Types of Morphisms in Categories

December 24, 2023

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Contents

1 Monomorphisms

1.1 Foundation 80UV

Let C be a category.

Definition 1.1.1.1. A morphism m: A - GODD of C is a **monomorphism** if for every commutative diagram of the form

$$C \xrightarrow{f} A \xrightarrow{m} B,$$

we have f = g.

Example 1.1.1.2. Let $f: A \to B$ be a **COLL***tion. The following conditions are equivalent:

- 1. The function f is injective.
- 2. The function f is a monomorphism in Sets.

Proof. Suppose that f is a monomorphism and consider the following diagram:

$$\{*\} \xrightarrow{[y]} A \xrightarrow{f} B,$$

¹That is, with $m \circ f = m \circ g$.

where [x] and [y] are the morphisms picking the elements x and y of A. Then f(x) = f(y) iff $f \circ [x] = f \circ [y]$, implying [x] = [y], and hence x = y. Therefore f is injective.

Conversely, suppose that f is injective. Proceeding by contrapositive, we claim that given a pair of maps $g, h \colon C \rightrightarrows A$ such that $g \neq h$, then $f \circ g \neq f \circ h$. Indeed, as g and h are different maps, there exists must exist at least one element $x \in C$ such that $g(x) \neq h(x)$. But then we have $f(g(x)) \neq f(h(x))$, as f is injective. Thus $f \circ g \neq f \circ h$, and we are done.

Proposition 1.1.1.3. Let C be a category \emptyset with pullbacks and $f: A \to B$ be a morphism of C.

- 1. Characterisations. The following conditions are equivalent:
 - (a) The morphism f is a monomorphism.
 - (b) For each $X \in \text{Obj}(\mathcal{C})$, the map of sets

$$f_* : \operatorname{Hom}_{\mathsf{Sets}}(X, A) \to \operatorname{Hom}_{\mathsf{Sets}}(X, B)$$

is injective.

(c) The kernel pair of f is trivial, i.e. we have

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$$A \times_B A \cong A, \qquad A \xrightarrow{\operatorname{id}_A} A \\ \downarrow f \\ A \xrightarrow{f} B.$$

- 2. Monomorphisms vs. Injective Maps. Let
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- C be a concrete category;
- $\overline{aborange}$: $C \to \mathsf{Sets}$ be the forgetful functor from C to Sets ;
- $f: A \to B$ be a morphism of C.

If 忘 preserves pullbacks, then the following conditions are equivalent:

- (a) The morphism f is a monomorphism.
- (b) The morphism f is injective.
- 3. Stability Properties. The class of all monophisms of C is stable under the following operations:

(a) Composition. If f and g are monomorphisms, then so is $g \circ f$.

(b) Pullbacks. Let

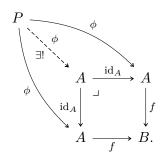
$$\begin{array}{ccc}
A \times_C B \longrightarrow B \\
\downarrow^{m'} & \downarrow^{m} \\
A \longrightarrow C
\end{array}$$

be a diagram in C. If m is a monomorphism in C, then so is m'.

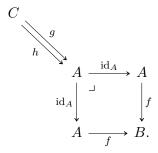
4. Morphisms From the Terminal Object Are Monomorphisms. If C has a terminal object \mathbb{K}_C , then every morphism of C from \mathbb{K}_C is a monomorphism.

Proof. ??, Characterisations: The equivalence between ???? is clear. We claim that ???? are equivalent:

1. $?? \implies ??$: Suppose that f is a monomorphism. Then A satisfies the universal property of the pullback:



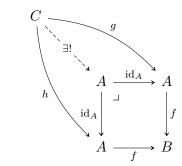
2. ?? \Longrightarrow ??: Suppose that $A \cong A \times_B A$ and let $g, h \colon C \rightrightarrows A$ be a pair of morphisms. Consider the diagram



²Conversely, if $g \circ f$ is a monomorphism, then so is f.

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The universal property of the pullback says that there exists a unique morphism $C \to A$ making the diagram



commute, which implies g = h. Therefore, f is a monomorphism.

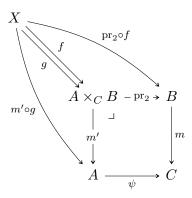
??, Monomorphisms vs. Injective Maps: Assume that f is injective. As the forgetful functor from C to Sets is faithful, we see that ?? together with ?? imply that f is a monomorphism.

Conversely, assume that f is a monomorphism. As F preserves pullbacks, it also preserves kernel pairs. By $\ref{eq:fig:prop}$, we see that F preserves monomorphisms. Thus F_f is a monomorphism, and hence is injective by $\ref{eq:fig:prop}$?

??, Stability Properties: Let $f,g\colon X\rightrightarrows A\times_C B$ be two morphisms such that the diagram

$$X \xrightarrow{f} A \times_C B \xrightarrow{m'} A$$

commutes. It follows that the diagram



also commutes. From the universal property of the pullback, it follows that there must be precisely one morphism from X to $A \times_C B$ making the above diagram commute. Thus f = g and m' is a monomorphism.

??, Morphisms From the Terminal Object Are Monomorphisms: Clear. \Box

1.2 Monomorphism-Reflecting Functors

Definition 1.2.1.1. A functor $F: C \to \mathfrak{DVF}$ flects monomorphisms if, for each morphism f of C, whenever F_f is a monomorphism, so is f.

Proposition 1.2.1.2. Let $F: C \to \mathcal{D}$ be a define tor. If F is faithful, then it reflects monomorphisms.

Proof. Let $f: A \to B$ be a morphism of C and suppose that $F_f: F_A \to F_B$ is a monomorphism. Let $g, h: B \rightrightarrows C$ be two morphisms of C such that $g \circ f = h \circ f$. As F is faithful, we must have

$$F_q \circ F_f = F_{q \circ f} = F_{h \circ f} = F_h \circ F_f,$$

but as F_f is a monomorphism, it must be that $F_g = F_h$. Using the faithfulness of F again, we see that g = h. Therefore f is a monomorphism. \Box

1.3 Split Monomorphisms

Let C be a category.

Definition 1.3.1.1. A morphism $f: A \oplus \mathcal{D} \setminus \mathcal{B}$ of C is a **split monomorphism**³ if there exists a morphism $g: B \to A$ of \mathcal{B} such that⁴

$$g \circ f = \mathrm{id}_A$$
.

Proposition 1.3.1.2. Let C be a category 00 VB

1. Split Monomorphisms are Monomorphisms. If m is a split monomorphism, then m is a monomorphism.

Proof. ??, Split Monomorphisms are Monomorphisms: Let $m: A \to B$ be a split monomorphism of C, let $e: B \to A$ be a morphism of C with

$$e \circ m = \mathrm{id}_A$$

and let $f, g: C \Rightarrow A$ be two morphisms of C such that the diagram

$$C \xrightarrow{g} A \xrightarrow{m} B$$

³Further Terminology: Also called a **section**, or a **split monic** morphism.

Warning: There exist monomorphisms which are not split monomorphisms, e.g. $\mathbb{Z}/2 \hookrightarrow \mathbb{Z}/4$ in Ring.

commutes. Then we have

$$f = id_A \circ f$$

$$= (e \circ m) \circ f$$

$$= e \circ (m \circ f)$$

$$= e \circ (m \circ g)$$

$$= (e \circ m) \circ g$$

$$= id_A \circ g$$

$$= g,$$

showing m to be a monomorphism.

2 Epimorphisms

2.1 Foundation®0VE

Let C be a category.

Definition 2.1.1.1. A morphism $f: A \rightarrow \mathcal{D}$ for C is an **epimorphism** if for every commutative diagram of the form

$$A \xrightarrow{f} B \xrightarrow{g} C,$$

we have g = h.

Example 2.1.1.2. Let $f: A \to B$ be a **Gavic**tion. The following conditions are equivalent:

- 1. The function f is injective.
- 2. The function f is an epimorphism in Sets.

Proof. Suppose that f is surjective and let $g,h \colon B \rightrightarrows C$ be morphisms such that $g \circ f = h \circ f$. Then for each $a \in A$, we have

$$g(f(a)) = h(f(a)),$$

but this implies that

$$g(b) = h(b)$$

⁵That is, with $g \circ f = h \circ f$.

for each $b \in B$, as f is surjective. Thus g = h and f is an epimorphism. To prove the converse, we proceed by contrapositive. So suppose that f is not surjective and consider the diagram

$$A \xrightarrow{f} B \xrightarrow{g} C,$$

where h is the map defined by h(b) = 0 for each $b \in B$ and g is the map defined by

$$g(b) = \begin{cases} 1 & \text{if } b \in \text{Im}(f), \\ 0 & \text{otherwise.} \end{cases}$$

Then $h \circ f = g \circ f$, as h(f(a)) = 1 = g(f(a)) for each $a \in A$. However, for any $b \in B \setminus \text{Im}(f)$, we have

$$q(b) = 0 \neq 1 = h(b).$$

Therefore $g \neq h$ and f is not an epimorphism.

Proposition 2.1.1.3. Let C be a category 00VH

- 1. Characterisations. Let C be a category with pullbacks and $f: A \to B$ be a morphism of C. The following conditions are equivalent:
 - (a) The morphism f is an epimorphism.

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(b) For each $X \in \text{Obj}(\mathcal{C})$, the map of sets

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$$f^* : \operatorname{Hom}_{\mathsf{Sets}}(B, X) \to \operatorname{Hom}_{\mathsf{Sets}}(A, X)$$

is injective.

(c) The cokernel pair of f is trivial, i.e. we have

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$$B \coprod_A B \cong B \qquad \begin{array}{c} B \longleftarrow B \\ \uparrow \\ B \longleftarrow A. \end{array}$$

2. Epimorphisms vs. Surjective Maps. Let

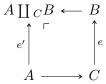
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- C be a concrete category;
- $\overline{\Sigma}$: $C \to \mathsf{Sets}$ be the forgetful functor from C to Sets ;

• $f: A \to B$ be a morphism of C.

If 忘 preserves pushouts, then the following conditions are equivalent:

- (a) The morphism f is a epimorphism.
- (b) The morphism f is surjective.
- 3. Stability Properties. The class of all epim**OPyR**isms of C is stable under the following operations:
 - (a) Composition. If f and g are epimorphisms, then so is $g \circ f$.
 - (b) Pushouts. Let



be a diagram in C. If m is an epimorphism in C, then so is e'.

4. Morphisms to the Initial Object Are Monomorphisms. If C has an initial object \varnothing_C , then every morphism of C to \varnothing_C is a epimorphism.

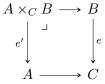
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Proof. This is dual to ??.

2.2 Regular Epinorphisms

Proposition 2.2.1.1. Let C be a category 00VS

1. Stability Under Pullbacks. Consider the diagram



in C. If e is a regular epimorphism, then so is e'.

Proof. Epimorphisms Need Not Be Stable Under Pullback. : Regular Epimorphisms Are Stable Under Pullback. : \Box

⁶Conversely, if $q \circ f$ is a epimorphism, then so is q.

2.3 Effective Ephatorphisms

Let C be a category.

Definition 2.3.1.1. An epimorphism $f: \mathfrak{AVU} B$ of C is **effective** if we have an isomorphism

$$B \cong \operatorname{CoEq}(A \times_B A \rightrightarrows A).$$

2.4 Split Epimorphisms

Let C be a category.

Definition 2.4.1.1. A morphism $f: A - \Theta CB Wof C$ is a **retraction**⁷ if there is an arrow $g: B \to A$ such that $f \circ g = \mathrm{id}_B$.

Proposition 2.4.1.2. Let $f: A \to B$ be a **Many** hism of C.

1. Every split epimorphism is an epimorphism.⁸

Proof. This is dual to ??.

Appendices

A Other Chapters

Sets

- 1. Sets
- 2. Constructions With Sets
- 3. Pointed Sets
- 4. Tensor Products of Pointed Sets
- 5. Relations
- 6. Spans
- 7. Posets

Indexed and Fibred Sets

- 7. Indexed Sets
- 8. Fibred Sets
- 9. Un/Straightening for Indexed and Fibred Sets

Category Theory

- 11. Categories
- 12. Types of Morphisms in Categories

 $^{^7\}mathit{Further\ Terminology:}$ Also called a $\mathbf{split\ epimorphism}.$

⁸ Warning: There are epimorphisms which are not split epimorphisms, however, e.g. $\mathbb{Z} \hookrightarrow \mathbb{Z}/2$.

- 13. Adjunctions and the Yoneda Lemma
- 14. Constructions With Categories
- 15. Kan Extensions

Bicategories

- 17. Bicategories
- 18. Internal Adjunctions

Internal Category Theory

19. Internal Categories

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20. The Cycle Category

Cubical Stuff

21. The Cube Category

Globular Stuff

22. The Globe Category

Cellular Stuff

23. The Cell Category

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- 24. Monoids
- 25. Constructions With Monoids

Monoids With Zero

- 26. Monoids With Zero
- 27. Constructions With Monoids With Zero

Groups

- 28. Groups
- 29. Constructions With Groups

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- 34. Near-Semirings
- 35. Near-Rings

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- 36. Real Analysis in One Variable
- 37. Real Analysis in Several Variables

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- 39. Measures and Integration

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39. Probability Theory

Stochastic Analysis

- 40. Stochastic Processes, Martingales, and Brownian Motion
- 41. Itô Calculus
- 42. Stochastic Differential Equations

Differential Geometry

43. Topological and Smooth Manifolds

Schemes

44. Schemes