Tensor Products of Pointed Sets

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This chapter contains some material on tensor products of pointed sets.

Contents

1 Bilinear Morphisms of Pointed Sets

1.1 Left Bilinear Morphisms of Pointed Sets

Let (X, x_0), (Y, y_0), and (Z, z_0) be pointed sets.

Definition 1.1.1.1. A **left bilinear morphism of pointed sets from** $(X \times Y, (x_0, y_0))$ **to** (Z, z_0) is a map of sets

$$f\colon (X \times Y, (x_0, y_0)) \longrightarrow (Z, z_0)$$

satisfying the following condition:[1,2]

($\star$) *Left Unital Bilinearity.* The diagram

$$\begin{tikzcd}
& \mathrm{pt}\times\mathrm{pt} \arrow[dr, dashed, "\sim"] & \\
\mathrm{pt}\times Y \arrow[ur, "\mathrm{id}_{\mathrm{pt}}\times\epsilon_Y"] \arrow[d, "{[x_0]\times\mathrm{id}_Y}"'] & & \mathrm{pt} \arrow[d, "{[z_0]}"] \\
X\times Y \arrow[rr, "f"'] & & Z
\end{tikzcd}$$

commutes, i.e. for each $y \in Y$, we have

$$f(x_0, y) = z_0.$$

Definition 1.1.1.2. The **set of left bilinear morphisms of pointed sets from** $(X \times Y, (x_0, y_0))$ **to** (Z, z_0) is the set $\mathrm{Hom}^{\otimes,\mathrm{L}}_{\mathsf{Sets}_*}(X \times Y, Z)$ defined by

$$\mathrm{Hom}^{\otimes,\mathrm{L}}_{\mathsf{Sets}_*}(X \times Y, Z) \overset{\mathrm{def}}{=} \{f \in \mathsf{Sets}_*(A \times B, C) \mid f \text{ is left bilinear}\}.$$

1.2 Right Bilinear Morphisms of Pointed Sets

Let (X, x_0), (Y, y_0), and (Z, z_0) be pointed sets.

[1] *Slogan:* f is left bilinear if it preserves basepoints in its first argument.

[2] Succinctly, f is bilinear if we have

$$f(x_0, y) = z_0$$

for each $y \in Y$.

Definition 1.2.1.1. A **right bilinear morphism of pointed sets from** $(X \times Y, (x_0, y_0))$ **to** (Z, z_0) is a map of sets

$$f\colon (X \times Y, (x_0, y_0)) \to (Z, z_0)$$

satisfying the following condition:[3,4]

($\star$) *Right Unital Bilinearity.* The diagram

$$\begin{tikzcd}
& \mathrm{pt} \times \mathrm{pt} \arrow[dr, dashed, "\sim"] & \\
X \times \mathrm{pt} \arrow[ur, "\epsilon_X \times \mathrm{id}_{\mathrm{pt}}"] \arrow[d, "\mathrm{id}_X \times [y_0]"'] & & \mathrm{pt} \arrow[d, "{[z_0]}"] \\
X \times Y \arrow[rr, "f"'] & & Z
\end{tikzcd}$$

commutes, i.e. for each $x \in X$, we have

$$f(x, y_0) = z_0.$$

Definition 1.2.1.2. The **set of right bilinear morphisms of pointed sets from** $(X \times Y, (x_0, y_0))$ **to** (Z, z_0) is the set $\mathrm{Hom}^{\otimes,\mathrm{R}}_{\mathsf{Sets}_*}(X \times Y, Z)$ defined by

$$\mathrm{Hom}^{\otimes,\mathrm{R}}_{\mathsf{Sets}_*}(X \times Y, Z) \overset{\mathrm{def}}{=} \left\{ f \in \mathsf{Sets}_*(A \times B, C) \,\middle|\, f \text{ is right bilinear} \right\}.$$

1.3 Bilinear Morphisms of Pointed Sets

Let (X, x_0), (Y, y_0), and (Z, z_0) be pointed sets.

Definition 1.3.1.1. A **bilinear morphism of pointed sets from** $(X \times Y, (x_0, y_0))$ **to** (Z, z_0) is a map of sets

$$f\colon (X \times Y, (x_0, y_0)) \to (Z, z_0)$$

that is both left bilinear and right bilinear.

[3] *Slogan:* f is right bilinear if it preserves basepoints in its second argument.

[4] Succinctly, f is bilinear if we have

$$f(x, y_0) = z_0$$

for each $x \in X$.

Remark 1.3.1.2. In detail, a **bilinear morphism of pointed sets from** $(X \times Y, (x_0, y_0))$ **to** (Z, z_0) is a map of sets

$$f\colon (X \times Y, (x_0, y_0)) \to (Z, z_0)$$

satisfying the following conditions:[5,6]

1. *Left Unital Bilinearity.* The diagram

$$\begin{tikzcd}
& \mathrm{pt}\times\mathrm{pt} \arrow[dr, dashed, "\sim"] & \\
\mathrm{pt}\times Y \arrow[ur, "\mathrm{id}_{\mathrm{pt}}\times\epsilon_Y"] \arrow[d, "{[x_0]\times\mathrm{id}_Y}"'] & & \mathrm{pt} \arrow[d, "{[z_0]}"] \\
X\times Y \arrow[rr, "f"'] & & Z
\end{tikzcd}$$

   commutes, i.e. for each $y \in Y$, we have

$$f(x_0, y) = z_0.$$

2. *Right Unital Bilinearity.* The diagram

$$\begin{tikzcd}
& \mathrm{pt}\times\mathrm{pt} \arrow[dr, dashed, "\sim"] & \\
X\times\mathrm{pt} \arrow[ur, "\epsilon_X\times\mathrm{id}_{\mathrm{pt}}"] \arrow[d, "{\mathrm{id}_X\times[y_0]}"'] & & \mathrm{pt} \arrow[d, "{[z_0]}"] \\
X\times Y \arrow[rr, "f"'] & & Z
\end{tikzcd}$$

   commutes, i.e. for each $x \in X$, we have

$$f(x, y_0) = z_0.$$

Definition 1.3.1.3. The **set of bilinear morphisms of pointed sets from** $(X \times Y, (x_0, y_0))$ **to** (Z, z_0) is the set $\mathrm{Hom}^{\otimes}_{\mathsf{Sets}_*}(X \times Y, Z)$ defined by

$$\mathrm{Hom}^{\otimes}_{\mathsf{Sets}_*}(X \times Y, Z) \overset{\text{def}}{=} \{f \in \mathsf{Sets}_*(A \times B, C) \mid f \text{ is bilinear}\}.$$

[5] *Slogan:* f is bilinear if it preserves basepoints in each argument.

[6] Succinctly, f is bilinear if we have

$$\begin{aligned} f(x_0, y) &= z_0, \\ f(x, y_0) &= z_0 \end{aligned}$$

2 Tensors and Cotensors of Pointed Sets by Sets

2.1 Tensors of Pointed Sets by Sets

Let (X, x_0) be a pointed set and let A be a set.

Definition 2.1.1.1. The **tensor of** (X, x_0) **by** A is the pointed set $A \odot (X, x_0)$ satisfying the following universal property:

(UP) We have a bijection

$$\mathsf{Sets}_*(A \odot X, K) \cong \mathsf{Sets}(A, \mathbf{Sets}_*(X, K)),$$

natural in $(K, k_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

Remark 2.1.1.2. The tensor of (X, x_0) by A satisfies the following universal property:

$$\mathsf{Sets}_*(A \odot X, K) \cong \mathsf{Sets}^{\otimes}_{\mathbb{E}_0}(A \times X, K),$$

where $\mathsf{Sets}^{\otimes}_{\mathbb{E}_0}(A \times X, K)$ is the set defined by

$$\mathsf{Sets}^{\otimes}_{\mathbb{E}_0}(A \times X, K) \stackrel{\text{def}}{=} \left\{ f \in \mathsf{Sets}(A \times X, K) \;\middle|\; \begin{array}{l}\text{for each } a \in A, \text{ we have} \\ f(a, x_0) = k_0\end{array} \right\}.$$

Construction 2.1.1.3. Concretely, the **tensor of** (X, x_0) **by** A is the pointed set $A \odot (X, x_0)$ consisting of

- *The Underlying Set.* The set $A \odot X$ given by

$$A \odot X \cong \bigvee_{a \in A} (X, x_0);$$

- *The Basepoint.* The point $[x_0]$ of $\bigvee_{a \in A}(X, x_0)$.

2.2 Cotensors of Pointed Sets by Sets

Let (X, x_0) be a pointed set and let A be a set.

Definition 2.2.1.1. The **cotensor of** (X, x_0) **by** A is the pointed set $A \pitchfork (X, x_0)$ satisfying the following universal property:

(UP) We have a bijection

$$\mathsf{Sets}_*(K, A \pitchfork X) \cong \mathsf{Sets}(A, \mathbf{Sets}_*(K, X)),$$

natural in $(K, k_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

Remark 2.2.1.2. The cotensor of (X, x_0) by A satisfies the following universal property:

$$\mathsf{Sets}_*(K, A \pitchfork X) \cong \mathsf{Sets}^{\otimes}_{\mathbb{E}_0}(A \times K, X),$$

where $\mathsf{Sets}^{\otimes}_{\mathbb{E}_0}(A \times K, X)$ is the set defined by

$$\mathsf{Sets}^{\otimes}_{\mathbb{E}_0}(A \times K, X) \stackrel{\text{def}}{=} \left\{ f \in \mathsf{Sets}(A \times K, X) \,\middle|\, \begin{array}{l} \text{for each } a \in A, \text{ we have} \\ f(a, k_0) = x_0 \end{array} \right\}.$$

Construction 2.2.1.3. Concretely, the **cotensor of** (X, x_0) **by** A is the pointed set $A \pitchfork (X, x_0)$ consisting of

- *The Underlying Set.* The set $A \pitchfork X$ given by

$$A \pitchfork X \cong \bigwedge_{a \in A} (X, x_0);$$

- *The Basepoint.* The point $[(x_0, x_0, x_0, \ldots)]$ of $\bigwedge_{a \in A}(X, x_0)$.

3 The Left Tensor Product of Pointed Sets

3.1 Foundations

Let (X, x_0) and (Y, y_0) be pointed sets.

Definition 3.1.1.1. The **left tensor product of pointed sets** is the functor

$$\lhd_{\mathsf{Sets}_*} \colon \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*$$

defined as the composition

$$\mathsf{Sets}_* \times \mathsf{Sets}_* \xrightarrow{\mathrm{id} \times 忘} \mathsf{Sets}_* \times \mathsf{Sets} \xrightarrow{\beta^{\mathsf{Cats}_2}_{\mathsf{Sets}_*, \mathsf{Sets}}} \mathsf{Sets} \times \mathsf{Sets}_* \xrightarrow{\odot} \mathsf{Sets}_*.$$

Remark 3.1.1.2. The left tensor product of pointed sets satisfies the following universal property:[7]

$$\mathsf{Sets}_*\big(X \lhd_{\mathsf{Sets}_*} Y, Z\big) \cong \mathrm{Hom}^{\otimes, \mathrm{L}}_{\mathsf{Sets}_*}(X \times Y, Z).$$

for each $x \in X$ and each $y \in Y$.

[7] Namely, a pointed map $f \colon X \lhd_{\mathsf{Sets}_*} Y \to Z$ is the same as a map $f^{\dagger} \colon X \times Y \to Z$ such that

$$f^{\dagger}(x_0, y) = z_0$$

for each $y \in Y$.

Remark 3.1.1.3. In detail, the **left tensor product of** (X, x_0) **and** (Y, y_0) is the pointed set $\left(X \lhd_{\mathsf{Sets}_*} Y, [x_0]\right)$ consisting of[8]

- *The Underlying Set.* The set $X \lhd_{\mathsf{Sets}_*} Y$ defined by

$$\begin{aligned} X \lhd_{\mathsf{Sets}_*} Y &\overset{\text{def}}{=} |Y| \odot X \\ &\cong \bigvee_{y \in Y} (X, x_0); \end{aligned}$$

- *The Underlying Basepoint.* The point $[x_0]$ of $\bigvee_{y \in Y}(X, x_0)$.

Proposition 3.1.1.4. Let (X, x_0) and (Y, y_0) be pointed sets.

1. *Functoriality.* The assignments $X, Y, (X, Y) \mapsto X \lhd_{\mathsf{Sets}_*} Y$ define functors

$$\begin{aligned} X \lhd_{\mathsf{Sets}_*} - &\colon \mathsf{Sets}_* \to \mathsf{Sets}_*, \\ - \lhd_{\mathsf{Sets}_*} Y &\colon \mathsf{Sets}_* \to \mathsf{Sets}_*, \\ -_1 \lhd_{\mathsf{Sets}_*} -_2 &\colon \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*. \end{aligned}$$

Proof. *Item 1*, *Functoriality*: Omitted. □

3.2 The Skew Associator

Definition 3.2.1.1. The **skew associator of the left tensor product of pointed sets** is the natural isomorphism

$$\alpha^{\mathsf{Sets}_*, \lhd} \colon \lhd_{\mathsf{Sets}_*} \circ \left(\lhd_{\mathsf{Sets}_*} \times \mathrm{id}_{\mathsf{Sets}_*}\right) \overset{\cong}{\Longrightarrow} \lhd_{\mathsf{Sets}_*} \circ \left(\mathrm{id}_{\mathsf{Sets}_*} \times \lhd_{\mathsf{Sets}_*}\right),$$

whose component

$$\alpha^{\mathsf{Sets}_*, \lhd}_{X,Y,Z} \colon \left(X \lhd_{\mathsf{Sets}_*} Y\right) \lhd_{\mathsf{Sets}_*} Z \xrightarrow{\cong} X \lhd_{\mathsf{Sets}_*} \left(Y \lhd_{\mathsf{Sets}_*} Z\right)$$

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[8] *Further Notation:* We write $x \lhd_{\mathsf{Sets}_*} y$ for the image of (x, y) under the map

$$X \times Y \to \underbrace{X \lhd_{\mathsf{Sets}_*} Y}_{\cong \bigvee_{y \in Y}(X, x_0)}.$$

sending (x, y) to the element $x \in X$ in the yth copy of X in $\bigvee_{y \in Y}(X, x_0)$. Note that we have

$$x_0 \lhd_{\mathsf{Sets}_*} y = x_0 \lhd_{\mathsf{Sets}_*} y',$$

for each $y, y' \in Y$.

at (X, Y, Z) is given by the composition[9]

$$
\begin{aligned}
(X \lhd_{\mathsf{Sets}_*} Y) \lhd_{\mathsf{Sets}_*} Z &\overset{\text{def}}{=} |Z| \odot (X \lhd_{\mathsf{Sets}_*} Y) \\
&\overset{\text{def}}{=} |Z| \odot (|Y| \odot X) \\
&\cong \bigvee_{z \in Z} (|Y| \odot X, [x_0]) \\
&\overset{\text{def}}{=} \bigvee_{z \in Z} \left(\bigvee_{y \in Y} (X, x_0) \right) \\
&\cong \bigvee_{(z,y) \in \bigvee_{z \in Z} (Y, y_0)} (X, x_0) \\
&\overset{\text{def}}{=} \bigvee_{(z,y) \in |Z| \odot Y} (X, x_0) \\
&\cong ||Z| \odot Y| \odot X \\
&\overset{\text{def}}{=} |Y \lhd_{\mathsf{Sets}_*} Z| \odot X \\
&\overset{\text{def}}{=} X \lhd_{\mathsf{Sets}_*} (Y \lhd_{\mathsf{Sets}_*} Z),
\end{aligned}
$$

where the isomorphism

$$
\bigvee_{z \in Z} \left(\bigvee_{y \in Y} (X, x_0) \right) \cong \bigvee_{(y,z) \in \bigvee_{z \in Z} (Y, y_0)} (X, x_0)
$$

is given by $[(z, (y, x))] \mapsto [((z, y), x)]$.

3.3 The Skew Left Unitor

Definition 3.3.1.1. The **skew left unitor of the left tensor product of pointed sets** is the natural transformation

$$
\lambda^{\mathsf{Sets}_*, \lhd} \colon \lhd_{\mathsf{Sets}_*} \circ \left(\mathbb{1}^{\mathsf{Sets}_*} \times \mathrm{id}_{\mathsf{Sets}_*} \right) \Longrightarrow \mathrm{id}_{\mathsf{Sets}_*},
$$

whose component

$$
\lambda^{\mathsf{Sets}_*, \lhd}_X \colon S^0 \lhd_{\mathsf{Sets}_*} X \to X
$$

[9] In other words, $\alpha^{\mathsf{Sets}_*, \lhd}_{X,Y,Z}$ acts on elements as

$$
\alpha^{\mathsf{Sets}_*, \lhd}_{X,Y,Z} ((x \lhd_{\mathsf{Sets}_*} y) \lhd_{\mathsf{Sets}_*} z) \overset{\text{def}}{=} x \lhd_{\mathsf{Sets}_*} (y \lhd_{\mathsf{Sets}_*} z)
$$

for each $(x \lhd_{\mathsf{Sets}_*} y) \lhd_{\mathsf{Sets}_*} z \in (X \lhd_{\mathsf{Sets}_*} Y) \lhd_{\mathsf{Sets}_*} Z$.

at X is given by the composition[10]

$$\begin{aligned} S^0 \lhd_{\mathsf{Sets}_*} X &\cong |X| \odot S^0 \\ &\cong \bigvee_{x\in X} S^0 \\ &\to X \end{aligned}$$

where $\bigvee_{x\in X} S^0 \to X$ is the map given by

$$\begin{aligned} (x,0) &\mapsto x, \\ (x,1) &\mapsto x. \end{aligned}$$

3.4 The Skew Right Unitor

Definition 3.4.1.1. The **skew right unitor of the left tensor product of pointed sets** is the natural transformation

$$\rho^{\mathsf{Sets}_*,\lhd} \colon \mathrm{id}_{\mathsf{Sets}_*} \Longrightarrow \lhd_{\mathsf{Sets}_*} \circ \left(\mathrm{id}_{\mathsf{Sets}_*} \times \mathbb{1}^{\mathsf{Sets}_*}\right),$$

whose component

$$\rho_X^{\mathsf{Sets}_*,\lhd} \colon X \to X \lhd_{\mathsf{Sets}_*} S^0$$

at X is given by the composition[11]

$$\begin{aligned} X &\to X \vee X \\ &\cong |S^0| \odot X \\ &\cong S^0 \lhd_{\mathsf{Sets}_*} X, \end{aligned}$$

where $X \to X \vee X$ is the map sending X to the first factor of X in $X \vee X$.

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[10] In other words, $\lambda_X^{\mathsf{Sets}_*,\lhd}$ acts on elements as

$$\begin{aligned} \lambda_X^{\mathsf{Sets}_*,\lhd}(x \lhd_{\mathsf{Sets}_*} 0) &\stackrel{\text{def}}{=} x, \\ \lambda_X^{\mathsf{Sets}_*,\lhd}(x \lhd_{\mathsf{Sets}_*} 1) &\stackrel{\text{def}}{=} x, \end{aligned}$$

for each $x \in X$.

[11] In other words, $\rho_X^{\mathsf{Sets}_*,\lhd}$ acts on elements as

$$\rho_X^{\mathsf{Sets}_*,\lhd}(x) \stackrel{\text{def}}{=} x \lhd_{\mathsf{Sets}_*} 0$$

for each $x \in X$.

3.5 The Left-Skew Monoidal Category Structure on Pointed Sets

Proposition 3.5.1.1. The category Sets_* admits a left-skew monoidal category structure consisting of[12]

- *The Skew Monoidal Product.* The left tensor product functor

$$\lhd_{\mathsf{Sets}_*} \colon \mathsf{Sets}_* \times \mathsf{Sets}_* \longrightarrow \mathsf{Sets}_*,$$

  of Proposition 3.1.1.4;

- *The Skew Monoidal Unit.* The functor

$$\mathbb{1}^{\mathsf{Sets}_*} \colon \mathsf{pt} \longrightarrow \mathsf{Sets}_*$$

  defined by

$$\mathbb{1}_{\mathsf{Sets}_*} \overset{\text{def}}{=} S^0;$$

- *The Skew Associators.* The natural isomorphism

$$\alpha^{\mathsf{Sets}_*,\lhd} \colon \lhd_{\mathsf{Sets}_*} \circ \left(\lhd_{\mathsf{Sets}_*} \times \mathrm{id}_{\mathsf{Sets}_*}\right) \overset{\cong}{\Longrightarrow} \lhd_{\mathsf{Sets}_*} \circ \left(\mathrm{id}_{\mathsf{Sets}_*} \times \lhd_{\mathsf{Sets}_*}\right),$$

  of Definition 3.2.1.1;

- *The Skew Left Unitors.* The natural transformation

$$\lambda^{\mathsf{Sets}_*,\lhd} \colon \lhd_{\mathsf{Sets}_*} \circ \left(\mathbb{1}^{\mathsf{Sets}_*} \times \mathrm{id}_{\mathsf{Sets}_*}\right) \Longrightarrow \mathrm{id}_{\mathsf{Sets}_*},$$

  of Definition 3.3.1.1;

- *The Skew Right Unitors.* The natural transformation

$$\rho^{\mathsf{Sets}_*,\lhd} \colon \mathrm{id}_{\mathsf{Sets}_*} \Longrightarrow \lhd_{\mathsf{Sets}_*} \circ \left(\mathrm{id}_{\mathsf{Sets}_*} \times \mathbb{1}^{\mathsf{Sets}_*}\right),$$

  of Definition 3.4.1.1.

Proof. Omitted. □

[12] Note in particular that, differently from general left-skew monoidal categories, the skew associator of $(\mathsf{Sets}_*, \lhd_{\mathsf{Sets}_*}, S^0)$ is a natural isomorphism.

4 The Right Tensor Product of Pointed Sets

4.1 Foundations

Let (X, x_0) and (Y, y_0) be pointed sets.

Definition 4.1.1.1. The **right tensor product of pointed sets** is the functor

$$\rhd_{\mathsf{Sets}_*} \colon \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*$$

defined as the composition

$$\mathsf{Sets}_* \times \mathsf{Sets}_* \xrightarrow{忘\times\mathrm{id}} \mathsf{Sets} \times \mathsf{Sets}_* \xrightarrow{\odot} \mathsf{Sets}_*.$$

Remark 4.1.1.2. The right tensor product of pointed sets satisfies the following universal property:[13]

$$\mathsf{Sets}_*\big(X \rhd_{\mathsf{Sets}_*} Y, Z\big) \cong \mathrm{Hom}^{\otimes,\mathrm{R}}_{\mathsf{Sets}_*}(X \times Y, Z).$$

Remark 4.1.1.3. In detail, the **right tensor product of** (X, x_0) **and** (Y, y_0) is the pointed set $\big(X \rhd_{\mathsf{Sets}_*} Y, [y_0]\big)$ consisting of [14]

- *The Underlying Set.* The set $X \rhd_{\mathsf{Sets}_*} Y$ defined by

$$\begin{aligned} X \rhd_{\mathsf{Sets}_*} Y &\stackrel{\text{def}}{=} |X| \odot Y \\ &\cong \bigvee_{x\in X} (Y, y_0); \end{aligned}$$

- *The Underlying Basepoint.* The point $[y_0]$ of $\bigvee_{x\in X}(Y, y_0)$.

Proposition 4.1.1.4. Let (X, x_0) and (Y, y_0) be pointed sets.

[13] Namely, a pointed map $f\colon X \lhd_{\mathsf{Sets}_*} Y \to Z$ is the same as a map $f^\dagger\colon X \times Y \to Z$ such that

$$f^\dagger(x, y_0) = z_0$$

for each $y \in Y$.

[14] *Further Notation:* We write $x \rhd_{\mathsf{Sets}_*} y$ for the image of (x, y) under the map

$$X \times Y \to \underbrace{X \rhd_{\mathsf{Sets}_*} Y}_{\cong \bigvee_{x\in X}(Y, y_0)}.$$

sending (x, y) to the element $y \in Y$ in the xth copy of Y in $\bigvee_{x\in X}(Y, y_0)$. Note that we have

$$x \rhd_{\mathsf{Sets}_*} y_0 = x' \rhd_{\mathsf{Sets}_*} y_0,$$

for each $x, x' \in X$.

1. *Functoriality.* The assignments $X, Y, (X, Y) \mapsto X \rhd_{\mathsf{Sets}_*} Y$ define functors

$$\begin{aligned} X \rhd_{\mathsf{Sets}_*} - &\colon \mathsf{Sets}_* \to \mathsf{Sets}_*, \\ - \rhd_{\mathsf{Sets}_*} Y &\colon \mathsf{Sets}_* \to \mathsf{Sets}_*, \\ -_1 \rhd_{\mathsf{Sets}_*} -_2 &\colon \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*. \end{aligned}$$

Proof. *Item 1, Functoriality*: Omitted. □

4.2 The Skew Associator

Definition 4.2.1.1. The **skew associator of the right tensor product of pointed sets** is the natural isomorphism

$$\alpha^{\mathsf{Sets}_*, \rhd} \colon \rhd_{\mathsf{Sets}_*} \circ \left(\mathrm{id}_{\mathsf{Sets}_*} \times \rhd_{\mathsf{Sets}_*}\right) \overset{\cong}{\Longrightarrow} \rhd_{\mathsf{Sets}_*} \circ \left(\rhd_{\mathsf{Sets}_*} \times \mathrm{id}_{\mathsf{Sets}_*}\right),$$

whose component

$$\alpha^{\mathsf{Sets}_*, \rhd}_{X,Y,Z} \colon X \rhd_{\mathsf{Sets}_*} \left(Y \rhd_{\mathsf{Sets}_*} Z\right) \xrightarrow{\cong} \left(X \rhd_{\mathsf{Sets}_*} Y\right) \rhd_{\mathsf{Sets}_*} Z$$

at (X, Y, Z) is given by the composition[15]

$$\begin{aligned} X \rhd_{\mathsf{Sets}_*} \left(Y \rhd_{\mathsf{Sets}_*} Z\right) &\overset{\text{def}}{=} |X| \odot \left(Y \rhd_{\mathsf{Sets}_*} Z\right) \\ &\overset{\text{def}}{=} |X| \odot (|Y| \odot Z) \\ &\cong |X| \odot \left(\bigvee_{y \in Y} (Z, z_0)\right) \\ &\cong \bigvee_{x \in X} \left(\bigvee_{y \in Y} (Z, z_0)\right) \\ &\cong \bigvee_{(x,y) \in \bigvee_{x \in X} (Y, y_0)} (Z, z_0) \\ &\cong \left|\bigvee_{x \in X} (Y, y_0)\right| \odot Z \\ &\overset{\text{def}}{=} |X \odot Y| \odot Z \\ &\overset{\text{def}}{=} |X \rhd_{\mathsf{Sets}_*} Y| \odot Z \\ &\overset{\text{def}}{=} \left(X \rhd_{\mathsf{Sets}_*} Y\right) \rhd_{\mathsf{Sets}_*} Z \end{aligned}$$

[15] In other words, $\alpha^{\mathsf{Sets}_*, \rhd}_{X,Y,Z}$ acts on elements as

$$\alpha^{\mathsf{Sets}_*, \rhd}_{X,Y,Z} \left(x \rhd_{\mathsf{Sets}_*} \left(y \rhd_{\mathsf{Sets}_*} z\right)\right) \overset{\text{def}}{=} \left(x \rhd_{\mathsf{Sets}_*} y\right) \rhd_{\mathsf{Sets}_*} z$$

where the isomorphism

$$\bigvee_{x\in X}\left(\bigvee_{y\in Y}(Z,z_0)\right)\cong\bigvee_{(x,y)\in\bigvee_{x\in X}(Y,y_0)}(Z,z_0)$$

is given by $[(x,(y,z))]\mapsto[((x,y),z)]$.

4.3 The Skew Left Unitor

Definition 4.3.1.1. The **skew left unitor of the right tensor product of pointed sets** is the natural transformation

$$\lambda^{\mathsf{Sets}_*,\triangleright}\colon \mathrm{id}_{\mathsf{Sets}_*}\Longrightarrow \triangleright_{\mathsf{Sets}_*}\circ\left(\mathbb{1}^{\mathsf{Sets}_*}\times\mathrm{id}_{\mathsf{Sets}_*}\right),$$

whose component

$$\lambda^{\mathsf{Sets}_*,\triangleright}_X\colon X\to S^0\triangleright_{\mathsf{Sets}_*}X$$

at X is given by the composition[16]

$$\begin{aligned}X&\to X\vee X\\&\cong|S^0|\odot X\\&\cong S^0\triangleright_{\mathsf{Sets}_*}X,\end{aligned}$$

where $X\to X\vee X$ is the map sending X to the first factor of X in $X\vee X$.

4.4 The Skew Right Unitor

Definition 4.4.1.1. The **skew right unitor of the right tensor product of pointed sets** is the natural transformation

$$\rho^{\mathsf{Sets}_*,\triangleright}\colon \triangleright_{\mathsf{Sets}_*}\circ\left(\mathrm{id}_{\mathsf{Sets}_*}\times\mathbb{1}^{\mathsf{Sets}_*}\right)\Longrightarrow\mathrm{id}_{\mathsf{Sets}_*},$$

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for each $x\triangleright_{\mathsf{Sets}_*}(y\triangleright_{\mathsf{Sets}_*}z)\in X\triangleright_{\mathsf{Sets}_*}(Y\triangleright_{\mathsf{Sets}_*}Z)$.

[16] In other words, $\lambda^{\mathsf{Sets}_*,\triangleright}_X$ acts on elements as

$$\lambda^{\mathsf{Sets}_*,\triangleright}_X(x)\overset{\mathrm{def}}{=}0\triangleright_{\mathsf{Sets}_*}x$$

for each $x\in X$.

whose component[17]

$$\rho^{\mathsf{Sets}_*,\triangleright}_X \colon X \triangleright_{\mathsf{Sets}_*} S^0 \longrightarrow X$$

at X is given by the composition

$$\begin{aligned} X \triangleright_{\mathsf{Sets}_*} S^0 &\cong |X| \odot S^0 \\ &\cong \bigvee_{x \in X} S^0 \\ &\longrightarrow X \end{aligned}$$

where $\bigvee_{x \in X} S^0 \longrightarrow X$ is the map given by

$$\begin{aligned} (x, 0) &\mapsto x, \\ (x, 1) &\mapsto x. \end{aligned}$$

4.5 The Right-Skew Monoidal Category Structure on Pointed Sets

Proposition 4.5.1.1. The category Sets_* admits a right-skew monoidal category structure consisting of[18]

- *The Skew Monoidal Product.* The right tensor product functor

  $$\triangleright_{\mathsf{Sets}_*} \colon \mathsf{Sets}_* \times \mathsf{Sets}_* \longrightarrow \mathsf{Sets}_*,$$

  of Item 1;

- *The Skew Monoidal Unit.* The functor

  $$\mathbb{1}^{\mathsf{Sets}_*} \colon \mathsf{pt} \longrightarrow \mathsf{Sets}_*$$

  defined by

  $$\mathbb{1}_{\mathsf{Sets}_*} \overset{\text{def}}{=} S^0;$$

- *The Skew Associators.* The natural isomorphism

  $$\alpha^{\mathsf{Sets}_*,\triangleright} \colon \triangleright_{\mathsf{Sets}_*} \circ \left(\triangleright_{\mathsf{Sets}_*} \times \mathrm{id}_{\mathsf{Sets}_*}\right) \overset{\cong}{\Longrightarrow} \triangleright_{\mathsf{Sets}_*} \circ \left(\mathrm{id}_{\mathsf{Sets}_*} \times \triangleright_{\mathsf{Sets}_*}\right),$$

  of Definition 4.2.1.1;

[17] In other words, $\rho^{\mathsf{Sets}_*,\triangleright}_X$ acts on elements as

$$\begin{aligned} \rho^{\mathsf{Sets}_*,\triangleright}_X\left(x \triangleright_{\mathsf{Sets}_*} 0\right) &\overset{\text{def}}{=} x, \\ \rho^{\mathsf{Sets}_*,\triangleright}_X\left(x \triangleright_{\mathsf{Sets}_*} 1\right) &\overset{\text{def}}{=} x \end{aligned}$$

for each $x \in X$.

[18] Note in particular that, differently from general right-skew monoidal categories, the skew associator of

- *The Skew Left Unitors.* The natural transformation

$$\lambda^{\mathsf{Sets}_*,\rhd}\colon \mathrm{id}_{\mathsf{Sets}_*} \Longrightarrow \rhd_{\mathsf{Sets}_*} \circ \left(\not\Bbbk^{\mathsf{Sets}_*} \times \mathrm{id}_{\mathsf{Sets}_*}\right),$$

  of Definition 3.3.1.1;

- *The Skew Right Unitors.* The natural transformation

$$\rho^{\mathsf{Sets}_*,\rhd}\colon \rhd_{\mathsf{Sets}_*} \circ \left(\mathrm{id}_{\mathsf{Sets}_*} \times \not\Bbbk^{\mathsf{Sets}_*}\right) \Longrightarrow \mathrm{id}_{\mathsf{Sets}_*},$$

  of Definition 3.4.1.1.

Proof. Omitted. □

5 Smash Products of Pointed Sets

5.1 Foundations

Let (X, x_0) and (Y, y_0) be pointed sets.

Definition 5.1.1.1. The **smash product of** (X, x_0) **and** (Y, y_0)[19] is the pointed set $X \wedge Y$[20] such that we have a bijection

$$\mathsf{Sets}_*(X \wedge Y, Z) \cong \mathrm{Hom}^{\otimes}_{\mathsf{Sets}_*}(X \times Y, Z),$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

Remark 5.1.1.2. In detail, the **smash product of** (X, x_0) **and** (Y, y_0) is the pair $((X \wedge Y, [(x_0, y_0)]), \iota)$ consisting of

- A pointed set $(X \wedge Y, [(x_0, y_0)])$;
- A bilinear morphism of pointed sets $\iota\colon (X \times Y, (x_0, y_0)) \to X \wedge Y$;

satisfying the following universal property:

(UP) Given another such pair $((Z, z_0), f)$ consisting of

  - A pointed set (Z, z_0);
  - A bilinear morphism of pointed sets $f\colon (X \times Y, (x_0, y_0)) \to X \wedge Y$;

$(\mathsf{Sets}_*, \rhd_{\mathsf{Sets}_*}, S^0)$ is a natural isomorphism.

[19] *Further Terminology:* Also called the **tensor product of** $\mathbb{F}_1$**-modules of** (X, x_0) **and** (Y, y_0) or the **tensor product of** (X, x_0) **and** (Y, y_0) **over** $\mathbb{F}_1$.

[20] *Further Notation:* Also written $X \otimes_{\mathbb{F}_1} Y$.

there exists a unique morphism of pointed sets $X \wedge Y \xrightarrow{\exists!} Z$ making the diagram

$$\begin{tikzcd} & X \wedge Y \arrow[d, dashed, "\exists!"] \\ X \times Y \arrow[ur, "\iota"] \arrow[r, "f"'] & Z \end{tikzcd}$$

commute.

Construction 5.1.1.3. Concretely, the **smash product of** (X, x_0) **and** (Y, y_0) is the pointed set $(X \wedge Y, [(x_0, y_0)])$ consisting of[21]

- *The Underlying Set.* The set $X \wedge Y$ defined by

$$\begin{aligned} X \wedge Y &\cong \mathrm{pt} \coprod_{X \vee Y} (X \times Y) \\ &\stackrel{\text{def}}{=} \frac{X \times Y}{X \vee Y} \\ &\cong X \times Y/\sim, \end{aligned} \qquad \begin{tikzcd} X \wedge Y \arrow[dr, phantom, "\ulcorner", very near start] & X \times Y \arrow[l] \\ \mathrm{pt} \arrow[u] & X \vee Y, \arrow[l, "!"] \arrow[u, hook] \end{tikzcd}$$

  where $\sim$ is the equivalence relation of $X \times Y$ obtained by declaring $(x, y) \sim (x', y')$ iff $(x, y), (x', y') \in X \vee Y$, i.e. by declaring

$$\begin{aligned} (x_0, y) &\sim (x_0, y'), \\ (x, y_0) &\sim (x', y_0) \end{aligned}$$

  for all $x \in X$ and all $y \in Y$;

- *The Basepoint.* The element $[(x_0, y_0)]$ of $X \wedge Y$ given by the equivalence class of (x_0, y_0) under the equivalence relation $\sim$ on $X \times Y$.

Proof. Clear. □

[21] *Further Notation:* We write $x \wedge y$ for the image of (x, y) under the quotient map

$$X \times Y \twoheadrightarrow \underbrace{\frac{X \times Y}{X \vee Y}}_{\stackrel{\text{def}}{=} X \wedge Y}$$

Note that we have

$$\begin{aligned} x \wedge y_0 &= x' \wedge y_0, \\ x_0 \wedge y &= x_0 \wedge y' \end{aligned}$$

for each $x, x' \in X$ and each $y, y' \in Y$.

Example 5.1.1.4. Here are some examples of smash products of pointed sets.

1. *Smashing With S^0.* For any pointed set X, we have isomorphisms of pointed sets

$$\begin{aligned} S^0 \wedge X &\cong X, \\ X \wedge S^0 &\cong X. \end{aligned}$$

Proposition 5.1.1.5. Let (X, x_0) and (Y, y_0) be pointed sets.

1. *Functoriality.* The assignments (X, x_0), (Y, y_0), $((X, x_0), (Y, y_0)) \mapsto X \wedge Y$ define functors

$$\begin{aligned} X \wedge -\colon \mathsf{Sets}_* &\longrightarrow \mathsf{Sets}_*, \\ - \wedge Y\colon \mathsf{Sets}_* &\longrightarrow \mathsf{Sets}_*, \\ -_1 \wedge -_2\colon \mathsf{Sets}_* \times \mathsf{Sets}_* &\longrightarrow \mathsf{Sets}_*. \end{aligned}$$

2. *Adjointness.* We have adjunctions

$$\begin{aligned} (X \wedge - \dashv \mathbf{Sets}_*(X, -))\colon &\quad \begin{tikzcd} \mathsf{Sets}_* \arrow[r, bend left, "X\wedge -"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_*, \arrow[l, bend left, "\mathbf{Sets}_*(X,-)"] \end{tikzcd} \\ (- \wedge Y \dashv \mathbf{Sets}_*(Y, -))\colon &\quad \begin{tikzcd} \mathsf{Sets}_* \arrow[r, bend left, "-\wedge Y"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_*, \arrow[l, bend left, "\mathbf{Sets}_*(Y,-)"] \end{tikzcd} \end{aligned}$$

witnessed by bijections

$$\begin{aligned} \mathsf{Sets}_*(X \wedge Y, Z) &\cong \mathsf{Sets}_*(X, \mathbf{Sets}_*(Y, Z)), \\ \mathsf{Sets}_*(X \wedge Y, Z) &\cong \mathsf{Sets}_*(X, \mathbf{Sets}_*(A, Z)), \end{aligned}$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, which internalise to isomorphisms of pointed sets

$$\begin{aligned} \mathbf{Sets}_*(X \wedge Y, Z) &\cong \mathbf{Sets}_*(X, \mathbf{Sets}_*(Y, Z)), \\ \mathbf{Sets}_*(X \wedge Y, Z) &\cong \mathbf{Sets}_*(X, \mathbf{Sets}_*(A, Z)), \end{aligned}$$

again natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

3. *Closed Symmetric Monoidality.* The quadruple $\left(\mathsf{Sets}_*, \wedge, S^0, \mathbf{Sets}_*\right)$ is a closed symmetric monoidal category.

4. *Morphisms From the Monoidal Unit.* We have a bijection of sets[22]

$$\mathsf{Sets}_*(S^0, X) \cong X,$$

natural in $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, internalising also to an isomorphism of pointed sets

$$\mathbf{Sets}_*(S^0, X) \cong (X, x_0),$$

again natural in $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

5. *Symmetric Strong Monoidality With Respect to Free Pointed Sets.* The free pointed set functor of Pointed Sets, Item 1 of Proposition 4.2.1.2 has a symmetric strong monoidal structure

$$\left((-)^+, (-)^{+,\times}, (-)^{+,\times}_{\mathbb{K}}\right) : (\mathsf{Sets}, \times, \mathrm{pt}) \to \left(\mathsf{Sets}_*, \wedge, S^0\right),$$

being equipped with isomorphisms of pointed sets

$$(-)^{+,\times}_{X,Y} : X^+ \wedge Y^+ \xrightarrow{\cong} (X \times Y)^+,$$
$$(-)^{+,\times}_{\mathbb{K}} : S^0 \xrightarrow{\cong} \mathrm{pt}^+,$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

6. *Distributivity Over Wedge Sums.* We have isomorphisms of pointed sets

$$X \wedge (Y \vee Z) \cong (X \wedge Y) \vee (X \wedge Z),$$
$$(X \vee Y) \wedge Z \cong (X \wedge Z) \vee (Y \wedge Z),$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

7. *Universal Property I.* The symmetric monoidal structure on the category Sets_* is uniquely determined by the following requirements:

   (a) *Two-Sided Preservation of Colimits.* The smash product

   $$\wedge : \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*$$

   of Sets_* preserves colimits separately in each variable.

   (b) *The Unit Object Is S^0.* We have $\mathbb{K}_{\mathsf{Sets}_*} = S^0$.

[22] In other words, the forgetful functor

$$忘 : \mathsf{Sets}_* \to \mathsf{Sets}$$

8. *Universal Property II.* The symmetric monoidal structure on the category Sets_* is the unique symmetric monoidal structure on Sets_* such that the free pointed set functor

$$(-)^+ \colon \mathsf{Sets} \to \mathsf{Sets}_*$$

admits a symmetric monoidal structure.

9. *Existence of Monoidal Diagonals.* The triple $\left(\mathsf{Sets}_*, \wedge, S^0\right)$ is a monoidal category with diagonals:

   (a) *Monoidal Diagonals.* The natural transformation

$$\Delta \colon \mathrm{id}_{\mathsf{Sets}_*} \Longrightarrow \wedge \circ \Delta^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}, \qquad \begin{tikzcd} \mathsf{Sets}_* \arrow[rr, bend left, "\mathrm{id}_{\mathsf{Sets}_*}"] \arrow[dr, bend right, "\Delta^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}"'] \arrow[rr, phantom, shift right=4ex, "\Downarrow\Delta"] & & \mathsf{Sets}_* \\ & \mathsf{Sets}_* \times \mathsf{Sets}_*, \arrow[ur, bend right, "\wedge"'] & \end{tikzcd}$$

   whose component

$$\Delta_X \colon (X, x_0) \to (X \wedge X, [(x_0, x_0)])$$

   at $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$ is given by the composition

$$\begin{aligned} (X, x_0) &\xrightarrow{\Delta_X} (X \times X, (x_0, x_0)) \\ &\twoheadrightarrow \left(\tfrac{X \times X}{X \vee X}, [(x_0, x_0)]\right) \\ &\stackrel{\text{def}}{=\!=} (X \wedge X, [(x_0, x_0)]) \end{aligned}$$

   in Sets_*, is a monoidal natural transformation:

   i. *Naturality.* For each morphism $f \colon X \to Y$ of pointed sets, the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\Delta_X"'] & Y \arrow[d, "\Delta_Y"] \\ X \wedge X \arrow[r, "f \wedge f"'] & Y \wedge Y \end{tikzcd}$$

   commutes.

   ii. *Compatibility With Strong Monoidality Constraints.* For each $(X, x_0), (Y, y_0) \in$

defined on objects by sending a pointed set to its underlying set is corepresentable by S^0.

Obj(Sets$_*$), the diagram

$$\begin{tikzcd} X \wedge Y \arrow[r, "\Delta_X \wedge \Delta_Y"] \arrow[d, equal] & (X \wedge X) \wedge (Y \wedge Y) \arrow[d, dashed, "\wr"] \\ X \wedge Y \arrow[r, "\Delta_{X \wedge Y}"'] & (X \wedge Y) \wedge (X \wedge Y) \end{tikzcd}$$

commutes.

iii. *Compatibility With Strong Unitality Constraints.* The diagram

$$\begin{tikzcd} S^0 \arrow[d, equal] \arrow[dr, "(\lambda^{\mathsf{Sets}_*}_{S^0})^{-1} = (\rho^{\mathsf{Sets}_*}_{S^0})^{-1}"] & \\ S^0 \arrow[r, "\Delta_{S^0}"'] & S^0 \wedge S^0 \end{tikzcd}$$

commutes.

(b) *The Diagonal of the Unit.* The component

$$\Delta^{\mathsf{Sets}_*}_{S^0} \colon S^0 \xrightarrow{\cong} S^0 \wedge S^0$$

of the diagonal natural transformation of Sets$_*$ at S^0 is an isomorphism.

10. *Comonoids in* Sets$_*$. The symmetric monoidal functor

$$\left((-)^+, (-)^{+,\times}, (-)^{+,\times}_{\mathbb{1}}\right) \colon (\mathsf{Sets}, \times, \mathrm{pt}) \longrightarrow \left(\mathsf{Sets}_*, \wedge, S^0\right),$$

of Pointed Sets, Item 4 of Proposition 4.2.1.2 lifts to an equivalence of categories

$$\begin{aligned} \mathsf{CoMon}\left(\mathsf{Sets}_*, \wedge, S^0\right) &\overset{\text{eq.}}{\cong} \mathsf{CoMon}(\mathsf{Sets}, \times, \mathrm{pt}) \\ &\cong \mathsf{Sets}. \end{aligned}$$

Proof. *Item 1, Functoriality*: Omitted.
Item 2, Adjointness: Omitted.
Item 3, Closed Symmetric Monoidality: Omitted.
Item 4, Morphisms From the Monoidal Unit: Omitted.
Item 5, Symmetric Strong Monoidality With Respect to Free Pointed Sets: Omitted.
Item 6, Distributivity Over Wedge Sums: This follows from Item 3, Monoidal Categories, **??** of **??**, and the fact that $\vee$ is the coproduct in Sets$_*$.
Item 7, Universal Property I: Omitted.
Item 8, Universal Property II: See [GGN15, Theorem 5.1].
Item 9, Existence of Monoidal Diagonals: Omitted.
Item 10, Comonoids in Sets$_*$: See [PS19, Lemma 2.4]. □

Appendices

A Other Chapters