Types of Morphisms in Bicategories

The Clowder Project Authors

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In this chapter, we study special kinds of morphisms in bicategories:

1. *Monomorphisms and Epimorphisms in Bicategories (Sections 1 and 2).* There is a large number of different notions capturing the idea of a "monomorphism" or of an "epimorphism" in a bicategory.

   Arguably, the notion that best captures these concepts is that of a *pseudomonic morphism* (Definition 1.10.1) and of a *pseudoepic morphism* (Definition 2.10.1), although the other notions introduced in Sections 1 and 2 are also interesting on their own.

Contents

1 Monomorphisms in Bicategories

1.1 Representably Faithful Morphisms

Let $\mathcal{C}$ be a bicategory.

DEFINITION 1.1.1 ► REPRESENTABLY FAITHFUL MORPHISMS

A 1-morphism $f: A \to B$ of $\mathcal{C}$ is **representably faithful**[1] if, for each $X \in \mathrm{Obj}(\mathcal{C})$, the functor

$$f_*: \mathrm{Hom}_{\mathcal{C}}(X, A) \to \mathrm{Hom}_{\mathcal{C}}(X, B)$$

given by postcomposition by f is faithful.

[1] *Further Terminology:* Also called simply a **faithful morphism**, based on Item 1 of Example 1.1.3.

REMARK 1.1.2 ► UNWINDING DEFINITION 1.1.1

In detail, f is representably faithful if, for all diagrams in $\mathcal{C}$ of the form

$$\begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & A \arrow[r, "f"] & B, \end{tikzcd}$$

if we have

$$\mathrm{id}_f \star \alpha = \mathrm{id}_f \star \beta,$$

then $\alpha = \beta$.

Example 1.1.3 ► Examples of Representably Faithful Morphisms

Here are some examples of representably faithful morphisms.

1. *Representably Faithful Morphisms in* Cats_2. The representably faithful morphisms in Cats_2 are precisely the faithful functors; see Categories, Item 1 of Proposition 5.1.2.

2. *Representably Faithful Morphisms in* **Rel**. Every morphism of **Rel** is representably faithful; see Relations, Item 1 of Proposition 3.8.1.

1.2 Representably Full Morphisms

Let C be a bicategory.

Definition 1.2.1 ► Representably Full Morphisms

A 1-morphism $f: A \to B$ of C is **representably full**[1] if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f_*: \mathrm{Hom}_C(X, A) \to \mathrm{Hom}_C(X, B)$$

given by postcomposition by f is full.

[1] *Further Terminology:* Also called simply a **full morphism**, based on Item 1 of Example 1.2.3.

Remark 1.2.2 ► Unwinding Definition 1.2.1

In detail, f is representably full if, for each $X \in \mathrm{Obj}(C)$ and each 2-morphism

$$\beta: f \circ \phi \Longrightarrow f \circ \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"] \arrow[r, bend right, "f\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & B \end{tikzcd}$$

of C, there exists a 2-morphism

$$\alpha: \phi \Longrightarrow \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & A \end{tikzcd}$$

of C such that we have an equality

$$\begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & A \arrow[r, "f"] & B \end{tikzcd} = \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"] \arrow[r, bend right, "f\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & B \end{tikzcd}$$

of pasting diagrams in C, i.e. such that we have

$$\beta = \mathrm{id}_f \star \alpha.$$

Example 1.2.3 ▶ Examples of Representably Full Morphisms

Here are some examples of representably full morphisms.

1. *Representably Full Morphisms in* Cats_2. The representably full morphisms in Cats_2 are precisely the full functors; see Categories, Item 1 of Proposition 5.2.2.

2. *Representably Full Morphisms in* **Rel**. The representably full morphisms in **Rel** are characterised in Relations, Item 2 of Proposition 3.8.1.

1.3 Representably Fully Faithful Morphisms

Let C be a bicategory.

Definition 1.3.1 ▶ Representably Fully Faithful Morphisms

A 1-morphism $f\colon A \to B$ of C is **representably fully faithful**[1] if the following equivalent conditions are satisfied:

1. The 1-morphism f is representably faithful (Definition 1.1.1) and representably full (Definition 1.2.1).

2. For each $X \in \mathrm{Obj}(C)$, the functor

$$f_*\colon \mathrm{Hom}_C(X, A) \to \mathrm{Hom}_C(X, B)$$

given by postcomposition by f is fully faithful.

[1] *Further Terminology:* Also called simply a **fully faithful morphism**, based on Item 1 of Example 1.3.3.

Remark 1.3.2 ► Unwinding Representably Fully Faithful Morphisms

In detail, f is representably fully faithful if the conditions in Remark 1.1.2 and Remark 1.2.2 hold:

1. For all diagrams in $\mathcal{C}$ of the form

$$\begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & A \arrow[r, "f"] & B, \end{tikzcd}$$

   if we have

$$\mathrm{id}_f \star \alpha = \mathrm{id}_f \star \beta,$$

   then $\alpha = \beta$.

2. For each $X \in \mathrm{Obj}(\mathcal{C})$ and each 2-morphism

$$\beta\colon f \circ \phi \Longrightarrow f \circ \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"] \arrow[r, bend right, "f\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & B \end{tikzcd}$$

   of $\mathcal{C}$, there exists a 2-morphism

$$\alpha\colon \phi \Longrightarrow \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & A \end{tikzcd}$$

   of $\mathcal{C}$ such that we have an equality

$$\begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & A \arrow[r, "f"] & B \end{tikzcd} = \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"] \arrow[r, bend right, "f\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & B \end{tikzcd}$$

   of pasting diagrams in $\mathcal{C}$, i.e. such that we have

$$\beta = \mathrm{id}_f \star \alpha.$$

Example 1.3.3 ▶ Examples of Representably Fully Faithful Morphisms

Here are some examples of representably fully faithful morphisms.

1. *Representably Fully Faithful Morphisms in* Cats_2. The representably fully faithful morphisms in Cats_2 are precisely the fully faithful functors; see Categories, Item 5 of Proposition 5.3.2.

2. *Representably Fully Faithful Morphisms in* **Rel**. The representably fully faithful morphisms of **Rel** coincide (Relations, Item 3 of Proposition 3.8.1) with the representably full morphisms in **Rel**, which are characterised in Relations, Item 2 of Proposition 3.8.1.

1.4 Morphisms Representably Faithful on Cores

Let C be a bicategory.

Definition 1.4.1 ▶ Morphisms Representably Faithful on Cores

A 1-morphism $f\colon A \to B$ of C is **representably faithful on cores** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f_*\colon \mathsf{Core}(\mathsf{Hom}_C(X,A)) \to \mathsf{Core}(\mathsf{Hom}_C(X,B))$$

given by postcomposition by f is faithful.

Remark 1.4.2 ▶ Unwinding Definition 1.4.1

In detail, f is representably faithful on cores if, for all diagrams in C of the form

$$\begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha \Downarrow\,\Downarrow \beta"] & A \arrow[r, "f"] & B, \end{tikzcd}$$

if α and β are 2-isomorphisms and we have

$$\mathrm{id}_f \star \alpha = \mathrm{id}_f \star \beta,$$

then $\alpha = \beta$.

1.5 Morphisms Representably Full on Cores

Let C be a bicategory.

Definition 1.5.1 ► Morphisms Representably Full on Cores

A 1-morphism $f: A \longrightarrow B$ of C is **representably full on cores** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f_*: \mathrm{Core}(\mathrm{Hom}_C(X, A)) \to \mathrm{Core}(\mathrm{Hom}_C(X, B))$$

given by postcomposition by f is full.

Remark 1.5.2 ► Unwinding Definition 1.5.1

In detail, f is representably full on cores if, for each $X \in \mathrm{Obj}(C)$ and each 2-isomorphism

$$\beta: f \circ \phi \overset{\sim}{\Longrightarrow} f \circ \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"] \arrow[r, bend right, "f\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & B \end{tikzcd}$$

of C, there exists a 2-isomorphism

$$\alpha: \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & A \end{tikzcd}$$

of C such that we have an equality

$$\begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & A \arrow[r, "f"] & B \end{tikzcd} = \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"] \arrow[r, bend right, "f\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & B \end{tikzcd}$$

of pasting diagrams in C, i.e. such that we have

$$\beta = \mathrm{id}_f \star \alpha.$$

1.6 Morphisms Representably Fully Faithful on Cores

Let C be a bicategory.

Definition 1.6.1 ▶ Morphisms Representably Fully Faithful on Cores

A 1-morphism $f\colon A \longrightarrow B$ of $\mathcal{C}$ is **representably fully faithful on cores** if the following equivalent conditions are satisfied:

1. The 1-morphism f is representably faithful on cores (Definition 1.5.1) and representably full on cores (Definition 1.4.1).
2. For each $X \in \mathrm{Obj}(\mathcal{C})$, the functor
$$f_*\colon \mathrm{Core}(\mathrm{Hom}_{\mathcal{C}}(X, A)) \to \mathrm{Core}(\mathrm{Hom}_{\mathcal{C}}(X, B))$$
given by postcomposition by f is fully faithful.

Remark 1.6.2 ▶ Unwinding Definition 1.6.1

In detail, f is representably fully faithful on cores if the conditions in Remark 1.4.2 and Remark 1.5.2 hold:

1. For all diagrams in $\mathcal{C}$ of the form
$$\begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & A \arrow[r, "f"] & B, \end{tikzcd}$$
if α and β are 2-isomorphisms and we have
$$\mathrm{id}_f \star \alpha = \mathrm{id}_f \star \beta,$$
then $\alpha = \beta$.
2. For each $X \in \mathrm{Obj}(\mathcal{C})$ and each 2-isomorphism
$$\beta\colon f \circ \phi \overset{\sim}{\Longrightarrow} f \circ \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"] \arrow[r, bend right, "f\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & B \end{tikzcd}$$
of $\mathcal{C}$, there exists a 2-isomorphism
$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & A \end{tikzcd}$$

of C such that we have an equality

$$\begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & A \arrow[r, "f"] & B \end{tikzcd} = \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"] \arrow[r, bend right, "f\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & B \end{tikzcd}$$

of pasting diagrams in C, i.e. such that we have

$$\beta = \mathrm{id}_f \star \alpha.$$

1.7 Representably Essentially Injective Morphisms

Let C be a bicategory.

Definition 1.7.1 ▶ Representably Essentially Injective Morphisms

A 1-morphism $f\colon A \to B$ of C is **representably essentially injective** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f_*\colon \mathrm{Hom}_C(X, A) \to \mathrm{Hom}_C(X, B)$$

given by postcomposition by f is essentially injective.

Remark 1.7.2 ▶ Unwinding Definition 1.7.1

In detail, f is representably essentially injective if, for each pair of morphisms $\phi, \psi\colon X \rightrightarrows A$ of C, the following condition is satisfied:

($\star$) If $f \circ \phi \cong f \circ \psi$, then $\phi \cong \psi$.

1.8 Representably Conservative Morphisms

Let C be a bicategory.

Definition 1.8.1 ▶ Representably Conservative Morphisms

A 1-morphism $f\colon A \to B$ of C is **representably conservative** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f_*\colon \mathrm{Hom}_C(X,A) \to \mathrm{Hom}_C(X,B)$$

given by postcomposition by f is conservative.

Remark 1.8.2 ▶ Unwinding Definition 1.8.1

In detail, f is representably conservative if, for each pair of morphisms $\phi, \psi\colon X \rightrightarrows A$ and each 2-morphism

$$\alpha\colon \phi \Longrightarrow \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & A \end{tikzcd}$$

of C, if the 2-morphism

$$\mathrm{id}_f \star \alpha\colon f\circ\phi \Longrightarrow f\circ\psi, \qquad \begin{tikzcd} X \arrow[rr, bend left, "f\circ\phi"] \arrow[rr, bend right, "f\circ\psi"'] \arrow[rr, phantom, "\Downarrow\,\mathrm{id}_f\star\alpha"] & & B \end{tikzcd}$$

is a 2-isomorphism, then so is α.

1.9 Strict Monomorphisms

Let C be a bicategory.

Definition 1.9.1 ▶ Strict Monomorphisms

A 1-morphism $f\colon A \to B$ of C is a **strict monomorphism** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f_*\colon \mathrm{Hom}_C(X,A) \to \mathrm{Hom}_C(X,B)$$

given by postcomposition by f is injective on objects, i.e. its action on objects

$$f_*\colon \mathrm{Obj}(\mathrm{Hom}_C(X,A)) \to \mathrm{Obj}(\mathrm{Hom}_C(X,B))$$

is injective.

Remark 1.9.2 ▶ Unwinding Definition 1.9.1

In detail, f is a strict monomorphism in C if, for each diagram in C of the form

$$X \underset{\psi}{\overset{\phi}{\rightrightarrows}} A \xrightarrow{f} B,$$

if $f \circ \phi = f \circ \psi$, then $\phi = \psi$.

Example 1.9.3 ▶ Examples of Strict Monomorphisms

Here are some examples of strict monomorphisms.

1. *Strict Monomorphisms in* Cats_2. The strict monomorphisms in Cats_2 are precisely the functors which are injective on objects and injective on morphisms; see Categories, Item 1 of Proposition 6.2.2.

2. *Strict Monomorphisms in* **Rel**. The strict monomorphisms in **Rel** are characterised in Relations, Proposition 3.7.1.

1.10 Pseudomonic Morphisms

Let C be a bicategory.

Definition 1.10.1 ▶ Pseudomonic Morphisms

A 1-morphism $f: A \to B$ of C is **pseudomonic** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f_*: \mathrm{Hom}_C(X, A) \to \mathrm{Hom}_C(X, B)$$

given by postcomposition by f is pseudomonic.

Remark 1.10.2 ▶ Unwinding Definition 1.10.1

In detail, a 1-morphism $f: A \to B$ of C is pseudomonic if it satisfies the following conditions:

1. For all diagrams in C of the form

$$\begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & A \arrow[r, "f"] & B, \end{tikzcd}$$

   if we have

$$\mathrm{id}_f \star \alpha = \mathrm{id}_f \star \beta,$$

   then $\alpha = \beta$.

2. For each $X \in \mathrm{Obj}(C)$ and each 2-isomorphism

$$\beta \colon f \circ \phi \overset{\sim}{\Longrightarrow} f \circ \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"] \arrow[r, bend right, "f\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & B \end{tikzcd}$$

   of C, there exists a 2-isomorphism

$$\alpha \colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & A \end{tikzcd}$$

   of C such that we have an equality

$$\begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & A \arrow[r, "f"] & B \end{tikzcd} = \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"] \arrow[r, bend right, "f\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & B \end{tikzcd}$$

   of pasting diagrams in C, i.e. such that we have

$$\beta = \mathrm{id}_f \star \alpha.$$

Proposition 1.10.3 ► Properties of Pseudomonic Morphisms

Let $f \colon A \to B$ be a 1-morphism of C.

1. *Characterisations.* The following conditions are equivalent:

(a) The morphism f is pseudomonic.

(b) The morphism f is representably full on cores and representably faithful.

(c) We have an isocomma square of the form

$$A \overset{\text{eq.}}{\cong} A \overset{\leftrightarrow}{\times}_B A, \qquad \begin{tikzcd} A \arrow[r, "\mathrm{id}_A"] \arrow[d, "\mathrm{id}_A"'] & A \arrow[d, "F"] \\ A \arrow[r, "F"'] & B \arrow[ul, phantom, "\Leftrightarrow"] \end{tikzcd}$$

in C up to equivalence.

2. *Interaction With Cotensors.* If C has cotensors with $\mathbb{1}$, then the following conditions are equivalent:

(a) The morphism f is pseudomonic.

(b) We have an isocomma square of the form

$$A \overset{\text{eq.}}{\cong} A \overset{\leftrightarrow}{\times}_{\mathbb{1} \pitchfork F} B, \qquad \begin{tikzcd} A \arrow[r, hook] \arrow[d, "F"'] & \mathbb{1} \pitchfork A \arrow[d, "\mathbb{1} \pitchfork F"] \\ B \arrow[r, hook] & \mathbb{1} \pitchfork B \arrow[ul, phantom, "\Leftrightarrow"] \end{tikzcd}$$

in C up to equivalence.

Proof 1.10.4 ► Proof of Proposition 1.10.3

Item 1: Characterisations

Omitted.

Item 2: Interaction With Cotensors

Omitted.

2 Epimorphisms in Bicategories

2.1 Corepresentably Faithful Morphisms

Let $\mathcal{C}$ be a bicategory.

Definition 2.1.1 ► Corepresentably Faithful Morphisms

A 1-morphism $f\colon A \to B$ of $\mathcal{C}$ is **corepresentably faithful** if, for each $X \in \mathrm{Obj}(\mathcal{C})$, the functor

$$f^*\colon \mathrm{Hom}_{\mathcal{C}}(B,X) \to \mathrm{Hom}_{\mathcal{C}}(A,X)$$

given by precomposition by f is faithful.

Remark 2.1.2 ► Unwinding Definition 2.1.1

In detail, f is corepresentably faithful if, for all diagrams in $\mathcal{C}$ of the form

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha \Downarrow\Downarrow \beta"] & X, \end{tikzcd}$$

if we have

$$\alpha \star \mathrm{id}_f = \beta \star \mathrm{id}_f,$$

then $\alpha = \beta$.

Example 2.1.3 ► Examples of Corepresentably Faithful Morphisms

Here are some examples of corepresentably faithful morphisms.

1. *Corepresentably Faithful Morphisms in* Cats_2. The corepresentably faithful morphisms in Cats_2 are characterised in Categories, Item 4 of Proposition 5.1.2.

2. *Corepresentably Faithful Morphisms in* **Rel**. Every morphism of **Rel** is corepresentably faithful; see Relations, Item 1 of Proposition 3.10.1.

2.2 Corepresentably Full Morphisms

Let $\mathcal{C}$ be a bicategory.

Definition 2.2.1 ▶ Corepresentably Full Morphisms

A 1-morphism $f\colon A \to B$ of $\mathcal{C}$ is **corepresentably full** if, for each $X \in \mathrm{Obj}(\mathcal{C})$, the functor

$$f^*\colon \mathrm{Hom}_{\mathcal{C}}(B,X) \to \mathrm{Hom}_{\mathcal{C}}(A,X)$$

given by precomposition by f is full.

Remark 2.2.2 ▶ Unwinding Definition 2.2.1

In detail, f is corepresentably full if, for each $X \in \mathrm{Obj}(\mathcal{C})$ and each 2-morphism

$$\beta\colon \phi \circ f \Longrightarrow \psi \circ f, \qquad \begin{tikzcd} A \arrow[r, bend left, "\phi\circ f"{name=U}] \arrow[r, bend right, "\psi\circ f"'{name=D}] & X \arrow[from=U, to=D, Rightarrow, "\beta"'] \end{tikzcd}$$

of $\mathcal{C}$, there exists a 2-morphism

$$\alpha\colon \phi \Longrightarrow \psi, \qquad \begin{tikzcd} B \arrow[r, bend left, "\phi"{name=U}] \arrow[r, bend right, "\psi"'{name=D}] & X \arrow[from=U, to=D, Rightarrow, "\alpha"'] \end{tikzcd}$$

of $\mathcal{C}$ such that we have an equality

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "\phi"{name=U}] \arrow[r, bend right, "\psi"'{name=D}] & X \arrow[from=U, to=D, Rightarrow, "\alpha"'] \end{tikzcd} = \begin{tikzcd} A \arrow[r, bend left, "\phi\circ f"{name=U}] \arrow[r, bend right, "\psi\circ f"'{name=D}] & X \arrow[from=U, to=D, Rightarrow, "\beta"'] \end{tikzcd}$$

of pasting diagrams in $\mathcal{C}$, i.e. such that we have

$$\beta = \alpha \star \mathrm{id}_f.$$

Example 2.2.3 ▶ Examples of Corepresentably Full Morphisms

Here are some examples of corepresentably full morphisms.

1. *Corepresentably Full Morphisms in* Cats_2. The corepresentably full morphisms in Cats_2 are characterised in Categories, Item 5 of Proposition 5.2.2.

2. *Corepresentably Full Morphisms in* **Rel**. The corepresentably full morphisms

in **Rel** are characterised in Relations, Item 2 of Proposition 3.10.1.

2.3 Corepresentably Fully Faithful Morphisms

Let C be a bicategory.

Definition 2.3.1 ▶ Corepresentably Fully Faithful Morphisms

A 1-morphism $f\colon A \to B$ of C is **corepresentably fully faithful**[1] if the following equivalent conditions are satisfied:

1. The 1-morphism f is corepresentably full (Definition 2.2.1) and corepresentably faithful (Definition 2.1.1).

2. For each $X \in \mathrm{Obj}(C)$, the functor

$$f^*\colon \mathrm{Hom}_C(B,X) \to \mathrm{Hom}_C(A,X)$$

given by precomposition by f is fully faithful.

[1] *Further Terminology:* Corepresentably fully faithful morphisms have also been called **lax epimorphisms** in the literature (e.g. in [Adá+01]), though we will always use the name "corepresentably fully faithful morphism" instead in this work.

Remark 2.3.2 ▶ Unwinding Definition 2.3.1

In detail, f is corepresentably fully faithful if the conditions in Remark 2.1.2 and Remark 2.2.2 hold:

1. For all diagrams in C of the form

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & X, \end{tikzcd}$$

if we have

$$\alpha \star \mathrm{id}_f = \beta \star \mathrm{id}_f,$$

then $\alpha = \beta$.

2. For each $X \in \mathrm{Obj}(\mathcal{C})$ and each 2-morphism

$$\beta\colon \phi \circ f \Longrightarrow \psi \circ f, \qquad \begin{tikzcd} A \arrow[r, bend left, "\phi\circ f"{name=U}] \arrow[r, bend right, "\psi\circ f"'{name=D}] & X \arrow[from=U, to=D, phantom, "\beta\Downarrow"] \end{tikzcd}$$

of $\mathcal{C}$, there exists a 2-morphism

$$\alpha\colon \phi \Longrightarrow \psi, \qquad \begin{tikzcd} B \arrow[r, bend left, "\phi"{name=U}] \arrow[r, bend right, "\psi"'{name=D}] & X \arrow[from=U, to=D, phantom, "\alpha\Downarrow"] \end{tikzcd}$$

of $\mathcal{C}$ such that we have an equality

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "\phi"{name=U}] \arrow[r, bend right, "\psi"'{name=D}] & X \arrow[from=U, to=D, phantom, "\alpha\Downarrow"] \end{tikzcd} = \begin{tikzcd} A \arrow[r, bend left, "\phi\circ f"{name=U}] \arrow[r, bend right, "\psi\circ f"'{name=D}] & X \arrow[from=U, to=D, phantom, "\beta\Downarrow"] \end{tikzcd}$$

of pasting diagrams in $\mathcal{C}$, i.e. such that we have

$$\beta = \alpha \star \mathrm{id}_f.$$

EXAMPLE 2.3.3 ► EXAMPLES OF COREPRESENTABLY FULLY FAITHFUL MORPHISMS

Here are some examples of corepresentably fully faithful morphisms.

1. *Corepresentably Fully Faithful Morphisms in* Cats_2. The fully faithful epimorphisms in Cats_2 are characterised in Categories, Item 9 of Proposition 5.3.2.

2. *Corepresentably Fully Faithful Morphisms in* **Rel**. The corepresentably fully faithful morphisms of **Rel** coincide (Relations, Item 3 of Proposition 3.10.1) with the corepresentably full morphisms in **Rel**, which are characterised in Relations, Item 2 of Proposition 3.10.1.

2.4 Morphisms Corepresentably Faithful on Cores

Let $\mathcal{C}$ be a bicategory.

Definition 2.4.1 ▶ Morphisms Corepresentably Faithful on Cores

A 1-morphism $f \colon A \to B$ of C is **corepresentably faithful on cores** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f^* \colon \mathrm{Core}(\mathrm{Hom}_C(B, X)) \to \mathrm{Core}(\mathrm{Hom}_C(A, X))$$

given by precomposition by f is faithful.

Remark 2.4.2 ▶ Unwinding Definition 2.4.1

In detail, f is corepresentably faithful on cores if, for all diagrams in C of the form

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha \Downarrow \Downarrow \beta"] & X, \end{tikzcd}$$

if α and β are 2-isomorphisms and we have

$$\alpha \star \mathrm{id}_f = \beta \star \mathrm{id}_f,$$

then $\alpha = \beta$.

2.5 Morphisms Corepresentably Full on Cores

Let C be a bicategory.

Definition 2.5.1 ▶ Morphisms Corepresentably Full on Cores

A 1-morphism $f \colon A \to B$ of C is **corepresentably full on cores** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f^* \colon \mathrm{Core}(\mathrm{Hom}_C(B, X)) \to \mathrm{Core}(\mathrm{Hom}_C(A, X))$$

given by precomposition by f is full.

Remark 2.5.2 ▶ Unwinding Definition 2.5.1

In detail, f is corepresentably full on cores if, for each $X \in \mathrm{Obj}(C)$ and each 2-isomorphism

$$\beta\colon \phi \circ f \overset{\sim}{\Longrightarrow} \psi \circ f, \qquad \begin{tikzcd} A \arrow[r, bend left, "\phi\circ f"{name=U}] \arrow[r, bend right, "\psi\circ f"'{name=D}] \arrow[from=U, to=D, Rightarrow, "\beta"'] & X \end{tikzcd}$$

of C, there exists a 2-isomorphism

$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} B \arrow[r, bend left, "\phi"{name=U}] \arrow[r, bend right, "\psi"'{name=D}] \arrow[from=U, to=D, Rightarrow, "\alpha"'] & X \end{tikzcd}$$

of C such that we have an equality

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "\phi"{name=U}] \arrow[r, bend right, "\psi"'{name=D}] \arrow[from=U, to=D, Rightarrow, "\alpha"'] & X \end{tikzcd} = \begin{tikzcd} A \arrow[r, bend left, "\phi\circ f"{name=U}] \arrow[r, bend right, "\psi\circ f"'{name=D}] \arrow[from=U, to=D, Rightarrow, "\beta"'] & X \end{tikzcd}$$

of pasting diagrams in C, i.e. such that we have

$$\beta = \alpha \star \mathrm{id}_f.$$

2.6 Morphisms Corepresentably Fully Faithful on Cores

Let C be a bicategory.

Definition 2.6.1 ▶ Morphisms Corepresentably Fully Faithful on Cores

A 1-morphism $f\colon A \to B$ of C is **corepresentably fully faithful on cores** if the following equivalent conditions are satisfied:

1. The 1-morphism f is corepresentably full on cores (Definition 2.5.1) and corepresentably faithful on cores (Definition 2.1.1).
2. For each $X \in \mathrm{Obj}(C)$, the functor

$$f^*\colon \mathrm{Core}(\mathrm{Hom}_C(B, X)) \to \mathrm{Core}(\mathrm{Hom}_C(A, X))$$

given by precomposition by f is fully faithful.

Remark 2.6.2 ▶ Unwinding Definition 2.6.1

In detail, f is corepresentably fully faithful on cores if the conditions in Remark 2.4.2 and Remark 2.5.2 hold:

1. For all diagrams in $\mathcal{C}$ of the form
$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & X, \end{tikzcd}$$
if α and β are 2-isomorphisms and we have
$$\alpha \star \mathrm{id}_f = \beta \star \mathrm{id}_f,$$
then $\alpha = \beta$.

2. For each $X \in \mathrm{Obj}(\mathcal{C})$ and each 2-isomorphism
$$\beta\colon \phi \circ f \overset{\sim}{\Longrightarrow} \psi \circ f, \qquad \begin{tikzcd} A \arrow[r, bend left, "\phi\circ f"] \arrow[r, bend right, "\psi\circ f"'] \arrow[r, phantom, "\beta\Downarrow"] & X \end{tikzcd}$$
of $\mathcal{C}$, there exists a 2-isomorphism
$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & X \end{tikzcd}$$
of $\mathcal{C}$ such that we have an equality
$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & X \end{tikzcd} = \begin{tikzcd} A \arrow[r, bend left, "\phi\circ f"] \arrow[r, bend right, "\psi\circ f"'] \arrow[r, phantom, "\beta\Downarrow"] & X \end{tikzcd}$$
of pasting diagrams in $\mathcal{C}$, i.e. such that we have
$$\beta = \alpha \star \mathrm{id}_f.$$

2.7 Corepresentably Essentially Injective Morphisms

Let C be a bicategory.

Definition 2.7.1 ► Corepresentably Essentially Injective Morphisms

A 1-morphism $f\colon A \to B$ of C is **corepresentably essentially injective** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f^*\colon \mathrm{Hom}_C(B, X) \to \mathrm{Hom}_C(A, X)$$

given by precomposition by f is essentially injective.

Remark 2.7.2 ► Unwinding Definition 2.7.1

In detail, f is corepresentably essentially injective if, for each pair of morphisms $\phi, \psi\colon B \rightrightarrows X$ of C, the following condition is satisfied:

(★) If $\phi \circ f \cong \psi \circ f$, then $\phi \cong \psi$.

2.8 Corepresentably Conservative Morphisms

Let C be a bicategory.

Definition 2.8.1 ► Corepresentably Conservative Morphisms

A 1-morphism $f\colon A \to B$ of C is **corepresentably conservative** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f^*\colon \mathrm{Hom}_C(B, X) \to \mathrm{Hom}_C(A, X)$$

given by precomposition by f is conservative.

Remark 2.8.2 ▶ Unwinding Definition 2.8.1

In detail, f is corepresentably conservative if, for each pair of morphisms $\phi, \psi\colon B \rightrightarrows X$ and each 2-morphism

$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & X \end{tikzcd}$$

of C, if the 2-morphism

$$\alpha \star \mathrm{id}_f\colon \phi \circ f \Longrightarrow \psi \circ f, \qquad \begin{tikzcd} A \arrow[r, bend left, "\phi\circ f"] \arrow[r, bend right, "\psi\circ f"'] \arrow[r, phantom, "\overset{\|}{\underset{\Downarrow}{\alpha\star\mathrm{id}_f}}"] & X \end{tikzcd}$$

is a 2-isomorphism, then so is α.

2.9 Strict Epimorphisms

Let C be a bicategory.

Definition 2.9.1 ▶ Strict Epimorphisms

A 1-morphism $f\colon A \to B$ is a **strict epimorphism in** C if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f^*\colon \mathrm{Hom}_C(B, X) \to \mathrm{Hom}_C(A, X)$$

given by precomposition by f is injective on objects, i.e. its action on objects

$$f_*\colon \mathrm{Obj}(\mathrm{Hom}_C(B, X)) \to \mathrm{Obj}(\mathrm{Hom}_C(A, X))$$

is injective.

Remark 2.9.2 ▶ Unwinding Definition 2.9.1

In detail, f is a strict epimorphism if, for each diagram in $\mathcal{C}$ of the form

$$A \xrightarrow{f} B \underset{\psi}{\overset{\phi}{\rightrightarrows}} X,$$

if $\phi \circ f = \psi \circ f$, then $\phi = \psi$.

Example 2.9.3 ▶ Examples of Strict Epimorphisms

Here are some examples of strict epimorphisms.

1. *Strict Epimorphisms in* Cats_2. The strict epimorphisms in Cats_2 are characterised in Categories, Item 1 of Proposition 6.3.2.

2. *Strict Epimorphisms in* **Rel**. The strict epimorphisms in **Rel** are characterised in Relations, Proposition 3.9.1.

2.10 Pseudoepic Morphisms

Let $\mathcal{C}$ be a bicategory.

Definition 2.10.1 ▶ Pseudoepic Morphisms

A 1-morphism $f : A \to B$ of $\mathcal{C}$ is **pseudoepic** if, for each $X \in \mathrm{Obj}(\mathcal{C})$, the functor

$$f^* : \mathrm{Hom}_{\mathcal{C}}(B, X) \to \mathrm{Hom}_{\mathcal{C}}(A, X)$$

given by precomposition by f is pseudomonic.

Remark 2.10.2 ▶ Unwinding Definition 2.10.1

In detail, a 1-morphism $f : A \to B$ of $\mathcal{C}$ is pseudoepic if it satisfies the following conditions:

1. For all diagrams in C of the form

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & X, \end{tikzcd}$$

if we have

$$\alpha \star \mathrm{id}_f = \beta \star \mathrm{id}_f,$$

then $\alpha = \beta$.

2. For each $X \in \mathrm{Obj}(C)$ and each 2-isomorphism

$$\beta \colon \phi \circ f \overset{\sim}{\Longrightarrow} \psi \circ f, \qquad \begin{tikzcd} A \arrow[r, bend left, "\phi\circ f"] \arrow[r, bend right, "\psi\circ f"'] \arrow[r, phantom, "\beta\Downarrow"] & X \end{tikzcd}$$

of C, there exists a 2-isomorphism

$$\alpha \colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & X \end{tikzcd}$$

of C such that we have an equality

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & X \end{tikzcd} = \begin{tikzcd} A \arrow[r, bend left, "\phi\circ f"] \arrow[r, bend right, "\psi\circ f"'] \arrow[r, phantom, "\beta\Downarrow"] & X \end{tikzcd}$$

of pasting diagrams in C, i.e. such that we have

$$\beta = \alpha \star \mathrm{id}_f.$$

Proposition 2.10.3 ▶ Properties of Pseudoepic Morphisms

Let $f \colon A \to B$ be a 1-morphism of C.

1. *Characterisations.* The following conditions are equivalent:

(a) The morphism f is pseudoepic.

(b) The morphism f is corepresentably full on cores and corepresentably faithful.

(c) We have an isococomma square of the form

$$B \overset{\text{eq.}}{\cong} B \overset{\leftrightarrow}{\coprod}_A B, \qquad \begin{tikzcd} B & B \arrow[l, "\mathrm{id}_B"'] \\ B \arrow[u, "\mathrm{id}_B"] \arrow[ur, Leftrightarrow, dashed] & A \arrow[l, "F"] \arrow[u, "F"'] \end{tikzcd}$$

in $\mathcal{C}$ up to equivalence.

Proof 2.10.4 ► Proof of Proposition 2.10.3

Item 1: Characterisations

Omitted.

Appendices

A Other Chapters

Sets

Relations

Category Theory

Bicategories

References

[Adá+01] Jiří Adámek, Robert El Bashir, Manuela Sobral, and Jiří Velebil. “On Functors Which Are Lax Epimorphisms”. In: *Theory Appl. Categ.* 8 (2001), pp. 509–521. ISSN: 1201-561X (cit. on p. 17).