Constructions With Sets

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This chapter develops some material relating to constructions with sets with an eye towards its categorical and higher-categorical counterparts to be introduced later in this work. In particular, it contains:

1. Explicit descriptions of the major types of co/limits in Sets, including in particular explicit descriptions of pushouts and coequalisers (see Definitions 2.4.1.1 and 2.5.1.1 and Remarks 2.4.1.2 and 2.5.1.2).

2. A discussion of powersets as decategorifications of categories of presheaves (Remarks 4.1.1.2 and 4.3.1.2), including a (-1)-categorical analogue of un/straightening, described in Items 1 and 2 of Proposition 4.3.1.6 and Remark 4.3.1.7.

3. A lengthy discussion of the adjoint triple

$$f_* \dashv f^{-1} \dashv f_! \colon \mathcal{P}(A) \overset{\rightleftarrows}{\rightarrow} \mathcal{P}(B)$$

   of functors (morphisms of posets) between $\mathcal{P}(A)$ and $\mathcal{P}(B)$ induced by a map of sets $f \colon A \longrightarrow B$, along with a discussion of the properties of f_*, f^{-1}, and $f_!$.

   In line with the categorical viewpoint developed here, this adjoint triple may be described in terms of Kan extensions, and, as it turns out, it also shows up in some definitions and results in point-set topology, such as in e.g. notions of continuity for functions (**??**, **??**).

Contents

1 Limits of Sets

1.1 The Terminal Set

Definition 1.1.1.1. The **terminal set** is the pair $(\mathrm{pt}, \{!_A\}_{A\in\mathrm{Obj}(\mathsf{Sets})})$ consisting of:

- *The Limit.* The punctual set $\mathrm{pt} \overset{\text{def}}{=} \{\star\}$.
- *The Cone.* The collection of maps

  $$\{!_A \colon A \longrightarrow \mathrm{pt}\}_{A\in\mathrm{Obj}(\mathsf{Sets})}$$

  defined by

  $$!_A(a) \overset{\text{def}}{=} \star$$

  for each $a \in A$ and each $A \in \mathrm{Obj}(\mathsf{Sets})$.

Proof. We claim that pt is the terminal object of Sets. Indeed, suppose we have a diagram of the form

$$A \qquad \mathrm{pt}$$

in Sets. Then there exists a unique map $\phi \colon A \longrightarrow \mathrm{pt}$ making the diagram

$$A \overset{\phi}{\underset{\exists!}{\dashrightarrow}} \mathrm{pt}$$

commute, namely $!_A$. □

1.2 Products of Families of Sets

Let $\{A_i\}_{i\in I}$ be a family of sets.

Definition 1.2.1.1. The **product**[1] **of** $\{A_i\}_{i\in I}$ is the pair $(\prod_{i\in I} A_i, \{\mathrm{pr}_i\}_{i\in I})$ consisting of:

- *The Limit.* The set $\prod_{i\in I} A_i$ defined by[2]

$$\prod_{i\in I} A_i \overset{\text{def}}{=} \left\{ f \in \mathsf{Sets}(I, \bigcup_{i\in I} A_i) \;\middle|\; \begin{array}{l}\text{for each } i \in I\text{, we}\\ \text{have } f(i) \in A_i\end{array} \right\}.$$

[1] *Further Terminology:* Also called the **Cartesian product of** $\{A_i\}_{i\in I}$.

[2] Less formally, $\prod_{i\in I} A_i$ is the set whose elements are I-indexed collections $(a_i)_{i\in I}$ with $a_i \in$

- *The Cone.* The collection

$$\left\{\mathrm{pr}_i\colon \prod_{i\in I} A_i \to A_i\right\}_{i\in I}$$

of maps given by

$$\mathrm{pr}_i(f) \stackrel{\text{def}}{=} f(i)$$

for each $f \in \prod_{i\in I} A_i$ and each $i \in I$.

Proof. We claim that $\prod_{i\in I} A_i$ is the categorical product of $\{A_i\}_{i\in I}$ in Sets. Indeed, suppose we have, for each $i \in I$, a diagram of the form

$$\begin{tikzcd} P \arrow[dr, "p_i"] & \\ \prod_{i\in I} A_i \arrow[r, "\mathrm{pr}_i"'] & A_i \end{tikzcd}$$

in Sets. Then there exists a unique map $\phi\colon P \to \prod_{i\in I} A_i$ making the diagram

$$\begin{tikzcd} P \arrow[d, dashed, "\phi"', "\exists!"] \arrow[dr, "p_i"] & \\ \prod_{i\in I} A_i \arrow[r, "\mathrm{pr}_i"'] & A_i \end{tikzcd}$$

commute, being uniquely determined by the condition $\mathrm{pr}_i \circ \phi = p_i$ for each $i \in I$ via

$$\phi(x) = (p_i(x))_{i\in I}$$

for each $x \in P$. □

A_i for each $i \in I$. The projection maps

$$\left\{\mathrm{pr}_i\colon \prod_{i\in I} A_i \to A_i\right\}_{i\in I}$$

are then given by

$$\mathrm{pr}_i((a_j)_{j\in I}) \stackrel{\text{def}}{=} a_i$$

Proposition 1.2.1.2. Let $\{A_i\}_{i\in I}$ be a family of sets.

1. *Functoriality.* The assignment $\{A_i\}_{i\in I} \mapsto \prod_{i\in I} A_i$ defines a functor

$$\prod_{i\in I} \colon \mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}) \longrightarrow \mathsf{Sets}$$

where

   - *Action on Objects.* For each $(A_i)_{i\in I} \in \mathrm{Obj}(\mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}))$, we have

$$\left[\prod_{i\in I}\right]((A_i)_{i\in I}) \stackrel{\text{def}}{=} \prod_{i\in I} A_i$$

   - *Action on Morphisms.* For each $(A_i)_{i\in I}, (B_i)_{i\in I} \in \mathrm{Obj}(\mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}))$, the action on Hom-sets

$$(\prod_{i\in I})_{(A_i)_{i\in I},(B_i)_{i\in I}} \colon \mathrm{Nat}((A_i)_{i\in I}, (B_i)_{i\in I}) \longrightarrow \mathsf{Sets}(\prod_{i\in I} A_i, \prod_{i\in I} B_i)$$

of $\prod_{i\in I}$ at $((A_i)_{i\in I}, (B_i)_{i\in I})$ is defined by sending a map

$$\{f_i \colon A_i \longrightarrow B_i\}_{i\in I}$$

in $\mathrm{Nat}((A_i)_{i\in I}, (B_i)_{i\in I})$ to the map of sets

$$\prod_{i\in I} f_i \colon \prod_{i\in I} A_i \to \prod_{i\in I} B_i$$

defined by

$$\left[\prod_{i\in I} f_i\right]((a_i)_{i\in I}) \stackrel{\text{def}}{=} (f_i(a_i))_{i\in I}$$

for each $(a_i)_{i\in I} \in \prod_{i\in I} A_i$.

Proof. Item 1, *Functoriality*: This follows from **??**, **??** of **??**. □

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for each $(a_j)_{j\in I} \in \prod_{i\in I} A_i$ and each $i \in I$.

1.3 Binary Products of Sets

Let A and B be sets.

Definition 1.3.1.1. The **product**[3] **of A and B** is the pair $(A \times B, \{\mathrm{pr}_1, \mathrm{pr}_2\})$ consisting of:

- *The Limit.* The set $A \times B$ defined by[4]

$$\begin{aligned} A \times B &\overset{\text{def}}{=} \prod_{z \in \{A,B\}} z \\ &\overset{\text{def}}{=} \{f \in \mathsf{Sets}(\{0,1\}, A \cup B) \mid \text{we have } f(0) \in A \text{ and } f(1) \in B\} \\ &\cong \{\{\{a\},\{a,b\}\} \in \mathcal{P}(\mathcal{P}(A \cup B)) \mid \text{we have } a \in A \text{ and } b \in B\}. \end{aligned}$$

- *The Cone.* The maps

$$\begin{aligned} \mathrm{pr}_1 &\colon A \times B \to A, \\ \mathrm{pr}_2 &\colon A \times B \to B \end{aligned}$$

  defined by

$$\begin{aligned} \mathrm{pr}_1(a,b) &\overset{\text{def}}{=} a, \\ \mathrm{pr}_2(a,b) &\overset{\text{def}}{=} b \end{aligned}$$

  for each $(a,b) \in A \times B$.

Proof. We claim that $A \times B$ is the categorical product of A and B in Sets. Indeed, suppose we have a diagram of the form

$$\begin{tikzcd} & P \arrow[dl, bend right, "p_1"'] \arrow[dr, bend left, "p_2"] & \\ A & A \times B \arrow[l, "\mathrm{pr}_1"] \arrow[r, "\mathrm{pr}_2"'] & B \end{tikzcd}$$

[3] *Further Terminology:* Also called the **Cartesian product of A and B** or the **binary Cartesian product of A and B**, for emphasis.

This can also be thought of as the $(\mathbb{E}_{-1}, \mathbb{E}_{-1})$**-tensor product of A and B**.

[4] In other words, $A \times B$ is the set whose elements are ordered pairs (a,b) with $a \in A$ and $b \in B$ as in Definition 3.4.1.1

in Sets. Then there exists a unique map $\phi\colon P \to A \times B$ making the diagram

$$\begin{tikzcd}
 & P \arrow[dl, bend right, "p_1"'] \arrow[dr, bend left, "p_2"] \arrow[d, dashed, "\phi"', "\exists!"] & \\
A & A \times B \arrow[l, "\mathrm{pr}_1"] \arrow[r, "\mathrm{pr}_2"'] & B
\end{tikzcd}$$

commute, being uniquely determined by the conditions

$$\begin{aligned}
\mathrm{pr}_1 \circ \phi &= p_1,\\
\mathrm{pr}_2 \circ \phi &= p_2
\end{aligned}$$

via

$$\phi(x) = (p_1(x), p_2(x))$$

for each $x \in P$. □

Proposition 1.3.1.2. Let A, B, C, and X be sets.

1. *Functoriality.* The assignments $A, B, (A, B) \mapsto A \times B$ define functors

$$\begin{aligned}
A \times -\colon \mathsf{Sets} &\to \mathsf{Sets},\\
- \times B\colon \mathsf{Sets} &\to \mathsf{Sets},\\
-_1 \times -_2\colon \mathsf{Sets} \times \mathsf{Sets} &\to \mathsf{Sets},
\end{aligned}$$

   where $-_1 \times -_2$ is the functor where

   - *Action on Objects.* For each $(A, B) \in \mathrm{Obj}(\mathsf{Sets} \times \mathsf{Sets})$, we have

$$[-_1 \times -_2](A, B) \stackrel{\text{def}}{=} A \times B.$$

   - *Action on Morphisms.* For each $(A, B), (X, Y) \in \mathrm{Obj}(\mathsf{Sets})$, the action on Hom-sets

$$\times_{(A,B),(X,Y)}\colon \mathsf{Sets}(A, X) \times \mathsf{Sets}(B, Y) \to \mathsf{Sets}(A \times B, X \times Y)$$

     of $\times$ at $((A, B), (X, Y))$ is defined by sending (f, g) to the function

$$f \times g\colon A \times B \to X \times Y$$

     defined by

$$[f \times g](a, b) \stackrel{\text{def}}{=} (f(a), g(b))$$

     for each $(a, b) \in A \times B$.

and where $A \times -$ and $- \times B$ are the partial functors of $-_1 \times -_2$ at $A, B \in \mathrm{Obj}(\mathsf{Sets})$.

2. *Adjointness.* We have adjunctions

$$\begin{tikzcd}[ampersand replacement=\&] (A \times - \dashv \mathrm{Hom}_{\mathsf{Sets}}(A,-)): \quad \mathsf{Sets} \arrow[r, bend left, "A\times -"] \arrow[r, phantom, "\perp"] \& \mathsf{Sets}, \arrow[l, bend left, "\mathrm{Hom}_{\mathsf{Sets}}(A,-)"] \end{tikzcd}$$

$$\begin{tikzcd}[ampersand replacement=\&] (- \times B \dashv \mathrm{Hom}_{\mathsf{Sets}}(B,-)): \quad \mathsf{Sets} \arrow[r, bend left, "-\times B"] \arrow[r, phantom, "\perp"] \& \mathsf{Sets}, \arrow[l, bend left, "\mathrm{Hom}_{\mathsf{Sets}}(B,-)"] \end{tikzcd}$$

witnessed by bijections

$$\mathrm{Hom}_{\mathsf{Sets}}(A \times B, C) \cong \mathrm{Hom}_{\mathsf{Sets}}(A, \mathrm{Hom}_{\mathsf{Sets}}(B, C)),$$
$$\mathrm{Hom}_{\mathsf{Sets}}(A \times B, C) \cong \mathrm{Hom}_{\mathsf{Sets}}(B, \mathrm{Hom}_{\mathsf{Sets}}(A, C)),$$

natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

3. *Associativity.* We have an isomorphism of sets

$$(A \times B) \times C \cong A \times (B \times C),$$

natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

4. *Unitality.* We have isomorphisms of sets

$$\mathrm{pt} \times A \cong A,$$
$$A \times \mathrm{pt} \cong A,$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$.

5. *Commutativity.* We have an isomorphism of sets

$$A \times B \cong B \times A,$$

natural in $A, B \in \mathrm{Obj}(\mathsf{Sets})$.

6. *Annihilation With the Empty Set.* We have isomorphisms of sets

$$A \times \emptyset \cong \emptyset,$$
$$\emptyset \times A \cong \emptyset,$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$.

7. *Distributivity Over Unions.* We have isomorphisms of sets
$$A \times (B \cup C) = (A \times B) \cup (A \times C),$$
$$(A \cup B) \times C = (A \times C) \cup (B \times C).$$

8. *Distributivity Over Intersections.* We have isomorphisms of sets
$$A \times (B \cap C) = (A \times B) \cap (A \times C),$$
$$(A \cap B) \times C = (A \times C) \cap (B \times C).$$

9. *Middle-Four Exchange with Respect to Intersections.* We have an isomorphism of sets
$$(A \times B) \cap (C \times D) \cong (A \cap B) \times (C \cap D).$$

10. *Distributivity Over Differences.* We have isomorphisms of sets
$$A \times (B \setminus C) = (A \times B) \setminus (A \times C),$$
$$(A \setminus B) \times C = (A \times C) \setminus (B \times C),$$
natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

11. *Distributivity Over Symmetric Differences.* We have isomorphisms of sets
$$A \times (B \mathbin{\triangle} C) = (A \times B) \mathbin{\triangle} (A \times C),$$
$$(A \mathbin{\triangle} B) \times C = (A \times C) \mathbin{\triangle} (B \times C),$$
natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

12. *Symmetric Monoidality.* The triple $(\mathsf{Sets}, \times, \mathrm{pt})$ is a symmetric monoidal category.

13. *Symmetric Bimonoidality.* The quintuple $(\mathsf{Sets}, \coprod, \emptyset, \times, \mathrm{pt})$ is a symmetric bimonoidal category.

Proof. *Item 1*, *Functoriality*: This follows from **??**, **??** of **??**.
Item 2, *Adjointness*: We prove only that there's an adjunction $- \times B \dashv \mathrm{Hom}_{\mathsf{Sets}}(B, -)$, witnessed by a bijection
$$\mathrm{Hom}_{\mathsf{Sets}}(A \times B, C) \cong \mathrm{Hom}_{\mathsf{Sets}}(A, \mathrm{Hom}_{\mathsf{Sets}}(B, C)),$$
natural in $B, C \in \mathrm{Obj}(\mathsf{Sets})$, as the proof of the existence of the adjunction $A \times - \dashv \mathrm{Hom}_{\mathsf{Sets}}(A, -)$ follows almost exactly in the same way.

- *Map I.* We define a map

  $$\Phi_{B,C}\colon \mathsf{Hom}_{\mathsf{Sets}}(A\times B,C)\to \mathsf{Hom}_{\mathsf{Sets}}(A,\mathsf{Hom}_{\mathsf{Sets}}(B,C)),$$

  by sending a function

  $$\xi\colon A\times B\to C$$

  to the function

  $$\begin{aligned}\xi^{\dagger}\colon A &\longrightarrow \mathsf{Hom}_{\mathsf{Sets}}(B,C),\\ a &\longmapsto (\xi^{\dagger}_{a}\colon B\to C),\end{aligned}$$

  where we define

  $$\xi^{\dagger}_{a}(b)\overset{\text{def}}{=}\xi(a,b)$$

  for each $b \in B$. In terms of the $[\![a \mapsto f(a)]\!]$ notation of Sets, Notation 1.1.1.2, we have

  $$\xi^{\dagger}\overset{\text{def}}{=}[\![a\mapsto[\![b\mapsto\xi(a,b)]\!]]\!].$$

- *Map II.* We define a map

  $$\Psi_{B,C}\colon \mathsf{Hom}_{\mathsf{Sets}}(A,\mathsf{Hom}_{\mathsf{Sets}}(B,C)),\to \mathsf{Hom}_{\mathsf{Sets}}(A\times B,C)$$

  given by sending a function

  $$\begin{aligned}\xi\colon A &\longrightarrow \mathsf{Hom}_{\mathsf{Sets}}(B,C),\\ a &\longmapsto (\xi_{a}\colon B\to C),\end{aligned}$$

  to the function

  $$\xi^{\dagger}\colon A\times B\to C$$

  defined by

  $$\begin{aligned}\xi^{\dagger}(a,b) &\overset{\text{def}}{=} \mathsf{ev}_b(\mathsf{ev}_a(\xi))\\ &\overset{\text{def}}{=} \mathsf{ev}_b(\xi_a)\\ &\overset{\text{def}}{=} \xi_a(b)\end{aligned}$$

  for each $(a,b)\in A\times B$.

- *Invertibility I.* We claim that

$$\Psi_{A,B} \circ \Phi_{A,B} = \mathrm{id}_{\mathrm{Hom}_{\mathsf{Sets}}(A \times B, C)}.$$

Indeed, given a function $\xi \colon A \times B \to C$, we have

$$\begin{aligned}
[\Psi_{A,B} \circ \Phi_{A,B}](\xi) &= \Psi_{A,B}(\Phi_{A,B}(\xi)) \\
&= \Psi_{A,B}(\Phi_{A,B}(\llbracket (a,b) \mapsto \xi(a,b) \rrbracket)) \\
&= \Psi_{A,B}(\llbracket a \mapsto \llbracket b \mapsto \xi(a,b) \rrbracket \rrbracket) \\
&= \Psi_{A,B}(\llbracket a' \mapsto \llbracket b' \mapsto \xi(a',b') \rrbracket \rrbracket) \\
&= \llbracket (a,b) \mapsto \mathrm{ev}_b(\mathrm{ev}_a(\llbracket a' \mapsto \llbracket b' \mapsto \xi(a',b') \rrbracket \rrbracket)) \rrbracket \\
&= \llbracket (a,b) \mapsto \mathrm{ev}_b(\llbracket b' \mapsto \xi(a,b') \rrbracket) \rrbracket \\
&= \llbracket (a,b) \mapsto \xi(a,b) \rrbracket \\
&= \xi.
\end{aligned}$$

- *Invertibility II.* We claim that

$$\Phi_{A,B} \circ \Psi_{A,B} = \mathrm{id}_{\mathrm{Hom}_{\mathsf{Sets}}(A, \mathrm{Hom}_{\mathsf{Sets}}(B,C))}.$$

Indeed, given a function

$$\begin{aligned}
\xi \colon A &\longrightarrow \mathrm{Hom}_{\mathsf{Sets}}(B,C), \\
a &\longmapsto (\xi_a \colon B \to C),
\end{aligned}$$

we have

$$\begin{aligned}
[\Phi_{A,B} \circ \Psi_{A,B}](\xi) &\overset{\text{def}}{=} \Phi_{A,B}(\Psi_{A,B}(\xi)) \\
&\overset{\text{def}}{=} \Phi_{A,B}(\llbracket (a,b) \mapsto \xi_a(b) \rrbracket) \\
&\overset{\text{def}}{=} \Phi_{A,B}(\llbracket (a',b') \mapsto \xi_{a'}(b') \rrbracket) \\
&\overset{\text{def}}{=} \llbracket a \mapsto \llbracket b \mapsto \mathrm{ev}_{(a,b)}(\llbracket (a',b') \mapsto \xi_{a'}(b') \rrbracket) \rrbracket \rrbracket \\
&\overset{\text{def}}{=} \llbracket a \mapsto \llbracket b \mapsto \xi_a(b) \rrbracket \rrbracket \\
&\overset{\text{def}}{=} \llbracket a \mapsto \xi_a \rrbracket \\
&\overset{\text{def}}{=} \xi.
\end{aligned}$$

- *Naturality for Φ, Part I.* We need to show that, given a function $g \colon B \to B'$, the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathsf{Sets}}(A \times B', C) \arrow[r, "\Phi_{B',C}"] \arrow[d, "\mathrm{id}_A \times g^*"'] & \mathrm{Hom}_{\mathsf{Sets}}(A, \mathrm{Hom}_{\mathsf{Sets}}(B', C)), \arrow[d, "(g^*)_*"] \\
\mathrm{Hom}_{\mathsf{Sets}}(A \times B, C) \arrow[r, "\Phi_{B,C}"'] & \mathrm{Hom}_{\mathsf{Sets}}(A, \mathrm{Hom}_{\mathsf{Sets}}(B, C))
\end{tikzcd}$$

commutes. Indeed, given a function

$$\xi \colon A \times B' \to C,$$

we have

$$\begin{aligned}
[\Phi_{B,C} \circ (\mathrm{id}_A \times g^*)](\xi) &= \Phi_{B,C}([\mathrm{id}_A \times g^*](\xi)) \\
&= \Phi_{B,C}(\xi(-_1, g(-_2))) \\
&= [\xi(-_1, g(-_2))]^\dagger \\
&= \xi^\dagger_{-_1}(g(-_2)) \\
&= (g^*)_*(\xi^\dagger) \\
&= (g^*)_*(\Phi_{B',C}(\xi)) \\
&= [(g^*)_* \circ \Phi_{B',C}](\xi).
\end{aligned}$$

Alternatively, using the $[\![a \mapsto f(a)]\!]$ notation of Sets, Notation 1.1.1.2, we have

$$\begin{aligned}
[\Phi_{B,C} \circ (\mathrm{id}_A \times g^*)](\xi) &= \Phi_{B,C}([\mathrm{id}_A \times g^*](\xi)) \\
&= \Phi_{B,C}([\mathrm{id}_A \times g^*]([\![(a, b') \mapsto \xi(a, b')]\!])) \\
&= \Phi_{B,C}([\![(a, b) \mapsto \xi(a, g(b))]\!]) \\
&= [\![a \mapsto [\![b \mapsto \xi(a, g(b))]\!]]\!] \\
&= [\![a \mapsto g^*([\![b' \mapsto \xi(a, b')]\!])]\!] \\
&= (g^*)_*([\![a \mapsto [\![b' \mapsto \xi(a, b')]\!]]\!]) \\
&= (g^*)_*(\Phi_{B',C}([\![(a, b') \mapsto \xi(a, b')]\!])) \\
&= (g^*)_*(\Phi_{B',C}(\xi)) \\
&= [(g^*)_* \circ \Phi_{B',C}](\xi).
\end{aligned}$$

- *Naturality for Φ, Part II.* We need to show that, given a function $h\colon C \to C'$, the diagram

$$\begin{tikzcd}
\mathsf{Hom}_{\mathsf{Sets}}(A \times B, C) \arrow[r, "\Phi_{B,C}"] \arrow[d, "h_*"'] & \mathsf{Hom}_{\mathsf{Sets}}(A, \mathsf{Hom}_{\mathsf{Sets}}(B, C)), \arrow[d, "(h_*)_*"] \\
\mathsf{Hom}_{\mathsf{Sets}}(A \times B, C') \arrow[r, "\Phi_{B,C'}"'] & \mathsf{Hom}_{\mathsf{Sets}}(A, \mathsf{Hom}_{\mathsf{Sets}}(B, C'))
\end{tikzcd}$$

  commutes. Indeed, given a function

$$\xi\colon A \times B \to C,$$

  we have

$$\begin{aligned}
[\Phi_{B,C} \circ h_*](\xi) &= \Phi_{B,C}(h_*(\xi)) \\
&= \Phi_{B,C}(h_*(\llbracket (a,b) \mapsto \xi(a,b) \rrbracket)) \\
&= \Phi_{B,C}(\llbracket (a,b) \mapsto h(\xi(a,b)) \rrbracket) \\
&= \llbracket a \mapsto \llbracket b \mapsto h(\xi(a,b)) \rrbracket \rrbracket \\
&= \llbracket a \mapsto h_*(\llbracket b \mapsto \xi(a,b) \rrbracket) \rrbracket \\
&= (h_*)_*(\llbracket a \mapsto \llbracket b \mapsto \xi(a,b) \rrbracket \rrbracket) \\
&= (h_*)_*(\Phi_{B,C}(\llbracket (a,b) \mapsto \xi(a,b) \rrbracket)) \\
&= (h_*)_*(\Phi_{B,C}(\xi)) \\
&= [(h_*)_* \circ \Phi_{B,C}](\xi).
\end{aligned}$$

- *Naturality for Ψ.* Since Φ is natural in each argument and Φ is a componentwise inverse to Ψ in each argument, it follows from Categories, Item 2 of Proposition 8.6.1.2 that Ψ is also natural in each argument.

Item 3, *Associativity*: See [Pro24a].
Item 4, *Unitality*: Clear.
Item 5, *Commutativity*: See [Pro24b].
Item 6, *Annihilation With the Empty Set*: See [Pro24f].
Item 7, *Distributivity Over Unions*: See [Pro24e].
Item 8, *Distributivity Over Intersections*: See [Pro24g, Corollary 1].
Item 9, *Middle-Four Exchange With Respect to Intersections*: See [Pro24g, Corollary 1].
Item 10, *Distributivity Over Differences*: See [Pro24c].

Item 11, Distributivity Over Symmetric Differences: See [Pro24d].
Item 12, Symmetric Monoidality: See [MO 382264].
Item 13, Symmetric Bimonoidality: Omitted. □

1.4 Pullbacks

Let A, B, and C be sets and let $f\colon A \longrightarrow C$ and $g\colon B \longrightarrow C$ be functions.

Definition 1.4.1.1. The **pullback of A and B over C along f and g**[5] is the pair[6] $(A \times_C B, \{\mathrm{pr}_1, \mathrm{pr}_2\})$ consisting of:

- *The Limit.* The set $A \times_C B$ defined by

$$A \times_C B \overset{\text{def}}{=} \{(a, b) \in A \times B \mid f(a) = g(b)\}.$$

- *The Cone.* The maps

$$\begin{aligned}\mathrm{pr}_1\colon A \times_C B &\longrightarrow A,\\ \mathrm{pr}_2\colon A \times_C B &\longrightarrow B\end{aligned}$$

  defined by

$$\begin{aligned}\mathrm{pr}_1(a, b) &\overset{\text{def}}{=} a,\\ \mathrm{pr}_2(a, b) &\overset{\text{def}}{=} b\end{aligned}$$

  for each $(a, b) \in A \times_C B$.

Proof. We claim that $A \times_C B$ is the categorical pullback of A and B over C with respect to (f, g) in Sets. First we need to check that the relevant pullback diagram commutes, i.e. that we have

$$f \circ \mathrm{pr}_1 = g \circ \mathrm{pr}_2, \qquad \begin{tikzcd} A \times_C B \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C. \end{tikzcd}$$

[5] *Further Terminology:* Also called the **fibre product of A and B over C along f and g**.
[6] *Further Notation:* Also written $A \times_{f,C,g} B$.

Indeed, given $(a,b) \in A \times_C B$, we have

$$\begin{aligned}
[f \circ \mathrm{pr}_1](a,b) &= f(\mathrm{pr}_1(a,b)) \\
&= f(a) \\
&= g(b) \\
&= g(\mathrm{pr}_2(a,b)) \\
&= [g \circ \mathrm{pr}_2](a,b),
\end{aligned}$$

where $f(a) = g(b)$ since $(a,b) \in A \times_C B$. Next, we prove that $A \times_C B$ satisfies the universal property of the pullback. Suppose we have a diagram of the form

$$\begin{tikzcd}
P \arrow[drr, bend left, "p_2"] \arrow[ddr, bend right, "p_1"'] & & \\
& A \times_C B \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "g"] \\
& A \arrow[r, "f"'] & C
\end{tikzcd}$$

in Sets. Then there exists a unique map $\phi \colon P \to A \times_C B$ making the diagram

$$\begin{tikzcd}
P \arrow[drr, bend left, "p_2"] \arrow[ddr, bend right, "p_1"'] \arrow[dr, dashed, "\phi", "\exists!"'] & & \\
& A \times_C B \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "g"] \\
& A \arrow[r, "f"'] & C
\end{tikzcd}$$

commute, being uniquely determined by the conditions

$$\begin{aligned}
\mathrm{pr}_1 \circ \phi &= p_1, \\
\mathrm{pr}_2 \circ \phi &= p_2
\end{aligned}$$

via

$$\phi(x) = (p_1(x), p_2(x))$$

for each $x \in P$, where we note that $(p_1(x), p_2(x)) \in A \times B$ indeed lies in $A \times_C B$ by the condition

$$f \circ p_1 = g \circ p_2,$$

which gives

$$f(p_1(x)) = g(p_2(x))$$

for each $x \in P$, so that $(p_1(x), p_2(x)) \in A \times_C B$. □

Example 1.4.1.2. Here are some examples of pullbacks of sets.

1. *Unions via Intersections.* Let $A, B \subset X$. We have a bijection of sets

$$A \cap B \cong A \times_{A \cup B} B, \qquad \begin{tikzcd} A \cap B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "\iota_B"] \\ A \arrow[r, "\iota_A"'] & A \cup B. \end{tikzcd}$$

Proof. Item 1, *Unions via Intersections*: Indeed, we have

$$\begin{aligned} A \times_{A \cup B} B &\cong \{(x, y) \in A \times B \mid x = y\} \\ &\cong A \cap B. \end{aligned}$$

This finishes the proof. □

Proposition 1.4.1.3. Let $A, B, C,$ and X be sets.

1. *Functoriality.* The assignment $(A, B, C, f, g) \mapsto A \times_{f,C,g} B$ defines a functor

$$-_1 \times_{-_3} -_1 \colon \mathsf{Fun}(\mathcal{P}, \mathsf{Sets}) \to \mathsf{Sets},$$

where $\mathcal{P}$ is the category that looks like this:

$$\begin{tikzcd} & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet. \end{tikzcd}$$

In particular, the action on morphisms of $-_1 \times_{-_3} -_1$ is given by sending a morphism

$$\begin{tikzcd}
A \times_C B \arrow[rr] \arrow[dd] \arrow[dr, phantom, very near start, "\lrcorner"] & & B \arrow[dd, "g" near start] \arrow[dr, "\psi"] & \\
& A' \times_{C'} B' \arrow[rr] \arrow[dd] \arrow[dr, phantom, very near start, "\lrcorner"] & & B' \arrow[dd, "g'"] \\
A \arrow[rr, "f" near start] \arrow[dr, "\phi"'] & & C \arrow[dr, "\chi"] & \\
& A' \arrow[rr, "f'"'] & & C'
\end{tikzcd}$$

in $\mathrm{Fun}(\mathcal{P}, \mathrm{Sets})$ to the map $\xi\colon A \times_C B \xrightarrow{\exists!} A' \times_{C'} B'$ given by

$$\xi(a, b) \overset{\text{def}}{=} (\phi(a), \psi(b))$$

for each $(a, b) \in A \times_C B$, which is the unique map making the diagram

$$\begin{tikzcd}
A \times_C B \arrow[rr] \arrow[dd] \arrow[dr, dashed] & & B \arrow[dd, "g" near start] \arrow[dr, "\psi"] & \\
& A' \times_{C'} B' \arrow[rr] \arrow[dd] \arrow[dr, phantom, very near start, "\lrcorner"] & & B' \arrow[dd, "g'"] \\
A \arrow[rr, "f" near start] \arrow[dr, "\phi"'] & & C \arrow[dr, "\chi"] & \\
& A' \arrow[rr, "f'"'] & & C'
\end{tikzcd}$$

commute.

2. *Associativity.* Given a diagram

$$\begin{tikzcd}
A \arrow[dr, "f"'] & & B \arrow[dl, "g"] \arrow[dr, "h"'] & & C \arrow[dl, "k"] \\
& X & & Y &
\end{tikzcd}$$

in Sets, we have isomorphisms of sets

$$(A \times_X B) \times_Y C \cong (A \times_X B) \times_B (B \times_Y C) \cong A \times_X (B \times_Y C),$$

where these pullbacks are built as in the diagrams

$$
\begin{tikzcd}[column sep=tiny]
& & (A\times_X B)\times_Y C \arrow[dl] \arrow[ddrr] \arrow[d, phantom, "\vee"] & & \\
& A\times_X B \arrow[dl] \arrow[dr] \arrow[d, phantom, "\vee"] & {} & & \\
A \arrow[dr, "f"'] & {} & B \arrow[dl, "g"] \arrow[dr, "h"'] & & C, \arrow[dl, "k"] \\
& X & & Y &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=tiny]
& & (A\times_X B)\times_B (B\times_Y C) \arrow[dl] \arrow[dr] \arrow[d, phantom, "\vee"] & & \\
& A\times_X B \arrow[dl] \arrow[dr] \arrow[d, phantom, "\vee"] & {} & B\times_Y C \arrow[dl] \arrow[dr] \arrow[d, phantom, "\vee"] & \\
A \arrow[dr, "f"'] & {} & B \arrow[dl, "g"] \arrow[dr, "h"'] & {} & C, \arrow[dl, "k"] \\
& X & & Y &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=tiny]
& & & A\times_X (B\times_Y C) \arrow[ddlll] \arrow[dr] \arrow[d, phantom, "\vee"] & \\
& & & B\times_Y C \arrow[dl] \arrow[dr] \arrow[d, phantom, "\vee"] & \\
A \arrow[dr, "f"'] & & B \arrow[dl, "g"] \arrow[dr, "h"'] & {} & C. \arrow[dl, "k"] \\
& X & & Y &
\end{tikzcd}
$$

3. *Unitality.* We have isomorphisms of sets

$$
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "f"'] \arrow[dr, phantom, very near start, "\lrcorner"] & A \arrow[d, "f"] \\
X \arrow[r, equal] & X
\end{tikzcd}
\qquad
\begin{aligned}
X \times_X A &\cong A, \\
A \times_X X &\cong A,
\end{aligned}
\qquad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, equal] \arrow[dr, phantom, very near start, "\lrcorner"] & X \arrow[d, equal] \\
X \arrow[r, "f"'] & X.
\end{tikzcd}
$$

4. *Commutativity.* We have an isomorphism of sets

$$
\begin{tikzcd}
A\times_X B \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & B \arrow[d, "g"] \\
A \arrow[r, "f"'] & X,
\end{tikzcd}
\qquad
A \times_X B \cong B \times_X A
\qquad
\begin{tikzcd}
B\times_X A \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & A \arrow[d, "f"] \\
B \arrow[r, "g"'] & X.
\end{tikzcd}
$$

5. *Annihilation With the Empty Set.* We have isomorphisms of sets

$$
\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \emptyset \arrow[d] \\
A \arrow[r, "f"'] & X,
\end{tikzcd}
\qquad
\begin{aligned}
A \times_X \emptyset &\cong \emptyset, \\
\emptyset \times_X A &\cong \emptyset,
\end{aligned}
\qquad
\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & A \arrow[d, "f"] \\
\emptyset \arrow[r] & X.
\end{tikzcd}
$$

6. *Interaction With Products.* We have an isomorphism of sets

$$
A \times_{\mathrm{pt}} B \cong A \times B,
\qquad
\begin{tikzcd}
A\times B \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & B \arrow[d, "!_B"] \\
A \arrow[r, "!_A"'] & \mathrm{pt}.
\end{tikzcd}
$$

7. *Symmetric Monoidality.* The triple $(\mathsf{Sets}, \times_X, X)$ is a symmetric monoidal category.

Proof. *Item 1, Functoriality*: This is a special case of functoriality of co/limits, **??**, **??** of **??**, with the explicit expression for ξ following from the commutativity of the cube pullback diagram.
Item 2, Associativity: Indeed, we have

$$\begin{aligned}
(A \times_X B) \times_Y C &\cong \{((a,b),c) \in (A \times_X B) \times C \mid h(b) = k(c)\} \\
&\cong \{((a,b),c) \in (A \times B) \times C \mid f(a) = g(b) \text{ and } h(b) = k(c)\} \\
&\cong \{(a,(b,c)) \in A \times (B \times C) \mid f(a) = g(b) \text{ and } h(b) = k(c)\} \\
&\cong \{(a,(b,c)) \in A \times (B \times_Y C) \mid f(a) = g(b)\} \\
&\cong A \times_X (B \times_Y C)
\end{aligned}$$

and

$$\begin{aligned}
(A \times_X B) \times_B (B \times_Y C) &\cong \left\{((a,b),(b',c)) \in (A \times_X B) \times (B \times_Y C) \,\middle|\, b = b'\right\} \\
&\cong \left\{((a,b),(b',c)) \in (A \times B) \times (B \times C) \,\middle|\, \begin{array}{l} f(a) = g(b), b = b', \\ \text{and } h(b') = k(c) \end{array}\right\} \\
&\cong \left\{(a,(b,(b',c))) \in A \times (B \times (B \times C)) \,\middle|\, \begin{array}{l} f(a) = g(b), b = b', \\ \text{and } h(b') = k(c) \end{array}\right\} \\
&\cong \left\{(a,((b,b'),c)) \in A \times ((B \times B) \times C) \,\middle|\, \begin{array}{l} f(a) = g(b), b = b', \\ \text{and } h(b') = k(c) \end{array}\right\} \\
&\cong \left\{(a,((b,b'),c)) \in A \times ((B \times_B B) \times C) \,\middle|\, \begin{array}{l} f(a) = g(b) \text{ and} \\ h(b') = k(c) \end{array}\right\} \\
&\cong \{(a,(b,c)) \in A \times (B \times C) \mid f(a) = g(b) \text{ and } h(b) = k(c)\} \\
&\cong A \times_X (B \times_Y C),
\end{aligned}$$

where we have used Item 3 for the isomorphism $B \times_B B \cong B$.
Item 3, Unitality: Indeed, we have

$$\begin{aligned}
X \times_X A &\cong \{(x,a) \in X \times A \mid f(a) = x\}, \\
A \times_X X &\cong \{(a,x) \in X \times A \mid f(a) = x\},
\end{aligned}$$

which are isomorphic to A via the maps $(x,a) \mapsto a$ and $(a,x) \mapsto a$.
Item 4, Commutativity: Clear.
Item 5, Annihilation With the Empty Set: Clear.
Item 6, Interaction With Products: Clear.
Item 7, Symmetric Monoidality: Omitted. □

1.5 Equalisers

Let A and B be sets and let $f, g \colon A \rightrightarrows B$ be functions.

Definition 1.5.1.1. The **equaliser of** f **and** g is the pair $(\mathrm{Eq}(f,g), \mathrm{eq}(f,g))$ consisting of:

- *The Limit.* The set $\mathrm{Eq}(f,g)$ defined by
$$\mathrm{Eq}(f,g) \stackrel{\text{def}}{=} \{a \in A \mid f(a) = g(a)\}.$$
- *The Cone.* The inclusion map
$$\mathrm{eq}(f,g) \colon \mathrm{Eq}(f,g) \hookrightarrow A.$$

Proof. We claim that $\mathrm{Eq}(f,g)$ is the categorical equaliser of f and g in Sets. First we need to check that the relevant equaliser diagram commutes, i.e. that we have
$$f \circ \mathrm{eq}(f,g) = g \circ \mathrm{eq}(f,g),$$
which indeed holds by the definition of the set $\mathrm{Eq}(f,g)$. Next, we prove that $\mathrm{Eq}(f,g)$ satisfies the universal property of the equaliser. Suppose we have a diagram of the form

$$\begin{tikzcd}
\mathrm{Eq}(f,g) \arrow[r, hook, "\mathrm{eq}(f,g)"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\
E \arrow[ur, "e"'] & &
\end{tikzcd}$$

in Sets. Then there exists a unique map $\phi \colon E \to \mathrm{Eq}(f,g)$ making the diagram

$$\begin{tikzcd}
\mathrm{Eq}(f,g) \arrow[r, hook, "\mathrm{eq}(f,g)"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\
E \arrow[u, dashed, "\phi", "\exists!"'] \arrow[ur, "e"'] & &
\end{tikzcd}$$

commute, being uniquely determined by the condition
$$\mathrm{eq}(f,g) \circ \phi = e$$

via

$$\phi(x) = e(x)$$

for each $x \in E$, where we note that $e(x) \in A$ indeed lies in $\mathrm{Eq}(f, g)$ by the condition

$$f \circ e = g \circ e,$$

which gives

$$f(e(x)) = g(e(x))$$

for each $x \in E$, so that $e(x) \in \mathrm{Eq}(f, g)$. □

Proposition 1.5.1.2. Let A, B, and C be sets.

1. *Associativity.* We have isomorphisms of sets[7]

$$\underbrace{\mathrm{Eq}(f \circ \mathrm{eq}(g,h), g \circ \mathrm{eq}(g,h))}_{=\mathrm{Eq}(f \circ \mathrm{eq}(g,h), h \circ \mathrm{eq}(g,h))} \cong \mathrm{Eq}(f,g,h) \cong \underbrace{\mathrm{Eq}(f \circ \mathrm{eq}(f,g), h \circ \mathrm{eq}(f,g))}_{=\mathrm{Eq}(g \circ \mathrm{eq}(f,g), h \circ \mathrm{eq}(f,g))},$$

[7] That is, the following three ways of forming "the" equaliser of (f, g, h) agree:

1. Take the equaliser of (f, g, h), i.e. the limit of the diagram

$$\begin{tikzcd} A \arrow[r, shift left=2, "f"] \arrow[r, "g" description] \arrow[r, shift right=2, "h"'] & B \end{tikzcd}$$

   in Sets.

2. First take the equaliser of f and g, forming a diagram

$$\mathrm{Eq}(f,g) \overset{\mathrm{eq}(f,g)}{\hookrightarrow} A \underset{g}{\overset{f}{\rightrightarrows}} B$$

   and then take the equaliser of the composition

$$\mathrm{Eq}(f,g) \overset{\mathrm{eq}(f,g)}{\hookrightarrow} A \underset{h}{\overset{f}{\rightrightarrows}} B,$$

   obtaining a subset

$$\mathrm{Eq}(f \circ \mathrm{eq}(f,g), h \circ \mathrm{eq}(f,g)) = \mathrm{Eq}(g \circ \mathrm{eq}(f,g), h \circ \mathrm{eq}(f,g))$$

   of $\mathrm{Eq}(f, g)$.

3. First take the equaliser of g and h, forming a diagram

$$\mathrm{Eq}(g,h) \overset{\mathrm{eq}(g,h)}{\hookrightarrow} A \underset{h}{\overset{g}{\rightrightarrows}} B$$

where $\mathrm{Eq}(f,g,h)$ is the limit of the diagram

$$A \overset{f}{\underset{h}{\overset{- g \rightarrow}{\rightrightarrows}}} B$$

in Sets, being explicitly given by

$$\mathrm{Eq}(f,g,h) \cong \{a \in A \,|\, f(a) = g(a) = h(a)\}.$$

4. *Unitality.* We have an isomorphism of sets

$$\mathrm{Eq}(f,f) \cong A.$$

5. *Commutativity.* We have an isomorphism of sets

$$\mathrm{Eq}(f,g) \cong \mathrm{Eq}(g,f).$$

6. *Interaction With Composition.* Let

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \underset{k}{\overset{h}{\rightrightarrows}} C$$

be functions. We have an inclusion of sets

$$\mathrm{Eq}(h \circ f \circ \mathrm{eq}(f,g), k \circ g \circ \mathrm{eq}(f,g)) \subset \mathrm{Eq}(h \circ f, k \circ g),$$

where $\mathrm{Eq}(h \circ f \circ \mathrm{eq}(f,g), k \circ g \circ \mathrm{eq}(f,g))$ is the equaliser of the composition

$$\mathrm{Eq}(f,g) \overset{\mathrm{eq}(f,g)}{\hookrightarrow} A \underset{g}{\overset{f}{\rightrightarrows}} B \underset{k}{\overset{h}{\rightrightarrows}} C.$$

---

and then take the equaliser of the composition

$$\mathrm{Eq}(g,h) \overset{\mathrm{eq}(g,h)}{\hookrightarrow} A \underset{g}{\overset{f}{\rightrightarrows}} B,$$

obtaining a subset

$$\mathrm{Eq}(f \circ \mathrm{eq}(g,h), g \circ \mathrm{eq}(g,h)) = \mathrm{Eq}(f \circ \mathrm{eq}(g,h), h \circ \mathrm{eq}(g,h))$$

of $\mathrm{Eq}(g,h)$.

Proof. *Item 1, Associativity*: We first prove that $\mathsf{Eq}(f,g,h)$ is indeed given by

$$\mathsf{Eq}(f,g,h) \cong \{a \in A \mid f(a) = g(a) = h(a)\}.$$

Indeed, suppose we have a diagram of the form

$$\begin{tikzcd}
\mathsf{Eq}(f,g,h) \arrow[r, hook, "\mathsf{eq}(f,g,h)"] & A \arrow[r, shift left=2, "f"] \arrow[r, "g" description] \arrow[r, shift right=2, "h"'] & B \\
E \arrow[ur, "e"'] & &
\end{tikzcd}$$

in Sets. Then there exists a unique map $\phi\colon E \longrightarrow \mathsf{Eq}(f,g,h)$, uniquely determined by the condition

$$\mathsf{eq}(f,g) \circ \phi = e$$

being necessarily given by

$$\phi(x) = e(x)$$

for each $x \in E$, where we note that $e(x) \in A$ indeed lies in $\mathsf{Eq}(f,g,h)$ by the condition

$$f \circ e = g \circ e = h \circ e,$$

which gives

$$f(e(x)) = g(e(x)) = h(e(x))$$

for each $x \in E$, so that $e(x) \in \mathsf{Eq}(f,g,h)$.
We now check the equalities

$$\mathsf{Eq}(f \circ \mathsf{eq}(g,h), g \circ \mathsf{eq}(g,h)) \cong \mathsf{Eq}(f,g,h) \cong \mathsf{Eq}(f \circ \mathsf{eq}(f,g), h \circ \mathsf{eq}(f,g)).$$

Indeed, we have

$$\begin{aligned}
\mathsf{Eq}(f \circ \mathsf{eq}(g,h), g \circ \mathsf{eq}(g,h)) &\cong \{x \in \mathsf{Eq}(g,h) \mid [f \circ \mathsf{eq}(g,h)](a) = [g \circ \mathsf{eq}(g,h)](a)\} \\
&\cong \{x \in \mathsf{Eq}(g,h) \mid f(a) = g(a)\} \\
&\cong \{x \in A \mid f(a) = g(a) \text{ and } g(a) = h(a)\} \\
&\cong \{x \in A \mid f(a) = g(a) = h(a)\} \\
&\cong \mathsf{Eq}(f,g,h).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}\mathrm{Eq}(f\circ \mathrm{eq}(f,g), h\circ \mathrm{eq}(f,g)) &\cong \{x\in \mathrm{Eq}(f,g) \mid [f\circ \mathrm{eq}(f,g)](a) = [h\circ \mathrm{eq}(f,g)](a)\}\\ &\cong \{x\in \mathrm{Eq}(f,g) \mid f(a) = h(a)\}\\ &\cong \{x\in A \mid f(a) = h(a) \text{ and } f(a) = g(a)\}\\ &\cong \{x\in A \mid f(a) = g(a) = h(a)\}\\ &\cong \mathrm{Eq}(f,g,h).\end{aligned}$$

Item 4, Unitality: Clear.
Item 5, Commutativity: Clear.
Item 6, Interaction With Composition: Indeed, we have

$$\begin{aligned}\mathrm{Eq}(h\circ f\circ \mathrm{eq}(f,g), k\circ g\circ \mathrm{eq}(f,g)) &\cong \{a\in \mathrm{Eq}(f,g) \mid h(f(a)) = k(g(a))\}\\ &\cong \{a\in A \mid f(a) = g(a) \text{ and } h(f(a)) = k(g(a))\}.\end{aligned}$$

and

$$\mathrm{Eq}(h\circ f, k\circ g) \cong \{a\in A \mid h(f(a)) = k(g(a))\},$$

and thus there's an inclusion from $\mathrm{Eq}(h\circ f\circ \mathrm{eq}(f,g), k\circ g\circ \mathrm{eq}(f,g))$ to $\mathrm{Eq}(h\circ f, k\circ g)$. □

2 Colimits of Sets

2.1 The Initial Set

Definition 2.1.1.1. The **initial set** is the pair $(\emptyset, \{\iota_A\}_{A\in\mathrm{Obj}(\mathsf{Sets})})$ consisting of:

- *The Limit.* The empty set $\emptyset$ of Definition 3.1.1.1.
- *The Cone.* The collection of maps

  $$\{\iota_A\colon \emptyset \longrightarrow A\}_{A\in\mathrm{Obj}(\mathsf{Sets})}$$

  given by the inclusion maps from $\emptyset$ to A.

Proof. We claim that $\emptyset$ is the initial object of Sets. Indeed, suppose we have a diagram of the form

$$\emptyset \qquad A$$

in Sets. Then there exists a unique map $\phi\colon \emptyset \longrightarrow A$ making the diagram

$$\begin{tikzcd}\emptyset \arrow[r, dashed, "\phi", "\exists!"'] & A\end{tikzcd}$$

commute, namely the inclusion map ι_A. □

2.2 Coproducts of Families of Sets

Let $\{A_i\}_{i\in I}$ be a family of sets.

Definition 2.2.1.1. The **disjoint union of the family** $\{A_i\}_{i\in I}$ is the pair $(\coprod_{i\in I} A_i, \{\mathrm{inj}_i\}_{i\in I})$ consisting of:

- *The Colimit.* The set $\coprod_{i\in I} A_i$ defined by

$$\coprod_{i\in I} A_i \stackrel{\text{def}}{=} \left\{ (i,x) \in I \times (\bigcup_{i\in I} A_i) \;\middle|\; x \in A_i \right\}.$$

- *The Cocone.* The collection

$$\left\{ \mathrm{inj}_i \colon A_i \to \coprod_{i\in I} A_i \right\}_{i\in I}$$

  of maps given by

$$\mathrm{inj}_i(x) \stackrel{\text{def}}{=} (i,x)$$

  for each $x \in A_i$ and each $i \in I$.

Proof. We claim that $\coprod_{i\in I} A_i$ is the categorical coproduct of $\{A_i\}_{i\in I}$ in Sets. Indeed, suppose we have, for each $i \in I$, a diagram of the form

$$\begin{tikzcd} & C \\ A_i \arrow[ur, "\iota_i"] \arrow[r, "\mathrm{inj}_i"'] & \coprod_{i\in I} A_i \end{tikzcd}$$

in Sets. Then there exists a unique map $\phi \colon \coprod_{i\in I} A_i \to C$ making the diagram

$$\begin{tikzcd} & C \\ A_i \arrow[ur, "\iota_i"] \arrow[r, "\mathrm{inj}_i"'] & \coprod_{i\in I} A_i \arrow[u, dashed, "\phi", "\exists!"'] \end{tikzcd}$$

commute, being uniquely determined by the condition $\phi \circ \mathrm{inj}_i = \iota_i$ for each $i \in I$ via

$$\phi((i,x)) = \iota_i(x)$$

for each $(i,x) \in \coprod_{i\in I} A_i$. □

Proposition 2.2.1.2. Let $\{A_i\}_{i\in I}$ be a family of sets.

1. *Functoriality.* The assignment $\{A_i\}_{i\in I} \mapsto \coprod_{i\in I} A_i$ defines a functor

$$\coprod_{i\in I} \colon \mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}) \longrightarrow \mathsf{Sets}$$

where

- *Action on Objects.* For each $(A_i)_{i\in I} \in \mathrm{Obj}(\mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}))$, we have

$$\left[\coprod_{i\in I}\right]((A_i)_{i\in I}) \stackrel{\text{def}}{=} \coprod_{i\in I} A_i$$

- *Action on Morphisms.* For each $(A_i)_{i\in I}, (B_i)_{i\in I} \in \mathrm{Obj}(\mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}))$, the action on Hom-sets

$$(\coprod_{i\in I})_{(A_i)_{i\in I},(B_i)_{i\in I}} \colon \mathrm{Nat}((A_i)_{i\in I}, (B_i)_{i\in I}) \longrightarrow \mathsf{Sets}(\coprod_{i\in I} A_i, \coprod_{i\in I} B_i)$$

of $\coprod_{i\in I}$ at $((A_i)_{i\in I}, (B_i)_{i\in I})$ is defined by sending a map

$$\{f_i \colon A_i \longrightarrow B_i\}_{i\in I}$$

in $\mathrm{Nat}((A_i)_{i\in I}, (B_i)_{i\in I})$ to the map of sets

$$\coprod_{i\in I} f_i \colon \coprod_{i\in I} A_i \longrightarrow \coprod_{i\in I} B_i$$

defined by

$$\left[\coprod_{i\in I} f_i\right](i,a) \stackrel{\text{def}}{=} f_i(a)$$

for each $(i,a) \in \coprod_{i\in I} A_i$.

Proof. *Item 1*, *Functoriality*: This follows from **??**, **??** of **??**. □

2.3 Binary Coproducts

Let A and B be sets.

Definition 2.3.1.1. The **coproduct**[8] **of** A **and** B is the pair $(A \coprod B, \{\mathrm{inj}_1, \mathrm{inj}_2\})$ consisting of:

- *The Colimit.* The set $A \coprod B$ defined by

$$\begin{aligned} A \coprod B &\stackrel{\text{def}}{=} \coprod_{z \in \{A,B\}} z \\ &\cong \{(0,a) \mid a \in A\} \cup \{(1,b) \mid b \in B\}. \end{aligned}$$

- *The Cocone.* The maps

$$\begin{aligned} \mathrm{inj}_1 &\colon A \to A \coprod B, \\ \mathrm{inj}_2 &\colon B \to A \coprod B, \end{aligned}$$

  given by

$$\begin{aligned} \mathrm{inj}_1(a) &\stackrel{\text{def}}{=} (0,a), \\ \mathrm{inj}_2(b) &\stackrel{\text{def}}{=} (1,b), \end{aligned}$$

  for each $a \in A$ and each $b \in B$.

Proof. We claim that $A \coprod B$ is the categorical coproduct of A and B in Sets. Indeed, suppose we have a diagram of the form

$$\begin{tikzcd} & C & \\ A \arrow[ur, bend left, "\iota_A"] \arrow[r, "\mathrm{inj}_A"'] & A \coprod B & B \arrow[ul, bend right, "\iota_B"'] \arrow[l, "\mathrm{inj}_B"] \end{tikzcd}$$

in Sets. Then there exists a unique map $\phi \colon A \coprod B \to C$ making the diagram

$$\begin{tikzcd} & C & \\ A \arrow[ur, bend left, "\iota_A"] \arrow[r, "\mathrm{inj}_A"'] & A \coprod B \arrow[u, dashed, "\phi", "\exists!"'] & B \arrow[ul, bend right, "\iota_B"'] \arrow[l, "\mathrm{inj}_B"] \end{tikzcd}$$

[8] *Further Terminology:* Also called the **disjoint union of** A **and** B, or the **binary disjoint union of**

commute, being uniquely determined by the conditions

$$\begin{aligned}\phi \circ \mathrm{inj}_A &= \iota_A,\\ \phi \circ \mathrm{inj}_B &= \iota_B\end{aligned}$$

via

$$\phi(x) = \begin{cases} \iota_A(a) & \text{if } x = (0, a), \\ \iota_B(b) & \text{if } x = (1, b) \end{cases}$$

for each $x \in A \coprod B$. □

Proposition 2.3.1.2. Let A, B, C, and X be sets.

1. *Functoriality.* The assignment $A, B, (A, B) \mapsto A \coprod B$ defines functors

$$\begin{aligned} A \coprod - \colon \mathsf{Sets} &\to \mathsf{Sets}, \\ - \coprod B \colon \mathsf{Sets} &\to \mathsf{Sets}, \\ -_1 \coprod -_2 \colon \mathsf{Sets} \times \mathsf{Sets} &\to \mathsf{Sets}, \end{aligned}$$

where $-_1 \coprod -_2$ is the functor where

- *Action on Objects.* For each $(A, B) \in \mathrm{Obj}(\mathsf{Sets} \times \mathsf{Sets})$, we have

$$[-_1 \coprod -_2](A, B) \stackrel{\text{def}}{=} A \coprod B.$$

- *Action on Morphisms.* For each $(A, B), (X, Y) \in \mathrm{Obj}(\mathsf{Sets})$, the action on Hom-sets

$$\coprod_{(A,B),(X,Y)} \colon \mathsf{Sets}(A, X) \times \mathsf{Sets}(B, Y) \to \mathsf{Sets}(A \coprod B, X \coprod Y)$$

of $\coprod$ at $((A, B), (X, Y))$ is defined by sending (f, g) to the function

$$f \coprod g \colon A \coprod B \to X \coprod Y$$

defined by

$$[f \coprod g](x) \stackrel{\text{def}}{=} \begin{cases} (0, f(a)) & \text{if } x = (0, a), \\ (1, g(b)) & \text{if } x = (1, b), \end{cases}$$

for each $x \in A \coprod B$.

and where $A \coprod -$ and $- \coprod B$ are the partial functors of $-_1 \coprod -_2$ at $A, B \in \mathrm{Obj}(\mathsf{Sets})$.

2. *Associativity.* We have an isomorphism of sets

$$(A \coprod B) \coprod C \cong A \coprod (B \coprod C),$$

natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

3. *Unitality.* We have isomorphisms of sets

$$A \coprod \emptyset \cong A,$$
$$\emptyset \coprod A \cong A,$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$.

4. *Commutativity.* We have an isomorphism of sets

$$A \coprod B \cong B \coprod A,$$

natural in $A, B \in \mathrm{Obj}(\mathsf{Sets})$.

5. *Symmetric Monoidality.* The triple $(\mathsf{Sets}, \coprod, \emptyset)$ is a symmetric monoidal category.

Proof. *Item 1*, *Functoriality*: This follows from **??**, **??** of **??**.
Item 2, *Associativity*: Clear.
Item 3, *Unitality*: Clear.
Item 4, *Commutativity*: Clear.
Item 5, *Symmetric Monoidality*: Omitted. □

2.4 Pushouts

Let A, B, and C be sets and let $f: C \to A$ and $g: C \to B$ be functions.

Definition 2.4.1.1. The **pushout of A and B over C along f and g**[9] is the pair[10] $(A \coprod_C B, \{\mathrm{inj}_1, \mathrm{inj}_2\})$ consisting of:

A **and** B, for emphasis.

[9] *Further Terminology:* Also called the **fibre coproduct of A and B over C along f and g**.

[10] *Further Notation:* Also written $A \coprod_{f,C,g} B$.

- *The Colimit.* The set $A \coprod_C B$ defined by

$$A \coprod_C B \overset{\text{def}}{=} A \coprod B/{\sim_C},$$

where $\sim_C$ is the equivalence relation on $A \coprod B$ generated by $(0, f(c)) \sim_C (1, g(c))$.

- *The Cocone.* The maps

$$\begin{aligned} \mathsf{inj}_1 \colon A &\to A \coprod_C B, \\ \mathsf{inj}_2 \colon B &\to A \coprod_C B \end{aligned}$$

given by

$$\begin{aligned} \mathsf{inj}_1(a) &\overset{\text{def}}{=} [(0, a)] \\ \mathsf{inj}_2(b) &\overset{\text{def}}{=} [(1, b)] \end{aligned}$$

for each $a \in A$ and each $b \in B$.

Proof. We claim that $A \coprod_C B$ is the categorical pushout of A and B over C with respect to (f, g) in Sets. First we need to check that the relevant pushout diagram commutes, i.e. that we have

$$\mathsf{inj}_1 \circ f = \mathsf{inj}_2 \circ g, \qquad \begin{tikzcd} A \coprod_C B & B \arrow[l, "\mathsf{inj}_2"'] \\ A \arrow[u, "\mathsf{inj}_1"] & C. \arrow[l, "f"] \arrow[u, "g"'] \end{tikzcd}$$

Indeed, given $c \in C$, we have

$$\begin{aligned} [\mathsf{inj}_1 \circ f](c) &= \mathsf{inj}_1(f(c)) \\ &= [(0, f(c))] \\ &= [(1, g(c))] \\ &= \mathsf{inj}_2(g(c)) \\ &= [\mathsf{inj}_2 \circ g](c), \end{aligned}$$

where $[(0, f(c))] = [(1, g(c))]$ by the definition of the relation $\sim$ on $A \coprod B$. Next, we prove that $A \coprod_C B$ satisfies the universal property of the pushout.

Suppose we have a diagram of the form

$$\begin{tikzcd}
P & & \\
& A \coprod_C B \arrow[dr, phantom, very near start, "\ulcorner"] & B \arrow[l, "\mathsf{inj}_2"'] \arrow[ull, bend right, "\iota_2"'] \\
& A \arrow[u, "\mathsf{inj}_1"] \arrow[uul, bend left, "\iota_1"] & C \arrow[l, "f"] \arrow[u, "g"']
\end{tikzcd}$$

in Sets. Then there exists a unique map $\phi\colon A \coprod_C B \to P$ making the diagram

$$\begin{tikzcd}
P & & \\
& A \coprod_C B \arrow[ul, dashed, "\exists!"', "\phi"] \arrow[dr, phantom, very near start, "\ulcorner"] & B \arrow[l, "\mathsf{inj}_2"'] \arrow[ull, bend right, "\iota_2"'] \\
& A \arrow[u, "\mathsf{inj}_1"] \arrow[uul, bend left, "\iota_1"] & C \arrow[l, "f"] \arrow[u, "g"']
\end{tikzcd}$$

commute, being uniquely determined by the conditions

$$\phi \circ \mathsf{inj}_1 = \iota_1,$$
$$\phi \circ \mathsf{inj}_2 = \iota_2$$

via

$$\phi(x) = \begin{cases} \iota_1(a) & \text{if } x = [(0,a)], \\ \iota_2(b) & \text{if } x = [(1,b)] \end{cases}$$

for each $x \in A \coprod_C B$, where the well-definedness of ϕ is guaranteed by the equality $\iota_1 \circ f = \iota_2 \circ g$ and the definition of the relation $\sim$ on $A \coprod B$ as follows:

1. *Case 1:* Suppose we have $x = [(0,a)] = [(0,a')]$ for some $a, a' \in A$. Then, by Remark 2.4.1.2, we have a sequence

$$(0,a) \sim' x_1 \sim' \cdots \sim' x_n \sim' (0,a').$$

2. *Case 2:* Suppose we have $x = [(1, b)] = [(1, b')]$ for some $b, b' \in B$. Then, by Remark 2.4.1.2, we have a sequence
$$(1, b) \sim' x_1 \sim' \cdots \sim' x_n \sim' (1, b').$$
3. *Case 3:* Suppose we have $x = [(0, a)] = [(1, b)]$ for some $a \in A$ and $b \in B$. Then, by Remark 2.4.1.2, we have a sequence
$$(0, a) \sim' x_1 \sim' \cdots \sim' x_n \sim' (1, b).$$

In all these cases, we declare $x \sim' y$ iff there exists some $c \in C$ such that $x = (0, f(c))$ and $y = (1, g(c))$ or $x = (1, g(c))$ and $y = (0, f(c))$. Then, the equality $\iota_1 \circ f = \iota_2 \circ g$ gives

$$\begin{aligned}\phi([x]) &= \phi([(0, f(c))]) \\ &\overset{\text{def}}{=} \iota_1(f(c)) \\ &= \iota_2(g(c)) \\ &\overset{\text{def}}{=} \phi([(1, g(c))]) \\ &= \phi([y]),\end{aligned}$$

with the case where $x = (1, g(c))$ and $y = (0, f(c))$ similarly giving $\phi([x]) = \phi([y])$. Thus, if $x \sim' y$, then $\phi([x]) = \phi([y])$. Applying this equality pairwise to the sequences

$$\begin{aligned}(0, a) &\sim' x_1 \sim' \cdots \sim' x_n \sim' (0, a'), \\ (1, b) &\sim' x_1 \sim' \cdots \sim' x_n \sim' (1, b'), \\ (0, a) &\sim' x_1 \sim' \cdots \sim' x_n \sim' (1, b)\end{aligned}$$

gives

$$\begin{aligned}\phi([(0, a)]) &= \phi([(0, a')]), \\ \phi([(1, b)]) &= \phi([(1, b')]), \\ \phi([(0, a)]) &= \phi([(1, b)]),\end{aligned}$$

showing ϕ to be well-defined. □

Remark 2.4.1.2. In detail, by Equivalence Relations and Apartness Relations, Construction 4.2.1.2, the relation $\sim$ of Definition 2.4.1.1 is given by declaring $a \sim b$ iff one of the following conditions is satisfied:

- We have $a, b \in A$ and $a = b$;
- We have $a, b \in B$ and $a = b$;
- There exist $x_1, \ldots, x_n \in A \coprod B$ such that $a \sim' x_1 \sim' \cdots \sim' x_n \sim' b$, where we declare $x \sim' y$ if one of the following conditions is satisfied:
  1. There exists $c \in C$ such that $x = (0, f(c))$ and $y = (1, g(c))$.
  2. There exists $c \in C$ such that $x = (1, g(c))$ and $y = (0, f(c))$.

  That is: we require the following condition to be satisfied:

  $(\star)$ There exist $x_1, \ldots, x_n \in A \coprod B$ satisfying the following conditions:
  1. There exists $c_0 \in C$ satisfying one of the following conditions:
     (a) We have $a = f(c_0)$ and $x_1 = g(c_0)$.
     (b) We have $a = g(c_0)$ and $x_1 = f(c_0)$.
  2. For each $1 \leq i \leq n-1$, there exists $c_i \in C$ satisfying one of the following conditions:
     (a) We have $x_i = f(c_i)$ and $x_{i+1} = g(c_i)$.
     (b) We have $x_i = g(c_i)$ and $x_{i+1} = f(c_i)$.
  3. There exists $c_n \in C$ satisfying one of the following conditions:
     (a) We have $x_n = f(c_n)$ and $b = g(c_n)$.
     (b) We have $x_n = g(c_n)$ and $b = f(c_n)$.

Example 2.4.1.3. Here are some examples of pushouts of sets.

1. *Wedge Sums of Pointed Sets.* The wedge sum of two pointed sets of Pointed Sets, Definition 3.3.1.1 is an example of a pushout of sets.
2. *Intersections via Unions.* Let $A, B \subset X$. We have a bijection of sets

$$A \cup B \cong A \coprod_{A \cap B} B, \qquad \begin{tikzcd} A \cup B \arrow[dr, phantom, "\ulcorner", very near start] & B \arrow[l] \\ A \arrow[u] & A \cap B. \arrow[l, hook'] \arrow[u, hook] \end{tikzcd}$$

Proof. *Item 1, Wedge Sums of Pointed Sets*: Follows by definition.
Item 2, Intersections via Unions: Indeed, $A \coprod_{A \cap B} B$ is the quotient of $A \coprod B$ by

the equivalence relation obtained by declaring $(0,a) \sim (1,b)$ iff $a = b \in A \cap B$, which is in bijection with $A \cup B$ via the map with $[(0,a)] \mapsto a$ and $[(1,b)] \mapsto b$. $\square$

Proposition 2.4.1.4. Let A, B, C, and X be sets.

1. *Functoriality.* The assignment $(A, B, C, f, g) \mapsto A \coprod_{f,C,g} B$ defines a functor

$$-_1 \coprod_{-_3} -_1 \colon \mathsf{Fun}(\mathcal{P}, \mathsf{Sets}) \to \mathsf{Sets},$$

where $\mathcal{P}$ is the category that looks like this:

$$\begin{tikzcd} & \bullet \\ \bullet & \bullet. \arrow[l] \arrow[u] \end{tikzcd}$$

In particular, the action on morphisms of $-_1 \coprod_{-_3} -_1$ is given by sending a morphism

$$\begin{tikzcd}
A \coprod_C B \arrow[dr, phantom, very near start, "\ulcorner"] & & B \arrow[ll] \arrow[dr, "\psi"] & \\
& A' \coprod_{C'} B' \arrow[dr, phantom, very near start, "\ulcorner"] & & B' \arrow[ll] \\
A \arrow[uu] \arrow[dr, "\phi"'] & & C \arrow[ll, "f"'] \arrow[uu, "g"] \arrow[dr, "\chi"] & \\
& A' \arrow[uu] & & C' \arrow[ll, "f'"] \arrow[uu, "g'"']
\end{tikzcd}$$

in $\mathsf{Fun}(\mathcal{P}, \mathsf{Sets})$ to the map $\xi \colon A \coprod_C B \xrightarrow{\exists!} A' \coprod_{C'} B'$ given by

$$\xi(x) \stackrel{\text{def}}{=} \begin{cases} \phi(a) & \text{if } x = [(0,a)], \\ \psi(b) & \text{if } x = [(1,b)] \end{cases}$$

for each $x \in A \coprod_C B$, which is the unique map making the diagram

$$\begin{tikzcd}
A \coprod_C B \arrow[dr, dashed] \arrow[dr, phantom, very near start, "\ulcorner"] & & B \arrow[ll] \arrow[dr, "\psi"] & \\
& A' \coprod_{C'} B' \arrow[dr, phantom, very near start, "\ulcorner"] & & B' \arrow[ll] \\
A \arrow[uu] \arrow[dr, "\phi"'] & & C \arrow[ll, "f"'] \arrow[uu, "g"] \arrow[dr, "\chi"] & \\
& A' \arrow[uu] & & C' \arrow[ll, "f'"] \arrow[uu, "g'"']
\end{tikzcd}$$

commute.

2. *Associativity.* Given a diagram

$$\begin{tikzcd}
A & & B & & C \\
& X \arrow[ul, "f"] \arrow[ur, "g"'] & & Y \arrow[ul, "h"] \arrow[ur, "k"'] &
\end{tikzcd}$$

in Sets, we have isomorphisms of sets

$$(A \coprod_X B) \coprod_Y C \cong (A \coprod_X B) \coprod_B (B \coprod_Y C) \cong A \coprod_X (B \coprod_Y C),$$

where these pullbacks are built as in the diagrams

$$\begin{tikzcd}[column sep=tiny, row sep=small]
& & (A \coprod_X B) \coprod_Y C & & \\
& A \coprod_X B \arrow[ur] & & & \\
A \arrow[ur] & & B \arrow[ul] \arrow[uu, phantom, "\wedge"] & & C, \arrow[uull] \\
& X \arrow[ul, "f"] \arrow[ur, "g"'] \arrow[uu, phantom, "\wedge"] & & Y \arrow[ul, "h"] \arrow[ur, "k"'] &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=tiny, row sep=small]
& & (A \coprod_X B) \coprod_B (B \coprod_Y C) & & \\
& A \coprod_X B \arrow[ur] & & B \coprod_Y C \arrow[ul] & \\
A \arrow[ur] & & B \arrow[ul] \arrow[ur] \arrow[uu, phantom, "\wedge"] & & C, \arrow[ul] \\
& X \arrow[ul, "f"] \arrow[ur, "g"'] \arrow[uu, phantom, "\wedge"] & & Y \arrow[ul, "h"] \arrow[ur, "k"'] \arrow[uu, phantom, "\wedge"] &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=tiny, row sep=small]
& & A \coprod_X (B \coprod_Y C) & & \\
& & & B \coprod_Y C \arrow[ul] & \\
A \arrow[uurr] & & B \arrow[ur] & & C. \arrow[ul] \\
& X \arrow[ul, "f"] \arrow[ur, "g"'] & & Y \arrow[ul, "h"] \arrow[ur, "k"'] \arrow[uu, phantom, "\wedge"] &
\end{tikzcd}$$

3. *Unitality.* We have isomorphisms of sets

$$\begin{tikzcd}
A \arrow[r, equal] \arrow[dr, phantom, very near start, "\ulcorner"] & A \\
X \arrow[u, "f"] \arrow[r, equal] & X \arrow[u, "f"']
\end{tikzcd}
\qquad
\begin{aligned}
X \coprod_X A &\cong A, \\
A \coprod_X X &\cong A,
\end{aligned}
\qquad
\begin{tikzcd}
A \arrow[dr, phantom, very near start, "\ulcorner"] & X \arrow[l, "f"'] \\
X \arrow[u, equal] & X. \arrow[l, "f"] \arrow[u, equal]
\end{tikzcd}$$

4. *Commutativity.* We have an isomorphism of sets

$$\begin{tikzcd} A \coprod_X B & B \arrow[l] \\ A \arrow[u] \arrow[ur, phantom, "\ulcorner"] & X, \arrow[l, "f"] \arrow[u, "g"'] \end{tikzcd} \qquad A \coprod_X B \cong B \coprod_X A \qquad \begin{tikzcd} B \coprod_X A & A \arrow[l] \\ B \arrow[u] \arrow[ur, phantom, "\ulcorner"] & X. \arrow[l, "g"] \arrow[u, "f"'] \end{tikzcd}$$

5. *Interaction With Coproducts.* We have

$$A \coprod_{\emptyset} B \cong A \coprod B, \qquad \begin{tikzcd} A \coprod B & B \arrow[l] \\ A \arrow[u] \arrow[ur, phantom, "\ulcorner"] & \emptyset. \arrow[l, hook', "\iota_A"] \arrow[u, hook, "\iota_B"'] \end{tikzcd}$$

6. *Symmetric Monoidality.* The triple $(\mathsf{Sets}, \coprod_X, X)$ is a symmetric monoidal category.

Proof. Item 1, *Functoriality*: This is a special case of functoriality of co/limits, **??**, **??** of **??**, with the explicit expression for ξ following from the commutativity of the cube pushout diagram.
Item 2, *Associativity*: Omitted.
Item 3, *Unitality*: Omitted.
Item 4, *Commutativity*: Clear.
Item 5, *Interaction With Coproducts*: Clear.
Item 6, *Symmetric Monoidality*: Omitted. □

2.5 Coequalisers

Let A and B be sets and let $f, g\colon A \rightrightarrows B$ be functions.

Definition 2.5.1.1. The **coequaliser of** f **and** g is the pair $(\mathrm{CoEq}(f,g), \mathrm{coeq}(f,g))$ consisting of:

- *The Colimit.* The set $\mathrm{CoEq}(f,g)$ defined by

$$\mathrm{CoEq}(f,g) \overset{\text{def}}{=} B/{\sim},$$

where $\sim$ is the equivalence relation on B generated by $f(a) \sim g(a)$.

- *The Cocone.* The map

$$\mathrm{coeq}(f,g)\colon B \longrightarrow \mathrm{CoEq}(f,g)$$

given by the quotient map $\pi\colon B \twoheadrightarrow B/{\sim}$ with respect to the equivalence relation generated by $f(a) \sim g(a)$.

Proof. We claim that $\mathrm{CoEq}(f,g)$ is the categorical coequaliser of f and g in Sets. First we need to check that the relevant coequaliser diagram commutes, i.e. that we have

$$\mathrm{coeq}(f,g) \circ f = \mathrm{coeq}(f,g) \circ g.$$

Indeed, we have

$$\begin{aligned}
[\mathrm{coeq}(f,g) \circ f](a) &\stackrel{\text{def}}{=} [\mathrm{coeq}(f,g)](f(a)) \\
&\stackrel{\text{def}}{=} [f(a)] \\
&= [g(a)] \\
&\stackrel{\text{def}}{=} [\mathrm{coeq}(f,g)](g(a)) \\
&\stackrel{\text{def}}{=} [\mathrm{coeq}(f,g) \circ g](a)
\end{aligned}$$

for each $a \in A$. Next, we prove that $\mathrm{CoEq}(f,g)$ satisfies the universal property of the coequaliser. Suppose we have a diagram of the form

$$\begin{tikzcd}
A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, hook, "\mathrm{coeq}(f,g)"] \arrow[dr, "c"'] & \mathrm{CoEq}(f,g) \\
& & C
\end{tikzcd}$$

in Sets. Then, since $c(f(a)) = c(g(a))$ for each $a \in A$, it follows from Equivalence Relations and Apartness Relations, Items 4 and 5 of Proposition 5.2.1.3 that there exists a unique map $\mathrm{CoEq}(f,g) \xrightarrow{\exists!} C$ making the diagram

$$\begin{tikzcd}
A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, hook, "\mathrm{coeq}(f,g)"] \arrow[dr, "c"'] & \mathrm{CoEq}(f,g) \arrow[d, dashed, "\exists!"] \\
& & C
\end{tikzcd}$$

commute. ∎

Remark 2.5.1.2. In detail, by Equivalence Relations and Apartness Relations, Construction 4.2.1.2, the relation $\sim$ of Definition 2.5.1.1 is given by declaring $a \sim b$ iff one of the following conditions is satisfied:

- We have $a = b$;
- There exist $x_1, \dots, x_n \in B$ such that $a \sim' x_1 \sim' \cdots \sim' x_n \sim' b$, where we declare $x \sim' y$ if one of the following conditions is satisfied:
  1. There exists $z \in A$ such that $x = f(z)$ and $y = g(z)$.
  2. There exists $z \in A$ such that $x = g(z)$ and $y = f(z)$.

  That is: we require the following condition to be satisfied:

  $(\star)$ There exist $x_1, \dots, x_n \in B$ satisfying the following conditions:
  1. There exists $z_0 \in A$ satisfying one of the following conditions:
     (a) We have $a = f(z_0)$ and $x_1 = g(z_0)$.
     (b) We have $a = g(z_0)$ and $x_1 = f(z_0)$.
  2. For each $1 \leq i \leq n-1$, there exists $z_i \in A$ satisfying one of the following conditions:
     (a) We have $x_i = f(z_i)$ and $x_{i+1} = g(z_i)$.
     (b) We have $x_i = g(z_i)$ and $x_{i+1} = f(z_i)$.
  3. There exists $z_n \in A$ satisfying one of the following conditions:
     (a) We have $x_n = f(z_n)$ and $b = g(z_n)$.
     (b) We have $x_n = g(z_n)$ and $b = f(z_n)$.

Example 2.5.1.3. Here are some examples of coequalisers of sets.

1. *Quotients by Equivalence Relations.* Let R be an equivalence relation on a set X. We have a bijection of sets

$$X/{\sim_R} \cong \mathrm{CoEq}(R \hookrightarrow X \times X \underset{\mathrm{pr}_2}{\overset{\mathrm{pr}_1}{\rightrightarrows}} X).$$

Proof. *Item 1, Quotients by Equivalence Relations*: See [Pro24z]. □

Proposition 2.5.1.4. Let A, B, and C be sets.

1. *Associativity.* We have isomorphisms of sets[11]

$$\underbrace{\mathrm{CoEq}(\mathrm{coeq}(f,g)\circ f,\mathrm{coeq}(f,g)\circ h)}_{=\mathrm{CoEq}(\mathrm{coeq}(f,g)\circ g,\mathrm{coeq}(f,g)\circ h)} \cong \mathrm{CoEq}(f,g,h) \cong \underbrace{\mathrm{CoEq}(\mathrm{coeq}(g,h)\circ f,\mathrm{coeq}(g,h)\circ g)}_{=\mathrm{CoEq}(\mathrm{coeq}(g,h)\circ f,\mathrm{coeq}(g,h)\circ h)},$$

where $\mathrm{CoEq}(f,g,h)$ is the colimit of the diagram

$$\begin{tikzcd} A \arrow[r, shift left=2, "f"] \arrow[r, "g" description] \arrow[r, shift right=2, "h"'] & B \end{tikzcd}$$

[11] That is, the following three ways of forming "the" coequaliser of (f,g,h) agree:

1. Take the coequaliser of (f,g,h), i.e. the colimit of the diagram

$$\begin{tikzcd} A \arrow[r, shift left=2, "f"] \arrow[r, "g" description] \arrow[r, shift right=2, "h"'] & B \end{tikzcd}$$

in Sets.

2. First take the coequaliser of f and g, forming a diagram

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \overset{\mathrm{coeq}(f,g)}{\twoheadrightarrow} \mathrm{CoEq}(f,g)$$

and then take the coequaliser of the composition

$$A \underset{h}{\overset{f}{\rightrightarrows}} B \overset{\mathrm{coeq}(f,g)}{\twoheadrightarrow} \mathrm{CoEq}(f,g),$$

obtaining a quotient

$$\mathrm{CoEq}(\mathrm{coeq}(f,g)\circ f,\mathrm{coeq}(f,g)\circ h) = \mathrm{CoEq}(\mathrm{coeq}(f,g)\circ g,\mathrm{coeq}(f,g)\circ h)$$

of $\mathrm{CoEq}(f,g)$

3. First take the coequaliser of g and h, forming a diagram

$$A \underset{h}{\overset{g}{\rightrightarrows}} B \overset{\mathrm{coeq}(g,h)}{\twoheadrightarrow} \mathrm{CoEq}(g,h)$$

and then take the coequaliser of the composition

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \overset{\mathrm{coeq}(g,h)}{\twoheadrightarrow} \mathrm{CoEq}(g,h),$$

obtaining a quotient

$$\mathrm{CoEq}(\mathrm{coeq}(g,h)\circ f,\mathrm{coeq}(g,h)\circ g) = \mathrm{CoEq}(\mathrm{coeq}(g,h)\circ f,\mathrm{coeq}(g,h)\circ h)$$

of $\mathrm{CoEq}(g,h)$.

in Sets.

4. *Unitality.* We have an isomorphism of sets

$$\mathrm{CoEq}(f,f) \cong B.$$

5. *Commutativity.* We have an isomorphism of sets

$$\mathrm{CoEq}(f,g) \cong \mathrm{CoEq}(g,f).$$

6. *Interaction With Composition.* Let

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \underset{k}{\overset{h}{\rightrightarrows}} C$$

be functions. We have a surjection

$$\mathrm{CoEq}(h \circ f, k \circ g) \twoheadrightarrow \mathrm{CoEq}(\mathrm{coeq}(h,k) \circ h \circ f, \mathrm{coeq}(h,k) \circ k \circ g)$$

exhibiting $\mathrm{CoEq}(\mathrm{coeq}(h,k) \circ h \circ f, \mathrm{coeq}(h,k) \circ k \circ g)$ as a quotient of $\mathrm{CoEq}(h \circ f, k \circ g)$ by the relation generated by declaring $h(y) \sim k(y)$ for each $y \in B$.

Proof. *Item 1*, *Associativity*: Omitted.
Item 4, *Unitality*: Clear.
Item 5, *Commutativity*: Clear.
Item 6, *Interaction With Composition*: Omitted. □

3 Operations With Sets

3.1 The Empty Set

Definition 3.1.1.1. The **empty set** is the set $\emptyset$ defined by

$$\emptyset \overset{\mathrm{def}}{=} \{x \in X \mid x \neq x\},$$

where A is the set in the set existence axiom, **??** of **??**.

3.2 Singleton Sets

Let X be a set.

Definition 3.2.1.1. The **singleton set containing** X is the set $\{X\}$ defined by

$$\{X\} \overset{\text{def}}{=} \{X, X\},$$

where $\{X, X\}$ is the pairing of X with itself (Definition 3.3.1.1).

3.3 Pairings of Sets

Let X and Y be sets.

Definition 3.3.1.1. The **pairing of** X **and** Y is the set $\{X, Y\}$ defined by

$$\{X, Y\} \overset{\text{def}}{=} \{x \in A \mid x = X \text{ or } x = Y\},$$

where A is the set in the axiom of pairing, **??** of **??**.

3.4 Ordered Pairs

Let A and B be sets.

Definition 3.4.1.1. The **ordered pair associated to** A **and** B is the set (A, B) defined by

$$(A, B) \overset{\text{def}}{=} \{\{A\}, \{A, B\}\}.$$

Proposition 3.4.1.2. Let A and B be sets.

1. *Uniqueness.* Let A, B, C, and D be sets. The following conditions are equivalent:
   (a) We have $(A, B) = (C, D)$.
   (b) We have $A = C$ and $B = D$.

Proof. *Item 1*, *Uniqueness*: See [Cie97, Theorem 1.2.3]. □

3.5 Sets of Maps

Let A and B be sets.

Definition 3.5.1.1. The **set of maps from A to B**[12] is the set $\mathrm{Hom}_{\mathsf{Sets}}(A, B)$[13] whose elements are the functions from A to B.

Proposition 3.5.1.2. Let A and B be sets.

1. *Functoriality.* The assignments $X, Y, (X, Y) \mapsto \mathrm{Hom}_{\mathsf{Sets}}(X, Y)$ define functors

$$\begin{aligned} \mathrm{Hom}_{\mathsf{Sets}}(X, -)&\colon \mathsf{Sets} \to \mathsf{Sets}, \\ \mathrm{Hom}_{\mathsf{Sets}}(-, Y)&\colon \mathsf{Sets}^{\mathsf{op}} \to \mathsf{Sets}, \\ \mathrm{Hom}_{\mathsf{Sets}}(-_1, -_2)&\colon \mathsf{Sets}^{\mathsf{op}} \times \mathsf{Sets} \to \mathsf{Sets}. \end{aligned}$$

Proof. *Item 1, Functoriality*: This follows from Categories, Items 2 and 5 of Proposition 1.6.1.2. ☐

3.6 Unions of Families

Let $\{A_i\}_{i\in I}$ be a family of sets.

Definition 3.6.1.1. The **union of the family $\{A_i\}_{i\in I}$** is the set $\bigcup_{i\in I} A_i$ defined by

$$\bigcup_{i\in I} A_i \stackrel{\text{def}}{=} \{x \in F \mid \text{there exists some } i \in I \text{ such that } x \in A_i\},$$

where F is the set in the axiom of union, **??** of **??**.

3.7 Binary Unions

Let A and B be sets.

Definition 3.7.1.1. The **union**[14] **of A and B** is the set $A \cup B$ defined by

$$A \cup B \stackrel{\text{def}}{=} \bigcup_{z\in\{A,B\}} z.$$

Proposition 3.7.1.2. Let X be a set.

[12] *Further Terminology:* Also called the **Hom set from A to B**.
[13] *Further Notation:* Also written $\mathsf{Sets}(A, B)$.
[14] *Further Terminology:* Also called the **binary union of A and B**, for emphasis.

1. *Functoriality.* The assignments $U, V, (U, V) \mapsto U \cup V$ define functors

$$\begin{aligned} U \cup -\colon (\mathcal{P}(X), \subset) &\to (\mathcal{P}(X), \subset), \\ - \cup V\colon (\mathcal{P}(X), \subset) &\to (\mathcal{P}(X), \subset), \\ -_1 \cup -_2\colon (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \subset) &\to (\mathcal{P}(X), \subset), \end{aligned}$$

   where $-_1 \cup -_2$ is the functor where

   - *Action on Objects.* For each $(U, V) \in \mathcal{P}(X) \times \mathcal{P}(X)$, we have

   $$[-_1 \cup -_2](U, V) \overset{\text{def}}{=} U \cup V.$$

   - *Action on Morphisms.* For each pair of morphisms

   $$\begin{aligned} \iota_U &\colon U \hookrightarrow U', \\ \iota_V &\colon V \hookrightarrow V' \end{aligned}$$

     of $\mathcal{P}(X) \times \mathcal{P}(X)$, the image

   $$\iota_U \cup \iota_V \colon U \cup V \hookrightarrow U' \cup V'$$

     of (ι_U, ι_V) by $\cup$ is the inclusion

   $$U \cup V \subset U' \cup V'$$

     i.e. where we have

     $(\star)$ If $U \subset U'$ and $V \subset V'$, then $U \cup V \subset U' \cup V'$.

   and where $U \cup -$ and $- \cup V$ are the partial functors of $-_1 \cup -_2$ at $U, V \in \mathcal{P}(X)$.

2. *Via Intersections and Symmetric Differences.* We have an equality of sets

$$U \cup V = (U \bigtriangleup V) \bigtriangleup (U \cap V)$$

   for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

3. *Associativity.* We have an equality of sets

$$(U \cup V) \cup W = U \cup (V \cup W)$$

   for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

4. *Unitality.* We have equalities of sets

$$U \cup \emptyset = U,$$
$$\emptyset \cup U = U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

5. *Commutativity.* We have an equality of sets

$$U \cup V = V \cup U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

6. *Idempotency.* We have an equality of sets

$$U \cup U = U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

7. *Distributivity Over Intersections.* We have equalities of sets

$$U \cup (V \cap W) = (U \cup V) \cap (U \cup W),$$
$$(U \cap V) \cup W = (U \cup W) \cap (V \cup W)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

8. *Interaction With Characteristic Functions I.* We have

$$\chi_{U \cup V} = \max(\chi_U, \chi_V)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

9. *Interaction With Characteristic Functions II.* We have

$$\chi_{U \cup V} = \chi_U + \chi_V - \chi_{U \cap V}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

10. *Interaction With Powersets and Semirings.* The quintuple $(\mathcal{P}(X), \cup, \cap, \emptyset, X)$ is an idempotent commutative semiring.

Proof. *Item 1*, *Functoriality*: See [Pro24an].
Item 2, *Via Intersections and Symmetric Differences*: See [Pro24ay].
Item 3, *Associativity*: See [Pro24ba].
Item 4, *Unitality*: This follows from [Pro24bd] and Item 5.
Item 5, *Commutativity*: See [Pro24bb].
Item 6, *Idempotency*: See [Pro24am].
Item 7, *Distributivity Over Intersections*: See [Pro24az].
Item 8, *Interaction With Characteristic Functions I*: See [Pro24k].
Item 9, *Interaction With Characteristic Functions II*: See [Pro24k].
Item 10, *Interaction With Powersets and Semirings*: This follows from Items 3 to 6 and Items 3 to 5, 7 and 8 of Proposition 3.9.1.2. □

3.8 Intersections of Families

Let $\mathcal{F}$ be a family of sets.

Definition 3.8.1.1. The **intersection of a family $\mathcal{F}$ of sets** is the set $\bigcap_{X\in\mathcal{F}} X$ defined by

$$\bigcap_{X\in\mathcal{F}} X \overset{\text{def}}{=} \left\{ z \in \bigcup_{X\in\mathcal{F}} X \;\middle|\; \text{for each } X \in \mathcal{F}\text{, we have } z \in X \right\}.$$

3.9 Binary Intersections

Let X and Y be sets.

Definition 3.9.1.1. The **intersection**[15] **of X and Y** is the set $X \cap Y$ defined by

$$X \cap Y \overset{\text{def}}{=} \bigcap_{z\in\{X,Y\}} z.$$

Proposition 3.9.1.2. Let X be a set.

1. *Functoriality.* The assignments $U, V, (U, V) \mapsto U \cap V$ define functors

$$\begin{aligned} U \cap -\colon (\mathcal{P}(X), \subset) &\to (\mathcal{P}(X), \subset), \\ - \cap V\colon (\mathcal{P}(X), \subset) &\to (\mathcal{P}(X), \subset), \\ -_1 \cap -_2\colon (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \subset) &\to (\mathcal{P}(X), \subset), \end{aligned}$$

   where $-_1 \cap -_2$ is the functor where

[15] *Further Terminology:* Also called the **binary intersection of X and Y**, for emphasis.

- *Action on Objects.* For each $(U, V) \in \mathcal{P}(X) \times \mathcal{P}(X)$, we have

$$[-_1 \cap -_2](U, V) \stackrel{\text{def}}{=} U \cap V.$$

- *Action on Morphisms.* For each pair of morphisms

$$\iota_U \colon U \hookrightarrow U',$$
$$\iota_V \colon V \hookrightarrow V'$$

of $\mathcal{P}(X) \times \mathcal{P}(X)$, the image

$$\iota_U \cap \iota_V \colon U \cap V \hookrightarrow U' \cap V'$$

of (ι_U, ι_V) by $\cap$ is the inclusion

$$U \cap V \subset U' \cap V'$$

i.e. where we have

$(\star)$ If $U \subset U'$ and $V \subset V'$, then $U \cap V \subset U' \cap V'$.

and where $U \cap -$ and $- \cap V$ are the partial functors of $-_1 \cap -_2$ at $U, V \in \mathcal{P}(X)$.

2. *Adjointness.* We have adjunctions

$$\big(U \cap - \dashv \mathbf{Hom}_{\mathcal{P}(X)}(U, -)\big)\colon \quad \begin{tikzcd} \mathcal{P}(X) \arrow[r, bend left, "U\cap-"] \arrow[r, phantom, "\perp"] & \mathcal{P}(X) \arrow[l, bend left, "\mathbf{Hom}_{\mathcal{P}(X)}(U,-)"] \end{tikzcd},$$

$$\big(- \cap V \dashv \mathbf{Hom}_{\mathcal{P}(X)}(V, -)\big)\colon \quad \begin{tikzcd} \mathcal{P}(X) \arrow[r, bend left, "-\cap V"] \arrow[r, phantom, "\perp"] & \mathcal{P}(X) \arrow[l, bend left, "\mathbf{Hom}_{\mathcal{P}(X)}(V,-)"] \end{tikzcd},$$

where

$$\mathbf{Hom}_{\mathcal{P}(X)}(-_1, -_2)\colon \mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \to \mathcal{P}(X)$$

is the bifunctor defined by[16]

$$\mathbf{Hom}_{\mathcal{P}(X)}(U, V) \stackrel{\text{def}}{=} (X \setminus U) \cup V$$

[16] For intuition regarding the expression defining $\mathbf{Hom}_{\mathcal{P}(X)}(U, V)$, see Remark 3.9.1.3.

witnessed by bijections

$$\begin{aligned}\mathrm{Hom}_{\mathcal{P}(X)}(U \cap V, W) &\cong \mathrm{Hom}_{\mathcal{P}(X)}(U, \mathbf{Hom}_{\mathcal{P}(X)}(V, W)),\\ \mathrm{Hom}_{\mathcal{P}(X)}(U \cap V, W) &\cong \mathrm{Hom}_{\mathcal{P}(X)}(V, \mathbf{Hom}_{\mathcal{P}(X)}(U, W)),\end{aligned}$$

natural in $U, V, W \in \mathcal{P}(X)$, i.e. where:

(a) The following conditions are equivalent:

   i. We have $U \cap V \subset W$.
   ii. We have $U \subset \mathbf{Hom}_{\mathcal{P}(X)}(V, W)$.
   iii. We have $U \subset (X \setminus V) \cup W$.

(b) The following conditions are equivalent:

   i. We have $V \cap U \subset W$.
   ii. We have $V \subset \mathbf{Hom}_{\mathcal{P}(X)}(U, W)$.
   iii. We have $V \subset (X \setminus U) \cup W$.

3. *Associativity.* We have an equality of sets

$$(U \cap V) \cap W = U \cap (V \cap W)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

4. *Unitality.* Let X be a set and let $U \in \mathcal{P}(X)$. We have equalities of sets

$$\begin{aligned}X \cap U &= U,\\ U \cap X &= U\end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

5. *Commutativity.* We have an equality of sets

$$U \cap V = V \cap U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

6. *Idempotency.* We have an equality of sets

$$U \cap U = U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

7. *Distributivity Over Unions.* We have equalities of sets

$$U \cap (V \cup W) = (U \cap V) \cup (U \cap W),$$
$$(U \cup V) \cap W = (U \cap W) \cup (V \cap W)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

8. *Annihilation With the Empty Set.* We have an equality of sets

$$\emptyset \cap X = \emptyset,$$
$$X \cap \emptyset = \emptyset$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

9. *Interaction With Characteristic Functions I.* We have

$$\chi_{U \cap V} = \chi_U \chi_V$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

10. *Interaction With Characteristic Functions II.* We have

$$\chi_{U \cap V} = \min(\chi_U, \chi_V)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

11. *Interaction With Powersets and Monoids With Zero.* The quadruple $((\mathcal{P}(X), \emptyset), \cap, X)$ is a commutative monoid with zero.

12. *Interaction With Powersets and Semirings.* The quintuple $(\mathcal{P}(X), \cup, \cap, \emptyset, X)$ is an idempotent commutative semiring.

Proof. *Item 1*, *Functoriality*: See [Pro24al].
Item 2, *Adjointness*: See [MSE 267469].
Item 3, *Associativity*: See [Pro24s].
Item 4, *Unitality*: This follows from [Pro24w] and Item 5.
Item 5, *Commutativity*: See [Pro24t].
Item 6, *Idempotency*: See [Pro24ak].
Item 7, *Distributivity Over Unions*: See [Pro24aj].
Item 8, *Annihilation With the Empty Set*: This follows from [Pro24u] and Item 5.
Item 9, *Interaction With Characteristic Functions I*: See [Pro24h].

Item 10, Interaction With Characteristic Functions II: See [Pro24h].
Item 11, Interaction With Powersets and Monoids With Zero: This follows from Items 3 to 5 and 8.
Item 12, Interaction With Powersets and Semirings: This follows from Items 3 to 6 and Items 3 to 5, 7 and 8 of Proposition 3.9.1.2. □

Remark 3.9.1.3. Since intersections are the products in $\mathcal{P}(X)$ (Item 1 of Proposition 4.3.1.3), the left adjoint $\mathbf{Hom}_{\mathcal{P}(X)}(U,V)$ may be thought of as a function type $[U,V]$.
Then, under the Curry–Howard correspondence, the function type $[U,V]$ corresponds to implication $U \implies V$, which is logically equivalent to the statement $\neg U \vee V$. This in turn corresponds to the set $U^{\mathsf{c}} \vee V = (X \setminus U) \cup V$.

3.10 Differences

Let X and Y be sets.

Definition 3.10.1.1. The **difference of X and Y** is the set $X \setminus Y$ defined by

$$X \setminus Y \overset{\text{def}}{=} \{a \in X \mid a \notin Y\}.$$

Proposition 3.10.1.2. Let X be a set.

1. *Functoriality.* The assignments $U, V, (U,V) \mapsto U \cap V$ define functors

$$\begin{aligned} U \setminus -\colon (\mathcal{P}(X), \supset) &\to (\mathcal{P}(X), \subset), \\ - \setminus V\colon (\mathcal{P}(X), \subset) &\to (\mathcal{P}(X), \subset), \\ -_1 \setminus -_2\colon (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \supset) &\to (\mathcal{P}(X), \subset), \end{aligned}$$

   where $-_1 \setminus -_2$ is the functor where

   - *Action on Objects.* For each $(U,V) \in \mathcal{P}(X) \times \mathcal{P}(X)$, we have

$$[-_1 \setminus -_2](U,V) \overset{\text{def}}{=} U \setminus V.$$

   - *Action on Morphisms.* For each pair of morphisms

$$\begin{aligned} \iota_A\colon A &\hookrightarrow B, \\ \iota_U\colon U &\hookrightarrow V \end{aligned}$$

of $\mathcal{P}(X) \times \mathcal{P}(X)$, the image

$$\iota_U \setminus \iota_V \colon A \setminus V \hookrightarrow B \setminus U$$

of (ι_U, ι_V) by $\setminus$ is the inclusion

$$A \setminus V \subset B \setminus U$$

i.e. where we have

($\star$) If $A \subset B$ and $U \subset V$, then $A \setminus V \subset B \setminus U$.

and where $U \setminus -$ and $- \setminus V$ are the partial functors of $-_1 \setminus -_2$ at $U, V \in \mathcal{P}(X)$.

2. *De Morgan's Laws.* We have equalities of sets

$$\begin{aligned} X \setminus (U \cup V) &= (X \setminus U) \cap (X \setminus V), \\ X \setminus (U \cap V) &= (X \setminus U) \cup (X \setminus V) \end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

3. *Interaction With Unions I.* We have equalities of sets

$$U \setminus (V \cup W) = (U \setminus V) \cap (U \setminus W)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

4. *Interaction With Unions II.* We have equalities of sets

$$(U \setminus V) \cup W = (U \cup W) \setminus (V \setminus W)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

5. *Interaction With Unions III.* We have equalities of sets

$$\begin{aligned} U \setminus (V \cup W) &= (U \cup W) \setminus (V \cup W) \\ &= (U \setminus V) \setminus W \\ &= (U \setminus W) \setminus V \end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

6. *Interaction With Unions IV.* We have equalities of sets

$$(U \cup V) \setminus W = (U \setminus W) \cup (V \setminus W)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

7. *Interaction With Intersections.* We have equalities of sets

$$\begin{aligned}(U \setminus V) \cap W &= (U \cap W) \setminus V\\ &= U \cap (W \setminus V)\end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

8. *Interaction With Complements.* We have an equality of sets

$$U \setminus V = U \cap V^{\mathsf{c}}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

9. *Interaction With Symmetric Differences.* We have an equality of sets

$$U \setminus V = U \mathbin{\triangle} (U \cap V)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

10. *Triple Differences.* We have

$$U \setminus (V \setminus W) = (U \cap W) \cup (U \setminus V)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

11. *Left Annihilation.* We have

$$\emptyset \setminus U = \emptyset$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

12. *Right Unitality.* We have

$$U \setminus \emptyset = U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

13. *Invertibility.* We have
$$U \setminus U = \emptyset$$
for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

14. *Interaction With Containment.* The following conditions are equivalent:
    (a) We have $V \setminus U \subset W$.
    (b) We have $V \setminus W \subset U$.

15. *Interaction With Characteristic Functions.* We have
$$\chi_{U \setminus V} = \chi_U - \chi_{U \cap V}$$
for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

Proof. *Item 1*, *Functoriality*: See [Pro24ad] and [Pro24ah].
Item 2, *De Morgan's Laws*: See [Pro24m].
Item 3, *Interaction With Unions I*: See [Pro24n].
Item 4, *Interaction With Unions II*: Omitted.
Item 5, *Interaction With Unions III*: See [Pro24ai].
Item 6, *Interaction With Unions IV*: See [Pro24ac].
Item 7, *Interaction With Intersections*: See [Pro24v].
Item 8, *Interaction With Complements*: See [Pro24aa].
Item 9, *Interaction With Symmetric Differences*: See [Pro24ab].
Item 10, *Triple Differences*: See [Pro24ag].
Item 11, *Left Annihilation*: Clear.
Item 12, *Right Unitality*: See [Pro24ae].
Item 13, *Invertibility*: See [Pro24af].
Item 14, *Interaction With Containment*: Omitted.
Item 15, *Interaction With Characteristic Functions*: See [Pro24i]. □

3.11 Complements

Let X be a set and let $U \in \mathcal{P}(X)$.

Definition 3.11.1.1. The **complement of** U is the set U^{c} defined by
$$\begin{aligned} U^{\mathsf{c}} &\overset{\text{def}}{=} X \setminus U \\ &\overset{\text{def}}{=} \{a \in X \mid a \notin U\}. \end{aligned}$$

Proposition 3.11.1.2. Let X be a set.

1. *Functoriality.* The assignment $U \mapsto U^{\mathsf{c}}$ defines a functor
$$(-)^{\mathsf{c}} \colon \mathcal{P}(X)^{\mathsf{op}} \to \mathcal{P}(X),$$
where
   - *Action on Objects.* For each $U \in \mathcal{P}(X)$, we have
   $$[(-)^{\mathsf{c}}](U) \overset{\text{def}}{=} U^{\mathsf{c}}.$$
   - *Action on Morphisms.* For each morphism $\iota_U \colon U \hookrightarrow V$ of $\mathcal{P}(X)$, the image
   $$\iota_U^{\mathsf{c}} \colon V^{\mathsf{c}} \hookrightarrow U^{\mathsf{c}}$$
   of ι_U by $(-)^{\mathsf{c}}$ is the inclusion
   $$V^{\mathsf{c}} \subset U^{\mathsf{c}}$$
   i.e. where we have
   $(\star)$ If $U \subset V$, then $V^{\mathsf{c}} \subset U^{\mathsf{c}}$.
2. *De Morgan's Laws.* We have equalities of sets
$$(U \cup V)^{\mathsf{c}} = U^{\mathsf{c}} \cap V^{\mathsf{c}},$$
$$(U \cap V)^{\mathsf{c}} = U^{\mathsf{c}} \cup V^{\mathsf{c}}$$
for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.
3. *Involutority.* We have
$$(U^{\mathsf{c}})^{\mathsf{c}} = U$$
for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.
4. *Interaction With Characteristic Functions.* We have
$$\chi_{U^{\mathsf{c}}} = 1 - \chi_U$$
for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

Proof. *Item 1*, *Functoriality*: This follows from Item 1 of Proposition 3.10.1.2.
Item 2, *De Morgan's Laws*: See [Pro24m].
Item 3, *Involutority*: See [Pro24l].
Item 4, *Interaction With Characteristic Functions*: Clear. □

3.12 Symmetric Differences

Let A and B be sets.

Definition 3.12.1.1. The **symmetric difference of** A **and** B is the set $A \triangle B$ defined by

$$A \triangle B \stackrel{\text{def}}{=} (A \setminus B) \cup (B \setminus A).$$

Proposition 3.12.1.2. Let X be a set.

1. *Lack of Functoriality.* The assignment $(U, V) \mapsto U \triangle V$ ***need not*** define functors

$$\begin{aligned} U \triangle -\colon (\mathcal{P}(X), \subset) &\to (\mathcal{P}(X), \subset), \\ - \triangle V\colon (\mathcal{P}(X), \subset) &\to (\mathcal{P}(X), \subset), \\ -_1 \triangle -_2\colon (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \subset) &\to (\mathcal{P}(X), \subset). \end{aligned}$$

2. *Via Unions and Intersections.* We have[17]

$$U \triangle V = (U \cup V) \setminus (U \cap V)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

3. *Associativity.* We have[18]

$$(U \triangle V) \triangle W = U \triangle (V \triangle W)$$

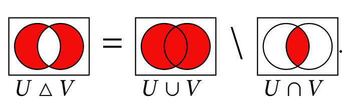
for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

4. *Commutativity.* We have

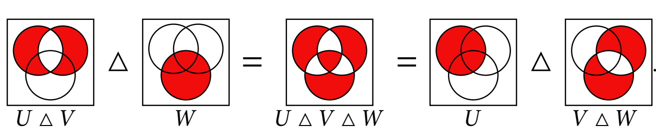
$$U \triangle V = V \triangle U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

[17] *Illustration:*



[18] *Illustration:*



5. *Unitality*. We have

$$\begin{aligned} U \triangle \emptyset &= U, \\ \emptyset \triangle U &= U \end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

6. *Invertibility*. We have

$$U \triangle U = \emptyset$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

7. *Interaction With Unions*. We have

$$(U \triangle V) \cup (V \triangle T) = (U \cup V \cup W) \setminus (U \cap V \cap W)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

8. *Interaction With Complements I*. We have

$$U \triangle U^{\mathsf{c}} = X$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

9. *Interaction With Complements II*. We have

$$\begin{aligned} U \triangle X &= U^{\mathsf{c}}, \\ X \triangle U &= U^{\mathsf{c}} \end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

10. *Interaction With Complements III*. We have

$$U^{\mathsf{c}} \triangle V^{\mathsf{c}} = U \triangle V$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

11. *"Transitivity"*. We have

$$(U \triangle V) \triangle (V \triangle W) = U \triangle W$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

12. *The Triangle Inequality for Symmetric Differences.* We have

$$U \triangle W \subset U \triangle V \cup V \triangle W$$

for each $X \in \mathsf{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

13. *Distributivity Over Intersections.* We have

$$\begin{aligned} U \cap (V \triangle W) &= (U \cap V) \triangle (U \cap W), \\ (U \triangle V) \cap W &= (U \cap W) \triangle (V \cap W) \end{aligned}$$

for each $X \in \mathsf{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

14. *Interaction With Characteristic Functions.* We have

$$\chi_{U \triangle V} = \chi_U + \chi_V - 2\chi_{U \cap V}$$

and thus, in particular, we have

$$\chi_{U \triangle V} \equiv \chi_U + \chi_V \mod 2$$

for each $X \in \mathsf{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

15. *Bijectivity.* Given $A, B \subset \mathcal{P}(X)$, the maps

$$\begin{aligned} A \triangle -&\colon \mathcal{P}(X) \to \mathcal{P}(X), \\ - \triangle B&\colon \mathcal{P}(X) \to \mathcal{P}(X) \end{aligned}$$

are bijections with inverses given by

$$\begin{aligned} (A \triangle -)^{-1} &= - \cup (A \cap -), \\ (- \triangle B)^{-1} &= - \cup (B \cap -). \end{aligned}$$

Moreover, the map

$$C \mapsto C \triangle (A \triangle B)$$

is a bijection of $\mathcal{P}(X)$ onto itself sending A to B and B to A.

16. *Interaction With Powersets and Groups.* Let X be a set.

(a) The quadruple $(\mathcal{P}(X), \triangle, \emptyset, \mathrm{id}_{\mathcal{P}(X)})$ is an abelian group.[19]

(b) Every element of $\mathcal{P}(X)$ has order 2 with respect to $\triangle$, and thus $\mathcal{P}(X)$ is a *Boolean group* (i.e. an abelian 2-group).

4. *Interaction With Powersets and Vector Spaces I.* The pair $(\mathcal{P}(X), \alpha_{\mathcal{P}(X)})$ consisting of

   - The group $\mathcal{P}(X)$ of **??**;
   - The map $\alpha_{\mathcal{P}(X)} \colon \mathbb{F}_2 \times \mathcal{P}(X) \to \mathcal{P}(X)$ defined by

$$\begin{aligned} 0 \cdot U &\overset{\text{def}}{=} \emptyset, \\ 1 \cdot U &\overset{\text{def}}{=} U; \end{aligned}$$

   is an $\mathbb{F}_2$-vector space.

5. *Interaction With Powersets and Vector Spaces II.* If X is finite, then:

   (a) The set of singletons sets on the elements of X forms a basis for the $\mathbb{F}_2$-vector space $(\mathcal{P}(X), \alpha_{\mathcal{P}(X)})$ of Item 4.

   (b) We have
$$\dim(\mathcal{P}(X)) = \#\mathcal{P}(X).$$

6. *Interaction With Powersets and Rings.* The quintuple $(\mathcal{P}(X), \triangle, \cap, \emptyset, X)$ is a commutative ring.[20]

---

[19] Here are some examples:

1. When $X = \emptyset$, we have an isomorphism of groups between $\mathcal{P}(\emptyset)$ and the trivial group:
$$(\mathcal{P}(\emptyset), \triangle, \emptyset, \mathrm{id}_{\mathcal{P}(\emptyset)}) \cong \mathrm{pt}.$$
2. When $X = \mathrm{pt}$, we have an isomorphism of groups between $\mathcal{P}(\mathrm{pt})$ and $\mathbb{Z}_{/2}$:
$$(\mathcal{P}(\mathrm{pt}), \triangle, \emptyset, \mathrm{id}_{\mathcal{P}(\mathrm{pt})}) \cong \mathbb{Z}_{/2}.$$
3. When $X = \{0, 1\}$, we have an isomorphism of groups between $\mathcal{P}(\{0, 1\})$ and $\mathbb{Z}_{/2} \times \mathbb{Z}_{/2}$:
$$(\mathcal{P}(\{0, 1\}), \triangle, \emptyset, \mathrm{id}_{\mathcal{P}(\{0,1\})}) \cong \mathbb{Z}_{/2} \times \mathbb{Z}_{/2}.$$

[20] *Warning:* The analogous statement replacing intersections by unions (i.e. that the quintuple $(\mathcal{P}(X), \triangle, \cup, \emptyset, X)$ is a ring) is false, however. See [Pro24aw] for a proof.

Proof. *Item 1, Lack of Functoriality*: Omitted.
Item 2, Via Unions and Intersections: See [Pro24o].
Item 3, Associativity: See [Pro24ao].
Item 4, Commutativity: See [Pro24ap].
Item 5, Unitality: This follows from Item 4 and [Pro24at].
Item 6, Invertibility: See [Pro24av].
Item 7, Interaction With Unions: See [Pro24bc].
Item 8, Interaction With Complements I: See [Pro24as].
Item 9, Interaction With Complements II: This follows from Item 4 and [Pro24ax].
Item 10, Interaction With Complements III: See [Pro24aq].
Item 11, "Transitivity": We have

$$\begin{aligned}(U \triangle V) \triangle (V \triangle W) &= U \triangle (V \triangle (V \triangle W)) && \text{(by Item 3)}\\ &= U \triangle ((V \triangle V) \triangle W) && \text{(by Item 3)}\\ &= U \triangle (\emptyset \triangle W) && \text{(by Item 6)}\\ &= U \triangle W && \text{(by Item 5)}\end{aligned}$$

Item 12, The Triangle Inequality for Symmetric Differences: This follows from Items 2 and 11.
Item 13, Distributivity Over Intersections: See [Pro24r].
Item 14, Interaction With Characteristic Functions: See [Pro24j].
Item 15, Bijectivity: Clear.
Item 16, Interaction With Powersets and Groups: Item 16a follows from[21] Items 3 to 6, while Item 3b follows from Item 6.
Item 4, Interaction With Powersets and Vector Spaces I: Clear.
Item 5, Interaction With Powersets and Vector Spaces II: Omitted.
Item 6, Interaction With Powersets and Rings: This follows from Items 8 and 11 of Proposition 3.9.1.2 and Items 13 and 16.[22] □

4 Powersets

4.1 Characteristic Functions

Let X be a set.

[21] Reference: [Pro24ar].
[22] *Reference:* [Pro24au].

Definition 4.1.1.1. Let $U \subset X$ and let $x \in X$.

1. The **characteristic function of** U[23] is the function[24]

$$\chi_U \colon X \to \{\mathsf{t}, \mathsf{f}\}$$

defined by

$$\chi_U(x) \overset{\text{def}}{=} \begin{cases} \text{true} & \text{if } x \in U, \\ \text{false} & \text{if } x \notin U \end{cases}$$

for each $x \in X$.

2. The **characteristic function of** x is the function[25]

$$\chi_x \colon X \to \{\mathsf{t}, \mathsf{f}\}$$

defined by

$$\chi_x \overset{\text{def}}{=} \chi_{\{x\}},$$

i.e. by

$$\chi_x(y) \overset{\text{def}}{=} \begin{cases} \text{true} & \text{if } x = y, \\ \text{false} & \text{if } x \neq y \end{cases}$$

for each $y \in X$.

3. The **characteristic relation on** X[26] is the relation[27]

$$\chi_X(-_1, -_2) \colon X \times X \to \{\mathsf{t}, \mathsf{f}\}$$

on X defined by[28]

$$\chi_X(x, y) \overset{\text{def}}{=} \begin{cases} \text{true} & \text{if } x = y, \\ \text{false} & \text{if } x \neq y \end{cases}$$

for each $x, y \in X$.

[23] *Further Terminology:* Also called the **indicator function of** U.
[24] *Further Notation:* Also written $\chi_X(U, -)$ or $\chi_X(-, U)$.
[25] *Further Notation:* Also written χ^x, $\chi_X(x, -)$, or $\chi_X(-, x)$.
[26] *Further Terminology:* Also called the **identity relation on** X.
[27] *Further Notation:* Also written χ^{-1}_{-2}, or $\sim_{\mathrm{id}}$ in the context of relations.
[28] As a subset of $X \times X$, the relation χ_X corresponds to the diagonal $\Delta_X \subset X \times X$ of X.

4. The **characteristic embedding**[29] **of X into $\mathcal{P}(X)$** is the function

$$\chi_{(-)} \colon X \hookrightarrow \mathcal{P}(X)$$

defined by

$$\chi_{(-)}(x) \overset{\text{def}}{=} \chi_x$$

for each $x \in X$.

Remark 4.1.1.2. The definitions in Definition 4.1.1.1 are decategorifications of co/presheaves, representable co/presheaves, Hom profunctors, and the Yoneda embedding:[30]

1. A function

$$f \colon X \to \{\mathsf{t}, \mathsf{f}\}$$

[29]The name "characteristic *embedding*" comes from the fact that there is an analogue of fully faithfulness for $\chi_{(-)}$: given a set X, we have

$$\mathrm{Hom}_{\mathcal{P}(X)}(\chi_x, \chi_y) = \chi_X(x, y),$$

for each $x, y \in X$.

[30]These statements can be made precise by using the embeddings

$$\begin{aligned} (-)_{\mathsf{disc}} &\colon \mathsf{Sets} \hookrightarrow \mathsf{Cats}, \\ (-)_{\mathsf{disc}} &\colon \{\mathsf{t}, \mathsf{f}\}_{\mathsf{disc}} \hookrightarrow \mathsf{Sets} \end{aligned}$$

of sets into categories and of classical truth values into sets.

For instance, in this approach the characteristic function

$$\chi_x \colon X \to \{\mathsf{t}, \mathsf{f}\}$$

of an element x of X, defined by

$$\chi_x(y) \overset{\text{def}}{=} \begin{cases} \mathsf{true} & \text{if } x = y, \\ \mathsf{false} & \text{if } x \neq y \end{cases}$$

for each $y \in X$, is recovered as the representable presheaf

$$\mathrm{Hom}_{X_{\mathsf{disc}}}(-, x) \colon X_{\mathsf{disc}} \to \mathsf{Sets}$$

of the corresponding object x of X_{disc}, defined on objects by

$$\mathrm{Hom}_{X_{\mathsf{disc}}}(y, x) \overset{\text{def}}{=} \begin{cases} \mathrm{pt} & \text{if } x = y, \\ \emptyset & \text{if } x \neq y \end{cases}$$

for each $y \in \mathrm{Obj}(X_{\mathsf{disc}})$.

is a decategorification of a presheaf

$$\mathcal{F}\colon \mathcal{C}^{\mathsf{op}} \to \mathsf{Sets},$$

with the characteristic functions χ_U of the subsets of X being the primordial examples (and, in fact, all examples) of these.

2. The characteristic function

$$\chi_x\colon X \to \{\mathsf{t},\mathsf{f}\}$$

of an *element* x of X is a decategorification of the representable presheaf

$$h_X\colon \mathcal{C}^{\mathsf{op}} \to \mathsf{Sets}$$

of an *object* x of a category $\mathcal{C}$.

3. The characteristic relation

$$\chi_X(-_1,-_2)\colon X\times X \to \{\mathsf{t},\mathsf{f}\}$$

of X is a decategorification of the Hom profunctor

$$\mathsf{Hom}_{\mathcal{C}}(-_1,-_2)\colon \mathcal{C}^{\mathsf{op}}\times\mathcal{C} \to \mathsf{Sets}$$

of a category $\mathcal{C}$.

4. The characteristic embedding

$$\chi_{(-)}\colon X \hookrightarrow \mathcal{P}(X)$$

of X into $\mathcal{P}(X)$ is a decategorification of the Yoneda embedding

$$よ\colon \mathcal{C}^{\mathsf{op}} \hookrightarrow \mathsf{PSh}(\mathcal{C})$$

of a category $\mathcal{C}$ into $\mathsf{PSh}(\mathcal{C})$.

5. There is also a direct parallel between unions and colimits:

   - An element of $\mathcal{P}(X)$ is a union of elements of X, viewed as one-point subsets $\{x\}\in\mathcal{P}(A)$.
   - An object of $\mathsf{PSh}(\mathcal{C})$ is a colimit of objects of $\mathcal{C}$, viewed as representable presheaves $h_X\in\mathsf{Obj}(\mathsf{PSh}(\mathcal{C}))$.

Proposition 4.1.1.3. Let X be a set.

1. *The Inclusion of Characteristic Relations Associated to a Function.* Let $f\colon A \to B$ be a function. We have an inclusion[31]

$$\chi_B \circ (f \times f) \subset \chi_A, \qquad \begin{tikzcd} A \times A \arrow[rr, "f\times f"] \arrow[dr, "\chi_A"'] & \arrow[d, phantom, "\supset"] & B \times B \arrow[dl, "\chi_B"] \\ & \{\mathrm{t}, \mathrm{f}\}. & \end{tikzcd}$$

2. *Interaction With Unions I.* We have

$$\chi_{U \cup V} = \max(\chi_U, \chi_V)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

3. *Interaction With Unions II.* We have

$$\chi_{U \cup V} = \chi_U + \chi_V - \chi_{U \cap V}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

4. *Interaction With Intersections I.* We have

$$\chi_{U \cap V} = \chi_U \chi_V$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

5. *Interaction With Intersections II.* We have

$$\chi_{U \cap V} = \min(\chi_U, \chi_V)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

6. *Interaction With Differences.* We have

$$\chi_{U \setminus V} = \chi_U - \chi_{U \cap V}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

[31] This is the 0-categorical version of Categories, Definition 4.4.1.1.

7. *Interaction With Complements.* We have

$$\chi_{U^{\mathsf{c}}} = 1 - \chi_U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

8. *Interaction With Symmetric Differences.* We have

$$\chi_{U \triangle V} = \chi_U + \chi_V - 2\chi_{U \cap V}$$

and thus, in particular, we have

$$\chi_{U \triangle V} \equiv \chi_U + \chi_V \mod 2$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

9. *Interaction Between the Characteristic Embedding and Morphisms.* Let $f \colon X \to Y$ be a map of sets. The diagram

$$f_* \circ \chi_X = \chi_{X'} \circ f, \qquad \begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\chi_X"'] & X' \arrow[d, "\chi_{X'}"] \\ \mathcal{P}(X) \arrow[r, "f_*"'] & \mathcal{P}(X'). \end{tikzcd}$$

commutes.

Proof. *Item 1, The Inclusion of Characteristic Relations Associated to a Function*: The inclusion $\chi_B(f(a), f(b)) \subset \chi_A(a, b)$ is equivalent to the statement "if $a = b$, then $f(a) = f(b)$", which is true.

Item 2, Interaction With Unions I: This is a repetition of Item 8 of Proposition 3.7.1.2 and is proved there.

Item 3, Interaction With Unions II: This is a repetition of Item 9 of Proposition 3.7.1.2 and is proved there.

Item 4, Interaction With Intersections I: This is a repetition of Item 9 of Proposition 3.9.1.2 and is proved there.

Item 5, Interaction With Intersections II: This is a repetition of Item 10 of Proposition 3.9.1.2 and is proved there.

Item 6, Interaction With Differences: This is a repetition of Item 15 of Proposition 3.10.1.2 and is proved there.

Item 7, Interaction With Complements: This is a repetition of Item 4 of Proposition 3.11.1.2 and is proved there.
Item 8, Interaction With Symmetric Differences: This is a repetition of Item 14 of Proposition 3.12.1.2 and is proved there.
Item 9, Interaction Between the Characteristic Embedding and Morphisms: Indeed, we have

$$\begin{aligned}[f_* \circ \chi_X](x) &\stackrel{\text{def}}{=} f_*(\chi_X(x)) \\ &\stackrel{\text{def}}{=} f_*(\{x\}) \\ &= \{f(x)\} \\ &\stackrel{\text{def}}{=} \chi_{X'}(f(x)) \\ &\stackrel{\text{def}}{=} [\chi_{X'} \circ f](x),\end{aligned}$$

for each $x \in X$, showing the desired equality. □

4.2 The Yoneda Lemma for Sets

Let X be a set and let $U \subset X$ be a subset of X.

Proposition 4.2.1.1. We have

$$\chi_{\mathcal{P}(X)}(\chi_x, \chi_U) = \chi_U(x)$$

for each $x \in X$, giving an equality of functions

$$\chi_{\mathcal{P}(X)}(\chi_{(-)}, \chi_U) = \chi_U.$$

Proof. Clear. □

Corollary 4.2.1.2. The characteristic embedding is fully faithful, i.e., we have

$$\chi_{\mathcal{P}(X)}(\chi_x, \chi_y) = \chi_X(x, y)$$

for each $x, y \in X$.

Proof. This follows from Proposition 4.2.1.1. □

4.3 Powersets

Let X be a set.

Definition 4.3.1.1. The **powerset of** X is the set $\mathcal{P}(X)$ defined by

$$\mathcal{P}(X) \overset{\text{def}}{=} \{U \in P \mid U \subset X\},$$

where P is the set in the axiom of powerset, **??** of **??**.

Remark 4.3.1.2. The powerset of a set is a decategorification of the category of presheaves of a category: while[32]

- The powerset of a set X is equivalently (Items 1 and 2 of Proposition 4.3.1.6) the set

  $$\mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\})$$

  of functions from X to the set $\{\mathsf{t}, \mathsf{f}\}$ of classical truth values.

- The category of presheaves on a category $\mathcal{C}$ is the category

  $$\mathsf{Fun}(\mathcal{C}^{\mathsf{op}}, \mathsf{Sets})$$

  of functors from $\mathcal{C}^{\mathsf{op}}$ to the category Sets of sets.

Proposition 4.3.1.3. Let X be a set.

1. *Co/Completeness.* The (posetal) category (associated to) $(\mathcal{P}(X), \subset)$ is complete and cocomplete:

   (a) *Products.* The products in $\mathcal{P}(X)$ are given by intersection of subsets.

---

[32] This parallel is based on the following comparison:

- A category is enriched over the category

  $$\mathsf{Sets} \overset{\text{def}}{=} \mathsf{Cats}_0$$

  of sets (i.e. “0-categories”), with presheaves taking values on it.

- A set is enriched over the set

  $$\{\mathsf{t}, \mathsf{f}\} \overset{\text{def}}{=} \mathsf{Cats}_{-1}$$

  of classical truth values (i.e. “(-1)-categories”), with characteristic functions taking values on it.

(b) *Coproducts.* The coproducts in $\mathcal{P}(X)$ are given by union of subsets.

(c) *Co/Equalisers.* Being a posetal category, $\mathcal{P}(X)$ only has at most one morphisms between any two objects, so co/equalisers are trivial.

2. *Cartesian Closedness.* The category $\mathcal{P}(X)$ is Cartesian closed with internal Hom

$$\mathbf{Hom}_{\mathcal{P}(X)}(-_1, -_2) \colon \mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \to \mathcal{P}(X)$$

given by[33]

$$\mathbf{Hom}_{\mathcal{P}(X)}(U, V) \overset{\text{def}}{=} (X \setminus U) \cup V$$

for each $U, V \in \mathrm{Obj}(\mathcal{P}(X))$.

Proof. *Item 1*, *Co/Completeness*: Clear.
Item 2, *Cartesian Closedness*: This follows from Item 2 of Proposition 3.9.1.2. □

Proposition 4.3.1.4. Let X be a set.

1. *Functoriality I.* The assignment $X \mapsto \mathcal{P}(X)$ defines a functor

$$\mathcal{P}_* \colon \mathsf{Sets} \to \mathsf{Sets},$$

where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$\mathcal{P}_*(A) \overset{\text{def}}{=} \mathcal{P}(A).$$

- *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathsf{Sets})$, the action on morphisms

$$\mathcal{P}_{*|A,B} \colon \mathsf{Sets}(A, B) \to \mathsf{Sets}(\mathcal{P}(A), \mathcal{P}(B))$$

of $\mathcal{P}_*$ at (A, B) is the map defined by by sending a map of sets $f \colon A \to B$ to the map

$$\mathcal{P}_*(f) \colon \mathcal{P}(A) \to \mathcal{P}(B)$$

defined by

$$\mathcal{P}_*(f) \overset{\text{def}}{=} f_*,$$

as in Definition 4.4.1.1.

[33] For intuition regarding the expression defining $\mathbf{Hom}_{\mathcal{P}(X)}(U, V)$, see Remark 3.9.1.3.

2. *Functoriality II.* The assignment $X \mapsto \mathcal{P}(X)$ defines a functor

$$\mathcal{P}^{-1}\colon \mathsf{Sets}^{\mathsf{op}} \to \mathsf{Sets},$$

where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$\mathcal{P}^{-1}(A) \stackrel{\text{def}}{=} \mathcal{P}(A).$$

- *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathsf{Sets})$, the action on morphisms

$$\mathcal{P}^{-1}_{A,B}\colon \mathsf{Sets}(A, B) \to \mathsf{Sets}(\mathcal{P}(B), \mathcal{P}(A))$$

  of $\mathcal{P}^{-1}$ at (A, B) is the map defined by by sending a map of sets $f\colon A \to B$ to the map

$$\mathcal{P}^{-1}(f)\colon \mathcal{P}(B) \to \mathcal{P}(A)$$

  defined by

$$\mathcal{P}^{-1}(f) \stackrel{\text{def}}{=} f^{-1},$$

  as in Definition 4.5.1.1.

3. *Functoriality III.* The assignment $X \mapsto \mathcal{P}(X)$ defines a functor

$$\mathcal{P}_!\colon \mathsf{Sets} \to \mathsf{Sets},$$

where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$\mathcal{P}_!(A) \stackrel{\text{def}}{=} \mathcal{P}(A).$$

- *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathsf{Sets})$, the action on morphisms

$$\mathcal{P}_{!|A,B}\colon \mathsf{Sets}(A, B) \to \mathsf{Sets}(\mathcal{P}(A), \mathcal{P}(B))$$

  of $\mathcal{P}_!$ at (A, B) is the map defined by by sending a map of sets $f\colon A \to B$ to the map

$$\mathcal{P}_!(f)\colon \mathcal{P}(A) \to \mathcal{P}(B)$$

  defined by

$$\mathcal{P}_!(f) \stackrel{\text{def}}{=} f_!,$$

  as in Definition 4.6.1.1.

4. *Adjointness I.* We have an adjunction

$$\left(\mathcal{P}^{-1} \dashv \mathcal{P}^{-1,\mathsf{op}}\right)\colon \quad \begin{tikzcd} \mathsf{Sets}^{\mathsf{op}} \arrow[r, bend left, "\mathcal{P}^{-1}"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}, \arrow[l, bend left, "\mathcal{P}^{-1,\mathsf{op}}"] \end{tikzcd}$$

witnessed by a bijection

$$\underbrace{\mathsf{Sets}^{\mathsf{op}}(\mathcal{P}(A), B)}_{\overset{\text{def}}{=}\mathsf{Sets}(B,\mathcal{P}(A))} \cong \mathsf{Sets}(A, \mathcal{P}(B)),$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$ and $B \in \mathrm{Obj}(\mathsf{Sets}^{\mathsf{op}})$.

5. *Adjointness II.* We have an adjunction

$$(\mathsf{Gr} \dashv \mathcal{P}_*)\colon \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "\mathsf{Gr}"] \arrow[r, phantom, "\perp"] & \mathsf{Rel}, \arrow[l, bend left, "\mathcal{P}_*"] \end{tikzcd}$$

witnessed by a bijection of sets

$$\mathsf{Rel}(\mathsf{Gr}(A), B) \cong \mathsf{Sets}(A, \mathcal{P}(B))$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$ and $B \in \mathrm{Obj}(\mathsf{Rel})$, where Gr is the graph functor of Constructions With Relations, Item 1 of Proposition 3.1.1.2 and $\mathcal{P}_*$ is the functor of Constructions With Relations, Proposition 4.5.1.1.

Proof. *Item 1, Functoriality I*: This follows from Items 3 and 4 of Proposition 4.4.1.5.
Item 2, Functoriality II: This follows Items 3 and 4 of Proposition 4.5.1.4.
Item 3, Functoriality III: This follows Items 3 and 4 of Proposition 4.6.1.7.
Item 4, Adjointness I: We have

$$\begin{aligned} \mathsf{Sets}^{\mathsf{op}}(\mathcal{P}(A), B) &\overset{\text{def}}{=} \mathsf{Sets}(B, \mathcal{P}(A)) \\ &\cong \mathsf{Sets}(B, \mathsf{Sets}(A, \{\mathsf{t}, \mathsf{f}\})) && \text{(by Item 1 of Proposition 4.3.1.6)} \\ &\cong \mathsf{Sets}(A \times B, \{\mathsf{t}, \mathsf{f}\}) && \text{(by Item 2 of Proposition 1.3.1.2)} \\ &\cong \mathsf{Sets}(A, \mathsf{Sets}(B, \{\mathsf{t}, \mathsf{f}\})) && \text{(by Item 2 of Proposition 1.3.1.2)} \\ &\cong \mathsf{Sets}(A, \mathcal{P}(B)) && \text{(by Item 1 of Proposition 4.3.1.6)} \end{aligned}$$

with all bijections natural in A and B (where we use Item 2 of Proposition 4.3.1.6 here).

Item 5, Adjointness II: We have

$$\begin{aligned}\mathsf{Rel}(\mathsf{Gr}(A), B) &\cong \mathcal{P}(A \times B)\\ &\cong \mathsf{Sets}(A \times B, \{\mathsf{t}, \mathsf{f}\}) && \text{(by Item 1 of Proposition 4.3.1.6)}\\ &\cong \mathsf{Sets}(A, \mathsf{Sets}(B, \{\mathsf{t}, \mathsf{f}\})) && \text{(by Item 2 of Proposition 1.3.1.2)}\\ &\cong \mathsf{Sets}(A, \mathcal{P}(B)) && \text{(by Item 1 of Proposition 4.3.1.6)}\end{aligned}$$

with all bijections natural in A (where we use Item 2 of Proposition 4.3.1.6 here). Explicitly, this isomorphism is given by sending a relation $R\colon \mathsf{Gr}(A) \nrightarrow B$ to the map $R^{\dagger}\colon A \to \mathcal{P}(B)$ sending a to the subset $R(a)$ of B, as in Relations, Remark 1.1.1.4.

Naturality in B is then the statement that given a relation $R\colon B \nrightarrow B'$, the diagram

$$\begin{tikzcd} \mathsf{Rel}(\mathsf{Gr}(A), B) \arrow[r, "R \diamond -"] \arrow[d, dashed, "\wr"] & \mathsf{Rel}(\mathsf{Gr}(A), B') \arrow[d, dashed, "\wr"'] \\ \mathsf{Sets}(A, \mathcal{P}(B)) \arrow[r, "R_*"'] & \mathsf{Sets}(A, \mathcal{P}(B')) \end{tikzcd}$$

commutes, which follows from Constructions With Relations, Remark 4.1.1.2. □

Proposition 4.3.1.5. Let X be a set.

1. *Symmetric Strong Monoidality With Respect to Coproducts I.* The powerset functor $\mathcal{P}_*$ of Item 1 of Proposition 4.3.1.4 has a symmetric strong monoidal structure

$$(\mathcal{P}_*, \mathcal{P}_*^{\coprod}, \mathcal{P}_{*|\mathbb{1}}^{\coprod})\colon (\mathsf{Sets}, \times, \mathrm{pt}) \to (\mathsf{Sets}, \coprod, \emptyset)$$

   being equipped with isomorphisms

$$\mathcal{P}^{\coprod}_{*|X,Y}\colon \mathcal{P}(X) \times \mathcal{P}(Y) \xrightarrow{\cong} \mathcal{P}(X \coprod Y),$$
$$\mathcal{P}^{\coprod}_{*|\mathbb{1}}\colon \mathrm{pt} \xrightarrow{\cong} \mathcal{P}(\emptyset),$$

   natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

2. *Symmetric Strong Monoidality With Respect to Coproducts II.* The powerset functor $\mathcal{P}^{-1}$ of Item 2 of Proposition 4.3.1.4 has a symmetric strong monoidal structure

$$(\mathcal{P}^{-1}, \mathcal{P}^{-1|\coprod}, \mathcal{P}_{\mathbb{1}}^{-1|\coprod})\colon (\mathsf{Sets}^{\mathsf{op}}, \times^{\mathsf{op}}, \mathrm{pt}) \to (\mathsf{Sets}, \coprod, \emptyset)$$

being equipped with isomorphisms

$$\mathcal{P}^{-1|\coprod}_{X,Y} \colon \mathcal{P}(X) \times \mathcal{P}(Y) \xrightarrow{\cong} \mathcal{P}(X \textstyle\coprod Y),$$
$$\mathcal{P}^{-1|\coprod}_{\mathbb{1}} \colon \mathrm{pt} \xrightarrow{\cong} \mathcal{P}(\emptyset),$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

3. *Symmetric Strong Monoidality With Respect to Coproducts III.* The powerset functor $\mathcal{P}_!$ of Item 3 of Proposition 4.3.1.4 has a symmetric strong monoidal structure

$$(\mathcal{P}_!, \mathcal{P}^{\coprod}_!, \mathcal{P}^{\coprod}_{!|\mathbb{1}}) \colon (\mathsf{Sets}, \times, \mathrm{pt}) \longrightarrow (\mathsf{Sets}, \textstyle\coprod, \emptyset)$$

being equipped with isomorphisms

$$\mathcal{P}^{\coprod}_{!|X,Y} \colon \mathcal{P}(X) \times \mathcal{P}(Y) \xrightarrow{\cong} \mathcal{P}(X \textstyle\coprod Y),$$
$$\mathcal{P}^{\coprod}_{!|\mathbb{1}} \colon \mathrm{pt} \xrightarrow{\cong} \mathcal{P}(\emptyset),$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

4. *Symmetric Lax Monoidality With Respect to Products I.* The powerset functor $\mathcal{P}_*$ of Item 1 of Proposition 4.3.1.4 has a symmetric lax monoidal structure

$$(\mathcal{P}_*, \mathcal{P}^{\otimes}_*, \mathcal{P}^{\otimes}_{*|\mathbb{1}}) \colon (\mathsf{Sets}, \times, \mathrm{pt}) \longrightarrow (\mathsf{Sets}, \times, \mathrm{pt})$$

being equipped with morphisms

$$\mathcal{P}^{\times}_{*|X,Y} \colon \mathcal{P}(X) \times \mathcal{P}(Y) \to \mathcal{P}(X \times Y),$$
$$\mathcal{P}^{\times}_{*|\mathbb{1}} \colon \mathrm{pt} \to \mathcal{P}(\mathrm{pt}),$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$, where

- The map $\mathcal{P}^{\times}_{*|X,Y}$ is given by

$$\mathcal{P}^{\times}_{*|X,Y}(U, V) \stackrel{\mathrm{def}}{=} U \times V$$

for each $(U, V) \in \mathcal{P}(X) \times \mathcal{P}(Y)$,

- The map $\mathcal{P}^{\times}_{*|\mathbb{1}}$ is given by

$$\mathcal{P}^{\times}_{*|\mathbb{1}}(\star) = \mathrm{pt}.$$

5. *Symmetric Lax Monoidality With Respect to Products II.* The powerset functor $\mathcal{P}^{-1}$ of Item 2 of Proposition 4.3.1.4 has a symmetric lax monoidal structure
$$(\mathcal{P}^{-1}, \mathcal{P}^{-1|\otimes}, \mathcal{P}^{-1|\otimes}_{\mathbb{1}}) \colon (\mathsf{Sets}^{\mathsf{op}}, \times^{\mathsf{op}}, \mathrm{pt}) \to (\mathsf{Sets}, \times, \mathrm{pt})$$
being equipped with morphisms
$$\mathcal{P}^{-1|\times}_{X,Y} \colon \mathcal{P}(X) \times \mathcal{P}(Y) \to \mathcal{P}(X \times Y),$$
$$\mathcal{P}^{\times}_{\mathbb{1}} \colon \mathrm{pt} \to \mathcal{P}(\emptyset),$$
natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$, defined as in Item 4.

6. *Symmetric Lax Monoidality With Respect to Products III.* The powerset functor $\mathcal{P}_!$ of Item 3 of Proposition 4.3.1.4 has a symmetric lax monoidal structure
$$(\mathcal{P}_!, \mathcal{P}^{\otimes}_!, \mathcal{P}^{\otimes}_{!|\mathbb{1}}) \colon (\mathsf{Sets}, \times, \mathrm{pt}) \to (\mathsf{Sets}, \times, \mathrm{pt})$$
being equipped with morphisms
$$\mathcal{P}^{\times}_{!|X,Y} \colon \mathcal{P}(X) \times \mathcal{P}(Y) \to \mathcal{P}(X \times Y),$$
$$\mathcal{P}^{\times}_{!|\mathbb{1}} \colon \mathrm{pt} \to \mathcal{P}(\emptyset),$$
natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$, defined as in Item 4.

Proof. *Item 1, Symmetric Strong Monoidality With Respect to Coproducts I*: The isomorphism
$$\mathcal{P}^{\coprod}_{*|X,Y} \colon \mathcal{P}(X) \times \mathcal{P}(Y) \to \mathcal{P}(X \coprod Y)$$
is given by sending $(U, V) \in \mathcal{P}(X) \times \mathcal{P}(Y)$ to $U \coprod V$, with inverse given by sending a subset S of $X \coprod Y$ to the pair $(S_X, S_Y) \in \mathcal{P}(X) \times \mathcal{P}(Y)$ with
$$S_X \stackrel{\text{def}}{=} \{x \in X \mid (0, x) \in S\}$$
$$S_Y \stackrel{\text{def}}{=} \{y \in Y \mid (1, y) \in S\}.$$
The isomorphism $\mathrm{pt} \cong \mathcal{P}(\emptyset)$ is given by $\star \mapsto \emptyset \in \mathcal{P}(\emptyset)$.
Naturality for the isomorphism $\mathcal{P}^{\coprod}_{*|X,Y}$ is the statement that, given maps of sets $f \colon X \to X'$ and $g \colon Y \to Y'$, the diagram
$$\begin{tikzcd}
\mathcal{P}(X) \times \mathcal{P}(Y) \arrow[r, "f_* \times g_*"] \arrow[d, dashed, "\wr"] & \mathcal{P}(X') \times \mathcal{P}(Y') \arrow[d, dashed, "\wr"'] \\
\mathcal{P}(X \coprod Y) \arrow[r, "(f \coprod g)_*"'] & \mathcal{P}(X' \coprod Y')
\end{tikzcd}$$

commutes, which is clear, as it acts on elements as

$$\begin{tikzcd}
(U,V) \arrow[r, mapsto] \arrow[d, mapsto] & (f_*(U), g_*(V)) \arrow[d, mapsto] \\
U \coprod V \arrow[r, mapsto] & (f \coprod g)_*(U \coprod V) = f_*(U) \coprod g_*(V),
\end{tikzcd}$$

where we are using Item 7 of Proposition 4.4.1.4.

Finally, monoidality, unity, and symmetry of $\mathcal{P}_*$ as a monoidal functor follow by checking the commutativity of the relevant diagrams on elements.

Item 2, Symmetric Strong Monoidality With Respect to Coproducts II: The proof is similar to Item 1, and is hence omitted.

Item 3, Symmetric Strong Monoidality With Respect to Coproducts III: The proof is similar to Item 1, and is hence omitted.

Item 4, Symmetric Lax Monoidality With Respect to Products I: Naturality for the morphism $\mathcal{P}^{\times}_{*|X,Y}$ is the statement that, given maps of sets $f\colon X \to X'$ and $g\colon Y \to Y'$, the diagram

$$\begin{tikzcd}
\mathcal{P}(X) \times \mathcal{P}(Y) \arrow[r, "f_* \times g_*"] \arrow[d, dashed, "\wr"] & \mathcal{P}(X') \times \mathcal{P}(Y') \arrow[d, dashed, "\wr"'] \\
\mathcal{P}(X \times Y) \arrow[r, "(f \times g)_*"'] & \mathcal{P}(X' \times Y')
\end{tikzcd}$$

commutes, which is clear, as it acts on elements as

$$\begin{tikzcd}
(U,V) \arrow[r, mapsto] \arrow[d, mapsto] & (f_*(U), g_*(V)) \arrow[d, mapsto] \\
U \times V \arrow[r, mapsto] & (f \times g)_*(U \times V) = f_*(U) \times g_*(V),
\end{tikzcd}$$

where we are using Item 8 of Proposition 4.4.1.4.

Finally, monoidality, unity, and symmetry of $\mathcal{P}_*$ as a monoidal functor follow by checking the commutativity of the relevant diagrams on elements.

Item 5, Symmetric Lax Monoidality With Respect to Products II: The proof is similar to Item 4, and is hence omitted.

Item 6, Symmetric Lax Monoidality With Respect to Products III: The proof is similar to Item 4, and is hence omitted. □

Proposition 4.3.1.6. Let X be a set.

1. *Powersets as Sets of Functions I.* The assignment $U \mapsto \chi_U$ defines a bijection

$$\chi_{(-)} \colon \mathcal{P}(X) \xrightarrow{\cong} \mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\}),$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$.

2. *Powersets as Sets of Functions II.* The bijection

$$\mathcal{P}(X) \cong \mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\})$$

of Item 1 is natural in $X \in \mathrm{Obj}(\mathsf{Sets})$, refining to a natural isomorphism of functors

$$\mathcal{P}^{-1} \cong \mathsf{Sets}(-, \{\mathsf{t}, \mathsf{f}\}).$$

3. *Powersets as Sets of Relations.* We have bijections

$$\begin{aligned} \mathcal{P}(X) &\cong \mathrm{Rel}(\mathrm{pt}, X), \\ \mathcal{P}(X) &\cong \mathrm{Rel}(X, \mathrm{pt}), \end{aligned}$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets})$.

Proof. *Item 1, Powersets as Sets of Functions I*: Indeed, the inverse of $\chi_{(-)}$ is given by sending a function $f \colon X \to \{\mathsf{t}, \mathsf{f}\}$ to the subset U_f of $\mathcal{P}(X)$ defined by

$$U_f \overset{\mathrm{def}}{=} \{x \in X \mid f(x) = \mathsf{true}\},$$

i.e. by $U_f = f^{-1}(\mathsf{true})$. That $\chi_{(-)}$ and $f \mapsto U_f$ are inverses is then straightforward to check.

Item 2, Powersets as Sets of Functions II: We need to check that, given a function $f \colon X \to Y$, the diagram

$$\begin{tikzcd} \mathcal{P}(Y) \arrow[r, "f^{-1}"] \arrow[d, dashed, "\chi_{(-)}"', "\wr"] & \mathcal{P}(X) \arrow[d, dashed, "\chi_{(-)}", "\wr"'] \\ \mathsf{Sets}(Y, \{\mathsf{t}, \mathsf{f}\}) \arrow[r, "f^*"'] & \mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\}) \end{tikzcd}$$

commutes, i.e. that for each $V \in \mathcal{P}(Y)$, we have

$$\chi_V \circ f = \chi_{f^{-1}(V)}.$$

And indeed, we have

$$\begin{aligned}[\chi_V \circ f](v) &\overset{\text{def}}{=} \chi_V(f(v)) \\ &= \begin{cases} \text{true} & \text{if } f(v) \in V, \\ \text{false} & \text{otherwise} \end{cases} \\ &= \begin{cases} \text{true} & \text{if } v \in f^{-1}(V), \\ \text{false} & \text{otherwise} \end{cases} \\ &\overset{\text{def}}{=} \chi_{f^{-1}(V)}(v)\end{aligned}$$

for each $v \in V$.

Item 3, *Powersets as Sets of Relations*: Indeed, we have

$$\begin{aligned}\mathrm{Rel}(\mathrm{pt}, X) &\overset{\text{def}}{=} \mathcal{P}(\mathrm{pt} \times X) \\ &\cong \mathcal{P}(X)\end{aligned}$$

and

$$\begin{aligned}\mathrm{Rel}(X, \mathrm{pt}) &\overset{\text{def}}{=} \mathcal{P}(X \times \mathrm{pt}) \\ &\cong \mathcal{P}(X),\end{aligned}$$

where we have used Item 4 of Proposition 1.3.1.2. □

Remark 4.3.1.7. The bijection

$$\mathcal{P}(X) \cong \mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\})$$

of Item 1 of Proposition 4.3.1.6, which

- Takes a subset $U \hookrightarrow X$ of X and *straightens* it to a function $\chi_U \colon X \to \{\text{true}, \text{false}\}$;
- Takes a function $f \colon X \to \{\text{true}, \text{false}\}$ and *unstraightens* it to a subset $f^{-1}(\text{true}) \hookrightarrow X$ of X;

may be viewed as the (-1)-categorical version of the un/straightening isomorphism for indexed and fibred sets

$$\underbrace{\mathsf{FibSets}(X)}_{\overset{\text{def}}{=} \mathsf{Sets}_{/X}} \cong \underbrace{\mathsf{ISets}(X)}_{\overset{\text{def}}{=} \mathsf{Fun}(X_{\mathsf{disc}}, \mathsf{Sets})}$$

of **??**, **??**, where we view:

- Subsets $U \hookrightarrow X$ as analogous to X-fibred sets $\phi_X : A \to X$.
- Functions $f : X \to \{\mathsf{t}, \mathsf{f}\}$ as analogous to X-indexed sets $A : X_{\mathsf{disc}} \to \mathsf{Sets}$.

Proposition 4.3.1.8. Let X be a set.

1. *Universal Property.* The pair $(\mathcal{P}(X), \chi_{(-)})$ consisting of

   - The powerset $\mathcal{P}(X)$ of X;
   - The characteristic embedding $\chi_{(-)} : X \hookrightarrow \mathcal{P}(X)$ of X into $\mathcal{P}(X)$;

   satisfies the following universal property:

   (★) Given another pair (Y, f) consisting of
   - A cocomplete poset $(Y, \preceq)$;
   - A function $f : X \to Y$;

   there exists a unique cocontinuous morphism of posets

$$(\mathcal{P}(X), \subset) \xrightarrow{\exists!} (Y, \preceq)$$

   making the diagram

$$\begin{tikzcd}
 & \mathcal{P}(X) \arrow[d, dashed, "\exists!"] \\
X \arrow[ur, "\chi_X"] \arrow[r, "f"'] & Y
\end{tikzcd}$$

   commute.

2. *Adjointness.* We have an adjunction[34]

$$(\mathcal{P} \dashv 忘) : \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "\mathcal{P}"] \arrow[r, phantom, "\perp"] & \mathsf{Pos}^{\mathsf{cocomp.}} \arrow[l, bend left, "忘"] \end{tikzcd},$$

[34] In this sense, $\mathcal{P}(A)$ is the free cocompletion of A. (Note that, despite its name, however, this is not an idempotent operation, as we have $\mathcal{P}(\mathcal{P}(A)) \neq \mathcal{P}(A)$.)

witnessed by a bijection

$$\mathsf{Pos}^{\mathsf{cocomp.}}((\mathcal{P}(X),\subset),(Y,\preceq)) \cong \mathsf{Sets}(X,Y),$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets})$ and $(Y,\preceq) \in \mathrm{Obj}(\mathsf{Pos}^{\mathsf{cocomp.}})$, where the maps witnessing this bijection are given by

- The map

  $$\chi_X^* \colon \mathsf{Pos}^{\mathsf{cocomp.}}((\mathcal{P}(X),\subset),(Y,\preceq)) \to \mathsf{Sets}(X,Y)$$

  defined by

  $$\chi_X^*(f) \overset{\text{def}}{=} f \circ \chi_X,$$

  i.e. by sending a cocontinuous morphism of posets $f \colon \mathcal{P}(X) \to Y$ to the composition

  $$X \overset{\chi_X}{\hookrightarrow} \mathcal{P}(X) \xrightarrow{f} Y.$$

- The map

  $$\mathrm{Lan}_{\chi_X} \colon \mathsf{Sets}(X,Y) \to \mathsf{Pos}^{\mathsf{cocomp.}}((\mathcal{P}(X),\subset),(Y,\preceq))$$

  is given by sending a function $f \colon X \to Y$ to its left Kan extension along χ_X,

  $$\mathrm{Lan}_{\chi_X}(f) \colon \mathcal{P}(X) \to Y, \qquad \begin{tikzcd} & \mathcal{P}(X) \arrow[d, dashed, "\mathrm{Lan}_{\chi_X}(f)"] \\ X \arrow[ur, "\chi_X"] \arrow[r, "f"'] \arrow[ur, phantom, shift right=3, "\Uparrow"] & Y. \end{tikzcd}$$

  Moreover, $\mathrm{Lan}_{\chi_X}(f)$ can be explicitly computed by

  $$\begin{aligned} [\mathrm{Lan}_{\chi_X}(f)](U) &\cong \int^{x\in X} \chi_{\mathcal{P}(X)}(\chi_x, U) \odot f(x) \\ &\cong \int^{x\in X} \chi_U(x) \odot f(x) && \text{(by Proposition 4.2.1.1)} \\ &\cong \bigvee_{x\in X} (\chi_U(x) \odot f(x)) \end{aligned}$$

  for each $U \in \mathcal{P}(X)$, where:

– $\bigvee$ is the join in $(Y, \preceq)$.
– We have

$$\mathsf{true} \odot f(x) \stackrel{\mathrm{def}}{=} f(x),$$
$$\mathsf{false} \odot f(x) \stackrel{\mathrm{def}}{=} \varnothing_Y,$$

where $\varnothing_Y$ is the minimal element of $(Y, \preceq)$.

Proof. *Item 1*, *Universal Property*: This is a rephrasing of Item 2.
Item 2, *Adjointness*: We claim we have adjunction $\mathcal{P} \dashv 忘$, witnessed by a bijection

$$\mathsf{Pos}^{\mathsf{cocomp.}}((\mathcal{P}(X), \subset), (Y, \preceq)) \cong \mathsf{Sets}(X, Y),$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets})$ and $(Y, \preceq) \in \mathrm{Obj}(\mathsf{Pos}^{\mathsf{cocomp.}})$.

- *Map I*. We define a map

$$\Phi_{X,Y} \colon \mathsf{Pos}^{\mathsf{cocomp.}}((\mathcal{P}(X), \subset), (Y, \preceq)) \to \mathsf{Sets}(X, Y)$$

  as in the statement, by

$$\Phi_{X,Y}(f) \stackrel{\mathrm{def}}{=} f \circ \chi_X$$

  for each $f \in \mathsf{Pos}^{\mathsf{cocomp.}}((\mathcal{P}(X), \subset), (Y, \preceq))$.

- *Map II*. We define a map

$$\Psi_{X,Y} \colon \mathsf{Sets}(X, Y) \to \mathsf{Pos}^{\mathsf{cocomp.}}((\mathcal{P}(X), \subset), (Y, \preceq))$$

  as in the statement, by

$$\Psi_{X,Y}(f) \stackrel{\mathrm{def}}{=} \mathrm{Lan}_{\chi_X}(f), \qquad \begin{tikzcd} & \mathcal{P}(X) \arrow[d, dashed, "\mathrm{Lan}_{\chi_X}(f)"] \\ X \arrow[ur, "\chi_X"] \arrow[r, "f"'] \arrow[ur, phantom, shift right=3, "\Uparrow"] & Y, \end{tikzcd}$$

  for each $f \in \mathsf{Sets}(X, Y)$.

- *Invertibility I.* We claim that

$$\Psi_{X,Y} \circ \Phi_{X,Y} = \mathrm{id}_{\mathsf{Pos}^{\text{cocomp.}}((\mathcal{P}(X),\subset),(Y,\preceq))}.$$

Indeed, given a cocontinuous morphism of posets

$$\xi\colon (\mathcal{P}(X),\subset) \to (Y,\preceq),$$

we have

$$\begin{aligned}
[\Psi_{X,Y} \circ \Phi_{X,Y}](\xi) &\overset{\text{def}}{=} \Psi_{X,Y}(\Phi_{X,Y}(\xi)) \\
&\overset{\text{def}}{=} \Psi_{X,Y}(\xi \circ \chi_X) \\
&\overset{\text{def}}{=} \mathrm{Lan}_{\chi_X}(\xi \circ \chi_X) \\
&\cong \bigvee_{x\in X} \chi_{(-)}(x) \odot \xi(\chi_X(x)) \\
&\overset{\text{clm}}{=} \xi,
\end{aligned}$$

where indeed

$$\begin{aligned}
\left[\bigvee_{x\in X} \chi_{(-)}(x) \odot \xi(\chi_X(x))\right](U) &\overset{\text{def}}{=} \bigvee_{x\in X} \chi_U(x) \odot \xi(\chi_X(x)) \\
&= (\bigvee_{x\in U} \chi_U(x) \odot \xi(\chi_X(x))) \vee (\bigvee_{x\in X\setminus U} \chi_U(x) \odot \xi(\chi_X(x))) \\
&= (\bigvee_{x\in U} \xi(\chi_X(x))) \vee (\bigvee_{x\in X\setminus U} \varnothing_Y) \\
&= \bigvee_{x\in U} \xi(\chi_X(x)) \\
&\overset{(\dagger)}{=} \xi(\bigvee_{x\in U} \chi_X(x)) \\
&= \xi(U)
\end{aligned}$$

for each $U \in \mathcal{P}(X)$, where we have used that ξ is cocontinuous for the equality $\overset{(\dagger)}{=}$.

- *Invertibility II.* We claim that

$$\Phi_{X,Y} \circ \Psi_{X,Y} = \mathrm{id}_{\mathsf{Sets}(X,Y)}.$$

Indeed, given a function $f\colon X \to Y$, we have

$$\begin{aligned}
[\Phi_{X,Y} \circ \Psi_{X,Y}](f) &\overset{\text{def}}{=} \Phi_{X,Y}(\Psi_{X,Y}(f))\\
&\overset{\text{def}}{=} \Phi_{X,Y}(\mathrm{Lan}_{\chi_X}(f))\\
&\overset{\text{def}}{=} \mathrm{Lan}_{\chi_X}(f) \circ \chi_X\\
&\overset{\text{clm}}{=} f,
\end{aligned}$$

where indeed

$$\begin{aligned}
[\mathrm{Lan}_{\chi_X}(f) \circ \chi_X](x) &\overset{\text{def}}{=} \bigvee_{y \in X} \chi_{\{x\}}(y) \odot f(y)\\
&= (\chi_{\{x\}}(x) \odot f(x)) \vee \Big(\bigvee_{y \in X \setminus \{x\}} \chi_{\{x\}}(y) \odot f(y)\Big)\\
&= f(x) \vee \Big(\bigvee_{y \in X \setminus \{x\}} \varnothing_Y\Big)\\
&= f(x) \vee \varnothing_Y\\
&= f(x)
\end{aligned}$$

for each $x \in X$.

- *Naturality for Φ, Part I.* We need to show that, given a function $f\colon X \to X'$, the diagram

$$\begin{tikzcd}
\mathsf{Pos}^{\mathsf{cocomp.}}((\mathcal{P}(X'), \subset), (Y, \preceq)) \arrow[r, "\Phi_{X',Y}"] \arrow[d, "\mathcal{P}_*(f)^*"'] & \mathsf{Sets}(X', Y) \arrow[d, "f^*"]\\
\mathsf{Pos}^{\mathsf{cocomp.}}((\mathcal{P}(X), \subset), (Y, \preceq)) \arrow[r, "\Phi_{X,Y}"'] & \mathsf{Sets}(X, Y)
\end{tikzcd}$$

commutes. Indeed, given a cocontinuous morphism of posets

$$\xi\colon (\mathcal{P}(X'), \subset) \to (Y, \preceq),$$

we have

$$\begin{aligned}[\Phi_{X,Y}\circ\mathcal{P}_*(f)^*](\xi) &\overset{\text{def}}{=} \Phi_{X,Y}(\mathcal{P}_*(f)^*(\xi))\\ &\overset{\text{def}}{=} \Phi_{X,Y}(\xi\circ f_*)\\ &\overset{\text{def}}{=} (\xi\circ f_*)\circ\chi_X\\ &= \xi\circ(f_*\circ\chi_X)\\ &\overset{(\dagger)}{=} \xi\circ(\chi_{X'}\circ f)\\ &= (\xi\circ\chi_{X'})\circ f\\ &\overset{\text{def}}{=} \Phi_{X',Y}(\xi)\circ f\\ &\overset{\text{def}}{=} f^*(\Phi_{X',Y}(\xi))\\ &\overset{\text{def}}{=} [f^*\circ\Phi_{X',Y}](\xi),\end{aligned}$$

where we have used Item 9 of Proposition 4.1.1.3 for the equality $\overset{(\dagger)}{=}$ above.

- *Naturality for* Φ*, Part II*. We need to show that, given a cocontinuous morphism of posets

$$g\colon (Y,\preceq_Y)\to(Y',\preceq_{Y'}),$$

the diagram

$$\begin{array}{ccc}\mathsf{Pos}^{\mathsf{cocomp.}}((\mathcal{P}(X),\subset),(Y,\preceq)) & \xrightarrow{\Phi_{X,Y}} & \mathsf{Sets}(X,Y)\\ {\scriptstyle g_*}\big\downarrow & & \big\downarrow{\scriptstyle g_*}\\ \mathsf{Pos}^{\mathsf{cocomp.}}((\mathcal{P}(X),\subset),(Y',\preceq)) & \xrightarrow[\Phi_{X,Y'}]{} & \mathsf{Sets}(X,Y')\end{array}$$

commutes. Indeed, given a cocontinuous morphism of posets

$$\xi\colon (\mathcal{P}(X),\subset)\to(Y,\preceq),$$

we have

$$\begin{aligned}[\Phi_{X,Y'}\circ g_*](\xi) &\overset{\text{def}}{=} \Phi_{X,Y'}(g_*(\xi))\\ &\overset{\text{def}}{=} \Phi_{X,Y'}(g\circ\xi)\\ &\overset{\text{def}}{=} (g\circ\xi)\circ\chi_X\\ &= g\circ(\xi\circ\chi_X)\\ &\overset{\text{def}}{=} g\circ(\Phi_{X,Y}(\xi))\\ &\overset{\text{def}}{=} g_*(\Phi_{X,Y}(\xi))\\ &\overset{\text{def}}{=} [g_*\circ\Phi_{X,Y}](\xi).\end{aligned}$$

- *Naturality for* Ψ. Since Φ is natural in each argument and Φ is a componentwise inverse to Ψ in each argument, it follows from Categories, Item 2 of Proposition 8.6.1.2 that Ψ is also natural in each argument.

This finishes the proof. $\square$

4.4 Direct Images

Let A and B be sets and let $f\colon A \to B$ be a function.

Definition 4.4.1.1. The **direct image function associated to** f is the function

$$f_*\colon \mathcal{P}(A) \to \mathcal{P}(B)$$

defined by[35,36]

$$\begin{aligned} f_*(U) &\overset{\text{def}}{=} f(U) \\ &\overset{\text{def}}{=} \left\{ b \in B \,\middle|\, \begin{array}{l} \text{there exists some } a \in U \\ \text{such that } b = f(a) \end{array} \right\} \\ &= \{ f(a) \in B \mid a \in U \} \end{aligned}$$

for each $U \in \mathcal{P}(A)$.

Notation 4.4.1.2. Sometimes one finds the notation

$$\exists_f\colon \mathcal{P}(A) \to \mathcal{P}(B)$$

for f_*. This notation comes from the fact that the following statements are equivalent, where $b \in B$ and $U \in \mathcal{P}(A)$:

- We have $b \in \exists_f(U)$.
- There exists some $a \in U$ such that $f(a) = b$.

Remark 4.4.1.3. Identifying subsets of A with functions from A to $\{\text{true}, \text{false}\}$

[35] *Further Terminology:* The set $f(U)$ is called the **direct image of** U **by** f.

[36] We also have

$$f_*(U) = B \setminus f_!(A \setminus U);$$

see Item 9 of Proposition 4.4.1.4.

via Items 1 and 2 of Proposition 4.3.1.6, we see that the direct image function associated to f is equivalently the function

$$f_*\colon \mathcal{P}(A) \to \mathcal{P}(B)$$

defined by

$$\begin{aligned}
f_*(\chi_U) &\stackrel{\text{def}}{=} \mathrm{Lan}_f(\chi_U)\\
&= \operatorname{colim}((f \overrightarrow{\times} \underline{(-_1)}) \stackrel{\text{pr}}{\twoheadrightarrow} A \xrightarrow{\chi_U} \{\mathsf{t},\mathsf{f}\})\\
&= \operatorname*{colim}_{\substack{a\in A\\ f(a)=-_1}} (\chi_U(a))\\
&= \bigvee_{\substack{a\in A\\ f(a)=-_1}} (\chi_U(a)),
\end{aligned}$$

where we have used **??**, **??** for the second equality. In other words, we have

$$\begin{aligned}
[f_*(\chi_U)](b) &= \bigvee_{\substack{a\in A\\ f(a)=b}} (\chi_U(a))\\
&= \begin{cases} \text{true} & \text{if there exists some } a \in A \text{ such}\\ & \text{that } f(a) = b \text{ and } a \in U,\\ \text{false} & \text{otherwise}\end{cases}\\
&= \begin{cases} \text{true} & \text{if there exists some } a \in U\\ & \text{such that } f(a) = b,\\ \text{false} & \text{otherwise}\end{cases}
\end{aligned}$$

for each $b \in B$.

Proposition 4.4.1.4. Let $f\colon A \to B$ be a function.

1. *Functoriality.* The assignment $U \mapsto f_*(U)$ defines a functor

$$f_*\colon (\mathcal{P}(A), \subset) \to (\mathcal{P}(B), \subset)$$

where

- *Action on Objects.* For each $U \in \mathcal{P}(A)$, we have

$$[f_*](U) \stackrel{\text{def}}{=} f_*(U).$$

- *Action on Morphisms.* For each $U, V \in \mathcal{P}(A)$:

  ($\star$) If $U \subset V$, then $f_*(U) \subset f_*(V)$.

2. *Triple Adjointness.* We have a triple adjunction

$$\left(f_* \dashv f^{-1} \dashv f_!\right)\colon\quad \begin{tikzcd} \mathcal{P}(A) \arrow[r, bend left=50, "f_*"] \arrow[r, bend right=50, "f_!"'] & \mathcal{P}(B), \arrow[l, "f^{-1}" description] \end{tikzcd}$$

witnessed by bijections of sets

$$\begin{aligned} \mathrm{Hom}_{\mathcal{P}(B)}(f_*(U), V) &\cong \mathrm{Hom}_{\mathcal{P}(A)}(U, f^{-1}(V)), \\ \mathrm{Hom}_{\mathcal{P}(A)}(f^{-1}(U), V) &\cong \mathrm{Hom}_{\mathcal{P}(A)}(U, f_!(V)), \end{aligned}$$

natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$ and (respectively) $V \in \mathcal{P}(A)$ and $U \in \mathcal{P}(B)$, i.e. where:

(a) The following conditions are equivalent:

  i. We have $f_*(U) \subset V$.

  ii. We have $U \subset f^{-1}(V)$.

(b) The following conditions are equivalent:

  i. We have $f^{-1}(U) \subset V$.

  ii. We have $U \subset f_!(V)$.

3. *Preservation of Colimits.* We have an equality of sets

$$f_*\Big(\bigcup_{i \in I} U_i\Big) = \bigcup_{i \in I} f_*(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have equalities

$$\begin{aligned} f_*(U) \cup f_*(V) &= f_*(U \cup V), \\ f_*(\emptyset) &= \emptyset, \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

4. *Oplax Preservation of Limits.* We have an inclusion of sets

$$f_*(\bigcap_{i\in I} U_i) \subset \bigcap_{i\in I} f_*(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have inclusions

$$\begin{aligned} f_*(U \cap V) &\subset f_*(U) \cap f_*(V), \\ f_*(A) &\subset B, \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

5. *Symmetric Strict Monoidality With Respect to Unions.* The direct image function of Item 1 has a symmetric strict monoidal structure

$$(f_*, f_*^{\otimes}, f_{*|\mathbb{1}}^{\otimes}) \colon (\mathcal{P}(A), \cup, \emptyset) \to (\mathcal{P}(B), \cup, \emptyset),$$

being equipped with equalities

$$\begin{aligned} f_{*|U,V}^{\otimes} &\colon f_*(U) \cup f_*(V) \xrightarrow{=} f_*(U \cup V), \\ f_{*|\mathbb{1}}^{\otimes} &\colon \emptyset \xrightarrow{=} \emptyset, \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

6. *Symmetric Oplax Monoidality With Respect to Intersections.* The direct image function of Item 1 has a symmetric oplax monoidal structure

$$(f_*, f_*^{\otimes}, f_{*|\mathbb{1}}^{\otimes}) \colon (\mathcal{P}(A), \cap, A) \to (\mathcal{P}(B), \cap, B),$$

being equipped with inclusions

$$\begin{aligned} f_{*|U,V}^{\otimes} &\colon f_*(U \cap V) \hookrightarrow f_*(U) \cap f_*(V), \\ f_{*|\mathbb{1}}^{\otimes} &\colon f_*(A) \hookrightarrow B, \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

7. *Interaction With Coproducts.* Let $f \colon A \to A'$ and $g \colon B \to B'$ be maps of sets. We have

$$(f \coprod g)_*(U \coprod V) = f_*(U) \coprod g_*(V)$$

for each $U \in \mathcal{P}(A)$ and each $V \in \mathcal{P}(B)$.

8. *Interaction With Products.* Let $f: A \longrightarrow A'$ and $g: B \longrightarrow B'$ be maps of sets. We have
$$(f \times g)_*(U \times V) = f_*(U) \times g_*(V)$$
for each $U \in \mathcal{P}(A)$ and each $V \in \mathcal{P}(B)$.

9. *Relation to Direct Images With Compact Support.* We have
$$f_*(U) = B \setminus f_!(A \setminus U)$$
for each $U \in \mathcal{P}(A)$.

Proof. *Item 1, Functoriality*: Clear.
Item 2, Triple Adjointness: This follows from Remark 4.4.1.3, Remark 4.5.1.2, Remark 4.6.1.3, and **??**, **??** of **??**.
Item 3, Preservation of Colimits: This follows from Item 2 and **??**, **??** of **??**.[37]
Item 4, Oplax Preservation of Limits: The inclusion $f_*(A) \subset B$ is clear. See [Pro24p] for the other inclusions.
Item 5, Symmetric Strict Monoidality With Respect to Unions: This follows from Item 3.
Item 6, Symmetric Oplax Monoidality With Respect to Intersections: This follows from Item 4.
Item 7, Interaction With Coproducts: Clear.
Item 8, Interaction With Products: Clear.
Item 9, Relation to Direct Images With Compact Support: Applying Item 9 of Proposition 4.6.1.6 to $A \setminus U$, we have
$$\begin{aligned} f_!(A \setminus U) &= B \setminus f_*(A \setminus (A \setminus U)) \\ &= B \setminus f_*(U). \end{aligned}$$
Taking complements, we then obtain
$$\begin{aligned} f_*(U) &= B \setminus (B \setminus f_*(U)), \\ &= B \setminus f_!(A \setminus U), \end{aligned}$$
which finishes the proof. □

Proposition 4.4.1.5. Let $f: A \longrightarrow B$ be a function.

[37] See also [Pro24q].

1. *Functionality I*. The assignment $f \mapsto f_*$ defines a function
$$(-)_{*|A,B}\colon \mathsf{Sets}(A,B) \to \mathsf{Sets}(\mathcal{P}(A),\mathcal{P}(B)).$$

2. *Functionality II*. The assignment $f \mapsto f_*$ defines a function
$$(-)_{*|A,B}\colon \mathsf{Sets}(A,B) \to \mathsf{Pos}((\mathcal{P}(A),\subset),(\mathcal{P}(B),\subset)).$$

3. *Interaction With Identities*. For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have
$$(\mathrm{id}_A)_* = \mathrm{id}_{\mathcal{P}(A)}.$$

4. *Interaction With Composition*. For each pair of composable functions $f\colon A \to B$ and $g\colon B \to C$, we have
$$(g \circ f)_* = g_* \circ f_*, \qquad \begin{tikzcd} \mathcal{P}(A) \arrow[r, "f_*"] \arrow[dr, "(g\circ f)_*"'] & \mathcal{P}(B) \arrow[d, "g_*"] \\ & \mathcal{P}(C). \end{tikzcd}$$

Proof. *Item 1*, *Functionality I*: Clear.
Item 2, *Functionality II*: Clear.
Item 3, *Interaction With Identities*: This follows from Remark 4.4.1.3 and **??**, **??** of **??**.
Item 4, *Interaction With Composition*: This follows from Remark 4.4.1.3 and **??**, **??** of **??**. □

4.5 Inverse Images

Let A and B be sets and let $f\colon A \to B$ be a function.

Definition 4.5.1.1. The **inverse image function associated to** f is the function[38]
$$f^{-1}\colon \mathcal{P}(B) \to \mathcal{P}(A)$$
defined by[39]
$$f^{-1}(V) \overset{\text{def}}{=} \{a \in A \mid \text{we have } f(a) \in V\}$$
for each $V \in \mathcal{P}(B)$.

[38] *Further Notation:* Also written $f^*\colon \mathcal{P}(B) \to \mathcal{P}(A)$.

[39] *Further Terminology:* The set $f^{-1}(V)$ is called the **inverse image of** V **by** f.

Remark 4.5.1.2. Identifying subsets of B with functions from B to $\{\text{true}, \text{false}\}$ via Items 1 and 2 of Proposition 4.3.1.6, we see that the inverse image function associated to f is equivalently the function

$$f^*\colon \mathcal{P}(B) \to \mathcal{P}(A)$$

defined by

$$f^*(\chi_V) \overset{\text{def}}{=} \chi_V \circ f$$

for each $\chi_V \in \mathcal{P}(B)$, where $\chi_V \circ f$ is the composition

$$A \xrightarrow{f} B \xrightarrow{\chi_V} \{\text{true}, \text{false}\}$$

in Sets.

Proposition 4.5.1.3. Let $f\colon A \to B$ be a function.

1. *Functoriality.* The assignment $V \mapsto f^{-1}(V)$ defines a functor

   $$f^{-1}\colon (\mathcal{P}(B), \subset) \to (\mathcal{P}(A), \subset)$$

   where

   - *Action on Objects.* For each $V \in \mathcal{P}(B)$, we have

     $$[f^{-1}](V) \overset{\text{def}}{=} f^{-1}(V).$$

   - *Action on Morphisms.* For each $U, V \in \mathcal{P}(B)$:

     ($\star$) If $U \subset V$, then $f^{-1}(U) \subset f^{-1}(V)$.

2. *Triple Adjointness.* We have a triple adjunction

   $$\left(f_* \dashv f^{-1} \dashv f_!\right)\colon \quad \begin{tikzcd} \mathcal{P}(A) \arrow[r, bend left=50, "f_*"] \arrow[r, bend right=50, "f_!"'] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, phantom, shift right=3, "\perp"] & \mathcal{P}(B), \arrow[l, "f^{-1}" description] \end{tikzcd}$$

   witnessed by bijections of sets

   $$\begin{aligned} \mathrm{Hom}_{\mathcal{P}(B)}(f_*(U), V) &\cong \mathrm{Hom}_{\mathcal{P}(A)}(U, f^{-1}(V)), \\ \mathrm{Hom}_{\mathcal{P}(A)}(f^{-1}(U), V) &\cong \mathrm{Hom}_{\mathcal{P}(A)}(U, f_!(V)), \end{aligned}$$

   natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$ and (respectively) $V \in \mathcal{P}(A)$ and $U \in \mathcal{P}(B)$, i.e. where:

(a) The following conditions are equivalent:

i. We have $f_*(U) \subset V$;

ii. We have $U \subset f^{-1}(V)$;

(b) The following conditions are equivalent:

i. We have $f^{-1}(U) \subset V$.

ii. We have $U \subset f_!(V)$.

3. *Preservation of Colimits.* We have an equality of sets

$$f^{-1}(\bigcup_{i\in I} U_i) = \bigcup_{i\in I} f^{-1}(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(B)^{\times I}$. In particular, we have equalities

$$\begin{aligned} f^{-1}(U) \cup f^{-1}(V) &= f^{-1}(U \cup V), \\ f^{-1}(\emptyset) &= \emptyset, \end{aligned}$$

natural in $U, V \in \mathcal{P}(B)$.

4. *Preservation of Limits.* We have an equality of sets

$$f^{-1}(\bigcap_{i\in I} U_i) = \bigcap_{i\in I} f^{-1}(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(B)^{\times I}$. In particular, we have equalities

$$\begin{aligned} f^{-1}(U) \cap f^{-1}(V) &= f^{-1}(U \cap V), \\ f^{-1}(B) &= A, \end{aligned}$$

natural in $U, V \in \mathcal{P}(B)$.

5. *Symmetric Strict Monoidality With Respect to Unions.* The inverse image function of Item 1 has a symmetric strict monoidal structure

$$(f^{-1}, f^{-1,\otimes}, f_{\mathbb{1}}^{-1,\otimes}) \colon (\mathcal{P}(B), \cup, \emptyset) \to (\mathcal{P}(A), \cup, \emptyset),$$

being equipped with equalities

$$\begin{aligned} f_{U,V}^{-1,\otimes} &\colon f^{-1}(U) \cup f^{-1}(V) \xrightarrow{=} f^{-1}(U \cup V), \\ f_{\mathbb{1}}^{-1,\otimes} &\colon \emptyset \xrightarrow{=} f^{-1}(\emptyset), \end{aligned}$$

natural in $U, V \in \mathcal{P}(B)$.

6. *Symmetric Strict Monoidality With Respect to Intersections.* The inverse image function of Item 1 has a symmetric strict monoidal structure

$$(f^{-1}, f^{-1,\otimes}, f_{\mathbb{1}}^{-1,\otimes}) \colon (\mathcal{P}(B), \cap, B) \longrightarrow (\mathcal{P}(A), \cap, A),$$

being equipped with equalities

$$f_{U,V}^{-1,\otimes} \colon f^{-1}(U) \cap f^{-1}(V) \xrightarrow{=} f^{-1}(U \cap V),$$
$$f_{\mathbb{1}}^{-1,\otimes} \colon A \xrightarrow{=} f^{-1}(B),$$

natural in $U, V \in \mathcal{P}(B)$.

7. *Interaction With Coproducts.* Let $f \colon A \longrightarrow A'$ and $g \colon B \longrightarrow B'$ be maps of sets. We have

$$(f \coprod g)^{-1}(U' \coprod V') = f^{-1}(U') \coprod g^{-1}(V')$$

for each $U' \in \mathcal{P}(A')$ and each $V' \in \mathcal{P}(B')$.

8. *Interaction With Products.* Let $f \colon A \longrightarrow A'$ and $g \colon B \longrightarrow B'$ be maps of sets. We have

$$(f \times g)^{-1}(U' \times V') = f^{-1}(U') \times g^{-1}(V')$$

for each $U' \in \mathcal{P}(A')$ and each $V' \in \mathcal{P}(B')$.

Proof. *Item 1*, *Functoriality*: Clear.
Item 2, *Triple Adjointness*: This follows from Remark 4.4.1.3, Remark 4.5.1.2, Remark 4.6.1.3, and **??**, **??** of **??**.
Item 3, *Preservation of Colimits*: This follows from Item 2 and **??**, **??** of **??**.[40]
Item 4, *Preservation of Limits*: This follows from Item 2 and **??**, **??** of **??**.[41]
Item 5, *Symmetric Strict Monoidality With Respect to Unions*: This follows from Item 3.
Item 6, *Symmetric Strict Monoidality With Respect to Intersections*: This follows from Item 4.
Item 7, *Interaction With Coproducts*: Clear.
Item 8, *Interaction With Products*: Clear. □

Proposition 4.5.1.4. Let $f \colon A \longrightarrow B$ be a function.

[40] See also [Pro24y].
[41] See also [Pro24x].

1. *Functionality I.* The assignment $f \mapsto f^{-1}$ defines a function

$$(-)^{-1}_{A,B} \colon \mathsf{Sets}(A, B) \to \mathsf{Sets}(\mathcal{P}(B), \mathcal{P}(A)).$$

2. *Functionality II.* The assignment $f \mapsto f^{-1}$ defines a function

$$(-)^{-1}_{A,B} \colon \mathsf{Sets}(A, B) \to \mathsf{Pos}((\mathcal{P}(B), \subset), (\mathcal{P}(A), \subset)).$$

3. *Interaction With Identities.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$\mathrm{id}^{-1}_A = \mathrm{id}_{\mathcal{P}(A)}.$$

4. *Interaction With Composition.* For each pair of composable functions $f \colon A \to B$ and $g \colon B \to C$, we have

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}, \qquad \begin{tikzcd} \mathcal{P}(C) \arrow[r, "g^{-1}"] \arrow[dr, "(g\circ f)^{-1}"'] & \mathcal{P}(B) \arrow[d, "f^{-1}"] \\ & \mathcal{P}(A). \end{tikzcd}$$

Proof. *Item 1, Functionality I*: Clear.
Item 2, Functionality II: Clear.
Item 3, Interaction With Identities: This follows from Remark 4.5.1.2 and Categories, Item 5 of Proposition 1.6.1.2.
Item 4, Interaction With Composition: This follows from Remark 4.5.1.2 and Categories, Item 2 of Proposition 1.6.1.2. □

4.6 Direct Images With Compact Support

Let A and B be sets and let $f \colon A \to B$ be a function.

Definition 4.6.1.1. The **direct image with compact support function associated to** f is the function

$$f_! \colon \mathcal{P}(A) \to \mathcal{P}(B)$$

defined by[42,43]

$$
\begin{aligned}
f_!(U) &\overset{\text{def}}{=} \left\{ b \in B \;\middle|\; \begin{array}{l} \text{for each } a \in A, \text{ if we have} \\ f(a) = b, \text{ then } a \in U \end{array} \right\} \\
&= \left\{ b \in B \;\middle|\; \text{we have } f^{-1}(b) \subset U \right\}
\end{aligned}
$$

for each $U \in \mathcal{P}(A)$.

Notation 4.6.1.2. Sometimes one finds the notation

$$\forall_f \colon \mathcal{P}(A) \to \mathcal{P}(B)$$

for f_*. This notation comes from the fact that the following statements are equivalent, where $b \in B$ and $U \in \mathcal{P}(A)$:

- We have $b \in \forall_f(U)$.
- For each $a \in A$, if $b = f(a)$, then $a \in U$.

Remark 4.6.1.3. Identifying subsets of A with functions from A to $\{\text{true}, \text{false}\}$ via Items 1 and 2 of Proposition 4.3.1.6, we see that the direct image with compact support function associated to f is equivalently the function

$$f_! \colon \mathcal{P}(A) \to \mathcal{P}(B)$$

defined by

$$
\begin{aligned}
f_!(\chi_U) &\overset{\text{def}}{=} \mathrm{Ran}_f(\chi_U) \\
&= \lim\left((\underline{(-_1)} \overset{\to}{\times} f) \overset{\text{pr}}{\twoheadrightarrow} A \xrightarrow{\chi_U} \{\text{true}, \text{false}\} \right) \\
&= \lim_{\substack{a \in A \\ f(a) = -_1}} (\chi_U(a)) \\
&= \bigwedge_{\substack{a \in A \\ f(a) = -_1}} (\chi_U(a)).
\end{aligned}
$$

[42] *Further Terminology*: The set $f_!(U)$ is called the **direct image with compact support of U by f**.

[43] We also have

$$f_!(U) = B \setminus f_*(A \setminus U);$$

see Item 9 of Proposition 4.6.1.6.

where we have used **??**, **??** for the second equality. In other words, we have

$$
\begin{aligned}
[f_!(\chi_U)](b) &= \bigwedge_{\substack{a\in A\\ f(a)=b}} (\chi_U(a)) \\
&= \begin{cases} \mathrm{true} & \text{if, for each } a \in A \text{ such that} \\ & f(a) = b\text{, we have } a \in U, \\ \mathrm{false} & \text{otherwise} \end{cases} \\
&= \begin{cases} \mathrm{true} & \text{if } f^{-1}(b) \subset U \\ \mathrm{false} & \text{otherwise} \end{cases}
\end{aligned}
$$

for each $b \in B$.

Definition 4.6.1.4. Let U be a subset of A.[44,45]

1. The **image part of the direct image with compact support** $f_!(U)$ **of** U is the set $f_{!,\mathrm{im}}(U)$ defined by

$$
\begin{aligned}
f_{!,\mathrm{im}}(U) &\overset{\mathrm{def}}{=} f_!(U) \cap \mathrm{Im}(f) \\
&= \left\{ b \in B \;\middle|\; \begin{array}{l} \text{we have } f^{-1}(b) \subset U \\ \text{and } f^{-1}(b) \neq \emptyset \end{array} \right\}.
\end{aligned}
$$

---

[44] Note that we have

$$f_!(U) = f_{!,\mathrm{im}}(U) \cup f_{!,\mathrm{cp}}(U),$$

as

$$
\begin{aligned}
f_!(U) &= f_!(U) \cap B \\
&= f_!(U) \cap (\mathrm{Im}(f) \cup (B \setminus \mathrm{Im}(f))) \\
&= (f_!(U) \cap \mathrm{Im}(f)) \cup (f_!(U) \cap (B \setminus \mathrm{Im}(f))) \\
&\overset{\mathrm{def}}{=} f_{!,\mathrm{im}}(U) \cup f_{!,\mathrm{cp}}(U).
\end{aligned}
$$

[45] In terms of the meet computation of $f_!(U)$ of Remark 4.6.1.3, namely

$$f_!(\chi_U) = \bigwedge_{\substack{a\in A\\ f(a)=-_1}} (\chi_U(a)),$$

we see that $f_{!,\mathrm{im}}$ corresponds to meets indexed over nonempty sets, while $f_{!,\mathrm{cp}}$ corresponds to meets indexed over the empty set.

2. The **complement part of the direct image with compact support $f_!(U)$ of** U is the set $f_{!,\mathrm{cp}}(U)$ defined by

$$\begin{aligned} f_{!,\mathrm{cp}}(U) &\overset{\text{def}}{=} f_!(U) \cap (B \setminus \mathrm{Im}(f)) \\ &= B \setminus \mathrm{Im}(f) \\ &= \left\{ b \in B \;\middle|\; \begin{array}{l} \text{we have } f^{-1}(b) \subset U \\ \text{and } f^{-1}(b) = \emptyset \end{array} \right\} \\ &= \left\{ b \in B \;\middle|\; f^{-1}(b) = \emptyset \right\}. \end{aligned}$$

Example 4.6.1.5. Here are some examples of direct images with compact support.

1. *The Multiplication by Two Map on the Natural Numbers.* Consider the function $f\colon \mathbb{N} \to \mathbb{N}$ given by

$$f(n) \overset{\text{def}}{=} 2n$$

for each $n \in \mathbb{N}$. Since f is injective, we have

$$\begin{aligned} f_{!,\mathrm{im}}(U) &= f_*(U) \\ f_{!,\mathrm{cp}}(U) &= \{\text{odd natural numbers}\} \end{aligned}$$

for any $U \subset \mathbb{N}$.

2. *Parabolas.* Consider the function $f\colon \mathbb{R} \to \mathbb{R}$ given by

$$f(x) \overset{\text{def}}{=} x^2$$

for each $x \in \mathbb{R}$. We have

$$f_{!,\mathrm{cp}}(U) = \mathbb{R}_{<0}$$

for any $U \subset \mathbb{R}$. Moreover, since $f^{-1}(x) = \left\{-\sqrt{x}, \sqrt{x}\right\}$, we have e.g.:

$$\begin{aligned} f_{!,\mathrm{im}}([0,1]) &= \{0\}, \\ f_{!,\mathrm{im}}([-1,1]) &= [0,1], \\ f_{!,\mathrm{im}}([1,2]) &= \emptyset, \\ f_{!,\mathrm{im}}([-2,-1] \cup [1,2]) &= [1,4]. \end{aligned}$$

3. *Circles.* Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x,y) \overset{\text{def}}{=} x^2 + y^2$$

for each $(x,y) \in \mathbb{R}^2$. We have

$$f_{!,\mathrm{cp}}(U) = \mathbb{R}_{<0}$$

for any $U \subset \mathbb{R}^2$, and since

$$f^{-1}(r) = \begin{cases} \text{a circle of radius } r \text{ about the origin} & \text{if } r > 0, \\ \{(0,0)\} & \text{if } r = 0, \\ \emptyset & \text{if } r < 0, \end{cases}$$

we have e.g.:

$$\begin{aligned} f_{!,\mathrm{im}}([-1,1] \times [-1,1]) &= [0,1], \\ f_{!,\mathrm{im}}(([-1,1] \times [-1,1]) \setminus [-1,1] \times \{0\}) &= \emptyset. \end{aligned}$$

Proposition 4.6.1.6. Let $f\colon A \to B$ be a function.

1. *Functoriality.* The assignment $U \mapsto f_!(U)$ defines a functor

$$f_!\colon (\mathcal{P}(A), \subset) \to (\mathcal{P}(B), \subset)$$

where

- *Action on Objects.* For each $U \in \mathcal{P}(A)$, we have

$$[f_!](U) \overset{\text{def}}{=} f_!(U).$$

- *Action on Morphisms.* For each $U, V \in \mathcal{P}(A)$:

  ($\star$) If $U \subset V$, then $f_!(U) \subset f_!(V)$.

2. *Triple Adjointness.* We have a triple adjunction

$$\left(f_* \dashv f^{-1} \dashv f_!\right)\colon \quad \begin{tikzcd} \mathcal{P}(A) \arrow[r, bend left=45, "f_*"] \arrow[r, bend right=45, "f_!"'] & \mathcal{P}(B), \arrow[l, "f^{-1}" description] \end{tikzcd}$$

witnessed by bijections of sets

$$\mathrm{Hom}_{\mathcal{P}(B)}(f_*(U),V) \cong \mathrm{Hom}_{\mathcal{P}(A)}(U,f^{-1}(V)),$$
$$\mathrm{Hom}_{\mathcal{P}(A)}(f^{-1}(U),V) \cong \mathrm{Hom}_{\mathcal{P}(A)}(U,f_!(V)),$$

natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$ and (respectively) $V \in \mathcal{P}(A)$ and $U \in \mathcal{P}(B)$, i.e. where:

(a) The following conditions are equivalent:

  i. We have $f_*(U) \subset V$.
  ii. We have $U \subset f^{-1}(V)$.

(b) The following conditions are equivalent:

  i. We have $f^{-1}(U) \subset V$.
  ii. We have $U \subset f_!(V)$.

3. *Lax Preservation of Colimits.* We have an inclusion of sets

$$\bigcup_{i\in I} f_!(U_i) \subset f_!(\bigcup_{i\in I} U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have inclusions

$$f_!(U) \cup f_!(V) \hookrightarrow f_!(U \cup V),$$
$$\emptyset \hookrightarrow f_!(\emptyset),$$

natural in $U, V \in \mathcal{P}(A)$.

4. *Preservation of Limits.* We have an equality of sets

$$f_!(\bigcap_{i\in I} U_i) = \bigcap_{i\in I} f_!(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have equalities

$$f^{-1}(U \cap V) = f_!(U) \cap f^{-1}(V),$$
$$f_!(A) = B,$$

natural in $U, V \in \mathcal{P}(A)$.

5. *Symmetric Lax Monoidality With Respect to Unions.* The direct image with compact support function of Item 1 has a symmetric lax monoidal structure

$$(f_!, f_!^{\otimes}, f_{!|\mathbb{1}}^{\otimes}) \colon (\mathcal{P}(A), \cup, \emptyset) \longrightarrow (\mathcal{P}(B), \cup, \emptyset),$$

being equipped with inclusions

$$f_{!|U,V}^{\otimes} \colon f_!(U) \cup f_!(V) \hookrightarrow f_!(U \cup V),$$
$$f_{!|\mathbb{1}}^{\otimes} \colon \emptyset \hookrightarrow f_!(\emptyset),$$

natural in $U, V \in \mathcal{P}(A)$.

6. *Symmetric Strict Monoidality With Respect to Intersections.* The direct image function of Item 1 has a symmetric strict monoidal structure

$$(f_!, f_!^{\otimes}, f_{!|\mathbb{1}}^{\otimes}) \colon (\mathcal{P}(A), \cap, A) \longrightarrow (\mathcal{P}(B), \cap, B),$$

being equipped with equalities

$$f_{!|U,V}^{\otimes} \colon f_!(U \cap V) \xrightarrow{=} f_!(U) \cap f_!(V),$$
$$f_{!|\mathbb{1}}^{\otimes} \colon f_!(A) \xrightarrow{=} B,$$

natural in $U, V \in \mathcal{P}(A)$.

7. *Interaction With Coproducts.* Let $f \colon A \longrightarrow A'$ and $g \colon B \longrightarrow B'$ be maps of sets. We have

$$(f \coprod g)_!(U \coprod V) = f_!(U) \coprod g_!(V)$$

for each $U \in \mathcal{P}(A)$ and each $V \in \mathcal{P}(B)$.

8. *Interaction With Products.* Let $f \colon A \longrightarrow A'$ and $g \colon B \longrightarrow B'$ be maps of sets. We have

$$(f \times g)_!(U \times V) = f_!(U) \times g_!(V)$$

for each $U \in \mathcal{P}(A)$ and each $V \in \mathcal{P}(B)$.

9. *Relation to Direct Images.* We have

$$f_!(U) = B \setminus f_*(A \setminus U)$$

for each $U \in \mathcal{P}(A)$.

10. *Interaction With Injections.* If f is injective, then we have

$$\begin{aligned} f_{!,\mathrm{im}}(U) &= f_*(U), \\ f_{!,\mathrm{cp}}(U) &= B \setminus \mathrm{Im}(f), \\ f_!(U) &= f_{!,\mathrm{im}}(U) \cup f_{!,\mathrm{cp}}(U) \\ &= f_*(U) \cup (B \setminus \mathrm{Im}(f)) \end{aligned}$$

    for each $U \in \mathcal{P}(A)$.

11. *Interaction With Surjections.* If f is surjective, then we have

$$\begin{aligned} f_{!,\mathrm{im}}(U) &\subset f_*(U), \\ f_{!,\mathrm{cp}}(U) &= \emptyset, \\ f_!(U) &\subset f_*(U) \end{aligned}$$

    for each $U \in \mathcal{P}(A)$.

Proof. *Item 1*, *Functoriality*: Clear.
Item 2, *Triple Adjointness*: This follows from Remark 4.4.1.3, Remark 4.5.1.2, Remark 4.6.1.3, and **??**, **??** of **??**.
Item 3, *Lax Preservation of Colimits*: Omitted.
Item 4, *Preservation of Limits*: This follows from Item 2 and **??**, **??** of **??**.
Item 5, *Symmetric Lax Monoidality With Respect to Unions*: This follows from Item 3.
Item 6, *Symmetric Strict Monoidality With Respect to Intersections*: This follows from Item 4.
Item 7, *Interaction With Coproducts*: Clear.
Item 8, *Interaction With Products*: Clear.
Item 9, *Relation to Direct Images*: We claim that $f_!(U) = B \setminus f_*(A \setminus U)$.

- *The First Implication.* We claim that

$$f_!(U) \subset B \setminus f_*(A \setminus U).$$

  Let $b \in f_!(U)$. We need to show that $b \notin f_*(A \setminus U)$, i.e. that there is no $a \in A \setminus U$ such that $f(a) = b$.

  This is indeed the case, as otherwise we would have $a \in f^{-1}(b)$ and $a \notin U$, contradicting $f^{-1}(b) \subset U$ (which holds since $b \in f_!(U)$).

  Thus $b \in B \setminus f_*(A \setminus U)$.

- *The Second Implication.* We claim that

$$B \setminus f_*(A \setminus U) \subset f_!(U).$$

Let $b \in B \setminus f_*(A \setminus U)$. We need to show that $b \in f_!(U)$, i.e. that $f^{-1}(b) \subset U$.

Since $b \notin f_*(A \setminus U)$, there exists no $a \in A \setminus U$ such that $b = f(a)$, and hence $f^{-1}(b) \subset U$.

Thus $b \in f_!(U)$.

This finishes the proof of Item 9.
Item 10, Interaction With Injections: Clear.
Item 11, Interaction With Surjections: Clear. □

Proposition 4.6.1.7. Let $f\colon A \longrightarrow B$ be a function.

1. *Functionality I.* The assignment $f \mapsto f_!$ defines a function

$$(-)_{!|A,B}\colon \mathsf{Sets}(A, B) \longrightarrow \mathsf{Sets}(\mathcal{P}(A), \mathcal{P}(B)).$$

2. *Functionality II.* The assignment $f \mapsto f_!$ defines a function

$$(-)_{!|A,B}\colon \mathsf{Sets}(A, B) \longrightarrow \mathsf{Pos}((\mathcal{P}(A), \subset), (\mathcal{P}(B), \subset)).$$

3. *Interaction With Identities.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$(\mathrm{id}_A)_! = \mathrm{id}_{\mathcal{P}(A)}.$$

4. *Interaction With Composition.* For each pair of composable functions $f\colon A \longrightarrow B$ and $g\colon B \longrightarrow C$, we have

$$(g \circ f)_! = g_! \circ f_!, \qquad \begin{tikzcd} \mathcal{P}(A) \arrow[r, "f_!"] \arrow[dr, "(g\circ f)_!"'] & \mathcal{P}(B) \arrow[d, "g_!"] \\ & \mathcal{P}(C). \end{tikzcd}$$

Proof. *Item 1, Functionality I*: Clear.
Item 2, Functionality II: Clear.
Item 3, Interaction With Identities: This follows from Remark 4.6.1.3 and **??**, **??** of **??**.
Item 4, Interaction With Composition: This follows from Remark 4.6.1.3 and **??**, **??** of **??**. □

Appendices

A Other Chapters

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