Sets

The Clowder Project Authors

May 3, 2024

This chapter (will eventually) contain material on axiomatic set theory, as well as a couple other things.

Contents

1 Sets and Functions

1.1 Functions

Definition 1.1.1.1. A **function** is a functional and total relation.

Notation 1.1.1.2. Throughout this work, we will sometimes denote a function $f\colon X \to Y$ by

$$f \overset{\text{def}}{=} [\![x \mapsto f(x)]\!].$$

1. For example, given a function

$$\Phi\colon \mathrm{Hom}_{\mathsf{Sets}}(X, Y) \to K$$

taking values on a set of functions such as $\mathrm{Hom}_{\mathsf{Sets}}(X, Y)$, we will sometimes also write

$$\Phi(f) \stackrel{\text{def}}{=} \Phi(\llbracket x \mapsto f(x) \rrbracket).$$

2. This notational choice is based on the lambda notation

$$f \stackrel{\text{def}}{=} (\lambda x.\ f(x)),$$

but uses a "$\mapsto$" symbol for better spacing and double brackets instead of either:

   (a) Square brackets $[x \mapsto f(x)]$;
   (b) Parentheses $(x \mapsto f(x))$;

hoping to improve readability when dealing with e.g.:

   (a) Equivalence classes, cf.:
      i. $\llbracket [x] \mapsto f([x]) \rrbracket$
      ii. $[[x] \mapsto f([x])]$
      iii. $(\lambda [x].\ f([x]))$
   (b) Function evaluations, cf.:
      i. $\Phi(\llbracket x \mapsto f(x) \rrbracket)$
      ii. $\Phi((x \mapsto f(x)))$
      iii. $\Phi((\lambda x.\ f(x)))$

3. We will also sometimes write $-_1$, $-_2$, etc. for the arguments of a function. Some examples include:

   (a) Writing $f(-_1)$ for a function $f\colon A \to B$.
   (b) Writing $f(-_1, -_2)$ for a function $f\colon A \times B \to C$.
   (c) Given a function $f\colon A \times B \to C$, writing

$$f(a, -)\colon B \to C$$

   for the function $\llbracket b \mapsto f(a, b) \rrbracket$.
   (d) Denoting a composition of the form

$$A \times B \xrightarrow{\phi \times \mathrm{id}_B} A' \times B \xrightarrow{f} C$$

   by $f(\phi(-_1), -_2)$.

4. Finally, given a function $f\colon A \to B$, we write

$$\mathrm{ev}_a(f) \overset{\text{def}}{=} f(a)$$

for the value of f at some $a \in A$.

For an example of the above notations being used in practice, see the proof of the adjunction

$$(A \times - \dashv \mathrm{Hom}_{\mathsf{Sets}}(A, -))\colon\ \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "A\times-"] \arrow[r, phantom, "\perp"] & \mathsf{Sets} \arrow[l, bend left, "\mathrm{Hom}_{\mathsf{Sets}}(A,-)"] \end{tikzcd},$$

stated in Constructions With Sets, Item 2 of Proposition 1.3.1.2.

2 The Enrichment of Sets in Classical Truth Values

2.1 (-2)-Categories

Definition 2.1.1.1. A (-2)**-category** is the "necessarily true" truth value.[1,2,3]

2.2 (-1)-Categories

Definition 2.2.1.1. A (-1)**-category** is a classical truth value.

Remark 2.2.1.2. [4](-1)-categories should be thought of as being "categories enriched in (-2)-categories", having a collection of objects and, for each pair of objects, a Hom-object $\mathrm{Hom}(x, y)$ that is a (-2)-category (i.e. trivial). Therefore, a (-1)-category $\mathcal{C}$ is either ([BS10, pp. 33–34]):

1. *Empty*, having no objects;

2. *Contractible*, having a collection of objects $\{a, b, c, \ldots\}$, but with $\mathrm{Hom}_{\mathcal{C}}(a, b)$ being a (-2)-category (i.e. trivial) for all $a, b \in \mathrm{Obj}(\mathcal{C})$, forcing all objects of $\mathcal{C}$ to be uniquely isomorphic to each other.

As such, there are only two (-1)-categories, up to equivalence:

[1] Thus, there is only one (-2)-category.

[2] A $(-n)$-category for $n = 3, 4, \ldots$ is also the "necessarily true" truth value, coinciding with a (-2)-category.

[3] For motivation, see [BS10, p. 13].

[4] For more motivation, see [BS10, p. 13].

- The (-1)-category false (the empty one);
- The (-1)-category true (the contractible one).

Definition 2.2.1.3. The **poset of truth values**[5] is the poset $(\{\mathsf{true},\mathsf{false}\},\preceq)$ consisting of

- *The Underlying Set.* The set $\{\mathsf{true},\mathsf{false}\}$ whose elements are the truth values true and false.
- *The Partial Order.* The partial order

$$\preceq\colon \{\mathsf{true},\mathsf{false}\}\times\{\mathsf{true},\mathsf{false}\}\to\{\mathsf{true},\mathsf{false}\}$$

  on $\{\mathsf{true},\mathsf{false}\}$ defined by[6]

$$\begin{aligned}\mathsf{false}\preceq\mathsf{false}&\stackrel{\text{def}}{=}\mathsf{true},\\ \mathsf{true}\preceq\mathsf{false}&\stackrel{\text{def}}{=}\mathsf{false},\\ \mathsf{false}\preceq\mathsf{true}&\stackrel{\text{def}}{=}\mathsf{true},\\ \mathsf{true}\preceq\mathsf{true}&\stackrel{\text{def}}{=}\mathsf{true}.\end{aligned}$$

Notation 2.2.1.4. We also write $\{\mathsf{t},\mathsf{f}\}$ for the poset $\{\mathsf{true},\mathsf{false}\}$.

Proposition 2.2.1.5. The poset of truth values $\{\mathsf{t},\mathsf{f}\}$ is Cartesian closed with product given by[7]

$$\begin{aligned}\mathsf{t}\times\mathsf{t}&=\mathsf{t},\\ \mathsf{t}\times\mathsf{f}&=\mathsf{f},\\ \mathsf{f}\times\mathsf{t}&=\mathsf{f},\\ \mathsf{f}\times\mathsf{f}&=\mathsf{f},\end{aligned}$$

and internal Hom $\mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}$ given by the partial order of $\{\mathsf{t},\mathsf{f}\}$, i.e. by

$$\begin{aligned}\mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t},\mathsf{t})&=\mathsf{t},\\ \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t},\mathsf{f})&=\mathsf{f},\\ \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f},\mathsf{t})&=\mathsf{t},\\ \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f},\mathsf{f})&=\mathsf{t}.\end{aligned}$$

[5] *Further Terminology:* Also called the **poset of** (-1)**-categories**.

[6] This partial order coincides with logical implication.

[7] Note that $\times$ coincides with the "and" operator, while $\mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}$ coincides with the

Proof. Existence of Products: We claim that the products $\mathsf{t}\times\mathsf{t}$, $\mathsf{t}\times\mathsf{f}$, $\mathsf{f}\times\mathsf{t}$, and $\mathsf{f}\times\mathsf{f}$ satisfy the universal property of the product in $\{\mathsf{t},\mathsf{f}\}$. Indeed, consider the diagrams

$$\begin{tikzcd} & P_1 \arrow[dl, bend right, "p_1^1"'] \arrow[d, dashed, "\exists!"] \arrow[dr, bend left, "p_2^1"] & \\ \mathsf{t} & \underbrace{\mathsf{t}\times\mathsf{t}}_{=\mathsf{t}} \arrow[l, "\mathrm{pr}_1"] \arrow[r, "\mathrm{pr}_2"'] & \mathsf{t} \end{tikzcd} \quad \begin{tikzcd} & P_2 \arrow[dl, bend right, "p_1^2"'] \arrow[d, dashed, "\exists!"] \arrow[dr, bend left, "p_2^2"] & \\ \mathsf{t} & \underbrace{\mathsf{t}\times\mathsf{f}}_{=\mathsf{f}} \arrow[l, "\mathrm{pr}_1"] \arrow[r, "\mathrm{pr}_2"'] & \mathsf{f} \end{tikzcd} \quad \begin{tikzcd} & P_3 \arrow[dl, bend right, "p_1^3"'] \arrow[d, dashed, "\exists!"] \arrow[dr, bend left, "p_2^3"] & \\ \mathsf{f} & \underbrace{\mathsf{f}\times\mathsf{t}}_{=\mathsf{f}} \arrow[l, "\mathrm{pr}_1"] \arrow[r, "\mathrm{pr}_2"'] & \mathsf{t} \end{tikzcd} \quad \begin{tikzcd} & P_4 \arrow[dl, bend right, "p_1^4"'] \arrow[d, dashed, "\exists!"] \arrow[dr, bend left, "p_2^4"] & \\ \mathsf{f} & \underbrace{\mathsf{f}\times\mathsf{f}}_{=\mathsf{f}} \arrow[l, "\mathrm{pr}_1"] \arrow[r, "\mathrm{pr}_2"'] & \mathsf{f}. \end{tikzcd}$$

Here:

1. If $P_1 = \mathsf{t}$, then $p_1^1 = p_2^1 = \mathrm{id}_{\mathsf{t}}$, and there's indeed a unique morphism from P_1 to t making the diagram commute, namely id_{t};
2. If $P_1 = \mathsf{f}$, then $p_1^1 = p_2^1$ are given by the unique morphism from f to t, and there's indeed a unique morphism from P_1 to t making the diagram commute, namely the unique morphism from f to t;
3. If $P_2 = \mathsf{t}$, then there is no morphism p_2^2.
4. If $P_2 = \mathsf{f}$, then p_1^2 is the unique morphism from f to t while $p_2^2 = \mathrm{id}_{\mathsf{f}}$, and there's indeed a unique morphism from P_2 to f making the diagram commute, namely id_{f};
5. The proof for P_3 is similar to the one for P_2;
6. If $P_4 = \mathsf{t}$, then there is no morphism p_1^4 or p_2^4.
7. If $P_4 = \mathsf{f}$, then $p_1^4 = p_2^4 = \mathrm{id}_{\mathsf{f}}$, and there's indeed a unique morphism from P_4 to f making the diagram commute, namely id_{f}.

Cartesian Closedness: We claim there's a bijection

$$\mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(A\times B, C) \cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(A, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(B, C))$$

natural in $A, B, C \in \{\mathsf{t},\mathsf{f}\}$. Indeed:

- For $(A, B, C) = (\mathsf{t},\mathsf{t},\mathsf{t})$, we have

$$\begin{aligned} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t}\times\mathsf{t},\mathsf{t}) &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t},\mathsf{t}) \\ &= \{\mathrm{id}_{\mathsf{true}}\} \\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t},\mathsf{t}) \\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t}, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t},\mathsf{t})). \end{aligned}$$

- For $(A, B, C) = (\mathsf{t}, \mathsf{t}, \mathsf{f})$, we have

$$\begin{aligned}\mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t} \times \mathsf{t}, \mathsf{f}) &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t}, \mathsf{f}) \\ &= \emptyset \\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t}, \mathsf{f}) \\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t}, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t}, \mathsf{f})).\end{aligned}$$

- For $(A, B, C) = (\mathsf{t}, \mathsf{f}, \mathsf{t})$, we have

$$\begin{aligned}\mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t} \times \mathsf{f}, \mathsf{t}) &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{t}) \\ &\cong \mathrm{pt} \\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{t}) \\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{t})).\end{aligned}$$

- For $(A, B, C) = (\mathsf{t}, \mathsf{f}, \mathsf{f})$, we have

$$\begin{aligned}\mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t} \times \mathsf{f}, \mathsf{f}) &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{f}) \\ &\cong \{\mathrm{id}_{\mathsf{false}}\} \\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{f}) \\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t}, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{f})).\end{aligned}$$

- For $(A, B, C) = (\mathsf{f}, \mathsf{t}, \mathsf{t})$, we have

$$\begin{aligned}\mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f} \times \mathsf{t}, \mathsf{t}) &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{t}) \\ &\cong \mathrm{pt} \\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{t}) \\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t}, \mathsf{t})).\end{aligned}$$

- For $(A, B, C) = (\mathsf{f}, \mathsf{t}, \mathsf{f})$, we have

$$\begin{aligned}\mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f} \times \mathsf{t}, \mathsf{f}) &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{f}) \\ &\cong \{\mathrm{id}_{\mathsf{false}}\} \\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{f}) \\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t}, \mathsf{f})).\end{aligned}$$

logical implication operator.

- For $(A,B,C)=(\mathsf{f},\mathsf{f},\mathsf{t})$, we have

$$\begin{aligned}\mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}\times\mathsf{f},\mathsf{t}) &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f},\mathsf{t})\\ &\cong \mathrm{pt}\\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f},\mathsf{t})\\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f},\mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f},\mathsf{t})).\end{aligned}$$

- For $(A,B,C)=(\mathsf{f},\mathsf{f},\mathsf{f})$, we have

$$\begin{aligned}\mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}\times\mathsf{f},\mathsf{f}) &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f},\mathsf{f})\\ &= \{\mathrm{id}_{\mathsf{false}}\}\\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f},\mathsf{f})\\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f},\mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f},\mathsf{f})).\end{aligned}$$

The proof of naturality is omitted. □

2.3 0-Categories

Definition 2.3.1.1. A **0-category** is a poset.[8]

Definition 2.3.1.2. A **0-groupoid** is a 0-category in which every morphism is invertible.[9]

2.4 Tables of Analogies Between Set Theory and Category Theory

Here we record some analogies between notions in set theory and category theory. Note that the analogies relating to presheaves relate equally well to copresheaves, as the opposite X^{op} of a set X is just X again.
Basics:

[8] *Motivation:* A 0-category is precisely a category enriched in the poset of (-1)-categories.

[9] That is, a *set*.

Set Theory	Category Theory
Enrichment in $\{\mathsf{true}, \mathsf{false}\}$	Enrichment in Sets
Set X	Category $\mathcal{C}$
Element $x \in X$	Object $X \in \mathrm{Obj}(\mathcal{C})$
Function	Functor
Function $X \to \{\mathsf{true}, \mathsf{false}\}$	Functor $\mathcal{C} \to \mathsf{Sets}$
Function $X \to \{\mathsf{true}, \mathsf{false}\}$	Presheaf $\mathcal{C}^{\mathsf{op}} \to \mathsf{Sets}$

Powersets and categories of presheaves:

Set Theory	Category Theory
Powerset $\mathcal{P}(X)$	Presheaf category $\mathsf{PSh}(\mathcal{C})$
Characteristic function $\chi_{\{x\}}$	Representable presheaf h_X
Characteristic embedding $\chi_{(-)}\colon X \hookrightarrow \mathcal{P}(X)$	Yoneda embedding よ$\colon \mathcal{C}^{\mathsf{op}} \hookrightarrow \mathsf{PSh}(\mathcal{C})$
Characteristic relation $\chi_X(-_1, -_2)$	Hom profunctor $\mathrm{Hom}_{\mathcal{C}}(-_1, -_2)$
The Yoneda lemma for sets $\mathrm{Hom}_{\mathcal{P}(X)}(\chi_x, \chi_U) = \chi_U(x)$	The Yoneda lemma for categories $\mathrm{Nat}(h_X, \mathcal{F}) \cong \mathcal{F}(X)$
The characteristic embedding is fully faithful, $\mathrm{Hom}_{\mathcal{P}(X)}(\chi_x, \chi_y) = \chi_X(x, y)$	The Yoneda embedding is fully faithful, $\mathrm{Nat}(h_X, h_Y) \cong \mathrm{Hom}_{\mathcal{C}}(X, Y)$
Subsets are unions of their elements $U = \bigcup_{x \in U} \{x\}$ or $\chi_U = \operatorname*{colim}_{\chi_x \in \mathsf{Sets}(U, \{\mathsf{t}, \mathsf{f}\})} (\chi_x)$	Presheaves are colimits of representables, $\mathcal{F} \cong \operatorname*{colim}_{h_X \in \int_{\mathcal{C}} \mathcal{F}} (h_X)$

Categories of elements:

Set Theory	Category Theory
Assignment $U \mapsto \chi_U$	Assignment $\mathcal{F} \mapsto \int_{\mathcal{C}} \mathcal{F}$ (the category of elements)
Assignment $U \mapsto \chi_U$ giving an isomorphism $\mathcal{P}(X) \cong \mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\})$	Assignment $\mathcal{F} \mapsto \int_{\mathcal{C}} \mathcal{F}$ giving an equivalence $\mathsf{PSh}(\mathcal{C}) \overset{\text{eq.}}{\cong} \mathsf{DFib}(\mathcal{C})$

Functions between powersets and functors between presheaf categories:

Set Theory	Category Theory
Direct image function $f_*: \mathcal{P}(X) \to \mathcal{P}(Y)$	Inverse image functor $f^{-1}: \mathsf{PSh}(\mathcal{C}) \to \mathsf{PSh}(\mathcal{D})$
Inverse image function $f^{-1}: \mathcal{P}(Y) \to \mathcal{P}(X)$	Direct image functor $f_*: \mathsf{PSh}(\mathcal{D}) \to \mathsf{PSh}(\mathcal{C})$
Direct image with compact support function $f_!: \mathcal{P}(X) \to \mathcal{P}(Y)$	Direct image with compact support functor $f_!: \mathsf{PSh}(\mathcal{C}) \to \mathsf{PSh}(\mathcal{D})$

Relations and profunctors:

Set Theory	Category Theory
Relation $R: X \times Y \to \{\mathsf{t}, \mathsf{f}\}$	Profunctor $\mathfrak{p}: \mathcal{D}^{\mathsf{op}} \times \mathcal{C} \to \mathsf{Sets}$
Relation $R: X \to \mathcal{P}(Y)$	Profunctor $\mathfrak{p}: \mathcal{C} \to \mathsf{PSh}(\mathcal{D})$
Relation as a cocontinuous morphism of posets $R: (\mathcal{P}(X), \subset) \to (\mathcal{P}(Y), \subset)$	Profunctor as a colimit-preserving functor $\mathfrak{p}: \mathsf{PSh}(\mathcal{C}) \to \mathsf{PSh}(\mathcal{D})$

Appendices

A Other Chapters

Sets

1. Sets
2. Constructions With Sets
3. Pointed Sets
4. Tensor Products of Pointed Sets

Relations

5. Relations
6. Constructions With Relations
7. Equivalence Relations and Apartness Relations

Category Theory

8. Categories

Bicategories

9. Types of Morphisms in Bicategories

References

[BS10] John C. Baez and Michael Shulman. "Lectures on n-Categories and Cohomology". In: *Towards higher categories*. Vol. 152. IMA Vol. Math. Appl. Springer, New York, 2010, pp. 1–68. DOI: 10.1007/978-1-4419-1524-5_1. URL: https://doi.org/10.1007/978-1-4419-1524-5_1 (cit. on p. 3).