Constructions With Relations

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This chapter contains some material about constructions with relations. Notably, we discuss and explore:

1. The existence or non-existence of Kan extensions and Kan lifts in the 2-category **Rel** (Section 2).

2. The various kinds of constructions involving relations, such as graphs, domains, ranges, unions, intersections, products, inverse relations, composition of relations, and collages (Section 3).

3. The adjoint pairs

   $$\begin{aligned} R_* \dashv R_{-1} &: \mathcal{P}(A) \rightleftarrows \mathcal{P}(B), \\ R^{-1} \dashv R_! &: \mathcal{P}(B) \rightleftarrows \mathcal{P}(A) \end{aligned}$$

   of functors (morphisms of posets) between $\mathcal{P}(A)$ and $\mathcal{P}(B)$ induced by a relation $R\colon A \nrightarrow B$, as well as the properties of R_*, R_{-1}, R^{-1}, and $R_!$ (Section 4).

   Of particular note are the following points:

   (a) These two pairs of adjoint functors are the counterpart for relations of the adjoint triple $f_* \dashv f^{-1} \dashv f_!$ induced by a function $f\colon A \to B$ studied in Constructions With Sets, Section 4.

   (b) We have $R_{-1} = R^{-1}$ iff R is total and functional (Item 8 of Proposition 4.2.1.3).

   (c) As a consequence of the previous item, when R comes from a function f, the pair of adjunctions

   $$R_* \dashv R_{-1} = R^{-1} \dashv R_!$$

reduces to the triple adjunction

$$f_* \dashv f^{-1} \dashv f_!$$

from Constructions With Sets, Section 4.

(d) The pairs $R_* \dashv R_{-1}$ and $R^{-1} \dashv R_!$ turn out to be rather important later on, as they appear in the definition and study of continuous, open, and closed relations between topological spaces (**??**, **??**).

Contents

1 Co/Limits in the Category of Relations

This section is currently just a stub, and will be properly developed later on.

2 Kan Extensions and Kan Lifts in the 2-Category of Relations

2.1 Left Kan Extensions in Rel

Proposition 2.1.1.1. Let $R\colon A \nrightarrow B$ be a relation.

1. *Non-Existence of All Left Kan Extensions in* **Rel**. Not all relations in **Rel** admit left Kan extensions.

2. *Characterisation of Relations Admitting Left Kan Extensions Along Them.* The following conditions are equivalent:

   (a) The left Kan extension

   $$\mathrm{Lan}_R\colon \mathbf{Rel}(A, X) \to \mathbf{Rel}(B, X)$$

   along R exists.

   (b) The relation R admits a left adjoint in **Rel**.

   (c) The relation R is of the form f^{-1} (as in Definition 3.2.1.1) for some function f.

Proof. *Item 1, Non-Existence of All Left Kan Extensions in* **Rel**: Omitted, but will eventually follow Fosco Loregian's comment on [MO 460656].
Item 2, Characterisation of Relations Admitting Left Kan Extensions Along Them: Omitted, but will eventually follow Tim Campion's answer to to [MO 460656]. □

Question 2.1.1.2. Given relations $S\colon A \nrightarrow X$ and $R\colon A \nrightarrow B$, is there a characterisation of when the left Kan extension

$$\mathrm{Lan}_S(R)\colon B \nrightarrow X$$

exists in terms of properties of R and S?
This question also appears as [MO 461592].

Question 2.1.1.3. As shown in Item 2 of Proposition 2.1.1.1, the left Kan extension

$$\mathrm{Lan}_R\colon \mathbf{Rel}(A,X) \to \mathbf{Rel}(B,X)$$

along a relation of the form $R = f^{-1}$ exists. Is there a explicit description of it, similarly to the explicit description of right Kan extensions given in Proposition 2.3.1.1?
This question also appears as [MO 461592].

2.2 Left Kan Lifts in **Rel**

Proposition 2.2.1.1. Let $R\colon A \nrightarrow B$ be a relation.

1. *Non-Existence of All Left Kan Lifts in* **Rel**. Not all relations in **Rel** admit left Kan lifts.
2. *Characterisation of Relations Admitting Left Kan Lifts Along Them.* The following conditions are equivalent:
   (a) The left Kan lift
   $$\mathrm{Lift}_R\colon \mathbf{Rel}(X,B) \to \mathbf{Rel}(X,A)$$
   along R exists.
   (b) The relation R admits a right adjoint in **Rel**.
   (c) The relation R is of the form $\mathrm{Gr}(f)$ (as in Definition 3.1.1.1) for some function f.

Proof. *Item 1, Non-Existence of All Left Kan Lifts in* **Rel**: Omitted, but will eventually follow (the dual of) Fosco Loregian's comment on [MO 460656].
Item 2, Characterisation of Relations Admitting Left Kan Lifts Along Them: Omitted, but will eventually follow Tim Campion's answer to to [MO 460656]. □

Question 2.2.1.2. Given relations $S\colon A \nrightarrow X$ and $R\colon A \nrightarrow B$, is there a characterisation of when the left Kan lift

$$\mathrm{Lift}_S(R)\colon X \nrightarrow A$$

exists in terms of properties of R and S?
This question also appears as [MO 461592].

Question 2.2.1.3. As shown in Item 2 of Proposition 2.2.1.1, the left Kan lift

$$\mathrm{Lift}_R\colon \mathbf{Rel}(X,B) \to \mathbf{Rel}(X,A)$$

along a relation of the form $R = \mathrm{Gr}(f)$ exists. Is there a explicit description of it, similarly to the explicit description of right Kan lifts given in Proposition 2.4.1.1?
This question also appears as [MO 461592].

2.3 Right Kan Extensions in **Rel**

Let $R\colon A \nrightarrow B$ be a relation.

Proposition 2.3.1.1. The right Kan extension

$$\mathrm{Ran}_R\colon \mathrm{Rel}(A,X) \to \mathrm{Rel}(B,X)$$

along R in **Rel** exists and is given by

$$\mathrm{Ran}_R(S) \overset{\mathrm{def}}{=} \int_{a\in A} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R_a^{-2}, S_a^{-1})$$

for each $S \in \mathrm{Rel}(A,X)$, so that the following conditions are equivalent:

1. We have $b \sim_{\mathrm{Ran}_R(S)} x$.
2. For each $a \in A$, if $a \sim_R b$, then $a \sim_S x$.

Proof. We have

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{Rel}(A,X)}(S \diamond R, T) &\cong \int_{a\in A}\int_{x\in X} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}((S\diamond R)^x_a, T^x_a)\\
&\cong \int_{a\in A}\int_{x\in X} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}((\int^{b\in B} S^x_b \times R^b_a), T^x_a)\\
&\cong \int_{a\in A}\int_{x\in X}\int_{b\in B} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(S^x_b \times R^b_a, T^x_a)\\
&\cong \int_{a\in A}\int_{x\in X}\int_{b\in B} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(S^x_b, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R^b_a, T^x_a))\\
&\cong \int_{b\in B}\int_{x\in X}\int_{a\in A} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(S^x_b, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R^b_a, T^x_a))\\
&\cong \int_{b\in B}\int_{x\in X} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(S^x_b, \int_{a\in A} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R^b_a, T^x_a))\\
&\cong \mathrm{Hom}_{\mathbf{Rel}(B,X)}(S, \int_{a\in A} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R_a^{-2}, T_a^{-1}))
\end{aligned}$$

naturally in each $S \in \mathbf{Rel}(B, X)$ and each $T \in \mathbf{Rel}(A, X)$, showing that

$$\int_{a \in A} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R^{-_2}_{a}, T^{-_1}_{a})$$

is right adjoint to the precomposition functor $- \diamond R$, being thus the right Kan extension along R. Here we have used the following results, respectively (i.e. for each $\cong$ sign):

1. Relations, Item 1 of Proposition 1.1.1.5.
2. Definition 3.12.1.1.
3. **??**, **??** of **??**.
4. Sets, Proposition 2.2.1.5.
5. **??**, **??** of **??**.
6. **??**, **??** of **??**.
7. Relations, Item 1 of Proposition 1.1.1.5.

This finishes the proof. □

2.4 Right Kan Lifts in **Rel**

Let $R \colon A \nrightarrow B$ be a relation.

Proposition 2.4.1.1. The right Kan lift

$$\mathrm{Rift}_R \colon \mathrm{Rel}(X, B) \to \mathrm{Rel}(X, A)$$

along R in **Rel** exists and is given by

$$\mathrm{Rift}_R(S) \stackrel{\text{def}}{=} \int_{b \in B} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R^{b}_{-_1}, S^{b}_{-_2})$$

for each $S \in \mathrm{Rel}(X, B)$, so that the following conditions are equivalent:

1. We have $x \sim_{\mathrm{Rift}_R(S)} a$.
2. For each $b \in B$, if $a \sim_R b$, then $x \sim_S b$.

Proof. We have

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{Rel}(X,B)}(R \diamond S, T) &\cong \int_{x \in X} \int_{b \in B} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}((R \diamond S)^b_x, T^b_x) \\
&\cong \int_{x \in X} \int_{b \in B} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}((\int^{a \in A} R^b_a \times S^a_x), T^b_x) \\
&\cong \int_{x \in X} \int_{b \in B} \int_{a \in A} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R^b_a \times S^a_x, T^b_x) \\
&\cong \int_{x \in X} \int_{b \in B} \int_{a \in A} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(S^a_x, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R^b_a, T^b_x)) \\
&\cong \int_{x \in X} \int_{a \in A} \int_{b \in B} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(S^a_x, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R^b_a, T^b_x)) \\
&\cong \int_{x \in X} \int_{a \in A} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(S^a_x, \int_{b \in B} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R^b_a, T^b_x)) \\
&\cong \mathrm{Hom}_{\mathbf{Rel}(X,A)}(S, \int_{b \in B} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R^b_{-_1}, T^b_{-_2}))
\end{aligned}$$

naturally in each $S \in \mathbf{Rel}(X, A)$ and each $T \in \mathbf{Rel}(X, B)$, showing that

$$\int_{b \in B} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R^b_{-_1}, S^b_{-_2})$$

is right adjoint to the postcomposition functor $R \diamond -$, being thus the right Kan lift along R. Here we have used the following results, respectively (i.e. for each $\cong$ sign):

1. Relations, Item 1 of Proposition 1.1.1.5.
2. Definition 3.12.1.1.
3. **??**, **??** of **??**.
4. Sets, Proposition 2.2.1.5.
5. **??**, **??** of **??**.
6. **??**, **??** of **??**.
7. Relations, Item 1 of Proposition 1.1.1.5.

This finishes the proof. □

3 More Constructions With Relations

3.1 The Graph of a Function

Let $f \colon A \to B$ be a function.

Definition 3.1.1.1. The **graph of** f is the relation $\mathrm{Gr}(f) \colon A \nrightarrow B$ defined as follows:[1]

- Viewing relations from A to B as subsets of $A \times B$, we define

$$\mathrm{Gr}(f) \overset{\text{def}}{=} \{(a, f(a)) \in A \times B \mid a \in A\}.$$

- Viewing relations from A to B as functions $A \times B \to \{\mathsf{true}, \mathsf{false}\}$, we define

$$[\mathrm{Gr}(f)](a, b) \overset{\text{def}}{=} \begin{cases} \mathsf{true} & \text{if } b = f(a), \\ \mathsf{false} & \text{otherwise} \end{cases}$$

  for each $(a, b) \in A \times B$.

- Viewing relations from A to B as functions $A \to \mathcal{P}(B)$, we define

$$[\mathrm{Gr}(f)](a) \overset{\text{def}}{=} \{f(a)\}$$

  for each $a \in A$, i.e. we define $\mathrm{Gr}(f)$ as the composition

$$A \xrightarrow{f} B \xrightarrow{\chi_B} \mathcal{P}(B).$$

Proposition 3.1.1.2. Let $f \colon A \to B$ be a function.

1. *Functoriality.* The assignment $A \mapsto \mathrm{Gr}(A)$ defines a functor

$$\mathrm{Gr} \colon \mathsf{Sets} \to \mathrm{Rel}$$

   where

   - *Action on Objects.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$\mathrm{Gr}(A) \overset{\text{def}}{=} A.$$

   - *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathsf{Sets})$, the action on

[1] *Further Notation:* We write $\mathrm{Gr}(A)$ for $\mathrm{Gr}(\mathrm{id}_A)$, and call it the **graph** of A.

Hom-sets

$$\mathrm{Gr}_{A,B}\colon \mathsf{Sets}(A,B) \to \underbrace{\mathrm{Rel}(\mathrm{Gr}(A),\mathrm{Gr}(B))}_{\overset{\mathrm{def}}{=}\mathrm{Rel}(A,B)}$$

of Gr at (A,B) is defined by

$$\mathrm{Gr}_{A,B}(f) \overset{\mathrm{def}}{=} \mathrm{Gr}(f),$$

where $\mathrm{Gr}(f)$ is the graph of f as in Definition 3.1.1.1.

In particular:

- *Preservation of Identities.* We have

$$\mathrm{Gr}(\mathrm{id}_A) = \chi_A$$

for each $A \in \mathrm{Obj}(\mathsf{Sets})$.

- *Preservation of Composition.* We have

$$\mathrm{Gr}(g \circ f) = \mathrm{Gr}(g) \diamond \mathrm{Gr}(f)$$

for each pair of functions $f\colon A \to B$ and $g\colon B \to C$.

2. *Adjointness Inside* **Rel**. We have an adjunction

$$\Big(\mathrm{Gr}(f) \dashv f^{-1}\Big)\colon \quad \begin{tikzcd} A \arrow[r, bend left, "\mathrm{Gr}(f)", "\shortmid"{marking}] \arrow[r, phantom, "\perp"] & B \arrow[l, bend left, "f^{-1}", "\shortmid"{marking}] \end{tikzcd}$$

in **Rel**, where f^{-1} is the inverse of f of Definition 3.2.1.1.

3. *Adjointness.* We have an adjunction

$$(\mathrm{Gr} \dashv \mathcal{P}_*)\colon \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "\mathrm{Gr}"] \arrow[r, phantom, "\perp"] & \mathrm{Rel}, \arrow[l, bend left, "\mathcal{P}_*"] \end{tikzcd}$$

witnessed by a bijection of sets

$$\mathrm{Rel}(\mathrm{Gr}(A),B) \cong \mathsf{Sets}(A,\mathcal{P}(B))$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$ and $B \in \mathrm{Obj}(\mathrm{Rel})$.

4. *Interaction With Inverses.* We have

$$\begin{aligned}\mathrm{Gr}(f)^\dagger &= f^{-1},\\ (f^{-1})^\dagger &= \mathrm{Gr}(f).\end{aligned}$$

5. *Cocontinuity.* The functor $\mathrm{Gr}\colon \mathsf{Sets} \to \mathrm{Rel}$ of Item 1 preserves colimits.

6. *Characterisations.* Let $R\colon A \nrightarrow B$ be a relation. The following conditions are equivalent:

   (a) There exists a function $f\colon A \to B$ such that $R = \mathrm{Gr}(f)$.

   (b) The relation R is total and functional.

   (c) The weak and strong inverse images of R agree, i.e. we have $R^{-1} = R_{-1}$.

   (d) The relation R has a right adjoint $R^\dagger$ in Rel.

Proof. *Item 1, Functoriality*: Clear.
Item 2, Adjointness Inside **Rel**: We need to check that there are inclusions

$$\begin{aligned}\chi_A &\subset f^{-1} \diamond \mathrm{Gr}(f),\\ \mathrm{Gr}(f) \diamond f^{-1} &\subset \chi_B.\end{aligned}$$

These correspond respectively to the following conditions:

1. For each $a \in A$, there exists some $b \in B$ such that $a \sim_{\mathrm{Gr}(f)} b$ and $b \sim_{f^{-1}} a$.

2. For each $a, b \in A$, if $a \sim_{\mathrm{Gr}(f)} b$ and $b \sim_{f^{-1}} a$, then $a = b$.

In other words, the first condition states that the image of any $a \in A$ by f is nonempty, whereas the second condition states that f is not multivalued. As f is a function, both of these statements are true, and we are done.
Item 3, Adjointness: The stated bijection follows from Relations, Remark 1.1.1.4, with naturality being clear.
Item 4, Interaction With Inverses: Clear.
Item 5, Cocontinuity: Omitted.
Item 6, Characterisations: We claim that Items 6a to 6d are indeed equivalent:

- *Item 6a* $\Longleftrightarrow$ *Item 6b.* This is shown in the proof of **??** of **??**.

- *Item 6b* $\implies$ *Item 6c*. If R is total and functional, then, for each $a \in A$, the set $R(a)$ is a singleton, implying that

$$\begin{aligned} R^{-1}(V) &\stackrel{\text{def}}{=} \{a \in A \mid R(a) \cap V \neq \emptyset\}, \\ R_{-1}(V) &\stackrel{\text{def}}{=} \{a \in A \mid R(a) \subset V\} \end{aligned}$$

  are equal for all $V \in \mathcal{P}(B)$, as the conditions $R(a) \cap V \neq \emptyset$ and $R(a) \subset V$ are equivalent when $R(a)$ is a singleton.

- *Item 6c* $\implies$ *Item 6b*. We claim that R is indeed total and functional:

  - *Totality.* If we had $R(a) = \emptyset$ for some $a \in A$, then we would have $a \in R_{-1}(\emptyset)$, so that $R_{-1}(\emptyset) \neq \emptyset$. But since $R^{-1}(\emptyset) = \emptyset$, this would imply $R_{-1}(\emptyset) \neq R^{-1}(\emptyset)$, a contradiction. Thus $R(a) \neq \emptyset$ for all $a \in A$ and R is total.
  - *Functionality.* If $R^{-1} = R_{-1}$, then we have

$$\begin{aligned} \{a\} &= R^{-1}(\{b\}) \\ &= R_{-1}(\{b\}) \end{aligned}$$

    for each $b \in R(a)$ and each $a \in A$, and thus $R(a) \subset \{b\}$. But since R is total, we must have $R(a) = \{b\}$, and thus we see that R is functional.

- *Item 6a* $\iff$ *Item 6d*. This follows from Relations, Proposition 3.3.1.1.

This finishes the proof. □

3.2 The Inverse of a Function

Let $f \colon A \to B$ be a function.

Definition 3.2.1.1. The **inverse of** f is the relation $f^{-1} \colon B \nrightarrow A$ defined as follows:

- Viewing relations from B to A as subsets of $B \times A$, we define

$$f^{-1} \stackrel{\text{def}}{=} \left\{ (b, f^{-1}(b)) \in B \times A \,\middle|\, a \in A \right\},$$

  where

$$f^{-1}(b) \stackrel{\text{def}}{=} \{a \in A \mid f(a) = b\}$$

  for each $b \in B$.

- Viewing relations from B to A as functions $B \times A \to \{\mathsf{true}, \mathsf{false}\}$, we define

$$f^{-1}(b,a) \stackrel{\text{def}}{=} \begin{cases} \mathsf{true} & \text{if there exists } a \in A \text{ with } f(a) = b, \\ \mathsf{false} & \text{otherwise} \end{cases}$$

  for each $(b,a) \in B \times A$.

- Viewing relations from B to A as functions $B \to \mathcal{P}(A)$, we define

$$f^{-1}(b) \stackrel{\text{def}}{=} \{a \in A \mid f(a) = b\}$$

  for each $b \in B$.

Proposition 3.2.1.2. Let $f\colon A \to B$ be a function.

1. *Functoriality.* The assignment $A \mapsto A$, $f \mapsto f^{-1}$ defines a functor

$$(-)^{-1}\colon \mathsf{Sets} \to \mathrm{Rel}$$

   where

   - *Action on Objects.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$\left[(-)^{-1}\right](A) \stackrel{\text{def}}{=} A.$$

   - *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathsf{Sets})$, the action on Hom-sets

$$(-)^{-1}_{A,B}\colon \mathsf{Sets}(A,B) \to \mathrm{Rel}(A,B)$$

     of $(-)^{-1}$ at (A,B) is defined by

$$(-)^{-1}_{A,B}(f) \stackrel{\text{def}}{=} \left[(-)^{-1}\right](f),$$

     where f^{-1} is the inverse of f as in Definition 3.2.1.1.

   In particular:

   - *Preservation of Identities.* We have

$$\mathrm{id}_A^{-1} = \chi_A$$

     for each $A \in \mathrm{Obj}(\mathsf{Sets})$.

- *Preservation of Composition.* We have

$$(g \circ f)^{-1} = g^{-1} \diamond f^{-1}$$

for pair of functions $f : A \to B$ and $g : B \to C$.

2. *Adjointness Inside* **Rel**. We have an adjunction

$$\Big(\mathrm{Gr}(f) \dashv f^{-1}\Big) \colon \quad \begin{tikzcd} A \arrow[r, bend left, "\mathrm{Gr}(f)", "\shortmid"{marking}] \arrow[r, phantom, "\perp"] & B \arrow[l, bend left, "f^{-1}", "\shortmid"{marking}] \end{tikzcd}$$

in **Rel**.

3. *Interaction With Inverses of Relations.* We have

$$\begin{aligned} (f^{-1})^{\dagger} &= \mathrm{Gr}(f), \\ \mathrm{Gr}(f)^{\dagger} &= f^{-1}. \end{aligned}$$

Proof. *Item 1, Functoriality*: Clear.
Item 2, Adjointness Inside **Rel**: This is proved in Item 2 of Proposition 3.1.1.2.
Item 3, Interaction With Inverses of Relations: Clear. □

3.3 Representable Relations

Let A and B be sets.

Definition 3.3.1.1. Let $f : A \to B$ and $g : B \to A$ be functions.[2]

1. The **representable relation associated to** f is the relation $\chi_f : A \nrightarrow B$ defined as the composition

$$A \times B \xrightarrow{f \times \mathrm{id}_B} B \times B \xrightarrow{\chi_B} \{\mathsf{true}, \mathsf{false}\},$$

i.e. given by declaring $a \sim_{\chi_f} b$ iff $f(a) = b$.

[2]More generally, given functions

$$\begin{aligned} f &: A \to C, \\ g &: B \to D \end{aligned}$$

and a relation $B \nrightarrow D$, we may consider the composite relation

$$A \times B \xrightarrow{f \times g} C \times D \xrightarrow{R} \{\mathsf{true}, \mathsf{false}\},$$

2. The **corepresentable relation associated to** g is the relation $\chi^g \colon B \nrightarrow A$ defined as the composition

$$B \times A \xrightarrow{g\times \mathrm{id}_A} A \times A \xrightarrow{\chi_A} \{\mathsf{true}, \mathsf{false}\},$$

i.e. given by declaring $b \sim_{\chi^g} a$ iff $g(b) = a$.

3.4 The Domain and Range of a Relation

Let A and B be sets.

Definition 3.4.1.1. Let $R \subset A \times B$ be a relation.[3,4]

1. The **domain of** R is the subset $\mathrm{dom}(R)$ of A defined by

$$\mathrm{dom}(R) \overset{\mathrm{def}}{=} \left\{ a \in A \;\middle|\; \begin{array}{l} \text{there exists some } b \in B \\ \text{such that } a \sim_R b \end{array} \right\}.$$

2. The **range of** R is the subset $\mathrm{range}(R)$ of B defined by

$$\mathrm{range}(R) \overset{\mathrm{def}}{=} \left\{ b \in B \;\middle|\; \begin{array}{l} \text{there exists some } a \in A \\ \text{such that } a \sim_R b \end{array} \right\}.$$

for which we have $a \sim_{R\circ(f\times g)} b$ iff $f(a) \sim_R g(b)$.

[3]Following **??**, **??**, we may compute the (characteristic functions associated to the) domain and range of a relation using the following colimit formulas:

$$\begin{aligned} \chi_{\mathrm{dom}(R)}(a) &\cong \operatorname*{colim}_{b\in B}(R^b_a) \qquad (a \in A) \\ &\cong \bigvee_{b\in B} R^b_a, \\ \chi_{\mathrm{range}(R)}(b) &\cong \operatorname*{colim}_{a\in A}(R^b_a) \qquad (b \in B) \\ &\cong \bigvee_{a\in A} R^b_a, \end{aligned}$$

where the join $\bigvee$ is taken in the poset $(\{\mathsf{true}, \mathsf{false}\}, \preceq)$ of Constructions With Sets, Definition 2.2.1.3.

[4]Viewing R as a function $R \colon A \to \mathcal{P}(B)$, we have

$$\begin{aligned} \mathrm{dom}(R) &\cong \operatorname*{colim}_{y\in Y}(R(y)) \\ &\cong \bigcup_{y\in Y} R(y), \\ \mathrm{range}(R) &\cong \operatorname*{colim}_{x\in X}(R(x)) \\ &\cong \bigcup_{x\in X} R(x), \end{aligned}$$

3.5 Binary Unions of Relations

Let A and B be sets and let R and S be relations from A to B.

Definition 3.5.1.1. The **union of** R **and** S[5] is the relation $R \cup S$ from A to B defined as follows:

- Viewing relations from A to B as subsets of $A \times B$, we define[6]

$$R \cup S \stackrel{\text{def}}{=} \{(a, b) \in B \times A \mid \text{we have } a \sim_R b \text{ or } a \sim_S b\}.$$

- Viewing relations from A to B as functions $A \to \mathcal{P}(B)$, we define

$$[R \cup S](a) \stackrel{\text{def}}{=} R(a) \cup S(a)$$

  for each $a \in A$.

Proposition 3.5.1.2. Let R, S, R_1, and R_2 be relations from A to B, and let S_1 and S_2 be relations from B to C.

1. *Interaction With Inverses.* We have

$$(R \cup S)^\dagger = R^\dagger \cup S^\dagger.$$

2. *Interaction With Composition.* We have

$$(S_1 \diamond R_1) \cup (S_2 \diamond R_2) \stackrel{\text{poss.}}{\neq} (S_1 \cup S_2) \diamond (R_1 \cup R_2).$$

Proof. *Item 1*, *Interaction With Inverses*: Clear.
Item 2, *Interaction With Composition*: Unwinding the definitions, we see that:

1. The condition for $(S_1 \diamond R_1) \cup (S_2 \diamond R_2)$ is:
   (a) There exists some $b \in B$ such that:
      i. $a \sim_{R_1} b$ and $b \sim_{S_1} c$;

   or

      i. $a \sim_{R_2} b$ and $b \sim_{S_2} c$;
3. The condition for $(S_1 \cup S_2) \diamond (R_1 \cup R_2)$ is:

[5] *Further Terminology:* Also called the **binary union of** R **and** S, for emphasis.
[6] This is the same as the union of R and S as subsets of $A \times B$.

(a) There exists some $b \in B$ such that:

i. $a \sim_{R_1} b$ or $a \sim_{R_2} b$;

and

i. $b \sim_{S_1} c$ or $b \sim_{S_2} c$.

These two conditions may fail to agree (counterexample omitted), and thus the two resulting relations on $A \times C$ may differ. □

3.6 Unions of Families of Relations

Let A and B be sets and let $\{R_i\}_{i \in I}$ be a family of relations from A to B.

Definition 3.6.1.1. The **union of the family** $\{R_i\}_{i \in I}$ is the relation $\bigcup_{i \in I} R_i$ from A to B defined as follows:

- Viewing relations from A to B as subsets of $A \times B$, we define[7]

$$\bigcup_{i \in I} R_i \overset{\text{def}}{=} \left\{ (a, b) \in (A \times B)^{\times I} \;\middle|\; \begin{array}{l} \text{there exists some } i \in I \\ \text{such that } a \sim_{R_i} b \end{array} \right\}.$$

- Viewing relations from A to B as functions $A \to \mathcal{P}(B)$, we define

$$\left[\bigcup_{i \in I} R_i\right](a) \overset{\text{def}}{=} \bigcup_{i \in I} R_i(a)$$

  for each $a \in A$.

Proposition 3.6.1.2. Let A and B be sets and let $\{R_i\}_{i \in I}$ be a family of relations from A to B.

1. *Interaction With Inverses.* We have

$$(\bigcup_{i \in I} R_i)^{\dagger} = \bigcup_{i \in I} R_i^{\dagger}.$$

Proof. Item 1, *Interaction With Inverses*: Clear. □

3.7 Binary Intersections of Relations

Let A and B be sets and let R and S be relations from A to B.

[7] This is the same as the union of $\{R_i\}_{i \in I}$ as a collection of subsets of $A \times B$.

Definition 3.7.1.1. The **intersection of** R **and** S[8] is the relation $R \cap S$ from A to B defined as follows:

- Viewing relations from A to B as subsets of $A \times B$, we define[9]

  $$R \cap S \stackrel{\text{def}}{=} \{(a, b) \in B \times A \mid \text{we have } a \sim_R b \text{ and } a \sim_S b\}.$$

- Viewing relations from A to B as functions $A \to \mathcal{P}(B)$, we define

  $$[R \cap S](a) \stackrel{\text{def}}{=} R(a) \cap S(a)$$

  for each $a \in A$.

Proposition 3.7.1.2. Let R, S, R_1, and R_2 be relations from A to B, and let S_1 and S_2 be relations from B to C.

1. *Interaction With Inverses.* We have

   $$(R \cap S)^\dagger = R^\dagger \cap S^\dagger.$$

2. *Interaction With Composition.* We have

   $$(S_1 \diamond R_1) \cap (S_2 \diamond R_2) = (S_1 \cap S_2) \diamond (R_1 \cap R_2).$$

Proof. *Item 1*, *Interaction With Inverses*: Clear.
Item 2, *Interaction With Composition*: Unwinding the definitions, we see that:

1. The condition for $(S_1 \diamond R_1) \cap (S_2 \diamond R_2)$ is:

   (a) There exists some $b \in B$ such that:

   i. $a \sim_{R_1} b$ and $b \sim_{S_1} c$;

   and

   i. $a \sim_{R_2} b$ and $b \sim_{S_2} c$;

3. The condition for $(S_1 \cap S_2) \diamond (R_1 \cap R_2)$ is:

   (a) There exists some $b \in B$ such that:

   i. $a \sim_{R_1} b$ and $a \sim_{R_2} b$;

   and

   i. $b \sim_{S_1} c$ and $b \sim_{S_2} c$.

These two conditions agree, and thus so do the two resulting relations on $A \times C$. □

[8] *Further Terminology:* Also called the **binary intersection of** R **and** S, for emphasis.
[9] This is the same as the intersection of R and S as subsets of $A \times B$.

3.8 Intersections of Families of Relations

Let A and B be sets and let $\{R_i\}_{i \in I}$ be a family of relations from A to B.

Definition 3.8.1.1. The **intersection of the family** $\{R_i\}_{i \in I}$ is the relation $\bigcup_{i \in I} R_i$ defined as follows:

- Viewing relations from A to B as subsets of $A \times B$, we define[10]

$$\bigcup_{i \in I} R_i \stackrel{\text{def}}{=} \left\{ (a, b) \in (A \times B)^{\times I} \;\middle|\; \begin{array}{l} \text{for each } i \in I, \\ \text{we have } a \sim_{R_i} b \end{array} \right\}.$$

- Viewing relations from A to B as functions $A \to \mathcal{P}(B)$, we define

$$\left[\bigcap_{i \in I} R_i\right](a) \stackrel{\text{def}}{=} \bigcap_{i \in I} R_i(a)$$

  for each $a \in A$.

Proposition 3.8.1.2. Let A and B be sets and let $\{R_i\}_{i \in I}$ be a family of relations from A to B.

1. *Interaction With Inverses.* We have

$$(\bigcap_{i \in I} R_i)^\dagger = \bigcap_{i \in I} R_i^\dagger.$$

Proof. *Item 1*, *Interaction With Inverses*: Clear. □

3.9 Binary Products of Relations

Let A, B, X, and Y be sets, let $R \colon A \mathrel{\mapstochar\rightarrow} B$ be a relation from A to B, and let $S \colon X \mathrel{\mapstochar\rightarrow} Y$ be a relation from X to Y.

Definition 3.9.1.1. The **product of** R **and** S[11] is the relation $R \times S$ from $A \times X$ to $B \times Y$ defined as follows:

- Viewing relations from $A \times X$ to $B \times Y$ as subsets of $(A \times X) \times (B \times Y)$, we define $R \times S$ as the Cartesian product of R and S as subsets of $A \times X$ and $B \times Y$.[12]

[10] This is the same as the intersection of $\{R_i\}_{i \in I}$ as a collection of subsets of $A \times B$.

[11] *Further Terminology:* Also called the **binary product of** R **and** S, for emphasis.

[12] That is, $R \times S$ is the relation given by declaring $(a, x) \sim_{R \times S} (b, y)$ iff $a \sim_R b$ and $x \sim_S y$.

- Viewing relations from $A \times X$ to $B \times Y$ as functions $A \times X \to \mathcal{P}(B \times Y)$, we define $R \times S$ as the composition

$$A \times X \xrightarrow{R \times S} \mathcal{P}(B) \times \mathcal{P}(Y) \overset{\mathcal{P}^{\otimes}_{B,Y}}{\hookrightarrow} \mathcal{P}(B \times Y)$$

  in Sets, i.e. by

$$[R \times S](a, x) \overset{\text{def}}{=} R(a) \times S(x)$$

  for each $(a, x) \in A \times X$.

Proposition 3.9.1.2. Let A, B, X, and Y be sets.

1. *Interaction With Inverses.* Let

$$R \colon A \nrightarrow A, \\ S \colon X \nrightarrow X$$

   We have
$$(R \times S)^{\dagger} = R^{\dagger} \times S^{\dagger}.$$

2. *Interaction With Composition.* Let

$$R_1 \colon A \nrightarrow B, \\ S_1 \colon B \nrightarrow C, \\ R_2 \colon X \nrightarrow Y, \\ S_2 \colon Y \nrightarrow Z$$

   be relations. We have

$$(S_1 \diamond R_1) \times (S_2 \diamond R_2) = (S_1 \times S_2) \diamond (R_1 \times R_2).$$

Proof. *Item 1, Interaction With Inverses*: Unwinding the definitions, we see that:

1. We have $(a, x) \sim_{(R \times S)^{\dagger}} (b, y)$ iff:
   - We have $(b, y) \sim_{R \times S} (a, x)$, i.e. iff:
     - We have $b \sim_R a$;
     - We have $y \sim_S x$;
2. We have $(a, x) \sim_{R^{\dagger} \times S^{\dagger}} (b, y)$ iff:

- We have $a \sim_{R^\dagger} b$ and $x \sim_{S^\dagger} y$, i.e. iff:
  - We have $b \sim_R a$;
  - We have $y \sim_S x$.

These two conditions agree, and thus the two resulting relations on $A \times X$ are equal.
Item 2, Interaction With Composition: Unwinding the definitions, we see that:

1. We have $(a, x) \sim_{(S_1 \diamond R_1) \times (S_2 \diamond R_2)} (c, z)$ iff:
   (a) We have $a \sim_{S_1 \diamond R_1} c$ and $x \sim_{S_2 \diamond R_2} z$, i.e. iff:
      i. There exists some $b \in B$ such that $a \sim_{R_1} b$ and $b \sim_{S_1} c$;
      ii. There exists some $y \in Y$ such that $x \sim_{R_2} y$ and $y \sim_{S_2} z$;
2. We have $(a, x) \sim_{(S_1 \times S_2) \diamond (R_1 \times R_2)} (c, z)$ iff:
   (a) There exists some $(b, y) \in B \times Y$ such that $(a, x) \sim_{R_1 \times R_2} (b, y)$ and $(b, y) \sim_{S_1 \times S_2} (c, z)$, i.e. such that:
      i. We have $a \sim_{R_1} b$ and $x \sim_{R_2} y$;
      ii. We have $b \sim_{S_1} c$ and $y \sim_{S_2} z$.

These two conditions agree, and thus the two resulting relations from $A \times X$ to $C \times Z$ are equal. $\square$

3.10 Products of Families of Relations

Let $\{A_i\}_{i \in I}$ and $\{B_i\}_{i \in I}$ be families of sets, and let $\{R_i : A_i \nrightarrow B_i\}_{i \in I}$ be a family of relations.

Definition 3.10.1.1. The **product of the family** $\{R_i\}_{i \in I}$ is the relation $\prod_{i \in I} R_i$ from $\prod_{i \in I} A_i$ to $\prod_{i \in I} B_i$ defined as follows:

- Viewing relations as subsets, we define $\prod_{i \in I} R_i$ as its product as a family of sets, i.e. we have
$$\prod_{i \in I} R_i \overset{\text{def}}{=} \left\{ (a_i, b_i)_{i \in I} \in \prod_{i \in I} (A_i \times B_i) \;\middle|\; \begin{array}{l} \text{for each } i \in I, \\ \text{we have } a_i \sim_{R_i} b_i \end{array} \right\}.$$
- Viewing relations as functions to powersets, we define
$$\left[\prod_{i \in I} R_i \right] ((a_i)_{i \in I}) \overset{\text{def}}{=} \prod_{i \in I} R_i(a_i)$$
  for each $(a_i)_{i \in I} \in \prod_{i \in I} R_i$.

3.11 The Inverse of a Relation

Let A, B, and C be sets and let $R \subset A \times B$ be a relation.

Definition 3.11.1.1. The **inverse of** R[13] is the relation $R^\dagger$ defined as follows:

- Viewing relations as subsets, we define

$$R^\dagger \stackrel{\text{def}}{=} \{(b,a) \in B \times A \mid \text{we have } b \sim_R a\}.$$

- Viewing relations as functions $A \times B \to \{\mathsf{true}, \mathsf{false}\}$, we define

$$[R^\dagger]^a_b \stackrel{\text{def}}{=} R^b_a$$

  for each $(b,a) \in B \times A$.

- Viewing relations as functions $A \to \mathcal{P}(B)$, we define

$$\begin{aligned}[R^\dagger](b) &\stackrel{\text{def}}{=} R^\dagger(\{b\}) \\ &\stackrel{\text{def}}{=} \{a \in A \mid b \in R(a)\}\end{aligned}$$

  for each $b \in B$, where $R^\dagger(\{b\})$ is the fibre of R over $\{b\}$.

Example 3.11.1.2. Here are some examples of inverses of relations.

1. *Less Than Equal Signs.* We have $(\leq)^\dagger = \geq$.
2. *Greater Than Equal Signs.* Dually to Item 1, we have $(\geq)^\dagger = \leq$.
3. *Functions.* Let $f \colon A \to B$ be a function. We have

$$\begin{aligned}\mathrm{Gr}(f)^\dagger &= f^{-1}, \\ (f^{-1})^\dagger &= \mathrm{Gr}(f).\end{aligned}$$

Proposition 3.11.1.3. Let $R \colon A \nrightarrow B$ and $S \colon B \nrightarrow C$ be relations.

1. *Functoriality.* The assignment $R \mapsto R^\dagger$ defines a functor (i.e. morphism of posets)

$$(-)^\dagger \colon \mathbf{Rel}(A,B) \to \mathbf{Rel}(B,A).$$

   In particular, given relations $R, S \colon A \rightrightarrows B$, we have:

[13] *Further Terminology:* Also called the **opposite of** R, the **transpose of** R, or the

(★) If $R \subset S$, then $R^\dagger \subset S^\dagger$.

2. *Interaction With Ranges and Domains.* We have

$$\begin{aligned}\mathrm{dom}(R^\dagger) &= \mathrm{range}(R),\\ \mathrm{range}(R^\dagger) &= \mathrm{dom}(R).\end{aligned}$$

3. *Interaction With Composition I.* We have

$$(S \diamond R)^\dagger = R^\dagger \diamond S^\dagger.$$

4. *Interaction With Composition II.* We have

$$\begin{aligned}\chi_B &\subset R \diamond R^\dagger,\\ \chi_A &\subset R^\dagger \diamond R.\end{aligned}$$

5. *Invertibility.* We have

$$(R^\dagger)^\dagger = R.$$

6. *Identity.* We have

$$\chi_A^\dagger = \chi_A.$$

Proof. *Item 1*, *Functoriality*: Clear.
Item 2, *Interaction With Ranges and Domains*: Clear.
Item 3, *Interaction With Composition I*: Clear.
Item 4, *Interaction With Composition II*: Clear.
Item 5, *Invertibility*: Clear.
Item 6, *Identity*: Clear. □

3.12 Composition of Relations

Let A, B, and C be sets and let $R\colon A \nrightarrow B$ and $S\colon B \nrightarrow C$ be relations.

Definition 3.12.1.1. The **composition of R and S** is the relation $S \diamond R$ defined as follows:

- Viewing relations from A to C as subsets of $A \times C$, we define

$$S \diamond R \overset{\text{def}}{=} \left\{ (a,c) \in A \times C \,\middle|\, \begin{array}{l}\text{there exists some } b \in B \text{ such}\\ \text{that } a \sim_R b \text{ and } b \sim_S c\end{array} \right\}.$$

- Viewing relations as functions $A \times B \to \{\mathsf{true}, \mathsf{false}\}$, we define

$$
\begin{aligned}
(S \diamond R)^{-1}_{-2} &\stackrel{\text{def}}{=} \int^{b \in B} S^{-1}_{b} \times R^{b}_{-2} \\
&= \bigvee_{b \in B} S^{-1}_{b} \times R^{b}_{-2},
\end{aligned}
$$

  where the join $\bigvee$ is taken in the poset $(\{\mathsf{true}, \mathsf{false}\}, \preceq)$ of Sets, Definition 2.2.1.3.

- Viewing relations as functions $A \to \mathcal{P}(B)$, we define

$$
S \diamond R \stackrel{\text{def}}{=} \mathrm{Lan}_{\chi_B}(S) \circ R, \qquad
\begin{tikzcd}
 & B \arrow[r, "S"] \arrow[d, hook, "\chi_B"'] & \mathcal{P}(C), \\
A \arrow[r, "R"'] & \mathcal{P}(B) \arrow[ur, bend right, "\mathrm{Lan}_{\chi_B}(S)"'] \arrow[ur, phantom, "\Downarrow"] &
\end{tikzcd}
$$

  where $\mathrm{Lan}_{\chi_B}(S)$ is computed by the formula

$$
\begin{aligned}
[\mathrm{Lan}_{\chi_B}(S)](V) &\cong \int^{y \in B} \chi_{\mathcal{P}(B)}(\chi_y, V) \odot S_y \\
&\cong \int^{y \in B} \chi_V(y) \odot S_y \\
&\cong \bigcup_{y \in B} \chi_V(y) \odot S_y \\
&\cong \bigcup_{y \in V} S_y
\end{aligned}
$$

  for each $V \in \mathcal{P}(B)$. In other words, $S \diamond R$ is defined by[14]

$$
\begin{aligned}
[S \diamond R](a) &\stackrel{\text{def}}{=} S(R(a)) \\
&\stackrel{\text{def}}{=} \bigcup_{x \in R(a)} S(x).
\end{aligned}
$$

  for each $a \in A$.

Example 3.12.1.2. Here are some examples of composition of relations.

---

converse of R.

[14]That is: the relation R may send $a \in A$ to a number of elements $\{b_i\}_{i \in I}$ in B, and then the relation S may send the image of each of the b_i's to a number of elements

1. *Composing Less/Greater Than Equal With Greater/Less Than Equal Signs.* We have

$$\leq \diamond \geq = \sim_{\mathrm{triv}},$$
$$\geq \diamond \leq = \sim_{\mathrm{triv}}.$$

2. *Composing Less/Greater Than Equal Signs With Less/Greater Than Equal Signs.* We have

$$\leq \diamond \leq = \leq,$$
$$\geq \diamond \geq = \geq.$$

Proposition 3.12.1.3. Let $R\colon A \nrightarrow B$, $S\colon B \nrightarrow C$, and $T\colon C \nrightarrow D$ be relations.

1. *Interaction With Ranges and Domains.* We have

$$\mathrm{dom}(S \diamond R) \subset \mathrm{dom}(R),$$
$$\mathrm{range}(S \diamond R) \subset \mathrm{range}(S).$$

2. *Associativity.* We have

$$(T \diamond S) \diamond R = T \diamond (S \diamond R).$$

3. *Unitality.* We have

$$\chi_B \diamond R = R,$$
$$R \diamond \chi_A = R.$$

4. *Interaction With Inverses.* We have

$$(S \diamond R)^{\dagger} = R^{\dagger} \diamond S^{\dagger}.$$

5. *Interaction With Composition.* We have

$$\chi_B \subset R \diamond R^{\dagger},$$
$$\chi_A \subset R^{\dagger} \diamond R.$$

$\{S(b_i)\}_{i\in I} = \left\{\{c_{j_i}\}_{j_i\in J_i}\right\}_{i\in I}$ in C.

Proof. *Item 1*, *Interaction With Ranges and Domains*: Clear.
Item 2, *Associativity*: Indeed, we have

$$\begin{aligned}
(T \diamond S) \diamond R &\stackrel{\text{def}}{=} \left(\int^{c\in C} T_c^{-1} \times S^c_{-_2}\right) \diamond R \\
&\stackrel{\text{def}}{=} \int^{b\in B} \left(\int^{c\in C} T_c^{-1} \times S^c_b\right) \diamond R^b_{-_2} \\
&= \int^{b\in B} \int^{c\in C} (T_c^{-1} \times S^c_b) \diamond R^b_{-_2} \\
&= \int^{c\in C} \int^{b\in B} (T_c^{-1} \times S^c_b) \diamond R^b_{-_2} \\
&= \int^{c\in C} \int^{b\in B} T_c^{-1} \times (S^c_b \diamond R^b_{-_2}) \\
&= \int^{c\in C} T_c^{-1} \times \left(\int^{b\in B} S^c_b \diamond R^b_{-_2}\right) \\
&\stackrel{\text{def}}{=} \int^{c\in C} T_c^{-1} \times (S \diamond R)^c_{-_2} \\
&\stackrel{\text{def}}{=} T \diamond (S \diamond R).
\end{aligned}$$

In the language of relations, given $a \in A$ and $d \in D$, the stated equality witnesses the equivalence of the following two statements:

1. We have $a \sim_{(T\diamond S)\diamond R} d$, i.e. there exists some $b \in B$ such that:
    (a) We have $a \sim_R b$;
    (b) We have $b \sim_{T\diamond S} d$, i.e. there exists some $c \in C$ such that:
        i. We have $b \sim_S c$;
        ii. We have $c \sim_T d$;
2. We have $a \sim_{T\diamond(S\diamond R)} d$, i.e. there exists some $c \in C$ such that:
    (a) We have $a \sim_{S\diamond R} c$, i.e. there exists some $b \in B$ such that:
        i. We have $a \sim_R b$;
        ii. We have $b \sim_S c$;
    (b) We have $c \sim_T d$;

both of which are equivalent to the statement

- There exist $b \in B$ and $c \in C$ such that $a \sim_R b \sim_S c \sim_T d$.

Item 3, *Unitality*: Indeed, we have

$$\begin{aligned}\chi_B \diamond R &\stackrel{\text{def}}{=} \int^{x\in B} (\chi_B)^{-1}_{x} \times R^{x}_{-2}\\ &= \bigvee_{x\in B} (\chi_B)^{-1}_{x} \times R^{x}_{-2}\\ &= \bigvee_{\substack{x\in B\\ x=-1}} R^{x}_{-2}\\ &= R^{-1}_{-2},\end{aligned}$$

and

$$\begin{aligned}R \diamond \chi_A &\stackrel{\text{def}}{=} \int^{x\in A} R^{-1}_{x} \times (\chi_A)^{x}_{-2}\\ &= \bigvee_{x\in B} R^{-1}_{x} \times (\chi_A)^{x}_{-2}\\ &= \bigvee_{\substack{x\in B\\ x=-2}} R^{-1}_{x}\\ &= R^{-1}_{-2}.\end{aligned}$$

In the language of relations, given $a \in A$ and $b \in B$:

- The equality
$$\chi_B \diamond R = R$$
witnesses the equivalence of the following two statements:
  1. We have $a \sim_b B$.
  2. There exists some $b' \in B$ such that:
     (a) We have $a \sim_R b'$
     (b) We have $b' \sim_{\chi_B} b$, i.e. $b' = b$.
- The equality
$$R \diamond \chi_A = R$$
witnesses the equivalence of the following two statements:
  1. There exists some $a' \in A$ such that:
     (a) We have $a \sim_{\chi_B} a'$, i.e. $a = a'$.
     (b) We have $a' \sim_R b$
  2. We have $a \sim_b B$.

Item 4, *Interaction With Inverses*: Clear.
Item 5, *Interaction With Composition*: Clear. □

3.13 The Collage of a Relation

Let A and B be sets and let $R\colon A \mathrel{-\!\!\!\!+\!\!\!\!\to} B$ be a relation from A to B.

Definition 3.13.1.1. The **collage of** R[15] is the poset $\mathbf{Coll}(R) \stackrel{\text{def}}{=} (\mathrm{Coll}(R), \preceq_{\mathbf{Coll}(R)})$ consisting of:

- *The Underlying Set.* The set $\mathrm{Coll}(R)$ defined by
$$\mathrm{Coll}(R) \stackrel{\text{def}}{=} A \coprod B.$$
- *The Partial Order.* The partial order
$$\preceq_{\mathbf{Coll}(R)}\colon \mathrm{Coll}(R) \times \mathrm{Coll}(R) \to \{\mathsf{true}, \mathsf{false}\}$$
on $\mathrm{Coll}(R)$ defined by
$$\preceq(a,b) \stackrel{\text{def}}{=} \begin{cases} \mathsf{true} & \text{if } a = b \text{ or } a \sim_R b, \\ \mathsf{false} & \text{otherwise.} \end{cases}$$

Proposition 3.13.1.2. Let A and B be sets and let $R\colon A \mathrel{-\!\!\!\!+\!\!\!\!\to} B$ be a relation from A to B.

1. *Functoriality I.* The assignment $R \mapsto \mathbf{Coll}(R)$ defines a functor[16]
$$\mathbf{Coll}\colon \mathbf{Rel}(A,B) \to \mathsf{Pos}_{/\Delta^1}(A,B),$$

[15] *Further Terminology:* Also called the **cograph of** R.

[16] Here $\mathsf{Pos}_{/\Delta^1}(A,B)$ is the category defined as the pullback
$$\mathsf{Pos}_{/\Delta^1}(A,B) \stackrel{\text{def}}{=} \mathsf{pt} \underset{[A],\mathsf{Pos},\mathrm{fib}_0}{\times} \mathsf{Pos}_{/\Delta^1} \underset{\mathrm{fib}_1,\mathsf{Pos},[B]}{\times} \mathsf{pt},$$
as in the diagram
$$\begin{tikzcd}
 & & \mathsf{Pos}_{/\Delta^1}(A,B) \arrow[dl] \arrow[dr] \arrow[dd, phantom, very near start, "\lrcorner"] & & \\
 & \mathsf{Pos}_{/\Delta^1} \underset{\mathsf{Pos}}{\times} \mathsf{pt} \arrow[dl] \arrow[dr] \arrow[dd, phantom, very near start, "\lrcorner"] & & \mathsf{pt} \underset{\mathsf{Pos}}{\times} \mathsf{Pos}_{/\Delta^1} \arrow[dl] \arrow[dr] \arrow[dd, phantom, very near start, "\lrcorner"] & \\
\mathsf{pt} \arrow[dr, "{[A]}"'] & & \mathsf{Pos}_{/\Delta^1} \arrow[dl, "\mathrm{fib}_{[0]}"] \arrow[dr, "\mathrm{fib}_{[1]}"'] & & \mathsf{pt}. \arrow[dl, "{[B]}"] \\
 & \mathsf{Pos} & & \mathsf{Pos} &
\end{tikzcd}$$

where

- *Action on Objects.* For each $R \in \text{Obj}(\mathbf{Rel}(A, B))$, we have

$$[\mathbf{Coll}](R) \stackrel{\text{def}}{=} (\mathbf{Coll}(R), \phi_R)$$

for each $R \in \mathbf{Rel}(A, B)$, where
  - The poset $\mathbf{Coll}(R)$ is the collage of R of Definition 3.13.1.1.
  - The morphism $\phi_R \colon \mathbf{Coll}(R) \to \Delta^1$ is given by

$$\phi_R(x) \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if } x \in A, \\ 1 & \text{if } x \in B \end{cases}$$

  for each $x \in \mathbf{Coll}(R)$.
- *Action on Morphisms.* For each $R, S \in \text{Obj}(\mathbf{Rel}(A, B))$, the action on Hom-sets

$$\mathbf{Coll}_{R,S} \colon \text{Hom}_{\mathbf{Rel}(A,B)}(R, S) \to \mathsf{Pos}(\mathbf{Coll}(R), \mathbf{Coll}(S))$$

of $\mathbf{Coll}$ at (R, S) is given by sending an inclusion

$$\iota \colon R \subset S$$

to the morphism

$$\mathbf{Coll}(\iota) \colon \mathbf{Coll}(R) \to \mathbf{Coll}(S)$$

of posets over Δ^1 defined by

$$[\mathbf{Coll}(\iota)](x) \stackrel{\text{def}}{=} x$$

for each $x \in \mathbf{Coll}(R)$.[17]

2. *Equivalence.* The functor of Item 1 is an equivalence of categories.

Proof. *Item 1, Functoriality*: Clear.
Item 2, Equivalence: Omitted. □

Explicitly, an object of $\mathsf{Pos}_{/\Delta^1}(A, B)$ is a pair (X, ϕ_X) consisting of

- A poset X;
- A morphism $\phi_X \colon X \to \Delta^1$;

such that $\phi_X^{-1}(0) = A$ and $\phi_X^{-1}(0) = B$, with morphisms between such objects being morphisms of posets over Δ^1.

[17] Note that this is indeed a morphism of posets: if $x \preceq_{\mathbf{Coll}(R)} y$, then $x = y$ or $x \sim_R y$,

4 Functoriality of Powersets

4.1 Direct Images

Let A and B be sets and let $R\colon A \nrightarrow B$ be a relation.

Definition 4.1.1.1. The **direct image function associated to** R is the function

$$R_*\colon \mathcal{P}(A) \to \mathcal{P}(B)$$

defined by[18,19]

$$\begin{aligned} R_*(U) &\overset{\text{def}}{=} R(U) \\ &\overset{\text{def}}{=} \bigcup_{a\in U} R(a) \\ &= \left\{ b \in B \,\middle|\, \begin{array}{l} \text{there exists some } a \in U \\ \text{such that } b \in R(a) \end{array} \right\} \end{aligned}$$

for each $U \in \mathcal{P}(A)$.

Remark 4.1.1.2. Identifying subsets of A with relations from pt to A via Constructions With Sets, Item 3 of Proposition 4.3.1.6, we see that the direct image function associated to R is equivalently the function

$$R_*\colon \underbrace{\mathcal{P}(A)}_{\cong \mathrm{Rel}(\mathrm{pt},A)} \to \underbrace{\mathcal{P}(B)}_{\cong \mathrm{Rel}(\mathrm{pt},B)}$$

defined by

$$R_*(U) \overset{\text{def}}{=} R \diamond U$$

for each $U \in \mathcal{P}(A)$, where $R \diamond U$ is the composition

$$\mathrm{pt} \overset{U}{\nrightarrow} A \overset{R}{\nrightarrow} B.$$

Proposition 4.1.1.3. Let $R\colon A \nrightarrow B$ be a relation.

---

so we have either $x = y$ or $x \sim_S y$ (as $R \subset S$), and thus $x \preceq_{\mathbf{Coll}(S)} y$.

[18] *Further Terminology:* The set $R(U)$ is called the **direct image of** U **by** R.

[19] We also have

$$R_*(U) = B \setminus R_!(A \setminus U);$$

see Item 7 of Proposition 4.1.1.3.

1. *Functoriality.* The assignment $U \mapsto R_*(U)$ defines a functor

$$R_* \colon (\mathcal{P}(A), \subset) \to (\mathcal{P}(B), \subset)$$

where

- *Action on Objects.* For each $U \in \mathcal{P}(A)$, we have

$$[R_*](U) \overset{\text{def}}{=} R_*(U).$$

- *Action on Morphisms.* For each $U, V \in \mathcal{P}(A)$:
  - If $U \subset V$, then $R_*(U) \subset R_*(V)$.

2. *Adjointness.* We have an adjunction

$$(R_* \dashv R_{-1})\colon \quad \begin{tikzcd} \mathcal{P}(A) \arrow[r, bend left, "R_*"] \arrow[r, phantom, "\perp"] & \mathcal{P}(B), \arrow[l, bend left, "R_{-1}"] \end{tikzcd}$$

witnessed by a bijections of sets

$$\mathrm{Hom}_{\mathcal{P}(A)}(R_*(U), V) \cong \mathrm{Hom}_{\mathcal{P}(A)}(U, R_{-1}(V)),$$

natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$, i.e. such that:

($\star$) The following conditions are equivalent:
  - We have $R_*(U) \subset V$.
  - We have $U \subset R_{-1}(V)$.

3. *Preservation of Colimits.* We have an equality of sets

$$R_*(\bigcup_{i \in I} U_i) = \bigcup_{i \in I} R_*(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have equalities

$$R_*(U) \cup R_*(V) = R_*(U \cup V),$$
$$R_*(\emptyset) = \emptyset,$$

natural in $U, V \in \mathcal{P}(A)$.

4. *Oplax Preservation of Limits.* We have an inclusion of sets

$$R_*(\bigcap_{i\in I} U_i) \subset \bigcap_{i\in I} R_*(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have inclusions

$$\begin{gathered} R_*(U \cap V) \subset R_*(U) \cap R_*(V), \\ R_*(A) \subset B, \end{gathered}$$

natural in $U, V \in \mathcal{P}(A)$.

5. *Symmetric Strict Monoidality With Respect to Unions.* The direct image function of Item 1 has a symmetric strict monoidal structure

$$(R_*, R_*^{\otimes}, R_{*|\mathbb{1}}^{\otimes}) \colon (\mathcal{P}(A), \cup, \emptyset) \to (\mathcal{P}(B), \cup, \emptyset),$$

being equipped with equalities

$$\begin{gathered} R_{*|U,V}^{\otimes} \colon R_*(U) \cup R_*(V) \xrightarrow{=} R_*(U \cup V), \\ R_{*|\mathbb{1}}^{\otimes} \colon \emptyset \xrightarrow{=} \emptyset, \end{gathered}$$

natural in $U, V \in \mathcal{P}(A)$.

6. *Symmetric Oplax Monoidality With Respect to Intersections.* The direct image function of Item 1 has a symmetric oplax monoidal structure

$$(R_*, R_*^{\otimes}, R_{*|\mathbb{1}}^{\otimes}) \colon (\mathcal{P}(A), \cap, A) \to (\mathcal{P}(B), \cap, B),$$

being equipped with inclusions

$$\begin{gathered} R_{*|U,V}^{\otimes} \colon R_*(U \cap V) \subset R_*(U) \cap R_*(V), \\ R_{*|\mathbb{1}}^{\otimes} \colon R_*(A) \subset B, \end{gathered}$$

natural in $U, V \in \mathcal{P}(A)$.

7. *Relation to Direct Images With Compact Support.* We have

$$R_*(U) = B \setminus R_!(A \setminus U)$$

for each $U \in \mathcal{P}(A)$.

Proof. *Item 1, Functoriality*: Clear.
Item 2, Adjointness: This follows from **??**, **??** of **??**.
Item 3, Preservation of Colimits: This follows from Item 2 and **??**, **??** of **??**.
Item 4, Oplax Preservation of Limits: Omitted.
Item 5, Symmetric Strict Monoidality With Respect to Unions: This follows from Item 3.
Item 6, Symmetric Oplax Monoidality With Respect to Intersections: This follows from Item 4.
Item 7, Relation to Direct Images With Compact Support: The proof proceeds in the same way as in the case of functions (Constructions With Sets, Item 9 of Proposition 4.4.1.4): applying Item 7 of Proposition 4.4.1.3 to $A \setminus U$, we have

$$\begin{aligned} R_!(A \setminus U) &= B \setminus R_*(A \setminus (A \setminus U)) \\ &= B \setminus R_*(U). \end{aligned}$$

Taking complements, we then obtain

$$\begin{aligned} R_*(U) &= B \setminus (B \setminus R_*(U)), \\ &= B \setminus R_!(A \setminus U), \end{aligned}$$

which finishes the proof. □

Proposition 4.1.1.4. Let $R\colon A \nrightarrow B$ be a relation.

1. *Functionality I.* The assignment $R \mapsto R_*$ defines a function

$$(-)_*\colon \mathrm{Rel}(A, B) \to \mathsf{Sets}(\mathcal{P}(A), \mathcal{P}(B)).$$

2. *Functionality II.* The assignment $R \mapsto R_*$ defines a function

$$(-)_*\colon \mathrm{Rel}(A, B) \to \mathsf{Pos}((\mathcal{P}(A), \subset), (\mathcal{P}(B), \subset)).$$

3. *Interaction With Identities.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have[20]

$$(\chi_A)_* = \mathrm{id}_{\mathcal{P}(A)}.$$

[20]That is, the postcomposition function

$$(\chi_A)_*\colon \mathrm{Rel}(\mathrm{pt}, A) \to \mathrm{Rel}(\mathrm{pt}, A)$$

is equal to $\mathrm{id}_{\mathrm{Rel}(\mathrm{pt},A)}$.

4. *Interaction With Composition.* For each pair of composable relations $R\colon A \nrightarrow B$ and $S\colon B \nrightarrow C$, we have[21]

$$(S \diamond R)_* = S_* \circ R_*, \qquad \begin{tikzcd} \mathcal{P}(A) \arrow[r, "R_*"] \arrow[dr, "(S\diamond R)_*"'] & \mathcal{P}(B) \arrow[d, "S_*"] \\ & \mathcal{P}(C). \end{tikzcd}$$

Proof. *Item 1, Functionality I*: Clear.
Item 2, Functionality II: Clear.
Item 3, Interaction With Identities: Indeed, we have

$$\begin{aligned} (\chi_A)_*(U) &\overset{\text{def}}{=} \bigcup_{a\in U} \chi_A(a) \\ &\overset{\text{def}}{=} \bigcup_{a\in U} \{a\} \\ &= U \\ &\overset{\text{def}}{=} \mathrm{id}_{\mathcal{P}(A)}(U) \end{aligned}$$

for each $U \in \mathcal{P}(A)$. Thus $(\chi_A)_* = \mathrm{id}_{\mathcal{P}(A)}$.
Item 4, Interaction With Composition: Indeed, we have

$$\begin{aligned} (S \diamond R)_*(U) &\overset{\text{def}}{=} \bigcup_{a\in U} [S \diamond R](a) \\ &\overset{\text{def}}{=} \bigcup_{a\in U} S(R(a)) \\ &\overset{\text{def}}{=} \bigcup_{a\in U} S_*(R(a)) \\ &= S_*\Big(\bigcup_{a\in U} R(a)\Big) \\ &\overset{\text{def}}{=} S_*(R_*(U)) \\ &\overset{\text{def}}{=} [S_* \circ R_*](U) \end{aligned}$$

[21] That is, we have

$$(S \diamond R)_* = S_* \circ R_*, \qquad \begin{tikzcd} \mathrm{Rel}(\mathrm{pt}, A) \arrow[r, "R_*"] \arrow[dr, "(S\diamond R)_*"'] & \mathrm{Rel}(\mathrm{pt}, B) \arrow[d, "S_*"] \\ & \mathrm{Rel}(\mathrm{pt}, C). \end{tikzcd}$$

for each $U \in \mathcal{P}(A)$, where we used Item 3 of Proposition 4.1.1.3. Thus $(S \diamond R)_* = S_* \circ R_*$. □

4.2 Strong Inverse Images

Let A and B be sets and let $R \colon A \nrightarrow B$ be a relation.

Definition 4.2.1.1. The **strong inverse image function associated to R** is the function

$$R_{-1} \colon \mathcal{P}(B) \to \mathcal{P}(A)$$

defined by[22]

$$R_{-1}(V) \overset{\text{def}}{=} \{a \in A \mid R(a) \subset V\}$$

for each $V \in \mathcal{P}(B)$.

Remark 4.2.1.2. Identifying subsets of B with relations from pt to B via Constructions With Sets, Item 3 of Proposition 4.3.1.6, we see that the inverse image function associated to R is equivalently the function

$$R_{-1} \colon \underbrace{\mathcal{P}(B)}_{\cong \mathrm{Rel}(\mathrm{pt}, B)} \to \underbrace{\mathcal{P}(A)}_{\cong \mathrm{Rel}(\mathrm{pt}, A)}$$

defined by

$$R_{-1}(V) \overset{\text{def}}{=} \mathrm{Rift}_R(V), \qquad \begin{tikzcd} & A \arrow[d, "R", "\mid" marking] \\ \mathrm{pt} \arrow[ur, dashed, "\mathrm{Rift}_R(V)", "\times" marking] \arrow[r, "V"', "\mid" marking] & B, \end{tikzcd}$$

and being explicitly computed by

$$\begin{aligned} R_{-1}(V) &\overset{\text{def}}{=} \mathrm{Rift}_R(V) \\ &\cong \int_{b \in B} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R^b_{-1}, V^b_{-2}), \end{aligned}$$

where we have used Proposition 2.4.1.1.

[22] *Further Terminology:* The set $R_{-1}(V)$ is called the **strong inverse image of V by R**.

Proof. We have

$$
\begin{aligned}
\mathrm{Rift}_R(V) &\cong \int_{b\in B} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R^b_{-_1}, V^b_{-_2})\\
&= \left\{ a\in A \;\middle|\; \int_{b\in B} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R^b_a, V^b_\star) = \mathsf{true} \right\}\\
&= \left\{ a\in A \;\middle|\; \begin{array}{l} \text{for each } b\in B\text{, at least one of the}\\ \text{following conditions hold:}\\ \quad 1.\ \text{We have } R^b_a = \mathsf{false}\\ \quad 2.\ \text{The following conditions hold:}\\ \qquad \text{(a) We have } R^b_a = \mathsf{true}\\ \qquad \text{(b) We have } V^b_\star = \mathsf{true} \end{array} \right\}\\
&= \left\{ a\in A \;\middle|\; \begin{array}{l} \text{for each } b\in B\text{, at least one of the}\\ \text{following conditions hold:}\\ \quad 1.\ \text{We have } b\notin R(a)\\ \quad 2.\ \text{The following conditions hold:}\\ \qquad \text{(a) We have } b\in R(a)\\ \qquad \text{(b) We have } b\in V \end{array} \right\}\\
&= \{a\in A \mid \text{for each } b\in R(a)\text{, we have } b\in V\}\\
&= \{a\in A \mid R(a)\subset V\}\\
&\stackrel{\text{def}}{=} R_{-1}(V).
\end{aligned}
$$

This finishes the proof. □

Proposition 4.2.1.3. Let $R\colon A \nrightarrow B$ be a relation.

1. *Functoriality.* The assignment $V \mapsto R_{-1}(V)$ defines a functor

$$R_{-1}\colon (\mathcal{P}(B),\subset) \to (\mathcal{P}(A),\subset)$$

where

- *Action on Objects.* For each $V\in\mathcal{P}(B)$, we have

$$[R_{-1}](V) \stackrel{\text{def}}{=} R_{-1}(V).$$

- *Action on Morphisms.* For each $U,V\in\mathcal{P}(B)$:

  - If $U \subset V$, then $R_{-1}(U) \subset R_{-1}(V)$.

2. *Adjointness.* We have an adjunction

$$(R_* \dashv R_{-1})\colon \quad \begin{tikzcd} \mathcal{P}(A) \arrow[r, bend left, "R_*"] \arrow[r, phantom, "\perp"] & \mathcal{P}(B), \arrow[l, bend left, "R_{-1}"] \end{tikzcd}$$

   witnessed by a bijections of sets

$$\mathrm{Hom}_{\mathcal{P}(A)}(R_*(U), V) \cong \mathrm{Hom}_{\mathcal{P}(A)}(U, R_{-1}(V)),$$

   natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$, i.e. such that:

   ($\star$) The following conditions are equivalent:
   - We have $R_*(U) \subset V$.
   - We have $U \subset R_{-1}(V)$.

3. *Lax Preservation of Colimits.* We have an inclusion of sets

$$\bigcup_{i \in I} R_{-1}(U_i) \subset R_{-1}(\bigcup_{i \in I} U_i),$$

   natural in $\{U_i\}_{i \in I} \in \mathcal{P}(B)^{\times I}$. In particular, we have inclusions

$$\begin{gathered} R_{-1}(U) \cup R_{-1}(V) \subset R_{-1}(U \cup V), \\ \emptyset \subset R_{-1}(\emptyset), \end{gathered}$$

   natural in $U, V \in \mathcal{P}(B)$.

4. *Preservation of Limits.* We have an equality of sets

$$R_{-1}(\bigcap_{i \in I} U_i) = \bigcap_{i \in I} R_{-1}(U_i),$$

   natural in $\{U_i\}_{i \in I} \in \mathcal{P}(B)^{\times I}$. In particular, we have equalities

$$\begin{gathered} R_{-1}(U \cap V) = R_{-1}(U) \cap R_{-1}(V), \\ R_{-1}(B) = B, \end{gathered}$$

   natural in $U, V \in \mathcal{P}(B)$.

5. *Symmetric Lax Monoidality With Respect to Unions.* The direct image

with compact support function of Item 1 has a symmetric lax monoidal structure

$$(R_{-1}, R^{\otimes}_{-1}, R^{\otimes}_{-1|\mathbb{1}}) \colon (\mathcal{P}(A), \cup, \emptyset) \to (\mathcal{P}(B), \cup, \emptyset),$$

being equipped with inclusions

$$R^{\otimes}_{-1|U,V} \colon R_{-1}(U) \cup R_{-1}(V) \subset R_{-1}(U \cup V),$$
$$R^{\otimes}_{-1|\mathbb{1}} \colon \emptyset \subset R_{-1}(\emptyset),$$

natural in $U, V \in \mathcal{P}(B)$.

6. *Symmetric Strict Monoidality With Respect to Intersections.* The direct image function of Item 1 has a symmetric strict monoidal structure

$$(R_{-1}, R^{\otimes}_{-1}, R^{\otimes}_{-1|\mathbb{1}}) \colon (\mathcal{P}(A), \cap, A) \to (\mathcal{P}(B), \cap, B),$$

being equipped with equalities

$$R^{\otimes}_{-1|U,V} \colon R_{-1}(U \cap V) \xrightarrow{=} R_{-1}(U) \cap R_{-1}(V),$$
$$R^{\otimes}_{-1|\mathbb{1}} \colon R_{-1}(A) \xrightarrow{=} B,$$

natural in $U, V \in \mathcal{P}(B)$.

7. *Interaction With Weak Inverse Images I.* We have

$$R_{-1}(V) = A \setminus R^{-1}(B \setminus V)$$

for each $V \in \mathcal{P}(B)$.

8. *Interaction With Weak Inverse Images II.* Let $R \colon A \nrightarrow B$ be a relation from A to B.

   (a) If R is a total relation, then we have an inclusion of sets

   $$R_{-1}(V) \subset R^{-1}(V)$$

   natural in $V \in \mathcal{P}(B)$.

   (b) If R is total and functional, then the above inclusion is in fact an equality.

   (c) Conversely, if we have $R_{-1} = R^{-1}$, then R is total and functional.

Proof. *Item 1, Functoriality*: Clear.
Item 2, Adjointness: This follows from **??**, **??** of **??**.
Item 3, Lax Preservation of Colimits: Omitted.
Item 4, Preservation of Limits: This follows from Item 2 and **??**, **??** of **??**.
Item 5, Symmetric Lax Monoidality With Respect to Unions: This follows from Item 3.
Item 6, Symmetric Strict Monoidality With Respect to Intersections: This follows from Item 4.
Item 7, Interaction With Weak Inverse Images I: We claim we have an equality

$$R_{-1}(B \setminus V) = A \setminus R^{-1}(V).$$

Indeed, we have

$$\begin{aligned} R_{-1}(B \setminus V) &= \{a \in A \mid R(a) \subset B \setminus V\}, \\ A \setminus R^{-1}(V) &= \{a \in A \mid R(a) \cap V = \emptyset\}. \end{aligned}$$

Taking $V = B \setminus V$ then implies the original statement.
Item 8, Interaction With Weak Inverse Images II: Item 8a is clear, while Items 8b and 8c follow from Item 6 of Proposition 3.1.1.2. □

Proposition 4.2.1.4. Let $R\colon A \nrightarrow B$ be a relation.

1. *Functionality I.* The assignment $R \mapsto R_{-1}$ defines a function

$$(-)_{-1}\colon \mathsf{Sets}(A, B) \to \mathsf{Sets}(\mathcal{P}(A), \mathcal{P}(B)).$$

2. *Functionality II.* The assignment $R \mapsto R_{-1}$ defines a function

$$(-)_{-1}\colon \mathsf{Sets}(A, B) \to \mathsf{Pos}((\mathcal{P}(A), \subset), (\mathcal{P}(B), \subset)).$$

3. *Interaction With Identities.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$(\mathrm{id}_A)_{-1} = \mathrm{id}_{\mathcal{P}(A)}.$$

4. *Interaction With Composition.* For each pair of composable relations $R\colon A \nrightarrow B$ and $S\colon B \nrightarrow C$, we have

$$(S \diamond R)_{-1} = R_{-1} \circ S_{-1}, \qquad \begin{tikzcd} \mathcal{P}(C) \arrow[r, "S_{-1}"] \arrow[dr, "(S\diamond R)_{-1}"'] & \mathcal{P}(B) \arrow[d, "R_{-1}"] \\ & \mathcal{P}(A). \end{tikzcd}$$

Proof. *Item 1, Functionality I*: Clear.
Item 2, Functionality II: Clear.
Item 3, Interaction With Identities: Indeed, we have

$$\begin{aligned}(\chi_A)_{-1}(U) &\stackrel{\text{def}}{=} \{a \in A \mid \chi_A(a) \subset U\} \\ &\stackrel{\text{def}}{=} \{a \in A \mid \{a\} \subset U\} \\ &= U\end{aligned}$$

for each $U \in \mathcal{P}(A)$. Thus $(\chi_A)_{-1} = \mathrm{id}_{\mathcal{P}(A)}$.
Item 4, Interaction With Composition: Indeed, we have

$$\begin{aligned}(S \diamond R)_{-1}(U) &\stackrel{\text{def}}{=} \{a \in A \mid [S \diamond R](a) \subset U\} \\ &\stackrel{\text{def}}{=} \{a \in A \mid S(R(a)) \subset U\} \\ &\stackrel{\text{def}}{=} \{a \in A \mid S_*(R(a)) \subset U\} \\ &= \{a \in A \mid R(a) \subset S_{-1}(U)\} \\ &\stackrel{\text{def}}{=} R_{-1}(S_{-1}(U)) \\ &\stackrel{\text{def}}{=} [R_{-1} \circ S_{-1}](U)\end{aligned}$$

for each $U \in \mathcal{P}(C)$, where we used Item 2 of Proposition 4.2.1.3, which implies that the conditions

- We have $S_*(R(a)) \subset U$.
- We have $R(a) \subset S_{-1}(U)$.

are equivalent. Thus $(S \diamond R)_{-1} = R_{-1} \circ S_{-1}$. □

4.3 Weak Inverse Images

Let A and B be sets and let $R\colon A \nrightarrow B$ be a relation.

Definition 4.3.1.1. The **weak inverse image function associated to** R[23] is the function

$$R^{-1}\colon \mathcal{P}(B) \to \mathcal{P}(A)$$

defined by[24]

$$R^{-1}(V) \stackrel{\text{def}}{=} \{a \in A \mid R(a) \cap V \neq \emptyset\}$$

for each $V \in \mathcal{P}(B)$.

[23] *Further Terminology:* Also called simply the **inverse image function associated to** R.
[24] *Further Terminology:* The set $R^{-1}(V)$ is called the **weak inverse image of** V **by** R

Remark 4.3.1.2. Identifying subsets of B with relations from B to pt via Constructions With Sets, Item 3 of Proposition 4.3.1.6, we see that the weak inverse image function associated to R is equivalently the function

$$R^{-1}\colon \underbrace{\mathcal{P}(B)}_{\cong \mathrm{Rel}(B,\mathrm{pt})} \to \underbrace{\mathcal{P}(A)}_{\cong \mathrm{Rel}(A,\mathrm{pt})}$$

defined by

$$R^{-1}(V) \stackrel{\text{def}}{=} V \diamond R$$

for each $V \in \mathcal{P}(A)$, where $R \diamond V$ is the composition

$$A \overset{R}{\nrightarrow} B \overset{V}{\nrightarrow} \mathrm{pt}.$$

Explicitly, we have

$$\begin{aligned} R^{-1}(V) &\stackrel{\text{def}}{=} V \diamond R \\ &\stackrel{\text{def}}{=} \int^{b\in B} V_b^{-1} \times R^b_{-2}. \end{aligned}$$

Proof. We have

$$\begin{aligned} V \diamond R &\stackrel{\text{def}}{=} \int^{b\in B} V_b^{-1} \times R^b_{-2} \\ &= \left\{ a \in A \;\middle|\; \int^{b\in B} V_b^{\star} \times R^b_a = \mathsf{true} \right\} \\ &= \left\{ a \in A \;\middle|\; \begin{array}{l} \text{there exists } b \in B \text{ such that the} \\ \text{following conditions hold:} \\ \quad 1.\ \text{We have } V_b^{\star} = \mathsf{true} \\ \quad 2.\ \text{We have } R^b_a = \mathsf{true} \end{array} \right\} \\ &= \left\{ a \in A \;\middle|\; \begin{array}{l} \text{there exists } b \in B \text{ such that the} \\ \text{following conditions hold:} \\ \quad 1.\ \text{We have } b \in V \\ \quad 2.\ \text{We have } b \in R(a) \end{array} \right\} \\ &= \{a \in A \mid \text{there exists } b \in V \text{ such that } b \in R(a)\} \\ &= \{a \in A \mid R(a) \cap V \neq \emptyset\} \\ &\stackrel{\text{def}}{=} R^{-1}(V) \end{aligned}$$

This finishes the proof. □

Proposition 4.3.1.3. Let $R\colon A \nrightarrow B$ be a relation.

1. *Functoriality.* The assignment $V \mapsto R^{-1}(V)$ defines a functor

$$R^{-1}\colon (\mathcal{P}(B), \subset) \to (\mathcal{P}(A), \subset)$$

where

- *Action on Objects.* For each $V \in \mathcal{P}(B)$, we have

$$[R^{-1}](V) \overset{\text{def}}{=} R^{-1}(V).$$

- *Action on Morphisms.* For each $U, V \in \mathcal{P}(B)$:
  - If $U \subset V$, then $R^{-1}(U) \subset R^{-1}(V)$.

2. *Adjointness.* We have an adjunction

$$\Big(R^{-1} \dashv R_!\Big)\colon \quad \begin{tikzcd} \mathcal{P}(B) \arrow[r, bend left, "R^{-1}"] \arrow[r, phantom, "\perp"] & \mathcal{P}(A), \arrow[l, bend left, "R_!"] \end{tikzcd}$$

witnessed by a bijections of sets

$$\mathrm{Hom}_{\mathcal{P}(A)}(R^{-1}(U), V) \cong \mathrm{Hom}_{\mathcal{P}(A)}(U, R_!(V)),$$

natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$, i.e. such that:

($\star$) The following conditions are equivalent:
  - We have $R^{-1}(U) \subset V$.
  - We have $U \subset R_!(V)$.

3. *Preservation of Colimits.* We have an equality of sets

$$R^{-1}(\bigcup_{i \in I} U_i) = \bigcup_{i \in I} R^{-1}(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(B)^{\times I}$. In particular, we have equalities

$$R^{-1}(U) \cup R^{-1}(V) = R^{-1}(U \cup V),$$
$$R^{-1}(\emptyset) = \emptyset,$$

natural in $U, V \in \mathcal{P}(B)$.

4. *Oplax Preservation of Limits.* We have an inclusion of sets

$$R^{-1}(\bigcap_{i\in I} U_i) \subset \bigcap_{i\in I} R^{-1}(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(B)^{\times I}$. In particular, we have inclusions

$$R^{-1}(U \cap V) \subset R^{-1}(U) \cap R^{-1}(V),$$
$$R^{-1}(A) \subset B,$$

natural in $U, V \in \mathcal{P}(B)$.

5. *Symmetric Strict Monoidality With Respect to Unions.* The direct image function of Item 1 has a symmetric strict monoidal structure

$$(R^{-1}, R^{-1,\otimes}, R^{-1,\otimes}_{\mathbb{1}}) \colon (\mathcal{P}(A), \cup, \emptyset) \to (\mathcal{P}(B), \cup, \emptyset),$$

being equipped with equalities

$$R^{-1,\otimes}_{U,V} \colon R^{-1}(U) \cup R^{-1}(V) \xrightarrow{=} R^{-1}(U \cup V),$$
$$R^{-1,\otimes}_{\mathbb{1}} \colon \emptyset \xrightarrow{=} \emptyset,$$

natural in $U, V \in \mathcal{P}(B)$.

6. *Symmetric Oplax Monoidality With Respect to Intersections.* The direct image function of Item 1 has a symmetric oplax monoidal structure

$$(R^{-1}, R^{-1,\otimes}, R^{-1,\otimes}_{\mathbb{1}}) \colon (\mathcal{P}(A), \cap, A) \to (\mathcal{P}(B), \cap, B),$$

being equipped with inclusions

$$R^{-1,\otimes}_{U,V} \colon R^{-1}(U \cap V) \subset R^{-1}(U) \cap R^{-1}(V),$$
$$R^{-1,\otimes}_{\mathbb{1}} \colon R^{-1}(A) \subset B,$$

natural in $U, V \in \mathcal{P}(B)$.

7. *Interaction With Strong Inverse Images I.* We have

$$R^{-1}(V) = A \setminus R_{-1}(B \setminus V)$$

for each $V \in \mathcal{P}(B)$.

---

or simply the **inverse image of V by** R.

8. *Interaction With Strong Inverse Images II.* Let $R\colon A \nrightarrow B$ be a relation from A to B.

   (a) If R is a total relation, then we have an inclusion of sets

   $$R_{-1}(V) \subset R^{-1}(V)$$

   natural in $V \in \mathcal{P}(B)$.

   (b) If R is total and functional, then the above inclusion is in fact an equality.

   (c) Conversely, if we have $R_{-1} = R^{-1}$, then R is total and functional.

Proof. *Item 1, Functoriality*: Clear.
Item 2, Adjointness: This follows from **??**, **??** of **??**.
Item 3, Preservation of Colimits: This follows from Item 2 and **??**, **??** of **??**.
Item 4, Oplax Preservation of Limits: Omitted.
Item 5, Symmetric Strict Monoidality With Respect to Unions: This follows from Item 3.
Item 6, Symmetric Oplax Monoidality With Respect to Intersections: This follows from Item 4.
Item 7, Interaction With Strong Inverse Images I: This follows from Item 7 of Proposition 4.2.1.3.
Item 8, Interaction With Strong Inverse Images II: This was proved in Item 8 of Proposition 4.2.1.3. □

Proposition 4.3.1.4. Let $R\colon A \nrightarrow B$ be a relation.

1. *Functionality I.* The assignment $R \mapsto R^{-1}$ defines a function

   $$(-)^{-1}\colon \mathrm{Rel}(A,B) \to \mathsf{Sets}(\mathcal{P}(A),\mathcal{P}(B)).$$

2. *Functionality II.* The assignment $R \mapsto R^{-1}$ defines a function

   $$(-)^{-1}\colon \mathrm{Rel}(A,B) \to \mathsf{Pos}((\mathcal{P}(A),\subset),(\mathcal{P}(B),\subset)).$$

3. *Interaction With Identities.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have[25]

   $$(\chi_A)^{-1} = \mathrm{id}_{\mathcal{P}(A)}.$$

[25]That is, the postcomposition

$$(\chi_A)^{-1}\colon \mathrm{Rel}(\mathrm{pt},A) \to \mathrm{Rel}(\mathrm{pt},A)$$

4. *Interaction With Composition.* For each pair of composable relations $R\colon A \mathrel{\mkern3mu\vcenter{\hbox{$\scriptscriptstyle+$}}\mkern-12mu{\to}} B$ and $S\colon B \mathrel{\mkern3mu\vcenter{\hbox{$\scriptscriptstyle+$}}\mkern-12mu{\to}} C$, we have[26]

$$
(S \diamond R)^{-1} = R^{-1} \circ S^{-1}, \qquad
\begin{tikzcd}
\mathcal{P}(C) \arrow[r, "S^{-1}"] \arrow[dr, swap, "(S\diamond R)^{-1}"] & \mathcal{P}(B) \arrow[d, "R^{-1}"] \\
& \mathcal{P}(A).
\end{tikzcd}
$$

Proof. *Item 1, Functionality I*: Clear.
Item 2, Functionality II: Clear.
Item 3, Interaction With Identities: This follows from Categories, Item 5 of Proposition 1.6.1.2.
Item 4, Interaction With Composition: This follows from Categories, Item 2 of Proposition 1.6.1.2. □

4.4 Direct Images With Compact Support

Let A and B be sets and let $R\colon A \mathrel{\mkern3mu\vcenter{\hbox{$\scriptscriptstyle+$}}\mkern-12mu{\to}} B$ be a relation.

Definition 4.4.1.1. The **direct image with compact support function associated to** R is the function

$$R_!\colon \mathcal{P}(A) \to \mathcal{P}(B)$$

---

is equal to $\mathrm{id}_{\mathrm{Rel}(\mathrm{pt},A)}$.

[26]That is, we have

$$
(S \diamond R)^{-1} = R^{-1} \circ S^{-1}, \qquad
\begin{tikzcd}
\mathrm{Rel}(\mathrm{pt}, C) \arrow[r, "R^{-1}"] \arrow[dr, swap, "(S\diamond R)^{-1}"] & \mathrm{Rel}(\mathrm{pt}, B) \arrow[d, "S^{-1}"] \\
& \mathrm{Rel}(\mathrm{pt}, A).
\end{tikzcd}
$$

defined by[27,28]

$$
\begin{aligned}
R_!(U) &\overset{\text{def}}{=} \left\{ b \in B \;\middle|\; \begin{array}{l} \text{for each } a \in A, \text{ if we have} \\ b \in R(a), \text{ then } a \in U \end{array} \right\} \\
&= \left\{ b \in B \;\middle|\; R^{-1}(b) \subset U \right\}
\end{aligned}
$$

for each $U \in \mathcal{P}(A)$.

Remark 4.4.1.2. Identifying subsets of B with relations from pt to B via Constructions With Sets, Item 3 of Proposition 4.3.1.6, we see that the direct image with compact support function associated to R is equivalently the function

$$
R_! \colon \underbrace{\mathcal{P}(A)}_{\cong \mathrm{Rel}(A,\mathrm{pt})} \to \underbrace{\mathcal{P}(B)}_{\cong \mathrm{Rel}(B,\mathrm{pt})}
$$

defined by

$$
R_!(U) \overset{\text{def}}{=} \mathrm{Ran}_R(U), \qquad
\begin{tikzcd}
 & B \arrow[d, dashed, "\mathrm{Ran}_R(U)", "\shortmid"{marking}] \\
A \arrow[ur, "R", "\shortmid"{marking}] \arrow[r, "U"', "\shortmid"{marking}] \arrow[ur, phantom, shift right=3, "\Downarrow"] & \mathrm{pt},
\end{tikzcd}
$$

being explicitly computed by

$$
\begin{aligned}
R^*(U) &\overset{\text{def}}{=} \mathrm{Ran}_R(U) \\
&\cong \int_{a \in A} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R^{-2}_a, U^{-1}_a),
\end{aligned}
$$

where we have used Proposition 2.3.1.1.

[27] *Further Terminology:* The set $R_!(U)$ is called the **direct image with compact support of U by R**.

[28] We also have

$$
R_!(U) = B \setminus R_*(A \setminus U);
$$

see Item 7 of Proposition 4.4.1.3.

Proof. We have

$$
\begin{aligned}
\mathrm{Ran}_R(V) &\cong \int_{a\in A} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^{-_2}_a, U^{-_1}_a\right) \\
&= \left\{ b \in B \;\middle|\; \int_{a\in A} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R^b_a, U^\star_a) = \mathsf{true} \right\} \\
&= \left\{ b \in B \;\middle|\; \begin{array}{l} \text{for each } a \in A \text{, at least one of the} \\ \text{following conditions hold:} \\ \quad 1.\ \text{We have } R^b_a = \mathsf{false} \\ \quad 2.\ \text{The following conditions hold:} \\ \qquad \text{(a) We have } R^b_a = \mathsf{true} \\ \qquad \text{(b) We have } U^\star_a = \mathsf{true} \end{array} \right\} \\
&= \left\{ b \in B \;\middle|\; \begin{array}{l} \text{for each } a \in A \text{, at least one of the} \\ \text{following conditions hold:} \\ \quad 1.\ \text{We have } b \notin R(A) \\ \quad 2.\ \text{The following conditions hold:} \\ \qquad \text{(a) We have } b \in R(a) \\ \qquad \text{(b) We have } a \in U \end{array} \right\} \\
&= \left\{ b \in B \;\middle|\; \begin{array}{l} \text{for each } a \in A \text{, if we have} \\ b \in R(a) \text{, then } a \in U \end{array} \right\} \\
&= \left\{ b \in B \;\middle|\; R^{-1}(b) \subset U \right\} \\
&\overset{\text{def}}{=} R^{-1}(U).
\end{aligned}
$$

This finishes the proof. ∎

Proposition 4.4.1.3. Let $R\colon A \nrightarrow B$ be a relation.

1. *Functoriality.* The assignment $U \mapsto R_!(U)$ defines a functor

$$R_!\colon (\mathcal{P}(A), \subset) \to (\mathcal{P}(B), \subset)$$

where

- *Action on Objects.* For each $U \in \mathcal{P}(A)$, we have

$$[R_!](U) \overset{\text{def}}{=} R_!(U).$$

- *Action on Morphisms.* For each $U, V \in \mathcal{P}(A)$:
  - If $U \subset V$, then $R_!(U) \subset R_!(V)$.

2. *Adjointness.* We have an adjunction

$$\left(R^{-1} \dashv R_!\right)\colon \quad \begin{tikzcd} \mathcal{P}(B) \arrow[r, bend left, "R^{-1}"] \arrow[r, phantom, "\perp"] & \mathcal{P}(A), \arrow[l, bend left, "R_!"] \end{tikzcd}$$

   witnessed by a bijections of sets

$$\mathrm{Hom}_{\mathcal{P}(A)}(R^{-1}(U), V) \cong \mathrm{Hom}_{\mathcal{P}(A)}(U, R_!(V)),$$

   natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$, i.e. such that:

   ($\star$) The following conditions are equivalent:
   - We have $R^{-1}(U) \subset V$.
   - We have $U \subset R_!(V)$.

3. *Lax Preservation of Colimits.* We have an inclusion of sets

$$\bigcup_{i \in I} R_!(U_i) \subset R_!(\bigcup_{i \in I} U_i),$$

   natural in $\{U_i\}_{i \in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have inclusions

$$\begin{gathered} R_!(U) \cup R_!(V) \subset R_!(U \cup V), \\ \emptyset \subset R_!(\emptyset), \end{gathered}$$

   natural in $U, V \in \mathcal{P}(A)$.

4. *Preservation of Limits.* We have an equality of sets

$$R_!(\bigcap_{i \in I} U_i) = \bigcap_{i \in I} R_!(U_i),$$

   natural in $\{U_i\}_{i \in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have equalities

$$\begin{gathered} R_!(U \cap V) = R_!(U) \cap R_!(V), \\ R_!(A) = B, \end{gathered}$$

   natural in $U, V \in \mathcal{P}(A)$.

5. *Symmetric Lax Monoidality With Respect to Unions.* The direct image with compact support function of Item 1 has a symmetric lax monoidal structure

$$(R_!, R_!^{\otimes}, R_{!|\mathbb{1}}^{\otimes}) \colon (\mathcal{P}(A), \cup, \emptyset) \to (\mathcal{P}(B), \cup, \emptyset),$$

being equipped with inclusions

$$R_{!|U,V}^{\otimes} \colon R_!(U) \cup R_!(V) \subset R_!(U \cup V),$$
$$R_{!|\mathbb{1}}^{\otimes} \colon \emptyset \subset R_!(\emptyset),$$

natural in $U, V \in \mathcal{P}(A)$.

6. *Symmetric Strict Monoidality With Respect to Intersections.* The direct image function of Item 1 has a symmetric strict monoidal structure

$$(R_!, R_!^{\otimes}, R_{!|\mathbb{1}}^{\otimes}) \colon (\mathcal{P}(A), \cap, A) \to (\mathcal{P}(B), \cap, B),$$

being equipped with equalities

$$R_{!|U,V}^{\otimes} \colon R_!(U \cap V) \xrightarrow{=} R_!(U) \cap R_!(V),$$
$$R_{!|\mathbb{1}}^{\otimes} \colon R_!(A) \xrightarrow{=} B,$$

natural in $U, V \in \mathcal{P}(A)$.

7. *Relation to Direct Images.* We have

$$R_!(U) = B \setminus R_*(A \setminus U)$$

for each $U \in \mathcal{P}(A)$.

Proof. *Item 1, Functoriality*: Clear.
Item 2, Adjointness: This follows from **??**, **??** of **??**.
Item 3, Lax Preservation of Colimits: Omitted.
Item 4, Preservation of Limits: This follows from Item 2 and **??**, **??** of **??**.
Item 5, Symmetric Lax Monoidality With Respect to Unions: This follows from Item 3.
Item 6, Symmetric Strict Monoidality With Respect to Intersections: This follows from Item 4.
Item 7, Relation to Direct Images: This follows from Item 7 of Proposition 4.1.1.3. Alternatively, we may prove it directly as follows, with the proof proceeding in the same way as in the case of functions (Constructions With Sets, Item 9 of Proposition 4.6.1.6).
We claim that $R_!(U) = B \setminus R_*(A \setminus U)$:

- *The First Implication.* We claim that

$$R_!(U) \subset B \setminus R_*(A \setminus U).$$

Let $b \in R_!(U)$. We need to show that $b \notin R_*(A \setminus U)$, i.e. that there is no $a \in A \setminus U$ such that $b \in R(a)$.

This is indeed the case, as otherwise we would have $a \in R^{-1}(b)$ and $a \notin U$, contradicting $R^{-1}(b) \subset U$ (which holds since $b \in R_!(U)$).

Thus $b \in B \setminus R_*(A \setminus U)$.

- *The Second Implication.* We claim that

$$B \setminus R_*(A \setminus U) \subset R_!(U).$$

Let $b \in B \setminus R_*(A \setminus U)$. We need to show that $b \in R_!(U)$, i.e. that $R^{-1}(b) \subset U$.

Since $b \notin R_*(A \setminus U)$, there exists no $a \in A \setminus U$ such that $b \in R(a)$, and hence $R^{-1}(b) \subset U$.

Thus $b \in R_!(U)$.

This finishes the proof. □

Proposition 4.4.1.4. Let $R\colon A \nrightarrow B$ be a relation.

1. *Functionality I.* The assignment $R \mapsto R_!$ defines a function

$$(-)_!\colon \mathsf{Sets}(A, B) \to \mathsf{Sets}(\mathcal{P}(A), \mathcal{P}(B)).$$

2. *Functionality II.* The assignment $R \mapsto R_!$ defines a function

$$(-)_!\colon \mathsf{Sets}(A, B) \to \mathrm{Hom}_{\mathsf{Pos}}((\mathcal{P}(A), \subset), (\mathcal{P}(B), \subset)).$$

3. *Interaction With Identities.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$(\mathrm{id}_A)_! = \mathrm{id}_{\mathcal{P}(A)}.$$

4. *Interaction With Composition.* For each pair of composable relations $R\colon A \nrightarrow B$ and $S\colon B \nrightarrow C$, we have

$$(S \diamond R)_! = S_! \circ R_!, \qquad \begin{tikzcd} \mathcal{P}(A) \arrow[r, "R_!"] \arrow[dr, "(S\diamond R)_!"'] & \mathcal{P}(B) \arrow[d, "S_!"] \\ & \mathcal{P}(C). \end{tikzcd}$$

Proof. *Item 1, Functionality I*: Clear.
Item 2, Functionality II: Clear.
Item 3, Interaction With Identities: Indeed, we have

$$\begin{aligned}(\chi_A)_!(U) &\stackrel{\text{def}}{=} \left\{a \in A \;\middle|\; \chi_A^{-1}(a) \subset U\right\}\\ &\stackrel{\text{def}}{=} \{a \in A \mid \{a\} \subset U\}\\ &= U\end{aligned}$$

for each $U \in \mathcal{P}(A)$. Thus $(\chi_A)_! = \mathrm{id}_{\mathcal{P}(A)}$.
Item 4, Interaction With Composition: Indeed, we have

$$\begin{aligned}(S \diamond R)_!(U) &\stackrel{\text{def}}{=} \left\{c \in C \;\middle|\; [S \diamond R]^{-1}(c) \subset U\right\}\\ &\stackrel{\text{def}}{=} \left\{c \in C \;\middle|\; S^{-1}(R^{-1}(c)) \subset U\right\}\\ &= \left\{c \in C \;\middle|\; R^{-1}(c) \subset S_!(U)\right\}\\ &\stackrel{\text{def}}{=} R_!(S_!(U))\\ &\stackrel{\text{def}}{=} [R_! \circ S_!](U)\end{aligned}$$

for each $U \in \mathcal{P}(C)$, where we used Item 2 of Proposition 4.4.1.3, which implies that the conditions

- We have $S^{-1}(R^{-1}(c)) \subset U$.
- We have $R^{-1}(c) \subset S_!(U)$.

are equivalent. Thus $(S \diamond R)_! = S_! \circ R_!$. □

4.5 Functoriality of Powersets

Proposition 4.5.1.1. The assignment $X \mapsto \mathcal{P}(X)$ defines functors[29]

$$\begin{aligned}\mathcal{P}_* &\colon \mathrm{Rel} \to \mathsf{Sets},\\ \mathcal{P}_{-1} &\colon \mathrm{Rel}^{\mathsf{op}} \to \mathsf{Sets},\\ \mathcal{P}^{-1} &\colon \mathrm{Rel}^{\mathsf{op}} \to \mathsf{Sets},\\ \mathcal{P}_! &\colon \mathrm{Rel} \to \mathsf{Sets}\end{aligned}$$

where

[29] The functor $\mathcal{P}_* \colon \mathrm{Rel} \to \mathsf{Sets}$ admits a left adjoint; see Item 3 of Proposition 3.1.1.2.

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathrm{Rel})$, we have

$$\begin{aligned} \mathcal{P}_*(A) &\stackrel{\text{def}}{=} \mathcal{P}(A), \\ \mathcal{P}_{-1}(A) &\stackrel{\text{def}}{=} \mathcal{P}(A), \\ \mathcal{P}^{-1}(A) &\stackrel{\text{def}}{=} \mathcal{P}(A), \\ \mathcal{P}_!(A) &\stackrel{\text{def}}{=} \mathcal{P}(A). \end{aligned}$$

- *Action on Morphisms.* For each morphism $R\colon A \nrightarrow B$ of Rel, the images

$$\begin{aligned} \mathcal{P}_*(R)&\colon \mathcal{P}(A) \to \mathcal{P}(B), \\ \mathcal{P}_{-1}(R)&\colon \mathcal{P}(B) \to \mathcal{P}(A), \\ \mathcal{P}^{-1}(R)&\colon \mathcal{P}(B) \to \mathcal{P}(A), \\ \mathcal{P}_!(R)&\colon \mathcal{P}(A) \to \mathcal{P}(B) \end{aligned}$$

  of R by $\mathcal{P}_*$, $\mathcal{P}_{-1}$, $\mathcal{P}^{-1}$, and $\mathcal{P}_!$ are defined by

$$\begin{aligned} \mathcal{P}_*(R) &\stackrel{\text{def}}{=} R_*, \\ \mathcal{P}_{-1}(R) &\stackrel{\text{def}}{=} R_{-1}, \\ \mathcal{P}^{-1}(R) &\stackrel{\text{def}}{=} R^{-1}, \\ \mathcal{P}_!(R) &\stackrel{\text{def}}{=} R_!, \end{aligned}$$

  as in Definitions 4.1.1.1, 4.2.1.1, 4.3.1.1 and 4.4.1.1.

Proof. This follows from Items 3 and 4 of Proposition 4.1.1.4, Items 3 and 4 of Proposition 4.2.1.4, Items 3 and 4 of Proposition 4.3.1.4, and Items 3 and 4 of Proposition 4.4.1.4. □

4.6 Functoriality of Powersets: Relations on Powersets

Let A and B be sets and let $R\colon A \nrightarrow B$ be a relation.

Definition 4.6.1.1. The **relation on powersets associated to** R is the relation

$$\mathcal{P}(R)\colon \mathcal{P}(A) \nrightarrow \mathcal{P}(B)$$

defined by[30]

$$\mathcal{P}(R)^V_U \overset{\text{def}}{=} \mathbf{Rel}(\chi_{\mathrm{pt}}, V \diamond R \diamond U)$$

for each $U \in \mathcal{P}(A)$ and each $V \in \mathcal{P}(B)$.

Remark 4.6.1.2. In detail, we have $U \sim_{\mathcal{P}(R)} V$ iff the following equivalent conditions hold:

- We have $\chi_{\mathrm{pt}} \subset V \diamond R \diamond U$.
- We have $(V \diamond R \diamond U)^{\star}_{\star} = \mathsf{true}$, i.e. we have

$$\int^{a \in A} \int^{b \in B} V^{\star}_{b} \times R^{b}_{a} \times U^{a}_{\star} = \mathsf{true}.$$

- There exists some $a \in A$ and some $b \in B$ such that:
  - We have $U^{a}_{\star} = \mathsf{true}$.
  - We have $R^{b}_{a} = \mathsf{true}$.
  - We have $V^{\star}_{b} = \mathsf{true}$.
- There exists some $a \in A$ and some $b \in B$ such that:
  - We have $a \in U$.
  - We have $a \sim_R b$.
  - We have $b \in V$.

Proposition 4.6.1.3. The assignment $R \mapsto \mathcal{P}(R)$ defines a functor

$$\mathcal{P} \colon \mathrm{Rel} \to \mathrm{Rel}.$$

Proof. Omitted. □

Appendices

[30]Illustration:

$$\begin{tikzcd} \mathrm{pt} \arrow[r, "U"', "|" marking] \arrow[rrr, bend left, "\chi_{\mathrm{pt}}", "|" marking] & A \arrow[r, "R"', "|" marking] & B \arrow[r, "V"', "|" marking] & \mathrm{pt}. \end{tikzcd}$$

A Other Chapters

References

[MO 460656] Emily de Oliveira Santos. *Existence and characterisations of left Kan extensions and liftings in the bicategory of relations I*. MathOverflow. URL: https://mathoverflow.net/q/460656 (cit. on pp. 3, 4).

[MO 461592] Emily de Oliveira Santos. *Existence and characterisations of left Kan extensions and liftings in the bicategory of relations II*. MathOverflow. URL: https://mathoverflow.net/q/461592 (cit. on pp. 4, 5).