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Part I

Sets

Chapter 1

Sets

This chapter (will eventually) contain material on axiomatic set theory, as well as a couple other things.

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1.1 Sets and Functions

1.1.1 Functions

Definition 1.1.1.1 ▶ Functions

A **function** is a functional and total relation.

Notation 1.1.1.2 ▶ Additional Notation for Functions

Throughout this work, we will sometimes denote a function $f\colon X \to Y$ by

$$f \overset{\text{def}}{=} [\![x \mapsto f(x)]\!].$$

1. For example, given a function

$$\Phi\colon \mathrm{Hom}_{\mathsf{Sets}}(X, Y) \to K$$

taking values on a set of functions such as $\mathrm{Hom}_{\mathsf{Sets}}(X, Y)$, we will sometimes also write

$$\Phi(f) \overset{\text{def}}{=} \Phi\big([\![x \mapsto f(x)]\!]\big).$$

2. This notational choice is based on the lambda notation

$$f \overset{\text{def}}{=} (\lambda x.\, f(x)),$$

but uses a "$\mapsto$" symbol for better spacing and double brackets instead of either:

   (a) Square brackets $[x \mapsto f(x)]$;
   (b) Parentheses $(x \mapsto f(x))$;

hoping to improve readability when dealing with e.g.:

   (a) Equivalence classes, cf.:
      i. $[\![[x] \mapsto f([x])]\!]$
      ii. $[[x] \mapsto f([x])]$
      iii. $(\lambda[x].\, f([x]))$
   (b) Function evaluations, cf.:
      i. $\Phi\big([\![x \mapsto f(x)]\!]\big)$
      ii. $\Phi((x \mapsto f(x)))$
      iii. $\Phi((\lambda x.\, f(x)))$

3. We will also sometimes write $-_1, -_2$, etc. for the arguments of a function. Some examples include:

   (a) Writing $f(-_1)$ for a function $f\colon A \to B$.
   (b) Writing $f(-_1, -_2)$ for a function $f\colon A \times B \to C$.
   (c) Given a function $f\colon A \times B \to C$, writing

$$f(a, -)\colon B \to C$$

   for the function $[\![b \mapsto f(a, b)]\!]$.

(d) Denoting a composition of the form

$$A \times B \xrightarrow{\phi \times \mathrm{id}_B} A' \times B \xrightarrow{f} C$$

by $f(\phi(-_1), -_2)$.

4. Finally, given a function $f: A \longrightarrow B$, we write

$$\mathrm{ev}_a(f) \stackrel{\text{def}}{=} f(a)$$

for the value of f at some $a \in A$.

For an example of the above notations being used in practice, see the proof of the adjunction

$$(A \times - \dashv \mathrm{Hom}_{\mathsf{Sets}}(A, -)): \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "A\times -"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}, \arrow[l, bend left, "\mathrm{Hom}_{\mathsf{Sets}}(A,-)"] \end{tikzcd}$$

stated in Item 2 of Proposition 2.1.3.3.

1.2 The Enrichment of Sets in Classical Truth Values

1.2.1 (-2)-Categories

Definition 1.2.1.1 ▶ (-2)-Categories

A (-2)**-category** is the "necessarily true" truth value.[1,2,3]

[1] Thus, there is only one (-2)-category.

[2] A $(-n)$-category for $n = 3, 4, \ldots$ is also the "necessarily true" truth value, coinciding with a (-2)-category.

[3] For motivation, see [BS10, p. 13].

1.2.2 (-1)-Categories

Definition 1.2.2.1 ▶ (-1)-Categories

A (-1)**-category** is a classical truth value.

REMARK 1.2.2.2 ► MOTIVATION FOR (-1)-CATEGORIES

[1] (-1)-categories should be thought of as being "categories enriched in (-2)-categories", having a collection of objects and, for each pair of objects, a Hom-object $\mathrm{Hom}(x, y)$ that is a (-2)-category (i.e. trivial).
Therefore, a (-1)-category C is either ([BS10, pp. 33–34]):

1. *Empty*, having no objects;
2. *Contractible*, having a collection of objects $\{a, b, c, \ldots\}$, but with $\mathrm{Hom}_C(a, b)$ being a (-2)-category (i.e. trivial) for all $a, b \in \mathrm{Obj}(C)$, forcing all objects of C to be uniquely isomorphic to each other.

As such, there are only two (-1)-categories, up to equivalence:

- The (-1)-category false (the empty one);
- The (-1)-category true (the contractible one).

[1] For more motivation, see [BS10, p. 13].

DEFINITION 1.2.2.3 ► THE POSET OF TRUTH VALUES

The **poset of truth values**[1] is the poset $(\{\text{true}, \text{false}\}, \preceq)$ consisting of

- *The Underlying Set.* The set $\{\text{true}, \text{false}\}$ whose elements are the truth values true and false.
- *The Partial Order.* The partial order

$$\preceq\colon \{\text{true}, \text{false}\} \times \{\text{true}, \text{false}\} \longrightarrow \{\text{true}, \text{false}\}$$

  on $\{\text{true}, \text{false}\}$ defined by[2]

$$\begin{aligned}
\text{false} \preceq \text{false} &\stackrel{\text{def}}{=} \text{true},\\
\text{true} \preceq \text{false} &\stackrel{\text{def}}{=} \text{false},\\
\text{false} \preceq \text{true} &\stackrel{\text{def}}{=} \text{true},\\
\text{true} \preceq \text{true} &\stackrel{\text{def}}{=} \text{true}.
\end{aligned}$$

[1] *Further Terminology:* Also called the **poset of (-1)-categories**.
[2] This partial order coincides with logical implication.

NOTATION 1.2.2.4 ► FURTHER NOTATION FOR THE POSET OF TRUTH VALUES

We also write $\{\text{t}, \text{f}\}$ for the poset $\{\text{true}, \text{false}\}$.

Proposition 1.2.2.5 ► Cartesian Closedness of the Poset of Truth Values

The poset of truth values $\{\mathsf{t}, \mathsf{f}\}$ is Cartesian closed with product given by[1]

$$\begin{aligned} \mathsf{t} \times \mathsf{t} &= \mathsf{t}, \\ \mathsf{t} \times \mathsf{f} &= \mathsf{f}, \\ \mathsf{f} \times \mathsf{t} &= \mathsf{f}, \\ \mathsf{f} \times \mathsf{f} &= \mathsf{f}, \end{aligned}$$

and internal Hom $\mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}$ given by the partial order of $\{\mathsf{t}, \mathsf{f}\}$, i.e. by

$$\begin{aligned} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t}, \mathsf{t}) &= \mathsf{t}, \\ \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t}, \mathsf{f}) &= \mathsf{f}, \\ \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{t}) &= \mathsf{t}, \\ \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{f}) &= \mathsf{t}. \end{aligned}$$

[1] Note that $\times$ coincides with the "and" operator, while $\mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}$ coincides with the logical implication operator.

Proof 1.2.2.6 ► Proof of Proposition 1.2.2.5

Existence of Products

We claim that the products $\mathsf{t} \times \mathsf{t}$, $\mathsf{t} \times \mathsf{f}$, $\mathsf{f} \times \mathsf{t}$, and $\mathsf{f} \times \mathsf{f}$ satisfy the universal property of the product in $\{\mathsf{t}, \mathsf{f}\}$. Indeed, consider the diagrams

$$\begin{tikzcd} & P_1 \arrow[dl, bend right, "p_1^1"'] \arrow[dr, bend left, "p_2^1"] \arrow[d, dashed, "\exists!"] & \\ \mathsf{t} & \underbrace{\mathsf{t} \times \mathsf{t}}_{=\mathsf{t}} \arrow[l, two heads, "\mathrm{pr}_1"] \arrow[r, two heads, "\mathrm{pr}_2"'] & \mathsf{t} \end{tikzcd} \quad \begin{tikzcd} & P_2 \arrow[dl, bend right, "p_1^2"'] \arrow[dr, bend left, "p_2^2"] \arrow[d, dashed, "\exists!"] & \\ \mathsf{t} & \underbrace{\mathsf{t} \times \mathsf{f}}_{=\mathsf{f}} \arrow[l, two heads, "\mathrm{pr}_1"] \arrow[r, two heads, "\mathrm{pr}_2"'] & \mathsf{f} \end{tikzcd} \quad \begin{tikzcd} & P_3 \arrow[dl, bend right, "p_1^3"'] \arrow[dr, bend left, "p_2^3"] \arrow[d, dashed, "\exists!"] & \\ \mathsf{f} & \underbrace{\mathsf{f} \times \mathsf{t}}_{=\mathsf{f}} \arrow[l, two heads, "\mathrm{pr}_1"] \arrow[r, two heads, "\mathrm{pr}_2"'] & \mathsf{t} \end{tikzcd} \quad \begin{tikzcd} & P_4 \arrow[dl, bend right, "p_1^4"'] \arrow[dr, bend left, "p_2^4"] \arrow[d, dashed, "\exists!"] & \\ \mathsf{f} & \underbrace{\mathsf{f} \times \mathsf{f}}_{=\mathsf{f}} \arrow[l, two heads, "\mathrm{pr}_1"] \arrow[r, two heads, "\mathrm{pr}_2"'] & \mathsf{f}. \end{tikzcd}$$

Here:

1. If $P_1 = \mathsf{t}$, then $p_1^1 = p_2^1 = \mathrm{id}_{\mathsf{t}}$, and there's indeed a unique morphism from P_1 to t making the diagram commute, namely id_{t};
2. If $P_1 = \mathsf{f}$, then $p_1^1 = p_2^1$ are given by the unique morphism from f to t, and there's indeed a unique morphism from P_1 to t making the diagram commute, namely the unique morphism from f to t;
3. If $P_2 = \mathsf{t}$, then there is no morphism p_2^2.
4. If $P_2 = \mathsf{f}$, then p_1^2 is the unique morphism from f to t while $p_2^2 = \mathrm{id}_{\mathsf{f}}$, and there's indeed a unique morphism from P_2 to f making the diagram commute, namely id_{f};

5. The proof for P_3 is similar to the one for P_2;
6. If $P_4 = \mathsf{t}$, then there is no morphism p_1^4 or p_2^4.
7. If $P_4 = \mathsf{f}$, then $p_1^4 = p_2^4 = \mathrm{id}_{\mathsf{f}}$, and there's indeed a unique morphism from P_4 to f making the diagram commute, namely id_{f}.

Cartesian Closedness

We claim there's a bijection

$$\mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(A \times B, C) \cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\big(A, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(B, C)\big)$$

natural in $A, B, C \in \{\mathsf{t}, \mathsf{f}\}$. Indeed:

- For $(A, B, C) = (\mathsf{t}, \mathsf{t}, \mathsf{t})$, we have

$$\begin{aligned} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t} \times \mathsf{t}, \mathsf{t}) &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t}, \mathsf{t}) \\ &= \{\mathrm{id}_{\mathsf{true}}\} \\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t}, \mathsf{t}) \\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\big(\mathsf{t}, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t}, \mathsf{t})\big). \end{aligned}$$

- For $(A, B, C) = (\mathsf{t}, \mathsf{t}, \mathsf{f})$, we have

$$\begin{aligned} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t} \times \mathsf{t}, \mathsf{f}) &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t}, \mathsf{f}) \\ &= \emptyset \\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t}, \mathsf{f}) \\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\big(\mathsf{t}, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t}, \mathsf{f})\big). \end{aligned}$$

- For $(A, B, C) = (\mathsf{t}, \mathsf{f}, \mathsf{t})$, we have

$$\begin{aligned} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t} \times \mathsf{f}, \mathsf{t}) &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{t}) \\ &\cong \mathrm{pt} \\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{t}) \\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\big(\mathsf{f}, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{t})\big). \end{aligned}$$

- For $(A, B, C) = (\mathsf{t}, \mathsf{f}, \mathsf{f})$, we have

$$\begin{aligned} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t} \times \mathsf{f}, \mathsf{f}) &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{f}) \\ &\cong \{\mathrm{id}_{\mathsf{false}}\} \\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{f}) \\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\big(\mathsf{t}, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{f})\big). \end{aligned}$$

- For $(A, B, C) = (\mathsf{f}, \mathsf{t}, \mathsf{t})$, we have

$$\begin{aligned}\mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f} \times \mathsf{t}, \mathsf{t}) &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{t})\\ &\cong \mathrm{pt}\\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{t})\\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\big(\mathsf{f}, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t}, \mathsf{t})\big).\end{aligned}$$

- For $(A, B, C) = (\mathsf{f}, \mathsf{t}, \mathsf{f})$, we have

$$\begin{aligned}\mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f} \times \mathsf{t}, \mathsf{f}) &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{f})\\ &\cong \{\mathrm{id}_{\mathsf{false}}\}\\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{f})\\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\big(\mathsf{f}, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t}, \mathsf{f})\big).\end{aligned}$$

- For $(A, B, C) = (\mathsf{f}, \mathsf{f}, \mathsf{t})$, we have

$$\begin{aligned}\mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f} \times \mathsf{f}, \mathsf{t}) &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{t})\\ &\cong \mathrm{pt}\\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{t})\\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\big(\mathsf{f}, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{t})\big).\end{aligned}$$

- For $(A, B, C) = (\mathsf{f}, \mathsf{f}, \mathsf{f})$, we have

$$\begin{aligned}\mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f} \times \mathsf{f}, \mathsf{f}) &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{f})\\ &= \{\mathrm{id}_{\mathsf{false}}\}\\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{f})\\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\big(\mathsf{f}, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{f})\big).\end{aligned}$$

The proof of naturality is omitted.

1.2.3 0-Categories

Definition 1.2.3.1 ► 0-Categories

A 0-**category** is a poset.[1]

[1] *Motivation:* A 0-category is precisely a category enriched in the poset of (-1)-categories.

Definition 1.2.3.2 ► 0-Groupoids

A 0-**groupoid** is a 0-category in which every morphism is invertible.[1]

[1] That is, a *set*.

1.2.4 Tables of Analogies Between Set Theory and Category Theory

Here we record some analogies between notions in set theory and category theory. Note that the analogies relating to presheaves relate equally well to copresheaves, as the opposite X^{op} of a set X is just X again.
Basics:

Set Theory	Category Theory
Enrichment in $\{\text{true}, \text{false}\}$	Enrichment in Sets
Set X	Category $\mathcal{C}$
Element $x \in X$	Object $X \in \mathrm{Obj}(\mathcal{C})$
Function	Functor
Function $X \longrightarrow \{\text{true}, \text{false}\}$	Functor $\mathcal{C} \longrightarrow \mathsf{Sets}$
Function $X \longrightarrow \{\text{true}, \text{false}\}$	Presheaf $\mathcal{C}^{\mathsf{op}} \longrightarrow \mathsf{Sets}$

Powersets and categories of presheaves:

Set Theory	Category Theory
Powerset $\mathcal{P}(X)$	Presheaf category $\mathsf{PSh}(\mathcal{C})$
Characteristic function $\chi_{\{x\}}$	Representable presheaf h_X
Characteristic embedding $\chi_{(-)} \colon X \hookrightarrow \mathcal{P}(X)$	Yoneda embedding $よ \colon \mathcal{C}^{\mathsf{op}} \hookrightarrow \mathsf{PSh}(\mathcal{C})$
Characteristic relation $\chi_X(-_1, -_2)$	Hom profunctor $\mathrm{Hom}_{\mathcal{C}}(-_1, -_2)$
The Yoneda lemma for sets $\mathrm{Hom}_{\mathcal{P}(X)}(\chi_x, \chi_U) = \chi_U(x)$	The Yoneda lemma for categories $\mathrm{Nat}(h_X, \mathcal{F}) \cong \mathcal{F}(X)$
The characteristic embedding is fully faithful, $\mathrm{Hom}_{\mathcal{P}(X)}\big(\chi_x, \chi_y\big) = \chi_X(x, y)$	The Yoneda embedding is fully faithful, $\mathrm{Nat}(h_X, h_Y) \cong \mathrm{Hom}_{\mathcal{C}}(X, Y)$
Subsets are unions of their elements $U = \bigcup_{x \in U} \{x\}$ or $\chi_U = \operatorname*{colim}_{\chi_x \in \mathsf{Sets}(U, \{\mathsf{t}, \mathsf{f}\})} (\chi_x)$	Presheaves are colimits of representables, $\mathcal{F} \cong \operatorname*{colim}_{h_X \in \int_{\mathcal{C}} \mathcal{F}} (h_X)$

Categories of elements:

Set Theory	Category Theory
Assignment $U \mapsto \chi_U$	Assignment $\mathcal{F} \mapsto \int_C \mathcal{F}$ (the category of elements)
Assignment $U \mapsto \chi_U$ giving an isomorphism $\mathcal{P}(X) \cong \mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\})$	Assignment $\mathcal{F} \mapsto \int_C \mathcal{F}$ giving an equivalence $\mathsf{PSh}(C) \overset{\text{eq.}}{\cong} \mathsf{DFib}(C)$

Functions between powersets and functors between presheaf categories:

Set Theory	Category Theory
Direct image function $f_*\colon \mathcal{P}(X) \to \mathcal{P}(Y)$	Inverse image functor $f^{-1}\colon \mathsf{PSh}(C) \to \mathsf{PSh}(\mathcal{D})$
Inverse image function $f^{-1}\colon \mathcal{P}(Y) \to \mathcal{P}(X)$	Direct image functor $f_*\colon \mathsf{PSh}(\mathcal{D}) \to \mathsf{PSh}(C)$
Direct image with compact support function $f_!\colon \mathcal{P}(X) \to \mathcal{P}(Y)$	Direct image with compact support functor $f_!\colon \mathsf{PSh}(C) \to \mathsf{PSh}(\mathcal{D})$

Relations and profunctors:

Set Theory	Category Theory
Relation $R\colon X \times Y \to \{\mathsf{t}, \mathsf{f}\}$	Profunctor $\mathfrak{p}\colon \mathcal{D}^{\mathsf{op}} \times C \to \mathsf{Sets}$
Relation $R\colon X \to \mathcal{P}(Y)$	Profunctor $\mathfrak{p}\colon C \to \mathsf{PSh}(\mathcal{D})$
Relation as a cocontinuous morphism of posets $R\colon (\mathcal{P}(X), \subset) \to (\mathcal{P}(Y), \subset)$	Profunctor as a colimit-preserving functor $\mathfrak{p}\colon \mathsf{PSh}(C) \to \mathsf{PSh}(\mathcal{D})$

Appendices

1.A Other Chapters

Sets

1. Sets
2. Constructions With Sets
3. Pointed Sets
4. Tensor Products of Pointed Sets

Relations

5. Relations
6. Constructions With Relations
7. Equivalence Relations and Apartness Relations

Category Theory

8. Categories

Bicategories

9. Types of Morphisms in Bicategories

Chapter 2

Constructions With Sets

This chapter develops some material relating to constructions with sets with an eye towards its categorical and higher-categorical counterparts to be introduced later in this work. In particular, it contains:

1. Explicit descriptions of the major types of co/limits in Sets, including in particular explicit descriptions of pushouts and coequalisers (see Definitions 2.2.4.1 and 2.2.5.1 and Remarks 2.2.4.3 and 2.2.5.3).

2. A discussion of powersets as decategorifications of categories of presheaves (Remarks 2.4.1.2 and 2.4.3.2), including a (-1)-categorical analogue of un/straightening, described in Items 1 and 2 of Proposition 2.4.3.9 and Remark 2.4.3.11.

3. A lengthy discussion of the adjoint triple

$$f_* \dashv f^{-1} \dashv f_! \colon \mathcal{P}(A) \overset{\rightleftarrows}{\rightarrow} \mathcal{P}(B)$$

   of functors (morphisms of posets) between $\mathcal{P}(A)$ and $\mathcal{P}(B)$ induced by a map of sets $f \colon A \to B$, along with a discussion of the properties of f_*, f^{-1}, and $f_!$.

   In line with the categorical viewpoint developed here, this adjoint triple may be described in terms of Kan extensions, and, as it turns out, it also shows up in some definitions and results in point-set topology, such as in e.g. notions of continuity for functions (**??**).

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2.1 Limits of Sets

2.1.1 The Terminal Set

Definition 2.1.1.1 ► The Terminal Set

The **terminal set** is the pair $\left(\mathrm{pt}, \{!_A\}_{A\in\mathrm{Obj}(\mathsf{Sets})}\right)$ consisting of:

- *The Limit.* The punctual set $\mathrm{pt} \overset{\text{def}}{=} \{\star\}$.

- *The Cone*. The collection of maps

$$\{!_A \colon A \to \mathrm{pt}\}_{A \in \mathrm{Obj}(\mathsf{Sets})}$$

defined by

$$!_A(a) \stackrel{\mathrm{def}}{=} \star$$

for each $a \in A$ and each $A \in \mathrm{Obj}(\mathsf{Sets})$.

Proof 2.1.1.2 ▶ Proof of Definition 2.1.1.1

We claim that pt is the terminal object of Sets. Indeed, suppose we have a diagram of the form

$$A \qquad \mathrm{pt}$$

in Sets. Then there exists a unique map $\phi \colon A \to \mathrm{pt}$ making the diagram

$$A \overset{\phi}{\underset{\exists !}{\dashrightarrow}} \mathrm{pt}$$

commute, namely $!_A$. ∎

2.1.2 Products of Families of Sets

Let $\{A_i\}_{i \in I}$ be a family of sets.

Definition 2.1.2.1 ▶ The Product of a Family of Sets

The **product**[1] **of** $\{A_i\}_{i \in I}$ is the pair $\left(\prod_{i \in I} A_i, \{\mathrm{pr}_i\}_{i \in I}\right)$ consisting of:

- *The Limit*. The set $\prod_{i \in I} A_i$ defined by[2]

$$\prod_{i \in I} A_i \stackrel{\mathrm{def}}{=} \left\{ f \in \mathsf{Sets}\left(I, \bigcup_{i \in I} A_i\right) \;\middle|\; \begin{array}{l} \text{for each } i \in I \text{, we} \\ \text{have } f(i) \in A_i \end{array} \right\}.$$

- *The Cone*. The collection

$$\left\{ \mathrm{pr}_i \colon \prod_{i \in I} A_i \to A_i \right\}_{i \in I}$$

of maps given by

$$\mathrm{pr}_i(f) \stackrel{\mathrm{def}}{=} f(i)$$

for each $f \in \prod_{i \in I} A_i$ and each $i \in I$.

[1] *Further Terminology:* Also called the **Cartesian product of** $\{A_i\}_{i\in I}$.

[2] Less formally, $\prod_{i\in I} A_i$ is the set whose elements are I-indexed collections $(a_i)_{i\in I}$ with $a_i \in A_i$ for each $i \in I$. The projection maps

$$\left\{\mathrm{pr}_i \colon \prod_{i\in I} A_i \to A_i\right\}_{i\in I}$$

are then given by

$$\mathrm{pr}_i\Big((a_j)_{j\in I}\Big) \overset{\text{def}}{=} a_i$$

for each $(a_j)_{j\in I} \in \prod_{i\in I} A_i$ and each $i \in I$.

Proof 2.1.2.2 ► Proof of Definition 2.1.2.1

We claim that $\prod_{i\in I} A_i$ is the categorical product of $\{A_i\}_{i\in I}$ in Sets. Indeed, suppose we have, for each $i \in I$, a diagram of the form

$$\begin{tikzcd} P \arrow[dr, "p_i"] & \\ \prod_{i\in I} A_i \arrow[r, "\mathrm{pr}_i"'] & A_i \end{tikzcd}$$

in Sets. Then there exists a unique map $\phi\colon P \to \prod_{i\in I} A_i$ making the diagram

$$\begin{tikzcd} P \arrow[d, dashed, "\phi"', "\exists!"] \arrow[dr, "p_i"] & \\ \prod_{i\in I} A_i \arrow[r, "\mathrm{pr}_i"'] & A_i \end{tikzcd}$$

commute, being uniquely determined by the condition $\mathrm{pr}_i \circ \phi = p_i$ for each $i \in I$ via

$$\phi(x) = (p_i(x))_{i\in I}$$

for each $x \in P$.

Proposition 2.1.2.3 ► Properties of Products of Families of Sets

Let $\{A_i\}_{i\in I}$ be a family of sets.

1. *Functoriality.* The assignment $\{A_i\}_{i\in I} \mapsto \prod_{i\in I} A_i$ defines a functor

$$\prod_{i\in I} \colon \mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}) \longrightarrow \mathsf{Sets}$$

where

- *Action on Objects.* For each $(A_i)_{i\in I} \in \mathrm{Obj}(\mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}))$, we have

$$\left[\prod_{i\in I}\right]((A_i)_{i\in I}) \stackrel{\text{def}}{=} \prod_{i\in I} A_i$$

- *Action on Morphisms.* For each $(A_i)_{i\in I}, (B_i)_{i\in I} \in \mathrm{Obj}(\mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}))$, the action on Hom-sets

$$\left(\prod_{i\in I}\right)_{(A_i)_{i\in I},(B_i)_{i\in I}} \colon \mathrm{Nat}((A_i)_{i\in I}, (B_i)_{i\in I}) \longrightarrow \mathsf{Sets}\left(\prod_{i\in I} A_i, \prod_{i\in I} B_i\right)$$

of $\prod_{i\in I}$ at $((A_i)_{i\in I}, (B_i)_{i\in I})$ is defined by sending a map

$$\{f_i \colon A_i \longrightarrow B_i\}_{i\in I}$$

in $\mathrm{Nat}((A_i)_{i\in I}, (B_i)_{i\in I})$ to the map of sets

$$\prod_{i\in I} f_i \colon \prod_{i\in I} A_i \longrightarrow \prod_{i\in I} B_i$$

defined by

$$\left[\prod_{i\in I} f_i\right]((a_i)_{i\in I}) \stackrel{\text{def}}{=} (f_i(a_i))_{i\in I}$$

for each $(a_i)_{i\in I} \in \prod_{i\in I} A_i$.

Proof 2.1.2.4 ► Proof of Proposition 2.1.2.3

Item 1: Functoriality

This follows from **??** of **??**. □

2.1.3 Binary Products of Sets

Let A and B be sets.

Definition 2.1.3.1 ► Products of Sets

The **product**[1] **of A and B** is the pair $(A \times B, \{\mathrm{pr}_1, \mathrm{pr}_2\})$ consisting of:

- *The Limit.* The set $A \times B$ defined by[2]

$$\begin{aligned} A \times B &\overset{\text{def}}{=} \prod_{z \in \{A,B\}} z \\ &\overset{\text{def}}{=} \{f \in \mathsf{Sets}(\{0,1\}, A \cup B) \mid \text{we have } f(0) \in A \text{ and } f(1) \in B\} \\ &\cong \{\{\{a\},\{a,b\}\} \in \mathcal{P}(\mathcal{P}(A \cup B)) \mid \text{we have } a \in A \text{ and } b \in B\}. \end{aligned}$$

- *The Cone.* The maps

$$\begin{aligned} \mathrm{pr}_1 &\colon A \times B \to A, \\ \mathrm{pr}_2 &\colon A \times B \to B \end{aligned}$$

  defined by

$$\begin{aligned} \mathrm{pr}_1(a,b) &\overset{\text{def}}{=} a, \\ \mathrm{pr}_2(a,b) &\overset{\text{def}}{=} b \end{aligned}$$

  for each $(a,b) \in A \times B$.

[1] *Further Terminology:* Also called the **Cartesian product of A and B** or the **binary Cartesian product of A and B**, for emphasis.

This can also be thought of as the $(\mathbb{E}_{-1}, \mathbb{E}_{-1})$**-tensor product of A and B**.

[2] In other words, $A \times B$ is the set whose elements are ordered pairs (a,b) with $a \in A$ and $b \in B$ as in Definition 2.3.4.1

Proof 2.1.3.2 ► Proof of Definition 2.1.3.1

We claim that $A \times B$ is the categorical product of A and B in Sets. Indeed, suppose we have a diagram of the form

$$\begin{tikzcd} & P \arrow[dl, bend right, "p_1"'] \arrow[dr, bend left, "p_2"] & \\ A & A \times B \arrow[l, "\mathrm{pr}_1"] \arrow[r, "\mathrm{pr}_2"'] & B \end{tikzcd}$$

in Sets. Then there exists a unique map $\phi\colon P \to A \times B$ making the diagram

$$\begin{tikzcd} & P \arrow[ld, bend right, "p_1"'] \arrow[d, dashed, "\phi", "\exists!"'] \arrow[rd, bend left, "p_2"] & \\ A & A \times B \arrow[l, "\mathrm{pr}_1"] \arrow[r, "\mathrm{pr}_2"'] & B \end{tikzcd}$$

commute, being uniquely determined by the conditions

$$\mathrm{pr}_1 \circ \phi = p_1,$$
$$\mathrm{pr}_2 \circ \phi = p_2$$

via

$$\phi(x) = (p_1(x), p_2(x))$$

for each $x \in P$.

Proposition 2.1.3.3 ► Properties of Products of Sets

Let A, B, C, and X be sets.

1. *Functoriality.* The assignments $A, B, (A, B) \mapsto A \times B$ define functors

$$A \times -\colon \mathsf{Sets} \to \mathsf{Sets},$$
$$- \times B\colon \mathsf{Sets} \to \mathsf{Sets},$$
$$-_1 \times -_2\colon \mathsf{Sets} \times \mathsf{Sets} \to \mathsf{Sets},$$

where $-_1 \times -_2$ is the functor where

- *Action on Objects.* For each $(A, B) \in \mathrm{Obj}(\mathsf{Sets} \times \mathsf{Sets})$, we have

$$[-_1 \times -_2](A, B) \stackrel{\text{def}}{=} A \times B.$$

- *Action on Morphisms.* For each $(A, B), (X, Y) \in \mathrm{Obj}(\mathsf{Sets})$, the action on Hom-sets

$$\times_{(A,B),(X,Y)}\colon \mathsf{Sets}(A, X) \times \mathsf{Sets}(B, Y) \to \mathsf{Sets}(A \times B, X \times Y)$$

of $\times$ at $((A, B), (X, Y))$ is defined by sending (f, g) to the function

$$f \times g\colon A \times B \to X \times Y$$

defined by

$$[f \times g](a, b) \stackrel{\text{def}}{=} (f(a), g(b))$$

for each $(a, b) \in A \times B$.

and where $A \times -$ and $- \times B$ are the partial functors of $-_1 \times -_2$ at $A, B \in \mathrm{Obj}(\mathsf{Sets})$.

2. *Adjointness.* We have adjunctions

$$(A \times - \dashv \mathrm{Hom}_{\mathsf{Sets}}(A, -)) \colon \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "A\times -"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}, \arrow[l, bend left, "\mathrm{Hom}_{\mathsf{Sets}}(A,-)"] \end{tikzcd}$$

$$(- \times B \dashv \mathrm{Hom}_{\mathsf{Sets}}(B, -)) \colon \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "-\times B"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}, \arrow[l, bend left, "\mathrm{Hom}_{\mathsf{Sets}}(B,-)"] \end{tikzcd}$$

witnessed by bijections

$$\begin{aligned} \mathrm{Hom}_{\mathsf{Sets}}(A \times B, C) &\cong \mathrm{Hom}_{\mathsf{Sets}}(A, \mathrm{Hom}_{\mathsf{Sets}}(B, C)), \\ \mathrm{Hom}_{\mathsf{Sets}}(A \times B, C) &\cong \mathrm{Hom}_{\mathsf{Sets}}(B, \mathrm{Hom}_{\mathsf{Sets}}(A, C)), \end{aligned}$$

natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

3. *Associativity.* We have an isomorphism of sets

$$(A \times B) \times C \cong A \times (B \times C),$$

natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

4. *Unitality.* We have isomorphisms of sets

$$\begin{aligned} \mathrm{pt} \times A &\cong A, \\ A \times \mathrm{pt} &\cong A, \end{aligned}$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$.

5. *Commutativity.* We have an isomorphism of sets

$$A \times B \cong B \times A,$$

natural in $A, B \in \mathrm{Obj}(\mathsf{Sets})$.

6. *Annihilation With the Empty Set.* We have isomorphisms of sets

$$\begin{aligned} A \times \emptyset &\cong \emptyset, \\ \emptyset \times A &\cong \emptyset, \end{aligned}$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$.

7. *Distributivity Over Unions.* We have isomorphisms of sets

$$
\begin{aligned}
A \times (B \cup C) &= (A \times B) \cup (A \times C), \\
(A \cup B) \times C &= (A \times C) \cup (B \times C).
\end{aligned}
$$

8. *Distributivity Over Intersections.* We have isomorphisms of sets

$$
\begin{aligned}
A \times (B \cap C) &= (A \times B) \cap (A \times C), \\
(A \cap B) \times C &= (A \times C) \cap (B \times C).
\end{aligned}
$$

9. *Middle-Four Exchange with Respect to Intersections.* We have an isomorphism of sets

$$(A \times B) \cap (C \times D) \cong (A \cap B) \times (C \cap D).$$

10. *Distributivity Over Differences.* We have isomorphisms of sets

$$
\begin{aligned}
A \times (B \setminus C) &= (A \times B) \setminus (A \times C), \\
(A \setminus B) \times C &= (A \times C) \setminus (B \times C),
\end{aligned}
$$

natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

11. *Distributivity Over Symmetric Differences.* We have isomorphisms of sets

$$
\begin{aligned}
A \times (B \bigtriangleup C) &= (A \times B) \bigtriangleup (A \times C), \\
(A \bigtriangleup B) \times C &= (A \times C) \bigtriangleup (B \times C),
\end{aligned}
$$

natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

12. *Symmetric Monoidality.* The triple $(\mathsf{Sets}, \times, \mathrm{pt})$ is a symmetric monoidal category.

13. *Symmetric Bimonoidality.* The quintuple $(\mathsf{Sets}, \coprod, \emptyset, \times, \mathrm{pt})$ is a symmetric bimonoidal category.

Proof 2.1.3.4 ▶ Proof of Proposition 2.1.3.3

Item 1: Functoriality

This follows from **??** of **??**.

Item 2: Adjointness

We prove only that there's an adjunction $- \times B \dashv \mathrm{Hom}_{\mathsf{Sets}}(B, -)$, witnessed by a bijection

$$\mathrm{Hom}_{\mathsf{Sets}}(A \times B, C) \cong \mathrm{Hom}_{\mathsf{Sets}}(A, \mathrm{Hom}_{\mathsf{Sets}}(B, C)),$$

natural in $B, C \in \mathrm{Obj}(\mathsf{Sets})$, as the proof of the existence of the adjunction $A \times - \dashv \mathrm{Hom}_{\mathsf{Sets}}(A, -)$ follows almost exactly in the same way.

- *Map I.* We define a map

  $$\Phi_{B,C} \colon \mathrm{Hom}_{\mathsf{Sets}}(A \times B, C) \longrightarrow \mathrm{Hom}_{\mathsf{Sets}}(A, \mathrm{Hom}_{\mathsf{Sets}}(B, C)),$$

  by sending a function

  $$\xi \colon A \times B \to C$$

  to the function

  $$\begin{aligned} \xi^{\dagger} \colon A &\longrightarrow \mathrm{Hom}_{\mathsf{Sets}}(B, C), \\ a &\longmapsto \left(\xi^{\dagger}_{a} \colon B \to C\right), \end{aligned}$$

  where we define

  $$\xi^{\dagger}_{a}(b) \stackrel{\text{def}}{=} \xi(a, b)$$

  for each $b \in B$. In terms of the $[\![a \mapsto f(a)]\!]$ notation of Notation 1.1.1.2, we have

  $$\xi^{\dagger} \stackrel{\text{def}}{=} [\![a \mapsto [\![b \mapsto \xi(a, b)]\!]]\!].$$

- *Map II.* We define a map

  $$\Psi_{B,C} \colon \mathrm{Hom}_{\mathsf{Sets}}(A, \mathrm{Hom}_{\mathsf{Sets}}(B, C)), \longrightarrow \mathrm{Hom}_{\mathsf{Sets}}(A \times B, C)$$

  given by sending a function

  $$\begin{aligned} \xi \colon A &\longrightarrow \mathrm{Hom}_{\mathsf{Sets}}(B, C), \\ a &\longmapsto (\xi_{a} \colon B \to C), \end{aligned}$$

  to the function

  $$\xi^{\dagger} \colon A \times B \to C$$

  defined by

  $$\begin{aligned} \xi^{\dagger}(a, b) &\stackrel{\text{def}}{=} \mathrm{ev}_{b}(\mathrm{ev}_{a}(\xi)) \\ &\stackrel{\text{def}}{=} \mathrm{ev}_{b}(\xi_{a}) \\ &\stackrel{\text{def}}{=} \xi_{a}(b) \end{aligned}$$

  for each $(a, b) \in A \times B$.

- *Invertibility I.* We claim that

$$\Psi_{A,B} \circ \Phi_{A,B} = \mathrm{id}_{\mathrm{Hom}_{\mathsf{Sets}}(A\times B,C)}.$$

Indeed, given a function $\xi \colon A \times B \to C$, we have

$$\begin{aligned}
\left[\Psi_{A,B} \circ \Phi_{A,B}\right](\xi) &= \Psi_{A,B}(\Phi_{A,B}(\xi)) \\
&= \Psi_{A,B}(\Phi_{A,B}(\llbracket (a,b) \mapsto \xi(a,b)\rrbracket)) \\
&= \Psi_{A,B}(\llbracket a \mapsto \llbracket b \mapsto \xi(a,b)\rrbracket\rrbracket) \\
&= \Psi_{A,B}(\llbracket a' \mapsto \llbracket b' \mapsto \xi(a',b')\rrbracket\rrbracket) \\
&= \llbracket (a,b) \mapsto \mathrm{ev}_b(\mathrm{ev}_a(\llbracket a' \mapsto \llbracket b' \mapsto \xi(a',b')\rrbracket\rrbracket))\rrbracket \\
&= \llbracket (a,b) \mapsto \mathrm{ev}_b(\llbracket b' \mapsto \xi(a,b')\rrbracket)\rrbracket \\
&= \llbracket (a,b) \mapsto \xi(a,b)\rrbracket \\
&= \xi.
\end{aligned}$$

- *Invertibility II.* We claim that

$$\Phi_{A,B} \circ \Psi_{A,B} = \mathrm{id}_{\mathrm{Hom}_{\mathsf{Sets}}(A,\mathrm{Hom}_{\mathsf{Sets}}(B,C))}.$$

Indeed, given a function

$$\begin{aligned}
\xi \colon A &\longrightarrow \mathrm{Hom}_{\mathsf{Sets}}(B,C), \\
a &\longmapsto (\xi_a \colon B \to C),
\end{aligned}$$

we have

$$\begin{aligned}
\left[\Phi_{A,B} \circ \Psi_{A,B}\right](\xi) &\overset{\mathrm{def}}{=} \Phi_{A,B}(\Psi_{A,B}(\xi)) \\
&\overset{\mathrm{def}}{=} \Phi_{A,B}(\llbracket (a,b) \mapsto \xi_a(b)\rrbracket) \\
&\overset{\mathrm{def}}{=} \Phi_{A,B}(\llbracket (a',b') \mapsto \xi_{a'}(b')\rrbracket) \\
&\overset{\mathrm{def}}{=} \llbracket a \mapsto \llbracket b \mapsto \mathrm{ev}_{(a,b)}(\llbracket (a',b') \mapsto \xi_{a'}(b')\rrbracket)\rrbracket\rrbracket \\
&\overset{\mathrm{def}}{=} \llbracket a \mapsto \llbracket b \mapsto \xi_a(b)\rrbracket\rrbracket \\
&\overset{\mathrm{def}}{=} \llbracket a \mapsto \xi_a\rrbracket \\
&\overset{\mathrm{def}}{=} \xi.
\end{aligned}$$

- *Naturality for Φ, Part I.* We need to show that, given a function $g \colon B \to B'$, the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathsf{Sets}}(A\times B',C) \arrow[r, "\Phi_{B',C}"] \arrow[d, "\mathrm{id}_A\times g^*"'] & \mathrm{Hom}_{\mathsf{Sets}}(A,\mathrm{Hom}_{\mathsf{Sets}}(B',C)), \arrow[d, "(g^*)_*"] \\
\mathrm{Hom}_{\mathsf{Sets}}(A\times B,C) \arrow[r, "\Phi_{B,C}"'] & \mathrm{Hom}_{\mathsf{Sets}}(A,\mathrm{Hom}_{\mathsf{Sets}}(B,C))
\end{tikzcd}$$

commutes. Indeed, given a function

$$\xi\colon A\times B'\to C,$$

we have

$$\begin{aligned}
\big[\Phi_{B,C}\circ(\mathrm{id}_A\times g^*)\big](\xi) &= \Phi_{B,C}([\mathrm{id}_A\times g^*](\xi))\\
&= \Phi_{B,C}(\xi(-_1,g(-_2)))\\
&= [\xi(-_1,g(-_2))]^\dagger\\
&= \xi^\dagger_{-_1}(g(-_2))\\
&= (g^*)_*\left(\xi^\dagger\right)\\
&= (g^*)_*(\Phi_{B',C}(\xi))\\
&= \big[(g^*)_*\circ\Phi_{B',C}\big](\xi).
\end{aligned}$$

Alternatively, using the $[\![a\mapsto f(a)]\!]$ notation of Notation 1.1.1.2, we have

$$\begin{aligned}
\big[\Phi_{B,C}\circ(\mathrm{id}_A\times g^*)\big](\xi) &= \Phi_{B,C}([\mathrm{id}_A\times g^*](\xi))\\
&= \Phi_{B,C}\big([\mathrm{id}_A\times g^*]([\![(a,b')\mapsto\xi(a,b')]\!])\big)\\
&= \Phi_{B,C}\big([\![(a,b)\mapsto\xi(a,g(b))]\!]\big)\\
&= [\![a\mapsto[\![b\mapsto\xi(a,g(b))]\!]]\!]\\
&= [\![a\mapsto g^*([\![b'\mapsto\xi(a,b')]\!])]\!]\\
&= (g^*)_*([\![a\mapsto[\![b'\mapsto\xi(a,b')]\!]]\!])\\
&= (g^*)_*(\Phi_{B',C}([\![(a,b')\mapsto\xi(a,b')]\!]))\\
&= (g^*)_*(\Phi_{B',C}(\xi))\\
&= \big[(g^*)_*\circ\Phi_{B',C}\big](\xi).
\end{aligned}$$

- *Naturality for Φ, Part II.* We need to show that, given a function $h\colon C\to C'$, the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathsf{Sets}}(A\times B,C) \arrow[r, "\Phi_{B,C}"] \arrow[d, "h_*"'] & \mathrm{Hom}_{\mathsf{Sets}}(A,\mathrm{Hom}_{\mathsf{Sets}}(B,C)), \arrow[d, "(h_*)_*"]\\
\mathrm{Hom}_{\mathsf{Sets}}(A\times B,C') \arrow[r, "\Phi_{B,C'}"'] & \mathrm{Hom}_{\mathsf{Sets}}(A,\mathrm{Hom}_{\mathsf{Sets}}(B,C'))
\end{tikzcd}$$

commutes. Indeed, given a function

$$\xi\colon A\times B\to C,$$

we have

$$\begin{aligned}
\left[\Phi_{B,C} \circ h_*\right](\xi) &= \Phi_{B,C}(h_*(\xi)) \\
&= \Phi_{B,C}\left(h_*\left(\llbracket (a,b) \mapsto \xi(a,b) \rrbracket\right)\right) \\
&= \Phi_{B,C}\left(\llbracket (a,b) \mapsto h(\xi(a,b)) \rrbracket\right) \\
&= \llbracket a \mapsto \llbracket b \mapsto h(\xi(a,b)) \rrbracket \rrbracket \\
&= \llbracket a \mapsto h_*\left(\llbracket b \mapsto \xi(a,b) \rrbracket\right) \rrbracket \\
&= (h_*)_*\left(\llbracket a \mapsto \llbracket b \mapsto \xi(a,b) \rrbracket \rrbracket\right) \\
&= (h_*)_*\left(\Phi_{B,C}\left(\llbracket (a,b) \mapsto \xi(a,b) \rrbracket\right)\right) \\
&= (h_*)_*\left(\Phi_{B,C}(\xi)\right) \\
&= \left[(h_*)_* \circ \Phi_{B,C}\right](\xi).
\end{aligned}$$

- *Naturality for* Ψ. Since Φ is natural in each argument and Φ is a componentwise inverse to Ψ in each argument, it follows from Item 2 of Proposition 8.8.6.2 that Ψ is also natural in each argument.

Item 3: Associativity

See [Pro24a].

Item 4: Unitality

Clear.

Item 5: Commutativity

See [Pro24b].

Item 6: Annihilation With the Empty Set

See [Pro24f].

Item 7: Distributivity Over Unions

See [Pro24e].

Item 8: Distributivity Over Intersections

See [Pro24g, Corollary 1].

Item 9: Middle-Four Exchange With Respect to Intersections

See [Pro24g, Corollary 1].

Item 10: Distributivity Over Differences

See [Pro24c].

Item 11: Distributivity Over Symmetric Differences

See [Pro24d].

Item 12: Symmetric Monoidality

See [MO 382264].

Item 13: Symmetric Bimonoidality

Omitted.

2.1.4 Pullbacks

Let A, B, and C be sets and let $f\colon A \to C$ and $g\colon B \to C$ be functions.

Definition 2.1.4.1 ► Pullbacks of Sets

The **pullback of A and B over C along f and g**[1] is the pair[2] $(A \times_C B, \{\mathrm{pr}_1, \mathrm{pr}_2\})$ consisting of:

- *The Limit.* The set $A \times_C B$ defined by

$$A \times_C B \overset{\text{def}}{=} \{(a, b) \in A \times B \mid f(a) = g(b)\}.$$

- *The Cone.* The maps

$$\begin{aligned}\mathrm{pr}_1 &: A \times_C B \to A,\\ \mathrm{pr}_2 &: A \times_C B \to B\end{aligned}$$

  defined by

$$\begin{aligned}\mathrm{pr}_1(a, b) &\overset{\text{def}}{=} a,\\ \mathrm{pr}_2(a, b) &\overset{\text{def}}{=} b\end{aligned}$$

  for each $(a, b) \in A \times_C B$.

[1] *Further Terminology:* Also called the **fibre product of A and B over C along f and g**.
[2] *Further Notation:* Also written $A \times_{f,C,g} B$.

Proof 2.1.4.2 ► Proof of Definition 2.1.4.1

We claim that $A \times_C B$ is the categorical pullback of A and B over C with respect to (f, g) in Sets. First we need to check that the relevant pullback diagram commutes, i.e. that we have

$$f \circ \mathrm{pr}_1 = g \circ \mathrm{pr}_2, \qquad \begin{tikzcd} A \times_C B \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C. \end{tikzcd}$$

Indeed, given $(a, b) \in A \times_C B$, we have

$$\begin{aligned} [f \circ \mathrm{pr}_1](a, b) &= f(\mathrm{pr}_1(a, b)) \\ &= f(a) \\ &= g(b) \\ &= g(\mathrm{pr}_2(a, b)) \\ &= [g \circ \mathrm{pr}_2](a, b), \end{aligned}$$

where $f(a) = g(b)$ since $(a, b) \in A \times_C B$. Next, we prove that $A \times_C B$ satisfies the universal property of the pullback. Suppose we have a diagram of the form

$$\begin{tikzcd} P \arrow[drr, bend left, "p_2"] \arrow[ddr, bend right, "p_1"'] & & \\ & A \times_C B \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "g"] \\ & A \arrow[r, "f"'] & C \end{tikzcd}$$

in Sets. Then there exists a unique map $\phi \colon P \to A \times_C B$ making the diagram

$$\begin{tikzcd} P \arrow[drr, bend left, "p_2"] \arrow[ddr, bend right, "p_1"'] \arrow[dr, dashed, "\phi", "\exists!"'] & & \\ & A \times_C B \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "g"] \\ & A \arrow[r, "f"'] & C \end{tikzcd}$$

commute, being uniquely determined by the conditions

$$\mathrm{pr}_1 \circ \phi = p_1,$$
$$\mathrm{pr}_2 \circ \phi = p_2$$

via

$$\phi(x) = (p_1(x), p_2(x))$$

for each $x \in P$, where we note that $(p_1(x), p_2(x)) \in A \times B$ indeed lies in $A \times_C B$ by the condition

$$f \circ p_1 = g \circ p_2,$$

which gives

$$f(p_1(x)) = g(p_2(x))$$

for each $x \in P$, so that $(p_1(x), p_2(x)) \in A \times_C B$.

EXAMPLE 2.1.4.3 ► EXAMPLES OF PULLBACKS OF SETS

Here are some examples of pullbacks of sets.

1. *Unions via Intersections.* Let $A, B \subset X$. We have a bijection of sets

$$A \cap B \cong A \times_{A \cup B} B, \qquad \begin{tikzcd} A \cap B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "\iota_B"] \\ A \arrow[r, "\iota_A"'] & A \cup B. \end{tikzcd}$$

PROOF 2.1.4.4 ► PROOF OF EXAMPLE 2.1.4.3

Item 1: Unions via Intersections

Indeed, we have

$$\begin{aligned} A \times_{A \cup B} B &\cong \{(x, y) \in A \times B \mid x = y\} \\ &\cong A \cap B. \end{aligned}$$

This finishes the proof.

PROPOSITION 2.1.4.5 ► PROPERTIES OF PULLBACKS OF SETS

Let A, B, C, and X be sets.

1. *Functoriality.* The assignment $(A, B, C, f, g) \mapsto A \times_{f,C,g} B$ defines a functor
$$-_1 \times_{-_3} -_1 \colon \mathrm{Fun}(\mathcal{P}, \mathsf{Sets}) \to \mathsf{Sets},$$
where $\mathcal{P}$ is the category that looks like this:
$$\begin{tikzcd}
 & \bullet \arrow[d] \\
\bullet \arrow[r] & \bullet.
\end{tikzcd}$$
In particular, the action on morphisms of $-_1 \times_{-_3} -_1$ is given by sending a morphism
$$\begin{tikzcd}
A \times_C B \arrow[rr] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & & B \arrow[dd, "g" description] \arrow[dr, "\psi"] & \\
 & A' \times_{C'} B' \arrow[rr] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & & B' \arrow[dd, "g'"] \\
A \arrow[rr, "f" description] \arrow[dr, "\phi"'] & & C \arrow[dr, "\chi" description] & \\
 & A' \arrow[rr, "f'"'] & & C'
\end{tikzcd}$$
in $\mathrm{Fun}(\mathcal{P}, \mathsf{Sets})$ to the map $\xi \colon A \times_C B \xrightarrow{\exists!} A' \times_{C'} B'$ given by
$$\xi(a, b) \overset{\text{def}}{=} (\phi(a), \psi(b))$$
for each $(a, b) \in A \times_C B$, which is the unique map making the diagram
$$\begin{tikzcd}
A \times_C B \arrow[rr] \arrow[dd] \arrow[dr, dashed] \arrow[dr, phantom, "\lrcorner", very near start] & & B \arrow[dd, "g" description] \arrow[dr, "\psi"] & \\
 & A' \times_{C'} B' \arrow[rr] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & & B' \arrow[dd, "g'"] \\
A \arrow[rr, "f" description] \arrow[dr, "\phi"'] & & C \arrow[dr, "\chi" description] & \\
 & A' \arrow[rr, "f'"'] & & C'
\end{tikzcd}$$
commute.

2. *Associativity.* Given a diagram

$$\begin{tikzcd}
A \arrow[dr, "f"'] & & B \arrow[dl, "g"] \arrow[dr, "h"'] & & C \arrow[dl, "k"] \\
& X & & Y &
\end{tikzcd}$$

in Sets, we have isomorphisms of sets

$$(A \times_X B) \times_Y C \cong (A \times_X B) \times_B (B \times_Y C) \cong A \times_X (B \times_Y C),$$

where these pullbacks are built as in the diagrams

$$\begin{tikzcd}[column sep=tiny]
& & (A \times_X B) \times_Y C \arrow[dl] \arrow[ddrr] \arrow[dd, phantom, "\lrcorner", very near start] & & \\
& A \times_X B \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner", very near start] & & & \\
A \arrow[dr, "f"'] & & B \arrow[dl, "g"] \arrow[dr, "h"'] & & C, \arrow[dl, "k"] \\
& X & & Y &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=tiny]
& & (A \times_X B) \times_B (B \times_Y C) \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner", very near start] & & \\
& A \times_X B \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner", very near start] & & B \times_Y C \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner", very near start] & \\
A \arrow[dr, "f"'] & & B \arrow[dl, "g"] \arrow[dr, "h"'] & & C, \arrow[dl, "k"] \\
& X & & Y &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=tiny]
& & A \times_X (B \times_Y C) \arrow[ddll] \arrow[dr] \arrow[dd, phantom, "\lrcorner", very near start] & & \\
& & & B \times_Y C \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner", very near start] & \\
A \arrow[dr, "f"'] & & B \arrow[dl, "g"] \arrow[dr, "h"'] & & C. \arrow[dl, "k"] \\
& X & & Y &
\end{tikzcd}$$

3. *Unitality.* We have isomorphisms of sets

$$\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\
X \arrow[r, equal] & X
\end{tikzcd}
\qquad
\begin{aligned}
X \times_X A &\cong A, \\
A \times_X X &\cong A,
\end{aligned}
\qquad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, equal] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, equal] \\
X \arrow[r, "f"'] & X.
\end{tikzcd}$$

4. *Commutativity.* We have an isomorphism of sets

$$\begin{tikzcd}
A \times_X B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "g"] \\
A \arrow[r, "f"'] & X,
\end{tikzcd}
\qquad
A \times_X B \cong B \times_X A
\qquad
\begin{tikzcd}
B \times_X A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\
B \arrow[r, "g"'] & X.
\end{tikzcd}$$

5. *Annihilation With the Empty Set.* We have isomorphisms of sets

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \emptyset \arrow[d] \\
A \arrow[r, "f"'] & X,
\end{tikzcd}
\qquad
\begin{aligned}
A \times_X \emptyset &\cong \emptyset, \\
\emptyset \times_X A &\cong \emptyset,
\end{aligned}
\qquad
\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\
\emptyset \arrow[r] & X.
\end{tikzcd}$$

6. *Interaction With Products.* We have an isomorphism of sets

$$A \times_{\mathrm{pt}} B \cong A \times B, \qquad \begin{tikzcd} A \times B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "!_B"] \\ A \arrow[r, "!_A"'] & \mathrm{pt}. \end{tikzcd}$$

7. *Symmetric Monoidality.* The triple $(\mathsf{Sets}, \times_X, X)$ is a symmetric monoidal category.

Proof 2.1.4.6 ► Proof of Proposition 2.1.4.5

Item 1: Functoriality

This is a special case of functoriality of co/limits, **??** of **??**, with the explicit expression for ξ following from the commutativity of the cube pullback diagram.

Item 2: Associativity

Indeed, we have

$$\begin{aligned} (A \times_X B) \times_Y C &\cong \{((a,b),c) \in (A \times_X B) \times C \mid h(b) = k(c)\} \\ &\cong \{((a,b),c) \in (A \times B) \times C \mid f(a) = g(b) \text{ and } h(b) = k(c)\} \\ &\cong \{(a,(b,c)) \in A \times (B \times C) \mid f(a) = g(b) \text{ and } h(b) = k(c)\} \\ &\cong \{(a,(b,c)) \in A \times (B \times_Y C) \mid f(a) = g(b)\} \\ &\cong A \times_X (B \times_Y C) \end{aligned}$$

and

$$\begin{aligned} (A \times_X B) \times_B (B \times_Y C) &\cong \left\{((a,b),(b',c)) \in (A \times_X B) \times (B \times_Y C) \,\middle|\, b = b'\right\} \\ &\cong \left\{((a,b),(b',c)) \in (A \times B) \times (B \times C) \,\middle|\, \begin{array}{l} f(a) = g(b), b = b', \\ \text{and } h(b') = k(c) \end{array}\right\} \\ &\cong \left\{(a,(b,(b',c))) \in A \times (B \times (B \times C)) \,\middle|\, \begin{array}{l} f(a) = g(b), b = b', \\ \text{and } h(b') = k(c) \end{array}\right\} \\ &\cong \left\{(a,((b,b'),c)) \in A \times ((B \times B) \times C) \,\middle|\, \begin{array}{l} f(a) = g(b), b = b', \\ \text{and } h(b') = k(c) \end{array}\right\} \\ &\cong \left\{(a,((b,b'),c)) \in A \times ((B \times_B B) \times C) \,\middle|\, \begin{array}{l} f(a) = g(b) \text{ and} \\ h(b') = k(c) \end{array}\right\} \\ &\cong \{(a,(b,c)) \in A \times (B \times C) \mid f(a) = g(b) \text{ and } h(b) = k(c)\} \\ &\cong A \times_X (B \times_Y C), \end{aligned}$$

where we have used Item 3 for the isomorphism $B \times_B B \cong B$.

Item 3: Unitality

Indeed, we have

$$
\begin{aligned}
X \times_X A &\cong \{(x, a) \in X \times A \mid f(a) = x\},\\
A \times_X X &\cong \{(a, x) \in X \times A \mid f(a) = x\},
\end{aligned}
$$

which are isomorphic to A via the maps $(x, a) \mapsto a$ and $(a, x) \mapsto a$.

Item 4: Commutativity

Clear.

Item 5: Annihilation With the Empty Set

Clear.

Item 6: Interaction With Products

Clear.

Item 7: Symmetric Monoidality

Omitted.

2.1.5 Equalisers

Let A and B be sets and let $f, g \colon A \rightrightarrows B$ be functions.

Definition 2.1.5.1 ► Equalisers of Sets

The **equaliser of** f **and** g is the pair $(\mathrm{Eq}(f, g), \mathrm{eq}(f, g))$ consisting of:

- *The Limit.* The set $\mathrm{Eq}(f, g)$ defined by

$$\mathrm{Eq}(f, g) \overset{\text{def}}{=} \{a \in A \mid f(a) = g(a)\}.$$

- *The Cone.* The inclusion map

$$\mathrm{eq}(f, g) \colon \mathrm{Eq}(f, g) \hookrightarrow A.$$

Proof 2.1.5.2 ► Proof of Definition 2.1.5.1

We claim that $\mathrm{Eq}(f, g)$ is the categorical equaliser of f and g in Sets. First we need to check that the relevant equaliser diagram commutes, i.e. that we have

$$f \circ \mathrm{eq}(f, g) = g \circ \mathrm{eq}(f, g),$$

which indeed holds by the definition of the set $\mathrm{Eq}(f, g)$. Next, we prove that $\mathrm{Eq}(f, g)$ satisfies the universal property of the equaliser. Suppose we have a diagram of the form

$$\begin{tikzcd} \mathrm{Eq}(f,g) \arrow[r, hook, "\mathrm{eq}(f,g)"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\ E \arrow[ur, "e"'] & & \end{tikzcd}$$

in Sets. Then there exists a unique map $\phi\colon E \to \mathrm{Eq}(f, g)$ making the diagram

$$\begin{tikzcd} \mathrm{Eq}(f,g) \arrow[r, hook, "\mathrm{eq}(f,g)"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\ E \arrow[u, dashed, "\phi", "\exists!"'] \arrow[ur, "e"'] & & \end{tikzcd}$$

commute, being uniquely determined by the condition

$$\mathrm{eq}(f, g) \circ \phi = e$$

via

$$\phi(x) = e(x)$$

for each $x \in E$, where we note that $e(x) \in A$ indeed lies in $\mathrm{Eq}(f, g)$ by the condition

$$f \circ e = g \circ e,$$

which gives

$$f(e(x)) = g(e(x))$$

for each $x \in E$, so that $e(x) \in \mathrm{Eq}(f, g)$. ∎

Proposition 2.1.5.3 ▶ Properties of Equalisers of Sets

Let A, B, and C be sets.

1. *Associativity*. We have isomorphisms of sets[1]

$$\underbrace{\mathrm{Eq}(f \circ \mathrm{eq}(g,h), g \circ \mathrm{eq}(g,h))}_{=\mathrm{Eq}(f\circ\mathrm{eq}(g,h), h\circ\mathrm{eq}(g,h))} \cong \mathrm{Eq}(f, g, h) \cong \underbrace{\mathrm{Eq}(f \circ \mathrm{eq}(f,g), h \circ \mathrm{eq}(f,g))}_{=\mathrm{Eq}(g\circ\mathrm{eq}(f,g), h\circ\mathrm{eq}(f,g))},$$

where $\mathrm{Eq}(f,g,h)$ is the limit of the diagram

$$A \overset{f}{\underset{h}{\overset{-g\rightarrow}{\rightrightarrows}}} B$$

in Sets, being explicitly given by

$$\mathrm{Eq}(f,g,h) \cong \{a \in A \mid f(a) = g(a) = h(a)\}.$$

2. *Unitality.* We have an isomorphism of sets

$$\mathrm{Eq}(f,f) \cong A.$$

3. *Commutativity.* We have an isomorphism of sets

$$\mathrm{Eq}(f,g) \cong \mathrm{Eq}(g,f).$$

4. *Interaction With Composition.* Let

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \underset{k}{\overset{h}{\rightrightarrows}} C$$

be functions. We have an inclusion of sets

$$\mathrm{Eq}(h \circ f \circ \mathrm{eq}(f,g), k \circ g \circ \mathrm{eq}(f,g)) \subset \mathrm{Eq}(h \circ f, k \circ g),$$

where $\mathrm{Eq}(h \circ f \circ \mathrm{eq}(f,g), k \circ g \circ \mathrm{eq}(f,g))$ is the equaliser of the composition

$$\mathrm{Eq}(f,g) \overset{\mathrm{eq}(f,g)}{\hookrightarrow} A \underset{g}{\overset{f}{\rightrightarrows}} B \underset{k}{\overset{h}{\rightrightarrows}} C.$$

---

[1]That is, the following three ways of forming "the" equaliser of (f,g,h) agree:

(a) Take the equaliser of (f,g,h), i.e. the limit of the diagram

$$A \overset{f}{\underset{h}{\overset{-g\rightarrow}{\rightrightarrows}}} B$$

in Sets.

(b) First take the equaliser of f and g, forming a diagram

$$\mathrm{Eq}(f,g) \overset{\mathrm{eq}(f,g)}{\hookrightarrow} A \underset{g}{\overset{f}{\rightrightarrows}} B$$

and then take the equaliser of the composition

$$\mathrm{Eq}(f,g) \overset{\mathrm{eq}(f,g)}{\hookrightarrow} A \underset{h}{\overset{f}{\rightrightarrows}} B,$$

obtaining a subset

$$\mathrm{Eq}(f \circ \mathrm{eq}(f,g), h \circ \mathrm{eq}(f,g)) = \mathrm{Eq}(g \circ \mathrm{eq}(f,g), h \circ \mathrm{eq}(f,g))$$

of $\mathrm{Eq}(f,g)$.

(c) First take the equaliser of g and h, forming a diagram

$$\mathrm{Eq}(g,h) \overset{\mathrm{eq}(g,h)}{\hookrightarrow} A \underset{h}{\overset{g}{\rightrightarrows}} B$$

and then take the equaliser of the composition

$$\mathrm{Eq}(g,h) \overset{\mathrm{eq}(g,h)}{\hookrightarrow} A \underset{g}{\overset{f}{\rightrightarrows}} B,$$

obtaining a subset

$$\mathrm{Eq}(f \circ \mathrm{eq}(g,h), g \circ \mathrm{eq}(g,h)) = \mathrm{Eq}(f \circ \mathrm{eq}(g,h), h \circ \mathrm{eq}(g,h))$$

of $\mathrm{Eq}(g,h)$.

Proof 2.1.5.4 ► Proof of Proposition 2.1.5.3

Item 1: Associativity

We first prove that $\mathrm{Eq}(f,g,h)$ is indeed given by

$$\mathrm{Eq}(f,g,h) \cong \{a \in A \mid f(a) = g(a) = h(a)\}.$$

Indeed, suppose we have a diagram of the form

$$\begin{tikzcd} \mathrm{Eq}(f,g,h) \arrow[r, hook, "\mathrm{eq}(f,g,h)"] & A \arrow[r, shift left=2, "f"] \arrow[r, "g" description] \arrow[r, shift right=2, "h"'] & B \\ E \arrow[ur, "e"'] & & \end{tikzcd}$$

in Sets. Then there exists a unique map $\phi\colon E \longrightarrow \mathrm{Eq}(f,g,h)$, uniquely determined by the condition

$$\mathrm{eq}(f,g) \circ \phi = e$$

being necessarily given by

$$\phi(x) = e(x)$$

for each $x \in E$, where we note that $e(x) \in A$ indeed lies in $\mathrm{Eq}(f, g, h)$ by the condition

$$f \circ e = g \circ e = h \circ e,$$

which gives

$$f(e(x)) = g(e(x)) = h(e(x))$$

for each $x \in E$, so that $e(x) \in \mathrm{Eq}(f, g, h)$.
We now check the equalities

$$\mathrm{Eq}(f \circ \mathrm{eq}(g, h), g \circ \mathrm{eq}(g, h)) \cong \mathrm{Eq}(f, g, h) \cong \mathrm{Eq}(f \circ \mathrm{eq}(f, g), h \circ \mathrm{eq}(f, g)).$$

Indeed, we have

$$\begin{aligned}
\mathrm{Eq}(f \circ \mathrm{eq}(g, h), g \circ \mathrm{eq}(g, h)) &\cong \{x \in \mathrm{Eq}(g, h) \mid [f \circ \mathrm{eq}(g, h)](a) = [g \circ \mathrm{eq}(g, h)](a)\} \\
&\cong \{x \in \mathrm{Eq}(g, h) \mid f(a) = g(a)\} \\
&\cong \{x \in A \mid f(a) = g(a) \text{ and } g(a) = h(a)\} \\
&\cong \{x \in A \mid f(a) = g(a) = h(a)\} \\
&\cong \mathrm{Eq}(f, g, h).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\mathrm{Eq}(f \circ \mathrm{eq}(f, g), h \circ \mathrm{eq}(f, g)) &\cong \{x \in \mathrm{Eq}(f, g) \mid [f \circ \mathrm{eq}(f, g)](a) = [h \circ \mathrm{eq}(f, g)](a)\} \\
&\cong \{x \in \mathrm{Eq}(f, g) \mid f(a) = h(a)\} \\
&\cong \{x \in A \mid f(a) = h(a) \text{ and } f(a) = g(a)\} \\
&\cong \{x \in A \mid f(a) = g(a) = h(a)\} \\
&\cong \mathrm{Eq}(f, g, h).
\end{aligned}$$

Item 2: Unitality

Clear.

Item 3: Commutativity

Clear.

Item 4: Interaction With Composition

Indeed, we have

$$\begin{aligned}
\mathrm{Eq}(h \circ f \circ \mathrm{eq}(f, g), k \circ g \circ \mathrm{eq}(f, g)) &\cong \{a \in \mathrm{Eq}(f, g) \mid h(f(a)) = k(g(a))\} \\
&\cong \{a \in A \mid f(a) = g(a) \text{ and } h(f(a)) = k(g(a))\}.
\end{aligned}$$

and

$$\mathrm{Eq}(h \circ f, k \circ g) \cong \{a \in A \mid h(f(a)) = k(g(a))\},$$

and thus there's an inclusion from $\mathrm{Eq}(h \circ f \circ \mathrm{eq}(f, g), k \circ g \circ \mathrm{eq}(f, g))$ to $\mathrm{Eq}(h \circ f, k \circ g)$. ∎

2.2 Colimits of Sets

2.2.1 The Initial Set

The **initial set** is the pair $\left(\emptyset, \{\iota_A\}_{A\in\mathrm{Obj}(\mathsf{Sets})}\right)$ consisting of:

- *The Limit.* The empty set $\emptyset$ of Definition 2.3.1.1.
- *The Cone.* The collection of maps

$$\{\iota_A \colon \emptyset \to A\}_{A\in\mathrm{Obj}(\mathsf{Sets})}$$

given by the inclusion maps from $\emptyset$ to A.

Proof 2.2.1.2 ► Proof of Definition 2.2.1.1

We claim that $\emptyset$ is the initial object of Sets. Indeed, suppose we have a diagram of the form

$$\emptyset \qquad A$$

in Sets. Then there exists a unique map $\phi \colon \emptyset \to A$ making the diagram

$$\begin{tikzcd} \emptyset \arrow[r, dashed, "\phi", "\exists!"'] & A \end{tikzcd}$$

commute, namely the inclusion map ι_A. □

2.2.2 Coproducts of Families of Sets

Let $\{A_i\}_{i\in I}$ be a family of sets.

Definition 2.2.2.1 ► Disjoint Unions of Families

The **disjoint union of the family** $\{A_i\}_{i\in I}$ is the pair $\left(\coprod_{i\in I} A_i, \{\mathrm{inj}_i\}_{i\in I}\right)$ consisting of:

- *The Colimit.* The set $\coprod_{i\in I} A_i$ defined by

$$\coprod_{i\in I} A_i \stackrel{\text{def}}{=} \left\{(i,x) \in I \times \left(\bigcup_{i\in I} A_i\right) \middle| x \in A_i\right\}.$$

- *The Cocone.* The collection

$$\left\{\mathrm{inj}_i \colon A_i \to \coprod_{i\in I} A_i\right\}_{i\in I}$$

of maps given by

$$\mathrm{inj}_i(x) \stackrel{\text{def}}{=} (i,x)$$

for each $x \in A_i$ and each $i \in I$.

Proof 2.2.2.2 ▶ Proof of Definition 2.2.2.1

We claim that $\coprod_{i\in I} A_i$ is the categorical coproduct of $\{A_i\}_{i\in I}$ in Sets. Indeed, suppose we have, for each $i \in I$, a diagram of the form

$$\begin{tikzcd} & C \\ A_i \arrow[ur, "\iota_i"] \arrow[r, "\mathrm{inj}_i"'] & \coprod\limits_{i\in I} A_i \end{tikzcd}$$

in Sets. Then there exists a unique map $\phi\colon \coprod_{i\in I} A_i \to C$ making the diagram

$$\begin{tikzcd} & C \\ A_i \arrow[ur, "\iota_i"] \arrow[r, "\mathrm{inj}_i"'] & \coprod\limits_{i\in I} A_i \arrow[u, dashed, "\phi", "\exists!"'] \end{tikzcd}$$

commute, being uniquely determined by the condition $\phi \circ \mathrm{inj}_i = \iota_i$ for each $i \in I$ via

$$\phi((i,x)) = \iota_i(x)$$

for each $(i,x) \in \coprod_{i\in I} A_i$.

Proposition 2.2.2.3 ▶ Properties of Coproducts of Families of Sets

Let $\{A_i\}_{i\in I}$ be a family of sets.

1. *Functoriality.* The assignment $\{A_i\}_{i\in I} \mapsto \coprod_{i\in I} A_i$ defines a functor

$$\coprod_{i\in I}\colon \mathrm{Fun}(I_{\mathrm{disc}}, \mathrm{Sets}) \to \mathrm{Sets}$$

where

- *Action on Objects.* For each $(A_i)_{i\in I} \in \mathrm{Obj}(\mathrm{Fun}(I_{\mathrm{disc}}, \mathrm{Sets}))$, we have

$$\left[\coprod_{i\in I}\right]((A_i)_{i\in I}) \overset{\mathrm{def}}{=} \coprod_{i\in I} A_i$$

- *Action on Morphisms.* For each $(A_i)_{i\in I}, (B_i)_{i\in I} \in \mathrm{Obj}(\mathrm{Fun}(I_{\mathrm{disc}}, \mathsf{Sets}))$, the action on Hom-sets

$$\left(\coprod_{i\in I}\right)_{(A_i)_{i\in I},(B_i)_{i\in I}} : \mathrm{Nat}((A_i)_{i\in I}, (B_i)_{i\in I}) \to \mathsf{Sets}\left(\coprod_{i\in I} A_i, \coprod_{i\in I} B_i\right)$$

of $\coprod_{i\in I}$ at $((A_i)_{i\in I}, (B_i)_{i\in I})$ is defined by sending a map

$$\{f_i : A_i \to B_i\}_{i\in I}$$

in $\mathrm{Nat}((A_i)_{i\in I}, (B_i)_{i\in I})$ to the map of sets

$$\coprod_{i\in I} f_i : \coprod_{i\in I} A_i \to \coprod_{i\in I} B_i$$

defined by

$$\left[\coprod_{i\in I} f_i\right](i, a) \overset{\mathrm{def}}{=} f_i(a)$$

for each $(i, a) \in \coprod_{i\in I} A_i$.

Proof 2.2.2.4 ▶ Proof of Proposition 2.2.2.3

Item 1: Functoriality

This follows from **??** of **??**.

2.2.3 Binary Coproducts

Let A and B be sets.

Definition 2.2.3.1 ▶ Coproducts of Sets

The **coproduct**[1] **of** A **and** B is the pair $(A \coprod B, \{\mathrm{inj}_1, \mathrm{inj}_2\})$ consisting of:

- *The Colimit.* The set $A \coprod B$ defined by

$$\begin{aligned} A \coprod B &\overset{\mathrm{def}}{=} \coprod_{z\in\{A,B\}} z \\ &\cong \{(0, a) \mid a \in A\} \cup \{(1, b) \mid b \in B\}. \end{aligned}$$

- *The Cocone.* The maps

$$\begin{aligned}\mathrm{inj}_1 &\colon A \to A \coprod B,\\ \mathrm{inj}_2 &\colon B \to A \coprod B,\end{aligned}$$

given by

$$\begin{aligned}\mathrm{inj}_1(a) &\stackrel{\text{def}}{=} (0, a),\\ \mathrm{inj}_2(b) &\stackrel{\text{def}}{=} (1, b),\end{aligned}$$

for each $a \in A$ and each $b \in B$.

[1] *Further Terminology:* Also called the **disjoint union of** A **and** B, or the **binary disjoint union of** A **and** B, for emphasis.

PROOF 2.2.3.2 ► PROOF OF DEFINITION 2.2.3.1

We claim that $A \coprod B$ is the categorical coproduct of A and B in Sets. Indeed, suppose we have a diagram of the form

$$\begin{tikzcd} & C & \\ A \arrow[ur, bend left, "\iota_A"] \arrow[r, "\mathrm{inj}_A"'] & A \coprod B & B \arrow[ul, bend right, "\iota_B"'] \arrow[l, "\mathrm{inj}_B"] \end{tikzcd}$$

in Sets. Then there exists a unique map $\phi\colon A \coprod B \to C$ making the diagram

$$\begin{tikzcd} & C & \\ A \arrow[ur, bend left, "\iota_A"] \arrow[r, "\mathrm{inj}_A"'] & A \coprod B \arrow[u, dashed, "\phi", "\exists!"'] & B \arrow[ul, bend right, "\iota_B"'] \arrow[l, "\mathrm{inj}_B"] \end{tikzcd}$$

commute, being uniquely determined by the conditions

$$\begin{aligned}\phi \circ \mathrm{inj}_A &= \iota_A,\\ \phi \circ \mathrm{inj}_B &= \iota_B\end{aligned}$$

via

$$\phi(x) = \begin{cases} \iota_A(a) & \text{if } x = (0, a),\\ \iota_B(b) & \text{if } x = (1, b)\end{cases}$$

for each $x \in A \coprod B$. □

PROPOSITION 2.2.3.3 ► PROPERTIES OF COPRODUCTS OF SETS

Let $A, B, C,$ and X be sets.

1. *Functoriality.* The assignment $A, B, (A, B) \mapsto A \coprod B$ defines functors

$$\begin{aligned} A \coprod - &\colon \mathsf{Sets} \to \mathsf{Sets}, \\ - \coprod B &\colon \mathsf{Sets} \to \mathsf{Sets}, \\ -_1 \coprod -_2 &\colon \mathsf{Sets} \times \mathsf{Sets} \to \mathsf{Sets}, \end{aligned}$$

   where $-_1 \coprod -_2$ is the functor where

   - *Action on Objects.* For each $(A, B) \in \mathrm{Obj}(\mathsf{Sets} \times \mathsf{Sets})$, we have

   $$[-_1 \coprod -_2](A, B) \stackrel{\text{def}}{=} A \coprod B.$$

   - *Action on Morphisms.* For each $(A, B), (X, Y) \in \mathrm{Obj}(\mathsf{Sets})$, the action on Hom-sets

   $$\coprod_{(A,B),(X,Y)} \colon \mathsf{Sets}(A, X) \times \mathsf{Sets}(B, Y) \to \mathsf{Sets}(A \coprod B, X \coprod Y)$$

   of $\coprod$ at $((A, B), (X, Y))$ is defined by sending (f, g) to the function

   $$f \coprod g \colon A \coprod B \to X \coprod Y$$

   defined by

   $$[f \coprod g](x) \stackrel{\text{def}}{=} \begin{cases} (0, f(a)) & \text{if } x = (0, a), \\ (1, g(b)) & \text{if } x = (1, b), \end{cases}$$

   for each $x \in A \coprod B$.

   and where $A \coprod -$ and $- \coprod B$ are the partial functors of $-_1 \coprod -_2$ at $A, B \in \mathrm{Obj}(\mathsf{Sets})$.

2. *Associativity.* We have an isomorphism of sets

$$(A \coprod B) \coprod C \cong A \coprod (B \coprod C),$$

   natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

3. *Unitality.* We have isomorphisms of sets

$$\begin{aligned} A \coprod \emptyset &\cong A, \\ \emptyset \coprod A &\cong A, \end{aligned}$$

   natural in $A \in \mathrm{Obj}(\mathsf{Sets})$.

4. *Commutativity.* We have an isomorphism of sets

$$A \coprod B \cong B \coprod A,$$

natural in $A, B \in \mathrm{Obj}(\mathsf{Sets})$.

5. *Symmetric Monoidality.* The triple $(\mathsf{Sets}, \coprod, \emptyset)$ is a symmetric monoidal category.

Proof 2.2.3.4 ▶ Proof of Proposition 2.2.3.3

Item 1: Functoriality

This follows from **??** of **??**.

Item 2: Associativity

Clear.

Item 3: Unitality

Clear.

Item 4: Commutativity

Clear.

Item 5: Symmetric Monoidality

Omitted. ∎

2.2.4 Pushouts

Let A, B, and C be sets and let $f: C \to A$ and $g: C \to B$ be functions.

Definition 2.2.4.1 ▶ Pushouts of Sets

The **pushout of A and B over C along f and g**[1] is the pair[2] $(A \coprod_C B, \{\mathrm{inj}_1, \mathrm{inj}_2\})$ consisting of:

- *The Colimit.* The set $A \coprod_C B$ defined by

$$A \coprod_C B \overset{\mathrm{def}}{=} A \coprod B/{\sim_C},$$

where $\sim_C$ is the equivalence relation on $A \coprod B$ generated by $(0, f(c)) \sim_C (1, g(c))$.

- *The Cocone.* The maps

$$\begin{aligned}\mathsf{inj}_1 &\colon A \to A \textstyle\coprod_C B,\\ \mathsf{inj}_2 &\colon B \to A \textstyle\coprod_C B\end{aligned}$$

given by

$$\begin{aligned}\mathsf{inj}_1(a) &\overset{\text{def}}{=} [(0,a)]\\ \mathsf{inj}_2(b) &\overset{\text{def}}{=} [(1,b)]\end{aligned}$$

for each $a \in A$ and each $b \in B$.

[1] *Further Terminology:* Also called the **fibre coproduct of** A **and** B **over** C **along** f **and** g.
[2] *Further Notation:* Also written $A \coprod_{f,C,g} B$.

Proof 2.2.4.2 ▶ Proof of Definition 2.2.4.1

We claim that $A \coprod_C B$ is the categorical pushout of A and B over C with respect to (f,g) in Sets. First we need to check that the relevant pushout diagram commutes, i.e. that we have

$$\mathsf{inj}_1 \circ f = \mathsf{inj}_2 \circ g, \qquad \begin{tikzcd} A \coprod_C B & B \arrow[l, "\mathsf{inj}_2"'] \\ A \arrow[u, "\mathsf{inj}_1"] & C. \arrow[l, "f"] \arrow[u, "g"'] \end{tikzcd}$$

Indeed, given $c \in C$, we have

$$\begin{aligned}[\mathsf{inj}_1 \circ f](c) &= \mathsf{inj}_1(f(c))\\ &= [(0,f(c))]\\ &= [(1,g(c))]\\ &= \mathsf{inj}_2(g(c))\\ &= [\mathsf{inj}_2 \circ g](c),\end{aligned}$$

where $[(0,f(c))] = [(1,g(c))]$ by the definition of the relation $\sim$ on $A \coprod B$. Next, we prove that $A \coprod_C B$ satisfies the universal property of the pushout.

Suppose we have a diagram of the form

$$\begin{tikzcd}
P & & \\
& A \coprod_C B \arrow[ul, phantom] \arrow[r, phantom] & B \arrow[l, "\mathrm{inj}_2"'] \arrow[ull, bend right, "\iota_2"'] \\
& A \arrow[u, "\mathrm{inj}_1"] \arrow[uul, bend left, "\iota_1"] \arrow[ur, phantom, "\ulcorner", very near end] & C \arrow[l, "f"] \arrow[u, "g"']
\end{tikzcd}$$

in Sets. Then there exists a unique map $\phi\colon A\coprod_C B \longrightarrow P$ making the diagram

$$\begin{tikzcd}
P & & \\
& A \coprod_C B \arrow[ul, dashed, "\exists!", "\phi"'] & B \arrow[l, "\mathrm{inj}_2"'] \arrow[ull, bend right, "\iota_2"'] \\
& A \arrow[u, "\mathrm{inj}_1"] \arrow[uul, bend left, "\iota_1"] \arrow[ur, phantom, "\ulcorner", very near end] & C \arrow[l, "f"] \arrow[u, "g"']
\end{tikzcd}$$

commute, being uniquely determined by the conditions

$$\phi \circ \mathrm{inj}_1 = \iota_1,$$
$$\phi \circ \mathrm{inj}_2 = \iota_2$$

via

$$\phi(x) = \begin{cases} \iota_1(a) & \text{if } x = [(0,a)], \\ \iota_2(b) & \text{if } x = [(1,b)] \end{cases}$$

for each $x \in A\coprod_C B$, where the well-definedness of ϕ is guaranteed by the equality $\iota_1 \circ f = \iota_2 \circ g$ and the definition of the relation $\sim$ on $A \coprod B$ as follows:

1. *Case 1:* Suppose we have $x = [(0,a)] = [(0,a')]$ for some $a, a' \in A$. Then, by Remark 2.2.4.3, we have a sequence
$$(0,a) \sim' x_1 \sim' \cdots \sim' x_n \sim' (0,a').$$
2. *Case 2:* Suppose we have $x = [(1,b)] = [(1,b')]$ for some $b, b' \in B$. Then, by Remark 2.2.4.3, we have a sequence
$$(1,b) \sim' x_1 \sim' \cdots \sim' x_n \sim' (1,b').$$

3. *Case 3:* Suppose we have $x = [(0, a)] = [(1, b)]$ for some $a \in A$ and $b \in B$. Then, by Remark 2.2.4.3, we have a sequence

$$(0, a) \sim' x_1 \sim' \cdots \sim' x_n \sim' (1, b).$$

In all these cases, we declare $x \sim' y$ iff there exists some $c \in C$ such that $x = (0, f(c))$ and $y = (1, g(c))$ or $x = (1, g(c))$ and $y = (0, f(c))$. Then, the equality $\iota_1 \circ f = \iota_2 \circ g$ gives

$$\begin{aligned} \phi([x]) &= \phi([(0, f(c))]) \\ &\overset{\text{def}}{=} \iota_1(f(c)) \\ &= \iota_2(g(c)) \\ &\overset{\text{def}}{=} \phi([(1, g(c))]) \\ &= \phi([y]), \end{aligned}$$

with the case where $x = (1, g(c))$ and $y = (0, f(c))$ similarly giving $\phi([x]) = \phi([y])$. Thus, if $x \sim' y$, then $\phi([x]) = \phi([y])$. Applying this equality pair-wise to the sequences

$$\begin{aligned} (0, a) \sim' x_1 \sim' \cdots \sim' x_n \sim' (0, a'), \\ (1, b) \sim' x_1 \sim' \cdots \sim' x_n \sim' (1, b'), \\ (0, a) \sim' x_1 \sim' \cdots \sim' x_n \sim' (1, b) \end{aligned}$$

gives

$$\begin{aligned} \phi([(0, a)]) &= \phi([(0, a')]), \\ \phi([(1, b)]) &= \phi([(1, b')]), \\ \phi([(0, a)]) &= \phi([(1, b)]), \end{aligned}$$

showing ϕ to be well-defined. □

Remark 2.2.4.3 ▶ Unwinding Definition 2.2.4.1

In detail, by Construction 7.4.2.2, the relation $\sim$ of Definition 2.2.4.1 is given by declaring $a \sim b$ iff one of the following conditions is satisfied:

- We have $a, b \in A$ and $a = b$;
- We have $a, b \in B$ and $a = b$;
- There exist $x_1, \ldots, x_n \in A \coprod B$ such that $a \sim' x_1 \sim' \cdots \sim' x_n \sim' b$,

where we declare $x \sim' y$ if one of the following conditions is satisfied:

1. There exists $c \in C$ such that $x = (0, f(c))$ and $y = (1, g(c))$.
2. There exists $c \in C$ such that $x = (1, g(c))$ and $y = (0, f(c))$.

That is: we require the following condition to be satisfied:

($\star$) There exist $x_1, \ldots, x_n \in A \coprod B$ satisfying the following conditions:

1. There exists $c_0 \in C$ satisfying one of the following conditions:
   (a) We have $a = f(c_0)$ and $x_1 = g(c_0)$.
   (b) We have $a = g(c_0)$ and $x_1 = f(c_0)$.
2. For each $1 \leq i \leq n-1$, there exists $c_i \in C$ satisfying one of the following conditions:
   (a) We have $x_i = f(c_i)$ and $x_{i+1} = g(c_i)$.
   (b) We have $x_i = g(c_i)$ and $x_{i+1} = f(c_i)$.
3. There exists $c_n \in C$ satisfying one of the following conditions:
   (a) We have $x_n = f(c_n)$ and $b = g(c_n)$.
   (b) We have $x_n = g(c_n)$ and $b = f(c_n)$.

Example 2.2.4.4 ▶ Examples of Pushouts of Sets

Here are some examples of pushouts of sets.

1. *Wedge Sums of Pointed Sets.* The wedge sum of two pointed sets of Definition 3.3.3.1 is an example of a pushout of sets.

2. *Intersections via Unions.* Let $A, B \subset X$. We have a bijection of sets

$$A \cup B \cong A \textstyle\coprod_{A \cap B} B, \qquad \begin{tikzcd} A \cup B \arrow[dr, phantom, "\ulcorner", very near start] & B \arrow[l, hook'] \\ A \arrow[u, hook] & A \cap B. \arrow[l, hook'] \arrow[u, hook] \end{tikzcd}$$

Proof 2.2.4.5 ► Proof of Example 2.2.4.4

Item 1: Wedge Sums of Pointed Sets

Follows by definition.

Item 2: Intersections via Unions

Indeed, $A \coprod_{A\cap B} B$ is the quotient of $A \coprod B$ by the equivalence relation obtained by declaring $(0, a) \sim (1, b)$ iff $a = b \in A \cap B$, which is in bijection with $A \cup B$ via the map with $[(0, a)] \mapsto a$ and $[(1, b)] \mapsto b$. ∎

Proposition 2.2.4.6 ► Properties of Pushouts of Sets

Let $A, B, C,$ and X be sets.

1. *Functoriality.* The assignment $(A, B, C, f, g) \mapsto A \coprod_{f,C,g} B$ defines a functor

$$-_1 \coprod_{-_3} -_1 \colon \mathsf{Fun}(\mathcal{P}, \mathsf{Sets}) \to \mathsf{Sets},$$

where $\mathcal{P}$ is the category that looks like this:

$$\begin{tikzcd}
 & \bullet \\
\bullet & \bullet. \arrow[l] \arrow[u]
\end{tikzcd}$$

In particular, the action on morphisms of $-_1 \coprod_{-_3} -_1$ is given by sending a morphism

$$\begin{tikzcd}
A \coprod_C B \arrow[dr, phantom, "\ulcorner", very near start] & & B \arrow[ll] \arrow[dr, "\psi"] & \\
& A' \coprod_{C'} B' \arrow[dr, phantom, "\ulcorner", very near start] & & B' \arrow[ll] \\
A \arrow[uu] \arrow[dr, "\phi"'] & & C \arrow[ll, "f"'] \arrow[uu, "g"] \arrow[dr, "\chi"] & \\
& A' \arrow[uu] & & C' \arrow[ll, "f'"] \arrow[uu, "g'"']
\end{tikzcd}$$

in $\mathsf{Fun}(\mathcal{P}, \mathsf{Sets})$ to the map $\xi \colon A \coprod_C B \xrightarrow{\exists!} A' \coprod_{C'} B'$ given by

$$\xi(x) \stackrel{\text{def}}{=} \begin{cases} \phi(a) & \text{if } x = [(0, a)], \\ \psi(b) & \text{if } x = [(1, b)] \end{cases}$$

for each $x \in A \coprod_C B$, which is the unique map making the diagram

$$\begin{tikzcd}
A \coprod_C B \arrow[dr, dashed] & & B \arrow[ll] \arrow[dr, "\psi"] & \\
& A' \coprod_{C'} B' & & B' \arrow[ll] \\
A \arrow[uu] \arrow[dr, "\phi"'] & & C \arrow[ll, "f"'] \arrow[uu, "g"] \arrow[dr, "\chi"] & \\
& A' \arrow[uu] & & C' \arrow[ll, "f'"] \arrow[uu, "g'"']
\end{tikzcd}$$

commute.

2. *Associativity.* Given a diagram

$$\begin{tikzcd}
A & & B & & C \\
& X \arrow[ul, "f"] \arrow[ur, "g"'] & & Y \arrow[ul, "h"] \arrow[ur, "k"'] &
\end{tikzcd}$$

in Sets, we have isomorphisms of sets

$$(A \coprod_X B) \coprod_Y C \cong (A \coprod_X B) \coprod_B (B \coprod_Y C) \cong A \coprod_X (B \coprod_Y C),$$

where these pullbacks are built as in the diagrams

$$\begin{tikzcd}[column sep=tiny, row sep=small]
& & (A \coprod_X B) \coprod_Y C & & \\
& A \coprod_X B \arrow[ur] & & & \\
A \arrow[ur] & & B \arrow[ul] & & C, \arrow[uull] \\
& X \arrow[ul, "f"] \arrow[ur, "g"'] & & Y \arrow[ul, "h"] \arrow[ur, "k"'] &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=tiny, row sep=small]
& & (A \coprod_X B) \coprod_B (B \coprod_Y C) & & \\
& A \coprod_X B \arrow[ur] & & B \coprod_Y C \arrow[ul] & \\
A \arrow[ur] & & B \arrow[ul] \arrow[ur] & & C, \arrow[ul] \\
& X \arrow[ul, "f"] \arrow[ur, "g"'] & & Y \arrow[ul, "h"] \arrow[ur, "k"'] &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=tiny, row sep=small]
& & A \coprod_X (B \coprod_Y C) & & \\
& & & B \coprod_Y C \arrow[ul] & \\
A \arrow[uurr] & & B \arrow[ur] & & C. \arrow[ul] \\
& X \arrow[ul, "f"] \arrow[ur, "g"'] & & Y \arrow[ul, "h"] \arrow[ur, "k"'] &
\end{tikzcd}$$

3. *Unitality.* We have isomorphisms of sets

$$\begin{tikzcd}
A \arrow[r, equal] & A \\
X \arrow[u, "f"] \arrow[r, equal] & X \arrow[u, "f"']
\end{tikzcd}
\qquad
\begin{aligned}
X \coprod_X A &\cong A, \\
A \coprod_X X &\cong A,
\end{aligned}
\qquad
\begin{tikzcd}
A & X \arrow[l, "f"'] \\
X \arrow[u, equal] & X. \arrow[l, "f"] \arrow[u, equal]
\end{tikzcd}$$

4. *Commutativity.* We have an isomorphism of sets

$$\begin{tikzcd} A \coprod_X B & B \arrow[l] \\ A \arrow[u] \arrow[ur, phantom, "\ulcorner", very near start] & X, \arrow[l, "f"] \arrow[u, "g"'] \end{tikzcd} \qquad A \coprod_X B \cong B \coprod_X A \qquad \begin{tikzcd} B \coprod_X A & A \arrow[l] \\ B \arrow[u] \arrow[ur, phantom, "\ulcorner", very near start] & X. \arrow[l, "g"] \arrow[u, "f"'] \end{tikzcd}$$

5. *Interaction With Coproducts.* We have

$$A \coprod_{\emptyset} B \cong A \coprod B, \qquad \begin{tikzcd} A \coprod B & B \arrow[l] \\ A \arrow[u] \arrow[ur, phantom, "\ulcorner", very near start] & \emptyset. \arrow[l, hook', "\iota_A"] \arrow[u, hook, "\iota_B"'] \end{tikzcd}$$

6. *Symmetric Monoidality.* The triple $(\mathsf{Sets}, \coprod_X, X)$ is a symmetric monoidal category.

Proof 2.2.4.7 ► Proof of Proposition 2.2.4.6

Item 1: Functoriality

This is a special case of functoriality of co/limits, **??** of **??**, with the explicit expression for ξ following from the commutativity of the cube pushout diagram.

Item 2: Associativity

Omitted.

Item 3: Unitality

Omitted.

Item 4: Commutativity

Clear.

Item 5: Interaction With Coproducts

Clear.

Item 6: Symmetric Monoidality

Omitted.

2.2.5 Coequalisers

Let A and B be sets and let $f, g\colon A \rightrightarrows B$ be functions.

Definition 2.2.5.1 ► Coequalisers of Sets

The **coequaliser of f and g** is the pair $(\mathrm{CoEq}(f,g), \mathrm{coeq}(f,g))$ consisting of:

- *The Colimit.* The set $\mathrm{CoEq}(f,g)$ defined by
$$\mathrm{CoEq}(f,g) \stackrel{\text{def}}{=} B/{\sim},$$
where $\sim$ is the equivalence relation on B generated by $f(a) \sim g(a)$.
- *The Cocone.* The map
$$\mathrm{coeq}(f,g)\colon B \longrightarrow \mathrm{CoEq}(f,g)$$
given by the quotient map $\pi\colon B \twoheadrightarrow B/{\sim}$ with respect to the equivalence relation generated by $f(a) \sim g(a)$.

Proof 2.2.5.2 ► Proof of Definition 2.2.5.1

We claim that $\mathrm{CoEq}(f,g)$ is the categorical coequaliser of f and g in Sets. First we need to check that the relevant coequaliser diagram commutes, i.e. that we have
$$\mathrm{coeq}(f,g) \circ f = \mathrm{coeq}(f,g) \circ g.$$
Indeed, we have
$$\begin{aligned}
[\mathrm{coeq}(f,g) \circ f](a) &\stackrel{\text{def}}{=} [\mathrm{coeq}(f,g)](f(a))\\
&\stackrel{\text{def}}{=} [f(a)]\\
&= [g(a)]\\
&\stackrel{\text{def}}{=} [\mathrm{coeq}(f,g)](g(a))\\
&\stackrel{\text{def}}{=} [\mathrm{coeq}(f,g) \circ g](a)
\end{aligned}$$
for each $a \in A$. Next, we prove that $\mathrm{CoEq}(f,g)$ satisfies the universal property of the coequaliser. Suppose we have a diagram of the form
$$\begin{tikzcd}
A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "\mathrm{coeq}(f,g)"] \arrow[drr, "c"'] & \mathrm{CoEq}(f,g) & \\
 & & & C
\end{tikzcd}$$
in Sets. Then, since $c(f(a)) = c(g(a))$ for each $a \in A$, it follows from Items 4 and 5 of Proposition 7.5.2.3 that there exists a unique map $\mathrm{CoEq}(f,g) \xrightarrow{\exists!} C$

making the diagram

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, hook, "\mathrm{coeq}(f,g)"] \arrow[dr, "c"'] & \mathrm{CoEq}(f,g) \arrow[d, dashed, "\exists!"] \\ & & C \end{tikzcd}$$

commute.

Remark 2.2.5.3 ▶ Unwinding Definition 2.2.5.1

In detail, by Construction 7.4.2.2, the relation $\sim$ of Definition 2.2.5.1 is given by declaring $a \sim b$ iff one of the following conditions is satisfied:

- We have $a = b$;
- There exist $x_1, \ldots, x_n \in B$ such that $a \sim' x_1 \sim' \cdots \sim' x_n \sim' b$, where we declare $x \sim' y$ if one of the following conditions is satisfied:

  1. There exists $z \in A$ such that $x = f(z)$ and $y = g(z)$.
  2. There exists $z \in A$ such that $x = g(z)$ and $y = f(z)$.

  That is: we require the following condition to be satisfied:

  ($\star$) There exist $x_1, \ldots, x_n \in B$ satisfying the following conditions:
  1. There exists $z_0 \in A$ satisfying one of the following conditions:
     (a) We have $a = f(z_0)$ and $x_1 = g(z_0)$.
     (b) We have $a = g(z_0)$ and $x_1 = f(z_0)$.
  2. For each $1 \leq i \leq n-1$, there exists $z_i \in A$ satisfying one of the following conditions:
     (a) We have $x_i = f(z_i)$ and $x_{i+1} = g(z_i)$.
     (b) We have $x_i = g(z_i)$ and $x_{i+1} = f(z_i)$.
  3. There exists $z_n \in A$ satisfying one of the following conditions:
     (a) We have $x_n = f(z_n)$ and $b = g(z_n)$.
     (b) We have $x_n = g(z_n)$ and $b = f(z_n)$.

Example 2.2.5.4 ► Examples of Coequalisers of Sets

Here are some examples of coequalisers of sets.

1. *Quotients by Equivalence Relations.* Let R be an equivalence relation on a set X. We have a bijection of sets

$$X/{\sim_R} \cong \mathsf{CoEq}\left(R \hookrightarrow X \times X \underset{\mathrm{pr}_2}{\overset{\mathrm{pr}_1}{\rightrightarrows}} X\right).$$

Proof 2.2.5.5 ► Proof of Example 2.2.5.4

Item 1: Quotients by Equivalence Relations

See [Pro24ad]. ▭

Proposition 2.2.5.6 ► Properties of Coequalisers of Sets

Let A, B, and C be sets.

1. *Associativity.* We have isomorphisms of sets[1]

$$\underbrace{\mathsf{CoEq}(\mathsf{coeq}(f,g)\circ f,\mathsf{coeq}(f,g)\circ h)}_{=\mathsf{CoEq}(\mathsf{coeq}(f,g)\circ g,\mathsf{coeq}(f,g)\circ h)} \cong \mathsf{CoEq}(f,g,h) \cong \underbrace{\mathsf{CoEq}(\mathsf{coeq}(g,h)\circ f,\mathsf{coeq}(g,h)\circ g)}_{=\mathsf{CoEq}(\mathsf{coeq}(g,h)\circ f,\mathsf{coeq}(g,h)\circ h)},$$

   where $\mathsf{CoEq}(f,g,h)$ is the colimit of the diagram

$$\begin{tikzcd} A \arrow[r, shift left=2, "f"] \arrow[r, "g" description] \arrow[r, shift right=2, "h"'] & B \end{tikzcd}$$

   in Sets.

2. *Unitality.* We have an isomorphism of sets

$$\mathsf{CoEq}(f,f) \cong B.$$

3. *Commutativity.* We have an isomorphism of sets

$$\mathsf{CoEq}(f,g) \cong \mathsf{CoEq}(g,f).$$

4. *Interaction With Composition.* Let

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \underset{k}{\overset{h}{\rightrightarrows}} C$$

be functions. We have a surjection

$$\mathrm{CoEq}(h \circ f, k \circ g) \twoheadrightarrow \mathrm{CoEq}(\mathrm{coeq}(h, k) \circ h \circ f, \mathrm{coeq}(h, k) \circ k \circ g)$$

exhibiting $\mathrm{CoEq}(\mathrm{coeq}(h, k) \circ h \circ f, \mathrm{coeq}(h, k) \circ k \circ g)$ as a quotient of $\mathrm{CoEq}(h \circ f, k \circ g)$ by the relation generated by declaring $h(y) \sim k(y)$ for each $y \in B$.

[1] That is, the following three ways of forming "the" coequaliser of (f, g, h) agree:

(a) Take the coequaliser of (f, g, h), i.e. the colimit of the diagram

$$\begin{tikzcd} A \arrow[r, shift left=2, "f"] \arrow[r, "g" description] \arrow[r, shift right=2, "h"'] & B \end{tikzcd}$$

in Sets.

(b) First take the coequaliser of f and g, forming a diagram

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \overset{\mathrm{coeq}(f,g)}{\twoheadrightarrow} \mathrm{CoEq}(f, g)$$

and then take the coequaliser of the composition

$$A \underset{h}{\overset{f}{\rightrightarrows}} B \overset{\mathrm{coeq}(f,g)}{\twoheadrightarrow} \mathrm{CoEq}(f, g),$$

obtaining a quotient

$$\mathrm{CoEq}(\mathrm{coeq}(f, g) \circ f, \mathrm{coeq}(f, g) \circ h) = \mathrm{CoEq}(\mathrm{coeq}(f, g) \circ g, \mathrm{coeq}(f, g) \circ h)$$

of $\mathrm{CoEq}(f, g)$

(c) First take the coequaliser of g and h, forming a diagram

$$A \underset{h}{\overset{g}{\rightrightarrows}} B \overset{\mathrm{coeq}(g,h)}{\twoheadrightarrow} \mathrm{CoEq}(g, h)$$

and then take the coequaliser of the composition

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \overset{\mathrm{coeq}(g,h)}{\twoheadrightarrow} \mathrm{CoEq}(g, h),$$

obtaining a quotient

$$\mathrm{CoEq}(\mathrm{coeq}(g, h) \circ f, \mathrm{coeq}(g, h) \circ g) = \mathrm{CoEq}(\mathrm{coeq}(g, h) \circ f, \mathrm{coeq}(g, h) \circ h)$$

of $\mathrm{CoEq}(g, h)$.

Proof 2.2.5.7 ► Proof of Proposition 2.2.5.6

Item 1: Associativity

Omitted.

Item 2: Unitality

Clear.

Item 3: Commutativity

Clear.

Item 4: Interaction With Composition

Omitted.

2.3 Operations With Sets

2.3.1 The Empty Set

Definition 2.3.1.1 ► The Empty Set

The **empty set** is the set $\emptyset$ defined by

$$\emptyset \overset{\text{def}}{=} \{x \in X \mid x \neq x\},$$

where A is the set in the set existence axiom, **??** of **??**.

2.3.2 Singleton Sets

Let X be a set.

Definition 2.3.2.1 ► Singleton Sets

The **singleton set containing** X is the set $\{X\}$ defined by

$$\{X\} \overset{\text{def}}{=} \{X, X\},$$

where $\{X, X\}$ is the pairing of X with itself (Definition 2.3.3.1).

2.3.3 Pairings of Sets

Let X and Y be sets.

Definition 2.3.3.1 ► Pairings of Sets

The **pairing of** X **and** Y is the set $\{X, Y\}$ defined by

$$\{X, Y\} \overset{\text{def}}{=} \{x \in A \mid x = X \text{ or } x = Y\},$$

where A is the set in the axiom of pairing, **??** of **??**.

2.3.4 Ordered Pairs

Let A and B be sets.

Definition 2.3.4.1 ► Ordered Pairs

The **ordered pair associated to** A **and** B is the set (A, B) defined by

$$(A, B) \overset{\text{def}}{=} \{\{A\}, \{A, B\}\}.$$

Proposition 2.3.4.2 ► Properties of Ordered Pairs

Let A and B be sets.

1. *Uniqueness.* Let A, B, C, and D be sets. The following conditions are equivalent:

   (a) We have $(A, B) = (C, D)$.

   (b) We have $A = C$ and $B = D$.

Proof 2.3.4.3 ► Proof of Proposition 2.3.4.2

Item 1: Uniqueness

See [Cie97, Theorem 1.2.3].

2.3.5 Sets of Maps

Let A and B be sets.

Definition 2.3.5.1 ► Sets of Maps

The **set of maps from** A **to** B[1] is the set $\mathrm{Hom}_{\mathsf{Sets}}(A, B)$[2] whose elements are the functions from A to B.

[1] *Further Terminology:* Also called the **Hom set from** A **to** B.
[2] *Further Notation:* Also written $\mathsf{Sets}(A, B)$.

Proposition 2.3.5.2 ▶ Properties of Sets of Maps

Let A and B be sets.

1. *Functoriality.* The assignments $X, Y, (X, Y) \mapsto \mathrm{Hom}_{\mathsf{Sets}}(X, Y)$ define functors

$$\begin{aligned} \mathrm{Hom}_{\mathsf{Sets}}(X, -) &\colon \mathsf{Sets} \to \mathsf{Sets}, \\ \mathrm{Hom}_{\mathsf{Sets}}(-, Y) &\colon \mathsf{Sets}^{\mathsf{op}} \to \mathsf{Sets}, \\ \mathrm{Hom}_{\mathsf{Sets}}(-_1, -_2) &\colon \mathsf{Sets}^{\mathsf{op}} \times \mathsf{Sets} \to \mathsf{Sets}. \end{aligned}$$

Proof 2.3.5.3 ▶ Proof of Proposition 2.3.5.2

Item 1: Functoriality

This follows from Items 2 and 5 of Proposition 8.1.6.2. ∎

2.3.6 Unions of Families

Let $\{A_i\}_{i\in I}$ be a family of sets.

Definition 2.3.6.1 ▶ Unions of Families

The **union of the family** $\{A_i\}_{i\in I}$ is the set $\bigcup_{i\in I} A_i$ defined by

$$\bigcup_{i\in I} A_i \overset{\text{def}}{=} \{x \in F \mid \text{there exists some } i \in I \text{ such that } x \in A_i\},$$

where F is the set in the axiom of union, **??** of **??**.

2.3.7 Binary Unions

Let A and B be sets.

Definition 2.3.7.1 ▶ Binary Unions

The **union**[1] **of A and B** is the set $A \cup B$ defined by

$$A \cup B \overset{\text{def}}{=} \bigcup_{z\in\{A,B\}} z.$$

[1] *Further Terminology:* Also called the **binary union of A and B**, for emphasis.

Proposition 2.3.7.2 ▶ Properties of Binary Unions

Let X be a set.

1. *Functoriality.* The assignments $U, V, (U, V) \mapsto U \cup V$ define functors

$$\begin{aligned} U \cup -\colon (\mathcal{P}(X), \subset) &\to (\mathcal{P}(X), \subset), \\ - \cup V\colon (\mathcal{P}(X), \subset) &\to (\mathcal{P}(X), \subset), \\ -_1 \cup -_2\colon (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \subset) &\to (\mathcal{P}(X), \subset), \end{aligned}$$

   where $-_1 \cup -_2$ is the functor where

   - *Action on Objects.* For each $(U, V) \in \mathcal{P}(X) \times \mathcal{P}(X)$, we have

   $$[-_1 \cup -_2](U, V) \overset{\text{def}}{=} U \cup V.$$

   - *Action on Morphisms.* For each pair of morphisms

   $$\begin{aligned} \iota_U &: U \hookrightarrow U', \\ \iota_V &: V \hookrightarrow V' \end{aligned}$$

     of $\mathcal{P}(X) \times \mathcal{P}(X)$, the image

   $$\iota_U \cup \iota_V : U \cup V \hookrightarrow U' \cup V'$$

     of (ι_U, ι_V) by $\cup$ is the inclusion

   $$U \cup V \subset U' \cup V'$$

     i.e. where we have

     $(\star)$ If $U \subset U'$ and $V \subset V'$, then $U \cup V \subset U' \cup V'$.

   and where $U \cup -$ and $- \cup V$ are the partial functors of $-_1 \cup -_2$ at $U, V \in \mathcal{P}(X)$.

2. *Via Intersections and Symmetric Differences.* We have an equality of sets

$$U \cup V = (U \vartriangle V) \vartriangle (U \cap V)$$

   for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

3. *Associativity.* We have an equality of sets

$$(U \cup V) \cup W = U \cup (V \cup W)$$

   for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

4. *Unitality.* We have equalities of sets

$$U \cup \emptyset = U,$$
$$\emptyset \cup U = U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

5. *Commutativity.* We have an equality of sets

$$U \cup V = V \cup U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

6. *Idempotency.* We have an equality of sets

$$U \cup U = U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

7. *Distributivity Over Intersections.* We have equalities of sets

$$U \cup (V \cap W) = (U \cup V) \cap (U \cup W),$$
$$(U \cap V) \cup W = (U \cup W) \cap (V \cup W)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

8. *Interaction With Characteristic Functions I.* We have

$$\chi_{U \cup V} = \max(\chi_U, \chi_V)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

9. *Interaction With Characteristic Functions II.* We have

$$\chi_{U \cup V} = \chi_U + \chi_V - \chi_{U \cap V}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

10. *Interaction With Powersets and Semirings.* The quintuple $(\mathcal{P}(X), \cup, \cap, \emptyset, X)$ is an idempotent commutative semiring.

Proof 2.3.7.3 ► Proof of Proposition 2.3.7.2

Item 1: Functoriality

See [Pro24ar].

Item 2: Via Intersections and Symmetric Differences

See [Pro24bc].

Item 3: Associativity

See [Pro24be].

Item 4: Unitality

This follows from [Pro24bh] and Item 5.

Item 5: Commutativity

See [Pro24bf].

Item 6: Idempotency

See [Pro24aq].

Item 7: Distributivity Over Intersections

See [Pro24bd].

Item 8: Interaction With Characteristic Functions I

See [Pro24k].

Item 9: Interaction With Characteristic Functions II

See [Pro24k].

Item 10: Interaction With Powersets and Semirings

This follows from Items 3 to 6 and Items 3 to 5, 7 and 8 of Proposition 2.3.9.2. □

2.3.8 Intersections of Families

Let $\mathcal{F}$ be a family of sets.

Definition 2.3.8.1 ► Intersections of Families

The **intersection of a family $\mathcal{F}$ of sets** is the set $\bigcap_{X\in\mathcal{F}} X$ defined by

$$\bigcap_{X\in\mathcal{F}} X \overset{\text{def}}{=} \left\{ z \in \bigcup_{X\in\mathcal{F}} X \;\middle|\; \text{for each } X \in \mathcal{F}\text{, we have } z \in X \right\}.$$

2.3.9 Binary Intersections

Let X and Y be sets.

Definition 2.3.9.1 ▶ Binary Intersections

The **intersection**[1] **of** X **and** Y is the set $X \cap Y$ defined by

$$X \cap Y \overset{\text{def}}{=} \bigcap_{z \in \{X,Y\}} z.$$

[1] *Further Terminology:* Also called the **binary intersection of** X **and** Y, for emphasis.

Proposition 2.3.9.2 ▶ Properties of Binary Intersections

Let X be a set.

1. *Functoriality.* The assignments $U, V, (U, V) \mapsto U \cap V$ define functors

$$\begin{aligned} U \cap -\colon (\mathcal{P}(X), \subset) &\to (\mathcal{P}(X), \subset), \\ - \cap V\colon (\mathcal{P}(X), \subset) &\to (\mathcal{P}(X), \subset), \\ -_1 \cap -_2\colon (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \subset) &\to (\mathcal{P}(X), \subset), \end{aligned}$$

where $-_1 \cap -_2$ is the functor where

- *Action on Objects.* For each $(U, V) \in \mathcal{P}(X) \times \mathcal{P}(X)$, we have

$$[-_1 \cap -_2](U, V) \overset{\text{def}}{=} U \cap V.$$

- *Action on Morphisms.* For each pair of morphisms

$$\begin{aligned} \iota_U &\colon U \hookrightarrow U', \\ \iota_V &\colon V \hookrightarrow V' \end{aligned}$$

of $\mathcal{P}(X) \times \mathcal{P}(X)$, the image

$$\iota_U \cap \iota_V \colon U \cap V \hookrightarrow U' \cap V'$$

of (ι_U, ι_V) by $\cap$ is the inclusion

$$U \cap V \subset U' \cap V'$$

i.e. where we have

($\star$) If $U \subset U'$ and $V \subset V'$, then $U \cap V \subset U' \cap V'$.

and where $U \cap -$ and $- \cap V$ are the partial functors of $-_1 \cap -_2$ at $U, V \in \mathcal{P}(X)$.

2. *Adjointness.* We have adjunctions

$$\left(U \cap - \dashv \mathbf{Hom}_{\mathcal{P}(X)}(U,-)\right)\colon \quad \begin{tikzcd} \mathcal{P}(X) \arrow[r, bend left, "U\cap-"] \arrow[r, phantom, "\perp"] & \mathcal{P}(X), \arrow[l, bend left, "\mathbf{Hom}_{\mathcal{P}(X)}(U,-)"] \end{tikzcd}$$

$$\left(- \cap V \dashv \mathbf{Hom}_{\mathcal{P}(X)}(V,-)\right)\colon \quad \begin{tikzcd} \mathcal{P}(X) \arrow[r, bend left, "-\cap V"] \arrow[r, phantom, "\perp"] & \mathcal{P}(X), \arrow[l, bend left, "\mathbf{Hom}_{\mathcal{P}(X)}(V,-)"] \end{tikzcd}$$

where

$$\mathbf{Hom}_{\mathcal{P}(X)}(-_1,-_2)\colon \mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \to \mathcal{P}(X)$$

is the bifunctor defined by[1]

$$\mathbf{Hom}_{\mathcal{P}(X)}(U,V) \overset{\text{def}}{=} (X \setminus U) \cup V$$

witnessed by bijections

$$\begin{aligned} \mathrm{Hom}_{\mathcal{P}(X)}(U \cap V, W) &\cong \mathrm{Hom}_{\mathcal{P}(X)}\left(U, \mathbf{Hom}_{\mathcal{P}(X)}(V,W)\right), \\ \mathrm{Hom}_{\mathcal{P}(X)}(U \cap V, W) &\cong \mathrm{Hom}_{\mathcal{P}(X)}\left(V, \mathbf{Hom}_{\mathcal{P}(X)}(U,W)\right), \end{aligned}$$

natural in $U, V, W \in \mathcal{P}(X)$, i.e. where:

(a) The following conditions are equivalent:
   i. We have $U \cap V \subset W$.
   ii. We have $U \subset \mathbf{Hom}_{\mathcal{P}(X)}(V,W)$.
   iii. We have $U \subset (X \setminus V) \cup W$.

(b) The following conditions are equivalent:
   i. We have $V \cap U \subset W$.
   ii. We have $V \subset \mathbf{Hom}_{\mathcal{P}(X)}(U,W)$.
   iii. We have $V \subset (X \setminus U) \cup W$.

3. *Associativity.* We have an equality of sets

$$(U \cap V) \cap W = U \cap (V \cap W)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

4. *Unitality.* Let X be a set and let $U \in \mathcal{P}(X)$. We have equalities of sets

$$\begin{aligned} X \cap U &= U, \\ U \cap X &= U \end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

5. *Commutativity.* We have an equality of sets
$$U \cap V = V \cap U$$
for each $X \in \text{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

6. *Idempotency.* We have an equality of sets
$$U \cap U = U$$
for each $X \in \text{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

7. *Distributivity Over Unions.* We have equalities of sets
$$\begin{aligned} U \cap (V \cup W) &= (U \cap V) \cup (U \cap W), \\ (U \cup V) \cap W &= (U \cap W) \cup (V \cap W) \end{aligned}$$
for each $X \in \text{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

8. *Annihilation With the Empty Set.* We have an equality of sets
$$\begin{aligned} \emptyset \cap X &= \emptyset, \\ X \cap \emptyset &= \emptyset \end{aligned}$$
for each $X \in \text{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

9. *Interaction With Characteristic Functions I.* We have
$$\chi_{U \cap V} = \chi_U \chi_V$$
for each $X \in \text{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

10. *Interaction With Characteristic Functions II.* We have
$$\chi_{U \cap V} = \min(\chi_U, \chi_V)$$
for each $X \in \text{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

11. *Interaction With Powersets and Monoids With Zero.* The quadruple $((\mathcal{P}(X), \emptyset), \cap, X)$ is a commutative monoid with zero.

12. *Interaction With Powersets and Semirings.* The quintuple $(\mathcal{P}(X), \cup, \cap, \emptyset, X)$ is an idempotent commutative semiring.

[1] For intuition regarding the expression defining $\mathbf{Hom}_{\mathcal{P}(X)}(U, V)$, see Remark 2.3.9.4.

Proof 2.3.9.3 ► Proof of Proposition 2.3.9.2

Item 1: Functoriality

See [Pro24ap].

Item 2: Adjointness

See [MSE 267469].

Item 3: Associativity

See [Pro24v].

Item 4: Unitality

This follows from [Pro24z] and Item 5.

Item 5: Commutativity

See [Pro24w].

Item 6: Idempotency

See [Pro24ao].

Item 7: Distributivity Over Unions

See [Pro24an].

Item 8: Annihilation With the Empty Set

This follows from [Pro24x] and Item 5.

Item 9: Interaction With Characteristic Functions I

See [Pro24h].

Item 10: Interaction With Characteristic Functions II

See [Pro24h].

Item 11: Interaction With Powersets and Monoids With Zero

This follows from Items 3 to 5 and 8.

Item 12: Interaction With Powersets and Semirings

This follows from Items 3 to 6 and Items 3 to 5, 7 and 8 of Proposition 2.3.9.2. □

Remark 2.3.9.4 ► Intuition for the Internal Hom of $\mathcal{P}(X)$

Since intersections are the products in $\mathcal{P}(X)$ (Item 1 of Proposition 2.4.3.3), the left adjoint $\mathbf{Hom}_{\mathcal{P}(X)}(U,V)$ may be thought of as a function type $[U,V]$. Then, under the Curry–Howard correspondence, the function type $[U,V]$ corresponds to implication $U \implies V$, which is logically equivalent to the statement $\neg U \vee V$. This in turn corresponds to the set $U^{\mathsf{c}} \vee V = (X \setminus U) \cup V$.

2.3.10 Differences

Let X and Y be sets.

Definition 2.3.10.1 ► Differences

The **difference of** X **and** Y is the set $X \setminus Y$ defined by

$$X \setminus Y \overset{\text{def}}{=} \{a \in X \mid a \notin Y\}.$$

Proposition 2.3.10.2 ► Properties of Differences

Let X be a set.

1. *Functoriality.* The assignments $U, V, (U, V) \mapsto U \cap V$ define functors

$$\begin{aligned} U \setminus -\colon (\mathcal{P}(X), \supset) &\to (\mathcal{P}(X), \subset), \\ - \setminus V\colon (\mathcal{P}(X), \subset) &\to (\mathcal{P}(X), \subset), \\ -_1 \setminus -_2\colon (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \supset) &\to (\mathcal{P}(X), \subset), \end{aligned}$$

   where $-_1 \setminus -_2$ is the functor where

   - *Action on Objects.* For each $(U, V) \in \mathcal{P}(X) \times \mathcal{P}(X)$, we have

$$[-_1 \setminus -_2](U, V) \overset{\text{def}}{=} U \setminus V.$$

   - *Action on Morphisms.* For each pair of morphisms

$$\begin{aligned} \iota_A &\colon A \hookrightarrow B, \\ \iota_U &\colon U \hookrightarrow V \end{aligned}$$

     of $\mathcal{P}(X) \times \mathcal{P}(X)$, the image

$$\iota_U \setminus \iota_V \colon A \setminus V \hookrightarrow B \setminus U$$

     of (ι_U, ι_V) by $\setminus$ is the inclusion

$$A \setminus V \subset B \setminus U$$

     i.e. where we have

     ($\star$) If $A \subset B$ and $U \subset V$, then $A \setminus V \subset B \setminus U$.

   and where $U \setminus -$ and $- \setminus V$ are the partial functors of $-_1 \setminus -_2$ at $U, V \in \mathcal{P}(X)$.

2. *De Morgan's Laws.* We have equalities of sets

$$\begin{aligned} X \setminus (U \cup V) &= (X \setminus U) \cap (X \setminus V), \\ X \setminus (U \cap V) &= (X \setminus U) \cup (X \setminus V) \end{aligned}$$

   for each $X \in \text{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

3. *Interaction With Unions I.* We have equalities of sets

$$U \setminus (V \cup W) = (U \setminus V) \cap (U \setminus W)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

4. *Interaction With Unions II.* We have equalities of sets

$$(U \setminus V) \cup W = (U \cup W) \setminus (V \setminus W)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

5. *Interaction With Unions III.* We have equalities of sets

$$\begin{aligned} U \setminus (V \cup W) &= (U \cup W) \setminus (V \cup W) \\ &= (U \setminus V) \setminus W \\ &= (U \setminus W) \setminus V \end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

6. *Interaction With Unions IV.* We have equalities of sets

$$(U \cup V) \setminus W = (U \setminus W) \cup (V \setminus W)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

7. *Interaction With Intersections.* We have equalities of sets

$$\begin{aligned} (U \setminus V) \cap W &= (U \cap W) \setminus V \\ &= U \cap (W \setminus V) \end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

8. *Interaction With Complements.* We have an equality of sets

$$U \setminus V = U \cap V^{\mathsf{c}}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

9. *Interaction With Symmetric Differences.* We have an equality of sets

$$U \setminus V = U \mathbin{\triangle} (U \cap V)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

10. *Triple Differences.* We have

$$U \setminus (V \setminus W) = (U \cap W) \cup (U \setminus V)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

11. *Left Annihilation.* We have

$$\emptyset \setminus U = \emptyset$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

12. *Right Unitality.* We have

$$U \setminus \emptyset = U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

13. *Invertibility.* We have

$$U \setminus U = \emptyset$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

14. *Interaction With Containment.* The following conditions are equivalent:

    (a) We have $V \setminus U \subset W$.

    (b) We have $V \setminus W \subset U$.

15. *Interaction With Characteristic Functions.* We have

$$\chi_{U \setminus V} = \chi_U - \chi_{U \cap V}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

PROOF 2.3.10.3 ► PROOF OF PROPOSITION 2.3.10.2

Item 1: Functoriality

See [Pro24ah] and [Pro24al].

Item 2: De Morgan's Laws

See [Pro24p].

Item 3: Interaction With Unions I

See [Pro24q].

Item 4: Interaction With Unions II

Omitted.

Item 5: Interaction With Unions III

See [Pro24am].

Item 6: Interaction With Unions IV

See [Pro24ag].

Item 7: Interaction With Intersections

See [Pro24y].

Item 8: Interaction With Complements

See [Pro24ae].

Item 9: Interaction With Symmetric Differences

See [Pro24af].

Item 10: Triple Differences

See [Pro24ak].

Item 11: Left Annihilation

Clear.

Item 12: Right Unitality

See [Pro24ai].

Item 13: Invertibility

See [Pro24aj].

Item 14: Interaction With Containment

Omitted.

Item 15: Interaction With Characteristic Functions

See [Pro24i].

2.3.11 Complements

Let X be a set and let $U \in \mathcal{P}(X)$.

Definition 2.3.11.1 ▶ Complements

The **complement of** U is the set U^{c} defined by

$$\begin{aligned} U^{\mathsf{c}} &\overset{\text{def}}{=} X \setminus U \\ &\overset{\text{def}}{=} \{a \in X \mid a \notin U\}. \end{aligned}$$

Proposition 2.3.11.2 ▶ Properties of Complements

Let X be a set.

1. *Functoriality.* The assignment $U \mapsto U^{\mathsf{c}}$ defines a functor

   $$(-)^{\mathsf{c}} \colon \mathcal{P}(X)^{\mathsf{op}} \to \mathcal{P}(X),$$

   where

   - *Action on Objects.* For each $U \in \mathcal{P}(X)$, we have

     $$\left[(-)^{\mathsf{c}}\right](U) \overset{\text{def}}{=} U^{\mathsf{c}}.$$

   - *Action on Morphisms.* For each morphism $\iota_U \colon U \hookrightarrow V$ of $\mathcal{P}(X)$, the image

     $$\iota_U^{\mathsf{c}} \colon V^{\mathsf{c}} \hookrightarrow U^{\mathsf{c}}$$

     of ι_U by $(-)^{\mathsf{c}}$ is the inclusion

     $$V^{\mathsf{c}} \subset U^{\mathsf{c}}$$

     i.e. where we have

     $(\star)$ If $U \subset V$, then $V^{\mathsf{c}} \subset U^{\mathsf{c}}$.

2. *De Morgan's Laws.* We have equalities of sets

   $$\begin{aligned} (U \cup V)^{\mathsf{c}} &= U^{\mathsf{c}} \cap V^{\mathsf{c}}, \\ (U \cap V)^{\mathsf{c}} &= U^{\mathsf{c}} \cup V^{\mathsf{c}} \end{aligned}$$

   for each $X \in \mathsf{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

3. *Involutority.* We have

   $$(U^{\mathsf{c}})^{\mathsf{c}} = U$$

   for each $X \in \mathsf{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

4. *Interaction With Characteristic Functions.* We have

   $$\chi_{U^{\mathsf{c}}} = 1 - \chi_U$$

   for each $X \in \mathsf{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

Proof 2.3.11.3 ▶ Proof of Proposition 2.3.11.2

Item 1: Functoriality

This follows from Item 1 of Proposition 2.3.10.2.

Item 2: De Morgan's Laws

See [Pro24p].

Item 3: Involutority

See [Pro24l].

Item 4: Interaction With Characteristic Functions

Clear. □

2.3.12 Symmetric Differences

Let A and B be sets.

Definition 2.3.12.1 ▶ Symmetric Differences

The **symmetric difference of** A **and** B is the set $A \triangle B$ defined by

$$A \triangle B \overset{\text{def}}{=} (A \setminus B) \cup (B \setminus A).$$

Proposition 2.3.12.2 ▶ Properties of Symmetric Differences

Let X be a set.

1. *Lack of Functoriality.* The assignment $(U, V) \mapsto U \triangle V$ ***need not*** define functors

$$\begin{aligned} U \triangle -&: (\mathcal{P}(X), \subset) \to (\mathcal{P}(X), \subset), \\ - \triangle V&: (\mathcal{P}(X), \subset) \to (\mathcal{P}(X), \subset), \\ -_1 \triangle -_2&: (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \subset) \to (\mathcal{P}(X), \subset). \end{aligned}$$

2. *Via Unions and Intersections.* We have[1]

$$U \triangle V = (U \cup V) \setminus (U \cap V)$$

for each $X \in \mathsf{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

3. *Associativity.* We have[2]

$$(U \triangle V) \triangle W = U \triangle (V \triangle W)$$

for each $X \in \mathsf{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

4. *Commutativity.* We have

$$U \triangle V = V \triangle U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

5. *Unitality.* We have

$$\begin{aligned} U \triangle \emptyset &= U, \\ \emptyset \triangle U &= U \end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

6. *Invertibility.* We have

$$U \triangle U = \emptyset$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

7. *Interaction With Unions.* We have

$$(U \triangle V) \cup (V \triangle T) = (U \cup V \cup W) \setminus (U \cap V \cap W)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

8. *Interaction With Complements I.* We have

$$U \triangle U^{\mathsf{c}} = X$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

9. *Interaction With Complements II.* We have

$$\begin{aligned} U \triangle X &= U^{\mathsf{c}}, \\ X \triangle U &= U^{\mathsf{c}} \end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

10. *Interaction With Complements III.* We have

$$U^{\mathsf{c}} \triangle V^{\mathsf{c}} = U \triangle V$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

11. *“Transitivity”.* We have

$$(U \triangle V) \triangle (V \triangle W) = U \triangle W$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

12. *The Triangle Inequality for Symmetric Differences.* We have

$$U \mathbin{\triangle} W \subset U \mathbin{\triangle} V \cup V \mathbin{\triangle} W$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

13. *Distributivity Over Intersections.* We have

$$\begin{aligned} U \cap (V \mathbin{\triangle} W) &= (U \cap V) \mathbin{\triangle} (U \cap W), \\ (U \mathbin{\triangle} V) \cap W &= (U \cap W) \mathbin{\triangle} (V \cap W) \end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

14. *Interaction With Characteristic Functions.* We have

$$\chi_{U \triangle V} = \chi_U + \chi_V - 2\chi_{U \cap V}$$

and thus, in particular, we have

$$\chi_{U \triangle V} \equiv \chi_U + \chi_V \mod 2$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

15. *Bijectivity.* Given $A, B \subset \mathcal{P}(X)$, the maps

$$\begin{aligned} A \mathbin{\triangle} -&\colon \mathcal{P}(X) \to \mathcal{P}(X), \\ - \mathbin{\triangle} B&\colon \mathcal{P}(X) \to \mathcal{P}(X) \end{aligned}$$

are bijections with inverses given by

$$\begin{aligned} (A \mathbin{\triangle} -)^{-1} &= - \cup (A \cap -), \\ (- \mathbin{\triangle} B)^{-1} &= - \cup (B \cap -). \end{aligned}$$

Moreover, the map

$$C \mapsto C \mathbin{\triangle} (A \mathbin{\triangle} B)$$

is a bijection of $\mathcal{P}(X)$ onto itself sending A to B and B to A.

16. *Interaction With Powersets and Groups.* Let X be a set.

    (a) The quadruple $(\mathcal{P}(X), \triangle, \emptyset, \mathrm{id}_{\mathcal{P}(X)})$ is an abelian group.[3]

    (b) Every element of $\mathcal{P}(X)$ has order 2 with respect to $\triangle$, and thus $\mathcal{P}(X)$ is a *Boolean group* (i.e. an abelian 2-group).

17. *Interaction With Powersets and Vector Spaces I.* The pair $\left(\mathcal{P}(X), \alpha_{\mathcal{P}(X)}\right)$ consisting of

- The group $\mathcal{P}(X)$ of **??**;
- The map $\alpha_{\mathcal{P}(X)} \colon \mathbb{F}_2 \times \mathcal{P}(X) \to \mathcal{P}(X)$ defined by

$$\begin{aligned} 0 \cdot U &\stackrel{\text{def}}{=} \emptyset, \\ 1 \cdot U &\stackrel{\text{def}}{=} U; \end{aligned}$$

is an $\mathbb{F}_2$-vector space.

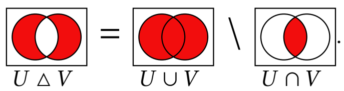
18. *Interaction With Powersets and Vector Spaces II.* If X is finite, then:

(a) The set of singletons sets on the elements of X forms a basis for the $\mathbb{F}_2$-vector space $\left(\mathcal{P}(X), \alpha_{\mathcal{P}(X)}\right)$ of Item 17.

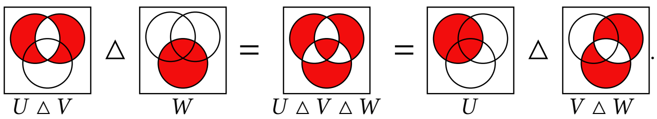
(b) We have
$$\dim(\mathcal{P}(X)) = \#\mathcal{P}(X).$$

19. *Interaction With Powersets and Rings.* The quintuple $(\mathcal{P}(X), \triangle, \cap, \emptyset, X)$ is a commutative ring.[4]

[1] *Illustration:*



[2] *Illustration:*



[3] Here are some examples:

i. When $X = \emptyset$, we have an isomorphism of groups between $\mathcal{P}(\emptyset)$ and the trivial group:
$$\left(\mathcal{P}(\emptyset), \triangle, \emptyset, \mathrm{id}_{\mathcal{P}(\emptyset)}\right) \cong \mathrm{pt}.$$

ii. When $X = \mathrm{pt}$, we have an isomorphism of groups between $\mathcal{P}(\mathrm{pt})$ and $\mathbb{Z}_{/2}$:
$$\left(\mathcal{P}(\mathrm{pt}), \triangle, \emptyset, \mathrm{id}_{\mathcal{P}(\mathrm{pt})}\right) \cong \mathbb{Z}_{/2}.$$

iii. When $X = \{0, 1\}$, we have an isomorphism of groups between $\mathcal{P}(\{0, 1\})$ and $\mathbb{Z}_{/2} \times \mathbb{Z}_{/2}$:
$$\left(\mathcal{P}(\{0, 1\}), \triangle, \emptyset, \mathrm{id}_{\mathcal{P}(\{0,1\})}\right) \cong \mathbb{Z}_{/2} \times \mathbb{Z}_{/2}.$$

[4] *Warning:* The analogous statement replacing intersections by unions (i.e. that the quintuple $(\mathcal{P}(X), \triangle, \cup, \emptyset, X)$ is a ring) is false, however. See [Pro24ba] for a proof.

END TEXTDBEND

Proof 2.3.12.3 ► Proof of Proposition 2.3.12.2

Item 1: Lack of Functoriality

Omitted.

Item 2: Via Unions and Intersections

See [Pro24r].

Item 3: Associativity

See [Pro24as].

Item 4: Commutativity

See [Pro24at].

Item 5: Unitality

This follows from Item 4 and [Pro24ax].

Item 6: Invertibility

See [Pro24az].

Item 7: Interaction With Unions

See [Pro24bg].

Item 8: Interaction With Complements I

See [Pro24aw].

Item 9: Interaction With Complements II

This follows from Item 4 and [Pro24bb].

Item 10: Interaction With Complements III

See [Pro24au].

Item 11: "Transitivity"

We have

$$\begin{aligned}(U \vartriangle V) \vartriangle (V \vartriangle W) &= U \vartriangle (V \vartriangle (V \vartriangle W)) && \text{(by Item 3)}\\ &= U \vartriangle ((V \vartriangle V) \vartriangle W) && \text{(by Item 3)}\\ &= U \vartriangle (\emptyset \vartriangle W) && \text{(by Item 6)}\\ &= U \vartriangle W && \text{(by Item 5)}\end{aligned}$$

Item 12: The Triangle Inequality for Symmetric Differences

This follows from Items 2 and 11.

Item 13: Distributivity Over Intersections

See [Pro24u].

Item 14: Interaction With Characteristic Functions

See [Pro24j].

Item 15: Bijectivity

Clear.

Item 16: Interaction With Powersets and Groups

Item 16a follows from[1] Items 3 to 6, while Item 16b follows from Item 6.

Item 17: Interaction With Powersets and Vector Spaces I

Clear.

Item 18: Interaction With Powersets and Vector Spaces II

Omitted.

Item 19: Interaction With Powersets and Rings

This follows from Items 8 and 11 of Proposition 2.3.9.2 and Items 13 and 16.[2]

[1] Reference: [Pro24av].
[2] *Reference:* [Pro24ay].

2.4 Powersets

2.4.1 Characteristic Functions

Let X be a set.

Definition 2.4.1.1 ► Characteristic Functions

Let $U \subset X$ and let $x \in X$.

1. The **characteristic function of** U[1] is the function[2]

$$\chi_U : X \to \{\mathrm{t}, \mathrm{f}\}$$

defined by

$$\chi_U(x) \overset{\text{def}}{=} \begin{cases} \text{true} & \text{if } x \in U, \\ \text{false} & \text{if } x \notin U \end{cases}$$

for each $x \in X$.

2. The **characteristic function of** x is the function[3]

$$\chi_x : X \to \{\mathrm{t}, \mathrm{f}\}$$

defined by

$$\chi_x \stackrel{\text{def}}{=} \chi_{\{x\}},$$

i.e. by

$$\chi_x(y) \stackrel{\text{def}}{=} \begin{cases} \text{true} & \text{if } x = y, \\ \text{false} & \text{if } x \neq y \end{cases}$$

for each $y \in X$.

3. The **characteristic relation on** X[4] is the relation[5]

$$\chi_X(-_1, -_2) \colon X \times X \to \{\mathsf{t}, \mathsf{f}\}$$

on X defined by[6]

$$\chi_X(x, y) \stackrel{\text{def}}{=} \begin{cases} \text{true} & \text{if } x = y, \\ \text{false} & \text{if } x \neq y \end{cases}$$

for each $x, y \in X$.

4. The **characteristic embedding**[7] **of** X **into** $\mathcal{P}(X)$ is the function

$$\chi_{(-)} \colon X \hookrightarrow \mathcal{P}(X)$$

defined by

$$\chi_{(-)}(x) \stackrel{\text{def}}{=} \chi_x$$

for each $x \in X$.

---

[1] *Further Terminology:* Also called the **indicator function of** U.

[2] *Further Notation:* Also written $\chi_X(U, -)$ or $\chi_X(-, U)$.

[3] *Further Notation:* Also written χ^x, $\chi_X(x, -)$, or $\chi_X(-, x)$.

[4] *Further Terminology:* Also called the **identity relation on** X.

[5] *Further Notation:* Also written $\chi_{-_2}^{-_1}$, or $\sim_{\text{id}}$ in the context of relations.

[6] As a subset of $X \times X$, the relation χ_X corresponds to the diagonal $\Delta_X \subset X \times X$ of X.

[7] The name "characteristic *embedding*" comes from the fact that there is an analogue of fully faithfulness for $\chi_{(-)}$: given a set X, we have

$$\mathrm{Hom}_{\mathcal{P}(X)}(\chi_x, \chi_y) = \chi_X(x, y),$$

for each $x, y \in X$.

Remark 2.4.1.2 ► Characteristic Functions as Decategorifications of Presheaves

The definitions in Definition 2.4.1.1 are decategorifications of co/presheaves, representable co/presheaves, Hom profunctors, and the Yoneda embedding:[1]

1. A function
$$f\colon X \to \{\mathsf{t}, \mathsf{f}\}$$
is a decategorification of a presheaf
$$\mathcal{F}\colon \mathcal{C}^{\mathsf{op}} \to \mathsf{Sets},$$
with the characteristic functions χ_U of the subsets of X being the primordial examples (and, in fact, all examples) of these.

2. The characteristic function
$$\chi_x\colon X \to \{\mathsf{t}, \mathsf{f}\}$$
of an *element* x of X is a decategorification of the representable presheaf
$$h_X\colon \mathcal{C}^{\mathsf{op}} \to \mathsf{Sets}$$
of an *object* x of a category $\mathcal{C}$.

3. The characteristic relation
$$\chi_X(-_1, -_2)\colon X \times X \to \{\mathsf{t}, \mathsf{f}\}$$
of X is a decategorification of the Hom profunctor
$$\mathsf{Hom}_{\mathcal{C}}(-_1, -_2)\colon \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathsf{Sets}$$
of a category $\mathcal{C}$.

4. The characteristic embedding
$$\chi_{(-)}\colon X \hookrightarrow \mathcal{P}(X)$$
of X into $\mathcal{P}(X)$ is a decategorification of the Yoneda embedding
$$よ\colon \mathcal{C}^{\mathsf{op}} \hookrightarrow \mathsf{PSh}(\mathcal{C})$$
of a category $\mathcal{C}$ into $\mathsf{PSh}(\mathcal{C})$.

5. There is also a direct parallel between unions and colimits:

- An element of $\mathcal{P}(X)$ is a union of elements of X, viewed as one-point subsets $\{x\} \in \mathcal{P}(A)$.
- An object of $\mathsf{PSh}(\mathcal{C})$ is a colimit of objects of $\mathcal{C}$, viewed as representable presheaves $h_X \in \mathrm{Obj}(\mathsf{PSh}(\mathcal{C}))$.

[1]These statements can be made precise by using the embeddings

$$\begin{aligned}(-)_{\mathsf{disc}}&\colon \mathsf{Sets} \hookrightarrow \mathsf{Cats},\\ (-)_{\mathsf{disc}}&\colon \{\mathsf{t},\mathsf{f}\}_{\mathsf{disc}} \hookrightarrow \mathsf{Sets}\end{aligned}$$

of sets into categories and of classical truth values into sets.

For instance, in this approach the characteristic function

$$\chi_x\colon X \to \{\mathsf{t},\mathsf{f}\}$$

of an element x of X, defined by

$$\chi_x(y) \stackrel{\text{def}}{=} \begin{cases}\mathsf{true} & \text{if } x = y,\\ \mathsf{false} & \text{if } x \neq y\end{cases}$$

for each $y \in X$, is recovered as the representable presheaf

$$\mathrm{Hom}_{X_{\mathsf{disc}}}(-,x)\colon X_{\mathsf{disc}} \to \mathsf{Sets}$$

of the corresponding object x of X_{disc}, defined on objects by

$$\mathrm{Hom}_{X_{\mathsf{disc}}}(y,x) \stackrel{\text{def}}{=} \begin{cases}\mathrm{pt} & \text{if } x = y,\\ \emptyset & \text{if } x \neq y\end{cases}$$

for each $y \in \mathrm{Obj}(X_{\mathsf{disc}})$.

Proposition 2.4.1.3 ► Properties of Characteristic Functions

Let X be a set.

1. *The Inclusion of Characteristic Relations Associated to a Function.* Let $f\colon A \to B$ be a function. We have an inclusion[1]

$$\chi_B \circ (f \times f) \subset \chi_A, \qquad \begin{tikzcd} A \times A \arrow[rr, "f\times f"] \arrow[dr, "\chi_A"'] & \arrow[d, phantom, "\supset"] & B \times B \arrow[dl, "\chi_B"] \\ & \{\mathsf{t},\mathsf{f}\}. & \end{tikzcd}$$

2. *Interaction With Unions I.* We have

$$\chi_{U\cup V} = \max(\chi_U, \chi_V)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

3. *Interaction With Unions II.* We have

$$\chi_{U\cup V} = \chi_U + \chi_V - \chi_{U\cap V}$$

for each $X \in \mathsf{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

4. *Interaction With Intersections I.* We have

$$\chi_{U\cap V} = \chi_U \chi_V$$

for each $X \in \mathsf{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

5. *Interaction With Intersections II.* We have

$$\chi_{U\cap V} = \min(\chi_U, \chi_V)$$

for each $X \in \mathsf{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

6. *Interaction With Differences.* We have

$$\chi_{U\setminus V} = \chi_U - \chi_{U\cap V}$$

for each $X \in \mathsf{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

7. *Interaction With Complements.* We have

$$\chi_{U^\mathsf{c}} = 1 - \chi_U$$

for each $X \in \mathsf{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

8. *Interaction With Symmetric Differences.* We have

$$\chi_{U\triangle V} = \chi_U + \chi_V - 2\chi_{U\cap V}$$

and thus, in particular, we have

$$\chi_{U\triangle V} \equiv \chi_U + \chi_V \mod 2$$

for each $X \in \mathsf{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

9. *Interaction Between the Characteristic Embedding and Morphisms.* Let $f : X \to Y$ be a map of sets. The diagram

$$f_* \circ \chi_X = \chi_{X'} \circ f, \qquad \begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\chi_X"'] & X' \arrow[d, "\chi_{X'}"] \\ \mathcal{P}(X) \arrow[r, "f_*"'] & \mathcal{P}(X'). \end{tikzcd}$$

commutes.

[1]This is the 0-categorical version of Definition 8.4.4.1.

PROOF 2.4.1.4 ► PROOF OF PROPOSITION 2.4.1.3

Item 1: The Inclusion of Characteristic Relations Associated to a Function

The inclusion $\chi_B(f(a), f(b)) \subset \chi_A(a, b)$ is equivalent to the statement "if $a = b$, then $f(a) = f(b)$", which is true.

Item 2: Interaction With Unions I

This is a repetition of Item 8 of Proposition 2.3.7.2 and is proved there.

Item 3: Interaction With Unions II

This is a repetition of Item 9 of Proposition 2.3.7.2 and is proved there.

Item 4: Interaction With Intersections I

This is a repetition of Item 9 of Proposition 2.3.9.2 and is proved there.

Item 5: Interaction With Intersections II

This is a repetition of Item 10 of Proposition 2.3.9.2 and is proved there.

Item 6: Interaction With Differences

This is a repetition of Item 15 of Proposition 2.3.10.2 and is proved there.

Item 7: Interaction With Complements

This is a repetition of Item 4 of Proposition 2.3.11.2 and is proved there.

Item 8: Interaction With Symmetric Differences

This is a repetition of Item 14 of Proposition 2.3.12.2 and is proved there.

Item 9: Interaction Between the Characteristic Embedding and Morphisms

Indeed, we have

$$\begin{aligned}
[f_* \circ \chi_X](x) &\overset{\text{def}}{=} f_*(\chi_X(x)) \\
&\overset{\text{def}}{=} f_*(\{x\}) \\
&= \{f(x)\} \\
&\overset{\text{def}}{=} \chi_{X'}(f(x)) \\
&\overset{\text{def}}{=} [\chi_{X'} \circ f](x),
\end{aligned}$$

for each $x \in X$, showing the desired equality. □

2.4.2 The Yoneda Lemma for Sets

Let X be a set and let $U \subset X$ be a subset of X.

Proposition 2.4.2.1 ► The Yoneda Lemma for Sets

We have

$$\chi_{\mathcal{P}(X)}(\chi_x, \chi_U) = \chi_U(x)$$

for each $x \in X$, giving an equality of functions

$$\chi_{\mathcal{P}(X)}(\chi_{(-)}, \chi_U) = \chi_U.$$

Proof 2.4.2.2 ► Proof of Proposition 2.4.2.1

Clear. ∎

Corollary 2.4.2.3 ► The Characteristic Embedding Is Fully Faithful

The characteristic embedding is fully faithful, i.e., we have

$$\chi_{\mathcal{P}(X)}(\chi_x, \chi_y) = \chi_X(x, y)$$

for each $x, y \in X$.

Proof 2.4.2.4 ► Proof of Corollary 2.4.2.3

This follows from Proposition 2.4.2.1. ∎

2.4.3 Powersets

Let X be a set.

Definition 2.4.3.1 ► Powersets

The **powerset of** X is the set $\mathcal{P}(X)$ defined by

$$\mathcal{P}(X) \overset{\text{def}}{=} \{U \in P \mid U \subset X\},$$

where P is the set in the axiom of powerset, **??** of **??**.

Remark 2.4.3.2 ► Powersets as Decategorifications of Co/Presheaf Categories

The powerset of a set is a decategorification of the category of presheaves of a category: while[1]

- The powerset of a set X is equivalently (Items 1 and 2 of Proposition 2.4.3.9) the set

$$\mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\})$$

of functions from X to the set $\{\mathsf{t}, \mathsf{f}\}$ of classical truth values.

- The category of presheaves on a category $\mathcal{C}$ is the category

$$\mathsf{Fun}(\mathcal{C}^{\mathsf{op}}, \mathsf{Sets})$$

of functors from $\mathcal{C}^{\mathsf{op}}$ to the category Sets of sets.

[1] This parallel is based on the following comparison:

- A category is enriched over the category

$$\mathsf{Sets} \overset{\text{def}}{=} \mathsf{Cats}_0$$

of sets (i.e. "0-categories"), with presheaves taking values on it.

- A set is enriched over the set

$$\{\mathsf{t}, \mathsf{f}\} \overset{\text{def}}{=} \mathsf{Cats}_{-1}$$

of classical truth values (i.e. "(-1)-categories"), with characteristic functions taking values on it.

Proposition 2.4.3.3 ▶ Properties of Powersets: As Categories

Let X be a set.

1. *Co/Completeness.* The (posetal) category (associated to) $(\mathcal{P}(X), \subset)$ is complete and cocomplete:

   (a) *Products.* The products in $\mathcal{P}(X)$ are given by intersection of subsets.

   (b) *Coproducts.* The coproducts in $\mathcal{P}(X)$ are given by union of subsets.

   (c) *Co/Equalisers.* Being a posetal category, $\mathcal{P}(X)$ only has at most one morphisms between any two objects, so co/equalisers are trivial.

2. *Cartesian Closedness.* The category $\mathcal{P}(X)$ is Cartesian closed with internal Hom

$$\mathbf{Hom}_{\mathcal{P}(X)}(-_1, -_2) \colon \mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \to \mathcal{P}(X)$$

given by[1]

$$\mathbf{Hom}_{\mathcal{P}(X)}(U, V) \overset{\text{def}}{=} (X \setminus U) \cup V$$

for each $U, V \in \mathrm{Obj}(\mathcal{P}(X))$.

[1] For intuition regarding the expression defining $\mathbf{Hom}_{\mathcal{P}(X)}(U, V)$, see Remark 2.3.9.4.

Proof 2.4.3.4 ► Proof of Proposition 2.4.3.3

Item 1: Co/Completeness

Clear.

Item 2: Cartesian Closedness

This follows from Item 2 of Proposition 2.3.9.2. □

Proposition 2.4.3.5 ► Properties of Powersets: Functoriality and Adjointness

Let X be a set.

1. *Functoriality I.* The assignment $X \mapsto \mathcal{P}(X)$ defines a functor

$$\mathcal{P}_* \colon \mathsf{Sets} \to \mathsf{Sets},$$

   where

   - *Action on Objects.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

   $$\mathcal{P}_*(A) \stackrel{\text{def}}{=} \mathcal{P}(A).$$

   - *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathsf{Sets})$, the action on morphisms

   $$\mathcal{P}_{*|A,B} \colon \mathsf{Sets}(A, B) \to \mathsf{Sets}(\mathcal{P}(A), \mathcal{P}(B))$$

   of $\mathcal{P}_*$ at (A, B) is the map defined by by sending a map of sets $f \colon A \to B$ to the map

   $$\mathcal{P}_*(f) \colon \mathcal{P}(A) \to \mathcal{P}(B)$$

   defined by

   $$\mathcal{P}_*(f) \stackrel{\text{def}}{=} f_*,$$

   as in Definition 2.4.4.1.

2. *Functoriality II.* The assignment $X \mapsto \mathcal{P}(X)$ defines a functor

$$\mathcal{P}^{-1} \colon \mathsf{Sets}^{\mathsf{op}} \to \mathsf{Sets},$$

   where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$\mathcal{P}^{-1}(A) \stackrel{\text{def}}{=} \mathcal{P}(A).$$

- *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathsf{Sets})$, the action on morphisms

$$\mathcal{P}^{-1}_{A,B} \colon \mathsf{Sets}(A, B) \to \mathsf{Sets}(\mathcal{P}(B), \mathcal{P}(A))$$

of $\mathcal{P}^{-1}$ at (A, B) is the map defined by by sending a map of sets $f \colon A \to B$ to the map

$$\mathcal{P}^{-1}(f) \colon \mathcal{P}(B) \to \mathcal{P}(A)$$

defined by

$$\mathcal{P}^{-1}(f) \stackrel{\text{def}}{=} f^{-1},$$

as in Definition 2.4.5.1.

3. *Functoriality III.* The assignment $X \mapsto \mathcal{P}(X)$ defines a functor

$$\mathcal{P}_! \colon \mathsf{Sets} \to \mathsf{Sets},$$

where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$\mathcal{P}_!(A) \stackrel{\text{def}}{=} \mathcal{P}(A).$$

- *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathsf{Sets})$, the action on morphisms

$$\mathcal{P}_{!|A,B} \colon \mathsf{Sets}(A, B) \to \mathsf{Sets}(\mathcal{P}(A), \mathcal{P}(B))$$

of $\mathcal{P}_!$ at (A, B) is the map defined by by sending a map of sets $f \colon A \to B$ to the map

$$\mathcal{P}_!(f) \colon \mathcal{P}(A) \to \mathcal{P}(B)$$

defined by

$$\mathcal{P}_!(f) \stackrel{\text{def}}{=} f_!,$$

as in Definition 2.4.6.1.

4. *Adjointness I.* We have an adjunction

$$
\left(\mathcal{P}^{-1} \dashv \mathcal{P}^{-1,\mathrm{op}}\right)\colon \quad
\begin{tikzcd}
\mathsf{Sets}^{\mathsf{op}} \arrow[r, bend left, "\mathcal{P}^{-1}"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}, \arrow[l, bend left, "\mathcal{P}^{-1,\mathrm{op}}"]
\end{tikzcd}
$$

witnessed by a bijection

$$
\underbrace{\mathsf{Sets}^{\mathsf{op}}(\mathcal{P}(A), B)}_{\stackrel{\text{def}}{=}\mathsf{Sets}(B,\mathcal{P}(A))} \cong \mathsf{Sets}(A, \mathcal{P}(B)),
$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$ and $B \in \mathrm{Obj}(\mathsf{Sets}^{\mathsf{op}})$.

5. *Adjointness II.* We have an adjunction

$$
(\mathrm{Gr} \dashv \mathcal{P}_*)\colon \quad
\begin{tikzcd}
\mathsf{Sets} \arrow[r, bend left, "\mathrm{Gr}"] \arrow[r, phantom, "\perp"] & \mathsf{Rel}, \arrow[l, bend left, "\mathcal{P}_*"]
\end{tikzcd}
$$

witnessed by a bijection of sets

$$
\mathsf{Rel}(\mathrm{Gr}(A), B) \cong \mathsf{Sets}(A, \mathcal{P}(B))
$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$ and $B \in \mathrm{Obj}(\mathsf{Rel})$, where Gr is the graph functor of Item 1 of Proposition 6.3.1.2 and $\mathcal{P}_*$ is the functor of Proposition 6.4.5.1.

Proof 2.4.3.6 ▶ Proof of Proposition 2.4.3.5

Item 1: Functoriality I

This follows from Items 3 and 4 of Proposition 2.4.4.6.

Item 2: Functoriality II

This follows Items 3 and 4 of Proposition 2.4.5.5.

Item 3: Functoriality III

This follows Items 3 and 4 of Proposition 2.4.6.8.

Item 4: Adjointness I

We have

$$
\begin{aligned}
\mathsf{Sets}^{\mathsf{op}}(\mathcal{P}(A), B) &\stackrel{\text{def}}{=} \mathsf{Sets}(B, \mathcal{P}(A)) \\
&\cong \mathsf{Sets}(B, \mathsf{Sets}(A, \{\mathsf{t}, \mathsf{f}\})) \\
&\qquad\qquad \text{(by Item 1 of Proposition 2.4.3.9)} \\
&\cong \mathsf{Sets}(A \times B, \{\mathsf{t}, \mathsf{f}\}) \quad \text{(by Item 2 of Proposition 2.1.3.3)} \\
&\cong \mathsf{Sets}(A, \mathsf{Sets}(B, \{\mathsf{t}, \mathsf{f}\})) \\
&\qquad\qquad \text{(by Item 2 of Proposition 2.1.3.3)} \\
&\cong \mathsf{Sets}(A, \mathcal{P}(B)) \quad \text{(by Item 1 of Proposition 2.4.3.9)}
\end{aligned}
$$

with all bijections natural in A and B (where we use Item 2 of Proposition 2.4.3.9 here).

Item 5: Adjointness II

We have

$$
\begin{aligned}
\mathsf{Rel}(\mathsf{Gr}(A), B) &\cong \mathcal{P}(A \times B) \\
&\cong \mathsf{Sets}(A \times B, \{\mathsf{t}, \mathsf{f}\}) \quad \text{(by Item 1 of Proposition 2.4.3.9)} \\
&\cong \mathsf{Sets}(A, \mathsf{Sets}(B, \{\mathsf{t}, \mathsf{f}\})) \quad \text{(by Item 2 of Proposition 2.1.3.3)} \\
&\cong \mathsf{Sets}(A, \mathcal{P}(B)) \quad \text{(by Item 1 of Proposition 2.4.3.9)}
\end{aligned}
$$

with all bijections natural in A (where we use Item 2 of Proposition 2.4.3.9 here). Explicitly, this isomorphism is given by sending a relation $R\colon \mathsf{Gr}(A) \nrightarrow B$ to the map $R^{\dagger}\colon A \longrightarrow \mathcal{P}(B)$ sending a to the subset $R(a)$ of B, as in Remark 5.1.1.4.

Naturality in B is then the statement that given a relation $R\colon B \nrightarrow B'$, the diagram

$$
\begin{tikzcd}
\mathsf{Rel}(\mathsf{Gr}(A), B) \arrow[r, "R \diamond -"] \arrow[d, dashed, "\wr"] & \mathsf{Rel}(\mathsf{Gr}(A), B') \arrow[d, dashed, "\wr"'] \\
\mathsf{Sets}(A, \mathcal{P}(B)) \arrow[r, "R_*"'] & \mathsf{Sets}(A, \mathcal{P}(B'))
\end{tikzcd}
$$

commutes, which follows from Remark 6.4.1.2. □

PROPOSITION 2.4.3.7 ▶ PROPERTIES OF POWERSETS: MONOIDALITY

Let X be a set.

1. *Symmetric Strong Monoidality With Respect to Coproducts I.* The power-

set functor $\mathcal{P}_*$ of Item 1 of Proposition 2.4.3.5 has a symmetric strong monoidal structure

$$\left(\mathcal{P}_*, \mathcal{P}_*^{\coprod}, \mathcal{P}_{*|\mathbb{1}}^{\coprod}\right) \colon (\mathsf{Sets}, \times, \mathrm{pt}) \to (\mathsf{Sets}, \coprod, \emptyset)$$

being equipped with isomorphisms

$$\mathcal{P}_{*|X,Y}^{\coprod} \colon \mathcal{P}(X) \times \mathcal{P}(Y) \xrightarrow{\cong} \mathcal{P}(X \coprod Y),$$
$$\mathcal{P}_{*|\mathbb{1}}^{\coprod} \colon \mathrm{pt} \xrightarrow{\cong} \mathcal{P}(\emptyset),$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

2. *Symmetric Strong Monoidality With Respect to Coproducts II.* The powerset functor $\mathcal{P}^{-1}$ of Item 2 of Proposition 2.4.3.5 has a symmetric strong monoidal structure

$$\left(\mathcal{P}^{-1}, \mathcal{P}^{-1|\coprod}, \mathcal{P}_{\mathbb{1}}^{-1|\coprod}\right) \colon (\mathsf{Sets}^{\mathsf{op}}, \times^{\mathsf{op}}, \mathrm{pt}) \to (\mathsf{Sets}, \coprod, \emptyset)$$

being equipped with isomorphisms

$$\mathcal{P}_{X,Y}^{-1|\coprod} \colon \mathcal{P}(X) \times \mathcal{P}(Y) \xrightarrow{\cong} \mathcal{P}(X \coprod Y),$$
$$\mathcal{P}_{\mathbb{1}}^{-1|\coprod} \colon \mathrm{pt} \xrightarrow{\cong} \mathcal{P}(\emptyset),$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

3. *Symmetric Strong Monoidality With Respect to Coproducts III.* The powerset functor $\mathcal{P}_!$ of Item 3 of Proposition 2.4.3.5 has a symmetric strong monoidal structure

$$\left(\mathcal{P}_!, \mathcal{P}_!^{\coprod}, \mathcal{P}_{!|\mathbb{1}}^{\coprod}\right) \colon (\mathsf{Sets}, \times, \mathrm{pt}) \to (\mathsf{Sets}, \coprod, \emptyset)$$

being equipped with isomorphisms

$$\mathcal{P}_{!|X,Y}^{\coprod} \colon \mathcal{P}(X) \times \mathcal{P}(Y) \xrightarrow{\cong} \mathcal{P}(X \coprod Y),$$
$$\mathcal{P}_{!|\mathbb{1}}^{\coprod} \colon \mathrm{pt} \xrightarrow{\cong} \mathcal{P}(\emptyset),$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

4. *Symmetric Lax Monoidality With Respect to Products I.* The powerset functor $\mathcal{P}_*$ of Item 1 of Proposition 2.4.3.5 has a symmetric lax monoidal structure

$$\left(\mathcal{P}_*, \mathcal{P}_*^{\otimes}, \mathcal{P}_{*|\mathbb{1}}^{\otimes}\right) \colon (\mathsf{Sets}, \times, \mathrm{pt}) \longrightarrow (\mathsf{Sets}, \times, \mathrm{pt})$$

being equipped with morphisms

$$\mathcal{P}_{*|X,Y}^{\times} \colon \mathcal{P}(X) \times \mathcal{P}(Y) \to \mathcal{P}(X \times Y),$$
$$\mathcal{P}_{*|\mathbb{1}}^{\times} \colon \mathrm{pt} \to \mathcal{P}(\mathrm{pt}),$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$, where

- The map $\mathcal{P}_{*|X,Y}^{\times}$ is given by

$$\mathcal{P}_{*|X,Y}^{\times}(U, V) \stackrel{\text{def}}{=} U \times V$$

  for each $(U, V) \in \mathcal{P}(X) \times \mathcal{P}(Y)$,
- The map $\mathcal{P}_{*|\mathbb{1}}^{\times}$ is given by

$$\mathcal{P}_{*|\mathbb{1}}^{\times}(\star) = \mathrm{pt}.$$

5. *Symmetric Lax Monoidality With Respect to Products II.* The powerset functor $\mathcal{P}^{-1}$ of Item 2 of Proposition 2.4.3.5 has a symmetric lax monoidal structure

$$\left(\mathcal{P}^{-1}, \mathcal{P}^{-1|\otimes}, \mathcal{P}_{\mathbb{1}}^{-1|\otimes}\right) \colon (\mathsf{Sets}^{\mathsf{op}}, \times^{\mathsf{op}}, \mathrm{pt}) \longrightarrow (\mathsf{Sets}, \times, \mathrm{pt})$$

being equipped with morphisms

$$\mathcal{P}_{X,Y}^{-1|\times} \colon \mathcal{P}(X) \times \mathcal{P}(Y) \to \mathcal{P}(X \times Y),$$
$$\mathcal{P}_{\mathbb{1}}^{\times} \colon \mathrm{pt} \to \mathcal{P}(\emptyset),$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$, defined as in Item 4.

6. *Symmetric Lax Monoidality With Respect to Products III.* The powerset functor $\mathcal{P}_!$ of Item 3 of Proposition 2.4.3.5 has a symmetric lax monoidal structure

$$\left(\mathcal{P}_!, \mathcal{P}_!^{\otimes}, \mathcal{P}_{!|\mathbb{1}}^{\otimes}\right) \colon (\mathsf{Sets}, \times, \mathrm{pt}) \longrightarrow (\mathsf{Sets}, \times, \mathrm{pt})$$

being equipped with morphisms

$$\mathcal{P}_{!|X,Y}^{\times} \colon \mathcal{P}(X) \times \mathcal{P}(Y) \to \mathcal{P}(X \times Y),$$
$$\mathcal{P}_{!|\mathbb{1}}^{\times} \colon \mathrm{pt} \to \mathcal{P}(\emptyset),$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$, defined as in Item 4.

Proof 2.4.3.8 ► Proof of Proposition 2.4.3.7

Item 1: Symmetric Strong Monoidality With Respect to Coproducts I

The isomorphism

$$\mathcal{P}^{\coprod}_{*|X,Y} \colon \mathcal{P}(X) \times \mathcal{P}(Y) \to \mathcal{P}(X \coprod Y)$$

is given by sending $(U, V) \in \mathcal{P}(X) \times \mathcal{P}(Y)$ to $U \coprod V$, with inverse given by sending a subset S of $X \coprod Y$ to the pair $(S_X, S_Y) \in \mathcal{P}(X) \times \mathcal{P}(Y)$ with

$$\begin{aligned} S_X &\stackrel{\text{def}}{=} \{x \in X \mid (0, x) \in S\} \\ S_Y &\stackrel{\text{def}}{=} \{y \in Y \mid (1, y) \in S\}. \end{aligned}$$

The isomorphism $\mathrm{pt} \cong \mathcal{P}(\emptyset)$ is given by $\star \mapsto \emptyset \in \mathcal{P}(\emptyset)$.
Naturality for the isomorphism $\mathcal{P}^{\coprod}_{*|X,Y}$ is the statement that, given maps of sets $f \colon X \to X'$ and $g \colon Y \to Y'$, the diagram

$$\begin{tikzcd} \mathcal{P}(X) \times \mathcal{P}(Y) \arrow[r, "f_* \times g_*"] \arrow[d, dashed, "\wr"] & \mathcal{P}(X') \times \mathcal{P}(Y') \arrow[d, dashed, "\wr"'] \\ \mathcal{P}(X \coprod Y) \arrow[r, "(f \coprod g)_*"'] & \mathcal{P}(X' \coprod Y') \end{tikzcd}$$

commutes, which is clear, as it acts on elements as

$$\begin{tikzcd} (U, V) \arrow[r, mapsto] \arrow[d, mapsto] & (f_*(U), g_*(V)) \arrow[d, mapsto] \\ U \coprod V \arrow[r, mapsto] & (f \coprod g)_*(U \coprod V) = f_*(U) \coprod g_*(V), \end{tikzcd}$$

where we are using Item 7 of Proposition 2.4.4.4.
Finally, monoidality, unity, and symmetry of $\mathcal{P}_*$ as a monoidal functor follow by checking the commutativity of the relevant diagrams on elements.

Item 2: Symmetric Strong Monoidality With Respect to Coproducts II

The proof is similar to Item 1, and is hence omitted.

Item 3: Symmetric Strong Monoidality With Respect to Coproducts III

The proof is similar to Item 1, and is hence omitted.

Item 4: Symmetric Lax Monoidality With Respect to Products I

Naturality for the morphism $\mathcal{P}^{\times}_{*|X,Y}$ is the statement that, given maps of sets $f\colon X \to X'$ and $g\colon Y \to Y'$, the diagram

$$\begin{tikzcd}
\mathcal{P}(X) \times \mathcal{P}(Y) \arrow[r, "f_* \times g_*"] \arrow[d, dashed, "\wr"] & \mathcal{P}(X') \times \mathcal{P}(Y') \arrow[d, dashed, "\wr"'] \\
\mathcal{P}(X \times Y) \arrow[r, "(f \times g)_*"'] & \mathcal{P}(X' \times Y')
\end{tikzcd}$$

commutes, which is clear, as it acts on elements as

$$\begin{tikzcd}
(U, V) \arrow[r, mapsto] \arrow[d, mapsto] & (f_*(U), g_*(V)) \arrow[d, mapsto] \\
U \times V \arrow[r, mapsto] & (f \times g)_*(U \times V) = f_*(U) \times g_*(V),
\end{tikzcd}$$

where we are using Item 8 of Proposition 2.4.4.4.
Finally, monoidality, unity, and symmetry of $\mathcal{P}_*$ as a monoidal functor follow by checking the commutativity of the relevant diagrams on elements.

Item 5: Symmetric Lax Monoidality With Respect to Products II

The proof is similar to Item 4, and is hence omitted.

Item 6: Symmetric Lax Monoidality With Respect to Products III

The proof is similar to Item 4, and is hence omitted.

Proposition 2.4.3.9 ► Properties of Powersets: As Sets of Functions/Relations

Let X be a set.

1. *Powersets as Sets of Functions I.* The assignment $U \mapsto \chi_U$ defines a bijection
$$\chi_{(-)} \colon \mathcal{P}(X) \xrightarrow{\cong} \mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\}),$$
for each $X \in \mathrm{Obj}(\mathsf{Sets})$.

2. *Powersets as Sets of Functions II.* The bijection
$$\mathcal{P}(X) \cong \mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\})$$
of Item 1 is natural in $X \in \mathrm{Obj}(\mathsf{Sets})$, refining to a natural isomorphism

of functors

$$\mathcal{P}^{-1} \cong \mathsf{Sets}(-, \{\mathsf{t}, \mathsf{f}\}).$$

3. *Powersets as Sets of Relations.* We have bijections

$$\mathcal{P}(X) \cong \mathsf{Rel}(\mathrm{pt}, X),$$
$$\mathcal{P}(X) \cong \mathsf{Rel}(X, \mathrm{pt}),$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets})$.

Proof 2.4.3.10 ► Proof of Proposition 2.4.3.9

Item 1: Powersets as Sets of Functions I

Indeed, the inverse of $\chi_{(-)}$ is given by sending a function $f \colon X \to \{\mathsf{t}, \mathsf{f}\}$ to the subset U_f of $\mathcal{P}(X)$ defined by

$$U_f \overset{\text{def}}{=} \{x \in X \mid f(x) = \mathsf{true}\},$$

i.e. by $U_f = f^{-1}(\mathsf{true})$. That $\chi_{(-)}$ and $f \mapsto U_f$ are inverses is then straightforward to check.

Item 2: Powersets as Sets of Functions II

We need to check that, given a function $f \colon X \to Y$, the diagram

$$\begin{tikzcd} \mathcal{P}(Y) \arrow[r, "f^{-1}"] \arrow[d, dashed, "\chi_{(-)}"', "\wr"] & \mathcal{P}(X) \arrow[d, dashed, "\chi_{(-)}", "\wr"'] \\ \mathsf{Sets}(Y, \{\mathsf{t}, \mathsf{f}\}) \arrow[r, "f^*"'] & \mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\}) \end{tikzcd}$$

commutes, i.e. that for each $V \in \mathcal{P}(Y)$, we have

$$\chi_V \circ f = \chi_{f^{-1}(V)}.$$

And indeed, we have

$$\begin{aligned} [\chi_V \circ f](v) &\overset{\text{def}}{=} \chi_V(f(v)) \\ &= \begin{cases} \mathsf{true} & \text{if } f(v) \in V, \\ \mathsf{false} & \text{otherwise} \end{cases} \\ &= \begin{cases} \mathsf{true} & \text{if } v \in f^{-1}(V), \\ \mathsf{false} & \text{otherwise} \end{cases} \\ &\overset{\text{def}}{=} \chi_{f^{-1}(V)}(v) \end{aligned}$$

for each $v \in V$.

Item 3: Powersets as Sets of Relations

Indeed, we have

$$\begin{aligned}\mathrm{Rel}(\mathrm{pt}, X) &\stackrel{\text{def}}{=} \mathcal{P}(\mathrm{pt} \times X)\\ &\cong \mathcal{P}(X)\end{aligned}$$

and

$$\begin{aligned}\mathrm{Rel}(X, \mathrm{pt}) &\stackrel{\text{def}}{=} \mathcal{P}(X \times \mathrm{pt})\\ &\cong \mathcal{P}(X),\end{aligned}$$

where we have used Item 4 of Proposition 2.1.3.3.

Remark 2.4.3.11 ▶ Powersets as Sets of Functions and Un/Straightening

The bijection

$$\mathcal{P}(X) \cong \mathrm{Sets}(X, \{\mathrm{t}, \mathrm{f}\})$$

of Item 1 of Proposition 2.4.3.9, which

- Takes a subset $U \hookrightarrow X$ of X and *straightens* it to a function $\chi_U : X \to \{\mathrm{true}, \mathrm{false}\}$;
- Takes a function $f : X \to \{\mathrm{true}, \mathrm{false}\}$ and *unstraightens* it to a subset $f^{-1}(\mathrm{true}) \hookrightarrow X$ of X;

may be viewed as the (-1)-categorical version of the un/straightening isomorphism for indexed and fibred sets

$$\underbrace{\mathrm{FibSets}(X)}_{\stackrel{\text{def}}{=}\mathrm{Sets}_{/X}} \cong \underbrace{\mathrm{ISets}(X)}_{\stackrel{\text{def}}{=}\mathrm{Fun}(X_{\mathrm{disc}}, \mathrm{Sets})}$$

of **??**, where we view:

- Subsets $U \hookrightarrow X$ as analogous to X-fibred sets $\phi_X : A \to X$.
- Functions $f : X \to \{\mathrm{t}, \mathrm{f}\}$ as analogous to X-indexed sets $A : X_{\mathrm{disc}} \to \mathrm{Sets}$.

Proposition 2.4.3.12 ▶ Properties of Powersets: As Free Cocompletions

Let X be a set.

1. *Universal Property.* The pair $(\mathcal{P}(X), \chi_{(-)})$ consisting of

   - The powerset $\mathcal{P}(X)$ of X;
   - The characteristic embedding $\chi_{(-)} \colon X \hookrightarrow \mathcal{P}(X)$ of X into $\mathcal{P}(X)$;

   satisfies the following universal property:

   (★) Given another pair (Y, f) consisting of

   – A cocomplete poset $(Y, \preceq)$;
   – A function $f \colon X \to Y$;

   there exists a unique cocontinuous morphism of posets

   $$(\mathcal{P}(X), \subset) \xrightarrow{\exists!} (Y, \preceq)$$

   making the diagram

   $$\begin{tikzcd} & \mathcal{P}(X) \arrow[d, dashed, "\exists!"] \\ X \arrow[ur, "\chi_X"] \arrow[r, "f"'] & Y \end{tikzcd}$$

   commute.

2. *Adjointness.* We have an adjunction[1]

   $$(\mathcal{P} \dashv 忘) \colon \quad \mathsf{Sets} \underset{忘}{\overset{\mathcal{P}}{\rightleftarrows}} \mathsf{Pos}^{\mathsf{cocomp.}}, \quad (\perp)$$

   witnessed by a bijection

   $$\mathsf{Pos}^{\mathsf{cocomp.}}((\mathcal{P}(X), \subset), (Y, \preceq)) \cong \mathsf{Sets}(X, Y),$$

   natural in $X \in \mathrm{Obj}(\mathsf{Sets})$ and $(Y, \preceq) \in \mathrm{Obj}(\mathsf{Pos}^{\mathsf{cocomp.}})$, where the maps witnessing this bijection are given by

- The map

  $$\chi_X^*\colon \mathsf{Pos}^{\mathsf{cocomp.}}((\mathcal{P}(X),\subset),(Y,\preceq)) \to \mathsf{Sets}(X,Y)$$

  defined by

  $$\chi_X^*(f) \overset{\mathrm{def}}{=} f \circ \chi_X,$$

  i.e. by sending a cocontinuous morphism of posets $f\colon \mathcal{P}(X) \to Y$ to the composition

  $$X \overset{\chi_X}{\hookrightarrow} \mathcal{P}(X) \xrightarrow{f} Y.$$

- The map

  $$\mathsf{Lan}_{\chi_X}\colon \mathsf{Sets}(X,Y) \to \mathsf{Pos}^{\mathsf{cocomp.}}((\mathcal{P}(X),\subset),(Y,\preceq))$$

  is given by sending a function $f\colon X \to Y$ to its left Kan extension along χ_X,

  $$\mathsf{Lan}_{\chi_X}(f)\colon \mathcal{P}(X) \to Y, \qquad \begin{tikzcd} & \mathcal{P}(X) \arrow[d, dashed, "\mathsf{Lan}_{\chi_X}(f)"] \\ X \arrow[ur, "\chi_X"] \arrow[r, "f"'] \arrow[ur, phantom, "\Uparrow", shift left=3] & Y. \end{tikzcd}$$

  Moreover, $\mathsf{Lan}_{\chi_X}(f)$ can be explicitly computed by

  $$\begin{aligned}\big[\mathsf{Lan}_{\chi_X}(f)\big](U) &\cong \int^{x\in X} \chi_{\mathcal{P}(X)}(\chi_x, U) \odot f(x)\\ &\cong \int^{x\in X} \chi_U(x) \odot f(x) && \text{(by Proposition 2.4.2.1)}\\ &\cong \bigvee_{x\in X} (\chi_U(x) \odot f(x))\end{aligned}$$

  for each $U \in \mathcal{P}(X)$, where:

  - $\bigvee$ is the join in $(Y,\preceq)$.
  - We have

    $$\begin{aligned}\mathsf{true} \odot f(x) &\overset{\mathrm{def}}{=} f(x),\\ \mathsf{false} \odot f(x) &\overset{\mathrm{def}}{=} \varnothing_Y,\end{aligned}$$

    where $\varnothing_Y$ is the minimal element of $(Y,\preceq)$.

[1] In this sense, $\mathcal{P}(A)$ is the free cocompletion of A. (Note that, despite its name, however, this is not an idempotent operation, as we have $\mathcal{P}(\mathcal{P}(A)) \neq \mathcal{P}(A)$.)

Proof 2.4.3.13 ▶ Proof of Proposition 2.4.3.7

Item 1: Universal Property

This is a rephrasing of Item 2.

Item 2: Adjointness

We claim we have adjunction $\mathcal{P} \dashv$ 忘, witnessed by a bijection

$$\mathsf{Pos}^{\mathsf{cocomp.}}((\mathcal{P}(X), \subset), (Y, \preceq)) \cong \mathsf{Sets}(X, Y),$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets})$ and $(Y, \preceq) \in \mathrm{Obj}(\mathsf{Pos}^{\mathsf{cocomp.}})$.

- *Map I.* We define a map

  $$\Phi_{X,Y} \colon \mathsf{Pos}^{\mathsf{cocomp.}}((\mathcal{P}(X), \subset), (Y, \preceq)) \to \mathsf{Sets}(X, Y)$$

  as in the statement, by

  $$\Phi_{X,Y}(f) \stackrel{\text{def}}{=} f \circ \chi_X$$

  for each $f \in \mathsf{Pos}^{\mathsf{cocomp.}}((\mathcal{P}(X), \subset), (Y, \preceq))$.

- *Map II.* We define a map

  $$\Psi_{X,Y} \colon \mathsf{Sets}(X, Y) \to \mathsf{Pos}^{\mathsf{cocomp.}}((\mathcal{P}(X), \subset), (Y, \preceq))$$

  as in the statement, by

  $$\Psi_{X,Y}(f) \stackrel{\text{def}}{=} \mathsf{Lan}_{\chi_X}(f), \qquad \begin{tikzcd} & \mathcal{P}(X) \arrow[d, dashed, "\mathsf{Lan}_{\chi_X}(f)"] \\ X \arrow[ur, "\chi_X"] \arrow[r, "f"'] \arrow[r, phantom, shift left=3, "\Uparrow"] & Y, \end{tikzcd}$$

  for each $f \in \mathsf{Sets}(X, Y)$.

- *Invertibility I.* We claim that

  $$\Psi_{X,Y} \circ \Phi_{X,Y} = \mathrm{id}_{\mathsf{Pos}^{\mathsf{cocomp.}}((\mathcal{P}(X),\subset),(Y,\preceq))}.$$

  Indeed, given a cocontinuous morphism of posets

  $$\xi \colon (\mathcal{P}(X), \subset) \to (Y, \preceq),$$

we have

$$\begin{aligned}\left[\Psi_{X,Y}\circ\Phi_{X,Y}\right](\xi) &\overset{\text{def}}{=} \Psi_{X,Y}\big(\Phi_{X,Y}(\xi)\big)\\ &\overset{\text{def}}{=} \Psi_{X,Y}(\xi\circ\chi_X)\\ &\overset{\text{def}}{=} \mathrm{Lan}_{\chi_X}(\xi\circ\chi_X)\\ &\cong \bigvee_{x\in X}\chi_{(-)}(x)\odot\xi(\chi_X(x))\\ &\overset{\text{clm}}{=} \xi,\end{aligned}$$

where indeed

$$\begin{aligned}\left[\bigvee_{x\in X}\chi_{(-)}(x)\odot\xi(\chi_X(x))\right](U) &\overset{\text{def}}{=} \bigvee_{x\in X}\chi_U(x)\odot\xi(\chi_X(x))\\ &= \left(\bigvee_{x\in U}\chi_U(x)\odot\xi(\chi_X(x))\right)\vee\left(\bigvee_{x\in X\setminus U}\chi_U(x)\odot\xi(\chi_X(x))\right)\\ &= \left(\bigvee_{x\in U}\xi(\chi_X(x))\right)\vee\left(\bigvee_{x\in X\setminus U}\emptyset_Y\right)\\ &= \bigvee_{x\in U}\xi(\chi_X(x))\\ &\overset{(\dagger)}{=} \xi\left(\bigvee_{x\in U}\chi_X(x)\right)\\ &= \xi(U)\end{aligned}$$

for each $U\in\mathcal{P}(X)$, where we have used that ξ is cocontinuous for the equality $\overset{(\dagger)}{=}$.

- *Invertibility II.* We claim that

$$\Phi_{X,Y}\circ\Psi_{X,Y} = \mathrm{id}_{\mathsf{Sets}(X,Y)}.$$

Indeed, given a function $f\colon X\longrightarrow Y$, we have

$$\begin{aligned}\left[\Phi_{X,Y}\circ\Psi_{X,Y}\right](f) &\overset{\text{def}}{=} \Phi_{X,Y}\big(\Psi_{X,Y}(f)\big)\\ &\overset{\text{def}}{=} \Phi_{X,Y}\big(\mathrm{Lan}_{\chi_X}(f)\big)\\ &\overset{\text{def}}{=} \mathrm{Lan}_{\chi_X}(f)\circ\chi_X\\ &\overset{\text{clm}}{=} f,\end{aligned}$$

where indeed

$$
\begin{aligned}
\left[\mathrm{Lan}_{\chi_X}(f)\circ\chi_X\right](x) &\overset{\text{def}}{=} \bigvee_{y\in X}\chi_{\{x\}}(y)\odot f(y)\\
&= \left(\chi_{\{x\}}(x)\odot f(x)\right)\vee\left(\bigvee_{y\in X\setminus\{x\}}\chi_{\{x\}}(y)\odot f(y)\right)\\
&= f(x)\vee\left(\bigvee_{y\in X\setminus\{x\}}\varnothing_Y\right)\\
&= f(x)\vee\varnothing_Y\\
&= f(x)
\end{aligned}
$$

for each $x\in X$.

- *Naturality for Φ, Part I.* We need to show that, given a function $f\colon X\to X'$, the diagram

$$
\begin{tikzcd}
\mathsf{Pos}^{\mathsf{cocomp.}}((\mathcal{P}(X'),\subset),(Y,\preceq)) \arrow[r, "\Phi_{X',Y}"] \arrow[d, "\mathcal{P}_*(f)^*"'] & \mathsf{Sets}(X',Y) \arrow[d, "f^*"]\\
\mathsf{Pos}^{\mathsf{cocomp.}}((\mathcal{P}(X),\subset),(Y,\preceq)) \arrow[r, "\Phi_{X,Y}"'] & \mathsf{Sets}(X,Y)
\end{tikzcd}
$$

commutes. Indeed, given a cocontinuous morphism of posets

$$\xi\colon(\mathcal{P}(X'),\subset)\to(Y,\preceq),$$

we have

$$
\begin{aligned}
\left[\Phi_{X,Y}\circ\mathcal{P}_*(f)^*\right](\xi) &\overset{\text{def}}{=} \Phi_{X,Y}\left(\mathcal{P}_*(f)^*(\xi)\right)\\
&\overset{\text{def}}{=} \Phi_{X,Y}(\xi\circ f_*)\\
&\overset{\text{def}}{=} (\xi\circ f_*)\circ\chi_X\\
&= \xi\circ(f_*\circ\chi_X)\\
&\overset{(\dagger)}{=} \xi\circ(\chi_{X'}\circ f)\\
&= (\xi\circ\chi_{X'})\circ f\\
&\overset{\text{def}}{=} \Phi_{X',Y}(\xi)\circ f\\
&\overset{\text{def}}{=} f^*\left(\Phi_{X',Y}(\xi)\right)\\
&\overset{\text{def}}{=} \left[f^*\circ\Phi_{X',Y}\right](\xi),
\end{aligned}
$$

where we have used Item 9 of Proposition 2.4.1.3 for the equality $\overset{(\dagger)}{=}$ above.

- *Naturality for Φ, Part II.* We need to show that, given a cocontinuous morphism of posets

$$g\colon (Y, \preceq_Y) \to (Y', \preceq_{Y'}),$$

the diagram

$$\begin{tikzcd}
\mathsf{Pos}^{\mathsf{cocomp.}}((\mathcal{P}(X), \subset), (Y, \preceq)) \arrow[r, "\Phi_{X,Y}"] \arrow[d, "g_*"'] & \mathsf{Sets}(X, Y) \arrow[d, "g_*"] \\
\mathsf{Pos}^{\mathsf{cocomp.}}((\mathcal{P}(X), \subset), (Y', \preceq)) \arrow[r, "\Phi_{X,Y'}"'] & \mathsf{Sets}(X, Y')
\end{tikzcd}$$

commutes. Indeed, given a cocontinuous morphism of posets

$$\xi\colon (\mathcal{P}(X), \subset) \to (Y, \preceq),$$

we have

$$\begin{aligned}
\left[\Phi_{X,Y'} \circ g_*\right](\xi) &\overset{\text{def}}{=} \Phi_{X,Y'}(g_*(\xi)) \\
&\overset{\text{def}}{=} \Phi_{X,Y'}(g \circ \xi) \\
&\overset{\text{def}}{=} (g \circ \xi) \circ \chi_X \\
&= g \circ (\xi \circ \chi_X) \\
&\overset{\text{def}}{=} g \circ \left(\Phi_{X,Y}(\xi)\right) \\
&\overset{\text{def}}{=} g_*\left(\Phi_{X,Y}(\xi)\right) \\
&\overset{\text{def}}{=} \left[g_* \circ \Phi_{X,Y}\right](\xi).
\end{aligned}$$

- *Naturality for Ψ.* Since Φ is natural in each argument and Φ is a componentwise inverse to Ψ in each argument, it follows from Item 2 of Proposition 8.8.6.2 that Ψ is also natural in each argument.

This finishes the proof. □

2.4.4 Direct Images

Let A and B be sets and let $f\colon A \to B$ be a function.

Definition 2.4.4.1 ► Direct Images

The **direct image function associated to** f is the function

$$f_*\colon \mathcal{P}(A) \to \mathcal{P}(B)$$

defined by[1,2]

$$\begin{aligned} f_*(U) &\stackrel{\text{def}}{=} f(U) \\ &\stackrel{\text{def}}{=} \left\{ b \in B \;\middle|\; \begin{array}{l} \text{there exists some } a \in U \\ \text{such that } b = f(a) \end{array} \right\} \\ &= \{ f(a) \in B \mid a \in U \} \end{aligned}$$

for each $U \in \mathcal{P}(A)$.

[1] *Further Terminology:* The set $f(U)$ is called the **direct image of** U **by** f.
[2] We also have

$$f_*(U) = B \setminus f_!(A \setminus U);$$

see Item 9 of Proposition 2.4.4.4.

Notation 2.4.4.2 ► Further Notation for Direct Images

Sometimes one finds the notation

$$\exists_f\colon \mathcal{P}(A) \to \mathcal{P}(B)$$

for f_*. This notation comes from the fact that the following statements are equivalent, where $b \in B$ and $U \in \mathcal{P}(A)$:

- We have $b \in \exists_f(U)$.
- There exists some $a \in U$ such that $f(a) = b$.

Remark 2.4.4.3 ► Unwinding Definition 2.4.4.1

Identifying subsets of A with functions from A to $\{\text{true}, \text{false}\}$ via Items 1 and 2 of Proposition 2.4.3.9, we see that the direct image function associated to f is equivalently the function

$$f_*\colon \mathcal{P}(A) \to \mathcal{P}(B)$$

defined by

$$\begin{aligned}
f_*(\chi_U) &\stackrel{\text{def}}{=} \mathrm{Lan}_f(\chi_U)\\
&= \mathrm{colim}\left(\left(f \overset{\rightarrow}{\times} \underline{(-_1)}\right) \overset{\mathrm{pr}}{\twoheadrightarrow} A \xrightarrow{\chi_U} \{\mathsf{t},\mathsf{f}\}\right)\\
&= \operatorname*{colim}_{\substack{a\in A\\ f(a)=-_1}} (\chi_U(a))\\
&= \bigvee_{\substack{a\in A\\ f(a)=-_1}} (\chi_U(a)),
\end{aligned}$$

where we have used **??** for the second equality. In other words, we have

$$\begin{aligned}
[f_*(\chi_U)](b) &= \bigvee_{\substack{a\in A\\ f(a)=b}} (\chi_U(a))\\
&= \begin{cases} \mathsf{true} & \text{if there exists some } a \in A \text{ such} \\ & \text{that } f(a) = b \text{ and } a \in U,\\ \mathsf{false} & \text{otherwise}\end{cases}\\
&= \begin{cases} \mathsf{true} & \text{if there exists some } a \in U \\ & \text{such that } f(a) = b,\\ \mathsf{false} & \text{otherwise}\end{cases}
\end{aligned}$$

for each $b \in B$.

Proposition 2.4.4.4 ► Properties of Direct Images I

Let $f : A \to B$ be a function.

1. *Functoriality.* The assignment $U \mapsto f_*(U)$ defines a functor

$$f_* : (\mathcal{P}(A), \subset) \to (\mathcal{P}(B), \subset)$$

where

- *Action on Objects.* For each $U \in \mathcal{P}(A)$, we have

$$[f_*](U) \stackrel{\text{def}}{=} f_*(U).$$

- *Action on Morphisms.* For each $U, V \in \mathcal{P}(A)$:

  (★) If $U \subset V$, then $f_*(U) \subset f_*(V)$.

2. *Triple Adjointness.* We have a triple adjunction

$$\left(f_* \dashv f^{-1} \dashv f_!\right)\colon\quad \begin{tikzcd} \mathcal{P}(A) \arrow[r, bend left=45, "f_*"] \arrow[r, bend right=45, "f_!"'] & \mathcal{P}(B), \arrow[l, "f^{-1}"'] \end{tikzcd}$$

witnessed by bijections of sets

$$\begin{aligned} \mathrm{Hom}_{\mathcal{P}(B)}(f_*(U), V) &\cong \mathrm{Hom}_{\mathcal{P}(A)}\left(U, f^{-1}(V)\right), \\ \mathrm{Hom}_{\mathcal{P}(A)}\left(f^{-1}(U), V\right) &\cong \mathrm{Hom}_{\mathcal{P}(A)}(U, f_!(V)), \end{aligned}$$

natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$ and (respectively) $V \in \mathcal{P}(A)$ and $U \in \mathcal{P}(B)$, i.e. where:

   (a) The following conditions are equivalent:
      i. We have $f_*(U) \subset V$.
      ii. We have $U \subset f^{-1}(V)$.

   (b) The following conditions are equivalent:
      i. We have $f^{-1}(U) \subset V$.
      ii. We have $U \subset f_!(V)$.

3. *Preservation of Colimits.* We have an equality of sets

$$f_*\left(\bigcup_{i \in I} U_i\right) = \bigcup_{i \in I} f_*(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have equalities

$$\begin{aligned} f_*(U) \cup f_*(V) &= f_*(U \cup V), \\ f_*(\emptyset) &= \emptyset, \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

4. *Oplax Preservation of Limits.* We have an inclusion of sets

$$f_*\left(\bigcap_{i \in I} U_i\right) \subset \bigcap_{i \in I} f_*(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have inclusions

$$\begin{aligned} f_*(U \cap V) &\subset f_*(U) \cap f_*(V), \\ f_*(A) &\subset B, \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

5. *Symmetric Strict Monoidality With Respect to Unions.* The direct image function of Item 1 has a symmetric strict monoidal structure

$$\left(f_*, f_*^{\otimes}, f_{*|\mathbb{1}}^{\otimes}\right) \colon (\mathcal{P}(A), \cup, \emptyset) \to (\mathcal{P}(B), \cup, \emptyset),$$

being equipped with equalities

$$f_{*|U,V}^{\otimes} \colon f_*(U) \cup f_*(V) \xrightarrow{=} f_*(U \cup V),$$
$$f_{*|\mathbb{1}}^{\otimes} \colon \emptyset \xrightarrow{=} \emptyset,$$

natural in $U, V \in \mathcal{P}(A)$.

6. *Symmetric Oplax Monoidality With Respect to Intersections.* The direct image function of Item 1 has a symmetric oplax monoidal structure

$$\left(f_*, f_*^{\otimes}, f_{*|\mathbb{1}}^{\otimes}\right) \colon (\mathcal{P}(A), \cap, A) \to (\mathcal{P}(B), \cap, B),$$

being equipped with inclusions

$$f_{*|U,V}^{\otimes} \colon f_*(U \cap V) \hookrightarrow f_*(U) \cap f_*(V),$$
$$f_{*|\mathbb{1}}^{\otimes} \colon f_*(A) \hookrightarrow B,$$

natural in $U, V \in \mathcal{P}(A)$.

7. *Interaction With Coproducts.* Let $f \colon A \to A'$ and $g \colon B \to B'$ be maps of sets. We have

$$(f \coprod g)_*(U \coprod V) = f_*(U) \coprod g_*(V)$$

for each $U \in \mathcal{P}(A)$ and each $V \in \mathcal{P}(B)$.

8. *Interaction With Products.* Let $f \colon A \to A'$ and $g \colon B \to B'$ be maps of sets. We have

$$(f \times g)_*(U \times V) = f_*(U) \times g_*(V)$$

for each $U \in \mathcal{P}(A)$ and each $V \in \mathcal{P}(B)$.

9. *Relation to Direct Images With Compact Support.* We have

$$f_*(U) = B \setminus f_!(A \setminus U)$$

for each $U \in \mathcal{P}(A)$.

Proof 2.4.4.5 ► Proof of Proposition 2.4.4.4

Item 1: Functoriality

Clear.

Item 2: Triple Adjointness

This follows from Remark 2.4.4.3, Remark 2.4.5.2, Remark 2.4.6.3, and ?? of ??.

Item 3: Preservation of Colimits

This follows from Item 2 and ?? of ??.[1]

Item 4: Oplax Preservation of Limits

The inclusion $f_*(A) \subset B$ is clear. See [Pro24s] for the other inclusions.

Item 5: Symmetric Strict Monoidality With Respect to Unions

This follows from Item 3.

Item 6: Symmetric Oplax Monoidality With Respect to Intersections

This follows from Item 4.

Item 7: Interaction With Coproducts

Clear.

Item 8: Interaction With Products

Clear.

Item 9: Relation to Direct Images With Compact Support

Applying Item 9 of Proposition 2.4.6.6 to $A \setminus U$, we have

$$\begin{aligned} f_!(A \setminus U) &= B \setminus f_*(A \setminus (A \setminus U)) \\ &= B \setminus f_*(U). \end{aligned}$$

Taking complements, we then obtain

$$\begin{aligned} f_*(U) &= B \setminus (B \setminus f_*(U)), \\ &= B \setminus f_!(A \setminus U), \end{aligned}$$

which finishes the proof. □

[1] See also [Pro24t].

Proposition 2.4.4.6 ► Properties of Direct Images II

Let $f \colon A \longrightarrow B$ be a function.

1. *Functionality I.* The assignment $f \mapsto f_*$ defines a function
$$(-)_{*|A,B}\colon \mathsf{Sets}(A,B) \to \mathsf{Sets}(\mathcal{P}(A), \mathcal{P}(B)).$$

2. *Functionality II.* The assignment $f \mapsto f_*$ defines a function
$$(-)_{*|A,B}\colon \mathsf{Sets}(A,B) \to \mathsf{Pos}((\mathcal{P}(A), \subset), (\mathcal{P}(B), \subset)).$$

3. *Interaction With Identities.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have
$$(\mathrm{id}_A)_* = \mathrm{id}_{\mathcal{P}(A)}.$$

4. *Interaction With Composition.* For each pair of composable functions $f\colon A \to B$ and $g\colon B \to C$, we have
$$(g \circ f)_* = g_* \circ f_*, \qquad \begin{tikzcd} \mathcal{P}(A) \arrow[r, "f_*"] \arrow[dr, "(g\circ f)_*"'] & \mathcal{P}(B) \arrow[d, "g_*"] \\ & \mathcal{P}(C). \end{tikzcd}$$

Proof 2.4.4.7 ▶ Proof of Proposition 2.4.4.6

Item 1: Functionality I

Clear.

Item 2: Functionality II

Clear.

Item 3: Interaction With Identities

This follows from Remark 2.4.4.3 and **??** of **??**.

Item 4: Interaction With Composition

This follows from Remark 2.4.4.3 and **??** of **??**. □

2.4.5 Inverse Images

Let A and B be sets and let $f\colon A \to B$ be a function.

Definition 2.4.5.1 ▶ Inverse Images

The **inverse image function associated to** f is the function[1]

$$f^{-1}\colon \mathcal{P}(B) \to \mathcal{P}(A)$$

defined by[2]

$$f^{-1}(V) \stackrel{\text{def}}{=} \{a \in A \mid \text{we have } f(a) \in V\}$$

for each $V \in \mathcal{P}(B)$.

[1] *Further Notation:* Also written $f^*\colon \mathcal{P}(B) \to \mathcal{P}(A)$.
[2] *Further Terminology:* The set $f^{-1}(V)$ is called the **inverse image of** V **by** f.

Remark 2.4.5.2 ▶ Unwinding Definition 2.4.5.1

Identifying subsets of B with functions from B to $\{\text{true}, \text{false}\}$ via Items 1 and 2 of Proposition 2.4.3.9, we see that the inverse image function associated to f is equivalently the function

$$f^*\colon \mathcal{P}(B) \to \mathcal{P}(A)$$

defined by

$$f^*(\chi_V) \stackrel{\text{def}}{=} \chi_V \circ f$$

for each $\chi_V \in \mathcal{P}(B)$, where $\chi_V \circ f$ is the composition

$$A \xrightarrow{f} B \xrightarrow{\chi_V} \{\text{true}, \text{false}\}$$

in Sets.

Proposition 2.4.5.3 ▶ Properties of Inverse Images I

Let $f\colon A \to B$ be a function.

1. *Functoriality.* The assignment $V \mapsto f^{-1}(V)$ defines a functor

$$f^{-1}\colon (\mathcal{P}(B), \subset) \to (\mathcal{P}(A), \subset)$$

   where

   - *Action on Objects.* For each $V \in \mathcal{P}(B)$, we have

   $$[f^{-1}](V) \stackrel{\text{def}}{=} f^{-1}(V).$$

   - *Action on Morphisms.* For each $U, V \in \mathcal{P}(B)$:

(★) If $U \subset V$, then $f^{-1}(U) \subset f^{-1}(V)$.

2. *Triple Adjointness.* We have a triple adjunction

$$\left(f_* \dashv f^{-1} \dashv f_!\right)\colon \quad \begin{tikzcd} \mathcal{P}(A) \arrow[r, bend left=45, "f_*"] \arrow[r, bend right=45, "f_!"'] & \mathcal{P}(B) \arrow[l, "f^{-1}"'] \end{tikzcd},$$

witnessed by bijections of sets

$$\begin{aligned} \mathrm{Hom}_{\mathcal{P}(B)}(f_*(U), V) &\cong \mathrm{Hom}_{\mathcal{P}(A)}\left(U, f^{-1}(V)\right), \\ \mathrm{Hom}_{\mathcal{P}(A)}\left(f^{-1}(U), V\right) &\cong \mathrm{Hom}_{\mathcal{P}(A)}(U, f_!(V)), \end{aligned}$$

natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$ and (respectively) $V \in \mathcal{P}(A)$ and $U \in \mathcal{P}(B)$, i.e. where:

(a) The following conditions are equivalent:
   i. We have $f_*(U) \subset V$;
   ii. We have $U \subset f^{-1}(V)$;

(b) The following conditions are equivalent:
   i. We have $f^{-1}(U) \subset V$.
   ii. We have $U \subset f_!(V)$.

3. *Preservation of Colimits.* We have an equality of sets

$$f^{-1}\left(\bigcup_{i\in I} U_i\right) = \bigcup_{i\in I} f^{-1}(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(B)^{\times I}$. In particular, we have equalities

$$\begin{aligned} f^{-1}(U) \cup f^{-1}(V) &= f^{-1}(U \cup V), \\ f^{-1}(\emptyset) &= \emptyset, \end{aligned}$$

natural in $U, V \in \mathcal{P}(B)$.

4. *Preservation of Limits.* We have an equality of sets

$$f^{-1}\left(\bigcap_{i\in I} U_i\right) = \bigcap_{i\in I} f^{-1}(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(B)^{\times I}$. In particular, we have equalities

$$\begin{aligned} f^{-1}(U) \cap f^{-1}(V) &= f^{-1}(U \cap V), \\ f^{-1}(B) &= A, \end{aligned}$$

natural in $U, V \in \mathcal{P}(B)$.

5. *Symmetric Strict Monoidality With Respect to Unions.* The inverse image function of Item 1 has a symmetric strict monoidal structure

$$\left(f^{-1}, f^{-1,\otimes}, f_{\mathbb{1}}^{-1,\otimes}\right) \colon (\mathcal{P}(B), \cup, \emptyset) \to (\mathcal{P}(A), \cup, \emptyset),$$

being equipped with equalities

$$\begin{aligned} f_{U,V}^{-1,\otimes} &\colon f^{-1}(U) \cup f^{-1}(V) \xrightarrow{=} f^{-1}(U \cup V), \\ f_{\mathbb{1}}^{-1,\otimes} &\colon \emptyset \xrightarrow{=} f^{-1}(\emptyset), \end{aligned}$$

natural in $U, V \in \mathcal{P}(B)$.

6. *Symmetric Strict Monoidality With Respect to Intersections.* The inverse image function of Item 1 has a symmetric strict monoidal structure

$$\left(f^{-1}, f^{-1,\otimes}, f_{\mathbb{1}}^{-1,\otimes}\right) \colon (\mathcal{P}(B), \cap, B) \to (\mathcal{P}(A), \cap, A),$$

being equipped with equalities

$$\begin{aligned} f_{U,V}^{-1,\otimes} &\colon f^{-1}(U) \cap f^{-1}(V) \xrightarrow{=} f^{-1}(U \cap V), \\ f_{\mathbb{1}}^{-1,\otimes} &\colon A \xrightarrow{=} f^{-1}(B), \end{aligned}$$

natural in $U, V \in \mathcal{P}(B)$.

7. *Interaction With Coproducts.* Let $f \colon A \to A'$ and $g \colon B \to B'$ be maps of sets. We have

$$(f \textstyle\coprod g)^{-1}(U' \textstyle\coprod V') = f^{-1}(U') \textstyle\coprod g^{-1}(V')$$

for each $U' \in \mathcal{P}(A')$ and each $V' \in \mathcal{P}(B')$.

8. *Interaction With Products.* Let $f \colon A \to A'$ and $g \colon B \to B'$ be maps of sets. We have

$$(f \times g)^{-1}(U' \times V') = f^{-1}(U') \times g^{-1}(V')$$

for each $U' \in \mathcal{P}(A')$ and each $V' \in \mathcal{P}(B')$.

Proof 2.4.5.4 ► Proof of Proposition 2.4.5.3

Item 1: Functoriality

Clear.

Item 2: Triple Adjointness

This follows from Remark 2.4.4.3, Remark 2.4.5.2, Remark 2.4.6.3, and ?? of ??.

Item 3: Preservation of Colimits

This follows from Item 2 and ?? of ??.[1]

Item 4: Preservation of Limits

This follows from Item 2 and ?? of ??.[2]

Item 5: Symmetric Strict Monoidality With Respect to Unions

This follows from Item 3.

Item 6: Symmetric Strict Monoidality With Respect to Intersections

This follows from Item 4.

Item 7: Interaction With Coproducts

Clear.

Item 8: Interaction With Products

Clear. □

[1]See also [Pro24ac].
[2]See also [Pro24ab].

Proposition 2.4.5.5 ► Properties of Inverse Images II

Let $f\colon A \to B$ be a function.

1. *Functionality I.* The assignment $f \mapsto f^{-1}$ defines a function

$$(-)^{-1}_{A,B}\colon \mathsf{Sets}(A,B) \to \mathsf{Sets}(\mathcal{P}(B),\mathcal{P}(A)).$$

2. *Functionality II.* The assignment $f \mapsto f^{-1}$ defines a function

$$(-)^{-1}_{A,B}\colon \mathsf{Sets}(A,B) \to \mathsf{Pos}((\mathcal{P}(B),\subset),(\mathcal{P}(A),\subset)).$$

3. *Interaction With Identities.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$\mathrm{id}_A^{-1} = \mathrm{id}_{\mathcal{P}(A)}.$$

4. *Interaction With Composition.* For each pair of composable functions $f\colon A \to B$ and $g\colon B \to C$, we have

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}, \qquad \begin{tikzcd} \mathcal{P}(C) \arrow[r, "g^{-1}"] \arrow[dr, "(g\circ f)^{-1}"'] & \mathcal{P}(B) \arrow[d, "f^{-1}"] \\ & \mathcal{P}(A). \end{tikzcd}$$

Proof 2.4.5.6 ▶ Proof of Proposition 2.4.5.5

Item 1: Functionality I

Clear.

Item 2: Functionality II

Clear.

Item 3: Interaction With Identities

This follows from Remark 2.4.5.2 and Item 5 of Proposition 8.1.6.2.

Item 4: Interaction With Composition

This follows from Remark 2.4.5.2 and Item 2 of Proposition 8.1.6.2. ∎

2.4.6 Direct Images With Compact Support

Let A and B be sets and let $f\colon A \to B$ be a function.

Definition 2.4.6.1 ▶ Direct Images With Compact Support

The **direct image with compact support function associated to** f is the function

$$f_!\colon \mathcal{P}(A) \to \mathcal{P}(B)$$

defined by[1,2]

$$\begin{aligned} f_!(U) &\stackrel{\text{def}}{=} \left\{ b \in B \;\middle|\; \begin{array}{l} \text{for each } a \in A \text{, if we have} \\ f(a) = b \text{, then } a \in U \end{array} \right\} \\ &= \left\{ b \in B \;\middle|\; \text{we have } f^{-1}(b) \subset U \right\} \end{aligned}$$

for each $U \in \mathcal{P}(A)$.

[1] *Further Terminology:* The set $f_!(U)$ is called the **direct image with compact support of** U **by** f.

[2]We also have

$$f_!(U) = B \setminus f_*(A \setminus U);$$

see Item 9 of Proposition 2.4.6.6.

Notation 2.4.6.2 ▶ Further Notation for Direct Images With Compact Support

Sometimes one finds the notation

$$\forall_f \colon \mathcal{P}(A) \to \mathcal{P}(B)$$

for f_*. This notation comes from the fact that the following statements are equivalent, where $b \in B$ and $U \in \mathcal{P}(A)$:

- We have $b \in \forall_f(U)$.
- For each $a \in A$, if $b = f(a)$, then $a \in U$.

Remark 2.4.6.3 ▶ Unwinding Definition 2.4.6.1

Identifying subsets of A with functions from A to $\{\text{true}, \text{false}\}$ via Items 1 and 2 of Proposition 2.4.3.9, we see that the direct image with compact support function associated to f is equivalently the function

$$f_! \colon \mathcal{P}(A) \to \mathcal{P}(B)$$

defined by

$$\begin{aligned}
f_!(\chi_U) &\overset{\text{def}}{=} \mathrm{Ran}_f(\chi_U) \\
&= \lim\left(\left(\underline{(-_1)} \overset{\to}{\times} f\right) \overset{\mathrm{pr}}{\twoheadrightarrow} A \xrightarrow{\chi_U} \{\text{true}, \text{false}\}\right) \\
&= \lim_{\substack{a \in A \\ f(a) = -_1}} (\chi_U(a)) \\
&= \bigwedge_{\substack{a \in A \\ f(a) = -_1}} (\chi_U(a)).
\end{aligned}$$

where we have used **??** for the second equality. In other words, we have

$$
\begin{aligned}
[f_!(\chi_U)](b) &= \bigwedge_{\substack{a\in A\\ f(a)=b}} (\chi_U(a))\\
&= \begin{cases} \text{true} & \text{if, for each } a\in A \text{ such that}\\ & f(a)=b, \text{ we have } a\in U,\\ \text{false} & \text{otherwise}\end{cases}\\
&= \begin{cases} \text{true} & \text{if } f^{-1}(b)\subset U\\ \text{false} & \text{otherwise}\end{cases}
\end{aligned}
$$

for each $b \in B$.

Definition 2.4.6.4 ▶ The Image and Complement Parts of $f_!$

Let U be a subset of A.[1,2]

1. The **image part of the direct image with compact support** $f_!(U)$ **of** U is the set $f_{!,\mathrm{im}}(U)$ defined by

$$
\begin{aligned}
f_{!,\mathrm{im}}(U) &\overset{\text{def}}{=} f_!(U)\cap \mathrm{Im}(f)\\
&= \left\{ b\in B \,\middle|\, \begin{array}{l}\text{we have } f^{-1}(b)\ \subset\\ U \text{ and } f^{-1}(b)\neq\emptyset\end{array}\right\}.
\end{aligned}
$$

2. The **complement part of the direct image with compact support** $f_!(U)$ **of** U is the set $f_{!,\mathrm{cp}}(U)$ defined by

$$
\begin{aligned}
f_{!,\mathrm{cp}}(U) &\overset{\text{def}}{=} f_!(U)\cap (B\setminus \mathrm{Im}(f))\\
&= B\setminus \mathrm{Im}(f)\\
&= \left\{ b\in B \,\middle|\, \begin{array}{l}\text{we have } f^{-1}(b)\ \subset\\ U \text{ and } f^{-1}(b)=\emptyset\end{array}\right\}\\
&= \left\{ b\in B \,\middle|\, f^{-1}(b)=\emptyset\right\}.
\end{aligned}
$$

[1] Note that we have

$$f_!(U) = f_{!,\mathrm{im}}(U)\cup f_{!,\mathrm{cp}}(U),$$

as

$$
\begin{aligned}
f_!(U) &= f_!(U)\cap B\\
&= f_!(U)\cap(\mathrm{Im}(f)\cup(B\setminus\mathrm{Im}(f)))\\
&= (f_!(U)\cap\mathrm{Im}(f))\cup(f_!(U)\cap(B\setminus\mathrm{Im}(f)))\\
&\overset{\text{def}}{=} f_{!,\mathrm{im}}(U)\cup f_{!,\mathrm{cp}}(U).
\end{aligned}
$$

[2] In terms of the meet computation of $f_!(U)$ of Remark 2.4.6.3, namely

$$f_!(\chi_U) = \bigwedge_{\substack{a\in A\\ f(a)=-_1}} (\chi_U(a)),$$

we see that $f_{!,\mathrm{im}}$ corresponds to meets indexed over nonempty sets, while $f_{!,\mathrm{cp}}$ corresponds to meets indexed over the empty set.

Example 2.4.6.5 ► Examples of Direct Images With Compact Support

Here are some examples of direct images with compact support.

1. *The Multiplication by Two Map on the Natural Numbers.* Consider the function $f\colon \mathbb{N} \to \mathbb{N}$ given by

$$f(n) \overset{\mathrm{def}}{=} 2n$$

for each $n \in \mathbb{N}$. Since f is injective, we have

$$\begin{aligned} f_{!,\mathrm{im}}(U) &= f_*(U) \\ f_{!,\mathrm{cp}}(U) &= \{\text{odd natural numbers}\} \end{aligned}$$

for any $U \subset \mathbb{N}$.

2. *Parabolas.* Consider the function $f\colon \mathbb{R} \to \mathbb{R}$ given by

$$f(x) \overset{\mathrm{def}}{=} x^2$$

for each $x \in \mathbb{R}$. We have

$$f_{!,\mathrm{cp}}(U) = \mathbb{R}_{<0}$$

for any $U \subset \mathbb{R}$. Moreover, since $f^{-1}(x) = \left\{-\sqrt{x}, \sqrt{x}\right\}$, we have e.g.:

$$\begin{aligned} f_{!,\mathrm{im}}([0,1]) &= \{0\}, \\ f_{!,\mathrm{im}}([-1,1]) &= [0,1], \\ f_{!,\mathrm{im}}([1,2]) &= \emptyset, \\ f_{!,\mathrm{im}}([-2,-1]\cup[1,2]) &= [1,4]. \end{aligned}$$

3. *Circles.* Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x,y) \overset{\mathrm{def}}{=} x^2 + y^2$$

for each $(x,y) \in \mathbb{R}^2$. We have

$$f_{!,\mathrm{cp}}(U) = \mathbb{R}_{<0}$$

for any $U \subset \mathbb{R}^2$, and since

$$f^{-1}(r) = \begin{cases} \text{a circle of radius } r \text{ about the origin} & \text{if } r > 0, \\ \{(0,0)\} & \text{if } r = 0, \\ \emptyset & \text{if } r < 0, \end{cases}$$

we have e.g.:

$$\begin{aligned} f_{!,\mathrm{im}}([-1,1] \times [-1,1]) &= [0,1], \\ f_{!,\mathrm{im}}(([-1,1] \times [-1,1]) \setminus [-1,1] \times \{0\}) &= \emptyset. \end{aligned}$$

Proposition 2.4.6.6 ► Properties of Direct Images With Compact Support I

Let $f \colon A \longrightarrow B$ be a function.

1. *Functoriality.* The assignment $U \mapsto f_!(U)$ defines a functor

   $$f_! \colon (\mathcal{P}(A), \subset) \to (\mathcal{P}(B), \subset)$$

   where

   - *Action on Objects.* For each $U \in \mathcal{P}(A)$, we have

     $$[f_!](U) \overset{\text{def}}{=} f_!(U).$$

   - *Action on Morphisms.* For each $U, V \in \mathcal{P}(A)$:

     ($\star$) If $U \subset V$, then $f_!(U) \subset f_!(V)$.

2. *Triple Adjointness.* We have a triple adjunction

   $$(f_* \dashv f^{-1} \dashv f_!)\colon \quad \begin{tikzcd} \mathcal{P}(A) \arrow[r, bend left=50, "f_*"] \arrow[r, bend right=50, "f_!"'] & \mathcal{P}(B), \arrow[l, "f^{-1}"'] \end{tikzcd}$$

   witnessed by bijections of sets

   $$\begin{aligned} \mathrm{Hom}_{\mathcal{P}(B)}(f_*(U), V) &\cong \mathrm{Hom}_{\mathcal{P}(A)}(U, f^{-1}(V)), \\ \mathrm{Hom}_{\mathcal{P}(A)}(f^{-1}(U), V) &\cong \mathrm{Hom}_{\mathcal{P}(A)}(U, f_!(V)), \end{aligned}$$

   natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$ and (respectively) $V \in \mathcal{P}(A)$ and $U \in \mathcal{P}(B)$, i.e. where:

(a) The following conditions are equivalent:

i. We have $f_*(U) \subset V$.

ii. We have $U \subset f^{-1}(V)$.

(b) The following conditions are equivalent:

i. We have $f^{-1}(U) \subset V$.

ii. We have $U \subset f_!(V)$.

3. *Lax Preservation of Colimits.* We have an inclusion of sets

$$\bigcup_{i \in I} f_!(U_i) \subset f_!\left(\bigcup_{i \in I} U_i\right),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have inclusions

$$\begin{aligned} f_!(U) \cup f_!(V) &\hookrightarrow f_!(U \cup V), \\ \emptyset &\hookrightarrow f_!(\emptyset), \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

4. *Preservation of Limits.* We have an equality of sets

$$f_!\left(\bigcap_{i \in I} U_i\right) = \bigcap_{i \in I} f_!(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have equalities

$$\begin{aligned} f^{-1}(U \cap V) &= f_!(U) \cap f^{-1}(V), \\ f_!(A) &= B, \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

5. *Symmetric Lax Monoidality With Respect to Unions.* The direct image with compact support function of Item 1 has a symmetric lax monoidal structure

$$\left(f_!, f_!^{\otimes}, f_{!|\mathbb{1}}^{\otimes}\right) : (\mathcal{P}(A), \cup, \emptyset) \to (\mathcal{P}(B), \cup, \emptyset),$$

being equipped with inclusions

$$\begin{aligned} f_{!|U,V}^{\otimes} : f_!(U) \cup f_!(V) &\hookrightarrow f_!(U \cup V), \\ f_{!|\mathbb{1}}^{\otimes} : \emptyset &\hookrightarrow f_!(\emptyset), \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

6. *Symmetric Strict Monoidality With Respect to Intersections.* The direct image function of Item 1 has a symmetric strict monoidal structure

$$\left(f_!, f_!^{\otimes}, f_{!|\mathbb{1}}^{\otimes}\right) \colon (\mathcal{P}(A), \cap, A) \to (\mathcal{P}(B), \cap, B),$$

being equipped with equalities

$$f_{!|U,V}^{\otimes} \colon f_!(U \cap V) \xrightarrow{=} f_!(U) \cap f_!(V),$$
$$f_{!|\mathbb{1}}^{\otimes} \colon f_!(A) \xrightarrow{=} B,$$

natural in $U, V \in \mathcal{P}(A)$.

7. *Interaction With Coproducts.* Let $f \colon A \to A'$ and $g \colon B \to B'$ be maps of sets. We have

$$(f \coprod g)_!(U \coprod V) = f_!(U) \coprod g_!(V)$$

for each $U \in \mathcal{P}(A)$ and each $V \in \mathcal{P}(B)$.

8. *Interaction With Products.* Let $f \colon A \to A'$ and $g \colon B \to B'$ be maps of sets. We have

$$(f \times g)_!(U \times V) = f_!(U) \times g_!(V)$$

for each $U \in \mathcal{P}(A)$ and each $V \in \mathcal{P}(B)$.

9. *Relation to Direct Images.* We have

$$f_!(U) = B \setminus f_*(A \setminus U)$$

for each $U \in \mathcal{P}(A)$.

10. *Interaction With Injections.* If f is injective, then we have

$$\begin{aligned} f_{!,\mathrm{im}}(U) &= f_*(U), \\ f_{!,\mathrm{cp}}(U) &= B \setminus \mathrm{Im}(f), \\ f_!(U) &= f_{!,\mathrm{im}}(U) \cup f_{!,\mathrm{cp}}(U) \\ &= f_*(U) \cup (B \setminus \mathrm{Im}(f)) \end{aligned}$$

for each $U \in \mathcal{P}(A)$.

11. *Interaction With Surjections.* If f is surjective, then we have

$$\begin{aligned} f_{!,\mathrm{im}}(U) &\subset f_*(U), \\ f_{!,\mathrm{cp}}(U) &= \emptyset, \\ f_!(U) &\subset f_*(U) \end{aligned}$$

for each $U \in \mathcal{P}(A)$.

Proof 2.4.6.7 ► Proof of Proposition 2.4.6.6

Item 1: Functoriality

Clear.

Item 2: Triple Adjointness

This follows from Remark 2.4.4.3, Remark 2.4.5.2, Remark 2.4.6.3, and **??** of **??**.

Item 3: Lax Preservation of Colimits

Omitted.

Item 4: Preservation of Limits

This follows from Item 2 and **??** of **??**.

Item 5: Symmetric Lax Monoidality With Respect to Unions

This follows from Item 3.

Item 6: Symmetric Strict Monoidality With Respect to Intersections

This follows from Item 4.

Item 7: Interaction With Coproducts

Clear.

Item 8: Interaction With Products

Clear.

Item 9: Relation to Direct Images

We claim that $f_!(U) = B \setminus f_*(A \setminus U)$.

- *The First Implication.* We claim that

  $$f_!(U) \subset B \setminus f_*(A \setminus U).$$

  Let $b \in f_!(U)$. We need to show that $b \notin f_*(A \setminus U)$, i.e. that there is no $a \in A \setminus U$ such that $f(a) = b$.

  This is indeed the case, as otherwise we would have $a \in f^{-1}(b)$ and $a \notin U$, contradicting $f^{-1}(b) \subset U$ (which holds since $b \in f_!(U)$).

  Thus $b \in B \setminus f_*(A \setminus U)$.

- *The Second Implication.* We claim that

  $$B \setminus f_*(A \setminus U) \subset f_!(U).$$

  Let $b \in B \setminus f_*(A \setminus U)$. We need to show that $b \in f_!(U)$, i.e. that $f^{-1}(b) \subset U$.

Since $b \notin f_*(A \setminus U)$, there exists no $a \in A \setminus U$ such that $b = f(a)$, and hence $f^{-1}(b) \subset U$.

Thus $b \in f_!(U)$.

This finishes the proof of Item 9.

Item 10: Interaction With Injections

Clear.

Item 11: Interaction With Surjections

Clear.

Proposition 2.4.6.8 ► Properties of Direct Images With Compact Support II

Let $f: A \to B$ be a function.

1. *Functionality I*. The assignment $f \mapsto f_!$ defines a function

$$(-)_{!|A,B}\colon \mathsf{Sets}(A, B) \to \mathsf{Sets}(\mathcal{P}(A), \mathcal{P}(B)).$$

2. *Functionality II*. The assignment $f \mapsto f_!$ defines a function

$$(-)_{!|A,B}\colon \mathsf{Sets}(A, B) \to \mathsf{Pos}((\mathcal{P}(A), \subset), (\mathcal{P}(B), \subset)).$$

3. *Interaction With Identities*. For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$(\mathrm{id}_A)_! = \mathrm{id}_{\mathcal{P}(A)}.$$

4. *Interaction With Composition*. For each pair of composable functions $f: A \to B$ and $g: B \to C$, we have

$$(g \circ f)_! = g_! \circ f_!, \qquad \begin{tikzcd} \mathcal{P}(A) \arrow[r, "f_!"] \arrow[dr, "(g\circ f)_!"'] & \mathcal{P}(B) \arrow[d, "g_!"] \\ & \mathcal{P}(C). \end{tikzcd}$$

Proof 2.4.6.9 ► Proof of Proposition 2.4.6.8

Item 1: Functionality I

Clear.

Item 2: Functionality II

Clear.

Item 3: Interaction With Identities

This follows from Remark 2.4.6.3 and ?? of ??.

Item 4: Interaction With Composition

This follows from Remark 2.4.6.3 and ?? of ??.

Appendices

2.A Other Chapters

Sets

1. Sets
2. Constructions With Sets
3. Pointed Sets
4. Tensor Products of Pointed Sets

Relations

5. Relations
6. Constructions With Relations
7. Equivalence Relations and Apartness Relations

Category Theory

8. Categories

Bicategories

9. Types of Morphisms in Bicategories

Chapter 3

Pointed Sets

This chapter contains some foundational material on pointed sets.

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3.1 Pointed Sets

3.1.1 Foundations

Definition 3.1.1.1 ▶ Pointed Sets

A **pointed set**[1] is equivalently:

- An $\mathbb{E}_0$-monoid in $(\mathsf{N}_\bullet(\mathsf{Sets}), \mathrm{pt})$.
- A pointed object in $(\mathsf{Sets}, \mathrm{pt})$.

[1] *Further Terminology:* In the context of monoids with zero as models for $\mathbb{F}_1$-algebras, pointed sets are viewed as **$\mathbb{F}_1$-modules**.

Remark 3.1.1.2 ▶ Unwinding Definition 3.1.1.1

In detail, a **pointed set** is a pair (X, x_0) consisting of:

- *The Underlying Set.* A set X, called the **underlying set of** (X, x_0).
- *The Basepoint.* A morphism

$$[x_0]\colon \mathrm{pt} \to X$$

in Sets, determining an element $x_0 \in X$, called the **basepoint of** X.

Example 3.1.1.3 ▶ The Zero Sphere

The **0-sphere**[1] is the pointed set $(S^0, 0)$[2] consisting of:

- *The Underlying Set.* The set S^0 defined by

$$S^0 \overset{\mathrm{def}}{=} \{0, 1\}.$$

- *The Basepoint.* The element 0 of S^0.

[1] *Further Terminology:* In the context of monoids with zero as models for $\mathbb{F}_1$-algebras, the 0-sphere is viewed as the **underlying pointed set of the field with one element**.

[2] *Further Notation:* In the context of monoids with zero as models for $\mathbb{F}_1$-algebras, S^0 is also denoted $(\mathbb{F}_1, 0)$.

Example 3.1.1.4 ▶ The Trivial Pointed Set

The **trivial pointed set** is the pointed set $(\mathrm{pt}, \star)$ consisting of:

- *The Underlying Set.* The punctual set $\mathrm{pt} \overset{\mathrm{def}}{=} \{\star\}$.
- *The Basepoint.* The element $\star$ of pt.

Example 3.1.1.5 ▶ The Underlying Pointed Set of a Semimodule

The **underlying pointed set** of a semimodule (M, α_M) is the pointed set $(M, 0_M)$.

Example 3.1.1.6 ▶ The Underlying Pointed Set of a Module

The **underlying pointed set** of a module (M, α_M) is the pointed set $(M, 0_M)$.

3.1.2 Morphisms of Pointed Sets

Definition 3.1.2.1 ▶ Morphisms of Pointed Sets

A **morphism of pointed sets**[1,2] is equivalently:

- A morphism of $\mathbb{E}_0$-monoids in $(\mathsf{N}_\bullet(\mathsf{Sets}), \mathrm{pt})$.
- A morphism of pointed objects in $(\mathsf{Sets}, \mathrm{pt})$.

[1] *Further Terminology:* Also called a **pointed function**.

[2] *Further Terminology:* In the context of monoids with zero as models for $\mathbb{F}_1$-algebras, morphisms of pointed sets are also called **morphism of $\mathbb{F}_1$-modules**.

Remark 3.1.2.2 ▶ Unwinding Definition 3.1.2.1

In detail, a **morphism of pointed sets** $f\colon (X, x_0) \to (Y, y_0)$ is a morphism of sets $f\colon X \to Y$ such that the diagram

$$\begin{tikzcd} & \mathrm{pt} \arrow[dl, "{[x_0]}"'] \arrow[dr, "{[y_0]}"] & \\ X \arrow[rr, "f"'] & & Y \end{tikzcd}$$

commutes, i.e. such that

$$f(x_0) = y_0.$$

3.1.3 The Category of Pointed Sets

Definition 3.1.3.1 ▶ The Category of Pointed Sets

The **category of pointed sets** is the category Sets_* defined equivalently as

- The homotopy category of the ∞-category $\mathsf{Mon}_{\mathbb{E}_0}(\mathsf{N}_\bullet(\mathsf{Sets}), \mathrm{pt})$ of **??**;

- The category Sets_* of **??**.

Remark 3.1.3.2 ▶ Unwinding Definition 3.1.3.1

In detail, the **category of pointed sets** is the category Sets_* where

- *Objects.* The objects of Sets_* are pointed sets;
- *Morphisms.* The morphisms of Sets_* are morphisms of pointed sets;
- *Identities.* For each $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, the unit map

$$\mathbb{1}^{\mathsf{Sets}_*}_{(X,x_0)} \colon \mathrm{pt} \to \mathsf{Sets}_*((X, x_0), (X, x_0))$$

  of Sets_* at (X, x_0) is defined by[1]

$$\mathrm{id}^{\mathsf{Sets}_*}_{(X,x_0)} \overset{\text{def}}{=} \mathrm{id}_X;$$

- *Composition.* For each $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, the composition map

$$\circ^{\mathsf{Sets}_*}_{(X,x_0),(Y,y_0),(Z,z_0)} \colon \mathsf{Sets}_*((Y, y_0), (Z, z_0)) \times \mathsf{Sets}_*((X, x_0), (Y, y_0)) \to \mathsf{Sets}_*((X, x_0), (Z, z_0))$$

  of Sets_* at $((X, x_0), (Y, y_0), (Z, z_0))$ is defined by[2]

$$g \circ^{\mathsf{Sets}_*}_{(X,x_0),(Y,y_0),(Z,z_0)} f \overset{\text{def}}{=} g \circ f.$$

[1]Note that id_X is indeed a morphism of pointed sets, as we have $\mathrm{id}_X(x_0) = x_0$.

[2]Note that the composition of two morphisms of pointed sets is indeed a morphism of pointed sets, as we have

$$\begin{aligned} g(f(x_0)) &= g(y_0) \\ &= z_0, \end{aligned}$$

or

$$\begin{tikzcd} & \mathrm{pt} \arrow[dl, "{[x_0]}"'] \arrow[d, "{[y_0]}"] \arrow[dr, "{[z_0]}"] & \\ X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z \end{tikzcd}$$

in terms of diagrams.

3.1.4 Elementary Properties of Pointed Sets

PROPOSITION 3.1.4.1 ► ELEMENTARY PROPERTIES OF POINTED SETS

Let (X, x_0) be a pointed set.

1. *Completeness.* The category Sets_* of pointed sets and morphisms between them is complete, having in particular:
   (a) Products, described as in Definition 3.2.3.1;
   (b) Pullbacks, described as in Definition 3.2.4.1;
   (c) Equalisers, described as in Definition 3.2.5.1.
2. *Cocompleteness.* The category Sets_* of pointed sets and morphisms between them is cocomplete, having in particular:
   (a) Coproducts, described as in Definition 3.3.3.1;
   (b) Pushouts, described as in Definition 3.3.4.1;
   (c) Coequalisers, described as in Definition 3.3.5.1.
3. *Failure To Be Cartesian Closed.* The category Sets_* is not Cartesian closed.[1]
4. *Morphisms From the Monoidal Unit.* We have a bijection of sets[2]

$$\mathsf{Sets}_*(S^0, X) \cong X,$$

   natural in $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, internalising also to an isomorphism of pointed sets

$$\mathbf{Sets}_*(S^0, X) \cong (X, x_0),$$

   again natural in $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.
5. *Relation to Partial Functions.* We have an equivalence of categories[3]

$$\mathsf{Sets}_* \overset{\text{eq.}}{\cong} \mathsf{Sets}^{\text{part.}}$$

   between the category of pointed sets and pointed functions between them and the category of sets and partial functions between them, where:
   (a) *From Pointed Sets to Sets With Partial Functions.* The equivalence

$$\xi\colon \mathsf{Sets}_* \xrightarrow{\cong} \mathsf{Sets}^{\text{part.}}$$

   sends:

i. A pointed set (X, x_0) to X.

ii. A pointed function

$$f\colon (X, x_0) \longrightarrow (Y, y_0)$$

to the partial function

$$\xi_f\colon X \longrightarrow Y$$

defined on $f^{-1}(Y \setminus y_0)$ and given by

$$\xi_f(x) \stackrel{\text{def}}{=} f(x)$$

for each $x \in f^{-1}(Y \setminus y_0)$.

(b) *From Sets With Partial Functions to Pointed Sets.* The equivalence

$$\xi^{-1}\colon \mathsf{Sets}^{\mathsf{part.}} \xrightarrow{\cong} \mathsf{Sets}_*$$

sends:

i. A set X is to the pointed set $(X, \star)$ with $\star$ an element that is not in X.

ii. A partial function

$$f\colon X \longrightarrow Y$$

defined on $U \subset X$ to the pointed function

$$\xi_f^{-1}\colon (X, x_0) \longrightarrow (Y, y_0)$$

defined by

$$\xi_f(x) \stackrel{\text{def}}{=} \begin{cases} f(x) & \text{if } x \in U, \\ y_0 & \text{otherwise.} \end{cases}$$

for each $x \in X$.

[1]The category Sets_* does admit monoidal closed structures however; see Tensor Products of Pointed Sets.

[2]In other words, the forgetful functor

$$忘\colon \mathsf{Sets}_* \to \mathsf{Sets}$$

defined on objects by sending a pointed set to its underlying set is corepresentable by S^0.

[3] *Warning:* This is not an isomorphism of categories, only an equivalence.

END TEXTDBEND

Proof 3.1.4.2 ▶ Proof of Proposition 3.1.4.1

Item 1: Completeness

This follows from (the proofs) of Definitions 3.2.3.1, 3.2.4.1 and 3.2.5.1 and **??**.

Item 2: Cocompleteness

This follows from (the proofs) of Definitions 3.3.3.1, 3.3.4.1 and 3.3.5.1 and **??**.

Item 3: Failure To Be Cartesian Closed

See [MSE 2855868].

Item 4: Morphisms From the Monoidal Unit

Since a morphism from S^0 to a pointed set (X, x_0) sends $0 \in S^0$ to x_0 and then can send $1 \in S^0$ to any element of X, we obtain a bijection between pointed maps $S^0 \to X$ and the elements of X.
The isomorphism then

$$\mathbf{Sets}_*(S^0, X) \cong (X, x_0)$$

follows by noting that $\Delta_{x_0} \colon S^0 \to X$, the basepoint of $\mathbf{Sets}_*(S^0, X)$, corresponds to the pointed map $S^0 \to X$ picking the element x_0 of X, and thus we see that the bijection between pointed maps $S^0 \to X$ and elements of X is compatible with basepoints, lifting to an isomorphism of pointed sets.

Item 5: Relation to Partial Functions

See [MSE 884460].

3.2 Limits of Pointed Sets

3.2.1 The Terminal Pointed Set

Definition 3.2.1.1 ▶ The Terminal Pointed Set

The **terminal pointed set** is the pair $\left((\mathrm{pt}, \star), \{!_X\}_{(X,x_0)\in\mathrm{Obj}(\mathsf{Sets}_*)}\right)$ consisting of:

- *The Limit.* The pointed set $(\mathrm{pt}, \star)$.
- *The Cone.* The collection of morphisms of pointed sets

  $$\{!_X \colon (X, x_0) \to (\mathrm{pt}, \star)\}_{(X,x_0)\in\mathrm{Obj}(\mathsf{Sets})}$$

  defined by

  $$!_X(x) \stackrel{\text{def}}{=} \star$$

for each $x \in X$ and each $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets})$.

PROOF 3.2.1.2 ▶ PROOF OF DEFINITION 3.2.1.1

We claim that $(\mathrm{pt}, \star)$ is the terminal object of Sets_*. Indeed, suppose we have a diagram of the form

$$(X, x_0) \qquad (\mathrm{pt}, \star)$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi\colon (X, x_0) \longrightarrow (\mathrm{pt}, \star)$$

making the diagram

$$(X, x_0) \overset{\phi}{\underset{\exists!}{\dashrightarrow}} (\mathrm{pt}, \star)$$

commute, namely $!_X$.

3.2.2 Products of Families of Pointed Sets

Let $\left\{(X_i, x_0^i)\right\}_{i \in I}$ be a family of pointed sets.

DEFINITION 3.2.2.1 ▶ THE PRODUCT OF A FAMILY OF POINTED SETS

The **product of** $\left\{(X_i, x_0^i)\right\}_{i \in I}$ is the pair $\left(\left(\prod_{i \in I} X_i, \left(x_0^i\right)_{i \in I}\right), \{\mathrm{pr}_i\}_{i \in I}\right)$ consisting of:

- *The Limit.* The pointed set $\left(\prod_{i \in I} X_i, \left(x_0^i\right)_{i \in I}\right)$.
- *The Cone.* The collection

$$\left\{\mathrm{pr}_i : \left(\prod_{i \in I} X_i, \left(x_0^i\right)_{i \in I}\right) \to \left(X_i, x_0^i\right)\right\}_{i \in I}$$

of maps given by

$$\mathrm{pr}_i\left(\left(x_j\right)_{j \in I}\right) \overset{\text{def}}{=} x_i$$

for each $\left(x_j\right)_{j \in I} \in \prod_{i \in I} X_i$ and each $i \in I$.

Proof 3.2.2.2 ► Proof of Definition 3.2.2.1

We claim that $\left(\prod_{i\in I} X_i, \left(x_0^i\right)_{i\in I}\right)$ is the categorical product of $\left\{\left(X_i, x_0^i\right)\right\}_{i\in I}$ in Sets_*. Indeed, suppose we have, for each $i \in I$, a diagram of the form

$$\begin{tikzcd}
(P,*) \arrow[dr, "p_i"] & \\
\left(\prod_{i\in I} X_i, \left(x_0^i\right)_{i\in I}\right) \arrow[r, "\mathrm{pr}_i"'] & \left(X_i, x_0^i\right)
\end{tikzcd}$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi\colon (P,*) \to \left(\prod_{i\in I} X_i, \left(x_0^i\right)_{i\in I}\right)$$

making the diagram

$$\begin{tikzcd}
(P,*) \arrow[d, dashed, "\phi"', "\exists!"] \arrow[dr, "p_i"] & \\
\left(\prod_{i\in I} X_i, \left(x_0^i\right)_{i\in I}\right) \arrow[r, "\mathrm{pr}_i"'] & \left(X_i, x_0^i\right)
\end{tikzcd}$$

commute, being uniquely determined by the condition $\mathrm{pr}_i \circ \phi = p_i$ for each $i \in I$ via

$$\phi(x) = (p_i(x))_{i\in I}$$

for each $x \in P$. Note that this is indeed a morphism of pointed sets, as we have

$$\begin{aligned}
\phi(*) &= (p_i(*))_{i\in I} \\
&= \left(x_0^i\right)_{i\in I},
\end{aligned}$$

where we have used that p_i is a morphism of pointed sets for each $i \in I$. ∎

Proposition 3.2.2.3 ► Properties of Products of Families of Pointed Sets

Let $\left\{\left(X_i, x_0^i\right)\right\}_{i\in I}$ be a family of pointed sets.

1. *Functoriality*. The assignment $\left\{\left(X_i, x_0^i\right)\right\}_{i\in I} \mapsto \left(\prod_{i\in I} X_i, \left(x_0^i\right)_{i\in I}\right)$ de-

fines a functor

$$\prod_{i \in I} \colon \mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}_*) \to \mathsf{Sets}_*.$$

Proof 3.2.2.4 ▶ Proof of Proposition 3.2.2.3

Item 1: Functoriality

This follows from **??** of **??**.

3.2.3 Products

Let (X, x_0) and (Y, y_0) be pointed sets.

Definition 3.2.3.1 ▶ Products of Pointed Sets

The **product of** (X, x_0) **and** (Y, y_0) is the pair consisting of:

- *The Limit.* The pointed set $(X \times Y, (x_0, y_0))$.
- *The Cone.* The morphisms of pointed sets

  $$\begin{aligned} \mathrm{pr}_1 &\colon (X \times Y, (x_0, y_0)) \to (X, x_0), \\ \mathrm{pr}_2 &\colon (X \times Y, (x_0, y_0)) \to (Y, y_0) \end{aligned}$$

  defined by

  $$\begin{aligned} \mathrm{pr}_1(x, y) &\overset{\text{def}}{=} x, \\ \mathrm{pr}_2(x, y) &\overset{\text{def}}{=} y \end{aligned}$$

  for each $(x, y) \in X \times Y$.

Proof 3.2.3.2 ▶ Proof of Definition 3.2.3.1

We claim that $(X \times Y, (x_0, y_0))$ is the categorical product of (X, x_0) and (Y, y_0) in Sets_*. Indeed, suppose we have a diagram of the form

$$\begin{tikzcd} & (P, *) \arrow[dl, bend right, "p_1"'] \arrow[dr, bend left, "p_2"] & \\ (X, x_0) & (X \times Y, (x_0, y_0)) \arrow[l, "\mathrm{pr}_1"] \arrow[r, "\mathrm{pr}_2"'] & (Y, y_0) \end{tikzcd}$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi\colon (P, *) \to (X \times Y, (x_0, y_0))$$

making the diagram

$$\begin{tikzcd}
 & (P, *) \arrow[dl, bend right, "p_1"'] \arrow[d, dashed, "\phi", "\exists!"'] \arrow[dr, bend left, "p_2"] & \\
(X, x_0) & (X \times Y, (x_0, y_0)) \arrow[l, "\mathrm{pr}_1"] \arrow[r, "\mathrm{pr}_2"'] & (Y, y_0)
\end{tikzcd}$$

commute, being uniquely determined by the conditions

$$\begin{aligned}
\mathrm{pr}_1 \circ \phi &= p_1,\\
\mathrm{pr}_2 \circ \phi &= p_2
\end{aligned}$$

via

$$\phi(x) = (p_1(x), p_2(x))$$

for each $x \in P$. Note that this is indeed a morphism of pointed sets, as we have

$$\begin{aligned}
\phi(*) &= (p_1(*), p_2(*))\\
&= (x_0, y_0),
\end{aligned}$$

where we have used that p_1 and p_2 are morphisms of pointed sets.

Proposition 3.2.3.3 ▶ Properties of Products of Pointed Sets

Let (X, x_0), (Y, y_0), and (Z, z_0) be pointed sets.

1. *Functoriality.* The assignments

   $$(X, x_0), (Y, y_0), ((X, x_0), (Y, y_0)) \mapsto (X \times Y, (x_0, y_0))$$

   define functors

   $$\begin{aligned}
   X \times -\colon \mathsf{Sets}_* &\to \mathsf{Sets}_*,\\
   - \times Y\colon \mathsf{Sets}_* &\to \mathsf{Sets}_*,\\
   -_1 \times -_2\colon \mathsf{Sets}_* \times \mathsf{Sets}_* &\to \mathsf{Sets}_*,
   \end{aligned}$$

   defined in the same way as the functors of Item 1 of Proposition 2.1.3.3.

2. *Associativity.* We have an isomorphism of pointed sets

$$((X \times Y) \times Z, ((x_0, y_0), z_0)) \cong (X \times (Y \times Z), (x_0, (y_0, z_0)))$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

3. *Unitality.* We have isomorphisms of pointed sets

$$\begin{aligned}(\mathrm{pt}, \star) \times (X, x_0) &\cong (X, x_0),\\ (X, x_0) \times (\mathrm{pt}, \star) &\cong (X, x_0),\end{aligned}$$

natural in $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

4. *Commutativity.* We have an isomorphism of pointed sets

$$(X \times Y, (x_0, y_0)) \cong (Y \times X, (y_0, x_0)),$$

natural in $(X, x_0), (Y, y_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

5. *Symmetric Monoidality.* The triple $(\mathsf{Sets}_*, \times, (\mathrm{pt}, \star))$ is a symmetric monoidal category.

Proof 3.2.3.4 ▶ Proof of Proposition 3.2.3.3

Item 1: Functoriality

This is a special case of functoriality of limits, **??** of **??**.

Item 2: Associativity

This follows from Item 3 of Proposition 2.1.3.3.

Item 3: Unitality

This follows from Item 4 of Proposition 2.1.3.3.

Item 4: Commutativity

This follows from Item 5 of Proposition 2.1.3.3.

Item 5: Symmetric Monoidality

This follows from Item 12 of Proposition 2.1.3.3.

3.2.4 Pullbacks

Let (X, x_0), (Y, y_0), and (Z, z_0) be pointed sets and let $f : (X, x_0) \to (Z, z_0)$ and $g : (Y, y_0) \to (Z, z_0)$ be morphisms of pointed sets.

Definition 3.2.4.1 ▶ Pullbacks of Pointed Sets

The **pullback of** (X, x_0) **and** (Y, y_0) **over** (Z, z_0) **along** (f, g) is the pair consisting of:

- *The Limit.* The pointed set $(X \times_Z Y, (x_0, y_0))$.
- *The Cone.* The morphisms of pointed sets

$$\begin{aligned} \mathrm{pr}_1 \colon (X \times_Z Y, (x_0, y_0)) &\longrightarrow (X, x_0), \\ \mathrm{pr}_2 \colon (X \times_Z Y, (x_0, y_0)) &\longrightarrow (Y, y_0) \end{aligned}$$

  defined by

$$\begin{aligned} \mathrm{pr}_1(x, y) &\stackrel{\text{def}}{=} x, \\ \mathrm{pr}_2(x, y) &\stackrel{\text{def}}{=} y \end{aligned}$$

  for each $(x, y) \in X \times_Z Y$.

Proof 3.2.4.2 ▶ Proof of Definition 3.2.4.1

We claim that $X \times_Z Y$ is the categorical pullback of (X, x_0) and (Y, y_0) over (Z, z_0) with respect to (f, g) in Sets_*. First we need to check that the relevant pullback diagram commutes, i.e. that we have

$$f \circ \mathrm{pr}_1 = g \circ \mathrm{pr}_2, \qquad \begin{tikzcd} (X \times_Z Y, (x_0, y_0)) \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] & (Y, y_0) \arrow[d, "g"] \\ (X, x_0) \arrow[r, "f"'] & (Z, z_0). \end{tikzcd}$$

Indeed, given $(x, y) \in X \times_Z Y$, we have

$$\begin{aligned} [f \circ \mathrm{pr}_1](x, y) &= f(\mathrm{pr}_1(x, y)) \\ &= f(x) \\ &= g(y) \\ &= g(\mathrm{pr}_2(x, y)) \\ &= [g \circ \mathrm{pr}_2](x, y), \end{aligned}$$

where $f(x) = g(y)$ since $(x, y) \in X \times_Z Y$. Next, we prove that $X \times_Z Y$ satisfies the universal property of the pullback. Suppose we have a diagram of the form

$$\begin{tikzcd}
(P,*) \arrow[drr, bend left, "p_2"] \arrow[ddr, bend right, "p_1"'] & & \\
& (X \times_Z Y, (x_0, y_0)) \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & (Y, y_0) \arrow[d, "g"] \\
& (X, x_0) \arrow[r, "f"'] & (Z, z_0)
\end{tikzcd}$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi\colon (P,*) \to (X \times_Z Y, (x_0, y_0))$$

making the diagram

$$\begin{tikzcd}
(P,*) \arrow[drr, bend left, "p_2"] \arrow[ddr, bend right, "p_1"'] \arrow[dr, dashed, "\phi", "\exists!"'] & & \\
& (X \times_Z Y, (x_0, y_0)) \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & (Y, y_0) \arrow[d, "g"] \\
& (X, x_0) \arrow[r, "f"'] & (Z, z_0)
\end{tikzcd}$$

commute, being uniquely determined by the conditions

$$\mathrm{pr}_1 \circ \phi = p_1,$$
$$\mathrm{pr}_2 \circ \phi = p_2$$

via

$$\phi(x) = (p_1(x), p_2(x))$$

for each $x \in P$, where we note that $(p_1(x), p_2(x)) \in X \times Y$ indeed lies in $X \times_Z Y$ by the condition

$$f \circ p_1 = g \circ p_2,$$

which gives

$$f(p_1(x)) = g(p_2(x))$$

for each $x \in P$, so that $(p_1(x), p_2(x)) \in X \times_Z Y$. Lastly, we note that ϕ is indeed a morphism of pointed sets, as we have

$$\begin{aligned}\phi(*) &= (p_1(*), p_2(*)) \\ &= (x_0, y_0),\end{aligned}$$

where we have used that p_1 and p_2 are morphisms of pointed sets. ∎

Proposition 3.2.4.3 ► Properties of Pullbacks of Pointed Sets

Let (X, x_0), (Y, y_0), (Z, z_0), and (A, a_0) be pointed sets.

1. *Functoriality.* The assignment $(X, Y, Z, f, g) \mapsto X \times_{f,Z,g} Y$ defines a functor

$$-_1 \times_{-_3} -_1 \colon \mathsf{Fun}(\mathcal{P}, \mathsf{Sets}_*) \to \mathsf{Sets}_*,$$

where $\mathcal{P}$ is the category that looks like this:

$$\begin{tikzcd} & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet. \end{tikzcd}$$

In particular, the action on morphisms of $-_1 \times_{-_3} -_1$ is given by sending a morphism

$$\begin{tikzcd}
X \times_Z Y \arrow[rr] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & & Y \arrow[dr, "\psi"] \arrow[dd, "g" near start] & \\
& X' \times_{Z'} Y' \arrow[rr] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & & Y' \arrow[dd, "g'"] \\
X \arrow[rr, "f" near start] \arrow[dr, "\phi"'] & & Z \arrow[dr, "\chi"] & \\
& X' \arrow[rr, "f'"'] & & Z'
\end{tikzcd}$$

in $\mathsf{Fun}(\mathcal{P}, \mathsf{Sets}_*)$ to the morphism of pointed sets

$$\xi \colon (X \times_Z Y, (x_0, y_0)) \xrightarrow{\exists!} (X' \times_{Z'} Y', (x'_0, y'_0))$$

given by

$$\xi(x, y) \stackrel{\text{def}}{=} (\phi(x), \psi(y))$$

for each $(x, y) \in X \times_Z Y$, which is the unique morphism of pointed sets making the diagram

$$
\begin{tikzcd}
X \times_Z Y \arrow[rr] \arrow[dr, dashed] \arrow[dd] \arrow[dr, phantom, very near start, "\lrcorner"] & & Y \arrow[dr, "\psi"] \arrow[dd, "g" near start] & \\
& X' \times_{Z'} Y' \arrow[rr] \arrow[dd] \arrow[dr, phantom, very near start, "\lrcorner"] & & Y' \arrow[dd, "g'"] \\
X \arrow[rr, "f" near start] \arrow[dr, "\phi"'] & & Z \arrow[dr, "\chi"] & \\
& X' \arrow[rr, "f'"'] & & Z'
\end{tikzcd}
$$

commute.

2. *Associativity*. Given a diagram

$$
\begin{tikzcd}
X \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \arrow[dr, "h"'] & & Z \arrow[dl, "k"] \\
& W & & V &
\end{tikzcd}
$$

in Sets_*, we have isomorphisms of pointed sets

$$
(X \times_W Y) \times_V Z \cong (X \times_W Y) \times_Y (Y \times_V Z) \cong X \times_W (Y \times_V Z),
$$

where these pullbacks are built as in the diagrams

$$
\begin{tikzcd}[column sep=tiny, row sep=small]
& & (X \times_W Y) \times_Y Z \arrow[dl] \arrow[ddrr] \arrow[d, phantom, "\lrcorner" rotate=-45] & & \\
& X \times_W Y \arrow[dl] \arrow[dr] \arrow[d, phantom, "\lrcorner" rotate=-45] & & & \\
X \arrow[dr, "f"'] & & Y \arrow[dl, "g"'] \arrow[dr, "h"] & & Z, \arrow[dl, "k"] \\
& W & & V &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=tiny, row sep=small]
& & (X \times_W Y) \times_Y (Y \times_V Z) \arrow[dl] \arrow[dr] \arrow[d, phantom, "\lrcorner" rotate=-45] & & \\
& X \times_W Y \arrow[dl] \arrow[dr] \arrow[d, phantom, "\lrcorner" rotate=-45] & & Y \times_V Z \arrow[dl] \arrow[dr] \arrow[d, phantom, "\lrcorner" rotate=-45] & \\
X \arrow[dr, "f"'] & & Y \arrow[dl, "g"'] \arrow[dr, "h"] & & Z, \arrow[dl, "k"] \\
& W & & V &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=tiny, row sep=small]
& & X \times_W (Y \times_V Z) \arrow[ddll] \arrow[dr] \arrow[d, phantom, "\lrcorner" rotate=-45] & & \\
& & & Y \times_V Z \arrow[dl] \arrow[dr] \arrow[d, phantom, "\lrcorner" rotate=-45] & \\
X \arrow[dr, "f"'] & & Y \arrow[dl, "g"'] \arrow[dr, "h"] & & Z. \arrow[dl, "k"] \\
& W & & V &
\end{tikzcd}
$$

3. *Unitality*. We have isomorphisms of pointed sets

$$
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "f"'] \arrow[dr, phantom, very near start, "\lrcorner"] & A \arrow[d, "f"] \\
X \arrow[r, equal] & X
\end{tikzcd}
\qquad
\begin{aligned}
X \times_X A &\cong A, \\
A \times_X X &\cong A,
\end{aligned}
\qquad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, equal] \arrow[dr, phantom, very near start, "\lrcorner"] & X \arrow[d, equal] \\
X \arrow[r, "f"'] & X.
\end{tikzcd}
$$

4. *Commutativity.* We have an isomorphism of pointed sets

$$
\begin{tikzcd}
A \times_X B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "g"] \\
A \arrow[r, "f"'] & X,
\end{tikzcd}
\qquad A \times_X B \cong B \times_X A \qquad
\begin{tikzcd}
B \times_X A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\
B \arrow[r, "g"'] & X.
\end{tikzcd}
$$

5. *Interaction With Products.* We have an isomorphism of pointed sets

$$
X \times_{\mathrm{pt}} Y \cong X \times Y, \qquad
\begin{tikzcd}
X \times Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "!_Y"] \\
X \arrow[r, "!_X"'] & \mathrm{pt}.
\end{tikzcd}
$$

6. *Symmetric Monoidality.* The triple $(\mathsf{Sets}_*, \times_X, X)$ is a symmetric monoidal category.

Proof 3.2.4.4 ▶ Proof of Proposition 3.2.4.3

Item 1: Functoriality

This is a special case of functoriality of co/limits, **??** of **??**, with the explicit expression for ξ following from the commutativity of the cube pullback diagram.

Item 2: Associativity

This follows from Item 2 of Proposition 3.2.4.3.

Item 3: Unitality

This follows from Item 3 of Proposition 2.1.4.5.

Item 4: Commutativity

This follows from Item 4 of Proposition 2.1.4.5.

Item 5: Interaction With Products

This follows from Item 6 of Proposition 2.1.4.5.

Item 6: Symmetric Monoidality

This follows from Item 7 of Proposition 2.1.4.5. □

3.2.5 Equalisers

Let $f, g\colon (X, x_0) \rightrightarrows (Y, y_0)$ be morphisms of pointed sets.

Definition 3.2.5.1 ► Equalisers of Pointed Sets

The **equaliser of** (f, g) is the pair consisting of:

- *The Limit.* The pointed set $(\mathrm{Eq}(f, g), x_0)$.
- *The Cone.* The morphism of pointed sets

$$\mathrm{eq}(f, g) \colon (\mathrm{Eq}(f, g), x_0) \hookrightarrow (X, x_0)$$

given by the canonical inclusion $\mathrm{eq}(f, g) \hookrightarrow \mathrm{Eq}(f, g) \hookrightarrow X$.

Proof 3.2.5.2 ► Proof of Definition 3.2.5.1

We claim that $(\mathrm{Eq}(f, g), x_0)$ is the categorical equaliser of f and g in Sets_*. First we need to check that the relevant equaliser diagram commutes, i.e. that we have

$$f \circ \mathrm{eq}(f, g) = g \circ \mathrm{eq}(f, g),$$

which indeed holds by the definition of the set $\mathrm{Eq}(f, g)$. Next, we prove that $\mathrm{Eq}(f, g)$ satisfies the universal property of the equaliser. Suppose we have a diagram of the form

$$\begin{tikzcd}
(\mathrm{Eq}(f,g), x_0) \arrow[r, hook, "\mathrm{eq}(f,g)"] & (X, x_0) \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & (Y, y_0) \\
(E, *) \arrow[ur, "e"'] & &
\end{tikzcd}$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi \colon (E, *) \to (\mathrm{Eq}(f, g), x_0)$$

making the diagram

$$\begin{tikzcd}
(\mathrm{Eq}(f,g), x_0) \arrow[r, hook, "\mathrm{eq}(f,g)"] & (X, x_0) \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & (Y, y_0) \\
(E, *) \arrow[u, dashed, "\phi", "\exists!"'] \arrow[ur, "e"'] & &
\end{tikzcd}$$

commute, being uniquely determined by the condition

$$\mathrm{eq}(f, g) \circ \phi = e$$

via

$$\phi(x) = e(x)$$

for each $x \in E$, where we note that $e(x) \in A$ indeed lies in $\mathrm{Eq}(f,g)$ by the condition

$$f \circ e = g \circ e,$$

which gives

$$f(e(x)) = g(e(x))$$

for each $x \in E$, so that $e(x) \in \mathrm{Eq}(f,g)$. Lastly, we note that ϕ is indeed a morphism of pointed sets, as we have

$$\begin{aligned}\phi(*) &= e(*)\\ &= x_0,\end{aligned}$$

where we have used that e is a morphism of pointed sets.

PROPOSITION 3.2.5.3 ▶ PROPERTIES OF EQUALISERS OF POINTED SETS

Let (X,x_0) and (Y,y_0) be pointed sets and let $f,g,h\colon (X,x_0) \to (Y,y_0)$ be morphisms of pointed sets.

1. *Associativity.* We have isomorphisms of pointed sets

$$\underbrace{\mathrm{Eq}(f \circ \mathrm{eq}(g,h), g \circ \mathrm{eq}(g,h))}_{=\mathrm{Eq}(f\circ\mathrm{eq}(g,h),h\circ\mathrm{eq}(g,h))} \cong \mathrm{Eq}(f,g,h) \cong \underbrace{\mathrm{Eq}(f \circ \mathrm{eq}(f,g), h \circ \mathrm{eq}(f,g))}_{=\mathrm{Eq}(g\circ\mathrm{eq}(f,g),h\circ\mathrm{eq}(f,g))},$$

   where $\mathrm{Eq}(f,g,h)$ is the limit of the diagram

$$\begin{tikzcd}(X,x_0) \arrow[r, shift left=2, "f"] \arrow[r, dashed, "g" description] \arrow[r, shift right=2, "h"'] & (Y,y_0)\end{tikzcd}$$

   in Sets_*, being explicitly given by

$$\mathrm{Eq}(f,g,h) \cong \{a \in A \mid f(a) = g(a) = h(a)\}.$$

2. *Unitality.* We have an isomorphism of pointed sets

$$\mathrm{Eq}(f,f) \cong X.$$

3. *Commutativity.* We have an isomorphism of pointed sets

$$\mathrm{Eq}(f,g) \cong \mathrm{Eq}(g,f).$$

Proof 3.2.5.4 ► Proof of Proposition 3.2.5.3

Item 1: Associativity

This follows from Item 1 of Proposition 2.1.5.3.

Item 2: Unitality

This follows from Item 2 of Proposition 2.1.5.3.

Item 3: Commutativity

This follows from Item 3 of Proposition 2.1.5.3. □

3.3 Colimits of Pointed Sets

3.3.1 The Initial Pointed Set

Definition 3.3.1.1 ► The Initial Pointed Set

The **initial pointed set** is the pair $\left((\mathrm{pt}, \star), \{\iota_X\}_{(X,x_0)\in\mathrm{Obj}(\mathsf{Sets}_*)}\right)$ consisting of:

- *The Limit.* The pointed set $(\mathrm{pt}, \star)$.
- *The Cone.* The collection of morphisms of pointed sets

  $$\{\iota_X : (\mathrm{pt}, \star) \longrightarrow (X, x_0)\}_{(X,x_0)\in\mathrm{Obj}(\mathsf{Sets})}$$

  defined by

  $$\iota_X(\star) \stackrel{\text{def}}{=} x_0.$$

Proof 3.3.1.2 ► Proof of Definition 3.3.1.1

We claim that $(\mathrm{pt}, \star)$ is the initial object of Sets_*. Indeed, suppose we have a diagram of the form

$$(\mathrm{pt}, \star) \qquad (X, x_0)$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi : (\mathrm{pt}, \star) \longrightarrow (X, x_0)$$

making the diagram

$$(\mathrm{pt}, \star) \overset{\phi}{\underset{\exists!}{\dashrightarrow}} (X, x_0)$$

commute, namely ι_X. □

3.3.2 Coproducts of Families of Pointed Sets

Let $\left\{(X_i, x_0^i)\right\}_{i\in I}$ be a family of pointed sets.

Definition 3.3.2.1 ▶ Coproducts of Families of Pointed Sets

The **coproduct of the family** $\left\{(X_i, x_0^i)\right\}_{i\in I}$, also called their **wedge sum**, is the pair consisting of:

- *The Colimit.* The pointed set $(\bigvee_{i\in I} X_i, p_0)$ consisting of:
  - *The Underlying Set.* The set $\bigvee_{i\in I} X_i$ defined by

    $$\bigvee_{i\in I} X_i \overset{\text{def}}{=} \left(\coprod_{i\in I} X_i\right)/\sim,$$

    where $\sim$ is the equivalence relation on $\coprod_{i\in I} X_i$ given by declaring

    $$(i, x_0^i) \sim \left(j, x_0^j\right)$$

    for each $i, j \in I$.
  - *The Basepoint.* The element p_0 of $\bigvee_{i\in I} X_i$ defined by

    $$\begin{aligned} p_0 &\overset{\text{def}}{=} \left[(i, x_0^i)\right] \\ &= \left[\left(j, x_0^j\right)\right] \end{aligned}$$

    for any $i, j \in I$.
- *The Cocone.* The collection

  $$\left\{\mathrm{inj}_i \colon (X_i, x_0^i) \to \left(\bigvee_{i\in I} X_i, p_0\right)\right\}_{i\in I}$$

  of morphism of pointed sets given by

  $$\mathrm{inj}_i(x) \overset{\text{def}}{=} (i, x)$$

  for each $x \in X_i$ and each $i \in I$.

Proof 3.3.2.2 ▶ Proof of Definition 3.3.2.1

We claim that $(\bigvee_{i\in I} X_i, p_0)$ is the categorical coproduct of $\{(X_i, x_0^i)\}_{i\in I}$ in Sets_*. Indeed, suppose we have, for each $i \in I$, a diagram of the form

$$\begin{tikzcd} & (C,*) \\ (X_i, x_0^i) \arrow[ur, "\iota_i"] \arrow[r, "\mathrm{inj}_i"'] & \left(\bigvee_{i\in I} X_i, p_0\right) \end{tikzcd}$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi\colon \left(\bigvee_{i\in I} X_i, p_0\right) \to (C,*)$$

making the diagram

$$\begin{tikzcd} & (C,*) \\ (X_i, x_0^i) \arrow[ur, "\iota_i"] \arrow[r, "\mathrm{inj}_i"'] & \left(\bigvee_{i\in I} X_i, p_0\right) \arrow[u, dashed, "\phi", "\exists!"'] \end{tikzcd}$$

commute, being uniquely determined by the condition $\phi \circ \mathrm{inj}_i = \iota_i$ for each $i \in I$ via

$$\phi([(i,x)]) = \iota_i(x)$$

for each $[(i,x)] \in \bigvee_{i\in I} X_i$, where we note that ϕ is indeed a morphism of pointed sets, as we have

$$\begin{aligned} \phi(p_0) &= \iota_i([(i, x_0^i)]) \\ &= *, \end{aligned}$$

as ι_i is a morphism of pointed sets. □

Proposition 3.3.2.3 ▶ Properties of Coproducts of Families of Pointed Sets

Let $\{(X_i, x_0^i)\}_{i\in I}$ be a family of pointed sets.

1. *Functoriality*. The assignment $\{(X_i, x_0^i)\}_{i\in I} \mapsto (\bigvee_{i\in I} X_i, p_0)$ defines a

functor

$$\bigvee_{i\in I}\colon \mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}_*) \to \mathsf{Sets}_*.$$

Proof 3.3.2.4 ► Proof of Proposition 3.3.2.3

Item 1: Functoriality

This follows from **??** of **??**.

3.3.3 Coproducts

Let (X, x_0) and (Y, y_0) be pointed sets.

Definition 3.3.3.1 ► Coproducts of Pointed Sets

The **coproduct of** (X, x_0) **and** (Y, y_0), also called their **wedge sum**, is the pair consisting of:

- *The Colimit.* The pointed set $(X \vee Y, p_0)$ consisting of:
  - *The Underlying Set.* The set $X \vee Y$ defined by

$$\begin{aligned}(X \vee Y, p_0) &\overset{\text{def}}{=} (X, x_0) \textstyle\coprod (Y, y_0)\\ &\cong \left(X \textstyle\coprod_{\mathrm{pt}} Y, p_0\right)\\ &\cong (X \textstyle\coprod Y/\sim, p_0),\end{aligned} \qquad \begin{tikzcd} X \vee Y \arrow[dr, phantom, "\ulcorner", very near start] & Y \arrow[l] \\ X \arrow[u] & \mathrm{pt}, \arrow[l, hook', "{[x_0]}"] \arrow[u, hook', "{[y_0]}"'] \end{tikzcd}$$

    where $\sim$ is the equivalence relation on $X \coprod Y$ obtained by declaring $(0, x_0) \sim (1, y_0)$.
  - *The Basepoint.* The element p_0 of $X \vee Y$ defined by

$$\begin{aligned}p_0 &\overset{\text{def}}{=} [(0, x_0)]\\ &= [(1, y_0)].\end{aligned}$$

- *The Cocone.* The morphisms of pointed sets

$$\begin{aligned}\mathrm{inj}_1 &\colon (X, x_0) \to (X \vee Y, p_0),\\ \mathrm{inj}_2 &\colon (Y, y_0) \to (X \vee Y, p_0),\end{aligned}$$

  given by

$$\begin{aligned}\mathrm{inj}_1(x) &\overset{\text{def}}{=} [(0, x)],\\ \mathrm{inj}_2(y) &\overset{\text{def}}{=} [(1, y)],\end{aligned}$$

for each $x \in X$ and each $y \in Y$.

Proof 3.3.3.2 ► Proof of Definition 3.3.3.1

We claim that $(X \vee Y, p_0)$ is the categorical coproduct of (X, x_0) and (Y, y_0) in Sets_*. Indeed, suppose we have a diagram of the form

$$\begin{tikzcd} & (C,*) & \\ (X,x_0) \arrow[ur, bend left, "\iota_X"] \arrow[r, "\mathrm{inj}_X"'] & (X \vee Y, p_0) & (Y,y_0) \arrow[ul, bend right, "\iota_Y"'] \arrow[l, "\mathrm{inj}_Y"] \end{tikzcd}$$

in Sets. Then there exists a unique morphism of pointed sets

$$\phi\colon (X \vee Y, p_0) \to (C, *)$$

making the diagram

$$\begin{tikzcd} & (C,*) & \\ (X,x_0) \arrow[ur, bend left, "\iota_X"] \arrow[r, "\mathrm{inj}_X"'] & (X \vee Y, p_0) \arrow[u, dashed, "\phi", "\exists!"'] & (Y,y_0) \arrow[ul, bend right, "\iota_Y"'] \arrow[l, "\mathrm{inj}_Y"] \end{tikzcd}$$

commute, being uniquely determined by the conditions

$$\begin{aligned} \phi \circ \mathrm{inj}_X &= \iota_X, \\ \phi \circ \mathrm{inj}_Y &= \iota_Y \end{aligned}$$

via

$$\phi(z) = \begin{cases} \iota_X(x) & \text{if } z = [(0,x)] \text{ with } x \in X, \\ \iota_Y(y) & \text{if } z = [(1,y)] \text{ with } y \in Y \end{cases}$$

for each $z \in X \vee Y$, where we note that ϕ is indeed a morphism of pointed sets, as we have

$$\begin{aligned} \phi(p_0) &= \iota_X([(0,x_0)]) \\ &= \iota_Y([(1,y_0)]) \\ &= *, \end{aligned}$$

as ι_X and ι_Y are morphisms of pointed sets. ∎

Proposition 3.3.3.3 ▶ Properties of Wedge Sums of Pointed Sets

Let (X, x_0) and (Y, y_0) be pointed sets.

1. *Functoriality.* The assignments
$$(X, x_0), (Y, y_0), ((X, x_0), (Y, y_0)) \mapsto (X \vee Y, p_0)$$
define functors
$$\begin{aligned} X \vee -\colon \mathsf{Sets}_* &\to \mathsf{Sets}_*, \\ - \vee Y\colon \mathsf{Sets}_* &\to \mathsf{Sets}_*, \\ -_1 \vee -_2\colon \mathsf{Sets}_* \times \mathsf{Sets}_* &\to \mathsf{Sets}_*. \end{aligned}$$

2. *Associativity.* We have an isomorphism of pointed sets
$$(X \vee Y) \vee Z \cong X \vee (Y \vee Z),$$
natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathsf{Sets}_*$.

3. *Unitality.* We have isomorphisms of pointed sets
$$\begin{aligned} (\mathrm{pt}, *) \vee (X, x_0) &\cong (X, x_0), \\ (X, x_0) \vee (\mathrm{pt}, *) &\cong (X, x_0), \end{aligned}$$
natural in $(X, x_0) \in \mathsf{Sets}_*$.

4. *Commutativity.* We have an isomorphism of pointed sets
$$X \vee Y \cong Y \vee X,$$
natural in $(X, x_0), (Y, y_0) \in \mathsf{Sets}_*$.

5. *Symmetric Monoidality.* The triple $(\mathsf{Sets}_*, \vee, \mathrm{pt})$ is a symmetric monoidal category.

6. *The Fold Map.* We have a natural transformation
$$\nabla\colon \vee \circ \Delta^{\mathsf{Cats}}_{\mathsf{Sets}_*} \Longrightarrow \mathrm{id}_{\mathsf{Sets}_*}, \qquad \begin{tikzcd} & \mathsf{Sets}_* \times \mathsf{Sets}_* \arrow[dr, "\vee"] \arrow[d, phantom, "\Downarrow\nabla"] & \\ \mathsf{Sets}_* \arrow[ur, "\Delta^{\mathsf{Cats}}_{\mathsf{Sets}_*}"] \arrow[rr, bend right, "\mathrm{id}_{\mathsf{Sets}_*}"'] & & \mathsf{Sets}_*, \end{tikzcd}$$

called the **fold map**, whose component

$$\nabla_X : X \vee X \to X$$

at X is given by

$$\nabla_X(p) \overset{\text{def}}{=} \begin{cases} x & \text{if } p = [(0,x)], \\ x & \text{if } p = [(1,x)] \end{cases}$$

for each $p \in X \vee X$.

Proof 3.3.3.4 ► Proof of Proposition 3.3.3.3

Item 1: Functoriality

This follows from **??** of **??**.

Item 2: Associativity

Clear.

Item 3: Unitality

Clear.

Item 4: Commutativity

Clear.

Item 5: Symmetric Monoidality

Omitted.

Item 6: The Fold Map

Naturality for the transformation ∇ is the statement that, given a morphism of pointed sets $f : (X, x_0) \to (Y, y_0)$, we have

$$\nabla_Y \circ (f \vee f) = f \circ \nabla_X, \qquad \begin{tikzcd} X \vee X \arrow[r, "\nabla_X"] \arrow[d, "f \vee f"'] & X \arrow[d, "f"] \\ Y \vee Y \arrow[r, "\nabla_Y"'] & Y. \end{tikzcd}$$

Indeed, we have

$$
\begin{aligned}
[\nabla_Y \circ (f \vee f)]([(i,x)]) &= \nabla_Y([(i,f(x))]) \\
&= f(x) \\
&= f(\nabla_X([(i,x)])) \\
&= [f \circ \nabla_X]([(i,x)])
\end{aligned}
$$

for each $[(i,x)] \in X \vee X$, and thus ∇ is indeed a natural transformation. ∎

3.3.4 Pushouts

Let (X, x_0), (Y, y_0), and (Z, z_0) be pointed sets and let $f\colon (Z, z_0) \to (X, x_0)$ and $g\colon (Z, z_0) \to (Y, y_0)$ be morphisms of pointed sets.

Definition 3.3.4.1 ▶ Pushouts of Pointed Sets

The **pushout of** (X, x_0) **and** (Y, y_0) **over** (Z, z_0) **along** (f, g) is the pair consisting of:

- *The Colimit.* The pointed set $\left(X \coprod_{f,Z,g} Y, p_0\right)$, where:
  - The set $X \coprod_{f,Z,g} Y$ is the pushout (of unpointed sets) of X and Y over Z with respect to f and g;
  - We have $p_0 = [x_0] = [y_0]$.
- *The Cocone.* The morphisms of pointed sets

  $$
  \begin{aligned}
  \mathrm{inj}_1 &\colon (X, x_0) \to (X \textstyle\coprod_Z Y, p_0), \\
  \mathrm{inj}_2 &\colon (Y, y_0) \to (X \textstyle\coprod_Z Y, p_0)
  \end{aligned}
  $$

  given by

  $$
  \begin{aligned}
  \mathrm{inj}_1(x) &\stackrel{\text{def}}{=} [(0, x)] \\
  \mathrm{inj}_2(y) &\stackrel{\text{def}}{=} [(1, y)]
  \end{aligned}
  $$

  for each $x \in X$ and each $y \in Y$.

PROOF 3.3.4.2 ► PROOF OF DEFINITION 3.3.4.1

Firstly, we note that indeed $[x_0] = [y_0]$, as we have

$$\begin{aligned} x_0 &= f(z_0), \\ y_0 &= g(z_0) \end{aligned}$$

since f and g are morphisms of pointed sets, with the relation $\sim$ on $X \coprod_Z Y$ then identifying $x_0 = f(z_0) \sim g(z_0) = y_0$.
We now claim that $(X \coprod_Z Y, p_0)$ is the categorical pushout of (X, x_0) and (Y, y_0) over (Z, z_0) with respect to (f, g) in Sets_*. First we need to check that the relevant pushout diagram commutes, i.e. that we have

$$\mathrm{inj}_1 \circ f = \mathrm{inj}_2 \circ g, \qquad \begin{tikzcd} (X \coprod_Z Y, p_0) & (Y, y_0) \arrow[l, "\mathrm{inj}_2"'] \\ (X, x_0) \arrow[u, "\mathrm{inj}_1"] & (Z, z_0). \arrow[l, "f"] \arrow[u, "g"'] \end{tikzcd}$$

Indeed, given $z \in Z$, we have

$$\begin{aligned} [\mathrm{inj}_1 \circ f](z) &= \mathrm{inj}_1(f(z)) \\ &= [(0, f(z))] \\ &= [(1, g(z))] \\ &= \mathrm{inj}_2(g(z)) \\ &= [\mathrm{inj}_2 \circ g](z), \end{aligned}$$

where $[(0, f(z))] = [(1, g(z))]$ by the definition of the relation $\sim$ on $X \coprod Y$ (the coproduct of unpointed sets of X and Y). Next, we prove that $X \coprod_Z Y$ satisfies the universal property of the pushout. Suppose we have a diagram of the form

$$\begin{tikzcd} (P, *) & & \\ & (X \coprod_Z Y, p_0) \arrow[ul, phantom] & (Y, y_0) \arrow[l, "\mathrm{inj}_2"'] \arrow[ull, bend right, "\iota_2"'] \\ & (X, x_0) \arrow[u, "\mathrm{inj}_1"] \arrow[uul, bend left, "\iota_1"] \arrow[ur, phantom, "\ulcorner"] & (Z, z_0) \arrow[l, "f"] \arrow[u, "g"'] \end{tikzcd}$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi\colon (X \textstyle\coprod_Z Y, p_0) \to (P, *)$$

making the diagram

$$\begin{tikzcd}
(P,*) & & \\
& (X \coprod_Z Y, p_0) \arrow[ul, dashed, "\exists!" , "\phi"'] & (Y, y_0) \arrow[l, "\mathrm{inj}_2"'] \arrow[ull, bend right, "\iota_2"'] \\
& (X, x_0) \arrow[u, "\mathrm{inj}_1"] \arrow[uul, bend left, "\iota_1"] & (Z, z_0) \arrow[l, "f"] \arrow[u, "g"']
\end{tikzcd}$$

commute, being uniquely determined by the conditions

$$\begin{aligned}
\phi \circ \mathrm{inj}_1 &= \iota_1,\\
\phi \circ \mathrm{inj}_2 &= \iota_2
\end{aligned}$$

via

$$\phi(p) = \begin{cases} \iota_1(x) & \text{if } x = [(0, x)], \\ \iota_2(y) & \text{if } x = [(1, y)] \end{cases}$$

for each $p \in X \coprod_Z Y$, where the well-definedness of ϕ is proven in the same way as in the proof of Definition 2.2.4.1. Finally, we show that ϕ is indeed a morphism of pointed sets, as we have

$$\begin{aligned}
\phi(p_0) &= \phi([(0, x_0)])\\
&= \iota_1(x_0)\\
&= *,
\end{aligned}$$

or alternatively

$$\begin{aligned}
\phi(p_0) &= \phi([(1, y_0)])\\
&= \iota_2(y_0)\\
&= *,
\end{aligned}$$

where we use that ι_1 (resp. ι_2) is a morphism of pointed sets. ∎

Proposition 3.3.4.3 ▶ Properties of Pushouts of Pointed Sets

Let (X, x_0), (Y, y_0), (Z, z_0), and (A, a_0) be pointed sets.

1. *Functoriality.* The assignment $(X, Y, Z, f, g) \mapsto X \coprod_{f,Z,g} Y$ defines a functor
$$-_1 \coprod_{-_3} -_1 \colon \mathsf{Fun}(\mathcal{P}, \mathsf{Sets}) \to \mathsf{Sets}_*,$$
where $\mathcal{P}$ is the category that looks like this:
$$\begin{tikzcd} & \bullet \\ \bullet & \bullet. \arrow[l] \arrow[u] \end{tikzcd}$$
In particular, the action on morphisms of $-_1 \coprod_{-_3} -_1$ is given by sending a morphism
$$\begin{tikzcd}
X \coprod_Z Y \arrow[dr, phantom, very near start, "\ulcorner"] & & Y \arrow[ll] \arrow[dr, "\psi"] & \\
& X' \coprod_{Z'} Y' \arrow[dr, phantom, very near start, "\ulcorner"] & & Y' \arrow[ll] \\
X \arrow[uu] \arrow[dr, "\phi"'] & & Z \arrow[ll, "f"'] \arrow[uu, "g"] \arrow[dr, "\chi"] & \\
& X' \arrow[uu] & & Z' \arrow[ll, "f'"] \arrow[uu, "g'"']
\end{tikzcd}$$
in $\mathsf{Fun}(\mathcal{P}, \mathsf{Sets}_*)$ to the morphism of pointed sets
$$\xi \colon (X \coprod_Z Y, p_0) \xrightarrow{\exists!} (X' \coprod_{Z'} Y', p_0')$$
given by
$$\xi(p) \overset{\text{def}}{=} \begin{cases} \phi(x) & \text{if } p = [(0, x)], \\ \psi(y) & \text{if } p = [(1, y)] \end{cases}$$
for each $p \in X \coprod_Z Y$, which is the unique morphism of pointed sets

making the diagram

$$\begin{tikzcd}
X \coprod_Z Y \arrow[dr, dashed] \arrow[dr, phantom, very near start, "\ulcorner"] & & Y \arrow[ll] \arrow[dr, "\psi"] & \\
& X' \coprod_{Z'} Y' \arrow[dr, phantom, very near start, "\ulcorner"] & & Y' \arrow[ll] \\
X \arrow[uu] \arrow[dr, "\phi"'] & & Z \arrow[ll, "f"'] \arrow[uu, "g"] \arrow[dr, "\chi"] & \\
& X' \arrow[uu] & & Z' \arrow[ll, "f'"] \arrow[uu, "g'"']
\end{tikzcd}$$

commute.

2. *Associativity.* Given a diagram

$$\begin{tikzcd}
X & & Y & & Z \\
& W \arrow[ul, "f"] \arrow[ur, "g"'] & & V \arrow[ul, "h"] \arrow[ur, "k"'] &
\end{tikzcd}$$

in Sets, we have isomorphisms of pointed sets

$$(X \coprod_W Y) \coprod_V Z \cong (X \coprod_W Y) \coprod_Y (Y \coprod_V Z) \cong X \coprod_W (Y \coprod_V Z),$$

where these pullbacks are built as in the diagrams

$$\begin{tikzcd}[column sep=tiny, row sep=small]
& & (X \coprod_W Y) \coprod_V Z & & \\
& X \coprod_W Y \arrow[ur] & & & \\
X \arrow[ur] & & Y \arrow[ul] & & Z, \arrow[uull] \\
& W \arrow[ul, "f"] \arrow[ur, "g"'] & & V \arrow[ul, "h"] \arrow[ur, "k"'] &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=tiny, row sep=small]
& & (X \coprod_W Y) \coprod_Y (Y \coprod_V Z) & & \\
& X \coprod_W Y \arrow[ur] & & Y \coprod_V Z \arrow[ul] & \\
X \arrow[ur] & & Y \arrow[ul] \arrow[ur] & & Z, \arrow[ul] \\
& W \arrow[ul, "f"] \arrow[ur, "g"'] & & V \arrow[ul, "h"] \arrow[ur, "k"'] &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=tiny, row sep=small]
& & X \coprod_W (Y \coprod_V Z) & & \\
& & & Y \coprod_V Z \arrow[ul] & \\
X \arrow[uurr] & & Y \arrow[ur] & & Z. \arrow[ul] \\
& W \arrow[ul, "f"] \arrow[ur, "g"'] & & V \arrow[ul, "h"] \arrow[ur, "k"'] &
\end{tikzcd}$$

3. *Unitality.* We have isomorphisms of sets

$$\begin{tikzcd}
A \arrow[r, equal] \arrow[dr, phantom, very near start, "\ulcorner"] & A \\
X \arrow[u, "f"] \arrow[r, equal] & X \arrow[u, "f"']
\end{tikzcd}
\qquad
\begin{aligned}
X \coprod_X A &\cong A, \\
A \coprod_X X &\cong A,
\end{aligned}
\qquad
\begin{tikzcd}
A \arrow[dr, phantom, very near start, "\ulcorner"] & X \arrow[l, "f"'] \\
X \arrow[u, equal] & X. \arrow[l, "f"] \arrow[u, equal]
\end{tikzcd}$$

4. *Commutativity.* We have an isomorphism of sets

$$\begin{tikzcd} X \coprod_Z Y \arrow[dr, phantom, very near start, "\ulcorner"] & Y \arrow[l] \\ X \arrow[u] & Z, \arrow[l, "f"] \arrow[u, "g"'] \end{tikzcd} \qquad X \coprod_Z Y \cong Y \coprod_Z X \qquad \begin{tikzcd} Y \coprod_Z X \arrow[dr, phantom, very near start, "\ulcorner"] & X \arrow[l] \\ Y \arrow[u] & Z. \arrow[l, "g"] \arrow[u, "f"'] \end{tikzcd}$$

5. *Interaction With Coproducts.* We have

$$X \coprod_{\mathrm{pt}} Y \cong X \vee Y, \qquad \begin{tikzcd} X \vee Y \arrow[dr, phantom, very near start, "\ulcorner"] & Y \arrow[l] \\ X \arrow[u] & \mathrm{pt}. \arrow[l, hook, "{[x_0]}"] \arrow[u, hook, "{[y_0]}"'] \end{tikzcd}$$

6. *Symmetric Monoidality.* The triple $(\mathsf{Sets}_*, \coprod_X, (X, x_0))$ is a symmetric monoidal category.

Proof 3.3.4.4 ▶ Proof of Proposition 3.3.4.3

Item 1: Functoriality

This is a special case of functoriality of co/limits, ?? of ??, with the explicit expression for ξ following from the commutativity of the cube pushout diagram.

Item 2: Associativity

This follows from Item 2 of Proposition 2.2.4.6.

Item 3: Unitality

This follows from Item 3 of Proposition 2.2.4.6.

Item 4: Commutativity

This follows from Item 4 of Proposition 2.2.4.6.

Item 5: Interaction With Coproducts

Clear.

Item 6: Symmetric Monoidality

Omitted. □

3.3.5 Coequalisers

Let $f, g \colon (X, x_0) \rightrightarrows (Y, y_0)$ be morphisms of pointed sets.

Definition 3.3.5.1 ► Coequalisers of Pointed Sets

The **coequaliser of** (f, g) is the pointed set $(\mathrm{CoEq}(f, g), [y_0])$.

Proof 3.3.5.2 ► Proof of Definition 3.3.5.1

We claim that $(\mathrm{CoEq}(f, g), [y_0])$ is the categorical coequaliser of f and g in Sets_*. First we need to check that the relevant coequaliser diagram commutes, i.e. that we have

$$\mathrm{coeq}(f, g) \circ f = \mathrm{coeq}(f, g) \circ g.$$

Indeed, we have

$$\begin{aligned}
[\mathrm{coeq}(f, g) \circ f](x) &\stackrel{\text{def}}{=} [\mathrm{coeq}(f, g)](f(x)) \\
&\stackrel{\text{def}}{=} [f(x)] \\
&= [g(x)] \\
&\stackrel{\text{def}}{=} [\mathrm{coeq}(f, g)](g(x)) \\
&\stackrel{\text{def}}{=} [\mathrm{coeq}(f, g) \circ g](x)
\end{aligned}$$

for each $x \in X$. Next, we prove that $\mathrm{CoEq}(f, g)$ satisfies the universal property of the coequaliser. Suppose we have a diagram of the form

$$\begin{tikzcd}
(X, x_0) \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & (Y, y_0) \arrow[r, hook, "\mathrm{coeq}(f,g)"] \arrow[dr, "c"'] & (\mathrm{CoEq}(f, g), [y_0]) \\
& & (C, *)
\end{tikzcd}$$

in Sets. Then, since $c(f(a)) = c(g(a))$ for each $a \in A$, it follows from Items 4 and 5 of Proposition 7.5.2.3 that there exists a unique map $\phi\colon \mathrm{CoEq}(f, g) \xrightarrow{\exists!} C$ making the diagram

$$\begin{tikzcd}
(X, x_0) \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & (Y, y_0) \arrow[r, hook, "\mathrm{coeq}(f,g)"] \arrow[dr, "c"'] & (\mathrm{CoEq}(f, g), [y_0]) \arrow[d, dashed, "\phi"', "\exists!"] \\
& & (C, *)
\end{tikzcd}$$

commute, where we note that ϕ is indeed a morphism of pointed sets since

$$\begin{aligned}
\phi([y_0]) &= [\phi \circ \mathrm{coeq}(f, g)]([y_0]) \\
&= c([y_0]) \\
&= *,
\end{aligned}$$

where we have used that c is a morphism of pointed sets. ∎

Proposition 3.3.5.3 ▶ Properties of Coequalisers of Pointed Sets

Let (X, x_0) and (Y, y_0) be pointed sets and let $f, g, h \colon (X, x_0) \to (Y, y_0)$ be morphisms of pointed sets.

1. *Associativity.* We have isomorphisms of pointed sets

$$\underbrace{\mathrm{CoEq}(\mathrm{coeq}(f,g) \circ f, \mathrm{coeq}(f,g) \circ h)}_{=\mathrm{CoEq}(\mathrm{coeq}(f,g)\circ g, \mathrm{coeq}(f,g)\circ h)} \cong \mathrm{CoEq}(f, g, h) \cong \underbrace{\mathrm{CoEq}(\mathrm{coeq}(g,h) \circ f, \mathrm{coeq}(g,h) \circ g)}_{=\mathrm{CoEq}(\mathrm{coeq}(g,h)\circ f, \mathrm{coeq}(g,h)\circ h)},$$

   where $\mathrm{CoEq}(f, g, h)$ is the colimit of the diagram

$$\begin{tikzcd} (X, x_0) \arrow[r, shift left=2, "f"] \arrow[r, dashed, "g" description] \arrow[r, shift right=2, "h"'] & (Y, y_0) \end{tikzcd}$$

   in Sets_*.

2. *Unitality.* We have an isomorphism of pointed sets

$$\mathrm{CoEq}(f, f) \cong B.$$

3. *Commutativity.* We have an isomorphism of pointed sets

$$\mathrm{CoEq}(f, g) \cong \mathrm{CoEq}(g, f).$$

Proof 3.3.5.4 ▶ Proof of Proposition 3.3.5.3

Item 1: Associativity

This follows from Item 1 of Proposition 2.2.5.6.

Item 2: Unitality

This follows from Item 2 of Proposition 2.2.5.6.

Item 3: Commutativity

This follows from Item 3 of Proposition 2.2.5.6. ∎

3.4 Constructions With Pointed Sets

3.4.1 Free Pointed Sets

Let X be a set.

Definition 3.4.1.1 ► Free Pointed Sets

The **free pointed set on** X is the pointed set X^+ consisting of:

- *The Underlying Set.* The set X^+ defined by[1]

$$\begin{aligned} X^+ &\overset{\text{def}}{=} X \coprod \mathrm{pt} \\ &\overset{\text{def}}{=} X \coprod \{\star\}. \end{aligned}$$

- *The Basepoint.* The element $\star$ of X^+.

[1] *Further Notation:* We sometimes write $\star_X$ for the basepoint of X^+ for clarity when there are multiple free pointed sets involved in the current discussion.

Proposition 3.4.1.2 ► Properties of Free Pointed Sets

Let X be a set.

1. *Functoriality.* The assignment $X \mapsto X^+$ defines a functor

$$(-)^+ \colon \mathsf{Sets} \to \mathsf{Sets}_*,$$

where

- *Action on Objects.* For each $X \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$[(-)^+](X) \overset{\text{def}}{=} X^+,$$

where X^+ is the pointed set of Definition 3.4.1.1;

- *Action on Morphisms.* For each morphism $f \colon X \to Y$ of Sets, the image

$$f^+ \colon X^+ \to Y^+$$

of f by $(-)^+$ is the map of pointed sets defined by

$$f^+(x) \overset{\text{def}}{=} \begin{cases} f(x) & \text{if } x \in X, \\ \star_Y & \text{if } x = \star_X. \end{cases}$$

2. *Adjointness.* We have an adjunction

$$\left((-)^+ \dashv 忘\right) \colon \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "(-)^+"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_*, \arrow[l, bend left, "忘"] \end{tikzcd}$$

witnessed by a bijection of sets

$$\mathsf{Sets}_*\left((X^+, \star_X), (Y, y_0)\right) \cong \mathsf{Sets}(X, Y),$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets})$ and $(Y, y_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

3. *Symmetric Strong Monoidality With Respect to Wedge Sums.* The free pointed set functor of Item 1 has a symmetric strong monoidal structure

$$\left((-)^{+}, (-)^{+,\coprod}, (-)^{+,\coprod}_{\mathbb{1}}\right) \colon (\mathsf{Sets}, \coprod, \emptyset) \to (\mathsf{Sets}_{*}, \vee, \mathrm{pt}),$$

being equipped with isomorphisms of pointed sets

$$(-)^{+,\coprod}_{X,Y} \colon X^{+} \vee Y^{+} \xrightarrow{\cong} (X \coprod Y)^{+},$$
$$(-)^{+,\coprod}_{\mathbb{1}} \colon \mathrm{pt} \xrightarrow{\cong} \emptyset^{+},$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

4. *Symmetric Strong Monoidality With Respect to Smash Products.* The free pointed set functor of Item 1 has a symmetric strong monoidal structure

$$\left((-)^{+}, (-)^{+,\times}, (-)^{+,\times}_{\mathbb{1}}\right) \colon (\mathsf{Sets}, \times, \mathrm{pt}) \to \left(\mathsf{Sets}_{*}, \wedge, S^{0}\right),$$

being equipped with isomorphisms of pointed sets

$$(-)^{+,\times}_{X,Y} \colon X^{+} \wedge Y^{+} \xrightarrow{\cong} (X \times Y)^{+},$$
$$(-)^{+,\times}_{\mathbb{1}} \colon S^{0} \xrightarrow{\cong} \mathrm{pt}^{+},$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

Proof 3.4.1.3 ► Proof of Proposition 3.4.1.2

Item 1: Functoriality

Clear.

Item 2: Adjointness

We claim there's an adjunction $(-)^{+} \dashv 忘$, witnessed by a bijection of sets

$$\mathsf{Sets}_{*}\left(\left(X^{+}, \star_{X}\right), \left(Y, y_{0}\right)\right) \cong \mathsf{Sets}(X, Y),$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets})$ and $(Y, y_0) \in \mathrm{Obj}(\mathsf{Sets}_{*})$.

- *Map I.* We define a map

  $$\Phi_{X,Y} \colon \mathsf{Sets}_{*}\left(\left(X^{+}, \star_{X}\right), \left(Y, y_{0}\right)\right) \to \mathsf{Sets}(X, Y)$$

  by sending a pointed function

  $$\xi \colon \left(X^{+}, \star_{X}\right) \to \left(Y, y_{0}\right)$$

to the function

$$\xi^{\dagger}\colon X \to Y$$

given by

$$\xi^{\dagger}(x) \overset{\text{def}}{=} \xi(x)$$

for each $x \in X$.

- *Map II.* We define a map

$$\Psi_{X,Y}\colon \mathsf{Sets}(X,Y) \to \mathsf{Sets}_*\big((X^+,\star_X),(Y,y_0)\big)$$

given by sending a function $\xi\colon X \to Y$ to the pointed function

$$\xi^{\dagger}\colon \big(X^+,\star_X\big) \to (Y,y_0)$$

defined by

$$\xi^{\dagger}(x) \overset{\text{def}}{=} \begin{cases} \xi(x) & \text{if } x \in X, \\ y_0 & \text{if } x = \star_X \end{cases}$$

for each $x \in X^+$.

- *Invertibility I.* We claim that

$$\Psi_{X,Y} \circ \Phi_{X,Y} = \mathrm{id}_{\mathsf{Sets}_*((X^+,\star_X),(Y,y_0))},$$

which is clear.

- *Invertibility II.* We claim that

$$\Phi_{X,Y} \circ \Psi_{X,Y} = \mathrm{id}_{\mathsf{Sets}(X,Y)},$$

which is clear.

- *Naturality for Φ, Part I.* We need to show that, given a pointed function $g\colon (Y,y_0) \to \big(Y',y'_0\big)$, the diagram

$$\begin{tikzcd}
\mathsf{Sets}_*((X^+,\star_X),(Y,y_0)) \arrow[r, "\Phi_{X,Y}"] \arrow[d, "g_*"'] & \mathsf{Sets}(X,Y) \arrow[d, "g_*"] \\
\mathsf{Sets}_*\big((X^+,\star_X),\big(Y',y'_0\big)\big), \arrow[r, "\Phi_{X,Y'}"'] & \mathsf{Sets}(X,Y')
\end{tikzcd}$$

commutes. Indeed, given a pointed function

$$\xi^{\dagger}\colon \big(X^+,\star_X\big) \to (Y,y_0)$$

we have

$$
\begin{aligned}
\left[\Phi_{X,Y'} \circ g_*\right](\xi) &= \Phi_{X,Y'}(g_*(\xi))\\
&= \Phi_{X,Y'}(g \circ \xi)\\
&= g \circ \xi\\
&= g \circ \Phi_{X,Y'}(\xi)\\
&= g_*\left(\Phi_{X,Y'}(\xi)\right)\\
&= \left[g_* \circ \Phi_{X,Y'}\right](\xi).
\end{aligned}
$$

- *Naturality for Φ, Part II.* We need to show that, given a pointed function $f\colon (X,x_0) \to (X',x'_0)$, the diagram

$$
\begin{tikzcd}
\mathsf{Sets}_*\left(\left(X'^{,+},\star_X\right),(Y,y_0)\right) \arrow[r, "\Phi_{X',Y}"] \arrow[d, "f^*"'] & \mathsf{Sets}(X',Y) \arrow[d, "f^*"]\\
\mathsf{Sets}_*((X^+,\star_X),(Y,y_0)) \arrow[r, "\Phi_{X,Y}"'] & \mathsf{Sets}(X,Y)
\end{tikzcd}
$$

commutes. Indeed, given a function

$$\xi\colon X' \to Y,$$

we have

$$
\begin{aligned}
\left[\Phi_{X,Y} \circ f^*\right](\xi) &= \Phi_{X,Y}(f^*(\xi))\\
&= \Phi_{X,Y}(\xi \circ f)\\
&= \xi \circ f\\
&= \Phi_{X',Y}(\xi) \circ f\\
&= f^*\left(\Phi_{X',Y}(\xi)\right)\\
&= f^*\left(\Phi_{X',Y}(\xi)\right)\\
&= \left[f^* \circ \Phi_{X',Y}\right](\xi).
\end{aligned}
$$

- *Naturality for Ψ.* Since Φ is natural in each argument and Φ is a componentwise inverse to Ψ in each argument, it follows from Item 2 of Proposition 8.8.6.2 that Ψ is also natural in each argument.

Item 3: Symmetric Strong Monoidality With Respect to Wedge Sums

The isomorphism

$$\phi\colon X^+ \vee Y^+ \xrightarrow{\cong} (X \textstyle\coprod Y)^+$$

is given by

$$\phi(z) = \begin{cases} x & \text{if } z = [(0,x)] \text{ with } x \in X, \\ y & \text{if } z = [(1,y)] \text{ with } y \in Y, \\ \star_{X\coprod Y} & \text{if } z = [(0,\star_X)], \\ \star_{X\coprod Y} & \text{if } z = [(1,\star_Y)] \end{cases}$$

for each $z \in X^+ \vee Y^+$, with inverse

$$\phi^{-1}\colon (X \textstyle\coprod Y)^+ \xrightarrow{\cong} X^+ \vee Y^+$$

given by

$$\phi^{-1}(z) \stackrel{\text{def}}{=} \begin{cases} [(0,x)] & \text{if } z = [(0,x)], \\ [(0,y)] & \text{if } z = [(1,y)], \\ p_0 & \text{if } z = \star_{X\coprod Y} \end{cases}$$

for each $z \in (X \coprod Y)^+$.

Meanwhile, the isomorphism $\mathrm{pt} \cong \emptyset^+$ is given by sending $\star_X$ to $\star_\emptyset$.

That these isomorphisms satisfy the coherence conditions making the functor $(-)^+$ symmetric strong monoidal can be directly checked element by element.

Item 4: Symmetric Strong Monoidality With Respect to Smash Products

The isomorphism

$$\phi\colon X^+ \wedge Y^+ \xrightarrow{\cong} (X \times Y)^+$$

is given by

$$\phi(x \wedge y) = \begin{cases} (x,y) & \text{if } x \neq \star_X \text{ and } y \neq \star_Y \\ \star_{X\times Y} & \text{otherwise} \end{cases}$$

for each $x \wedge y \in X^+ \wedge Y^+$, with inverse

$$\phi^{-1}\colon (X \times Y)^+ \xrightarrow{\cong} X^+ \wedge Y^+$$

given by

$$\phi^{-1}(z) \stackrel{\text{def}}{=} \begin{cases} x \wedge y & \text{if } z = (x,y) \text{ with } (x,y) \in X \times Y, \\ \star_X \wedge \star_Y & \text{if } z = \star_{X\times Y}, \end{cases}$$

for each $z \in (X \coprod Y)^+$.

Meanwhile, the isomorphism $S^0 \cong \mathrm{pt}^+$ is given by sending $\star$ to $1 \in S^0 = \{0, 1\}$ and $\star_{\mathrm{pt}}$ to $0 \in S^0$.

That these isomorphisms satisfy the coherence conditions making the functor $(-)^+$ symmetric strong monoidal can be directly checked element by element.

Appendices

3.A Other Chapters

Sets

1. Sets
2. Constructions With Sets
3. Pointed Sets
4. Tensor Products of Pointed Sets

Relations

5. Relations
6. Constructions With Relations
7. Equivalence Relations and Apartness Relations

Category Theory

8. Categories

Bicategories

9. Types of Morphisms in Bicategories

Chapter 4

Tensor Products of Pointed Sets

In this chapter we introduce, construct, and study tensor products of pointed sets. The most well-known among these is the *smash product of pointed sets*

$$\wedge\colon \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*,$$

introduced in Section 4.5.1, defined via a universal property as inducing a bijection between the following data:

- Pointed maps $f\colon X \wedge Y \to Z$.
- Maps of sets $f\colon X \times Y \to Z$ satisfying

  $$\begin{aligned} f(x_0, y) &= z_0, \\ f(x, y_0) &= z_0 \end{aligned}$$

  for each $x \in X$ and each $y \in Y$.

As it turns out, however, dropping either of the *bilinearity* conditions

$$\begin{aligned} f(x_0, y) &= z_0, \\ f(x, y_0) &= z_0 \end{aligned}$$

while retaining the other leads to two other tensor products of pointed sets,

$$\begin{aligned} \lhd\colon \mathsf{Sets}_* \times \mathsf{Sets}_* &\to \mathsf{Sets}_*, \\ \rhd\colon \mathsf{Sets}_* \times \mathsf{Sets}_* &\to \mathsf{Sets}_*, \end{aligned}$$

called the *left* and *right tensor products of pointed sets*. In contrast to $\wedge$, which turns out to endow Sets_* with a monoidal category structure (Proposition 4.5.9.1), these do not admit invertible associators and unitors, but do endow Sets_* with the structure of a skew monoidal category, however (Propositions 4.3.8.1 and 4.4.8.1).
Finally, in addition to the tensor products $\lhd$, $\rhd$, and $\wedge$, we also have a "tensor product" of the form

$$\odot\colon \mathsf{Sets} \times \mathsf{Sets}_* \to \mathsf{Sets}_*,$$

called the *tensor* of sets with pointed sets. All in all, these tensor products assemble into a family of functors of the form

$$\begin{aligned}
\otimes_{k,\ell} &\colon \mathsf{Mon}_{\mathbb{E}_k}(\mathsf{Sets}) \times \mathsf{Mon}_{\mathbb{E}_\ell}(\mathsf{Sets}) \longrightarrow \mathsf{Mon}_{\mathbb{E}_{k+\ell}}(\mathsf{Sets}),\\
\lhd_{i,k} &\colon \mathsf{Mon}_{\mathbb{E}_k}(\mathsf{Sets}) \times \mathsf{Mon}_{\mathbb{E}_k}(\mathsf{Sets}) \longrightarrow \mathsf{Mon}_{\mathbb{E}_k}(\mathsf{Sets}),\\
\rhd_{i,k} &\colon \mathsf{Mon}_{\mathbb{E}_k}(\mathsf{Sets}) \times \mathsf{Mon}_{\mathbb{E}_k}(\mathsf{Sets}) \longrightarrow \mathsf{Mon}_{\mathbb{E}_k}(\mathsf{Sets}),
\end{aligned}$$

where $k, \ell, i \in \mathbb{N}$ with $i \leq k-1$. Together with the Cartesian product $\times$ of Sets, the tensor products studied in this chapter form the cases:

- $(k, \ell) = (-1, -1)$ for the Cartesian product of Sets;
- $(k, \ell) = (0, -1)$ and $(-1, 0)$ for the tensor of sets with pointed sets of Definition 4.2.1.1;
- $(i, k) = (-1, 0)$ for the left and right tensor products of pointed sets of Sections 4.3 and 4.4;
- $(k, \ell) = (-1, -1)$ for the smash product of pointed sets of Section 4.5.

In this chapter, we will carefully define and study bilinearity for pointed sets, as well as all the tensor products described above. Then, in **??**, we will extend these to tensor products involving also monoids and commutative monoids, which will end up covering all cases up to $k, \ell \leq 2$, and hence *all* cases since $\mathbb{E}_k$-monoids on Sets are the same as $\mathbb{E}_2$-monoids on Sets when $k \geq 2$.

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4.1 Bilinear Morphisms of Pointed Sets

4.1.1 Left Bilinear Morphisms of Pointed Sets

Let (X, x_0), (Y, y_0), and (Z, z_0) be pointed sets.

Definition 4.1.1.1 ► Left Bilinear Morphisms of Pointed Sets

A **left bilinear morphism of pointed sets from** $(X \times Y, (x_0, y_0))$ **to** (Z, z_0) is a map of sets

$$f : X \times Y \to Z$$

satisfying the following condition:[1,2]

($\star$) *Left Unital Bilinearity.* The diagram

$$\begin{tikzcd}
& \mathrm{pt}\times\mathrm{pt} \arrow[dr, dashed, "\sim"] & \\
\mathrm{pt}\times Y \arrow[ur, "\mathrm{id}_{\mathrm{pt}}\times\epsilon_Y"] \arrow[d, "{[x_0]\times\mathrm{id}_Y}"'] & & \mathrm{pt} \arrow[d, "{[z_0]}"] \\
X\times Y \arrow[rr, "f"'] & & Z
\end{tikzcd}$$

commutes, i.e. for each $y \in Y$, we have

$$f(x_0, y) = z_0.$$

[1] *Slogan:* The map f is left bilinear if it preserves basepoints in its first argument.

[2] Succinctly, f is bilinear if we have

$$f(x_0, y) = z_0$$

for each $y \in Y$.

Definition 4.1.1.2 ► The Set of Left Bilinear Morphisms of Pointed Sets

The **set of left bilinear morphisms of pointed sets from** $(X \times Y, (x_0, y_0))$ **to** (Z, z_0) is the set $\mathrm{Hom}^{\otimes,\mathrm{L}}_{\mathsf{Sets}_*}(X \times Y, Z)$ defined by

$$\mathrm{Hom}^{\otimes,\mathrm{L}}_{\mathsf{Sets}_*}(X \times Y, Z) \stackrel{\text{def}}{=} \{f \in \mathrm{Hom}_{\mathsf{Sets}}(X \times Y, Z) \mid f \text{ is left bilinear}\}.$$

4.1.2 Right Bilinear Morphisms of Pointed Sets

Let (X, x_0), (Y, y_0), and (Z, z_0) be pointed sets.

Definition 4.1.2.1 ► Right Bilinear Morphisms of pointed sets

A **right bilinear morphism of pointed sets from** $(X \times Y, (x_0, y_0))$ **to** (Z, z_0) is a map of sets

$$f\colon X \times Y \to Z$$

satisfying the following condition:[1,2]

(★) *Right Unital Bilinearity.* The diagram

$$\begin{tikzcd} & \mathrm{pt} \times \mathrm{pt} \arrow[dr, dashed, "\sim"] & \\ X \times \mathrm{pt} \arrow[ur, "\epsilon_X \times \mathrm{id}_{\mathrm{pt}}"] \arrow[d, "\mathrm{id}_X \times [y_0]"'] & & \mathrm{pt} \arrow[d, "{[z_0]}"] \\ X \times Y \arrow[rr, "f"'] & & Z \end{tikzcd}$$

commutes, i.e. for each $x \in X$, we have

$$f(x, y_0) = z_0.$$

[1] *Slogan:* The map f is right bilinear if it preserves basepoints in its second argument.
[2] Succinctly, f is bilinear if we have

$$f(x, y_0) = z_0$$

for each $x \in X$.

Definition 4.1.2.2 ► The Set of Right Bilinear Morphisms of Pointed Sets

The **set of right bilinear morphisms of pointed sets from** $(X \times Y, (x_0, y_0))$ **to** (Z, z_0) is the set $\mathrm{Hom}^{\otimes, \mathrm{R}}_{\mathsf{Sets}_*}(X \times Y, Z)$ defined by

$$\mathrm{Hom}^{\otimes, \mathrm{R}}_{\mathsf{Sets}_*}(X \times Y, Z) \stackrel{\mathrm{def}}{=} \{f \in \mathrm{Hom}_{\mathsf{Sets}}(X \times Y, Z) \mid f \text{ is right bilinear}\}.$$

4.1.3 Bilinear Morphisms of Pointed Sets

Let (X, x_0), (Y, y_0), and (Z, z_0) be pointed sets.

Definition 4.1.3.1 ▶ Bilinear Morphisms of pointed sets

A **bilinear morphism of pointed sets from** $(X \times Y, (x_0, y_0))$ **to** (Z, z_0) is a map of sets

$$f: X \times Y \to Z$$

that is both left bilinear and right bilinear.

Remark 4.1.3.2 ▶ Unwinding Definition 4.1.3.1

In detail, a **bilinear morphism of pointed sets from** $(X \times Y, (x_0, y_0))$ **to** (Z, z_0) is a map of sets

$$f: (X \times Y, (x_0, y_0)) \to (Z, z_0)$$

satisfying the following conditions:[1,2]

1. *Left Unital Bilinearity.* The diagram

$$\begin{tikzcd} & \mathrm{pt} \times \mathrm{pt} \arrow[dr, dashed, "\sim"] & \\ \mathrm{pt} \times Y \arrow[ur, "\mathrm{id}_{\mathrm{pt}} \times \epsilon_Y"] \arrow[d, "{[x_0] \times \mathrm{id}_Y}"'] & & \mathrm{pt} \arrow[d, "{[z_0]}"] \\ X \times Y \arrow[rr, "f"'] & & Z \end{tikzcd}$$

commutes, i.e. for each $y \in Y$, we have

$$f(x_0, y) = z_0.$$

2. *Right Unital Bilinearity.* The diagram

$$\begin{tikzcd} & \mathrm{pt} \times \mathrm{pt} \arrow[dr, dashed, "\sim"] & \\ X \times \mathrm{pt} \arrow[ur, "\epsilon_X \times \mathrm{id}_{\mathrm{pt}}"] \arrow[d, "{\mathrm{id}_X \times [y_0]}"'] & & \mathrm{pt} \arrow[d, "{[z_0]}"] \\ X \times Y \arrow[rr, "f"'] & & Z \end{tikzcd}$$

commutes, i.e. for each $x \in X$, we have

$$f(x, y_0) = z_0.$$

[1] *Slogan:* The map f is bilinear if it preserves basepoints in each argument.
[2] Succinctly, f is bilinear if we have

$$\begin{aligned} f(x_0, y) &= z_0, \\ f(x, y_0) &= z_0 \end{aligned}$$

for each $x \in X$ and each $y \in Y$.

Definition 4.1.3.3 ▶ The Set of Bilinear Morphisms of Pointed Sets

The **set of bilinear morphisms of pointed sets from** $(X \times Y, (x_0, y_0))$ **to** (Z, z_0) is the set $\mathrm{Hom}^{\otimes}_{\mathsf{Sets}_*}(X \times Y, Z)$ defined by

$$\mathrm{Hom}^{\otimes}_{\mathsf{Sets}_*}(X \times Y, Z) \overset{\text{def}}{=} \{f \in \mathrm{Hom}_{\mathsf{Sets}}(X \times Y, Z) \mid f \text{ is bilinear}\}.$$

4.2 Tensors and Cotensors of Pointed Sets by Sets

4.2.1 Tensors of Pointed Sets by Sets

Let (X, x_0) be a pointed set and let A be a set.

Definition 4.2.1.1 ▶ Tensors of Pointed Sets by Sets

The **tensor of** (X, x_0) **by** A[1] is the pointed set[2] $A \odot (X, x_0)$ satisfying the following universal property:

(UP) We have a bijection

$$\mathsf{Sets}_*(A \odot X, K) \cong \mathsf{Sets}(A, \mathsf{Sets}_*(X, K)),$$

natural in $(K, k_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

[1] *Further Terminology:* Also called the **copower of** (X, x_0) **by** A.
[2] *Further Notation:* Often written $A \odot X$ for simplicity.

Remark 4.2.1.2 ▶ Unwinding Definition 4.2.1.1

The universal property in Definition 4.2.1.1 is equivalent to the following one:

(UP) We have a bijection

$$\mathsf{Sets}_*(A \odot X, K) \cong \mathsf{Sets}^{\otimes}_{\mathbb{E}_0}(A \times X, K),$$

natural in $(K, k_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, where $\mathsf{Sets}^{\otimes}_{\mathbb{E}_0}(A \times X, K)$ is the set

defined by

$$\mathsf{Sets}^{\otimes}_{\mathbb{E}_0}(A \times X, K) \stackrel{\text{def}}{=} \left\{ f \in \mathsf{Sets}(A \times X, K) \,\middle|\, \begin{array}{l}\text{for each } a \in A\text{, we}\\ \text{have } f(a, x_0) = k_0\end{array} \right\}.$$

Proof 4.2.1.3 ▶ Proof of Remark 4.2.1.2

We claim we have a bijection

$$\mathsf{Sets}(A, \mathsf{Sets}_*(X, K)) \cong \mathsf{Sets}^{\otimes}_{\mathbb{E}_0}(A \times X, K)$$

natural in $(K, k_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$. Indeed, this bijection is a restriction of the bijection

$$\mathsf{Sets}(A, \mathsf{Sets}(X, K)) \cong \mathsf{Sets}(A \times X, K)$$

of Item 2 of Proposition 2.1.3.3:

- A map
$$\begin{aligned} \xi \colon A &\longrightarrow \mathsf{Sets}_*(X, K), \\ a &\longmapsto (\xi_a \colon X \to K), \end{aligned}$$
in $\mathsf{Sets}(A, \mathsf{Sets}_*(X, K))$ gets sent to the map
$$\xi^\dagger \colon A \times X \to K$$
defined by
$$\xi^\dagger(a, x) \stackrel{\text{def}}{=} \xi_a(x)$$
for each $(a, x) \in A \times X$, which indeed lies in $\mathsf{Sets}^{\otimes}_{\mathbb{E}_0}(A \times X, K)$, as we have
$$\begin{aligned} \xi^\dagger(a, x_0) &\stackrel{\text{def}}{=} \xi_a(x_0) \\ &\stackrel{\text{def}}{=} k_0 \end{aligned}$$
for each $a \in A$, where we have used that $\xi_a \in \mathsf{Sets}_*(X, K)$ is a morphism of pointed sets.

- Conversely, a map
$$\xi \colon A \times X \to K$$
in $\mathsf{Sets}^{\otimes}_{\mathbb{E}_0}(A \times X, K)$ gets sent to the map
$$\begin{aligned} \xi^\dagger \colon A &\longrightarrow \mathsf{Sets}_*(X, K), \\ a &\longmapsto \left(\xi^\dagger_a \colon X \to K\right), \end{aligned}$$

where

$$\xi_a^\dagger \colon X \to K$$

is the map defined by

$$\xi_a^\dagger(x) \overset{\text{def}}{=} \xi(a, x)$$

for each $x \in X$, and indeed lies in $\mathsf{Sets}_*(X, K)$, as we have

$$\begin{aligned}\xi_a^\dagger(x_0) &\overset{\text{def}}{=} \xi(a, x_0)\\ &\overset{\text{def}}{=} k_0.\end{aligned}$$

This finishes the proof.

Construction 4.2.1.4 ► Construction of Tensors of Pointed Sets by Sets

Concretely, the **tensor of** (X, x_0) **by** A is the pointed set $A \odot (X, x_0)$ consisting of:

- *The Underlying Set.* The set $A \odot X$ given by

$$A \odot X \cong \bigvee_{a \in A} (X, x_0),$$

where $\bigvee_{a \in A}(X, x_0)$ is the wedge product of the A-indexed family $((X, x_0))_{a \in A}$ of Definition 3.3.2.1.

- *The Basepoint.* The point $[(a, x_0)] = [(a', x_0)]$ of $\bigvee_{a \in A}(X, x_0)$.

Proof 4.2.1.5 ► Proof of Construction 4.2.1.4

(Proven below in a bit.)

Notation 4.2.1.6 ► Elements of Tensors of Pointed Sets by Sets

We write $a \odot x$ for the element $[(a, x)]$ of

$$\begin{aligned}A \odot X &\cong \bigvee_{a \in A}(X, x_0)\\ &\overset{\text{def}}{=} \left(\coprod_{i \in I} X_i\right)\Big/{\sim}.\end{aligned}$$

Remark 4.2.1.7 ► Basepoints of Tensors of Pointed Sets by Sets

Taking the tensor of any element of A with the basepoint x_0 of X leads to the same element in $A \odot X$, i.e. we have

$$a \odot x_0 = a' \odot x_0,$$

for each $a, a' \in A$. This is due to the equivalence relation $\sim$ on

$$\bigvee_{a \in A} (X, x_0) \overset{\text{def}}{=} \coprod_{a \in A} X/\sim$$

identifying (a, x_0) with (a', x_0), so that the equivalence class $a \odot x_0$ is independent from the choice of $a \in A$.

Proof 4.2.1.8 ► Proof of Construction 4.2.1.4

We claim we have a bijection

$$\mathsf{Sets}_*(A \odot X, K) \cong \mathsf{Sets}(A, \mathsf{Sets}_*(X, K))$$

natural in $(K, k_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

- *Map I.* We define a map

  $$\Phi_K : \mathsf{Sets}_*(A \odot X, K) \to \mathsf{Sets}(A, \mathsf{Sets}_*(X, K))$$

  by sending a morphism of pointed sets

  $$\xi : (A \odot X, a \odot x_0) \to (K, k_0)$$

  to the map of sets

  $$\begin{aligned} \xi^\dagger : A &\longrightarrow \mathsf{Sets}_*(X, K), \\ a &\longmapsto (\xi_a : X \to K), \end{aligned}$$

  where

  $$\xi_a : (X, x_0) \to (K, k_0)$$

  is the morphism of pointed sets defined by

  $$\xi_a(x) \overset{\text{def}}{=} \xi(a \odot x)$$

  for each $x \in X$. Note that we have

  $$\begin{aligned} \xi_a(x_0) &\overset{\text{def}}{=} \xi(a \odot x_0) \\ &= k_0, \end{aligned}$$

so that ξ_a is indeed a morphism of pointed sets, where we have used that ξ is a morphism of pointed sets.

- *Map II.* We define a map

$$\Psi_K \colon \mathsf{Sets}(A, \mathsf{Sets}_*(X, K)) \to \mathsf{Sets}_*(A \odot X, K)$$

given by sending a map

$$\begin{aligned} \xi \colon A &\longrightarrow \mathsf{Sets}_*(X, K), \\ a &\longmapsto (\xi_a \colon X \to K), \end{aligned}$$

to the morphism of pointed sets

$$\xi^\dagger \colon (A \odot X, a \odot x_0) \to (K, k_0)$$

defined by

$$\xi^\dagger(a \odot x) \overset{\text{def}}{=} \xi_a(x)$$

for each $a \odot x \in A \odot X$. Note that $\xi^\dagger$ is indeed a morphism of pointed sets, as we have

$$\begin{aligned} \xi^\dagger(a \odot x_0) &\overset{\text{def}}{=} \xi_a(x_0) \\ &= k_0, \end{aligned}$$

where we have used that $\xi(a) \in \mathsf{Sets}_*(X, K)$ is a morphism of pointed sets.

- *Invertibility I.* We claim that

$$\Psi_K \circ \Phi_K = \mathrm{id}_{\mathsf{Sets}_*(A \odot X, K)}.$$

Indeed, given a morphism of pointed sets

$$\xi \colon (A \odot X, a \odot x_0) \to (K, k_0),$$

we have

$$\begin{aligned} [\Psi_K \circ \Phi_K](\xi) &= \Psi_K(\Phi_K(\xi)) \\ &= \Psi_K\big([\![a \mapsto [\![x \mapsto \xi(a \odot x)]\!]]\!]\big) \\ &= \Psi_K\big([\![a' \mapsto [\![x' \mapsto \xi(a' \odot x')]\!]]\!]\big) \\ &= [\![a \odot x \mapsto \mathrm{ev}_x\big(\mathrm{ev}_a\big([\![a' \mapsto [\![x' \mapsto \xi(a' \odot x')]\!]]\!]\big)\big)]\!] \\ &= [\![a \odot x \mapsto \mathrm{ev}_x\big([\![x' \mapsto \xi(a \odot x')]\!]\big)]\!] \\ &= [\![a \odot x \mapsto \xi(a \odot x)]\!] \\ &= \xi. \end{aligned}$$

- *Invertibility II.* We claim that

$$\Phi_K \circ \Psi_K = \mathrm{id}_{\mathsf{Sets}(A,\mathsf{Sets}_*(X,K))}.$$

Indeed, given a morphism $\xi \colon A \to \mathsf{Sets}_*(X,K)$, we have

$$\begin{aligned}
[\Phi_K \circ \Psi_K](\xi) &= \Phi_K(\Psi_K(\xi)) \\
&= \Phi_K(\llbracket a \odot x \mapsto \xi_a(x) \rrbracket) \\
&= \llbracket a \mapsto \llbracket x \mapsto \xi_a(x) \rrbracket \rrbracket \\
&= \llbracket a \mapsto \xi(a) \rrbracket \\
&= \xi.
\end{aligned}$$

- *Naturality of* Φ. We need to show that, given a morphism of pointed sets

$$\phi \colon (K, k_0) \to (K', k'_0),$$

the diagram

$$\begin{tikzcd}
\mathsf{Sets}_*(A \odot X, K) \arrow[r, "\Phi_K"] \arrow[d, "\phi_*"'] & \mathsf{Sets}(A, \mathsf{Sets}_*(X, K)) \arrow[d, "(\phi_*)_*"] \\
\mathsf{Sets}_*(A \odot X, K') \arrow[r, "\Phi_{K'}"'] & \mathsf{Sets}(A, \mathsf{Sets}_*(X, K'))
\end{tikzcd}$$

commutes. Indeed, given a morphism of pointed sets

$$\xi \colon (A \odot X, a \odot x_0) \to (K, k_0),$$

we have

$$\begin{aligned}
[\Phi_{K'} \circ \phi_*](\xi) &= \Phi_{K'}(\phi_*(\xi)) \\
&= \Phi_{K'}(\phi \circ \xi) \\
&= (\phi \circ \xi)^\dagger \\
&= \llbracket a \mapsto \phi \circ \xi(a \odot -) \rrbracket \\
&= \llbracket a \mapsto \phi_*(\xi(a \odot -)) \rrbracket \\
&= (\phi_*)_*(\llbracket a \mapsto \xi(a \odot -\rrbracket)) \\
&= (\phi_*)_*(\Phi_K(\xi)) \\
&= \big[(\phi_*)_* \circ \Phi_K\big](\xi).
\end{aligned}$$

- *Naturality of* Ψ. Since Φ is natural and Φ is a componentwise inverse to Ψ, it follows from Item 2 of Proposition 8.8.6.2 that Ψ is also natural.

This finishes the proof. $\square$

Proposition 4.2.1.9 ► Properties of Tensors of Pointed Sets by Sets

Let (X, x_0) be a pointed set and let A be a set.

1. *Functoriality.* The assignments A, (X, x_0), $(A, (X, x_0))$ define functors

$$A \odot -\colon \mathsf{Sets}_* \to \mathsf{Sets}_*,$$
$$- \odot X\colon \mathsf{Sets} \to \mathsf{Sets}_*,$$
$$-_1 \odot -_2\colon \mathsf{Sets} \times \mathsf{Sets}_* \to \mathsf{Sets}_*.$$

In particular, given:

- A map of sets $f\colon A \to B$;
- A pointed map $\phi\colon (X, x_0) \to (Y, y_0)$;

the induced map

$$f \odot \phi\colon A \odot X \to B \odot Y$$

is given by

$$[f \odot \phi](a \odot x) \overset{\text{def}}{=} f(a) \odot \phi(x)$$

for each $a \odot x \in A \odot X$.

2. *Adjointness I.* We have an adjunction

$$(- \odot X \dashv \mathsf{Sets}_*(X, -))\colon\quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "-\odot X"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_*, \arrow[l, bend left, "\mathsf{Sets}_*(X,-)"] \end{tikzcd}$$

witnessed by a bijection

$$\mathsf{Sets}_*(A \odot X, K) \cong \mathsf{Sets}(A, \mathsf{Sets}_*(X, K)),$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$ and $X, Y \in \mathrm{Obj}(\mathsf{Sets}_*)$.

3. *Adjointness II.* We have an adjunctions

$$(A \odot - \dashv A \pitchfork -)\colon\quad \begin{tikzcd} \mathsf{Sets}_* \arrow[r, bend left, "A\odot -"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_*, \arrow[l, bend left, "A\pitchfork -"] \end{tikzcd}$$

witnessed by a bijection

$$\mathrm{Hom}_{\mathsf{Sets}_*}(A \odot X, Y) \cong \mathrm{Hom}_{\mathsf{Sets}_*}(X, A \pitchfork Y),$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$ and $X, Y \in \mathrm{Obj}(\mathsf{Sets}_*)$.

4. *As a Weighted Colimit.* We have

$$A \odot X \cong \mathrm{colim}^{[A]}(X),$$

where in the right hand side we write:

- A for the functor $A \colon \mathrm{pt} \longrightarrow \mathsf{Sets}$ picking $A \in \mathrm{Obj}(\mathsf{Sets})$;
- X for the functor $X \colon \mathrm{pt} \longrightarrow \mathsf{Sets}_*$ picking $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

5. *Iterated Tensors.* We have an isomorphism of pointed sets

$$A \odot (B \odot X) \cong (A \times B) \odot X,$$

natural in $A, B \in \mathrm{Obj}(\mathsf{Sets})$ and $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

6. *Interaction With Homs.* We have a natural isomorphism

$$\mathsf{Sets}_*(A \odot X, -) \cong A \pitchfork \mathsf{Sets}_*(X, -).$$

7. *The Tensor Evaluation Map.* For each $X, Y \in \mathrm{Obj}(\mathsf{Sets}_*)$, we have a map

$$\mathrm{ev}^{\odot}_{X,Y} \colon \mathsf{Sets}_*(X, Y) \odot X \longrightarrow Y,$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets}_*)$, and given by

$$\mathrm{ev}^{\odot}_{X,Y}(f \odot x) \stackrel{\mathrm{def}}{=} f(x)$$

for each $f \odot x \in \mathsf{Sets}_*(X, Y) \odot X$.

8. *The Tensor Coevaluation Map.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$ and each $X \in \mathrm{Obj}(\mathsf{Sets}_*)$, we have a map

$$\mathrm{coev}^{\odot}_{A,X} \colon A \longrightarrow \mathsf{Sets}_*(X, A \odot X),$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$ and $X \in \mathrm{Obj}(\mathsf{Sets}_*)$, and given by

$$\mathrm{coev}^{\odot}_{A,X}(a) \stackrel{\mathrm{def}}{=} [\![x \mapsto a \odot x]\!]$$

for each $a \in A$.

Proof 4.2.1.10 ▶ Proof of Proposition 4.2.1.9

Item 1: Functoriality

This is the special case of **??** of **??** for when $\mathcal{C} = \mathsf{Sets}_*$.

Item 2: Adjointness I

This is simply a rephrasing of Definition 4.2.1.1.

Item 3: : Adjointness II

This is the special case of **??** of **??** for when $\mathcal{C} = \mathsf{Sets}_*$.

Item 4: As a Weighted Colimit

This is the special case of **??** of **??** for when $\mathcal{C} = \mathsf{Sets}_*$.

Item 5: Iterated Tensors

This is the special case of **??** of **??** for when $\mathcal{C} = \mathsf{Sets}_*$.

Item 6: Interaction With Homs

This is the special case of **??** of **??** for when $\mathcal{C} = \mathsf{Sets}_*$.

Item 7: The Tensor Evaluation Map

This is the special case of **??** of **??** for when $\mathcal{C} = \mathsf{Sets}_*$.

Item 8: The Tensor Coevaluation Map

This is the special case of **??** of **??** for when $\mathcal{C} = \mathsf{Sets}_*$. □

4.2.2 Cotensors of Pointed Sets by Sets

Let (X, x_0) be a pointed set and let A be a set.

Definition 4.2.2.1 ▶ Cotensors of Pointed Sets by Sets

The **cotensor of** (X, x_0) **by** A[1] is the pointed set[2] $A \pitchfork (X, x_0)$ satisfying the following universal property:

(UP) We have a bijection

$$\mathsf{Sets}_*(K, A \pitchfork X) \cong \mathsf{Sets}(A, \mathsf{Sets}_*(K, X)),$$

natural in $(K, k_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

[1] *Further Terminology:* Also called the **power of** (X, x_0) **by** A.
[2] *Further Notation:* Often written $A \pitchfork X$ for simplicity.

Remark 4.2.2.2 ► Unwinding Definition 4.2.2.1

The universal property of Definition 4.2.2.1 is equivalent to the following one:

(**UP**) We have a bijection

$$\mathsf{Sets}_*(K, A \pitchfork X) \cong \mathsf{Sets}^{\otimes}_{\mathbb{E}_0}(A \times K, X),$$

natural in $(K, k_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, where $\mathsf{Sets}^{\otimes}_{\mathbb{E}_0}(A \times K, X)$ is the set defined by

$$\mathsf{Sets}^{\otimes}_{\mathbb{E}_0}(A \times K, X) \stackrel{\text{def}}{=} \left\{ f \in \mathsf{Sets}(A \times K, X) \;\middle|\; \begin{array}{l}\text{for each } a \in A, \text{ we}\\ \text{have } f(a, k_0) = x_0\end{array} \right\}.$$

Proof 4.2.2.3 ► Proof of Remark 4.2.2.2

This follows from the bijection

$$\mathsf{Sets}(A, \mathsf{Sets}_*(K, X)) \cong \mathsf{Sets}^{\otimes}_{\mathbb{E}_0}(A \times K, X),$$

natural in $(K, k_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$ constructed in the proof of Remark 4.2.1.2. ∎

Construction 4.2.2.4 ► Construction of Cotensors of Pointed Sets by Sets

Concretely, the **cotensor of** (X, x_0) **by** A is the pointed set $A \pitchfork (X, x_0)$ consisting of:

- *The Underlying Set.* The set $A \pitchfork X$ given by

  $$A \pitchfork X \cong \bigwedge_{a \in A} (X, x_0),$$

  where $\bigwedge_{a \in A}(X, x_0)$ is the smash product of the A-indexed family $((X, x_0))_{a \in A}$ of Definition 4.6.1.1.

- *The Basepoint.* The point $[(x_0)_{a \in A}] = [(x_0, x_0, x_0, \ldots)]$ of $\bigwedge_{a \in A}(X, x_0)$.

Proof 4.2.2.5 ► Proof of Construction 4.2.2.4

We claim we have a bijection

$$\mathsf{Sets}_*(K, A \pitchfork X) \cong \mathsf{Sets}(A, \mathsf{Sets}_*(K, X)),$$

natural in $(K, k_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

- *Map I.* We define a map

$$\Phi_K : \mathsf{Sets}_*(K, A \pitchfork X) \to \mathsf{Sets}(A, \mathsf{Sets}_*(K, X)),$$

by sending a morphism of pointed sets

$$\xi : (K, k_0) \to (A \pitchfork X, [(x_0)_{a\in A}])$$

to the map of sets

$$\begin{aligned} \xi^\dagger : A &\longrightarrow \mathsf{Sets}_*(K, X), \\ a &\longmapsto (\xi_a : K \to X), \end{aligned}$$

where

$$\xi_a : (K, k_0) \to (X, x_0)$$

is the morphism of pointed sets defined by

$$\xi_a(k) = \begin{cases} x_a^k & \text{if } \xi(k) \neq [(x_0)_{a\in A}], \\ x_0 & \text{if } \xi(k) = [(x_0)_{a\in A}] \end{cases}$$

for each $k \in K$, where x_a^k is the ath component of $\xi(k) = \left[(x_a^k)_{a\in A}\right]$. Note that:

1. The definition of $\xi_a(k)$ is independent of the choice of equivalence class. Indeed, suppose we have

$$\begin{aligned} \xi(k) &= \left[\left(x_a^k\right)_{a\in A}\right] \\ &= \left[\left(y_a^k\right)_{a\in A}\right] \end{aligned}$$

with $x_a^k \neq y_a^k$ for some $a \in A$. Then there exist $a_x, a_y \in A$ such that $x_{a_x}^k = y_{a_y}^k = x_0$. The equivalence relation $\sim$ on $\prod_{a\in A} X$ then forces

$$\begin{aligned} \left[\left(x_a^k\right)_{a\in A}\right] &= [(x_0)_{a\in A}], \\ \left[\left(y_a^k\right)_{a\in A}\right] &= [(x_0)_{a\in A}], \end{aligned}$$

however, and $\xi_a(k)$ is defined to be x_0 in this case.

2. The map ξ_a is indeed a morphism of pointed sets, as we have

$$\xi_a(k_0) = x_0$$

since $\xi(k_0) = [(x_0)_{a\in A}]$ as ξ is a morphism of pointed sets and $\xi_a(k_0)$, defined to be the ath component of $[(x_0)_{a\in A}]$, is equal to x_0.

- *Map II*. We define a map

$$\Psi_K \colon \mathsf{Sets}(A, \mathsf{Sets}_*(K, X)) \to \mathsf{Sets}_*(K, A \pitchfork X),$$

given by sending a map

$$\begin{aligned} \xi \colon A &\longrightarrow \mathsf{Sets}_*(K, X), \\ a &\longmapsto (\xi_a \colon K \to X), \end{aligned}$$

to the morphism of pointed sets

$$\xi^\dagger \colon (K, k_0) \to (A \pitchfork X, [(x_0)_{a\in A}])$$

defined by

$$\xi^\dagger(k) \overset{\text{def}}{=} \left[(\xi_a(k))_{a\in A}\right]$$

for each $k \in K$. Note that $\xi^\dagger$ is indeed a morphism of pointed sets, as we have

$$\begin{aligned} \xi^\dagger(k_0) &\overset{\text{def}}{=} \left[(\xi_a(k_0))_{a\in A}\right] \\ &= x_0, \end{aligned}$$

where we have used that $\xi_a \in \mathsf{Sets}_*(K, X)$ is a morphism of pointed sets for each $a \in A$.

- *Naturality of* Ψ. We need to show that, given a morphism of pointed sets

$$\phi \colon (K, k_0) \to (K', k_0'),$$

the diagram

$$\begin{tikzcd} \mathsf{Sets}(A, \mathsf{Sets}_*(K', X)) \arrow[r, "\Psi_{K'}"] \arrow[d, "(\phi^*)_*"'] & \mathsf{Sets}_*(K', A \pitchfork X) \arrow[d, "\phi^*"] \\ \mathsf{Sets}(A, \mathsf{Sets}_*(K, X)) \arrow[r, "\Psi_K"'] & \mathsf{Sets}_*(K, A \pitchfork X) \end{tikzcd}$$

commutes. Indeed, given a map of sets

$$\begin{aligned} \xi \colon A &\longrightarrow \mathsf{Sets}_*(K', X), \\ a &\longmapsto (\xi_a \colon K' \to X), \end{aligned}$$

we have

$$\begin{aligned} \big[\Psi_K \circ (\phi^*)_*\big](\xi) &= \Psi_K\big((\phi^*)_*(\xi)\big) \\ &= \Psi_K\big((\phi^*)_*(\llbracket a \mapsto \xi_a \rrbracket)\big) \\ &= \Psi_K\big((\llbracket a \mapsto \phi^*(\xi_a) \rrbracket)\big) \\ &= \Psi_K\big((\llbracket a \mapsto \llbracket k \mapsto \xi_a(\phi(k)) \rrbracket \rrbracket)\big) \\ &= \llbracket k \mapsto \big[(\xi_a(\phi(k)))_{a\in A}\big] \rrbracket \\ &= \phi^*(\llbracket k' \mapsto \big[(\xi_a(k'))_{a\in A}\big] \rrbracket) \\ &= \phi^*(\Psi_{K'}(\xi)) \\ &= [\phi^* \circ \Psi_{K'}](\xi). \end{aligned}$$

- *Naturality of* Φ. Since Ψ is natural and Ψ is a componentwise inverse to Φ, it follows from Item 2 of Proposition 8.8.6.2 that Φ is also natural.
- *Invertibility I*. We claim that

$$\Psi_K \circ \Phi_K = \mathrm{id}_{\mathsf{Sets}_*(K, A \pitchfork X)}.$$

Indeed, given a morphism of pointed sets

$$\xi \colon (K, k_0) \to (A \pitchfork X, [(x_0)_{a\in A}])$$

we have

$$\begin{aligned} [\Psi_K \circ \Phi_K](\xi) &= \Psi_K(\Phi_K(\xi)) \\ &= \Psi_K(\llbracket a \mapsto \xi_a \rrbracket) \\ &= \Psi_K(\llbracket a' \mapsto \xi_{a'} \rrbracket) \\ &= \llbracket k \mapsto \big[\big(\mathrm{ev}_a(\llbracket a' \mapsto \xi_{a'}(k) \rrbracket)\big)_{a\in A}\big] \rrbracket \\ &= \llbracket k \mapsto \big[(\xi_a(k))_{a\in A}\big] \rrbracket. \end{aligned}$$

Now, we have two cases:

1. If $\xi(k) = [(x_0)_{a\in A}]$, we have

$$
\begin{aligned}
[\Psi_K \circ \Phi_K](\xi) &= \cdots \\
&= [\![k \mapsto \big[(\xi_a(k))_{a\in A}\big]]\!] \\
&= [\![k \mapsto [(x_0)_{a\in A}]]\!] \\
&= [\![k \mapsto \xi(k)]\!] \\
&= \xi.
\end{aligned}
$$

2. If $\xi(k) \neq [(x_0)_{a\in A}]$ and $\xi(k) = \big[(x_a^k)_{a\in A}\big]$ instead, we have

$$
\begin{aligned}
[\Psi_K \circ \Phi_K](\xi) &= \cdots \\
&= [\![k \mapsto \big[(\xi_a(k))_{a\in A}\big]]\!] \\
&= [\![k \mapsto \left[\left(x_a^k\right)_{a\in A}\right]]\!] \\
&= [\![k \mapsto \xi(k)]\!] \\
&= \xi.
\end{aligned}
$$

In both cases, we have $[\Psi_K \circ \Phi_K](\xi) = \xi$, and thus we are done.

- *Invertibility II.* We claim that

$$\Phi_K \circ \Psi_K = \mathrm{id}_{\mathsf{Sets}(A,\mathsf{Sets}_*(K,X))}.$$

Indeed, given a morphism $\xi \colon A \longrightarrow \mathsf{Sets}_*(K, X)$, we have

$$
\begin{aligned}
[\Phi_K \circ \Psi_K](\xi) &= \Phi_K(\Psi_K(\xi)) \\
&= \Phi_K\big([\![k \mapsto \big[(\xi_a(k))_{a\in A}\big]]\!]\big) \\
&= [\![a \mapsto [\![k \mapsto \xi_a(k)]\!]]\!] \\
&= \xi
\end{aligned}
$$

This finishes the proof. □

Proposition 4.2.2.6 ► Properties of Cotensors of Pointed Sets by Sets

Let (X, x_0) be a pointed set and let A be a set.

1. *Functoriality.* The assignments A, (X, x_0), $(A, (X, x_0))$ define functors

$$\begin{aligned} A \pitchfork - &\colon \mathsf{Sets}_* \to \mathsf{Sets}_*, \\ - \pitchfork X &\colon \mathsf{Sets}^{\mathsf{op}} \to \mathsf{Sets}_*, \\ -_1 \pitchfork -_2 &\colon \mathsf{Sets}^{\mathsf{op}} \times \mathsf{Sets}_* \to \mathsf{Sets}_*. \end{aligned}$$

   In particular, given:

   - A map of sets $f\colon A \to B$;
   - A pointed map $\phi\colon (X, x_0) \to (Y, y_0)$;

   the induced map

$$f \odot \phi\colon A \pitchfork X \to B \pitchfork Y$$

   is given by

$$[f \odot \phi]([(x_a)_{a\in A}]) \overset{\text{def}}{=} \big[\big(\phi\big(x_{f(a)}\big)\big)_{a\in A}\big]$$

   for each $[(x_a)_{a\in A}] \in A \pitchfork X$.

2. *Adjointness I.* We have an adjunction

$$(- \pitchfork X \dashv \mathsf{Sets}_*(-, X))\colon \quad \begin{tikzcd} \mathsf{Sets}^{\mathsf{op}} \arrow[r, bend left, "- \pitchfork X"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_*, \arrow[l, bend left, "\mathsf{Sets}_*(-, X)"] \end{tikzcd}$$

   witnessed by a bijection

$$\mathsf{Sets}^{\mathsf{op}}_*(A \pitchfork X, K) \cong \mathsf{Sets}(A, \mathsf{Sets}_*(K, X)),$$

   i.e. by a bijection

$$\mathsf{Sets}_*(K, A \pitchfork X) \cong \mathsf{Sets}(A, \mathsf{Sets}_*(K, X)),$$

   natural in $A \in \mathrm{Obj}(\mathsf{Sets})$ and $X, Y \in \mathrm{Obj}(\mathsf{Sets}_*)$.

3. *Adjointness II.* We have an adjunctions

$$(A \odot - \dashv A \pitchfork -)\colon \quad \begin{tikzcd} \mathsf{Sets}_* \arrow[r, bend left, "A \odot -"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_*, \arrow[l, bend left, "A \pitchfork -"] \end{tikzcd}$$

   witnessed by a bijection

$$\mathrm{Hom}_{\mathsf{Sets}_*}(A \odot X, Y) \cong \mathrm{Hom}_{\mathsf{Sets}_*}(X, A \pitchfork Y),$$

   natural in $A \in \mathrm{Obj}(\mathsf{Sets})$ and $X, Y \in \mathrm{Obj}(\mathsf{Sets}_*)$.

4. *As a Weighted Limit.* We have

$$A \pitchfork X \cong \lim^{[A]}(X),$$

where in the right hand side we write:

- A for the functor $A\colon \mathrm{pt} \to \mathsf{Sets}$ picking $A \in \mathrm{Obj}(\mathsf{Sets})$;
- X for the functor $X\colon \mathrm{pt} \to \mathsf{Sets}_*$ picking $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

5. *Iterated Cotensors.* We have an isomorphism of pointed sets

$$A \pitchfork (B \pitchfork X) \cong (A \times B) \pitchfork X,$$

natural in $A, B \in \mathrm{Obj}(\mathsf{Sets})$ and $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

6. *Commutativity With Homs.* We have natural isomorphisms

$$A \pitchfork \mathsf{Sets}_*(X, -) \cong \mathsf{Sets}_*(A \odot X, -),$$
$$A \pitchfork \mathsf{Sets}_*(-, Y) \cong \mathsf{Sets}_*(-, A \pitchfork Y).$$

7. *The Cotensor Evaluation Map.* For each $X, Y \in \mathrm{Obj}(\mathsf{Sets}_*)$, we have a map

$$\mathrm{ev}^{\pitchfork}_{X,Y}\colon X \to \mathsf{Sets}_*(X, Y) \pitchfork Y,$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets}_*)$, and given by

$$\mathrm{ev}^{\pitchfork}_{X,Y}(x) \stackrel{\text{def}}{=} \Big[(f(x))_{f \in \mathsf{Sets}_*(X,Y)}\Big]$$

for each $x \in X$.

8. *The Cotensor Coevaluation Map.* For each $X \in \mathrm{Obj}(\mathsf{Sets}_*)$ and each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have a map

$$\mathrm{coev}^{\pitchfork}_{A,X}\colon A \to \mathsf{Sets}_*(A \pitchfork X, X),$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets}_*)$ and $A \in \mathrm{Obj}(\mathsf{Sets})$, and given by

$$\mathrm{coev}^{\pitchfork}_{A,X}(a) \stackrel{\text{def}}{=} [\![[(x_b)_{b \in A}] \mapsto x_a]\!]$$

for each $a \in A$.

Proof 4.2.2.7 ▶ Proof of Proposition 4.2.2.6

Item 1: Functoriality

This is the special case of ?? of ?? for when $\mathcal{C} = \mathsf{Sets}_*$.

Item 2: Adjointness I

This is simply a rephrasing of Definition 4.2.2.1.

Item 3: : Adjointness II

This is the special case of ?? of ?? for when $\mathcal{C} = \mathsf{Sets}_*$.

Item 4: As a Weighted Limit

This is the special case of ?? of ?? for when $\mathcal{C} = \mathsf{Sets}_*$.

Item 5: Iterated Cotensors

This is the special case of ?? of ?? for when $\mathcal{C} = \mathsf{Sets}_*$.

Item 6: Commutativity With Homs

This is the special case of ?? of ?? for when $\mathcal{C} = \mathsf{Sets}_*$.

Item 7: The Cotensor Evaluation Map

This is the special case of ?? of ?? for when $\mathcal{C} = \mathsf{Sets}_*$.

Item 8: The Cotensor Coevaluation Map

This is the special case of ?? of ?? for when $\mathcal{C} = \mathsf{Sets}_*$. □

4.3 The Left Tensor Product of Pointed Sets

4.3.1 Foundations

Let (X, x_0) and (Y, y_0) be pointed sets.

Definition 4.3.1.1 ▶ The Left Tensor Product of Pointed Sets

The **left tensor product of pointed sets** is the functor[1]

$$\lhd \colon \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*$$

defined as the composition

$$\mathsf{Sets}_* \times \mathsf{Sets}_* \xrightarrow{\mathrm{id} \times 忘} \mathsf{Sets}_* \times \mathsf{Sets} \xrightarrow{\beta^{\mathsf{Cats}_2}_{\mathsf{Sets}_*, \mathsf{Sets}}} \mathsf{Sets} \times \mathsf{Sets}_* \xrightarrow{\odot} \mathsf{Sets}_*,$$

where:

- $忘 \colon \mathsf{Sets}_* \to \mathsf{Sets}$ is the forgetful functor from pointed sets to sets.

- $\beta^{\mathsf{Cats}_2}_{\mathsf{Sets}_*,\mathsf{Sets}} \colon \mathsf{Sets}_* \times \mathsf{Sets} \xrightarrow{\cong} \mathsf{Sets} \times \mathsf{Sets}_*$ is the braiding of Cats_2, i.e. the functor witnessing the isomorphism

$$\mathsf{Sets}_* \times \mathsf{Sets} \cong \mathsf{Sets} \times \mathsf{Sets}_*.$$

- $\odot \colon \mathsf{Sets} \times \mathsf{Sets}_* \to \mathsf{Sets}_*$ is the tensor functor of Item 1 of Proposition 4.2.1.9.

[1] *Further Notation:* Also written $\lhd_{\mathsf{Sets}_*}$.

Remark 4.3.1.2 ▶ Unwinding Definition 4.3.1.1: Universal Property I

The left tensor product of pointed sets satisfies the following natural bijection:

$$\mathsf{Sets}_*(X \lhd Y, Z) \cong \mathrm{Hom}^{\otimes,\mathrm{L}}_{\mathsf{Sets}_*}(X \times Y, Z).$$

That is to say, the following data are in natural bijection:

1. Pointed maps $f \colon X \lhd Y \to Z$.
2. Maps of sets $f \colon X \times Y \to Z$ satisfying $f(x_0, y) = z_0$ for each $y \in Y$.

Remark 4.3.1.3 ▶ Unwinding Definition 4.3.1.1: Universal Property II

The left tensor product of pointed sets may be described as follows:

- The left tensor product of (X, x_0) and (Y, y_0) is the pair $((X \lhd Y, x_0 \lhd y_0), \iota)$ consisting of
  - A pointed set $(X \lhd Y, x_0 \lhd y_0)$;
  - A left bilinear morphism of pointed sets $\iota \colon (X \times Y, (x_0, y_0)) \to X \lhd Y$;

  satisfying the following universal property:

  (UP) Given another such pair $((Z, z_0), f)$ consisting of
  - ⁂ A pointed set (Z, z_0);
  - ⁂ A left bilinear morphism of pointed sets $f \colon (X \times Y, (x_0, y_0)) \to X \lhd Y$;

  there exists a unique morphism of pointed sets $X \lhd Y \xrightarrow{\exists!} Z$

making the diagram

$$
\begin{tikzcd}
 & X \lhd Y \arrow[d, dashed, "\exists!"] \\
X \times Y \arrow[ur, "\iota"] \arrow[r, "f"'] & Z
\end{tikzcd}
$$

commute.

Construction 4.3.1.4 ▶ The Left Tensor Product of Pointed Sets

In detail, the **left tensor product of** (X, x_0) **and** (Y, y_0) is the pointed set $(X \lhd Y, [x_0])$ consisting of

- *The Underlying Set.* The set $X \lhd Y$ defined by

$$
\begin{aligned}
X \lhd Y &\stackrel{\text{def}}{=} |Y| \odot X \\
&\cong \bigvee_{y \in Y} (X, x_0),
\end{aligned}
$$

  where $|Y|$ denotes the underlying set of (Y, y_0);

- *The Underlying Basepoint.* The point $[(y_0, x_0)]$ of $\bigvee_{y\in Y}(X, x_0)$, which is equal to $[(y, x_0)]$ for any $y \in Y$.

Notation 4.3.1.5 ▶ Elements of Left Tensor Products of Pointed Sets

We write[1] $x \lhd y$ for the element $[(y, x)]$ of

$$X \lhd Y \cong |Y| \odot X.$$

[1] *Further Notation:* Also written $x \lhd_{\mathsf{Sets}_*} y$.

Remark 4.3.1.6 ▶ Basepoints of Left Tensor Products of Pointed Sets

Employing the notation introduced in Notation 4.3.1.5, we have

$$x_0 \lhd y_0 = x_0 \lhd y$$

for each $y \in Y$, and

$$x_0 \lhd y = x_0 \lhd y'$$

for each $y, y' \in Y$.

Proposition 4.3.1.7 ▶ Properties of Left Tensor Products of Pointed Sets

Let (X, x_0) and (Y, y_0) be pointed sets.

1. *Functoriality.* The assignments $X, Y, (X, Y) \mapsto X \lhd Y$ define functors

$$X \lhd - \colon \mathsf{Sets}_* \to \mathsf{Sets}_*,$$
$$- \lhd Y \colon \mathsf{Sets}_* \to \mathsf{Sets}_*,$$
$$-_1 \lhd -_2 \colon \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*.$$

In particular, given pointed maps

$$f \colon (X, x_0) \to (A, a_0),$$
$$g \colon (Y, y_0) \to (B, b_0),$$

the induced map

$$f \lhd g \colon X \lhd Y \to A \lhd B$$

is given by

$$[f \lhd g](x \lhd y) \overset{\text{def}}{=} f(x) \lhd g(y)$$

for each $x \lhd y \in X \lhd Y$.

2. *Adjointness I.* We have an adjunction

$$\Big(- \lhd Y \dashv [Y, -]^{\lhd}_{\mathsf{Sets}_*}\Big) \colon \quad \begin{tikzcd} \mathsf{Sets}_* \arrow[r, bend left, "- \lhd Y"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_*, \arrow[l, bend left, "{[Y,-]^{\lhd}_{\mathsf{Sets}_*}}"] \end{tikzcd}$$

witnessed by a bijection of sets

$$\mathrm{Hom}_{\mathsf{Sets}_*}(X \lhd Y, Z) \cong \mathrm{Hom}_{\mathsf{Sets}_*}\Big(X, [Y, Z]^{\lhd}_{\mathsf{Sets}_*}\Big)$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, where $[X, Y]^{\lhd}_{\mathsf{Sets}_*}$ is the pointed set of Definition 4.3.2.1.

3. *Adjointness II.* The functor

$$X \lhd - \colon \mathsf{Sets}_* \to \mathsf{Sets}_*$$

does not admit a right adjoint.

4. *Adjointness III.* We have a bijection of sets

$$\mathrm{Hom}_{\mathsf{Sets}_*}(X \lhd Y, Z) \cong \mathrm{Hom}_{\mathsf{Sets}}(|Y|, \mathsf{Sets}_*(X, Z))$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

Proof 4.3.1.8 ▶ Proof of Proposition 4.3.1.7

Item 1: Functoriality

Clear.

Item 2: Adjointness I

This follows from Item 3 of Proposition 4.2.1.9.

Item 3: Adjointness II

For $X \lhd -$ to admit a right adjoint would require it to preserve colimits by **??** of **??**. However, we have

$$\begin{aligned} X \lhd \mathrm{pt} &\overset{\text{def}}{=} |\mathrm{pt}| \odot X \\ &\cong X \\ &\ncong \mathrm{pt}, \end{aligned}$$

and thus we see that $X \lhd -$ does not have a right adjoint.

Item 4: Adjointness III

This follows from Item 2 of Proposition 4.2.1.9.

Remark 4.3.1.9 ▶ On the Failure of $X \lhd -$ to Be a Left Adjoint

Here is some intuition on why $X \lhd -$ fails to be a left adjoint. Item 4 of Proposition 4.3.1.7 states that we have a natural bijection

$$\mathrm{Hom}_{\mathsf{Sets}_*}(X \lhd Y, Z) \cong \mathrm{Hom}_{\mathsf{Sets}}(|Y|, \mathrm{Sets}_*(X, Z)),$$

so it would be reasonable to wonder whether a natural bijection of the form

$$\mathrm{Hom}_{\mathsf{Sets}_*}(X \lhd Y, Z) \cong \mathrm{Hom}_{\mathsf{Sets}_*}(Y, \mathbf{Sets}_*(X, Z)),$$

also holds, which would give $X \lhd - \dashv \mathbf{Sets}_*(X, -)$. However, such a bijection would require every map

$$f\colon X \lhd Y \to Z$$

to satisfy

$$f(x \lhd y_0) = z_0$$

for each $x \in X$, whereas we are imposing such a basepoint preservation condition only for elements of the form $x_0 \lhd y$. Thus $\mathbf{Sets}_*(X, -)$ can't be a right adjoint for $X \lhd -$, and as shown by Item 3 of Proposition 4.3.1.7, no functor can.[1]

[1] The functor $\mathbf{Sets}_*(X, -)$ is instead right adjoint to $X \wedge -$, the smash product of pointed sets of Definition 4.5.1.1. See Item 2 of Proposition 4.5.1.10.

4.3.2 The Left Internal Hom of Pointed Sets

Let (X, x_0) and (Y, y_0) be pointed sets.

Definition 4.3.2.1 ▶ The Left Internal Hom of Pointed Sets

The **left internal Hom of pointed sets** is the functor

$$[-,-]^{\lhd}_{\mathsf{Sets}_*} \colon \mathsf{Sets}_*^{\mathsf{op}} \times \mathsf{Sets}_* \to \mathsf{Sets}_*$$

defined as the composition

$$\mathsf{Sets}_*^{\mathsf{op}} \times \mathsf{Sets}_* \xrightarrow{忘\times\mathrm{id}} \mathsf{Sets}^{\mathsf{op}} \times \mathsf{Sets}_* \xrightarrow{\pitchfork} \mathsf{Sets}_*,$$

where:

- $忘\colon \mathsf{Sets}_* \to \mathsf{Sets}$ is the forgetful functor from pointed sets to sets.
- $\pitchfork\colon \mathsf{Sets}^{\mathsf{op}} \times \mathsf{Sets}_* \to \mathsf{Sets}_*$ is the cotensor functor of Item 1 of Proposition 4.2.2.6.

Proof 4.3.2.2 ▶ Proof of Definition 4.3.2.1

For a proof that $[-,-]^{\lhd}_{\mathsf{Sets}_*}$ is indeed the left internal Hom of Sets_* with respect to the left tensor product of pointed sets, see Item 2 of Proposition 4.3.1.7.
□

Remark 4.3.2.3 ▶ Unwinding Definition 4.3.2.1, I: Universal Property

The left internal Hom of pointed sets satisfies the following universal property:

$$\mathsf{Sets}_*(X \lhd Y, Z) \cong \mathsf{Sets}_*\left(X, [Y, Z]^{\lhd}_{\mathsf{Sets}_*}\right)$$

That is to say, the following data are in bijection:

1. Pointed maps $f\colon X \lhd Y \to Z$.
2. Pointed maps $f\colon X \to [Y, Z]^{\lhd}_{\mathsf{Sets}_*}$.

Remark 4.3.2.4 ▶ Unwinding Definition 4.3.2.1, II: Explicit Description

In detail, the **left internal Hom of** (X, x_0) **and** (Y, y_0) is the pointed set $\left([X, Y]^{\lhd}_{\mathsf{Sets}_*}, \left[(y_0)_{x\in X}\right]\right)$ consisting of

- *The Underlying Set.* The set $[X,Y]^{\lhd}_{\mathsf{Sets}_*}$ defined by

$$\begin{aligned}[X,Y]^{\lhd}_{\mathsf{Sets}_*} &\overset{\text{def}}{=} |X| \pitchfork Y\\ &\cong \bigwedge_{x\in X}(Y,y_0),\end{aligned}$$

where $|X|$ denotes the underlying set of (X,x_0);

- *The Underlying Basepoint.* The point $\left[(y_0)_{x\in X}\right]$ of $\bigwedge_{x\in X}(Y,y_0)$.

Proposition 4.3.2.5 ▶ Properties of Left Internal Homs of Pointed Sets

Let (X,x_0) and (Y,y_0) be pointed sets.

1. *Functoriality.* The assignments $X, Y, (X,Y) \mapsto [X,Y]^{\lhd}_{\mathsf{Sets}_*}$ define functors

$$\begin{aligned}[X,-]^{\lhd}_{\mathsf{Sets}_*} &\colon \mathsf{Sets}_* \to \mathsf{Sets}_*,\\ [-,Y]^{\lhd}_{\mathsf{Sets}_*} &\colon \mathsf{Sets}_*^{\mathsf{op}} \to \mathsf{Sets}_*,\\ [-_1,-_2]^{\lhd}_{\mathsf{Sets}_*} &\colon \mathsf{Sets}_*^{\mathsf{op}} \times \mathsf{Sets}_* \to \mathsf{Sets}_*.\end{aligned}$$

In particular, given pointed maps

$$\begin{aligned}f &\colon (X,x_0) \to (A,a_0),\\ g &\colon (Y,y_0) \to (B,b_0),\end{aligned}$$

the induced map

$$[f,g]^{\lhd}_{\mathsf{Sets}_*} \colon [A,Y]^{\lhd}_{\mathsf{Sets}_*} \to [X,B]^{\lhd}_{\mathsf{Sets}_*}$$

is given by

$$[f,g]^{\lhd}_{\mathsf{Sets}_*}\left(\left[(y_a)_{a\in A}\right]\right) \overset{\text{def}}{=} \left[\left(g\left(y_{f(x)}\right)\right)_{x\in X}\right]$$

for each $\left[(y_a)_{a\in A}\right] \in [A,Y]^{\lhd}_{\mathsf{Sets}_*}$.

2. *Adjointness I.* We have an adjunction

$$\left(- \lhd Y \dashv [Y,-]^{\lhd}_{\mathsf{Sets}_*}\right)\colon \quad \begin{tikzcd} \mathsf{Sets}_* \arrow[r, bend left, "- \lhd Y"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_*, \arrow[l, bend left, "{[Y,-]^{\lhd}_{\mathsf{Sets}_*}}"] \end{tikzcd}$$

witnessed by a bijection of sets

$$\mathrm{Hom}_{\mathsf{Sets}_*}(X \lhd Y, Z) \cong \mathrm{Hom}_{\mathsf{Sets}_*}\left(X, [Y,Z]^{\lhd}_{\mathsf{Sets}_*}\right)$$

natural in $(X,x_0), (Y,y_0), (Z,z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

3. *Adjointness II.* The functor

$$X \lhd -\colon \mathsf{Sets}_* \longrightarrow \mathsf{Sets}_*$$

does not admit a right adjoint.

Proof 4.3.2.6 ▶ Proof of Proposition 4.3.2.5

Item 1: Functoriality

Clear.

Item 2: Adjointness I

This is a repetition of Item 2 of Proposition 4.3.1.7, and is proved there.

Item 3: Adjointness II

This is a repetition of Item 3 of Proposition 4.3.1.7, and is proved there. □

4.3.3 The Left Skew Unit

Definition 4.3.3.1 ▶ The Left Skew Unit of $\lhd$

The **left skew unit of the left tensor product of pointed sets** is the functor

$$\mathbb{1}^{\mathsf{Sets}_*,\lhd}\colon \mathrm{pt} \longrightarrow \mathsf{Sets}_*$$

defined by

$$\mathbb{1}^{\lhd}_{\mathsf{Sets}_*} \stackrel{\text{def}}{=} S^0.$$

4.3.4 The Left Skew Associator

Definition 4.3.4.1 ▶ The Left Skew Associator of $\lhd$

The **skew associator of the left tensor product of pointed sets** is the natural transformation

$$\alpha^{\mathsf{Sets}_*,\lhd}\colon \lhd \circ (\lhd \times \mathrm{id}_{\mathsf{Sets}_*}) \Longrightarrow \lhd \circ (\mathrm{id}_{\mathsf{Sets}_*} \times \lhd) \circ \boldsymbol{\alpha}^{\mathsf{Cats}}_{\mathsf{Sets}_*,\mathsf{Sets}_*,\mathsf{Sets}_*}$$

as in the diagram

$$\begin{tikzcd}
 & & \mathsf{Sets}_* \times (\mathsf{Sets}_* \times \mathsf{Sets}_*) \arrow[drr, "\mathrm{id}\times\lhd"] & & \\
(\mathsf{Sets}_* \times \mathsf{Sets}_*) \times \mathsf{Sets}_* \arrow[urr, dashed, "\boldsymbol{\alpha}^{\mathsf{Cats}}_{\mathsf{Sets}_*,\mathsf{Sets}_*,\mathsf{Sets}_*}", "\sim"'] \arrow[dr, "\lhd\times\mathrm{id}"'] & & \arrow[d, phantom, "\Uparrow\alpha^{\mathsf{Sets}_*,\lhd}"] & & \mathsf{Sets}_* \times \mathsf{Sets}_* \arrow[dl, "\lhd"] \\
 & \mathsf{Sets}_* \times \mathsf{Sets}_* \arrow[rr, "\lhd"'] & {} & \mathsf{Sets}_*, &
\end{tikzcd}$$

whose component

$$\alpha^{\mathsf{Sets}_*,\lhd}_{X,Y,Z} \colon (X \lhd Y) \lhd Z \to X \lhd (Y \lhd Z)$$

at $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$ is given by

$$\begin{aligned}
(X \lhd Y) \lhd Z &\stackrel{\text{def}}{=} |Z| \odot (X \lhd Y) \\
&\stackrel{\text{def}}{=} |Z| \odot (|Y| \odot X) \\
&\cong \bigvee_{z\in Z} |Y| \odot X \\
&\cong \bigvee_{z\in Z}\left(\bigvee_{y\in Y} X\right) \\
&\to \bigvee_{[(z,y)]\in\bigvee_{z\in Z} Y} X \\
&\cong \bigvee_{[(z,y)]\in|Z|\odot Y} X \\
&\cong ||Z| \odot Y| \odot X \\
&\stackrel{\text{def}}{=} |Y \lhd Z| \odot X \\
&\stackrel{\text{def}}{=} X \lhd (Y \lhd Z),
\end{aligned}$$

where the map

$$\bigvee_{z\in Z}\left(\bigvee_{y\in Y} X\right) \to \bigvee_{(z,y)\in\bigvee_{z\in Z} Y} X$$

is given by $[(z, [(y, x)])] \mapsto [([(z, y)], x)]$.

Proof 4.3.4.2 ► Proof of Definition 4.3.4.1

(Proven below in a bit.) □

Remark 4.3.4.3 ► Unwinding Definition 4.3.4.1

Unwinding the notation for elements, we have

$$\begin{aligned}[(z,[(y,x)])] &\overset{\text{def}}{=} [(z, x \lhd y)] \\ &\overset{\text{def}}{=} (x \lhd y) \lhd z\end{aligned}$$

and

$$\begin{aligned}[([(z,y)],x)] &\overset{\text{def}}{=} [(y \lhd z, x)] \\ &\overset{\text{def}}{=} x \lhd (y \lhd z).\end{aligned}$$

So, in other words, $\alpha^{\mathsf{Sets}_*,\lhd}_{X,Y,Z}$ acts on elements via

$$\alpha^{\mathsf{Sets}_*,\lhd}_{X,Y,Z}((x \lhd y) \lhd z) \overset{\text{def}}{=} x \lhd (y \lhd z)$$

for each $(x \lhd y) \lhd z \in (X \lhd Y) \lhd Z$.

Remark 4.3.4.4 ► Non-Invertibility of the Skew Associator of $\lhd$

Taking $y = y_0$, we see that the morphism $\alpha^{\mathsf{Sets}_*,\lhd}_{X,Y,Z}$ acts on elements as

$$\alpha^{\mathsf{Sets}_*,\lhd}_{X,Y,Z}((x \lhd y_0) \lhd z) \overset{\text{def}}{=} x \lhd (y_0 \lhd z).$$

However, by the definition of $\lhd$, we have $y_0 \lhd z = y_0 \lhd z'$ for all $z, z' \in Z$, preventing $\alpha^{\mathsf{Sets}_*,\lhd}_{X,Y,Z}$ from being non-invertible.

Proof 4.3.4.5 ► Proof of Definition 4.3.4.1

Firstly, note that, given $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, the map

$$\alpha^{\mathsf{Sets}_*,\lhd}_{X,Y,Z} \colon (X \lhd Y) \lhd Z \to X \lhd (Y \lhd Z)$$

is indeed a morphism of pointed sets, as we have

$$\alpha^{\mathsf{Sets}_*,\lhd}_{X,Y,Z}((x_0 \lhd y_0) \lhd z_0) = x_0 \lhd (y_0 \lhd z_0).$$

Next, we claim that $\alpha^{\mathsf{Sets}_*,\lhd}$ is a natural transformation. We need to show that, given morphisms of pointed sets

$$
\begin{aligned}
f &\colon (X,x_0) \to \left(X',x'_0\right),\\
g &\colon (Y,y_0) \to \left(Y',y'_0\right),\\
h &\colon (Z,z_0) \to \left(Z',z'_0\right)
\end{aligned}
$$

the diagram

$$
\begin{tikzcd}
(X \lhd Y) \lhd Z \arrow[r, "(f\lhd g)\lhd h"] \arrow[d, "\alpha^{\mathsf{Sets}_*,\lhd}_{X,Y,Z}"'] & (X' \lhd Y') \lhd Z' \arrow[d, "\alpha^{\mathsf{Sets}_*,\lhd}_{X',Y',Z'}"] \\
X \lhd (Y \lhd Z) \arrow[r, "f\lhd(g\lhd h)"'] & X' \lhd (Y' \lhd Z')
\end{tikzcd}
$$

commutes. Indeed, this diagram acts on elements as

$$
\begin{tikzcd}
(x \lhd y) \lhd z \arrow[r, mapsto] \arrow[d, mapsto] & (f(x) \lhd g(y)) \lhd h(z) \arrow[d, mapsto] \\
x \lhd (y \lhd z) \arrow[r, mapsto] & f(x) \lhd (g(y) \lhd h(z))
\end{tikzcd}
$$

and hence indeed commutes, showing $\alpha^{\mathsf{Sets}_*,\lhd}$ to be a natural transformation. This finishes the proof. □

4.3.5 The Left Skew Left Unitor

Definition 4.3.5.1 ► The Left Skew Left Unitor of ◁

The **skew left unitor of the left tensor product of pointed sets** is the natural transformation

$$\lambda^{\mathsf{Sets}_*,\lhd}\colon \lhd\circ\left(\mathbb{1}^{\mathsf{Sets}_*}\times\mathrm{id}_{\mathsf{Sets}_*}\right)\overset{\sim}{\Longrightarrow}\lambda^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}\qquad\begin{tikzcd} \mathrm{pt}\times\mathsf{Sets}_* \arrow[r, "\mathbb{1}^{\mathsf{Sets}_*}\times\mathrm{id}"] \arrow[dr, dashed, bend right, "\lambda^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}"'] & \mathsf{Sets}_*\times\mathsf{Sets}_* \arrow[d, "\lhd"] \arrow[dl, phantom, "\Swarrow\lambda^{\mathsf{Sets}_*,\lhd}"] \\ & \mathsf{Sets}_*, \end{tikzcd}$$

whose component

$$\lambda^{\mathsf{Sets}_*,\lhd}_X\colon S^0\lhd X\to X$$

at $(X,x_0)\in\mathrm{Obj}(\mathsf{Sets}_*)$ is given by the composition

$$\begin{aligned} S^0\lhd X &\cong |X|\odot S^0\\ &\cong \bigvee_{x\in X} S^0\\ &\to X, \end{aligned}$$

where $\bigvee_{x\in X} S^0\to X$ is the map given by

$$\begin{aligned} [(x,0)] &\mapsto x_0,\\ [(x,1)] &\mapsto x. \end{aligned}$$

Proof 4.3.5.2 ► Proof of Definition 4.3.5.1

(Proven below in a bit.)

Remark 4.3.5.3 ► Unwinding Definition 4.3.5.1

In other words, $\lambda^{\mathsf{Sets}_*,\lhd}_X$ acts on elements as

$$\begin{aligned} \lambda^{\mathsf{Sets}_*,\lhd}_X(0\lhd x) &\overset{\mathrm{def}}{=} x_0,\\ \lambda^{\mathsf{Sets}_*,\lhd}_X(1\lhd x) &\overset{\mathrm{def}}{=} x \end{aligned}$$

for each $1\lhd x\in S^0\lhd X$.

Remark 4.3.5.4 ▶ Non-Invertibility of the Skew Left Unitor of $\lhd$

The morphism $\lambda^{\mathsf{Sets}_*,\lhd}_X$ is almost invertible, with its would-be-inverse

$$\phi_X \colon X \to S^0 \lhd X$$

given by

$$\phi_X(x) \overset{\text{def}}{=} 1 \lhd x$$

for each $x \in X$. Indeed, we have

$$\begin{aligned}\left[\lambda^{\mathsf{Sets}_*,\lhd}_X \circ \phi\right](x) &= \lambda^{\mathsf{Sets}_*,\lhd}_X(\phi(x))\\ &= \lambda^{\mathsf{Sets}_*,\lhd}_X(1 \lhd x)\\ &= x\\ &= [\mathrm{id}_X](x)\end{aligned}$$

so that

$$\lambda^{\mathsf{Sets}_*,\lhd}_X \circ \phi = \mathrm{id}_X$$

and

$$\begin{aligned}\left[\phi \circ \lambda^{\mathsf{Sets}_*,\lhd}_X\right](1 \lhd x) &= \phi\left(\lambda^{\mathsf{Sets}_*,\lhd}_X(1 \lhd x)\right)\\ &= \phi(x)\\ &= 1 \lhd x\\ &= [\mathrm{id}_{S^0 \lhd X}](1 \lhd x),\end{aligned}$$

but

$$\begin{aligned}\left[\phi \circ \lambda^{\mathsf{Sets}_*,\lhd}_X\right](0 \lhd x) &= \phi\left(\lambda^{\mathsf{Sets}_*,\lhd}_X(0 \lhd x)\right)\\ &= \phi(x_0)\\ &= 1 \lhd x_0,\end{aligned}$$

where $0 \lhd x \neq 1 \lhd x_0$. Thus

$$\phi \circ \lambda^{\mathsf{Sets}_*,\lhd}_X \overset{?}{=} \mathrm{id}_{S^0 \lhd X}$$

holds for all elements in $S^0 \lhd X$ except one.

Proof 4.3.5.5 ► Proof of Definition 4.3.5.1

Firstly, note that, given $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, the map

$$\lambda_X^{\mathsf{Sets}_*,\lhd} : S^0 \lhd X \to X$$

is indeed a morphism of pointed sets, as we have

$$\lambda_X^{\mathsf{Sets}_*,\lhd}(0 \lhd x_0) = x_0.$$

Next, we claim that $\lambda^{\mathsf{Sets}_*,\lhd}$ is a natural transformation. We need to show that, given a morphism of pointed sets

$$f : (X, x_0) \to (Y, y_0),$$

the diagram

$$\begin{tikzcd}
S^0 \lhd X \arrow[r, "\mathrm{id}_{S^0} \lhd f"] \arrow[d, "\lambda_X^{\mathsf{Sets}_*,\lhd}"'] & S^0 \lhd Y \arrow[d, "\lambda_Y^{\mathsf{Sets}_*,\lhd}"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
0 \lhd x \arrow[d, mapsto] & \\
x_0 \arrow[r, mapsto] & f(x_0)
\end{tikzcd}
\qquad
\begin{tikzcd}
0 \lhd x \arrow[r, mapsto] & 0 \lhd f(x) \arrow[d, mapsto] \\
 & y_0
\end{tikzcd}$$

and

$$\begin{tikzcd}
1 \lhd x \arrow[r, mapsto] \arrow[d, mapsto] & 1 \lhd f(x) \arrow[d, mapsto] \\
x \arrow[r, mapsto] & f(x)
\end{tikzcd}$$

and hence indeed commutes, showing $\lambda^{\mathsf{Sets}_*,\lhd}$ to be a natural transformation. This finishes the proof.

4.3.6 The Left Skew Right Unitor

Definition 4.3.6.1 ▶ The Left Skew Right Unitor of $\lhd$

The **skew right unitor of the left tensor product of pointed sets** is the natural transformation

$$\rho^{\mathsf{Sets}_*,\lhd}\colon \rho^{\mathsf{Cats}_2}_{\mathsf{Sets}_*} \overset{\sim}{\Longrightarrow} \lhd \circ \left(\mathrm{id} \times \mathbb{1}^{\mathsf{Sets}_*}\right), \qquad \begin{tikzcd} \mathsf{Sets}_* \times \mathrm{pt} \arrow[r, "\mathrm{id}\times\mathbb{1}^{\mathsf{Sets}_*}"] \arrow[dr, dashed, bend right, "\rho^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}"'] & \mathsf{Sets}_* \times \mathsf{Sets}_* \arrow[d, "\lhd"] \arrow[dl, phantom, "\Uparrow \rho^{\mathsf{Sets}_*,\lhd}"] \\ & \mathsf{Sets}_*, \end{tikzcd}$$

whose component

$$\rho^{\mathsf{Sets}_*,\lhd}_X \colon X \to X \lhd S^0$$

at $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$ is given by the composition

$$\begin{aligned} X &\to X \vee X \\ &\cong |S^0| \odot X \\ &\cong X \lhd S^0, \end{aligned}$$

where $X \to X \vee X$ is the map sending X to the second factor of X in $X \vee X$.

Proof 4.3.6.2 ▶ Proof of Definition 4.3.6.1

(Proven below in a bit.)

Remark 4.3.6.3 ▶ Unwinding Definition 4.3.6.1

In other words, $\rho^{\mathsf{Sets}_*,\lhd}_X$ acts on elements as

$$\rho^{\mathsf{Sets}_*,\lhd}_X(x) \overset{\text{def}}{=} [(1, x)]$$

i.e. by

$$\rho^{\mathsf{Sets}_*,\lhd}_X(x) \overset{\text{def}}{=} x \lhd 1$$

for each $x \in X$.

Remark 4.3.6.4 ► Non-Invertibility of the Skew Right Unitor of $\lhd$

The morphism $\rho_X^{\mathsf{Sets}_*,\lhd}$ is non-invertible, as it is non-surjective when viewed as a map of sets, since the elements $x \lhd 0$ of $X \lhd S^0$ with $x \neq x_0$ are outside the image of $\rho_X^{\mathsf{Sets}_*,\lhd}$, which sends x to $x \lhd 1$.

Proof 4.3.6.5 ► Proof of Definition 4.3.6.1

Firstly, note that, given $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, the map

$$\rho_X^{\mathsf{Sets}_*,\lhd} \colon X \to X \lhd S^0$$

is indeed a morphism of pointed sets as we have

$$\begin{aligned}\rho_X^{\mathsf{Sets}_*,\lhd}(x_0) &= x_0 \lhd 1\\ &= x_0 \lhd 0.\end{aligned}$$

Next, we claim that $\rho^{\mathsf{Sets}_*,\lhd}$ is a natural transformation. We need to show that, given a morphism of pointed sets

$$f \colon (X, x_0) \to (Y, y_0),$$

the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\rho_X^{\mathsf{Sets}_*,\lhd}"'] & Y \arrow[d, "\rho_Y^{\mathsf{Sets}_*,\lhd}"] \\ X \lhd S^0 \arrow[r, "f \lhd \mathrm{id}_{S^0}"'] & Y \lhd S^0 \end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd} x \arrow[r, mapsto] \arrow[d, mapsto] & f(x) \arrow[d, mapsto] \\ x \lhd 0 \arrow[r, mapsto] & f(x) \lhd 0 \end{tikzcd}$$

and hence indeed commutes, showing $\rho^{\mathsf{Sets}_*,\lhd}$ to be a natural transformation. This finishes the proof. □

4.3.7 The Diagonal

Definition 4.3.7.1 ► The Diagonal of ◁

The **diagonal of the left tensor product of pointed sets** is the natural transformation

$$\Delta^{\triangleleft}\colon \mathrm{id}_{\mathsf{Sets}_*} \Longrightarrow \triangleleft \circ \Delta^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}, \qquad \begin{tikzcd} \mathsf{Sets}_* \arrow[rr, bend left, "\mathrm{id}_{\mathsf{Sets}_*}"] \arrow[dr, swap, "\Delta^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}"] \arrow[rr, phantom, shift right=3ex, "\Downarrow \Delta^{\triangleleft}"] & & \mathsf{Sets}_* \\ & \mathsf{Sets}_* \times \mathsf{Sets}_*, \arrow[ur, swap, "\triangleleft"] & \end{tikzcd}$$

whose component

$$\Delta^{\triangleleft}_X\colon (X, x_0) \to (X \triangleleft X, x_0 \triangleleft x_0)$$

at $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$ is given by

$$\Delta^{\triangleleft}_X(x) \overset{\mathrm{def}}{=} x \triangleleft x$$

for each $x \in X$.

Proof 4.3.7.2 ► Proof of Definition 4.3.7.1

Being a Morphism of Pointed Sets

We have

$$\Delta^{\triangleleft}_X(x_0) \overset{\mathrm{def}}{=} x_0 \triangleleft x_0,$$

and thus $\Delta^{\triangleleft}_X$ is a morphism of pointed sets.

Naturality

We need to show that, given a morphism of pointed sets

$$f\colon (X, x_0) \to (Y, y_0),$$

the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, swap, "\Delta^{\triangleleft}_X"] & Y \arrow[d, "\Delta^{\triangleleft}_Y"] \\ X \triangleleft X \arrow[r, swap, "f \triangleleft f"] & Y \triangleleft Y \end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
x \arrow[r, mapsto] \arrow[d, mapsto] & f(x) \arrow[d, mapsto] \\
x \lhd x \arrow[r, mapsto] & f(x) \lhd f(x)
\end{tikzcd}$$

and hence indeed commutes, showing $\Delta^{\lhd}$ to be natural. □

4.3.8 The Left Skew Monoidal Structure on Pointed Sets Associated to $\lhd$

Proposition 4.3.8.1 ▶ The Left Skew Monoidal Structure on Pointed Sets Associated to $\lhd$

The category Sets_* admits a left-closed left skew monoidal category structure consisting of

- *The Underlying Category.* The category Sets_* of pointed sets;
- *The Left Skew Monoidal Product.* The left tensor product functor

  $$\lhd \colon \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*$$

  of Definition 4.3.1.1;
- *The Left Internal Skew Hom.* The left internal Hom functor

  $$[-,-]^{\lhd}_{\mathsf{Sets}_*} \colon \mathsf{Sets}_*^{\mathsf{op}} \times \mathsf{Sets}_* \to \mathsf{Sets}_*$$

  of Definition 4.3.2.1;
- *The Left Skew Monoidal Unit.* The functor

  $$\mathbb{1}^{\mathsf{Sets}_*,\lhd} \colon \mathrm{pt} \to \mathsf{Sets}_*$$

  of Definition 4.3.3.1;
- *The Left Skew Associators.* The natural transformation

  $$\alpha^{\mathsf{Sets}_*,\lhd} \colon \lhd \circ (\lhd \times \mathrm{id}_{\mathsf{Sets}_*}) \Longrightarrow \lhd \circ (\mathrm{id}_{\mathsf{Sets}_*} \times \lhd) \circ \boldsymbol{\alpha}^{\mathsf{Cats}}_{\mathsf{Sets}_*,\mathsf{Sets}_*,\mathsf{Sets}_*}$$

  of Definition 4.3.4.1;

- *The Left Skew Left Unitors.* The natural transformation

$$\lambda^{\mathsf{Sets}_*,\lhd}\colon \lhd\circ\left(\mathbb{1}^{\mathsf{Sets}_*}\times \mathrm{id}_{\mathsf{Sets}_*}\right)\overset{\sim}{\Longrightarrow}\boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}$$

of Definition 4.3.5.1;

- *The Left Skew Right Unitors.* The natural transformation

$$\rho^{\mathsf{Sets}_*,\lhd}\colon \boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}\overset{\sim}{\Longrightarrow}\lhd\circ\left(\mathrm{id}\times\mathbb{1}^{\mathsf{Sets}_*}\right)$$

of Definition 4.3.6.1.

Proof 4.3.8.2 ► Proof of Proposition 4.3.8.1

The Pentagon Identity

Let (W,w_0), (X,x_0), (Y,y_0) and (Z,z_0) be pointed sets. We have to show that the diagram

$$\begin{tikzcd}
 & (W\lhd(X\lhd Y))\lhd Z \arrow[dr, "\alpha^{\mathsf{Sets}_*,\lhd}_{W,X\lhd Y,Z}"] & \\
((W\lhd X)\lhd Y)\lhd Z \arrow[ur, "\alpha^{\mathsf{Sets}_*,\lhd}_{W,X,Y}\lhd\mathrm{id}_Z"] \arrow[d, "\alpha^{\mathsf{Sets}_*,\lhd}_{W\lhd X,Y,Z}"'] & & W\lhd((X\lhd Y)\lhd Z) \arrow[d, "\mathrm{id}_W\lhd\alpha^{\mathsf{Sets}_*,\lhd}_{X,Y,Z}"] \\
(W\lhd X)\lhd(Y\lhd Z) \arrow[rr, "\alpha^{\mathsf{Sets}_*,\lhd}_{W,X,Y\lhd Z}"'] & & W\lhd(X\lhd(Y\lhd Z))
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$
\begin{tikzcd}
 & (w \lhd (x \lhd y)) \lhd z \arrow[dr, mapsto] & \\
((w \lhd x) \lhd y) \lhd z \arrow[ur, mapsto] \arrow[d, mapsto] & & w \lhd ((x \lhd y) \lhd z) \arrow[d, mapsto] \\
(w \lhd x) \lhd (y \lhd z) \arrow[rr, mapsto] & & w \lhd (x \lhd (y \lhd z))
\end{tikzcd}
$$

and thus we see that the pentagon identity is satisfied.

The Left Skew Left Triangle Identity

Let (X, x_0) and (Y, y_0) be pointed sets. We have to show that the diagram

$$
\begin{tikzcd}
\left(S^0 \lhd X\right) \lhd Y \arrow[r, "\alpha^{\mathsf{Sets}_*,\lhd}_{S^0,X,Y}"] \arrow[dr, "\lambda^{\mathsf{Sets}_*,\lhd}_{X} \lhd \mathrm{id}_Y"'] & S^0 \lhd (X \lhd Y) \arrow[d, "\lambda^{\mathsf{Sets}_*,\lhd}_{X \lhd Y}"] \\
 & X \lhd Y
\end{tikzcd}
$$

commutes. Indeed, this diagram acts on elements as

$$
\begin{tikzcd}
(0 \lhd x) \lhd y \arrow[r, mapsto] \arrow[dr, mapsto] & 0 \lhd (x \lhd y) \arrow[d, mapsto] \\
 & x_0 \lhd y = x_0 \lhd y_0
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
(1 \lhd x) \lhd y \arrow[r, mapsto] \arrow[dr, mapsto] & 1 \lhd (x \lhd y) \arrow[d, mapsto] \\
 & x \lhd y
\end{tikzcd}
$$

and hence indeed commutes. Thus the left skew triangle identity is satisfied.

The Left Skew Right Triangle Identity

Let (X, x_0) and (Y, y_0) be pointed sets. We have to show that the diagram

$$\begin{tikzcd}
X \lhd Y \arrow[d, "\rho^{\mathsf{Sets}_*,\lhd}_{X \lhd Y}"'] \arrow[dr, "\mathrm{id}_X \lhd \rho^{\mathsf{Sets}_*,\lhd}_{Y}"] & \\
(X \lhd Y) \lhd S^0 \arrow[r, "\alpha^{\mathsf{Sets}_*,\lhd}_{X,Y,S^0}"'] & X \lhd \left(Y \lhd S^0\right)
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
x \lhd y \arrow[d, mapsto] \arrow[dr, mapsto] & \\
(x \lhd y) \lhd 1 \arrow[r, mapsto] & x \lhd (y \lhd 1)
\end{tikzcd}$$

and hence indeed commutes. Thus the right skew triangle identity is satisfied.

The Left Skew Middle Triangle Identity

Let (X, x_0) and (Y, y_0) be pointed sets. We have to show that the diagram

$$\begin{tikzcd}
X \lhd Y \arrow[r, equal] \arrow[d, "\rho^{\mathsf{Sets}_*,\lhd}_{X} \lhd \mathrm{id}_Y"'] & X \lhd Y \\
\left(X \lhd S^0\right) \lhd Y \arrow[r, "\alpha^{\mathsf{Sets}_*,\lhd}_{X,S^0,Y}"'] & X \lhd \left(S^0 \lhd Y\right) \arrow[u, "\mathrm{id}_A \lhd \lambda^{\mathsf{Sets}_*,\lhd}_{Y}"']
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
x \lhd y \arrow[r, mapsto] \arrow[d, mapsto] & x \lhd y \\
(x \lhd 1) \lhd y \arrow[r, mapsto] & x \lhd (1 \lhd y) \arrow[u, mapsto]
\end{tikzcd}$$

and hence indeed commutes. Thus the right skew triangle identity is satisfied.

The Zig-Zag Identity

We have to show that the diagram

$$\begin{tikzcd} S^0 \arrow[r, "\rho^{\mathsf{Sets}_*,\lhd}_{S^0}"] \arrow[dr, equal] & S^0 \lhd S^0 \arrow[d, "\lambda^{\mathsf{Sets}_*,\lhd}_{S^0}"] \\ & S^0 \end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd} 0 \arrow[r, mapsto] \arrow[dr, mapsto] & 0 \lhd 1 \arrow[d, mapsto] \\ & 0 \end{tikzcd}$$

and

$$\begin{tikzcd} 1 \arrow[r, mapsto] \arrow[dr, mapsto] & 1 \lhd 1 \arrow[d, mapsto] \\ & 1 \end{tikzcd}$$

and hence indeed commutes. Thus the zig-zag identity is satisfied.

Left Skew Monoidal Left-Closedness

This follows from Item 2 of Proposition 4.3.1.7.

4.3.9 Monoids With Respect to the Left Tensor Product of Pointed Sets

Proposition 4.3.9.1 ▶ Monoids With Respect to $\lhd$

The category of monoids on $(\mathsf{Sets}_*, \lhd, S^0)$ is isomorphic to the category of "monoids with left zero"[1] and morphisms between them.

[1] A monoid with left zero is defined similarly as the monoids with zero of **??**. Succinctly, they are monoids (A, μ_A, η_A) with a special element 0_A satisfying

$$0_A a = 0_A$$

for each $a \in A$.

Proof 4.3.9.2 ► Proof of Proposition 4.3.9.1

Monoids on $(\mathsf{Sets}_*, \lhd, S^0)$

A monoid on $(\mathsf{Sets}_*, \lhd, S^0)$ consists of:

- *The Underlying Object.* A pointed set $(A, 0_A)$.
- *The Multiplication Morphism.* A morphism of pointed sets
$$\mu_A \colon A \lhd A \to A,$$
determining a left bilinear morphism of pointed sets
$$\begin{aligned} A \times A &\longrightarrow A \\ (a, b) &\longmapsto ab. \end{aligned}$$
- *The Unit Morphism.* A morphism of pointed sets
$$\eta_A \colon S^0 \to A$$
picking an element 1_A of A.

satisfying the following conditions:

1. *Associativity.* The diagram
$$\begin{tikzcd} & A \lhd (A \lhd A) \arrow[dr, "\mathrm{id}_A \lhd \mu_A"] & \\ (A \lhd A) \lhd A \arrow[ur, "\alpha^{\mathsf{Sets}_*, \lhd}_{A,A,A}"] \arrow[d, "\mu_A \lhd \mathrm{id}_A"'] & & A \lhd A \arrow[d, "\mu_A"] \\ A \lhd A \arrow[rr, "\mu_A"'] & & A \end{tikzcd}$$
2. *Left Unitality.* The diagram
$$\begin{tikzcd} S^0 \lhd A \arrow[r, "\eta_A \times \mathrm{id}_A"] \arrow[dr, "\lambda^{\mathsf{Sets}_*, \lhd}_A"'] & A \lhd A \arrow[d, "\mu_A"] \\ & A \end{tikzcd}$$
commutes.

3. *Right Unitality.* The diagram

$$\begin{tikzcd} A \arrow[r, "\rho_A^{\mathsf{Sets}_*,\lhd}"] \arrow[d, equal] & A \lhd S^0 \arrow[d, "\mathrm{id}_A \times \eta_A"] \\ A & A \lhd A \arrow[l, "\mu_A"] \end{tikzcd}$$

commutes.

Being a left-bilinear morphism of pointed sets, the multiplication map satisfies

$$0_A a = 0_A$$

for each $a \in A$. Now, the associativity, left unitality, and right unitality conditions act on elements as follows:

1. *Associativity.* The associativity condition acts as

$$\begin{tikzcd} (a \lhd b) \lhd c \arrow[d, mapsto] & \\ ab \lhd c \arrow[r, mapsto] & (ab)c \end{tikzcd} \qquad \begin{tikzcd} & a \lhd (b \lhd c) \arrow[dr, mapsto] & \\ (a \lhd b) \lhd c \arrow[ur, mapsto] & & a \lhd bc \arrow[d, mapsto] \\ & & a(bc) \end{tikzcd}$$

This gives

$$(ab)c = a(bc)$$

for each $a, b, c \in A$.

2. *Left Unitality.* The left unitality condition acts:

(a) On $0 \lhd a$ as

$$\begin{tikzcd} 0 \lhd a \arrow[dr, mapsto] & \\ & 0_A \end{tikzcd} \qquad \begin{tikzcd} 0 \lhd a \arrow[r, mapsto] \arrow[dr, mapsto] & 0_A \lhd a \arrow[d, mapsto] \\ & 0_A a. \end{tikzcd}$$

(b) On $1 \lhd a$ as

$$\begin{tikzcd} 1 \lhd a \arrow[dr, mapsto] & \\ & a \end{tikzcd} \qquad \begin{tikzcd} 1 \lhd a \arrow[r, mapsto] \arrow[dr, mapsto] & 1_A \lhd a \arrow[d, mapsto] \\ & 1_A a. \end{tikzcd}$$

This gives

$$\begin{aligned} 1_A a &= a, \\ 0_A a &= 0_A \end{aligned}$$

for each $a \in A$.

3. *Right Unitality.* The right unitality condition acts as

$$\begin{tikzcd} a \arrow[d, mapsto] \\ a \end{tikzcd} \qquad \begin{tikzcd} a \arrow[r, mapsto] & a \lhd 1 \arrow[d, mapsto] \\ a1_A & a \lhd 1_A \arrow[l, mapsto] \end{tikzcd}$$

This gives

$$a1_A = a$$

for each $a \in A$.

Thus we see that monoids with respect to $\lhd$ are exactly monoids with left zero.

Morphisms of Monoids on $(\mathsf{Sets}_*, \lhd, S^0)$

A morphism of monoids on $(\mathsf{Sets}_*, \lhd, S^0)$ from $(A, \mu_A, \eta_A, 0_A)$ to $(B, \mu_B, \eta_B, 0_B)$ is a morphism of pointed sets

$$f: (A, 0_A) \to (B, 0_B)$$

satisfying the following conditions:

1. *Compatibility With the Multiplication Morphisms.* The diagram

$$\begin{tikzcd} A \lhd A \arrow[r, "f \lhd f"] \arrow[d, "\mu_A"'] & B \lhd B \arrow[d, "\mu_B"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

commutes.

2. *Compatibility With the Unit Morphisms.* The diagram

$$\begin{tikzcd} S^0 \arrow[r, "\eta_A"] \arrow[dr, "\eta_B"'] & A \arrow[d, "f"] \\ & B \end{tikzcd}$$

commutes.

These act on elements as

$$\begin{tikzcd} a \lhd b \arrow[d, mapsto] & \\ ab \arrow[r, mapsto] & f(ab) \end{tikzcd} \qquad \begin{tikzcd} a \lhd b \arrow[r, mapsto] & f(a) \lhd f(b) \arrow[d, mapsto] \\ & f(a)f(b) \end{tikzcd}$$

and

$$\begin{tikzcd} 0 \arrow[dr, mapsto] & \\ & 0_B \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[r, mapsto] & 0_A \arrow[d, mapsto] \\ & f(0_A) \end{tikzcd}$$

and

$$\begin{tikzcd} 1 \arrow[dr, mapsto] & \\ & 1_B \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[r, mapsto] & 1_A \arrow[d, mapsto] \\ & f(1_A) \end{tikzcd}$$

giving

$$\begin{aligned} f(ab) &= f(a)f(b), \\ f(0_A) &= 0_B, \\ f(1_A) &= 1_B, \end{aligned}$$

for each $a, b \in A$, which is exactly a morphism of monoids with left zero.

Identities and Composition

Similarly, the identities and composition of $\mathrm{Mon}(\mathsf{Sets}_*, \lhd, S^0)$ can be easily seen to agree with those of monoids with left zero, which finishes the proof. ∎

4.4 The Right Tensor Product of Pointed Sets

4.4.1 Foundations

Let (X, x_0) and (Y, y_0) be pointed sets.

Definition 4.4.1.1 ▶ The Right Tensor Product of Pointed Sets

The **right tensor product of pointed sets** is the functor[1]

$$\rhd \colon \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*$$

defined as the composition

$$\mathsf{Sets}_* \times \mathsf{Sets}_* \xrightarrow{忘 \times \mathrm{id}} \mathsf{Sets} \times \mathsf{Sets}_* \xrightarrow{\odot} \mathsf{Sets}_*,$$

where:

- $忘 \colon \mathsf{Sets}_* \to \mathsf{Sets}$ is the forgetful functor from pointed sets to sets.
- $\odot \colon \mathsf{Sets} \times \mathsf{Sets}_* \to \mathsf{Sets}_*$ is the tensor functor of Item 1 of Proposition 4.2.1.9.

[1] *Further Notation:* Also written $\rhd_{\mathsf{Sets}_*}$.

Remark 4.4.1.2 ▶ Unwinding Definition 4.4.1.1: Universal Property I

The right tensor product of pointed sets satisfies the following natural bijection:

$$\mathsf{Sets}_*(X \rhd Y, Z) \cong \mathrm{Hom}^{\otimes,\mathrm{R}}_{\mathsf{Sets}_*}(X \times Y, Z).$$

That is to say, the following data are in natural bijection:

1. Pointed maps $f \colon X \rhd Y \to Z$.
2. Maps of sets $f \colon X \times Y \to Z$ satisfying $f(x, y_0) = z_0$ for each $x \in X$.

Remark 4.4.1.3 ▶ Unwinding Definition 4.4.1.1: Universal Property II

The right tensor product of pointed sets may be described as follows:

- The right tensor product of (X, x_0) and (Y, y_0) is the pair $((X \rhd Y, x_0 \rhd y_0), \iota)$ consisting of
  - A pointed set $(X \rhd Y, x_0 \rhd y_0)$;
  - A right bilinear morphism of pointed sets $\iota \colon (X \times Y, (x_0, y_0)) \to$

$$X \rhd Y;$$

satisfying the following universal property:

(UP) Given another such pair $((Z, z_0), f)$ consisting of

- A pointed set (Z, z_0);
- A right bilinear morphism of pointed sets $f\colon (X \times Y, (x_0, y_0)) \to X \rhd Y$;

there exists a unique morphism of pointed sets $X \rhd Y \xrightarrow{\exists!} Z$ making the diagram

$$\begin{tikzcd} & X \rhd Y \arrow[d, dashed, "\exists!"] \\ X \times Y \arrow[ur, "\iota"] \arrow[r, "f"'] & Z \end{tikzcd}$$

commute.

Construction 4.4.1.4 ► The Right Tensor Product of Pointed Sets

In detail, the **right tensor product of** (X, x_0) **and** (Y, y_0) is the pointed set $(X \rhd Y, [y_0])$ consisting of:

- *The Underlying Set.* The set $X \rhd Y$ defined by

$$\begin{aligned} X \rhd Y &\stackrel{\text{def}}{=} |X| \odot Y \\ &\cong \bigvee_{x \in X} (Y, y_0), \end{aligned}$$

  where $|X|$ denotes the underlying set of (X, x_0).

- *The Underlying Basepoint.* The point $[(x_0, y_0)]$ of $\bigvee_{x\in X}(Y, y_0)$, which is equal to $[(x, y_0)]$ for any $x \in X$.

Notation 4.4.1.5 ► Elements of Right Tensor Products of Pointed Sets

We write[1] $x \rhd y$ for the element $[(x, y)]$ of

$$X \rhd Y \cong |X| \odot Y.$$

[1] *Further Notation:* Also written $x \rhd_{\mathsf{Sets}_*} y$.

Remark 4.4.1.6 ▶ Basepoints of Right Tensor Products of Pointed Sets

Employing the notation introduced in Notation 4.4.1.5, we have

$$x_0 \rhd y_0 = x \rhd y_0$$

for each $x \in X$, and

$$x \rhd y_0 = x' \rhd y_0$$

for each $x, x' \in X$.

Proposition 4.4.1.7 ▶ Properties of Right Tensor Products of Pointed Sets

Let (X, x_0) and (Y, y_0) be pointed sets.

1. *Functoriality.* The assignments $X, Y, (X, Y) \mapsto X \rhd Y$ define functors

$$\begin{aligned} X \rhd -\colon \mathsf{Sets}_* &\to \mathsf{Sets}_*, \\ - \rhd Y\colon \mathsf{Sets}_* &\to \mathsf{Sets}_*, \\ -_1 \rhd -_2\colon \mathsf{Sets}_* \times \mathsf{Sets}_* &\to \mathsf{Sets}_*. \end{aligned}$$

   In particular, given pointed maps

$$\begin{aligned} f\colon (X, x_0) &\to (A, a_0), \\ g\colon (Y, y_0) &\to (B, b_0), \end{aligned}$$

   the induced map

$$f \rhd g\colon X \rhd Y \to A \rhd B$$

   is given by

$$[f \rhd g](x \rhd y) \overset{\text{def}}{=} f(x) \rhd g(y)$$

   for each $x \rhd y \in X \rhd Y$.

2. *Adjointness I.* We have an adjunction

$$\left(X \rhd - \dashv [X, -]^{\rhd}_{\mathsf{Sets}_*}\right)\colon \quad \begin{tikzcd} \mathsf{Sets}_* \arrow[r, bend left, "X \rhd -"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_*, \arrow[l, bend left, "{[X,-]^{\rhd}_{\mathsf{Sets}_*}}"] \end{tikzcd}$$

   witnessed by a bijection of sets

$$\mathrm{Hom}_{\mathsf{Sets}_*}(X \rhd Y, Z) \cong \mathrm{Hom}_{\mathsf{Sets}_*}\left(Y, [X, Z]^{\rhd}_{\mathsf{Sets}_*}\right)$$

   natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, where $[X, Y]^{\rhd}_{\mathsf{Sets}_*}$ is the pointed set of Definition 4.4.2.1.

3. *Adjointness II.* The functor

$$- \triangleright Y : \mathsf{Sets}_* \to \mathsf{Sets}_*$$

does not admit a right adjoint.

4. *Adjointness III.* We have a bijection of sets

$$\mathrm{Hom}_{\mathsf{Sets}_*}(X \triangleright Y, Z) \cong \mathrm{Hom}_{\mathsf{Sets}}(|X|, \mathsf{Sets}_*(Y, Z))$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

Proof 4.4.1.8 ► Proof of Proposition 4.4.1.7

Item 1: Functoriality

Clear.

Item 2: Adjointness I

This follows from Item 3 of Proposition 4.2.1.9.

Item 3: Adjointness II

For $- \triangleright Y$ to admit a right adjoint would require it to preserve colimits by **??** of **??**. However, we have

$$\begin{aligned} \mathrm{pt} \triangleright X &\stackrel{\text{def}}{=} |\mathrm{pt}| \odot X \\ &\cong X \\ &\ncong \mathrm{pt}, \end{aligned}$$

and thus we see that $- \triangleright Y$ does not have a right adjoint.

Item 4: Adjointness III

This follows from Item 2 of Proposition 4.2.1.9. □

Remark 4.4.1.9 ► On the Failure of $- \triangleright Y$ To Be a Left Adjoint

Here is some intuition on why $- \triangleright Y$ fails to be a left adjoint. Item 4 of Proposition 4.3.1.7 states that we have a natural bijection

$$\mathrm{Hom}_{\mathsf{Sets}_*}(X \triangleright Y, Z) \cong \mathrm{Hom}_{\mathsf{Sets}}(|X|, \mathsf{Sets}_*(Y, Z)),$$

so it would be reasonable to wonder whether a natural bijection of the form

$$\mathrm{Hom}_{\mathsf{Sets}_*}(X \triangleright Y, Z) \cong \mathrm{Hom}_{\mathsf{Sets}_*}(X, \mathbf{Sets}_*(Y, Z)),$$

also holds, which would give $- \rhd Y \dashv \mathbf{Sets}_*(Y, -)$. However, such a bijection would require every map

$$f : X \rhd Y \to Z$$

to satisfy

$$f(x_0 \rhd y) = z_0$$

for each $x \in X$, whereas we are imposing such a basepoint preservation condition only for elements of the form $x \rhd y_0$. Thus $\mathbf{Sets}_*(Y, -)$ can't be a right adjoint for $- \rhd Y$, and as shown by Item 3 of Proposition 4.4.1.7, no functor can.[1]

[1]The functor $\mathbf{Sets}_*(Y, -)$ is instead right adjoint to $- \wedge Y$, the smash product of pointed sets of Definition 4.5.1.1. See Item 2 of Proposition 4.5.1.10.

4.4.2 The Right Internal Hom of Pointed Sets

Let (X, x_0) and (Y, y_0) be pointed sets.

Definition 4.4.2.1 ► The Right Internal Hom of Pointed Sets

The **right internal Hom of pointed sets** is the functor

$$[-,-]^{\rhd}_{\mathsf{Sets}_*} : \mathsf{Sets}_*^{\mathsf{op}} \times \mathsf{Sets}_* \to \mathsf{Sets}_*$$

defined as the composition

$$\mathsf{Sets}_*^{\mathsf{op}} \times \mathsf{Sets}_* \xrightarrow{忘 \times \mathrm{id}} \mathsf{Sets}^{\mathsf{op}} \times \mathsf{Sets}_* \xrightarrow{\pitchfork} \mathsf{Sets}_*,$$

where:

- $忘 : \mathsf{Sets}_* \to \mathsf{Sets}$ is the forgetful functor from pointed sets to sets.
- $\pitchfork : \mathsf{Sets}^{\mathsf{op}} \times \mathsf{Sets}_* \to \mathsf{Sets}_*$ is the cotensor functor of Item 1 of Proposition 4.2.2.6.

Proof 4.4.2.2 ► Proof of Definition 4.4.2.1

For a proof that $[-,-]^{\rhd}_{\mathsf{Sets}_*}$ is indeed the right internal Hom of Sets_* with respect to the right tensor product of pointed sets, see Item 2 of Proposition 4.4.1.7. ∎

Remark 4.4.2.3 ▶ Unwinding Definition 4.4.2.1, I: Comparison With $[-,-]^{\lhd}_{\mathsf{Sets}_*}$

We have

$$[-,-]^{\lhd}_{\mathsf{Sets}_*} = [-,-]^{\rhd}_{\mathsf{Sets}_*}.$$

Remark 4.4.2.4 ▶ Unwinding Definition 4.4.2.1, II: Universal Property

The right internal Hom of pointed sets satisfies the following universal property:

$$\mathsf{Sets}_*(X \rhd Y, Z) \cong \mathsf{Sets}_*\Big(Y, [X,Z]^{\rhd}_{\mathsf{Sets}_*}\Big)$$

That is to say, the following data are in bijection:

1. Pointed maps $f\colon X \rhd Y \to Z$.
2. Pointed maps $f\colon Y \to [X,Z]^{\rhd}_{\mathsf{Sets}_*}$.

Remark 4.4.2.5 ▶ Unwinding Definition 4.4.2.1, III: Explicit Description

In detail, the **right internal Hom of** (X, x_0) **and** (Y, y_0) is the pointed set $\Big([X,Y]^{\rhd}_{\mathsf{Sets}_*}, \big[(y_0)_{x\in X}\big]\Big)$ consisting of

- *The Underlying Set*. The set $[X,Y]^{\rhd}_{\mathsf{Sets}_*}$ defined by

$$\begin{aligned}[X,Y]^{\rhd}_{\mathsf{Sets}_*} &\overset{\text{def}}{=} |X| \pitchfork Y\\ &\cong \bigwedge_{x\in X} (Y, y_0),\end{aligned}$$

  where $|X|$ denotes the underlying set of (X, x_0);
- *The Underlying Basepoint*. The point $\big[(y_0)_{x\in X}\big]$ of $\bigwedge_{x\in X}(Y, y_0)$.

Proposition 4.4.2.6 ▶ Properties of Right Internal Homs of Pointed Sets

Let (X, x_0) and (Y, y_0) be pointed sets.

1. *Functoriality*. The assignments $X, Y, (X,Y) \mapsto [X,Y]^{\rhd}_{\mathsf{Sets}_*}$ define functors

$$\begin{aligned}[X,-]^{\rhd}_{\mathsf{Sets}_*} &\colon \mathsf{Sets}_* \to \mathsf{Sets}_*,\\ [-,Y]^{\rhd}_{\mathsf{Sets}_*} &\colon \mathsf{Sets}_*^{\mathsf{op}} \to \mathsf{Sets}_*,\\ [-_1,-_2]^{\rhd}_{\mathsf{Sets}_*} &\colon \mathsf{Sets}_*^{\mathsf{op}} \times \mathsf{Sets}_* \to \mathsf{Sets}_*.\end{aligned}$$

In particular, given pointed maps

$$
\begin{aligned}
f &\colon (X, x_0) \to (A, a_0),\\
g &\colon (Y, y_0) \to (B, b_0),
\end{aligned}
$$

the induced map

$$[f, g]^{\rhd}_{\mathsf{Sets}_*} \colon [A, Y]^{\rhd}_{\mathsf{Sets}_*} \to [X, B]^{\rhd}_{\mathsf{Sets}_*}$$

is given by

$$[f, g]^{\rhd}_{\mathsf{Sets}_*}\left(\left[(y_a)_{a \in A}\right]\right) \overset{\text{def}}{=} \left[\left(g\left(y_{f(x)}\right)\right)_{x \in X}\right]$$

for each $\left[(y_a)_{a \in A}\right] \in [A, Y]^{\rhd}_{\mathsf{Sets}_*}$.

2. *Adjointness I*. We have an adjunction

$$\left(X \rhd - \dashv [X, -]^{\rhd}_{\mathsf{Sets}_*}\right) \colon \begin{tikzcd} \mathsf{Sets}_* \arrow[r, bend left, "X \rhd -"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_*, \arrow[l, bend left, "{[X,-]^{\rhd}_{\mathsf{Sets}_*}}"] \end{tikzcd}$$

witnessed by a bijection of sets

$$\mathrm{Hom}_{\mathsf{Sets}_*}(X \rhd Y, Z) \cong \mathrm{Hom}_{\mathsf{Sets}_*}\left(Y, [X, Z]^{\rhd}_{\mathsf{Sets}_*}\right)$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, where $[X, Y]^{\rhd}_{\mathsf{Sets}_*}$ is the pointed set of Definition 4.4.2.1.

3. *Adjointness II*. The functor

$$- \rhd Y \colon \mathsf{Sets}_* \to \mathsf{Sets}_*$$

does not admit a right adjoint.

Proof 4.4.2.7 ► Proof of Proposition 4.4.2.6

Item 1: Functoriality

Clear.

Item 2: Adjointness I

This is a repetition of Item 2 of Proposition 4.4.1.7, and is proved there.

Item 3: Adjointness II

This is a repetition of Item 3 of Proposition 4.4.1.7, and is proved there. □

4.4.3 The Right Skew Unit

Definition 4.4.3.1 ▶ The Right Skew Unit of $\rhd$

The **right skew unit of the right tensor product of pointed sets** is the functor

$$\mathbb{1}^{\mathsf{Sets}_*,\rhd}\colon \mathrm{pt} \longrightarrow \mathsf{Sets}_*$$

defined by

$$\mathbb{1}^{\rhd}_{\mathsf{Sets}_*} \stackrel{\text{def}}{=} S^0.$$

4.4.4 The Right Skew Associator

Definition 4.4.4.1 ▶ The Right Skew Associator of $\rhd$

The **skew associator of the right tensor product of pointed sets** is the natural transformation

$$\alpha^{\mathsf{Sets}_*,\rhd}\colon \rhd\circ\left(\mathrm{id}_{\mathsf{Sets}_*}\times\rhd\right) \Longrightarrow \rhd\circ\left(\rhd\times\mathrm{id}_{\mathsf{Sets}_*}\right)\circ\boldsymbol{\alpha}^{\mathsf{Cats},-1}_{\mathsf{Sets}_*,\mathsf{Sets}_*,\mathsf{Sets}_*}$$

as in the diagram

$$\begin{tikzcd}
 & & (\mathsf{Sets}_*\times\mathsf{Sets}_*)\times\mathsf{Sets}_* \arrow[dr, "\rhd\times\mathrm{id}"] & \\
\mathsf{Sets}_*\times(\mathsf{Sets}_*\times\mathsf{Sets}_*) \arrow[urr, dashed, "\boldsymbol{\alpha}^{\mathsf{Cats},-1}_{\mathsf{Sets}_*,\mathsf{Sets}_*,\mathsf{Sets}_*}", "\sim"'] \arrow[dr, "\mathrm{id}\times\rhd"'] & & & \mathsf{Sets}_*\times\mathsf{Sets}_* \arrow[dl, "\rhd"] \\
 & \mathsf{Sets}_*\times\mathsf{Sets}_* \arrow[r, "\rhd"'] \arrow[urr, phantom, "\Nearrow\alpha^{\mathsf{Sets}_*,\rhd}"] & \mathsf{Sets}_*, &
\end{tikzcd}$$

whose component

$$\alpha^{\mathsf{Sets}_*,\rhd}_{X,Y,Z}\colon X\rhd(Y\rhd Z)\longrightarrow(X\rhd Y)\rhd Z$$

at $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$ is given by

$$\begin{aligned} X \rhd (Y \rhd Z) &\stackrel{\text{def}}{=} |X| \odot (Y \rhd Z) \\ &\stackrel{\text{def}}{=} |X| \odot (|Y| \odot Z) \\ &\cong \bigvee_{x \in X} (|Y| \odot Z) \\ &\cong \bigvee_{x \in X} \left(\bigvee_{y \in Y} Z \right) \\ &\to \bigvee_{[(x,y)] \in \bigvee_{x \in X} Y} Z \\ &\cong \bigvee_{[(x,y)] \in |X| \odot Y} Z \\ &\cong ||X| \odot Y| \odot Z \\ &\stackrel{\text{def}}{=} |X \rhd Y| \odot Z \\ &\stackrel{\text{def}}{=} (X \rhd Y) \rhd Z, \end{aligned}$$

where the map

$$\bigvee_{x \in X} \left(\bigvee_{y \in Y} Z \right) \to \bigvee_{[(x,y)] \in \bigvee_{x \in X} Y} Z$$

is given by $[(x, [(y, z)])] \mapsto [([(x, y)], z)]$.

Proof 4.4.4.2 ► Proof of Definition 4.4.4.1

(Proven below in a bit.)

Remark 4.4.4.3 ► Unwinding Definition 4.4.4.1

Unwinding the notation for elements, we have

$$\begin{aligned} [(x, [(y, z)])] &\stackrel{\text{def}}{=} [(x, y \rhd z)] \\ &\stackrel{\text{def}}{=} x \rhd (y \rhd z) \end{aligned}$$

and

$$\begin{aligned} [([(x, y)], z)] &\stackrel{\text{def}}{=} [(x \rhd y, z)] \\ &\stackrel{\text{def}}{=} (x \rhd y) \rhd z. \end{aligned}$$

So, in other words, $\alpha^{\mathsf{Sets}_*,\triangleright}_{X,Y,Z}$ acts on elements via

$$\alpha^{\mathsf{Sets}_*,\triangleright}_{X,Y,Z}(x \triangleright (y \triangleright z)) \overset{\text{def}}{=} (x \triangleright y) \triangleright z$$

for each $x \triangleright (y \triangleright z) \in X \triangleright (Y \triangleright Z)$.

Remark 4.4.4.4 ▶ Non-Invertibility of the Skew Associator of $\triangleright$

Taking $y = y_0$, we see that the morphism $\alpha^{\mathsf{Sets}_*,\triangleright}_{X,Y,Z}$ acts on elements as

$$\alpha^{\mathsf{Sets}_*,\triangleright}_{X,Y,Z}(x \triangleright (y_0 \triangleright z)) \overset{\text{def}}{=} (x \triangleright y_0) \triangleright z.$$

However, by the definition of $\triangleright$, we have $x \triangleright y_0 = x' \triangleright y_0$ for all $x, x' \in X$, preventing $\alpha^{\mathsf{Sets}_*,\triangleright}_{X,Y,Z}$ from being non-invertible.

Proof 4.4.4.5 ▶ Proof of Definition 4.4.4.1

Firstly, note that, given $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, the map

$$\alpha^{\mathsf{Sets}_*,\triangleright}_{X,Y,Z} \colon X \triangleright (Y \triangleright Z) \to (X \triangleright Y) \triangleright Z$$

is indeed a morphism of pointed sets, as we have

$$\alpha^{\mathsf{Sets}_*,\triangleright}_{X,Y,Z}(x_0 \triangleright (y_0 \triangleright z_0)) = (x_0 \triangleright y_0) \triangleright z_0.$$

Next, we claim that $\alpha^{\mathsf{Sets}_*,\triangleright}$ is a natural transformation. We need to show that, given morphisms of pointed sets

$$\begin{aligned} f &\colon (X, x_0) \to (X', x_0'), \\ g &\colon (Y, y_0) \to (Y', y_0'), \\ h &\colon (Z, z_0) \to (Z', z_0') \end{aligned}$$

the diagram

$$\begin{tikzcd} X \triangleright (Y \triangleright Z) \arrow[r, "f \triangleright (g \triangleright h)"] \arrow[d, "\alpha^{\mathsf{Sets}_*,\triangleright}_{X,Y,Z}"'] & X' \triangleright (Y' \triangleright Z') \arrow[d, "\alpha^{\mathsf{Sets}_*,\triangleright}_{X',Y',Z'}"] \\ (X \triangleright Y) \triangleright Z \arrow[r, "(f \triangleright g) \triangleright h"'] & (X' \triangleright Y') \triangleright Z' \end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
x \triangleright (y \triangleright z) \arrow[r, mapsto] \arrow[d, mapsto] & f(x) \triangleright (g(y) \triangleright h(z)) \arrow[d, mapsto] \\
(x \triangleright y) \triangleright z \arrow[r, mapsto] & (f(x) \triangleright g(y)) \triangleright h(z)
\end{tikzcd}$$

and hence indeed commutes, showing $\alpha^{\mathsf{Sets}_*,\triangleright}$ to be a natural transformation. This finishes the proof. ∎

4.4.5 The Right Skew Left Unitor

Definition 4.4.5.1 ► The Right Skew Left Unitor of $\triangleright$

The **skew left unitor of the right tensor product of pointed sets** is the natural transformation

$$\lambda^{\mathsf{Sets}_*,\triangleright} \colon \lambda^{\mathsf{Cats}_2}_{\mathsf{Sets}_*} \overset{\sim}{\Longrightarrow} \triangleright \circ \left(\mathbb{1}^{\mathsf{Sets}_*} \times \mathrm{id}_{\mathsf{Sets}_*}\right) \qquad \begin{tikzcd}
\mathrm{pt} \times \mathsf{Sets}_* \arrow[r, "\mathbb{1}^{\mathsf{Sets}_*} \times \mathrm{id}"] \arrow[dr, dashed, bend right, "\lambda^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}"'] & \mathsf{Sets}_* \times \mathsf{Sets}_* \arrow[d, "\triangleright"] \arrow[dl, phantom, "\Uparrow \lambda^{\mathsf{Sets}_*,\triangleright}"] \\
& \mathsf{Sets}_*,
\end{tikzcd}$$

whose component

$$\lambda^{\mathsf{Sets}_*,\triangleright}_X \colon X \to S^0 \triangleright X$$

at $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$ is given by the composition

$$\begin{aligned}
X &\to X \vee X \\
&\cong |S^0| \odot X \\
&\cong S^0 \triangleright X,
\end{aligned}$$

where $X \to X \vee X$ is the map sending X to the second factor of X in $X \vee X$.

Proof 4.4.5.2 ► Proof of Definition 4.4.5.1

(Proven below in a bit.)

Remark 4.4.5.3 ► Unwinding Definition 4.4.5.1

In other words, $\lambda_X^{\mathsf{Sets}_*,\rhd}$ acts on elements as

$$\lambda_X^{\mathsf{Sets}_*,\rhd}(x) \stackrel{\text{def}}{=} [(1,x)]$$

i.e. by

$$\lambda_X^{\mathsf{Sets}_*,\rhd}(x) \stackrel{\text{def}}{=} 1 \rhd x$$

for each $x \in X$.

Remark 4.4.5.4 ► Non-Invertibility of the Skew Left Unitor of $\rhd$

The morphism $\lambda_X^{\mathsf{Sets}_*,\rhd}$ is non-invertible, as it is non-surjective when viewed as a map of sets, since the elements $0 \rhd x$ of $S^0 \rhd X$ with $x \neq x_0$ are outside the image of $\lambda_X^{\mathsf{Sets}_*,\rhd}$, which sends x to $1 \rhd x$.

Proof 4.4.5.5 ► Proof of Definition 4.4.5.1

Firstly, note that, given $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, the map

$$\lambda_X^{\mathsf{Sets}_*,\rhd} : X \to S^0 \rhd X$$

is indeed a morphism of pointed sets, as we have

$$\begin{aligned} \lambda_X^{\mathsf{Sets}_*,\rhd}(x_0) &= 1 \rhd x_0 \\ &= 0 \rhd x_0. \end{aligned}$$

Next, we claim that $\lambda^{\mathsf{Sets}_*,\rhd}$ is a natural transformation. We need to show that, given a morphism of pointed sets

$$f : (X, x_0) \to (Y, y_0),$$

the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\lambda_X^{\mathsf{Sets}_*,\rhd}"'] & Y \arrow[d, "\lambda_Y^{\mathsf{Sets}_*,\rhd}"] \\ S^0 \rhd X \arrow[r, "\mathrm{id}_{S^0} \rhd f"'] & S^0 \rhd Y \end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
x \arrow[r, mapsto] \arrow[d, mapsto] & f(x) \arrow[d, mapsto] \\
1 \rhd x \arrow[r, mapsto] & 1 \rhd f(x)
\end{tikzcd}$$

and hence indeed commutes, showing $\lambda^{\mathsf{Sets}_*,\rhd}$ to be a natural transformation. This finishes the proof.

4.4.6 The Right Skew Right Unitor

Definition 4.4.6.1 ► The Right Skew Right Unitor of $\rhd$

The **skew right unitor of the right tensor product of pointed sets** is the natural transformation

$$\rho^{\mathsf{Sets}_*,\rhd} \colon \rhd \circ \left(\mathrm{id} \times \mathbb{1}^{\mathsf{Sets}_*}\right) \overset{\sim}{\Longrightarrow} \rho^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}, \qquad \begin{tikzcd}
\mathsf{Sets}_* \times \mathrm{pt} \arrow[r, "\mathrm{id}\times\mathbb{1}^{\mathsf{Sets}_*}"] \arrow[dr, dashed, bend right, "\rho^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}"'] & \mathsf{Sets}_* \times \mathsf{Sets}_* \arrow[d, "\rhd"] \arrow[dl, phantom, "\Swarrow \rho^{\mathsf{Sets}_*,\rhd}"] \\
 & \mathsf{Sets}_*,
\end{tikzcd}$$

whose component

$$\rho^{\mathsf{Sets}_*,\rhd}_X \colon X \rhd S^0 \to X$$

at $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$ is given by the composition

$$\begin{aligned}
X \rhd S^0 &\cong |X| \odot S^0 \\
&\cong \bigvee_{x\in X} S^0 \\
&\to X,
\end{aligned}$$

where $\bigvee_{x\in X} S^0 \to X$ is the map given by

$$\begin{aligned}
[(x, 0)] &\mapsto x_0, \\
[(x, 1)] &\mapsto x.
\end{aligned}$$

Proof 4.4.6.2 ► Proof of Definition 4.4.6.1

(Proven below in a bit.)

Remark 4.4.6.3 ► Unwinding Definition 4.4.6.1

In other words, $\rho_X^{\mathsf{Sets}_*,\triangleright}$ acts on elements as

$$\begin{aligned}\rho_X^{\mathsf{Sets}_*,\triangleright}(x\triangleright 0) &\overset{\text{def}}{=} x_0,\\ \rho_X^{\mathsf{Sets}_*,\triangleright}(x\triangleright 1) &\overset{\text{def}}{=} x\end{aligned}$$

for each $x\triangleright 1\in X\triangleright S^0$.

Remark 4.4.6.4 ► Non-Invertibility of the Skew Right Unitor of $\triangleright$

The morphism $\rho_X^{\mathsf{Sets}_*,\triangleright}$ is almost invertible, with its would-be-inverse

$$\phi_X\colon X\to X\triangleright S^0$$

given by

$$\phi_X(x)\overset{\text{def}}{=} x\triangleright 1$$

for each $x\in X$. Indeed, we have

$$\begin{aligned}\left[\rho_X^{\mathsf{Sets}_*,\triangleright}\circ\phi\right](x) &= \rho_X^{\mathsf{Sets}_*,\triangleright}(\phi(x))\\ &= \rho_X^{\mathsf{Sets}_*,\triangleright}(x\triangleright 1)\\ &= x\\ &= [\mathrm{id}_X](x)\end{aligned}$$

so that

$$\rho_X^{\mathsf{Sets}_*,\triangleright}\circ\phi=\mathrm{id}_X$$

and

$$\begin{aligned}\left[\phi\circ\rho_X^{\mathsf{Sets}_*,\triangleright}\right](x\triangleright 1) &= \phi\left(\rho_X^{\mathsf{Sets}_*,\triangleright}(x\triangleright 1)\right)\\ &= \phi(x)\\ &= x\triangleright 1\\ &= [\mathrm{id}_{X\triangleright S^0}](x\triangleright 1),\end{aligned}$$

but

$$\begin{aligned}\left[\phi\circ\rho_X^{\mathsf{Sets}_*,\triangleright}\right](x\triangleright 0) &= \phi\left(\rho_X^{\mathsf{Sets}_*,\triangleright}(x\triangleright 0)\right)\\ &= \phi(x_0)\\ &= 1\triangleright x_0,\end{aligned}$$

where $x \rhd 0 \neq 1 \rhd x_0$. Thus

$$\phi \circ \rho_X^{\mathsf{Sets}_*,\rhd} \stackrel{?}{=} \mathrm{id}_{X \rhd S^0}$$

holds for all elements in $X \rhd S^0$ except one.

Proof 4.4.6.5 ► Proof of Definition 4.4.6.1

Firstly, note that, given $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, the map

$$\rho_X^{\mathsf{Sets}_*,\rhd} : X \rhd S^0 \to X$$

is indeed a morphism of pointed sets as we have

$$\rho_X^{\mathsf{Sets}_*,\rhd}(x_0 \rhd 0) = x_0.$$

Next, we claim that $\rho^{\mathsf{Sets}_*,\rhd}$ is a natural transformation. We need to show that, given a morphism of pointed sets

$$f : (X, x_0) \to (Y, y_0),$$

the diagram

$$\begin{tikzcd} X \rhd S^0 \arrow[r, "f \rhd \mathrm{id}_{S^0}"] \arrow[d, "\rho_X^{\mathsf{Sets}_*,\rhd}"'] & Y \rhd S^0 \arrow[d, "\rho_Y^{\mathsf{Sets}_*,\rhd}"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd} x \rhd 0 \arrow[d, mapsto] & \\ x_0 \arrow[r, mapsto] & f(x_0) \end{tikzcd} \qquad \begin{tikzcd} x \rhd 0 \arrow[r, mapsto] & f(x) \rhd 0 \arrow[d, mapsto] \\ & y_0 \end{tikzcd}$$

and

$$\begin{tikzcd} x \rhd 1 \arrow[r, mapsto] \arrow[d, mapsto] & f(x) \rhd 1 \arrow[d, mapsto] \\ x \arrow[r, mapsto] & f(x) \end{tikzcd}$$

and hence indeed commutes, showing $\rho^{\mathsf{Sets}_*,\rhd}$ to be a natural transformation. This finishes the proof. □

4.4.7 The Diagonal

The **diagonal of the right tensor product of pointed sets** is the natural transformation

$$\Delta^{\rhd}\colon \mathrm{id}_{\mathsf{Sets}_*} \Longrightarrow \rhd \circ \Delta^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}, \qquad \begin{tikzcd} \mathsf{Sets}_* \arrow[rr, bend left, "\mathrm{id}_{\mathsf{Sets}_*}"] \arrow[dr, swap, "\Delta^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}"] \arrow[rr, phantom, "\Downarrow \Delta^{\rhd}"] & & \mathsf{Sets}_* \\ & \mathsf{Sets}_* \times \mathsf{Sets}_*, \arrow[ur, swap, "\rhd"] & \end{tikzcd}$$

whose component

$$\Delta^{\rhd}_X\colon (X, x_0) \to (X \rhd X, x_0 \rhd x_0)$$

at $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$ is given by

$$\Delta^{\rhd}_X(x) \stackrel{\text{def}}{=} x \rhd x$$

for each $x \in X$.

Proof 4.4.7.2 ▶ Proof of Definition 4.4.7.1

Being a Morphism of Pointed Sets

We have

$$\Delta^{\rhd}_X(x_0) \stackrel{\text{def}}{=} x_0 \rhd x_0,$$

and thus $\Delta^{\rhd}_X$ is a morphism of pointed sets.

Naturality

We need to show that, given a morphism of pointed sets

$$f\colon (X, x_0) \to (Y, y_0),$$

the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, swap, "\Delta^{\rhd}_X"] & Y \arrow[d, "\Delta^{\rhd}_Y"] \\ X \rhd X \arrow[r, swap, "f \rhd f"] & Y \rhd Y \end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
x \arrow[r, mapsto] \arrow[d, mapsto] & f(x) \arrow[d, mapsto] \\
x \triangleright x \arrow[r, mapsto] & f(x) \triangleright f(x)
\end{tikzcd}$$

and hence indeed commutes, showing $\Delta^{\triangleright}$ to be natural.

4.4.8 The Right Skew Monoidal Structure on Pointed Sets Associated to $\triangleright$

Proposition 4.4.8.1 ► The Right Skew Monoidal Structure on Pointed Sets Associated to $\triangleright$

The category Sets_* admits a right-closed right skew monoidal category structure consisting of

- *The Underlying Category.* The category Sets_* of pointed sets;
- *The Right Skew Monoidal Product.* The right tensor product functor

$$\triangleright\colon \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*$$

  of Definition 4.4.1.1;
- *The Right Internal Skew Hom.* The right internal Hom functor

$$[-,-]^{\triangleright}_{\mathsf{Sets}_*}\colon \mathsf{Sets}^{\mathsf{op}}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*$$

  of Definition 4.4.2.1;
- *The Right Skew Monoidal Unit.* The functor

$$\mathbb{1}^{\mathsf{Sets}_*,\triangleright}\colon \mathsf{pt} \to \mathsf{Sets}_*$$

  of Definition 4.4.3.1;
- *The Right Skew Associators.* The natural transformation

$$\alpha^{\mathsf{Sets}_*,\triangleright}\colon \triangleright \circ (\mathsf{id}_{\mathsf{Sets}_*} \times \triangleright) \Longrightarrow \triangleright \circ (\triangleright \times \mathsf{id}_{\mathsf{Sets}_*}) \circ \boldsymbol{\alpha}^{\mathsf{Cats},-1}_{\mathsf{Sets}_*,\mathsf{Sets}_*,\mathsf{Sets}_*}$$

  of Definition 4.4.4.1;

- *The Right Skew Left Unitors.* The natural transformation

$$\lambda^{\mathsf{Sets}_*,\triangleright} \colon \boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*} \overset{\sim}{\Longrightarrow} \triangleright \circ \left(\mathbb{1}^{\mathsf{Sets}_*} \times \mathrm{id}_{\mathsf{Sets}_*}\right)$$

of Definition 4.4.5.1;

- *The Right Skew Right Unitors.* The natural transformation

$$\rho^{\mathsf{Sets}_*,\triangleright} \colon \triangleright \circ \left(\mathrm{id} \times \mathbb{1}^{\mathsf{Sets}_*}\right) \overset{\sim}{\Longrightarrow} \boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}$$

of Definition 4.4.6.1.

PROOF 4.4.8.2 ► PROOF OF PROPOSITION 4.4.8.1

The Pentagon Identity

Let (W, w_0), (X, x_0), (Y, y_0) and (Z, z_0) be pointed sets. We have to show that the diagram

$$\begin{tikzcd}
 & W \triangleright ((X \triangleright Y) \triangleright Z) \arrow[dr, "\alpha^{\mathsf{Sets}_*,\triangleright}_{W,X\triangleright Y,Z}"] & \\
W \triangleright (X \triangleright (Y \triangleright Z)) \arrow[ur, "\alpha^{\mathsf{Sets}_*,\triangleright}_{W,X,Y} \triangleright \mathrm{id}_Z"] \arrow[d, "\alpha^{\mathsf{Sets}_*,\triangleright}_{W\triangleright X,Y,Z}"'] & & (W \triangleright (X \triangleright Y)) \triangleright Z \arrow[d, "\mathrm{id}_W \triangleright \alpha^{\mathsf{Sets}_*,\triangleright}_{X,Y,Z}"] \\
(W \triangleright X) \triangleright (Y \triangleright Z) \arrow[rr, "\alpha^{\mathsf{Sets}_*,\triangleright}_{W,X,Y\triangleright Z}"'] & & ((W \triangleright X) \triangleright Y) \triangleright Z
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
 & w \rhd ((x \rhd y) \rhd z) \arrow[dr, mapsto] & \\
w \rhd (x \rhd (y \rhd z)) \arrow[ur, mapsto] \arrow[d, mapsto] & & (w \rhd (x \rhd y)) \rhd z \arrow[d, mapsto] \\
(w \rhd x) \rhd (y \rhd z) \arrow[rr, mapsto] & & ((w \rhd x) \rhd y) \rhd z
\end{tikzcd}$$

and thus we see that the pentagon identity is satisfied.

The Right Skew Left Triangle Identity

Let (X, x_0) and (Y, y_0) be pointed sets. We have to show that the diagram

$$\begin{tikzcd}
X \rhd Y \arrow[d, "\lambda^{\mathsf{Sets}_*,\rhd}_{X \rhd Y}"'] \arrow[dr, "\lambda^{\mathsf{Sets}_*,\rhd}_{X} \rhd \mathrm{id}_Y"] & \\
S^0 \rhd (X \rhd Y) \arrow[r, "\alpha^{\mathsf{Sets}_*,\rhd}_{S^0,X,Y}"'] & \left(S^0 \rhd X\right) \rhd Y
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
x \rhd y \arrow[d, mapsto] \arrow[dr, mapsto] & \\
1 \rhd (x \rhd y) \arrow[r, mapsto] & (1 \rhd x) \rhd y
\end{tikzcd}$$

and hence indeed commutes. Thus the left skew triangle identity is satisfied.

The Right Skew Right Triangle Identity

Let (X, x_0) and (Y, y_0) be pointed sets. We have to show that the diagram

$$\begin{tikzcd}
X \rhd \left(Y \rhd S^0\right) \arrow[r, "\mathrm{id}_X \rhd \rho^{\mathsf{Sets}_*,\rhd}_Y"] \arrow[dr, "\alpha^{\mathsf{Sets}_*,\rhd}_{S^0,X,Y}"'] & (X \rhd Y) \rhd S^0 \arrow[d, "\rho^{\mathsf{Sets}_*,\rhd}_{X\rhd Y}"] \\
 & X \rhd Y
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
x \rhd (y \rhd 0) \arrow[r, mapsto] \arrow[dr, mapsto] & (x \rhd y) \rhd 0 \arrow[d, mapsto] \\
 & x \rhd y_0 = x_0 \rhd y_0
\end{tikzcd}$$

and

$$\begin{tikzcd}
x \rhd (y \rhd 1) \arrow[r, mapsto] \arrow[dr, mapsto] & (x \rhd y) \rhd 1 \arrow[d, mapsto] \\
 & x \rhd y
\end{tikzcd}$$

and hence indeed commutes. Thus the right skew triangle identity is satisfied.

The Right Skew Middle Triangle Identity

Let (X, x_0) and (Y, y_0) be pointed sets. We have to show that the diagram

$$\begin{tikzcd}
X \rhd Y \arrow[r, equal] \arrow[d, "\mathrm{id}_X \rhd \lambda^{\mathsf{Sets}_*,\rhd}_Y"'] & X \rhd Y \\
X \rhd \left(S^0 \rhd Y\right) \arrow[r, "\alpha^{\mathsf{Sets}_*,\rhd}_{X,S^0,Y}"'] & \left(X \rhd S^0\right) \rhd Y \arrow[u, "\rho^{\mathsf{Sets}_*,\rhd}_X \rhd \mathrm{id}_Y"']
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
x \rhd y \arrow[r, mapsto] \arrow[d, mapsto] & x \rhd y \\
x \rhd (1 \rhd y) \arrow[r, mapsto] & (x \rhd 1) \rhd y \arrow[u, mapsto]
\end{tikzcd}$$

and hence indeed commutes. Thus the right skew triangle identity is satisfied.

The Zig-Zag Identity

We have to show that the diagram

$$\begin{tikzcd} S^0 \arrow[r, "\lambda^{\mathsf{Sets}_*,\rhd}_{S^0}"] \arrow[dr, equal] & S^0 \rhd S^0 \arrow[d, "\rho^{\mathsf{Sets}_*,\rhd}_{S^0}"] \\ & S^0 \end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd} 0 \arrow[r, mapsto] & 1 \rhd 0 \arrow[d, mapsto] \\ & 0 \arrow[ul, mapsto] \end{tikzcd}$$

and

$$\begin{tikzcd} 1 \arrow[r, mapsto] & 1 \rhd 1 \arrow[d, mapsto] \\ & 1 \arrow[ul, mapsto] \end{tikzcd}$$

and hence indeed commutes. Thus the zig-zag identity is satisfied.

Right Skew Monoidal Right-Closedness

This follows from Item 2 of Proposition 4.4.1.7.

4.4.9 Monoids With Respect to the Right Tensor Product of Pointed Sets

Proposition 4.4.9.1 ▶ Monoids With Respect to $\rhd$

The category of monoids on $(\mathsf{Sets}_*, \rhd, S^0)$ is isomorphic to the category of "monoids with right zero"[1] and morphisms between them.

[1] A monoid with right zero is defined similarly as the monoids with zero of **??**. Succinctly, they are monoids (A, μ_A, η_A) with a special element 0_A satisfying

$$0_A a = 0_A$$

for each $a \in A$.

PROOF 4.4.9.2 ► PROOF OF PROPOSITION 4.4.9.1

Monoids on $(\mathsf{Sets}_*, \rhd, S^0)$

A monoid on $(\mathsf{Sets}_*, \rhd, S^0)$ consists of:

- *The Underlying Object.* A pointed set $(A, 0_A)$.
- *The Multiplication Morphism.* A morphism of pointed sets
$$\mu_A \colon A \rhd A \to A,$$
determining a right bilinear morphism of pointed sets
$$\begin{aligned} A \times A &\longrightarrow A \\ (a, b) &\longmapsto ab. \end{aligned}$$
- *The Unit Morphism.* A morphism of pointed sets
$$\eta_A \colon S^0 \to A$$
picking an element 1_A of A.

satisfying the following conditions:

1. *Associativity.* The diagram
$$\begin{tikzcd} & A \rhd (A \rhd A) \arrow[dr, "\mathrm{id}_A \rhd \mu_A"] & \\ (A \rhd A) \rhd A \arrow[ur, "\alpha^{\mathsf{Sets}_*, \rhd}_{A,A,A}"] \arrow[d, "\mu_A \rhd \mathrm{id}_A"'] & & A \rhd A \arrow[d, "\mu_A"] \\ A \rhd A \arrow[rr, "\mu_A"'] & & A \end{tikzcd}$$
2. *Left Unitality.* The diagram
$$\begin{tikzcd} A \arrow[r, "\lambda^{\mathsf{Sets}_*, \rhd}_A"] \arrow[d, equal] & S^0 \rhd A \arrow[d, "\eta_A \times \mathrm{id}_A"] \\ A & A \rhd A \arrow[l, "\mu_A"] \end{tikzcd}$$
commutes.

3. *Right Unitality.* The diagram

$$\begin{tikzcd} A \rhd S^0 \arrow[r, "\mathrm{id}_A \times \eta_A"] \arrow[dr, "\rho_A^{\mathsf{Sets}_*,\rhd}"'] & A \rhd A \arrow[d, "\mu_A"] \\ & A \end{tikzcd}$$

   commutes.

Being a right-bilinear morphism of pointed sets, the multiplication map satisfies

$$0_A a = 0_A$$

for each $a \in A$. Now, the associativity, left unitality, and right unitality conditions act on elements as follows:

1. *Associativity.* The associativity condition acts as

$$\begin{tikzcd} (a \rhd b) \rhd c \arrow[d, mapsto] & & \\ ab \rhd c \arrow[r, mapsto] & (ab)c & \end{tikzcd} \qquad \begin{tikzcd} & a \rhd (b \rhd c) \arrow[dr, mapsto] & \\ (a \rhd b) \rhd c \arrow[ur, mapsto] & & a \rhd bc \arrow[d, mapsto] \\ & & a(bc) \end{tikzcd}$$

   This gives

$$(ab)c = a(bc)$$

   for each $a, b, c \in A$.

2. *Left Unitality.* The left unitality condition acts as

$$\begin{tikzcd} a \arrow[d, mapsto] \\ a \end{tikzcd} \qquad \begin{tikzcd} a \arrow[r, mapsto] & 1 \rhd a \arrow[d, mapsto] \\ 1_A a & 1_A \rhd a \arrow[l, mapsto] \end{tikzcd}$$

   This gives

$$1_A a = a$$

   for each $a \in A$.

3. *Right Unitality.* The right unitality condition acts:

   (a) On $1 \rhd 0$ as

$$\begin{tikzcd} 1 \rhd 0 \arrow[dr, mapsto] & \\ & 0_A \end{tikzcd} \qquad \begin{tikzcd} a \rhd 0 \arrow[r, mapsto] \arrow[dr, mapsto] & a \rhd 0_A \arrow[d, mapsto] \\ & a0_A. \end{tikzcd}$$

   (b) On $a \rhd 1$ as

$$\begin{tikzcd} a \rhd 1 \arrow[dr, mapsto] & \\ & a \end{tikzcd} \qquad \begin{tikzcd} a \rhd 1 \arrow[r, mapsto] \arrow[dr, mapsto] & a \rhd 1_A \arrow[d, mapsto] \\ & a1_A. \end{tikzcd}$$

   This gives

$$\begin{aligned} a1_A &= a, \\ a0_A &= 0_A \end{aligned}$$

   for each $a \in A$.

Thus we see that monoids with respect to $\rhd$ are exactly monoids with right zero.

Morphisms of Monoids on $(\mathsf{Sets}_*, \rhd, S^0)$

A morphism of monoids on $(\mathsf{Sets}_*, \rhd, S^0)$ from $(A, \mu_A, \eta_A, 0_A)$ to $(B, \mu_B, \eta_B, 0_B)$ is a morphism of pointed sets

$$f \colon (A, 0_A) \to (B, 0_B)$$

satisfying the following conditions:

1. *Compatibility With the Multiplication Morphisms.* The diagram

$$\begin{tikzcd} A \rhd A \arrow[r, "f \rhd f"] \arrow[d, "\mu_A"'] & B \rhd B \arrow[d, "\mu_B"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

   commutes.

2. *Compatibility With the Unit Morphisms.* The diagram

$$\begin{tikzcd} S^0 \arrow[r, "\eta_A"] \arrow[dr, "\eta_B"'] & A \arrow[d, "f"] \\ & B \end{tikzcd}$$

commutes.

These act on elements as

$$\begin{tikzcd} a \triangleright b \arrow[d, mapsto] & \\ ab \arrow[r, mapsto] & f(ab) \end{tikzcd} \qquad \begin{tikzcd} a \triangleright b \arrow[r, mapsto] & f(a) \triangleright f(b) \arrow[d, mapsto] \\ & f(a)f(b) \end{tikzcd}$$

and

$$\begin{tikzcd} 0 \arrow[dr, mapsto] & \\ & 0_B \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[r, mapsto] & 0_A \arrow[d, mapsto] \\ & f(0_A) \end{tikzcd}$$

and

$$\begin{tikzcd} 1 \arrow[dr, mapsto] & \\ & 1_B \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[r, mapsto] & 1_A \arrow[d, mapsto] \\ & f(1_A) \end{tikzcd}$$

giving

$$\begin{aligned} f(ab) &= f(a)f(b), \\ f(0_A) &= 0_B, \\ f(1_A) &= 1_B, \end{aligned}$$

for each $a, b \in A$, which is exactly a morphism of monoids with right zero.

Identities and Composition

Similarly, the identities and composition of $\mathrm{Mon}(\mathsf{Sets}_*, \triangleright, S^0)$ can be easily seen to agree with those of monoids with right zero, which finishes the proof. ∎

4.5 The Smash Product of Pointed Sets

4.5.1 Foundations

Let (X, x_0) and (Y, y_0) be pointed sets.

Definition 4.5.1.1 ► Smash Products of Pointed Sets

The **smash product of** (X, x_0) **and** (Y, y_0)[1] is the pointed set $X \wedge Y$[2] satisfying the bijection

$$\mathsf{Sets}_*(X \wedge Y, Z) \cong \mathrm{Hom}^{\otimes}_{\mathsf{Sets}_*}(X \times Y, Z),$$

naturally in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

[1] *Further Terminology:* In the context of monoids with zero as models for $\mathbb{F}_1$-algebras, the smash product $X \wedge Y$ is also called the **tensor product of** $\mathbb{F}_1$**-modules of** (X, x_0) **and** (Y, y_0) or the **tensor product of** (X, x_0) **and** (Y, y_0) **over** $\mathbb{F}_1$.

[2] *Further Notation:* In the context of monoids with zero as models for $\mathbb{F}_1$-algebras, the smash product $X \wedge Y$ is also denoted $X \otimes_{\mathbb{F}_1} Y$.

Remark 4.5.1.2 ► Unwinding Definition 4.5.1.1: The Universal Property I

That is to say, the smash product of pointed sets is defined so as to induce a bijection between the following data:

- Pointed maps $f : X \wedge Y \to Z$.
- Maps of sets $f : X \times Y \to Z$ satisfying

$$\begin{aligned} f(x_0, y) &= z_0, \\ f(x, y_0) &= z_0 \end{aligned}$$

  for each $x \in X$ and each $y \in Y$.

Remark 4.5.1.3 ► Unwinding Definition 4.5.1.1: The Universal Property II

The smash product of pointed sets may be described as follows:

- The smash product of (X, x_0) and (Y, y_0) is the pair $((X \wedge Y, x_0 \wedge y_0), \iota)$ consisting of
  - A pointed set $(X \wedge Y, x_0 \wedge y_0)$;
  - A bilinear morphism of pointed sets $\iota : (X \times Y, (x_0, y_0)) \to X \wedge Y$;

satisfying the following universal property:

(UP) Given another such pair $((Z, z_0), f)$ consisting of

- A pointed set (Z, z_0);
- A bilinear morphism of pointed sets $f\colon (X \times Y, (x_0, y_0)) \to X \wedge Y$;

there exists a unique morphism of pointed sets $X \wedge Y \xrightarrow{\exists!} Z$ making the diagram

$$\begin{tikzcd}
 & X \wedge Y \arrow[d, dashed, "\exists!"] \\
X \times Y \arrow[ur, "\iota"] \arrow[r, "f"'] & Z
\end{tikzcd}$$

commute.

Construction 4.5.1.4 ► Smash Products of Pointed Sets

Concretely, the **smash product of** (X, x_0) **and** (Y, y_0) is the pointed set $(X \wedge Y, x_0 \wedge y_0)$ consisting of

- *The Underlying Set.* The set $X \wedge Y$ defined by

$$X \wedge Y \cong (X \times Y)/{\sim_R},$$

where $\sim_R$ is the equivalence relation on $X \times Y$ obtained by declaring

$$\begin{aligned} (x_0, y) &\sim_R (x_0, y'), \\ (x, y_0) &\sim_R (x', y_0) \end{aligned}$$

for each $x, x' \in X$ and each $y, y' \in Y$;

- *The Basepoint.* The element $[(x_0, y_0)]$ of $X \wedge Y$ given by the equivalence class of (x_0, y_0) under the equivalence relation $\sim$ on $X \times Y$.

Proof 4.5.1.5 ► Proof of Construction 4.5.1.4

By Item 6 of Proposition 7.5.2.3, we have a natural bijection

$$\mathsf{Sets}_*(X \wedge Y, Z) \cong \mathrm{Hom}^R_{\mathsf{Sets}}(X \times Y, Z).$$

Now, by definition, $\mathrm{Hom}^R_{\mathsf{Sets}}(X \times Y, Z)$ is the set

$$\mathrm{Hom}^R_{\mathsf{Sets}}(X \times Y, Z) \stackrel{\text{def}}{=} \left\{ f \in \mathrm{Hom}_{\mathsf{Sets}}(X \times Y, Z) \;\middle|\; \begin{array}{l} \text{for each } x, y \in X, \text{ if} \\ (x, y) \sim_R (x', y'), \text{ then} \\ f(x, y) = f(x', y') \end{array} \right\}.$$

However, the condition $(x, y) \sim_R (x', y')$ only holds when:

1. We have $x = x'$ and $y = y'$.
2. The following conditions are satisfied:
   (a) We have $x = x_0$ or $y = y_0$.
   (b) We have $x' = x_0$ or $y' = y_0$.

So, given $f \in \mathrm{Hom}_{\mathsf{Sets}}(X \times Y, Z)$ with a corresponding $\overline{f} \colon X \wedge Y \to Z$, the latter case above implies

$$\begin{aligned} f(x_0, y) &= f(x, y_0) \\ &= f(x_0, y_0), \end{aligned}$$

and since $\overline{f} \colon X \wedge Y \to Z$ is a pointed map, we have

$$\begin{aligned} f(x_0, y_0) &= \overline{f}(x_0, y_0) \\ &= z_0. \end{aligned}$$

Thus the elements f in $\mathrm{Hom}_{\mathsf{Sets}}(X \times Y, Z)$ are precisely those functions $f \colon X \times Y \to Z$ satisfying the equalities

$$\begin{aligned} f(x_0, y) &= z_0, \\ f(x, y_0) &= z_0 \end{aligned}$$

for each $x \in X$ and each $y \in Y$, giving an equality

$$\mathrm{Hom}^R_{\mathsf{Sets}}(X \times Y, Z) = \mathrm{Hom}^{\otimes}_{\mathsf{Sets}_*}(X \times Y, Z)$$

of sets, which when composed with our earlier isomorphism

$$\mathsf{Sets}_*(X \wedge Y, Z) \cong \mathrm{Hom}^R_{\mathsf{Sets}}(X \times Y, Z)$$

gives our desired natural bijection, finishing the proof. ∎

Remark 4.5.1.6 ▶ On the Construction of the Smash Product of Pointed Sets

It is also somewhat common to write

$$X \wedge Y \stackrel{\text{def}}{=} \frac{X \times Y}{X \vee Y},$$

identifying $X \vee Y$ with the subspace $(\{x_0\} \times Y) \cup (X \times \{y_0\})$ of $X \times Y$, and having the quotient be defined by declaring $(x, y) \sim (x', y')$ iff we have $(x, y), (x', y') \in X \vee Y$.

Notation 4.5.1.7 ▶ Elements of Smash Products of Pointed Sets

We write $x \wedge y$ for the element $[(x, y)]$ of

$$X \wedge Y \cong X \times Y/\sim.$$

Remark 4.5.1.8 ▶ Basepoints of Smash Products of Pointed Sets

Employing the notation introduced in Notation 4.5.1.7, we have

$$\begin{aligned} x_0 \wedge y_0 &= x \wedge y_0, \\ &= x_0 \wedge y \end{aligned}$$

for each $x \in X$ and each $y \in Y$, and

$$\begin{aligned} x \wedge y_0 &= x' \wedge y_0, \\ x_0 \wedge y &= x_0 \wedge y' \end{aligned}$$

for each $x, x' \in X$ and each $y, y' \in Y$.

Example 4.5.1.9 ▶ Examples of Smash Products of Pointed Sets

Here are some examples of smash products of pointed sets.

1. *Smashing With pt.* For any pointed set X, we have isomorphisms of pointed sets

$$\begin{aligned} \mathrm{pt} \wedge X &\cong \mathrm{pt}, \\ X \wedge \mathrm{pt} &\cong \mathrm{pt}. \end{aligned}$$

2. *Smashing With S^0.* For any pointed set X, we have isomorphisms of

pointed sets

$$\begin{aligned} S^0 \wedge X &\cong X, \\ X \wedge S^0 &\cong X. \end{aligned}$$

PROPOSITION 4.5.1.10 ▶ PROPERTIES OF SMASH PRODUCTS OF POINTED SETS

Let (X, x_0) and (Y, y_0) be pointed sets.

1. *Functoriality.* The assignments $X, Y, (X, Y) \mapsto X \wedge Y$ define functors

$$\begin{aligned} X \wedge -\colon \mathsf{Sets}_* &\to \mathsf{Sets}_*, \\ - \wedge Y\colon \mathsf{Sets}_* &\to \mathsf{Sets}_*, \\ -_1 \wedge -_2\colon \mathsf{Sets}_* \times \mathsf{Sets}_* &\to \mathsf{Sets}_*. \end{aligned}$$

   In particular, given pointed maps

$$\begin{aligned} f\colon (X, x_0) &\to (A, a_0), \\ g\colon (Y, y_0) &\to (B, b_0), \end{aligned}$$

   the induced map

$$f \wedge g\colon X \wedge Y \to A \wedge B$$

   is given by

$$[f \wedge g](x \wedge y) \stackrel{\text{def}}{=} f(x) \wedge g(y)$$

   for each $x \wedge y \in X \wedge Y$.

2. *Adjointness.* We have adjunctions

$$(X \wedge - \dashv \mathbf{Sets}_*(X, -))\colon \quad \begin{tikzcd} \mathsf{Sets}_* \arrow[r, bend left, "X\wedge -"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_*, \arrow[l, bend left, "\mathbf{Sets}_*(X,-)"] \end{tikzcd}$$

$$(- \wedge Y \dashv \mathbf{Sets}_*(Y, -))\colon \quad \begin{tikzcd} \mathsf{Sets}_* \arrow[r, bend left, "-\wedge Y"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_*, \arrow[l, bend left, "\mathbf{Sets}_*(Y,-)"] \end{tikzcd}$$

   witnessed by bijections

$$\begin{aligned} \mathrm{Hom}_{\mathsf{Sets}_*}(X \wedge Y, Z) &\cong \mathrm{Hom}_{\mathsf{Sets}_*}(X, \mathbf{Sets}_*(Y, Z)), \\ \mathrm{Hom}_{\mathsf{Sets}_*}(X \wedge Y, Z) &\cong \mathrm{Hom}_{\mathsf{Sets}_*}(X, \mathbf{Sets}_*(A, Z)), \end{aligned}$$

   natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

3. *Enriched Adjointness.* We have Sets$_*$-enriched adjunctions

$$(X \wedge - \dashv \mathbf{Sets}_*(X, -)) \colon \quad \begin{tikzcd} \mathbf{Sets}_* \arrow[r, bend left, "X\wedge -"] \arrow[r, phantom, "\perp"] & \mathbf{Sets}_*, \arrow[l, bend left, "\mathbf{Sets}_*(X,-)"] \end{tikzcd}$$

$$(- \wedge Y \dashv \mathbf{Sets}_*(Y, -)) \colon \quad \begin{tikzcd} \mathbf{Sets}_* \arrow[r, bend left, "-\wedge Y"] \arrow[r, phantom, "\perp"] & \mathbf{Sets}_*, \arrow[l, bend left, "\mathbf{Sets}_*(Y,-)"] \end{tikzcd}$$

witnessed by isomorphisms of pointed sets

$$\mathbf{Sets}_*(X \wedge Y, Z) \cong \mathbf{Sets}_*(X, \mathbf{Sets}_*(Y, Z)),$$
$$\mathbf{Sets}_*(X \wedge Y, Z) \cong \mathbf{Sets}_*(X, \mathbf{Sets}_*(A, Z)),$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathbf{Sets}_*)$.

4. *As a Pushout.* We have an isomorphism

$$X \wedge Y \cong \mathrm{pt} \coprod_{X \vee Y} (X \times Y), \qquad \begin{tikzcd} X \wedge Y & X \times Y \arrow[l] \\ \mathrm{pt} \arrow[u] & X \vee Y, \arrow[l, "!"] \arrow[u, hook, "\iota"'] \end{tikzcd}$$

natural in $X, Y \in \mathrm{Obj}(\mathrm{Sets}_*)$, where the pushout is taken in Sets, and the embedding $\iota \colon X \vee Y \hookrightarrow X \times Y$ is defined following Remark 4.5.1.6.

5. *Distributivity Over Wedge Sums.* We have isomorphisms of pointed sets

$$X \wedge (Y \vee Z) \cong (X \wedge Y) \vee (X \wedge Z),$$
$$(X \vee Y) \wedge Z \cong (X \wedge Z) \vee (Y \wedge Z),$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathrm{Sets}_*)$.

Proof 4.5.1.11 ▶ Proof of Proposition 4.5.1.10

Item 1: Functoriality

The map $f \wedge g$ comes from Item 4 of Proposition 7.5.2.3 via the map

$$f \wedge g \colon X \times Y \to A \wedge B$$

sending (x, y) to $f(x) \wedge g(y)$, which we need to show satisfies

$$[f \wedge g](x, y) = [f \wedge g](x', y')$$

for each $(x, y), (x', y') \in X \times Y$ with $(x, y) \sim_R (x', y')$, where $\sim_R$ is the relation constructing $X \wedge Y$ as

$$X \wedge Y \cong (X \times Y)/{\sim_R}$$

in Construction 4.5.1.4. The condition defining $\sim$ is that at least one of the following conditions is satisfied:

1. We have $x = x'$ and $y = y'$;
2. Both of the following conditions are satisfied:
   (a) We have $x = x_0$ or $y = y_0$.
   (b) We have $x' = x_0$ or $y' = y_0$.

We have five cases:

1. In the first case, we clearly have
$$[f \wedge g](x, y) = [f \wedge g](x', y')$$
since $x = x'$ and $y = y'$.
2. If $x = x_0$ and $x' = x_0$, we have
$$\begin{aligned} [f \wedge g](x_0, y) &\stackrel{\text{def}}{=} f(x_0) \wedge g(y) \\ &= a_0 \wedge g(y) \\ &= a_0 \wedge g(y') \\ &= f(x_0) \wedge g(y') \\ &\stackrel{\text{def}}{=} [f \wedge g](x_0, y'). \end{aligned}$$
3. If $x = x_0$ and $y' = y_0$, we have
$$\begin{aligned} [f \wedge g](x_0, y) &\stackrel{\text{def}}{=} f(x_0) \wedge g(y) \\ &= a_0 \wedge g(y) \\ &= a_0 \wedge b_0 \\ &= f(x') \wedge b_0 \\ &= f(x') \wedge g(y_0) \\ &\stackrel{\text{def}}{=} [f \wedge g](x', y_0). \end{aligned}$$

4. If $y = y_0$ and $x' = x_0$, we have

$$\begin{aligned}[f \wedge g](x, y_0) &\overset{\text{def}}{=} f(x) \wedge g(y_0)\\ &= f(x) \wedge b_0\\ &= a_0 \wedge b_0\\ &= a_0 \wedge g(y')\\ &= f(x_0) \wedge g(y')\\ &\overset{\text{def}}{=} [f \wedge g](x_0, y').\end{aligned}$$

5. If $y = y_0$ and $y' = y_0$, we have

$$\begin{aligned}[f \wedge g](x, y_0) &\overset{\text{def}}{=} f(x) \wedge g(y_0)\\ &= f(x) \wedge b_0\\ &= f(x') \wedge b_0\\ &= f(x) \wedge g(y_0)\\ &\overset{\text{def}}{=} [f \wedge g](x', y_0).\end{aligned}$$

Thus $f \wedge g$ is well-defined. Next, we claim that $\wedge$ preserves identities and composition:

- *Preservation of Identities.* We have

$$\begin{aligned}[\mathrm{id}_X \wedge \mathrm{id}_Y](x \wedge y) &\overset{\text{def}}{=} \mathrm{id}_X(x) \wedge \mathrm{id}_Y(y)\\ &= x \wedge y\\ &= [\mathrm{id}_{X \wedge Y}](x \wedge y)\end{aligned}$$

  for each $x \wedge y \in X \wedge Y$, and thus

$$\mathrm{id}_X \wedge \mathrm{id}_Y = \mathrm{id}_{X \wedge Y}.$$

- *Preservation of Composition.* Given pointed maps

$$\begin{aligned}f &\colon (X, x_0) \to (X', x_0'),\\ h &\colon (X', x_0') \to (X'', x_0''),\\ g &\colon (Y, y_0) \to (Y', y_0'),\\ k &\colon (Y', y_0') \to (Y'', y_0''),\end{aligned}$$

we have

$$\begin{aligned}[(h \circ f) \wedge (k \circ g)](x \wedge y) &\overset{\text{def}}{=} h(f(x)) \wedge k(g(y)) \\ &\overset{\text{def}}{=} [h \wedge k](f(x) \wedge g(y)) \\ &\overset{\text{def}}{=} [h \wedge k]([f \wedge g](x \wedge y)) \\ &\overset{\text{def}}{=} [(h \wedge k) \circ (f \wedge g)](x \wedge y)\end{aligned}$$

for each $x \wedge y \in X \wedge Y$, and thus

$$(h \circ f) \wedge (k \circ g) = (h \wedge k) \circ (f \wedge g).$$

This finishes the proof.

Item 2: Adjointness

We prove only the adjunction $- \wedge Y \dashv \mathbf{Sets}_*(Y, -)$, witnessed by a natural bijection

$$\mathrm{Hom}_{\mathsf{Sets}_*}(X \wedge Y, Z) \cong \mathrm{Hom}_{\mathsf{Sets}_*}(X, \mathbf{Sets}_*(Y, Z)),$$

as the proof of the adjunction $X \wedge - \dashv \mathbf{Sets}_*(X, -)$ is similar. We claim we have a bijection

$$\mathrm{Hom}^{\otimes}_{\mathsf{Sets}_*}(X \times Y, Z) \cong \mathrm{Hom}_{\mathsf{Sets}_*}(X, \mathbf{Sets}_*(Y, Z))$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, impliying the desired adjunction. Indeed, this bijection is a restriction of the bijection

$$\mathsf{Sets}(X \times Y, Z) \cong \mathsf{Sets}(X, \mathsf{Sets}(Y, Z))$$

of Item 2 of Proposition 2.1.3.3:

- A map
$$\xi \colon X \times Y \to Z$$
in $\mathrm{Hom}^{\otimes}_{\mathsf{Sets}_*}(X \times Y, Z)$ gets sent to the pointed map
$$\begin{aligned} \xi^\dagger \colon (X, x_0) &\to (\mathbf{Sets}_*(Y, Z), \Delta_{z_0}), \\ x &\longmapsto \left(\xi^\dagger_x \colon Y \to Z\right), \end{aligned}$$
where $\xi^\dagger_x \colon Y \to Z$ is the map defined by
$$\xi^\dagger_x(y) \overset{\text{def}}{=} \xi(x, y)$$
for each $y \in Y$, where:

- The map $\xi^\dagger$ is indeed pointed, as we have

$$\begin{aligned}\xi^\dagger_{x_0}(y) &\overset{\text{def}}{=} \xi(x_0, y)\\ &\overset{\text{def}}{=} z_0\end{aligned}$$

  for each $y \in Y$. Thus $\xi^\dagger_{x_0} = \Delta_{z_0}$ and $\xi^\dagger$ is pointed.
- The map $\xi^\dagger_x$ indeed lies in $\mathbf{Sets}_*(Y, Z)$, as we have

$$\begin{aligned}\xi^\dagger_x(y_0) &\overset{\text{def}}{=} \xi(x, y_0)\\ &\overset{\text{def}}{=} z_0.\end{aligned}$$

· Conversely, a map

$$\begin{aligned}\xi\colon (X, x_0) &\to \left(\mathbf{Sets}_*(Y, Z), \Delta_{z_0}\right),\\ x &\longmapsto (\xi_x \colon Y \to Z),\end{aligned}$$

in $\mathrm{Hom}_{\mathsf{Sets}_*}(X, \mathbf{Sets}_*(Y, Z))$ gets sent to the map

$$\xi^\dagger \colon X \times Y \to Z$$

defined by

$$\xi^\dagger(x, y) \overset{\text{def}}{=} \xi_x(y)$$

for each $(x, y) \in X \times Y$, which indeed lies in $\mathrm{Hom}^{\otimes}_{\mathsf{Sets}_*}(X \times Y, Z)$, as:

- *Left Bilinearity.* We have

$$\begin{aligned}\xi^\dagger(x_0, y) &\overset{\text{def}}{=} \xi_{x_0}(y)\\ &\overset{\text{def}}{=} \Delta_{z_0}(y)\\ &\overset{\text{def}}{=} z_0\end{aligned}$$

  for each $y \in Y$, since $\xi_{x_0} = \Delta_{z_0}$ as ξ is assumed to be a pointed map.
- *Right Bilinearity.* We have

$$\begin{aligned}\xi^\dagger(x, y_0) &\overset{\text{def}}{=} \xi_x(y_0)\\ &\overset{\text{def}}{=} z_0\end{aligned}$$

  for each $x \in X$, since $\xi_x \in \mathbf{Sets}_*(Y, Z)$ is a morphism of pointed sets.

This finishes the proof.

Item 3: Enriched Adjointness

This follows from Item 2 and ?? of ??.

Item 4: As a Pushout

Following the description of Remark 2.2.4.3, we have

$$\mathrm{pt} \textstyle\coprod_{X \vee Y} (X \times Y) \cong (\mathrm{pt} \times (X \times Y))/\sim,$$

where $\sim$ identifies the elemenet $\star$ in pt with all elements of the form (x_0, y) and (x, y_0) in $X \times Y$. Thus Item 4 of Proposition 7.5.2.3 coupled with Remark 4.5.1.8 then gives us a well-defined map

$$\mathrm{pt} \textstyle\coprod_{X \vee Y} (X \times Y) \to X \wedge Y$$

via $[(\star, (x, y))] \mapsto x \wedge y$, with inverse

$$X \wedge Y \to \mathrm{pt} \textstyle\coprod_{X \vee Y} (X \times Y)$$

given by $x \wedge y \mapsto [(\star, (x, y))]$.

Item 5: Distributivity Over Wedge Sums

This follows from Proposition 4.5.9.1, ?? of ??, and the fact that $\vee$ is the coproduct in Sets_* (Definition 3.3.3.1).

4.5.2 The Internal Hom of Pointed Sets

Let (X, x_0) and (Y, y_0) be pointed sets.

DEFINITION 4.5.2.1 ▶ THE INTERNAL HOM OF POINTED SETS

The **internal Hom**[1] **of pointed sets from** (X, x_0) **to** (Y, y_0) is the pointed set $\mathbf{Sets}_*((X, x_0), (Y, y_0))$[2] consisting of:

- *The Underlying Set.* The set $\mathsf{Sets}_*((X, x_0), (Y, y_0))$ of morphisms of pointed sets from (X, x_0) to (Y, y_0).
- *The Basepoint.* The element

  $$\Delta_{y_0} : (X, x_0) \to (Y, y_0)$$

  of $\mathsf{Sets}_*((X, x_0), (Y, y_0))$ given by

  $$\Delta_{y_0}(x) \stackrel{\text{def}}{=} y_0$$

for each $x \in X$.

[1]The pointed set $\mathbf{Sets}_*(X, Y)$ is the internal **Hom** of Sets_* with respect to the smash product of Definition 4.5.1.1; see Item 2 of Proposition 4.5.1.10.

[2]*Further Notation:* Also written $\mathbf{Hom}_{\mathbf{Sets}_*}(X, Y)$.

Proof 4.5.2.2 ▶ Proof of Definition 4.5.2.1

For a proof that $\mathbf{Sets}_*$ is indeed the internal Hom of Sets_* with respect to the smash product of pointed sets, see Item 2 of Proposition 4.5.1.10.

Proposition 4.5.2.3 ▶ Properties of the Internal Hom of Pointed Sets

Let (X, x_0) and (Y, y_0) be pointed sets.

1. *Functoriality.* The assignments $X, Y, (X, Y) \mapsto \mathbf{Sets}_*(X, Y)$ define functors

$$\begin{aligned} \mathbf{Sets}_*(X, -) &\colon \mathsf{Sets}_* \to \mathsf{Sets}_*, \\ \mathbf{Sets}_*(-, Y) &\colon \mathsf{Sets}_*^{\mathsf{op}} \to \mathsf{Sets}_*, \\ \mathbf{Sets}_*(-_1, -_2) &\colon \mathsf{Sets}_*^{\mathsf{op}} \times \mathsf{Sets}_* \to \mathsf{Sets}_*. \end{aligned}$$

   In particular, given pointed maps

$$\begin{aligned} f &\colon (X, x_0) \to (A, a_0), \\ g &\colon (Y, y_0) \to (B, b_0), \end{aligned}$$

   the induced map

$$\mathbf{Sets}_*(f, g) \colon \mathbf{Sets}_*(A, Y) \to \mathbf{Sets}_*(X, B)$$

   is given by

$$[\mathbf{Sets}_*(f, g)](\phi) \stackrel{\text{def}}{=} g \circ \phi \circ f$$

   for each $\phi \in \mathbf{Sets}_*(A, Y)$.

2. *Adjointness.* We have adjunctions

$$(X \wedge - \dashv \mathbf{Sets}_*(X, -)) \colon \quad \begin{tikzcd} \mathsf{Sets}_* \arrow[r, bend left, "X \wedge -"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_*, \arrow[l, bend left, "\mathbf{Sets}_*(X, -)"] \end{tikzcd}$$

$$(- \wedge Y \dashv \mathbf{Sets}_*(Y, -)) \colon \quad \begin{tikzcd} \mathsf{Sets}_* \arrow[r, bend left, "- \wedge Y"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_*, \arrow[l, bend left, "\mathbf{Sets}_*(Y, -)"] \end{tikzcd}$$

witnessed by bijections

$$\begin{aligned}\mathrm{Hom}_{\mathsf{Sets}_*}(X \wedge Y, Z) &\cong \mathrm{Hom}_{\mathsf{Sets}_*}(X, \mathbf{Sets}_*(Y, Z)),\\ \mathrm{Hom}_{\mathsf{Sets}_*}(X \wedge Y, Z) &\cong \mathrm{Hom}_{\mathsf{Sets}_*}(X, \mathbf{Sets}_*(A, Z)),\end{aligned}$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

3. *Enriched Adjointness.* We have Sets_*-enriched adjunctions

$$(X \wedge - \dashv \mathbf{Sets}_*(X, -)): \quad \begin{tikzcd} \mathbf{Sets}_* \arrow[r, bend left, "X\wedge-"] \arrow[r, phantom, "\perp"] & \mathbf{Sets}_*, \arrow[l, bend left, "\mathbf{Sets}_*(X,-)"] \end{tikzcd}$$

$$(- \wedge Y \dashv \mathbf{Sets}_*(Y, -)): \quad \begin{tikzcd} \mathbf{Sets}_* \arrow[r, bend left, "-\wedge Y"] \arrow[r, phantom, "\perp"] & \mathbf{Sets}_*, \arrow[l, bend left, "\mathbf{Sets}_*(Y,-)"] \end{tikzcd}$$

witnessed by isomorphisms of pointed sets

$$\begin{aligned}\mathbf{Sets}_*(X \wedge Y, Z) &\cong \mathbf{Sets}_*(X, \mathbf{Sets}_*(Y, Z)),\\ \mathbf{Sets}_*(X \wedge Y, Z) &\cong \mathbf{Sets}_*(X, \mathbf{Sets}_*(A, Z)),\end{aligned}$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathbf{Sets}_*)$.

Proof 4.5.2.4 ▶ Proof of Proposition 4.5.2.3

Item 1: Functoriality

This follows from Item 1 of Proposition 2.3.5.2 and from the equalities

$$\begin{aligned}g \circ \Delta_{y_0} &= \Delta_{z_0},\\ \Delta_{y_0} \circ f &= \Delta_{y_0}\end{aligned}$$

for morphisms $f: (K, k_0) \to (X, x_0)$ and $g: (Y, y_0) \to (Z, z_0)$, which guarantee pre- and postcomposition by morphisms of pointed sets to also be morphisms of pointed sets.

Item 2: Adjointness

This is a repetition of Item 2 of Proposition 4.5.1.10, and is proved there.

Item 3: Enriched Adjointness

This is a repetition of Item 3 of Proposition 4.5.1.10, and is proved there. □

4.5.3 The Monoidal Unit

Definition 4.5.3.1 ► The Monoidal Unit of ∧

The **monoidal unit of the smash product of pointed sets** is the functor

$$\mathbb{1}^{\mathsf{Sets}_*}\colon \mathrm{pt} \longrightarrow \mathsf{Sets}_*$$

defined by

$$\mathbb{1}_{\mathsf{Sets}_*} \overset{\text{def}}{=} S^0.$$

4.5.4 The Associator

Definition 4.5.4.1 ► The Associator of ∧

The **associator of the smash product of pointed sets** is the natural isomorphism

$$\alpha^{\mathsf{Sets}_*}\colon \wedge \circ \left(\wedge \times \mathrm{id}_{\mathsf{Sets}_*}\right) \overset{\sim}{\Longrightarrow} \wedge \circ \left(\mathrm{id}_{\mathsf{Sets}_*} \times \wedge\right) \circ \boldsymbol{\alpha}^{\mathsf{Cats}}_{\mathsf{Sets}_*,\mathsf{Sets}_*,\mathsf{Sets}_*},$$

as in the diagram

$$\begin{tikzcd}
 & \mathsf{Sets}_* \times (\mathsf{Sets}_* \times \mathsf{Sets}_*) \arrow[dr, "\mathrm{id}\times\wedge"] & \\
(\mathsf{Sets}_* \times \mathsf{Sets}_*) \times \mathsf{Sets}_* \arrow[ur, dashed, "\boldsymbol{\alpha}^{\mathsf{Cats}}_{\mathsf{Sets}_*,\mathsf{Sets}_*,\mathsf{Sets}_*}", "\sim"'] \arrow[d, "\wedge\times\mathrm{id}"'] & & \mathsf{Sets}_* \times \mathsf{Sets}_* \arrow[d, "\wedge"] \\
\mathsf{Sets}_* \times \mathsf{Sets}_* \arrow[rr, "\wedge"'] \arrow[urr, Rightarrow, "\alpha^{\mathsf{Sets}_*}"'] & & \mathsf{Sets}_*,
\end{tikzcd}$$

whose component

$$\alpha^{\mathsf{Sets}_*}_{X,Y,Z}\colon (X \wedge Y) \wedge Z \xrightarrow{\cong} X \wedge (Y \wedge Z)$$

at $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$ is given by

$$\alpha^{\mathsf{Sets}_*}_{X,Y,Z}((x \wedge y) \wedge z) \overset{\text{def}}{=} x \wedge (y \wedge z)$$

for each $(x \wedge y) \wedge z \in (X \wedge Y) \wedge Z$.

Proof 4.5.4.2 ► Proof of Definition 4.5.4.1

Well-Definedness

Let $[((x,y),z)] = [((x',y'),z')]$ be an element in $(X \wedge Y) \wedge Z$. Then either:

1. We have $x = x'$, $y = y'$, and $z = z'$.
2. Both of the following conditions are satisfied:
   (a) We have $x = x_0$ or $y = y_0$ or $z = z_0$.
   (b) We have $x' = x_0$ or $y' = y_0$ or $z' = z_0$.

In the first case, $\alpha^{\mathsf{Sets}_*}_{X,Y,Z}$ clearly sends both elements to the same element in $X \wedge (Y \wedge Z)$. Meanwhile, in the latter case both elements are equal to the basepoint $(x_0 \wedge y_0) \wedge z_0$ of $(X \wedge Y) \wedge Z$, which gets sent to the basepoint $x_0 \wedge (y_0 \wedge z_0)$ of $X \wedge (Y \wedge Z)$.

Being a Morphism of Pointed Sets

As just mentioned, we have

$$\alpha^{\mathsf{Sets}_*}_{X,Y,Z}((x_0 \wedge y_0) \wedge z_0) \stackrel{\text{def}}{=} x_0 \wedge (y_0 \wedge z_0),$$

and thus $\alpha^{\mathsf{Sets}_*}_{X,Y,Z}$ is a morphism of pointed sets.

Invertibility

Clearly, the inverse of $\alpha^{\mathsf{Sets}_*}_{X,Y,Z}$ is given by the morphism

$$\alpha^{\mathsf{Sets}_*,-1}_{X,Y,Z} \colon X \wedge (Y \wedge Z) \xrightarrow{\cong} (X \wedge Y) \wedge Z$$

defined by

$$\alpha^{\mathsf{Sets}_*,-1}_{X,Y,Z}(x \wedge (y \wedge z)) \stackrel{\text{def}}{=} (x \wedge y) \wedge z$$

for each $x \wedge (y \wedge z) \in X \wedge (Y \wedge Z)$.

Naturality

We need to show that, given morphisms of pointed sets

$$\begin{aligned} f &\colon (X, x_0) \to (X', x_0'), \\ g &\colon (Y, y_0) \to (Y', y_0'), \\ h &\colon (Z, z_0) \to (Z', z_0') \end{aligned}$$

the diagram

$$\begin{tikzcd} (X \wedge Y) \wedge Z \arrow[r, "(f\wedge g)\wedge h"] \arrow[d, "\alpha^{\mathsf{Sets}_*}_{X,Y,Z}"'] & (X' \wedge Y') \wedge Z' \arrow[d, "\alpha^{\mathsf{Sets}_*}_{X',Y',Z'}"] \\ X \wedge (Y \wedge Z) \arrow[r, "f\wedge(g\wedge h)"'] & X' \wedge (Y' \wedge Z') \end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd} (x \wedge y) \wedge z \arrow[r, mapsto] \arrow[d, mapsto] & (f(x) \wedge g(y)) \wedge h(z) \arrow[d, mapsto] \\ x \wedge (y \wedge z) \arrow[r, mapsto] & f(x) \wedge (g(y) \wedge h(z)) \end{tikzcd}$$

and hence indeed commutes, showing α^{Sets_*} to be a natural transformation.

Being a Natural Isomorphism

Since α^{Sets_*} is natural and $\alpha^{\mathsf{Sets}_*,-1}$ is a componentwise inverse to α^{Sets_*}, it follows from Item 2 of Proposition 8.8.6.2 that $\alpha^{\mathsf{Sets}_*,-1}$ is also natural. Thus α^{Sets_*} is a natural isomorphism.

4.5.5 The Left Unitor

Definition 4.5.5.1 ► The Left Unitor of $\wedge$

The **left unitor of the smash product of pointed sets** is the natural isomorphism

$$\lambda^{\mathsf{Sets}_*} \colon \wedge \circ \left(\mathbb{1}^{\mathsf{Sets}_*} \times \mathrm{id}_{\mathsf{Sets}_*}\right) \overset{\sim}{\Longrightarrow} \lambda^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}$$

$$\begin{tikzcd} \mathrm{pt} \times \mathsf{Sets}_* \arrow[r, "\mathbb{1}^{\mathsf{Sets}_*} \times \mathrm{id}"] \arrow[dr, dashed, bend right, "\lambda^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}"'] & \mathsf{Sets}_* \times \mathsf{Sets}_* \arrow[d, "\wedge"] \arrow[dl, phantom, "\Swarrow \lambda^{\mathsf{Sets}_*}"] \\ & \mathsf{Sets}_*, \end{tikzcd}$$

whose component

$$\lambda^{\mathsf{Sets}_*}_X \colon S^0 \wedge X \xrightarrow{\cong} X$$

at $X \in \mathrm{Obj}(\mathsf{Sets}_*)$ is given by

$$\begin{aligned} 0 \wedge x &\mapsto x_0, \\ 1 \wedge x &\mapsto x. \end{aligned}$$

PROOF 4.5.5.2 ▶ PROOF OF DEFINITION 4.5.5.1

Well-Definedness

Let $[(x, y)] = [(x', y')]$ be an element in $S^0 \wedge X$. Then either:

1. We have $x = x'$ and $y = y'$.
2. Both of the following conditions are satisfied:
   (a) We have $x = 0$ or $y = x_0$.
   (b) We have $x' = 0$ or $y' = x_0$.

In the first case, $\lambda^{\mathsf{Sets}_*}_X$ clearly sends both elements to the same element in X. Meanwhile, in the latter case both elements are equal to the basepoint $0 \wedge x_0$ of $S^0 \wedge X$, which gets sent to the basepoint x_0 of X.

Being a Morphism of Pointed Sets

As just mentioned, we have

$$\lambda^{\mathsf{Sets}_*}_X(0 \wedge x_0) \stackrel{\text{def}}{=} x_0,$$

and thus $\lambda^{\mathsf{Sets}_*}_X$ is a morphism of pointed sets.

Invertibility

The inverse of $\lambda^{\mathsf{Sets}_*}_X$ is the morphism

$$\lambda^{\mathsf{Sets}_*,-1}_X \colon X \xrightarrow{\cong} S^0 \wedge X$$

defined by

$$\lambda^{\mathsf{Sets}_*,-1}_X(x) \stackrel{\text{def}}{=} 1 \wedge x$$

for each $x \in X$. Indeed:

- *Invertibility I.* We have

$$\begin{aligned} \left[\lambda^{\mathsf{Sets}_*,-1}_X \circ \lambda^{\mathsf{Sets}_*}_X\right](0 \wedge x) &= \lambda^{\mathsf{Sets}_*,-1}_X\left(\lambda^{\mathsf{Sets}_*}_X(0 \wedge x)\right) \\ &= \lambda^{\mathsf{Sets}_*,-1}_X(x_0) \\ &= 1 \wedge x_0 \\ &= 0 \wedge x, \end{aligned}$$

and

$$\begin{aligned}\left[\lambda_X^{\mathsf{Sets}_*,-1} \circ \lambda_X^{\mathsf{Sets}_*}\right](1 \wedge x) &= \lambda_X^{\mathsf{Sets}_*,-1}\Big(\lambda_X^{\mathsf{Sets}_*}(1 \wedge x)\Big)\\ &= \lambda_X^{\mathsf{Sets}_*,-1}(x)\\ &= 1 \wedge x\end{aligned}$$

for each $x \in X$, and thus we have

$$\lambda_X^{\mathsf{Sets}_*,-1} \circ \lambda_X^{\mathsf{Sets}_*} = \mathrm{id}_{S^0 \wedge X}.$$

- *Invertibility II.* We have

$$\begin{aligned}\left[\lambda_X^{\mathsf{Sets}_*} \circ \lambda_X^{\mathsf{Sets}_*,-1}\right](x) &= \lambda_X^{\mathsf{Sets}_*}\Big(\lambda_X^{\mathsf{Sets}_*,-1}(x)\Big)\\ &= \lambda_X^{\mathsf{Sets}_*,-1}(1 \wedge x)\\ &= x\end{aligned}$$

for each $x \in X$, and thus we have

$$\lambda_X^{\mathsf{Sets}_*} \circ \lambda_X^{\mathsf{Sets}_*,-1} = \mathrm{id}_X.$$

This shows $\lambda_X^{\mathsf{Sets}_*}$ to be invertible.

Naturality

We need to show that, given a morphism of pointed sets

$$f\colon (X, x_0) \to (Y, y_0),$$

the diagram

$$\begin{tikzcd} S^0 \wedge X \arrow[r, "\mathrm{id}_{S^0} \wedge f"] \arrow[d, "\lambda_X^{\mathsf{Sets}_*}"'] & S^0 \wedge Y \arrow[d, "\lambda_Y^{\mathsf{Sets}_*}"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd} 0 \wedge x \arrow[d, mapsto] & \\ x_0 \arrow[r, mapsto] & f(x_0) \end{tikzcd} \qquad \begin{tikzcd} 0 \wedge x \arrow[r, mapsto] & 0 \wedge f(x) \arrow[d, mapsto] \\ & y_0 \end{tikzcd}$$

and

$$\begin{tikzcd} 1 \wedge x \arrow[r, mapsto] \arrow[d, mapsto] & 1 \wedge f(x) \arrow[d, mapsto] \\ x \arrow[r, mapsto] & f(x) \end{tikzcd}$$

and hence indeed commutes, showing $\lambda^{\mathsf{Sets}_*}$ to be a natural transformation.

Being a Natural Isomorphism

Since $\lambda^{\mathsf{Sets}_*}$ is natural and $\lambda^{\mathsf{Sets}_*,-1}$ is a componentwise inverse to $\lambda^{\mathsf{Sets}_*}$, it follows from Item 2 of Proposition 8.8.6.2 that $\lambda^{\mathsf{Sets}_*,-1}$ is also natural. Thus $\lambda^{\mathsf{Sets}_*}$ is a natural isomorphism.

4.5.6 The Right Unitor

Definition 4.5.6.1 ▶ The Right Unitor of $\wedge$

The **right unitor of the smash product of pointed sets** is the natural isomorphism

$$\rho^{\mathsf{Sets}_*} \colon \wedge \circ \left(\mathrm{id} \times \mathbb{1}^{\mathsf{Sets}_*}\right) \overset{\sim}{\Longrightarrow} \rho^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}, \qquad \begin{tikzcd} \mathsf{Sets}_* \times \mathrm{pt} \arrow[r, "\mathrm{id}\times\mathbb{1}^{\mathsf{Sets}_*}"] \arrow[rd, dashed, bend right, "\rho^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}"'] & \mathsf{Sets}_* \times \mathsf{Sets}_* \arrow[d, "\wedge"] \arrow[ld, phantom, "\Swarrow \rho^{\mathsf{Sets}_*}"] \\ & \mathsf{Sets}_*, \end{tikzcd}$$

whose component

$$\rho^{\mathsf{Sets}_*}_X \colon X \wedge S^0 \xrightarrow{\cong} X$$

at $X \in \mathrm{Obj}(\mathsf{Sets}_*)$ is given by

$$x \wedge 0 \mapsto x_0,$$
$$x \wedge 1 \mapsto x.$$

Proof 4.5.6.2 ► Proof of Definition 4.5.6.1

Well-Definedness

Let $[(x, y)] = [(x', y')]$ be an element in $X \wedge S^0$. Then either:

1. We have $x = x'$ and $y = y'$.
2. Both of the following conditions are satisfied:
   (a) We have $x = x_0$ or $y = 0$.
   (b) We have $x' = x_0$ or $y' = 0$.

In the first case, $\rho_X^{\mathsf{Sets}_*}$ clearly sends both elements to the same element in X. Meanwhile, in the latter case both elements are equal to the basepoint $x_0 \wedge 0$ of $X \wedge S^0$, which gets sent to the basepoint x_0 of X.

Being a Morphism of Pointed Sets

As just mentioned, we have

$$\rho_X^{\mathsf{Sets}_*}(x_0 \wedge 0) \overset{\text{def}}{=} x_0,$$

and thus $\rho_X^{\mathsf{Sets}_*}$ is a morphism of pointed sets.

Invertibility

The inverse of $\rho_X^{\mathsf{Sets}_*}$ is the morphism

$$\rho_X^{\mathsf{Sets}_*,-1} \colon X \xrightarrow{\cong} X \wedge S^0$$

defined by

$$\rho_X^{\mathsf{Sets}_*,-1}(x) \overset{\text{def}}{=} x \wedge 1$$

for each $x \in X$. Indeed:

- *Invertibility I.* We have

$$\begin{aligned}
\left[\rho_X^{\mathsf{Sets}_*,-1} \circ \rho_X^{\mathsf{Sets}_*}\right](x \wedge 0) &= \rho_X^{\mathsf{Sets}_*,-1}\left(\rho_X^{\mathsf{Sets}_*}(x \wedge 0)\right) \\
&= \rho_X^{\mathsf{Sets}_*,-1}(x_0) \\
&= x_0 \wedge 1 \\
&= x \wedge 0,
\end{aligned}$$

  and

$$\begin{aligned}
\left[\rho_X^{\mathsf{Sets}_*,-1} \circ \rho_X^{\mathsf{Sets}_*}\right](x \wedge 1) &= \rho_X^{\mathsf{Sets}_*,-1}\left(\rho_X^{\mathsf{Sets}_*}(x \wedge 1)\right) \\
&= \rho_X^{\mathsf{Sets}_*,-1}(x) \\
&= x \wedge 1
\end{aligned}$$

for each $x \in X$, and thus we have

$$\rho_X^{\mathsf{Sets}_*,-1} \circ \rho_X^{\mathsf{Sets}_*} = \mathrm{id}_{X \wedge S^0}.$$

- *Invertibility II.* We have

$$\begin{aligned}
\left[\rho_X^{\mathsf{Sets}_*} \circ \rho_X^{\mathsf{Sets}_*,-1}\right](x) &= \rho_X^{\mathsf{Sets}_*}\left(\rho_X^{\mathsf{Sets}_*,-1}(x)\right)\\
&= \rho_X^{\mathsf{Sets}_*,-1}(x \wedge 1)\\
&= x
\end{aligned}$$

for each $x \in X$, and thus we have

$$\rho_X^{\mathsf{Sets}_*} \circ \rho_X^{\mathsf{Sets}_*,-1} = \mathrm{id}_X.$$

This shows $\rho_X^{\mathsf{Sets}_*}$ to be invertible.

Naturality

We need to show that, given a morphism of pointed sets

$$f\colon (X, x_0) \to (Y, y_0),$$

the diagram

$$\begin{tikzcd}
X \wedge S^0 \arrow[r, "f \wedge \mathrm{id}_{S^0}"] \arrow[d, "\rho_X^{\mathsf{Sets}_*}"'] & Y \wedge S^0 \arrow[d, "\rho_Y^{\mathsf{Sets}_*}"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
x \wedge 0 \arrow[d, mapsto] & \\
x_0 \arrow[r, mapsto] & f(x_0)
\end{tikzcd}
\qquad
\begin{tikzcd}
x \wedge 0 \arrow[r, mapsto] & f(x) \wedge 0 \arrow[d, mapsto] \\
 & y_0
\end{tikzcd}$$

and

$$\begin{tikzcd}
x \wedge 1 \arrow[r, mapsto] \arrow[d, mapsto] & f(x) \wedge 1 \arrow[d, mapsto] \\
x \arrow[r, mapsto] & f(x)
\end{tikzcd}$$

and hence indeed commutes, showing ρ^{Sets_*} to be a natural transformation.

Being a Natural Isomorphism

Since ρ^{Sets_*} is natural and $\rho^{\mathsf{Sets}_*,-1}$ is a componentwise inverse to ρ^{Sets_*}, it follows from Item 2 of Proposition 8.8.6.2 that $\rho^{\mathsf{Sets}_*,-1}$ is also natural. Thus ρ^{Sets_*} is a natural isomorphism.

4.5.7 The Symmetry

Definition 4.5.7.1 ► The Symmetry of $\wedge$

The **symmetry of the smash product of pointed sets** is the natural isomorphism

$$\sigma^{\mathsf{Sets}_*} \colon \wedge \overset{\sim}{\Longrightarrow} \wedge \circ \boldsymbol{\sigma}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*,\mathsf{Sets}_*}, \qquad \begin{tikzcd} \mathsf{Sets}_* \times \mathsf{Sets}_* \arrow[rr, "\wedge"] \arrow[dr, "\boldsymbol{\sigma}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*,\mathsf{Sets}_*}"'] & \arrow[d, phantom, "\overset{\|}{\Downarrow}\sigma^{\mathsf{Sets}_*}"] & \mathsf{Sets}_*, \\ & \mathsf{Sets}_* \times \mathsf{Sets}_* \arrow[ur, "\wedge"'] & \end{tikzcd}$$

whose component

$$\sigma^{\mathsf{Sets}_*}_{X,Y} \colon X \wedge Y \xrightarrow{\cong} Y \wedge X$$

at $X, Y \in \mathrm{Obj}(\mathsf{Sets}_*)$ is defined by

$$\sigma^{\mathsf{Sets}_*}_{X,Y}(x \wedge y) \overset{\text{def}}{=} y \wedge x$$

for each $x \wedge y \in X \wedge Y$.

Proof 4.5.7.2 ► Proof of Definition 4.5.6.1

Well-Definedness

Let $[(x, y)] = [(x', y')]$ be an element in $X \wedge Y$. Then either:

1. We have $x = x'$ and $y = y'$.
2. Both of the following conditions are satisfied:
   (a) We have $x = x_0$ or $y = y_0$.
   (b) We have $x' = x_0$ or $y' = y_0$.

In the first case, $\sigma^{\mathsf{Sets}_*}_X$ clearly sends both elements to the same element in

X. Meanwhile, in the latter case both elements are equal to the basepoint $x_0 \wedge y_0$ of $X \wedge Y$, which gets sent to the basepoint $y_0 \wedge x_0$ of $Y \wedge X$.

Being a Morphism of Pointed Sets

As just mentioned, we have

$$\sigma^{\mathsf{Sets}_*}_{X}(x_0 \wedge y_0) \overset{\text{def}}{=} y_0 \wedge x_0,$$

and thus $\sigma^{\mathsf{Sets}_*}_{X}$ is a morphism of pointed sets.

Invertibility

Clearly, the inverse of $\sigma^{\mathsf{Sets}_*}_{X,Y}$ is given by the morphism

$$\sigma^{\mathsf{Sets}_*,-1}_{X,Y} \colon Y \wedge X \xrightarrow{\cong} X \wedge Y$$

defined by

$$\sigma^{\mathsf{Sets}_*,-1}_{X,Y}(y \wedge x) \overset{\text{def}}{=} x \wedge y$$

for each $y \wedge x \in Y \wedge X$.

Naturality

We need to show that, given morphisms of pointed sets

$$f \colon (X, x_0) \to (A, a_0),$$
$$g \colon (Y, y_0) \to (B, b_0)$$

the diagram

$$\begin{tikzcd}
X \wedge Y \arrow[r, "f \wedge g"] \arrow[d, "\sigma^{\mathsf{Sets}_*}_{X,Y}"'] & A \wedge B \arrow[d, "\sigma^{\mathsf{Sets}_*}_{A,B}"] \\
Y \wedge X \arrow[r, "g \wedge f"'] & B \wedge A
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
x \wedge y \arrow[r, mapsto] \arrow[d, mapsto] & f(x) \wedge g(y) \arrow[d, mapsto] \\
y \wedge x \arrow[r, mapsto] & g(y) \wedge f(x)
\end{tikzcd}$$

and hence indeed commutes, showing σ^{Sets_*} to be a natural transformation.

Being a Natural Isomorphism

Since σ^{Sets_*} is natural and $\sigma^{\mathsf{Sets}_*,-1}$ is a componentwise inverse to σ^{Sets_*}, it follows from Item 2 of Proposition 8.8.6.2 that $\sigma^{\mathsf{Sets}_*,-1}$ is also natural. Thus σ^{Sets_*} is a natural isomorphism.

4.5.8 The Diagonal

Definition 4.5.8.1 ► The Diagonal of $\wedge$

The **diagonal of the smash product of pointed sets** is the natural transformation

$$\Delta^{\wedge} \colon \mathrm{id}_{\mathsf{Sets}_*} \Longrightarrow \wedge \circ \Delta^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}, \qquad \begin{tikzcd} \mathsf{Sets}_* \arrow[rr, bend left, "\mathrm{id}_{\mathsf{Sets}_*}"] \arrow[dr, swap, "\Delta^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}"] & \Downarrow \Delta^{\wedge} & \mathsf{Sets}_* \\ & \mathsf{Sets}_* \wedge \mathsf{Sets}_*, \arrow[ur, swap, "\wedge"] & \end{tikzcd}$$

whose component

$$\Delta^{\wedge}_X \colon (X, x_0) \to (X \wedge X, x_0 \wedge x_0)$$

at $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$ is given by the composition

$$\begin{aligned} (X, x_0) &\xrightarrow{\Delta^{\wedge}_X} (X \times X, (x_0, x_0)) \\ &\twoheadrightarrow ((X \times X)/\sim, [(x_0, x_0)]) \\ &\overset{\text{def}}{=\!=} (X \wedge X, x_0 \wedge x_0) \end{aligned}$$

in Sets_*, and thus by

$$\Delta^{\wedge}_X(x) \overset{\text{def}}{=} x \wedge x$$

for each $x \in X$.

Proof 4.5.8.2 ► Proof of Definition 4.5.8.1

Being a Morphism of Pointed Sets

We have

$$\Delta^{\wedge}_X(x_0) \overset{\text{def}}{=} x_0 \wedge x_0,$$

and thus $\Delta^{\wedge}_X$ is a morphism of pointed sets.

Naturality

We need to show that, given a morphism of pointed sets

$$f\colon (X, x_0) \to (Y, y_0),$$

the diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\Delta^{\wedge}_X"'] & Y \arrow[d, "\Delta^{\wedge}_Y"] \\
X \wedge X \arrow[r, "f \wedge f"'] & Y \wedge Y
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
x \arrow[r, mapsto] \arrow[d, mapsto] & f(x) \arrow[d, mapsto] \\
x \wedge x \arrow[r, mapsto] & f(x) \wedge f(x)
\end{tikzcd}$$

and hence indeed commutes, showing $\Delta^{\wedge}$ to be natural. □

Proposition 4.5.8.3 ▶ Properties of the Diagonal of $\wedge$

Let $(X, x_0) \in \mathsf{Obj}(\mathsf{Sets}_*)$.

1. *Monoidality.* The diagonal

$$\Delta^{\wedge}\colon \mathrm{id}_{\mathsf{Sets}_*} \Longrightarrow \wedge \circ \Delta^{\mathsf{Cats}_2}_{\mathsf{Sets}_*},$$

of the smash product of pointed sets is a monoidal natural transformation:

(a) *Compatibility With Strong Monoidality Constraints.* For each $(X, x_0), (Y, y_0) \in \mathsf{Obj}(\mathsf{Sets}_*)$, the diagram

$$\begin{tikzcd}
X \wedge Y \arrow[r, "\Delta^{\wedge}_X \wedge \Delta^{\wedge}_Y"] \arrow[dr, "\Delta^{\wedge}_{X \wedge Y}"'] & (X \wedge X) \wedge (Y \wedge Y) \arrow[d, dashed, "\wr"] \\
& (X \wedge Y) \wedge (X \wedge Y)
\end{tikzcd}$$

commutes.

(b) *Compatibility With Strong Unitality Constraints.* The diagrams

$$
\begin{tikzcd}
S^0 \arrow[r, "\Delta^{\wedge}_{S^0}"] \arrow[dr, equal] & S^0 \wedge S^0 \arrow[d, "\lambda^{\mathsf{Sets}_*}_{S^0}"] \\
& S^0
\end{tikzcd}
\qquad
\begin{tikzcd}
S^0 \arrow[r, "\Delta^{\wedge}_{S^0}"] \arrow[dr, equal] & S^0 \wedge S^0 \arrow[d, "\rho^{\mathsf{Sets}_*}_{S^0}"] \\
& S^0
\end{tikzcd}
$$

commute, i.e. we have

$$
\begin{aligned}
\Delta^{\wedge}_{S^0} &= \lambda^{\mathsf{Sets}_*,-1}_{S^0} \\
&= \rho^{\mathsf{Sets}_*,-1}_{S^0},
\end{aligned}
$$

where we recall that the equalities

$$
\begin{aligned}
\lambda^{\mathsf{Sets}_*}_{S^0} &= \rho^{\mathsf{Sets}_*}_{S^0}, \\
\lambda^{\mathsf{Sets}_*,-1}_{S^0} &= \rho^{\mathsf{Sets}_*,-1}_{S^0}
\end{aligned}
$$

are always true in any monoidal category by **??** of **??**.

2. *The Diagonal of the Unit.* The component

$$\Delta^{\wedge}_{S^0} \colon S^0 \xrightarrow{\cong} S^0 \wedge S^0$$

of $\Delta^{\wedge}$ at S^0 is an isomorphism.

Proof 4.5.8.4 ► Proof of Proposition 4.5.8.3

Item 1: Monoidality

We claim that $\Delta^{\wedge}$ is indeed monoidal:

1. *Item 1a*: *Compatibility With Strong Monoidality Constraints:* We need to show that the diagram

$$
\begin{tikzcd}
X \wedge Y \arrow[r, "\Delta^{\wedge}_X \wedge \Delta^{\wedge}_Y"] \arrow[dr, "\Delta^{\wedge}_{X \wedge Y}"'] & (X \wedge X) \wedge (Y \wedge Y) \arrow[d, dashed, "\wr"] \\
& (X \wedge Y) \wedge (X \wedge Y)
\end{tikzcd}
$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd} x \wedge y \arrow[r, mapsto] \arrow[dr, mapsto] & (x \wedge x) \wedge (y \wedge y) \arrow[d, mapsto] \\ & (x \wedge y) \wedge (x \wedge y) \end{tikzcd}$$

and hence indeed commutes.

2. *Item 1b: Compatibility With Strong Unitality Constraints:* As shown in the proof of Definition 4.5.5.1, the inverse of the left unitor of Sets_* with respect to to the smash product of pointed sets at $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$ is given by

$$\lambda_X^{\mathsf{Sets}_*,-1}(x) \stackrel{\text{def}}{=} 1 \wedge x$$

for each $x \in X$, so when $X = S^0$, we have

$$\begin{aligned} \lambda_{S^0}^{\mathsf{Sets}_*,-1}(0) &\stackrel{\text{def}}{=} 1 \wedge 0, \\ \lambda_{S^0}^{\mathsf{Sets}_*,-1}(1) &\stackrel{\text{def}}{=} 1 \wedge 1. \end{aligned}$$

But since $1 \wedge 0 = 0 \wedge 0$ and

$$\begin{aligned} \Delta_{S^0}^{\wedge}(0) &\stackrel{\text{def}}{=} 0 \wedge 0, \\ \Delta_{S^0}^{\wedge}(1) &\stackrel{\text{def}}{=} 1 \wedge 1, \end{aligned}$$

it follows that we indeed have $\Delta_{S^0}^{\wedge} = \lambda_{S^0}^{\mathsf{Sets}_*,-1}$.

This finishes the proof.

Item 2: The Diagonal of the Unit

This follows from Item 1 and the invertibility of the left/right unitor of Sets_* with respect to $\wedge$, proved in the proof of Definition 4.5.5.1 for the left unitor or the proof of Definition 4.5.6.1 for the right unitor.

4.5.9 The Monoidal Structure on Pointed Sets Associated to $\wedge$

Proposition 4.5.9.1 ► The Monoidal Structure on Pointed Sets Associated to $\wedge$

The category Sets_* admits a closed monoidal category with diagonals structure consisting of

- *The Underlying Category.* The category Sets_* of pointed sets;
- *The Monoidal Product.* The smash product functor

$$\wedge\colon \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*$$

of Item 1 of Proposition 4.5.1.10;
- *The Internal Hom.* The internal Hom functor

$$\mathbf{Sets}_*\colon \mathsf{Sets}_*^{\mathsf{op}} \times \mathsf{Sets}_* \to \mathsf{Sets}_*$$

of Item 1 of Proposition 4.5.2.3;
- *The Monoidal Unit.* The functor

$$\mathbb{1}^{\mathsf{Sets}_*}\colon \mathrm{pt} \to \mathsf{Sets}_*$$

of Definition 4.5.3.1;
- *The Associators.* The natural isomorphism

$$\alpha^{\mathsf{Sets}_*}\colon \wedge \circ (\wedge \times \mathrm{id}_{\mathsf{Sets}_*}) \xrightarrow{\sim} \wedge \circ (\mathrm{id}_{\mathsf{Sets}_*} \times \wedge) \circ \boldsymbol{\alpha}^{\mathsf{Cats}}_{\mathsf{Sets}_*,\mathsf{Sets}_*,\mathsf{Sets}_*}$$

of Definition 4.5.4.1;
- *The Left Unitors.* The natural isomorphism

$$\lambda^{\mathsf{Sets}_*}\colon \wedge \circ \left(\mathbb{1}^{\mathsf{Sets}_*} \times \mathrm{id}_{\mathsf{Sets}_*}\right) \xrightarrow{\sim} \boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}$$

of Definition 4.5.5.1;
- *The Right Unitors.* The natural isomorphism

$$\rho^{\mathsf{Sets}_*}\colon \wedge \circ \left(\mathrm{id} \times \mathbb{1}^{\mathsf{Sets}_*}\right) \xrightarrow{\sim} \boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}$$

of Definition 4.5.6.1;
- *The Symmetry.* The natural isomorphism

$$\sigma^{\mathsf{Sets}_*}\colon \wedge \xrightarrow{\sim} \wedge \circ \boldsymbol{\sigma}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*,\mathsf{Sets}_*}$$

of Definition 4.5.7.1;
- *The Diagonals.* The monoidal natural transformation

$$\Delta^{\wedge}\colon \mathrm{id}_{\mathsf{Sets}_*} \Longrightarrow \wedge \circ \Delta^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}$$

of Definition 4.5.8.1.

Proof 4.5.9.2 ► Proof of Proposition 4.5.9.1

The Pentagon Identity

Let (W, w_0), (X, x_0), (Y, y_0) and (Z, z_0) be pointed sets. We have to show that the diagram

$$\begin{tikzcd}
 & (W \wedge (X \wedge Y)) \wedge Z \arrow[dr, "\alpha^{\mathsf{Sets}_*}_{W,X\wedge Y,Z}"] & \\
((W \wedge X) \wedge Y) \wedge Z \arrow[ur, "\alpha^{\mathsf{Sets}_*}_{W,X,Y} \wedge \mathrm{id}_Z"] \arrow[d, "\alpha^{\mathsf{Sets}_*}_{W\wedge X,Y,Z}"'] & & W \wedge ((X \wedge Y) \wedge Z) \arrow[d, "\mathrm{id}_W \wedge \alpha^{\mathsf{Sets}_*}_{X,Y,Z}"] \\
(W \wedge X) \wedge (Y \wedge Z) \arrow[rr, "\alpha^{\mathsf{Sets}_*}_{W,X,Y\wedge Z}"'] & & W \wedge (X \wedge (Y \wedge Z))
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
 & (w \wedge (x \wedge y)) \wedge z \arrow[dr, mapsto] & \\
((w \wedge x) \wedge y) \wedge z \arrow[ur, mapsto] \arrow[d, mapsto] & & w \wedge ((x \wedge y) \wedge z) \arrow[d, mapsto] \\
(w \wedge x) \wedge (y \wedge z) \arrow[rr, mapsto] & & w \wedge (x \wedge (y \wedge z))
\end{tikzcd}$$

and thus we see that the pentagon identity is satisfied.

The Triangle Identity

Let (X, x_0) and (Y, y_0) be pointed sets. We have to show that the diagram

$$\begin{tikzcd}
(X \wedge S^0) \wedge Y \arrow[rr, "\alpha^{\mathsf{Sets}_*}_{X,S^0,Y}"] \arrow[dr, "\rho^{\mathsf{Sets}_*}_X \wedge \mathrm{id}_Y"'] & & X \wedge (S^0 \wedge Y) \arrow[dl, "\mathrm{id}_X \wedge \lambda^{\mathsf{Sets}_*}_Y"] \\
& X \wedge Y &
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
(x \wedge 0) \wedge y \arrow[dr, mapsto] & \\
& x_0 \wedge y
\end{tikzcd}
\qquad
\begin{tikzcd}
(x \wedge 0) \wedge y \arrow[r, mapsto] & x \wedge (0 \wedge y) \arrow[dl, mapsto] \\
x \wedge y_0 &
\end{tikzcd}$$

and

$$\begin{tikzcd}
(x \wedge 1) \wedge y \arrow[rr, mapsto] \arrow[dr, mapsto] & & x \wedge (1 \wedge y) \arrow[dl, mapsto] \\
& x \wedge y, &
\end{tikzcd}$$

and thus we see that the triangle identity is satisfied.

The Left Hexagon Identity

Let (X, x_0), (Y, y_0), and (Z, z_0) be pointed sets. We have to show that the diagram

$$\begin{tikzcd}
& (X \wedge Y) \wedge Z \arrow[dl, "\alpha^{\mathsf{Sets}_*}_{X,Y,Z}"'] \arrow[dr, "\beta^{\mathsf{Sets}_*}_{X,Y} \wedge \mathrm{id}_Z"] & \\
X \wedge (Y \wedge Z) \arrow[d, "\beta^{\mathsf{Sets}_*}_{X,Y \wedge Z}"'] & & (Y \wedge X) \wedge Z \arrow[d, "\alpha^{\mathsf{Sets}_*}_{Y,X,Z}"] \\
(Y \wedge Z) \wedge X \arrow[dr, "\alpha^{\mathsf{Sets}_*}_{Y,Z,X}"'] & & Y \wedge (X \wedge Z) \arrow[dl, "\mathrm{id}_Y \wedge \beta^{\mathsf{Sets}_*}_{X,Z}"] \\
& Y \wedge (Z \wedge X) &
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$
\begin{tikzcd}
 & (x \wedge y) \wedge z \arrow[dl, mapsto] \arrow[dr, mapsto] & \\
x \wedge (y \wedge z) \arrow[d, mapsto] & & (y \wedge x) \wedge z \arrow[d, mapsto] \\
(y \wedge z) \wedge x \arrow[dr, mapsto] & & y \wedge (x \wedge z) \arrow[dl, mapsto] \\
 & y \wedge (z \wedge x) &
\end{tikzcd}
$$

and thus we see that the left hexagon identity is satisfied.

The Right Hexagon Identity

Let (X, x_0), (Y, y_0), and (Z, z_0) be pointed sets. We have to show that the diagram

$$
\begin{tikzcd}
 & X \wedge (Y \wedge Z) \arrow[dl, "\left(\alpha^{\mathsf{Sets}_*}_{X,Y,Z}\right)^{-1}"'] \arrow[dr, "\mathrm{id}_X \wedge \beta^{\mathsf{Sets}_*}_{Y,Z}"] & \\
(X \wedge Y) \wedge Z \arrow[d, "\beta^{\mathsf{Sets}_*}_{X \wedge Y, Z}"'] & & X \wedge (Z \wedge Y) \arrow[d, "\left(\alpha^{\mathsf{Sets}_*}_{X,Z,Y}\right)^{-1}"] \\
Z \wedge (X \wedge Y) \arrow[dr, "\left(\alpha^{\mathsf{Sets}_*}_{Z,X,Y}\right)^{-1}"'] & & (X \wedge Z) \wedge Y \arrow[dl, "\beta^{\mathsf{Sets}_*}_{X,Z} \wedge \mathrm{id}_Y"] \\
 & (Z \wedge X) \wedge Y &
\end{tikzcd}
$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
 & x \wedge (y \wedge z) \arrow[dl, mapsto] \arrow[dr, mapsto] & \\
(x \wedge y) \wedge z \arrow[d, mapsto] & & x \wedge (z \wedge y) \arrow[d, mapsto] \\
z \wedge (x \wedge y) \arrow[dr, mapsto] & & (x \wedge z) \wedge y \arrow[dl, mapsto] \\
 & (z \wedge x) \wedge y &
\end{tikzcd}$$

and thus we see that the right hexagon identity is satisfied.

Monoidal Closedness

This follows from Item 2 of Proposition 4.5.1.10.

Existence of Monoidal Diagonals

This follows from Items 1 and 2 of Proposition 4.5.8.3.

4.5.10 Universal Properties of the Smash Product of Pointed Sets I

Theorem 4.5.10.1 ► Universal Properties of the Smash Product of Pointed Sets I

The symmetric monoidal structure on the category Sets_* is uniquely determined by the following requirements:

1. *Two-Sided Preservation of Colimits.* The smash product

$$\wedge\colon \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*$$

   of Sets_* preserves colimits separately in each variable.

2. *The Unit Object Is S^0.* We have $\mathbb{1}_{\mathsf{Sets}_*} = S^0$.

Proof 4.5.10.2 ► Proof of Theorem 4.5.10.1

Omitted.

4.5.11 Universal Properties of the Smash Product of Pointed Sets II

Theorem 4.5.11.1 ▶ Universal Properties of the Smash Product of Pointed Sets II

The symmetric monoidal structure on the category Sets_* is the unique symmetric monoidal structure on Sets_* such that the free pointed set functor

$$(-)^+ \colon \mathsf{Sets} \to \mathsf{Sets}_*$$

admits a symmetric monoidal structure.

Proof 4.5.11.2 ▶ Proof of Theorem 4.5.11.1

See [GGN15, Theorem 5.1].

4.5.12 Monoids With Respect to the Smash Product of Pointed Sets

Proposition 4.5.12.1 ▶ Monoids With Respect to $\wedge$

The category of monoids on $(\mathsf{Sets}_*, \wedge, S^0)$ is isomorphic to the category of monoids with zero and morphisms between them.

Proof 4.5.12.2 ▶ Proof of Proposition 4.5.12.1

See **??**, in particular **??**, **??**, and **??**.

4.5.13 Comonoids With Respect to the Smash Product of Pointed Sets

Proposition 4.5.13.1 ▶ Comonoids With Respect to $\wedge$

The symmetric monoidal functor

$$\left((-)^+, (-)^{+,\times}, (-)^{+,\times}_{\mathbb{1}}\right) \colon (\mathsf{Sets}, \times, \mathrm{pt}) \to \left(\mathsf{Sets}_*, \wedge, S^0\right),$$

of Item 4 of Proposition 3.4.1.2 lifts to an equivalence of categories

$$\begin{aligned}\mathsf{CoMon}\left(\mathsf{Sets}_*, \wedge, S^0\right) &\overset{\text{eq.}}{\cong} \mathsf{CoMon}(\mathsf{Sets}, \times, \mathrm{pt})\\ &\cong \mathsf{Sets}.\end{aligned}$$

Proof 4.5.13.2 ▶ Proof of Proposition 4.5.13.1

See [PS19, Lemma 2.4].

4.6 Miscellany

4.6.1 The Smash Product of a Family of Pointed Sets

Let $\left\{(X_i, x_0^i)\right\}_{i\in I}$ be a family of pointed sets.

Definition 4.6.1.1 ► The Smash Product of a Family of Pointed Sets

The **smash product of the family** $\left\{(X_i, x_0^i)\right\}_{i\in I}$ is the pointed set $\bigwedge_{i\in I} X_i$ consisting of:

- *The Underlying Set.* The set $\bigwedge_{i\in I} X_i$ defined by

$$\bigwedge_{i\in I} X_i \stackrel{\text{def}}{=} \left(\prod_{i\in I} X_i\right)/\sim,$$

  where $\sim$ is the equivalence relation on $\prod_{i\in I} X_i$ obtained by declaring

$$(x_i)_{i\in I} \sim (y_i)_{i\in I}$$

  if there exist $i_0 \in I$ such that $x_{i_0} = x_0$ and $y_{i_0} = y_0$, for each $(x_i)_{i\in I}, (y_i)_{i\in I} \in \prod_{i\in I} X_i$.

- *The Basepoint.* The element $[(x_0)_{i\in I}]$ of $\bigwedge_{i\in I} X_i$.

Appendices

4.A Other Chapters

Sets

1. Sets
2. Constructions With Sets
3. Pointed Sets
4. Tensor Products of Pointed Sets

Relations

5. Relations
6. Constructions With Relations
7. Equivalence Relations and Apartness Relations

Category Theory

8. Categories

Bicategories

9. Types of Morphisms in Bicategories

Part II

Relations

Chapter 5

Relations

This chapter contains some material about relations. Notably, we discuss and explore:

1. The definition of relations (Section 5.1.1).
2. How relations may be viewed as decategorification of profunctors (Section 5.1.2).
3. The various kind of categories that relations form, namely:
    (a) A category (Section 5.2.1).
    (b) A monoidal category (Section 5.2.2).
    (c) A 2-category (Section 5.2.3).
    (d) A double category (Section 5.2.4).
4. The various categorical properties of the 2-category of relations, including:
    (a) The self-duality of Rel and **Rel** (Proposition 5.3.1.1).
    (b) Identifications of equivalences and isomorphisms in **Rel** with bijections (Proposition 5.3.2.1).
    (c) Identifications of adjunctions in **Rel** with functions (Proposition 5.3.3.1).
    (d) Identifications of monads in **Rel** with preorders (Proposition 5.3.4.1).
    (e) Identifications of comonads in **Rel** with subsets (Proposition 5.3.5.1).
    (f) A description of the monoids and comonoids in **Rel** with respect to the Cartesian product (Remark 5.3.6.1).
    (g) Characterisations of monomorphisms in Rel (Proposition 5.3.7.1).
    (h) Characterisations of 2-categorical notions of monomorphisms in **Rel** (Proposition 5.3.8.1).
    (i) Characterisations of epimorphisms in Rel (Proposition 5.3.9.1).

(j) Characterisations of 2-categorical notions of epimorphisms in **Rel** (Proposition 5.3.10.1).

(k) The partial co/completeness of Rel (Proposition 5.3.11.1).

(l) The existence or non-existence of Kan extensions and Kan lifts in Rel (Remark 5.3.12.1).

(m) The closedness of **Rel** (Proposition 5.3.13.1).

(n) The identification of **Rel** with the category of free algebras of the powerset monad on Sets (Proposition 5.3.14.1).

5. A description of two notions of "skew composition" on $\mathbf{Rel}(A, B)$, giving rise to left and right skew monoidal structures analogous to the left skew monoidal structure on $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$ appearing in the definition of a relative monad (Sections 5.4 and 5.5).

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5.1 Relations

5.1.1 Foundations

Let A and B be sets.

Definition 5.1.1.1 ► Relations

A **relation** $R\colon A \nrightarrow B$ **from** A **to** B[1,2] is a subset R of $A \times B$.

[1] *Further Terminology:* Also called a **multivalued function from** A **to** B, a **relation over** A **and** B, **relation on** A **and** B, a **binary relation over** A **and** B, or a **binary relation on** A **and** B.
[2] *Further Terminology:* When $A = B$, we also call $R \subset A \times A$ a **relation on** A.

Notation 5.1.1.2 ► Further Notation for Relations

Let $R\colon A \nrightarrow B$ be a relation.

1. Given elements $a \in A$ and $b \in B$ and a relation $R\colon A \nrightarrow B$, we write $a \sim_R b$ to mean $(a, b) \in R$.
2. Viewing R as a function

$$R\colon A \times B \longrightarrow \{\mathrm{t}, \mathrm{f}\}$$

via Remark 5.1.1.4, we write R^b_a for the value of R at (a, b).[1]

[1]The choice R^b_a in place of R^a_b is to keep the notation consistent with the notation we will later employ for profunctors.

Definition 5.1.1.3 ► The Po/Set of Relations Over Two Sets

Let A and B be sets.

1. The **set of relations from A to B** is the set $\mathrm{Rel}(A, B)$ defined by

$$\mathrm{Rel}(A, B) \overset{\text{def}}{=} \{\text{Relations from } A \text{ to } B\}.$$

2. The **poset of relations from A to B** is the poset

$$\mathbf{Rel}(A, B) \overset{\text{def}}{=} (\mathrm{Rel}(A, B), \subset)$$

consisting of:

- *The Underlying Set.* The set $\mathrm{Rel}(A, B)$ of Item 1.
- *The Partial Order.* The partial order

$$\subset\colon \mathrm{Rel}(A, B) \times \mathrm{Rel}(A, B) \longrightarrow \{\text{true}, \text{false}\}$$

on $\mathrm{Rel}(A, B)$ given by inclusion of relations.

3. The **category of relations from A to B** is the posetal category $\mathbf{Rel}(A, B)$[1] associated to the poset $\mathbf{Rel}(A, B)$ of Item 2 via Definition 8.1.3.1.

[1]Here we choose to slightly abuse notation by writing $\mathbf{Rel}(A, B)$ (instead of e.g. $\mathbf{Rel}(A, B)_{\mathrm{pos}}$) for the posetal category of relations from A to B, even though the same notation is used for the poset of relations from A to B.

Remark 5.1.1.4 ► Equivalent Definitions of Relations

A relation from A to B is equivalently:[1]

1. A subset of $A \times B$.
2. A function from $A \times B$ to $\{\text{true}, \text{false}\}$.
3. A function from A to $\mathcal{P}(B)$.
4. A function from B to $\mathcal{P}(A)$.
5. A cocontinuous morphism of posets from $(\mathcal{P}(A), \subset)$ to $(\mathcal{P}(B), \subset)$.

That is: we have bijections of sets

$$\begin{aligned}\mathsf{Rel}(A,B) &\overset{\text{def}}{=} \mathcal{P}(A\times B),\\ &\cong \mathrm{Hom}_{\mathsf{Sets}}(A\times B,\{\mathsf{true},\mathsf{false}\}),\\ &\cong \mathrm{Hom}_{\mathsf{Sets}}(A,\mathcal{P}(B)),\\ &\cong \mathrm{Hom}_{\mathsf{Sets}}(B,\mathcal{P}(A)),\\ &\cong \mathrm{Hom}^{\mathrm{cocont}}_{\mathsf{Pos}}(\mathcal{P}(A),\mathcal{P}(B)),\end{aligned}$$

natural in $A, B \in \mathrm{Obj}(\mathsf{Sets})$.

[1] *Intuition:* In particular, we may think of a relation $R\colon A \to \mathcal{P}(B)$ from A to B as a multivalued function from A to B (including the possibility of a given $a \in A$ having no value at all).

Proof 5.1.1.5 ▶ Proof of Remark 5.1.1.4

We claim that Items 1 to 5 are indeed equivalent:

- *Item 1* $\iff$ *Item 2*: This is a special case of Items 1 and 2 of Proposition 2.4.3.9.
- *Item 2* $\iff$ *Item 3*: This follows from the bijections

$$\begin{aligned}\mathrm{Hom}_{\mathsf{Sets}}(A\times B,\{\mathsf{true},\mathsf{false}\}) &\cong \mathrm{Hom}_{\mathsf{Sets}}(A,\mathrm{Hom}_{\mathsf{Sets}}(B,\{\mathsf{true},\mathsf{false}\}))\\ &\cong \mathrm{Hom}_{\mathsf{Sets}}(A,\mathcal{P}(B)),\end{aligned}$$

  where the last bijection is from Items 1 and 2 of Proposition 2.4.3.9.
- *Item 2* $\iff$ *Item 4*: This follows from the bijections

$$\begin{aligned}\mathrm{Hom}_{\mathsf{Sets}}(A\times B,\{\mathsf{true},\mathsf{false}\}) &\cong \mathrm{Hom}_{\mathsf{Sets}}(B,\mathrm{Hom}_{\mathsf{Sets}}(B,\{\mathsf{true},\mathsf{false}\}))\\ &\cong \mathrm{Hom}_{\mathsf{Sets}}(B,\mathcal{P}(A)),\end{aligned}$$

  where again the last bijection is from Items 1 and 2 of Proposition 2.4.3.9.
- *Item 2* $\iff$ *Item 5*: This follows from the universal property of the powerset $\mathcal{P}(X)$ of a set X as the free cocompletion of X via the characteristic embedding

$$\chi_X\colon X \hookrightarrow \mathcal{P}(X)$$

  of X into $\mathcal{P}(X)$, Item 2 of Proposition 2.4.3.12.

  In particular, the bijection

$$\mathsf{Rel}(A,B) \cong \mathrm{Hom}^{\mathrm{cocont}}_{\mathsf{Pos}}(\mathcal{P}(A),\mathcal{P}(B))$$

is given by taking a relation $R\colon A \nrightarrow B$, passing to its associated function $f\colon A \to \mathcal{P}(B)$ from A to B and then extending f from A to all of $\mathcal{P}(A)$ by taking its left Kan extension along χ_X.

This coincides with the direct image function $f_*\colon \mathcal{P}(A) \to \mathcal{P}(B)$ of Definition 2.4.4.1.

This finishes the proof. □

Proposition 5.1.1.6 ► Properties of Relations

Let A and B be sets and let $R, S\colon A \nrightarrow B$ be relations.

1. *End Formula for the Set of Inclusions of Relations.* We have

$$\mathrm{Hom}_{\mathbf{Rel}(A,B)}(R,S) \cong \int_{a\in A}\int_{b\in B} \mathrm{Hom}_{\{\mathrm{t},\mathrm{f}\}}\left(R^b_a, S^b_a\right).$$

Proof 5.1.1.7 ► Proof of Proposition 5.1.1.6

Item 1: End Formula for the Set of Inclusions of Relations

Unwinding the expression inside the end on the right hand side, we have

$$\int_{a\in A}\int_{b\in B} \mathrm{Hom}_{\{\mathrm{t},\mathrm{f}\}}\left(R^b_a, S^b_a\right) \cong \begin{cases} \mathrm{pt} & \text{if, for each } a \in A \text{ and each } b \in B, \\ & \text{we have } \mathrm{Hom}_{\{\mathrm{t},\mathrm{f}\}}(R^b_a, S^b_a) \cong \mathrm{pt} \\ \emptyset & \text{otherwise.} \end{cases}$$

Since we have $\mathrm{Hom}_{\{\mathrm{t},\mathrm{f}\}}(R^b_a, S^b_a) = \{\mathrm{true}\} \cong \mathrm{pt}$ exactly when $R^b_a = \mathrm{false}$ or $R^b_a = S^b_a = \mathrm{true}$, we get

$$\int_{a\in A}\int_{b\in B} \mathrm{Hom}_{\{\mathrm{t},\mathrm{f}\}}\left(R^b_a, S^b_a\right) \cong \begin{cases} \mathrm{pt} & \text{if, for each } a \in A \text{ and each } b \in B, \\ & \text{if } a \sim_R b, \text{ then } a \sim_S b, \\ \emptyset & \text{otherwise.} \end{cases}$$

On the left hand-side, we have

$$\mathrm{Hom}_{\mathbf{Rel}(A,B)}(R,S) \cong \begin{cases} \mathrm{pt} & \text{if } R \subset S, \\ \emptyset & \text{otherwise.} \end{cases}$$

It is then clear that the conditions for each set to evaluate to pt (up to isomorphism) are equivalent, implying that those two sets are isomorphic. □

5.1.2 Relations as Decategorifications of Profunctors

The notion of a relation is a decategorification of that of a profunctor:

1. A profunctor from a category $\mathcal{C}$ to a category $\mathcal{D}$ is a functor

$$\mathfrak{p}\colon \mathcal{D}^{\mathsf{op}} \times \mathcal{C} \longrightarrow \mathsf{Sets}.$$

2. A relation on sets A and B is a function

$$R\colon A \times B \longrightarrow \{\mathsf{true}, \mathsf{false}\}.$$

Here we notice that:

- The opposite X^{op} of a set X is itself, as $(-)^{\mathsf{op}}\colon \mathsf{Cats} \longrightarrow \mathsf{Cats}$ restricts to the identity endofunctor on Sets.
- The values that profunctors and relations take are analogous:
  - A category is enriched over the category

    $$\mathsf{Sets} \stackrel{\text{def}}{=} \mathsf{Cats}_0$$

    of sets, with profunctors taking values on it.
  - A set is enriched over the set

    $$\{\mathsf{true}, \mathsf{false}\} \stackrel{\text{def}}{=} \mathsf{Cats}_{-1}$$

    of classical truth values, with relations taking values on it.

Remark 5.1.2.2 ▶ Relations as Decategorifications of Profunctors II

Extending Remark 5.1.2.1, the equivalent definitions of relations in Remark 5.1.1.4 are also related to the corresponding ones for profunctors (**??**), which state that a profunctor $\mathfrak{p}\colon \mathcal{C} \nrightarrow \mathcal{D}$ is equivalently:

1. A functor $\mathfrak{p}\colon \mathcal{D}^{\mathsf{op}} \times \mathcal{C} \longrightarrow \mathsf{Sets}$.
2. A functor $\mathfrak{p}\colon \mathcal{C} \longrightarrow \mathsf{PSh}(\mathcal{D})$.
3. A functor $\mathfrak{p}\colon \mathcal{D}^{\mathsf{op}} \longrightarrow \mathsf{Fun}(\mathcal{C}, \mathsf{Sets})$.
4. A colimit-preserving functor $\mathfrak{p}\colon \mathsf{PSh}(\mathcal{C}) \longrightarrow \mathsf{PSh}(\mathcal{D})$.

Indeed:

- The equivalence between Items 1 and 2 (and also that between Items 1 and 3, which is proved analogously) is an instance of currying, both for profunctors as well as for relations, using the isomorphisms

$$\begin{aligned}\mathsf{Sets}(A \times B, \{\mathsf{true}, \mathsf{false}\}) &\cong \mathsf{Sets}(A, \mathsf{Sets}(B, \{\mathsf{true}, \mathsf{false}\}))\\ &\cong \mathsf{Sets}(A, \mathcal{P}(B)),\\ \mathsf{Fun}(\mathcal{D}^{\mathsf{op}} \times \mathcal{D}, \mathsf{Sets}) &\cong \mathsf{Fun}(\mathcal{C}, \mathsf{Fun}(\mathcal{D}^{\mathsf{op}}, \mathsf{Sets}))\\ &\cong \mathsf{Fun}(\mathcal{C}, \mathsf{PSh}(\mathcal{D})).\end{aligned}$$

- The equivalence between Items 1 and 3 follows from the universal properties of:
  - The powerset $\mathcal{P}(X)$ of a set X as the free cocompletion of X via the characteristic embedding

    $$\chi_{(-)} : X \hookrightarrow \mathcal{P}(X)$$

    of X into $\mathcal{P}(X)$, as stated and proved in Item 2 of Proposition 2.4.3.12.
  - The category $\mathsf{PSh}(\mathcal{C})$ of presheaves on a category $\mathcal{C}$ as the free cocompletion of $\mathcal{C}$ via the Yoneda embedding

    $$よ : \mathcal{C} \hookrightarrow \mathsf{PSh}(\mathcal{C})$$

    of $\mathcal{C}$ into $\mathsf{PSh}(\mathcal{C})$, as stated and proved in **??** of **??**.

5.1.3 Examples of Relations

Example 5.1.3.1 ▶ The Trivial Relation

The **trivial relation on A and B** is the relation $\sim_{\mathsf{triv}}$ defined equivalently as follows:

1. As a subset of $A \times B$, we have

   $$\sim_{\mathsf{triv}} \overset{\mathrm{def}}{=} A \times B.$$

2. As a function from $A \times B$ to $\{\mathsf{true}, \mathsf{false}\}$, the relation $\sim_{\mathsf{triv}}$ is the constant function

   $$\Delta_{\mathsf{true}} : A \times B \longrightarrow \{\mathsf{true}, \mathsf{false}\}$$

   from $A \times B$ to $\{\mathsf{true}, \mathsf{false}\}$ taking the value true.

3. As a function from A to $\mathcal{P}(B)$, the relation $\sim_{\text{triv}}$ is the function

$$\Delta_{\text{true}} \colon A \to \mathcal{P}(B)$$

defined by

$$\Delta_{\text{true}}(a) \stackrel{\text{def}}{=} B$$

for each $a \in A$.

4. Lastly, it is the unique relation R on A and B such that we have $a \sim_R b$ for each $a \in A$ and each $b \in B$.

Example 5.1.3.2 ► The Cotrivial Relation

The **cotrivial relation on** A **and** B is the relation $\sim_{\text{cotriv}}$ defined equivalently as follows:

1. As a subset of $A \times B$, we have

$$\sim_{\text{cotriv}} \stackrel{\text{def}}{=} \emptyset.$$

2. As a function from $A \times B$ to $\{\text{true}, \text{false}\}$, the relation $\sim_{\text{cotriv}}$ is the constant function

$$\Delta_{\text{false}} \colon A \times B \to \{\text{true}, \text{false}\}$$

from $A \times B$ to $\{\text{true}, \text{false}\}$ taking the value false.

3. As a function from A to $\mathcal{P}(B)$, the relation $\sim_{\text{cotriv}}$ is the function

$$\Delta_{\text{false}} \colon A \to \mathcal{P}(B)$$

defined by

$$\Delta_{\text{false}}(a) \stackrel{\text{def}}{=} \emptyset$$

for each $a \in A$.

4. Lastly, it is the unique relation R on A and B such that we have $a \nsim_R b$ for each $a \in A$ and each $b \in B$.

Example 5.1.3.3 ► The Characteristic Relation of a Set

The characteristic relation

$$\chi_X(-_1, -_2) \colon X \times X \to \{\mathsf{t}, \mathsf{f}\}$$

on X of Item 3 of Definition 2.4.1.1, defined by

$$\chi_X(x, y) \overset{\text{def}}{=} \begin{cases} \text{true} & \text{if } x = y, \\ \text{false} & \text{if } x \neq y \end{cases}$$

for each $x, y \in X$, is another example of a relation.

Example 5.1.3.4 ► Square Roots

Square roots are examples of relations:

1. *Square Roots in* $\mathbb{R}$. The assignment $x \mapsto \sqrt{x}$ defines a relation

   $$\sqrt{-} \colon \mathbb{R} \to \mathcal{P}(\mathbb{R})$$

   from $\mathbb{R}$ to itself, being explicitly given by

   $$\sqrt{x} \overset{\text{def}}{=} \begin{cases} 0 & \text{if } x = 0, \\ \left\{-\sqrt{|x|}, \sqrt{|x|}\right\} & \text{if } x \neq 0. \end{cases}$$

2. *Square Roots in* $\mathbb{Q}$. Square roots in $\mathbb{Q}$ are similar to square roots in $\mathbb{R}$, though now additionally it may also occur that $\sqrt{-} \colon \mathbb{Q} \to \mathcal{P}(\mathbb{Q})$ sends a rational number x (e.g. 2) to the empty set (since $\sqrt{2} \notin \mathbb{Q}$).

Example 5.1.3.5 ► Complex Logarithms

The complex logarithm defines a relation

$$\log \colon \mathbb{C} \to \mathcal{P}(\mathbb{C})$$

from $\mathbb{C}$ to itself, where we have

$$\log(a + bi) \overset{\text{def}}{=} \left\{ \log\left(\sqrt{a^2 + b^2}\right) + i \arg(a + bi) + (2\pi i)k \,\middle|\, k \in \mathbb{Z} \right\}$$

for each $a + bi \in \mathbb{C}$.

Example 5.1.3.6 ▶ More Examples of Relations

See [Wik24] for more examples of relations, such as antiderivation, inverse trigonometric functions, and inverse hyperbolic functions.

5.1.4 Functional Relations

Let A and B be sets.

Definition 5.1.4.1 ▶ Functional Relations

A relation $R\colon A \nrightarrow B$ is **functional** if, for each $a \in A$, the set $R(a)$ is either empty or a singleton.

Proposition 5.1.4.2 ▶ Properties of Functional Relations

Let $R\colon A \nrightarrow B$ be a relation.

1. *Characterisations.* The following conditions are equivalent:
   (a) The relation R is functional.
   (b) We have $R \diamond R^{\dagger} \subset \chi_B$.

Proof 5.1.4.3 ▶ Proof of Proposition 5.1.4.2

Item 1: Characterisations

We claim that Items 1a and 1b are indeed equivalent:

- *Item 1a* $\implies$ *Item 1b*: Let $(b, b') \in B \times B$. We need to show that
$$\left[R \diamond R^{\dagger}\right](b, b') \preceq_{\{\mathsf{t},\mathsf{f}\}} \chi_B(b, b'),$$
i.e. that if there exists some $a \in A$ such that $b \sim_{R^{\dagger}} a$ and $a \sim_R b'$, then $b = b'$. But since $b \sim_{R^{\dagger}} a$ is the same as $a \sim_R b$, we have both $a \sim_R b$ and $a \sim_R b'$ at the same time, which implies $b = b'$ since R is functional.
- *Item 1b* $\implies$ *Item 1a*: Suppose that we have $a \sim_R b$ and $a \sim_R b'$ for $b, b' \in B$. We claim that $b = b'$:
  1. Since $a \sim_R b$, we have $b \sim_{R^{\dagger}} a$.
  2. Since $R \diamond R^{\dagger} \subset \chi_B$, we have
  $$\left[R \diamond R^{\dagger}\right](b, b') \preceq_{\{\mathsf{t},\mathsf{f}\}} \chi_B(b, b'),$$

and since $b \sim_{R^\dagger} a$ and $a \sim_R b'$, it follows that $\left[R \diamond R^\dagger\right](b, b') =$ true, and thus $\chi_B(b, b') =$ true as well, i.e. $b = b'$.

This finishes the proof.

5.1.5 Total Relations

Let A and B be sets.

Definition 5.1.5.1 ► Total Relations

A relation $R \colon A \nrightarrow B$ is **total** if, for each $a \in A$, we have $R(a) \neq \emptyset$.

Proposition 5.1.5.2 ► Properties of Total Relations

Let $R \colon A \nrightarrow B$ be a relation.

1. *Characterisations.* The following conditions are equivalent:
   (a) The relation R is total.
   (b) We have $\chi_A \subset R^\dagger \diamond R$.

Proof 5.1.5.3 ► Proof of Proposition 5.1.5.2

Item 1: Characterisations

We claim that Items 1a and 1b are indeed equivalent:

- *Item 1a* $\Longrightarrow$ *Item 1b*: We have to show that, for each $(a, a') \in A$, we have

$$\chi_A(a, a') \preceq_{\{\mathrm{t},\mathrm{f}\}} \left[R^\dagger \diamond R\right](a, a'),$$

  i.e. that if $a = a'$, then there exists some $b \in B$ such that $a \sim_R b$ and $b \sim_{R^\dagger} a'$ (i.e. $a \sim_R b$ again), which follows from the totality of R.

- *Item 1b* $\Longrightarrow$ *Item 1a*: Given $a \in A$, since $\chi_A \subset R^\dagger \diamond R$, we must have

$$\{a\} \subset \left[R^\dagger \diamond R\right](a),$$

  implying that there must exist some $b \in B$ such that $a \sim_R b$ and $b \sim_{R^\dagger} a$ (i.e. $a \sim_R b$) and thus $R(a) \neq \emptyset$, as $b \in R(a)$.

This finishes the proof.

5.2 Categories of Relations

5.2.1 The Category of Relations

Definition 5.2.1.1 ► The Category of Relations

The **category of relations** is the category Rel where

- *Objects.* The objects of Rel are sets.
- *Morphisms.* For each $A, B \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$\mathsf{Rel}(A,B) \overset{\mathrm{def}}{=} \mathrm{Rel}(A,B).$$

- *Identities.* For each $A \in \mathrm{Obj}(\mathsf{Rel})$, the unit map

$$\mathbb{1}^{\mathsf{Rel}}_{A} \colon \mathrm{pt} \longrightarrow \mathsf{Rel}(A,A)$$

of Rel at A is defined by

$$\mathrm{id}^{\mathsf{Rel}}_{A} \overset{\mathrm{def}}{=} \chi_A(-_1,-_2),$$

where $\chi_A(-_1,-_2)$ is the characteristic relation of A of Item 3 of Definition 2.4.1.1.

- *Composition.* For each $A, B, C \in \mathrm{Obj}(\mathsf{Rel})$, the composition map

$$\circ^{\mathsf{Rel}}_{A,B,C} \colon \mathsf{Rel}(B,C) \times \mathsf{Rel}(A,B) \longrightarrow \mathsf{Rel}(A,C)$$

of Rel at (A,B,C) is defined by

$$S \circ^{\mathsf{Rel}}_{A,B,C} R \overset{\mathrm{def}}{=} S \diamond R$$

for each $(S,R) \in \mathsf{Rel}(B,C) \times \mathsf{Rel}(A,B)$, where $S \diamond R$ is the composition of S and R of Definition 6.3.12.1.

5.2.2 The Closed Symmetric Monoidal Category of Relations

5.2.2.1 The Monoidal Product

Definition 5.2.2.1 ► The Monoidal Product of Rel

The **monoidal product of** Rel is the functor

$$\times \colon \mathsf{Rel} \times \mathsf{Rel} \longrightarrow \mathsf{Rel}$$

where

- *Action on Objects.* For each $A, B \in \mathrm{Obj}(\mathsf{Rel})$, we have

$$\times(A, B) \stackrel{\text{def}}{=} A \times B,$$

where $A \times B$ is the Cartesian product of sets of Definition 2.1.3.1.

- *Action on Morphisms.* For each $(A, C), (B, D) \in \mathrm{Obj}(\mathsf{Rel} \times \mathsf{Rel})$, the action on morphisms

$$\times_{(A,C),(B,D)} \colon \mathsf{Rel}(A, B) \times \mathsf{Rel}(C, D) \longrightarrow \mathsf{Rel}(A \times C, B \times D)$$

of $\times$ is given by sending a pair of morphisms (R, S) of the form

$$R \colon A \nrightarrow B,$$
$$S \colon C \nrightarrow D$$

to the relation

$$R \times S \colon A \times C \nrightarrow B \times D$$

of Definition 6.3.9.1.

5.2.2.2 The Monoidal Unit

Definition 5.2.2.2 ► The Monoidal Unit of Rel

The **monoidal unit of** Rel is the functor

$$\mathbb{1}^{\mathsf{Rel}} \colon \mathrm{pt} \longrightarrow \mathsf{Rel}$$

picking the set

$$\mathbb{1}_{\mathsf{Rel}} \stackrel{\text{def}}{=} \mathrm{pt}$$

of Rel.

5.2.2.3 The Associator

Definition 5.2.2.3 ► The Associator of Rel

The **associator of** Rel is the natural isomorphism

$$\alpha^{\mathsf{Rel}} \colon \times \circ ((\times) \times \mathrm{id}) \stackrel{\sim}{\Longrightarrow} \times \circ (\mathrm{id} \times (\times)) \circ \boldsymbol{\alpha}^{\mathsf{Cats}}_{\mathsf{Rel},\mathsf{Rel},\mathsf{Rel}},$$

as in the diagram

$$
\begin{tikzcd}
 & \mathsf{Rel} \times (\mathsf{Rel} \times \mathsf{Rel}) \arrow[dr, "\mathrm{id} \times (\times)"] & \\
(\mathsf{Rel} \times \mathsf{Rel}) \times \mathsf{Rel} \arrow[ur, dashed, "\alpha^{\mathsf{Cats}}_{\mathsf{Rel},\mathsf{Rel},\mathsf{Rel}}", "\sim"'] \arrow[d, "(\times) \times \mathrm{id}"'] & & \mathsf{Rel} \times \mathsf{Rel} \arrow[d, "\times"] \\
\mathsf{Rel} \times \mathsf{Rel} \arrow[rr, "\times"'] \arrow[rr, phantom, bend left, "\Nearrow \alpha^{\mathsf{Rel}}"] & & \mathsf{Rel},
\end{tikzcd}
$$

whose component

$$\alpha^{\mathsf{Rel}}_{A,B,C} \colon (A \times B) \times C \nrightarrow A \times (B \times C)$$

at $A, B, C \in \mathrm{Obj}(\mathsf{Rel})$ is the relation defined by declaring

$$((a, b), c) \sim_{\alpha^{\mathsf{Rel}}_{A,B,C}} (a', (b', c'))$$

iff $a = a'$, $b = b'$, and $c = c'$.

5.2.2.4 The Left Unitor

Definition 5.2.2.4 ► The Left Unitor of Rel

The **left unitor of** Rel is the natural isomorphism

$$
\lambda^{\mathsf{Rel}} \colon \times \circ \left(\mathbb{1}^{\mathsf{Rel}} \times \mathrm{id}\right) \overset{\sim}{\Longrightarrow} \lambda^{\mathsf{Cats}_2}_{\mathsf{Rel}},
\qquad
\begin{tikzcd}
\mathrm{pt} \times \mathsf{Rel} \arrow[r, "\mathbb{1}^{\mathsf{Rel}} \times \mathrm{id}"] \arrow[dr, dashed, bend right, "\lambda^{\mathsf{Cats}_2}_{\mathsf{Rel}}"'] & \mathsf{Rel} \times \mathsf{Rel}, \arrow[d, "\times"] \arrow[dl, phantom, "\Swarrow \lambda^{\mathsf{Rel}}"] \\
 & \mathsf{Rel}
\end{tikzcd}
$$

whose component

$$\lambda^{\mathsf{Rel}}_A \colon \mathbb{1}_{\mathsf{Rel}} \times A \nrightarrow A$$

at A is defined by declaring

$$(\star, a) \sim_{\lambda^{\mathsf{Rel}}_A} b$$

iff $a = b$.

5.2.2.5 The Right Unitor

Definition 5.2.2.5 ▶ The Right Unitor of Rel

The **right unitor of** Rel is the natural isomorphism

$$\rho^{\mathsf{Rel}} \colon \times \circ \left(\mathrm{id} \times \mathbb{1}^{\mathsf{Rel}}\right) \overset{\sim}{\Longrightarrow} \rho^{\mathsf{Cats}_2}_{\mathsf{Rel}}, \qquad \begin{tikzcd} \mathsf{Rel} \times \mathrm{pt} \arrow[r, "\mathrm{id} \times \mathbb{1}^{\mathsf{Rel}}"] \arrow[dr, dashed, bend right, "\rho^{\mathsf{Cats}_2}_{\mathsf{Rel}}"'] & \mathsf{Rel} \times \mathsf{Rel}, \arrow[d, "\times"] \arrow[dl, phantom, "\Swarrow \rho^{\mathsf{Rel}}"] \\ & \mathsf{Rel} \end{tikzcd}$$

whose component

$$\rho^{\mathsf{Rel}}_A \colon A \times \mathbb{1}_{\mathsf{Rel}} \nrightarrow A$$

at A is defined by declaring

$$(a, \star) \sim_{\rho^{\mathsf{Rel}}_A} b$$

iff $a = b$.

5.2.2.6 The Symmetry

Definition 5.2.2.6 ▶ The Symmetry of Rel

The **symmetry of** Rel is the natural isomorphism

$$\sigma^{\mathsf{Rel}} \colon \times \Longrightarrow \times \circ \sigma^{\mathsf{Cats}_2}_{\mathsf{Rel},\mathsf{Rel}}, \qquad \begin{tikzcd} \mathsf{Rel} \times \mathsf{Rel} \arrow[rr, "\times"] \arrow[dr, "\sigma^{\mathsf{Cats}_2}_{\mathsf{Rel},\mathsf{Rel}}"'] & \arrow[d, phantom, "\Downarrow \sigma^{\mathsf{Rel}}"] & \mathsf{Rel}, \\ & \mathsf{Rel} \times \mathsf{Rel} \arrow[ur, "\times"'] & \end{tikzcd}$$

whose component

$$\sigma^{\mathsf{Rel}}_{A,B} \colon A \times B \to B \times A$$

at (A, B) is defined by declaring

$$(a, b) \sim_{\sigma^{\mathsf{Rel}}_{A,B}} (b', a')$$

iff $a = a'$ and $b = b'$.

5.2.2.7 The Internal Hom

Definition 5.2.2.7 ► The Internal Hom of Rel

The **internal Hom of** Rel is the functor

$$\mathrm{Rel}\colon \mathrm{Rel}^{\mathrm{op}} \times \mathrm{Rel} \longrightarrow \mathrm{Rel}$$

defined

- On objects by sending $A, B \in \mathrm{Obj}(\mathrm{Rel})$ to the set $\mathrm{Rel}(A, B)$ of Item 1 of Definition 5.1.1.3.
- On morphisms by pre/post-composition defined as in Definition 6.3.12.1.

Proposition 5.2.2.8 ► Properties of the Internal Hom of Rel

Let $A, B, C \in \mathrm{Obj}(\mathrm{Rel})$.

1. *Adjointness.* We have adjunctions

$$(A \times - \dashv \mathrm{Rel}(A, -))\colon \quad \begin{tikzcd} \mathrm{Rel} \arrow[r, bend left, "A\times -"] \arrow[r, phantom, "\perp"] & \mathrm{Rel}, \arrow[l, bend left, "\mathrm{Rel}(A,-)"] \end{tikzcd}$$

$$(- \times B \dashv \mathrm{Rel}(B, -))\colon \quad \begin{tikzcd} \mathrm{Rel} \arrow[r, bend left, "-\times B"] \arrow[r, phantom, "\perp"] & \mathrm{Rel}, \arrow[l, bend left, "\mathrm{Rel}(B,-)"] \end{tikzcd}$$

witnessed by bijections

$$\begin{aligned} \mathrm{Rel}(A \times B, C) &\cong \mathrm{Rel}(A, \mathrm{Rel}(B, C)), \\ \mathrm{Rel}(A \times B, C) &\cong \mathrm{Rel}(B, \mathrm{Rel}(A, C)), \end{aligned}$$

natural in $A, B, C \in \mathrm{Obj}(\mathrm{Rel})$.

Proof 5.2.2.9 ► Proof of Proposition 5.2.2.8

Item 1: Adjointness

Indeed, we have

$$\begin{aligned} \mathrm{Rel}(A \times B, C) &\overset{\text{def}}{=} \mathrm{Sets}(A \times B \times C, \{\mathrm{true}, \mathrm{false}\}) \\ &\overset{\text{def}}{=} \mathrm{Rel}(A, B \times C) \\ &\overset{\text{def}}{=} \mathrm{Rel}(A, \mathrm{Rel}(B, C)), \end{aligned}$$

and similarly for the bijection $\mathrm{Rel}(A \times B, C) \cong \mathrm{Rel}(B, \mathrm{Rel}(A, C))$.

5.2.2.8 The Closed Symmetric Monoidal Category of Relations

PROPOSITION 5.2.2.10 ▸ THE CLOSED SYMMETRIC MONOIDAL CATEGORY OF RELATIONS

The category Rel admits a closed symmetric monoidal category structure consisting of[1]

- *The Underlying Category.* The category Rel of sets and relations of Definition 5.2.1.1.
- *The Monoidal Product.* The functor

  $$\times \colon \mathrm{Rel} \times \mathrm{Rel} \longrightarrow \mathrm{Rel}$$

  of Definition 5.2.2.1.
- *The Internal Hom.* The internal Hom functor

  $$\mathbf{Rel} \colon \mathrm{Rel}^{\mathrm{op}} \times \mathrm{Rel} \longrightarrow \mathrm{Rel}$$

  of Definition 5.2.2.7.
- *The Monoidal Unit.* The functor

  $$\mathbb{1}^{\mathrm{Rel}} \colon \mathrm{pt} \longrightarrow \mathrm{Rel}$$

  of Definition 5.2.2.2.
- *The Associators.* The natural isomorphism

  $$\alpha^{\mathrm{Rel}} \colon \times \circ (\times \times \mathrm{id}_{\mathrm{Rel}}) \stackrel{\sim}{\Longrightarrow} \times \circ (\mathrm{id}_{\mathrm{Rel}} \times \times) \circ \boldsymbol{\alpha}^{\mathsf{Cats}}_{\mathrm{Rel},\mathrm{Rel},\mathrm{Rel}}$$

  of Definition 5.2.2.3.
- *The Left Unitors.* The natural isomorphism

  $$\lambda^{\mathrm{Rel}} \colon \times \circ \left(\mathbb{1}^{\mathrm{Rel}} \times \mathrm{id}_{\mathrm{Rel}}\right) \stackrel{\sim}{\Longrightarrow} \boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathrm{Rel}}$$

  of Definition 5.2.2.4.

- *The Right Unitors.* The natural isomorphism

$$\rho^{\mathsf{Rel}}\colon \times \circ \left(\mathrm{id} \times \mathbb{1}^{\mathsf{Rel}}\right) \overset{\sim}{\Longrightarrow} \boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathsf{Rel}}$$

of Definition 5.2.2.5.

- *The Symmetry.* The natural isomorphism

$$\sigma^{\mathsf{Rel}}\colon \times \overset{\sim}{\Longrightarrow} \times \circ \boldsymbol{\sigma}^{\mathsf{Cats}_2}_{\mathsf{Rel},\mathsf{Rel}}$$

of Definition 5.2.2.6.

---

[1] *Warning:* This is not a Cartesian monoidal structure, as the product on Rel is in fact given by the disjoint union of sets; see **??**.
END TEXTDBEND

PROOF 5.2.2.11 ▶ PROOF OF PROPOSITION 5.2.2.10

Omitted. □

5.2.3 The 2-Category of Relations

DEFINITION 5.2.3.1 ▶ THE 2-CATEGORY OF RELATIONS

The 2-**category of relations** is the locally posetal 2-category **Rel** where

- *Objects.* The objects of **Rel** are sets.
- ***Hom**-Objects.* For each $A, B \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$\begin{aligned}\mathrm{Hom}_{\mathbf{Rel}}(A, B) &\overset{\mathrm{def}}{=} \mathbf{Rel}(A, B)\\ &\overset{\mathrm{def}}{=} (\mathrm{Rel}(A, B), \subset).\end{aligned}$$

- *Identities.* For each $A \in \mathrm{Obj}(\mathbf{Rel})$, the unit map

$$\mathbb{1}^{\mathbf{Rel}}_{A}\colon \mathrm{pt} \to \mathbf{Rel}(A, A)$$

of **Rel** at A is defined by

$$\mathrm{id}^{\mathbf{Rel}}_{A} \overset{\mathrm{def}}{=} \chi_A(-_1, -_2),$$

where $\chi_A(-_1, -_2)$ is the characteristic relation of A of Item 3 of Definition 2.4.1.1.

- *Composition*. For each $A, B, C \in \mathrm{Obj}(\mathbf{Rel})$, the composition map[1]

$$\circ^{\mathbf{Rel}}_{A,B,C} \colon \mathbf{Rel}(B, C) \times \mathbf{Rel}(A, B) \longrightarrow \mathbf{Rel}(A, C)$$

  of **Rel** at (A, B, C) is defined by

$$S \circ^{\mathbf{Rel}}_{A,B,C} R \stackrel{\text{def}}{=} S \diamond R$$

  for each $(S, R) \in \mathbf{Rel}(B, C) \times \mathbf{Rel}(A, B)$, where $S \diamond R$ is the composition of S and R of Definition 6.3.12.1.

[1]Note that this is indeed a morphism of posets: given relations $R_1, R_2 \in \mathbf{Rel}(A, B)$ and $S_1, S_2 \in \mathbf{Rel}(B, C)$ such that

$$R_1 \subset R_2,$$
$$S_1 \subset S_2,$$

we have also $S_1 \diamond R_1 \subset S_2 \diamond R_2$.

5.2.4 The Double Category of Relations

5.2.4.1 The Double Category of Relations

Definition 5.2.4.1 ► The Double Category of Relations

The **double category of relations** is the locally posetal double category $\mathrm{Rel}^{\mathrm{dbl}}$ where

- *Objects*. The objects of $\mathrm{Rel}^{\mathrm{dbl}}$ are sets.
- *Vertical Morphisms*. The vertical morphisms of $\mathrm{Rel}^{\mathrm{dbl}}$ are maps of sets $f \colon A \to B$.
- *Horizontal Morphisms*. The horizontal morphisms of $\mathrm{Rel}^{\mathrm{dbl}}$ are relations $R \colon A \nrightarrow X$.
- *2-Morphisms*. A 2-cell

$$\begin{tikzcd} A \arrow[r, "R", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & B \arrow[d, "g"] \\ X \arrow[r, "S"', "\shortmid"{marking}] & Y \end{tikzcd}$$

of $\mathsf{Rel}^{\mathsf{dbl}}$ is either non-existent or an inclusion of relations of the form

$$R \subset S \circ (f \times g), \qquad \begin{tikzcd} A \times B \arrow[r, "R"] \arrow[d, "f\times g"'] \arrow[dr, phantom, "\circlearrowleft"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ X \times Y \arrow[r, "S"'] & \{\mathsf{true}, \mathsf{false}\}. \end{tikzcd}$$

- *Horizontal Identities.* The horizontal unit functor of $\mathsf{Rel}^{\mathsf{dbl}}$ is the functor of Definition 5.2.4.2.
- *Vertical Identities.* For each $A \in \mathrm{Obj}\left(\mathsf{Rel}^{\mathsf{dbl}}\right)$, we have
$$\mathrm{id}_A^{\mathsf{Rel}^{\mathsf{dbl}}} \stackrel{\text{def}}{=} \mathrm{id}_A.$$
- *Identity 2-Morphisms.* For each horizontal morphism $R\colon A \nrightarrow B$ of $\mathsf{Rel}^{\mathsf{dbl}}$, the identity 2-morphism
$$\begin{tikzcd} A \arrow[r, "R", "\shortmid"{marking}] \arrow[d, "\mathrm{id}_A"'] \arrow[dr, phantom, "\Downarrow \mathrm{id}_R"] & B \arrow[d, "\mathrm{id}_B"] \\ A \arrow[r, "R"', "\shortmid"{marking}] & B \end{tikzcd}$$
of R is the identity inclusion
$$R \subset R, \qquad \begin{tikzcd} B \times A \arrow[r, "R"] \arrow[d, "\mathrm{id}_B\times\mathrm{id}_A"'] \arrow[dr, phantom, "\circlearrowleft"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ B \times A \arrow[r, "R"'] & \{\mathsf{true}, \mathsf{false}\}. \end{tikzcd}$$
- *Horizontal Composition.* The horizontal composition functor of $\mathsf{Rel}^{\mathsf{dbl}}$ is the functor of Definition 5.2.4.3.
- *Vertical Composition of 1-Morphisms.* For each composable pair $A \xrightarrow{F} B \xrightarrow{G} C$ of vertical morphisms of $\mathsf{Rel}^{\mathsf{dbl}}$, i.e. maps of sets, we have
$$g \circ^{\mathsf{Rel}^{\mathsf{dbl}}} f \stackrel{\text{def}}{=} g \circ f.$$

- *Vertical Composition of 2-Morphisms.* The vertical composition of 2-morphisms in $\mathsf{Rel}^{\mathsf{dbl}}$ is defined as in Definition 5.2.4.4.
- *Associators.* The associators of $\mathsf{Rel}^{\mathsf{dbl}}$ is defined as in Definition 5.2.4.5.
- *Left Unitors.* The left unitors of $\mathsf{Rel}^{\mathsf{dbl}}$ is defined as in Definition 5.2.4.6.
- *Right Unitors.* The right unitors of $\mathsf{Rel}^{\mathsf{dbl}}$ is defined as in Definition 5.2.4.7.

5.2.4.2 Horizontal Identities

Definition 5.2.4.2 ► The Horizontal Identities of $\mathsf{Rel}^{\mathsf{dbl}}$

The **horizontal unit functor** of $\mathsf{Rel}^{\mathsf{dbl}}$ is the functor

$$\mathbb{1}^{\mathsf{Rel}^{\mathsf{dbl}}} \colon \mathsf{Rel}^{\mathsf{dbl}}_0 \to \mathsf{Rel}^{\mathsf{dbl}}_1$$

of $\mathsf{Rel}^{\mathsf{dbl}}$ is the functor where

- *Action on Objects.* For each $A \in \mathrm{Obj}\left(\mathsf{Rel}^{\mathsf{dbl}}_0\right)$, we have

$$\mathbb{1}_A \overset{\text{def}}{=} \chi_A(-_1, -_2).$$

- *Action on Morphisms.* For each vertical morphism $f \colon A \to B$ of $\mathsf{Rel}^{\mathsf{dbl}}$, i.e. each map of sets f from A to B, the identity 2-morphism

$$\begin{tikzcd}
A \arrow[r, "\mathbb{1}_A", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\overset{\Vert}{\underset{\Downarrow}{\mathbb{1}_f}}"] & A \arrow[d, "f"] \\
B \arrow[r, "\mathbb{1}_B"', "\shortmid"{marking}] & B
\end{tikzcd}$$

of f is the inclusion

$$\chi_B \circ (f \times f) \subset \chi_A, \qquad \begin{tikzcd}
A \times A \arrow[r, "\chi_A(-_1,-_2)"] \arrow[d, "f \times f"'] \arrow[dr, phantom, "\subset"] & \{\mathrm{true}, \mathrm{false}\} \arrow[d, "\mathrm{id}_{\{\mathrm{true},\mathrm{false}\}}"] \\
B \times B \arrow[r, "\chi_B(-_1,-_2)"'] & \{\mathrm{true}, \mathrm{false}\}
\end{tikzcd}$$

of Item 1 of Proposition 2.4.1.3.

5.2.4.3 Horizontal Composition

Definition 5.2.4.3 ► The Horizontal Composition of $\mathsf{Rel}^{\mathsf{dbl}}$

The **horizontal composition functor** of $\mathsf{Rel}^{\mathsf{dbl}}$ is the functor

$$\odot^{\mathsf{Rel}^{\mathsf{dbl}}} \colon \mathsf{Rel}^{\mathsf{dbl}}_1 \underset{\mathsf{Rel}^{\mathsf{dbl}}_0}{\times} \mathsf{Rel}^{\mathsf{dbl}}_1 \to \mathsf{Rel}^{\mathsf{dbl}}_1$$

of $\mathsf{Rel}^{\mathsf{dbl}}$ is the functor where

- *Action on Objects.* For each composable pair $A \overset{R}{\nrightarrow} B \overset{S}{\nrightarrow} C$ of horizontal morphisms of $\mathsf{Rel}^{\mathsf{dbl}}$, we have

$$S \odot R \overset{\mathrm{def}}{=} S \diamond R,$$

where $S \diamond R$ is the composition of R and S of Definition 6.3.12.1.

- *Action on Morphisms.* For each horizontally composable pair

$$\begin{tikzcd} A \arrow[r, "R", "\mapstochar"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & B \arrow[d, "g"] \\ X \arrow[r, "T"', "\mapstochar"{marking}] & Y \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "S", "\mapstochar"{marking}] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\beta"] & C \arrow[d, "h"] \\ Y \arrow[r, "U"', "\mapstochar"{marking}] & Z \end{tikzcd}$$

of 2-morphisms of $\mathsf{Rel}^{\mathsf{dbl}}$, i.e. for each pair

$$\begin{tikzcd} A \times B \arrow[r, "R"] \arrow[d, "f\times g"'] \arrow[dr, phantom, "\subset"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ X \times Y \arrow[r, "T"'] & \{\mathsf{true}, \mathsf{false}\} \end{tikzcd} \qquad \begin{tikzcd} B \times C \arrow[r, "S"] \arrow[d, "g\times h"'] \arrow[dr, phantom, "\subset"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ Y \times Z \arrow[r, "U"'] & \{\mathsf{true}, \mathsf{false}\} \end{tikzcd}$$

of inclusions of relations, the horizontal composition

$$\begin{tikzcd} A \arrow[r, "S\odot R", "\mapstochar"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\beta\odot\alpha"] & C \arrow[d, "h"] \\ X \arrow[r, "U\odot T"', "\mapstochar"{marking}] & Z \end{tikzcd}$$

of α and β is the inclusion of relations[1]

$$(U \diamond T) \circ (f \times h) \subset (S \diamond R) \qquad \begin{tikzcd} A \times C \arrow[r, "S\diamond R"] \arrow[d, "f\times h"'] \arrow[dr, phantom, "\subset"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ X \times Z \arrow[r, "U\diamond T"'] & \{\mathsf{true}, \mathsf{false}\}. \end{tikzcd}$$

[1]This is justified by noting that, given $(a,c) \in A \times C$, the statement

- We have $a \sim_{(U \diamond T) \circ (f \times h)} c$, i.e. $f(a) \sim_{U \diamond T} h(c)$, i.e. there exists some $y \in Y$ such that:
  1. We have $f(a) \sim_T y$;
  2. We have $y \sim_U h(c)$;

is implied by the statement

- We have $a \sim_{S \diamond R} c$, i.e. there exists some $b \in B$ such that:
  1. We have $a \sim_R b$;
  2. We have $b \sim_S c$;

since:

- If $a \sim_R b$, then $f(a) \sim_T g(b)$, as $T \circ (f \times g) \subset R$;
- If $b \sim_S c$, then $g(b) \sim_U h(c)$, as $U \circ (g \times h) \subset S$.

5.2.4.4 Vertical Composition of 2-Morphisms

Definition 5.2.4.4 ► The Vertical Composition of 2-Morphisms in $\mathsf{Rel}^{\mathsf{dbl}}$

The **vertical composition** in $\mathsf{Rel}^{\mathsf{dbl}}$ is defined as follows: for each vertically composable pair

$$\begin{tikzcd} A \arrow[r, "R", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & X \arrow[d, "g"] \\ B \arrow[r, "S"', "\shortmid"{marking}] & Y \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "S", "\shortmid"{marking}] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow\beta"] & Y \arrow[d, "k"] \\ C \arrow[r, "T"', "\shortmid"{marking}] & Z \end{tikzcd}$$

of 2-morphisms of $\mathsf{Rel}^{\mathsf{dbl}}$, i.e. for each each pair

$$\begin{tikzcd} A \times X \arrow[r, "R"] \arrow[d, "f \times g"'] \arrow[dr, phantom, "\subset"] & \{\text{true}, \text{false}\} \arrow[d, "\mathrm{id}_{\{\text{true},\text{false}\}}"] \\ B \times Y \arrow[r, "S"'] & \{\text{true}, \text{false}\} \end{tikzcd} \qquad \begin{tikzcd} B \times Y \arrow[r, "S"] \arrow[d, "h \times k"'] \arrow[dr, phantom, "\subset"] & \{\text{true}, \text{false}\} \arrow[d, "\mathrm{id}_{\{\text{true},\text{false}\}}"] \\ C \times Z \arrow[r, "T"'] & \{\text{true}, \text{false}\} \end{tikzcd}$$

of inclusions of relations, we define the vertical composition

$$\begin{tikzcd} A \arrow[r, "R", "\shortmid"{marking}] \arrow[d, "h\circ f"'] \arrow[dr, phantom, "\overset{\|}{\underset{\Downarrow}{\beta\circ\alpha}}"] & X \arrow[d, "k\circ g"] \\ C \arrow[r, "T"', "\shortmid"{marking}] & Z \end{tikzcd}$$

of α and β as the inclusion of relations

$$T \circ [(h \circ f) \times (k \circ g)] \subset R, \qquad \begin{tikzcd} A \times X \arrow[r, "R"] \arrow[d, "(h\circ f)\times(k\circ g)"'] \arrow[dr, phantom, "\subset"] & \{\text{true, false}\} \arrow[d, "\mathrm{id}_{\{\text{true,false}\}}"] \\ C \times Z \arrow[r, "T"'] & \{\text{true, false}\} \end{tikzcd}$$

given by the pasting of inclusions[1]

$$\begin{tikzcd} A \times X \arrow[r, "R"] \arrow[d, "f\times g"'] \arrow[dr, phantom, "\subset"] & \{\text{true, false}\} \arrow[d, "\mathrm{id}_{\{\text{true,false}\}}"] \\ B \times Y \arrow[r, "S"] \arrow[d, "h\times k"'] \arrow[dr, phantom, "\subset"] & \{\text{true, false}\} \arrow[d, "\mathrm{id}_{\{\text{true,false}\}}"] \\ C \times Z \arrow[r, "T"'] & \{\text{true, false}\}. \end{tikzcd}$$

---

[1] This is justified by noting that, given $(a, x) \in A \times X$, the statement

- We have $h(f(a)) \sim_T k(g(x))$;

is implied by the statement

- We have $a \sim_R x$;

since

- If $a \sim_R x$, then $f(a) \sim_S g(x)$, as $S \circ (f \times g) \subset R$;
- If $b \sim_S y$, then $h(b) \sim_T k(y)$, as $T \circ (h \times k) \subset S$, and thus, in particular:
  - If $f(a) \sim_S g(x)$, then $h(f(a)) \sim_T k(g(x))$.

5.2.4.5 The Associators

Definition 5.2.4.5 ▶ The Associators of $\mathsf{Rel}^{\mathsf{dbl}}$

For each composable triple

$$A \overset{R}{\nrightarrow} B \overset{S}{\nrightarrow} C \overset{T}{\nrightarrow} D$$

of horizontal morphisms of $\mathsf{Rel}^{\mathsf{dbl}}$, the component

$$\alpha^{\mathsf{Rel}^{\mathsf{dbl}}}_{T,S,R} \colon (T \odot S) \odot R \overset{\sim}{\Longrightarrow} T \odot (S \odot R), \qquad \begin{tikzcd} A \arrow[r, "R", "|"{marking}] \arrow[d, "\mathrm{id}_A"'] & B \arrow[r, "S", "|"{marking}] \arrow[d, phantom, "\Downarrow \alpha^{\mathsf{Rel}^{\mathsf{dbl}}}_{T,S,R}"] & C \arrow[r, "T", "|"{marking}] & D \arrow[d, "\mathrm{id}_D"] \\ A \arrow[r, "R"', "|"{marking}] & B \arrow[r, "S"', "|"{marking}] & C \arrow[r, "T"', "|"{marking}] & D \end{tikzcd}$$

of the associator of $\mathsf{Rel}^{\mathsf{dbl}}$ at (R, S, T) is the identity inclusion[1]

$$(T \diamond S) \diamond R = T \diamond (S \diamond R) \qquad \begin{tikzcd} A \times B \arrow[r, "(T \diamond S) \diamond R"] \arrow[d, equal] \arrow[dr, phantom, "\not\Rightarrow"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ A \times B \arrow[r, "T \diamond (S \diamond R)"'] & \{\mathsf{true}, \mathsf{false}\}. \end{tikzcd}$$

[1] This is justified by Item 2 of Proposition 6.3.12.3.

5.2.4.6 The Left Unitors

Definition 5.2.4.6 ▶ The Left Unitors of $\mathsf{Rel}^{\mathsf{dbl}}$

For each horizontal morphism $R \colon A \nrightarrow B$ of $\mathsf{Rel}^{\mathsf{dbl}}$, the component

$$\lambda^{\mathsf{Rel}^{\mathsf{dbl}}}_R \colon \mathbb{1}_B \odot R \overset{\sim}{\Longrightarrow} R, \qquad \begin{tikzcd} A \arrow[r, "R", "|"{marking}] \arrow[d, "\mathrm{id}_A"'] & B \arrow[r, "\mathbb{1}_B", "|"{marking}] \arrow[d, phantom, "\Downarrow \lambda^{\mathsf{Rel}^{\mathsf{dbl}}}_R"] & B \arrow[d, "\mathrm{id}_B"] \\ A \arrow[rr, "R"', "|"{marking}] & & B \end{tikzcd}$$

of the left unitor of $\mathsf{Rel}^{\mathsf{dbl}}$ at R is the identity inclusion[1]

$$R = \chi_B \diamond R, \qquad \begin{tikzcd} A \times B \arrow[r, "\chi_B \diamond R"] \arrow[d, equal] \arrow[dr, phantom, "\not\Rightarrow"] & \{\mathrm{true}, \mathrm{false}\} \arrow[d, "\mathrm{id}_{\{\mathrm{true},\mathrm{false}\}}"] \\ A \times B \arrow[r, "R"'] & \{\mathrm{true}, \mathrm{false}\}. \end{tikzcd}$$

[1]This is justified by Item 3 of Proposition 6.3.12.3.

5.2.4.7 The Right Unitors

Definition 5.2.4.7 ▶ The Right Unitors of $\mathsf{Rel}^{\mathsf{dbl}}$

For each horizontal morphism $R\colon A \nrightarrow B$ of $\mathsf{Rel}^{\mathsf{dbl}}$, the component

$$\rho^{\mathsf{Rel}^{\mathsf{dbl}}}_R \colon R \odot \mathbb{1}_A \overset{\sim}{\Longrightarrow} R, \qquad \begin{tikzcd} A \arrow[r, "\mathbb{1}_A", "\shortmid"{marking}] \arrow[d, "\mathrm{id}_A"'] & A \arrow[r, "R", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow \rho^{\mathsf{Rel}^{\mathsf{dbl}}}_R"] & B \arrow[d, "\mathrm{id}_B"] \\ A \arrow[rr, "R"', "\shortmid"{marking}] & & B \end{tikzcd}$$

of the right unitor of $\mathsf{Rel}^{\mathsf{dbl}}$ at R is the identity inclusion[1]

$$R = R \diamond \chi_A, \qquad \begin{tikzcd} A \times B \arrow[r, "R \diamond \chi_A"] \arrow[d, equal] \arrow[dr, phantom, "\not\Rightarrow"] & \{\mathrm{true}, \mathrm{false}\} \arrow[d, "\mathrm{id}_{\{\mathrm{true},\mathrm{false}\}}"] \\ A \times B \arrow[r, "R"'] & \{\mathrm{true}, \mathrm{false}\}. \end{tikzcd}$$

[1]This is justified by Item 3 of Proposition 6.3.12.3.

5.3 Properties of the 2-Category of Relations

5.3.1 Self-Duality

Proposition 5.3.1.1 ▶ Self-Duality for the (2-)Category of Relations

The (2-)category of relations is self-dual:

1. *Self-Duality I.* We have an isomorphism

$$\mathsf{Rel}^{\mathsf{op}} \overset{\text{eq.}}{\cong} \mathsf{Rel}$$

of categories.

2. *Self-Duality II.* We have a 2-isomorphism

$$\mathbf{Rel}^{\mathsf{op}} \overset{\text{eq.}}{\cong} \mathbf{Rel}$$

of 2-categories.

Proof 5.3.1.2 ▶ Proof of Proposition 5.3.1.1

Item 1: Self-Duality I

We claim that the functor

$$F \colon \mathsf{Rel}^{\mathsf{op}} \to \mathsf{Rel}$$

given by the identity on objects and by $R \mapsto R^{\dagger}$ on morphisms is an isomorphism of categories.
By Item 1 of Proposition 8.5.8.3, it suffices to show that F is bijective on objects (which is clear) and fully faithful. Indeed, the map

$$(-)^{\dagger} \colon \mathsf{Rel}(A, B) \to \mathsf{Rel}(B, A)$$

defined by the assignment $R \mapsto R^{\dagger}$ is a bijection by Item 5 of Proposition 6.3.11.3, showing F to be fully faithful.

Item 2: Self-Duality II

We claim that the 2-functor

$$F \colon \mathsf{Rel}^{\mathsf{op}} \to \mathsf{Rel}$$

given by the identity on objects, by $R \mapsto R^{\dagger}$ on morphisms, and by preserving inclusions on 2-morphisms via Item 1 of Proposition 6.3.11.3, is an isomorphism of categories.
By **??** of **??**, it suffices to show that F is:

- Bijective on objects, which is clear.
- Bijective on 1-morphisms, which was shown in Item 1.
- Bijective on 2-morphisms, which follows from Item 1 of Proposition 6.3.11.3.

Thus F is indeed a 2-isomorphism of categories. ∎

5.3.2 Isomorphisms and Equivalences in Rel

Let $R\colon A \nrightarrow B$ be a relation from A to B.

Proposition 5.3.2.1 ▶ Isomorphisms and Equivalences in Rel

The following conditions are equivalent:

1. The relation $R\colon A \nrightarrow B$ is an equivalence in **Rel**, i.e.:

   ($\star$) There exists a relation $R^{-1}\colon B \nrightarrow A$ from B to A together with isomorphisms

   $$\begin{aligned} R^{-1} \diamond R &\cong \chi_A, \\ R \diamond R^{-1} &\cong \chi_B. \end{aligned}$$

2. The relation $R\colon A \nrightarrow B$ is an isomorphism in Rel, i.e.:

   ($\star$) There exists a relation $R^{-1}\colon B \nrightarrow A$ from B to A such that we have

   $$\begin{aligned} R^{-1} \diamond R &= \chi_A, \\ R \diamond R^{-1} &= \chi_B. \end{aligned}$$

3. There exists a bijection $f\colon A \xrightarrow{\cong} B$ with $R = \mathrm{Gr}(f)$.

Proof 5.3.2.2 ▶ Proof of Proposition 5.3.2.1

We claim that Items 1 to 3 are indeed equivalent:

- *Item 1* $\Longleftrightarrow$ *Item 2*: This follows from the fact that **Rel** is locally posetal, so that natural isomorphisms and equalities of 1-morphisms in **Rel** coincide.

- *Item 2* $\Longrightarrow$ *Item 3*: The equalities in Item 2 imply $R \dashv R^{-1}$, and thus by Proposition 5.3.3.1, there exists a function $f_R\colon A \to B$ associated to R, where, for each $a \in A$, the image $f_R(a)$ of a by f_R is the unique element of $R(a)$, which implies $R = \mathrm{Gr}(f_R)$ in particular. Furthermore, we have $R^{-1} = f_R^{-1}$ (as in Definition 6.3.2.1). The conditions from Item 2 then become the following:

  $$\begin{aligned} f_R^{-1} \diamond f_R &= \chi_A, \\ f_R \diamond f_R^{-1} &= \chi_B. \end{aligned}$$

All that is left is to show then is that f_R is a bijection:

- *The Function f_R Is Injective.* Let $a, b \in A$ and suppose that $f_R(a) = f_R(b)$. Since $a \sim_R f_R(a)$ and $f_R(a) = f_R(b) \sim_{R^{-1}} b$, the condition $f_R^{-1} \diamond f_R = \chi_A$ implies that $a = b$, showing f_R to be injective.
- *The Function f_R Is Surjective.* Let $b \in B$. Applying the condition $f_R \diamond f_R^{-1} = \chi_B$ to (b, b), it follows that there exists some $a \in A$ such that $f_R^{-1}(b) = a$ and $f_R(a) = b$. This shows f_R to be surjective.

· *Item 3 $\implies$ Item 2:* By Item 2 of Proposition 6.3.1.2, we have an adjunction $\mathrm{Gr}(f) \dashv f^{-1}$, giving inclusions

$$\chi_A \subset f^{-1} \diamond \mathrm{Gr}(f),$$
$$\mathrm{Gr}(f) \diamond f^{-1} \subset \chi_B.$$

We claim the reverse inclusions are also true:

- $f^{-1} \diamond \mathrm{Gr}(f) \subset \chi_A$: This is equivalent to the statement that if $f(a) = b$ and $f^{-1}(b) = a'$, then $a = a'$, which follows from the injectivity of f.
- $\chi_B \subset \mathrm{Gr}(f) \diamond f^{-1}$: This is equivalent to the statement that given $b \in B$ there exists some $a \in A$ such that $f^{-1}(b) = a$ and $f(a) = b$, which follows from the surjectivity of f.

This finishes the proof. □

5.3.3 Adjunctions in Rel

Let A and B be sets.

Proposition 5.3.3.1 ► Adjunctions in Rel

We have a natural bijection

$$\left\{\begin{matrix}\text{Adjunctions in } \mathbf{Rel}\\ \text{from } A \text{ to } B\end{matrix}\right\} \cong \left\{\begin{matrix}\text{Functions}\\ \text{from } A \text{ to } B\end{matrix}\right\},$$

with every adjunction in **Rel** being of the form $\mathrm{Gr}(f) \dashv f^{-1}$ for some function f.

Proof 5.3.3.2 ► Proof of Proposition 5.3.3.1

We proceed step by step:

1. *From Adjunctions in* **Rel** *to Functions.* An adjunction in **Rel** from A to B consists of a pair of relations

$$R\colon A \nrightarrow B,$$
$$S\colon B \nrightarrow A,$$

together with inclusions

$$\chi_A \subset S \diamond R,$$
$$R \diamond S \subset \chi_B.$$

We claim that these conditions imply that R is total and functional, i.e. that $R(a)$ is a singleton for each $a \in A$:

(a) $R(a)$ *Has an Element.* Given $a \in A$, since $\chi_A \subset S \diamond R$, we must have $\{a\} \subset S(R(a))$, implying that there exists some $b \in B$ such that $a \sim_R b$ and $b \sim_S a$, and thus $R(a) \neq \emptyset$, as $b \in R(a)$.

(b) $R(a)$ *Has No More Than One Element.* Suppose that we have $a \sim_R b$ and $a \sim_R b'$ for $b, b' \in B$. We claim that $b = b'$:

   i. Since $\chi_A \subset S \diamond R$, there exists some $k \in B$ such that $a \sim_R k$ and $k \sim_S a$.
   ii. Since $R \diamond S \subset \chi_B$, if $b'' \sim_S a'$ and $a' \sim_R b'''$, then $b'' = b'''$.
   iii. Applying the above to $b'' = k$, $b''' = b$, and $a' = a$, since $k \sim_S a$ and $a \sim_R b'$, we have $k = b$.
   iv. Similarly $k = b'$.
   v. Thus $b = b'$.

Together, the above two items show $R(a)$ to be a singleton, being thus given by $\mathrm{Gr}(f)$ for some function $f\colon A \to B$, which gives a map

$$\left\{\begin{array}{c}\text{Adjunctions in } \mathbf{Rel}\\ \text{from } A \text{ to } B\end{array}\right\} \to \left\{\begin{array}{c}\text{Functions}\\ \text{from } A \text{ to } B\end{array}\right\}.$$

Moreover, by uniqueness of adjoints (**??** of **??**), this implies also that $S = f^{-1}$.

2. *From Functions to Adjunctions in* **Rel**. By Item 2 of Proposition 6.3.1.2, every function $f : A \to B$ gives rise to an adjunction $\mathrm{Gr}(f) \dashv f^{-1}$ in Rel, giving a map
$$\begin{Bmatrix} \text{Functions} \\ \text{from } A \text{ to } B \end{Bmatrix} \to \begin{Bmatrix} \text{Adjunctions in } \mathbf{Rel} \\ \text{from } A \text{ to } B \end{Bmatrix}.$$

3. *Invertibility: From Functions to Adjunctions Back to Functions.* We need to show that starting with a function $f : A \to B$, passing to $\mathrm{Gr}(f) \dashv f^{-1}$, and then passing again to a function gives f again. This is clear however, since we have $a \sim_{\mathrm{Gr}(f)} b$ iff $f(a) = b$.

4. *Invertibility: From Adjunctions to Functions Back to Adjunctions.* We need to show that, given an adjunction $R \dashv S$ in **Rel** giving rise to a function $f_{R,S} : A \to B$, we have
$$\begin{aligned} \mathrm{Gr}(f_{R,S}) &= R, \\ f_{R,S}^{-1} &= S. \end{aligned}$$
We check these explicitly:

   - $\mathrm{Gr}(f_{R,S}) = R$. We have
   $$\begin{aligned} \mathrm{Gr}(f_{R,S}) &\overset{\text{def}}{=} \{(a, f_{R,S}(a)) \in A \times B \mid a \in A\} \\ &\overset{\text{def}}{=} \{(a, R(a)) \in A \times B \mid a \in A\} \\ &= R. \end{aligned}$$
   - $f_{R,S}^{-1} = S$. We first claim that, given $a \in A$ and $b \in B$, the following conditions are equivalent:
     – We have $a \sim_R b$.
     – We have $b \sim_S a$.

     Indeed:
     – *If $a \sim_R b$, then $b \sim_S a$:* Since $\chi_A \subset S \diamond R$, there exists $k \in B$ such that $a \sim_R k$ and $k \sim_S a$, but since $a \sim_R b$ and R is functional, we have $k = b$ and thus $b \sim_S a$.
     – *If $b \sim_S a$, then $a \sim_R b$:* First note that since R is total we have $a \sim_R b'$ for some $b' \in B$. Now, since $R \diamond S \subset \chi_B$, $b \sim_S a$, and $a \sim_R b'$, we have $b = b'$, and thus $a \sim_R b$.

Having show this, we now have

$$\begin{aligned} f_{R,S}^{-1}(b) &\overset{\text{def}}{=} \{a \in A \mid f_{R,S}(a) = b\} \\ &\overset{\text{def}}{=} \{a \in A \mid a \sim_R b\} \\ &= \{a \in A \mid b \sim_S a\} \\ &\overset{\text{def}}{=} S(b). \end{aligned}$$

for each $b \in B$, showing $f_{R,S}^{-1} = S$.

This finishes the proof. ∎

5.3.4 Monads in **Rel**

Let A be a set.

Proposition 5.3.4.1 ▶ Monads in Rel

We have a natural identification[1]

$$\left\{\begin{matrix}\text{Monads in} \\ \textbf{Rel} \text{ on } A\end{matrix}\right\} \cong \{\text{Preorders on } A\}.$$

[1] See also ?? for an extension of this correspondence to "relative monads in **Rel**".

Proof 5.3.4.2 ▶ Proof of Proposition 5.3.4.1

A monad in **Rel** on A consists of a relation $R\colon A \nrightarrow A$ together with maps

$$\begin{aligned} \mu_R &\colon R \diamond R \subset R, \\ \eta_R &\colon \chi_A \subset R \end{aligned}$$

making the diagrams

$$\begin{tikzcd} \chi_A \diamond R \arrow[r, "\eta_R \diamond \mathrm{id}_R"] \arrow[dr, equals, "\lambda_R^{\mathbf{Rel}(A,B)}"'] & R \diamond R \arrow[d, "\mu_R"] \\ & R \end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small] & R \diamond (R \diamond R) \arrow[dr, "\mathrm{id}_R \diamond \mu_R"] & \\ (R \diamond R) \diamond R \arrow[ur, equals, "\alpha_{R,R,R}^{\mathbf{Rel}(A,B)}"] \arrow[d, "\mu_R \diamond \mathrm{id}_R"'] & & R \diamond R \arrow[d, "\mu_R"] \\ R \diamond R \arrow[rr, "\mu_R"'] & & R \end{tikzcd}
\qquad
\begin{tikzcd} R \diamond \chi_A \arrow[r, "\mathrm{id}_R \diamond \eta_R"] \arrow[dr, equals, "\rho_R^{\mathbf{Rel}(A,B)}"'] & R \diamond R \arrow[d, "\mu_R"] \\ & R \end{tikzcd}$$

commute. However, since all morphisms involved are inclusions, the commutativity of the above diagrams is automatic, and hence all that is left is the data of the two maps μ_R and η_R, which correspond respectively to the following conditions:

1. For each $a, b, c \in A$, if $a \sim_R b$ and $b \sim_R c$, then $a \sim_R c$.
2. For each $a \in A$, we have $a \sim_R a$.

These are exactly the requirements for R to be a preorder (**??**). Conversely any preorder $\preceq$ gives rise to a pair of maps $\mu_\preceq$ and $\eta_\preceq$, forming a monad on A. ∎

5.3.5 Comonads in Rel

Let A be a set.

PROPOSITION 5.3.5.1 ► COMONADS IN Rel

We have a natural identification

$$\left\{\begin{matrix}\text{Comonads in}\\ \textbf{Rel}\text{ on }A\end{matrix}\right\} \cong \{\text{Subsets of }A\}.$$

PROOF 5.3.5.2 ► PROOF OF PROPOSITION 5.3.5.1

A comonad in **Rel** on A consists of a relation $R\colon A \nrightarrow A$ together with maps

$$\Delta_R\colon R \subset R \diamond R,$$
$$\epsilon_R\colon R \subset \chi_A$$

making the diagrams

$$\begin{tikzcd}
R \arrow[r, "\Delta_R", "+" marking] \arrow[dr, equal, "\lambda^{\mathbf{Rel}(A,B),-1}_R"'] & R \diamond R \arrow[d, "\epsilon_R \diamond \mathrm{id}_R", "+" marking] \\
& \chi_A \diamond R
\end{tikzcd}
\qquad
\begin{tikzcd}
& R \diamond R \arrow[dr, "\mathrm{id}_R \diamond \Delta_R", "+" marking] & \\
R \arrow[ur, "\Delta_R", "+" marking] \arrow[d, "\Delta_R"', "+" marking] & & R \diamond (R \diamond R) \\
R \diamond R \arrow[rr, "\Delta_R \diamond \mathrm{id}_R"', "+" marking] & & (R \diamond R) \diamond R \arrow[u, equal, "\alpha^{\mathbf{Rel}(A,B),-1}_{R,R,R}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
R \arrow[r, "\Delta_R", "+" marking] \arrow[dr, equal, "\rho^{\mathbf{Rel}(A,B),-1}_R"'] & R \diamond R \arrow[d, "\mathrm{id}_R \diamond \epsilon_R", "+" marking] \\
& R \diamond \chi_A
\end{tikzcd}$$

commute. However, since all morphisms involved are inclusions, the commutativity of the above diagrams is automatic, and hence all that is left is the data of the two maps Δ_R and ϵ_R, which correspond respectively to the following conditions:

1. For each $a, b \in A$, if $a \sim_R b$, then there exists some $k \in A$ such that $a \sim_R k$ and $k \sim_R b$.
2. For each $a, b \in A$, if $a \sim_R b$, then $a = b$.

Taking $k = b$ in the first condition above shows it to be trivially satisfied, while the second condition implies $R \subset \Delta_A$, i.e. R must be a subset of A. Conversely, any subset U of A satisfies $U \subset \Delta_A$, defining a comonad as above.

5.3.6 Co/Monoids in Rel

Remark 5.3.6.1 ▶ Co/Monoids in Rel

The monoids in **Rel** with respect to the Cartesian monoidal structure of Proposition 5.2.2.10 are called *hypermonoids*, and their theory is explored in **??**. Similarly, the comonoids in **Rel** are called *hypercomonoids*, and they are defined and studied in **??**.

5.3.7 Monomorphisms in Rel

In this section we characterise the epimorphisms in the category Rel, following **??**.

Proposition 5.3.7.1 ▶ Characterisations of Monomorphisms in Rel

Let $R \colon A \nrightarrow B$ be a relation. The following conditions are equivalent:

1. The relation R is a monomorphism in Rel.
2. The direct image function
$$R_* \colon \mathcal{P}(A) \to \mathcal{P}(B)$$
associated to R is injective.
3. The direct image with compact support function
$$R_! \colon \mathcal{P}(A) \to \mathcal{P}(B)$$
associated to R is injective.

Moreover, if R is a monomorphism, then it satisfies the following condition, and the converse holds if R is total:

(★) For each $a, a' \in A$, if there exists some $b \in B$ such that

$$a \sim_R b,$$
$$a' \sim_R b,$$

then $a = a'$.

PROOF 5.3.7.2 ► PROOF OF PROPOSITION 5.3.7.1

Firstly note that Items 2 and 3 are equivalent by Item 7 of Proposition 6.4.1.3. We then claim that Items 1 and 2 are also equivalent:

- *Item 1* $\Longrightarrow$ *Item 2*: Let $U, V \in \mathcal{P}(A)$ and consider the diagram

$$\mathrm{pt} \underset{V}{\overset{U}{\rightrightarrows}} A \overset{R}{\longrightarrow} B.$$

By Remark 6.4.1.2, we have

$$R_*(U) = R \diamond U,$$
$$R_*(V) = R \diamond V.$$

Now, if $R \diamond U = R \diamond V$, i.e. $R_*(U) = R_*(V)$, then $U = V$ since R is assumed to be a monomorphism, showing R_* to be injective.

- *Item 2* $\Longrightarrow$ *Item 1*: Conversely, suppose that R_* is injective, consider the diagram

$$X \underset{T}{\overset{S}{\rightrightarrows}} A \overset{R}{\longrightarrow} B,$$

and suppose that $R \diamond S = R \diamond T$. Note that, since R_* is injective, given a diagram of the form

$$\mathrm{pt} \underset{V}{\overset{U}{\rightrightarrows}} A \overset{R}{\longrightarrow} B,$$

if $R_*(U) = R \diamond U = R \diamond V = R_*(V)$, then $U = V$. In particular, for each $x \in X$, we may consider the diagram

$$\mathrm{pt} \overset{[x]}{\longrightarrow} X \underset{T}{\overset{S}{\rightrightarrows}} A \overset{R}{\longrightarrow} B,$$

for which we have $R \diamond S \diamond [x] = R \diamond T \diamond [x]$, implying that we have

$$S(x) = S \diamond [x] = T \diamond [x] = T(x)$$

for each $x \in X$, implying $S = T$, and thus R is a monomorphism.

We can also prove this in a more abstract way, following [MSE 350788]:

- *Item 1* $\implies$ *Item 2*: Assume that R is a monomorphism.
  - We first notice that the functor $\mathrm{Rel}(\mathrm{pt}, -)\colon \mathrm{Rel} \to \mathrm{Sets}$ maps R to R_* by Remark 6.4.1.2.
  - Since $\mathrm{Rel}(\mathrm{pt}, -)$ preserves all limits by **??** of **??**, it follows by **??** of **??** that $\mathrm{Rel}(\mathrm{pt}, -)$ also preserves monomorphisms.
  - Since R is a monomorphism and $\mathrm{Rel}(\mathrm{pt}, -)$ maps R to R_*, it follows that R_* is also a monomorphism.
  - Since the monomorphisms in Sets are precisely the injections (**??** of **??**), it follows that R_* is injective.
- *Item 2* $\implies$ *Item 1*: Assume that R_* is injective.
  - We first notice that the functor $\mathrm{Rel}(\mathrm{pt}, -)\colon \mathrm{Rel} \to \mathrm{Sets}$ maps R to R_* by Remark 6.4.1.2.
  - Since the monomorphisms in Sets are precisely the injections (**??** of **??**), it follows that R_* is a monomorphism.
  - Since $\mathrm{Rel}(\mathrm{pt}, -)$ is faithful, it follows by **??** of **??** that $\mathrm{Rel}(\mathrm{pt}, -)$ reflects monomorphisms.
  - Since R_* is a monomorphism and $\mathrm{Rel}(\mathrm{pt}, -)$ maps R to R_*, it follows that R is also a monomorphism.

Finally, we prove the second part of the statement. Assume that R is a monomorphism, let $a, a' \in A$ such that $a \sim_R b$ and $a' \sim_R b$ for some $b \in B$, and consider the diagram

$$\begin{tikzcd} \mathrm{pt} \arrow[r, shift left, "{[a]}", "|" marking] \arrow[r, shift right, "{[a']}"', "|" marking] & A \arrow[r, "R", "|" marking] & B. \end{tikzcd}$$

Since $\star \sim_{[a]} a$ and $a \sim_R b$, we have $\star \sim_{R\diamond[a]} b$. Similarly, $\star \sim_{R\diamond[a']} b$. Thus $R \diamond [a] = R \diamond [a']$, and since R is a monomorphism, we have $[a] = [a']$, i.e. $a = a'$.

Conversely, assume the condition

($\star$) For each $a, a' \in A$, if there exists some $b \in B$ such that

$$a \sim_R b,$$
$$a' \sim_R b,$$

then $a = a'$.

consider the diagram

$$X \underset{T}{\overset{S}{\rightrightarrows}} A \xrightarrow{R} B,$$

and let $(x, a) \in S$. Since R is total and $a \in A$, there exists some $b \in B$ such that $a \sim_R b$. In this case, we have $x \sim_{R \diamond S} b$, and since $R \diamond S = R \diamond T$, we have also $x \sim_{R \diamond T} b$. Thus there must exist some $a' \in A$ such that $x \sim_T a'$ and $a' \sim_R b$. However, since $a, a' \sim_R b$, we must have $a = a'$, and thus $(x, a) \in T$ as well.

A similar argument shows that if $(x, a) \in T$, then $(x, a) \in S$, and thus $S = T$ and it follows that R is a monomorphism. ∎

5.3.8 2-Categorical Monomorphisms in Rel

In this section we characterise (for now, some of) the 2-categorical monomorphisms in **Rel**, following Section 9.1.

Proposition 5.3.8.1 ▶ 2-Categorical Monomorphisms in Rel

Let $R \colon A \nrightarrow B$ be a relation.

1. *Representably Faithful Morphisms in* **Rel**. Every morphism of **Rel** is a representably faithful morphism.
2. *Representably Full Morphisms in* **Rel**. The following conditions are equivalent:
   (a) The morphism $R \colon A \nrightarrow B$ is a representably full morphism.
   (b) For each pair of relations $S, T \colon X \rightrightarrows A$, the following condition is satisfied:
   ($\star$) If $R \diamond S \subset R \diamond T$, then $S \subset T$.
   (c) The functor

$$R_* \colon (\mathcal{P}(A), \subset) \to (\mathcal{P}(B), \subset)$$

is full.

(d) For each $U, V \in \mathcal{P}(A)$, if $R_*(U) \subset R_*(V)$, then $U \subset V$.

(e) The functor

$$R_!\colon (\mathcal{P}(A), \subset) \to (\mathcal{P}(B), \subset)$$

is full.

(f) For each $U, V \in \mathcal{P}(A)$, if $R_!(U) \subset R_!(V)$, then $U \subset V$.

3. *Representably Fully Faithful Morphisms in* **Rel**. Every representaly full morphism in **Rel** is a representably fully faithful morphism.

Proof 5.3.8.2 ► Proof of Proposition 5.3.8.1

Item 1: Representably Faithful Morphisms in Rel

The relation R is a representably faithful morphism in **Rel** iff, for each $X \in \mathrm{Obj}(\mathbf{Rel})$, the functor

$$R_*\colon \mathbf{Rel}(X, A) \to \mathbf{Rel}(X, B)$$

is faithful, i.e. iff the morphism

$$R_{*|S,T}\colon \mathrm{Hom}_{\mathbf{Rel}(X,A)}(S, T) \to \mathrm{Hom}_{\mathbf{Rel}(X,B)}(R \diamond S, R \diamond T)$$

is injective for each $S, T \in \mathrm{Obj}(\mathbf{Rel}(X, A))$. However, $\mathrm{Hom}_{\mathbf{Rel}(X,A)}(S, T)$ is either empty or a singleton, in either case of which the map $R_{*|S,T}$ is necessarily injective.

Item 2: Representably Full Morphisms in Rel

We claim Items 2a to 2f are indeed equivalent:

- *Item 2a* $\iff$ *Item 2b*: This is simply a matter of unwinding definitions: The relation R is a representably full morphism in **Rel** iff, for each $X \in \mathrm{Obj}(\mathbf{Rel})$, the functor

  $$R_*\colon \mathbf{Rel}(X, A) \to \mathbf{Rel}(X, B)$$

  is full, i.e. iff the morphism

  $$R_{*|S,T}\colon \mathrm{Hom}_{\mathbf{Rel}(X,A)}(S, T) \to \mathrm{Hom}_{\mathbf{Rel}(X,B)}(R \diamond S, R \diamond T)$$

  is surjective for each $S, T \in \mathrm{Obj}(\mathbf{Rel}(X, A))$, i.e. iff, whenever $R \diamond S \subset R \diamond T$, we also have $S \subset T$.

- *Item 2c* $\iff$ *Item 2d*: This is also simply a matter of unwinding definitions: The functor

$$R_* : (\mathcal{P}(A), \subset) \to (\mathcal{P}(B), \subset)$$

is full iff, for each $U, V \in \mathcal{P}(A)$, the morphism

$$R_{*|U,V} : \mathrm{Hom}_{\mathcal{P}(A)}(U, V) \to \mathrm{Hom}_{\mathcal{P}(B)}(R_*(U), R_*(V))$$

is surjective, i.e. iff whenever $R_*(U) \subset R_*(V)$, we also necessarily have $U \subset V$.

- *Item 2e* $\iff$ *Item 2f*: This is once again simply a matter of unwinding definitions, and proceeds exactly in the same way as in the proof of the equivalence between Items 2c and 2d given above.

- *Item 2d* $\implies$ *Item 2f*: Suppose that the following condition is true:

  ($\star$) For each $U, V \in \mathcal{P}(A)$, if $R_*(U) \subset R_*(V)$, then $U \subset V$.

  We need to show that the condition

  ($\star$) For each $U, V \in \mathcal{P}(A)$, if $R_!(U) \subset R_!(V)$, then $U \subset V$.

  is also true. We proceed step by step:

  1. Suppose we have $U, V \in \mathcal{P}(A)$ with $R_!(U) \subset R_!(V)$.
  2. By Item 7 of Proposition 6.4.4.4, we have

  $$\begin{aligned} R_!(U) &= B \setminus R_*(A \setminus U), \\ R_!(V) &= B \setminus R_*(A \setminus V). \end{aligned}$$

  3. By Item 1 of Proposition 2.3.10.2 we have $R_*(A \setminus V) \subset R_*(A \setminus U)$.
  4. By assumption, we then have $A \setminus V \subset A \setminus U$.
  5. By Item 1 of Proposition 2.3.10.2 again, we have $U \subset V$.

- *Item 2f* $\implies$ *Item 2d*: Suppose that the following condition is true:

  ($\star$) For each $U, V \in \mathcal{P}(A)$, if $R_!(U) \subset R_!(V)$, then $U \subset V$.

  We need to show that the condition

  ($\star$) For each $U, V \in \mathcal{P}(A)$, if $R_*(U) \subset R_*(V)$, then $U \subset V$.

is also true. We proceed step by step:

1. Suppose we have $U, V \in \mathcal{P}(A)$ with $R_*(U) \subset R_*(V)$.
2. By Item 7 of Proposition 6.4.1.3, we have
$$\begin{aligned} R_*(U) &= B \setminus R_!(A \setminus U), \\ R_*(V) &= B \setminus R_!(A \setminus V). \end{aligned}$$
3. By Item 1 of Proposition 2.3.10.2 we have $R_!(A \setminus V) \subset R_!(A \setminus U)$.
4. By assumption, we then have $A \setminus V \subset A \setminus U$.
5. By Item 1 of Proposition 2.3.10.2 again, we have $U \subset V$.

- *Item 2b* $\implies$ *Item 2d*: Consider the diagram

$$\begin{tikzcd} X \arrow[r, shift left, "S", "|"{marking}] \arrow[r, shift right, "T"', "|"{marking}] & A \arrow[r, "R", "|"{marking}] & B, \end{tikzcd}$$

and suppose that $R \diamond S \subset R \diamond T$. Note that, by assumption, given a diagram of the form

$$\begin{tikzcd} \mathrm{pt} \arrow[r, shift left, "U", "|"{marking}] \arrow[r, shift right, "V"', "|"{marking}] & A \arrow[r, "R", "|"{marking}] & B, \end{tikzcd}$$

if $R_*(U) = R \diamond U \subset R \diamond V = R_*(V)$, then $U \subset V$. In particular, for each $x \in X$, we may consider the diagram

$$\begin{tikzcd} \mathrm{pt} \arrow[r, "{[x]}", "|"{marking}] & X \arrow[r, shift left, "S", "|"{marking}] \arrow[r, shift right, "T"', "|"{marking}] & A \arrow[r, "R", "|"{marking}] & B, \end{tikzcd}$$

for which we have $R \diamond S \diamond [x] \subset R \diamond T \diamond [x]$, implying that we have

$$S(x) = S \diamond [x] \subset T \diamond [x] = T(x)$$

for each $x \in X$, implying $S \subset T$.

- *Item 2d* $\implies$ *Item 2b*: Let $U, V \in \mathcal{P}(A)$ and consider the diagram

$$\begin{tikzcd} \mathrm{pt} \arrow[r, shift left, "U", "|"{marking}] \arrow[r, shift right, "V"', "|"{marking}] & A \arrow[r, "R", "|"{marking}] & B. \end{tikzcd}$$

By Remark 6.4.1.2, we have

$$\begin{aligned} R_*(U) &= R \diamond U, \\ R_*(V) &= R \diamond V. \end{aligned}$$

Now, if $R_*(U) \subset R_*(V)$, i.e. $R \diamond U \subset R \diamond V$, then $U \subset V$ by assumption.

??: Fully Faithful Monomorphisms in Rel

This follows from Items 1 and 2.

Question 5.3.8.3 ▶ Better Characterisations of Representably Full Morphisms in Rel

Item 2 of Proposition 5.3.8.1 gives a characterisation of the representably full morphisms in **Rel**.
Are there other nice characterisations of these?
This question also appears as [MO 467527].

5.3.9 Epimorphisms in Rel

In this section we characterise the epimorphisms in the category Rel, following **??**.

Proposition 5.3.9.1 ▶ Characterisations of Epimorphisms in Rel

Let $R\colon A \nrightarrow B$ be a relation. The following conditions are equivalent:

1. The relation R is an epimorphism in Rel.
2. The weak inverse image function

$$R^{-1}\colon \mathcal{P}(B) \to \mathcal{P}(A)$$

   associated to R is injective.
3. The strong inverse image function

$$R_{-1}\colon \mathcal{P}(B) \to \mathcal{P}(A)$$

   associated to R is injective.
4. The function $R\colon A \to \mathcal{P}(B)$ is "surjective on singletons":

   ($\star$) For each $b \in B$, there exists some $a \in A$ such that $R(a) = \{b\}$.

Moreover, if R is total and an epimorphism, then it satisfies the following equivalent conditions:

1. For each $b \in B$, there exists some $a \in A$ such that $a \sim_R b$.
2. We have $\mathrm{Im}(R) = B$.

Proof 5.3.9.2 ▶ Proof of Proposition 5.3.9.1

Firstly note that Items 2 and 3 are equivalent by Item 7 of Proposition 6.4.2.4. We then claim that Items 1 and 2 are also equivalent:

- *Item 1* $\implies$ *Item 2*: Let $U, V \in \mathcal{P}(A)$ and consider the diagram

$$A \overset{R}{\longrightarrow\!\!\!\!\!\!\!+\;\;} B \underset{V}{\overset{U}{\rightrightarrows}} \mathrm{pt}.$$

  By Remark 6.4.1.2, we have

$$\begin{aligned} R^{-1}(U) &= U \diamond R, \\ R^{-1}(V) &= V \diamond R. \end{aligned}$$

  Now, if $U \diamond R = V \diamond R$, i.e. $R^{-1}(U) = R^{-1}(V)$, then $U = V$ since R is assumed to be an epimorphism, showing R^{-1} to be injective.

- *Item 2* $\implies$ *Item 1*: Conversely, suppose that R^{-1} is injective, consider the diagram

$$A \overset{R}{\longrightarrow\!\!\!\!\!\!\!+\;\;} B \underset{T}{\overset{S}{\rightrightarrows}} X,$$

  and suppose that $S \diamond R = T \diamond R$. Note that, since R^{-1} is injective, given a diagram of the form

$$A \overset{R}{\longrightarrow\!\!\!\!\!\!\!+\;\;} B \underset{V}{\overset{U}{\rightrightarrows}} \mathrm{pt},$$

  if $R^{-1}(U) = U \diamond R = V \diamond R = R^{-1}(V)$, then $U = V$. In particular, for each $x \in X$, we may consider the diagram

$$A \overset{R}{\longrightarrow\!\!\!\!\!\!\!+\;\;} B \underset{T}{\overset{S}{\rightrightarrows}} X \overset{[x]}{\longrightarrow\!\!\!\!\!\!\!+\;\;} \mathrm{pt},$$

  for which we have $[x] \diamond S \diamond R = [x] \diamond T \diamond R$, implying that we have

$$S^{-1}(x) = [x] \diamond S = [x] \diamond T = T^{-1}(x)$$

  for each $x \in X$, implying $S = T$, and thus R is an epimorphism.

We can also prove this in a more abstract way, following [MSE 350788]:

- *Item 1* $\implies$ *Item 2*: Assume that R is an epimorphism.
  - We first notice that the functor $\mathrm{Rel}(-,\mathrm{pt})\colon \mathrm{Rel}^{\mathrm{op}} \to \mathsf{Sets}$ maps R to R^{-1} by Remark 6.4.3.2.
  - Since $\mathrm{Rel}(-,\mathrm{pt})$ preserves limits by **??** of **??**, it follows by **??** of **??** that $\mathrm{Rel}(-,\mathrm{pt})$ also preserves monomorphisms.
  - That is: $\mathrm{Rel}(-,\mathrm{pt})$ sends monomorphisms in $\mathrm{Rel}^{\mathrm{op}}$ to monomorphisms in Sets.
  - The monomorphisms $\mathrm{Rel}^{\mathrm{op}}$ are precisely the epimorphisms in Rel by **??** of **??**.
  - Since R is an epimorphism and $\mathrm{Rel}(-,\mathrm{pt})$ maps R to R^{-1}, it follows that R^{-1} is a monomorphism.
  - Since the monomorphisms in Sets are precisely the injections (**??** of **??**), it follows that R^{-1} is injective.
- *Item 2* $\implies$ *Item 1*: Assume that R^{-1} is injective.
  - We first notice that the functor $\mathrm{Rel}(-,\mathrm{pt})\colon \mathrm{Rel}^{\mathrm{op}} \to \mathsf{Sets}$ maps R to R^{-1} by Remark 6.4.3.2.
  - Since the monomorphisms in Sets are precisely the injections (**??** of **??**), it follows that R^{-1} is a monomorphism.
  - Since $\mathrm{Rel}(-,\mathrm{pt})$ is faithful, it follows by **??** of **??** that $\mathrm{Rel}(,\mathrm{pt})$ reflects monomorphisms.
  - That is: $\mathrm{Rel}(-,\mathrm{pt})$ reflects monomorphisms in Sets to monomorphisms in $\mathrm{Rel}^{\mathrm{op}}$.
  - The monomorphisms $\mathrm{Rel}^{\mathrm{op}}$ are precisely the epimorphisms in Rel by **??** of **??**.
  - Since R^{-1} is a monomorphism and $\mathrm{Rel}(-,\mathrm{pt})$ maps R to R^{-1}, it follows that R is an epimorphism.

We also claim that Items 2 and 4 are equivalent, following [MO 350788]:

- *Item 2* $\implies$ *Item 4*: Since $B \setminus \{b\} \subset B$ and R^{-1} is injective, we have $R^{-1}(B \setminus \{b\}) \subsetneq R^{-1}(B)$. So taking some $a \in R^{-1}(B) \setminus R^{-1}(B \setminus \{b\})$ we get an element of A such that $R(a) = \{b\}$.
- *Item 4* $\implies$ *Item 2*: Let $U, V \subset B$ with $U \neq V$. Without loss of generality, we can assume $U \setminus V \neq \emptyset$; otherwise just swap U and V. Let then $b \in U \setminus V$. By assumption, there exists an $a \in A$ with $R(a) = \{b\}$. Then $a \in R^{-1}(U)$ but $a \notin R^{-1}(V)$, and thus $R^{-1}(U) \neq R^{-1}(V)$, showing R^{-1} to be injective.

Finally, we prove the second part of the statement. So assume R is a total epimorphism in Rel and consider the diagram

$$A \overset{R}{\rightarrow\!\!\!\!\!\!+} B \underset{T}{\overset{S}{\rightrightarrows}} \{0,1\},$$

where $b \sim_S 0$ for each $b \in B$ and where we have

$$b \sim_T \begin{cases} 0 & \text{if } b \in \mathrm{Im}(R), \\ 1 & \text{otherwise} \end{cases}$$

for each $b \in B$. Since R is total, we have $a \sim_{S \diamond R} 0$ and $a \sim_{T \diamond R} 0$ for all $a \in A$, and no element of A is related to 1 by $S \diamond R$ or $T \diamond R$. Thus $S \diamond R = T \diamond R$, and since R is an epimorphism, we have $S = T$. But by the definition of T, this implies $\mathrm{Im}(R) = B$. ∎

5.3.10 2-Categorical Epimorphisms in Rel

In this section we characterise (for now, some of) the 2-categorical epimorphisms in **Rel**, following Section 9.2.

Proposition 5.3.10.1 ▶ 2-Categorical Epimorphisms in Rel

Let $R \colon A \rightarrow\!\!\!\!\!\!+ B$ be a relation.

1. *Corepresentably Faithful Morphisms in* **Rel**. Every morphism of **Rel** is a corepresentably faithful morphism.
2. *Corepresentably Full Morphisms in* **Rel**. The following conditions are equivalent:
   (a) The morphism $R \colon A \rightarrow\!\!\!\!\!\!+ B$ is a corepresentably full morphism.
   (b) For each pair of relations $S, T \colon X \rightrightarrows A$, the following condition is satisfied:
      (★) If $S \diamond R \subset T \diamond R$, then $S \subset T$.
   (c) The functor
   $$R^{-1} \colon (\mathcal{P}(B), \subset) \to (\mathcal{P}(A), \subset)$$
   is full.
   (d) For each $U, V \in \mathcal{P}(B)$, if $R^{-1}(U) \subset R^{-1}(V)$, then $U \subset V$.

(e) The functor

$$R_{-1}\colon (\mathcal{P}(B), \subset) \to (\mathcal{P}(A), \subset)$$

is full.

(f) For each $U, V \in \mathcal{P}(B)$, if $R_{-1}(U) \subset R_{-1}(V)$, then $U \subset V$.

3. *Corepresentably Fully Faithful Morphisms in* **Rel**. Every corepresentably full morphism of **Rel** is a corepresentably fully faithful morphism.

Proof 5.3.10.2 ▶ Proof of Proposition 5.3.10.1

Item 1: Corepresentably Faithful Morphisms in Rel

The relation R is a corepresentably faithful morphism in **Rel** iff, for each $X \in \mathrm{Obj}(\mathbf{Rel})$, the functor

$$R^*\colon \mathbf{Rel}(B, X) \to \mathbf{Rel}(A, X)$$

is faithful, i.e. iff the morphism

$$R^*_{S,T}\colon \mathrm{Hom}_{\mathbf{Rel}(B,X)}(S, T) \to \mathrm{Hom}_{\mathbf{Rel}(A,X)}(S \diamond R, T \diamond R)$$

is injective for each $S, T \in \mathrm{Obj}(\mathbf{Rel}(B, X))$. However, $\mathrm{Hom}_{\mathbf{Rel}(B,X)}(S, T)$ is either empty or a singleton, in either case of which the map $R^*_{S,T}$ is necessarily injective.

Item 2: Corepresentably Full Morphisms in Rel

We claim Items 2a to 2f are indeed equivalent:

- *Item 2a* $\iff$ *Item 2b*: This is simply a matter of unwinding definitions: The relation R is a corepresentably full morphism in **Rel** iff, for each $X \in \mathrm{Obj}(\mathbf{Rel})$, the functor

  $$R^*\colon \mathbf{Rel}(B, X) \to \mathbf{Rel}(A, X)$$

  is full, i.e. iff the morphism

  $$R^*_{S,T}\colon \mathrm{Hom}_{\mathbf{Rel}(B,X)}(S, T) \to \mathrm{Hom}_{\mathbf{Rel}(A,X)}(S \diamond R, T \diamond R)$$

  is surjective for each $S, T \in \mathrm{Obj}(\mathbf{Rel}(B, X))$, i.e. iff, whenever $S \diamond R \subset T \diamond R$, we also have $S \subset T$.

- *Item 2c* $\iff$ *Item 2d*: This is also simply a matter of unwinding definitions: The functor

$$R^{-1}\colon (\mathcal{P}(B), \subset) \to (\mathcal{P}(A), \subset)$$

is full iff, for each $U, V \in \mathcal{P}(A)$, the morphism

$$R^{-1}_{U,V}\colon \mathrm{Hom}_{\mathcal{P}(B)}(U, V) \to \mathrm{Hom}_{\mathcal{P}(A)}\big(R^{-1}(U), R^{-1}(V)\big)$$

is surjective, i.e. iff whenever $R^{-1}(U) \subset R^{-1}(V)$, we also necessarily have $U \subset V$.

- *Item 2e* $\iff$ *Item 2f*: This is once again simply a matter of unwinding definitions, and proceeds exactly in the same way as in the proof of the equivalence between Items 2c and 2d given above.

- *Item 2d* $\implies$ *Item 2f*: Suppose that the following condition is true:

  ($\star$) For each $U, V \in \mathcal{P}(B)$, if $R^{-1}(U) \subset R^{-1}(V)$, then $U \subset V$.

  We need to show that the condition

  ($\star$) For each $U, V \in \mathcal{P}(B)$, if $R_{-1}(U) \subset R_{-1}(V)$, then $U \subset V$.

  is also true. We proceed step by step:

  1. Suppose we have $U, V \in \mathcal{P}(B)$ with $R_{-1}(U) \subset R_{-1}(V)$.
  2. By Item 7 of Proposition 6.4.2.4, we have
  $$\begin{aligned} R_{-1}(U) &= B \setminus R^{-1}(A \setminus U), \\ R_{-1}(V) &= B \setminus R^{-1}(A \setminus V). \end{aligned}$$
  3. By Item 1 of Proposition 2.3.10.2 we have $R^{-1}(A \setminus V) \subset R^{-1}(A \setminus U)$.
  4. By assumption, we then have $A \setminus V \subset A \setminus U$.
  5. By Item 1 of Proposition 2.3.10.2 again, we have $U \subset V$.

- *Item 2f* $\implies$ *Item 2d*: Suppose that the following condition is true:

  ($\star$) For each $U, V \in \mathcal{P}(B)$, if $R_{-1}(U) \subset R_{-1}(V)$, then $U \subset V$.

  We need to show that the condition

  ($\star$) For each $U, V \in \mathcal{P}(B)$, if $R^{-1}(U) \subset R^{-1}(V)$, then $U \subset V$.

is also true. We proceed step by step:

1. Suppose we have $U, V \in \mathcal{P}(B)$ with $R^{-1}(U) \subset R^{-1}(V)$.
2. By Item 7 of Proposition 6.4.3.4, we have
$$\begin{aligned} R^{-1}(U) &= B \setminus R_{-1}(A \setminus U), \\ R^{-1}(V) &= B \setminus R_{-1}(A \setminus V). \end{aligned}$$
3. By Item 1 of Proposition 2.3.10.2 we have $R_{-1}(A \setminus V) \subset R_{-1}(A \setminus U)$.
4. By assumption, we then have $A \setminus V \subset A \setminus U$.
5. By Item 1 of Proposition 2.3.10.2 again, we have $U \subset V$.

- *Item 2b* $\implies$ *Item 2d*: Consider the diagram
$$\begin{tikzcd} A \arrow[r, "R", "|" marking] & B \arrow[r, shift left, "S", "|" marking] \arrow[r, shift right, "T"', "|" marking] & X, \end{tikzcd}$$
and suppose that $S \diamond R \subset T \diamond R$. Note that, by assumption, given a diagram of the form
$$\begin{tikzcd} A \arrow[r, "R", "|" marking] & B \arrow[r, shift left, "U", "|" marking] \arrow[r, shift right, "V"', "|" marking] & \mathrm{pt}, \end{tikzcd}$$
if $R^{-1}(U) = R \diamond U \subset R \diamond V = R^{-1}(V)$, then $U \subset V$. In particular, for each $x \in X$, we may consider the diagram
$$\begin{tikzcd} A \arrow[r, "R", "|" marking] & B \arrow[r, shift left, "S", "|" marking] \arrow[r, shift right, "T"', "|" marking] & X \arrow[r, "{[x]}", "|" marking] & \mathrm{pt}, \end{tikzcd}$$
for which we have $[x] \diamond S \diamond R \subset [x] \diamond T \diamond R$, implying that we have
$$S^{-1}(x) = [x] \diamond S \subset [x] \diamond T = T^{-1}(x)$$
for each $x \in X$, implying $S \subset T$.
- *Item 2d* $\implies$ *Item 2b*: Let $U, V \in \mathcal{P}(B)$ and consider the diagram
$$\begin{tikzcd} A \arrow[r, "R", "|" marking] & B \arrow[r, shift left, "U", "|" marking] \arrow[r, shift right, "V"', "|" marking] & \mathrm{pt}. \end{tikzcd}$$

By Remark 6.4.1.2, we have

$$R^{-1}(U) = U \diamond R,$$
$$R^{-1}(V) = V \diamond R.$$

Now, if $R^{-1}(U) \subset R^{-1}(V)$, i.e. $U \diamond R \subset V \diamond R$, then $U \subset V$ by assumption.

Item 3: Corepresentably Fully Faithful Morphisms in **Rel**

This follows from Items 1 and 2.

Question 5.3.10.3 ▶ Better Characterisations of Corepresentably Full Morphisms in Rel

Item 2 of Proposition 5.3.10.1 gives a characterisation of the corepresentably full morphisms in **Rel**.
Are there other nice characterisations of these?
This question also appears as [MO 467527].

5.3.11 Co/Limits in Rel

Proposition 5.3.11.1 ▶ Co/Limits in Rel

This will be properly written later on.

Proof 5.3.11.2 ▶ Proof of Proposition 5.3.11.1

Omitted.

5.3.12 Kan Extensions and Kan Lifts in Rel

Remark 5.3.12.1 ▶ Kan Extensions and Kan Lifts in Rel

The 2-category **Rel** admits all right Kan extensions and right Kan lifts, though not all left Kan extensions and neither does it admit all left Kan lifts. See Section 6.2 for a detailed discussion of this.

5.3.13 Closedness of Rel

Proposition 5.3.13.1 ▶ Closedness of Rel

The 2-category **Rel** is a closed bicategory, there being, for each $R\colon A \nrightarrow B$ and set X, a pair of adjunctions

$$(R^* \dashv \mathrm{Ran}_R)\colon \quad \begin{tikzcd} \mathrm{Rel}(B, X) \arrow[r, bend left, "R^*"] \arrow[r, phantom, "\perp"] & \mathrm{Rel}(A, X), \arrow[l, bend left, "\mathrm{Ran}_R"] \end{tikzcd}$$

$$(R_* \dashv \mathrm{Rift}_R)\colon \quad \begin{tikzcd} \mathrm{Rel}(X, A) \arrow[r, bend left, "R_*"] \arrow[r, phantom, "\perp"] & \mathrm{Rel}(X, B), \arrow[l, bend left, "\mathrm{Rift}_R"] \end{tikzcd}$$

witnessed by bijections

$$\begin{aligned} \mathbf{Rel}(S \diamond R, T) &\cong \mathbf{Rel}(S, \mathrm{Ran}_R(T)), \\ \mathbf{Rel}(R \diamond U, V) &\cong \mathbf{Rel}(U, \mathrm{Rift}_R(V)), \end{aligned}$$

natural in $S \in \mathrm{Rel}(B, X)$, $T \in \mathrm{Rel}(A, X)$, $U \in \mathrm{Rel}(X, A)$, and $V \in \mathrm{Rel}(X, B)$.

Proof 5.3.13.2 ▶ Proof of Proposition 5.3.13.1

This follows from Propositions 6.2.3.1 and 6.2.4.1.

5.3.14 Rel as a Category of Free Algebras

Proposition 5.3.14.1 ▶ Rel as a Category of Free Algebras

We have an isomorphism of categories

$$\mathrm{Rel} \cong \mathrm{FreeAlg}_{\mathcal{P}_*}(\mathrm{Sets}),$$

where $\mathcal{P}_*$ is the powerset monad of **??**.

Proof 5.3.14.2 ▶ Proof of Proposition 5.3.14.1

Omitted.

5.4 The Left Skew Monoidal Structure on Rel(A, B)

5.4.1 The Left Skew Monoidal Product

Let A and B be sets and let $J\colon A \nrightarrow B$ be a relation.

Definition 5.4.1.1 ▶ The Left J-Skew Monoidal Product of Rel(A, B)

The **left J-skew monoidal product of Rel(A, B)** is the functor

$$\lhd_J \colon \mathbf{Rel}(A, B) \times \mathbf{Rel}(A, B) \to \mathbf{Rel}(A, B)$$

where

- *Action on Objects.* For each $R, S \in \mathrm{Obj}(\mathbf{Rel}(A, B))$, we have

$$S \lhd_J R \overset{\text{def}}{=} S \diamond \mathrm{Rift}_J(R), \qquad \begin{tikzcd} & A \arrow[r, "S", "\shortmid"{marking}] \arrow[d, "J", "\shortmid"{marking}] & B. \\ A \arrow[ur, dashed, "\mathrm{Rift}_J(R)", "\shortmid"{marking}] \arrow[r, "R"', "\shortmid"{marking}] & B \arrow[ul, phantom, "\Leftarrow"] & \end{tikzcd}$$

- *Action on Morphisms.* For each $R, S, R', S' \in \mathrm{Obj}(\mathbf{Rel}(A, B))$, the action on Hom-sets

$$(\lhd_J)_{(G,F),(G',F')} \colon \mathrm{Hom}_{\mathbf{Rel}(A,B)}(S, S') \times \mathrm{Hom}_{\mathbf{Rel}(A,B)}(R, R') \to \mathrm{Hom}_{\mathbf{Rel}(A,B)}(S \lhd_J R, S' \lhd_J R')$$

  of $\lhd_J$ at $((R, S), (R', S'))$ is defined by[1]

$$\beta \lhd_J \alpha \overset{\text{def}}{=} \beta \diamond \mathrm{Rift}_J(\alpha), \qquad \begin{tikzcd} & A \arrow[r, bend left, "S", "\shortmid"{marking}] \arrow[r, bend right, "S'"', "\shortmid"{marking}] \arrow[r, phantom, "\beta\Downarrow"] \arrow[d, "J", "\shortmid"{marking}] & B \\ A \arrow[ur, dashed, bend left, "\mathrm{Rift}_J(R)", "\shortmid"{marking}] \arrow[ur, dashed, "\mathrm{Rift}_J(R')"', "\shortmid"{marking}] \arrow[r, bend left, "R", "\shortmid"{marking}] \arrow[r, bend right, "R'"', "\shortmid"{marking}] \arrow[r, phantom, "\alpha\Downarrow"] & B \arrow[ul, phantom, "\Leftarrow"] & \end{tikzcd}$$

  for each $\beta \in \mathrm{Hom}_{\mathbf{Rel}(A,B)}(S, S')$ and each $\alpha \in \mathrm{Hom}_{\mathbf{Rel}(A,B)}(R, R')$.

[1] Since **Rel**(A, B) is posetal, this is to say that if $S \subset S'$ and $R \subset R'$, then $S \lhd_J R \subset S' \lhd_J R'$.

5.4.2 The Left Skew Monoidal Unit

Let A and B be sets and let $J \colon A \nrightarrow B$ be a relation.

Definition 5.4.2.1 ► The Left J-Skew Monoidal Unit of $\mathrm{Rel}(A, B)$

The **left J-skew monoidal unit of $\mathbf{Rel}(A, B)$** is the functor

$$\mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleleft_J} \colon \mathrm{pt} \longrightarrow \mathbf{Rel}(A, B)$$

picking the object

$$\mathbb{1}^{\triangleleft_J}_{\mathbf{Rel}(A,B)} \overset{\text{def}}{=} J$$

of $\mathbf{Rel}(A, B)$.

5.4.3 The Left Skew Associators

Let A and B be sets and let $J \colon A \nrightarrow B$ be a relation.

Definition 5.4.3.1 ► The Left J-Skew Associator of $\mathrm{Rel}(A, B)$

The **left J-skew associator of $\mathbf{Rel}(A, B)$** is the natural transformation

$$\alpha^{\mathbf{Rel}(A,B),\triangleleft_J} \colon \triangleleft_J \circ (\triangleleft_J \times \mathrm{id}) \Longrightarrow \triangleleft_J \circ (\mathrm{id} \times \triangleleft_J) \circ \boldsymbol{\alpha}^{\mathsf{Cats}}_{\mathbf{Rel}(A,B),\mathbf{Rel}(A,B),\mathbf{Rel}(A,B)},$$

as in the diagram

$$\begin{tikzcd}
 & \mathbf{Rel}(A, B) \times (\mathbf{Rel}(A, B) \times \mathbf{Rel}(A, B)) \arrow[dr, "\mathrm{id}\times\triangleleft_J"] & \\
(\mathbf{Rel}(A, B) \times \mathbf{Rel}(A, B)) \times \mathbf{Rel}(A, B) \arrow[ur, dashed, "\boldsymbol{\alpha}^{\mathsf{Cats}}_{\mathbf{Rel}(A,B),\mathbf{Rel}(A,B),\mathbf{Rel}(A,B)}", "\sim"'] \arrow[dr, "\triangleleft_J\times\mathrm{id}"'] & & \mathbf{Rel}(A, B) \times \mathbf{Rel}(A, B) \arrow[d, "\triangleleft_J"] \\
 & \mathbf{Rel}(A, B) \times \mathbf{Rel}(A, B) \arrow[r, "\triangleleft_J"'] \arrow[ur, phantom, "\Nearrow \alpha^{\mathbf{Rel}(A,B),\triangleleft_J}"] & \mathbf{Rel}(A, B),
\end{tikzcd}$$

whose component

$$\alpha^{\mathbf{Rel}(A,B),\triangleleft_J}_{T,S,R} \colon \underbrace{(T \triangleleft_J S) \triangleleft_J R}_{\overset{\text{def}}{=} T \diamond \mathrm{Rift}_J(S) \diamond \mathrm{Rift}_J(R)} \hookrightarrow \underbrace{T \triangleleft_J (S \triangleleft_J R)}_{\overset{\text{def}}{=} T \diamond \mathrm{Rift}_J(S \diamond \mathrm{Rift}_J(R))}$$

at (T, S, R) is given by

$$\alpha^{\mathbf{Rel}(A,B),\triangleleft_J}_{T,S,R} \overset{\text{def}}{=} \mathrm{id}_T \diamond \gamma,$$

where

$$\gamma\colon \mathrm{Rift}_J(S) \diamond \mathrm{Rift}_J(R) \hookrightarrow \mathrm{Rift}_J(S \diamond \mathrm{Rift}_J(R))$$

is the inclusion adjunct to the inclusion

$$\epsilon_S \star \mathrm{id}_{\mathrm{Rift}_J(R)}\colon \underbrace{J \diamond \mathrm{Rift}_J(S) \diamond \mathrm{Rift}_J(R)}_{\overset{\text{def}}{=} J_*(\mathrm{Rift}_J(S)\diamond \mathrm{Rift}_J(R))} \hookrightarrow S \diamond \mathrm{Rift}_J(R)$$

under the adjunction $J_* \dashv \mathrm{Rift}_J$, where $\epsilon\colon J \diamond \mathrm{Rift}_J \Longrightarrow \mathrm{id}_{\mathbf{Rel}(A,B)}$ is the counit of the adjunction $J_* \dashv \mathrm{Rift}_J$.

5.4.4 The Left Skew Left Unitors

Let A and B be sets and let $J\colon A \nrightarrow B$ be a relation.

Definition 5.4.4.1 ▶ The Left J-Skew Left Unitor of Rel(A, B)

The **left J-skew left unitor of Rel(A, B)** is the natural transformation

$$\lambda^{\mathbf{Rel}(A,B),\triangleleft_J}\colon \triangleleft_J \circ \left(\mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleleft_J} \times \mathrm{id}\right) \Longrightarrow \lambda^{\mathsf{Cats}_2}_{\mathbf{Rel}(A,B)}$$

as in the diagram

$$\begin{tikzcd}
\mathrm{pt} \times \mathbf{Rel}(A, B) \arrow[r, "\mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleleft_J} \times \mathrm{id}"] \arrow[dr, dashed, bend right, "\lambda^{\mathsf{Cats}_2}_{\mathbf{Rel}(A,B)}"'] & \mathbf{Rel}(A, B) \times \mathbf{Rel}(A, B) \arrow[d, "\triangleleft_J"] \arrow[dl, phantom, "\Swarrow \lambda^{\mathbf{Rel}(A,B),\triangleleft_J}"] \\
 & \mathbf{Rel}(A, B),
\end{tikzcd}$$

whose component

$$\lambda^{\mathbf{Rel}(A,B),\triangleleft_J}_R\colon \underbrace{J \triangleleft_J R}_{\overset{\text{def}}{=} J \diamond \mathrm{Rift}_J(R)} \hookrightarrow R$$

at R is given by

$$\lambda^{\mathbf{Rel}(A,B),\triangleleft_J}_R \overset{\text{def}}{=} \epsilon_R,$$

where $\epsilon\colon J_* \diamond \mathrm{Rift}_J \Longrightarrow \mathrm{id}_{\mathbf{Rel}(A,B)}$ is the counit of the adjunction $J_* \dashv \mathrm{Rift}_J$.

5.4.5 The Left Skew Right Unitors

Let A and B be sets and let $J\colon A \nrightarrow B$ be a relation.

Definition 5.4.5.1 ► The Left J-Skew Right Unitor of Rel(A, B)

The **left J-skew right unitor of Rel(A, B)** is the natural transformation

$$\rho^{\mathbf{Rel}(A,B),\lhd_J}\colon \rho^{\mathsf{Cats}_2}_{\mathbf{Rel}(A,B)} \Longrightarrow \lhd_J \circ \left(\mathrm{id} \times \mathbb{1}^{\mathbf{Rel}(A,B)}_{\lhd_J}\right)$$

as in the diagram

$$\begin{tikzcd} \mathbf{Rel}(A,B) \times \mathrm{pt} \arrow[r, "\mathrm{id}\times\mathbb{1}^{\mathbf{Rel}(A,B)}_{\lhd_J}"] \arrow[dr, dashed, bend right, "\rho^{\mathsf{Cats}_2}_{\mathbf{Rel}(A,B)}"'] & \mathbf{Rel}(A,B) \times \mathbf{Rel}(A,B), \arrow[d, "\lhd_J"] \\ & \mathbf{Rel}(A,B) \arrow[ul, phantom, "\Nearrow \rho^{\mathbf{Rel}(A,B),\lhd_J}"] \end{tikzcd}$$

whose component

$$\rho^{\mathbf{Rel}(A,B),\lhd_J}_R\colon R \hookrightarrow \underbrace{R \lhd_J J}_{\overset{\text{def}}{=} R\diamond \mathrm{Rift}_J(J)}$$

at R is given by the composition

$$\begin{aligned} R &\overset{\sim}{\Longrightarrow} R \diamond \chi_A \\ &\overset{\mathrm{id}_R \diamond \eta_{\chi_A}}{\Longrightarrow} R \diamond \mathrm{Rift}_J(J_*(\chi_A)) \\ &\overset{\text{def}}{=} R \diamond \mathrm{Rift}_J(J \diamond \chi_A) \\ &\overset{\sim}{\Longrightarrow} R \diamond \mathrm{Rift}_J(J) \\ &\overset{\text{def}}{=} R \lhd_J J, \end{aligned}$$

where $\eta\colon \mathrm{id}_{\mathbf{Rel}(A,A)} \Longrightarrow \mathrm{Rift}_J \circ J_*$ is the unit of the adjunction $J_* \dashv \mathrm{Rift}_J$.

5.4.6 The Left Skew Monoidal Structure on **Rel**(A, B)

Proposition 5.4.6.1 ► The Left J-Skew Monoidal Structure on $\mathbf{Rel}(A, B)$

The category $\mathbf{Rel}(A, B)$ admits a left skew monoidal category structure consisting of

- *The Underlying Category.* The posetal category associated to the poset $\mathbf{Rel}(A, B)$ of relations from A to B of Item 2 of Definition 5.1.1.3.
- *The Left Skew Monoidal Product.* The left J-skew monoidal product
$$\triangleleft_J \colon \mathbf{Rel}(A, B) \times \mathbf{Rel}(A, B) \longrightarrow \mathbf{Rel}(A, B)$$
of Definition 5.4.1.1.
- *The Left Skew Monoidal Unit.* The functor
$$\mathbb{1}^{\mathbf{Rel}(A,B),\triangleleft_J} \colon \mathrm{pt} \longrightarrow \mathbf{Rel}(A, B)$$
of Definition 5.4.2.1.
- *The Left Skew Associators.* The natural transformation
$$\alpha^{\mathbf{Rel}(A,B),\triangleleft_J} \colon \triangleleft_J \circ (\triangleleft_J \times \mathrm{id}) \Longrightarrow \triangleleft_J \circ (\mathrm{id} \times \triangleleft_J) \circ \boldsymbol{\alpha}^{\mathsf{Cats}}_{\mathbf{Rel}(A,B),\mathbf{Rel}(A,B),\mathbf{Rel}(A,B)}$$
of Definition 5.4.3.1.
- *The Left Skew Left Unitors.* The natural transformation
$$\lambda^{\mathbf{Rel}(A,B),\triangleleft_J} \colon \triangleleft_J \circ \left(\mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleleft_J} \times \mathrm{id}\right) \Longrightarrow \boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathbf{Rel}(A,B)}$$
of Definition 5.4.4.1.
- *The Left Skew Right Unitors.* The natural transformation
$$\rho^{\mathbf{Rel}(A,B),\triangleleft_J} \colon \boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathbf{Rel}(A,B)} \Longrightarrow \triangleleft_J \circ \left(\mathrm{id} \times \mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleleft_J}\right)$$
of Definition 5.4.5.1.

Proof 5.4.6.2 ► Proof of Proposition 5.4.6.1

Since $\mathbf{Rel}(A, B)$ is posetal, the commutativity of the pentagon identity, the left skew left triangle identity, the left skew right triangle identity, the left skew middle triangle identity, and the zigzag identity is automatic, and thus $\mathbf{Rel}(A, B)$ together with the data in the statement forms a left skew monoidal category.

5.5 The Right Skew Monoidal Structure on **Rel**(A, B)

Let A and B be sets and let $J : A \nrightarrow B$ be a relation.

5.5.1 The Right Skew Monoidal Product

Definition 5.5.1.1 ► The Right J-Skew Monoidal Product of Rel(A, B)

The **right J-skew monoidal product of Rel**(A, B) is the functor

$$\triangleright_J : \mathbf{Rel}(A, B) \times \mathbf{Rel}(A, B) \to \mathbf{Rel}(A, B)$$

where

- *Action on Objects.* For each $R, S \in \mathrm{Obj}(\mathbf{Rel}(A, B))$, we have

$$S \triangleright_J R \overset{\mathrm{def}}{=} \mathrm{Ran}_J(S) \diamond R, \qquad \begin{tikzcd} A \arrow[r, "R", "|"] & B \arrow[r, dashed, "\mathrm{Ran}_J(S)", "|"] & B. \\ & A \arrow[u, "J", "|"'] \arrow[ur, "S"', "|"] \arrow[ur, phantom, "\Nwarrow"] & \end{tikzcd}$$

- *Action on Morphisms.* For each $R, S, R', S' \in \mathrm{Obj}(\mathbf{Rel}(A, B))$, the action on Hom-sets

$$(\triangleright_J)_{(S,R),(S',R')} : \mathrm{Hom}_{\mathbf{Rel}(A,B)}(S, S') \times \mathrm{Hom}_{\mathbf{Rel}(A,B)}(R, R') \to \mathrm{Hom}_{\mathbf{Rel}(A,B)}(S \triangleright_J R, S' \triangleright_J R')$$

of $\triangleright_J$ at $((S, R), (S', R'))$ is defined by[1]

$$\beta \triangleright_J \alpha \overset{\mathrm{def}}{=} \mathrm{Ran}_J(\beta) \diamond \alpha, \qquad \begin{tikzcd} A \arrow[r, bend left, "R'", "|"'] \arrow[r, bend right, "R"', "|"] \arrow[r, phantom, "\alpha\Uparrow"] & B \arrow[r, dashed, bend left, "\mathrm{Ran}_J(S')", "|"'] \arrow[r, dashed, "\mathrm{Ran}_J(S)"', "|"] \arrow[r, phantom, shift left=3, "\mathrm{Ran}_J(\beta)\Uparrow"] & B \\ & A \arrow[u, "J", "|"'] \arrow[ur, "S'", "|"'] \arrow[ur, bend right, "S"', "|"] \arrow[ur, phantom, "\Nwarrow \beta"] & \end{tikzcd}$$

for each $\beta \in \mathrm{Hom}_{\mathbf{Rel}(A,B)}(S, S')$ and each $\alpha \in \mathrm{Hom}_{\mathbf{Rel}(A,B)}(R, R')$.

[1] Since **Rel**(A, B) is posetal, this is to say that if $S \subset S'$ and $R \subset R'$, then $S \triangleright_J R \subset S' \triangleright_J R'$.

5.5.2 The Right Skew Monoidal Unit

Definition 5.5.2.1 ▶ The Right J-Skew Monoidal Unit of $\mathrm{Rel}(A,B)$

The **right J-skew monoidal unit of $\mathbf{Rel}(A,B)$** is the functor

$$\mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleright_J}\colon \mathrm{pt} \longrightarrow \mathbf{Rel}(A,B)$$

picking the object

$$\mathbb{1}^{\triangleright_J}_{\mathbf{Rel}(A,B)} \stackrel{\text{def}}{=} J$$

of $\mathbf{Rel}(A,B)$.

5.5.3 The Right Skew Associators

Definition 5.5.3.1 ▶ The Right J-Skew Associator of $\mathrm{Rel}(A,B)$

The **right J-skew associator of $\mathbf{Rel}(A,B)$** is the natural transformation

$$\alpha^{\mathbf{Rel}(A,B),\triangleright_J}\colon \triangleright_J \circ (\mathrm{id}\times\triangleright_J) \Longrightarrow \triangleright_J \circ (\triangleright_J\times\mathrm{id}) \circ \boldsymbol{\alpha}^{\mathsf{Cats},-1}_{\mathbf{Rel}(A,B),\mathbf{Rel}(A,B),\mathbf{Rel}(A,B)},$$

as in the diagram

$$\begin{tikzcd}
 & (\mathbf{Rel}(A,B)\times\mathbf{Rel}(A,B))\times\mathbf{Rel}(A,B) \arrow[dr, "\triangleright_J\times\mathrm{id}"] & \\
\mathbf{Rel}(A,B)\times(\mathbf{Rel}(A,B)\times\mathbf{Rel}(A,B)) \arrow[ur, dashed, "\boldsymbol{\alpha}^{\mathsf{Cats},-1}_{\mathbf{Rel}(A,B),\mathbf{Rel}(A,B),\mathbf{Rel}(A,B)}", "\sim"'] \arrow[dr, "\mathrm{id}\times\triangleright_J"'] & \arrow[phantom, "\alpha^{\mathbf{Rel}(A,B),\triangleright_J}"] & \mathbf{Rel}(A,B)\times\mathbf{Rel}(A,B) \arrow[dl, "\triangleright_J"] \\
 & \mathbf{Rel}(A,B)\times\mathbf{Rel}(A,B) \xrightarrow[\triangleright_J]{} \mathbf{Rel}(A,B), &
\end{tikzcd}$$

whose component

$$\alpha^{\mathbf{Rel}(A,B),\triangleright_J}_{T,S,R}\colon \underbrace{T\triangleright_J(S\triangleright_J R)}_{\stackrel{\text{def}}{=}\mathrm{Ran}_J(T)\diamond\mathrm{Ran}_J(S)\diamond R} \hookrightarrow \underbrace{(T\triangleright_J S)\triangleright_J R}_{\stackrel{\text{def}}{=}\mathrm{Ran}_J(\mathrm{Ran}_J(T)\diamond S)\diamond R}$$

at (T,S,R) is given by

$$\alpha^{\mathbf{Rel}(A,B),\triangleright}_{T,S,R} \stackrel{\text{def}}{=} \gamma\diamond\mathrm{id}_R,$$

where

$$\gamma\colon \mathrm{Ran}_J(T)\diamond\mathrm{Ran}_J(S) \hookrightarrow \mathrm{Ran}_J(\mathrm{Ran}_J(T)\diamond S)$$

is the inclusion adjunct to the inclusion

$$\mathrm{id}_{\mathrm{Ran}_J(T)} \diamond \epsilon_S \colon \underbrace{\mathrm{Ran}_J(T) \diamond \mathrm{Ran}_J(S) \diamond J}_{\overset{\text{def}}{=} J^*(\mathrm{Ran}_J(T) \diamond \mathrm{Ran}_J(S))} \hookrightarrow \mathrm{Ran}_J(T) \diamond S$$

under the adjunction $J^* \dashv \mathrm{Ran}_J$, where $\epsilon \colon \mathrm{Ran}_J \diamond J \Longrightarrow \mathrm{id}_{\mathbf{Rel}(A,B)}$ is the counit of the adjunction $J^* \dashv \mathrm{Ran}_J$.

5.5.4 The Right Skew Left Unitors

Definition 5.5.4.1 ► The Right J-Skew Left Unitor of Rel(A, B)

The **right J-skew left unitor of Rel**(A, B) is the natural transformation

$$\lambda^{\mathbf{Rel}(A,B),\triangleright_J} \colon \lambda^{\mathsf{Cats}_2}_{\mathbf{Rel}(A,B)} \Longrightarrow \triangleright_J \circ \left(\mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleright} \times \mathrm{id}\right),$$

as in the diagram

$$\begin{tikzcd}
\mathrm{pt} \times \mathbf{Rel}(A,B) \arrow[r, "\mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleright_J} \times \mathrm{id}"] \arrow[dr, dashed, bend right, "\lambda^{\mathsf{Cats}_2}_{\mathbf{Rel}(A,B)}"'] \arrow[r, phantom, shift right=6ex, "\lambda^{\mathbf{Rel}(A,B),\triangleright_J}\ \Nearrow"] & \mathbf{Rel}(A,B) \times \mathbf{Rel}(A,B) \arrow[d, "\triangleright_J"] \\
 & \mathbf{Rel}(A,B),
\end{tikzcd}$$

whose component

$$\lambda^{\mathbf{Rel}(A,B),\triangleright_J}_R \colon R \hookrightarrow \underbrace{J \triangleright_J R}_{\overset{\text{def}}{=} \mathrm{Ran}_J(J) \diamond R}$$

at R is given by the composition

$$\begin{aligned}
R &\overset{\sim}{\Longrightarrow} \chi_B \diamond R \\
&\xRightarrow{\eta_{\chi_B}} \diamond \mathrm{id}_R \mathrm{Ran}_J(J^*(\chi_A)) \diamond R \\
&\overset{\text{def}}{=} \mathrm{Ran}_J(J^* \diamond \chi_A) \diamond R \\
&\overset{\sim}{\Longrightarrow} \mathrm{Ran}_J(J) \diamond R \\
&\overset{\text{def}}{=} R \triangleright_J J,
\end{aligned}$$

where $\eta \colon \mathrm{id}_{\mathbf{Rel}(B,B)} \Longrightarrow \mathrm{Ran}_J \circ J^*$ is the unit of the adjunction $J^* \dashv \mathrm{Ran}_J$.

5.5.5 The Right Skew Right Unitors

Definition 5.5.5.1 ▶ The Right J-Skew Right Unitor of Rel(A, B)

The **right J-skew right unitor of Rel(A, B)** is the natural transformation

$$\rho^{\mathbf{Rel}(A,B),\triangleright_J} \colon \triangleright_J \circ \left(\mathrm{id} \times \mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleright}\right) \Longrightarrow \rho^{\mathsf{Cats}_2}_{\mathbf{Rel}(A,B)},$$

as in the diagram

$$\begin{tikzcd}
\mathbf{Rel}(A,B) \times \mathrm{pt} \arrow[r, "\mathrm{id}\times\mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleright_J}"] \arrow[dr, dashed, bend right, "\rho^{\mathsf{Cats}_2}_{\mathbf{Rel}(A,B)}"'] & \mathbf{Rel}(A,B) \times \mathbf{Rel}(A,B), \arrow[d, "\triangleright_J"] \arrow[dl, phantom, "\Swarrow \rho^{\mathbf{Rel}(A,B),\triangleright_J}"] \\
& \mathbf{Rel}(A,B)
\end{tikzcd}$$

whose component

$$\rho^{\mathbf{Rel}(A,B),\triangleright_J}_S \colon \underbrace{S \triangleright_J J}_{\overset{\text{def}}{=} \mathrm{Ran}_J(S)\diamond J} \hookrightarrow S$$

at S is given by

$$\rho^{\mathbf{Rel}(A,B),\triangleright_J}_S \overset{\text{def}}{=} \epsilon_R,$$

where $\epsilon \colon J^* \circ \mathrm{Ran}_J \Longrightarrow \mathrm{id}_{\mathbf{Rel}(A,B)}$ is the counit of the adjunction $J^* \dashv \mathrm{Ran}_J$.

5.5.6 The Right Skew Monoidal Structure on **Rel**(A, B)

Proposition 5.5.6.1 ▶ The Right J-Skew Monoidal Structure on Rel(A, B)

The category **Rel**(A, B) admits a right skew monoidal category structure consisting of

- *The Underlying Category.* The posetal category associated to the poset **Rel**(A, B) of relations from A to B of Item 2 of Definition 5.1.1.3.
- *The Right Skew Monoidal Product.* The right J-skew monoidal product

$$\triangleleft_J \colon \mathbf{Rel}(A,B) \times \mathbf{Rel}(A,B) \to \mathbf{Rel}(A,B)$$

  of Definition 5.5.1.1.

- *The Right Skew Monoidal Unit.* The functor

$$\mathbb{1}^{\mathbf{Rel}(A,B),\lhd_J}\colon \mathrm{pt} \to \mathbf{Rel}(A,B)$$

of Definition 5.5.2.1.

- *The Right Skew Associators.* The natural transformation

$$\alpha^{\mathbf{Rel}(A,B),\rhd_J}\colon \rhd_J \circ (\mathrm{id} \times \rhd_J) \Longrightarrow \rhd_J \circ (\rhd_J \times \mathrm{id}) \circ \boldsymbol{\alpha}^{\mathsf{Cats},-1}_{\mathbf{Rel}(A,B),\mathbf{Rel}(A,B),\mathbf{Rel}(A,B)}$$

of Definition 5.5.3.1.

- *The Right Skew Left Unitors.* The natural transformation

$$\lambda^{\mathbf{Rel}(A,B),\rhd_J}\colon \boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathbf{Rel}(A,B)} \Longrightarrow \rhd_J \circ \left(\mathbb{1}^{\mathbf{Rel}(A,B)}_{\rhd} \times \mathrm{id}\right)$$

of Definition 5.5.4.1.

- *The Right Skew Right Unitors.* The natural transformation

$$\rho^{\mathbf{Rel}(A,B),\rhd_J}\colon \rhd_J \circ \left(\mathrm{id} \times \mathbb{1}^{\mathbf{Rel}(A,B)}_{\rhd}\right) \Longrightarrow \boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathbf{Rel}(A,B)}$$

of Definition 5.5.5.1.

Proof 5.5.6.2 ▶ Proof of Proposition 5.5.6.1

Since $\mathbf{Rel}(A,B)$ is posetal, the commutativity of the pentagon identity, the right skew left triangle identity, the right skew right triangle identity, the right skew middle triangle identity, and the zigzag identity is automatic, and thus $\mathbf{Rel}(A,B)$ together with the data in the statement forms a right skew monoidal category. □

Appendices

5.A Other Chapters

Sets

1. Sets
2. Constructions With Sets
3. Pointed Sets
4. Tensor Products of Pointed Sets

Relations

5. Relations
6. Constructions With Relations
7. Equivalence Relations and Apartness Relations

Category Theory

8. Categories

Bicategories

9. Types of Morphisms in Bicategories

Chapter 6

Constructions With Relations

This chapter contains some material about constructions with relations. Notably, we discuss and explore:

1. The existence or non-existence of Kan extensions and Kan lifts in the 2-category **Rel** (Section 6.2).

2. The various kinds of constructions involving relations, such as graphs, domains, ranges, unions, intersections, products, inverse relations, composition of relations, and collages (Section 6.3).

3. The adjoint pairs

$$\begin{aligned} R_* \dashv R_{-1} &: \mathcal{P}(A) \rightleftarrows \mathcal{P}(B), \\ R^{-1} \dashv R_! &: \mathcal{P}(B) \rightleftarrows \mathcal{P}(A) \end{aligned}$$

   of functors (morphisms of posets) between $\mathcal{P}(A)$ and $\mathcal{P}(B)$ induced by a relation $R\colon A \nrightarrow B$, as well as the properties of R_*, R_{-1}, R^{-1}, and $R_!$ (Section 6.4).

   Of particular note are the following points:

   (a) These two pairs of adjoint functors are the counterpart for relations of the adjoint triple $f_* \dashv f^{-1} \dashv f_!$ induced by a function $f\colon A \to B$ studied in Section 2.4.

   (b) We have $R_{-1} = R^{-1}$ iff R is total and functional (Item 8 of Proposition 6.4.2.4).

   (c) As a consequence of the previous item, when R comes from a function f, the pair of adjunctions

   $$R_* \dashv R_{-1} = R^{-1} \dashv R_!$$

   reduces to the triple adjunction

   $$f_* \dashv f^{-1} \dashv f_!$$

from Section 2.4.

(d) The pairs $R_* \dashv R_{-1}$ and $R^{-1} \dashv R_!$ turn out to be rather important later on, as they appear in the definition and study of continuous, open, and closed relations between topological spaces (**??**).

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6.A Other Chapters **389**

6.1 Co/Limits in the Category of Relations

This section is currently just a stub, and will be properly developed later on.

6.2 Kan Extensions and Kan Lifts in the 2-Category of Relations

6.2.1 Left Kan Extensions in Rel

Proposition 6.2.1.1 ▶ Left Kan Extensions in Rel

Let $R\colon A \nrightarrow B$ be a relation.

1. *Non-Existence of All Left Kan Extensions in* **Rel**. Not all relations in **Rel** admit left Kan extensions.

2. *Characterisation of Relations Admitting Left Kan Extensions Along Them*. The following conditions are equivalent:

   (a) The left Kan extension

   $$\mathrm{Lan}_R\colon \mathbf{Rel}(A, X) \to \mathbf{Rel}(B, X)$$

   along R exists.

   (b) The relation R admits a left adjoint in **Rel**.

   (c) The relation R is of the form f^{-1} (as in Definition 6.3.2.1) for some function f.

Proof 6.2.1.2 ▶ Proof of Proposition 6.2.1.1

Item 1: Non-Existence of All Left Kan Extensions in **Rel**

Omitted, but will eventually follow Fosco Loregian's comment on [MO 460656].

Item 2: Characterisation of Relations Admitting Left Kan Extensions Along T

Omitted, but will eventually follow Tim Campion's answer to to [MO 460656].

Question 6.2.1.3 ▶ Existence of Specific Left Kan Extensions of Relations

Given relations $S\colon A \nrightarrow X$ and $R\colon A \nrightarrow B$, is there a characterisation of when the left Kan extension

$$\mathrm{Lan}_S(R)\colon B \nrightarrow X$$

exists in terms of properties of R and S?

This question also appears as [MO 461592].

Question 6.2.1.4 ► Explicit Description of Left Kan Extensions Along Functions

As shown in Item 2 of Proposition 6.2.1.1, the left Kan extension

$$\mathrm{Lan}_R\colon \mathbf{Rel}(A,X) \longrightarrow \mathbf{Rel}(B,X)$$

along a relation of the form $R = f^{-1}$ exists. Is there a explicit description of it, similarly to the explicit description of right Kan extensions given in Proposition 6.2.3.1?

This question also appears as [MO 461592].

6.2.2 Left Kan Lifts in Rel

Proposition 6.2.2.1 ► Left Kan Lifts in Rel

Let $R\colon A \nrightarrow B$ be a relation.

1. *Non-Existence of All Left Kan Lifts in* **Rel**. Not all relations in **Rel** admit left Kan lifts.
2. *Characterisation of Relations Admitting Left Kan Lifts Along Them*. The following conditions are equivalent:
   (a) The left Kan lift

   $$\mathrm{Lift}_R\colon \mathbf{Rel}(X,B) \longrightarrow \mathbf{Rel}(X,A)$$

   along R exists.

   (b) The relation R admits a right adjoint in **Rel**.

   (c) The relation R is of the form $\mathrm{Gr}(f)$ (as in Definition 6.3.1.1) for some function f.

Proof 6.2.2.2 ► Proof of Proposition 6.2.2.1

Item 1: Non-Existence of All Left Kan Lifts in Rel

Omitted, but will eventually follow (the dual of) Fosco Loregian's comment on [MO 460656].

Item 2: Characterisation of Relations Admitting Left Kan Lifts Along Them

Omitted, but will eventually follow Tim Campion's answer to to [MO 460656].

QUESTION 6.2.2.3 ► EXISTENCE OF SPECIFIC LEFT KAN LIFTS OF RELATIONS

Given relations $S\colon A \nrightarrow X$ and $R\colon A \nrightarrow B$, is there a characterisation of when the left Kan lift

$$\mathrm{Lift}_S(R)\colon X \nrightarrow A$$

exists in terms of properties of R and S?

This question also appears as [MO 461592].

QUESTION 6.2.2.4 ► EXPLICIT DESCRIPTION OF LEFT KAN LIFTS ALONG FUNCTIONS

As shown in Item 2 of Proposition 6.2.2.1, the left Kan lift

$$\mathrm{Lift}_R\colon \mathbf{Rel}(X, B) \longrightarrow \mathbf{Rel}(X, A)$$

along a relation of the form $R = \mathrm{Gr}(f)$ exists. Is there a explicit description of it, similarly to the explicit description of right Kan lifts given in Proposition 6.2.4.1?

This question also appears as [MO 461592].

6.2.3 Right Kan Extensions in Rel

Let $R\colon A \nrightarrow B$ be a relation.

PROPOSITION 6.2.3.1 ► EXISTENCE OF RIGHT KAN EXTENSIONS IN Rel

The right Kan extension

$$\mathrm{Ran}_R\colon \mathrm{Rel}(A, X) \longrightarrow \mathrm{Rel}(B, X)$$

along R in **Rel** exists and is given by

$$\mathrm{Ran}_R(S) \stackrel{\text{def}}{=} \int_{a\in A} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^{-2}_a, S^{-1}_a\right)$$

for each $S \in \mathrm{Rel}(A, X)$, so that the following conditions are equivalent:

1. We have $b \sim_{\mathrm{Ran}_R(S)} x$.
2. For each $a \in A$, if $a \sim_R b$, then $a \sim_S x$.

Proof 6.2.3.2 ► Proof of Proposition 6.2.3.1

We have

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{Rel}(A,X)}(S \diamond R, T) &\cong \int_{a\in A}\int_{x\in X} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left((S\diamond R)^x_a, T^x_a\right) \\
&\cong \int_{a\in A}\int_{x\in X} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(\left(\int^{b\in B} S^x_b \times R^b_a\right), T^x_a\right) \\
&\cong \int_{a\in A}\int_{x\in X}\int_{b\in B} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(S^x_b \times R^b_a, T^x_a\right) \\
&\cong \int_{a\in A}\int_{x\in X}\int_{b\in B} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(S^x_b, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^b_a, T^x_a\right)\right) \\
&\cong \int_{b\in B}\int_{x\in X}\int_{a\in A} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(S^x_b, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^b_a, T^x_a\right)\right) \\
&\cong \int_{b\in B}\int_{x\in X} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(S^x_b, \int_{a\in A}\mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^b_a, T^x_a\right)\right) \\
&\cong \mathrm{Hom}_{\mathbf{Rel}(B,X)}\left(S, \int_{a\in A}\mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^{-_2}_a, T^{-_1}_a\right)\right)
\end{aligned}$$

naturally in each $S \in \mathbf{Rel}(B, X)$ and each $T \in \mathbf{Rel}(A, X)$, showing that

$$\int_{a\in A}\mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^{-_2}_a, T^{-_1}_a\right)$$

is right adjoint to the precomposition functor $- \diamond R$, being thus the right Kan extension along R. Here we have used the following results, respectively (i.e. for each $\cong$ sign):

1. Item 1 of Proposition 5.1.1.6.
2. Definition 6.3.12.1.
3. **??** of **??**.
4. Proposition 1.2.2.5.
5. **??** of **??**.
6. **??** of **??**.
7. Item 1 of Proposition 5.1.1.6.

This finishes the proof. □

6.2.4 Right Kan Lifts in Rel

Let $R \colon A \nrightarrow B$ be a relation.

Proposition 6.2.4.1 ▶ Existence of Right Kan Lifts in Rel

The right Kan lift

$$\mathrm{Rift}_R \colon \mathrm{Rel}(X,B) \longrightarrow \mathrm{Rel}(X,A)$$

along R in **Rel** exists and is given by

$$\mathrm{Rift}_R(S) \overset{\text{def}}{=} \int_{b\in B} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^b_{-_1}, S^b_{-_2}\right)$$

for each $S \in \mathrm{Rel}(X,B)$, so that the following conditions are equivalent:

1. We have $x \sim_{\mathrm{Rift}_R(S)} a$.
2. For each $b \in B$, if $a \sim_R b$, then $x \sim_S b$.

Proof 6.2.4.2 ▶ Proof of Proposition 6.2.4.1

We have

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{Rel}(X,B)}(R \diamond S, T) &\cong \int_{x\in X}\int_{b\in B} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left((R\diamond S)^b_x, T^b_x\right)\\
&\cong \int_{x\in X}\int_{b\in B} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(\left(\int^{a\in A} R^b_a \times S^a_x\right), T^b_x\right)\\
&\cong \int_{x\in X}\int_{b\in B}\int_{a\in A} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^b_a \times S^a_x, T^b_x\right)\\
&\cong \int_{x\in X}\int_{b\in B}\int_{a\in A} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(S^a_x, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^b_a, T^b_x\right)\right)\\
&\cong \int_{x\in X}\int_{a\in A}\int_{b\in B} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(S^a_x, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^b_a, T^b_x\right)\right)\\
&\cong \int_{x\in X}\int_{a\in A} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(S^a_x, \int_{b\in B}\mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^b_a, T^b_x\right)\right)\\
&\cong \mathrm{Hom}_{\mathbf{Rel}(X,A)}\left(S, \int_{b\in B}\mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^b_{-_1}, T^b_{-_2}\right)\right)
\end{aligned}$$

naturally in each $S \in \mathbf{Rel}(X,A)$ and each $T \in \mathbf{Rel}(X,B)$, showing that

$$\int_{b\in B} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^b_{-_1}, S^b_{-_2}\right)$$

is right adjoint to the postcomposition functor $R \diamond -$, being thus the right Kan lift along R. Here we have used the following results, respectively (i.e. for each $\cong$ sign):

1. Item 1 of Proposition 5.1.1.6.

2. Definition 6.3.12.1.
3. ?? of ??.
4. Proposition 1.2.2.5.
5. ?? of ??.
6. ?? of ??.
7. Item 1 of Proposition 5.1.1.6.

This finishes the proof. ∎

6.3 More Constructions With Relations

6.3.1 The Graph of a Function

Let $f\colon A\to B$ be a function.

Definition 6.3.1.1 ► The Graph of a Function

The **graph of** f is the relation $\mathrm{Gr}(f)\colon A \nrightarrow B$ defined as follows:[1]

- Viewing relations from A to B as subsets of $A\times B$, we define
$$\mathrm{Gr}(f) \overset{\text{def}}{=} \{(a, f(a)) \in A\times B \mid a\in A\}.$$
- Viewing relations from A to B as functions $A\times B\to\{\text{true},\text{false}\}$, we define
$$[\mathrm{Gr}(f)](a,b) \overset{\text{def}}{=} \begin{cases}\text{true} & \text{if } b=f(a),\\ \text{false} & \text{otherwise}\end{cases}$$
for each $(a,b)\in A\times B$.
- Viewing relations from A to B as functions $A\to\mathcal{P}(B)$, we define
$$[\mathrm{Gr}(f)](a) \overset{\text{def}}{=} \{f(a)\}$$
for each $a\in A$, i.e. we define $\mathrm{Gr}(f)$ as the composition
$$A\xrightarrow{f} B\xrightarrow{\chi_B}\mathcal{P}(B).$$

[1] *Further Notation:* We write $\mathrm{Gr}(A)$ for $\mathrm{Gr}(\mathrm{id}_A)$, and call it the **graph** of A.

PROPOSITION 6.3.1.2 ► PROPERTIES OF GRAPHS OF FUNCTIONS

Let $f\colon A \to B$ be a function.

1. *Functoriality.* The assignment $A \mapsto \mathrm{Gr}(A)$ defines a functor

$$\mathrm{Gr}\colon \mathsf{Sets} \to \mathrm{Rel}$$

where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$\mathrm{Gr}(A) \stackrel{\text{def}}{=} A.$$

- *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathsf{Sets})$, the action on Hom-sets

$$\mathrm{Gr}_{A,B}\colon \mathsf{Sets}(A, B) \to \underbrace{\mathrm{Rel}(\mathrm{Gr}(A), \mathrm{Gr}(B))}_{\stackrel{\text{def}}{=}\mathrm{Rel}(A,B)}$$

of Gr at (A, B) is defined by

$$\mathrm{Gr}_{A,B}(f) \stackrel{\text{def}}{=} \mathrm{Gr}(f),$$

where $\mathrm{Gr}(f)$ is the graph of f as in Definition 6.3.1.1.

In particular:

- *Preservation of Identities.* We have

$$\mathrm{Gr}(\mathrm{id}_A) = \chi_A$$

for each $A \in \mathrm{Obj}(\mathsf{Sets})$.

- *Preservation of Composition.* We have

$$\mathrm{Gr}(g \circ f) = \mathrm{Gr}(g) \diamond \mathrm{Gr}(f)$$

for each pair of functions $f\colon A \to B$ and $g\colon B \to C$.

2. *Adjointness Inside* **Rel**. We have an adjunction

$$(\mathrm{Gr}(f) \dashv f^{-1})\colon\quad \begin{tikzcd} A \arrow[r, bend left, "\mathrm{Gr}(f)"] \arrow[r, phantom, "\perp"] & B \arrow[l, bend left, "f^{-1}"] \end{tikzcd}$$

in **Rel**, where f^{-1} is the inverse of f of Definition 6.3.2.1.

3. *Adjointness.* We have an adjunction

$$(\mathrm{Gr} \dashv \mathcal{P}_*)\colon \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "\mathrm{Gr}"] \arrow[r, phantom, "\perp"] & \mathsf{Rel}, \arrow[l, bend left, "\mathcal{P}_*"] \end{tikzcd}$$

witnessed by a bijection of sets

$$\mathsf{Rel}(\mathrm{Gr}(A), B) \cong \mathsf{Sets}(A, \mathcal{P}(B))$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$ and $B \in \mathrm{Obj}(\mathsf{Rel})$.

4. *Interaction With Inverses.* We have

$$\begin{aligned} \mathrm{Gr}(f)^\dagger &= f^{-1}, \\ \left(f^{-1}\right)^\dagger &= \mathrm{Gr}(f). \end{aligned}$$

5. *Cocontinuity.* The functor $\mathrm{Gr}\colon \mathsf{Sets} \to \mathsf{Rel}$ of Item 1 preserves colimits.

6. *Characterisations.* Let $R\colon A \nrightarrow B$ be a relation. The following conditions are equivalent:

   (a) There exists a function $f\colon A \to B$ such that $R = \mathrm{Gr}(f)$.

   (b) The relation R is total and functional.

   (c) The weak and strong inverse images of R agree, i.e. we have $R^{-1} = R_{-1}$.

   (d) The relation R has a right adjoint $R^\dagger$ in Rel.

PROOF 6.3.1.3 ► PROOF OF PROPOSITION 6.3.1.2

Item 1: Functoriality

Clear.

Item 2: Adjointness Inside Rel

We need to check that there are inclusions

$$\begin{aligned} \chi_A &\subset f^{-1} \diamond \mathrm{Gr}(f), \\ \mathrm{Gr}(f) \diamond f^{-1} &\subset \chi_B. \end{aligned}$$

These correspond respectively to the following conditions:

1. For each $a \in A$, there exists some $b \in B$ such that $a \sim_{\mathrm{Gr}(f)} b$ and

$b \sim_{f^{-1}} a$.

2. For each $a, b \in A$, if $a \sim_{\mathrm{Gr}(f)} b$ and $b \sim_{f^{-1}} a$, then $a = b$.

In other words, the first condition states that the image of any $a \in A$ by f is nonempty, whereas the second condition states that f is not multivalued. As f is a function, both of these statements are true, and we are done.

Item 3: Adjointness

The stated bijection follows from Remark 5.1.1.4, with naturality being clear.

Item 4: Interaction With Inverses

Clear.

Item 5: Cocontinuity

Omitted.

Item 6: Characterisations

We claim that Items 6a to 6d are indeed equivalent:

- *Item 6a* $\iff$ *Item 6b*. This is shown in the proof of **??** of **??**.
- *Item 6b* $\implies$ *Item 6c*. If R is total and functional, then, for each $a \in A$, the set $R(a)$ is a singleton, implying that

$$\begin{aligned} R^{-1}(V) &\overset{\text{def}}{=} \{a \in A \mid R(a) \cap V \neq \emptyset\}, \\ R_{-1}(V) &\overset{\text{def}}{=} \{a \in A \mid R(a) \subset V\} \end{aligned}$$

  are equal for all $V \in \mathcal{P}(B)$, as the conditions $R(a) \cap V \neq \emptyset$ and $R(a) \subset V$ are equivalent when $R(a)$ is a singleton.
- *Item 6c* $\implies$ *Item 6b*. We claim that R is indeed total and functional:
  - *Totality*. If we had $R(a) = \emptyset$ for some $a \in A$, then we would have $a \in R_{-1}(\emptyset)$, so that $R_{-1}(\emptyset) \neq \emptyset$. But since $R^{-1}(\emptyset) = \emptyset$, this would imply $R_{-1}(\emptyset) \neq R^{-1}(\emptyset)$, a contradiction. Thus $R(a) \neq \emptyset$ for all $a \in A$ and R is total.
  - *Functionality*. If $R^{-1} = R_{-1}$, then we have

$$\begin{aligned} \{a\} &= R^{-1}(\{b\}) \\ &= R_{-1}(\{b\}) \end{aligned}$$

    for each $b \in R(a)$ and each $a \in A$, and thus $R(a) \subset \{b\}$. But since R is total, we must have $R(a) = \{b\}$, and thus we see that R is functional.

- *Item 6a* $\iff$ *Item 6d*. This follows from Proposition 5.3.3.1.

This finishes the proof. ∎

6.3.2 The Inverse of a Function

Let $f\colon A \to B$ be a function.

Definition 6.3.2.1 ► The Inverse of a Function

The **inverse of** f is the relation $f^{-1}\colon B \nrightarrow A$ defined as follows:

- Viewing relations from B to A as subsets of $B \times A$, we define

$$f^{-1} \overset{\text{def}}{=} \left\{\left(b, f^{-1}(b)\right) \in B \times A \,\middle|\, a \in A\right\},$$

  where

$$f^{-1}(b) \overset{\text{def}}{=} \{a \in A \mid f(a) = b\}$$

  for each $b \in B$.

- Viewing relations from B to A as functions $B \times A \to \{\text{true}, \text{false}\}$, we define

$$f^{-1}(b, a) \overset{\text{def}}{=} \begin{cases} \text{true} & \text{if there exists } a \in A \text{ with } f(a) = b, \\ \text{false} & \text{otherwise} \end{cases}$$

  for each $(b, a) \in B \times A$.

- Viewing relations from B to A as functions $B \to \mathcal{P}(A)$, we define

$$f^{-1}(b) \overset{\text{def}}{=} \{a \in A \mid f(a) = b\}$$

  for each $b \in B$.

Proposition 6.3.2.2 ► Properties of Inverses of Functions

Let $f\colon A \to B$ be a function.

1. *Functoriality*. The assignment $A \mapsto A, f \mapsto f^{-1}$ defines a functor

$$(-)^{-1}\colon \mathsf{Sets} \to \mathsf{Rel}$$

   where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$\left[(-)^{-1}\right](A) \overset{\text{def}}{=} A.$$

- *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathsf{Sets})$, the action on Hom-sets

$$(-)^{-1}_{A,B} \colon \mathsf{Sets}(A, B) \to \mathrm{Rel}(A, B)$$

of $(-)^{-1}$ at (A, B) is defined by

$$(-)^{-1}_{A,B}(f) \overset{\text{def}}{=} \left[(-)^{-1}\right](f),$$

where f^{-1} is the inverse of f as in Definition 6.3.2.1.

In particular:

- *Preservation of Identities.* We have

$$\mathrm{id}^{-1}_A = \chi_A$$

for each $A \in \mathrm{Obj}(\mathsf{Sets})$.

- *Preservation of Composition.* We have

$$(g \circ f)^{-1} = g^{-1} \diamond f^{-1}$$

for pair of functions $f \colon A \to B$ and $g \colon B \to C$.

2. *Adjointness Inside* **Rel**. We have an adjunction

$$(\mathrm{Gr}(f) \dashv f^{-1})\colon \quad \begin{tikzcd} A \arrow[r, bend left, "\mathrm{Gr}(f)", "|" marking] \arrow[r, phantom, "\perp"] & B \arrow[l, bend left, "f^{-1}", "|" marking] \end{tikzcd}$$

in **Rel**.

3. *Interaction With Inverses of Relations.* We have

$$\begin{aligned} \left(f^{-1}\right)^{\dagger} &= \mathrm{Gr}(f), \\ \mathrm{Gr}(f)^{\dagger} &= f^{-1}. \end{aligned}$$

Proof 6.3.2.3 ▶ Proof of Proposition 6.3.2.2

Item 1: Functoriality

Clear.

Item 2: Adjointness Inside Rel

This is proved in Item 2 of Proposition 6.3.1.2.

Item 3: Interaction With Inverses of Relations

Clear. □

6.3.3 Representable Relations

Let A and B be sets.

Definition 6.3.3.1 ▶ Representable Relations

Let $f\colon A \to B$ and $g\colon B \to A$ be functions.[1]

1. The **representable relation associated to** f is the relation $\chi_f\colon A \nrightarrow B$ defined as the composition

$$A \times B \xrightarrow{f\times \mathrm{id}_B} B \times B \xrightarrow{\chi_B} \{\mathsf{true}, \mathsf{false}\},$$

i.e. given by declaring $a \sim_{\chi_f} b$ iff $f(a) = b$.

2. The **corepresentable relation associated to** g is the relation $\chi^g\colon B \nrightarrow A$ defined as the composition

$$B \times A \xrightarrow{g\times \mathrm{id}_A} A \times A \xrightarrow{\chi_A} \{\mathsf{true}, \mathsf{false}\},$$

i.e. given by declaring $b \sim_{\chi^g} a$ iff $g(b) = a$.

[1] More generally, given functions

$$f\colon A \to C,$$
$$g\colon B \to D$$

and a relation $B \nrightarrow D$, we may consider the composite relation

$$A \times B \xrightarrow{f\times g} C \times D \xrightarrow{R} \{\mathsf{true}, \mathsf{false}\},$$

for which we have $a \sim_{R\circ(f\times g)} b$ iff $f(a) \sim_R g(b)$.

6.3.4 The Domain and Range of a Relation

Let A and B be sets.

Definition 6.3.4.1 ▶ The Domain and Range of a Relation

Let $R \subset A \times B$ be a relation.[1,2]

1. The **domain of** R is the subset $\mathrm{dom}(R)$ of A defined by

$$\mathrm{dom}(R) \stackrel{\mathrm{def}}{=} \left\{ a \in A \;\middle|\; \begin{array}{l}\text{there exists some } b \in B\\ \text{such that } a \sim_R b\end{array} \right\}.$$

2. The **range of** R is the subset $\mathrm{range}(R)$ of B defined by

$$\mathrm{range}(R) \stackrel{\mathrm{def}}{=} \left\{ b \in B \;\middle|\; \begin{array}{l}\text{there exists some } a \in A\\ \text{such that } a \sim_R b\end{array} \right\}.$$

[1]Following **??**, we may compute the (characteristic functions associated to the) domain and range of a relation using the following colimit formulas:

$$\begin{aligned} \chi_{\mathrm{dom}(R)}(a) &\cong \operatorname*{colim}_{b \in B}\left(R^b_a\right) && (a \in A) \\ &\cong \bigvee_{b \in B} R^b_a, \\ \chi_{\mathrm{range}(R)}(b) &\cong \operatorname*{colim}_{a \in A}\left(R^b_a\right) && (b \in B) \\ &\cong \bigvee_{a \in A} R^b_a, \end{aligned}$$

where the join $\bigvee$ is taken in the poset $(\{\mathrm{true}, \mathrm{false}\}, \preceq)$ of Definition 1.2.2.3.

[2]Viewing R as a function $R\colon A \to \mathcal{P}(B)$, we have

$$\begin{aligned} \mathrm{dom}(R) &\cong \operatorname*{colim}_{y \in Y}(R(y)) \\ &\cong \bigcup_{y \in Y} R(y), \\ \mathrm{range}(R) &\cong \operatorname*{colim}_{x \in X}(R(x)) \\ &\cong \bigcup_{x \in X} R(x), \end{aligned}$$

6.3.5 Binary Unions of Relations

Let A and B be sets and let R and S be relations from A to B.

Definition 6.3.5.1 ▶ Binary Unions of Relations

The **union of** R **and** S[1] is the relation $R \cup S$ from A to B defined as follows:

- Viewing relations from A to B as subsets of $A \times B$, we define[2]

$$R \cup S \overset{\text{def}}{=} \{(a,b) \in B \times A \mid \text{we have } a \sim_R b \text{ or } a \sim_S b\}.$$

- Viewing relations from A to B as functions $A \to \mathcal{P}(B)$, we define

$$[R \cup S](a) \overset{\text{def}}{=} R(a) \cup S(a)$$

for each $a \in A$.

[1] *Further Terminology:* Also called the **binary union of R and S**, for emphasis.
[2] This is the same as the union of R and S as subsets of $A \times B$.

Proposition 6.3.5.2 ▶ Properties of Binary Unions of Relations

Let R, S, R_1, and R_2 be relations from A to B, and let S_1 and S_2 be relations from B to C.

1. *Interaction With Inverses.* We have

$$(R \cup S)^\dagger = R^\dagger \cup S^\dagger.$$

2. *Interaction With Composition.* We have

$$(S_1 \diamond R_1) \cup (S_2 \diamond R_2) \overset{\text{poss.}}{\neq} (S_1 \cup S_2) \diamond (R_1 \cup R_2).$$

Proof 6.3.5.3 ▶ Proof of Proposition 6.3.5.2

Item 1: Interaction With Inverses

Clear.

Item 2: Interaction With Composition

Unwinding the definitions, we see that:

1. The condition for $(S_1 \diamond R_1) \cup (S_2 \diamond R_2)$ is:
   (a) There exists some $b \in B$ such that:
      i. $a \sim_{R_1} b$ and $b \sim_{S_1} c$;

   or
      i. $a \sim_{R_2} b$ and $b \sim_{S_2} c$;
3. The condition for $(S_1 \cup S_2) \diamond (R_1 \cup R_2)$ is:
   (a) There exists some $b \in B$ such that:

i. $a \sim_{R_1} b$ or $a \sim_{R_2} b$;

and

i. $b \sim_{S_1} c$ or $b \sim_{S_2} c$.

These two conditions may fail to agree (counterexample omitted), and thus the two resulting relations on $A \times C$ may differ.

6.3.6 Unions of Families of Relations

Let A and B be sets and let $\{R_i\}_{i\in I}$ be a family of relations from A to B.

Definition 6.3.6.1 ► The Union of a Family of Relations

The **union of the family** $\{R_i\}_{i\in I}$ is the relation $\bigcup_{i\in I} R_i$ from A to B defined as follows:

- Viewing relations from A to B as subsets of $A \times B$, we define[1]

$$\bigcup_{i\in I} R_i \overset{\text{def}}{=} \left\{ (a,b) \in (A \times B)^{\times I} \;\middle|\; \begin{array}{l}\text{there exists some } i \in I \\ \text{such that } a \sim_{R_i} b\end{array} \right\}.$$

- Viewing relations from A to B as functions $A \to \mathcal{P}(B)$, we define

$$\left[\bigcup_{i\in I} R_i\right](a) \overset{\text{def}}{=} \bigcup_{i\in I} R_i(a)$$

  for each $a \in A$.

[1] This is the same as the union of $\{R_i\}_{i\in I}$ as a collection of subsets of $A \times B$.

Proposition 6.3.6.2 ► Properties of Unions of Families of Relations

Let A and B be sets and let $\{R_i\}_{i\in I}$ be a family of relations from A to B.

1. *Interaction With Inverses.* We have

$$\left(\bigcup_{i\in I} R_i\right)^{\dagger} = \bigcup_{i\in I} R_i^{\dagger}.$$

Proof 6.3.6.3 ► Proof of Proposition 6.3.6.2

Item 1: Interaction With Inverses

Clear.

6.3.7 Binary Intersections of Relations

Let A and B be sets and let R and S be relations from A to B.

Definition 6.3.7.1 ► Binary Intersections of Relations

The **intersection of R and S**[1] is the relation $R \cap S$ from A to B defined as follows:

- Viewing relations from A to B as subsets of $A \times B$, we define[2]

$$R \cap S \stackrel{\text{def}}{=} \{(a, b) \in B \times A \mid \text{we have } a \sim_R b \text{ and } a \sim_S b\}.$$

- Viewing relations from A to B as functions $A \longrightarrow \mathcal{P}(B)$, we define

$$[R \cap S](a) \stackrel{\text{def}}{=} R(a) \cap S(a)$$

for each $a \in A$.

[1] *Further Terminology:* Also called the **binary intersection of R and S**, for emphasis.
[2] This is the same as the intersection of R and S as subsets of $A \times B$.

Proposition 6.3.7.2 ► Properties of Binary Intersections of Relations

Let R, S, R_1, and R_2 be relations from A to B, and let S_1 and S_2 be relations from B to C.

1. *Interaction With Inverses.* We have

$$(R \cap S)^{\dagger} = R^{\dagger} \cap S^{\dagger}.$$

2. *Interaction With Composition.* We have

$$(S_1 \diamond R_1) \cap (S_2 \diamond R_2) = (S_1 \cap S_2) \diamond (R_1 \cap R_2).$$

Proof 6.3.7.3 ► Proof of Proposition 6.3.7.2

Item 1: Interaction With Inverses

Clear.

Item 2: Interaction With Composition

Unwinding the definitions, we see that:

1. The condition for $(S_1 \diamond R_1) \cap (S_2 \diamond R_2)$ is:

   (a) There exists some $b \in B$ such that:

      i. $a \sim_{R_1} b$ and $b \sim_{S_1} c$;

   and

      i. $a \sim_{R_2} b$ and $b \sim_{S_2} c$;

3. The condition for $(S_1 \cap S_2) \diamond (R_1 \cap R_2)$ is:

   (a) There exists some $b \in B$ such that:

      i. $a \sim_{R_1} b$ and $a \sim_{R_2} b$;

   and

      i. $b \sim_{S_1} c$ and $b \sim_{S_2} c$.

These two conditions agree, and thus so do the two resulting relations on $A \times C$.

6.3.8 Intersections of Families of Relations

Let A and B be sets and let $\{R_i\}_{i \in I}$ be a family of relations from A to B.

Definition 6.3.8.1 ► The Intersection of a Family of Relations

The **intersection of the family** $\{R_i\}_{i \in I}$ is the relation $\bigcup_{i \in I} R_i$ defined as follows:

- Viewing relations from A to B as subsets of $A \times B$, we define[1]

$$\bigcup_{i \in I} R_i \overset{\text{def}}{=} \left\{ (a,b) \in (A \times B)^{\times I} \,\middle|\, \begin{array}{l} \text{for each } i \in I, \\ \text{we have } a \sim_{R_i} b \end{array} \right\}.$$

- Viewing relations from A to B as functions $A \to \mathcal{P}(B)$, we define

$$\left[\bigcap_{i \in I} R_i\right](a) \overset{\text{def}}{=} \bigcap_{i \in I} R_i(a)$$

  for each $a \in A$.

[1] This is the same as the intersection of $\{R_i\}_{i \in I}$ as a collection of subsets of $A \times B$.

Proposition 6.3.8.2 ► Properties of Intersections of Families of Relations

Let A and B be sets and let $\{R_i\}_{i\in I}$ be a family of relations from A to B.

1. *Interaction With Inverses.* We have

$$\left(\bigcap_{i\in I} R_i\right)^{\dagger} = \bigcap_{i\in I} R_i^{\dagger}.$$

Proof 6.3.8.3 ► Proof of Proposition 6.3.8.2

Item 1: Interaction With Inverses

Clear. ▭

6.3.9 Binary Products of Relations

Let A, B, X, and Y be sets, let $R\colon A \nrightarrow B$ be a relation from A to B, and let $S\colon X \nrightarrow Y$ be a relation from X to Y.

Definition 6.3.9.1 ► Binary Products of Relations

The **product of R and S**[1] is the relation $R \times S$ from $A \times X$ to $B \times Y$ defined as follows:

- Viewing relations from $A\times X$ to $B\times Y$ as subsets of $(A \times X) \times (B \times Y)$, we define $R \times S$ as the Cartesian product of R and S as subsets of $A \times X$ and $B \times Y$.[2]
- Viewing relations from $A\times X$ to $B\times Y$ as functions $A\times X \to \mathcal{P}(B \times Y)$, we define $R \times S$ as the composition

$$A \times X \xrightarrow{R\times S} \mathcal{P}(B) \times \mathcal{P}(Y) \overset{\mathcal{P}^{\otimes}_{B,Y}}{\hookrightarrow} \mathcal{P}(B \times Y)$$

in Sets, i.e. by

$$[R \times S](a, x) \overset{\text{def}}{=} R(a) \times S(x)$$

for each $(a, x) \in A \times X$.

[1] *Further Terminology:* Also called the **binary product of R and S**, for emphasis.

[2] That is, $R \times S$ is the relation given by declaring $(a, x) \sim_{R\times S} (b, y)$ iff $a \sim_R b$ and $x \sim_S y$.

Proposition 6.3.9.2 ▶ Properties of Binary Products of Relations

Let A, B, X, and Y be sets.

1. *Interaction With Inverses.* Let

$$\begin{aligned} R &\colon A \nrightarrow A, \\ S &\colon X \nrightarrow X \end{aligned}$$

   We have
$$(R \times S)^{\dagger} = R^{\dagger} \times S^{\dagger}.$$

2. *Interaction With Composition.* Let

$$\begin{aligned} R_1 &\colon A \nrightarrow B, \\ S_1 &\colon B \nrightarrow C, \\ R_2 &\colon X \nrightarrow Y, \\ S_2 &\colon Y \nrightarrow Z \end{aligned}$$

   be relations. We have

$$(S_1 \diamond R_1) \times (S_2 \diamond R_2) = (S_1 \times S_2) \diamond (R_1 \times R_2).$$

Proof 6.3.9.3 ▶ Proof of Proposition 6.3.9.2

Item 1: Interaction With Inverses

Unwinding the definitions, we see that:

1. We have $(a, x) \sim_{(R\times S)^{\dagger}} (b, y)$ iff:
   - We have $(b, y) \sim_{R\times S} (a, x)$, i.e. iff:
     - We have $b \sim_R a$;
     - We have $y \sim_S x$;
2. We have $(a, x) \sim_{R^{\dagger}\times S^{\dagger}} (b, y)$ iff:
   - We have $a \sim_{R^{\dagger}} b$ and $x \sim_{S^{\dagger}} y$, i.e. iff:
     - We have $b \sim_R a$;
     - We have $y \sim_S x$.

These two conditions agree, and thus the two resulting relations on $A \times X$ are equal.

Item 2: Interaction With Composition

Unwinding the definitions, we see that:

1. We have $(a,x) \sim_{(S_1 \diamond R_1)\times(S_2 \diamond R_2)} (c,z)$ iff:
   (a) We have $a \sim_{S_1 \diamond R_1} c$ and $x \sim_{S_2 \diamond R_2} z$, i.e. iff:
      i. There exists some $b \in B$ such that $a \sim_{R_1} b$ and $b \sim_{S_1} c$;
      ii. There exists some $y \in Y$ such that $x \sim_{R_2} y$ and $y \sim_{S_2} z$;
2. We have $(a,x) \sim_{(S_1 \times S_2)\diamond(R_1 \times R_2)} (c,z)$ iff:
   (a) There exists some $(b,y) \in B \times Y$ such that $(a,x) \sim_{R_1 \times R_2} (b,y)$ and $(b,y) \sim_{S_1 \times S_2} (c,z)$, i.e. such that:
      i. We have $a \sim_{R_1} b$ and $x \sim_{R_2} y$;
      ii. We have $b \sim_{S_1} c$ and $y \sim_{S_2} z$.

These two conditions agree, and thus the two resulting relations from $A \times X$ to $C \times Z$ are equal. □

6.3.10 Products of Families of Relations

Let $\{A_i\}_{i\in I}$ and $\{B_i\}_{i\in I}$ be families of sets, and let $\{R_i \colon A_i \nrightarrow B_i\}_{i\in I}$ be a family of relations.

Definition 6.3.10.1 ► The Product of a Family of Relations

The **product of the family** $\{R_i\}_{i\in I}$ is the relation $\prod_{i\in I} R_i$ from $\prod_{i\in I} A_i$ to $\prod_{i\in I} B_i$ defined as follows:

- Viewing relations as subsets, we define $\prod_{i\in I} R_i$ as its product as a family of sets, i.e. we have

$$\prod_{i\in I} R_i \stackrel{\text{def}}{=} \left\{ (a_i, b_i)_{i\in I} \in \prod_{i\in I}(A_i \times B_i) \,\middle|\, \begin{aligned}&\text{for each } i \in I,\\ &\text{we have } a_i \sim_{R_i} b_i\end{aligned} \right\}.$$

- Viewing relations as functions to powersets, we define

$$\left[\prod_{i\in I} R_i\right]((a_i)_{i\in I}) \stackrel{\text{def}}{=} \prod_{i\in I} R_i(a_i)$$

for each $(a_i)_{i\in I} \in \prod_{i\in I} R_i$.

6.3.11 The Inverse of a Relation

Let A, B, and C be sets and let $R \subset A \times B$ be a relation.

Definition 6.3.11.1 ► The Inverse of a Relation

The **inverse of** R[1] is the relation $R^\dagger$ defined as follows:

- Viewing relations as subsets, we define

$$R^\dagger \stackrel{\text{def}}{=} \{(b, a) \in B \times A \mid \text{we have } b \sim_R a\}.$$

- Viewing relations as functions $A \times B \to \{\text{true}, \text{false}\}$, we define

$$\left[R^\dagger\right]_b^a \stackrel{\text{def}}{=} R_a^b$$

  for each $(b, a) \in B \times A$.

- Viewing relations as functions $A \to \mathcal{P}(B)$, we define

$$\begin{aligned}\left[R^\dagger\right](b) &\stackrel{\text{def}}{=} R^\dagger(\{b\}) \\ &\stackrel{\text{def}}{=} \{a \in A \mid b \in R(a)\}\end{aligned}$$

  for each $b \in B$, where $R^\dagger(\{b\})$ is the fibre of R over $\{b\}$.

[1] *Further Terminology:* Also called the **opposite of** R, the **transpose of** R, or the **converse of** R.

Example 6.3.11.2 ► Examples of Inverses of Relations

Here are some examples of inverses of relations.

1. *Less Than Equal Signs.* We have $(\leq)^\dagger = \geq$.
2. *Greater Than Equal Signs.* Dually to Item 1, we have $(\geq)^\dagger = \leq$.
3. *Functions.* Let $f : A \to B$ be a function. We have

$$\begin{aligned}\mathrm{Gr}(f)^\dagger &= f^{-1}, \\ \left(f^{-1}\right)^\dagger &= \mathrm{Gr}(f).\end{aligned}$$

Proposition 6.3.11.3 ► Properties of Inverses of Relations

Let $R : A \nrightarrow B$ and $S : B \nrightarrow C$ be relations.

1. *Functoriality.* The assignment $R \mapsto R^\dagger$ defines a functor (i.e. morphism

of posets)

$$(-)^{\dagger}\colon \mathbf{Rel}(A,B) \longrightarrow \mathbf{Rel}(B,A).$$

In particular, given relations $R, S\colon A \nrightarrow B$, we have:

($\star$) If $R \subset S$, then $R^{\dagger} \subset S^{\dagger}$.

2. *Interaction With Ranges and Domains.* We have

$$\begin{aligned}\mathrm{dom}\left(R^{\dagger}\right) &= \mathrm{range}(R),\\ \mathrm{range}\left(R^{\dagger}\right) &= \mathrm{dom}(R).\end{aligned}$$

3. *Interaction With Composition I.* We have

$$(S \diamond R)^{\dagger} = R^{\dagger} \diamond S^{\dagger}.$$

4. *Interaction With Composition II.* We have

$$\begin{aligned}\chi_B &\subset R \diamond R^{\dagger},\\ \chi_A &\subset R^{\dagger} \diamond R.\end{aligned}$$

5. *Invertibility.* We have

$$\left(R^{\dagger}\right)^{\dagger} = R.$$

6. *Identity.* We have

$$\chi_A^{\dagger} = \chi_A.$$

Proof 6.3.11.4 ► Proof of Proposition 6.3.11.3

Item 1: Functoriality

Clear.

Item 2: Interaction With Ranges and Domains

Clear.

Item 3: Interaction With Composition I

Clear.

Item 4: Interaction With Composition II

Clear.

Item 5: Invertibility

Clear.

Item 6: Identity

Clear. □

6.3.12 Composition of Relations

Let A, B, and C be sets and let $R\colon A \nrightarrow B$ and $S\colon B \nrightarrow C$ be relations.

Definition 6.3.12.1 ► Composition of Relations

The **composition of** R **and** S is the relation $S \diamond R$ defined as follows:

- Viewing relations from A to C as subsets of $A \times C$, we define

$$S \diamond R \overset{\text{def}}{=} \left\{ (a, c) \in A \times C \;\middle|\; \begin{array}{l}\text{there exists some } b \in B \text{ such} \\ \text{that } a \sim_R b \text{ and } b \sim_S c\end{array} \right\}.$$

- Viewing relations as functions $A \times B \to \{\text{true}, \text{false}\}$, we define

$$\begin{aligned}(S \diamond R)^{-1}_{-2} &\overset{\text{def}}{=} \int^{b \in B} S^{-1}_{b} \times R^{b}_{-2}\\ &= \bigvee_{b \in B} S^{-1}_{b} \times R^{b}_{-2},\end{aligned}$$

  where the join $\bigvee$ is taken in the poset $(\{\text{true}, \text{false}\}, \preceq)$ of Definition 1.2.2.3.

- Viewing relations as functions $A \to \mathcal{P}(B)$, we define

$$S \diamond R \overset{\text{def}}{=} \mathrm{Lan}_{\chi_B}(S) \circ R, \qquad \begin{tikzcd} & B \arrow[r, "S"] \arrow[d, hook, "\chi_B"'] & \mathcal{P}(C), \\ A \arrow[r, "R"'] & \mathcal{P}(B) \arrow[ur, "\mathrm{Lan}_{\chi_B}(S)"'] \arrow[u, phantom, "\Swarrow"] & \end{tikzcd}$$

where $\mathrm{Lan}_{\chi_B}(S)$ is computed by the formula

$$\begin{aligned}\left[\mathrm{Lan}_{\chi_B}(S)\right](V) &\cong \int^{y\in B} \chi_{\mathcal{P}(B)}\left(\chi_y, V\right) \odot S_y\\ &\cong \int^{y\in B} \chi_V(y) \odot S_y\\ &\cong \bigcup_{y\in B} \chi_V(y) \odot S_y\\ &\cong \bigcup_{y\in V} S_y\end{aligned}$$

for each $V \in \mathcal{P}(B)$. In other words, $S \diamond R$ is defined by[1]

$$\begin{aligned}[S \diamond R](a) &\stackrel{\text{def}}{=} S(R(a))\\ &\stackrel{\text{def}}{=} \bigcup_{x\in R(a)} S(x).\end{aligned}$$

for each $a \in A$.

[1]That is: the relation R may send $a \in A$ to a number of elements $\{b_i\}_{i\in I}$ in B, and then the relation S may send the image of each of the b_i's to a number of elements $\{S(b_i)\}_{i\in I} = \left\{\{c_{j_i}\}_{j_i\in J_i}\right\}_{i\in I}$ in C.

Example 6.3.12.2 ► Examples of Composition of Relations

Here are some examples of composition of relations.

1. *Composing Less/Greater Than Equal With Greater/Less Than Equal Signs.* We have

$$\begin{aligned}\leq \diamond \geq &= \sim_{\text{triv}},\\ \geq \diamond \leq &= \sim_{\text{triv}}.\end{aligned}$$

2. *Composing Less/Greater Than Equal Signs With Less/Greater Than Equal Signs.* We have

$$\begin{aligned}\leq \diamond \leq &= \leq,\\ \geq \diamond \geq &= \geq.\end{aligned}$$

Proposition 6.3.12.3 ▶ Properties of Composition of Relations

Let $R \colon A \nrightarrow B$, $S \colon B \nrightarrow C$, and $T \colon C \nrightarrow D$ be relations.

1. *Interaction With Ranges and Domains.* We have

$$\begin{aligned}\mathrm{dom}(S \diamond R) &\subset \mathrm{dom}(R),\\ \mathrm{range}(S \diamond R) &\subset \mathrm{range}(S).\end{aligned}$$

2. *Associativity.* We have

$$(T \diamond S) \diamond R = T \diamond (S \diamond R).$$

3. *Unitality.* We have

$$\begin{aligned}\chi_B \diamond R &= R,\\ R \diamond \chi_A &= R.\end{aligned}$$

4. *Interaction With Inverses.* We have

$$(S \diamond R)^{\dagger} = R^{\dagger} \diamond S^{\dagger}.$$

5. *Interaction With Composition.* We have

$$\begin{aligned}\chi_B &\subset R \diamond R^{\dagger},\\ \chi_A &\subset R^{\dagger} \diamond R.\end{aligned}$$

Proof 6.3.12.4 ▶ Proof of Proposition 6.3.12.3

Item 1: Interaction With Ranges and Domains

Clear.

Item 2: Associativity

Indeed, we have

$$
\begin{aligned}
(T \diamond S) \diamond R &\overset{\text{def}}{=} \left(\int^{c\in C} T^{-1}_{c} \times S^{c}_{-_2}\right) \diamond R \\
&\overset{\text{def}}{=} \int^{b\in B} \left(\int^{c\in C} T^{-1}_{c} \times S^{c}_{b}\right) \diamond R^{b}_{-_2} \\
&= \int^{b\in B} \int^{c\in C} \left(T^{-1}_{c} \times S^{c}_{b}\right) \diamond R^{b}_{-_2} \\
&= \int^{c\in C} \int^{b\in B} \left(T^{-1}_{c} \times S^{c}_{b}\right) \diamond R^{b}_{-_2} \\
&= \int^{c\in C} \int^{b\in B} T^{-1}_{c} \times \left(S^{c}_{b} \diamond R^{b}_{-_2}\right) \\
&= \int^{c\in C} T^{-1}_{c} \times \left(\int^{b\in B} S^{c}_{b} \diamond R^{b}_{-_2}\right) \\
&\overset{\text{def}}{=} \int^{c\in C} T^{-1}_{c} \times (S \diamond R)^{c}_{-_2} \\
&\overset{\text{def}}{=} T \diamond (S \diamond R).
\end{aligned}
$$

In the language of relations, given $a \in A$ and $d \in D$, the stated equality witnesses the equivalence of the following two statements:

1. We have $a \sim_{(T \diamond S)\diamond R} d$, i.e. there exists some $b \in B$ such that:
    (a) We have $a \sim_R b$;
    (b) We have $b \sim_{T\diamond S} d$, i.e. there exists some $c \in C$ such that:
        i. We have $b \sim_S c$;
        ii. We have $c \sim_T d$;
2. We have $a \sim_{T\diamond(S\diamond R)} d$, i.e. there exists some $c \in C$ such that:
    (a) We have $a \sim_{S\diamond R} c$, i.e. there exists some $b \in B$ such that:
        i. We have $a \sim_R b$;
        ii. We have $b \sim_S c$;
    (b) We have $c \sim_T d$;

both of which are equivalent to the statement

- There exist $b \in B$ and $c \in C$ such that $a \sim_R b \sim_S c \sim_T d$.

Item 3: Unitality

Indeed, we have

$$\begin{aligned}\chi_B \diamond R &\stackrel{\text{def}}{=} \int^{x\in B} (\chi_B)^{-_1}_{x} \times R^{x}_{-_2}\\ &= \bigvee_{x\in B} (\chi_B)^{-_1}_{x} \times R^{x}_{-_2}\\ &= \bigvee_{\substack{x\in B\\ x=-_1}} R^{x}_{-_2}\\ &= R^{-_1}_{-_2},\end{aligned}$$

and

$$\begin{aligned}R \diamond \chi_A &\stackrel{\text{def}}{=} \int^{x\in A} R^{-_1}_{x} \times (\chi_A)^{x}_{-_2}\\ &= \bigvee_{x\in B} R^{-_1}_{x} \times (\chi_A)^{x}_{-_2}\\ &= \bigvee_{\substack{x\in B\\ x=-_2}} R^{-_1}_{x}\\ &= R^{-_1}_{-_2}.\end{aligned}$$

In the language of relations, given $a \in A$ and $b \in B$:

- The equality
$$\chi_B \diamond R = R$$
witnesses the equivalence of the following two statements:
    1. We have $a \sim_b B$.
    2. There exists some $b' \in B$ such that:
        (a) We have $a \sim_R b'$
        (b) We have $b' \sim_{\chi_B} b$, i.e. $b' = b$.
- The equality
$$R \diamond \chi_A = R$$
witnesses the equivalence of the following two statements:
    1. There exists some $a' \in A$ such that:
        (a) We have $a \sim_{\chi_B} a'$, i.e. $a = a'$.

(b) We have $a' \sim_R b$

2. We have $a \sim_b B$.

Item 4: Interaction With Inverses

Clear.

Item 5: Interaction With Composition

Clear.

6.3.13 The Collage of a Relation

Let A and B be sets and let $R\colon A \nrightarrow B$ be a relation from A to B.

DEFINITION 6.3.13.1 ▶ THE COLLAGE OF A RELATION

The **collage of** R[1] is the poset $\mathbf{Coll}(R) \overset{\text{def}}{=} (\mathrm{Coll}(R), \preceq_{\mathbf{Coll}(R)})$ consisting of:

- *The Underlying Set.* The set $\mathrm{Coll}(R)$ defined by

$$\mathrm{Coll}(R) \overset{\text{def}}{=} A \coprod B.$$

- *The Partial Order.* The partial order

$$\preceq_{\mathbf{Coll}(R)} \colon \mathrm{Coll}(R) \times \mathrm{Coll}(R) \to \{\mathrm{true}, \mathrm{false}\}$$

on $\mathrm{Coll}(R)$ defined by

$$\preceq(a, b) \overset{\text{def}}{=} \begin{cases} \mathrm{true} & \text{if } a = b \text{ or } a \sim_R b, \\ \mathrm{false} & \text{otherwise.} \end{cases}$$

[1] *Further Terminology:* Also called the **cograph of** R.

PROPOSITION 6.3.13.2 ▶ PROPERTIES OF COLLAGES OF RELATIONS

Let A and B be sets and let $R\colon A \nrightarrow B$ be a relation from A to B.

1. *Functoriality I.* The assignment $R \mapsto \mathbf{Coll}(R)$ defines a functor[1]

$$\mathbf{Coll}\colon \mathbf{Rel}(A, B) \to \mathrm{Pos}_{/\Delta^1}(A, B),$$

where

- *Action on Objects.* For each $R \in \mathrm{Obj}(\mathbf{Rel}(A, B))$, we have

$$[\mathbf{Coll}](R) \overset{\text{def}}{=} (\mathbf{Coll}(R), \phi_R)$$

for each $R \in \mathbf{Rel}(A, B)$, where

– The poset $\mathbf{Coll}(R)$ is the collage of R of Definition 6.3.13.1.

– The morphism $\phi_R \colon \mathbf{Coll}(R) \to \Delta^1$ is given by

$$\phi_R(x) \overset{\text{def}}{=} \begin{cases} 0 & \text{if } x \in A, \\ 1 & \text{if } x \in B \end{cases}$$

for each $x \in \mathbf{Coll}(R)$.

· *Action on Morphisms.* For each $R, S \in \mathrm{Obj}(\mathbf{Rel}(A, B))$, the action on Hom-sets

$$\mathbf{Coll}_{R,S} \colon \mathrm{Hom}_{\mathbf{Rel}(A,B)}(R, S) \to \mathrm{Pos}(\mathbf{Coll}(R), \mathbf{Coll}(S))$$

of $\mathbf{Coll}$ at (R, S) is given by sending an inclusion

$$\iota \colon R \subset S$$

to the morphism

$$\mathbf{Coll}(\iota) \colon \mathbf{Coll}(R) \to \mathbf{Coll}(S)$$

of posets over Δ^1 defined by

$$[\mathbf{Coll}(\iota)](x) \overset{\text{def}}{=} x$$

for each $x \in \mathbf{Coll}(R)$.[2]

2. *Equivalence.* The functor of Item 1 is an equivalence of categories.

---

[1] Here $\mathrm{Pos}_{/\Delta^1}(A, B)$ is the category defined as the pullback

$$\mathrm{Pos}_{/\Delta^1}(A, B) \overset{\text{def}}{=} \mathrm{pt} \underset{[A],\mathrm{Pos},\mathrm{fib}_0}{\times} \mathrm{Pos}_{/\Delta^1} \underset{\mathrm{fib}_1,\mathrm{Pos},[B]}{\times} \mathrm{pt},$$

as in the diagram

$$\begin{tikzcd}
 & & \mathbf{Pos}_{/\Delta^1}(A, B) \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & & \\
 & \mathbf{Pos}_{/\Delta^1} \underset{\mathbf{Pos}}{\times} \mathrm{pt} \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & & \mathrm{pt} \underset{\mathbf{Pos}}{\times} \mathbf{Pos}_{/\Delta^1} \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & \\
\mathrm{pt} \arrow[dr, "{[A]}"'] & & \mathbf{Pos}_{/\Delta^1} \arrow[dl, "\mathrm{fib}_{[0]}"] \arrow[dr, "\mathrm{fib}_{[1]}"'] & & \mathrm{pt}. \arrow[dl, "{[B]}"] \\
 & \mathbf{Pos} & & \mathbf{Pos} &
\end{tikzcd}$$

Explicitly, an object of $\mathrm{Pos}_{/\Delta^1}(A, B)$ is a pair (X, ϕ_X) consisting of

- A poset X;
- A morphism $\phi_X : X \to \Delta^1$;

such that $\phi_X^{-1}(0) = A$ and $\phi_X^{-1}(0) = B$, with morphisms between such objects being morphisms of posets over Δ^1.

[2]Note that this is indeed a morphism of posets: if $x \preceq_{\mathsf{Coll}(R)} y$, then $x = y$ or $x \sim_R y$, so we have either $x = y$ or $x \sim_S y$ (as $R \subset S$), and thus $x \preceq_{\mathsf{Coll}(S)} y$.

Proof 6.3.13.3 ▶ Proof of Proposition 6.3.13.2

Item 1: Functoriality

Clear.

Item 2: Equivalence

Omitted. □

6.4 Functoriality of Powersets

6.4.1 Direct Images

Let A and B be sets and let $R : A \nrightarrow B$ be a relation.

Definition 6.4.1.1 ▶ Direct Images

The **direct image function associated to** R is the function

$$R_* : \mathcal{P}(A) \to \mathcal{P}(B)$$

defined by[1,2]

$$\begin{aligned} R_*(U) &\overset{\text{def}}{=} R(U) \\ &\overset{\text{def}}{=} \bigcup_{a \in U} R(a) \\ &= \left\{ b \in B \,\middle|\, \begin{array}{l} \text{there exists some } a \in U \\ \text{such that } b \in R(a) \end{array} \right\} \end{aligned}$$

for each $U \in \mathcal{P}(A)$.

[1]*Further Terminology:* The set $R(U)$ is called the **direct image of** U **by** R.

[2]We also have

$$R_*(U) = B \setminus R_!(A \setminus U);$$

see Item 7 of Proposition 6.4.1.3.

Remark 6.4.1.2 ▶ Unwinding Definition 6.4.1.1

Identifying subsets of A with relations from pt to A via Item 3 of Proposition 2.4.3.9, we see that the direct image function associated to R is equivalently the function

$$R_* \colon \underbrace{\mathcal{P}(A)}_{\cong \mathrm{Rel}(\mathrm{pt}, A)} \to \underbrace{\mathcal{P}(B)}_{\cong \mathrm{Rel}(\mathrm{pt}, B)}$$

defined by

$$R_*(U) \overset{\mathrm{def}}{=} R \diamond U$$

for each $U \in \mathcal{P}(A)$, where $R \diamond U$ is the composition

$$\mathrm{pt} \overset{U}{\nrightarrow} A \overset{R}{\nrightarrow} B.$$

Proposition 6.4.1.3 ▶ Properties of Direct Image Functions

Let $R \colon A \nrightarrow B$ be a relation.

1. *Functoriality.* The assignment $U \mapsto R_*(U)$ defines a functor

   $$R_* \colon (\mathcal{P}(A), \subset) \to (\mathcal{P}(B), \subset)$$

   where

   - *Action on Objects.* For each $U \in \mathcal{P}(A)$, we have

     $$[R_*](U) \overset{\mathrm{def}}{=} R_*(U).$$

   - *Action on Morphisms.* For each $U, V \in \mathcal{P}(A)$:
     - If $U \subset V$, then $R_*(U) \subset R_*(V)$.

2. *Adjointness.* We have an adjunction

   $$(R_* \dashv R_{-1}) \colon \begin{tikzcd} \mathcal{P}(A) \arrow[r, bend left, "R_*"] \arrow[r, phantom, "\perp"] & \mathcal{P}(B) \arrow[l, bend left, "R_{-1}"] \end{tikzcd},$$

   witnessed by a bijections of sets

   $$\mathrm{Hom}_{\mathcal{P}(A)}(R_*(U), V) \cong \mathrm{Hom}_{\mathcal{P}(A)}(U, R_{-1}(V)),$$

   natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$, i.e. such that:

   ($\star$) The following conditions are equivalent:

- We have $R_*(U) \subset V$.
- We have $U \subset R_{-1}(V)$.

3. *Preservation of Colimits.* We have an equality of sets

$$R_*\left(\bigcup_{i\in I} U_i\right) = \bigcup_{i\in I} R_*(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have equalities

$$\begin{aligned} R_*(U) \cup R_*(V) &= R_*(U \cup V), \\ R_*(\emptyset) &= \emptyset, \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

4. *Oplax Preservation of Limits.* We have an inclusion of sets

$$R_*\left(\bigcap_{i\in I} U_i\right) \subset \bigcap_{i\in I} R_*(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have inclusions

$$\begin{aligned} R_*(U \cap V) &\subset R_*(U) \cap R_*(V), \\ R_*(A) &\subset B, \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

5. *Symmetric Strict Monoidality With Respect to Unions.* The direct image function of Item 1 has a symmetric strict monoidal structure

$$\left(R_*, R_*^{\otimes}, R_{*|\mathbb{1}}^{\otimes}\right) \colon (\mathcal{P}(A), \cup, \emptyset) \longrightarrow (\mathcal{P}(B), \cup, \emptyset),$$

being equipped with equalities

$$\begin{aligned} R_{*|U,V}^{\otimes} \colon R_*(U) \cup R_*(V) &\xrightarrow{=} R_*(U \cup V), \\ R_{*|\mathbb{1}}^{\otimes} \colon \emptyset &\xrightarrow{=} \emptyset, \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

6. *Symmetric Oplax Monoidality With Respect to Intersections.* The direct image function of Item 1 has a symmetric oplax monoidal structure

$$\left(R_*, R_*^{\otimes}, R_{*|\mathbb{1}}^{\otimes}\right) \colon (\mathcal{P}(A), \cap, A) \longrightarrow (\mathcal{P}(B), \cap, B),$$

being equipped with inclusions

$$R^{\otimes}_{*|U,V} \colon R_*(U \cap V) \subset R_*(U) \cap R_*(V),$$
$$R^{\otimes}_{*|\mathbb{1}} \colon R_*(A) \subset B,$$

natural in $U, V \in \mathcal{P}(A)$.

7. *Relation to Direct Images With Compact Support.* We have

$$R_*(U) = B \setminus R_!(A \setminus U)$$

for each $U \in \mathcal{P}(A)$.

Proof 6.4.1.4 ► Proof of Proposition 6.4.1.3

Item 1: Functoriality

Clear.

Item 2: Adjointness

This follows from **??** of **??**.

Item 3: Preservation of Colimits

This follows from Item 2 and **??** of **??**.

Item 4: Oplax Preservation of Limits

Omitted.

Item 5: Symmetric Strict Monoidality With Respect to Unions

This follows from Item 3.

Item 6: Symmetric Oplax Monoidality With Respect to Intersections

This follows from Item 4.

Item 7: Relation to Direct Images With Compact Support

The proof proceeds in the same way as in the case of functions (**??** of Proposition 2.4.4.4): applying Item 7 of Proposition 6.4.4.4 to $A \setminus U$, we have

$$\begin{aligned} R_!(A \setminus U) &= B \setminus R_*(A \setminus (A \setminus U)) \\ &= B \setminus R_*(U). \end{aligned}$$

Taking complements, we then obtain

$$\begin{aligned} R_*(U) &= B \setminus (B \setminus R_*(U)), \\ &= B \setminus R_!(A \setminus U), \end{aligned}$$

which finishes the proof. ∎

Proposition 6.4.1.5 ▶ Properties of the Direct Image Function Operation

Let $R\colon A \nrightarrow B$ be a relation.

1. *Functionality I*. The assignment $R \mapsto R_*$ defines a function
$$(-)_*\colon \mathsf{Rel}(A,B) \to \mathsf{Sets}(\mathcal{P}(A),\mathcal{P}(B)).$$
2. *Functionality II*. The assignment $R \mapsto R_*$ defines a function
$$(-)_*\colon \mathsf{Rel}(A,B) \to \mathsf{Pos}((\mathcal{P}(A),\subset),(\mathcal{P}(B),\subset)).$$
3. *Interaction With Identities*. For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have[1]
$$(\chi_A)_* = \mathrm{id}_{\mathcal{P}(A)}.$$
4. *Interaction With Composition*. For each pair of composable relations $R\colon A \nrightarrow B$ and $S\colon B \nrightarrow C$, we have[2]
$$(S \diamond R)_* = S_* \circ R_*, \qquad \begin{tikzcd} \mathcal{P}(A) \arrow[r, "R_*"] \arrow[dr, "(S\diamond R)_*"'] & \mathcal{P}(B) \arrow[d, "S_*"] \\ & \mathcal{P}(C). \end{tikzcd}$$

---

[1] That is, the postcomposition function
$$(\chi_A)_*\colon \mathsf{Rel}(\mathrm{pt},A) \to \mathsf{Rel}(\mathrm{pt},A)$$
is equal to $\mathrm{id}_{\mathsf{Rel}(\mathrm{pt},A)}$.

[2] That is, we have
$$(S \diamond R)_* = S_* \circ R_*, \qquad \begin{tikzcd} \mathsf{Rel}(\mathrm{pt},A) \arrow[r, "R_*"] \arrow[dr, "(S\diamond R)_*"'] & \mathsf{Rel}(\mathrm{pt},B) \arrow[d, "S_*"] \\ & \mathsf{Rel}(\mathrm{pt},C). \end{tikzcd}$$

Proof 6.4.1.6 ▶ Proof of Proposition 6.4.1.5

Item 1: Functionality I

Clear.

Item 2: Functionality II

Clear.

Item 3: Interaction With Identities

Indeed, we have

$$\begin{aligned}
(\chi_A)_*(U) &\stackrel{\text{def}}{=} \bigcup_{a\in U} \chi_A(a) \\
&\stackrel{\text{def}}{=} \bigcup_{a\in U} \{a\} \\
&= U \\
&\stackrel{\text{def}}{=} \mathrm{id}_{\mathcal{P}(A)}(U)
\end{aligned}$$

for each $U \in \mathcal{P}(A)$. Thus $(\chi_A)_* = \mathrm{id}_{\mathcal{P}(A)}$.

Item 4: Interaction With Composition

Indeed, we have

$$\begin{aligned}
(S \diamond R)_*(U) &\stackrel{\text{def}}{=} \bigcup_{a\in U} [S \diamond R](a) \\
&\stackrel{\text{def}}{=} \bigcup_{a\in U} S(R(a)) \\
&\stackrel{\text{def}}{=} \bigcup_{a\in U} S_*(R(a)) \\
&= S_*\left(\bigcup_{a\in U} R(a)\right) \\
&\stackrel{\text{def}}{=} S_*(R_*(U)) \\
&\stackrel{\text{def}}{=} [S_* \circ R_*](U)
\end{aligned}$$

for each $U \in \mathcal{P}(A)$, where we used Item 3 of Proposition 6.4.1.3. Thus $(S \diamond R)_* = S_* \circ R_*$. □

6.4.2 Strong Inverse Images

Let A and B be sets and let $R\colon A \nrightarrow B$ be a relation.

Definition 6.4.2.1 ► Strong Inverse Images

The **strong inverse image function associated to** R is the function

$$R_{-1}\colon \mathcal{P}(B) \to \mathcal{P}(A)$$

defined by[1]

$$R_{-1}(V) \overset{\text{def}}{=} \{a \in A \mid R(a) \subset V\}$$

for each $V \in \mathcal{P}(B)$.

[1] *Further Terminology:* The set $R_{-1}(V)$ is called the **strong inverse image of** V **by** R.

Remark 6.4.2.2 ► Unwinding Definition 6.4.2.1

Identifying subsets of B with relations from pt to B via Item 3 of Proposition 2.4.3.9, we see that the inverse image function associated to R is equivalently the function

$$R_{-1}\colon \underbrace{\mathcal{P}(B)}_{\cong \mathrm{Rel}(\mathrm{pt},B)} \to \underbrace{\mathcal{P}(A)}_{\cong \mathrm{Rel}(\mathrm{pt},A)}$$

defined by

$$R_{-1}(V) \overset{\text{def}}{=} \mathrm{Rift}_R(V), \qquad \begin{tikzcd} & A \arrow[d, "R", "|" marking] \arrow[dl, phantom, "\Downarrow"] \\ \mathrm{pt} \arrow[ur, dashed, "\mathrm{Rift}_R(V)", "|" marking] \arrow[r, "V"', "|" marking] & B, \end{tikzcd}$$

and being explicitly computed by

$$\begin{aligned} R_{-1}(V) &\overset{\text{def}}{=} \mathrm{Rift}_R(V) \\ &\cong \int_{b\in B} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^b_{-1}, V^b_{-2}\right), \end{aligned}$$

where we have used Proposition 6.2.4.1.

Proof 6.4.2.3 ▶ Proof of Remark 6.4.2.2

We have

$$
\begin{aligned}
\mathrm{Rift}_R(V) &\cong \int_{b\in B} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^b_{-_1}, V^b_{-_2}\right)\\
&= \left\{a \in A \,\middle|\, \int_{b\in B} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^b_a, V^b_\star\right) = \mathsf{true}\right\}\\
&= \left\{a \in A \,\middle|\, \begin{array}{l}\text{for each } b \in B\text{, at least one of the}\\ \text{following conditions hold:}\\ \quad 1.\ \text{We have } R^b_a = \mathsf{false}\\ \quad 2.\ \text{The following conditions hold:}\\ \qquad \text{(a) We have } R^b_a = \mathsf{true}\\ \qquad \text{(b) We have } V^b_\star = \mathsf{true}\end{array}\right\}\\
&= \left\{a \in A \,\middle|\, \begin{array}{l}\text{for each } b \in B\text{, at least one of the}\\ \text{following conditions hold:}\\ \quad 1.\ \text{We have } b \notin R(a)\\ \quad 2.\ \text{The following conditions hold:}\\ \qquad \text{(a) We have } b \in R(a)\\ \qquad \text{(b) We have } b \in V\end{array}\right\}\\
&= \{a \in A \mid \text{for each } b \in R(a)\text{, we have } b \in V\}\\
&= \{a \in A \mid R(a) \subset V\}\\
&\overset{\text{def}}{=} R_{-1}(V).
\end{aligned}
$$

This finishes the proof. □

Proposition 6.4.2.4 ▶ Properties of Strong Inverse Images

Let $R\colon A \nrightarrow B$ be a relation.

1. *Functoriality.* The assignment $V \mapsto R_{-1}(V)$ defines a functor

$$R_{-1}\colon (\mathcal{P}(B), \subset) \to (\mathcal{P}(A), \subset)$$

where

- *Action on Objects.* For each $V \in \mathcal{P}(B)$, we have

$$[R_{-1}](V) \overset{\text{def}}{=} R_{-1}(V).$$

- *Action on Morphisms.* For each $U, V \in \mathcal{P}(B)$:
  - If $U \subset V$, then $R_{-1}(U) \subset R_{-1}(V)$.

2. *Adjointness.* We have an adjunction

$$(R_* \dashv R_{-1})\colon \quad \begin{tikzcd} \mathcal{P}(A) \arrow[r, bend left, "R_*"] \arrow[r, phantom, "\perp"] & \mathcal{P}(B) \arrow[l, bend left, "R_{-1}"] \end{tikzcd},$$

witnessed by a bijections of sets

$$\mathrm{Hom}_{\mathcal{P}(A)}(R_*(U), V) \cong \mathrm{Hom}_{\mathcal{P}(A)}(U, R_{-1}(V)),$$

natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$, i.e. such that:

($\star$) The following conditions are equivalent:
  - We have $R_*(U) \subset V$.
  - We have $U \subset R_{-1}(V)$.

3. *Lax Preservation of Colimits.* We have an inclusion of sets

$$\bigcup_{i \in I} R_{-1}(U_i) \subset R_{-1}\left(\bigcup_{i \in I} U_i\right),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(B)^{\times I}$. In particular, we have inclusions

$$\begin{gathered} R_{-1}(U) \cup R_{-1}(V) \subset R_{-1}(U \cup V), \\ \emptyset \subset R_{-1}(\emptyset), \end{gathered}$$

natural in $U, V \in \mathcal{P}(B)$.

4. *Preservation of Limits.* We have an equality of sets

$$R_{-1}\left(\bigcap_{i \in I} U_i\right) = \bigcap_{i \in I} R_{-1}(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(B)^{\times I}$. In particular, we have equalities

$$\begin{gathered} R_{-1}(U \cap V) = R_{-1}(U) \cap R_{-1}(V), \\ R_{-1}(B) = B, \end{gathered}$$

natural in $U, V \in \mathcal{P}(B)$.

5. *Symmetric Lax Monoidality With Respect to Unions.* The direct image with compact support function of Item 1 has a symmetric lax monoidal structure

$$\left(R_{-1}, R_{-1}^{\otimes}, R_{-1|\mathbb{1}}^{\otimes}\right) \colon (\mathcal{P}(A), \cup, \emptyset) \to (\mathcal{P}(B), \cup, \emptyset),$$

being equipped with inclusions

$$R_{-1|U,V}^{\otimes} \colon R_{-1}(U) \cup R_{-1}(V) \subset R_{-1}(U \cup V),$$
$$R_{-1|\mathbb{1}}^{\otimes} \colon \emptyset \subset R_{-1}(\emptyset),$$

natural in $U, V \in \mathcal{P}(B)$.

6. *Symmetric Strict Monoidality With Respect to Intersections.* The direct image function of Item 1 has a symmetric strict monoidal structure

$$\left(R_{-1}, R_{-1}^{\otimes}, R_{-1|\mathbb{1}}^{\otimes}\right) \colon (\mathcal{P}(A), \cap, A) \to (\mathcal{P}(B), \cap, B),$$

being equipped with equalities

$$R_{-1|U,V}^{\otimes} \colon R_{-1}(U \cap V) \xrightarrow{=} R_{-1}(U) \cap R_{-1}(V),$$
$$R_{-1|\mathbb{1}}^{\otimes} \colon R_{-1}(A) \xrightarrow{=} B,$$

natural in $U, V \in \mathcal{P}(B)$.

7. *Interaction With Weak Inverse Images I.* We have

$$R_{-1}(V) = A \setminus R^{-1}(B \setminus V)$$

for each $V \in \mathcal{P}(B)$.

8. *Interaction With Weak Inverse Images II.* Let $R \colon A \nrightarrow B$ be a relation from A to B.

   (a) If R is a total relation, then we have an inclusion of sets

   $$R_{-1}(V) \subset R^{-1}(V)$$

   natural in $V \in \mathcal{P}(B)$.

   (b) If R is total and functional, then the above inclusion is in fact an equality.

   (c) Conversely, if we have $R_{-1} = R^{-1}$, then R is total and functional.

Proof 6.4.2.5 ▶ Proof of Proposition 6.4.2.4

Item 1: Functoriality

Clear.

Item 2: Adjointness

This follows from ?? of ??.

Item 3: Lax Preservation of Colimits

Omitted.

Item 4: Preservation of Limits

This follows from Item 2 and ?? of ??.

Item 5: Symmetric Lax Monoidality With Respect to Unions

This follows from Item 3.

Item 6: Symmetric Strict Monoidality With Respect to Intersections

This follows from Item 4.

Item 7: Interaction With Weak Inverse Images I

We claim we have an equality

$$R_{-1}(B \setminus V) = A \setminus R^{-1}(V).$$

Indeed, we have

$$\begin{aligned} R_{-1}(B \setminus V) &= \{a \in A \mid R(a) \subset B \setminus V\}, \\ A \setminus R^{-1}(V) &= \{a \in A \mid R(a) \cap V = \emptyset\}. \end{aligned}$$

Taking $V = B \setminus V$ then implies the original statement.

Item 8: Interaction With Weak Inverse Images II

Item 8a is clear, while Items 8b and 8c follow from Item 6 of Proposition 6.3.1.2.

□

Proposition 6.4.2.6 ▶ Properties of the Strong Inverse Image Function Operation

Let $R\colon A \nrightarrow B$ be a relation.

1. *Functionality I.* The assignment $R \mapsto R_{-1}$ defines a function

$$(-)_{-1}\colon \mathsf{Sets}(A, B) \to \mathsf{Sets}(\mathcal{P}(A), \mathcal{P}(B)).$$

2. *Functionality II.* The assignment $R \mapsto R_{-1}$ defines a function

$$(-)_{-1}\colon \mathsf{Sets}(A, B) \to \mathsf{Pos}((\mathcal{P}(A), \subset), (\mathcal{P}(B), \subset)).$$

3. *Interaction With Identities.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$(\mathrm{id}_A)_{-1} = \mathrm{id}_{\mathcal{P}(A)}.$$

4. *Interaction With Composition.* For each pair of composable relations $R\colon A \nrightarrow B$ and $S\colon B \nrightarrow C$, we have

$$(S \diamond R)_{-1} = R_{-1} \circ S_{-1}, \qquad \begin{tikzcd} \mathcal{P}(C) \arrow[r, "S_{-1}"] \arrow[dr, "(S\diamond R)_{-1}"'] & \mathcal{P}(B) \arrow[d, "R_{-1}"] \\ & \mathcal{P}(A). \end{tikzcd}$$

Proof 6.4.2.7 ▶ Proof of Proposition 6.4.2.6

Item 1: Functionality I

Clear.

Item 2: Functionality II

Clear.

Item 3: Interaction With Identities

Indeed, we have

$$\begin{aligned} (\chi_A)_{-1}(U) &\overset{\text{def}}{=} \{a \in A \mid \chi_A(a) \subset U\} \\ &\overset{\text{def}}{=} \{a \in A \mid \{a\} \subset U\} \\ &= U \end{aligned}$$

for each $U \in \mathcal{P}(A)$. Thus $(\chi_A)_{-1} = \mathrm{id}_{\mathcal{P}(A)}$.

Item 4: Interaction With Composition

Indeed, we have

$$\begin{aligned}
(S \diamond R)_{-1}(U) &\stackrel{\text{def}}{=} \{a \in A \mid [S \diamond R](a) \subset U\} \\
&\stackrel{\text{def}}{=} \{a \in A \mid S(R(a)) \subset U\} \\
&\stackrel{\text{def}}{=} \{a \in A \mid S_*(R(a)) \subset U\} \\
&= \{a \in A \mid R(a) \subset S_{-1}(U)\} \\
&\stackrel{\text{def}}{=} R_{-1}(S_{-1}(U)) \\
&\stackrel{\text{def}}{=} [R_{-1} \circ S_{-1}](U)
\end{aligned}$$

for each $U \in \mathcal{P}(C)$, where we used Item 2 of Proposition 6.4.2.4, which implies that the conditions

- We have $S_*(R(a)) \subset U$.
- We have $R(a) \subset S_{-1}(U)$.

are equivalent. Thus $(S \diamond R)_{-1} = R_{-1} \circ S_{-1}$.

6.4.3 Weak Inverse Images

Let A and B be sets and let $R\colon A \nrightarrow B$ be a relation.

Definition 6.4.3.1 ► Weak Inverse Images

The **weak inverse image function associated to** R[1] is the function

$$R^{-1}\colon \mathcal{P}(B) \to \mathcal{P}(A)$$

defined by[2]

$$R^{-1}(V) \stackrel{\text{def}}{=} \{a \in A \mid R(a) \cap V \neq \emptyset\}$$

for each $V \in \mathcal{P}(B)$.

[1] *Further Terminology:* Also called simply the **inverse image function associated to** R.

[2] *Further Terminology:* The set $R^{-1}(V)$ is called the **weak inverse image of** V **by** R or simply the **inverse image of** V **by** R.

Remark 6.4.3.2 ► Unwinding Definition 6.4.3.1

Identifying subsets of B with relations from B to pt via Item 3 of Proposition 2.4.3.9, we see that the weak inverse image function associated to R is equivalently the function

$$R^{-1}\colon \underbrace{\mathcal{P}(B)}_{\cong\mathrm{Rel}(B,\mathrm{pt})} \to \underbrace{\mathcal{P}(A)}_{\cong\mathrm{Rel}(A,\mathrm{pt})}$$

defined by

$$R^{-1}(V) \stackrel{\text{def}}{=} V \diamond R$$

for each $V \in \mathcal{P}(A)$, where $R \diamond V$ is the composition

$$A \overset{R}{\nrightarrow} B \overset{V}{\nrightarrow} \mathrm{pt}.$$

Explicitly, we have

$$\begin{aligned} R^{-1}(V) &\stackrel{\text{def}}{=} V \diamond R \\ &\stackrel{\text{def}}{=} \int^{b\in B} V_b^{-1} \times R^b_{-_2}. \end{aligned}$$

Proof 6.4.3.3 ► Proof of Remark 6.4.3.2

We have

$$
\begin{aligned}
V \diamond R &\stackrel{\text{def}}{=} \int^{b \in B} V^{-1}_{b} \times R^{b}_{-_2} \\
&= \left\{ a \in A \,\middle|\, \int^{b \in B} V^{\star}_{b} \times R^{b}_{a} = \text{true} \right\} \\
&= \left\{ a \in A \,\middle|\, \begin{array}{l} \text{there exists } b \in B \text{ such that the} \\ \text{following conditions hold:} \\ \quad 1.\ \text{We have } V^{\star}_{b} = \text{true} \\ \quad 2.\ \text{We have } R^{b}_{a} = \text{true} \end{array} \right\} \\
&= \left\{ a \in A \,\middle|\, \begin{array}{l} \text{there exists } b \in B \text{ such that the} \\ \text{following conditions hold:} \\ \quad 1.\ \text{We have } b \in V \\ \quad 2.\ \text{We have } b \in R(a) \end{array} \right\} \\
&= \{ a \in A \mid \text{there exists } b \in V \text{ such that } b \in R(a) \} \\
&= \{ a \in A \mid R(a) \cap V \neq \emptyset \} \\
&\stackrel{\text{def}}{=} R^{-1}(V)
\end{aligned}
$$

This finishes the proof.

Proposition 6.4.3.4 ► Properties of Weak Inverse Image Functions

Let $R \colon A \nrightarrow B$ be a relation.

1. *Functoriality.* The assignment $V \mapsto R^{-1}(V)$ defines a functor

   $$R^{-1} \colon (\mathcal{P}(B), \subset) \to (\mathcal{P}(A), \subset)$$

   where

   - *Action on Objects.* For each $V \in \mathcal{P}(B)$, we have

     $$[R^{-1}](V) \stackrel{\text{def}}{=} R^{-1}(V).$$

   - *Action on Morphisms.* For each $U, V \in \mathcal{P}(B)$:

- If $U \subset V$, then $R^{-1}(U) \subset R^{-1}(V)$.

2. *Adjointness.* We have an adjunction

$$\left(R^{-1} \dashv R_!\right)\colon \quad \begin{tikzcd} \mathcal{P}(B) \arrow[r, bend left, "R^{-1}"] \arrow[r, phantom, "\perp"] & \mathcal{P}(A) \arrow[l, bend left, "R_!"] \end{tikzcd},$$

witnessed by a bijections of sets

$$\mathrm{Hom}_{\mathcal{P}(A)}\left(R^{-1}(U), V\right) \cong \mathrm{Hom}_{\mathcal{P}(A)}(U, R_!(V)),$$

natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$, i.e. such that:

($\star$) The following conditions are equivalent:
- We have $R^{-1}(U) \subset V$.
- We have $U \subset R_!(V)$.

3. *Preservation of Colimits.* We have an equality of sets

$$R^{-1}\left(\bigcup_{i \in I} U_i\right) = \bigcup_{i \in I} R^{-1}(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(B)^{\times I}$. In particular, we have equalities

$$\begin{aligned} R^{-1}(U) \cup R^{-1}(V) &= R^{-1}(U \cup V), \\ R^{-1}(\emptyset) &= \emptyset, \end{aligned}$$

natural in $U, V \in \mathcal{P}(B)$.

4. *Oplax Preservation of Limits.* We have an inclusion of sets

$$R^{-1}\left(\bigcap_{i \in I} U_i\right) \subset \bigcap_{i \in I} R^{-1}(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(B)^{\times I}$. In particular, we have inclusions

$$\begin{aligned} R^{-1}(U \cap V) &\subset R^{-1}(U) \cap R^{-1}(V), \\ R^{-1}(A) &\subset B, \end{aligned}$$

natural in $U, V \in \mathcal{P}(B)$.

5. *Symmetric Strict Monoidality With Respect to Unions.* The direct image function of Item 1 has a symmetric strict monoidal structure

$$\left(R^{-1}, R^{-1,\otimes}, R^{-1,\otimes}_{\mathbb{1}}\right) \colon (\mathcal{P}(A), \cup, \emptyset) \to (\mathcal{P}(B), \cup, \emptyset),$$

being equipped with equalities

$$R^{-1,\otimes}_{U,V} \colon R^{-1}(U) \cup R^{-1}(V) \xrightarrow{=} R^{-1}(U \cup V),$$
$$R^{-1,\otimes}_{\mathbb{1}} \colon \emptyset \xrightarrow{=} \emptyset,$$

natural in $U, V \in \mathcal{P}(B)$.

6. *Symmetric Oplax Monoidality With Respect to Intersections.* The direct image function of Item 1 has a symmetric oplax monoidal structure

$$\left(R^{-1}, R^{-1,\otimes}, R^{-1,\otimes}_{\mathbb{1}}\right) \colon (\mathcal{P}(A), \cap, A) \to (\mathcal{P}(B), \cap, B),$$

being equipped with inclusions

$$R^{-1,\otimes}_{U,V} \colon R^{-1}(U \cap V) \subset R^{-1}(U) \cap R^{-1}(V),$$
$$R^{-1,\otimes}_{\mathbb{1}} \colon R^{-1}(A) \subset B,$$

natural in $U, V \in \mathcal{P}(B)$.

7. *Interaction With Strong Inverse Images I.* We have

$$R^{-1}(V) = A \setminus R_{-1}(B \setminus V)$$

for each $V \in \mathcal{P}(B)$.

8. *Interaction With Strong Inverse Images II.* Let $R \colon A \nrightarrow B$ be a relation from A to B.

   (a) If R is a total relation, then we have an inclusion of sets

   $$R_{-1}(V) \subset R^{-1}(V)$$

   natural in $V \in \mathcal{P}(B)$.

   (b) If R is total and functional, then the above inclusion is in fact an equality.

   (c) Conversely, if we have $R_{-1} = R^{-1}$, then R is total and functional.

Proof 6.4.3.5 ► Proof of Proposition 6.4.3.4

Item 1: Functoriality

Clear.

Item 2: Adjointness

This follows from ?? of ??.

Item 3: Preservation of Colimits

This follows from Item 2 and ?? of ??.

Item 4: Oplax Preservation of Limits

Omitted.

Item 5: Symmetric Strict Monoidality With Respect to Unions

This follows from Item 3.

Item 6: Symmetric Oplax Monoidality With Respect to Intersections

This follows from Item 4.

Item 7: Interaction With Strong Inverse Images I

This follows from Item 7 of Proposition 6.4.2.4.

Item 8: Interaction With Strong Inverse Images II

This was proved in Item 8 of Proposition 6.4.2.4.

Proposition 6.4.3.6 ► Properties of the Weak Inverse Image Function Operation

Let $R\colon A \nrightarrow B$ be a relation.

1. *Functionality I.* The assignment $R \mapsto R^{-1}$ defines a function

$$(-)^{-1}\colon \mathrm{Rel}(A,B) \to \mathrm{Sets}(\mathcal{P}(A),\mathcal{P}(B)).$$

2. *Functionality II.* The assignment $R \mapsto R^{-1}$ defines a function

$$(-)^{-1}\colon \mathrm{Rel}(A,B) \to \mathrm{Pos}((\mathcal{P}(A),\subset),(\mathcal{P}(B),\subset)).$$

3. *Interaction With Identities.* For each $A \in \mathrm{Obj}(\mathrm{Sets})$, we have[1]

$$(\chi_A)^{-1} = \mathrm{id}_{\mathcal{P}(A)}.$$

4. *Interaction With Composition.* For each pair of composable relations $R\colon A \nrightarrow B$ and $S\colon B \nrightarrow C$, we have[2]

$$(S \diamond R)^{-1} = R^{-1} \circ S^{-1}, \qquad \begin{tikzcd} \mathcal{P}(C) \arrow[r, "S^{-1}"] \arrow[dr, "(S\diamond R)^{-1}"'] & \mathcal{P}(B) \arrow[d, "R^{-1}"] \\ & \mathcal{P}(A). \end{tikzcd}$$

[1]That is, the postcomposition

$$(\chi_A)^{-1}\colon \mathrm{Rel}(\mathrm{pt}, A) \to \mathrm{Rel}(\mathrm{pt}, A)$$

is equal to $\mathrm{id}_{\mathrm{Rel}(\mathrm{pt},A)}$.

[2]That is, we have

$$(S \diamond R)^{-1} = R^{-1} \circ S^{-1}, \qquad \begin{tikzcd} \mathrm{Rel}(\mathrm{pt}, C) \arrow[r, "R^{-1}"] \arrow[dr, "(S\diamond R)^{-1}"'] & \mathrm{Rel}(\mathrm{pt}, B) \arrow[d, "S^{-1}"] \\ & \mathrm{Rel}(\mathrm{pt}, A). \end{tikzcd}$$

Proof 6.4.3.7 ► Proof of Proposition 6.4.3.6

Item 1: Functionality I

Clear.

Item 2: Functionality II

Clear.

Item 3: Interaction With Identities

This follows from Item 5 of Proposition 8.1.6.2.

Item 4: Interaction With Composition

This follows from Item 2 of Proposition 8.1.6.2. □

6.4.4 Direct Images With Compact Support

Let A and B be sets and let $R\colon A \nrightarrow B$ be a relation.

Definition 6.4.4.1 ▶ Direct Images With Compact Support

The **direct image with compact support function associated to** R is the function

$$R_! \colon \mathcal{P}(A) \to \mathcal{P}(B)$$

defined by[1,2]

$$\begin{aligned} R_!(U) &\overset{\text{def}}{=} \left\{ b \in B \,\middle|\, \begin{array}{l} \text{for each } a \in A \text{, if we have} \\ b \in R(a) \text{, then } a \in U \end{array} \right\} \\ &= \left\{ b \in B \,\middle|\, R^{-1}(b) \subset U \right\} \end{aligned}$$

for each $U \in \mathcal{P}(A)$.

[1] *Further Terminology:* The set $R_!(U)$ is called the **direct image with compact support of** U **by** R.

[2] We also have

$$R_!(U) = B \setminus R_*(A \setminus U);$$

see Item 7 of Proposition 6.4.4.4.

Remark 6.4.4.2 ▶ Unwinding Definition 6.4.4.1

Identifying subsets of B with relations from pt to B via Item 3 of Proposition 2.4.3.9, we see that the direct image with compact support function associated to R is equivalently the function

$$R_! \colon \underbrace{\mathcal{P}(A)}_{\cong \mathrm{Rel}(A,\mathrm{pt})} \to \underbrace{\mathcal{P}(B)}_{\cong \mathrm{Rel}(B,\mathrm{pt})}$$

defined by

$$R_!(U) \overset{\text{def}}{=} \mathrm{Ran}_R(U), \qquad \begin{tikzcd} & B \arrow[d, dashed, "\mathrm{Ran}_R(U)"] \\ A \arrow[ru, "R", "\shortmid" marking] \arrow[r, "U"', "\shortmid" marking] & \mathrm{pt}, \end{tikzcd}$$

being explicitly computed by

$$\begin{aligned} R^*(U) &\overset{\text{def}}{=} \mathrm{Ran}_R(U) \\ &\cong \int_{a \in A} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R_a^{-2}, U_a^{-1}\right), \end{aligned}$$

where we have used Proposition 6.2.3.1.

Proof 6.4.4.3 ► Proof of Remark 6.4.4.2

We have

$$\begin{aligned}
\mathrm{Ran}_R(V) &\cong \int_{a\in A} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^{-2}_a, U^{-1}_a\right)\\
&= \left\{b \in B \,\middle|\, \int_{a\in A} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^b_a, U^\star_a\right) = \mathsf{true}\right\}\\
&= \left\{b \in B \,\middle|\, \begin{array}{l}\text{for each } a \in A\text{, at least one of the}\\ \text{following conditions hold:}\\ \quad 1.\ \text{We have } R^b_a = \mathsf{false}\\ \quad 2.\ \text{The following conditions hold:}\\ \qquad \text{(a) We have } R^b_a = \mathsf{true}\\ \qquad \text{(b) We have } U^\star_a = \mathsf{true}\end{array}\right\}\\
&= \left\{b \in B \,\middle|\, \begin{array}{l}\text{for each } a \in A\text{, at least one of the}\\ \text{following conditions hold:}\\ \quad 1.\ \text{We have } b \notin R(A)\\ \quad 2.\ \text{The following conditions hold:}\\ \qquad \text{(a) We have } b \in R(a)\\ \qquad \text{(b) We have } a \in U\end{array}\right\}\\
&= \left\{b \in B \,\middle|\, \begin{array}{l}\text{for each } a \in A\text{, if we have}\\ b \in R(a)\text{, then } a \in U\end{array}\right\}\\
&= \left\{b \in B \,\middle|\, R^{-1}(b) \subset U\right\}\\
&\overset{\text{def}}{=} R^{-1}(U).
\end{aligned}$$

This finishes the proof. ∎

Proposition 6.4.4.4 ► Properties of Direct Images With Compact Support

Let $R\colon A \nrightarrow B$ be a relation.

1. *Functoriality.* The assignment $U \mapsto R_!(U)$ defines a functor

$$R_!\colon (\mathcal{P}(A), \subset) \to (\mathcal{P}(B), \subset)$$

   where

- *Action on Objects.* For each $U \in \mathcal{P}(A)$, we have
$$[R_!](U) \stackrel{\text{def}}{=} R_!(U).$$
- *Action on Morphisms.* For each $U, V \in \mathcal{P}(A)$:
  - If $U \subset V$, then $R_!(U) \subset R_!(V)$.

2. *Adjointness.* We have an adjunction
$$\left(R^{-1} \dashv R_!\right)\colon \quad \begin{tikzcd} \mathcal{P}(B) \arrow[r, bend left, "R^{-1}"] \arrow[r, phantom, "\perp"] & \mathcal{P}(A) \arrow[l, bend left, "R_!"] \end{tikzcd},$$
witnessed by a bijections of sets
$$\mathrm{Hom}_{\mathcal{P}(A)}\left(R^{-1}(U), V\right) \cong \mathrm{Hom}_{\mathcal{P}(A)}(U, R_!(V)),$$
natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$, i.e. such that:

   ($\star$) The following conditions are equivalent:
   - We have $R^{-1}(U) \subset V$.
   - We have $U \subset R_!(V)$.

3. *Lax Preservation of Colimits.* We have an inclusion of sets
$$\bigcup_{i \in I} R_!(U_i) \subset R_!\left(\bigcup_{i \in I} U_i\right),$$
natural in $\{U_i\}_{i \in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have inclusions
$$R_!(U) \cup R_!(V) \subset R_!(U \cup V),$$
$$\emptyset \subset R_!(\emptyset),$$
natural in $U, V \in \mathcal{P}(A)$.

4. *Preservation of Limits.* We have an equality of sets
$$R_!\left(\bigcap_{i \in I} U_i\right) = \bigcap_{i \in I} R_!(U_i),$$
natural in $\{U_i\}_{i \in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have equalities
$$R_!(U \cap V) = R_!(U) \cap R_!(V),$$
$$R_!(A) = B,$$
natural in $U, V \in \mathcal{P}(A)$.

5. *Symmetric Lax Monoidality With Respect to Unions.* The direct image with compact support function of Item 1 has a symmetric lax monoidal structure

$$\left(R_!, R_!^{\otimes}, R_{!|\mathbb{1}}^{\otimes}\right) : (\mathcal{P}(A), \cup, \emptyset) \to (\mathcal{P}(B), \cup, \emptyset),$$

being equipped with inclusions

$$R_{!|U,V}^{\otimes} : R_!(U) \cup R_!(V) \subset R_!(U \cup V),$$
$$R_{!|\mathbb{1}}^{\otimes} : \emptyset \subset R_!(\emptyset),$$

natural in $U, V \in \mathcal{P}(A)$.

6. *Symmetric Strict Monoidality With Respect to Intersections.* The direct image function of Item 1 has a symmetric strict monoidal structure

$$\left(R_!, R_!^{\otimes}, R_{!|\mathbb{1}}^{\otimes}\right) : (\mathcal{P}(A), \cap, A) \to (\mathcal{P}(B), \cap, B),$$

being equipped with equalities

$$R_{!|U,V}^{\otimes} : R_!(U \cap V) \xrightarrow{=} R_!(U) \cap R_!(V),$$
$$R_{!|\mathbb{1}}^{\otimes} : R_!(A) \xrightarrow{=} B,$$

natural in $U, V \in \mathcal{P}(A)$.

7. *Relation to Direct Images.* We have

$$R_!(U) = B \setminus R_*(A \setminus U)$$

for each $U \in \mathcal{P}(A)$.

Proof 6.4.4.5 ► Proof of Proposition 6.4.4.4

Item 1: Functoriality

Clear.

Item 2: Adjointness

This follows from **??** of **??**.

Item 3: Lax Preservation of Colimits

Omitted.

Item 4: Preservation of Limits

This follows from Item 2 and ?? of ??.

Item 5: Symmetric Lax Monoidality With Respect to Unions

This follows from Item 3.

Item 6: Symmetric Strict Monoidality With Respect to Intersections

This follows from Item 4.

Item 7: Relation to Direct Images

This follows from Item 7 of Proposition 6.4.1.3. Alternatively, we may prove it directly as follows, with the proof proceeding in the same way as in the case of functions (Item 9 of Proposition 2.4.6.6).

We claim that $R_!(U) = B \setminus R_*(A \setminus U)$:

- *The First Implication.* We claim that

$$R_!(U) \subset B \setminus R_*(A \setminus U).$$

  Let $b \in R_!(U)$. We need to show that $b \notin R_*(A \setminus U)$, i.e. that there is no $a \in A \setminus U$ such that $b \in R(a)$.

  This is indeed the case, as otherwise we would have $a \in R^{-1}(b)$ and $a \notin U$, contradicting $R^{-1}(b) \subset U$ (which holds since $b \in R_!(U)$).

  Thus $b \in B \setminus R_*(A \setminus U)$.

- *The Second Implication.* We claim that

$$B \setminus R_*(A \setminus U) \subset R_!(U).$$

  Let $b \in B \setminus R_*(A \setminus U)$. We need to show that $b \in R_!(U)$, i.e. that $R^{-1}(b) \subset U$.

  Since $b \notin R_*(A \setminus U)$, there exists no $a \in A \setminus U$ such that $b \in R(a)$, and hence $R^{-1}(b) \subset U$.

  Thus $b \in R_!(U)$.

This finishes the proof. □

Proposition 6.4.4.6 ▶ Properties of the Direct Image With Compact Support Function Operation

Let $R\colon A \nrightarrow B$ be a relation.

1. *Functionality I*. The assignment $R \mapsto R_!$ defines a function
$$(-)_!\colon \mathsf{Sets}(A, B) \to \mathsf{Sets}(\mathcal{P}(A), \mathcal{P}(B)).$$
2. *Functionality II*. The assignment $R \mapsto R_!$ defines a function
$$(-)_!\colon \mathsf{Sets}(A, B) \to \mathsf{Hom}_{\mathsf{Pos}}((\mathcal{P}(A), \subset), (\mathcal{P}(B), \subset)).$$
3. *Interaction With Identities*. For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have
$$(\mathrm{id}_A)_! = \mathrm{id}_{\mathcal{P}(A)}.$$
4. *Interaction With Composition*. For each pair of composable relations $R\colon A \nrightarrow B$ and $S\colon B \nrightarrow C$, we have
$$(S \diamond R)_! = S_! \circ R_!, \qquad \begin{tikzcd} \mathcal{P}(A) \arrow[r, "R_!"] \arrow[dr, "(S\diamond R)_!"'] & \mathcal{P}(B) \arrow[d, "S_!"] \\ & \mathcal{P}(C). \end{tikzcd}$$

Proof 6.4.4.7 ▶ Proof of Proposition 6.4.4.6

Item 1: Functionality I

Clear.

Item 2: Functionality II

Clear.

Item 3: Interaction With Identities

Indeed, we have
$$\begin{aligned}(\chi_A)_!(U) &\stackrel{\text{def}}{=} \left\{a \in A \,\middle|\, \chi_A^{-1}(a) \subset U\right\} \\ &\stackrel{\text{def}}{=} \{a \in A \mid \{a\} \subset U\} \\ &= U\end{aligned}$$
for each $U \in \mathcal{P}(A)$. Thus $(\chi_A)_! = \mathrm{id}_{\mathcal{P}(A)}$.

Item 4: Interaction With Composition

Indeed, we have

$$\begin{aligned}(S \diamond R)_!(U) &\overset{\text{def}}{=} \left\{c \in C \,\middle|\, [S \diamond R]^{-1}(c) \subset U\right\}\\ &\overset{\text{def}}{=} \left\{c \in C \,\middle|\, S^{-1}\left(R^{-1}(c)\right) \subset U\right\}\\ &= \left\{c \in C \,\middle|\, R^{-1}(c) \subset S_!(U)\right\}\\ &\overset{\text{def}}{=} R_!(S_!(U))\\ &\overset{\text{def}}{=} [R_! \circ S_!](U)\end{aligned}$$

for each $U \in \mathcal{P}(C)$, where we used Item 2 of Proposition 6.4.4.4, which implies that the conditions

- We have $S^{-1}\left(R^{-1}(c)\right) \subset U$.
- We have $R^{-1}(c) \subset S_!(U)$.

are equivalent. Thus $(S \diamond R)_! = S_! \circ R_!$.

6.4.5 Functoriality of Powersets

Proposition 6.4.5.1 ► Functoriality of Powersets I

The assignment $X \mapsto \mathcal{P}(X)$ defines functors[1]

$$\begin{aligned}\mathcal{P}_* &\colon \mathsf{Rel} \to \mathsf{Sets},\\ \mathcal{P}_{-1} &\colon \mathsf{Rel}^{\mathsf{op}} \to \mathsf{Sets},\\ \mathcal{P}^{-1} &\colon \mathsf{Rel}^{\mathsf{op}} \to \mathsf{Sets},\\ \mathcal{P}_! &\colon \mathsf{Rel} \to \mathsf{Sets}\end{aligned}$$

where

- *Action on Objects*. For each $A \in \mathrm{Obj}(\mathsf{Rel})$, we have

$$\begin{aligned}\mathcal{P}_*(A) &\overset{\text{def}}{=} \mathcal{P}(A),\\ \mathcal{P}_{-1}(A) &\overset{\text{def}}{=} \mathcal{P}(A),\\ \mathcal{P}^{-1}(A) &\overset{\text{def}}{=} \mathcal{P}(A),\\ \mathcal{P}_!(A) &\overset{\text{def}}{=} \mathcal{P}(A).\end{aligned}$$

- *Action on Morphisms.* For each morphism $R \colon A \nrightarrow B$ of Rel, the images

$$\begin{aligned}\mathcal{P}_*(R) &\colon \mathcal{P}(A) \to \mathcal{P}(B),\\ \mathcal{P}_{-1}(R) &\colon \mathcal{P}(B) \to \mathcal{P}(A),\\ \mathcal{P}^{-1}(R) &\colon \mathcal{P}(B) \to \mathcal{P}(A),\\ \mathcal{P}_!(R) &\colon \mathcal{P}(A) \to \mathcal{P}(B)\end{aligned}$$

of R by $\mathcal{P}_*$, $\mathcal{P}_{-1}$, $\mathcal{P}^{-1}$, and $\mathcal{P}_!$ are defined by

$$\begin{aligned}\mathcal{P}_*(R) &\overset{\text{def}}{=} R_*,\\ \mathcal{P}_{-1}(R) &\overset{\text{def}}{=} R_{-1},\\ \mathcal{P}^{-1}(R) &\overset{\text{def}}{=} R^{-1},\\ \mathcal{P}_!(R) &\overset{\text{def}}{=} R_!,\end{aligned}$$

as in Definitions 6.4.1.1, 6.4.2.1, 6.4.3.1 and 6.4.4.1.

[1]The functor $\mathcal{P}_* \colon \mathsf{Rel} \to \mathsf{Sets}$ admits a left adjoint; see Item 3 of Proposition 6.3.1.2.

Proof 6.4.5.2 ► Proof of Proposition 6.4.5.1

This follows from Items 3 and 4 of Proposition 6.4.1.5, Items 3 and 4 of Proposition 6.4.2.6, Items 3 and 4 of Proposition 6.4.3.6, and Items 3 and 4 of Proposition 6.4.4.6. □

6.4.6 Functoriality of Powersets: Relations on Powersets

Let A and B be sets and let $R \colon A \nrightarrow B$ be a relation.

Definition 6.4.6.1 ► The Relation on Powersets Associated to a Relation

The **relation on powersets associated to** R is the relation

$$\mathcal{P}(R) \colon \mathcal{P}(A) \nrightarrow \mathcal{P}(B)$$

defined by[1]

$$\mathcal{P}(R)^V_U \overset{\text{def}}{=} \mathbf{Rel}\big(\chi_{\mathrm{pt}}, V \diamond R \diamond U\big)$$

for each $U \in \mathcal{P}(A)$ and each $V \in \mathcal{P}(B)$.

[1]Illustration:

$$\begin{tikzcd} \mathrm{pt} \arrow[r, "U"'] \arrow[rrr, bend left, "\chi_{\mathrm{pt}}"] & A \arrow[r, "R"'] & B \arrow[r, "V"'] & \mathrm{pt}. \end{tikzcd}$$

Remark 6.4.6.2 ▶ Unwinding Definition 6.4.6.1

In detail, we have $U \sim_{\mathcal{P}(R)} V$ iff the following equivalent conditions hold:

- We have $\chi_{\mathrm{pt}} \subset V \diamond R \diamond U$.
- We have $(V \diamond R \diamond U)^{\star}_{\star} = \mathrm{true}$, i.e. we have

$$\int^{a \in A} \int^{b \in B} V^{\star}_{b} \times R^{b}_{a} \times U^{a}_{\star} = \mathrm{true}.$$

- There exists some $a \in A$ and some $b \in B$ such that:
  - We have $U^{a}_{\star} = \mathrm{true}$.
  - We have $R^{b}_{a} = \mathrm{true}$.
  - We have $V^{\star}_{b} = \mathrm{true}$.
- There exists some $a \in A$ and some $b \in B$ such that:
  - We have $a \in U$.
  - We have $a \sim_R b$.
  - We have $b \in V$.

Proposition 6.4.6.3 ▶ Functoriality of Powersets II

The assignment $R \mapsto \mathcal{P}(R)$ defines a functor

$$\mathcal{P} \colon \mathsf{Rel} \longrightarrow \mathsf{Rel}.$$

Proof 6.4.6.4 ▶ Proof of Proposition 6.4.6.3

Omitted. □

Appendices

6.A Other Chapters

Sets

1. Sets
2. Constructions With Sets
3. Pointed Sets
4. Tensor Products of Pointed Sets

Relations

5. Relations
6. Constructions With Relations
7. Equivalence Relations and Apartness Relations

Category Theory

8. Categories

Bicategories

9. Types of Morphisms in Bicategories

Chapter 7

Equivalence Relations and Apartness Relations

This chapter contains some material about reflexive, symmetric, transitive, equivalence, and apartness relations.

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7.1 Reflexive Relations

7.1.1 Foundations

Let A be a set.

Definition 7.1.1.1 ► Reflexive Relations

A **reflexive relation** is equivalently:[1]

- An $\mathbb{E}_0$-monoid in $(\mathrm{N}_\bullet(\mathbf{Rel}(A, A)), \chi_A)$.
- A pointed object in $(\mathbf{Rel}(A, A), \chi_A)$.

[1]Note that since $\mathbf{Rel}(A, A)$ is posetal, reflexivity is a property of a relation, rather than extra structure.

Remark 7.1.1.2 ► Unwinding Definition 7.1.1.1

In detail, a relation R on A is **reflexive** if we have an inclusion

$$\eta_R : \chi_A \subset R$$

of relations in $\mathbf{Rel}(A, A)$, i.e. if, for each $a \in A$, we have $a \sim_R a$.

Definition 7.1.1.3 ► The Po/Set of Reflexive Relations on a Set

Let A be a set.

1. The **set of reflexive relations on** A is the subset $\mathrm{Rel}^{\mathrm{refl}}(A, A)$ of $\mathrm{Rel}(A, A)$ spanned by the reflexive relations.
2. The **poset of relations on** A is is the subposet $\mathbf{Rel}^{\mathrm{refl}}(A, A)$ of $\mathbf{Rel}(A, A)$ spanned by the reflexive relations.

Proposition 7.1.1.4 ► Properties of Reflexive Relations

Let R and S be relations on A.

1. *Interaction With Inverses.* If R is reflexive, then so is $R^\dagger$.
2. *Interaction With Composition.* If R and S are reflexive, then so is $S \diamond R$.

Proof 7.1.1.5 ► Proof of Proposition 7.1.1.4

Item 1: Interaction With Inverses

Clear.

Item 2: Interaction With Composition

Clear. □

7.1.2 The Reflexive Closure of a Relation

Let R be a relation on A.

The **reflexive closure** of $\sim_R$ is the relation $\sim_R^{\text{refl}}$[1] satisfying the following universal property:[2]

(★) Given another reflexive relation $\sim_S$ on A such that $R \subset S$, there exists an inclusion $\sim_R^{\text{refl}} \subset \sim_S$.

[1] *Further Notation:* Also written R^{refl}.
[2] *Slogan:* The reflexive closure of R is the smallest reflexive relation containing R.

Construction 7.1.2.2 ▶ The Reflexive Closure of a Relation

Concretely, $\sim_R^{\text{refl}}$ is the free pointed object on R in $(\mathbf{Rel}(A, A), \chi_A)$[1], being given by

$$\begin{aligned} R^{\text{refl}} &\overset{\text{def}}{=} R \coprod^{\mathbf{Rel}(A,A)} \Delta_A \\ &= R \cup \Delta_A \\ &= \{(a, b) \in A \times A \mid \text{we have } a \sim_R b \text{ or } a = b\}. \end{aligned}$$

[1] Or, equivalently, the free $\mathbb{E}_0$-monoid on R in $(\mathsf{N}_\bullet(\mathbf{Rel}(A, A)), \chi_A)$.

Proof 7.1.2.3 ▶ Proof of Construction 7.1.2.2

Clear. □

Proposition 7.1.2.4 ▶ Properties of the Reflexive Closure of a Relation

Let R be a relation on A.

1. *Adjointness.* We have an adjunction

$$\Big((-)^{\text{refl}} \dashv 忘\Big)\colon \quad \begin{tikzcd} \mathbf{Rel}(A,A) \arrow[r, bend left, "(-)^{\text{refl}}"] \arrow[r, phantom, "\perp"] & \mathbf{Rel}^{\text{refl}}(A,A), \arrow[l, bend left, "忘"] \end{tikzcd}$$

witnessed by a bijection of sets

$$\mathbf{Rel}^{\text{refl}}\Big(R^{\text{refl}}, S\Big) \cong \mathbf{Rel}(R, S),$$

natural in $R \in \mathrm{Obj}\Big(\mathbf{Rel}^{\text{refl}}(A, A)\Big)$ and $S \in \mathrm{Obj}(\mathbf{Rel}(A, A))$.

2. *The Reflexive Closure of a Reflexive Relation.* If R is reflexive, then $R^{\text{refl}} = R$.

3. *Idempotency.* We have

$$\Big(R^{\text{refl}}\Big)^{\text{refl}} = R^{\text{refl}}.$$

4. *Interaction With Inverses.* We have

$$\left(R^{\dagger}\right)^{\mathrm{refl}} = \left(R^{\mathrm{refl}}\right)^{\dagger}, \qquad \begin{tikzcd} \mathrm{Rel}(A,A) \arrow[r, "(-)^{\mathrm{refl}}"] \arrow[d, "(-)^{\dagger}"'] & \mathrm{Rel}(A,A) \arrow[d, "(-)^{\dagger}"] \\ \mathrm{Rel}(A,A) \arrow[r, "(-)^{\mathrm{refl}}"'] & \mathrm{Rel}(A,A). \end{tikzcd}$$

5. *Interaction With Composition.* We have

$$(S \diamond R)^{\mathrm{refl}} = S^{\mathrm{refl}} \diamond R^{\mathrm{refl}}, \qquad \begin{tikzcd} \mathrm{Rel}(A,A) \times \mathrm{Rel}(A,A) \arrow[r, "\diamond"] \arrow[d, "(-)^{\mathrm{refl}} \times (-)^{\mathrm{refl}}"'] & \mathrm{Rel}(A,A) \arrow[d, "(-)^{\mathrm{refl}}"] \\ \mathrm{Rel}(A,A) \times \mathrm{Rel}(A,A) \arrow[r, "\diamond"'] & \mathrm{Rel}(A,A). \end{tikzcd}$$

Proof 7.1.2.5 ► Proof of Proposition 7.1.2.4

Item 1: Adjointness

This is a rephrasing of the universal property of the reflexive closure of a relation, stated in Definition 7.1.2.1.

Item 2: The Reflexive Closure of a Reflexive Relation

Clear.

Item 3: Idempotency

This follows from Item 2.

Item 4: Interaction With Inverses

Clear.

Item 5: Interaction With Composition

This follows from Item 2 of Proposition 7.1.1.4.

7.2 Symmetric Relations

7.2.1 Foundations

Let A be a set.

Definition 7.2.1.1 ▶ Symmetric Relations

A relation R on A is **symmetric** if we have $R^{\dagger} = R$.

Remark 7.2.1.2 ▶ Unwinding Definition 7.2.1.1

In detail, a relation R is symmetric if it satisfies the following condition:

($\star$) For each $a, b \in A$, if $a \sim_R b$, then $b \sim_R a$.

Definition 7.2.1.3 ▶ The Po/Set of Symmetric Relations on a Set

Let A be a set.

1. The **set of symmetric relations on** A is the subset $\mathrm{Rel}^{\mathrm{symm}}(A, A)$ of $\mathrm{Rel}(A, A)$ spanned by the symmetric relations.
2. The **poset of relations on** A is is the subposet $\mathbf{Rel}^{\mathrm{symm}}(A, A)$ of $\mathbf{Rel}(A, A)$ spanned by the symmetric relations.

Proposition 7.2.1.4 ▶ Properties of Symmetric Relations

Let R and S be relations on A.

1. *Interaction With Inverses.* If R is symmetric, then so is $R^{\dagger}$.
2. *Interaction With Composition.* If R and S are symmetric, then so is $S \diamond R$.

Proof 7.2.1.5 ▶ Proof of Proposition 7.2.1.4

Item 1: Interaction With Inverses

Clear.

Item 2: Interaction With Composition

Clear. ▭

7.2.2 The Symmetric Closure of a Relation

Let R be a relation on A.

Definition 7.2.2.1 ▶ The Symmetric Closure of a Relation

The **symmetric closure** of $\sim_R$ is the relation $\sim_R^{\mathrm{symm}}$[1] satisfying the following universal property:[2]

(★) Given another symmetric relation $\sim_S$ on A such that $R \subset S$, there exists an inclusion $\sim_R^{\mathrm{symm}} \subset \sim_S$.

[1] *Further Notation:* Also written R^{symm}.

[2] *Slogan:* The symmetric closure of R is the smallest symmetric relation containing R.

Construction 7.2.2.2 ▶ The Symmetric Closure of a Relation

Concretely, $\sim_R^{\mathrm{symm}}$ is the symmetric relation on A defined by

$$\begin{aligned} R^{\mathrm{symm}} &\overset{\mathrm{def}}{=} R \cup R^{\dagger} \\ &= \{(a,b) \in A \times A \mid \text{we have } a \sim_R b \text{ or } b \sim_R a\}. \end{aligned}$$

Proof 7.2.2.3 ▶ Proof of Construction 7.2.2.2

Clear. □

Proposition 7.2.2.4 ▶ Properties of the Symmetric Closure of a Relation

Let R be a relation on A.

1. *Adjointness.* We have an adjunction

$$\left((-)^{\mathrm{symm}} \dashv 忘\right) \colon \quad \begin{tikzcd} \mathbf{Rel}(A,A) \arrow[r, bend left, "(-)^{\mathrm{symm}}"] \arrow[r, phantom, "\perp"] & \mathbf{Rel}^{\mathrm{symm}}(A,A), \arrow[l, bend left, "忘"] \end{tikzcd}$$

   witnessed by a bijection of sets

$$\mathbf{Rel}^{\mathrm{symm}}(R^{\mathrm{symm}}, S) \cong \mathbf{Rel}(R, S),$$

   natural in $R \in \mathrm{Obj}(\mathbf{Rel}^{\mathrm{symm}}(A,A))$ and $S \in \mathrm{Obj}(\mathbf{Rel}(A,A))$.

2. *The Symmetric Closure of a Symmetric Relation.* If R is symmetric, then $R^{\mathrm{symm}} = R$.

3. *Idempotency.* We have

$$(R^{\mathrm{symm}})^{\mathrm{symm}} = R^{\mathrm{symm}}.$$

4. *Interaction With Inverses.* We have

$$\left(R^{\dagger}\right)^{\mathsf{symm}} = \left(R^{\mathsf{symm}}\right)^{\dagger},\qquad \begin{tikzcd} \mathsf{Rel}(A,A) \arrow[r, "(-)^{\mathsf{symm}}"] \arrow[d, "(-)^{\dagger}"'] & \mathsf{Rel}(A,A) \arrow[d, "(-)^{\dagger}"] \\ \mathsf{Rel}(A,A) \arrow[r, "(-)^{\mathsf{symm}}"'] & \mathsf{Rel}(A,A). \end{tikzcd}$$

5. *Interaction With Composition.* We have

$$(S \diamond R)^{\mathsf{symm}} = S^{\mathsf{symm}} \diamond R^{\mathsf{symm}},\qquad \begin{tikzcd} \mathsf{Rel}(A,A) \times \mathsf{Rel}(A,A) \arrow[r, "\diamond"] \arrow[d, "(-)^{\mathsf{symm}} \times (-)^{\mathsf{symm}}"'] & \mathsf{Rel}(A,A) \arrow[d, "(-)^{\mathsf{symm}}"] \\ \mathsf{Rel}(A,A) \times \mathsf{Rel}(A,A) \arrow[r, "\diamond"'] & \mathsf{Rel}(A,A). \end{tikzcd}$$

PROOF 7.2.2.5 ▶ PROOF OF PROPOSITION 7.2.2.4

Item 1: Adjointness

This is a rephrasing of the universal property of the symmetric closure of a relation, stated in Definition 7.2.2.1.

Item 2: The Symmetric Closure of a Symmetric Relation

Clear.

Item 3: Idempotency

This follows from Item 2.

Item 4: Interaction With Inverses

Clear.

Item 5: Interaction With Composition

This follows from Item 2 of Proposition 7.2.1.4. □

7.3 Transitive Relations

7.3.1 Foundations

Let A be a set.

Definition 7.3.1.1 ► Transitive Relations

A **transitive relation** is equivalently:[1]

- A non-unital $\mathbb{E}_1$-monoid in $(\mathsf{N}_\bullet(\mathbf{Rel}(A,A)), \diamond)$.
- A non-unital monoid in $(\mathbf{Rel}(A,A), \diamond)$.

[1]Note that since $\mathbf{Rel}(A,A)$ is posetal, transitivity is a property of a relation, rather than extra structure.

Remark 7.3.1.2 ► Unwinding Definition 7.3.1.1

In detail, a relation R on A is **transitive** if we have an inclusion

$$\mu_R : R \diamond R \subset R$$

of relations in $\mathbf{Rel}(A,A)$, i.e. if, for each $a, c \in A$, the following condition is satisfied:

($\star$) If there exists some $b \in A$ such that $a \sim_R b$ and $b \sim_R c$, then $a \sim_R c$.

Definition 7.3.1.3 ► The Po/Set of Transitive Relations on a Set

Let A be a set.

1. The **set of transitive relations from** A **to** B is the subset $\mathrm{Rel}^{\mathrm{trans}}(A)$ of $\mathrm{Rel}(A,A)$ spanned by the transitive relations.
2. The **poset of relations from** A **to** B is is the subposet $\mathbf{Rel}^{\mathrm{trans}}(A)$ of $\mathbf{Rel}(A,A)$ spanned by the transitive relations.

Proposition 7.3.1.4 ► Properties of Transitive Relations

Let R and S be relations on A.

1. *Interaction With Inverses.* If R is transitive, then so is $R^\dagger$.
2. *Interaction With Composition.* If R and S are transitive, then $S \diamond R$ ***may fail to be transitive***.

Proof 7.3.1.5 ► Proof of Proposition 7.3.1.4

Item 1: Interaction With Inverses

Clear.

Item 2: Interaction With Composition

See [MSE 2096272].[1]

[1] *Intuition:* Transitivity for R and S fails to imply that of $S \diamond R$ because the composition operation for relations intertwines R and S in an incompatible way:

1. If $a \sim_{S\diamond R} c$ and $c \sim_{S\diamond r} e$, then:
   (a) There is some $b \in A$ such that:
      i. $a \sim_R b$;
      ii. $b \sim_S c$;
   (b) There is some $d \in A$ such that:
      i. $c \sim_R d$;
      ii. $d \sim_S e$.

7.3.2 The Transitive Closure of a Relation

Let R be a relation on A.

Definition 7.3.2.1 ▶ The Transitive Closure of a Relation

The **transitive closure** of $\sim_R$ is the relation $\sim_R^{\text{trans}}$[1] satisfying the following universal property:[2]

($\star$) Given another transitive relation $\sim_S$ on A such that $R \subset S$, there exists an inclusion $\sim_R^{\text{trans}} \subset \sim_S$.

[1] *Further Notation:* Also written R^{trans}.

[2] *Slogan:* The transitive closure of R is the smallest transitive relation containing R.

Construction 7.3.2.2 ▶ The Transitive Closure of a Relation

Concretely, $\sim_R^{\text{trans}}$ is the free non-unital monoid on R in $(\mathbf{Rel}(A, A), \diamond)$[1], being given by

$$\begin{aligned} R^{\text{trans}} &\stackrel{\text{def}}{=} \coprod_{n=1}^{\infty} R^{\diamond n} \\ &\stackrel{\text{def}}{=} \bigcup_{n=1}^{\infty} R^{\diamond n} \\ &\stackrel{\text{def}}{=} \left\{ (a,b) \in A \times B \;\middle|\; \begin{array}{l} \text{there exists some } (x_1, \ldots, x_n) \in R^{\times n} \\ \text{such that } a \sim_R x_1 \sim_R \cdots \sim_R x_n \sim_R b \end{array} \right\}. \end{aligned}$$

[1] Or, equivalently, the free non-unital $\mathbb{E}_1$-monoid on R in $(\mathsf{N}_\bullet(\mathbf{Rel}(A, A)), \diamond)$.

Proof 7.3.2.3 ▶ Proof of Construction 7.3.2.2

Clear. ∎

Proposition 7.3.2.4 ▶ Properties of the Transitive Closure of a Relation

Let R be a relation on A.

1. *Adjointness.* We have an adjunction

$$\left((-)^{\text{trans}} \dashv 忘\right) \colon \quad \begin{tikzcd} \mathbf{Rel}(A,A) \arrow[r, bend left, "(-)^{\text{trans}}"] \arrow[r, phantom, "\perp"] & \mathbf{Rel}^{\text{trans}}(A,A), \arrow[l, bend left, "忘"] \end{tikzcd}$$

   witnessed by a bijection of sets

$$\mathbf{Rel}^{\text{trans}}\left(R^{\text{trans}}, S\right) \cong \mathbf{Rel}(R, S),$$

   natural in $R \in \mathrm{Obj}\left(\mathbf{Rel}^{\text{trans}}(A,A)\right)$ and $S \in \mathrm{Obj}(\mathbf{Rel}(A,B))$.

2. *The Transitive Closure of a Transitive Relation.* If R is transitive, then $R^{\text{trans}} = R$.

3. *Idempotency.* We have

$$\left(R^{\text{trans}}\right)^{\text{trans}} = R^{\text{trans}}.$$

4. *Interaction With Inverses.* We have

$$\left(R^{\dagger}\right)^{\text{trans}} = \left(R^{\text{trans}}\right)^{\dagger}, \qquad \begin{tikzcd} \mathrm{Rel}(A,A) \arrow[r, "(-)^{\text{trans}}"] \arrow[d, "(-)^{\dagger}"'] & \mathrm{Rel}(A,A) \arrow[d, "(-)^{\dagger}"] \\ \mathrm{Rel}(A,A) \arrow[r, "(-)^{\text{trans}}"'] & \mathrm{Rel}(A,A). \end{tikzcd}$$

5. *Interaction With Composition.* We have

$$(S \diamond R)^{\text{trans}} \overset{\text{poss.}}{\neq} S^{\text{trans}} \diamond R^{\text{trans}}, \qquad \begin{tikzcd} \mathrm{Rel}(A,A) \times \mathrm{Rel}(A,A) \arrow[r, "\diamond"] \arrow[d, "(-)^{\text{trans}} \times (-)^{\text{trans}}"'] \arrow[dr, phantom, "\times"] & \mathrm{Rel}(A,A) \arrow[d, "(-)^{\text{trans}}"] \\ \mathrm{Rel}(A,A) \times \mathrm{Rel}(A,A) \arrow[r, "\diamond"'] & \mathrm{Rel}(A,A). \end{tikzcd}$$

Proof 7.3.2.5 ► Proof of Proposition 7.3.2.4

Item 1: Adjointness

This is a rephrasing of the universal property of the transitive closure of a relation, stated in Definition 7.3.2.1.

Item 2: The Transitive Closure of a Transitive Relation

Clear.

Item 3: Idempotency

This follows from Item 2.

Item 4: Interaction With Inverses

We have

$$\begin{aligned}\left(R^{\dagger}\right)^{\mathrm{trans}} &= \bigcup_{n=1}^{\infty}\left(R^{\dagger}\right)^{\diamond n} \\ &= \bigcup_{n=1}^{\infty}\left(R^{\diamond n}\right)^{\dagger} \\ &= \left(\bigcup_{n=1}^{\infty} R^{\diamond n}\right)^{\dagger} \\ &= \left(R^{\mathrm{trans}}\right)^{\dagger},\end{aligned}$$

where we have used, respectively:

1. Construction 7.3.2.2.
2. Item 4 of Proposition 6.3.12.3.
3. Item 1 of Proposition 6.3.6.2.
4. Construction 7.3.2.2.

Item 5: Interaction With Composition

This follows from Item 2 of Proposition 7.3.1.4.

7.4 Equivalence Relations

7.4.1 Foundations

Let A be a set.

Definition 7.4.1.1 ▶ Equivalence Relations

A relation R is an **equivalence relation** if it is reflexive, symmetric, and transitive.[1]

[1] *Further Terminology:* If instead R is just symmetric and transitive, then it is called a **partial equivalence relation**.

Example 7.4.1.2 ▶ The Kernel of a Function

The **kernel of a function** $f\colon A \to B$ is the equivalence relation $\sim_{\mathrm{Ker}(f)}$ on A obtained by declaring $a \sim_{\mathrm{Ker}(f)} b$ iff $f(a) = f(b)$.[1]

[1] The kernel $\mathrm{Ker}(f)\colon A \nrightarrow A$ of f is the underlying functor of the monad induced by the adjunction $\mathrm{Gr}(f) \dashv f^{-1}\colon A \rightleftarrows B$ in **Rel** of Item 2 of Proposition 6.3.1.2.

Definition 7.4.1.3 ▶ The Po/Set of Equivalence Relations on a Set

Let A and B be sets.

1. The **set of equivalence relations from A to B** is the subset $\mathrm{Rel}^{\mathrm{eq}}(A, B)$ of $\mathrm{Rel}(A, B)$ spanned by the equivalence relations.
2. The **poset of relations from A to B** is is the subposet $\mathbf{Rel}^{\mathrm{eq}}(A, B)$ of $\mathbf{Rel}(A, B)$ spanned by the equivalence relations.

7.4.2 The Equivalence Closure of a Relation

Let R be a relation on A.

Definition 7.4.2.1 ▶ The Equivalence Closure of a Relation

The **equivalence closure**[1] of $\sim_R$ is the relation $\sim_R^{\mathrm{eq}}$[2] satisfying the following universal property:[3]

(★) Given another equivalence relation $\sim_S$ on A such that $R \subset S$, there exists an inclusion $\sim_R^{\mathrm{eq}} \subset \sim_S$.

[1] *Further Terminology:* Also called the **equivalence relation associated to** $\sim_R$.
[2] *Further Notation:* Also written R^{eq}.
[3] *Slogan:* The equivalence closure of R is the smallest equivalence relation containing R.

Construction 7.4.2.2 ► The Equivalence Closure of a Relation

Concretely, $\sim_R^{\mathrm{eq}}$ is the equivalence relation on A defined by

$$\begin{aligned} R^{\mathrm{eq}} &\overset{\mathrm{def}}{=} \left(\left(R^{\mathrm{refl}}\right)^{\mathrm{symm}}\right)^{\mathrm{trans}} \\ &= \left((R^{\mathrm{symm}})^{\mathrm{trans}}\right)^{\mathrm{refl}} \\ &= \left\{ (a,b) \in A \times B \;\middle|\; \begin{array}{l} \text{there exists } (x_1, \ldots, x_n) \in R^{\times n} \text{ satisfying at} \\ \text{least one of the following conditions:} \\ \quad 1. \text{ The following conditions are satisfied:} \\ \qquad \text{(a) We have } a \sim_R x_1 \text{ or } x_1 \sim_R a; \\ \qquad \text{(b) We have } x_i \sim_R x_{i+1} \text{ or } x_{i+1} \sim_R x_i \\ \qquad\quad \text{for each } 1 \leq i \leq n-1; \\ \qquad \text{(c) We have } b \sim_R x_n \text{ or } x_n \sim_R b; \\ \quad 2. \text{ We have } a = b. \end{array} \right\}. \end{aligned}$$

Proof 7.4.2.3 ► Proof of Construction 7.4.2.2

From the universal properties of the reflexive, symmetric, and transitive closures of a relation (Definitions 7.1.2.1, 7.2.2.1 and 7.3.2.1), we see that it suffices to prove that:

1. The symmetric closure of a reflexive relation is still reflexive.
2. The transitive closure of a symmetric relation is still symmetric.

which are both clear. ∎

Proposition 7.4.2.4 ► Properties of Equivalence Relations

Let R be a relation on A.

1. *Adjointness.* We have an adjunction

$$\left((-)^{\mathrm{eq}} \dashv 忘\right) \colon \quad \begin{tikzcd} \mathbf{Rel}(A,B) \arrow[r, bend left, "(-)^{\mathrm{eq}}"] \arrow[r, phantom, "\perp"] & \mathbf{Rel}^{\mathrm{eq}}(A,B), \arrow[l, bend left, "忘"] \end{tikzcd}$$

   witnessed by a bijection of sets

$$\mathbf{Rel}^{\mathrm{eq}}(R^{\mathrm{eq}}, S) \cong \mathbf{Rel}(R, S),$$

natural in $R \in \mathrm{Obj}(\mathbf{Rel}^{\mathrm{eq}}(A, B))$ and $S \in \mathrm{Obj}(\mathbf{Rel}(A, B))$.

2. *The Equivalence Closure of an Equivalence Relation.* If R is an equivalence relation, then $R^{\mathrm{eq}} = R$.

3. *Idempotency.* We have

$$(R^{\mathrm{eq}})^{\mathrm{eq}} = R^{\mathrm{eq}}.$$

Proof 7.4.2.5 ▶ Proof of Proposition 7.4.2.4

Item 1: Adjointness

This is a rephrasing of the universal property of the equivalence closure of a relation, stated in Definition 7.4.2.1.

Item 2: The Equivalence Closure of an Equivalence Relation

Clear.

Item 3: Idempotency

This follows from Item 2.

7.5 Quotients by Equivalence Relations

7.5.1 Equivalence Classes

Let A be a set, let R be a relation on A, and let $a \in A$.

Definition 7.5.1.1 ▶ Equivalence Classes

The **equivalence class associated to** a is the set $[a]$ defined by

$$\begin{aligned} [a] &\stackrel{\text{def}}{=} \{x \in X \mid x \sim_R a\} \\ &= \{x \in X \mid a \sim_R x\}. && \text{(since } R \text{ is symmetric)} \end{aligned}$$

7.5.2 Quotients of Sets by Equivalence Relations

Let A be a set and let R be a relation on A.

Definition 7.5.2.1 ▶ Quotients of Sets by Equivalence Relations

The **quotient of** X **by** R is the set $X/{\sim_R}$ defined by

$$X/{\sim_R} \stackrel{\text{def}}{=} \{[a] \in \mathcal{P}(X) \mid a \in X\}.$$

Remark 7.5.2.2 ▶ Why Use "Equivalence" Relations for Quotient Sets

The reason we define quotient sets for equivalence relations only is that each of the properties of being an equivalence relation—reflexivity, symmetry, and transitivity—ensures that the equivalences classes $[a]$ of X under R are well-behaved:

- *Reflexivity.* If R is reflexive, then, for each $a \in X$, we have $a \in [a]$.
- *Symmetry.* The equivalence class $[a]$ of an element a of X is defined by

  $$[a] \stackrel{\text{def}}{=} \{x \in X \mid x \sim_R a\},$$

  but we could equally well define

  $$[a]' \stackrel{\text{def}}{=} \{x \in X \mid a \sim_R x\}$$

  instead. This is not a problem when R is symmetric, as we then have $[a] = [a]'$.[1]
- *Transitivity.* If R is transitive, then $[a]$ and $[b]$ are disjoint iff $a \nsim_R b$, and equal otherwise.

[1]When categorifying equivalence relations, one finds that $[a]$ and $[a]'$ correspond to presheaves and copresheaves; see ??.

Proposition 7.5.2.3 ▶ Properties of Quotient Sets

Let $f\colon X \to Y$ be a function and let R be a relation on X.

1. *As a Coequaliser.* We have an isomorphism of sets

   $$X/{\sim_R^{\text{eq}}} \cong \mathrm{CoEq}\left(R \hookrightarrow X \times X \overset{\mathrm{pr}_1}{\underset{\mathrm{pr}_2}{\rightrightarrows}} X\right),$$

   where $\sim_R^{\text{eq}}$ is the equivalence relation generated by $\sim_R$.

2. *As a Pushout.* We have an isomorphism of sets[1]

$$X/{\sim}_R^{\mathrm{eq}} \cong X \textstyle\coprod_{\mathrm{Eq}(\mathrm{pr}_1,\mathrm{pr}_2)} X, \qquad \begin{tikzcd} X/{\sim}_R^{\mathrm{eq}} & X \arrow[l] \\ X \arrow[u] & \mathrm{Eq}(\mathrm{pr}_1,\mathrm{pr}_2). \arrow[l] \arrow[u] \arrow[ul, phantom, "\ulcorner", very near end] \end{tikzcd}$$

   where ${\sim}_R^{\mathrm{eq}}$ is the equivalence relation generated by ${\sim}_R$.

3. *The First Isomorphism Theorem for Sets.* We have an isomorphism of sets[2,3]

$$X/{\sim}_{\mathrm{Ker}(f)} \cong \mathrm{Im}(f).$$

4. *Descending Functions to Quotient Sets, I.* Let R be an equivalence relation on X. The following conditions are equivalent:

   (a) There exists a map

$$\overline{f}\colon X/{\sim}_R \to Y$$

   making the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "q"'] & Y \\ X/{\sim}_R \arrow[ur, dotted, "\exists", "\overline{f}"'] \end{tikzcd}$$

   commute.

   (b) We have $R \subset \mathrm{Ker}(f)$.

   (c) For each $x, y \in X$, if $x \sim_R y$, then $f(x) = f(y)$.

5. *Descending Functions to Quotient Sets, II.* Let R be an equivalence relation on X. If the conditions of Item 4 hold, then $\overline{f}$ is the *unique* map making the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "q"'] & Y \\ X/{\sim}_R \arrow[ur, dashed, "\exists!", "\overline{f}"'] \end{tikzcd}$$

   commute.

6. *Descending Functions to Quotient Sets, III.* Let R be an equivalence relation on X. We have a bijection

$$\mathrm{Hom}_{\mathsf{Sets}}(X/\sim_R, Y) \cong \mathrm{Hom}^R_{\mathsf{Sets}}(X, Y),$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$, given by the assignment $f \mapsto \overline{f}$ of Items 4 and 5, where $\mathrm{Hom}^R_{\mathsf{Sets}}(X, Y)$ is the set defined by

$$\mathrm{Hom}^R_{\mathsf{Sets}}(X, Y) \overset{\text{def}}{=} \left\{ f \in \mathrm{Hom}_{\mathsf{Sets}}(X, Y) \;\middle|\; \begin{array}{l} \text{for each } x, y \in X, \\ \text{if } x \sim_R y, \text{ then} \\ f(x) = f(y) \end{array} \right\}.$$

7. *Descending Functions to Quotient Sets, IV.* Let R be an equivalence relation on X. If the conditions of Item 4 hold, then the following conditions are equivalent:

   (a) The map $\overline{f}$ is an injection.

   (b) We have $R = \mathrm{Ker}(f)$.

   (c) For each $x, y \in X$, we have $x \sim_R y$ iff $f(x) = f(y)$.

8. *Descending Functions to Quotient Sets, V.* Let R be an equivalence relation on X. If the conditions of Item 4 hold, then the following conditions are equivalent:

   (a) The map $f \colon X \to Y$ is surjective.

   (b) The map $\overline{f} \colon X/\sim_R \to Y$ is surjective.

9. *Descending Functions to Quotient Sets, VI.* Let R be a relation on X and let $\sim^{\mathrm{eq}}_R$ be the equivalence relation associated to R. The following conditions are equivalent:

   (a) The map f satisfies the equivalent conditions of Item 4:

      - There exists a map

        $$\overline{f} \colon X/\sim^{\mathrm{eq}}_R \to Y$$

        making the diagram

        $$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "q"'] & Y \\ X/\sim^{\mathrm{eq}}_R \arrow[ur, dotted, "\exists"', "\overline{f}"] & \end{tikzcd}$$

        commute.

- For each $x, y \in X$, if $x \sim_R^{\mathrm{eq}} y$, then $f(x) = f(y)$.

(b) For each $x, y \in X$, if $x \sim_R y$, then $f(x) = f(y)$.

---

[1] Dually, we also have an isomorphism of sets

$$\mathrm{Eq}(\mathrm{pr}_1, \mathrm{pr}_2) \cong X \times_{X/\sim_R^{\mathrm{eq}}} X, \qquad \begin{tikzcd} \mathrm{Eq}(\mathrm{pr}_1, \mathrm{pr}_2) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d] \\ X \arrow[r] & X/\sim_R^{\mathrm{eq}}. \end{tikzcd}$$

[2] *Further Terminology:* The set $X/\sim_{\mathrm{Ker}(f)}$ is often called the **coimage of** f, and denoted by $\mathrm{Coim}(f)$.

[3] In a sense this is a result relating the monad in **Rel** induced by f with the comonad in **Rel** induced by f, as the kernel and image

$$\mathrm{Ker}(f) \colon X \nrightarrow X,$$
$$\mathrm{Im}(f) \subset Y$$

of f are the underlying functors of (respectively) the induced monad and comonad of the adjunction

$$\left(\mathrm{Gr}(f) \dashv f^{-1}\right)\colon \quad \begin{tikzcd} A \arrow[r, bend left, "\mathrm{Gr}(f)", "\shortmid"{marking}] \arrow[r, phantom, "\perp"] & B \arrow[l, bend left, "f^{-1}", "\shortmid"{marking}] \end{tikzcd}$$

of Item 2 of Proposition 6.3.1.2.

PROOF 7.5.2.4 ▶ PROOF OF PROPOSITION 7.5.2.3

Item 1: As a Coequaliser

Omitted.

Item 2: As a Pushout

Omitted.

Item 3: The First Isomorphism Theorem for Sets

Clear.

Item 4: Descending Functions to Quotient Sets, I

See [Pro240].

Item 5: Descending Functions to Quotient Sets, II

See [Pro24aa].

Item 6: Descending Functions to Quotient Sets, III

This follows from Items 5 and 6.

Item 7: Descending Functions to Quotient Sets, IV

See [Pro24n].

Item 8: Descending Functions to Quotient Sets, V

See [Pro24m].

Item 9: Descending Functions to Quotient Sets, VI

The implication Item 9a $\implies$ Item 9b is clear.
Conversely, suppose that, for each $x, y \in X$, if $x \sim_R y$, then $f(x) = f(y)$. Spelling out the definition of the equivalence closure of R, we see that the condition $x \sim_R^{\mathrm{eq}} y$ unwinds to the following:

($\star$) There exist $(x_1, \ldots, x_n) \in R^{\times n}$ satisfying at least one of the following conditions:

1. The following conditions are satisfied:
    (a) We have $x \sim_R x_1$ or $x_1 \sim_R x$;
    (b) We have $x_i \sim_R x_{i+1}$ or $x_{i+1} \sim_R x_i$ for each $1 \leq i \leq n-1$;
    (c) We have $y \sim_R x_n$ or $x_n \sim_R y$;
2. We have $x = y$.

Now, if $x = y$, then $f(x) = f(y)$ trivially; otherwise, we have

$$\begin{aligned} f(x) &= f(x_1), \\ f(x_1) &= f(x_2), \\ &\vdots \\ f(x_{n-1}) &= f(x_n), \\ f(x_n) &= f(y), \end{aligned}$$

and $f(x) = f(y)$, as we wanted to show. □

Appendices

7.A Other Chapters

Sets

1. Sets

2. Constructions With Sets

3. Pointed Sets

4. Tensor Products of Pointed Sets

Relations

5. Relations

6. Constructions With Relations

7. Equivalence Relations and Apartness Relations

Category Theory

8. Categories

Bicategories

9. Types of Morphisms in Bicategories

Part III

Category Theory

Chapter 8

Categories

This chapter contains some elementary material about categories, functors, and natural transformations. Notably, we discuss and explore:

1. Categories (Section 8.1).
2. The quadruple adjunction $\pi_0 \dashv (-)_{\mathrm{disc}} \dashv \mathrm{Obj} \dashv (-)_{\mathrm{indisc}}$ between the category of categories and the category of sets (Section 8.2).
3. Groupoids, categories in which all morphisms admit inverses (Section 8.3).
4. Functors (Section 8.4).
5. The conditions one may impose on functors in decreasing order of importance:
   (a) Section 8.5 introduces the foundationally important conditions one may impose on functors, such as faithfulness, conservativity, essential surjectivity, etc.
   (b) Section 8.6 introduces more conditions one may impose on functors that are still important but less omni-present than those of Section 8.5, such as being dominant, being a monomorphism, being pseudomonic, etc.
   (c) Section 8.7 introduces some rather rare or uncommon conditions one may impose on functors that are nevertheless still useful to explicit record in this chapter.
6. Natural transformations (Section 8.8).
7. The various categorical and 2-categorical structures formed by categories, functors, and natural transformations (Section 8.9).

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8.1 Categories

8.1.1 Foundations

DEFINITION 8.1.1.1 ► CATEGORIES

A **category** $(\mathcal{C}, \circ^{\mathcal{C}}, \mathbb{1}^{\mathcal{C}})$ consists of:

- *Objects.* A class $\mathrm{Obj}(\mathcal{C})$ of **objects**.
- *Morphisms.* For each $A, B \in \mathrm{Obj}(\mathcal{C})$, a class $\mathrm{Hom}_{\mathcal{C}}(A, B)$, called the **class of morphisms of** $\mathcal{C}$ **from** A **to** B.
- *Identities.* For each $A \in \mathrm{Obj}(\mathcal{C})$, a map of sets

$$\mathbb{1}_A^{\mathcal{C}} \colon \mathrm{pt} \to \mathrm{Hom}_{\mathcal{C}}(A, A),$$

called the **unit map of $\mathcal{C}$ at A**, determining a morphism

$$\mathrm{id}_A \colon A \to A$$

of $\mathcal{C}$, called the **identity morphism of A**.

- *Composition*. For each $A, B, C \in \mathrm{Obj}(\mathcal{C})$, a map of sets

$$\circ^{\mathcal{C}}_{A,B,C} \colon \mathrm{Hom}_{\mathcal{C}}(B, C) \times \mathrm{Hom}_{\mathcal{C}}(A, B) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(A, C),$$

called the **composition map of $\mathcal{C}$ at (A, B, C)**.

such that the following conditions are satisfied:

1. *Associativity*. The diagram

$$\begin{tikzcd}
 & \mathrm{Hom}_{\mathcal{C}}(C,D) \times (\mathrm{Hom}_{\mathcal{C}}(B,C) \times \mathrm{Hom}_{\mathcal{C}}(A,B)) \arrow[dr, "\mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(C,D)} \times \circ^{\mathcal{C}}_{A,B,C}"] & \\
(\mathrm{Hom}_{\mathcal{C}}(C,D) \times \mathrm{Hom}_{\mathcal{C}}(B,C)) \times \mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[ur, dashed, "\alpha^{\mathsf{Sets}}_{\mathrm{Hom}_{\mathcal{C}}(C,D),\mathrm{Hom}_{\mathcal{C}}(B,C),\mathrm{Hom}_{\mathcal{C}}(A,B)}", "\sim"'] \arrow[d, "\circ^{\mathcal{C}}_{B,C,D} \times \mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(A,B)}"'] & & \mathrm{Hom}_{\mathcal{C}}(C,D) \times \mathrm{Hom}_{\mathcal{C}}(A,C) \arrow[d, "\circ^{\mathcal{C}}_{A,C,D}"] \\
\mathrm{Hom}_{\mathcal{C}}(B,D) \times \mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[rr, "\circ^{\mathcal{C}}_{A,B,D}"'] & & \mathrm{Hom}_{\mathcal{C}}(A,D)
\end{tikzcd}$$

   commutes, i.e. for each composable triple (f, g, h) of morphisms of $\mathcal{C}$, we have

$$(f \circ g) \circ h = f \circ (g \circ h).$$

2. *Left Unitality*. The diagram

$$\begin{tikzcd}
\mathrm{pt} \times \mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[d, "\mathbb{1}^{\mathcal{C}}_B \times \mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(A,B)}"'] \arrow[dr, dashed, "\lambda^{\mathsf{Sets}}_{\mathrm{Hom}_{\mathcal{C}}(A,B)}", "\sim"'] & \\
\mathrm{Hom}_{\mathcal{C}}(B,B) \times \mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[r, "\circ^{\mathcal{C}}_{A,B,B}"'] & \mathrm{Hom}_{\mathcal{C}}(A,B)
\end{tikzcd}$$

   commutes, i.e. for each morphism $f \colon A \longrightarrow B$ of $\mathcal{C}$, we have

$$\mathrm{id}_B \circ f = f.$$

3. *Right Unitality.* The diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(A,B)\times\mathrm{pt} \arrow[d, "\mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(A,B)}\times\mathbb{1}^{\mathcal{C}}_{A}"'] \arrow[drr, dashed, "\rho^{\mathsf{Sets}}_{\mathrm{Hom}_{\mathcal{C}}(A,B)}", "\sim"'] & & \\
\mathrm{Hom}_{\mathcal{C}}(A,B)\times\mathrm{Hom}_{\mathcal{C}}(A,A) \arrow[rr, "\circ^{\mathcal{C}}_{A,A,B}"'] & & \mathrm{Hom}_{\mathcal{C}}(A,B)
\end{tikzcd}$$

commutes, i.e. for each morphism $f\colon A\longrightarrow B$ of $\mathcal{C}$, we have

$$f\circ\mathrm{id}_A=f.$$

Notation 8.1.1.2 ► Further Notation for Morphisms in Categories

Let $\mathcal{C}$ be a category.

1. We also write $\mathcal{C}(A,B)$ for $\mathrm{Hom}_{\mathcal{C}}(A,B)$.
2. We write $\mathrm{Mor}(\mathcal{C})$ for the class of all morphisms of $\mathcal{C}$.

Definition 8.1.1.3 ► Size Conditions on Categories

Let κ be a regular cardinal. A category $\mathcal{C}$ is

1. **Locally small** if, for each $A,B\in\mathrm{Obj}(\mathcal{C})$, the class $\mathrm{Hom}_{\mathcal{C}}(A,B)$ is a set.
2. **Locally essentially small** if, for each $A,B\in\mathrm{Obj}(\mathcal{C})$, the class

$$\mathrm{Hom}_{\mathcal{C}}(A,B)/\{\text{isomorphisms}\}$$

   is a set.
3. **Small** if $\mathcal{C}$ is locally small and $\mathrm{Obj}(\mathcal{C})$ is a set.
4. **κ-Small** if $\mathcal{C}$ is locally small, $\mathrm{Obj}(\mathcal{C})$ is a set, and we have $\#\mathrm{Obj}(\mathcal{C})<\kappa$.

8.1.2 Examples of Categories

Example 8.1.2.1 ► The Punctual Category

The **punctual category**[1] is the category pt where

- *Objects.* We have
$$\mathrm{Obj}(\mathrm{pt}) \overset{\mathrm{def}}{=} \{\star\}.$$
- *Morphisms.* The unique Hom-set of pt is defined by
$$\mathrm{Hom}_{\mathrm{pt}}(\star,\star) \overset{\mathrm{def}}{=} \{\mathrm{id}_\star\}.$$
- *Identities.* The unit map
$$\mathbb{1}^{\mathrm{pt}}_\star \colon \mathrm{pt} \longrightarrow \mathrm{Hom}_{\mathrm{pt}}(\star,\star)$$
of pt at $\star$ is defined by
$$\mathrm{id}^{\mathrm{pt}}_\star \overset{\mathrm{def}}{=} \mathrm{id}_\star.$$
- *Composition.* The composition map
$$\circ^{\mathrm{pt}}_{\star,\star,\star} \colon \mathrm{Hom}_{\mathrm{pt}}(\star,\star) \times \mathrm{Hom}_{\mathrm{pt}}(\star,\star) \longrightarrow \mathrm{Hom}_{\mathrm{pt}}(\star,\star)$$
of pt at $(\star,\star,\star)$ is given by the bijection $\mathrm{pt} \times \mathrm{pt} \cong \mathrm{pt}$.

[1] *Further Terminology:* Also called the **singleton category**.

Example 8.1.2.2 ▶ Monoids as One-Object Categories

We have an isomorphism of categories[1]

$$\mathrm{Mon} \cong \mathrm{pt} \underset{\mathrm{Sets}}{\times} \mathrm{Cats}, \qquad \begin{tikzcd} \mathrm{Mon} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Cats} \arrow[d, "\mathrm{Obj}"] \\ \mathrm{pt} \arrow[r, "{[\mathrm{pt}]}"'] & \mathrm{Sets} \end{tikzcd}$$

via the delooping functor $\mathsf{B} \colon \mathrm{Mon} \longrightarrow \mathrm{Cats}$ of ?? of ??, exhibiting monoids as exactly those categories having a single object.

[1] This can be enhanced to an isomorphism of 2-categories

$$\mathrm{Mon}_{2\mathrm{disc}} \cong \mathrm{pt}_{\mathrm{bi}} \underset{\mathrm{Sets}_{2\mathrm{disc}}}{\times} \mathrm{Cats}_{2,*}, \qquad \begin{tikzcd} \mathrm{Mon}_{2\mathrm{disc}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Cats}_{2,*} \arrow[d, "\mathrm{Obj}"] \\ \mathrm{pt}_{\mathrm{bi}} \arrow[r, "{[\mathrm{pt}]}"'] & \mathrm{Sets}_{2\mathrm{disc}} \end{tikzcd}$$

between the discrete 2-category $\mathrm{Mon}_{2\mathrm{disc}}$ on Mon and the 2-category of pointed categories with one object.

Proof 8.1.2.3 ▶ Proof of Example 8.1.2.2

Omitted. □

Example 8.1.2.4 ▶ The Empty Category

The **empty category** is the category $\emptyset_{\mathsf{cat}}$ where

- *Objects.* We have
$$\mathrm{Obj}(\emptyset_{\mathsf{cat}}) \stackrel{\mathrm{def}}{=} \emptyset.$$
- *Morphisms.* We have
$$\mathrm{Mor}(\emptyset_{\mathsf{cat}}) \stackrel{\mathrm{def}}{=} \emptyset.$$
- *Identities and Composition.* Having no objects, $\emptyset_{\mathsf{cat}}$ has no unit nor composition maps.

Example 8.1.2.5 ▶ Ordinal Categories

The ***n*th ordinal category** is the category $\mathbb{n}$ where[1]

- *Objects.* We have
$$\mathrm{Obj}(\mathbb{n}) \stackrel{\mathrm{def}}{=} \{[0], \ldots, [n]\}.$$
- *Morphisms.* For each $[i], [j] \in \mathrm{Obj}(\mathbb{n})$, we have
$$\mathrm{Hom}_{\mathbb{n}}([i],[j]) \stackrel{\mathrm{def}}{=} \begin{cases} \{\mathrm{id}_{[i]}\} & \text{if } [i] = [j], \\ \{[i] \to [j]\} & \text{if } [j] < [i], \\ \emptyset & \text{if } [j] > [i]. \end{cases}$$
- *Identities.* For each $[i] \in \mathrm{Obj}(\mathbb{n})$, the unit map
$$\mathbb{1}^{\mathbb{n}}_{[i]} \colon \mathrm{pt} \to \mathrm{Hom}_{\mathbb{n}}([i],[i])$$
of $\mathbb{n}$ at $[i]$ is defined by
$$\mathrm{id}^{\mathbb{n}}_{[i]} \stackrel{\mathrm{def}}{=} \mathrm{id}_{[i]}.$$
- *Composition.* For each $[i], [j], [k] \in \mathrm{Obj}(\mathbb{n})$, the composition map
$$\circ^{\mathbb{n}}_{[i],[j],[k]} \colon \mathrm{Hom}_{\mathbb{n}}([j],[k]) \times \mathrm{Hom}_{\mathbb{n}}([i],[j]) \to \mathrm{Hom}_{\mathbb{n}}([i],[k])$$

of $\mathbb{n}$ at $([i],[j],[k])$ is defined by

$$\begin{aligned}\mathrm{id}_{[i]}\circ\mathrm{id}_{[i]}&=\mathrm{id}_{[i]},\\([j]\to[k])\circ([i]\to[j])&=([i]\to[k]).\end{aligned}$$

[1]In other words, $\mathbb{n}$ is the category associated to the poset

$$[0]\to[1]\to\cdots\to[n-1]\to[n].$$

The category $\mathbb{n}$ for $n\geq 2$ may also be defined in terms of $\mathbb{0}$ and joins (**??**): we have isomorphisms of categories

$$\begin{aligned}\mathbb{1}&\cong\mathbb{0}\star\mathbb{0},\\\mathbb{2}&\cong\mathbb{1}\star\mathbb{0}\\&\cong(\mathbb{0}\star\mathbb{0})\star\mathbb{0},\\\mathbb{3}&\cong\mathbb{2}\star\mathbb{0}\\&\cong(\mathbb{1}\star\mathbb{0})\star\mathbb{0}\\&\cong((\mathbb{0}\star\mathbb{0})\star\mathbb{0})\star\mathbb{0},\\\mathbb{4}&\cong\mathbb{3}\star\mathbb{0}\\&\cong(\mathbb{2}\star\mathbb{0})\star\mathbb{0}\\&\cong((\mathbb{1}\star\mathbb{0})\star\mathbb{0})\star\mathbb{0}\\&\cong(((\mathbb{0}\star\mathbb{0})\star\mathbb{0})\star\mathbb{0})\star\mathbb{0},\end{aligned}$$

and so on.

Example 8.1.2.6 ▶ More Examples of Categories

Here we list some of the other categories appearing throughout this work.

1. The category Sets_* of pointed sets of Definition 3.1.3.1.
2. The category Rel of sets and relations of Definition 5.2.1.1.
3. The category $\mathsf{Span}(A,B)$ of spans from a set A to a set B of **??**.
4. The category $\mathsf{ISets}(K)$ of K-indexed sets of **??**.
5. The category ISets of indexed sets of **??**.
6. The category $\mathsf{FibSets}(K)$ of K-fibred sets of **??**.
7. The category $\mathsf{FibSets}$ of fibred sets of **??**.
8. Categories of functors $\mathsf{Fun}(\mathcal{C},\mathcal{D})$ as in Definition 8.9.1.1.
9. The category of categories Cats of Definition 8.9.2.1.
10. The category of groupoids Grpd of Definition 8.9.4.1.

8.1.3 Posetal Categories

Let $(X, \preceq_X)$ be a poset.

1. The **posetal category associated to** $(X, \preceq_X)$ is the category X_{pos} where

   - *Objects.* We have
   $$\mathrm{Obj}(X_{\mathrm{pos}}) \stackrel{\mathrm{def}}{=} X.$$
   - *Morphisms.* For each $a, b \in \mathrm{Obj}(X_{\mathrm{pos}})$, we have
   $$\mathrm{Hom}_{X_{\mathrm{pos}}}(a, b) \stackrel{\mathrm{def}}{=} \begin{cases} \mathrm{pt} & \text{if } a \preceq_X b, \\ \emptyset & \text{otherwise.} \end{cases}$$
   - *Identities.* For each $a \in \mathrm{Obj}(X_{\mathrm{pos}})$, the unit map
   $$\mathbb{1}^{X_{\mathrm{pos}}}_a \colon \mathrm{pt} \to \mathrm{Hom}_{X_{\mathrm{pos}}}(a, a)$$
   of X_{pos} at a is given by the identity map.
   - *Composition.* For each $a, b, c \in \mathrm{Obj}(X_{\mathrm{pos}})$, the composition map
   $$\circ^{X_{\mathrm{pos}}}_{a,b,c} \colon \mathrm{Hom}_{X_{\mathrm{pos}}}(b, c) \times \mathrm{Hom}_{X_{\mathrm{pos}}}(a, b) \to \mathrm{Hom}_{X_{\mathrm{pos}}}(a, c)$$
   of X_{pos} at (a, b, c) is defined as either the inclusion $\emptyset \hookrightarrow \mathrm{pt}$ or the identity map of pt, depending on whether we have $a \preceq_X b$, $b \preceq_X c$, and $a \preceq_X c$.

2. A category $\mathcal{C}$ is **posetal**[1] if $\mathcal{C}$ is equivalent to X_{pos} for some poset $(X, \preceq_X)$.

[1] *Further Terminology:* Also called a **thin** category or a $(0, 1)$**-category**.

Proposition 8.1.3.2 ▶ Properties of Posetal Categories

Let $(X, \preceq_X)$ be a poset and let $\mathcal{C}$ be a category.

1. *Functoriality.* The assignment $(X, \preceq_X) \mapsto X_{\mathrm{pos}}$ defines a functor
$$(-)_{\mathrm{pos}} \colon \mathsf{Pos} \to \mathsf{Cats}.$$
2. *Fully Faithfulness.* The functor $(-)_{\mathrm{pos}}$ of Item 1 is fully faithful.
3. *Characterisations.* The following conditions are equivalent:
   (a) The category $\mathcal{C}$ is posetal.

(b) For each $A, B \in \mathrm{Obj}(\mathcal{C})$ and each $f, g \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, we have $f = g$.

Proof 8.1.3.3 ► Proof of Proposition 8.1.3.2

Item 1: Functoriality

Omitted.

Item 2: Fully Faithfulness

Omitted.

Item 3: Characterisations

Clear.

8.1.4 Subcategories

Let $\mathcal{C}$ be a category.

Definition 8.1.4.1 ► Subcategories

A **subcategory** of $\mathcal{C}$ is a category $\mathcal{A}$ satisfying the following conditions:

1. *Objects.* We have $\mathrm{Obj}(\mathcal{A}) \subset \mathrm{Obj}(\mathcal{C})$.
2. *Morphisms.* For each $A, B \in \mathrm{Obj}(\mathcal{A})$, we have
$$\mathrm{Hom}_{\mathcal{A}}(A, B) \subset \mathrm{Hom}_{\mathcal{C}}(A, B).$$
3. *Identities.* For each $A \in \mathrm{Obj}(\mathcal{A})$, we have
$$\mathbb{1}_A^{\mathcal{A}} = \mathbb{1}_A^{\mathcal{C}}.$$
4. *Composition.* For each $A, B, C \in \mathrm{Obj}(\mathcal{A})$, we have
$$\circ^{\mathcal{A}}_{A,B,C} = \circ^{\mathcal{C}}_{A,B,C}.$$

Definition 8.1.4.2 ► Full Subcategories

A subcategory $\mathcal{A}$ of $\mathcal{C}$ is **full** if the canonical inclusion functor $\mathcal{A} \to \mathcal{C}$ is full, i.e. if, for each $A, B \in \mathrm{Obj}(\mathcal{A})$, the inclusion

$$\iota_{A,B} \colon \mathrm{Hom}_{\mathcal{A}}(A, B) \hookrightarrow \mathrm{Hom}_{\mathcal{C}}(A, B)$$

is surjective (and thus bijective).

Definition 8.1.4.3 ► Strictly Full Subcategories

A subcategory $\mathcal{A}$ of a category C is **strictly full** if it satisfies the following conditions:

1. *Fullness*. The subcategory $\mathcal{A}$ is full.
2. *Closedness Under Isomorphisms*. The class $\mathrm{Obj}(\mathcal{A})$ is closed under isomorphisms.[1]

[1] That is, given $A \in \mathrm{Obj}(\mathcal{A})$ and $C \in \mathrm{Obj}(C)$, if $C \cong A$, then $C \in \mathrm{Obj}(\mathcal{A})$.

Definition 8.1.4.4 ► Wide Subcategories

A subcategory $\mathcal{A}$ of C is **wide**[1] if $\mathrm{Obj}(\mathcal{A}) = \mathrm{Obj}(C)$.

[1] *Further Terminology:* Also called **lluf**.

8.1.5 Skeletons of Categories

Definition 8.1.5.1 ► Skeletons of Categories

A[1] **skeleton** of a category C is a full subcategory $\mathrm{Sk}(C)$ with one object from each isomorphism class of objects of C.

[1] Due to Item 3 of Proposition 8.1.5.3, we often refer to any such full subcategory $\mathrm{Sk}(C)$ of C as *the* skeleton of C.

Definition 8.1.5.2 ► Skeletal Categories

A category C is **skeletal** if $C \cong \mathrm{Sk}(C)$.[1]

[1] That is, C is **skeletal** if isomorphic objects of C are equal.

Proposition 8.1.5.3 ► Properties of Skeletons of Categories

Let C be a category.

1. *Existence*. Assuming the axiom of choice, $\mathrm{Sk}(C)$ always exists.
2. *Pseudofunctoriality*. The assignment $C \mapsto \mathrm{Sk}(C)$ defines a pseudofunctor
$$\mathrm{Sk}\colon \mathsf{Cats}_2 \to \mathsf{Cats}_2.$$
3. *Uniqueness Up to Equivalence*. Any two skeletons of C are equivalent.

4. *Inclusions of Skeletons Are Equivalences.* The inclusion

$$\iota_{\mathcal{C}} : \mathsf{Sk}(\mathcal{C}) \hookrightarrow \mathcal{C}$$

of a skeleton of $\mathcal{C}$ into $\mathcal{C}$ is an equivalence of categories.

Proof 8.1.5.4 ▶ Proof of Proposition 8.1.5.3

Item 1: Existence

See [nLab23, Section "Existence of Skeletons of Categories"].

Item 2: Pseudofunctoriality

See [nLab23, Section "Skeletons as an Endo-Pseudofunctor on $\mathfrak{Cat}$"].

Item 3: Uniqueness Up to Equivalence

Clear.

Item 4: Inclusions of Skeletons Are Equivalences

Clear.

8.1.6 Precomposition and Postcomposition

Let $\mathcal{C}$ be a category and let $A, B, C \in \mathrm{Obj}(\mathcal{C})$.

Definition 8.1.6.1 ▶ Precomposition and Postcomposition Functions

Let $f : A \to B$ and $g : B \to C$ be morphisms of $\mathcal{C}$.

1. The **precomposition function associated to** f is the function

$$f^* : \mathrm{Hom}_{\mathcal{C}}(B, C) \to \mathrm{Hom}_{\mathcal{C}}(A, C)$$

defined by

$$f^*(\phi) \overset{\text{def}}{=} \phi \circ f$$

for each $\phi \in \mathrm{Hom}_{\mathcal{C}}(B, C)$.

2. The **postcomposition function associated to** g is the function

$$g_* : \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{C}}(A, C)$$

defined by

$$g_*(\phi) \overset{\text{def}}{=} g \circ \phi$$

for each $\phi \in \mathrm{Hom}_{\mathcal{C}}(A, B)$.

Proposition 8.1.6.2 ► Properties of Pre/Postcomposition

Let $A, B, C, D \in \mathrm{Obj}(\mathcal{C})$ and let $f\colon A \to B$ and $g\colon B \to C$ be morphisms of $\mathcal{C}$.

1. *Interaction Between Precomposition and Postcomposition*. We have

$$g_* \circ f^* = f^* \circ g_*, \qquad \begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(B, C) \arrow[r, "g_*"] \arrow[d, "f^*"'] & \mathrm{Hom}_{\mathcal{C}}(B, D) \arrow[d, "f^*"] \\ \mathrm{Hom}_{\mathcal{C}}(A, C) \arrow[r, "g_*"'] & \mathrm{Hom}_{\mathcal{C}}(A, D). \end{tikzcd}$$

2. *Interaction With Composition I*. We have

$$(g \circ f)^* = f^* \circ g^*, \qquad \begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(X, A) \arrow[r, "f_*"] \arrow[dr, "(g\circ f)_*"'] & \mathrm{Hom}_{\mathcal{C}}(X, B) \arrow[d, "g_*"] \\ & \mathrm{Hom}_{\mathcal{C}}(X, C), \end{tikzcd}$$

$$(g \circ f)_* = g_* \circ f_*, \qquad \begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(C, X) \arrow[r, "g^*"] \arrow[dr, "(g\circ f)^*"'] & \mathrm{Hom}_{\mathcal{C}}(B, X) \arrow[d, "f^*"] \\ & \mathrm{Hom}_{\mathcal{C}}(A, X). \end{tikzcd}$$

3. *Interaction With Composition II*. We have

$$\begin{tikzcd} \mathrm{pt} \arrow[r, "{[f]}"] \arrow[dr, "{[g\circ f]}"'] & \mathrm{Hom}_{\mathcal{C}}(A, B) \arrow[d, "g_*"] \\ & \mathrm{Hom}_{\mathcal{C}}(A, C) \end{tikzcd} \qquad \begin{aligned} [g \circ f] &= g_* \circ [f], \\ [g \circ f] &= f^* \circ [g], \end{aligned} \qquad \begin{tikzcd} \mathrm{pt} \arrow[r, "{[g]}"] \arrow[dr, "{[g\circ f]}"'] & \mathrm{Hom}_{\mathcal{C}}(B, C) \arrow[d, "f^*"] \\ & \mathrm{Hom}_{\mathcal{C}}(A, C). \end{tikzcd}$$

4. *Interaction With Composition III.* We have

$$f^* \circ \circ^C_{A,B,C} = \circ^C_{X,B,C} \circ (f^* \times \mathrm{id}),\qquad \begin{tikzcd} \mathrm{Hom}_C(B,C) \times \mathrm{Hom}_C(A,B) \arrow[r, "\circ^C_{A,B,C}"] \arrow[d, "\mathrm{id}\times f^*"'] & \mathrm{Hom}_C(A,C) \arrow[d, "f^*"] \\ \mathrm{Hom}_C(B,C) \times \mathrm{Hom}_C(X,B) \arrow[r, "\circ^C_{X,B,C}"'] & \mathrm{Hom}_C(X,C), \end{tikzcd}$$

$$g_* \circ \circ^C_{A,B,C} = \circ^C_{A,B,D} \circ (\mathrm{id} \times g_*),\qquad \begin{tikzcd} \mathrm{Hom}_C(B,C) \times \mathrm{Hom}_C(A,B) \arrow[r, "\circ^C_{A,B,C}"] \arrow[d, "g_*\times\mathrm{id}"'] & \mathrm{Hom}_C(A,C) \arrow[d, "g^*"] \\ \mathrm{Hom}_C(B,D) \times \mathrm{Hom}_C(A,B) \arrow[r, "\circ^C_{A,B,D}"'] & \mathrm{Hom}_C(A,D). \end{tikzcd}$$

5. *Interaction With Identities.* We have

$$(\mathrm{id}_A)^* = \mathrm{id}_{\mathrm{Hom}_C(A,B)},$$
$$(\mathrm{id}_B)_* = \mathrm{id}_{\mathrm{Hom}_C(A,B)}.$$

Proof 8.1.6.3 ► Proof of Proposition 8.1.6.2

Item 1: Interaction Between Precomposition and Postcomposition

Clear.

Item 2: Interaction With Composition I

Clear.

Item 3: Interaction With Composition II

Clear.

Item 4: Interaction With Composition III

Clear.

Item 5: Interaction With Identities

Clear.

8.2 The Quadruple Adjunction With Sets

8.2.1 Statement

Let C be a category.

Proposition 8.2.1.1 ► The Quadruple Adjunction Between Sets and Cats

We have a quadruple adjunction

$$(\pi_0 \dashv (-)_{\mathsf{disc}} \dashv \mathsf{Obj} \dashv (-)_{\mathsf{indisc}})\colon \quad \begin{tikzcd} \mathsf{Sets} \arrow[rr, bend left=25, "(-)_{\mathsf{disc}}" description] \arrow[rr, bend right=80, "(-)_{\mathsf{indisc}}"'] & & \mathsf{Cats}, \arrow[ll, bend right=80, "\pi_0"'] \arrow[ll, bend left=25, "\mathsf{Obj}" description] \end{tikzcd}$$

witnessed by bijections of sets

$$\begin{aligned} \mathsf{Hom}_{\mathsf{Sets}}(\pi_0(\mathcal{C}), X) &\cong \mathsf{Hom}_{\mathsf{Cats}}(\mathcal{C}, X_{\mathsf{disc}}), \\ \mathsf{Hom}_{\mathsf{Cats}}(X_{\mathsf{disc}}, \mathcal{C}) &\cong \mathsf{Hom}_{\mathsf{Sets}}(X, \mathsf{Obj}(\mathcal{C})), \\ \mathsf{Hom}_{\mathsf{Sets}}(\mathsf{Obj}(\mathcal{C}), X) &\cong \mathsf{Hom}_{\mathsf{Cats}}(\mathcal{C}, X_{\mathsf{indisc}}), \end{aligned}$$

natural in $\mathcal{C} \in \mathsf{Obj}(\mathsf{Cats})$ and $X \in \mathsf{Obj}(\mathsf{Sets})$, where

- The functor
$$\pi_0\colon \mathsf{Cats} \to \mathsf{Sets},$$
the **connected components functor**, is the functor sending a category to its set of connected components of Definition 8.2.2.2.
- The functor
$$(-)_{\mathsf{disc}}\colon \mathsf{Sets} \to \mathsf{Cats},$$
the **discrete category functor**, is the functor sending a set to its associated discrete category of Item 1.
- The functor
$$\mathsf{Obj}\colon \mathsf{Cats} \to \mathsf{Sets},$$
the **object functor**, is the functor sending a category to its set of objects.
- The functor
$$(-)_{\mathsf{indisc}}\colon \mathsf{Sets} \to \mathsf{Cats},$$
the **indiscrete category functor**, is the functor sending a set to its associated indiscrete category of Item 1.

Proof 8.2.1.2 ► Proof of Proposition 8.2.1.1

Omitted.

8.2.2 Connected Components and Connected Categories

8.2.2.1 Connected Components of Categories

Let $\mathcal{C}$ be a category.

Definition 8.2.2.1 ► Connected Components of Categories

A **connected component** of $\mathcal{C}$ is a full subcategory $\mathcal{I}$ of $\mathcal{C}$ satisfying the following conditions:[1]

1. *Non-Emptiness.* We have $\mathrm{Obj}(\mathcal{I}) \neq \emptyset$.

2. *Connectedness.* There exists a zigzag of arrows between any two objects of $\mathcal{I}$.

[1] In other words, a **connected component** of $\mathcal{C}$ is an element of the set $\mathrm{Obj}(\mathcal{C})/{\sim}$ with $\sim$ the equivalence relation generated by the relation $\sim'$ obtained by declaring $A \sim' B$ iff there exists a morphism of $\mathcal{C}$ from A to B.

8.2.2.2 Sets of Connected Components of Categories

Let $\mathcal{C}$ be a category.

Definition 8.2.2.2 ► Sets of Connected Components of Categories

The **set of connected components of** $\mathcal{C}$ is the set $\pi_0(\mathcal{C})$ whose elements are the connected components of $\mathcal{C}$.

Proposition 8.2.2.3 ► Properties of Sets of Connected Components

Let $\mathcal{C}$ be a category.

1. *Functoriality.* The assignment $\mathcal{C} \mapsto \pi_0(\mathcal{C})$ defines a functor

$$\pi_0 \colon \mathsf{Cats} \longrightarrow \mathsf{Sets}.$$

2. *Adjointness.* We have a quadruple adjunction

$$(\pi_0 \dashv (-)_{\mathsf{disc}} \dashv \mathrm{Obj} \dashv (-)_{\mathsf{indisc}})\colon \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left=20, "(-)_{\mathsf{disc}}" description] \arrow[r, bend right=60, "(-)_{\mathsf{indisc}}"'] \arrow[r, phantom, shift left=6, "\perp"] \arrow[r, phantom, "\perp"] \arrow[r, phantom, shift right=6, "\perp"] & \mathsf{Cats}. \arrow[l, bend right=60, "\pi_0"'] \arrow[l, bend left=20, "\mathrm{Obj}" description] \end{tikzcd}$$

3. *Interaction With Groupoids.* If $\mathcal{C}$ is a groupoid, then we have an isomorphism of categories

$$\pi_0(\mathcal{C}) \cong \mathsf{K}(\mathcal{C}),$$

where $\mathsf{K}(\mathcal{C})$ is the set of isomorphism classes of $\mathcal{C}$ of **??**.

4. *Preservation of Colimits.* The functor π_0 of Item 1 preserves colimits. In particular, we have bijections of sets

$$\pi_0(\mathcal{C} \coprod \mathcal{D}) \cong \pi_0(\mathcal{C}) \coprod \pi_0(\mathcal{D}),$$
$$\pi_0(\mathcal{C} \textstyle\coprod_{\mathcal{E}} \mathcal{D}) \cong \pi_0(\mathcal{C}) \coprod_{\pi_0(\mathcal{E})} \pi_0(\mathcal{D}),$$
$$\pi_0\left(\mathsf{CoEq}\left(\mathcal{C} \underset{G}{\overset{F}{\rightrightarrows}} \mathcal{D}\right)\right) \cong \mathsf{CoEq}\left(\pi_0(\mathcal{C}) \underset{\pi_0(G)}{\overset{\pi_0(F)}{\rightrightarrows}} \pi_0(\mathcal{D})\right),$$

natural in $\mathcal{C}, \mathcal{D}, \mathcal{E} \in \mathrm{Obj}(\mathsf{Cats})$.

5. *Symmetric Strong Monoidality With Respect to Coproducts.* The connected components functor of Item 1 has a symmetric strong monoidal structure

$$\left(\pi_0, \pi_0^{\coprod}, \pi_{0|\mathbb{1}}^{\coprod}\right) : (\mathsf{Cats}, \textstyle\coprod, \emptyset_{\mathsf{cat}}) \longrightarrow (\mathsf{Sets}, \textstyle\coprod, \emptyset),$$

being equipped with isomorphisms

$$\pi_{0|\mathcal{C},\mathcal{D}}^{\coprod} : \pi_0(\mathcal{C}) \textstyle\coprod \pi_0(\mathcal{D}) \xrightarrow{\cong} \pi_0(\mathcal{C} \textstyle\coprod \mathcal{D}),$$
$$\pi_{0|\mathbb{1}}^{\coprod} : \emptyset \xrightarrow{\cong} \pi_0(\emptyset_{\mathsf{cat}}),$$

natural in $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$.

6. *Symmetric Strong Monoidality With Respect to Products.* The connected components functor of Item 1 has a symmetric strong monoidal structure

$$\left(\pi_0, \pi_0^{\times}, \pi_{0|\mathbb{1}}^{\times}\right) : (\mathsf{Cats}, \times, \mathrm{pt}) \longrightarrow (\mathsf{Sets}, \times, \mathrm{pt}),$$

being equipped with isomorphisms

$$\pi_{0|\mathcal{C},\mathcal{D}}^{\times} : \pi_0(\mathcal{C}) \times \pi_0(\mathcal{D}) \xrightarrow{\cong} \pi_0(\mathcal{C} \times \mathcal{D}),$$
$$\pi_{0|\mathbb{1}}^{\times} : \mathrm{pt} \xrightarrow{\cong} \pi_0(\mathrm{pt}),$$

natural in $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$.

Proof 8.2.2.4 ▶ Proof of Proposition 8.2.2.3

Item 1: Functoriality

Clear.

Item 2: Adjointness

This is proved in Proposition 8.2.1.1.

Item 3: Interaction With Groupoids

Clear.

Item 4: Preservation of Colimits

This follows from Item 2 and **??** of **??**.

Item 5: Symmetric Strong Monoidality With Respect to Coproducts

Clear.

Item 6: Symmetric Strong Monoidality With Respect to Products

Clear. □

8.2.2.3 Connected Categories

Definition 8.2.2.5 ▶ Connected Categories

A category $\mathcal{C}$ is **connected** if $\pi_0(\mathcal{C}) \cong \mathrm{pt}$.[1,2]

[1] *Further Terminology:* A category is **disconnected** if it is not connected.

[2] *Example:* A groupoid is connected iff any two of its objects are isomorphic.

8.2.3 Discrete Categories

Definition 8.2.3.1 ▶ Discrete Categories

Let X be a set.

1. The **discrete category on** X is the category X_{disc} where
   - *Objects.* We have
   $$\mathrm{Obj}(X_{\mathsf{disc}}) \stackrel{\text{def}}{=} X.$$
   - *Morphisms.* For each $A, B \in \mathrm{Obj}(X_{\mathsf{disc}})$, we have
   $$\mathrm{Hom}_{X_{\mathsf{disc}}}(A, B) \stackrel{\text{def}}{=} \begin{cases} \mathrm{id}_A & \text{if } A = B, \\ \emptyset & \text{if } A \neq B. \end{cases}$$

- *Identities.* For each $A \in \mathrm{Obj}(X_{\mathsf{disc}})$, the unit map

$$\mathbb{1}^{X_{\mathsf{disc}}}_{A} \colon \mathrm{pt} \to \mathrm{Hom}_{X_{\mathsf{disc}}}(A, A)$$

of X_{disc} at A is defined by

$$\mathrm{id}^{X_{\mathsf{disc}}}_{A} \overset{\mathrm{def}}{=} \mathrm{id}_{A}.$$

- *Composition.* For each $A, B, C \in \mathrm{Obj}(X_{\mathsf{disc}})$, the composition map

$$\circ^{X_{\mathsf{disc}}}_{A,B,C} \colon \mathrm{Hom}_{X_{\mathsf{disc}}}(B, C) \times \mathrm{Hom}_{X_{\mathsf{disc}}}(A, B) \to \mathrm{Hom}_{X_{\mathsf{disc}}}(A, C)$$

of X_{disc} at (A, B, C) is defined by

$$\mathrm{id}_{A} \circ \mathrm{id}_{A} \overset{\mathrm{def}}{=} \mathrm{id}_{A}.$$

2. A category C is **discrete** if it is equivalent to X_{disc} for some set X.

Proposition 8.2.3.2 ► Properties of Discrete Categories on Sets

Let X be a set.

1. *Functoriality.* The assignment $X \mapsto X_{\mathsf{disc}}$ defines a functor

$$(-)_{\mathsf{disc}} \colon \mathsf{Sets} \to \mathsf{Cats}.$$

2. *Adjointness.* We have a quadruple adjunction

$$(\pi_0 \dashv (-)_{\mathsf{disc}} \dashv \mathrm{Obj} \dashv (-)_{\mathsf{indisc}}) \colon \begin{tikzcd} \mathsf{Sets} \arrow[r, "(-)_{\mathsf{disc}}" description, bend left=20] \arrow[r, phantom, bend left=35, "\perp"] \arrow[r, phantom, "\perp"] \arrow[r, phantom, bend right=35, "\perp"] \arrow[r, "(-)_{\mathsf{indisc}}"', bend right=60] & \mathsf{Cats}. \arrow[l, "\pi_0"', bend right=60] \arrow[l, "\mathrm{Obj}" description, bend left=20] \end{tikzcd}$$

3. *Symmetric Strong Monoidality With Respect to Coproducts.* The functor of Item 1 has a symmetric strong monoidal structure

$$\left((-)_{\mathsf{disc}}, (-)^{\coprod}_{\mathsf{disc}}, (-)^{\coprod}_{\mathsf{disc}|\mathbb{1}}\right) \colon (\mathsf{Sets}, \coprod, \emptyset) \to (\mathsf{Cats}, \coprod, \emptyset_{\mathsf{cat}}),$$

being equipped with isomorphisms

$$(-)^{\coprod}_{\mathsf{disc}|X,Y} \colon X_{\mathsf{disc}} \coprod Y_{\mathsf{disc}} \xrightarrow{\cong} (X \coprod Y)_{\mathsf{disc}},$$
$$(-)^{\coprod}_{\mathsf{disc}|\mathbb{1}} \colon \emptyset_{\mathsf{cat}} \xrightarrow{\cong} \emptyset_{\mathsf{disc}},$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

4. *Symmetric Strong Monoidality With Respect to Products.* The functor of Item 1 has a symmetric strong monoidal structure

$$\left((-)_{\mathsf{disc}}, (-)^{\times}_{\mathsf{disc}}, (-)^{\times}_{\mathsf{disc}|\mathbb{1}}\right) \colon (\mathsf{Sets}, \times, \mathrm{pt}) \to (\mathsf{Cats}, \times, \mathrm{pt}),$$

being equipped with isomorphisms

$$(-)^{\times}_{\mathsf{disc}|X,Y} \colon X_{\mathsf{disc}} \times Y_{\mathsf{disc}} \xrightarrow{\cong} (X \times Y)_{\mathsf{disc}},$$
$$(-)^{\times}_{\mathsf{disc}|\mathbb{1}} \colon \mathrm{pt} \xrightarrow{\cong} \mathrm{pt}_{\mathsf{disc}},$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

Proof 8.2.3.3 ► Proof of Proposition 8.2.3.2

Item 1: Functoriality

Clear.

Item 2: Adjointness

This is proved in Proposition 8.2.1.1.

Item 3: Symmetric Strong Monoidality With Respect to Coproducts

Clear.

Item 4: Symmetric Strong Monoidality With Respect to Products

Clear. □

8.2.4 Indiscrete Categories

Definition 8.2.4.1 ► Indiscrete Categories

Let X be a set.

1. The **indiscrete category on** X[1] is the category X_{indisc} where
   - *Objects.* We have
   $$\mathrm{Obj}(X_{\mathsf{indisc}}) \overset{\mathrm{def}}{=} X.$$
   - *Morphisms.* For each $A, B \in \mathrm{Obj}(X_{\mathsf{indisc}})$, we have
   $$\begin{aligned}\mathrm{Hom}_{X_{\mathsf{disc}}}(A, B) &\overset{\mathrm{def}}{=} \{[A] \to [B]\}\\ &\cong \mathrm{pt}.\end{aligned}$$

- *Identities.* For each $A \in \mathrm{Obj}(X_{\mathrm{indisc}})$, the unit map

$$\mathbb{1}_A^{X_{\mathrm{indisc}}} \colon \mathrm{pt} \longrightarrow \mathrm{Hom}_{X_{\mathrm{indisc}}}(A, A)$$

of X_{indisc} at A is defined by

$$\mathrm{id}_A^{X_{\mathrm{indisc}}} \overset{\mathrm{def}}{=} \{[A] \to [A]\}.$$

- *Composition.* For each $A, B, C \in \mathrm{Obj}(X_{\mathrm{indisc}})$, the composition map

$$\circ_{A,B,C}^{X_{\mathrm{indisc}}} \colon \mathrm{Hom}_{X_{\mathrm{indisc}}}(B, C) \times \mathrm{Hom}_{X_{\mathrm{indisc}}}(A, B) \longrightarrow \mathrm{Hom}_{X_{\mathrm{indisc}}}(A, C)$$

of X_{disc} at (A, B, C) is defined by

$$([B] \to [C]) \circ ([A] \to [B]) \overset{\mathrm{def}}{=} ([A] \to [C]).$$

2. A category $\mathcal{C}$ is **indiscrete** if it is equivalent to X_{indisc} for some set X.

[1] *Further Terminology:* Sometimes called the **chaotic category on** X.

Proposition 8.2.4.2 ▶ Properties of Indiscrete Categories on Sets

Let X be a set.

1. *Functoriality.* The assignment $X \mapsto X_{\mathrm{indisc}}$ defines a functor

$$(-)_{\mathrm{indisc}} \colon \mathsf{Sets} \longrightarrow \mathsf{Cats}.$$

2. *Adjointness.* We have a quadruple adjunction

$$(\pi_0 \dashv (-)_{\mathrm{disc}} \dashv \mathrm{Obj} \dashv (-)_{\mathrm{indisc}})\colon \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left=20, "(-)_{\mathrm{disc}}"] \arrow[r, bend right=50, "(-)_{\mathrm{indisc}}"'] & \mathsf{Cats}. \arrow[l, bend right=50, "\pi_0"'] \arrow[l, bend left=20, "\mathrm{Obj}"] \end{tikzcd}$$

3. *Symmetric Strong Monoidality With Respect to Products.* The functor of Item 1 has a symmetric strong monoidal structure

$$\left((-)_{\mathrm{indisc}}, (-)_{\mathrm{indisc}}^{\times}, (-)_{\mathrm{indisc}|\mathbb{1}}^{\times}\right) \colon (\mathsf{Sets}, \times, \mathrm{pt}) \longrightarrow (\mathsf{Cats}, \times, \mathrm{pt}),$$

being equipped with isomorphisms

$$(-)^{\times}_{\mathrm{indisc}|X,Y}\colon X_{\mathrm{indisc}} \times Y_{\mathrm{indisc}} \xrightarrow{\cong} (X \times Y)_{\mathrm{indisc}},$$
$$(-)^{\times}_{\mathrm{indisc}|\mathbb{1}}\colon \mathrm{pt} \xrightarrow{\cong} \mathrm{pt}_{\mathrm{indisc}},$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

Proof 8.2.4.3 ► Proof of Proposition 8.2.4.2

Item 1: Functoriality

Clear.

Item 2: Adjointness

This is proved in Proposition 8.2.1.1.

Item 3: Symmetric Strong Monoidality With Respect to Products

Clear. □

8.3 Groupoids

8.3.1 Foundations

Let $\mathcal{C}$ be a category.

Definition 8.3.1.1 ► Isomorphisms

A morphism $f\colon A \to B$ of $\mathcal{C}$ is an **isomorphism** if there exists a morphism $f^{-1}\colon B \to A$ of $\mathcal{C}$ such that

$$f \circ f^{-1} = \mathrm{id}_B,$$
$$f^{-1} \circ f = \mathrm{id}_A.$$

Notation 8.3.1.2 ► The Set of Isomorphisms Between Two Objects in a Category

We write $\mathrm{Iso}_{\mathcal{C}}(A, B)$ for the set of all isomorphisms in $\mathcal{C}$ from A to B.

Definition 8.3.1.3 ► Groupoids

A **groupoid** is a category in which every morphism is an isomorphism.

8.3.2 The Groupoid Completion of a Category

Let $\mathcal{C}$ be a category.

Definition 8.3.2.1 ▶ The Groupoid Completion of a Category

The **groupoid completion of** $\mathcal{C}$[1] is the pair $(\mathsf{K}_0(\mathcal{C}), \iota_{\mathcal{C}})$ consisting of

- A groupoid $\mathsf{K}_0(\mathcal{C})$;
- A functor $\iota_{\mathcal{C}} \colon \mathcal{C} \to \mathsf{K}_0(\mathcal{C})$;

satisfying the following universal property:[2]

(**UP**) Given another such pair $(\mathcal{G}, i)$, there exists a unique functor $\mathsf{K}_0(\mathcal{C}) \xrightarrow{\exists!} \mathcal{G}$ making the diagram

$$\begin{tikzcd} & \mathsf{K}_0(\mathcal{C}) \arrow[d, dashed, "\exists!"] \\ \mathcal{C} \arrow[ur, "\iota_{\mathcal{C}}"] \arrow[r, "i"'] & \mathcal{G} \end{tikzcd}$$

commute.

[1] *Further Terminology:* Also called the **Grothendieck groupoid of** $\mathcal{C}$ or the **Grothendieck groupoid completion of** $\mathcal{C}$.

[2] See Item 3 of Proposition 8.3.2.4 for an explicit construction.

Construction 8.3.2.2 ▶ Construction of the Groupoid Completion of a Category

Concretely, the groupoid completion of $\mathcal{C}$ is the Gabriel–Zisman localisation $\mathrm{Mor}(\mathcal{C})^{-1}\mathcal{C}$ of $\mathcal{C}$ at the set $\mathrm{Mor}(\mathcal{C})$ of all morphisms of $\mathcal{C}$; see **??**.
(To be expanded upon later on.)

Proof 8.3.2.3 ▶ Proof of Construction 8.3.2.2

Omitted.

Proposition 8.3.2.4 ▶ Properties of Groupoid Completion

Let $\mathcal{C}$ be a category.

1. *Functoriality.* The assignment $\mathcal{C} \mapsto \mathsf{K}_0(\mathcal{C})$ defines a functor

$$\mathsf{K}_0 \colon \mathsf{Cats} \to \mathsf{Grpd}.$$

2. *2-Functoriality.* The assignment $\mathcal{C} \mapsto \mathsf{K}_0(\mathcal{C})$ defines a 2-functor

$$\mathsf{K}_0 \colon \mathsf{Cats}_2 \to \mathsf{Grpd}_2.$$

3. *Adjointness.* We have an adjunction

$$(\mathsf{K}_0 \dashv \iota)\colon\quad \begin{tikzcd} \mathsf{Cats} \arrow[r, bend left, "\mathsf{K}_0"] \arrow[r, phantom, "\perp"] & \mathsf{Grpd}, \arrow[l, bend left, hook, "\iota"] \end{tikzcd}$$

witnessed by a bijection of sets

$$\mathrm{Hom}_{\mathsf{Grpd}}(\mathsf{K}_0(\mathcal{C}), \mathcal{G}) \cong \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, \mathcal{G}),$$

natural in $\mathcal{C} \in \mathrm{Obj}(\mathsf{Cats})$ and $\mathcal{G} \in \mathrm{Obj}(\mathsf{Grpd})$, forming, together with the functor Core of Item 1 of Proposition 8.3.3.5, a triple adjunction

$$(\mathsf{K}_0 \dashv \iota \dashv \mathrm{Core})\colon\quad \begin{tikzcd} \mathsf{Cats} \arrow[r, bend left=50, "\mathsf{K}_0"] \arrow[r, bend right=50, "\mathrm{Core}"'] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, phantom, shift right=3, "\perp"] & \mathsf{Grpd}, \arrow[l, hook', "\iota" description] \end{tikzcd}$$

witnessed by bijections of sets

$$\begin{aligned} \mathrm{Hom}_{\mathsf{Grpd}}(\mathsf{K}_0(\mathcal{C}), \mathcal{G}) &\cong \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, \mathcal{G}), \\ \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{G}, \mathcal{D}) &\cong \mathrm{Hom}_{\mathsf{Grpd}}(\mathcal{G}, \mathrm{Core}(\mathcal{D})), \end{aligned}$$

natural in $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$ and $\mathcal{G} \in \mathrm{Obj}(\mathsf{Grpd})$.

4. *2-Adjointness.* We have a 2-adjunction

$$(\mathsf{K}_0 \dashv \iota)\colon\quad \begin{tikzcd} \mathsf{Cats} \arrow[r, bend left, "\mathsf{K}_0"] \arrow[r, phantom, "\perp_2"] & \mathsf{Grpd}, \arrow[l, bend left, hook, "\iota"] \end{tikzcd}$$

witnessed by an isomorphism of categories

$$\mathsf{Fun}(\mathsf{K}_0(\mathcal{C}), \mathcal{G}) \cong \mathsf{Fun}(\mathcal{C}, \mathcal{G}),$$

natural in $\mathcal{C} \in \mathrm{Obj}(\mathsf{Cats})$ and $\mathcal{G} \in \mathrm{Obj}(\mathsf{Grpd})$, forming, together with the 2-functor Core of Item 2 of Proposition 8.3.3.5, a triple 2-adjunction

$$(\mathsf{K}_0 \dashv \iota \dashv \mathrm{Core})\colon\quad \begin{tikzcd} \mathsf{Cats} \arrow[r, bend left=50, "\mathsf{K}_0"] \arrow[r, bend right=50, "\mathrm{Core}"'] \arrow[r, phantom, shift left=3, "\perp_2"] \arrow[r, phantom, shift right=3, "\perp_2"] & \mathsf{Grpd}, \arrow[l, hook', "\iota" description] \end{tikzcd}$$

witnessed by isomorphisms of categories

$$\begin{aligned}\mathsf{Fun}(\mathsf{K}_0(\mathcal{C}), \mathcal{G}) &\cong \mathsf{Fun}(\mathcal{C}, \mathcal{G}),\\ \mathsf{Fun}(\mathcal{G}, \mathcal{D}) &\cong \mathsf{Fun}(\mathcal{G}, \mathsf{Core}(\mathcal{D})),\end{aligned}$$

natural in $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$ and $\mathcal{G} \in \mathrm{Obj}(\mathsf{Grpd})$.

5. *Interaction With Classifying Spaces.* We have an isomorphism of groupoids

$$\mathsf{K}_0(\mathcal{C}) \cong \Pi_{\leq 1}(|\mathsf{N}_\bullet(\mathcal{C})|),$$

natural in $\mathcal{C} \in \mathrm{Obj}(\mathsf{Cats})$; i.e. the diagram

$$\begin{tikzcd} \mathsf{Cats} \arrow[r, "\mathsf{K}_0"] \arrow[d, "\mathsf{N}_\bullet"'] \arrow[dr, phantom, "\wr"] & \mathsf{Grp} \\ \mathsf{sSets} \arrow[r, "|-|"'] & \mathsf{Top} \arrow[u, "\Pi_{\leq 1}"'] \end{tikzcd}$$

commutes up to natural isomorphism.

6. *Symmetric Strong Monoidality With Respect to Coproducts.* The groupoid completion functor of Item 1 has a symmetric strong monoidal structure

$$\left(\mathsf{K}_0, \mathsf{K}_0^{\coprod}, \mathsf{K}_{0|\mathbb{1}}^{\coprod}\right) \colon (\mathsf{Cats}, \coprod, \emptyset_{\mathsf{cat}}) \to (\mathsf{Grpd}, \coprod, \emptyset_{\mathsf{cat}})$$

being equipped with isomorphisms

$$\begin{aligned}\mathsf{K}_{0|\mathcal{C},\mathcal{D}}^{\coprod} &\colon \mathsf{K}_0(\mathcal{C}) \coprod \mathsf{K}_0(\mathcal{D}) \xrightarrow{\cong} \mathsf{K}_0(\mathcal{C} \coprod \mathcal{D}),\\ \mathsf{K}_{0|\mathbb{1}}^{\coprod} &\colon \emptyset_{\mathsf{cat}} \xrightarrow{\cong} \mathsf{K}_0(\emptyset_{\mathsf{cat}}),\end{aligned}$$

natural in $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$.

7. *Symmetric Strong Monoidality With Respect to Products.* The groupoid completion functor of Item 1 has a symmetric strong monoidal structure

$$\left(\mathsf{K}_0, \mathsf{K}_0^{\times}, \mathsf{K}_{0|\mathbb{1}}^{\times}\right) \colon (\mathsf{Cats}, \times, \mathrm{pt}) \to (\mathsf{Grpd}, \times, \mathrm{pt})$$

being equipped with isomorphisms

$$\begin{aligned}\mathsf{K}_{0|\mathcal{C},\mathcal{D}}^{\times} &\colon \mathsf{K}_0(\mathcal{C}) \times \mathsf{K}_0(\mathcal{D}) \xrightarrow{\cong} \mathsf{K}_0(\mathcal{C} \times \mathcal{D}),\\ \mathsf{K}_{0|\mathbb{1}}^{\times} &\colon \mathrm{pt} \xrightarrow{\cong} \mathsf{K}_0(\mathrm{pt}),\end{aligned}$$

natural in $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$.

Proof 8.3.2.5 ► Proof of Proposition 8.3.2.4

Item 1: Functoriality

Omitted.

Item 2: 2-Functoriality

Omitted.

Item 3: Adjointness

Omitted.

Item 4: 2-Adjointness

Omitted.

Item 5: Interaction With Classifying Spaces

See Corollary 18.33 of https://web.ma.utexas.edu/users/dafr/M392C-2012/Notes/lecture18.pdf.

Item 6: Symmetric Strong Monoidality With Respect to Coproducts

Omitted.

Item 7: Symmetric Strong Monoidality With Respect to Products

Omitted.

8.3.3 The Core of a Category

Let $\mathcal{C}$ be a category.

Definition 8.3.3.1 ► The Core of a Category

The **core** of $\mathcal{C}$ is the pair $(\mathrm{Core}(\mathcal{C}), \iota_{\mathcal{C}})$ consisting of

- A groupoid $\mathrm{Core}(\mathcal{C})$;
- A functor $\iota_{\mathcal{C}} \colon \mathrm{Core}(\mathcal{C}) \hookrightarrow \mathcal{C}$;

satisfying the following universal property:

(UP) Given another such pair $(\mathcal{G}, i)$, there exists a unique functor $\mathcal{G} \xrightarrow{\exists!} \mathrm{Core}(\mathcal{C})$ making the diagram

$$\begin{tikzcd}
 & \mathrm{Core}(\mathcal{C}) \arrow[d, hook, "\iota_{\mathcal{C}}"] \\
\mathcal{G} \arrow[ur, dashed, "\exists!"] \arrow[r, "i"'] & \mathcal{C}
\end{tikzcd}$$

commute.

Notation 8.3.3.2 ▶ Alternative Notation for the Core of a Category

We also write $\mathcal{C}^{\simeq}$ for $\mathsf{Core}(\mathcal{C})$.

Construction 8.3.3.3 ▶ Construction of the Core of a Category

The core of $\mathcal{C}$ is the wide subcategory of $\mathcal{C}$ spanned by the isomorphisms of $\mathcal{C}$, i.e. the category $\mathsf{Core}(\mathcal{C})$ where[1]

1. *Objects.* We have
$$\mathrm{Obj}(\mathsf{Core}(\mathcal{C})) \stackrel{\text{def}}{=} \mathrm{Obj}(\mathcal{C}).$$
2. *Morphisms.* The morphisms of $\mathsf{Core}(\mathcal{C})$ are the isomorphisms of $\mathcal{C}$.

[1] *Slogan:* The groupoid $\mathsf{Core}(\mathcal{C})$ is the maximal subgroupoid of $\mathcal{C}$.

Proof 8.3.3.4 ▶ Proof of Construction 8.3.3.3

This follows from the fact that functors preserve isomorphisms (Item 1 of Proposition 8.4.1.8). □

Proposition 8.3.3.5 ▶ Properties of the Core of a Category

Let $\mathcal{C}$ be a category.

1. *Functoriality.* The assignment $\mathcal{C} \mapsto \mathsf{Core}(\mathcal{C})$ defines a functor
$$\mathsf{Core}\colon \mathsf{Cats} \to \mathsf{Grpd}.$$
2. *2-Functoriality.* The assignment $\mathcal{C} \mapsto \mathsf{Core}(\mathcal{C})$ defines a 2-functor
$$\mathsf{Core}\colon \mathsf{Cats}_2 \to \mathsf{Grpd}_2.$$
3. *Adjointness.* We have an adjunction
$$(\iota \dashv \mathsf{Core})\colon\quad \begin{tikzcd} \mathsf{Grpd} \arrow[r, bend left, "\iota"] \arrow[r, phantom, "\perp"] & \mathsf{Cats} \arrow[l, bend left, "\mathsf{Core}"] \end{tikzcd},$$
witnessed by a bijection of sets
$$\mathrm{Hom}_{\mathsf{Cats}}(\mathcal{G}, \mathcal{D}) \cong \mathrm{Hom}_{\mathsf{Grpd}}(\mathcal{G}, \mathsf{Core}(\mathcal{D})),$$

natural in $\mathcal{G} \in \mathrm{Obj}(\mathsf{Grpd})$ and $\mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$, forming, together with the functor K_0 of Item 1 of Proposition 8.3.2.4, a triple adjunction

$$(\mathsf{K}_0 \dashv \iota \dashv \mathsf{Core})\colon\quad \begin{tikzcd} \mathsf{Cats} \arrow[r, bend left=50, "\mathsf{K}_0"] \arrow[r, bend right=50, "\mathsf{Core}"'] & \mathsf{Grpd}, \arrow[l, "\iota" description] \end{tikzcd}$$

witnessed by bijections of sets

$$\begin{aligned} \mathrm{Hom}_{\mathsf{Grpd}}(\mathsf{K}_0(\mathcal{C}), \mathcal{G}) &\cong \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, \mathcal{G}),\\ \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{G}, \mathcal{D}) &\cong \mathrm{Hom}_{\mathsf{Grpd}}(\mathcal{G}, \mathsf{Core}(\mathcal{D})), \end{aligned}$$

natural in $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$ and $\mathcal{G} \in \mathrm{Obj}(\mathsf{Grpd})$.

4. *2-Adjointness.* We have an adjunction

$$(\iota \dashv \mathsf{Core})\colon\quad \begin{tikzcd} \mathsf{Grpd} \arrow[r, bend left, "\iota"] \arrow[r, phantom, "\perp_2"] & \mathsf{Cats}, \arrow[l, bend left, "\mathsf{Core}"] \end{tikzcd}$$

witnessed by an isomorphism of categories

$$\mathsf{Fun}(\mathcal{G}, \mathcal{D}) \cong \mathsf{Fun}(\mathcal{G}, \mathsf{Core}(\mathcal{D})),$$

natural in $\mathcal{G} \in \mathrm{Obj}(\mathsf{Grpd})$ and $\mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$, forming, together with the 2-functor K_0 of Item 2 of Proposition 8.3.2.4, a triple 2-adjunction

$$(\mathsf{K}_0 \dashv \iota \dashv \mathsf{Core})\colon\quad \begin{tikzcd} \mathsf{Cats} \arrow[r, bend left=50, "\mathsf{K}_0"] \arrow[r, bend right=50, "\mathsf{Core}"'] & \mathsf{Grpd}, \arrow[l, "\iota" description] \end{tikzcd}$$

witnessed by isomorphisms of categories

$$\begin{aligned} \mathsf{Fun}(\mathsf{K}_0(\mathcal{C}), \mathcal{G}) &\cong \mathsf{Fun}(\mathcal{C}, \mathcal{G}),\\ \mathsf{Fun}(\mathcal{G}, \mathcal{D}) &\cong \mathsf{Fun}(\mathcal{G}, \mathsf{Core}(\mathcal{D})), \end{aligned}$$

natural in $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$ and $\mathcal{G} \in \mathrm{Obj}(\mathsf{Grpd})$.

5. *Symmetric Strong Monoidality With Respect to Products.* The core functor of Item 1 has a symmetric strong monoidal structure

$$(\mathsf{Core}, \mathsf{Core}^{\times}, \mathsf{Core}^{\times}_{\mathbb{1}})\colon (\mathsf{Cats}, \times, \mathrm{pt}) \longrightarrow (\mathsf{Grpd}, \times, \mathrm{pt})$$

being equipped with isomorphisms

$$\mathrm{Core}^{\times}_{\mathcal{C},\mathcal{D}} \colon \mathrm{Core}(\mathcal{C}) \times \mathrm{Core}(\mathcal{D}) \xrightarrow{\cong} \mathrm{Core}(\mathcal{C} \times \mathcal{D}),$$
$$\mathrm{Core}^{\times}_{\mathbb{1}} \colon \mathrm{pt} \xrightarrow{\cong} \mathrm{Core}(\mathrm{pt}),$$

natural in $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$.

6. *Symmetric Strong Monoidality With Respect to Coproducts.* The core functor of Item 1 has a symmetric strong monoidal structure

$$\left(\mathrm{Core}, \mathrm{Core}^{\coprod}, \mathrm{Core}^{\coprod}_{\mathbb{1}}\right) \colon (\mathsf{Cats}, \coprod, \emptyset_{\mathsf{cat}}) \to (\mathsf{Grpd}, \coprod, \emptyset_{\mathsf{cat}})$$

being equipped with isomorphisms

$$\mathrm{Core}^{\coprod}_{\mathcal{C},\mathcal{D}} \colon \mathrm{Core}(\mathcal{C}) \coprod \mathrm{Core}(\mathcal{D}) \xrightarrow{\cong} \mathrm{Core}(\mathcal{C} \coprod \mathcal{D}),$$
$$\mathrm{Core}^{\coprod}_{\mathbb{1}} \colon \emptyset_{\mathsf{cat}} \xrightarrow{\cong} \mathrm{Core}(\emptyset_{\mathsf{cat}}),$$

natural in $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$.

Proof 8.3.3.6 ► Proof of Proposition 8.3.3.5

Item 1: Functoriality

Omitted.

Item 2: 2-Functoriality

Omitted.

Item 3: Adjointness

Omitted.

Item 4: 2-Adjointness

Omitted.

Item 5: Symmetric Strong Monoidality With Respect to Products

Omitted.

Item 6: Symmetric Strong Monoidality With Respect to Coproducts

Omitted.

8.4 Functors

8.4.1 Foundations

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

A **functor** $F\colon \mathcal{C} \to \mathcal{D}$ **from** $\mathcal{C}$ **to** $\mathcal{D}$[1] consists of:

1. *Action on Objects.* A map of sets
$$F\colon \mathrm{Obj}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{D}),$$
called the **action on objects of** F.

2. *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathcal{C})$, a map
$$F_{A,B}\colon \mathrm{Hom}_{\mathcal{C}}(A,B) \to \mathrm{Hom}_{\mathcal{D}}(F(A),F(B)),$$
called the **action on morphisms of** F **at** (A,B)[2].

satisfying the following conditions:

1. *Preservation of Identities.* For each $A \in \mathrm{Obj}(\mathcal{C})$, the diagram
$$\begin{tikzcd} \mathrm{pt} \arrow[d, "\mathbb{1}^{\mathcal{C}}_A"'] \arrow[dr, "\mathbb{1}^{\mathcal{D}}_{F(A)}"] & \\ \mathrm{Hom}_{\mathcal{C}}(A,A) \arrow[r, "F_{A,A}"'] & \mathrm{Hom}_{\mathcal{D}}(F(A),F(A)) \end{tikzcd}$$
commutes, i.e. we have
$$F(\mathrm{id}_A) = \mathrm{id}_{F(A)}.$$

2. *Preservation of Composition.* For each $A, B, C \in \mathrm{Obj}(\mathcal{C})$, the diagram
$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(B,C) \times \mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[r, "\circ^{\mathcal{C}}_{A,B,C}"] \arrow[d, "F_{B,C}\times F_{A,B}"'] & \mathrm{Hom}_{\mathcal{C}}(A,C) \arrow[d, "F_{A,C}"] \\ \mathrm{Hom}_{\mathcal{D}}(F(B),F(C)) \times \mathrm{Hom}_{\mathcal{D}}(F(A),F(B)) \arrow[r, "\circ^{\mathcal{D}}_{F(A),F(B),F(C)}"'] & \mathrm{Hom}_{\mathcal{D}}(F(A),F(C)) \end{tikzcd}$$
commutes, i.e. for each composable pair (g, f) of morphisms of $\mathcal{C}$, we have
$$F(g \circ f) = F(g) \circ F(f).$$

[1] *Further Terminology:* Also called a **covariant functor**.
[2] *Further Terminology:* Also called **action on Hom-sets of** F **at** (A, B).

Notation 8.4.1.2 ▶ Subscript and Superscript Notation for Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories, and write $\mathcal{C}^{\mathsf{op}}$ for the opposite category of $\mathcal{C}$ of **??**.

1. Given a functor
$$F\colon \mathcal{C} \to \mathcal{D},$$
we also write F_A for $F(A)$.

2. Given a functor
$$F\colon \mathcal{C}^{\mathsf{op}} \to \mathcal{D},$$
we also write F^A for $F(A)$.

3. Given a functor
$$F\colon \mathcal{C} \times \mathcal{C} \to \mathcal{D},$$
we also write $F_{A,B}$ for $F(A, B)$.

4. Given a functor
$$F\colon \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathcal{D},$$
we also write F^A_B for $F(A, B)$.

We employ a similar notation for morphisms, writing e.g. F_f for $F(f)$ given a functor $F\colon \mathcal{C} \to \mathcal{D}$.

Notation 8.4.1.3 ▶ Additional Notation for Functors

Following the notation $[\![x \mapsto f(x)]\!]$ for a function $f\colon X \to Y$ introduced in Notation 1.1.1.2, we will sometimes denote a functor $F\colon \mathcal{C} \to \mathcal{D}$ by
$$F \overset{\text{def}}{=} [\![A \mapsto F(A)]\!],$$
specially when the action on morphisms of F is clear from its action on objects.

Example 8.4.1.4 ▶ Identity Functors

The **identity functor** of a category $\mathcal{C}$ is the functor $\mathrm{id}_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{C}$ where

1. *Action on Objects.* For each $A \in \mathrm{Obj}(\mathcal{C})$, we have
$$\mathrm{id}_{\mathcal{C}}(A) \overset{\text{def}}{=} A.$$

2. *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathcal{C})$, the action on morphisms

$$(\mathrm{id}_{\mathcal{C}})_{A,B} \colon \mathrm{Hom}_{\mathcal{C}}(A,B) \to \underbrace{\mathrm{Hom}_{\mathcal{C}}(\mathrm{id}_{\mathcal{C}}(A), \mathrm{id}_{\mathcal{C}}(B))}_{\stackrel{\text{def}}{=}\mathrm{Hom}_{\mathcal{C}}(A,B)}$$

of $\mathrm{id}_{\mathcal{C}}$ at (A, B) is defined by

$$(\mathrm{id}_{\mathcal{C}})_{A,B} \stackrel{\text{def}}{=} \mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(A,B)}.$$

Proof 8.4.1.5 ► Proof of Example 8.4.1.4

Preservation of Identities

We have $\mathrm{id}_{\mathcal{C}}(\mathrm{id}_A) \stackrel{\text{def}}{=} \mathrm{id}_A$ for each $A \in \mathrm{Obj}(\mathcal{C})$ by definition.

Preservation of Compositions

For each composable pair $A \xrightarrow{f} B \xrightarrow{g} B$ of morphisms of $\mathcal{C}$, we have

$$\begin{aligned} \mathrm{id}_{\mathcal{C}}(g \circ f) &\stackrel{\text{def}}{=} g \circ f \\ &\stackrel{\text{def}}{=} \mathrm{id}_{\mathcal{C}}(g) \circ \mathrm{id}_{\mathcal{C}}(f). \end{aligned}$$

This finishes the proof. ∎

Definition 8.4.1.6 ► Composition of Functors

The **composition** of two functors $F \colon \mathcal{C} \to \mathcal{D}$ and $G \colon \mathcal{D} \to \mathcal{E}$ is the functor $G \circ F$ where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathcal{C})$, we have

$$[G \circ F](A) \stackrel{\text{def}}{=} G(F(A)).$$

- *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathcal{C})$, the action on morphisms

$$(G \circ F)_{A,B} \colon \mathrm{Hom}_{\mathcal{C}}(A,B) \to \mathrm{Hom}_{\mathcal{E}}(G_{F_A}, G_{F_B})$$

of $G \circ F$ at (A, B) is defined by

$$[G \circ F](f) \stackrel{\text{def}}{=} G(F(f)).$$

Proof 8.4.1.7 ► Proof of Definition 8.4.1.6

Preservation of Identities

For each $A \in \mathrm{Obj}(\mathcal{C})$, we have

$$\begin{aligned} G_{F_{\mathrm{id}_A}} &= G_{\mathrm{id}_{F_A}} && \text{(functoriality of } F\text{)} \\ &= \mathrm{id}_{G_{F_A}}. && \text{(functoriality of } G\text{)} \end{aligned}$$

Preservation of Composition

For each composable pair (g, f) of morphisms of $\mathcal{C}$, we have

$$\begin{aligned} G_{F_{g\circ f}} &= G_{F_g \circ F_f} && \text{(functoriality of } F\text{)} \\ &= G_{F_g} \circ G_{F_f}. && \text{(functoriality of } G\text{)} \end{aligned}$$

This finishes the proof. ∎

Proposition 8.4.1.8 ► Elementary Properties of Functors

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor.

1. *Preservation of Isomorphisms.* If f is an isomorphism in $\mathcal{C}$, then $F(f)$ is an isomorphism in $\mathcal{D}$.[1]

[1] When the converse holds, we call F *conservative*, see Definition 8.5.4.1.

Proof 8.4.1.9 ► Proof of Proposition 8.4.1.8

Item 1: Preservation of Isomorphisms

Indeed, we have

$$\begin{aligned} F(f)^{-1} \circ F(f) &= F\left(f^{-1} \circ f\right) \\ &= F(\mathrm{id}_A) \\ &= \mathrm{id}_{F(A)} \end{aligned}$$

and

$$\begin{aligned} F(f) \circ F(f)^{-1} &= F\left(f \circ f^{-1}\right) \\ &= F(\mathrm{id}_B) \\ &= \mathrm{id}_{F(B)}, \end{aligned}$$

showing $F(f)$ to be an isomorphism. ∎

8.4.2 Contravariant Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories, and let $\mathcal{C}^{\mathsf{op}}$ denote the opposite category of $\mathcal{C}$ of **??**.

Definition 8.4.2.1 ▶ Contravariant Functors

A **contravariant functor** from $\mathcal{C}$ to $\mathcal{D}$ is a functor from $\mathcal{C}^{\mathsf{op}}$ to $\mathcal{D}$.

Remark 8.4.2.2 ▶ Unwinding Definition 8.4.2.1

In detail, a **contravariant functor** from $\mathcal{C}$ to $\mathcal{D}$ consists of:

1. *Action on Objects.* A map of sets

$$F\colon \mathrm{Obj}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{D}),$$

called the **action on objects of** F.

2. *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathcal{C})$, a map

$$F_{A,B}\colon \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F(B), F(A)),$$

called the **action on morphisms of** F **at** (A, B).

satisfying the following conditions:

1. *Preservation of Identities.* For each $A \in \mathrm{Obj}(\mathcal{C})$, the diagram

$$\begin{tikzcd} \mathrm{pt} \arrow[d, "\mathbb{1}^{\mathcal{C}}_{A}"'] \arrow[dr, "\mathbb{1}^{\mathcal{D}}_{F(A)}"] & \\ \mathrm{Hom}_{\mathcal{C}}(A, A) \arrow[r, "F_{A,A}"'] & \mathrm{Hom}_{\mathcal{D}}(F(A), F(A)) \end{tikzcd}$$

commutes, i.e. we have

$$F(\mathrm{id}_A) = \mathrm{id}_{F(A)}.$$

2. *Preservation of Composition.* For each $A, B, C \in \mathrm{Obj}(\mathcal{C})$, the diagram

$$\begin{tikzcd}
 & \mathrm{Hom}_{\mathcal{D}}(F(C), F(B)) \times \mathrm{Hom}_{\mathcal{D}}(F(B), F(A)) \arrow[dr, dashed, "\sigma^{\mathsf{Sets}}_{\mathrm{Hom}_{\mathcal{D}}(F(C),F(B)),\mathrm{Hom}_{\mathcal{D}}(F(B),F(A))}", "\sim"'] & \\
\mathrm{Hom}_{\mathcal{C}}(B, C) \times \mathrm{Hom}_{\mathcal{C}}(A, B) \arrow[ur, "F_{B,C} \times F_{A,B}"] \arrow[d, "\circ^{\mathcal{C}}_{A,B,C}"'] & & \mathrm{Hom}_{\mathcal{D}}(F(B), F(A)) \times \mathrm{Hom}_{\mathcal{D}}(F(C), F(B)) \arrow[d, "\circ^{\mathcal{D}}_{F(C),F(B),F(A)}"] \\
\mathrm{Hom}_{\mathcal{C}}(A, C) \arrow[rr, "F_{A,C}"'] & & \mathrm{Hom}_{\mathcal{D}}(F(C), F(A))
\end{tikzcd}$$

commutes, i.e. for each composable pair (g, f) of morphisms of $\mathcal{C}$, we have

$$F(g \circ f) = F(f) \circ F(g).$$

REMARK 8.4.2.3 ► ON THE TERM CONTRAVARIANT FUNCTOR

Throughout this work we will not use the term "contravariant" functor, speaking instead simply of functors $F : \mathcal{C}^{\mathsf{op}} \to \mathcal{D}$. We will usually, however, write

$$F_{A,B} : \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F(B), F(A))$$

for the action on morphisms

$$F_{A,B} : \mathrm{Hom}_{\mathcal{C}^{\mathsf{op}}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F(A), F(B))$$

of F, as well as write $F(g \circ f) = F(f) \circ F(g)$.

8.4.3 Forgetful Functors

DEFINITION 8.4.3.1 ► FORGETFUL FUNCTORS

There isn't a precise definition of a **forgetful functor**.

Remark 8.4.3.2 ▶ Unwinding Definition 8.4.3.1

Despite there not being a formal or precise definition of a forgetful functor, the term is often very useful in practice, similarly to the word "canonical". The idea is that a "forgetful functor" is a functor that forgets structure or properties, and is best explained through examples, such as the ones below (see Examples 8.4.3.3 and 8.4.3.4).

Example 8.4.3.3 ▶ Forgetful Functors That Forget Structure

Examples of forgetful functors that forget structure include:

1. *Forgetting Group Structures.* The functor $\mathsf{Grp} \longrightarrow \mathsf{Sets}$ sending a group (G, μ_G, η_G) to its underlying set G, forgetting the multiplication and unit maps μ_G and η_G of G.

2. *Forgetting Topologies.* The functor $\mathsf{Top} \longrightarrow \mathsf{Sets}$ sending a topological space $(X, \mathcal{T}_X)$ to its underlying set X, forgetting the topology $\mathcal{T}_X$.

3. *Forgetting Fibrations.* The functor $\mathsf{FibSets}(K) \longrightarrow \mathsf{Sets}$ sending a K-fibred set $\phi_X \colon X \longrightarrow K$ to the set X, forgetting the map ϕ_X and the base set K.

Example 8.4.3.4 ▶ Forgetful Functors That Forget Properties

Examples of forgetful functors that forget properties include:

1. *Forgetting Commutativity.* The inclusion functor $\iota \colon \mathsf{CMon} \hookrightarrow \mathsf{Mon}$ which forgets the property of being commutative.

2. *Forgetting Inverses.* The inclusion functor $\iota \colon \mathsf{Grp} \hookrightarrow \mathsf{Mon}$ which forgets the property of having inverses.

Notation 8.4.3.5 ▶ Notation For Forgetful Functors That Forget Structure

Throughout this work, we will denote forgetful functors that forget structure by 忘, e.g. as in

$$忘\colon \mathsf{Grp} \longrightarrow \mathsf{Sets}.$$

The symbol 忘, pronounced *wasureru* (see Item 1 of Remark 8.4.3.6 below), means *to forget*, and is a kanji found in the following words in Japanese and Chinese:

1. 忘れる, transcribed as *wasureru*, meaning *to forget*.

2. 忘却関手, transcribed as *boukyaku kanshu*, meaning *forgetful functor*.
3. 忘记 or 忘記, transcribed as *wàngjì*, meaning *to forget*.
4. 遗忘函子 or 遺忘函子, transcribed as *yíwàng hánzǐ*, meaning *forgetful functor*.

Remark 8.4.3.6 ► Pronunciation of the Words in Notation 8.4.3.5

Here we collect the pronunciation of the words in Notation 8.4.3.5 for accuracy and completeness.

1. Pronunciation of 忘れる:
   - Audio: see https://topological-modular-forms.github.io/the-clowder-project/static/sounds/wasureru-01.mp3
   - IPA broad transcription: [wäsɯɾe̞ɾɯ].
   - IPA narrow transcription: [ɰᵝäsɨᵝɾe̞ɾɯ̟ᵝ].
2. Pronunciation of 忘却関手: Pronunciation:
   - Audio: see https://topological-modular-forms.github.io/the-clowder-project/static/sounds/wasureru-02.mp3
   - IPA broad transcription: [bo̞ːkʲäkɯ kä̃ɰ̃ɕɯ].
   - IPA narrow transcription: [bo̞ːkʲäkɯ̟ᵝ kä̃ɰ̃ɕɯᵝ].
3. Pronunciation of 忘记:
   - Audio: see https://topological-modular-forms.github.io/the-clowder-project/static/sounds/wasureru-03.ogg
   - Broad IPA transcription: [wɑŋtɕi].
   - Sinological IPA transcription: [wɑŋ$^{51-53}$t͡ɕi^{51}].
4. Pronunciation of 遗忘函子:
   - Audio: see https://topological-modular-forms.github.io/the-clowder-project/static/sounds/wasureru-04.mp3
   - Broad IPA transcription: [iwɑŋ xänt͡sz̩ɨ].
   - Sinological IPA transcription: [i^{35}wɑŋ^{51} xän^{35}t͡sz̩$^{214-21(4)}$].

8.4.4 The Natural Transformation Associated to a Functor

Every functor $F: \mathcal{C} \to \mathcal{D}$ defines a natural transformation[1]

$$F^{\dagger}\colon \mathrm{Hom}_{\mathcal{C}} \Longrightarrow \mathrm{Hom}_{\mathcal{D}} \circ (F^{\mathrm{op}} \times F), \qquad \begin{tikzcd} \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \arrow[rr, "F^{\mathrm{op}}\times F"] \arrow[dr, "\mathrm{Hom}_{\mathcal{C}}"', ""{name=L}] & & \mathcal{D}^{\mathrm{op}} \times \mathcal{D} \arrow[dl, "\mathrm{Hom}_{\mathcal{D}}", ""{name=R}] \\ & \mathrm{Sets}, & \arrow[from=L, to=R, Rightarrow, "F^{\dagger}"] \end{tikzcd}$$

called the **natural transformation associated to** F, consisting of the collection

$$\left\{F^{\dagger}_{A,B}\colon \mathrm{Hom}_{\mathcal{C}}(A,B) \to \mathrm{Hom}_{\mathcal{D}}(F_A, F_B)\right\}_{(A,B)\in\mathrm{Obj}(\mathcal{C}^{\mathrm{op}}\times\mathcal{C})}$$

with

$$F^{\dagger}_{A,B} \stackrel{\mathrm{def}}{=} F_{A,B}.$$

[1] This is the 1-categorical version of Item 1 of Proposition 2.4.1.3.

PROOF 8.4.4.2 ▶ PROOF OF DEFINITION 8.4.4.1

The naturality condition for $F^{\dagger}$ is the requirement that for each morphism

$$(\phi,\psi)\colon (X,Y) \to (A,B)$$

of $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}$, the diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(X,Y) \arrow[r, "\phi^*\circ\psi_*=\psi_*\circ\phi^*"] \arrow[d, "F_{X,Y}"'] & \mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[d, "F_{A,B}"] \\ \mathrm{Hom}_{\mathcal{D}}(F_X,F_Y) \arrow[r, "F(\phi)^*\circ F(\psi)_*=F(\psi)_*\circ F(\phi)^*"'] & \mathrm{Hom}_{\mathcal{D}}(F_A,F_B), \end{tikzcd}$$

acting on elements as

$$\begin{tikzcd} f \arrow[r, mapsto] \arrow[d, mapsto] & \psi\circ f\circ\phi \arrow[d, mapsto] \\ F(f) \arrow[r, mapsto] & F(\psi)\circ F(f)\circ F(\psi) = F(\psi\circ f\circ\phi) \end{tikzcd}$$

commutes, which follows from the functoriality of F. ∎

Proposition 8.4.4.3 ▶ Properties of Natural Transformations Associated to Functors

Let $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{E}$ be functors.

1. *Interaction With Natural Isomorphisms.* The following conditions are equivalent:
   (a) The natural transformation $F^{\dagger}\colon \mathrm{Hom}_{\mathcal{C}} \Longrightarrow \mathrm{Hom}_{\mathcal{D}} \circ (F^{\mathsf{op}} \times F)$ associated to F is a natural isomorphism.
   (b) The functor F is fully faithful.
2. *Interaction With Composition.* We have an equality of pasting diagrams

$$
\begin{tikzcd}
\mathcal{C}^{\mathsf{op}} \times \mathcal{C} \arrow[r, "F^{\mathsf{op}}\times F"] \arrow[dr, "\mathrm{Hom}_{\mathcal{C}}"'] & \mathcal{D}^{\mathsf{op}} \times \mathcal{D} \arrow[r, "G^{\mathsf{op}}\times G"] \arrow[d, "\mathrm{Hom}_{\mathcal{D}}" description] \arrow[dl, phantom, "\overset{F^{\dagger}}{\Longrightarrow}"] & \mathcal{E}^{\mathsf{op}} \times \mathcal{E} \arrow[dl, "\mathrm{Hom}_{\mathcal{E}}"] \arrow[dl, phantom, shift right=4, "\overset{G^{\dagger}}{\Longrightarrow}"] \\
 & \mathrm{Sets} &
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{C}^{\mathsf{op}} \times \mathcal{C} \arrow[rr, "(G\circ F)^{\mathsf{op}}\times(G\circ F)"] \arrow[dr, "\mathrm{Hom}_{\mathcal{C}}"'] & \arrow[d, phantom, "\overset{(G\circ F)^{\dagger}}{\Longrightarrow}"] & \mathcal{E}^{\mathsf{op}} \times \mathcal{E}, \arrow[dl, "\mathrm{Hom}_{\mathcal{E}}"] \\
 & \mathrm{Sets} &
\end{tikzcd}
$$

   in Cats_2, i.e. we have

$$(G \circ F)^{\dagger} = \left(G^{\dagger} \star \mathrm{id}_{F^{\mathsf{op}}\times F}\right) \circ F^{\dagger}.$$

3. *Interaction With Identities.* We have

$$\mathrm{id}^{\dagger}_{\mathcal{C}} = \mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(-_1,-_2)},$$

   i.e. the natural transformation associated to $\mathrm{id}_{\mathcal{C}}$ is the identity natural transformation of the functor $\mathrm{Hom}_{\mathcal{C}}(-_1, -_2)$.

Proof 8.4.4.4 ▶ Proof of Proposition 8.4.4.3

Item 1: Interaction With Natural Isomorphisms

Clear.

Item 2: Interaction With Composition

Clear.

Item 3: Interaction With Identities

Clear. $\square$

8.5 Conditions on Functors

8.5.1 Faithful Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 8.5.1.1 ▶ Faithful Functors

A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **faithful** if, for each $A, B \in \mathrm{Obj}(\mathcal{C})$, the action on morphisms

$$F_{A,B}\colon \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F_A, F_B)$$

of F at (A, B) is injective.

Proposition 8.5.1.2 ▶ Properties of Faithful Functors

Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor.

1. *Interaction With Postcomposition.* The following conditions are equivalent:

   (a) The functor $F\colon \mathcal{C} \to \mathcal{D}$ is faithful.

   (b) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor

   $$F_*\colon \mathrm{Fun}(\mathcal{X}, \mathcal{C}) \to \mathrm{Fun}(\mathcal{X}, \mathcal{D})$$

   is faithful.

   (c) The functor $F\colon \mathcal{C} \to \mathcal{D}$ is a representably faithful morphism in Cats_2 in the sense of Definition 9.1.1.1.

2. *Interaction With Precomposition I.* Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor.

   (a) If F is faithful, then the precomposition functor

   $$F^*\colon \mathrm{Fun}(\mathcal{D}, \mathcal{X}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{X})$$

   can fail to be faithful.

   (b) Conversely, if the precomposition functor

   $$F^*\colon \mathrm{Fun}(\mathcal{D}, \mathcal{X}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{X})$$

   is faithful, then F *can fail* to be faithful.

3. *Interaction With Precomposition II.* If F is essentially surjective, then the precomposition functor

$$F^* \colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

is faithful.

4. *Interaction With Precomposition III.* The following conditions are equivalent:

   (a) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor

   $$F^* \colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

   is faithful.

   (b) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor

   $$F^* \colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

   is conservative.

   (c) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor

   $$F^* \colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

   is monadic.

   (d) The functor $F \colon \mathcal{C} \to \mathcal{D}$ is a corepresentably faithful morphism in Cats_2 in the sense of Definition 9.2.1.1.

   (e) The components

   $$\eta_G \colon G \Longrightarrow \mathrm{Ran}_F(G \circ F)$$

   of the unit

   $$\eta \colon \mathrm{id}_{\mathsf{Fun}(\mathcal{D},\mathcal{X})} \Longrightarrow \mathrm{Ran}_F \circ F^*$$

   of the adjunction $F^* \dashv \mathrm{Ran}_F$ are all monomorphisms.

   (f) The components

   $$\epsilon_G \colon \mathrm{Lan}_F(G \circ F) \Longrightarrow G$$

   of the counit

   $$\epsilon \colon \mathrm{Lan}_F \circ F^* \Longrightarrow \mathrm{id}_{\mathsf{Fun}(\mathcal{D},\mathcal{X})}$$

   of the adjunction $\mathrm{Lan}_F \dashv F^*$ are all epimorphisms.

(g) The functor F is dominant (Definition 8.6.1.1), i.e. every object of $\mathcal{D}$ is a retract of some object in $\mathrm{Im}(F)$:

($\star$) For each $B \in \mathrm{Obj}(\mathcal{D})$, there exist:
- An object A of $\mathcal{C}$;
- A morphism $s \colon B \to F(A)$ of $\mathcal{D}$;
- A morphism $r \colon F(A) \to B$ of $\mathcal{D}$;

such that $r \circ s = \mathrm{id}_B$.

Proof 8.5.1.3 ▸ Proof of Proposition 8.5.1.2

Item 1: Interaction With Postcomposition

Omitted.

Item 2: Interaction With Precomposition I

See [MSE 733163] for Item 2a. Item 2b follows from Item 3 and the fact that there are essentially surjective functors that are not faithful.

Item 3: Interaction With Precomposition II

Omitted, but see `https://unimath.github.io/doc/UniMath/d4de26f//UniMath.CategoryTheory.precomp_fully_faithful.html` for a formalised proof.

Item 4: Interaction With Precomposition III

We claim Items 4a to 4g are equivalent:

- *Items 4a and 4d Are Equivalent:* This is true by the definition of corepresentably faithful morphism; see Definition 9.2.1.1.
- *Items 4a to 4c and 4g Are Equivalent:* See [Adá+01, Proposition 4.1] or alternatively [Fre09, Lemmas 3.1 and 3.2] for the equivalence between Items 4a and 4g.
- *Items 4a, 4e and 4f Are Equivalent:* See **??** of **??**.

This finishes the proof. □

8.5.2 Full Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 8.5.2.1 ► Full Functors

A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **full** if, for each $A, B \in \mathrm{Obj}(\mathcal{C})$, the action on morphisms

$$F_{A,B}\colon \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F_A, F_B)$$

of F at (A, B) is surjective.

Proposition 8.5.2.2 ► Properties of Full Functors

Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor.

1. *Interaction With Postcomposition.* The following conditions are equivalent:

   (a) The functor $F\colon \mathcal{C} \to \mathcal{D}$ is full.

   (b) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor

   $$F_*\colon \mathrm{Fun}(\mathcal{X}, \mathcal{C}) \to \mathrm{Fun}(\mathcal{X}, \mathcal{D})$$

   is full.

   (c) The functor $F\colon \mathcal{C} \to \mathcal{D}$ is a representably full morphism in Cats_2 in the sense of Definition 9.1.2.1.

2. *Interaction With Precomposition I.* If F is full, then the precomposition functor

   $$F^*\colon \mathrm{Fun}(\mathcal{D}, \mathcal{X}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{X})$$

   can fail to be full.

3. *Interaction With Precomposition II.* If the precomposition functor

   $$F^*\colon \mathrm{Fun}(\mathcal{D}, \mathcal{X}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{X})$$

   is full, then F *can fail* to be full.

4. *Interaction With Precomposition III.* If F is essentially surjective and full, then the precomposition functor

   $$F^*\colon \mathrm{Fun}(\mathcal{D}, \mathcal{X}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{X})$$

   is full (and also faithful by Item 3 of Proposition 8.5.1.2).

5. *Interaction With Precomposition IV.* The following conditions are equivalent:

(a) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor

$$F^* \colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

is full.

(b) The functor $F \colon \mathcal{C} \to \mathcal{D}$ is a corepresentably full morphism in Cats_2 in the sense of Definition 9.2.1.1.

(c) The components

$$\eta_G \colon G \Longrightarrow \mathrm{Ran}_F(G \circ F)$$

of the unit

$$\eta \colon \mathrm{id}_{\mathsf{Fun}(\mathcal{D},\mathcal{X})} \Longrightarrow \mathrm{Ran}_F \circ F^*$$

of the adjunction $F^* \dashv \mathrm{Ran}_F$ are all retractions/split epimorphisms.

(d) The components

$$\epsilon_G \colon \mathrm{Lan}_F(G \circ F) \Longrightarrow G$$

of the counit

$$\epsilon \colon \mathrm{Lan}_F \circ F^* \Longrightarrow \mathrm{id}_{\mathsf{Fun}(\mathcal{D},\mathcal{X})}$$

of the adjunction $\mathrm{Lan}_F \dashv F^*$ are all sections/split monomorphisms.

(e) For each $B \in \mathrm{Obj}(\mathcal{D})$, there exist:

- An object A_B of $\mathcal{C}$;
- A morphism $s_B \colon B \to F(A_B)$ of $\mathcal{D}$;
- A morphism $r_B \colon F(A_B) \to B$ of $\mathcal{D}$;

satisfying the following condition:

(★) For each $A \in \mathrm{Obj}(\mathcal{C})$ and each pair of morphisms

$$r \colon F(A) \to B,$$
$$s \colon B \to F(A)$$

of $\mathcal{D}$, we have

$$[(A_B, s_B, r_B)] = [(A, s, r \circ s_B \circ r_B)]$$

in $\int^{A \in \mathcal{C}} h^{B'}_{F_A} \times h^{F_A}_{B}$.

Proof 8.5.2.3 ► Proof of Proposition 8.5.2.2

Item 1: Interaction With Postcomposition

Omitted.

Item 2: Interaction With Precomposition I

Omitted.

Item 3: Interaction With Precomposition II

See [BS10, p. 47].

Item 4: Interaction With Precomposition III

Omitted, but see `https://unimath.github.io/doc/UniMath/d4de26f//UniMath.CategoryTheory.precomp_fully_faithful.html` for a formalised proof.

Item 5: Interaction With Precomposition IV

We claim Items 5a to 5e are equivalent:

- *Items 5a and 5b Are Equivalent:* This is true by the definition of corepresentably full morphism; see Definition 9.2.2.1.
- *Items 5a, 5c and 5d Are Equivalent:* See **??** of **??**.
- *Items 5a and 5e Are Equivalent:* See [Adá+01, Item (b) of Remark 4.3].

This finishes the proof. □

Question 8.5.2.4 ► Better Characterisations of Functors With Full Precomposition

Item 5 of Proposition 8.5.2.2 gives a characterisation of the functors F for which F^* is full, but the characterisations given there are really messy. Are there better ones?
This question also appears as [MO 468121b].

8.5.3 Fully Faithful Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 8.5.3.1 ▶ Fully Faithful Functors

A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **fully faithful** if F is full and faithful, i.e. if, for each $A, B \in \mathrm{Obj}(\mathcal{C})$, the action on morphisms

$$F_{A,B}\colon \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F_A, F_B)$$

of F at (A, B) is bijective.

Proposition 8.5.3.2 ▶ Properties of Fully Faithful Functors

Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor.

1. *Characterisations.* The following conditions are equivalent:

   (a) The functor F is fully faithful.

   (b) We have a pullback square

$$\mathrm{Arr}(\mathcal{C}) \cong (\mathcal{C} \times \mathcal{C}) \times_{\mathcal{D}\times\mathcal{D}} \mathrm{Arr}(\mathcal{D}), \qquad \begin{tikzcd} \mathrm{Arr}(\mathcal{C}) \arrow[r, "\mathrm{Arr}(F)"] \arrow[d, "\mathrm{src}\times\mathrm{tgt}"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathrm{Arr}(\mathcal{D}) \arrow[d, "\mathrm{src}\times\mathrm{tgt}"] \\ \mathcal{C}\times\mathcal{C} \arrow[r, "F\times F"'] & \mathcal{D}\times\mathcal{D} \end{tikzcd}$$

   in Cats.

2. *Conservativity.* If F is fully faithful, then F is conservative.

3. *Essential Injectivity.* If F is fully faithful, then F is essentially injective.

4. *Interaction With Co/Limits.* If F is fully faithful, then F reflects co/limits.

5. *Interaction With Postcomposition.* The following conditions are equivalent:

   (a) The functor $F\colon \mathcal{C} \to \mathcal{D}$ is fully faithful.

   (b) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor

$$F_*\colon \mathrm{Fun}(\mathcal{X}, \mathcal{C}) \to \mathrm{Fun}(\mathcal{X}, \mathcal{D})$$

   is fully faithful.

   (c) The functor $F\colon \mathcal{C} \to \mathcal{D}$ is a representably fully faithful morphism in Cats_2 in the sense of Definition 9.1.3.1.

6. *Interaction With Precomposition I*. If F is fully faithful, then the precomposition functor

$$F^* \colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

can fail to be fully faithful.

7. *Interaction With Precomposition II*. If the precomposition functor

$$F^* \colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

is fully faithful, then F *can fail* to be fully faithful (and in fact it can also fail to be either full or faithful).

8. *Interaction With Precomposition III*. If F is essentially surjective and full, then the precomposition functor

$$F^* \colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

is fully faithful.

9. *Interaction With Precomposition IV*. The following conditions are equivalent:

   (a) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor

   $$F^* \colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

   is fully faithful.

   (b) The precomposition functor

   $$F^* \colon \mathsf{Fun}(\mathcal{D}, \mathsf{Sets}) \to \mathsf{Fun}(\mathcal{C}, \mathsf{Sets})$$

   is fully faithful.

   (c) The functor

   $$\mathrm{Lan}_F \colon \mathsf{Fun}(\mathcal{C}, \mathsf{Sets}) \to \mathsf{Fun}(\mathcal{D}, \mathsf{Sets})$$

   is fully faithful.

   (d) The functor F is a corepresentably fully faithful morphism in Cats_2 in the sense of Definition 9.2.3.1.

   (e) The functor F is absolutely dense.

(f) The components

$$\eta_G \colon G \Longrightarrow \mathrm{Ran}_F(G \circ F)$$

of the unit

$$\eta \colon \mathrm{id}_{\mathsf{Fun}(\mathcal{D},\mathcal{X})} \Longrightarrow \mathrm{Ran}_F \circ F^*$$

of the adjunction $F^* \dashv \mathrm{Ran}_F$ are all isomorphisms.

(g) The components

$$\epsilon_G \colon \mathrm{Lan}_F(G \circ F) \Longrightarrow G$$

of the counit

$$\epsilon \colon \mathrm{Lan}_F \circ F^* \Longrightarrow \mathrm{id}_{\mathsf{Fun}(\mathcal{D},\mathcal{X})}$$

of the adjunction $\mathrm{Lan}_F \dashv F^*$ are all isomorphisms.

(h) The natural transformation

$$\alpha \colon \mathrm{Lan}_{h_F}\left(h^F\right) \Longrightarrow h$$

with components

$$\alpha_{B',B} \colon \int^{A \in C} h^{B'}_{F_A} \times h^{F_A}_{B} \to h^{B'}_{B}$$

given by

$$\alpha_{B',B}([(\phi, \psi)]) = \psi \circ \phi$$

is a natural isomorphism.

(i) For each $B \in \mathrm{Obj}(\mathcal{D})$, there exist:

- An object A_B of C;
- A morphism $s_B \colon B \to F(A_B)$ of $\mathcal{D}$;
- A morphism $r_B \colon F(A_B) \to B$ of $\mathcal{D}$;

satisfying the following conditions:

i. The triple $(F(A_B), r_B, s_B)$ is a retract of B, i.e. we have $r_B \circ s_B = \mathrm{id}_B$.

ii. For each morphism $f \colon B' \to B$ of $\mathcal{D}$, we have

$$[(A_B, s_{B'}, f \circ r_{B'})] = [(A_B, s_B \circ f, r_B)]$$

in $\int^{A \in C} h^{B'}_{F_A} \times h^{F_A}_{B}$.

Proof 8.5.3.3 ▶ Proof of Proposition 8.5.3.2

Item 1: Characterisations

Omitted.

Item 2: Conservativity

This is a repetition of Item 2 of Proposition 8.5.4.2, and is proved there.

Item 3: Essential Injectivity

Omitted.

Item 4: Interaction With Co/Limits

Omitted.

Item 5: Interaction With Postcomposition

This follows from Item 1 of Proposition 8.5.1.2 and Item 1 of Proposition 8.5.2.2.

Item 6: Interaction With Precomposition I

See [MSE 733161] for an example of a fully faithful functor whose precomposition with which fails to be full.

Item 7: Interaction With Precomposition II

See [MSE 749304, Item 3].

Item 8: Interaction With Precomposition III

Omitted, but see `https://unimath.github.io/doc/UniMath/d4de26f//UniMath.CategoryTheory.precomp_fully_faithful.html` for a formalised proof.

Item 9: Interaction With Precomposition IV

We claim Items 9a to 9i are equivalent:

- *Items 9a and 9d Are Equivalent:* This is true by the definition of corepresentably fully faithful morphism; see Definition 9.2.3.1.
- *Items 9a, 9f and 9g Are Equivalent:* See **??** of **??**.
- *Items 9a to 9c Are Equivalent:* This follows from [Low15, Proposition A.1.5].
- *Items 9a, 9e, 9h and 9i Are Equivalent:* See [Fre09, Theorem 4.1] and [Adá+01, Theorem 1.1].

This finishes the proof. □

8.5.4 Conservative Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 8.5.4.1 ► Conservative Functors

A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **conservative** if it satisfies the following condition:[1]

($\star$) For each $f \in \mathrm{Mor}(\mathcal{C})$, if $F(f)$ is an isomorphism in $\mathcal{D}$, then f is an isomorphism in $\mathcal{C}$.

[1] *Slogan:* A functor F is **conservative** if it reflects isomorphisms.

Proposition 8.5.4.2 ► Properties of Conservative Functors

Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor.

1. *Characterisations.* The following conditions are equivalent:
   (a) The functor F is conservative.
   (b) For each $f \in \mathrm{Mor}(\mathcal{C})$, the morphism $F(f)$ is an isomorphism in $\mathcal{D}$ iff f is an isomorphism in $\mathcal{C}$.
2. *Interaction With Fully Faithfulness.* Every fully faithful functor is conservative.
3. *Interaction With Precomposition.* The following conditions are equivalent:
   (a) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor
   $$F^*\colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$
   is conservative.
   (b) The equivalent conditions of Item 4 of Proposition 8.5.1.2 are satisfied.

Proof 8.5.4.3 ► Proof of Proposition 8.5.4.2

Item 1: Characterisations

This follows from Item 1 of Proposition 8.4.1.8.

Item 2: Interaction With Fully Faithfulness

Let $F\colon \mathcal{C} \to \mathcal{D}$ be a fully faithful functor, let $f\colon A \to B$ be a morphism of $\mathcal{C}$, and suppose that F_f is an isomorphism. We have

$$\begin{aligned} F(\mathrm{id}_B) &= \mathrm{id}_{F(B)} \\ &= F(f) \circ F(f)^{-1} \\ &= F\big(f \circ f^{-1}\big). \end{aligned}$$

Similarly, $F(\mathrm{id}_A) = F(f^{-1} \circ f)$. But since F is fully faithful, we must have

$$\begin{aligned} f \circ f^{-1} &= \mathrm{id}_B, \\ f^{-1} \circ f &= \mathrm{id}_A, \end{aligned}$$

showing f to be an isomorphism. Thus F is conservative.

Question 8.5.4.4 ► Characterisations of Functors With Conservative Pre/-Postcomposition

Is there a characterisation of functors $F: \mathcal{C} \to \mathcal{D}$ satisfying the following condition:

(★) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor

$$F_*: \mathsf{Fun}(\mathcal{X}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{X}, \mathcal{D})$$

is conservative?

This question also appears as [MO 468121a].

8.5.5 Essentially Injective Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 8.5.5.1 ► Essentially Injective Functors

A functor $F: \mathcal{C} \to \mathcal{D}$ is **essentially injective** if it satisfies the following condition:

(★) For each $A, B \in \mathrm{Obj}(\mathcal{C})$, if $F(A) \cong F(B)$, then $A \cong B$.

Question 8.5.5.2 ► Characterisations of Functors With Essentially Injective Pre/Postcomposition

Is there a characterisation of functors $F: \mathcal{C} \to \mathcal{D}$ such that:

1. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor

$$F^*: \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

is essentially injective, i.e. if $\phi \circ F \cong \psi \circ F$, then $\phi \cong \psi$ for all functors ϕ and ψ?

2. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor

$$F_*\colon \mathsf{Fun}(\mathcal{X}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{X}, \mathcal{D})$$

is essentially injective, i.e. if $F \circ \phi \cong F \circ \psi$, then $\phi \cong \psi$?

This question also appears as [MO 468121a].

8.5.6 Essentially Surjective Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 8.5.6.1 ▶ Essentially Surjective Functors

A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **essentially surjective**[1] if it satisfies the following condition:

(★) For each $D \in \mathrm{Obj}(\mathcal{D})$, there exists some object A of $\mathcal{C}$ such that $F(A) \cong D$.

[1] *Further Terminology:* Also called an **eso** functor, where the name "eso" comes from *essentially surjective on objects*.

Question 8.5.6.2 ▶ Characterisations of Functors With Essentially Surjective Pre/Postcomposition

Is there a characterisation of functors $F\colon \mathcal{C} \to \mathcal{D}$ such that:

1. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor

$$F^*\colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

is essentially surjective?

2. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor

$$F_*\colon \mathsf{Fun}(\mathcal{X}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{X}, \mathcal{D})$$

is essentially surjective?

This question also appears as [MO 468121a].

8.5.7 Equivalences of Categories

Definition 8.5.7.1 ► Equivalences of Categories

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

1. An **equivalence of categories** between $\mathcal{C}$ and $\mathcal{D}$ consists of a pair of functors

$$F\colon \mathcal{C} \to \mathcal{D},$$
$$G\colon \mathcal{D} \to \mathcal{C}$$

together with natural isomorphisms

$$\eta\colon \mathrm{id}_{\mathcal{C}} \overset{\sim}{\Longrightarrow} G \circ F,$$
$$\epsilon\colon F \circ G \overset{\sim}{\Longrightarrow} \mathrm{id}_{\mathcal{D}}.$$

2. An **adjoint equivalence of categories** between $\mathcal{C}$ and $\mathcal{D}$ is an equivalence (F, G, η, ϵ) between $\mathcal{C}$ and $\mathcal{D}$ which is also an adjunction.

Proposition 8.5.7.2 ► Properties of Equivalences of Categories

Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor.

1. *Characterisations.* If $\mathcal{C}$ and $\mathcal{D}$ are small[1], then the following conditions are equivalent:[2]

   (a) The functor F is an equivalence of categories.

   (b) The functor F is fully faithful and essentially surjective.

   (c) The induced functor

   $$\upharpoonright F\mathsf{Sk}(\mathcal{C})\colon \mathsf{Sk}(\mathcal{C}) \to \mathsf{Sk}(\mathcal{D})$$

   is an *isomorphism* of categories.

   (d) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor

   $$F^*\colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

   is an equivalence of categories.

   (e) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor

   $$F_*\colon \mathsf{Fun}(\mathcal{X}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{X}, \mathcal{D})$$

   is an equivalence of categories.

2. *Two-Out-of-Three.* Let

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "G\circ F"] \arrow[dr, "F"'] & & \mathcal{E} \\ & \mathcal{D} \arrow[ur, "G"'] & \end{tikzcd}$$

be a diagram in Cats. If two out of the three functors among F, G, and $G \circ F$ are equivalences of categories, then so is the third.

3. *Stability Under Composition.* Let

$$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D} \underset{G'}{\overset{F'}{\rightleftarrows}} \mathcal{E}$$

be a diagram in Cats. If (F, G) and (F', G') are equivalences of categories, then so is their composite $(F' \circ F, G' \circ G)$.

4. *Equivalences vs.Adjoint Equivalences.* Every equivalence of categories can be promoted to an adjoint equivalence.[3]

5. *Interaction With Groupoids.* If $\mathcal{C}$ and $\mathcal{D}$ are groupoids, then the following conditions are equivalent:

   (a) The functor F is an equivalence of groupoids.

   (b) The following conditions are satisfied:

      i. The functor F induces a bijection

      $$\pi_0(F) \colon \pi_0(\mathcal{C}) \to \pi_0(\mathcal{D})$$

      of sets.

      ii. For each $A \in \mathrm{Obj}(\mathcal{C})$, the induced map

      $$F_{x,x} \colon \mathrm{Aut}_{\mathcal{C}}(A) \to \mathrm{Aut}_{\mathcal{D}}(F_A)$$

      is an isomorphism of groups.

[1]Otherwise there will be size issues. One can also work with large categories and universes, or require F to be *constructively* essentially surjective; see [MSE 1465107].

[2]In ZFC, the equivalence between Item 1a and Item 1b is equivalent to the axiom of choice; see [MO 119454].

In Univalent Foundations, this is true without requiring neither the axiom of choice nor the law of excluded middle.

[3]More precisely, we can promote an equivalence of categories (F, G, η, ϵ) to adjoint equivalences (F, G, η', ϵ) and (F, G, η, ϵ').

PROOF 8.5.7.3 ▶ PROOF OF PROPOSITION 8.5.7.2

Item 1: Characterisations

We claim that Items 1a to 1e are indeed equivalent:

1. *Item 1a* $\implies$ *Item 1b*: Clear.
2. *Item 1b* $\implies$ *Item 1a*: Since F is essentially surjective and $\mathcal{C}$ and $\mathcal{D}$ are small, we can choose, using the axiom of choice, for each $B \in \mathrm{Obj}(\mathcal{D})$, an object j_B of $\mathcal{C}$ and an isomorphism $i_B : B \to F_{j_B}$ of $\mathcal{D}$.

   Since F is fully faithful, we can extend the assignment $B \mapsto j_B$ to a *unique* functor $j : \mathcal{D} \to \mathcal{C}$ such that the isomorphisms $i_B : B \to F_{j_B}$ assemble into a natural isomorphism $\eta : \mathrm{id}_{\mathcal{D}} \overset{\sim}{\Longrightarrow} F \circ j$, with a similar natural isomorphism $\epsilon : \mathrm{id}_{\mathcal{C}} \overset{\sim}{\Longrightarrow} j \circ F$. Hence F is an equivalence.
3. *Item 1a* $\implies$ *Item 1c*: This follows from Item 4 of Proposition 8.1.5.3.
4. *Item 1c* $\implies$ *Item 1a*: Omitted.
5. *Items 1a, 1d and 1e Are Equivalent*: This follows from **??**.

This finishes the proof of Item 1.

Item 2: Two-Out-of-Three

Omitted.

Item 3: Stability Under Composition

Clear.

Item 4: Equivalences vs.Adjoint Equivalences

See [Rie17, Proposition 4.4.5].

Item 5: Interaction With Groupoids

See [nLa24, Proposition 4.4]. □

8.5.8 Isomorphisms of Categories

DEFINITION 8.5.8.1 ▶ ISOMORPHISMS OF CATEGORIES

An **isomorphism of categories** is a pair of functors

$$
\begin{aligned}
F &: \mathcal{C} \to \mathcal{D},\\
G &: \mathcal{D} \to \mathcal{C}
\end{aligned}
$$

such that we have

$$G \circ F = \mathrm{id}_{\mathcal{C}},$$

$$F \circ G = \mathrm{id}_{\mathcal{D}}.$$

Example 8.5.8.2 ▶ Equivalent But Non-Isomorphic Categories

Categories can be equivalent but non-isomorphic. For example, the category consisting of two isomorphic objects is equivalent to pt, but not isomorphic to it.

Proposition 8.5.8.3 ▶ Properties of Isomorphisms of Categories

Let $F: \mathcal{C} \to \mathcal{D}$ be a functor.

1. *Characterisations.* If $\mathcal{C}$ and $\mathcal{D}$ are small, then the following conditions are equivalent:

   (a) The functor F is an isomorphism of categories.

   (b) The functor F is fully faithful and bijective on objects.

   (c) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor

   $$F^*: \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

   is an isomorphism of categories.

   (d) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor

   $$F_*: \mathsf{Fun}(\mathcal{X}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{X}, \mathcal{D})$$

   is an isomorphism of categories.

Proof 8.5.8.4 ▶ Proof of Proposition 8.5.8.3

Item 1: Characterisations

We claim that Items 1a to 1d are indeed equivalent:

1. *Items 1a and 1b Are Equivalent:* Omitted, but similar to Item 1 of Proposition 8.5.7.2.

2. *Items 1a, 1c and 1d Are Equivalent:* This follows from **??**.

This finishes the proof. □

8.6 More Conditions on Functors

8.6.1 Dominant Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 8.6.1.1 ▶ Dominant Functors

A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **dominant** if every object of $\mathcal{D}$ is a retract of some object in $\mathrm{Im}(F)$, i.e.:

($\star$) For each $B \in \mathrm{Obj}(\mathcal{D})$, there exist:

- An object A of $\mathcal{C}$;
- A morphism $r\colon F(A) \to B$ of $\mathcal{D}$;
- A morphism $s\colon B \to F(A)$ of $\mathcal{D}$;

such that we have

$$r \circ s = \mathrm{id}_B, \qquad \begin{tikzcd} B \arrow[r, "s"] \arrow[dr, "\mathrm{id}_B"'] & F(A) \arrow[d, "r"] \\ & B. \end{tikzcd}$$

Proposition 8.6.1.2 ▶ Properties of Dominant Functors

Let $F, G\colon \mathcal{C} \rightrightarrows \mathcal{D}$ be functors and let $I\colon \mathcal{X} \to \mathcal{C}$ be a functor.

1. *Interaction With Right Whiskering.* If I is full and dominant, then the map

$$- \star \mathrm{id}_I\colon \mathrm{Nat}(F, G) \to \mathrm{Nat}(F \circ I, G \circ I)$$

is a bijection.

2. *Interaction With Adjunctions.* Let $(F, G)\colon \mathcal{C} \rightleftarrows \mathcal{D}$ be an adjunction.
   (a) If F is dominant, then G is faithful.
   (b) The following conditions are equivalent:
      i. The functor G is full.

ii. The restriction

$$\restriction G\mathrm{Im}_F \colon \mathrm{Im}(F) \to \mathcal{C}$$

of G to $\mathrm{Im}(F)$ is full.

PROOF 8.6.1.3 ▶ PROOF OF PROPOSITION 8.6.1.2

Item 1: Interaction With Right Whiskering

See [DFH75, Proposition 1.4].

Item 2: Interaction With Adjunctions

See [DFH75, Proposition 1.7].

QUESTION 8.6.1.4 ▶ CHARACTERISATIONS OF FUNCTORS WITH DOMINANT PRE/POST-COMPOSITION

Is there a characterisation of functors $F\colon \mathcal{C} \to \mathcal{D}$ such that:

1. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor

$$F^*\colon \mathrm{Fun}(\mathcal{D}, \mathcal{X}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{X})$$

is dominant?

2. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor

$$F_*\colon \mathrm{Fun}(\mathcal{X}, \mathcal{C}) \to \mathrm{Fun}(\mathcal{X}, \mathcal{D})$$

is dominant?

This question also appears as [MO 468121a].

8.6.2 Monomorphisms of Categories

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

DEFINITION 8.6.2.1 ▶ MONOMORPHISMS OF CATEGORIES

A functor $F\colon \mathcal{C} \to \mathcal{D}$ is a **monomorphism of categories** if it is a monomorphism in Cats (see **??**).

Proposition 8.6.2.2 ▶ Properties of Monomorphisms of Categories

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor.

1. *Characterisations.* The following conditions are equivalent:

   (a) The functor F is a monomorphism of categories.

   (b) The functor F is injective on objects and morphisms, i.e. F is injective on objects and the map

   $$F : \mathrm{Mor}(\mathcal{C}) \to \mathrm{Mor}(\mathcal{D})$$

   is injective.

Proof 8.6.2.3 ▶ Proof of Proposition 8.6.2.2

Item 1: Characterisations

Omitted. □

Question 8.6.2.4 ▶ Characterisations of Functors With Monic Pre/Postcomposition

Is there a characterisation of functors $F : \mathcal{C} \to \mathcal{D}$ such that:

1. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor

   $$F^* : \mathrm{Fun}(\mathcal{D}, \mathcal{X}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{X})$$

   is a monomorphism of categories?

2. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor

   $$F_* : \mathrm{Fun}(\mathcal{X}, \mathcal{C}) \to \mathrm{Fun}(\mathcal{X}, \mathcal{D})$$

   is a monomorphism of categories?

This question also appears as [MO 468121a].

8.6.3 Epimorphisms of Categories

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 8.6.3.1 ► Epimorphisms of Categories

A functor $F\colon \mathcal{C} \to \mathcal{D}$ is a **epimorphism of categories** if it is a epimorphism in Cats (see **??**).

Proposition 8.6.3.2 ► Properties of Epimorphisms of Categories

Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor.

1. *Characterisations.* The following conditions are equivalent:[1]

   (a) The functor F is a epimorphism of categories.

   (b) For each morphism $f\colon A \to B$ of $\mathcal{D}$, we have a diagram

$$
\begin{tikzcd}
 & & & & A \arrow[dllll, "\phi_1"'] \arrow[dll, "\phi_2"'] \arrow[d, "\phi_3"] \arrow[drr, "\phi_m"] \arrow[drrrr, dashed, "\mathrm{id}_A"] & & & & \\
X_1 \arrow[dr, dashed, "\alpha_1"] \arrow[ddrrrr, dashed, bend right, "\alpha_0"'] & & X_2 \arrow[dl, dashed, "\alpha_2"'] \arrow[dr, dashed, "\alpha_3"] & & X_3 \arrow[dl, dashed, "\alpha_4"'] \arrow[dr, dashed, "\alpha_6"] & \cdots & X_m \arrow[dl, dashed, "\alpha_{2m-2}"'] \arrow[dr, dashed, "\alpha_{2m-1}"] & & A \arrow[dl, dashed, "\alpha_{2m}"'] \\
 & Y_1 \arrow[drrr, "\psi_1"] & & Y_2 \arrow[dr, "\psi_2"] & & \cdots \arrow[dl, "\psi_3"'] & & Y_m \arrow[dlll, "\psi_m"'] & \\
 & & & & B & & & &
\end{tikzcd}
$$

   in $\mathcal{D}$ satisfying the following conditions:

   i. We have $f = \alpha_0 \circ \phi_1$.
   ii. We have $f = \psi_m \circ \alpha_{2m}$.
   iii. For each $0 \leq i \leq 2m$, we have $\alpha_i \in \mathrm{Mor}(\mathrm{Im}(F))$.

2. *Surjectivity on Objects.* If F is an epimorphism of categories, then F is surjective on objects.

[1] *Further Terminology:* This statement is known as **Isbell's zigzag theorem**.

Proof 8.6.3.3 ► Proof of Proposition 8.6.3.2

Item 1: Characterisations

See [Isb68].

Item 2: Surjectivity on Objects

Omitted.

Question 8.6.3.4 ▶ Characterisations of Functors With Epic Pre/Postcomposition

Is there a characterisation of functors $F: \mathcal{C} \to \mathcal{D}$ such that:

1. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor
$$F^*: \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$
is an epimorphism of categories?
2. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor
$$F_*: \mathsf{Fun}(\mathcal{X}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{X}, \mathcal{D})$$
is an epimorphism of categories?

This question also appears as [MO 468121a].

8.6.4 Pseudomonic Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 8.6.4.1 ▶ Pseudomonic Functors

A functor $F: \mathcal{C} \to \mathcal{D}$ is **pseudomonic** if it satisfies the following conditions:

1. For all diagrams of the form
$$\begin{tikzcd} \mathcal{X} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & \mathcal{C} \arrow[r, "F"] & \mathcal{D}, \end{tikzcd}$$
if we have
$$\mathrm{id}_F \star \alpha = \mathrm{id}_F \star \beta,$$
then $\alpha = \beta$.

2. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$ and each natural isomorphism

$$\beta\colon F \circ \phi \overset{\sim}{\Longrightarrow} F \circ \psi, \qquad \begin{tikzcd} \mathcal{X} \arrow[r, bend left, "F\circ\phi"] \arrow[r, bend right, "F\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{D}, \end{tikzcd}$$

there exists a natural isomorphism

$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} \mathcal{X} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{C} \end{tikzcd}$$

such that we have an equality

$$\begin{tikzcd} \mathcal{X} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{C} \arrow[r, "F"] & \mathcal{D} \end{tikzcd} = \begin{tikzcd} \mathcal{X} \arrow[r, bend left, "F\circ\phi"] \arrow[r, bend right, "F\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{D} \end{tikzcd}$$

of pasting diagrams, i.e. such that we have

$$\beta = \mathrm{id}_F \star \alpha.$$

Proposition 8.6.4.2 ▶ Properties of Pseudomonic Functors

Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor.

1. *Characterisations.* The following conditions are equivalent:
   (a) The functor F is pseudomonic.
   (b) The functor F satisfies the following conditions:
      i. The functor F is faithful, i.e. for each $A, B \in \mathrm{Obj}(\mathcal{C})$, the action on morphisms

      $$F_{A,B}\colon \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F_A, F_B)$$

      of F at (A, B) is injective.
      ii. For each $A, B \in \mathrm{Obj}(\mathcal{C})$, the restriction

      $$F^{\mathrm{iso}}_{A,B}\colon \mathrm{Iso}_{\mathcal{C}}(A, B) \to \mathrm{Iso}_{\mathcal{D}}(F_A, F_B)$$

      of the action on morphisms of F at (A, B) to isomorphisms is surjective.

(c) We have an isocomma square of the form

$$\mathcal{C} \overset{\text{eq.}}{\cong} \mathcal{C} \overset{\leftrightarrow}{\times}_{\mathcal{D}} \mathcal{C}, \qquad \begin{tikzcd} \mathcal{C} \arrow[r, "\mathrm{id}_{\mathcal{C}}"] \arrow[d, "\mathrm{id}_{\mathcal{C}}"'] \arrow[dr, phantom, "\Leftrightarrow"] & \mathcal{C} \arrow[d, "F"] \\ \mathcal{C} \arrow[r, "F"'] & \mathcal{D} \end{tikzcd}$$

in Cats_2 up to equivalence.

(d) We have an isocomma square of the form

$$\mathcal{C} \overset{\text{eq.}}{\cong} \mathcal{C} \overset{\leftrightarrow}{\times}_{\mathsf{Arr}(\mathcal{D})} \mathcal{D}, \qquad \begin{tikzcd} \mathcal{C} \arrow[r, hook] \arrow[d, "F"'] \arrow[dr, phantom, "\Leftrightarrow"] & \mathsf{Arr}(\mathcal{C}) \arrow[d, "\mathsf{Arr}(F)"] \\ \mathcal{D} \arrow[r, hook] & \mathsf{Arr}(\mathcal{D}) \end{tikzcd}$$

in Cats_2 up to equivalence.

(e) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition[1] functor

$$F_* \colon \mathsf{Fun}(\mathcal{X}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{X}, \mathcal{D})$$

is pseudomonic.

2. *Conservativity.* If F is pseudomonic, then F is conservative.

3. *Essential Injectivity.* If F is pseudomonic, then F is essentially injective.

[1]Asking the precomposition functors

$$F^* \colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

to be pseudomonic leads to pseudoepic functors; see Item 1b of Item 1 of Proposition 8.6.5.2.

Proof 8.6.4.3 ► Proof of Proposition 8.6.4.2

Item 1: Characterisations

Omitted.

Item 2: Conservativity

Omitted.

Item 3: Essential Injectivity

Omitted.

8.6.5 Pseudoepic Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 8.6.5.1 ► Pseudoepic Functors

A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **pseudoepic** if it satisfies the following conditions:

1. For all diagrams of the form
$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] & \mathcal{D} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & \mathcal{X}, \end{tikzcd}$$
if we have
$$\alpha \star \mathrm{id}_F = \beta \star \mathrm{id}_F,$$
then $\alpha = \beta$.

2. For each $X \in \mathrm{Obj}(\mathcal{C})$ and each 2-isomorphism
$$\beta\colon \phi \circ F \overset{\sim}{\Longrightarrow} \psi \circ F, \qquad \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "\phi\circ F"] \arrow[r, bend right, "\psi\circ F"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{X} \end{tikzcd}$$
of $\mathcal{C}$, there exists a 2-isomorphism
$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{X} \end{tikzcd}$$
of $\mathcal{C}$ such that we have an equality
$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] & \mathcal{D} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{X} \end{tikzcd} = \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "\phi\circ F"] \arrow[r, bend right, "\psi\circ F"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{X} \end{tikzcd}$$
of pasting diagrams in $\mathcal{C}$, i.e. such that we have
$$\beta = \alpha \star \mathrm{id}_F.$$

Proposition 8.6.5.2 ▶ Properties of Pseudoepic Functors

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor.

1. *Characterisations.* The following conditions are equivalent:

   (a) The functor F is pseudoepic.

   (b) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the functor

   $$F^* : \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

   given by precomposition by F is pseudomonic.

   (c) We have an isococomma square of the form

   $$\mathcal{D} \overset{\text{eq.}}{\cong} \mathcal{D} \overset{\leftrightarrow}{\coprod}_{\mathcal{C}} \mathcal{D}, \qquad \begin{tikzcd} \mathcal{D} & \mathcal{D} \arrow[l, "\mathrm{id}_{\mathcal{D}}"'] \\ \mathcal{D} \arrow[u, "\mathrm{id}_{\mathcal{D}}"] & \mathcal{C} \arrow[l, "F"] \arrow[u, "F"'] \arrow[ul, phantom, "\Leftrightarrow"] \end{tikzcd}$$

   in Cats_2 up to equivalence.

2. *Dominance.* If F is pseudoepic, then F is dominant (Definition 8.6.1.1).

Proof 8.6.5.3 ▶ Proof of Proposition 8.6.5.2

Item 1: Characterisations

Omitted.

Item 2: Dominance

If F is pseudoepic, then

$$F^* : \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

is pseudomonic for all $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, and thus in particular faithful. By Item 4g of Item 4 of Proposition 8.5.1.2, this is equivalent to requiring F to be dominant. □

Question 8.6.5.4 ▶ Characterisations of Pseudoepic Functors

Is there a nice characterisation of the pseudoepic functors, similarly to the characterisaiton of pseudomonic functors given in Item 1b of Item 1 of Propo-

sition 8.6.4.2?
This question also appears as [MO 321971].

Question 8.6.5.5 ▶ Must a Pseudomonic and Pseudoepic Functor Be an Equivalence of Categories

A pseudomonic and pseudoepic functor is dominant, faithful, essentially injective, and full on isomorphisms. Is it necessarily an equivalence of categories? If not, how bad can this fail, i.e. how far can a pseudomonic and pseudoepic functor be from an equivalence of categories?
This question also appears as [MO 468334].

Question 8.6.5.6 ▶ Characterisations of Functors With Pseudoepic Pre/-Postcomposition

Is there a characterisation of functors $F: \mathcal{C} \to \mathcal{D}$ such that:

1. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor
$$F^*: \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$
is pseudoepic?
2. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor
$$F_*: \mathsf{Fun}(\mathcal{X}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{X}, \mathcal{D})$$
is pseudoepic?

This question also appears as [MO 468121a].

8.7 Even More Conditions on Functors

8.7.1 Injective on Objects Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 8.7.1.1 ▶ Injective on Objects Functors

A functor $F: \mathcal{C} \to \mathcal{D}$ is **injective on objects** if the action on objects
$$F: \mathrm{Obj}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{D})$$

of F is injective.

Proposition 8.7.1.2 ► Properties of Injective on Objects Functors

Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor.

1. *Characterisations.* The following conditions are equivalent:
   (a) The functor F is injective on objects.
   (b) The functor F is an isocofibration in Cats_2.

Proof 8.7.1.3 ► Proof of Proposition 8.7.1.2

Item 1: Characterisations

Omitted.

8.7.2 Surjective on Objects Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 8.7.2.1 ► Surjective on Objects Functors

A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **surjective on objects** if the action on objects

$$F\colon \mathrm{Obj}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{D})$$

of F is surjective.

8.7.3 Bijective on Objects Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 8.7.3.1 ► Bijective on Objects Functors

A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **bijective on objects**[1] if the action on objects

$$F\colon \mathrm{Obj}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{D})$$

of F is a bijection.

[1] *Further Terminology:* Also called a **bo** functor.

8.7.4 Functors Representably Faithful on Cores

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 8.7.4.1 ▶ Functors Representably Faithful on Cores

A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **representably faithful on cores** if, for each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition by F functor

$$F_*\colon \mathsf{Core}(\mathsf{Fun}(\mathcal{X}, \mathcal{C})) \to \mathsf{Core}(\mathsf{Fun}(\mathcal{X}, \mathcal{D}))$$

is faithful.

Remark 8.7.4.2 ▶ Unwinding Definition 8.7.4.1

In detail, a functor $F\colon \mathcal{C} \to \mathcal{D}$ is **representably faithful on cores** if, given a diagram of the form

$$\begin{tikzcd} \mathcal{X} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & \mathcal{C} \arrow[r, "F"] & \mathcal{D}, \end{tikzcd}$$

if α and β are natural isomorphisms and we have

$$\mathrm{id}_F \star \alpha = \mathrm{id}_F \star \beta,$$

then $\alpha = \beta$.

Question 8.7.4.3 ▶ Characterisation of Functors Representably Faithful on Cores

Is there a characterisation of functors representably faithful on cores?

8.7.5 Functors Representably Full on Cores

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 8.7.5.1 ▶ Functors Representably Full on Cores

A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **representably full on cores** if, for each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition by F functor

$$F_*\colon \mathsf{Core}(\mathsf{Fun}(\mathcal{X}, \mathcal{C})) \to \mathsf{Core}(\mathsf{Fun}(\mathcal{X}, \mathcal{D}))$$

is full.

Remark 8.7.5.2 ▶ Unwinding Definition 8.7.5.1

In detail, a functor $F\colon \mathcal{C} \to \mathcal{D}$ is **representably full on cores** if, for each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$ and each natural isomorphism

$$\beta\colon F\circ\phi \overset{\sim}{\Longrightarrow} F\circ\psi, \qquad \begin{tikzcd} \mathcal{X} \arrow[r, bend left, "F\circ\phi"] \arrow[r, bend right, "F\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{D}, \end{tikzcd}$$

there exists a natural isomorphism

$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} \mathcal{X} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{C} \end{tikzcd}$$

such that we have an equality

$$\begin{tikzcd} \mathcal{X} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{C} \arrow[r, "F"] & \mathcal{D} \end{tikzcd} = \begin{tikzcd} \mathcal{X} \arrow[r, bend left, "F\circ\phi"] \arrow[r, bend right, "F\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{D} \end{tikzcd}$$

of pasting diagrams in Cats_2, i.e. such that we have

$$\beta = \mathrm{id}_F \star \alpha.$$

Question 8.7.5.3 ▶ Characterisation of Functors Representably Full on Cores

Is there a characterisation of functors representably full on cores?
This question also appears as [MO 468121a].

8.7.6 Functors Representably Fully Faithful on Cores

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 8.7.6.1 ▶ Functors Representably Fully Faithful on Cores

A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **representably fully faithful on cores** if, for each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition by F functor

$$F_*\colon \mathrm{Core}(\mathrm{Fun}(\mathcal{X},\mathcal{C})) \to \mathrm{Core}(\mathrm{Fun}(\mathcal{X},\mathcal{D}))$$

is fully faithful.

Remark 8.7.6.2 ▶ Unwinding Definition 8.7.6.1

In detail, a functor $F\colon \mathcal{C} \to \mathcal{D}$ is **representably fully faithful on cores** if it satisfies the conditions in Remarks 8.7.4.2 and 8.7.5.2, i.e.:

1. For all diagrams of the form

$$\begin{tikzcd} \mathcal{X} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & \mathcal{C} \arrow[r, "F"] & \mathcal{D}, \end{tikzcd}$$

with α and β natural isomorphisms, if we have $\mathrm{id}_F \star \alpha = \mathrm{id}_F \star \beta$, then $\alpha = \beta$.

2. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$ and each natural isomorphism

$$\beta\colon F \circ \phi \overset{\sim}{\Longrightarrow} F \circ \psi, \qquad \begin{tikzcd} \mathcal{X} \arrow[r, bend left, "F\circ\phi"] \arrow[r, bend right, "F\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{D} \end{tikzcd}$$

of $\mathcal{C}$, there exists a natural isomorphism

$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} \mathcal{X} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{C} \end{tikzcd}$$

of $\mathcal{C}$ such that we have an equality

$$\begin{tikzcd} \mathcal{X} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{C} \arrow[r, "F"] & \mathcal{D} \end{tikzcd} = \begin{tikzcd} \mathcal{X} \arrow[r, bend left, "F\circ\phi"] \arrow[r, bend right, "F\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{D} \end{tikzcd}$$

of pasting diagrams in Cats_2, i.e. such that we have

$$\beta = \mathrm{id}_F \star \alpha.$$

Question 8.7.6.3 ▶ Characterisation of Functors Representably Fully Faithful on Cores

Is there a characterisation of functors representably fully faithful on cores?

8.7.7 Functors Corepresentably Faithful on Cores

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 8.7.7.1 ▶ Functors Corepresentably Faithful on Cores

A functor $F: \mathcal{C} \to \mathcal{D}$ is **corepresentably faithful on cores** if, for each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition by F functor

$$F_*: \mathsf{Core}(\mathsf{Fun}(\mathcal{X}, \mathcal{C})) \to \mathsf{Core}(\mathsf{Fun}(\mathcal{X}, \mathcal{D}))$$

is faithful.

Remark 8.7.7.2 ▶ Unwinding Definition 8.7.7.1

In detail, a functor $F: \mathcal{C} \to \mathcal{D}$ is **corepresentably faithful on cores** if, given a diagram of the form

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] & \mathcal{D} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & \mathcal{X}, \end{tikzcd}$$

if α and β are natural isomorphisms and we have

$$\alpha \star \mathrm{id}_F = \beta \star \mathrm{id}_F,$$

then $\alpha = \beta$.

Question 8.7.7.3 ▶ Characterisation of Functors Corepresentably Faithful on Cores

Is there a characterisation of functors corepresentably faithful on cores?

8.7.8 Functors Corepresentably Full on Cores

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 8.7.8.1 ▶ Functors Corepresentably Full on Cores

A functor $F: \mathcal{C} \to \mathcal{D}$ is **corepresentably full on cores** if, for each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition by F functor

$$F_*: \mathsf{Core}(\mathsf{Fun}(\mathcal{X}, \mathcal{C})) \to \mathsf{Core}(\mathsf{Fun}(\mathcal{X}, \mathcal{D}))$$

is full.

Remark 8.7.8.2 ▶ Unwinding Definition 8.7.8.1

In detail, a functor $F\colon \mathcal{C} \to \mathcal{D}$ is **corepresentably full on cores** if, for each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$ and each natural isomorphism

$$\beta\colon \phi \circ F \overset{\sim}{\Longrightarrow} \psi \circ F, \qquad \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "\phi\circ F"] \arrow[r, bend right, "\psi\circ F"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{X}, \end{tikzcd}$$

there exists a natural isomorphism

$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{X} \end{tikzcd}$$

such that we have an equality

$$\begin{tikzcd} \mathcal{X} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{C} \arrow[r, "F"] & \mathcal{D} \end{tikzcd} = \begin{tikzcd} \mathcal{X} \arrow[r, bend left, "F\circ\phi"] \arrow[r, bend right, "F\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{D} \end{tikzcd}$$

of pasting diagrams in Cats_2, i.e. such that we have

$$\beta = \alpha \star \mathrm{id}_F.$$

Question 8.7.8.3 ▶ Characterisation of Functors Corepresentably Full on Cores

Is there a characterisation of functors corepresentably full on cores?
This question also appears as [MO 468121a].

8.7.9 Functors Corepresentably Fully Faithful on Cores

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 8.7.9.1 ▶ Functors Corepresentably Fully Faithful on Cores

A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **corepresentably fully faithful on cores** if, for each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition by F functor

$$F_*\colon \mathrm{Core}(\mathrm{Fun}(\mathcal{X}, \mathcal{C})) \to \mathrm{Core}(\mathrm{Fun}(\mathcal{X}, \mathcal{D}))$$

is fully faithful.

Remark 8.7.9.2 ► Unwinding Definition 8.7.9.1

In detail, a functor $F\colon \mathcal{C} \to \mathcal{D}$ is **corepresentably fully faithful on cores** if it satisfies the conditions in Remarks 8.7.7.2 and 8.7.8.2, i.e.:

1. For all diagrams of the form

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] & \mathcal{D} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & \mathcal{X}, \end{tikzcd}$$

   if α and β are natural isomorphisms and we have

$$\alpha \star \mathrm{id}_F = \beta \star \mathrm{id}_F,$$

   then $\alpha = \beta$.

2. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$ and each natural isomorphism

$$\beta\colon \phi \circ F \overset{\sim}{\Longrightarrow} \psi \circ F, \qquad \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "\phi\circ F"] \arrow[r, bend right, "\psi\circ F"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{X}, \end{tikzcd}$$

   there exists a natural isomorphism

$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{X} \end{tikzcd}$$

   such that we have an equality

$$\begin{tikzcd} \mathcal{X} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{C} \arrow[r, "F"] & \mathcal{D} \end{tikzcd} = \begin{tikzcd} \mathcal{X} \arrow[r, bend left, "F\circ\phi"] \arrow[r, bend right, "F\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{D} \end{tikzcd}$$

   of pasting diagrams in Cats_2, i.e. such that we have

$$\beta = \alpha \star \mathrm{id}_F.$$

Question 8.7.9.3 ► Characterisation of Functors Corepresentably Fully Faithful on Cores

Is there a characterisation of functors corepresentably fully faithful on cores?

8.8 Natural Transformations

8.8.1 Transformations

Let $\mathcal{C}$ and $\mathcal{D}$ be categories and $F, G\colon \mathcal{C} \rightrightarrows \mathcal{D}$ be functors.

Definition 8.8.1.1 ► Transformations

A **transformation**[1] $\alpha\colon F \Rightarrow G$ **from** F **to** G is a collection

$$\{\alpha_A\colon F(A) \to G(A)\}_{A\in\mathrm{Obj}(\mathcal{C})}$$

of morphisms of $\mathcal{D}$.

[1] *Further Terminology:* Also called an **unnatural transformation** for emphasis.

Notation 8.8.1.2 ► The Set of Transformations Between Two Functors

We write $\mathrm{Trans}(F, G)$ for the set of transformations from F to G.

8.8.2 Natural Transformations

Let $\mathcal{C}$ and $\mathcal{D}$ be categories and $F, G\colon \mathcal{C} \rightrightarrows \mathcal{D}$ be functors.

Definition 8.8.2.1 ► Natural Transformations

A **natural transformation** $\alpha\colon F \Longrightarrow G$ **from** F **to** G is a transformation

$$\{\alpha_A\colon F(A) \to G(A)\}_{A\in\mathrm{Obj}(\mathcal{C})}$$

from F to G such that, for each morphism $f\colon A \to B$ of $\mathcal{C}$, the diagram

$$\begin{tikzcd}
F(A) \arrow[r, "F(f)"] \arrow[d, "\alpha_A"'] & F(B) \arrow[d, "\alpha_B"] \\
G(A) \arrow[r, "G(f)"'] & G(B)
\end{tikzcd}$$

commutes.[1]

[1] *Further Terminology:* The morphism $\alpha_A\colon F_A \to G_A$ is called the **component of** α **at** A.

Remark 8.8.2.2 ► Picturing Natural Transformations in Diagrams

We denote natural transformations in diagrams as

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"{name=U}] \arrow[r, bend right, "G"'{name=D}] & \mathcal{D}. \arrow[from=U, to=D, Rightarrow, "\alpha"'] \end{tikzcd}$$

Notation 8.8.2.3 ► The Set of Natural Transformations Between Two Functors

We write $\mathrm{Nat}(F, G)$ for the set of natural transformations from F to G.

Example 8.8.2.4 ► Identity Natural Transformations

The **identity natural transformation** $\mathrm{id}_F \colon F \Longrightarrow F$ **of** F is the natural transformation consisting of the collection

$$\left\{\mathrm{id}_{F(A)} \colon F(A) \to F(A)\right\}_{A \in \mathrm{Obj}(\mathcal{C})}.$$

Proof 8.8.2.5 ► Proof of Example 8.8.2.4

The naturality condition for id_F is the requirement that, for each morphism $f \colon A \to B$ of $\mathcal{C}$, the diagram

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\mathrm{id}_{F(A)}"'] & F(B) \arrow[d, "\mathrm{id}_{F(B)}"] \\ F(A) \arrow[r, "F(f)"'] & F(B) \end{tikzcd}$$

commutes, which follows from unitality of the composition of $\mathcal{C}$. □

Definition 8.8.2.6 ► Equality of Natural Transformations

Two natural transformations $\alpha, \beta \colon F \Longrightarrow G$ are **equal** if we have

$$\alpha_A = \beta_A$$

for each $A \in \mathrm{Obj}(\mathcal{C})$.

8.8.3 Vertical Composition of Natural Transformations

Definition 8.8.3.1 ► Vertical Composition of Natural Transformations

The **vertical composition** of two natural transformations $\alpha\colon F \implies G$ and $\beta\colon G \implies H$ as in the diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=60, "F"] \arrow[r, "G" description] \arrow[r, bend right=60, "H"'] & \mathcal{D} \end{tikzcd} \quad \alpha\Downarrow,\ \beta\Downarrow$$

is the natural transformation $\beta \circ \alpha\colon F \implies H$ consisting of the collection

$$\left\{(\beta \circ \alpha)_A\colon F(A) \to H(A)\right\}_{A \in \mathrm{Obj}(\mathcal{C})}$$

with

$$(\beta \circ \alpha)_A \stackrel{\text{def}}{=} \beta_A \circ \alpha_A$$

for each $A \in \mathrm{Obj}(\mathcal{C})$.

Proof 8.8.3.2 ► Proof of Definition 8.8.3.1

The naturality condition for $\beta \circ \alpha$ is the requirement that the boundary of the diagram

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\alpha_A"'] \arrow[dr, phantom, "(1)"] & F(B) \arrow[d, "\alpha_B"] \\ G(A) \arrow[r, "G(f)" description] \arrow[d, "\beta_A"'] \arrow[dr, phantom, "(2)"] & G(B) \arrow[d, "\beta_B"] \\ H(A) \arrow[r, "H(f)"'] & H(B) \end{tikzcd}$$

commutes. Since

1. Subdiagram (1) commutes by the naturality of α.
2. Subdiagram (2) commutes by the naturality of β.

so does the boundary diagram. Hence $\beta \circ \alpha$ is a natural transformation. ∎

Proposition 8.8.3.3 ► Properties of Vertical Composition of Natural Transformations

Let $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ be categories.

1. *Functionality.* The assignment $(\beta, \alpha) \mapsto \beta \circ \alpha$ defines a function
$$\circ_{F,G,H} \colon \mathsf{Nat}(G, H) \times \mathsf{Nat}(F, G) \longrightarrow \mathsf{Nat}(F, H).$$
2. *Associativity.* Let $F, G, H, K \colon \mathcal{C} \overset{\rightrightarrows}{\rightrightarrows} \mathcal{D}$ be functors. The diagram
$$\begin{tikzcd}
 & \mathsf{Nat}(H,K) \times (\mathsf{Nat}(G,H) \times \mathsf{Nat}(F,G)) \arrow[dr, "\mathrm{id}_{\mathsf{Nat}(H,K)} \times \circ_{F,G,H}"] & \\
(\mathsf{Nat}(H,K) \times \mathsf{Nat}(G,H)) \times \mathsf{Nat}(F,G) \arrow[ur, dashed, "\alpha^{\mathsf{Sets}}_{\mathsf{Nat}(H,K),\mathsf{Nat}(G,H),\mathsf{Nat}(F,G)}", "\sim"'] \arrow[d, "\circ_{G,H,K} \times \mathrm{id}_{\mathsf{Nat}(F,G)}"'] & & \mathsf{Nat}(H,K) \times \mathsf{Nat}(F,H) \arrow[d, "\circ_{F,H,K}"] \\
\mathsf{Nat}(G,K) \times \mathsf{Nat}(F,G) \arrow[rr, "\circ_{F,G,K}"'] & & \mathsf{Nat}(F,K)
\end{tikzcd}$$
commutes, i.e. given natural transformations
$$F \overset{\alpha}{\Longrightarrow} G \overset{\beta}{\Longrightarrow} H \overset{\gamma}{\Longrightarrow} K,$$
we have
$$(\gamma \circ \beta) \circ \alpha = \gamma \circ (\beta \circ \alpha).$$
3. *Unitality.* Let $F, G \colon \mathcal{C} \rightrightarrows \mathcal{D}$ be functors.
   (a) *Left Unitality.* The diagram
$$\begin{tikzcd}
\mathrm{pt} \times \mathsf{Nat}(F,G) \arrow[d, "{[\mathrm{id}_G] \times \mathrm{id}_{\mathsf{Nat}(F,G)}}"'] \arrow[dr, dashed, "\lambda^{\mathsf{Sets}}_{\mathsf{Nat}(F,G)}", "\sim"'] & \\
\mathsf{Nat}(G,G) \times \mathsf{Nat}(F,G) \arrow[r, "\circ_{F,G,G}"'] & \mathsf{Nat}(F,G)
\end{tikzcd}$$
   commutes, i.e. given a natural transformation $\alpha \colon F \Longrightarrow G$, we have
$$\mathrm{id}_G \circ \alpha = \alpha.$$

(b) *Right Unitality.* The diagram

$$\begin{tikzcd}
\mathrm{Nat}(F,G)\times\mathrm{pt} \arrow[d, "\mathrm{id}_{\mathrm{Nat}(F,G)}\times[\mathrm{id}_F]"'] \arrow[dr, dashed, "\rho^{\mathsf{Sets}}_{\mathrm{Nat}(F,G)}", "\sim"'] & \\
\mathrm{Nat}(F,G)\times\mathrm{Nat}(F,F) \arrow[r, "\circ^{\mathcal{C}}_{F,F,G}"'] & \mathrm{Nat}(F,G)
\end{tikzcd}$$

commutes, i.e. given a natural transformation $\alpha\colon F \Longrightarrow G$, we have

$$\alpha \circ \mathrm{id}_F = \alpha.$$

4. *Middle Four Exchange.* Let $F_1, F_2, F_3 \colon \mathcal{C} \to \mathcal{D}$ and $G_1, G_2, G_3 \colon \mathcal{D} \to \mathcal{E}$ be functors. The diagram

$$\begin{tikzcd}
(\mathrm{Nat}(G_2,G_3)\times\mathrm{Nat}(G_1,G_2))\times(\mathrm{Nat}(F_2,F_3)\times\mathrm{Nat}(F_1,F_2)) \arrow[r, dashed, leftrightarrow, "\mu_4", "\sim"'] \arrow[d, "\circ_{G_1,G_2,G_3}\times\circ_{F_1,F_2,F_3}"'] & (\mathrm{Nat}(G_2,G_3)\times\mathrm{Nat}(F_2,F_3))\times(\mathrm{Nat}(G_1,G_2)\times\mathrm{Nat}(F_1,F_2)) \arrow[d, "\star_{F_2,F_3,G_2,G_3}\times\star_{F_1,F_2,G_1,G_2}"] \\
\mathrm{Nat}(G_1,G_3)\times\mathrm{Nat}(F_1,F_3) \arrow[dr, "\star_{F_1,F_3,G_1,G_3}"'] & \mathrm{Nat}(G_2\circ F_2,G_3\circ F_3)\times\mathrm{Nat}(G_1\circ F_1,G_2\circ F_2) \arrow[d, "\circ_{G_1\circ F_1,G_2\circ F_2,G_3\circ F_3}"] \\
& \mathrm{Nat}(G_1\circ F_1,G_3\circ F_3)
\end{tikzcd}$$

commutes, i.e. given a diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=60, "F_1"{name=A}] \arrow[r, "F_2"{name=B, description}] \arrow[r, bend right=60, "F_3"'{name=C}] & \mathcal{D} \arrow[r, bend left=60, "G_1"{name=D}] \arrow[r, "G_2"{name=E, description}] \arrow[r, bend right=60, "G_3"'{name=F}] & \mathcal{E}
\arrow[from=A, to=B, phantom, "\alpha\Downarrow"]
\arrow[from=B, to=C, phantom, "\alpha'\Downarrow"]
\arrow[from=D, to=E, phantom, "\beta\Downarrow"]
\arrow[from=E, to=F, phantom, "\beta'\Downarrow"]
\end{tikzcd}$$

in Cats_2, we have

$$(\beta' \star \alpha') \circ (\beta \star \alpha) = (\beta' \circ \beta) \star (\alpha' \circ \alpha).$$

Proof 8.8.3.4 ► Proof of Proposition 8.8.3.3

Item 1: Functionality

Clear.

Item 2: Associativity

Indeed, we have

$$\begin{aligned}((\gamma \circ \beta) \circ \alpha)_A &\overset{\text{def}}{=} (\gamma \circ \beta)_A \circ \alpha_A \\ &\overset{\text{def}}{=} (\gamma_A \circ \beta_A) \circ \alpha_A \\ &= \gamma_A \circ (\beta_A \circ \alpha_A) \\ &\overset{\text{def}}{=} \gamma_A \circ (\beta \circ \alpha)_A \\ &\overset{\text{def}}{=} (\gamma \circ (\beta \circ \alpha))_A\end{aligned}$$

for each $A \in \text{Obj}(\mathcal{C})$, showing the desired equality.

Item 3: Unitality

We have

$$\begin{aligned}(\text{id}_G \circ \alpha)_A &= \text{id}_G \circ \alpha_A \\ &= \alpha_A, \\ (\alpha \circ \text{id}_F)_A &= \alpha_A \circ \text{id}_F \\ &= \alpha_A\end{aligned}$$

for each $A \in \text{Obj}(\mathcal{C})$, showing the desired equality.

Item 4: Middle Four Exchange

This is proved in Item 4 of Proposition 8.8.4.4.

8.8.4 Horizontal Composition of Natural Transformations

Definition 8.8.4.1 ► Horizontal Composition of Natural Transformations

The **horizontal composition**[1,2] of two natural transformations $\alpha\colon F \Longrightarrow G$ and $\beta\colon H \Longrightarrow K$ as in the diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{D} \arrow[r, bend left, "H"] \arrow[r, bend right, "K"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{E} \end{tikzcd}$$

of α and β is the natural transformation

$$\beta \star \alpha\colon (H \circ F) \Longrightarrow (K \circ G),$$

as in the diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "H\circ F"] \arrow[r, bend right, "K\circ G"'] \arrow[r, phantom, "\Downarrow \beta\star\alpha"] & \mathcal{E}, \end{tikzcd}$$

consisting of the collection

$$\left\{(\beta \star \alpha)_A \colon H(F(A)) \to K(G(A))\right\}_{A\in\mathrm{Obj}(\mathcal{C})},$$

of morphisms of $\mathcal{E}$ with

$$\begin{aligned}(\beta \star \alpha)_A &\overset{\text{def}}{=} \beta_{G(A)} \circ H(\alpha_A)\\ &= K(\alpha_A) \circ \beta_{F(A)},\end{aligned} \qquad \begin{tikzcd} H(F(A)) \arrow[r, "H(\alpha_A)"] \arrow[d, "\beta_{F(A)}"'] & H(G(A)) \arrow[d, "\beta_{G(A)}"] \\ K(F(A)) \arrow[r, "K(\alpha_A)"'] & K(G(A)). \end{tikzcd}$$

---

[1] *Further Terminology:* Also called the **Godement product** of α and β.
[2] Horizontal composition forms a map

$$\star_{(F,H),(G,K)} \colon \mathsf{Nat}(H,K) \times \mathsf{Nat}(F,G) \to \mathsf{Nat}(H\circ F, K\circ G).$$

PROOF 8.8.4.2 ► PROOF OF DEFINITION 8.8.4.1

First, we claim that we indeed have

$$\beta_{G(A)} \circ H(\alpha_A) = K(\alpha_A) \circ \beta_{F(A)}, \qquad \begin{tikzcd} H(F(A)) \arrow[r, "H(\alpha_A)"] \arrow[d, "\beta_{F(A)}"'] & H(G(A)) \arrow[d, "\beta_{G(A)}"] \\ K(F(A)) \arrow[r, "K(\alpha_A)"'] & K(G(A)). \end{tikzcd}$$

This is, however, simply the naturality square for β applied to the morphism $\alpha_A \colon F(A) \to G(A)$. Next, we check the naturality condition for $\beta \star \alpha$, which is the requirement that the boundary of the diagram

$$\begin{tikzcd}
H(F(A)) \arrow[r, "H(F(f))"] \arrow[d, "H(\alpha_A)"'] \arrow[dr, phantom, "(1)"] & H(F(B)) \arrow[d, "H(\alpha_B)"] \\
H(G(A)) \arrow[r, "H(G(f))"] \arrow[d, "\beta_{G(A)}"'] \arrow[dr, phantom, "(2)"] & H(G(B)) \arrow[d, "\beta_{G(B)}"] \\
K(G(A)) \arrow[r, "K(G(f))"'] & K(G(B))
\end{tikzcd}$$

commutes. Since

1. Subdiagram (1) commutes by the naturality of α.
2. Subdiagram (2) commutes by the naturality of β.

so does the boundary diagram. Hence $\beta \circ \alpha$ is a natural transformation.[1] ▭

[1] *Reference:* [Bor94, Proposition 1.3.4].

Definition 8.8.4.3 ▶ Whiskering of Functors With Natural Transformations

Let

$$\begin{tikzcd}
\mathcal{X} \arrow[r, "F"] & \mathcal{C} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{D} \arrow[r, "G"] & \mathcal{Y}
\end{tikzcd}$$

be a diagram in Cats_2.

1. The **left whiskering of** α **with** G is the natural transformation[1]
$$\mathrm{id}_G \star \alpha \colon G \circ \phi \Longrightarrow G \circ \psi.$$
2. The **right whiskering of** α **with** F is the natural transformation[2]
$$\alpha \star \mathrm{id}_F \colon \phi \circ F \Longrightarrow \psi \circ F.$$

[1] *Further Notation:* Also written $G\alpha$ or $G \star \alpha$, although we won't use either of these notations in this work.
[2] *Further Notation:* Also written αF or $\alpha \star F$, although we won't use either of these notations in this work.

Proposition 8.8.4.4 ▶ Properties of Horizontal Composition of Natural Transformations

Let $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ be categories.

1. *Functionality.* The assignment $(\beta, \alpha) \mapsto \beta \star \alpha$ defines a function
$$\star_{(F,G),(H,K)} \colon \mathrm{Nat}(H, K) \times \mathrm{Nat}(F, G) \to \mathrm{Nat}(H \circ F, K \circ G).$$

2. *Associativity.* Let
$$\mathcal{C} \underset{G_1}{\overset{F_1}{\rightrightarrows}} \mathcal{D} \underset{G_2}{\overset{F_2}{\rightrightarrows}} \mathcal{E} \underset{G_3}{\overset{F_3}{\rightrightarrows}} \mathcal{F}$$
be a diagram in Cats_2. The diagram
$$\begin{tikzcd} \mathrm{Nat}(F_3, G_3) \times \mathrm{Nat}(F_2, G_2) \times \mathrm{Nat}(F_1, G_1) \arrow[r, "\star_{(F_2,G_2),(F_3,G_3)} \times \mathrm{id}"] \arrow[d, "\mathrm{id} \times \star_{(F_1,G_1),(F_2,G_2)}"'] & \mathrm{Nat}(F_3 \circ F_2, G_3 \circ G_2) \times \mathrm{Nat}(F_1, G_1) \arrow[d, "\star_{(F_3 \circ F_2),(G_3 \circ G_2, F_1, G_1)}"] \\ \mathrm{Nat}(F_3, G_3) \times \mathrm{Nat}(F_2 \circ F_1, G_2 \circ G_1) \arrow[r, "\star_{(F_2 \circ F_1),(G_2 \circ G_1, F_3, G_3)}"'] & \mathrm{Nat}(F_3 \circ F_2 \circ F_1, G_3 \circ G_2 \circ G_1) \end{tikzcd}$$
commutes, i.e. given natural transformations
$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F_1"] \arrow[r, bend right, "G_1"'] \arrow[r, phantom, "\alpha \Downarrow"] & \mathcal{D} \arrow[r, bend left, "F_2"] \arrow[r, bend right, "G_2"'] \arrow[r, phantom, "\beta \Downarrow"] & \mathcal{E} \arrow[r, bend left, "F_3"] \arrow[r, bend right, "G_3"'] \arrow[r, phantom, "\gamma \Downarrow"] & \mathcal{F}, \end{tikzcd}$$
we have
$$(\gamma \star \beta) \star \alpha = \gamma \star (\beta \star \alpha).$$

3. *Interaction With Identities.* Let $F \colon \mathcal{C} \to \mathcal{D}$ and $G \colon \mathcal{D} \to \mathcal{E}$ be functors. The diagram
$$\begin{tikzcd} \mathrm{pt} \times \mathrm{pt} \arrow[r, "{[\mathrm{id}_G] \times [\mathrm{id}_F]}"] & \mathrm{Nat}(G, G) \times \mathrm{Nat}(F, F) \arrow[d, "\star_{(F,F),(G,G)}"] \\ \mathrm{pt} \arrow[u, dashed, "\wr"] \arrow[r, "{[\mathrm{id}_{G \circ F}]}"'] & \mathrm{Nat}(G \circ F, G \circ F) \end{tikzcd}$$
commutes, i.e. we have
$$\mathrm{id}_G \star \mathrm{id}_F = \mathrm{id}_{G \circ F}.$$

4. *Middle Four Exchange.* Let $F_1, F_2, F_3 \colon \mathcal{C} \to \mathcal{D}$ and $G_1, G_2, G_3 \colon \mathcal{D} \to \mathcal{E}$ be functors. The diagram

$$\begin{tikzcd}
(\mathsf{Nat}(G_2, G_3) \times \mathsf{Nat}(G_1, G_2)) \times (\mathsf{Nat}(F_2, F_3) \times \mathsf{Nat}(F_1, F_2)) \arrow[rr, dashed, leftrightarrow, "\mu_4", "\sim"'] \arrow[d, "\circ_{G_1,G_2,G_3} \times \circ_{F_1,F_2,F_3}"'] & & (\mathsf{Nat}(G_2, G_3) \times \mathsf{Nat}(F_2, F_3)) \times (\mathsf{Nat}(G_1, G_2) \times \mathsf{Nat}(F_1, F_2)) \arrow[d, "\star_{F_2,F_3,G_2,G_3} \times \star_{F_1,F_2,G_1,G_2}"] \\
\mathsf{Nat}(G_1, G_3) \times \mathsf{Nat}(F_1, F_3) \arrow[dr, "\star_{F_1,F_3,G_1,G_3}"'] & & \mathsf{Nat}(G_2 \circ F_2, G_3 \circ F_3) \times \mathsf{Nat}(G_1 \circ F_1, G_2 \circ F_2) \arrow[dl, "\circ_{G_1 \circ F_1, G_2 \circ F_2, G_3 \circ F_3}"] \\
& \mathsf{Nat}(G_1 \circ F_1, G_3 \circ F_3) &
\end{tikzcd}$$

commutes, i.e. given a diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=60, "F_1"{name=A}] \arrow[r, "F_2"{name=B, description}] \arrow[r, bend right=60, "F_3"'{name=C}] & \mathcal{D} \arrow[r, bend left=60, "G_1"{name=D}] \arrow[r, "G_2"{name=E, description}] \arrow[r, bend right=60, "G_3"'{name=F}] & \mathcal{E}
\arrow[from=A, to=B, phantom, "\alpha\Downarrow"]
\arrow[from=B, to=C, phantom, "\alpha'\Downarrow"]
\arrow[from=D, to=E, phantom, "\beta\Downarrow"]
\arrow[from=E, to=F, phantom, "\beta'\Downarrow"]
\end{tikzcd}$$

in Cats_2, we have

$$(\beta' \star \alpha') \circ (\beta \star \alpha) = (\beta' \circ \beta) \star (\alpha' \circ \alpha).$$

Proof 8.8.4.5 ► Proof of Proposition 8.8.4.4

Item 1: Functionality

Clear.

Item 2: Associativity

Omitted.

Item 3: Interaction With Identities

We have

$$\begin{aligned}
(\mathrm{id}_G \star \mathrm{id}_F)_A &\overset{\text{def}}{=} (\mathrm{id}_G)_{F_A} \circ G_{(\mathrm{id}_F)_A} \\
&\overset{\text{def}}{=} \mathrm{id}_{G_{F_A}} \circ G_{\mathrm{id}_{F_A}} \\
&= \mathrm{id}_{G_{F_A}} \circ \mathrm{id}_{G_{F_A}} \\
&= \mathrm{id}_{G_{F_A}} \\
&\overset{\text{def}}{=} (\mathrm{id}_{G \circ F})_A
\end{aligned}$$

for each $A \in \mathsf{Obj}(\mathcal{C})$, showing the desired equality.

Item 4: Middle Four Exchange

Let $A \in \mathsf{Obj}(\mathcal{C})$ and consider the diagram

$$\begin{tikzcd}
 & & G_1(F_3(A)) \arrow[dr, "\beta_{F_3(A)}"] & & \\
G_1(F_1(A)) \arrow[r, "G_1(\alpha_A)"] & G_1(F_2(A)) \arrow[ur, "G_1(\alpha'_A)"] \arrow[dr, "\beta_{F_2(A)}"'] & (1) & G_2(F_3(A)) \arrow[r, "\beta'_{F_3(A)}"] & G_3(F_3(A)). \\
 & & G_2(F_2(A)) \arrow[ur, "G_2(\alpha'_A)"'] & &
\end{tikzcd}$$

The top composition

$$\begin{tikzcd}
 & & G_1(F_3(A)) \arrow[dr, "\beta_{F_3(A)}"] & & \\
G_1(F_1(A)) \arrow[r, "G_1(\alpha_A)"] & G_1(F_2(A)) \arrow[ur, "G_1(\alpha'_A)"] \arrow[dr, "\beta_{F_2(A)}"'] & (1) & G_2(F_3(A)) \arrow[r, "\beta'_{F_3(A)}"] & G_3(F_3(A)). \\
 & & G_2(F_2(A)) \arrow[ur, "G_2(\alpha'_A)"'] & &
\end{tikzcd}$$

is given by $((\beta' \circ \beta) \star (\alpha' \circ \alpha))_A$, while the bottom composition

$$\begin{tikzcd}
 & & G_1(F_3(A)) \arrow[dr, "\beta_{F_3(A)}"] & & \\
G_1(F_1(A)) \arrow[r, "G_1(\alpha_A)"] & G_1(F_2(A)) \arrow[ur, "G_1(\alpha'_A)"] \arrow[dr, "\beta_{F_2(A)}"'] & (1) & G_2(F_3(A)) \arrow[r, "\beta'_{F_3(A)}"] & G_3(F_3(A)). \\
 & & G_2(F_2(A)) \arrow[ur, "G_2(\alpha'_A)"'] & &
\end{tikzcd}$$

is given by $((\beta' \star \alpha') \circ (\beta \star \alpha))_A$. Now, Subdiagram (1) corresponds to the naturality condition

$$G_2(\alpha'_A) \circ \beta_{F_2(A)} = \beta_{F_3}(A) \circ G_1(\alpha'_A), \qquad \begin{tikzcd} G_1(F_2(A)) \arrow[r, "G_1(\alpha'_A)"] \arrow[d, "\beta_{F_2(A)}"'] & G_1(F_3(A)) \arrow[d, "\beta_{F_3(A)}"] \\ G_2(F_2(A)) \arrow[r, "G_2(\alpha'_A)"'] & G_2(F_3(A)) \end{tikzcd}$$

for $\beta\colon G_1 \Longrightarrow G_2$ at $\alpha'_A\colon F_2(A) \to F_3(A)$, and thus commutes. Thus we have

$$((\beta' \circ \beta) \star (\alpha' \circ \alpha))_A = ((\beta' \star \alpha') \circ (\beta \star \alpha))_A$$

for each $A \in \mathrm{Obj}(\mathcal{C})$ and therefore

$$(\beta' \star \alpha') \circ (\beta \star \alpha) = (\beta' \circ \beta) \star (\alpha' \circ \alpha).$$

This finishes the proof. □

8.8.5 Properties of Natural Transformations

Proposition 8.8.5.1 ► Natural Transformations as Categorical Homotopies

Let $F, G\colon \mathcal{C} \rightrightarrows \mathcal{D}$ be functors. The following data are equivalent:[1]

1. A natural transformation $\alpha\colon F \Longrightarrow G$.
2. A functor $[\alpha]\colon \mathcal{C} \to \mathcal{D}^{\mathbb{1}}$ filling the diagram

$$\begin{tikzcd} & \mathcal{D} \\ \mathcal{C} \arrow[ur, "F"] \arrow[r, "{[\alpha]}"] \arrow[dr, "G"'] & \mathcal{D}^{\mathbb{1}}. \arrow[u, "\mathrm{ev}_0"'] \arrow[d, "\mathrm{ev}_1"] \\ & \mathcal{D} \end{tikzcd}$$

3. A functor $[\alpha]\colon \mathcal{C}\times\mathbb{1}\to\mathcal{D}$ filling the diagram

$$\begin{tikzcd} & \mathcal{C} \arrow[dr, "F"] & \\ & \mathcal{C}\times\mathbb{1} \arrow[u, "\mathrm{ev}_0"] \arrow[r, "{[\alpha]}"] \arrow[d, "\mathrm{ev}_1"'] & \mathcal{D}. \\ & \mathcal{C} \arrow[ur, "G"'] & \end{tikzcd}$$

[1]Taken from [MO 64365].

Proof 8.8.5.2 ▶ Proof of Proposition 8.8.5.1

From Item 1 to Item 2 and Back

We may identify $\mathcal{D}^{\mathbb{1}}$ with $\mathrm{Arr}(\mathcal{D})$. Given a natural transformation $\alpha\colon F\Longrightarrow G$, we have a functor

$$\begin{aligned} [\alpha]\colon \mathcal{C} &\longrightarrow \mathcal{D}^{\mathbb{1}} \\ A &\longmapsto \alpha_A \\ (f\colon A\to B) &\longmapsto \left(\begin{tikzcd} F_A \arrow[r, "F_f"] \arrow[d, "\alpha_A"'] & F_B \arrow[d, "\alpha_B"] \\ G_A \arrow[r, "G_f"'] & G_B \end{tikzcd}\right) \end{aligned}$$

making the diagram in Item 2 commute. Conversely, every such functor gives rise to a natural transformation from F to G, and these constructions are inverse to each other.

From Item 2 to Item 3 and Back

This follows from Item 3 of Proposition 8.9.1.2.

8.8.6 Natural Isomorphisms

Let $\mathcal{C}$ and $\mathcal{D}$ be categories and let $F, G\colon \mathcal{C}\rightrightarrows\mathcal{D}$ be functors.

Definition 8.8.6.1 ► Natural Isomorphisms

A natural transformation $\alpha\colon F \Longrightarrow G$ is a **natural isomorphism** if there exists a natural transformation $\alpha^{-1}\colon G \Longrightarrow F$ such that

$$\alpha^{-1} \circ \alpha = \mathrm{id}_F,$$
$$\alpha \circ \alpha^{-1} = \mathrm{id}_G.$$

Proposition 8.8.6.2 ► Properties of Natural Isomorphisms

Let $\alpha\colon F \Longrightarrow G$ be a natural transformation.

1. *Characterisations.* The following conditions are equivalent:
   (a) The natural transformation α is a natural isomorphism.
   (b) For each $A \in \mathrm{Obj}(\mathcal{C})$, the morphism $\alpha_A\colon F_A \to G_A$ is an isomorphism.
2. *Componentwise Inverses of Natural Transformations Assemble Into Natural Transformations.* Let $\alpha^{-1}\colon G \Longrightarrow F$ be a transformation such that, for each $A \in \mathrm{Obj}(\mathcal{C})$, we have

$$\alpha_A^{-1} \circ \alpha_A = \mathrm{id}_{F(A)},$$
$$\alpha_A \circ \alpha_A^{-1} = \mathrm{id}_{G(A)}.$$

   Then α^{-1} is a natural transformation.

Proof 8.8.6.3 ► Proof of Proposition 8.8.6.2

Item 1: Characterisations

The implication Item 1a $\Longrightarrow$ Item 1b is clear, whereas the implication Item 1b $\Longrightarrow$ Item 1a follows from Item 2.

Item 2: Componentwise Inverses of Natural Transformations Assemble Into

The naturality condition for α^{-1} corresponds to the commutativity of the diagram

$$\begin{tikzcd} G(A) \arrow[r, "G(f)"] \arrow[d, "\alpha_A^{-1}"'] & G(B) \arrow[d, "\alpha_B^{-1}"] \\ F(A) \arrow[r, "F(f)"'] & F(B) \end{tikzcd}$$

for each $A, B \in \mathrm{Obj}(C)$ and each $f \in \mathrm{Hom}_C(A, B)$. Considering the diagram

$$\begin{tikzcd}
G(A) \arrow[r, "G(f)"] \arrow[d, "\alpha_A^{-1}"'] \arrow[dr, phantom, "(1)"] & G(B) \arrow[d, "\alpha_B^{-1}"] \\
F(A) \arrow[r, "F(f)"] \arrow[d, "\alpha_A"'] \arrow[dr, phantom, "(2)"] & F(B) \arrow[d, "\alpha_B"] \\
G(A) \arrow[r, "G(f)"'] & G(B),
\end{tikzcd}$$

where the boundary diagram as well as Subdiagram (2) commute, we have

$$\begin{aligned}
G(f) &= G(f) \circ \mathrm{id}_{G(A)} \\
&= G(f) \circ \alpha_A \circ \alpha_A^{-1} \\
&= \alpha_B \circ F(f) \circ \alpha_A^{-1}.
\end{aligned}$$

Postcomposing both sides with α_B^{-1}, we get

$$\begin{aligned}
\alpha_B^{-1} \circ G(f) &= \alpha_B^{-1} \circ \alpha_B \circ F(f) \circ \alpha_A^{-1} \\
&= \mathrm{id}_{F(B)} \circ F(f) \circ \alpha_A^{-1} \\
&= F(f) \circ \alpha_A^{-1},
\end{aligned}$$

which is the naturality condition we wanted to show. Thus α^{-1} is a natural transformation.

8.9 Categories of Categories

8.9.1 Functor Categories

Let C be a category and $\mathcal{D}$ be a small category.

DEFINITION 8.9.1.1 ► FUNCTOR CATEGORIES

The **category of functors from C to $\mathcal{D}$**[1] is the category $\mathrm{Fun}(C, \mathcal{D})$[2] where

- *Objects.* The objects of $\mathrm{Fun}(C, \mathcal{D})$ are functors from C to $\mathcal{D}$.
- *Morphisms.* For each $F, G \in \mathrm{Obj}(\mathrm{Fun}(C, \mathcal{D}))$, we have

$$\mathrm{Hom}_{\mathrm{Fun}(C,\mathcal{D})}(F, G) \stackrel{\text{def}}{=} \mathrm{Nat}(F, G).$$

- *Identities.* For each $F \in \mathrm{Obj}(\mathsf{Fun}(\mathcal{C}, \mathcal{D}))$, the unit map

$$\mathbb{1}_F^{\mathsf{Fun}(\mathcal{C},\mathcal{D})} : \mathrm{pt} \to \mathrm{Nat}(F, F)$$

  of $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$ at F is given by

$$\mathrm{id}_F^{\mathsf{Fun}(\mathcal{C},\mathcal{D})} \stackrel{\text{def}}{=} \mathrm{id}_F,$$

  where $\mathrm{id}_F : F \Longrightarrow F$ is the identity natural transformation of F of Example 8.8.2.4.

- *Composition.* For each $F, G, H \in \mathrm{Obj}(\mathsf{Fun}(\mathcal{C}, \mathcal{D}))$, the composition map

$$\circ_{F,G,H}^{\mathsf{Fun}(\mathcal{C},\mathcal{D})} : \mathrm{Nat}(G, H) \times \mathrm{Nat}(F, G) \to \mathrm{Nat}(F, H)$$

  of $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$ at (F, G, H) is given by

$$\beta \circ_{F,G,H}^{\mathsf{Fun}(\mathcal{C},\mathcal{D})} \alpha \stackrel{\text{def}}{=} \beta \circ \alpha,$$

  where $\beta \circ \alpha$ is the vertical composition of α and β of Item 1 of Proposition 8.8.3.3.

[1] *Further Terminology:* Also called the **functor category** $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$.
[2] *Further Notation:* Also written $\mathcal{D}^{\mathcal{C}}$ and $[\mathcal{C}, \mathcal{D}]$.

PROPOSITION 8.9.1.2 ► PROPERTIES OF FUNCTOR CATEGORIES

Let $\mathcal{C}$ and $\mathcal{D}$ be categories and let $F : \mathcal{C} \to \mathcal{D}$ be a functor.

1. *Functoriality.* The assignments $\mathcal{C}, \mathcal{D}, (\mathcal{C}, \mathcal{D}) \mapsto \mathsf{Fun}(\mathcal{C}, \mathcal{D})$ define functors

$$\begin{aligned} \mathsf{Fun}(\mathcal{C}, -_2) &: \mathsf{Cats} \to \mathsf{Cats}, \\ \mathsf{Fun}(-_1, \mathcal{D}) &: \mathsf{Cats}^{\mathsf{op}} \to \mathsf{Cats}, \\ \mathsf{Fun}(-_1, -_2) &: \mathsf{Cats}^{\mathsf{op}} \times \mathsf{Cats} \to \mathsf{Cats}. \end{aligned}$$

2. *2-Functoriality.* The assignments $\mathcal{C}, \mathcal{D}, (\mathcal{C}, \mathcal{D}) \mapsto \mathsf{Fun}(\mathcal{C}, \mathcal{D})$ define 2-functors

$$\begin{aligned} \mathsf{Fun}(\mathcal{C}, -_2) &: \mathsf{Cats}_2 \to \mathsf{Cats}_2, \\ \mathsf{Fun}(-_1, \mathcal{D}) &: \mathsf{Cats}_2^{\mathsf{op}} \to \mathsf{Cats}_2, \\ \mathsf{Fun}(-_1, -_2) &: \mathsf{Cats}_2^{\mathsf{op}} \times \mathsf{Cats}_2 \to \mathsf{Cats}_2. \end{aligned}$$

3. *Adjointness.* We have adjunctions

$$(\mathcal{C} \times - \dashv \mathrm{Fun}(\mathcal{C}, -))\colon \quad \begin{tikzcd} \mathsf{Cats} \arrow[r, bend left, "\mathcal{C}\times -"] \arrow[r, phantom, "\perp"] & \mathsf{Cats}, \arrow[l, bend left, "\mathrm{Fun}(\mathcal{C},-)"] \end{tikzcd}$$

$$(- \times \mathcal{D} \dashv \mathrm{Fun}(\mathcal{D}, -))\colon \quad \begin{tikzcd} \mathsf{Cats} \arrow[r, bend left, "-\times\mathcal{D}"] \arrow[r, phantom, "\perp"] & \mathsf{Cats}, \arrow[l, bend left, "\mathrm{Fun}(\mathcal{D},-)"] \end{tikzcd}$$

witnessed by bijections of sets

$$\begin{aligned} \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) &\cong \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{D}, \mathrm{Fun}(\mathcal{C}, \mathcal{E})), \\ \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) &\cong \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, \mathrm{Fun}(\mathcal{D}, \mathcal{E})), \end{aligned}$$

natural in $\mathcal{C}, \mathcal{D}, \mathcal{E} \in \mathrm{Obj}(\mathsf{Cats})$.

4. *2-Adjointness.* We have 2-adjunctions

$$(\mathcal{C} \times - \dashv \mathrm{Fun}(\mathcal{C}, -))\colon \quad \begin{tikzcd} \mathsf{Cats}_2 \arrow[r, bend left, "\mathcal{C}\times -"] \arrow[r, phantom, "\perp_2"] & \mathsf{Cats}_2, \arrow[l, bend left, "\mathrm{Fun}(\mathcal{C},-)"] \end{tikzcd}$$

$$(- \times \mathcal{D} \dashv \mathrm{Fun}(\mathcal{D}, -))\colon \quad \begin{tikzcd} \mathsf{Cats}_2 \arrow[r, bend left, "-\times\mathcal{D}"] \arrow[r, phantom, "\perp_2"] & \mathsf{Cats}_2, \arrow[l, bend left, "\mathrm{Fun}(\mathcal{D},-)"] \end{tikzcd}$$

witnessed by isomorphisms of categories

$$\begin{aligned} \mathrm{Fun}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) &\cong \mathrm{Fun}(\mathcal{D}, \mathrm{Fun}(\mathcal{C}, \mathcal{E})), \\ \mathrm{Fun}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) &\cong \mathrm{Fun}(\mathcal{C}, \mathrm{Fun}(\mathcal{D}, \mathcal{E})), \end{aligned}$$

natural in $\mathcal{C}, \mathcal{D}, \mathcal{E} \in \mathrm{Obj}(\mathsf{Cats}_2)$.

5. *Interaction With Punctual Categories.* We have a canonical isomorphism of categories

$$\mathrm{Fun}(\mathrm{pt}, \mathcal{C}) \cong \mathcal{C},$$

natural in $\mathcal{C} \in \mathrm{Obj}(\mathsf{Cats})$.

6. *Objectwise Computation of Co/Limits.* Let

$$D\colon \mathcal{I} \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

be a diagram in $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$. We have isomorphisms

$$\lim(D)_A \cong \lim_{i \in I}(D_i(A)),$$
$$\operatorname{colim}(D)_A \cong \operatorname*{colim}_{i \in I}(D_i(A)),$$

naturally in $A \in \mathrm{Obj}(\mathcal{C})$.

7. *Interaction With Co/Completeness.* If $\mathcal{E}$ is co/complete, then so is $\mathsf{Fun}(\mathcal{C}, \mathcal{E})$.

8. *Monomorphisms and Epimorphisms.* Let $\alpha \colon F \Longrightarrow G$ be a morphism of $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$. The following conditions are equivalent:

   (a) The natural transformation

   $$\alpha \colon F \Longrightarrow G$$

   is a monomorphism (resp. epimorphism) in $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$.

   (b) For each $A \in \mathrm{Obj}(\mathcal{C})$, the morphism

   $$\alpha_A \colon F_A \to G_A$$

   is a monomorphism (resp. epimorphism) in $\mathcal{D}$.

Proof 8.9.1.3 ► Proof of Proposition 8.9.1.2

Item 1: Functoriality

Omitted.

Item 2: 2-Functoriality

Omitted.

Item 3: Adjointness

Omitted.

Item 4: 2-Adjointness

Omitted.

Item 5: Interaction With Punctual Categories

Omitted.

Item 6: Objectwise Computation of Co/Limits

Omitted.

Item 7: Interaction With Co/Completeness

This follows from **??**.

Item 8: Monomorphisms and Epimorphisms

Omitted.

8.9.2 The Category of Categories and Functors

Definition 8.9.2.1 ► The Category of Categories and Functors

The **category of** (**small**) **categories and functors** is the category Cats where

- *Objects.* The objects of Cats are small categories.
- *Morphisms.* For each $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$, we have

$$\mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, \mathcal{D}) \overset{\mathrm{def}}{=} \mathrm{Obj}(\mathsf{Fun}(\mathcal{C}, \mathcal{D})).$$

- *Identities.* For each $\mathcal{C} \in \mathrm{Obj}(\mathsf{Cats})$, the unit map

$$\mathbb{1}^{\mathsf{Cats}}_{\mathcal{C}} : \mathrm{pt} \to \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, \mathcal{C})$$

  of Cats at $\mathcal{C}$ is defined by

$$\mathrm{id}^{\mathsf{Cats}}_{\mathcal{C}} \overset{\mathrm{def}}{=} \mathrm{id}_{\mathcal{C}},$$

  where $\mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ is the identity functor of $\mathcal{C}$ of Example 8.4.1.4.
- *Composition.* For each $\mathcal{C}, \mathcal{D}, \mathcal{E} \in \mathrm{Obj}(\mathsf{Cats})$, the composition map

$$\circ^{\mathsf{Cats}}_{\mathcal{C},\mathcal{D},\mathcal{E}} : \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{D}, \mathcal{E}) \times \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, \mathcal{D}) \to \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, \mathcal{E})$$

  of Cats at $(\mathcal{C}, \mathcal{D}, \mathcal{E})$ is given by

$$G \circ^{\mathsf{Cats}}_{\mathcal{C},\mathcal{D},\mathcal{E}} F \overset{\mathrm{def}}{=} G \circ F,$$

  where $G \circ F : \mathcal{C} \to \mathcal{E}$ is the composition of F and G of Definition 8.4.1.6.

Proposition 8.9.2.2 ► Properties of the Category Cats

Let $\mathcal{C}$ be a category.

1. *Co/Completeness.* The category Cats is complete and cocomplete.

2. *Cartesian Monoidal Structure.* The quadruple $(\mathsf{Cats}, \times, \mathrm{pt}, \mathsf{Fun})$ is a Cartesian closed monoidal category.

Proof 8.9.2.3 ► Proof of Proposition 8.9.2.2

Item 1: Co/Completeness

Omitted.

Item 2: Cartesian Monoidal Structure

Omitted.

8.9.3 The 2-Category of Categories, Functors, and Natural Transformations

Definition 8.9.3.1 ► The 2-Category of Categories

The 2-**category of (small) categories, functors, and natural transformations** is the 2-category Cats_2 where

- *Objects.* The objects of Cats_2 are small categories.
- Hom-*Categories.* For each $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats}_2)$, we have

$$\mathrm{Hom}_{\mathsf{Cats}_2}(\mathcal{C}, \mathcal{D}) \stackrel{\text{def}}{=} \mathsf{Fun}(\mathcal{C}, \mathcal{D}).$$

- *Identities.* For each $\mathcal{C} \in \mathrm{Obj}(\mathsf{Cats}_2)$, the unit functor

$$\mathbb{1}^{\mathsf{Cats}_2}_{\mathcal{C}} \colon \mathrm{pt} \to \mathsf{Fun}(\mathcal{C}, \mathcal{C})$$

  of Cats_2 at $\mathcal{C}$ is the functor picking the identity functor $\mathrm{id}_{\mathcal{C}} \colon \mathcal{C} \to \mathcal{C}$ of $\mathcal{C}$.

- *Composition.* For each $\mathcal{C}, \mathcal{D}, \mathcal{E} \in \mathrm{Obj}(\mathsf{Cats}_2)$, the composition bifunctor

$$\circ^{\mathsf{Cats}_2}_{\mathcal{C},\mathcal{D},\mathcal{E}} \colon \mathrm{Hom}_{\mathsf{Cats}_2}(\mathcal{D}, \mathcal{E}) \times \mathrm{Hom}_{\mathsf{Cats}_2}(\mathcal{C}, \mathcal{D}) \to \mathrm{Hom}_{\mathsf{Cats}_2}(\mathcal{C}, \mathcal{E})$$

  of Cats_2 at $(\mathcal{C}, \mathcal{D}, \mathcal{E})$ is the functor where

  - *Action on Objects.* For each object $(G, F) \in \mathrm{Obj}\big(\mathrm{Hom}_{\mathsf{Cats}_2}(\mathcal{D}, \mathcal{E}) \times \mathrm{Hom}_{\mathsf{Cats}_2}(\mathcal{C}, \mathcal{D})\big)$, we have

$$\circ^{\mathsf{Cats}_2}_{\mathcal{C},\mathcal{D},\mathcal{E}}(G, F) \stackrel{\text{def}}{=} G \circ F.$$

– *Action on Morphisms.* For each morphism $(\beta, \alpha)\colon (K, H) \Longrightarrow (G, F)$ of $\mathsf{Hom}_{\mathsf{Cats}_2}(\mathcal{D}, \mathcal{E}) \times \mathsf{Hom}_{\mathsf{Cats}_2}(\mathcal{C}, \mathcal{D})$, we have

$$\circ^{\mathsf{Cats}_2}_{\mathcal{C},\mathcal{D},\mathcal{E}}(\beta, \alpha) \stackrel{\text{def}}{=} \beta \star \alpha,$$

where $\beta \star \alpha$ is the horizontal composition of α and β of Definition 8.8.4.1.

PROPOSITION 8.9.3.2 ► PROPERTIES OF THE 2-CATEGORY Cats_2

Let $\mathcal{C}$ be a category.

1. *2-Categorical Co/Completeness.* The 2-category Cats_2 is complete and cocomplete as a 2-category, having all 2-categorical and bicategorical co/limits.

PROOF 8.9.3.3 ► PROOF OF PROPOSITION 8.9.3.2

Item 1: Co/Completeness

Omitted. □

8.9.4 The Category of Groupoids

DEFINITION 8.9.4.1 ► THE CATEGORY OF SMALL GROUPOIDS

The **category of** (**small**) **groupoids** is the full subcategory Grpd of Cats spanned by the groupoids.

8.9.5 The 2-Category of Groupoids

DEFINITION 8.9.5.1 ► THE 2-CATEGORY OF SMALL GROUPOIDS

The 2-**category of** (**small**) **groupoids** is the full sub-2-category Grpd_2 of Cats_2 spanned by the groupoids.

Appendices

8.A Other Chapters

Sets

1. Sets
2. Constructions With Sets
3. Pointed Sets
4. Tensor Products of Pointed Sets

Relations

5. Relations
6. Constructions With Relations
7. Equivalence Relations and Apartness Relations

Category Theory

8. Categories

Bicategories

9. Types of Morphisms in Bicategories

Part IV

Bicategories

Chapter 9

Types of Morphisms in Bicategories

In this chapter, we study special kinds of morphisms in bicategories:

1. *Monomorphisms and Epimorphisms in Bicategories (Sections 9.1 and 9.2).* There is a large number of different notions capturing the idea of a "monomorphism" or of an "epimorphism" in a bicategory.

   Arguably, the notion that best captures these concepts is that of a *pseudomonic morphism* (Definition 9.1.10.1) and of a *pseudoepic morphism* (Definition 9.2.10.1), although the other notions introduced in Sections 9.1 and 9.2 are also interesting on their own.

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9.1 Monomorphisms in Bicategories

9.1.1 Representably Faithful Morphisms

Let C be a bicategory.

Definition 9.1.1.1 ► Representably Faithful Morphisms

A 1-morphism $f\colon A \longrightarrow B$ of C is **representably faithful**[1] if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f_*\colon \mathrm{Hom}_C(X, A) \longrightarrow \mathrm{Hom}_C(X, B)$$

given by postcomposition by f is faithful.

[1] *Further Terminology:* Also called simply a **faithful morphism**, based on Item 1 of Example 9.1.1.3.

Remark 9.1.1.2 ► Unwinding Definition 9.1.1.1

In detail, f is representably faithful if, for all diagrams in C of the form

$$\begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & A \arrow[r, "f"] & B, \end{tikzcd}$$

if we have

$$\mathrm{id}_f \star \alpha = \mathrm{id}_f \star \beta,$$

then $\alpha = \beta$.

Example 9.1.1.3 ► Examples of Representably Faithful Morphisms

Here are some examples of representably faithful morphisms.

1. *Representably Faithful Morphisms in* Cats_2. The representably faithful

morphisms in Cats_2 are precisely the faithful functors; see Item 1 of Proposition 8.5.1.2.

2. *Representably Faithful Morphisms in* **Rel**. Every morphism of **Rel** is representably faithful; see Item 1 of Proposition 5.3.8.1.

9.1.2 Representably Full Morphisms

Let $\mathcal{C}$ be a bicategory.

Definition 9.1.2.1 ▶ Representably Full Morphisms

A 1-morphism $f\colon A \to B$ of $\mathcal{C}$ is **representably full**[1] if, for each $X \in \mathrm{Obj}(\mathcal{C})$, the functor

$$f_*\colon \mathrm{Hom}_{\mathcal{C}}(X, A) \to \mathrm{Hom}_{\mathcal{C}}(X, B)$$

given by postcomposition by f is full.

[1] *Further Terminology:* Also called simply a **full morphism**, based on Item 1 of Example 9.1.2.3.

Remark 9.1.2.2 ▶ Unwinding Definition 9.1.2.1

In detail, f is representably full if, for each $X \in \mathrm{Obj}(\mathcal{C})$ and each 2-morphism

$$\beta\colon f \circ \phi \Longrightarrow f \circ \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"] \arrow[r, bend right, "f\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & B \end{tikzcd}$$

of $\mathcal{C}$, there exists a 2-morphism

$$\alpha\colon \phi \Longrightarrow \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & A \end{tikzcd}$$

of $\mathcal{C}$ such that we have an equality

$$\begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & A \arrow[r, "f"] & B \end{tikzcd} = \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"] \arrow[r, bend right, "f\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & B \end{tikzcd}$$

of pasting diagrams in $\mathcal{C}$, i.e. such that we have

$$\beta = \mathrm{id}_f \star \alpha.$$

EXAMPLE 9.1.2.3 ► EXAMPLES OF REPRESENTABLY FULL MORPHISMS

Here are some examples of representably full morphisms.

1. *Representably Full Morphisms in* Cats_2. The representably full morphisms in Cats_2 are precisely the full functors; see Item 1 of Proposition 8.5.2.2.

2. *Representably Full Morphisms in* **Rel**. The representably full morphisms in **Rel** are characterised in Item 2 of Proposition 5.3.8.1.

9.1.3 Representably Fully Faithful Morphisms

Let $\mathcal{C}$ be a bicategory.

DEFINITION 9.1.3.1 ► REPRESENTABLY FULLY FAITHFUL MORPHISMS

A 1-morphism $f: A \to B$ of $\mathcal{C}$ is **representably fully faithful**[1] if the following equivalent conditions are satisfied:

1. The 1-morphism f is representably faithful (Definition 9.1.1.1) and representably full (Definition 9.1.2.1).

2. For each $X \in \mathrm{Obj}(\mathcal{C})$, the functor

$$f_*: \mathrm{Hom}_{\mathcal{C}}(X, A) \to \mathrm{Hom}_{\mathcal{C}}(X, B)$$

given by postcomposition by f is fully faithful.

[1] *Further Terminology:* Also called simply a **fully faithful morphism**, based on Item 1 of Example 9.1.3.3.

REMARK 9.1.3.2 ► UNWINDING REPRESENTABLY FULLY FAITHFUL MORPHISMS

In detail, f is representably fully faithful if the conditions in Remark 9.1.1.2 and Remark 9.1.2.2 hold:

1. For all diagrams in $\mathcal{C}$ of the form

$$\begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha \Downarrow \Downarrow \beta"] & A \arrow[r, "f"] & B, \end{tikzcd}$$

if we have

$$\mathrm{id}_f \star \alpha = \mathrm{id}_f \star \beta,$$

then $\alpha = \beta$.

2. For each $X \in \mathrm{Obj}(C)$ and each 2-morphism

$$\beta\colon f \circ \phi \Longrightarrow f \circ \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"] \arrow[r, bend right, "f\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & B \end{tikzcd}$$

of C, there exists a 2-morphism

$$\alpha\colon \phi \Longrightarrow \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & A \end{tikzcd}$$

of C such that we have an equality

$$\begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & A \arrow[r, "f"] & B \end{tikzcd} = \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"] \arrow[r, bend right, "f\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & B \end{tikzcd}$$

of pasting diagrams in C, i.e. such that we have

$$\beta = \mathrm{id}_f \star \alpha.$$

EXAMPLE 9.1.3.3 ► EXAMPLES OF REPRESENTABLY FULLY FAITHFUL MORPHISMS

Here are some examples of representably fully faithful morphisms.

1. *Representably Fully Faithful Morphisms in* Cats_2. The representably fully faithful morphisms in Cats_2 are precisely the fully faithful functors; see Item 5 of Proposition 8.5.3.2.

2. *Representably Fully Faithful Morphisms in* **Rel**. The representably fully faithful morphisms of **Rel** coincide (Item 3 of Proposition 5.3.8.1) with the representably full morphisms in **Rel**, which are characterised in Item 2 of Proposition 5.3.8.1.

9.1.4 Morphisms Representably Faithful on Cores

Let C be a bicategory.

Definition 9.1.4.1 ▶ Morphisms Representably Faithful on Cores

A 1-morphism $f\colon A \to B$ of C is **representably faithful on cores** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f_*\colon \mathsf{Core}(\mathsf{Hom}_C(X, A)) \to \mathsf{Core}(\mathsf{Hom}_C(X, B))$$

given by postcomposition by f is faithful.

Remark 9.1.4.2 ▶ Unwinding Definition 9.1.4.1

In detail, f is representably faithful on cores if, for all diagrams in C of the form

$$\begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & A \arrow[r, "f"] & B, \end{tikzcd}$$

if α and β are 2-isomorphisms and we have

$$\mathrm{id}_f \star \alpha = \mathrm{id}_f \star \beta,$$

then $\alpha = \beta$.

9.1.5 Morphisms Representably Full on Cores

Let C be a bicategory.

Definition 9.1.5.1 ▶ Morphisms Representably Full on Cores

A 1-morphism $f\colon A \to B$ of C is **representably full on cores** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f_*\colon \mathsf{Core}(\mathsf{Hom}_C(X, A)) \to \mathsf{Core}(\mathsf{Hom}_C(X, B))$$

given by postcomposition by f is full.

Remark 9.1.5.2 ► Unwinding Definition 9.1.5.1

In detail, f is representably full on cores if, for each $X \in \mathrm{Obj}(\mathcal{C})$ and each 2-isomorphism

$$\beta\colon f \circ \phi \overset{\sim}{\Longrightarrow} f \circ \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"{name=U}] \arrow[r, bend right, "f\circ\psi"'{name=D}] & B \arrow[Rightarrow, from=U, to=D, "\beta"'] \end{tikzcd}$$

of $\mathcal{C}$, there exists a 2-isomorphism

$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "\phi"{name=U}] \arrow[r, bend right, "\psi"'{name=D}] & A \arrow[Rightarrow, from=U, to=D, "\alpha"'] \end{tikzcd}$$

of $\mathcal{C}$ such that we have an equality

$$\begin{tikzcd} X \arrow[r, bend left, "\phi"{name=U}] \arrow[r, bend right, "\psi"'{name=D}] & A \arrow[r, "f"] & B \arrow[Rightarrow, from=U, to=D, "\alpha"'] \end{tikzcd} = \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"{name=U}] \arrow[r, bend right, "f\circ\psi"'{name=D}] & B \arrow[Rightarrow, from=U, to=D, "\beta"'] \end{tikzcd}$$

of pasting diagrams in $\mathcal{C}$, i.e. such that we have

$$\beta = \mathrm{id}_f \star \alpha.$$

9.1.6 Morphisms Representably Fully Faithful on Cores

Let $\mathcal{C}$ be a bicategory.

Definition 9.1.6.1 ► Morphisms Representably Fully Faithful on Cores

A 1-morphism $f\colon A \to B$ of $\mathcal{C}$ is **representably fully faithful on cores** if the following equivalent conditions are satisfied:

1. The 1-morphism f is representably faithful on cores (Definition 9.1.5.1) and representably full on cores (Definition 9.1.4.1).
2. For each $X \in \mathrm{Obj}(\mathcal{C})$, the functor

$$f_*\colon \mathrm{Core}(\mathrm{Hom}_{\mathcal{C}}(X, A)) \to \mathrm{Core}(\mathrm{Hom}_{\mathcal{C}}(X, B))$$

given by postcomposition by f is fully faithful.

Remark 9.1.6.2 ▶ Unwinding Definition 9.1.6.1

In detail, f is representably fully faithful on cores if the conditions in Remark 9.1.4.2 and Remark 9.1.5.2 hold:

1. For all diagrams in C of the form
$$\begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & A \arrow[r, "f"] & B, \end{tikzcd}$$
if α and β are 2-isomorphisms and we have
$$\mathrm{id}_f \star \alpha = \mathrm{id}_f \star \beta,$$
then $\alpha = \beta$.

2. For each $X \in \mathrm{Obj}(C)$ and each 2-isomorphism
$$\beta\colon f \circ \phi \overset{\sim}{\Longrightarrow} f \circ \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"] \arrow[r, bend right, "f\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & B \end{tikzcd}$$
of C, there exists a 2-isomorphism
$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & A \end{tikzcd}$$
of C such that we have an equality
$$\begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & A \arrow[r, "f"] & B \end{tikzcd} = \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"] \arrow[r, bend right, "f\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & B \end{tikzcd}$$
of pasting diagrams in C, i.e. such that we have
$$\beta = \mathrm{id}_f \star \alpha.$$

9.1.7 Representably Essentially Injective Morphisms

Let C be a bicategory.

Definition 9.1.7.1 ▶ Representably Essentially Injective Morphisms

A 1-morphism $f\colon A \to B$ of C is **representably essentially injective** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f_*\colon \mathrm{Hom}_C(X, A) \to \mathrm{Hom}_C(X, B)$$

given by postcomposition by f is essentially injective.

Remark 9.1.7.2 ▶ Unwinding Definition 9.1.7.1

In detail, f is representably essentially injective if, for each pair of morphisms $\phi, \psi\colon X \rightrightarrows A$ of C, the following condition is satisfied:

($\star$) If $f \circ \phi \cong f \circ \psi$, then $\phi \cong \psi$.

9.1.8 Representably Conservative Morphisms

Let C be a bicategory.

Definition 9.1.8.1 ▶ Representably Conservative Morphisms

A 1-morphism $f\colon A \to B$ of C is **representably conservative** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f_*\colon \mathrm{Hom}_C(X, A) \to \mathrm{Hom}_C(X, B)$$

given by postcomposition by f is conservative.

Remark 9.1.8.2 ▶ Unwinding Definition 9.1.8.1

In detail, f is representably conservative if, for each pair of morphisms $\phi, \psi\colon X \rightrightarrows A$ and each 2-morphism

$$\alpha\colon \phi \Longrightarrow \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "\phi"{name=U}] \arrow[r, bend right, "\psi"'{name=D}] & A \arrow[from=U, to=D, Rightarrow, "\alpha"'] \end{tikzcd}$$

of C, if the 2-morphism

$$\mathrm{id}_f \star \alpha\colon f \circ \phi \Longrightarrow f \circ \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"{name=U}] \arrow[r, bend right, "f\circ\psi"'{name=D}] & B \arrow[from=U, to=D, Rightarrow, "\mathrm{id}_f\star\alpha"'] \end{tikzcd}$$

is a 2-isomorphism, then so is α.

9.1.9 Strict Monomorphisms

Let C be a bicategory.

DEFINITION 9.1.9.1 ▶ STRICT MONOMORPHISMS

A 1-morphism $f : A \longrightarrow B$ of C is a **strict monomorphism** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f_* : \mathrm{Hom}_C(X, A) \longrightarrow \mathrm{Hom}_C(X, B)$$

given by postcomposition by f is injective on objects, i.e. its action on objects

$$f_* : \mathrm{Obj}(\mathrm{Hom}_C(X, A)) \longrightarrow \mathrm{Obj}(\mathrm{Hom}_C(X, B))$$

is injective.

REMARK 9.1.9.2 ▶ UNWINDING DEFINITION 9.1.9.1

In detail, f is a strict monomorphism in C if, for each diagram in C of the form

$$X \underset{\psi}{\overset{\phi}{\rightrightarrows}} A \xrightarrow{f} B,$$

if $f \circ \phi = f \circ \psi$, then $\phi = \psi$.

EXAMPLE 9.1.9.3 ▶ EXAMPLES OF STRICT MONOMORPHISMS

Here are some examples of strict monomorphisms.

1. *Strict Monomorphisms in* Cats_2. The strict monomorphisms in Cats_2 are precisely the functors which are injective on objects and injective on morphisms; see Item 1 of Proposition 8.6.2.2.

2. *Strict Monomorphisms in* **Rel**. The strict monomorphisms in **Rel** are characterised in Proposition 5.3.7.1.

9.1.10 Pseudomonic Morphisms

Let C be a bicategory.

Definition 9.1.10.1 ▶ Pseudomonic Morphisms

A 1-morphism $f\colon A \to B$ of $\mathcal{C}$ is **pseudomonic** if, for each $X \in \mathrm{Obj}(\mathcal{C})$, the functor

$$f_*\colon \mathrm{Hom}_{\mathcal{C}}(X, A) \to \mathrm{Hom}_{\mathcal{C}}(X, B)$$

given by postcomposition by f is pseudomonic.

Remark 9.1.10.2 ▶ Unwinding Definition 9.1.10.1

In detail, a 1-morphism $f\colon A \to B$ of $\mathcal{C}$ is pseudomonic if it satisfies the following conditions:

1. For all diagrams in $\mathcal{C}$ of the form

$$\begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & A \arrow[r, "f"] & B, \end{tikzcd}$$

   if we have

$$\mathrm{id}_f \star \alpha = \mathrm{id}_f \star \beta,$$

   then $\alpha = \beta$.

2. For each $X \in \mathrm{Obj}(\mathcal{C})$ and each 2-isomorphism

$$\beta\colon f \circ \phi \overset{\sim}{\Longrightarrow} f \circ \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"] \arrow[r, bend right, "f\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & B \end{tikzcd}$$

   of $\mathcal{C}$, there exists a 2-isomorphism

$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & A \end{tikzcd}$$

   of $\mathcal{C}$ such that we have an equality

$$\begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & A \arrow[r, "f"] & B \end{tikzcd} = \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"] \arrow[r, bend right, "f\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & B \end{tikzcd}$$

   of pasting diagrams in $\mathcal{C}$, i.e. such that we have

$$\beta = \mathrm{id}_f \star \alpha.$$

Proposition 9.1.10.3 ► Properties of Pseudomonic Morphisms

Let $f\colon A \longrightarrow B$ be a 1-morphism of C.

1. *Characterisations.* The following conditions are equivalent:

    (a) The morphism f is pseudomonic.

    (b) The morphism f is representably full on cores and representably faithful.

    (c) We have an isocomma square of the form

$$A \overset{\text{eq.}}{\cong} A \overset{\leftrightarrow}{\times}_B A, \qquad \begin{tikzcd} A \arrow[r, "\mathrm{id}_A"] \arrow[d, "\mathrm{id}_A"'] & A \arrow[d, "F"] \arrow[dl, phantom, "\Leftrightarrow"] \\ A \arrow[r, "F"'] & B \end{tikzcd}$$

    in C up to equivalence.

2. *Interaction With Cotensors.* If C has cotensors with $\mathbb{1}$, then the following conditions are equivalent:

    (a) The morphism f is pseudomonic.

    (b) We have an isocomma square of the form

$$A \overset{\text{eq.}}{\cong} A \overset{\leftrightarrow}{\times}_{\mathbb{1}\pitchfork F} B, \qquad \begin{tikzcd} A \arrow[r, hook] \arrow[d, "F"'] & \mathbb{1} \pitchfork A \arrow[d, "\mathbb{1}\pitchfork F"] \arrow[dl, phantom, "\Leftrightarrow"] \\ B \arrow[r, hook] & \mathbb{1} \pitchfork B \end{tikzcd}$$

    in C up to equivalence.

Proof 9.1.10.4 ► Proof of Proposition 9.1.10.3

Item 1: Characterisations

Omitted.

Item 2: Interaction With Cotensors

Omitted.

9.2 Epimorphisms in Bicategories

9.2.1 Corepresentably Faithful Morphisms

Let C be a bicategory.

Definition 9.2.1.1 ▶ Corepresentably Faithful Morphisms

A 1-morphism $f \colon A \to B$ of C is **corepresentably faithful** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f^* \colon \mathrm{Hom}_C(B, X) \to \mathrm{Hom}_C(A, X)$$

given by precomposition by f is faithful.

Remark 9.2.1.2 ▶ Unwinding Definition 9.2.1.1

In detail, f is corepresentably faithful if, for all diagrams in C of the form

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & X, \end{tikzcd}$$

if we have

$$\alpha \star \mathrm{id}_f = \beta \star \mathrm{id}_f,$$

then $\alpha = \beta$.

Example 9.2.1.3 ▶ Examples of Corepresentably Faithful Morphisms

Here are some examples of corepresentably faithful morphisms.

1. *Corepresentably Faithful Morphisms in* Cats_2. The corepresentably faithful morphisms in Cats_2 are characterised in Item 4 of Proposition 8.5.1.2.

2. *Corepresentably Faithful Morphisms in* **Rel**. Every morphism of **Rel** is corepresentably faithful; see Item 1 of Proposition 5.3.10.1.

9.2.2 Corepresentably Full Morphisms

Let C be a bicategory.

Definition 9.2.2.1 ▶ Corepresentably Full Morphisms

A 1-morphism $f : A \to B$ of C is **corepresentably full** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f^* : \mathrm{Hom}_C(B, X) \to \mathrm{Hom}_C(A, X)$$

given by precomposition by f is full.

Remark 9.2.2.2 ▶ Unwinding Definition 9.2.2.1

In detail, f is corepresentably full if, for each $X \in \mathrm{Obj}(C)$ and each 2-morphism

$$\beta : \phi \circ f \Longrightarrow \psi \circ f, \qquad \begin{tikzcd} A \arrow[r, bend left, "\phi\circ f"] \arrow[r, bend right, "\psi\circ f"'] \arrow[r, phantom, "\beta\Downarrow"] & X \end{tikzcd}$$

of C, there exists a 2-morphism

$$\alpha : \phi \Longrightarrow \psi, \qquad \begin{tikzcd} B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & X \end{tikzcd}$$

of C such that we have an equality

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & X \end{tikzcd} = \begin{tikzcd} A \arrow[r, bend left, "\phi\circ f"] \arrow[r, bend right, "\psi\circ f"'] \arrow[r, phantom, "\beta\Downarrow"] & X \end{tikzcd}$$

of pasting diagrams in C, i.e. such that we have

$$\beta = \alpha \star \mathrm{id}_f.$$

Example 9.2.2.3 ▶ Examples of Corepresentably Full Morphisms

Here are some examples of corepresentably full morphisms.

1. *Corepresentably Full Morphisms in* Cats_2. The corepresentably full morphisms in Cats_2 are characterised in Item 5 of Proposition 8.5.2.2.

2. *Corepresentably Full Morphisms in* **Rel**. The corepresentably full morphisms in **Rel** are characterised in **??** of Proposition 5.3.8.1.

9.2.3 Corepresentably Fully Faithful Morphisms

Let C be a bicategory.

Definition 9.2.3.1 ► Corepresentably Fully Faithful Morphisms

A 1-morphism $f\colon A \to B$ of C is **corepresentably fully faithful**[1] if the following equivalent conditions are satisfied:

1. The 1-morphism f is corepresentably full (Definition 9.2.2.1) and corepresentably faithful (Definition 9.2.1.1).
2. For each $X \in \mathrm{Obj}(C)$, the functor
$$f^*\colon \mathrm{Hom}_C(B,X) \to \mathrm{Hom}_C(A,X)$$
given by precomposition by f is fully faithful.

[1] *Further Terminology:* Corepresentably fully faithful morphisms have also been called **lax epimorphisms** in the literature (e.g. in [Adá+01]), though we will always use the name "corepresentably fully faithful morphism" instead in this work.

Remark 9.2.3.2 ► Unwinding Definition 9.2.3.1

In detail, f is corepresentably fully faithful if the conditions in Remark 9.2.1.2 and Remark 9.2.2.2 hold:

1. For all diagrams in C of the form
$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & X, \end{tikzcd}$$
if we have
$$\alpha \star \mathrm{id}_f = \beta \star \mathrm{id}_f,$$
then $\alpha = \beta$.
2. For each $X \in \mathrm{Obj}(C)$ and each 2-morphism
$$\beta\colon \phi \circ f \Longrightarrow \psi \circ f, \qquad \begin{tikzcd} A \arrow[r, bend left, "\phi\circ f"] \arrow[r, bend right, "\psi\circ f"'] \arrow[r, phantom, "\beta\Downarrow"] & X \end{tikzcd}$$
of C, there exists a 2-morphism
$$\alpha\colon \phi \Longrightarrow \psi, \qquad \begin{tikzcd} B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & X \end{tikzcd}$$

of C such that we have an equality

$$
\begin{tikzcd}
A \arrow[r, "f"] & B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & X
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, bend left, "\phi\circ f"] \arrow[r, bend right, "\psi\circ f"'] \arrow[r, phantom, "\beta\Downarrow"] & X
\end{tikzcd}
$$

of pasting diagrams in C, i.e. such that we have

$$\beta = \alpha \star \mathrm{id}_f.$$

Example 9.2.3.3 ▶ Examples of Corepresentably Fully Faithful Morphisms

Here are some examples of corepresentably fully faithful morphisms.

1. *Corepresentably Fully Faithful Morphisms in* Cats_2. The fully faithful epimorphisms in Cats_2 are characterised in Item 9 of Proposition 8.5.3.2.
2. *Corepresentably Fully Faithful Morphisms in* **Rel**. The corepresentably fully faithful morphisms of **Rel** coincide (Item 3 of Proposition 5.3.10.1) with the corepresentably full morphisms in **Rel**, which are characterised in Item 2 of Proposition 5.3.10.1.

9.2.4 Morphisms Corepresentably Faithful on Cores

Let C be a bicategory.

Definition 9.2.4.1 ▶ Morphisms Corepresentably Faithful on Cores

A 1-morphism $f\colon A \longrightarrow B$ of C is **corepresentably faithful on cores** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f^*\colon \mathrm{Core}(\mathrm{Hom}_C(B,X)) \longrightarrow \mathrm{Core}(\mathrm{Hom}_C(A,X))$$

given by precomposition by f is faithful.

Remark 9.2.4.2 ► Unwinding Definition 9.2.4.1

In detail, f is corepresentably faithful on cores if, for all diagrams in $\mathcal{C}$ of the form

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "\phi"{name=U}] \arrow[r, bend right, "\psi"'{name=D}] \arrow[from=U, to=D, phantom, "\alpha\Downarrow\Downarrow\beta"] & X, \end{tikzcd}$$

if α and β are 2-isomorphisms and we have

$$\alpha \star \mathrm{id}_f = \beta \star \mathrm{id}_f,$$

then $\alpha = \beta$.

9.2.5 Morphisms Corepresentably Full on Cores

Let $\mathcal{C}$ be a bicategory.

Definition 9.2.5.1 ► Morphisms Corepresentably Full on Cores

A 1-morphism $f\colon A \to B$ of $\mathcal{C}$ is **corepresentably full on cores** if, for each $X \in \mathrm{Obj}(\mathcal{C})$, the functor

$$f^*\colon \mathrm{Core}(\mathrm{Hom}_{\mathcal{C}}(B, X)) \to \mathrm{Core}(\mathrm{Hom}_{\mathcal{C}}(A, X))$$

given by precomposition by f is full.

Remark 9.2.5.2 ► Unwinding Definition 9.2.5.1

In detail, f is corepresentably full on cores if, for each $X \in \mathrm{Obj}(\mathcal{C})$ and each 2-isomorphism

$$\beta\colon \phi \circ f \overset{\sim}{\Longrightarrow} \psi \circ f, \qquad \begin{tikzcd} A \arrow[r, bend left, "\phi\circ f"{name=U}] \arrow[r, bend right, "\psi\circ f"'{name=D}] \arrow[from=U, to=D, phantom, "\beta\Downarrow"] & X \end{tikzcd}$$

of $\mathcal{C}$, there exists a 2-isomorphism

$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} B \arrow[r, bend left, "\phi"{name=U}] \arrow[r, bend right, "\psi"'{name=D}] \arrow[from=U, to=D, phantom, "\alpha\Downarrow"] & X \end{tikzcd}$$

of C such that we have an equality

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & X \end{tikzcd} = \begin{tikzcd} A \arrow[r, bend left, "\phi\circ f"] \arrow[r, bend right, "\psi\circ f"'] \arrow[r, phantom, "\beta\Downarrow"] & X \end{tikzcd}$$

of pasting diagrams in C, i.e. such that we have

$$\beta = \alpha \star \mathrm{id}_f.$$

9.2.6 Morphisms Corepresentably Fully Faithful on Cores

Let C be a bicategory.

Definition 9.2.6.1 ▶ Morphisms Corepresentably Fully Faithful on Cores

A 1-morphism $f\colon A \to B$ of C is **corepresentably fully faithful on cores** if the following equivalent conditions are satisfied:

1. The 1-morphism f is corepresentably full on cores (Definition 9.2.5.1) and corepresentably faithful on cores (Definition 9.2.1.1).
2. For each $X \in \mathrm{Obj}(C)$, the functor
$$f^*\colon \mathrm{Core}(\mathrm{Hom}_C(B, X)) \to \mathrm{Core}(\mathrm{Hom}_C(A, X))$$
given by precomposition by f is fully faithful.

Remark 9.2.6.2 ▶ Unwinding Definition 9.2.6.1

In detail, f is corepresentably fully faithful on cores if the conditions in Remark 9.2.4.2 and Remark 9.2.5.2 hold:

1. For all diagrams in C of the form
$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & X, \end{tikzcd}$$
if α and β are 2-isomorphisms and we have
$$\alpha \star \mathrm{id}_f = \beta \star \mathrm{id}_f,$$
then $\alpha = \beta$.

2. For each $X \in \mathrm{Obj}(\mathcal{C})$ and each 2-isomorphism

$$\beta\colon \phi \circ f \overset{\sim}{\Longrightarrow} \psi \circ f, \qquad \begin{tikzcd} A \arrow[r, bend left, "\phi\circ f"] \arrow[r, bend right, "\psi\circ f"'] \arrow[r, phantom, "\beta\Downarrow"] & X \end{tikzcd}$$

of $\mathcal{C}$, there exists a 2-isomorphism

$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & X \end{tikzcd}$$

of $\mathcal{C}$ such that we have an equality

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & X \end{tikzcd} = \begin{tikzcd} A \arrow[r, bend left, "\phi\circ f"] \arrow[r, bend right, "\psi\circ f"'] \arrow[r, phantom, "\beta\Downarrow"] & X \end{tikzcd}$$

of pasting diagrams in $\mathcal{C}$, i.e. such that we have

$$\beta = \alpha \star \mathrm{id}_f.$$

9.2.7 Corepresentably Essentially Injective Morphisms

Let $\mathcal{C}$ be a bicategory.

Definition 9.2.7.1 ▶ Corepresentably Essentially Injective Morphisms

A 1-morphism $f\colon A \to B$ of $\mathcal{C}$ is **corepresentably essentially injective** if, for each $X \in \mathrm{Obj}(\mathcal{C})$, the functor

$$f^*\colon \mathrm{Hom}_{\mathcal{C}}(B, X) \to \mathrm{Hom}_{\mathcal{C}}(A, X)$$

given by precomposition by f is essentially injective.

Remark 9.2.7.2 ▶ Unwinding Definition 9.2.7.1

In detail, f is corepresentably essentially injective if, for each pair of morphisms $\phi, \psi\colon B \rightrightarrows X$ of $\mathcal{C}$, the following condition is satisfied:

($\star$) If $\phi \circ f \cong \psi \circ f$, then $\phi \cong \psi$.

9.2.8 Corepresentably Conservative Morphisms

Let $\mathcal{C}$ be a bicategory.

Definition 9.2.8.1 ▶ Corepresentably Conservative Morphisms

A 1-morphism $f \colon A \to B$ of $\mathcal{C}$ is **corepresentably conservative** if, for each $X \in \mathrm{Obj}(\mathcal{C})$, the functor

$$f^* \colon \mathrm{Hom}_{\mathcal{C}}(B, X) \to \mathrm{Hom}_{\mathcal{C}}(A, X)$$

given by precomposition by f is conservative.

Remark 9.2.8.2 ▶ Unwinding Definition 9.2.8.1

In detail, f is corepresentably conservative if, for each pair of morphisms $\phi, \psi \colon B \rightrightarrows X$ and each 2-morphism

$$\alpha \colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha \Downarrow"] & X \end{tikzcd}$$

of $\mathcal{C}$, if the 2-morphism

$$\alpha \star \mathrm{id}_f \colon \phi \circ f \Longrightarrow \psi \circ f, \qquad \begin{tikzcd} A \arrow[r, bend left, "\phi \circ f"] \arrow[r, bend right, "\psi \circ f"'] \arrow[r, phantom, "\alpha \star \mathrm{id}_f \Downarrow"] & X \end{tikzcd}$$

is a 2-isomorphism, then so is α.

9.2.9 Strict Epimorphisms

Let $\mathcal{C}$ be a bicategory.

Definition 9.2.9.1 ▶ Strict Epimorphisms

A 1-morphism $f \colon A \to B$ is a **strict epimorphism in** $\mathcal{C}$ if, for each $X \in \mathrm{Obj}(\mathcal{C})$, the functor

$$f^* \colon \mathrm{Hom}_{\mathcal{C}}(B, X) \to \mathrm{Hom}_{\mathcal{C}}(A, X)$$

given by precomposition by f is injective on objects, i.e. its action on objects

$$f_* \colon \mathrm{Obj}(\mathrm{Hom}_{\mathcal{C}}(B, X)) \to \mathrm{Obj}(\mathrm{Hom}_{\mathcal{C}}(A, X))$$

is injective.

Remark 9.2.9.2 ▶ Unwinding Definition 9.2.9.1

In detail, f is a strict epimorphism if, for each diagram in C of the form

$$A \xrightarrow{f} B \underset{\psi}{\overset{\phi}{\rightrightarrows}} X,$$

if $\phi \circ f = \psi \circ f$, then $\phi = \psi$.

Example 9.2.9.3 ▶ Examples of Strict Epimorphisms

Here are some examples of strict epimorphisms.

1. *Strict Epimorphisms in* Cats_2. The strict epimorphisms in Cats_2 are characterised in Item 1 of Proposition 8.6.3.2.

2. *Strict Epimorphisms in* **Rel**. The strict epimorphisms in **Rel** are characterised in Proposition 5.3.9.1.

9.2.10 Pseudoepic Morphisms

Let C be a bicategory.

Definition 9.2.10.1 ▶ Pseudoepic Morphisms

A 1-morphism $f\colon A \to B$ of C is **pseudoepic** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f^*\colon \mathrm{Hom}_C(B, X) \to \mathrm{Hom}_C(A, X)$$

given by precomposition by f is pseudomonic.

Remark 9.2.10.2 ▶ Unwinding Definition 9.2.10.1

In detail, a 1-morphism $f\colon A \to B$ of C is pseudoepic if it satisfies the following conditions:

1. For all diagrams in C of the form

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & X, \end{tikzcd}$$

if we have

$$\alpha \star \mathrm{id}_f = \beta \star \mathrm{id}_f,$$

then $\alpha = \beta$.

2. For each $X \in \mathrm{Obj}(\mathcal{C})$ and each 2-isomorphism

$$\beta \colon \phi \circ f \overset{\sim}{\Longrightarrow} \psi \circ f, \qquad \begin{tikzcd} A \arrow[r, bend left, "\phi\circ f"] \arrow[r, bend right, "\psi\circ f"'] \arrow[r, phantom, "\beta\Downarrow"] & X \end{tikzcd}$$

of $\mathcal{C}$, there exists a 2-isomorphism

$$\alpha \colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & X \end{tikzcd}$$

of $\mathcal{C}$ such that we have an equality

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & X \end{tikzcd} = \begin{tikzcd} A \arrow[r, bend left, "\phi\circ f"] \arrow[r, bend right, "\psi\circ f"'] \arrow[r, phantom, "\beta\Downarrow"] & X \end{tikzcd}$$

of pasting diagrams in $\mathcal{C}$, i.e. such that we have

$$\beta = \alpha \star \mathrm{id}_f.$$

Proposition 9.2.10.3 ► Properties of Pseudoepic Morphisms

Let $f \colon A \to B$ be a 1-morphism of $\mathcal{C}$.

1. *Characterisations.* The following conditions are equivalent:
   (a) The morphism f is pseudoepic.
   (b) The morphism f is corepresentably full on cores and corepresentably faithful.

(c) We have an isococomma square of the form

$$B \overset{\text{eq.}}{\cong} B \overset{\leftrightarrow}{\coprod}_A B, \qquad \begin{tikzcd} B & B \arrow[l, "\mathrm{id}_B"'] \\ B \arrow[u, "\mathrm{id}_B"] \arrow[ur, phantom, "\Leftarrow\!\!\Rightarrow"] & A \arrow[l, "F"] \arrow[u, "F"'] \end{tikzcd}$$

in $\mathcal{C}$ up to equivalence.

Proof 9.2.10.4 ▶ **Proof of Proposition 9.2.10.3**

Item 1: Characterisations

Omitted. □

Appendices

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Part V

Extra Part

Chapter 10

Miscellaneous Notes

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10.1 To Do List

10.1.1 Omitted Proofs To Add

Не так благотворна истина, как зловредна ее видимость.

Даниил Данковский

Truth does not do as much good in the world as the appearance of truth does evil.

Daniil Dankovsky

There's a very large number of omitted proofs throughout these notes. Here I list them in decreasing order of how nice it would be to add them.

Remark 10.1.1.1 ► Omitted Proofs To Add

Proofs that *need* to be added at some point:

1. **??**.
2. **??**.
3. Horizontal composition of natural transformations is associative: **??** of

??.

4. Fully faithful functors are essentially injective: **??** of **??**.

Proofs that *would be very nice* to be added at some point:

1. Properties of pseudomonic functors: **??**.
2. Characterisation of fully faithful functors: **??** of **??**.

Proofs that *would be nice* to be added at some point:

1. Properties of posetal categories: **??**.
2. The quadruple adjunction between categories and sets: **??**.
3. Properties of groupoid completions: **??**.
4. Properties of cores: **??**.
5. F_* faithful iff F faithful: **??** of **??**.
6. F_* full iff F full: **??** of **??**.
7. Injective on objects functors are precisely the isocofibrations in Cats_2: **??** of **??**.
8. Characterisations of monomorphisms of categories: **??** of **??**.
9. Epimorphisms of categories are surjective on objects: **??** of **??**.
10. Properties of pseudoepic functors: **??**.

10.1.2 Things To Explore/Add

Here we list things to be explored/added to this work in the future.

REMARK 10.1.2.1 ► THINGS TO EXPLORE/ADD

Set theory through a category theory lens:

1. Isbell duality for sets.
2. Density comonads and codensity monads for sets.

Relations:

1. 2-Categorical monomorphisms and epimorphisms in **Rel**.

2. Co/limits in **Rel**.

3. Apartness composition, categorical properties of **Rel** with apartness, and apartness relations.

4. Apartness defines a composition for relations, but its analogue

$$\mathfrak{q} \mathbin{\Box} \mathfrak{p} \overset{\text{def}}{=} \int_{A \in C} \mathfrak{p}^{-1}_{A} \coprod \mathfrak{q}^{A}_{-_2}$$

fails to be unital for profunctors. Is there a less obvious analogue of apartness composition for profunctors?

5. Codensity monad $\mathrm{Ran}_J(J)$ of a relation (What about $\mathrm{Rift}_J(J)$?)

6. Relative comonads in the 2-category of relations

7. Discrete fibrations and Street fibrations in **Rel**.

8. Consider adding the sections

    - The Monoidal Bicategory of Relations
    - The Monoidal Double Category of Relations

    to Relations.

Spans:

1. Universal property of the bicategory of spans, `https://ncatlab.org/nlab/show/span`

2. Write about cospans.

Un/Straightening:

1. Write proper sections on straightening for lax functors from sets to Rel or Span (displayed sets)

Categories:

1. Expand **??** and add a proof to it.

2. Sections and retractions; retracts, `https://ncatlab.org/nlab/show/retract`.

3. Regular categories: `https://arxiv.org/pdf/2004.08964.pdf`.

4. Are pseudoepic functors those functors whose restricted Yoneda embedding is pseudomonic and Yoneda preserves absolute colimits?

5. Absolutely dense functors enriched over $\mathbb{R}^+$ apparently reduce to topological density

Types of Morphisms in Categories:

1. Behaviour in $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$, e.g. pointwise sections vs. sections in $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$.

2. A faithful functor from balanced category is conservative

Yoneda stuff:

1. Properties of restricted Yoneda embedding, e.g. if the restricted Yoneda embedding is full, then what can we conclude? Related: https://qchu.wordpress.com/2015/05/17/generators/

Adjunctions:

1. Adjunctions, units, counits, and fully faithfulness as in https://mathoverflow.net/questions/100808/properties-of-functors-and-their-adjoints.

2. Morphisms between adjunctions and bicategory $\mathsf{Adj}(\mathcal{C})$.

3. https://ncatlab.org/nlab/show/transformation+of+adjoints

Constructions With Categories:

1. Comparison between pseudopullbacks and isocomma categories: the "evident" functor $\mathcal{C} \times^{\mathsf{ps}}_{\mathcal{E}} \mathcal{D} \to \mathcal{C} \overset{\leftrightarrow}{\times}_{\mathcal{E}} \mathcal{D}$ is essentially surjective and full, but not faithful in general.

Co/limits:

1. Add the characterisations of absolutely dense functors given in **??** to **??**.

2. Absolutely dense functors, https://ncatlab.org/nlab/show/absolutely+dense+functor. Also theorem 1.1 here: http://www.tac.mta.ca/tac/volumes/8/n20/n20.pdf.

3. Dense functors, codense functors, and absolutely codense functors.

Co/ends:

1. Examples of co/ends: https://mathoverflow.net/a/461814
2. Cofinality for co/ends, https://mathoverflow.net/questions/353876

Fibred category theory:

1. Internal **Hom** in categories of co/Cartesian fibrations.
2. *Tensor structures on fibered categories* by Luca Terenzi: https://arxiv.org/abs/2401.13491. Check also the other papers by Luca Terenzi.
3. https://ncatlab.org/nlab/show/cartesian+natural+transformation (this is a cartesian morphism in $\mathsf{Fun}(C, \mathcal{D})$ apparently)
4. CoCartesian fibration classifying $\mathsf{Fun}(F, G)$, https://mathoverflow.net/questions/457533/cocartesian-fibration-classifying-mathrmfunf-g

Monoidal categories:

1. Free braided monoidal category with a braided monoid: https://ncatlab.org/nlab/show/vine

Skew monoidal categories:

1. Does the $\mathbb{E}_1$ tensor product of monoids admit a skew monoidal category structure?
2. Is there a (right?) skew monoidal category structure on $\mathsf{Fun}(C, \mathcal{D})$ using right Kan extensions instead of left Kan extensions?
3. Similarly, are there skew monoidal category structures on the subcategory of **Rel**(A, B) spanned by the functions using left Kan extensions and left Kan lifts?

Higher categories:

1. Internal adjunctions in Mod as in [JY21, Section 6.3]; see [JY21, Example 6.2.6].
2. Comonads in the bicategory of profunctors.

Monoids:

1. Isbell's zigzag theorem for semigroups: the following conditions are equivalent:

   (a) A morphism $f\colon A \to B$ of semigroups is an epimorphism.

   (b) For each $b \in B$, one of the following conditions is satisfied:

      - We have $f(a) = b$.
      - There exist some $m \in \mathbb{N}_{\geq 1}$ and two factorisations

$$\begin{aligned} b &= a_0 y_1, \\ b &= x_m a_{2m} \end{aligned}$$

        connected by relations

$$\begin{aligned} a_0 &= x_1 a_1, \\ a_1 y_1 &= a_2 y_2, \\ x_1 a_2 &= x_2 a_3, \\ a_{2m-1} y_m &= a_{2m} \end{aligned}$$

        such that, for each $1 \leq i \leq m$, we have $a_i \in \mathrm{Im}(f)$.

   Wikipedia says in https://en.wikipedia.org/wiki/Isbell%27s_zigzag_theorem:

   > For monoids, this theorem can be written more concisely:

Types of morphisms in bicategories:

1. Behaviour in 2-categories of pseudofunctors (or lax functors, etc.), e.g. pointwise pseudoepic morphisms in vs. pseudoepic morphisms in 2-categories of pseudofunctors.

2. Statements like "coequifiers are lax epimorphisms", Item 2 of Examples 2.4 of https://arxiv.org/abs/2109.09836, along with most of the other statements/examples there.

3. Dense, absolutely dense, etc. morphisms in bicategories

Other:

1. https://qchu.wordpress.com/

2. https://aroundtoposes.com/

3. https://ncatlab.org/nlab/show/essentially+surjective+and+full+functor
4. https://mathoverflow.net/questions/415363/objects-whose-representable-presheaf-is-a-fibration
5. https://mathoverflow.net/questions/460146/universal-property-of-isbell-duality
6. http://www.tac.mta.ca/tac/volumes/36/12/36-12abs.html (Isbell conjugacy and the reflexive completion)
7. https://ncatlab.org/nlab/show/enrichment+versus+internalisation
8. The works of Philip Saville, https://philipsaville.co.uk/
9. https://golem.ph.utexas.edu/category/2024/02/from_cartesian_to_symmetric_mo.html
10. https://mathoverflow.net/q/463855 (One-object lax transformations)
11. https://ncatlab.org/nlab/show/analytic+completion+of+a+ring
12. https://en.wikipedia.org/wiki/Quaternionic_analysis
13. https://arxiv.org/abs/2401.15051 (The Norm Functor over Schemes)
14. https://mathoverflow.net/questions/407291/ (Adjunctions with respect to profunctors)
15. https://mathoverflow.net/a/462726 (Prof is free completion of Cats under right extensions)
16. there's some cool stuff in https://arxiv.org/abs/2312.00990 (Polynomial Functors: A Mathematical Theory of Interaction), e.g. on cofunctors.
17. https://ncatlab.org/nlab/show/adjoint+lifting+theorem
18. https://ncatlab.org/nlab/show/Gabriel%E2%80%93Ulmer+duality

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