

# Equivalence Relations and Apartness Relations

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**00TJ** This chapter contains some material about reflexive, symmetric, transitive, equivalence, and apartness relations.

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## 00TK 1 Reflexive Relations

### 00TL 1.1 Foundations

Let  $A$  be a set.

#### 00TM DEFINITION 1.1.1 ► REFLEXIVE RELATIONS

A **reflexive relation** is equivalently:<sup>1</sup>

- An  $\mathbb{E}_0$ -monoid in  $(\mathbf{N}_\bullet(\mathbf{Rel}(A, A)), \chi_A)$ .
- A pointed object in  $(\mathbf{Rel}(A, A), \chi_A)$ .

<sup>1</sup>Note that since  $\mathbf{Rel}(A, A)$  is posetal, reflexivity is a property of a relation, rather than extra structure.

#### 00TN REMARK 1.1.2 ► UNWINDING DEFINITION 1.1.1

In detail, a relation  $R$  on  $A$  is **reflexive** if we have an inclusion

$$\eta_R: \chi_A \subset R$$

of relations in  $\mathbf{Rel}(A, A)$ , i.e. if, for each  $a \in A$ , we have  $a \sim_R a$ .

#### 00TP DEFINITION 1.1.3 ► THE PO/SET OF REFLEXIVE RELATIONS ON A SET

Let  $A$  be a set.

1. The **set of reflexive relations on  $A$**  is the subset  $\mathbf{Rel}^{\text{refl}}(A, A)$  of  $\mathbf{Rel}(A, A)$  spanned by the reflexive relations.
2. The **poset of relations on  $A$**  is the subposet  $\mathbf{Rel}^{\text{refl}}(A, A)$  of  $\mathbf{Rel}(A, A)$  spanned by the reflexive relations.

#### 00TS PROPOSITION 1.1.4 ► PROPERTIES OF REFLEXIVE RELATIONS

Let  $R$  and  $S$  be relations on  $A$ .

1. *Interaction With Inverses.* If  $R$  is reflexive, then so is  $R^\dagger$ .
2. *Interaction With Composition.* If  $R$  and  $S$  are reflexive, then so is  $S \diamond R$ .

## PROOF 1.1.5 ► PROOF OF PROPOSITION 1.1.4

Item 1: Interaction With Inverses

Clear.

Item 2: Interaction With Composition

Clear. 

## 00TV 1.2 The Reflexive Closure of a Relation

Let  $R$  be a relation on  $A$ .

## 00TW DEFINITION 1.2.1 ► THE REFLEXIVE CLOSURE OF A RELATION

The **reflexive closure** of  $\sim_R$  is the relation  $\sim_R^{\text{refl}}$ <sup>1</sup> satisfying the following universal property:<sup>2</sup>

- (★) Given another reflexive relation  $\sim_S$  on  $A$  such that  $R \subset S$ , there exists an inclusion  $\sim_R^{\text{refl}} \subset \sim_S$ .

<sup>1</sup>Further Notation: Also written  $R^{\text{refl}}$ .

<sup>2</sup>Slogan: The reflexive closure of  $R$  is the smallest reflexive relation containing  $R$ .

## 00TX CONSTRUCTION 1.2.2 ► THE REFLEXIVE CLOSURE OF A RELATION

Concretely,  $\sim_R^{\text{refl}}$  is the free pointed object on  $R$  in  $(\mathbf{Rel}(A, A), \chi_A)$ <sup>1</sup>, being given by

$$\begin{aligned} R^{\text{refl}} &\stackrel{\text{def}}{=} R \coprod^{\mathbf{Rel}(A, A)} \Delta_A \\ &= R \cup \Delta_A \\ &= \{(a, b) \in A \times A \mid \text{we have } a \sim_R b \text{ or } a = b\}. \end{aligned}$$

<sup>1</sup>Or, equivalently, the free  $\mathbb{E}_0$ -monoid on  $R$  in  $(\mathbf{N}_\bullet(\mathbf{Rel}(A, A)), \chi_A)$ .

## PROOF 1.2.3 ► PROOF OF CONSTRUCTION 1.2.2

Clear. 

00TY

**PROPOSITION 1.2.4 ► PROPERTIES OF THE REFLEXIVE CLOSURE OF A RELATION**

Let  $R$  be a relation on  $A$ .

00TZ

1. *Adjointness.* We have an adjunction

$$\left( (-)^{\text{refl}} \dashv \text{忘} \right): \mathbf{Rel}(A, A) \begin{matrix} \xrightarrow{(-)^{\text{refl}}} \\ \perp \\ \xleftarrow{\text{忘}} \end{matrix} \mathbf{Rel}^{\text{refl}}(A, A),$$

witnessed by a bijection of sets

$$\mathbf{Rel}^{\text{refl}}(R^{\text{refl}}, S) \cong \mathbf{Rel}(R, S),$$

natural in  $R \in \text{Obj}(\mathbf{Rel}^{\text{refl}}(A, A))$  and  $S \in \text{Obj}(\mathbf{Rel}(A, A))$ .

00U0

2. *The Reflexive Closure of a Reflexive Relation.* If  $R$  is reflexive, then  $R^{\text{refl}} = R$ .

00U1

3. *Idempotency.* We have

$$(R^{\text{refl}})^{\text{refl}} = R^{\text{refl}}.$$

00U2

4. *Interaction With Inverses.* We have

$$\left( R^{\dagger} \right)^{\text{refl}} = \left( R^{\text{refl}} \right)^{\dagger}, \quad \begin{array}{ccc} \mathbf{Rel}(A, A) & \xrightarrow{(-)^{\text{refl}}} & \mathbf{Rel}(A, A) \\ (-)^{\dagger} \downarrow & & \downarrow (-)^{\dagger} \\ \mathbf{Rel}(A, A) & \xrightarrow{(-)^{\text{refl}}} & \mathbf{Rel}(A, A). \end{array}$$

00U3

5. *Interaction With Composition.* We have

$$(S \diamond R)^{\text{refl}} = S^{\text{refl}} \diamond R^{\text{refl}}, \quad \begin{array}{ccc} \mathbf{Rel}(A, A) \times \mathbf{Rel}(A, A) & \xrightarrow{\diamond} & \mathbf{Rel}(A, A) \\ (-)^{\text{refl}} \times (-)^{\text{refl}} \downarrow & & \downarrow (-)^{\text{refl}} \\ \mathbf{Rel}(A, A) \times \mathbf{Rel}(A, A) & \xrightarrow{\diamond} & \mathbf{Rel}(A, A). \end{array}$$

### PROOF 1.2.5 ► PROOF OF PROPOSITION 1.2.4

#### Item 1: Adjointness

This is a rephrasing of the universal property of the reflexive closure of a relation, stated in [Definition 1.2.1](#).

#### Item 2: The Reflexive Closure of a Reflexive Relation

Clear.

#### Item 3: Idempotency

This follows from [Item 2](#).

#### Item 4: Interaction With Inverses

Clear.

#### Item 5: Interaction With Composition

This follows from [Item 2](#) of [Proposition 1.1.4](#). 

## 00U4 2 Symmetric Relations

### 00U5 2.1 Foundations

Let  $A$  be a set.

#### 00U6 DEFINITION 2.1.1 ► SYMMETRIC RELATIONS

A relation  $R$  on  $A$  is **symmetric** if we have  $R^\dagger = R$ .

#### 00U7 REMARK 2.1.2 ► UNWINDING DEFINITION 2.1.1

In detail, a relation  $R$  is symmetric if it satisfies the following condition:

(★) For each  $a, b \in A$ , if  $a \sim_R b$ , then  $b \sim_R a$ .

#### 00U8 DEFINITION 2.1.3 ► THE PO/SET OF SYMMETRIC RELATIONS ON A SET

Let  $A$  be a set.

1. The **set of symmetric relations on  $A$**  is the subset  $\text{Rel}^{\text{symm}}(A, A)$  of  $\text{Rel}(A, A)$  spanned by the symmetric relations.

00UA

2. The **poset of relations on  $A$**  is the subposet  $\mathbf{Rel}^{\text{symm}}(A, A)$  of  $\mathbf{Rel}(A, A)$  spanned by the symmetric relations.

00UB

#### PROPOSITION 2.1.4 ► PROPERTIES OF SYMMETRIC RELATIONS

Let  $R$  and  $S$  be relations on  $A$ .

00UC

1. *Interaction With Inverses.* If  $R$  is symmetric, then so is  $R^\dagger$ .

00UD

2. *Interaction With Composition.* If  $R$  and  $S$  are symmetric, then so is  $S \diamond R$ .

#### PROOF 2.1.5 ► PROOF OF PROPOSITION 2.1.4

Item 1: Interaction With Inverses

Clear.

Item 2: Interaction With Composition

Clear.



## 00UE 2.2 The Symmetric Closure of a Relation

Let  $R$  be a relation on  $A$ .

00UF

#### DEFINITION 2.2.1 ► THE SYMMETRIC CLOSURE OF A RELATION

The **symmetric closure** of  $\sim_R$  is the relation  $\sim_R^{\text{symm}}$ <sup>1</sup> satisfying the following universal property:<sup>2</sup>

- (★) Given another symmetric relation  $\sim_S$  on  $A$  such that  $R \subset S$ , there exists an inclusion  $\sim_R^{\text{symm}} \subset \sim_S$ .

<sup>1</sup>Further Notation: Also written  $R^{\text{symm}}$ .

<sup>2</sup>Slogan: The symmetric closure of  $R$  is the smallest symmetric relation containing  $R$ .

00UG

#### CONSTRUCTION 2.2.2 ► THE SYMMETRIC CLOSURE OF A RELATION

Concretely,  $\sim_R^{\text{symm}}$  is the symmetric relation on  $A$  defined by

$$\begin{aligned} R^{\text{symm}} &\stackrel{\text{def}}{=} R \cup R^\dagger \\ &= \{(a, b) \in A \times A \mid \text{we have } a \sim_R b \text{ or } b \sim_R a\}. \end{aligned}$$

#### PROOF 2.2.3 ► PROOF OF CONSTRUCTION 2.2.2

Clear. 

00UH

#### PROPOSITION 2.2.4 ► PROPERTIES OF THE SYMMETRIC CLOSURE OF A RELATION

Let  $R$  be a relation on  $A$ .

00UJ

1. *Adjointness.* We have an adjunction

$$\left( (-)^{\text{symm}} \dashv \overset{\sim}{\text{forget}} \right): \mathbf{Rel}(A, A) \begin{array}{c} \xrightarrow{(-)^{\text{symm}}} \\ \perp \\ \xleftarrow{\overset{\sim}{\text{forget}}} \end{array} \mathbf{Rel}^{\text{symm}}(A, A),$$

witnessed by a bijection of sets

$$\mathbf{Rel}^{\text{symm}}(R^{\text{symm}}, S) \cong \mathbf{Rel}(R, S),$$

natural in  $R \in \text{Obj}(\mathbf{Rel}^{\text{symm}}(A, A))$  and  $S \in \text{Obj}(\mathbf{Rel}(A, A))$ .

00UK

2. *The Symmetric Closure of a Symmetric Relation.* If  $R$  is symmetric, then  $R^{\text{symm}} = R$ .

00UL

3. *Idempotency.* We have

$$(R^{\text{symm}})^{\text{symm}} = R^{\text{symm}}.$$

00UM

4. *Interaction With Inverses.* We have

$$\left( R^\dagger \right)^{\text{symm}} = \left( R^{\text{symm}} \right)^\dagger, \quad \begin{array}{ccc} \mathbf{Rel}(A, A) & \xrightarrow{(-)^{\text{symm}}} & \mathbf{Rel}(A, A) \\ (-)^\dagger \downarrow & & \downarrow (-)^\dagger \\ \mathbf{Rel}(A, A) & \xrightarrow{(-)^{\text{symm}}} & \mathbf{Rel}(A, A). \end{array}$$

00UN

5. *Interaction With Composition.* We have

$$\begin{array}{ccc}
 & \text{Rel}(A, A) \times \text{Rel}(A, A) \xrightarrow{\diamond} \text{Rel}(A, A) & \\
 (S \diamond R)^{\text{symm}} = S^{\text{symm}} \diamond R^{\text{symm}}, & \begin{array}{c} \downarrow (-)^{\text{symm}} \times (-)^{\text{symm}} \\ \text{Rel}(A, A) \times \text{Rel}(A, A) \end{array} & \begin{array}{c} \downarrow (-)^{\text{symm}} \\ \text{Rel}(A, A) \end{array} \\
 & \text{Rel}(A, A) \times \text{Rel}(A, A) \xrightarrow{\diamond} \text{Rel}(A, A). &
 \end{array}$$

#### PROOF 2.2.5 ► PROOF OF PROPOSITION 2.2.4

##### Item 1: Adjointness

This is a rephrasing of the universal property of the symmetric closure of a relation, stated in [Definition 2.2.1](#).

##### Item 2: The Symmetric Closure of a Symmetric Relation

Clear.

##### Item 3: Idempotency

This follows from [Item 2](#).

##### Item 4: Interaction With Inverses

Clear.

##### Item 5: Interaction With Composition

This follows from [Item 2](#) of [Proposition 2.1.4](#). 

## 00UP 3 Transitive Relations

### 00UQ 3.1 Foundations

Let  $A$  be a set.

00UR

#### DEFINITION 3.1.1 ► TRANSITIVE RELATIONS

A **transitive relation** is equivalently:<sup>1</sup>

- A non-unital  $\mathbb{E}_1$ -monoid in  $(\mathbf{N}_\bullet(\mathbf{Rel}(A, A)), \diamond)$ .
- A non-unital monoid in  $(\mathbf{Rel}(A, A), \diamond)$ .



<sup>1</sup>Note that since  $\mathbf{Rel}(A, A)$  is posetal, transitivity is a property of a relation, rather than extra structure.

00US

**REMARK 3.1.2 ► UNWINDING DEFINITION 3.1.1**

In detail, a relation  $R$  on  $A$  is **transitive** if we have an inclusion

$$\mu_R: R \diamond R \subset R$$

of relations in  $\mathbf{Rel}(A, A)$ , i.e. if, for each  $a, c \in A$ , the following condition is satisfied:

( $\star$ ) If there exists some  $b \in A$  such that  $a \sim_R b$  and  $b \sim_R c$ , then  $a \sim_R c$ .

00UT

**DEFINITION 3.1.3 ► THE PO/SET OF TRANSITIVE RELATIONS ON A SET**

Let  $A$  be a set.

00UU

1. The **set of transitive relations from  $A$  to  $B$**  is the subset  $\mathbf{Rel}^{\text{trans}}(A)$  of  $\mathbf{Rel}(A, A)$  spanned by the transitive relations.

00UV

2. The **poset of relations from  $A$  to  $B$**  is the subposet  $\mathbf{Rel}^{\text{trans}}(A)$  of  $\mathbf{Rel}(A, A)$  spanned by the transitive relations.

00UW

**PROPOSITION 3.1.4 ► PROPERTIES OF TRANSITIVE RELATIONS**

Let  $R$  and  $S$  be relations on  $A$ .

00UX

1. *Interaction With Inverses.* If  $R$  is transitive, then so is  $R^\dagger$ .

00UY

2. *Interaction With Composition.* If  $R$  and  $S$  are transitive, then  $S \diamond R$  **may fail to be transitive**.

**PROOF 3.1.5 ► PROOF OF PROPOSITION 3.1.4**

Item 1: Interaction With Inverses

Clear.

Item 2: Interaction With Composition

See [MSE 2096272].<sup>1</sup>



<sup>1</sup>*Intuition:* Transitivity for  $R$  and  $S$  fails to imply that of  $S \circ R$  because the composition operation for relations intertwines  $R$  and  $S$  in an incompatible way:

1. If  $a \sim_{S \circ R} c$  and  $c \sim_{S \circ R} e$ , then:
  - (a) There is some  $b \in A$  such that:
    - i.  $a \sim_R b$ ;
    - ii.  $b \sim_S c$ ;
  - (b) There is some  $d \in A$  such that:
    - i.  $c \sim_R d$ ;
    - ii.  $d \sim_S e$ .

### 00UZ 3.2 The Transitive Closure of a Relation

Let  $R$  be a relation on  $A$ .

#### 00V0 DEFINITION 3.2.1 ► THE TRANSITIVE CLOSURE OF A RELATION

The **transitive closure** of  $\sim_R$  is the relation  $\sim_R^{\text{trans}}$ <sup>1</sup> satisfying the following universal property:<sup>2</sup>

- (★) Given another transitive relation  $\sim_S$  on  $A$  such that  $R \subset S$ , there exists an inclusion  $\sim_R^{\text{trans}} \subset \sim_S$ .

<sup>1</sup>*Further Notation:* Also written  $R^{\text{trans}}$ .

<sup>2</sup>*Slogan:* The transitive closure of  $R$  is the smallest transitive relation containing  $R$ .

#### 00V1 CONSTRUCTION 3.2.2 ► THE TRANSITIVE CLOSURE OF A RELATION

Concretely,  $\sim_R^{\text{trans}}$  is the free non-unital monoid on  $R$  in  $(\mathbf{Rel}(A, A), \diamond)^1$ , being given by

$$\begin{aligned} R^{\text{trans}} &\stackrel{\text{def}}{=} \coprod_{n=1}^{\infty} R^{\diamond n} \\ &\stackrel{\text{def}}{=} \bigcup_{n=1}^{\infty} R^{\diamond n} \\ &\stackrel{\text{def}}{=} \left\{ (a, b) \in A \times B \mid \begin{array}{l} \text{there exists some } (x_1, \dots, x_n) \in R^{\times n} \\ \text{such that } a \sim_R x_1 \sim_R \dots \sim_R x_n \sim_R b \end{array} \right\}. \end{aligned}$$

<sup>1</sup>Or, equivalently, the free non-unital  $\mathbb{B}_1$ -monoid on  $R$  in  $(\mathbf{N}_\bullet(\mathbf{Rel}(A, A)), \diamond)$ .

#### PROOF 3.2.3 ► PROOF OF CONSTRUCTION 3.2.2

Clear. 

00V2

#### PROPOSITION 3.2.4 ► PROPERTIES OF THE TRANSITIVE CLOSURE OF A RELATION

Let  $R$  be a relation on  $A$ .

00V3

1. *Adjointness.* We have an adjunction

$$((-)^{\text{trans}} \dashv \overset{\sim}{\text{忘}}): \mathbf{Rel}(A, A) \overset{((-)^{\text{trans}})}{\underset{\overset{\sim}{\text{忘}}}{\rightleftarrows}} \mathbf{Rel}^{\text{trans}}(A, A),$$

witnessed by a bijection of sets

$$\mathbf{Rel}^{\text{trans}}(R^{\text{trans}}, S) \cong \mathbf{Rel}(R, S),$$

natural in  $R \in \text{Obj}(\mathbf{Rel}^{\text{trans}}(A, A))$  and  $S \in \text{Obj}(\mathbf{Rel}(A, B))$ .

00V4

2. *The Transitive Closure of a Transitive Relation.* If  $R$  is transitive, then  $R^{\text{trans}} = R$ .

00V5

3. *Idempotency.* We have

$$(R^{\text{trans}})^{\text{trans}} = R^{\text{trans}}.$$

00V6

4. *Interaction With Inverses.* We have

$$\begin{array}{ccc} \text{Rel}(A, A) & \xrightarrow{(-)^{\text{trans}}} & \text{Rel}(A, A) \\ \downarrow (-)^{\dagger} & & \downarrow (-)^{\dagger} \\ \text{Rel}(A, A) & \xrightarrow{(-)^{\text{trans}}} & \text{Rel}(A, A). \end{array}$$

$$(R^{\dagger})^{\text{trans}} = (R^{\text{trans}})^{\dagger},$$

00V7

5. *Interaction With Composition.* We have

$$\begin{array}{ccc} \text{Rel}(A, A) \times \text{Rel}(A, A) & \xrightarrow{\diamond} & \text{Rel}(A, A) \\ \downarrow (-)^{\text{trans}} \times (-)^{\text{trans}} & \text{X} & \downarrow (-)^{\text{trans}} \\ \text{Rel}(A, A) \times \text{Rel}(A, A) & \xrightarrow{\diamond} & \text{Rel}(A, A). \end{array}$$

$$(S \diamond R)^{\text{trans}} \stackrel{\text{poss.}}{\neq} S^{\text{trans}} \diamond R^{\text{trans}},$$

## PROOF 3.2.5 ► PROOF OF PROPOSITION 3.2.4

## Item 1: Adjointness

This is a rephrasing of the universal property of the transitive closure of a relation, stated in [Definition 3.2.1](#).

## Item 2: The Transitive Closure of a Transitive Relation

Clear.

## Item 3: Idempotency

This follows from [Item 2](#).

## Item 4: Interaction With Inverses

We have

$$\begin{aligned}
 (R^\dagger)^{\text{trans}} &= \bigcup_{n=1}^{\infty} (R^\dagger)^{\diamond n} \\
 &= \bigcup_{n=1}^{\infty} (R^{\diamond n})^\dagger \\
 &= \left( \bigcup_{n=1}^{\infty} R^{\diamond n} \right)^\dagger \\
 &= (R^{\text{trans}})^\dagger,
 \end{aligned}$$

where we have used, respectively:

1. Construction 3.2.2.
2. Constructions With Relations, Item 4 of Proposition 3.12.3.
3. Constructions With Relations, Item 1 of Proposition 3.6.2.
4. Construction 3.2.2.

Item 5: Interaction With Composition

This follows from Item 2 of Proposition 3.1.4.



## 00V8 4 Equivalence Relations

### 00V9 4.1 Foundations

Let  $A$  be a set.

#### 00VA DEFINITION 4.1.1 ► EQUIVALENCE RELATIONS

A relation  $R$  is an **equivalence relation** if it is reflexive, symmetric, and transitive.<sup>1</sup>

<sup>1</sup>*Further Terminology:* If instead  $R$  is just symmetric and transitive, then it is called a **partial equivalence relation**.

## 00VB EXAMPLE 4.1.2 ► THE KERNEL OF A FUNCTION

The **kernel of a function**  $f: A \rightarrow B$  is the equivalence relation  $\sim_{\text{Ker}(f)}$  on  $A$  obtained by declaring  $a \sim_{\text{Ker}(f)} b$  iff  $f(a) = f(b)$ .<sup>1</sup>

<sup>1</sup>The kernel  $\text{Ker}(f): A \dashv A$  of  $f$  is the underlying functor of the monad induced by the adjunction  $\text{Gr}(f) \dashv f^{-1}: A \rightleftarrows B$  in **Rel** of **Constructions With Relations**, Item 2 of **Proposition 3.1.2**.

## 00VC DEFINITION 4.1.3 ► THE PO/SET OF EQUIVALENCE RELATIONS ON A SET

Let  $A$  and  $B$  be sets.

1. The **set of equivalence relations from  $A$  to  $B$**  is the subset  $\text{Rel}^{\text{eq}}(A, B)$  of  $\text{Rel}(A, B)$  spanned by the equivalence relations.
2. The **poset of relations from  $A$  to  $B$**  is the subposet  $\mathbf{Rel}^{\text{eq}}(A, B)$  of  $\mathbf{Rel}(A, B)$  spanned by the equivalence relations.

## 00VF 4.2 The Equivalence Closure of a Relation

Let  $R$  be a relation on  $A$ .

## 00VG DEFINITION 4.2.1 ► THE EQUIVALENCE CLOSURE OF A RELATION

The **equivalence closure**<sup>1</sup> of  $\sim_R$  is the relation  $\sim_R^{\text{eq}}$ <sup>2</sup> satisfying the following universal property:<sup>3</sup>

- (★) Given another equivalence relation  $\sim_S$  on  $A$  such that  $R \subset S$ , there exists an inclusion  $\sim_R^{\text{eq}} \subset \sim_S$ .

<sup>1</sup>*Further Terminology:* Also called the **equivalence relation associated to  $\sim_R$** .

<sup>2</sup>*Further Notation:* Also written  $R^{\text{eq}}$ .

<sup>3</sup>*Slogan:* The equivalence closure of  $R$  is the smallest equivalence relation containing  $R$ .

## 00VH CONSTRUCTION 4.2.2 ► THE EQUIVALENCE CLOSURE OF A RELATION


Concretely,  $\sim_R^{\text{eq}}$  is the equivalence relation on  $A$  defined by

$$\begin{aligned}
 R^{\text{eq}} &\stackrel{\text{def}}{=} \left( \left( R^{\text{refl}} \right)^{\text{symm}} \right)^{\text{trans}} \\
 &= \left( \left( R^{\text{symm}} \right)^{\text{trans}} \right)^{\text{refl}} \\
 &= \left\{ (a, b) \in A \times B \mid \begin{array}{l} \text{there exists } (x_1, \dots, x_n) \in R^{\times n} \text{ satisfying at} \\ \text{least one of the following conditions:} \\ \\ 1. \text{ The following conditions are satisfied:} \\ \quad (a) \text{ We have } a \sim_R x_1 \text{ or } x_1 \sim_R a; \\ \quad (b) \text{ We have } x_i \sim_R x_{i+1} \text{ or } x_{i+1} \sim_R x_i \\ \quad \quad \text{for each } 1 \leq i \leq n-1; \\ \quad (c) \text{ We have } b \sim_R x_n \text{ or } x_n \sim_R b; \\ \\ 2. \text{ We have } a = b. \end{array} \right\}.
 \end{aligned}$$

#### PROOF 4.2.3 ► PROOF OF CONSTRUCTION 4.2.2

From the universal properties of the reflexive, symmetric, and transitive closures of a relation (**Definitions 1.2.1, 2.2.1 and 3.2.1**), we see that it suffices to prove that:

- 00VJ 1. The symmetric closure of a reflexive relation is still reflexive.
- 00VK 2. The transitive closure of a symmetric relation is still symmetric.

which are both clear. 

#### 00VL PROPOSITION 4.2.4 ► PROPERTIES OF EQUIVALENCE RELATIONS

Let  $R$  be a relation on  $A$ .

- 00VM 1. *Adjointness.* We have an adjunction

$$\left( (-)^{\text{eq}} \dashv \overset{\sim}{\text{忘}} \right): \mathbf{Rel}(A, B) \overset{(-)^{\text{eq}}}{\overset{\perp}{\rightleftarrows}} \mathbf{Rel}^{\text{eq}}(A, B),$$

witnessed by a bijection of sets

$$\mathbf{Rel}^{\text{eq}}(R^{\text{eq}}, S) \cong \mathbf{Rel}(R, S),$$

natural in  $R \in \text{Obj}(\mathbf{Rel}^{\text{eq}}(A, B))$  and  $S \in \text{Obj}(\mathbf{Rel}(A, B))$ .

00VN

2. *The Equivalence Closure of an Equivalence Relation.* If  $R$  is an equivalence relation, then  $R^{\text{eq}} = R$ .

00VP

3. *Idempotency.* We have

$$(R^{\text{eq}})^{\text{eq}} = R^{\text{eq}}.$$

#### PROOF 4.2.5 ► PROOF OF PROPOSITION 4.2.4

Item 1: Adjointness

This is a rephrasing of the universal property of the equivalence closure of a relation, stated in [Definition 4.2.1](#).

Item 2: The Equivalence Closure of an Equivalence Relation

Clear.

Item 3: Idempotency

This follows from [Item 2](#). 

## 00VQ 5 Quotients by Equivalence Relations

### 00VR 5.1 Equivalence Classes

Let  $A$  be a set, let  $R$  be a relation on  $A$ , and let  $a \in A$ .

00VS

#### DEFINITION 5.1.1 ► EQUIVALENCE CLASSES

The **equivalence class associated to**  $a$  is the set  $[a]$  defined by

$$\begin{aligned} [a] &\stackrel{\text{def}}{=} \{x \in X \mid x \sim_R a\} \\ &= \{x \in X \mid a \sim_R x\}. \end{aligned} \quad (\text{since } R \text{ is symmetric})$$

### 00VT 5.2 Quotients of Sets by Equivalence Relations

Let  $A$  be a set and let  $R$  be a relation on  $A$ .



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**DEFINITION 5.2.1 ► QUOTIENTS OF SETS BY EQUIVALENCE RELATIONS**

The **quotient of  $X$  by  $R$**  is the set  $X/\sim_R$  defined by

$$X/\sim_R \stackrel{\text{def}}{=} \{[a] \in \mathcal{P}(X) \mid a \in X\}.$$

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**REMARK 5.2.2 ► WHY USE “EQUIVALENCE” RELATIONS FOR QUOTIENT SETS**

The reason we define quotient sets for equivalence relations only is that each of the properties of being an equivalence relation—reflexivity, symmetry, and transitivity—ensures that the equivalence classes  $[a]$  of  $X$  under  $R$  are well-behaved:

- *Reflexivity.* If  $R$  is reflexive, then, for each  $a \in X$ , we have  $a \in [a]$ .
- *Symmetry.* The equivalence class  $[a]$  of an element  $a$  of  $X$  is defined by

$$[a] \stackrel{\text{def}}{=} \{x \in X \mid x \sim_R a\},$$

but we could equally well define

$$[a]' \stackrel{\text{def}}{=} \{x \in X \mid a \sim_R x\}$$

instead. This is not a problem when  $R$  is symmetric, as we then have  $[a] = [a]'$ .<sup>1</sup>

- *Transitivity.* If  $R$  is transitive, then  $[a]$  and  $[b]$  are disjoint iff  $a \not\sim_R b$ , and equal otherwise.

<sup>1</sup>When categorifying equivalence relations, one finds that  $[a]$  and  $[a]'$  correspond to presheaves and copresheaves; see ??, ??.

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**PROPOSITION 5.2.3 ► PROPERTIES OF QUOTIENT SETS**

Let  $f: X \rightarrow Y$  be a function and let  $R$  be a relation on  $X$ .

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1. *As a Coequaliser.* We have an isomorphism of sets

$$X/\sim_R^{\text{eq}} \cong \text{CoEq} \left( R \hookrightarrow X \times X \begin{array}{c} \xrightarrow{\text{pr}_1} \\ \xrightarrow{\text{pr}_2} \end{array} X \right),$$

where  $\sim_R^{\text{eq}}$  is the equivalence relation generated by  $\sim_R$ .

00VY

2. *As a Pushout.* We have an isomorphism of sets<sup>1</sup>

$$X/\sim_R^{\text{eq}} \cong X \amalg_{\text{Eq}(pr_1, pr_2)} X$$

$$\begin{array}{ccc} X/\sim_R^{\text{eq}} & \longleftarrow & X \\ \uparrow \scriptstyle \Gamma & & \uparrow \\ X & \longleftarrow & \text{Eq}(pr_1, pr_2) \end{array}$$

where  $\sim_R^{\text{eq}}$  is the equivalence relation generated by  $\sim_R$ .

00VZ

3. *The First Isomorphism Theorem for Sets.* We have an isomorphism of sets<sup>2,3</sup>

$$X/\sim_{\text{Ker}(f)} \cong \text{Im}(f).$$

00W0

4. *Descending Functions to Quotient Sets, I.* Let  $R$  be an equivalence relation on  $X$ . The following conditions are equivalent:

- (a) There exists a map

$$\bar{f}: X/\sim_R \rightarrow Y$$

making the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ q \downarrow & \searrow \exists \bar{f} & \\ X/\sim_R & & \end{array}$$

commute.

- (b) We have  $R \subset \text{Ker}(f)$ .

- (c) For each  $x, y \in X$ , if  $x \sim_R y$ , then  $f(x) = f(y)$ .

00W1

5. *Descending Functions to Quotient Sets, II.* Let  $R$  be an equivalence relation on  $X$ . If the conditions of **Item 4** hold, then  $\bar{f}$  is the *unique* map making the

diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ q \downarrow & \exists! \nearrow \bar{f} & \\ X/\sim_R & & \end{array}$$

commute.

00W2

6. *Descending Functions to Quotient Sets, III.* Let  $R$  be an equivalence relation on  $X$ . We have a bijection

$$\text{Hom}_{\text{Sets}}(X/\sim_R, Y) \cong \text{Hom}_{\text{Sets}}^R(X, Y),$$

natural in  $X, Y \in \text{Obj}(\text{Sets})$ , given by the assignment  $f \mapsto \bar{f}$  of [Items 4](#) and [5](#), where  $\text{Hom}_{\text{Sets}}^R(X, Y)$  is the set defined by

$$\text{Hom}_{\text{Sets}}^R(X, Y) \stackrel{\text{def}}{=} \left\{ f \in \text{Hom}_{\text{Sets}}(X, Y) \left| \begin{array}{l} \text{for each } x, y \in X, \\ \text{if } x \sim_R y, \text{ then} \\ f(x) = f(y) \end{array} \right. \right\}.$$

00W3

7. *Descending Functions to Quotient Sets, IV.* Let  $R$  be an equivalence relation on  $X$ . If the conditions of [Item 4](#) hold, then the following conditions are equivalent:

- (a) The map  $\bar{f}$  is an injection.
- (b) We have  $R = \text{Ker}(f)$ .
- (c) For each  $x, y \in X$ , we have  $x \sim_R y$  iff  $f(x) = f(y)$ .

00W4

8. *Descending Functions to Quotient Sets, V.* Let  $R$  be an equivalence relation on  $X$ . If the conditions of [Item 4](#) hold, then the following conditions are equivalent:

- (a) The map  $f: X \rightarrow Y$  is surjective.
- (b) The map  $\bar{f}: X/\sim_R \rightarrow Y$  is surjective.

00W5

9. *Descending Functions to Quotient Sets, VI.* Let  $R$  be a relation on  $X$  and let  $\sim_R^{\text{eq}}$  be the equivalence relation associated to  $R$ . The following conditions are equivalent:

00W6

(a) The map  $f$  satisfies the equivalent conditions of **Item 4**:

- There exists a map

$$\bar{f}: X/\sim_R^{\text{eq}} \rightarrow Y$$

making the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ q \downarrow & \exists \nearrow \bar{f} & \\ X/\sim_R^{\text{eq}} & & \end{array}$$

commute.

- For each  $x, y \in X$ , if  $x \sim_R^{\text{eq}} y$ , then  $f(x) = f(y)$ .

00W7

(b) For each  $x, y \in X$ , if  $x \sim_R y$ , then  $f(x) = f(y)$ .<sup>1</sup>Dually, we also have an isomorphism of sets

$$\begin{array}{ccc} \text{Eq}(pr_1, pr_2) & \longrightarrow & X \\ \downarrow \lrcorner & & \downarrow \\ X & \longrightarrow & X/\sim_R^{\text{eq}} \end{array}$$

$\text{Eq}(pr_1, pr_2) \cong X \times_{X/\sim_R^{\text{eq}}} X,$

<sup>2</sup>*Further Terminology:* The set  $X/\sim_{\text{Ker}(f)}$  is often called the **coimage** of  $f$ , and denoted by  $\text{Coim}(f)$ .<sup>3</sup>In a sense this is a result relating the monad in **Rel** induced by  $f$  with the comonad in **Rel** induced by  $f$ , as the kernel and image

$$\begin{aligned} \text{Ker}(f) &: X \rightharpoonup X, \\ \text{Im}(f) &\subset Y \end{aligned}$$

of  $f$  are the underlying functors of (respectively) the induced monad and comonad of the adjunction

$$\left( \text{Gr}(f) \dashv f^{-1} \right): \begin{array}{ccc} & \text{Gr}(f) & \\ \uparrow & \text{ } & \downarrow \\ A & \xrightarrow{\quad} & B \\ \downarrow & \text{ } & \uparrow \\ & f^{-1} & \end{array}$$

of **Constructions With Relations**, **Item 2** of **Proposition 3.1.2**.

## PROOF 5.2.4 ► PROOF OF PROPOSITION 5.2.3

Item 1: As a Coequaliser

Omitted.

Item 2: As a Pushout

Omitted.

Item 3: The First Isomorphism Theorem for Sets

Clear.

Item 4: Descending Functions to Quotient Sets, I

See [Pro24c].

Item 5: Descending Functions to Quotient Sets, II

See [Pro24d].

Item 6: Descending Functions to Quotient Sets, III

This follows from Items 5 and 6.

Item 7: Descending Functions to Quotient Sets, IV

See [Pro24b].

Item 8: Descending Functions to Quotient Sets, V

See [Pro24a].

Item 9: Descending Functions to Quotient Sets, VI

The implication Item 9a  $\implies$  Item 9b is clear.


Conversely, suppose that, for each  $x, y \in X$ , if  $x \sim_R y$ , then  $f(x) = f(y)$ . Spelling out the definition of the equivalence closure of  $R$ , we see that the condition  $x \sim_R^{\text{eq}} y$  unwinds to the following:

(★) There exist  $(x_1, \dots, x_n) \in R^{\times n}$  satisfying at least one of the following conditions:

1. The following conditions are satisfied:
  - (a) We have  $x \sim_R x_1$  or  $x_1 \sim_R x$ ;
  - (b) We have  $x_i \sim_R x_{i+1}$  or  $x_{i+1} \sim_R x_i$  for each  $1 \leq i \leq n - 1$ ;
  - (c) We have  $y \sim_R x_n$  or  $x_n \sim_R y$ ;
2. We have  $x = y$ .

Now, if  $x = y$ , then  $f(x) = f(y)$  trivially; otherwise, we have

$$\begin{aligned} f(x) &= f(x_1), \\ f(x_1) &= f(x_2), \\ &\vdots \\ f(x_{n-1}) &= f(x_n), \\ f(x_n) &= f(y), \end{aligned}$$

and  $f(x) = f(y)$ , as we wanted to show. 

## Appendices

### A Other Chapters

#### Sets

1. [Sets](#)
2. [Constructions With Sets](#)
3. [Pointed Sets](#)
4. [Tensor Products of Pointed Sets](#)

#### 6. [Constructions With Relations](#)

#### 7. [Equivalence Relations and Apartness Relations](#)

#### Category Theory

#### 8. [Categories](#)

#### Relations

#### 5. [Relations](#)

#### Bicategories

#### 9. [Types of Morphisms in Bicategories](#)

## References

- [MSE 2096272] [Akiva Weinberger](#). *Is composition of two transitive relations transitive? If not, can you give me a counterexample?* Mathematics Stack Exchange. URL: <https://math.stackexchange.com/q/2096272> (cit. on p. 10).
- [Pro24a] Proof Wiki Contributors. *Condition For Mapping from Quotient Set To Be A Surjection* — Proof Wiki. 2024. URL: <https://proofwiki.org/>

- wiki/Condition\_for\_Mapping\_from\_Quotient\_Set\_to\_be\_Surjection (cit. on p. 21).
- [Pro24b] Proof Wiki Contributors. *Condition For Mapping From Quotient Set To Be An Injection*—Proof Wiki. 2024. URL: [https://proofwiki.org/wiki/Condition\\_for\\_Mapping\\_from\\_Quotient\\_Set\\_to\\_be\\_Injection](https://proofwiki.org/wiki/Condition_for_Mapping_from_Quotient_Set_to_be_Injection) (cit. on p. 21).
- [Pro24c] Proof Wiki Contributors. *Condition For Mapping From Quotient Set To Be Well-Defined*—Proof Wiki. 2024. URL: [https://proofwiki.org/wiki/Condition\\_for\\_Mapping\\_from\\_Quotient\\_Set\\_to\\_be\\_Well-Defined](https://proofwiki.org/wiki/Condition_for_Mapping_from_Quotient_Set_to_be_Well-Defined) (cit. on p. 21).
- [Pro24d] Proof Wiki Contributors. *Mapping From Quotient Set When Defined Is Unique*—Proof Wiki. 2024. URL: [https://proofwiki.org/wiki/Mapping\\_from\\_Quotient\\_Set\\_when\\_Defined\\_is\\_Unique](https://proofwiki.org/wiki/Mapping_from_Quotient_Set_when_Defined_is_Unique) (cit. on p. 21).