

Bicategories

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Contents

1	Monomorphisms in Bicategories	1
1.1	Faithful Monomorphisms.....	1
1.2	Full Monomorphisms.....	2
1.3	Fully Faithful Monomorphisms.....	3
1.4	Strict Monomorphisms.....	3
2	Epimorphisms in Bicategories.....	4
2.1	Faithful Epimorphisms.....	4
2.2	Full Epimorphisms.....	5
2.3	Fully Faithful Epimorphisms.....	5
2.4	Strict Epimorphisms.....	6
A	Other Chapters	8

Create tags and TODO:

1. spans in bicategories: add Proposition 7 here: <https://arxiv.org/abs/1903.03890>
2. add fact: internal adjunctions in $\mathbf{PseudoFun}(\mathcal{C}, \mathcal{D})$ are precisely the invertible strong transformations as in [JY21, Example 6.2.7]. What are the internal adjunctions?

1 Monomorphisms in Bicategories

1.1 Faithful Monomorphisms

Let \mathcal{C} be a bicategory.

Definition 1.1.1.1. A 1-morphism $f: A \rightarrow B$ is a **faithful monomorphism in \mathcal{C}** if the following equivalent conditions are satisfied:

1. For each $X \in \text{Obj}(\mathcal{C})$, the functor

$$f_*: \text{Hom}_{\mathcal{C}}(X, A) \rightarrow \text{Hom}_{\mathcal{C}}(X, B)$$

given by postcomposition by f is faithful.

2. Given a diagram in \mathcal{C} of the form

$$X \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \Downarrow \beta \\ \xrightarrow{\psi} \end{array} A \xrightarrow{f} B,$$

if we have $\text{id}_f \circ \alpha = \text{id}_f \circ \beta$, then $\alpha = \beta$.

Example 1.1.1.2. Here are some examples of faithful monomorphisms.

1. *Full Monomorphisms in \mathbf{Cats}_2 .*
2. *Full Monomorphisms in \mathbf{Rel} .*
3. *Full Monomorphisms in \mathbf{Span} .*

1.2 Full Monomorphisms

Let \mathcal{C} be a bicategory.

Definition 1.2.1.1. A 1-morphism $f: A \rightarrow B$ is a **full monomorphism in \mathcal{C}** if the following equivalent conditions are satisfied:

1. For each $X \in \text{Obj}(\mathcal{C})$, the functor

$$f_*: \text{Hom}_{\mathcal{C}}(X, A) \rightarrow \text{Hom}_{\mathcal{C}}(X, B)$$

given by postcomposition by f is full.

2. For each $X \in \text{Obj}(\mathcal{C})$ and each 2-morphism

$$\gamma: f \circ \phi \Longrightarrow f \circ \psi, \quad X \begin{array}{c} \xrightarrow{f \circ \phi} \\ \gamma \Downarrow \\ \xrightarrow{f \circ \psi} \end{array} B$$

of \mathcal{C} , there exists a 2-morphism $\alpha: \phi \Rightarrow \psi$ of \mathcal{C} such that we have an equality

$$X \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} A \xrightarrow{f} B = X \begin{array}{c} \xrightarrow{f \circ \phi} \\ \gamma \Downarrow \\ \xrightarrow{f \circ \psi} \end{array} B$$

of pasting diagrams in \mathcal{C} , i.e. such that we have

$$\gamma = \text{id}_f \circ \alpha.$$

Example 1.2.1.2. Here are some examples of full monomorphisms.

1. *Full Monomorphisms in \mathbf{Cats}_2 .*
2. *Full Monomorphisms in \mathbf{Rel} .*
3. *Full Monomorphisms in \mathbf{Span} .*

1.3 Fully Faithful Monomorphisms

Let \mathcal{C} be a bicategory.

Definition 1.3.1.1. A 1-morphism $f: A \rightarrow B$ is a **fully faithful monomorphism in \mathcal{C}** if the following equivalent conditions are satisfied:

1. The 1-morphism f is fully and faithful.
2. For each $X \in \text{Obj}(\mathcal{C})$, the functor

$$f_*: \text{Hom}_{\mathcal{C}}(X, A) \rightarrow \text{Hom}_{\mathcal{C}}(X, B)$$

given by postcomposition by f is fully faithful.

3. The conditions in **Item 1** of **Definition 1.1.1.1** and **Item 1** of **Definition 1.2.1.1** hold.

Example 1.3.1.2. Here are some examples of fully faithful monomorphisms.

1. *Fully Faithful Monomorphisms in \mathbf{Cats}_2 .*
2. *Fully Faithful Monomorphisms in \mathbf{Rel} .*
3. *Fully Faithful Monomorphisms in \mathbf{Span} .*

1.4 Strict Monomorphisms

Let \mathcal{C} be a bicategory.

Definition 1.4.1.1. A 1-morphism $f: A \rightarrow B$ is a **strict monomorphism in \mathcal{C}** if the following equivalent conditions are satisfied:

1. For each $X \in \text{Obj}(\mathcal{C})$, the action on objects

$$f_*: \text{Hom}_{\mathcal{C}}(X, A) \rightarrow \text{Hom}_{\mathcal{C}}(X, B)$$

of the functor

$$f_*: \text{Hom}_{\mathcal{C}}(X, A) \rightarrow \text{Hom}_{\mathcal{C}}(X, B)$$

given by postcomposition by f is injective.

2. For each diagram in \mathcal{C} of the form

$$X \begin{array}{c} \xrightarrow{\phi} \\ \xrightarrow{\psi} \end{array} A \xrightarrow{f} B,$$

if $f \circ \phi = f \circ \psi$, then $\phi = \psi$.

Example 1.4.1.2. Here are some examples of strict monomorphisms.

1. *Strict Monomorphisms in \mathbf{Cats}_2 .*
2. *Strict Monomorphisms in \mathbf{Rel} .*
3. *Strict Monomorphisms in \mathbf{Span} .*

2 Epimorphisms in Bicategories

2.1 Faithful Epimorphisms

Let \mathcal{C} be a bicategory.

Definition 2.1.1.1. A 1-morphism $f: A \rightarrow B$ is a **faithful epimorphism in \mathcal{C}** if the following equivalent conditions are satisfied:

1. For each $X \in \text{Obj}(\mathcal{C})$, the functor

$$f^*: \text{Hom}_{\mathcal{C}}(B, X) \rightarrow \text{Hom}_{\mathcal{C}}(A, X)$$

given by precomposition by f is faithful.

2. Given a diagram in \mathcal{C} of the form

$$A \xrightarrow{f} B \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \beta \\ \xrightarrow{\psi} \end{array} X,$$

if we have $\alpha \circ \text{id}_f = \beta \circ \text{id}_f$, then $\alpha = \beta$.

Example 2.1.1.2. Here are some examples of faithful epimorphisms.

1. *Full Epimorphisms in \mathbf{Cats}_2 .*
2. *Full Epimorphisms in \mathbf{Rel} .*
3. *Full Epimorphisms in \mathbf{Span} .*

2.2 Full Epimorphisms

Let \mathcal{C} be a bicategory.

Definition 2.2.1.1. A 1-morphism $f: A \rightarrow B$ is a **full epimorphism in \mathcal{C}** if the following equivalent conditions are satisfied:

1. For each $X \in \text{Obj}(\mathcal{C})$, the functor

$$f^*: \text{Hom}_{\mathcal{C}}(B, X) \rightarrow \text{Hom}_{\mathcal{C}}(A, X)$$

given by precomposition by f is full.

2. For each $X \in \text{Obj}(\mathcal{C})$ and each 2-morphism

$$\gamma: \phi \circ f \Longrightarrow \psi \circ f, \quad X \begin{array}{c} \xrightarrow{\phi \circ f} \\ \gamma \Downarrow \\ \xrightarrow{\psi \circ f} \end{array} B$$

of \mathcal{C} , there exists a 2-morphism $\alpha: \phi \Longrightarrow \psi$ of \mathcal{C} such that we have an equality

$$A \xrightarrow{f} B \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} X \quad = \quad A \begin{array}{c} \xrightarrow{\phi \circ f} \\ \gamma \Downarrow \\ \xrightarrow{\psi \circ f} \end{array} X$$

of pasting diagrams in \mathcal{C} , i.e. such that we have

$$\gamma = \alpha \circ \text{id}_f.$$

Example 2.2.1.2. Here are some examples of full epimorphisms.

1. *Full Epimorphisms in \mathbf{Cats}_2 .*
2. *Full Epimorphisms in \mathbf{Rel} .*
3. *Full Epimorphisms in \mathbf{Span} .*

2.3 Fully Faithful Epimorphisms

Let \mathcal{C} be a bicategory.

Definition 2.3.1.1. A 1-morphism $f: A \rightarrow B$ is a **fully faithful epimorphism in \mathcal{C}** if the following equivalent conditions are satisfied:

1. The 1-morphism f is fully and faithful.
2. For each $X \in \mathbf{Obj}(\mathcal{C})$, the functor

$$f^*: \mathbf{Hom}_{\mathcal{C}}(B, X) \rightarrow \mathbf{Hom}_{\mathcal{C}}(A, X)$$

given by precomposition by f is fully faithful.

3. The conditions in [Item 1](#) of [Definition 2.1.1.1](#) and [Item 1](#) of [Definition 2.2.1.1](#) hold.

Example 2.3.1.2. Here are some examples of fully faithful epimorphisms.

1. *Fully Faithful Epimorphisms in \mathbf{Cats}_2 .*
2. *Fully Faithful Epimorphisms in \mathbf{Rel} .*
3. *Fully Faithful Epimorphisms in \mathbf{Span} .*

2.4 Strict Epimorphisms

Let \mathcal{C} be a bicategory.

Definition 2.4.1.1. A 1-morphism $f: A \rightarrow B$ is a **strict epimorphism in \mathcal{C}** if the following equivalent conditions are satisfied:

1. For each $X \in \mathbf{Obj}(\mathcal{C})$, the action on objects

$$f^*: \mathbf{Obj}(\mathbf{Hom}_{\mathcal{C}}(B, X)) \rightarrow \mathbf{Obj}(\mathbf{Hom}_{\mathcal{C}}(A, X))$$

of the functor

$$f^*: \mathbf{Hom}_{\mathcal{C}}(B, X) \rightarrow \mathbf{Hom}_{\mathcal{C}}(A, X)$$

given by precomposition by f is injective.

2. For each diagram in \mathcal{C} of the form

$$A \xrightarrow{f} B \begin{array}{c} \xrightarrow{\phi} \\ \xrightarrow{\psi} \end{array} X,$$

if $\phi \circ f = \psi \circ f$, then $\phi = \psi$.

Example 2.4.1.2. Here are some examples of strict epimorphisms.

1. *Strict Epimorphisms in \mathbf{Cats}_2 .*
2. *Strict Epimorphisms in \mathbf{Rel} .*
3. *Strict Epimorphisms in \mathbf{Span} .*

Appendices

A Other Chapters

Set Theory

1. [Sets](#)
2. [Constructions With Sets](#)
3. [Pointed Sets](#)
4. [Tensor Products of Pointed Sets](#)
5. [Indexed and Fibred Sets](#)
6. [Relations](#)
7. [Spans](#)
8. [Posets](#)

Category Theory

9. [Categories](#)
10. [Constructions With Categories](#)
11. [Kan Extensions](#)

Bicategories

12. [Bicategories](#)
13. [Internal Adjunctions](#)

Internal Category Theory

14. [Internal Categories](#)

Cyclic Stuff

15. [The Cycle Category](#)

Cubical Stuff

16. [The Cube Category](#)

Globular Stuff

17. [The Globe Category](#)

Cellular Stuff

18. [The Cell Category](#)

Monoids

19. Monoids

20. Constructions With Monoids

Monoids With Zero

21. Monoids With Zero

22. Constructions With Monoids
With Zero

Groups

23. Groups

24. Constructions With Groups

Hyper Algebra

25. Hypermonoids

26. Hypergroups

27. Hypersemirings and Hyperrings

28. Quantaes

Near-Rings

29. Near-Semirings

30. Near-Rings

Real Analysis

31. Real Analysis in One Variable

32. Real Analysis in Several Variables

Measure Theory

33. Measurable Spaces

34. Measures and Integration

Probability Theory

34. Probability Theory

Stochastic Analysis

35. Stochastic Processes, Martingales, and Brownian Motion

36. Itô Calculus

37. Stochastic Differential Equations

Differential Geometry

38. Topological and Smooth Manifolds

Schemes

39. Schemes