Indexed and Fibred Sets

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This chapter contains a discussion of the un/straightening equivalence in the context of sets, as well as a general discussion of indexed and fibred sets. In particular, it contains:

1. A discussion of indexed sets (i.e. functors $K_{\mathsf{disc}} \longrightarrow \mathsf{Sets}$ with K a set), constructions with them like dependent sums and dependent products, and their properties (Sections 1 and 2);
2. A discussion of fibred sets (i.e. maps of sets $X \longrightarrow K$), constructions with them like dependent sums and dependent products, and their properties (Sections 3 and 4);
3. A discussion of the un/straightening equivalence for indexed and fibred sets (Section 5).

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1 Indexed Sets

1.1 Foundations

Let K be a set.

Definition 1.1.1 ► Indexed Sets

A K**-indexed set** is a functor $X\colon K_{\mathrm{disc}} \longrightarrow \mathrm{Sets}$.

Remark 1.1.2 ► Unwinding Definition 1.1.1

By Categories, **??**, a K**-indexed set** consists of a K-indexed collection

$$X^{\dagger}\colon K \longrightarrow \mathrm{Obj}(\mathrm{Sets}),$$

of sets, assigning a set $X^\dagger_x \overset{\text{def}}{=} X_x$ to each element x of K.

1.2 Morphisms of Indexed Sets

Let $X\colon K_{\mathsf{disc}} \longrightarrow \mathsf{Sets}$ and $Y\colon K_{\mathsf{disc}} \longrightarrow \mathsf{Sets}$ be indexed sets.

Definition 1.2.1 ► Morphisms of Indexed Sets

A **morphism of K-indexed sets from X to Y**[1] is a natural transformation

$$f\colon X \Longrightarrow Y, \qquad \begin{tikzcd} K_{\mathsf{disc}} \arrow[r, bend left, "X"{name=U}] \arrow[r, bend right, "Y"'{name=D}] & \mathsf{Sets} \arrow[from=U, to=D, Rightarrow, "f"'] \end{tikzcd}$$

from X to Y.

[1] *Further Terminology:* Also called a K**-indexed map of sets from** X **to** Y.

Remark 1.2.2 ► Unwinding Definition 1.2.1

In detail, a **morphism of K-indexed sets** consists of a K-indexed collection

$$\{f_x\colon X_x \longrightarrow Y_x\}_{x\in K}$$

of maps of sets.

1.3 The Category of Sets Indexed by a Fixed Set

Let K be a set.

Definition 1.3.1 ► The Category of K-Indexed Sets

The **category of K-indexed sets** is the category $\mathsf{ISets}(K)$ defined by

$$\mathsf{ISets}(K) \overset{\text{def}}{=} \mathsf{Fun}\big(K_{\mathsf{disc}}, \mathsf{Sets}\big).$$

Remark 1.3.2 ▶ Unwinding Definition 1.3.1

In detail, the **category of K-indexed sets** is the category $\mathsf{ISets}(K)$ where

- *Objects.* The objects of $\mathsf{ISets}(K)$ are K-indexed sets as in Definition 1.1.1;
- *Morphisms.* The morphisms of $\mathsf{ISets}(K)$ are morphisms of K-indexed sets as in Definition 1.2.1;
- *Identities.* For each $X \in \mathrm{Obj}(\mathsf{ISets}(K))$, the unit map
$$\mathbb{1}_X^{\mathsf{ISets}(K)} \colon \mathrm{pt} \longrightarrow \mathrm{Hom}_{\mathsf{ISets}(K)}(X, X)$$
of $\mathsf{ISets}(K)$ at X is defined by
$$\mathrm{id}_X^{\mathsf{ISets}(K)} \overset{\mathrm{def}}{=} \{\mathrm{id}_{X_x}\}_{x \in K};$$
- *Composition.* For each $X, Y, Z \in \mathrm{Obj}(\mathsf{ISets}(K))$, the composition map
$$\circ_{X,Y,Z}^{\mathsf{ISets}(K)} \colon \mathrm{Hom}_{\mathsf{ISets}(K)}(Y, Z) \times \mathrm{Hom}_{\mathsf{ISets}(K)}(X, Y) \longrightarrow \mathrm{Hom}_{\mathsf{ISets}(K)}(X, Z)$$
of $\mathsf{ISets}(K)$ at (X, Y, Z) is defined by
$$\{g_x\}_{x \in K} \circ_{X,Y,Z}^{\mathsf{ISets}(K)} \{f_x\}_{x \in K} \overset{\mathrm{def}}{=} \{g_x \circ f_x\}_{x \in K}.$$

1.4 The Category of Indexed Sets

Definition 1.4.1 ▶ The Category of Indexed Sets

The **category of indexed sets** is the category ISets defined as the Grothendieck construction of the functor $\mathsf{ISets} \colon \mathsf{Sets}^{\mathsf{op}} \longrightarrow \mathsf{Cats}$ of Proposition 2.1.5:
$$\mathsf{ISets} \overset{\mathrm{def}}{=} \int^{\mathsf{Sets}} \mathsf{ISets}.$$

Remark 1.4.2 ▶ Unwinding Definition 1.4.1

In detail, the **category of indexed sets** is the category ISets where

- *Objects.* The objects of ISets are pairs (K, X) consisting of

- *The Indexing Set.* A set K;
- *The Indexed Set.* A K-indexed set $X\colon K_{\mathsf{disc}} \to \mathsf{Sets}$;

· *Morphisms.* A morphism of ISets from (K,X) to (K',Y) is a pair (ϕ,f) consisting of

- *The Reindexing Map.* A map of sets $\phi\colon K \to K'$;
- *The Morphism of Indexed Sets.* A morphism of K-indexed sets $f\colon X \to \phi_*(Y)$ as in the diagram

$$f\colon X \to \phi_*(Y), \qquad \begin{tikzcd} K_{\mathsf{disc}} \arrow[rr, "\phi"] \arrow[dr, "X"'] & \arrow[d, phantom, "\overset{f}{\Rightarrow}"] & K'_{\mathsf{disc}} \arrow[dl, "Y"] \\ & \mathsf{Sets}; & \end{tikzcd}$$

· *Identities.* For each $(K,X) \in \mathrm{Obj}(\mathsf{ISets})$, the unit map

$$\mathbb{1}^{\mathsf{ISets}}_{(K,X)}\colon \mathrm{pt} \to \mathsf{ISets}((K,X),(K,X))$$

of ISets at (K,X) is defined by

$$\mathrm{id}^{\mathsf{ISets}}_{(K,X)} \overset{\mathrm{def}}{=} (\mathrm{id}_K, \mathrm{id}_X).$$

· *Composition.* For each $\mathbf{X} = (K,X)$, $\mathbf{Y} = (K',Y)$, $\mathbf{Z} = (K'',Z) \in \mathrm{Obj}(\mathsf{ISets})$, the composition map

$$\circ^{\mathsf{ISets}}_{\mathbf{X},\mathbf{Y},\mathbf{Z}}\colon \mathsf{ISets}(\mathbf{Y},\mathbf{Z}) \times \mathsf{ISets}(\mathbf{X},\mathbf{Y}) \to \mathsf{ISets}(\mathbf{X},\mathbf{Z})$$

of ISets at $(\mathbf{X},\mathbf{Y},\mathbf{Z})$ is defined by

$$(\psi,g)\circ(\phi,f) \overset{\mathrm{def}}{=} \big(\psi\circ\phi, \big(g\circ \mathrm{id}_\phi\big)\circ f\big),$$

as in the diagram

$$\begin{tikzcd} K_{\mathsf{disc}} \arrow[r, "\phi"] \arrow[dr, "X"'] \arrow[r, phantom, shift right=4, "\overset{f}{\Rightarrow}"] & K'_{\mathsf{disc}} \arrow[r, "\psi"] \arrow[d, "Y"] \arrow[r, phantom, shift right=4, "\overset{g}{\Rightarrow}"] & K''_{\mathsf{disc}} \arrow[dl, "Z"] \\ & \mathsf{Sets}; & \end{tikzcd}$$

for each $(\phi,f) \in \mathsf{ISets}(\mathbf{X},\mathbf{Y})$ and each $(\psi,g) \in \mathsf{ISets}(\mathbf{Y},\mathbf{Z})$.

2 Constructions With Indexed Sets

2.1 Change of Indexing

Let $\phi\colon K \to K'$ be a function and let X be a K'-indexed set.

Definition 2.1.1 ▶ Change of Indexing of Indexed Sets

The **change of indexing of** X **to** K is the K-indexed set $\phi^*(X)$ defined by

$$\phi^*(X) \overset{\text{def}}{=} X \circ \phi_{\mathsf{disc}}.$$

Remark 2.1.2 ▶ Unwinding Definition 2.1.1

In detail, the **change of indexing of** X **to** K is the K-indexed set $\phi^*(X)$ defined by

$$\phi^*(X)_x \overset{\text{def}}{=} X_{\phi(x)}$$

for each $x \in K$.

Proposition 2.1.3 ▶ Functoriality of Change of Indexing

The assignment $X \mapsto \phi^*(X)$ defines a functor

$$\phi^*\colon \mathsf{ISets}(K') \to \mathsf{ISets}(K),$$

where

- *Action on Objects.* For each $X \in \mathrm{Obj}(\mathsf{ISets}(K'))$, we have

$$[\phi^*](X) \overset{\text{def}}{=} \phi^*(X);$$

- *Action on Morphisms.* For each $X, Y \in \mathrm{Obj}(\mathsf{ISets}(K'))$, the action on Hom-sets

$$\phi^*_{X,Y}\colon \mathrm{Hom}_{\mathsf{ISets}(K')}(X, Y) \to \mathrm{Hom}_{\mathsf{ISets}(K)}(\phi^*(X), \phi^*(Y))$$

  of ϕ^* at (X, Y) is the map sending a morphism of K'-indexed sets

$$f = \{f_x\colon X_x \to Y_x\}_{x \in K'}$$

  from X to Y to the morphism of K-indexed sets defined by

$$\phi^*(f) \overset{\text{def}}{=} \left\{f_{\phi(x)}\colon X_{\phi(x)} \to Y_{\phi(x)}\right\}_{x \in K}.$$

Proof 2.1.4 ▶ Proof of Proposition 2.1.3

Omitted.

Proposition 2.1.5 ▶ Functoriality of Categories of K-Indexed Sets

The assignment $K \mapsto \mathsf{ISets}(K)$ defines a functor

$$\mathsf{ISets}\colon \mathsf{Sets}^{\mathsf{op}} \to \mathsf{Cats},$$

where

- *Action on Objects*. For each $K \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$[\mathsf{ISets}](K) \stackrel{\text{def}}{=} \mathsf{ISets}(K);$$

- *Action on Morphisms*. For each $K, K' \in \mathrm{Obj}(\mathsf{Sets})$, the action on Hom-sets

$$\mathsf{ISets}_{K,K'}\colon \mathsf{Sets}^{\mathsf{op}}(K, K') \to \mathsf{Fun}(\mathsf{ISets}(K), \mathsf{ISets}(K'))$$

  of ISets at (K, K') is the map defined by

$$\mathsf{ISets}_{K,K'}(\phi) \stackrel{\text{def}}{=} \phi^*$$

  for each $\phi \in \mathsf{Sets}^{\mathsf{op}}(K, K')$.

Proof 2.1.6 ▶ Proof of Proposition 2.1.5

Omitted.

2.2 Dependent Sums

Let $\phi\colon K \to K'$ be a function and let X be a K-indexed set.

Definition 2.2.1 ▶ Dependent Sums of Indexed Sets

The **dependent sum of** X is the K'-indexed set $\Sigma_\phi(X)$[1] defined by

$$\Sigma_\phi(X) \stackrel{\text{def}}{=} \mathrm{Lan}_\phi(X),$$

and hence given by

$$\Sigma_\phi(X)_x \cong \coprod_{y\in\phi^{-1}(x)} X_y$$

for each $x \in K'$.

[1] *Further Notation:* Also written $\phi_*(X)$.

Proposition 2.2.2 ▶ Functoriality of Dependent Sums

The assignment $X \mapsto \Sigma_\phi(X)$ defines a functor

$$\Sigma_\phi\colon \mathsf{ISets}(K) \to \mathsf{ISets}(K'),$$

where

- *Action on Objects.* For each $X \in \mathrm{Obj}(\mathsf{ISets}(K))$, we have

$$[\Sigma_\phi](X) \stackrel{\text{def}}{=} \Sigma_\phi(X);$$

- *Action on Morphisms.* For each $X, Y \in \mathrm{Obj}(\mathsf{ISets}(K))$, the action on Hom-sets

$$\Sigma_{\phi|X,Y}\colon \mathrm{Hom}_{\mathsf{ISets}(K)}(X,Y) \to \mathrm{Hom}_{\mathsf{ISets}(K')}(\Sigma_\phi(X), \Sigma_\phi(Y))$$

of Σ_ϕ at (X,Y) is the map sending a morphism of K-indexed sets

$$f\colon X \to Y$$

to the morphism of K'-indexed sets defined by

$$\begin{aligned}\Sigma_\phi(f) &\stackrel{\text{def}}{=} \mathrm{Lan}_\phi(f);\\ &\cong \coprod_{y\in\phi^{-1}(X)} f_y.\end{aligned}$$

Proof 2.2.3 ▶ Proof of Proposition 2.2.2

Omitted. □

2.3 Dependent Products

Let $\phi\colon K \to K'$ be a function and let X be a K-indexed set.

Definition 2.3.1 ▶ Dependent Products of Indexed Sets

The **dependent product of** X is the K'-indexed set $\Pi_\phi(X)$[1] defined by

$$\Pi_\phi(X) \stackrel{\text{def}}{=} \mathrm{Ran}_\phi(X),$$

and hence given by

$$\Pi_\phi(X)_x \cong \prod_{y\in\phi^{-1}(x)} X_y$$

for each $x \in K'$.

[1] *Further Notation:* Also written $\phi_!(X)$.

Proposition 2.3.2 ▶ Functoriality of Dependent Products

The assignment $X \mapsto \Pi_\phi(X)$ defines a functor

$$\Pi_\phi \colon \mathsf{ISets}(K) \to \mathsf{ISets}(K'),$$

where

- *Action on Objects.* For each $X \in \mathrm{Obj}(\mathsf{ISets}(K))$, we have

$$[\Pi_\phi](X) \stackrel{\text{def}}{=} \Pi_\phi(X);$$

- *Action on Morphisms.* For each $X, Y \in \mathrm{Obj}(\mathsf{ISets}(K))$, the action on Hom-sets

$$\Pi_{\phi|X,Y} \colon \mathrm{Hom}_{\mathsf{ISets}(K)}(X, Y) \to \mathrm{Hom}_{\mathsf{ISets}(K')}\big(\Pi_\phi(X), \Pi_\phi(Y)\big)$$

  of Π_ϕ at (X, Y) is the map sending a morphism of K-indexed sets

$$f \colon X \to Y$$

  to the morphism of K'-indexed sets defined by

$$\begin{aligned}\Pi_\phi(f) &\stackrel{\text{def}}{=} \mathrm{Ran}_\phi(f);\\ &\cong \prod_{y\in\phi^{-1}(x)} f_y.\end{aligned}$$

Proof 2.3.3 ▶ Proof of Proposition 2.3.2

Omitted. ∎

2.4 Internal Homs

Let K be a set and let X and Y be K-indexed sets.

Definition 2.4.1 ▶ Internal Hom of Indexed Sets

The **internal Hom of indexed sets from** X **to** Y is the indexed set $\mathbf{Hom}_{\mathsf{ISets}(K)}(X, Y)$ defined by

$$\mathbf{Hom}_{\mathsf{ISets}(K)}(X, Y) \overset{\text{def}}{=} \mathsf{Sets}(X_x, Y_x)$$

for each $x \in K$.

2.5 Adjointness of Indexed Sets

Let $\phi\colon K \to K'$ be a map of sets.

Proposition 2.5.1 ▶ Adjointness of Indexed Sets

We have a triple adjunction

$$(\Sigma_\phi \dashv \phi^* \dashv \Pi_\phi)\colon\quad \begin{tikzcd} \mathsf{ISets}(K) \arrow[r, bend left=50, "\Sigma_\phi"] \arrow[r, bend right=50, "\Pi_\phi"'] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, phantom, shift right=3, "\perp"] & \mathsf{ISets}(K') \arrow[l, "\phi^*" description] \end{tikzcd}.$$

Proof 2.5.2 ▶ Proof of Proposition 2.5.1

This follows from Kan Extensions, ?? of ??. ∎

3 Fibred Sets

3.1 Foundations

Let K be a set.

Definition 3.1.1 ► Fibred Sets

A K**-fibred set** is a pair (X, ϕ) consisting of[1]

- *The Underlying Set.* A set X, called the **underlying set of** (X, ϕ);
- *The Fibration.* A map of sets $\phi \colon X \to K$.

[1] *Further Terminology:* The **fibre of** (X, ϕ) **over** $x \in K$ is the set $\phi^{-1}(x)$ (also written ϕ_x) defined by

$$\phi^{-1}(x) \overset{\text{def}}{=} \mathrm{pt} \times_{[x],K,\phi} X, \qquad \begin{tikzcd} \phi^{-1}(x) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "\phi"] \\ \mathrm{pt} \arrow[r, "{[x]}"'] & K. \end{tikzcd}$$

3.2 Morphisms of Fibred Sets

Definition 3.2.1 ► Morphisms of Fibred Sets

A **morphism of** K**-fibred sets from** (X, ϕ) **to** (Y, ψ) is a function $f \colon X \to Y$ such that the diagram[1]

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\phi"'] & & Y \arrow[dl, "\psi"] \\ & K & \end{tikzcd}$$

commutes.

[1] *Further Terminology:* The **transport map associated to** f **at** $x \in K$ is the function

$$f^*_x \colon \phi^{-1}(x) \to \psi^{-1}(x)$$

given by the dashed map in the diagram

$$\begin{tikzcd} \phi^{-1}(x) \arrow[rr] \arrow[dd] \arrow[dr, dashed] & & X \arrow[dd, "\phi" near start] \arrow[dr, "f"] & \\ & \psi^{-1}(x) \arrow[rr] \arrow[dd] & & Y \arrow[dd, "\psi"] \\ \mathrm{pt} \arrow[rr, "{[x]}" near start] \arrow[dr, equal] & & K \arrow[dr, equal] & \\ & \mathrm{pt} \arrow[rr, "{[x]}"'] & & K. \end{tikzcd}$$

3.3 The Category of Fibred Sets Over a Fixed Base

Definition 3.3.1 ▶ The Category of K-Fibred Sets

The **category of K-fibred sets** is the category $\mathsf{FibSets}(K)$ defined as the slice category $\mathsf{Sets}_{/K}$ of Sets over K:

$$\mathsf{FibSets}(K) \overset{\text{def}}{=} \mathsf{Sets}_{/K}.$$

Remark 3.3.2 ▶ Unwinding Definition 3.3.1

In detail $\mathsf{FibSets}(K)$ is the category where

- *Objects.* The objects of $\mathsf{FibSets}(K)$ are pairs (X, ϕ) consisting of
  - *The Fibred Set.* A set X;
  - *The Fibration.* A function $\phi \colon X \to K$;
- *Morphisms.* A morphism of $\mathsf{FibSets}(K)$ from (X, ϕ) to (Y, ψ) is a function $f \colon X \to Y$ making the diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\phi"'] & & Y \arrow[dl, "\psi"] \\ & X & \end{tikzcd}$$

  commute;
- *Identities.* For each $(X, \phi) \in \mathrm{Obj}(\mathsf{FibSets}(K))$, the unit map

$$\mathbb{1}^{\mathsf{FibSets}(K)}_{(X,\phi)} \colon \mathrm{pt} \to \mathrm{Hom}_{\mathsf{FibSets}(K)}((X, \phi), (X, \phi))$$

  of $\mathsf{FibSets}(K)$ at (X, ϕ) is given by

$$\mathrm{id}^{\mathsf{FibSets}(K)}_{(X,\phi)} \overset{\text{def}}{=} \mathrm{id}_X,$$

  as witnessed by the commutativity of the diagram

$$\begin{tikzcd} X \arrow[rr, equal, "\mathrm{id}_X"] \arrow[dr, "\phi"'] & & X \arrow[dl, "\phi"] \\ & K & \end{tikzcd}$$

  in Sets;

- *Composition.* For each $\mathbf{X} = (X, \phi)$, $\mathbf{Y} = (Y, \psi)$, $\mathbf{Z} = (Z, \chi) \in \mathrm{Obj}(\mathsf{FibSets}(K))$, the composition map

$$\circ^{\mathsf{FibSets}(K)}_{\mathbf{X},\mathbf{Y},\mathbf{Z}} \colon \mathrm{Hom}_{\mathsf{FibSets}(K)}(\mathbf{Y}, \mathbf{Z}) \times \mathrm{Hom}_{\mathsf{FibSets}(K)}(\mathbf{X}, \mathbf{Y}) \to \mathrm{Hom}_{\mathsf{FibSets}(K)}(\mathbf{X}, \mathbf{Z})$$

of $\mathsf{FibSets}(K)$ at $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ is defined by

$$\circ^{\mathsf{FibSets}(K)}_{\mathbf{X},\mathbf{Y},\mathbf{Z}} \overset{\text{def}}{=} \circ^{\mathsf{Sets}}_{X,Y,Z},$$

as witnessed by the commutativity of the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "\phi"'] & Y \arrow[r, "g"] \arrow[d, "\psi"] & Z \arrow[dl, "\chi"] \\ & K & \end{tikzcd}$$

in Sets.

3.4 The Category of Fibred Sets

Definition 3.4.1 ► The Category of Fibred Sets

The **category of fibred sets** is the category $\mathsf{FibSets}$ defined as the Grothendieck construction of the functor $\mathsf{FibSets} \colon \mathsf{Sets}^{\mathsf{op}} \to \mathsf{Cats}$ of Proposition 4.1.4:

$$\mathsf{FibSets} \overset{\text{def}}{=} \int^{\mathsf{Sets}} \mathsf{FibSets}.$$

Remark 3.4.2 ► Unwinding Definition 3.4.1

In detail, the **category of fibred sets** is the category $\mathsf{FibSets}$ where

- *Objects.* The objects of $\mathsf{FibSets}$ are pairs $(K, (X, \phi_X))$ consisting of
  - *The Base Set.* A set K;
  - *The Fibred Set.* A K-fibred set $\phi_X \colon X \to K$;
- *Morphisms.* A morphism of $\mathsf{FibSets}$ from $(K, (X, \phi_X))$ to $(K', (Y, \phi_Y))$ is a pair (ϕ, f) consisting of

- *The Base Map.* A map of sets $\phi\colon K \to K'$;
- *The Morphism of Fibred Sets.* A morphism of K-fibred sets

$$f\colon (X,\phi_X) \to \phi_Y^*(Y), \qquad \begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\phi_X"'] & & Y \times_{K'} K \arrow[dl, "\mathrm{pr}_2"] \\ & K; & \end{tikzcd}$$

· *Identities.* For each $(K,X) \in \mathrm{Obj}(\mathsf{FibSets})$, the unit map

$$\mathbb{1}^{\mathsf{FibSets}}_{(K,X)}\colon \mathrm{pt} \to \mathsf{FibSets}((K,X),(K,X))$$

of FibSets at (K,X) is defined by

$$\mathrm{id}^{\mathsf{FibSets}}_{(K,X)} \stackrel{\mathrm{def}}{=} (\mathrm{id}_K, \sim),$$

where $\sim$ is the isomorphism $X \to X \times_K K$ as in the diagram

$$\begin{tikzcd} X \arrow[rr, dashed, "\sim"] \arrow[dr, "\phi_X"'] & & X \times_K K \arrow[dl, "\mathrm{pr}_2"] \\ & K; & \end{tikzcd}$$

· *Composition.* For each $\mathbf{X} = (K,X), \mathbf{Y} = (K',Y), \mathbf{Z} = (K'',Z) \in \mathrm{Obj}(\mathsf{FibSets})$, the composition map

$$\circ^{\mathsf{FibSets}}_{\mathbf{X},\mathbf{Y},\mathbf{Z}}\colon \mathsf{FibSets}(\mathbf{Y},\mathbf{Z}) \times \mathsf{FibSets}(\mathbf{X},\mathbf{Y}) \to \mathsf{FibSets}(\mathbf{X},\mathbf{Z})$$

of FibSets at $(\mathbf{X},\mathbf{Y},\mathbf{Z})$ is defined by

$$g \circ^{\mathsf{FibSets}}_{\mathbf{X},\mathbf{Y},\mathbf{Z}} f \stackrel{\mathrm{def}}{=} (g \times_{K'} \mathrm{id}_K) \circ f$$

as in the diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[drr, "\phi_X"'] & & Y \times_{K'} K \arrow[r, "g \times_{K'} \mathrm{id}_K"] \arrow[d, "\mathrm{pr}_2"] & \overbrace{(Z \times_{K''} K') \times_{K'} K}^{\cong Z \times_{K''} K} \arrow[dl, "\mathrm{pr}_2"] \\ & & K; & \end{tikzcd}$$

for each $f \in \mathsf{FibSets}(\mathbf{X},\mathbf{Y})$ and each $g \in \mathsf{FibSets}(\mathbf{Y},\mathbf{Z})$.

4 Constructions With Fibred Sets

4.1 Change of Base

Let $f\colon K \longrightarrow K'$ be a function and let (X, ϕ) be a K'-fibred set.

DEFINITION 4.1.1 ► CHANGE OF BASE FOR FIBRED SETS

The **change of base of** (X, ϕ) **to** K is the K-fibred set $f^*(X)$ defined by

$$f^*(X) \overset{\text{def}}{=} (K \times_{K'} X, \mathrm{pr}_1), \qquad \begin{tikzcd} f^*(X) \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "\phi"] \\ K \arrow[r, "f"'] & K'. \end{tikzcd}$$

PROPOSITION 4.1.2 ► FUNCTORIALITY OF CHANGE OF BASE

The assignment $X \mapsto f^*(X)$ defines a functor

$$f^*\colon \mathsf{FibSets}(K') \longrightarrow \mathsf{FibSets}(K),$$

where

- *Action on Objects.* For each $(X, \phi) \in \mathrm{Obj}(\mathsf{FibSets}(K'))$, we have

$$f^*(X, \phi) \overset{\text{def}}{=} f^*(X);$$

- *Action on Morphisms.* For each $(X, \phi), (Y, \psi) \in \mathrm{Obj}(\mathsf{FibSets}(K'))$, the action on Hom-sets

$$f^*_{X,Y}\colon \mathrm{Hom}_{\mathsf{FibSets}(K')}(X, Y) \longrightarrow \mathrm{Hom}_{\mathsf{FibSets}(K)}(f^*(X), f^*(Y))$$

of f^* at $((X, \phi), (Y, \psi))$ is the map sending a morphism of K'-fibred sets

$$g\colon (X, \phi) \to (Y, \psi)$$

to the morphism of K-fibred sets given by the dashed morphism in the diagram

$$
\begin{tikzcd}
f^*(X) \arrow[rr] \arrow[dr, dashed] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & & X \arrow[dr, "g"] \arrow[dd, dashed, "\phi"] & \\
& f^*(Y) \arrow[rr] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & & Y \arrow[dd, "\psi"] \\
K \arrow[rr, "f"] \arrow[dr, equal] & & K' \arrow[dr, equal] & \\
& K \arrow[rr, "f"'] & & K'.
\end{tikzcd}
$$

Proof 4.1.3 ► Proof of Proposition 4.1.2

Omitted. □

Proposition 4.1.4 ► Functoriality of Categories of K-Fibred Sets

The assignment $K \mapsto \mathsf{FibSets}(K)$ defines a functor

$$\mathsf{FibSets}\colon \mathsf{Sets}^{\mathsf{op}} \to \mathsf{Cats},$$

where

- *Action on Objects.* For each $K \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$[\mathsf{FibSets}](K) \stackrel{\text{def}}{=} \mathsf{FibSets}(K);$$

- *Action on Morphisms.* For each $K, K' \in \mathrm{Obj}(\mathsf{Sets})$, the action on Hom-sets

$$\mathsf{Sets}_{/(-)|K,K'}\colon \mathsf{Sets}^{\mathsf{op}}(K,K') \to \mathsf{Fun}(\mathsf{FibSets}(K), \mathsf{FibSets}(K'))$$

 of $\mathsf{Sets}_{/(-)}$ at (K,K') is the map sending a map of sets $f\colon K \to K'$ to the functor

$$\mathsf{Sets}_{/f}\colon \mathsf{FibSets}(K') \to \mathsf{FibSets}(K)$$

 defined by

$$\mathsf{Sets}_{/f} \stackrel{\text{def}}{=} f^*.$$

Proof 4.1.5 ▶ Proof of Proposition 4.1.4

Omitted.

4.2 Dependent Sums

Let $f\colon K \to K'$ be a function and let (X, ϕ) be a K-fibred set.

Definition 4.2.1 ▶ Dependent Sums for Fibred Sets

The **dependent sum**[1] **of** (X, ϕ) is the K'-fibred set $\Sigma_f(X)$[2] defined by

$$\begin{aligned}\Sigma_f(X) &\overset{\text{def}}{=} \left(\Sigma_f(X), \Sigma_f(\phi)\right)\\ &\overset{\text{def}}{=} (X, f \circ \phi).\end{aligned}$$

[1]The name "dependent sum" comes from the fact that the fibre $\Sigma_f(\phi)^{-1}(x)$ of $\Sigma_f(X)$ at $x \in K'$ is given by

$$\Sigma_f(\phi)^{-1}(x) \cong \coprod_{y \in f^{-1}(x)} \phi^{-1}(y);$$

see Item 2 of Proposition 4.2.2.

[2]*Further Notation:* Also written $f_*(X)$.

Proposition 4.2.2 ▶ Properties of Dependent Sums of Fibred Sets

Let $f\colon K \to K'$ be a function.

1. *Functoriality.* The assignment $X \mapsto \Sigma_f(X)$ defines a functor

$$\Sigma_f\colon \mathsf{FibSets}(K) \to \mathsf{FibSets}(K'),$$

where

- *Action on Objects.* For each $(X, \phi) \in \mathrm{Obj}(\mathsf{FibSets}(K))$, we have

$$\Sigma_f(X, \phi) \overset{\text{def}}{=} \left(\Sigma_f(X), \Sigma_f(\phi)\right);$$

- *Action on Morphisms.* For each $(X, \phi), (Y, \psi) \in \mathrm{Obj}(\mathsf{FibSets}(K))$, the action on Hom-sets

$$\Sigma_{f|X,Y}\colon \mathrm{Hom}_{\mathsf{FibSets}(K)}(X, Y) \to \mathrm{Hom}_{\mathsf{FibSets}(K)}\left(\Sigma_f(X), \Sigma_f(Y)\right)$$

of Σ_f at $((X,\phi),(Y,\psi))$ is the map sending a morphism of K-fibred sets

$$g\colon (X,\phi) \longrightarrow (Y,\psi)$$

to the morphism of K'-fibred sets defined by

$$\Sigma_f(g) \stackrel{\text{def}}{=} g.$$

2. *Interaction With Fibres.* We have a bijection of sets

$$\Sigma_f(\phi)^{-1}(x) \cong \coprod_{y\in f^{-1}(x)} \phi^{-1}(y)$$

for each $x \in K'$.

Proof 4.2.3 ► Proof of Proposition 4.2.2

Item 1: Functoriality

Omitted.

Item 2: Interaction With Fibres

Indeed, we have

$$\begin{aligned}\Sigma_f(\phi)^{-1}(x) &\stackrel{\text{def}}{=} \mathrm{pt} \times_{[x],K',f\circ\phi} X\\ &\cong \{(a,y) \in X \times K \mid f(\phi(a)) = x\}\\ &\cong \coprod_{y\in f^{-1}(x)} \phi^{-1}(y)\end{aligned}$$

for each $x \in K'$. □

4.3 Dependent Products

Let $f: K \longrightarrow K'$ be a function and let (X,ϕ) be a K-fibred set.

Definition 4.3.1 ► Dependent Products for Fibred Sets

The **dependent product**[1] **of** (X,ϕ) is the K'-fibred set $\Pi_f(X)$[2] consisting of[3]

- *The Underlying Set.* The set $\Pi_f(X)$ defined by

$$\begin{aligned}\Pi_f(X) &\overset{\text{def}}{=} \coprod_{x\in K'} \Gamma^{\phi}_{f^{-1}(x)}\left(\phi^{-1}\left(f^{-1}(x)\right)\right)\\ &\overset{\text{def}}{=} \left\{(x,h) \in \coprod_{x\in K'} \mathsf{Sets}\left(f^{-1}(x), \phi^{-1}\left(f^{-1}(x)\right)\right) \,\middle|\, \phi \circ h = \mathrm{id}_{f^{-1}(x)}\right\};\end{aligned}$$

- *The Fibration.* The map of sets

$$\Pi_f(\phi)\colon \coprod_{x\in K'} \Gamma^{\phi}_{f^{-1}(x)}\left(\phi^{-1}\left(f^{-1}(x)\right)\right) \to K$$

defined by sending a map $h\colon f^{-1}(x) \to \phi^{-1}\left(f^{-1}(x)\right)$ to its index $x \in K$.

[1]The name "dependent product" comes from the fact that the fibre $\Pi_f(\phi)^{-1}(x)$ of $\Pi_f(X)$ at $x \in K'$ is given by

$$\Pi_f(\phi)^{-1}(x) \cong \prod_{y\in f^{-1}(x)} \phi^{-1}(y);$$

see Item 2 of Proposition 4.3.3.

[2]*Further Notation:* Also written $f_!(X)$.

[3]We can also define dependent products via the internal **Hom** in $\mathsf{FibSets}(K')$; see Item 3 of Proposition 4.3.3.

Example 4.3.2 ▶ Examples of Dependent Products of Sets

Here are some examples of dependent products of sets.

1. *Spaces of Sections.* Let $K = X$, $K' = \mathrm{pt}$, and let $\phi\colon E \to X$ be a map of sets. We have a bijection of sets

$$\begin{aligned}\Pi_{!_X}(\phi) &\cong \Gamma_X(\phi)\\ &\cong \{h \in \mathsf{Sets}(X,E) \mid \phi \circ h = \mathrm{id}_X\}.\end{aligned}$$

2. *Function Spaces.* Let $K = K' = \mathrm{pt}$. We have a bijection of sets

$$\mathsf{Sets}(X,Y) \cong \Pi_{!_X}\left(!_X^*(Y)\right).$$

Proposition 4.3.3 ► Properties of Dependent Products of Fibred Sets

Let $f\colon K \to K'$ be a function.

1. *Functoriality.* The assignment $X \mapsto \Pi_f(X)$ defines a functor

$$\Pi_f\colon \mathsf{FibSets}(K) \to \mathsf{FibSets}(K'),$$

where

- *Action on Objects.* For each $(X, \phi) \in \mathrm{Obj}(\mathsf{FibSets}(K))$, we have

$$\Pi_f(X, \phi) \stackrel{\text{def}}{=} \Pi_f(X);$$

- *Action on Morphisms.* For each $(X, \phi), (Y, \psi) \in \mathrm{Obj}(\mathsf{FibSets}(K))$, the action on Hom-sets

$$\Pi_{f|X,Y}\colon \mathrm{Hom}_{\mathsf{FibSets}(K)}(X, Y) \to \mathrm{Hom}_{\mathsf{FibSets}(K')}(\Pi_f(X), \Pi_f(Y))$$

of Π_f at $((X, \phi), (Y, \psi))$ is the map sending a morphism of K-fibred sets

$$g\colon (X, \phi) \to (Y, \psi)$$

to the morphism of K'-fibred sets from

$$\Pi_f(X) \stackrel{\text{def}}{=} \left\{ (x, h) \in \coprod_{x \in K'} \mathsf{Sets}(f^{-1}(x), \phi^{-1}(f^{-1}(x))) \,\middle|\, \phi \circ h = \mathrm{id}_{f^{-1}(x)} \right\};$$

to

$$\Pi_f(Y) \stackrel{\text{def}}{=} \left\{ (x, h) \in \coprod_{x \in K'} \mathsf{Sets}(f^{-1}(x), \psi^{-1}(f^{-1}(x))) \,\middle|\, \psi \circ h = \mathrm{id}_{f^{-1}(x)} \right\};$$

induced by the composition

$$\begin{aligned}
\mathsf{Sets}(f^{-1}(x), \phi^{-1}(f^{-1}(x))) &= \mathsf{Sets}(f^{-1}(x), [\psi \circ g]^{-1}(f^{-1}(x))) \\
&= \mathsf{Sets}(f^{-1}(x), g^{-1}(\psi^{-1}(f^{-1}(x)))) \\
&\xrightarrow{g_*} \mathsf{Sets}(f^{-1}(x), g(g^{-1}(\psi^{-1}(f^{-1}(x))))) \\
&\xrightarrow{\iota_*} \mathsf{Sets}(f^{-1}(x), \psi^{-1}(f^{-1}(x))),
\end{aligned}$$

where $\iota\colon g(g^{-1}(\psi^{-1}(f^{-1}(x)))) \hookrightarrow \psi^{-1}(f^{-1}(x))$ is the canonical inclusion.[1]

2. *Interaction With Fibres.* We have a bijection of sets

$$\Pi_f(\phi)^{-1}(x) \cong \prod_{y \in f^{-1}(x)} \phi^{-1}(y)$$

for each $x \in K'$.

3. *Construction Using the Internal Hom.* We have

$$\Pi_f(X,\phi) = (K' \times_{\mathbf{Hom}_{\mathsf{FibSets}(K')}(f,f)} \mathbf{Hom}_{\mathsf{Sets}/K'}(f, f \circ \phi), \mathrm{pr}_1),$$

$$\begin{tikzcd} \Pi_f(X,\phi) \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Hom}_{\mathsf{Sets}/K'}(f, f \circ \phi) \arrow[d] \\ K' \arrow[r, "I"'] & \mathbf{Hom}_{\mathsf{FibSets}(K')}(f,f), \end{tikzcd}$$

where the bottom map is defined by

$$I(x) \overset{\text{def}}{=} \mathrm{id}_{f^{-1}(x)}$$

for each $x \in K'$.

[1] Note that the section condition is satisfied: given $(x,h) \in \Pi_f(X)$, we have

$$\begin{aligned} \psi \circ [\Pi_f(g)](h) &\overset{\text{def}}{=} \psi \circ (g \circ h) \\ &= (\psi \circ g) \circ h \\ &= \phi \circ h \\ &= \mathrm{id}_{f^{-1}(x)}. \end{aligned}$$

PROOF 4.3.4 ► PROOF OF PROPOSITION 4.3.3

Item 1: Functoriality

Omitted.

Item 2: Interaction With Fibres

Indeed, we have

$$\begin{aligned}\Pi_f(\phi)^{-1}(x) &\overset{\text{def}}{=} \left\{(y,h)\in\Pi_f(X)\,\middle|\,[\Pi_f(\phi)](h)=x\right\}\\ &\overset{\text{def}}{=} \left\{(y,h)\in\Pi_f(X)\,\middle|\,y=x\right\}\\ &\cong \left\{h\in\mathsf{Sets}\big(f^{-1}(x),\phi^{-1}\big(f^{-1}(x)\big)\big)\,\middle|\,\phi\circ h=\mathrm{id}_{f^{-1}(x)}\right\}\\ &\cong \prod_{y\in f^{-1}(x)}\phi^{-1}(y)\end{aligned}$$

for each $x \in K'$.

Item 3: Construction Using the Internal Hom

Omitted.

4.4 Internal Homs

Let K be a set and let (X, ϕ) and (Y, ψ) be K-fibred sets.

Definition 4.4.1 ► Internal Hom of Fibred Sets

The **internal Hom of fibred sets from** (X, ϕ) **to** (Y, ψ) is the fibred set $\mathbf{Hom}_{\mathsf{FibSets}(K)}(X, Y)$ consisting of

- *The Underlying Set.* The set $\mathbf{Hom}_{\mathsf{FibSets}(K)}(X, Y)$ defined by

$$\mathbf{Hom}_{\mathsf{FibSets}(K)}(X,Y) \overset{\text{def}}{=} \coprod_{x\in K}\mathsf{Sets}\big(\phi^{-1}(x),\psi^{-1}(x)\big);$$

- *The Fibration.* The map of sets[1]

$$\phi_{\mathbf{Hom}_{\mathsf{FibSets}(K)}(X,Y)}\colon \underbrace{\mathbf{Hom}_{\mathsf{FibSets}(K)}(X,Y)}_{\coprod_{x\in K}\mathsf{Sets}(\phi^{-1}(x),\psi^{-1}(x))} \longrightarrow K$$

  defined by sending a map $f\colon \phi^{-1}(x) \longrightarrow \psi^{-1}(x)$ to its index $x \in K$.

[1] The fibres of the internal **Hom** of $\mathsf{FibSets}(K)$ are precisely the sets $\mathsf{Sets}\big(\phi^{-1}(x),\psi^{-1}(x)\big)$, i.e. we have

$$\phi_{\mathbf{Hom}_{\mathsf{FibSets}(K)}(X,Y)|x} \cong \mathsf{Sets}\Big(\phi^{-1}(x),\psi^{-1}(x)\Big)$$

for each $x \in K$.

4.5 Adjointness for Fibred Sets

Let $f\colon K \to K'$ be a map of sets.

Proposition 4.5.1 ▶ Adjointness for Fibred Sets

We have a triple adjunction

$$
(\Sigma_f \dashv f^* \dashv \Pi_f)\colon\quad
\begin{tikzcd}
\mathsf{FibSets}(K) \arrow[r, bend left, "\Sigma_f"] \arrow[r, bend right, "\Pi_f"'] & \mathsf{FibSets}(K') \arrow[l, "f^*"']
\end{tikzcd}.
$$

Proof 4.5.2 ▶ Proof of Proposition 4.5.1

Omitted. ∎

5 Un/Straightening for Indexed and Fibred Sets

5.1 Straightening for Fibred Sets

Let K be a set and let (X, ϕ) be a K-fibred set.

Definition 5.1.1 ▶ The Straightening of a Fibred Set

The **straightening of** (X, ϕ) is the K-indexed set

$$\mathsf{St}_K(X, \phi)\colon K_{\mathsf{disc}} \to \mathsf{Sets}$$

defined by

$$\mathsf{St}_K(X, \phi)_x \overset{\text{def}}{=} \phi^{-1}(x)$$

for each $x \in K$.

Proposition 5.1.2 ▶ Properties of Straightening for Fibred Sets

Let K be a set.

1. *Functoriality.* The assignment $(X, \phi) \mapsto \mathsf{St}_K(X, \phi)$ defines a functor

$$\mathsf{St}_K : \mathsf{FibSets}(K) \longrightarrow \mathsf{ISets}(K)$$

   - *Action on Objects.* For each $(X, \phi) \in \mathrm{Obj}(\mathsf{FibSets}(K))$, we have

$$[\mathsf{St}_K](X, \phi) \stackrel{\text{def}}{=} \mathsf{St}_K(X, \phi);$$

   - *Action on Morphisms.* For each $(X, \phi), (Y, \psi) \in \mathrm{Obj}(\mathsf{FibSets}(K))$, the action on Hom-sets

$$\mathsf{St}_{K|X,Y} : \mathrm{Hom}_{\mathsf{FibSets}(K)}(X, Y) \to \mathrm{Hom}_{\mathsf{ISets}(K)}(\mathsf{St}_K(X), \mathsf{St}_K(Y))$$

     of St_K at (X, Y) is given by sending a morphism

$$f : (X, \phi) \to (Y, \psi)$$

     of K-fibred sets to the morphism

$$\mathsf{St}_K(f) : \mathsf{St}_K(X, \phi) \to \mathsf{St}_K(Y, \psi)$$

     of K-indexed sets defined by

$$\mathsf{St}_K(f) \stackrel{\text{def}}{=} \{f^*_x\}_{x \in K},$$

     where f^*_x is the transport map associated to f at $x \in K$ of Definition 3.2.1.

2. *Interaction With Change of Base/Indexing.* Let $f : K \to K'$ be a map of sets. The diagram

$$\begin{tikzcd}
\mathsf{FibSets}(K') \arrow[r, "f^*"] \arrow[d, "\mathsf{St}_{K'}"'] & \mathsf{FibSets}(K) \arrow[d, "\mathsf{St}_K"] \\
\mathsf{ISets}(K') \arrow[r, "f^*"'] & \mathsf{ISets}(K)
\end{tikzcd}$$

   commutes.

3. *Interaction With Dependent Sums.* Let $f\colon K \to K'$ be a map of sets. The diagram

$$\begin{tikzcd}
\mathsf{FibSets}(K) \arrow[r, "\Sigma_f"] \arrow[d, "\mathrm{St}_K"'] & \mathsf{FibSets}(K') \arrow[d, "\mathrm{St}_{K'}"] \\
\mathsf{ISets}(K) \arrow[r, "\Sigma_f"'] & \mathsf{ISets}(K')
\end{tikzcd}$$

commutes.

4. *Interaction With Dependent Products.* Let $f\colon K \to K'$ be a map of sets. The diagram

$$\begin{tikzcd}
\mathsf{Sets}_{/K} \arrow[r, "\Pi_f"] \arrow[d, "\mathrm{St}_K"'] & \mathsf{FibSets}(K') \arrow[d, "\mathrm{St}_{K'}"] \\
\mathsf{ISets}(K) \arrow[r, "\Pi_f"'] & \mathsf{ISets}(K')
\end{tikzcd}$$

commutes.

Proof 5.1.3 ► Proof of Proposition 5.1.2

Item 1: Functoriality

Omitted.

Item 2: Interaction With Change of Base/Indexing

Indeed, we have

$$
\begin{aligned}
\mathsf{St}_K(f^*(X,\phi))_x &\overset{\text{def}}{=} \mathsf{St}_K(K \times_{K'} X)_x \\
&\overset{\text{def}}{=} \left(\mathsf{pr}_1^{K\times_{K'}X}\right)^{-1}(x) \\
&= \left\{(k,y) \in K \times_{K'} X \,\middle|\, \mathsf{pr}_1^{K\times_{K'}X}(k,y) = x\right\} \\
&= \{(k,y) \in K \times_{K'} X \mid k = x\} \\
&= \{(k,y) \in K \times X \mid k = x \text{ and } f(k) = \phi(y)\} \\
&\cong \{y \in X \mid \phi(y) = f(x)\} \\
&= \phi^{-1}(f(x)) \\
&\overset{\text{def}}{=} f^*\big(\phi^{-1}(x)\big) \\
&\overset{\text{def}}{=} f^*\big(\mathsf{St}_{K'}(X,\phi)_x\big)
\end{aligned}
$$

for each $(X,\phi) \in \mathsf{Obj}(\mathsf{FibSets}(K'))$ and each $x \in K$, and similarly for morphisms.

Item 3: Interaction With Dependent Sums

Indeed, we have

$$
\begin{aligned}
\mathsf{St}_{K'}\big(\Sigma_f(X,\phi)\big)_x &\overset{\text{def}}{=} \Sigma_f(\phi)^{-1}(x) \\
&\cong \coprod_{\substack{y\in X\\ f(y)=x}} \phi^{-1}(y) \\
&\cong \Sigma_f\big(\phi^{-1}(x)\big) \\
&\overset{\text{def}}{=} \Sigma_f\big(\mathsf{St}_K(X,\phi)_x\big)
\end{aligned}
$$

for each $(X,\phi) \in \mathsf{Obj}(\mathsf{FibSets}(K))$ and each $x \in K'$, where we have used Item 2 of Proposition 4.2.2 for the first bijection, and similarly for morphisms.

Item 4: Interaction With Dependent Products

Indeed, we have

$$
\begin{aligned}
\mathsf{St}_{K'}\big(\Pi_f(X,\phi)\big)_x &\overset{\text{def}}{=} \Pi_f(\phi)^{-1}(x) \\
&\cong \prod_{\substack{y\in X\\ f(y)=x}} \phi^{-1}(y) \\
&\cong \Pi_f\big(\phi^{-1}(x)\big) \\
&\overset{\text{def}}{=} \Pi_f\big(\mathsf{St}_K(X,\phi)_x\big)
\end{aligned}
$$

for each $(X, \phi) \in \mathrm{Obj}(\mathsf{FibSets}(K))$ and each $x \in K'$, where we have used Item 2 of Proposition 4.3.3 for the first bijection, and similarly for morphisms.

5.2 Unstraightening for Indexed Sets

Let K be a set and let X be a K-indexed set.

Definition 5.2.1 ► The Unstraightening of an Indexed Set

The **unstraightening of** X is the K-fibred set

$$\phi_{\mathsf{Un}_K} \colon \mathsf{Un}_K(X) \to K$$

consisting of

- *The Underlying Set.* The set $\mathsf{Un}_K(X)$ defined by

$$\mathsf{Un}_K(X) \stackrel{\text{def}}{=} \coprod_{x \in K} X_x;$$

- *The Fibration.* The map of sets

$$\phi_{\mathsf{Un}_K} \colon \mathsf{Un}_K(X) \to K$$

  defined by sending an element of $\coprod_{x \in K} X_x$ to its index in K.

Proposition 5.2.2 ► Properties of Unstraightening for Indexed Sets

Let K be a set.

1. *Functoriality.* The assignment $X \mapsto \mathsf{Un}_K(X)$ defines a functor

$$\mathsf{Un}_K \colon \mathsf{ISets}(K) \to \mathsf{FibSets}(K)$$

   - *Action on Objects.* For each $X \in \mathrm{Obj}(\mathsf{ISets}(K))$, we have

$$[\mathsf{Un}_K](X) \stackrel{\text{def}}{=} \mathsf{Un}_K(X);$$

   - *Action on Morphisms.* For each $X, Y \in \mathrm{Obj}(\mathsf{ISets}(K))$, the action on Hom-sets

$$\mathsf{Un}_{K|X,Y} \colon \mathrm{Hom}_{\mathsf{ISets}(K)}(X, Y) \to \mathrm{Hom}_{\mathsf{FibSets}(K)}(\mathsf{Un}_K(X), \mathsf{Un}_K(Y))$$

of Un_K at (X, Y) is defined by

$$\mathsf{Un}_{K|X,Y}(f) \stackrel{\text{def}}{=} \coprod_{x \in K} f_x^*.$$

2. *Interaction With Fibres.* We have a bijection of sets

$$\phi_{\mathsf{Un}_K}^{-1}(x) \cong X_x$$

for each $x \in K$.

3. *As a Pullback.* We have a bijection of sets

$$\mathsf{Un}_K(X) \cong K_{\mathsf{disc}} \times_{\mathsf{Sets}} \mathsf{Sets}_*, \qquad \begin{tikzcd} \mathsf{Un}_K(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Sets}_* \arrow[d, "\mathrm{忘}"] \\ K_{\mathsf{disc}} \arrow[r, "X"'] & \mathsf{Sets}. \end{tikzcd}$$

4. *As a Colimit.* We have a bijection of sets

$$\mathsf{Un}_K(X) \cong \mathrm{colim}(X).$$

5. *Interaction With Change of Indexing/Base.* Let $f : K \to K'$ be a map of sets. The diagram

$$\begin{tikzcd} \mathsf{ISets}(K') \arrow[r, "f^*"] \arrow[d, "\mathsf{Un}_{K'}"'] & \mathsf{ISets}(K) \arrow[d, "\mathsf{Un}_K"] \\ \mathsf{FibSets}(K') \arrow[r, "f^*"'] & \mathsf{FibSets}(K) \end{tikzcd}$$

commutes.

6. *Interaction With Dependent Sums.* Let $f : K \to K'$ be a map of sets. The diagram

$$\begin{tikzcd} \mathsf{ISets}(K) \arrow[r, "\Sigma_f"] \arrow[d, "\mathsf{Un}_K"'] & \mathsf{ISets}(K') \arrow[d, "\mathsf{Un}_{K'}"] \\ \mathsf{FibSets}(K) \arrow[r, "\Sigma_f"'] & \mathsf{FibSets}(K') \end{tikzcd}$$

commutes.

7. *Interaction With Dependent Products.* Let $f\colon K \to K'$ be a map of sets. The diagram

$$\begin{tikzcd}
\mathsf{ISets}(K) \arrow[r, "\Pi_f"] \arrow[d, "\mathsf{Un}_K"'] & \mathsf{ISets}(K') \arrow[d, "\mathsf{Un}_{K'}"] \\
\mathsf{FibSets}(K) \arrow[r, "\Pi_f"'] & \mathsf{FibSets}(K')
\end{tikzcd}$$

commutes.

Proof 5.2.3 ► Proof of Proposition 5.2.2

Item 1: Functoriality

Omitted.

Item 2: Interaction With Fibres

Omitted.

Item 3: As a Pullback

Omitted.

Item 4: As a Colimit

Clear.

Item 5: Interaction With Change of Indexing/Base

Indeed, we have

$$\begin{aligned}
\mathsf{Un}_K(f^*(X)) &\overset{\text{def}}{=} \mathsf{Un}_K(X \circ f) \\
&\overset{\text{def}}{=} \coprod_{x\in K} X_{f(x)} \\
&\cong \left\{ (x,(y,a)) \in K \times \coprod_{y\in K'} X_y \,\middle|\, f(x) = y \right\} \\
&\cong K \times_{K'} \coprod_{y\in K'} X_y \\
&\overset{\text{def}}{=} K \times_{K'} \mathsf{Un}_{K'}(X) \\
&\overset{\text{def}}{=} f^*(\mathsf{Un}_{K'}(X))
\end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{ISets}(K'))$. Similarly, it can be shown that we also have $\mathsf{Un}_K(f^*(\phi)) = f^*(\mathsf{Un}_{K'}(\phi))$ and that $\mathsf{Un}_K \circ f^* = f^* \circ \mathsf{Un}_{K'}$ also holds on morphisms.

Item 6: Interaction With Dependent Sums

Indeed, we have

$$\begin{aligned}
\mathsf{Un}_{K'}\big(\Sigma_f(X)\big) &\overset{\text{def}}{=} \coprod_{x\in K'} \Sigma_f(X)_x \\
&\cong \coprod_{x\in K'} \coprod_{y\in f^{-1}(x)} X_y \\
&\cong \coprod_{y\in K} X_y \\
&\cong \mathsf{Un}_K(X) \\
&\overset{\text{def}}{=} \Sigma_f(\mathsf{Un}_K(X))
\end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{ISets}(K))$, where we have used Item 2 of Proposition 4.2.2 for the first bijection. Similarly, it can be shown that we also have $\mathsf{Un}_{K'}\big(\Sigma_f(\phi)\big) = \Sigma_f\big(\phi_{\mathsf{Un}_K}\big)$ and that $\mathsf{Un}_{K'} \circ \Sigma_f = \Sigma_f \circ \mathsf{Un}_K$ also holds on morphisms.

Item 7: Interaction With Dependent Products

Indeed, we have

$$\begin{aligned}
\mathsf{Un}_{K'}\big(\Pi_f(X)\big) &\overset{\text{def}}{=} \coprod_{x\in K'} \Pi_f(X)_x \\
&\cong \coprod_{x\in K'} \prod_{y\in f^{-1}(x)} X_y \\
&\cong \left\{ (x,h) \in \coprod_{x\in K'} \mathsf{Sets}\Big(f^{-1}(x), \phi^{-1}_{\mathsf{Un}_K}\big(f^{-1}(x)\big)\Big) \,\middle|\, \phi \circ h = \mathrm{id}_{f^{-1}(x)} \right\} \\
&\overset{\text{def}}{=} \Pi_f\left(\coprod_{y\in K} X_y\right) \\
&\overset{\text{def}}{=} \Pi_f(\mathsf{Un}_K(X))
\end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{ISets}(K))$, where we have used Item 2 of Proposition 4.3.3 for the first bijection. Similarly, it can be shown that we also have $\mathsf{Un}_{K'}\big(\Pi_f(\phi)\big) = \Pi_f\big(\phi_{\mathsf{Un}_K}\big)$ and that $\mathsf{Un}_{K'} \circ \Pi_f = \Pi_f \circ \mathsf{Un}_K$ also holds on morphisms. □

5.3 The Un/Straightening Equivalence

We have an isomorphism of categories

$$(\mathsf{St}_K \dashv \mathsf{Un}_K) \colon \quad \begin{tikzcd} \mathsf{FibSets}(K) \arrow[r, bend left, "\mathsf{St}_K"] \arrow[r, phantom, "\perp"] & \mathsf{ISets}(K). \arrow[l, bend left, "\mathsf{Un}_K"] \end{tikzcd}$$

PROOF 5.3.2 ► PROOF OF THEOREM 5.3.1

Omitted.

6 Miscellany

6.1 Other Kinds of Un/Straightening

REMARK 6.1.1 ► OTHER KINDS OF UN/STRAIGHTENING

There are also other kinds of un/straightening for sets, where Sets is replaced by **Rel** or Span:

- *Un/Straightening With **Rel**, I.* We have an isomorphism of sets

  $$\mathrm{Rel}(A, B) \cong \mathrm{Sets}(B \times A, \{\mathrm{true}, \mathrm{false}\}).$$

  by the definition of a relation from A to B, Relations, Definition 1.1.1.

- *Un/Straightening With **Rel**, II.* We have an equivalence of categories

  $$\mathrm{LaxFun}(K_{\mathrm{disc}}, \mathbf{Rel}) \overset{\text{eq.}}{\cong} \mathrm{Cats}^{\mathrm{fth}}_{/K_{\mathrm{disc}}},$$

  where $\mathrm{Cats}^{\mathrm{fth}}_{/K_{\mathrm{disc}}}$ is the full subcategory of $\mathrm{Cats}_{/K_{\mathrm{disc}}}$ spanned by the faithful functors; see [Nie04, Theorem 3.1].

- *Un/Straightening With* Span*, I.* For each $A, B \in \mathrm{Obj}(\mathrm{Sets})$, we have a morphism of sets

  $$\mathrm{Span}(A, B) \longrightarrow \mathrm{Sets}(A \times B, \mathbb{N} \cup \{\infty\})$$

  which assemble into an equivalence of categories between Span(Sets) and the category MRel of "multirelations"; see Spans, Remark 7.5.1.

- *Un/Straightening With* Span, *II.* We have an equivalence of categories

$$\mathsf{LaxFun}(K_{\mathrm{disc}}, \mathsf{Span}) \overset{\text{eq.}}{\cong} \mathsf{Cats}_{/K_{\mathrm{disc}}};$$

see [nLa23, Section 3].

Appendices

A Other Chapters

Set Theory

1. Sets
2. Constructions With Sets
3. Pointed Sets
4. Tensor Products of Pointed Sets
5. Indexed and Fibred Sets
6. Relations
7. Spans
8. Posets

Category Theory

9. Categories
10. Constructions With Categories
11. Kan Extensions

Bicategories

12. Bicategories
13. Internal Adjunctions

Internal Category Theory

14. Internal Categories

Cyclic Stuff

15. The Cycle Category

Cubical Stuff

16. The Cube Category

Globular Stuff

17. The Globe Category

Cellular Stuff

18. The Cell Category

Monoids

19. Monoids
20. Constructions With Monoids

Monoids With Zero

21. Monoids With Zero
22. Constructions With Monoids With Zero

Groups

23. Groups
24. Constructions With Groups

Hyper Algebra

25. Hypermonoids
26. Hypergroups
27. Hypersemirings and Hyperrings
28. Quantales

Near-Rings

29. Near-Semirings
30. Near-Rings

Real Analysis

31. Real Analysis in One Variable
32. Real Analysis in Several Variables

Measure Theory

33. Measurable Spaces
34. Measures and Integration

Probability Theory

34. Probability Theory

Stochastic Analysis

35. Stochastic Processes, Martingales, and Brownian Motion
36. Itô Calculus
37. Stochastic Differential Equations

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38. Topological and Smooth Manifolds

Schemes

39. Schemes