Pointed Sets

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This chapter contains some foundational material on pointed sets.

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1 Pointed Sets

1.1 Foundations

Definition 1.1.1 ▶ Pointed Sets

A **pointed set**[1] is equivalently:

- An $\mathbb{E}_0$-monoid in $(\mathsf{N}_\bullet(\mathsf{Sets}), \mathrm{pt})$.
- A pointed object in $(\mathsf{Sets}, \mathrm{pt})$.

[1] *Further Terminology:* In the context of monoids with zero as models for $\mathbb{F}_1$-algebras, pointed sets are viewed as **$\mathbb{F}_1$-modules**.

Remark 1.1.2 ▶ Unwinding Definition 1.1.1

In detail, a **pointed set** is a pair (X, x_0) consisting of:

- *The Underlying Set.* A set X, called the **underlying set of** (X, x_0).
- *The Basepoint.* A morphism

$$[x_0]\colon \mathrm{pt} \to X$$

  in Sets, determining an element $x_0 \in X$, called the **basepoint of** X.

Example 1.1.3 ▶ The Zero Sphere

The 0-**sphere**[1] is the pointed set $(S^0, 0)$[2] consisting of:

- *The Underlying Set.* The set S^0 defined by

$$S^0 \overset{\mathrm{def}}{=} \{0, 1\}.$$

- *The Basepoint.* The element 0 of S^0.

[1] *Further Terminology:* In the context of monoids with zero as models for $\mathbb{F}_1$-algebras, the 0-sphere is viewed as the **underlying pointed set of the field with one element**.

[2] *Further Notation:* In the context of monoids with zero as models for $\mathbb{F}_1$-algebras, S^0 is also denoted $(\mathbb{F}_1, 0)$.

Example 1.1.4 ▶ The Trivial Pointed Set

The **trivial pointed set** is the pointed set $(\mathrm{pt}, \star)$ consisting of:

- *The Underlying Set.* The punctual set $\mathrm{pt} \overset{\mathrm{def}}{=} \{\star\}$.
- *The Basepoint.* The element $\star$ of pt.

Example 1.1.5 ▶ The Underlying Pointed Set of a Semimodule

The **underlying pointed set** of a semimodule (M, α_M) is the pointed set $(M, 0_M)$.

Example 1.1.6 ▶ The Underlying Pointed Set of a Module

The **underlying pointed set** of a module (M, α_M) is the pointed set $(M, 0_M)$.

1.2 Morphisms of Pointed Sets

Definition 1.2.1 ▶ Morphisms of Pointed Sets

A **morphism of pointed sets**[1,2] is equivalently:

- A morphism of $\mathbb{E}_0$-monoids in $(\mathsf{N}_\bullet(\mathsf{Sets}), \mathrm{pt})$.
- A morphism of pointed objects in $(\mathsf{Sets}, \mathrm{pt})$.

[1] *Further Terminology:* Also called a **pointed function**.

[2] *Further Terminology:* In the context of monoids with zero as models for $\mathbb{F}_1$-algebras, morphisms of pointed sets are also called **morphism of $\mathbb{F}_1$-modules**.

Remark 1.2.2 ▶ Unwinding Definition 1.2.1

In detail, a **morphism of pointed sets** $f\colon (X, x_0) \longrightarrow (Y, y_0)$ is a morphism of sets $f\colon X \longrightarrow Y$ such that the diagram

$$\begin{tikzcd} & \mathrm{pt} \arrow[dl, "{[x_0]}"'] \arrow[dr, "{[y_0]}"] & \\ X \arrow[rr, "f"'] & & Y \end{tikzcd}$$

commutes, i.e. such that

$$f(x_0) = y_0.$$

1.3 The Category of Pointed Sets

Definition 1.3.1 ► The Category of Pointed Sets

The **category of pointed sets** is the category Sets_* defined equivalently as

- The homotopy category of the ∞-category $\mathsf{Mon}_{\mathbb{E}_0}(\mathsf{N}_\bullet(\mathsf{Sets}), \mathrm{pt})$ of **??**, **??**;
- The category Sets_* of **??**, **??**.

Remark 1.3.2 ► Unwinding Definition 1.3.1

In detail, the **category of pointed sets** is the category Sets_* where

- *Objects.* The objects of Sets_* are pointed sets;
- *Morphisms.* The morphisms of Sets_* are morphisms of pointed sets;
- *Identities.* For each $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, the unit map

  $$\mathbb{1}^{\mathsf{Sets}_*}_{(X,x_0)} : \mathrm{pt} \longrightarrow \mathsf{Sets}_*((X, x_0), (X, x_0))$$

  of Sets_* at (X, x_0) is defined by[1]

  $$\mathrm{id}^{\mathsf{Sets}_*}_{(X,x_0)} \stackrel{\text{def}}{=} \mathrm{id}_X;$$

- *Composition.* For each $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, the composition map

  $$\circ^{\mathsf{Sets}_*}_{(X,x_0),(Y,y_0),(Z,z_0)} : \mathsf{Sets}_*((Y, y_0), (Z, z_0)) \times \mathsf{Sets}_*((X, x_0), (Y, y_0)) \to \mathsf{Sets}_*((X, x_0), (Z, z_0))$$

  of Sets_* at $((X, x_0), (Y, y_0), (Z, z_0))$ is defined by[2]

  $$g \circ^{\mathsf{Sets}_*}_{(X,x_0),(Y,y_0),(Z,z_0)} f \stackrel{\text{def}}{=} g \circ f.$$

[1] Note that id_X is indeed a morphism of pointed sets, as we have $\mathrm{id}_X(x_0) = x_0$.

[2]Note that the composition of two morphisms of pointed sets is indeed a morphism of pointed sets, as we have

$$\begin{aligned} g(f(x_0)) &= g(y_0) \\ &= z_0, \end{aligned}$$

or

$$\begin{tikzcd} & \mathrm{pt} \arrow[dl, "{[x_0]}"'] \arrow[d, "{[y_0]}"] \arrow[dr, "{[z_0]}"] & \\ X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z \end{tikzcd}$$

in terms of diagrams.

1.4 Elementary Properties of Pointed Sets

PROPOSITION 1.4.1 ► ELEMENTARY PROPERTIES OF POINTED SETS

Let (X, x_0) be a pointed set.

1. *Completeness.* The category Sets_* of pointed sets and morphisms between them is complete, having in particular:
   (a) Products, described as in Definition 2.3.1;
   (b) Pullbacks, described as in Definition 2.4.1;
   (c) Equalisers, described as in Definition 2.5.1.
2. *Cocompleteness.* The category Sets_* of pointed sets and morphisms between them is cocomplete, having in particular:
   (a) Coproducts, described as in Definition 3.3.1;
   (b) Pushouts, described as in Definition 3.4.1;
   (c) Coequalisers, described as in Definition 3.5.1.
3. *Failure To Be Cartesian Closed.* The category Sets_* is not Cartesian closed.[1]
4. *Morphisms From the Monoidal Unit.* We have a bijection of sets[2]

$$\mathsf{Sets}_*(S^0, X) \cong X,$$

natural in $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, internalising also to an isomorphism of pointed sets

$$\mathbf{Sets}_*(S^0, X) \cong (X, x_0),$$

again natural in $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

5. *Relation to Partial Functions.* We have an equivalence of categories[3]

$$\mathsf{Sets}_* \overset{\text{eq.}}{\cong} \mathsf{Sets}^{\text{part.}}$$

between the category of pointed sets and pointed functions between them and the category of sets and partial functions between them, where:

(a) *From Pointed Sets to Sets With Partial Functions.* The equivalence

$$\xi \colon \mathsf{Sets}_* \xrightarrow{\cong} \mathsf{Sets}^{\text{part.}}$$

sends:

i. A pointed set (X, x_0) to X.
ii. A pointed function

$$f \colon (X, x_0) \to (Y, y_0)$$

to the partial function

$$\xi_f \colon X \to Y$$

defined on $f^{-1}(Y \setminus y_0)$ and given by

$$\xi_f(x) \overset{\text{def}}{=} f(x)$$

for each $x \in f^{-1}(Y \setminus y_0)$.

(b) *From Sets With Partial Functions to Pointed Sets.* The equivalence

$$\xi^{-1} \colon \mathsf{Sets}^{\text{part.}} \xrightarrow{\cong} \mathsf{Sets}_*$$

sends:

i. A set X is to the pointed set $(X, \star)$ with $\star$ an element that is not in X.

ii. A partial function

$$f\colon X \to Y$$

defined on $U \subset X$ to the pointed function

$$\xi_f^{-1}\colon (X, x_0) \to (Y, y_0)$$

defined by

$$\xi_f(x) \overset{\text{def}}{=} \begin{cases} f(x) & \text{if } x \in U, \\ y_0 & \text{otherwise.} \end{cases}$$

for each $x \in X$.

[1]The category Sets_* does admit monoidal closed structures however; see Tensor Products of Pointed Sets.

[2]In other words, the forgetful functor

$$忘\colon \mathsf{Sets}_* \to \mathsf{Sets}$$

defined on objects by sending a pointed set to its underlying set is corepresentable by S^0.

[3] *Warning*: This is not an isomorphism of categories, only an equivalence.

Proof 1.4.2 ▶ Proof of Proposition 1.4.1

Item 1: Completeness

This follows from (the proofs) of Definitions 2.3.1, 2.4.1 and 2.5.1 and **??**, **??**.

Item 2: Cocompleteness

This follows from (the proofs) of Definitions 3.3.1, 3.4.1 and 3.5.1 and **??**, **??**.

Item 3: Failure To Be Cartesian Closed

See [MSE 2855868].

Item 4: Morphisms From the Monoidal Unit

Since a morphism from S^0 to a pointed set (X, x_0) sends $0 \in S^0$ to x_0 and then can send $1 \in S^0$ to any element of X, we obtain a bijection between pointed maps $S^0 \to X$ and the elements of X.

The isomorphism then

$$\mathbf{Sets}_*(S^0, X) \cong (X, x_0)$$

follows by noting that $\Delta_{x_0}: S^0 \to X$, the basepoint of **Sets**$_*(S^0, X)$, corresponds to the pointed map $S^0 \to X$ picking the element x_0 of X, and thus we see that the bijection between pointed maps $S^0 \to X$ and elements of X is compatible with basepoints, lifting to an isomorphism of pointed sets.

Item 5: Relation to Partial Functions

See [MSE 884460].

2 Limits of Pointed Sets

2.1 The Terminal Pointed Set

Definition 2.1.1 ► The Terminal Pointed Set

The **terminal pointed set** is the pair $((\mathrm{pt}, \star), \{!_X\}_{(X,x_0)\in\mathrm{Obj}(\mathsf{Sets}_*)})$ consisting of:

- *The Limit.* The pointed set $(\mathrm{pt}, \star)$.
- *The Cone.* The collection of morphisms of pointed sets

$$\{!_X \colon (X, x_0) \to (\mathrm{pt}, \star)\}_{(X,x_0)\in\mathrm{Obj}(\mathsf{Sets})}$$

defined by

$$!_X(x) \overset{\text{def}}{=} \star$$

for each $x \in X$ and each $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets})$.

Proof 2.1.2 ► Proof of Definition 2.1.1

We claim that $(\mathrm{pt}, \star)$ is the terminal object of Sets_*. Indeed, suppose we have a diagram of the form

$$(X, x_0) \qquad (\mathrm{pt}, \star)$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi \colon (X, x_0) \to (\mathrm{pt}, \star)$$

making the diagram

$$(X, x_0) \overset{\phi}{\underset{\exists!}{\dashrightarrow}} (\mathrm{pt}, \star)$$

commute, namely $!_X$.

2.2 Products of Families of Pointed Sets

Let $\left\{(X_i, x_0^i)\right\}_{i\in I}$ be a family of pointed sets.

Definition 2.2.1 ► The Product of a Family of Pointed Sets

The **product of** $\left\{(X_i, x_0^i)\right\}_{i\in I}$ is the pair $((\prod_{i\in I} X_i, (x_0^i)_{i\in I}), \{\mathrm{pr}_i\}_{i\in I})$ consisting of:

- *The Limit.* The pointed set $(\prod_{i\in I} X_i, (x_0^i)_{i\in I})$.
- *The Cone.* The collection

$$\left\{\mathrm{pr}_i\colon (\prod_{i\in I} X_i, (x_0^i)_{i\in I}) \longrightarrow (X_i, x_0^i)\right\}_{i\in I}$$

  of maps given by

$$\mathrm{pr}_i((x_j)_{j\in I}) \overset{\text{def}}{=} x_i$$

  for each $(x_j)_{j\in I} \in \prod_{i\in I} X_i$ and each $i \in I$.

Proof 2.2.2 ► Proof of Definition 2.2.1

We claim that $(\prod_{i\in I} X_i, (x_0^i)_{i\in I})$ is the categorical product of $\left\{(X_i, x_0^i)\right\}_{i\in I}$ in Sets_*. Indeed, suppose we have, for each $i \in I$, a diagram of the form

$$\begin{tikzcd}
& (P, *) \arrow[dr, "p_i"] & \\
(\prod_{i\in I} X_i, (x_0^i)_{i\in I}) \arrow[rr, "\mathrm{pr}_i"'] & & (X_i, x_0^i)
\end{tikzcd}$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi\colon (P, *) \longrightarrow (\prod_{i\in I} X_i, (x_0^i)_{i\in I})$$

making the diagram

$$\begin{tikzcd}
(P, *) \arrow[d, dashed, "\phi"', "\exists!"] \arrow[dr, "p_i"] & \\
\left(\prod_{i\in I} X_i, (x_0^i)_{i\in I}\right) \arrow[r, "\mathrm{pr}_i"'] & (X_i, x_0^i)
\end{tikzcd}$$

commute, being uniquely determined by the condition $\mathrm{pr}_i \circ \phi = p_i$ for each $i \in I$ via

$$\phi(x) = (p_i(x))_{i\in I}$$

for each $x \in P$. Note that this is indeed a morphism of pointed sets, as we have

$$\begin{aligned}
\phi(*) &= (p_i(*))_{i\in I} \\
&= (x_0^i)_{i\in I},
\end{aligned}$$

where we have used that p_i is a morphism of pointed sets for each $i \in I$.

Proposition 2.2.3 ► Properties of Products of Families of Pointed Sets

Let $\left\{(X_i, x_0^i)\right\}_{i\in I}$ be a family of pointed sets.

1. *Functoriality.* The assignment $\left\{(X_i, x_0^i)\right\}_{i\in I} \mapsto \left(\prod_{i\in I} X_i, (x_0^i)_{i\in I}\right)$ defines a functor
$$\prod_{i\in I} : \mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}_*) \longrightarrow \mathsf{Sets}_*.$$

Proof 2.2.4 ► Proof of Proposition 2.2.3

Item 1: Functoriality

This follows from **??**, **??** of **??**.

2.3 Products

Let (X, x_0) and (Y, y_0) be pointed sets.

Definition 2.3.1 ► Products of Pointed Sets

The **product of** (X, x_0) **and** (Y, y_0) is the pair consisting of:

- *The Limit.* The pointed set $(X \times Y, (x_0, y_0))$.
- *The Cone.* The morphisms of pointed sets

$$\begin{aligned} \mathrm{pr}_1 &\colon (X \times Y, (x_0, y_0)) \to (X, x_0), \\ \mathrm{pr}_2 &\colon (X \times Y, (x_0, y_0)) \to (Y, y_0) \end{aligned}$$

defined by

$$\begin{aligned} \mathrm{pr}_1(x, y) &\stackrel{\text{def}}{=} x, \\ \mathrm{pr}_2(x, y) &\stackrel{\text{def}}{=} y \end{aligned}$$

for each $(x, y) \in X \times Y$.

Proof 2.3.2 ► Proof of Definition 2.3.1

We claim that $(X \times Y, (x_0, y_0))$ is the categorical product of (X, x_0) and (Y, y_0) in Sets_*. Indeed, suppose we have a diagram of the form

$$\begin{tikzcd} & (P, *) \arrow[dl, bend right, "p_1"'] \arrow[dr, bend left, "p_2"] & \\ (X, x_0) & (X \times Y, (x_0, y_0)) \arrow[l, "\mathrm{pr}_1"] \arrow[r, "\mathrm{pr}_2"'] & (Y, y_0) \end{tikzcd}$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi \colon (P, *) \to (X \times Y, (x_0, y_0))$$

making the diagram

$$\begin{tikzcd} & (P, *) \arrow[dl, bend right, "p_1"'] \arrow[dr, bend left, "p_2"] \arrow[d, dashed, "\phi"', "\exists!"] & \\ (X, x_0) & (X \times Y, (x_0, y_0)) \arrow[l, "\mathrm{pr}_1"] \arrow[r, "\mathrm{pr}_2"'] & (Y, y_0) \end{tikzcd}$$

commute, being uniquely determined by the conditions

$$\begin{aligned}\mathrm{pr}_1 \circ \phi &= p_1,\\ \mathrm{pr}_2 \circ \phi &= p_2\end{aligned}$$

via

$$\phi(x) = (p_1(x), p_2(x))$$

for each $x \in P$. Note that this is indeed a morphism of pointed sets, as we have

$$\begin{aligned}\phi(*) &= (p_1(*), p_2(*))\\ &= (x_0, y_0),\end{aligned}$$

where we have used that p_1 and p_2 are morphisms of pointed sets.

Proposition 2.3.3 ► Properties of Products of Pointed Sets

Let (X, x_0), (Y, y_0), and (Z, z_0) be pointed sets.

1. *Functoriality.* The assignments

   $$(X, x_0), (Y, y_0), ((X, x_0), (Y, y_0)) \mapsto (X \times Y, (x_0, y_0))$$

   define functors

   $$\begin{aligned}X \times -\colon \mathsf{Sets}_* &\to \mathsf{Sets}_*,\\ - \times Y\colon \mathsf{Sets}_* &\to \mathsf{Sets}_*,\\ -_1 \times -_2\colon \mathsf{Sets}_* \times \mathsf{Sets}_* &\to \mathsf{Sets}_*,\end{aligned}$$

   defined in the same way as the functors of Constructions With Sets, Item 1 of Proposition 1.3.3.

2. *Associativity.* We have an isomorphism of pointed sets

   $$((X \times Y) \times Z, ((x_0, y_0), z_0)) \cong (X \times (Y \times Z), (x_0, (y_0, z_0)))$$

   natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

3. *Unitality.* We have isomorphisms of pointed sets

$$(\mathrm{pt}, \star) \times (X, x_0) \cong (X, x_0),$$
$$(X, x_0) \times (\mathrm{pt}, \star) \cong (X, x_0),$$

natural in $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

4. *Commutativity.* We have an isomorphism of pointed sets

$$(X \times Y, (x_0, y_0)) \cong (Y \times X, (y_0, x_0)),$$

natural in $(X, x_0), (Y, y_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

5. *Symmetric Monoidality.* The triple $(\mathsf{Sets}_*, \times, (\mathrm{pt}, \star))$ is a symmetric monoidal category.

Proof 2.3.4 ▶ Proof of Proposition 2.3.3

Item 1: Functoriality

This is a special case of functoriality of limits, **??**, **??** of **??**.

Item 2: Associativity

This follows from Constructions With Sets, Item 3 of Proposition 1.3.3.

Item 3: Unitality

This follows from Constructions With Sets, Item 4 of Proposition 1.3.3.

Item 4: Commutativity

This follows from Constructions With Sets, Item 5 of Proposition 1.3.3.

Item 5: Symmetric Monoidality

This follows from Constructions With Sets, Item 12 of Proposition 1.3.3. □

2.4 Pullbacks

Let (X, x_0), (Y, y_0), and (Z, z_0) be pointed sets and let $f\colon (X, x_0) \to (Z, z_0)$ and $g\colon (Y, y_0) \to (Z, z_0)$ be morphisms of pointed sets.

Definition 2.4.1 ► Pullbacks of Pointed Sets

The **pullback of** (X, x_0) **and** (Y, y_0) **over** (Z, z_0) **along** (f, g) is the pair consisting of:

- *The Limit.* The pointed set $(X \times_Z Y, (x_0, y_0))$.
- *The Cone.* The morphisms of pointed sets

$$\begin{aligned}\mathrm{pr}_1 &\colon (X \times_Z Y, (x_0, y_0)) \to (X, x_0),\\ \mathrm{pr}_2 &\colon (X \times_Z Y, (x_0, y_0)) \to (Y, y_0)\end{aligned}$$

defined by

$$\begin{aligned}\mathrm{pr}_1(x, y) &\overset{\text{def}}{=} x,\\ \mathrm{pr}_2(x, y) &\overset{\text{def}}{=} y\end{aligned}$$

for each $(x, y) \in X \times_Z Y$.

Proof 2.4.2 ► Proof of Definition 2.4.1

We claim that $X \times_Z Y$ is the categorical pullback of (X, x_0) and (Y, y_0) over (Z, z_0) with respect to (f, g) in Sets_*. First we need to check that the relevant pullback diagram commutes, i.e. that we have

$$f \circ \mathrm{pr}_1 = g \circ \mathrm{pr}_2, \qquad \begin{tikzcd} (X \times_Z Y, (x_0, y_0)) \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] & (Y, y_0) \arrow[d, "g"] \\ (X, x_0) \arrow[r, "f"'] & (Z, z_0). \end{tikzcd}$$

Indeed, given $(x, y) \in X \times_Z Y$, we have

$$\begin{aligned}[f \circ \mathrm{pr}_1](x, y) &= f(\mathrm{pr}_1(x, y))\\ &= f(x)\\ &= g(y)\\ &= g(\mathrm{pr}_2(x, y))\\ &= [g \circ \mathrm{pr}_2](x, y),\end{aligned}$$

where $f(x) = g(y)$ since $(x, y) \in X \times_Z Y$. Next, we prove that $X \times_Z Y$ satisfies the universal property of the pullback. Suppose we have a diagram of the form

$$\begin{tikzcd}
(P,*) \arrow[rrd, bend left, "p_2"] \arrow[rdd, bend right, "p_1"'] & & \\
& (X \times_Z Y, (x_0, y_0)) \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] \arrow[dr, phantom, "\lrcorner", very near start] & (Y, y_0) \arrow[d, "g"] \\
& (X, x_0) \arrow[r, "f"'] & (Z, z_0)
\end{tikzcd}$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi \colon (P, *) \longrightarrow (X \times_Z Y, (x_0, y_0))$$

making the diagram

$$\begin{tikzcd}
(P,*) \arrow[rrd, bend left, "p_2"] \arrow[rdd, bend right, "p_1"'] \arrow[rd, dashed, "\phi", "\exists!"'] & & \\
& (X \times_Z Y, (x_0, y_0)) \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] \arrow[dr, phantom, "\lrcorner", very near start] & (Y, y_0) \arrow[d, "g"] \\
& (X, x_0) \arrow[r, "f"'] & (Z, z_0)
\end{tikzcd}$$

commute, being uniquely determined by the conditions

$$\begin{aligned}
\mathrm{pr}_1 \circ \phi &= p_1, \\
\mathrm{pr}_2 \circ \phi &= p_2
\end{aligned}$$

via

$$\phi(x) = (p_1(x), p_2(x))$$

for each $x \in P$, where we note that $(p_1(x), p_2(x)) \in X \times Y$ indeed lies in $X \times_Z Y$ by the condition

$$f \circ p_1 = g \circ p_2,$$

which gives

$$f(p_1(x)) = g(p_2(x))$$

for each $x \in P$, so that $(p_1(x), p_2(x)) \in X \times_Z Y$. Lastly, we note that ϕ is indeed a morphism of pointed sets, as we have

$$\begin{aligned}\phi(*) &= (p_1(*), p_2(*))\\ &= (x_0, y_0),\end{aligned}$$

where we have used that p_1 and p_2 are morphisms of pointed sets. □

PROPOSITION 2.4.3 ▶ PROPERTIES OF PULLBACKS OF POINTED SETS

Let (X, x_0), (Y, y_0), (Z, z_0), and (A, a_0) be pointed sets.

1. *Functoriality.* The assignment $(X, Y, Z, f, g) \mapsto X \times_{f,Z,g} Y$ defines a functor

$$-_1 \times_{-_3} -_1 \colon \mathsf{Fun}(\mathcal{P}, \mathsf{Sets}_*) \to \mathsf{Sets}_*,$$

where $\mathcal{P}$ is the category that looks like this:

$$\begin{tikzcd} & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet. \end{tikzcd}$$

In particular, the action on morphisms of $-_1 \times_{-_3} -_1$ is given by sending a morphism

$$\begin{tikzcd}
X \times_Z Y \arrow[rr] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & & Y \arrow[dd, "g" near start] \arrow[dr, "\psi"] & \\
& X' \times_{Z'} Y' \arrow[rr, crossing over] \arrow[dr, phantom, "\lrcorner", very near start] & & Y' \arrow[dd, "g'"] \\
X \arrow[rr, "f" near start] \arrow[dr, "\phi"'] & & Z \arrow[dr, "\chi"] & \\
& X' \arrow[rr, "f'"'] \arrow[from=uu, crossing over] & & Z'
\end{tikzcd}$$

in $\mathrm{Fun}(\mathcal{P}, \mathsf{Sets}_*)$ to the morphism of pointed sets

$$\xi\colon (X \times_Z Y, (x_0, y_0)) \xrightarrow{\exists!} (X' \times_{Z'} Y', (x'_0, y'_0))$$

given by

$$\xi(x, y) \overset{\text{def}}{=} (\phi(x), \psi(y))$$

for each $(x, y) \in X \times_Z Y$, which is the unique morphism of pointed sets making the diagram

$$\begin{tikzcd}
X \times_Z Y \arrow[rr] \arrow[dr, dashed] \arrow[dd] \arrow[dr, phantom, very near start, "\lrcorner"] & & Y \arrow[dd, "g" near start] \arrow[dr, "\psi"] & \\
& X' \times_{Z'} Y' \arrow[rr, crossing over] \arrow[dr, phantom, very near start, "\lrcorner"] & & Y' \arrow[dd, "g'"] \\
X \arrow[rr, "f" near start] \arrow[dr, "\phi"'] & & Z \arrow[dr, "\chi"] & \\
& X' \arrow[rr, "f'"'] \arrow[from=uu, crossing over] & & Z'
\end{tikzcd}$$

commute.

2. *Associativity.* Given a diagram

$$\begin{tikzcd}
X \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \arrow[dr, "h"'] & & Z \arrow[dl, "k"] \\
& W & & V &
\end{tikzcd}$$

in Sets_*, we have isomorphisms of pointed sets

$$(X \times_W Y) \times_V Z \cong (X \times_W Y) \times_Y (Y \times_V Z) \cong X \times_W (Y \times_V Z),$$

where these pullbacks are built as in the diagrams

$$\begin{tikzcd}
& & (X \times_W Y) \times_Y Z \arrow[dl] \arrow[ddrr] \arrow[dd, phantom, very near start, "\lrcorner" rotate=-45] & & \\
& X \times_W Y \arrow[dl] \arrow[dr] \arrow[dd, phantom, very near start, "\lrcorner" rotate=-45] & & & \\
X \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \arrow[dr, "h"'] & & Z, \arrow[dl, "k"] \\
& W & & V &
\end{tikzcd}
\quad
\begin{tikzcd}
& & (X \times_W Y) \times_Y (Y \times_V Z) \arrow[dl] \arrow[dr] \arrow[dd, phantom, very near start, "\lrcorner" rotate=-45] & & \\
& X \times_W Y \arrow[dl] \arrow[dr] \arrow[dd, phantom, very near start, "\lrcorner" rotate=-45] & & Y \times_V Z \arrow[dl] \arrow[dr] \arrow[dd, phantom, very near start, "\lrcorner" rotate=-45] & \\
X \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \arrow[dr, "h"'] & & Z, \arrow[dl, "k"] \\
& W & & V &
\end{tikzcd}
\quad
\begin{tikzcd}
& & X \times_W (Y \times_V Z) \arrow[ddll] \arrow[dr] \arrow[dd, phantom, very near start, "\lrcorner" rotate=-45] & & \\
& & & Y \times_V Z \arrow[dl] \arrow[dr] \arrow[dd, phantom, very near start, "\lrcorner" rotate=-45] & \\
X \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \arrow[dr, "h"'] & & Z. \arrow[dl, "k"] \\
& W & & V &
\end{tikzcd}$$

3. *Unitality.* We have isomorphisms of pointed sets

$$
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\
X \arrow[r, equal] & X
\end{tikzcd}
\qquad
\begin{aligned}
X \times_X A &\cong A, \\
A \times_X X &\cong A,
\end{aligned}
\qquad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, equal] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, equal] \\
X \arrow[r, "f"'] & X.
\end{tikzcd}
$$

4. *Commutativity.* We have an isomorphism of pointed sets

$$
\begin{tikzcd}
A \times_X B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "g"] \\
A \arrow[r, "f"'] & X,
\end{tikzcd}
\qquad
A \times_X B \cong B \times_X A
\qquad
\begin{tikzcd}
B \times_X A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\
B \arrow[r, "g"'] & X.
\end{tikzcd}
$$

5. *Interaction With Products.* We have an isomorphism of pointed sets

$$
X \times_{\mathrm{pt}} Y \cong X \times Y,
\qquad
\begin{tikzcd}
X \times Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "!_Y"] \\
X \arrow[r, "!_X"'] & \mathrm{pt}.
\end{tikzcd}
$$

6. *Symmetric Monoidality.* The triple $(\mathsf{Sets}_*, \times_X, X)$ is a symmetric monoidal category.

Proof 2.4.4 ▶ Proof of Proposition 2.4.3

Item 1: Functoriality

This is a special case of functoriality of co/limits, **??**, **??** of **??**, with the explicit expression for ξ following from the commutativity of the cube pullback diagram.

Item 2: Associativity

This follows from Constructions With Sets, Item 2 of Proposition 1.4.5.

Item 3: Unitality

This follows from Constructions With Sets, Item 3 of Proposition 1.4.5.

Item 4: Commutativity

This follows from Constructions With Sets, Item 4 of Proposition 1.4.5.

Item 5: Interaction With Products

This follows from Constructions With Sets, Item 6 of Proposition 1.4.5.

Item 6: Symmetric Monoidality

This follows from Constructions With Sets, Item 7 of Proposition 1.4.5.

2.5 Equalisers

Let $f, g \colon (X, x_0) \rightrightarrows (Y, y_0)$ be morphisms of pointed sets.

Definition 2.5.1 ► Equalisers of Pointed Sets

The **equaliser of** (f, g) is the pair consisting of:

- *The Limit.* The pointed set $(\mathrm{Eq}(f, g), x_0)$.
- *The Cone.* The morphism of pointed sets

  $$\mathrm{eq}(f, g) \colon (\mathrm{Eq}(f, g), x_0) \hookrightarrow (X, x_0)$$

  given by the canonical inclusion $\mathrm{eq}(f, g) \hookrightarrow \mathrm{Eq}(f, g) \hookrightarrow X$.

Proof 2.5.2 ► Proof of Definition 2.5.1

We claim that $(\mathrm{Eq}(f, g), x_0)$ is the categorical equaliser of f and g in Sets_*. First we need to check that the relevant equaliser diagram commutes, i.e. that we have

$$f \circ \mathrm{eq}(f, g) = g \circ \mathrm{eq}(f, g),$$

which indeed holds by the definition of the set $\mathrm{Eq}(f, g)$. Next, we prove that $\mathrm{Eq}(f, g)$ satisfies the universal property of the equaliser. Suppose we have a diagram of the form

$$\begin{tikzcd}
(\mathsf{Eq}(f,g), x_0) \arrow[r, hook, "\mathsf{eq}(f,g)"] & (X, x_0) \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & (Y, y_0) \\
(E, *) \arrow[ur, "e"'] & &
\end{tikzcd}$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi\colon (E,*) \longrightarrow (\mathsf{Eq}(f,g), x_0)$$

making the diagram

$$\begin{tikzcd}
(\mathsf{Eq}(f,g), x_0) \arrow[r, hook, "\mathsf{eq}(f,g)"] & (X, x_0) \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & (Y, y_0) \\
(E, *) \arrow[u, dashed, "\phi", "\exists!"'] \arrow[ur, "e"'] & &
\end{tikzcd}$$

commute, being uniquely determined by the condition

$$\mathsf{eq}(f,g) \circ \phi = e$$

via

$$\phi(x) = e(x)$$

for each $x \in E$, where we note that $e(x) \in A$ indeed lies in $\mathsf{Eq}(f,g)$ by the condition

$$f \circ e = g \circ e,$$

which gives

$$f(e(x)) = g(e(x))$$

for each $x \in E$, so that $e(x) \in \mathsf{Eq}(f,g)$. Lastly, we note that ϕ is indeed a morphism of pointed sets, as we have

$$\begin{aligned} \phi(*) &= e(*) \\ &= x_0, \end{aligned}$$

where we have used that e is a morphism of pointed sets. ∎

Proposition 2.5.3 ▶ Properties of Equalisers of Pointed Sets

Let (X, x_0) and (Y, y_0) be pointed sets and let $f, g, h \colon (X, x_0) \to (Y, y_0)$ be morphisms of pointed sets.

1. *Associativity.* We have isomorphisms of pointed sets

$$\underbrace{\mathrm{Eq}(f \circ \mathrm{eq}(g,h), g \circ \mathrm{eq}(g,h))}_{=\mathrm{Eq}(f\circ\mathrm{eq}(g,h),h\circ\mathrm{eq}(g,h))} \cong \mathrm{Eq}(f,g,h) \cong \underbrace{\mathrm{Eq}(f \circ \mathrm{eq}(f,g), h \circ \mathrm{eq}(f,g))}_{=\mathrm{Eq}(g\circ\mathrm{eq}(f,g),h\circ\mathrm{eq}(f,g))},$$

   where $\mathrm{Eq}(f, g, h)$ is the limit of the diagram

$$\begin{tikzcd} (X, x_0) \arrow[r, shift left=2, "f"] \arrow[r, dashed, "g" description] \arrow[r, shift right=2, "h"'] & (Y, y_0) \end{tikzcd}$$

   in Sets_*, being explicitly given by

$$\mathrm{Eq}(f,g,h) \cong \{a \in A \mid f(a) = g(a) = h(a)\}.$$

2. *Unitality.* We have an isomorphism of pointed sets

$$\mathrm{Eq}(f,f) \cong X.$$

3. *Commutativity.* We have an isomorphism of pointed sets

$$\mathrm{Eq}(f,g) \cong \mathrm{Eq}(g,f).$$

Proof 2.5.4 ▶ Proof of Proposition 2.5.3

Item 1: Associativity

This follows from Constructions With Sets, Item 1 of Proposition 1.5.3.

Item 2: Unitality

This follows from Constructions With Sets, Item 2 of Proposition 1.5.3.

Item 3: Commutativity

This follows from Constructions With Sets, Item 3 of Proposition 1.5.3. ☐

3 Colimits of Pointed Sets

3.1 The Initial Pointed Set

The **initial pointed set** is the pair $((\mathrm{pt},\star),\{\iota_X\}_{(X,x_0)\in\mathrm{Obj}(\mathsf{Sets}_*)})$ consisting of:

- *The Limit.* The pointed set $(\mathrm{pt},\star)$.
- *The Cone.* The collection of morphisms of pointed sets

  $$\{\iota_X\colon(\mathrm{pt},\star)\longrightarrow(X,x_0)\}_{(X,x_0)\in\mathrm{Obj}(\mathsf{Sets})}$$

  defined by

  $$\iota_X(\star)\overset{\mathrm{def}}{=}x_0.$$

PROOF 3.1.2 ► PROOF OF DEFINITION 3.1.1

We claim that $(\mathrm{pt},\star)$ is the initial object of Sets_*. Indeed, suppose we have a diagram of the form

$$(\mathrm{pt},\star) \qquad (X,x_0)$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi\colon(\mathrm{pt},\star)\longrightarrow(X,x_0)$$

making the diagram

$$(\mathrm{pt},\star)\ \underset{\exists!}{\overset{\phi}{\dashrightarrow}}\ (X,x_0)$$

commute, namely ι_X. ∎

3.2 Coproducts of Families of Pointed Sets

Let $\left\{(X_i,x_0^i)\right\}_{i\in I}$ be a family of pointed sets.

DEFINITION 3.2.1 ► COPRODUCTS OF FAMILIES OF POINTED SETS

The **coproduct of the family** $\left\{(X_i,x_0^i)\right\}_{i\in I}$, also called their **wedge sum**, is the pair consisting of:

- *The Colimit.* The pointed set $(\bigvee_{i\in I}X_i,p_0)$ consisting of:

- *The Underlying Set.* The set $\bigvee_{i\in I} X_i$ defined by

  $$\bigvee_{i\in I} X_i \overset{\text{def}}{=} \Bigl(\coprod_{i\in I} X_i\Bigr)/\!\sim,$$

  where $\sim$ is the equivalence relation on $\coprod_{i\in I} X_i$ given by declaring

  $$(i, x^i_0) \sim (j, x^j_0)$$

  for each $i, j \in I$.
- *The Basepoint.* The element p_0 of $\bigvee_{i\in I} X_i$ defined by

  $$\begin{aligned} p_0 &\overset{\text{def}}{=} [(i, x^i_0)] \\ &= [(j, x^j_0)] \end{aligned}$$

  for any $i, j \in I$.

· *The Cocone.* The collection

$$\left\{ \mathrm{inj}_i \colon (X_i, x^i_0) \to \Bigl(\bigvee_{i\in I} X_i, p_0\Bigr) \right\}_{i\in I}$$

of morphism of pointed sets given by

$$\mathrm{inj}_i(x) \overset{\text{def}}{=} (i, x)$$

for each $x \in X_i$ and each $i \in I$.

Proof 3.2.2 ► Proof of Definition 3.2.1

We claim that $(\bigvee_{i\in I} X_i, p_0)$ is the categorical coproduct of $\{(X_i, x_0^i)\}_{i\in I}$ in Sets_*. Indeed, suppose we have, for each $i \in I$, a diagram of the form

$$\begin{tikzcd} & (C,*) \\ (X_i, x_0^i) \arrow[r, "\mathrm{inj}_i"'] \arrow[ur, "\iota_i"] & (\bigvee_{i\in I} X_i, p_0) \end{tikzcd}$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi\colon (\bigvee_{i\in I} X_i, p_0) \to (C,*)$$

making the diagram

$$\begin{tikzcd} & (C,*) \\ (X_i, x_0^i) \arrow[r, "\mathrm{inj}_i"'] \arrow[ur, "\iota_i"] & (\bigvee_{i\in I} X_i, p_0) \arrow[u, dashed, "\phi", "\exists!"'] \end{tikzcd}$$

commute, being uniquely determined by the condition $\phi \circ \mathrm{inj}_i = \iota_i$ for each $i \in I$ via

$$\phi([(i,x)]) = \iota_i(x)$$

for each $[(i,x)] \in \bigvee_{i\in I} X_i$, where we note that ϕ is indeed a morphism of pointed sets, as we have

$$\begin{aligned} \phi(p_0) &= \iota_i([(i,x_0^i)]) \\ &= *, \end{aligned}$$

as ι_i is a morphism of pointed sets. ∎

PROPOSITION 3.2.3 ► PROPERTIES OF COPRODUCTS OF FAMILIES OF POINTED SETS

Let $\left\{(X_i, x_0^i)\right\}_{i\in I}$ be a family of pointed sets.

1. *Functoriality.* The assignment $\left\{(X_i, x_0^i)\right\}_{i\in I} \mapsto (\bigvee_{i\in I} X_i, p_0)$ defines a functor

$$\bigvee_{i\in I}\colon \mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}_*) \longrightarrow \mathsf{Sets}_*.$$

PROOF 3.2.4 ► PROOF OF PROPOSITION 3.2.3

Item 1: Functoriality

This follows from **??**, **??** of **??**.

3.3 Coproducts

Let (X, x_0) and (Y, y_0) be pointed sets.

DEFINITION 3.3.1 ► COPRODUCTS OF POINTED SETS

The **coproduct of** (X, x_0) **and** (Y, y_0), also called their **wedge sum**, is the pair consisting of:

- *The Colimit.* The pointed set $(X \vee Y, p_0)$ consisting of:
  - *The Underlying Set.* The set $X \vee Y$ defined by

$$\begin{aligned}(X \vee Y, p_0) &\overset{\text{def}}{=} (X, x_0) \textstyle\coprod (Y, y_0)\\ &\cong (X \textstyle\coprod_{\mathsf{pt}} Y, p_0)\\ &\cong (X \textstyle\coprod Y/\sim, p_0),\end{aligned}
\qquad
\begin{tikzcd} X \vee Y \arrow[dr, phantom, "\ulcorner", very near start] & Y \arrow[l] \\ X \arrow[u, hook] & \mathsf{pt}, \arrow[l, hook, "{[x_0]}"] \arrow[u, hook, "{[y_0]}"'] \end{tikzcd}$$

    where $\sim$ is the equivalence relation on $X \coprod Y$ obtained by declaring $(0, x_0) \sim (1, y_0)$.
  - *The Basepoint.* The element p_0 of $X \vee Y$ defined by

$$\begin{aligned}p_0 &\overset{\text{def}}{=} [(0, x_0)]\\ &= [(1, y_0)].\end{aligned}$$

- *The Cocone.* The morphisms of pointed sets

$$
\begin{aligned}
\mathrm{inj}_1 &: (X, x_0) \longrightarrow (X \vee Y, p_0), \\
\mathrm{inj}_2 &: (Y, y_0) \longrightarrow (X \vee Y, p_0),
\end{aligned}
$$

given by

$$
\begin{aligned}
\mathrm{inj}_1(x) &\stackrel{\text{def}}{=} [(0, x)], \\
\mathrm{inj}_2(y) &\stackrel{\text{def}}{=} [(1, y)],
\end{aligned}
$$

for each $x \in X$ and each $y \in Y$.

Proof 3.3.2 ► Proof of Definition 3.3.1

We claim that $(X \vee Y, p_0)$ is the categorical coproduct of (X, x_0) and (Y, y_0) in Sets_*. Indeed, suppose we have a diagram of the form

$$
\begin{tikzcd}
 & (C, *) & \\
(X, x_0) \arrow[r, "\mathrm{inj}_X"'] \arrow[ur, bend left, "\iota_X"] & (X \vee Y, p_0) & (Y, y_0) \arrow[l, "\mathrm{inj}_Y"] \arrow[ul, bend right, "\iota_Y"']
\end{tikzcd}
$$

in Sets. Then there exists a unique morphism of pointed sets

$$
\phi \colon (X \vee Y, p_0) \longrightarrow (C, *)
$$

making the diagram

$$
\begin{tikzcd}
 & (C, *) & \\
(X, x_0) \arrow[r, "\mathrm{inj}_X"'] \arrow[ur, bend left, "\iota_X"] & (X \vee Y, p_0) \arrow[u, dashed, "\phi", "\exists!"'] & (Y, y_0) \arrow[l, "\mathrm{inj}_Y"] \arrow[ul, bend right, "\iota_Y"']
\end{tikzcd}
$$

commute, being uniquely determined by the conditions

$$
\begin{aligned}
\phi \circ \mathrm{inj}_X &= \iota_X, \\
\phi \circ \mathrm{inj}_Y &= \iota_Y
\end{aligned}
$$

via

$$\phi(z) = \begin{cases} \iota_X(x) & \text{if } z = [(0, x)] \text{ with } x \in X, \\ \iota_Y(y) & \text{if } z = [(1, y)] \text{ with } y \in Y \end{cases}$$

for each $z \in X \vee Y$, where we note that ϕ is indeed a morphism of pointed sets, as we have

$$\begin{aligned} \phi(p_0) &= \iota_X([(0, x_0)]) \\ &= \iota_Y([(1, y_0)]) \\ &= *, \end{aligned}$$

as ι_X and ι_Y are morphisms of pointed sets. □

PROPOSITION 3.3.3 ► PROPERTIES OF WEDGE SUMS OF POINTED SETS

Let (X, x_0) and (Y, y_0) be pointed sets.

1. *Functoriality.* The assignments

   $$(X, x_0), (Y, y_0), ((X, x_0), (Y, y_0)) \mapsto (X \vee Y, p_0)$$

   define functors

   $$\begin{aligned} X \vee -\colon \mathsf{Sets}_* &\longrightarrow \mathsf{Sets}_*, \\ - \vee Y\colon \mathsf{Sets}_* &\longrightarrow \mathsf{Sets}_*, \\ -_1 \vee -_2\colon \mathsf{Sets}_* \times \mathsf{Sets}_* &\longrightarrow \mathsf{Sets}_*. \end{aligned}$$

2. *Associativity.* We have an isomorphism of pointed sets

   $$(X \vee Y) \vee Z \cong X \vee (Y \vee Z),$$

   natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathsf{Sets}_*$.

3. *Unitality.* We have isomorphisms of pointed sets

   $$\begin{aligned} (\mathrm{pt}, *) \vee (X, x_0) &\cong (X, x_0), \\ (X, x_0) \vee (\mathrm{pt}, *) &\cong (X, x_0), \end{aligned}$$

   natural in $(X, x_0) \in \mathsf{Sets}_*$.

4. *Commutativity.* We have an isomorphism of pointed sets

$$X \vee Y \cong Y \vee X,$$

natural in $(X, x_0), (Y, y_0) \in \mathsf{Sets}_*$.

5. *Symmetric Monoidality.* The triple $(\mathsf{Sets}_*, \vee, \mathrm{pt})$ is a symmetric monoidal category.

6. *The Fold Map.* We have a natural transformation

$$\nabla \colon \vee \circ \Delta^{\mathsf{Cats}}_{\mathsf{Sets}_*} \Longrightarrow \mathrm{id}_{\mathsf{Sets}_*}, \qquad \begin{tikzcd} & \mathsf{Sets}_* \times \mathsf{Sets}_* \arrow[dr, "\vee"] \arrow[d, phantom, "\Downarrow \nabla"] & \\ \mathsf{Sets}_* \arrow[ur, "\Delta^{\mathsf{Cats}}_{\mathsf{Sets}_*}"] \arrow[rr, bend right, "\mathrm{id}_{\mathsf{Sets}_*}"'] & & \mathsf{Sets}_*, \end{tikzcd}$$

called the **fold map**, whose component

$$\nabla_X \colon X \vee X \to X$$

at X is given by

$$\nabla_X(p) \overset{\mathrm{def}}{=} \begin{cases} x & \text{if } p = [(0, x)], \\ x & \text{if } p = [(1, x)] \end{cases}$$

for each $p \in X \vee X$.

Proof 3.3.4 ► Proof of Proposition 3.3.3

Item 1: Functoriality

This follows from **??**, **??** of **??**.

Item 2: Associativity

Clear.

Item 3: Unitality

Clear.

Item 4: Commutativity

Clear.

Item 5: Symmetric Monoidality

Omitted.

Item 6: The Fold Map

Naturality for the transformation ∇ is the statement that, given a morphism of pointed sets $f\colon (X, x_0) \to (Y, y_0)$, we have

$$\nabla_Y \circ (f \vee f) = f \circ \nabla_X, \qquad \begin{tikzcd} X \vee X \arrow[r, "\nabla_X"] \arrow[d, "f \vee f"'] & X \arrow[d, "f"] \\ Y \vee Y \arrow[r, "\nabla_Y"'] & Y. \end{tikzcd}$$

Indeed, we have

$$\begin{aligned} [\nabla_Y \circ (f \vee f)]([(i, x)]) &= \nabla_Y([(i, f(x))]) \\ &= f(x) \\ &= f(\nabla_X([(i, x)])) \\ &= [f \circ \nabla_X]([(i, x)]) \end{aligned}$$

for each $[(i, x)] \in X \vee X$, and thus ∇ is indeed a natural transformation. ∎

3.4 Pushouts

Let (X, x_0), (Y, y_0), and (Z, z_0) be pointed sets and let $f\colon (Z, z_0) \to (X, x_0)$ and $g\colon (Z, z_0) \to (Y, y_0)$ be morphisms of pointed sets.

Definition 3.4.1 ▶ Pushouts of Pointed Sets

The **pushout of** (X, x_0) **and** (Y, y_0) **over** (Z, z_0) **along** (f, g) is the pair consisting of:

- *The Colimit.* The pointed set $(X \coprod_{f,Z,g} Y, p_0)$, where:
  - The set $X \coprod_{f,Z,g} Y$ is the pushout (of unpointed sets) of X and Y over

Z with respect to f and g;

- We have $p_0 = [x_0] = [y_0]$.

· *The Cocone.* The morphisms of pointed sets

$$\begin{aligned}\mathrm{inj}_1 &\colon (X, x_0) \to (X \textstyle\coprod_Z Y, p_0),\\ \mathrm{inj}_2 &\colon (Y, y_0) \to (X \textstyle\coprod_Z Y, p_0)\end{aligned}$$

given by

$$\begin{aligned}\mathrm{inj}_1(x) &\stackrel{\text{def}}{=} [(0, x)]\\ \mathrm{inj}_2(y) &\stackrel{\text{def}}{=} [(1, y)]\end{aligned}$$

for each $x \in X$ and each $y \in Y$.

Proof 3.4.2 ► Proof of Definition 3.4.1

Firstly, we note that indeed $[x_0] = [y_0]$, as we have

$$\begin{aligned}x_0 &= f(z_0),\\ y_0 &= g(z_0)\end{aligned}$$

since f and g are morphisms of pointed sets, with the relation $\sim$ on $X \coprod_Z Y$ then identifying $x_0 = f(z_0) \sim g(z_0) = y_0$.

We now claim that $(X \coprod_Z Y, p_0)$ is the categorical pushout of (X, x_0) and (Y, y_0) over (Z, z_0) with respect to (f, g) in Sets_*. First we need to check that the relevant pushout diagram commutes, i.e. that we have

$$\mathrm{inj}_1 \circ f = \mathrm{inj}_2 \circ g, \qquad \begin{tikzcd} (X \coprod_Z Y, p_0) & (Y, y_0) \arrow[l, "\mathrm{inj}_2"'] \\ (X, x_0) \arrow[u, "\mathrm{inj}_1"] & (Z, z_0). \arrow[l, "f"] \arrow[u, "g"'] \end{tikzcd}$$

Indeed, given $z \in Z$, we have

$$\begin{aligned}
[\mathsf{inj}_1 \circ f](z) &= \mathsf{inj}_1(f(z)) \\
&= [(0, f(z))] \\
&= [(1, g(z))] \\
&= \mathsf{inj}_2(g(z)) \\
&= [\mathsf{inj}_2 \circ g](z),
\end{aligned}$$

where $[(0, f(z))] = [(1, g(z))]$ by the definition of the relation $\sim$ on $X \coprod Y$ (the coproduct of unpointed sets of X and Y). Next, we prove that $X \coprod_Z Y$ satisfies the universal property of the pushout. Suppose we have a diagram of the form

$$\begin{tikzcd}
(P, *) & & \\
& (X \coprod_Z Y, p_0) \arrow[dr, phantom, "\ulcorner", very near start] & (Y, y_0) \arrow[l, "\mathsf{inj}_2"'] \arrow[ull, bend right, "\iota_2"'] \\
& (X, x_0) \arrow[u, "\mathsf{inj}_1"] \arrow[uul, bend left, "\iota_1"] & (Z, z_0) \arrow[l, "f"] \arrow[u, "g"']
\end{tikzcd}$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi \colon (X \textstyle\coprod_Z Y, p_0) \to (P, *)$$

making the diagram

$$\begin{tikzcd}
(P, *) & & \\
& (X \coprod_Z Y, p_0) \arrow[ul, dashed, "\exists!"', "\phi"] \arrow[dr, phantom, "\ulcorner", very near start] & (Y, y_0) \arrow[l, "\mathsf{inj}_2"'] \arrow[ull, bend right, "\iota_2"'] \\
& (X, x_0) \arrow[u, "\mathsf{inj}_1"] \arrow[uul, bend left, "\iota_1"] & (Z, z_0) \arrow[l, "f"] \arrow[u, "g"']
\end{tikzcd}$$

commute, being uniquely determined by the conditions

$$\begin{aligned}\phi \circ \mathrm{inj}_1 &= \iota_1,\\ \phi \circ \mathrm{inj}_2 &= \iota_2\end{aligned}$$

via

$$\phi(p) = \begin{cases} \iota_1(x) & \text{if } x = [(0, x)], \\ \iota_2(y) & \text{if } x = [(1, y)] \end{cases}$$

for each $p \in X \coprod_Z Y$, where the well-definedness of ϕ is proven in the same way as in the proof of Constructions With Sets, Definition 2.4.1. Finally, we show that ϕ is indeed a morphism of pointed sets, as we have

$$\begin{aligned}\phi(p_0) &= \phi([(0, x_0)])\\ &= \iota_1(x_0)\\ &= *,\end{aligned}$$

or alternatively

$$\begin{aligned}\phi(p_0) &= \phi([(1, y_0)])\\ &= \iota_2(y_0)\\ &= *,\end{aligned}$$

where we use that ι_1 (resp. ι_2) is a morphism of pointed sets.

Proposition 3.4.3 ▶ Properties of Pushouts of Pointed Sets

Let (X, x_0), (Y, y_0), (Z, z_0), and (A, a_0) be pointed sets.

1. *Functoriality*. The assignment $(X, Y, Z, f, g) \mapsto X \coprod_{f,Z,g} Y$ defines a functor

$$-_1 \coprod_{-_3} -_1 \colon \mathsf{Fun}(\mathcal{P}, \mathsf{Sets}) \to \mathsf{Sets}_*,$$

   where $\mathcal{P}$ is the category that looks like this:

$$\begin{tikzcd} & \bullet \\ \bullet & \bullet \arrow[l] \arrow[u] \end{tikzcd}.$$

In particular, the action on morphisms of $-_1 \coprod_{-_3} -_1$ is given by sending a morphism

$$\begin{tikzcd}
X \coprod_Z Y \arrow[dr, phantom, very near start, "\ulcorner"] & & Y \arrow[ll] \arrow[dr, "\psi"] & \\
& X' \coprod_{Z'} Y' \arrow[dr, phantom, very near start, "\ulcorner"] & & Y' \arrow[ll] \\
X \arrow[uu] \arrow[dr, "\phi"'] & & Z \arrow[ll, "f"'] \arrow[uu, "g"] \arrow[dr, "\chi"] & \\
& X' \arrow[uu] & & Z' \arrow[ll, "f'"] \arrow[uu, "g'"']
\end{tikzcd}$$

in $\mathsf{Fun}(\mathcal{P}, \mathsf{Sets}_*)$ to the morphism of pointed sets

$$\xi \colon (X \coprod_Z Y, p_0) \xrightarrow{\exists!} (X' \coprod_{Z'} Y', p_0')$$

given by

$$\xi(p) \overset{\text{def}}{=} \begin{cases} \phi(x) & \text{if } p = [(0, x)], \\ \psi(y) & \text{if } p = [(1, y)] \end{cases}$$

for each $p \in X \coprod_Z Y$, which is the unique morphism of pointed sets making the diagram

$$\begin{tikzcd}
X \coprod_Z Y \arrow[dr, phantom, very near start, "\ulcorner"] \arrow[dr, dashed] & & Y \arrow[ll] \arrow[dr, "\psi"] & \\
& X' \coprod_{Z'} Y' \arrow[dr, phantom, very near start, "\ulcorner"] & & Y' \arrow[ll] \\
X \arrow[uu] \arrow[dr, "\phi"'] & & Z \arrow[ll, "f"'] \arrow[uu, "g"] \arrow[dr, "\chi"] & \\
& X' \arrow[uu] & & Z' \arrow[ll, "f'"] \arrow[uu, "g'"']
\end{tikzcd}$$

commute.

2. *Associativity.* Given a diagram

$$
\begin{tikzcd}
X & & Y & & Z \\
& W \arrow[ul, "f"] \arrow[ur, "g"'] & & V \arrow[ul, "h"] \arrow[ur, "k"'] &
\end{tikzcd}
$$

in Sets, we have isomorphisms of pointed sets

$$(X \textstyle\coprod_W Y) \coprod_V Z \cong (X \coprod_W Y) \coprod_Y (Y \coprod_V Z) \cong X \coprod_W (Y \coprod_V Z),$$

where these pullbacks are built as in the diagrams

$$
\begin{tikzcd}[column sep=tiny]
& & (X \coprod_W Y) \coprod_V Z & & \\
& X \coprod_W Y \arrow[ur] \arrow[d, phantom, "\wedge" near start] & & & \\
X \arrow[ur] & & Y \arrow[ul] & & Z, \arrow[uull] \\
& W \arrow[ul, "f"] \arrow[ur, "g"'] & & V \arrow[ul, "h"] \arrow[ur, "k"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=tiny]
& & (X \coprod_W Y) \coprod_Y (Y \coprod_V Z) & & \\
& X \coprod_W Y \arrow[ur] \arrow[d, phantom, "\wedge" near start] & & Y \coprod_V Z \arrow[ul] \arrow[d, phantom, "\wedge" near start] & \\
X \arrow[ur] & & Y \arrow[ul] \arrow[ur] & & Z, \arrow[ul] \\
& W \arrow[ul, "f"] \arrow[ur, "g"'] & & V \arrow[ul, "h"] \arrow[ur, "k"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=tiny]
& & X \coprod_W (Y \coprod_V Z) & & \\
& & & Y \coprod_V Z \arrow[ul] \arrow[d, phantom, "\wedge" near start] & \\
X \arrow[uurr] & & Y \arrow[ur] & & Z. \arrow[ul] \\
& W \arrow[ul, "f"] \arrow[ur, "g"'] & & V \arrow[ul, "h"] \arrow[ur, "k"'] &
\end{tikzcd}
$$

3. *Unitality.* We have isomorphisms of sets

$$
\begin{tikzcd}
A \arrow[r, equal] & A \\
X \arrow[u, "f"] \arrow[r, equal] \arrow[ur, phantom, "\ulcorner" very near end] & X \arrow[u, "f"']
\end{tikzcd}
\qquad
\begin{aligned}
X \textstyle\coprod_X A &\cong A, \\
A \textstyle\coprod_X X &\cong A,
\end{aligned}
\qquad
\begin{tikzcd}
A & X \arrow[l, "f"'] \\
X \arrow[u, equal] \arrow[ur, phantom, "\ulcorner" very near end] & X. \arrow[l, "f"] \arrow[u, equal]
\end{tikzcd}
$$

4. *Commutativity.* We have an isomorphism of sets

$$
\begin{tikzcd}
X \coprod_Z Y & Y \arrow[l] \\
X \arrow[u] \arrow[ur, phantom, "\ulcorner" very near end] & Z, \arrow[l, "f"] \arrow[u, "g"']
\end{tikzcd}
\qquad
X \textstyle\coprod_Z Y \cong Y \coprod_Z X
\qquad
\begin{tikzcd}
Y \coprod_Z X & X \arrow[l] \\
Y \arrow[u] \arrow[ur, phantom, "\ulcorner" very near end] & Z. \arrow[l, "g"] \arrow[u, "f"']
\end{tikzcd}
$$

5. *Interaction With Coproducts.* We have

$$X \textstyle\coprod_{\mathrm{pt}} Y \cong X \vee Y, \qquad \begin{tikzcd} X \vee Y \arrow[dr, phantom, "\ulcorner", very near start] & Y \arrow[l] \\ X \arrow[u] & \mathrm{pt}. \arrow[l, hook', "{[x_0]}"] \arrow[u, hook, "{[y_0]}"'] \end{tikzcd}$$

6. *Symmetric Monoidality.* The triple $(\mathsf{Sets}_*, \coprod_X, (X, x_0))$ is a symmetric monoidal category.

Proof 3.4.4 ▶ Proof of Proposition 3.4.3

Item 1: Functoriality

This is a special case of functoriality of co/limits, **??**, **??** of **??**, with the explicit expression for ξ following from the commutativity of the cube pushout diagram.

Item 2: Associativity

This follows from Constructions With Sets, Item 2 of Proposition 2.4.6.

Item 3: Unitality

This follows from Constructions With Sets, Item 3 of Proposition 2.4.6.

Item 4: Commutativity

This follows from Constructions With Sets, Item 4 of Proposition 2.4.6.

Item 5: Interaction With Coproducts

Clear.

Item 6: Symmetric Monoidality

Omitted.

3.5 Coequalisers

Let $f, g\colon (X, x_0) \rightrightarrows (Y, y_0)$ be morphisms of pointed sets.

Definition 3.5.1 ▶ Coequalisers of Pointed Sets

The **coequaliser of** (f, g) is the pointed set $(\mathrm{CoEq}(f, g), [y_0])$.

PROOF 3.5.2 ▶ PROOF OF DEFINITION 3.5.1

We claim that $(\mathrm{CoEq}(f,g), [y_0])$ is the categorical coequaliser of f and g in Sets_*. First we need to check that the relevant coequaliser diagram commutes, i.e. that we have

$$\mathrm{coeq}(f,g) \circ f = \mathrm{coeq}(f,g) \circ g.$$

Indeed, we have

$$\begin{aligned}
[\mathrm{coeq}(f,g) \circ f](x) &\stackrel{\text{def}}{=} [\mathrm{coeq}(f,g)](f(x)) \\
&\stackrel{\text{def}}{=} [f(x)] \\
&= [g(x)] \\
&\stackrel{\text{def}}{=} [\mathrm{coeq}(f,g)](g(x)) \\
&\stackrel{\text{def}}{=} [\mathrm{coeq}(f,g) \circ g](x)
\end{aligned}$$

for each $x \in X$. Next, we prove that $\mathrm{CoEq}(f,g)$ satisfies the universal property of the coequaliser. Suppose we have a diagram of the form

$$\begin{tikzcd}
(X, x_0) \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & (Y, y_0) \arrow[r, "\mathrm{coeq}(f,g)"] \arrow[dr, "c"'] & (\mathrm{CoEq}(f,g), [y_0]) \\
& & (C, *)
\end{tikzcd}$$

in Sets. Then, since $c(f(a)) = c(g(a))$ for each $a \in A$, it follows from Equivalence Relations and Apartness Relations, Items 4 and 5 of Proposition 5.2.3 that there exists a unique map $\phi \colon \mathrm{CoEq}(f,g) \xrightarrow{\exists!} C$ making the diagram

$$\begin{tikzcd}
(X, x_0) \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & (Y, y_0) \arrow[r, "\mathrm{coeq}(f,g)"] \arrow[dr, "c"'] & (\mathrm{CoEq}(f,g), [y_0]) \arrow[d, dashed, "\phi"', "\exists!"] \\
& & (C, *)
\end{tikzcd}$$

commute, where we note that ϕ is indeed a morphism of pointed sets since

$$\begin{aligned}
\phi([y_0]) &= [\phi \circ \mathrm{coeq}(f,g)]([y_0]) \\
&= c([y_0]) \\
&= *,
\end{aligned}$$

where we have used that c is a morphism of pointed sets. ▭

Proposition 3.5.3 ▶ Properties of Coequalisers of Pointed Sets

Let (X, x_0) and (Y, y_0) be pointed sets and let $f, g, h\colon (X, x_0) \longrightarrow (Y, y_0)$ be morphisms of pointed sets.

1. *Associativity.* We have isomorphisms of pointed sets

$$\underbrace{\mathrm{CoEq}(\mathrm{coeq}(f,g)\circ f, \mathrm{coeq}(f,g)\circ h)}_{=\mathrm{CoEq}(\mathrm{coeq}(f,g)\circ g, \mathrm{coeq}(f,g)\circ h)} \cong \mathrm{CoEq}(f,g,h) \cong \underbrace{\mathrm{CoEq}(\mathrm{coeq}(g,h)\circ f, \mathrm{coeq}(g,h)\circ g)}_{=\mathrm{CoEq}(\mathrm{coeq}(g,h)\circ f, \mathrm{coeq}(g,h)\circ h)},$$

   where $\mathrm{CoEq}(f,g,h)$ is the colimit of the diagram

$$(X, x_0) \overset{f}{\underset{h}{\overset{\xrightarrow{\;g\;}}{\rightrightarrows}}} (Y, y_0)$$

   in Sets_*.

2. *Unitality.* We have an isomorphism of pointed sets

$$\mathrm{CoEq}(f,f) \cong B.$$

3. *Commutativity.* We have an isomorphism of pointed sets

$$\mathrm{CoEq}(f,g) \cong \mathrm{CoEq}(g,f).$$

Proof 3.5.4 ▶ Proof of Proposition 3.5.3

Item 1: Associativity

This follows from Constructions With Sets, Item 1 of Proposition 2.5.6.

Item 2: Unitality

This follows from Constructions With Sets, Item 2 of Proposition 2.5.6.

Item 3: Commutativity

This follows from Constructions With Sets, Item 3 of Proposition 2.5.6. ▭

4 Constructions With Pointed Sets

4.1 Free Pointed Sets

Let X be a set.

Definition 4.1.1 ▶ Free Pointed Sets

The **free pointed set on** X is the pointed set X^+ consisting of:

- *The Underlying Set.* The set X^+ defined by[1]

$$\begin{aligned} X^+ &\overset{\text{def}}{=} X \coprod \mathrm{pt} \\ &\overset{\text{def}}{=} X \coprod \{\star\}. \end{aligned}$$

- *The Basepoint.* The element $\star$ of X^+.

[1] *Further Notation:* We sometimes write $\star_X$ for the basepoint of X^+ for clarity when there are multiple free pointed sets involved in the current discussion.

Proposition 4.1.2 ▶ Properties of Free Pointed Sets

Let X be a set.

1. *Functoriality.* The assignment $X \mapsto X^+$ defines a functor

$$(-)^+ \colon \mathsf{Sets} \longrightarrow \mathsf{Sets}_*,$$

where

- *Action on Objects.* For each $X \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$[(-)^+](X) \overset{\text{def}}{=} X^+,$$

where X^+ is the pointed set of Definition 4.1.1;

- *Action on Morphisms.* For each morphism $f \colon X \longrightarrow Y$ of Sets, the image

$$f^+ \colon X^+ \longrightarrow Y^+$$

of f by $(-)^+$ is the map of pointed sets defined by

$$f^+(x) \overset{\text{def}}{=} \begin{cases} f(x) & \text{if } x \in X, \\ \star_Y & \text{if } x = \star_X. \end{cases}$$

2. *Adjointness.* We have an adjunction

$$\bigl((-)^{+} \dashv 忘\bigr)\colon\quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "(-)^{+}"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_{*}, \arrow[l, bend left, "忘"] \end{tikzcd}$$

witnessed by a bijection of sets

$$\mathsf{Sets}_{*}((X^{+}, \star_X), (Y, y_0)) \cong \mathsf{Sets}(X, Y),$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets})$ and $(Y, y_0) \in \mathrm{Obj}(\mathsf{Sets}_{*})$.

3. *Symmetric Strong Monoidality With Respect to Wedge Sums.* The free pointed set functor of Item 1 has a symmetric strong monoidal structure

$$\left((-)^{+}, (-)^{+,\coprod}, (-)^{+,\coprod}_{\mathbb{1}}\right)\colon (\mathsf{Sets}, \textstyle\coprod, \emptyset) \longrightarrow (\mathsf{Sets}_{*}, \vee, \mathrm{pt}),$$

being equipped with isomorphisms of pointed sets

$$(-)^{+,\coprod}_{X,Y}\colon X^{+} \vee Y^{+} \xrightarrow{\cong} (X \textstyle\coprod Y)^{+},$$
$$(-)^{+,\coprod}_{\mathbb{1}}\colon \mathrm{pt} \xrightarrow{\cong} \emptyset^{+},$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

4. *Symmetric Strong Monoidality With Respect to Smash Products.* The free pointed set functor of Item 1 has a symmetric strong monoidal structure

$$\left((-)^{+}, (-)^{+,\times}, (-)^{+,\times}_{\mathbb{1}}\right)\colon (\mathsf{Sets}, \times, \mathrm{pt}) \longrightarrow (\mathsf{Sets}_{*}, \wedge, S^{0}),$$

being equipped with isomorphisms of pointed sets

$$(-)^{+,\times}_{X,Y}\colon X^{+} \wedge Y^{+} \xrightarrow{\cong} (X \times Y)^{+},$$
$$(-)^{+,\times}_{\mathbb{1}}\colon S^{0} \xrightarrow{\cong} \mathrm{pt}^{+},$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

Proof 4.1.3 ▶ Proof of Proposition 4.1.2

Item 1: Functoriality

Clear.

Item 2: Adjointness

We claim there's an adjunction $(-)^{+} \dashv 忘$, witnessed by a bijection of sets

$$\mathsf{Sets}_*((X^+, \star_X), (Y, y_0)) \cong \mathsf{Sets}(X, Y),$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets})$ and $(Y, y_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

- *Map I.* We define a map

  $$\Phi_{X,Y} \colon \mathsf{Sets}_*((X^+, \star_X), (Y, y_0)) \longrightarrow \mathsf{Sets}(X, Y)$$

  by sending a pointed function

  $$\xi \colon (X^+, \star_X) \to (Y, y_0)$$

  to the function

  $$\xi^{\dagger} \colon X \to Y$$

  given by

  $$\xi^{\dagger}(x) \overset{\text{def}}{=} \xi(x)$$

  for each $x \in X$.

- *Map II.* We define a map

  $$\Psi_{X,Y} \colon \mathsf{Sets}(X, Y) \longrightarrow \mathsf{Sets}_*((X^+, \star_X), (Y, y_0))$$

  given by sending a function $\xi \colon X \to Y$ to the pointed function

  $$\xi^{\dagger} \colon (X^+, \star_X) \to (Y, y_0)$$

  defined by

  $$\xi^{\dagger}(x) \overset{\text{def}}{=} \begin{cases} \xi(x) & \text{if } x \in X, \\ y_0 & \text{if } x = \star_X \end{cases}$$

  for each $x \in X^+$.

- *Invertibility I.* We claim that

$$\Psi_{X,Y} \circ \Phi_{X,Y} = \mathrm{id}_{\mathsf{Sets}_*((X^+,\star_X),(Y,y_0))},$$

which is clear.

- *Invertibility II.* We claim that

$$\Phi_{X,Y} \circ \Psi_{X,Y} = \mathrm{id}_{\mathsf{Sets}(X,Y)},$$

which is clear.

- *Naturality for* Φ*, Part I.* We need to show that, given a pointed function $g\colon (Y,y_0) \to (Y',y'_0)$, the diagram

$$\begin{tikzcd}
\mathsf{Sets}_*((X^+,\star_X),(Y,y_0)) \arrow[r, "\Phi_{X,Y}"] \arrow[d, "g_*"'] & \mathsf{Sets}(X,Y) \arrow[d, "g_*"] \\
\mathsf{Sets}_*((X^+,\star_X),(Y',y'_0)), \arrow[r, "\Phi_{X,Y'}"'] & \mathsf{Sets}(X,Y')
\end{tikzcd}$$

commutes. Indeed, given a pointed function

$$\xi^\dagger\colon (X^+,\star_X) \to (Y,y_0)$$

we have

$$\begin{aligned}
[\Phi_{X,Y'} \circ g_*](\xi) &= \Phi_{X,Y'}(g_*(\xi)) \\
&= \Phi_{X,Y'}(g \circ \xi) \\
&= g \circ \xi \\
&= g \circ \Phi_{X,Y'}(\xi) \\
&= g_*(\Phi_{X,Y'}(\xi)) \\
&= [g_* \circ \Phi_{X,Y'}](\xi).
\end{aligned}$$

- *Naturality for* Φ*, Part II.* We need to show that, given a pointed function

$f\colon (X, x_0) \to (X', x'_0)$, the diagram

$$\begin{tikzcd}
\mathsf{Sets}_*((X'^{,+}, \star_X), (Y, y_0)) \arrow[r, "\Phi_{X',Y}"] \arrow[d, "f^*"'] & \mathsf{Sets}(X', Y) \arrow[d, "f^*"] \\
\mathsf{Sets}_*((X^+, \star_X), (Y, y_0)) \arrow[r, "\Phi_{X,Y}"'] & \mathsf{Sets}(X, Y)
\end{tikzcd}$$

commutes. Indeed, given a function

$$\xi\colon X' \to Y,$$

we have

$$\begin{aligned}
[\Phi_{X,Y} \circ f^*](\xi) &= \Phi_{X,Y}(f^*(\xi)) \\
&= \Phi_{X,Y}(\xi \circ f) \\
&= \xi \circ f \\
&= \Phi_{X',Y}(\xi) \circ f \\
&= f^*(\Phi_{X',Y}(\xi)) \\
&= f^*(\Phi_{X',Y}(\xi)) \\
&= [f^* \circ \Phi_{X',Y}](\xi).
\end{aligned}$$

- *Naturality for* Ψ. Since Φ is natural in each argument and Φ is a componentwise inverse to Ψ in each argument, it follows from Categories, Item 2 of Proposition 8.6.2 that Ψ is also natural in each argument.

Item 3: Symmetric Strong Monoidality With Respect to Wedge Sums

The isomorphism

$$\phi\colon X^+ \vee Y^+ \xrightarrow{\cong} (X \textstyle\coprod Y)^+$$

is given by

$$\phi(z) = \begin{cases} x & \text{if } z = [(0, x)] \text{ with } x \in X, \\ y & \text{if } z = [(1, y)] \text{ with } y \in Y, \\ \star_{X \coprod Y} & \text{if } z = [(0, \star_X)], \\ \star_{X \coprod Y} & \text{if } z = [(1, \star_Y)] \end{cases}$$

for each $z \in X^+ \vee Y^+$, with inverse

$$\phi^{-1}\colon (X \coprod Y)^+ \xrightarrow{\cong} X^+ \vee Y^+$$

given by

$$\phi^{-1}(z) \overset{\text{def}}{=} \begin{cases} [(0,x)] & \text{if } z = [(0,x)], \\ [(0,y)] & \text{if } z = [(1,y)], \\ p_0 & \text{if } z = \star_{X \coprod Y} \end{cases}$$

for each $z \in (X \coprod Y)^+$.
Meanwhile, the isomorphism $\mathrm{pt} \cong \emptyset^+$ is given by sending $\star_X$ to $\star_\emptyset$.
That these isomorphisms satisfy the coherence conditions making the functor $(-)^+$ symmetric strong monoidal can be directly checked element by element.

Item 4: Symmetric Strong Monoidality With Respect to Smash Products

The isomorphism

$$\phi\colon X^+ \wedge Y^+ \xrightarrow{\cong} (X \times Y)^+$$

is given by

$$\phi(x \wedge y) = \begin{cases} (x,y) & \text{if } x \neq \star_X \text{ and } y \neq \star_Y \\ \star_{X \times Y} & \text{otherwise} \end{cases}$$

for each $x \wedge y \in X^+ \wedge Y^+$, with inverse

$$\phi^{-1}\colon (X \times Y)^+ \xrightarrow{\cong} X^+ \wedge Y^+$$

given by

$$\phi^{-1}(z) \overset{\text{def}}{=} \begin{cases} x \wedge y & \text{if } z = (x,y) \text{ with } (x,y) \in X \times Y, \\ \star_X \wedge \star_Y & \text{if } z = \star_{X \times Y}, \end{cases}$$

for each $z \in (X \coprod Y)^+$.
Meanwhile, the isomorphism $S^0 \cong \mathrm{pt}^+$ is given by sending $\star$ to $1 \in S^0 = \{0,1\}$ and $\star_{\mathrm{pt}}$ to $0 \in S^0$.
That these isomorphisms satisfy the coherence conditions making the functor $(-)^+$ symmetric strong monoidal can be directly checked element by element. □

Appendices

A Other Chapters

Sets

1. Sets

2. Constructions With Sets

3. Pointed Sets

4. Tensor Products of Pointed Sets

Relations

5. Relations

6. Constructions With Relations

7. Equivalence Relations and Apartness Relations

Category Theory

8. Categories

Bicategories

9. Types of Morphisms in Bicategories

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