Constructions With Sets

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This chapter develops some material relating to constructions with sets with an eye towards its categorical and higher-categorical counterparts to be introduced later in this work. In particular, it contains:

1. Explicit descriptions of the major types of co/limits in Sets, including in particular explicit descriptions of pushouts and coequalisers (see Definitions 2.4.1 and 2.5.1 and Remarks 2.4.3 and 2.5.3).

2. A discussion of powersets as decategorifications of categories of presheaves (Remarks 4.1.2 and 4.3.2), including a (-1)-categorical analogue of un/straightening, described in Items 1 and 2 of Proposition 4.3.9 and Remark 4.3.11.

3. A lengthy discussion of the adjoint triple

$$f_* \dashv f^{-1} \dashv f_! \colon \mathcal{P}(A) \overset{\rightleftarrows}{\rightarrow} \mathcal{P}(B)$$

   of functors (morphisms of posets) between $\mathcal{P}(A)$ and $\mathcal{P}(B)$ induced by a map of sets $f \colon A \longrightarrow B$, along with a discussion of the properties of f_*, f^{-1}, and $f_!$.

   In line with the categorical viewpoint developed here, this adjoint triple may be described in terms of Kan extensions, and, as it turns out, it also shows up in some definitions and results in point-set topology, such as in e.g. notions of continuity for functions (**??**, **??**).

Contents

1 Limits of Sets

1.1 The Terminal Set

Definition 1.1.1 ► The Terminal Set

The **terminal set** is the pair $(\mathrm{pt}, \{!_A\}_{A\in\mathrm{Obj}(\mathsf{Sets})})$ consisting of:

- *The Limit.* The punctual set $\mathrm{pt} \overset{\mathrm{def}}{=} \{\star\}$.
- *The Cone.* The collection of maps

  $$\{!_A\colon A \longrightarrow \mathrm{pt}\}_{A\in\mathrm{Obj}(\mathsf{Sets})}$$

  defined by

  $$!_A(a) \overset{\mathrm{def}}{=} \star$$

  for each $a \in A$ and each $A \in \mathrm{Obj}(\mathsf{Sets})$.

Proof 1.1.2 ► Proof of Definition 1.1.1

We claim that pt is the terminal object of Sets. Indeed, suppose we have a diagram of the form

$$\begin{tikzcd} A & \mathrm{pt} \end{tikzcd}$$

in Sets. Then there exists a unique map $\phi\colon A \longrightarrow \mathrm{pt}$ making the diagram

$$\begin{tikzcd} A \arrow[r, dashed, "\phi", "\exists!"'] & \mathrm{pt} \end{tikzcd}$$

commute, namely $!_A$. □

1.2 Products of Families of Sets

Let $\{A_i\}_{i\in I}$ be a family of sets.

Definition 1.2.1 ► The Product of a Family of Sets

The **product**[1] **of** $\{A_i\}_{i\in I}$ is the pair $(\prod_{i\in I} A_i, \{\mathrm{pr}_i\}_{i\in I})$ consisting of:

- *The Limit.* The set $\prod_{i\in I} A_i$ defined by[2]

  $$\prod_{i\in I} A_i \overset{\mathrm{def}}{=} \left\{ f \in \mathsf{Sets}(I, \bigcup_{i\in I} A_i) \;\middle|\; \begin{array}{l}\text{for each } i \in I\text{, we}\\ \text{have } f(i) \in A_i\end{array} \right\}.$$

- *The Cone.* The collection

$$\left\{\mathrm{pr}_i\colon \prod_{i\in I} A_i \to A_i\right\}_{i\in I}$$

of maps given by

$$\mathrm{pr}_i(f) \overset{\text{def}}{=} f(i)$$

for each $f \in \prod_{i\in I} A_i$ and each $i \in I$.

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[1] *Further Terminology*: Also called the **Cartesian product of** $\{A_i\}_{i\in I}$.

[2] Less formally, $\prod_{i\in I} A_i$ is the set whose elements are I-indexed collections $(a_i)_{i\in I}$ with $a_i \in A_i$ for each $i \in I$. The projection maps

$$\left\{\mathrm{pr}_i\colon \prod_{i\in I} A_i \to A_i\right\}_{i\in I}$$

are then given by

$$\mathrm{pr}_i((a_j)_{j\in I}) \overset{\text{def}}{=} a_i$$

for each $(a_j)_{j\in I} \in \prod_{i\in I} A_i$ and each $i \in I$.

Proof 1.2.2 ► Proof of Definition 1.2.1

We claim that $\prod_{i\in I} A_i$ is the categorical product of $\{A_i\}_{i\in I}$ in Sets. Indeed, suppose we have, for each $i \in I$, a diagram of the form

$$\begin{tikzcd}
P \arrow[rd, "p_i"] & \\
\prod_{i\in I} A_i \arrow[r, "\mathrm{pr}_i"'] & A_i
\end{tikzcd}$$

in Sets. Then there exists a unique map $\phi\colon P \to \prod_{i\in I} A_i$ making the diagram

$$\begin{tikzcd}
P \arrow[d, dashed, "\phi"', "\exists!"] \arrow[rd, "p_i"] & \\
\prod_{i\in I} A_i \arrow[r, "\mathrm{pr}_i"'] & A_i
\end{tikzcd}$$

commute, being uniquely determined by the condition $\mathrm{pr}_i \circ \phi = p_i$ for each $i \in I$ via

$$\phi(x) = (p_i(x))_{i\in I}$$

for each $x \in P$.

PROPOSITION 1.2.3 ▶ PROPERTIES OF PRODUCTS OF FAMILIES OF SETS

Let $\{A_i\}_{i\in I}$ be a family of sets.

1. *Functoriality.* The assignment $\{A_i\}_{i\in I} \mapsto \prod_{i\in I} A_i$ defines a functor

   $$\prod_{i\in I} \colon \mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}) \to \mathsf{Sets}$$

   where

   - *Action on Objects.* For each $(A_i)_{i\in I} \in \mathrm{Obj}(\mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}))$, we have

     $$\left[\prod_{i\in I}\right]((A_i)_{i\in I}) \stackrel{\text{def}}{=} \prod_{i\in I} A_i$$

   - *Action on Morphisms.* For each $(A_i)_{i\in I}, (B_i)_{i\in I} \in \mathrm{Obj}(\mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}))$, the action on Hom-sets

     $$\Big(\prod_{i\in I}\Big)_{(A_i)_{i\in I},(B_i)_{i\in I}} \colon \mathrm{Nat}((A_i)_{i\in I}, (B_i)_{i\in I}) \to \mathsf{Sets}\Big(\prod_{i\in I} A_i, \prod_{i\in I} B_i\Big)$$

     of $\prod_{i\in I}$ at $((A_i)_{i\in I}, (B_i)_{i\in I})$ is defined by sending a map

     $$\{f_i \colon A_i \to B_i\}_{i\in I}$$

     in $\mathrm{Nat}((A_i)_{i\in I}, (B_i)_{i\in I})$ to the map of sets

     $$\prod_{i\in I} f_i \colon \prod_{i\in I} A_i \to \prod_{i\in I} B_i$$

     defined by

     $$\left[\prod_{i\in I} f_i\right]((a_i)_{i\in I}) \stackrel{\text{def}}{=} (f_i(a_i))_{i\in I}$$

     for each $(a_i)_{i\in I} \in \prod_{i\in I} A_i$.

Proof 1.2.4 ► Proof of Proposition 1.2.3

Item 1: Functoriality

This follows from **??**, **??** of **??**. □

1.3 Binary Products of Sets

Let A and B be sets.

Definition 1.3.1 ► Products of Sets

The **product**[1] **of** A **and** B is the pair $(A \times B, \{\mathrm{pr}_1, \mathrm{pr}_2\})$ consisting of:

- *The Limit.* The set $A \times B$ defined by[2]

$$\begin{aligned} A \times B &\overset{\text{def}}{=} \prod_{z \in \{A,B\}} z \\ &\overset{\text{def}}{=} \{f \in \mathsf{Sets}(\{0,1\}, A \cup B) \mid \text{we have } f(0) \in A \text{ and } f(1) \in B\} \\ &\cong \{\{\{a\},\{a,b\}\} \in \mathcal{P}(\mathcal{P}(A \cup B)) \mid \text{we have } a \in A \text{ and } b \in B\}. \end{aligned}$$

- *The Cone.* The maps

$$\begin{aligned} \mathrm{pr}_1 &\colon A \times B \longrightarrow A, \\ \mathrm{pr}_2 &\colon A \times B \longrightarrow B \end{aligned}$$

defined by

$$\begin{aligned} \mathrm{pr}_1(a,b) &\overset{\text{def}}{=} a, \\ \mathrm{pr}_2(a,b) &\overset{\text{def}}{=} b \end{aligned}$$

for each $(a,b) \in A \times B$.

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[1] *Further Terminology:* Also called the **Cartesian product of** A **and** B or the **binary Cartesian product of** A **and** B, for emphasis.

This can also be thought of as the $(\mathbb{E}_{-1}, \mathbb{E}_{-1})$**-tensor product of** A **and** B.

[2] In other words, $A \times B$ is the set whose elements are ordered pairs (a,b) with $a \in A$ and $b \in B$ as in Definition 3.4.1

Proof 1.3.2 ► Proof of Definition 1.3.1

We claim that $A \times B$ is the categorical product of A and B in Sets. Indeed, suppose we have a diagram of the form

$$\begin{tikzcd} & P \arrow[ld, bend right, "p_1"'] \arrow[rd, bend left, "p_2"] & \\ A & A \times B \arrow[l, "\mathsf{pr}_1"] \arrow[r, "\mathsf{pr}_2"'] & B \end{tikzcd}$$

in Sets. Then there exists a unique map $\phi \colon P \longrightarrow A \times B$ making the diagram

$$\begin{tikzcd} & P \arrow[ld, bend right, "p_1"'] \arrow[rd, bend left, "p_2"] \arrow[d, dashed, "\phi"', "\exists!"] & \\ A & A \times B \arrow[l, "\mathsf{pr}_1"] \arrow[r, "\mathsf{pr}_2"'] & B \end{tikzcd}$$

commute, being uniquely determined by the conditions

$$\begin{aligned} \mathsf{pr}_1 \circ \phi &= p_1, \\ \mathsf{pr}_2 \circ \phi &= p_2 \end{aligned}$$

via

$$\phi(x) = (p_1(x), p_2(x))$$

for each $x \in P$. ∎

Proposition 1.3.3 ► Properties of Products of Sets

Let A, B, C, and X be sets.

1. *Functoriality.* The assignments $A, B, (A, B) \mapsto A \times B$ define functors

$$\begin{aligned} A \times - \colon \mathsf{Sets} &\longrightarrow \mathsf{Sets}, \\ - \times B \colon \mathsf{Sets} &\longrightarrow \mathsf{Sets}, \\ -_1 \times -_2 \colon \mathsf{Sets} \times \mathsf{Sets} &\longrightarrow \mathsf{Sets}, \end{aligned}$$

   where $-_1 \times -_2$ is the functor where

- *Action on Objects.* For each $(A, B) \in \mathrm{Obj}(\mathsf{Sets} \times \mathsf{Sets})$, we have

$$[-_1 \times -_2](A, B) \overset{\mathrm{def}}{=} A \times B.$$

- *Action on Morphisms.* For each $(A, B), (X, Y) \in \mathrm{Obj}(\mathsf{Sets})$, the action on Hom-sets

$$\times_{(A,B),(X,Y)} \colon \mathsf{Sets}(A, X) \times \mathsf{Sets}(B, Y) \to \mathsf{Sets}(A \times B, X \times Y)$$

  of $\times$ at $((A, B), (X, Y))$ is defined by sending (f, g) to the function

$$f \times g \colon A \times B \to X \times Y$$

  defined by

$$[f \times g](a, b) \overset{\mathrm{def}}{=} (f(a), g(b))$$

  for each $(a, b) \in A \times B$.

and where $A \times -$ and $- \times B$ are the partial functors of $-_1 \times -_2$ at $A, B \in \mathrm{Obj}(\mathsf{Sets})$.

2. *Adjointness.* We have adjunctions

$$\begin{tikzcd}(A \times - \dashv \mathrm{Hom}_{\mathsf{Sets}}(A, -)) \colon & \mathsf{Sets} \arrow[r, bend left, "A\times-"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}, \arrow[l, bend left, "\mathrm{Hom}_{\mathsf{Sets}}(A,-)"]\end{tikzcd}$$

$$\begin{tikzcd}(- \times B \dashv \mathrm{Hom}_{\mathsf{Sets}}(B, -)) \colon & \mathsf{Sets} \arrow[r, bend left, "-\times B"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}, \arrow[l, bend left, "\mathrm{Hom}_{\mathsf{Sets}}(B,-)"]\end{tikzcd}$$

witnessed by bijections

$$\mathrm{Hom}_{\mathsf{Sets}}(A \times B, C) \cong \mathrm{Hom}_{\mathsf{Sets}}(A, \mathrm{Hom}_{\mathsf{Sets}}(B, C)),$$
$$\mathrm{Hom}_{\mathsf{Sets}}(A \times B, C) \cong \mathrm{Hom}_{\mathsf{Sets}}(B, \mathrm{Hom}_{\mathsf{Sets}}(A, C)),$$

natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

3. *Associativity.* We have an isomorphism of sets

   $$(A \times B) \times C \cong A \times (B \times C),$$

   natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

4. *Unitality.* We have isomorphisms of sets

   $$\begin{aligned} \mathrm{pt} \times A &\cong A, \\ A \times \mathrm{pt} &\cong A, \end{aligned}$$

   natural in $A \in \mathrm{Obj}(\mathsf{Sets})$.

5. *Commutativity.* We have an isomorphism of sets

   $$A \times B \cong B \times A,$$

   natural in $A, B \in \mathrm{Obj}(\mathsf{Sets})$.

6. *Annihilation With the Empty Set.* We have isomorphisms of sets

   $$\begin{aligned} A \times \emptyset &\cong \emptyset, \\ \emptyset \times A &\cong \emptyset, \end{aligned}$$

   natural in $A \in \mathrm{Obj}(\mathsf{Sets})$.

7. *Distributivity Over Unions.* We have isomorphisms of sets

   $$\begin{aligned} A \times (B \cup C) &= (A \times B) \cup (A \times C), \\ (A \cup B) \times C &= (A \times C) \cup (B \times C). \end{aligned}$$

8. *Distributivity Over Intersections.* We have isomorphisms of sets

   $$\begin{aligned} A \times (B \cap C) &= (A \times B) \cap (A \times C), \\ (A \cap B) \times C &= (A \times C) \cap (B \times C). \end{aligned}$$

9. *Middle-Four Exchange with Respect to Intersections.* We have an isomorphism of sets

   $$(A \times B) \cap (C \times D) \cong (A \cap B) \times (C \cap D).$$

10. *Distributivity Over Differences.* We have isomorphisms of sets

$$\begin{aligned} A \times (B \setminus C) &= (A \times B) \setminus (A \times C), \\ (A \setminus B) \times C &= (A \times C) \setminus (B \times C), \end{aligned}$$

natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

11. *Distributivity Over Symmetric Differences.* We have isomorphisms of sets

$$\begin{aligned} A \times (B \vartriangle C) &= (A \times B) \vartriangle (A \times C), \\ (A \vartriangle B) \times C &= (A \times C) \vartriangle (B \times C), \end{aligned}$$

natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

12. *Symmetric Monoidality.* The triple $(\mathsf{Sets}, \times, \mathrm{pt})$ is a symmetric monoidal category.

13. *Symmetric Bimonoidality.* The quintuple $(\mathsf{Sets}, \coprod, \emptyset, \times, \mathrm{pt})$ is a symmetric bimonoidal category.

PROOF 1.3.4 ► PROOF OF PROPOSITION 1.3.3

Item 1: Functoriality

This follows from **??**, **??** of **??**.

Item 2: Adjointness

We prove only that there's an adjunction $- \times B \dashv \mathrm{Hom}_{\mathsf{Sets}}(B, -)$, witnessed by a bijection

$$\mathrm{Hom}_{\mathsf{Sets}}(A \times B, C) \cong \mathrm{Hom}_{\mathsf{Sets}}(A, \mathrm{Hom}_{\mathsf{Sets}}(B, C)),$$

natural in $B, C \in \mathrm{Obj}(\mathsf{Sets})$, as the proof of the existence of the adjunction $A \times - \dashv \mathrm{Hom}_{\mathsf{Sets}}(A, -)$ follows almost exactly in the same way.

- *Map I.* We define a map

  $$\Phi_{B,C} \colon \mathrm{Hom}_{\mathsf{Sets}}(A \times B, C) \to \mathrm{Hom}_{\mathsf{Sets}}(A, \mathrm{Hom}_{\mathsf{Sets}}(B, C)),$$

  by sending a function

  $$\xi \colon A \times B \to C$$

to the function

$$\begin{aligned} \xi^{\dagger}\colon A &\longrightarrow \mathrm{Hom}_{\mathsf{Sets}}(B,C),\\ a &\longmapsto (\xi^{\dagger}_{a}\colon B \to C), \end{aligned}$$

where we define

$$\xi^{\dagger}_{a}(b) \overset{\text{def}}{=} \xi(a,b)$$

for each $b \in B$. In terms of the $[\![a \mapsto f(a)]\!]$ notation of Sets, Notation 1.1.2, we have

$$\xi^{\dagger} \overset{\text{def}}{=} [\![a \mapsto [\![b \mapsto \xi(a,b)]\!]]\!].$$

- *Map II.* We define a map

  $$\Psi_{B,C}\colon \mathrm{Hom}_{\mathsf{Sets}}(A,\mathrm{Hom}_{\mathsf{Sets}}(B,C)), \to \mathrm{Hom}_{\mathsf{Sets}}(A\times B,C)$$

  given by sending a function

  $$\begin{aligned} \xi\colon A &\longrightarrow \mathrm{Hom}_{\mathsf{Sets}}(B,C),\\ a &\longmapsto (\xi_{a}\colon B \to C), \end{aligned}$$

  to the function

  $$\xi^{\dagger}\colon A\times B \to C$$

  defined by

  $$\begin{aligned} \xi^{\dagger}(a,b) &\overset{\text{def}}{=} \mathrm{ev}_{b}(\mathrm{ev}_{a}(\xi))\\ &\overset{\text{def}}{=} \mathrm{ev}_{b}(\xi_{a})\\ &\overset{\text{def}}{=} \xi_{a}(b) \end{aligned}$$

  for each $(a,b) \in A\times B$.

- *Invertibility I.* We claim that

  $$\Psi_{A,B}\circ\Phi_{A,B} = \mathrm{id}_{\mathrm{Hom}_{\mathsf{Sets}}(A\times B,C)}.$$

Indeed, given a function $\xi\colon A\times B\to C$, we have

$$\begin{aligned}
[\Psi_{A,B}\circ\Phi_{A,B}](\xi) &= \Psi_{A,B}(\Phi_{A,B}(\xi))\\
&= \Psi_{A,B}(\Phi_{A,B}(\llbracket (a,b)\mapsto \xi(a,b)\rrbracket))\\
&= \Psi_{A,B}(\llbracket a\mapsto \llbracket b\mapsto \xi(a,b)\rrbracket\rrbracket)\\
&= \Psi_{A,B}(\llbracket a'\mapsto \llbracket b'\mapsto \xi(a',b')\rrbracket\rrbracket)\\
&= \llbracket (a,b)\mapsto \mathrm{ev}_b(\mathrm{ev}_a(\llbracket a'\mapsto \llbracket b'\mapsto \xi(a',b')\rrbracket\rrbracket))\rrbracket\\
&= \llbracket (a,b)\mapsto \mathrm{ev}_b(\llbracket b'\mapsto \xi(a,b')\rrbracket)\rrbracket\\
&= \llbracket (a,b)\mapsto \xi(a,b)\rrbracket\\
&= \xi.
\end{aligned}$$

- *Invertibility II.* We claim that

$$\Phi_{A,B}\circ\Psi_{A,B} = \mathrm{id}_{\mathrm{Hom}_{\mathsf{Sets}}(A,\mathrm{Hom}_{\mathsf{Sets}}(B,C))}.$$

Indeed, given a function

$$\begin{aligned}
\xi\colon A &\longrightarrow \mathrm{Hom}_{\mathsf{Sets}}(B,C),\\
a &\longmapsto (\xi_a\colon B\to C),
\end{aligned}$$

we have

$$\begin{aligned}
[\Phi_{A,B}\circ\Psi_{A,B}](\xi) &\overset{\text{def}}{=} \Phi_{A,B}(\Psi_{A,B}(\xi))\\
&\overset{\text{def}}{=} \Phi_{A,B}(\llbracket (a,b)\mapsto \xi_a(b)\rrbracket)\\
&\overset{\text{def}}{=} \Phi_{A,B}(\llbracket (a',b')\mapsto \xi_{a'}(b')\rrbracket)\\
&\overset{\text{def}}{=} \llbracket a\mapsto \llbracket b\mapsto \mathrm{ev}_{(a,b)}(\llbracket (a',b')\mapsto \xi_{a'}(b')\rrbracket)\rrbracket\rrbracket\\
&\overset{\text{def}}{=} \llbracket a\mapsto \llbracket b\mapsto \xi_a(b)\rrbracket\rrbracket\\
&\overset{\text{def}}{=} \llbracket a\mapsto \xi_a\rrbracket\\
&\overset{\text{def}}{=} \xi.
\end{aligned}$$

- *Naturality for Φ, Part I.* We need to show that, given a function $g\colon B\to B'$, the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathsf{Sets}}(A\times B',C) \arrow[r, "\Phi_{B',C}"] \arrow[d, "\mathrm{id}_A\times g^*"'] & \mathrm{Hom}_{\mathsf{Sets}}(A,\mathrm{Hom}_{\mathsf{Sets}}(B',C)), \arrow[d, "(g^*)_*"] \\
\mathrm{Hom}_{\mathsf{Sets}}(A\times B,C) \arrow[r, "\Phi_{B,C}"'] & \mathrm{Hom}_{\mathsf{Sets}}(A,\mathrm{Hom}_{\mathsf{Sets}}(B,C))
\end{tikzcd}$$

commutes. Indeed, given a function

$$\xi\colon A\times B'\to C,$$

we have

$$\begin{aligned}
[\Phi_{B,C}\circ(\mathrm{id}_A\times g^*)](\xi) &= \Phi_{B,C}([\mathrm{id}_A\times g^*](\xi))\\
&= \Phi_{B,C}(\xi(-_1,g(-_2)))\\
&= [\xi(-_1,g(-_2))]^\dagger\\
&= \xi^\dagger_{-_1}(g(-_2))\\
&= (g^*)_*(\xi^\dagger)\\
&= (g^*)_*(\Phi_{B',C}(\xi))\\
&= [(g^*)_*\circ\Phi_{B',C}](\xi).
\end{aligned}$$

Alternatively, using the $[\![a\mapsto f(a)]\!]$ notation of Sets, Notation 1.1.2, we have

$$\begin{aligned}
[\Phi_{B,C}\circ(\mathrm{id}_A\times g^*)](\xi) &= \Phi_{B,C}([\mathrm{id}_A\times g^*](\xi))\\
&= \Phi_{B,C}([\mathrm{id}_A\times g^*]([\![(a,b')\mapsto\xi(a,b')]\!]))\\
&= \Phi_{B,C}([\![(a,b)\mapsto\xi(a,g(b))]\!])\\
&= [\![a\mapsto[\![b\mapsto\xi(a,g(b))]\!]]\!]\\
&= [\![a\mapsto g^*([\![b'\mapsto\xi(a,b')]\!])]\!]\\
&= (g^*)_*([\![a\mapsto[\![b'\mapsto\xi(a,b')]\!]]\!])\\
&= (g^*)_*(\Phi_{B',C}([\![(a,b')\mapsto\xi(a,b')]\!]))\\
&= (g^*)_*(\Phi_{B',C}(\xi))\\
&= [(g^*)_*\circ\Phi_{B',C}](\xi).
\end{aligned}$$

- *Naturality for Φ, Part II.* We need to show that, given a function $h\colon C \to C'$, the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathsf{Sets}}(A \times B, C) \arrow[r, "\Phi_{B,C}"] \arrow[d, "h_*"'] & \mathrm{Hom}_{\mathsf{Sets}}(A, \mathrm{Hom}_{\mathsf{Sets}}(B, C)), \arrow[d, "(h_*)_*"] \\
\mathrm{Hom}_{\mathsf{Sets}}(A \times B, C') \arrow[r, "\Phi_{B,C'}"'] & \mathrm{Hom}_{\mathsf{Sets}}(A, \mathrm{Hom}_{\mathsf{Sets}}(B, C'))
\end{tikzcd}$$

commutes. Indeed, given a function

$$\xi\colon A \times B \to C,$$

we have

$$\begin{aligned}
[\Phi_{B,C} \circ h_*](\xi) &= \Phi_{B,C}(h_*(\xi)) \\
&= \Phi_{B,C}(h_*([\![(a, b) \mapsto \xi(a, b)]\!])) \\
&= \Phi_{B,C}([\![(a, b) \mapsto h(\xi(a, b))]\!]) \\
&= [\![a \mapsto [\![b \mapsto h(\xi(a, b))]\!]]\!] \\
&= [\![a \mapsto h_*([\![b \mapsto \xi(a, b)]\!]]\!]) \\
&= (h_*)_*([\![a \mapsto [\![b \mapsto \xi(a, b)]\!]]\!]) \\
&= (h_*)_*(\Phi_{B,C}([\![(a, b) \mapsto \xi(a, b)]\!])) \\
&= (h_*)_*(\Phi_{B,C}(\xi)) \\
&= [(h_*)_* \circ \Phi_{B,C}](\xi).
\end{aligned}$$

- *Naturality for Ψ.* Since Φ is natural in each argument and Φ is a componentwise inverse to Ψ in each argument, it follows from Categories, Item 2 of Proposition 8.6.2 that Ψ is also natural in each argument.

Item 3: Associativity

See [Pro24a].

Item 4: Unitality

Clear.

Item 5: Commutativity

See [Pro24b].

Item 6: Annihilation With the Empty Set

See [Pro24f].

Item 7: Distributivity Over Unions

See [Pro24e].

Item 8: Distributivity Over Intersections

See [Pro24g, Corollary 1].

Item 9: Middle-Four Exchange With Respect to Intersections

See [Pro24g, Corollary 1].

Item 10: Distributivity Over Differences

See [Pro24c].

Item 11: Distributivity Over Symmetric Differences

See [Pro24d].

Item 12: Symmetric Monoidality

See [`MO 382264`].

Item 13: Symmetric Bimonoidality

Omitted. □

1.4 Pullbacks

Let A, B, and C be sets and let $f\colon A \to C$ and $g\colon B \to C$ be functions.

DEFINITION 1.4.1 ► PULLBACKS OF SETS

The **pullback of A and B over C along f and g**[1] is the pair[2] $(A \times_C B, \{\mathrm{pr}_1, \mathrm{pr}_2\})$ consisting of:

- *The Limit.* The set $A \times_C B$ defined by

$$A \times_C B \overset{\text{def}}{=} \{(a, b) \in A \times B \mid f(a) = g(b)\}.$$

- *The Cone.* The maps

$$\begin{aligned}\mathrm{pr}_1 \colon A \times_C B &\longrightarrow A,\\ \mathrm{pr}_2 \colon A \times_C B &\longrightarrow B\end{aligned}$$

defined by

$$\begin{aligned}\mathrm{pr}_1(a, b) &\overset{\mathrm{def}}{=} a,\\ \mathrm{pr}_2(a, b) &\overset{\mathrm{def}}{=} b\end{aligned}$$

for each $(a, b) \in A \times_C B$.

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[1] *Further Terminology:* Also called the **fibre product of** A **and** B **over** C **along** f **and** g.

[2] *Further Notation:* Also written $A \times_{f,C,g} B$.

PROOF 1.4.2 ► PROOF OF DEFINITION 1.4.1

We claim that $A \times_C B$ is the categorical pullback of A and B over C with respect to (f, g) in Sets. First we need to check that the relevant pullback diagram commutes, i.e. that we have

$$f \circ \mathrm{pr}_1 = g \circ \mathrm{pr}_2, \qquad \begin{tikzcd} A \times_C B \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C. \end{tikzcd}$$

Indeed, given $(a, b) \in A \times_C B$, we have

$$\begin{aligned}[f \circ \mathrm{pr}_1](a, b) &= f(\mathrm{pr}_1(a, b))\\ &= f(a)\\ &= g(b)\\ &= g(\mathrm{pr}_2(a, b))\\ &= [g \circ \mathrm{pr}_2](a, b),\end{aligned}$$

where $f(a) = g(b)$ since $(a,b) \in A \times_C B$. Next, we prove that $A \times_C B$ satisfies the universal property of the pullback. Suppose we have a diagram of the form

$$\begin{tikzcd}
P \arrow[drr, bend left, "p_2"] \arrow[ddr, bend right, "p_1"'] & & \\
& A \times_C B \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "g"] \\
& A \arrow[r, "f"'] & C
\end{tikzcd}$$

in Sets. Then there exists a unique map $\phi\colon P \to A \times_C B$ making the diagram

$$\begin{tikzcd}
P \arrow[drr, bend left, "p_2"] \arrow[ddr, bend right, "p_1"'] \arrow[dr, dashed, "\phi", "\exists!"'] & & \\
& A \times_C B \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "g"] \\
& A \arrow[r, "f"'] & C
\end{tikzcd}$$

commute, being uniquely determined by the conditions

$$\mathrm{pr}_1 \circ \phi = p_1,$$
$$\mathrm{pr}_2 \circ \phi = p_2$$

via

$$\phi(x) = (p_1(x), p_2(x))$$

for each $x \in P$, where we note that $(p_1(x), p_2(x)) \in A \times B$ indeed lies in $A \times_C B$ by the condition

$$f \circ p_1 = g \circ p_2,$$

which gives

$$f(p_1(x)) = g(p_2(x))$$

for each $x \in P$, so that $(p_1(x), p_2(x)) \in A \times_C B$. ∎

Example 1.4.3 ▶ Examples of Pullbacks of Sets

Here are some examples of pullbacks of sets.

1. *Unions via Intersections.* Let $A, B \subset X$. We have a bijection of sets

$$A \cap B \cong A \times_{A \cup B} B,$$

$$\begin{tikzcd} A \cap B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "\iota_B"] \\ A \arrow[r, "\iota_A"'] & A \cup B. \end{tikzcd}$$

Proof 1.4.4 ▶ Proof of Example 1.4.3

Item 1: Unions via Intersections

Indeed, we have

$$\begin{aligned} A \times_{A \cup B} B &\cong \{(x, y) \in A \times B \mid x = y\} \\ &\cong A \cap B. \end{aligned}$$

This finishes the proof. □

Proposition 1.4.5 ▶ Properties of Pullbacks of Sets

Let A, B, C, and X be sets.

1. *Functoriality.* The assignment $(A, B, C, f, g) \mapsto A \times_{f,C,g} B$ defines a functor

$$-_1 \times_{-_3} -_1 \colon \mathrm{Fun}(\mathcal{P}, \mathrm{Sets}) \longrightarrow \mathrm{Sets},$$

where $\mathcal{P}$ is the category that looks like this:

$$\begin{tikzcd} & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet. \end{tikzcd}$$

In particular, the action on morphisms of $-_1 \times_{-_3} -_1$ is given by sending a morphism

$$\begin{tikzcd}
A \times_C B \arrow[rr] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & & B \arrow[dd, "g", near start] \arrow[dr, "\psi"] & \\
& A' \times_{C'} B' \arrow[rr] \arrow[dr, phantom, "\lrcorner", very near start] & & B' \arrow[dd, "g'"] \\
A \arrow[rr, "f", near start] \arrow[dr, "\phi"'] & & C \arrow[dr, "\chi"] & \\
& A' \arrow[rr, "f'"'] \arrow[from=uu, crossing over] & & C'
\end{tikzcd}$$

in Fun($\mathcal{P}$, Sets) to the map $\xi\colon A \times_C B \xrightarrow{\exists!} A' \times_{C'} B'$ given by

$$\xi(a, b) \overset{\text{def}}{=} (\phi(a), \psi(b))$$

for each $(a, b) \in A \times_C B$, which is the unique map making the diagram

$$\begin{tikzcd}
A \times_C B \arrow[rr] \arrow[dd] \arrow[dr, dashed] \arrow[dr, phantom, "\lrcorner", very near start] & & B \arrow[dd, "g", near start] \arrow[dr, "\psi"] & \\
& A' \times_{C'} B' \arrow[rr] \arrow[dr, phantom, "\lrcorner", very near start] & & B' \arrow[dd, "g'"] \\
A \arrow[rr, "f", near start] \arrow[dr, "\phi"'] & & C \arrow[dr, "\chi"] & \\
& A' \arrow[rr, "f'"'] \arrow[from=uu, crossing over] & & C'
\end{tikzcd}$$

commute.

2. *Associativity.* Given a diagram

$$\begin{tikzcd}
A \arrow[dr, "f"'] & & B \arrow[dl, "g"] \arrow[dr, "h"'] & & C \arrow[dl, "k"] \\
& X & & Y &
\end{tikzcd}$$

in Sets, we have isomorphisms of sets

$$(A \times_X B) \times_Y C \cong (A \times_X B) \times_B (B \times_Y C) \cong A \times_X (B \times_Y C),$$

where these pullbacks are built as in the diagrams

$$\begin{tikzcd}[column sep=small]
 & & (A \times_X B) \times_Y C \arrow[dl] \arrow[ddrr] \arrow[d, phantom, "\lrcorner" rotate=-45, very near start] & & \\
 & A \times_X B \arrow[dl] \arrow[dr] \arrow[d, phantom, "\lrcorner" rotate=-45, very near start] & & & \\
A \arrow[dr, "f"'] & & B \arrow[dl, "g"] \arrow[dr, "h"'] & & C, \arrow[dl, "k"] \\
 & X & & Y &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=small]
 & & (A \times_X B) \times_B (B \times_Y C) \arrow[dl] \arrow[dr] \arrow[d, phantom, "\lrcorner" rotate=-45, very near start] & & \\
 & A \times_X B \arrow[dl] \arrow[dr] \arrow[d, phantom, "\lrcorner" rotate=-45, very near start] & & B \times_Y C \arrow[dl] \arrow[dr] \arrow[d, phantom, "\lrcorner" rotate=-45, very near start] & \\
A \arrow[dr, "f"'] & & B \arrow[dl, "g"] \arrow[dr, "h"'] & & C, \arrow[dl, "k"] \\
 & X & & Y &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=small]
 & & A \times_X (B \times_Y C) \arrow[ddll] \arrow[dr] \arrow[d, phantom, "\lrcorner" rotate=-45, very near start] & & \\
 & & & B \times_Y C \arrow[dl] \arrow[dr] \arrow[d, phantom, "\lrcorner" rotate=-45, very near start] & \\
A \arrow[dr, "f"'] & & B \arrow[dl, "g"] \arrow[dr, "h"'] & & C. \arrow[dl, "k"] \\
 & X & & Y &
\end{tikzcd}$$

3. *Unitality.* We have isomorphisms of sets

$$\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\
X \arrow[r, equal] & X
\end{tikzcd}
\qquad
\begin{aligned}
X \times_X A &\cong A, \\
A \times_X X &\cong A,
\end{aligned}
\qquad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, equal] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, equal] \\
X \arrow[r, "f"'] & X.
\end{tikzcd}$$

4. *Commutativity.* We have an isomorphism of sets

$$\begin{tikzcd}
A \times_X B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "g"] \\
A \arrow[r, "f"'] & X,
\end{tikzcd}
\qquad
A \times_X B \cong B \times_X A
\qquad
\begin{tikzcd}
B \times_X A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\
B \arrow[r, "g"'] & X.
\end{tikzcd}$$

5. *Annihilation With the Empty Set.* We have isomorphisms of sets

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \emptyset \arrow[d] \\
A \arrow[r, "f"'] & X,
\end{tikzcd}
\qquad
\begin{aligned}
A \times_X \emptyset &\cong \emptyset, \\
\emptyset \times_X A &\cong \emptyset,
\end{aligned}
\qquad
\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\
\emptyset \arrow[r] & X.
\end{tikzcd}$$

6. *Interaction With Products.* We have an isomorphism of sets

$$A \times_{\mathrm{pt}} B \cong A \times B, \qquad \begin{tikzcd} A \times B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "!_B"] \\ A \arrow[r, "!_A"'] & \mathrm{pt}. \end{tikzcd}$$

7. *Symmetric Monoidality.* The triple $(\mathsf{Sets}, \times_X, X)$ is a symmetric monoidal category.

PROOF 1.4.6 ► PROOF OF PROPOSITION 1.4.5

Item 1: Functoriality

This is a special case of functoriality of co/limits, **??**, **??** of **??**, with the explicit expression for ξ following from the commutativity of the cube pullback diagram.

Item 2: Associativity

Indeed, we have

$$\begin{aligned} (A \times_X B) \times_Y C &\cong \{((a,b),c) \in (A \times_X B) \times C \mid h(b) = k(c)\} \\ &\cong \{((a,b),c) \in (A \times B) \times C \mid f(a) = g(b) \text{ and } h(b) = k(c)\} \\ &\cong \{(a,(b,c)) \in A \times (B \times C) \mid f(a) = g(b) \text{ and } h(b) = k(c)\} \\ &\cong \{(a,(b,c)) \in A \times (B \times_Y C) \mid f(a) = g(b)\} \\ &\cong A \times_X (B \times_Y C) \end{aligned}$$

and

$$\begin{aligned}
(A \times_X B) \times_B (B \times_Y C) &\cong \left\{((a,b),(b',c)) \in (A \times_X B) \times (B \times_Y C) \,\middle|\, b = b'\right\}\\
&\cong \left\{((a,b),(b',c)) \in (A \times B) \times (B \times C) \,\middle|\, \begin{array}{l} f(a) = g(b), b = b', \\ \text{and } h(b') = k(c) \end{array}\right\}\\
&\cong \left\{(a,(b,(b',c))) \in A \times (B \times (B \times C)) \,\middle|\, \begin{array}{l} f(a) = g(b), b = b', \\ \text{and } h(b') = k(c) \end{array}\right\}\\
&\cong \left\{(a,((b,b'),c)) \in A \times ((B \times B) \times C) \,\middle|\, \begin{array}{l} f(a) = g(b), b = b', \\ \text{and } h(b') = k(c) \end{array}\right\}\\
&\cong \left\{(a,((b,b'),c)) \in A \times ((B \times_B B) \times C) \,\middle|\, \begin{array}{l} f(a) = g(b) \text{ and} \\ h(b') = k(c) \end{array}\right\}\\
&\cong \{(a,(b,c)) \in A \times (B \times C) \mid f(a) = g(b) \text{ and } h(b) = k(c)\}\\
&\cong A \times_X (B \times_Y C),
\end{aligned}$$

where we have used Item 3 for the isomorphism $B \times_B B \cong B$.

Item 3: Unitality

Indeed, we have

$$\begin{aligned}
X \times_X A &\cong \{(x,a) \in X \times A \mid f(a) = x\},\\
A \times_X X &\cong \{(a,x) \in X \times A \mid f(a) = x\},
\end{aligned}$$

which are isomorphic to A via the maps $(x,a) \mapsto a$ and $(a,x) \mapsto a$.

Item 4: Commutativity

Clear.

Item 5: Annihilation With the Empty Set

Clear.

Item 6: Interaction With Products

Clear.

Item 7: Symmetric Monoidality

Omitted. ∎

1.5 Equalisers

Let A and B be sets and let $f, g \colon A \rightrightarrows B$ be functions.

DEFINITION 1.5.1 ► EQUALISERS OF SETS

The **equaliser of f and g** is the pair $(\mathrm{Eq}(f,g), \mathrm{eq}(f,g))$ consisting of:

- *The Limit.* The set $\mathrm{Eq}(f,g)$ defined by

$$\mathrm{Eq}(f,g) \overset{\text{def}}{=} \{a \in A \mid f(a) = g(a)\}.$$

- *The Cone.* The inclusion map

$$\mathrm{eq}(f,g)\colon \mathrm{Eq}(f,g) \hookrightarrow A.$$

PROOF 1.5.2 ► PROOF OF DEFINITION 1.5.1

We claim that $\mathrm{Eq}(f,g)$ is the categorical equaliser of f and g in Sets. First we need to check that the relevant equaliser diagram commutes, i.e. that we have

$$f \circ \mathrm{eq}(f,g) = g \circ \mathrm{eq}(f,g),$$

which indeed holds by the definition of the set $\mathrm{Eq}(f,g)$. Next, we prove that $\mathrm{Eq}(f,g)$ satisfies the universal property of the equaliser. Suppose we have a diagram of the form

$$\begin{tikzcd}
\mathrm{Eq}(f,g) \arrow[r, hook, "\mathrm{eq}(f,g)"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\
E \arrow[ru, "e"'] & &
\end{tikzcd}$$

in Sets. Then there exists a unique map $\phi\colon E \longrightarrow \mathrm{Eq}(f,g)$ making the diagram

$$\begin{tikzcd}
\mathrm{Eq}(f,g) \arrow[r, hook, "\mathrm{eq}(f,g)"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\
E \arrow[u, dashed, "\phi", "\exists!"'] \arrow[ru, "e"'] & &
\end{tikzcd}$$

commute, being uniquely determined by the condition

$$\mathrm{eq}(f,g) \circ \phi = e$$

via

$$\phi(x) = e(x)$$

for each $x \in E$, where we note that $e(x) \in A$ indeed lies in $\mathrm{Eq}(f, g)$ by the condition

$$f \circ e = g \circ e,$$

which gives

$$f(e(x)) = g(e(x))$$

for each $x \in E$, so that $e(x) \in \mathrm{Eq}(f, g)$. □

PROPOSITION 1.5.3 ► PROPERTIES OF EQUALISERS OF SETS

Let A, B, and C be sets.

1. *Associativity.* We have isomorphisms of sets[1]

$$\underbrace{\mathrm{Eq}(f \circ \mathrm{eq}(g, h), g \circ \mathrm{eq}(g, h))}_{=\mathrm{Eq}(f \circ \mathrm{eq}(g,h), h \circ \mathrm{eq}(g,h))} \cong \mathrm{Eq}(f, g, h) \cong \underbrace{\mathrm{Eq}(f \circ \mathrm{eq}(f, g), h \circ \mathrm{eq}(f, g))}_{=\mathrm{Eq}(g \circ \mathrm{eq}(f,g), h \circ \mathrm{eq}(f,g))},$$

   where $\mathrm{Eq}(f, g, h)$ is the limit of the diagram

$$\begin{tikzcd} A \arrow[r, shift left=2, "f"] \arrow[r, "g" description] \arrow[r, shift right=2, "h"'] & B \end{tikzcd}$$

   in Sets, being explicitly given by

$$\mathrm{Eq}(f, g, h) \cong \{a \in A \mid f(a) = g(a) = h(a)\}.$$

2. *Unitality.* We have an isomorphism of sets

$$\mathrm{Eq}(f, f) \cong A.$$

3. *Commutativity.* We have an isomorphism of sets

$$\mathrm{Eq}(f, g) \cong \mathrm{Eq}(g, f).$$

4. *Interaction With Composition.* Let

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, shift left, "h"] \arrow[r, shift right, "k"'] & C \end{tikzcd}$$

be functions. We have an inclusion of sets

$$\mathrm{Eq}(h \circ f \circ \mathrm{eq}(f,g), k \circ g \circ \mathrm{eq}(f,g)) \subset \mathrm{Eq}(h \circ f, k \circ g),$$

where $\mathrm{Eq}(h \circ f \circ \mathrm{eq}(f,g), k \circ g \circ \mathrm{eq}(f,g))$ is the equaliser of the composition

$$\begin{tikzcd} \mathrm{Eq}(f,g) \arrow[r, hook, "\mathrm{eq}(f,g)"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, shift left, "h"] \arrow[r, shift right, "k"'] & C. \end{tikzcd}$$

---

[1] That is, the following three ways of forming "the" equaliser of (f,g,h) agree:

(a) Take the equaliser of (f,g,h), i.e. the limit of the diagram

$$\begin{tikzcd} A \arrow[r, shift left=2, "f"] \arrow[r, "g" description] \arrow[r, shift right=2, "h"'] & B \end{tikzcd}$$

in Sets.

(b) First take the equaliser of f and g, forming a diagram

$$\begin{tikzcd} \mathrm{Eq}(f,g) \arrow[r, hook, "\mathrm{eq}(f,g)"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

and then take the equaliser of the composition

$$\begin{tikzcd} \mathrm{Eq}(f,g) \arrow[r, hook, "\mathrm{eq}(f,g)"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "h"'] & B, \end{tikzcd}$$

obtaining a subset

$$\mathrm{Eq}(f \circ \mathrm{eq}(f,g), h \circ \mathrm{eq}(f,g)) = \mathrm{Eq}(g \circ \mathrm{eq}(f,g), h \circ \mathrm{eq}(f,g))$$

of $\mathrm{Eq}(f,g)$.

(c) First take the equaliser of g and h, forming a diagram

$$\begin{tikzcd} \mathrm{Eq}(g,h) \arrow[r, hook, "\mathrm{eq}(g,h)"] & A \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & B \end{tikzcd}$$

and then take the equaliser of the composition

$$\begin{tikzcd} \mathrm{Eq}(g,h) \arrow[r, hook, "\mathrm{eq}(g,h)"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B, \end{tikzcd}$$

obtaining a subset

$$\mathrm{Eq}(f \circ \mathrm{eq}(g,h), g \circ \mathrm{eq}(g,h)) = \mathrm{Eq}(f \circ \mathrm{eq}(g,h), h \circ \mathrm{eq}(g,h))$$

of $\mathrm{Eq}(g,h)$.

PROOF 1.5.4 ► PROOF OF PROPOSITION 1.5.3

Item 1: Associativity

We first prove that $\mathsf{Eq}(f,g,h)$ is indeed given by

$$\mathsf{Eq}(f,g,h) \cong \{a \in A \mid f(a) = g(a) = h(a)\}.$$

Indeed, suppose we have a diagram of the form

$$\begin{tikzcd}
\mathsf{Eq}(f,g,h) \arrow[r, hook, "\mathsf{eq}(f,g,h)"] & A \arrow[r, shift left=2, "f"] \arrow[r, "g" description] \arrow[r, shift right=2, "h"'] & B \\
E \arrow[ur, "e"'] & &
\end{tikzcd}$$

in Sets. Then there exists a unique map $\phi\colon E \longrightarrow \mathsf{Eq}(f,g,h)$, uniquely determined by the condition

$$\mathsf{eq}(f,g) \circ \phi = e$$

being necessarily given by

$$\phi(x) = e(x)$$

for each $x \in E$, where we note that $e(x) \in A$ indeed lies in $\mathsf{Eq}(f,g,h)$ by the condition

$$f \circ e = g \circ e = h \circ e,$$

which gives

$$f(e(x)) = g(e(x)) = h(e(x))$$

for each $x \in E$, so that $e(x) \in \mathsf{Eq}(f,g,h)$.

We now check the equalities

$$\mathsf{Eq}(f \circ \mathsf{eq}(g,h), g \circ \mathsf{eq}(g,h)) \cong \mathsf{Eq}(f,g,h) \cong \mathsf{Eq}(f \circ \mathsf{eq}(f,g), h \circ \mathsf{eq}(f,g)).$$

Indeed, we have

$$\begin{aligned}
\mathsf{Eq}(f \circ \mathsf{eq}(g,h), g \circ \mathsf{eq}(g,h)) &\cong \{x \in \mathsf{Eq}(g,h) \mid [f \circ \mathsf{eq}(g,h)](a) = [g \circ \mathsf{eq}(g,h)](a)\} \\
&\cong \{x \in \mathsf{Eq}(g,h) \mid f(a) = g(a)\} \\
&\cong \{x \in A \mid f(a) = g(a) \text{ and } g(a) = h(a)\} \\
&\cong \{x \in A \mid f(a) = g(a) = h(a)\} \\
&\cong \mathsf{Eq}(f,g,h).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}\mathrm{Eq}(f\circ \mathrm{eq}(f,g), h\circ \mathrm{eq}(f,g)) &\cong \{x\in \mathrm{Eq}(f,g) \mid [f\circ \mathrm{eq}(f,g)](a)=[h\circ \mathrm{eq}(f,g)](a)\}\\ &\cong \{x\in \mathrm{Eq}(f,g) \mid f(a)=h(a)\}\\ &\cong \{x\in A \mid f(a)=h(a) \text{ and } f(a)=g(a)\}\\ &\cong \{x\in A \mid f(a)=g(a)=h(a)\}\\ &\cong \mathrm{Eq}(f,g,h).\end{aligned}$$

Item 2: Unitality

Clear.

Item 3: Commutativity

Clear.

Item 4: Interaction With Composition

Indeed, we have

$$\begin{aligned}\mathrm{Eq}(h\circ f\circ \mathrm{eq}(f,g), k\circ g\circ \mathrm{eq}(f,g)) &\cong \{a\in \mathrm{Eq}(f,g) \mid h(f(a))=k(g(a))\}\\ &\cong \{a\in A \mid f(a)=g(a) \text{ and } h(f(a))=k(g(a))\}.\end{aligned}$$

and

$$\mathrm{Eq}(h\circ f, k\circ g) \cong \{a\in A \mid h(f(a))=k(g(a))\},$$

and thus there's an inclusion from $\mathrm{Eq}(h\circ f\circ \mathrm{eq}(f,g), k\circ g\circ \mathrm{eq}(f,g))$ to $\mathrm{Eq}(h\circ f, k\circ g)$. □

2 Colimits of Sets

2.1 The Initial Set

DEFINITION 2.1.1 ► THE INITIAL SET

The **initial set** is the pair $(\emptyset, \{\iota_A\}_{A\in \mathrm{Obj}(\mathsf{Sets})})$ consisting of:

- *The Limit.* The empty set $\emptyset$ of Definition 3.1.1.
- *The Cone.* The collection of maps

  $$\{\iota_A\colon \emptyset \longrightarrow A\}_{A\in \mathrm{Obj}(\mathsf{Sets})}$$

  given by the inclusion maps from $\emptyset$ to A.

Proof 2.1.2 ► Proof of Definition 2.1.1

We claim that $\emptyset$ is the initial object of Sets. Indeed, suppose we have a diagram of the form

$$\begin{tikzcd}\emptyset & A\end{tikzcd}$$

in Sets. Then there exists a unique map $\phi\colon \emptyset \to A$ making the diagram

$$\begin{tikzcd}\emptyset \arrow[r, dashed, "\phi", "\exists!"'] & A\end{tikzcd}$$

commute, namely the inclusion map ι_A. ∎

2.2 Coproducts of Families of Sets

Let $\{A_i\}_{i\in I}$ be a family of sets.

Definition 2.2.1 ► Disjoint Unions of Families

The **disjoint union of the family** $\{A_i\}_{i\in I}$ is the pair $(\coprod_{i\in I} A_i, \{\mathrm{inj}_i\}_{i\in I})$ consisting of:

- *The Colimit.* The set $\coprod_{i\in I} A_i$ defined by
$$\coprod_{i\in I} A_i \overset{\text{def}}{=} \left\{ (i,x) \in I \times (\bigcup_{i\in I} A_i) \;\middle|\; x \in A_i \right\}.$$
- *The Cocone.* The collection
$$\left\{ \mathrm{inj}_i\colon A_i \longrightarrow \coprod_{i\in I} A_i \right\}_{i\in I}$$
of maps given by
$$\mathrm{inj}_i(x) \overset{\text{def}}{=} (i,x)$$
for each $x \in A_i$ and each $i \in I$.

Proof 2.2.2 ▶ Proof of Definition 2.2.1

We claim that $\coprod_{i\in I} A_i$ is the categorical coproduct of $\{A_i\}_{i\in I}$ in Sets. Indeed, suppose we have, for each $i \in I$, a diagram of the form

$$\begin{tikzcd}
& C \\
A_i \arrow[ur, "\iota_i"] \arrow[r, "\mathrm{inj}_i"'] & \coprod\limits_{i\in I} A_i
\end{tikzcd}$$

in Sets. Then there exists a unique map $\phi\colon \coprod_{i\in I} A_i \to C$ making the diagram

$$\begin{tikzcd}
& C \\
A_i \arrow[ur, "\iota_i"] \arrow[r, "\mathrm{inj}_i"'] & \coprod\limits_{i\in I} A_i \arrow[u, dashed, "\phi", "\exists!"']
\end{tikzcd}$$

commute, being uniquely determined by the condition $\phi \circ \mathrm{inj}_i = \iota_i$ for each $i \in I$ via

$$\phi((i, x)) = \iota_i(x)$$

for each $(i, x) \in \coprod_{i\in I} A_i$. □

Proposition 2.2.3 ▶ Properties of Coproducts of Families of Sets

Let $\{A_i\}_{i\in I}$ be a family of sets.

1. *Functoriality*. The assignment $\{A_i\}_{i\in I} \mapsto \coprod_{i\in I} A_i$ defines a functor

$$\coprod_{i\in I}\colon \mathrm{Fun}(I_{\mathrm{disc}}, \mathsf{Sets}) \to \mathsf{Sets}$$

   where

   - *Action on Objects*. For each $(A_i)_{i\in I} \in \mathrm{Obj}(\mathrm{Fun}(I_{\mathrm{disc}}, \mathsf{Sets}))$, we have

$$\left[\coprod_{i\in I}\right]((A_i)_{i\in I}) \overset{\mathrm{def}}{=} \coprod_{i\in I} A_i$$

- *Action on Morphisms.* For each $(A_i)_{i\in I}, (B_i)_{i\in I} \in \mathrm{Obj}(\mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}))$, the action on Hom-sets

$$\Big(\coprod_{i\in I}\Big)_{(A_i)_{i\in I},(B_i)_{i\in I}} \colon \mathrm{Nat}((A_i)_{i\in I}, (B_i)_{i\in I}) \to \mathsf{Sets}\Big(\coprod_{i\in I} A_i, \coprod_{i\in I} B_i\Big)$$

of $\coprod_{i\in I}$ at $((A_i)_{i\in I}, (B_i)_{i\in I})$ is defined by sending a map

$$\{f_i \colon A_i \to B_i\}_{i\in I}$$

in $\mathrm{Nat}((A_i)_{i\in I}, (B_i)_{i\in I})$ to the map of sets

$$\coprod_{i\in I} f_i \colon \coprod_{i\in I} A_i \to \coprod_{i\in I} B_i$$

defined by

$$\left[\coprod_{i\in I} f_i\right](i, a) \overset{\text{def}}{=} f_i(a)$$

for each $(i, a) \in \coprod_{i\in I} A_i$.

PROOF 2.2.4 ► PROOF OF PROPOSITION 2.2.3

Item 1: Functoriality

This follows from **??**, **??** of **??**. □

2.3 Binary Coproducts

Let A and B be sets.

DEFINITION 2.3.1 ► COPRODUCTS OF SETS

The **coproduct**[1] **of** A **and** B is the pair $(A \coprod B, \{\mathrm{inj}_1, \mathrm{inj}_2\})$ consisting of:

- *The Colimit.* The set $A \coprod B$ defined by

$$\begin{aligned} A \coprod B &\overset{\text{def}}{=} \coprod_{z\in\{A,B\}} z \\ &\cong \{(0, a) \mid a \in A\} \cup \{(1, b) \mid b \in B\}. \end{aligned}$$

- *The Cocone.* The maps

$$\begin{aligned}\mathsf{inj}_1\colon A &\longrightarrow A \coprod B,\\ \mathsf{inj}_2\colon B &\longrightarrow A \coprod B,\end{aligned}$$

given by

$$\begin{aligned}\mathsf{inj}_1(a) &\overset{\text{def}}{=} (0, a),\\ \mathsf{inj}_2(b) &\overset{\text{def}}{=} (1, b),\end{aligned}$$

for each $a \in A$ and each $b \in B$.

---

[1] *Further Terminology:* Also called the **disjoint union of** A **and** B, or the **binary disjoint union of** A **and** B, for emphasis.

Proof 2.3.2 ► Proof of Definition 2.3.1

We claim that $A \coprod B$ is the categorical coproduct of A and B in Sets. Indeed, suppose we have a diagram of the form

$$\begin{tikzcd} & C & \\ A \arrow[ur, bend left, "\iota_A"] \arrow[r, "\mathsf{inj}_A"'] & A \coprod B & B \arrow[ul, bend right, "\iota_B"'] \arrow[l, "\mathsf{inj}_B"] \end{tikzcd}$$

in Sets. Then there exists a unique map $\phi\colon A \coprod B \to C$ making the diagram

$$\begin{tikzcd} & C & \\ A \arrow[ur, bend left, "\iota_A"] \arrow[r, "\mathsf{inj}_A"'] & A \coprod B \arrow[u, dashed, "\phi", "\exists!"'] & B \arrow[ul, bend right, "\iota_B"'] \arrow[l, "\mathsf{inj}_B"] \end{tikzcd}$$

commute, being uniquely determined by the conditions

$$\begin{aligned}\phi \circ \mathsf{inj}_A &= \iota_A,\\ \phi \circ \mathsf{inj}_B &= \iota_B\end{aligned}$$

via

$$\phi(x) = \begin{cases} \iota_A(a) & \text{if } x = (0, a),\\ \iota_B(b) & \text{if } x = (1, b)\end{cases}$$

for each $x \in A \coprod B$.

PROPOSITION 2.3.3 ► PROPERTIES OF COPRODUCTS OF SETS

Let A, B, C, and X be sets.

1. *Functoriality.* The assignment $A, B, (A, B) \mapsto A \coprod B$ defines functors

$$\begin{aligned} A \coprod - &\colon \mathsf{Sets} \to \mathsf{Sets}, \\ - \coprod B &\colon \mathsf{Sets} \to \mathsf{Sets}, \\ -_1 \coprod -_2 &\colon \mathsf{Sets} \times \mathsf{Sets} \to \mathsf{Sets}, \end{aligned}$$

   where $-_1 \coprod -_2$ is the functor where

   - *Action on Objects.* For each $(A, B) \in \mathrm{Obj}(\mathsf{Sets} \times \mathsf{Sets})$, we have

$$[-_1 \coprod -_2](A, B) \overset{\text{def}}{=} A \coprod B.$$

   - *Action on Morphisms.* For each $(A, B), (X, Y) \in \mathrm{Obj}(\mathsf{Sets})$, the action on Hom-sets

$$\coprod_{(A,B),(X,Y)} \colon \mathsf{Sets}(A, X) \times \mathsf{Sets}(B, Y) \to \mathsf{Sets}(A \coprod B, X \coprod Y)$$

     of $\coprod$ at $((A, B), (X, Y))$ is defined by sending (f, g) to the function

$$f \coprod g \colon A \coprod B \to X \coprod Y$$

     defined by

$$[f \coprod g](x) \overset{\text{def}}{=} \begin{cases} (0, f(a)) & \text{if } x = (0, a), \\ (1, g(b)) & \text{if } x = (1, b), \end{cases}$$

     for each $x \in A \coprod B$.

   and where $A \coprod -$ and $- \coprod B$ are the partial functors of $-_1 \coprod -_2$ at $A, B \in \mathrm{Obj}(\mathsf{Sets})$.

2. *Associativity.* We have an isomorphism of sets

$$(A \coprod B) \coprod C \cong A \coprod (B \coprod C),$$

   natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

3. *Unitality.* We have isomorphisms of sets

$$\begin{aligned} A \coprod \emptyset &\cong A, \\ \emptyset \coprod A &\cong A, \end{aligned}$$

   natural in $A \in \mathrm{Obj}(\mathsf{Sets})$.

4. *Commutativity.* We have an isomorphism of sets

$$A \coprod B \cong B \coprod A,$$

   natural in $A, B \in \mathrm{Obj}(\mathsf{Sets})$.

5. *Symmetric Monoidality.* The triple $(\mathsf{Sets}, \coprod, \emptyset)$ is a symmetric monoidal category.

Proof 2.3.4 ► Proof of Proposition 2.3.3

Item 1: Functoriality

This follows from **??**, **??** of **??**.

Item 2: Associativity

Clear.

Item 3: Unitality

Clear.

Item 4: Commutativity

Clear.

Item 5: Symmetric Monoidality

Omitted. □

2.4 Pushouts

Let A, B, and C be sets and let $f\colon C \to A$ and $g\colon C \to B$ be functions.

DEFINITION 2.4.1 ► PUSHOUTS OF SETS

The **pushout of** A **and** B **over** C **along** f **and** g[1] is the pair[2] $(A \coprod_C B, \{\mathrm{inj}_1, \mathrm{inj}_2\})$ consisting of:

- *The Colimit.* The set $A \coprod_C B$ defined by

  $$A \textstyle\coprod_C B \overset{\text{def}}{=} A \coprod B/{\sim_C},$$

  where $\sim_C$ is the equivalence relation on $A \coprod B$ generated by $(0, f(c)) \sim_C (1, g(c))$.

- *The Cocone.* The maps

  $$\begin{aligned}\mathrm{inj}_1 &: A \longrightarrow A \textstyle\coprod_C B,\\ \mathrm{inj}_2 &: B \longrightarrow A \textstyle\coprod_C B\end{aligned}$$

  given by

  $$\begin{aligned}\mathrm{inj}_1(a) &\overset{\text{def}}{=} [(0, a)]\\ \mathrm{inj}_2(b) &\overset{\text{def}}{=} [(1, b)]\end{aligned}$$

  for each $a \in A$ and each $b \in B$.

---

[1] *Further Terminology:* Also called the **fibre coproduct of** A **and** B **over** C **along** f **and** g.

[2] *Further Notation:* Also written $A \coprod_{f,C,g} B$.

PROOF 2.4.2 ► PROOF OF DEFINITION 2.4.1

We claim that $A \coprod_C B$ is the categorical pushout of A and B over C with respect to (f, g) in Sets. First we need to check that the relevant pushout diagram commutes, i.e. that we have

$$\mathrm{inj}_1 \circ f = \mathrm{inj}_2 \circ g, \qquad \begin{tikzcd} A \coprod_C B & B \arrow[l, "\mathrm{inj}_2"'] \\ A \arrow[u, "\mathrm{inj}_1"] & C. \arrow[l, "f"] \arrow[u, "g"'] \end{tikzcd}$$

Indeed, given $c \in C$, we have

$$\begin{aligned}[\mathsf{inj}_1 \circ f](c) &= \mathsf{inj}_1(f(c)) \\ &= [(0, f(c))] \\ &= [(1, g(c))] \\ &= \mathsf{inj}_2(g(c)) \\ &= [\mathsf{inj}_2 \circ g](c),\end{aligned}$$

where $[(0, f(c))] = [(1, g(c))]$ by the definition of the relation $\sim$ on $A \coprod B$. Next, we prove that $A \coprod_C B$ satisfies the universal property of the pushout. Suppose we have a diagram of the form

$$\begin{tikzcd}
P & & \\
& A \coprod_C B \arrow[dr, phantom, "\ulcorner", very near start] & B \arrow[l, "\mathsf{inj}_2" description] \arrow[ull, bend right, "\iota_2"'] \\
& A \arrow[u, "\mathsf{inj}_1" description] \arrow[uul, bend left, "\iota_1"] & C \arrow[l, "f"] \arrow[u, "g"']
\end{tikzcd}$$

in Sets. Then there exists a unique map $\phi \colon A \coprod_C B \longrightarrow P$ making the diagram

$$\begin{tikzcd}
P & & \\
& A \coprod_C B \arrow[ul, dashed, "\exists!"', "\phi"] \arrow[dr, phantom, "\ulcorner", very near start] & B \arrow[l, "\mathsf{inj}_2" description] \arrow[ull, bend right, "\iota_2"'] \\
& A \arrow[u, "\mathsf{inj}_1" description] \arrow[uul, bend left, "\iota_1"] & C \arrow[l, "f"] \arrow[u, "g"']
\end{tikzcd}$$

commute, being uniquely determined by the conditions

$$\begin{aligned}\phi \circ \mathsf{inj}_1 &= \iota_1, \\ \phi \circ \mathsf{inj}_2 &= \iota_2\end{aligned}$$

via

$$\phi(x) = \begin{cases} \iota_1(a) & \text{if } x = [(0,a)], \\ \iota_2(b) & \text{if } x = [(1,b)] \end{cases}$$

for each $x \in A \coprod_C B$, where the well-definedness of ϕ is guaranteed by the equality $\iota_1 \circ f = \iota_2 \circ g$ and the definition of the relation $\sim$ on $A \coprod B$ as follows:

1. *Case 1:* Suppose we have $x = [(0,a)] = [(0,a')]$ for some $a, a' \in A$. Then, by Remark 2.4.3, we have a sequence
$$(0,a) \sim' x_1 \sim' \cdots \sim' x_n \sim' (0,a').$$
2. *Case 2:* Suppose we have $x = [(1,b)] = [(1,b')]$ for some $b, b' \in B$. Then, by Remark 2.4.3, we have a sequence
$$(1,b) \sim' x_1 \sim' \cdots \sim' x_n \sim' (1,b').$$
3. *Case 3:* Suppose we have $x = [(0,a)] = [(1,b)]$ for some $a \in A$ and $b \in B$. Then, by Remark 2.4.3, we have a sequence
$$(0,a) \sim' x_1 \sim' \cdots \sim' x_n \sim' (1,b).$$

In all these cases, we declare $x \sim' y$ iff there exists some $c \in C$ such that $x = (0, f(c))$ and $y = (1, g(c))$ or $x = (1, g(c))$ and $y = (0, f(c))$. Then, the equality $\iota_1 \circ f = \iota_2 \circ g$ gives

$$\begin{aligned} \phi([x]) &= \phi([(0, f(c))]) \\ &\overset{\text{def}}{=} \iota_1(f(c)) \\ &= \iota_2(g(c)) \\ &\overset{\text{def}}{=} \phi([(1, g(c))]) \\ &= \phi([y]), \end{aligned}$$

with the case where $x = (1, g(c))$ and $y = (0, f(c))$ similarly giving $\phi([x]) = \phi([y])$. Thus, if $x \sim' y$, then $\phi([x]) = \phi([y])$. Applying this equality pairwise to the sequences

$$\begin{aligned} (0,a) \sim' x_1 \sim' \cdots \sim' x_n \sim' (0,a'), \\ (1,b) \sim' x_1 \sim' \cdots \sim' x_n \sim' (1,b'), \\ (0,a) \sim' x_1 \sim' \cdots \sim' x_n \sim' (1,b) \end{aligned}$$

gives

$$\begin{aligned}\phi([(0,a)]) &= \phi([(0,a')]),\\ \phi([(1,b)]) &= \phi([(1,b')]),\\ \phi([(0,a)]) &= \phi([(1,b)]),\end{aligned}$$

showing ϕ to be well-defined. □

Remark 2.4.3 ▶ Unwinding Definition 2.4.1

In detail, by Equivalence Relations and Apartness Relations, Construction 4.2.2, the relation $\sim$ of Definition 2.4.1 is given by declaring $a \sim b$ iff one of the following conditions is satisfied:

- We have $a, b \in A$ and $a = b$;
- We have $a, b \in B$ and $a = b$;
- There exist $x_1, \ldots, x_n \in A \coprod B$ such that $a \sim' x_1 \sim' \cdots \sim' x_n \sim' b$, where we declare $x \sim' y$ if one of the following conditions is satisfied:
  1. There exists $c \in C$ such that $x = (0, f(c))$ and $y = (1, g(c))$.
  2. There exists $c \in C$ such that $x = (1, g(c))$ and $y = (0, f(c))$.

  That is: we require the following condition to be satisfied:

  (★) There exist $x_1, \ldots, x_n \in A \coprod B$ satisfying the following conditions:
    1. There exists $c_0 \in C$ satisfying one of the following conditions:
       (a) We have $a = f(c_0)$ and $x_1 = g(c_0)$.
       (b) We have $a = g(c_0)$ and $x_1 = f(c_0)$.
    2. For each $1 \leq i \leq n-1$, there exists $c_i \in C$ satisfying one of the following conditions:
       (a) We have $x_i = f(c_i)$ and $x_{i+1} = g(c_i)$.
       (b) We have $x_i = g(c_i)$ and $x_{i+1} = f(c_i)$.
    3. There exists $c_n \in C$ satisfying one of the following conditions:
       (a) We have $x_n = f(c_n)$ and $b = g(c_n)$.
       (b) We have $x_n = g(c_n)$ and $b = f(c_n)$.

EXAMPLE 2.4.4 ► EXAMPLES OF PUSHOUTS OF SETS

Here are some examples of pushouts of sets.

1. *Wedge Sums of Pointed Sets.* The wedge sum of two pointed sets of Pointed Sets, Definition 3.3.1 is an example of a pushout of sets.
2. *Intersections via Unions.* Let $A, B \subset X$. We have a bijection of sets

$$A \cup B \cong A \coprod_{A \cap B} B, \qquad \begin{tikzcd} A \cup B \arrow[dr, phantom, "\ulcorner", very near start] & B \arrow[l] \\ A \arrow[u] & A \cap B. \arrow[l, hook'] \arrow[u, hook] \end{tikzcd}$$

PROOF 2.4.5 ► PROOF OF EXAMPLE 2.4.4

Item 1: Wedge Sums of Pointed Sets

Follows by definition.

Item 2: Intersections via Unions

Indeed, $A \coprod_{A \cap B} B$ is the quotient of $A \coprod B$ by the equivalence relation obtained by declaring $(0, a) \sim (1, b)$ iff $a = b \in A \cap B$, which is in bijection with $A \cup B$ via the map with $[(0, a)] \mapsto a$ and $[(1, b)] \mapsto b$. □

PROPOSITION 2.4.6 ► PROPERTIES OF PUSHOUTS OF SETS

Let $A, B, C,$ and X be sets.

1. *Functoriality.* The assignment $(A, B, C, f, g) \mapsto A \coprod_{f,C,g} B$ defines a functor

$$-_1 \coprod_{-_3} -_1 \colon \mathsf{Fun}(\mathcal{P}, \mathsf{Sets}) \to \mathsf{Sets},$$

where $\mathcal{P}$ is the category that looks like this:

$$\begin{tikzcd} & \bullet \\ \bullet & \bullet. \arrow[l] \arrow[u] \end{tikzcd}$$

In particular, the action on morphisms of $-_1 \coprod_{-_3} -_1$ is given by sending a morphism

$$\begin{tikzcd}
A \coprod_C B \arrow[dr, phantom, "\ulcorner", very near start] & & B \arrow[ll] \arrow[dr, "\psi"] & \\
& A' \coprod_{C'} B' \arrow[dr, phantom, "\ulcorner", very near start] & & B' \arrow[ll] \\
A \arrow[uu] \arrow[dr, "\phi"'] & & C \arrow[uu, "g"] \arrow[ll, "f"'] \arrow[dr, "\chi"] & \\
& A' \arrow[uu] & & C' \arrow[uu, "g'"'] \arrow[ll, "f'"]
\end{tikzcd}$$

in Fun($\mathcal{P}$, Sets) to the map $\xi\colon A \coprod_C B \xrightarrow{\exists!} A' \coprod_{C'} B'$ given by

$$\xi(x) \overset{\text{def}}{=} \begin{cases} \phi(a) & \text{if } x = [(0, a)], \\ \psi(b) & \text{if } x = [(1, b)] \end{cases}$$

for each $x \in A \coprod_C B$, which is the unique map making the diagram

$$\begin{tikzcd}
A \coprod_C B \arrow[dr, phantom, "\ulcorner", very near start] \arrow[dr, dashed] & & B \arrow[ll] \arrow[dr, "\psi"] & \\
& A' \coprod_{C'} B' \arrow[dr, phantom, "\ulcorner", very near start] & & B' \arrow[ll] \\
A \arrow[uu] \arrow[dr, "\phi"'] & & C \arrow[uu, "g"] \arrow[ll, "f"'] \arrow[dr, "\chi"] & \\
& A' \arrow[uu] & & C' \arrow[uu, "g'"'] \arrow[ll, "f'"]
\end{tikzcd}$$

commute.

2. *Associativity.* Given a diagram

$$\begin{tikzcd}
A & & B & & C \\
& X \arrow[ul, "f"] \arrow[ur, "g"'] & & Y \arrow[ul, "h"] \arrow[ur, "k"'] &
\end{tikzcd}$$

in Sets, we have isomorphisms of sets

$$(A \coprod_X B) \coprod_Y C \cong (A \coprod_X B) \coprod_B (B \coprod_Y C) \cong A \coprod_X (B \coprod_Y C),$$

where these pullbacks are built as in the diagrams

$$\begin{tikzcd}
& & (A \coprod_X B) \coprod_Y C \arrow[d, phantom, "\wedge" near start] & & \\
& A \coprod_X B \arrow[ur] \arrow[d, phantom, "\wedge" near start] & & & \\
A \arrow[ur] & & B \arrow[ul] & & C, \arrow[uull] \\
& X \arrow[ul, "f"] \arrow[ur, "g"'] & & Y \arrow[ul, "h"] \arrow[ur, "k"'] &
\end{tikzcd}
\quad
\begin{tikzcd}
& & (A \coprod_X B) \coprod_B (B \coprod_Y C) \arrow[d, phantom, "\wedge" near start] & & \\
& A \coprod_X B \arrow[ur] \arrow[d, phantom, "\wedge" near start] & & B \coprod_Y C \arrow[ul] \arrow[d, phantom, "\wedge" near start] & \\
A \arrow[ur] & & B \arrow[ul] \arrow[ur] & & C, \arrow[ul] \\
& X \arrow[ul, "f"] \arrow[ur, "g"'] & & Y \arrow[ul, "h"] \arrow[ur, "k"'] &
\end{tikzcd}
\quad
\begin{tikzcd}
& & A \coprod_X (B \coprod_Y C) \arrow[d, phantom, "\wedge" near start] & & \\
& & & B \coprod_Y C \arrow[ul] \arrow[d, phantom, "\wedge" near start] & \\
A \arrow[uurr] & & B \arrow[ur] & & C. \arrow[ul] \\
& X \arrow[ul, "f"] \arrow[ur, "g"'] & & Y \arrow[ul, "h"] \arrow[ur, "k"'] &
\end{tikzcd}$$

3. *Unitality.* We have isomorphisms of sets

$$\begin{tikzcd}
A \arrow[r, equal] & A \\
X \arrow[u, "f"] \arrow[r, equal] & X \arrow[u, "f"'] \arrow[ul, phantom, "\ulcorner" very near end]
\end{tikzcd}
\qquad
\begin{aligned}
X \coprod_X A &\cong A, \\
A \coprod_X X &\cong A,
\end{aligned}
\qquad
\begin{tikzcd}
A & X \arrow[l, "f"'] \\
X \arrow[u, equal] & X. \arrow[l, "f"] \arrow[u, equal] \arrow[ul, phantom, "\ulcorner" very near end]
\end{tikzcd}$$

4. *Commutativity.* We have an isomorphism of sets

$$\begin{tikzcd}
A \coprod_X B & B \arrow[l] \\
A \arrow[u] & X, \arrow[l, "f"] \arrow[u, "g"'] \arrow[ul, phantom, "\ulcorner" very near end]
\end{tikzcd}
\qquad
A \coprod_X B \cong B \coprod_X A
\qquad
\begin{tikzcd}
B \coprod_X A & A \arrow[l] \\
B \arrow[u] & X. \arrow[l, "g"] \arrow[u, "f"'] \arrow[ul, phantom, "\ulcorner" very near end]
\end{tikzcd}$$

5. *Interaction With Coproducts.* We have

$$A \coprod_{\emptyset} B \cong A \coprod B,
\qquad
\begin{tikzcd}
A \coprod B & B \arrow[l] \\
A \arrow[u] & \emptyset. \arrow[l, hook', "\iota_A"] \arrow[u, hook, "\iota_B"'] \arrow[ul, phantom, "\ulcorner" very near end]
\end{tikzcd}$$

6. *Symmetric Monoidality.* The triple $(\text{Sets}, \coprod_X, X)$ is a symmetric monoidal category.

Proof 2.4.7 ► Proof of Proposition 2.4.6

Item 1: Functoriality

This is a special case of functoriality of co/limits, **??**, **??** of **??**, with the explicit expression for ξ following from the commutativity of the cube pushout diagram.

Item 2: Associativity

Omitted.

Item 3: Unitality

Omitted.

Item 4: Commutativity

Clear.

Item 5: Interaction With Coproducts

Clear.

Item 6: Symmetric Monoidality

Omitted. □

2.5 Coequalisers

Let A and B be sets and let $f, g\colon A \rightrightarrows B$ be functions.

Definition 2.5.1 ► Coequalisers of Sets

The **coequaliser of** f **and** g is the pair $(\mathrm{CoEq}(f,g), \mathrm{coeq}(f,g))$ consisting of:

- *The Colimit.* The set $\mathrm{CoEq}(f,g)$ defined by

$$\mathrm{CoEq}(f,g) \overset{\text{def}}{=} B/{\sim},$$

where $\sim$ is the equivalence relation on B generated by $f(a) \sim g(a)$.

- *The Cocone.* The map

$$\mathrm{coeq}(f,g)\colon B \to \mathrm{CoEq}(f,g)$$

given by the quotient map $\pi\colon B \twoheadrightarrow B/{\sim}$ with respect to the equivalence relation generated by $f(a) \sim g(a)$.

PROOF 2.5.2 ▶ PROOF OF DEFINITION 2.5.1

We claim that $\mathrm{CoEq}(f,g)$ is the categorical coequaliser of f and g in Sets. First we need to check that the relevant coequaliser diagram commutes, i.e. that we have

$$\mathrm{coeq}(f,g) \circ f = \mathrm{coeq}(f,g) \circ g.$$

Indeed, we have

$$\begin{aligned}
[\mathrm{coeq}(f,g) \circ f](a) &\overset{\text{def}}{=} [\mathrm{coeq}(f,g)](f(a)) \\
&\overset{\text{def}}{=} [f(a)] \\
&= [g(a)] \\
&\overset{\text{def}}{=} [\mathrm{coeq}(f,g)](g(a)) \\
&\overset{\text{def}}{=} [\mathrm{coeq}(f,g) \circ g](a)
\end{aligned}$$

for each $a \in A$. Next, we prove that $\mathrm{CoEq}(f,g)$ satisfies the universal property of the coequaliser. Suppose we have a diagram of the form

$$\begin{tikzcd}
A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, hook, "\mathrm{coeq}(f,g)"] \arrow[rd, "c"'] & \mathrm{CoEq}(f,g) \\
& & C
\end{tikzcd}$$

in Sets. Then, since $c(f(a)) = c(g(a))$ for each $a \in A$, it follows from Equivalence Relations and Apartness Relations, Items 4 and 5 of Proposition 5.2.3 that there exists a unique map $\mathrm{CoEq}(f,g) \xrightarrow{\exists!} C$ making the diagram

$$\begin{tikzcd}
A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, hook, "\mathrm{coeq}(f,g)"] \arrow[rd, "c"'] & \mathrm{CoEq}(f,g) \arrow[d, dashed, "\exists!"] \\
& & C
\end{tikzcd}$$

commute. ∎

REMARK 2.5.3 ► UNWINDING DEFINITION 2.5.1

In detail, by Equivalence Relations and Apartness Relations, Construction 4.2.2, the relation $\sim$ of Definition 2.5.1 is given by declaring $a \sim b$ iff one of the following conditions is satisfied:

- We have $a = b$;
- There exist $x_1, \ldots, x_n \in B$ such that $a \sim' x_1 \sim' \cdots \sim' x_n \sim' b$, where we declare $x \sim' y$ if one of the following conditions is satisfied:

  1. There exists $z \in A$ such that $x = f(z)$ and $y = g(z)$.
  2. There exists $z \in A$ such that $x = g(z)$ and $y = f(z)$.

  That is: we require the following condition to be satisfied:

  ($\star$) There exist $x_1, \ldots, x_n \in B$ satisfying the following conditions:

  1. There exists $z_0 \in A$ satisfying one of the following conditions:
     (a) We have $a = f(z_0)$ and $x_1 = g(z_0)$.
     (b) We have $a = g(z_0)$ and $x_1 = f(z_0)$.
  2. For each $1 \leq i \leq n - 1$, there exists $z_i \in A$ satisfying one of the following conditions:
     (a) We have $x_i = f(z_i)$ and $x_{i+1} = g(z_i)$.
     (b) We have $x_i = g(z_i)$ and $x_{i+1} = f(z_i)$.
  3. There exists $z_n \in A$ satisfying one of the following conditions:
     (a) We have $x_n = f(z_n)$ and $b = g(z_n)$.
     (b) We have $x_n = g(z_n)$ and $b = f(z_n)$.

EXAMPLE 2.5.4 ► EXAMPLES OF COEQUALISERS OF SETS

Here are some examples of coequalisers of sets.

1. *Quotients by Equivalence Relations.* Let R be an equivalence relation on a set X. We have a bijection of sets

$$X/{\sim_R} \cong \mathrm{CoEq}(R \hookrightarrow X \times X \underset{\mathrm{pr}_2}{\overset{\mathrm{pr}_1}{\rightrightarrows}} X).$$

PROOF 2.5.5 ▶ PROOF OF EXAMPLE 2.5.4

Item 1: Quotients by Equivalence Relations

See [Pro24z].

PROPOSITION 2.5.6 ▶ PROPERTIES OF COEQUALISERS OF SETS

Let A, B, and C be sets.

1. *Associativity.* We have isomorphisms of sets[1]

$$\underbrace{\mathrm{CoEq}(\mathrm{coeq}(f,g)\circ f,\mathrm{coeq}(f,g)\circ h)}_{=\mathrm{CoEq}(\mathrm{coeq}(f,g)\circ g,\mathrm{coeq}(f,g)\circ h)} \cong \mathrm{CoEq}(f,g,h) \cong \underbrace{\mathrm{CoEq}(\mathrm{coeq}(g,h)\circ f,\mathrm{coeq}(g,h)\circ g)}_{=\mathrm{CoEq}(\mathrm{coeq}(g,h)\circ f,\mathrm{coeq}(g,h)\circ h)},$$

where $\mathrm{CoEq}(f,g,h)$ is the colimit of the diagram

$$\begin{tikzcd} A \arrow[r, shift left=2, "f"] \arrow[r, "g" description] \arrow[r, shift right=2, "h"'] & B \end{tikzcd}$$

in Sets.

2. *Unitality.* We have an isomorphism of sets

$$\mathrm{CoEq}(f,f) \cong B.$$

3. *Commutativity.* We have an isomorphism of sets

$$\mathrm{CoEq}(f,g) \cong \mathrm{CoEq}(g,f).$$

4. *Interaction With Composition.* Let

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, shift left, "h"] \arrow[r, shift right, "k"'] & C \end{tikzcd}$$

be functions. We have a surjection

$$\mathrm{CoEq}(h\circ f,k\circ g) \twoheadrightarrow \mathrm{CoEq}(\mathrm{coeq}(h,k)\circ h\circ f,\mathrm{coeq}(h,k)\circ k\circ g)$$

exhibiting $\mathrm{CoEq}(\mathrm{coeq}(h,k)\circ h\circ f,\mathrm{coeq}(h,k)\circ k\circ g)$ as a quotient of $\mathrm{CoEq}(h\circ f,k\circ g)$ by the relation generated by declaring $h(y)\sim k(y)$ for each $y\in B$.

---

[1] That is, the following three ways of forming "the" coequaliser of (f, g, h) agree:

(a) Take the coequaliser of (f, g, h), i.e. the colimit of the diagram

$$\begin{tikzcd} A \arrow[r, shift left=2, "f"] \arrow[r, "g" description] \arrow[r, shift right=2, "h"'] & B \end{tikzcd}$$

in Sets.

(b) First take the coequaliser of f and g, forming a diagram

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, two heads, "\mathrm{coeq}(f,g)"] & \mathrm{CoEq}(f, g) \end{tikzcd}$$

and then take the coequaliser of the composition

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "h"'] & B \arrow[r, two heads, "\mathrm{coeq}(f,g)"] & \mathrm{CoEq}(f, g), \end{tikzcd}$$

obtaining a quotient

$$\mathrm{CoEq}(\mathrm{coeq}(f, g) \circ f, \mathrm{coeq}(f, g) \circ h) = \mathrm{CoEq}(\mathrm{coeq}(f, g) \circ g, \mathrm{coeq}(f, g) \circ h)$$

of $\mathrm{CoEq}(f, g)$

(c) First take the coequaliser of g and h, forming a diagram

$$\begin{tikzcd} A \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & B \arrow[r, two heads, "\mathrm{coeq}(g,h)"] & \mathrm{CoEq}(g, h) \end{tikzcd}$$

and then take the coequaliser of the composition

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, two heads, "\mathrm{coeq}(g,h)"] & \mathrm{CoEq}(g, h), \end{tikzcd}$$

obtaining a quotient

$$\mathrm{CoEq}(\mathrm{coeq}(g, h) \circ f, \mathrm{coeq}(g, h) \circ g) = \mathrm{CoEq}(\mathrm{coeq}(g, h) \circ f, \mathrm{coeq}(g, h) \circ h)$$

of $\mathrm{CoEq}(g, h)$.

Proof 2.5.7 ► Proof of Proposition 2.5.6

Item 1: Associativity

Omitted.

Item 2: Unitality

Clear.

Item 3: Commutativity

Clear.

Item 4: Interaction With Composition

Omitted. □

3 Operations With Sets

3.1 The Empty Set

Definition 3.1.1 ► The Empty Set

The **empty set** is the set $\emptyset$ defined by

$$\emptyset \overset{\text{def}}{=} \{x \in X \mid x \neq x\},$$

where A is the set in the set existence axiom, **??** of **??**.

3.2 Singleton Sets

Let X be a set.

Definition 3.2.1 ► Singleton Sets

The **singleton set containing** X is the set $\{X\}$ defined by

$$\{X\} \overset{\text{def}}{=} \{X, X\},$$

where $\{X, X\}$ is the pairing of X with itself (Definition 3.3.1).

3.3 Pairings of Sets

Let X and Y be sets.

Definition 3.3.1 ► Pairings of Sets

The **pairing of X and Y** is the set $\{X, Y\}$ defined by

$$\{X, Y\} \overset{\text{def}}{=} \{x \in A \mid x = X \text{ or } x = Y\},$$

where A is the set in the axiom of pairing, **??** of **??**.

3.4 Ordered Pairs

Let A and B be sets.

Definition 3.4.1 ► Ordered Pairs

The **ordered pair associated to A and B** is the set (A, B) defined by

$$(A, B) \overset{\text{def}}{=} \{\{A\}, \{A, B\}\}.$$

Proposition 3.4.2 ► Properties of Ordered Pairs

Let A and B be sets.

1. *Uniqueness.* Let $A, B, C,$ and D be sets. The following conditions are equivalent:
    (a) We have $(A, B) = (C, D)$.
    (b) We have $A = C$ and $B = D$.

Proof 3.4.3 ► Proof of Proposition 3.4.2

Item 1: Uniqueness

See [Cie97, Theorem 1.2.3]. □

3.5 Sets of Maps

Let A and B be sets.

Definition 3.5.1 ► Sets of Maps

The **set of maps from A to B**[1] is the set $\mathrm{Hom}_{\mathsf{Sets}}(A, B)$[2] whose elements are the functions from A to B.

---

[1] *Further Terminology:* Also called the **Hom set from A to B**.

[2] *Further Notation:* Also written $\mathsf{Sets}(A, B)$.

Proposition 3.5.2 ► Properties of Sets of Maps

Let A and B be sets.

1. *Functoriality.* The assignments $X, Y, (X, Y) \mapsto \mathrm{Hom}_{\mathsf{Sets}}(X, Y)$ define functors

$$\begin{aligned} \mathrm{Hom}_{\mathsf{Sets}}(X, -)&\colon \mathsf{Sets} \to \mathsf{Sets},\\ \mathrm{Hom}_{\mathsf{Sets}}(-, Y)&\colon \mathsf{Sets}^{\mathsf{op}} \to \mathsf{Sets},\\ \mathrm{Hom}_{\mathsf{Sets}}(-_1, -_2)&\colon \mathsf{Sets}^{\mathsf{op}} \times \mathsf{Sets} \to \mathsf{Sets}. \end{aligned}$$

Proof 3.5.3 ► Proof of Proposition 3.5.2

Item 1: Functoriality

This follows from Categories, Items 2 and 5 of Proposition 1.6.2. □

3.6 Unions of Families

Let $\{A_i\}_{i\in I}$ be a family of sets.

Definition 3.6.1 ► Unions of Families

The **union of the family** $\{A_i\}_{i\in I}$ is the set $\bigcup_{i\in I} A_i$ defined by

$$\bigcup_{i\in I} A_i \overset{\text{def}}{=} \{x \in F \mid \text{there exists some } i \in I \text{ such that } x \in A_i\},$$

where F is the set in the axiom of union, **??** of **??**.

3.7 Binary Unions

Let A and B be sets.

Definition 3.7.1 ► Binary Unions

The **union**[1] **of A and B** is the set $A \cup B$ defined by

$$A \cup B \overset{\text{def}}{=} \bigcup_{z \in \{A,B\}} z.$$

---

[1] *Further Terminology*: Also called the **binary union of A and B**, for emphasis.

Proposition 3.7.2 ► Properties of Binary Unions

Let X be a set.

1. *Functoriality*. The assignments $U, V, (U,V) \mapsto U \cup V$ define functors

$$\begin{aligned} U \cup -\colon (\mathcal{P}(X), \subset) &\to (\mathcal{P}(X), \subset), \\ - \cup V\colon (\mathcal{P}(X), \subset) &\to (\mathcal{P}(X), \subset), \\ -_1 \cup -_2\colon (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \subset) &\to (\mathcal{P}(X), \subset), \end{aligned}$$

   where $-_1 \cup -_2$ is the functor where

   - *Action on Objects*. For each $(U,V) \in \mathcal{P}(X) \times \mathcal{P}(X)$, we have

$$[-_1 \cup -_2](U,V) \overset{\text{def}}{=} U \cup V.$$

   - *Action on Morphisms*. For each pair of morphisms

$$\begin{aligned} \iota_U &\colon U \hookrightarrow U', \\ \iota_V &\colon V \hookrightarrow V' \end{aligned}$$

     of $\mathcal{P}(X) \times \mathcal{P}(X)$, the image

$$\iota_U \cup \iota_V \colon U \cup V \hookrightarrow U' \cup V'$$

     of (ι_U, ι_V) by $\cup$ is the inclusion

$$U \cup V \subset U' \cup V'$$

     i.e. where we have

     ($\star$) If $U \subset U'$ and $V \subset V'$, then $U \cup V \subset U' \cup V'$.

and where $U \cup -$ and $- \cup V$ are the partial functors of $-_1 \cup -_2$ at $U, V \in \mathcal{P}(X)$.

2. *Via Intersections and Symmetric Differences.* We have an equality of sets

$$U \cup V = (U \vartriangle V) \vartriangle (U \cap V)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

3. *Associativity.* We have an equality of sets

$$(U \cup V) \cup W = U \cup (V \cup W)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

4. *Unitality.* We have equalities of sets

$$\begin{aligned} U \cup \emptyset &= U, \\ \emptyset \cup U &= U \end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

5. *Commutativity.* We have an equality of sets

$$U \cup V = V \cup U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

6. *Idempotency.* We have an equality of sets

$$U \cup U = U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

7. *Distributivity Over Intersections.* We have equalities of sets

$$\begin{aligned} U \cup (V \cap W) &= (U \cup V) \cap (U \cup W), \\ (U \cap V) \cup W &= (U \cup W) \cap (V \cup W) \end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

8. *Interaction With Characteristic Functions I.* We have

$$\chi_{U\cup V} = \max(\chi_U, \chi_V)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

9. *Interaction With Characteristic Functions II.* We have

$$\chi_{U\cup V} = \chi_U + \chi_V - \chi_{U\cap V}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

10. *Interaction With Powersets and Semirings.* The quintuple $(\mathcal{P}(X), \cup, \cap, \emptyset, X)$ is an idempotent commutative semiring.

PROOF 3.7.3 ► PROOF OF PROPOSITION 3.7.2

Item 1: Functoriality

See [Pro24an].

Item 2: Via Intersections and Symmetric Differences

See [Pro24ay].

Item 3: Associativity

See [Pro24ba].

Item 4: Unitality

This follows from [Pro24bd] and Item 5.

Item 5: Commutativity

See [Pro24bb].

Item 6: Idempotency

See [Pro24am].

Item 7: Distributivity Over Intersections

See [Pro24az].

Item 8: Interaction With Characteristic Functions I

See [Pro24k].

Item 9: Interaction With Characteristic Functions II

See [Pro24k].

Item 10: Interaction With Powersets and Semirings

This follows from Items 3 to 6 and Items 3 to 5, 7 and 8 of Proposition 3.9.2. ∎

3.8 Intersections of Families

Let $\mathcal{F}$ be a family of sets.

Definition 3.8.1 ► Intersections of Families

The **intersection of a family $\mathcal{F}$ of sets** is the set $\bigcap_{X\in\mathcal{F}} X$ defined by

$$\bigcap_{X\in\mathcal{F}} X \overset{\text{def}}{=} \left\{ z \in \bigcup_{X\in\mathcal{F}} X \,\middle|\, \text{for each } X \in \mathcal{F}\text{, we have } z \in X \right\}.$$

3.9 Binary Intersections

Let X and Y be sets.

Definition 3.9.1 ► Binary Intersections

The **intersection**[1] **of X and Y** is the set $X \cap Y$ defined by

$$X \cap Y \overset{\text{def}}{=} \bigcap_{z\in\{X,Y\}} z.$$

---

[1] *Further Terminology:* Also called the **binary intersection of X and Y**, for emphasis.

Proposition 3.9.2 ► Properties of Binary Intersections

Let X be a set.

1. *Functoriality.* The assignments $U, V, (U, V) \mapsto U \cap V$ define functors

$$\begin{aligned} U \cap - &\colon (\mathcal{P}(X), \subset) \to (\mathcal{P}(X), \subset), \\ - \cap V &\colon (\mathcal{P}(X), \subset) \to (\mathcal{P}(X), \subset), \\ -_1 \cap -_2 &\colon (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \subset) \to (\mathcal{P}(X), \subset), \end{aligned}$$

where $-_1 \cap -_2$ is the functor where

- *Action on Objects.* For each $(U, V) \in \mathcal{P}(X) \times \mathcal{P}(X)$, we have

$$[-_1 \cap -_2](U, V) \overset{\text{def}}{=} U \cap V.$$

- *Action on Morphisms.* For each pair of morphisms

$$\begin{aligned} \iota_U &\colon U \hookrightarrow U', \\ \iota_V &\colon V \hookrightarrow V' \end{aligned}$$

of $\mathcal{P}(X) \times \mathcal{P}(X)$, the image

$$\iota_U \cap \iota_V \colon U \cap V \hookrightarrow U' \cap V'$$

of (ι_U, ι_V) by $\cap$ is the inclusion

$$U \cap V \subset U' \cap V'$$

i.e. where we have

(★) If $U \subset U'$ and $V \subset V'$, then $U \cap V \subset U' \cap V'$.

and where $U \cap -$ and $- \cap V$ are the partial functors of $-_1 \cap -_2$ at $U, V \in \mathcal{P}(X)$.

2. *Adjointness.* We have adjunctions

$$\big(U \cap - \dashv \mathbf{Hom}_{\mathcal{P}(X)}(U, -)\big)\colon \quad \begin{tikzcd} \mathcal{P}(X) \arrow[r, bend left, "U\cap-"] \arrow[r, phantom, "\perp"] & \mathcal{P}(X), \arrow[l, bend left, "\mathbf{Hom}_{\mathcal{P}(X)}(U,-)"] \end{tikzcd}$$

$$\big(- \cap V \dashv \mathbf{Hom}_{\mathcal{P}(X)}(V, -)\big)\colon \quad \begin{tikzcd} \mathcal{P}(X) \arrow[r, bend left, "-\cap V"] \arrow[r, phantom, "\perp"] & \mathcal{P}(X), \arrow[l, bend left, "\mathbf{Hom}_{\mathcal{P}(X)}(V,-)"] \end{tikzcd}$$

where

$$\mathbf{Hom}_{\mathcal{P}(X)}(-_1, -_2)\colon \mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \to \mathcal{P}(X)$$

is the bifunctor defined by[1]

$$\mathbf{Hom}_{\mathcal{P}(X)}(U, V) \overset{\text{def}}{=} (X \setminus U) \cup V$$

witnessed by bijections

$$\begin{aligned}\mathrm{Hom}_{\mathcal{P}(X)}(U\cap V,W)&\cong\mathrm{Hom}_{\mathcal{P}(X)}(U,\mathbf{Hom}_{\mathcal{P}(X)}(V,W)),\\ \mathrm{Hom}_{\mathcal{P}(X)}(U\cap V,W)&\cong\mathrm{Hom}_{\mathcal{P}(X)}(V,\mathbf{Hom}_{\mathcal{P}(X)}(U,W)),\end{aligned}$$

natural in $U,V,W\in\mathcal{P}(X)$, i.e. where:

(a) The following conditions are equivalent:
  i. We have $U\cap V\subset W$.
  ii. We have $U\subset\mathbf{Hom}_{\mathcal{P}(X)}(V,W)$.
  iii. We have $U\subset(X\setminus V)\cup W$.

(b) The following conditions are equivalent:
  i. We have $V\cap U\subset W$.
  ii. We have $V\subset\mathbf{Hom}_{\mathcal{P}(X)}(U,W)$.
  iii. We have $V\subset(X\setminus U)\cup W$.

3. *Associativity.* We have an equality of sets

$$(U\cap V)\cap W=U\cap(V\cap W)$$

for each $X\in\mathrm{Obj}(\mathsf{Sets})$ and each $U,V,W\in\mathcal{P}(X)$.

4. *Unitality.* Let X be a set and let $U\in\mathcal{P}(X)$. We have equalities of sets

$$\begin{aligned}X\cap U&=U,\\ U\cap X&=U\end{aligned}$$

for each $X\in\mathrm{Obj}(\mathsf{Sets})$ and each $U\in\mathcal{P}(X)$.

5. *Commutativity.* We have an equality of sets

$$U\cap V=V\cap U$$

for each $X\in\mathrm{Obj}(\mathsf{Sets})$ and each $U,V\in\mathcal{P}(X)$.

6. *Idempotency.* We have an equality of sets

$$U\cap U=U$$

for each $X\in\mathrm{Obj}(\mathsf{Sets})$ and each $U\in\mathcal{P}(X)$.

7. *Distributivity Over Unions.* We have equalities of sets

$$U \cap (V \cup W) = (U \cap V) \cup (U \cap W),$$
$$(U \cup V) \cap W = (U \cap W) \cup (V \cap W)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

8. *Annihilation With the Empty Set.* We have an equality of sets

$$\emptyset \cap X = \emptyset,$$
$$X \cap \emptyset = \emptyset$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

9. *Interaction With Characteristic Functions I.* We have

$$\chi_{U \cap V} = \chi_U \chi_V$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

10. *Interaction With Characteristic Functions II.* We have

$$\chi_{U \cap V} = \min(\chi_U, \chi_V)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

11. *Interaction With Powersets and Monoids With Zero.* The quadruple $((\mathcal{P}(X), \emptyset), \cap, X)$ is a commutative monoid with zero.

12. *Interaction With Powersets and Semirings.* The quintuple $(\mathcal{P}(X), \cup, \cap, \emptyset, X)$ is an idempotent commutative semiring.

---

[1] For intuition regarding the expression defining $\mathbf{Hom}_{\mathcal{P}(X)}(U, V)$, see Remark 3.9.4.

PROOF 3.9.3 ► PROOF OF PROPOSITION 3.9.2

Item 1: Functoriality

See [Pro24al].

Item 2: Adjointness

See [MSE 267469].

Item 3: Associativity

See [Pro24s].

Item 4: Unitality

This follows from [Pro24w] and Item 5.

Item 5: Commutativity

See [Pro24t].

Item 6: Idempotency

See [Pro24ak].

Item 7: Distributivity Over Unions

See [Pro24aj].

Item 8: Annihilation With the Empty Set

This follows from [Pro24u] and Item 5.

Item 9: Interaction With Characteristic Functions I

See [Pro24h].

Item 10: Interaction With Characteristic Functions II

See [Pro24h].

Item 11: Interaction With Powersets and Monoids With Zero

This follows from Items 3 to 5 and 8.

Item 12: Interaction With Powersets and Semirings

This follows from Items 3 to 6 and Items 3 to 5, 7 and 8 of Proposition 3.9.2. ∎

REMARK 3.9.4 ► INTUITION FOR THE INTERNAL HOM OF $\mathcal{P}(X)$

Since intersections are the products in $\mathcal{P}(X)$ (Item 1 of Proposition 4.3.3), the left adjoint $\mathbf{Hom}_{\mathcal{P}(X)}(U, V)$ may be thought of as a function type $[U, V]$.
Then, under the Curry–Howard correspondence, the function type $[U, V]$ corresponds to implication $U \implies V$, which is logically equivalent to the statement $\neg U \vee V$. This in turn corresponds to the set $U^{\mathsf{c}} \vee V = (X \setminus U) \cup V$.

3.10 Differences

Let X and Y be sets.

DEFINITION 3.10.1 ► DIFFERENCES

The **difference of X and Y** is the set $X \setminus Y$ defined by

$$X \setminus Y \overset{\text{def}}{=} \{a \in X \mid a \notin Y\}.$$

PROPOSITION 3.10.2 ► PROPERTIES OF DIFFERENCES

Let X be a set.

1. *Functoriality.* The assignments $U, V, (U, V) \mapsto U \cap V$ define functors

$$\begin{aligned} U \setminus -&\colon (\mathcal{P}(X), \supset) \to (\mathcal{P}(X), \subset), \\ - \setminus V&\colon (\mathcal{P}(X), \subset) \to (\mathcal{P}(X), \subset), \\ -_1 \setminus -_2&\colon (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \supset) \to (\mathcal{P}(X), \subset), \end{aligned}$$

   where $-_1 \setminus -_2$ is the functor where

   - *Action on Objects.* For each $(U, V) \in \mathcal{P}(X) \times \mathcal{P}(X)$, we have

$$[-_1 \setminus -_2](U, V) \overset{\text{def}}{=} U \setminus V.$$

   - *Action on Morphisms.* For each pair of morphisms

$$\begin{aligned} \iota_A &\colon A \hookrightarrow B, \\ \iota_U &\colon U \hookrightarrow V \end{aligned}$$

     of $\mathcal{P}(X) \times \mathcal{P}(X)$, the image

$$\iota_U \setminus \iota_V \colon A \setminus V \hookrightarrow B \setminus U$$

     of (ι_U, ι_V) by $\setminus$ is the inclusion

$$A \setminus V \subset B \setminus U$$

     i.e. where we have

     (★) If $A \subset B$ and $U \subset V$, then $A \setminus V \subset B \setminus U$.

   and where $U \setminus -$ and $- \setminus V$ are the partial functors of $-_1 \setminus -_2$ at $U, V \in \mathcal{P}(X)$.

2. *De Morgan's Laws.* We have equalities of sets

$$\begin{aligned}X \setminus (U \cup V) &= (X \setminus U) \cap (X \setminus V),\\ X \setminus (U \cap V) &= (X \setminus U) \cup (X \setminus V)\end{aligned}$$

   for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

3. *Interaction With Unions I.* We have equalities of sets

$$U \setminus (V \cup W) = (U \setminus V) \cap (U \setminus W)$$

   for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

4. *Interaction With Unions II.* We have equalities of sets

$$(U \setminus V) \cup W = (U \cup W) \setminus (V \setminus W)$$

   for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

5. *Interaction With Unions III.* We have equalities of sets

$$\begin{aligned}U \setminus (V \cup W) &= (U \cup W) \setminus (V \cup W)\\ &= (U \setminus V) \setminus W\\ &= (U \setminus W) \setminus V\end{aligned}$$

   for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

6. *Interaction With Unions IV.* We have equalities of sets

$$(U \cup V) \setminus W = (U \setminus W) \cup (V \setminus W)$$

   for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

7. *Interaction With Intersections.* We have equalities of sets

$$\begin{aligned}(U \setminus V) \cap W &= (U \cap W) \setminus V\\ &= U \cap (W \setminus V)\end{aligned}$$

   for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

8. *Interaction With Complements.* We have an equality of sets

$$U \setminus V = U \cap V^{\mathsf{c}}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

9. *Interaction With Symmetric Differences.* We have an equality of sets

$$U \setminus V = U \vartriangle (U \cap V)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

10. *Triple Differences.* We have

$$U \setminus (V \setminus W) = (U \cap W) \cup (U \setminus V)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

11. *Left Annihilation.* We have

$$\emptyset \setminus U = \emptyset$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

12. *Right Unitality.* We have

$$U \setminus \emptyset = U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

13. *Invertibility.* We have

$$U \setminus U = \emptyset$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

14. *Interaction With Containment.* The following conditions are equivalent:
    (a) We have $V \setminus U \subset W$.
    (b) We have $V \setminus W \subset U$.

15. *Interaction With Characteristic Functions.* We have

$$\chi_{U \setminus V} = \chi_U - \chi_{U \cap V}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

PROOF 3.10.3 ► PROOF OF PROPOSITION 3.10.2

Item 1: Functoriality

See [Pro24ad] and [Pro24ah].

Item 2: De Morgan's Laws

See [Pro24m].

Item 3: Interaction With Unions I

See [Pro24n].

Item 4: Interaction With Unions II

Omitted.

Item 5: Interaction With Unions III

See [Pro24ai].

Item 6: Interaction With Unions IV

See [Pro24ac].

Item 7: Interaction With Intersections

See [Pro24v].

Item 8: Interaction With Complements

See [Pro24aa].

Item 9: Interaction With Symmetric Differences

See [Pro24ab].

Item 10: Triple Differences

See [Pro24ag].

Item 11: Left Annihilation

Clear.

Item 12: Right Unitality

See [Pro24ae].

Item 13: Invertibility

See [Pro24af].

Item 14: Interaction With Containment

Omitted.

Item 15: Interaction With Characteristic Functions

See [Pro24i].

3.11 Complements

Let X be a set and let $U \in \mathcal{P}(X)$.

DEFINITION 3.11.1 ► COMPLEMENTS

The **complement of** U is the set U^{c} defined by

$$\begin{aligned} U^{\mathsf{c}} &\overset{\text{def}}{=} X \setminus U \\ &\overset{\text{def}}{=} \{a \in X \mid a \notin U\}. \end{aligned}$$

PROPOSITION 3.11.2 ► PROPERTIES OF COMPLEMENTS

Let X be a set.

1. *Functoriality.* The assignment $U \mapsto U^{\mathsf{c}}$ defines a functor

   $$(-)^{\mathsf{c}} \colon \mathcal{P}(X)^{\mathsf{op}} \to \mathcal{P}(X),$$

   where

   - *Action on Objects.* For each $U \in \mathcal{P}(X)$, we have

     $$[(-)^{\mathsf{c}}](U) \overset{\text{def}}{=} U^{\mathsf{c}}.$$

   - *Action on Morphisms.* For each morphism $\iota_U \colon U \hookrightarrow V$ of $\mathcal{P}(X)$, the image

     $$\iota_U^{\mathsf{c}} \colon V^{\mathsf{c}} \hookrightarrow U^{\mathsf{c}}$$

     of ι_U by $(-)^{\mathsf{c}}$ is the inclusion

     $$V^{\mathsf{c}} \subset U^{\mathsf{c}}$$

     i.e. where we have

     ($\star$) If $U \subset V$, then $V^{\mathsf{c}} \subset U^{\mathsf{c}}$.

2. *De Morgan's Laws.* We have equalities of sets
$$\begin{aligned}(U\cup V)^{\mathsf{c}}&=U^{\mathsf{c}}\cap V^{\mathsf{c}},\\(U\cap V)^{\mathsf{c}}&=U^{\mathsf{c}}\cup V^{\mathsf{c}}\end{aligned}$$
for each $X\in\mathrm{Obj}(\mathsf{Sets})$ and each $U,V\in\mathcal{P}(X)$.

3. *Involutority.* We have
$$(U^{\mathsf{c}})^{\mathsf{c}}=U$$
for each $X\in\mathrm{Obj}(\mathsf{Sets})$ and each $U\in\mathcal{P}(X)$.

4. *Interaction With Characteristic Functions.* We have
$$\chi_{U^{\mathsf{c}}}=1-\chi_{U}$$
for each $X\in\mathrm{Obj}(\mathsf{Sets})$ and each $U\in\mathcal{P}(X)$.

PROOF 3.11.3 ► PROOF OF PROPOSITION 3.11.2

Item 1: Functoriality

This follows from Item 1 of Proposition 3.10.2.

Item 2: De Morgan's Laws

See [Pro24m].

Item 3: Involutority

See [Pro24l].

Item 4: Interaction With Characteristic Functions

Clear. ▭

3.12 Symmetric Differences

Let A and B be sets.

DEFINITION 3.12.1 ► SYMMETRIC DIFFERENCES

The **symmetric difference of** A **and** B is the set $A \bigtriangleup B$ defined by

$$A \bigtriangleup B \overset{\text{def}}{=} (A \setminus B) \cup (B \setminus A).$$

PROPOSITION 3.12.2 ► PROPERTIES OF SYMMETRIC DIFFERENCES

Let X be a set.

1. *Lack of Functoriality.* The assignment $(U, V) \mapsto U \bigtriangleup V$ ***need not*** define functors

$$\begin{aligned} U \bigtriangleup -&\colon (\mathcal{P}(X), \subset) \to (\mathcal{P}(X), \subset),\\ - \bigtriangleup V&\colon (\mathcal{P}(X), \subset) \to (\mathcal{P}(X), \subset),\\ -_1 \bigtriangleup -_2&\colon (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \subset) \to (\mathcal{P}(X), \subset). \end{aligned}$$

2. *Via Unions and Intersections.* We have[1]

$$U \bigtriangleup V = (U \cup V) \setminus (U \cap V)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

3. *Associativity.* We have[2]

$$(U \bigtriangleup V) \bigtriangleup W = U \bigtriangleup (V \bigtriangleup W)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

4. *Commutativity.* We have

$$U \bigtriangleup V = V \bigtriangleup U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

5. *Unitality.* We have

$$\begin{aligned} U \bigtriangleup \emptyset &= U,\\ \emptyset \bigtriangleup U &= U \end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

6. *Invertibility.* We have
$$U \mathbin{\triangle} U = \emptyset$$
for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

7. *Interaction With Unions.* We have
$$(U \mathbin{\triangle} V) \cup (V \mathbin{\triangle} T) = (U \cup V \cup W) \setminus (U \cap V \cap W)$$
for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

8. *Interaction With Complements I.* We have
$$U \mathbin{\triangle} U^{\mathsf{c}} = X$$
for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

9. *Interaction With Complements II.* We have
$$\begin{aligned} U \mathbin{\triangle} X &= U^{\mathsf{c}}, \\ X \mathbin{\triangle} U &= U^{\mathsf{c}} \end{aligned}$$
for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

10. *Interaction With Complements III.* We have
$$U^{\mathsf{c}} \mathbin{\triangle} V^{\mathsf{c}} = U \mathbin{\triangle} V$$
for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

11. *“Transitivity”.* We have
$$(U \mathbin{\triangle} V) \mathbin{\triangle} (V \mathbin{\triangle} W) = U \mathbin{\triangle} W$$
for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

12. *The Triangle Inequality for Symmetric Differences.* We have
$$U \mathbin{\triangle} W \subset U \mathbin{\triangle} V \cup V \mathbin{\triangle} W$$
for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

13. *Distributivity Over Intersections.* We have

$$U \cap (V \triangle W) = (U \cap V) \triangle (U \cap W),$$
$$(U \triangle V) \cap W = (U \cap W) \triangle (V \cap W)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

14. *Interaction With Characteristic Functions.* We have

$$\chi_{U \triangle V} = \chi_U + \chi_V - 2\chi_{U \cap V}$$

and thus, in particular, we have

$$\chi_{U \triangle V} \equiv \chi_U + \chi_V \mod 2$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

15. *Bijectivity.* Given $A, B \subset \mathcal{P}(X)$, the maps

$$A \triangle - \colon \mathcal{P}(X) \to \mathcal{P}(X),$$
$$- \triangle B \colon \mathcal{P}(X) \to \mathcal{P}(X)$$

are bijections with inverses given by

$$(A \triangle -)^{-1} = - \cup (A \cap -),$$
$$(- \triangle B)^{-1} = - \cup (B \cap -).$$

Moreover, the map

$$C \mapsto C \triangle (A \triangle B)$$

is a bijection of $\mathcal{P}(X)$ onto itself sending A to B and B to A.

16. *Interaction With Powersets and Groups.* Let X be a set.

    (a) The quadruple $(\mathcal{P}(X), \triangle, \emptyset, \mathrm{id}_{\mathcal{P}(X)})$ is an abelian group.[3]

    (b) Every element of $\mathcal{P}(X)$ has order 2 with respect to $\triangle$, and thus $\mathcal{P}(X)$ is a *Boolean group* (i.e. an abelian 2-group).

17. *Interaction With Powersets and Vector Spaces I.* The pair $(\mathcal{P}(X), \alpha_{\mathcal{P}(X)})$ consisting of

- The group $\mathcal{P}(X)$ of **??**;
- The map $\alpha_{\mathcal{P}(X)}\colon \mathbb{F}_2 \times \mathcal{P}(X) \to \mathcal{P}(X)$ defined by

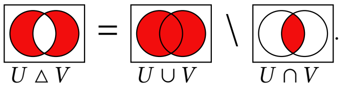
$$\begin{aligned} 0 \cdot U &\overset{\text{def}}{=} \emptyset, \\ 1 \cdot U &\overset{\text{def}}{=} U; \end{aligned}$$

is an $\mathbb{F}_2$-vector space.

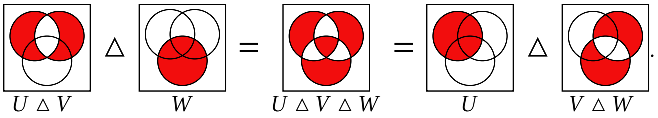
18. *Interaction With Powersets and Vector Spaces II.* If X is finite, then:
    (a) The set of singletons sets on the elements of X forms a basis for the $\mathbb{F}_2$-vector space $(\mathcal{P}(X), \alpha_{\mathcal{P}(X)})$ of Item 17.
    (b) We have
    $$\dim(\mathcal{P}(X)) = \#\mathcal{P}(X).$$
19. *Interaction With Powersets and Rings.* The quintuple $(\mathcal{P}(X), \triangle, \cap, \emptyset, X)$ is a commutative ring.[4]

---

[1] *Illustration:*



[2] *Illustration:*



[3] Here are some examples:

i. When $X = \emptyset$, we have an isomorphism of groups between $\mathcal{P}(\emptyset)$ and the trivial group:

$$(\mathcal{P}(\emptyset), \triangle, \emptyset, \mathrm{id}_{\mathcal{P}(\emptyset)}) \cong \mathrm{pt}.$$

ii. When $X = \mathrm{pt}$, we have an isomorphism of groups between $\mathcal{P}(\mathrm{pt})$ and $\mathbb{Z}_{/2}$:

$$(\mathcal{P}(\mathrm{pt}), \triangle, \emptyset, \mathrm{id}_{\mathcal{P}(\mathrm{pt})}) \cong \mathbb{Z}_{/2}.$$

iii. When $X = \{0, 1\}$, we have an isomorphism of groups between $\mathcal{P}(\{0, 1\})$ and $\mathbb{Z}_{/2} \times \mathbb{Z}_{/2}$:

$$(\mathcal{P}(\{0, 1\}), \triangle, \emptyset, \mathrm{id}_{\mathcal{P}(\{0,1\})}) \cong \mathbb{Z}_{/2} \times \mathbb{Z}_{/2}.$$

[4] *Warning:* The analogous statement replacing intersections by unions (i.e. that the quintuple $(\mathcal{P}(X), \triangle, \cup, \emptyset, X)$ is a ring) is false, however. See [Pro24aw] for a proof.

PROOF 3.12.3 ▶ PROOF OF PROPOSITION 3.12.2

Item 1: Lack of Functoriality

Omitted.

Item 2: Via Unions and Intersections

See [Pro24o].

Item 3: Associativity

See [Pro24ao].

Item 4: Commutativity

See [Pro24ap].

Item 5: Unitality

This follows from Item 4 and [Pro24at].

Item 6: Invertibility

See [Pro24av].

Item 7: Interaction With Unions

See [Pro24bc].

Item 8: Interaction With Complements I

See [Pro24as].

Item 9: Interaction With Complements II

This follows from Item 4 and [Pro24ax].

Item 10: Interaction With Complements III

See [Pro24aq].

Item 11: "Transitivity"

We have

$$\begin{aligned}(U \mathbin{\triangle} V) \mathbin{\triangle} (V \mathbin{\triangle} W) &= U \mathbin{\triangle} (V \mathbin{\triangle} (V \mathbin{\triangle} W)) && \text{(by Item 3)}\\ &= U \mathbin{\triangle} ((V \mathbin{\triangle} V) \mathbin{\triangle} W) && \text{(by Item 3)}\\ &= U \mathbin{\triangle} (\emptyset \mathbin{\triangle} W) && \text{(by Item 6)}\\ &= U \mathbin{\triangle} W && \text{(by Item 5)}\end{aligned}$$

Item 12: The Triangle Inequality for Symmetric Differences

This follows from Items 2 and 11.

Item 13: Distributivity Over Intersections

See [Pro24r].

Item 14: Interaction With Characteristic Functions

See [Pro24j].

Item 15: Bijectivity

Clear.

Item 16: Interaction With Powersets and Groups

Item 16a follows from[1] Items 3 to 6, while Item 16b follows from Item 6.

Item 17: Interaction With Powersets and Vector Spaces I

Clear.

Item 18: Interaction With Powersets and Vector Spaces II

Omitted.

Item 19: Interaction With Powersets and Rings

This follows from Items 8 and 11 of Proposition 3.9.2 and Items 13 and 16.[2] □

---

[1] Reference: [Pro24ar].

[2] *Reference*: [Pro24au].

4 Powersets

4.1 Characteristic Functions

Let X be a set.

Definition 4.1.1 ► Characteristic Functions

Let $U \subset X$ and let $x \in X$.

1. The **characteristic function of** U[1] is the function[2]

$$\chi_U : X \longrightarrow \{\mathrm{t}, \mathrm{f}\}$$

defined by

$$\chi_U(x) \overset{\text{def}}{=} \begin{cases} \text{true} & \text{if } x \in U, \\ \text{false} & \text{if } x \notin U \end{cases}$$

for each $x \in X$.

2. The **characteristic function of** x is the function[3]

$$\chi_x \colon X \longrightarrow \{\mathsf{t}, \mathsf{f}\}$$

defined by

$$\chi_x \overset{\text{def}}{=} \chi_{\{x\}},$$

i.e. by

$$\chi_x(y) \overset{\text{def}}{=} \begin{cases} \text{true} & \text{if } x = y, \\ \text{false} & \text{if } x \neq y \end{cases}$$

for each $y \in X$.

3. The **characteristic relation on** X[4] is the relation[5]

$$\chi_X(-_1, -_2) \colon X \times X \longrightarrow \{\mathsf{t}, \mathsf{f}\}$$

on X defined by[6]

$$\chi_X(x, y) \overset{\text{def}}{=} \begin{cases} \text{true} & \text{if } x = y, \\ \text{false} & \text{if } x \neq y \end{cases}$$

for each $x, y \in X$.

4. The **characteristic embedding**[7] **of** X **into** $\mathcal{P}(X)$ is the function

$$\chi_{(-)} \colon X \hookrightarrow \mathcal{P}(X)$$

defined by

$$\chi_{(-)}(x) \overset{\text{def}}{=} \chi_x$$

for each $x \in X$.

---

[1] *Further Terminology:* Also called the **indicator function of U**.

[2] *Further Notation:* Also written $\chi_X(U, -)$ or $\chi_X(-, U)$.

[3] *Further Notation:* Also written χ^x, $\chi_X(x, -)$, or $\chi_X(-, x)$.

[4] *Further Terminology:* Also called the **identity relation on X**.

[5] *Further Notation:* Also written χ_{-2}^{-1}, or $\sim_{\mathrm{id}}$ in the context of relations.

[6] As a subset of $X \times X$, the relation χ_X corresponds to the diagonal $\Delta_X \subset X \times X$ of X.

[7] The name "characteristic *embedding*" comes from the fact that there is an analogue of fully faithfulness for $\chi_{(-)}$: given a set X, we have

$$\mathrm{Hom}_{\mathcal{P}(X)}(\chi_x, \chi_y) = \chi_X(x, y),$$

for each $x, y \in X$.

REMARK 4.1.2 ► CHARACTERISTIC FUNCTIONS AS DECATEGORIFICATIONS OF PRESHEAVES

The definitions in Definition 4.1.1 are decategorifications of co/presheaves, representable co/presheaves, Hom profunctors, and the Yoneda embedding:[1]

1. A function
$$f : X \longrightarrow \{\mathrm{t}, \mathrm{f}\}$$
is a decategorification of a presheaf
$$\mathcal{F} : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathsf{Sets},$$
with the characteristic functions χ_U of the subsets of X being the primordial examples (and, in fact, all examples) of these.

2. The characteristic function
$$\chi_x : X \longrightarrow \{\mathrm{t}, \mathrm{f}\}$$
of an *element* x of X is a decategorification of the representable presheaf
$$h_X : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathsf{Sets}$$
of an *object* x of a category $\mathcal{C}$.

3. The characteristic relation
$$\chi_X(-_1, -_2) : X \times X \longrightarrow \{\mathrm{t}, \mathrm{f}\}$$
of X is a decategorification of the Hom profunctor
$$\mathrm{Hom}_{\mathcal{C}}(-_1, -_2) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \longrightarrow \mathsf{Sets}$$
of a category $\mathcal{C}$.

4. The characteristic embedding

$$\chi_{(-)}\colon X \hookrightarrow \mathcal{P}(X)$$

of X into $\mathcal{P}(X)$ is a decategorification of the Yoneda embedding

$$よ\colon \mathcal{C}^{\mathsf{op}} \hookrightarrow \mathsf{PSh}(\mathcal{C})$$

of a category $\mathcal{C}$ into $\mathsf{PSh}(\mathcal{C})$.

5. There is also a direct parallel between unions and colimits:

   - An element of $\mathcal{P}(X)$ is a union of elements of X, viewed as one-point subsets $\{x\} \in \mathcal{P}(A)$.
   - An object of $\mathsf{PSh}(\mathcal{C})$ is a colimit of objects of $\mathcal{C}$, viewed as representable presheaves $h_X \in \mathrm{Obj}(\mathsf{PSh}(\mathcal{C}))$.

---

[1] These statements can be made precise by using the embeddings

$$(-)_{\mathsf{disc}}\colon \mathsf{Sets} \hookrightarrow \mathsf{Cats},$$
$$(-)_{\mathsf{disc}}\colon \{\mathsf{t},\mathsf{f}\}_{\mathsf{disc}} \hookrightarrow \mathsf{Sets}$$

of sets into categories and of classical truth values into sets.

For instance, in this approach the characteristic function

$$\chi_x\colon X \longrightarrow \{\mathsf{t},\mathsf{f}\}$$

of an element x of X, defined by

$$\chi_x(y) \overset{\text{def}}{=} \begin{cases} \mathsf{true} & \text{if } x = y, \\ \mathsf{false} & \text{if } x \neq y \end{cases}$$

for each $y \in X$, is recovered as the representable presheaf

$$\mathrm{Hom}_{X_{\mathsf{disc}}}(-,x)\colon X_{\mathsf{disc}} \longrightarrow \mathsf{Sets}$$

of the corresponding object x of X_{disc}, defined on objects by

$$\mathrm{Hom}_{X_{\mathsf{disc}}}(y,x) \overset{\text{def}}{=} \begin{cases} \mathrm{pt} & \text{if } x = y, \\ \emptyset & \text{if } x \neq y \end{cases}$$

for each $y \in \mathrm{Obj}(X_{\mathsf{disc}})$.

Proposition 4.1.3 ▶ Properties of Characteristic Functions

Let X be a set.

1. *The Inclusion of Characteristic Relations Associated to a Function.* Let $f\colon A \longrightarrow B$ be a function. We have an inclusion[1]

$$\chi_B \circ (f \times f) \subset \chi_A, \qquad \begin{tikzcd} A \times A \arrow[rr, "f\times f"] \arrow[dr, "\chi_A"'] & \supset & B \times B \arrow[dl, "\chi_B"] \\ & \{\mathsf{t}, \mathsf{f}\}. & \end{tikzcd}$$

2. *Interaction With Unions I.* We have

$$\chi_{U\cup V} = \max(\chi_U, \chi_V)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

3. *Interaction With Unions II.* We have

$$\chi_{U\cup V} = \chi_U + \chi_V - \chi_{U\cap V}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

4. *Interaction With Intersections I.* We have

$$\chi_{U\cap V} = \chi_U \chi_V$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

5. *Interaction With Intersections II.* We have

$$\chi_{U\cap V} = \min(\chi_U, \chi_V)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

6. *Interaction With Differences.* We have

$$\chi_{U\setminus V} = \chi_U - \chi_{U\cap V}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

7. *Interaction With Complements.* We have
$$\chi_{U^{\mathsf{c}}} = 1 - \chi_U$$
for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

8. *Interaction With Symmetric Differences.* We have
$$\chi_{U \triangle V} = \chi_U + \chi_V - 2\chi_{U \cap V}$$
and thus, in particular, we have
$$\chi_{U \triangle V} \equiv \chi_U + \chi_V \mod 2$$
for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

9. *Interaction Between the Characteristic Embedding and Morphisms.* Let $f \colon X \to Y$ be a map of sets. The diagram
$$f_* \circ \chi_X = \chi_{X'} \circ f, \qquad \begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\chi_X"'] & X' \arrow[d, "\chi_{X'}"] \\ \mathcal{P}(X) \arrow[r, "f_*"'] & \mathcal{P}(X'). \end{tikzcd}$$
commutes.

---

[1] This is the 0-categorical version of Categories, Definition 4.4.1.

Proof 4.1.4 ► Proof of Proposition 4.1.3

Item 1: The Inclusion of Characteristic Relations Associated to a Function

The inclusion $\chi_B(f(a), f(b)) \subset \chi_A(a, b)$ is equivalent to the statement "if $a = b$, then $f(a) = f(b)$", which is true.

Item 2: Interaction With Unions I

This is a repetition of Item 8 of Proposition 3.7.2 and is proved there.

Item 3: Interaction With Unions II

This is a repetition of Item 9 of Proposition 3.7.2 and is proved there.

Item 4: Interaction With Intersections I

This is a repetition of Item 9 of Proposition 3.9.2 and is proved there.

Item 5: Interaction With Intersections II

This is a repetition of Item 10 of Proposition 3.9.2 and is proved there.

Item 6: Interaction With Differences

This is a repetition of Item 15 of Proposition 3.10.2 and is proved there.

Item 7: Interaction With Complements

This is a repetition of Item 4 of Proposition 3.11.2 and is proved there.

Item 8: Interaction With Symmetric Differences

This is a repetition of Item 14 of Proposition 3.12.2 and is proved there.

Item 9: Interaction Between the Characteristic Embedding and Morphisms

Indeed, we have

$$\begin{aligned} [f_* \circ \chi_X](x) &\overset{\text{def}}{=} f_*(\chi_X(x)) \\ &\overset{\text{def}}{=} f_*(\{x\}) \\ &= \{f(x)\} \\ &\overset{\text{def}}{=} \chi_{X'}(f(x)) \\ &\overset{\text{def}}{=} [\chi_{X'} \circ f](x), \end{aligned}$$

for each $x \in X$, showing the desired equality. □

4.2 The Yoneda Lemma for Sets

Let X be a set and let $U \subset X$ be a subset of X.

PROPOSITION 4.2.1 ► THE YONEDA LEMMA FOR SETS

We have

$$\chi_{\mathcal{P}(X)}(\chi_x, \chi_U) = \chi_U(x)$$

for each $x \in X$, giving an equality of functions

$$\chi_{\mathcal{P}(X)}(\chi_{(-)}, \chi_U) = \chi_U.$$

Proof 4.2.2 ► Proof of Proposition 4.2.1

Clear. ∎

Corollary 4.2.3 ► The Characteristic Embedding Is Fully Faithful

The characteristic embedding is fully faithful, i.e., we have

$$\chi_{\mathcal{P}(X)}(\chi_x, \chi_y) = \chi_X(x, y)$$

for each $x, y \in X$.

Proof 4.2.4 ► Proof of Corollary 4.2.3

This follows from Proposition 4.2.1. ∎

4.3 Powersets

Let X be a set.

Definition 4.3.1 ► Powersets

The **powerset of** X is the set $\mathcal{P}(X)$ defined by

$$\mathcal{P}(X) \overset{\text{def}}{=} \{U \in P \mid U \subset X\},$$

where P is the set in the axiom of powerset, **??** of **??**.

Remark 4.3.2 ► Powersets as Decategorifications of Co/Presheaf Categories

The powerset of a set is a decategorification of the category of presheaves of a category: while[1]

- The powerset of a set X is equivalently (Items 1 and 2 of Proposition 4.3.9) the set

  $$\mathrm{Sets}(X, \{\mathsf{t}, \mathsf{f}\})$$

  of functions from X to the set $\{\mathsf{t}, \mathsf{f}\}$ of classical truth values.

- The category of presheaves on a category C is the category

$$\mathsf{Fun}(C^{\mathsf{op}}, \mathsf{Sets})$$

of functors from C^{op} to the category Sets of sets.

---

[1] This parallel is based on the following comparison:

- A category is enriched over the category

$$\mathsf{Sets} \overset{\text{def}}{=} \mathsf{Cats}_0$$

of sets (i.e. "0-categories"), with presheaves taking values on it.

- A set is enriched over the set

$$\{\mathsf{t}, \mathsf{f}\} \overset{\text{def}}{=} \mathsf{Cats}_{-1}$$

of classical truth values (i.e. "(-1)-categories"), with characteristic functions taking values on it.

Proposition 4.3.3 ► Properties of Powersets: As Categories

Let X be a set.

1. *Co/Completeness.* The (posetal) category (associated to) $(\mathcal{P}(X), \subset)$ is complete and cocomplete:
    (a) *Products.* The products in $\mathcal{P}(X)$ are given by intersection of subsets.
    (b) *Coproducts.* The coproducts in $\mathcal{P}(X)$ are given by union of subsets.
    (c) *Co/Equalisers.* Being a posetal category, $\mathcal{P}(X)$ only has at most one morphisms between any two objects, so co/equalisers are trivial.
2. *Cartesian Closedness.* The category $\mathcal{P}(X)$ is Cartesian closed with internal Hom

$$\mathbf{Hom}_{\mathcal{P}(X)}(-_1, -_2) \colon \mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \to \mathcal{P}(X)$$

given by[1]

$$\mathbf{Hom}_{\mathcal{P}(X)}(U, V) \overset{\text{def}}{=} (X \setminus U) \cup V$$

for each $U, V \in \mathrm{Obj}(\mathcal{P}(X))$.

---

[1] For intuition regarding the expression defining $\mathbf{Hom}_{\mathcal{P}(X)}(U, V)$, see Remark 3.9.4.

Proof 4.3.4 ► Proof of Proposition 4.3.3

Item 1: Co/Completeness

Clear.

Item 2: Cartesian Closedness

This follows from Item 2 of Proposition 3.9.2. □

Proposition 4.3.5 ► Properties of Powersets: Functoriality and Adjointness

Let X be a set.

1. *Functoriality I.* The assignment $X \mapsto \mathcal{P}(X)$ defines a functor

   $$\mathcal{P}_* \colon \mathsf{Sets} \to \mathsf{Sets},$$

   where

   - *Action on Objects.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

     $$\mathcal{P}_*(A) \overset{\text{def}}{=} \mathcal{P}(A).$$

   - *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathsf{Sets})$, the action on morphisms

     $$\mathcal{P}_{*|A,B} \colon \mathsf{Sets}(A, B) \to \mathsf{Sets}(\mathcal{P}(A), \mathcal{P}(B))$$

     of $\mathcal{P}_*$ at (A, B) is the map defined by by sending a map of sets $f \colon A \to B$ to the map

     $$\mathcal{P}_*(f) \colon \mathcal{P}(A) \to \mathcal{P}(B)$$

     defined by

     $$\mathcal{P}_*(f) \overset{\text{def}}{=} f_*,$$

     as in Definition 4.4.1.

2. *Functoriality II.* The assignment $X \mapsto \mathcal{P}(X)$ defines a functor

   $$\mathcal{P}^{-1} \colon \mathsf{Sets}^{\mathsf{op}} \to \mathsf{Sets},$$

   where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

  $$\mathcal{P}^{-1}(A) \overset{\text{def}}{=} \mathcal{P}(A).$$

- *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathsf{Sets})$, the action on morphisms

  $$\mathcal{P}^{-1}_{A,B} \colon \mathsf{Sets}(A, B) \to \mathsf{Sets}(\mathcal{P}(B), \mathcal{P}(A))$$

  of $\mathcal{P}^{-1}$ at (A, B) is the map defined by by sending a map of sets $f \colon A \to B$ to the map

  $$\mathcal{P}^{-1}(f) \colon \mathcal{P}(B) \to \mathcal{P}(A)$$

  defined by

  $$\mathcal{P}^{-1}(f) \overset{\text{def}}{=} f^{-1},$$

  as in Definition 4.5.1.

3. *Functoriality III.* The assignment $X \mapsto \mathcal{P}(X)$ defines a functor

   $$\mathcal{P}_! \colon \mathsf{Sets} \to \mathsf{Sets},$$

   where

   - *Action on Objects.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

     $$\mathcal{P}_!(A) \overset{\text{def}}{=} \mathcal{P}(A).$$

   - *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathsf{Sets})$, the action on morphisms

     $$\mathcal{P}_{!|A,B} \colon \mathsf{Sets}(A, B) \to \mathsf{Sets}(\mathcal{P}(A), \mathcal{P}(B))$$

     of $\mathcal{P}_!$ at (A, B) is the map defined by by sending a map of sets $f \colon A \to B$ to the map

     $$\mathcal{P}_!(f) \colon \mathcal{P}(A) \to \mathcal{P}(B)$$

     defined by

     $$\mathcal{P}_!(f) \overset{\text{def}}{=} f_!,$$

     as in Definition 4.6.1.

4. *Adjointness I.* We have an adjunction

$$\begin{tikzcd}(\mathcal{P}^{-1} \dashv \mathcal{P}^{-1,\mathsf{op}}) \colon \quad \mathsf{Sets}^{\mathsf{op}} \arrow[r, bend left, "\mathcal{P}^{-1}"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}, \arrow[l, bend left, "\mathcal{P}^{-1,\mathsf{op}}"]\end{tikzcd}$$

witnessed by a bijection

$$\underbrace{\mathsf{Sets}^{\mathsf{op}}(\mathcal{P}(A), B)}_{\overset{\mathrm{def}}{=}\mathsf{Sets}(B,\mathcal{P}(A))} \cong \mathsf{Sets}(A, \mathcal{P}(B)),$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$ and $B \in \mathrm{Obj}(\mathsf{Sets}^{\mathsf{op}})$.

5. *Adjointness II.* We have an adjunction

$$\begin{tikzcd}(\mathrm{Gr} \dashv \mathcal{P}_*) \colon \quad \mathsf{Sets} \arrow[r, bend left, "\mathrm{Gr}"] \arrow[r, phantom, "\perp"] & \mathsf{Rel}, \arrow[l, bend left, "\mathcal{P}_*"]\end{tikzcd}$$

witnessed by a bijection of sets

$$\mathsf{Rel}(\mathrm{Gr}(A), B) \cong \mathsf{Sets}(A, \mathcal{P}(B))$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$ and $B \in \mathrm{Obj}(\mathsf{Rel})$, where Gr is the graph functor of Constructions With Relations, Item 1 of Proposition 3.1.2 and $\mathcal{P}_*$ is the functor of Constructions With Relations, Proposition 4.5.1.

PROOF 4.3.6 ► PROOF OF PROPOSITION 4.3.5

Item 1: Functoriality I

This follows from Items 3 and 4 of Proposition 4.4.6.

Item 2: Functoriality II

This follows Items 3 and 4 of Proposition 4.5.5.

Item 3: Functoriality III

This follows Items 3 and 4 of Proposition 4.6.8.

Item 4: Adjointness I

We have

$$\begin{aligned}\mathsf{Sets}^{\mathsf{op}}(\mathcal{P}(A),B) &\overset{\text{def}}{=} \mathsf{Sets}(B,\mathcal{P}(A))\\ &\cong \mathsf{Sets}(B,\mathsf{Sets}(A,\{\mathsf{t},\mathsf{f}\})) && \text{(by Item 1 of Proposition 4.3.9)}\\ &\cong \mathsf{Sets}(A\times B,\{\mathsf{t},\mathsf{f}\}) && \text{(by Item 2 of Proposition 1.3.3)}\\ &\cong \mathsf{Sets}(A,\mathsf{Sets}(B,\{\mathsf{t},\mathsf{f}\})) && \text{(by Item 2 of Proposition 1.3.3)}\\ &\cong \mathsf{Sets}(A,\mathcal{P}(B)) && \text{(by Item 1 of Proposition 4.3.9)}\end{aligned}$$

with all bijections natural in A and B (where we use Item 2 of Proposition 4.3.9 here).

Item 5: Adjointness II

We have

$$\begin{aligned}\mathsf{Rel}(\mathsf{Gr}(A),B) &\cong \mathcal{P}(A\times B)\\ &\cong \mathsf{Sets}(A\times B,\{\mathsf{t},\mathsf{f}\}) && \text{(by Item 1 of Proposition 4.3.9)}\\ &\cong \mathsf{Sets}(A,\mathsf{Sets}(B,\{\mathsf{t},\mathsf{f}\})) && \text{(by Item 2 of Proposition 1.3.3)}\\ &\cong \mathsf{Sets}(A,\mathcal{P}(B)) && \text{(by Item 1 of Proposition 4.3.9)}\end{aligned}$$

with all bijections natural in A (where we use Item 2 of Proposition 4.3.9 here). Explicitly, this isomorphism is given by sending a relation $R\colon \mathsf{Gr}(A) \nrightarrow B$ to the map $R^{\dagger}\colon A \to \mathcal{P}(B)$ sending a to the subset $R(a)$ of B, as in Relations, Remark 1.1.4. Naturality in B is then the statement that given a relation $R\colon B \nrightarrow B'$, the diagram

$$\begin{tikzcd}\mathsf{Rel}(\mathsf{Gr}(A),B) \arrow[r, "R\diamond -"] \arrow[d, dashed, "\wr"] & \mathsf{Rel}(\mathsf{Gr}(A),B') \arrow[d, dashed, "\wr"'] \\ \mathsf{Sets}(A,\mathcal{P}(B)) \arrow[r, "R_*"'] & \mathsf{Sets}(A,\mathcal{P}(B'))\end{tikzcd}$$

commutes, which follows from Constructions With Relations, Remark 4.1.2. □

Proposition 4.3.7 ► Properties of Powersets: Monoidality

Let X be a set.

1. *Symmetric Strong Monoidality With Respect to Coproducts I.* The powerset func-

tor $\mathcal{P}_*$ of Item 1 of Proposition 4.3.5 has a symmetric strong monoidal structure

$$(\mathcal{P}_*, \mathcal{P}_*^{\coprod}, \mathcal{P}_{*|\mathbb{1}}^{\coprod}) \colon (\mathsf{Sets}, \times, \mathrm{pt}) \longrightarrow (\mathsf{Sets}, \coprod, \emptyset)$$

being equipped with isomorphisms

$$\mathcal{P}_{*|X,Y}^{\coprod} \colon \mathcal{P}(X) \times \mathcal{P}(Y) \xrightarrow{\cong} \mathcal{P}(X \coprod Y),$$
$$\mathcal{P}_{*|\mathbb{1}}^{\coprod} \colon \mathrm{pt} \xrightarrow{\cong} \mathcal{P}(\emptyset),$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

2. *Symmetric Strong Monoidality With Respect to Coproducts II.* The powerset functor $\mathcal{P}^{-1}$ of Item 2 of Proposition 4.3.5 has a symmetric strong monoidal structure

$$(\mathcal{P}^{-1}, \mathcal{P}^{-1|\coprod}, \mathcal{P}_{\mathbb{1}}^{-1|\coprod}) \colon (\mathsf{Sets}^{\mathsf{op}}, \times^{\mathsf{op}}, \mathrm{pt}) \longrightarrow (\mathsf{Sets}, \coprod, \emptyset)$$

being equipped with isomorphisms

$$\mathcal{P}_{X,Y}^{-1|\coprod} \colon \mathcal{P}(X) \times \mathcal{P}(Y) \xrightarrow{\cong} \mathcal{P}(X \coprod Y),$$
$$\mathcal{P}_{\mathbb{1}}^{-1|\coprod} \colon \mathrm{pt} \xrightarrow{\cong} \mathcal{P}(\emptyset),$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

3. *Symmetric Strong Monoidality With Respect to Coproducts III.* The powerset functor $\mathcal{P}_!$ of Item 3 of Proposition 4.3.5 has a symmetric strong monoidal structure

$$(\mathcal{P}_!, \mathcal{P}_!^{\coprod}, \mathcal{P}_{!|\mathbb{1}}^{\coprod}) \colon (\mathsf{Sets}, \times, \mathrm{pt}) \longrightarrow (\mathsf{Sets}, \coprod, \emptyset)$$

being equipped with isomorphisms

$$\mathcal{P}_{!|X,Y}^{\coprod} \colon \mathcal{P}(X) \times \mathcal{P}(Y) \xrightarrow{\cong} \mathcal{P}(X \coprod Y),$$
$$\mathcal{P}_{!|\mathbb{1}}^{\coprod} \colon \mathrm{pt} \xrightarrow{\cong} \mathcal{P}(\emptyset),$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

4. *Symmetric Lax Monoidality With Respect to Products I.* The powerset functor $\mathcal{P}_*$ of Item 1 of Proposition 4.3.5 has a symmetric lax monoidal structure

$$(\mathcal{P}_*, \mathcal{P}_*^{\otimes}, \mathcal{P}_{*|\mathbb{1}}^{\otimes})\colon (\mathsf{Sets}, \times, \mathrm{pt}) \longrightarrow (\mathsf{Sets}, \times, \mathrm{pt})$$

being equipped with morphisms

$$\mathcal{P}_{*|X,Y}^{\times}\colon \mathcal{P}(X) \times \mathcal{P}(Y) \to \mathcal{P}(X \times Y),$$
$$\mathcal{P}_{*|\mathbb{1}}^{\times}\colon \mathrm{pt} \to \mathcal{P}(\mathrm{pt}),$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$, where

- The map $\mathcal{P}_{*|X,Y}^{\times}$ is given by

  $$\mathcal{P}_{*|X,Y}^{\times}(U, V) \overset{\text{def}}{=} U \times V$$

  for each $(U, V) \in \mathcal{P}(X) \times \mathcal{P}(Y)$,
- The map $\mathcal{P}_{*|\mathbb{1}}^{\times}$ is given by

  $$\mathcal{P}_{*|\mathbb{1}}^{\times}(\star) = \mathrm{pt}.$$

5. *Symmetric Lax Monoidality With Respect to Products II.* The powerset functor $\mathcal{P}^{-1}$ of Item 2 of Proposition 4.3.5 has a symmetric lax monoidal structure

$$(\mathcal{P}^{-1}, \mathcal{P}^{-1|\otimes}, \mathcal{P}_{\mathbb{1}}^{-1|\otimes})\colon (\mathsf{Sets}^{\mathsf{op}}, \times^{\mathsf{op}}, \mathrm{pt}) \longrightarrow (\mathsf{Sets}, \times, \mathrm{pt})$$

being equipped with morphisms

$$\mathcal{P}_{X,Y}^{-1|\times}\colon \mathcal{P}(X) \times \mathcal{P}(Y) \to \mathcal{P}(X \times Y),$$
$$\mathcal{P}_{\mathbb{1}}^{\times}\colon \mathrm{pt} \to \mathcal{P}(\emptyset),$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$, defined as in Item 4.

6. *Symmetric Lax Monoidality With Respect to Products III.* The powerset functor $\mathcal{P}_!$ of Item 3 of Proposition 4.3.5 has a symmetric lax monoidal structure

$$(\mathcal{P}_!, \mathcal{P}_!^{\otimes}, \mathcal{P}_{!|\mathbb{1}}^{\otimes})\colon (\mathsf{Sets}, \times, \mathrm{pt}) \longrightarrow (\mathsf{Sets}, \times, \mathrm{pt})$$

being equipped with morphisms

$$\mathcal{P}_{!|X,Y}^{\times}\colon \mathcal{P}(X) \times \mathcal{P}(Y) \to \mathcal{P}(X \times Y),$$
$$\mathcal{P}_{!|\mathbb{1}}^{\times}\colon \mathrm{pt} \to \mathcal{P}(\emptyset),$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$, defined as in Item 4.

Proof 4.3.8 ▶ Proof of Proposition 4.3.7

Item 1: Symmetric Strong Monoidality With Respect to Coproducts I

The isomorphism

$$\mathcal{P}^{\coprod}_{*|X,Y}\colon \mathcal{P}(X)\times\mathcal{P}(Y)\to\mathcal{P}(X\coprod Y)$$

is given by sending $(U,V)\in\mathcal{P}(X)\times\mathcal{P}(Y)$ to $U\coprod V$, with inverse given by sending a subset S of $X\coprod Y$ to the pair $(S_X,S_Y)\in\mathcal{P}(X)\times\mathcal{P}(Y)$ with

$$\begin{aligned} S_X &\overset{\text{def}}{=} \{x\in X \mid (0,x)\in S\}\\ S_Y &\overset{\text{def}}{=} \{y\in Y \mid (1,y)\in S\}. \end{aligned}$$

The isomorphism $\mathrm{pt}\cong\mathcal{P}(\emptyset)$ is given by $\star\mapsto\emptyset\in\mathcal{P}(\emptyset)$.

Naturality for the isomorphism $\mathcal{P}^{\coprod}_{*|X,Y}$ is the statement that, given maps of sets $f\colon X\to X'$ and $g\colon Y\to Y'$, the diagram

$$\begin{tikzcd}
\mathcal{P}(X)\times\mathcal{P}(Y) \arrow[r, "f_*\times g_*"] \arrow[d, dashed, "\wr"] & \mathcal{P}(X')\times\mathcal{P}(Y') \arrow[d, dashed, "\wr"'] \\
\mathcal{P}(X\coprod Y) \arrow[r, "(f\coprod g)_*"'] & \mathcal{P}(X'\coprod Y')
\end{tikzcd}$$

commutes, which is clear, as it acts on elements as

$$\begin{tikzcd}
(U,V) \arrow[r, mapsto] \arrow[d, mapsto] & (f_*(U),g_*(V)) \arrow[d, mapsto] \\
U\coprod V \arrow[r, mapsto] & (f\coprod g)_*(U\coprod V)=f_*(U)\coprod g_*(V),
\end{tikzcd}$$

where we are using Item 7 of Proposition 4.4.4.

Finally, monoidality, unity, and symmetry of $\mathcal{P}_*$ as a monoidal functor follow by checking the commutativity of the relevant diagrams on elements.

Item 2: Symmetric Strong Monoidality With Respect to Coproducts II

The proof is similar to Item 1, and is hence omitted.

Item 3: Symmetric Strong Monoidality With Respect to Coproducts III

The proof is similar to Item 1, and is hence omitted.

Item 4: Symmetric Lax Monoidality With Respect to Products I

Naturality for the morphism $\mathcal{P}^{\times}_{*|X,Y}$ is the statement that, given maps of sets $f\colon X \to X'$ and $g\colon Y \to Y'$, the diagram

$$\begin{tikzcd}
\mathcal{P}(X)\times\mathcal{P}(Y) \arrow[r, "f_*\times g_*"] \arrow[d, dashed, "\wr"] & \mathcal{P}(X')\times\mathcal{P}(Y') \arrow[d, dashed, "\wr"'] \\
\mathcal{P}(X\times Y) \arrow[r, "(f\times g)_*"'] & \mathcal{P}(X'\times Y')
\end{tikzcd}$$

commutes, which is clear, as it acts on elements as

$$\begin{tikzcd}
(U,V) \arrow[r, mapsto] \arrow[d, mapsto] & (f_*(U),g_*(V)) \arrow[d, mapsto] \\
U\times V \arrow[r, mapsto] & (f\times g)_*(U\times V)=f_*(U)\times g_*(V),
\end{tikzcd}$$

where we are using Item 8 of Proposition 4.4.4.

Finally, monoidality, unity, and symmetry of $\mathcal{P}_*$ as a monoidal functor follow by checking the commutativity of the relevant diagrams on elements.

Item 5: Symmetric Lax Monoidality With Respect to Products II

The proof is similar to Item 4, and is hence omitted.

Item 6: Symmetric Lax Monoidality With Respect to Products III

The proof is similar to Item 4, and is hence omitted. □

PROPOSITION 4.3.9 ► PROPERTIES OF POWERSETS: AS SETS OF FUNCTIONS/RELATIONS

Let X be a set.

1. *Powersets as Sets of Functions I.* The assignment $U \mapsto \chi_U$ defines a bijection

$$\chi_{(-)}\colon \mathcal{P}(X) \xrightarrow{\cong} \mathsf{Sets}(X,\{\mathsf{t},\mathsf{f}\}),$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$.

2. *Powersets as Sets of Functions II.* The bijection

$$\mathcal{P}(X) \cong \mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\})$$

of Item 1 is natural in $X \in \mathrm{Obj}(\mathsf{Sets})$, refining to a natural isomorphism of functors

$$\mathcal{P}^{-1} \cong \mathsf{Sets}(-, \{\mathsf{t}, \mathsf{f}\}).$$

3. *Powersets as Sets of Relations.* We have bijections

$$\begin{aligned}\mathcal{P}(X) &\cong \mathrm{Rel}(\mathrm{pt}, X),\\ \mathcal{P}(X) &\cong \mathrm{Rel}(X, \mathrm{pt}),\end{aligned}$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets})$.

PROOF 4.3.10 ► PROOF OF PROPOSITION 4.3.9

Item 1: Powersets as Sets of Functions I

Indeed, the inverse of $\chi_{(-)}$ is given by sending a function $f\colon X \to \{\mathsf{t}, \mathsf{f}\}$ to the subset U_f of $\mathcal{P}(X)$ defined by

$$U_f \overset{\mathrm{def}}{=} \{x \in X \mid f(x) = \mathsf{true}\},$$

i.e. by $U_f = f^{-1}(\mathsf{true})$. That $\chi_{(-)}$ and $f \mapsto U_f$ are inverses is then straightforward to check.

Item 2: Powersets as Sets of Functions II

We need to check that, given a function $f\colon X \to Y$, the diagram

$$\begin{tikzcd}
\mathcal{P}(Y) \arrow[r, "f^{-1}"] \arrow[d, dashed, "\chi_{(-)}"', "\wr"] & \mathcal{P}(X) \arrow[d, dashed, "\chi_{(-)}", "\wr"'] \\
\mathsf{Sets}(Y, \{\mathsf{t}, \mathsf{f}\}) \arrow[r, "f^*"'] & \mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\})
\end{tikzcd}$$

commutes, i.e. that for each $V \in \mathcal{P}(Y)$, we have

$$\chi_V \circ f = \chi_{f^{-1}(V)}.$$

And indeed, we have

$$\begin{aligned}[\chi_V \circ f](v) &\overset{\text{def}}{=} \chi_V(f(v)) \\ &= \begin{cases} \text{true} & \text{if } f(v) \in V, \\ \text{false} & \text{otherwise} \end{cases} \\ &= \begin{cases} \text{true} & \text{if } v \in f^{-1}(V), \\ \text{false} & \text{otherwise} \end{cases} \\ &\overset{\text{def}}{=} \chi_{f^{-1}(V)}(v)\end{aligned}$$

for each $v \in V$.

Item 3: Powersets as Sets of Relations

Indeed, we have

$$\begin{aligned}\mathrm{Rel}(\mathrm{pt}, X) &\overset{\text{def}}{=} \mathcal{P}(\mathrm{pt} \times X) \\ &\cong \mathcal{P}(X)\end{aligned}$$

and

$$\begin{aligned}\mathrm{Rel}(X, \mathrm{pt}) &\overset{\text{def}}{=} \mathcal{P}(X \times \mathrm{pt}) \\ &\cong \mathcal{P}(X),\end{aligned}$$

where we have used Item 4 of Proposition 1.3.3. □

Remark 4.3.11 ► Powersets as Sets of Functions and Un/Straightening

The bijection

$$\mathcal{P}(X) \cong \mathrm{Sets}(X, \{\mathrm{t}, \mathrm{f}\})$$

of Item 1 of Proposition 4.3.9, which

- Takes a subset $U \hookrightarrow X$ of X and *straightens* it to a function $\chi_U : X \to \{\text{true}, \text{false}\}$;
- Takes a function $f : X \to \{\text{true}, \text{false}\}$ and *unstraightens* it to a subset $f^{-1}(\text{true}) \hookrightarrow X$ of X;

may be viewed as the (-1)-categorical version of the un/straightening isomorphism for indexed and fibred sets

$$\underbrace{\mathsf{FibSets}(X)}_{\overset{\text{def}}{=}\mathsf{Sets}_{/X}} \cong \underbrace{\mathsf{ISets}(X)}_{\overset{\text{def}}{=}\mathsf{Fun}(X_{\mathsf{disc}},\mathsf{Sets})}$$

of **??**, **??**, where we view:

- Subsets $U \hookrightarrow X$ as analogous to X-fibred sets $\phi_X \colon A \to X$.
- Functions $f \colon X \to \{\mathsf{t}, \mathsf{f}\}$ as analogous to X-indexed sets $A \colon X_{\mathsf{disc}} \to \mathsf{Sets}$.

Proposition 4.3.12 ▶ Properties of Powersets: As Free Cocompletions

Let X be a set.

1. *Universal Property.* The pair $(\mathcal{P}(X), \chi_{(-)})$ consisting of
   - The powerset $\mathcal{P}(X)$ of X;
   - The characteristic embedding $\chi_{(-)} \colon X \hookrightarrow \mathcal{P}(X)$ of X into $\mathcal{P}(X)$;

   satisfies the following universal property:

   (★) Given another pair (Y, f) consisting of
   - A cocomplete poset $(Y, \preceq)$;
   - A function $f \colon X \to Y$;

   there exists a unique cocontinuous morphism of posets

   $$(\mathcal{P}(X), \subset) \xrightarrow{\exists!} (Y, \preceq)$$

   making the diagram

   $$\begin{tikzcd} & \mathcal{P}(X) \arrow[d, dashed, "\exists!"] \\ X \arrow[ru, "\chi_X"] \arrow[r, "f"'] & Y \end{tikzcd}$$

   commute.

2. *Adjointness.* We have an adjunction[1]

$$(\mathcal{P} \dashv 忘)\colon \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "\mathcal{P}"] \arrow[r, phantom, "\perp"] & \mathsf{Pos}^{\mathsf{cocomp.}}, \arrow[l, bend left, "忘"] \end{tikzcd}$$

witnessed by a bijection

$$\mathsf{Pos}^{\mathsf{cocomp.}}((\mathcal{P}(X), \subset), (Y, \preceq)) \cong \mathsf{Sets}(X, Y),$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets})$ and $(Y, \preceq) \in \mathrm{Obj}(\mathsf{Pos}^{\mathsf{cocomp.}})$, where the maps witnessing this bijection are given by

- The map

  $$\chi_X^* \colon \mathsf{Pos}^{\mathsf{cocomp.}}((\mathcal{P}(X), \subset), (Y, \preceq)) \to \mathsf{Sets}(X, Y)$$

  defined by

  $$\chi_X^*(f) \overset{\mathrm{def}}{=} f \circ \chi_X,$$

  i.e. by sending a cocontinuous morphism of posets $f\colon \mathcal{P}(X) \to Y$ to the composition

  $$X \overset{\chi_X}{\hookrightarrow} \mathcal{P}(X) \xrightarrow{f} Y.$$

- The map

  $$\mathrm{Lan}_{\chi_X} \colon \mathsf{Sets}(X, Y) \to \mathsf{Pos}^{\mathsf{cocomp.}}((\mathcal{P}(X), \subset), (Y, \preceq))$$

  is given by sending a function $f\colon X \to Y$ to its left Kan extension along χ_X,

  $$\mathrm{Lan}_{\chi_X}(f)\colon \mathcal{P}(X) \to Y, \qquad \begin{tikzcd} & \mathcal{P}(X) \arrow[d, dashed, "\mathrm{Lan}_{\chi_X}(f)"] \\ X \arrow[ru, "\chi_X"] \arrow[r, "f"'] \arrow[ru, Rightarrow, shift right=3, shorten=3mm] & Y. \end{tikzcd}$$

Moreover, $\mathrm{Lan}_{\chi_X}(f)$ can be explicitly computed by

$$\begin{aligned}[\mathrm{Lan}_{\chi_X}(f)](U) &\cong \int^{x\in X} \chi_{\mathcal{P}(X)}(\chi_x, U) \odot f(x)\\ &\cong \int^{x\in X} \chi_U(x) \odot f(x) && \text{(by Proposition 4.2.1)}\\ &\cong \bigvee_{x\in X} (\chi_U(x) \odot f(x))\end{aligned}$$

for each $U \in \mathcal{P}(X)$, where:

- $\bigvee$ is the join in $(Y, \preceq)$.
- We have

$$\begin{aligned}\mathsf{true} \odot f(x) &\overset{\text{def}}{=} f(x),\\ \mathsf{false} \odot f(x) &\overset{\text{def}}{=} \varnothing_Y,\end{aligned}$$

where $\varnothing_Y$ is the minimal element of $(Y, \preceq)$.

---

[1] In this sense, $\mathcal{P}(A)$ is the free cocompletion of A. (Note that, despite its name, however, this is not an idempotent operation, as we have $\mathcal{P}(\mathcal{P}(A)) \neq \mathcal{P}(A)$.)

PROOF 4.3.13 ▶ PROOF OF PROPOSITION 4.3.7

Item 1: Universal Property

This is a rephrasing of Item 2.

Item 2: Adjointness

We claim we have adjunction $\mathcal{P} \dashv 忘$, witnessed by a bijection

$$\mathsf{Pos}^{\mathsf{cocomp.}}((\mathcal{P}(X), \subset), (Y, \preceq)) \cong \mathsf{Sets}(X, Y),$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets})$ and $(Y, \preceq) \in \mathrm{Obj}(\mathsf{Pos}^{\mathsf{cocomp.}})$.

- *Map I.* We define a map

  $$\Phi_{X,Y} \colon \mathsf{Pos}^{\mathsf{cocomp.}}((\mathcal{P}(X), \subset), (Y, \preceq)) \to \mathsf{Sets}(X, Y)$$

  as in the statement, by

  $$\Phi_{X,Y}(f) \overset{\text{def}}{=} f \circ \chi_X$$

for each $f \in \mathsf{Pos}^{\mathsf{cocomp.}}((\mathcal{P}(X), \subset), (Y, \preceq))$.

- *Map II.* We define a map

  $$\Psi_{X,Y} \colon \mathsf{Sets}(X, Y) \to \mathsf{Pos}^{\mathsf{cocomp.}}((\mathcal{P}(X), \subset), (Y, \preceq))$$

  as in the statement, by

  $$\Psi_{X,Y}(f) \overset{\text{def}}{=} \mathsf{Lan}_{\chi_X}(f), \qquad \begin{tikzcd} & \mathcal{P}(X) \arrow[d, dashed, "\mathsf{Lan}_{\chi_X}(f)", "|" marking] \\ X \arrow[ru, "\chi_X", "/" marking] \arrow[r, "f"', "|" marking] & Y, \arrow[u, Rightarrow, shift right=3ex, shorten=2ex] \end{tikzcd}$$

  for each $f \in \mathsf{Sets}(X, Y)$.

- *Invertibility I.* We claim that

  $$\Psi_{X,Y} \circ \Phi_{X,Y} = \mathrm{id}_{\mathsf{Pos}^{\mathsf{cocomp.}}((\mathcal{P}(X),\subset),(Y,\preceq))}.$$

  Indeed, given a cocontinuous morphism of posets

  $$\xi \colon (\mathcal{P}(X), \subset) \to (Y, \preceq),$$

  we have

  $$\begin{aligned} [\Psi_{X,Y} \circ \Phi_{X,Y}](\xi) &\overset{\text{def}}{=} \Psi_{X,Y}(\Phi_{X,Y}(\xi)) \\ &\overset{\text{def}}{=} \Psi_{X,Y}(\xi \circ \chi_X) \\ &\overset{\text{def}}{=} \mathsf{Lan}_{\chi_X}(\xi \circ \chi_X) \\ &\cong \bigvee_{x \in X} \chi_{(-)}(x) \odot \xi(\chi_X(x)) \\ &\overset{\text{clm}}{=} \xi, \end{aligned}$$

where indeed

$$
\begin{aligned}
\left[\bigvee_{x\in X}\chi_{(-)}(x)\odot\xi(\chi_X(x))\right](U) &\overset{\text{def}}{=} \bigvee_{x\in X}\chi_U(x)\odot\xi(\chi_X(x))\\
&= \Big(\bigvee_{x\in U}\chi_U(x)\odot\xi(\chi_X(x))\Big)\vee\Big(\bigvee_{x\in X\setminus U}\chi_U(x)\odot\xi(\chi_X(x))\Big)\\
&= \Big(\bigvee_{x\in U}\xi(\chi_X(x))\Big)\vee\Big(\bigvee_{x\in X\setminus U}\varnothing_Y\Big)\\
&= \bigvee_{x\in U}\xi(\chi_X(x))\\
&\overset{(\dagger)}{=} \xi\Big(\bigvee_{x\in U}\chi_X(x)\Big)\\
&= \xi(U)
\end{aligned}
$$

for each $U \in \mathcal{P}(X)$, where we have used that ξ is cocontinuous for the equality $\overset{(\dagger)}{=}$.

- *Invertibility II.* We claim that

$$\Phi_{X,Y}\circ\Psi_{X,Y} = \mathrm{id}_{\mathsf{Sets}(X,Y)}.$$

Indeed, given a function $f\colon X\to Y$, we have

$$
\begin{aligned}
[\Phi_{X,Y}\circ\Psi_{X,Y}](f) &\overset{\text{def}}{=} \Phi_{X,Y}(\Psi_{X,Y}(f))\\
&\overset{\text{def}}{=} \Phi_{X,Y}(\mathrm{Lan}_{\chi_X}(f))\\
&\overset{\text{def}}{=} \mathrm{Lan}_{\chi_X}(f)\circ\chi_X\\
&\overset{\text{clm}}{=} f,
\end{aligned}
$$

where indeed

$$
\begin{aligned}
[\mathrm{Lan}_{\chi_X}(f)\circ\chi_X](x) &\overset{\text{def}}{=} \bigvee_{y\in X}\chi_{\{x\}}(y)\odot f(y)\\
&= (\chi_{\{x\}}(x)\odot f(x))\vee\Big(\bigvee_{y\in X\setminus\{x\}}\chi_{\{x\}}(y)\odot f(y)\Big)\\
&= f(x)\vee\Big(\bigvee_{y\in X\setminus\{x\}}\varnothing_Y\Big)\\
&= f(x)\vee\varnothing_Y\\
&= f(x)
\end{aligned}
$$

for each $x\in X$.

- *Naturality for* Φ*, Part I.* We need to show that, given a function $f\colon X \to X'$, the diagram

$$\begin{tikzcd}
\mathsf{Pos}^{\mathsf{cocomp.}}((\mathcal{P}(X'),\subset),(Y,\preceq)) \arrow[r, "\Phi_{X',Y}"] \arrow[d, "\mathcal{P}_*(f)^*"'] & \mathsf{Sets}(X',Y) \arrow[d, "f^*"] \\
\mathsf{Pos}^{\mathsf{cocomp.}}((\mathcal{P}(X),\subset),(Y,\preceq)) \arrow[r, "\Phi_{X,Y}"'] & \mathsf{Sets}(X,Y)
\end{tikzcd}$$

  commutes. Indeed, given a cocontinuous morphism of posets

$$\xi\colon (\mathcal{P}(X'),\subset) \to (Y,\preceq),$$

  we have

$$\begin{aligned}
[\Phi_{X,Y}\circ \mathcal{P}_*(f)^*](\xi) &\overset{\text{def}}{=} \Phi_{X,Y}(\mathcal{P}_*(f)^*(\xi))\\
&\overset{\text{def}}{=} \Phi_{X,Y}(\xi\circ f_*)\\
&\overset{\text{def}}{=} (\xi\circ f_*)\circ \chi_X\\
&= \xi\circ(f_*\circ\chi_X)\\
&\overset{(\dagger)}{=} \xi\circ(\chi_{X'}\circ f)\\
&= (\xi\circ\chi_{X'})\circ f\\
&\overset{\text{def}}{=} \Phi_{X',Y}(\xi)\circ f\\
&\overset{\text{def}}{=} f^*(\Phi_{X',Y}(\xi))\\
&\overset{\text{def}}{=} [f^*\circ\Phi_{X',Y}](\xi),
\end{aligned}$$

  where we have used Item 9 of Proposition 4.1.3 for the equality $\overset{(\dagger)}{=}$ above.

- *Naturality for* Φ*, Part II.* We need to show that, given a cocontinuous morphism of posets

$$g\colon (Y,\preceq_Y) \to (Y',\preceq_{Y'}),$$

  the diagram

$$\begin{tikzcd}
\mathsf{Pos}^{\mathsf{cocomp.}}((\mathcal{P}(X),\subset),(Y,\preceq)) \arrow[r, "\Phi_{X,Y}"] \arrow[d, "g_*"'] & \mathsf{Sets}(X,Y) \arrow[d, "g_*"] \\
\mathsf{Pos}^{\mathsf{cocomp.}}((\mathcal{P}(X),\subset),(Y',\preceq)) \arrow[r, "\Phi_{X,Y'}"'] & \mathsf{Sets}(X,Y')
\end{tikzcd}$$

commutes. Indeed, given a cocontinuous morphism of posets

$$\xi\colon (\mathcal{P}(X), \subset) \to (Y, \preceq),$$

we have

$$\begin{aligned}
[\Phi_{X,Y'} \circ g_*](\xi) &\overset{\text{def}}{=} \Phi_{X,Y'}(g_*(\xi))\\
&\overset{\text{def}}{=} \Phi_{X,Y'}(g \circ \xi)\\
&\overset{\text{def}}{=} (g \circ \xi) \circ \chi_X\\
&= g \circ (\xi \circ \chi_X)\\
&\overset{\text{def}}{=} g \circ (\Phi_{X,Y}(\xi))\\
&\overset{\text{def}}{=} g_*(\Phi_{X,Y}(\xi))\\
&\overset{\text{def}}{=} [g_* \circ \Phi_{X,Y}](\xi).
\end{aligned}$$

- *Naturality for* Ψ. Since Φ is natural in each argument and Φ is a componentwise inverse to Ψ in each argument, it follows from Categories, Item 2 of Proposition 8.6.2 that Ψ is also natural in each argument.

This finishes the proof. □

4.4 Direct Images

Let A and B be sets and let $f\colon A \to B$ be a function.

Definition 4.4.1 ► Direct Images

The **direct image function associated to** f is the function

$$f_*\colon \mathcal{P}(A) \to \mathcal{P}(B)$$

defined by[1,2]

$$\begin{aligned}
f_*(U) &\overset{\text{def}}{=} f(U)\\
&\overset{\text{def}}{=} \left\{ b \in B \,\middle|\, \begin{array}{l}\text{there exists some } a \in U\\ \text{such that } b = f(a)\end{array} \right\}\\
&= \{ f(a) \in B \mid a \in U \}
\end{aligned}$$

for each $U \in \mathcal{P}(A)$.

---

[1] *Further Terminology:* The set $f(U)$ is called the **direct image of U by f**.

[2] We also have

$$f_*(U) = B \setminus f_!(A \setminus U);$$

see Item 9 of Proposition 4.4.4.

NOTATION 4.4.2 ► FURTHER NOTATION FOR DIRECT IMAGES

Sometimes one finds the notation

$$\exists_f \colon \mathcal{P}(A) \to \mathcal{P}(B)$$

for f_*. This notation comes from the fact that the following statements are equivalent, where $b \in B$ and $U \in \mathcal{P}(A)$:

- We have $b \in \exists_f(U)$.
- There exists some $a \in U$ such that $f(a) = b$.

REMARK 4.4.3 ► UNWINDING DEFINITION 4.4.1

Identifying subsets of A with functions from A to $\{\text{true}, \text{false}\}$ via Items 1 and 2 of Proposition 4.3.9, we see that the direct image function associated to f is equivalently the function

$$f_* \colon \mathcal{P}(A) \to \mathcal{P}(B)$$

defined by

$$\begin{aligned} f_*(\chi_U) &\overset{\text{def}}{=} \mathrm{Lan}_f(\chi_U) \\ &= \mathrm{colim}\Big(\big(f \overset{\rightarrow}{\times} \underline{(-_1)}\big) \overset{\text{pr}}{\twoheadrightarrow} A \xrightarrow{\chi_U} \{\mathsf{t}, \mathsf{f}\}\Big) \\ &= \operatorname*{colim}_{\substack{a \in A \\ f(a) = -_1}} (\chi_U(a)) \\ &= \bigvee_{\substack{a \in A \\ f(a) = -_1}} (\chi_U(a)), \end{aligned}$$

where we have used **??**, **??** for the second equality. In other words, we have

$$
\begin{aligned}
[f_*(\chi_U)](b) &= \bigvee_{\substack{a\in A\\ f(a)=b}} (\chi_U(a))\\
&= \begin{cases} \text{true} & \text{if there exists some } a \in A \text{ such}\\ & \text{that } f(a) = b \text{ and } a \in U,\\ \text{false} & \text{otherwise} \end{cases}\\
&= \begin{cases} \text{true} & \text{if there exists some } a \in U\\ & \text{such that } f(a) = b,\\ \text{false} & \text{otherwise} \end{cases}
\end{aligned}
$$

for each $b \in B$.

Proposition 4.4.4 ► Properties of Direct Images I

Let $f\colon A \longrightarrow B$ be a function.

1. *Functoriality*. The assignment $U \mapsto f_*(U)$ defines a functor

$$f_*\colon (\mathcal{P}(A), \subset) \longrightarrow (\mathcal{P}(B), \subset)$$

where

- *Action on Objects*. For each $U \in \mathcal{P}(A)$, we have

$$[f_*](U) \overset{\text{def}}{=} f_*(U).$$

- *Action on Morphisms*. For each $U, V \in \mathcal{P}(A)$:

  ($\star$) If $U \subset V$, then $f_*(U) \subset f_*(V)$.

2. *Triple Adjointness*. We have a triple adjunction

$$(f_* \dashv f^{-1} \dashv f_!)\colon \quad \begin{tikzcd} \mathcal{P}(A) \arrow[r, bend left=50, "f_*"] \arrow[r, bend right=50, "f_!"'] \arrow[r, phantom, shift left=2.5ex, "\perp"] \arrow[r, phantom, shift right=2.5ex, "\perp"] & \mathcal{P}(B), \arrow[l, "f^{-1}" description] \end{tikzcd}$$

witnessed by bijections of sets

$$\begin{aligned}\mathrm{Hom}_{\mathcal{P}(B)}(f_*(U),V) &\cong \mathrm{Hom}_{\mathcal{P}(A)}(U,f^{-1}(V)),\\ \mathrm{Hom}_{\mathcal{P}(A)}(f^{-1}(U),V) &\cong \mathrm{Hom}_{\mathcal{P}(A)}(U,f_!(V)),\end{aligned}$$

natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$ and (respectively) $V \in \mathcal{P}(A)$ and $U \in \mathcal{P}(B)$, i.e. where:

(a) The following conditions are equivalent:
    i. We have $f_*(U) \subset V$.
    ii. We have $U \subset f^{-1}(V)$.
(b) The following conditions are equivalent:
    i. We have $f^{-1}(U) \subset V$.
    ii. We have $U \subset f_!(V)$.

3. *Preservation of Colimits.* We have an equality of sets

$$f_*(\bigcup_{i\in I} U_i) = \bigcup_{i\in I} f_*(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have equalities

$$\begin{aligned}f_*(U) \cup f_*(V) &= f_*(U \cup V),\\ f_*(\emptyset) &= \emptyset,\end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

4. *Oplax Preservation of Limits.* We have an inclusion of sets

$$f_*(\bigcap_{i\in I} U_i) \subset \bigcap_{i\in I} f_*(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have inclusions

$$\begin{aligned}f_*(U \cap V) &\subset f_*(U) \cap f_*(V),\\ f_*(A) &\subset B,\end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

5. *Symmetric Strict Monoidality With Respect to Unions.* The direct image function of Item 1 has a symmetric strict monoidal structure

$$(f_*, f_*^{\otimes}, f_{*|\mathbb{1}}^{\otimes}) \colon (\mathcal{P}(A), \cup, \emptyset) \longrightarrow (\mathcal{P}(B), \cup, \emptyset),$$

being equipped with equalities

$$f_{*|U,V}^{\otimes} \colon f_*(U) \cup f_*(V) \xrightarrow{=} f_*(U \cup V),$$
$$f_{*|\mathbb{1}}^{\otimes} \colon \emptyset \xrightarrow{=} \emptyset,$$

natural in $U, V \in \mathcal{P}(A)$.

6. *Symmetric Oplax Monoidality With Respect to Intersections.* The direct image function of Item 1 has a symmetric oplax monoidal structure

$$(f_*, f_*^{\otimes}, f_{*|\mathbb{1}}^{\otimes}) \colon (\mathcal{P}(A), \cap, A) \longrightarrow (\mathcal{P}(B), \cap, B),$$

being equipped with inclusions

$$f_{*|U,V}^{\otimes} \colon f_*(U \cap V) \hookrightarrow f_*(U) \cap f_*(V),$$
$$f_{*|\mathbb{1}}^{\otimes} \colon f_*(A) \hookrightarrow B,$$

natural in $U, V \in \mathcal{P}(A)$.

7. *Interaction With Coproducts.* Let $f \colon A \longrightarrow A'$ and $g \colon B \longrightarrow B'$ be maps of sets. We have

$$(f \coprod g)_*(U \coprod V) = f_*(U) \coprod g_*(V)$$

for each $U \in \mathcal{P}(A)$ and each $V \in \mathcal{P}(B)$.

8. *Interaction With Products.* Let $f \colon A \longrightarrow A'$ and $g \colon B \longrightarrow B'$ be maps of sets. We have

$$(f \times g)_*(U \times V) = f_*(U) \times g_*(V)$$

for each $U \in \mathcal{P}(A)$ and each $V \in \mathcal{P}(B)$.

9. *Relation to Direct Images With Compact Support.* We have

$$f_*(U) = B \setminus f_!(A \setminus U)$$

for each $U \in \mathcal{P}(A)$.

PROOF 4.4.5 ► PROOF OF PROPOSITION 4.4.4

Item 1: Functoriality

Clear.

Item 2: Triple Adjointness

This follows from Remark 4.4.3, Remark 4.5.2, Remark 4.6.3, and **??**, **??** of **??**.

Item 3: Preservation of Colimits

This follows from Item 2 and **??**, **??** of **??**.[1]

Item 4: Oplax Preservation of Limits

The inclusion $f_*(A) \subset B$ is clear. See [Pro24p] for the other inclusions.

Item 5: Symmetric Strict Monoidality With Respect to Unions

This follows from Item 3.

Item 6: Symmetric Oplax Monoidality With Respect to Intersections

This follows from Item 4.

Item 7: Interaction With Coproducts

Clear.

Item 8: Interaction With Products

Clear.

Item 9: Relation to Direct Images With Compact Support

Applying Item 9 of Proposition 4.6.6 to $A \setminus U$, we have

$$\begin{aligned} f_!(A \setminus U) &= B \setminus f_*(A \setminus (A \setminus U)) \\ &= B \setminus f_*(U). \end{aligned}$$

Taking complements, we then obtain

$$\begin{aligned} f_*(U) &= B \setminus (B \setminus f_*(U)), \\ &= B \setminus f_!(A \setminus U), \end{aligned}$$

which finishes the proof. □

---

[1] See also [Pro24q].

Proposition 4.4.6 ► Properties of Direct Images II

Let $f\colon A \longrightarrow B$ be a function.

1. *Functionality I.* The assignment $f \mapsto f_*$ defines a function

$$(-)_{*|A,B}\colon \mathsf{Sets}(A, B) \longrightarrow \mathsf{Sets}(\mathcal{P}(A), \mathcal{P}(B)).$$

2. *Functionality II.* The assignment $f \mapsto f_*$ defines a function

$$(-)_{*|A,B}\colon \mathsf{Sets}(A, B) \longrightarrow \mathsf{Pos}((\mathcal{P}(A), \subset), (\mathcal{P}(B), \subset)).$$

3. *Interaction With Identities.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$(\mathrm{id}_A)_* = \mathrm{id}_{\mathcal{P}(A)}.$$

4. *Interaction With Composition.* For each pair of composable functions $f\colon A \longrightarrow B$ and $g\colon B \longrightarrow C$, we have

$$(g \circ f)_* = g_* \circ f_*, \qquad \begin{tikzcd} \mathcal{P}(A) \arrow[r, "f_*"] \arrow[dr, "(g\circ f)_*"'] & \mathcal{P}(B) \arrow[d, "g_*"] \\ & \mathcal{P}(C). \end{tikzcd}$$

Proof 4.4.7 ► Proof of Proposition 4.4.6

Item 1: Functionality I

Clear.

Item 2: Functionality II

Clear.

Item 3: Interaction With Identities

This follows from Remark 4.4.3 and ??, ?? of ??.

Item 4: Interaction With Composition

This follows from Remark 4.4.3 and ??, ?? of ??. □

4.5 Inverse Images

Let A and B be sets and let $f\colon A \to B$ be a function.

DEFINITION 4.5.1 ► INVERSE IMAGES

The **inverse image function associated to** f is the function[1]

$$f^{-1}\colon \mathcal{P}(B) \to \mathcal{P}(A)$$

defined by[2]

$$f^{-1}(V) \overset{\text{def}}{=} \{a \in A \mid \text{we have } f(a) \in V\}$$

for each $V \in \mathcal{P}(B)$.

---

[1] *Further Notation:* Also written $f^*\colon \mathcal{P}(B) \to \mathcal{P}(A)$.

[2] *Further Terminology:* The set $f^{-1}(V)$ is called the **inverse image of** V **by** f.

REMARK 4.5.2 ► UNWINDING DEFINITION 4.5.1

Identifying subsets of B with functions from B to $\{\text{true}, \text{false}\}$ via Items 1 and 2 of Proposition 4.3.9, we see that the inverse image function associated to f is equivalently the function

$$f^*\colon \mathcal{P}(B) \to \mathcal{P}(A)$$

defined by

$$f^*(\chi_V) \overset{\text{def}}{=} \chi_V \circ f$$

for each $\chi_V \in \mathcal{P}(B)$, where $\chi_V \circ f$ is the composition

$$A \xrightarrow{f} B \xrightarrow{\chi_V} \{\text{true}, \text{false}\}$$

in Sets.

PROPOSITION 4.5.3 ► PROPERTIES OF INVERSE IMAGES I

Let $f\colon A \to B$ be a function.

1. *Functoriality.* The assignment $V \mapsto f^{-1}(V)$ defines a functor

$$f^{-1}\colon (\mathcal{P}(B), \subset) \to (\mathcal{P}(A), \subset)$$

where

   - *Action on Objects.* For each $V \in \mathcal{P}(B)$, we have

$$[f^{-1}](V) \overset{\text{def}}{=} f^{-1}(V).$$

   - *Action on Morphisms.* For each $U, V \in \mathcal{P}(B)$:

     ($\star$) If $U \subset V$, then $f^{-1}(U) \subset f^{-1}(V)$.

2. *Triple Adjointness.* We have a triple adjunction

$$\left(f_* \dashv f^{-1} \dashv f_!\right)\colon \quad \begin{tikzcd}[column sep=large] \mathcal{P}(A) \arrow[r, bend left=40, "f_*"] \arrow[r, bend right=40, "f_!"'] \arrow[r, phantom, shift left=2.5ex, "\perp"] \arrow[r, phantom, shift right=2.5ex, "\perp"] & \mathcal{P}(B) \arrow[l, "f^{-1}" description] \end{tikzcd},$$

witnessed by bijections of sets

$$\begin{aligned} \mathrm{Hom}_{\mathcal{P}(B)}(f_*(U), V) &\cong \mathrm{Hom}_{\mathcal{P}(A)}(U, f^{-1}(V)), \\ \mathrm{Hom}_{\mathcal{P}(A)}(f^{-1}(U), V) &\cong \mathrm{Hom}_{\mathcal{P}(A)}(U, f_!(V)), \end{aligned}$$

natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$ and (respectively) $V \in \mathcal{P}(A)$ and $U \in \mathcal{P}(B)$, i.e. where:

   (a) The following conditions are equivalent:
      i. We have $f_*(U) \subset V$;
      ii. We have $U \subset f^{-1}(V)$;

   (b) The following conditions are equivalent:
      i. We have $f^{-1}(U) \subset V$.
      ii. We have $U \subset f_!(V)$.

3. *Preservation of Colimits.* We have an equality of sets

$$f^{-1}\Big(\bigcup_{i\in I} U_i\Big) = \bigcup_{i\in I} f^{-1}(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(B)^{\times I}$. In particular, we have equalities

$$\begin{gathered} f^{-1}(U) \cup f^{-1}(V) = f^{-1}(U \cup V), \\ f^{-1}(\emptyset) = \emptyset, \end{gathered}$$

natural in $U, V \in \mathcal{P}(B)$.

4. *Preservation of Limits.* We have an equality of sets

$$f^{-1}\Big(\bigcap_{i\in I} U_i\Big) = \bigcap_{i\in I} f^{-1}(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(B)^{\times I}$. In particular, we have equalities

$$\begin{gathered} f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V), \\ f^{-1}(B) = A, \end{gathered}$$

natural in $U, V \in \mathcal{P}(B)$.

5. *Symmetric Strict Monoidality With Respect to Unions.* The inverse image function of Item 1 has a symmetric strict monoidal structure

$$\big(f^{-1}, f^{-1,\otimes}, f^{-1,\otimes}_{\mathbb{1}}\big)\colon (\mathcal{P}(B), \cup, \emptyset) \longrightarrow (\mathcal{P}(A), \cup, \emptyset),$$

being equipped with equalities

$$\begin{gathered} f^{-1,\otimes}_{U,V}\colon f^{-1}(U) \cup f^{-1}(V) \xrightarrow{=} f^{-1}(U \cup V), \\ f^{-1,\otimes}_{\mathbb{1}}\colon \emptyset \xrightarrow{=} f^{-1}(\emptyset), \end{gathered}$$

natural in $U, V \in \mathcal{P}(B)$.

6. *Symmetric Strict Monoidality With Respect to Intersections.* The inverse image function of Item 1 has a symmetric strict monoidal structure

$$\big(f^{-1}, f^{-1,\otimes}, f^{-1,\otimes}_{\mathbb{1}}\big)\colon (\mathcal{P}(B), \cap, B) \longrightarrow (\mathcal{P}(A), \cap, A),$$

being equipped with equalities

$$f^{-1,\otimes}_{U,V}\colon f^{-1}(U)\cap f^{-1}(V)\xrightarrow{=}f^{-1}(U\cap V),$$
$$f^{-1,\otimes}_{\mathbb{1}}\colon A\xrightarrow{=}f^{-1}(B),$$

natural in $U, V\in\mathcal{P}(B)$.

7. *Interaction With Coproducts.* Let $f\colon A\longrightarrow A'$ and $g\colon B\longrightarrow B'$ be maps of sets. We have

$$(f\coprod g)^{-1}(U'\coprod V')=f^{-1}(U')\coprod g^{-1}(V')$$

for each $U'\in\mathcal{P}(A')$ and each $V'\in\mathcal{P}(B')$.

8. *Interaction With Products.* Let $f\colon A\longrightarrow A'$ and $g\colon B\longrightarrow B'$ be maps of sets. We have

$$(f\times g)^{-1}(U'\times V')=f^{-1}(U')\times g^{-1}(V')$$

for each $U'\in\mathcal{P}(A')$ and each $V'\in\mathcal{P}(B')$.

PROOF 4.5.4 ▶ PROOF OF PROPOSITION 4.5.3

Item 1: Functoriality

Clear.

Item 2: Triple Adjointness

This follows from Remark 4.4.3, Remark 4.5.2, Remark 4.6.3, and **??**, **??** of **??**.

Item 3: Preservation of Colimits

This follows from Item 2 and **??**, **??** of **??**.[1]

Item 4: Preservation of Limits

This follows from Item 2 and **??**, **??** of **??**.[2]

Item 5: Symmetric Strict Monoidality With Respect to Unions

This follows from Item 3.

Item 6: Symmetric Strict Monoidality With Respect to Intersections

This follows from Item 4.

Item 7: Interaction With Coproducts

Clear.

Item 8: Interaction With Products

Clear. □

---

[1] See also [Pro24y].

[2] See also [Pro24x].

Proposition 4.5.5 ► Properties of Inverse Images II

Let $f\colon A \longrightarrow B$ be a function.

1. *Functionality I.* The assignment $f \mapsto f^{-1}$ defines a function
$$(-)^{-1}_{A,B}\colon \mathsf{Sets}(A,B) \longrightarrow \mathsf{Sets}(\mathcal{P}(B),\mathcal{P}(A)).$$
2. *Functionality II.* The assignment $f \mapsto f^{-1}$ defines a function
$$(-)^{-1}_{A,B}\colon \mathsf{Sets}(A,B) \longrightarrow \mathsf{Pos}((\mathcal{P}(B),\subset),(\mathcal{P}(A),\subset)).$$
3. *Interaction With Identities.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have
$$\mathrm{id}_A^{-1} = \mathrm{id}_{\mathcal{P}(A)}.$$
4. *Interaction With Composition.* For each pair of composable functions $f\colon A \longrightarrow B$ and $g\colon B \longrightarrow C$, we have
$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}, \qquad \begin{tikzcd} \mathcal{P}(C) \arrow[r, "g^{-1}"] \arrow[dr, "(g\circ f)^{-1}"'] & \mathcal{P}(B) \arrow[d, "f^{-1}"] \\ & \mathcal{P}(A). \end{tikzcd}$$

PROOF 4.5.6 ► PROOF OF PROPOSITION 4.5.5

Item 1: Functionality I

Clear.

Item 2: Functionality II

Clear.

Item 3: Interaction With Identities

This follows from Remark 4.5.2 and Categories, Item 5 of Proposition 1.6.2.

Item 4: Interaction With Composition

This follows from Remark 4.5.2 and Categories, Item 2 of Proposition 1.6.2. ∎

4.6 Direct Images With Compact Support

Let A and B be sets and let $f\colon A \longrightarrow B$ be a function.

DEFINITION 4.6.1 ► DIRECT IMAGES WITH COMPACT SUPPORT

The **direct image with compact support function associated to** f is the function

$$f_!\colon \mathcal{P}(A) \longrightarrow \mathcal{P}(B)$$

defined by[1,2]

$$\begin{aligned} f_!(U) &\overset{\text{def}}{=} \left\{ b \in B \,\middle|\, \begin{array}{l} \text{for each } a \in A, \text{ if we have} \\ f(a) = b, \text{ then } a \in U \end{array} \right\} \\ &= \left\{ b \in B \,\middle|\, \text{we have } f^{-1}(b) \subset U \right\} \end{aligned}$$

for each $U \in \mathcal{P}(A)$.

---

[1] *Further Terminology*: The set $f_!(U)$ is called the **direct image with compact support of** U **by** f.

[2] We also have

$$f_!(U) = B \setminus f_*(A \setminus U);$$

see Item 9 of Proposition 4.6.6.

NOTATION 4.6.2 ► FURTHER NOTATION FOR DIRECT IMAGES WITH COMPACT SUPPORT

Sometimes one finds the notation

$$\forall_f \colon \mathcal{P}(A) \to \mathcal{P}(B)$$

for f_*. This notation comes from the fact that the following statements are equivalent, where $b \in B$ and $U \in \mathcal{P}(A)$:

- We have $b \in \forall_f(U)$.
- For each $a \in A$, if $b = f(a)$, then $a \in U$.

REMARK 4.6.3 ► UNWINDING DEFINITION 4.6.1

Identifying subsets of A with functions from A to $\{\text{true}, \text{false}\}$ via Items 1 and 2 of Proposition 4.3.9, we see that the direct image with compact support function associated to f is equivalently the function

$$f_! \colon \mathcal{P}(A) \to \mathcal{P}(B)$$

defined by

$$\begin{aligned} f_!(\chi_U) &\overset{\text{def}}{=} \mathrm{Ran}_f(\chi_U) \\ &= \lim((\underline{(-_1)} \overset{\rightarrow}{\times} f) \overset{\text{pr}}{\twoheadrightarrow} A \xrightarrow{\chi_U} \{\text{true}, \text{false}\}) \\ &= \lim_{\substack{a \in A \\ f(a) = -_1}} (\chi_U(a)) \\ &= \bigwedge_{\substack{a \in A \\ f(a) = -_1}} (\chi_U(a)). \end{aligned}$$

where we have used **??**, **??** for the second equality. In other words, we have

$$\begin{aligned}[f_!(\chi_U)](b) &= \bigwedge_{\substack{a\in A\\ f(a)=b}} (\chi_U(a))\\ &= \begin{cases}\text{true} & \text{if, for each } a \in A \text{ such that}\\ & f(a) = b\text{, we have } a \in U,\\ \text{false} & \text{otherwise}\end{cases}\\ &= \begin{cases}\text{true} & \text{if } f^{-1}(b) \subset U\\ \text{false} & \text{otherwise}\end{cases}\end{aligned}$$

for each $b \in B$.

Definition 4.6.4 ► The Image and Complement Parts of $f_!$

Let U be a subset of A.[1,2]

1. The **image part of the direct image with compact support $f_!(U)$ of U** is the set $f_{!,\mathrm{im}}(U)$ defined by

$$\begin{aligned}f_{!,\mathrm{im}}(U) &\overset{\text{def}}{=} f_!(U) \cap \mathrm{Im}(f)\\ &= \left\{b \in B \,\middle|\, \begin{array}{l}\text{we have } f^{-1}(b) \subset\\ U \text{ and } f^{-1}(b) \neq \emptyset\end{array}\right\}.\end{aligned}$$

2. The **complement part of the direct image with compact support $f_!(U)$ of U** is the set $f_{!,\mathrm{cp}}(U)$ defined by

$$\begin{aligned}f_{!,\mathrm{cp}}(U) &\overset{\text{def}}{=} f_!(U) \cap (B \setminus \mathrm{Im}(f))\\ &= B \setminus \mathrm{Im}(f)\\ &= \left\{b \in B \,\middle|\, \begin{array}{l}\text{we have } f^{-1}(b) \subset\\ U \text{ and } f^{-1}(b) = \emptyset\end{array}\right\}\\ &= \left\{b \in B \,\middle|\, f^{-1}(b) = \emptyset\right\}.\end{aligned}$$

---

[1] Note that we have

$$f_!(U) = f_{!,\mathrm{im}}(U) \cup f_{!,\mathrm{cp}}(U),$$

as

$$\begin{aligned} f_!(U) &= f_!(U) \cap B \\ &= f_!(U) \cap (\mathrm{Im}(f) \cup (B \setminus \mathrm{Im}(f))) \\ &= (f_!(U) \cap \mathrm{Im}(f)) \cup (f_!(U) \cap (B \setminus \mathrm{Im}(f))) \\ &\overset{\text{def}}{=} f_{!,\mathrm{im}}(U) \cup f_{!,\mathrm{cp}}(U). \end{aligned}$$

[2] In terms of the meet computation of $f_!(U)$ of Remark 4.6.3, namely

$$f_!(\chi_U) = \bigwedge_{\substack{a \in A \\ f(a) = -_1}} (\chi_U(a)),$$

we see that $f_{!,\mathrm{im}}$ corresponds to meets indexed over nonempty sets, while $f_{!,\mathrm{cp}}$ corresponds to meets indexed over the empty set.

Example 4.6.5 ▶ Examples of Direct Images With Compact Support

Here are some examples of direct images with compact support.

1. *The Multiplication by Two Map on the Natural Numbers.* Consider the function $f\colon \mathbb{N} \longrightarrow \mathbb{N}$ given by

$$f(n) \overset{\text{def}}{=} 2n$$

for each $n \in \mathbb{N}$. Since f is injective, we have

$$\begin{aligned} f_{!,\mathrm{im}}(U) &= f_*(U) \\ f_{!,\mathrm{cp}}(U) &= \{\text{odd natural numbers}\} \end{aligned}$$

for any $U \subset \mathbb{N}$.

2. *Parabolas.* Consider the function $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ given by

$$f(x) \overset{\text{def}}{=} x^2$$

for each $x \in \mathbb{R}$. We have

$$f_{!,\mathrm{cp}}(U) = \mathbb{R}_{<0}$$

for any $U \subset \mathbb{R}$. Moreover, since $f^{-1}(x) = \left\{-\sqrt{x}, \sqrt{x}\right\}$, we have e.g.:

$$\begin{aligned} f_{!,\mathrm{im}}([0,1]) &= \{0\}, \\ f_{!,\mathrm{im}}([-1,1]) &= [0,1], \\ f_{!,\mathrm{im}}([1,2]) &= \emptyset, \\ f_{!,\mathrm{im}}([-2,-1] \cup [1,2]) &= [1,4]. \end{aligned}$$

3. *Circles.* Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x,y) \overset{\text{def}}{=} x^2 + y^2$$

for each $(x,y) \in \mathbb{R}^2$. We have

$$f_{!,\mathrm{cp}}(U) = \mathbb{R}_{<0}$$

for any $U \subset \mathbb{R}^2$, and since

$$f^{-1}(r) = \begin{cases} \text{a circle of radius } r \text{ about the origin} & \text{if } r > 0, \\ \{(0,0)\} & \text{if } r = 0, \\ \emptyset & \text{if } r < 0, \end{cases}$$

we have e.g.:

$$\begin{aligned} f_{!,\mathrm{im}}([-1,1] \times [-1,1]) &= [0,1], \\ f_{!,\mathrm{im}}(([-1,1] \times [-1,1]) \setminus [-1,1] \times \{0\}) &= \emptyset. \end{aligned}$$

Proposition 4.6.6 ► Properties of Direct Images With Compact Support I

Let $f\colon A \to B$ be a function.

1. *Functoriality.* The assignment $U \mapsto f_!(U)$ defines a functor

$$f_!\colon (\mathcal{P}(A), \subset) \to (\mathcal{P}(B), \subset)$$

where

- *Action on Objects.* For each $U \in \mathcal{P}(A)$, we have

$$[f_!](U) \overset{\text{def}}{=} f_!(U).$$

- *Action on Morphisms.* For each $U, V \in \mathcal{P}(A)$:

  ($\star$) If $U \subset V$, then $f_!(U) \subset f_!(V)$.

2. *Triple Adjointness.* We have a triple adjunction

$$\left(f_* \dashv f^{-1} \dashv f_!\right)\colon \quad \begin{tikzcd} \mathcal{P}(A) \arrow[r, bend left=45, "f_*"] \arrow[r, bend right=45, "f_!"'] & \mathcal{P}(B), \arrow[l, "f^{-1}"' description] \arrow[l, phantom, bend right=25, "\perp"] \arrow[l, phantom, bend left=25, "\perp"] \end{tikzcd}$$

   witnessed by bijections of sets

$$\begin{aligned} \mathrm{Hom}_{\mathcal{P}(B)}(f_*(U), V) &\cong \mathrm{Hom}_{\mathcal{P}(A)}(U, f^{-1}(V)), \\ \mathrm{Hom}_{\mathcal{P}(A)}(f^{-1}(U), V) &\cong \mathrm{Hom}_{\mathcal{P}(A)}(U, f_!(V)), \end{aligned}$$

   natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$ and (respectively) $V \in \mathcal{P}(A)$ and $U \in \mathcal{P}(B)$, i.e. where:

   (a) The following conditions are equivalent:
      i. We have $f_*(U) \subset V$.
      ii. We have $U \subset f^{-1}(V)$.

   (b) The following conditions are equivalent:
      i. We have $f^{-1}(U) \subset V$.
      ii. We have $U \subset f_!(V)$.

3. *Lax Preservation of Colimits.* We have an inclusion of sets

$$\bigcup_{i \in I} f_!(U_i) \subset f_!(\bigcup_{i \in I} U_i),$$

   natural in $\{U_i\}_{i \in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have inclusions

$$\begin{aligned} f_!(U) \cup f_!(V) &\hookrightarrow f_!(U \cup V), \\ \emptyset &\hookrightarrow f_!(\emptyset), \end{aligned}$$

   natural in $U, V \in \mathcal{P}(A)$.

4. *Preservation of Limits.* We have an equality of sets

$$f_!(\bigcap_{i \in I} U_i) = \bigcap_{i \in I} f_!(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have equalities

$$\begin{aligned} f^{-1}(U \cap V) &= f_!(U) \cap f^{-1}(V), \\ f_!(A) &= B, \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

5. *Symmetric Lax Monoidality With Respect to Unions.* The direct image with compact support function of Item 1 has a symmetric lax monoidal structure

   $$(f_!, f_!^{\otimes}, f_{!|\mathbb{1}}^{\otimes}) \colon (\mathcal{P}(A), \cup, \emptyset) \to (\mathcal{P}(B), \cup, \emptyset),$$

   being equipped with inclusions

   $$\begin{aligned} f_{!|U,V}^{\otimes} \colon f_!(U) \cup f_!(V) &\hookrightarrow f_!(U \cup V), \\ f_{!|\mathbb{1}}^{\otimes} \colon \emptyset &\hookrightarrow f_!(\emptyset), \end{aligned}$$

   natural in $U, V \in \mathcal{P}(A)$.

6. *Symmetric Strict Monoidality With Respect to Intersections.* The direct image function of Item 1 has a symmetric strict monoidal structure

   $$(f_!, f_!^{\otimes}, f_{!|\mathbb{1}}^{\otimes}) \colon (\mathcal{P}(A), \cap, A) \to (\mathcal{P}(B), \cap, B),$$

   being equipped with equalities

   $$\begin{aligned} f_{!|U,V}^{\otimes} \colon f_!(U \cap V) &\xrightarrow{=} f_!(U) \cap f_!(V), \\ f_{!|\mathbb{1}}^{\otimes} \colon f_!(A) &\xrightarrow{=} B, \end{aligned}$$

   natural in $U, V \in \mathcal{P}(A)$.

7. *Interaction With Coproducts.* Let $f \colon A \to A'$ and $g \colon B \to B'$ be maps of sets. We have

   $$(f \textstyle\coprod g)_!(U \coprod V) = f_!(U) \coprod g_!(V)$$

   for each $U \in \mathcal{P}(A)$ and each $V \in \mathcal{P}(B)$.

8. *Interaction With Products.* Let $f \colon A \to A'$ and $g \colon B \to B'$ be maps of sets. We have

   $$(f \times g)_!(U \times V) = f_!(U) \times g_!(V)$$

   for each $U \in \mathcal{P}(A)$ and each $V \in \mathcal{P}(B)$.

9. *Relation to Direct Images.* We have

$$f_!(U) = B \setminus f_*(A \setminus U)$$

for each $U \in \mathcal{P}(A)$.

10. *Interaction With Injections.* If f is injective, then we have

$$\begin{aligned} f_{!,\mathrm{im}}(U) &= f_*(U), \\ f_{!,\mathrm{cp}}(U) &= B \setminus \mathrm{Im}(f), \\ f_!(U) &= f_{!,\mathrm{im}}(U) \cup f_{!,\mathrm{cp}}(U) \\ &= f_*(U) \cup (B \setminus \mathrm{Im}(f)) \end{aligned}$$

for each $U \in \mathcal{P}(A)$.

11. *Interaction With Surjections.* If f is surjective, then we have

$$\begin{aligned} f_{!,\mathrm{im}}(U) &\subset f_*(U), \\ f_{!,\mathrm{cp}}(U) &= \emptyset, \\ f_!(U) &\subset f_*(U) \end{aligned}$$

for each $U \in \mathcal{P}(A)$.

PROOF 4.6.7 ► PROOF OF PROPOSITION 4.6.6

Item 1: Functoriality

Clear.

Item 2: Triple Adjointness

This follows from Remark 4.4.3, Remark 4.5.2, Remark 4.6.3, and **??**, **??** of **??**.

Item 3: Lax Preservation of Colimits

Omitted.

Item 4: Preservation of Limits

This follows from Item 2 and **??**, **??** of **??**.

Item 5: Symmetric Lax Monoidality With Respect to Unions

This follows from Item 3.

Item 6: Symmetric Strict Monoidality With Respect to Intersections

This follows from Item 4.

Item 7: Interaction With Coproducts

Clear.

Item 8: Interaction With Products

Clear.

Item 9: Relation to Direct Images

We claim that $f_!(U) = B \setminus f_*(A \setminus U)$.

- *The First Implication.* We claim that

  $$f_!(U) \subset B \setminus f_*(A \setminus U).$$

  Let $b \in f_!(U)$. We need to show that $b \notin f_*(A \setminus U)$, i.e. that there is no $a \in A \setminus U$ such that $f(a) = b$.

  This is indeed the case, as otherwise we would have $a \in f^{-1}(b)$ and $a \notin U$, contradicting $f^{-1}(b) \subset U$ (which holds since $b \in f_!(U)$).

  Thus $b \in B \setminus f_*(A \setminus U)$.

- *The Second Implication.* We claim that

  $$B \setminus f_*(A \setminus U) \subset f_!(U).$$

  Let $b \in B \setminus f_*(A \setminus U)$. We need to show that $b \in f_!(U)$, i.e. that $f^{-1}(b) \subset U$.

  Since $b \notin f_*(A \setminus U)$, there exists no $a \in A \setminus U$ such that $b = f(a)$, and hence $f^{-1}(b) \subset U$.

  Thus $b \in f_!(U)$.

This finishes the proof of Item 9.

Item 10: Interaction With Injections

Clear.

Item 11: Interaction With Surjections

Clear. ∎

PROPOSITION 4.6.8 ► PROPERTIES OF DIRECT IMAGES WITH COMPACT SUPPORT II

Let $f\colon A \longrightarrow B$ be a function.

1. *Functionality I.* The assignment $f \longmapsto f_!$ defines a function
$$(-)_{!|A,B}\colon \mathsf{Sets}(A, B) \longrightarrow \mathsf{Sets}(\mathcal{P}(A), \mathcal{P}(B)).$$
2. *Functionality II.* The assignment $f \longmapsto f_!$ defines a function
$$(-)_{!|A,B}\colon \mathsf{Sets}(A, B) \longrightarrow \mathsf{Pos}((\mathcal{P}(A), \subset), (\mathcal{P}(B), \subset)).$$
3. *Interaction With Identities.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have
$$(\mathrm{id}_A)_! = \mathrm{id}_{\mathcal{P}(A)}.$$
4. *Interaction With Composition.* For each pair of composable functions $f\colon A \longrightarrow B$ and $g\colon B \longrightarrow C$, we have
$$(g \circ f)_! = g_! \circ f_!, \qquad \begin{tikzcd} \mathcal{P}(A) \arrow[r, "f_!"] \arrow[rd, "(g\circ f)_!"'] & \mathcal{P}(B) \arrow[d, "g_!"] \\ & \mathcal{P}(C). \end{tikzcd}$$

PROOF 4.6.9 ► PROOF OF PROPOSITION 4.6.8

Item 1: Functionality I

Clear.

Item 2: Functionality II

Clear.

Item 3: Interaction With Identities

This follows from Remark 4.6.3 and **??**, **??** of **??**.

Item 4: Interaction With Composition

This follows from Remark 4.6.3 and **??**, **??** of **??**.

Appendices

A Other Chapters

Sets

Relations

Category Theory

Bicategories

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