Categories

The Clowder Project Authors

May 3, 2024

This chapter contains some elementary material about categories, functors, and natural transformations. Notably, we discuss and explore:

1. Categories (Section 1).
2. The quadruple adjunction $\pi_0 \dashv (-)_{\mathsf{disc}} \dashv \mathrm{Obj} \dashv (-)_{\mathsf{indisc}}$ between the category of categories and the category of sets (Section 2).
3. Groupoids, categories in which all morphisms admit inverses (Section 3).
4. Functors (Section 4).
5. The conditions one may impose on functors in decreasing order of importance:
    (a) Section 5 introduces the foundationally important conditions one may impose on functors, such as faithfulness, conservativity, essential surjectivity, etc.
    (b) Section 6 introduces more conditions one may impose on functors that are still important but less omni-present than those of Section 5, such as being dominant, being a monomorphism, being pseudomonic, etc.
    (c) Section 7 introduces some rather rare or uncommon conditions one may impose on functors that are nevertheless still useful to explicit record in this chapter.
6. Natural transformations (Section 8).
7. The various categorical and 2-categorical structures formed by categories, functors, and natural transformations (Section 9).

Contents

1 Categories

1.1 Foundations

Definition 1.1.1 ▶ Categories

A **category** $(\mathcal{C}, \circ^{\mathcal{C}}, \mathbb{1}^{\mathcal{C}})$ consists of:

- *Objects.* A class $\mathrm{Obj}(\mathcal{C})$ of **objects**.
- *Morphisms.* For each $A, B \in \mathrm{Obj}(\mathcal{C})$, a class $\mathrm{Hom}_{\mathcal{C}}(A, B)$, called the **class of morphisms of $\mathcal{C}$ from A to B**.
- *Identities.* For each $A \in \mathrm{Obj}(\mathcal{C})$, a map of sets
$$\mathbb{1}^{\mathcal{C}}_A \colon \mathrm{pt} \to \mathrm{Hom}_{\mathcal{C}}(A, A),$$
called the **unit map of $\mathcal{C}$ at A**, determining a morphism
$$\mathrm{id}_A \colon A \to A$$
of $\mathcal{C}$, called the **identity morphism of A**.
- *Composition.* For each $A, B, C \in \mathrm{Obj}(\mathcal{C})$, a map of sets
$$\circ^{\mathcal{C}}_{A,B,C} \colon \mathrm{Hom}_{\mathcal{C}}(B, C) \times \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{C}}(A, C),$$
called the **composition map of $\mathcal{C}$ at (A, B, C)**.

such that the following conditions are satisfied:

1. *Associativity.* The diagram

$$\begin{tikzcd}
& \mathrm{Hom}_{\mathcal{C}}(C, D) \times (\mathrm{Hom}_{\mathcal{C}}(B, C) \times \mathrm{Hom}_{\mathcal{C}}(A, B)) \arrow[dr, "\mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(C,D)} \times \circ^{\mathcal{C}}_{A,B,C}"] & \\
(\mathrm{Hom}_{\mathcal{C}}(C, D) \times \mathrm{Hom}_{\mathcal{C}}(B, C)) \times \mathrm{Hom}_{\mathcal{C}}(A, B) \arrow[ur, dashed, "\alpha^{\mathsf{Sets}}_{\mathrm{Hom}_{\mathcal{C}}(C,D),\mathrm{Hom}_{\mathcal{C}}(B,C),\mathrm{Hom}_{\mathcal{C}}(A,B)}", "\sim"'] \arrow[d, "\circ^{\mathcal{C}}_{B,C,D} \times \mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(A,B)}"'] & & \mathrm{Hom}_{\mathcal{C}}(C, D) \times \mathrm{Hom}_{\mathcal{C}}(A, C) \arrow[d, "\circ^{\mathcal{C}}_{A,C,D}"] \\
\mathrm{Hom}_{\mathcal{C}}(B, D) \times \mathrm{Hom}_{\mathcal{C}}(A, B) \arrow[rr, "\circ^{\mathcal{C}}_{A,B,D}"'] & & \mathrm{Hom}_{\mathcal{C}}(A, D)
\end{tikzcd}$$

commutes, i.e. for each composable triple (f, g, h) of morphisms of $\mathcal{C}$, we have

$$(f \circ g) \circ h = f \circ (g \circ h).$$

2. *Left Unitality.* The diagram

$$\begin{tikzcd}
\mathrm{pt} \times \mathrm{Hom}_{\mathcal{C}}(A, B) \arrow[d, "\mathbb{1}^{\mathcal{C}}_{B} \times \mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(A,B)}"'] \arrow[dr, dashed, "\lambda^{\mathsf{Sets}}_{\mathrm{Hom}_{\mathcal{C}}(A,B)}", "\sim"'] & \\
\mathrm{Hom}_{\mathcal{C}}(B, B) \times \mathrm{Hom}_{\mathcal{C}}(A, B) \arrow[r, "\circ^{\mathcal{C}}_{A,B,B}"'] & \mathrm{Hom}_{\mathcal{C}}(A, B)
\end{tikzcd}$$

commutes, i.e. for each morphism $f \colon A \longrightarrow B$ of $\mathcal{C}$, we have

$$\mathrm{id}_B \circ f = f.$$

3. *Right Unitality.* The diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(A, B) \times \mathrm{pt} \arrow[d, "\mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(A,B)} \times \mathbb{1}^{\mathcal{C}}_{A}"'] \arrow[dr, dashed, "\rho^{\mathsf{Sets}}_{\mathrm{Hom}_{\mathcal{C}}(A,B)}", "\sim"'] & \\
\mathrm{Hom}_{\mathcal{C}}(A, B) \times \mathrm{Hom}_{\mathcal{C}}(A, A) \arrow[r, "\circ^{\mathcal{C}}_{A,A,B}"'] & \mathrm{Hom}_{\mathcal{C}}(A, B)
\end{tikzcd}$$

commutes, i.e. for each morphism $f \colon A \longrightarrow B$ of $\mathcal{C}$, we have

$$f \circ \mathrm{id}_A = f.$$

NOTATION 1.1.2 ► FURTHER NOTATION FOR MORPHISMS IN CATEGORIES

Let $\mathcal{C}$ be a category.

1. We also write $\mathcal{C}(A, B)$ for $\mathrm{Hom}_{\mathcal{C}}(A, B)$.
2. We write $\mathrm{Mor}(\mathcal{C})$ for the class of all morphisms of $\mathcal{C}$.

Definition 1.1.3 ► Size Conditions on Categories

Let κ be a regular cardinal. A category $\mathcal{C}$ is

1. **Locally small** if, for each $A, B \in \mathrm{Obj}(\mathcal{C})$, the class $\mathrm{Hom}_{\mathcal{C}}(A, B)$ is a set.
2. **Locally essentially small** if, for each $A, B \in \mathrm{Obj}(\mathcal{C})$, the class

$$\mathrm{Hom}_{\mathcal{C}}(A, B)/\{\text{isomorphisms}\}$$

   is a set.
3. **Small** if $\mathcal{C}$ is locally small and $\mathrm{Obj}(\mathcal{C})$ is a set.
4. **κ-Small** if $\mathcal{C}$ is locally small, $\mathrm{Obj}(\mathcal{C})$ is a set, and we have $\#\mathrm{Obj}(\mathcal{C}) < \kappa$.

1.2 Examples of Categories

Example 1.2.1 ► The Punctual Category

The **punctual category**[1] is the category pt where

- *Objects.* We have
$$\mathrm{Obj}(\mathrm{pt}) \overset{\mathrm{def}}{=} \{\star\}.$$
- *Morphisms.* The unique Hom-set of pt is defined by
$$\mathrm{Hom}_{\mathrm{pt}}(\star, \star) \overset{\mathrm{def}}{=} \{\mathrm{id}_\star\}.$$
- *Identities.* The unit map
$$\mathbb{1}^{\mathrm{pt}}_\star \colon \mathrm{pt} \longrightarrow \mathrm{Hom}_{\mathrm{pt}}(\star, \star)$$
  of pt at $\star$ is defined by
$$\mathrm{id}^{\mathrm{pt}}_\star \overset{\mathrm{def}}{=} \mathrm{id}_\star.$$
- *Composition.* The composition map
$$\circ^{\mathrm{pt}}_{\star,\star,\star} \colon \mathrm{Hom}_{\mathrm{pt}}(\star, \star) \times \mathrm{Hom}_{\mathrm{pt}}(\star, \star) \longrightarrow \mathrm{Hom}_{\mathrm{pt}}(\star, \star)$$
  of pt at $(\star, \star, \star)$ is given by the bijection $\mathrm{pt} \times \mathrm{pt} \cong \mathrm{pt}$.

[1] *Further Terminology:* Also called the **singleton category**.

EXAMPLE 1.2.2 ▶ MONOIDS AS ONE-OBJECT CATEGORIES

We have an isomorphism of categories[1]

$$\mathsf{Mon} \cong \mathrm{pt} \underset{\mathsf{Sets}}{\times} \mathsf{Cats}, \qquad \begin{tikzcd} \mathsf{Mon} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Cats} \arrow[d, "\mathrm{Obj}"] \\ \mathrm{pt} \arrow[r, "{[\mathrm{pt}]}"'] & \mathsf{Sets} \end{tikzcd}$$

via the delooping functor $\mathbf{B}\colon \mathsf{Mon} \to \mathsf{Cats}$ of **??** of **??**, exhibiting monoids as exactly those categories having a single object.

[1] This can be enhanced to an isomorphism of 2-categories

$$\mathsf{Mon}_{\mathsf{2disc}} \cong \mathrm{pt}_{\mathsf{bi}} \underset{\mathsf{Sets}_{\mathsf{2disc}}}{\times} \mathsf{Cats}_{2,*}, \qquad \begin{tikzcd} \mathsf{Mon}_{\mathsf{2disc}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Cats}_{2,*} \arrow[d, "\mathrm{Obj}"] \\ \mathrm{pt}_{\mathsf{bi}} \arrow[r, "{[\mathrm{pt}]}"'] & \mathsf{Sets}_{\mathsf{2disc}} \end{tikzcd}$$

between the discrete 2-category $\mathsf{Mon}_{\mathsf{2disc}}$ on Mon and the 2-category of pointed categories with one object.

PROOF 1.2.3 ▶ PROOF OF EXAMPLE 1.2.2

Omitted.

EXAMPLE 1.2.4 ▶ THE EMPTY CATEGORY

The **empty category** is the category $\emptyset_{\mathsf{cat}}$ where

- *Objects.* We have
$$\mathrm{Obj}(\emptyset_{\mathsf{cat}}) \overset{\mathrm{def}}{=} \emptyset.$$
- *Morphisms.* We have
$$\mathrm{Mor}(\emptyset_{\mathsf{cat}}) \overset{\mathrm{def}}{=} \emptyset.$$
- *Identities and Composition.* Having no objects, $\emptyset_{\mathsf{cat}}$ has no unit nor composition maps.

Example 1.2.5 ▶ Ordinal Categories

The ***n*th ordinal category** is the category $\mathbb{n}$ where[1]

- *Objects.* We have

$$\mathrm{Obj}(\mathbb{n}) \stackrel{\text{def}}{=} \{[0], \dots, [n]\}.$$

- *Morphisms.* For each $[i], [j] \in \mathrm{Obj}(\mathbb{n})$, we have

$$\mathrm{Hom}_{\mathbb{n}}([i],[j]) \stackrel{\text{def}}{=} \begin{cases} \{\mathrm{id}_{[i]}\} & \text{if } [i] = [j], \\ \{[i] \to [j]\} & \text{if } [j] < [i], \\ \emptyset & \text{if } [j] > [i]. \end{cases}$$

- *Identities.* For each $[i] \in \mathrm{Obj}(\mathbb{n})$, the unit map

$$\mathbb{1}^{\mathbb{n}}_{[i]} \colon \mathrm{pt} \to \mathrm{Hom}_{\mathbb{n}}([i],[i])$$

  of $\mathbb{n}$ at $[i]$ is defined by

$$\mathrm{id}^{\mathbb{n}}_{[i]} \stackrel{\text{def}}{=} \mathrm{id}_{[i]}.$$

- *Composition.* For each $[i], [j], [k] \in \mathrm{Obj}(\mathbb{n})$, the composition map

$$\circ^{\mathbb{n}}_{[i],[j],[k]} \colon \mathrm{Hom}_{\mathbb{n}}([j],[k]) \times \mathrm{Hom}_{\mathbb{n}}([i],[j]) \to \mathrm{Hom}_{\mathbb{n}}([i],[k])$$

  of $\mathbb{n}$ at $([i],[j],[k])$ is defined by

$$\begin{aligned} \mathrm{id}_{[i]} \circ \mathrm{id}_{[i]} &= \mathrm{id}_{[i]}, \\ ([j] \to [k]) \circ ([i] \to [j]) &= ([i] \to [k]). \end{aligned}$$

[1] In other words, $\mathbb{n}$ is the category associated to the poset

$$[0] \to [1] \to \cdots \to [n-1] \to [n].$$

The category $\mathbb{n}$ for $n \geq 2$ may also be defined in terms of $\mathbb{0}$ and joins (**??**, **??**): we have isomorphisms of categories

$$
\begin{aligned}
\mathbb{1} &\cong \mathbb{0} \star \mathbb{0}, \\
\mathbb{2} &\cong \mathbb{1} \star \mathbb{0} \\
&\cong (\mathbb{0} \star \mathbb{0}) \star \mathbb{0}, \\
\mathbb{3} &\cong \mathbb{2} \star \mathbb{0} \\
&\cong (\mathbb{1} \star \mathbb{0}) \star \mathbb{0} \\
&\cong ((\mathbb{0} \star \mathbb{0}) \star \mathbb{0}) \star \mathbb{0}, \\
\mathbb{4} &\cong \mathbb{3} \star \mathbb{0} \\
&\cong (\mathbb{2} \star \mathbb{0}) \star \mathbb{0} \\
&\cong ((\mathbb{1} \star \mathbb{0}) \star \mathbb{0}) \star \mathbb{0} \\
&\cong (((\mathbb{0} \star \mathbb{0}) \star \mathbb{0}) \star \mathbb{0}) \star \mathbb{0},
\end{aligned}
$$

and so on.

Example 1.2.6 ► More Examples of Categories

Here we list some of the other categories appearing throughout this work.

1. The category Sets_* of pointed sets of Pointed Sets, Definition 1.3.1.
2. The category Rel of sets and relations of Relations, Definition 2.1.1.
3. The category $\mathsf{Span}(A, B)$ of spans from a set A to a set B of **??**, **??**.
4. The category $\mathsf{ISets}(K)$ of K-indexed sets of **??**, **??**.
5. The category ISets of indexed sets of **??**, **??**.
6. The category $\mathsf{FibSets}(K)$ of K-fibred sets of **??**, **??**.
7. The category $\mathsf{FibSets}$ of fibred sets of **??**, **??**.
8. Categories of functors $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$ as in Definition 9.1.1.
9. The category of categories Cats of Definition 9.2.1.
10. The category of groupoids Grpd of Definition 9.4.1.

1.3 Posetal Categories

Definition 1.3.1 ▶ Posetal Categories

Let $(X, \preceq_X)$ be a poset.

1. The **posetal category associated to** $(X, \preceq_X)$ is the category X_{pos} where
   - *Objects.* We have
     $$\mathrm{Obj}(X_{\mathsf{pos}}) \overset{\text{def}}{=} X.$$
   - *Morphisms.* For each $a, b \in \mathrm{Obj}(X_{\mathsf{pos}})$, we have
     $$\mathrm{Hom}_{X_{\mathsf{pos}}}(a, b) \overset{\text{def}}{=} \begin{cases} \mathrm{pt} & \text{if } a \preceq_X b, \\ \emptyset & \text{otherwise.} \end{cases}$$
   - *Identities.* For each $a \in \mathrm{Obj}(X_{\mathsf{pos}})$, the unit map
     $$\mathbb{1}^{X_{\mathsf{pos}}}_a \colon \mathrm{pt} \to \mathrm{Hom}_{X_{\mathsf{pos}}}(a, a)$$
     of X_{pos} at a is given by the identity map.
   - *Composition.* For each $a, b, c \in \mathrm{Obj}(X_{\mathsf{pos}})$, the composition map
     $$\circ^{X_{\mathsf{pos}}}_{a,b,c} \colon \mathrm{Hom}_{X_{\mathsf{pos}}}(b, c) \times \mathrm{Hom}_{X_{\mathsf{pos}}}(a, b) \to \mathrm{Hom}_{X_{\mathsf{pos}}}(a, c)$$
     of X_{pos} at (a, b, c) is defined as either the inclusion $\emptyset \hookrightarrow \mathrm{pt}$ or the identity map of pt, depending on whether we have $a \preceq_X b$, $b \preceq_X c$, and $a \preceq_X c$.
2. A category C is **posetal**[1] if C is equivalent to X_{pos} for some poset $(X, \preceq_X)$.

[1] *Further Terminology:* Also called a **thin** category or a $(0, 1)$**-category**.

Proposition 1.3.2 ▶ Properties of Posetal Categories

Let $(X, \preceq_X)$ be a poset and let C be a category.

1. *Functoriality.* The assignment $(X, \preceq_X) \mapsto X_{\mathsf{pos}}$ defines a functor
   $$(-)_{\mathsf{pos}} \colon \mathsf{Pos} \to \mathsf{Cats}.$$
2. *Fully Faithfulness.* The functor $(-)_{\mathsf{pos}}$ of Item 1 is fully faithful.

3. *Characterisations.* The following conditions are equivalent:
   (a) The category $\mathcal{C}$ is posetal.
   (b) For each $A, B \in \mathrm{Obj}(\mathcal{C})$ and each $f, g \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, we have $f = g$.

Proof 1.3.3 ► Proof of Proposition 1.3.2

Item 1: Functoriality

Omitted.

Item 2: Fully Faithfulness

Omitted.

Item 3: Characterisations

Clear.

1.4 Subcategories

Let $\mathcal{C}$ be a category.

Definition 1.4.1 ► Subcategories

A **subcategory** of $\mathcal{C}$ is a category $\mathcal{A}$ satisfying the following conditions:

1. *Objects.* We have $\mathrm{Obj}(\mathcal{A}) \subset \mathrm{Obj}(\mathcal{C})$.
2. *Morphisms.* For each $A, B \in \mathrm{Obj}(\mathcal{A})$, we have
$$\mathrm{Hom}_{\mathcal{A}}(A, B) \subset \mathrm{Hom}_{\mathcal{C}}(A, B).$$
3. *Identities.* For each $A \in \mathrm{Obj}(\mathcal{A})$, we have
$$\mathbb{1}_A^{\mathcal{A}} = \mathbb{1}_A^{\mathcal{C}}.$$
4. *Composition.* For each $A, B, C \in \mathrm{Obj}(\mathcal{A})$, we have
$$\circ_{A,B,C}^{\mathcal{A}} = \circ_{A,B,C}^{\mathcal{C}}.$$

Definition 1.4.2 ► Full Subcategories

A subcategory $\mathcal{A}$ of C is **full** if the canonical inclusion functor $\mathcal{A} \longrightarrow C$ is full, i.e. if, for each $A, B \in \mathrm{Obj}(\mathcal{A})$, the inclusion

$$\iota_{A,B} \colon \mathrm{Hom}_{\mathcal{A}}(A, B) \hookrightarrow \mathrm{Hom}_{C}(A, B)$$

is surjective (and thus bijective).

Definition 1.4.3 ► Strictly Full Subcategories

A subcategory $\mathcal{A}$ of a category C is **strictly full** if it satisfies the following conditions:

1. *Fullness.* The subcategory $\mathcal{A}$ is full.
2. *Closedness Under Isomorphisms.* The class $\mathrm{Obj}(\mathcal{A})$ is closed under isomorphisms.[1]

[1] That is, given $A \in \mathrm{Obj}(\mathcal{A})$ and $C \in \mathrm{Obj}(C)$, if $C \cong A$, then $C \in \mathrm{Obj}(\mathcal{A})$.

Definition 1.4.4 ► Wide Subcategories

A subcategory $\mathcal{A}$ of C is **wide**[1] if $\mathrm{Obj}(\mathcal{A}) = \mathrm{Obj}(C)$.

[1] *Further Terminology:* Also called **lluf**.

1.5 Skeletons of Categories

Definition 1.5.1 ► Skeletons of Categories

A[1] **skeleton** of a category C is a full subcategory $\mathrm{Sk}(C)$ with one object from each isomorphism class of objects of C.

[1] Due to Item 3 of Proposition 1.5.3, we often refer to any such full subcategory $\mathrm{Sk}(C)$ of C as *the* skeleton of C.

Definition 1.5.2 ► Skeletal Categories

A category C is **skeletal** if $C \cong \mathrm{Sk}(C)$.[1]

[1] That is, C is **skeletal** if isomorphic objects of C are equal.

Proposition 1.5.3 ▸ Properties of Skeletons of Categories

Let $\mathcal{C}$ be a category.

1. *Existence.* Assuming the axiom of choice, $\mathsf{Sk}(\mathcal{C})$ always exists.
2. *Pseudofunctoriality.* The assignment $\mathcal{C} \mapsto \mathsf{Sk}(\mathcal{C})$ defines a pseudofunctor
$$\mathsf{Sk}\colon \mathsf{Cats}_2 \to \mathsf{Cats}_2.$$
3. *Uniqueness Up to Equivalence.* Any two skeletons of $\mathcal{C}$ are equivalent.
4. *Inclusions of Skeletons Are Equivalences.* The inclusion
$$\iota_{\mathcal{C}}\colon \mathsf{Sk}(\mathcal{C}) \hookrightarrow \mathcal{C}$$
of a skeleton of $\mathcal{C}$ into $\mathcal{C}$ is an equivalence of categories.

Proof 1.5.4 ▸ Proof of Proposition 1.5.3

Item 1: Existence

See [nLab23, Section “Existence of Skeletons of Categories”].

Item 2: Pseudofunctoriality

See [nLab23, Section “Skeletons as an Endo-Pseudofunctor on $\mathfrak{Cat}$”].

Item 3: Uniqueness Up to Equivalence

Clear.

Item 4: Inclusions of Skeletons Are Equivalences

Clear. ∎

1.6 Precomposition and Postcomposition

Let $\mathcal{C}$ be a category and let $A, B, C \in \mathrm{Obj}(\mathcal{C})$.

Definition 1.6.1 ▸ Precomposition and Postcomposition Functions

Let $f\colon A \to B$ and $g\colon B \to C$ be morphisms of $\mathcal{C}$.

1. The **precomposition function associated to** f is the function

$$f^*\colon \mathrm{Hom}_{\mathcal{C}}(B,C) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(A,C)$$

   defined by

$$f^*(\phi) \stackrel{\text{def}}{=} \phi \circ f$$

   for each $\phi \in \mathrm{Hom}_{\mathcal{C}}(B,C)$.

2. The **postcomposition function associated to** g is the function

$$g_*\colon \mathrm{Hom}_{\mathcal{C}}(A,B) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(A,C)$$

   defined by

$$g_*(\phi) \stackrel{\text{def}}{=} g \circ \phi$$

   for each $\phi \in \mathrm{Hom}_{\mathcal{C}}(A,B)$.

Proposition 1.6.2 ► Properties of Pre/Postcomposition

Let $A, B, C, D \in \mathrm{Obj}(\mathcal{C})$ and let $f\colon A \longrightarrow B$ and $g\colon B \longrightarrow C$ be morphisms of $\mathcal{C}$.

1. *Interaction Between Precomposition and Postcomposition.* We have

$$g_* \circ f^* = f^* \circ g_*, \qquad \begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(B,C) \arrow[r, "g_*"] \arrow[d, "f^*"'] & \mathrm{Hom}_{\mathcal{C}}(B,D) \arrow[d, "f^*"] \\ \mathrm{Hom}_{\mathcal{C}}(A,C) \arrow[r, "g_*"'] & \mathrm{Hom}_{\mathcal{C}}(A,D). \end{tikzcd}$$

2. *Interaction With Composition I.* We have

$$(g \circ f)^* = f^* \circ g^*, \qquad \begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(X,A) \arrow[r, "f_*"] \arrow[dr, "(g\circ f)_*"'] & \mathrm{Hom}_{\mathcal{C}}(X,B) \arrow[d, "g_*"] \\ & \mathrm{Hom}_{\mathcal{C}}(X,C), \end{tikzcd}$$

$$(g \circ f)_* = g_* \circ f_*, \qquad \begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(C,X) \arrow[r, "g^*"] \arrow[dr, "(g\circ f)^*"'] & \mathrm{Hom}_{\mathcal{C}}(B,X) \arrow[d, "f^*"] \\ & \mathrm{Hom}_{\mathcal{C}}(A,X). \end{tikzcd}$$

3. *Interaction With Composition II.* We have

$$\begin{tikzcd} \mathrm{pt} \arrow[r, "{[f]}"] \arrow[dr, "{[g\circ f]}"'] & \mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[d, "g_*"] \\ & \mathrm{Hom}_{\mathcal{C}}(A,C) \end{tikzcd} \qquad \begin{aligned} [g \circ f] &= g_* \circ [f], \\ [g \circ f] &= f^* \circ [g], \end{aligned} \qquad \begin{tikzcd} \mathrm{pt} \arrow[r, "{[g]}"] \arrow[dr, "{[g\circ f]}"'] & \mathrm{Hom}_{\mathcal{C}}(B,C) \arrow[d, "f^*"] \\ & \mathrm{Hom}_{\mathcal{C}}(A,C). \end{tikzcd}$$

4. *Interaction With Composition III.* We have

$$f^* \circ \circ^{\mathcal{C}}_{A,B,C} = \circ^{\mathcal{C}}_{X,B,C} \circ (f^* \times \mathrm{id}), \qquad \begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(B,C) \times \mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[r, "\circ^{\mathcal{C}}_{A,B,C}"] \arrow[d, "\mathrm{id}\times f^*"'] & \mathrm{Hom}_{\mathcal{C}}(A,C) \arrow[d, "f^*"] \\ \mathrm{Hom}_{\mathcal{C}}(B,C) \times \mathrm{Hom}_{\mathcal{C}}(X,B) \arrow[r, "\circ^{\mathcal{C}}_{X,B,C}"'] & \mathrm{Hom}_{\mathcal{C}}(X,C), \end{tikzcd}$$

$$g_* \circ \circ^{\mathcal{C}}_{A,B,C} = \circ^{\mathcal{C}}_{A,B,D} \circ (\mathrm{id} \times g_*), \qquad \begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(B,C) \times \mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[r, "\circ^{\mathcal{C}}_{A,B,C}"] \arrow[d, "g_*\times \mathrm{id}"'] & \mathrm{Hom}_{\mathcal{C}}(A,C) \arrow[d, "g^*"] \\ \mathrm{Hom}_{\mathcal{C}}(B,D) \times \mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[r, "\circ^{\mathcal{C}}_{A,B,D}"'] & \mathrm{Hom}_{\mathcal{C}}(A,D). \end{tikzcd}$$

5. *Interaction With Identities.* We have

$$
\begin{aligned}
(\mathrm{id}_A)^* &= \mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(A,B)},\\
(\mathrm{id}_B)_* &= \mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(A,B)}.
\end{aligned}
$$

PROOF 1.6.3 ► PROOF OF PROPOSITION 1.6.2

Item 1: Interaction Between Precomposition and Postcomposition

Clear.

Item 2: Interaction With Composition I

Clear.

Item 3: Interaction With Composition II

Clear.

Item 4: Interaction With Composition III

Clear.

Item 5: Interaction With Identities

Clear. □

2 The Quadruple Adjunction With Sets

2.1 Statement

Let $\mathcal{C}$ be a category.

PROPOSITION 2.1.1 ► THE QUADRUPLE ADJUNCTION BETWEEN Sets AND Cats

We have a quadruple adjunction

$$
(\pi_0 \dashv (-)_{\mathsf{disc}} \dashv \mathsf{Obj} \dashv (-)_{\mathsf{indisc}})\colon
\begin{tikzcd}
\mathsf{Sets} \arrow[r, bend left=20, "(-)_{\mathsf{disc}}" description] \arrow[r, bend right=70, "(-)_{\mathsf{indisc}}"'] \arrow[r, phantom, shift left=9, "\perp"] \arrow[r, phantom, "\perp"] \arrow[r, phantom, shift right=9, "\perp"] & \mathsf{Cats}, \arrow[l, bend right=70, "\pi_0"'] \arrow[l, bend left=20, "\mathsf{Obj}" description]
\end{tikzcd}
$$

witnessed by bijections of sets

$$\begin{aligned} \mathrm{Hom}_{\mathsf{Sets}}(\pi_0(\mathcal{C}), X) &\cong \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, X_{\mathsf{disc}}), \\ \mathrm{Hom}_{\mathsf{Cats}}(X_{\mathsf{disc}}, \mathcal{C}) &\cong \mathrm{Hom}_{\mathsf{Sets}}(X, \mathrm{Obj}(\mathcal{C})), \\ \mathrm{Hom}_{\mathsf{Sets}}(\mathrm{Obj}(\mathcal{C}), X) &\cong \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, X_{\mathsf{indisc}}), \end{aligned}$$

natural in $\mathcal{C} \in \mathrm{Obj}(\mathsf{Cats})$ and $X \in \mathrm{Obj}(\mathsf{Sets})$, where

- The functor
$$\pi_0 \colon \mathsf{Cats} \longrightarrow \mathsf{Sets},$$
the **connected components functor**, is the functor sending a category to its set of connected components of Definition 2.2.2.
- The functor
$$(-)_{\mathsf{disc}} \colon \mathsf{Sets} \longrightarrow \mathsf{Cats},$$
the **discrete category functor**, is the functor sending a set to its associated discrete category of Item 1.
- The functor
$$\mathrm{Obj} \colon \mathsf{Cats} \longrightarrow \mathsf{Sets},$$
the **object functor**, is the functor sending a category to its set of objects.
- The functor
$$(-)_{\mathsf{indisc}} \colon \mathsf{Sets} \longrightarrow \mathsf{Cats},$$
the **indiscrete category functor**, is the functor sending a set to its associated indiscrete category of Item 1.

PROOF 2.1.2 ► PROOF OF PROPOSITION 2.1.1

Omitted.

2.2 Connected Components and Connected Categories

2.2.1 Connected Components of Categories

Let $\mathcal{C}$ be a category.

Definition 2.2.1 ▶ Connected Components of Categories

A **connected component** of $\mathcal{C}$ is a full subcategory $\mathcal{I}$ of $\mathcal{C}$ satisfying the following conditions:[1]

1. *Non-Emptiness.* We have $\mathrm{Obj}(\mathcal{I}) \neq \emptyset$.
2. *Connectedness.* There exists a zigzag of arrows between any two objects of $\mathcal{I}$.

[1] In other words, a **connected component** of $\mathcal{C}$ is an element of the set $\mathrm{Obj}(\mathcal{C})/\sim$ with $\sim$ the equivalence relation generated by the relation $\sim'$ obtained by declaring $A \sim' B$ iff there exists a morphism of $\mathcal{C}$ from A to B.

2.2.2 Sets of Connected Components of Categories

Let $\mathcal{C}$ be a category.

Definition 2.2.2 ▶ Sets of Connected Components of Categories

The **set of connected components of** $\mathcal{C}$ is the set $\pi_0(\mathcal{C})$ whose elements are the connected components of $\mathcal{C}$.

Proposition 2.2.3 ▶ Properties of Sets of Connected Components

Let $\mathcal{C}$ be a category.

1. *Functoriality.* The assignment $\mathcal{C} \mapsto \pi_0(\mathcal{C})$ defines a functor
$$\pi_0 \colon \mathsf{Cats} \to \mathsf{Sets}.$$
2. *Adjointness.* We have a quadruple adjunction
$$(\pi_0 \dashv (-)_{\mathsf{disc}} \dashv \mathsf{Obj} \dashv (-)_{\mathsf{indisc}}) \colon \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left=20, "(-)_{\mathsf{disc}}"{description}] \arrow[r, bend right=60, "(-)_{\mathsf{indisc}}"'] & \mathsf{Cats}. \arrow[l, bend right=60, "\pi_0"'] \arrow[l, bend left=20, "\mathsf{Obj}"{description}] \end{tikzcd}$$
3. *Interaction With Groupoids.* If $\mathcal{C}$ is a groupoid, then we have an isomorphism of categories
$$\pi_0(\mathcal{C}) \cong \mathsf{K}(\mathcal{C}),$$

where $\mathsf{K}(\mathcal{C})$ is the set of isomorphism classes of $\mathcal{C}$ of **??**.

4. *Preservation of Colimits.* The functor π_0 of Item 1 preserves colimits. In particular, we have bijections of sets

$$\begin{aligned}\pi_0(\mathcal{C} \coprod \mathcal{D}) &\cong \pi_0(\mathcal{C}) \coprod \pi_0(\mathcal{D}),\\ \pi_0(\mathcal{C} \textstyle\coprod_{\mathcal{E}} \mathcal{D}) &\cong \pi_0(\mathcal{C}) \textstyle\coprod_{\pi_0(\mathcal{E})} \pi_0(\mathcal{D}),\\ \pi_0(\mathsf{CoEq}(\mathcal{C} \underset{G}{\overset{F}{\rightrightarrows}} \mathcal{D})) &\cong \mathsf{CoEq}(\pi_0(\mathcal{C}) \underset{\pi_0(G)}{\overset{\pi_0(F)}{\rightrightarrows}} \pi_0(\mathcal{D})),\end{aligned}$$

natural in $\mathcal{C}, \mathcal{D}, \mathcal{E} \in \mathrm{Obj}(\mathsf{Cats})$.

5. *Symmetric Strong Monoidality With Respect to Coproducts.* The connected components functor of Item 1 has a symmetric strong monoidal structure

$$(\pi_0, \pi_0^{\coprod}, \pi_{0|\mathbb{1}}^{\coprod}) \colon (\mathsf{Cats}, \textstyle\coprod, \emptyset_{\mathsf{cat}}) \longrightarrow (\mathsf{Sets}, \textstyle\coprod, \emptyset),$$

being equipped with isomorphisms

$$\begin{aligned}\pi_{0|\mathcal{C},\mathcal{D}}^{\coprod} \colon \pi_0(\mathcal{C}) \textstyle\coprod \pi_0(\mathcal{D}) &\xrightarrow{\cong} \pi_0(\mathcal{C} \textstyle\coprod \mathcal{D}),\\ \pi_{0|\mathbb{1}}^{\coprod} \colon \emptyset &\xrightarrow{\cong} \pi_0(\emptyset_{\mathsf{cat}}),\end{aligned}$$

natural in $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$.

6. *Symmetric Strong Monoidality With Respect to Products.* The connected components functor of Item 1 has a symmetric strong monoidal structure

$$(\pi_0, \pi_0^{\times}, \pi_{0|\mathbb{1}}^{\times}) \colon (\mathsf{Cats}, \times, \mathrm{pt}) \longrightarrow (\mathsf{Sets}, \times, \mathrm{pt}),$$

being equipped with isomorphisms

$$\begin{aligned}\pi_{0|\mathcal{C},\mathcal{D}}^{\times} \colon \pi_0(\mathcal{C}) \times \pi_0(\mathcal{D}) &\xrightarrow{\cong} \pi_0(\mathcal{C} \times \mathcal{D}),\\ \pi_{0|\mathbb{1}}^{\times} \colon \mathrm{pt} &\xrightarrow{\cong} \pi_0(\mathrm{pt}),\end{aligned}$$

natural in $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$.

Proof 2.2.4 ▶ Proof of Proposition 2.2.3

Item 1: Functoriality

Clear.

Item 2: Adjointness

This is proved in Proposition 2.1.1.

Item 3: Interaction With Groupoids

Clear.

Item 4: Preservation of Colimits

This follows from Item 2 and **??** of **??**.

Item 5: Symmetric Strong Monoidality With Respect to Coproducts

Clear.

Item 6: Symmetric Strong Monoidality With Respect to Products

Clear.

2.2.3 Connected Categories

Definition 2.2.5 ▶ Connected Categories

A category C is **connected** if $\pi_0(C) \cong \mathrm{pt}$.[1,2]

[1] *Further Terminology:* A category is **disconnected** if it is not connected.

[2] *Example:* A groupoid is connected iff any two of its objects are isomorphic.

2.3 Discrete Categories

Definition 2.3.1 ▶ Discrete Categories

Let X be a set.

1. The **discrete category on** X is the category X_{disc} where
   - *Objects.* We have

$$\mathrm{Obj}(X_{\mathsf{disc}}) \overset{\mathrm{def}}{=} X.$$

- *Morphisms.* For each $A, B \in \mathrm{Obj}(X_{\mathsf{disc}})$, we have

$$\mathrm{Hom}_{X_{\mathsf{disc}}}(A, B) \overset{\text{def}}{=} \begin{cases} \mathrm{id}_A & \text{if } A = B, \\ \emptyset & \text{if } A \neq B. \end{cases}$$

- *Identities.* For each $A \in \mathrm{Obj}(X_{\mathsf{disc}})$, the unit map

$$\mathbb{1}^{X_{\mathsf{disc}}}_A \colon \mathrm{pt} \longrightarrow \mathrm{Hom}_{X_{\mathsf{disc}}}(A, A)$$

  of X_{disc} at A is defined by

$$\mathrm{id}^{X_{\mathsf{disc}}}_A \overset{\text{def}}{=} \mathrm{id}_A.$$

- *Composition.* For each $A, B, C \in \mathrm{Obj}(X_{\mathsf{disc}})$, the composition map

$$\circ^{X_{\mathsf{disc}}}_{A,B,C} \colon \mathrm{Hom}_{X_{\mathsf{disc}}}(B, C) \times \mathrm{Hom}_{X_{\mathsf{disc}}}(A, B) \longrightarrow \mathrm{Hom}_{X_{\mathsf{disc}}}(A, C)$$

  of X_{disc} at (A, B, C) is defined by

$$\mathrm{id}_A \circ \mathrm{id}_A \overset{\text{def}}{=} \mathrm{id}_A.$$

2. A category $\mathcal{C}$ is **discrete** if it is equivalent to X_{disc} for some set X.

Proposition 2.3.2 ▶ Properties of Discrete Categories on Sets

Let X be a set.

1. *Functoriality.* The assignment $X \mapsto X_{\mathsf{disc}}$ defines a functor

$$(-)_{\mathsf{disc}} \colon \mathsf{Sets} \longrightarrow \mathsf{Cats}.$$

2. *Adjointness.* We have a quadruple adjunction

$$(\pi_0 \dashv (-)_{\mathsf{disc}} \dashv \mathrm{Obj} \dashv (-)_{\mathsf{indisc}})\colon \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left=20, "(-)_{\mathsf{disc}}"{description}] \arrow[r, bend right=60, "(-)_{\mathsf{indisc}}"'] & \mathsf{Cats} \arrow[l, bend right=60, "\pi_0"'] \arrow[l, bend left=20, "\mathrm{Obj}"{description}] \end{tikzcd}.$$

3. *Symmetric Strong Monoidality With Respect to Coproducts.* The functor of Item 1 has a symmetric strong monoidal structure

$$\left((-)_{\mathsf{disc}}, (-)^{\coprod}_{\mathsf{disc}}, (-)^{\coprod}_{\mathsf{disc}|\mathbb{1}}\right) \colon (\mathsf{Sets}, \textstyle\coprod, \emptyset) \to (\mathsf{Cats}, \textstyle\coprod, \emptyset_{\mathsf{cat}}),$$

being equipped with isomorphisms

$$(-)^{\coprod}_{\mathsf{disc}|X,Y} \colon X_{\mathsf{disc}} \textstyle\coprod Y_{\mathsf{disc}} \xrightarrow{\cong} (X \textstyle\coprod Y)_{\mathsf{disc}},$$
$$(-)^{\coprod}_{\mathsf{disc}|\mathbb{1}} \colon \emptyset_{\mathsf{cat}} \xrightarrow{\cong} \emptyset_{\mathsf{disc}},$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

4. *Symmetric Strong Monoidality With Respect to Products.* The functor of Item 1 has a symmetric strong monoidal structure

$$\left((-)_{\mathsf{disc}}, (-)^{\times}_{\mathsf{disc}}, (-)^{\times}_{\mathsf{disc}|\mathbb{1}}\right) \colon (\mathsf{Sets}, \times, \mathrm{pt}) \to (\mathsf{Cats}, \times, \mathrm{pt}),$$

being equipped with isomorphisms

$$(-)^{\times}_{\mathsf{disc}|X,Y} \colon X_{\mathsf{disc}} \times Y_{\mathsf{disc}} \xrightarrow{\cong} (X \times Y)_{\mathsf{disc}},$$
$$(-)^{\times}_{\mathsf{disc}|\mathbb{1}} \colon \mathrm{pt} \xrightarrow{\cong} \mathrm{pt}_{\mathsf{disc}},$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

Proof 2.3.3 ► Proof of Proposition 2.3.2

Item 1: Functoriality

Clear.

Item 2: Adjointness

This is proved in Proposition 2.1.1.

Item 3: Symmetric Strong Monoidality With Respect to Coproducts

Clear.

Item 4: Symmetric Strong Monoidality With Respect to Products

Clear. □

2.4 Indiscrete Categories

Definition 2.4.1 ▶ Indiscrete Categories

Let X be a set.

1. The **indiscrete category on** X[1] is the category X_{indisc} where
   - *Objects.* We have
   $$\mathrm{Obj}(X_{\mathsf{indisc}}) \overset{\text{def}}{=} X.$$
   - *Morphisms.* For each $A, B \in \mathrm{Obj}(X_{\mathsf{indisc}})$, we have
   $$\begin{aligned}\mathrm{Hom}_{X_{\mathsf{disc}}}(A, B) &\overset{\text{def}}{=} \{[A] \longrightarrow [B]\}\\ &\cong \mathrm{pt}.\end{aligned}$$
   - *Identities.* For each $A \in \mathrm{Obj}(X_{\mathsf{indisc}})$, the unit map
   $$\mathbb{1}^{X_{\mathsf{indisc}}}_{A} \colon \mathrm{pt} \longrightarrow \mathrm{Hom}_{X_{\mathsf{indisc}}}(A, A)$$
   of X_{indisc} at A is defined by
   $$\mathrm{id}^{X_{\mathsf{indisc}}}_{A} \overset{\text{def}}{=} \{[A] \longrightarrow [A]\}.$$
   - *Composition.* For each $A, B, C \in \mathrm{Obj}(X_{\mathsf{indisc}})$, the composition map
   $$\circ^{X_{\mathsf{indisc}}}_{A,B,C} \colon \mathrm{Hom}_{X_{\mathsf{indisc}}}(B, C) \times \mathrm{Hom}_{X_{\mathsf{indisc}}}(A, B) \longrightarrow \mathrm{Hom}_{X_{\mathsf{indisc}}}(A, C)$$
   of X_{disc} at (A, B, C) is defined by
   $$([B] \longrightarrow [C]) \circ ([A] \longrightarrow [B]) \overset{\text{def}}{=} ([A] \longrightarrow [C]).$$
2. A category $\mathcal{C}$ is **indiscrete** if it is equivalent to X_{indisc} for some set X.

[1] *Further Terminology:* Sometimes called the **chaotic category on** X.

Proposition 2.4.2 ▶ Properties of Indiscrete Categories on Sets

Let X be a set.

1. *Functoriality.* The assignment $X \mapsto X_{\mathrm{indisc}}$ defines a functor

$$(-)_{\mathrm{indisc}} \colon \mathsf{Sets} \to \mathsf{Cats}.$$

2. *Adjointness.* We have a quadruple adjunction

$$(\pi_0 \dashv (-)_{\mathrm{disc}} \dashv \mathrm{Obj} \dashv (-)_{\mathrm{indisc}}) \colon \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left=20, "(-)_{\mathrm{disc}}"] \arrow[r, bend right=60, "(-)_{\mathrm{indisc}}"'] & \mathsf{Cats}. \arrow[l, bend right=60, "\pi_0"'] \arrow[l, bend left=20, "\mathrm{Obj}"] \end{tikzcd}$$

3. *Symmetric Strong Monoidality With Respect to Products.* The functor of Item 1 has a symmetric strong monoidal structure

$$\left((-)_{\mathrm{indisc}}, (-)^{\times}_{\mathrm{indisc}}, (-)^{\times}_{\mathrm{indisc}|\mathbb{1}}\right) \colon (\mathsf{Sets}, \times, \mathrm{pt}) \to (\mathsf{Cats}, \times, \mathrm{pt}),$$

being equipped with isomorphisms

$$(-)^{\times}_{\mathrm{indisc}|X,Y} \colon X_{\mathrm{indisc}} \times Y_{\mathrm{indisc}} \xrightarrow{\cong} (X \times Y)_{\mathrm{indisc}},$$
$$(-)^{\times}_{\mathrm{indisc}|\mathbb{1}} \colon \mathrm{pt} \xrightarrow{\cong} \mathrm{pt}_{\mathrm{indisc}},$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

PROOF 2.4.3 ► PROOF OF PROPOSITION 2.4.2

Item 1: Functoriality

Clear.

Item 2: Adjointness

This is proved in Proposition 2.1.1.

Item 3: Symmetric Strong Monoidality With Respect to Products

Clear.

3 Groupoids

3.1 Foundations

Let $\mathcal{C}$ be a category.

Definition 3.1.1 ► Isomorphisms

A morphism $f\colon A \to B$ of $\mathcal{C}$ is an **isomorphism** if there exists a morphism $f^{-1}\colon B \to A$ of $\mathcal{C}$ such that

$$\begin{aligned} f \circ f^{-1} &= \mathrm{id}_B, \\ f^{-1} \circ f &= \mathrm{id}_A. \end{aligned}$$

Notation 3.1.2 ► The Set of Isomorphisms Between Two Objects in a Category

We write $\mathrm{Iso}_{\mathcal{C}}(A, B)$ for the set of all isomorphisms in $\mathcal{C}$ from A to B.

Definition 3.1.3 ► Groupoids

A **groupoid** is a category in which every morphism is an isomorphism.

3.2 The Groupoid Completion of a Category

Let $\mathcal{C}$ be a category.

Definition 3.2.1 ► The Groupoid Completion of a Category

The **groupoid completion of** $\mathcal{C}$[1] is the pair $(\mathrm{K}_0(\mathcal{C}), \iota_{\mathcal{C}})$ consisting of

- A groupoid $\mathrm{K}_0(\mathcal{C})$;
- A functor $\iota_{\mathcal{C}}\colon \mathcal{C} \to \mathrm{K}_0(\mathcal{C})$;

satisfying the following universal property:[2]

(**UP**) Given another such pair $(\mathcal{G}, i)$, there exists a unique functor $\mathrm{K}_0(\mathcal{C}) \xrightarrow{\exists!} \mathcal{G}$

making the diagram

$$
\begin{tikzcd}
 & \mathrm{K}_0(\mathcal{C}) \arrow[d, dashed, "\exists!"] \\
\mathcal{C} \arrow[ur, "\iota_{\mathcal{C}}"] \arrow[r, "i"'] & \mathcal{G}
\end{tikzcd}
$$

commute.

[1] *Further Terminology:* Also called the **Grothendieck groupoid of $\mathcal{C}$** or the **Grothendieck groupoid completion of $\mathcal{C}$**.

See Item 5 of Proposition 3.2.4 for an explicit construction.

Construction 3.2.2 ► Construction of the Groupoid Completion of a Category

Concretely, the groupoid completion of $\mathcal{C}$ is the Gabriel–Zisman localisation $\mathrm{Mor}(\mathcal{C})^{-1}\mathcal{C}$ of $\mathcal{C}$ at the set $\mathrm{Mor}(\mathcal{C})$ of all morphisms of $\mathcal{C}$; see **??**, **??**.
(To be expanded upon later on.)

Proof 3.2.3 ► Proof of Construction 3.2.2

Omitted.

Proposition 3.2.4 ► Properties of Groupoid Completion

Let $\mathcal{C}$ be a category.

1. *Functoriality.* The assignment $\mathcal{C} \mapsto \mathrm{K}_0(\mathcal{C})$ defines a functor
$$\mathrm{K}_0\colon \mathsf{Cats} \longrightarrow \mathsf{Grpd}.$$
2. *2-Functoriality.* The assignment $\mathcal{C} \mapsto \mathrm{K}_0(\mathcal{C})$ defines a 2-functor
$$\mathrm{K}_0\colon \mathsf{Cats}_2 \longrightarrow \mathsf{Grpd}_2.$$
3. *Adjointness.* We have an adjunction
$$(\mathrm{K}_0 \dashv \iota)\colon \quad \mathsf{Cats} \underset{\iota}{\overset{\mathrm{K}_0}{\underset{\perp}{\rightleftarrows}}} \mathsf{Grpd},$$

witnessed by a bijection of sets

$$\mathrm{Hom}_{\mathsf{Grpd}}(\mathrm{K}_0(\mathcal{C}), \mathcal{G}) \cong \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, \mathcal{G}),$$

natural in $\mathcal{C} \in \mathrm{Obj}(\mathsf{Cats})$ and $\mathcal{G} \in \mathrm{Obj}(\mathsf{Grpd})$, forming, together with the functor Core of Item 1 of Proposition 3.3.5, a triple adjunction

$$(\mathrm{K}_0 \dashv \iota \dashv \mathrm{Core}): \quad \begin{tikzcd} \mathsf{Cats} \arrow[r, bend left, "\mathrm{K}_0"] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, phantom, shift right=2, "\perp"] \arrow[r, bend right, "\mathrm{Core}"'] & \mathsf{Grpd} \arrow[l, hook', "\iota"] \end{tikzcd},$$

witnessed by bijections of sets

$$\begin{aligned} \mathrm{Hom}_{\mathsf{Grpd}}(\mathrm{K}_0(\mathcal{C}), \mathcal{G}) &\cong \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, \mathcal{G}), \\ \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{G}, \mathcal{D}) &\cong \mathrm{Hom}_{\mathsf{Grpd}}(\mathcal{G}, \mathrm{Core}(\mathcal{D})), \end{aligned}$$

natural in $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$ and $\mathcal{G} \in \mathrm{Obj}(\mathsf{Grpd})$.

4. *2-Adjointness.* We have a 2-adjunction

$$(\mathrm{K}_0 \dashv \iota): \quad \begin{tikzcd} \mathsf{Cats} \arrow[r, shift left, "\mathrm{K}_0"] \arrow[r, phantom, "\perp_2"] & \mathsf{Grpd} \arrow[l, hook', shift left, "\iota"] \end{tikzcd},$$

witnessed by an isomorphism of categories

$$\mathrm{Fun}(\mathrm{K}_0(\mathcal{C}), \mathcal{G}) \cong \mathrm{Fun}(\mathcal{C}, \mathcal{G}),$$

natural in $\mathcal{C} \in \mathrm{Obj}(\mathsf{Cats})$ and $\mathcal{G} \in \mathrm{Obj}(\mathsf{Grpd})$, forming, together with the 2-functor Core of Item 2 of Proposition 3.3.5, a triple 2-adjunction

$$(\mathrm{K}_0 \dashv \iota \dashv \mathrm{Core}): \quad \begin{tikzcd} \mathsf{Cats} \arrow[r, bend left, "\mathrm{K}_0"] \arrow[r, phantom, shift left=2, "\perp_2"] \arrow[r, phantom, shift right=2, "\perp_2"] \arrow[r, bend right, "\mathrm{Core}"'] & \mathsf{Grpd} \arrow[l, hook', "\iota"] \end{tikzcd},$$

witnessed by isomorphisms of categories

$$\begin{aligned} \mathrm{Fun}(\mathrm{K}_0(\mathcal{C}), \mathcal{G}) &\cong \mathrm{Fun}(\mathcal{C}, \mathcal{G}), \\ \mathrm{Fun}(\mathcal{G}, \mathcal{D}) &\cong \mathrm{Fun}(\mathcal{G}, \mathrm{Core}(\mathcal{D})), \end{aligned}$$

natural in $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$ and $\mathcal{G} \in \mathrm{Obj}(\mathsf{Grpd})$.

5. *Interaction With Classifying Spaces.* We have an isomorphism of groupoids

$$\mathsf{K}_0(\mathcal{C}) \cong \Pi_{\leq 1}(|\mathsf{N}_\bullet(\mathcal{C})|),$$

natural in $\mathcal{C} \in \mathrm{Obj}(\mathsf{Cats})$; i.e. the diagram

$$\begin{tikzcd} \mathsf{Cats} \arrow[r, "\mathsf{K}_0"] \arrow[d, "\mathsf{N}_\bullet"'] \arrow[dr, phantom, "\wr"] & \mathsf{Grp} \\ \mathsf{sSets} \arrow[r, "|-|"'] & \mathsf{Top} \arrow[u, "\Pi_{\leq 1}"'] \end{tikzcd}$$

commutes up to natural isomorphism.

6. *Symmetric Strong Monoidality With Respect to Coproducts.* The groupoid completion functor of Item 1 has a symmetric strong monoidal structure

$$(\mathsf{K}_0, \mathsf{K}_0^{\coprod}, \mathsf{K}_{0|\mathbb{1}}^{\coprod}) \colon (\mathsf{Cats}, \coprod, \emptyset_{\mathsf{cat}}) \to (\mathsf{Grpd}, \coprod, \emptyset_{\mathsf{cat}})$$

being equipped with isomorphisms

$$\mathsf{K}_{0|\mathcal{C},\mathcal{D}}^{\coprod} \colon \mathsf{K}_0(\mathcal{C}) \textstyle\coprod \mathsf{K}_0(\mathcal{D}) \xrightarrow{\cong} \mathsf{K}_0(\mathcal{C} \textstyle\coprod \mathcal{D}),$$
$$\mathsf{K}_{0|\mathbb{1}}^{\coprod} \colon \emptyset_{\mathsf{cat}} \xrightarrow{\cong} \mathsf{K}_0(\emptyset_{\mathsf{cat}}),$$

natural in $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$.

7. *Symmetric Strong Monoidality With Respect to Products.* The groupoid completion functor of Item 1 has a symmetric strong monoidal structure

$$(\mathsf{K}_0, \mathsf{K}_0^{\times}, \mathsf{K}_{0|\mathbb{1}}^{\times}) \colon (\mathsf{Cats}, \times, \mathrm{pt}) \to (\mathsf{Grpd}, \times, \mathrm{pt})$$

being equipped with isomorphisms

$$\mathsf{K}_{0|\mathcal{C},\mathcal{D}}^{\times} \colon \mathsf{K}_0(\mathcal{C}) \times \mathsf{K}_0(\mathcal{D}) \xrightarrow{\cong} \mathsf{K}_0(\mathcal{C} \times \mathcal{D}),$$
$$\mathsf{K}_{0|\mathbb{1}}^{\times} \colon \mathrm{pt} \xrightarrow{\cong} \mathsf{K}_0(\mathrm{pt}),$$

natural in $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$.

Proof 3.2.5 ▶ Proof of Proposition 3.2.4

Item 1: Functoriality

Omitted.

Item 2: 2-Functoriality

Omitted.

Item 3: Adjointness

Omitted.

Item 4: 2-Adjointness

Omitted.

Item 5: Interaction With Classifying Spaces

See Corollary 18.33 of `https://web.ma.utexas.edu/users/dafr/M392C-2012/Notes/lecture18.pdf`.

Item 6: Symmetric Strong Monoidality With Respect to Coproducts

Omitted.

Item 7: Symmetric Strong Monoidality With Respect to Products

Omitted.

3.3 The Core of a Category

Let $\mathcal{C}$ be a category.

Definition 3.3.1 ▶ The Core of a Category

The **core** of $\mathcal{C}$ is the pair $(\mathrm{Core}(\mathcal{C}), \iota_{\mathcal{C}})$ consisting of

- A groupoid $\mathrm{Core}(\mathcal{C})$;
- A functor $\iota_{\mathcal{C}} \colon \mathrm{Core}(\mathcal{C}) \hookrightarrow \mathcal{C}$;

satisfying the following universal property:

(**UP**) Given another such pair $(\mathcal{G}, i)$, there exists a unique functor $\mathcal{G} \xrightarrow{\exists!}$

$\mathrm{Core}(\mathcal{C})$ making the diagram

$$\begin{tikzcd} & \mathrm{Core}(\mathcal{C}) \arrow[d, hook, "\iota_{\mathcal{C}}"] \\ \mathcal{G} \arrow[ur, dashed, "\exists!"] \arrow[r, "i"'] & \mathcal{C} \end{tikzcd}$$

commute.

Notation 3.3.2 ► Alternative Notation for the Core of a Category

We also write $\mathcal{C}^{\simeq}$ for $\mathrm{Core}(\mathcal{C})$.

Construction 3.3.3 ► Construction of the Core of a Category

The core of $\mathcal{C}$ is the wide subcategory of $\mathcal{C}$ spanned by the isomorphisms of $\mathcal{C}$, i.e. the category $\mathrm{Core}(\mathcal{C})$ where[1]

1. *Objects.* We have

$$\mathrm{Obj}(\mathrm{Core}(\mathcal{C})) \stackrel{\text{def}}{=} \mathrm{Obj}(\mathcal{C}).$$

2. *Morphisms.* The morphisms of $\mathrm{Core}(\mathcal{C})$ are the isomorphisms of $\mathcal{C}$.

[1] *Slogan:* The groupoid $\mathrm{Core}(\mathcal{C})$ is the maximal subgroupoid of $\mathcal{C}$.

Proof 3.3.4 ► Proof of Construction 3.3.3

This follows from the fact that functors preserve isomorphisms (Item 1 of Proposition 4.1.8).

Proposition 3.3.5 ► Properties of the Core of a Category

Let $\mathcal{C}$ be a category.

1. *Functoriality.* The assignment $\mathcal{C} \mapsto \mathrm{Core}(\mathcal{C})$ defines a functor

$$\mathrm{Core}\colon \mathsf{Cats} \to \mathsf{Grpd}.$$

2. *2-Functoriality.* The assignment $\mathcal{C} \mapsto \mathrm{Core}(\mathcal{C})$ defines a 2-functor

$$\mathrm{Core}\colon \mathsf{Cats}_2 \longrightarrow \mathsf{Grpd}_2.$$

3. *Adjointness.* We have an adjunction

$$(\iota \dashv \mathrm{Core})\colon\quad \begin{tikzcd} \mathsf{Grpd} \arrow[r, bend left, "\iota"] \arrow[r, phantom, "\perp"] & \mathsf{Cats}, \arrow[l, bend left, "\mathrm{Core}"] \end{tikzcd}$$

witnessed by a bijection of sets

$$\mathrm{Hom}_{\mathsf{Cats}}(\mathcal{G}, \mathcal{D}) \cong \mathrm{Hom}_{\mathsf{Grpd}}(\mathcal{G}, \mathrm{Core}(\mathcal{D})),$$

natural in $\mathcal{G} \in \mathrm{Obj}(\mathsf{Grpd})$ and $\mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$, forming, together with the functor K_0 of Item 1 of Proposition 3.2.4, a triple adjunction

$$(\mathrm{K}_0 \dashv \iota \dashv \mathrm{Core})\colon\quad \begin{tikzcd} \mathsf{Cats} \arrow[r, bend left=50, "\mathrm{K}_0"] \arrow[r, bend right=50, "\mathrm{Core}"'] & \mathsf{Grpd}, \arrow[l, "\iota" description] \end{tikzcd}$$

witnessed by bijections of sets

$$\begin{aligned} \mathrm{Hom}_{\mathsf{Grpd}}(\mathrm{K}_0(\mathcal{C}), \mathcal{G}) &\cong \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, \mathcal{G}), \\ \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{G}, \mathcal{D}) &\cong \mathrm{Hom}_{\mathsf{Grpd}}(\mathcal{G}, \mathrm{Core}(\mathcal{D})), \end{aligned}$$

natural in $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$ and $\mathcal{G} \in \mathrm{Obj}(\mathsf{Grpd})$.

4. *2-Adjointness.* We have an adjunction

$$(\iota \dashv \mathrm{Core})\colon\quad \begin{tikzcd} \mathsf{Grpd} \arrow[r, bend left, "\iota"] \arrow[r, phantom, "\perp_2"] & \mathsf{Cats}, \arrow[l, bend left, "\mathrm{Core}"] \end{tikzcd}$$

witnessed by an isomorphism of categories

$$\mathsf{Fun}(\mathcal{G}, \mathcal{D}) \cong \mathsf{Fun}(\mathcal{G}, \mathrm{Core}(\mathcal{D})),$$

natural in $\mathcal{G} \in \mathrm{Obj}(\mathsf{Grpd})$ and $\mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$, forming, together with the 2-functor K_0 of Item 2 of Proposition 3.2.4, a triple 2-adjunction

$$(\mathrm{K}_0 \dashv \iota \dashv \mathrm{Core}): \quad \begin{tikzcd} \mathsf{Cats} \arrow[r, leftarrow, "\iota" description] \arrow[r, bend left=50, leftarrow, "\mathrm{K}_0"] \arrow[r, bend right=50, "\mathrm{Core}"'] & \mathsf{Grpd}, \end{tikzcd}$$

witnessed by isomorphisms of categories

$$\begin{aligned} \mathsf{Fun}(\mathrm{K}_0(\mathcal{C}), \mathcal{G}) &\cong \mathsf{Fun}(\mathcal{C}, \mathcal{G}), \\ \mathsf{Fun}(\mathcal{G}, \mathcal{D}) &\cong \mathsf{Fun}(\mathcal{G}, \mathrm{Core}(\mathcal{D})), \end{aligned}$$

natural in $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$ and $\mathcal{G} \in \mathrm{Obj}(\mathsf{Grpd})$.

5. *Symmetric Strong Monoidality With Respect to Products.* The core functor of Item 1 has a symmetric strong monoidal structure

$$(\mathrm{Core}, \mathrm{Core}^{\times}, \mathrm{Core}^{\times}_{\mathbb{1}}) \colon (\mathsf{Cats}, \times, \mathrm{pt}) \longrightarrow (\mathsf{Grpd}, \times, \mathrm{pt})$$

being equipped with isomorphisms

$$\begin{aligned} \mathrm{Core}^{\times}_{\mathcal{C},\mathcal{D}} &\colon \mathrm{Core}(\mathcal{C}) \times \mathrm{Core}(\mathcal{D}) \xrightarrow{\cong} \mathrm{Core}(\mathcal{C} \times \mathcal{D}), \\ \mathrm{Core}^{\times}_{\mathbb{1}} &\colon \mathrm{pt} \xrightarrow{\cong} \mathrm{Core}(\mathrm{pt}), \end{aligned}$$

natural in $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$.

6. *Symmetric Strong Monoidality With Respect to Coproducts.* The core functor of Item 1 has a symmetric strong monoidal structure

$$(\mathrm{Core}, \mathrm{Core}^{\coprod}, \mathrm{Core}^{\coprod}_{\mathbb{1}}) \colon (\mathsf{Cats}, \coprod, \emptyset_{\mathsf{cat}}) \longrightarrow (\mathsf{Grpd}, \coprod, \emptyset_{\mathsf{cat}})$$

being equipped with isomorphisms

$$\begin{aligned} \mathrm{Core}^{\coprod}_{\mathcal{C},\mathcal{D}} &\colon \mathrm{Core}(\mathcal{C}) \textstyle\coprod \mathrm{Core}(\mathcal{D}) \xrightarrow{\cong} \mathrm{Core}(\mathcal{C} \textstyle\coprod \mathcal{D}), \\ \mathrm{Core}^{\coprod}_{\mathbb{1}} &\colon \emptyset_{\mathsf{cat}} \xrightarrow{\cong} \mathrm{Core}(\emptyset_{\mathsf{cat}}), \end{aligned}$$

natural in $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$.

PROOF 3.3.6 ▶ PROOF OF PROPOSITION 3.3.5

Item 1: Functoriality

Omitted.

Item 2: 2-Functoriality

Omitted.

Item 3: Adjointness

Omitted.

Item 4: 2-Adjointness

Omitted.

Item 5: Symmetric Strong Monoidality With Respect to Products

Omitted.

Item 6: Symmetric Strong Monoidality With Respect to Coproducts

Omitted.

4 Functors

4.1 Foundations

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

DEFINITION 4.1.1 ▶ FUNCTORS

A **functor** $F\colon \mathcal{C} \to \mathcal{D}$ **from** $\mathcal{C}$ **to** $\mathcal{D}$[1] consists of:

1. *Action on Objects.* A map of sets

$$F\colon \mathrm{Obj}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{D}),$$

called the **action on objects of** F.

2. *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathcal{C})$, a map

$$F_{A,B}\colon \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F(A), F(B)),$$

called the **action on morphisms of** F **at** (A, B)[2].

satisfying the following conditions:

1. *Preservation of Identities.* For each $A \in \mathrm{Obj}(\mathcal{C})$, the diagram

$$\begin{tikzcd}
\mathrm{pt} \arrow[d, "\mathbb{1}^{\mathcal{C}}_{A}"'] \arrow[dr, "\mathbb{1}^{\mathcal{D}}_{F(A)}"] & \\
\mathrm{Hom}_{\mathcal{C}}(A, A) \arrow[r, "F_{A,A}"'] & \mathrm{Hom}_{\mathcal{D}}(F(A), F(A))
\end{tikzcd}$$

   commutes, i.e. we have

$$F(\mathrm{id}_A) = \mathrm{id}_{F(A)}.$$

2. *Preservation of Composition.* For each $A, B, C \in \mathrm{Obj}(\mathcal{C})$, the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(B, C) \times \mathrm{Hom}_{\mathcal{C}}(A, B) \arrow[r, "\circ^{\mathcal{C}}_{A,B,C}"] \arrow[d, "F_{B,C} \times F_{A,B}"'] & \mathrm{Hom}_{\mathcal{C}}(A, C) \arrow[d, "F_{A,C}"] \\
\mathrm{Hom}_{\mathcal{D}}(F(B), F(C)) \times \mathrm{Hom}_{\mathcal{D}}(F(A), F(B)) \arrow[r, "\circ^{\mathcal{D}}_{F(A),F(B),F(C)}"'] & \mathrm{Hom}_{\mathcal{D}}(F(A), F(C))
\end{tikzcd}$$

   commutes, i.e. for each composable pair (g, f) of morphisms of $\mathcal{C}$, we have

$$F(g \circ f) = F(g) \circ F(f).$$

[1] *Further Terminology:* Also called a **covariant functor**.
[2] *Further Terminology:* Also called **action on Hom-sets of F at** (A, B).

NOTATION 4.1.2 ► SUBSCRIPT AND SUPERSCRIPT NOTATION FOR FUNCTORS

Let $\mathcal{C}$ and $\mathcal{D}$ be categories, and write $\mathcal{C}^{\mathsf{op}}$ for the opposite category of $\mathcal{C}$ of **??**, **??**.

1. Given a functor

$$F : \mathcal{C} \to \mathcal{D},$$

   we also write F_A for $F(A)$.

2. Given a functor

$$F : \mathcal{C}^{\mathsf{op}} \to \mathcal{D},$$

we also write F^A for $F(A)$.

3. Given a functor
$$F: \mathcal{C} \times \mathcal{C} \to \mathcal{D},$$
we also write $F_{A,B}$ for $F(A, B)$.

4. Given a functor
$$F: \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathcal{D},$$
we also write F^A_B for $F(A, B)$.

We employ a similar notation for morphisms, writing e.g. F_f for $F(f)$ given a functor $F: \mathcal{C} \to \mathcal{D}$.

Notation 4.1.3 ▶ Additional Notation for Functors

Following the notation $[\![x \mapsto f(x)]\!]$ for a function $f: X \to Y$ introduced in Sets, Notation 1.1.2, we will sometimes denote a functor $F: \mathcal{C} \to \mathcal{D}$ by

$$F \overset{\text{def}}{=} [\![A \mapsto F(A)]\!],$$

specially when the action on morphisms of F is clear from its action on objects.

Example 4.1.4 ▶ Identity Functors

The **identity functor** of a category $\mathcal{C}$ is the functor $\mathrm{id}_{\mathcal{C}}: \mathcal{C} \to \mathcal{C}$ where

1. *Action on Objects.* For each $A \in \mathrm{Obj}(\mathcal{C})$, we have
$$\mathrm{id}_{\mathcal{C}}(A) \overset{\text{def}}{=} A.$$

2. *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathcal{C})$, the action on morphisms
$$(\mathrm{id}_{\mathcal{C}})_{A,B}: \mathrm{Hom}_{\mathcal{C}}(A, B) \to \underbrace{\mathrm{Hom}_{\mathcal{C}}(\mathrm{id}_{\mathcal{C}}(A), \mathrm{id}_{\mathcal{C}}(B))}_{\overset{\text{def}}{=} \mathrm{Hom}_{\mathcal{C}}(A,B)}$$
of $\mathrm{id}_{\mathcal{C}}$ at (A, B) is defined by
$$(\mathrm{id}_{\mathcal{C}})_{A,B} \overset{\text{def}}{=} \mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(A,B)}.$$

Proof 4.1.5 ► Proof of Example 4.1.4

Preservation of Identities

We have $\mathrm{id}_{\mathcal{C}}(\mathrm{id}_A) \overset{\text{def}}{=} \mathrm{id}_A$ for each $A \in \mathrm{Obj}(\mathcal{C})$ by definition.

Preservation of Compositions

For each composable pair $A \xrightarrow{f} B \xrightarrow{g} B$ of morphisms of $\mathcal{C}$, we have

$$\begin{aligned}\mathrm{id}_{\mathcal{C}}(g \circ f) &\overset{\text{def}}{=} g \circ f\\ &\overset{\text{def}}{=} \mathrm{id}_{\mathcal{C}}(g) \circ \mathrm{id}_{\mathcal{C}}(f).\end{aligned}$$

This finishes the proof.

Definition 4.1.6 ► Composition of Functors

The **composition** of two functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ is the functor $G \circ F$ where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathcal{C})$, we have

$$[G \circ F](A) \overset{\text{def}}{=} G(F(A)).$$

- *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathcal{C})$, the action on morphisms

$$(G \circ F)_{A,B} : \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{E}}(G_{F_A}, G_{F_B})$$

  of $G \circ F$ at (A, B) is defined by

$$[G \circ F](f) \overset{\text{def}}{=} G(F(f)).$$

Proof 4.1.7 ► Proof of Definition 4.1.6

Preservation of Identities

For each $A \in \mathrm{Obj}(\mathcal{C})$, we have

$$\begin{aligned}G_{F_{\mathrm{id}_A}} &= G_{\mathrm{id}_{F_A}} && \text{(functoriality of } F)\\ &= \mathrm{id}_{G_{F_A}}. && \text{(functoriality of } G)\end{aligned}$$

Preservation of Composition

For each composable pair (g, f) of morphisms of $\mathcal{C}$, we have

$$\begin{aligned} G_{F_{g \circ f}} &= G_{F_g \circ F_f} && \text{(functoriality of } F\text{)} \\ &= G_{F_g} \circ G_{F_f}. && \text{(functoriality of } G\text{)} \end{aligned}$$

This finishes the proof.

PROPOSITION 4.1.8 ► ELEMENTARY PROPERTIES OF FUNCTORS

Let $F \colon \mathcal{C} \to \mathcal{D}$ be a functor.

1. *Preservation of Isomorphisms.* If f is an isomorphism in $\mathcal{C}$, then $F(f)$ is an isomorphism in $\mathcal{D}$.[1]

[1]When the converse holds, we call F *conservative*, see Definition 5.4.1.

PROOF 4.1.9 ► PROOF OF PROPOSITION 4.1.8

Item 1: Preservation of Isomorphisms

Indeed, we have

$$\begin{aligned} F(f)^{-1} \circ F(f) &= F(f^{-1} \circ f) \\ &= F(\mathrm{id}_A) \\ &= \mathrm{id}_{F(A)} \end{aligned}$$

and

$$\begin{aligned} F(f) \circ F(f)^{-1} &= F(f \circ f^{-1}) \\ &= F(\mathrm{id}_B) \\ &= \mathrm{id}_{F(B)}, \end{aligned}$$

showing $F(f)$ to be an isomorphism.

4.2 Contravariant Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories, and let $\mathcal{C}^{\mathsf{op}}$ denote the opposite category of $\mathcal{C}$ of **??**, **??**.

Definition 4.2.1 ► Contravariant Functors

A **contravariant functor** from $\mathcal{C}$ to $\mathcal{D}$ is a functor from $\mathcal{C}^{\mathsf{op}}$ to $\mathcal{D}$.

Remark 4.2.2 ► Unwinding Definition 4.2.1

In detail, a **contravariant functor** from $\mathcal{C}$ to $\mathcal{D}$ consists of:

1. *Action on Objects.* A map of sets

$$F\colon \mathrm{Obj}(\mathcal{C}) \longrightarrow \mathrm{Obj}(\mathcal{D}),$$

called the **action on objects of** F.

2. *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathcal{C})$, a map

$$F_{A,B}\colon \mathrm{Hom}_{\mathcal{C}}(A,B) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(B),F(A)),$$

called the **action on morphisms of** F **at** (A,B).

satisfying the following conditions:

1. *Preservation of Identities.* For each $A \in \mathrm{Obj}(\mathcal{C})$, the diagram

$$\begin{tikzcd}
\mathrm{pt} \arrow[d, "\mathbb{1}^{\mathcal{C}}_{A}"'] \arrow[dr, "\mathbb{1}^{\mathcal{D}}_{F(A)}"] & \\
\mathrm{Hom}_{\mathcal{C}}(A,A) \arrow[r, "F_{A,A}"'] & \mathrm{Hom}_{\mathcal{D}}(F(A),F(A))
\end{tikzcd}$$

commutes, i.e. we have

$$F(\mathrm{id}_A) = \mathrm{id}_{F(A)}.$$

2. *Preservation of Composition.* For each $A, B, C \in \mathrm{Obj}(\mathcal{C})$, the diagram

$$\begin{tikzcd}
& & \mathrm{Hom}_{\mathcal{D}}(F(C), F(B)) \times \mathrm{Hom}_{\mathcal{D}}(F(B), F(A)) \arrow[drr, dashed, "\sigma^{\mathsf{Sets}}_{\mathrm{Hom}_{\mathcal{D}}(F(C),F(B)),\mathrm{Hom}_{\mathcal{D}}(F(B),F(A))}", "\sim"'] & & \\
\mathrm{Hom}_{\mathcal{C}}(B, C) \times \mathrm{Hom}_{\mathcal{C}}(A, B) \arrow[urr, "F_{B,C}\times F_{A,B}"] \arrow[dr, "\circ^{\mathcal{C}}_{A,B,C}"'] & & & & \mathrm{Hom}_{\mathcal{D}}(F(B), F(A)) \times \mathrm{Hom}_{\mathcal{D}}(F(C), F(B)) \arrow[dl, "\circ^{\mathcal{D}}_{F(C),F(B),F(A)}"] \\
& \mathrm{Hom}_{\mathcal{C}}(A, C) \arrow[rr, "F_{A,C}"'] & & \mathrm{Hom}_{\mathcal{D}}(F(C), F(A)) &
\end{tikzcd}$$

commutes, i.e. for each composable pair (g, f) of morphisms of $\mathcal{C}$, we have

$$F(g \circ f) = F(f) \circ F(g).$$

REMARK 4.2.3 ▶ ON THE TERM CONTRAVARIANT FUNCTOR

Throughout this work we will not use the term "contravariant" functor, speaking instead simply of functors $F : \mathcal{C}^{\mathsf{op}} \to \mathcal{D}$. We will usually, however, write

$$F_{A,B} : \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F(B), F(A))$$

for the action on morphisms

$$F_{A,B} : \mathrm{Hom}_{\mathcal{C}^{\mathsf{op}}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F(A), F(B))$$

of F, as well as write $F(g \circ f) = F(f) \circ F(g)$.

4.3 Forgetful Functors

Definition 4.3.1 ► Forgetful Functors

There isn't a precise definition of a **forgetful functor**.

Remark 4.3.2 ► Unwinding Definition 4.3.1

Despite there not being a formal or precise definition of a forgetful functor, the term is often very useful in practice, similarly to the word "canonical". The idea is that a "forgetful functor" is a functor that forgets structure or properties, and is best explained through examples, such as the ones below (see Examples 4.3.3 and 4.3.4).

Example 4.3.3 ► Forgetful Functors That Forget Structure

Examples of forgetful functors that forget structure include:

1. *Forgetting Group Structures.* The functor $\mathsf{Grp} \longrightarrow \mathsf{Sets}$ sending a group (G, μ_G, η_G) to its underlying set G, forgetting the multiplication and unit maps μ_G and η_G of G.
2. *Forgetting Topologies.* The functor $\mathsf{Top} \longrightarrow \mathsf{Sets}$ sending a topological space $(X, \mathcal{T}_X)$ to its underlying set X, forgetting the topology $\mathcal{T}_X$.
3. *Forgetting Fibrations.* The functor $\mathsf{FibSets}(K) \longrightarrow \mathsf{Sets}$ sending a K-fibred set $\phi_X \colon X \longrightarrow K$ to the set X, forgetting the map ϕ_X and the base set K.

Example 4.3.4 ► Forgetful Functors That Forget Properties

Examples of forgetful functors that forget properties include:

1. *Forgetting Commutativity.* The inclusion functor $\iota\colon \mathsf{CMon} \hookrightarrow \mathsf{Mon}$ which forgets the property of being commutative.
2. *Forgetting Inverses.* The inclusion functor $\iota\colon \mathsf{Grp} \hookrightarrow \mathsf{Mon}$ which forgets the property of having inverses.

NOTATION 4.3.5 ▶ NOTATION FOR FORGETFUL FUNCTORS THAT FORGET STRUCTURE

Throughout this work, we will denote forgetful functors that forget structure by 忘, e.g. as in

$$忘\colon \mathsf{Grp} \to \mathsf{Sets}.$$

The symbol 忘, pronounced *wasureru* (see Item 1 of Remark 4.3.6 below), means *to forget*, and is a kanji found in the following words in Japanese and Chinese:

1. 忘れる, transcribed as *wasureru*, meaning *to forget*.
2. 忘却関手, transcribed as *boukyaku kanshu*, meaning *forgetful functor*.
3. 忘记 or 忘記, transcribed as *wàngjì*, meaning *to forget*.
4. 遗忘函子 or 遺忘函子, transcribed as *yíwàng hánzǐ*, meaning *forgetful functor*.

REMARK 4.3.6 ▶ PRONUNCIATION OF THE WORDS IN NOTATION 4.3.5

Here we collect the pronunciation of the words in Notation 4.3.5 for accuracy and completeness.

1. Pronunciation of 忘れる:
    - Audio: see `https://topological-modular-forms.github.io/the-clowder-project/static/sounds/wasureru-01.mp3`
    - IPA broad transcription: [wäsɯɾe̞ɾɯ].
    - IPA narrow transcription: [ɰᵝäsɨᵝɾe̞ɾɯ̟ᵝ].
2. Pronunciation of 忘却関手: Pronunciation:
    - Audio: see `https://topological-modular-forms.github.io/the-clowder-project/static/sounds/wasureru-02.mp3`
    - IPA broad transcription: [bo̞ːkʲäkɯ kä̃ɰ̃ɕɯ].
    - IPA narrow transcription: [bo̞ːkʲäkɯ̟ᵝ kä̃ɰ̃ɕɯᵝ].
3. Pronunciation of 忘记:
    - Audio: see `https://topological-modular-forms.github.io/the-clowder-project/static/sounds/wasureru-03.ogg`

- Broad IPA transcription: [waŋtɕi].
- Sinological IPA transcription: [waŋ$^{51-53}$t͡ɕi^{51}].

4. Pronunciation of 遗忘函子:

   - Audio: see https://topological-modular-forms.github.io/the-clowder-project/static/sounds/wasureru-04.mp3
   - Broad IPA transcription: [iwaŋ xänt͡sz̩ɨ].
   - Sinological IPA transcription: [i^{35}waŋ^{51} xän^{35}t͡sz̩$^{214-21(4)}$].

4.4 The Natural Transformation Associated to a Functor

Definition 4.4.1 ► The Natural Transformation Associated to a Functor

Every functor $F\colon \mathcal{C} \to \mathcal{D}$ defines a natural transformation[1]

$$F^{\dagger}\colon \mathrm{Hom}_{\mathcal{C}} \Longrightarrow \mathrm{Hom}_{\mathcal{D}} \circ (F^{\mathsf{op}} \times F), \qquad \begin{tikzcd} \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \arrow[rr, "F^{\mathsf{op}} \times F"] \arrow[dr, "\mathrm{Hom}_{\mathcal{C}}"'] & \arrow[d, phantom, "\overset{F^{\dagger}}{\Longrightarrow}"] & \mathcal{D}^{\mathsf{op}} \times \mathcal{D} \arrow[dl, "\mathrm{Hom}_{\mathcal{D}}"] \\ & \mathsf{Sets}, & \end{tikzcd}$$

called the **natural transformation associated to** F, consisting of the collection

$$\left\{F^{\dagger}_{A,B}\colon \mathrm{Hom}_{\mathcal{C}}(A,B) \to \mathrm{Hom}_{\mathcal{D}}(F_A, F_B)\right\}_{(A,B)\in\mathrm{Obj}(\mathcal{C}^{\mathsf{op}}\times\mathcal{C})}$$

with

$$F^{\dagger}_{A,B} \overset{\mathrm{def}}{=} F_{A,B}.$$

[1] This is the 1-categorical version of Constructions With Sets, Item 1 of Proposition 4.1.3.

Proof 4.4.2 ► Proof of Definition 4.4.1

The naturality condition for $F^{\dagger}$ is the requirement that for each morphism

$$(\phi,\psi)\colon (X,Y) \to (A,B)$$

of $\mathcal{C}^{\mathsf{op}} \times \mathcal{C}$, the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(X, Y) \arrow[r, "\phi^* \circ \psi_* = \psi_* \circ \phi^*"] \arrow[d, "F_{X,Y}"'] & \mathrm{Hom}_{\mathcal{C}}(A, B) \arrow[d, "F_{A,B}"] \\
\mathrm{Hom}_{\mathcal{D}}(F_X, F_Y) \arrow[r, "F(\phi)^* \circ F(\psi)_* = F(\psi)_* \circ F(\phi)^*"'] & \mathrm{Hom}_{\mathcal{D}}(F_A, F_B),
\end{tikzcd}$$

acting on elements as

$$\begin{tikzcd}
f \arrow[r, mapsto] \arrow[d, mapsto] & \psi \circ f \circ \phi \arrow[d, mapsto] \\
F(f) \arrow[r, mapsto] & F(\psi) \circ F(f) \circ F(\psi) = F(\psi \circ f \circ \phi)
\end{tikzcd}$$

commutes, which follows from the functoriality of F.

Proposition 4.4.3 ▶ Properties of Natural Transformations Associated to Functors

Let $F \colon \mathcal{C} \to \mathcal{D}$ and $G \colon \mathcal{D} \to \mathcal{E}$ be functors.

1. *Interaction With Natural Isomorphisms.* The following conditions are equivalent:

   (a) The natural transformation $F^\dagger \colon \mathrm{Hom}_{\mathcal{C}} \Longrightarrow \mathrm{Hom}_{\mathcal{D}} \circ (F^{\mathsf{op}} \times F)$ associated to F is a natural isomorphism.

   (b) The functor F is fully faithful.

2. *Interaction With Composition.* We have an equality of pasting diagrams

$$\begin{tikzcd}
\mathcal{C}^{\mathsf{op}} \times \mathcal{C} \arrow[r, "F^{\mathsf{op}} \times F"] \arrow[dr, "\mathrm{Hom}_{\mathcal{C}}"'] & \mathcal{D}^{\mathsf{op}} \times \mathcal{D} \arrow[r, "G^{\mathsf{op}} \times G"] \arrow[d, "\mathrm{Hom}_{\mathcal{D}}"] & \mathcal{E}^{\mathsf{op}} \times \mathcal{E} \arrow[dl, "\mathrm{Hom}_{\mathcal{E}}"] \\
 & \mathrm{Sets} &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathcal{C}^{\mathsf{op}} \times \mathcal{C} \arrow[rr, "(G \circ F)^{\mathsf{op}} \times (G \circ F)"] \arrow[dr, "\mathrm{Hom}_{\mathcal{C}}"'] & & \mathcal{E}^{\mathsf{op}} \times \mathcal{E}, \arrow[dl, "\mathrm{Hom}_{\mathcal{E}}"] \\
 & \mathrm{Sets} &
\end{tikzcd}$$

in Cats_2, i.e. we have

$$(G \circ F)^\dagger = (G^\dagger \star \mathrm{id}_{F^{\mathrm{op}} \times F}) \circ F^\dagger.$$

3. *Interaction With Identities.* We have

$$\mathrm{id}_{\mathcal{C}}^\dagger = \mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(-_1,-_2)},$$

i.e. the natural transformation associated to $\mathrm{id}_{\mathcal{C}}$ is the identity natural transformation of the functor $\mathrm{Hom}_{\mathcal{C}}(-_1, -_2)$.

Proof 4.4.4 ► Proof of Proposition 4.4.3

Item 1: Interaction With Natural Isomorphisms

Clear.

Item 2: Interaction With Composition

Clear.

Item 3: Interaction With Identities

Clear.

5 Conditions on Functors

5.1 Faithful Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 5.1.1 ► Faithful Functors

A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **faithful** if, for each $A, B \in \mathrm{Obj}(\mathcal{C})$, the action on morphisms

$$F_{A,B}\colon \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F_A, F_B)$$

of F at (A, B) is injective.

Proposition 5.1.2 ► Properties of Faithful Functors

Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor.

1. *Interaction With Postcomposition.* The following conditions are equivalent:

   (a) The functor $F\colon \mathcal{C} \to \mathcal{D}$ is faithful.

   (b) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor

   $$F_*\colon \mathsf{Fun}(\mathcal{X}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{X}, \mathcal{D})$$

   is faithful.

   (c) The functor $F\colon \mathcal{C} \to \mathcal{D}$ is a representably faithful morphism in Cats_2 in the sense of Types of Morphisms in Bicategories, Definition 1.1.1.

2. *Interaction With Precomposition I.* Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor.

   (a) If F is faithful, then the precomposition functor

   $$F^*\colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

   can fail to be faithful.

   (b) Conversely, if the precomposition functor

   $$F^*\colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

   is faithful, then F *can fail* to be faithful.

3. *Interaction With Precomposition II.* If F is essentially surjective, then the precomposition functor

   $$F^*\colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

   is faithful.

4. *Interaction With Precomposition III.* The following conditions are equivalent:

   (a) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor

   $$F^*\colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

   is faithful.

(b) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor

$$F^*\colon \mathsf{Fun}(\mathcal{D},\mathcal{X}) \to \mathsf{Fun}(\mathcal{C},\mathcal{X})$$

is conservative.

(c) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor

$$F^*\colon \mathsf{Fun}(\mathcal{D},\mathcal{X}) \to \mathsf{Fun}(\mathcal{C},\mathcal{X})$$

is monadic.

(d) The functor $F\colon \mathcal{C} \to \mathcal{D}$ is a corepresentably faithful morphism in Cats_2 in the sense of Types of Morphisms in Bicategories, Definition 2.1.1.

(e) The components

$$\eta_G\colon G \Longrightarrow \mathrm{Ran}_F(G \circ F)$$

of the unit

$$\eta\colon \mathrm{id}_{\mathsf{Fun}(\mathcal{D},\mathcal{X})} \Longrightarrow \mathrm{Ran}_F \circ F^*$$

of the adjunction $F^* \dashv \mathrm{Ran}_F$ are all monomorphisms.

(f) The components

$$\epsilon_G\colon \mathrm{Lan}_F(G \circ F) \Longrightarrow G$$

of the counit

$$\epsilon\colon \mathrm{Lan}_F \circ F^* \Longrightarrow \mathrm{id}_{\mathsf{Fun}(\mathcal{D},\mathcal{X})}$$

of the adjunction $\mathrm{Lan}_F \dashv F^*$ are all epimorphisms.

(g) The functor F is dominant (Definition 6.1.1), i.e. every object of $\mathcal{D}$ is a retract of some object in $\mathrm{Im}(F)$:

($\star$) For each $B \in \mathrm{Obj}(\mathcal{D})$, there exist:
- An object A of $\mathcal{C}$;
- A morphism $s\colon B \to F(A)$ of $\mathcal{D}$;
- A morphism $r\colon F(A) \to B$ of $\mathcal{D}$;

such that $r \circ s = \mathrm{id}_B$.

Proof 5.1.3 ► Proof of Proposition 5.1.2

Item 1: Interaction With Postcomposition

Omitted.

Item 2: Interaction With Precomposition I

See [MSE 733163] for Item 2a. Item 2b follows from Item 3 and the fact that there are essentially surjective functors that are not faithful.

Item 3: Interaction With Precomposition II

Omitted, but see `https://unimath.github.io/doc/UniMath/d4de26f//UniMath.CategoryTheory.precomp_fully_faithful.html` for a formalised proof.

Item 4: Interaction With Precomposition III

We claim Items 4a to 4g are equivalent:

- *Items 4a and 4d Are Equivalent:* This is true by the definition of corepresentably faithful morphism; see Types of Morphisms in Bicategories, Definition 2.1.1.
- *Items 4a to 4c and 4g Are Equivalent:* See [Adá+01, Proposition 4.1] or alternatively [Fre09, Lemmas 3.1 and 3.2] for the equivalence between Items 4a and 4g.
- *Items 4a, 4e and 4f Are Equivalent:* See **??**, **??** of **??**.

This finishes the proof. ∎

5.2 Full Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 5.2.1 ► Full Functors

A functor $F: \mathcal{C} \to \mathcal{D}$ is **full** if, for each $A, B \in \mathrm{Obj}(\mathcal{C})$, the action on morphisms

$$F_{A,B}: \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F_A, F_B)$$

of F at (A, B) is surjective.

Proposition 5.2.2 ► Properties of Full Functors

Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor.

1. *Interaction With Postcomposition.* The following conditions are equivalent:

   (a) The functor $F\colon \mathcal{C} \to \mathcal{D}$ is full.

   (b) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor

   $$F_*\colon \mathsf{Fun}(\mathcal{X}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{X}, \mathcal{D})$$

   is full.

   (c) The functor $F\colon \mathcal{C} \to \mathcal{D}$ is a representably full morphism in Cats_2 in the sense of Types of Morphisms in Bicategories, Definition 1.2.1.

2. *Interaction With Precomposition I.* If F is full, then the precomposition functor

   $$F^*\colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

   can fail to be full.

3. *Interaction With Precomposition II.* If the precomposition functor

   $$F^*\colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

   is full, then F *can fail* to be full.

4. *Interaction With Precomposition III.* If F is essentially surjective and full, then the precomposition functor

   $$F^*\colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

   is full (and also faithful by Item 3 of Proposition 5.1.2).

5. *Interaction With Precomposition IV.* The following conditions are equivalent:

   (a) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor

   $$F^*\colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

   is full.

(b) The functor $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ is a corepresentably full morphism in Cats_2 in the sense of Types of Morphisms in Bicategories, Definition 2.1.1.

(c) The components

$$\eta_G\colon G \Longrightarrow \mathrm{Ran}_F(G \circ F)$$

of the unit

$$\eta\colon \mathrm{id}_{\mathsf{Fun}(\mathcal{D},\mathcal{X})} \Longrightarrow \mathrm{Ran}_F \circ F^*$$

of the adjunction $F^* \dashv \mathrm{Ran}_F$ are all retractions/split epimorphisms.

(d) The components

$$\epsilon_G\colon \mathrm{Lan}_F(G \circ F) \Longrightarrow G$$

of the counit

$$\epsilon\colon \mathrm{Lan}_F \circ F^* \Longrightarrow \mathrm{id}_{\mathsf{Fun}(\mathcal{D},\mathcal{X})}$$

of the adjunction $\mathrm{Lan}_F \dashv F^*$ are all sections/split monomorphisms.

(e) For each $B \in \mathrm{Obj}(\mathcal{D})$, there exist:

- An object A_B of $\mathcal{C}$;
- A morphism $s_B\colon B \longrightarrow F(A_B)$ of $\mathcal{D}$;
- A morphism $r_B\colon F(A_B) \longrightarrow B$ of $\mathcal{D}$;

satisfying the following condition:

($\star$) For each $A \in \mathrm{Obj}(\mathcal{C})$ and each pair of morphisms

$$\begin{aligned} r&\colon F(A) \longrightarrow B, \\ s&\colon B \longrightarrow F(A) \end{aligned}$$

of $\mathcal{D}$, we have

$$[(A_B, s_B, r_B)] = [(A, s, r \circ s_B \circ r_B)]$$

in $\int^{A \in \mathcal{C}} h^{B'}_{F_A} \times h^{F_A}_{B}$.

Proof 5.2.3 ► Proof of Proposition 5.2.2

Item 1: Interaction With Postcomposition

Omitted.

Item 2: Interaction With Precomposition I

Omitted.

Item 3: Interaction With Precomposition II

See [BS10, p. 47].

Item 4: Interaction With Precomposition III

Omitted, but see `https://unimath.github.io/doc/UniMath/d4de26f//UniMath.CategoryTheory.precomp_fully_faithful.html` for a formalised proof.

Item 5: Interaction With Precomposition IV

We claim Items 5a to 5e are equivalent:

- *Items 5a and 5b Are Equivalent:* This is true by the definition of corepresentably full morphism; see Types of Morphisms in Bicategories, Definition 2.2.1.
- *Items 5a, 5c and 5d Are Equivalent:* See **??**, **??** of **??**.
- *Items 5a and 5e Are Equivalent:* See [Adá+01, Item (b) of Remark 4.3].

This finishes the proof.

Question 5.2.4 ► Better Characterisations of Functors With Full Precomposition

Item 5 of Proposition 5.2.2 gives a characterisation of the functors F for which F^* is full, but the characterisations given there are really messy. Are there better ones? This question also appears as [MO 468121b].

5.3 Fully Faithful Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 5.3.1 ► Fully Faithful Functors

A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **fully faithful** if F is full and faithful, i.e. if, for each $A, B \in \mathrm{Obj}(\mathcal{C})$, the action on morphisms

$$F_{A,B}\colon \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F_A, F_B)$$

of F at (A, B) is bijective.

Proposition 5.3.2 ► Properties of Fully Faithful Functors

Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor.

1. *Characterisations.* The following conditions are equivalent:
   (a) The functor F is fully faithful.
   (b) We have a pullback square

$$\mathrm{Arr}(\mathcal{C}) \cong (\mathcal{C} \times \mathcal{C}) \times_{\mathcal{D}\times\mathcal{D}} \mathrm{Arr}(\mathcal{D}), \qquad \begin{tikzcd} \mathrm{Arr}(\mathcal{C}) \arrow[r, "\mathrm{Arr}(F)"] \arrow[d, "\mathrm{src}\times\mathrm{tgt}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Arr}(\mathcal{D}) \arrow[d, "\mathrm{src}\times\mathrm{tgt}"] \\ \mathcal{C} \times \mathcal{C} \arrow[r, "F\times F"'] & \mathcal{D} \times \mathcal{D} \end{tikzcd}$$

   in Cats.

2. *Conservativity.* If F is fully faithful, then F is conservative.

3. *Essential Injectivity.* If F is fully faithful, then F is essentially injective.

4. *Interaction With Co/Limits.* If F is fully faithful, then F reflects co/limits.

5. *Interaction With Postcomposition.* The following conditions are equivalent:
   (a) The functor $F\colon \mathcal{C} \to \mathcal{D}$ is fully faithful.
   (b) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor

$$F_*\colon \mathrm{Fun}(\mathcal{X}, \mathcal{C}) \to \mathrm{Fun}(\mathcal{X}, \mathcal{D})$$

   is fully faithful.

(c) The functor $F : \mathcal{C} \to \mathcal{D}$ is a representably fully faithful morphism in Cats_2 in the sense of Types of Morphisms in Bicategories, Definition 1.3.1.

6. *Interaction With Precomposition I.* If F is fully faithful, then the precomposition functor
$$F^* : \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$
can fail to be fully faithful.

7. *Interaction With Precomposition II.* If the precomposition functor
$$F^* : \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$
is fully faithful, then F *can fail* to be fully faithful (and in fact it can also fail to be either full or faithful).

8. *Interaction With Precomposition III.* If F is essentially surjective and full, then the precomposition functor
$$F^* : \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$
is fully faithful.

9. *Interaction With Precomposition IV.* The following conditions are equivalent:

   (a) For each $\mathcal{X} \in \mathsf{Obj}(\mathsf{Cats})$, the precomposition functor
   $$F^* : \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$
   is fully faithful.

   (b) The precomposition functor
   $$F^* : \mathsf{Fun}(\mathcal{D}, \mathsf{Sets}) \to \mathsf{Fun}(\mathcal{C}, \mathsf{Sets})$$
   is fully faithful.

   (c) The functor
   $$\mathsf{Lan}_F : \mathsf{Fun}(\mathcal{C}, \mathsf{Sets}) \to \mathsf{Fun}(\mathcal{D}, \mathsf{Sets})$$
   is fully faithful.

(d) The functor F is a corepresentably fully faithful morphism in Cats_2 in the sense of Types of Morphisms in Bicategories, Definition 2.3.1.

(e) The functor F is absolutely dense.

(f) The components

$$\eta_G \colon G \Longrightarrow \mathrm{Ran}_F(G \circ F)$$

of the unit

$$\eta \colon \mathrm{id}_{\mathsf{Fun}(\mathcal{D},\mathcal{X})} \Longrightarrow \mathrm{Ran}_F \circ F^*$$

of the adjunction $F^* \dashv \mathrm{Ran}_F$ are all isomorphisms.

(g) The components

$$\epsilon_G \colon \mathrm{Lan}_F(G \circ F) \Longrightarrow G$$

of the counit

$$\epsilon \colon \mathrm{Lan}_F \circ F^* \Longrightarrow \mathrm{id}_{\mathsf{Fun}(\mathcal{D},\mathcal{X})}$$

of the adjunction $\mathrm{Lan}_F \dashv F^*$ are all isomorphisms.

(h) The natural transformation

$$\alpha \colon \mathrm{Lan}_{h_F}(h^F) \Longrightarrow h$$

with components

$$\alpha_{B',B} \colon \int^{A \in C} h^{B'}_{F_A} \times h^{F_A}_{B} \to h^{B'}_{B}$$

given by

$$\alpha_{B',B}([(\phi,\psi)]) = \psi \circ \phi$$

is a natural isomorphism.

(i) For each $B \in \mathrm{Obj}(\mathcal{D})$, there exist:

- An object A_B of C;
- A morphism $s_B \colon B \to F(A_B)$ of $\mathcal{D}$;
- A morphism $r_B \colon F(A_B) \to B$ of $\mathcal{D}$;

satisfying the following conditions:

i. The triple $(F(A_B), r_B, s_B)$ is a retract of B, i.e. we have $r_B \circ s_B = \mathrm{id}_B$.

ii. For each morphism $f\colon B' \to B$ of $\mathcal{D}$, we have

$$[(A_B, s_{B'}, f \circ r_{B'})] = [(A_B, s_B \circ f, r_B)]$$

in $\int^{A \in C} h^{B'}_{F_A} \times h^{F_A}_{B}$.

Proof 5.3.3 ► Proof of Proposition 5.3.2

Item 1: Characterisations

Omitted.

Item 2: Conservativity

This is a repetition of Item 2 of Proposition 5.4.2, and is proved there.

Item 3: Essential Injectivity

Omitted.

Item 4: Interaction With Co/Limits

Omitted.

Item 5: Interaction With Postcomposition

This follows from Item 1 of Proposition 5.1.2 and Item 1 of Proposition 5.2.2.

Item 6: Interaction With Precomposition I

See [MSE 733161] for an example of a fully faithful functor whose precomposition with which fails to be full.

Item 7: Interaction With Precomposition II

See [MSE 749304, Item 3].

Item 8: Interaction With Precomposition III

Omitted, but see `https://unimath.github.io/doc/UniMath/d4de26f//UniMath.CategoryTheory.precomp_fully_faithful.html` for a formalised proof.

Item 9: Interaction With Precomposition IV

We claim Items 9a to 9i are equivalent:

- *Items 9a and 9d Are Equivalent:* This is true by the definition of corepresentably fully faithful morphism; see Types of Morphisms in Bicategories, Definition 2.3.1.
- *Items 9a, 9f and 9g Are Equivalent:* See **??**, **??** of **??**.
- *Items 9a to 9c Are Equivalent:* This follows from [Low15, Proposition A.1.5].
- *Items 9a, 9e, 9h and 9i Are Equivalent:* See [Fre09, Theorem 4.1] and [Adá+01, Theorem 1.1].

This finishes the proof.

5.4 Conservative Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 5.4.1 ► Conservative Functors

A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **conservative** if it satisfies the following condition:[1]

($\star$) For each $f \in \mathrm{Mor}(\mathcal{C})$, if $F(f)$ is an isomorphism in $\mathcal{D}$, then f is an isomorphism in $\mathcal{C}$.

[1] *Slogan:* A functor F is **conservative** if it reflects isomorphisms.

Proposition 5.4.2 ► Properties of Conservative Functors

Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor.

1. *Characterisations.* The following conditions are equivalent:
   (a) The functor F is conservative.
   (b) For each $f \in \mathrm{Mor}(\mathcal{C})$, the morphism $F(f)$ is an isomorphism in $\mathcal{D}$ iff f is an isomorphism in $\mathcal{C}$.
2. *Interaction With Fully Faithfulness.* Every fully faithful functor is conservative.
3. *Interaction With Precomposition.* The following conditions are equivalent:

(a) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor

$$F^* \colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

is conservative.

(b) The equivalent conditions of Item 4 of Proposition 5.1.2 are satisfied.

Proof 5.4.3 ▶ Proof of Proposition 5.4.2

Item 1: Characterisations

This follows from Item 1 of Proposition 4.1.8.

Item 2: Interaction With Fully Faithfulness

Let $F \colon \mathcal{C} \to \mathcal{D}$ be a fully faithful functor, let $f \colon A \to B$ be a morphism of $\mathcal{C}$, and suppose that F_f is an isomorphism. We have

$$\begin{aligned} F(\mathrm{id}_B) &= \mathrm{id}_{F(B)} \\ &= F(f) \circ F(f)^{-1} \\ &= F(f \circ f^{-1}). \end{aligned}$$

Similarly, $F(\mathrm{id}_A) = F(f^{-1} \circ f)$. But since F is fully faithful, we must have

$$\begin{aligned} f \circ f^{-1} &= \mathrm{id}_B, \\ f^{-1} \circ f &= \mathrm{id}_A, \end{aligned}$$

showing f to be an isomorphism. Thus F is conservative. □

Question 5.4.4 ▶ Characterisations of Functors With Conservative Pre/Post-composition

Is there a characterisation of functors $F \colon \mathcal{C} \to \mathcal{D}$ satisfying the following condition:

($\star$) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor

$$F_* \colon \mathsf{Fun}(\mathcal{X}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{X}, \mathcal{D})$$

is conservative?

This question also appears as [MO 468121a].

5.5 Essentially Injective Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 5.5.1 ▶ Essentially Injective Functors

A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **essentially injective** if it satisfies the following condition:

(★) For each $A, B \in \mathrm{Obj}(\mathcal{C})$, if $F(A) \cong F(B)$, then $A \cong B$.

Question 5.5.2 ▶ Characterisations of Functors With Essentially Injective Pre/-Postcomposition

Is there a characterisation of functors $F\colon \mathcal{C} \to \mathcal{D}$ such that:

1. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor

$$F^*\colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

   is essentially injective, i.e. if $\phi \circ F \cong \psi \circ F$, then $\phi \cong \psi$ for all functors ϕ and ψ?

2. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor

$$F_*\colon \mathsf{Fun}(\mathcal{X}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{X}, \mathcal{D})$$

   is essentially injective, i.e. if $F \circ \phi \cong F \circ \psi$, then $\phi \cong \psi$?

This question also appears as [MO 468121a].

5.6 Essentially Surjective Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 5.6.1 ► Essentially Surjective Functors

A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **essentially surjective**[1] if it satisfies the following condition:

(★) For each $D \in \mathrm{Obj}(\mathcal{D})$, there exists some object A of $\mathcal{C}$ such that $F(A) \cong D$.

[1] *Further Terminology:* Also called an **eso** functor, where the name "eso" comes from *essentially surjective on objects*.

Question 5.6.2 ► Characterisations of Functors With Essentially Surjective Pre/Postcomposition

Is there a characterisation of functors $F\colon \mathcal{C} \to \mathcal{D}$ such that:

1. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor

$$F^{*}\colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

is essentially surjective?

2. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor

$$F_{*}\colon \mathsf{Fun}(\mathcal{X}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{X}, \mathcal{D})$$

is essentially surjective?

This question also appears as [MO 468121a].

5.7 Equivalences of Categories

Definition 5.7.1 ► Equivalences of Categories

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

1. An **equivalence of categories** between $\mathcal{C}$ and $\mathcal{D}$ consists of a pair of functors

$$\begin{aligned} F&\colon \mathcal{C} \to \mathcal{D},\\ G&\colon \mathcal{D} \to \mathcal{C} \end{aligned}$$

together with natural isomorphisms

$$\eta\colon \mathrm{id}_{\mathcal{C}} \overset{\sim}{\Longrightarrow} G \circ F,$$
$$\epsilon\colon F \circ G \overset{\sim}{\Longrightarrow} \mathrm{id}_{\mathcal{D}}.$$

2. An **adjoint equivalence of categories** between $\mathcal{C}$ and $\mathcal{D}$ is an equivalence (F, G, η, ϵ) between $\mathcal{C}$ and $\mathcal{D}$ which is also an adjunction.

Proposition 5.7.2 ▶ Properties of Equivalences of Categories

Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor.

1. *Characterisations.* If $\mathcal{C}$ and $\mathcal{D}$ are small[1], then the following conditions are equivalent:[2]

   (a) The functor F is an equivalence of categories.

   (b) The functor F is fully faithful and essentially surjective.

   (c) The induced functor

   $$\upharpoonright F\mathsf{Sk}(\mathcal{C})\colon \mathsf{Sk}(\mathcal{C}) \to \mathsf{Sk}(\mathcal{D})$$

   is an *isomorphism* of categories.

   (d) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor

   $$F^*\colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

   is an equivalence of categories.

   (e) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor

   $$F_*\colon \mathsf{Fun}(\mathcal{X}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{X}, \mathcal{D})$$

   is an equivalence of categories.

2. *Two-Out-of-Three.* Let

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "G\circ F"] \arrow[dr, "F"'] & & \mathcal{E} \\ & \mathcal{D} \arrow[ur, "G"'] & \end{tikzcd}$$

   be a diagram in Cats. If two out of the three functors among F, G, and $G \circ F$ are equivalences of categories, then so is the third.

3. *Stability Under Composition.* Let

$$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D} \underset{G'}{\overset{F'}{\rightleftarrows}} \mathcal{E}$$

be a diagram in Cats. If (F, G) and (F', G') are equivalences of categories, then so is their composite $(F' \circ F, G' \circ G)$.

4. *Equivalences vs.Adjoint Equivalences.* Every equivalence of categories can be promoted to an adjoint equivalence.[3]

5. *Interaction With Groupoids.* If $\mathcal{C}$ and $\mathcal{D}$ are groupoids, then the following conditions are equivalent:

   (a) The functor F is an equivalence of groupoids.

   (b) The following conditions are satisfied:

      i. The functor F induces a bijection

      $$\pi_0(F) \colon \pi_0(\mathcal{C}) \to \pi_0(\mathcal{D})$$

      of sets.

      ii. For each $A \in \mathrm{Obj}(\mathcal{C})$, the induced map

      $$F_{x,x} \colon \mathrm{Aut}_{\mathcal{C}}(A) \to \mathrm{Aut}_{\mathcal{D}}(F_A)$$

      is an isomorphism of groups.

[1]Otherwise there will be size issues. One can also work with large categories and universes, or require F to be *constructively* essentially surjective; see [MSE 1465107].

[2]In ZFC, the equivalence between Item 1a and Item 1b is equivalent to the axiom of choice; see [MO 119454].

In Univalent Foundations, this is true without requiring neither the axiom of choice nor the law of excluded middle.

[3]More precisely, we can promote an equivalence of categories (F, G, η, ϵ) to adjoint equivalences (F, G, η', ϵ) and (F, G, η, ϵ').

Proof 5.7.3 ▶ Proof of Proposition 5.7.2

Item 1: Characterisations

We claim that Items 1a to 1e are indeed equivalent:

1. *Item 1a* $\implies$ *Item 1b*: Clear.

2. *Item 1b* $\implies$ *Item 1a*: Since F is essentially surjective and $\mathcal{C}$ and $\mathcal{D}$ are small, we can choose, using the axiom of choice, for each $B \in \mathrm{Obj}(\mathcal{D})$, an object j_B of $\mathcal{C}$ and an isomorphism $i_B : B \to F_{j_B}$ of $\mathcal{D}$.

   Since F is fully faithful, we can extend the assignment $B \mapsto j_B$ to a *unique* functor $j : \mathcal{D} \to \mathcal{C}$ such that the isomorphisms $i_B : B \to F_{j_B}$ assemble into a natural isomorphism $\eta : \mathrm{id}_{\mathcal{D}} \overset{\sim}{\Longrightarrow} F \circ j$, with a similar natural isomorphism $\epsilon : \mathrm{id}_{\mathcal{C}} \overset{\sim}{\Longrightarrow} j \circ F$. Hence F is an equivalence.

3. *Item 1a* $\implies$ *Item 1c*: This follows from Item 4 of Proposition 1.5.3.

4. *Item 1c* $\implies$ *Item 1a*: Omitted.

5. *Items 1a, 1d and 1e Are Equivalent*: This follows from **??**.

This finishes the proof of Item 1.

Item 2: Two-Out-of-Three

Omitted.

Item 3: Stability Under Composition

Clear.

Item 4: Equivalences vs.Adjoint Equivalences

See [Rie17, Proposition 4.4.5].

Item 5: Interaction With Groupoids

See [nLa24, Proposition 4.4].

5.8 Isomorphisms of Categories

Definition 5.8.1 ► Isomorphisms of Categories

An **isomorphism of categories** is a pair of functors

$$F : \mathcal{C} \to \mathcal{D},$$
$$G : \mathcal{D} \to \mathcal{C}$$

such that we have

$$G \circ F = \mathrm{id}_{\mathcal{C}},$$

$$F \circ G = \mathrm{id}_{\mathcal{D}}.$$

Example 5.8.2 ▶ Equivalent But Non-Isomorphic Categories

Categories can be equivalent but non-isomorphic. For example, the category consisting of two isomorphic objects is equivalent to pt, but not isomorphic to it.

Proposition 5.8.3 ▶ Properties of Isomorphisms of Categories

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor.

1. *Characterisations.* If $\mathcal{C}$ and $\mathcal{D}$ are small, then the following conditions are equivalent:

    (a) The functor F is an isomorphism of categories.

    (b) The functor F is fully faithful and bijective on objects.

    (c) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor

    $$F^* : \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

    is an isomorphism of categories.

    (d) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor

    $$F_* : \mathsf{Fun}(\mathcal{X}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{X}, \mathcal{D})$$

    is an isomorphism of categories.

Proof 5.8.4 ▶ Proof of Proposition 5.8.3

Item 1: Characterisations

We claim that Items 1a to 1d are indeed equivalent:

1. *Items 1a and 1b Are Equivalent:* Omitted, but similar to Item 1 of Proposi-

tion 5.7.2.

2. *Items 1a, 1c and 1d Are Equivalent:* This follows from **??**.

This finishes the proof. □

6 More Conditions on Functors

6.1 Dominant Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 6.1.1 ► Dominant Functors

A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **dominant** if every object of $\mathcal{D}$ is a retract of some object in $\mathrm{Im}(F)$, i.e.:

($\star$) For each $B \in \mathrm{Obj}(\mathcal{D})$, there exist:

- An object A of $\mathcal{C}$;
- A morphism $r\colon F(A) \to B$ of $\mathcal{D}$;
- A morphism $s\colon B \to F(A)$ of $\mathcal{D}$;

such that we have

$$r \circ s = \mathrm{id}_B, \qquad \begin{tikzcd} B \arrow[r, "s"] \arrow[dr, "\mathrm{id}_B"'] & F(A) \arrow[d, "r"] \\ & B. \end{tikzcd}$$

Proposition 6.1.2 ► Properties of Dominant Functors

Let $F, G\colon \mathcal{C} \rightrightarrows \mathcal{D}$ be functors and let $I\colon \mathcal{X} \to \mathcal{C}$ be a functor.

1. *Interaction With Right Whiskering.* If I is full and dominant, then the map

$$- \star \mathrm{id}_I\colon \mathrm{Nat}(F, G) \to \mathrm{Nat}(F \circ I, G \circ I)$$

is a bijection.

2. *Interaction With Adjunctions.* Let $(F, G) : \mathcal{C} \rightleftarrows \mathcal{D}$ be an adjunction.

   (a) If F is dominant, then G is faithful.

   (b) The following conditions are equivalent:

      i. The functor G is full.

      ii. The restriction

      $$\restriction G\mathrm{Im}_F : \mathrm{Im}(F) \to \mathcal{C}$$

      of G to $\mathrm{Im}(F)$ is full.

Proof 6.1.3 ▶ Proof of Proposition 6.1.2

Item 1: Interaction With Right Whiskering

See [DFH75, Proposition 1.4].

Item 2: Interaction With Adjunctions

See [DFH75, Proposition 1.7]. ∎

Question 6.1.4 ▶ Characterisations of Functors With Dominant Pre/Postcomposition

Is there a characterisation of functors $F : \mathcal{C} \to \mathcal{D}$ such that:

1. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor

   $$F^* : \mathrm{Fun}(\mathcal{D}, \mathcal{X}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{X})$$

   is dominant?

2. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor

   $$F_* : \mathrm{Fun}(\mathcal{X}, \mathcal{C}) \to \mathrm{Fun}(\mathcal{X}, \mathcal{D})$$

   is dominant?

This question also appears as [MO 468121a].

6.2 Monomorphisms of Categories

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

A functor $F: \mathcal{C} \to \mathcal{D}$ is a **monomorphism of categories** if it is a monomorphism in Cats (see **??**, **??**).

Proposition 6.2.2 ▶ Properties of Monomorphisms of Categories

Let $F: \mathcal{C} \to \mathcal{D}$ be a functor.

1. *Characterisations.* The following conditions are equivalent:
   (a) The functor F is a monomorphism of categories.
   (b) The functor F is injective on objects and morphisms, i.e. F is injective on objects and the map
   $$F: \mathrm{Mor}(\mathcal{C}) \to \mathrm{Mor}(\mathcal{D})$$
   is injective.

Proof 6.2.3 ▶ Proof of Proposition 6.2.2

Item 1: Characterisations

Omitted.

Question 6.2.4 ▶ Characterisations of Functors With Monic Pre/Postcomposition

Is there a characterisation of functors $F: \mathcal{C} \to \mathcal{D}$ such that:

1. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor
   $$F^*: \mathrm{Fun}(\mathcal{D}, \mathcal{X}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{X})$$
   is a monomorphism of categories?
2. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor
   $$F_*: \mathrm{Fun}(\mathcal{X}, \mathcal{C}) \to \mathrm{Fun}(\mathcal{X}, \mathcal{D})$$
   is a monomorphism of categories?

This question also appears as [MO 468121a].

6.3 Epimorphisms of Categories

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 6.3.1 ▶ Epimorphisms of Categories

A functor $F\colon \mathcal{C} \to \mathcal{D}$ is a **epimorphism of categories** if it is a epimorphism in Cats (see **??**, **??**).

Proposition 6.3.2 ▶ Properties of Epimorphisms of Categories

Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor.

1. *Characterisations.* The following conditions are equivalent:[1]
   (a) The functor F is a epimorphism of categories.
   (b) For each morphism $f\colon A \to B$ of $\mathcal{D}$, we have a diagram

$$
\begin{tikzcd}
 & & & & A \arrow[dllll, "\phi_1"'] \arrow[dll, "\phi_2"'] \arrow[d, "\phi_3"] \arrow[drr, "\phi_m"] \arrow[drrrr, dashed, "\mathrm{id}_A"] & & & & \\
X_1 \arrow[dr, dashed, "\alpha_1"] \arrow[ddrrrr, dashed, bend right, "\alpha_0"'] & & X_2 \arrow[dl, dashed, "\alpha_2"'] \arrow[dr, dashed, "\alpha_3"] & & X_3 \arrow[dl, dashed, "\alpha_4"'] \arrow[dr, dashed, "\alpha_6"] & \cdots & X_m \arrow[dl, dashed, "\alpha_{2m-2}"'] \arrow[dr, dashed, "\alpha_{2m-1}"] & & A \arrow[dl, dashed, "\alpha_{2m}"'] \\
 & Y_1 \arrow[drrr, "\psi_1"] & & Y_2 \arrow[dr, "\psi_2"] & & \cdots \arrow[dl, "\psi_3"'] & & Y_m \arrow[dlll, "\psi_m"'] & \\
 & & & & B & & & &
\end{tikzcd}
$$

   in $\mathcal{D}$ satisfying the following conditions:
   i. We have $f = \alpha_0 \circ \phi_1$.
   ii. We have $f = \psi_m \circ \alpha_{2m}$.
   iii. For each $0 \leq i \leq 2m$, we have $\alpha_i \in \mathrm{Mor}(\mathrm{Im}(F))$.
2. *Surjectivity on Objects.* If F is an epimorphism of categories, then F is surjective on objects.

[1] *Further Terminology:* This statement is known as **Isbell's zigzag theorem**.

PROOF 6.3.3 ► PROOF OF PROPOSITION 6.3.2

Item 1: Characterisations

See [Isb68].

Item 2: Surjectivity on Objects

Omitted.

QUESTION 6.3.4 ► CHARACTERISATIONS OF FUNCTORS WITH EPIC PRE/POSTCOMPOSITION

Is there a characterisation of functors $F\colon \mathcal{C} \to \mathcal{D}$ such that:

1. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor

$$F^*\colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

   is an epimorphism of categories?

2. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor

$$F_*\colon \mathsf{Fun}(\mathcal{X}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{X}, \mathcal{D})$$

   is an epimorphism of categories?

This question also appears as [MO 468121a].

6.4 Pseudomonic Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

DEFINITION 6.4.1 ► PSEUDOMONIC FUNCTORS

A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **pseudomonic** if it satisfies the following conditions:

1. For all diagrams of the form

$$\begin{tikzcd} \mathcal{X} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & \mathcal{C} \arrow[r, "F"] & \mathcal{D}, \end{tikzcd}$$

if we have

$$\mathrm{id}_F \star \alpha = \mathrm{id}_F \star \beta,$$

then $\alpha = \beta$.

2. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$ and each natural isomorphism

$$\beta\colon F \circ \phi \overset{\sim}{\Longrightarrow} F \circ \psi, \qquad \begin{tikzcd} \mathcal{X} \arrow[r, bend left, "F\circ\phi"] \arrow[r, bend right, "F\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{D}, \end{tikzcd}$$

there exists a natural isomorphism

$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} \mathcal{X} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{C} \end{tikzcd}$$

such that we have an equality

$$\begin{tikzcd} \mathcal{X} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{C} \arrow[r, "F"] & \mathcal{D} \end{tikzcd} = \begin{tikzcd} \mathcal{X} \arrow[r, bend left, "F\circ\phi"] \arrow[r, bend right, "F\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{D} \end{tikzcd}$$

of pasting diagrams, i.e. such that we have

$$\beta = \mathrm{id}_F \star \alpha.$$

PROPOSITION 6.4.2 ► PROPERTIES OF PSEUDOMONIC FUNCTORS

Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor.

1. *Characterisations.* The following conditions are equivalent:
   (a) The functor F is pseudomonic.
   (b) The functor F satisfies the following conditions:
      i. The functor F is faithful, i.e. for each $A, B \in \mathrm{Obj}(\mathcal{C})$, the action on morphisms

$$F_{A,B}\colon \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F_A, F_B)$$

of F at (A, B) is injective.

ii. For each $A, B \in \mathrm{Obj}(\mathcal{C})$, the restriction

$$F^{\mathrm{iso}}_{A,B} \colon \mathrm{Iso}_{\mathcal{C}}(A, B) \longrightarrow \mathrm{Iso}_{\mathcal{D}}(F_A, F_B)$$

of the action on morphisms of F at (A, B) to isomorphisms is surjective.

(c) We have an isocomma square of the form

$$\mathcal{C} \overset{\text{eq.}}{\cong} \mathcal{C} \overset{\leftrightarrow}{\times}_{\mathcal{D}} \mathcal{C}, \qquad \begin{tikzcd} \mathcal{C} \arrow[r, "\mathrm{id}_{\mathcal{C}}"] \arrow[d, "\mathrm{id}_{\mathcal{C}}"'] \arrow[dr, phantom, "\Leftrightarrow"] & \mathcal{C} \arrow[d, "F"] \\ \mathcal{C} \arrow[r, "F"'] & \mathcal{D} \end{tikzcd}$$

in Cats_2 up to equivalence.

(d) We have an isocomma square of the form

$$\mathcal{C} \overset{\text{eq.}}{\cong} \mathcal{C} \overset{\leftrightarrow}{\times}_{\mathrm{Arr}(\mathcal{D})} \mathcal{D}, \qquad \begin{tikzcd} \mathcal{C} \arrow[r, hook] \arrow[d, "F"'] \arrow[dr, phantom, "\Leftrightarrow"] & \mathrm{Arr}(\mathcal{C}) \arrow[d, "\mathrm{Arr}(F)"] \\ \mathcal{D} \arrow[r, hook] & \mathrm{Arr}(\mathcal{D}) \end{tikzcd}$$

in Cats_2 up to equivalence.

(e) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition[1] functor

$$F_* \colon \mathrm{Fun}(\mathcal{X}, \mathcal{C}) \to \mathrm{Fun}(\mathcal{X}, \mathcal{D})$$

is pseudomonic.

2. *Conservativity.* If F is pseudomonic, then F is conservative.

3. *Essential Injectivity.* If F is pseudomonic, then F is essentially injective.

[1]Asking the precomposition functors

$$F^* \colon \mathrm{Fun}(\mathcal{D}, \mathcal{X}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{X})$$

to be pseudomonic leads to pseudoepic functors; see Item 1b of Item 1 of Proposition 6.5.2.

Proof 6.4.3 ► Proof of Proposition 6.4.2

Item 1: Characterisations

Omitted.

Item 2: Conservativity

Omitted.

Item 3: Essential Injectivity

Omitted.

6.5 Pseudoepic Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 6.5.1 ► Pseudoepic Functors

A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **pseudoepic** if it satisfies the following conditions:

1. For all diagrams of the form
$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] & \mathcal{D} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & \mathcal{X}, \end{tikzcd}$$
if we have
$$\alpha \star \mathrm{id}_F = \beta \star \mathrm{id}_F,$$
then $\alpha = \beta$.
2. For each $X \in \mathrm{Obj}(\mathcal{C})$ and each 2-isomorphism
$$\beta\colon \phi \circ F \overset{\sim}{\Longrightarrow} \psi \circ F, \qquad \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "\phi\circ F"] \arrow[r, bend right, "\psi\circ F"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{X} \end{tikzcd}$$
of $\mathcal{C}$, there exists a 2-isomorphism
$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{X} \end{tikzcd}$$

of $\mathcal{C}$ such that we have an equality

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] & \mathcal{D} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{X} \end{tikzcd} \quad = \quad \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "\phi\circ F"] \arrow[r, bend right, "\psi\circ F"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{X} \end{tikzcd}$$

of pasting diagrams in $\mathcal{C}$, i.e. such that we have

$$\beta = \alpha \star \mathrm{id}_F.$$

Proposition 6.5.2 ► Properties of Pseudoepic Functors

Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor.

1. *Characterisations.* The following conditions are equivalent:
   (a) The functor F is pseudoepic.
   (b) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the functor
   $$F^*\colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$
   given by precomposition by F is pseudomonic.
   (c) We have an isococomma square of the form
   $$\mathcal{D} \overset{\text{eq.}}{\cong} \mathcal{D} \overset{\leftrightarrow}{\coprod}_{\mathcal{C}} \mathcal{D}, \qquad \begin{tikzcd} \mathcal{D} & \mathcal{D} \arrow[l, "\mathrm{id}_{\mathcal{D}}"'] \arrow[dl, phantom, "\Leftrightarrow"] \\ \mathcal{D} \arrow[u, "\mathrm{id}_{\mathcal{D}}"] & \mathcal{C} \arrow[l, "F"] \arrow[u, "F"'] \end{tikzcd}$$
   in Cats_2 up to equivalence.
2. *Dominance.* If F is pseudoepic, then F is dominant (Definition 6.1.1).

Proof 6.5.3 ▶ Proof of Proposition 6.5.2

Item 1: Characterisations

Omitted.

Item 2: Dominance

If F is pseudoepic, then

$$F^*\colon \mathsf{Fun}(\mathcal{D},\mathcal{X}) \to \mathsf{Fun}(\mathcal{C},\mathcal{X})$$

is pseudomonic for all $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, and thus in particular faithful. By Item 4g of Item 4 of Proposition 5.1.2, this is equivalent to requiring F to be dominant. ∎

Question 6.5.4 ▶ Characterisations of Pseudoepic Functors

Is there a nice characterisation of the pseudoepic functors, similarly to the characterisaiton of pseudomonic functors given in Item 1b of Item 1 of Proposition 6.4.2? This question also appears as [MO 321971].

Question 6.5.5 ▶ Must a Pseudomonic and Pseudoepic Functor Be an Equivalence of Categories

A pseudomonic and pseudoepic functor is dominant, faithful, essentially injective, and full on isomorphisms. Is it necessarily an equivalence of categories? If not, how bad can this fail, i.e. how far can a pseudomonic and pseudoepic functor be from an equivalence of categories?
This question also appears as [MO 468334].

Question 6.5.6 ▶ Characterisations of Functors With Pseudoepic Pre/Postcomposition

Is there a characterisation of functors $F\colon \mathcal{C} \to \mathcal{D}$ such that:

1. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor

$$F^*\colon \mathsf{Fun}(\mathcal{D},\mathcal{X}) \to \mathsf{Fun}(\mathcal{C},\mathcal{X})$$

is pseudoepic?

2. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor

$$F_* \colon \mathsf{Fun}(\mathcal{X}, C) \to \mathsf{Fun}(\mathcal{X}, \mathcal{D})$$

is pseudoepic?

This question also appears as [MO 468121a].

7 Even More Conditions on Functors

7.1 Injective on Objects Functors

Let C and $\mathcal{D}$ be categories.

Definition 7.1.1 ► Injective on Objects Functors

A functor $F \colon C \to \mathcal{D}$ is **injective on objects** if the action on objects

$$F \colon \mathrm{Obj}(C) \to \mathrm{Obj}(\mathcal{D})$$

of F is injective.

Proposition 7.1.2 ► Properties of Injective on Objects Functors

Let $F \colon C \to \mathcal{D}$ be a functor.

1. *Characterisations.* The following conditions are equivalent:

   (a) The functor F is injective on objects.

   (b) The functor F is an isocofibration in Cats_2.

Proof 7.1.3 ► Proof of Proposition 7.1.2

Item 1: Characterisations

Omitted.

7.2 Surjective on Objects Functors

Let C and $\mathcal{D}$ be categories.

Definition 7.2.1 ► Surjective on Objects Functors

A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **surjective on objects** if the action on objects

$$F\colon \mathrm{Obj}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{D})$$

of F is surjective.

7.3 Bijective on Objects Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 7.3.1 ► Bijective on Objects Functors

A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **bijective on objects**[1] if the action on objects

$$F\colon \mathrm{Obj}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{D})$$

of F is a bijection.

[1] *Further Terminology:* Also called a **bo** functor.

7.4 Functors Representably Faithful on Cores

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 7.4.1 ► Functors Representably Faithful on Cores

A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **representably faithful on cores** if, for each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition by F functor

$$F_*\colon \mathrm{Core}(\mathrm{Fun}(\mathcal{X}, \mathcal{C})) \to \mathrm{Core}(\mathrm{Fun}(\mathcal{X}, \mathcal{D}))$$

is faithful.

Remark 7.4.2 ▶ Unwinding Definition 7.4.1

In detail, a functor $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ is **representably faithful on cores** if, given a diagram of the form

$$\begin{tikzcd} \mathcal{X} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & \mathcal{C} \arrow[r, "F"] & \mathcal{D}, \end{tikzcd}$$

if α and β are natural isomorphisms and we have

$$\mathrm{id}_F \star \alpha = \mathrm{id}_F \star \beta,$$

then $\alpha = \beta$.

Question 7.4.3 ▶ Characterisation of Functors Representably Faithful on Cores

Is there a characterisation of functors representably faithful on cores?

7.5 Functors Representably Full on Cores

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 7.5.1 ▶ Functors Representably Full on Cores

A functor $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ is **representably full on cores** if, for each $X \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition by F functor

$$F_*\colon \mathrm{Core}(\mathrm{Fun}(\mathcal{X}, \mathcal{C})) \longrightarrow \mathrm{Core}(\mathrm{Fun}(\mathcal{X}, \mathcal{D}))$$

is full.

Remark 7.5.2 ▶ Unwinding Definition 7.5.1

In detail, a functor $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ is **representably full on cores** if, for each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$ and each natural isomorphism

$$\beta\colon F \circ \phi \overset{\sim}{\Longrightarrow} F \circ \psi, \qquad \begin{tikzcd} \mathcal{X} \arrow[r, bend left, "F\circ\phi"] \arrow[r, bend right, "F\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{D}, \end{tikzcd}$$

there exists a natural isomorphism

$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} \mathcal{X} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{C} \end{tikzcd}$$

such that we have an equality

$$\begin{tikzcd} \mathcal{X} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{C} \arrow[r, "F"] & \mathcal{D} \end{tikzcd} = \begin{tikzcd} \mathcal{X} \arrow[r, bend left, "F\circ\phi"] \arrow[r, bend right, "F\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{D} \end{tikzcd}$$

of pasting diagrams in Cats_2, i.e. such that we have

$$\beta = \mathrm{id}_F \star \alpha.$$

Question 7.5.3 ► Characterisation of Functors Representably Full on Cores

Is there a characterisation of functors representably full on cores?
This question also appears as [MO 468121a].

7.6 Functors Representably Fully Faithful on Cores

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 7.6.1 ► Functors Representably Fully Faithful on Cores

A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **representably fully faithful on cores** if, for each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition by F functor

$$F_*\colon \mathrm{Core}(\mathrm{Fun}(\mathcal{X}, \mathcal{C})) \to \mathrm{Core}(\mathrm{Fun}(\mathcal{X}, \mathcal{D}))$$

is fully faithful.

Remark 7.6.2 ▶ Unwinding Definition 7.6.1

In detail, a functor $F\colon \mathcal{C} \to \mathcal{D}$ is **representably fully faithful on cores** if it satisfies the conditions in Remarks 7.4.2 and 7.5.2, i.e.:

1. For all diagrams of the form

$$\begin{tikzcd} \mathcal{X} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha \Downarrow\Downarrow \beta"] & \mathcal{C} \arrow[r, "F"] & \mathcal{D}, \end{tikzcd}$$

   with α and β natural isomorphisms, if we have $\mathrm{id}_F \star \alpha = \mathrm{id}_F \star \beta$, then $\alpha = \beta$.

2. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$ and each natural isomorphism

$$\beta\colon F \circ \phi \overset{\sim}{\Longrightarrow} F \circ \psi, \qquad \begin{tikzcd} \mathcal{X} \arrow[r, bend left, "F\circ\phi"] \arrow[r, bend right, "F\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{D} \end{tikzcd}$$

   of $\mathcal{C}$, there exists a natural isomorphism

$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} \mathcal{X} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{C} \end{tikzcd}$$

   of $\mathcal{C}$ such that we have an equality

$$\begin{tikzcd} \mathcal{X} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{C} \arrow[r, "F"] & \mathcal{D} \end{tikzcd} = \begin{tikzcd} \mathcal{X} \arrow[r, bend left, "F\circ\phi"] \arrow[r, bend right, "F\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{D} \end{tikzcd}$$

   of pasting diagrams in Cats_2, i.e. such that we have

$$\beta = \mathrm{id}_F \star \alpha.$$

Question 7.6.3 ▶ Characterisation of Functors Representably Fully Faithful on Cores

Is there a characterisation of functors representably fully faithful on cores?

7.7 Functors Corepresentably Faithful on Cores

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 7.7.1 ▶ Functors Corepresentably Faithful on Cores

A functor $F \colon \mathcal{C} \to \mathcal{D}$ is **corepresentably faithful on cores** if, for each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition by F functor

$$F_* \colon \mathsf{Core}(\mathsf{Fun}(\mathcal{X}, \mathcal{C})) \to \mathsf{Core}(\mathsf{Fun}(\mathcal{X}, \mathcal{D}))$$

is faithful.

Remark 7.7.2 ▶ Unwinding Definition 7.7.1

In detail, a functor $F \colon \mathcal{C} \to \mathcal{D}$ is **corepresentably faithful on cores** if, given a diagram of the form

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] & \mathcal{D} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha \Downarrow \Downarrow \beta"] & \mathcal{X}, \end{tikzcd}$$

if α and β are natural isomorphisms and we have

$$\alpha \star \mathrm{id}_F = \beta \star \mathrm{id}_F,$$

then $\alpha = \beta$.

Question 7.7.3 ▶ Characterisation of Functors Corepresentably Faithful on Cores

Is there a characterisation of functors corepresentably faithful on cores?

7.8 Functors Corepresentably Full on Cores

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 7.8.1 ► Functors Corepresentably Full on Cores

A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **corepresentably full on cores** if, for each $X \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition by F functor

$$F_*\colon \mathsf{Core}(\mathsf{Fun}(\mathcal{X}, \mathcal{C})) \to \mathsf{Core}(\mathsf{Fun}(\mathcal{X}, \mathcal{D}))$$

is full.

Remark 7.8.2 ► Unwinding Definition 7.8.1

In detail, a functor $F\colon \mathcal{C} \to \mathcal{D}$ is **corepresentably full on cores** if, for each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$ and each natural isomorphism

$$\beta\colon \phi \circ F \overset{\sim}{\Longrightarrow} \psi \circ F, \qquad \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "\phi\circ F"] \arrow[r, bend right, "\psi\circ F"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{X}, \end{tikzcd}$$

there exists a natural isomorphism

$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{X} \end{tikzcd}$$

such that we have an equality

$$\begin{tikzcd} \mathcal{X} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{C} \arrow[r, "F"] & \mathcal{D} \end{tikzcd} \quad = \quad \begin{tikzcd} \mathcal{X} \arrow[r, bend left, "F\circ\phi"] \arrow[r, bend right, "F\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{D} \end{tikzcd}$$

of pasting diagrams in Cats_2, i.e. such that we have

$$\beta = \alpha \star \mathrm{id}_F.$$

Question 7.8.3 ► Characterisation of Functors Corepresentably Full on Cores

Is there a characterisation of functors corepresentably full on cores?
This question also appears as [MO 468121a].

7.9 Functors Corepresentably Fully Faithful on Cores

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 7.9.1 ▶ Functors Corepresentably Fully Faithful on Cores

A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **corepresentably fully faithful on cores** if, for each $X \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition by F functor

$$F_*\colon \mathsf{Core}(\mathsf{Fun}(\mathcal{X}, \mathcal{C})) \to \mathsf{Core}(\mathsf{Fun}(\mathcal{X}, \mathcal{D}))$$

is fully faithful.

Remark 7.9.2 ▶ Unwinding Definition 7.9.1

In detail, a functor $F\colon \mathcal{C} \to \mathcal{D}$ is **corepresentably fully faithful on cores** if it satisfies the conditions in Remarks 7.7.2 and 7.8.2, i.e.:

1. For all diagrams of the form

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] & \mathcal{D} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & \mathcal{X}, \end{tikzcd}$$

if α and β are natural isomorphisms and we have

$$\alpha \star \mathrm{id}_F = \beta \star \mathrm{id}_F,$$

then $\alpha = \beta$.

2. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$ and each natural isomorphism

$$\beta\colon \phi \circ F \overset{\sim}{\Longrightarrow} \psi \circ F, \qquad \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "\phi\circ F"] \arrow[r, bend right, "\psi\circ F"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{X}, \end{tikzcd}$$

there exists a natural isomorphism

$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{X} \end{tikzcd}$$

such that we have an equality

$$
\begin{tikzcd}
\mathcal{X} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{C} \arrow[r, "F"] & \mathcal{D}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathcal{X} \arrow[r, bend left, "F\circ\phi"] \arrow[r, bend right, "F\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{D}
\end{tikzcd}
$$

of pasting diagrams in Cats_2, i.e. such that we have

$$\beta = \alpha \star \mathrm{id}_F.$$

QUESTION 7.9.3 ▶ CHARACTERISATION OF FUNCTORS COREPRESENTABLY FULLY FAITHFUL ON CORES

Is there a characterisation of functors corepresentably fully faithful on cores?

8 Natural Transformations

8.1 Transformations

Let $\mathcal{C}$ and $\mathcal{D}$ be categories and $F, G\colon \mathcal{C} \rightrightarrows \mathcal{D}$ be functors.

DEFINITION 8.1.1 ▶ TRANSFORMATIONS

A **transformation**[1] $\alpha\colon F \Rightarrow G$ **from** F **to** G is a collection

$$\{\alpha_A\colon F(A) \to G(A)\}_{A\in\mathrm{Obj}(\mathcal{C})}$$

of morphisms of $\mathcal{D}$.

[1] *Further Terminology:* Also called an **unnatural transformation** for emphasis.

NOTATION 8.1.2 ▶ THE SET OF TRANSFORMATIONS BETWEEN TWO FUNCTORS

We write $\mathrm{Trans}(F, G)$ for the set of transformations from F to G.

8.2 Natural Transformations

Let $\mathcal{C}$ and $\mathcal{D}$ be categories and $F, G\colon \mathcal{C} \rightrightarrows \mathcal{D}$ be functors.

Definition 8.2.1 ▶ Natural Transformations

A **natural transformation** $\alpha\colon F \Longrightarrow G$ **from** F **to** G is a transformation

$$\{\alpha_A\colon F(A) \longrightarrow G(A)\}_{A\in\mathsf{Obj}(\mathcal{C})}$$

from F to G such that, for each morphism $f\colon A \longrightarrow B$ of $\mathcal{C}$, the diagram

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\alpha_A"'] & F(B) \arrow[d, "\alpha_B"] \\ G(A) \arrow[r, "G(f)"'] & G(B) \end{tikzcd}$$

commutes.[1]

[1] *Further Terminology:* The morphism $\alpha_A\colon F_A \longrightarrow G_A$ is called the **component of** α **at** A.

Remark 8.2.2 ▶ Picturing Natural Transformations in Diagrams

We denote natural transformations in diagrams as

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"{name=U}] \arrow[r, bend right, "G"'{name=D}] & \mathcal{D}. \arrow[from=U, to=D, Rightarrow, "\alpha"'] \end{tikzcd}$$

Notation 8.2.3 ▶ The Set of Natural Transformations Between Two Functors

We write $\mathsf{Nat}(F, G)$ for the set of natural transformations from F to G.

Example 8.2.4 ▶ Identity Natural Transformations

The **identity natural transformation** $\mathrm{id}_F\colon F \Longrightarrow F$ **of** F is the natural transformation consisting of the collection

$$\left\{\mathrm{id}_{F(A)}\colon F(A) \longrightarrow F(A)\right\}_{A\in\mathsf{Obj}(\mathcal{C})}.$$

Proof 8.2.5 ▶ Proof of Example 8.2.4

The naturality condition for id_F is the requirement that, for each morphism $f: A \to B$ of C, the diagram

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\mathrm{id}_{F(A)}"'] & F(B) \arrow[d, "\mathrm{id}_{F(B)}"] \\ F(A) \arrow[r, "F(f)"'] & F(B) \end{tikzcd}$$

commutes, which follows from unitality of the composition of C.

Definition 8.2.6 ▶ Equality of Natural Transformations

Two natural transformations $\alpha, \beta: F \Longrightarrow G$ are **equal** if we have

$$\alpha_A = \beta_A$$

for each $A \in \mathrm{Obj}(C)$.

8.3 Vertical Composition of Natural Transformations

Definition 8.3.1 ▶ Vertical Composition of Natural Transformations

The **vertical composition** of two natural transformations $\alpha: F \Longrightarrow G$ and $\beta: G \Longrightarrow H$ as in the diagram

$$\begin{tikzcd}[row sep=small] {} & {} \arrow[d, phantom, "\alpha\Downarrow"] & {} \\ C \arrow[rr, bend left=60, "F"] \arrow[rr, "G" description] \arrow[rr, bend right=60, "H"'] & {} \arrow[d, phantom, "\beta\Downarrow"] & \mathcal{D} \\ {} & {} & {} \end{tikzcd}$$

is the natural transformation $\beta \circ \alpha: F \Longrightarrow H$ consisting of the collection

$$\{(\beta \circ \alpha)_A: F(A) \to H(A)\}_{A \in \mathrm{Obj}(C)}$$

with

$$(\beta \circ \alpha)_A \stackrel{\mathrm{def}}{=} \beta_A \circ \alpha_A$$

for each $A \in \mathrm{Obj}(\mathcal{C})$.

Proof 8.3.2 ▶ Proof of Definition 8.3.1

The naturality condition for $\beta \circ \alpha$ is the requirement that the boundary of the diagram

$$\begin{tikzcd}
F(A) \arrow[r, "F(f)"] \arrow[d, "\alpha_A"'] \arrow[dr, phantom, "(1)"] & F(B) \arrow[d, "\alpha_B"] \\
G(A) \arrow[r, "G(f)" description] \arrow[d, "\beta_A"'] \arrow[dr, phantom, "(2)"] & G(B) \arrow[d, "\beta_B"] \\
H(A) \arrow[r, "H(f)"'] & H(B)
\end{tikzcd}$$

commutes. Since

1. Subdiagram (1) commutes by the naturality of α.
2. Subdiagram (2) commutes by the naturality of β.

so does the boundary diagram. Hence $\beta \circ \alpha$ is a natural transformation.

Proposition 8.3.3 ▶ Properties of Vertical Composition of Natural Transformations

Let $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ be categories.

1. *Functionality.* The assignment $(\beta, \alpha) \mapsto \beta \circ \alpha$ defines a function

$$\circ_{F,G,H} \colon \mathrm{Nat}(G, H) \times \mathrm{Nat}(F, G) \longrightarrow \mathrm{Nat}(F, H).$$

2. *Associativity.* Let $F, G, H, K \colon \mathcal{C} \overset{\rightrightarrows}{\rightrightarrows} \mathcal{D}$ be functors. The diagram

$$\begin{tikzcd}
 & & \mathsf{Nat}(H,K) \times (\mathsf{Nat}(G,H) \times \mathsf{Nat}(F,G)) \arrow[drr, "\mathrm{id}_{\mathsf{Nat}(H,K)} \times \circ_{F,G,H}"] & & \\
(\mathsf{Nat}(H,K) \times \mathsf{Nat}(G,H)) \times \mathsf{Nat}(F,G) \arrow[urr, dashed, "\alpha^{\mathsf{Sets}}_{\mathsf{Nat}(H,K),\mathsf{Nat}(G,H),\mathsf{Nat}(F,G)}", "\sim"'] \arrow[dr, "\circ_{G,H,K} \times \mathrm{id}_{\mathsf{Nat}(F,G)}"'] & & & & \mathsf{Nat}(H,K) \times \mathsf{Nat}(F,H) \arrow[dl, "\circ_{F,H,K}"] \\
 & \mathsf{Nat}(G,K) \times \mathsf{Nat}(F,G) \arrow[rr, "\circ_{F,G,K}"'] & & \mathsf{Nat}(F,K) &
\end{tikzcd}$$

commutes, i.e. given natural transformations

$$F \overset{\alpha}{\Longrightarrow} G \overset{\beta}{\Longrightarrow} H \overset{\gamma}{\Longrightarrow} K,$$

we have

$$(\gamma \circ \beta) \circ \alpha = \gamma \circ (\beta \circ \alpha).$$

3. *Unitality.* Let $F, G \colon \mathcal{C} \rightrightarrows \mathcal{D}$ be functors.

   (a) *Left Unitality.* The diagram

$$\begin{tikzcd}
\mathrm{pt} \times \mathsf{Nat}(F,G) \arrow[d, "{[\mathrm{id}_G] \times \mathrm{id}_{\mathsf{Nat}(F,G)}}"'] \arrow[dr, dashed, "\lambda^{\mathsf{Sets}}_{\mathsf{Nat}(F,G)}", "\sim"'] & \\
\mathsf{Nat}(G,G) \times \mathsf{Nat}(F,G) \arrow[r, "\circ_{F,G,G}"'] & \mathsf{Nat}(F,G)
\end{tikzcd}$$

   commutes, i.e. given a natural transformation $\alpha \colon F \Longrightarrow G$, we have

$$\mathrm{id}_G \circ \alpha = \alpha.$$

(b) *Right Unitality.* The diagram

$$
\begin{tikzcd}
\mathsf{Nat}(F,G)\times\mathrm{pt} \arrow[d, "\mathrm{id}_{\mathsf{Nat}(F,G)}\times[\mathrm{id}_F]"'] \arrow[dr, dashed, "\rho^{\mathsf{Sets}}_{\mathsf{Nat}(F,G)}", "\sim"'] & \\
\mathsf{Nat}(F,G)\times\mathsf{Nat}(F,F) \arrow[r, "\circ^{\mathcal{C}}_{F,F,G}"'] & \mathsf{Nat}(F,G)
\end{tikzcd}
$$

commutes, i.e. given a natural transformation $\alpha\colon F \Longrightarrow G$, we have

$$\alpha \circ \mathrm{id}_F = \alpha.$$

4. *Middle Four Exchange.* Let $F_1, F_2, F_3\colon \mathcal{C} \to \mathcal{D}$ and $G_1, G_2, G_3\colon \mathcal{D} \to \mathcal{E}$ be functors. The diagram

$$
\begin{tikzcd}
(\mathsf{Nat}(G_2,G_3)\times\mathsf{Nat}(G_1,G_2))\times(\mathsf{Nat}(F_2,F_3)\times\mathsf{Nat}(F_1,F_2)) \arrow[rr, dashed, leftrightarrow, "\mu_4", "\sim"'] \arrow[d, "\circ_{G_1,G_2,G_3}\times\circ_{F_1,F_2,F_3}"'] & & (\mathsf{Nat}(G_2,G_3)\times\mathsf{Nat}(F_2,F_3))\times(\mathsf{Nat}(G_1,G_2)\times\mathsf{Nat}(F_1,F_2)) \arrow[d, "\star_{F_2,F_3,G_2,G_3}\times\star_{F_1,F_2,G_1,G_2}"] \\
\mathsf{Nat}(G_1,G_3)\times\mathsf{Nat}(F_1,F_3) \arrow[dr, "\star_{F_1,F_3,G_1,G_3}"'] & & \mathsf{Nat}(G_2\circ F_2, G_3\circ F_3)\times\mathsf{Nat}(G_1\circ F_1, G_2\circ F_2) \arrow[dl, "\circ_{G_1\circ F_1, G_2\circ F_2, G_3\circ F_3}"] \\
& \mathsf{Nat}(G_1\circ F_1, G_3\circ F_3) &
\end{tikzcd}
$$

commutes, i.e. given a diagram

$$
\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=60, "F_1"] \arrow[r, "F_2" description] \arrow[r, bend right=60, "F_3"'] \arrow[r, phantom, shift left=3, "\alpha\Downarrow"] \arrow[r, phantom, shift right=3, "\alpha'\Downarrow"] & \mathcal{D} \arrow[r, bend left=60, "G_1"] \arrow[r, "G_2" description] \arrow[r, bend right=60, "G_3"'] \arrow[r, phantom, shift left=3, "\beta\Downarrow"] \arrow[r, phantom, shift right=3, "\beta'\Downarrow"] & \mathcal{E}
\end{tikzcd}
$$

in Cats_2, we have

$$(\beta' \star \alpha') \circ (\beta \star \alpha) = (\beta' \circ \beta) \star (\alpha' \circ \alpha).$$

Proof 8.3.4 ▶ Proof of Proposition 8.3.3

Item 1: Functionality

Clear.

Item 2: Associativity

Indeed, we have

$$\begin{aligned}((\gamma\circ\beta)\circ\alpha)_A &\overset{\text{def}}{=} (\gamma\circ\beta)_A\circ\alpha_A\\ &\overset{\text{def}}{=} (\gamma_A\circ\beta_A)\circ\alpha_A\\ &= \gamma_A\circ(\beta_A\circ\alpha_A)\\ &\overset{\text{def}}{=} \gamma_A\circ(\beta\circ\alpha)_A\\ &\overset{\text{def}}{=} (\gamma\circ(\beta\circ\alpha))_A\end{aligned}$$

for each $A\in\mathrm{Obj}(C)$, showing the desired equality.

Item 3: Unitality

We have

$$\begin{aligned}(\mathrm{id}_G\circ\alpha)_A &= \mathrm{id}_G\circ\alpha_A\\ &= \alpha_A,\\ (\alpha\circ\mathrm{id}_F)_A &= \alpha_A\circ\mathrm{id}_F\\ &= \alpha_A\end{aligned}$$

for each $A\in\mathrm{Obj}(C)$, showing the desired equality.

Item 4: Middle Four Exchange

This is proved in Item 4 of Proposition 8.4.4.

8.4 Horizontal Composition of Natural Transformations

Definition 8.4.1 ► Horizontal Composition of Natural Transformations

The **horizontal composition**[1,2] of two natural transformations $\alpha\colon F \Longrightarrow G$ and $\beta\colon H \Longrightarrow K$ as in the diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{D} \arrow[r, bend left, "H"] \arrow[r, bend right, "K"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{E} \end{tikzcd}$$

of α and β is the natural transformation

$$\beta \star \alpha\colon (H \circ F) \Longrightarrow (K \circ G),$$

as in the diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "H\circ F"] \arrow[r, bend right, "K\circ G"'] \arrow[r, phantom, "\beta\star\alpha \Downarrow"] & \mathcal{E}, \end{tikzcd}$$

consisting of the collection

$$\{(\beta \star \alpha)_A\colon H(F(A)) \to K(G(A))\}_{A\in\mathrm{Obj}(\mathcal{C})},$$

of morphisms of $\mathcal{E}$ with

$$\begin{aligned}(\beta \star \alpha)_A &\overset{\text{def}}{=} \beta_{G(A)} \circ H(\alpha_A)\\ &= K(\alpha_A) \circ \beta_{F(A)},\end{aligned} \qquad \begin{tikzcd} H(F(A)) \arrow[r, "H(\alpha_A)"] \arrow[d, "\beta_{F(A)}"'] & H(G(A)) \arrow[d, "\beta_{G(A)}"] \\ K(F(A)) \arrow[r, "K(\alpha_A)"'] & K(G(A)). \end{tikzcd}$$

[1] *Further Terminology:* Also called the **Godement product** of α and β.

[2] Horizontal composition forms a map

$$\star_{(F,H),(G,K)}\colon \mathsf{Nat}(H,K) \times \mathsf{Nat}(F,G) \to \mathsf{Nat}(H \circ F, K \circ G).$$

Proof 8.4.2 ▶ Proof of Definition 8.4.1

First, we claim that we indeed have

$$\beta_{G(A)} \circ H(\alpha_A) = K(\alpha_A) \circ \beta_{F(A)}, \qquad \begin{tikzcd} H(F(A)) \arrow[r, "H(\alpha_A)"] \arrow[d, "\beta_{F(A)}"'] & H(G(A)) \arrow[d, "\beta_{G(A)}"] \\ K(F(A)) \arrow[r, "K(\alpha_A)"'] & K(G(A)). \end{tikzcd}$$

This is, however, simply the naturality square for β applied to the morphism $\alpha_A\colon F(A) \to G(A)$. Next, we check the naturality condition for $\beta \star \alpha$, which is the requirement that the boundary of the diagram

$$\begin{tikzcd} H(F(A)) \arrow[r, "H(F(f))"] \arrow[d, "H(\alpha_A)"'] \arrow[dr, phantom, "(1)"] & H(F(B)) \arrow[d, "H(\alpha_B)"] \\ H(G(A)) \arrow[r, "H(G(f))"] \arrow[d, "\beta_{G(A)}"'] \arrow[dr, phantom, "(2)"] & H(G(B)) \arrow[d, "\beta_{G(B)}"] \\ K(G(A)) \arrow[r, "K(G(f))"'] & K(G(B)) \end{tikzcd}$$

commutes. Since

1. Subdiagram (1) commutes by the naturality of α.
2. Subdiagram (2) commutes by the naturality of β.

so does the boundary diagram. Hence $\beta \circ \alpha$ is a natural transformation.[1]

[1] *Reference:* [Bor94, Proposition 1.3.4].

Definition 8.4.3 ▶ Whiskering of Functors With Natural Transformations

Let

$$\begin{tikzcd} \mathcal{X} \arrow[r, "F"] & \mathcal{C} \arrow[r, bend left, "\phi"{name=U}] \arrow[r, bend right, "\psi"'{name=D}] \arrow[from=U, to=D, Rightarrow, "\alpha"'] & \mathcal{D} \arrow[r, "G"] & \mathcal{Y} \end{tikzcd}$$

be a diagram in Cats_2.

1. The **left whiskering of** α **with** G is the natural transformation[1]

$$\mathrm{id}_G \star \alpha \colon G \circ \phi \Longrightarrow G \circ \psi.$$

2. The **right whiskering of** α **with** F is the natural transformation[2]

$$\alpha \star \mathrm{id}_F \colon \phi \circ F \Longrightarrow \psi \circ F.$$

[1] *Further Notation:* Also written $G\alpha$ or $G \star \alpha$, although we won't use either of these notations in this work.

[2] *Further Notation:* Also written αF or $\alpha \star F$, although we won't use either of these notations in this work.

Proposition 8.4.4 ▶ Properties of Horizontal Composition of Natural Transformations

Let $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ be categories.

1. *Functionality.* The assignment $(\beta, \alpha) \mapsto \beta \star \alpha$ defines a function

$$\star_{(F,G),(H,K)} \colon \mathsf{Nat}(H, K) \times \mathsf{Nat}(F, G) \longrightarrow \mathsf{Nat}(H \circ F, K \circ G).$$

2. *Associativity.* Let

$$\mathcal{C} \underset{G_1}{\overset{F_1}{\rightrightarrows}} \mathcal{D} \underset{G_2}{\overset{F_2}{\rightrightarrows}} \mathcal{E} \underset{G_3}{\overset{F_3}{\rightrightarrows}} \mathcal{F}$$

be a diagram in Cats_2. The diagram

$$\begin{tikzcd} \mathsf{Nat}(F_3, G_3) \times \mathsf{Nat}(F_2, G_2) \times \mathsf{Nat}(F_1, G_1) \arrow[r, "\star_{(F_2,G_2),(F_3,G_3)} \times \mathrm{id}"] \arrow[d, "\mathrm{id} \times \star_{(F_1,G_1),(F_2,G_2)}"'] & \mathsf{Nat}(F_3 \circ F_2, G_3 \circ G_2) \times \mathsf{Nat}(F_1, G_1) \arrow[d, "\star_{(F_3 \circ F_2),(G_3 \circ G_2,F_1,G_1)}"] \\ \mathsf{Nat}(F_3, G_3) \times \mathsf{Nat}(F_2 \circ F_1, G_2 \circ G_1) \arrow[r, "\star_{(F_2 \circ F_1),(G_2 \circ G_1,F_3,G_3)}"'] & \mathsf{Nat}(F_3 \circ F_2 \circ F_1, G_3 \circ G_2 \circ G_1) \end{tikzcd}$$

commutes, i.e. given natural transformations

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F_1"{name=U1}] \arrow[r, bend right, "G_1"'{name=D1}] & \mathcal{D} \arrow[r, bend left, "F_2"{name=U2}] \arrow[r, bend right, "G_2"'{name=D2}] & \mathcal{E} \arrow[r, bend left, "F_3"{name=U3}] \arrow[r, bend right, "G_3"'{name=D3}] & \mathcal{F}, \arrow[from=U1, to=D1, Rightarrow, "\alpha"', phantom] \arrow[from=U2, to=D2, Rightarrow, "\beta"', phantom] \arrow[from=U3, to=D3, Rightarrow, "\gamma"', phantom] \end{tikzcd}$$

we have

$$(\gamma \star \beta) \star \alpha = \gamma \star (\beta \star \alpha).$$

3. *Interaction With Identities.* Let $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{E}$ be functors. The diagram

$$\begin{tikzcd} \mathrm{pt} \times \mathrm{pt} \arrow[r, "{[\mathrm{id}_G] \times [\mathrm{id}_F]}"] & \mathrm{Nat}(G, G) \times \mathrm{Nat}(F, F) \arrow[d, "\star_{(F,F),(G,G)}"] \\ \mathrm{pt} \arrow[u, dashed, "\wr"] \arrow[r, "{[\mathrm{id}_{G \circ F}]}"'] & \mathrm{Nat}(G \circ F, G \circ F) \end{tikzcd}$$

commutes, i.e. we have

$$\mathrm{id}_G \star \mathrm{id}_F = \mathrm{id}_{G \circ F}.$$

4. *Middle Four Exchange.* Let $F_1, F_2, F_3\colon \mathcal{C} \to \mathcal{D}$ and $G_1, G_2, G_3\colon \mathcal{D} \to \mathcal{E}$ be functors. The diagram

$$\begin{tikzcd}[column sep=tiny] (\mathrm{Nat}(G_2, G_3) \times \mathrm{Nat}(G_1, G_2)) \times (\mathrm{Nat}(F_2, F_3) \times \mathrm{Nat}(F_1, F_2)) \arrow[rr, dashed, leftrightarrow, "\mu_4", "\sim"'] \arrow[d, "\circ_{G_1,G_2,G_3} \times \circ_{F_1,F_2,F_3}"'] & & (\mathrm{Nat}(G_2, G_3) \times \mathrm{Nat}(F_2, F_3)) \times (\mathrm{Nat}(G_1, G_2) \times \mathrm{Nat}(F_1, F_2)) \arrow[d, "\star_{F_2,F_3,G_2,G_3} \times \star_{F_1,F_2,G_1,G_2}"] \\ \mathrm{Nat}(G_1, G_3) \times \mathrm{Nat}(F_1, F_3) \arrow[dr, "\star_{F_1,F_3,G_1,G_3}"'] & & \mathrm{Nat}(G_2 \circ F_2, G_3 \circ F_3) \times \mathrm{Nat}(G_1 \circ F_1, G_2 \circ F_2) \arrow[dl, "\circ_{G_1 \circ F_1, G_2 \circ F_2, G_3 \circ F_3}"] \\ & \mathrm{Nat}(G_1 \circ F_1, G_3 \circ F_3) & \end{tikzcd}$$

commutes, i.e. given a diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=50, "F_1"{name=A}] \arrow[r, "F_2"{name=B, description}] \arrow[r, bend right=50, "F_3"'{name=C}] & \mathcal{D} \arrow[r, bend left=50, "G_1"{name=D}] \arrow[r, "G_2"{name=E, description}] \arrow[r, bend right=50, "G_3"'{name=F}] & \mathcal{E} \arrow[from=A, to=B, Rightarrow, "\alpha"', shorten=2pt] \arrow[from=B, to=C, Rightarrow, "\alpha'"', shorten=2pt] \arrow[from=D, to=E, Rightarrow, "\beta"', shorten=2pt] \arrow[from=E, to=F, Rightarrow, "\beta'"', shorten=2pt] \end{tikzcd}$$

in Cats_2, we have

$$(\beta' \star \alpha') \circ (\beta \star \alpha) = (\beta' \circ \beta) \star (\alpha' \circ \alpha).$$

Proof 8.4.5 ▶ Proof of Proposition 8.4.4

Item 1: Functionality

Clear.

Item 2: Associativity

Omitted.

Item 3: Interaction With Identities

We have

$$\begin{aligned}(\mathrm{id}_G \star \mathrm{id}_F)_A &\overset{\text{def}}{=} (\mathrm{id}_G)_{F_A} \circ G_{(\mathrm{id}_F)_A} \\ &\overset{\text{def}}{=} \mathrm{id}_{G_{F_A}} \circ G_{\mathrm{id}_{F_A}} \\ &= \mathrm{id}_{G_{F_A}} \circ \mathrm{id}_{G_{F_A}} \\ &= \mathrm{id}_{G_{F_A}} \\ &\overset{\text{def}}{=} (\mathrm{id}_{G\circ F})_A\end{aligned}$$

for each $A \in \mathrm{Obj}(\mathcal{C})$, showing the desired equality.

Item 4: Middle Four Exchange

Let $A \in \mathrm{Obj}(\mathcal{C})$ and consider the diagram

$$\begin{tikzcd} & & G_1(F_3(A)) \arrow[dr, "\beta_{F_3(A)}"] & & \\ G_1(F_1(A)) \arrow[r, "G_1(\alpha_A)"] & G_1(F_2(A)) \arrow[ur, "G_1(\alpha'_A)"] \arrow[dr, "\beta_{F_2(A)}"'] \arrow[rr, phantom, "(1)"] & & G_2(F_3(A)) \arrow[r, "\beta'_{F_3(A)}"] & G_3(F_3(A)). \\ & & G_2(F_2(A)) \arrow[ur, "G_2(\alpha'_A)"'] & & \end{tikzcd}$$

The top composition

$$
\begin{tikzcd}
 & & G_1(F_3(A)) \arrow[dr, "\beta_{F_3(A)}"] & & \\
G_1(F_1(A)) \arrow[r, "G_1(\alpha_A)"] & G_1(F_2(A)) \arrow[ur, "G_1(\alpha'_A)"] \arrow[dr, "\beta_{F_2(A)}"'] & (1) & G_2(F_3(A)) \arrow[r, "\beta'_{F_3(A)}"] & G_3(F_3(A)). \\
 & & G_2(F_2(A)) \arrow[ur, "G_2(\alpha'_A)"'] & &
\end{tikzcd}
$$

is given by $((\beta' \circ \beta) \star (\alpha' \circ \alpha))_A$, while the bottom composition

$$
\begin{tikzcd}
 & & G_1(F_3(A)) \arrow[dr, "\beta_{F_3(A)}"] & & \\
G_1(F_1(A)) \arrow[r, "G_1(\alpha_A)"] & G_1(F_2(A)) \arrow[ur, "G_1(\alpha'_A)"] \arrow[dr, "\beta_{F_2(A)}"'] & (1) & G_2(F_3(A)) \arrow[r, "\beta'_{F_3(A)}"] & G_3(F_3(A)). \\
 & & G_2(F_2(A)) \arrow[ur, "G_2(\alpha'_A)"'] & &
\end{tikzcd}
$$

is given by $((\beta' \star \alpha') \circ (\beta \star \alpha))_A$. Now, Subdiagram (1) corresponds to the naturality condition

$$
G_2(\alpha'_A) \circ \beta_{F_2(A)} = \beta_{F_3}(A) \circ G_1(\alpha'_A), \qquad
\begin{tikzcd}
G_1(F_2(A)) \arrow[r, "G_1(\alpha'_A)"] \arrow[d, "\beta_{F_2(A)}"'] & G_1(F_3(A)) \arrow[d, "\beta_{F_3(A)}"] \\
G_2(F_2(A)) \arrow[r, "G_2(\alpha'_A)"'] & G_2(F_3(A))
\end{tikzcd}
$$

for $\beta \colon G_1 \Longrightarrow G_2$ at $\alpha'_A \colon F_2(A) \to F_3(A)$, and thus commutes. Thus we have

$$((\beta' \circ \beta) \star (\alpha' \circ \alpha))_A = ((\beta' \star \alpha') \circ (\beta \star \alpha))_A$$

for each $A \in \mathrm{Obj}(\mathcal{C})$ and therefore

$$(\beta' \star \alpha') \circ (\beta \star \alpha) = (\beta' \circ \beta) \star (\alpha' \circ \alpha).$$

This finishes the proof.

8.5 Properties of Natural Transformations

Proposition 8.5.1 ► Natural Transformations as Categorical Homotopies

Let $F, G \colon \mathcal{C} \rightrightarrows \mathcal{D}$ be functors. The following data are equivalent:[1]

1. A natural transformation $\alpha \colon F \Longrightarrow G$.
2. A functor $[\alpha] \colon \mathcal{C} \to \mathcal{D}^{\mathbb{1}}$ filling the diagram

$$\begin{tikzcd} & \mathcal{D} \\ \mathcal{C} \arrow[ur, "F"] \arrow[r, "{[\alpha]}"] \arrow[dr, "G"'] & \mathcal{D}^{\mathbb{1}}. \arrow[u, two heads, "\mathrm{ev}_0"'] \arrow[d, two heads, "\mathrm{ev}_1"] \\ & \mathcal{D} \end{tikzcd}$$

3. A functor $[\alpha] \colon \mathcal{C} \times \mathbb{1} \to \mathcal{D}$ filling the diagram

$$\begin{tikzcd} \mathcal{C} \arrow[dr, "F"] & \\ \mathcal{C} \times \mathbb{1} \arrow[u, two heads, "\mathrm{ev}_0"] \arrow[d, two heads, "\mathrm{ev}_1"'] \arrow[r, "{[\alpha]}"] & \mathcal{D}. \\ \mathcal{C} \arrow[ur, "G"'] & \end{tikzcd}$$

[1] Taken from [MO 64365].

Proof 8.5.2 ▶ Proof of Proposition 8.5.1

From Item 1 to Item 2 and Back

We may identify $\mathcal{D}^{\mathbb{1}}$ with $\mathrm{Arr}(\mathcal{D})$. Given a natural transformation $\alpha\colon F \Longrightarrow G$, we have a functor

$$
\begin{aligned}
[\alpha]\colon \mathcal{C} &\longrightarrow \mathcal{D}^{\mathbb{1}}\\
A &\longmapsto \alpha_A\\
(f\colon A \to B) &\longmapsto \left(\begin{tikzcd} F_A \arrow[r, "F_f"] \arrow[d, "\alpha_A"'] & F_B \arrow[d, "\alpha_B"] \\ G_A \arrow[r, "G_f"'] & G_B \end{tikzcd}\right)
\end{aligned}
$$

making the diagram in Item 2 commute. Conversely, every such functor gives rise to a natural transformation from F to G, and these constructions are inverse to each other.

From Item 2 to Item 3 and Back

This follows from Item 3 of Proposition 9.1.2.

8.6 Natural Isomorphisms

Let $\mathcal{C}$ and $\mathcal{D}$ be categories and let $F, G\colon \mathcal{C} \rightrightarrows \mathcal{D}$ be functors.

Definition 8.6.1 ▶ Natural Isomorphisms

A natural transformation $\alpha\colon F \Longrightarrow G$ is a **natural isomorphism** if there exists a natural transformation $\alpha^{-1}\colon G \Longrightarrow F$ such that

$$
\begin{aligned}
\alpha^{-1} \circ \alpha &= \mathrm{id}_F,\\
\alpha \circ \alpha^{-1} &= \mathrm{id}_G.
\end{aligned}
$$

Proposition 8.6.2 ► Properties of Natural Isomorphisms

Let $\alpha \colon F \Longrightarrow G$ be a natural transformation.

1. *Characterisations.* The following conditions are equivalent:
   (a) The natural transformation α is a natural isomorphism.
   (b) For each $A \in \mathrm{Obj}(\mathcal{C})$, the morphism $\alpha_A \colon F_A \longrightarrow G_A$ is an isomorphism.

2. *Componentwise Inverses of Natural Transformations Assemble Into Natural Transformations.* Let $\alpha^{-1} \colon G \Longrightarrow F$ be a transformation such that, for each $A \in \mathrm{Obj}(\mathcal{C})$, we have

$$\begin{aligned} \alpha_A^{-1} \circ \alpha_A &= \mathrm{id}_{F(A)}, \\ \alpha_A \circ \alpha_A^{-1} &= \mathrm{id}_{G(A)}. \end{aligned}$$

   Then α^{-1} is a natural transformation.

Proof 8.6.3 ► Proof of Proposition 8.6.2

Item 1: Characterisations

The implication Item 1a $\Longrightarrow$ Item 1b is clear, whereas the implication Item 1b $\Longrightarrow$ Item 1a follows from Item 2.

Item 2: Componentwise Inverses of Natural Transformations Assemble Into Nat

The naturality condition for α^{-1} corresponds to the commutativity of the diagram

$$\begin{tikzcd} G(A) \arrow[r, "G(f)"] \arrow[d, "\alpha_A^{-1}"'] & G(B) \arrow[d, "\alpha_B^{-1}"] \\ F(A) \arrow[r, "F(f)"'] & F(B) \end{tikzcd}$$

for each $A, B \in \mathrm{Obj}(C)$ and each $f \in \mathrm{Hom}_C(A, B)$. Considering the diagram

$$\begin{tikzcd}
G(A) \arrow[r, "G(f)"] \arrow[d, "\alpha_A^{-1}"'] \arrow[dr, phantom, "(1)"] & G(B) \arrow[d, "\alpha_B^{-1}"] \\
F(A) \arrow[r, "F(f)" description] \arrow[d, "\alpha_A"'] \arrow[dr, phantom, "(2)"] & F(B) \arrow[d, "\alpha_B"] \\
G(A) \arrow[r, "G(f)"'] & G(B),
\end{tikzcd}$$

where the boundary diagram as well as Subdiagram (2) commute, we have

$$\begin{aligned}
G(f) &= G(f) \circ \mathrm{id}_{G(A)} \\
&= G(f) \circ \alpha_A \circ \alpha_A^{-1} \\
&= \alpha_B \circ F(f) \circ \alpha_A^{-1}.
\end{aligned}$$

Postcomposing both sides with α_B^{-1}, we get

$$\begin{aligned}
\alpha_B^{-1} \circ G(f) &= \alpha_B^{-1} \circ \alpha_B \circ F(f) \circ \alpha_A^{-1} \\
&= \mathrm{id}_{F(B)} \circ F(f) \circ \alpha_A^{-1} \\
&= F(f) \circ \alpha_A^{-1},
\end{aligned}$$

which is the naturality condition we wanted to show. Thus α^{-1} is a natural transformation. ∎

9 Categories of Categories

9.1 Functor Categories

Let C be a category and $\mathcal{D}$ be a small category.

Definition 9.1.1 ► Functor Categories

The **category of functors from $\mathcal{C}$ to $\mathcal{D}$**[1] is the category $\mathsf{Fun}(\mathcal{C},\mathcal{D})$[2] where

- *Objects.* The objects of $\mathsf{Fun}(\mathcal{C},\mathcal{D})$ are functors from $\mathcal{C}$ to $\mathcal{D}$.
- *Morphisms.* For each $F, G \in \mathrm{Obj}(\mathsf{Fun}(\mathcal{C},\mathcal{D}))$, we have

$$\mathrm{Hom}_{\mathsf{Fun}(\mathcal{C},\mathcal{D})}(F,G) \overset{\mathrm{def}}{=} \mathrm{Nat}(F,G).$$

- *Identities.* For each $F \in \mathrm{Obj}(\mathsf{Fun}(\mathcal{C},\mathcal{D}))$, the unit map

$$\mathbb{1}^{\mathsf{Fun}(\mathcal{C},\mathcal{D})}_F \colon \mathrm{pt} \longrightarrow \mathrm{Nat}(F,F)$$

  of $\mathsf{Fun}(\mathcal{C},\mathcal{D})$ at F is given by

$$\mathrm{id}^{\mathsf{Fun}(\mathcal{C},\mathcal{D})}_F \overset{\mathrm{def}}{=} \mathrm{id}_F,$$

  where $\mathrm{id}_F \colon F \Longrightarrow F$ is the identity natural transformation of F of Example 8.2.4.
- *Composition.* For each $F, G, H \in \mathrm{Obj}(\mathsf{Fun}(\mathcal{C},\mathcal{D}))$, the composition map

$$\circ^{\mathsf{Fun}(\mathcal{C},\mathcal{D})}_{F,G,H} \colon \mathrm{Nat}(G,H) \times \mathrm{Nat}(F,G) \longrightarrow \mathrm{Nat}(F,H)$$

  of $\mathsf{Fun}(\mathcal{C},\mathcal{D})$ at (F,G,H) is given by

$$\beta \circ^{\mathsf{Fun}(\mathcal{C},\mathcal{D})}_{F,G,H} \alpha \overset{\mathrm{def}}{=} \beta \circ \alpha,$$

  where $\beta \circ \alpha$ is the vertical composition of α and β of Item 1 of Proposition 8.3.3.

[1] *Further Terminology:* Also called the **functor category** $\mathsf{Fun}(\mathcal{C},\mathcal{D})$.
[2] *Further Notation:* Also written $\mathcal{D}^{\mathcal{C}}$ and $[\mathcal{C},\mathcal{D}]$.

Proposition 9.1.2 ► Properties of Functor Categories

Let $\mathcal{C}$ and $\mathcal{D}$ be categories and let $F \colon \mathcal{C} \longrightarrow \mathcal{D}$ be a functor.

1. *Functoriality.* The assignments $\mathcal{C}, \mathcal{D}, (\mathcal{C},\mathcal{D}) \mapsto \mathsf{Fun}(\mathcal{C},\mathcal{D})$ define func-

tors

$$\begin{aligned}\mathsf{Fun}(\mathcal{C},-_2)\colon \mathsf{Cats} &\to \mathsf{Cats},\\ \mathsf{Fun}(-_1,\mathcal{D})\colon \mathsf{Cats}^{\mathsf{op}} &\to \mathsf{Cats},\\ \mathsf{Fun}(-_1,-_2)\colon \mathsf{Cats}^{\mathsf{op}}\times\mathsf{Cats} &\to \mathsf{Cats}.\end{aligned}$$

2. *2-Functoriality.* The assignments $\mathcal{C},\mathcal{D},(\mathcal{C},\mathcal{D}) \mapsto \mathsf{Fun}(\mathcal{C},\mathcal{D})$ define 2-functors

$$\begin{aligned}\mathsf{Fun}(\mathcal{C},-_2)\colon \mathsf{Cats}_2 &\to \mathsf{Cats}_2,\\ \mathsf{Fun}(-_1,\mathcal{D})\colon \mathsf{Cats}_2^{\mathsf{op}} &\to \mathsf{Cats}_2,\\ \mathsf{Fun}(-_1,-_2)\colon \mathsf{Cats}_2^{\mathsf{op}}\times\mathsf{Cats}_2 &\to \mathsf{Cats}_2.\end{aligned}$$

3. *Adjointness.* We have adjunctions

$$(\mathcal{C}\times - \dashv \mathsf{Fun}(\mathcal{C},-))\colon \quad \begin{tikzcd} \mathsf{Cats} \arrow[r, bend left, "\mathcal{C}\times -"] \arrow[r, phantom, "\perp"] & \mathsf{Cats}, \arrow[l, bend left, "\mathsf{Fun}(\mathcal{C},-)"] \end{tikzcd}$$

$$(-\times\mathcal{D} \dashv \mathsf{Fun}(\mathcal{D},-))\colon \quad \begin{tikzcd} \mathsf{Cats} \arrow[r, bend left, "-\times\mathcal{D}"] \arrow[r, phantom, "\perp"] & \mathsf{Cats}, \arrow[l, bend left, "\mathsf{Fun}(\mathcal{D},-)"] \end{tikzcd}$$

witnessed by bijections of sets

$$\begin{aligned}\mathsf{Hom}_{\mathsf{Cats}}(\mathcal{C}\times\mathcal{D},\mathcal{E}) &\cong \mathsf{Hom}_{\mathsf{Cats}}(\mathcal{D},\mathsf{Fun}(\mathcal{C},\mathcal{E})),\\ \mathsf{Hom}_{\mathsf{Cats}}(\mathcal{C}\times\mathcal{D},\mathcal{E}) &\cong \mathsf{Hom}_{\mathsf{Cats}}(\mathcal{C},\mathsf{Fun}(\mathcal{D},\mathcal{E})),\end{aligned}$$

natural in $\mathcal{C},\mathcal{D},\mathcal{E}\in\mathrm{Obj}(\mathsf{Cats})$.

4. *2-Adjointness.* We have 2-adjunctions

$$(\mathcal{C}\times - \dashv \mathsf{Fun}(\mathcal{C},-))\colon \quad \begin{tikzcd} \mathsf{Cats}_2 \arrow[r, bend left, "\mathcal{C}\times -"] \arrow[r, phantom, "\perp_2"] & \mathsf{Cats}_2, \arrow[l, bend left, "\mathsf{Fun}(\mathcal{C},-)"] \end{tikzcd}$$

$$(-\times\mathcal{D} \dashv \mathsf{Fun}(\mathcal{D},-))\colon \quad \begin{tikzcd} \mathsf{Cats}_2 \arrow[r, bend left, "-\times\mathcal{D}"] \arrow[r, phantom, "\perp_2"] & \mathsf{Cats}_2, \arrow[l, bend left, "\mathsf{Fun}(\mathcal{D},-)"] \end{tikzcd}$$

witnessed by isomorphisms of categories

$$\begin{aligned}\mathsf{Fun}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) &\cong \mathsf{Fun}(\mathcal{D}, \mathsf{Fun}(\mathcal{C}, \mathcal{E})),\\ \mathsf{Fun}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) &\cong \mathsf{Fun}(\mathcal{C}, \mathsf{Fun}(\mathcal{D}, \mathcal{E})),\end{aligned}$$

natural in $\mathcal{C}, \mathcal{D}, \mathcal{E} \in \mathrm{Obj}(\mathsf{Cats}_2)$.

5. *Interaction With Punctual Categories.* We have a canonical isomorphism of categories

$$\mathsf{Fun}(\mathrm{pt}, \mathcal{C}) \cong \mathcal{C},$$

natural in $\mathcal{C} \in \mathrm{Obj}(\mathsf{Cats})$.

6. *Objectwise Computation of Co/Limits.* Let

$$D\colon \mathcal{I} \to \mathsf{Fun}(\mathcal{C}, \mathcal{D})$$

be a diagram in $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$. We have isomorphisms

$$\begin{aligned}\lim(D)_A &\cong \lim_{i \in \mathcal{I}}(D_i(A)),\\ \operatorname{colim}(D)_A &\cong \operatorname*{colim}_{i \in \mathcal{I}}(D_i(A)),\end{aligned}$$

naturally in $A \in \mathrm{Obj}(\mathcal{C})$.

7. *Interaction With Co/Completeness.* If $\mathcal{E}$ is co/complete, then so is $\mathsf{Fun}(\mathcal{C}, \mathcal{E})$.

8. *Monomorphisms and Epimorphisms.* Let $\alpha\colon F \Longrightarrow G$ be a morphism of $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$. The following conditions are equivalent:

   (a) The natural transformation

   $$\alpha\colon F \Longrightarrow G$$

   is a monomorphism (resp. epimorphism) in $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$.

   (b) For each $A \in \mathrm{Obj}(\mathcal{C})$, the morphism

   $$\alpha_A\colon F_A \to G_A$$

   is a monomorphism (resp. epimorphism) in $\mathcal{D}$.

PROOF 9.1.3 ► PROOF OF PROPOSITION 9.1.2

Item 1: Functoriality

Omitted.

Item 2: 2-Functoriality

Omitted.

Item 3: Adjointness

Omitted.

Item 4: 2-Adjointness

Omitted.

Item 5: Interaction With Punctual Categories

Omitted.

Item 6: Objectwise Computation of Co/Limits

Omitted.

Item 7: Interaction With Co/Completeness

This follows from **??**.

Item 8: Monomorphisms and Epimorphisms

Omitted.

9.2 The Category of Categories and Functors

DEFINITION 9.2.1 ► THE CATEGORY OF CATEGORIES AND FUNCTORS

The **category of** (**small**) **categories and functors** is the category Cats where

- *Objects.* The objects of Cats are small categories.
- *Morphisms.* For each $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$, we have

$$\mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, \mathcal{D}) \stackrel{\text{def}}{=} \mathrm{Obj}(\mathrm{Fun}(\mathcal{C}, \mathcal{D})).$$

- *Identities.* For each $\mathcal{C} \in \mathrm{Obj}(\mathsf{Cats})$, the unit map

$$\mathbb{1}_{\mathcal{C}}^{\mathsf{Cats}} \colon \mathrm{pt} \to \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, \mathcal{C})$$

of Cats at $\mathcal{C}$ is defined by

$$\mathrm{id}^{\mathsf{Cats}}_{\mathcal{C}} \overset{\text{def}}{=} \mathrm{id}_{\mathcal{C}},$$

where $\mathrm{id}_{\mathcal{C}} \colon \mathcal{C} \to \mathcal{C}$ is the identity functor of $\mathcal{C}$ of Example 4.1.4.

- *Composition.* For each $\mathcal{C}, \mathcal{D}, \mathcal{E} \in \mathrm{Obj}(\mathsf{Cats})$, the composition map

$$\circ^{\mathsf{Cats}}_{\mathcal{C},\mathcal{D},\mathcal{E}} \colon \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{D}, \mathcal{E}) \times \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, \mathcal{D}) \to \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, \mathcal{E})$$

of Cats at $(\mathcal{C}, \mathcal{D}, \mathcal{E})$ is given by

$$G \circ^{\mathsf{Cats}}_{\mathcal{C},\mathcal{D},\mathcal{E}} F \overset{\text{def}}{=} G \circ F,$$

where $G \circ F \colon \mathcal{C} \to \mathcal{E}$ is the composition of F and G of Definition 4.1.6.

Proposition 9.2.2 ► Properties of the Category Cats

Let $\mathcal{C}$ be a category.

1. *Co/Completeness.* The category Cats is complete and cocomplete.
2. *Cartesian Monoidal Structure.* The quadruple $(\mathsf{Cats}, \times, \mathrm{pt}, \mathsf{Fun})$ is a Cartesian closed monoidal category.

Proof 9.2.3 ► Proof of Proposition 9.2.2

Item 1: Co/Completeness

Omitted.

Item 2: Cartesian Monoidal Structure

Omitted. ∎

9.3 The 2-Category of Categories, Functors, and Natural Transformations

Definition 9.3.1 ► The 2-Category of Categories

The 2-**category of** (**small**) **categories, functors, and natural transformations** is the 2-category Cats_2 where

- *Objects.* The objects of Cats_2 are small categories.
- Hom-*Categories.* For each $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats}_2)$, we have

$$\mathrm{Hom}_{\mathsf{Cats}_2}(\mathcal{C}, \mathcal{D}) \stackrel{\text{def}}{=} \mathrm{Fun}(\mathcal{C}, \mathcal{D}).$$

- *Identities.* For each $\mathcal{C} \in \mathrm{Obj}(\mathsf{Cats}_2)$, the unit functor

$$\mathbb{1}^{\mathsf{Cats}_2}_{\mathcal{C}} \colon \mathrm{pt} \to \mathrm{Fun}(\mathcal{C}, \mathcal{C})$$

  of Cats_2 at $\mathcal{C}$ is the functor picking the identity functor $\mathrm{id}_{\mathcal{C}} \colon \mathcal{C} \to \mathcal{C}$ of $\mathcal{C}$.
- *Composition.* For each $\mathcal{C}, \mathcal{D}, \mathcal{E} \in \mathrm{Obj}(\mathsf{Cats}_2)$, the composition bifunctor

$$\circ^{\mathsf{Cats}_2}_{\mathcal{C},\mathcal{D},\mathcal{E}} \colon \mathrm{Hom}_{\mathsf{Cats}_2}(\mathcal{D}, \mathcal{E}) \times \mathrm{Hom}_{\mathsf{Cats}_2}(\mathcal{C}, \mathcal{D}) \to \mathrm{Hom}_{\mathsf{Cats}_2}(\mathcal{C}, \mathcal{E})$$

  of Cats_2 at $(\mathcal{C}, \mathcal{D}, \mathcal{E})$ is the functor where
  - *Action on Objects.* For each object $(G, F) \in \mathrm{Obj}(\mathrm{Hom}_{\mathsf{Cats}_2}(\mathcal{D}, \mathcal{E}) \times \mathrm{Hom}_{\mathsf{Cats}_2}(\mathcal{C}, \mathcal{D}))$, we have

$$\circ^{\mathsf{Cats}_2}_{\mathcal{C},\mathcal{D},\mathcal{E}}(G, F) \stackrel{\text{def}}{=} G \circ F.$$

  - *Action on Morphisms.* For each morphism $(\beta, \alpha) \colon (K, H) \Longrightarrow (G, F)$ of $\mathrm{Hom}_{\mathsf{Cats}_2}(\mathcal{D}, \mathcal{E}) \times \mathrm{Hom}_{\mathsf{Cats}_2}(\mathcal{C}, \mathcal{D})$, we have

$$\circ^{\mathsf{Cats}_2}_{\mathcal{C},\mathcal{D},\mathcal{E}}(\beta, \alpha) \stackrel{\text{def}}{=} \beta \star \alpha,$$

    where $\beta \star \alpha$ is the horizontal composition of α and β of Definition 8.4.1.

Proposition 9.3.2 ► Properties of the 2-Category Cats_2

Let $\mathcal{C}$ be a category.

1. *2-Categorical Co/Completeness.* The 2-category Cats_2 is complete and cocom-

plete as a 2-category, having all 2-categorical and bicategorical co/limits.

Proof 9.3.3 ▶ Proof of Proposition 9.3.2

Item 1: Co/Completeness

Omitted. □

9.4 The Category of Groupoids

Definition 9.4.1 ▶ The Category of Small Groupoids

The **category of** (**small**) **groupoids** is the full subcategory Grpd of Cats spanned by the groupoids.

9.5 The 2-Category of Groupoids

Definition 9.5.1 ▶ The 2-Category of Small Groupoids

The 2**-category of** (**small**) **groupoids** is the full sub-2-category Grpd_2 of Cats_2 spanned by the groupoids.

Appendices

A Other Chapters

References

[MO 119454] user30818. *Category and the axiom of choice*. MathOverflow. URL: https://mathoverflow.net/q/119454 (cit. on p. 62).

[MO 321971] Ivan Di Liberti. *Characterization of pseudo monomorphisms and pseudo epimorphisms in Cat*. MathOverflow. URL: https://mathoverflow.net/q/321971 (cit. on p. 75).

[MO 468121a] Emily de Oliveira Santos. *Characterisations of functors F such that F^* or F_* is [property], e.g. faithful, conservative, etc*. MathOverflow. URL: https://mathoverflow.net/q/468125 (cit. on pp. 59, 60, 66, 68, 70, 76, 79, 82).

[MO 468121b] Emily de Oliveira Santos. *Looking for a nice characterisation of functors F whose precomposition functor F^* is full*. MathOverflow. URL: https://mathoverflow.net/q/468121 (cit. on p. 52).

[MO 468334] Emily de Oliveira Santos. *Is a pseudomonic and pseudoepic functor necessarily an equivalence of categories?* MathOverflow. URL: https://mathoverflow.net/q/468334 (cit. on p. 75).

[MO 64365] Giorgio Mossa. *Natural transformations as categorical homotopies*. MathOverflow. URL: https://mathoverflow.net/q/64365 (cit. on p. 97).

[MSE 1465107] kilian. *Equivalence of categories and axiom of choice*. Mathematics Stack Exchange. URL: https://math.stackexchange.com/q/1465107 (cit. on p. 62).

[MSE 733161] Stefan Hamcke. *Precomposition with a faithful functor*. Mathematics Stack Exchange. URL: https://math.stackexchange.com/q/733161 (cit. on p. 56).

[MSE 733163] Zhen Lin. *Precomposition with a faithful functor*. Mathematics Stack Exchange. URL: https://math.stackexchange.com/q/733163 (cit. on p. 49).

[MSE 749304] Zhen Lin. *If the functor on presheaf categories given by precomposition by F is ff, is F full? faithful?* Mathematics Stack Exchange. URL: https://math.stackexchange.com/q/749304 (cit. on p. 56).

[Adá+01] Jiří Adámek, Robert El Bashir, Manuela Sobral, and Jiří Velebil. "On Functors Which Are Lax Epimorphisms". In: *Theory Appl. Categ.* 8 (2001), pp. 509–521. ISSN: 1201-561X (cit. on pp. 49, 52, 57).

[Bor94] Francis Borceux. *Handbook of Categorical Algebra I*. Vol. 50. Encyclopedia of Mathematics and its Applications. Basic Category Theory. Cambridge University Press, Cambridge, 1994, pp. xvi+345. ISBN: 0-521-44178-1 (cit. on p. 92).

[BS10] John C. Baez and Michael Shulman. "Lectures on n-Categories and Cohomology". In: *Towards higher categories*. Vol. 152. IMA Vol. Math. Appl. Springer, New York, 2010, pp. 1–68. DOI: 10.1007/978-1-4419-1524-5_1. URL: https://doi.org/10.1007/978-1-4419-1524-5_1 (cit. on p. 52).

[DFH75] Aristide Deleanu, Armin Frei, and Peter Hilton. "Idempotent Triples and Completion". In: *Math. Z.* 143 (1975), pp. 91–104. ISSN: 0025-5874,1432-1823. DOI: 10.1007/BF01173053. URL: https://doi.org/10.1007/BF01173053 (cit. on p. 66).

[Fre09] Jonas Frey. *On the 2-Categorical Duals of (Full and) Faithful Functors*. https://citeseerx.ist.psu.edu/document?repid=rep1&type=pdf&doi=4c289321d622f8fcf947e7a7cfd1bdf75c95ca33. Archived at https://web.archive.org/web/20240331195546/https://citeseerx.ist.psu.edu/document?repid=rep1&type=pdf&doi=4c289321d622f8fcf947e7a7cfd1bdf75c95ca33. July 2009. URL: https://citeseerx.ist.psu.edu/document?repid=rep1%5C&type=pdf%5C&doi=4c289321d622f8fcf947e7a7cfd1bdf75c95ca33 (cit. on pp. 49, 57).

[Isb68] John R. Isbell. "Epimorphisms and Dominions. III". In: *Amer. J. Math.* 90 (1968), pp. 1025–1030. ISSN: 0002-9327,1080-6377. DOI: 10.2307/2373286. URL: https://doi.org/10.2307/2373286 (cit. on p. 70).

[Low15] Zhen Lin Low. *Notes on Homotopical Algebra*. Nov. 2015. URL: https://zll22.user.srcf.net/writing/homotopical-algebra/2015-11-10-Main.pdf (cit. on p. 57).

[nLa24] nLab Authors. *Groupoid*. https://ncatlab.org/nlab/show/groupoid. Oct. 2024 (cit. on p. 63).

[nLab23] nLab Authors. *Skeleton*. 2024. URL: https://ncatlab.org/nlab/show/skeleton (cit. on p. 14).

[Rie17] Emily Riehl. *Category Theory in Context*. Vol. 10. Aurora: Dover Modern Math Originals. Courier Dover Publications, 2017, pp. xviii+240. ISBN: 978-0486809038. URL: http://www.math.jhu.edu/~eriehl/context.pdf (cit. on p. 63).