Tensor Products of Pointed Sets

December 3, 2023

008H This chapter contains some material on tensor products of pointed sets.

Contents

008J 1 Bilinear Morphisms of Pointed Sets

008K 1.1 Left Bilinear Morphisms of Pointed Sets

Let (X, x_0), (Y, y_0), and (Z, z_0) be pointed sets.

008L **Definition 1.1.1 ► Left Bilinear Morphisms of pointed sets**

A **left bilinear morphism of pointed sets from** $(X \times Y, (x_0, y_0))$ **to** (Z, z_0) is a map of sets

$$f\colon (X \times Y, (x_0, y_0)) \longrightarrow (Z, z_0)$$

satisfying the following condition:[1,2]

($\star$) *Left Unital Bilinearity.* The diagram

$$\begin{tikzcd}
 & \mathrm{pt}\times\mathrm{pt} \arrow[dr, dashed, "\sim"] & \\
\mathrm{pt}\times Y \arrow[ur, "\mathrm{id}_{\mathrm{pt}}\times\epsilon_Y"] \arrow[d, "{[x_0]\times\mathrm{id}_Y}"'] & & \mathrm{pt} \arrow[d, "{[z_0]}"] \\
X\times Y \arrow[rr, "f"'] & & Z
\end{tikzcd}$$

commutes, i.e. for each $y \in Y$, we have

$$f(x_0, y) = z_0.$$

[1] *Slogan:* f is left bilinear if it preserves basepoints in its first argument.

[2] Succinctly, f is bilinear if we have

$$f(x_0, y) = z_0$$

for each $y \in Y$.

008M **Definition 1.1.2 ▶ The Set of Left Bilinear Morphisms of Pointed Sets**

The **set of left bilinear morphisms of pointed sets from** $(X \times Y, (x_0, y_0))$ **to** (Z, z_0) is the set $\mathrm{Hom}^{\otimes,\mathrm{L}}_{\mathsf{Sets}_*}(X \times Y, Z)$ defined by

$$\mathrm{Hom}^{\otimes,\mathrm{L}}_{\mathsf{Sets}_*}(X \times Y, Z) \overset{\mathrm{def}}{=} \{f \in \mathsf{Sets}_*(A \times B, C) \mid f \text{ is left bilinear}\}.$$

008N 1.2 Right Bilinear Morphisms of Pointed Sets

Let (X, x_0), (Y, y_0), and (Z, z_0) be pointed sets.

008P **Definition 1.2.1 ▶ Right Bilinear Morphisms of pointed sets**

A **right bilinear morphism of pointed sets from** $(X \times Y, (x_0, y_0))$ **to** (Z, z_0) is a map of sets

$$f\colon (X \times Y, (x_0, y_0)) \to (Z, z_0)$$

satisfying the following condition:[1,2]

($\star$) *Right Unital Bilinearity.* The diagram

$$\begin{tikzcd}
& \mathrm{pt}\times\mathrm{pt} \arrow[dr, dashed, "\sim"] & \\
X\times\mathrm{pt} \arrow[ur, "\epsilon_X\times\mathrm{id}_{\mathrm{pt}}"] \arrow[d, "\mathrm{id}_X\times[y_0]"'] & & \mathrm{pt} \arrow[d, "{[z_0]}"] \\
X\times Y \arrow[rr, "f"'] & & Z
\end{tikzcd}$$

commutes, i.e. for each $x \in X$, we have

$$f(x, y_0) = z_0.$$

[1] *Slogan:* f is right bilinear if it preserves basepoints in its second argument.

[2] Succinctly, f is bilinear if we have

$$f(x, y_0) = z_0$$

for each $x \in X$.

008Q **DEFINITION 1.2.2 ► THE SET OF RIGHT BILINEAR MORPHISMS OF POINTED SETS**

The **set of right bilinear morphisms of pointed sets from** $(X \times Y, (x_0, y_0))$ **to** (Z, z_0) is the set $\mathrm{Hom}^{\otimes,\mathrm{R}}_{\mathsf{Sets}_*}(X \times Y, Z)$ defined by

$$\mathrm{Hom}^{\otimes,\mathrm{R}}_{\mathsf{Sets}_*}(X \times Y, Z) \overset{\mathrm{def}}{=} \{f \in \mathsf{Sets}_*(A \times B, C) \mid f \text{ is right bilinear}\}.$$

008R 1.3 Bilinear Morphisms of Pointed Sets

Let (X, x_0), (Y, y_0), and (Z, z_0) be pointed sets.

008S **DEFINITION 1.3.1 ► BILINEAR MORPHISMS OF POINTED SETS**

A **bilinear morphism of pointed sets from** $(X \times Y, (x_0, y_0))$ **to** (Z, z_0) is a map of sets

$$f : (X \times Y, (x_0, y_0)) \longrightarrow (Z, z_0)$$

that is both left bilinear and right bilinear.

008T **REMARK 1.3.2 ► UNWINDING DEFINITION 1.3.1**

In detail, a **bilinear morphism of pointed sets from** $(X \times Y, (x_0, y_0))$ **to** (Z, z_0) is a map of sets

$$f : (X \times Y, (x_0, y_0)) \longrightarrow (Z, z_0)$$

satisfying the following conditions:[1,2]

1. *Left Unital Bilinearity.* The diagram

$$\begin{tikzcd}
& \mathrm{pt} \times \mathrm{pt} \arrow[dr, dashed, "\sim"] & \\
\mathrm{pt} \times Y \arrow[ur, "\mathrm{id}_{\mathrm{pt}} \times \epsilon_Y"] \arrow[d, "{[x_0] \times \mathrm{id}_Y}"'] & & \mathrm{pt} \arrow[d, "{[z_0]}"] \\
X \times Y \arrow[rr, "f"'] & & Z
\end{tikzcd}$$

   commutes, i.e. for each $y \in Y$, we have

$$f(x_0, y) = z_0.$$

2. *Right Unital Bilinearity.* The diagram

$$\begin{tikzcd}
 & \mathrm{pt}\times\mathrm{pt} \arrow[dr, dashed, "\sim"] & \\
X\times\mathrm{pt} \arrow[ur, "\epsilon_X\times\mathrm{id}_{\mathrm{pt}}"] \arrow[d, "\mathrm{id}_X\times[y_0]"'] & & \mathrm{pt} \arrow[d, "{[z_0]}"] \\
X\times Y \arrow[rr, "f"'] & & Z
\end{tikzcd}$$

commutes, i.e. for each $x \in X$, we have

$$f(x, y_0) = z_0.$$

[1] *Slogan:* f is bilinear if it preserves basepoints in each argument.

[2] Succinctly, f is bilinear if we have

$$\begin{aligned} f(x_0, y) &= z_0, \\ f(x, y_0) &= z_0 \end{aligned}$$

for each $x \in X$ and each $y \in Y$.

008U **Definition 1.3.3 ► The Set of Bilinear Morphisms of Pointed Sets**

The **set of bilinear morphisms of pointed sets from** $(X \times Y, (x_0, y_0))$ **to** (Z, z_0) is the set $\mathrm{Hom}^{\otimes}_{\mathsf{Sets}_*}(X \times Y, Z)$ defined by

$$\mathrm{Hom}^{\otimes}_{\mathsf{Sets}_*}(X \times Y, Z) \overset{\text{def}}{=} \{f \in \mathsf{Sets}_*(A \times B, C) \mid f \text{ is bilinear}\}.$$

008V 2 Tensors and Cotensors of Pointed Sets by Sets

008W 2.1 Tensors of Pointed Sets by Sets

Let (X, x_0) be a pointed set and let A be a set.

008X **Definition 2.1.1 ▶ Tensors of Pointed Sets by Sets**

The **tensor of** (X, x_0) **by** A is the pointed set $A \odot (X, x_0)$ satisfying the following universal property:

(UP) We have a bijection

$$\mathsf{Sets}_*(A \odot X, K) \cong \mathsf{Sets}(A, \mathbf{Sets}_*(X, K)),$$

natural in $(K, k_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

008Y **Remark 2.1.2 ▶ Unwinding Definition 2.1.1**

The tensor of (X, x_0) by A satisfies the following universal property:

$$\mathsf{Sets}_*(A \odot X, K) \cong \mathsf{Sets}^{\otimes}_{\mathbb{E}_0}(A \times X, K),$$

where $\mathsf{Sets}^{\otimes}_{\mathbb{E}_0}(A \times X, K)$ is the set defined by

$$\mathsf{Sets}^{\otimes}_{\mathbb{E}_0}(A \times X, K) \stackrel{\text{def}}{=} \left\{ f \in \mathsf{Sets}(A \times X, K) \,\middle|\, \begin{array}{l} \text{for each } a \in A, \text{ we have} \\ f(a, x_0) = k_0 \end{array} \right\}.$$

008Z **Construction 2.1.3 ▶ Construction of Tensors of Pointed Sets by Sets**

Concretely, the **tensor of** (X, x_0) **by** A is the pointed set $A \odot (X, x_0)$ consisting of

- *The Underlying Set.* The set $A \odot X$ given by

$$A \odot X \cong \bigvee_{a \in A} (X, x_0);$$

- *The Basepoint.* The point $[x_0]$ of $\bigvee_{a \in A}(X, x_0)$.

0090 2.2 Cotensors of Pointed Sets by Sets

Let (X, x_0) be a pointed set and let A be a set.

0091 **Definition 2.2.1 ▶ Cotensors of Pointed Sets by Sets**

The **cotensor of** (X, x_0) **by** A is the pointed set $A \pitchfork (X, x_0)$ satisfying the following universal property:

(UP) We have a bijection

$$\mathsf{Sets}_*(K, A \pitchfork X) \cong \mathsf{Sets}(A, \mathbf{Sets}_*(K, X)),$$

natural in $(K, k_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

0092 **Remark 2.2.2 ▶ Unwinding Definition 2.2.1**

The cotensor of (X, x_0) by A satisfies the following universal property:

$$\mathsf{Sets}_*(K, A \pitchfork X) \cong \mathsf{Sets}^{\otimes}_{\mathbb{E}_0}(A \times K, X),$$

where $\mathsf{Sets}^{\otimes}_{\mathbb{E}_0}(A \times K, X)$ is the set defined by

$$\mathsf{Sets}^{\otimes}_{\mathbb{E}_0}(A \times K, X) \overset{\mathrm{def}}{=} \left\{ f \in \mathsf{Sets}(A \times K, X) \;\middle|\; \begin{array}{l}\text{for each } a \in A, \text{ we have}\\ f(a, k_0) = x_0\end{array} \right\}.$$

0093 **Construction 2.2.3 ▶ Construction of Cotensors of Pointed Sets by Sets**

Concretely, the **cotensor of** (X, x_0) **by** A is the pointed set $A \pitchfork (X, x_0)$ consisting of

- *The Underlying Set.* The set $A \pitchfork X$ given by

$$A \pitchfork X \cong \bigwedge_{a \in A} (X, x_0);$$

- *The Basepoint.* The point $[(x_0, x_0, x_0, \ldots)]$ of $\bigwedge_{a \in A}(X, x_0)$.

0094 3 The Left Tensor Product of Pointed Sets

0095 3.1 Foundations

Let (X, x_0) and (Y, y_0) be pointed sets.

0096 **Definition 3.1.1 ▶ The Left Tensor Product of Pointed Sets**

The **left tensor product of pointed sets** is the functor

$$\lhd_{\mathsf{Sets}_*} \colon \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*$$

defined as the composition

$$\mathsf{Sets}_* \times \mathsf{Sets}_* \xrightarrow{\mathrm{id} \times 忘} \mathsf{Sets}_* \times \mathsf{Sets} \xrightarrow{\beta^{\mathsf{Cats}_2}_{\mathsf{Sets}_*, \mathsf{Sets}}} \mathsf{Sets} \times \mathsf{Sets}_* \xrightarrow{\odot} \mathsf{Sets}_*.$$

0097 **Remark 3.1.2 ▶ Unwinding Definition 3.1.1, I: Universal Property**

The left tensor product of pointed sets satisfies the following universal property:[1]

$$\mathsf{Sets}_*\left(X \lhd_{\mathsf{Sets}_*} Y, Z\right) \cong \mathrm{Hom}^{\otimes, \mathsf{L}}_{\mathsf{Sets}_*}(X \times Y, Z).$$

[1] Namely, a pointed map $f \colon X \lhd_{\mathsf{Sets}_*} Y \to Z$ is the same as a map $f^\dagger \colon X \times Y \to Z$ such that

$$f^\dagger(x_0, y) = z_0$$

for each $y \in Y$.

0098 **Remark 3.1.3 ▶ Unwinding Definition 3.1.1, II: Explicit Description**

In detail, the **left tensor product of** (X, x_0) **and** (Y, y_0) is the pointed set $\left(X \lhd_{\mathsf{Sets}_*} Y, [x_0]\right)$ consisting of[1]

- *The Underlying Set.* The set $X \lhd_{\mathsf{Sets}_*} Y$ defined by

$$\begin{aligned} X \lhd_{\mathsf{Sets}_*} Y &\stackrel{\text{def}}{=} |Y| \odot X \\ &\cong \bigvee_{y \in Y} (X, x_0); \end{aligned}$$

- *The Underlying Basepoint.* The point $[x_0]$ of $\bigvee_{y \in Y}(X, x_0)$.

[1] *Further Notation:* We write $x \lhd_{\mathsf{Sets}_*} y$ for the image of (x, y) under the map

$$X \times Y \to \underbrace{X \lhd_{\mathsf{Sets}_*} Y}_{\cong \bigvee_{y \in Y}(X, x_0)}.$$

sending (x, y) to the element $x \in X$ in the yth copy of X in $\bigvee_{y \in Y}(X, x_0)$. Note that we have

$$x_0 \lhd_{\mathsf{Sets}_*} y = x_0 \lhd_{\mathsf{Sets}_*} y',$$

for each $y, y' \in Y$.

0099 **Proposition 3.1.4 ► Properties of Left Tensor Products of Pointed Sets**

Let (X, x_0) and (Y, y_0) be pointed sets.

009A 1. *Functoriality.* The assignments $X, Y, (X, Y) \mapsto X \lhd_{\mathsf{Sets}_*} Y$ define functors

$$\begin{aligned} X \lhd_{\mathsf{Sets}_*} - &\colon \mathsf{Sets}_* \longrightarrow \mathsf{Sets}_*, \\ - \lhd_{\mathsf{Sets}_*} Y &\colon \mathsf{Sets}_* \longrightarrow \mathsf{Sets}_*, \\ -_1 \lhd_{\mathsf{Sets}_*} -_2 &\colon \mathsf{Sets}_* \times \mathsf{Sets}_* \longrightarrow \mathsf{Sets}_*. \end{aligned}$$

Proof 3.1.5 ► Proof of Proposition 3.1.4

Item 1: Functoriality

Omitted. □

009B
3.2 The Skew Associator

009C **Definition 3.2.1 ► The Skew Associator of $\lhd_{\mathsf{Sets}_*}$**

The **skew associator of the left tensor product of pointed sets** is the natural isomorphism

$$\alpha^{\mathsf{Sets}_*, \lhd} \colon \lhd_{\mathsf{Sets}_*} \circ \left(\lhd_{\mathsf{Sets}_*} \times \mathrm{id}_{\mathsf{Sets}_*}\right) \overset{\cong}{\Longrightarrow} \lhd_{\mathsf{Sets}_*} \circ \left(\mathrm{id}_{\mathsf{Sets}_*} \times \lhd_{\mathsf{Sets}_*}\right),$$

whose component

$$\alpha^{\mathsf{Sets}_*, \lhd}_{X,Y,Z} \colon \left(X \lhd_{\mathsf{Sets}_*} Y\right) \lhd_{\mathsf{Sets}_*} Z \xrightarrow{\cong} X \lhd_{\mathsf{Sets}_*} \left(Y \lhd_{\mathsf{Sets}_*} Z\right)$$

at (X, Y, Z) is given by the composition[1]

$$
\begin{aligned}
\left(X \lhd_{\mathsf{Sets}_*} Y\right) \lhd_{\mathsf{Sets}_*} Z &\overset{\text{def}}{=} |Z| \odot \left(X \lhd_{\mathsf{Sets}_*} Y\right) \\
&\overset{\text{def}}{=} |Z| \odot (|Y| \odot X) \\
&\cong \bigvee_{z \in Z} (|Y| \odot X, [x_0]) \\
&\overset{\text{def}}{=} \bigvee_{z \in Z} \left(\bigvee_{y \in Y} (X, x_0) \right) \\
&\cong \bigvee_{(z,y) \in \bigvee_{z \in Z}(Y, y_0)} (X, x_0) \\
&\overset{\text{def}}{=} \bigvee_{(z,y) \in |Z| \odot Y} (X, x_0) \\
&\cong ||Z| \odot Y| \odot X \\
&\overset{\text{def}}{=} |Y \lhd_{\mathsf{Sets}_*} Z| \odot X \\
&\overset{\text{def}}{=} X \lhd_{\mathsf{Sets}_*} \left(Y \lhd_{\mathsf{Sets}_*} Z\right),
\end{aligned}
$$

where the isomorphism

$$
\bigvee_{z \in Z} \left(\bigvee_{y \in Y} (X, x_0) \right) \cong \bigvee_{(y,z) \in \bigvee_{z \in Z}(Y, y_0)} (X, x_0)
$$

is given by $[(z, (y, x))] \mapsto [((z, y), x)]$.

[1] In other words, $\alpha^{\mathsf{Sets}_*, \lhd}_{X,Y,Z}$ acts on elements as

$$
\alpha^{\mathsf{Sets}_*, \lhd}_{X,Y,Z}\left(\left(x \lhd_{\mathsf{Sets}_*} y\right) \lhd_{\mathsf{Sets}_*} z\right) \overset{\text{def}}{=} x \lhd_{\mathsf{Sets}_*} \left(y \lhd_{\mathsf{Sets}_*} z\right)
$$

for each $\left(x \lhd_{\mathsf{Sets}_*} y\right) \lhd_{\mathsf{Sets}_*} z \in \left(X \lhd_{\mathsf{Sets}_*} Y\right) \lhd_{\mathsf{Sets}_*} Z$.

009D 3.3 The Skew Left Unitor

009E **Definition 3.3.1 ► The Skew Left Unitor of $\lhd_{\mathsf{Sets}_*}$**

The **skew left unitor of the left tensor product of pointed sets** is the natural transformation

$$\lambda^{\mathsf{Sets}_*,\lhd}\colon \lhd_{\mathsf{Sets}_*} \circ \left(\mathbb{1}^{\mathsf{Sets}_*} \times \mathrm{id}_{\mathsf{Sets}_*}\right) \Longrightarrow \mathrm{id}_{\mathsf{Sets}_*},$$

whose component

$$\lambda^{\mathsf{Sets}_*,\lhd}_X\colon S^0 \lhd_{\mathsf{Sets}_*} X \to X$$

at X is given by the composition[1]

$$\begin{aligned} S^0 \lhd_{\mathsf{Sets}_*} X &\cong |X| \odot S^0 \\ &\cong \bigvee_{x\in X} S^0 \\ &\to X \end{aligned}$$

where $\bigvee_{x\in X} S^0 \to X$ is the map given by

$$\begin{aligned} (x,0) &\mapsto x, \\ (x,1) &\mapsto x. \end{aligned}$$

[1] In other words, $\lambda^{\mathsf{Sets}_*,\lhd}_X$ acts on elements as

$$\begin{aligned} \lambda^{\mathsf{Sets}_*,\lhd}_X(x \lhd_{\mathsf{Sets}_*} 0) &\overset{\mathrm{def}}{=} x, \\ \lambda^{\mathsf{Sets}_*,\lhd}_X(x \lhd_{\mathsf{Sets}_*} 1) &\overset{\mathrm{def}}{=} x, \end{aligned}$$

for each $x \in X$.

009F 3.4 The Skew Right Unitor

009G **Definition 3.4.1 ► The Skew Right Unitor of $\lhd_{\mathsf{Sets}_*}$**

The **skew right unitor of the left tensor product of pointed sets** is the natural transformation

$$\rho^{\mathsf{Sets}_*,\lhd}\colon \mathrm{id}_{\mathsf{Sets}_*} \Longrightarrow \lhd_{\mathsf{Sets}_*} \circ \left(\mathrm{id}_{\mathsf{Sets}_*} \times \mathbb{1}^{\mathsf{Sets}_*}\right),$$

whose component

$$\rho^{\mathsf{Sets}_*,\lhd}_X\colon X \to X \lhd_{\mathsf{Sets}_*} S^0$$

at X is given by the composition[1]

$$\begin{aligned} X &\to X \vee X \\ &\cong |S^0| \odot X \\ &\cong S^0 \lhd_{\mathsf{Sets}_*} X, \end{aligned}$$

where $X \to X \vee X$ is the map sending X to the first factor of X in $X \vee X$.

[1] In other words, $\rho_X^{\mathsf{Sets}_*,\lhd}$ acts on elements as

$$\rho_X^{\mathsf{Sets}_*,\lhd}(x) \stackrel{\text{def}}{=} x \lhd_{\mathsf{Sets}_*} 0$$

for each $x \in X$.

009H 3.5 The Left-Skew Monoidal Category Structure on Pointed Sets

009J **Proposition 3.5.1 ► The Left-Skew Monoidal Category Structure on Pointed Sets**

The category Sets_* admits a left-skew monoidal category structure consisting of[1]

- *The Skew Monoidal Product.* The left tensor product functor

$$\lhd_{\mathsf{Sets}_*} : \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*,$$

  of Proposition 3.1.4;

- *The Skew Monoidal Unit.* The functor

$$\mathbb{1}^{\mathsf{Sets}_*} : \mathrm{pt} \to \mathsf{Sets}_*$$

  defined by

$$\mathbb{1}_{\mathsf{Sets}_*} \stackrel{\text{def}}{=} S^0;$$

- *The Skew Associators.* The natural isomorphism

$$\alpha^{\mathsf{Sets}_*,\lhd} : \lhd_{\mathsf{Sets}_*} \circ (\lhd_{\mathsf{Sets}_*} \times \mathrm{id}_{\mathsf{Sets}_*}) \stackrel{\cong}{\Longrightarrow} \lhd_{\mathsf{Sets}_*} \circ (\mathrm{id}_{\mathsf{Sets}_*} \times \lhd_{\mathsf{Sets}_*}),$$

  of Definition 3.2.1;

- *The Skew Left Unitors.* The natural transformation

$$\lambda^{\mathsf{Sets}_*,\lhd} : \lhd_{\mathsf{Sets}_*} \circ \left(\mathbb{1}^{\mathsf{Sets}_*} \times \mathrm{id}_{\mathsf{Sets}_*}\right) \Longrightarrow \mathrm{id}_{\mathsf{Sets}_*},$$

  of Definition 3.3.1;

- *The Skew Right Unitors.* The natural transformation

$$\rho^{\mathsf{Sets}_*,\lhd}\colon \mathrm{id}_{\mathsf{Sets}_*} \Longrightarrow \lhd_{\mathsf{Sets}_*} \circ \left(\mathrm{id}_{\mathsf{Sets}_*} \times \Bbbk^{\mathsf{Sets}_*}\right),$$

of Definition 3.4.1.

[1] Note in particular that, differently from general left-skew monoidal categories, the skew associator of $(\mathsf{Sets}_*, \lhd_{\mathsf{Sets}_*}, S^0)$ is a natural isomorphism.

Proof 3.5.2 ► Proof of Proposition 3.5.1

Omitted.

009K # 4 The Right Tensor Product of Pointed Sets

009L ## 4.1 Foundations

Let (X, x_0) and (Y, y_0) be pointed sets.

009M **Definition 4.1.1 ► The Right Tensor Product of Pointed Sets**

The **right tensor product of pointed sets** is the functor

$$\rhd_{\mathsf{Sets}_*}\colon \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*$$

defined as the composition

$$\mathsf{Sets}_* \times \mathsf{Sets}_* \xrightarrow{忘\times\mathrm{id}} \mathsf{Sets} \times \mathsf{Sets}_* \xrightarrow{\odot} \mathsf{Sets}_*.$$

009N **Remark 4.1.2 ► Unwinding Definition 4.1.1, I: Universal Property**

The right tensor product of pointed sets satisfies the following universal property:[1]

$$\mathsf{Sets}_*(X \rhd_{\mathsf{Sets}_*} Y, Z) \cong \mathrm{Hom}^{\otimes,\mathsf{R}}_{\mathsf{Sets}_*}(X \times Y, Z).$$

[1] Namely, a pointed map $f\colon X \lhd_{\mathsf{Sets}_*} Y \to Z$ is the same as a map $f^\dagger\colon X \times Y \to Z$ such that

$$f^\dagger(x, y_0) = z_0$$

for each $y \in Y$.

009P **Remark 4.1.3 ▶ Unwinding Definition 4.1.1, II: Explicit Description**

In detail, the **right tensor product of** (X, x_0) **and** (Y, y_0) is the pointed set $(X \rhd_{\mathsf{Sets}_*} Y, [y_0])$ consisting of[1]

- *The Underlying Set.* The set $X \rhd_{\mathsf{Sets}_*} Y$ defined by

$$\begin{aligned} X \rhd_{\mathsf{Sets}_*} Y &\overset{\text{def}}{=} |X| \odot Y \\ &\cong \bigvee_{x \in X} (Y, y_0); \end{aligned}$$

- *The Underlying Basepoint.* The point $[y_0]$ of $\bigvee_{x \in X}(Y, y_0)$.

[1] *Further Notation:* We write $x \rhd_{\mathsf{Sets}_*} y$ for the image of (x, y) under the map

$$X \times Y \to \underbrace{X \rhd_{\mathsf{Sets}_*} Y}_{\cong \bigvee_{x \in X}(Y, y_0)}.$$

sending (x, y) to the element $y \in Y$ in the xth copy of Y in $\bigvee_{x \in X}(Y, y_0)$. Note that we have

$$x \rhd_{\mathsf{Sets}_*} y_0 = x' \rhd_{\mathsf{Sets}_*} y_0,$$

for each $x, x' \in X$.

009Q **Proposition 4.1.4 ▶ Properties of Right Tensor Products of Pointed Sets**

Let (X, x_0) and (Y, y_0) be pointed sets.

009R 1. *Functoriality.* The assignments $X, Y, (X, Y) \mapsto X \rhd_{\mathsf{Sets}_*} Y$ define functors

$$\begin{aligned} X \rhd_{\mathsf{Sets}_*} - &\colon \mathsf{Sets}_* \longrightarrow \mathsf{Sets}_*, \\ - \rhd_{\mathsf{Sets}_*} Y &\colon \mathsf{Sets}_* \longrightarrow \mathsf{Sets}_*, \\ -_1 \rhd_{\mathsf{Sets}_*} -_2 &\colon \mathsf{Sets}_* \times \mathsf{Sets}_* \longrightarrow \mathsf{Sets}_*. \end{aligned}$$

Proof 4.1.5 ▶ Proof of Proposition 4.1.4

Item 1: Functoriality

Omitted. □

009S 4.2 The Skew Associator

009T **Definition 4.2.1 ▶ The Skew Associator of $\rhd_{\mathsf{Sets}_*}$**

The **skew associator of the right tensor product of pointed sets** is the natural isomorphism

$$\alpha^{\mathsf{Sets}_*,\rhd} : \rhd_{\mathsf{Sets}_*} \circ \left(\mathrm{id}_{\mathsf{Sets}_*} \times \rhd_{\mathsf{Sets}_*}\right) \overset{\cong}{\Longrightarrow} \rhd_{\mathsf{Sets}_*} \circ \left(\rhd_{\mathsf{Sets}_*} \times \mathrm{id}_{\mathsf{Sets}_*}\right),$$

whose component

$$\alpha^{\mathsf{Sets}_*,\rhd}_{X,Y,Z} : X \rhd_{\mathsf{Sets}_*} \left(Y \rhd_{\mathsf{Sets}_*} Z\right) \xrightarrow{\cong} \left(X \rhd_{\mathsf{Sets}_*} Y\right) \rhd_{\mathsf{Sets}_*} Z$$

at (X, Y, Z) is given by the composition[1]

$$\begin{aligned}
X \rhd_{\mathsf{Sets}_*} \left(Y \rhd_{\mathsf{Sets}_*} Z\right) &\overset{\text{def}}{=} |X| \odot \left(Y \rhd_{\mathsf{Sets}_*} Z\right) \\
&\overset{\text{def}}{=} |X| \odot (|Y| \odot Z) \\
&\cong |X| \odot \left(\bigvee_{y \in Y} (Z, z_0)\right) \\
&\cong \bigvee_{x \in X} \left(\bigvee_{y \in Y} (Z, z_0)\right) \\
&\cong \bigvee_{(x,y) \in \bigvee_{x \in X} (Y, y_0)} (Z, z_0) \\
&\cong \left|\bigvee_{x \in X} (Y, y_0)\right| \odot Z \\
&\overset{\text{def}}{=} |X \odot Y| \odot Z \\
&\overset{\text{def}}{=} |X \rhd_{\mathsf{Sets}_*} Y| \odot Z \\
&\overset{\text{def}}{=} \left(X \rhd_{\mathsf{Sets}_*} Y\right) \rhd_{\mathsf{Sets}_*} Z
\end{aligned}$$

where the isomorphism

$$\bigvee_{x \in X} \left(\bigvee_{y \in Y} (Z, z_0)\right) \cong \bigvee_{(x,y) \in \bigvee_{x \in X} (Y, y_0)} (Z, z_0)$$

is given by $[(x, (y, z))] \mapsto [((x, y), z)]$.

[1] In other words, $\alpha^{\mathsf{Sets}_*,\rhd}_{X,Y,Z}$ acts on elements as

$$\alpha^{\mathsf{Sets}_*,\rhd}_{X,Y,Z}\left(x \rhd_{\mathsf{Sets}_*} \left(y \rhd_{\mathsf{Sets}_*} z\right)\right) \overset{\text{def}}{=} \left(x \rhd_{\mathsf{Sets}_*} y\right) \rhd_{\mathsf{Sets}_*} z$$

for each $x \rhd_{\mathsf{Sets}_*} \left(y \rhd_{\mathsf{Sets}_*} z\right) \in X \rhd_{\mathsf{Sets}_*} \left(Y \rhd_{\mathsf{Sets}_*} Z\right)$.

009U
4.3 The Skew Left Unitor

009V
Definition 4.3.1 ► The Skew Left Unitor of $\rhd_{\mathsf{Sets}_*}$

The **skew left unitor of the right tensor product of pointed sets** is the natural transformation

$$\lambda^{\mathsf{Sets}_*,\rhd} \colon \mathrm{id}_{\mathsf{Sets}_*} \Longrightarrow \rhd_{\mathsf{Sets}_*} \circ \left(\mathbb{1}^{\mathsf{Sets}_*} \times \mathrm{id}_{\mathsf{Sets}_*}\right),$$

whose component

$$\lambda^{\mathsf{Sets}_*,\rhd}_X \colon X \to S^0 \rhd_{\mathsf{Sets}_*} X$$

at X is given by the composition[1]

$$\begin{aligned} X &\to X \vee X \\ &\cong |S^0| \odot X \\ &\cong S^0 \rhd_{\mathsf{Sets}_*} X, \end{aligned}$$

where $X \to X \vee X$ is the map sending X to the first factor of X in $X \vee X$.

[1] In other words, $\lambda^{\mathsf{Sets}_*,\rhd}_X$ acts on elements as

$$\lambda^{\mathsf{Sets}_*,\rhd}_X(x) \stackrel{\text{def}}{=} 0 \rhd_{\mathsf{Sets}_*} x$$

for each $x \in X$.

009W
4.4 The Skew Right Unitor

009X
Definition 4.4.1 ► The Skew Right Unitor of $\rhd_{\mathsf{Sets}_*}$

The **skew right unitor of the right tensor product of pointed sets** is the natural transformation

$$\rho^{\mathsf{Sets}_*,\rhd} \colon \rhd_{\mathsf{Sets}_*} \circ \left(\mathrm{id}_{\mathsf{Sets}_*} \times \mathbb{1}^{\mathsf{Sets}_*}\right) \Longrightarrow \mathrm{id}_{\mathsf{Sets}_*},$$

whose component[1]

$$\rho^{\mathsf{Sets}_*,\rhd}_X \colon X \rhd_{\mathsf{Sets}_*} S^0 \to X$$

at X is given by the composition

$$\begin{aligned} X \triangleright_{\mathsf{Sets}_*} S^0 &\cong |X| \odot S^0 \\ &\cong \bigvee_{x \in X} S^0 \\ &\to X \end{aligned}$$

where $\bigvee_{x \in X} S^0 \to X$ is the map given by

$$\begin{aligned} (x, 0) &\mapsto x, \\ (x, 1) &\mapsto x. \end{aligned}$$

[1]In other words, $\rho_X^{\mathsf{Sets}_*, \triangleright}$ acts on elements as

$$\begin{aligned} \rho_X^{\mathsf{Sets}_*, \triangleright}(x \triangleright_{\mathsf{Sets}_*} 0) &\stackrel{\text{def}}{=} x, \\ \rho_X^{\mathsf{Sets}_*, \triangleright}(x \triangleright_{\mathsf{Sets}_*} 1) &\stackrel{\text{def}}{=} x \end{aligned}$$

for each $x \in X$.

009Y
4.5 The Right-Skew Monoidal Category Structure on Pointed Sets

009Z
Proposition 4.5.1 ► The Right-Skew Monoidal Category Structure on Pointed Sets

The category Sets_* admits a right-skew monoidal category structure consisting of[1]

- *The Skew Monoidal Product.* The right tensor product functor

  $$\triangleright_{\mathsf{Sets}_*} \colon \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*,$$

  of Item 1;

- *The Skew Monoidal Unit.* The functor

  $$\mathbb{1}^{\mathsf{Sets}_*} \colon \mathrm{pt} \to \mathsf{Sets}_*$$

  defined by

  $$\mathbb{1}_{\mathsf{Sets}_*} \stackrel{\text{def}}{=} S^0;$$

- *The Skew Associators.* The natural isomorphism

$$\alpha^{\mathsf{Sets}_*,\rhd} : \rhd_{\mathsf{Sets}_*} \circ (\rhd_{\mathsf{Sets}_*} \times \mathrm{id}_{\mathsf{Sets}_*}) \xrightarrow{\cong} \rhd_{\mathsf{Sets}_*} \circ (\mathrm{id}_{\mathsf{Sets}_*} \times \rhd_{\mathsf{Sets}_*}),$$

  of Definition 4.2.1;

- *The Skew Left Unitors.* The natural transformation

$$\lambda^{\mathsf{Sets}_*,\rhd} : \mathrm{id}_{\mathsf{Sets}_*} \Longrightarrow \rhd_{\mathsf{Sets}_*} \circ \left(\mathbb{1}^{\mathsf{Sets}_*} \times \mathrm{id}_{\mathsf{Sets}_*}\right),$$

  of Definition 3.3.1;

- *The Skew Right Unitors.* The natural transformation

$$\rho^{\mathsf{Sets}_*,\rhd} : \rhd_{\mathsf{Sets}_*} \circ \left(\mathrm{id}_{\mathsf{Sets}_*} \times \mathbb{1}^{\mathsf{Sets}_*}\right) \Longrightarrow \mathrm{id}_{\mathsf{Sets}_*},$$

  of Definition 3.4.1.

[1]Note in particular that, differently from general right-skew monoidal categories, the skew associator of $(\mathsf{Sets}_*, \rhd_{\mathsf{Sets}_*}, S^0)$ is a natural isomorphism.

Proof 4.5.2 ▶ Proof of Proposition 3.5.1

Omitted.

00A0 # 5 Smash Products of Pointed Sets

00A1 ## 5.1 Foundations

Let (X, x_0) and (Y, y_0) be pointed sets.

00A2 **Definition 5.1.1 ▶ Smash Products of Pointed Sets**

The **smash product of** (X, x_0) **and** (Y, y_0)[1] is the pointed set $X \wedge Y$[2] such that we have a bijection

$$\mathsf{Sets}_*(X \wedge Y, Z) \cong \mathrm{Hom}^{\otimes}_{\mathsf{Sets}_*}(X \times Y, Z),$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

[1] *Further Terminology:* Also called the **tensor product of $\mathbb{F}_1$-modules of** (X, x_0) **and** (Y, y_0) or the **tensor product of** (X, x_0) **and** (Y, y_0) **over** $\mathbb{F}_1$.
[2] *Further Notation:* Also written $X \otimes_{\mathbb{F}_1} Y$.

00A3 **Remark 5.1.2 ▶ Unwinding Definition 5.1.1**

In detail, the **smash product of** (X, x_0) **and** (Y, y_0) is the pair $((X \wedge Y, [(x_0, y_0)]), \iota)$ consisting of

- A pointed set $(X \wedge Y, [(x_0, y_0)])$;
- A bilinear morphism of pointed sets $\iota\colon (X \times Y, (x_0, y_0)) \to X \wedge Y$;

satisfying the following universal property:

(**UP**) Given another such pair $((Z, z_0), f)$ consisting of

- A pointed set (Z, z_0);
- A bilinear morphism of pointed sets $f\colon (X \times Y, (x_0, y_0)) \to X \wedge Y$;

there exists a unique morphism of pointed sets $X \wedge Y \xrightarrow{\exists!} Z$ making the diagram

$$\begin{tikzcd} & X \wedge Y \arrow[d, dashed, "\exists!"] \\ X \times Y \arrow[ur, "\iota"] \arrow[r, "f"'] & Z \end{tikzcd}$$

commute.

00A4 **Construction 5.1.3 ▶ Smash Products of Pointed Sets**

Concretely, the **smash product of** (X, x_0) **and** (Y, y_0) is the pointed set $(X \wedge Y, [(x_0, y_0)])$ consisting of[1]

- *The Underlying Set.* The set $X \wedge Y$ defined by

$$
\begin{aligned}
X \wedge Y &\cong \mathrm{pt} \coprod_{X \vee Y} (X \times Y) \\
&\stackrel{\text{def}}{=} \frac{X \times Y}{X \vee Y} \\
&\cong X \times Y/\sim,
\end{aligned}
\qquad
\begin{tikzcd}
X \wedge Y \arrow[dr, phantom, "\ulcorner", very near start] & X \times Y \arrow[l] \\
\mathrm{pt} \arrow[u] & X \vee Y, \arrow[l, "!"] \arrow[u, hook]
\end{tikzcd}
$$

where $\sim$ is the equivalence relation of $X \times Y$ obtained by declaring $(x, y) \sim (x', y')$ iff $(x, y), (x', y') \in X \vee Y$, i.e. by declaring

$$
\begin{aligned}
(x_0, y) &\sim (x_0, y'), \\
(x, y_0) &\sim (x', y_0)
\end{aligned}
$$

for all $x \in X$ and all $y \in Y$;

- *The Basepoint.* The element $[(x_0, y_0)]$ of $X \wedge Y$ given by the equivalence class of (x_0, y_0) under the equivalence relation $\sim$ on $X \times Y$.

---

[1] *Further Notation:* We write $x \wedge y$ for the image of (x, y) under the quotient map

$$X \times Y \twoheadrightarrow \underbrace{\frac{X \times Y}{X \vee Y}}_{\stackrel{\text{def}}{=} X \wedge Y}$$

Note that we have

$$
\begin{aligned}
x \wedge y_0 &= x' \wedge y_0, \\
x_0 \wedge y &= x_0 \wedge y'
\end{aligned}
$$

for each $x, x' \in X$ and each $y, y' \in Y$.

PROOF 5.1.4 ▶ PROOF OF CONSTRUCTION 5.1.3

Clear. □

00A5 **EXAMPLE 5.1.5 ▶ EXAMPLES OF SMASH PRODUCTS OF POINTED SETS**

Here are some examples of smash products of pointed sets.

1. *Smashing With S^0.* For any pointed set X, we have isomorphisms of pointed

sets

$$S^0 \wedge X \cong X,$$
$$X \wedge S^0 \cong X.$$

00A6 **Proposition 5.1.6 ► Properties of Smash Products of Pointed Sets**

Let (X, x_0) and (Y, y_0) be pointed sets.

00A7 1. *Functoriality.* The assignments $(X, x_0), (Y, y_0), ((X, x_0), (Y, y_0)) \mapsto X \wedge Y$ define functors

$$X \wedge - \colon \mathsf{Sets}_* \longrightarrow \mathsf{Sets}_*,$$
$$- \wedge Y \colon \mathsf{Sets}_* \longrightarrow \mathsf{Sets}_*,$$
$$-_1 \wedge -_2 \colon \mathsf{Sets}_* \times \mathsf{Sets}_* \longrightarrow \mathsf{Sets}_*.$$

00A8 2. *Adjointness.* We have adjunctions

$$(X \wedge - \dashv \mathbf{Sets}_*(X, -)) \colon \quad \begin{tikzcd} \mathsf{Sets}_* \arrow[r, bend left, "X\wedge -"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_*, \arrow[l, bend left, "\mathbf{Sets}_*(X,-)"] \end{tikzcd}$$

$$(- \wedge Y \dashv \mathbf{Sets}_*(Y, -)) \colon \quad \begin{tikzcd} \mathsf{Sets}_* \arrow[r, bend left, "-\wedge Y"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_*, \arrow[l, bend left, "\mathbf{Sets}_*(Y,-)"] \end{tikzcd}$$

witnessed by bijections

$$\mathsf{Sets}_*(X \wedge Y, Z) \cong \mathsf{Sets}_*(X, \mathbf{Sets}_*(Y, Z)),$$
$$\mathsf{Sets}_*(X \wedge Y, Z) \cong \mathsf{Sets}_*(X, \mathbf{Sets}_*(A, Z)),$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, which internalise to isomorphisms of pointed sets

$$\mathbf{Sets}_*(X \wedge Y, Z) \cong \mathbf{Sets}_*(X, \mathbf{Sets}_*(Y, Z)),$$
$$\mathbf{Sets}_*(X \wedge Y, Z) \cong \mathbf{Sets}_*(X, \mathbf{Sets}_*(A, Z)),$$

again natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

00A9 3. *Closed Symmetric Monoidality.* The quadruple $(\mathsf{Sets}_*, \wedge, S^0, \mathbf{Sets}_*)$ is a closed symmetric monoidal category.

00AA 4. *Morphisms From the Monoidal Unit.* We have a bijection of sets[1]

$$\mathsf{Sets}_*(S^0, X) \cong X,$$

natural in $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, internalising also to an isomorphism of pointed sets

$$\mathbf{Sets}_*(S^0, X) \cong (X, x_0),$$

again natural in $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

00AB 5. *Symmetric Strong Monoidality With Respect to Free Pointed Sets.* The free pointed set functor of Pointed Sets, Item 1 of Proposition 4.2.2 has a symmetric strong monoidal structure

$$\left((-)^+, (-)^{+,\times}, (-)^{+,\times}_{\mathbb{1}}\right) : (\mathsf{Sets}, \times, \mathrm{pt}) \to \left(\mathsf{Sets}_*, \wedge, S^0\right),$$

being equipped with isomorphisms of pointed sets

$$(-)^{+,\times}_{X,Y} : X^+ \wedge Y^+ \xrightarrow{\cong} (X \times Y)^+,$$
$$(-)^{+,\times}_{\mathbb{1}} : S^0 \xrightarrow{\cong} \mathrm{pt}^+,$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

00AC 6. *Distributivity Over Wedge Sums.* We have isomorphisms of pointed sets

$$X \wedge (Y \vee Z) \cong (X \wedge Y) \vee (X \wedge Z),$$
$$(X \vee Y) \wedge Z \cong (X \wedge Z) \vee (Y \wedge Z),$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

00AD 7. *Universal Property I.* The symmetric monoidal structure on the category Sets_* is uniquely determined by the following requirements:

   (a) *Two-Sided Preservation of Colimits.* The smash product

$$\wedge : \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*$$

   of Sets_* preserves colimits separately in each variable.

(b) *The Unit Object Is S^0.* We have $\mathbb{1}_{\mathsf{Sets}_*} = S^0$.

00AE 8. *Universal Property II.* The symmetric monoidal structure on the category Sets_* is the unique symmetric monoidal structure on Sets_* such that the free pointed set functor

$$(-)^+ \colon \mathsf{Sets} \to \mathsf{Sets}_*$$

admits a symmetric monoidal structure.

00AF 9. *Existence of Monoidal Diagonals.* The triple $(\mathsf{Sets}_*, \wedge, S^0)$ is a monoidal category with diagonals:

(a) *Monoidal Diagonals.* The natural transformation

$$\Delta \colon \mathrm{id}_{\mathsf{Sets}_*} \Longrightarrow \wedge \circ \Delta^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}, \qquad \begin{tikzcd} \mathsf{Sets}_* \arrow[rr, bend left, "\mathrm{id}_{\mathsf{Sets}_*}"] \arrow[dr, "\Delta^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}"'] & \arrow[d, phantom, "\Downarrow \Delta"] & \mathsf{Sets}_* \\ & \mathsf{Sets}_* \times \mathsf{Sets}_*, \arrow[ur, "\wedge"'] & \end{tikzcd}$$

whose component

$$\Delta_X \colon (X, x_0) \to (X \wedge X, [(x_0, x_0)])$$

at $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$ is given by the composition

$$\begin{aligned} (X, x_0) &\xrightarrow{\Delta_X} (X \times X, (x_0, x_0)) \\ &\twoheadrightarrow \left(\tfrac{X \times X}{X \vee X}, [(x_0, x_0)]\right) \\ &\overset{\text{def}}{=\!=} (X \wedge X, [(x_0, x_0)]) \end{aligned}$$

in Sets_*, is a monoidal natural transformation:

i. *Naturality.* For each morphism $f \colon X \to Y$ of pointed sets, the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\Delta_X"'] & Y \arrow[d, "\Delta_Y"] \\ X \wedge X \arrow[r, "f \wedge f"'] & Y \wedge Y \end{tikzcd}$$

commutes.

ii. *Compatibility With Strong Monoidality Constraints.* For each $(X,x_0),(Y,y_0)\in\mathrm{Obj}(\mathsf{Sets}_*)$, the diagram

$$\begin{tikzcd}
X \wedge Y \arrow[r, "\Delta_X \wedge \Delta_Y"] \arrow[d, equal] & (X \wedge X) \wedge (Y \wedge Y) \arrow[d, dashed, "\wr"] \\
X \wedge Y \arrow[r, "\Delta_{X \wedge Y}"'] & (X \wedge Y) \wedge (X \wedge Y)
\end{tikzcd}$$

commutes.

iii. *Compatibility With Strong Unitality Constraints.* The diagram

$$\begin{tikzcd}
S^0 \arrow[d, equal] \arrow[dr, "(\lambda^{\mathsf{Sets}_*}_{S^0})^{-1}=(\rho^{\mathsf{Sets}_*}_{S^0})^{-1}"] & \\
S^0 \arrow[r, "\Delta_{S^0}"'] & S^0 \wedge S^0
\end{tikzcd}$$

commutes.

(b) *The Diagonal of the Unit.* The component

$$\Delta^{\mathsf{Sets}_*}_{S^0}\colon S^0 \xrightarrow{\cong} S^0 \wedge S^0$$

of the diagonal natural transformation of Sets_* at S^0 is an isomorphism.

00AG 10. *Comonoids in* Sets_*. The symmetric monoidal functor

$$\left((-)^+,(-)^{+,\times},(-)^{+,\times}_{\mathbb{1}}\right)\colon (\mathsf{Sets},\times,\mathrm{pt}) \longrightarrow \left(\mathsf{Sets}_*,\wedge,S^0\right),$$

of Pointed Sets, Item 4 of Proposition 4.2.2 lifts to an equivalence of categories

$$\begin{aligned}
\mathsf{CoMon}\left(\mathsf{Sets}_*,\wedge,S^0\right) &\overset{\text{eq.}}{\cong} \mathsf{CoMon}(\mathsf{Sets},\times,\mathrm{pt})\\
&\cong \mathsf{Sets}.
\end{aligned}$$

[1] In other words, the forgetful functor

$$忘\colon \mathsf{Sets}_* \to \mathsf{Sets}$$

defined on objects by sending a pointed set to its underlying set is corepresentable by S^0.

Proof 5.1.7 ▶ Proof of Proposition 5.1.6

Item 1: Functoriality

Omitted.

Item 2: Adjointness

Omitted.

Item 3: Closed Symmetric Monoidality

Omitted.

Item 4: Morphisms From the Monoidal Unit

Omitted.

Item 5: Symmetric Strong Monoidality With Respect to Free Pointed Sets

Omitted.

Item 6: Distributivity Over Wedge Sums

This follows from Item 3, Monoidal Categories, **??** of **??**, and the fact that $\vee$ is the coproduct in Sets_*.

Item 7: Universal Property I

Omitted.

Item 8: Universal Property II

See [GGN15, Theorem 5.1].

Item 9: Existence of Monoidal Diagonals

Omitted.

Item 10: Comonoids in Sets_*

See [PS19, Lemma 2.4].

Appendices

A Other Chapters