Relations

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This chapter contains some material about relations. Notably, we discuss and explore:

1. The definition of relations (Section 1.1).
2. How relations may be viewed as decategorification of profunctors (Section 1.2).
3. The various kind of categories that relations form, namely:
    (a) A category (Section 2.1).
    (b) A monoidal category (Section 2.2).
    (c) A 2-category (Section 2.3).
    (d) A double category (Section 2.4).
4. The various categorical properties of the 2-category of relations, including:
    (a) The self-duality of Rel and $\mathbf{Rel}$ (Proposition 3.1.1.1).
    (b) Identifications of equivalences and isomorphisms in $\mathbf{Rel}$ with bijections (Proposition 3.2.1.1).
    (c) Identifications of adjunctions in $\mathbf{Rel}$ with functions (Proposition 3.3.1.1).
    (d) Identifications of monads in $\mathbf{Rel}$ with preorders (Proposition 3.4.1.1).
    (e) Identifications of comonads in $\mathbf{Rel}$ with subsets (Proposition 3.5.1.1).
    (f) A description of the monoids and comonoids in $\mathbf{Rel}$ with respect to the Cartesian product (Remark 3.6.1.1).
    (g) Characterisations of monomorphisms in Rel (Proposition 3.7.1.1).

(h) Characterisations of 2-categorical notions of monomorphisms in **Rel** (Proposition 3.8.1.1).
(i) Characterisations of epimorphisms in Rel (Proposition 3.9.1.1).
(j) Characterisations of 2-categorical notions of epimorphisms in **Rel** (Proposition 3.10.1.1).
(k) The partial co/completeness of Rel (Proposition 3.11.1.1).
(l) The existence or non-existence of Kan extensions and Kan lifts in Rel (Remark 3.12.1.1).
(m) The closedness of **Rel** (Proposition 3.13.1.1).
(n) The identification of **Rel** with the category of free algebras of the powerset monad on Sets (Proposition 3.14.1.1).

5. A description of two notions of "skew composition" on $\mathbf{Rel}(A, B)$, giving rise to left and right skew monoidal structures analogous to the left skew monoidal structure on $\mathsf{Fun}(C, \mathcal{D})$ appearing in the definition of a relative monad (Sections 4 and 5).

Contents

1 Relations

1.1 Foundations

Let A and B be sets.

Definition 1.1.1.1. A **relation** $R\colon A \nrightarrow B$ **from** A **to** B[1,2] is a subset R of $A \times B$.

Notation 1.1.1.2. Let $R\colon A \nrightarrow B$ be a relation.

1. Given elements $a \in A$ and $b \in B$ and a relation $R\colon A \nrightarrow B$, we write $a \sim_R b$ to mean $(a, b) \in R$.

2. Viewing R as a function

$$R\colon A \times B \longrightarrow \{\mathsf{t}, \mathsf{f}\}$$

via Remark 1.1.1.4, we write R^b_a for the value of R at (a, b).[3]

Definition 1.1.1.3. Let A and B be sets.

1. The **set of relations from** A **to** B is the set $\mathrm{Rel}(A, B)$ defined by

$$\mathrm{Rel}(A, B) \overset{\text{def}}{=} \{\text{Relations from } A \text{ to } B\}.$$

2. The **poset of relations from** A **to** B is the poset

$$\mathbf{Rel}(A, B) \overset{\text{def}}{=} (\mathrm{Rel}(A, B), \subset)$$

consisting of:

- *The Underlying Set.* The set $\mathrm{Rel}(A, B)$ of Item 1.
- *The Partial Order.* The partial order

$$\subset\colon \mathrm{Rel}(A, B) \times \mathrm{Rel}(A, B) \longrightarrow \{\mathsf{true}, \mathsf{false}\}$$

on $\mathrm{Rel}(A, B)$ given by inclusion of relations.

3. The **category of relations from** A **to** B is the posetal category $\mathbf{Rel}(A, B)$[4] associated to the poset $\mathbf{Rel}(A, B)$ of Item 2 via Categories, Definition 1.3.1.1.

[1] *Further Terminology:* Also called a **multivalued function from** A **to** B, a **relation over** A **and** B, **relation on** A **and** B, a **binary relation over** A **and** B, or a **binary relation on** A **and** B.

[2] *Further Terminology:* When $A = B$, we also call $R \subset A \times A$ a **relation on** A.

[3] The choice R^b_a in place of R^a_b is to keep the notation consistent with the notation we will later employ for profunctors.

[4] Here we choose to slightly abuse notation by writing $\mathbf{Rel}(A, B)$ (instead of e.g. $\mathbf{Rel}(A, B)_{\mathsf{pos}}$) for the posetal category of relations from A to B, even though the same notation is used for the

Remark 1.1.1.4. A relation from A to B is equivalently:[5]

1. A subset of $A \times B$.
2. A function from $A \times B$ to $\{\mathsf{true}, \mathsf{false}\}$.
3. A function from A to $\mathcal{P}(B)$.
4. A function from B to $\mathcal{P}(A)$.
5. A cocontinuous morphism of posets from $(\mathcal{P}(A), \subset)$ to $(\mathcal{P}(B), \subset)$.

That is: we have bijections of sets

$$\begin{aligned} \mathrm{Rel}(A, B) &\overset{\mathrm{def}}{=} \mathcal{P}(A \times B), \\ &\cong \mathrm{Hom}_{\mathsf{Sets}}(A \times B, \{\mathsf{true}, \mathsf{false}\}), \\ &\cong \mathrm{Hom}_{\mathsf{Sets}}(A, \mathcal{P}(B)), \\ &\cong \mathrm{Hom}_{\mathsf{Sets}}(B, \mathcal{P}(A)), \\ &\cong \mathrm{Hom}^{\mathsf{cocont}}_{\mathsf{Pos}}(\mathcal{P}(A), \mathcal{P}(B)), \end{aligned}$$

natural in $A, B \in \mathrm{Obj}(\mathsf{Sets})$.

Proof. We claim that Items 1 to 5 are indeed equivalent:

- *Item 1* $\iff$ *Item 2:* This is a special case of Constructions With Sets, Items 1 and 2 of Proposition 4.3.1.6.
- *Item 2* $\iff$ *Item 3:* This follows from the bijections

$$\begin{aligned} \mathrm{Hom}_{\mathsf{Sets}}(A \times B, \{\mathsf{true}, \mathsf{false}\}) &\cong \mathrm{Hom}_{\mathsf{Sets}}(A, \mathrm{Hom}_{\mathsf{Sets}}(B, \{\mathsf{true}, \mathsf{false}\})) \\ &\cong \mathrm{Hom}_{\mathsf{Sets}}(A, \mathcal{P}(B)), \end{aligned}$$

  where the last bijection is from Constructions With Sets, Items 1 and 2 of Proposition 4.3.1.6.

- *Item 2* $\iff$ *Item 4:* This follows from the bijections

$$\begin{aligned} \mathrm{Hom}_{\mathsf{Sets}}(A \times B, \{\mathsf{true}, \mathsf{false}\}) &\cong \mathrm{Hom}_{\mathsf{Sets}}(B, \mathrm{Hom}_{\mathsf{Sets}}(B, \{\mathsf{true}, \mathsf{false}\})) \\ &\cong \mathrm{Hom}_{\mathsf{Sets}}(B, \mathcal{P}(A)), \end{aligned}$$

poset of relations from A to B.

[5] *Intuition:* In particular, we may think of a relation $R\colon A \to \mathcal{P}(B)$ from A to B as a multivalued

where again the last bijection is from Constructions With Sets, Items 1 and 2 of Proposition 4.3.1.6.

- *Item 2* $\iff$ *Item 5*: This follows from the universal property of the powerset $\mathcal{P}(X)$ of a set X as the free cocompletion of X via the characteristic embedding
$$\chi_X \colon X \hookrightarrow \mathcal{P}(X)$$
of X into $\mathcal{P}(X)$, Constructions With Sets, Item 2 of Proposition 4.3.1.8.

  In particular, the bijection
$$\mathrm{Rel}(A, B) \cong \mathrm{Hom}^{\mathsf{cocont}}_{\mathsf{Pos}}(\mathcal{P}(A), \mathcal{P}(B))$$
is given by taking a relation $R \colon A \nrightarrow B$, passing to its associated function $f \colon A \to \mathcal{P}(B)$ from A to B and then extending f from A to all of $\mathcal{P}(A)$ by taking its left Kan extension along χ_X.

  This coincides with the direct image function $f_* \colon \mathcal{P}(A) \to \mathcal{P}(B)$ of Constructions With Sets, Definition 4.4.1.1.

This finishes the proof. □

Proposition 1.1.1.5. Let A and B be sets and let $R, S \colon A \nrightarrow B$ be relations.

1. *End Formula for the Set of Inclusions of Relations.* We have
$$\mathrm{Hom}_{\mathbf{Rel}(A,B)}(R, S) \cong \int_{a \in A} \int_{b \in B} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R^b_a, S^b_a).$$

Proof. *Item 1, End Formula for the Set of Inclusions of Relations*: Unwinding the expression inside the end on the right hand side, we have
$$\int_{a \in A} \int_{b \in B} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R^b_a, S^b_a) \cong \begin{cases} \mathrm{pt} & \text{if, for each } a \in A \text{ and each } b \in B, \\ & \text{we have } \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R^b_a, S^b_a) \cong \mathrm{pt} \\ \emptyset & \text{otherwise.} \end{cases}$$
Since we have $\mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R^b_a, S^b_a) = \{\mathsf{true}\} \cong \mathrm{pt}$ exactly when $R^b_a = \mathsf{false}$ or $R^b_a = S^b_a = \mathsf{true}$, we get
$$\int_{a \in A} \int_{b \in B} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R^b_a, S^b_a) \cong \begin{cases} \mathrm{pt} & \text{if, for each } a \in A \text{ and each } b \in B, \\ & \text{if } a \sim_R b, \text{ then } a \sim_S b, \\ \emptyset & \text{otherwise.} \end{cases}$$

On the left hand-side, we have

$$\mathrm{Hom}_{\mathbf{Rel}(A,B)}(R,S) \cong \begin{cases} \mathrm{pt} & \text{if } R \subset S, \\ \emptyset & \text{otherwise.} \end{cases}$$

It is then clear that the conditions for each set to evaluate to pt (up to isomorphism) are equivalent, implying that those two sets are isomorphic. □

1.2 Relations as Decategorifications of Profunctors

Remark 1.2.1.1. The notion of a relation is a decategorification of that of a profunctor:

1. A profunctor from a category $\mathcal{C}$ to a category $\mathcal{D}$ is a functor

$$\mathfrak{p}\colon \mathcal{D}^{\mathsf{op}} \times \mathcal{C} \longrightarrow \mathsf{Sets}.$$

2. A relation on sets A and B is a function

$$R\colon A \times B \longrightarrow \{\mathsf{true}, \mathsf{false}\}.$$

Here we notice that:

- The opposite X^{op} of a set X is itself, as $(-)^{\mathsf{op}}\colon \mathsf{Cats} \to \mathsf{Cats}$ restricts to the identity endofunctor on Sets.
- The values that profunctors and relations take are analogous:
  - A category is enriched over the category

$$\mathsf{Sets} \stackrel{\text{def}}{=} \mathsf{Cats}_0$$

    of sets, with profunctors taking values on it.
  - A set is enriched over the set

$$\{\mathsf{true}, \mathsf{false}\} \stackrel{\text{def}}{=} \mathsf{Cats}_{-1}$$

    of classical truth values, with relations taking values on it.

function from A to B (including the possibility of a given $a \in A$ having no value at all).

Remark 1.2.1.2. Extending Remark 1.2.1.1, the equivalent definitions of relations in Remark 1.1.1.4 are also related to the corresponding ones for profunctors (**??**), which state that a profunctor $\mathfrak{p}\colon \mathcal{C} \nrightarrow \mathcal{D}$ is equivalently:

1. A functor $\mathfrak{p}\colon \mathcal{D}^{\mathsf{op}} \times \mathcal{C} \to \mathsf{Sets}$.
2. A functor $\mathfrak{p}\colon \mathcal{C} \to \mathsf{PSh}(\mathcal{D})$.
3. A functor $\mathfrak{p}\colon \mathcal{D}^{\mathsf{op}} \to \mathsf{Fun}(\mathcal{C}, \mathsf{Sets})$.
4. A colimit-preserving functor $\mathfrak{p}\colon \mathsf{PSh}(\mathcal{C}) \to \mathsf{PSh}(\mathcal{D})$.

Indeed:

- The equivalence between Items 1 and 2 (and also that between Items 1 and 3, which is proved analogously) is an instance of currying, both for profunctors as well as for relations, using the isomorphisms

$$\begin{aligned}
\mathsf{Sets}(A \times B, \{\mathsf{true}, \mathsf{false}\}) &\cong \mathsf{Sets}(A, \mathsf{Sets}(B, \{\mathsf{true}, \mathsf{false}\}))\\
&\cong \mathsf{Sets}(A, \mathcal{P}(B)),\\
\mathsf{Fun}(\mathcal{D}^{\mathsf{op}} \times \mathcal{D}, \mathsf{Sets}) &\cong \mathsf{Fun}(\mathcal{C}, \mathsf{Fun}(\mathcal{D}^{\mathsf{op}}, \mathsf{Sets}))\\
&\cong \mathsf{Fun}(\mathcal{C}, \mathsf{PSh}(\mathcal{D})).
\end{aligned}$$

- The equivalence between Items 1 and 3 follows from the universal properties of:
  - The powerset $\mathcal{P}(X)$ of a set X as the free cocompletion of X via the characteristic embedding

    $$\chi_{(-)}\colon X \hookrightarrow \mathcal{P}(X)$$

    of X into $\mathcal{P}(X)$, as stated and proved in Constructions With Sets, Item 2 of Proposition 4.3.1.8.
  - The category $\mathsf{PSh}(\mathcal{C})$ of presheaves on a category $\mathcal{C}$ as the free cocompletion of $\mathcal{C}$ via the Yoneda embedding

    $$よ\colon \mathcal{C} \hookrightarrow \mathsf{PSh}(\mathcal{C})$$

    of $\mathcal{C}$ into $\mathsf{PSh}(\mathcal{C})$, as stated and proved in **??**, **??** of **??**.

1.3 Examples of Relations

Example 1.3.1.1. The **trivial relation on** A **and** B is the relation $\sim_{\mathrm{triv}}$ defined equivalently as follows:

1. As a subset of $A \times B$, we have
$$\sim_{\mathrm{triv}} \overset{\mathrm{def}}{=} A \times B.$$
2. As a function from $A \times B$ to $\{\mathsf{true}, \mathsf{false}\}$, the relation $\sim_{\mathrm{triv}}$ is the constant function
$$\Delta_{\mathsf{true}} \colon A \times B \longrightarrow \{\mathsf{true}, \mathsf{false}\}$$
from $A \times B$ to $\{\mathsf{true}, \mathsf{false}\}$ taking the value true.
3. As a function from A to $\mathcal{P}(B)$, the relation $\sim_{\mathrm{triv}}$ is the function
$$\Delta_{\mathsf{true}} \colon A \longrightarrow \mathcal{P}(B)$$
defined by
$$\Delta_{\mathsf{true}}(a) \overset{\mathrm{def}}{=} B$$
for each $a \in A$.
4. Lastly, it is the unique relation R on A and B such that we have $a \sim_R b$ for each $a \in A$ and each $b \in B$.

Example 1.3.1.2. The **cotrivial relation on** A **and** B is the relation $\sim_{\mathrm{cotriv}}$ defined equivalently as follows:

1. As a subset of $A \times B$, we have
$$\sim_{\mathrm{cotriv}} \overset{\mathrm{def}}{=} \emptyset.$$
2. As a function from $A \times B$ to $\{\mathsf{true}, \mathsf{false}\}$, the relation $\sim_{\mathrm{cotriv}}$ is the constant function
$$\Delta_{\mathsf{false}} \colon A \times B \longrightarrow \{\mathsf{true}, \mathsf{false}\}$$
from $A \times B$ to $\{\mathsf{true}, \mathsf{false}\}$ taking the value false.

3. As a function from A to $\mathcal{P}(B)$, the relation $\sim_{\text{cotriv}}$ is the function
$$\Delta_{\mathsf{false}} \colon A \to \mathcal{P}(B)$$
defined by
$$\Delta_{\mathsf{false}}(a) \overset{\text{def}}{=} \emptyset$$
for each $a \in A$.

4. Lastly, it is the unique relation R on A and B such that we have $a \nsim_R b$ for each $a \in A$ and each $b \in B$.

Example 1.3.1.3. The characteristic relation
$$\chi_X(-_1, -_2) \colon X \times X \to \{\mathsf{t}, \mathsf{f}\}$$
on X of Constructions With Sets, Item 3 of Definition 4.1.1.1, defined by
$$\chi_X(x, y) \overset{\text{def}}{=} \begin{cases} \mathsf{true} & \text{if } x = y, \\ \mathsf{false} & \text{if } x \neq y \end{cases}$$
for each $x, y \in X$, is another example of a relation.

Example 1.3.1.4. Square roots are examples of relations:

1. *Square Roots in* $\mathbb{R}$. The assignment $x \mapsto \sqrt{x}$ defines a relation
$$\sqrt{-} \colon \mathbb{R} \to \mathcal{P}(\mathbb{R})$$
from $\mathbb{R}$ to itself, being explicitly given by
$$\sqrt{x} \overset{\text{def}}{=} \begin{cases} 0 & \text{if } x = 0, \\ \left\{-\sqrt{|x|}, \sqrt{|x|}\right\} & \text{if } x \neq 0. \end{cases}$$

2. *Square Roots in* $\mathbb{Q}$. Square roots in $\mathbb{Q}$ are similar to square roots in $\mathbb{R}$, though now additionally it may also occur that $\sqrt{-} \colon \mathbb{Q} \to \mathcal{P}(\mathbb{Q})$ sends a rational number x (e.g. 2) to the empty set (since $\sqrt{2} \notin \mathbb{Q}$).

Example 1.3.1.5. The complex logarithm defines a relation
$$\log \colon \mathbb{C} \to \mathcal{P}(\mathbb{C})$$
from $\mathbb{C}$ to itself, where we have
$$\log(a + bi) \overset{\text{def}}{=} \left\{\log(\sqrt{a^2 + b^2}) + i \arg(a + bi) + (2\pi i)k \;\middle|\; k \in \mathbb{Z}\right\}$$
for each $a + bi \in \mathbb{C}$.

Example 1.3.1.6. See [Wik24] for more examples of relations, such as antiderivation, inverse trigonometric functions, and inverse hyperbolic functions.

1.4 Functional Relations

Let A and B be sets.

Definition 1.4.1.1. A relation $R\colon A \nrightarrow B$ is **functional** if, for each $a \in A$, the set $R(a)$ is either empty or a singleton.

Proposition 1.4.1.2. Let $R\colon A \nrightarrow B$ be a relation.

1. *Characterisations.* The following conditions are equivalent:

   (a) The relation R is functional.

   (b) We have $R \diamond R^{\dagger} \subset \chi_B$.

Proof. *Item 1*, *Characterisations*: We claim that Items 1a and 1b are indeed equivalent:

- *Item 1a $\implies$ Item 1b:* Let $(b, b') \in B \times B$. We need to show that

  $$[R \diamond R^{\dagger}](b, b') \preceq_{\{\mathsf{t},\mathsf{f}\}} \chi_B(b, b'),$$

  i.e. that if there exists some $a \in A$ such that $b \sim_{R^{\dagger}} a$ and $a \sim_R b'$, then $b = b'$. But since $b \sim_{R^{\dagger}} a$ is the same as $a \sim_R b$, we have both $a \sim_R b$ and $a \sim_R b'$ at the same time, which implies $b = b'$ since R is functional.

- *Item 1b $\implies$ Item 1a:* Suppose that we have $a \sim_R b$ and $a \sim_R b'$ for $b, b' \in B$. We claim that $b = b'$:

  1. Since $a \sim_R b$, we have $b \sim_{R^{\dagger}} a$.
  2. Since $R \diamond R^{\dagger} \subset \chi_B$, we have

     $$[R \diamond R^{\dagger}](b, b') \preceq_{\{\mathsf{t},\mathsf{f}\}} \chi_B(b, b'),$$

     and since $b \sim_{R^{\dagger}} a$ and $a \sim_R b'$, it follows that $[R \diamond R^{\dagger}](b, b') = \mathsf{true}$, and thus $\chi_B(b, b') = \mathsf{true}$ as well, i.e. $b = b'$.

This finishes the proof. □

1.5 Total Relations

Let A and B be sets.

Definition 1.5.1.1. A relation $R\colon A \nrightarrow B$ is **total** if, for each $a \in A$, we have $R(a) \neq \emptyset$.

Proposition 1.5.1.2. Let $R\colon A \nrightarrow B$ be a relation.

1. *Characterisations.* The following conditions are equivalent:
   (a) The relation R is total.
   (b) We have $\chi_A \subset R^\dagger \diamond R$.

Proof. *Item 1*, *Characterisations*: We claim that Items 1a and 1b are indeed equivalent:

- *Item 1a* $\implies$ *Item 1b:* We have to show that, for each $(a, a') \in A$, we have

$$\chi_A(a, a') \preceq_{\{\mathsf{t},\mathsf{f}\}} [R^\dagger \diamond R](a, a'),$$

  i.e. that if $a = a'$, then there exists some $b \in B$ such that $a \sim_R b$ and $b \sim_{R^\dagger} a'$ (i.e. $a \sim_R b$ again), which follows from the totality of R.

- *Item 1b* $\implies$ *Item 1a:* Given $a \in A$, since $\chi_A \subset R^\dagger \diamond R$, we must have

$$\{a\} \subset [R^\dagger \diamond R](a),$$

  implying that there must exist some $b \in B$ such that $a \sim_R b$ and $b \sim_{R^\dagger} a$ (i.e. $a \sim_R b$) and thus $R(a) \neq \emptyset$, as $b \in R(a)$.

This finishes the proof. □

2 Categories of Relations

2.1 The Category of Relations

Definition 2.1.1.1. The **category of relations** is the category Rel where

- *Objects.* The objects of Rel are sets.
- *Morphisms.* For each $A, B \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$\mathsf{Rel}(A, B) \overset{\text{def}}{=} \mathrm{Rel}(A, B).$$

- *Identities.* For each $A \in \mathrm{Obj}(\mathsf{Rel})$, the unit map

$$\mathbb{1}^{\mathsf{Rel}}_A \colon \mathrm{pt} \longrightarrow \mathrm{Rel}(A, A)$$

  of Rel at A is defined by

$$\mathrm{id}^{\mathsf{Rel}}_A \overset{\mathrm{def}}{=} \chi_A(-_1, -_2),$$

  where $\chi_A(-_1, -_2)$ is the characteristic relation of A of Constructions With Sets, Item 3 of Definition 4.1.1.1.

- *Composition.* For each $A, B, C \in \mathrm{Obj}(\mathsf{Rel})$, the composition map

$$\circ^{\mathsf{Rel}}_{A,B,C} \colon \mathrm{Rel}(B, C) \times \mathrm{Rel}(A, B) \longrightarrow \mathrm{Rel}(A, C)$$

  of Rel at (A, B, C) is defined by

$$S \circ^{\mathsf{Rel}}_{A,B,C} R \overset{\mathrm{def}}{=} S \diamond R$$

  for each $(S, R) \in \mathrm{Rel}(B, C) \times \mathrm{Rel}(A, B)$, where $S \diamond R$ is the composition of S and R of Constructions With Relations, Definition 3.12.1.1.

2.2 The Closed Symmetric Monoidal Category of Relations

2.2.1 The Monoidal Product

Definition 2.2.1.1. The **monoidal product of** Rel is the functor

$$\times \colon \mathsf{Rel} \times \mathsf{Rel} \longrightarrow \mathsf{Rel}$$

where

- *Action on Objects.* For each $A, B \in \mathrm{Obj}(\mathsf{Rel})$, we have

$$\times(A, B) \overset{\mathrm{def}}{=} A \times B,$$

  where $A \times B$ is the Cartesian product of sets of Constructions With Sets, Definition 1.3.1.1.

- *Action on Morphisms.* For each $(A, C), (B, D) \in \mathrm{Obj}(\mathsf{Rel} \times \mathsf{Rel})$, the action on morphisms

$$\times_{(A,C),(B,D)} \colon \mathrm{Rel}(A, B) \times \mathrm{Rel}(C, D) \longrightarrow \mathrm{Rel}(A \times C, B \times D)$$

of $\times$ is given by sending a pair of morphisms (R, S) of the form

$$R\colon A \nrightarrow B,$$
$$S\colon C \nrightarrow D$$

to the relation

$$R \times S\colon A \times C \nrightarrow B \times D$$

of Constructions With Relations, Definition 3.9.1.1.

2.2.2 The Monoidal Unit

Definition 2.2.2.1. The **monoidal unit of** Rel is the functor

$$\mathbb{1}^{\mathsf{Rel}}\colon \mathrm{pt} \to \mathsf{Rel}$$

picking the set

$$\mathbb{1}_{\mathsf{Rel}} \overset{\mathrm{def}}{=} \mathrm{pt}$$

of Rel.

2.2.3 The Associator

Definition 2.2.3.1. The **associator of** Rel is the natural isomorphism

$$\alpha^{\mathsf{Rel}}\colon \times \circ ((\times) \times \mathrm{id}) \overset{\sim}{\Longrightarrow} \times \circ (\mathrm{id} \times (\times)) \circ \boldsymbol{\alpha}^{\mathsf{Cats}}_{\mathsf{Rel},\mathsf{Rel},\mathsf{Rel}},$$

as in the diagram

$$\begin{tikzcd}
 & \mathsf{Rel}\times(\mathsf{Rel}\times\mathsf{Rel}) \arrow[dr, "\mathrm{id}\times(\times)"] & \\
(\mathsf{Rel}\times\mathsf{Rel})\times\mathsf{Rel} \arrow[ur, dashed, "\boldsymbol{\alpha}^{\mathsf{Cats}}_{\mathsf{Rel},\mathsf{Rel},\mathsf{Rel}}", "\sim"'] \arrow[d, "(\times)\times\mathrm{id}"'] & & \mathsf{Rel}\times\mathsf{Rel} \arrow[d, "\times"] \\
\mathsf{Rel}\times\mathsf{Rel} \arrow[rr, "\times"'] \arrow[urr, Rightarrow, "\alpha^{\mathsf{Rel}}"] & & \mathsf{Rel},
\end{tikzcd}$$

whose component

$$\alpha^{\mathsf{Rel}}_{A,B,C}\colon (A \times B) \times C \nrightarrow A \times (B \times C)$$

at $A, B, C \in \mathrm{Obj}(\mathsf{Rel})$ is the relation defined by declaring

$$((a, b), c) \sim_{\alpha^{\mathsf{Rel}}_{A,B,C}} (a', (b', c'))$$

iff $a = a'$, $b = b'$, and $c = c'$.

2.2.4 The Left Unitor

Definition 2.2.4.1. The **left unitor of** Rel is the natural isomorphism

$$\lambda^{\mathsf{Rel}}\colon \times\circ(\mathbb{1}^{\mathsf{Rel}}\times\mathrm{id}) \overset{\sim}{\Longrightarrow} \boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathsf{Rel}}, \qquad \begin{tikzcd} \mathrm{pt}\times\mathsf{Rel} \arrow[r, "\mathbb{1}^{\mathsf{Rel}}\times\mathrm{id}"] \arrow[dr, dashed, bend right, "\boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathsf{Rel}}"'] & \mathsf{Rel}\times\mathsf{Rel}, \arrow[d, "\times"] \arrow[dl, phantom, "\Swarrow\lambda^{\mathsf{Rel}}"] \\ & \mathsf{Rel} \end{tikzcd}$$

whose component

$$\lambda^{\mathsf{Rel}}_{A}\colon \mathbb{1}_{\mathsf{Rel}}\times A \nrightarrow A$$

at A is defined by declaring

$$(\star, a) \sim_{\lambda^{\mathsf{Rel}}_{A}} b$$

iff $a = b$.

2.2.5 The Right Unitor

Definition 2.2.5.1. The **right unitor of** Rel is the natural isomorphism

$$\rho^{\mathsf{Rel}}\colon \times\circ(\mathrm{id}\times\mathbb{1}^{\mathsf{Rel}}) \overset{\sim}{\Longrightarrow} \boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathsf{Rel}}, \qquad \begin{tikzcd} \mathsf{Rel}\times\mathrm{pt} \arrow[r, "\mathrm{id}\times\mathbb{1}^{\mathsf{Rel}}"] \arrow[dr, dashed, bend right, "\boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathsf{Rel}}"'] & \mathsf{Rel}\times\mathsf{Rel}, \arrow[d, "\times"] \arrow[dl, phantom, "\Swarrow\rho^{\mathsf{Rel}}"] \\ & \mathsf{Rel} \end{tikzcd}$$

whose component

$$\rho^{\mathsf{Rel}}_{A}\colon A\times\mathbb{1}_{\mathsf{Rel}} \nrightarrow A$$

at A is defined by declaring

$$(a, \star) \sim_{\rho^{\mathsf{Rel}}_{A}} b$$

iff $a = b$.

2.2.6 The Symmetry

Definition 2.2.6.1. The **symmetry of** Rel is the natural isomorphism

$$\sigma^{\mathsf{Rel}} \colon \times \Longrightarrow \times \circ \boldsymbol{\sigma}^{\mathsf{Cats}_2}_{\mathsf{Rel},\mathsf{Rel}}, \qquad \begin{tikzcd} \mathsf{Rel} \times \mathsf{Rel} \arrow[rr, "\times"] \arrow[dr, "\boldsymbol{\sigma}^{\mathsf{Cats}_2}_{\mathsf{Rel},\mathsf{Rel}}"'] & \arrow[d, phantom, "\overset{\|}{\underset{\Downarrow}{\sigma^{\mathsf{Rel}}}}"] & \mathsf{Rel}, \\ & \mathsf{Rel} \times \mathsf{Rel} \arrow[ur, "\times"'] & \end{tikzcd}$$

whose component

$$\sigma^{\mathsf{Rel}}_{A,B} \colon A \times B \to B \times A$$

at (A, B) is defined by declaring

$$(a, b) \sim_{\sigma^{\mathsf{Rel}}_{A,B}} (b', a')$$

iff $a = a'$ and $b = b'$.

2.2.7 The Internal Hom

Definition 2.2.7.1. The **internal Hom of** Rel is the functor

$$\mathrm{Rel} \colon \mathsf{Rel}^{\mathsf{op}} \times \mathsf{Rel} \to \mathsf{Rel}$$

defined

- On objects by sending $A, B \in \mathrm{Obj}(\mathsf{Rel})$ to the set $\mathrm{Rel}(A, B)$ of Item 1 of Definition 1.1.1.3.
- On morphisms by pre/post-composition defined as in Constructions With Relations, Definition 3.12.1.1.

Proposition 2.2.7.2. Let $A, B, C \in \mathrm{Obj}(\mathsf{Rel})$.

1. *Adjointness.* We have adjunctions

$$(A \times - \dashv \mathrm{Rel}(A, -)) \colon \quad \begin{tikzcd} \mathsf{Rel} \arrow[r, bend left, "A\times-"] \arrow[r, phantom, "\perp"] & \mathsf{Rel}, \arrow[l, bend left, "\mathrm{Rel}(A,-)"] \end{tikzcd}$$

$$(- \times B \dashv \mathrm{Rel}(B, -)) \colon \quad \begin{tikzcd} \mathsf{Rel} \arrow[r, bend left, "-\times B"] \arrow[r, phantom, "\perp"] & \mathsf{Rel}, \arrow[l, bend left, "\mathrm{Rel}(B,-)"] \end{tikzcd}$$

witnessed by bijections

$$
\begin{aligned}
\mathrm{Rel}(A \times B, C) &\cong \mathrm{Rel}(A, \mathbf{Rel}(B, C)), \\
\mathrm{Rel}(A \times B, C) &\cong \mathrm{Rel}(B, \mathbf{Rel}(A, C)),
\end{aligned}
$$

natural in $A, B, C \in \mathrm{Obj}(\mathsf{Rel})$.

Proof. *Item 1, Adjointness*: Indeed, we have

$$
\begin{aligned}
\mathrm{Rel}(A \times B, C) &\stackrel{\text{def}}{=} \mathsf{Sets}(A \times B \times C, \{\mathsf{true}, \mathsf{false}\}) \\
&\stackrel{\text{def}}{=} \mathrm{Rel}(A, B \times C) \\
&\stackrel{\text{def}}{=} \mathrm{Rel}(A, \mathbf{Rel}(B, C)),
\end{aligned}
$$

and similarly for the bijection $\mathrm{Rel}(A \times B, C) \cong \mathrm{Rel}(B, \mathbf{Rel}(A, C))$. □

2.2.8 The Closed Symmetric Monoidal Category of Relations

Proposition 2.2.8.1. The category Rel admits a closed symmetric monoidal category structure consisting of[6]

- *The Underlying Category.* The category Rel of sets and relations of Definition 2.1.1.1.

- *The Monoidal Product.* The functor

$$\times \colon \mathrm{Rel} \times \mathrm{Rel} \longrightarrow \mathrm{Rel}$$

  of Definition 2.2.1.1.

- *The Internal Hom.* The internal Hom functor

$$\mathbf{Rel} \colon \mathrm{Rel}^{\mathsf{op}} \times \mathrm{Rel} \longrightarrow \mathrm{Rel}$$

  of Definition 2.2.7.1.

- *The Monoidal Unit.* The functor

$$\mathbb{1}^{\mathrm{Rel}} \colon \mathsf{pt} \longrightarrow \mathrm{Rel}$$

  of Definition 2.2.2.1.

[6] *Warning:* This is not a Cartesian monoidal structure, as the product on Rel is in fact given

- *The Associators.* The natural isomorphism

  $$\alpha^{\mathrm{Rel}}\colon \times \circ (\times \times \mathrm{id}_{\mathrm{Rel}}) \overset{\sim}{\Longrightarrow} \times \circ (\mathrm{id}_{\mathrm{Rel}} \times \times) \circ \boldsymbol{\alpha}^{\mathsf{Cats}}_{\mathrm{Rel},\mathrm{Rel},\mathrm{Rel}}$$

  of Definition 2.2.3.1.

- *The Left Unitors.* The natural isomorphism

  $$\lambda^{\mathrm{Rel}}\colon \times \circ (\mathbb{1}^{\mathrm{Rel}} \times \mathrm{id}_{\mathrm{Rel}}) \overset{\sim}{\Longrightarrow} \boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathrm{Rel}}$$

  of Definition 2.2.4.1.

- *The Right Unitors.* The natural isomorphism

  $$\rho^{\mathrm{Rel}}\colon \times \circ (\mathsf{id} \times \mathbb{1}^{\mathrm{Rel}}) \overset{\sim}{\Longrightarrow} \boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathrm{Rel}}$$

  of Definition 2.2.5.1.

- *The Symmetry.* The natural isomorphism

  $$\sigma^{\mathrm{Rel}}\colon \times \overset{\sim}{\Longrightarrow} \times \circ \boldsymbol{\sigma}^{\mathsf{Cats}_2}_{\mathrm{Rel},\mathrm{Rel}}$$

  of Definition 2.2.6.1.

Proof. Omitted. □

2.3 The 2-Category of Relations

Definition 2.3.1.1. The 2-**category of relations** is the locally posetal 2-category Rel where

- *Objects.* The objects of Rel are sets.

- ***Hom**-Objects.* For each $A, B \in \mathrm{Obj}(\mathsf{Sets})$, we have

  $$\begin{aligned}\mathrm{Hom}_{\mathsf{Rel}}(A, B) &\overset{\mathrm{def}}{=} \mathbf{Rel}(A, B)\\ &\overset{\mathrm{def}}{=} (\mathrm{Rel}(A, B), \subset).\end{aligned}$$

by the disjoint union of sets; see Constructions With Relations, **??**.

- *Identities.* For each $A \in \mathrm{Obj}(\mathsf{Rel})$, the unit map

$$\mathbb{1}^{\mathsf{Rel}}_A \colon \mathrm{pt} \to \mathbf{Rel}(A, A)$$

of Rel at A is defined by

$$\mathrm{id}^{\mathsf{Rel}}_A \stackrel{\text{def}}{=} \chi_A(-_1, -_2),$$

where $\chi_A(-_1, -_2)$ is the characteristic relation of A of Constructions With Sets, Item 3 of Definition 4.1.1.1.

- *Composition.* For each $A, B, C \in \mathrm{Obj}(\mathsf{Rel})$, the composition map[7]

$$\circ^{\mathsf{Rel}}_{A,B,C} \colon \mathbf{Rel}(B, C) \times \mathbf{Rel}(A, B) \to \mathbf{Rel}(A, C)$$

of Rel at (A, B, C) is defined by

$$S \circ^{\mathsf{Rel}}_{A,B,C} R \stackrel{\text{def}}{=} S \diamond R$$

for each $(S, R) \in \mathsf{Rel}(B, C) \times \mathsf{Rel}(A, B)$, where $S \diamond R$ is the composition of S and R of Constructions With Relations, Definition 3.12.1.1.

2.4 The Double Category of Relations

2.4.1 The Double Category of Relations

Definition 2.4.1.1. The **double category of relations** is the locally posetal double category $\mathsf{Rel}^{\mathsf{dbl}}$ where

- *Objects.* The objects of $\mathsf{Rel}^{\mathsf{dbl}}$ are sets.
- *Vertical Morphisms.* The vertical morphisms of $\mathsf{Rel}^{\mathsf{dbl}}$ are maps of sets $f \colon A \to B$.
- *Horizontal Morphisms.* The horizontal morphisms of $\mathsf{Rel}^{\mathsf{dbl}}$ are relations $R \colon A \nrightarrow X$.

[7] Note that this is indeed a morphism of posets: given relations $R_1, R_2 \in \mathbf{Rel}(A, B)$ and $S_1, S_2 \in \mathbf{Rel}(B, C)$ such that

$$R_1 \subset R_2,$$
$$S_1 \subset S_2,$$

- *2-Morphisms.* A 2-cell

$$\begin{tikzcd} A \arrow[r, "R", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\overset{\alpha}{\Longrightarrow}"] & B \arrow[d, "g"] \\ X \arrow[r, "S"', "\shortmid"{marking}] & Y \end{tikzcd}$$

  of $\mathsf{Rel}^{\mathsf{dbl}}$ is either non-existent or an inclusion of relations of the form

$$R \subset S \circ (f \times g), \qquad \begin{tikzcd} A \times B \arrow[r, "R"] \arrow[d, "f \times g"'] \arrow[dr, phantom, "\subset"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true}, \mathsf{false}\}}"] \\ X \times Y \arrow[r, "S"'] & \{\mathsf{true}, \mathsf{false}\}. \end{tikzcd}$$

- *Horizontal Identities.* The horizontal unit functor of $\mathsf{Rel}^{\mathsf{dbl}}$ is the functor of Definition 2.4.2.1.

- *Vertical Identities.* For each $A \in \mathrm{Obj}(\mathsf{Rel}^{\mathsf{dbl}})$, we have

$$\mathrm{id}_A^{\mathsf{Rel}^{\mathsf{dbl}}} \overset{\mathrm{def}}{=} \mathrm{id}_A.$$

- *Identity 2-Morphisms.* For each horizontal morphism $R \colon A \nrightarrow B$ of $\mathsf{Rel}^{\mathsf{dbl}}$, the identity 2-morphism

$$\begin{tikzcd} A \arrow[r, "R", "\shortmid"{marking}] \arrow[d, "\mathrm{id}_A"'] \arrow[dr, phantom, "\overset{\mathrm{id}_R}{\Longrightarrow}"] & B \arrow[d, "\mathrm{id}_B"] \\ A \arrow[r, "R"', "\shortmid"{marking}] & B \end{tikzcd}$$

  of R is the identity inclusion

$$R \subset R, \qquad \begin{tikzcd} B \times A \arrow[r, "R"] \arrow[d, "\mathrm{id}_B \times \mathrm{id}_A"'] \arrow[dr, phantom, "\subset"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true}, \mathsf{false}\}}"] \\ B \times A \arrow[r, "R"'] & \{\mathsf{true}, \mathsf{false}\}. \end{tikzcd}$$

- *Horizontal Composition.* The horizontal composition functor of $\mathsf{Rel}^{\mathsf{dbl}}$ is the functor of Definition 2.4.3.1.
- *Vertical Composition of 1-Morphisms.* For each composable pair $A \xrightarrow{F} B \xrightarrow{G} C$ of vertical morphisms of $\mathsf{Rel}^{\mathsf{dbl}}$, i.e. maps of sets, we have
$$g \circ^{\mathsf{Rel}^{\mathsf{dbl}}} f \stackrel{\text{def}}{=} g \circ f.$$
- *Vertical Composition of 2-Morphisms.* The vertical composition of 2-morphisms in $\mathsf{Rel}^{\mathsf{dbl}}$ is defined as in Definition 2.4.4.1.
- *Associators.* The associators of $\mathsf{Rel}^{\mathsf{dbl}}$ is defined as in Definition 2.4.5.1.
- *Left Unitors.* The left unitors of $\mathsf{Rel}^{\mathsf{dbl}}$ is defined as in Definition 2.4.6.1.
- *Right Unitors.* The right unitors of $\mathsf{Rel}^{\mathsf{dbl}}$ is defined as in Definition 2.4.7.1.

2.4.2 Horizontal Identities

Definition 2.4.2.1. The **horizontal unit functor** of $\mathsf{Rel}^{\mathsf{dbl}}$ is the functor
$$\mathbb{1}^{\mathsf{Rel}^{\mathsf{dbl}}} \colon \mathsf{Rel}_0^{\mathsf{dbl}} \to \mathsf{Rel}_1^{\mathsf{dbl}}$$
of $\mathsf{Rel}^{\mathsf{dbl}}$ is the functor where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathsf{Rel}_0^{\mathsf{dbl}})$, we have
$$\mathbb{1}_A \stackrel{\text{def}}{=} \chi_A(-_1, -_2).$$
- *Action on Morphisms.* For each vertical morphism $f \colon A \to B$ of $\mathsf{Rel}^{\mathsf{dbl}}$, i.e. each map of sets f from A to B, the identity 2-morphism
$$\begin{tikzcd} A \arrow[r, "\mathbb{1}_A", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "{\overset{\parallel}{\mathbb{1}_f}\,\Downarrow}"] & A \arrow[d, "f"] \\ B \arrow[r, "\mathbb{1}_B"', "\shortmid"{marking}] & B \end{tikzcd}$$

we have also $S_1 \diamond R_1 \subset S_2 \diamond R_2$.

of f is the inclusion

$$\chi_B \circ (f \times f) \subset \chi_A, \qquad \begin{tikzcd} A \times A \arrow[r, "\chi_A(-_1,-_2)"] \arrow[d, "f\times f"'] \arrow[dr, phantom, "\subset"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ B \times B \arrow[r, "\chi_B(-_1,-_2)"'] & \{\mathsf{true}, \mathsf{false}\} \end{tikzcd}$$

of Constructions With Sets, Item 1 of Proposition 4.1.1.3.

2.4.3 Horizontal Composition

Definition 2.4.3.1. The **horizontal composition functor** of $\mathsf{Rel}^{\mathsf{dbl}}$ is the functor

$$\odot^{\mathsf{Rel}^{\mathsf{dbl}}} \colon \mathsf{Rel}^{\mathsf{dbl}}_1 \underset{\mathsf{Rel}^{\mathsf{dbl}}_0}{\times} \mathsf{Rel}^{\mathsf{dbl}}_1 \to \mathsf{Rel}^{\mathsf{dbl}}_1$$

of $\mathsf{Rel}^{\mathsf{dbl}}$ is the functor where

- *Action on Objects.* For each composable pair $A \overset{R}{\nrightarrow} B \overset{S}{\nrightarrow} C$ of horizontal morphisms of $\mathsf{Rel}^{\mathsf{dbl}}$, we have

$$S \odot R \overset{\mathrm{def}}{=} S \diamond R,$$

  where $S \diamond R$ is the composition of R and S of Constructions With Relations, Definition 3.12.1.1.

- *Action on Morphisms.* For each horizontally composable pair

$$\begin{tikzcd} A \arrow[r, "R", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & B \arrow[d, "g"] \\ X \arrow[r, "T"', "\shortmid"{marking}] & Y \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "S", "\shortmid"{marking}] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\beta"] & C \arrow[d, "h"] \\ Y \arrow[r, "U"', "\shortmid"{marking}] & Z \end{tikzcd}$$

  of 2-morphisms of $\mathsf{Rel}^{\mathsf{dbl}}$, i.e. for each pair

$$\begin{tikzcd} A \times B \arrow[r, "R"] \arrow[d, "f\times g"'] \arrow[dr, phantom, "\subset"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ X \times Y \arrow[r, "T"'] & \{\mathsf{true}, \mathsf{false}\} \end{tikzcd} \qquad \begin{tikzcd} B \times C \arrow[r, "S"] \arrow[d, "g\times h"'] \arrow[dr, phantom, "\subset"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ Y \times Z \arrow[r, "U"'] & \{\mathsf{true}, \mathsf{false}\} \end{tikzcd}$$

of inclusions of relations, the horizontal composition

$$\begin{tikzcd} A \arrow[r, "S\odot R", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow \beta\odot\alpha"] & C \arrow[d, "h"] \\ X \arrow[r, "U\odot T"', "|" marking] & Z \end{tikzcd}$$

of α and β is the inclusion of relations[8]

$$(U \diamond T) \circ (f \times h) \subset (S \diamond R) \qquad \begin{tikzcd} A \times C \arrow[r, "S\diamond R"] \arrow[d, "f\times h"'] \arrow[dr, phantom, "\subset"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ X \times Z \arrow[r, "U\diamond T"'] & \{\mathsf{true}, \mathsf{false}\}. \end{tikzcd}$$

2.4.4 Vertical Composition of 2-Morphisms

Definition 2.4.4.1. The **vertical composition** in $\mathsf{Rel}^{\mathsf{dbl}}$ is defined as follows: for each vertically composable pair

$$\begin{tikzcd} A \arrow[r, "R", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow \alpha"] & X \arrow[d, "g"] \\ B \arrow[r, "S"', "|" marking] & Y \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "S", "|" marking] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow \beta"] & Y \arrow[d, "k"] \\ C \arrow[r, "T"', "|" marking] & Z \end{tikzcd}$$

---

[8] This is justified by noting that, given $(a, c) \in A \times C$, the statement

- We have $a \sim_{(U\diamond T)\circ(f\times h)} c$, i.e. $f(a) \sim_{U\diamond T} h(c)$, i.e. there exists some $y \in Y$ such that:
  1. We have $f(a) \sim_T y$;
  2. We have $y \sim_U h(c)$;

is implied by the statement

- We have $a \sim_{S\diamond R} c$, i.e. there exists some $b \in B$ such that:
  1. We have $a \sim_R b$;
  2. We have $b \sim_S c$;

since:

- If $a \sim_R b$, then $f(a) \sim_T g(b)$, as $T \circ (f \times g) \subset R$;

of 2-morphisms of $\mathsf{Rel}^{\mathsf{dbl}}$, i.e. for each each pair

$$\begin{tikzcd} A \times X \arrow[r, "R"] \arrow[d, "f\times g"'] \arrow[dr, phantom, "\subset"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ B \times Y \arrow[r, "S"'] & \{\mathsf{true}, \mathsf{false}\} \end{tikzcd} \qquad \begin{tikzcd} B \times Y \arrow[r, "S"] \arrow[d, "h\times k"'] \arrow[dr, phantom, "\subset"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ C \times Z \arrow[r, "T"'] & \{\mathsf{true}, \mathsf{false}\} \end{tikzcd}$$

of inclusions of relations, we define the vertical composition

$$\begin{tikzcd} A \arrow[r, "R", "|" marking] \arrow[d, "h\circ f"'] \arrow[dr, phantom, "\overset{\beta\circ\alpha}{\Downarrow}"] & X \arrow[d, "k\circ g"] \\ C \arrow[r, "T"', "|" marking] & Z \end{tikzcd}$$

of α and β as the inclusion of relations

$$T \circ [(h \circ f) \times (k \circ g)] \subset R, \qquad \begin{tikzcd} A \times X \arrow[r, "R"] \arrow[d, "(h\circ f)\times(k\circ g)"'] \arrow[dr, phantom, "\subset"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ C \times Z \arrow[r, "T"'] & \{\mathsf{true}, \mathsf{false}\} \end{tikzcd}$$

given by the pasting of inclusions[9]

$$\begin{tikzcd} A \times X \arrow[r, "R"] \arrow[d, "f\times g"'] \arrow[dr, phantom, "\subset"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ B \times Y \arrow[r, "S"] \arrow[d, "h\times k"'] \arrow[dr, phantom, "\subset"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ C \times Z \arrow[r, "T"'] & \{\mathsf{true}, \mathsf{false}\}. \end{tikzcd}$$

- If $b \sim_S c$, then $g(b) \sim_U h(c)$, as $U \circ (g \times h) \subset S$.

[9]This is justified by noting that, given $(a, x) \in A \times X$, the statement

2.4.5 The Associators

Definition 2.4.5.1. For each composable triple

$$A \overset{R}{\nrightarrow} B \overset{S}{\nrightarrow} C \overset{T}{\nrightarrow} D$$

of horizontal morphisms of $\mathsf{Rel}^{\mathsf{dbl}}$, the component

$$\alpha^{\mathsf{Rel}^{\mathsf{dbl}}}_{T,S,R} \colon (T \odot S) \odot R \overset{\sim}{\Longrightarrow} T \odot (S \odot R), \qquad \begin{tikzcd} A \arrow[r, "R", "\shortmid"{marking}] \arrow[d, "\mathrm{id}_A"'] & B \arrow[r, "S", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow \alpha^{\mathsf{Rel}^{\mathsf{dbl}}}_{T,S,R}"] & C \arrow[r, "T", "\shortmid"{marking}] & D \arrow[d, "\mathrm{id}_D"] \\ A \arrow[r, "R"', "\shortmid"{marking}] & B \arrow[r, "S"', "\shortmid"{marking}] & C \arrow[r, "T"', "\shortmid"{marking}] & D \end{tikzcd}$$

of the associator of $\mathsf{Rel}^{\mathsf{dbl}}$ at (R, S, T) is the identity inclusion[10]

$$(T \diamond S) \diamond R = T \diamond (S \diamond R) \qquad \begin{tikzcd} A \times B \arrow[r, "(T \diamond S) \diamond R"] \arrow[d, equal] \arrow[dr, phantom, "\rotatebox{45}{=}"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ A \times B \arrow[r, "T \diamond (S \diamond R)"'] & \{\mathsf{true}, \mathsf{false}\}. \end{tikzcd}$$

- We have $h(f(a)) \sim_T k(g(x))$;

is implied by the statement

- We have $a \sim_R x$;

since

- If $a \sim_R x$, then $f(a) \sim_S g(x)$, as $S \circ (f \times g) \subset R$;
- If $b \sim_S y$, then $h(b) \sim_T k(y)$, as $T \circ (h \times k) \subset S$, and thus, in particular:
  - If $f(a) \sim_S g(x)$, then $h(f(a)) \sim_T k(g(x))$.

[10]This is justified by Constructions With Relations, Item 2 of Proposition 3.12.1.3.

2.4.6 The Left Unitors

Definition 2.4.6.1. For each horizontal morphism $R\colon A \nrightarrow B$ of $\mathsf{Rel}^{\mathsf{dbl}}$, the component

$$\lambda^{\mathsf{Rel}^{\mathsf{dbl}}}_R : \mathbb{1}_B \odot R \xRightarrow{\sim} R, \qquad \begin{tikzcd} A \arrow[r, "R", "|"{marking}] \arrow[d, "\mathrm{id}_A"'] & B \arrow[r, "\mathbb{1}_B", "|"{marking}] \arrow[d, phantom, "\Downarrow \lambda^{\mathsf{Rel}^{\mathsf{dbl}}}_R"] & B \arrow[d, "\mathrm{id}_B"] \\ A \arrow[rr, "R"', "|"{marking}] & & B \end{tikzcd}$$

of the left unitor of $\mathsf{Rel}^{\mathsf{dbl}}$ at R is the identity inclusion[11]

$$R = \chi_B \diamond R, \qquad \begin{tikzcd} A \times B \arrow[r, "\chi_B \diamond R"] \arrow[d, equal] \arrow[dr, phantom, "\mathbin{/\!\!/}"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ A \times B \arrow[r, "R"'] & \{\mathsf{true}, \mathsf{false}\}. \end{tikzcd}$$

2.4.7 The Right Unitors

Definition 2.4.7.1. For each horizontal morphism $R\colon A \nrightarrow B$ of $\mathsf{Rel}^{\mathsf{dbl}}$, the component

$$\rho^{\mathsf{Rel}^{\mathsf{dbl}}}_R : R \odot \mathbb{1}_A \xRightarrow{\sim} R, \qquad \begin{tikzcd} A \arrow[r, "\mathbb{1}_A", "|"{marking}] \arrow[d, "\mathrm{id}_A"'] & A \arrow[r, "R", "|"{marking}] \arrow[d, phantom, "\Downarrow \rho^{\mathsf{Rel}^{\mathsf{dbl}}}_R"] & B \arrow[d, "\mathrm{id}_B"] \\ A \arrow[rr, "R"', "|"{marking}] & & B \end{tikzcd}$$

of the right unitor of $\mathsf{Rel}^{\mathsf{dbl}}$ at R is the identity inclusion[12]

$$R = R \diamond \chi_A, \qquad \begin{tikzcd} A \times B \arrow[r, "R \diamond \chi_A"] \arrow[d, equal] \arrow[dr, phantom, "\mathbin{/\!\!/}"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ A \times B \arrow[r, "R"'] & \{\mathsf{true}, \mathsf{false}\}. \end{tikzcd}$$

[11] This is justified by Constructions With Relations, Item 3 of Proposition 3.12.1.3.

[12] This is justified by Constructions With Relations, Item 3 of Proposition 3.12.1.3.

3 Properties of the 2-Category of Relations

3.1 Self-Duality

Proposition 3.1.1.1. The (2-)category of relations is self-dual:

1. *Self-Duality I.* We have an isomorphism
$$\mathrm{Rel}^{\mathsf{op}} \overset{\text{eq.}}{\cong} \mathrm{Rel}$$
of categories.

2. *Self-Duality II.* We have a 2-isomorphism
$$\mathbf{Rel}^{\mathsf{op}} \overset{\text{eq.}}{\cong} \mathbf{Rel}$$
of 2-categories.

Proof. *Item 1*, *Self-Duality I*: We claim that the functor
$$F\colon \mathsf{Rel}^{\mathsf{op}} \longrightarrow \mathsf{Rel}$$
given by the identity on objects and by $R \mapsto R^{\dagger}$ on morphisms is an isomorphism of categories.
By Categories, Item 1 of Proposition 5.8.1.3, it suffices to show that F is bijective on objects (which is clear) and fully faithful. Indeed, the map
$$(-)^{\dagger}\colon \mathrm{Rel}(A, B) \longrightarrow \mathrm{Rel}(B, A)$$
defined by the assignment $R \mapsto R^{\dagger}$ is a bijection by Constructions With Relations, Item 5 of Proposition 3.11.1.3, showing F to be fully faithful.
Item 2, *Self-Duality II*: We claim that the 2-functor
$$F\colon \mathsf{Rel}^{\mathsf{op}} \longrightarrow \mathsf{Rel}$$
given by the identity on objects, by $R \mapsto R^{\dagger}$ on morphisms, and by preserving inclusions on 2-morphisms via Constructions With Relations, Item 1 of Proposition 3.11.1.3, is an isomorphism of categories.
By **??**, **??** of **??**, it suffices to show that F is:

- Bijective on objects, which is clear.
- Bijective on 1-morphisms, which was shown in Item 1.
- Bijective on 2-morphisms, which follows from Constructions With Relations, Item 1 of Proposition 3.11.1.3.

Thus F is indeed a 2-isomorphism of categories. □

3.2 Isomorphisms and Equivalences in Rel

Let $R\colon A \nrightarrow B$ be a relation from A to B.

Proposition 3.2.1.1. The following conditions are equivalent:

1. The relation $R\colon A \nrightarrow B$ is an equivalence in **Rel**, i.e.:

   ($\star$) There exists a relation $R^{-1}\colon B \nrightarrow A$ from B to A together with isomorphisms

$$
\begin{aligned}
R^{-1} \diamond R &\cong \chi_A, \\
R \diamond R^{-1} &\cong \chi_B.
\end{aligned}
$$

2. The relation $R\colon A \nrightarrow B$ is an isomorphism in Rel, i.e.:

   ($\star$) There exists a relation $R^{-1}\colon B \nrightarrow A$ from B to A such that we have

$$
\begin{aligned}
R^{-1} \diamond R &= \chi_A, \\
R \diamond R^{-1} &= \chi_B.
\end{aligned}
$$

3. There exists a bijection $f\colon A \xrightarrow{\cong} B$ with $R = \mathrm{Gr}(f)$.

Proof. We claim that Items 1 to 3 are indeed equivalent:

- *Item 1* $\iff$ *Item 2*: This follows from the fact that **Rel** is locally posetal, so that natural isomorphisms and equalities of 1-morphisms in **Rel** coincide.

- *Item 2* $\implies$ *Item 3*: The equalities in Item 2 imply $R \dashv R^{-1}$, and thus by Proposition 3.3.1.1, there exists a function $f_R\colon A \to B$ associated to R, where, for each $a \in A$, the image $f_R(a)$ of a by f_R is the unique element of $R(a)$, which implies $R = \mathrm{Gr}(f_R)$ in particular. Furthermore, we have $R^{-1} = f_R^{-1}$ (as in Constructions With Relations, Definition 3.2.1.1). The conditions from Item 2 then become the following:

$$
\begin{aligned}
f_R^{-1} \diamond f_R &= \chi_A, \\
f_R \diamond f_R^{-1} &= \chi_B.
\end{aligned}
$$

  All that is left is to show then is that f_R is a bijection:

- *The Function f_R Is Injective.* Let $a, b \in A$ and suppose that $f_R(a) = f_R(b)$. Since $a \sim_R f_R(a)$ and $f_R(a) = f_R(b) \sim_{R^{-1}} b$, the condition $f_R^{-1} \diamond f_R = \chi_A$ implies that $a = b$, showing f_R to be injective.
- *The Function f_R Is Surjective.* Let $b \in B$. Applying the condition $f_R \diamond f_R^{-1} = \chi_B$ to (b, b), it follows that there exists some $a \in A$ such that $f_R^{-1}(b) = a$ and $f_R(a) = b$. This shows f_R to be surjective.

- *Item 3* $\Longrightarrow$ *Item 2:* By Constructions With Relations, Item 2 of Proposition 3.1.1.2, we have an adjunction $\mathrm{Gr}(f) \dashv f^{-1}$, giving inclusions

$$\chi_A \subset f^{-1} \diamond \mathrm{Gr}(f),$$
$$\mathrm{Gr}(f) \diamond f^{-1} \subset \chi_B.$$

  We claim the reverse inclusions are also true:

  - $f^{-1} \diamond Gr(f) \subset \chi_A$: This is equivalent to the statement that if $f(a) = b$ and $f^{-1}(b) = a'$, then $a = a'$, which follows from the injectivity of f.
  - $\chi_B \subset Gr(f) \diamond f^{-1}$: This is equivalent to the statement that given $b \in B$ there exists some $a \in A$ such that $f^{-1}(b) = a$ and $f(a) = b$, which follows from the surjectivity of f.

This finishes the proof. □

3.3 Adjunctions in **Rel**

Let A and B be sets.

Proposition 3.3.1.1. We have a natural bijection

$$\left\{\begin{matrix}\text{Adjunctions in } \mathsf{Rel} \\ \text{from } A \text{ to } B\end{matrix}\right\} \cong \left\{\begin{matrix}\text{Functions} \\ \text{from } A \text{ to } B\end{matrix}\right\},$$

with every adjunction in **Rel** being of the form $\mathrm{Gr}(f) \dashv f^{-1}$ for some function f.

Proof. We proceed step by step:

1. *From Adjunctions in* **Rel** *to Functions.* An adjunction in **Rel** from A to B

consists of a pair of relations

$$R\colon A \nrightarrow B,$$
$$S\colon B \nrightarrow A,$$

together with inclusions

$$\chi_A \subset S \diamond R,$$
$$R \diamond S \subset \chi_B.$$

We claim that these conditions imply that R is total and functional, i.e. that $R(a)$ is a singleton for each $a \in A$:

(a) *$R(a)$ Has an Element.* Given $a \in A$, since $\chi_A \subset S \diamond R$, we must have $\{a\} \subset S(R(a))$, implying that there exists some $b \in B$ such that $a \sim_R b$ and $b \sim_S a$, and thus $R(a) \neq \emptyset$, as $b \in R(a)$.

(b) *$R(a)$ Has No More Than One Element.* Suppose that we have $a \sim_R b$ and $a \sim_R b'$ for $b, b' \in B$. We claim that $b = b'$:

   i. Since $\chi_A \subset S \diamond R$, there exists some $k \in B$ such that $a \sim_R k$ and $k \sim_S a$.

   ii. Since $R \diamond S \subset \chi_B$, if $b'' \sim_S a'$ and $a' \sim_R b'''$, then $b'' = b'''$.

   iii. Applying the above to $b'' = k$, $b''' = b$, and $a' = a$, since $k \sim_S a$ and $a \sim_R b'$, we have $k = b$.

   iv. Similarly $k = b'$.

   v. Thus $b = b'$.

Together, the above two items show $R(a)$ to be a singleton, being thus given by $\mathrm{Gr}(f)$ for some function $f\colon A \to B$, which gives a map

$$\left\{\begin{matrix}\text{Adjunctions in } \mathbf{Rel} \\ \text{from } A \text{ to } B\end{matrix}\right\} \to \left\{\begin{matrix}\text{Functions} \\ \text{from } A \text{ to } B\end{matrix}\right\}.$$

Moreover, by uniqueness of adjoints (**??**, **??** of **??**), this implies also that $S = f^{-1}$.

2. *From Functions to Adjunctions in* **Rel**. By Constructions With Relations, Item 2 of Proposition 3.1.1.2, every function $f\colon A \to B$ gives rise to an adjunction $\mathrm{Gr}(f) \dashv f^{-1}$ in Rel, giving a map

$$\left\{\begin{matrix}\text{Functions} \\ \text{from } A \text{ to } B\end{matrix}\right\} \to \left\{\begin{matrix}\text{Adjunctions in } \mathbf{Rel} \\ \text{from } A \text{ to } B\end{matrix}\right\}.$$

3. *Invertibility: From Functions to Adjunctions Back to Functions.* We need to show that starting with a function $f \colon A \to B$, passing to $\mathrm{Gr}(f) \dashv f^{-1}$, and then passing again to a function gives f again. This is clear however, since we have $a \sim_{\mathrm{Gr}(f)} b$ iff $f(a) = b$.

4. *Invertibility: From Adjunctions to Functions Back to Adjunctions.* We need to show that, given an adjunction $R \dashv S$ in **Rel** giving rise to a function $f_{R,S} \colon A \to B$, we have

$$\begin{aligned} \mathrm{Gr}(f_{R,S}) &= R, \\ f^{-1}_{R,S} &= S. \end{aligned}$$

We check these explicitly:

- $Gr(f_{R,S}) = R$. We have

$$\begin{aligned} \mathrm{Gr}(f_{R,S}) &\stackrel{\text{def}}{=} \left\{(a, f_{R,S}(a)) \in A \times B \,\middle|\, a \in A\right\} \\ &\stackrel{\text{def}}{=} \{(a, R(a)) \in A \times B \mid a \in A\} \\ &= R. \end{aligned}$$

- $f^{-1}_{R,S} = S$. We first claim that, given $a \in A$ and $b \in B$, the following conditions are equivalent:
  - We have $a \sim_R b$.
  - We have $b \sim_S a$.

  Indeed:
  - *If $a \sim_R b$, then $b \sim_S a$:* Since $\chi_A \subset S \diamond R$, there exists $k \in B$ such that $a \sim_R k$ and $k \sim_S a$, but since $a \sim_R b$ and R is functional, we have $k = b$ and thus $b \sim_S a$.
  - *If $b \sim_S a$, then $a \sim_R b$:* First note that since R is total we have $a \sim_R b'$ for some $b' \in B$. Now, since $R \diamond S \subset \chi_B$, $b \sim_S a$, and $a \sim_R b'$, we have $b = b'$, and thus $a \sim_R b$.

  Having show this, we now have

$$\begin{aligned} f^{-1}_{R,S}(b) &\stackrel{\text{def}}{=} \left\{a \in A \,\middle|\, f_{R,S}(a) = b\right\} \\ &\stackrel{\text{def}}{=} \{a \in A \mid a \sim_R b\} \\ &= \{a \in A \mid b \sim_S a\} \\ &\stackrel{\text{def}}{=} S(b). \end{aligned}$$

  for each $b \in B$, showing $f^{-1}_{R,S} = S$.

This finishes the proof. □

3.4 Monads in Rel

Let A be a set.

Proposition 3.4.1.1. We have a natural identification[13]

$$\left\{\begin{matrix}\text{Monads in}\\ \textbf{Rel} \text{ on } A\end{matrix}\right\} \cong \{\text{Preorders on } A\}.$$

Proof. A monad in **Rel** on A consists of a relation $R\colon A \nrightarrow A$ together with maps

$$\begin{aligned}\mu_R &\colon R \diamond R \subset R,\\ \eta_R &\colon \chi_A \subset R\end{aligned}$$

making the diagrams

$$\begin{tikzcd}
\chi_A \diamond R \arrow[r, "\eta_R \diamond \mathrm{id}_R", "\shortmid"{marking}] \arrow[dr, equal, "\lambda^{\mathbf{Rel}(A,B)}_R"'] & R \diamond R \arrow[d, "\mu_R", "\shortmid"{marking}] \\
 & R
\end{tikzcd}
\qquad
\begin{tikzcd}
 & R \diamond (R \diamond R) \arrow[dr, "\mathrm{id}_R \diamond \mu_R", "\shortmid"{marking}] & \\
(R \diamond R) \diamond R \arrow[ur, equal, "\alpha^{\mathbf{Rel}(A,B)}_{R,R,R}"] \arrow[d, "\mu_R \diamond \mathrm{id}_R"', "\shortmid"{marking}] & & R \diamond R \arrow[d, "\mu_R", "\shortmid"{marking}] \\
R \diamond R \arrow[rr, "\mu_R"', "\shortmid"{marking}] & & R
\end{tikzcd}
\qquad
\begin{tikzcd}
R \diamond \chi_A \arrow[r, "\mathrm{id}_R \diamond \eta_R", "\shortmid"{marking}] \arrow[dr, equal, "\rho^{\mathbf{Rel}(A,B)}_R"'] & R \diamond R \arrow[d, "\mu_R", "\shortmid"{marking}] \\
 & R
\end{tikzcd}$$

commute. However, since all morphisms involved are inclusions, the commutativity of the above diagrams is automatic, and hence all that is left is the data of the two maps μ_R and η_R, which correspond respectively to the following conditions:

1. For each $a, b, c \in A$, if $a \sim_R b$ and $b \sim_R c$, then $a \sim_R c$.
2. For each $a \in A$, we have $a \sim_R a$.

These are exactly the requirements for R to be a preorder (**??**, **??**). Conversely any preorder $\preceq$ gives rise to a pair of maps $\mu_{\preceq}$ and $\eta_{\preceq}$, forming a monad on A. □

[13] See also **??** for an extension of this correspondence to "relative monads in **Rel**".

3.5 Comonads in Rel

Let A be a set.

Proposition 3.5.1.1. We have a natural identification

$$\left\{\begin{matrix}\text{Comonads in}\\ \textsf{Rel} \text{ on } A\end{matrix}\right\} \cong \{\text{Subsets of } A\}.$$

Proof. A comonad in **Rel** on A consists of a relation $R\colon A \nrightarrow A$ together with maps

$$\begin{aligned}\Delta_R &\colon R \subset R \diamond R,\\ \epsilon_R &\colon R \subset \chi_A\end{aligned}$$

making the diagrams

$$\begin{tikzcd}
R \arrow[r, "\Delta_R", "\shortmid"{marking}] \arrow[dr, equals, "\lambda^{\mathbf{Rel}(A,B),-1}_R"'] & R \diamond R \arrow[d, "\epsilon_R \diamond \mathrm{id}_R", "\shortmid"{marking}] \\
& \chi_A \diamond R
\end{tikzcd}
\quad
\begin{tikzcd}
& & R \diamond R \arrow[drr, "\mathrm{id}_R \diamond \Delta_R", "\shortmid"{marking}] & & \\
R \arrow[urr, "\Delta_R", "\shortmid"{marking}] \arrow[dr, "\Delta_R"', "\shortmid"{marking}] & & & & R \diamond (R \diamond R) \arrow[dl, equals, "\alpha^{\mathbf{Rel}(A,B),-1}_{R,R,R}"] \\
& R \diamond R \arrow[rr, "\Delta_R \diamond \mathrm{id}_R"', "\shortmid"{marking}] & & (R \diamond R) \diamond R &
\end{tikzcd}
\quad
\begin{tikzcd}
R \arrow[r, "\Delta_R", "\shortmid"{marking}] \arrow[dr, equals, "\rho^{\mathbf{Rel}(A,B),-1}_R"'] & R \diamond R \arrow[d, "\mathrm{id}_R \diamond \epsilon_R", "\shortmid"{marking}] \\
& R \diamond \chi_A
\end{tikzcd}$$

commute. However, since all morphisms involved are inclusions, the commutativity of the above diagrams is automatic, and hence all that is left is the data of the two maps Δ_R and ϵ_R, which correspond respectively to the following conditions:

1. For each $a, b \in A$, if $a \sim_R b$, then there exists some $k \in A$ such that $a \sim_R k$ and $k \sim_R b$.
2. For each $a, b \in A$, if $a \sim_R b$, then $a = b$.

Taking $k = b$ in the first condition above shows it to be trivially satisfied, while the second condition implies $R \subset \Delta_A$, i.e. R must be a subset of A. Conversely, any subset U of A satisfies $U \subset \Delta_A$, defining a comonad as above. $\square$

3.6 Co/Monoids in **Rel**

Remark 3.6.1.1. The monoids in **Rel** with respect to the Cartesian monoidal structure of Proposition 2.2.8.1 are called *hypermonoids*, and their theory is explored in **??**. Similarly, the comonoids in **Rel** are called *hypercomonoids*, and they are defined and studied in **??**.

3.7 Monomorphisms in Rel

In this section we characterise the epimorphisms in the category Rel, following **??**, **??**.

Proposition 3.7.1.1. Let $R\colon A \nrightarrow B$ be a relation. The following conditions are equivalent:

1. The relation R is a monomorphism in Rel.
2. The direct image function
$$R_*\colon \mathcal{P}(A) \to \mathcal{P}(B)$$
associated to R is injective.
3. The direct image with compact support function
$$R_!\colon \mathcal{P}(A) \to \mathcal{P}(B)$$
associated to R is injective.

Moreover, if R is a monomorphism, then it satisfies the following condition, and the converse holds if R is total:

(★) For each $a, a' \in A$, if there exists some $b \in B$ such that
$$\begin{aligned} a &\sim_R b, \\ a' &\sim_R b, \end{aligned}$$
then $a = a'$.

Proof. Firstly note that Items 2 and 3 are equivalent by Constructions With Relations, Item 7 of Proposition 4.1.1.3. We then claim that Items 1 and 2 are also equivalent:

- *Item 1* $\implies$ *Item 2:* Let $U, V \in \mathcal{P}(A)$ and consider the diagram

$$\mathrm{pt} \underset{V}{\overset{U}{\rightrightarrows}} A \xrightarrow{R} B.$$

By Constructions With Relations, Remark 4.1.1.2, we have

$$\begin{aligned} R_*(U) &= R \diamond U, \\ R_*(V) &= R \diamond V. \end{aligned}$$

Now, if $R \diamond U = R \diamond V$, i.e. $R_*(U) = R_*(V)$, then $U = V$ since R is assumed to be a monomorphism, showing R_* to be injective.

- *Item 2* $\implies$ *Item 1:* Conversely, suppose that R_* is injective, consider the diagram

$$X \underset{T}{\overset{S}{\rightrightarrows}} A \xrightarrow{R} B,$$

and suppose that $R \diamond S = R \diamond T$. Note that, since R_* is injective, given a diagram of the form

$$\mathrm{pt} \underset{V}{\overset{U}{\rightrightarrows}} A \xrightarrow{R} B,$$

if $R_*(U) = R \diamond U = R \diamond V = R_*(V)$, then $U = V$. In particular, for each $x \in X$, we may consider the diagram

$$\mathrm{pt} \xrightarrow{[x]} X \underset{T}{\overset{S}{\rightrightarrows}} A \xrightarrow{R} B,$$

for which we have $R \diamond S \diamond [x] = R \diamond T \diamond [x]$, implying that we have

$$S(x) = S \diamond [x] = T \diamond [x] = T(x)$$

for each $x \in X$, implying $S = T$, and thus R is a monomorphism.

We can also prove this in a more abstract way, following [MSE 350788]:

- *Item 1* $\implies$ *Item 2:* Assume that R is a monomorphism.
  - We first notice that the functor $\mathrm{Rel}(\mathrm{pt}, -) \colon \mathrm{Rel} \to \mathsf{Sets}$ maps R to R_* by Constructions With Relations, Remark 4.1.1.2.

- Since $\mathrm{Rel}(\mathrm{pt}, -)$ preserves all limits by **??**, **??** of **??**, it follows by **??**, **??** of **??** that $\mathrm{Rel}(\mathrm{pt}, -)$ also preserves monomorphisms.
- Since R is a monomorphism and $\mathrm{Rel}(\mathrm{pt}, -)$ maps R to R_*, it follows that R_* is also a monomorphism.
- Since the monomorphisms in Sets are precisely the injections (**??**, **??** of **??**), it follows that R_* is injective.

• *Item 2* $\Longrightarrow$ *Item 1*: Assume that R_* is injective.

- We first notice that the functor $\mathrm{Rel}(\mathrm{pt}, -) \colon \mathrm{Rel} \to \mathsf{Sets}$ maps R to R_* by Constructions With Relations, Remark 4.1.1.2.
- Since the monomorphisms in Sets are precisely the injections (**??**, **??** of **??**), it follows that R_* is a monomorphism.
- Since $\mathrm{Rel}(\mathrm{pt}, -)$ is faithful, it follows by **??**, **??** of **??** that $\mathrm{Rel}(\mathrm{pt}, -)$ reflects monomorphisms.
- Since R_* is a monomorphism and $\mathrm{Rel}(\mathrm{pt}, -)$ maps R to R_*, it follows that R is also a monomorphism.

Finally, we prove the second part of the statement. Assume that R is a monomorphism, let $a, a' \in A$ such that $a \sim_R b$ and $a' \sim_R b$ for some $b \in B$, and consider the diagram

$$\mathrm{pt} \underset{[a']}{\overset{[a]}{\rightrightarrows}} A \xrightarrow{R} B.$$

Since $\star \sim_{[a]} a$ and $a \sim_R b$, we have $\star \sim_{R \diamond [a]} b$. Similarly, $\star \sim_{R \diamond [a']} b$. Thus $R \diamond [a] = R \diamond [a']$, and since R is a monomorphism, we have $[a] = [a']$, i.e. $a = a'$.

Conversely, assume the condition

($\star$) For each $a, a' \in A$, if there exists some $b \in B$ such that

$$a \sim_R b,$$
$$a' \sim_R b,$$

then $a = a'$.

consider the diagram

$$X \underset{T}{\overset{S}{\rightrightarrows}} A \xrightarrow{R} B,$$

and let $(x, a) \in S$. Since R is total and $a \in A$, there exists some $b \in B$ such that $a \sim_R b$. In this case, we have $x \sim_{R \diamond S} b$, and since $R \diamond S = R \diamond T$, we have also $x \sim_{R \diamond T} b$. Thus there must exist some $a' \in A$ such that $x \sim_T a'$ and $a' \sim_R b$. However, since $a, a' \sim_R b$, we must have $a = a'$, and thus $(x, a) \in T$ as well.
A similar argument shows that if $(x, a) \in T$, then $(x, a) \in S$, and thus $S = T$ and it follows that R is a monomorphism. □

3.8 2-Categorical Monomorphisms in **Rel**

In this section we characterise (for now, some of) the 2-categorical monomorphisms in **Rel**, following Types of Morphisms in Bicategories, Section 1.

Proposition 3.8.1.1. Let $R\colon A \nrightarrow B$ be a relation.

1. *Representably Faithful Morphisms in* **Rel**. Every morphism of **Rel** is a representably faithful morphism.

2. *Representably Full Morphisms in* **Rel**. The following conditions are equivalent:

   (a) The morphism $R\colon A \nrightarrow B$ is a representably full morphism.

   (b) For each pair of relations $S, T\colon X \rightrightarrows A$, the following condition is satisfied:

   ($\star$) If $R \diamond S \subset R \diamond T$, then $S \subset T$.

   (c) The functor

   $$R_*\colon (\mathcal{P}(A), \subset) \to (\mathcal{P}(B), \subset)$$

   is full.

   (d) For each $U, V \in \mathcal{P}(A)$, if $R_*(U) \subset R_*(V)$, then $U \subset V$.

   (e) The functor

   $$R_!\colon (\mathcal{P}(A), \subset) \to (\mathcal{P}(B), \subset)$$

   is full.

   (f) For each $U, V \in \mathcal{P}(A)$, if $R_!(U) \subset R_!(V)$, then $U \subset V$.

3. *Representably Fully Faithful Morphisms in* **Rel**. Every representaly full morphism in **Rel** is a representably fully faithful morphism.

Proof. *Item 1, Representably Faithful Morphisms in* **Rel**: The relation R is a representably faithful morphism in **Rel** iff, for each $X \in \mathrm{Obj}(\mathsf{Rel})$, the functor

$$R_* \colon \mathbf{Rel}(X, A) \to \mathbf{Rel}(X, B)$$

is faithful, i.e. iff the morphism

$$R_{*|S,T} \colon \mathrm{Hom}_{\mathbf{Rel}(X,A)}(S, T) \to \mathrm{Hom}_{\mathbf{Rel}(X,B)}(R \diamond S, R \diamond T)$$

is injective for each $S, T \in \mathrm{Obj}(\mathbf{Rel}(X, A))$. However, $\mathrm{Hom}_{\mathbf{Rel}(X,A)}(S, T)$ is either empty or a singleton, in either case of which the map $R_{*|S,T}$ is necessarily injective.

Item 2, Representably Full Morphisms in **Rel**: We claim Items 2a to 2f are indeed equivalent:

- *Item 2a* $\iff$ *Item 2b:* This is simply a matter of unwinding definitions: The relation R is a representably full morphism in **Rel** iff, for each $X \in \mathrm{Obj}(\mathsf{Rel})$, the functor

  $$R_* \colon \mathbf{Rel}(X, A) \to \mathbf{Rel}(X, B)$$

  is full, i.e. iff the morphism

  $$R_{*|S,T} \colon \mathrm{Hom}_{\mathbf{Rel}(X,A)}(S, T) \to \mathrm{Hom}_{\mathbf{Rel}(X,B)}(R \diamond S, R \diamond T)$$

  is surjective for each $S, T \in \mathrm{Obj}(\mathbf{Rel}(X, A))$, i.e. iff, whenever $R \diamond S \subset R \diamond T$, we also have $S \subset T$.

- *Item 2c* $\iff$ *Item 2d:* This is also simply a matter of unwinding definitions: The functor

  $$R_* \colon (\mathcal{P}(A), \subset) \to (\mathcal{P}(B), \subset)$$

  is full iff, for each $U, V \in \mathcal{P}(A)$, the morphism

  $$R_{*|U,V} \colon \mathrm{Hom}_{\mathcal{P}(A)}(U, V) \to \mathrm{Hom}_{\mathcal{P}(B)}(R_*(U), R_*(V))$$

  is surjective, i.e. iff whenever $R_*(U) \subset R_*(V)$, we also necessarily have $U \subset V$.

- *Item 2e* $\iff$ *Item 2f:* This is once again simply a matter of unwinding definitions, and proceeds exactly in the same way as in the proof of the equivalence between Items 2c and 2d given above.

- *Item 2d* $\implies$ *Item 2f*: Suppose that the following condition is true:

  (★) For each $U, V \in \mathcal{P}(A)$, if $R_*(U) \subset R_*(V)$, then $U \subset V$.

  We need to show that the condition

  (★) For each $U, V \in \mathcal{P}(A)$, if $R_!(U) \subset R_!(V)$, then $U \subset V$.

  is also true. We proceed step by step:

  1. Suppose we have $U, V \in \mathcal{P}(A)$ with $R_!(U) \subset R_!(V)$.
  2. By Constructions With Relations, Item 7 of Proposition 4.4.1.3, we have

  $$\begin{aligned} R_!(U) &= B \setminus R_*(A \setminus U), \\ R_!(V) &= B \setminus R_*(A \setminus V). \end{aligned}$$

  3. By Constructions With Sets, Item 1 of Proposition 3.10.1.2 we have $R_*(A \setminus V) \subset R_*(A \setminus U)$.
  4. By assumption, we then have $A \setminus V \subset A \setminus U$.
  5. By Constructions With Sets, Item 1 of Proposition 3.10.1.2 again, we have $U \subset V$.

- *Item 2f* $\implies$ *Item 2d*: Suppose that the following condition is true:

  (★) For each $U, V \in \mathcal{P}(A)$, if $R_!(U) \subset R_!(V)$, then $U \subset V$.

  We need to show that the condition

  (★) For each $U, V \in \mathcal{P}(A)$, if $R_*(U) \subset R_*(V)$, then $U \subset V$.

  is also true. We proceed step by step:

  1. Suppose we have $U, V \in \mathcal{P}(A)$ with $R_*(U) \subset R_*(V)$.
  2. By Constructions With Relations, Item 7 of Proposition 4.1.1.3, we have

  $$\begin{aligned} R_*(U) &= B \setminus R_!(A \setminus U), \\ R_*(V) &= B \setminus R_!(A \setminus V). \end{aligned}$$

3. By Constructions With Sets, Item 1 of Proposition 3.10.1.2 we have $R_!(A \setminus V) \subset R_!(A \setminus U)$.
4. By assumption, we then have $A \setminus V \subset A \setminus U$.
5. By Constructions With Sets, Item 1 of Proposition 3.10.1.2 again, we have $U \subset V$.

- *Item 2b* $\Longrightarrow$ *Item 2d:* Consider the diagram

$$X \underset{T}{\overset{S}{\rightrightarrows}} A \xrightarrow{R} B,$$

and suppose that $R \diamond S \subset R \diamond T$. Note that, by assumption, given a diagram of the form

$$\mathrm{pt} \underset{V}{\overset{U}{\rightrightarrows}} A \xrightarrow{R} B,$$

if $R_*(U) = R \diamond U \subset R \diamond V = R_*(V)$, then $U \subset V$. In particular, for each $x \in X$, we may consider the diagram

$$\mathrm{pt} \xrightarrow{[x]} X \underset{T}{\overset{S}{\rightrightarrows}} A \xrightarrow{R} B,$$

for which we have $R \diamond S \diamond [x] \subset R \diamond T \diamond [x]$, implying that we have

$$S(x) = S \diamond [x] \subset T \diamond [x] = T(x)$$

for each $x \in X$, implying $S \subset T$.

- *Item 2d* $\Longrightarrow$ *Item 2b:* Let $U, V \in \mathcal{P}(A)$ and consider the diagram

$$\mathrm{pt} \underset{V}{\overset{U}{\rightrightarrows}} A \xrightarrow{R} B.$$

By **??**, we have

$$\begin{aligned} R_*(U) &= R \diamond U, \\ R_*(V) &= R \diamond V. \end{aligned}$$

Now, if $R_*(U) \subset R_*(V)$, i.e. $R \diamond U \subset R \diamond V$, then $U \subset V$ by assumption.

??, *Fully Faithful Monomorphisms in* **Rel**: This follows from Items 1 and 2. □

Question 3.8.1.2. Item 2 of Proposition 3.8.1.1 gives a characterisation of the representably full morphisms in **Rel**.
Are there other nice characterisations of these?
This question also appears as [MO 467527].

3.9 Epimorphisms in Rel

In this section we characterise the epimorphisms in the category Rel, following **??**, **??**.

Proposition 3.9.1.1. Let $R\colon A \nrightarrow B$ be a relation. The following conditions are equivalent:

1. The relation R is an epimorphism in Rel.
2. The weak inverse image function
$$R^{-1}\colon \mathcal{P}(B) \to \mathcal{P}(A)$$
associated to R is injective.
3. The strong inverse image function
$$R_{-1}\colon \mathcal{P}(B) \to \mathcal{P}(A)$$
associated to R is injective.
4. The function $R\colon A \to \mathcal{P}(B)$ is "surjective on singletons":
   (★) For each $b \in B$, there exists some $a \in A$ such that $R(a) = \{b\}$.

Moreover, if R is total and an epimorphism, then it satisfies the following equivalent conditions:

1. For each $b \in B$, there exists some $a \in A$ such that $a \sim_R b$.
2. We have $\mathrm{Im}(R) = B$.

Proof. Firstly note that Items 2 and 3 are equivalent by Constructions With Relations, Item 7 of Proposition 4.2.1.3. We then claim that Items 1 and 2 are also equivalent:

- *Item 1* $\implies$ *Item 2*: Let $U, V \in \mathcal{P}(A)$ and consider the diagram

$$A \xrightarrow{R} B \overset{U}{\underset{V}{\rightrightarrows}} \mathrm{pt}.$$

  By Constructions With Relations, Remark 4.1.1.2, we have

$$\begin{aligned} R^{-1}(U) &= U \diamond R, \\ R^{-1}(V) &= V \diamond R. \end{aligned}$$

  Now, if $U \diamond R = V \diamond R$, i.e. $R^{-1}(U) = R^{-1}(V)$, then $U = V$ since R is assumed to be an epimorphism, showing R^{-1} to be injective.

- *Item 2* $\implies$ *Item 1*: Conversely, suppose that R^{-1} is injective, consider the diagram

$$A \xrightarrow{R} B \overset{S}{\underset{T}{\rightrightarrows}} X,$$

  and suppose that $S \diamond R = T \diamond R$. Note that, since R^{-1} is injective, given a diagram of the form

$$A \xrightarrow{R} B \overset{U}{\underset{V}{\rightrightarrows}} \mathrm{pt},$$

  if $R^{-1}(U) = U \diamond R = V \diamond R = R^{-1}(V)$, then $U = V$. In particular, for each $x \in X$, we may consider the diagram

$$A \xrightarrow{R} B \overset{S}{\underset{T}{\rightrightarrows}} X \xrightarrow{[x]} \mathrm{pt},$$

  for which we have $[x] \diamond S \diamond R = [x] \diamond T \diamond R$, implying that we have

$$S^{-1}(x) = [x] \diamond S = [x] \diamond T = T^{-1}(x)$$

  for each $x \in X$, implying $S = T$, and thus R is an epimorphism.

We can also prove this in a more abstract way, following [MSE 350788]:

- *Item 1* $\implies$ *Item 2*: Assume that R is an epimorphism.
  - We first notice that the functor $\mathrm{Rel}(-, \mathrm{pt}) \colon \mathrm{Rel}^{\mathrm{op}} \to \mathsf{Sets}$ maps R to R^{-1} by Constructions With Relations, Remark 4.3.1.2.

- Since $\mathrm{Rel}(-,\mathrm{pt})$ preserves limits by **??**, **??** of **??**, it follows by **??**, **??** of **??** that $\mathrm{Rel}(-,\mathrm{pt})$ also preserves monomorphisms.
- That is: $\mathrm{Rel}(-,\mathrm{pt})$ sends monomorphisms in $\mathrm{Rel}^{\mathrm{op}}$ to monomorphisms in Sets.
- The monomorphisms $\mathrm{Rel}^{\mathrm{op}}$ are precisely the epimorphisms in Rel by **??**, **??** of **??**.
- Since R is an epimorphism and $\mathrm{Rel}(-,\mathrm{pt})$ maps R to R^{-1}, it follows that R^{-1} is a monomorphism.
- Since the monomorphisms in Sets are precisely the injections (**??**, **??** of **??**), it follows that R^{-1} is injective.

• *Item 2* $\Longrightarrow$ *Item 1:* Assume that R^{-1} is injective.

- We first notice that the functor $\mathrm{Rel}(-,\mathrm{pt})\colon \mathrm{Rel}^{\mathrm{op}} \to \mathsf{Sets}$ maps R to R^{-1} by Constructions With Relations, Remark 4.3.1.2.
- Since the monomorphisms in Sets are precisely the injections (**??**, **??** of **??**), it follows that R^{-1} is a monomorphism.
- Since $\mathrm{Rel}(-,\mathrm{pt})$ is faithful, it follows by **??**, **??** of **??** that $\mathrm{Rel}(,\mathrm{pt})$ reflects monomorphisms.
- That is: $\mathrm{Rel}(-,\mathrm{pt})$ reflects monomorphisms in Sets to monomorphisms in $\mathrm{Rel}^{\mathrm{op}}$.
- The monomorphisms $\mathrm{Rel}^{\mathrm{op}}$ are precisely the epimorphisms in Rel by **??**, **??** of **??**.
- Since R^{-1} is a monomorphism and $\mathrm{Rel}(-,\mathrm{pt})$ maps R to R^{-1}, it follows that R is an epimorphism.

We also claim that Items 2 and 4 are equivalent, following [`MO 350788`]:

• *Item 2* $\Longrightarrow$ *Item 4:* Since $B \setminus \{b\} \subset B$ and R^{-1} is injective, we have $R^{-1}(B \setminus \{b\}) \subsetneq R^{-1}(B)$. So taking some $a \in R^{-1}(B) \setminus R^{-1}(B \setminus \{b\})$ we get an element of A such that $R(a) = \{b\}$.

• *Item 4* $\Longrightarrow$ *Item 2:* Let $U, V \subset B$ with $U \neq V$. Without loss of generality, we can assume $U \setminus V \neq \emptyset$; otherwise just swap U and V. Let then $b \in U \setminus V$. By assumption, there exists an $a \in A$ with $R(a) = \{b\}$. Then $a \in R^{-1}(U)$ but $a \notin R^{-1}(V)$, and thus $R^{-1}(U) \neq R^{-1}(V)$, showing R^{-1} to be injective.

Finally, we prove the second part of the statement. So assume R is a total epimorphism in Rel and consider the diagram

$$A \xrightarrow{R} B \underset{T}{\overset{S}{\rightrightarrows}} \{0,1\},$$

where $b \sim_S 0$ for each $b \in B$ and where we have

$$b \sim_T \begin{cases} 0 & \text{if } b \in \mathrm{Im}(R), \\ 1 & \text{otherwise} \end{cases}$$

for each $b \in B$. Since R is total, we have $a \sim_{S \diamond R} 0$ and $a \sim_{T \diamond R} 0$ for all $a \in A$, and no element of A is related to 1 by $S \diamond R$ or $T \diamond R$. Thus $S \diamond R = T \diamond R$, and since R is an epimorphism, we have $S = T$. But by the definition of T, this implies $\mathrm{Im}(R) = B$. □

3.10 2-Categorical Epimorphisms in **Rel**

In this section we characterise (for now, some of) the 2-categorical epimorphisms in **Rel**, following Types of Morphisms in Bicategories, Section 2.

Proposition 3.10.1.1. Let $R \colon A \nrightarrow B$ be a relation.

1. *Corepresentably Faithful Morphisms in* **Rel**. Every morphism of **Rel** is a corepresentably faithful morphism.

2. *Corepresentably Full Morphisms in* **Rel**. The following conditions are equivalent:

   (a) The morphism $R \colon A \nrightarrow B$ is a corepresentably full morphism.

   (b) For each pair of relations $S, T \colon X \rightrightarrows A$, the following condition is satisfied:

   ($\star$) If $S \diamond R \subset T \diamond R$, then $S \subset T$.

   (c) The functor

   $$R^{-1} \colon (\mathcal{P}(B), \subset) \to (\mathcal{P}(A), \subset)$$

   is full.

   (d) For each $U, V \in \mathcal{P}(B)$, if $R^{-1}(U) \subset R^{-1}(V)$, then $U \subset V$.

(e) The functor

$$R_{-1}\colon (\mathcal{P}(B), \subset) \to (\mathcal{P}(A), \subset)$$

is full.

(f) For each $U, V \in \mathcal{P}(B)$, if $R_{-1}(U) \subset R_{-1}(V)$, then $U \subset V$.

3. *Corepresentably Fully Faithful Morphisms in* **Rel**. Every corepresentably full morphism of **Rel** is a corepresentably fully faithful morphism.

Proof. Item 1, Corepresentably Faithful Morphisms in **Rel**: The relation R is a corepresentably faithful morphism in **Rel** iff, for each $X \in \mathrm{Obj}(\mathbf{Rel})$, the functor

$$R^*\colon \mathbf{Rel}(B, X) \to \mathbf{Rel}(A, X)$$

is faithful, i.e. iff the morphism

$$R^*_{S,T}\colon \mathrm{Hom}_{\mathbf{Rel}(B,X)}(S, T) \to \mathrm{Hom}_{\mathbf{Rel}(A,X)}(S \diamond R, T \diamond R)$$

is injective for each $S, T \in \mathrm{Obj}(\mathbf{Rel}(B, X))$. However, $\mathrm{Hom}_{\mathbf{Rel}(B,X)}(S, T)$ is either empty or a singleton, in either case of which the map $R^*_{S,T}$ is necessarily injective.

Item 2, Corepresentably Full Morphisms in **Rel**: We claim Items 2a to 2f are indeed equivalent:

- *Item 2a* $\iff$ *Item 2b:* This is simply a matter of unwinding definitions: The relation R is a corepresentably full morphism in **Rel** iff, for each $X \in \mathrm{Obj}(\mathbf{Rel})$, the functor

$$R^*\colon \mathbf{Rel}(B, X) \to \mathbf{Rel}(A, X)$$

  is full, i.e. iff the morphism

$$R^*_{S,T}\colon \mathrm{Hom}_{\mathbf{Rel}(B,X)}(S, T) \to \mathrm{Hom}_{\mathbf{Rel}(A,X)}(S \diamond R, T \diamond R)$$

  is surjective for each $S, T \in \mathrm{Obj}(\mathbf{Rel}(B, X))$, i.e. iff, whenever $S \diamond R \subset T \diamond R$, we also have $S \subset T$.

- *Item 2c* $\iff$ *Item 2d:* This is also simply a matter of unwinding definitions: The functor

$$R^{-1}\colon (\mathcal{P}(B), \subset) \to (\mathcal{P}(A), \subset)$$

is full iff, for each $U, V \in \mathcal{P}(A)$, the morphism

$$R^{-1}_{U,V} \colon \mathrm{Hom}_{\mathcal{P}(B)}(U, V) \to \mathrm{Hom}_{\mathcal{P}(A)}(R^{-1}(U), R^{-1}(V))$$

is surjective, i.e. iff whenever $R^{-1}(U) \subset R^{-1}(V)$, we also necessarily have $U \subset V$.

- *Item 2e* $\iff$ *Item 2f*: This is once again simply a matter of unwinding definitions, and proceeds exactly in the same way as in the proof of the equivalence between Items 2c and 2d given above.

- *Item 2d* $\implies$ *Item 2f*: Suppose that the following condition is true:

  ($\star$) For each $U, V \in \mathcal{P}(B)$, if $R^{-1}(U) \subset R^{-1}(V)$, then $U \subset V$.

  We need to show that the condition

  ($\star$) For each $U, V \in \mathcal{P}(B)$, if $R_{-1}(U) \subset R_{-1}(V)$, then $U \subset V$.

  is also true. We proceed step by step:

  1. Suppose we have $U, V \in \mathcal{P}(B)$ with $R_{-1}(U) \subset R_{-1}(V)$.
  2. By Constructions With Relations, Item 7 of Proposition 4.2.1.3, we have
  $$\begin{aligned} R_{-1}(U) &= B \setminus R^{-1}(A \setminus U), \\ R_{-1}(V) &= B \setminus R^{-1}(A \setminus V). \end{aligned}$$
  3. By Constructions With Sets, Item 1 of Proposition 3.10.1.2 we have $R^{-1}(A \setminus V) \subset R^{-1}(A \setminus U)$.
  4. By assumption, we then have $A \setminus V \subset A \setminus U$.
  5. By Constructions With Sets, Item 1 of Proposition 3.10.1.2 again, we have $U \subset V$.

- *Item 2f* $\implies$ *Item 2d*: Suppose that the following condition is true:

  ($\star$) For each $U, V \in \mathcal{P}(B)$, if $R_{-1}(U) \subset R_{-1}(V)$, then $U \subset V$.

  We need to show that the condition

  ($\star$) For each $U, V \in \mathcal{P}(B)$, if $R^{-1}(U) \subset R^{-1}(V)$, then $U \subset V$.

is also true. We proceed step by step:

1. Suppose we have $U, V \in \mathcal{P}(B)$ with $R^{-1}(U) \subset R^{-1}(V)$.
2. By Constructions With Relations, Item 7 of Proposition 4.3.1.3, we have

$$\begin{aligned} R^{-1}(U) &= B \setminus R_{-1}(A \setminus U), \\ R^{-1}(V) &= B \setminus R_{-1}(A \setminus V). \end{aligned}$$

3. By Constructions With Sets, Item 1 of Proposition 3.10.1.2 we have $R_{-1}(A \setminus V) \subset R_{-1}(A \setminus U)$.
4. By assumption, we then have $A \setminus V \subset A \setminus U$.
5. By Constructions With Sets, Item 1 of Proposition 3.10.1.2 again, we have $U \subset V$.

- *Item 2b* $\Longrightarrow$ *Item 2d:* Consider the diagram

$$A \overset{R}{\longrightarrow\!\!\!\!\!\!\!\!\mapsto} B \underset{T}{\overset{S}{\rightrightarrows}} X,$$

and suppose that $S \diamond R \subset T \diamond R$. Note that, by assumption, given a diagram of the form

$$A \overset{R}{\longrightarrow\!\!\!\!\!\!\!\!\mapsto} B \underset{V}{\overset{U}{\rightrightarrows}} \mathrm{pt},$$

if $R^{-1}(U) = R \diamond U \subset R \diamond V = R^{-1}(V)$, then $U \subset V$. In particular, for each $x \in X$, we may consider the diagram

$$A \overset{R}{\longrightarrow\!\!\!\!\!\!\!\!\mapsto} B \underset{T}{\overset{S}{\rightrightarrows}} X \overset{[x]}{\longrightarrow\!\!\!\!\!\!\!\!\mapsto} \mathrm{pt},$$

for which we have $[x] \diamond S \diamond R \subset [x] \diamond T \diamond R$, implying that we have

$$S^{-1}(x) = [x] \diamond S \subset [x] \diamond T = T^{-1}(x)$$

for each $x \in X$, implying $S \subset T$.

- *Item 2d* $\Longrightarrow$ *Item 2b:* Let $U, V \in \mathcal{P}(B)$ and consider the diagram

$$A \overset{R}{\longrightarrow\!\!\!\!\!\!\!\!\mapsto} B \underset{V}{\overset{U}{\rightrightarrows}} \mathrm{pt}.$$

By **??**, we have

$$R^{-1}(U) = U \diamond R,$$
$$R^{-1}(V) = V \diamond R.$$

Now, if $R^{-1}(U) \subset R^{-1}(V)$, i.e. $U \diamond R \subset V \diamond R$, then $U \subset V$ by assumption.

Item 3, Corepresentably Fully Faithful Morphisms in **Rel**: This follows from Items 1 and 2. □

Question 3.10.1.2. Item 2 of Proposition 3.10.1.1 gives a characterisation of the corepresentably full morphisms in **Rel**.
Are there other nice characterisations of these?
This question also appears as [MO 467527].

3.11 Co/Limits in Rel

Proposition 3.11.1.1. This will be properly written later on.

Proof. Omitted. □

3.12 Kan Extensions and Kan Lifts in Rel

Remark 3.12.1.1. The 2-category **Rel** admits all right Kan extensions and right Kan lifts, though not all left Kan extensions and neither does it admit all left Kan lifts. See Constructions With Relations, Section 2 for a detailed discussion of this.

3.13 Closedness of Rel

Proposition 3.13.1.1. The 2-category **Rel** is a closed bicategory, there being, for each $R\colon A \nrightarrow B$ and set X, a pair of adjunctions

$$(R^* \dashv \mathrm{Ran}_R)\colon \quad \begin{tikzcd} \mathrm{Rel}(B,X) \arrow[r, bend left, "R^*"] \arrow[r, phantom, "\perp"] & \mathrm{Rel}(A,X), \arrow[l, bend left, "\mathrm{Ran}_R"] \end{tikzcd}$$

$$(R_* \dashv \mathrm{Rift}_R)\colon \quad \begin{tikzcd} \mathrm{Rel}(X,A) \arrow[r, bend left, "R_*"] \arrow[r, phantom, "\perp"] & \mathrm{Rel}(X,B), \arrow[l, bend left, "\mathrm{Rift}_R"] \end{tikzcd}$$

witnessed by bijections

$$\begin{aligned}\mathbf{Rel}(S \diamond R, T) &\cong \mathbf{Rel}(S, \mathrm{Ran}_R(T)),\\ \mathbf{Rel}(R \diamond U, V) &\cong \mathbf{Rel}(U, \mathrm{Rift}_R(V)),\end{aligned}$$

natural in $S \in \mathrm{Rel}(B, X)$, $T \in \mathrm{Rel}(A, X)$, $U \in \mathrm{Rel}(X, A)$, and $V \in \mathrm{Rel}(X, B)$.

Proof. This follows from Constructions With Relations, Propositions 2.3.1.1 and 2.4.1.1. □

3.14 Rel as a Category of Free Algebras

Proposition 3.14.1.1. We have an isomorphism of categories

$$\mathrm{Rel} \cong \mathsf{FreeAlg}_{\mathcal{P}_*}(\mathsf{Sets}),$$

where $\mathcal{P}_*$ is the powerset monad of **??**, **??**.

Proof. Omitted. □

4 The Left Skew Monoidal Structure on Rel(A, B)

4.1 The Left Skew Monoidal Product

Let A and B be sets and let $J\colon A \nrightarrow B$ be a relation.

Definition 4.1.1.1. The **left J-skew monoidal product of Rel(A, B)** is the functor

$$\triangleleft_J \colon \mathbf{Rel}(A, B) \times \mathbf{Rel}(A, B) \longrightarrow \mathbf{Rel}(A, B)$$

where

- *Action on Objects.* For each $R, S \in \mathrm{Obj}(\mathbf{Rel}(A, B))$, we have

$$S \triangleleft_J R \overset{\text{def}}{=} S \diamond \mathrm{Rift}_J(R), \qquad \begin{tikzcd} & A \arrow[r, "S", "\shortmid"{marking}] \arrow[d, "J", "\shortmid"{marking}] & B. \\ A \arrow[r, "R"', "\shortmid"{marking}] \arrow[ur, dashed, "\mathrm{Rift}_J(R)", "\shortmid"{marking}] & B \end{tikzcd}$$

- *Action on Morphisms.* For each $R, S, R', S' \in \mathrm{Obj}(\mathbf{Rel}(A, B))$, the action on Hom-sets

$$(\triangleleft_J)_{(G,F),(G',F')} \colon \mathrm{Hom}_{\mathbf{Rel}(A,B)}(S, S') \times \mathrm{Hom}_{\mathbf{Rel}(A,B)}(R, R') \to \mathrm{Hom}_{\mathbf{Rel}(A,B)}(S \triangleleft_J R, S' \triangleleft_J R')$$

of $\triangleleft_J$ at $((R, S), (R', S'))$ is defined by[14]

$$\beta \triangleleft_J \alpha \overset{\mathrm{def}}{=} \beta \diamond \mathrm{Rift}_J(\alpha), \qquad \begin{tikzcd} {} & A \arrow[d, "J"] \arrow[r, bend left, "S"{name=S}] \arrow[r, bend right, "S'"'{name=Sp}] & B \\ A \arrow[ur, dashed, bend left=40, "\mathrm{Rift}_J(R)"{name=RR}] \arrow[ur, dashed, "\mathrm{Rift}_J(R')"'{name=RRp}] \arrow[r, "R"{name=Rn}] \arrow[r, bend right=50, "R'"'{name=Rpn}] & B & \arrow[from=S, to=Sp, Rightarrow, "\beta"'] \arrow[from=Rn, to=Rpn, Rightarrow, "\alpha"'] \arrow[from=RR, to=RRp, Rightarrow, "\mathrm{Rift}_J(\alpha)"] \arrow[from=1-2, to=Rn, Rightarrow] \end{tikzcd}$$

for each $\beta \in \mathrm{Hom}_{\mathbf{Rel}(A,B)}(S, S')$ and each $\alpha \in \mathrm{Hom}_{\mathbf{Rel}(A,B)}(R, R')$.

4.2 The Left Skew Monoidal Unit

Let A and B be sets and let $J \colon A \nrightarrow B$ be a relation.

Definition 4.2.1.1. The **left J-skew monoidal unit of $\mathbf{Rel}(A, B)$** is the functor

$$\mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleleft_J} \colon \mathrm{pt} \to \mathbf{Rel}(A, B)$$

picking the object

$$\mathbb{1}^{\triangleleft_J}_{\mathbf{Rel}(A,B)} \overset{\mathrm{def}}{=} J$$

of $\mathbf{Rel}(A, B)$.

4.3 The Left Skew Associators

Let A and B be sets and let $J \colon A \nrightarrow B$ be a relation.

Definition 4.3.1.1. The **left J-skew associator of $\mathbf{Rel}(A, B)$** is the natural transformation

$$\alpha^{\mathbf{Rel}(A,B),\triangleleft_J} \colon \triangleleft_J \circ (\triangleleft_J \times \mathrm{id}) \Longrightarrow \triangleleft_J \circ (\mathrm{id} \times \triangleleft_J) \circ \boldsymbol{\alpha}^{\mathsf{Cats}}_{\mathbf{Rel}(A,B),\mathbf{Rel}(A,B),\mathbf{Rel}(A,B)},$$

[14] Since $\mathbf{Rel}(A, B)$ is posetal, this is to say that if $S \subset S'$ and $R \subset R'$, then $S \triangleleft_J R \subset S' \triangleleft_J R'$.

as in the diagram

$$
\begin{tikzcd}
 & \mathbf{Rel}(A,B)\times(\mathbf{Rel}(A,B)\times\mathbf{Rel}(A,B)) \arrow[dr, "\mathsf{id}\times\triangleleft_J"] & \\
(\mathbf{Rel}(A,B)\times\mathbf{Rel}(A,B))\times\mathbf{Rel}(A,B) \arrow[ur, dashed, "\alpha^{\mathsf{Cats}}_{\mathbf{Rel}(A,B),\mathbf{Rel}(A,B),\mathbf{Rel}(A,B)}", "\sim"'] \arrow[d, "\triangleleft_J\times\mathsf{id}"'] & & \mathbf{Rel}(A,B)\times\mathbf{Rel}(A,B) \arrow[d, "\triangleleft_J"] \\
\mathbf{Rel}(A,B)\times\mathbf{Rel}(A,B) \arrow[rr, "\triangleleft_J"'] \arrow[urr, phantom, "\Uparrow\alpha^{\mathbf{Rel}(A,B),\triangleleft_J}"] & & \mathbf{Rel}(A,B),
\end{tikzcd}
$$

whose component

$$\alpha^{\mathbf{Rel}(A,B),\triangleleft_J}_{T,S,R}\colon \underbrace{(T\triangleleft_J S)\triangleleft_J R}_{\overset{\text{def}}{=}T\diamond\mathrm{Rift}_J(S)\diamond\mathrm{Rift}_J(R)} \hookrightarrow \underbrace{T\triangleleft_J(S\triangleleft_J R)}_{\overset{\text{def}}{=}T\diamond\mathrm{Rift}_J(S\diamond\mathrm{Rift}_J(R))}$$

at (T,S,R) is given by

$$\alpha^{\mathbf{Rel}(A,B),\triangleleft_J}_{T,S,R}\overset{\text{def}}{=}\mathrm{id}_T\diamond\gamma,$$

where

$$\gamma\colon \mathrm{Rift}_J(S)\diamond\mathrm{Rift}_J(R)\hookrightarrow\mathrm{Rift}_J(S\diamond\mathrm{Rift}_J(R))$$

is the inclusion adjunct to the inclusion

$$\epsilon_S\star\mathrm{id}_{\mathrm{Rift}_J(R)}\colon \underbrace{J\diamond\mathrm{Rift}_J(S)\diamond\mathrm{Rift}_J(R)}_{\overset{\text{def}}{=}J_*(\mathrm{Rift}_J(S)\diamond\mathrm{Rift}_J(R))}\hookrightarrow S\diamond\mathrm{Rift}_J(R)$$

under the adjunction $J_*\dashv\mathrm{Rift}_J$, where $\epsilon\colon J\diamond\mathrm{Rift}_J\Longrightarrow\mathrm{id}_{\mathbf{Rel}(A,B)}$ is the counit of the adjunction $J_*\dashv\mathrm{Rift}_J$.

4.4 The Left Skew Left Unitors

Let A and B be sets and let $J\colon A\nrightarrow B$ be a relation.

Definition 4.4.1.1. The **left J-skew left unitor of $\mathbf{Rel}(A,B)$** is the natural transformation

$$\lambda^{\mathbf{Rel}(A,B),\triangleleft_J}\colon \triangleleft_J\circ(\mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleleft_J}\times\mathsf{id})\Longrightarrow\lambda^{\mathsf{Cats}_2}_{\mathbf{Rel}(A,B)}$$

as in the diagram

$$
\begin{tikzcd}
\mathsf{pt}\times\mathbf{Rel}(A,B) \arrow[r, "\mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleleft_J}\times\mathsf{id}"] \arrow[dr, dashed, bend right, "\lambda^{\mathsf{Cats}_2}_{\mathbf{Rel}(A,B)}"'] & \mathbf{Rel}(A,B)\times\mathbf{Rel}(A,B) \arrow[d, "\triangleleft_J"] \arrow[dl, phantom, "\Leftarrow\,\lambda^{\mathbf{Rel}(A,B),\triangleleft_J}"] \\
 & \mathbf{Rel}(A,B),
\end{tikzcd}
$$

whose component

$$\lambda^{\mathbf{Rel}(A,B),\triangleleft_J}_R\colon \underbrace{J\triangleleft_J R}_{\overset{\mathrm{def}}{=}J\diamond\mathrm{Rift}_J(R)} \hookrightarrow R$$

at R is given by

$$\lambda^{\mathbf{Rel}(A,B),\triangleleft_J}_R \overset{\mathrm{def}}{=} \epsilon_R,$$

where $\epsilon\colon J_*\diamond\mathrm{Rift}_J \Longrightarrow \mathrm{id}_{\mathbf{Rel}(A,B)}$ is the counit of the adjunction $J_*\dashv\mathrm{Rift}_J$.

4.5 The Left Skew Right Unitors

Let A and B be sets and let $J\colon A\nrightarrow B$ be a relation.

Definition 4.5.1.1. The **left J-skew right unitor of $\mathbf{Rel}(A,B)$** is the natural transformation

$$\rho^{\mathbf{Rel}(A,B),\triangleleft_J}\colon \rho^{\mathsf{Cats}_2}_{\mathbf{Rel}(A,B)} \Longrightarrow \triangleleft_J\circ(\mathsf{id}\times\mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleleft_J})$$

as in the diagram

$$
\begin{tikzcd}
\mathbf{Rel}(A,B)\times\mathsf{pt} \arrow[r, "\mathsf{id}\times\mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleleft_J}"] \arrow[dr, dashed, bend right, "\rho^{\mathsf{Cats}_2}_{\mathbf{Rel}(A,B)}"'] & \mathbf{Rel}(A,B)\times\mathbf{Rel}(A,B), \arrow[d, "\triangleleft_J"] \arrow[dl, phantom, "\Rightarrow\,\rho^{\mathbf{Rel}(A,B),\triangleleft_J}"] \\
 & \mathbf{Rel}(A,B)
\end{tikzcd}
$$

whose component

$$\rho_R^{\mathbf{Rel}(A,B),\lhd_J}\colon R \hookrightarrow \underbrace{R \lhd_J J}_{\overset{\mathrm{def}}{=} R\diamond \mathrm{Rift}_J(J)}$$

at R is given by the composition

$$\begin{aligned} R &\overset{\sim}{\Longrightarrow} R\diamond \chi_A\\ &\overset{\mathrm{id}_R\diamond\eta_{\chi_A}}{\Longrightarrow} R\diamond \mathrm{Rift}_J(J_*(\chi_A))\\ &\overset{\mathrm{def}}{=} R\diamond \mathrm{Rift}_J(J\diamond\chi_A)\\ &\overset{\sim}{\Longrightarrow} R\diamond \mathrm{Rift}_J(J)\\ &\overset{\mathrm{def}}{=} R\lhd_J J, \end{aligned}$$

where $\eta\colon \mathrm{id}_{\mathbf{Rel}(A,A)} \Longrightarrow \mathrm{Rift}_J \circ J_*$ is the unit of the adjunction $J_* \dashv \mathrm{Rift}_J$.

4.6 The Left Skew Monoidal Structure on $\mathbf{Rel}(A,B)$

Proposition 4.6.1.1. The category $\mathbf{Rel}(A,B)$ admits a left skew monoidal category structure consisting of

- *The Underlying Category.* The posetal category associated to the poset $\mathbf{Rel}(A,B)$ of relations from A to B of Item 2 of Definition 1.1.1.3.

- *The Left Skew Monoidal Product.* The left J-skew monoidal product

$$\lhd_J\colon \mathbf{Rel}(A,B)\times\mathbf{Rel}(A,B) \longrightarrow \mathbf{Rel}(A,B)$$

  of Definition 4.1.1.1.

- *The Left Skew Monoidal Unit.* The functor

$$\mathbb{1}^{\mathbf{Rel}(A,B),\lhd_J}\colon \mathrm{pt} \longrightarrow \mathbf{Rel}(A,B)$$

  of Definition 4.2.1.1.

- *The Left Skew Associators.* The natural transformation

$$\alpha^{\mathbf{Rel}(A,B),\lhd_J}\colon \lhd_J\circ(\lhd_J\times\mathrm{id}) \Longrightarrow \lhd_J\circ(\mathrm{id}\times\lhd_J)\circ\boldsymbol{\alpha}^{\mathsf{Cats}}_{\mathbf{Rel}(A,B),\mathbf{Rel}(A,B),\mathbf{Rel}(A,B)}$$

  of Definition 4.3.1.1.

- *The Left Skew Left Unitors.* The natural transformation

$$\lambda^{\mathbf{Rel}(A,B),\triangleleft_J}\colon \triangleleft_J \circ (\mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleleft_J} \times \mathrm{id}) \Longrightarrow \lambda^{\mathsf{Cats}_2}_{\mathbf{Rel}(A,B)}$$

  of Definition 4.4.1.1.

- *The Left Skew Right Unitors.* The natural transformation

$$\rho^{\mathbf{Rel}(A,B),\triangleleft_J}\colon \rho^{\mathsf{Cats}_2}_{\mathbf{Rel}(A,B)} \Longrightarrow \triangleleft_J \circ (\mathrm{id} \times \mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleleft_J})$$

  of Definition 4.5.1.1.

Proof. Since $\mathbf{Rel}(A, B)$ is posetal, the commutativity of the pentagon identity, the left skew left triangle identity, the left skew right triangle identity, the left skew middle triangle identity, and the zigzag identity is automatic, and thus $\mathbf{Rel}(A, B)$ together with the data in the statement forms a left skew monoidal category. □

5 The Right Skew Monoidal Structure on $\mathbf{Rel}(A, B)$

Let A and B be sets and let $J\colon A \nrightarrow B$ be a relation.

5.1 The Right Skew Monoidal Product

Definition 5.1.1.1. The **right J-skew monoidal product of $\mathbf{Rel}(A, B)$** is the functor

$$\triangleright_J\colon \mathbf{Rel}(A, B) \times \mathbf{Rel}(A, B) \longrightarrow \mathbf{Rel}(A, B)$$

where

- *Action on Objects.* For each $R, S \in \mathrm{Obj}(\mathbf{Rel}(A, B))$, we have

$$S \triangleright_J R \overset{\text{def}}{=} \mathrm{Ran}_J(S) \diamond R, \qquad \begin{tikzcd} A \arrow[r, "R", "\mapstochar"{anchor=center}] & B \arrow[r, dashed, "\mathrm{Ran}_J(S)", "\mapstochar"{anchor=center}] & B. \\ & A \arrow[u, "J", "\mapstochar"{anchor=center}] \arrow[ur, "S"'] \arrow[u, phantom, shift left=4ex, "\Leftarrow"] & \end{tikzcd}$$

- *Action on Morphisms.* For each $R, S, R', S' \in \mathrm{Obj}(\mathbf{Rel}(A, B))$, the action on Hom-sets

$$(\triangleright_J)_{(S,R),(S',R')}\colon \mathrm{Hom}_{\mathbf{Rel}(A,B)}(S, S') \times \mathrm{Hom}_{\mathbf{Rel}(A,B)}(R, R') \to \mathrm{Hom}_{\mathbf{Rel}(A,B)}(S \triangleright_J R, S' \triangleright_J R')$$

of $\triangleright_J$ at $((S, R), (S', R'))$ is defined by[15]

$$\beta \triangleright_J \alpha \overset{\text{def}}{=} \mathrm{Ran}_J(\beta) \diamond \alpha,$$

$$\begin{tikzcd}
A \arrow[r, bend left, "R'"] \arrow[r, bend right, "R"'] \arrow[r, phantom, "\alpha\Uparrow"] & B \arrow[rr, dashed, bend left, "\mathrm{Ran}_J(S')"] \arrow[rr, dashed, "\mathrm{Ran}_J(S)"'] \arrow[rr, phantom, bend left=15, "\mathrm{Ran}_J(\beta)\Uparrow"] & & B \\
& A \arrow[u, "J"] \arrow[urr, "S'"] \arrow[urr, bend right, "S"'] \arrow[urr, phantom, bend right=15, "\beta"] & &
\end{tikzcd}$$

for each $\beta \in \mathrm{Hom}_{\mathbf{Rel}(A,B)}(S, S')$ and each $\alpha \in \mathrm{Hom}_{\mathbf{Rel}(A,B)}(R, R')$.

5.2 The Right Skew Monoidal Unit

Definition 5.2.1.1. The **right J-skew monoidal unit of $\mathbf{Rel}(A, B)$** is the functor

$$\mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleright_J} \colon \mathsf{pt} \to \mathbf{Rel}(A, B)$$

picking the object

$$\mathbb{1}^{\triangleright_J}_{\mathbf{Rel}(A,B)} \overset{\text{def}}{=} J$$

of $\mathbf{Rel}(A, B)$.

5.3 The Right Skew Associators

Definition 5.3.1.1. The **right J-skew associator of $\mathbf{Rel}(A, B)$** is the natural transformation

$$\alpha^{\mathbf{Rel}(A,B),\triangleright_J} \colon \triangleright_J \circ (\mathsf{id} \times \triangleright_J) \Longrightarrow \triangleright_J \circ (\triangleright_J \times \mathsf{id}) \circ \boldsymbol{\alpha}^{\mathsf{Cats},-1}_{\mathbf{Rel}(A,B),\mathbf{Rel}(A,B),\mathbf{Rel}(A,B)},$$

[15] Since $\mathbf{Rel}(A, B)$ is posetal, this is to say that if $S \subset S'$ and $R \subset R'$, then $S \triangleright_J R \subset S' \triangleright_J R'$.

as in the diagram

$$\begin{tikzcd}
& (\mathbf{Rel}(A,B)\times\mathbf{Rel}(A,B))\times\mathbf{Rel}(A,B) \arrow[dr, "\triangleright_J\times\mathsf{id}"] & \\
\mathbf{Rel}(A,B)\times(\mathbf{Rel}(A,B)\times\mathbf{Rel}(A,B)) \arrow[ur, dashed, "\alpha^{\mathsf{Cats},-1}_{\mathbf{Rel}(A,B),\mathbf{Rel}(A,B),\mathbf{Rel}(A,B)}", "\sim"'] \arrow[d, "\mathsf{id}\times\triangleright_J"'] & & \mathbf{Rel}(A,B)\times\mathbf{Rel}(A,B) \arrow[d, "\triangleright_J"] \\
\mathbf{Rel}(A,B)\times\mathbf{Rel}(A,B) \arrow[rr, "\triangleright_J"'] \arrow[urr, phantom, "\Uparrow\alpha^{\mathbf{Rel}(A,B),\triangleright_J}"] & & \mathbf{Rel}(A,B),
\end{tikzcd}$$

whose component

$$\alpha^{\mathbf{Rel}(A,B),\triangleright_J}_{T,S,R}\colon \underbrace{T\triangleright_J(S\triangleright_J R)}_{\overset{\text{def}}{=}\mathrm{Ran}_J(T)\diamond\mathrm{Ran}_J(S)\diamond R} \hookrightarrow \underbrace{(T\triangleright_J S)\triangleright_J R}_{\overset{\text{def}}{=}\mathrm{Ran}_J(\mathrm{Ran}_J(T)\diamond S)\diamond R}$$

at (T,S,R) is given by

$$\alpha^{\mathbf{Rel}(A,B),\triangleright}_{T,S,R}\overset{\text{def}}{=}\gamma\diamond\mathrm{id}_R,$$

where

$$\gamma\colon \mathrm{Ran}_J(T)\diamond\mathrm{Ran}_J(S)\hookrightarrow\mathrm{Ran}_J(\mathrm{Ran}_J(T)\diamond S)$$

is the inclusion adjunct to the inclusion

$$\mathrm{id}_{\mathrm{Ran}_J(T)}\diamond\epsilon_S\colon \underbrace{\mathrm{Ran}_J(T)\diamond\mathrm{Ran}_J(S)\diamond J}_{\overset{\text{def}}{=}J^*(\mathrm{Ran}_J(T)\diamond\mathrm{Ran}_J(S))}\hookrightarrow\mathrm{Ran}_J(T)\diamond S$$

under the adjunction $J^*\dashv\mathrm{Ran}_J$, where $\epsilon\colon\mathrm{Ran}_J\diamond J\Longrightarrow\mathrm{id}_{\mathbf{Rel}(A,B)}$ is the counit of the adjunction $J^*\dashv\mathrm{Ran}_J$.

5.4 The Right Skew Left Unitors

Definition 5.4.1.1. The **right J-skew left unitor of $\mathbf{Rel}(A,B)$** is the natural transformation

$$\lambda^{\mathbf{Rel}(A,B),\triangleright_J}\colon\lambda^{\mathsf{Cats}_2}_{\mathbf{Rel}(A,B)}\Longrightarrow\triangleright_J\circ(\mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleright}\times\mathsf{id}),$$

as in the diagram

$$
\begin{tikzcd}
\mathsf{pt} \times \mathbf{Rel}(A,B) \arrow[r, "\mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleright_J} \times \mathsf{id}"] \arrow[dr, dashed, bend right, "\lambda^{\mathsf{Cats}_2}_{\mathbf{Rel}(A,B)}"'] \arrow[r, phantom, shift right=6ex, "\Nearrow \lambda^{\mathbf{Rel}(A,B),\triangleright_J}"] & \mathbf{Rel}(A,B) \times \mathbf{Rel}(A,B) \arrow[d, "\triangleright_J"] \\
 & \mathbf{Rel}(A,B),
\end{tikzcd}
$$

whose component

$$\lambda^{\mathbf{Rel}(A,B),\triangleright_J}_R \colon R \hookrightarrow \underbrace{J \triangleright_J R}_{\overset{\text{def}}{=} \mathrm{Ran}_J(J) \diamond R}$$

at R is given by the composition

$$
\begin{aligned}
R & \overset{\sim}{\Longrightarrow} \chi_B \diamond R \\
& \overset{\eta_{\chi_B}}{\Longrightarrow} \diamond \, \mathrm{id}_R \, \mathrm{Ran}_J(J^*(\chi_A)) \diamond R \\
& \overset{\text{def}}{=} \mathrm{Ran}_J(J^* \diamond \chi_A) \diamond R \\
& \overset{\sim}{\Longrightarrow} \mathrm{Ran}_J(J) \diamond R \\
& \overset{\text{def}}{=} R \triangleright_J J,
\end{aligned}
$$

where $\eta \colon \mathrm{id}_{\mathbf{Rel}(B,B)} \Longrightarrow \mathrm{Ran}_J \circ J^*$ is the unit of the adjunction $J^* \dashv \mathrm{Ran}_J$.

5.5 The Right Skew Right Unitors

Definition 5.5.1.1. The **right J-skew right unitor of $\mathbf{Rel}(A,B)$** is the natural transformation

$$\rho^{\mathbf{Rel}(A,B),\triangleright_J} \colon \triangleright_J \circ (\mathsf{id} \times \mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleright}) \Longrightarrow \rho^{\mathsf{Cats}_2}_{\mathbf{Rel}(A,B)},$$

as in the diagram

$$
\begin{tikzcd}
\mathbf{Rel}(A,B)\times\mathsf{pt} \arrow[r, "\mathrm{id}\times\mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleright_J}"] \arrow[dr, dashed, bend right, "\rho^{\mathsf{Cats}_2}_{\mathbf{Rel}(A,B)}"'] \arrow[dr, phantom, "\Swarrow\rho^{\mathbf{Rel}(A,B),\triangleright_J}"] & \mathbf{Rel}(A,B)\times\mathbf{Rel}(A,B), \arrow[d, "\triangleright_J"] \\
& \mathbf{Rel}(A,B)
\end{tikzcd}
$$

whose component

$$\rho^{\mathbf{Rel}(A,B),\triangleright_J}_S\colon \underbrace{S\triangleright_J J}_{\overset{\text{def}}{=}\mathrm{Ran}_J(S)\diamond J} \hookrightarrow S$$

at S is given by

$$\rho^{\mathbf{Rel}(A,B),\triangleright_J}_S \overset{\text{def}}{=} \epsilon_R,$$

where $\epsilon\colon J^*\circ\mathrm{Ran}_J \Longrightarrow \mathrm{id}_{\mathbf{Rel}(A,B)}$ is the counit of the adjunction $J^*\dashv\mathrm{Ran}_J$.

5.6 The Right Skew Monoidal Structure on $\mathbf{Rel}(A, B)$

Proposition 5.6.1.1. The category $\mathbf{Rel}(A, B)$ admits a right skew monoidal category structure consisting of

- *The Underlying Category.* The posetal category associated to the poset $\mathbf{Rel}(A, B)$ of relations from A to B of Item 2 of Definition 1.1.1.3.
- *The Right Skew Monoidal Product.* The right J-skew monoidal product

  $$\triangleleft_J\colon \mathbf{Rel}(A,B)\times\mathbf{Rel}(A,B)\longrightarrow\mathbf{Rel}(A,B)$$

  of Definition 5.1.1.1.
- *The Right Skew Monoidal Unit.* The functor

  $$\mathbb{1}^{\mathbf{Rel}(A,B),\triangleleft_J}\colon \mathsf{pt}\longrightarrow\mathbf{Rel}(A,B)$$

  of Definition 5.2.1.1.

- *The Right Skew Associators.* The natural transformation

$$\alpha^{\mathbf{Rel}(A,B),\triangleright_J}\colon \triangleright_J \circ (\mathrm{id} \times \triangleright_J) \Longrightarrow \triangleright_J \circ (\triangleright_J \times \mathrm{id}) \circ \boldsymbol{\alpha}^{\mathsf{Cats},-1}_{\mathbf{Rel}(A,B),\mathbf{Rel}(A,B),\mathbf{Rel}(A,B)}$$

  of Definition 5.3.1.1.

- *The Right Skew Left Unitors.* The natural transformation

$$\lambda^{\mathbf{Rel}(A,B),\triangleright_J}\colon \boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathbf{Rel}(A,B)} \Longrightarrow \triangleright_J \circ (\mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleright} \times \mathrm{id})$$

  of Definition 5.4.1.1.

- *The Right Skew Right Unitors.* The natural transformation

$$\rho^{\mathbf{Rel}(A,B),\triangleright_J}\colon \triangleright_J \circ (\mathrm{id} \times \mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleright}) \Longrightarrow \boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathbf{Rel}(A,B)}$$

  of Definition 5.5.1.1.

Proof. Since $\mathbf{Rel}(A, B)$ is posetal, the commutativity of the pentagon identity, the right skew left triangle identity, the right skew right triangle identity, the right skew middle triangle identity, and the zigzag identity is automatic, and thus $\mathbf{Rel}(A, B)$ together with the data in the statement forms a right skew monoidal category. □

Appendices

A Other Chapters

References

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