \arrow[dl, mapsto] \\
 & (z \wedge x) \wedge y &
\end{tikzcd}$$

and thus we see that the right hexagon identity is satisfied.
Monoidal Closedness: This follows from Item 2 of Proposition 5.1.1.9.
Existence of Monoidal Diagonals: This follows from Items 1 and 2 of Proposition 5.8.1.2. □

5.10 Universal Properties of the Smash Product of Pointed Sets I

Theorem 5.10.1.1. The symmetric monoidal structure on the category Sets_* is uniquely determined by the following requirements:

1. *Two-Sided Preservation of Colimits.* The smash product

$$\wedge \colon \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*$$

   of Sets_* preserves colimits separately in each variable.

2. *The Unit Object Is S^0.* We have $\mathbb{1}_{\mathsf{Sets}_*} = S^0$.

Proof. Omitted. □

5.11 Universal Properties of the Smash Product of Pointed Sets II

Theorem 5.11.1.1. The symmetric monoidal structure on the category Sets_* is the unique symmetric monoidal structure on Sets_* such that the free pointed set functor

$$(-)^+ \colon \mathsf{Sets} \to \mathsf{Sets}_*$$

admits a symmetric monoidal structure.

Proof. See [GGN15, Theorem 5.1]. □

5.12 Monoids With Respect to the Smash Product of Pointed Sets

Proposition 5.12.1.1. The category of monoids on $(\mathsf{Sets}_*, \wedge, S^0)$ is isomorphic to the category of monoids with zero and morphisms between them.

Proof. See **??**, in particular **??**, **??**, **??**, and **??**. □

5.13 Comonoids With Respect to the Smash Product of Pointed Sets

Proposition 5.13.1.1. The symmetric monoidal functor

$$\left((-)^+, (-)^{+,\times}, (-)^{+,\times}_{\mathbb{1}}\right) \colon (\mathsf{Sets}, \times, \mathrm{pt}) \to (\mathsf{Sets}_*, \wedge, S^0),$$

of Pointed Sets, Item 4 of Proposition 4.1.1.2 lifts to an equivalence of categories

$$\begin{aligned} \mathsf{CoMon}(\mathsf{Sets}_*, \wedge, S^0) &\overset{\text{eq.}}{\cong} \mathsf{CoMon}(\mathsf{Sets}, \times, \mathrm{pt}) \\ &\cong \mathsf{Sets}. \end{aligned}$$

Proof. See [PS19, Lemma 2.4]. □

6 Miscellany

6.1 The Smash Product of a Family of Pointed Sets

Let $\left\{(X_i, x_0^i)\right\}_{i\in I}$ be a family of pointed sets.

Definition 6.1.1.1. The **smash product of the family** $\left\{(X_i, x_0^i)\right\}_{i\in I}$ is the pointed set $\bigwedge_{i\in I} X_i$ consisting of:

- *The Underlying Set.* The set $\bigwedge_{i\in I} X_i$ defined by

  $$\bigwedge_{i\in I} X_i \overset{\text{def}}{=} (\prod_{i\in I} X_i)/\sim,$$

  where $\sim$ is the equivalence relation on $\prod_{i\in I} X_i$ obtained by declaring

  $$(x_i)_{i\in I} \sim (y_i)_{i\in I}$$

  if there exist $i_0 \in I$ such that $x_{i_0} = x_0$ and $y_{i_0} = y_0$, for each $(x_i)_{i\in I}, (y_i)_{i\in I} \in \prod_{i\in I} X_i$.

- *The Basepoint.* The element $[(x_0)_{i\in I}]$ of $\bigwedge_{i\in I} X_i$.

Appendices

A Other Chapters

References

[GGN15] David Gepner, Moritz Groth, and Thomas Nikolaus. "Universality of Multiplicative Infinite Loop Space Machines". In: *Algebr. Geom. Topol.* 15.6 (2015), pp. 3107–3153. ISSN: 1472-2747. DOI: 10.2140/agt.2015.15.3107. URL: https://doi.org/10.2140/agt.2015.15.3107 (cit. on p. 100).

[PS19] Maximilien Péroux and Brooke Shipley. "Coalgebras in Symmetric Monoidal Categories of Spectra". In: *Homology Homotopy Appl.* 21.1 (2019), pp. 1–18. ISSN: 1532-0073. DOI: 10.4310/HHA.2019.v21.n1.a1. URL: https://doi.org/10.4310/HHA.2019.v21.n1.a1 (cit. on p. 100).