

Types of Morphisms in Bicategories

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019H In this chapter, we study special kinds of morphisms in bicategories:

1. *Monomorphisms and Epimorphisms in Bicategories* ([Sections 1 and 2](#)). There is a large number of different notions capturing the idea of a “monomorphism” or of an “epimorphism” in a bicategory.

Arguably, the notion that best captures these concepts is that of a *pseudomononic morphism* ([Definition 1.10.1.1](#)) and of a *pseudoepic morphism* ([Definition 2.10.1.1](#)), although the other notions introduced in [Sections 1 and 2](#) are also interesting on their own.

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019J 1 Monomorphisms in Bicategories

019K 1.1 Representably Faithful Morphisms

Let C be a bicategory.

019L Definition 1.1.1.1. A 1-morphism $f: A \rightarrow B$ of C is **representably faithful**¹ if, for each $X \in \text{Obj}(C)$, the functor

$$f_*: \text{Hom}_C(X, A) \rightarrow \text{Hom}_C(X, B)$$

given by postcomposition by f is faithful.

019M Remark 1.1.1.2. In detail, f is representably faithful if, for all diagrams in C of the form

$$X \begin{array}{c} \xrightarrow{\phi} \\ \alpha \downarrow \downarrow \downarrow \beta \\ \xrightarrow{\psi} \end{array} A \xrightarrow{f} B,$$

if we have

$$\text{id}_f \star \alpha = \text{id}_f \star \beta,$$

then $\alpha = \beta$.

019N Example 1.1.1.3. Here are some examples of representably faithful morphisms.

019P 1. *Representably Faithful Morphisms in Cats_2 .* The representably faithful mor-

¹*Further Terminology:* Also called simply a **faithful morphism**, based on Item 1 of Example 1.1.1.3.

phisms in \mathbf{Cats}_2 are precisely the faithful functors; see [Categories, Item 1](#) of [Proposition 5.1.1.2](#).

- 019Q 2. *Representably Faithful Morphisms in \mathbf{Rel} .* Every morphism of \mathbf{Rel} is representably faithful; see [Relations, Item 1](#) of [Proposition 3.8.1.1](#).

019R 1.2 Representably Full Morphisms

Let C be a bicategory.

- 019S **Definition 1.2.1.1.** A 1-morphism $f: A \rightarrow B$ of C is **representably full**² if, for each $X \in \text{Obj}(C)$, the functor

$$f_*: \text{Hom}_C(X, A) \rightarrow \text{Hom}_C(X, B)$$

given by postcomposition by f is full.

- 019T **Remark 1.2.1.2.** In detail, f is representably full if, for each $X \in \text{Obj}(C)$ and each 2-morphism

$$\beta: f \circ \phi \Rightarrow f \circ \psi, \quad X \begin{array}{c} \xrightarrow{f \circ \phi} \\ \Downarrow \beta \\ \xrightarrow{f \circ \psi} \end{array} B$$

of C , there exists a 2-morphism

$$\alpha: \phi \Rightarrow \psi, \quad X \begin{array}{c} \xrightarrow{\phi} \\ \Downarrow \alpha \\ \xrightarrow{\psi} \end{array} A$$

of C such that we have an equality

$$X \begin{array}{c} \xrightarrow{\phi} \\ \Downarrow \alpha \\ \xrightarrow{\psi} \end{array} A \xrightarrow{f} B = X \begin{array}{c} \xrightarrow{f \circ \phi} \\ \Downarrow \beta \\ \xrightarrow{f \circ \psi} \end{array} B$$

of pasting diagrams in C , i.e. such that we have

$$\beta = \text{id}_f \star \alpha.$$

- 019U **Example 1.2.1.3.** Here are some examples of representably full morphisms.

²Further Terminology: Also called simply a **full morphism**, based on [Item 1](#) of [Example 1.2.1.3](#).

- 019V 1. *Representably Full Morphisms in \mathbf{Cats}_2* . The representably full morphisms in \mathbf{Cats}_2 are precisely the full functors; see [Categories, Item 1 of Proposition 5.2.1.2](#).
- 019W 2. *Representably Full Morphisms in \mathbf{Rel}* . The representably full morphisms in \mathbf{Rel} are characterised in [Relations, Item 2 of Proposition 3.8.1.1](#).

019X 1.3 Representably Fully Faithful Morphisms

Let C be a bicategory.

019Y **Definition 1.3.1.1.** A 1-morphism $f : A \rightarrow B$ of C is **representably fully faithful**³ if the following equivalent conditions are satisfied:

- 019Z 1. The 1-morphism f is representably faithful ([Definition 1.1.1.1](#)) and representably full ([Definition 1.2.1.1](#)).
- 01A0 2. For each $X \in \text{Obj}(C)$, the functor

$$f_* : \text{Hom}_C(X, A) \rightarrow \text{Hom}_C(X, B)$$

given by postcomposition by f is fully faithful.

01A1 **Remark 1.3.1.2.** In detail, f is representably fully faithful if the conditions in [Remark 1.1.1.2](#) and [Remark 1.2.1.2](#) hold:

1. For all diagrams in C of the form

$$X \begin{array}{c} \xrightarrow{\phi} \\ \Downarrow \alpha \quad \Downarrow \beta \\ \xrightarrow{\psi} \end{array} A \xrightarrow{f} B,$$

if we have

$$\text{id}_f \star \alpha = \text{id}_f \star \beta,$$

then $\alpha = \beta$.

³*Further Terminology:* Also called simply a **fully faithful morphism**, based on [Item 1 of Example 1.3.1.3](#).

2. For each $X \in \text{Obj}(C)$ and each 2-morphism

$$\beta: f \circ \phi \Rightarrow f \circ \psi, \quad X \begin{array}{c} \xrightarrow{f \circ \phi} \\ \beta \Downarrow \\ \xrightarrow{f \circ \psi} \end{array} B$$

of C , there exists a 2-morphism

$$\alpha: \phi \Rightarrow \psi, \quad X \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} A$$

of C such that we have an equality

$$X \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} A \xrightarrow{f} B = X \begin{array}{c} \xrightarrow{f \circ \phi} \\ \beta \Downarrow \\ \xrightarrow{f \circ \psi} \end{array} B$$

of pasting diagrams in C , i.e. such that we have

$$\beta = \text{id}_f \star \alpha.$$

01A2 Example 1.3.1.3. Here are some examples of representably fully faithful morphisms.

- 01A3** 1. *Representably Fully Faithful Morphisms in \mathbf{Cats}_2 .* The representably fully faithful morphisms in \mathbf{Cats}_2 are precisely the fully faithful functors; see [Categories, Item 5 of Proposition 5.3.1.2](#).
- 01A4** 2. *Representably Fully Faithful Morphisms in \mathbf{Rel} .* The representably fully faithful morphisms of \mathbf{Rel} coincide ([Relations, Item 3 of Proposition 3.8.1.1](#)) with the representably full morphisms in \mathbf{Rel} , which are characterised in [Relations, Item 2 of Proposition 3.8.1.1](#).

01A5 1.4 Morphisms Representably Faithful on Cores

Let C be a bicategory.

01A6 Definition 1.4.1.1. A 1-morphism $f: A \rightarrow B$ of C is **representably faithful on cores** if, for each $X \in \text{Obj}(C)$, the functor

$$f_*: \text{Core}(\text{Hom}_C(X, A)) \rightarrow \text{Core}(\text{Hom}_C(X, B))$$

given by postcomposition by f is faithful.

01A7 Remark 1.4.1.2. In detail, f is representably faithful on cores if, for all diagrams in C of the form

$$X \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \beta \\ \xrightarrow{\psi} \end{array} A \xrightarrow{f} B,$$

if α and β are 2-isomorphisms and we have

$$\mathrm{id}_f \star \alpha = \mathrm{id}_f \star \beta,$$

then $\alpha = \beta$.

01A8 1.5 Morphisms Representably Full on Cores

Let C be a bicategory.

01A9 Definition 1.5.1.1. A 1-morphism $f: A \rightarrow B$ of C is **representably full on cores** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f_*: \mathrm{Core}(\mathrm{Hom}_C(X, A)) \rightarrow \mathrm{Core}(\mathrm{Hom}_C(X, B))$$

given by postcomposition by f is full.

01AA Remark 1.5.1.2. In detail, f is representably full on cores if, for each $X \in \mathrm{Obj}(C)$ and each 2-isomorphism

$$\beta: f \circ \phi \xrightarrow{\sim} f \circ \psi, \quad X \begin{array}{c} \xrightarrow{f \circ \phi} \\ \beta \Downarrow \\ \xrightarrow{f \circ \psi} \end{array} B$$

of C , there exists a 2-isomorphism

$$\alpha: \phi \xrightarrow{\sim} \psi, \quad X \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} A$$

of C such that we have an equality

$$X \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} A \xrightarrow{f} B = X \begin{array}{c} \xrightarrow{f \circ \phi} \\ \beta \Downarrow \\ \xrightarrow{f \circ \psi} \end{array} B$$

of pasting diagrams in C , i.e. such that we have

$$\beta = \mathrm{id}_f \star \alpha.$$

01AB 1.6 Morphisms Representably Fully Faithful on Cores

Let C be a bicategory.

01AC Definition 1.6.1.1. A 1-morphism $f: A \rightarrow B$ of C is **representably fully faithful on cores** if the following equivalent conditions are satisfied:

- 01AD** 1. The 1-morphism f is representably faithful on cores (**Definition 1.5.1.1**) and representably full on cores (**Definition 1.4.1.1**).
- 01AE** 2. For each $X \in \text{Obj}(C)$, the functor

$$f_*: \text{Core}(\text{Hom}_C(X, A)) \rightarrow \text{Core}(\text{Hom}_C(X, B))$$

given by postcomposition by f is fully faithful.

01AF Remark 1.6.1.2. In detail, f is representably fully faithful on cores if the conditions in **Remark 1.4.1.2** and **Remark 1.5.1.2** hold:

1. For all diagrams in C of the form

$$X \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \beta \\ \xrightarrow{\psi} \end{array} A \xrightarrow{f} B,$$

if α and β are 2-isomorphisms and we have

$$\text{id}_f \star \alpha = \text{id}_f \star \beta,$$

then $\alpha = \beta$.

2. For each $X \in \text{Obj}(C)$ and each 2-isomorphism

$$\beta: f \circ \phi \xrightarrow{\sim} f \circ \psi, \quad X \begin{array}{c} \xrightarrow{f \circ \phi} \\ \beta \Downarrow \\ \xrightarrow{f \circ \psi} \end{array} B$$

of C , there exists a 2-isomorphism

$$\alpha: \phi \xrightarrow{\sim} \psi, \quad X \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} A$$

of C such that we have an equality

$$X \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} A \xrightarrow{f} B = X \begin{array}{c} \xrightarrow{f \circ \phi} \\ \beta \Downarrow \\ \xrightarrow{f \circ \psi} \end{array} B$$

of pasting diagrams in C , i.e. such that we have

$$\beta = \text{id}_f \star \alpha.$$

01AG 1.7 Representably Essentially Injective Morphisms

Let C be a bicategory.

01AH Definition 1.7.1.1. A 1-morphism $f: A \rightarrow B$ of C is **representably essentially injective** if, for each $X \in \text{Obj}(C)$, the functor

$$f_*: \text{Hom}_C(X, A) \rightarrow \text{Hom}_C(X, B)$$

given by postcomposition by f is essentially injective.

01AJ Remark 1.7.1.2. In detail, f is representably essentially injective if, for each pair of morphisms $\phi, \psi: X \rightrightarrows A$ of C , the following condition is satisfied:

(\star) If $f \circ \phi \cong f \circ \psi$, then $\phi \cong \psi$.

01AK 1.8 Representably Conservative Morphisms

Let C be a bicategory.

01AL Definition 1.8.1.1. A 1-morphism $f: A \rightarrow B$ of C is **representably conservative** if, for each $X \in \text{Obj}(C)$, the functor

$$f_*: \text{Hom}_C(X, A) \rightarrow \text{Hom}_C(X, B)$$

given by postcomposition by f is conservative.

01AM Remark 1.8.1.2. In detail, f is representably conservative if, for each pair of morphisms $\phi, \psi: X \rightrightarrows A$ and each 2-morphism

$$\alpha: \phi \Rightarrow \psi, \quad X \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} A$$

of C , if the 2-morphism

$$\text{id}_f \star \alpha : f \circ \phi \Rightarrow f \circ \psi, \quad \begin{array}{ccc} & f \circ \phi & \\ \curvearrowright & \parallel & \curvearrowright \\ X & \text{id}_f \star \alpha & B \\ \curvearrowleft & \Downarrow & \curvearrowleft \\ & f \circ \psi & \end{array}$$

is a 2-isomorphism, then so is α .

01AN 1.9 Strict Monomorphisms

Let C be a bicategory.

01AP Definition 1.9.1.1. A 1-morphism $f : A \rightarrow B$ of C is a **strict monomorphism** if, for each $X \in \text{Obj}(C)$, the functor

$$f_* : \text{Hom}_C(X, A) \rightarrow \text{Hom}_C(X, B)$$

given by postcomposition by f is injective on objects, i.e. its action on objects

$$f_* : \text{Obj}(\text{Hom}_C(X, A)) \rightarrow \text{Obj}(\text{Hom}_C(X, B))$$

is injective.

01AQ Remark 1.9.1.2. In detail, f is a strict monomorphism in C if, for each diagram in C of the form

$$X \begin{array}{c} \xrightarrow{\phi} \\ \Downarrow \psi \end{array} A \xrightarrow{f} B,$$

if $f \circ \phi = f \circ \psi$, then $\phi = \psi$.

01AR Example 1.9.1.3. Here are some examples of strict monomorphisms.

01AS 1. *Strict Monomorphisms in \mathbf{Cats}_2 .* The strict monomorphisms in \mathbf{Cats}_2 are precisely the functors which are injective on objects and injective on morphisms; see [Categories](#), [Item 1](#) of [Proposition 6.2.1.2](#).

01AT 2. *Strict Monomorphisms in \mathbf{Rel} .* The strict monomorphisms in \mathbf{Rel} are characterised in [Relations](#), [Proposition 3.7.1.1](#).

01AU **1.10 Pseudomonic Morphisms**

Let C be a bicategory.

01AV **Definition 1.10.1.1.** A 1-morphism $f: A \rightarrow B$ of C is **pseudomonic** if, for each $X \in \text{Obj}(C)$, the functor

$$f_*: \text{Hom}_C(X, A) \rightarrow \text{Hom}_C(X, B)$$

given by postcomposition by f is pseudomonic.

01AW **Remark 1.10.1.2.** In detail, a 1-morphism $f: A \rightarrow B$ of C is pseudomonic if it satisfies the following conditions:

01AX 1. For all diagrams in C of the form

$$X \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \beta \\ \xrightarrow{\psi} \end{array} A \xrightarrow{f} B,$$

if we have

$$\text{id}_f \star \alpha = \text{id}_f \star \beta,$$

then $\alpha = \beta$.

01AY 2. For each $X \in \text{Obj}(C)$ and each 2-isomorphism

$$\beta: f \circ \phi \xRightarrow{\sim} f \circ \psi, \quad X \begin{array}{c} \xrightarrow{f \circ \phi} \\ \beta \Downarrow \\ \xrightarrow{f \circ \psi} \end{array} B$$

of C , there exists a 2-isomorphism

$$\alpha: \phi \xRightarrow{\sim} \psi, \quad X \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} A$$

of C such that we have an equality

$$X \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} A \xrightarrow{f} B = X \begin{array}{c} \xrightarrow{f \circ \phi} \\ \beta \Downarrow \\ \xrightarrow{f \circ \psi} \end{array} B$$

of pasting diagrams in C , i.e. such that we have

$$\beta = \text{id}_f \star \alpha.$$

01AZ Proposition 1.10.1.3. Let $f: A \rightarrow B$ be a 1-morphism of C .

01B0 1. *Characterisations.* The following conditions are equivalent:

- 01B1** (a) The morphism f is pseudomonadic.
01B2 (b) The morphism f is representably full on cores and representably faithful.
01B3 (c) We have an isocomma square of the form

$$A \stackrel{\text{eq.}}{\cong} A \times_B A, \quad \begin{array}{ccc} A & \xrightarrow{\text{id}_A} & A \\ \text{id}_A \downarrow & \nearrow \text{dashed} & \downarrow F \\ A & \xrightarrow{F} & B \end{array}$$

in C up to equivalence.

01B4 2. *Interaction With Cotensors.* If C has cotensors with $\mathbb{1}$, then the following conditions are equivalent:

- (a) The morphism f is pseudomonadic.
(b) We have an isocomma square of the form

$$A \stackrel{\text{eq.}}{\cong} A \times_{\mathbb{1} \pitchfork F} B, \quad \begin{array}{ccc} A & \hookrightarrow \mathbb{1} \pitchfork A & \\ F \downarrow & \nearrow \text{dashed} & \downarrow \mathbb{1} \pitchfork F \\ B & \hookrightarrow \mathbb{1} \pitchfork B & \end{array}$$

in C up to equivalence.

Proof. **Item 1**, *Characterisations*: Omitted.

Item 2, *Interaction With Cotensors*: Omitted. □

01B5 2 Epimorphisms in Bicategories

01B6 2.1 Corepresentably Faithful Morphisms

Let C be a bicategory.

01B7 Definition 2.1.1.1. A 1-morphism $f: A \rightarrow B$ of C is **corepresentably faithful** if, for each $X \in \text{Obj}(C)$, the functor

$$f^*: \text{Hom}_C(B, X) \rightarrow \text{Hom}_C(A, X)$$

given by precomposition by f is faithful.

01B8 Remark 2.1.1.2. In detail, f is corepresentably faithful if, for all diagrams in C of the form

$$A \xrightarrow{f} B \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \beta \\ \xrightarrow{\psi} \end{array} X,$$

if we have

$$\alpha \star \text{id}_f = \beta \star \text{id}_f,$$

then $\alpha = \beta$.

01B9 Example 2.1.1.3. Here are some examples of corepresentably faithful morphisms.

01BA 1. *Corepresentably Faithful Morphisms in \mathbf{Cats}_2 .* The corepresentably faithful morphisms in \mathbf{Cats}_2 are characterised in [Categories, Item 4 of Proposition 5.1.1.2](#).

01BB 2. *Corepresentably Faithful Morphisms in \mathbf{Rel} .* Every morphism of \mathbf{Rel} is corepresentably faithful; see [Relations, Item 1 of Proposition 3.10.1.1](#).

01BC 2.2 Corepresentably Full Morphisms

Let C be a bicategory.

01BD Definition 2.2.1.1. A 1-morphism $f: A \rightarrow B$ of C is **corepresentably full** if, for each $X \in \text{Obj}(C)$, the functor

$$f^*: \text{Hom}_C(B, X) \rightarrow \text{Hom}_C(A, X)$$

given by precomposition by f is full.

01BE Remark 2.2.1.2. In detail, f is corepresentably full if, for each $X \in \text{Obj}(C)$ and each 2-morphism

$$\beta: \phi \circ f \Rightarrow \psi \circ f, \quad A \begin{array}{c} \xrightarrow{\phi \circ f} \\ \beta \Downarrow \\ \xrightarrow{\psi \circ f} \end{array} X$$

of C , there exists a 2-morphism

$$\alpha : \phi \Rightarrow \psi, \quad B \begin{array}{c} \xrightarrow{\phi} \\ \Downarrow \alpha \\ \xrightarrow{\psi} \end{array} X$$

of C such that we have an equality

$$A \xrightarrow{f} B \begin{array}{c} \xrightarrow{\phi} \\ \Downarrow \alpha \\ \xrightarrow{\psi} \end{array} X = A \begin{array}{c} \xrightarrow{\phi \circ f} \\ \Downarrow \beta \\ \xrightarrow{\psi \circ f} \end{array} X$$

of pasting diagrams in C , i.e. such that we have

$$\beta = \alpha \star \text{id}_f.$$

01BF Example 2.2.1.3. Here are some examples of corepresentably full morphisms.

01BG 1. *Corepresentably Full Morphisms in \mathbf{Cats}_2 .* The corepresentably full morphisms in \mathbf{Cats}_2 are characterised in [Categories, Item 5](#) of [Proposition 5.2.1.2](#).

01BH 2. *Corepresentably Full Morphisms in \mathbf{Rel} .* The corepresentably full morphisms in \mathbf{Rel} are characterised in [Relations, Item 2](#) of [Proposition 3.10.1.1](#).

01BJ 2.3 Corepresentably Fully Faithful Morphisms

Let C be a bicategory.

01BK Definition 2.3.1.1. A 1-morphism $f : A \rightarrow B$ of C is **corepresentably fully faithful**⁴ if the following equivalent conditions are satisfied:

- 01BL** 1. The 1-morphism f is corepresentably full ([Definition 2.2.1.1](#)) and corepresentably faithful ([Definition 2.1.1.1](#)).
- 01BM** 2. For each $X \in \text{Obj}(C)$, the functor

$$f^* : \text{Hom}_C(B, X) \rightarrow \text{Hom}_C(A, X)$$

given by precomposition by f is fully faithful.

⁴*Further Terminology:* Corepresentably fully faithful morphisms have also been called **lax epi-**

01BN Remark 2.3.1.2. In detail, f is corepresentably fully faithful if the conditions in Remark 2.1.1.2 and Remark 2.2.1.2 hold:

1. For all diagrams in C of the form

$$A \xrightarrow{f} B \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \beta \\ \xrightarrow{\psi} \end{array} X,$$

if we have

$$\alpha \star \text{id}_f = \beta \star \text{id}_f,$$

then $\alpha = \beta$.

2. For each $X \in \text{Obj}(C)$ and each 2-morphism

$$\beta: \phi \circ f \Rightarrow \psi \circ f, \quad A \begin{array}{c} \xrightarrow{\phi \circ f} \\ \beta \Downarrow \\ \xrightarrow{\psi \circ f} \end{array} X$$

of C , there exists a 2-morphism

$$\alpha: \phi \Rightarrow \psi, \quad B \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} X$$

of C such that we have an equality

$$A \xrightarrow{f} B \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} X = A \begin{array}{c} \xrightarrow{\phi \circ f} \\ \beta \Downarrow \\ \xrightarrow{\psi \circ f} \end{array} X$$

of pasting diagrams in C , i.e. such that we have

$$\beta = \alpha \star \text{id}_f.$$

01BP Example 2.3.1.3. Here are some examples of corepresentably fully faithful morphisms.

- 01BQ 1. *Corepresentably Fully Faithful Morphisms in \mathbf{Cats}_2 .* The fully faithful epimorphisms in \mathbf{Cats}_2 are characterised in **Categories**, Item 9 of **Proposition 5.3.1.2**.
- 01BR 2. *Corepresentably Fully Faithful Morphisms in \mathbf{Rel} .* The corepresentably fully faithful morphisms of \mathbf{Rel} coincide (**Relations**, Item 3 of **Proposition 3.10.1.1**) with the corepresentably full morphisms in \mathbf{Rel} , which are characterised in **Relations**, Item 2 of **Proposition 3.10.1.1**.

01BS 2.4 Morphisms Corepresentably Faithful on Cores

Let C be a bicategory.

- 01BT **Definition 2.4.1.1.** A 1-morphism $f: A \rightarrow B$ of C is **corepresentably faithful on cores** if, for each $X \in \text{Obj}(C)$, the functor

$$f^*: \text{Core}(\text{Hom}_C(B, X)) \rightarrow \text{Core}(\text{Hom}_C(A, X))$$

given by precomposition by f is faithful.

- 01BU **Remark 2.4.1.2.** In detail, f is corepresentably faithful on cores if, for all diagrams in C of the form

$$A \xrightarrow{f} B \begin{array}{c} \xrightarrow{\phi} \\ \alpha \downarrow \Downarrow \beta \\ \xrightarrow{\psi} \end{array} X,$$

if α and β are 2-isomorphisms and we have

$$\alpha \star \text{id}_f = \beta \star \text{id}_f,$$

then $\alpha = \beta$.

01BV 2.5 Morphisms Corepresentably Full on Cores

Let C be a bicategory.

- 01BW **Definition 2.5.1.1.** A 1-morphism $f: A \rightarrow B$ of C is **corepresentably full on cores** if, for each $X \in \text{Obj}(C)$, the functor

$$f^*: \text{Core}(\text{Hom}_C(B, X)) \rightarrow \text{Core}(\text{Hom}_C(A, X))$$

given by precomposition by f is full.

01BX Remark 2.5.1.2. In detail, f is corepresentably full on cores if, for each $X \in \text{Obj}(C)$ and each 2-isomorphism

$$\beta: \phi \circ f \xRightarrow{\sim} \psi \circ f, \quad A \begin{array}{c} \xrightarrow{\phi \circ f} \\ \beta \Downarrow \\ \xrightarrow{\psi \circ f} \end{array} X$$

of C , there exists a 2-isomorphism

$$\alpha: \phi \xRightarrow{\sim} \psi, \quad B \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} X$$

of C such that we have an equality

$$A \xrightarrow{f} B \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} X = A \begin{array}{c} \xrightarrow{\phi \circ f} \\ \beta \Downarrow \\ \xrightarrow{\psi \circ f} \end{array} X$$

of pasting diagrams in C , i.e. such that we have

$$\beta = \alpha \star \text{id}_f.$$

01BY 2.6 Morphisms Corepresentably Fully Faithful on Cores

Let C be a bicategory.

01BZ Definition 2.6.1.1. A 1-morphism $f: A \rightarrow B$ of C is **corepresentably fully faithful on cores** if the following equivalent conditions are satisfied:

1. The 1-morphism f is corepresentably full on cores ([Definition 2.5.1.1](#)) and corepresentably faithful on cores ([Definition 2.1.1.1](#)).
2. For each $X \in \text{Obj}(C)$, the functor

$$f^*: \text{Core}(\text{Hom}_C(B, X)) \rightarrow \text{Core}(\text{Hom}_C(A, X))$$

given by precomposition by f is fully faithful.

morphisms in the literature (e.g. in [\[Ad  +01\]](#)), though we will always use the name “corepresentably fully faithful morphism” instead in this work.

01C2 Remark 2.6.1.2. In detail, f is corepresentably fully faithful on cores if the conditions in **Remark 2.4.1.2** and **Remark 2.5.1.2** hold:

1. For all diagrams in C of the form

$$A \xrightarrow{f} B \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \beta \\ \xrightarrow{\psi} \end{array} X,$$

if α and β are 2-isomorphisms and we have

$$\alpha \star \text{id}_f = \beta \star \text{id}_f,$$

then $\alpha = \beta$.

2. For each $X \in \text{Obj}(C)$ and each 2-isomorphism

$$\beta: \phi \circ f \xRightarrow{\sim} \psi \circ f, \quad A \begin{array}{c} \xrightarrow{\phi \circ f} \\ \beta \Downarrow \\ \xrightarrow{\psi \circ f} \end{array} X$$

of C , there exists a 2-isomorphism

$$\alpha: \phi \xRightarrow{\sim} \psi, \quad B \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} X$$

of C such that we have an equality

$$A \xrightarrow{f} B \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} X = A \begin{array}{c} \xrightarrow{\phi \circ f} \\ \beta \Downarrow \\ \xrightarrow{\psi \circ f} \end{array} X$$

of pasting diagrams in C , i.e. such that we have

$$\beta = \alpha \star \text{id}_f.$$

01C3 2.7 Corepresentably Essentially Injective Morphisms

Let C be a bicategory.

01C4 Definition 2.7.1.1. A 1-morphism $f: A \rightarrow B$ of C is **corepresentably essentially injective** if, for each $X \in \text{Obj}(C)$, the functor

$$f^*: \text{Hom}_C(B, X) \rightarrow \text{Hom}_C(A, X)$$

given by precomposition by f is essentially injective.

01C5 Remark 2.7.1.2. In detail, f is corepresentably essentially injective if, for each pair of morphisms $\phi, \psi: B \rightrightarrows X$ of C , the following condition is satisfied:

$$(\star) \text{ If } \phi \circ f \cong \psi \circ f, \text{ then } \phi \cong \psi.$$

01C6 2.8 Corepresentably Conservative Morphisms

Let C be a bicategory.

01C7 Definition 2.8.1.1. A 1-morphism $f: A \rightarrow B$ of C is **corepresentably conservative** if, for each $X \in \text{Obj}(C)$, the functor

$$f^*: \text{Hom}_C(B, X) \rightarrow \text{Hom}_C(A, X)$$

given by precomposition by f is conservative.

01C8 Remark 2.8.1.2. In detail, f is corepresentably conservative if, for each pair of morphisms $\phi, \psi: B \rightrightarrows X$ and each 2-morphism

$$\alpha: \phi \xRightarrow{\sim} \psi, \quad B \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} X$$

of C , if the 2-morphism

$$\alpha \star \text{id}_f: \phi \circ f \xRightarrow{\sim} \psi \circ f, \quad A \begin{array}{c} \xrightarrow{\phi \circ f} \\ \parallel \\ \alpha \star \text{id}_f \\ \Downarrow \\ \xrightarrow{\psi \circ f} \end{array} X$$

is a 2-isomorphism, then so is α .

01C9 2.9 Strict Epimorphisms

Let C be a bicategory.

01CA Definition 2.9.1.1. A 1-morphism $f: A \rightarrow B$ is a **strict epimorphism in C** if, for each $X \in \text{Obj}(C)$, the functor

$$f^*: \text{Hom}_C(B, X) \rightarrow \text{Hom}_C(A, X)$$

given by precomposition by f is injective on objects, i.e. its action on objects

$$f_*: \text{Obj}(\text{Hom}_C(B, X)) \rightarrow \text{Obj}(\text{Hom}_C(A, X))$$

is injective.

01CB Remark 2.9.1.2. In detail, f is a strict epimorphism if, for each diagram in C of the form

$$A \xrightarrow{f} B \begin{array}{c} \xrightarrow{\phi} \\ \xrightarrow{\psi} \end{array} X,$$

if $\phi \circ f = \psi \circ f$, then $\phi = \psi$.

01CC Example 2.9.1.3. Here are some examples of strict epimorphisms.

01CD 1. *Strict Epimorphisms in Cats_2* . The strict epimorphisms in Cats_2 are characterised in [Categories, Item 1](#) of [Proposition 6.3.1.2](#).

01CE 2. *Strict Epimorphisms in **Rel***. The strict epimorphisms in **Rel** are characterised in [Relations, Proposition 3.9.1.1](#).

01CF 2.10 Pseudoepic Morphisms

Let C be a bicategory.

01CG Definition 2.10.1.1. A 1-morphism $f: A \rightarrow B$ of C is **pseudoepic** if, for each $X \in \text{Obj}(C)$, the functor

$$f^*: \text{Hom}_C(B, X) \rightarrow \text{Hom}_C(A, X)$$

given by precomposition by f is pseudomonic.

01CH Remark 2.10.1.2. In detail, a 1-morphism $f: A \rightarrow B$ of C is pseudoepic if it satisfies the following conditions:

- 01CJ 1. For all diagrams in C of the form

$$A \xrightarrow{f} B \begin{array}{c} \xrightarrow{\phi} \\ \Downarrow \alpha \\ \Downarrow \beta \\ \xrightarrow{\psi} \end{array} X,$$

if we have

$$\alpha \star \text{id}_f = \beta \star \text{id}_f,$$

then $\alpha = \beta$.

- 01CK 2. For each $X \in \text{Obj}(C)$ and each 2-isomorphism

$$\beta: \phi \circ f \xRightarrow{\sim} \psi \circ f, \quad A \begin{array}{c} \xrightarrow{\phi \circ f} \\ \Downarrow \beta \\ \xrightarrow{\psi \circ f} \end{array} X$$

of C , there exists a 2-isomorphism

$$\alpha: \phi \xRightarrow{\sim} \psi, \quad B \begin{array}{c} \xrightarrow{\phi} \\ \Downarrow \alpha \\ \xrightarrow{\psi} \end{array} X$$

of C such that we have an equality

$$A \xrightarrow{f} B \begin{array}{c} \xrightarrow{\phi} \\ \Downarrow \alpha \\ \xrightarrow{\psi} \end{array} X = A \begin{array}{c} \xrightarrow{\phi \circ f} \\ \Downarrow \beta \\ \xrightarrow{\psi \circ f} \end{array} X$$

of pasting diagrams in C , i.e. such that we have

$$\beta = \alpha \star \text{id}_f.$$

01CL **Proposition 2.10.1.3.** Let $f: A \rightarrow B$ be a 1-morphism of C .

01CM 1. *Characterisations.* The following conditions are equivalent:

- 01CN (a) The morphism f is pseudoepic.
 (b) The morphism f is corepresentably full on cores and corepresentably faithful.
- 01CP

01CQ

(c) We have an isococcomma square of the form

$$B \stackrel{\text{eq.}}{\cong} B \overset{\leftrightarrow}{\coprod}_A B,$$

$$\begin{array}{ccc} B & \xleftarrow{\text{id}_B} & B \\ \text{id}_B \uparrow & \nearrow & \uparrow F \\ B & \xleftarrow{F} & A \end{array}$$

in \mathcal{C} up to equivalence.*Proof.* **Item 1**, *Characterisations*: Omitted. □

Appendices

A Other Chapters

Sets

1. **Sets**
2. **Constructions With Sets**
3. **Pointed Sets**
4. **Tensor Products of Pointed Sets**

6. **Constructions With Relations**

7. **Equivalence Relations and Apartness Relations**

Category Theory

8. **Categories**

Bicategories

9. **Types of Morphisms in Bicat-** **egories**

Relations

5. **Relations**

References

- [Adá+01] Jiří Adámek, Robert El Bashir, Manuela Sobral, and Jiří Velebil. “On Functors Which Are Lax Epimorphisms”. In: *Theory Appl. Categ.* 8 (2001), pp. 509–521. ISSN: 1201-561X (cit. on p. **16**).