Pointed Sets

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This chapter contains some foundational material on pointed sets.

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1 Pointed Sets

1.1 Foundations

Definition 1.1.1.1. A **pointed set**[1] is equivalently:

- An $\mathbb{E}_0$-monoid in $(\mathrm{N}_\bullet(\mathsf{Sets}), \mathrm{pt})$.
- A pointed object in $(\mathsf{Sets}, \mathrm{pt})$.

Remark 1.1.1.2. In detail, a **pointed set** is a pair (X, x_0) consisting of:

- *The Underlying Set.* A set X, called the **underlying set of** (X, x_0).
- *The Basepoint.* A morphism

$$[x_0] \colon \mathrm{pt} \to X$$

  in Sets, determining an element $x_0 \in X$, called the **basepoint of** X.

Example 1.1.1.3. The 0**-sphere**[2] is the pointed set $(S^0, 0)$[3] consisting of:

- *The Underlying Set.* The set S^0 defined by

$$S^0 \stackrel{\text{def}}{=} \{0, 1\}.$$

- *The Basepoint.* The element 0 of S^0.

Example 1.1.1.4. The **trivial pointed set** is the pointed set $(\mathrm{pt}, \star)$ consisting of:

- *The Underlying Set.* The punctual set $\mathrm{pt} \stackrel{\text{def}}{=} \{\star\}$.
- *The Basepoint.* The element $\star$ of pt.

Example 1.1.1.5. The **underlying pointed set** of a semimodule (M, α_M) is the pointed set $(M, 0_M)$.

Example 1.1.1.6. The **underlying pointed set** of a module (M, α_M) is the pointed set $(M, 0_M)$.

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[1] *Further Terminology:* In the context of monoids with zero as models for $\mathbb{F}_1$-algebras, pointed sets are viewed as $\mathbb{F}_1$**-modules**.

[2] *Further Terminology:* In the context of monoids with zero as models for $\mathbb{F}_1$-algebras, the 0-sphere is viewed as the **underlying pointed set of the field with one element**.

[3] *Further Notation:* In the context of monoids with zero as models for $\mathbb{F}_1$-algebras, S^0

1.2 Morphisms of Pointed Sets

Definition 1.2.1.1. A **morphism of pointed sets**[4,5] is equivalently:

- A morphism of $\mathbb{E}_0$-monoids in $(\mathrm{N}_\bullet(\mathsf{Sets}), \mathrm{pt})$.
- A morphism of pointed objects in $(\mathsf{Sets}, \mathrm{pt})$.

Remark 1.2.1.2. In detail, a **morphism of pointed sets** $f\colon (X, x_0) \to (Y, y_0)$ is a morphism of sets $f\colon X \to Y$ such that the diagram

$$\begin{tikzcd} & \mathrm{pt} \arrow[dl, "{[x_0]}"'] \arrow[dr, "{[y_0]}"] & \\ X \arrow[rr, "f"'] & & Y \end{tikzcd}$$

commutes, i.e. such that

$$f(x_0) = y_0.$$

1.3 The Category of Pointed Sets

Definition 1.3.1.1. The **category of pointed sets** is the category Sets_* defined equivalently as

- The homotopy category of the ∞-category $\mathsf{Mon}_{\mathbb{E}_0}(\mathrm{N}_\bullet(\mathsf{Sets}), \mathrm{pt})$ of **??**, **??**;
- The category Sets_* of **??**, **??**.

Remark 1.3.1.2. In detail, the **category of pointed sets** is the category Sets_* where

- *Objects.* The objects of Sets_* are pointed sets;
- *Morphisms.* The morphisms of Sets_* are morphisms of pointed sets;
- *Identities.* For each $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, the unit map

  $$\mathbb{1}^{\mathsf{Sets}_*}_{(X,x_0)}\colon \mathrm{pt} \to \mathsf{Sets}_*((X, x_0), (X, x_0))$$

  of Sets_* at (X, x_0) is defined by[6]

  $$\mathrm{id}^{\mathsf{Sets}_*}_{(X,x_0)} \stackrel{\text{def}}{=} \mathrm{id}_X;$$

is also denoted $(\mathbb{F}_1, 0)$.

[4] *Further Terminology:* Also called a **pointed function**.

[5] *Further Terminology:* In the context of monoids with zero as models for $\mathbb{F}_1$-algebras, morphisms of pointed sets are also called **morphism of $\mathbb{F}_1$-modules**.

[6] Note that id_X is indeed a morphism of pointed sets, as we have $\mathrm{id}_X(x_0) = x_0$.

- *Composition.* For each $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, the composition map

$$\circ^{\mathsf{Sets}_*}_{(X,x_0),(Y,y_0),(Z,z_0)} \colon \mathsf{Sets}_*((Y, y_0), (Z, z_0)) \times \mathsf{Sets}_*((X, x_0), (Y, y_0)) \to \mathsf{Sets}_*((X, x_0), (Z, z_0))$$

of Sets_* at $((X, x_0), (Y, y_0), (Z, z_0))$ is defined by[7]

$$g \circ^{\mathsf{Sets}_*}_{(X,x_0),(Y,y_0),(Z,z_0)} f \stackrel{\text{def}}{=} g \circ f.$$

1.4 Elementary Properties of Pointed Sets

Proposition 1.4.1.1. Let (X, x_0) be a pointed set.

1. *Completeness.* The category Sets_* of pointed sets and morphisms between them is complete, having in particular:

   (a) Products, described as in Definition 2.3.1.1;

   (b) Pullbacks, described as in Definition 2.4.1.1;

   (c) Equalisers, described as in Definition 2.5.1.1.

2. *Cocompleteness.* The category Sets_* of pointed sets and morphisms between them is cocomplete, having in particular:

   (a) Coproducts, described as in Definition 3.3.1.1;

   (b) Pushouts, described as in Definition 3.4.1.1;

   (c) Coequalisers, described as in Definition 3.5.1.1.

3. *Failure To Be Cartesian Closed.* The category Sets_* is not Cartesian closed.[8]

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[7] Note that the composition of two morphisms of pointed sets is indeed a morphism of pointed sets, as we have

$$\begin{aligned} g(f(x_0)) &= g(y_0) \\ &= z_0, \end{aligned}$$

or

$$\begin{tikzcd} & \mathrm{pt} \arrow[dl, "{[x_0]}"'] \arrow[d, "{[y_0]}"] \arrow[dr, "{[z_0]}"] & \\ X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z \end{tikzcd}$$

in terms of diagrams.

[8] The category Sets_* does admit monoidal closed structures however; see Tensor Products

4. *Morphisms From the Monoidal Unit.* We have a bijection of sets[9]

$$\mathsf{Sets}_*(S^0, X) \cong X,$$

natural in $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, internalising also to an isomorphism of pointed sets

$$\mathbf{Sets}_*(S^0, X) \cong (X, x_0),$$

again natural in $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

5. *Relation to Partial Functions.* We have an equivalence of categories[10]

$$\mathsf{Sets}_* \overset{\text{eq.}}{\cong} \mathsf{Sets}^{\text{part.}}$$

between the category of pointed sets and pointed functions between them and the category of sets and partial functions between them, where:

(a) *From Pointed Sets to Sets With Partial Functions.* The equivalence

$$\xi \colon \mathsf{Sets}_* \xrightarrow{\cong} \mathsf{Sets}^{\text{part.}}$$

sends:

i. A pointed set (X, x_0) to X.

ii. A pointed function

$$f \colon (X, x_0) \to (Y, y_0)$$

to the partial function

$$\xi_f \colon X \to Y$$

defined on $f^{-1}(Y \setminus y_0)$ and given by

$$\xi_f(x) \overset{\text{def}}{=} f(x)$$

for each $x \in f^{-1}(Y \setminus y_0)$.

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of Pointed Sets.

[9] In other words, the forgetful functor

$$忘 \colon \mathsf{Sets}_* \to \mathsf{Sets}$$

defined on objects by sending a pointed set to its underlying set is corepresentable by S^0.

[10] *Warning:* This is not an isomorphism of categories, only an equivalence.

(b) *From Sets With Partial Functions to Pointed Sets.* The equivalence

$$\xi^{-1}\colon \mathsf{Sets}^{\text{part.}} \xrightarrow{\cong} \mathsf{Sets}_*$$

sends:

i. A set X is to the pointed set $(X, \star)$ with $\star$ an element that is not in X.
ii. A partial function

$$f\colon X \to Y$$

defined on $U \subset X$ to the pointed function

$$\xi_f^{-1}\colon (X, x_0) \to (Y, y_0)$$

defined by

$$\xi_f(x) \stackrel{\text{def}}{=} \begin{cases} f(x) & \text{if } x \in U, \\ y_0 & \text{otherwise.} \end{cases}$$

for each $x \in X$.

Proof. Item 1, Completeness: This follows from (the proofs) of Definitions 2.3.1.1, 2.4.1.1 and 2.5.1.1 and **??**, **??**.
Item 2, Cocompleteness: This follows from (the proofs) of Definitions 3.3.1.1, 3.4.1.1 and 3.5.1.1 and **??**, **??**.
Item 3, Failure To Be Cartesian Closed: See [MSE 2855868].
Item 4, Morphisms From the Monoidal Unit: Since a morphism from S^0 to a pointed set (X, x_0) sends $0 \in S^0$ to x_0 and then can send $1 \in S^0$ to any element of X, we obtain a bijection between pointed maps $S^0 \to X$ and the elements of X.
The isomorphism then

$$\mathbf{Sets}_*(S^0, X) \cong (X, x_0)$$

follows by noting that $\Delta_{x_0}\colon S^0 \to X$, the basepoint of $\mathbf{Sets}_*(S^0, X)$, corresponds to the pointed map $S^0 \to X$ picking the element x_0 of X, and thus we see that the bijection between pointed maps $S^0 \to X$ and elements of X is compatible with basepoints, lifting to an isomorphism of pointed sets.
Item 5, Relation to Partial Functions: See [MSE 884460]. □

2 Limits of Pointed Sets

2.1 The Terminal Pointed Set

Definition 2.1.1.1. The **terminal pointed set** is the pair $((\mathrm{pt}, \star), \{!_X\}_{(X,x_0)\in\mathrm{Obj}(\mathsf{Sets}_*)})$ consisting of:

- *The Limit.* The pointed set $(\mathrm{pt}, \star)$.
- *The Cone.* The collection of morphisms of pointed sets

$$\{!_X : (X, x_0) \to (\mathrm{pt}, \star)\}_{(X,x_0)\in\mathrm{Obj}(\mathsf{Sets})}$$

  defined by

$$!_X(x) \stackrel{\mathrm{def}}{=} \star$$

  for each $x \in X$ and each $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets})$.

Proof. We claim that $(\mathrm{pt}, \star)$ is the terminal object of Sets_*. Indeed, suppose we have a diagram of the form

$$(X, x_0) \qquad (\mathrm{pt}, \star)$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi\colon (X, x_0) \to (\mathrm{pt}, \star)$$

making the diagram

$$\begin{tikzcd} (X, x_0) \arrow[r, dashed, "\phi", "\exists!"'] & (\mathrm{pt}, \star) \end{tikzcd}$$

commute, namely $!_X$. $\square$

2.2 Products of Families of Pointed Sets

Let $\{(X_i, x_0^i)\}_{i\in I}$ be a family of pointed sets.

Definition 2.2.1.1. The **product of** $\{(X_i, x_0^i)\}_{i\in I}$ is the pair $((\prod_{i\in I} X_i, (x_0^i)_{i\in I}), \{\mathrm{pr}_i\}_{i\in I})$ consisting of:

- *The Limit.* The pointed set $(\prod_{i\in I} X_i, (x_0^i)_{i\in I})$.
- *The Cone.* The collection

$$\left\{\mathrm{pr}_i : (\prod_{i\in I} X_i, (x_0^i)_{i\in I}) \to (X_i, x_0^i)\right\}_{i\in I}$$

of maps given by

$$\mathrm{pr}_i((x_j)_{j\in I}) \overset{\mathrm{def}}{=} x_i$$

for each $(x_j)_{j\in I} \in \prod_{i\in I} X_i$ and each $i \in I$.

Proof. We claim that $(\prod_{i\in I} X_i, (x_0^i)_{i\in I})$ is the categorical product of $\{(X_i, x_0^i)\}_{i\in I}$ in Sets_*. Indeed, suppose we have, for each $i \in I$, a diagram of the form

$$\begin{tikzcd} (P,*) \arrow[dr, "p_i"] & \\ (\prod_{i\in I} X_i, (x_0^i)_{i\in I}) \arrow[r, "\mathrm{pr}_i"'] & (X_i, x_0^i) \end{tikzcd}$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi\colon (P,*) \to (\prod_{i\in I} X_i, (x_0^i)_{i\in I})$$

making the diagram

$$\begin{tikzcd} (P,*) \arrow[d, dashed, "\phi"', "\exists!"] \arrow[dr, "p_i"] & \\ (\prod_{i\in I} X_i, (x_0^i)_{i\in I}) \arrow[r, "\mathrm{pr}_i"'] & (X_i, x_0^i) \end{tikzcd}$$

commute, being uniquely determined by the condition $\mathrm{pr}_i \circ \phi = p_i$ for each $i \in I$ via

$$\phi(x) = (p_i(x))_{i\in I}$$

for each $x \in P$. Note that this is indeed a morphism of pointed sets, as we have

$$\begin{aligned} \phi(*) &= (p_i(*))_{i\in I} \\ &= (x_0^i)_{i\in I}, \end{aligned}$$

where we have used that p_i is a morphism of pointed sets for each $i \in I$. □

Proposition 2.2.1.2. Let $\{(X_i, x_0^i)\}_{i\in I}$ be a family of pointed sets.

1. *Functoriality.* The assignment $\{(X_i, x_0^i)\}_{i\in I} \mapsto (\prod_{i\in I} X_i, (x_0^i)_{i\in I})$ defines a functor

$$\prod_{i\in I}\colon \mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}_*) \to \mathsf{Sets}_*.$$

Proof. Item 1, *Functoriality*: This follows from **??**, **??** of **??**. □

2.3 Products

Let (X, x_0) and (Y, y_0) be pointed sets.

Definition 2.3.1.1. The **product of** (X, x_0) **and** (Y, y_0) is the pair consisting of:

- *The Limit.* The pointed set $(X \times Y, (x_0, y_0))$.
- *The Cone.* The morphisms of pointed sets

  $$\begin{aligned} \mathrm{pr}_1 &\colon (X \times Y, (x_0, y_0)) \to (X, x_0), \\ \mathrm{pr}_2 &\colon (X \times Y, (x_0, y_0)) \to (Y, y_0) \end{aligned}$$

  defined by

  $$\begin{aligned} \mathrm{pr}_1(x, y) &\overset{\text{def}}{=} x, \\ \mathrm{pr}_2(x, y) &\overset{\text{def}}{=} y \end{aligned}$$

  for each $(x, y) \in X \times Y$.

Proof. We claim that $(X \times Y, (x_0, y_0))$ is the categorical product of (X, x_0) and (Y, y_0) in Sets_*. Indeed, suppose we have a diagram of the form

$$\begin{tikzcd} & (P, *) \arrow[dl, bend right, "p_1"'] \arrow[dr, bend left, "p_2"] & \\ (X, x_0) & (X \times Y, (x_0, y_0)) \arrow[l, "\mathrm{pr}_1"] \arrow[r, "\mathrm{pr}_2"'] & (Y, y_0) \end{tikzcd}$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi \colon (P, *) \to (X \times Y, (x_0, y_0))$$

making the diagram

$$\begin{tikzcd} & (P, *) \arrow[dl, bend right, "p_1"'] \arrow[dr, bend left, "p_2"] \arrow[d, dashed, "\phi"', "\exists !"] & \\ (X, x_0) & (X \times Y, (x_0, y_0)) \arrow[l, "\mathrm{pr}_1"] \arrow[r, "\mathrm{pr}_2"'] & (Y, y_0) \end{tikzcd}$$

commute, being uniquely determined by the conditions

$$\begin{aligned} \mathrm{pr}_1 \circ \phi &= p_1, \\ \mathrm{pr}_2 \circ \phi &= p_2 \end{aligned}$$

via

$$\phi(x) = (p_1(x), p_2(x))$$

for each $x \in P$. Note that this is indeed a morphism of pointed sets, as we have

$$\begin{aligned}\phi(*) &= (p_1(*), p_2(*))\\ &= (x_0, y_0),\end{aligned}$$

where we have used that p_1 and p_2 are morphisms of pointed sets. □

Proposition 2.3.1.2. Let (X, x_0), (Y, y_0), and (Z, z_0) be pointed sets.

1. *Functoriality.* The assignments

$$(X, x_0), (Y, y_0), ((X, x_0), (Y, y_0)) \mapsto (X \times Y, (x_0, y_0))$$

   define functors

$$\begin{aligned}X \times -&\colon \mathsf{Sets}_* \to \mathsf{Sets}_*,\\ - \times Y&\colon \mathsf{Sets}_* \to \mathsf{Sets}_*,\\ -_1 \times -_2&\colon \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*,\end{aligned}$$

   defined in the same way as the functors of Constructions With Sets, Item 1 of Proposition 1.3.1.2.

2. *Associativity.* We have an isomorphism of pointed sets

$$((X \times Y) \times Z, ((x_0, y_0), z_0)) \cong (X \times (Y \times Z), (x_0, (y_0, z_0)))$$

   natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

3. *Unitality.* We have isomorphisms of pointed sets

$$\begin{aligned}(\mathrm{pt}, \star) \times (X, x_0) &\cong (X, x_0),\\ (X, x_0) \times (\mathrm{pt}, \star) &\cong (X, x_0),\end{aligned}$$

   natural in $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

4. *Commutativity.* We have an isomorphism of pointed sets

$$(X \times Y, (x_0, y_0)) \cong (Y \times X, (y_0, x_0)),$$

   natural in $(X, x_0), (Y, y_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

5. *Symmetric Monoidality.* The triple $(\mathsf{Sets}_*, \times, (\mathrm{pt}, \star))$ is a symmetric monoidal category.

Proof. *Item 1, Functoriality*: This is a special case of functoriality of limits, **??**, **??** of **??**.
Item 2, Associativity: This follows from Constructions With Sets, Item 3 of Proposition 1.3.1.2.
Item 3, Unitality: This follows from Constructions With Sets, Item 4 of Proposition 1.3.1.2.
Item 4, Commutativity: This follows from Constructions With Sets, Item 5 of Proposition 1.3.1.2.
Item 5, Symmetric Monoidality: This follows from Constructions With Sets, Item 12 of Proposition 1.3.1.2. □

2.4 Pullbacks

Let (X, x_0), (Y, y_0), and (Z, z_0) be pointed sets and let $f\colon (X, x_0) \to (Z, z_0)$ and $g\colon (Y, y_0) \to (Z, z_0)$ be morphisms of pointed sets.

Definition 2.4.1.1. The **pullback of** (X, x_0) **and** (Y, y_0) **over** (Z, z_0) **along** (f, g) is the pair consisting of:

- *The Limit.* The pointed set $(X \times_Z Y, (x_0, y_0))$.
- *The Cone.* The morphisms of pointed sets

$$\begin{aligned}\mathrm{pr}_1 &\colon (X \times_Z Y, (x_0, y_0)) \to (X, x_0),\\ \mathrm{pr}_2 &\colon (X \times_Z Y, (x_0, y_0)) \to (Y, y_0)\end{aligned}$$

  defined by

$$\begin{aligned}\mathrm{pr}_1(x, y) &\stackrel{\text{def}}{=} x,\\ \mathrm{pr}_2(x, y) &\stackrel{\text{def}}{=} y\end{aligned}$$

  for each $(x, y) \in X \times_Z Y$.

Proof. We claim that $X \times_Z Y$ is the categorical pullback of (X, x_0) and (Y, y_0) over (Z, z_0) with respect to (f, g) in Sets_*. First we need to check that the relevant pullback diagram commutes, i.e. that we have

$$f \circ \mathrm{pr}_1 = g \circ \mathrm{pr}_2, \qquad \begin{tikzcd} (X \times_Z Y, (x_0, y_0)) \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] & (Y, y_0) \arrow[d, "g"] \\ (X, x_0) \arrow[r, "f"'] & (Z, z_0). \end{tikzcd}$$

Indeed, given $(x, y) \in X \times_Z Y$, we have

$$\begin{aligned}[f \circ \mathrm{pr}_1](x, y) &= f(\mathrm{pr}_1(x, y)) \\ &= f(x) \\ &= g(y) \\ &= g(\mathrm{pr}_2(x, y)) \\ &= [g \circ \mathrm{pr}_2](x, y),\end{aligned}$$

where $f(x) = g(y)$ since $(x, y) \in X \times_Z Y$. Next, we prove that $X \times_Z Y$ satisfies the universal property of the pullback. Suppose we have a diagram of the form

$$\begin{tikzcd} (P, *) \arrow[drr, bend left, "p_2"] \arrow[ddr, bend right, "p_1"'] & & \\ & (X \times_Z Y, (x_0, y_0)) \arrow[r, dashed, "\mathrm{pr}_2"] \arrow[d, dashed, "\mathrm{pr}_1"] \arrow[dr, phantom, "\lrcorner", very near start] & (Y, y_0) \arrow[d, "g"] \\ & (X, x_0) \arrow[r, "f"'] & (Z, z_0) \end{tikzcd}$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi \colon (P, *) \to (X \times_Z Y, (x_0, y_0))$$

making the diagram

$$\begin{tikzcd} (P, *) \arrow[drr, bend left, "p_2"] \arrow[ddr, bend right, "p_1"'] \arrow[dr, dashed, "\phi", "\exists!"'] & & \\ & (X \times_Z Y, (x_0, y_0)) \arrow[r, dashed, "\mathrm{pr}_2"] \arrow[d, dashed, "\mathrm{pr}_1"] \arrow[dr, phantom, "\lrcorner", very near start] & (Y, y_0) \arrow[d, "g"] \\ & (X, x_0) \arrow[r, "f"'] & (Z, z_0) \end{tikzcd}$$

commute, being uniquely determined by the conditions

$$\begin{aligned}\mathrm{pr}_1 \circ \phi &= p_1, \\ \mathrm{pr}_2 \circ \phi &= p_2\end{aligned}$$

via

$$\phi(x) = (p_1(x), p_2(x))$$

for each $x \in P$, where we note that $(p_1(x), p_2(x)) \in X \times Y$ indeed lies in $X \times_Z Y$ by the condition

$$f \circ p_1 = g \circ p_2,$$

which gives

$$f(p_1(x)) = g(p_2(x))$$

for each $x \in P$, so that $(p_1(x), p_2(x)) \in X \times_Z Y$. Lastly, we note that ϕ is indeed a morphism of pointed sets, as we have

$$\begin{aligned}\phi(*) &= (p_1(*), p_2(*))\\ &= (x_0, y_0),\end{aligned}$$

where we have used that p_1 and p_2 are morphisms of pointed sets. □

Proposition 2.4.1.2. Let (X, x_0), (Y, y_0), (Z, z_0), and (A, a_0) be pointed sets.

1. *Functoriality.* The assignment $(X, Y, Z, f, g) \mapsto X \times_{f,Z,g} Y$ defines a functor

   $$-_1 \times_{-_3} -_1 \colon \mathsf{Fun}(\mathcal{P}, \mathsf{Sets}_*) \to \mathsf{Sets}_*,$$

   where $\mathcal{P}$ is the category that looks like this:

   $$\begin{tikzcd} & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet. \end{tikzcd}$$

   In particular, the action on morphisms of $-_1 \times_{-_3} -_1$ is given by sending a morphism

   $$\begin{tikzcd} X \times_Z Y \arrow[rr] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & & Y \arrow[dd, "g" near start] \arrow[dr, "\psi"] & \\ & X' \times_{Z'} Y' \arrow[rr, crossing over] \arrow[dr, phantom, "\lrcorner", very near start] & & Y' \arrow[dd, "g'"] \\ X \arrow[rr, "f" near start] \arrow[dr, "\phi"'] & & Z \arrow[dr, "\chi"] & \\ & X' \arrow[rr, "f'"'] \arrow[from=uu, crossing over] & & Z' \end{tikzcd}$$

in $\mathsf{Fun}(\mathcal{P}, \mathsf{Sets}_*)$ to the morphism of pointed sets

$$\xi\colon (X \times_Z Y, (x_0, y_0)) \xrightarrow{\exists!} (X' \times_{Z'} Y', (x'_0, y'_0))$$

given by

$$\xi(x, y) \overset{\text{def}}{=} (\phi(x), \psi(y))$$

for each $(x, y) \in X \times_Z Y$, which is the unique morphism of pointed sets making the diagram

$$\begin{tikzcd}
X \times_Z Y \arrow[rr] \arrow[dr, dashed] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & & Y \arrow[dr, "\psi"] \arrow[dd, "g"' near start] & \\
& X' \times_{Z'} Y' \arrow[rr] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & & Y' \arrow[dd, "g'"] \\
X \arrow[rr, "f" near start] \arrow[dr, "\phi"'] & & Z \arrow[dr, "\chi"] & \\
& X' \arrow[rr, "f'"'] & & Z'
\end{tikzcd}$$

commute.

2. *Associativity.* Given a diagram

$$\begin{tikzcd}
X \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \arrow[dr, "h"'] & & Z \arrow[dl, "k"] \\
& W & & V &
\end{tikzcd}$$

in Sets_*, we have isomorphisms of pointed sets

$$(X \times_W Y) \times_V Z \cong (X \times_W Y) \times_Y (Y \times_V Z) \cong X \times_W (Y \times_V Z),$$

where these pullbacks are built as in the diagrams

$$\begin{tikzcd}[column sep=small]
& & (X \times_W Y) \times_Y Z \arrow[dl] \arrow[ddrr] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & & \\
& X \times_W Y \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & & & \\
X \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \arrow[dr, "h"'] & & Z, \arrow[dl, "k"] \\
& W & & V &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=small]
& & (X \times_W Y) \times_Y (Y \times_V Z) \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & & \\
& X \times_W Y \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & & Y \times_V Z \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & \\
X \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \arrow[dr, "h"'] & & Z, \arrow[dl, "k"] \\
& W & & V &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=small]
& & X \times_W (Y \times_V Z) \arrow[ddll] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & & \\
& & & Y \times_V Z \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & \\
X \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \arrow[dr, "h"'] & & Z. \arrow[dl, "k"] \\
& W & & V &
\end{tikzcd}$$

3. *Unitality.* We have isomorphisms of pointed sets

$$
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\
X \arrow[r, equal] & X
\end{tikzcd}
\qquad
\begin{aligned}
X \times_X A &\cong A, \\
A \times_X X &\cong A,
\end{aligned}
\qquad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, equal] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, equal] \\
X \arrow[r, "f"'] & X.
\end{tikzcd}
$$

4. *Commutativity.* We have an isomorphism of pointed sets

$$
\begin{tikzcd}
A \times_X B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "g"] \\
A \arrow[r, "f"'] & X,
\end{tikzcd}
\qquad
A \times_X B \cong B \times_X A
\qquad
\begin{tikzcd}
B \times_X A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\
B \arrow[r, "g"'] & X.
\end{tikzcd}
$$

5. *Interaction With Products.* We have an isomorphism of pointed sets

$$
X \times_{\mathrm{pt}} Y \cong X \times Y,
\qquad
\begin{tikzcd}
X \times Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "!_Y"] \\
X \arrow[r, "!_X"'] & \mathrm{pt}.
\end{tikzcd}
$$

6. *Symmetric Monoidality.* The triple $(\mathsf{Sets}_*, \times_X, X)$ is a symmetric monoidal category.

Proof. *Item 1, Functoriality*: This is a special case of functoriality of co/limits, **??**, **??** of **??**, with the explicit expression for ξ following from the commutativity of the cube pullback diagram.
Item 2, Associativity: This follows from Constructions With Sets, Item 2 of Proposition 1.4.1.3.
Item 3, Unitality: This follows from Constructions With Sets, Item 3 of Proposition 1.4.1.3.
Item 4, Commutativity: This follows from Constructions With Sets, Item 4 of Proposition 1.4.1.3.
Item 5, Interaction With Products: This follows from Constructions With Sets, Item 6 of Proposition 1.4.1.3.
Item 6, Symmetric Monoidality: This follows from Constructions With Sets, Item 7 of Proposition 1.4.1.3. □

2.5 Equalisers

Let $f, g \colon (X, x_0) \rightrightarrows (Y, y_0)$ be morphisms of pointed sets.

Definition 2.5.1.1. The **equaliser of** (f, g) is the pair consisting of:

- *The Limit.* The pointed set $(\mathrm{Eq}(f, g), x_0)$.
- *The Cone.* The morphism of pointed sets

$$\mathrm{eq}(f, g) \colon (\mathrm{Eq}(f, g), x_0) \hookrightarrow (X, x_0)$$

  given by the canonical inclusion $\mathrm{eq}(f, g) \hookrightarrow \mathrm{Eq}(f, g) \hookrightarrow X$.

Proof. We claim that $(\mathrm{Eq}(f, g), x_0)$ is the categorical equaliser of f and g in Sets_*. First we need to check that the relevant equaliser diagram commutes, i.e. that we have

$$f \circ \mathrm{eq}(f, g) = g \circ \mathrm{eq}(f, g),$$

which indeed holds by the definition of the set $\mathrm{Eq}(f, g)$. Next, we prove that $\mathrm{Eq}(f, g)$ satisfies the universal property of the equaliser. Suppose we have a diagram of the form

$$\begin{tikzcd}
(\mathrm{Eq}(f,g), x_0) \arrow[r, hook, "\mathrm{eq}(f,g)"] & (X, x_0) \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & (Y, y_0) \\
(E, *) \arrow[ur, "e"'] & &
\end{tikzcd}$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi \colon (E, *) \to (\mathrm{Eq}(f, g), x_0)$$

making the diagram

$$\begin{tikzcd}
(\mathrm{Eq}(f,g), x_0) \arrow[r, hook, "\mathrm{eq}(f,g)"] & (X, x_0) \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & (Y, y_0) \\
(E, *) \arrow[u, dashed, "\phi", "\exists !"'] \arrow[ur, "e"'] & &
\end{tikzcd}$$

commute, being uniquely determined by the condition

$$\mathrm{eq}(f, g) \circ \phi = e$$

via

$$\phi(x) = e(x)$$

for each $x \in E$, where we note that $e(x) \in A$ indeed lies in $\mathrm{Eq}(f, g)$ by the condition

$$f \circ e = g \circ e,$$

which gives

$$f(e(x)) = g(e(x))$$

for each $x \in E$, so that $e(x) \in \mathrm{Eq}(f, g)$. Lastly, we note that ϕ is indeed a morphism of pointed sets, as we have

$$\begin{aligned} \phi(*) &= e(*) \\ &= x_0, \end{aligned}$$

where we have used that e is a morphism of pointed sets. □

Proposition 2.5.1.2. Let (X, x_0) and (Y, y_0) be pointed sets and let $f, g, h \colon (X, x_0) \to (Y, y_0)$ be morphisms of pointed sets.

1. *Associativity.* We have isomorphisms of pointed sets

$$\underbrace{\mathrm{Eq}(f \circ \mathrm{eq}(g, h), g \circ \mathrm{eq}(g, h))}_{=\mathrm{Eq}(f \circ \mathrm{eq}(g,h), h \circ \mathrm{eq}(g,h))} \cong \mathrm{Eq}(f, g, h) \cong \underbrace{\mathrm{Eq}(f \circ \mathrm{eq}(f, g), h \circ \mathrm{eq}(f, g))}_{=\mathrm{Eq}(g \circ \mathrm{eq}(f,g), h \circ \mathrm{eq}(f,g))},$$

   where $\mathrm{Eq}(f, g, h)$ is the limit of the diagram

$$\begin{tikzcd} (X, x_0) \arrow[r, shift left=2, "f"] \arrow[r, "g" description] \arrow[r, shift right=2, "h"'] & (Y, y_0) \end{tikzcd}$$

   in Sets_*, being explicitly given by

$$\mathrm{Eq}(f, g, h) \cong \{a \in A \mid f(a) = g(a) = h(a)\}.$$

2. *Unitality.* We have an isomorphism of pointed sets

$$\mathrm{Eq}(f, f) \cong X.$$

3. *Commutativity.* We have an isomorphism of pointed sets

$$\mathrm{Eq}(f, g) \cong \mathrm{Eq}(g, f).$$

Proof. Item 1, Associativity: This follows from Constructions With Sets, Item 1 of Proposition 1.5.1.2.
Item 2, Unitality: This follows from Constructions With Sets, Item 4 of Proposition 1.5.1.2.
Item 3, Commutativity: This follows from Constructions With Sets, Item 5 of Proposition 1.5.1.2. □

3 Colimits of Pointed Sets

3.1 The Initial Pointed Set

Definition 3.1.1.1. The **initial pointed set** is the pair $\left((\mathrm{pt},\star),\{\iota_X\}_{(X,x_0)\in\mathrm{Obj}(\mathsf{Sets}_*)}\right)$ consisting of:

- *The Limit.* The pointed set $(\mathrm{pt},\star)$.
- *The Cone.* The collection of morphisms of pointed sets

  $$\{\iota_X\colon(\mathrm{pt},\star)\to(X,x_0)\}_{(X,x_0)\in\mathrm{Obj}(\mathsf{Sets})}$$

  defined by

  $$\iota_X(\star)\stackrel{\mathrm{def}}{=}x_0.$$

Proof. We claim that $(\mathrm{pt},\star)$ is the initial object of Sets_*. Indeed, suppose we have a diagram of the form

$$(\mathrm{pt},\star)\qquad\quad(X,x_0)$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi\colon(\mathrm{pt},\star)\to(X,x_0)$$

making the diagram

$$(\mathrm{pt},\star)\underset{\exists!}{\stackrel{\phi}{\dashrightarrow}}(X,x_0)$$

commute, namely ι_X. $\square$

3.2 Coproducts of Families of Pointed Sets

Let $\{(X_i,x_0^i)\}_{i\in I}$ be a family of pointed sets.

Definition 3.2.1.1. The **coproduct of the family** $\{(X_i,x_0^i)\}_{i\in I}$, also called their **wedge sum**, is the pair consisting of:

- *The Colimit.* The pointed set $(\bigvee_{i\in I}X_i,p_0)$ consisting of:
  - *The Underlying Set.* The set $\bigvee_{i\in I}X_i$ defined by

    $$\bigvee_{i\in I}X_i\stackrel{\mathrm{def}}{=}\Big(\coprod_{i\in I}X_i\Big)/\sim,$$

    where $\sim$ is the equivalence relation on $\coprod_{i\in I}X_i$ given by declaring

    $$(i,x_0^i)\sim(j,x_0^j)$$

    for each $i,j\in I$.

- *The Basepoint.* The element p_0 of $\bigvee_{i\in I} X_i$ defined by

$$\begin{aligned} p_0 &\overset{\text{def}}{=} [(i, x_0^i)] \\ &= [(j, x_0^j)] \end{aligned}$$

for any $i, j \in I$.

- *The Cocone.* The collection

$$\left\{\mathrm{inj}_i \colon (X_i, x_0^i) \to (\bigvee_{i\in I} X_i, p_0)\right\}_{i\in I}$$

of morphism of pointed sets given by

$$\mathrm{inj}_i(x) \overset{\text{def}}{=} (i, x)$$

for each $x \in X_i$ and each $i \in I$.

Proof. We claim that $(\bigvee_{i\in I} X_i, p_0)$ is the categorical coproduct of $\{(X_i, x_0^i)\}_{i\in I}$ in Sets_*. Indeed, suppose we have, for each $i \in I$, a diagram of the form

$$\begin{tikzcd} & (C, *) \\ (X_i, x_0^i) \arrow[ur, "\iota_i"] \arrow[r, "\mathrm{inj}_i"'] & (\bigvee_{i\in I} X_i, p_0) \end{tikzcd}$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi \colon (\bigvee_{i\in I} X_i, p_0) \to (C, *)$$

making the diagram

$$\begin{tikzcd} & (C, *) \\ (X_i, x_0^i) \arrow[ur, "\iota_i"] \arrow[r, "\mathrm{inj}_i"'] & (\bigvee_{i\in I} X_i, p_0) \arrow[u, dashed, "\phi", "\exists!"'] \end{tikzcd}$$

commute, being uniquely determined by the condition $\phi \circ \mathrm{inj}_i = \iota_i$ for each $i \in I$ via

$$\phi([(i, x)]) = \iota_i(x)$$

for each $[(i,x)] \in \bigvee_{i\in I} X_i$, where we note that ϕ is indeed a morphism of pointed sets, as we have

$$\begin{aligned}\phi(p_0) &= \iota_i([(i,x_0^i)])\\ &= *,\end{aligned}$$

as ι_i is a morphism of pointed sets. □

Proposition 3.2.1.2. Let $\{(X_i, x_0^i)\}_{i\in I}$ be a family of pointed sets.

1. *Functoriality.* The assignment $\{(X_i, x_0^i)\}_{i\in I} \mapsto (\bigvee_{i\in I} X_i, p_0)$ defines a functor

$$\bigvee_{i\in I}\colon \mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}_*) \to \mathsf{Sets}_*.$$

Proof. *Item 1, Functoriality*: This follows from **??**, **??** of **??**. □

3.3 Coproducts

Let (X, x_0) and (Y, y_0) be pointed sets.

Definition 3.3.1.1. The **coproduct of** (X, x_0) **and** (Y, y_0), also called their **wedge sum**, is the pair consisting of:

- *The Colimit.* The pointed set $(X \vee Y, p_0)$ consisting of:
  - *The Underlying Set.* The set $X \vee Y$ defined by

$$\begin{aligned}(X\vee Y, p_0) &\overset{\text{def}}{=} (X, x_0) \textstyle\coprod (Y, y_0)\\ &\cong (X \textstyle\coprod_{\mathrm{pt}} Y, p_0)\\ &\cong (X \textstyle\coprod Y/\sim, p_0),\end{aligned} \qquad \begin{tikzcd} X \vee Y \arrow[dr, phantom, "\ulcorner", very near start] & Y \arrow[l] \\ X \arrow[u] & \mathrm{pt}, \arrow[l, hook', "{[x_0]}"'] \arrow[u, hook, "{[y_0]}"'] \end{tikzcd}$$

    where $\sim$ is the equivalence relation on $X \coprod Y$ obtained by declaring $(0, x_0) \sim (1, y_0)$.
  - *The Basepoint.* The element p_0 of $X \vee Y$ defined by

$$\begin{aligned}p_0 &\overset{\text{def}}{=} [(0, x_0)]\\ &= [(1, y_0)].\end{aligned}$$

- *The Cocone.* The morphisms of pointed sets

$$\begin{aligned}\mathrm{inj}_1 &\colon (X, x_0) \to (X \vee Y, p_0),\\ \mathrm{inj}_2 &\colon (Y, y_0) \to (X \vee Y, p_0),\end{aligned}$$

  given by

$$\begin{aligned}\mathrm{inj}_1(x) &\stackrel{\text{def}}{=} [(0, x)],\\ \mathrm{inj}_2(y) &\stackrel{\text{def}}{=} [(1, y)],\end{aligned}$$

  for each $x \in X$ and each $y \in Y$.

Proof. We claim that $(X \vee Y, p_0)$ is the categorical coproduct of (X, x_0) and (Y, y_0) in Sets_*. Indeed, suppose we have a diagram of the form

$$\begin{tikzcd} & (C,*) & \\ (X, x_0) \arrow[ur, bend left, "\iota_X"] \arrow[r, "\mathrm{inj}_X"'] & (X \vee Y, p_0) & (Y, y_0) \arrow[ul, bend right, "\iota_Y"'] \arrow[l, "\mathrm{inj}_Y"] \end{tikzcd}$$

in Sets. Then there exists a unique morphism of pointed sets

$$\phi\colon (X \vee Y, p_0) \to (C, *)$$

making the diagram

$$\begin{tikzcd} & (C,*) & \\ (X, x_0) \arrow[ur, bend left, "\iota_X"] \arrow[r, "\mathrm{inj}_X"'] & (X \vee Y, p_0) \arrow[u, dashed, "\phi", "\exists!"'] & (Y, y_0) \arrow[ul, bend right, "\iota_Y"'] \arrow[l, "\mathrm{inj}_Y"] \end{tikzcd}$$

commute, being uniquely determined by the conditions

$$\begin{aligned}\phi \circ \mathrm{inj}_X &= \iota_X,\\ \phi \circ \mathrm{inj}_Y &= \iota_Y\end{aligned}$$

via

$$\phi(z) = \begin{cases} \iota_X(x) & \text{if } z = [(0, x)] \text{ with } x \in X,\\ \iota_Y(y) & \text{if } z = [(1, y)] \text{ with } y \in Y\end{cases}$$

for each $z \in X \vee Y$, where we note that ϕ is indeed a morphism of pointed sets, as we have

$$\begin{aligned}\phi(p_0) &= \iota_X([(0, x_0)]) \\ &= \iota_Y([(1, y_0)]) \\ &= *,\end{aligned}$$

as ι_X and ι_Y are morphisms of pointed sets. □

Proposition 3.3.1.2. Let (X, x_0) and (Y, y_0) be pointed sets.

1. *Functoriality.* The assignments

   $$(X, x_0), (Y, y_0), ((X, x_0), (Y, y_0)) \mapsto (X \vee Y, p_0)$$

   define functors

   $$\begin{aligned} X \vee - \colon \mathsf{Sets}_* &\to \mathsf{Sets}_*, \\ - \vee Y \colon \mathsf{Sets}_* &\to \mathsf{Sets}_*, \\ -_1 \vee -_2 \colon \mathsf{Sets}_* \times \mathsf{Sets}_* &\to \mathsf{Sets}_*. \end{aligned}$$

2. *Associativity.* We have an isomorphism of pointed sets

   $$(X \vee Y) \vee Z \cong X \vee (Y \vee Z),$$

   natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathsf{Sets}_*$.

3. *Unitality.* We have isomorphisms of pointed sets

   $$\begin{aligned}(\mathrm{pt}, *) \vee (X, x_0) &\cong (X, x_0), \\ (X, x_0) \vee (\mathrm{pt}, *) &\cong (X, x_0),\end{aligned}$$

   natural in $(X, x_0) \in \mathsf{Sets}_*$.

4. *Commutativity.* We have an isomorphism of pointed sets

   $$X \vee Y \cong Y \vee X,$$

   natural in $(X, x_0), (Y, y_0) \in \mathsf{Sets}_*$.

5. *Symmetric Monoidality.* The triple $(\mathsf{Sets}_*, \vee, \mathrm{pt})$ is a symmetric monoidal category.

6. *The Fold Map.* We have a natural transformation

$$\nabla\colon \vee \circ \Delta^{\mathsf{Cats}}_{\mathsf{Sets}_*} \Longrightarrow \mathrm{id}_{\mathsf{Sets}_*}, \qquad \begin{tikzcd} & \mathsf{Sets}_* \times \mathsf{Sets}_* \arrow[dr, "\vee"] \arrow[d, phantom, "\Downarrow \nabla"] & \\ \mathsf{Sets}_* \arrow[ur, "\Delta^{\mathsf{Cats}}_{\mathsf{Sets}_*}"] \arrow[rr, bend right, "\mathrm{id}_{\mathsf{Sets}_*}"'] & & \mathsf{Sets}_*, \end{tikzcd}$$

called the **fold map**, whose component

$$\nabla_X\colon X \vee X \to X$$

at X is given by

$$\nabla_X(p) \stackrel{\text{def}}{=} \begin{cases} x & \text{if } p = [(0,x)], \\ x & \text{if } p = [(1,x)] \end{cases}$$

for each $p \in X \vee X$.

Proof. *Item 1*, *Functoriality*: This follows from **??**, **??** of **??**.
Item 2, *Associativity*: Clear.
Item 3, *Unitality*: Clear.
Item 4, *Commutativity*: Clear.
Item 5, *Symmetric Monoidality*: Omitted.
Item 6, *The Fold Map*: Naturality for the transformation ∇ is the statement that, given a morphism of pointed sets $f\colon (X,x_0) \to (Y,y_0)$, we have

$$\nabla_Y \circ (f \vee f) = f \circ \nabla_X, \qquad \begin{tikzcd} X \vee X \arrow[r, "\nabla_X"] \arrow[d, "f \vee f"'] & X \arrow[d, "f"] \\ Y \vee Y \arrow[r, "\nabla_Y"'] & Y. \end{tikzcd}$$

Indeed, we have

$$\begin{aligned} [\nabla_Y \circ (f \vee f)]([(i,x)]) &= \nabla_Y([(i, f(x))]) \\ &= f(x) \\ &= f(\nabla_X([(i,x)])) \\ &= [f \circ \nabla_X]([(i,x)]) \end{aligned}$$

for each $[(i,x)] \in X \vee X$, and thus ∇ is indeed a natural transformation. □

3.4 Pushouts

Let (X, x_0), (Y, y_0), and (Z, z_0) be pointed sets and let $f\colon (Z, z_0) \to (X, x_0)$ and $g\colon (Z, z_0) \to (Y, y_0)$ be morphisms of pointed sets.

Definition 3.4.1.1. The **pushout of** (X, x_0) **and** (Y, y_0) **over** (Z, z_0) **along** (f, g) is the pair consisting of:

- *The Colimit.* The pointed set $(X \coprod_{f,Z,g} Y, p_0)$, where:
  - The set $X \coprod_{f,Z,g} Y$ is the pushout (of unpointed sets) of X and Y over Z with respect to f and g;
  - We have $p_0 = [x_0] = [y_0]$.
- *The Cocone.* The morphisms of pointed sets

$$\begin{aligned} \mathrm{inj}_1 &\colon (X, x_0) \to (X \textstyle\coprod_Z Y, p_0), \\ \mathrm{inj}_2 &\colon (Y, y_0) \to (X \textstyle\coprod_Z Y, p_0) \end{aligned}$$

  given by

$$\begin{aligned} \mathrm{inj}_1(x) &\overset{\text{def}}{=} [(0, x)] \\ \mathrm{inj}_2(y) &\overset{\text{def}}{=} [(1, y)] \end{aligned}$$

  for each $x \in X$ and each $y \in Y$.

Proof. Firstly, we note that indeed $[x_0] = [y_0]$, as we have

$$\begin{aligned} x_0 &= f(z_0), \\ y_0 &= g(z_0) \end{aligned}$$

since f and g are morphisms of pointed sets, with the relation $\sim$ on $X \coprod_Z Y$ then identifying $x_0 = f(z_0) \sim g(z_0) = y_0$.
We now claim that $(X \coprod_Z Y, p_0)$ is the categorical pushout of (X, x_0) and (Y, y_0) over (Z, z_0) with respect to (f, g) in Sets_*. First we need to check that the relevant pushout diagram commutes, i.e. that we have

$$\mathrm{inj}_1 \circ f = \mathrm{inj}_2 \circ g, \qquad \begin{tikzcd} (X \coprod_Z Y, p_0) & (Y, y_0) \arrow[l, "\mathrm{inj}_2"'] \\ (X, x_0) \arrow[u, "\mathrm{inj}_1"] & (Z, z_0). \arrow[l, "f"] \arrow[u, "g"'] \end{tikzcd}$$

Indeed, given $z \in Z$, we have

$$\begin{aligned}
[\mathrm{inj}_1 \circ f](z) &= \mathrm{inj}_1(f(z)) \\
&= [(0, f(z))] \\
&= [(1, g(z))] \\
&= \mathrm{inj}_2(g(z)) \\
&= [\mathrm{inj}_2 \circ g](z),
\end{aligned}$$

where $[(0, f(z))] = [(1, g(z))]$ by the definition of the relation $\sim$ on $X \coprod Y$ (the coproduct of unpointed sets of X and Y). Next, we prove that $X \coprod_Z Y$ satisfies the universal property of the pushout. Suppose we have a diagram of the form

$$\begin{tikzcd}
(P, *) & & \\
& (X \coprod_Z Y, p_0) \arrow[dr, phantom, "\ulcorner"] & (Y, y_0) \arrow[l, "\mathrm{inj}_2"'] \arrow[ull, bend right, "\iota_2"'] \\
& (X, x_0) \arrow[u, "\mathrm{inj}_1"] \arrow[uul, bend left, "\iota_1"] & (Z, z_0) \arrow[l, "f"] \arrow[u, "g"']
\end{tikzcd}$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi \colon (X \textstyle\coprod_Z Y, p_0) \to (P, *)$$

making the diagram

$$\begin{tikzcd}
(P, *) & & \\
& (X \coprod_Z Y, p_0) \arrow[ul, dashed, "\exists!"', "\phi"] \arrow[dr, phantom, "\ulcorner"] & (Y, y_0) \arrow[l, "\mathrm{inj}_2"'] \arrow[ull, bend right, "\iota_2"'] \\
& (X, x_0) \arrow[u, "\mathrm{inj}_1"] \arrow[uul, bend left, "\iota_1"] & (Z, z_0) \arrow[l, "f"] \arrow[u, "g"']
\end{tikzcd}$$

commute, being uniquely determined by the conditions

$$\begin{aligned}
\phi \circ \mathrm{inj}_1 &= \iota_1, \\
\phi \circ \mathrm{inj}_2 &= \iota_2
\end{aligned}$$

via

$$\phi(p) = \begin{cases} \iota_1(x) & \text{if } x = [(0,x)], \\ \iota_2(y) & \text{if } x = [(1,y)] \end{cases}$$

for each $p \in X \coprod_Z Y$, where the well-definedness of ϕ is proven in the same way as in the proof of Constructions With Sets, Definition 2.4.1.1. Finally, we show that ϕ is indeed a morphism of pointed sets, as we have

$$\begin{aligned} \phi(p_0) &= \phi([(0,x_0)]) \\ &= \iota_1(x_0) \\ &= *, \end{aligned}$$

or alternatively

$$\begin{aligned} \phi(p_0) &= \phi([(1,y_0)]) \\ &= \iota_2(y_0) \\ &= *, \end{aligned}$$

where we use that ι_1 (resp. ι_2) is a morphism of pointed sets. □

Proposition 3.4.1.2. Let (X, x_0), (Y, y_0), (Z, z_0), and (A, a_0) be pointed sets.

1. *Functoriality.* The assignment $(X, Y, Z, f, g) \mapsto X \coprod_{f,Z,g} Y$ defines a functor

   $$-_1 \coprod_{-_3} -_1 \colon \mathsf{Fun}(\mathcal{P}, \mathsf{Sets}) \to \mathsf{Sets}_*,$$

   where $\mathcal{P}$ is the category that looks like this:

   $$\begin{tikzcd} & \bullet \\ \bullet & \bullet. \arrow[l] \arrow[u] \end{tikzcd}$$

   In particular, the action on morphisms of $-_1 \coprod_{-_3} -_1$ is given by sending a morphism

$$
\begin{tikzcd}
X \coprod_Z Y \arrow[dr, phantom, very near start, "\ulcorner"] & & Y \arrow[ll] \arrow[dr, "\psi"] & \\
& X' \coprod_{Z'} Y' \arrow[dr, phantom, very near start, "\ulcorner"] & & Y' \arrow[ll] \\
X \arrow[uu] \arrow[dr, "\phi"'] & & Z \arrow[ll, "f"'] \arrow[uu, "g"] \arrow[dr, "\chi"] & \\
& X' \arrow[uu] & & Z' \arrow[ll, "f'"] \arrow[uu, "g'"']
\end{tikzcd}
$$

in $\mathsf{Fun}(\mathcal{P}, \mathsf{Sets}_*)$ to the morphism of pointed sets

$$\xi \colon \left(X \coprod_Z Y, p_0\right) \xrightarrow{\exists!} \left(X' \coprod_{Z'} Y', p_0'\right)$$

given by

$$\xi(p) \overset{\text{def}}{=} \begin{cases} \phi(x) & \text{if } p = [(0, x)], \\ \psi(y) & \text{if } p = [(1, y)] \end{cases}$$

for each $p \in X \coprod_Z Y$, which is the unique morphism of pointed sets making the diagram

$$
\begin{tikzcd}
X \coprod_Z Y \arrow[dr, phantom, very near start, "\ulcorner"] \arrow[dr, dashed] & & Y \arrow[ll] \arrow[dr, "\psi"] & \\
& X' \coprod_{Z'} Y' \arrow[dr, phantom, very near start, "\ulcorner"] & & Y' \arrow[ll] \\
X \arrow[uu] \arrow[dr, "\phi"'] & & Z \arrow[ll, "f"'] \arrow[uu, "g"] \arrow[dr, "\chi"] & \\
& X' \arrow[uu] & & Z' \arrow[ll, "f'"] \arrow[uu, "g'"']
\end{tikzcd}
$$

commute.

2. *Associativity.* Given a diagram

$$
\begin{tikzcd}
X & & Y & & Z \\
& W \arrow[ul, "f"] \arrow[ur, "g"'] & & V \arrow[ul, "h"] \arrow[ur, "k"'] &
\end{tikzcd}
$$

in Sets, we have isomorphisms of pointed sets

$$(X \textstyle\coprod_W Y) \coprod_V Z \cong (X \coprod_W Y) \coprod_Y (Y \coprod_V Z) \cong X \coprod_W (Y \coprod_V Z),$$

where these pullbacks are built as in the diagrams

$$\begin{tikzcd}[column sep=tiny, row sep=small]
 & & (X \coprod_W Y) \coprod_V Z & & \\
 & X \coprod_W Y \arrow[ur] \arrow[dd, phantom, "\wedge" very near start] & & & \\
X \arrow[ur] & & Y \arrow[ul] & & Z, \arrow[uull] \\
 & W \arrow[ul, "f"] \arrow[ur, "g"'] & & V \arrow[ul, "h"] \arrow[ur, "k"'] &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=tiny, row sep=small]
 & & (X \coprod_W Y) \coprod_Y (Y \coprod_V Z) & & \\
 & X \coprod_W Y \arrow[ur] & & Y \coprod_V Z \arrow[ul] & \\
X \arrow[ur] & & Y \arrow[ul] \arrow[ur] & & Z, \arrow[ul] \\
 & W \arrow[ul, "f"] \arrow[ur, "g"'] & & V \arrow[ul, "h"] \arrow[ur, "k"'] &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=tiny, row sep=small]
 & & X \coprod_W (Y \coprod_V Z) & & \\
 & & & Y \coprod_V Z \arrow[ul] & \\
X \arrow[uurr] & & Y \arrow[ur] & & Z. \arrow[ul] \\
 & W \arrow[ul, "f"] \arrow[ur, "g"'] & & V \arrow[ul, "h"] \arrow[ur, "k"'] &
\end{tikzcd}$$

3. *Unitality.* We have isomorphisms of sets

$$\begin{tikzcd}
A \arrow[r, equal] & A \\
X \arrow[u, "f"] \arrow[r, equal] \arrow[ur, phantom, "\ulcorner" very near end] & X \arrow[u, "f"']
\end{tikzcd}
\qquad
\begin{aligned} X \textstyle\coprod_X A &\cong A, \\ A \textstyle\coprod_X X &\cong A, \end{aligned}
\qquad
\begin{tikzcd}
A & X \arrow[l, "f"'] \\
X \arrow[u, equal] \arrow[ur, phantom, "\ulcorner" very near end] & X. \arrow[l, "f"] \arrow[u, equal]
\end{tikzcd}$$

4. *Commutativity.* We have an isomorphism of sets

$$\begin{tikzcd}
X \coprod_Z Y & Y \arrow[l] \\
X \arrow[u] \arrow[ur, phantom, "\ulcorner" very near end] & Z, \arrow[l, "f"] \arrow[u, "g"']
\end{tikzcd}
\qquad
X \textstyle\coprod_Z Y \cong Y \coprod_Z X
\qquad
\begin{tikzcd}
Y \coprod_Z X & X \arrow[l] \\
Y \arrow[u] \arrow[ur, phantom, "\ulcorner" very near end] & Z. \arrow[l, "g"] \arrow[u, "f"']
\end{tikzcd}$$

5. *Interaction With Coproducts.* We have

$$X \textstyle\coprod_{\mathrm{pt}} Y \cong X \vee Y, \qquad
\begin{tikzcd}
X \vee Y & Y \arrow[l] \\
X \arrow[u] \arrow[ur, phantom, "\ulcorner" very near end] & \mathrm{pt}. \arrow[l, hook', "{[x_0]}"] \arrow[u, hook, "{[y_0]}"']
\end{tikzcd}$$

6. *Symmetric Monoidality.* The triple $(\mathsf{Sets}_*, \coprod_X, (X, x_0))$ is a symmetric monoidal category.

Proof. *Item 1, Functoriality*: This is a special case of functoriality of co/limits, **??**, **??** of **??**, with the explicit expression for ξ following from the commutativity of the cube pushout diagram.
Item 2, Associativity: This follows from Constructions With Sets, Item 2 of Proposition 2.4.1.4.
Item 3, Unitality: This follows from Constructions With Sets, Item 3 of Proposition 2.4.1.4.
Item 4, Commutativity: This follows from Constructions With Sets, Item 4 of Proposition 2.4.1.4.
Item 5, Interaction With Coproducts: Clear.
Item 6, Symmetric Monoidality: Omitted. □

3.5 Coequalisers

Let $f, g\colon (X, x_0) \rightrightarrows (Y, y_0)$ be morphisms of pointed sets.

Definition 3.5.1.1. The **coequaliser of** (f, g) is the pointed set $(\mathrm{CoEq}(f, g), [y_0])$.

Proof. We claim that $(\mathrm{CoEq}(f, g), [y_0])$ is the categorical coequaliser of f and g in Sets_*. First we need to check that the relevant coequaliser diagram commutes, i.e. that we have

$$\mathrm{coeq}(f, g) \circ f = \mathrm{coeq}(f, g) \circ g.$$

Indeed, we have

$$\begin{aligned}
[\mathrm{coeq}(f, g) \circ f](x) &\stackrel{\text{def}}{=} [\mathrm{coeq}(f, g)](f(x)) \\
&\stackrel{\text{def}}{=} [f(x)] \\
&= [g(x)] \\
&\stackrel{\text{def}}{=} [\mathrm{coeq}(f, g)](g(x)) \\
&\stackrel{\text{def}}{=} [\mathrm{coeq}(f, g) \circ g](x)
\end{aligned}$$

for each $x \in X$. Next, we prove that $\mathrm{CoEq}(f, g)$ satisfies the universal property of the coequaliser. Suppose we have a diagram of the form

$$\begin{tikzcd}
(X, x_0) \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & (Y, y_0) \arrow[r, "\mathrm{coeq}(f,g)"] \arrow[dr, "c"'] & (\mathrm{CoEq}(f, g), [y_0]) \\
& & (C, *)
\end{tikzcd}$$

in Sets. Then, since $c(f(a)) = c(g(a))$ for each $a \in A$, it follows from Equivalence Relations and Apartness Relations, Items 4 and 5 of Proposition 5.2.1.3 that there exists a unique map $\phi\colon \mathrm{CoEq}(f,g) \xrightarrow{\exists!} C$ making the diagram

$$\begin{tikzcd}
(X,x_0) \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & (Y,y_0) \arrow[r, "\mathrm{coeq}(f,g)"] \arrow[rd, "c"'] & (\mathrm{CoEq}(f,g),[y_0]) \arrow[d, dashed, "\phi"', "\exists!"] \\
 & & (C,*)
\end{tikzcd}$$

commute, where we note that ϕ is indeed a morphism of pointed sets since

$$\begin{aligned}
\phi([y_0]) &= [\phi \circ \mathrm{coeq}(f,g)]([y_0]) \\
&= c([y_0]) \\
&= *,
\end{aligned}$$

where we have used that c is a morphism of pointed sets. □

Proposition 3.5.1.2. Let (X,x_0) and (Y,y_0) be pointed sets and let $f,g,h\colon (X,x_0) \to (Y,y_0)$ be morphisms of pointed sets.

1. *Associativity.* We have isomorphisms of pointed sets

$$\underbrace{\mathrm{CoEq}(\mathrm{coeq}(f,g)\circ f, \mathrm{coeq}(f,g)\circ h)}_{=\mathrm{CoEq}(\mathrm{coeq}(f,g)\circ g, \mathrm{coeq}(f,g)\circ h)} \cong \mathrm{CoEq}(f,g,h) \cong \underbrace{\mathrm{CoEq}(\mathrm{coeq}(g,h)\circ f, \mathrm{coeq}(g,h)\circ g)}_{=\mathrm{CoEq}(\mathrm{coeq}(g,h)\circ f, \mathrm{coeq}(g,h)\circ h)},$$

   where $\mathrm{CoEq}(f,g,h)$ is the colimit of the diagram

$$\begin{tikzcd}
(X,x_0) \arrow[r, shift left=2, "f"] \arrow[r, dashed, "g" description] \arrow[r, shift right=2, "h"'] & (Y,y_0)
\end{tikzcd}$$

   in Sets_*.

2. *Unitality.* We have an isomorphism of pointed sets

$$\mathrm{CoEq}(f,f) \cong B.$$

3. *Commutativity.* We have an isomorphism of pointed sets

$$\mathrm{CoEq}(f,g) \cong \mathrm{CoEq}(g,f).$$

Proof. *Item 1, Associativity*: This follows from Constructions With Sets, Item 1 of Proposition 2.5.1.4.
Item 2, Unitality: This follows from Constructions With Sets, Item 4 of Proposition 2.5.1.4.
Item 3, Commutativity: This follows from Constructions With Sets, Item 5 of Proposition 2.5.1.4. □

4 Constructions With Pointed Sets

4.1 Free Pointed Sets

Let X be a set.

Definition 4.1.1.1. The **free pointed set on** X is the pointed set X^+ consisting of:

- *The Underlying Set.* The set X^+ defined by[11]

$$\begin{aligned} X^+ &\overset{\text{def}}{=} X \coprod \mathrm{pt} \\ &\overset{\text{def}}{=} X \coprod \{\star\}. \end{aligned}$$

- *The Basepoint.* The element $\star$ of X^+.

Proposition 4.1.1.2. Let X be a set.

1. *Functoriality.* The assignment $X \mapsto X^+$ defines a functor

$$(-)^+ \colon \mathsf{Sets} \to \mathsf{Sets}_*,$$

   where

   - *Action on Objects.* For each $X \in \mathrm{Obj}(\mathsf{Sets})$, we have

   $$[(-)^+](X) \overset{\text{def}}{=} X^+,$$

     where X^+ is the pointed set of Definition 4.1.1.1;

   - *Action on Morphisms.* For each morphism $f \colon X \to Y$ of Sets, the image

   $$f^+ \colon X^+ \to Y^+$$

     of f by $(-)^+$ is the map of pointed sets defined by

   $$f^+(x) \overset{\text{def}}{=} \begin{cases} f(x) & \text{if } x \in X, \\ \star_Y & \text{if } x = \star_X. \end{cases}$$

[11] *Further Notation:* We sometimes write $\star_X$ for the basepoint of X^+ for clarity when there are multiple free pointed sets involved in the current discussion.

2. *Adjointness.* We have an adjunction

$$\Big((-)^{+} \dashv 忘\Big)\colon \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "(-)^{+}"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_{*}, \arrow[l, bend left, "忘"] \end{tikzcd}$$

witnessed by a bijection of sets

$$\mathsf{Sets}_{*}((X^{+}, \star_X), (Y, y_0)) \cong \mathsf{Sets}(X, Y),$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets})$ and $(Y, y_0) \in \mathrm{Obj}(\mathsf{Sets}_{*})$.

3. *Symmetric Strong Monoidality With Respect to Wedge Sums.* The free pointed set functor of Item 1 has a symmetric strong monoidal structure

$$\left((-)^{+}, (-)^{+,\coprod}, (-)^{+,\coprod}_{\mathbb{1}}\right)\colon (\mathsf{Sets}, \textstyle\coprod, \emptyset) \to (\mathsf{Sets}_{*}, \vee, \mathrm{pt}),$$

being equipped with isomorphisms of pointed sets

$$(-)^{+,\coprod}_{X,Y}\colon X^{+} \vee Y^{+} \xrightarrow{\cong} (X \textstyle\coprod Y)^{+},$$
$$(-)^{+,\coprod}_{\mathbb{1}}\colon \mathrm{pt} \xrightarrow{\cong} \emptyset^{+},$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

4. *Symmetric Strong Monoidality With Respect to Smash Products.* The free pointed set functor of Item 1 has a symmetric strong monoidal structure

$$\left((-)^{+}, (-)^{+,\times}, (-)^{+,\times}_{\mathbb{1}}\right)\colon (\mathsf{Sets}, \times, \mathrm{pt}) \to (\mathsf{Sets}_{*}, \wedge, S^{0}),$$

being equipped with isomorphisms of pointed sets

$$(-)^{+,\times}_{X,Y}\colon X^{+} \wedge Y^{+} \xrightarrow{\cong} (X \times Y)^{+},$$
$$(-)^{+,\times}_{\mathbb{1}}\colon S^{0} \xrightarrow{\cong} \mathrm{pt}^{+},$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

Proof. *Item 1, Functoriality*: Clear.
Item 2, Adjointness: We claim there's an adjunction $(-)^{+} \dashv 忘$, witnessed by a bijection of sets

$$\mathsf{Sets}_{*}((X^{+}, \star_X), (Y, y_0)) \cong \mathsf{Sets}(X, Y),$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets})$ and $(Y, y_0) \in \mathrm{Obj}(\mathsf{Sets}_{*})$.

- *Map I.* We define a map

  $$\Phi_{X,Y}\colon \mathsf{Sets}_*((X^+,\star_X),(Y,y_0)) \to \mathsf{Sets}(X,Y)$$

  by sending a pointed function

  $$\xi\colon (X^+,\star_X) \to (Y,y_0)$$

  to the function

  $$\xi^\dagger\colon X \to Y$$

  given by

  $$\xi^\dagger(x) \overset{\text{def}}{=} \xi(x)$$

  for each $x \in X$.

- *Map II.* We define a map

  $$\Psi_{X,Y}\colon \mathsf{Sets}(X,Y) \to \mathsf{Sets}_*((X^+,\star_X),(Y,y_0))$$

  given by sending a function $\xi\colon X \to Y$ to the pointed function

  $$\xi^\dagger\colon (X^+,\star_X) \to (Y,y_0)$$

  defined by

  $$\xi^\dagger(x) \overset{\text{def}}{=} \begin{cases} \xi(x) & \text{if } x \in X, \\ y_0 & \text{if } x = \star_X \end{cases}$$

  for each $x \in X^+$.

- *Invertibility I.* We claim that

  $$\Psi_{X,Y} \circ \Phi_{X,Y} = \mathrm{id}_{\mathsf{Sets}_*((X^+,\star_X),(Y,y_0))},$$

  which is clear.

- *Invertibility II.* We claim that

  $$\Phi_{X,Y} \circ \Psi_{X,Y} = \mathrm{id}_{\mathsf{Sets}(X,Y)},$$

  which is clear.

- *Naturality for* Φ*, Part I.* We need to show that, given a pointed function

$g\colon (Y,y_0) \to (Y',y_0')$, the diagram

$$\begin{tikzcd}
\mathsf{Sets}_*((X^+,\star_X),(Y,y_0)) \arrow[r, "\Phi_{X,Y}"] \arrow[d, "g_*"'] & \mathsf{Sets}(X,Y) \arrow[d, "g_*"] \\
\mathsf{Sets}_*((X^+,\star_X),(Y',y_0')) \arrow[r, "\Phi_{X,Y'}"'] & \mathsf{Sets}(X,Y')
\end{tikzcd}$$

commutes. Indeed, given a pointed function

$$\xi^\dagger\colon (X^+,\star_X) \to (Y,y_0)$$

we have

$$\begin{aligned}
[\Phi_{X,Y'} \circ g_*](\xi) &= \Phi_{X,Y'}(g_*(\xi)) \\
&= \Phi_{X,Y'}(g \circ \xi) \\
&= g \circ \xi \\
&= g \circ \Phi_{X,Y'}(\xi) \\
&= g_*(\Phi_{X,Y'}(\xi)) \\
&= [g_* \circ \Phi_{X,Y'}](\xi).
\end{aligned}$$

- *Naturality for* Φ*, Part II.* We need to show that, given a pointed function $f\colon (X,x_0) \to (X',x_0')$, the diagram

$$\begin{tikzcd}
\mathsf{Sets}_*((X'^{,+},\star_X),(Y,y_0)) \arrow[r, "\Phi_{X',Y}"] \arrow[d, "f^*"'] & \mathsf{Sets}(X',Y) \arrow[d, "f^*"] \\
\mathsf{Sets}_*((X^+,\star_X),(Y,y_0)) \arrow[r, "\Phi_{X,Y}"'] & \mathsf{Sets}(X,Y)
\end{tikzcd}$$

commutes. Indeed, given a function

$$\xi\colon X' \to Y,$$

we have

$$
\begin{aligned}
[\Phi_{X,Y} \circ f^*](\xi) &= \Phi_{X,Y}(f^*(\xi)) \\
&= \Phi_{X,Y}(\xi \circ f) \\
&= \xi \circ f \\
&= \Phi_{X',Y}(\xi) \circ f \\
&= f^*(\Phi_{X',Y}(\xi)) \\
&= f^*(\Phi_{X',Y}(\xi)) \\
&= [f^* \circ \Phi_{X',Y}](\xi).
\end{aligned}
$$

- *Naturality for* Ψ. Since Φ is natural in each argument and Φ is a componentwise inverse to Ψ in each argument, it follows from Categories, Item 2 of Proposition 8.6.1.2 that Ψ is also natural in each argument.

Item 3, Symmetric Strong Monoidality With Respect to Wedge Sums: The isomorphism

$$\phi\colon X^+ \vee Y^+ \xrightarrow{\cong} (X \textstyle\coprod Y)^+$$

is given by

$$\phi(z) = \begin{cases} x & \text{if } z = [(0,x)] \text{ with } x \in X, \\ y & \text{if } z = [(1,y)] \text{ with } y \in Y, \\ \star_{X \coprod Y} & \text{if } z = [(0, \star_X)], \\ \star_{X \coprod Y} & \text{if } z = [(1, \star_Y)] \end{cases}$$

for each $z \in X^+ \vee Y^+$, with inverse

$$\phi^{-1}\colon (X \textstyle\coprod Y)^+ \xrightarrow{\cong} X^+ \vee Y^+$$

given by

$$\phi^{-1}(z) \stackrel{\text{def}}{=} \begin{cases} [(0,x)] & \text{if } z = [(0,x)], \\ [(0,y)] & \text{if } z = [(1,y)], \\ p_0 & \text{if } z = \star_{X \coprod Y} \end{cases}$$

for each $z \in (X \coprod Y)^+$.
Meanwhile, the isomorphism $\mathrm{pt} \cong \emptyset^+$ is given by sending $\star_X$ to $\star_\emptyset$.
That these isomorphisms satisfy the coherence conditions making the functor $(-)^+$ symmetric strong monoidal can be directly checked element by element.
Item 4, Symmetric Strong Monoidality With Respect to Smash Products: The isomorphism

$$\phi\colon X^+ \wedge Y^+ \xrightarrow{\cong} (X \times Y)^+$$

is given by

$$\phi(x \wedge y) = \begin{cases} (x, y) & \text{if } x \neq \star_X \text{ and } y \neq \star_Y \\ \star_{X \times Y} & \text{otherwise} \end{cases}$$

for each $x \wedge y \in X^+ \wedge Y^+$, with inverse

$$\phi^{-1} \colon (X \times Y)^+ \xrightarrow{\cong} X^+ \wedge Y^+$$

given by

$$\phi^{-1}(z) \stackrel{\text{def}}{=} \begin{cases} x \wedge y & \text{if } z = (x, y) \text{ with } (x, y) \in X \times Y, \\ \star_X \wedge \star_Y & \text{if } z = \star_{X \times Y}, \end{cases}$$

for each $z \in (X \coprod Y)^+$.
Meanwhile, the isomorphism $S^0 \cong \mathrm{pt}^+$ is given by sending $\star$ to $1 \in S^0 = \{0, 1\}$ and $\star_{\mathrm{pt}}$ to $0 \in S^0$.
That these isomorphisms satisfy the coherence conditions making the functor $(-)^+$ symmetric strong monoidal can be directly checked element by element.

□

Appendices

A Other Chapters

References

[MSE 2855868] Qiaochu Yuan. *Is the category of pointed sets Cartesian closed?* Mathematics Stack Exchange. URL: https://math.stackexchange.com/q/2855868 (cit. on p. 6).

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