Spans

December 20, 2023

This chapter contains some material about spans. Notably, we discuss and explore:

1. The basic definitions around spans (Section 1);
2. The relation between spans and functions (Proposition 7.1.1.1);
3. The relation between spans and relations (Propositions 7.2.2.1 and 7.3.1.1 and Remark 7.5.1.1).
4. “Hyperpointed sets” (**??**). I don’t know why I wrote this…

TODO:

1. https://www.sciencedirect.com/science/article/pii/0022404994900094?ref=pdf_download&fr=RR-2&rr=834107b75c906aa4
2. https://arxiv.org/abs/1605.08100
3. https://arxiv.org/abs/1603.08181
4. https://arxiv.org/abs/1601.02307
5. https://arxiv.org/abs/1507.01460
6. https://arxiv.org/abs/1506.08870
7. https://arxiv.org/abs/1505.00048
8. https://arxiv.org/abs/1501.07592
9. https://arxiv.org/abs/1501.04664
10. https://arxiv.org/abs/1501.00792

11. https://arxiv.org/abs/1412.6560
12. https://arxiv.org/abs/1412.0212
13. https://arxiv.org/abs/1409.0837
14. https://arxiv.org/abs/1408.5220
15. https://arxiv.org/abs/1308.6548
16. https://arxiv.org/abs/1304.0219
17. https://arxiv.org/abs/1210.8192
18. https://arxiv.org/abs/1210.1433
19. https://arxiv.org/abs/1201.3789
20. https://arxiv.org/abs/1112.0560
21. https://arxiv.org/abs/1109.1598
22. https://arxiv.org/abs/1101.4594
23. https://arxiv.org/abs/1012.6001
24. https://arxiv.org/abs/1011.3243
25. https://arxiv.org/abs/0910.2996
26. https://arxiv.org/abs/0810.2361
27. https://arxiv.org/abs/0803.2429
28. https://arxiv.org/abs/0712.2525
29. https://arxiv.org/abs/0706.1286
30. https://arxiv.org/abs/math/0611930
31. https://arxiv.org/abs/2311.15342
32. https://arxiv.org/abs/2310.19428
33. https://arxiv.org/abs/2309.08084
34. https://arxiv.org/abs/2308.01662

35. https://arxiv.org/abs/2301.11860
36. https://arxiv.org/abs/2301.01199
37. https://arxiv.org/abs/2212.09060
38. https://arxiv.org/abs/2208.07183
39. https://arxiv.org/abs/2205.06892
40. https://arxiv.org/abs/2203.16179
41. https://arxiv.org/abs/2201.09551
42. https://arxiv.org/abs/2112.04599
43. https://arxiv.org/abs/2111.10968
44. https://arxiv.org/abs/2107.07621
45. https://arxiv.org/abs/2106.14743
46. https://arxiv.org/abs/2105.14654
47. https://arxiv.org/abs/2102.08051
48. https://arxiv.org/abs/2102.04386
49. https://arxiv.org/abs/2101.06734
50. https://arxiv.org/abs/2011.11042
51. https://arxiv.org/abs/2010.15722
52. https://arxiv.org/abs/2006.10375
53. https://arxiv.org/abs/2006.10375
54. https://arxiv.org/abs/2005.10496
55. https://arxiv.org/abs/2003.11541
56. https://arxiv.org/abs/2002.10334
57. https://arxiv.org/abs/1909.00069
58. https://arxiv.org/abs/1907.02695

59. https://arxiv.org/abs/1905.06671
60. define a relational span
61. consider giving Ran and Rift their dedicated sections on the relations chapter, perhaps together with the other sections on co/limits
62. https://arxiv.org/abs/1710.02742
63. https://arxiv.org/search/math?searchtype=author&query=Walker,+Charles
64. https://arxiv.org/abs/1706.09575
65. https://arxiv.org/abs/1710.01465
66. fibred categories: https://arxiv.org/abs/1806.02376
67. https://arxiv.org/abs/1806.10477v2
68. double categorical limits in $\mathsf{Rel}^{\mathsf{dbl}}$
69. double categorical limits in $\mathsf{Span}^{\mathsf{dbl}}$
70. internal adjoint equivalences in $\mathbf{Rel}$
71. internal adjoint equivalences in Span
72. 2-categorical limits in $\mathbf{Rel}$;
73. morphism of internal adjunctions in $\mathbf{Rel}$;
74. morphism of internal adjunctions in Span;
75. morphism of co/monads in Span;
76. What is $\mathsf{Adj}(\mathsf{Span}(A, B))$?
77. monoids, comonoids, pseudomonoids, etc. in Span.
78. write down the dumb intuition about spans inducing morphisms $\mathsf{Sets}(S, A) \to \mathsf{Sets}(S, B)$ instead of $\mathcal{P}(A) \to \mathcal{P}(B)$ from the similarity between

$$S \to A \times B$$

and

$$A \times B \to \{\mathsf{t}, \mathsf{f}\}.$$

This intuition is justified by taking $A = \mathrm{pt}$ or $B = \mathrm{pt}$.

79. What about using the direct image with compact support in $g(f^{-1}(a))$?
80. Monads in Span | develop this in the level of morphisms too
81. Comonads in Span are spans whose legs are equal | develop this in the level of morphisms too
82. Does Span have an internal **Hom**?
83. Examples of spans
84. Functional and total spans
85. closed symmetric monoidal category of spans
86. double category of relations
87. collage of a span
88. equivalence spans?
89. functoriality of powersets for spans
90. Is Span a closed bicategory?
91. skew monoidal structure on $\mathsf{Span}(A, B)$
92. Adjunctions in Span
93. Isomorphisms in Span
94. Equivalences in Span
95. Interaction between the above notions in Span vs.in $\mathbf{\mathsf{Rel}}$ via the comparison functors
96. $\mathrm{Hom}_{\mathcal{C}}(S, A) \times \mathrm{Hom}_{\mathcal{C}}(f^*(S), A)$.
97. Proof of non-existence of left Kan extensions/lifts in $\mathbf{\mathsf{Rel}}$ (when do these exist btw?)
98. add intuition for spans as relations with multiple witnesses
99. description of unitors and associators of span

Contents

1 Spans

1.1 The Walking Span

Definition 1.1.1.1. The **walking span** is the category Λ that looks like this:

$$\begin{tikzcd} & {[0]} \arrow[loop above] \arrow[dl] \arrow[dr] & \\ {[-1]} \arrow[loop left] & & {[1]} \arrow[loop right] \end{tikzcd}$$

1.2 Spans

Let A and B be sets.

Definition 1.2.1.1. A **span from A to B**[1] is a functor $F\colon \Lambda \to \mathsf{Sets}$ such that

$$\begin{aligned} F([-1]) &= A, \\ F([1]) &= B. \end{aligned}$$

Remark 1.2.1.2. In detail, a **span from A to B** is a triple (S, f, g)

[1] *Further Terminology:* Also called a **roof from A to B** or a **correspondence from A to B**.

consisting of[2,3,4]

- *The Underlying Set.* A set S, called the **underlying set of** (S, f, g);
- *The Legs.* A pair of functions $f\colon S \to A$ and $g\colon S \to B$.

1.3 Morphisms of Spans

Definition 1.3.1.1. A **morphism of spans from** (R, f_1, g_1) **to** (S, f_2, g_2)[5] is a natural transformation $(R, f_1, g_1) \Longrightarrow (S, f_2, g_2)$.

Remark 1.3.1.2. In detail, a **morphism of spans from** (R, f_1, g_1) **to**

---

[2] *Picture:*

$$\begin{tikzcd} & S \arrow[dl, "f"'] \arrow[dr, "g"] & \\ A & & B. \end{tikzcd}$$

[3] Every span (S, f, g) from A to B determines in particular a relation $R\colon A \nrightarrow B$ via

$$R \overset{\text{def}}{=} \{(f(a), g(a)) \mid a \in A\},$$

i.e. where $R(a) = g\big(f^{-1}(a)\big)$ for each $a \in A$; see Proposition 7.2.2.1.

[4] In general, spans from A to B may be thought of as relations from A to B which can relate an element $a \in A$ to an element $b \in B$ in multiple ways via f and g, with the "set of witnesses of $a \sim_S b$" being given by

$$\mathrm{Wit}_S(a, b) \overset{\text{def}}{=} \{s \in S \mid a = f(s) \text{ and } g(s) = b\}.$$

This analogy is made precise by Remark 7.5.1.1. Note, however, that when making the passage from sets to categories, the suitable generalisation of an assignment $(a, b) \mapsto \mathrm{Wit}_S(a, b)$ for each $(a, b) \in A \times B$ becomes that of a profunctor (TODO), which differs from spans of categories.

[5] *Further Terminology:* Also called a **morphism of roofs from** (R, f_1, g_1) **to** (S, f_2, g_2) or a **morphism of correspondences from** (R, f_1, g_1) **to** (S, f_2, g_2).

(S, f_2, g_2) is a function $\phi\colon R \to S$ making the diagram[6]

$$
\begin{tikzcd}
 & R \arrow[dl, "f_1"'] \arrow[dr, "g_1"] \arrow[rrr, "\phi"] & & & S \arrow[dl, "f_2"'] \arrow[dr, "g_2"] & \\
A \arrow[rrr, equal] & & B \arrow[rrr, equal] & A & & B.
\end{tikzcd}
$$

commute.

1.4 Functional Spans

Let $\lambda = \left(A \xleftarrow{f} S \xrightarrow{g} B\right)$ be a span.

Definition 1.4.1.1. The span λ is **functional** if f the following equivalent conditions are satisfied:

1. The associated relation $g \circ f^{-1}$ of λ is functional.
2. For each $s, t \in S$, if $f(s) = f(t)$, then $g(s) = g(t)$.
3. this "f-relative injectivity" condition is the same as being a monomorphism/monoid/whatever in nice category | maybe this is the same as being a skew monoid in $\mathsf{Span}(A, B)$ or something?

1. a[7]

Definition 1.4.1.2. The span λ is **total** if f is surjective.

Let $\lambda = \left(A \xleftarrow{f} S \xrightarrow{g} B\right)$ be a span. A morphism of spans from id_A to $\lambda \diamond \lambda^\dagger$

[6] *Alternative Picture:*

$$
\begin{tikzcd}
 & R \arrow[dl, "f_1"'] \arrow[dr, "g_1"] \arrow[dd, "\phi"] & \\
A & & B. \\
 & S \arrow[ul, "f_2"] \arrow[ur, "g_2"'] &
\end{tikzcd}
$$

[7] Here we could perhaps also use the direct image with compact support $g_!$ of g (see Constructions With Sets, Definition 4.5.1.1) instead of the usual direct image, although

is a morphism

$$s\colon A \to S \times_B S$$

making the diagram

$$\begin{tikzcd}
 & A \arrow[dl, equal, "\mathrm{id}_A"'] \arrow[dr, equal, "\mathrm{id}_A"] \arrow[dd, "s"] & \\
A & & A \\
 & S \times_B S \arrow[ul, "f \circ \mathrm{pr}'_1"] \arrow[ur, "f \circ \mathrm{pr}'_2"'] &
\end{tikzcd}$$

commute, where $S \times_B S$ is the pullback

$$S \times_B S \cong \{(s,t) \in S \times S \mid g(s) = g(t)\} \qquad \begin{tikzcd}
S \times_B S \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & S \arrow[d, "g"] \\
S \arrow[r, "g"'] & B
\end{tikzcd}$$

of S with itself along g. In particular, $\mathrm{pr}_1 \circ s$ and $\mathrm{pr}_2 \circ s$ are both left-inverses/retractions for f, i.e. we have

$$\begin{aligned}
(\mathrm{pr}_1 \circ s) \circ f &\cong \mathrm{id}_A, \\
(\mathrm{pr}_2 \circ s) \circ f &\cong \mathrm{id}_A.
\end{aligned}$$

Thus, by Categories, **??** of **??**, f is injective if $A \neq \emptyset$.

---

the expression for $g_!\big(f^{-1}(a)\big)$ seems a bit weird. It can also actually be given as a right Kan extension (Relations, Item 11 of Proposition 2.5.1.1):

$$\begin{aligned}
g_!\big(f^{-1}(a)\big) &= g_!(\{s \in S \mid f(s) = a\}) \\
&= \left\{b \in B \;\middle|\; g^{-1}(b) \subset \{s \in S \mid f(s) = a\}\right\} \\
&= \{b \in B \mid \text{for each } s \in S, \text{ if } g(s) = b, \text{ then } f(s) = a\} \\
&= [\mathrm{Ran}^{\dagger}_g(f)](a)
\end{aligned}$$

as in the diagram

$$\mathrm{Ran}^{\dagger}_g(f)\colon A \nrightarrow B \qquad \begin{tikzcd}
 & B \arrow[d, dashed, "\mathrm{Ran}_g(f)", "|"{marking}] \\
S \arrow[ur, "g", "|"{marking}] \arrow[r, "f"', "|"{marking}] \arrow[ur, phantom, "\Leftarrow", shift right=3] & A.
\end{tikzcd}$$

1.5 Total Spans

2 Categories of Spans

2.1 The Category of Spans Between Two Sets

Let A and B be sets.

Definition 2.1.1.1. The **category of spans from** A **to** B is the category $\mathsf{Span}(A,B)$ defined by

$$\mathsf{Span}(A,B) \overset{\text{def}}{=} \mathsf{Fun}(\Lambda,\mathsf{Sets}) \underset{\mathrm{ev}_{[-1]},\mathsf{Sets},[A]}{\times} \mathsf{pt} \underset{[B],\mathsf{Sets},\mathrm{ev}_{[1]}}{\times} \mathsf{Fun}(\Lambda,\mathsf{Sets}),$$

as in the diagram

$$\begin{tikzcd}
 & & \mathsf{Span}(A,B) \arrow[dl] \arrow[dr] \arrow[d, phantom, "\lrcorner"] & & \\
 & \mathsf{Fun}(\Lambda,\mathsf{Sets}) \underset{\mathsf{Sets}}{\times} \mathsf{pt} \arrow[dl] \arrow[dr] \arrow[d, phantom, "\lrcorner"] & {} & \mathsf{pt} \underset{\mathsf{Sets}}{\times} \mathsf{Fun}(\Lambda,\mathsf{Sets}) \arrow[dl] \arrow[dr] \arrow[d, phantom, "\lrcorner"] & \\
\mathsf{Fun}(\Lambda,\mathsf{Sets}) \arrow[dr, "\mathrm{ev}_{[-1]}"'] & {} & \mathsf{pt} \arrow[dl, "{[A]}"] \arrow[dr, "{[B]}"'] & {} & \mathsf{Fun}(\Lambda,\mathsf{Sets}). \arrow[dl, "\mathrm{ev}_{[1]}"] \\
 & \mathsf{Sets} & & \mathsf{Sets} &
\end{tikzcd}$$

Remark 2.1.1.2. In detail, the **category of spans from** A **to** B is the category $\mathsf{Span}(A,B)$ where

- *Objects.* The objects of $\mathsf{Span}(A,B)$ are spans from A to B;
- *Morphisms.* The morphism of $\mathsf{Span}(A,B)$ are morphisms of spans;
- *Identities.* The unit map

$$\mathbb{1}^{\mathsf{Span}(A,B)}_{(S,f,g)} \colon \mathrm{pt} \to \mathrm{Hom}_{\mathsf{Span}(A,B)}((S,f,g),(S,f,g))$$

of $\mathsf{Span}(A, B)$ at (S, f, g) is defined by[8]

$$\mathrm{id}^{\mathsf{Span}(A,B)}_{(S,f,g)} \overset{\text{def}}{=} \mathrm{id}_S;$$

- *Composition.* The composition map

  $$\circ^{\mathsf{Span}(A,B)}_{R,S,T} \colon \mathrm{Hom}_{\mathsf{Span}(A,B)}(S, T) \times \mathrm{Hom}_{\mathsf{Span}(A,B)}(R, S) \to \mathrm{Hom}_{\mathsf{Span}(A,B)}(R, T)$$

  of $\mathsf{Span}(A, B)$ at $((R, f_1, g_1), (S, f_2, g_2), (T, f_3, g_3))$ is defined by[9]

  $$\psi \circ^{\mathsf{Span}(A,B)}_{R,S,T} \phi \overset{\text{def}}{=} \psi \circ \phi.$$

Proposition 2.1.1.3. Let A and B be sets.

1. *As a Pullback.* We have an isomorphism of categories

$$\mathsf{Span}(A, B) \cong \mathsf{Sets}_{/A} \times_{\mathsf{Sets}} \mathsf{Sets}_{/B}, \qquad \begin{tikzcd} \mathsf{Span}(A, B) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Sets}_{/B} \arrow[d, "忘"] \\ \mathsf{Sets}_{/A} \arrow[r, "忘"'] & \mathsf{Sets}. \end{tikzcd}$$

[8] *Picture:*

$$\begin{tikzcd} & S \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[rrr, "\mathrm{id}_S"] & & & S \arrow[dl, "f"'] \arrow[dr, "g"] & \\ A \arrow[rrr, equal] & & B \arrow[rrr, equal] & A & & B. \end{tikzcd}$$

[9] *Picture:*

$$\begin{tikzcd} & R \arrow[dl, "f_1"'] \arrow[dr, "g_1"] \arrow[rrr, "\phi"] & & & S \arrow[dl, "f_2"'] \arrow[dr, "g_2"] \arrow[rrr, "\psi"] & & & T \arrow[dl, "f_3"'] \arrow[dr, "g_3"] & \\ A \arrow[rrr, equal] & & B \arrow[rrr, equal] & A \arrow[rrr, equal] & & B \arrow[rrr, equal] & A & & B. \end{tikzcd}$$

Proof. *Item 1, As a Pullback*: In detail, the pullback $\mathsf{Sets}_{/A} \times_{\mathsf{Sets}} \mathsf{Sets}_{/B}$ is the category where

- *Objects.* The objects of $\mathsf{Sets}_{/A} \times_{\mathsf{Sets}} \mathsf{Sets}_{/B}$ consist of pairs $((S, f), (S', g))$ of objects of Sets consisting of
  - A pair (S, f) in $\mathrm{Obj}\big(\mathsf{Sets}_{/A}\big)$ consisting of a set S and a map $f\colon S \to A$;
  - A pair (S', g) in $\mathrm{Obj}\big(\mathsf{Sets}_{/B}\big)$ consisting of a set S' and a map $g\colon S \to B$;

  such that
  $$\underbrace{忘(S, f)}_{\overset{\text{def}}{=} S} = \underbrace{忘(S', g)}_{\overset{\text{def}}{=} S'}.$$
  Thus the objects of $\mathsf{Sets}_{/A} \times_{\mathsf{Sets}} \mathsf{Sets}_{/B}$ are the same as spans from A to B.

- *Morphisms.* A morphism of $\mathsf{Sets}_{/A} \times_{\mathsf{Sets}} \mathsf{Sets}_{/B}$ from (S, f, g) to (S', f', g') consists of a pair of morphisms
  $$\phi\colon S \to S'$$
  $$\psi\colon S \to S'$$
  such that the diagrams
  $$\begin{tikzcd} S \arrow[rr, "\phi"] \arrow[dr, "f"'] & & S' \arrow[dl, "f'"] \\ & A & \end{tikzcd} \qquad \begin{tikzcd} S \arrow[rr, "\psi"] \arrow[dr, "g"'] & & S' \arrow[dl, "g'"] \\ & B & \end{tikzcd}$$
  such that
  $$\underbrace{忘(\phi)}_{\overset{\text{def}}{=} \phi} = \underbrace{忘(\psi)}_{\overset{\text{def}}{=} \psi}.$$
  Thus the morphisms of $\mathsf{Sets}_{/A} \times_{\mathsf{Sets}} \mathsf{Sets}_{/B}$ are also the same as morphisms of spans from (S, f, g) to (S, f', g').

- *Identities and Composition.* The identities and composition of $\mathsf{Sets}_{/A} \times_{\mathsf{Sets}} \mathsf{Sets}_{/B}$ are also the same as those in $\mathsf{Span}(A, B)$.

This finishes the proof. □

2.2 The Bicategory of Spans

Definition 2.2.1.1. The **bicategory of spans** is the bicategory Span where

- *Objects.* The objects of Span are sets;
- Hom-*Categories.* For each $A, B \in \mathrm{Obj}(\mathsf{Span})$, we have
$$\mathsf{Hom}_{\mathsf{Span}}(A,B) \overset{\mathrm{def}}{=} \mathsf{Span}(A,B);$$
- *Identities.* For each $A \in \mathrm{Obj}(\mathsf{Span})$, the unit functor
$$\mathbb{1}^{\mathsf{Span}}_{A} \colon \mathsf{pt} \to \mathsf{Span}(A,A)$$
of Span at A is the functor picking the span $(A, \mathrm{id}_A, \mathrm{id}_A)$:
$$\begin{tikzcd}
 & A \arrow[dl, "\mathrm{id}_A"'] \arrow[dr, "\mathrm{id}_A"] & \\
A & & A.
\end{tikzcd}$$
- *Composition.* For each $A, B, C \in \mathrm{Obj}(\mathsf{Span})$, the composition bifunctor
$$\circ^{\mathsf{Span}}_{A,B,C} \colon \mathsf{Span}(B,C) \times \mathsf{Span}(A,B) \to \mathsf{Span}(A,C)$$
of Span at (A,B,C) is the bifunctor where
  - *Action on Objects.* The composition of two spans
$$\begin{tikzcd}
 & R \arrow[dl, "f_1"'] \arrow[dr, "g_1"] & & & & S \arrow[dl, "f_2"'] \arrow[dr, "g_2"] & \\
A & & B & \text{and} & B & & C
\end{tikzcd}$$
is the span $(R \times_B S, f_1 \circ \mathrm{pr}_1, g_2 \circ \mathrm{pr}_2)$, constructed as in the diagram
$$\begin{tikzcd}
 & & R \times_B S \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] \arrow[dd, phantom, very near start, "\lrcorner", rotate=-45] \arrow[ddll, bend right, "f_1 \circ \mathrm{pr}_1"'] \arrow[ddrr, bend left, "g_2 \circ \mathrm{pr}_2"] & & \\
 & R \arrow[dl, "f_1"'] \arrow[dr, "g_1"] & & S \arrow[dl, "f_2"'] \arrow[dr, "g_2"] & \\
A & & B & & C.
\end{tikzcd}$$

– *Action on Morphisms.* The horizontal composition of 2-morphisms is defined via functoriality of pullbacks: given morphisms of spans

$$
\begin{tikzcd}
 & R \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[dd, "\phi"] & \\
A & & B \\
 & R' \arrow[ul, "f'"] \arrow[ur, "g'"'] &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
 & S \arrow[dl, "h"'] \arrow[dr, "k"] \arrow[dd, "\psi"] & \\
B & & C, \\
 & S' \arrow[ul, "h'"] \arrow[ur, "k'"'] &
\end{tikzcd}
$$

their horizontal composition is the morphism of spans

$$
\begin{tikzcd}
 & R \times_B S \arrow[dl, "f \circ \mathrm{pr}_1"'] \arrow[dr, "k \circ \mathrm{pr}_2"] \arrow[dd, dashed, "\exists!"] & \\
A & & C, \\
 & R' \times_B S' \arrow[ul, "h' \circ \mathrm{pr}'_1"] \arrow[ur, "k' \circ \mathrm{pr}'_2"'] &
\end{tikzcd}
$$

constructed as in the diagram

$$
\begin{tikzcd}
 & & R \times_B S \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] \arrow[ddll, bend right, "f \circ \mathrm{pr}_1"'] \arrow[ddrr, bend left, "k \circ \mathrm{pr}_2"] \arrow[dddd, dashed, "\exists!"] & & \\
 & R \arrow[dl, "f"'] \arrow[dr, "g"] & & S \arrow[dl, "h"'] \arrow[dr, "k"] & \\
A & & B & & C; \\
 & R' \arrow[ul, "f'"] \arrow[ur, "g'"'] & & S' \arrow[ul, "h'"] \arrow[ur, "k'"'] & \\
 & & R' \times_B S' \arrow[ul, "\mathrm{pr}'_1"] \arrow[ur, "\mathrm{pr}'_2"'] \arrow[uull, bend left, "f' \circ \mathrm{pr}'_1"] \arrow[uurr, bend right, "k' \circ \mathrm{pr}'_2"'] & &
\end{tikzcd}
$$

- *Associators and Unitors.* The associator and unitors are defined using the universal property of the pullback.

2.3 The Monoidal Bicategory of Spans

2.4 The Double Category of Spans

Definition 2.4.1.1. The **double category of spans** is the double category $\mathsf{Span}^{\mathsf{dbl}}$ where

- *Objects.* The objects of $\mathsf{Span}^{\mathsf{dbl}}$ are sets;
- *Vertical Morphisms.* The vertical morphisms of $\mathsf{Span}^{\mathsf{dbl}}$ are functions $f\colon A \to B$;
- *Horizontal Morphisms.* The horizontal morphisms of $\mathsf{Span}^{\mathsf{dbl}}$ are spans $(S, \phi, \psi)\colon A \nrightarrow X$;
- 2-*Morphisms.* A 2-cell

$$\begin{tikzcd} A \arrow[r, "{(R,\phi_R,\psi_R)}", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & B \arrow[d, "g"] \\ X \arrow[r, "{(S,\phi_S,\psi_S)}"', "\shortmid"{marking}] & Y \end{tikzcd}$$

of $\mathsf{Span}^{\mathsf{dbl}}$ is a morphism of spans from the span

$$\begin{tikzcd} & & R \arrow[lld, "\phi_R"'] \arrow[d, "\psi_R"] & \\ & & B \arrow[d, "g"] & \\ A & & & Y \end{tikzcd}$$

to the span

$$\begin{tikzcd} & & A \times_X S \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner"{rotate=-45}] & & \\ & A \arrow[dl, "f"'] \arrow[dr, "f"] & & S \arrow[dl, "\phi_S"'] \arrow[dr, "\psi_S"] & \\ X & & X & & Y; \end{tikzcd}$$

- *Horizontal Identities.* The horizontal unit functor
$$\mathbb{1}^{\mathsf{Span}^{\mathsf{dbl}}}\colon \Big(\mathsf{Span}^{\mathsf{dbl}}\Big)_0 \to \Big(\mathsf{Span}^{\mathsf{dbl}}\Big)_1$$
of $\mathsf{Span}^{\mathsf{dbl}}$ is the functor where
  - *Action on Objects.* For each $A \in \mathrm{Obj}\Big(\Big(\mathsf{Span}^{\mathsf{dbl}}\Big)_0\Big)$, we have
$$\mathbb{1}_A \overset{\text{def}}{=} (A, \mathrm{id}_A, \mathrm{id}_A),$$
as in the diagram
$$\begin{tikzcd}
 & A \arrow[dl, equal, "\mathrm{id}_A"'] \arrow[dr, equal, "\mathrm{id}_A"] & \\
A & & A;
\end{tikzcd}$$
  - *Action on Morphisms.* For each vertical morphism $f\colon A \to B$ of $\mathsf{Span}^{\mathsf{dbl}}$, i.e. each map of sets f from A to B, the identity 2-morphism
$$\begin{tikzcd}
A \arrow[r, "\mathbb{1}_A", "|"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\overset{\|}{\mathbb{1}_f}\Downarrow"] & A \arrow[d, "f"] \\
B \arrow[r, "\mathbb{1}_B"', "|"{marking}] & B
\end{tikzcd}$$
of f is the morphism of spans from
$$\begin{tikzcd}
 & & A \arrow[dll, equal, "\mathrm{id}_A"'] \arrow[dr, equal, "\mathrm{id}_A"] & & \\
 & & & A \arrow[dr, "f"] & \\
A & & & & B
\end{tikzcd}$$
to
$$\begin{tikzcd}
 & & A \times_B B \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner"{rotate=-45}, very near start] & & \\
 & A \arrow[dl, "f"'] \arrow[dr, "f"] & & B \arrow[dl, equal, "\mathrm{id}_B"'] \arrow[dr, equal, "\mathrm{id}_B"] & \\
B & & B & & B
\end{tikzcd}$$
given by the isomorphism $A \xrightarrow{\cong} A \times_B B$;

- *Vertical Identities.* For each $A \in \mathrm{Obj}\left(\mathsf{Span}^{\mathsf{dbl}}\right)$, we have
$$\mathrm{id}_A^{\mathsf{Span}^{\mathsf{dbl}}} \overset{\mathrm{def}}{=} \mathrm{id}_A;$$
- *Identity 2-Morphisms.* For each horizontal morphism $R\colon A \nrightarrow B$ of $\mathsf{Span}^{\mathsf{dbl}}$, the identity 2-morphism
$$\begin{tikzcd} A \arrow[r, "S", "\shortmid"{marking}] \arrow[d, "\mathrm{id}_A"'] \arrow[dr, phantom, "{\Downarrow \mathrm{id}_S}"] & B \arrow[d, "\mathrm{id}_B"] \\ A \arrow[r, "S"', "\shortmid"{marking}] & B \end{tikzcd}$$
of R is the morphism of spans from
$$\begin{tikzcd} & S \arrow[dl, "\phi_S"'] \arrow[d, "\psi_S"] & \\ & B \arrow[dr, equal, "\mathrm{id}_B"] & \\ A & & B \end{tikzcd}$$
to
$$\begin{tikzcd} & & A \times_A S \arrow[dl] \arrow[dr] \arrow[d, phantom, "\lrcorner"{rotate=-45}] & & \\ & A \arrow[dl, equal, "\mathrm{id}_A"'] \arrow[dr, equal, "\mathrm{id}_A"] & & S \arrow[dl, "\phi_S"'] \arrow[dr, "\psi_S"] & \\ A & & A & & B \end{tikzcd}$$
given by the isomorphism $S \xrightarrow{\cong} A \times_A S$;
- *Horizontal Composition.* The horizontal composition functor
$$\odot^{\mathsf{Span}^{\mathsf{dbl}}} \colon \left(\mathsf{Span}^{\mathsf{dbl}}\right)_1 \times_{\left(\mathsf{Span}^{\mathsf{dbl}}\right)_0} \left(\mathsf{Span}^{\mathsf{dbl}}\right)_1 \to \left(\mathsf{Span}^{\mathsf{dbl}}\right)_1$$
of $\mathsf{Span}^{\mathsf{dbl}}$ is the functor where
  - *Action on Objects.* For each composable pair
$$A \overset{(R,\phi_R,\psi_R)}{\nrightarrow} B \overset{(S,\phi_S,\psi_S)}{\nrightarrow} C$$

of horizontal morphisms of $\mathsf{Span}^{\mathsf{dbl}}$, we have

$$(S, \phi_S, \psi_S) \odot (R, \phi_R, \psi_R) \stackrel{\text{def}}{=} S \circ^{\mathsf{Span}}_{A,B,C} R,$$

where $S \circ^{\mathsf{Span}}_{A,B,C} R$ is the composition of (R, ϕ_R, ψ_R) and (S, ϕ_S, ψ_S) defined as in Definition 2.2.1.1;

- *Action on Morphisms.* For each horizontally composable pair

$$\begin{tikzcd} A \arrow[r, "{(R,\phi_R,\psi_R)}", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\overset{\parallel}{\underset{\Downarrow}{\alpha}}"] & B \arrow[d, "g"] \\ X \arrow[r, "{(T,\phi_T,\psi_T)}"', "\shortmid"{marking}] & Y \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "{(S,\phi_S,\psi_S)}", "\shortmid"{marking}] \arrow[d, "g"'] \arrow[dr, phantom, "\overset{\parallel}{\underset{\Downarrow}{\beta}}"] & C \arrow[d, "h"] \\ Y \arrow[r, "{(U,\phi_U,\psi_U)}"', "\shortmid"{marking}] & Z \end{tikzcd}$$

of 2-morphisms of $\mathsf{Span}^{\mathsf{dbl}}$,

- *Vertical Composition of* 1*-Morphisms.* For each composable pair $A \xrightarrow{F} B \xrightarrow{G} C$ of vertical morphisms of $\mathsf{Span}^{\mathsf{dbl}}$, i.e. maps of sets, we have

$$g \circ^{\mathsf{Span}^{\mathsf{dbl}}} f \stackrel{\text{def}}{=} g \circ f;$$

- *Vertical Composition of* 2*-Morphisms.* For each vertically composable pair

$$\begin{tikzcd} A \arrow[r, "{(R,\phi_R,\psi_R)}", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\overset{\parallel}{\underset{\Downarrow}{\alpha}}"] & X \arrow[d, "g"] \\ B \arrow[r, "{(S,\phi_S,\psi_S)}"', "\shortmid"{marking}] & Y \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "{(S,\phi_S,\psi_S)}", "\shortmid"{marking}] \arrow[d, "h"'] \arrow[dr, phantom, "\overset{\parallel}{\underset{\Downarrow}{\beta}}"] & Y \arrow[d, "k"] \\ C \arrow[r, "{(T,\phi_T,\psi_T)}"', "\shortmid"{marking}] & Z \end{tikzcd}$$

of 2-morphisms of $\mathsf{Span}^{\mathsf{dbl}}$,

- *Associators and Unitors.* The associator and unitors of $\mathsf{Span}^{\mathsf{dbl}}$ are defined using the universal property of the pullback.

2.5 Properties of The Bicategory of Spans

Proposition 2.5.1.1. Let $\lambda = \left(A \overset{f}{\leftarrow} S \overset{g}{\rightarrow} B\right)$ be a span.

1. *Self-Duality.*

2. *Isomorphisms in* Span*.*

3. *Equivalences in* Span.
4. *Adjunctions in* Span. Let A and B be sets.[10]
   (a) We have a natural bijection
   $$\left\{\begin{matrix}\text{Adjunctions in } \mathsf{Span} \\ \text{from } A \text{ to } B\end{matrix}\right\} \cong \left\{\begin{matrix}\text{Spans } A \overset{f}{\leftarrow} S \overset{g}{\rightarrow} B \\ \text{from } A \text{ to } B \text{ with} \\ f \text{ an isomorphism}\end{matrix}\right\}.$$
   (b) We have an equivalence of categories
   $$\mathsf{MapSpan}(A, B) \overset{\text{eq.}}{\cong} \mathsf{Sets}(A, B)_{\mathsf{disc}},$$
   where $\mathsf{MapSpan}(A, B)$ is the full subcategory of $\mathsf{Span}(A, B)$ spanned by the spans $A \overset{f}{\leftarrow} S \overset{g}{\rightarrow} B$ from A to B with f an isomorphism.
   (c) We have a biequivalence of bicategories
   $$\mathsf{MapSpan} \overset{\text{eq.}}{\cong} \mathsf{Sets}_{\mathsf{bidisc}},$$
   where $\mathsf{MapSpan}$ is the sub-bicategory of Span whose Hom-categories are given by $\mathsf{MapSpan}(A, B)$.
5. *Monads in* Span.
6. *Comonads in* Span.
7. *Monomorphisms in* Span.
8. *Epimorphisms in* Span.
9. *Existence of Right Kan Extensions.*
10. *Existence of Right Kan Lifts.*
11. *Closedness.*

Proof. *Item 1, Self-Duality*:
Item 2, Isomorphisms in Span:
Item 3, Equivalences in Span:
Item 4, Adjunctions in Span: We first prove Item 4a.
We proceed step by step:

[10] In the literature (e.g. [**ref**]),...are called maps and denoted by $\mathsf{MapSpan}(A, B)$

1. *From Adjunctions in* Span *to Functions.* An adjunction in Span from A to B consists of a pair of spans

$$\begin{tikzcd} & R \arrow[dl, "f"'] \arrow[dr, "g"] & \\ A & & B \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & S' \arrow[dl, "h"'] \arrow[dr, "k"] & \\ B & & A \end{tikzcd}$$

together with maps

$$\begin{tikzcd} & A \arrow[dl, equal, "\mathrm{id}_A"'] \arrow[dr, equal, "\mathrm{id}_A"] \arrow[dd, "\phi"] & \\ A & & A \\ & S \times_B S' \arrow[ul, "f \circ \mathrm{pr}'_1"] \arrow[ur, "k \circ \mathrm{pr}'_2"'] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & S' \times_A S \arrow[dl, "h \circ \mathrm{pr}_1"'] \arrow[dr, "g \circ \mathrm{pr}_2"] \arrow[dd, "\psi"] & \\ B & & B. \\ & B \arrow[ul, equal, "\mathrm{id}_B"] \arrow[ur, equal, "\mathrm{id}_B"'] & \end{tikzcd}$$

We claim that these conditions

2. *From Functions to Adjunctions in* $\mathbf{Rel}$.

3. *Invertibility: From Functions to Adjunctions Back to Functions.*

4. *Invertibility: From Adjunctions to Functions Back to Adjunctions.*

We now proceed to the proof of Item 4b. For this, we will construct a functor

$$F \colon \mathsf{Sets}(A, B)_{\mathsf{disc}} \to \mathsf{MapSpan}(A, B)$$

and prove it to be essentially surjective and fully faithful, and thus an equivalence by Categories, **??** of **??**. Indeed, given a map $f \colon A \to B$, let $F(f)$ be the representable span associated to f of Definition 5.1.1.1, and let F send the unique (identity) morphism from f to itself to the identity morphism of $F(f)$ in $\mathsf{MapSpan}(A, B)$. We now prove that F is fully faithful and essentially surjective:

1. *F Is Fully Faithful:* Given maps $f, g \colon A \rightrightarrows B$, we need to show that

$$\mathrm{Hom}_{\mathsf{MapSpan}(A,B)}(F(f), F(g)) = \begin{cases} \mathrm{pt} & \text{if } f = g, \\ \emptyset & \text{otherwise.} \end{cases}$$

Indeed, a morphism from $F(f)$ to $F(g)$ takes the form

$$\begin{tikzcd}
& A \arrow[dl, equal] \arrow[dr, "f"] \arrow[dd, "\phi"] & \\
A \arrow[dr, equal] & & B. \\
& A \arrow[ur, "g"'] &
\end{tikzcd}$$

From the relations $\mathrm{id}_A = \mathrm{id}_A \circ \phi$ and $f = g \circ \phi$, we see that $\phi = \mathrm{id}_A$, and thus from the relation $f = g \circ \phi$ there is such a morphism iff $f = g$.

2. *F Is Essentially Surjective:* Let λ be a span of the form

$$\begin{tikzcd}
& S \arrow[dl, dashed, "\phi"', "\wr"] \arrow[dr, "f"] & \\
A & & B.
\end{tikzcd}$$

we claim that $\lambda \cong F\big(f \circ \phi^{-1}\big)$. Indeed, we have morphisms

$$\begin{tikzcd}
& S \arrow[dl, dashed, "\phi"', "\wr"] \arrow[dr, "f"] \arrow[dd, "\phi"] & & & & A \arrow[dl, equal] \arrow[dr, "f\circ\phi^{-1}"] \arrow[dd, "\phi^{-1}"] & \\
A \arrow[dr, equal] & & B & \text{and} & A & & B, \\
& A \arrow[ur, "f\circ\phi^{-1}"'] & & & & S \arrow[ul, dashed, "\phi", "\wr"'] \arrow[ur, "f"'] &
\end{tikzcd}$$

inverse to each other in $\mathsf{MapSpan}(A, B)$, and thus $\lambda \cong F\big(f \circ \phi^{-1}\big)$.

Finally, we prove Item 4c.
Item 5, Monads in Span:
Item 6, Comonads in Span:
Item 7, Monomorphisms in Span:
Item 8, Epimorphisms in Span:
Item 9, Existence of Right Kan Extensions:
Item 10, Existence of Right Kan Lifts:
Item 11, Closedness:

□

3 Limits of Spans

3.1 tmp2

$$\mathrm{Hom}_{\mathrm{Rel}(A,X)}(\mathrm{Lan}_S(R), T) \cong \mathrm{Hom}_{\mathrm{Rel}(B,X)}(R, T \diamond S)$$

1. $\mathrm{Lan}_S(R) \subset T$, i.e. if $a \sim_{\mathrm{Lan}_S(R)} x$, then $a \sim_T x$.
2. $R \subset T \diamond S$, i.e. if $b \sim_R x$, then there exists some $a \in A$ such that $a \sim_S b$ and $b \sim_T x$.

3.2 tmp

$$\begin{tikzcd} & S \arrow[dl, "f"'] \arrow[dr, "g"] & \\ A & & B \end{tikzcd}$$

$$\begin{tikzcd} & S' \arrow[dl, "\phi_{S'}"'] \arrow[dr, "\psi_{S'}"] & & \mapsto & & S \times_B S' \arrow[dl, "f \circ \mathrm{pr}_1"'] \arrow[dr, "\psi_{S'} \circ \mathrm{pr}_2"] & \\ B & & X & & A & & X \end{tikzcd}$$

$$\begin{tikzcd} & S'' \arrow[dl, "\phi_{S''}"'] \arrow[dr, "\psi_{S''}"] & & \mapsto & & ? \arrow[dl, "?"'] \arrow[dr, "?"] & \\ A & & X & & B & & X \end{tikzcd}$$

$$\mathrm{Hom}_{\mathsf{Span}(A,X)}(S \times_B S', K) \cong \mathrm{Hom}_{\mathsf{Span}(B,X)}(S', R(K))$$

$$\begin{tikzcd} & S \times_B S' \arrow[dl, "f \circ \mathrm{pr}_1"'] \arrow[dr, "\psi_{S'} \circ \mathrm{pr}_2"] \arrow[dd, "\xi"] & \\ A & & X \\ & S'' \arrow[ul, "\phi_{S''}"] \arrow[ur, "\psi_{S''}"'] & \end{tikzcd} \qquad \begin{tikzcd} & S' \arrow[dl, "\phi_{S'}"'] \arrow[dr, "\psi_{S'}"] \arrow[dd, "\xi^\dagger"] & \\ B & & X \\ & \Pi_g(S'') \arrow[ul, "?"] \arrow[ur, "?"'] & \end{tikzcd}$$

3.3 Left Kan Extensions

Let $\lambda = \left(A \xleftarrow{f} S \xrightarrow{g} B\right)$ be a span.

Proposition 3.3.1.1. The left Kan extension

$$\mathrm{Lan}_\lambda \colon \mathsf{Span}(A, X) \to \mathsf{Span}(B, X)$$

along λ in Span exists and is the functor given on objects by sending a span λ' in $\mathsf{Span}(A, X)$ as in

$$\begin{tikzcd} & S' \arrow[dl, "\phi"'] \arrow[dr, "\psi"] & \\ A & & X \end{tikzcd}$$

to the span

$$\mathrm{Lan}_\lambda(\lambda') \stackrel{\text{def}}{=} (\mathrm{Lan}_\lambda(S'), \mathrm{Lan}_\lambda(\phi), \mathrm{Lan}_\lambda(\psi)),$$

in $\mathsf{Span}(B, X)$ where

- The set $\mathrm{Lan}_\lambda(S')$ is given by

$$\begin{aligned}\mathrm{Lan}_\lambda(S') &\stackrel{\text{def}}{=} \Sigma_g(S') \\ &\stackrel{\text{def}}{=} S'\end{aligned}$$

  where $\Sigma_g(S')$ is the dependent sum of $\phi\colon S' \to A$ along g of Fibred Sets, Definition 2.3.1.1;

- The map $\mathrm{Lan}_\lambda(\phi)\colon \mathrm{Lan}_\lambda(S') \to B$ is given by $\Sigma_g(\phi)$;

- The map $\mathrm{Lan}_\lambda(\psi)\colon \mathrm{Lan}_\lambda(S') \to X$ is given by ψ.

Proof. □

3.4 Right Kan Extensions

Let $\lambda = \left(A \xleftarrow{f} S \xrightarrow{g} B\right)$ be a span.

Proposition 3.4.1.1. The right Kan extension

$$\mathrm{Ran}_\lambda\colon \mathsf{Span}(A, X) \to \mathsf{Span}(B, X)$$

along λ in Span exists and is the functor given on objects by sending a span λ' in $\mathsf{Span}(A, X)$ as in

$$\begin{tikzcd} & S' \arrow[dl, "\phi_{S'}"'] \arrow[dr, "\psi_{S'}"] & \\ A & & X \end{tikzcd}$$

to the span

$$\mathrm{Ran}_\lambda(\lambda') \stackrel{\text{def}}{=} (\mathrm{Ran}_\lambda(S'), \mathrm{Ran}_\lambda(\phi_{S'}), \mathrm{Ran}_\lambda(\psi_{S'})),$$

in $\mathsf{Span}(B, X)$ where

- The set $\mathrm{Ran}_\lambda(S')$ is given by

$$\mathrm{Ran}_\lambda(S') \stackrel{\text{def}}{=} \coprod_{b \in B} \prod_{s \in g^{-1}(b)} \phi_{S'}^{-1}(f(s));$$

- The map $\mathrm{Ran}_\lambda(\phi_{S'})\colon \mathrm{Ran}_\lambda(S') \to B$ is given by
$$[\mathrm{Ran}_\lambda(\phi_{S'})](b, (s'_s)_{s\in g^{-1}(b)}) \overset{\text{def}}{=} b;$$
for each $\left(b, (s'_s)_{s\in g^{-1}(b)}\right) \in \mathrm{Ran}_\lambda(S')$;
- The map $\mathrm{Ran}_\lambda(\psi_{S'})\colon \mathrm{Ran}_\lambda(S') \to X$ is given by
$$[\mathrm{Ran}_\lambda(\psi_{S'})]\left(b, (s'_s)_{s\in g^{-1}(b)}\right) \overset{\text{def}}{=} \psi_{S'}(s'_i)$$
for each $\left(b, (s'_s)_{s\in g^{-1}(b)}\right) \in \mathrm{Ran}_\lambda(S')$, where the i in s'_i denotes any $s \in g^{-1}(b)$, as we have $\psi_{S'}(s'_i) = \psi_{S'}\left(s'_j\right)$ for all $s \in g^{-1}(b)$.[11]

Proof. □

3.5 Right Kan Lifts

(Although right Kan lifts aren't really limits, this is probably the most appropriate to place this section.)
Let $\lambda = \left(A \overset{f}{\leftarrow} S \overset{g}{\rightarrow} B\right)$ be a span.

Proposition 3.5.1.1. The right Kan lift
$$\mathrm{Rift}_\lambda\colon \mathsf{Span}(X, B) \to \mathsf{Span}(X, A)$$
along λ in Span exists and is the functor given on objects by sending a span λ' in $\mathsf{Span}(X, B)$ as in
$$\begin{tikzcd} & S' \arrow[dl, "\phi"'] \arrow[dr, "\psi"] & \\ X & & B \end{tikzcd}$$
to the span
$$\mathrm{Rift}_\lambda(\lambda') \overset{\text{def}}{=} (\mathrm{Rift}_\lambda(S'), \mathrm{Rift}_\lambda(\phi), \mathrm{Rift}_\lambda(\psi)),$$
in $\mathsf{Span}(X, A)$ where

- The set $\mathrm{Rift}_\lambda(S')$ is given by
$$\mathrm{Rift}_\lambda(S') \overset{\text{def}}{=} \Pi_f(S'),$$
where $\Pi_f(S')$ is the dependent product of $\psi\colon S' \to A$ along f of Fibred Sets, Definition 2.3.1.1;

[11]Indeed

- The map $\mathrm{Rift}_\lambda(\phi)\colon \mathrm{Rift}_\lambda(S') \to X$ is given by ϕ;
- The map $\mathrm{Rift}_\lambda(\psi)\colon \mathrm{Rift}_\lambda(S') \to A$ is given by $\Pi_f(\psi)$.

Proof. □

4 Colimits of Spans

5 Constructions With Spans

5.1 Representable Spans

Definition 5.1.1.1. Let $f\colon A \to B$ be a function.

- The **representable span associated to** f is the span

$$\begin{tikzcd} & A \arrow[dl, equal, "\mathrm{id}_A"'] \arrow[dr, "f"] & \\ A & & B \end{tikzcd}$$

  from A to B.

- The **corepresentable span associated to** f is the span

$$\begin{tikzcd} & A \arrow[dl, "f"'] \arrow[dr, equal, "\mathrm{id}_A"] & \\ B & & A \end{tikzcd}$$

  from B to A.

5.2 Composition of Spans

Definition 5.2.1.1. The **composition** of two spans

$$\begin{tikzcd} & R \arrow[dl, "f_1"'] \arrow[dr, "g_1"] & \\ A & & B \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & S \arrow[dl, "f_2"'] \arrow[dr, "g_2"] & \\ B & & C \end{tikzcd}$$

is the span $(R \times_B S, f_1 \circ \mathrm{pr}_1, g_2 \circ \mathrm{pr}_2)$, constructed as in the diagram

$$
\begin{tikzcd}
 & & R \times_B S \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] \arrow[ddll, bend right, "f_1 \circ \mathrm{pr}_1"'] \arrow[ddrr, bend left, "g_2 \circ \mathrm{pr}_2"] \arrow[dd, phantom, "\lrcorner", very near start] & & \\
 & R \arrow[dl, "f_1"'] \arrow[dr, "g_1"] & & S \arrow[dl, "f_2"'] \arrow[dr, "g_2"] & \\
A & & B & & C.
\end{tikzcd}
$$

5.3 Horizontal Composition of Morphisms of Spans

Definition 5.3.1.1. The **horizontal composition** of a pair of 2-morphisms of spans

$$
\begin{tikzcd}
 & R \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[dd, "\phi"] & \\
A & & B \\
 & R' \arrow[ul, "f'"] \arrow[ur, "g'"'] &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
 & S \arrow[dl, "h"'] \arrow[dr, "k"] \arrow[dd, "\psi"] & \\
B & & C, \\
 & S' \arrow[ul, "h'"] \arrow[ur, "k'"'] &
\end{tikzcd}
$$

is the morphism of spans

$$
\begin{tikzcd}
 & R \times_B S \arrow[dl, "f \circ \mathrm{pr}_1"'] \arrow[dr, "k \circ \mathrm{pr}_2"] \arrow[dd, dashed, "\exists!"] & \\
A & & C, \\
 & R' \times_B S' \arrow[ul, "h' \circ \mathrm{pr}'_1"] \arrow[ur, "k' \circ \mathrm{pr}'_2"'] &
\end{tikzcd}
$$

constructed as in the diagram

$$\begin{tikzcd}
 & & R \times_B S \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] \arrow[ddll, bend right, "f \circ \mathrm{pr}_1"'] \arrow[ddrr, bend left, "k \circ \mathrm{pr}_2"] \arrow[dddd, dashed, "\exists!"'] & & \\
 & R \arrow[dl, "f"'] \arrow[dr, "g"] & & S \arrow[dl, "h"'] \arrow[dr, "k"] & \\
A & & B & & C; \\
 & R' \arrow[ul, "f'"] \arrow[ur, "g'"'] & & S' \arrow[ul, "h'"] \arrow[ur, "k'"'] & \\
 & & R' \times_B S' \arrow[ul, "\mathrm{pr}'_1"] \arrow[ur, "\mathrm{pr}'_2"'] \arrow[uull, bend left, "f' \circ \mathrm{pr}'_1"] \arrow[uurr, bend right, "k' \circ \mathrm{pr}'_2"'] & &
\end{tikzcd}$$

5.4 Properties of Composition of Spans

Proposition 5.4.1.1. Let $\lambda = \left(A \xleftarrow{f} S \xrightarrow{g} B\right)$ be a span.

1. *Functoriality.*

Proof. $\square$

5.5 The Inverse of a Span

6 Functoriality of Spans

6.1 Direct Images

6.2 Functoriality of Spans on Powersets

7 Comparison of Spans to Functions and Relations

7.1 Comparison to Functions

Proposition 7.1.1.1. We have a pseudofunctor

$$\iota\colon \mathsf{Sets}_{\mathsf{bidisc}} \to \mathsf{Span}$$

from $\mathsf{Sets}_{\mathsf{bidisc}}$ to Span where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathsf{Sets}_{\mathsf{bidisc}})$, we have
$$\iota(A) \overset{\text{def}}{=} A;$$
- *Action on* Hom*-Categories.* For each $A, B \in \mathrm{Obj}(\mathsf{Sets}_{\mathsf{bidisc}})$, the action on Hom-categories
$$\iota_{A,B}\colon \mathsf{Sets}(A,B)_{\mathsf{disc}} \to \mathsf{Span}(A,B)$$
of ι at (A,B) is the functor defined on objects by sending a function $f\colon A \to B$ to the span
$$\begin{tikzcd} & A \arrow[dl, equal, "\mathrm{id}_A"'] \arrow[dr, "f"] & \\ A & & B \end{tikzcd}$$
from A to B.
- *Strict Unity Constraints.* For each $A \in \mathrm{Obj}(\mathsf{Sets}_{\mathsf{bidisc}})$, the strict unity constraint
$$\iota^0_A\colon \mathrm{id}_{\iota(A)} \Longrightarrow \iota(\mathrm{id}_A)$$
of ι at A is given by the identity morphism of spans
$$\begin{tikzcd} & A \arrow[dl, equal, "\mathrm{id}_A"'] \arrow[dr, equal, "\mathrm{id}_A"] \arrow[dd, equal, "\mathrm{id}"] & \\ A & & A, \\ & A \arrow[ul, equal, "\mathrm{id}_A"] \arrow[ur, equal, "\mathrm{id}_A"'] & \end{tikzcd}$$

as indeed $\mathrm{id}_{\iota(A)} = \iota(\mathrm{id}_A)$;

- *Pseudofunctoriality Constraints.* For each $A, B, C \in \mathrm{Obj}(\mathsf{Sets}_{\mathsf{bidisc}})$, each $f \in \mathsf{Hom}_{\mathsf{Sets}_{\mathsf{bidisc}}}(A, B)$, and each $g \in \mathsf{Hom}_{\mathsf{Sets}_{\mathsf{bidisc}}}(B, C)$, the pseudofunctoriality constraint

$$\iota^2_{g,f} \colon \iota(g) \circ \iota(f) \Longrightarrow \iota(g \circ f)$$

of ι at (f, g) is the morphism of spans from the span

$$\begin{tikzcd}
 & & A \times_B B \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] \arrow[dd, phantom, very near start, "\lrcorner" rotate=-45] \arrow[ddll, bend right, "\mathrm{id}_A \circ \mathrm{pr}_1"'] \arrow[ddrr, bend left, "g \circ \mathrm{pr}_2"] & & \\
 & A \arrow[dl, equal, "\mathrm{id}_A"'] \arrow[dr, "f"] & & B \arrow[dl, equal, "\mathrm{id}_B"'] \arrow[dr, "g"] & \\
A & & B & & C.
\end{tikzcd}$$

to the span

$$\begin{tikzcd}
 & A \arrow[dl, equal, "\mathrm{id}_A"'] \arrow[dr, "g \circ f"] & \\
A & & C
\end{tikzcd}$$

given by the isomorphism $A \times_B B \cong A$.

Proof. Omitted. □

7.2 Comparison to Relations: From Span to $\mathbf{Rel}$

7.2.1 Relations Associated to Spans

Let $\lambda = \left(A \overset{f}{\leftarrow} S \overset{g}{\rightarrow} B\right)$ be a span.

Definition 7.2.1.1. The **relation associated to** λ is the relation

$$S(\lambda) \colon A \nrightarrow B$$

from A to B defined as follows:

- Viewing relations from A to B as functions $A \times B \to \{\mathsf{true}, \mathsf{false}\}$, we define

$$\iota_{A,B}(S)^a_b \overset{\text{def}}{=} \begin{cases} \mathsf{true} & \text{if there exists } x \in S \text{ such} \\ & \text{that } a = f(x) \text{ and } b = g(x), \\ \mathsf{false} & \text{otherwise} \end{cases}$$

for each $(a, b) \in A \times B$.

- Viewing relations from A to B as functions $A \to \mathcal{P}(B)$, we define

$$[\iota_{A,B}(S)](a) \overset{\text{def}}{=} g(f^{-1}(a))$$

for each $a \in A$.

- Viewing relations from A to B as subsets of $A \times B$, we define

$$\iota_{A,B}(S) \overset{\text{def}}{=} \{(f(x), g(x)) \mid x \in S\}.$$

Proposition 7.2.1.2. Let $\lambda = \left(A \overset{f}{\leftarrow} S \overset{g}{\rightarrow} B\right)$ be a span.

1. *Interaction With Identities.*
2. *Interaction With Composition.*
3. *Interaction With Inverses.*

Proof. □

7.2.2 The Comparison Functor from Span to $\mathbf{Rel}$

Proposition 7.2.2.1. We have a pseudofunctor

$$\iota \colon \mathsf{Span} \to \mathbf{Rel}$$

from Span to $\mathbf{Rel}$ where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathsf{Span})$, we have

$$\iota(A) \overset{\text{def}}{=} A;$$

- *Action on* Hom*-Categories.* For each $A, B \in \mathrm{Obj}(\mathsf{Span})$, the action on Hom-categories

$$\iota_{A,B} \colon \mathsf{Span}(A, B) \to \mathbf{Rel}(A, B)$$

of ι at (A, B) is the functor where

– *Action on Objects.* Given a span

$$\begin{tikzcd} & S \arrow[dl, "f"'] \arrow[dr, "g"] & \\ A & & B \end{tikzcd}$$

from A to B, the image

$$\iota_{A,B}(S) \colon A \nrightarrow B$$

of S by ι is the relation from A to B defined as follows:

 * Viewing relations as functions $A \times B \to \{\mathsf{true}, \mathsf{false}\}$, we define

$$\iota_{A,B}(S)^a_b \overset{\text{def}}{=} \begin{cases} \mathsf{true} & \text{if there exists } x \in S \\ & \text{such that } a = f(x) \\ & \text{and } b = g(x), \\ \mathsf{false} & \text{otherwise} \end{cases}$$

 for each $(a, b) \in A \times B$;
 * Viewing relations as functions $A \to \mathcal{P}(B)$, we define

$$[\iota_{A,B}(S)](a) \overset{\text{def}}{=} g(f^{-1}(a))$$

 for each $a \in A$;
 * Viewing relations as subsets of $A \times B$, we define

$$\iota_{A,B}(S) \overset{\text{def}}{=} \{(f(x), g(x)) \mid x \in S\}.$$

– *Action on Morphisms.* Given a morphism of spans

$$\begin{tikzcd} & R \arrow[dl, "f_R"'] \arrow[dr, "g_R"] \arrow[dd, "\phi"] & \\ A & & B, \\ & S \arrow[ul, "f_S"] \arrow[ur, "g_S"'] & \end{tikzcd}$$

we have a corresponding inclusion of relations

$$\iota_{A,B}(\phi) \colon \iota_{A,B}(R) \subset \iota_{A,B}(S),$$

since we have $a \sim_{\iota_{A,B}(R)} b$ iff there exists $x \in R$ such that $a = f_R(x)$ and $b = g_R(x)$, in which case we then have

$$\begin{aligned} a &= f_R(x) \\ &= f_S(\phi(x)), \\ b &= g_R(x) \\ &= g_S(\phi(x)), \end{aligned}$$

so that $a \sim_{\iota_{A,B}(S)} b$, and thus $\iota_{A,B}(R) \subset \iota_{A,B}(S)$.

Proof. Omitted. □

7.3 Comparison to Relations: From Rel to Span

Proposition 7.3.1.1. We have a lax functor

$$\left(\iota, \iota^2, \iota^0\right) \colon \mathsf{Rel} \to \mathsf{Span}$$

from **Rel** to Span where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathsf{Span})$, we have

$$\iota(A) \overset{\mathrm{def}}{=} A;$$

- *Action on* Hom*-Categories.* For each $A, B \in \mathrm{Obj}(\mathsf{Span})$, the action on Hom-categories

$$\iota_{A,B} \colon \mathbf{Rel}(A, B) \to \mathsf{Span}(A, B)$$

  of ι at (A, B) is the functor where

  – *Action on Objects.* Given a relation $R \colon A \nrightarrow B$ from A to B, we define a span

$$\iota_{A,B}(R) \colon A \nrightarrow B$$

  from A to B by

$$\iota_{A,B}(R) \overset{\mathrm{def}}{=} (R, \restriction \mathrm{pr}_1 R, \restriction \mathrm{pr}_2 R),$$

  where $R \subset A \times B$ and $\restriction \mathrm{pr}_1 R$ and $\restriction \mathrm{pr}_2 R$ are the restriction of the projections

$$\begin{aligned} \mathrm{pr}_1 &\colon A \times B \to A, \\ \mathrm{pr}_2 &\colon A \times B \to B \end{aligned}$$

  to R;

- *Action on Morphisms.* Given an inclusion $\phi\colon R \subset S$ of relations, we have a corresponding morphism of spans

$$\iota_{A,B}(\phi)\colon \iota_{A,B}(R) \to \iota_{A,B}(S)$$

as in the diagram

$$\begin{tikzcd}
 & R \arrow[dl, "\upharpoonright \mathrm{pr}_1 R"'] \arrow[dr, "\upharpoonright \mathrm{pr}_2 R"] \arrow[dd, hook] & \\
A & & B. \\
 & S \arrow[ul, "\upharpoonright \mathrm{pr}_1 S"] \arrow[ur, "\upharpoonright \mathrm{pr}_2 S"'] &
\end{tikzcd}$$

- *The Lax Functoriality Constraints.* The lax functoriality constraint

$$\iota^2_{R,S}\colon \iota(S) \circ \iota(R) \Longrightarrow \iota(S \diamond R)$$

of ι at (R, S) is given by the morphism of spans from

$$\begin{tikzcd}
 & & R \times_B S \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] \arrow[dd, phantom, "\lrcorner", very near start] \arrow[ddll, bend right, "\upharpoonright \mathrm{pr}_1 R \circ \mathrm{pr}_1"'] \arrow[ddrr, bend left, "\upharpoonright \mathrm{pr}_2 S \circ \mathrm{pr}_2"] & & \\
 & R \arrow[dl, "\upharpoonright \mathrm{pr}_1 R"'] \arrow[dr, "\upharpoonright \mathrm{pr}_2 R"] & & S \arrow[dl, "\upharpoonright \mathrm{pr}_1 S"'] \arrow[dr, "\upharpoonright \mathrm{pr}_2 S"] & \\
A & & B & & C
\end{tikzcd}$$

to

$$\begin{tikzcd}
 & S \diamond R \arrow[dl, "\upharpoonright \mathrm{pr}_1 S \diamond R"'] \arrow[dr, "\upharpoonright \mathrm{pr}_2 S \diamond R"] & \\
A & & C
\end{tikzcd}$$

given by the natural inclusion $R \times_B S \hookrightarrow S \diamond R$, since we have

$$R \times_B S = \{((a_R, b_R), (b_S, c_S)) \in R \times S \mid b_R = b_S\};$$
$$S \diamond R = \left\{ (a, c) \in A \times C \;\middle|\; \begin{array}{l} \text{there exists some } b \in B \text{ such} \\ \text{that } (a, b) \in R \text{ and } (b, c) \in S \end{array} \right\};$$

- *The Lax Unity Constraints.* The lax unity constraint[12]

$$\iota^0_A\colon \underbrace{\mathrm{id}_{\iota(A)}}_{(A,\mathrm{id}_A,\mathrm{id}_A)} \Longrightarrow \underbrace{\iota(\chi_A)}_{(\Delta_A,\restriction\mathrm{pr}_1\Delta_A,\restriction\mathrm{pr}_2\Delta_A)}$$

of ι at A is given by the diagonal morphism of A, as in the diagram

$$\begin{tikzcd}
& A \arrow[dl, equal, "\mathrm{id}_A"'] \arrow[dr, equal, "\mathrm{id}_A"] \arrow[dd, "\delta_A" description] & \\
A & & A. \\
& \Delta_A \arrow[ul, "\restriction\mathrm{pr}_1\Delta_A"] \arrow[ur, "\restriction\mathrm{pr}_2\Delta_A"'] &
\end{tikzcd}$$

Proof. Omitted. □

7.4 Comparison to Relations: The Wehrheim–Woodward Construction

7.5 Comparison to Multirelations

Remark 7.5.1.1. The pseudofunctor of Proposition 7.2.2.1 and the lax functor of Proposition 7.3.1.1 fail to be equivalences of bicategories. This happens essentially because a span $(S,f,g)\colon A \nrightarrow B$ from A to B may relate elements $a \in A$ and $b \in B$ by more than one element, e.g. there could be $s \neq s' \in S$ such that $a = f(s) = f(s')$ and $b = g(s) = g(s')$.
Thus, in a sense, spans may be thought of as "relations with multiplicity". And indeed, if instead of considering relations from A to B, i.e. functions

$$R\colon A \times B \to \{\mathsf{true}, \mathsf{false}\}$$

from $A \times B$ to $\{\mathsf{true}, \mathsf{false}\} \cong \{0,1\}$, we consider functions

$$R\colon A \times B \to \mathbb{N} \cup \{\infty\}$$

from $A \times B$ to $\mathbb{N} \cup \{\infty\}$, then we obtain the notion of a **multirelation from A to B**, and these turn out to assemble together with sets into a bicategory MRel that is biequivalent to Span; see [**some-algebraic-laws-for-spans-and-their-connections-with-mul**

[12] Which is in fact strong, as δ_A is an isomorphism.

7.6 Comparison to Relations via Double Categories

Remark 7.6.1.1. There are double functors between the double categories $\mathsf{Rel}^{\mathsf{dbl}}$ and $\mathsf{Span}^{\mathsf{dbl}}$ analogous to the functors of Propositions 7.2.2.1 and 7.3.1.1, assembling moreover into a strict-lax adjunction of double functors; see [**higher-dimensional-categories**].

Appendices

A Other Chapters