# Sets

#### December 24, 2023

This chapter (will eventually) contain material on axiomatic set theory, as well as a couple other things.

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# 1 The Enrichment of Sets in Classical Truth Values

#### 1.1 (-2)-Categories

#### **DEFINITION 1.1.1** $\blacktriangleright$ (-2)-CATEGORIES

A (-2)-category is the "necessarily true" truth value.<sup>1,2,3</sup>

# 1.2 (-1)-Categories

<sup>&</sup>lt;sup>1</sup>Thus, there is only one (-2)-category.

 $<sup>^2</sup>$ A (-n)-category for  $n=3,4,\ldots$  is also the "necessarily true" truth value, coinciding with a (-2)-category.

<sup>&</sup>lt;sup>3</sup>For motivation, see [BS10, p. 13].

#### **DEFINITION 1.2.1** $\triangleright$ (-1)-CATEGORIES

A (-1)-category is a classical truth value.

#### REMARK 1.2.2 $\blacktriangleright$ MOTIVATION FOR (-1)-CATEGORIES

 $^{1}(-1)$ -categories should be thought of as being "categories enriched in (-2)-categories", having a collection of objects and, for each pair of objects, a Homobject Hom(x,y) that is a (-2)-category (i.e. trivial).

Therefore, a (-1)-category C is either ([BS10, pp. 33–34]):

- 1. Empty, having no objects;
- 2. Contractible, having a collection of objects  $\{a,b,c,\ldots\}$ , but with  $\operatorname{Hom}_C(a,b)$  being a (-2)-category (i.e. trivial) for all  $a,b\in\operatorname{Obj}(C)$ , forcing all objects of C to be uniquely isomorphic to each other.

As such, there are only two (-1)-categories, up to equivalence:

- · The (-1)-category false (the empty one);
- · The (-1)-category true (the contractible one).

#### **DEFINITION 1.2.3** ► THE POSET OF TRUTH VALUES

The **poset of truth values**<sup>1</sup> is the poset  $(\{\text{true}, \text{false}\}, \leq)^2 \text{ consisting of }$ 

- The Underlying Set. The set {true, false} whose elements are the truth values true and false;
- · The Partial Order. The partial order

```
\leq: {true, false} \times {true, false} \rightarrow {true, false}
```

on {true, false} defined by<sup>3</sup>

```
false \leq false \stackrel{\text{def}}{=} true,
true \leq false \stackrel{\text{def}}{=} false,
false \leq true \stackrel{\text{def}}{=} true,
true \leq true \stackrel{\text{def}}{=} true.
```

<sup>&</sup>lt;sup>1</sup>For more motivation, see [BS10, p. 13].

(−1)-Categories 1.2

<sup>1</sup> Further Terminology: Also called the **poset of** (-1)-categories.

<sup>2</sup> Further Notation: Also written  $\{t, f\}$ .

<sup>3</sup>This partial order coincides with logical implication.

### PROPOSITION 1.2.4 ► CARTESIAN CLOSEDNESS OF THE POSET OF TRUTH VALUES

The poset of truth values  $\{t, f\}$  is Cartesian closed with product given by

$$t \times t = t$$
,

$$t \times f = f$$

$$f \times t = f$$

$$f \times f = f$$
.

and internal Hom  $\textbf{Hom}_{\{t,f\}}$  given by the partial order of  $\{t,f\},$  i.e. by

$$Hom_{\{t,f\}}(t,t)=t,$$

$$Hom_{\{t,f\}}(t,f) = f,$$

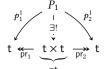
$$Hom_{\{t,f\}}(f,t) = t,$$

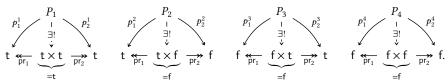
$$Hom_{\{t,f\}}(f,f) = t.$$

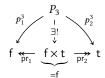
#### PROOF 1.2.5 ► PROOF OF PROPOSITION 1.2.4

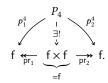
#### **Existence of Products**

We claim that the products  $t \times t$ ,  $t \times f$ ,  $f \times t$ , and  $f \times f$  satisfy the universal property of the product in {t, f}. Indeed, consider the diagrams









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Here:

1. If  $P_1 = t$ , then  $p_1^1 = p_2^1 = id_t$ , and there's indeed a unique morphism from  $P_1$  to t making the diagram commute, namely id<sub>t</sub>;

 $<sup>^1</sup>$ Note that  $\times$  coincides with the "and" operator, while  $\mathbf{Hom}_{\{\mathbf{t},\mathbf{f}\}}$  coincides with the logical implication operator.

- 2. If  $P_1 = f$ , then  $p_1^1 = p_2^1$  are given by the unique morphism from f to t, and there's indeed a unique morphism from  $P_1$  to t making the diagram commute, namely the unique morphism from f to t;
- 3. If  $P_2 = t$ , then there is no morphism  $p_2^2$ .
- 4. If  $P_2 = f$ , then  $p_1^2$  is the unique morphism from f to t while  $p_2^2 = id_f$ , and there's indeed a unique morphism from  $P_2$  to f making the diagram commute, namely  $id_f$ ;
- 5. The proof for  $P_3$  is similar to the one for  $P_2$ ;
- 6. If  $P_4 = t$ , then there is no morphism  $p_1^4$  or  $p_2^4$ .
- 7. If  $P_4 = f$ , then  $p_1^4 = p_2^4 = id_f$ , and there's indeed a unique morphism from  $P_4$  to f making the diagram commute, namely  $id_f$ .

#### Cartesian Closedness

We claim there's a bijection

$$\operatorname{Hom}_{\{\mathsf{t},\mathsf{f}\}}(A \times B,C) \cong \operatorname{Hom}_{\{\mathsf{t},\mathsf{f}\}}(A,\operatorname{Hom}_{\{\mathsf{t},\mathsf{f}\}}(B,C))$$

natural in  $A, B, C \in \{t, f\}$ . Indeed:

· For (A, B, C) = (t, t, t), we have

$$\begin{split} \mathsf{Hom}_{\{t,f\}}(t\times t,t) &\cong \mathsf{Hom}_{\{t,f\}}(t,t) \\ &= \{\mathsf{id}_{\mathsf{true}}\} \\ &\cong \mathsf{Hom}_{\{t,f\}}(t,t) \\ &\cong \mathsf{Hom}_{\{t,f\}}\big(t,\textbf{Hom}_{\{t,f\}}(t,t)\big). \end{split}$$

· For (A, B, C) = (t, t, f), we have

$$\begin{split} \mathsf{Hom}_{\{t,f\}}(t\times t,f) &\cong \mathsf{Hom}_{\{t,f\}}(t,f) \\ &= \emptyset \\ &\cong \mathsf{Hom}_{\{t,f\}}(t,f) \\ &\cong \mathsf{Hom}_{\{t,f\}}\big(t,\textbf{Hom}_{\{t,f\}}(t,f)\big). \end{split}$$

$$\begin{split} & \quad \text{For } (A,B,C) = (\mathsf{t},\mathsf{f},\mathsf{t}), \text{ we have} \\ & \quad \quad \text{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t}\times\mathsf{f},\mathsf{t}) \cong \text{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f},\mathsf{t}) \\ & \cong \mathsf{pt} \\ & \cong \text{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f},\mathsf{t}) \\ & \cong \text{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f},\mathsf{t}) \\ & \cong \text{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f},\mathsf{f}) \\ & \cong \text{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f},\mathsf{f}) \\ & \cong \{\mathsf{id}_{\mathsf{false}}\} \\ & \cong \text{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f},\mathsf{f}) \\ & \cong \text{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f},\mathsf{f}) \\ & \cong \text{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f},\mathsf{t}) \\ & \cong \mathsf{pt} \\ & \cong \text{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f},\mathsf{t}) \\ & \cong \text{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f},\mathsf{t}) \\ & \cong \text{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f},\mathsf{f}) \\ & \cong \text{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f},\mathsf{f})$$

1.3 0-Categories 6

#### 1.3 0-Categories

#### **DEFINITION 1.3.1** ► 0-CATEGORIES

A 0-category is a poset.<sup>1</sup>

<sup>1</sup> Motivation: A 0-category is precisely a category enriched in the poset of (-1)-categories.

#### **DEFINITION 1.3.2** ► 0-GROUPOIDS

A 0-groupoid is a 0-category in which every morphism is invertible.<sup>1</sup>

<sup>1</sup>That is, a set.

#### 1.4 Tables of Analogies Between Set Theory and Category Theory

Here we record some analogies between notions in set theory and category theory. Note that the analogies relating to presheaves relate equally well to copresheaves, as the opposite  $X^{\mathrm{op}}$  of a set X is just X again. Basics:

Set Theory	Category Theory
Enrichment in {true, false}	Enrichment in Sets
Set X	Category ${\mathcal C}$
Element $x \in X$	$ObjectX \in Obj(\mathcal{C})$
Function	Functor
Function $X \to \{\text{true}, \text{false}\}$	Functor $C \rightarrow Sets$
Function $X \to \{\text{true}, \text{false}\}$	Presheaf $C^{op} \rightarrow Sets$

Powersets and categories of presheaves:

SET THEORY	CATEGORY THEORY
Powerset $\mathcal{P}(X)$	Presheaf category $PSh(C)$
Characteristic function $\chi_{\{x\}}$	Representable presheaf $h_X$
Characteristic embedding $\chi_{(-)} \colon X \hookrightarrow \mathcal{P}(X)$	Yoneda embedding $\mathcal{L}: C^{\mathrm{op}} \hookrightarrow PSh(C)$
Characteristic relation $\chi_X(1,2)$	Hom profunctor $\operatorname{Hom}_{\mathcal{C}}(-_1,-_2)$
The Yoneda lemma for sets $\operatorname{Hom}_{\mathcal{P}(X)}(\chi_x,\chi_U)=\chi_U(x)$	The Yoneda lemma for categories $\operatorname{Nat}(h_X,\mathcal{F})\cong\mathcal{F}(X)$
The characteristic embedding is fully faithful, $\operatorname{Hom}_{\mathcal{P}(X)} \left( \chi_x, \chi_y \right) = \chi_X(x,y)$	The Yoneda embedding is fully faithful, $\operatorname{Nat}(h_X,h_Y)\cong\operatorname{Hom}_C(X,Y)$
Subsets are unions of their elements $U = \bigcup_{x \in U} \{x\}$ or $\chi_U = \operatorname*{colim}_{\chi_x \in Sets(U, \{t, f\})} (\chi_x)$	Presheaves are colimits of representables, $\mathcal{F}\cong \operatornamewithlimits{colim}_{h_X\in\int_{\mathcal{C}}\mathcal{F}}(h_X)$

# Categories of elements:

SET THEORY	Category Theory
Assignment $U\mapsto \chi_U$	Assignment $\mathcal{F} \mapsto \int_{\mathcal{C}} \mathcal{F}$ (the category of elements)
Assignment $U \mapsto \chi_U$ giving an isomorphism $\mathcal{P}(X) \cong Sets(X, \{t, f\})$	Assignment $\mathcal{F} \mapsto \int_{\mathcal{C}} \mathcal{F}$ giving an equivalence $PSh(\mathcal{C}) \stackrel{\mathrm{eq.}}{\cong} DFib(\mathcal{C})$

Functions between powersets and functors between presheaf categories:

SET THEORY	Category Theory
Direct image function $f_* \colon \mathcal{P}(X) \to \mathcal{P}(Y)$	Inverse image functor $f^{-1} \colon PSh(C) \to PSh(\mathcal{D})$
Inverse image function $f^{-1} \colon \mathcal{P}(Y) \to \mathcal{P}(X)$	Direct image functor $f_* \colon PSh(\mathcal{D}) \to PSh(C)$
Direct image with compact support function $f_! \colon \mathcal{P}(X) \to \mathcal{P}(Y)$	Direct image with compact support functor $f_i \colon PSh(\mathcal{C}) \to PSh(\mathcal{D})$

#### Relations and profunctors:

SET THEORY	Category Theory
Relation $R: X \times Y \to \{t, f\}$	Profunctor $\mathfrak{p} \colon \mathcal{D}^{op} \times \mathcal{C} \to Sets$
Relation $R: X \to \mathcal{P}(Y)$	$Profunctor \mathfrak{p} \colon \mathcal{C} \to PSh(\mathcal{D})$
Relation as a cocontinuous morphism of posets $R\colon (\mathcal{P}(X),\subset) \to (\mathcal{P}(Y),\subset)$	Profunctor as a colimit-preserving functor $\mathfrak{p} \colon PSh(\mathcal{C}) \to PSh(\mathcal{D})$

# **Appendices**

# A Other Chapters

#### Sets

- 1. Sets
- 2. Constructions With Sets
- 3. Pointed Sets
- 4. Tensor Products of Pointed Sets
- 5. Relations
- 6. Spans
- 7. Posets
- **Indexed and Fibred Sets**

- 7. Indexed Sets
- 8. Fibred Sets
- 9. Un/Straightening for Indexed and Fibred Sets

#### **Category Theory**

- 11. Categories
- 12. Types of Morphisms in Categories
- 13. Adjunctions and the Yoneda Lemma
- 14. Constructions With Categories
- 15. Kan Extensions

#### **Bicategories**

- 17. Bicategories
- 18. Internal Adjunctions

#### **Internal Category Theory**

19. Internal Categories

#### **Cyclic Stuff**

20. The Cycle Category

#### **Cubical Stuff**

21. The Cube Category

#### Globular Stuff

22. The Globe Category

#### Cellular Stuff

23. The Cell Category

#### Monoids

- 24. Monoids
- 25. Constructions With Monoids

#### Monoids With Zero

- 26. Monoids With Zero
- 27. Constructions With Monoids With Zero

#### Groups

- 28. Groups
- 29. Constructions With Groups

#### Hyper Algebra

30. Hypermonoids

- 31. Hypergroups
- 32. Hypersemirings and Hyperrings
- 33. Quantales

#### **Near-Rings**

- 34. Near-Semirings
- 35. Near-Rings

#### **Real Analysis**

- 36. Real Analysis in One Variable
- 37. Real Analysis in Several Variables

#### **Measure Theory**

- 38. Measurable Spaces
- 39. Measures and Integration

#### **Probability Theory**

39. Probability Theory

#### Stochastic Analysis

- 40. Stochastic Processes, Martingales, and Brownian Motion
- 41. Itô Calculus
- 42. Stochastic Differential Equations

#### **Differential Geometry**

43. Topological and Smooth Manifolds

#### Schemes

44. Schemes