

Constructions With Sets

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000J This chapter develops some material relating to constructions with sets with an eye towards its categorical and higher-categorical counterparts to be introduced later in this work. In particular, it contains:

1. Explicit descriptions of the major types of co/limits in **Sets**, including in particular explicit descriptions of pushouts and coequalisers (see **Definitions 2.4.1.1** and **2.5.1.1** and **Remarks 2.4.1.2** and **2.5.1.2**).
2. A discussion of powersets as decategorifications of categories of presheaves (**Remarks 4.1.1.2** and **4.3.1.2**), including a (-1) -categorical analogue of un/straightening, described in **Items 1** and **2** of **Proposition 4.3.1.6** and **Remark 4.3.1.7**.
3. A lengthy discussion of the adjoint triple

$$f_* \dashv f^{-1} \dashv f_! : \mathcal{P}(A) \xrightarrow{\cong} \mathcal{P}(B)$$

of functors (morphisms of posets) between $\mathcal{P}(A)$ and $\mathcal{P}(B)$ induced by a map of sets $f : A \rightarrow B$, along with a discussion of the properties of f_* , f^{-1} , and $f_!$.

In line with the categorical viewpoint developed here, this adjoint triple may be described in terms of Kan extensions, and, as it turns out, it also shows up in some definitions and results in point-set topology, such as in e.g. notions of continuity for functions (??, ??).

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000K 1 Limits of Sets

000L 1.1 The Terminal Set

000M Definition 1.1.1.1. The **terminal set** is the pair $(\text{pt}, \{!_A\}_{A \in \text{Obj}(\text{Sets})})$ consisting of:

- *The Limit.* The punctual set $\text{pt} \stackrel{\text{def}}{=} \{\star\}$.
- *The Cone.* The collection of maps

$$\{!_A: A \rightarrow \text{pt}\}_{A \in \text{Obj}(\text{Sets})}$$

defined by

$$!_A(a) \stackrel{\text{def}}{=} \star$$

for each $a \in A$ and each $A \in \text{Obj}(\text{Sets})$.

Proof. We claim that pt is the terminal object of **Sets**. Indeed, suppose we have a diagram of the form

$$A \quad \text{pt}$$

in **Sets**. Then there exists a unique map $\phi: A \rightarrow \text{pt}$ making the diagram

$$A \xrightarrow[\exists!]{\phi} \text{pt}$$

commute, namely $!_A$. □

000N 1.2 Products of Families of Sets

Let $\{A_i\}_{i \in I}$ be a family of sets.

000P Definition 1.2.1.1. The **product**¹ of $\{A_i\}_{i \in I}$ is the pair $(\prod_{i \in I} A_i, \{\text{pr}_i\}_{i \in I})$ consisting of:

- *The Limit.* The set $\prod_{i \in I} A_i$ defined by²

$$\prod_{i \in I} A_i \stackrel{\text{def}}{=} \left\{ f \in \text{Sets} \left(I, \bigcup_{i \in I} A_i \right) \mid \begin{array}{l} \text{for each } i \in I, \text{ we} \\ \text{have } f(i) \in A_i \end{array} \right\}.$$

¹ *Further Terminology:* Also called the **Cartesian product** of $\{A_i\}_{i \in I}$.

² Less formally, $\prod_{i \in I} A_i$ is the set whose elements are I -indexed collections $(a_i)_{i \in I}$ with

- *The Cone.* The collection

$$\left\{ \text{pr}_i : \prod_{i \in I} A_i \rightarrow A_i \right\}_{i \in I}$$

of maps given by

$$\text{pr}_i(f) \stackrel{\text{def}}{=} f(i)$$

for each $f \in \prod_{i \in I} A_i$ and each $i \in I$.

Proof. We claim that $\prod_{i \in I} A_i$ is the categorical product of $\{A_i\}_{i \in I}$ in **Sets**. Indeed, suppose we have, for each $i \in I$, a diagram of the form

$$\begin{array}{ccc} P & & \\ & \searrow p_i & \\ \prod_{i \in I} A_i & \xrightarrow{\text{pr}_i} & A_i \end{array}$$

in **Sets**. Then there exists a unique map $\phi: P \rightarrow \prod_{i \in I} A_i$ making the diagram

$$\begin{array}{ccc} P & & \\ \downarrow \phi \exists! & \searrow p_i & \\ \prod_{i \in I} A_i & \xrightarrow{\text{pr}_i} & A_i \end{array}$$

commute, being uniquely determined by the condition $\text{pr}_i \circ \phi = p_i$ for each $i \in I$ via

$$\phi(x) = (p_i(x))_{i \in I}$$

for each $x \in P$. □

000Q Proposition 1.2.1.2. Let $\{A_i\}_{i \in I}$ be a family of sets.

$a_i \in A_i$ for each $i \in I$. The projection maps

$$\left\{ \text{pr}_i : \prod_{i \in I} A_i \rightarrow A_i \right\}_{i \in I}$$

are then given by

$$\text{pr}_i((a_j)_{j \in I}) \stackrel{\text{def}}{=} a_i$$

for each $(a_j)_{j \in I} \in \prod_{i \in I} A_i$ and each $i \in I$.

000R 1. *Functoriality.* The assignment $\{A_i\}_{i \in I} \mapsto \prod_{i \in I} A_i$ defines a functor

$$\prod_{i \in I} : \text{Fun}(I_{\text{disc}}, \text{Sets}) \rightarrow \text{Sets}$$

where

- *Action on Objects.* For each $(A_i)_{i \in I} \in \text{Obj}(\text{Fun}(I_{\text{disc}}, \text{Sets}))$, we have

$$\left[\prod_{i \in I} \right] ((A_i)_{i \in I}) \stackrel{\text{def}}{=} \prod_{i \in I} A_i$$

- *Action on Morphisms.* For each $(A_i)_{i \in I}, (B_i)_{i \in I} \in \text{Obj}(\text{Fun}(I_{\text{disc}}, \text{Sets}))$, the action on Hom-sets

$$\left(\prod_{i \in I} \right)_{(A_i)_{i \in I}, (B_i)_{i \in I}} : \text{Nat}((A_i)_{i \in I}, (B_i)_{i \in I}) \rightarrow \text{Sets} \left(\prod_{i \in I} A_i, \prod_{i \in I} B_i \right)$$

of $\prod_{i \in I}$ at $((A_i)_{i \in I}, (B_i)_{i \in I})$ is defined by sending a map

$$\{f_i : A_i \rightarrow B_i\}_{i \in I}$$

in $\text{Nat}((A_i)_{i \in I}, (B_i)_{i \in I})$ to the map of sets

$$\prod_{i \in I} f_i : \prod_{i \in I} A_i \rightarrow \prod_{i \in I} B_i$$

defined by

$$\left[\prod_{i \in I} f_i \right] ((a_i)_{i \in I}) \stackrel{\text{def}}{=} (f_i(a_i))_{i \in I}$$

for each $(a_i)_{i \in I} \in \prod_{i \in I} A_i$.

Proof. **Item 1, Functoriality:** This follows from ??, ?? of ??.

□

000S 1.3 Binary Products of Sets

Let A and B be sets.

000T Definition 1.3.1.1. The **product**³ of A and B is the pair $(A \times B, \{\text{pr}_1, \text{pr}_2\})$ consisting of:

- *The Limit.* The set $A \times B$ defined by⁴

$$\begin{aligned} A \times B &\stackrel{\text{def}}{=} \prod_{z \in \{A, B\}} z \\ &\stackrel{\text{def}}{=} \{f \in \mathbf{Sets}(\{0, 1\}, A \cup B) \mid \text{we have } f(0) \in A \text{ and } f(1) \in B\} \\ &\cong \{\{\{a\}, \{a, b\}\} \in \mathcal{P}(\mathcal{P}(A \cup B)) \mid \text{we have } a \in A \text{ and } b \in B\}. \end{aligned}$$

- *The Cone.* The maps

$$\begin{aligned} \text{pr}_1 &: A \times B \rightarrow A, \\ \text{pr}_2 &: A \times B \rightarrow B \end{aligned}$$

defined by

$$\begin{aligned} \text{pr}_1(a, b) &\stackrel{\text{def}}{=} a, \\ \text{pr}_2(a, b) &\stackrel{\text{def}}{=} b \end{aligned}$$

for each $(a, b) \in A \times B$.

Proof. We claim that $A \times B$ is the categorical product of A and B in \mathbf{Sets} . Indeed, suppose we have a diagram of the form

$$\begin{array}{ccccc} & & P & & \\ & \swarrow p_1 & & \searrow p_2 & \\ A & \xleftarrow{\text{pr}_1} & A \times B & \xrightarrow{\text{pr}_2} & B \end{array}$$

in \mathbf{Sets} . Then there exists a unique map $\phi: P \rightarrow A \times B$ making the diagram

$$\begin{array}{ccccc} & & P & & \\ & \swarrow p_1 & \downarrow \phi \downarrow \exists! & \searrow p_2 & \\ A & \xleftarrow{\text{pr}_1} & A \times B & \xrightarrow{\text{pr}_2} & B \end{array}$$

³*Further Terminology:* Also called the **Cartesian product** of A and B or the **binary Cartesian product** of A and B , for emphasis.

This can also be thought of as the $(\mathbb{E}_{-1}, \mathbb{E}_{-1})$ -**tensor product** of A and B .

⁴In other words, $A \times B$ is the set whose elements are ordered pairs (a, b) with $a \in A$ and $b \in B$ as in [Definition 3.4.1.1](#)

commute, being uniquely determined by the conditions

$$\begin{aligned}\text{pr}_1 \circ \phi &= p_1, \\ \text{pr}_2 \circ \phi &= p_2\end{aligned}$$

via

$$\phi(x) = (p_1(x), p_2(x))$$

for each $x \in P$. □

000U Proposition 1.3.1.2. Let A , B , C , and X be sets.

000V 1. *Functoriality.* The assignments $A, B, (A, B) \mapsto A \times B$ define functors

$$\begin{aligned}A \times - &: \mathbf{Sets} \rightarrow \mathbf{Sets}, \\ - \times B &: \mathbf{Sets} \rightarrow \mathbf{Sets}, \\ -_1 \times -_2 &: \mathbf{Sets} \times \mathbf{Sets} \rightarrow \mathbf{Sets},\end{aligned}$$

where $-_1 \times -_2$ is the functor where

- *Action on Objects.* For each $(A, B) \in \text{Obj}(\mathbf{Sets} \times \mathbf{Sets})$, we have

$$[-_1 \times -_2](A, B) \stackrel{\text{def}}{=} A \times B.$$

- *Action on Morphisms.* For each $(A, B), (X, Y) \in \text{Obj}(\mathbf{Sets})$, the action on Hom-sets

$$\times_{(A,B),(X,Y)}: \mathbf{Sets}(A, X) \times \mathbf{Sets}(B, Y) \rightarrow \mathbf{Sets}(A \times B, X \times Y)$$

of \times at $((A, B), (X, Y))$ is defined by sending (f, g) to the function

$$f \times g: A \times B \rightarrow X \times Y$$

defined by

$$[f \times g](a, b) \stackrel{\text{def}}{=} (f(a), g(b))$$

for each $(a, b) \in A \times B$.

and where $A \times -$ and $- \times B$ are the partial functors of $-_1 \times -_2$ at $A, B \in \text{Obj}(\mathbf{Sets})$.

000W 2. *Adjointness.* We have adjunctions

$$(A \times - \dashv \text{Hom}_{\mathbf{Sets}}(A, -)): \text{Sets} \begin{array}{c} \xrightarrow{A \times -} \\ \perp \\ \xleftarrow{\text{Hom}_{\mathbf{Sets}}(A, -)} \end{array} \text{Sets},$$

$$(- \times B \dashv \text{Hom}_{\mathbf{Sets}}(B, -)): \text{Sets} \begin{array}{c} \xrightarrow{- \times B} \\ \perp \\ \xleftarrow{\text{Hom}_{\mathbf{Sets}}(B, -)} \end{array} \text{Sets},$$

witnessed by bijections

$$\begin{aligned} \text{Hom}_{\mathbf{Sets}}(A \times B, C) &\cong \text{Hom}_{\mathbf{Sets}}(A, \text{Hom}_{\mathbf{Sets}}(B, C)), \\ \text{Hom}_{\mathbf{Sets}}(A \times B, C) &\cong \text{Hom}_{\mathbf{Sets}}(B, \text{Hom}_{\mathbf{Sets}}(A, C)), \end{aligned}$$

natural in $A, B, C \in \text{Obj}(\mathbf{Sets})$.

000X 3. *Associativity.* We have an isomorphism of sets

$$(A \times B) \times C \cong A \times (B \times C),$$

natural in $A, B, C \in \text{Obj}(\mathbf{Sets})$.

000Y 4. *Unitality.* We have isomorphisms of sets

$$\begin{aligned} \text{pt} \times A &\cong A, \\ A \times \text{pt} &\cong A, \end{aligned}$$

natural in $A \in \text{Obj}(\mathbf{Sets})$.

000Z 5. *Commutativity.* We have an isomorphism of sets

$$A \times B \cong B \times A,$$

natural in $A, B \in \text{Obj}(\mathbf{Sets})$.

0010 6. *Annihilation With the Empty Set.* We have isomorphisms of sets

$$\begin{aligned} A \times \emptyset &\cong \emptyset, \\ \emptyset \times A &\cong \emptyset, \end{aligned}$$

natural in $A \in \text{Obj}(\mathbf{Sets})$.

0011 7. *Distributivity Over Unions.* We have isomorphisms of sets

$$\begin{aligned} A \times (B \cup C) &= (A \times B) \cup (A \times C), \\ (A \cup B) \times C &= (A \times C) \cup (B \times C). \end{aligned}$$

0012 8. *Distributivity Over Intersections.* We have isomorphisms of sets

$$\begin{aligned} A \times (B \cap C) &= (A \times B) \cap (A \times C), \\ (A \cap B) \times C &= (A \times C) \cap (B \times C). \end{aligned}$$

0013 9. *Middle-Four Exchange with Respect to Intersections.* We have an isomorphism of sets

$$(A \times B) \cap (C \times D) \cong (A \cap C) \times (B \cap D).$$

0014 10. *Distributivity Over Differences.* We have isomorphisms of sets

$$\begin{aligned} A \times (B \setminus C) &= (A \times B) \setminus (A \times C), \\ (A \setminus B) \times C &= (A \times C) \setminus (B \times C), \end{aligned}$$

natural in $A, B, C \in \text{Obj}(\text{Sets})$.

0015 11. *Distributivity Over Symmetric Differences.* We have isomorphisms of sets

$$\begin{aligned} A \times (B \triangle C) &= (A \times B) \triangle (A \times C), \\ (A \triangle B) \times C &= (A \times C) \triangle (B \times C), \end{aligned}$$

natural in $A, B, C \in \text{Obj}(\text{Sets})$.

0016 12. *Symmetric Monoidality.* The triple $(\text{Sets}, \times, \text{pt})$ is a symmetric monoidal category.

0017 13. *Symmetric Bimonoidality.* The quintuple $(\text{Sets}, \coprod, \emptyset, \times, \text{pt})$ is a symmetric bimonoidal category.

Proof. Item 1, Functoriality: This follows from ??, ?? of ??.

Item 2, Adjointness: We prove only that there's an adjunction $- \times B \dashv \text{Hom}_{\text{Sets}}(B, -)$, witnessed by a bijection

$$\text{Hom}_{\text{Sets}}(A \times B, C) \cong \text{Hom}_{\text{Sets}}(A, \text{Hom}_{\text{Sets}}(B, C)),$$

natural in $B, C \in \text{Obj}(\text{Sets})$, as the proof of the existence of the adjunction $A \times - \dashv \text{Hom}_{\text{Sets}}(A, -)$ follows almost exactly in the same way.

- *Map I.* We define a map

$$\Phi_{B,C}: \text{Hom}_{\text{Sets}}(A \times B, C) \rightarrow \text{Hom}_{\text{Sets}}(A, \text{Hom}_{\text{Sets}}(B, C)),$$

by sending a function

$$\xi: A \times B \rightarrow C$$

to the function

$$\begin{aligned} \xi^\dagger: A &\rightarrow \text{Hom}_{\text{Sets}}(B, C), \\ a &\mapsto (\xi_a^\dagger: B \rightarrow C), \end{aligned}$$

where we define

$$\xi_a^\dagger(b) \stackrel{\text{def}}{=} \xi(a, b)$$

for each $b \in B$. In terms of the $\llbracket a \mapsto f(a) \rrbracket$ notation of [Sets](#), [Notation 1.1.1.2](#), we have

$$\xi^\dagger \stackrel{\text{def}}{=} \llbracket a \mapsto \llbracket b \mapsto \xi(a, b) \rrbracket \rrbracket.$$

- *Map II.* We define a map

$$\Psi_{B,C}: \text{Hom}_{\text{Sets}}(A, \text{Hom}_{\text{Sets}}(B, C)) \rightarrow \text{Hom}_{\text{Sets}}(A \times B, C)$$

given by sending a function

$$\begin{aligned} \xi: A &\rightarrow \text{Hom}_{\text{Sets}}(B, C), \\ a &\mapsto (\xi_a: B \rightarrow C), \end{aligned}$$

to the function

$$\xi^\dagger: A \times B \rightarrow C$$

defined by

$$\begin{aligned} \xi^\dagger(a, b) &\stackrel{\text{def}}{=} \text{ev}_b(\text{ev}_a(\xi)) \\ &\stackrel{\text{def}}{=} \text{ev}_b(\xi_a) \\ &\stackrel{\text{def}}{=} \xi_a(b) \end{aligned}$$

for each $(a, b) \in A \times B$.

- *Invertibility I.* We claim that

$$\Psi_{A,B} \circ \Phi_{A,B} = \text{id}_{\text{Hom}_{\text{Sets}}(A \times B, C)}.$$

Indeed, given a function $\xi: A \times B \rightarrow C$, we have

$$\begin{aligned} [\Psi_{A,B} \circ \Phi_{A,B}](\xi) &= \Psi_{A,B}(\Phi_{A,B}(\xi)) \\ &= \Psi_{A,B}(\Phi_{A,B}(\llbracket (a, b) \mapsto \xi(a, b) \rrbracket)) \\ &= \Psi_{A,B}(\llbracket a \mapsto \llbracket b \mapsto \xi(a, b) \rrbracket \rrbracket) \\ &= \Psi_{A,B}(\llbracket a' \mapsto \llbracket b' \mapsto \xi(a', b') \rrbracket \rrbracket) \\ &= \llbracket (a, b) \mapsto \text{ev}_b(\text{ev}_a(\llbracket a' \mapsto \llbracket b' \mapsto \xi(a', b') \rrbracket) \rrbracket) \rrbracket \\ &= \llbracket (a, b) \mapsto \text{ev}_b(\llbracket b' \mapsto \xi(a, b') \rrbracket) \rrbracket \\ &= \llbracket (a, b) \mapsto \xi(a, b) \rrbracket \\ &= \xi. \end{aligned}$$

- *Invertibility II.* We claim that

$$\Phi_{A,B} \circ \Psi_{A,B} = \text{id}_{\text{Hom}_{\text{Sets}}(A, \text{Hom}_{\text{Sets}}(B, C))}.$$

Indeed, given a function

$$\begin{aligned} \xi: A &\rightarrow \text{Hom}_{\text{Sets}}(B, C), \\ a &\mapsto (\xi_a: B \rightarrow C), \end{aligned}$$

we have

$$\begin{aligned} [\Phi_{A,B} \circ \Psi_{A,B}](\xi) &\stackrel{\text{def}}{=} \Phi_{A,B}(\Psi_{A,B}(\xi)) \\ &\stackrel{\text{def}}{=} \Phi_{A,B}(\llbracket (a, b) \mapsto \xi_a(b) \rrbracket) \\ &\stackrel{\text{def}}{=} \Phi_{A,B}(\llbracket (a', b') \mapsto \xi_{a'}(b') \rrbracket) \\ &\stackrel{\text{def}}{=} \llbracket a \mapsto \llbracket b \mapsto \text{ev}_{(a,b)}(\llbracket (a', b') \mapsto \xi_{a'}(b') \rrbracket) \rrbracket \rrbracket \\ &\stackrel{\text{def}}{=} \llbracket a \mapsto \llbracket b \mapsto \xi_a(b) \rrbracket \rrbracket \\ &\stackrel{\text{def}}{=} \llbracket a \mapsto \xi_a \rrbracket \\ &\stackrel{\text{def}}{=} \xi. \end{aligned}$$

- *Naturality for Φ , Part I.* We need to show that, given a function $g: B \rightarrow B'$, the diagram

$$\begin{array}{ccc} \text{Hom}_{\text{Sets}}(A \times B', C) & \xrightarrow{\Phi_{B',C}} & \text{Hom}_{\text{Sets}}(A, \text{Hom}_{\text{Sets}}(B', C)), \\ \text{id}_A \times g^* \downarrow & & \downarrow (g^*)_* \\ \text{Hom}_{\text{Sets}}(A \times B, C) & \xrightarrow{\Phi_{B,C}} & \text{Hom}_{\text{Sets}}(A, \text{Hom}_{\text{Sets}}(B, C)) \end{array}$$

commutes. Indeed, given a function

$$\xi: A \times B' \rightarrow C,$$

we have

$$\begin{aligned} [\Phi_{B,C} \circ (\text{id}_A \times g^*)](\xi) &= \Phi_{B,C}([\text{id}_A \times g^*](\xi)) \\ &= \Phi_{B,C}(\xi(-1, g(-2))) \\ &= [\xi(-1, g(-2))]^\dagger \\ &= \xi_{-1}^\dagger(g(-2)) \\ &= (g^*)_* (\xi^\dagger) \\ &= (g^*)_* (\Phi_{B',C}(\xi)) \\ &= [(g^*)_* \circ \Phi_{B',C}](\xi). \end{aligned}$$

Alternatively, using the $\llbracket a \mapsto f(a) \rrbracket$ notation of [Sets, Notation 1.1.1.2](#), we have

$$\begin{aligned} [\Phi_{B,C} \circ (\text{id}_A \times g^*)](\xi) &= \Phi_{B,C}([\text{id}_A \times g^*](\xi)) \\ &= \Phi_{B,C}([\text{id}_A \times g^*](\llbracket (a, b') \mapsto \xi(a, b') \rrbracket)) \\ &= \Phi_{B,C}(\llbracket (a, b) \mapsto \xi(a, g(b)) \rrbracket) \\ &= \llbracket a \mapsto \llbracket b \mapsto \xi(a, g(b)) \rrbracket \rrbracket \\ &= \llbracket a \mapsto g^*(\llbracket b' \mapsto \xi(a, b') \rrbracket) \rrbracket \\ &= (g^*)_*(\llbracket a \mapsto \llbracket b' \mapsto \xi(a, b') \rrbracket \rrbracket) \\ &= (g^*)_*(\Phi_{B',C}(\llbracket (a, b') \mapsto \xi(a, b') \rrbracket)) \\ &= (g^*)_*(\Phi_{B',C}(\xi)) \\ &= [(g^*)_* \circ \Phi_{B',C}](\xi). \end{aligned}$$

- *Naturality for Φ , Part II.* We need to show that, given a function $h: C \rightarrow C'$, the diagram

$$\begin{array}{ccc} \text{Hom}_{\text{Sets}}(A \times B, C) & \xrightarrow{\Phi_{B,C}} & \text{Hom}_{\text{Sets}}(A, \text{Hom}_{\text{Sets}}(B, C)), \\ h_* \downarrow & & \downarrow (h_*)_* \\ \text{Hom}_{\text{Sets}}(A \times B, C') & \xrightarrow{\Phi_{B,C'}} & \text{Hom}_{\text{Sets}}(A, \text{Hom}_{\text{Sets}}(B, C')) \end{array}$$

commutes. Indeed, given a function

$$\xi: A \times B \rightarrow C,$$

we have

$$\begin{aligned}
[\Phi_{B,C} \circ h_*](\xi) &= \Phi_{B,C}(h_*(\xi)) \\
&= \Phi_{B,C}(h_*([[(a, b) \mapsto \xi(a, b)]])) \\
&= \Phi_{B,C}([[(a, b) \mapsto h(\xi(a, b))]]]) \\
&= [a \mapsto [b \mapsto h(\xi(a, b))]] \\
&= [a \mapsto h_*([b \mapsto \xi(a, b)])] \\
&= (h_*)_*([a \mapsto [b \mapsto \xi(a, b)]]) \\
&= (h_*)_*(\Phi_{B,C}([[(a, b) \mapsto \xi(a, b)]])) \\
&= (h_*)_*(\Phi_{B,C}(\xi)) \\
&= [(h_*)_* \circ \Phi_{B,C}](\xi).
\end{aligned}$$

- *Naturality for Ψ .* Since Φ is natural in each argument and Φ is a componentwise inverse to Ψ in each argument, it follows from [Categories, Item 2](#) of [Proposition 8.6.1.2](#) that Ψ is also natural in each argument.

Item 3, Associativity: See [\[Pro24a\]](#).

Item 4, Unitality: Clear.

Item 5, Commutativity: See [\[Pro24b\]](#).

Item 6, Annihilation With the Empty Set: See [\[Pro24f\]](#).

Item 7, Distributivity Over Unions: See [\[Pro24e\]](#).

Item 8, Distributivity Over Intersections: See [\[Pro24g, Corollary 1\]](#).

Item 9, Middle-Four Exchange With Respect to Intersections: See [\[Pro24g, Corollary 1\]](#).

Item 10, Distributivity Over Differences: See [\[Pro24c\]](#).

Item 11, Distributivity Over Symmetric Differences: See [\[Pro24d\]](#).

Item 12, Symmetric Monoidality: See [\[MO 382264\]](#).

Item 13, Symmetric Bimonoidality: Omitted. □

0018 1.4 Pullbacks

Let A , B , and C be sets and let $f: A \rightarrow C$ and $g: B \rightarrow C$ be functions.

0019 Definition 1.4.1.1. The **pullback of A and B over C along f and g** ⁵ is the pair⁶ $(A \times_C B, \{\text{pr}_1, \text{pr}_2\})$ consisting of:

⁵*Further Terminology:* Also called the **fibre product of A and B over C along f and g** .

⁶*Further Notation:* Also written $A \times_{f,C,g} B$.

- *The Limit.* The set $A \times_C B$ defined by

$$A \times_C B \stackrel{\text{def}}{=} \{(a, b) \in A \times B \mid f(a) = g(b)\}.$$

- *The Cone.* The maps

$$\begin{aligned} \text{pr}_1 &: A \times_C B \rightarrow A, \\ \text{pr}_2 &: A \times_C B \rightarrow B \end{aligned}$$

defined by

$$\begin{aligned} \text{pr}_1(a, b) &\stackrel{\text{def}}{=} a, \\ \text{pr}_2(a, b) &\stackrel{\text{def}}{=} b \end{aligned}$$

for each $(a, b) \in A \times_C B$.

Proof. We claim that $A \times_C B$ is the categorical pullback of A and B over C with respect to (f, g) in **Sets**. First we need to check that the relevant pullback diagram commutes, i.e. that we have

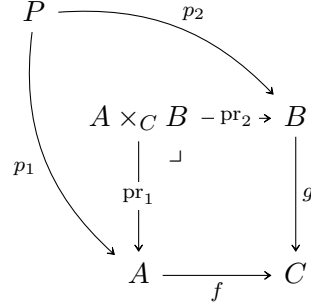
$$f \circ \text{pr}_1 = g \circ \text{pr}_2, \quad \begin{array}{ccc} A \times_C B & \xrightarrow{\text{pr}_2} & B \\ \text{pr}_1 \downarrow & & \downarrow g \\ A & \xrightarrow{f} & C. \end{array}$$

Indeed, given $(a, b) \in A \times_C B$, we have

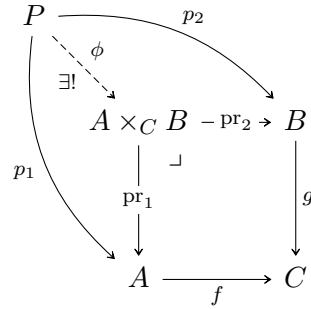
$$\begin{aligned} [f \circ \text{pr}_1](a, b) &= f(\text{pr}_1(a, b)) \\ &= f(a) \\ &= g(b) \\ &= g(\text{pr}_2(a, b)) \\ &= [g \circ \text{pr}_2](a, b), \end{aligned}$$

where $f(a) = g(b)$ since $(a, b) \in A \times_C B$. Next, we prove that $A \times_C B$ satisfies the universal property of the pullback. Suppose we have a diagram

of the form



in **Sets**. Then there exists a unique map $\phi: P \rightarrow A \times_C B$ making the diagram



commute, being uniquely determined by the conditions

$$\begin{aligned} \text{pr}_1 \circ \phi &= p_1, \\ \text{pr}_2 \circ \phi &= p_2 \end{aligned}$$

via

$$\phi(x) = (p_1(x), p_2(x))$$

for each $x \in P$, where we note that $(p_1(x), p_2(x)) \in A \times B$ indeed lies in $A \times_C B$ by the condition

$$f \circ p_1 = g \circ p_2,$$

which gives

$$f(p_1(x)) = g(p_2(x))$$

for each $x \in P$, so that $(p_1(x), p_2(x)) \in A \times_C B$. □

001A Example 1.4.1.2. Here are some examples of pullbacks of sets.

- 001B** 1. *Unions via Intersections.* Let $A, B \subset X$. We have a bijection of sets

$$A \cap B \cong A \times_{A \cup B} B,$$

$$\begin{array}{ccc} A \cap B & \longrightarrow & B \\ \downarrow \lrcorner & & \downarrow \iota_B \\ A & \xrightarrow{\iota_A} & A \cup B \end{array}$$

Proof. **Item 1, Unions via Intersections:** Indeed, we have

$$\begin{aligned} A \times_{A \cup B} B &\cong \{(x, y) \in A \times B \mid x = y\} \\ &\cong A \cap B. \end{aligned}$$

This finishes the proof. \square

- 001C Proposition 1.4.1.3.** Let A, B, C , and X be sets.

- 001D** 1. *Functoriality.* The assignment $(A, B, C, f, g) \mapsto A \times_{f, C, g} B$ defines a functor

$$-_1 \times_{-3} -_1: \mathbf{Fun}(\mathcal{P}, \mathbf{Sets}) \rightarrow \mathbf{Sets},$$

where \mathcal{P} is the category that looks like this:

$$\begin{array}{ccc} & \bullet & \\ & \downarrow & \\ \bullet & \longrightarrow & \bullet \end{array}$$

In particular, the action on morphisms of $-_1 \times_{-3} -_1$ is given by sending a morphism

$$\begin{array}{ccccc} A \times_C B & \longrightarrow & B & & \\ \downarrow \lrcorner & & \downarrow g & \searrow \psi & \\ & A' \times_{C'} B' & \longrightarrow & B' & \\ \downarrow \lrcorner & & \downarrow & & \downarrow g' \\ A & \xrightarrow{f} & C & & \\ \downarrow \phi & & \downarrow \chi & & \\ & A' & \xrightarrow{f'} & C' & \end{array}$$

in $\mathbf{Fun}(\mathcal{P}, \mathbf{Sets})$ to the map $\xi: A \times_C B \xrightarrow{\exists!} A' \times_{C'} B'$ given by

$$\xi(a, b) \stackrel{\text{def}}{=} (\phi(a), \psi(b))$$

for each $(a, b) \in A \times_C B$, which is the unique map making the diagram

$$\begin{array}{ccccc}
 A \times_C B & \longrightarrow & B & & \\
 \downarrow & \searrow \lrcorner & \downarrow g & \searrow \psi & \\
 & A' \times_{C'} B' & \longrightarrow & B' & \\
 & \downarrow & \downarrow & \downarrow g' & \\
 A & \xrightarrow{f} & C & \xrightarrow{\chi} & C' \\
 \downarrow \phi & & \downarrow & & \downarrow \\
 A' & \xrightarrow{f'} & C' & &
 \end{array}$$

commute.

001E 2. *Associativity.* Given a diagram

$$\begin{array}{ccccc}
 A & & B & & C \\
 & \searrow f & & \searrow h & \\
 & X & & Y &
 \end{array}$$

in **Sets**, we have isomorphisms of sets

$$(A \times_X B) \times_Y C \cong (A \times_X B) \times_B (B \times_Y C) \cong A \times_X (B \times_Y C),$$

where these pullbacks are built as in the diagrams

$$\begin{array}{ccc}
 \begin{array}{c} (A \times_X B) \times_Y C \\ \swarrow \quad \searrow \\ A \times_X B \quad C \\ \swarrow \quad \searrow \\ A \quad B \\ \searrow \quad \swarrow \\ X \quad Y \end{array} &
 \begin{array}{c} (A \times_X B) \times_B (B \times_Y C) \\ \swarrow \quad \searrow \\ A \times_X B \quad B \times_Y C \\ \swarrow \quad \searrow \\ A \quad B \\ \searrow \quad \swarrow \\ X \quad Y \end{array} &
 \begin{array}{c} A \times_X (B \times_Y C) \\ \swarrow \quad \searrow \\ A \quad B \times_Y C \\ \swarrow \quad \searrow \\ A \quad B \\ \searrow \quad \swarrow \\ X \quad Y \end{array}
 \end{array}$$

001F 3. *Unitality.* We have isomorphisms of sets

$$\begin{array}{ccc}
 \begin{array}{c} A \xlongequal{\quad} A \\ \downarrow f \quad \lrcorner \quad \downarrow f \\ X \xlongequal{\quad} X \end{array} &
 \begin{array}{c} X \times_X A \cong A, \\ A \times_X X \cong A, \end{array} &
 \begin{array}{c} A \xrightarrow{f} X \\ \parallel \quad \lrcorner \quad \parallel \\ X \xrightarrow{f} X. \end{array}
 \end{array}$$

001G 4. *Commutativity.* We have an isomorphism of sets

$$\begin{array}{ccc}
 A \times_X B & \longrightarrow & B \\
 \downarrow \lrcorner & & \downarrow g \\
 A & \xrightarrow{f} & X,
 \end{array}
 \quad A \times_X B \cong B \times_X A
 \quad
 \begin{array}{ccc}
 B \times_X A & \longrightarrow & A \\
 \downarrow \lrcorner & & \downarrow f \\
 B & \xrightarrow{g} & X.
 \end{array}$$

001H 5. *Annihilation With the Empty Set.* We have isomorphisms of sets

$$\begin{array}{ccc}
 \emptyset & \longrightarrow & \emptyset \\
 \downarrow \lrcorner & & \downarrow \\
 A & \xrightarrow{f} & X,
 \end{array}
 \quad
 \begin{array}{l}
 A \times_X \emptyset \cong \emptyset, \\
 \emptyset \times_X A \cong \emptyset,
 \end{array}
 \quad
 \begin{array}{ccc}
 \emptyset & \longrightarrow & A \\
 \downarrow \lrcorner & & \downarrow f \\
 \emptyset & \longrightarrow & X.
 \end{array}$$

001J 6. *Interaction With Products.* We have an isomorphism of sets

$$\begin{array}{ccc}
 A \times B & \longrightarrow & B \\
 \downarrow \lrcorner & & \downarrow !_B \\
 A & \xrightarrow{!_A} & \text{pt.}
 \end{array}
 \quad A \times_{\text{pt}} B \cong A \times B,$$

001K 7. *Symmetric Monoidality.* The triple $(\mathbf{Sets}, \times_X, X)$ is a symmetric monoidal category.

Proof. **Item 1, Functoriality:** This is a special case of functoriality of co/limits, ??, ?? of ??, with the explicit expression for ξ following from the commutativity of the cube pullback diagram.

Item 2, Associativity: Indeed, we have

$$\begin{aligned}
 (A \times_X B) \times_Y C &\cong \{(a, b), c) \in (A \times_X B) \times C \mid h(b) = k(c)\} \\
 &\cong \{(a, b), c) \in (A \times B) \times C \mid f(a) = g(b) \text{ and } h(b) = k(c)\} \\
 &\cong \{(a, (b, c)) \in A \times (B \times_Y C) \mid f(a) = g(b) \text{ and } h(b) = k(c)\} \\
 &\cong \{(a, (b, c)) \in A \times (B \times_Y C) \mid f(a) = g(b)\} \\
 &\cong A \times_X (B \times_Y C)
 \end{aligned}$$

and

$$\begin{aligned}
(A \times_X B) \times_B (B \times_Y C) &\cong \{((a, b), (b', c)) \in (A \times_X B) \times (B \times_Y C) \mid b = b'\} \\
&\cong \left\{ ((a, b), (b', c)) \in (A \times B) \times (B \times C) \left| \begin{array}{l} f(a) = g(b), b = b', \\ \text{and } h(b') = k(c) \end{array} \right. \right\} \\
&\cong \left\{ (a, (b, (b', c))) \in A \times (B \times (B \times C)) \left| \begin{array}{l} f(a) = g(b), b = b', \\ \text{and } h(b') = k(c) \end{array} \right. \right\} \\
&\cong \left\{ (a, ((b, b'), c)) \in A \times ((B \times B) \times C) \left| \begin{array}{l} f(a) = g(b), b = b', \\ \text{and } h(b') = k(c) \end{array} \right. \right\} \\
&\cong \left\{ (a, ((b, b'), c)) \in A \times ((B \times_B B) \times C) \left| \begin{array}{l} f(a) = g(b) \text{ and } \\ h(b') = k(c) \end{array} \right. \right\} \\
&\cong \{(a, (b, c)) \in A \times (B \times C) \mid f(a) = g(b) \text{ and } h(b) = k(c)\} \\
&\cong A \times_X (B \times_Y C),
\end{aligned}$$

where we have used [Item 3](#) for the isomorphism $B \times_B B \cong B$.

[Item 3, Unitality](#): Indeed, we have

$$\begin{aligned}
X \times_X A &\cong \{(x, a) \in X \times A \mid f(a) = x\}, \\
A \times_X X &\cong \{(a, x) \in X \times A \mid f(a) = x\},
\end{aligned}$$

which are isomorphic to A via the maps $(x, a) \mapsto a$ and $(a, x) \mapsto a$.

[Item 4, Commutativity](#): Clear.

[Item 5, Annihilation With the Empty Set](#): Clear.

[Item 6, Interaction With Products](#): Clear.

[Item 7, Symmetric Monoidality](#): Omitted. □

001L 1.5 Equalisers

Let A and B be sets and let $f, g: A \rightrightarrows B$ be functions.

001M Definition 1.5.1.1. The **equaliser of f and g** is the pair $(\text{Eq}(f, g), \text{eq}(f, g))$ consisting of:

- *The Limit.* The set $\text{Eq}(f, g)$ defined by

$$\text{Eq}(f, g) \stackrel{\text{def}}{=} \{a \in A \mid f(a) = g(a)\}.$$

- *The Cone.* The inclusion map

$$\text{eq}(f, g): \text{Eq}(f, g) \hookrightarrow A.$$

Proof. We claim that $\text{Eq}(f, g)$ is the categorical equaliser of f and g in **Sets**. First we need to check that the relevant equaliser diagram commutes, i.e. that we have

$$f \circ \text{eq}(f, g) = g \circ \text{eq}(f, g),$$

which indeed holds by the definition of the set $\text{Eq}(f, g)$. Next, we prove that $\text{Eq}(f, g)$ satisfies the universal property of the equaliser. Suppose we have a diagram of the form

$$\begin{array}{ccccc} \text{Eq}(f, g) & \xrightarrow{\text{eq}(f, g)} & A & \xrightleftharpoons[f]{f} & B \\ & \nearrow e & & & \\ E & & & & \end{array}$$

in **Sets**. Then there exists a unique map $\phi: E \rightarrow \text{Eq}(f, g)$ making the diagram

$$\begin{array}{ccccc} \text{Eq}(f, g) & \xrightarrow{\text{eq}(f, g)} & A & \xrightleftharpoons[f]{f} & B \\ \uparrow \phi \exists! & \nearrow e & & & \\ E & & & & \end{array}$$

commute, being uniquely determined by the condition

$$\text{eq}(f, g) \circ \phi = e$$

via

$$\phi(x) = e(x)$$

for each $x \in E$, where we note that $e(x) \in A$ indeed lies in $\text{Eq}(f, g)$ by the condition

$$f \circ e = g \circ e,$$

which gives

$$f(e(x)) = g(e(x))$$

for each $x \in E$, so that $e(x) \in \text{Eq}(f, g)$. □

001N Proposition 1.5.1.2. Let A , B , and C be sets.

001P 1. *Associativity.* We have isomorphisms of sets⁷

$$\underbrace{\text{Eq}(f \circ \text{eq}(g, h), g \circ \text{eq}(g, h))}_{=\text{Eq}(f \circ \text{eq}(g, h), h \circ \text{eq}(g, h))} \cong \text{Eq}(f, g, h) \cong \underbrace{\text{Eq}(f \circ \text{eq}(f, g), h \circ \text{eq}(f, g))}_{=\text{Eq}(g \circ \text{eq}(f, g), h \circ \text{eq}(f, g))},$$

where $\text{Eq}(f, g, h)$ is the limit of the diagram

$$A \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{-g} \\ \xrightarrow{h} \end{array} B$$

in **Sets**, being explicitly given by

$$\text{Eq}(f, g, h) \cong \{a \in A \mid f(a) = g(a) = h(a)\}.$$

⁷That is, the following three ways of forming “the” equaliser of (f, g, h) agree:

1. Take the equaliser of (f, g, h) , i.e. the limit of the diagram

$$A \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{-g} \\ \xrightarrow{h} \end{array} B$$

in **Sets**.

2. First take the equaliser of f and g , forming a diagram

$$\text{Eq}(f, g) \xrightarrow{\text{eq}(f, g)} A \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} B$$

and then take the equaliser of the composition

$$\text{Eq}(f, g) \xrightarrow{\text{eq}(f, g)} A \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{h} \end{array} B,$$

obtaining a subset

$$\text{Eq}(f \circ \text{eq}(f, g), h \circ \text{eq}(f, g)) = \text{Eq}(g \circ \text{eq}(f, g), h \circ \text{eq}(f, g))$$

of $\text{Eq}(f, g)$.

3. First take the equaliser of g and h , forming a diagram

$$\text{Eq}(g, h) \xrightarrow{\text{eq}(g, h)} A \begin{array}{c} \xrightarrow{g} \\ \xrightarrow{h} \end{array} B$$

and then take the equaliser of the composition

$$\text{Eq}(g, h) \xrightarrow{\text{eq}(g, h)} A \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} B,$$

obtaining a subset

$$\text{Eq}(f \circ \text{eq}(g, h), g \circ \text{eq}(g, h)) = \text{Eq}(f \circ \text{eq}(g, h), h \circ \text{eq}(g, h))$$

of $\text{Eq}(g, h)$.

001Q 4. *Unitality.* We have an isomorphism of sets

$$\text{Eq}(f, f) \cong A.$$

001R 5. *Commutativity.* We have an isomorphism of sets

$$\text{Eq}(f, g) \cong \text{Eq}(g, f).$$

001S 6. *Interaction With Composition.* Let

$$A \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} B \begin{array}{c} \xrightarrow{h} \\ \xrightarrow{k} \end{array} C$$

be functions. We have an inclusion of sets

$$\text{Eq}(h \circ f \circ \text{eq}(f, g), k \circ g \circ \text{eq}(f, g)) \subset \text{Eq}(h \circ f, k \circ g),$$

where $\text{Eq}(h \circ f \circ \text{eq}(f, g), k \circ g \circ \text{eq}(f, g))$ is the equaliser of the composition

$$\text{Eq}(f, g) \xrightarrow{\text{eq}(f, g)} A \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} B \begin{array}{c} \xrightarrow{h} \\ \xrightarrow{k} \end{array} C.$$

Proof. Item 1, Associativity: We first prove that $\text{Eq}(f, g, h)$ is indeed given by

$$\text{Eq}(f, g, h) \cong \{a \in A \mid f(a) = g(a) = h(a)\}.$$

Indeed, suppose we have a diagram of the form

$$\begin{array}{ccc} \text{Eq}(f, g, h) & \xrightarrow{\text{eq}(f, g, h)} & A \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \\ \xrightarrow{h} \end{array} B \\ & \nearrow e & \\ E & & \end{array}$$

in **Sets**. Then there exists a unique map $\phi: E \rightarrow \text{Eq}(f, g, h)$, uniquely determined by the condition

$$\text{eq}(f, g) \circ \phi = e$$

being necessarily given by

$$\phi(x) = e(x)$$

for each $x \in E$, where we note that $e(x) \in A$ indeed lies in $\text{Eq}(f, g, h)$ by the condition

$$f \circ e = g \circ e = h \circ e,$$

which gives

$$f(e(x)) = g(e(x)) = h(e(x))$$

for each $x \in E$, so that $e(x) \in \text{Eq}(f, g, h)$.

We now check the equalities

$$\text{Eq}(f \circ \text{eq}(g, h), g \circ \text{eq}(g, h)) \cong \text{Eq}(f, g, h) \cong \text{Eq}(f \circ \text{eq}(f, g), h \circ \text{eq}(f, g)).$$

Indeed, we have

$$\begin{aligned} \text{Eq}(f \circ \text{eq}(g, h), g \circ \text{eq}(g, h)) &\cong \{x \in \text{Eq}(g, h) \mid [f \circ \text{eq}(g, h)](a) = [g \circ \text{eq}(g, h)](a)\} \\ &\cong \{x \in \text{Eq}(g, h) \mid f(a) = g(a)\} \\ &\cong \{x \in A \mid f(a) = g(a) \text{ and } g(a) = h(a)\} \\ &\cong \{x \in A \mid f(a) = g(a) = h(a)\} \\ &\cong \text{Eq}(f, g, h). \end{aligned}$$

Similarly, we have

$$\begin{aligned} \text{Eq}(f \circ \text{eq}(f, g), h \circ \text{eq}(f, g)) &\cong \{x \in \text{Eq}(f, g) \mid [f \circ \text{eq}(f, g)](a) = [h \circ \text{eq}(f, g)](a)\} \\ &\cong \{x \in \text{Eq}(f, g) \mid f(a) = h(a)\} \\ &\cong \{x \in A \mid f(a) = h(a) \text{ and } f(a) = g(a)\} \\ &\cong \{x \in A \mid f(a) = g(a) = h(a)\} \\ &\cong \text{Eq}(f, g, h). \end{aligned}$$

Item 4, Unitality: Clear.

Item 5, Commutativity: Clear.

Item 6, Interaction With Composition: Indeed, we have

$$\begin{aligned} \text{Eq}(h \circ f \circ \text{eq}(f, g), k \circ g \circ \text{eq}(f, g)) &\cong \{a \in \text{Eq}(f, g) \mid h(f(a)) = k(g(a))\} \\ &\cong \{a \in A \mid f(a) = g(a) \text{ and } h(f(a)) = k(g(a))\}. \end{aligned}$$

and

$$\text{Eq}(h \circ f, k \circ g) \cong \{a \in A \mid h(f(a)) = k(g(a))\},$$

and thus there's an inclusion from $\text{Eq}(h \circ f \circ \text{eq}(f, g), k \circ g \circ \text{eq}(f, g))$ to $\text{Eq}(h \circ f, k \circ g)$. \square

001T 2 Colimits of Sets

001U 2.1 The Initial Set

001V Definition 2.1.1.1. The **initial set** is the pair $(\emptyset, \{\iota_A\}_{A \in \text{Obj}(\text{Sets})})$ consisting of:

- *The Limit.* The empty set \emptyset of **Definition 3.1.1.1**.
- *The Cone.* The collection of maps

$$\{\iota_A: \emptyset \rightarrow A\}_{A \in \text{Obj}(\mathbf{Sets})}$$

given by the inclusion maps from \emptyset to A .

Proof. We claim that \emptyset is the initial object of **Sets**. Indeed, suppose we have a diagram of the form

$$\emptyset \quad A$$

in **Sets**. Then there exists a unique map $\phi: \emptyset \rightarrow A$ making the diagram

$$\emptyset \xrightarrow[\exists!]{\phi} A$$

commute, namely the inclusion map ι_A . □

001W 2.2 Coproducts of Families of Sets

Let $\{A_i\}_{i \in I}$ be a family of sets.

001X Definition 2.2.1.1. The **disjoint union of the family** $\{A_i\}_{i \in I}$ is the pair $(\coprod_{i \in I} A_i, \{\text{inj}_i\}_{i \in I})$ consisting of:

- *The Colimit.* The set $\coprod_{i \in I} A_i$ defined by

$$\coprod_{i \in I} A_i \stackrel{\text{def}}{=} \left\{ (i, x) \in I \times \left(\bigcup_{i \in I} A_i \right) \mid x \in A_i \right\}.$$

- *The Cocone.* The collection

$$\left\{ \text{inj}_i: A_i \rightarrow \coprod_{i \in I} A_i \right\}_{i \in I}$$

of maps given by

$$\text{inj}_i(x) \stackrel{\text{def}}{=} (i, x)$$

for each $x \in A_i$ and each $i \in I$.

Proof. We claim that $\coprod_{i \in I} A_i$ is the categorical coproduct of $\{A_i\}_{i \in I}$ in **Sets**.

Indeed, suppose we have, for each $i \in I$, a diagram of the form

$$\begin{array}{ccc} & & C \\ & \nearrow \iota_i & \\ A_i & \xrightarrow{\text{inj}_i} & \coprod_{i \in I} A_i \end{array}$$

in **Sets**. Then there exists a unique map $\phi: \coprod_{i \in I} A_i \rightarrow C$ making the diagram

$$\begin{array}{ccc} & & C \\ & \nearrow \iota_i & \uparrow \phi \exists! \\ A_i & \xrightarrow{\text{inj}_i} & \coprod_{i \in I} A_i \end{array}$$

commute, being uniquely determined by the condition $\phi \circ \text{inj}_i = \iota_i$ for each $i \in I$ via

$$\phi((i, x)) = \iota_i(x)$$

for each $(i, x) \in \coprod_{i \in I} A_i$. □

001Y Proposition 2.2.1.2. Let $\{A_i\}_{i \in I}$ be a family of sets.

001Z 1. *Functoriality.* The assignment $\{A_i\}_{i \in I} \mapsto \coprod_{i \in I} A_i$ defines a functor

$$\coprod_{i \in I}: \text{Fun}(I_{\text{disc}}, \mathbf{Sets}) \rightarrow \mathbf{Sets}$$

where

- *Action on Objects.* For each $(A_i)_{i \in I} \in \text{Obj}(\text{Fun}(I_{\text{disc}}, \mathbf{Sets}))$, we have

$$\left[\coprod_{i \in I} \right] ((A_i)_{i \in I}) \stackrel{\text{def}}{=} \coprod_{i \in I} A_i$$

- *Action on Morphisms.* For each $(A_i)_{i \in I}, (B_i)_{i \in I} \in \text{Obj}(\text{Fun}(I_{\text{disc}}, \mathbf{Sets}))$, the action on Hom-sets

$$\left(\coprod_{i \in I} \right)_{(A_i)_{i \in I}, (B_i)_{i \in I}} : \text{Nat}((A_i)_{i \in I}, (B_i)_{i \in I}) \rightarrow \mathbf{Sets} \left(\coprod_{i \in I} A_i, \coprod_{i \in I} B_i \right)$$

of $\coprod_{i \in I}$ at $((A_i)_{i \in I}, (B_i)_{i \in I})$ is defined by sending a map

$$\{f_i: A_i \rightarrow B_i\}_{i \in I}$$

in $\text{Nat}((A_i)_{i \in I}, (B_i)_{i \in I})$ to the map of sets

$$\coprod_{i \in I} f_i: \coprod_{i \in I} A_i \rightarrow \coprod_{i \in I} B_i$$

defined by

$$\left[\coprod_{i \in I} f_i \right] (i, a) \stackrel{\text{def}}{=} f_i(a)$$

for each $(i, a) \in \coprod_{i \in I} A_i$.

Proof. Item 1, Functoriality: This follows from ??, ?? of ??. □

0020 2.3 Binary Coproducts

Let A and B be sets.

0021 Definition 2.3.1.1. The **coproduct**⁸ of A and B is the pair $(A \coprod B, \{\text{inj}_1, \text{inj}_2\})$ consisting of:

- *The Colimit.* The set $A \coprod B$ defined by

$$\begin{aligned} A \coprod B &\stackrel{\text{def}}{=} \coprod_{z \in \{A, B\}} z \\ &\cong \{(0, a) \mid a \in A\} \cup \{(1, b) \mid b \in B\}. \end{aligned}$$

- *The Cocone.* The maps

$$\begin{aligned} \text{inj}_1: A &\rightarrow A \coprod B, \\ \text{inj}_2: B &\rightarrow A \coprod B, \end{aligned}$$

given by

$$\begin{aligned} \text{inj}_1(a) &\stackrel{\text{def}}{=} (0, a), \\ \text{inj}_2(b) &\stackrel{\text{def}}{=} (1, b), \end{aligned}$$

for each $a \in A$ and each $b \in B$.

⁸*Further Terminology:* Also called the **disjoint union** of A and B , or the **binary**

Proof. We claim that $A \amalg B$ is the categorical coproduct of A and B in **Sets**. Indeed, suppose we have a diagram of the form

$$\begin{array}{ccccc} & & C & & \\ & \nearrow \iota_A & & \nwarrow \iota_B & \\ A & \xrightarrow{\text{inj}_A} & A \amalg B & \xleftarrow{\text{inj}_B} & B \end{array}$$

in **Sets**. Then there exists a unique map $\phi: A \amalg B \rightarrow C$ making the diagram

$$\begin{array}{ccccc} & & C & & \\ & \nearrow \iota_A & \uparrow \phi \exists! & \nwarrow \iota_B & \\ A & \xrightarrow{\text{inj}_A} & A \amalg B & \xleftarrow{\text{inj}_B} & B \end{array}$$

commute, being uniquely determined by the conditions

$$\begin{aligned} \phi \circ \text{inj}_A &= \iota_A, \\ \phi \circ \text{inj}_B &= \iota_B \end{aligned}$$

via

$$\phi(x) = \begin{cases} \iota_A(a) & \text{if } x = (0, a), \\ \iota_B(b) & \text{if } x = (1, b) \end{cases}$$

for each $x \in A \amalg B$. □

0022 Proposition 2.3.1.2. Let A , B , C , and X be sets.

0023 1. *Functoriality.* The assignment $A, B, (A, B) \mapsto A \amalg B$ defines functors

$$\begin{aligned} A \amalg - &: \mathbf{Sets} \rightarrow \mathbf{Sets}, \\ - \amalg B &: \mathbf{Sets} \rightarrow \mathbf{Sets}, \\ -_1 \amalg -_2 &: \mathbf{Sets} \times \mathbf{Sets} \rightarrow \mathbf{Sets}, \end{aligned}$$

where $-_1 \amalg -_2$ is the functor where

- *Action on Objects.* For each $(A, B) \in \text{Obj}(\mathbf{Sets} \times \mathbf{Sets})$, we have

$$[-_1 \amalg -_2](A, B) \stackrel{\text{def}}{=} A \amalg B.$$

disjoint union of A and B , for emphasis.

- *Action on Morphisms.* For each $(A, B), (X, Y) \in \text{Obj}(\text{Sets})$, the action on Hom-sets

$$\amalg_{(A,B),(X,Y)} : \text{Sets}(A, X) \times \text{Sets}(B, Y) \rightarrow \text{Sets}(A \amalg B, X \amalg Y)$$

of \amalg at $((A, B), (X, Y))$ is defined by sending (f, g) to the function

$$f \amalg g : A \amalg B \rightarrow X \amalg Y$$

defined by

$$[f \amalg g](x) \stackrel{\text{def}}{=} \begin{cases} (0, f(a)) & \text{if } x = (0, a), \\ (1, g(b)) & \text{if } x = (1, b), \end{cases}$$

for each $x \in A \amalg B$.

and where $A \amalg -$ and $- \amalg B$ are the partial functors of $-_1 \amalg -_2$ at $A, B \in \text{Obj}(\text{Sets})$.

- 0024** 2. *Associativity.* We have an isomorphism of sets

$$(A \amalg B) \amalg C \cong A \amalg (B \amalg C),$$

natural in $A, B, C \in \text{Obj}(\text{Sets})$.

- 0025** 3. *Unitality.* We have isomorphisms of sets

$$\begin{aligned} A \amalg \emptyset &\cong A, \\ \emptyset \amalg A &\cong A, \end{aligned}$$

natural in $A \in \text{Obj}(\text{Sets})$.

- 0026** 4. *Commutativity.* We have an isomorphism of sets

$$A \amalg B \cong B \amalg A,$$

natural in $A, B \in \text{Obj}(\text{Sets})$.

- 0027** 5. *Symmetric Monoidality.* The triple $(\text{Sets}, \amalg, \emptyset)$ is a symmetric monoidal category.

Proof. Item 1, Functoriality: This follows from ??, ?? of ??.

Item 2, Associativity: Clear.

Item 3, Unitality: Clear.

Item 4, Commutativity: Clear.

Item 5, Symmetric Monoidality: Omitted. □

0028 2.4 Pushouts

Let A , B , and C be sets and let $f: C \rightarrow A$ and $g: C \rightarrow B$ be functions.

0029 **Definition 2.4.1.1.** The **pushout of A and B over C along f and g** ⁹ is the pair¹⁰ $(A \amalg_C B, \{\text{inj}_1, \text{inj}_2\})$ consisting of:

- *The Colimit.* The set $A \amalg_C B$ defined by

$$A \amalg_C B \stackrel{\text{def}}{=} A \amalg B / \sim_C,$$

where \sim_C is the equivalence relation on $A \amalg B$ generated by $(0, f(c)) \sim_C (1, g(c))$.

- *The Cocone.* The maps

$$\text{inj}_1: A \rightarrow A \amalg_C B,$$

$$\text{inj}_2: B \rightarrow A \amalg_C B$$

given by

$$\text{inj}_1(a) \stackrel{\text{def}}{=} [(0, a)]$$

$$\text{inj}_2(b) \stackrel{\text{def}}{=} [(1, b)]$$

for each $a \in A$ and each $b \in B$.

Proof. We claim that $A \amalg_C B$ is the categorical pushout of A and B over C with respect to (f, g) in **Sets**. First we need to check that the relevant pushout diagram commutes, i.e. that we have

$$\begin{array}{ccc} & A \amalg_C B & \xleftarrow{\text{inj}_2} B \\ \text{inj}_1 \uparrow & & \uparrow g \\ A & \xleftarrow{f} & C. \end{array}$$

$\text{inj}_1 \circ f = \text{inj}_2 \circ g,$

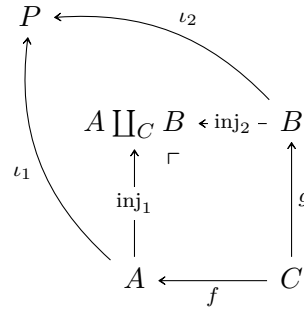
Indeed, given $c \in C$, we have

$$\begin{aligned} [\text{inj}_1 \circ f](c) &= \text{inj}_1(f(c)) \\ &= [(0, f(c))] \\ &= [(1, g(c))] \\ &= \text{inj}_2(g(c)) \\ &= [\text{inj}_2 \circ g](c), \end{aligned}$$

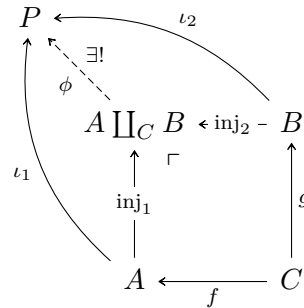
⁹ *Further Terminology:* Also called the **fibre coproduct of A and B over C along f and g** .

¹⁰ *Further Notation:* Also written $A \amalg_{f, C, g} B$.

where $[(0, f(c))] = [(1, g(c))]$ by the definition of the relation \sim on $A \amalg B$. Next, we prove that $A \amalg_C B$ satisfies the universal property of the pushout. Suppose we have a diagram of the form



in **Sets**. Then there exists a unique map $\phi: A \amalg_C B \rightarrow P$ making the diagram



commute, being uniquely determined by the conditions

$$\begin{aligned}\phi \circ \text{inj}_1 &= \iota_1, \\ \phi \circ \text{inj}_2 &= \iota_2\end{aligned}$$

via

$$\phi(x) = \begin{cases} \iota_1(a) & \text{if } x = [(0, a)], \\ \iota_2(b) & \text{if } x = [(1, b)] \end{cases}$$

for each $x \in A \amalg_C B$, where the well-definedness of ϕ is guaranteed by the equality $\iota_1 \circ f = \iota_2 \circ g$ and the definition of the relation \sim on $A \amalg B$ as follows:

1. *Case 1:* Suppose we have $x = [(0, a)] = [(0, a')]$ for some $a, a' \in A$. Then, by [Remark 2.4.1.2](#), we have a sequence

$$(0, a) \sim' x_1 \sim' \dots \sim' x_n \sim' (0, a').$$

2. *Case 2:* Suppose we have $x = [(1, b)] = [(1, b')]$ for some $b, b' \in B$. Then, by [Remark 2.4.1.2](#), we have a sequence

$$(1, b) \sim' x_1 \sim' \cdots \sim' x_n \sim' (1, b').$$

3. *Case 3:* Suppose we have $x = [(0, a)] = [(1, b)]$ for some $a \in A$ and $b \in B$. Then, by [Remark 2.4.1.2](#), we have a sequence

$$(0, a) \sim' x_1 \sim' \cdots \sim' x_n \sim' (1, b).$$

In all these cases, we declare $x \sim' y$ iff there exists some $c \in C$ such that $x = (0, f(c))$ and $y = (1, g(c))$ or $x = (1, g(c))$ and $y = (0, f(c))$. Then, the equality $\iota_1 \circ f = \iota_2 \circ g$ gives

$$\begin{aligned} \phi([x]) &= \phi([(0, f(c))]) \\ &\stackrel{\text{def}}{=} \iota_1(f(c)) \\ &= \iota_2(g(c)) \\ &\stackrel{\text{def}}{=} \phi([(1, g(c))]) \\ &= \phi([y]), \end{aligned}$$

with the case where $x = (1, g(c))$ and $y = (0, f(c))$ similarly giving $\phi([x]) = \phi([y])$. Thus, if $x \sim' y$, then $\phi([x]) = \phi([y])$. Applying this equality pairwise to the sequences

$$\begin{aligned} (0, a) \sim' x_1 \sim' \cdots \sim' x_n \sim' (0, a'), \\ (1, b) \sim' x_1 \sim' \cdots \sim' x_n \sim' (1, b'), \\ (0, a) \sim' x_1 \sim' \cdots \sim' x_n \sim' (1, b) \end{aligned}$$

gives

$$\begin{aligned} \phi([(0, a)]) &= \phi([(0, a')]), \\ \phi([(1, b)]) &= \phi([(1, b')]), \\ \phi([(0, a)]) &= \phi([(1, b)]), \end{aligned}$$

showing ϕ to be well-defined. □

002A Remark 2.4.1.2. In detail, by [Equivalence Relations and Apartness Relations](#), [Construction 4.2.1.2](#), the relation \sim of [Definition 2.4.1.1](#) is given by declaring $a \sim b$ iff one of the following conditions is satisfied:

- We have $a, b \in A$ and $a = b$;

- We have $a, b \in B$ and $a = b$;
- There exist $x_1, \dots, x_n \in A \amalg B$ such that $a \sim' x_1 \sim' \dots \sim' x_n \sim' b$, where we declare $x \sim' y$ if one of the following conditions is satisfied:
 1. There exists $c \in C$ such that $x = (0, f(c))$ and $y = (1, g(c))$.
 2. There exists $c \in C$ such that $x = (1, g(c))$ and $y = (0, f(c))$.

That is: we require the following condition to be satisfied:

- (\star) There exist $x_1, \dots, x_n \in A \amalg B$ satisfying the following conditions:
1. There exists $c_0 \in C$ satisfying one of the following conditions:
 - (a) We have $a = f(c_0)$ and $x_1 = g(c_0)$.
 - (b) We have $a = g(c_0)$ and $x_1 = f(c_0)$.
 2. For each $1 \leq i \leq n-1$, there exists $c_i \in C$ satisfying one of the following conditions:
 - (a) We have $x_i = f(c_i)$ and $x_{i+1} = g(c_i)$.
 - (b) We have $x_i = g(c_i)$ and $x_{i+1} = f(c_i)$.
 3. There exists $c_n \in C$ satisfying one of the following conditions:
 - (a) We have $x_n = f(c_n)$ and $b = g(c_n)$.
 - (b) We have $x_n = g(c_n)$ and $b = f(c_n)$.

002B Example 2.4.1.3. Here are some examples of pushouts of sets.

002C 1. *Wedge Sums of Pointed Sets.* The wedge sum of two pointed sets of **Pointed Sets**, [Definition 3.3.1.1](#) is an example of a pushout of sets.

002D 2. *Intersections via Unions.* Let $A, B \subset X$. We have a bijection of sets

$$\begin{array}{ccc}
 A \cup B & \longleftarrow & B \\
 \uparrow \lrcorner & & \uparrow \\
 A \cup B \cong A \amalg_{A \cap B} B & & A \\
 & \longleftarrow & A \cap B
 \end{array}$$

Proof. **Item 1**, *Wedge Sums of Pointed Sets*: Follows by definition.

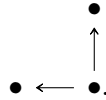
Item 2, *Intersections via Unions*: Indeed, $A \amalg_{A \cap B} B$ is the quotient of $A \amalg B$ by the equivalence relation obtained by declaring $(0, a) \sim (1, b)$ iff $a = b \in A \cap B$, which is in bijection with $A \cup B$ via the map with $[(0, a)] \mapsto a$ and $[(1, b)] \mapsto b$. \square

002E Proposition 2.4.1.4. Let A , B , C , and X be sets.

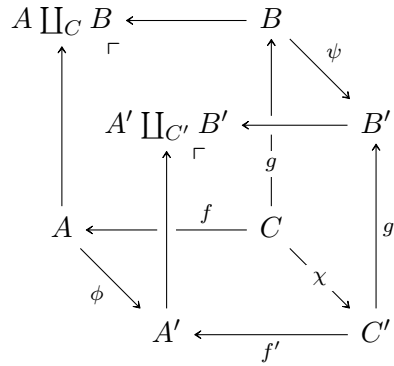
002F 1. *Functoriality.* The assignment $(A, B, C, f, g) \mapsto A \amalg_{f,C,g} B$ defines a functor

$$-_1 \amalg_{-3} -_1 : \text{Fun}(\mathcal{P}, \text{Sets}) \rightarrow \text{Sets},$$

where \mathcal{P} is the category that looks like this:



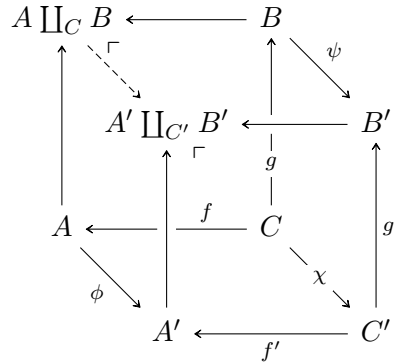
In particular, the action on morphisms of $-_1 \amalg_{-3} -_1$ is given by sending a morphism



in $\text{Fun}(\mathcal{P}, \text{Sets})$ to the map $\xi : A \amalg_C B \xrightarrow{\exists!} A' \amalg_{C'} B'$ given by

$$\xi(x) \stackrel{\text{def}}{=} \begin{cases} \phi(a) & \text{if } x = [(0, a)], \\ \psi(b) & \text{if } x = [(1, b)] \end{cases}$$

for each $x \in A \amalg_C B$, which is the unique map making the diagram



commute.

002G 2. *Associativity.* Given a diagram

$$\begin{array}{ccccc} & A & & B & & C \\ & \swarrow f & & \nearrow g & \swarrow h & \nearrow k \\ & X & & Y & & \end{array}$$

in **Sets**, we have isomorphisms of sets

$$(A \amalg_X B) \amalg_Y C \cong (A \amalg_X B) \amalg_B (B \amalg_Y C) \cong A \amalg_X (B \amalg_Y C),$$

where these pullbacks are built as in the diagrams

$$\begin{array}{ccc} \begin{array}{c} (A \amalg_X B) \amalg_Y C \\ \swarrow \quad \searrow \\ A \amalg_X B \quad B \amalg_Y C \\ \swarrow \quad \searrow \\ A \quad B \\ \swarrow f \quad \searrow g \\ X \end{array} & \begin{array}{c} (A \amalg_X B) \amalg_B (B \amalg_Y C) \\ \swarrow \quad \searrow \\ A \amalg_X B \quad B \amalg_Y C \\ \swarrow \quad \searrow \\ A \quad B \\ \swarrow f \quad \searrow g \\ X \end{array} & \begin{array}{c} A \amalg_X (B \amalg_Y C) \\ \swarrow \quad \searrow \\ A \quad B \amalg_Y C \\ \swarrow f \quad \searrow g \\ X \end{array} \end{array}$$

002H 3. *Unitality.* We have isomorphisms of sets

$$\begin{array}{ccc} \begin{array}{ccc} A & \xlongequal{\quad} & A \\ \uparrow f & \lrcorner & \uparrow f \\ X & \xlongequal{\quad} & X \end{array} & \begin{array}{c} X \amalg_X A \cong A, \\ A \amalg_X X \cong A, \end{array} & \begin{array}{ccc} A & \xleftarrow{f} & X \\ \parallel \lrcorner \parallel & & \parallel \\ X & \xleftarrow{f} & X. \end{array} \end{array}$$

002J 4. *Commutativity.* We have an isomorphism of sets

$$\begin{array}{ccc} \begin{array}{ccc} A \amalg_X B & \longleftarrow & B \\ \uparrow \lrcorner \uparrow g \\ A & \xleftarrow{f} & X, \end{array} & A \amalg_X B \cong B \amalg_X A & \begin{array}{ccc} B \amalg_X A & \longleftarrow & A \\ \uparrow \lrcorner \uparrow f \\ B & \xleftarrow{g} & X. \end{array} \end{array}$$

002K 5. *Interaction With Coproducts.* We have

$$\begin{array}{ccc} A \amalg_{\emptyset} B \cong A \amalg B, & \begin{array}{ccc} A \amalg B & \longleftarrow & B \\ \uparrow \lrcorner \uparrow \iota_B \\ A & \xleftarrow{\iota_A} & \emptyset. \end{array} \end{array}$$

- 002L 6. *Symmetric Monoidality.* The triple $(\mathbf{Sets}, \coprod_X, X)$ is a symmetric monoidal category.

Proof. *Item 1, Functoriality:* This is a special case of functoriality of co/limits, ??, ?? of ??, with the explicit expression for ξ following from the commutativity of the cube pushout diagram.

Item 2, Associativity: Omitted.

Item 3, Unitality: Omitted.

Item 4, Commutativity: Clear.

Item 5, Interaction With Coproducts: Clear.

Item 6, Symmetric Monoidality: Omitted. \square

002M 2.5 Coequalisers

Let A and B be sets and let $f, g: A \rightrightarrows B$ be functions.

- 002N **Definition 2.5.1.1.** The **coequaliser of f and g** is the pair $(\text{CoEq}(f, g), \text{coeq}(f, g))$ consisting of:

- *The Colimit.* The set $\text{CoEq}(f, g)$ defined by

$$\text{CoEq}(f, g) \stackrel{\text{def}}{=} B/\sim,$$

where \sim is the equivalence relation on B generated by $f(a) \sim g(a)$.

- *The Cocone.* The map

$$\text{coeq}(f, g): B \rightarrow \text{CoEq}(f, g)$$

given by the quotient map $\pi: B \twoheadrightarrow B/\sim$ with respect to the equivalence relation generated by $f(a) \sim g(a)$.

Proof. We claim that $\text{CoEq}(f, g)$ is the categorical coequaliser of f and g in \mathbf{Sets} . First we need to check that the relevant coequaliser diagram commutes, i.e. that we have

$$\text{coeq}(f, g) \circ f = \text{coeq}(f, g) \circ g.$$

Indeed, we have

$$\begin{aligned} [\text{coeq}(f, g) \circ f](a) &\stackrel{\text{def}}{=} [\text{coeq}(f, g)](f(a)) \\ &\stackrel{\text{def}}{=} [f(a)] \\ &= [g(a)] \\ &\stackrel{\text{def}}{=} [\text{coeq}(f, g)](g(a)) \\ &\stackrel{\text{def}}{=} [\text{coeq}(f, g) \circ g](a) \end{aligned}$$

for each $a \in A$. Next, we prove that $\text{CoEq}(f, g)$ satisfies the universal property of the coequaliser. Suppose we have a diagram of the form

$$\begin{array}{ccc} A & \xrightarrow[g]{f} & B \xrightarrow{\text{coeq}(f, g)} \text{CoEq}(f, g) \\ & & \searrow c \\ & & C \end{array}$$

in **Sets**. Then, since $c(f(a)) = c(g(a))$ for each $a \in A$, it follows from **Equivalence Relations and Apartness Relations**, **Items 4 and 5** of **Proposition 5.2.1.3** that there exists a unique map $\text{CoEq}(f, g) \xrightarrow{\exists!} C$ making the diagram

$$\begin{array}{ccc} A & \xrightarrow[g]{f} & B \xrightarrow{\text{coeq}(f, g)} \text{CoEq}(f, g) \\ & & \searrow c \quad \downarrow \exists! \\ & & C \end{array}$$

commute. □

002P Remark 2.5.1.2. In detail, by **Equivalence Relations and Apartness Relations**, **Construction 4.2.1.2**, the relation \sim of **Definition 2.5.1.1** is given by declaring $a \sim b$ iff one of the following conditions is satisfied:

- We have $a = b$;
- There exist $x_1, \dots, x_n \in B$ such that $a \sim' x_1 \sim' \dots \sim' x_n \sim' b$, where we declare $x \sim' y$ if one of the following conditions is satisfied:
 1. There exists $z \in A$ such that $x = f(z)$ and $y = g(z)$.
 2. There exists $z \in A$ such that $x = g(z)$ and $y = f(z)$.

That is: we require the following condition to be satisfied:

- (★) There exist $x_1, \dots, x_n \in B$ satisfying the following conditions:
 1. There exists $z_0 \in A$ satisfying one of the following conditions:
 - (a) We have $a = f(z_0)$ and $x_1 = g(z_0)$.
 - (b) We have $a = g(z_0)$ and $x_1 = f(z_0)$.
 2. For each $1 \leq i \leq n - 1$, there exists $z_i \in A$ satisfying one of the following conditions:

- (a) We have $x_i = f(z_i)$ and $x_{i+1} = g(z_i)$.
- (b) We have $x_i = g(z_i)$ and $x_{i+1} = f(z_i)$.
- 3. There exists $z_n \in A$ satisfying one of the following conditions:
 - (a) We have $x_n = f(z_n)$ and $b = g(z_n)$.
 - (b) We have $x_n = g(z_n)$ and $b = f(z_n)$.

002Q Example 2.5.1.3. Here are some examples of coequalisers of sets.

002R 1. *Quotients by Equivalence Relations.* Let R be an equivalence relation on a set X . We have a bijection of sets

$$X/\sim_R \cong \text{CoEq}\left(R \hookrightarrow X \times X \begin{matrix} \xrightarrow{\text{pr}_1} \\ \xrightarrow{\text{pr}_2} \end{matrix} X\right).$$

Proof. **Item 1, Quotients by Equivalence Relations:** See [Pro24z]. □

002S Proposition 2.5.1.4. Let A , B , and C be sets.

002T 1. *Associativity.* We have isomorphisms of sets¹¹

$$\underbrace{\text{CoEq}(\text{coeq}(f, g) \circ f, \text{coeq}(f, g) \circ h)}_{=\text{CoEq}(\text{coeq}(f, g) \circ g, \text{coeq}(f, g) \circ h)} \cong \text{CoEq}(f, g, h) \cong \underbrace{\text{CoEq}(\text{coeq}(g, h) \circ f, \text{coeq}(g, h) \circ g)}_{=\text{CoEq}(\text{coeq}(g, h) \circ f, \text{coeq}(g, h) \circ h)},$$

where $\text{CoEq}(f, g, h)$ is the colimit of the diagram

$$A \begin{matrix} \xrightarrow{f} \\ \xrightarrow[g]{\quad} \\ \xrightarrow{h} \end{matrix} B$$

¹¹That is, the following three ways of forming “the” coequaliser of (f, g, h) agree:

1. Take the coequaliser of (f, g, h) , i.e. the colimit of the diagram

$$A \begin{matrix} \xrightarrow{f} \\ \xrightarrow[g]{\quad} \\ \xrightarrow{h} \end{matrix} B$$

in **Sets**.

2. First take the coequaliser of f and g , forming a diagram

$$A \begin{matrix} \xrightarrow{f} \\ \xrightarrow[g]{\quad} \end{matrix} B \xrightarrow{\text{coeq}(f, g)} \text{CoEq}(f, g)$$

and then take the coequaliser of the composition

$$A \begin{matrix} \xrightarrow{f} \\ \xrightarrow[h]{\quad} \end{matrix} B \xrightarrow{\text{coeq}(f, g)} \text{CoEq}(f, g),$$

in Sets.

- 002U** 4. *Unitality.* We have an isomorphism of sets

$$\text{CoEq}(f, f) \cong B.$$

- 002V** 5. *Commutativity.* We have an isomorphism of sets

$$\text{CoEq}(f, g) \cong \text{CoEq}(g, f).$$

- 002W** 6. *Interaction With Composition.* Let

$$A \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} B \begin{array}{c} \xrightarrow{h} \\ \xrightarrow{k} \end{array} C$$

be functions. We have a surjection

$$\text{CoEq}(h \circ f, k \circ g) \twoheadrightarrow \text{CoEq}(\text{coeq}(h, k) \circ h \circ f, \text{coeq}(h, k) \circ k \circ g)$$

exhibiting $\text{CoEq}(\text{coeq}(h, k) \circ h \circ f, \text{coeq}(h, k) \circ k \circ g)$ as a quotient of $\text{CoEq}(h \circ f, k \circ g)$ by the relation generated by declaring $h(y) \sim k(y)$ for each $y \in B$.

Proof. Item 1, Associativity: Omitted.

Item 4, Unitality: Clear.

Item 5, Commutativity: Clear.

Item 6, Interaction With Composition: Omitted. □

obtaining a quotient

$$\text{CoEq}(\text{coeq}(f, g) \circ f, \text{coeq}(f, g) \circ h) = \text{CoEq}(\text{coeq}(f, g) \circ g, \text{coeq}(f, g) \circ h)$$

of $\text{CoEq}(f, g)$

3. First take the coequaliser of g and h , forming a diagram

$$A \begin{array}{c} \xrightarrow{g} \\ \xrightarrow{h} \end{array} B \xrightarrow{\text{coeq}(g, h)} \text{CoEq}(g, h)$$

and then take the coequaliser of the composition

$$A \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} B \xrightarrow{\text{coeq}(g, h)} \text{CoEq}(g, h),$$

obtaining a quotient

$$\text{CoEq}(\text{coeq}(g, h) \circ f, \text{coeq}(g, h) \circ g) = \text{CoEq}(\text{coeq}(g, h) \circ f, \text{coeq}(g, h) \circ h)$$

of $\text{CoEq}(g, h)$.

002X 3 Operations With Sets

002Y 3.1 The Empty Set

002Z **Definition 3.1.1.1.** The **empty set** is the set \emptyset defined by

$$\emptyset \stackrel{\text{def}}{=} \{x \in X \mid x \neq x\},$$

where A is the set in the set existence axiom, ?? of ??.

0030 3.2 Singleton Sets

Let X be a set.

0031 **Definition 3.2.1.1.** The **singleton set containing X** is the set $\{X\}$ defined by

$$\{X\} \stackrel{\text{def}}{=} \{X, X\},$$

where $\{X, X\}$ is the pairing of X with itself (Definition 3.3.1.1).

0032 3.3 Pairings of Sets

Let X and Y be sets.

0033 **Definition 3.3.1.1.** The **pairing of X and Y** is the set $\{X, Y\}$ defined by

$$\{X, Y\} \stackrel{\text{def}}{=} \{x \in A \mid x = X \text{ or } x = Y\},$$

where A is the set in the axiom of pairing, ?? of ??.

0034 3.4 Ordered Pairs

Let A and B be sets.

0035 **Definition 3.4.1.1.** The **ordered pair associated to A and B** is the set (A, B) defined by

$$(A, B) \stackrel{\text{def}}{=} \{\{A\}, \{A, B\}\}.$$

0036 **Proposition 3.4.1.2.** Let A and B be sets.

0037 1. *Uniqueness.* Let A , B , C , and D be sets. The following conditions are equivalent:

0038 (a) We have $(A, B) = (C, D)$.

0039 (b) We have $A = C$ and $B = D$.

Proof. *Item 1, Uniqueness:* See [Cie97, Theorem 1.2.3].

□

003A 3.5 Sets of Maps

Let A and B be sets.

003B Definition 3.5.1.1. The **set of maps from A to B** ¹² is the set $\text{Hom}_{\text{Sets}}(A, B)$ ¹³ whose elements are the functions from A to B .

003C Proposition 3.5.1.2. Let A and B be sets.

003D 1. *Functoriality.* The assignments $X, Y, (X, Y) \mapsto \text{Hom}_{\text{Sets}}(X, Y)$ define functors

$$\begin{aligned}\text{Hom}_{\text{Sets}}(X, -) &: \text{Sets} \rightarrow \text{Sets}, \\ \text{Hom}_{\text{Sets}}(-, Y) &: \text{Sets}^{\text{op}} \rightarrow \text{Sets}, \\ \text{Hom}_{\text{Sets}}(-, -) &: \text{Sets}^{\text{op}} \times \text{Sets} \rightarrow \text{Sets}.\end{aligned}$$

Proof. Item 1, Functoriality: This follows from **Categories**, **Items 2** and **5** of **Proposition 1.6.1.2**. \square

003E 3.6 Unions of Families

Let $\{A_i\}_{i \in I}$ be a family of sets.

003F Definition 3.6.1.1. The **union of the family $\{A_i\}_{i \in I}$** is the set $\bigcup_{i \in I} A_i$ defined by

$$\bigcup_{i \in I} A_i \stackrel{\text{def}}{=} \{x \in F \mid \text{there exists some } i \in I \text{ such that } x \in A_i\},$$

where F is the set in the axiom of union, ?? of ??.

003G 3.7 Binary Unions

Let A and B be sets.

003H Definition 3.7.1.1. The **union**¹⁴ **of A and B** is the set $A \cup B$ defined by

$$A \cup B \stackrel{\text{def}}{=} \bigcup_{z \in \{A, B\}} z.$$

003J Proposition 3.7.1.2. Let X be a set.

¹² *Further Terminology:* Also called the **Hom set from A to B** .

¹³ *Further Notation:* Also written **Sets**(A, B).

¹⁴ *Further Terminology:* Also called the **binary union of A and B** , for emphasis.

- 003K** 1. *Functoriality.* The assignments $U, V, (U, V) \mapsto U \cup V$ define functors

$$\begin{aligned} U \cup -: (\mathcal{P}(X), \subset) &\rightarrow (\mathcal{P}(X), \subset), \\ - \cup V: (\mathcal{P}(X), \subset) &\rightarrow (\mathcal{P}(X), \subset), \\ -_1 \cup -_2: (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \subset) &\rightarrow (\mathcal{P}(X), \subset), \end{aligned}$$

where $-_1 \cup -_2$ is the functor where

- *Action on Objects.* For each $(U, V) \in \mathcal{P}(X) \times \mathcal{P}(X)$, we have

$$[-_1 \cup -_2](U, V) \stackrel{\text{def}}{=} U \cup V.$$

- *Action on Morphisms.* For each pair of morphisms

$$\begin{aligned} \iota_U: U &\hookrightarrow U', \\ \iota_V: V &\hookrightarrow V' \end{aligned}$$

of $\mathcal{P}(X) \times \mathcal{P}(X)$, the image

$$\iota_U \cup \iota_V: U \cup V \hookrightarrow U' \cup V'$$

of (ι_U, ι_V) by \cup is the inclusion

$$U \cup V \subset U' \cup V'$$

i.e. where we have

$$(\star) \text{ If } U \subset U' \text{ and } V \subset V', \text{ then } U \cup V \subset U' \cup V'.$$

and where $U \cup -$ and $- \cup V$ are the partial functors of $-_1 \cup -_2$ at $U, V \in \mathcal{P}(X)$.

- 003L** 2. *Via Intersections and Symmetric Differences.* We have an equality of sets

$$U \cup V = (U \triangle V) \triangle (U \cap V)$$

for each $X \in \text{Obj}(\mathbf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

- 003M** 3. *Associativity.* We have an equality of sets

$$(U \cup V) \cup W = U \cup (V \cup W)$$

for each $X \in \text{Obj}(\mathbf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

- 003N 4. *Unitality.* We have equalities of sets

$$\begin{aligned} U \cup \emptyset &= U, \\ \emptyset \cup U &= U \end{aligned}$$

for each $X \in \text{Obj}(\mathbf{Sets})$ and each $U \in \mathcal{P}(X)$.

- 003P 5. *Commutativity.* We have an equality of sets

$$U \cup V = V \cup U$$

for each $X \in \text{Obj}(\mathbf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

- 003Q 6. *Idempotency.* We have an equality of sets

$$U \cup U = U$$

for each $X \in \text{Obj}(\mathbf{Sets})$ and each $U \in \mathcal{P}(X)$.

- 003R 7. *Distributivity Over Intersections.* We have equalities of sets

$$\begin{aligned} U \cup (V \cap W) &= (U \cup V) \cap (U \cup W), \\ (U \cap V) \cup W &= (U \cup W) \cap (V \cup W) \end{aligned}$$

for each $X \in \text{Obj}(\mathbf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

- 003S 8. *Interaction With Characteristic Functions I.* We have

$$\chi_{U \cup V} = \max(\chi_U, \chi_V)$$

for each $X \in \text{Obj}(\mathbf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

- 003T 9. *Interaction With Characteristic Functions II.* We have

$$\chi_{U \cup V} = \chi_U + \chi_V - \chi_{U \cap V}$$

for each $X \in \text{Obj}(\mathbf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

- 003U 10. *Interaction With Powersets and Semirings.* The quintuple $(\mathcal{P}(X), \cup, \cap, \emptyset, X)$ is an idempotent commutative semiring.

Proof. *Item 1, Functoriality:* See [Pro24an].

Item 2, Via Intersections and Symmetric Differences: See [Pro24ay].

Item 3, Associativity: See [Pro24ba].

Item 4, Unitality: This follows from [Pro24bd] and *Item 5*.

Item 5, Commutativity: See [Pro24bb].

Item 6, Idempotency: See [Pro24am].

Item 7, Distributivity Over Intersections: See [Pro24az].

Item 8, Interaction With Characteristic Functions I: See [Pro24k].

Item 9, Interaction With Characteristic Functions II: See [Pro24k].

Item 10, Interaction With Powersets and Semirings: This follows from Items 3 to 6 and Items 3 to 5, 7 and 8 of Proposition 3.9.1.2. \square

003V 3.8 Intersections of Families

Let \mathcal{F} be a family of sets.

003W Definition 3.8.1.1. The **intersection of a family \mathcal{F} of sets** is the set $\bigcap_{X \in \mathcal{F}} X$ defined by

$$\bigcap_{X \in \mathcal{F}} X \stackrel{\text{def}}{=} \left\{ z \in \bigcup_{X \in \mathcal{F}} X \mid \text{for each } X \in \mathcal{F}, \text{ we have } z \in X \right\}.$$

003X 3.9 Binary Intersections

Let X and Y be sets.

003Y Definition 3.9.1.1. The **intersection¹⁵ of X and Y** is the set $X \cap Y$ defined by

$$X \cap Y \stackrel{\text{def}}{=} \bigcap_{z \in \{X, Y\}} z.$$

003Z Proposition 3.9.1.2. Let X be a set.

0040 1. *Functoriality.* The assignments $U, V, (U, V) \mapsto U \cap V$ define functors

$$\begin{aligned} U \cap - &: (\mathcal{P}(X), \subset) \rightarrow (\mathcal{P}(X), \subset), \\ - \cap V &: (\mathcal{P}(X), \subset) \rightarrow (\mathcal{P}(X), \subset), \\ -_1 \cap -_2 &: (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \subset) \rightarrow (\mathcal{P}(X), \subset), \end{aligned}$$

where $-_1 \cap -_2$ is the functor where

- *Action on Objects.* For each $(U, V) \in \mathcal{P}(X) \times \mathcal{P}(X)$, we have

$$[-_1 \cap -_2](U, V) \stackrel{\text{def}}{=} U \cap V.$$

¹⁵ *Further Terminology:* Also called the **binary intersection of X and Y** , for emphasis.

- *Action on Morphisms.* For each pair of morphisms

$$\begin{aligned}\iota_U: U &\hookrightarrow U', \\ \iota_V: V &\hookrightarrow V'\end{aligned}$$

of $\mathcal{P}(X) \times \mathcal{P}(X)$, the image

$$\iota_U \cap \iota_V: U \cap V \hookrightarrow U' \cap V'$$

of (ι_U, ι_V) by \cap is the inclusion

$$U \cap V \subset U' \cap V'$$

i.e. where we have

$$(\star) \text{ If } U \subset U' \text{ and } V \subset V', \text{ then } U \cap V \subset U' \cap V'.$$

and where $U \cap -$ and $- \cap V$ are the partial functors of $-_1 \cap -_2$ at $U, V \in \mathcal{P}(X)$.

0041 2. *Adjointness.* We have adjunctions

$$\begin{aligned}\left(U \cap - \dashv \mathbf{Hom}_{\mathcal{P}(X)}(U, -) \right): \mathcal{P}(X) &\overset{U \cap -}{\underset{\mathbf{Hom}_{\mathcal{P}(X)}(U, -)}{\rightleftarrows}} \mathcal{P}(X), \\ \left(- \cap V \dashv \mathbf{Hom}_{\mathcal{P}(X)}(V, -) \right): \mathcal{P}(X) &\overset{- \cap V}{\underset{\mathbf{Hom}_{\mathcal{P}(X)}(V, -)}{\rightleftarrows}} \mathcal{P}(X),\end{aligned}$$

where

$$\mathbf{Hom}_{\mathcal{P}(X)}(-_1, -_2): \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X) \rightarrow \mathcal{P}(X)$$

is the bifunctor defined by¹⁶

$$\mathbf{Hom}_{\mathcal{P}(X)}(U, V) \stackrel{\text{def}}{=} (X \setminus U) \cup V$$

witnessed by bijections

$$\begin{aligned}\text{Hom}_{\mathcal{P}(X)}(U \cap V, W) &\cong \text{Hom}_{\mathcal{P}(X)}\left(U, \mathbf{Hom}_{\mathcal{P}(X)}(V, W)\right), \\ \text{Hom}_{\mathcal{P}(X)}(U \cap V, W) &\cong \text{Hom}_{\mathcal{P}(X)}\left(V, \mathbf{Hom}_{\mathcal{P}(X)}(U, W)\right),\end{aligned}$$

natural in $U, V, W \in \mathcal{P}(X)$, i.e. where:

¹⁶For intuition regarding the expression defining $\mathbf{Hom}_{\mathcal{P}(X)}(U, V)$, see [Remark 3.9.1.3](#).

(a) The following conditions are equivalent:

- i. We have $U \cap V \subset W$.
- ii. We have $U \subset \mathbf{Hom}_{\mathcal{P}(X)}(V, W)$.
- iii. We have $U \subset (X \setminus V) \cup W$.

(b) The following conditions are equivalent:

- i. We have $V \cap U \subset W$.
- ii. We have $V \subset \mathbf{Hom}_{\mathcal{P}(X)}(U, W)$.
- iii. We have $V \subset (X \setminus U) \cup W$.

0042 3. *Associativity.* We have an equality of sets

$$(U \cap V) \cap W = U \cap (V \cap W)$$

for each $X \in \mathbf{Obj}(\mathbf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

0043 4. *Unitality.* Let X be a set and let $U \in \mathcal{P}(X)$. We have equalities of sets

$$\begin{aligned} X \cap U &= U, \\ U \cap X &= U \end{aligned}$$

for each $X \in \mathbf{Obj}(\mathbf{Sets})$ and each $U \in \mathcal{P}(X)$.

0044 5. *Commutativity.* We have an equality of sets

$$U \cap V = V \cap U$$

for each $X \in \mathbf{Obj}(\mathbf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

0045 6. *Idempotency.* We have an equality of sets

$$U \cap U = U$$

for each $X \in \mathbf{Obj}(\mathbf{Sets})$ and each $U \in \mathcal{P}(X)$.

0046 7. *Distributivity Over Unions.* We have equalities of sets

$$\begin{aligned} U \cap (V \cup W) &= (U \cap V) \cup (U \cap W), \\ (U \cup V) \cap W &= (U \cap W) \cup (V \cap W) \end{aligned}$$

for each $X \in \mathbf{Obj}(\mathbf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

0047 8. *Annihilation With the Empty Set.* We have an equality of sets

$$\begin{aligned}\emptyset \cap X &= \emptyset, \\ X \cap \emptyset &= \emptyset\end{aligned}$$

for each $X \in \text{Obj}(\text{Sets})$ and each $U \in \mathcal{P}(X)$.

0048 9. *Interaction With Characteristic Functions I.* We have

$$\chi_{U \cap V} = \chi_U \chi_V$$

for each $X \in \text{Obj}(\text{Sets})$ and each $U, V \in \mathcal{P}(X)$.

0049 10. *Interaction With Characteristic Functions II.* We have

$$\chi_{U \cap V} = \min(\chi_U, \chi_V)$$

for each $X \in \text{Obj}(\text{Sets})$ and each $U, V \in \mathcal{P}(X)$.

004A 11. *Interaction With Powersets and Monoids With Zero.* The quadruple $((\mathcal{P}(X), \emptyset), \cap, X)$ is a commutative monoid with zero.

004B 12. *Interaction With Powersets and Semirings.* The quintuple $(\mathcal{P}(X), \cup, \cap, \emptyset, X)$ is an idempotent commutative semiring.

Proof. Item 1, Functoriality: See [Pro24al].

Item 2, Adjointness: See [MSE 267469].

Item 3, Associativity: See [Pro24s].

Item 4, Unitality: This follows from [Pro24w] and Item 5.

Item 5, Commutativity: See [Pro24t].

Item 6, Idempotency: See [Pro24ak].

Item 7, Distributivity Over Unions: See [Pro24aj].

Item 8, Annihilation With the Empty Set: This follows from [Pro24u] and Item 5.

Item 9, Interaction With Characteristic Functions I: See [Pro24h].

Item 10, Interaction With Characteristic Functions II: See [Pro24h].

Item 11, Interaction With Powersets and Monoids With Zero: This follows from Items 3 to 5 and 8.

Item 12, Interaction With Powersets and Semirings: This follows from Items 3 to 6 and Items 3 to 5, 7 and 8 of Proposition 3.9.1.2. \square

004C Remark 3.9.1.3. Since intersections are the products in $\mathcal{P}(X)$ (Item 1 of Proposition 4.3.1.3), the left adjoint $\mathbf{Hom}_{\mathcal{P}(X)}(U, V)$ may be thought of as a function type $[U, V]$.

Then, under the Curry–Howard correspondence, the function type $[U, V]$ corresponds to implication $U \implies V$, which is logically equivalent to the statement $\neg U \vee V$. This in turn corresponds to the set $U^c \vee V = (X \setminus U) \cup V$.

004D 3.10 Differences

Let X and Y be sets.

004E Definition 3.10.1.1. The **difference of X and Y** is the set $X \setminus Y$ defined by

$$X \setminus Y \stackrel{\text{def}}{=} \{a \in X \mid a \notin Y\}.$$

004F Proposition 3.10.1.2. Let X be a set.

004G 1. *Functoriality.* The assignments $U, V, (U, V) \mapsto U \cap V$ define functors

$$\begin{aligned} U \setminus - &: (\mathcal{P}(X), \supset) \rightarrow (\mathcal{P}(X), \subset), \\ - \setminus V &: (\mathcal{P}(X), \subset) \rightarrow (\mathcal{P}(X), \subset), \\ -_1 \setminus -_2 &: (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \supset) \rightarrow (\mathcal{P}(X), \subset), \end{aligned}$$

where $-_1 \setminus -_2$ is the functor where

- *Action on Objects.* For each $(U, V) \in \mathcal{P}(X) \times \mathcal{P}(X)$, we have

$$[-_1 \setminus -_2](U, V) \stackrel{\text{def}}{=} U \setminus V.$$

- *Action on Morphisms.* For each pair of morphisms

$$\begin{aligned} \iota_A &: A \hookrightarrow B, \\ \iota_U &: U \hookrightarrow V \end{aligned}$$

of $\mathcal{P}(X) \times \mathcal{P}(X)$, the image

$$\iota_U \setminus \iota_V: A \setminus V \hookrightarrow B \setminus U$$

of (ι_U, ι_V) by \setminus is the inclusion

$$A \setminus V \subset B \setminus U$$

i.e. where we have

(\star) If $A \subset B$ and $U \subset V$, then $A \setminus V \subset B \setminus U$.

and where $U \setminus -$ and $- \setminus V$ are the partial functors of $-_1 \setminus -_2$ at $U, V \in \mathcal{P}(X)$.

004H 2. *De Morgan's Laws.* We have equalities of sets

$$\begin{aligned} X \setminus (U \cup V) &= (X \setminus U) \cap (X \setminus V), \\ X \setminus (U \cap V) &= (X \setminus U) \cup (X \setminus V) \end{aligned}$$

for each $X \in \text{Obj}(\mathbf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

004J 3. *Interaction With Unions I.* We have equalities of sets

$$U \setminus (V \cup W) = (U \setminus V) \cap (U \setminus W)$$

for each $X \in \text{Obj}(\mathbf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

004K 4. *Interaction With Unions II.* We have equalities of sets

$$(U \setminus V) \cup W = (U \cup W) \setminus (V \setminus W)$$

for each $X \in \text{Obj}(\mathbf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

004L 5. *Interaction With Unions III.* We have equalities of sets

$$\begin{aligned} U \setminus (V \cup W) &= (U \cup W) \setminus (V \cup W) \\ &= (U \setminus V) \setminus W \\ &= (U \setminus W) \setminus V \end{aligned}$$

for each $X \in \text{Obj}(\mathbf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

004M 6. *Interaction With Unions IV.* We have equalities of sets

$$(U \cup V) \setminus W = (U \setminus W) \cup (V \setminus W)$$

for each $X \in \text{Obj}(\mathbf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

004N 7. *Interaction With Intersections.* We have equalities of sets

$$\begin{aligned} (U \setminus V) \cap W &= (U \cap W) \setminus V \\ &= U \cap (W \setminus V) \end{aligned}$$

for each $X \in \text{Obj}(\mathbf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

- 004P** 8. *Interaction With Complements.* We have an equality of sets

$$U \setminus V = U \cap V^c$$

for each $X \in \text{Obj}(\mathbf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

- 004Q** 9. *Interaction With Symmetric Differences.* We have an equality of sets

$$U \setminus V = U \Delta (U \cap V)$$

for each $X \in \text{Obj}(\mathbf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

- 004R** 10. *Triple Differences.* We have

$$U \setminus (V \setminus W) = (U \cap W) \cup (U \setminus V)$$

for each $X \in \text{Obj}(\mathbf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

- 004S** 11. *Left Annihilation.* We have

$$\emptyset \setminus U = \emptyset$$

for each $X \in \text{Obj}(\mathbf{Sets})$ and each $U \in \mathcal{P}(X)$.

- 004T** 12. *Right Unitality.* We have

$$U \setminus \emptyset = U$$

for each $X \in \text{Obj}(\mathbf{Sets})$ and each $U \in \mathcal{P}(X)$.

- 004U** 13. *Invertibility.* We have

$$U \setminus U = \emptyset$$

for each $X \in \text{Obj}(\mathbf{Sets})$ and each $U \in \mathcal{P}(X)$.

- 004V** 14. *Interaction With Containment.* The following conditions are equivalent:

004W (a) We have $V \setminus U \subset W$.

004X (b) We have $V \setminus W \subset U$.

- 004Y** 15. *Interaction With Characteristic Functions.* We have

$$\chi_{U \setminus V} = \chi_U - \chi_{U \cap V}$$

for each $X \in \text{Obj}(\mathbf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

Proof. Item 1, Functoriality: See [Pro24ad] and [Pro24ah].

Item 2, De Morgan's Laws: See [Pro24m].

Item 3, Interaction With Unions I: See [Pro24n].

Item 4, Interaction With Unions II: Omitted.

Item 5, Interaction With Unions III: See [Pro24ai].

Item 6, Interaction With Unions IV: See [Pro24ac].

Item 7, Interaction With Intersections: See [Pro24v].

Item 8, Interaction With Complements: See [Pro24aa].

Item 9, Interaction With Symmetric Differences: See [Pro24ab].

Item 10, Triple Differences: See [Pro24ag].

Item 11, Left Annihilation: Clear.

Item 12, Right Unitality: See [Pro24ae].

Item 13, Invertibility: See [Pro24af].

Item 14, Interaction With Containment: Omitted.

Item 15, Interaction With Characteristic Functions: See [Pro24i]. □

004Z 3.11 Complements

Let X be a set and let $U \in \mathcal{P}(X)$.

0050 Definition 3.11.1.1. The **complement of U** is the set U^c defined by

$$\begin{aligned} U^c &\stackrel{\text{def}}{=} X \setminus U \\ &\stackrel{\text{def}}{=} \{a \in X \mid a \notin U\}. \end{aligned}$$

0051 Proposition 3.11.1.2. Let X be a set.

0052 1. *Functoriality.* The assignment $U \mapsto U^c$ defines a functor

$$(-)^c: \mathcal{P}(X)^{\text{op}} \rightarrow \mathcal{P}(X),$$

where

- *Action on Objects.* For each $U \in \mathcal{P}(X)$, we have

$$[(-)^c](U) \stackrel{\text{def}}{=} U^c.$$

- *Action on Morphisms.* For each morphism $\iota_U: U \hookrightarrow V$ of $\mathcal{P}(X)$, the image

$$\iota_U^c: V^c \hookrightarrow U^c$$

of ι_U by $(-)^c$ is the inclusion

$$V^c \subset U^c$$

i.e. where we have

(\star) If $U \subset V$, then $V^c \subset U^c$.

0053 2. *De Morgan's Laws.* We have equalities of sets

$$\begin{aligned}(U \cup V)^c &= U^c \cap V^c, \\ (U \cap V)^c &= U^c \cup V^c\end{aligned}$$

for each $X \in \text{Obj}(\mathbf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

0054 3. *Involutority.* We have

$$(U^c)^c = U$$

for each $X \in \text{Obj}(\mathbf{Sets})$ and each $U \in \mathcal{P}(X)$.

0055 4. *Interaction With Characteristic Functions.* We have

$$\chi_{U^c} = 1 - \chi_U$$

for each $X \in \text{Obj}(\mathbf{Sets})$ and each $U \in \mathcal{P}(X)$.

Proof. **Item 1, Functoriality:** This follows from **Item 1** of **Proposition 3.10.1.2**.

Item 2, De Morgan's Laws: See [Pro24m].

Item 3, Involutority: See [Pro24l].

Item 4, Interaction With Characteristic Functions: Clear. \square

0056 3.12 Symmetric Differences

Let A and B be sets.

0057 **Definition 3.12.1.1.** The **symmetric difference of A and B** is the set $A \triangle B$ defined by

$$A \triangle B \stackrel{\text{def}}{=} (A \setminus B) \cup (B \setminus A).$$

0058 **Proposition 3.12.1.2.** Let X be a set.

0059 1. *Lack of Functoriality.* The assignment $(U, V) \mapsto U \triangle V$ **need not** define functors

$$\begin{aligned}U \triangle -: (\mathcal{P}(X), \subset) &\rightarrow (\mathcal{P}(X), \subset), \\ - \triangle V: (\mathcal{P}(X), \subset) &\rightarrow (\mathcal{P}(X), \subset), \\ -_1 \triangle -_2: (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \subset) &\rightarrow (\mathcal{P}(X), \subset).\end{aligned}$$

005A 2. *Via Unions and Intersections.* We have¹⁷

$$U \triangle V = (U \cup V) \setminus (U \cap V)$$

for each $X \in \text{Obj}(\text{Sets})$ and each $U, V \in \mathcal{P}(X)$.

005B 3. *Associativity.* We have¹⁸

$$(U \triangle V) \triangle W = U \triangle (V \triangle W)$$

for each $X \in \text{Obj}(\text{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

005C 4. *Commutativity.* We have

$$U \triangle V = V \triangle U$$

for each $X \in \text{Obj}(\text{Sets})$ and each $U, V \in \mathcal{P}(X)$.

005D 5. *Unitality.* We have

$$\begin{aligned} U \triangle \emptyset &= U, \\ \emptyset \triangle U &= U \end{aligned}$$

for each $X \in \text{Obj}(\text{Sets})$ and each $U \in \mathcal{P}(X)$.

005E 6. *Invertibility.* We have

$$U \triangle U = \emptyset$$

for each $X \in \text{Obj}(\text{Sets})$ and each $U \in \mathcal{P}(X)$.

005F 7. *Interaction With Unions.* We have

$$(U \triangle V) \cup (V \triangle W) = (U \cup V \cup W) \setminus (U \cap V \cap W)$$

for each $X \in \text{Obj}(\text{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

¹⁷Illustration:



¹⁸Illustration:



- 005G 8. *Interaction With Complements I.* We have

$$U \triangle U^c = X$$

for each $X \in \text{Obj}(\mathbf{Sets})$ and each $U \in \mathcal{P}(X)$.

- 005H 9. *Interaction With Complements II.* We have

$$\begin{aligned} U \triangle X &= U^c, \\ X \triangle U &= U^c \end{aligned}$$

for each $X \in \text{Obj}(\mathbf{Sets})$ and each $U \in \mathcal{P}(X)$.

- 005J 10. *Interaction With Complements III.* We have

$$U^c \triangle V^c = U \triangle V$$

for each $X \in \text{Obj}(\mathbf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

- 005K 11. *“Transitivity”.* We have

$$(U \triangle V) \triangle (V \triangle W) = U \triangle W$$

for each $X \in \text{Obj}(\mathbf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

- 005L 12. *The Triangle Inequality for Symmetric Differences.* We have

$$U \triangle W \subset U \triangle V \cup V \triangle W$$

for each $X \in \text{Obj}(\mathbf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

- 005M 13. *Distributivity Over Intersections.* We have

$$\begin{aligned} U \cap (V \triangle W) &= (U \cap V) \triangle (U \cap W), \\ (U \triangle V) \cap W &= (U \cap W) \triangle (V \cap W) \end{aligned}$$

for each $X \in \text{Obj}(\mathbf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

- 005N 14. *Interaction With Characteristic Functions.* We have

$$\chi_{U \triangle V} = \chi_U + \chi_V - 2\chi_{U \cap V}$$

and thus, in particular, we have

$$\chi_{U \triangle V} \equiv \chi_U + \chi_V \pmod{2}$$

for each $X \in \text{Obj}(\mathbf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

005P 15. *Bijectivity.* Given $A, B \subset \mathcal{P}(X)$, the maps

$$\begin{aligned} A \triangle - : \mathcal{P}(X) &\rightarrow \mathcal{P}(X), \\ - \triangle B : \mathcal{P}(X) &\rightarrow \mathcal{P}(X) \end{aligned}$$

are bijections with inverses given by

$$\begin{aligned} (A \triangle -)^{-1} &= - \cup (A \cap -), \\ (- \triangle B)^{-1} &= - \cup (B \cap -). \end{aligned}$$

Moreover, the map

$$C \mapsto C \triangle (A \triangle B)$$

is a bijection of $\mathcal{P}(X)$ onto itself sending A to B and B to A .

005Q 16. *Interaction With Powersets and Groups.* Let X be a set.

005R (a) The quadruple $(\mathcal{P}(X), \triangle, \emptyset, \text{id}_{\mathcal{P}(X)})$ is an abelian group. ¹⁹

(b) Every element of $\mathcal{P}(X)$ has order 2 with respect to \triangle , and thus $\mathcal{P}(X)$ is a *Boolean group* (i.e. an abelian 2-group).

005S

005T 4. *Interaction With Powersets and Vector Spaces I.* The pair $(\mathcal{P}(X), \alpha_{\mathcal{P}(X)})$ consisting of

- The group $\mathcal{P}(X)$ of ??;
- The map $\alpha_{\mathcal{P}(X)} : \mathbb{F}_2 \times \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ defined by

$$\begin{aligned} 0 \cdot U &\stackrel{\text{def}}{=} \emptyset, \\ 1 \cdot U &\stackrel{\text{def}}{=} U; \end{aligned}$$

¹⁹Here are some examples:

1. When $X = \emptyset$, we have an isomorphism of groups between $\mathcal{P}(\emptyset)$ and the trivial group:

$$(\mathcal{P}(\emptyset), \triangle, \emptyset, \text{id}_{\mathcal{P}(\emptyset)}) \cong \text{pt}.$$

2. When $X = \text{pt}$, we have an isomorphism of groups between $\mathcal{P}(\text{pt})$ and \mathbb{Z}_2 :

$$(\mathcal{P}(\text{pt}), \triangle, \emptyset, \text{id}_{\mathcal{P}(\text{pt})}) \cong \mathbb{Z}_2.$$

3. When $X = \{0, 1\}$, we have an isomorphism of groups between $\mathcal{P}(\{0, 1\})$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$:

$$(\mathcal{P}(\{0, 1\}), \triangle, \emptyset, \text{id}_{\mathcal{P}(\{0, 1\})}) \cong \mathbb{Z}_2 \times \mathbb{Z}_2.$$

is an \mathbb{F}_2 -vector space.

005U 5. *Interaction With Powersets and Vector Spaces II.* If X is finite, then:

(a) The set of singletons sets on the elements of X forms a basis for the \mathbb{F}_2 -vector space $(\mathcal{P}(X), \alpha_{\mathcal{P}(X)})$ of **Item 4**.

(b) We have

$$\dim(\mathcal{P}(X)) = \#\mathcal{P}(X).$$

005V 6. *Interaction With Powersets and Rings.* The quintuple $(\mathcal{P}(X), \triangle, \cap, \emptyset, X)$ is a commutative ring.²⁰

Proof. **Item 1**, *Lack of Functoriality*: Omitted.

Item 2, *Via Unions and Intersections*: See [Pro24o].

Item 3, *Associativity*: See [Pro24ao].

Item 4, *Commutativity*: See [Pro24ap].

Item 5, *Unitality*: This follows from **Item 4** and [Pro24at].

Item 6, *Invertibility*: See [Pro24av].

Item 7, *Interaction With Unions*: See [Pro24bc].

Item 8, *Interaction With Complements I*: See [Pro24as].

Item 9, *Interaction With Complements II*: This follows from **Item 4** and [Pro24ax].

Item 10, *Interaction With Complements III*: See [Pro24aq].

Item 11, “*Transitivity*”: We have


$$\begin{aligned} (U \triangle V) \triangle (V \triangle W) &= U \triangle (V \triangle (V \triangle W)) && \text{(by Item 3)} \\ &= U \triangle ((V \triangle V) \triangle W) && \text{(by Item 3)} \\ &= U \triangle (\emptyset \triangle W) && \text{(by Item 6)} \\ &= U \triangle W && \text{(by Item 5)} \end{aligned}$$

Item 12, *The Triangle Inequality for Symmetric Differences*: This follows from **Items 2** and **11**.

Item 13, *Distributivity Over Intersections*: See [Pro24r].

Item 14, *Interaction With Characteristic Functions*: See [Pro24j].

Item 15, *Bijectivity*: Clear.

²⁰  *Warning*: The analogous statement replacing intersections by unions (i.e. that the quintuple $(\mathcal{P}(X), \triangle, \cup, \emptyset, X)$ is a ring) is false, however. See [Pro24aw] for a proof.

Item 16, Interaction With Powersets and Groups: **Item 16a** follows from²¹ **Items 3 to 6**, while **Item 3b** follows from **Item 6**.

Item 4, Interaction With Powersets and Vector Spaces I: Clear.

Item 5, Interaction With Powersets and Vector Spaces II: Omitted.

Item 6, Interaction With Powersets and Rings: This follows from **Items 8 and 11** of **Proposition 3.9.1.2** and **Items 13 and 16**.²² \square

005W 4 Powersets

005X 4.1 Characteristic Functions

Let X be a set.

005Y **Definition 4.1.1.1.** Let $U \subset X$ and let $x \in X$.

005Z 1. The **characteristic function of U** ²³ is the function²⁴

$$\chi_U: X \rightarrow \{\mathbf{t}, \mathbf{f}\}$$

defined by

$$\chi_U(x) \stackrel{\text{def}}{=} \begin{cases} \mathbf{true} & \text{if } x \in U, \\ \mathbf{false} & \text{if } x \notin U \end{cases}$$

for each $x \in X$.

0060 2. The **characteristic function of x** is the function²⁵

$$\chi_x: X \rightarrow \{\mathbf{t}, \mathbf{f}\}$$

defined by

$$\chi_x \stackrel{\text{def}}{=} \chi_{\{x\}},$$

i.e. by

$$\chi_x(y) \stackrel{\text{def}}{=} \begin{cases} \mathbf{true} & \text{if } x = y, \\ \mathbf{false} & \text{if } x \neq y \end{cases}$$

for each $y \in X$.

²¹Reference: [Pro24ar].

²²Reference: [Pro24au].

²³Further Terminology: Also called the **indicator function of U** .

²⁴Further Notation: Also written $\chi_X(U, -)$ or $\chi_X(-, U)$.

²⁵Further Notation: Also written χ^x , $\chi_X(x, -)$, or $\chi_X(-, x)$.

0061 3. The **characteristic relation on X** ²⁶ is the relation²⁷

$$\chi_X(-1, -2): X \times X \rightarrow \{\mathbf{t}, \mathbf{f}\}$$

on X defined by²⁸

$$\chi_X(x, y) \stackrel{\text{def}}{=} \begin{cases} \mathbf{true} & \text{if } x = y, \\ \mathbf{false} & \text{if } x \neq y \end{cases}$$

for each $x, y \in X$.

0062 4. The **characteristic embedding**²⁹ of X into $\mathcal{P}(X)$ is the function

$$\chi_{(-)}: X \hookrightarrow \mathcal{P}(X)$$

defined by

$$\chi_{(-)}(x) \stackrel{\text{def}}{=} \chi_x$$

for each $x \in X$.

0063 **Remark 4.1.1.2.** The definitions in [Definition 4.1.1.1](#) are decategorifications of co/presheaves, representable co/presheaves, Hom profunctors, and the Yoneda embedding:³⁰

²⁶*Further Terminology:* Also called the **identity relation on X** .

²⁷*Further Notation:* Also written χ_{-2}^{-1} , or \sim_{id} in the context of relations.

²⁸As a subset of $X \times X$, the relation χ_X corresponds to the diagonal $\Delta_X \subset X \times X$ of X .

²⁹The name “characteristic *embedding*” comes from the fact that there is an analogue of fully faithfulness for $\chi_{(-)}$: given a set X , we have

$$\text{Hom}_{\mathcal{P}(X)}(\chi_x, \chi_y) = \chi_X(x, y),$$

for each $x, y \in X$.

³⁰These statements can be made precise by using the embeddings

$$\begin{aligned} (-)_{\text{disc}}: \mathbf{Sets} &\hookrightarrow \mathbf{Cats}, \\ (-)_{\text{disc}}: \{\mathbf{t}, \mathbf{f}\}_{\text{disc}} &\hookrightarrow \mathbf{Sets} \end{aligned}$$

of sets into categories and of classical truth values into sets.

For instance, in this approach the characteristic function

$$\chi_x: X \rightarrow \{\mathbf{t}, \mathbf{f}\}$$

of an element x of X , defined by

$$\chi_x(y) \stackrel{\text{def}}{=} \begin{cases} \mathbf{true} & \text{if } x = y, \\ \mathbf{false} & \text{if } x \neq y \end{cases}$$

- 0064 1. A function

$$f: X \rightarrow \{\mathbf{t}, \mathbf{f}\}$$

is a decategorification of a presheaf

$$\mathcal{F}: C^{\text{op}} \rightarrow \mathbf{Sets},$$

with the characteristic functions χ_U of the subsets of X being the primordial examples (and, in fact, all examples) of these.

- 0065 2. The characteristic function

$$\chi_x: X \rightarrow \{\mathbf{t}, \mathbf{f}\}$$

of an *element* x of X is a decategorification of the representable presheaf

$$h_X: C^{\text{op}} \rightarrow \mathbf{Sets}$$

of an *object* x of a category C .

- 0066 3. The characteristic relation

$$\chi_X(-_1, -_2): X \times X \rightarrow \{\mathbf{t}, \mathbf{f}\}$$

of X is a decategorification of the Hom profunctor

$$\text{Hom}_C(-_1, -_2): C^{\text{op}} \times C \rightarrow \mathbf{Sets}$$

of a category C .

- 0067 4. The characteristic embedding

$$\chi_{(-)}: X \hookrightarrow \mathcal{P}(X)$$

of X into $\mathcal{P}(X)$ is a decategorification of the Yoneda embedding

$$\mathcal{Y}: C^{\text{op}} \hookrightarrow \mathbf{PSh}(C)$$

of a category C into $\mathbf{PSh}(C)$.

for each $y \in X$, is recovered as the representable presheaf

$$\text{Hom}_{X_{\text{disc}}}(-, x): X_{\text{disc}} \rightarrow \mathbf{Sets}$$

- 0068 5. There is also a direct parallel between unions and colimits:
- An element of $\mathcal{P}(X)$ is a union of elements of X , viewed as one-point subsets $\{x\} \in \mathcal{P}(A)$.
 - An object of $\mathbf{PSh}(\mathcal{C})$ is a colimit of objects of \mathcal{C} , viewed as representable presheaves $h_X \in \mathbf{Obj}(\mathbf{PSh}(\mathcal{C}))$.

0069 **Proposition 4.1.1.3.** Let X be a set.

1. *The Inclusion of Characteristic Relations Associated to a Function.*

006A Let $f: A \rightarrow B$ be a function. We have an inclusion³¹

$$\chi_B \circ (f \times f) \subset \chi_A, \quad \begin{array}{ccc} A \times A & \xrightarrow{f \times f} & B \times B \\ \searrow \chi_A & \curvearrowright & \swarrow \chi_B \\ & \{t, f\}. & \end{array}$$

- 006B 2. *Interaction With Unions I.* We have

$$\chi_{U \cup V} = \max(\chi_U, \chi_V)$$

for each $X \in \mathbf{Obj}(\mathbf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

- 006C 3. *Interaction With Unions II.* We have

$$\chi_{U \cup V} = \chi_U + \chi_V - \chi_{U \cap V}$$

for each $X \in \mathbf{Obj}(\mathbf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

- 006D 4. *Interaction With Intersections I.* We have

$$\chi_{U \cap V} = \chi_U \chi_V$$

for each $X \in \mathbf{Obj}(\mathbf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

of the corresponding object x of X_{disc} , defined on objects by

$$\text{Hom}_{X_{\text{disc}}}(y, x) \stackrel{\text{def}}{=} \begin{cases} \text{pt} & \text{if } x = y, \\ \emptyset & \text{if } x \neq y \end{cases}$$

for each $y \in \mathbf{Obj}(X_{\text{disc}})$.

³¹This is the 0-categorical version of [Categories](#), [Definition 4.4.1.1](#).

006E 5. *Interaction With Intersections II.* We have

$$\chi_{U \cap V} = \min(\chi_U, \chi_V)$$

for each $X \in \text{Obj}(\text{Sets})$ and each $U, V \in \mathcal{P}(X)$.

006F 6. *Interaction With Differences.* We have

$$\chi_{U \setminus V} = \chi_U - \chi_{U \cap V}$$

for each $X \in \text{Obj}(\text{Sets})$ and each $U, V \in \mathcal{P}(X)$.

006G 7. *Interaction With Complements.* We have

$$\chi_{U^c} = 1 - \chi_U$$

for each $X \in \text{Obj}(\text{Sets})$ and each $U \in \mathcal{P}(X)$.

006H 8. *Interaction With Symmetric Differences.* We have

$$\chi_{U \Delta V} = \chi_U + \chi_V - 2\chi_{U \cap V}$$

and thus, in particular, we have

$$\chi_{U \Delta V} \equiv \chi_U + \chi_V \pmod{2}$$

for each $X \in \text{Obj}(\text{Sets})$ and each $U, V \in \mathcal{P}(X)$.

006J 9. *Interaction Between the Characteristic Embedding and Morphisms.* Let $f: X \rightarrow Y$ be a map of sets. The diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ \chi_X \downarrow & & \downarrow \chi_{X'} \\ \mathcal{P}(X) & \xrightarrow{f_*} & \mathcal{P}(X'). \end{array}$$

$f_* \circ \chi_X = \chi_{X'} \circ f,$

commutes.

Proof. Item 1, The Inclusion of Characteristic Relations Associated to a Function: The inclusion $\chi_B(f(a), f(b)) \subset \chi_A(a, b)$ is equivalent to the statement “if $a = b$, then $f(a) = f(b)$ ”, which is true.

Item 2, Interaction With Unions I: This is a repetition of *Item 8* of *Proposition 3.7.1.2* and is proved there.

Item 3, Interaction With Unions II: This is a repetition of **Item 9** of **Proposition 3.7.1.2** and is proved there.

Item 4, Interaction With Intersections I: This is a repetition of **Item 9** of **Proposition 3.9.1.2** and is proved there.

Item 5, Interaction With Intersections II: This is a repetition of **Item 10** of **Proposition 3.9.1.2** and is proved there.

Item 6, Interaction With Differences: This is a repetition of **Item 15** of **Proposition 3.10.1.2** and is proved there.

Item 7, Interaction With Complements: This is a repetition of **Item 4** of **Proposition 3.11.1.2** and is proved there.

Item 8, Interaction With Symmetric Differences: This is a repetition of **Item 14** of **Proposition 3.12.1.2** and is proved there.

Item 9, Interaction Between the Characteristic Embedding and Morphisms: Indeed, we have

$$\begin{aligned} [f_* \circ \chi_X](x) &\stackrel{\text{def}}{=} f_*(\chi_X(x)) \\ &\stackrel{\text{def}}{=} f_*({x}) \\ &= \{f(x)\} \\ &\stackrel{\text{def}}{=} \chi_{X'}(f(x)) \\ &\stackrel{\text{def}}{=} [\chi_{X'} \circ f](x), \end{aligned}$$

for each $x \in X$, showing the desired equality. \square

006K 4.2 The Yoneda Lemma for Sets

Let X be a set and let $U \subset X$ be a subset of X .

006L Proposition 4.2.1.1. We have

$$\chi_{\mathcal{P}(X)}(\chi_x, \chi_U) = \chi_U(x)$$

for each $x \in X$, giving an equality of functions

$$\chi_{\mathcal{P}(X)}(\chi_{(-)}, \chi_U) = \chi_U.$$

Proof. Clear. \square

006M Corollary 4.2.1.2. The characteristic embedding is fully faithful, i.e., we have

$$\chi_{\mathcal{P}(X)}(\chi_x, \chi_y) = \chi_X(x, y)$$

for each $x, y \in X$.

Proof. This follows from **Proposition 4.2.1.1**. \square

006N 4.3 Powersets

Let X be a set.

006P Definition 4.3.1.1. The **powerset** of X is the set $\mathcal{P}(X)$ defined by

$$\mathcal{P}(X) \stackrel{\text{def}}{=} \{U \in P \mid U \subset X\},$$

where P is the set in the axiom of powerset, ?? of ??.

006Q Remark 4.3.1.2. The powerset of a set is a decategorification of the category of presheaves of a category: while³²

- The powerset of a set X is equivalently (Items 1 and 2 of Proposition 4.3.1.6) the set

$$\mathbf{Sets}(X, \{\mathbf{t}, \mathbf{f}\})$$

of functions from X to the set $\{\mathbf{t}, \mathbf{f}\}$ of classical truth values.

- The category of presheaves on a category \mathcal{C} is the category

$$\mathbf{Fun}(\mathcal{C}^{\text{op}}, \mathbf{Sets})$$

of functors from \mathcal{C}^{op} to the category \mathbf{Sets} of sets.

006R Proposition 4.3.1.3. Let X be a set.

006S 1. *Co/Completeness.* The (posetal) category (associated to) $(\mathcal{P}(X), \subset)$ is complete and cocomplete:

- (a) *Products.* The products in $\mathcal{P}(X)$ are given by intersection of subsets.

³²This parallel is based on the following comparison:

- A category is enriched over the category

$$\mathbf{Sets} \stackrel{\text{def}}{=} \mathbf{Cats}_0$$

of sets (i.e. “0-categories”), with presheaves taking values on it.

- A set is enriched over the set

$$\{\mathbf{t}, \mathbf{f}\} \stackrel{\text{def}}{=} \mathbf{Cats}_{-1}$$

of classical truth values (i.e. “(−1)-categories”), with characteristic functions taking values on it.

- (b) *Coproducts*. The coproducts in $\mathcal{P}(X)$ are given by union of subsets.
- (c) *Co/Equalisers*. Being a posetal category, $\mathcal{P}(X)$ only has at most one morphisms between any two objects, so co/equalisers are trivial.

006T 2. *Cartesian Closedness*. The category $\mathcal{P}(X)$ is Cartesian closed with internal Hom

$$\mathbf{Hom}_{\mathcal{P}(X)}(-_1, -_2): \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X) \rightarrow \mathcal{P}(X)$$

given by³³

$$\mathbf{Hom}_{\mathcal{P}(X)}(U, V) \stackrel{\text{def}}{=} (X \setminus U) \cup V$$

for each $U, V \in \text{Obj}(\mathcal{P}(X))$.

Proof. **Item 1**, *Co/Completeness*: Clear.

Item 2, *Cartesian Closedness*: This follows from **Item 2** of **Proposition 3.9.1.2**. □

006U **Proposition 4.3.1.4**. Let X be a set.

006V 1. *Functoriality I*. The assignment $X \mapsto \mathcal{P}(X)$ defines a functor

$$\mathcal{P}_*: \mathbf{Sets} \rightarrow \mathbf{Sets},$$

where

- *Action on Objects*. For each $A \in \text{Obj}(\mathbf{Sets})$, we have

$$\mathcal{P}_*(A) \stackrel{\text{def}}{=} \mathcal{P}(A).$$

- *Action on Morphisms*. For each $A, B \in \text{Obj}(\mathbf{Sets})$, the action on morphisms

$$\mathcal{P}_{*|A,B}: \mathbf{Sets}(A, B) \rightarrow \mathbf{Sets}(\mathcal{P}(A), \mathcal{P}(B))$$

of \mathcal{P}_* at (A, B) is the map defined by sending a map of sets $f: A \rightarrow B$ to the map

$$\mathcal{P}_*(f): \mathcal{P}(A) \rightarrow \mathcal{P}(B)$$

defined by

$$\mathcal{P}_*(f) \stackrel{\text{def}}{=} f_*,$$

as in **Definition 4.4.1.1**.

³³For intuition regarding the expression defining $\mathbf{Hom}_{\mathcal{P}(X)}(U, V)$, see **Remark 3.9.1.3**.

- 006W 2. *Functoriality II.* The assignment $X \mapsto \mathcal{P}(X)$ defines a functor

$$\mathcal{P}^{-1} : \mathbf{Sets}^{\text{op}} \rightarrow \mathbf{Sets},$$

where

- *Action on Objects.* For each $A \in \text{Obj}(\mathbf{Sets})$, we have

$$\mathcal{P}^{-1}(A) \stackrel{\text{def}}{=} \mathcal{P}(A).$$

- *Action on Morphisms.* For each $A, B \in \text{Obj}(\mathbf{Sets})$, the action on morphisms

$$\mathcal{P}_{A,B}^{-1} : \mathbf{Sets}(A, B) \rightarrow \mathbf{Sets}(\mathcal{P}(B), \mathcal{P}(A))$$

of \mathcal{P}^{-1} at (A, B) is the map defined by sending a map of sets $f : A \rightarrow B$ to the map

$$\mathcal{P}^{-1}(f) : \mathcal{P}(B) \rightarrow \mathcal{P}(A)$$

defined by

$$\mathcal{P}^{-1}(f) \stackrel{\text{def}}{=} f^{-1},$$

as in [Definition 4.5.1.1](#).

- 006X 3. *Functoriality III.* The assignment $X \mapsto \mathcal{P}(X)$ defines a functor

$$\mathcal{P}_! : \mathbf{Sets} \rightarrow \mathbf{Sets},$$

where

- *Action on Objects.* For each $A \in \text{Obj}(\mathbf{Sets})$, we have

$$\mathcal{P}_!(A) \stackrel{\text{def}}{=} \mathcal{P}(A).$$

- *Action on Morphisms.* For each $A, B \in \text{Obj}(\mathbf{Sets})$, the action on morphisms

$$\mathcal{P}_{!A,B} : \mathbf{Sets}(A, B) \rightarrow \mathbf{Sets}(\mathcal{P}(A), \mathcal{P}(B))$$

of $\mathcal{P}_!$ at (A, B) is the map defined by sending a map of sets $f : A \rightarrow B$ to the map

$$\mathcal{P}_!(f) : \mathcal{P}(A) \rightarrow \mathcal{P}(B)$$

defined by

$$\mathcal{P}_!(f) \stackrel{\text{def}}{=} f_!,$$

as in [Definition 4.6.1.1](#).

006Y 4. *Adjointness I.* We have an adjunction

$$(\mathcal{P}^{-1} \dashv \mathcal{P}^{-1, \text{op}}): \text{Sets}^{\text{op}} \begin{array}{c} \xrightarrow{\mathcal{P}^{-1}} \\ \perp \\ \xleftarrow{\mathcal{P}^{-1, \text{op}}} \end{array} \text{Sets},$$

witnessed by a bijection

$$\underbrace{\text{Sets}^{\text{op}}(\mathcal{P}(A), B)}_{\stackrel{\text{def}}{=} \text{Sets}(B, \mathcal{P}(A))} \cong \text{Sets}(A, \mathcal{P}(B)),$$

natural in $A \in \text{Obj}(\text{Sets})$ and $B \in \text{Obj}(\text{Sets}^{\text{op}})$.

006Z 5. *Adjointness II.* We have an adjunction

$$(\text{Gr} \dashv \mathcal{P}_*): \text{Sets} \begin{array}{c} \xrightarrow{\text{Gr}} \\ \perp \\ \xleftarrow{\mathcal{P}_*} \end{array} \text{Rel},$$

witnessed by a bijection of sets

$$\text{Rel}(\text{Gr}(A), B) \cong \text{Sets}(A, \mathcal{P}(B))$$

natural in $A \in \text{Obj}(\text{Sets})$ and $B \in \text{Obj}(\text{Rel})$, where Gr is the graph functor of **Constructions With Relations**, **Item 1** of **Proposition 3.1.1.2** and \mathcal{P}_* is the functor of **Constructions With Relations**, **Proposition 4.5.1.1**.

Proof. Item 1, Functoriality I: This follows from **Items 3** and **4** of **Proposition 4.4.1.5**.

Item 2, Functoriality II: This follows **Items 3** and **4** of **Proposition 4.5.1.4**.

Item 3, Functoriality III: This follows **Items 3** and **4** of **Proposition 4.6.1.7**.

Item 4, Adjointness I: We have

$$\begin{aligned} \text{Sets}^{\text{op}}(\mathcal{P}(A), B) &\stackrel{\text{def}}{=} \text{Sets}(B, \mathcal{P}(A)) \\ &\cong \text{Sets}(B, \text{Sets}(A, \{\text{t}, \text{f}\})) && \text{(by Item 1 of Proposition 4.3.1.6)} \\ &\cong \text{Sets}(A \times B, \{\text{t}, \text{f}\}) && \text{(by Item 2 of Proposition 1.3.1.2)} \\ &\cong \text{Sets}(A, \text{Sets}(B, \{\text{t}, \text{f}\})) && \text{(by Item 2 of Proposition 1.3.1.2)} \\ &\cong \text{Sets}(A, \mathcal{P}(B)) && \text{(by Item 1 of Proposition 4.3.1.6)} \end{aligned}$$

with all bijections natural in A and B (where we use [Item 2 of Proposition 4.3.1.6](#) here).

Item 5, Adjointness II: We have

$$\begin{aligned}
 \text{Rel}(\text{Gr}(A), B) &\cong \mathcal{P}(A \times B) \\
 &\cong \text{Sets}(A \times B, \{\text{t}, \text{f}\}) \quad (\text{by } \text{Item 1 of Proposition 4.3.1.6}) \\
 &\cong \text{Sets}(A, \text{Sets}(B, \{\text{t}, \text{f}\})) \quad (\text{by } \text{Item 2 of Proposition 1.3.1.2}) \\
 &\cong \text{Sets}(A, \mathcal{P}(B)) \quad (\text{by } \text{Item 1 of Proposition 4.3.1.6})
 \end{aligned}$$

with all bijections natural in A (where we use [Item 2 of Proposition 4.3.1.6](#) here). Explicitly, this isomorphism is given by sending a relation $R: \text{Gr}(A) \dashv B$ to the map $R^\dagger: A \rightarrow \mathcal{P}(B)$ sending a to the subset $R(a)$ of B , as in [Relations, Remark 1.1.1.4](#).

Naturality in B is then the statement that given a relation $R: B \dashv B'$, the diagram

$$\begin{array}{ccc}
 \text{Rel}(\text{Gr}(A), B) & \xrightarrow{R \diamond -} & \text{Rel}(\text{Gr}(A), B') \\
 \downarrow \wr & & \downarrow \wr \\
 \text{Sets}(A, \mathcal{P}(B)) & \xrightarrow{R_*} & \text{Sets}(A, \mathcal{P}(B'))
 \end{array}$$

commutes, which follows from [Constructions With Relations, Remark 4.1.1.2](#). \square

0070 Proposition 4.3.1.5. Let X be a set.

- 0071** 1. *Symmetric Strong Monoidality With Respect to Coproducts I.* The powerset functor \mathcal{P}_* of [Item 1 of Proposition 4.3.1.4](#) has a symmetric strong monoidal structure

$$\left(\mathcal{P}_*, \mathcal{P}_*^{\amalg}, \mathcal{P}_{*|\mathbb{1}}^{\amalg} \right): (\text{Sets}, \times, \text{pt}) \rightarrow (\text{Sets}, \amalg, \emptyset)$$

being equipped with isomorphisms

$$\begin{aligned}
 \mathcal{P}_{*|X,Y}^{\amalg}: \mathcal{P}(X) \times \mathcal{P}(Y) &\xrightarrow{\cong} \mathcal{P}(X \amalg Y), \\
 \mathcal{P}_{*|\mathbb{1}}^{\amalg}: \text{pt} &\xrightarrow{\cong} \mathcal{P}(\emptyset),
 \end{aligned}$$

natural in $X, Y \in \text{Obj}(\text{Sets})$.

- 0072** 2. *Symmetric Strong Monoidality With Respect to Coproducts II.* The powerset functor \mathcal{P}^{-1} of **Item 2** of **Proposition 4.3.1.4** has a symmetric strong monoidal structure

$$\left(\mathcal{P}^{-1}, \mathcal{P}^{-1}\amalg, \mathcal{P}_{\mathbb{1}}^{-1}\amalg\right): (\mathbf{Sets}^{\mathrm{op}}, \times^{\mathrm{op}}, \mathrm{pt}) \rightarrow (\mathbf{Sets}, \amalg, \emptyset)$$

being equipped with isomorphisms

$$\begin{aligned} \mathcal{P}_{X,Y}^{-1}\amalg: \mathcal{P}(X) \times \mathcal{P}(Y) &\xrightarrow{\cong} \mathcal{P}(X \amalg Y), \\ \mathcal{P}_{\mathbb{1}}^{-1}\amalg: \mathrm{pt} &\xrightarrow{\cong} \mathcal{P}(\emptyset), \end{aligned}$$

natural in $X, Y \in \mathrm{Obj}(\mathbf{Sets})$.

- 0073** 3. *Symmetric Strong Monoidality With Respect to Coproducts III.* The powerset functor $\mathcal{P}_!$ of **Item 3** of **Proposition 4.3.1.4** has a symmetric strong monoidal structure

$$\left(\mathcal{P}_!, \mathcal{P}_!\amalg, \mathcal{P}_{\mathbb{1}}!\amalg\right): (\mathbf{Sets}, \times, \mathrm{pt}) \rightarrow (\mathbf{Sets}, \amalg, \emptyset)$$

being equipped with isomorphisms

$$\begin{aligned} \mathcal{P}_{\mathbb{1}|X,Y}!\amalg: \mathcal{P}(X) \times \mathcal{P}(Y) &\xrightarrow{\cong} \mathcal{P}(X \amalg Y), \\ \mathcal{P}_{\mathbb{1}}!\amalg: \mathrm{pt} &\xrightarrow{\cong} \mathcal{P}(\emptyset), \end{aligned}$$

natural in $X, Y \in \mathrm{Obj}(\mathbf{Sets})$.

- 0074** 4. *Symmetric Lax Monoidality With Respect to Products I.* The powerset functor \mathcal{P}_* of **Item 1** of **Proposition 4.3.1.4** has a symmetric lax monoidal structure

$$\left(\mathcal{P}_*, \mathcal{P}_*^{\otimes}, \mathcal{P}_{*\mathbb{1}}^{\otimes}\right): (\mathbf{Sets}, \times, \mathrm{pt}) \rightarrow (\mathbf{Sets}, \times, \mathrm{pt})$$

being equipped with morphisms

$$\begin{aligned} \mathcal{P}_{*|X,Y}^{\times}: \mathcal{P}(X) \times \mathcal{P}(Y) &\rightarrow \mathcal{P}(X \times Y), \\ \mathcal{P}_{*\mathbb{1}}^{\times}: \mathrm{pt} &\rightarrow \mathcal{P}(\mathrm{pt}), \end{aligned}$$

natural in $X, Y \in \mathrm{Obj}(\mathbf{Sets})$, where

- The map $\mathcal{P}_{*|X,Y}^\times$ is given by

$$\mathcal{P}_{*|X,Y}^\times(U, V) \stackrel{\text{def}}{=} U \times V$$

for each $(U, V) \in \mathcal{P}(X) \times \mathcal{P}(Y)$,

- The map $\mathcal{P}_{*|\mathbb{1}}^\times$ is given by

$$\mathcal{P}_{*|\mathbb{1}}^\times(\star) = \text{pt}.$$

- 0075** 5. *Symmetric Lax Monoidality With Respect to Products II.* The powerset functor \mathcal{P}^{-1} of **Item 2** of **Proposition 4.3.1.4** has a symmetric lax monoidal structure

$$(\mathcal{P}^{-1}, \mathcal{P}^{-1|\otimes}, \mathcal{P}_{\mathbb{1}}^{-1|\otimes}) : (\mathbf{Sets}^{\text{op}}, \times^{\text{op}}, \text{pt}) \rightarrow (\mathbf{Sets}, \times, \text{pt})$$

being equipped with morphisms

$$\begin{aligned} \mathcal{P}_{X,Y}^{-1|\times} : \mathcal{P}(X) \times \mathcal{P}(Y) &\rightarrow \mathcal{P}(X \times Y), \\ \mathcal{P}_{\mathbb{1}}^\times : \text{pt} &\rightarrow \mathcal{P}(\emptyset), \end{aligned}$$

natural in $X, Y \in \text{Obj}(\mathbf{Sets})$, defined as in **Item 4**.

- 0076** 6. *Symmetric Lax Monoidality With Respect to Products III.* The powerset functor $\mathcal{P}_!$ of **Item 3** of **Proposition 4.3.1.4** has a symmetric lax monoidal structure

$$(\mathcal{P}_!, \mathcal{P}_!^\otimes, \mathcal{P}_{!|\mathbb{1}}^\otimes) : (\mathbf{Sets}, \times, \text{pt}) \rightarrow (\mathbf{Sets}, \times, \text{pt})$$

being equipped with morphisms

$$\begin{aligned} \mathcal{P}_{!|X,Y}^\times : \mathcal{P}(X) \times \mathcal{P}(Y) &\rightarrow \mathcal{P}(X \times Y), \\ \mathcal{P}_{!|\mathbb{1}}^\times : \text{pt} &\rightarrow \mathcal{P}(\emptyset), \end{aligned}$$

natural in $X, Y \in \text{Obj}(\mathbf{Sets})$, defined as in **Item 4**.

Proof. **Item 1**, *Symmetric Strong Monoidality With Respect to Coproducts I*: The isomorphism

$$\mathcal{P}_{*|X,Y}^{\coprod} : \mathcal{P}(X) \times \mathcal{P}(Y) \rightarrow \mathcal{P}(X \coprod Y)$$

is given by sending $(U, V) \in \mathcal{P}(X) \times \mathcal{P}(Y)$ to $U \amalg V$, with inverse given by sending a subset S of $X \amalg Y$ to the pair $(S_X, S_Y) \in \mathcal{P}(X) \times \mathcal{P}(Y)$ with

$$\begin{aligned} S_X &\stackrel{\text{def}}{=} \{x \in X \mid (0, x) \in S\} \\ S_Y &\stackrel{\text{def}}{=} \{y \in Y \mid (1, y) \in S\}. \end{aligned}$$

The isomorphism $\text{pt} \cong \mathcal{P}(\emptyset)$ is given by $\star \mapsto \emptyset \in \mathcal{P}(\emptyset)$.

Naturality for the isomorphism $\mathcal{P}_{*|X,Y}^{\amalg}$ is the statement that, given maps of sets $f: X \rightarrow X'$ and $g: Y \rightarrow Y'$, the diagram

$$\begin{array}{ccc} \mathcal{P}(X) \times \mathcal{P}(Y) & \xrightarrow{f_* \times g_*} & \mathcal{P}(X') \times \mathcal{P}(Y') \\ \downarrow \wr & & \downarrow \wr \\ \mathcal{P}(X \amalg Y) & \xrightarrow{(f \amalg g)_*} & \mathcal{P}(X' \amalg Y') \end{array}$$

commutes, which is clear, as it acts on elements as

$$\begin{array}{ccc} (U, V) & \longmapsto & (f_*(U), g_*(V)) \\ \downarrow & & \downarrow \\ U \amalg V & \mapsto & (f \amalg g)_*(U \amalg V) = f_*(U) \amalg g_*(V), \end{array}$$

where we are using [Item 7 of Proposition 4.4.1.4](#).

Finally, monoidality, unity, and symmetry of \mathcal{P}_* as a monoidal functor follow by checking the commutativity of the relevant diagrams on elements.

[Item 2, Symmetric Strong Monoidality With Respect to Coproducts II](#): The proof is similar to [Item 1](#), and is hence omitted.

[Item 3, Symmetric Strong Monoidality With Respect to Coproducts III](#): The proof is similar to [Item 1](#), and is hence omitted.

[Item 4, Symmetric Lax Monoidality With Respect to Products I](#): Naturality for the morphism $\mathcal{P}_{*|X,Y}^{\times}$ is the statement that, given maps of sets $f: X \rightarrow X'$ and $g: Y \rightarrow Y'$, the diagram

$$\begin{array}{ccc} \mathcal{P}(X) \times \mathcal{P}(Y) & \xrightarrow{f_* \times g_*} & \mathcal{P}(X') \times \mathcal{P}(Y') \\ \downarrow \wr & & \downarrow \wr \\ \mathcal{P}(X \times Y) & \xrightarrow{(f \times g)_*} & \mathcal{P}(X' \times Y') \end{array}$$

commutes, which is clear, as it acts on elements as

$$\begin{array}{ccc} (U, V) & \longmapsto & (f_*(U), g_*(V)) \\ \downarrow & & \downarrow \\ U \times V & \mapsto & (f \times g)_*(U \times V) = f_*(U) \times g_*(V), \end{array}$$

where we are using **Item 8** of **Proposition 4.4.1.4**.

Finally, monoidality, unity, and symmetry of \mathcal{P}_* as a monoidal functor follow by checking the commutativity of the relevant diagrams on elements.

Item 5, *Symmetric Lax Monoidality With Respect to Products II*: The proof is similar to **Item 4**, and is hence omitted.

Item 6, *Symmetric Lax Monoidality With Respect to Products III*: The proof is similar to **Item 4**, and is hence omitted. \square

0077 **Proposition 4.3.1.6.** Let X be a set.

0078 1. *Powersets as Sets of Functions I.* The assignment $U \mapsto \chi_U$ defines a bijection

$$\chi_{(-)}: \mathcal{P}(X) \xrightarrow{\cong} \text{Sets}(X, \{\mathbf{t}, \mathbf{f}\}),$$

for each $X \in \text{Obj}(\text{Sets})$.

0079 2. *Powersets as Sets of Functions II.* The bijection

$$\mathcal{P}(X) \cong \text{Sets}(X, \{\mathbf{t}, \mathbf{f}\})$$

of **Item 1** is natural in $X \in \text{Obj}(\text{Sets})$, refining to a natural isomorphism of functors

$$\mathcal{P}^{-1} \cong \text{Sets}(-, \{\mathbf{t}, \mathbf{f}\}).$$

007A 3. *Powersets as Sets of Relations.* We have bijections

$$\mathcal{P}(X) \cong \text{Rel}(\text{pt}, X),$$

$$\mathcal{P}(X) \cong \text{Rel}(X, \text{pt}),$$

natural in $X \in \text{Obj}(\text{Sets})$.

Proof. **Item 1**, *Powersets as Sets of Functions I*: Indeed, the inverse of $\chi_{(-)}$ is given by sending a function $f: X \rightarrow \{\mathbf{t}, \mathbf{f}\}$ to the subset U_f of $\mathcal{P}(X)$ defined by

$$U_f \stackrel{\text{def}}{=} \{x \in X \mid f(x) = \mathbf{true}\},$$

i.e. by $U_f = f^{-1}(\text{true})$. That $\chi_{(-)}$ and $f \mapsto U_f$ are inverses is then straightforward to check.

Item 2, Powersets as Sets of Functions II: We need to check that, given a function $f: X \rightarrow Y$, the diagram

$$\begin{array}{ccc} \mathcal{P}(Y) & \xrightarrow{f^{-1}} & \mathcal{P}(X) \\ \chi_{(-)} \downarrow \wr & & \wr \downarrow \chi_{(-)} \\ \text{Sets}(Y, \{\mathbf{t}, \mathbf{f}\}) & \xrightarrow{f^*} & \text{Sets}(X, \{\mathbf{t}, \mathbf{f}\}) \end{array}$$

commutes, i.e. that for each $V \in \mathcal{P}(Y)$, we have

$$\chi_V \circ f = \chi_{f^{-1}(V)}.$$

And indeed, we have

$$\begin{aligned} [\chi_V \circ f](v) &\stackrel{\text{def}}{=} \chi_V(f(v)) \\ &= \begin{cases} \text{true} & \text{if } f(v) \in V, \\ \text{false} & \text{otherwise} \end{cases} \\ &= \begin{cases} \text{true} & \text{if } v \in f^{-1}(V), \\ \text{false} & \text{otherwise} \end{cases} \\ &\stackrel{\text{def}}{=} \chi_{f^{-1}(V)}(v) \end{aligned}$$

for each $v \in V$.

Item 3, Powersets as Sets of Relations: Indeed, we have

$$\begin{aligned} \text{Rel}(\text{pt}, X) &\stackrel{\text{def}}{=} \mathcal{P}(\text{pt} \times X) \\ &\cong \mathcal{P}(X) \end{aligned}$$

and

$$\begin{aligned} \text{Rel}(X, \text{pt}) &\stackrel{\text{def}}{=} \mathcal{P}(X \times \text{pt}) \\ &\cong \mathcal{P}(X), \end{aligned}$$

where we have used *Item 4 of Proposition 1.3.1.2*. □

007B Remark 4.3.1.7. The bijection

$$\mathcal{P}(X) \cong \text{Sets}(X, \{\mathbf{t}, \mathbf{f}\})$$

of *Item 1 of Proposition 4.3.1.6*, which

- Takes a subset $U \hookrightarrow X$ of X and *straightens* it to a function $\chi_U: X \rightarrow \{\text{true}, \text{false}\}$;
- Takes a function $f: X \rightarrow \{\text{true}, \text{false}\}$ and *unstraightens* it to a subset $f^{-1}(\text{true}) \hookrightarrow X$ of X ;

may be viewed as the (-1) -categorical version of the un/straightening isomorphism for indexed and fibred sets

$$\underbrace{\text{FibSets}(X)}_{\stackrel{\text{def}}{=} \text{Sets}_{/X}} \cong \underbrace{\text{ISets}(X)}_{\stackrel{\text{def}}{=} \text{Fun}(X_{\text{disc}}, \text{Sets})}$$

of ??, ??, where we view:

- Subsets $U \hookrightarrow X$ as analogous to X -fibred sets $\phi_X: A \rightarrow X$.
- Functions $f: X \rightarrow \{\text{t}, \text{f}\}$ as analogous to X -indexed sets $A: X_{\text{disc}} \rightarrow \text{Sets}$.

007C Proposition 4.3.1.8. Let X be a set.

007D 1. *Universal Property.* The pair $(\mathcal{P}(X), \chi_{(-)})$ consisting of

- The powerset $\mathcal{P}(X)$ of X ;
- The characteristic embedding $\chi_{(-)}: X \hookrightarrow \mathcal{P}(X)$ of X into $\mathcal{P}(X)$;

satisfies the following universal property:

- (\star) Given another pair (Y, f) consisting of
- A cocomplete poset (Y, \preceq) ;
 - A function $f: X \rightarrow Y$;

there exists a unique cocontinuous morphism of posets

$$(\mathcal{P}(X), \subset) \xrightarrow{\exists!} (Y, \preceq)$$

making the diagram

$$\begin{array}{ccc} & & \mathcal{P}(X) \\ & \nearrow \chi_X & \downarrow \exists! \\ X & \xrightarrow{f} & Y \end{array}$$

commute.

007E 2. *Adjointness.* We have an adjunction³⁴

$$(\mathcal{P} \dashv \mathfrak{L}): \mathbf{Sets} \begin{array}{c} \xrightarrow{\mathcal{P}} \\ \perp \\ \xleftarrow{\mathfrak{L}} \end{array} \mathbf{Pos}^{\text{cocomp.}},$$

witnessed by a bijection

$$\mathbf{Pos}^{\text{cocomp.}}((\mathcal{P}(X), \subset), (Y, \preceq)) \cong \mathbf{Sets}(X, Y),$$

natural in $X \in \mathbf{Obj}(\mathbf{Sets})$ and $(Y, \preceq) \in \mathbf{Obj}(\mathbf{Pos}^{\text{cocomp.}})$, where the maps witnessing this bijection are given by

- The map

$$\chi_X^*: \mathbf{Pos}^{\text{cocomp.}}((\mathcal{P}(X), \subset), (Y, \preceq)) \rightarrow \mathbf{Sets}(X, Y)$$

defined by

$$\chi_X^*(f) \stackrel{\text{def}}{=} f \circ \chi_X,$$

i.e. by sending a cocontinuous morphism of posets $f: \mathcal{P}(X) \rightarrow Y$ to the composition

$$X \xrightarrow{\chi_X} \mathcal{P}(X) \xrightarrow{f} Y.$$

- The map

$$\mathbf{Lan}_{\chi_X}: \mathbf{Sets}(X, Y) \rightarrow \mathbf{Pos}^{\text{cocomp.}}((\mathcal{P}(X), \subset), (Y, \preceq))$$

is given by sending a function $f: X \rightarrow Y$ to its left Kan extension along χ_X ,

$$\mathbf{Lan}_{\chi_X}(f): \mathcal{P}(X) \rightarrow Y, \quad \begin{array}{ccc} & \mathcal{P}(X) & \\ \chi_X \nearrow & \uparrow & \downarrow \mathbf{Lan}_{\chi_X}(f) \\ X & \xrightarrow{f} & Y. \end{array}$$

³⁴In this sense, $\mathcal{P}(A)$ is the free cocompletion of A . (Note that, despite its name, however, this is not an idempotent operation, as we have $\mathcal{P}(\mathcal{P}(A)) \neq \mathcal{P}(A)$.)

Moreover, $\text{Lan}_{\chi_X}(f)$ can be explicitly computed by

$$\begin{aligned} [\text{Lan}_{\chi_X}(f)](U) &\cong \int^{x \in X} \chi_{\mathcal{P}(X)}(\chi_x, U) \odot f(x) \\ &\cong \int^{x \in X} \chi_U(x) \odot f(x) && \text{(by Proposition 4.2.1.1)} \\ &\cong \bigvee_{x \in X} (\chi_U(x) \odot f(x)) \end{aligned}$$

for each $U \in \mathcal{P}(X)$, where:

- \bigvee is the join in (Y, \preceq) .
- We have

$$\begin{aligned} \text{true} \odot f(x) &\stackrel{\text{def}}{=} f(x), \\ \text{false} \odot f(x) &\stackrel{\text{def}}{=} \emptyset_Y, \end{aligned}$$

where \emptyset_Y is the minimal element of (Y, \preceq) .

Proof. Item 1, Universal Property: This is a rephrasing of **Item 2**.

Item 2, Adjointness: We claim we have adjunction $\mathcal{P} \dashv \mathfrak{Lan}$, witnessed by a bijection

$$\text{Pos}^{\text{cocomp}}((\mathcal{P}(X), \subset), (Y, \preceq)) \cong \text{Sets}(X, Y),$$

natural in $X \in \text{Obj}(\text{Sets})$ and $(Y, \preceq) \in \text{Obj}(\text{Pos}^{\text{cocomp}})$.

- *Map I.* We define a map

$$\Phi_{X,Y} : \text{Pos}^{\text{cocomp}}((\mathcal{P}(X), \subset), (Y, \preceq)) \rightarrow \text{Sets}(X, Y)$$

as in the statement, by

$$\Phi_{X,Y}(f) \stackrel{\text{def}}{=} f \circ \chi_X$$

for each $f \in \text{Pos}^{\text{cocomp}}((\mathcal{P}(X), \subset), (Y, \preceq))$.

- *Map II.* We define a map

$$\Psi_{X,Y} : \text{Sets}(X, Y) \rightarrow \text{Pos}^{\text{cocomp}}((\mathcal{P}(X), \subset), (Y, \preceq))$$

as in the statement, by

$$\Psi_{X,Y}(f) \stackrel{\text{def}}{=} \text{Lan}_{\chi_X}(f),$$

for each $f \in \text{Sets}(X, Y)$.

- *Invertibility I.* We claim that

$$\Psi_{X,Y} \circ \Phi_{X,Y} = \text{id}_{\mathbf{Pos}^{\text{cocomp}}((\mathcal{P}(X), \subset), (Y, \preceq))}.$$

Indeed, given a cocontinuous morphism of posets

$$\xi: (\mathcal{P}(X), \subset) \rightarrow (Y, \preceq),$$

we have

$$\begin{aligned} [\Psi_{X,Y} \circ \Phi_{X,Y}](\xi) &\stackrel{\text{def}}{=} \Psi_{X,Y}(\Phi_{X,Y}(\xi)) \\ &\stackrel{\text{def}}{=} \Psi_{X,Y}(\xi \circ \chi_X) \\ &\stackrel{\text{def}}{=} \text{Lan}_{\chi_X}(\xi \circ \chi_X) \\ &\cong \bigvee_{x \in X} \chi_{(-)}(x) \odot \xi(\chi_X(x)) \\ &\stackrel{\text{clm}}{=} \xi, \end{aligned}$$

where indeed

$$\begin{aligned} \left[\bigvee_{x \in X} \chi_{(-)}(x) \odot \xi(\chi_X(x)) \right] (U) &\stackrel{\text{def}}{=} \bigvee_{x \in X} \chi_U(x) \odot \xi(\chi_X(x)) \\ &= \left(\bigvee_{x \in U} \chi_U(x) \odot \xi(\chi_X(x)) \right) \vee \left(\bigvee_{x \in X \setminus U} \chi_U(x) \odot \xi(\chi_X(x)) \right) \\ &= \left(\bigvee_{x \in U} \xi(\chi_X(x)) \right) \vee \left(\bigvee_{x \in X \setminus U} \emptyset_Y \right) \\ &= \bigvee_{x \in U} \xi(\chi_X(x)) \\ &\stackrel{(\dagger)}{=} \xi \left(\bigvee_{x \in U} \chi_X(x) \right) \\ &= \xi(U) \end{aligned}$$

for each $U \in \mathcal{P}(X)$, where we have used that ξ is cocontinuous for the equality $\stackrel{(\dagger)}{=}$.

- *Invertibility II.* We claim that

$$\Phi_{X,Y} \circ \Psi_{X,Y} = \text{id}_{\mathbf{Sets}(X,Y)}.$$

Indeed, given a function $f: X \rightarrow Y$, we have

$$\begin{aligned} [\Phi_{X,Y} \circ \Psi_{X,Y}](f) &\stackrel{\text{def}}{=} \Phi_{X,Y}(\Psi_{X,Y}(f)) \\ &\stackrel{\text{def}}{=} \Phi_{X,Y}(\text{Lan}_{\chi_X}(f)) \\ &\stackrel{\text{def}}{=} \text{Lan}_{\chi_X}(f) \circ \chi_X \\ &\stackrel{\text{clm}}{=} f, \end{aligned}$$

where indeed

$$\begin{aligned} [\text{Lan}_{\chi_X}(f) \circ \chi_X](x) &\stackrel{\text{def}}{=} \bigvee_{y \in X} \chi_{\{x\}}(y) \odot f(y) \\ &= (\chi_{\{x\}}(x) \odot f(x)) \vee \left(\bigvee_{y \in X \setminus \{x\}} \chi_{\{x\}}(y) \odot f(y) \right) \\ &= f(x) \vee \left(\bigvee_{y \in X \setminus \{x\}} \emptyset_Y \right) \\ &= f(x) \vee \emptyset_Y \\ &= f(x) \end{aligned}$$

for each $x \in X$.

- *Naturality for Φ , Part I.* We need to show that, given a function $f: X \rightarrow X'$, the diagram

$$\begin{array}{ccc} \text{Pos}^{\text{cocomp.}}((\mathcal{P}(X'), \subset), (Y, \preceq)) & \xrightarrow{\Phi_{X',Y}} & \text{Sets}(X', Y) \\ \mathcal{P}_*(f)^* \downarrow & & \downarrow f^* \\ \text{Pos}^{\text{cocomp.}}((\mathcal{P}(X), \subset), (Y, \preceq)) & \xrightarrow{\Phi_{X,Y}} & \text{Sets}(X, Y) \end{array}$$

commutes. Indeed, given a cocontinuous morphism of posets

$$\xi: (\mathcal{P}(X'), \subset) \rightarrow (Y, \preceq),$$

we have

$$\begin{aligned}
[\Phi_{X,Y} \circ \mathcal{P}_*(f)^*](\xi) &\stackrel{\text{def}}{=} \Phi_{X,Y}(\mathcal{P}_*(f)^*(\xi)) \\
&\stackrel{\text{def}}{=} \Phi_{X,Y}(\xi \circ f_*) \\
&\stackrel{\text{def}}{=} (\xi \circ f_*) \circ \chi_X \\
&= \xi \circ (f_* \circ \chi_X) \\
&\stackrel{(\dagger)}{=} \xi \circ (\chi_{X'} \circ f) \\
&= (\xi \circ \chi_{X'}) \circ f \\
&\stackrel{\text{def}}{=} \Phi_{X',Y}(\xi) \circ f \\
&\stackrel{\text{def}}{=} f^*(\Phi_{X',Y}(\xi)) \\
&\stackrel{\text{def}}{=} [f^* \circ \Phi_{X',Y}](\xi),
\end{aligned}$$

where we have used [Item 9](#) of [Proposition 4.1.1.3](#) for the equality $\stackrel{(\dagger)}{=}$ above.

- *Naturality for Φ , Part II.* We need to show that, given a cocontinuous morphism of posets

$$g: (Y, \preceq_Y) \rightarrow (Y', \preceq_{Y'}),$$

the diagram

$$\begin{array}{ccc}
\text{Pos}^{\text{cocomp}}((\mathcal{P}(X), \subset), (Y, \preceq)) & \xrightarrow{\Phi_{X,Y}} & \text{Sets}(X, Y) \\
\downarrow g_* & & \downarrow g_* \\
\text{Pos}^{\text{cocomp}}((\mathcal{P}(X), \subset), (Y', \preceq)) & \xrightarrow{\Phi_{X,Y'}} & \text{Sets}(X, Y')
\end{array}$$

commutes. Indeed, given a cocontinuous morphism of posets

$$\xi: (\mathcal{P}(X), \subset) \rightarrow (Y, \preceq),$$

we have

$$\begin{aligned}
[\Phi_{X,Y'} \circ g_*](\xi) &\stackrel{\text{def}}{=} \Phi_{X,Y'}(g_*(\xi)) \\
&\stackrel{\text{def}}{=} \Phi_{X,Y'}(g \circ \xi) \\
&\stackrel{\text{def}}{=} (g \circ \xi) \circ \chi_X \\
&= g \circ (\xi \circ \chi_X) \\
&\stackrel{\text{def}}{=} g \circ (\Phi_{X,Y}(\xi)) \\
&\stackrel{\text{def}}{=} g_*(\Phi_{X,Y}(\xi)) \\
&\stackrel{\text{def}}{=} [g_* \circ \Phi_{X,Y}](\xi).
\end{aligned}$$

- *Naturality for Ψ .* Since Φ is natural in each argument and Φ is a componentwise inverse to Ψ in each argument, it follows from **Categories**, **Item 2** of **Proposition 8.6.1.2** that Ψ is also natural in each argument.

This finishes the proof. \square

007F 4.4 Direct Images

Let A and B be sets and let $f: A \rightarrow B$ be a function.

007G **Definition 4.4.1.1.** The **direct image function associated to f** is the function

$$f_*: \mathcal{P}(A) \rightarrow \mathcal{P}(B)$$

defined by^{35,36}

$$\begin{aligned} f_*(U) &\stackrel{\text{def}}{=} f(U) \\ &\stackrel{\text{def}}{=} \left\{ b \in B \mid \begin{array}{l} \text{there exists some } a \in U \\ \text{such that } b = f(a) \end{array} \right\} \\ &= \{f(a) \in B \mid a \in U\} \end{aligned}$$

for each $U \in \mathcal{P}(A)$.

007H **Notation 4.4.1.2.** Sometimes one finds the notation

$$\exists_f: \mathcal{P}(A) \rightarrow \mathcal{P}(B)$$

for f_* . This notation comes from the fact that the following statements are equivalent, where $b \in B$ and $U \in \mathcal{P}(A)$:

- We have $b \in \exists_f(U)$.
- There exists some $a \in U$ such that $f(a) = b$.

007J **Remark 4.4.1.3.** Identifying subsets of A with functions from A to $\{\text{true}, \text{false}\}$ via **Items 1** and **2** of **Proposition 4.3.1.6**, we see that the direct image function associated to f is equivalently the function

$$f_*: \mathcal{P}(A) \rightarrow \mathcal{P}(B)$$

³⁵*Further Terminology:* The set $f(U)$ is called the **direct image of U by f** .

³⁶We also have

$$f_*(U) = B \setminus f_*(A \setminus U);$$

defined by

$$\begin{aligned}
 f_*(\chi_U) &\stackrel{\text{def}}{=} \text{Lan}_f(\chi_U) \\
 &= \text{colim} \left(\left(f \overset{\rightarrow}{\times} (-1) \right) \overset{\text{pr}}{\twoheadrightarrow} A \xrightarrow{\chi_U} \{\mathbf{t}, \mathbf{f}\} \right) \\
 &= \text{colim}_{\substack{a \in A \\ f(a) = -1}} (\chi_U(a)) \\
 &= \bigvee_{\substack{a \in A \\ f(a) = -1}} (\chi_U(a)),
 \end{aligned}$$

where we have used ??, ?? for the second equality. In other words, we have

$$\begin{aligned}
 [f_*(\chi_U)](b) &= \bigvee_{\substack{a \in A \\ f(a) = b}} (\chi_U(a)) \\
 &= \begin{cases} \text{true} & \text{if there exists some } a \in A \text{ such} \\ & \text{that } f(a) = b \text{ and } a \in U, \\ \text{false} & \text{otherwise} \end{cases} \\
 &= \begin{cases} \text{true} & \text{if there exists some } a \in U \\ & \text{such that } f(a) = b, \\ \text{false} & \text{otherwise} \end{cases}
 \end{aligned}$$

for each $b \in B$.

007K Proposition 4.4.1.4. Let $f: A \rightarrow B$ be a function.

007L 1. *Functoriality.* The assignment $U \mapsto f_*(U)$ defines a functor

$$f_*: (\mathcal{P}(A), \subset) \rightarrow (\mathcal{P}(B), \subset)$$

where

- *Action on Objects.* For each $U \in \mathcal{P}(A)$, we have

$$[f_*](U) \stackrel{\text{def}}{=} f_*(U).$$

- *Action on Morphisms.* For each $U, V \in \mathcal{P}(A)$:

$$(\star) \text{ If } U \subset V, \text{ then } f_*(U) \subset f_*(V).$$

see Item 9 of Proposition 4.4.1.4.

007M 2. *Triple Adjointness.* We have a triple adjunction

$$(f_* \dashv f^{-1} \dashv f_!): \quad \mathcal{P}(A) \begin{array}{c} \xrightarrow{f_*} \\ \perp \\ \xleftarrow{f^{-1}} \\ \perp \\ \xrightarrow{f_!} \end{array} \mathcal{P}(B),$$

witnessed by bijections of sets

$$\begin{aligned} \mathrm{Hom}_{\mathcal{P}(B)}(f_*(U), V) &\cong \mathrm{Hom}_{\mathcal{P}(A)}(U, f^{-1}(V)), \\ \mathrm{Hom}_{\mathcal{P}(A)}(f^{-1}(U), V) &\cong \mathrm{Hom}_{\mathcal{P}(A)}(U, f_!(V)), \end{aligned}$$

natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$ and (respectively) $V \in \mathcal{P}(A)$ and $U \in \mathcal{P}(B)$, i.e. where:

- (a) The following conditions are equivalent:
 - i. We have $f_*(U) \subset V$.
 - ii. We have $U \subset f^{-1}(V)$.
- (b) The following conditions are equivalent:
 - i. We have $f^{-1}(U) \subset V$.
 - ii. We have $U \subset f_!(V)$.

007N 3. *Preservation of Colimits.* We have an equality of sets

$$f_*\left(\bigcup_{i \in I} U_i\right) = \bigcup_{i \in I} f_*(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have equalities

$$\begin{aligned} f_*(U) \cup f_*(V) &= f_*(U \cup V), \\ f_*(\emptyset) &= \emptyset, \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

007P 4. *Oplax Preservation of Limits.* We have an inclusion of sets

$$f_*\left(\bigcap_{i \in I} U_i\right) \subset \bigcap_{i \in I} f_*(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have inclusions

$$\begin{aligned} f_*(U \cap V) &\subset f_*(U) \cap f_*(V), \\ f_*(A) &\subset B, \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

- 007Q** 5. *Symmetric Strict Monoidality With Respect to Unions.* The direct image function of **Item 1** has a symmetric strict monoidal structure

$$(f_*, f_*^\otimes, f_{*|\mathbb{1}}^\otimes): (\mathcal{P}(A), \cup, \emptyset) \rightarrow (\mathcal{P}(B), \cup, \emptyset),$$

being equipped with equalities

$$\begin{aligned} f_{*|U,V}^\otimes: f_*(U) \cup f_*(V) &\xrightarrow{=} f_*(U \cup V), \\ f_{*|\mathbb{1}}^\otimes: \emptyset &\xrightarrow{=} \emptyset, \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

- 007R** 6. *Symmetric Oplax Monoidality With Respect to Intersections.* The direct image function of **Item 1** has a symmetric oplax monoidal structure

$$(f_*, f_*^\otimes, f_{*|\mathbb{1}}^\otimes): (\mathcal{P}(A), \cap, A) \rightarrow (\mathcal{P}(B), \cap, B),$$

being equipped with inclusions

$$\begin{aligned} f_{*|U,V}^\otimes: f_*(U \cap V) &\hookrightarrow f_*(U) \cap f_*(V), \\ f_{*|\mathbb{1}}^\otimes: f_*(A) &\hookrightarrow B, \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

- 007S** 7. *Interaction With Coproducts.* Let $f: A \rightarrow A'$ and $g: B \rightarrow B'$ be maps of sets. We have

$$(f \amalg g)_*(U \amalg V) = f_*(U) \amalg g_*(V)$$

for each $U \in \mathcal{P}(A)$ and each $V \in \mathcal{P}(B)$.

- 007T** 8. *Interaction With Products.* Let $f: A \rightarrow A'$ and $g: B \rightarrow B'$ be maps of sets. We have

$$(f \times g)_*(U \times V) = f_*(U) \times g_*(V)$$

for each $U \in \mathcal{P}(A)$ and each $V \in \mathcal{P}(B)$.

007U 9. *Relation to Direct Images With Compact Support.* We have

$$f_*(U) = B \setminus f_!(A \setminus U)$$

for each $U \in \mathcal{P}(A)$.

Proof. **Item 1, Functoriality:** Clear.

Item 2, Triple Adjointness: This follows from **Remark 4.4.1.3**, **Remark 4.5.1.2**, **Remark 4.6.1.3**, and ??, ?? of ??.

Item 3, Preservation of Colimits: This follows from **Item 2** and ??, ?? of ??.³⁷

Item 4, Oplax Preservation of Limits: The inclusion $f_*(A) \subset B$ is clear. See [Pro24p] for the other inclusions.

Item 5, Symmetric Strict Monoidality With Respect to Unions: This follows from **Item 3**.

Item 6, Symmetric Oplax Monoidality With Respect to Intersections: This follows from **Item 4**.

Item 7, Interaction With Coproducts: Clear.

Item 8, Interaction With Products: Clear.

Item 9, Relation to Direct Images With Compact Support: Applying **Item 9** of **Proposition 4.6.1.6** to $A \setminus U$, we have

$$\begin{aligned} f_!(A \setminus U) &= B \setminus f_*(A \setminus (A \setminus U)) \\ &= B \setminus f_*(U). \end{aligned}$$

Taking complements, we then obtain

$$\begin{aligned} f_*(U) &= B \setminus (B \setminus f_*(U)), \\ &= B \setminus f_!(A \setminus U), \end{aligned}$$

which finishes the proof. \square

007V **Proposition 4.4.1.5.** Let $f: A \rightarrow B$ be a function.

007W 1. *Functionality I.* The assignment $f \mapsto f_*$ defines a function

$$(-)_{*|A,B}: \mathbf{Sets}(A, B) \rightarrow \mathbf{Sets}(\mathcal{P}(A), \mathcal{P}(B)).$$

007X 2. *Functionality II.* The assignment $f \mapsto f_*$ defines a function

$$(-)_{*|A,B}: \mathbf{Sets}(A, B) \rightarrow \mathbf{Pos}((\mathcal{P}(A), \subset), (\mathcal{P}(B), \subset)).$$

³⁷See also [Pro24q].

007Y 3. *Interaction With Identities.* For each $A \in \text{Obj}(\text{Sets})$, we have

$$(\text{id}_A)_* = \text{id}_{\mathcal{P}(A)}.$$

007Z 4. *Interaction With Composition.* For each pair of composable functions $f: A \rightarrow B$ and $g: B \rightarrow C$, we have

$$(g \circ f)_* = g_* \circ f_*,$$

$$\begin{array}{ccc} \mathcal{P}(A) & \xrightarrow{f_*} & \mathcal{P}(B) \\ & \searrow (g \circ f)_* & \downarrow g_* \\ & & \mathcal{P}(C). \end{array}$$

Proof. **Item 1, Functionality I:** Clear.

Item 2, Functionality II: Clear.

Item 3, Interaction With Identities: This follows from **Remark 4.4.1.3** and ??, ?? of ??.

Item 4, Interaction With Composition: This follows from **Remark 4.4.1.3** and ??, ?? of ??. \square

0080 4.5 Inverse Images

Let A and B be sets and let $f: A \rightarrow B$ be a function.

0081 **Definition 4.5.1.1.** The **inverse image function** associated to f is the function³⁸

$$f^{-1}: \mathcal{P}(B) \rightarrow \mathcal{P}(A)$$

defined by³⁹

$$f^{-1}(V) \stackrel{\text{def}}{=} \{a \in A \mid \text{we have } f(a) \in V\}$$

for each $V \in \mathcal{P}(B)$.

0082 **Remark 4.5.1.2.** Identifying subsets of B with functions from B to $\{\text{true}, \text{false}\}$ via **Items 1** and **2** of **Proposition 4.3.1.6**, we see that the inverse image function associated to f is equivalently the function

$$f^*: \mathcal{P}(B) \rightarrow \mathcal{P}(A)$$

³⁸ *Further Notation:* Also written $f^*: \mathcal{P}(B) \rightarrow \mathcal{P}(A)$.

³⁹ *Further Terminology:* The set $f^{-1}(V)$ is called the **inverse image of V by f** .

defined by

$$f^*(\chi_V) \stackrel{\text{def}}{=} \chi_V \circ f$$

for each $\chi_V \in \mathcal{P}(B)$, where $\chi_V \circ f$ is the composition

$$A \xrightarrow{f} B \xrightarrow{\chi_V} \{\text{true}, \text{false}\}$$

in Sets.

0083 Proposition 4.5.1.3. Let $f: A \rightarrow B$ be a function.

0084 1. *Functoriality.* The assignment $V \mapsto f^{-1}(V)$ defines a functor

$$f^{-1}: (\mathcal{P}(B), \subset) \rightarrow (\mathcal{P}(A), \subset)$$

where

- *Action on Objects.* For each $V \in \mathcal{P}(B)$, we have

$$[f^{-1}](V) \stackrel{\text{def}}{=} f^{-1}(V).$$

- *Action on Morphisms.* For each $U, V \in \mathcal{P}(B)$:

$$(\star) \text{ If } U \subset V, \text{ then } f^{-1}(U) \subset f^{-1}(V).$$

0085 2. *Triple Adjointness.* We have a triple adjunction

$$(f_* \dashv f^{-1} \dashv f_!): \mathcal{P}(A) \begin{array}{c} \xrightarrow{f_*} \\ \perp \\ \xleftarrow{f^{-1}} \\ \perp \\ \xrightarrow{f_!} \end{array} \mathcal{P}(B),$$

witnessed by bijections of sets

$$\begin{aligned} \text{Hom}_{\mathcal{P}(B)}(f_*(U), V) &\cong \text{Hom}_{\mathcal{P}(A)}(U, f^{-1}(V)), \\ \text{Hom}_{\mathcal{P}(A)}(f^{-1}(U), V) &\cong \text{Hom}_{\mathcal{P}(A)}(U, f_!(V)), \end{aligned}$$

natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$ and (respectively) $V \in \mathcal{P}(A)$ and $U \in \mathcal{P}(B)$, i.e. where:

- (a) The following conditions are equivalent:
- We have $f_*(U) \subset V$;
 - We have $U \subset f^{-1}(V)$;

(b) The following conditions are equivalent:

- i. We have $f^{-1}(U) \subset V$.
- ii. We have $U \subset f_!(V)$.

0086 3. *Preservation of Colimits.* We have an equality of sets

$$f^{-1}\left(\bigcup_{i \in I} U_i\right) = \bigcup_{i \in I} f^{-1}(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(B)^{\times I}$. In particular, we have equalities

$$\begin{aligned} f^{-1}(U) \cup f^{-1}(V) &= f^{-1}(U \cup V), \\ f^{-1}(\emptyset) &= \emptyset, \end{aligned}$$

natural in $U, V \in \mathcal{P}(B)$.

0087 4. *Preservation of Limits.* We have an equality of sets

$$f^{-1}\left(\bigcap_{i \in I} U_i\right) = \bigcap_{i \in I} f^{-1}(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(B)^{\times I}$. In particular, we have equalities

$$\begin{aligned} f^{-1}(U) \cap f^{-1}(V) &= f^{-1}(U \cap V), \\ f^{-1}(B) &= A, \end{aligned}$$

natural in $U, V \in \mathcal{P}(B)$.

0088 5. *Symmetric Strict Monoidality With Respect to Unions.* The inverse image function of **Item 1** has a symmetric strict monoidal structure

$$\left(f^{-1}, f^{-1, \otimes}, f_{\mathbf{1}}^{-1, \otimes}\right): (\mathcal{P}(B), \cup, \emptyset) \rightarrow (\mathcal{P}(A), \cup, \emptyset),$$

being equipped with equalities

$$\begin{aligned} f_{U, V}^{-1, \otimes}: f^{-1}(U) \cup f^{-1}(V) &\xrightarrow{=} f^{-1}(U \cup V), \\ f_{\mathbf{1}}^{-1, \otimes}: \emptyset &\xrightarrow{=} f^{-1}(\emptyset), \end{aligned}$$

natural in $U, V \in \mathcal{P}(B)$.

- 0089** 6. *Symmetric Strict Monoidality With Respect to Intersections.* The inverse image function of **Item 1** has a symmetric strict monoidal structure

$$(f^{-1}, f^{-1, \otimes}, f_{\mathbb{1}}^{-1, \otimes}): (\mathcal{P}(B), \cap, B) \rightarrow (\mathcal{P}(A), \cap, A),$$

being equipped with equalities

$$\begin{aligned} f_{U,V}^{-1, \otimes}: f^{-1}(U) \cap f^{-1}(V) &\xrightarrow{\cong} f^{-1}(U \cap V), \\ f_{\mathbb{1}}^{-1, \otimes}: A &\xrightarrow{\cong} f^{-1}(B), \end{aligned}$$

natural in $U, V \in \mathcal{P}(B)$.

- 008A** 7. *Interaction With Coproducts.* Let $f: A \rightarrow A'$ and $g: B \rightarrow B'$ be maps of sets. We have

$$(f \amalg g)^{-1}(U' \amalg V') = f^{-1}(U') \amalg g^{-1}(V')$$

for each $U' \in \mathcal{P}(A')$ and each $V' \in \mathcal{P}(B')$.

- 008B** 8. *Interaction With Products.* Let $f: A \rightarrow A'$ and $g: B \rightarrow B'$ be maps of sets. We have

$$(f \times g)^{-1}(U' \times V') = f^{-1}(U') \times g^{-1}(V')$$

for each $U' \in \mathcal{P}(A')$ and each $V' \in \mathcal{P}(B')$.

Proof. Item 1, Functoriality: Clear.

Item 2, Triple Adjointness: This follows from **Remark 4.4.1.3**, **Remark 4.5.1.2**, **Remark 4.6.1.3**, and ??, ?? of ??.

Item 3, Preservation of Colimits: This follows from **Item 2** and ??, ?? of ??.⁴⁰

Item 4, Preservation of Limits: This follows from **Item 2** and ??, ?? of ??.⁴¹

Item 5, Symmetric Strict Monoidality With Respect to Unions: This follows from **Item 3**.

Item 6, Symmetric Strict Monoidality With Respect to Intersections: This follows from **Item 4**.

Item 7, Interaction With Coproducts: Clear.

Item 8, Interaction With Products: Clear. □

⁴⁰See also [Pro24y].

⁴¹See also [Pro24x].

008C Proposition 4.5.1.4. Let $f: A \rightarrow B$ be a function.

008D 1. *Functionality I.* The assignment $f \mapsto f^{-1}$ defines a function

$$(-)_{A,B}^{-1}: \mathbf{Sets}(A, B) \rightarrow \mathbf{Sets}(\mathcal{P}(B), \mathcal{P}(A)).$$

008E 2. *Functionality II.* The assignment $f \mapsto f^{-1}$ defines a function

$$(-)_{A,B}^{-1}: \mathbf{Sets}(A, B) \rightarrow \mathbf{Pos}((\mathcal{P}(B), \subset), (\mathcal{P}(A), \subset)).$$

008F 3. *Interaction With Identities.* For each $A \in \mathbf{Obj}(\mathbf{Sets})$, we have

$$\mathrm{id}_A^{-1} = \mathrm{id}_{\mathcal{P}(A)}.$$

008G 4. *Interaction With Composition.* For each pair of composable functions $f: A \rightarrow B$ and $g: B \rightarrow C$, we have

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1},$$

$$\begin{array}{ccc} \mathcal{P}(C) & \xrightarrow{g^{-1}} & \mathcal{P}(B) \\ & \searrow (g \circ f)^{-1} & \downarrow f^{-1} \\ & & \mathcal{P}(A). \end{array}$$

Proof. **Item 1, Functionality I:** Clear.

Item 2, Functionality II: Clear.

Item 3, Interaction With Identities: This follows from **Remark 4.5.1.2** and **Categories, Item 5** of **Proposition 1.6.1.2**.

Item 4, Interaction With Composition: This follows from **Remark 4.5.1.2** and **Categories, Item 2** of **Proposition 1.6.1.2**. \square

008H 4.6 Direct Images With Compact Support

Let A and B be sets and let $f: A \rightarrow B$ be a function.

008J Definition 4.6.1.1. The **direct image with compact support function** associated to f is the function

$$f_!: \mathcal{P}(A) \rightarrow \mathcal{P}(B)$$

defined by^{42,43}

$$\begin{aligned} f_!(U) &\stackrel{\text{def}}{=} \left\{ b \in B \mid \begin{array}{l} \text{for each } a \in A, \text{ if we have} \\ f(a) = b, \text{ then } a \in U \end{array} \right\} \\ &= \left\{ b \in B \mid \text{we have } f^{-1}(b) \subset U \right\} \end{aligned}$$

for each $U \in \mathcal{P}(A)$.

008K Notation 4.6.1.2. Sometimes one finds the notation

$$\forall_f: \mathcal{P}(A) \rightarrow \mathcal{P}(B)$$

for f_* . This notation comes from the fact that the following statements are equivalent, where $b \in B$ and $U \in \mathcal{P}(A)$:

- We have $b \in \forall_f(U)$.
- For each $a \in A$, if $b = f(a)$, then $a \in U$.

008L Remark 4.6.1.3. Identifying subsets of A with functions from A to $\{\text{true}, \text{false}\}$ via **Items 1 and 2** of **Proposition 4.3.1.6**, we see that the direct image with compact support function associated to f is equivalently the function

$$f_!: \mathcal{P}(A) \rightarrow \mathcal{P}(B)$$

defined by

$$\begin{aligned} f_!(\chi_U) &\stackrel{\text{def}}{=} \text{Ran}_f(\chi_U) \\ &= \lim \left(\left(\underline{(-1)} \overset{\rightarrow}{\times} f \right) \overset{\text{pr}}{\twoheadrightarrow} A \xrightarrow{\chi_U} \{\text{true}, \text{false}\} \right) \\ &= \lim_{\substack{a \in A \\ f(a) = -1}} (\chi_U(a)) \\ &= \bigwedge_{\substack{a \in A \\ f(a) = -1}} (\chi_U(a)). \end{aligned}$$

⁴²*Further Terminology:* The set $f_!(U)$ is called the **direct image with compact support of U by f** .

⁴³We also have

$$f_!(U) = B \setminus f_*(A \setminus U);$$

see **Item 9** of **Proposition 4.6.1.6**.

where we have used $??$, $??$ for the second equality. In other words, we have

$$\begin{aligned}
 [f_!(\chi_U)](b) &= \bigwedge_{\substack{a \in A \\ f(a)=b}} (\chi_U(a)) \\
 &= \begin{cases} \text{true} & \text{if, for each } a \in A \text{ such that} \\ & f(a) = b, \text{ we have } a \in U, \\ \text{false} & \text{otherwise} \end{cases} \\
 &= \begin{cases} \text{true} & \text{if } f^{-1}(b) \subset U \\ \text{false} & \text{otherwise} \end{cases}
 \end{aligned}$$

for each $b \in B$.

008M Definition 4.6.1.4. Let U be a subset of A .^{44,45}

1. The **image part of the direct image with compact support**
008N $f_!(U)$ of U is the set $f_{!,\text{im}}(U)$ defined by

$$\begin{aligned}
 f_{!,\text{im}}(U) &\stackrel{\text{def}}{=} f_!(U) \cap \text{Im}(f) \\
 &= \left\{ b \in B \mid \begin{array}{l} \text{we have } f^{-1}(b) \subset U \\ \text{and } f^{-1}(b) \neq \emptyset \end{array} \right\}.
 \end{aligned}$$

2. The **complement part of the direct image with compact sup-**

⁴⁴Note that we have

$$f_!(U) = f_{!,\text{im}}(U) \cup f_{!,\text{cp}}(U),$$

as

$$\begin{aligned}
 f_!(U) &= f_!(U) \cap B \\
 &= f_!(U) \cap (\text{Im}(f) \cup (B \setminus \text{Im}(f))) \\
 &= (f_!(U) \cap \text{Im}(f)) \cup (f_!(U) \cap (B \setminus \text{Im}(f))) \\
 &\stackrel{\text{def}}{=} f_{!,\text{im}}(U) \cup f_{!,\text{cp}}(U).
 \end{aligned}$$

⁴⁵In terms of the meet computation of $f_!(U)$ of **Remark 4.6.1.3**, namely

$$f_!(\chi_U) = \bigwedge_{\substack{a \in A \\ f(a)=-1}} (\chi_U(a)),$$

we see that $f_{!,\text{im}}$ corresponds to meets indexed over nonempty sets, while $f_{!,\text{cp}}$ corresponds to meets indexed over the empty set.

008P **port** $f_!(U)$ **of** U is the set $f_{!,\text{cp}}(U)$ defined by

$$\begin{aligned} f_{!,\text{cp}}(U) &\stackrel{\text{def}}{=} f_!(U) \cap (B \setminus \text{Im}(f)) \\ &= B \setminus \text{Im}(f) \\ &= \left\{ b \in B \mid \begin{array}{l} \text{we have } f^{-1}(b) \subset \\ U \text{ and } f^{-1}(b) = \emptyset \end{array} \right\} \\ &= \left\{ b \in B \mid f^{-1}(b) = \emptyset \right\}. \end{aligned}$$

008Q **Example 4.6.1.5.** Here are some examples of direct images with compact support.

1. *The Multiplication by Two Map on the Natural Numbers.* Consider the function $f: \mathbb{N} \rightarrow \mathbb{N}$ given by

$$f(n) \stackrel{\text{def}}{=} 2n$$

for each $n \in \mathbb{N}$. Since f is injective, we have

$$\begin{aligned} f_{!,\text{im}}(U) &= f_*(U) \\ f_{!,\text{cp}}(U) &= \{\text{odd natural numbers}\} \end{aligned}$$

for any $U \subset \mathbb{N}$.

2. *Parabolas.* Consider the function $f: \mathbb{R} \rightarrow \mathbb{R}$ given by

$$f(x) \stackrel{\text{def}}{=} x^2$$

for each $x \in \mathbb{R}$. We have

$$f_{!,\text{cp}}(U) = \mathbb{R}_{<0}$$

for any $U \subset \mathbb{R}$. Moreover, since $f^{-1}(x) = \{-\sqrt{x}, \sqrt{x}\}$, we have e.g.:

$$\begin{aligned} f_{!,\text{im}}([0, 1]) &= \{0\}, \\ f_{!,\text{im}}([-1, 1]) &= [0, 1], \\ f_{!,\text{im}}([1, 2]) &= \emptyset, \\ f_{!,\text{im}}([-2, -1] \cup [1, 2]) &= [1, 4]. \end{aligned}$$

3. *Circles.* Consider the function $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ given by

$$f(x, y) \stackrel{\text{def}}{=} x^2 + y^2$$

for each $(x, y) \in \mathbb{R}^2$. We have

$$f_{!,\text{cp}}(U) = \mathbb{R}_{<0}$$

for any $U \subset \mathbb{R}^2$, and since

$$f^{-1}(r) = \begin{cases} \text{a circle of radius } r \text{ about the origin} & \text{if } r > 0, \\ \{(0, 0)\} & \text{if } r = 0, \\ \emptyset & \text{if } r < 0, \end{cases}$$

we have e.g.:

$$\begin{aligned} f_{!,\text{im}}([-1, 1] \times [-1, 1]) &= [0, 1], \\ f_{!,\text{im}}([-1, 1] \times [-1, 1] \setminus [-1, 1] \times \{0\}) &= \emptyset. \end{aligned}$$

008R Proposition 4.6.1.6. Let $f: A \rightarrow B$ be a function.

008S 1. *Functoriality.* The assignment $U \mapsto f_!(U)$ defines a functor

$$f_!: (\mathcal{P}(A), \subset) \rightarrow (\mathcal{P}(B), \subset)$$

where

- *Action on Objects.* For each $U \in \mathcal{P}(A)$, we have

$$[f_!](U) \stackrel{\text{def}}{=} f_!(U).$$

- *Action on Morphisms.* For each $U, V \in \mathcal{P}(A)$:

$$(\star) \text{ If } U \subset V, \text{ then } f_!(U) \subset f_!(V).$$

008T 2. *Triple Adjointness.* We have a triple adjunction

$$(f_* \dashv f^{-1} \dashv f_!): \mathcal{P}(A) \begin{array}{c} \xrightarrow{f_*} \\ \perp \\ \xleftarrow{f^{-1}} \\ \perp \\ \xrightarrow{f_!} \end{array} \mathcal{P}(B),$$

witnessed by bijections of sets

$$\begin{aligned} \text{Hom}_{\mathcal{P}(B)}(f_*(U), V) &\cong \text{Hom}_{\mathcal{P}(A)}(U, f^{-1}(V)), \\ \text{Hom}_{\mathcal{P}(A)}(f^{-1}(U), V) &\cong \text{Hom}_{\mathcal{P}(A)}(U, f_!(V)), \end{aligned}$$

natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$ and (respectively) $V \in \mathcal{P}(A)$ and $U \in \mathcal{P}(B)$, i.e. where:

(a) The following conditions are equivalent:

- i. We have $f_*(U) \subset V$.
- ii. We have $U \subset f^{-1}(V)$.

(b) The following conditions are equivalent:

- i. We have $f^{-1}(U) \subset V$.
- ii. We have $U \subset f_!(V)$.

008U 3. *Lax Preservation of Colimits.* We have an inclusion of sets

$$\bigcup_{i \in I} f_!(U_i) \subset f_!\left(\bigcup_{i \in I} U_i\right),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have inclusions

$$\begin{aligned} f_!(U) \cup f_!(V) &\hookrightarrow f_!(U \cup V), \\ \emptyset &\hookrightarrow f_!(\emptyset), \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

008V 4. *Preservation of Limits.* We have an equality of sets

$$f_!\left(\bigcap_{i \in I} U_i\right) = \bigcap_{i \in I} f_!(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have equalities

$$\begin{aligned} f^{-1}(U \cap V) &= f_!(U) \cap f^{-1}(V), \\ f_!(A) &= B, \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

008W 5. *Symmetric Lax Monoidality With Respect to Unions.* The direct image with compact support function of **Item 1** has a symmetric lax monoidal structure

$$(f_!, f_!^{\otimes}, f_{!|\mathbb{1}}^{\otimes}) : (\mathcal{P}(A), \cup, \emptyset) \rightarrow (\mathcal{P}(B), \cup, \emptyset),$$

being equipped with inclusions

$$\begin{aligned} f_{!|U,V}^{\otimes} : f_!(U) \cup f_!(V) &\hookrightarrow f_!(U \cup V), \\ f_{!|\mathbb{1}}^{\otimes} : \emptyset &\hookrightarrow f_!(\emptyset), \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

- 008X** 6. *Symmetric Strict Monoidality With Respect to Intersections.* The direct image function of **Item 1** has a symmetric strict monoidal structure

$$(f_!, f_!^\otimes, f_{!1}^\otimes) : (\mathcal{P}(A), \cap, A) \rightarrow (\mathcal{P}(B), \cap, B),$$

being equipped with equalities

$$\begin{aligned} f_{!|U,V}^\otimes : f_!(U \cap V) &\xrightarrow{=} f_!(U) \cap f_!(V), \\ f_{!1}^\otimes : f_!(A) &\xrightarrow{=} B, \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

- 008Y** 7. *Interaction With Coproducts.* Let $f : A \rightarrow A'$ and $g : B \rightarrow B'$ be maps of sets. We have

$$(f \amalg g)_!(U \amalg V) = f_!(U) \amalg g_!(V)$$

for each $U \in \mathcal{P}(A)$ and each $V \in \mathcal{P}(B)$.

- 008Z** 8. *Interaction With Products.* Let $f : A \rightarrow A'$ and $g : B \rightarrow B'$ be maps of sets. We have

$$(f \times g)_!(U \times V) = f_!(U) \times g_!(V)$$

for each $U \in \mathcal{P}(A)$ and each $V \in \mathcal{P}(B)$.

- 0090** 9. *Relation to Direct Images.* We have

$$f_!(U) = B \setminus f_*(A \setminus U)$$

for each $U \in \mathcal{P}(A)$.

- 0091** 10. *Interaction With Injections.* If f is injective, then we have

$$\begin{aligned} f_{!,\text{im}}(U) &= f_*(U), \\ f_{!,\text{cp}}(U) &= B \setminus \text{Im}(f), \\ f_!(U) &= f_{!,\text{im}}(U) \cup f_{!,\text{cp}}(U) \\ &= f_*(U) \cup (B \setminus \text{Im}(f)) \end{aligned}$$

for each $U \in \mathcal{P}(A)$.

0092 11. *Interaction With Surjections.* If f is surjective, then we have

$$\begin{aligned} f_{!,\text{im}}(U) &\subset f_*(U), \\ f_{!,\text{cp}}(U) &= \emptyset, \\ f_!(U) &\subset f_*(U) \end{aligned}$$

for each $U \in \mathcal{P}(A)$.

Proof. **Item 1, Functoriality:** Clear.

Item 2, Triple Adjointness: This follows from **Remark 4.4.1.3**, **Remark 4.5.1.2**, **Remark 4.6.1.3**, and ??, ?? of ??.

Item 3, Lax Preservation of Colimits: Omitted.

Item 4, Preservation of Limits: This follows from **Item 2** and ??, ?? of ??.

Item 5, Symmetric Lax Monoidality With Respect to Unions: This follows from **Item 3**.

Item 6, Symmetric Strict Monoidality With Respect to Intersections: This follows from **Item 4**.

Item 7, Interaction With Coproducts: Clear.

Item 8, Interaction With Products: Clear.

Item 9, Relation to Direct Images: We claim that $f_!(U) = B \setminus f_*(A \setminus U)$.

- *The First Implication.* We claim that

$$f_!(U) \subset B \setminus f_*(A \setminus U).$$

Let $b \in f_!(U)$. We need to show that $b \notin f_*(A \setminus U)$, i.e. that there is no $a \in A \setminus U$ such that $f(a) = b$.

This is indeed the case, as otherwise we would have $a \in f^{-1}(b)$ and $a \notin U$, contradicting $f^{-1}(b) \subset U$ (which holds since $b \in f_!(U)$).

Thus $b \in B \setminus f_*(A \setminus U)$.

- *The Second Implication.* We claim that

$$B \setminus f_*(A \setminus U) \subset f_!(U).$$

Let $b \in B \setminus f_*(A \setminus U)$. We need to show that $b \in f_!(U)$, i.e. that $f^{-1}(b) \subset U$.

Since $b \notin f_*(A \setminus U)$, there exists no $a \in A \setminus U$ such that $b = f(a)$, and hence $f^{-1}(b) \subset U$.

Thus $b \in f_!(U)$.

This finishes the proof of **Item 9**.

Item 10, *Interaction With Injections*: Clear.

Item 11, *Interaction With Surjections*: Clear. \square

0093 Proposition 4.6.1.7. Let $f: A \rightarrow B$ be a function.

0094 1. *Functionality I*. The assignment $f \mapsto f_!$ defines a function

$$(-)_{!|A,B}: \mathbf{Sets}(A, B) \rightarrow \mathbf{Sets}(\mathcal{P}(A), \mathcal{P}(B)).$$

0095 2. *Functionality II*. The assignment $f \mapsto f_!$ defines a function

$$(-)_{!|A,B}: \mathbf{Sets}(A, B) \rightarrow \mathbf{Pos}((\mathcal{P}(A), \subset), (\mathcal{P}(B), \subset)).$$

0096 3. *Interaction With Identities*. For each $A \in \mathbf{Obj}(\mathbf{Sets})$, we have

$$(\mathrm{id}_A)_! = \mathrm{id}_{\mathcal{P}(A)}.$$

0097 4. *Interaction With Composition*. For each pair of composable functions $f: A \rightarrow B$ and $g: B \rightarrow C$, we have

$$(g \circ f)_! = g_! \circ f_!,$$

$$\begin{array}{ccc} \mathcal{P}(A) & \xrightarrow{f_!} & \mathcal{P}(B) \\ & \searrow (g \circ f)_! & \downarrow g_! \\ & & \mathcal{P}(C). \end{array}$$

Proof. **Item 1**, *Functionality I*: Clear.

Item 2, *Functionality II*: Clear.

Item 3, *Interaction With Identities*: This follows from **Remark 4.6.1.3** and ??, ?? of ??.

Item 4, *Interaction With Composition*: This follows from **Remark 4.6.1.3** and ??, ?? of ??. \square

Appendices

A Other Chapters

Sets

1. Sets
2. Constructions With Sets
3. Pointed Sets
4. Tensor Products of Pointed Sets

6. Constructions With Relations

7. Equivalence Relations and Apartness Relations

Category Theory

8. Categories

Bicategories

9. Types of Morphisms in Bicat-
egories

Relations

5. Relations

References

- [MO 382264] Neil Strickland. *Proof that a cartesian category is monoidal*. MathOverflow. URL: <https://mathoverflow.net/q/382264> (cit. on p. 13).
- [MSE 267469] Zhen Lin. *Show that the powerset partial order is a cartesian closed category*. Mathematics Stack Exchange. URL: <https://math.stackexchange.com/q/267469> (cit. on p. 46).
- [Cie97] Krzysztof Ciesielski. *Set Theory for the Working Mathematician*. Vol. 39. London Mathematical Society Student Texts. Cambridge University Press, Cambridge, 1997, pp. xii+236. ISBN: 0-521-59441-3; 0-521-59465-0. DOI: [10.1017/CB09781139173131](https://doi.org/10.1017/CB09781139173131). URL: <https://doi.org/10.1017/CB09781139173131> (cit. on p. 39).
- [Pro24a] Proof Wiki Contributors. *Bijection Between $R \times (S \times T)$ and $(R \times S) \times T$ — Proof Wiki*. 2024. URL: [https://proofwiki.org/wiki/Bijection_between_R_x_\(S_x_T\)_and_\(R_x_S\)_x_T](https://proofwiki.org/wiki/Bijection_between_R_x_(S_x_T)_and_(R_x_S)_x_T) (cit. on p. 13).
- [Pro24b] Proof Wiki Contributors. *Bijection Between $S \times T$ and $T \times S$ — Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Bijection_between_S_x_T_and_T_x_S (cit. on p. 13).

- [Pro24c] Proof Wiki Contributors. *Cartesian Product Distributes Over Set Difference* — *Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Cartesian_Product_Distributes_over_Set_Difference (cit. on p. 13).
- [Pro24d] Proof Wiki Contributors. *Cartesian Product Distributes Over Symmetric Difference* — *Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Cartesian_Product_Distributes_over_Symmetric_Difference (cit. on p. 13).
- [Pro24e] Proof Wiki Contributors. *Cartesian Product Distributes Over Union* — *Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Cartesian_Product_Distributes_over_Union (cit. on p. 13).
- [Pro24f] Proof Wiki Contributors. *Cartesian Product Is Empty Iff Factor Is Empty* — *Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Cartesian_Product_is_Empty_iff_Factor_is_Empty (cit. on p. 13).
- [Pro24g] Proof Wiki Contributors. *Cartesian Product of Intersections* — *Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Cartesian_Product_of_Intersections (cit. on p. 13).
- [Pro24h] Proof Wiki Contributors. *Characteristic Function of Intersection* — *Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Characteristic_Function_of_Intersection (cit. on p. 46).
- [Pro24i] Proof Wiki Contributors. *Characteristic Function of Set Difference* — *Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Characteristic_Function_of_Set_Difference (cit. on p. 50).
- [Pro24j] Proof Wiki Contributors. *Characteristic Function of Symmetric Difference* — *Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Characteristic_Function_of_Symmetric_Difference (cit. on p. 55).
- [Pro24k] Proof Wiki Contributors. *Characteristic Function of Union* — *Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Characteristic_Function_of_Union (cit. on p. 43).
- [Pro24l] Proof Wiki Contributors. *Complement of Complement* — *Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Complement_of_Complement (cit. on p. 51).

- [Pro24m] Proof Wiki Contributors. *De Morgan's Laws (Set Theory)* — *Proof Wiki*. 2024. URL: [https://proofwiki.org/wiki/De_Morgan%5C%27s_Laws_\(Set_Theory\)](https://proofwiki.org/wiki/De_Morgan%5C%27s_Laws_(Set_Theory)) (cit. on pp. 50, 51).
- [Pro24n] Proof Wiki Contributors. *De Morgan's Laws (Set Theory)/Set Difference/Difference with Union* — *Proof Wiki*. 2024. URL: [https://proofwiki.org/wiki/De_Morgan%5C%27s_Laws_\(Set_Theory\)/Set_Difference/Difference_with_Union](https://proofwiki.org/wiki/De_Morgan%5C%27s_Laws_(Set_Theory)/Set_Difference/Difference_with_Union) (cit. on p. 50).
- [Pro24o] Proof Wiki Contributors. *Equivalence of Definitions of Symmetric Difference* — *Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Equivalence_of_Definitions_of_Symmetric_Difference (cit. on p. 55).
- [Pro24p] Proof Wiki Contributors. *Image of Intersection Under Mapping* — *Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Image_of_Intersection_under_Mapping (cit. on p. 82).
- [Pro24q] Proof Wiki Contributors. *Image of Union Under Mapping* — *Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Image_of_Union_under_Mapping (cit. on p. 82).
- [Pro24r] Proof Wiki Contributors. *Intersection Distributes Over Symmetric Difference* — *Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Intersection_Distributes_over_Symmetric_Difference (cit. on p. 55).
- [Pro24s] Proof Wiki Contributors. *Intersection Is Associative* — *Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Intersection_is_Associative (cit. on p. 46).
- [Pro24t] Proof Wiki Contributors. *Intersection Is Commutative* — *Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Intersection_is_Commutative (cit. on p. 46).
- [Pro24u] Proof Wiki Contributors. *Intersection With Empty Set* — *Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Intersection_with_Empty_Set (cit. on p. 46).
- [Pro24v] Proof Wiki Contributors. *Intersection With Set Difference Is Set Difference With Intersection* — *Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Intersection_with_Set_Difference_is_Set_Difference_with_Intersection (cit. on p. 50).

- [Pro24w] Proof Wiki Contributors. *Intersection With Subset Is Subset* — *Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Intersection_with_Subset_is_Subset (cit. on p. 46).
- [Pro24x] Proof Wiki Contributors. *Preimage of Intersection Under Mapping* — *Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Preimage_of_Intersection_under_Mapping (cit. on p. 86).
- [Pro24y] Proof Wiki Contributors. *Preimage of Union Under Mapping* — *Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Preimage_of_Union_under_Mapping (cit. on p. 86).
- [Pro24z] Proof Wiki Contributors. *Quotient Mapping Is Coequalizer* — *Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Quotient_Mapping_is_Coequalizer (cit. on p. 37).
- [Pro24aa] Proof Wiki Contributors. *Set Difference as Intersection With Complement* — *Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Set_Difference_as_Intersection_with_Complement (cit. on p. 50).
- [Pro24ab] Proof Wiki Contributors. *Set Difference as Symmetric Difference With Intersection* — *Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Set_Difference_as_Symmetric_Difference_with_Intersection (cit. on p. 50).
- [Pro24ac] Proof Wiki Contributors. *Set Difference Is Right Distributive Over Union* — *Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Set_Difference_is_Right_Distributive_over_Union (cit. on p. 50).
- [Pro24ad] Proof Wiki Contributors. *Set Difference Over Subset* — *Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Set_Difference_over_Subset (cit. on p. 50).
- [Pro24ae] Proof Wiki Contributors. *Set Difference With Empty Set Is Self* — *Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Set_Difference_with_Empty_Set_is_Self (cit. on p. 50).
- [Pro24af] Proof Wiki Contributors. *Set Difference With Self Is Empty Set* — *Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Set_Difference_with_Self_is_Empty_Set (cit. on p. 50).

- [Pro24ag] Proof Wiki Contributors. *Set Difference With Set Difference Is Union of Set Difference With Intersection* — *Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Set_Difference_with_Set_Difference_is_Union_of_Set_Difference_with_Intersection (cit. on p. 50).
- [Pro24ah] Proof Wiki Contributors. *Set Difference With Subset Is Superset of Set Difference* — *Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Set_Difference_with_Subset_is_Superset_of_Set_Difference (cit. on p. 50).
- [Pro24ai] Proof Wiki Contributors. *Set Difference With Union* — *Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Set_Difference_with_Union (cit. on p. 50).
- [Pro24aj] Proof Wiki Contributors. *Set Intersection Distributes Over Union* — *Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Intersection_Distributes_over_Union (cit. on p. 46).
- [Pro24ak] Proof Wiki Contributors. *Set Intersection Is Idempotent* — *Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Set_Intersection_is_Idempotent (cit. on p. 46).
- [Pro24al] Proof Wiki Contributors. *Set Intersection Preserves Subsets* — *Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Set_Intersection_Preserves_Subsets (cit. on p. 46).
- [Pro24am] Proof Wiki Contributors. *Set Union Is Idempotent* — *Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Set_Union_is_Idempotent (cit. on p. 43).
- [Pro24an] Proof Wiki Contributors. *Set Union Preserves Subsets* — *Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Set_Union_Preserves_Subsets (cit. on p. 42).
- [Pro24ao] Proof Wiki Contributors. *Symmetric Difference Is Associative* — *Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Symmetric_Difference_is_Associative (cit. on p. 55).
- [Pro24ap] Proof Wiki Contributors. *Symmetric Difference Is Commutative* — *Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Symmetric_Difference_is_Commutative (cit. on p. 55).
- [Pro24aq] Proof Wiki Contributors. *Symmetric Difference of Complements* — *Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Symmetric_Difference_of_Complements (cit. on p. 55).

- [Pro24ar] Proof Wiki Contributors. *Symmetric Difference on Power Set Forms Abelian Group* — *Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Symmetric_Difference_on_Power_Set_forms_Abelian_Group (cit. on p. 56).
- [Pro24as] Proof Wiki Contributors. *Symmetric Difference With Complement* — *Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Symmetric_Difference_with_Complement (cit. on p. 55).
- [Pro24at] Proof Wiki Contributors. *Symmetric Difference With Empty Set* — *Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Symmetric_Difference_with_Empty_Set (cit. on p. 55).
- [Pro24au] Proof Wiki Contributors. *Symmetric Difference With Intersection Forms Ring* — *Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Symmetric_Difference_with_Intersection_forms_Ring (cit. on p. 56).
- [Pro24av] Proof Wiki Contributors. *Symmetric Difference With Self Is Empty Set* — *Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Symmetric_Difference_with_Self_is_Empty_Set (cit. on p. 55).
- [Pro24aw] Proof Wiki Contributors. *Symmetric Difference With Union Does Not Form Ring* — *Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Symmetric_Difference_with_Union_does_not_form_Ring (cit. on p. 55).
- [Pro24ax] Proof Wiki Contributors. *Symmetric Difference With Universe* — *Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Symmetric_Difference_with_Universe (cit. on p. 55).
- [Pro24ay] Proof Wiki Contributors. *Union as Symmetric Difference With Intersection* — *Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Union_as_Symmetric_Difference_with_Intersection (cit. on p. 42).
- [Pro24az] Proof Wiki Contributors. *Union Distributes Over Intersection* — *Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Union_Distributes_over_Intersection (cit. on p. 43).
- [Pro24ba] Proof Wiki Contributors. *Union Is Associative* — *Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Union_is_Associative (cit. on p. 42).

-
- [Pro24bb] Proof Wiki Contributors. *Union Is Commutative* — *Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Union_is_Commutative (cit. on p. 43).
- [Pro24bc] Proof Wiki Contributors. *Union of Symmetric Differences* — *Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Union_of_Symmetric_Differences (cit. on p. 55).
- [Pro24bd] Proof Wiki Contributors. *Union With Empty Set* — *Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Union_with_Empty_Set (cit. on p. 42).