Constructions With Sets

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This chapter contains some material relating to constructions with sets. Notably, it contains:

1. Explicit descriptions of the major types of co/limits in Sets, including in particular pushouts and coequalisers (see Definitions 2.3.1.1 and 2.4.1.1 and Remarks 2.3.1.2 and 2.4.1.2);

2. A discussion of powersets as decategorifications of categories of presheaves (Remarks 4.1.1.2 and 4.2.1.2);

3. A lengthy discussion of the adjoint triple

$$f_* \dashv f^{-1} \dashv f_! \colon \mathcal{P}(A) \overset{\rightleftarrows}{\to} \mathcal{P}(B)$$

of functors (morphisms of posets) between $\mathcal{P}(A)$ and $\mathcal{P}(B)$ induced by a map of sets $f\colon A \longrightarrow B$, along with a discussion of the properties of f_*, f^{-1}, and $f_!$.

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1 Limits of Sets

1.1 Products of Families of Sets

Let $\{A_i\}_{i\in I}$ be a family of sets.

Definition 1.1.1.1. The **product**[1] **of** $\{A_i\}_{i\in I}$ is the set $\prod_{i\in I} A_i$ defined by

$$\prod_{i\in I} A_i \overset{\text{def}}{=} \left\{ f \in \mathsf{Sets}\left(I, \bigcup_{i\in I} A_i\right) \,\middle|\, \begin{array}{l}\text{for each } i \in I\text{, we}\\ \text{have } f(i) \in A_i\end{array} \right\}.$$

1.2 Binary Products of Sets

Let A and B be sets.

[1] *Further Terminology:* Also called the **Cartesian product of** $\{A_i\}_{i\in I}$.

Definition 1.2.1.1. The **product**[2] **of** A **and** B is the set $A \times B$ defined by

$$\begin{aligned} A \times B &\overset{\text{def}}{=} \prod_{z \in \{A,B\}} z \\ &\overset{\text{def}}{=} \{f \in \mathsf{Sets}(\{0,1\}, A \cup B) \mid \text{we have } f(0) \in A \text{ and } f(1) \in B\} \\ &\cong \{\{\{a\},\{a,b\}\} \in \mathcal{P}(\mathcal{P}(A \cup B)) \mid \text{we have } a \in A \text{ and } b \in B\}. \end{aligned}$$

Proposition 1.2.1.2. Let A, B, C, and X be sets.

1. *Functoriality.* The assignments $A, B, (A,B) \mapsto A \times B$ define functors

$$\begin{aligned} A \times -_2 &\colon \mathsf{Sets} \to \mathsf{Sets}, \\ -_1 \times B &\colon \mathsf{Sets} \to \mathsf{Sets}, \\ -_1 \times -_2 &\colon \mathsf{Sets} \times \mathsf{Sets} \to \mathsf{Sets}, \end{aligned}$$

where $-_1 \times -_2$ is the functor where

- *Action on Objects.* For each $(A,B) \in \mathrm{Obj}(\mathsf{Sets} \times \mathsf{Sets})$, we have

$$[-_1 \times -_2](A,B) \overset{\text{def}}{=} A \times B;$$

- *Action on Morphisms.* For each $(A,B), (X,Y) \in \mathrm{Obj}(\mathsf{Sets})$, the action on Hom-sets

$$\times_{(A,B),(X,Y)} \colon \mathsf{Sets}(A,X) \times \mathsf{Sets}(B,Y) \to \mathsf{Sets}(A \times B, X \times Y)$$

of $\times$ at $((A,B),(X,Y))$ is defined by sending (f,g) to the function

$$f \times g \colon A \times B \to X \times Y$$

defined by

$$[f \times g](a,b) \overset{\text{def}}{=} (f(a), g(b))$$

for each $(a,b) \in A \times B$;

and where $A \times -$ and $- \times B$ are the partial functors of $-_1 \times -_2$ at $A, B \in \mathrm{Obj}(\mathsf{Sets})$.

[2] *Further Terminology:* Also called the **Cartesian product of** A **and** B or the **binary Cartesian product of** A **and** B, for emphasis.

This can also be thought of as the $(\mathbb{E}_{-1}, \mathbb{E}_{-1})$**-tensor product of** A **and** B.

2. *Adjointness.* We have adjunctions

$$(A \times - \dashv \mathsf{Sets}(A, -))\colon \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "A\times -"] \arrow[r, phantom, "\perp"] & \mathsf{Sets} \arrow[l, bend left, "\mathsf{Sets}(A,-)"] \end{tikzcd},$$

$$(- \times B \dashv \mathsf{Sets}(B, -))\colon \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "-\times B"] \arrow[r, phantom, "\perp"] & \mathsf{Sets} \arrow[l, bend left, "\mathsf{Sets}(B,-)"] \end{tikzcd},$$

witnessed by bijections

$$\begin{aligned} \mathsf{Sets}(A \times B, C) &\cong \mathsf{Sets}(A, \mathsf{Sets}(B, C)), \\ \mathsf{Sets}(A \times B, C) &\cong \mathsf{Sets}(B, \mathsf{Sets}(A, C)), \end{aligned}$$

natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

3. *Associativity.* We have an isomorphism of sets

$$(A \times B) \times C \cong A \times (B \times C),$$

natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

4. *Unitality.* We have isomorphisms of sets

$$\begin{aligned} \mathrm{pt} \times A &\cong A, \\ A \times \mathrm{pt} &\cong A, \end{aligned}$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$.

5. *Commutativity.* We have an isomorphism of sets

$$A \times B \cong B \times A,$$

natural in $A, B \in \mathrm{Obj}(\mathsf{Sets})$.

6. *Annihilation With the Empty Set.* We have isomorphisms of sets

$$\begin{aligned} A \times \emptyset &\cong \emptyset, \\ \emptyset \times A &\cong \emptyset, \end{aligned}$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$.

7. *Distributivity Over Unions.* We have isomorphisms of sets

$$\begin{aligned} A \times (B \cup C) &= (A \times B) \cup (A \times C), \\ (A \cup B) \times C &= (A \times C) \cup (B \times C), \end{aligned}$$

natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

8. *Distributivity Over Intersections.* We have isomorphisms of sets

$$\begin{aligned} A \times (B \cap C) &= (A \times B) \cap (A \times C), \\ (A \cap B) \times C &= (A \times C) \cap (B \times C), \end{aligned}$$

natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

9. *Distributivity Over Differences.* We have isomorphisms of sets

$$\begin{aligned} A \times (B \setminus C) &= (A \times B) \setminus (A \times C), \\ (A \setminus B) \times C &= (A \times C) \setminus (B \times C), \end{aligned}$$

natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

10. *Distributivity Over Symmetric Differences.* We have isomorphisms of sets

$$\begin{aligned} A \times (B \bigtriangleup C) &= (A \times B) \bigtriangleup (A \times C), \\ (A \bigtriangleup B) \times C &= (A \times C) \bigtriangleup (B \times C), \end{aligned}$$

natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

11. *Symmetric Monoidality.* The triple $(\mathsf{Sets}, \times, \mathrm{pt})$ is a symmetric monoidal category.

12. *Symmetric Bimonoidality.* The quintuple $(\mathsf{Sets}, \coprod, \emptyset, \times, \mathrm{pt})$ is a symmetric bimonoidal category.

Proof. *Item 1*, *Functoriality*: Omitted.
Item 2, *Adjointness*: Omitted.
Item 3, *Associativity*: Clear.
Item 4, *Unitality*: Clear.
Item 5, *Commutativity*: Clear.
Item 6, *Annihilation With the Empty Set*: Clear.
Item 7, *Distributivity Over Unions*: Omitted.
Item 8, *Distributivity Over Intersections*: Omitted.
Item 9, *Distributivity Over Differences*: Omitted.
Item 10, *Distributivity Over Symmetric Differences*: Omitted.
Item 11, *Symmetric Monoidality*: Omitted.
Item 12, *Symmetric Bimonoidality*: Omitted. □

1.3 Pullbacks

Let A, B, and C be sets and let $f\colon A \to C$ and $g\colon B \to C$ be functions.

Definition 1.3.1.1. The **pullback of A and B over C along f and g**[3] is the set $A \times_C B$ defined by

$$A \times_C B \overset{\text{def}}{=} \{(a,b) \in A \times B \mid f(a) = g(b)\}.$$

Example 1.3.1.2. Here are some examples of pullbacks of sets.

1. *Unions via Intersections.* Let $A, B \subset X$. We have a bijection of sets

$$A \cap B \cong A \times_{A \cup B} B.$$

Proposition 1.3.1.3. Let A, B, C, and X be sets.

1. *Associativity.* We have an isomorphism of sets

$$(A \times_X B) \times_X C \cong A \times_X (B \times_X C),$$

natural in $A, B, C, X \in \mathrm{Obj}(\mathsf{Sets})$.

2. *Unitality.* We have isomorphisms of sets

$$X \times_X A \cong A,$$
$$A \times_X X \cong A,$$

natural in $A, X \in \mathrm{Obj}(\mathsf{Sets})$.

3. *Commutativity.* We have an isomorphism of sets

$$A \times_X B \cong B \times_X A,$$

natural in $A, B, X \in \mathrm{Obj}(\mathsf{Sets})$.

4. *Annihilation With the Empty Set.* We have isomorphisms of sets

$$A \times_X \emptyset \cong \emptyset,$$
$$\emptyset \times_X A \cong \emptyset,$$

natural in $A, X \in \mathrm{Obj}(\mathsf{Sets})$.

5. *Symmetric Monoidality.* The triple $(\mathsf{Sets}, \times_X, X)$ is a symmetric monoidal category.

Proof. Item 1, *Associativity*: Clear.
Item 2, *Unitality*: Clear.
Item 3, *Commutativity*: Clear.
Item 4, *Annihilation With the Empty Set*: Clear.
Item 5, *Symmetric Monoidality*: Omitted. □

[3] *Further Terminology:* Also called the **fibre product of A and B over C along f and g**.

1.4 Equalisers

Let A and B be sets and let $f, g\colon A \rightrightarrows B$ be functions.

Definition 1.4.1.1. The **equaliser of** f **and** g is the set $\mathrm{Eq}(f, g)$ defined by

$$\mathrm{Eq}(f, g) \overset{\text{def}}{=} \{a \in A \mid f(a) = g(a)\}.$$

Proposition 1.4.1.2. Let A, B, and C be sets.

1. *Associativity.* We have an isomorphism of sets[4]

$$\underbrace{\mathrm{Eq}(f \circ \mathrm{eq}(g,h), g \circ \mathrm{eq}(g,h))}_{=\mathrm{Eq}(f \circ \mathrm{eq}(g,h), h \circ \mathrm{eq}(g,h))} \cong \mathrm{Eq}(f,g,h) \cong \underbrace{\mathrm{Eq}(f \circ \mathrm{eq}(f,g), h \circ \mathrm{eq}(f,g))}_{=\mathrm{Eq}(g \circ \mathrm{eq}(f,g), h \circ \mathrm{eq}(f,g))},$$

where $\mathrm{Eq}(f,g,h)$ is the limit of the diagram

$$A \overset{f}{\underset{h}{\overset{\xrightarrow{\quad}}{\underset{\xrightarrow{\quad}}{-g\to}}}} B$$

in Sets.

[4] That is: the following constructions give the same result:

1. Take the equaliser of (f,g,h), i.e. the limit of the diagram

$$A \overset{f}{\underset{h}{\overset{\xrightarrow{\quad}}{\underset{\xrightarrow{\quad}}{-g\to}}}} B$$

in Sets.

2. First take the equaliser of f and g, forming a diagram

$$\mathrm{Eq}(f,g) \overset{\mathrm{eq}(f,g)}{\hookrightarrow} A \underset{g}{\overset{f}{\rightrightarrows}} B$$

and then take the equaliser of the composition

$$\mathrm{Eq}(f,g) \overset{\mathrm{eq}(f,g)}{\hookrightarrow} A \underset{h}{\overset{f}{\rightrightarrows}} B,$$

obtaining a subset

$$\mathrm{Eq}(f \circ \mathrm{eq}(f,g), h \circ \mathrm{eq}(f,g)) = \mathrm{Eq}(g \circ \mathrm{eq}(f,g), h \circ \mathrm{eq}(f,g))$$

of $\mathrm{Eq}(f,g)$.

3. First take the equaliser of g and h, forming a diagram

$$\mathrm{Eq}(g,h) \overset{\mathrm{eq}(g,h)}{\hookrightarrow} A \underset{h}{\overset{g}{\rightrightarrows}} B$$

and then take the equaliser of the composition

$$\mathrm{Eq}(g,h) \overset{\mathrm{eq}(g,h)}{\hookrightarrow} A \underset{g}{\overset{f}{\rightrightarrows}} B,$$

obtaining a subset

$$\mathrm{Eq}(f \circ \mathrm{eq}(g,h), g \circ \mathrm{eq}(g,h)) = \mathrm{Eq}(f \circ \mathrm{eq}(g,h), h \circ \mathrm{eq}(g,h))$$

of $\mathrm{Eq}(g,h)$.

4. *Unitality.* We have an isomorphism of sets

$$\mathrm{Eq}(f, f) \cong A.$$

5. *Commutativity.* We have an isomorphism of sets

$$\mathrm{Eq}(f, g) \cong \mathrm{Eq}(g, f).$$

6. *Interaction With Composition.* Let

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \underset{k}{\overset{h}{\rightrightarrows}} C$$

be functions. We have an inclusion of sets

$$\mathrm{Eq}(h \circ f \circ \mathrm{eq}(f, g), k \circ g \circ \mathrm{eq}(f, g)) \subset \mathrm{Eq}(h \circ f, k \circ g),$$

where $\mathrm{Eq}(h \circ f \circ \mathrm{eq}(f, g), k \circ g \circ \mathrm{eq}(f, g))$ is the equaliser of the composition

$$\mathrm{Eq}(f, g) \overset{\mathrm{eq}(f,g)}{\hookrightarrow} A \underset{g}{\overset{f}{\rightrightarrows}} B \underset{k}{\overset{h}{\rightrightarrows}} C.$$

Proof. *Item 1, Associativity*: Clear.
Item 4, Unitality: Clear.
Item 5, Commutativity: Clear.
Item 6, Interaction With Composition: Omitted. □

2 Colimits of Sets

2.1 Coproducts of Families of Sets

Let $\{A_i\}_{i \in I}$ be a family of sets.

Definition 2.1.1.1. The **disjoint union of the family** $\{A_i\}_{i \in I}$ is the set $\coprod_{i \in I} A_i$ defined by

$$\coprod_{i \in I} A_i \overset{\mathrm{def}}{=} \bigcup_{i \in I} \left\{ (x, i) \in \left(\bigcup_{i \in I} A_i \right) \times I \,\middle|\, x \in A_i \right\}.$$

2.2 Binary Coproducts

Let A and B be sets.

Definition 2.2.1.1. The **coproduct**[5] **of** A **and** B is the set $A \coprod B$ defined by

$$\begin{aligned} A \coprod B &\overset{\text{def}}{=} \coprod_{z \in \{A,B\}} z \\ &\overset{\text{def}}{=} \{(a,0) \mid a \in A\} \cup \{(b,1) \mid b \in B\}. \end{aligned}$$

Proposition 2.2.1.2. Let A, B, C, and X be sets.

1. *Functoriality.* The assignment $A, B, (A,B) \mapsto A \coprod B$ defines functors

$$\begin{aligned} A \coprod -_2 &\colon \mathsf{Sets} \to \mathsf{Sets}, \\ -_1 \coprod B &\colon \mathsf{Sets} \to \mathsf{Sets}, \\ -_1 \coprod -_2 &\colon \mathsf{Sets} \times \mathsf{Sets} \to \mathsf{Sets}, \end{aligned}$$

where $-_1 \coprod -_2$ is the functor where

- *Action on Objects.* For each $(A,B) \in \mathrm{Obj}(\mathsf{Sets} \times \mathsf{Sets})$, we have

$$[-_1 \coprod -_2](A,B) \overset{\text{def}}{=} A \coprod B;$$

- *Action on Morphisms.* For each $(A,B), (X,Y) \in \mathrm{Obj}(\mathsf{Sets})$, the action on Hom-sets

$$\coprod_{(A,B),(X,Y)} \colon \mathsf{Sets}(A,X) \times \mathsf{Sets}(B,Y) \to \mathsf{Sets}(A \coprod B, X \coprod Y)$$

of $\coprod$ at $((A,B),(X,Y))$ is defined by sending (f,g) to the function

$$f \coprod g \colon A \coprod B \to X \coprod Y$$

defined by

$$[f \coprod g](x) \overset{\text{def}}{=} \begin{cases} f(x) & \text{if } x \in A, \\ g(x) & \text{if } x \in B \end{cases}$$

for each $x \in A \coprod B$;

and where $A \coprod -$ and $- \coprod B$ are the partial functors of $-_1 \coprod -_2$ at $A, B \in \mathrm{Obj}(\mathsf{Sets})$.

2. *Associativity.* We have an isomorphism of sets

$$(A \coprod B) \coprod C \cong A \coprod (B \coprod C),$$

natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

[5]*Further Terminology:* Also called the **disjoint union of** A **and** B, or the **binary disjoint union of** A **and** B,

3. *Unitality.* We have isomorphisms of sets

$$A \coprod \emptyset \cong A,$$
$$\emptyset \coprod A \cong A,$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$.

4. *Commutativity.* We have an isomorphism of sets

$$A \coprod B \cong B \coprod A,$$

natural in $A, B \in \mathrm{Obj}(\mathsf{Sets})$.

5. *Symmetric Monoidality.* The triple $(\mathsf{Sets}, \coprod, \emptyset)$ is a symmetric monoidal category.

Proof. *Item 1*, *Functoriality*: Omitted.
Item 2, *Associativity*: Clear.
Item 3, *Unitality*: Clear.
Item 4, *Commutativity*: Clear.
Item 5, *Symmetric Monoidality*: Omitted. □

2.3 Pushouts

Let A, B, and C be sets and let $f: C \to A$ and $g: C \to B$ be functions.

Definition 2.3.1.1. The **pushout of A and B over C along f and g**[6] is the set $A \coprod_C B$ defined by

$$A \coprod_C B \stackrel{\text{def}}{=} A \coprod B/{\sim_C},$$

where $\sim_C$ is the equivalence relation on $A \coprod B$ generated by $f(c) \sim_C g(c)$.

Remark 2.3.1.2. In detail, the relation $\sim$ of Definition 2.3.1.1 is given by declaring $a \sim b$ iff one of the following conditions is satisfied:

- We have $a, b \in A$ and $a = b$;
- We have $a, b \in B$ and $a = b$;
- There exist $x_1, \ldots, x_n \in A \coprod B$ such that $a \sim' x_1 \sim' \cdots \sim' x_n \sim' b$, where we declare $x \sim' y$ if one of the following conditions is satisfied:

  1. There exists $c \in C$ such that $x = f(c)$ and $y = g(c)$.

for emphasis.

[6] *Further Terminology:* Also called the **fibre coproduct of A and B over C along f and g**.

2. There exists $c \in C$ such that $x = g(c)$ and $y = f(c)$.

That is: we require the following condition to be satisfied:

($\star$) There exist $x_1, \ldots, x_n \in A \coprod B$ satisfying the following conditions:
   1. There exists $c_0 \in C$ satisfying one of the following conditions:
      (a) We have $a = f(c_0)$ and $x_1 = g(c_0)$.
      (b) We have $a = g(c_0)$ and $x_1 = f(c_0)$.
   2. For each $1 \leq i \leq n-1$, there exists $c_i \in C$ satisfying one of the following conditions:
      (a) We have $x_i = f(c_i)$ and $x_{i+1} = g(c_i)$.
      (b) We have $x_i = g(c_i)$ and $x_{i+1} = f(c_i)$.
   3. There exists $c_n \in C$ satisfying one of the following conditions:
      (a) We have $x_n = f(c_n)$ and $b = g(c_n)$.
      (b) We have $x_n = g(c_n)$ and $b = f(c_n)$.

Example 2.3.1.3. Here are some examples of pushouts of sets.

1. *Wedge Sums of Pointed Sets.* The wedge sum of two pointed sets of **??** is an example of a pushout of sets.

2. *Intersections via Unions.* Let $A, B \subset X$. We have a bijection of sets

$$A \cup B \cong A \coprod_{A \cap B} B.$$

Proposition 2.3.1.4. Let A, B, C, and X be sets.

1. *Associativity.* We have an isomorphism of sets

$$(A \coprod_X B) \coprod_X C \cong A \coprod_X (B \coprod_X C),$$

natural in $A, B, C, X \in \mathrm{Obj}(\mathsf{Sets})$.

2. *Unitality.* We have isomorphisms of sets

$$\emptyset \coprod_X A \cong A,$$
$$A \coprod_X \emptyset \cong A,$$

natural in $A, X \in \mathrm{Obj}(\mathsf{Sets})$.

3. *Commutativity.* We have an isomorphism of sets

$$A \coprod_X B \cong B \coprod_X A,$$

natural in $A, B, X \in \mathrm{Obj}(\mathsf{Sets})$.

4. *Annihilation With the Empty Set.* We have isomorphisms of sets

$$A \coprod_X \emptyset \cong \emptyset,$$
$$\emptyset \coprod_X A \cong \emptyset,$$

natural in $A, X \in \mathrm{Obj}(\mathsf{Sets})$.

5. *Symmetric Monoidality.* The triple $(\mathsf{Sets}, \coprod_X, \emptyset)$ is a symmetric monoidal category.

Proof. *Item 1*, *Associativity*: Clear.
Item 2, *Unitality*: Clear.
Item 3, *Commutativity*: Clear.
Item 4, *Annihilation With the Empty Set*: Clear.
Item 5, *Symmetric Monoidality*: Omitted. □

2.4 Coequalisers

Let A and B be sets and let $f, g \colon A \rightrightarrows B$ be functions.

Definition 2.4.1.1. The **coequaliser of** f **and** g is the set $\mathrm{CoEq}(f, g)$ defined by

$$\mathrm{CoEq}(f, g) \stackrel{\text{def}}{=} B/\sim,$$

where $\sim$ is the equivalence relation on B generated by $f(a) \sim g(a)$.

Remark 2.4.1.2. In detail, the relation $\sim$ of Definition 2.4.1.1 is given by declaring $a \sim b$ iff one of the following conditions is satisfied:

- We have $a = b$;
- There exist $x_1, \ldots, x_n \in B$ such that $a \sim' x_1 \sim' \cdots \sim' x_n \sim' b$, where we declare $x \sim' y$ if one of the following conditions is satisfied:
  1. There exists $z \in A$ such that $x = f(z)$ and $y = g(z)$.
  2. There exists $z \in A$ such that $x = g(z)$ and $y = f(z)$.

  That is: we require the following condition to be satisfied:

  $(\star)$ There exist $x_1, \ldots, x_n \in B$ satisfying the following conditions:
  1. There exists $z_0 \in A$ satisfying one of the following conditions:
     (a) We have $a = f(z_0)$ and $x_1 = g(z_0)$.
     (b) We have $a = g(z_0)$ and $x_1 = f(z_0)$.

2. For each $1 \leq i \leq n-1$, there exists $z_i \in A$ satisfying one of the following conditions:
   (a) We have $x_i = f(z_i)$ and $x_{i+1} = g(z_i)$.
   (b) We have $x_i = g(z_i)$ and $x_{i+1} = f(z_i)$.
3. There exists $z_n \in A$ satisfying one of the following conditions:
   (a) We have $x_n = f(z_n)$ and $b = g(z_n)$.
   (b) We have $x_n = g(z_n)$ and $b = f(z_n)$.

Example 2.4.1.3. Here are some examples of coequalisers of sets.

1. *Quotients by Equivalence Relations.* Let R be an equivalence relation on a set X. We have a bijection of sets

$$X/{\sim_R} \cong \mathsf{CoEq}\left(R \hookrightarrow X \times X \underset{\mathrm{pr}_2}{\overset{\mathrm{pr}_1}{\rightrightarrows}} X\right).$$

Proposition 2.4.1.4. Let A, B, and C be sets.

1. *Associativity.* We have an isomorphism of sets[7]

$$\underbrace{\mathsf{CoEq}(\mathrm{coeq}(f,g) \circ f, \mathrm{coeq}(f,g) \circ h)}_{=\mathsf{CoEq}(\mathrm{coeq}(f,g)\circ g, \mathrm{coeq}(f,g)\circ h)} \cong \mathsf{CoEq}(f,g,h) \cong \underbrace{\mathsf{CoEq}(\mathrm{coeq}(g,h) \circ f, \mathrm{coeq}(g,h) \circ g)}_{=\mathsf{CoEq}(\mathrm{coeq}(g,h)\circ f, \mathrm{coeq}(g,h)\circ h)},$$

where $\mathsf{CoEq}(f,g,h)$ is the colimit of the diagram

$$\begin{tikzcd} A \arrow[r, shift left=2, "f"] \arrow[r, "g" description] \arrow[r, shift right=2, "h"'] & B \end{tikzcd}$$

---

[7] That is: the following constructions give the same result:

1. Take the coequaliser of (f,g,h), i.e. the colimit of the diagram

$$\begin{tikzcd} A \arrow[r, shift left=2, "f"] \arrow[r, "g" description] \arrow[r, shift right=2, "h"'] & B \end{tikzcd}$$

in Sets.

2. First take the coequaliser of f and g, forming a diagram

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \overset{\mathrm{coeq}(f,g)}{\twoheadrightarrow} \mathsf{CoEq}(f,g)$$

and then take the coequaliser of the composition

$$A \underset{h}{\overset{f}{\rightrightarrows}} B \overset{\mathrm{coeq}(f,g)}{\twoheadrightarrow} \mathsf{CoEq}(f,g),$$

in Sets.

4. *Unitality.* We have an isomorphism of sets

$$\mathrm{CoEq}(f,f) \cong B.$$

5. *Commutativity.* We have an isomorphism of sets

$$\mathrm{CoEq}(f,g) \cong \mathrm{CoEq}(g,f).$$

6. *Interaction With Composition.* Let

$$A \overset{f}{\underset{g}{\rightrightarrows}} B \overset{h}{\underset{k}{\rightrightarrows}} C$$

be functions. We have a surjection

$$\mathrm{CoEq}(h \circ f, k \circ g) \twoheadrightarrow \mathrm{CoEq}(\mathrm{coeq}(h,k) \circ h \circ f, \mathrm{coeq}(h,k) \circ k \circ g)$$

exhibiting $\mathrm{CoEq}(\mathrm{coeq}(h,k) \circ h \circ f, \mathrm{coeq}(h,k) \circ k \circ g)$ as a quotient of $\mathrm{CoEq}(h \circ f, k \circ g)$ by the relation generated by declaring $h(y) \sim k(y)$ for each $y \in B$.

Proof. Item 1, *Associativity*: Omitted.
Item 4, *Unitality*: Clear.
Item 5, *Commutativity*: Clear.
Item 6, *Interaction With Composition*: Omitted. □

obtaining a quotient

$$\mathrm{CoEq}(\mathrm{coeq}(f,g) \circ f, \mathrm{coeq}(f,g) \circ h) = \mathrm{CoEq}(\mathrm{coeq}(f,g) \circ g, \mathrm{coeq}(f,g) \circ h)$$

of $\mathrm{CoEq}(f,g)$

3. First take the coequaliser of g and h, forming a diagram

$$A \overset{g}{\underset{h}{\rightrightarrows}} B \overset{\mathrm{coeq}(g,h)}{\twoheadrightarrow} \mathrm{CoEq}(g,h)$$

and then take the coequaliser of the composition

$$A \overset{f}{\underset{g}{\rightrightarrows}} B \overset{\mathrm{coeq}(g,h)}{\twoheadrightarrow} \mathrm{CoEq}(g,h),$$

obtaining a quotient

$$\mathrm{CoEq}(\mathrm{coeq}(g,h) \circ f, \mathrm{coeq}(g,h) \circ g) = \mathrm{CoEq}(\mathrm{coeq}(g,h) \circ f, \mathrm{coeq}(g,h) \circ h)$$

of $\mathrm{CoEq}(g,h)$.

3 Operations With Sets

3.1 The Empty Set

Definition 3.1.1.1. The **empty set** is the set $\emptyset$ defined by

$$\emptyset \stackrel{\text{def}}{=} \{x \in X \mid x \neq x\},$$

where A is the set in the set existence axiom, **??** of **??**.

3.2 Singleton Sets

Let X be a set.

Definition 3.2.1.1. The **singleton set containing** X is the set $\{X\}$ defined by

$$\{X\} \stackrel{\text{def}}{=} \{X, X\},$$

where $\{X, X\}$ is the pairing of X with itself (Definition 3.3.1.1).

3.3 Pairings of Sets

Let X and Y be sets.

Definition 3.3.1.1. The **pairing of** X **and** Y is the set $\{X, Y\}$ defined by

$$\{X, Y\} \stackrel{\text{def}}{=} \{x \in A \mid x = X \text{ or } x = Y\},$$

where A is the set in the axiom of pairing, **??** of **??**.

3.4 Unions of Families

Let $\{A_i\}_{i \in I}$ be a family of sets.

Definition 3.4.1.1. The **union of the family** $\{A_i\}_{i \in I}$ is the set $\bigcup_{i \in I} A_i$ defined by

$$\bigcup_{i \in I} A_i \stackrel{\text{def}}{=} \{x \in F \mid \text{there exists some } i \in I \text{ such that } x \in A_i\},$$

where F is the set in the axiom of union, **??** of **??**.

3.5 Binary Unions

Let A and B be sets.

Definition 3.5.1.1. The **union**[8] **of** A **and** B is the set $A \cup B$ defined by

$$A \cup B \overset{\text{def}}{=} \bigcup_{z \in \{A,B\}} z.$$

Proposition 3.5.1.2. Let X be a set.

1. *Functoriality.* The assignments $U, V, (U, V) \mapsto U \cup V$ define functors

$$\begin{aligned} U \cup -\colon (\mathcal{P}(X), \subset) &\to (\mathcal{P}(X), \subset), \\ - \cup V\colon (\mathcal{P}(X), \subset) &\to (\mathcal{P}(X), \subset), \\ -_1 \cup -_2\colon (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \subset) &\to (\mathcal{P}(X), \subset), \end{aligned}$$

where $-_1 \cup -_2$ is the functor where

- *Action on Objects.* For each $(U, V) \in \mathcal{P}(X) \times \mathcal{P}(X)$, we have

$$[-_1 \cup -_2](U, V) \overset{\text{def}}{=} U \cup V;$$

- *Action on Morphisms.* For each pair of morphisms

$$\begin{aligned} \iota_U &\colon U \hookrightarrow U', \\ \iota_V &\colon V \hookrightarrow V' \end{aligned}$$

of $\mathcal{P}(X) \times \mathcal{P}(X)$, the image

$$\iota_U \cup \iota_V \colon U \cup V \hookrightarrow U' \cup V'$$

of (ι_U, ι_V) by $\cup$ is the inclusion

$$U \cup V \subset U' \cup V'$$

i.e. where we have

($\star$) If $U \subset U'$ and $V \subset V'$, then $U \cup V \subset U' \cup V'$;

and where $U \cup -$ and $- \cup V$ are the partial functors of $-_1 \cup -_2$ at $U, V \in \mathcal{P}(X)$.

2. *Via Intersections and Symmetric Differences.* We have an equality of sets

$$U \cup V = (U \mathbin{\triangle} V) \mathbin{\triangle} (U \cap V)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

[8] *Further Terminology:* Also called the **binary union of** A **and** B, for emphasis.

3. *Associativity.* We have an equality of sets

$$(U \cup V) \cup W = U \cup (V \cup W)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

4. *Unitality.* We have equalities of sets

$$U \cup \emptyset = U,$$
$$\emptyset \cup U = U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

5. *Commutativity.* We have an equality of sets

$$U \cup V = V \cup U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

6. *Idempotency.* We have an equality of sets

$$U \cup U = U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

7. *Distributivity Over Intersections.* We have equalities of sets

$$U \cup (V \cap W) = (U \cup V) \cap (U \cup W),$$
$$(U \cap V) \cup W = (U \cup W) \cap (V \cup W)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

8. *Interaction With Powersets and Semirings.* The quintuple $(\mathcal{P}(X), \cup, \cap, \emptyset, X)$ is an idempotent commutative semiring.

Proof. *Item 1, Functoriality*: Omitted.
Item 2, Via Intersections and Symmetric Differences: Omitted.
Item 3, Associativity: Clear.
Item 4, Unitality: Clear.
Item 5, Commutativity: Clear.
Item 6, Idempotency: Clear.
Item 7, Distributivity Over Intersections: Omitted.
Item 8, Interaction With Powersets and Semirings: This follows from Items 3 to 6 and Items 3 to 5, 7 and 8 of Proposition 3.7.1.2. □

3.6 Intersections of Families

Let $\mathcal{F}$ be a family of sets.

Definition 3.6.1.1. The **intersection of a family $\mathcal{F}$ of sets** is the set $\bigcap_{X\in\mathcal{F}} X$ defined by

$$\bigcap_{X\in\mathcal{F}} X \stackrel{\text{def}}{=} \left\{ z \in \bigcup_{X\in\mathcal{F}} X \,\middle|\, \text{for each } X \in \mathcal{F}\text{, we have } z \in X \right\}.$$

3.7 Binary Intersections

Let X and Y be sets.

Definition 3.7.1.1. The **intersection**[9] **of X and Y** is the set $X \cap Y$ defined by

$$X \cap Y \stackrel{\text{def}}{=} \bigcap_{z\in\{X,Y\}} z.$$

Proposition 3.7.1.2. Let X be a set.

1. *Functoriality.* The assignments $U, V, (U, V) \mapsto U \cap V$ define functors

$$\begin{aligned} U \cap - \colon (\mathcal{P}(X), \subset) &\to (\mathcal{P}(X), \subset), \\ - \cap V \colon (\mathcal{P}(X), \subset) &\to (\mathcal{P}(X), \subset), \\ -_1 \cap -_2 \colon (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \subset) &\to (\mathcal{P}(X), \subset), \end{aligned}$$

   where $-_1 \cap -_2$ is the functor where

   - *Action on Objects.* For each $(U, V) \in \mathcal{P}(X) \times \mathcal{P}(X)$, we have

$$[-_1 \cap -_2](U, V) \stackrel{\text{def}}{=} U \cap V;$$

   - *Action on Morphisms.* For each pair of morphisms

$$\begin{aligned} \iota_U &\colon U \hookrightarrow U', \\ \iota_V &\colon V \hookrightarrow V' \end{aligned}$$

     of $\mathcal{P}(X) \times \mathcal{P}(X)$, the image

$$\iota_U \cap \iota_V \colon U \cap V \hookrightarrow U' \cap V'$$

     of (ι_U, ι_V) by $\cap$ is the inclusion

$$U \cap V \subset U' \cap V'$$

     i.e. where we have

[9] *Further Terminology:* Also called the **binary intersection of X and Y**, for emphasis.

($\star$) If $U \subset U'$ and $V \subset V'$, then $U \cap V \subset U' \cap V'$;

and where $U \cap -$ and $- \cap V$ are the partial functors of $-_1 \cap -_2$ at $U, V \in \mathcal{P}(X)$.

2. *Adjointness.* We have adjunctions

$$\big(U \cap - \dashv \mathbf{Hom}_{\mathcal{P}(X)}(U, -)\big)\colon \quad \begin{tikzcd} \mathcal{P}(X) \arrow[r, bend left, "U\cap -"] \arrow[r, phantom, "\perp"] & \mathcal{P}(X) \arrow[l, bend left, "\mathbf{Hom}_{\mathcal{P}(X)}(U,-)"] \end{tikzcd},$$

$$\big(- \cap V \dashv \mathbf{Hom}_{\mathcal{P}(X)}(V, -)\big)\colon \quad \begin{tikzcd} \mathcal{P}(X) \arrow[r, bend left, "-\cap V"] \arrow[r, phantom, "\perp"] & \mathcal{P}(X) \arrow[l, bend left, "\mathbf{Hom}_{\mathcal{P}(X)}(V,-)"] \end{tikzcd},$$

where

$$\mathbf{Hom}_{\mathcal{P}(X)}(-_1, -_2)\colon \mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \to \mathcal{P}(X)$$

is the bifunctor defined by[10]

$$\mathbf{Hom}_{\mathcal{P}(X)}(U, V) \stackrel{\text{def}}{=} (X \setminus U) \cup V$$

witnessed by bijections

$$\begin{aligned} \mathrm{Hom}_{\mathcal{P}(X)}(U \cap V, W) &\cong \mathrm{Hom}_{\mathcal{P}(X)}\big(U, \mathbf{Hom}_{\mathcal{P}(X)}(V, W)\big), \\ \mathrm{Hom}_{\mathcal{P}(X)}(U \cap V, W) &\cong \mathrm{Hom}_{\mathcal{P}(X)}\big(V, \mathbf{Hom}_{\mathcal{P}(X)}(U, W)\big), \end{aligned}$$

natural in $U, V, W \in \mathcal{P}(X)$, i.e. where:

(a) The following conditions are equivalent:
   i. We have $U \cap V \subset W$.
   ii. We have $U \subset \mathbf{Hom}_{\mathcal{P}(X)}(V, W)$.
   iii. We have $U \subset (X \setminus V) \cup W$.

(b) The following conditions are equivalent:
   i. We have $V \cap U \subset W$.
   ii. We have $V \subset \mathbf{Hom}_{\mathcal{P}(X)}(U, W)$.
   iii. We have $V \subset (X \setminus U) \cup W$.

[10] *Intuition:* Since intersections are the products in $\mathcal{P}(X)$, the left adjoint $\mathbf{Hom}_{\mathcal{P}(X)}(U, V)$ works as a function type $U \to V$.

Now, under the Curry–Howard correspondence, the function type $U \to V$ corresponds to implication $U \implies V$, which is logically equivalent to the statement $\neg U \vee V$, which in turn corresponds to the set $U^{\mathsf{c}} \vee V \stackrel{\text{def}}{=} (X \setminus U) \cup V$.

3. *Associativity.* We have an equality of sets

$$(U \cap V) \cap W = U \cap (V \cap W)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

4. *Unitality.* Let X be a set and let $U \in \mathcal{P}(X)$. We have equalities of sets

$$\begin{aligned} X \cap U &= U, \\ U \cap X &= U \end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

5. *Commutativity.* We have an equality of sets

$$U \cap V = V \cap U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

6. *Idempotency.* We have an equality of sets

$$U \cap U = U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

7. *Distributivity Over Unions.* We have equalities of sets

$$\begin{aligned} U \cap (V \cup W) &= (U \cap V) \cup (U \cap W), \\ (U \cup V) \cap W &= (U \cap W) \cup (V \cap W) \end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

8. *Annihilation With the Empty Set.* We have an equality of sets

$$\begin{aligned} \emptyset \cap X &= \emptyset, \\ X \cap \emptyset &= \emptyset \end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

9. *Interaction With Powersets and Monoids With Zero.* The quadruple $((\mathcal{P}(X), \emptyset), \cap, X)$ is a commutative monoid with zero.

10. *Interaction With Powersets and Semirings.* The quintuple $(\mathcal{P}(X), \cup, \cap, \emptyset, X)$ is an idempotent commutative semiring.

Proof. *Item 1*, *Functoriality*: Omitted.
Item 2, *Adjointness*: See [MSE 267469].
Item 3, *Associativity*: Clear.
Item 4, *Unitality*: Clear.
Item 5, *Commutativity*: Clear.
Item 6, *Idempotency*: Clear.
Item 7, *Distributivity Over Unions*: Omitted.
Item 8, *Annihilation With the Empty Set*: Clear.
Item 9, *Interaction With Powersets and Monoids With Zero*: This follows from Items 3 to 5 and 8.
Item 10, *Interaction With Powersets and Semirings*: This follows from Items 3 to 6 and Items 3 to 5, 7 and 8 of Proposition 3.7.1.2. □

3.8 Differences

Let X and Y be sets.

Definition 3.8.1.1. The **difference of** X **and** Y is the set $X \setminus Y$ defined by

$$X \setminus Y \overset{\text{def}}{=} \{a \in X \mid a \notin Y\}.$$

Proposition 3.8.1.2. Let X be a set.

1. *Functoriality.* The assignments $U, V, (U, V) \mapsto U \cap V$ define functors

$$\begin{aligned} U \setminus -\colon (\mathcal{P}(X), \supset) &\to (\mathcal{P}(X), \subset), \\ - \setminus V\colon (\mathcal{P}(X), \subset) &\to (\mathcal{P}(X), \subset), \\ -_1 \setminus -_2\colon (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \supset) &\to (\mathcal{P}(X), \subset), \end{aligned}$$

where $-_1 \setminus -_2$ is the functor where

- *Action on Objects.* For each $(U, V) \in \mathcal{P}(X) \times \mathcal{P}(X)$, we have

$$[-_1 \setminus -_2](U, V) \overset{\text{def}}{=} U \setminus V;$$

- *Action on Morphisms.* For each pair of morphisms

$$\begin{aligned} \iota_A &\colon A \hookrightarrow B, \\ \iota_U &\colon U \hookrightarrow V \end{aligned}$$

of $\mathcal{P}(X) \times \mathcal{P}(X)$, the image

$$\iota_U \setminus \iota_V \colon A \setminus V \hookrightarrow B \setminus U$$

of (ι_U, ι_V) by \ is the inclusion

$$A \setminus V \subset B \setminus U$$

i.e. where we have

($\star$) If $A \subset B$ and $U \subset V$, then $A \setminus V \subset B \setminus U$;

and where $U \setminus -$ and $- \setminus V$ are the partial functors of $-_1 \setminus -_2$ at $U, V \in \mathcal{P}(X)$.

2. *De Morgan's Laws.* We have equalities of sets

$$\begin{aligned} X \setminus (U \cup V) &= (X \setminus U) \cap (X \setminus V), \\ X \setminus (U \cap V) &= (X \setminus U) \cup (X \setminus V) \end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

3. *Interaction With Unions I.* We have equalities of sets

$$(U \setminus V) \cup W = (U \cup W) \setminus (V \setminus W)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

4. *Interaction With Unions II.* We have equalities of sets

$$(U \cup V) \setminus W = (U \setminus W) \cup (V \setminus W)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

5. *Interaction With Intersections.* We have equalities of sets

$$\begin{aligned} (U \setminus V) \cap W &= (U \cap W) \setminus V \\ &= U \cap (W \setminus V) \end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

6. *Triple Differences.* We have

$$U \setminus (V \setminus W) = (U \cap W) \cup (U \setminus V)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

7. *Left Annihilation.* We have

$$\emptyset \setminus U = \emptyset$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

8. *Right Unitality.* We have
$$U \setminus \emptyset = U$$
for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

9. *Invertibility.* We have
$$U \setminus U = \emptyset$$
for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

10. *Interaction With Containment.* The following conditions are equivalent:

    (a) We have $V \setminus U \subset W$.

    (b) We have $V \setminus W \subset U$.

Proof. *Item 1*, *Functoriality*: Omitted.
Item 2, *De Morgan's Laws*: Omitted.
Item 3, *Interaction With Unions I*: Omitted.
Item 4, *Interaction With Unions II*: Omitted.
Item 5, *Interaction With Intersections*: Omitted.
Item 6, *Triple Differences*: Omitted.
Item 7, *Left Annihilation*: Clear.
Item 8, *Right Unitality*: Clear.
Item 9, *Invertibility*: Clear.
Item 10, *Interaction With Containment*: Omitted. □

3.9 Complements

Let X be a set and let $U \in \mathcal{P}(X)$.

Definition 3.9.1.1. The **complement of** U is the set U^{c} defined by

$$\begin{aligned} U^{\mathsf{c}} &\overset{\text{def}}{=} X \setminus U \\ &\overset{\text{def}}{=} \{a \in X \mid a \notin U\}. \end{aligned}$$

Proposition 3.9.1.2. Let X be a set.

1. *Functoriality.* The assignment $U \mapsto U^{\mathsf{c}}$ defines a functor

$$(-)^{\mathsf{c}} \colon \mathcal{P}(X)^{\mathsf{op}} \to \mathcal{P}(X),$$

where

- *Action on Objects.* For each $U \in \mathcal{P}(X)$, we have

$$[(-)^{\mathsf{c}}](U) \stackrel{\text{def}}{=} U^{\mathsf{c}};$$

- *Action on Morphisms.* For each morphism $\iota_U \colon U \hookrightarrow V$ of $\mathcal{P}(X)$, the image

$$\iota_U^{\mathsf{c}} \colon V^{\mathsf{c}} \hookrightarrow U^{\mathsf{c}}$$

of ι_U by $(-)^{\mathsf{c}}$ is the inclusion

$$V^{\mathsf{c}} \subset U^{\mathsf{c}}$$

i.e. where we have

($\star$) If $U \subset V$, then $V^{\mathsf{c}} \subset U^{\mathsf{c}}$.

2. *De Morgan's Laws.* We have equalities of sets

$$(U \cup V)^{\mathsf{c}} = U^{\mathsf{c}} \cap V^{\mathsf{c}},$$
$$(U \cap V)^{\mathsf{c}} = U^{\mathsf{c}} \cup V^{\mathsf{c}}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

3. *Involutority.* We have

$$(U^{\mathsf{c}})^{\mathsf{c}} = U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

Proof. *Item 1*, *Functoriality*: Clear.
Item 2, *De Morgan's Laws*: Omitted.
Item 3, *Involutority*: Clear. □

3.10 Symmetric Differences

Let A and B be sets.

Definition 3.10.1.1. The **symmetric difference of A and B** is the set $A \triangle B$ defined by

$$A \triangle B \stackrel{\text{def}}{=} (A \setminus B) \cup (B \setminus A).$$

Proposition 3.10.1.2. Let X be a set.

1. *Lack of Functoriality.* The assignment $(U, V) \mapsto U \triangle V$ ***does not*** define a functor

$$-_1 \triangle -_2 \colon (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \subset) \to (\mathcal{P}(X), \subset).$$

2. *Via Unions and Intersections.* We have[11]

$$U \triangle V = (U \cup V) \setminus (U \cap V)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

3. *Associativity.* We have[12]

$$(U \triangle V) \triangle W = U \triangle (V \triangle W)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

4. *Unitality.* We have

$$\begin{aligned} U \triangle \emptyset &= U, \\ \emptyset \triangle U &= U \end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

5. *Invertibility.* We have

$$U \triangle U = \emptyset$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

6. *Commutativity.* We have

$$U \triangle V = V \triangle U$$

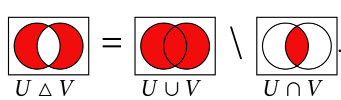
for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

7. *"Transitivity".* We have

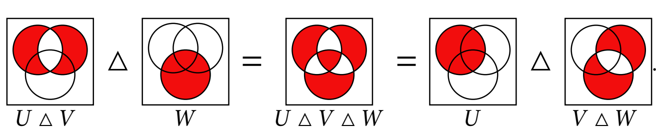
$$(U \triangle V) \triangle (V \triangle W) = U \triangle W$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

[11] *Illustration:*



[12] *Illustration:*



8. *The Triangle Inequality for Symmetric Differences.* We have

$$U \triangle W \subset U \triangle V \cup V \triangle W$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

9. *Distributivity Over Intersections.* We have

$$\begin{aligned} U \cap (V \triangle W) &= (U \cap V) \triangle (U \cap W), \\ (U \triangle V) \cap W &= (U \cap W) \triangle (V \cap W) \end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

10. *Interaction With Indicator Functions.* We have

$$\chi_{U \triangle V} \equiv \chi_U + \chi_V \mod 2$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

11. *Bijectivity.* Given $A, B \subset \mathcal{P}(X)$, the maps

$$\begin{aligned} A \triangle - &: \mathcal{P}(X) \to \mathcal{P}(X), \\ - \triangle B &: \mathcal{P}(X) \to \mathcal{P}(X) \end{aligned}$$

are bijections with inverses given by

$$\begin{aligned} (A \triangle -)^{-1} &= - \cup (A \cap -), \\ (- \triangle B)^{-1} &= - \cup (B \cap -). \end{aligned}$$

Moreover, the map

$$C \mapsto C \triangle (A \triangle B)$$

is a bijection of $\mathcal{P}(X)$ onto itself sending A to B and B to A.

12. *Interaction With Powersets and Groups I.* The quadruple $\left(\mathcal{P}(X), \triangle, \emptyset, \mathrm{id}_{\mathcal{P}(X)}\right)$ is an

abelian group.[13,14,15]

13. *Interaction With Powersets and Groups II*. Every element of $\mathcal{P}(X)$ has order 2 with respect to $\triangle$, and thus $\mathcal{P}(X)$ is a *Boolean group* (i.e. an abelian 2-group).

14. *Interaction With Powersets and Vector Spaces I*. The pair $\left(\mathcal{P}(X), \alpha_{\mathcal{P}(X)}\right)$ consisting of

    - The group $\mathcal{P}(X)$ of Item 12;
    - The map $\alpha_{\mathcal{P}(X)} \colon \mathbb{F}_2 \times \mathcal{P}(X) \to \mathcal{P}(X)$ defined by

    $$\begin{aligned} 0 \cdot U &\stackrel{\text{def}}{=} \emptyset, \\ 1 \cdot U &\stackrel{\text{def}}{=} U; \end{aligned}$$

    is an $\mathbb{F}_2$-vector space.

15. *Interaction With Powersets and Vector Spaces II*. If X is finite, then:

    (a) The set of singletons sets on the elements of X forms a basis for the $\mathbb{F}_2$-vector space $\left(\mathcal{P}(X), \alpha_{\mathcal{P}(X)}\right)$ of Item 14.

    (b) We have
    $$\dim(\mathcal{P}(X)) = \#\mathcal{P}(X).$$

16. *Interaction With Powersets and Rings*. The quintuple $(\mathcal{P}(X), \triangle, \cap, \emptyset, X)$ is a commutative ring.[16]

[13] *Example:* When $X = \emptyset$, we have an isomorphism of groups between $\mathcal{P}(\emptyset)$ and the trivial group:

$$\left(\mathcal{P}(\emptyset), \triangle, \emptyset, \mathrm{id}_{\mathcal{P}(\emptyset)}\right) \cong \mathrm{pt}.$$

[14] *Example:* When $X = \mathrm{pt}$, we have an isomorphism of groups between $\mathcal{P}(\mathrm{pt})$ and $\mathbb{Z}_{/2}$:

$$\left(\mathcal{P}(\mathrm{pt}), \triangle, \emptyset, \mathrm{id}_{\mathcal{P}(\mathrm{pt})}\right) \cong \mathbb{Z}_{/2}.$$

[15] *Example:* When $X = \{0,1\}$, we have an isomorphism of groups between $\mathcal{P}(\{0,1\})$ and $\mathbb{Z}_{/2} \times \mathbb{Z}_{/2}$:

$$\left(\mathcal{P}(\{0,1\}), \triangle, \emptyset, \mathrm{id}_{\mathcal{P}(\{0,1\})}\right) \cong \mathbb{Z}_{/2} \times \mathbb{Z}_{/2}.$$

[16] *Warning:* The analogous statement replacing intersections by unions (i.e. that the quintuple $(\mathcal{P}(X), \triangle, \cup, \emptyset, X)$ is a ring) is false, however. See [Pro23b] for a proof.

Proof. *Item 1, Lack of Functoriality*: Omitted.
Item 2, Via Unions and Intersections: Omitted.
Item 3, Associativity: Omitted.
Item 4, Unitality: Clear.
Item 5, Invertibility: Clear.
Item 6, Commutativity: Clear.
Item 7, "Transitivity": We have

$$\begin{aligned}(U \triangle V) \triangle (V \triangle W) &= U \triangle (V \triangle (V \triangle W)) && \text{(by Item 3)}\\ &= U \triangle ((V \triangle V) \triangle W) && \text{(by Item 3)}\\ &= U \triangle (\emptyset \triangle W) && \text{(by Item 5)}\\ &= U \triangle W && \text{(by Item 4)}\end{aligned}$$

Item 8, The Triangle Inequality for Symmetric Differences: This follows from Items 2 and 7.
Item 9, Distributivity Over Intersections: Omitted.
Item 10, Interaction With Indicator Functions: Clear.
Item 11, Bijectivity: Clear.
Item 12, Interaction With Powersets and Groups I: This follows from Items 3 to 6.
Item 13, Interaction With Powersets and Groups II: This follows from Item 5.
Item 14, Interaction With Powersets and Vector Spaces I: Clear.
Item 15, Interaction With Powersets and Vector Spaces II: Omitted.
Item 16, Interaction With Powersets and Rings: This follows from Items 9 and 12 and Items 8 and 9 of Proposition 3.7.1.2.[17] □

3.11 Ordered Pairs

Let A and B be sets.

Definition 3.11.1.1. The **ordered pair associated to** A **and** B is the set (A, B) defined by

$$(A, B) \stackrel{\text{def}}{=} \{\{A\}, \{A, B\}\}.$$

Proposition 3.11.1.2. Let A and B be sets.

1. *Uniqueness.* Let A, B, C, and D be sets. The following conditions are equivalent:

   (a) We have $(A, B) = (C, D)$.

   (b) We have $A = C$ and $B = D$.

Proof. *Item 1, Uniqueness*: See [Cie97, Theorem 1.2.3]. □

[17] *Reference:* [Pro23a].

4 Powersets

4.1 Characteristic Functions

Let X be a set.

Definition 4.1.1.1. Let $U \subset X$ and let $x \in X$.

1. The **characteristic function of** U[18] is the function[19]

$$\chi_U : X \longrightarrow \{\mathsf{t}, \mathsf{f}\}$$

defined by

$$\chi_U(x) \overset{\text{def}}{=} \begin{cases} \text{true} & \text{if } x \in U, \\ \text{false} & \text{if } x \notin U \end{cases}$$

for each $x \in X$.

2. The **characteristic function of** x is the function[20]

$$\chi_x : X \longrightarrow \{\mathsf{t}, \mathsf{f}\}$$

defined by

$$\chi_x \overset{\text{def}}{=} \chi_{\{x\}},$$

i.e. by

$$\chi_x(y) \overset{\text{def}}{=} \begin{cases} \text{true} & \text{if } x = y, \\ \text{false} & \text{if } x \neq y \end{cases}$$

for each $y \in X$.

3. The **characteristic relation on** X[21] is the relation[22]

$$\chi_X(-_1, -_2) : X \times X \longrightarrow \{\mathsf{t}, \mathsf{f}\}$$

on X defined by[23]

$$\chi_X(x, y) \overset{\text{def}}{=} \begin{cases} \text{true} & \text{if } x = y, \\ \text{false} & \text{if } x \neq y \end{cases}$$

for each $x, y \in X$.

---

[18] *Further Terminology:* Also called the **indicator function of** U.

[19] *Further Notation:* Also written $\chi_X(U, -)$ or $\chi_X(-, U)$.

[20] *Further Notation:* Also written χ_x, $\chi_X(x, -)$, or $\chi_X(-, x)$.

[21] *Further Terminology:* Also called the **identity relation on** X.

[22] *Further Notation:* Also written χ_{-2}^{-1}, or $\sim_{\text{id}}$ in the context of relations.

[23] As a subset of $X \times X$, the relation χ_X corresponds to the diagonal $\Delta_X \subset X \times X$ of X.

4. The **characteristic embedding**[24] **of** X **into** $\mathcal{P}(X)$ is the function

$$\chi_{(-)}\colon X \hookrightarrow \mathcal{P}(X)$$

defined by

$$\chi_{(-)}(x) \overset{\text{def}}{=} \chi_x$$

for each $x \in X$.

Remark 4.1.1.2. The definitions in Definition 4.1.1.1 are decategorifications of co/presheaves, representable co/presheaves, Hom profunctors, and the Yoneda embedding:[25]

1. A function

$$f\colon X \to \{\mathsf{t}, \mathsf{f}\}$$

is a decategorification of a presheaf

$$\mathcal{F}\colon \mathcal{C}^{\mathsf{op}} \to \mathsf{Sets},$$

with the characteristic functions χ_U of the subsets of X being the primordial examples (and, in fact, all examples) of these.

[24] The name "characteristic *embedding*" comes from the fact that there is an analogue of fully faithfulness for $\chi_{(-)}$: given a set X, we have

$$\mathrm{Hom}_{\mathcal{P}(X)}(\chi_x, \chi_y) = \chi_X(x, y),$$

for each $x, y \in X$.

[25] These statements can be made precise by using the embeddings

$$(-)_{\mathsf{disc}}\colon \mathsf{Sets} \hookrightarrow \mathsf{Cats},$$
$$(-)_{\mathsf{disc}}\colon \{\mathsf{t}, \mathsf{f}\}_{\mathsf{disc}} \hookrightarrow \mathsf{Sets}$$

of sets into categories and of classical truth values into sets.

For instance, in this approach the characteristic function

$$\chi_x\colon X \to \{\mathsf{t}, \mathsf{f}\}$$

of an element x of X, defined by

$$\chi_x(y) \overset{\text{def}}{=} \begin{cases} \mathsf{true} & \text{if } x = y, \\ \mathsf{false} & \text{if } x \neq y \end{cases}$$

for each $y \in X$, is recovered as the representable presheaf

$$\mathrm{Hom}_{X_{\mathsf{disc}}}(-, x)\colon X_{\mathsf{disc}} \to \mathsf{Sets}$$

of the corresponding object x of X_{disc}, defined on objects by

$$\mathrm{Hom}_{X_{\mathsf{disc}}}(y, x) \overset{\text{def}}{=} \begin{cases} \mathrm{pt} & \text{if } x = y, \\ \emptyset & \text{if } x \neq y \end{cases}$$

for each $y \in \mathrm{Obj}(X_{\mathsf{disc}})$.

2. The characteristic function

$$\chi_x \colon X \to \{\mathsf{t}, \mathsf{f}\}$$

of an *element* x of X is a decategorification of the representable presheaf

$$h_X \colon \mathcal{C}^{\mathsf{op}} \to \mathsf{Sets}$$

of an *object* x of a category $\mathcal{C}$.

3. The characteristic relation

$$\chi_X(-_1, -_2) \colon X \times X \to \{\mathsf{t}, \mathsf{f}\}$$

of X is a decategorification of the Hom profunctor

$$\mathsf{Hom}_{\mathcal{C}}(-_1, -_2) \colon \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathsf{Sets}$$

of a category $\mathcal{C}$.

4. The characteristic embedding

$$\chi_{(-)} \colon X \hookrightarrow \mathcal{P}(X)$$

of X into $\mathcal{P}(X)$ is a decategorification of the Yoneda embedding

$$よ \colon \mathcal{C}^{\mathsf{op}} \hookrightarrow \mathsf{PSh}(\mathcal{C})$$

of a category $\mathcal{C}$ into $\mathsf{PSh}(\mathcal{C})$.

5. There is also a direct parallel between unions and colimits:

   - An element of $\mathcal{P}(X)$ is a union of elements of X, viewed as one-point subsets $\{x\} \in \mathcal{P}(A)$;
   - An object of $\mathsf{PSh}(\mathcal{C})$ is a colimit of objects of $\mathcal{C}$, viewed as representable presheaves $h_X \in \mathsf{Obj}(\mathsf{PSh}(\mathcal{C}))$.

Proposition 4.1.1.3. Let $f \colon A \to B$ be a function. We have an inclusion

$$\chi_B \circ (f \times f) \subset \chi_A, \qquad \begin{tikzcd} A \times A \arrow[r, "\chi_A(-_1,-_2)"] \arrow[d, "f\times f"'] \arrow[dr, phantom, "\subset"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathsf{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ B \times B \arrow[r, "\chi_B(-_1,-_2)"'] & \{\mathsf{true}, \mathsf{false}\}. \end{tikzcd}$$

Proof. The inclusion $\chi_B(f(a), f(b)) \subset \chi_A(a, b)$ is equivalent to the statement "if $a = b$, then $f(a) = f(b)$", which is true. □

Proposition 4.1.1.4. Let X be a set and let $U \subset X$ be a subset of X. We have

$$\mathrm{Hom}_{\mathcal{P}(X)}(\chi_x, \chi_U) = \chi_U(x)$$

for each $x \in X$, giving an equality of functions

$$\mathrm{Hom}_{\mathcal{P}(X)}\big(\chi_{(-)}, \chi_U\big) = \chi_U.$$

Proof. Clear. □

Corollary 4.1.1.5. The characteristic embedding is fully faithful, i.e., we have

$$\mathrm{Hom}_{\mathcal{P}(X)}\big(\chi_x, \chi_y\big) = \chi_X(x, y)$$

for each $x, y \in X$.

Proof. This follows from Proposition 4.1.1.4. □

4.2 Powersets

Let X be a set.

Definition 4.2.1.1. The **powerset of** X is the set $\mathcal{P}(X)$ defined by

$$\mathcal{P}(X) \stackrel{\text{def}}{=} \{U \in P \mid U \subset X\},$$

where P is the set in the axiom of powerset, **??** of **??**.

Remark 4.2.1.2. The powerset of a set is a decategorification of the category of presheaves of a category: while[26]

[26]This parallel is based on the following comparison:

- A category is enriched over the category

$$\mathsf{Sets} \stackrel{\text{def}}{=} \mathsf{Cats}_0$$

  of sets (i.e. "0-categories"), with presheaves taking values on it;

- A set is enriched over the set

$$\{\mathsf{t}, \mathsf{f}\} \stackrel{\text{def}}{=} \mathsf{Cats}_{-1}$$

  of classical truth values (i.e. "(-1)-categories"), with characteristic functions taking values on it.

- The powerset of a set X is equivalently (Item 6 of Proposition 4.2.1.3) the set

$$\mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\})$$

of functions from X to the set $\{\mathsf{t}, \mathsf{f}\}$ of classical truth values;

- The category of presheaves on a category $\mathcal{C}$ is the category

$$\mathsf{Fun}(\mathcal{C}^{\mathsf{op}}, \mathsf{Sets})$$

of functors from $\mathcal{C}^{\mathsf{op}}$ to the category Sets of sets.

Proposition 4.2.1.3. Let X be a set.

1. *Functoriality.* The assignment $X \mapsto \mathcal{P}(X)$ defines functors

$$\begin{aligned} \mathcal{P}_* \colon \mathsf{Sets} &\to \mathsf{Sets}, \\ \mathcal{P}^{-1} \colon \mathsf{Sets}^{\mathsf{op}} &\to \mathsf{Sets}, \\ \mathcal{P}_! \colon \mathsf{Sets} &\to \mathsf{Sets} \end{aligned}$$

where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$\begin{aligned} \mathcal{P}_*(A) &\overset{\mathrm{def}}{=} \mathcal{P}(A), \\ \mathcal{P}^{-1}(A) &\overset{\mathrm{def}}{=} \mathcal{P}(A), \\ \mathcal{P}_!(A) &\overset{\mathrm{def}}{=} \mathcal{P}(A); \end{aligned}$$

- *Action on Morphisms.* For each morphism $f \colon A \to B$ of Sets, the images

$$\begin{aligned} \mathcal{P}_*(f) \colon \mathcal{P}(A) &\to \mathcal{P}(B), \\ \mathcal{P}^{-1}(f) \colon \mathcal{P}(B) &\to \mathcal{P}(A), \\ \mathcal{P}_!(f) \colon \mathcal{P}(A) &\to \mathcal{P}(B) \end{aligned}$$

of f by $\mathcal{P}_*$, $\mathcal{P}^{-1}$, and $\mathcal{P}_!$ are defined by

$$\begin{aligned} \mathcal{P}_*(f) &\overset{\mathrm{def}}{=} f_*, \\ \mathcal{P}^{-1}(f) &\overset{\mathrm{def}}{=} f^{-1}, \\ \mathcal{P}_!(f) &\overset{\mathrm{def}}{=} f_!, \end{aligned}$$

as in Definitions 4.3.1.1, 4.4.1.1 and 4.5.1.1.

2. *Adjointness I.* We have an adjunction

$$\left(\mathcal{P}^{-1} \dashv \mathcal{P}^{-1,\mathsf{op}}\right)\colon \quad \begin{tikzcd} \mathsf{Sets}^{\mathsf{op}} \arrow[r, bend left, "\mathcal{P}^{-1}"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}, \arrow[l, bend left, "\mathcal{P}^{-1,\mathsf{op}}"] \end{tikzcd}$$

witnessed by a bijection

$$\underbrace{\mathsf{Sets}^{\mathsf{op}}(\mathcal{P}(X), Y)}_{\stackrel{\text{def}}{=} \mathsf{Sets}(Y, \mathcal{P}(X))} \cong \mathsf{Sets}(X, \mathcal{P}(Y)),$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets})$ and $Y \in \mathrm{Obj}\left(\mathsf{Sets}^{\mathsf{op}}\right)$.

3. *Adjointness II.* We have an adjunction

$$(\mathrm{Gr} \dashv \mathcal{P}_*)\colon \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "\mathrm{Gr}"] \arrow[r, phantom, "\perp"] & \mathsf{Rel}, \arrow[l, bend left, "\mathcal{P}_*"] \end{tikzcd}$$

witnessed by a bijection of sets

$$\mathsf{Rel}(\mathrm{Gr}(A), B) \cong \mathsf{Sets}(A, \mathcal{P}(B))$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$ and $B \in \mathrm{Obj}(\mathsf{Rel})$, where Gr is the graph functor of Relations, Item 1 of Proposition 3.1.1.2.

4. *Symmetric Strong Monoidality With Respect to Coproducts.* The powerset functor $\mathcal{P}_*$ of Item 1 has a symmetric strong monoidal structure

$$\left(\mathcal{P}_*, \mathcal{P}_*^{\coprod}, \mathcal{P}_{*|\mathbb{1}}^{\coprod}\right)\colon (\mathsf{Sets}, \coprod, \emptyset) \longrightarrow (\mathsf{Sets}, \times, \mathrm{pt})$$

being equipped with isomorphisms

$$\mathcal{P}_{*|X,Y}^{\coprod}\colon \mathcal{P}(X) \times \mathcal{P}(Y) \xrightarrow{\cong} \mathcal{P}(X \coprod Y),$$
$$\mathcal{P}_{*|\mathbb{1}}^{\coprod}\colon \mathrm{pt} \xrightarrow{\cong} \mathcal{P}(\emptyset),$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

5. *Symmetric Lax Monoidality With Respect to Products.* The powerset functor $\mathcal{P}_*$ of Item 1 has a symmetric lax monoidal structure

$$\left(\mathcal{P}_*, \mathcal{P}_*^{\otimes}, \mathcal{P}_{*|\mathbb{1}}^{\otimes}\right)\colon (\mathsf{Sets}, \times, \mathrm{pt}) \longrightarrow (\mathsf{Sets}, \times, \mathrm{pt})$$

being equipped with isomorphisms

$$\mathcal{P}^{\otimes}_{*|X,Y}\colon \mathcal{P}(X)\times\mathcal{P}(Y)\to\mathcal{P}(X\times Y),$$
$$\mathcal{P}^{\otimes}_{*|\mathbb{1}}\colon \mathrm{pt}\xrightarrow{=}\mathcal{P}(\emptyset),$$

natural in $X,Y\in\mathrm{Obj}(\mathsf{Sets})$, where $\mathcal{P}^{\otimes}_{*|X,Y}$ is given by

$$\mathcal{P}^{\otimes}_{*|X,Y}(U,V)\stackrel{\text{def}}{=}U\times V$$

for each $(U,V)\in\mathcal{P}(X)\times\mathcal{P}(Y)$.

6. *Powersets as Sets of Functions.* The assignment $U\mapsto\chi_U$ defines a bijection[27]

$$\chi_{(-)}\colon\mathcal{P}(X)\xrightarrow{\cong}\mathsf{Sets}(X,\{\mathsf{t},\mathsf{f}\}),$$

natural in $X\in\mathrm{Obj}(\mathsf{Sets})$.

7. *Powersets as Sets of Relations.* We have bijections

$$\mathcal{P}(X)\cong\mathrm{Rel}(\mathrm{pt},X),$$
$$\mathcal{P}(X)\cong\mathrm{Rel}(X,\mathrm{pt}),$$

natural in $X\in\mathrm{Obj}(\mathsf{Sets})$.

8. *As a Free Cocompletion: Universal Property.* The pair $\big(\mathcal{P}(X),\chi_{(-)}\big)$ consisting of

   - The powerset $\mathcal{P}(X)$ of X;
   - The characteristic embedding $\chi_{(-)}\colon X\hookrightarrow\mathcal{P}(X)$ of X into $\mathcal{P}(X)$;

   satisfies the following universal property:

   (★) Given another pair (Y,f) consisting of
      - A cocomplete poset $(Y,\preceq)$;
      - A function $f\colon X\to Y$;

      there exists a unique cocontinuous morphism of posets $(\mathcal{P}(X),\subset)\xrightarrow{\exists!}$

[27] This bijection is a decategorified form of the equivalence

$$\mathsf{PSh}(\mathcal{C})\stackrel{\text{eq.}}{\cong}\mathsf{DFib}(\mathcal{C})$$

of Fibred Categories, **??** of **??**, with $\chi_{(-)}$ being a decategorified version of the category of elements construction of Fibred Categories, **??**.

See also **??** of **??**.

$(Y, \preceq)$ making the diagram

$$\begin{tikzcd} & \mathcal{P}(X) \arrow[d, dashed, "\exists!"] \\ X \arrow[ur, "\chi_X"] \arrow[r, "f"'] & Y \end{tikzcd}$$

commute.

9. *As a Free Cocompletion: Adjointness.* We have an adjunction[28]

$$(\chi_{(-)} \dashv 忘)\colon \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "\chi_{(-)}"] \arrow[r, phantom, "\perp"] & \mathsf{Pos}^{\mathsf{cocomp.}}, \arrow[l, bend left, "忘"] \end{tikzcd}$$

witnessed by a bijection

$$\mathsf{Pos}^{\mathsf{cocomp.}}((\mathcal{P}(X), \subset), (Y, \preceq)) \cong \mathsf{Sets}(X, Y),$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets})$ and $(Y, \preceq) \in \mathrm{Obj}(\mathsf{Pos})$, where

- We have a natural map
$$\chi_X^*\colon \mathsf{Pos}^{\mathsf{cocomp.}}((\mathcal{P}(X), \subset), (Y, \preceq)) \longrightarrow \mathsf{Sets}(X, Y)$$
defined by
$$\chi_X^*(f) \stackrel{\text{def}}{=} f \circ \chi_X,$$
i.e. by sending a cocontinuous morphism of posets $f\colon \mathcal{P}(X) \longrightarrow Y$ to the composition
$$X \overset{\chi_X}{\hookrightarrow} \mathcal{P}(X) \xrightarrow{f} Y;$$
- We have a natural map
$$\mathrm{Lan}_{\chi_X}\colon \mathsf{Sets}(X, Y) \longrightarrow \mathsf{Pos}^{\mathsf{cocomp.}}((\mathcal{P}(X), \subset), (Y, \preceq))$$
computed by
$$\begin{aligned} [\mathrm{Lan}_{\chi_X}(f)](U) &\cong \int^{x \in X} \chi_{\mathcal{P}(X)}(\chi_x, U) \odot f(x) \\ &\cong \int^{x \in X} \chi_U(x) \odot f(x) && \text{(by Proposition 4.1.1.4)} \\ &\cong \bigvee_{x \in X} (\chi_U(x) \odot f(x)) \end{aligned}$$
for each $U \in \mathcal{P}(X)$, where:

[28] In this sense, $\mathcal{P}(A)$ is the free cocompletion of A. (Note that, despite its name, however, this is not an

– $\bigvee$ is the join in $(Y, \preceq)$;
– We have

$$\begin{aligned}\mathsf{true} \odot f(x) &\stackrel{\text{def}}{=} f(x),\\ \mathsf{false} \odot f(x) &\stackrel{\text{def}}{=} \varnothing_Y,\end{aligned}$$

where $\varnothing_Y$ is the minimal element of $(Y, \preceq)$.

Proof. *Item 1*, *Functoriality*: This follows from Items 3 and 4 of Proposition 4.3.1.4, Items 3 and 4 of Proposition 4.4.1.4, and Items 3 and 4 of Proposition 4.5.1.6.
Item 2, *Adjointness I*: Omitted.
Item 3, *Adjointness II*: Omitted.
Item 4, *Symmetric Strong Monoidality With Respect to Coproducts*: Omitted.
Item 5, *Symmetric Lax Monoidality With Respect to Products*: Omitted.
Item 6, *Powersets as Sets of Functions*: Omitted.
Item 7, *Powersets as Sets of Relations*: Omitted.
Item 8, *As a Free Cocompletion: Universal Property*: This is a rephrasing of **??**.
Item 9, *As a Free Cocompletion: Adjointness*: Omitted. □

4.3 Direct Images

Let A and B be sets and let $f\colon A \to B$ be a function.

Definition 4.3.1.1. The **direct image function associated to** f is the function[29]

$$f_*\colon \mathcal{P}(A) \to \mathcal{P}(B)$$

defined by[30,31]

$$\begin{aligned} f_*(U) &\stackrel{\text{def}}{=} f(U)\\ &\stackrel{\text{def}}{=} \left\{ b \in B \;\middle|\; \begin{array}{l}\text{there exists some } a \in U\\ \text{such that } b = f(a)\end{array}\right\}\\ &= \{f(a) \in B \mid a \in U\}\end{aligned}$$

idempotent operation, as we have $\mathcal{P}(\mathcal{P}(A)) \neq \mathcal{P}(A)$.)

[29] *Further Notation:* Also written $\exists_f\colon \mathcal{P}(A) \to \mathcal{P}(B)$. This notation comes from the fact that the following statements are equivalent, where $b \in B$ and $U \in \mathcal{P}(A)$:

- We have $b \in \exists_f(U)$.
- There exists some $a \in U$ such that $f(a) = b$.

[30] *Further Terminology*: The set $f(U)$ is called the **direct image of** U **by** f.
[31] We also have

$$f_*(U) = B \setminus f_!(A \setminus U);$$

for each $U \in \mathcal{P}(A)$.

Remark 4.3.1.2. Identifying subsets of A with functions from A to $\{\text{true}, \text{false}\}$ via Item 6 of Proposition 4.2.1.3, we see that the direct image function associated to f is equivalently the function

$$f_*\colon \mathcal{P}(A) \to \mathcal{P}(B)$$

defined by

$$\begin{aligned} f_*(\chi_U) &\stackrel{\text{def}}{=} \mathrm{Lan}_f(\chi_U) \\ &= \mathrm{colim}\Big(\Big(f \overset{\rightarrow}{\times} \underline{(-_1)}\Big) \overset{\mathrm{pr}}{\twoheadrightarrow} A \xrightarrow{\chi_U} \{\mathrm{t}, \mathrm{f}\}\Big) \\ &= \underset{\substack{a\in A \\ f(a)=-_1}}{\mathrm{colim}}\, (\chi_U(a)) \\ &= \bigvee_{\substack{a\in A \\ f(a)=-_1}} (\chi_U(a)). \end{aligned}$$

So, in other words, we have

$$\begin{aligned} [f_*(\chi_U)](b) &= \bigvee_{\substack{a\in A \\ f(a)=b}} (\chi_U(a)) \\ &= \begin{cases} \text{true} & \text{if there exists some } a \in A \text{ such} \\ & \text{that } f(a) = b \text{ and } a \in U, \\ \text{false} & \text{otherwise} \end{cases} \\ &= \begin{cases} \text{true} & \text{if there exists some } a \in U \\ & \text{such that } f(a) = b, \\ \text{false} & \text{otherwise} \end{cases} \end{aligned}$$

for each $b \in B$.

Proposition 4.3.1.3. Let $f\colon A \to B$ be a function.

1. *Functoriality.* The assignment $U \mapsto f_*(U)$ defines a functor

$$f_*\colon (\mathcal{P}(A), \subset) \to (\mathcal{P}(B), \subset)$$

where

---

see Item 7 of Proposition 4.3.1.3.

- *Action on Objects.* For each $U \in \mathcal{P}(A)$, we have

$$[f_*](U) \stackrel{\text{def}}{=} f_*(U);$$

- *Action on Morphisms.* For each $U, V \in \mathcal{P}(A)$:

  ($\star$) If $U \subset V$, then $f_*(U) \subset f_*(V)$.

2. *Triple Adjointness.* We have a triple adjunction

$$\left(f_* \dashv f^{-1} \dashv f_!\right)\colon \quad \begin{tikzcd} \mathcal{P}(A) \arrow[r, bend left=50, "f_*"] \arrow[r, bend left=25, phantom, "\perp"] \arrow[r, bend right=25, phantom, "\perp"] \arrow[r, bend right=50, "f_!"'] & \mathcal{P}(B) \arrow[l, "f^{-1}"'] \end{tikzcd},$$

witnessed by bijections of sets

$$\mathrm{Hom}_{\mathcal{P}(B)}(f_*(U), V) \cong \mathrm{Hom}_{\mathcal{P}(A)}\left(U, f^{-1}(V)\right),$$
$$\mathrm{Hom}_{\mathcal{P}(A)}\left(f^{-1}(U), V\right) \cong \mathrm{Hom}_{\mathcal{P}(A)}(U, f_!(V)),$$

natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$ and (respectively) $V \in \mathcal{P}(A)$ and $U \in \mathcal{P}(B)$, i.e. where:

(a) The following conditions are equivalent:
   i. We have $f_*(U) \subset V$.
   ii. We have $U \subset f^{-1}(V)$.

(b) The following conditions are equivalent:
   i. We have $f^{-1}(U) \subset V$.
   ii. We have $U \subset f_!(V)$.

3. *Preservation of Colimits.* We have an equality of sets

$$f_*\left(\bigcup_{i \in I} U_i\right) = \bigcup_{i \in I} f_*(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have equalities

$$f_*(U) \cup f_*(V) = f_*(U \cup V),$$
$$f_*(\emptyset) = \emptyset,$$

natural in $U, V \in \mathcal{P}(A)$.

4. *Oplax Preservation of Limits.* We have an inclusion of sets

$$f_*\left(\bigcap_{i\in I} U_i\right) \subset \bigcap_{i\in I} f_*(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have inclusions

$$\begin{aligned} f_*(U \cap V) &\subset f_*(U) \cap f_*(V), \\ f_*(A) &\subset B, \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

5. *Symmetric Strict Monoidality With Respect to Unions.* The direct image function of Item 1 has a symmetric strict monoidal structure

$$\left(f_*, f_*^{\otimes}, f_{*|\mathbb{1}}^{\otimes}\right) : (\mathcal{P}(A), \cup, \emptyset) \to (\mathcal{P}(B), \cup, \emptyset),$$

being equipped with equalities

$$\begin{aligned} f_{*|U,V}^{\otimes} &: f_*(U) \cup f_*(V) \xrightarrow{=} f_*(U \cup V), \\ f_{*|\mathbb{1}}^{\otimes} &: \emptyset \xrightarrow{=} \emptyset, \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

6. *Symmetric Oplax Monoidality With Respect to Intersections.* The direct image function of Item 1 has a symmetric oplax monoidal structure

$$\left(f_*, f_*^{\otimes}, f_{*|\mathbb{1}}^{\otimes}\right) : (\mathcal{P}(A), \cap, A) \to (\mathcal{P}(B), \cap, B),$$

being equipped with inclusions

$$\begin{aligned} f_{*|U,V}^{\otimes} &: f_*(U \cap V) \hookrightarrow f_*(U) \cap f_*(V), \\ f_{*|\mathbb{1}}^{\otimes} &: f_*(A) \hookrightarrow B, \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

7. *Relation to Direct Images With Compact Support.* We have

$$f_*(U) = B \setminus f_!(A \setminus U)$$

for each $U \in \mathcal{P}(A)$.

Proof. *Item 1*, *Functoriality*: Clear.
Item 2, *Triple Adjointness*: This follows from Kan Extensions, **??** of **??**.
Item 3, *Preservation of Colimits*: This follows from Item 2 and Categories, **??** of **??**.
Item 4, *Oplax Preservation of Limits*: Omitted.
Item 5, *Symmetric Strict Monoidality With Respect to Unions*: This follows from Item 3.
Item 6, *Symmetric Oplax Monoidality With Respect to Intersections*: This follows from **??**.
Item 7, *Relation to Direct Images With Compact Support*: Applying **??** of **??** to $A \setminus U$, we have

$$\begin{aligned} f_!(A \setminus U) &= B \setminus f_*(A \setminus (A \setminus U)) \\ &= B \setminus f_*(U). \end{aligned}$$

Taking complements, we then obtain

$$\begin{aligned} f_*(U) &= B \setminus (B \setminus f_*(U)), \\ &= B \setminus f_!(A \setminus U), \end{aligned}$$

which finishes the proof. □

Proposition 4.3.1.4. Let $f\colon A \longrightarrow B$ be a function.

1. *Functionality I.* The assignment $f \longmapsto f_*$ defines a function

$$(-)_{*|A,B}\colon \mathsf{Sets}(A, B) \longrightarrow \mathsf{Sets}(\mathcal{P}(A), \mathcal{P}(B)).$$

2. *Functionality II.* The assignment $f \longmapsto f_*$ defines a function

$$(-)_{*|A,B}\colon \mathsf{Sets}(A, B) \longrightarrow \mathsf{Pos}((\mathcal{P}(A), \subset), (\mathcal{P}(B), \subset)).$$

3. *Interaction With Identities.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$(\mathrm{id}_A)_* = \mathrm{id}_{\mathcal{P}(A)};$$

4. *Interaction With Composition.* For each pair of composable functions $f\colon A \longrightarrow B$ and $g\colon B \longrightarrow C$, we have

$$(g \circ f)_* = g_* \circ f_*, \qquad \begin{tikzcd} \mathcal{P}(A) \arrow[r, "f_*"] \arrow[dr, "(g\circ f)_*"'] & \mathcal{P}(B) \arrow[d, "g_*"] \\ & \mathcal{P}(C). \end{tikzcd}$$

Proof. *Item 1*, *Functionality I*: Clear.
Item 2, *Functionality II*: Clear.
Item 3, *Interaction With Identities*: This follows from Kan Extensions, **??** of **??**.
Item 4, *Interaction With Composition*: This follows from Kan Extensions, **??** of **??**. □

4.4 Inverse Images

Let A and B be sets and let $f\colon A \to B$ be a function.

Definition 4.4.1.1. The **inverse image function associated to** f is the function[32]

$$f^{-1}\colon \mathcal{P}(B) \to \mathcal{P}(A)$$

defined by[33]

$$f^{-1}(V) \overset{\text{def}}{=} \{a \in A \mid \text{we have } f(a) \in V\}$$

for each $V \in \mathcal{P}(B)$.

Remark 4.4.1.2. Identifying subsets of B with functions from B to $\{\text{true}, \text{false}\}$ via Item 6 of Proposition 4.2.1.3, we see that the inverse image function associated to f is equivalently the function

$$f^*\colon \mathcal{P}(B) \to \mathcal{P}(A)$$

defined by

$$f^*(\chi_V) \overset{\text{def}}{=} \chi_V \circ f$$

for each $\chi_V \in \mathcal{P}(B)$, where $\chi_V \circ f$ is the composition

$$A \xrightarrow{f} B \xrightarrow{\chi_V} \{\text{true}, \text{false}\}$$

in Sets.

Proposition 4.4.1.3. Let $f\colon A \to B$ be a function.

1. *Functoriality.* The assignment $V \mapsto f^{-1}(V)$ defines a functor

   $$f^{-1}\colon (\mathcal{P}(B), \subset) \to (\mathcal{P}(A), \subset)$$

   where

   - *Action on Objects.* For each $V \in \mathcal{P}(B)$, we have

     $$[f^{-1}](V) \overset{\text{def}}{=} f^{-1}(V);$$

   - *Action on Morphisms.* For each $U, V \in \mathcal{P}(B)$:

     $(\star)$ If $U \subset V$, then $f^{-1}(U) \subset f^{-1}(V)$.

[32] *Further Notation:* Also written $f^*\colon \mathcal{P}(B) \to \mathcal{P}(A)$.

[33] *Further Terminology:* The set $f^{-1}(V)$ is called the **inverse image of** V **by** f.

2. *Triple Adjointness.* We have a triple adjunction

$$\left(f_* \dashv f^{-1} \dashv f_!\right)\colon\quad \begin{tikzcd} \mathcal{P}(A) \arrow[r, bend left=50, "f_*"] \arrow[r, bend right=50, "f_!"'] & \mathcal{P}(B) \arrow[l, "f^{-1}"'] \end{tikzcd},$$

witnessed by bijections of sets

$$\begin{aligned} \mathrm{Hom}_{\mathcal{P}(B)}(f_*(U), V) &\cong \mathrm{Hom}_{\mathcal{P}(A)}\left(U, f^{-1}(V)\right), \\ \mathrm{Hom}_{\mathcal{P}(A)}\left(f^{-1}(U), V\right) &\cong \mathrm{Hom}_{\mathcal{P}(A)}(U, f_!(V)), \end{aligned}$$

natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$ and (respectively) $V \in \mathcal{P}(A)$ and $U \in \mathcal{P}(B)$, i.e. where:

(a) The following conditions are equivalent:
   i. We have $f_*(U) \subset V$;
   ii. We have $U \subset f^{-1}(V)$;

(b) The following conditions are equivalent:
   i. We have $f^{-1}(U) \subset V$.
   ii. We have $U \subset f_!(V)$.

3. *Preservation of Colimits.* We have an equality of sets

$$f^{-1}\left(\bigcup_{i\in I} U_i\right) = \bigcup_{i\in I} f^{-1}(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(B)^{\times I}$. In particular, we have equalities

$$\begin{aligned} f^{-1}(U) \cup f^{-1}(V) &= f^{-1}(U \cup V), \\ f^{-1}(\emptyset) &= \emptyset, \end{aligned}$$

natural in $U, V \in \mathcal{P}(B)$.

4. *Preservation of Limits.* We have an equality of sets

$$f^{-1}\left(\bigcap_{i\in I} U_i\right) = \bigcap_{i\in I} f^{-1}(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(B)^{\times I}$. In particular, we have equalities

$$f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V),$$
$$f^{-1}(B) = A,$$

natural in $U, V \in \mathcal{P}(B)$.

5. *Symmetric Strict Monoidality With Respect to Unions.* The inverse image function of Item 1 has a symmetric strict monoidal structure

$$\left(f^{-1}, f^{-1,\otimes}, f^{-1,\otimes}_{\mathbb{1}}\right) : (\mathcal{P}(B), \cup, \emptyset) \to (\mathcal{P}(A), \cup, \emptyset),$$

being equipped with equalities

$$f^{-1,\otimes}_{U,V} : f^{-1}(U) \cup f^{-1}(V) \xrightarrow{=} f^{-1}(U \cup V),$$
$$f^{-1,\otimes}_{\mathbb{1}} : \emptyset \xrightarrow{=} f^{-1}(\emptyset),$$

natural in $U, V \in \mathcal{P}(B)$.

6. *Symmetric Strict Monoidality With Respect to Intersections.* The inverse image function of Item 1 has a symmetric strict monoidal structure

$$\left(f^{-1}, f^{-1,\otimes}, f^{-1,\otimes}_{\mathbb{1}}\right) : (\mathcal{P}(B), \cap, B) \to (\mathcal{P}(A), \cap, A),$$

being equipped with equalities

$$f^{-1,\otimes}_{U,V} : f^{-1}(U) \cap f^{-1}(V) \xrightarrow{=} f^{-1}(U \cap V),$$
$$f^{-1,\otimes}_{\mathbb{1}} : A \xrightarrow{=} f^{-1}(B),$$

natural in $U, V \in \mathcal{P}(B)$.

Proof. *Item 1, Functoriality*: Clear.
Item 2, Triple Adjointness: This follows from Kan Extensions, **??** of **??**.
Item 3, Preservation of Colimits: This follows from Item 2 and Categories, **??** of **??**.
Item 4, Preservation of Limits: This follows from Item 2 and Categories, **??** of **??**.
Item 5, Symmetric Strict Monoidality With Respect to Unions: This follows from Item 3.
Item 6, Symmetric Strict Monoidality With Respect to Intersections: This follows from Item 4. □

Proposition 4.4.1.4. Let $f : A \to B$ be a function.

1. *Functionality I.* The assignment $f \mapsto f^{-1}$ defines a function

$$(-)^{-1}_{A,B} : \mathsf{Sets}(A, B) \to \mathsf{Sets}(\mathcal{P}(B), \mathcal{P}(A)).$$

2. *Functionality II.* The assignment $f \mapsto f^{-1}$ defines a function

$$(-)^{-1}_{A,B} \colon \mathsf{Sets}(A, B) \to \mathsf{Pos}((\mathcal{P}(B), \subset), (\mathcal{P}(A), \subset)).$$

3. *Interaction With Identities.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$\mathrm{id}^{-1}_{A} = \mathrm{id}_{\mathcal{P}(A)};$$

4. *Interaction With Composition.* For each pair of composable functions $f \colon A \to B$ and $g \colon B \to C$, we have

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}, \qquad \begin{tikzcd} \mathcal{P}(C) \arrow[r, "g^{-1}"] \arrow[dr, "(g\circ f)^{-1}"'] & \mathcal{P}(B) \arrow[d, "f^{-1}"] \\ & \mathcal{P}(A). \end{tikzcd}$$

Proof. *Item 1*, *Functionality I*: Clear.
Item 2, *Functionality II*: Clear.
Item 3, *Interaction With Identities*: This follows from Categories, **??** of **??**.
Item 4, *Interaction With Composition*: This follows from Categories, **??** of **??**. □

4.5 Direct Images With Compact Support

Let A and B be sets and let $f \colon A \to B$ be a function.

Definition 4.5.1.1. The **direct image with compact support function associated to** f is the function[34]

$$f_! \colon \mathcal{P}(A) \to \mathcal{P}(B)$$

[34] *Further Notation:* Also written $\forall_f \colon \mathcal{P}(A) \to \mathcal{P}(B)$. This notation comes from the fact that the following statements are equivalent, where $b \in B$ and $U \in \mathcal{P}(A)$:

- We have $b \in \forall_f(U)$.
- For each $a \in A$, if $b = f(a)$, then $a \in U$.

defined by[35,36]

$$\begin{aligned} f_!(U) &\overset{\text{def}}{=} \left\{ b \in B \,\middle|\, \begin{array}{l} \text{for each } a \in A, \text{ if we have} \\ f(a) = b, \text{then } a \in U \end{array} \right\} \\ &= \left\{ b \in B \,\middle|\, \text{we have } f^{-1}(b) \subset U \right\} \end{aligned}$$

for each $U \in \mathcal{P}(A)$.

Remark 4.5.1.2. Identifying subsets of A with functions from A to $\{\text{true}, \text{false}\}$ via Item 6 of Proposition 4.2.1.3, we see that the direct image with compact support function associated to f is equivalently the function

$$f_! \colon \mathcal{P}(A) \to \mathcal{P}(B)$$

defined by

$$\begin{aligned} f_!(\chi_U) &\overset{\text{def}}{=} \mathrm{Ran}_f(\chi_U) \\ &= \lim\left(\left(\underline{(-_1)} \overset{\rightarrow}{\times} f\right) \overset{\text{pr}}{\twoheadrightarrow} A \xrightarrow{\chi_U} \{\text{true}, \text{false}\}\right) \\ &= \lim_{\substack{a \in A \\ f(a) = -_1}} (\chi_U(a)) \\ &= \bigwedge_{\substack{a \in A \\ f(a) = -_1}} (\chi_U(a)). \end{aligned}$$

So, in other words, we have

$$\begin{aligned} [f_!(\chi_U)](b) &= \bigwedge_{\substack{a \in A \\ f(a) = b}} (\chi_U(a)) \\ &= \begin{cases} \text{true} & \text{if, for each } a \in A \text{ such that} \\ & f(a) = b, \text{ we have } a \in U, \\ \text{false} & \text{otherwise} \end{cases} \\ &= \begin{cases} \text{true} & \text{if } f^{-1}(b) \subset U \\ \text{false} & \text{otherwise} \end{cases} \end{aligned}$$

for each $b \in B$.

[35] *Further Terminology:* The set $f_!(U)$ is called the **direct image with compact support of U by f**.

[36] We also have

$$f_!(U) = B \setminus f_*(A \setminus U);$$

Definition 4.5.1.3. Let U be a subset of A.[37,38]

1. The **image part of the direct image with compact support** $f_!(U)$ **of** U is the set $f_{!,\mathrm{im}}(U)$ defined by

$$\begin{aligned} f_{!,\mathrm{im}}(U) &\overset{\mathrm{def}}{=} f_!(U) \cap \mathrm{Im}(f) \\ &= \left\{ b \in B \,\middle|\, \begin{array}{l} \text{we have } f^{-1}(b) \subset U \\ \text{and } f^{-1}(b) \neq \emptyset \end{array} \right\}. \end{aligned}$$

2. The **complement part of the direct image with compact support** $f_!(U)$ **of** U is the set $f_{!,\mathrm{cp}}(U)$ defined by

$$\begin{aligned} f_{!,\mathrm{cp}}(U) &\overset{\mathrm{def}}{=} f_!(U) \cap (B \setminus \mathrm{Im}(f)) \\ &= B \setminus \mathrm{Im}(f) \\ &= \left\{ b \in B \,\middle|\, \begin{array}{l} \text{we have } f^{-1}(b) \subset U \\ \text{and } f^{-1}(b) = \emptyset \end{array} \right\} \\ &= \left\{ b \in B \,\middle|\, f^{-1}(b) = \emptyset \right\}. \end{aligned}$$

Example 4.5.1.4. Here are some examples of direct images with compact support.

1. *The Multiplication by Two Map on the Natural Numbers.* Consider the function $f \colon \mathbb{N} \to \mathbb{N}$ given by

$$f(n) \overset{\mathrm{def}}{=} 2n$$

see Item 7 of Proposition 4.5.1.5.

[37] Note that we have

$$f_!(U) = f_{!,\mathrm{im}}(U) \cup f_{!,\mathrm{cp}}(U),$$

as

$$\begin{aligned} f_!(U) &= f_!(U) \cap B \\ &= f_!(U) \cap (\mathrm{Im}(f) \cup (B \setminus \mathrm{Im}(f))) \\ &= (f_!(U) \cap \mathrm{Im}(f)) \cup (f_!(U) \cap (B \setminus \mathrm{Im}(f))) \\ &\overset{\mathrm{def}}{=} f_{!,\mathrm{im}}(U) \cup f_{!,\mathrm{cp}}(U). \end{aligned}$$

[38] In terms of the meet computation of $f_!(U)$ of Remark 4.5.1.2, namely

$$f_!(\chi_U) = \bigwedge_{\substack{a \in A \\ f(a) = -_1}} (\chi_U(a)),$$

we see that $f_{!,\mathrm{im}}$ corresponds to meets indexed over nonempty sets, while $f_{!,\mathrm{cp}}$ corresponds to meets indexed over the empty set.

for each $n \in \mathbb{N}$. Since f is injective, we have

$$f_{!,\mathrm{im}}(U) = f_*(U)$$
$$f_{!,\mathrm{cp}}(U) = \{\text{odd natural numbers}\}$$

for any $U \subset \mathbb{N}$.

2. *Parabolas.* Consider the function $f\colon \mathbb{R} \to \mathbb{R}$ given by

$$f(x) \overset{\text{def}}{=} x^2$$

for each $x \in \mathbb{R}$. We have

$$f_{!,\mathrm{cp}}(U) = \mathbb{R}_{<0}$$

for any $U \subset \mathbb{R}$. Moreover, since $f^{-1}(x) = \left\{-\sqrt{x}, \sqrt{x}\right\}$, we have e.g.:

$$\begin{aligned} f_{!,\mathrm{im}}([0,1]) &= \{0\}, \\ f_{!,\mathrm{im}}([-1,1]) &= [0,1], \\ f_{!,\mathrm{im}}([1,2]) &= \emptyset, \\ f_{!,\mathrm{im}}([-2,-1] \cup [1,2]) &= [1,4]. \end{aligned}$$

3. *Circles.* Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x,y) \overset{\text{def}}{=} x^2 + y^2$$

for each $(x,y) \in \mathbb{R}^2$. We have

$$f_{!,\mathrm{cp}}(U) = \mathbb{R}_{<0}$$

for any $U \subset \mathbb{R}^2$, and since

$$f^{-1}(r) = \begin{cases} \text{a circle of radius } r \text{ about the origin} & \text{if } r > 0, \\ \{(0,0)\} & \text{if } r = 0, \\ \emptyset & \text{if } r < 0, \end{cases}$$

we have e.g.:

$$\begin{aligned} f_{!,\mathrm{im}}([-1,1] \times [-1,1]) &= [0,1], \\ f_{!,\mathrm{im}}(([-1,1] \times [-1,1]) \setminus [-1,1] \times \{0\}) &= \emptyset. \end{aligned}$$

Proposition 4.5.1.5. Let $f\colon A \to B$ be a function.

1. *Functoriality.* The assignment $U \mapsto f_!(U)$ defines a functor

$$f_! \colon (\mathcal{P}(A), \subset) \to (\mathcal{P}(B), \subset)$$

where

- *Action on Objects.* For each $U \in \mathcal{P}(A)$, we have

$$[f_!](U) \overset{\text{def}}{=} f_!(U);$$

- *Action on Morphisms.* For each $U, V \in \mathcal{P}(A)$:

  $(\star)$ If $U \subset V$, then $f_!(U) \subset f_!(V)$.

2. *Triple Adjointness.* We have a triple adjunction

$$\left(f_* \dashv f^{-1} \dashv f_!\right)\colon \quad \begin{tikzcd} \mathcal{P}(A) \arrow[r, bend left=45, "f_*"] \arrow[r, phantom, bend left=22, "\perp"] \arrow[r, phantom, bend right=22, "\perp"] \arrow[r, bend right=45, "f_!"'] & \mathcal{P}(B), \arrow[l, "f^{-1}"'] \end{tikzcd}$$

witnessed by bijections of sets

$$\begin{aligned} \mathrm{Hom}_{\mathcal{P}(B)}(f_*(U), V) &\cong \mathrm{Hom}_{\mathcal{P}(A)}\left(U, f^{-1}(V)\right), \\ \mathrm{Hom}_{\mathcal{P}(A)}\left(f^{-1}(U), V\right) &\cong \mathrm{Hom}_{\mathcal{P}(A)}(U, f_!(V)), \end{aligned}$$

natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$ and (respectively) $V \in \mathcal{P}(A)$ and $U \in \mathcal{P}(B)$, i.e. where:

   (a) The following conditions are equivalent:
      i. We have $f_*(U) \subset V$;
      ii. We have $U \subset f^{-1}(V)$;

   (b) The following conditions are equivalent:
      i. We have $f^{-1}(U) \subset V$.
      ii. We have $U \subset f_!(V)$.

3. *Lax Preservation of Colimits.* We have an inclusion of sets

$$\bigcup_{i \in I} f_!(U_i) \subset f_!\left(\bigcup_{i \in I} U_i\right),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have inclusions

$$\begin{gathered} f_!(U) \cup f_!(V) \hookrightarrow f_!(U \cup V), \\ \emptyset \hookrightarrow f_!(\emptyset), \end{gathered}$$

natural in $U, V \in \mathcal{P}(A)$.

4. *Preservation of Limits.* We have an equality of sets

$$f_!\left(\bigcap_{i\in I} U_i\right) = \bigcap_{i\in I} f_!(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have equalities

$$\begin{gathered} f^{-1}(U \cap V) = f_!(U) \cap f^{-1}(V), \\ f_!(A) = B, \end{gathered}$$

natural in $U, V \in \mathcal{P}(A)$.

5. *Symmetric Lax Monoidality With Respect to Unions.* The direct image with compact support function of Item 1 has a symmetric lax monoidal structure

$$\left(f_!, f_!^{\otimes}, f_{!|\mathbb{1}}^{\otimes}\right) \colon (\mathcal{P}(A), \cup, \emptyset) \longrightarrow (\mathcal{P}(B), \cup, \emptyset),$$

being equipped with inclusions

$$\begin{gathered} f_{!|U,V}^{\otimes} \colon f_!(U) \cup f_!(V) \hookrightarrow f_!(U \cup V), \\ f_{!|\mathbb{1}}^{\otimes} \colon \emptyset \hookrightarrow f_!(\emptyset), \end{gathered}$$

natural in $U, V \in \mathcal{P}(A)$.

6. *Symmetric Strict Monoidality With Respect to Intersections.* The direct image function of Item 1 has a symmetric strict monoidal structure

$$\left(f_!, f_!^{\otimes}, f_{!|\mathbb{1}}^{\otimes}\right) \colon (\mathcal{P}(A), \cap, A) \longrightarrow (\mathcal{P}(B), \cap, B),$$

being equipped with equalities

$$\begin{gathered} f_{!|U,V}^{\otimes} \colon f_!(U \cap V) \xrightarrow{=} f_!(U) \cap f_!(V), \\ f_{!|\mathbb{1}}^{\otimes} \colon f_!(A) \xrightarrow{=} B, \end{gathered}$$

natural in $U, V \in \mathcal{P}(A)$.

7. *Relation to Direct Images.* We have

$$f_!(U) = B \setminus f_*(A \setminus U)$$

for each $U \in \mathcal{P}(A)$.

8. *Interaction With Injections.* If f is injective, then we have

$$\begin{aligned} f_{!,\mathrm{im}}(U) &= f_*(U), \\ f_{!,\mathrm{cp}}(U) &= B \setminus \mathrm{Im}(f), \\ f_!(U) &= f_{!,\mathrm{im}}(U) \cup f_{!,\mathrm{cp}}(U) \\ &= f_*(U) \cup (B \setminus \mathrm{Im}(f)) \end{aligned}$$

for each $U \in \mathcal{P}(A)$.

9. *Interaction With Surjections.* If f is surjective, then we have

$$\begin{aligned} f_{!,\mathrm{im}}(U) &\subset f_*(U), \\ f_{!,\mathrm{cp}}(U) &= \emptyset, \\ f_!(U) &\subset f_*(U) \end{aligned}$$

for each $U \in \mathcal{P}(A)$.

Proof. *Item 1*, *Functoriality*: Clear.
Item 2, *Triple Adjointness*: This follows from Kan Extensions, **??** of **??**.
Item 3, *Lax Preservation of Colimits*: Omitted.
Item 4, *Preservation of Limits*: Omitted. This follows from Item 2 and Categories, **??** of **??**.
Item 5, *Symmetric Lax Monoidality With Respect to Unions*: This follows from **??**.
Item 6, *Symmetric Strict Monoidality With Respect to Intersections*: This follows from Item 4.
Item 7, *Relation to Direct Images*: We claim that $f_!(U) = B \setminus f_*(A \setminus U)$.

- *The First Implication.* We claim that

  $$f_!(U) \subset B \setminus f_*(A \setminus U).$$

  Let $b \in f_!(U)$. We need to show that $b \notin f_*(A \setminus U)$, i.e. that there is no $a \in A \setminus U$ such that $f(a) = b$.

  This is indeed the case, as otherwise we would have $a \in f^{-1}(b)$ and $a \notin U$, contradicting $f^{-1}(b) \subset U$ (which holds since $b \in f_!(U)$).

  Thus $b \in B \setminus f_*(A \setminus U)$.

- *The Second Implication*. We claim that

$$B \setminus f_*(A \setminus U) \subset f_!(U).$$

  Let $b \in B \setminus f_*(A \setminus U)$. We need to show that $b \in f_!(U)$, i.e. that $f^{-1}(b) \subset U$.

  Since $b \notin f_*(A \setminus U)$, there exists no $a \in A \setminus U$ such that $b = f(a)$, and hence $f^{-1}(b) \subset U$.

  Thus $b \in f_!(U)$.

This finishes the proof of Item 7.
Item 8, *Interaction With Injections*: Clear.
Item 9, *Interaction With Surjections*: Clear. □

Proposition 4.5.1.6. Let $f: A \longrightarrow B$ be a function.

1. *Functionality I*. The assignment $f \mapsto f_!$ defines a function

$$(-)_{!|A,B}\colon \mathsf{Sets}(A, B) \longrightarrow \mathsf{Sets}(\mathcal{P}(A), \mathcal{P}(B)).$$

2. *Functionality II*. The assignment $f \mapsto f_!$ defines a function

$$(-)_{!|A,B}\colon \mathsf{Sets}(A, B) \longrightarrow \mathsf{Pos}((\mathcal{P}(A), \subset), (\mathcal{P}(B), \subset)).$$

3. *Interaction With Identities*. For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$(\mathrm{id}_A)_! = \mathrm{id}_{\mathcal{P}(A)};$$

4. *Interaction With Composition*. For each pair of composable functions $f: A \longrightarrow B$ and $g: B \longrightarrow C$, we have

$$(g \circ f)_! = g_! \circ f_!, \qquad \begin{tikzcd} \mathcal{P}(A) \arrow[r, "f_!"] \arrow[dr, "(g\circ f)_!"'] & \mathcal{P}(B) \arrow[d, "g_!"] \\ & \mathcal{P}(C). \end{tikzcd}$$

Proof. *Item 1*, *Functionality I*: Clear.
Item 2, *Functionality II*: Clear.
Item 3, *Interaction With Identities*: This follows from Kan Extensions, **??** of **??**.
Item 4, *Interaction With Composition*: This follows from Kan Extensions, **??** of **??**. □

Appendices

A Other Chapters

Set Theory

1. Sets
2. Constructions With Sets
3. Pointed Sets
4. Tensor Products of Pointed Sets
5. Indexed and Fibred Sets
6. Relations
7. Spans
8. Posets

Category Theory

9. Categories
10. Constructions With Categories
11. Kan Extensions

Bicategories

12. Bicategories
13. Internal Adjunctions

Internal Category Theory

14. Internal Categories

Cyclic Stuff

15. The Cycle Category

Cubical Stuff

16. The Cube Category

Globular Stuff

17. The Globe Category

Cellular Stuff

18. The Cell Category

Monoids

19. Monoids
20. Constructions With Monoids

Monoids With Zero

21. Monoids With Zero
22. Constructions With Monoids With Zero

Groups

23. Groups
24. Constructions With Groups

Hyper Algebra

25. Hypermonoids
26. Hypergroups
27. Hypersemirings and Hyperrings
28. Quantales

Near-Rings

29. Near-Semirings
30. Near-Rings

Real Analysis

31. Real Analysis in One Variable
32. Real Analysis in Several Variables

Measure Theory

33. Measurable Spaces

34. Measures and Integration

Probability Theory

34. Probability Theory

Stochastic Analysis

35. Stochastic Processes, Martingales, and Brownian Motion

36. Itô Calculus

37. Stochastic Differential Equations

Differential Geometry

38. Topological and Smooth Manifolds

Schemes

39. Schemes