The Gigantic Mess Project

2017–2023

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Version 1d560bdd, compiled on Jan 05, 2024.

Contents

Contents (detailed)

Part I

Sets

Chapter 1

Sets

This chapter (will eventually) contain material on axiomatic set theory, as well as a couple other things.

Contents

1.1 The Enrichment of Sets in Classical Truth Values

1.1.1 (-2)-Categories

Definition 1.1.1.1 ► (-2)-Categories

A (-2)**-category** is the "necessarily true" truth value.[1,2,3]

[1] Thus, there is only one (-2)-category.

[2] A $(-n)$-category for $n = 3, 4, \ldots$ is also the "necessarily true" truth value, coinciding with a (-2)-category.

[3] For motivation, see [BS10, p. 13].

1.1.2 (-1)-Categories

Definition 1.1.2.1 ▶ (-1)-Categories

A (-1)**-category** is a classical truth value.

Remark 1.1.2.2 ▶ Motivation for (-1)-Categories

[1](-1)-categories should be thought of as being "categories enriched in (-2)-categories", having a collection of objects and, for each pair of objects, a Hom-object $\mathrm{Hom}(x, y)$ that is a (-2)-category (i.e. trivial).

Therefore, a (-1)-category C is either ([BS10, pp. 33–34]):

1. *Empty*, having no objects;
2. *Contractible*, having a collection of objects $\{a, b, c, \ldots\}$, but with $\mathrm{Hom}_C(a, b)$ being a (-2)-category (i.e. trivial) for all $a, b \in \mathrm{Obj}(C)$, forcing all objects of C to be uniquely isomorphic to each other.

As such, there are only two (-1)-categories, up to equivalence:

- The (-1)-category false (the empty one);
- The (-1)-category true (the contractible one).

[1]For more motivation, see [BS10, p. 13].

Definition 1.1.2.3 ▶ The Poset of Truth Values

The **poset of truth values**[1] is the poset $(\{\mathrm{true}, \mathrm{false}\}, \preceq)$[2] consisting of

- *The Underlying Set.* The set $\{\mathrm{true}, \mathrm{false}\}$ whose elements are the truth values true and false;
- *The Partial Order.* The partial order

$$\preceq\colon \{\mathrm{true}, \mathrm{false}\} \times \{\mathrm{true}, \mathrm{false}\} \longrightarrow \{\mathrm{true}, \mathrm{false}\}$$

on $\{\mathrm{true}, \mathrm{false}\}$ defined by[3]

$$\begin{aligned} \mathrm{false} \preceq \mathrm{false} &\overset{\mathrm{def}}{=} \mathrm{true}, \\ \mathrm{true} \preceq \mathrm{false} &\overset{\mathrm{def}}{=} \mathrm{false}, \\ \mathrm{false} \preceq \mathrm{true} &\overset{\mathrm{def}}{=} \mathrm{true}, \\ \mathrm{true} \preceq \mathrm{true} &\overset{\mathrm{def}}{=} \mathrm{true}. \end{aligned}$$

[1]*Further Terminology:* Also called the **poset of** (-1)**-categories**.
[2]*Further Notation:* Also written $\{\mathrm{t}, \mathrm{f}\}$.
[3]This partial order coincides with logical implication.

PROPOSITION 1.1.2.4 ► CARTESIAN CLOSEDNESS OF THE POSET OF TRUTH VALUES

The poset of truth values $\{\mathrm{t}, \mathrm{f}\}$ is Cartesian closed with product given by[1]

$$
\begin{aligned}
\mathrm{t} \times \mathrm{t} &= \mathrm{t}, \\
\mathrm{t} \times \mathrm{f} &= \mathrm{f}, \\
\mathrm{f} \times \mathrm{t} &= \mathrm{f}, \\
\mathrm{f} \times \mathrm{f} &= \mathrm{f},
\end{aligned}
$$

and internal Hom $\mathbf{Hom}_{\{\mathrm{t},\mathrm{f}\}}$ given by the partial order of $\{\mathrm{t}, \mathrm{f}\}$, i.e. by

$$
\begin{aligned}
\mathbf{Hom}_{\{\mathrm{t},\mathrm{f}\}}(\mathrm{t}, \mathrm{t}) &= \mathrm{t}, \\
\mathbf{Hom}_{\{\mathrm{t},\mathrm{f}\}}(\mathrm{t}, \mathrm{f}) &= \mathrm{f}, \\
\mathbf{Hom}_{\{\mathrm{t},\mathrm{f}\}}(\mathrm{f}, \mathrm{t}) &= \mathrm{t}, \\
\mathbf{Hom}_{\{\mathrm{t},\mathrm{f}\}}(\mathrm{f}, \mathrm{f}) &= \mathrm{t}.
\end{aligned}
$$

[1] Note that $\times$ coincides with the "and" operator, while $\mathbf{Hom}_{\{\mathrm{t},\mathrm{f}\}}$ coincides with the logical implication operator.

PROOF 1.1.2.5 ► PROOF OF PROPOSITION 1.1.2.4

Existence of Products

We claim that the products $\mathrm{t} \times \mathrm{t}$, $\mathrm{t} \times \mathrm{f}$, $\mathrm{f} \times \mathrm{t}$, and $\mathrm{f} \times \mathrm{f}$ satisfy the universal property of the product in $\{\mathrm{t}, \mathrm{f}\}$. Indeed, consider the diagrams

$$
\begin{tikzcd}
 & P_1 \arrow[dl, bend right, "p^1_1"'] \arrow[d, dashed, "\exists!"] \arrow[dr, bend left, "p^1_2"] & \\
\mathrm{t} & \underbrace{\mathrm{t} \times \mathrm{t}}_{=\mathrm{t}} \arrow[l, two heads, "\mathrm{pr}_1"] \arrow[r, two heads, "\mathrm{pr}_2"'] & \mathrm{t}
\end{tikzcd}
\quad
\begin{tikzcd}
 & P_2 \arrow[dl, bend right, "p^2_1"'] \arrow[d, dashed, "\exists!"] \arrow[dr, bend left, "p^2_2"] & \\
\mathrm{t} & \underbrace{\mathrm{t} \times \mathrm{f}}_{=\mathrm{f}} \arrow[l, two heads, "\mathrm{pr}_1"] \arrow[r, two heads, "\mathrm{pr}_2"'] & \mathrm{f}
\end{tikzcd}
\quad
\begin{tikzcd}
 & P_3 \arrow[dl, bend right, "p^3_1"'] \arrow[d, dashed, "\exists!"] \arrow[dr, bend left, "p^3_2"] & \\
\mathrm{f} & \underbrace{\mathrm{f} \times \mathrm{t}}_{=\mathrm{f}} \arrow[l, two heads, "\mathrm{pr}_1"] \arrow[r, two heads, "\mathrm{pr}_2"'] & \mathrm{t}
\end{tikzcd}
\quad
\begin{tikzcd}
 & P_4 \arrow[dl, bend right, "p^4_1"'] \arrow[d, dashed, "\exists!"] \arrow[dr, bend left, "p^4_2"] & \\
\mathrm{f} & \underbrace{\mathrm{f} \times \mathrm{f}}_{=\mathrm{f}} \arrow[l, two heads, "\mathrm{pr}_1"] \arrow[r, two heads, "\mathrm{pr}_2"'] & \mathrm{f}.
\end{tikzcd}
$$

Here:

1. If $P_1 = \mathrm{t}$, then $p^1_1 = p^1_2 = \mathrm{id}_{\mathrm{t}}$, and there's indeed a unique morphism from P_1 to t making the diagram commute, namely id_{t};
2. If $P_1 = \mathrm{f}$, then $p^1_1 = p^1_2$ are given by the unique morphism from f to t, and there's indeed a unique morphism from P_1 to t making the diagram commute, namely the unique morphism from f to t;
3. If $P_2 = \mathrm{t}$, then there is no morphism p^2_2.
4. If $P_2 = \mathrm{f}$, then p^2_1 is the unique morphism from f to t while $p^2_2 = \mathrm{id}_{\mathrm{f}}$, and there's indeed a unique morphism from P_2 to f making the diagram commute, namely id_{f};

5. The proof for P_3 is similar to the one for P_2;
6. If $P_4 = \mathsf{t}$, then there is no morphism p_1^4 or p_2^4.
7. If $P_4 = \mathsf{f}$, then $p_1^4 = p_2^4 = \mathrm{id}_{\mathsf{f}}$, and there's indeed a unique morphism from P_4 to f making the diagram commute, namely id_{f}.

Cartesian Closedness

We claim there's a bijection

$$\mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(A \times B, C) \cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\big(A, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(B, C)\big)$$

natural in $A, B, C \in \{\mathsf{t}, \mathsf{f}\}$. Indeed:

- For $(A, B, C) = (\mathsf{t}, \mathsf{t}, \mathsf{t})$, we have
$$\begin{aligned} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t} \times \mathsf{t}, \mathsf{t}) &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t}, \mathsf{t}) \\ &= \{\mathrm{id}_{\mathsf{true}}\} \\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t}, \mathsf{t}) \\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\big(\mathsf{t}, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t}, \mathsf{t})\big). \end{aligned}$$
- For $(A, B, C) = (\mathsf{t}, \mathsf{t}, \mathsf{f})$, we have
$$\begin{aligned} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t} \times \mathsf{t}, \mathsf{f}) &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t}, \mathsf{f}) \\ &= \emptyset \\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t}, \mathsf{f}) \\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\big(\mathsf{t}, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t}, \mathsf{f})\big). \end{aligned}$$
- For $(A, B, C) = (\mathsf{t}, \mathsf{f}, \mathsf{t})$, we have
$$\begin{aligned} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t} \times \mathsf{f}, \mathsf{t}) &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{t}) \\ &\cong \mathrm{pt} \\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{t}) \\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\big(\mathsf{f}, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{t})\big). \end{aligned}$$
- For $(A, B, C) = (\mathsf{t}, \mathsf{f}, \mathsf{f})$, we have
$$\begin{aligned} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t} \times \mathsf{f}, \mathsf{f}) &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{f}) \\ &\cong \{\mathrm{id}_{\mathsf{false}}\} \\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{f}) \\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\big(\mathsf{t}, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{f})\big). \end{aligned}$$

- For $(A, B, C) = (\mathsf{f}, \mathsf{t}, \mathsf{t})$, we have

$$\begin{aligned}\mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f} \times \mathsf{t}, \mathsf{t}) &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{t})\\ &\cong \mathrm{pt}\\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{t})\\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(\mathsf{f}, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t}, \mathsf{t})\right).\end{aligned}$$

- For $(A, B, C) = (\mathsf{f}, \mathsf{t}, \mathsf{f})$, we have

$$\begin{aligned}\mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f} \times \mathsf{t}, \mathsf{f}) &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{f})\\ &\cong \{\mathrm{id}_{\mathsf{false}}\}\\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{f})\\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(\mathsf{f}, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t}, \mathsf{f})\right).\end{aligned}$$

- For $(A, B, C) = (\mathsf{f}, \mathsf{f}, \mathsf{t})$, we have

$$\begin{aligned}\mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f} \times \mathsf{f}, \mathsf{t}) &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{t})\\ &\cong \mathrm{pt}\\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{t})\\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(\mathsf{f}, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{t})\right).\end{aligned}$$

- For $(A, B, C) = (\mathsf{f}, \mathsf{f}, \mathsf{f})$, we have

$$\begin{aligned}\mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f} \times \mathsf{f}, \mathsf{f}) &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{f})\\ &= \{\mathrm{id}_{\mathsf{false}}\}\\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{f})\\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(\mathsf{f}, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{f})\right).\end{aligned}$$

The proof of naturality is omitted.

1.1.3 0-Categories

Definition 1.1.3.1 ► 0-Categories

A 0-**category** is a poset.[1]

[1] *Motivation:* A 0-category is precisely a category enriched in the poset of (-1)-categories.

Definition 1.1.3.2 ▶ 0-Groupoids

A 0-**groupoid** is a 0-category in which every morphism is invertible.[1]

[1] That is, a *set*.

1.1.4 Tables of Analogies Between Set Theory and Category Theory

Here we record some analogies between notions in set theory and category theory. Note that the analogies relating to presheaves relate equally well to copresheaves, as the opposite X^{op} of a set X is just X again.

Basics:

Set Theory	Category Theory
Enrichment in $\{\mathsf{true}, \mathsf{false}\}$	Enrichment in Sets
Set X	Category $\mathcal{C}$
Element $x \in X$	Object $X \in \mathsf{Obj}(\mathcal{C})$
Function	Functor
Function $X \longrightarrow \{\mathsf{true}, \mathsf{false}\}$	Functor $\mathcal{C} \longrightarrow \mathsf{Sets}$
Function $X \longrightarrow \{\mathsf{true}, \mathsf{false}\}$	Presheaf $\mathcal{C}^{\mathsf{op}} \longrightarrow \mathsf{Sets}$

Powersets and categories of presheaves:

Set Theory	Category Theory
Powerset $\mathcal{P}(X)$	Presheaf category $\mathsf{PSh}(\mathcal{C})$
Characteristic function $\chi_{\{x\}}$	Representable presheaf h_X
Characteristic embedding $\chi_{(-)} \colon X \hookrightarrow \mathcal{P}(X)$	Yoneda embedding $よ \colon \mathcal{C}^{\mathsf{op}} \hookrightarrow \mathsf{PSh}(\mathcal{C})$
Characteristic relation $\chi_X(-_1, -_2)$	Hom profunctor $\mathsf{Hom}_{\mathcal{C}}(-_1, -_2)$
The Yoneda lemma for sets $\mathsf{Hom}_{\mathcal{P}(X)}(\chi_x, \chi_U) = \chi_U(x)$	The Yoneda lemma for categories $\mathsf{Nat}(h_X, \mathcal{F}) \cong \mathcal{F}(X)$
The characteristic embedding is fully faithful, $\mathsf{Hom}_{\mathcal{P}(X)}\big(\chi_x, \chi_y\big) = \chi_X(x, y)$	The Yoneda embedding is fully faithful, $\mathsf{Nat}(h_X, h_Y) \cong \mathsf{Hom}_{\mathcal{C}}(X, Y)$
Subsets are unions of their elements $U = \bigcup_{x \in U} \{x\}$ or $\chi_U = \operatorname*{colim}_{\chi_x \in \mathsf{Sets}(U, \{\mathsf{t}, \mathsf{f}\})} (\chi_x)$	Presheaves are colimits of representables, $\mathcal{F} \cong \operatorname*{colim}_{h_X \in \int_{\mathcal{C}} \mathcal{F}} (h_X)$

Categories of elements:

Set Theory	Category Theory
Assignment $U \mapsto \chi_U$	Assignment $\mathcal{F} \mapsto \int_C \mathcal{F}$ (the category of elements)
Assignment $U \mapsto \chi_U$ giving an isomorphism $\mathcal{P}(X) \cong \mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\})$	Assignment $\mathcal{F} \mapsto \int_C \mathcal{F}$ giving an equivalence $\mathsf{PSh}(C) \overset{\text{eq.}}{\cong} \mathsf{DFib}(C)$

Functions between powersets and functors between presheaf categories:

Set Theory	Category Theory
Direct image function $f_*\colon \mathcal{P}(X) \to \mathcal{P}(Y)$	Inverse image functor $f^{-1}\colon \mathsf{PSh}(C) \to \mathsf{PSh}(\mathcal{D})$
Inverse image function $f^{-1}\colon \mathcal{P}(Y) \to \mathcal{P}(X)$	Direct image functor $f_*\colon \mathsf{PSh}(\mathcal{D}) \to \mathsf{PSh}(C)$
Direct image with compact support function $f_!\colon \mathcal{P}(X) \to \mathcal{P}(Y)$	Direct image with compact support functor $f_!\colon \mathsf{PSh}(C) \to \mathsf{PSh}(\mathcal{D})$

Relations and profunctors:

Set Theory	Category Theory
Relation $R\colon X \times Y \to \{\mathsf{t}, \mathsf{f}\}$	Profunctor $\mathfrak{p}\colon \mathcal{D}^{\mathsf{op}} \times C \to \mathsf{Sets}$
Relation $R\colon X \to \mathcal{P}(Y)$	Profunctor $\mathfrak{p}\colon C \to \mathsf{PSh}(\mathcal{D})$
Relation as a cocontinuous morphism of posets $R\colon (\mathcal{P}(X), \subset) \to (\mathcal{P}(Y), \subset)$	Profunctor as a colimit-preserving functor $\mathfrak{p}\colon \mathsf{PSh}(C) \to \mathsf{PSh}(\mathcal{D})$

Appendices

1.A Other Chapters

Chapter 2

Constructions With Sets

This chapter contains some material relating to constructions with sets. Notably, it contains:

1. Explicit descriptions of the major types of co/limits in Sets, including in particular pushouts and coequalisers (see Definitions 2.2.3.1 and 2.2.4.1 and Remarks 2.2.3.3 and 2.2.4.3);

2. A discussion of powersets as decategorifications of categories of presheaves (Remarks 2.4.1.2 and 2.4.3.2);

3. A lengthy discussion of the adjoint triple

$$f_* \dashv f^{-1} \dashv f_! \colon \mathcal{P}(A) \overset{\rightleftarrows}{\rightarrow} \mathcal{P}(B)$$

of functors (morphisms of posets) between $\mathcal{P}(A)$ and $\mathcal{P}(B)$ induced by a map of sets $f \colon A \to B$, along with a discussion of the properties of f_*, f^{-1}, and $f_!$.

Contents

2.1 Limits of Sets

2.1.1 Products of Families of Sets

Let $\{A_i\}_{i\in I}$ be a family of sets.

Definition 2.1.1.1 ► The Product of a Family of Sets

The **product**[1] **of** $\{A_i\}_{i\in I}$ is the pair $\left(\prod_{i\in I} A_i, \{\mathrm{pr}_i\}_{i\in I}\right)$ consisting of

- *The Limit.* The set $\prod_{i\in I} A_i$ defined by[2]

$$\prod_{i\in I} A_i \stackrel{\text{def}}{=} \left\{ f \in \mathsf{Sets}\left(I, \bigcup_{i\in I} A_i\right) \;\middle|\; \begin{array}{l}\text{for each } i \in I\text{, we}\\ \text{have } f(i) \in A_i\end{array} \right\}.$$

- *The Cone.* The collection

$$\left\{ \mathrm{pr}_i \colon \prod_{i\in I} A_i \to A_i \right\}_{i\in I}$$

of maps given by

$$\mathrm{pr}_i(f) \stackrel{\text{def}}{=} f(i)$$

for each $f \in \prod_{i\in I} A_i$ and each $i \in I$.

[1] *Further Terminology:* Also called the **Cartesian product of** $\{A_i\}_{i\in I}$.
[2] Less formally, $\prod_{i\in I} A_i$ is the set whose elements are I-indexed collections $(a_i)_{i\in I}$ with $a_i \in A_i$ for each $i \in I$.

Proof 2.1.1.2 ▶ Proof of Definition 2.1.1.1

We claim that $\prod_{i\in I} A_i$ is the categorical product of $\{A_i\}_{i\in I}$ in Sets. Indeed, suppose we have, for each $i \in I$, a diagram of the form

$$\begin{tikzcd} P \arrow[dr, "p_i"] & \\ \prod\limits_{i\in I} A_i \arrow[r, "\mathrm{pr}_i"'] & A_i \end{tikzcd}$$

in Sets. Then there exists a unique map $\phi \colon P \to \prod_{i\in I} A_i$, uniquely determined by the condition $\mathrm{pr}_i \circ \phi = p_i$ for each $i \in I$, being necessarily given by

$$\phi(x) = (p_i(x))_{i\in I}$$

for each $x \in P$.

Proposition 2.1.1.3 ▶ Properties of Products of Families of Sets

Let $\{A_i\}_{i\in I}$ be a family of sets.

1. *Functoriality.* The assignment $\{A_i\}_{i\in I} \mapsto \prod_{i\in I} A_i$ defines a functor

$$\prod_{i\in I} \colon \mathsf{Fun}(I_{\mathrm{disc}}, \mathsf{Sets}) \to \mathsf{Sets}$$

where

- *Action on Objects.* For each $(A_i)_{i\in I} \in \mathrm{Obj}(\mathsf{Fun}(I_{\mathrm{disc}}, \mathsf{Sets}))$, we have

$$\left[\prod_{i\in I}\right]((A_i)_{i\in I}) \stackrel{\mathrm{def}}{=} \prod_{i\in I} A_i$$

- *Action on Morphisms.* For each $(A_i)_{i\in I}, (B_i)_{i\in I} \in \mathrm{Obj}(\mathsf{Fun}(I_{\mathrm{disc}}, \mathsf{Sets}))$, the action on Hom-sets

$$\left(\prod_{i\in I}\right)_{(A_i)_{i\in I},(B_i)_{i\in I}} \colon \mathrm{Nat}((A_i)_{i\in I}, (B_i)_{i\in I}) \to \mathsf{Sets}\left(\prod_{i\in I} A_i, \prod_{i\in I} B_i\right)$$

of $\prod_{i\in I}$ at $((A_i)_{i\in I}, (B_i)_{i\in I})$ is defined by sending a map

$$\{f_i \colon A_i \to B_i\}_{i\in I}$$

in $\mathsf{Nat}((A_i)_{i\in I}, (B_i)_{i\in I})$ to the map of sets

$$\prod_{i\in I} f_i \colon \prod_{i\in I} A_i \to \prod_{i\in I} B_i$$

defined by

$$\left[\prod_{i\in I} f_i\right]((a_i)_{i\in I}) \stackrel{\text{def}}{=} (f_i(a_i))_{i\in I}$$

for each $(a_i)_{i\in I} \in \prod_{i\in I} A_i$.

Proof 2.1.1.4 ► Proof of Proposition 2.1.1.3

Item 1: Functoriality

Clear.

2.1.2 Binary Products of Sets

Let A and B be sets.

Definition 2.1.2.1 ► Products of Sets

The **product**[1] **of A and B** is the pair $(A \times B, \{\mathrm{pr}_1, \mathrm{pr}_2\})$ consisting of

- *The Limit.* The set $A \times B$ defined by[2]

$$\begin{aligned} A \times B &\stackrel{\text{def}}{=} \prod_{z\in\{A,B\}} z \\ &\stackrel{\text{def}}{=} \{f \in \mathsf{Sets}(\{0,1\}, A \cup B) \mid \text{we have } f(0) \in A \text{ and } f(1) \in B\} \\ &\cong \{\{\{a\}, \{a,b\}\} \in \mathcal{P}(\mathcal{P}(A \cup B)) \mid \text{we have } a \in A \text{ and } b \in B\}. \end{aligned}$$

- *The Cone.* The maps

$$\begin{aligned} \mathrm{pr}_1 &\colon A \times B \to A, \\ \mathrm{pr}_2 &\colon A \times B \to B \end{aligned}$$

defined by

$$\begin{aligned}\mathrm{pr}_1(a,b) &\overset{\text{def}}{=} a,\\ \mathrm{pr}_2(a,b) &\overset{\text{def}}{=} b\end{aligned}$$

for each $(a,b) \in A \times B$.

[1] *Further Terminology:* Also called the **Cartesian product of A and B** or the **binary Cartesian product of A and B**, for emphasis.

This can also be thought of as the $(\mathbb{E}_{-1}, \mathbb{E}_{-1})$**-tensor product of A and B**.

[2] In other words, $A \times B$ is the set whose elements are ordered pairs (a,b) with $a \in A$ and $b \in B$ as in Definition 2.3.4.1

Proof 2.1.2.2 ▶ Proof of Definition 2.1.2.1

We claim that $A \times B$ is the categorical product of A and B in Sets. Indeed, suppose we have a diagram of the form

$$\begin{tikzcd} & P \arrow[dl, bend right, "p_1"'] \arrow[dr, bend left, "p_2"] & \\ A & A \times B \arrow[l, "\mathrm{pr}_1"] \arrow[r, "\mathrm{pr}_2"'] & B \end{tikzcd}$$

in Sets. Then there exists a unique map $\phi \colon P \to A \times B$, uniquely determined by the conditions

$$\begin{aligned}\mathrm{pr}_1 \circ \phi &= p_1,\\ \mathrm{pr}_2 \circ \phi &= p_2,\end{aligned}$$

being necessarily given by

$$\phi(x) = (p_1(x), p_2(x))$$

for each $x \in P$.

Proposition 2.1.2.3 ▶ Properties of Products of Sets

Let A, B, C, and X be sets.

1. *Functoriality.* The assignments $A, B, (A,B) \mapsto A \times B$ define functors

$$\begin{aligned}A \times -_2 &\colon \mathsf{Sets} \to \mathsf{Sets},\\ -_1 \times B &\colon \mathsf{Sets} \to \mathsf{Sets},\\ -_1 \times -_2 &\colon \mathsf{Sets} \times \mathsf{Sets} \to \mathsf{Sets},\end{aligned}$$

where $-_1 \times -_2$ is the functor where

- *Action on Objects.* For each $(A, B) \in \mathrm{Obj}(\mathsf{Sets} \times \mathsf{Sets})$, we have

$$[-_1 \times -_2](A, B) \stackrel{\text{def}}{=} A \times B;$$

- *Action on Morphisms.* For each $(A, B), (X, Y) \in \mathrm{Obj}(\mathsf{Sets})$, the action on Hom-sets

$$\times_{(A,B),(X,Y)} \colon \mathsf{Sets}(A, X) \times \mathsf{Sets}(B, Y) \longrightarrow \mathsf{Sets}(A \times B, X \times Y)$$

of $\times$ at $((A, B), (X, Y))$ is defined by sending (f, g) to the function

$$f \times g \colon A \times B \longrightarrow X \times Y$$

defined by

$$[f \times g](a, b) \stackrel{\text{def}}{=} (f(a), g(b))$$

for each $(a, b) \in A \times B$;

and where $A \times -$ and $- \times B$ are the partial functors of $-_1 \times -_2$ at $A, B \in \mathrm{Obj}(\mathsf{Sets})$.

2. *Adjointness.* We have adjunctions

$$(A \times - \dashv \mathsf{Sets}(A, -)) \colon \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "A\times -"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}, \arrow[l, bend left, "\mathsf{Sets}(A,-)"] \end{tikzcd}$$

$$(- \times B \dashv \mathsf{Sets}(B, -)) \colon \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "-\times B"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}, \arrow[l, bend left, "\mathsf{Sets}(B,-)"] \end{tikzcd}$$

witnessed by bijections

$$\begin{aligned} \mathsf{Sets}(A \times B, C) &\cong \mathsf{Sets}(A, \mathsf{Sets}(B, C)), \\ \mathsf{Sets}(A \times B, C) &\cong \mathsf{Sets}(B, \mathsf{Sets}(A, C)), \end{aligned}$$

natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

3. *Associativity.* We have an isomorphism of sets

$$(A \times B) \times C \cong A \times (B \times C),$$

natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

4. *Unitality.* We have isomorphisms of sets
$$\begin{aligned} \mathrm{pt} \times A &\cong A, \\ A \times \mathrm{pt} &\cong A, \end{aligned}$$
natural in $A \in \mathrm{Obj}(\mathsf{Sets})$.

5. *Commutativity.* We have an isomorphism of sets
$$A \times B \cong B \times A,$$
natural in $A, B \in \mathrm{Obj}(\mathsf{Sets})$.

6. *Annihilation With the Empty Set.* We have isomorphisms of sets
$$\begin{aligned} A \times \emptyset &\cong \emptyset, \\ \emptyset \times A &\cong \emptyset, \end{aligned}$$
natural in $A \in \mathrm{Obj}(\mathsf{Sets})$.

7. *Distributivity Over Unions.* We have isomorphisms of sets
$$\begin{aligned} A \times (B \cup C) &= (A \times B) \cup (A \times C), \\ (A \cup B) \times C &= (A \times C) \cup (B \times C). \end{aligned}$$

8. *Distributivity Over Intersections.* We have isomorphisms of sets
$$\begin{aligned} A \times (B \cap C) &= (A \times B) \cap (A \times C), \\ (A \cap B) \times C &= (A \times C) \cap (B \times C). \end{aligned}$$

9. *Middle-Four Exchange with Respect to Intersections.* We have an isomorphism of sets
$$(A \times B) \cap (C \times D) \cong (A \cap B) \times (C \cap D).$$

10. *Distributivity Over Differences.* We have isomorphisms of sets
$$\begin{aligned} A \times (B \setminus C) &= (A \times B) \setminus (A \times C), \\ (A \setminus B) \times C &= (A \times C) \setminus (B \times C), \end{aligned}$$
natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

11. *Distributivity Over Symmetric Differences.* We have isomorphisms of sets

$$A \times (B \triangle C) = (A \times B) \triangle (A \times C),$$
$$(A \triangle B) \times C = (A \times C) \triangle (B \times C),$$

natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

12. *Symmetric Monoidality.* The triple $(\mathsf{Sets}, \times, \mathrm{pt})$ is a symmetric monoidal category.

13. *Symmetric Bimonoidality.* The quintuple $(\mathsf{Sets}, \coprod, \emptyset, \times, \mathrm{pt})$ is a symmetric bimonoidal category.

Proof 2.1.2.4 ▶ Proof of Proposition 2.1.2.3

Item 1: Functoriality

This follows by applying associativity and unitality componentwise.

Item 2: Adjointness

We prove only that there's an adjunction $X \times - \dashv \mathrm{Hom}_{\mathsf{Sets}}(-, Z)$, witnessed by a bijection

$$\mathrm{Hom}_{\mathsf{Sets}}(X \times Y, Z) \cong \mathrm{Hom}_{\mathsf{Sets}}(X, \mathrm{Hom}_{\mathsf{Sets}}(Y, Z)),$$

natural in $Y, Z \in \mathrm{Obj}(\mathsf{Sets})$, as the proof of the existence of the adjunction $- \times Y \dashv \mathrm{Hom}_{\mathsf{Sets}}(-, Z)$ follows almost exactly in the same way.[1]

- *Map I.* We define a map

  $$\Phi_{Y,Z} \colon \mathrm{Hom}_{\mathsf{Sets}}(X \times Y, Z) \longrightarrow \mathrm{Hom}_{\mathsf{Sets}}(X, \mathrm{Hom}_{\mathsf{Sets}}(Y, Z)),$$

  by sending a morphism $\xi \colon X \times Y \longrightarrow Z$ to the morphism

  $$\xi^{\dagger} \colon X \longrightarrow \mathrm{Hom}_{\mathsf{Sets}(Y,Z)}$$

  defined by

  $$\xi^{\dagger}(x) \overset{\text{def}}{=} \xi_x$$

  for each $x \in X$, where $\xi_x \colon Y \longrightarrow Z$ is the map defined by

  $$\xi_x(y) \overset{\text{def}}{=} \xi(x, y)$$

  for each $y \in Y$.

- *Map II.* We define a map

$$\Psi_{Y,Z}\colon \mathrm{Hom}_{\mathsf{Sets}}(X, \mathrm{Hom}_{\mathsf{Sets}}(Y, Z)), \longrightarrow \mathrm{Hom}_{\mathsf{Sets}}(X \times Y, Z)$$

given by sending a map $\xi\colon X \longrightarrow \mathrm{Hom}_{\mathsf{Sets}}(Y, Z)$ to the map

$$\xi^{\dagger}\colon X \times Y \to Z$$

defined by

$$\xi^{\dagger}(x, y) \overset{\text{def}}{=} [\xi(x)](y)$$

for each $(x, y) \in X \times Y$.

- *Naturality I.* We need to show that, given a function $g\colon Y \longrightarrow Y'$, the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathsf{Sets}}(X \times Y', Z) \arrow[r, "\Phi_{Y',Z}"] \arrow[d, "\mathrm{id}_X \times g^*"'] & \mathrm{Hom}_{\mathsf{Sets}}(X, \mathrm{Hom}_{\mathsf{Sets}}(Y', Z)), \arrow[d, "(g^*)_*"] \\
\mathrm{Hom}_{\mathsf{Sets}}(X \times Y, Z) \arrow[r, "\Phi_{Y,Z}"'] & \mathrm{Hom}_{\mathsf{Sets}}(X, \mathrm{Hom}_{\mathsf{Sets}}(Y, Z)),
\end{tikzcd}$$

commutes. Indeed, given a morphism $\xi\colon X' \times Y \longrightarrow Z$, we have

$$\begin{aligned}
[\Phi_{Y,Z} \circ (g^* \times \mathrm{id}_Y)](\xi) &\overset{\text{def}}{=} (\xi(-_1, g(-_2)))^{\dagger} \\
&\overset{\text{def}}{=} \xi_{-_1}(g(-_2)) \\
&\overset{\text{def}}{=} (g_*)^*(\xi_{-_1}(-_2)) \\
&\overset{\text{def}}{=} (g_*)^*\left(\xi^{\dagger}\right) \\
&\overset{\text{def}}{=} [(g_*)^* \circ \Phi_{Y',Z}](\xi).
\end{aligned}$$

- *Naturality II.* We need to show that, given a function $h\colon Z \longrightarrow Z'$, the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathsf{Sets}}(X \times Y, Z) \arrow[r, "\Phi_{Y,Z}"] \arrow[d, "h_*"'] & \mathrm{Hom}_{\mathsf{Sets}}(X, \mathrm{Hom}_{\mathsf{Sets}}(Y, Z)), \arrow[d, "(h_*)_*"] \\
\mathrm{Hom}_{\mathsf{Sets}}(X \times Y, Z') \arrow[r, "\Phi_{Y,Z'}"'] & \mathrm{Hom}_{\mathsf{Sets}}(X, \mathrm{Hom}_{\mathsf{Sets}}(Y, Z')),
\end{tikzcd}$$

commutes. Indeed, given a morphism $\xi\colon X \times Y \to Z$, we have

$$\begin{aligned}
[\Phi_{Y,Z} \circ h_*](\xi) &\stackrel{\text{def}}{=} (h(\xi(-_1, -_2)))^\dagger \\
&\stackrel{\text{def}}{=} [x \mapsto [y \mapsto h(\xi(x,y))]] \\
&\stackrel{\text{def}}{=} [x \mapsto h_*([y \mapsto \xi(x,y)])] \\
&\stackrel{\text{def}}{=} \left[x \mapsto h_*\left(\xi^\dagger(x)\right)\right] \\
&\stackrel{\text{def}}{=} h_*\left(\xi^\dagger\right) \\
&\stackrel{\text{def}}{=} [(h_*)_* \circ \Phi_{Y,Z}](\xi).
\end{aligned}$$

- *Invertibility I.* We claim that

$$\Psi_{X,Y} \circ \Phi_{X,Y} = \mathrm{id}_{\mathrm{Hom}_{\mathsf{Sets}}(X \times Y, Z)}.$$

Indeed, given a morphism $\xi\colon X \times Y \to Z$, we have

$$\begin{aligned}
[\Psi_{X,Y} \circ \Phi_{X,Y}](\xi) &\stackrel{\text{def}}{=} \Psi_{X,Y}(\Phi_{X,Y}(\xi)) \\
&\stackrel{\text{def}}{=} \Psi_{X,Y}([x \mapsto [y \mapsto \xi(x,y)]]) \\
&\stackrel{\text{def}}{=} [(x,y) \mapsto \mathrm{ev}_x([x \mapsto \mathrm{ev}_y([y \mapsto \xi(x,y)]])])] \\
&\stackrel{\text{def}}{=} [(x,y) \mapsto \mathrm{ev}_x([x \mapsto \xi(x,y)])] \\
&\stackrel{\text{def}}{=} [(x,y) \mapsto \xi(x,y)] \\
&\stackrel{\text{def}}{=} \xi.
\end{aligned}$$

- *Invertibility II.* We claim that

$$\Phi_{X,Y} \circ \Psi_{X,Y} = \mathrm{id}_{\mathrm{Hom}_{\mathsf{Sets}}(X, \mathrm{Hom}_{\mathsf{Sets}}(Y,Z))}.$$

Indeed, given a morphism $\xi\colon X \to \mathrm{Hom}_{\mathsf{Sets}}(Y, Z)$, we have

$$\begin{aligned}
[\Phi_{X,Y} \circ \Psi_{X,Y}](\xi) &\stackrel{\text{def}}{=} \Phi_{X,Y}(\Psi_{X,Y}(\xi)) \\
&\stackrel{\text{def}}{=} \Phi_{X,Y}(\Psi_{X,Y}([x \mapsto \xi(x)])) \\
&\stackrel{\text{def}}{=} \Phi_{X,Y}([(x,y) \mapsto \mathrm{ev}_x([x \mapsto \mathrm{ev}_y(\xi(x))])]) \\
&\stackrel{\text{def}}{=} \Phi_{X,Y}([(x,y) \mapsto \xi(x,y)]) \\
&\stackrel{\text{def}}{=} [x \mapsto [y \mapsto \xi(x,y)]] \\
&\stackrel{\text{def}}{=} \xi.
\end{aligned}$$

Item 3: Associativity

See [Pro24a].

Item 4: Unitality

Clear.

Item 5: Commutativity

See [Pro24b].

Item 6: Annihilation With the Empty Set

See [Pro24f].

Item 7: Distributivity Over Unions

See [Pro24e].

Item 8: Distributivity Over Intersections

See [Pro24g, Corollary 1].

Item 9: Middle-Four Exchange With Respect to Intersections

See [Pro24g, Corollary 1].

Item 10: Distributivity Over Differences

See [Pro24c].

Item 11: Distributivity Over Symmetric Differences

See [Pro24d].

Item 12: Symmetric Monoidality

See [MO 382264].

Item 13: Symmetric Bimonoidality

Omitted.

[1] Here we sometimes denote a map $f\colon X \to Y$ by $[x \mapsto f(x)]$, similar to the lambda notation $\lambda x.f(x)$.

2.1.3 Pullbacks

Let A, B, and C be sets and let $f\colon A \to C$ and $g\colon B \to C$ be functions.

Definition 2.1.3.1 ▶ Pullbacks of Sets

The **pullback of** A **and** B **over** C **along** f **and** g[1] is the pair[2] $(A \times_C B, \{\mathrm{pr}_1, \mathrm{pr}_2\})$ consisting of

- *The Limit.* The set $A \times_C B$ defined by

$$A \times_C B \overset{\text{def}}{=} \{(a, b) \in A \times B \mid f(a) = g(b)\}.$$

- *The Cone.* The maps

$$\begin{aligned} \mathrm{pr}_1 &\colon A \times_C B \to A, \\ \mathrm{pr}_2 &\colon A \times_C B \to B \end{aligned}$$

defined by

$$\begin{aligned} \mathrm{pr}_1(a, b) &\overset{\text{def}}{=} a, \\ \mathrm{pr}_2(a, b) &\overset{\text{def}}{=} b \end{aligned}$$

for each $(a, b) \in A \times_C B$.

[1] *Further Terminology:* Also called the **fibre product of A and B over C along f and g**.
[2] *Further Notation:* Also written $A \times_{f,C,g} B$.

PROOF 2.1.3.2 ► PROOF OF DEFINITION 2.1.3.1

We claim that $A \times_C B$ is the categorical pullback of A and B over C with respect to (f, g) in Sets. First we need to check that the relevant pullback diagram commutes, i.e. that we have

$$f \circ \mathrm{pr}_1 = g \circ \mathrm{pr}_2, \qquad \begin{tikzcd} A \times_C B \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C. \end{tikzcd}$$

Indeed, given $(a, b) \in A \times_C B$, we have

$$\begin{aligned} [f \circ \mathrm{pr}_1](a, b) &= f(\mathrm{pr}_1(a, b)) \\ &= f(a) \\ &= g(b) \\ &= g(\mathrm{pr}_2(a, b)) \\ &= [g \circ \mathrm{pr}_2](a, b), \end{aligned}$$

where $f(a) = g(b)$ since $(a, b) \in A \times_C B$. Next, we prove that $A \times_C B$ satisfies the universal property of the pullback. Suppose we have a diagram of the form

$$\begin{tikzcd}
P \arrow[rrd, bend left, "p_2"] \arrow[rdd, bend right, "p_1"'] & & \\
& A \times_C B \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "g"] \\
& A \arrow[r, "f"'] & C
\end{tikzcd}$$

in Sets. Then there exists a unique map $\phi \colon P \to A \times_C B$, uniquely determined by the conditions

$$\begin{aligned}
\mathrm{pr}_1 \circ \phi &= p_1, \\
\mathrm{pr}_2 \circ \phi &= p_2,
\end{aligned}$$

being necessarily given by

$$\phi(x) = (p_1(x), p_2(x))$$

for each $x \in P$, where we note that $(p_1(x), p_2(x)) \in A \times B$ indeed lies in $A \times_C B$ by the condition

$$f \circ p_1 = g \circ p_2,$$

which gives

$$f(p_1(x)) = g(p_2(x))$$

for each $x \in P$, so that $(p_1(x), p_2(x)) \in A \times_C B$.

Example 2.1.3.3 ▶ Examples of Pullbacks of Sets

Here are some examples of pullbacks of sets.

1. *Unions via Intersections.* Let $A, B \subset X$. We have a bijection of sets

$$A \cap B \cong A \times_{A \cup B} B, \qquad \begin{tikzcd}
A \cap B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "\iota_B"] \\
A \arrow[r, "\iota_A"'] & A \cup B.
\end{tikzcd}$$

PROOF 2.1.3.4 ▶ PROOF OF EXAMPLE 2.1.3.3

Item 1: Unions via Intersections

Indeed, we have

$$\begin{aligned} A \times_{A\cup B} B &\cong \{(x,y) \in A \times B \mid x = y\} \\ &\cong A \cap B. \end{aligned}$$

This finishes the proof. ∎

PROPOSITION 2.1.3.5 ▶ PROPERTIES OF PULLBACKS OF SETS

Let A, B, C, and X be sets.

1. *Functoriality.* The assignment $(A, B, C, f, g) \mapsto A \times_{f,C,g} B$ defines a functor

$$-_1 \times_{-_3} -_1 \colon \mathsf{Fun}(\mathcal{P}, \mathsf{Sets}) \longrightarrow \mathsf{Sets},$$

where $\mathcal{P}$ is the category that looks like this:

$$\begin{tikzcd} & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet. \end{tikzcd}$$

In particular, the action on morphisms of $-_1 \times_{-_3} -_1$ is given by sending a morphism

$$\begin{tikzcd} A \times_C B \arrow[rr] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & & B \arrow[dd, "g"] \arrow[dr, "\psi"] & \\ & A' \times_{C'} B' \arrow[rr] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & & B' \arrow[dd, "g'"] \\ A \arrow[rr, "f"] \arrow[dr, "\phi"'] & & C \arrow[dr, "\chi"] & \\ & A' \arrow[rr, "f'"'] & & C' \end{tikzcd}$$

in $\mathsf{Fun}(\mathcal{P}, \mathsf{Sets})$ to the map $\xi \colon A \times_C B \xrightarrow{\exists!} A' \times_{C'} B'$ given by

$$\xi(a, b) \overset{\text{def}}{=} (\phi(a), \psi(b))$$

for each $(a,b) \in A \times_C B$, which is the unique map making the diagram

$$
\begin{tikzcd}
A \times_C B \arrow[rr] \arrow[dd] \arrow[dr, dashed] \arrow[dr, phantom, very near start, "\lrcorner"] & & B \arrow[dd, "g" near start] \arrow[dr, "\psi"] & \\
& A' \times_{C'} B' \arrow[rr] \arrow[dd] \arrow[dr, phantom, very near start, "\lrcorner"] & & B' \arrow[dd, "g'"] \\
A \arrow[rr, "f" near start] \arrow[dr, "\phi"'] & & C \arrow[dr, "\chi"] & \\
& A' \arrow[rr, "f'"'] & & C'
\end{tikzcd}
$$

commute.

2. *Associativity.* Given a diagram

$$
\begin{tikzcd}
A \arrow[dr, "f"'] & & B \arrow[dl, "g"] \arrow[dr, "h"'] & & C \arrow[dl, "k"] \\
& X & & Y &
\end{tikzcd}
$$

in Sets, we have isomorphisms

$$(A \times_X B) \times_Y C \cong (A \times_X B) \times_B (B \times_Y C) \cong A \times_X (B \times_Y C),$$

where these pullbacks are built as in the diagrams

$$
\begin{tikzcd}[column sep=small, row sep=small]
& & (A \times_X B) \times_Y C \arrow[dl] \arrow[ddrr] \arrow[dd, phantom, very near start, "\lrcorner" rotate=-45] & & \\
& A \times_X B \arrow[dl] \arrow[dr] \arrow[dd, phantom, very near start, "\lrcorner" rotate=-45] & & & \\
A \arrow[dr, "f"'] & & B \arrow[dl, "g"] \arrow[dr, "h"'] & & C, \arrow[dl, "k"] \\
& X & & Y &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=small, row sep=small]
& & (A \times_X B) \times_B (B \times_Y C) \arrow[dl] \arrow[dr] \arrow[d, phantom, very near start, "\lrcorner" rotate=-45] & & \\
& A \times_X B \arrow[dl] \arrow[dr] \arrow[dd, phantom, very near start, "\lrcorner" rotate=-45] & & B \times_Y C \arrow[dl] \arrow[dr] \arrow[dd, phantom, very near start, "\lrcorner" rotate=-45] & \\
A \arrow[dr, "f"'] & & B \arrow[dl, "g"] \arrow[dr, "h"'] & & C, \arrow[dl, "k"] \\
& X & & Y &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=small, row sep=small]
& & A \times_X (B \times_Y C) \arrow[ddll] \arrow[dr] \arrow[dd, phantom, very near start, "\lrcorner" rotate=-45] & & \\
& & & B \times_Y C \arrow[dl] \arrow[dr] \arrow[dd, phantom, very near start, "\lrcorner" rotate=-45] & \\
A \arrow[dr, "f"'] & & B \arrow[dl, "g"] \arrow[dr, "h"'] & & C. \arrow[dl, "k"] \\
& X & & Y &
\end{tikzcd}
$$

3. *Unitality.* We have isomorphisms of sets

$$
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "f"'] \arrow[dr, phantom, very near start, "\lrcorner"] & A \arrow[d, "f"] \\
X \arrow[r, equal] & X
\end{tikzcd}
\qquad
\begin{aligned}
X \times_X A &\cong A, \\
A \times_X X &\cong A,
\end{aligned}
\qquad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, equal] \arrow[dr, phantom, very near start, "\lrcorner"] & X \arrow[d, equal] \\
X \arrow[r, "f"'] & X.
\end{tikzcd}
$$

4. *Commutativity.* We have an isomorphism of sets

$$\begin{tikzcd} A \times_X B \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & X, \end{tikzcd} \qquad A \times_X B \cong B \times_X A \qquad \begin{tikzcd} B \times_X A \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & A \arrow[d, "f"] \\ B \arrow[r, "g"'] & X. \end{tikzcd}$$

5. *Annihilation With the Empty Set.* We have isomorphisms of sets

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \emptyset \arrow[d] \\ A \arrow[r, "f"'] & X, \end{tikzcd} \qquad \begin{aligned} A \times_X \emptyset &\cong \emptyset, \\ \emptyset \times_X A &\cong \emptyset, \end{aligned} \qquad \begin{tikzcd} \emptyset \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & A \arrow[d, "f"] \\ \emptyset \arrow[r] & X. \end{tikzcd}$$

6. *Interaction With Products.* We have

$$A \times_{\mathrm{pt}} B \cong A \times B, \qquad \begin{tikzcd} A \times B \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & B \arrow[d, "!_B"] \\ A \arrow[r, "!_A"'] & \mathrm{pt}. \end{tikzcd}$$

7. *Symmetric Monoidality.* The triple $(\mathsf{Sets}, \times_X, X)$ is a symmetric monoidal category.

Proof 2.1.3.6 ► Proof of Proposition 2.1.3.5

Item 1: Functoriality

This is a special case of functoriality of co/limits, **??** of **??**, with the explicit expression for ξ following from the commutativity of the cube pullback diagram.

Item 2: Associativity

Indeed, we have

$$\begin{aligned} (A \times_X B) \times_Y C &\cong \{((a,b),c) \in (A \times_X B) \times C \mid h(b) = k(c)\} \\ &\cong \{((a,b),c) \in (A \times B) \times C \mid f(a) = g(b) \text{ and } h(b) = k(c)\} \\ &\cong \{(a,(b,c)) \in A \times (B \times C) \mid f(a) = g(b) \text{ and } h(b) = k(c)\} \\ &\cong \{(a,(b,c)) \in A \times (B \times_Y C) \mid f(a) = g(b)\} \\ &\cong A \times_X (B \times_Y C) \end{aligned}$$

and

$$\begin{aligned}
(A \times_X B) \times_B (B \times_Y C) &\cong \{((a,b),(b',c)) \in (A \times_X B) \times (B \times_Y C) \mid b = b'\}\\
&\cong \left\{ ((a,b),(b',c)) \in (A \times B) \times (B \times C) \,\middle|\, \begin{array}{l} f(a) = g(b), b = b', \\ \text{and } h(b') = k(c) \end{array} \right\}\\
&\cong \left\{ (a,(b,(b',c))) \in A \times (B \times (B \times C)) \,\middle|\, \begin{array}{l} f(a) = g(b), b = b', \\ \text{and } h(b') = k(c) \end{array} \right\}\\
&\cong \left\{ (a,((b,b'),c)) \in A \times ((B \times B) \times C) \,\middle|\, \begin{array}{l} f(a) = g(b), b = b', \\ \text{and } h(b') = k(c) \end{array} \right\}\\
&\cong \left\{ (a,((b,b'),c)) \in A \times ((B \times_B B) \times C) \,\middle|\, \begin{array}{l} f(a) = g(b) \text{ and} \\ h(b') = k(c) \end{array} \right\}\\
&\cong \{(a,(b,c)) \in A \times (B \times C) \mid f(a) = g(b) \text{ and } h(b) = k(c)\}\\
&\cong A \times_X (B \times_Y C),
\end{aligned}$$

where we have used Item 3 for the isomorphism $B \times_B B \cong B$.

Item 3: Unitality

Indeed, we have

$$\begin{aligned}
X \times_X A &\cong \{(x,a) \in X \times A \mid f(a) = x\},\\
A \times_X X &\cong \{(a,x) \in X \times A \mid f(a) = x\},
\end{aligned}$$

which are isomorphic to A via the maps $(x,a) \mapsto a$ and $(a,x) \mapsto a$.

Item 4: Commutativity

Clear.

Item 5: Annihilation With the Empty Set

Clear.

Item 6: Interaction With Products

Clear.

Item 7: Symmetric Monoidality

Omitted. □

2.1.4 Equalisers

Let A and B be sets and let $f, g \colon A \rightrightarrows B$ be functions.

Definition 2.1.4.1 ► Equalisers of Sets

The **equaliser of f and g** is the pair $(\mathrm{Eq}(f,g), \mathrm{eq}(f,g))$ consisting of

- *The Limit.* The set $\mathrm{Eq}(f,g)$ defined by

$$\mathrm{Eq}(f,g) \overset{\text{def}}{=} \{a \in A \mid f(a) = g(a)\}.$$

- *The Cone.* The inclusion map

$$\mathrm{eq}(f,g) \colon \mathrm{Eq}(f,g) \hookrightarrow A.$$

Proof 2.1.4.2 ► Proof of Definition 2.1.4.1

We claim that $\mathrm{Eq}(f,g)$ is the categorical equaliser of f and g in Sets. First we need to check that the relevant equaliser diagram commutes, i.e. that we have

$$f \circ \mathrm{eq}(f,g) = g \circ \mathrm{eq}(f,g),$$

which indeed holds by the definition of the set $\mathrm{Eq}(f,g)$. Next, we prove that $\mathrm{Eq}(f,g)$ satisfies the universal property of the equaliser. Suppose we have a diagram of the form

$$\begin{tikzcd}
\mathrm{Eq}(f,g) \arrow[r, hook, "\mathrm{eq}(f,g)"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\
E \arrow[ur, "e"'] & &
\end{tikzcd}$$

in Sets. Then there exists a unique map $\phi \colon E \longrightarrow \mathrm{Eq}(f,g)$, uniquely determined by the condition

$$\mathrm{eq}(f,g) \circ \phi = e$$

being necessarily given by

$$\phi(x) = e(x)$$

for each $x \in E$, where we note that $e(x) \in A$ indeed lies in $\mathrm{Eq}(f,g)$ by the condition

$$f \circ e = g \circ e,$$

which gives

$$f(e(x)) = g(e(x))$$

for each $x \in E$, so that $e(x) \in \mathrm{Eq}(f,g)$. ∎

Proposition 2.1.4.3 ► Properties of Equalisers of Sets

Let A, B, and C be sets.

1. *Associativity.* We have an isomorphism of sets[1]

$$\underbrace{\mathrm{Eq}(f\circ \mathrm{eq}(g,h), g\circ \mathrm{eq}(g,h))}_{=\mathrm{Eq}(f\circ \mathrm{eq}(g,h), h\circ \mathrm{eq}(g,h))} \cong \mathrm{Eq}(f,g,h) \cong \underbrace{\mathrm{Eq}(f\circ \mathrm{eq}(f,g), h\circ \mathrm{eq}(f,g))}_{=\mathrm{Eq}(g\circ \mathrm{eq}(f,g), h\circ \mathrm{eq}(f,g))},$$

where $\mathrm{Eq}(f,g,h)$ is the limit of the diagram

$$\begin{tikzcd} A \arrow[r, shift left=2, "f"] \arrow[r, "g" description] \arrow[r, shift right=2, "h"'] & B \end{tikzcd}$$

in Sets, being explicitly given by

$$\mathrm{Eq}(f,g,h) \cong \{a \in A \mid f(a) = g(a) = h(a)\}.$$

2. *Unitality.* We have an isomorphism of sets

$$\mathrm{Eq}(f,f) \cong A.$$

3. *Commutativity.* We have an isomorphism of sets

$$\mathrm{Eq}(f,g) \cong \mathrm{Eq}(g,f).$$

4. *Interaction With Composition.* Let

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \underset{k}{\overset{h}{\rightrightarrows}} C$$

be functions. We have an inclusion of sets

$$\mathrm{Eq}(h\circ f\circ \mathrm{eq}(f,g), k\circ g\circ \mathrm{eq}(f,g)) \subset \mathrm{Eq}(h\circ f, k\circ g),$$

where $\mathrm{Eq}(h\circ f\circ \mathrm{eq}(f,g), k\circ g\circ \mathrm{eq}(f,g))$ is the equaliser of the composition

$$\mathrm{Eq}(f,g) \overset{\mathrm{eq}(f,g)}{\hookrightarrow} A \underset{g}{\overset{f}{\rightrightarrows}} B \underset{k}{\overset{h}{\rightrightarrows}} C.$$

[1] That is, the following three ways of forming "the" equaliser of (f,g,h) agree:

(a) Take the equaliser of (f,g,h), i.e. the limit of the diagram

$$\begin{tikzcd} A \arrow[r, shift left=2, "f"] \arrow[r, "g" description] \arrow[r, shift right=2, "h"'] & B \end{tikzcd}$$

in Sets.

(b) First take the equaliser of f and g, forming a diagram

$$\mathrm{Eq}(f,g) \overset{\mathrm{eq}(f,g)}{\hookrightarrow} A \overset{f}{\underset{g}{\rightrightarrows}} B$$

and then take the equaliser of the composition

$$\mathrm{Eq}(f,g) \overset{\mathrm{eq}(f,g)}{\hookrightarrow} A \overset{f}{\underset{h}{\rightrightarrows}} B,$$

obtaining a subset

$$\mathrm{Eq}(f \circ \mathrm{eq}(f,g), h \circ \mathrm{eq}(f,g)) = \mathrm{Eq}(g \circ \mathrm{eq}(f,g), h \circ \mathrm{eq}(f,g))$$

of $\mathrm{Eq}(f,g)$.

(c) First take the equaliser of g and h, forming a diagram

$$\mathrm{Eq}(g,h) \overset{\mathrm{eq}(g,h)}{\hookrightarrow} A \overset{g}{\underset{h}{\rightrightarrows}} B$$

and then take the equaliser of the composition

$$\mathrm{Eq}(g,h) \overset{\mathrm{eq}(g,h)}{\hookrightarrow} A \overset{f}{\underset{g}{\rightrightarrows}} B,$$

obtaining a subset

$$\mathrm{Eq}(f \circ \mathrm{eq}(g,h), g \circ \mathrm{eq}(g,h)) = \mathrm{Eq}(f \circ \mathrm{eq}(g,h), h \circ \mathrm{eq}(g,h))$$

of $\mathrm{Eq}(g,h)$.

Proof 2.1.4.4 ► Proof of Proposition 2.1.4.3

Item 1: Associativity

We first prove that $\mathrm{Eq}(f,g,h)$ is indeed given by

$$\mathrm{Eq}(f,g,h) \cong \{a \in A \mid f(a) = g(a) = h(a)\}.$$

Indeed, suppose we have a diagram of the form

$$\begin{tikzcd}
\mathrm{Eq}(f,g,h) \arrow[r, hook, "\mathrm{eq}(f,g,h)"] & A \arrow[r, shift left=2, "f"] \arrow[r, "g"] \arrow[r, shift right=2, "h"'] & B \\
E \arrow[ur, "e"'] & &
\end{tikzcd}$$

in Sets. Then there exists a unique map $\phi \colon E \longrightarrow \mathrm{Eq}(f,g,h)$, uniquely determined by the condition

$$\mathrm{eq}(f,g) \circ \phi = e$$

being necessarily given by

$$\phi(x) = e(x)$$

for each $x \in E$, where we note that $e(x) \in A$ indeed lies in $\mathrm{Eq}(f, g, h)$ by the condition

$$f \circ e = g \circ e = h \circ e,$$

which gives

$$f(e(x)) = g(e(x)) = h(e(x))$$

for each $x \in E$, so that $e(x) \in \mathrm{Eq}(f, g, h)$.

We now check the equalities

$$\mathrm{Eq}(f \circ \mathrm{eq}(g, h), g \circ \mathrm{eq}(g, h)) \cong \mathrm{Eq}(f, g, h) \cong \mathrm{Eq}(f \circ \mathrm{eq}(f, g), h \circ \mathrm{eq}(f, g)).$$

Indeed, we have

$$\begin{aligned}
\mathrm{Eq}(f \circ \mathrm{eq}(g, h), g \circ \mathrm{eq}(g, h)) &\cong \{x \in \mathrm{Eq}(g, h) \mid [f \circ \mathrm{eq}(g, h)](a) = [g \circ \mathrm{eq}(g, h)](a)\} \\
&\cong \{x \in \mathrm{Eq}(g, h) \mid f(a) = g(a)\} \\
&\cong \{x \in A \mid f(a) = g(a) \text{ and } g(a) = h(a)\} \\
&\cong \{x \in A \mid f(a) = g(a) = h(a)\} \\
&\cong \mathrm{Eq}(f, g, h).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\mathrm{Eq}(f \circ \mathrm{eq}(f, g), h \circ \mathrm{eq}(f, g)) &\cong \{x \in \mathrm{Eq}(f, g) \mid [f \circ \mathrm{eq}(f, g)](a) = [h \circ \mathrm{eq}(f, g)](a)\} \\
&\cong \{x \in \mathrm{Eq}(f, g) \mid f(a) = h(a)\} \\
&\cong \{x \in A \mid f(a) = h(a) \text{ and } f(a) = g(a)\} \\
&\cong \{x \in A \mid f(a) = g(a) = h(a)\} \\
&\cong \mathrm{Eq}(f, g, h).
\end{aligned}$$

Item 2: Unitality

Clear.

Item 3: Commutativity

Clear.

Item 4: Interaction With Composition

Indeed, we have

$$\begin{aligned}
\mathrm{Eq}(h \circ f \circ \mathrm{eq}(f, g), k \circ g \circ \mathrm{eq}(f, g)) &\cong \{a \in \mathrm{Eq}(f, g) \mid h(f(a)) = k(g(a))\} \\
&\cong \{a \in A \mid f(a) = g(a) \text{ and } h(f(a)) = k(g(a))\}.
\end{aligned}$$

and

$$\mathrm{Eq}(h \circ f, k \circ g) \cong \{a \in A \mid h(f(a)) = k(g(a))\},$$

and thus there's an inclusion from $\mathrm{Eq}(h \circ f \circ \mathrm{eq}(f, g), k \circ g \circ \mathrm{eq}(f, g))$ to $\mathrm{Eq}(h \circ f, k \circ g)$.

2.2 Colimits of Sets

2.2.1 Coproducts of Families of Sets

Let $\{A_i\}_{i\in I}$ be a family of sets.

Definition 2.2.1.1 ► Disjoint Unions of Families

The **disjoint union of the family** $\{A_i\}_{i\in I}$ is the pair $\left(\coprod_{i\in I} A_i, \{\mathrm{inj}_i\}_{i\in I}\right)$ consisting of

- *The Colimit.* The set $\coprod_{i\in I} A_i$ defined by
$$\coprod_{i\in I} A_i \overset{\text{def}}{=} \left\{(i,x) \in I \times \left(\bigcup_{i\in I} A_i\right) \,\middle|\, x \in A_i\right\}.$$
- *The Cocone.* The collection
$$\left\{\mathrm{inj}_i \colon A_i \to \coprod_{i\in I} A_i\right\}_{i\in I}$$
of maps given by
$$\mathrm{inj}_i(x) \overset{\text{def}}{=} (i,x)$$
for each $x \in A_i$ and each $i \in I$.

Proof 2.2.1.2 ► Proof of Definition 2.2.1.1

We claim that $\coprod_{i\in I} A_i$ is the categorical coproduct of $\{A_i\}_{i\in I}$ in Sets. Indeed, suppose we have, for each $i \in I$, a diagram of the form

$$\begin{tikzcd} & C \\ A_i \arrow[ur, "\iota_i"] \arrow[r, "\mathrm{inj}_i"'] & \coprod_{i\in I} A_i \end{tikzcd}$$

in Sets. Then there exists a unique map $\phi\colon \coprod_{i\in I} A_i \to C$, uniquely determined by the condition $\phi \circ \mathrm{inj}_i = \iota_i$ for each $i \in I$, being necessarily given by

$$\phi(i,x) = \iota_i(x)$$

for each $(i,x) \in \coprod_{i\in I} A_i$.

Proposition 2.2.1.3 ▶ Properties of Coproducts of Families of Sets

Let $\{A_i\}_{i\in I}$ be a family of sets.

1. *Functoriality.* The assignment $\{A_i\}_{i\in I} \mapsto \coprod_{i\in I} A_i$ defines a functor

$$\coprod_{i\in I} \colon \mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}) \longrightarrow \mathsf{Sets}$$

where

- *Action on Objects.* For each $(A_i)_{i\in I} \in \mathrm{Obj}(\mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}))$, we have

$$\left[\coprod_{i\in I}\right]((A_i)_{i\in I}) \stackrel{\text{def}}{=} \coprod_{i\in I} A_i$$

- *Action on Morphisms.* For each $(A_i)_{i\in I}, (B_i)_{i\in I} \in \mathrm{Obj}(\mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}))$, the action on Hom-sets

$$\left(\coprod_{i\in I}\right)_{(A_i)_{i\in I},(B_i)_{i\in I}} \colon \mathrm{Nat}((A_i)_{i\in I}, (B_i)_{i\in I}) \longrightarrow \mathsf{Sets}\left(\coprod_{i\in I} A_i, \coprod_{i\in I} B_i\right)$$

of $\coprod_{i\in I}$ at $((A_i)_{i\in I}, (B_i)_{i\in I})$ is defined by sending a map

$$\{f_i \colon A_i \longrightarrow B_i\}_{i\in I}$$

in $\mathrm{Nat}((A_i)_{i\in I}, (B_i)_{i\in I})$ to the map of sets

$$\coprod_{i\in I} f_i \colon \coprod_{i\in I} A_i \to \coprod_{i\in I} B_i$$

defined by

$$\left[\coprod_{i\in I} f_i\right](i, a) \stackrel{\text{def}}{=} f_i(a)$$

for each $(i, a) \in \coprod_{i\in I} A_i$.

Proof 2.2.1.4 ▶ Proof of Proposition 2.2.1.3

Item 1: Functoriality

Clear. □

2.2.2 Binary Coproducts

Let A and B be sets.

Definition 2.2.2.1 ► Coproducts of Sets

The **coproduct**[1] **of** A **and** B is the pair $(A \coprod B, \{\mathrm{inj}_1, \mathrm{inj}_2\})$ consisting of

- *The Colimit.* The set $A \coprod B$ defined by

$$\begin{aligned} A \coprod B &\overset{\mathrm{def}}{=} \coprod_{z \in \{A,B\}} z \\ &\cong \{(0,a) \mid a \in A\} \cup \{(1,b) \mid b \in B\}. \end{aligned}$$

- *The Cocone.* The maps

$$\begin{aligned} \mathrm{inj}_1 &: A \to A \coprod B, \\ \mathrm{inj}_2 &: B \to A \coprod B, \end{aligned}$$

 given by

$$\begin{aligned} \mathrm{inj}_1(a) &\overset{\mathrm{def}}{=} (0,a), \\ \mathrm{inj}_2(b) &\overset{\mathrm{def}}{=} (1,b), \end{aligned}$$

 for each $a \in A$ and each $b \in B$.

[1] *Further Terminology:* Also called the **disjoint union of** A **and** B, or the **binary disjoint union of** A **and** B, for emphasis.

Proof 2.2.2.2 ► Proof of Definition 2.2.2.1

We claim that $A \coprod B$ is the categorical coproduct of A and B in Sets. Indeed, suppose we have a diagram of the form

$$\begin{tikzcd} & & C & & \\ A \arrow[r, "\mathrm{inj}_A"'] \arrow[urr, bend left, "i_A"] & A \coprod B & & B \arrow[ll, "\mathrm{inj}_B"] \arrow[ul, bend right, "i_B"'] \end{tikzcd}$$

in Sets. Then there exists a unique map $\phi\colon A \coprod B \to C$, uniquely determined by the conditions

$$\begin{aligned} \phi \circ \mathrm{inj}_A &= i_A, \\ \phi \circ \mathrm{inj}_B &= i_B, \end{aligned}$$

being necessarily given by

$$\phi(x) = \begin{cases} i_A(x) & \text{if } x \in A, \\ i_B(x) & \text{if } x \in B \end{cases}$$

for each $x \in C$.

Proposition 2.2.2.3 ▶ Properties of Coproducts of Sets

Let A, B, C, and X be sets.

1. *Functoriality.* The assignment $A, B, (A, B) \mapsto A \coprod B$ defines functors

$$A \coprod -_2 \colon \mathsf{Sets} \to \mathsf{Sets},$$
$$-_1 \coprod B \colon \mathsf{Sets} \to \mathsf{Sets},$$
$$-_1 \coprod -_2 \colon \mathsf{Sets} \times \mathsf{Sets} \to \mathsf{Sets},$$

where $-_1 \coprod -_2$ is the functor where

- *Action on Objects.* For each $(A, B) \in \mathrm{Obj}(\mathsf{Sets} \times \mathsf{Sets})$, we have

$$[-_1 \coprod -_2](A, B) \stackrel{\text{def}}{=} A \coprod B;$$

- *Action on Morphisms.* For each $(A, B), (X, Y) \in \mathrm{Obj}(\mathsf{Sets})$, the action on Hom-sets

$$\coprod_{(A,B),(X,Y)} \colon \mathsf{Sets}(A, X) \times \mathsf{Sets}(B, Y) \to \mathsf{Sets}(A \coprod B, X \coprod Y)$$

of $\coprod$ at $((A, B), (X, Y))$ is defined by sending (f, g) to the function

$$f \coprod g \colon A \coprod B \to X \coprod Y$$

defined by

$$[f \coprod g](x) \stackrel{\text{def}}{=} \begin{cases} f(x) & \text{if } x \in A, \\ g(x) & \text{if } x \in B \end{cases}$$

for each $x \in A \coprod B$;

and where $A \coprod -$ and $- \coprod B$ are the partial functors of $-_1 \coprod -_2$ at $A, B \in \mathrm{Obj}(\mathsf{Sets})$.

2. *Associativity.* We have an isomorphism of sets

$$(A \coprod B) \coprod C \cong A \coprod (B \coprod C),$$

natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

3. *Unitality.* We have isomorphisms of sets

$$A \coprod \emptyset \cong A,$$
$$\emptyset \coprod A \cong A,$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$.

4. *Commutativity.* We have an isomorphism of sets

$$A \coprod B \cong B \coprod A,$$

natural in $A, B \in \mathrm{Obj}(\mathsf{Sets})$.

5. *Symmetric Monoidality.* The triple $(\mathsf{Sets}, \coprod, \emptyset)$ is a symmetric monoidal category.

Proof 2.2.2.4 ► Proof of Proposition 2.2.2.3

Item 1: Functoriality

Clear.

Item 2: Associativity

Clear.

Item 3: Unitality

Clear.

Item 4: Commutativity

Clear.

Item 5: Symmetric Monoidality

Omitted.

2.2.3 Pushouts

Let A, B, and C be sets and let $f: C \to A$ and $g: C \to B$ be functions.

Definition 2.2.3.1 ► Pushouts of Sets

The **pushout of** A **and** B **over** C **along** f **and** g[1] is the pair $(A \coprod_C B, \{\mathrm{inj}_1, \mathrm{inj2}\})$ consisting of

- *The Colimit.* The set $A \coprod_C B$ defined by

$$A \coprod_C B \overset{\text{def}}{=} A \coprod B/{\sim_C},$$

where $\sim_C$ is the equivalence relation on $A \coprod B$ generated by $(0, f(c)) \sim_C (1, g(c))$.

- *The Cocone.* The maps

$$\begin{aligned} \mathrm{inj}_1 &: A \to A \textstyle\coprod_C B, \\ \mathrm{inj}_2 &: B \to A \textstyle\coprod_C B \end{aligned}$$

given by

$$\begin{aligned} \mathrm{inj}_1(a) &\stackrel{\mathrm{def}}{=} [(0, a)] \\ \mathrm{inj}_2(b) &\stackrel{\mathrm{def}}{=} [(1, b)] \end{aligned}$$

for each $a \in A$ and each $b \in B$.

[1]*Further Terminology:* Also called the **fibre coproduct of** A **and** B **over** C **along** f **and** g.

Proof 2.2.3.2 ► Proof of Definition 2.2.3.1

We claim that $A \coprod_C B$ is the categorical pushout of A and B over C with respect to (f, g) in Sets. First we need to check that the relevant pushout diagram commutes, i.e. that we have

$$\mathrm{inj}_1 \circ f = \mathrm{inj}_2 \circ g, \qquad \begin{tikzcd} A \coprod_C B & B \arrow[l, "\mathrm{inj}_2"'] \\ A \arrow[u, "\mathrm{inj}_1"] & C. \arrow[l, "f"] \arrow[u, "g"'] \end{tikzcd}$$

Indeed, given $c \in C$, we have

$$\begin{aligned} [\mathrm{inj}_1 \circ f](c) &= \mathrm{inj}_1(f(c)) \\ &= [(0, f(c))] \\ &= [(1, g(c))] \\ &= \mathrm{inj}_2(g(c)) \\ &= [\mathrm{inj}_2 \circ g](c), \end{aligned}$$

where $[(0, f(c))] = [(1, g(c))]$ by the definition of the relation $\sim$ on B. Next, we prove that $A \coprod{}_C B$ satisfies the universal property of the pushout. Suppose we have a diagram of the form

$$
\begin{tikzcd}
P & & \\
& A \coprod_C B \arrow[dr, phantom, "\ulcorner", very near start] & B \arrow[l, "\mathrm{inj}_2"'] \arrow[ull, bend right, "i_2"'] \\
& A \arrow[u, "\mathrm{inj}_1"] \arrow[uul, bend left, "i_1"] & C \arrow[l, "f"] \arrow[u, "g"']
\end{tikzcd}
$$

in Sets. Then there exists a unique map $\phi\colon A \coprod_C B \to P$, uniquely determined by the conditions

$$
\begin{aligned}
\phi \circ \mathrm{inj}_1 &= i_1,\\
\phi \circ \mathrm{inj}_2 &= i_2,
\end{aligned}
$$

being necessarily given by

$$
\phi(x) = \begin{cases} i_1(a) & \text{if } x = [(0,a)],\\ i_2(b) & \text{if } x = [(1,b)] \end{cases}
$$

for each $x \in A \coprod_C B$, where the well-definedness of ϕ is guaranteed by the equality $i_1 \circ f = i_2 \circ g$ and the definition of the relation $\sim$ on $A \coprod B$ as follows.

1. *Case 1:* Suppose we have $x = [(0,a)] = [(0,a')]$ for some $a, a' \in A$. Then, by Remark 2.2.3.3, we have a sequence

$$(0,a) \sim' x_1 \sim' \cdots \sim' x_n \sim' (0,a').$$

2. *Case 2:* Suppose we have $x = [(1,b)] = [(1,b')]$ for some $b, b' \in B$. Then, by Remark 2.2.3.3, we have a sequence

$$(1,b) \sim' x_1 \sim' \cdots \sim' x_n \sim' (1,b').$$

3. *Case 3:* Suppose we have $x = [(0,a)] = [(1,b)]$ for some $a \in A$ and $b \in B$. Then, by Remark 2.2.3.3, we have a sequence

$$(0,a) \sim' x_1 \sim' \cdots \sim' x_n \sim' (1,b).$$

In all these cases, we declare $x \sim' y$ iff there exists some $c \in C$ such that $x = (0, f(c))$ and $y = (1, g(c))$ or $x = (1, g(c))$ and $y = (0, f(c))$. Then, the equality $i_1 \circ f = i_2 \circ g$ gives

$$\begin{aligned}\phi([x]) &= \phi([(0, f(c))])\\ &\overset{\text{def}}{=} i_1(f(c))\\ &= i_2(g(c))\\ &\overset{\text{def}}{=} \phi([(1, g(c))])\\ &= \phi([y]),\end{aligned}$$

with the case where $x = (1, g(c))$ and $y = (0, f(c))$ similarly giving $\phi([x]) = \phi([y])$. Thus, if $x \sim' y$, then $\phi([x]) = \phi([y])$. Applying this equality pairwise to the sequences

$$\begin{aligned}(0, a) \sim' x_1 \sim' \cdots \sim' x_n \sim' (0, a'),\\ (1, b) \sim' x_1 \sim' \cdots \sim' x_n \sim' (1, b'),\\ (0, a) \sim' x_1 \sim' \cdots \sim' x_n \sim' (1, b)\end{aligned}$$

gives

$$\begin{aligned}\phi([(0, a)]) = \phi([(0, a')]),\\ \phi([(1, b)]) = \phi([(1, b')]),\\ \phi([(0, a)]) = \phi([(1, b)]),\end{aligned}$$

showing ϕ to be well-defined. ∎

Remark 2.2.3.3 ▶ Unwinding Definition 2.2.3.1

In detail, by **??**, the relation $\sim$ of Definition 2.2.3.1 is given by declaring $a \sim b$ iff one of the following conditions is satisfied:

- We have $a, b \in A$ and $a = b$;
- We have $a, b \in B$ and $a = b$;
- There exist $x_1, \ldots, x_n \in A \coprod B$ such that $a \sim' x_1 \sim' \cdots \sim' x_n \sim' b$, where we declare $x \sim' y$ if one of the following conditions is satisfied:
  1. There exists $c \in C$ such that $x = (0, f(c))$ and $y = (1, g(c))$.
  2. There exists $c \in C$ such that $x = (1, g(c))$ and $y = (0, f(c))$.

  That is: we require the following condition to be satisfied:

(★) There exist $x_1, \dots, x_n \in A \coprod B$ satisfying the following conditions:

1. There exists $c_0 \in C$ satisfying one of the following conditions:
   (a) We have $a = f(c_0)$ and $x_1 = g(c_0)$.
   (b) We have $a = g(c_0)$ and $x_1 = f(c_0)$.
2. For each $1 \leq i \leq n-1$, there exists $c_i \in C$ satisfying one of the following conditions:
   (a) We have $x_i = f(c_i)$ and $x_{i+1} = g(c_i)$.
   (b) We have $x_i = g(c_i)$ and $x_{i+1} = f(c_i)$.
3. There exists $c_n \in C$ satisfying one of the following conditions:
   (a) We have $x_n = f(c_n)$ and $b = g(c_n)$.
   (b) We have $x_n = g(c_n)$ and $b = f(c_n)$.

EXAMPLE 2.2.3.4 ► EXAMPLES OF PUSHOUTS OF SETS

Here are some examples of pushouts of sets.

1. *Wedge Sums of Pointed Sets.* The wedge sum of two pointed sets of **??** is an example of a pushout of sets.

2. *Intersections via Unions.* Let $A, B \subset X$. We have a bijection of sets

$$A \cup B \cong A \coprod_{A \cap B} B, \qquad \begin{tikzcd} A \cup B \arrow[dr, phantom, "\ulcorner", very near start] & B \arrow[l] \\ A \arrow[u] & A \cap B. \arrow[l, hook'] \arrow[u, hook] \end{tikzcd}$$

PROOF 2.2.3.5 ► PROOF OF EXAMPLE 2.2.3.4

Item 1: Wedge Sums of Pointed Sets

Follows by definition.

Item 2: Intersections via Unions

Indeed, $A \coprod_{A \cap B} B$ is the quotient of $A \coprod B$ by the equivalence relation obtained by declaring $(0, a) \sim (1, b)$ iff $a = b \in A \cap B$, which is in bijection with $A \cup B$ via the map with $[(0, a)] \mapsto a$ and $[(1, b)] \mapsto b$. ∎

Proposition 2.2.3.6 ► Properties of Pushouts of Sets

Let $A, B, C,$ and X be sets.

1. *Functoriality.* The assignment $(A, B, C, f, g) \mapsto A \coprod_{f,C,g} B$ defines a functor
$$-_1 \coprod_{-_3} -_1 \colon \mathrm{Fun}(\mathcal{P}, \mathsf{Sets}) \to \mathsf{Sets},$$
where $\mathcal{P}$ is the category that looks like this:
$$\begin{tikzcd} & \bullet \\ \bullet & \bullet. \arrow[l] \arrow[u] \end{tikzcd}$$
In particular, the action on morphisms of $-_1 \coprod_{-_3} -_1$ is given by sending a morphism
$$\begin{tikzcd}
A \coprod_C B \arrow[dr, phantom, very near start, "\ulcorner"] & & B \arrow[ll] \arrow[dr, "\psi"] & \\
& A' \coprod_{C'} B' \arrow[dr, phantom, very near start, "\ulcorner"] & & B' \arrow[ll] \\
A \arrow[uu] \arrow[dr, "\phi"'] & & C \arrow[ll, "f"'] \arrow[uu, "g"] \arrow[dr, "\chi"] & \\
& A' \arrow[uu] & & C' \arrow[ll, "f'"] \arrow[uu, "g'"']
\end{tikzcd}$$
in $\mathrm{Fun}(\mathcal{P}, \mathsf{Sets})$ to the map $\xi \colon A \coprod_C B \xrightarrow{\exists!} A' \coprod_{C'} B'$ given by
$$\xi(x) \overset{\text{def}}{=} \begin{cases} \phi(a) & \text{if } x = [(0, a)], \\ \psi(b) & \text{if } x = [(1, b)] \end{cases}$$
for each $x \in A \coprod_C B$, which is the unique map making the diagram
$$\begin{tikzcd}
A \coprod_C B \arrow[dr, dashed] \arrow[dr, phantom, very near start, "\ulcorner"] & & B \arrow[ll] \arrow[dr, "\psi"] & \\
& A' \coprod_{C'} B' \arrow[dr, phantom, very near start, "\ulcorner"] & & B' \arrow[ll] \\
A \arrow[uu] \arrow[dr, "\phi"'] & & C \arrow[ll, "f"'] \arrow[uu, "g"] \arrow[dr, "\chi"] & \\
& A' \arrow[uu] & & C' \arrow[ll, "f'"] \arrow[uu, "g'"']
\end{tikzcd}$$
commute.

2. *Associativity.* Given a diagram

$$\begin{tikzcd}
A & & B & & C \\
& X \arrow[ul, "f"] \arrow[ur, "g"'] & & Y \arrow[ul, "h"] \arrow[ur, "k"'] &
\end{tikzcd}$$

in Sets, we have isomorphisms

$$(A \coprod_X B) \coprod_Y C \cong (A \coprod_X B) \coprod_B (B \coprod_Y C) \cong A \coprod_X (B \coprod_Y C),$$

where these pullbacks are built as in the diagrams

$$\begin{tikzcd}[column sep=small, row sep=small]
& & (A \coprod_X B) \coprod_Y C & & \\
& A \coprod_X B \arrow[ur] \arrow[ur, phantom, "\wedge"] & & & \\
A \arrow[ur] \arrow[ur, phantom, "\wedge"] & & B \arrow[ul] & & C, \arrow[uull] \\
& X \arrow[ul, "f"] \arrow[ur, "g"'] & & Y \arrow[ul, "h"] \arrow[ur, "k"'] &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=small, row sep=small]
& & (A \coprod_X B) \coprod_B (B \coprod_Y C) & & \\
& A \coprod_X B \arrow[ur] \arrow[ur, phantom, "\wedge"] & & B \coprod_Y C \arrow[ul] & \\
A \arrow[ur] \arrow[ur, phantom, "\wedge"] & & B \arrow[ul] \arrow[ur] \arrow[ur, phantom, "\wedge"] & & C, \arrow[ul] \\
& X \arrow[ul, "f"] \arrow[ur, "g"'] & & Y \arrow[ul, "h"] \arrow[ur, "k"'] &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=small, row sep=small]
& & A \coprod_X (B \coprod_Y C) & & \\
& & & B \coprod_Y C \arrow[ul] \arrow[ul, phantom, "\wedge"] & \\
A \arrow[uurr] & & B \arrow[ur] \arrow[ur, phantom, "\wedge"] & & C. \arrow[ul] \\
& X \arrow[ul, "f"] \arrow[ur, "g"'] & & Y \arrow[ul, "h"] \arrow[ur, "k"'] &
\end{tikzcd}$$

3. *Unitality.* We have isomorphisms of sets

$$\begin{tikzcd}
A \arrow[r, equal] & A \\
X \arrow[u, "f"] \arrow[r, equal] \arrow[ur, phantom, very near end, "\ulcorner"] & X \arrow[u, "f"']
\end{tikzcd}
\qquad
\begin{aligned}
X \coprod_X A &\cong A, \\
A \coprod_X X &\cong A,
\end{aligned}
\qquad
\begin{tikzcd}
A & X \arrow[l, "f"'] \\
X \arrow[u, equal] \arrow[ur, phantom, very near end, "\ulcorner"] & X. \arrow[l, "f"] \arrow[u, equal]
\end{tikzcd}$$

4. *Commutativity.* We have an isomorphism of sets

$$\begin{tikzcd}
A \coprod_X B & B \arrow[l] \\
A \arrow[u] \arrow[ur, phantom, very near end, "\ulcorner"] & X, \arrow[l, "f"] \arrow[u, "g"']
\end{tikzcd}
\qquad
A \coprod_X B \cong B \coprod_X A
\qquad
\begin{tikzcd}
B \coprod_X A & A \arrow[l] \\
B \arrow[u] \arrow[ur, phantom, very near end, "\ulcorner"] & X. \arrow[l, "g"] \arrow[u, "f"']
\end{tikzcd}$$

5. *Interaction With Coproducts.* We have

$$A \coprod_{\emptyset} B \cong A \coprod B,
\qquad
\begin{tikzcd}
A \coprod B & B \arrow[l] \\
A \arrow[u] \arrow[ur, phantom, very near end, "\ulcorner"] & \emptyset. \arrow[l, hook', "\iota_A"] \arrow[u, hook, "\iota_B"']
\end{tikzcd}$$

6. *Symmetric Monoidality.* The triple $(\mathsf{Sets}, \coprod_X, \emptyset)$ is a symmetric monoidal category.

Proof 2.2.3.7 ► Proof of Proposition 2.2.3.6

Item 1: Functoriality

This is a special case of functoriality of co/limits, **??** of **??**, with the explicit expression for ξ following from the commutativity of the cube pushout diagram.

Item 2: Associativity

Omitted.

Item 3: Unitality

Omitted.

Item 4: Commutativity

Clear.

Item 5: Interaction With Coproducts

Clear.

Item 6: Symmetric Monoidality

Omitted.

2.2.4 Coequalisers

Let A and B be sets and let $f, g \colon A \rightrightarrows B$ be functions.

Definition 2.2.4.1 ► Coequalisers of Sets

The **coequaliser of f and g** is the pair $(\mathrm{CoEq}(f,g), \mathrm{coeq}(f,g))$ consisting of

- *The Colimit.* The set $\mathrm{CoEq}(f,g)$ defined by

$$\mathrm{CoEq}(f,g) \stackrel{\text{def}}{=} B/{\sim},$$

  where $\sim$ is the equivalence relation on B generated by $f(a) \sim g(a)$.

- *The Cocone.* The map

$$\mathrm{coeq}(f,g) \colon B \longrightarrow \mathrm{CoEq}(f,g)$$

  given by the quotient map $\pi \colon B \twoheadrightarrow B/{\sim}$ with respect to the equivalence relation generated by $f(a) \sim g(a)$.

Proof 2.2.4.2 ▶ Proof of Definition 2.2.4.1

We claim that $\mathrm{CoEq}(f,g)$ is the categorical coequaliser of f and g in Sets. First we need to check that the relevant coequaliser diagram commutes, i.e. that we have

$$\mathrm{coeq}(f,g)\circ f=\mathrm{coeq}(f,g)\circ g.$$

Indeed, we have

$$\begin{aligned}[\mathrm{coeq}(f,g)\circ f](a)&\overset{\text{def}}{=}[\mathrm{coeq}(f,g)](f(a))\\&\overset{\text{def}}{=}[f(a)]\\&=[g(a)]\\&\overset{\text{def}}{=}[\mathrm{coeq}(f,g)](g(a))\\&\overset{\text{def}}{=}[\mathrm{coeq}(f,g)\circ g](a)\end{aligned}$$

for each $a\in A$. Next, we prove that $\mathrm{CoEq}(f,g)$ satisfies the universal property of the coequaliser. Suppose we have a diagram of the form

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, two heads, "\mathrm{coeq}(f,g)"] \arrow[dr, "c"'] & \mathrm{CoEq}(f,g) \\ & & C \end{tikzcd}$$

in Sets. Then, since $c(f(a))=c(g(a))$ for each $a\in A$, it follows from **????** of **??** that there exists a unique map $\mathrm{CoEq}(f,g)\xrightarrow{\exists!}C$ making the diagram

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, two heads, "\mathrm{coeq}(f,g)"] \arrow[dr, "c"'] & \mathrm{CoEq}(f,g) \arrow[d, dashed, "\exists!"] \\ & & C \end{tikzcd}$$

commutes. ∎

Remark 2.2.4.3 ▶ Unwinding Definition 2.2.4.1

In detail, by **??**, the relation $\sim$ of Definition 2.2.4.1 is given by declaring $a\sim b$ iff one of the following conditions is satisfied:

- We have $a=b$;
- There exist $x_1,\ldots,x_n\in B$ such that $a\sim' x_1\sim'\cdots\sim' x_n\sim' b$, where we

declare $x \sim' y$ if one of the following conditions is satisfied:

1. There exists $z \in A$ such that $x = f(z)$ and $y = g(z)$.
2. There exists $z \in A$ such that $x = g(z)$ and $y = f(z)$.

That is: we require the following condition to be satisfied:

($\star$) There exist $x_1, \ldots, x_n \in B$ satisfying the following conditions:
   1. There exists $z_0 \in A$ satisfying one of the following conditions:
      (a) We have $a = f(z_0)$ and $x_1 = g(z_0)$.
      (b) We have $a = g(z_0)$ and $x_1 = f(z_0)$.
   2. For each $1 \leq i \leq n-1$, there exists $z_i \in A$ satisfying one of the following conditions:
      (a) We have $x_i = f(z_i)$ and $x_{i+1} = g(z_i)$.
      (b) We have $x_i = g(z_i)$ and $x_{i+1} = f(z_i)$.
   3. There exists $z_n \in A$ satisfying one of the following conditions:
      (a) We have $x_n = f(z_n)$ and $b = g(z_n)$.
      (b) We have $x_n = g(z_n)$ and $b = f(z_n)$.

Example 2.2.4.4 ▶ Examples of Coequalisers of Sets

Here are some examples of coequalisers of sets.

1. *Quotients by Equivalence Relations.* Let R be an equivalence relation on a set X. We have a bijection of sets

$$X/{\sim_R} \cong \mathrm{CoEq}\left(R \hookrightarrow X \times X \underset{\mathrm{pr}_2}{\overset{\mathrm{pr}_1}{\rightrightarrows}} X\right).$$

Proof 2.2.4.5 ▶ Proof of Example 2.2.4.4

Item 1: Quotients by Equivalence Relations

See [Pro24z]. □

Proposition 2.2.4.6 ▶ Properties of Coequalisers of Sets

Let A, B, and C be sets.

1. *Associativity.* We have an isomorphism of sets[1]

$$\underbrace{\mathrm{CoEq}(\mathrm{coeq}(f,g)\circ f, \mathrm{coeq}(f,g)\circ h)}_{=\mathrm{CoEq}(\mathrm{coeq}(f,g)\circ g, \mathrm{coeq}(f,g)\circ h)} \cong \mathrm{CoEq}(f,g,h) \cong \underbrace{\mathrm{CoEq}(\mathrm{coeq}(g,h)\circ f, \mathrm{coeq}(g,h)\circ g)}_{=\mathrm{CoEq}(\mathrm{coeq}(g,h)\circ f, \mathrm{coeq}(g,h)\circ h)},$$

where $\mathrm{CoEq}(f, g, h)$ is the colimit of the diagram

$$A \overset{f}{\underset{h}{\overset{-g\to}{\rightrightarrows}}} B$$

in Sets.

2. *Unitality.* We have an isomorphism of sets

$$\mathrm{CoEq}(f, f) \cong B.$$

3. *Commutativity.* We have an isomorphism of sets

$$\mathrm{CoEq}(f, g) \cong \mathrm{CoEq}(g, f).$$

4. *Interaction With Composition.* Let

$$A \overset{f}{\underset{g}{\rightrightarrows}} B \overset{h}{\underset{k}{\rightrightarrows}} C$$

be functions. We have a surjection

$$\mathrm{CoEq}(h \circ f, k \circ g) \twoheadrightarrow \mathrm{CoEq}(\mathrm{coeq}(h, k) \circ h \circ f, \mathrm{coeq}(h, k) \circ k \circ g)$$

exhibiting $\mathrm{CoEq}(\mathrm{coeq}(h, k) \circ h \circ f, \mathrm{coeq}(h, k) \circ k \circ g)$ as a quotient of $\mathrm{CoEq}(h \circ f, k \circ g)$ by the relation generated by declaring $h(y) \sim k(y)$ for each $y \in B$.

---

[1]That is, the following three ways of forming "the" coequaliser of (f, g, h) agree:

(a) Take the coequaliser of (f, g, h), i.e. the colimit of the diagram

$$A \overset{f}{\underset{h}{\overset{-g\to}{\rightrightarrows}}} B$$

in Sets.

(b) First take the coequaliser of f and g, forming a diagram

$$A \overset{f}{\underset{g}{\rightrightarrows}} B \overset{\mathrm{coeq}(f,g)}{\twoheadrightarrow} \mathrm{CoEq}(f, g)$$

and then take the coequaliser of the composition

$$A \overset{f}{\underset{h}{\rightrightarrows}} B \overset{\mathrm{coeq}(f,g)}{\twoheadrightarrow} \mathrm{CoEq}(f, g),$$

obtaining a quotient

$$\mathrm{CoEq}(\mathrm{coeq}(f, g) \circ f, \mathrm{coeq}(f, g) \circ h) = \mathrm{CoEq}(\mathrm{coeq}(f, g) \circ g, \mathrm{coeq}(f, g) \circ h)$$

of $\mathrm{CoEq}(f, g)$

(c) First take the coequaliser of g and h, forming a diagram

$$A \underset{h}{\overset{g}{\rightrightarrows}} B \overset{\mathrm{coeq}(g,h)}{\twoheadrightarrow} \mathrm{CoEq}(g,h)$$

and then take the coequaliser of the composition

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \overset{\mathrm{coeq}(g,h)}{\twoheadrightarrow} \mathrm{CoEq}(g,h),$$

obtaining a quotient

$$\mathrm{CoEq}(\mathrm{coeq}(g,h) \circ f, \mathrm{coeq}(g,h) \circ g) = \mathrm{CoEq}(\mathrm{coeq}(g,h) \circ f, \mathrm{coeq}(g,h) \circ h)$$

of $\mathrm{CoEq}(g,h)$.

PROOF 2.2.4.7 ► PROOF OF PROPOSITION 2.2.4.6

Item 1: Associativity

Omitted.

Item 2: Unitality

Clear.

Item 3: Commutativity

Clear.

Item 4: Interaction With Composition

Omitted. ∎

2.3 Operations With Sets

2.3.1 The Empty Set

DEFINITION 2.3.1.1 ► THE EMPTY SET

The **empty set** is the set $\emptyset$ defined by

$$\emptyset \overset{\mathrm{def}}{=} \{x \in X \mid x \neq x\},$$

where A is the set in the set existence axiom, **??** of **??**.

2.3.2 Singleton Sets

Let X be a set.

Definition 2.3.2.1 ► Singleton Sets

The **singleton set containing** X is the set $\{X\}$ defined by

$$\{X\} \overset{\text{def}}{=} \{X, X\},$$

where $\{X, X\}$ is the pairing of X with itself (Definition 2.3.3.1).

2.3.3 Pairings of Sets

Let X and Y be sets.

Definition 2.3.3.1 ► Pairings of Sets

The **pairing of** X **and** Y is the set $\{X, Y\}$ defined by

$$\{X, Y\} \overset{\text{def}}{=} \{x \in A \mid x = X \text{ or } x = Y\},$$

where A is the set in the axiom of pairing, **??** of **??**.

2.3.4 Ordered Pairs

Let A and B be sets.

Definition 2.3.4.1 ► Ordered Pairs

The **ordered pair associated to** A **and** B is the set (A, B) defined by

$$(A, B) \overset{\text{def}}{=} \{\{A\}, \{A, B\}\}.$$

Proposition 2.3.4.2 ► Properties of Ordered Pairs

Let A and B be sets.

1. *Uniqueness.* Let $A, B, C,$ and D be sets. The following conditions are equivalent:

   (a) We have $(A, B) = (C, D)$.

   (b) We have $A = C$ and $B = D$.

PROOF 2.3.4.3 ▶ PROOF OF PROPOSITION 2.3.4.2

Item 1: Uniqueness

See [Cie97, Theorem 1.2.3].

2.3.5 Unions of Families

Let $\{A_i\}_{i\in I}$ be a family of sets.

DEFINITION 2.3.5.1 ▶ UNIONS OF FAMILIES

The **union of the family** $\{A_i\}_{i\in I}$ is the set $\bigcup_{i\in I} A_i$ defined by

$$\bigcup_{i\in I} A_i \overset{\text{def}}{=} \{x \in F \mid \text{there exists some } i \in I \text{ such that } x \in A_i\},$$

where F is the set in the axiom of union, **??** of **??**.

2.3.6 Binary Unions

Let A and B be sets.

DEFINITION 2.3.6.1 ▶ BINARY UNIONS

The **union**[1] **of A and B** is the set $A \cup B$ defined by

$$A \cup B \overset{\text{def}}{=} \bigcup_{z\in\{A,B\}} z.$$

[1] *Further Terminology:* Also called the **binary union of A and B**, for emphasis.

PROPOSITION 2.3.6.2 ▶ PROPERTIES OF BINARY UNIONS

Let X be a set.

1. *Functoriality.* The assignments $U, V, (U, V) \mapsto U \cup V$ define functors

$$U \cup -\colon (\mathcal{P}(X), \subset) \to (\mathcal{P}(X), \subset),$$
$$- \cup V\colon (\mathcal{P}(X), \subset) \to (\mathcal{P}(X), \subset),$$
$$-_1 \cup -_2\colon (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \subset) \to (\mathcal{P}(X), \subset),$$

where $-_1 \cup -_2$ is the functor where

- *Action on Objects.* For each $(U, V) \in \mathcal{P}(X) \times \mathcal{P}(X)$, we have

$$[-_1 \cup -_2](U, V) \overset{\text{def}}{=} U \cup V;$$

- *Action on Morphisms.* For each pair of morphisms

$$\iota_U : U \hookrightarrow U',$$
$$\iota_V : V \hookrightarrow V'$$

of $\mathcal{P}(X) \times \mathcal{P}(X)$, the image

$$\iota_U \cup \iota_V : U \cup V \hookrightarrow U' \cup V'$$

of (ι_U, ι_V) by $\cup$ is the inclusion

$$U \cup V \subset U' \cup V'$$

i.e. where we have

(★) If $U \subset U'$ and $V \subset V'$, then $U \cup V \subset U' \cup V'$;

and where $U \cup -$ and $- \cup V$ are the partial functors of $-_1 \cup -_2$ at $U, V \in \mathcal{P}(X)$.

2. *Via Intersections and Symmetric Differences.* We have an equality of sets

$$U \cup V = (U \triangle V) \triangle (U \cap V)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

3. *Associativity.* We have an equality of sets

$$(U \cup V) \cup W = U \cup (V \cup W)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

4. *Unitality.* We have equalities of sets

$$U \cup \emptyset = U,$$
$$\emptyset \cup U = U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

5. *Commutativity.* We have an equality of sets

$$U \cup V = V \cup U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

6. *Idempotency.* We have an equality of sets

$$U \cup U = U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

7. *Distributivity Over Intersections.* We have equalities of sets

$$\begin{aligned} U \cup (V \cap W) &= (U \cup V) \cap (U \cup W), \\ (U \cap V) \cup W &= (U \cup W) \cap (V \cup W) \end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

8. *Interaction With Characteristic Functions I.* We have

$$\chi_{U \cup V} = \max(\chi_U, \chi_V)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

9. *Interaction With Characteristic Functions II.* We have

$$\chi_{U \cup V} = \chi_U + \chi_V - \chi_{U \cap V}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

10. *Interaction With Powersets and Semirings.* The quintuple $(\mathcal{P}(X), \cup, \cap, \emptyset, X)$ is an idempotent commutative semiring.

Proof 2.3.6.3 ► Proof of Proposition 2.3.6.2

Item 1: Functoriality

See [Pro24an].

Item 2: Via Intersections and Symmetric Differences

See [Pro24ay].

Item 3: Associativity

See [Pro24ba].

Item 4: Unitality

This follows from [Pro24bd] and Item 5.

Item 5: Commutativity

See [Pro24bb].

Item 6: Idempotency

See [Pro24am].

Item 7: Distributivity Over Intersections

See [Pro24az].

Item 8: Interaction With Characteristic Functions I

See [Pro24k].

Item 9: Interaction With Characteristic Functions II

See [Pro24k].

Item 10: Interaction With Powersets and Semirings

This follows from Items 3 to 6 and Items 3 to 5, 7 and 8 of Proposition 2.3.8.2.

2.3.7 Intersections of Families

Let $\mathcal{F}$ be a family of sets.

Definition 2.3.7.1 ► Intersections of Families

The **intersection of a family $\mathcal{F}$ of sets** is the set $\bigcap_{X\in\mathcal{F}} X$ defined by

$$\bigcap_{X\in\mathcal{F}} X \overset{\text{def}}{=} \left\{ z \in \bigcup_{X\in\mathcal{F}} X \,\middle|\, \text{for each } X \in \mathcal{F}, \text{ we have } z \in X \right\}.$$

2.3.8 Binary Intersections

Let X and Y be sets.

Definition 2.3.8.1 ► Binary Intersections

The **intersection**[1] **of X and Y** is the set $X \cap Y$ defined by

$$X \cap Y \overset{\text{def}}{=} \bigcap_{z\in\{X,Y\}} z.$$

[1] *Further Terminology:* Also called the **binary intersection of X and Y**, for emphasis.

Proposition 2.3.8.2 ► Properties of Binary Intersections

Let X be a set.

1. *Functoriality.* The assignments $U, V, (U, V) \mapsto U \cap V$ define functors

$$
\begin{aligned}
U \cap - &\colon (\mathcal{P}(X), \subset) \to (\mathcal{P}(X), \subset),\\
- \cap V &\colon (\mathcal{P}(X), \subset) \to (\mathcal{P}(X), \subset),\\
-_1 \cap -_2 &\colon (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \subset) \to (\mathcal{P}(X), \subset),
\end{aligned}
$$

where $-_1 \cap -_2$ is the functor where

- *Action on Objects.* For each $(U, V) \in \mathcal{P}(X) \times \mathcal{P}(X)$, we have

$$[-_1 \cap -_2](U, V) \stackrel{\text{def}}{=} U \cap V;$$

- *Action on Morphisms.* For each pair of morphisms

$$
\begin{aligned}
\iota_U &\colon U \hookrightarrow U',\\
\iota_V &\colon V \hookrightarrow V'
\end{aligned}
$$

of $\mathcal{P}(X) \times \mathcal{P}(X)$, the image

$$\iota_U \cap \iota_V \colon U \cap V \hookrightarrow U' \cap V'$$

of (ι_U, ι_V) by $\cap$ is the inclusion

$$U \cap V \subset U' \cap V'$$

i.e. where we have

(★) If $U \subset U'$ and $V \subset V'$, then $U \cap V \subset U' \cap V'$;

and where $U \cap -$ and $- \cap V$ are the partial functors of $-_1 \cap -_2$ at $U, V \in \mathcal{P}(X)$.

2. *Adjointness.* We have adjunctions

$$
\begin{aligned}
\left(U \cap - \dashv \mathbf{Hom}_{\mathcal{P}(X)}(U, -)\right) &\colon \quad
\begin{tikzcd}
\mathcal{P}(X) \arrow[r, bend left, "U\cap-"] \arrow[r, phantom, "\perp"] & \mathcal{P}(X) \arrow[l, bend left, "\mathbf{Hom}_{\mathcal{P}(X)}(U,-)"]
\end{tikzcd},\\
\left(- \cap V \dashv \mathbf{Hom}_{\mathcal{P}(X)}(V, -)\right) &\colon \quad
\begin{tikzcd}
\mathcal{P}(X) \arrow[r, bend left, "-\cap V"] \arrow[r, phantom, "\perp"] & \mathcal{P}(X) \arrow[l, bend left, "\mathbf{Hom}_{\mathcal{P}(X)}(V,-)"]
\end{tikzcd},
\end{aligned}
$$

where

$$\mathbf{Hom}_{\mathcal{P}(X)}(-_1, -_2) \colon \mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \to \mathcal{P}(X)$$

is the bifunctor defined by[1]

$$\mathbf{Hom}_{\mathcal{P}(X)}(U,V) \overset{\text{def}}{=} (X \setminus U) \cup V$$

witnessed by bijections

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{P}(X)}(U \cap V, W) &\cong \mathrm{Hom}_{\mathcal{P}(X)}\big(U, \mathbf{Hom}_{\mathcal{P}(X)}(V,W)\big),\\
\mathrm{Hom}_{\mathcal{P}(X)}(U \cap V, W) &\cong \mathrm{Hom}_{\mathcal{P}(X)}\big(V, \mathbf{Hom}_{\mathcal{P}(X)}(U,W)\big),
\end{aligned}$$

natural in $U, V, W \in \mathcal{P}(X)$, i.e. where:

(a) The following conditions are equivalent:

i. We have $U \cap V \subset W$.

ii. We have $U \subset \mathbf{Hom}_{\mathcal{P}(X)}(V,W)$.

iii. We have $U \subset (X \setminus V) \cup W$.

(b) The following conditions are equivalent:

i. We have $V \cap U \subset W$.

ii. We have $V \subset \mathbf{Hom}_{\mathcal{P}(X)}(U,W)$.

iii. We have $V \subset (X \setminus U) \cup W$.

3. *Associativity.* We have an equality of sets

$$(U \cap V) \cap W = U \cap (V \cap W)$$

for each $X \in \mathrm{Obj}(\mathrm{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

4. *Unitality.* Let X be a set and let $U \in \mathcal{P}(X)$. We have equalities of sets

$$\begin{aligned}
X \cap U &= U,\\
U \cap X &= U
\end{aligned}$$

for each $X \in \mathrm{Obj}(\mathrm{Sets})$ and each $U \in \mathcal{P}(X)$.

5. *Commutativity.* We have an equality of sets

$$U \cap V = V \cap U$$

for each $X \in \mathrm{Obj}(\mathrm{Sets})$ and each $U, V \in \mathcal{P}(X)$.

6. *Idempotency.* We have an equality of sets

$$U \cap U = U$$

for each $X \in \mathrm{Obj}(\mathrm{Sets})$ and each $U \in \mathcal{P}(X)$.

7. *Distributivity Over Unions.* We have equalities of sets

$$U \cap (V \cup W) = (U \cap V) \cup (U \cap W),$$
$$(U \cup V) \cap W = (U \cap W) \cup (V \cap W)$$

for each $X \in \text{Obj}(\text{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

8. *Annihilation With the Empty Set.* We have an equality of sets

$$\emptyset \cap X = \emptyset,$$
$$X \cap \emptyset = \emptyset$$

for each $X \in \text{Obj}(\text{Sets})$ and each $U \in \mathcal{P}(X)$.

9. *Interaction With Characteristic Functions I.* We have

$$\chi_{U \cap V} = \chi_U \chi_V$$

for each $X \in \text{Obj}(\text{Sets})$ and each $U, V \in \mathcal{P}(X)$.

10. *Interaction With Characteristic Functions II.* We have

$$\chi_{U \cap V} = \min(\chi_U, \chi_V)$$

for each $X \in \text{Obj}(\text{Sets})$ and each $U, V \in \mathcal{P}(X)$.

11. *Interaction With Powersets and Monoids With Zero.* The quadruple $((\mathcal{P}(X), \emptyset), \cap, X)$ is a commutative monoid with zero.

12. *Interaction With Powersets and Semirings.* The quintuple $(\mathcal{P}(X), \cup, \cap, \emptyset, X)$ is an idempotent commutative semiring.

---

[1] *Intuition:* Since intersections are the products in $\mathcal{P}(X)$, the left adjoint $\mathbf{Hom}_{\mathcal{P}(X)}(U, V)$ works as a function type $U \to V$.

Now, under the Curry–Howard correspondence, the function type $U \to V$ corresponds to implication $U \implies V$, which is logically equivalent to the statement $\neg U \vee V$, which in turn corresponds to the set $U^{\mathsf{c}} \vee V \stackrel{\text{def}}{=} (X \setminus U) \cup V$.

Proof 2.3.8.3 ► Proof of Proposition 2.3.8.2

Item 1: Functoriality

See [Pro24al].

Item 2: Adjointness

See [MSE 267469].

Item 3: Associativity

See [Pro24t].

Item 4: Unitality

This follows from [Pro24x] and Item 5.

Item 5: Commutativity

See [Pro24u].

Item 6: Idempotency

See [Pro24ak].

Item 7: Distributivity Over Unions

See [Pro24aj].

Item 8: Annihilation With the Empty Set

This follows from [Pro24v] and Item 5.

Item 9: Interaction With Characteristic Functions I

See [Pro24h].

Item 10: Interaction With Characteristic Functions II

See [Pro24h].

Item 11: Interaction With Powersets and Monoids With Zero

This follows from Items 3 to 5 and 8.

Item 12: Interaction With Powersets and Semirings

This follows from Items 3 to 6 and Items 3 to 5, 7 and 8 of Proposition 2.3.8.2.

2.3.9 Differences

Let X and Y be sets.

Definition 2.3.9.1 ► Differences

The **difference of X and Y** is the set $X \setminus Y$ defined by

$$X \setminus Y \stackrel{\text{def}}{=} \{a \in X \mid a \notin Y\}.$$

Proposition 2.3.9.2 ► Properties of Differences

Let X be a set.

1. *Functoriality.* The assignments $U, V, (U, V) \mapsto U \cap V$ define functors

$$U \setminus -: (\mathcal{P}(X), \supset) \to (\mathcal{P}(X), \subset),$$
$$- \setminus V: (\mathcal{P}(X), \subset) \to (\mathcal{P}(X), \subset),$$
$$-_1 \setminus -_2: (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \supset) \to (\mathcal{P}(X), \subset),$$

   where $-_1 \setminus -_2$ is the functor where

   - *Action on Objects.* For each $(U, V) \in \mathcal{P}(X) \times \mathcal{P}(X)$, we have

$$[-_1 \setminus -_2](U, V) \stackrel{\text{def}}{=} U \setminus V;$$

   - *Action on Morphisms.* For each pair of morphisms

$$\iota_A: A \hookrightarrow B,$$
$$\iota_U: U \hookrightarrow V$$

     of $\mathcal{P}(X) \times \mathcal{P}(X)$, the image

$$\iota_U \setminus \iota_V: A \setminus V \hookrightarrow B \setminus U$$

     of (ι_U, ι_V) by $\setminus$ is the inclusion

$$A \setminus V \subset B \setminus U$$

     i.e. where we have

     $(\star)$ If $A \subset B$ and $U \subset V$, then $A \setminus V \subset B \setminus U$;

   and where $U \setminus -$ and $- \setminus V$ are the partial functors of $-_1 \setminus -_2$ at $U, V \in \mathcal{P}(X)$.

2. *De Morgan's Laws.* We have equalities of sets

$$X \setminus (U \cup V) = (X \setminus U) \cap (X \setminus V),$$
$$X \setminus (U \cap V) = (X \setminus U) \cup (X \setminus V)$$

   for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

3. *Interaction With Unions I.* We have equalities of sets

$$U \setminus (V \cup W) = (U \setminus V) \cap (U \setminus W)$$

   for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

4. *Interaction With Unions II.* We have equalities of sets

$$(U \setminus V) \cup W = (U \cup W) \setminus (V \setminus W)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

5. *Interaction With Unions III.* We have equalities of sets

$$\begin{aligned} U \setminus (V \cup W) &= (U \cup W) \setminus (V \cup W) \\ &= (U \setminus V) \setminus W \\ &= (U \setminus W) \setminus V \end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

6. *Interaction With Unions IV.* We have equalities of sets

$$(U \cup V) \setminus W = (U \setminus W) \cup (V \setminus W)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

7. *Interaction With Intersections.* We have equalities of sets

$$\begin{aligned} (U \setminus V) \cap W &= (U \cap W) \setminus V \\ &= U \cap (W \setminus V) \end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

8. *Interaction With Complements.* We have an equality of sets

$$U \setminus V = U \cap V^{\mathsf{c}}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

9. *Interaction With Symmetric Differences.* We have an equality of sets

$$U \setminus V = U \vartriangle (U \cap V)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

10. *Triple Differences.* We have

$$U \setminus (V \setminus W) = (U \cap W) \cup (U \setminus V)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

11. *Left Annihilation.* We have

$$\emptyset \setminus U = \emptyset$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

12. *Right Unitality.* We have

$$U \setminus \emptyset = U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and $U \in \mathcal{P}(X)$.

13. *Invertibility.* We have

$$U \setminus U = \emptyset$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

14. *Interaction With Containment.* The following conditions are equivalent:

    (a) We have $V \setminus U \subset W$.

    (b) We have $V \setminus W \subset U$.

15. *Interaction With Characteristic Functions.* We have

$$\chi_{U \setminus V} = \chi_U - \chi_{U \cap V}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

Proof 2.3.9.3 ► Proof of Proposition 2.3.9.2

Item 1: Functoriality

See [Pro24ad] and [Pro24ah].

Item 2: De Morgan's Laws

See [Pro24p].

Item 3: Interaction With Unions I

See [Pro24q].

Item 4: Interaction With Unions II

Omitted.

Item 5: Interaction With Unions III

See [Pro24ai].

Item 6: Interaction With Unions IV

See [Pro24ac].

Item 7: Interaction With Intersections

See [Pro24w].

Item 8: Interaction With Complements

See [Pro24aa].

Item 9: Interaction With Symmetric Differences

See [Pro24ab].

Item 10: Triple Differences

See [Pro24ag].

Item 11: Left Annihilation

Clear.

Item 12: Right Unitality

See [Pro24ae].

Item 13: Invertibility

See [Pro24af].

Item 14: Interaction With Containment

Omitted.

Item 15: Interaction With Characteristic Functions

See [Pro24i].

2.3.10 Complements

Let X be a set and let $U \in \mathcal{P}(X)$.

Definition 2.3.10.1 ► Complements

The **complement of** U is the set U^{c} defined by

$$\begin{aligned} U^{\mathsf{c}} &\overset{\text{def}}{=} X \setminus U \\ &\overset{\text{def}}{=} \{a \in X \mid a \notin U\}. \end{aligned}$$

Proposition 2.3.10.2 ▶ Properties of Complements

Let X be a set.

1. *Functoriality.* The assignment $U \mapsto U^{\mathsf{c}}$ defines a functor

$$(-)^{\mathsf{c}} \colon \mathcal{P}(X)^{\mathsf{op}} \to \mathcal{P}(X),$$

   where

   - *Action on Objects.* For each $U \in \mathcal{P}(X)$, we have

$$[(-)^{\mathsf{c}}](U) \overset{\text{def}}{=} U^{\mathsf{c}};$$

   - *Action on Morphisms.* For each morphism $\iota_U \colon U \hookrightarrow V$ of $\mathcal{P}(X)$, the image

$$\iota_U^{\mathsf{c}} \colon V^{\mathsf{c}} \hookrightarrow U^{\mathsf{c}}$$

     of ι_U by $(-)^{\mathsf{c}}$ is the inclusion

$$V^{\mathsf{c}} \subset U^{\mathsf{c}}$$

     i.e. where we have

     $(\star)$ If $U \subset V$, then $V^{\mathsf{c}} \subset U^{\mathsf{c}}$.

2. *De Morgan's Laws.* We have equalities of sets

$$(U \cup V)^{\mathsf{c}} = U^{\mathsf{c}} \cap V^{\mathsf{c}},$$
$$(U \cap V)^{\mathsf{c}} = U^{\mathsf{c}} \cup V^{\mathsf{c}}$$

   for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

3. *Involutority.* We have

$$(U^{\mathsf{c}})^{\mathsf{c}} = U$$

   for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

4. *Interaction With Characteristic Functions.* We have

$$\chi_{U^{\mathsf{c}}} = 1 - \chi_U$$

   for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

Proof 2.3.10.3 ▶ Proof of Proposition 2.3.10.2

Item 1: Functoriality

This follows from Item 1 of Proposition 2.3.9.2.

Item 2: De Morgan's Laws

See [Pro24p].

Item 3: Involutority

See [Pro24l].

Item 4: Interaction With Characteristic Functions

Clear.

2.3.11 Symmetric Differences

Let A and B be sets.

Definition 2.3.11.1 ▶ Symmetric Differences

The **symmetric difference of** A **and** B is the set $A \mathbin{\triangle} B$ defined by

$$A \mathbin{\triangle} B \overset{\text{def}}{=} (A \setminus B) \cup (B \setminus A).$$

Proposition 2.3.11.2 ▶ Properties of Symmetric Differences

Let X be a set.

1. *Lack of Functoriality.* The assignment $(U, V) \mapsto U \mathbin{\triangle} V$ ***need not*** define functors

$$\begin{aligned} U \mathbin{\triangle} -_2 &\colon (\mathcal{P}(X), \subset) \to (\mathcal{P}(X), \subset), \\ -_1 \mathbin{\triangle} V &\colon (\mathcal{P}(X), \subset) \to (\mathcal{P}(X), \subset), \\ -_1 \mathbin{\triangle} -_2 &\colon (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \subset) \to (\mathcal{P}(X), \subset). \end{aligned}$$

2. *Via Unions and Intersections.* We have[1]

$$U \mathbin{\triangle} V = (U \cup V) \setminus (U \cap V)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

3. *Associativity.* We have[2]

$$(U \mathbin{\triangle} V) \mathbin{\triangle} W = U \mathbin{\triangle} (V \mathbin{\triangle} W)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

4. *Commutativity.* We have

$$U \triangle V = V \triangle U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

5. *Unitality.* We have

$$\begin{aligned} U \triangle \emptyset &= U, \\ \emptyset \triangle U &= U \end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

6. *Invertibility.* We have

$$U \triangle U = \emptyset$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

7. *Interaction With Unions.* We have

$$(U \triangle V) \cup (V \triangle T) = (U \cup V \cup W) \setminus (U \cap V \cap W)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

8. *Interaction With Complements I.* We have

$$U \triangle U^{\mathsf{c}} = X$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

9. *Interaction With Complements II.* We have

$$\begin{aligned} U \triangle X &= U^{\mathsf{c}}, \\ X \triangle U &= U^{\mathsf{c}} \end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

10. *Interaction With Complements III.* We have

$$U^{\mathsf{c}} \triangle V^{\mathsf{c}} = U \triangle V$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

11. *“Transitivity”.* We have

$$(U \triangle V) \triangle (V \triangle W) = U \triangle W$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

12. *The Triangle Inequality for Symmetric Differences.* We have

$$U \bigtriangleup W \subset U \bigtriangleup V \cup V \bigtriangleup W$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

13. *Distributivity Over Intersections.* We have

$$\begin{aligned} U \cap (V \bigtriangleup W) &= (U \cap V) \bigtriangleup (U \cap W), \\ (U \bigtriangleup V) \cap W &= (U \cap W) \bigtriangleup (V \cap W) \end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

14. *Interaction With Characteristic Functions.* We have

$$\chi_{U \bigtriangleup V} = \chi_U + \chi_V - 2\chi_{U \cap V}$$

and thus, in particular, we have

$$\chi_{U \bigtriangleup V} \equiv \chi_U + \chi_V \mod 2$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

15. *Bijectivity.* Given $A, B \subset \mathcal{P}(X)$, the maps

$$\begin{aligned} A \bigtriangleup - &\colon \mathcal{P}(X) \to \mathcal{P}(X), \\ - \bigtriangleup B &\colon \mathcal{P}(X) \to \mathcal{P}(X) \end{aligned}$$

are bijections with inverses given by

$$\begin{aligned} (A \bigtriangleup -)^{-1} &= - \cup (A \cap -), \\ (- \bigtriangleup B)^{-1} &= - \cup (B \cap -). \end{aligned}$$

Moreover, the map

$$C \mapsto C \bigtriangleup (A \bigtriangleup B)$$

is a bijection of $\mathcal{P}(X)$ onto itself sending A to B and B to A.

16. *Interaction With Powersets and Groups.* Let X be a set.

    (a) The quadruple $(\mathcal{P}(X), \bigtriangleup, \emptyset, \mathrm{id}_{\mathcal{P}(X)})$ is an abelian group.[3]

    (b) Every element of $\mathcal{P}(X)$ has order 2 with respect to $\bigtriangleup$, and thus $\mathcal{P}(X)$ is a *Boolean group* (i.e. an abelian 2-group).

17. *Interaction With Powersets and Vector Spaces I.* The pair $\left(\mathcal{P}(X), \alpha_{\mathcal{P}(X)}\right)$ consisting of

- The group $\mathcal{P}(X)$ of **??**;
- The map $\alpha_{\mathcal{P}(X)} \colon \mathbb{F}_2 \times \mathcal{P}(X) \to \mathcal{P}(X)$ defined by

$$0 \cdot U \overset{\text{def}}{=} \emptyset,$$
$$1 \cdot U \overset{\text{def}}{=} U;$$

is an $\mathbb{F}_2$-vector space.

18. *Interaction With Powersets and Vector Spaces II.* If X is finite, then:

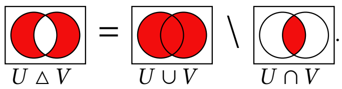
(a) The set of singletons sets on the elements of X forms a basis for the $\mathbb{F}_2$-vector space $\left(\mathcal{P}(X), \alpha_{\mathcal{P}(X)}\right)$ of Item 17.

(b) We have
$$\dim(\mathcal{P}(X)) = \#\mathcal{P}(X).$$

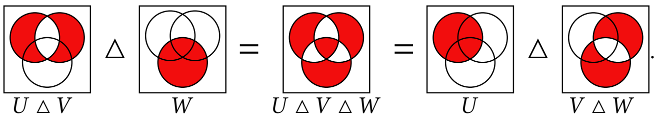
19. *Interaction With Powersets and Rings.* The quintuple $(\mathcal{P}(X), \triangle, \cap, \emptyset, X)$ is a commutative ring.[4]

---

[1] *Illustration:*



[2] *Illustration:*



[3] Here are some examples:

i. When $X = \emptyset$, we have an isomorphism of groups between $\mathcal{P}(\emptyset)$ and the trivial group:
$$\left(\mathcal{P}(\emptyset), \triangle, \emptyset, \mathrm{id}_{\mathcal{P}(\emptyset)}\right) \cong \mathrm{pt}.$$

ii. When $X = \mathrm{pt}$, we have an isomorphism of groups between $\mathcal{P}(\mathrm{pt})$ and $\mathbb{Z}_{/2}$:
$$\left(\mathcal{P}(\mathrm{pt}), \triangle, \emptyset, \mathrm{id}_{\mathcal{P}(\mathrm{pt})}\right) \cong \mathbb{Z}_{/2}.$$

iii. When $X = \{0, 1\}$, we have an isomorphism of groups between $\mathcal{P}(\{0, 1\})$ and $\mathbb{Z}_{/2} \times \mathbb{Z}_{/2}$:
$$\left(\mathcal{P}(\{0, 1\}), \triangle, \emptyset, \mathrm{id}_{\mathcal{P}(\{0,1\})}\right) \cong \mathbb{Z}_{/2} \times \mathbb{Z}_{/2}.$$

[4] *Warning:* The analogous statement replacing intersections by unions (i.e. that the quintuple $(\mathcal{P}(X), \triangle, \cup, \emptyset, X)$ is a ring) is false, however. See [Pro24aw] for a proof.

END TEXTDBEND

Proof 2.3.11.3 ▶ Proof of Proposition 2.3.11.2

Item 1: Lack of Functoriality

Omitted.

Item 2: Via Unions and Intersections

See [Pro24r].

Item 3: Associativity

See [Pro24ao].

Item 4: Commutativity

See [Pro24ap].

Item 5: Unitality

This follows from Item 4 and [Pro24at].

Item 6: Invertibility

See [Pro24av].

Item 7: Interaction With Unions

See [Pro24bc].

Item 8: Interaction With Complements I

See [Pro24as].

Item 9: Interaction With Complements II

This follows from Item 4 and [Pro24ax].

Item 10: Interaction With Complements III

See [Pro24aq].

Item 11: "Transitivity"

We have

$$\begin{aligned}(U \triangle V) \triangle (V \triangle W) &= U \triangle (V \triangle (V \triangle W)) && \text{(by Item 3)}\\ &= U \triangle ((V \triangle V) \triangle W) && \text{(by Item 3)}\\ &= U \triangle (\emptyset \triangle W) && \text{(by Item 6)}\\ &= U \triangle W && \text{(by Item 5)}\end{aligned}$$

Item 12: The Triangle Inequality for Symmetric Differences

This follows from Items 2 and 11.

Item 13: Distributivity Over Intersections

See [Pro24s].

Item 14: Interaction With Characteristic Functions

See [Pro24j].

Item 15: Bijectivity

Clear.

Item 16: Interaction With Powersets and Groups

Item 16a follows from[1] Items 3 to 6, while Item 16b follows from Item 6.

Item 17: Interaction With Powersets and Vector Spaces I

Clear.

Item 18: Interaction With Powersets and Vector Spaces II

Omitted.

Item 19: Interaction With Powersets and Rings

This follows from Items 8 and 11 of Proposition 2.3.8.2 and Items 13 and 16.[2]

[1] Reference: [Pro24ar].
[2] *Reference:* [Pro24au].

2.4 Powersets

2.4.1 Characteristic Functions

Let X be a set.

Definition 2.4.1.1 ▶ Characteristic Functions

Let $U \subset X$ and let $x \in X$.

1. The **characteristic function of** U[1] is the function[2]

$$\chi_U : X \longrightarrow \{\mathrm{t}, \mathrm{f}\}$$

defined by

$$\chi_U(x) \stackrel{\text{def}}{=} \begin{cases} \text{true} & \text{if } x \in U, \\ \text{false} & \text{if } x \notin U \end{cases}$$

for each $x \in X$.

2. The **characteristic function of** x is the function[3]

$$\chi_x : X \longrightarrow \{\mathrm{t}, \mathrm{f}\}$$

defined by

$$\chi_x \stackrel{\text{def}}{=} \chi_{\{x\}},$$

i.e. by

$$\chi_x(y) \overset{\text{def}}{=} \begin{cases} \text{true} & \text{if } x = y, \\ \text{false} & \text{if } x \neq y \end{cases}$$

for each $y \in X$.

3. The **characteristic relation on X**[4] is the relation[5]

$$\chi_X(-_1, -_2) \colon X \times X \to \{\mathsf{t}, \mathsf{f}\}$$

on X defined by[6]

$$\chi_X(x, y) \overset{\text{def}}{=} \begin{cases} \text{true} & \text{if } x = y, \\ \text{false} & \text{if } x \neq y \end{cases}$$

for each $x, y \in X$.

4. The **characteristic embedding**[7] **of X into $\mathcal{P}(X)$** is the function

$$\chi_{(-)} \colon X \hookrightarrow \mathcal{P}(X)$$

defined by

$$\chi_{(-)}(x) \overset{\text{def}}{=} \chi_x$$

for each $x \in X$.

---

[1] *Further Terminology:* Also called the **indicator function of U**.
[2] *Further Notation:* Also written $\chi_X(U, -)$ or $\chi_X(-, U)$.
[3] *Further Notation:* Also written χ_x, $\chi_X(x, -)$, or $\chi_X(-, x)$.
[4] *Further Terminology:* Also called the **identity relation on X**.
[5] *Further Notation:* Also written $\chi^{-1}_{-_2}$, or $\sim_{\mathrm{id}}$ in the context of relations.
[6] As a subset of $X \times X$, the relation χ_X corresponds to the diagonal $\Delta_X \subset X \times X$ of X.
[7] The name "characteristic *embedding*" comes from the fact that there is an analogue of fully faithfulness for $\chi_{(-)}$: given a set X, we have

$$\mathrm{Hom}_{\mathcal{P}(X)}(\chi_x, \chi_y) = \chi_X(x, y),$$

for each $x, y \in X$.

REMARK 2.4.1.2 ► CHARACTERISTIC FUNCTIONS AS DECATEGORIFICATIONS OF PRESHEAVES

The definitions in Definition 2.4.1.1 are decategorifications of co/presheaves, representable co/presheaves, Hom profunctors, and the Yoneda embedding:[1]

1. A function

$$f \colon X \to \{\mathsf{t}, \mathsf{f}\}$$

is a decategorification of a presheaf

$$\mathcal{F}\colon \mathcal{C}^{\mathsf{op}} \to \mathsf{Sets},$$

with the characteristic functions χ_U of the subsets of X being the primordial examples (and, in fact, all examples) of these.

2. The characteristic function

$$\chi_x\colon X \to \{\mathsf{t}, \mathsf{f}\}$$

of an *element* x of X is a decategorification of the representable presheaf

$$h_X\colon \mathcal{C}^{\mathsf{op}} \to \mathsf{Sets}$$

of an *object* x of a category $\mathcal{C}$.

3. The characteristic relation

$$\chi_X(-_1, -_2)\colon X \times X \to \{\mathsf{t}, \mathsf{f}\}$$

of X is a decategorification of the Hom profunctor

$$\mathrm{Hom}_{\mathcal{C}}(-_1, -_2)\colon \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathsf{Sets}$$

of a category $\mathcal{C}$.

4. The characteristic embedding

$$\chi_{(-)}\colon X \hookrightarrow \mathcal{P}(X)$$

of X into $\mathcal{P}(X)$ is a decategorification of the Yoneda embedding

$$よ\colon \mathcal{C}^{\mathsf{op}} \hookrightarrow \mathsf{PSh}(\mathcal{C})$$

of a category $\mathcal{C}$ into $\mathsf{PSh}(\mathcal{C})$.

5. There is also a direct parallel between unions and colimits:

   - An element of $\mathcal{P}(X)$ is a union of elements of X, viewed as one-point subsets $\{x\} \in \mathcal{P}(A)$;
   - An object of $\mathsf{PSh}(\mathcal{C})$ is a colimit of objects of $\mathcal{C}$, viewed as representable presheaves $h_X \in \mathrm{Obj}(\mathsf{PSh}(\mathcal{C}))$.

[1]These statements can be made precise by using the embeddings

$$
\begin{aligned}
(-)_{\mathrm{disc}} &\colon \mathsf{Sets} \hookrightarrow \mathsf{Cats},\\
(-)_{\mathrm{disc}} &\colon \{\mathsf{t},\mathsf{f}\}_{\mathrm{disc}} \hookrightarrow \mathsf{Sets}
\end{aligned}
$$

of sets into categories and of classical truth values into sets.

For instance, in this approach the characteristic function

$$\chi_x \colon X \to \{\mathsf{t},\mathsf{f}\}$$

of an element x of X, defined by

$$\chi_x(y) \overset{\text{def}}{=} \begin{cases} \mathsf{true} & \text{if } x = y, \\ \mathsf{false} & \text{if } x \neq y \end{cases}$$

for each $y \in X$, is recovered as the representable presheaf

$$\mathrm{Hom}_{X_{\mathrm{disc}}}(-, x) \colon X_{\mathrm{disc}} \to \mathsf{Sets}$$

of the corresponding object x of X_{disc}, defined on objects by

$$\mathrm{Hom}_{X_{\mathrm{disc}}}(y, x) \overset{\text{def}}{=} \begin{cases} \mathrm{pt} & \text{if } x = y, \\ \emptyset & \text{if } x \neq y \end{cases}$$

for each $y \in \mathrm{Obj}(X_{\mathrm{disc}})$.

Proposition 2.4.1.3 ▶ Properties of Characteristic Functions

Let X be a set.

1. *The Inclusion of Characteristic Relations Associated to a Function.* Let $f \colon A \to B$ be a function. We have an inclusion[1]

$$\chi_B \circ (f \times f) \subset \chi_A, \qquad \begin{tikzcd} A \times A \arrow[rr, "f \times f"] \arrow[dr, "\chi_A"'] & \arrow[d, phantom, "\supset"] & B \times B \arrow[dl, "\chi_B"] \\ & \{\mathsf{t},\mathsf{f}\}. & \end{tikzcd}$$

2. *Interaction With Unions I.* We have

$$\chi_{U \cup V} = \max(\chi_U, \chi_V)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

3. *Interaction With Unions II.* We have

$$\chi_{U \cup V} = \chi_U + \chi_V - \chi_{U \cap V}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

4. *Interaction With Intersections I.* We have

$$\chi_{U \cap V} = \chi_U \chi_V$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

5. *Interaction With Intersections II.* We have

$$\chi_{U \cap V} = \min(\chi_U, \chi_V)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

6. *Interaction With Differences.* We have

$$\chi_{U \setminus V} = \chi_U - \chi_{U \cap V}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

7. *Interaction With Complements.* We have

$$\chi_{U^{\mathsf{c}}} = 1 - \chi_U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

8. *Interaction With Symmetric Differences.* We have

$$\chi_{U \triangle V} = \chi_U + \chi_V - 2\chi_{U \cap V}$$

and thus, in particular, we have

$$\chi_{U \triangle V} \equiv \chi_U + \chi_V \mod 2$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

[1]This is the 0-categorical version of **??**.

Proof 2.4.1.4 ► Proof of Proposition 2.4.1.3

Item 1: The Inclusion of Characteristic Relations Associated to a Function

The inclusion $\chi_B(f(a), f(b)) \subset \chi_A(a, b)$ is equivalent to the statement "if $a = b$, then $f(a) = f(b)$", which is true.

Item 2: Interaction With Unions I

This is a repetition of Item 8 of Proposition 2.3.6.2 and is proved there.

Item 3: Interaction With Unions II

This is a repetition of Item 9 of Proposition 2.3.6.2 and is proved there.

Item 4: Interaction With Intersections I

This is a repetition of Item 9 of Proposition 2.3.8.2 and is proved there.

Item 5: Interaction With Intersections II

This is a repetition of Item 10 of Proposition 2.3.8.2 and is proved there.

Item 6: Interaction With Differences

This is a repetition of Item 15 of Proposition 2.3.9.2 and is proved there.

Item 7: Interaction With Complements

This is a repetition of Item 4 of Proposition 2.3.10.2 and is proved there.

Item 8: Interaction With Symmetric Differences

This is a repetition of Item 14 of Proposition 2.3.11.2 and is proved there. □

2.4.2 The Yoneda Lemma for Sets

Let X be a set and let $U \subset X$ be a subset of X.

Proposition 2.4.2.1 ▶ The Yoneda Lemma for Sets

We have
$$\chi_{\mathcal{P}(X)}(\chi_x, \chi_U) = \chi_U(x)$$
for each $x \in X$, giving an equality of functions
$$\chi_{\mathcal{P}(X)}\left(\chi_{(-)}, \chi_U\right) = \chi_U.$$

Proof 2.4.2.2 ▶ Proof of Proposition 2.4.2.1

Clear. □

Corollary 2.4.2.3 ▶ The Characteristic Embedding Is Fully Faithful

The characteristic embedding is fully faithful, i.e., we have
$$\chi_{\mathcal{P}(X)}\left(\chi_x, \chi_y\right) = \chi_X(x, y)$$
for each $x, y \in X$.

Proof 2.4.2.4 ► Proof of ??

This follows from Proposition 2.4.2.1. □

2.4.3 Powersets

Let X be a set.

Definition 2.4.3.1 ► Powersets

The **powerset of** X is the set $\mathcal{P}(X)$ defined by

$$\mathcal{P}(X) \overset{\text{def}}{=} \{U \in P \mid U \subset X\},$$

where P is the set in the axiom of powerset, **??** of **??**.

Remark 2.4.3.2 ► Powersets as Decategorifications of Co/Presheaf Categories

The powerset of a set is a decategorification of the category of presheaves of a category: while[1]

- The powerset of a set X is equivalently (Item 6 of Proposition 2.4.3.3) the set

$$\mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\})$$

of functions from X to the set $\{\mathsf{t}, \mathsf{f}\}$ of classical truth values;

- The category of presheaves on a category $\mathcal{C}$ is the category

$$\mathsf{Fun}(\mathcal{C}^{\mathsf{op}}, \mathsf{Sets})$$

of functors from $\mathcal{C}^{\mathsf{op}}$ to the category Sets of sets.

[1]This parallel is based on the following comparison:

- A category is enriched over the category

$$\mathsf{Sets} \overset{\text{def}}{=} \mathsf{Cats}_0$$

of sets (i.e. "0-categories"), with presheaves taking values on it;

- A set is enriched over the set

$$\{\mathsf{t}, \mathsf{f}\} \overset{\text{def}}{=} \mathsf{Cats}_{-1}$$

of classical truth values (i.e. "(-1)-categories"), with characteristic functions taking values on it.

Proposition 2.4.3.3 ► Properties of Powersets

Let X be a set.

1. *Functoriality.* The assignment $X \mapsto \mathcal{P}(X)$ defines functors

$$\begin{aligned}\mathcal{P}_* &\colon \mathsf{Sets} \to \mathsf{Sets},\\ \mathcal{P}^{-1} &\colon \mathsf{Sets}^{\mathsf{op}} \to \mathsf{Sets},\\ \mathcal{P}_! &\colon \mathsf{Sets} \to \mathsf{Sets}\end{aligned}$$

   where

   - *Action on Objects.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$\begin{aligned}\mathcal{P}_*(A) &\stackrel{\text{def}}{=} \mathcal{P}(A),\\ \mathcal{P}^{-1}(A) &\stackrel{\text{def}}{=} \mathcal{P}(A),\\ \mathcal{P}_!(A) &\stackrel{\text{def}}{=} \mathcal{P}(A);\end{aligned}$$

   - *Action on Morphisms.* For each morphism $f \colon A \to B$ of Sets, the images

$$\begin{aligned}\mathcal{P}_*(f) &\colon \mathcal{P}(A) \to \mathcal{P}(B),\\ \mathcal{P}^{-1}(f) &\colon \mathcal{P}(B) \to \mathcal{P}(A),\\ \mathcal{P}_!(f) &\colon \mathcal{P}(A) \to \mathcal{P}(B)\end{aligned}$$

     of f by $\mathcal{P}_*$, $\mathcal{P}^{-1}$, and $\mathcal{P}_!$ are defined by

$$\begin{aligned}\mathcal{P}_*(f) &\stackrel{\text{def}}{=} f_*,\\ \mathcal{P}^{-1}(f) &\stackrel{\text{def}}{=} f^{-1},\\ \mathcal{P}_!(f) &\stackrel{\text{def}}{=} f_!,\end{aligned}$$

     as in Definitions 2.4.4.1, 2.4.5.1 and 2.4.6.1.

2. *Adjointness I.* We have an adjunction

$$\left(\mathcal{P}^{-1} \dashv \mathcal{P}^{-1,\mathsf{op}}\right) \colon \quad \begin{tikzcd} \mathsf{Sets}^{\mathsf{op}} \arrow[r, bend left, "\mathcal{P}^{-1}"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}, \arrow[l, bend left, "\mathcal{P}^{-1,\mathsf{op}}"] \end{tikzcd}$$

   witnessed by a bijection

$$\underbrace{\mathsf{Sets}^{\mathsf{op}}(\mathcal{P}(X), Y)}_{\stackrel{\text{def}}{=} \mathsf{Sets}(Y, \mathcal{P}(X))} \cong \mathsf{Sets}(X, \mathcal{P}(Y)),$$

   natural in $X \in \mathrm{Obj}(\mathsf{Sets})$ and $Y \in \mathrm{Obj}(\mathsf{Sets}^{\mathsf{op}})$.

3. *Adjointness II.* We have an adjunction

$$(\mathrm{Gr} \dashv \mathcal{P}_*)\colon \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "\mathrm{Gr}"] \arrow[r, phantom, "\perp"] & \mathsf{Rel}, \arrow[l, bend left, "\mathcal{P}_*"] \end{tikzcd}$$

witnessed by a bijection of sets

$$\mathsf{Rel}(\mathrm{Gr}(A), B) \cong \mathsf{Sets}(A, \mathcal{P}(B))$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$ and $B \in \mathrm{Obj}(\mathsf{Rel})$, where Gr is the graph functor of **??** of **??**.

4. *Symmetric Strong Monoidality With Respect to Coproducts.* The powerset functor $\mathcal{P}_*$ of Item 1 has a symmetric strong monoidal structure

$$\left(\mathcal{P}_*, \mathcal{P}_*^{\coprod}, \mathcal{P}_{*|\mathbb{1}}^{\coprod}\right)\colon (\mathsf{Sets}, \coprod, \emptyset) \to (\mathsf{Sets}, \times, \mathrm{pt})$$

being equipped with isomorphisms

$$\mathcal{P}_{*|X,Y}^{\coprod}\colon \mathcal{P}(X) \times \mathcal{P}(Y) \xrightarrow{\cong} \mathcal{P}(X \textstyle\coprod Y),$$
$$\mathcal{P}_{*|\mathbb{1}}^{\coprod}\colon \mathrm{pt} \xrightarrow{=} \mathcal{P}(\emptyset),$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

5. *Symmetric Lax Monoidality With Respect to Products.* The powerset functor $\mathcal{P}_*$ of Item 1 has a symmetric lax monoidal structure

$$\left(\mathcal{P}_*, \mathcal{P}_*^{\otimes}, \mathcal{P}_{*|\mathbb{1}}^{\otimes}\right)\colon (\mathsf{Sets}, \times, \mathrm{pt}) \to (\mathsf{Sets}, \times, \mathrm{pt})$$

being equipped with isomorphisms

$$\mathcal{P}_{*|X,Y}^{\otimes}\colon \mathcal{P}(X) \times \mathcal{P}(Y) \to \mathcal{P}(X \times Y),$$
$$\mathcal{P}_{*|\mathbb{1}}^{\otimes}\colon \mathrm{pt} \xrightarrow{=} \mathcal{P}(\emptyset),$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$, where $\mathcal{P}_{*|X,Y}^{\otimes}$ is given by

$$\mathcal{P}_{*|X,Y}^{\otimes}(U, V) \stackrel{\text{def}}{=} U \times V$$

for each $(U, V) \in \mathcal{P}(X) \times \mathcal{P}(Y)$.

6. *Powersets as Sets of Functions.* The assignment $U \mapsto \chi_U$ defines a bijection[1]

$$\chi_{(-)}\colon \mathcal{P}(X) \xrightarrow{\cong} \mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\}),$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets})$.

7. *Powersets as Sets of Relations.* We have bijections

$$\begin{aligned}\mathcal{P}(X) &\cong \mathrm{Rel}(\mathrm{pt}, X),\\ \mathcal{P}(X) &\cong \mathrm{Rel}(X, \mathrm{pt}),\end{aligned}$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets})$.

8. *As a Free Cocompletion: Universal Property.* The pair $\left(\mathcal{P}(X), \chi_{(-)}\right)$ consisting of

   - The powerset $\mathcal{P}(X)$ of X;
   - The characteristic embedding $\chi_{(-)}\colon X \hookrightarrow \mathcal{P}(X)$ of X into $\mathcal{P}(X)$;

   satisfies the following universal property:

   ($\star$) Given another pair (Y, f) consisting of
   - A cocomplete poset $(Y, \preceq)$;
   - A function $f\colon X \to Y$;

   there exists a unique cocontinuous morphism of posets $(\mathcal{P}(X), \subset) \xrightarrow{\exists!} (Y, \preceq)$ making the diagram

$$\begin{tikzcd} & \mathcal{P}(X) \arrow[d, dashed, "\exists!"] \\ X \arrow[ur, "\chi_X"] \arrow[r, "f"'] & Y \end{tikzcd}$$

   commute.

9. *As a Free Cocompletion: Adjointness.* We have an adjunction[2]

$$\left(\chi_{(-)} \dashv 忘\right)\colon \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "\chi_{(-)}"] \arrow[r, phantom, "\perp"] & \mathsf{Pos}^{\mathsf{cocomp.}} \arrow[l, bend left, "忘"] \end{tikzcd},$$

witnessed by a bijection

$$\mathsf{Pos}^{\mathsf{cocomp.}}((\mathcal{P}(X), \subset), (Y, \preceq)) \cong \mathsf{Sets}(X, Y),$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets})$ and $(Y, \preceq) \in \mathrm{Obj}(\mathsf{Pos})$, where

- We have a natural map

$$\chi_X^* \colon \mathsf{Pos}^{\mathsf{cocomp.}}((\mathcal{P}(X), \subset), (Y, \preceq)) \to \mathsf{Sets}(X, Y)$$

defined by

$$\chi_X^*(f) \stackrel{\mathrm{def}}{=} f \circ \chi_X,$$

i.e. by sending a cocontinuous morphism of posets $f \colon \mathcal{P}(X) \to Y$ to the composition

$$X \stackrel{\chi_X}{\hookrightarrow} \mathcal{P}(X) \xrightarrow{f} Y;$$

- We have a natural map

$$\mathsf{Lan}_{\chi_X} \colon \mathsf{Sets}(X, Y) \to \mathsf{Pos}^{\mathsf{cocomp.}}((\mathcal{P}(X), \subset), (Y, \preceq))$$

computed by

$$\begin{aligned} [\mathsf{Lan}_{\chi_X}(f)](U) &\cong \int^{x \in X} \chi_{\mathcal{P}(X)}(\chi_x, U) \odot f(x) \\ &\cong \int^{x \in X} \chi_U(x) \odot f(x) \qquad \text{(by Proposition 2.4.2.1)} \\ &\cong \bigvee_{x \in X} (\chi_U(x) \odot f(x)) \end{aligned}$$

for each $U \in \mathcal{P}(X)$, where:

  - $\bigvee$ is the join in $(Y, \preceq)$;
  - We have

$$\begin{aligned} \mathsf{true} \odot f(x) &\stackrel{\mathrm{def}}{=} f(x), \\ \mathsf{false} \odot f(x) &\stackrel{\mathrm{def}}{=} \emptyset_Y, \end{aligned}$$

  where $\emptyset_Y$ is the minimal element of $(Y, \preceq)$.

[1]This bijection is a decategorified form of the equivalence

$$\mathsf{PSh}(\mathcal{C}) \stackrel{\text{eq.}}{\cong} \mathsf{DFib}(\mathcal{C})$$

of ?? of ??, with $\chi_{(-)}$ being a decategorified version of the category of elements construction of ??. See also ?? of ??.

[2]In this sense, $\mathcal{P}(A)$ is the free cocompletion of A. (Note that, despite its name, however, this is not an idempotent operation, as we have $\mathcal{P}(\mathcal{P}(A)) \neq \mathcal{P}(A)$.)

Proof 2.4.3.4 ▶ Proof of Proposition 2.4.3.3

Item 1: Functoriality

This follows from Items 3 and 4 of Proposition 2.4.4.5, Items 3 and 4 of Proposition 2.4.5.5, and Items 3 and 4 of Proposition 2.4.6.7.

Item 2: Adjointness I

Omitted.

Item 3: Adjointness II

We have

$$\begin{aligned}
\mathsf{Rel}(\mathsf{Gr}(A), B) &= \mathcal{P}(A \times B) \\
&= \mathsf{Sets}(A \times B, \{\mathsf{t}, \mathsf{f}\}) && \text{(by Item 6)} \\
&= \mathsf{Sets}(A, \mathsf{Sets}(B, \{\mathsf{t}, \mathsf{f}\})) && \text{(by Item 2 of Proposition 2.1.2.3)} \\
&= \mathsf{Sets}(A, \mathcal{P}(B)) && \text{(by Item 6)}
\end{aligned}$$

with all bijections natural in A and B.

Item 4: Symmetric Strong Monoidality With Respect to Coproducts

Omitted.

Item 5: Symmetric Lax Monoidality With Respect to Products

Omitted.

Item 6: Powersets as Sets of Functions

Omitted.

Item 7: Powersets as Sets of Relations

Omitted.

Item 8: As a Free Cocompletion: Universal Property

This is a rephrasing of **??**.

Item 9: As a Free Cocompletion: Adjointness

Omitted. ☐

2.4.4 Direct Images

Let A and B be sets and let $f : A \longrightarrow B$ be a function.

Definition 2.4.4.1 ▶ Direct Images

The **direct image function associated to** f is the function[1]

$$f_*\colon \mathcal{P}(A) \to \mathcal{P}(B)$$

defined by[2,3]

$$\begin{aligned} f_*(U) &\overset{\text{def}}{=} f(U) \\ &\overset{\text{def}}{=} \left\{ b \in B \,\middle|\, \begin{array}{l}\text{there exists some } a \in \\ U \text{ such that } b = f(a)\end{array} \right\} \\ &= \{f(a) \in B \mid a \in U\} \end{aligned}$$

for each $U \in \mathcal{P}(A)$.

[1] *Further Notation:* Also written $\exists_f\colon \mathcal{P}(A) \to \mathcal{P}(B)$. This notation comes from the fact that the following statements are equivalent, where $b \in B$ and $U \in \mathcal{P}(A)$:

- We have $b \in \exists_f(U)$.
- There exists some $a \in U$ such that $f(a) = b$.

[2] *Further Terminology:* The set $f(U)$ is called the **direct image of** U **by** f.

[3] We also have

$$f_*(U) = B \setminus f_!(A \setminus U);$$

see Item 7 of Proposition 2.4.4.3.

Remark 2.4.4.2 ▶ Unwinding Definition 2.4.4.1

Identifying subsets of A with functions from A to $\{\text{true}, \text{false}\}$ via Item 6 of Proposition 2.4.3.3, we see that the direct image function associated to f is equivalently the function

$$f_*\colon \mathcal{P}(A) \to \mathcal{P}(B)$$

defined by

$$\begin{aligned} f_*(\chi_U) &\overset{\text{def}}{=} \mathrm{Lan}_f(\chi_U) \\ &= \operatorname{colim}\left(\left(f \overset{\rightarrow}{\times} \underline{(-_1)}\right) \overset{\text{pr}}{\twoheadrightarrow} A \xrightarrow{\chi_U} \{\mathsf{t}, \mathsf{f}\}\right) \\ &= \operatorname*{colim}_{\substack{a \in A \\ f(a) = -_1}} (\chi_U(a)) \\ &= \bigvee_{\substack{a \in A \\ f(a) = -_1}} (\chi_U(a)). \end{aligned}$$

So, in other words, we have

$$
\begin{aligned}
[f_*(\chi_U)](b) &= \bigvee_{\substack{a\in A\\ f(a)=b}} (\chi_U(a)) \\
&= \begin{cases} \text{true} & \text{if there exists some } a \in A \text{ such} \\ & \text{that } f(a) = b \text{ and } a \in U, \\ \text{false} & \text{otherwise} \end{cases} \\
&= \begin{cases} \text{true} & \text{if there exists some } a \in U \\ & \text{such that } f(a) = b, \\ \text{false} & \text{otherwise} \end{cases}
\end{aligned}
$$

for each $b \in B$.

Proposition 2.4.4.3 ▶ Properties of Direct Images I

Let $f\colon A \to B$ be a function.

1. *Functoriality.* The assignment $U \mapsto f_*(U)$ defines a functor
$$f_*\colon (\mathcal{P}(A), \subset) \to (\mathcal{P}(B), \subset)$$
where
   - *Action on Objects.* For each $U \in \mathcal{P}(A)$, we have
   $$[f_*](U) \stackrel{\text{def}}{=} f_*(U);$$
   - *Action on Morphisms.* For each $U, V \in \mathcal{P}(A)$:
   ($\star$) If $U \subset V$, then $f_*(U) \subset f_*(V)$.
2. *Triple Adjointness.* We have a triple adjunction
$$
(f_* \dashv f^{-1} \dashv f_!)\colon \quad
\begin{tikzcd}
\mathcal{P}(A) \arrow[r, bend left=50, "f_*"{name=A}] \arrow[r, bend right=50, "f_!"'{name=B}] & \mathcal{P}(B) \arrow[l, "f^{-1}"' description] \arrow[from=A, to=1-1, phantom, "\perp"]
\end{tikzcd},
$$
witnessed by bijections of sets
$$
\begin{aligned}
\mathrm{Hom}_{\mathcal{P}(B)}(f_*(U), V) &\cong \mathrm{Hom}_{\mathcal{P}(A)}(U, f^{-1}(V)), \\
\mathrm{Hom}_{\mathcal{P}(A)}(f^{-1}(U), V) &\cong \mathrm{Hom}_{\mathcal{P}(A)}(U, f_!(V)),
\end{aligned}
$$
natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$ and (respectively) $V \in \mathcal{P}(A)$ and $U \in \mathcal{P}(B)$, i.e. where:

(a) The following conditions are equivalent:

i. We have $f_*(U) \subset V$.

ii. We have $U \subset f^{-1}(V)$.

(b) The following conditions are equivalent:

i. We have $f^{-1}(U) \subset V$.

ii. We have $U \subset f_!(V)$.

3. *Preservation of Colimits.* We have an equality of sets

$$f_*\left(\bigcup_{i\in I} U_i\right) = \bigcup_{i\in I} f_*(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have equalities

$$\begin{aligned} f_*(U) \cup f_*(V) &= f_*(U \cup V), \\ f_*(\emptyset) &= \emptyset, \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

4. *Oplax Preservation of Limits.* We have an inclusion of sets

$$f_*\left(\bigcap_{i\in I} U_i\right) \subset \bigcap_{i\in I} f_*(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have inclusions

$$\begin{aligned} f_*(U \cap V) &\subset f_*(U) \cap f_*(V), \\ f_*(A) &\subset B, \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

5. *Symmetric Strict Monoidality With Respect to Unions.* The direct image function of Item 1 has a symmetric strict monoidal structure

$$\left(f_*, f_*^{\otimes}, f_{*|\mathbb{1}}^{\otimes}\right) \colon (\mathcal{P}(A), \cup, \emptyset) \to (\mathcal{P}(B), \cup, \emptyset),$$

being equipped with equalities

$$\begin{aligned} f_{*|U,V}^{\otimes} &\colon f_*(U) \cup f_*(V) \xrightarrow{=} f_*(U \cup V), \\ f_{*|\mathbb{1}}^{\otimes} &\colon \emptyset \xrightarrow{=} \emptyset, \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

6. *Symmetric Oplax Monoidality With Respect to Intersections.* The direct image function of Item 1 has a symmetric oplax monoidal structure

$$\left(f_*, f_*^{\otimes}, f_{*|\mathbb{1}}^{\otimes}\right) : (\mathcal{P}(A), \cap, A) \to (\mathcal{P}(B), \cap, B),$$

being equipped with inclusions

$$f_{*|U,V}^{\otimes} : f_*(U \cap V) \hookrightarrow f_*(U) \cap f_*(V),$$
$$f_{*|\mathbb{1}}^{\otimes} : f_*(A) \hookrightarrow B,$$

natural in $U, V \in \mathcal{P}(A)$.

7. *Relation to Direct Images With Compact Support.* We have

$$f_*(U) = B \setminus f_!(A \setminus U)$$

for each $U \in \mathcal{P}(A)$.

Proof 2.4.4.4 ► Proof of Proposition 2.4.4.3

Item 1: Functoriality

Clear.

Item 2: Triple Adjointness

This follows from **??** of **??**.

Item 3: Preservation of Colimits

This follows from Item 2 and **??** of **??**.

Item 4: Oplax Preservation of Limits

Omitted.

Item 5: Symmetric Strict Monoidality With Respect to Unions

This follows from Item 3.

Item 6: Symmetric Oplax Monoidality With Respect to Intersections

This follows from **??**.

Item 7: Relation to Direct Images With Compact Support

Applying **??** of **??** to $A \setminus U$, we have

$$\begin{aligned} f_!(A \setminus U) &= B \setminus f_*(A \setminus (A \setminus U)) \\ &= B \setminus f_*(U). \end{aligned}$$

Taking complements, we then obtain

$$\begin{aligned} f_*(U) &= B \setminus (B \setminus f_*(U)), \\ &= B \setminus f_!(A \setminus U), \end{aligned}$$

which finishes the proof. ∎

Proposition 2.4.4.5 ► Properties of Direct Images II

Let $f : A \longrightarrow B$ be a function.

1. *Functionality I.* The assignment $f \mapsto f_*$ defines a function
$$(-)_{*|A,B} : \mathsf{Sets}(A, B) \longrightarrow \mathsf{Sets}(\mathcal{P}(A), \mathcal{P}(B)).$$
2. *Functionality II.* The assignment $f \mapsto f_*$ defines a function
$$(-)_{*|A,B} : \mathsf{Sets}(A, B) \longrightarrow \mathsf{Pos}((\mathcal{P}(A), \subset), (\mathcal{P}(B), \subset)).$$
3. *Interaction With Identities.* For each $A \in \mathsf{Obj}(\mathsf{Sets})$, we have
$$(\mathrm{id}_A)_* = \mathrm{id}_{\mathcal{P}(A)};$$
4. *Interaction With Composition.* For each pair of composable functions $f : A \longrightarrow B$ and $g : B \longrightarrow C$, we have
$$(g \circ f)_* = g_* \circ f_*, \qquad \begin{tikzcd} \mathcal{P}(A) \arrow[r, "f_*"] \arrow[dr, "(g\circ f)_*"'] & \mathcal{P}(B) \arrow[d, "g_*"] \\ & \mathcal{P}(C). \end{tikzcd}$$

Proof 2.4.4.6 ► Proof of Proposition 2.4.4.5

Item 1: Functionality I

Clear.

Item 2: Functionality II

Clear.

Item 3: Interaction With Identities

This follows from **??** of **??**.

Item 4: Interaction With Composition

This follows from **??** of **??**.

2.4.5 Inverse Images

Let A and B be sets and let $f : A \longrightarrow B$ be a function.

Definition 2.4.5.1 ▶ Inverse Images

The **inverse image function associated to** f is the function[1]

$$f^{-1} : \mathcal{P}(B) \to \mathcal{P}(A)$$

defined by[2]

$$f^{-1}(V) \stackrel{\text{def}}{=} \{a \in A \mid \text{we have } f(a) \in V\}$$

for each $V \in \mathcal{P}(B)$.

[1] *Further Notation:* Also written $f^* : \mathcal{P}(B) \to \mathcal{P}(A)$.
[2] *Further Terminology:* The set $f^{-1}(V)$ is called the **inverse image of** V **by** f.

Remark 2.4.5.2 ▶ Unwinding Definition 2.4.5.1

Identifying subsets of B with functions from B to $\{\text{true}, \text{false}\}$ via Item 6 of Proposition 2.4.3.3, we see that the inverse image function associated to f is equivalently the function

$$f^* : \mathcal{P}(B) \to \mathcal{P}(A)$$

defined by

$$f^*(\chi_V) \stackrel{\text{def}}{=} \chi_V \circ f$$

for each $\chi_V \in \mathcal{P}(B)$, where $\chi_V \circ f$ is the composition

$$A \xrightarrow{f} B \xrightarrow{\chi_V} \{\text{true}, \text{false}\}$$

in Sets.

Proposition 2.4.5.3 ▶ Properties of Inverse Images I

Let $f : A \to B$ be a function.

1. *Functoriality.* The assignment $V \mapsto f^{-1}(V)$ defines a functor

$$f^{-1} : (\mathcal{P}(B), \subset) \to (\mathcal{P}(A), \subset)$$

where

- *Action on Objects.* For each $V \in \mathcal{P}(B)$, we have

$$[f^{-1}](V) \stackrel{\text{def}}{=} f^{-1}(V);$$

- *Action on Morphisms.* For each $U, V \in \mathcal{P}(B)$:

  ($\star$) If $U \subset V$, then $f^{-1}(U) \subset f^{-1}(V)$.

2. *Triple Adjointness.* We have a triple adjunction

$$(f_* \dashv f^{-1} \dashv f_!)\colon\quad \begin{tikzcd} \mathcal{P}(A) \arrow[rr, bend left, "f_*"] \arrow[rr, phantom, bend left=20, "\perp"] \arrow[rr, bend right, "f_!"'] \arrow[rr, phantom, bend right=20, "\perp"] & & \mathcal{P}(B), \arrow[ll, "f^{-1}"'] \end{tikzcd}$$

witnessed by bijections of sets

$$\begin{aligned} \mathrm{Hom}_{\mathcal{P}(B)}(f_*(U), V) &\cong \mathrm{Hom}_{\mathcal{P}(A)}\bigl(U, f^{-1}(V)\bigr),\\ \mathrm{Hom}_{\mathcal{P}(A)}\bigl(f^{-1}(U), V\bigr) &\cong \mathrm{Hom}_{\mathcal{P}(A)}(U, f_!(V)), \end{aligned}$$

natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$ and (respectively) $V \in \mathcal{P}(A)$ and $U \in \mathcal{P}(B)$, i.e. where:

(a) The following conditions are equivalent:

   i. We have $f_*(U) \subset V$;

   ii. We have $U \subset f^{-1}(V)$;

(b) The following conditions are equivalent:

   i. We have $f^{-1}(U) \subset V$.

   ii. We have $U \subset f_!(V)$.

3. *Preservation of Colimits.* We have an equality of sets

$$f^{-1}\left(\bigcup_{i \in I} U_i\right) = \bigcup_{i \in I} f^{-1}(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(B)^{\times I}$. In particular, we have equalities

$$\begin{aligned} f^{-1}(U) \cup f^{-1}(V) &= f^{-1}(U \cup V),\\ f^{-1}(\emptyset) &= \emptyset, \end{aligned}$$

natural in $U, V \in \mathcal{P}(B)$.

4. *Preservation of Limits.* We have an equality of sets

$$f^{-1}\left(\bigcap_{i\in I} U_i\right) = \bigcap_{i\in I} f^{-1}(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(B)^{\times I}$. In particular, we have equalities

$$\begin{aligned} f^{-1}(U) \cap f^{-1}(V) &= f^{-1}(U \cap V), \\ f^{-1}(B) &= A, \end{aligned}$$

natural in $U, V \in \mathcal{P}(B)$.

5. *Symmetric Strict Monoidality With Respect to Unions.* The inverse image function of Item 1 has a symmetric strict monoidal structure

$$\left(f^{-1}, f^{-1,\otimes}, f_{\mathbb{1}}^{-1,\otimes}\right) \colon (\mathcal{P}(B), \cup, \emptyset) \to (\mathcal{P}(A), \cup, \emptyset),$$

being equipped with equalities

$$\begin{aligned} f_{U,V}^{-1,\otimes} \colon f^{-1}(U) \cup f^{-1}(V) &\xrightarrow{=} f^{-1}(U \cup V), \\ f_{\mathbb{1}}^{-1,\otimes} \colon \emptyset &\xrightarrow{=} f^{-1}(\emptyset), \end{aligned}$$

natural in $U, V \in \mathcal{P}(B)$.

6. *Symmetric Strict Monoidality With Respect to Intersections.* The inverse image function of Item 1 has a symmetric strict monoidal structure

$$\left(f^{-1}, f^{-1,\otimes}, f_{\mathbb{1}}^{-1,\otimes}\right) \colon (\mathcal{P}(B), \cap, B) \to (\mathcal{P}(A), \cap, A),$$

being equipped with equalities

$$\begin{aligned} f_{U,V}^{-1,\otimes} \colon f^{-1}(U) \cap f^{-1}(V) &\xrightarrow{=} f^{-1}(U \cap V), \\ f_{\mathbb{1}}^{-1,\otimes} \colon A &\xrightarrow{=} f^{-1}(B), \end{aligned}$$

natural in $U, V \in \mathcal{P}(B)$.

Proof 2.4.5.4 ► Proof of Proposition 2.4.5.3

Item 1: Functoriality

Clear.

Item 2: Triple Adjointness

This follows from ?? of ??.

Item 3: Preservation of Colimits

This follows from Item 2 and ?? of ??.

Item 4: Preservation of Limits

This follows from Item 2 and ?? of ??.

Item 5: Symmetric Strict Monoidality With Respect to Unions

This follows from Item 3.

Item 6: Symmetric Strict Monoidality With Respect to Intersections

This follows from Item 4.

PROPOSITION 2.4.5.5 ▶ PROPERTIES OF INVERSE IMAGES II

Let $f\colon A \to B$ be a function.

1. *Functionality I.* The assignment $f \mapsto f^{-1}$ defines a function

$$(-)^{-1}_{A,B}\colon \mathsf{Sets}(A,B) \to \mathsf{Sets}(\mathcal{P}(B),\mathcal{P}(A)).$$

2. *Functionality II.* The assignment $f \mapsto f^{-1}$ defines a function

$$(-)^{-1}_{A,B}\colon \mathsf{Sets}(A,B) \to \mathsf{Pos}((\mathcal{P}(B),\subset),(\mathcal{P}(A),\subset)).$$

3. *Interaction With Identities.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$\mathrm{id}_A^{-1} = \mathrm{id}_{\mathcal{P}(A)};$$

4. *Interaction With Composition.* For each pair of composable functions $f\colon A \to B$ and $g\colon B \to C$, we have

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}, \qquad \begin{tikzcd} \mathcal{P}(C) \arrow[r, "g^{-1}"] \arrow[dr, "(g\circ f)^{-1}"'] & \mathcal{P}(B) \arrow[d, "f^{-1}"] \\ & \mathcal{P}(A). \end{tikzcd}$$

Proof 2.4.5.6 ▶ Proof of Proposition 2.4.5.5

Item 1: Functionality I

Clear.

Item 2: Functionality II

Clear.

Item 3: Interaction With Identities

This follows from **??** of **??**.

Item 4: Interaction With Composition

This follows from **??** of **??**.

2.4.6 Direct Images With Compact Support

Let A and B be sets and let $f\colon A \to B$ be a function.

Definition 2.4.6.1 ▶ Direct Images With Compact Support

The **direct image with compact support function associated to** f is the function[1]

$$f_!\colon \mathcal{P}(A) \to \mathcal{P}(B)$$

defined by[2,3]

$$\begin{aligned} f_!(U) &\overset{\text{def}}{=} \left\{ b \in B \,\middle|\, \begin{array}{l} \text{for each } a \in A\text{, if we have} \\ f(a) = b\text{, then } a \in U \end{array} \right\} \\ &= \left\{ b \in B \,\middle|\, \text{we have } f^{-1}(b) \subset U \right\} \end{aligned}$$

for each $U \in \mathcal{P}(A)$.

[1] *Further Notation:* Also written $\forall_f\colon \mathcal{P}(A) \to \mathcal{P}(B)$. This notation comes from the fact that the following statements are equivalent, where $b \in B$ and $U \in \mathcal{P}(A)$:

- We have $b \in \forall_f(U)$.
- For each $a \in A$, if $b = f(a)$, then $a \in U$.

[2] *Further Terminology:* The set $f_!(U)$ is called the **direct image with compact support of** U **by** f.

[3] We also have

$$f_!(U) = B \setminus f_*(A \setminus U);$$

see Item 7 of Proposition 2.4.6.5.

Remark 2.4.6.2 ▶ Unwinding Definition 2.4.6.1

Identifying subsets of A with functions from A to $\{\text{true}, \text{false}\}$ via Item 6 of Proposition 2.4.3.3, we see that the direct image with compact support function associated to f is equivalently the function

$$f_! \colon \mathcal{P}(A) \to \mathcal{P}(B)$$

defined by

$$\begin{aligned} f_!(\chi_U) &\stackrel{\text{def}}{=} \mathrm{Ran}_f(\chi_U) \\ &= \lim\left(\left(\underline{(-_1)} \overset{\rightarrow}{\times} f\right) \overset{\mathrm{pr}}{\twoheadrightarrow} A \xrightarrow{\chi_U} \{\text{true}, \text{false}\}\right) \\ &= \lim_{\substack{a\in A \\ f(a)=-_1}} (\chi_U(a)) \\ &= \bigwedge_{\substack{a\in A \\ f(a)=-_1}} (\chi_U(a)). \end{aligned}$$

So, in other words, we have

$$\begin{aligned} [f_!(\chi_U)](b) &= \bigwedge_{\substack{a\in A \\ f(a)=b}} (\chi_U(a)) \\ &= \begin{cases} \text{true} & \text{if, for each } a \in A \text{ such that} \\ & f(a) = b, \text{ we have } a \in U, \\ \text{false} & \text{otherwise} \end{cases} \\ &= \begin{cases} \text{true} & \text{if } f^{-1}(b) \subset U \\ \text{false} & \text{otherwise} \end{cases} \end{aligned}$$

for each $b \in B$.

Definition 2.4.6.3 ▶ The Image and Complement Parts of $f_!$

Let U be a subset of A.[1,2]

1. The **image part of the direct image with compact support** $f_!(U)$ **of** U is the set $f_{!,\mathrm{im}}(U)$ defined by

$$\begin{aligned} f_{!,\mathrm{im}}(U) &\stackrel{\text{def}}{=} f_!(U) \cap \mathrm{Im}(f) \\ &= \left\{ b \in B \;\middle|\; \begin{array}{l} \text{we have } f^{-1}(b) \subset U \\ \text{and } f^{-1}(b) \neq \emptyset \end{array} \right\}. \end{aligned}$$

2. The **complement part of the direct image with compact support $f_!(U)$ of U** is the set $f_{!,\mathrm{cp}}(U)$ defined by

$$\begin{aligned} f_{!,\mathrm{cp}}(U) &\overset{\mathrm{def}}{=} f_!(U) \cap (B \setminus \mathrm{Im}(f)) \\ &= B \setminus \mathrm{Im}(f) \\ &= \left\{ b \in B \,\middle|\, \begin{array}{l} \text{we have } f^{-1}(b) \subset U \\ \text{and } f^{-1}(b) = \emptyset \end{array} \right\} \\ &= \left\{ b \in B \,\middle|\, f^{-1}(b) = \emptyset \right\}. \end{aligned}$$

[1]Note that we have

$$f_!(U) = f_{!,\mathrm{im}}(U) \cup f_{!,\mathrm{cp}}(U),$$

as

$$\begin{aligned} f_!(U) &= f_!(U) \cap B \\ &= f_!(U) \cap (\mathrm{Im}(f) \cup (B \setminus \mathrm{Im}(f))) \\ &= (f_!(U) \cap \mathrm{Im}(f)) \cup (f_!(U) \cap (B \setminus \mathrm{Im}(f))) \\ &\overset{\mathrm{def}}{=} f_{!,\mathrm{im}}(U) \cup f_{!,\mathrm{cp}}(U). \end{aligned}$$

[2]In terms of the meet computation of $f_!(U)$ of Remark 2.4.6.2, namely

$$f_!(\chi_U) = \bigwedge_{\substack{a \in A \\ f(a) = -_1}} (\chi_U(a)),$$

we see that $f_{!,\mathrm{im}}$ corresponds to meets indexed over nonempty sets, while $f_{!,\mathrm{cp}}$ corresponds to meets indexed over the empty set.

Example 2.4.6.4 ▶ Examples of Direct Images With Compact Support

Here are some examples of direct images with compact support.

1. *The Multiplication by Two Map on the Natural Numbers.* Consider the function $f : \mathbb{N} \to \mathbb{N}$ given by
$$f(n) \overset{\mathrm{def}}{=} 2n$$
for each $n \in \mathbb{N}$. Since f is injective, we have
$$\begin{aligned} f_{!,\mathrm{im}}(U) &= f_*(U) \\ f_{!,\mathrm{cp}}(U) &= \{\text{odd natural numbers}\} \end{aligned}$$
for any $U \subset \mathbb{N}$.

2. *Parabolas.* Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by
$$f(x) \overset{\mathrm{def}}{=} x^2$$

for each $x \in \mathbb{R}$. We have

$$f_{!,\mathrm{cp}}(U) = \mathbb{R}_{<0}$$

for any $U \subset \mathbb{R}$. Moreover, since $f^{-1}(x) = \left\{-\sqrt{x}, \sqrt{x}\right\}$, we have e.g.:

$$\begin{aligned} f_{!,\mathrm{im}}([0,1]) &= \{0\}, \\ f_{!,\mathrm{im}}([-1,1]) &= [0,1], \\ f_{!,\mathrm{im}}([1,2]) &= \emptyset, \\ f_{!,\mathrm{im}}([-2,-1] \cup [1,2]) &= [1,4]. \end{aligned}$$

3. *Circles.* Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x,y) \overset{\mathrm{def}}{=} x^2 + y^2$$

for each $(x,y) \in \mathbb{R}^2$. We have

$$f_{!,\mathrm{cp}}(U) = \mathbb{R}_{<0}$$

for any $U \subset \mathbb{R}^2$, and since

$$f^{-1}(r) = \begin{cases} \text{a circle of radius } r \text{ about the origin} & \text{if } r > 0, \\ \{(0,0)\} & \text{if } r = 0, \\ \emptyset & \text{if } r < 0, \end{cases}$$

we have e.g.:

$$\begin{aligned} f_{!,\mathrm{im}}([-1,1] \times [-1,1]) &= [0,1], \\ f_{!,\mathrm{im}}(([-1,1] \times [-1,1]) \setminus [-1,1] \times \{0\}) &= \emptyset. \end{aligned}$$

Proposition 2.4.6.5 ▶ Properties of Direct Images With Compact Support I

Let $f\colon A \to B$ be a function.

1. *Functoriality.* The assignment $U \mapsto f_!(U)$ defines a functor

$$f_!\colon (\mathcal{P}(A), \subset) \to (\mathcal{P}(B), \subset)$$

where

- *Action on Objects.* For each $U \in \mathcal{P}(A)$, we have

$$[f_!](U) \overset{\mathrm{def}}{=} f_!(U);$$

- *Action on Morphisms.* For each $U, V \in \mathcal{P}(A)$:
  ($\star$) If $U \subset V$, then $f_!(U) \subset f_!(V)$.

2. *Triple Adjointness.* We have a triple adjunction

$$(f_* \dashv f^{-1} \dashv f_!)\colon \quad \begin{tikzcd} \mathcal{P}(A) \arrow[r, bend left, "f_*"] \arrow[r, bend right, "f_!"'] & \mathcal{P}(B), \arrow[l, "f^{-1}"'] \end{tikzcd}$$

witnessed by bijections of sets

$$\begin{aligned} \mathrm{Hom}_{\mathcal{P}(B)}(f_*(U), V) &\cong \mathrm{Hom}_{\mathcal{P}(A)}\big(U, f^{-1}(V)\big), \\ \mathrm{Hom}_{\mathcal{P}(A)}\big(f^{-1}(U), V\big) &\cong \mathrm{Hom}_{\mathcal{P}(A)}(U, f_!(V)), \end{aligned}$$

natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$ and (respectively) $V \in \mathcal{P}(A)$ and $U \in \mathcal{P}(B)$, i.e. where:

   (a) The following conditions are equivalent:
      i. We have $f_*(U) \subset V$;
      ii. We have $U \subset f^{-1}(V)$;
   (b) The following conditions are equivalent:
      i. We have $f^{-1}(U) \subset V$.
      ii. We have $U \subset f_!(V)$.

3. *Lax Preservation of Colimits.* We have an inclusion of sets

$$\bigcup_{i \in I} f_!(U_i) \subset f_!\left(\bigcup_{i \in I} U_i\right),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have inclusions

$$\begin{aligned} f_!(U) \cup f_!(V) &\hookrightarrow f_!(U \cup V), \\ \emptyset &\hookrightarrow f_!(\emptyset), \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

4. *Preservation of Limits.* We have an equality of sets

$$f_!\left(\bigcap_{i \in I} U_i\right) = \bigcap_{i \in I} f_!(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have equalities

$$\begin{aligned} f^{-1}(U \cap V) &= f_!(U) \cap f^{-1}(V), \\ f_!(A) &= B, \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

5. *Symmetric Lax Monoidality With Respect to Unions.* The direct image with compact support function of Item 1 has a symmetric lax monoidal structure

$$\left(f_!, f_!^{\otimes}, f_{!|\mathbb{1}}^{\otimes}\right) \colon (\mathcal{P}(A), \cup, \emptyset) \to (\mathcal{P}(B), \cup, \emptyset),$$

being equipped with inclusions

$$\begin{aligned} f_{!|U,V}^{\otimes} &\colon f_!(U) \cup f_!(V) \hookrightarrow f_!(U \cup V), \\ f_{!|\mathbb{1}}^{\otimes} &\colon \emptyset \hookrightarrow f_!(\emptyset), \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

6. *Symmetric Strict Monoidality With Respect to Intersections.* The direct image function of Item 1 has a symmetric strict monoidal structure

$$\left(f_!, f_!^{\otimes}, f_{!|\mathbb{1}}^{\otimes}\right) \colon (\mathcal{P}(A), \cap, A) \to (\mathcal{P}(B), \cap, B),$$

being equipped with equalities

$$\begin{aligned} f_{!|U,V}^{\otimes} &\colon f_!(U \cap V) \xrightarrow{=} f_!(U) \cap f_!(V), \\ f_{!|\mathbb{1}}^{\otimes} &\colon f_!(A) \xrightarrow{=} B, \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

7. *Relation to Direct Images.* We have

$$f_!(U) = B \setminus f_*(A \setminus U)$$

for each $U \in \mathcal{P}(A)$.

8. *Interaction With Injections.* If f is injective, then we have

$$\begin{aligned} f_{!,\mathrm{im}}(U) &= f_*(U), \\ f_{!,\mathrm{cp}}(U) &= B \setminus \mathrm{Im}(f), \\ f_!(U) &= f_{!,\mathrm{im}}(U) \cup f_{!,\mathrm{cp}}(U) \\ &= f_*(U) \cup (B \setminus \mathrm{Im}(f)) \end{aligned}$$

for each $U \in \mathcal{P}(A)$.

9. *Interaction With Surjections*. If f is surjective, then we have

$$\begin{aligned} f_{!,\mathrm{im}}(U) &\subset f_*(U), \\ f_{!,\mathrm{cp}}(U) &= \emptyset, \\ f_!(U) &\subset f_*(U) \end{aligned}$$

for each $U \in \mathcal{P}(A)$.

Proof 2.4.6.6 ▸ Proof of Proposition 2.4.6.5

Item 1: Functoriality

Clear.

Item 2: Triple Adjointness

This follows from **??** of **??**.

Item 3: Lax Preservation of Colimits

Omitted.

Item 4: Preservation of Limits

Omitted. This follows from Item 2 and **??** of **??**.

Item 5: Symmetric Lax Monoidality With Respect to Unions

This follows from **??**.

Item 6: Symmetric Strict Monoidality With Respect to Intersections

This follows from Item 4.

Item 7: Relation to Direct Images

We claim that $f_!(U) = B \setminus f_*(A \setminus U)$.

- *The First Implication*. We claim that

$$f_!(U) \subset B \setminus f_*(A \setminus U).$$

  Let $b \in f_!(U)$. We need to show that $b \notin f_*(A \setminus U)$, i.e. that there is no $a \in A \setminus U$ such that $f(a) = b$.

  This is indeed the case, as otherwise we would have $a \in f^{-1}(b)$ and $a \notin U$, contradicting $f^{-1}(b) \subset U$ (which holds since $b \in f_!(U)$).

  Thus $b \in B \setminus f_*(A \setminus U)$.

- *The Second Implication*. We claim that

$$B \setminus f_*(A \setminus U) \subset f_!(U).$$

Let $b \in B \setminus f_*(A \setminus U)$. We need to show that $b \in f_!(U)$, i.e. that $f^{-1}(b) \subset U$.

Since $b \notin f_*(A \setminus U)$, there exists no $a \in A \setminus U$ such that $b = f(a)$, and hence $f^{-1}(b) \subset U$.

Thus $b \in f_!(U)$.

This finishes the proof of Item 7.

Item 8: Interaction With Injections

Clear.

Item 9: Interaction With Surjections

Clear.

Proposition 2.4.6.7 ► Properties of Direct Images With Compact Support II

Let $f : A \to B$ be a function.

1. *Functionality I.* The assignment $f \mapsto f_!$ defines a function

$$(-)_{!|A,B} : \mathsf{Sets}(A, B) \to \mathsf{Sets}(\mathcal{P}(A), \mathcal{P}(B)).$$

2. *Functionality II.* The assignment $f \mapsto f_!$ defines a function

$$(-)_{!|A,B} : \mathsf{Sets}(A, B) \to \mathsf{Pos}((\mathcal{P}(A), \subset), (\mathcal{P}(B), \subset)).$$

3. *Interaction With Identities.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$(\mathrm{id}_A)_! = \mathrm{id}_{\mathcal{P}(A)};$$

4. *Interaction With Composition.* For each pair of composable functions $f : A \to B$ and $g : B \to C$, we have

$$(g \circ f)_! = g_! \circ f_!, \qquad \begin{tikzcd} \mathcal{P}(A) \arrow[r, "f_!"] \arrow[dr, "(g\circ f)_!"'] & \mathcal{P}(B) \arrow[d, "g_!"] \\ & \mathcal{P}(C). \end{tikzcd}$$

Proof 2.4.6.8 ► Proof of Proposition 2.4.6.7

Item 1: Functionality I

Clear.

Item 2: Functionality II

Clear.

Item 3: Interaction With Identities

This follows from ?? of ??.

Item 4: Interaction With Composition

This follows from ?? of ??. ∎

Appendices

2.A Other Chapters

Chapter 3

Pointed Sets

This chapter contains some foundational material on pointed sets.

Contents

3.1 Pointed Sets

3.1.1 Foundations

Definition 3.1.1.1 ▶ Pointed Sets

A **pointed set**[1] is equivalently

- An $\mathbb{E}_0$-monoid in $(\mathsf{N}_\bullet(\mathsf{Sets}), \mathrm{pt})$;
- A pointed object in $(\mathsf{Sets}, \mathrm{pt})$.

[1] *Further Terminology:* Also called an $\mathbb{F}_1$**-module**.

Remark 3.1.1.2 ▶ Unwinding Definition 3.1.1.1

In detail, a **pointed set** is a pair (X, x_0) consisting of

- *The Underlying Set.* A set X, called the **underlying set of** (X, x_0);
- *The Basepoint.* A morphism

$$[x_0]\colon \mathrm{pt} \to X$$

  in Sets, determining an element $x_0 \in X$, called the **basepoint of** X.

Example 3.1.1.3 ▶ The Zero Sphere

The 0-**sphere**[1] is the pointed set $(S^0, 0)$[2] consisting of

- *The Underlying Set.* The set S^0 defined by

$$S^0 \overset{\text{def}}{=} \{0, 1\};$$

- *The Basepoint.* The element 0 of S^0.

[1] *Further Terminology:* Also called the **underlying pointed set of the field with one element**.
[2] *Further Notation:* Also denoted $(\mathbb{F}_1, 0)$.

Example 3.1.1.4 ▶ The Trivial Pointed Set

The **trivial pointed set** is the pointed set $(\mathrm{pt}, \star)$ consisting of

- *The Underlying Set.* The punctual set $\mathrm{pt} \overset{\text{def}}{=} \{\star\}$;
- *The Basepoint.* The element $\star$ of pt.

Example 3.1.1.5 ▶ The Underlying Pointed Set of a Semimodule

The **underlying pointed set** of a semimodule (M, α_M) is the pointed set $(M, 0_M)$.

Example 3.1.1.6 ► The Underlying Pointed Set of a Module

The **underlying pointed set** of a module (M, α_M) is the pointed set $(M, 0_M)$.

3.1.2 Morphisms of Pointed Sets

Definition 3.1.2.1 ► Morphisms of Pointed Sets

A **morphism of pointed sets**[1] is equivalently

- A morphism of $\mathbb{E}_0$-monoids in $(\mathrm{N}_\bullet(\mathsf{Sets}), \mathrm{pt})$.
- A morphism of pointed objects in $(\mathsf{Sets}, \mathrm{pt})$.

[1] *Further Terminology:* Also called a **pointed function** or a **morphism of $\mathbb{F}_1$-modules**.

Remark 3.1.2.2 ► Unwinding Definition 3.1.2.1

In detail, a **morphism of pointed sets** $f\colon (X, x_0) \to (Y, y_0)$ is a morphism of sets $f\colon X \to Y$ such that the diagram

$$\begin{tikzcd} & \mathrm{pt} \arrow[dl, "{[x_0]}"'] \arrow[dr, "{[y_0]}"] & \\ X \arrow[rr, "f"'] & & Y \end{tikzcd}$$

commutes, i.e. such that

$$f(x_0) = y_0.$$

3.1.3 The Category of Pointed Sets

Definition 3.1.3.1 ► The Category of Pointed Sets

The **category of pointed sets** is the category Sets_* defined equivalently as

- The homotopy category of the ∞-category $\mathsf{Mon}_{\mathbb{E}_0}(\mathrm{N}_\bullet(\mathsf{Sets}), \mathrm{pt})$ of **??**;
- The category Sets_* of **??**.

Remark 3.1.3.2 ► Unwinding Definition 3.1.3.1

In detail, the **category of pointed sets** is the category Sets_* where

- *Objects.* The objects of Sets_* are pointed sets;

- *Morphisms.* The morphisms of Sets_* are morphisms of pointed sets;
- *Identities.* For each $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, the unit map

  $$\mathbb{1}^{\mathsf{Sets}_*}_{(X,x_0)} \colon \mathrm{pt} \longrightarrow \mathsf{Sets}_*((X, x_0), (X, x_0))$$

  of Sets_* at (X, x_0) is defined by[1]

  $$\mathrm{id}^{\mathsf{Sets}_*}_{(X,x_0)} \overset{\mathrm{def}}{=} \mathrm{id}_X;$$

- *Composition.* For each $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, the composition map

  $$\circ^{\mathsf{Sets}_*}_{(X,x_0),(Y,y_0),(Z,z_0)} \colon \mathsf{Sets}_*((Y, y_0), (Z, z_0)) \times \mathsf{Sets}_*((X, x_0), (Y, y_0)) \to \mathsf{Sets}_*((X, x_0), (Z, z_0))$$

  of Sets_* at $((X, x_0), (Y, y_0), (Z, z_0))$ is defined by[2]

  $$g \circ^{\mathsf{Sets}_*}_{(X,x_0),(Y,y_0),(Z,z_0)} f \overset{\mathrm{def}}{=} g \circ f.$$

[1] Note that id_X is indeed a morphism of pointed sets, as we have $\mathrm{id}_X(x_0) = x_0$.

[2] Note that the composition of two morphisms of pointed sets is indeed a morphism of pointed sets, as we have

$$\begin{aligned} g(f(x_0)) &= g(y_0) \\ &= z_0, \end{aligned}$$

or

$$\begin{tikzcd}
 & \mathrm{pt} \arrow[dl, "{[x_0]}"'] \arrow[d, "{[y_0]}"] \arrow[dr, "{[z_0]}"] & \\
X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z
\end{tikzcd}$$

in terms of diagrams.

3.1.4 Elementary Properties of Pointed Sets

Proposition 3.1.4.1 ▶ Elementary Properties of Pointed Sets

Let (X, x_0) be a pointed set.

1. *Completeness.* The category Sets_* of pointed sets and morphisms between them is complete, having in particular products (Definition 3.2.1.1), pullbacks (Definition 3.2.3.1), and equalisers (Definition 3.2.2.1).

2. *Cocompleteness.* The category Sets_* of pointed sets and morphisms between

them is cocomplete, having in particular coproducts (Definition 3.3.1.1), pushouts (Definition 3.3.2.1), and coequalisers (Definition 3.3.3.1).

3. *Failure To Be Cartesian Closed*. The category Sets_* is not Cartesian closed.

4. *Relation to Partial Functions*. We have an equivalence of categories[1]

$$\mathsf{Sets}_* \overset{\text{eq.}}{\cong} \mathsf{Sets}^{\text{part.}}$$

between the category of pointed sets and pointed functions between them and the category of sets and partial functions between them.

[1] *Warning:* This is not an isomorphism of categories, only an equivalence.
END TEXTDBEND

PROOF 3.1.4.2 ► PROOF OF PROPOSITION 3.1.4.1

Item 1: Completeness

Omitted.

Item 2: Cocompleteness

Omitted.

Item 3: Failure To Be Cartesian Closed

See [**MSE2855868**].

Item 4: Relation to Partial Functions

Omitted. □

3.2 Limits of Pointed Sets

3.2.1 Products

Let (X, x_0) and (Y, y_0) be pointed sets.

DEFINITION 3.2.1.1 ► PRODUCTS OF POINTED SETS

The **product of** (X, x_0) **and** (Y, y_0) is the pointed set $(X \times Y, (x_0, y_0))$.

3.2.2 Equalisers

Let $f, g \colon (X, x_0) \rightrightarrows (Y, y_0)$ be morphisms of pointed sets.

Definition 3.2.2.1 ► Equalisers of Pointed Sets

The **equaliser of** (f, g) is the pointed set $(\mathrm{Eq}_*(f, g), x_0)$ consisting of

- *The Underlying Set.* The set $\mathrm{Eq}_*(f, g)$ defined by

$$\mathrm{Eq}_*(f, g) \stackrel{\text{def}}{=} \{x \in X \mid f(x) = y_0 = g(x)\};$$

- *The Basepoint.* The element x_0 of $\mathrm{Eq}_*(f, g)$.

3.2.3 Pullbacks

Let (X, x_0), (Y, y_0), and (Z, z_0) be pointed sets and let $f: (X, x_0) \to (Z, z_0)$ and $g: (Y, y_0) \to (Z, z_0)$ be morphisms of pointed sets.

Definition 3.2.3.1 ► Pullbacks of Pointed Sets

The **pullback of** (X, x_0) **and** (Y, y_0) **over** (Z, z_0) **along** (f, g) is the pointed set $\big((X, x_0) \times_{(z,z_0)} (Y, y_0), p_0\big)$ consisting of

- *The Underlying Set.* The set $(X, x_0) \times_{(z,z_0)} (Y, y_0)$ defined by

$$(X, x_0) \times_{(z,z_0)} (Y, y_0) \stackrel{\text{def}}{=} \{(x, y) \in X \times Y \mid f(x) = z_0 = g(y)\};$$

- *The Basepoint.* The element (x_0, y_0) of $(X, x_0) \times_{(z,z_0)} (Y, y_0)$.

3.3 Colimits of Pointed Sets

3.3.1 Coproducts

Let (X, x_0) and (Y, y_0) be pointed sets.

Definition 3.3.1.1 ► Coproducts of Pointed Sets

The **coproduct of** (X, x_0) **and** (Y, y_0) is their wedge sum $(X \vee Y, p_0)$ of Definition 3.4.3.1.

3.3.2 Pushouts

Let (X, x_0), (Y, y_0), and (Z, z_0) be pointed sets and let $f: (Z, z_0) \to (X, x_0)$ and $g: (Z, z_0) \to (Y, y_0)$ be morphisms of pointed sets.

Definition 3.3.2.1 ▶ Pushouts of Pointed Sets

The **pushout of** (X, x_0) **and** (Y, y_0) **over** (Z, z_0) **along** (f, g) is the pointed set $\left(X \coprod_{f,Z,g} Y, p_0\right)$, where $p_0 = [x_0] = [y_0]$.

3.3.3 Coequalisers

Let $f, g\colon (X, x_0) \rightrightarrows (Y, y_0)$ be morphisms of pointed sets.

Definition 3.3.3.1 ▶ Coequalisers of Pointed Sets

The **coequaliser of** (f, g) is the pointed set $(\mathrm{CoEq}(f, g), x_0)$.

3.4 Constructions With Pointed Sets

3.4.1 Internal Homs

Let (X, x_0) and (Y, y_0) be pointed sets.

Definition 3.4.1.1 ▶ Pointed Sets of Morphisms of Pointed Sets

The **pointed set of morphisms of pointed sets from** (X, x_0) **to** (Y, y_0) is the pointed set $\mathbf{Sets}_*(X, Y)$ consisting of

- *The Underlying Set.* The set $\mathbf{Sets}_*((X, x_0), (Y, y_0))$ of morphisms of pointed sets from (X, x_0) to (Y, y_0);
- *The Basepoint.* The element

$$\Delta_{y_0}\colon (X, x_0) \longrightarrow (Y, y_0)$$

  of $\mathbf{Sets}_*((X, x_0), (Y, y_0))$.

3.4.2 Free Pointed Sets

Let X be a set.

Definition 3.4.2.1 ▶ Free Pointed Sets

The **free pointed set on** X is the pointed set X^+ consisting of

- *The Underlying Set.* The set X^+ defined by

$$X^+ \stackrel{\text{def}}{=} X \coprod \mathrm{pt};$$

- *The Basepoint.* The element $\star$ of X^+.

Proposition 3.4.2.2 ▶ Properties of Free Pointed Sets

Let X be a set.

1. *Functoriality.* The assignment $X \mapsto X^+$ defines a functor
$$(-)^+ \colon \mathsf{Sets} \to \mathsf{Sets}_*,$$
where
   - *Action on Objects.* For each $X \in \mathrm{Obj}(\mathsf{Sets})$, we have
$$[(-)^+](X) \overset{\mathrm{def}}{=} X_+,$$
where X_+ is the pointed set of Definition 3.4.2.1;
   - *Action on Morphisms.* For each morphism $f \colon X \to Y$ of Sets, the image
$$f_+ \colon X_+ \to Y_+$$
of f by $(-)^+$ is the map of pointed sets defined by
$$f^+(x) \overset{\mathrm{def}}{=} \begin{cases} f(x) & \text{if } x \in X, \\ \star & \text{if } x = \star. \end{cases}$$
2. *Adjointness.* We have an adjunction
$$\left((-)^+ \dashv 忘\right) \colon \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "(-)^+"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_*, \arrow[l, bend left, "忘"] \end{tikzcd}$$
witnessed by a bijection of sets
$$\mathsf{Sets}_*((X_+, \star), (Y, y_0)) \cong \mathsf{Sets}(X, Y),$$
natural in $X \in \mathrm{Obj}(\mathsf{Sets})$ and $(Y, y_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.
3. *Symmetric Strong Monoidality With Respect to Wedge Sums.* The free pointed set functor of Item 1 has a symmetric strong monoidal structure
$$\left((-)^+, (-)^{+,\coprod}, (-)^{+,\coprod}_{\mathbb{1}}\right) \colon (\mathsf{Sets}, \coprod, \emptyset) \to (\mathsf{Sets}_*, \vee, \mathrm{pt}),$$
being equipped with isomorphisms of pointed sets
$$(-)^{+,\coprod}_{X,Y} \colon X^+ \vee Y^+ \xrightarrow{\cong} (X \coprod Y)^+,$$
$$(-)^{+,\coprod}_{\mathbb{1}} \colon \mathrm{pt} \xrightarrow{\cong} \emptyset^+,$$
natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

4. *Symmetric Strong Monoidality With Respect to Smash Products.* The free pointed set functor of Item 1 has a symmetric strong monoidal structure

$$\left((-)^{+},(-)^{+,\times},(-)^{+,\times}_{\mathbb{1}}\right)\colon(\mathsf{Sets},\times,\mathrm{pt})\to\left(\mathsf{Sets}_{*},\wedge,S^{0}\right),$$

being equipped with isomorphisms of pointed sets

$$(-)^{+,\times}_{X,Y}\colon X^{+}\wedge Y^{+}\xrightarrow{\cong}(X\times Y)^{+},$$
$$(-)^{+,\times}_{\mathbb{1}}\colon S^{0}\xrightarrow{\cong}\mathrm{pt}^{+},$$

natural in $X,Y\in\mathrm{Obj}(\mathsf{Sets})$.

Proof 3.4.2.3 ▶ Proof of Proposition 3.4.2.2

Item 1: Functoriality

Clear.

Item 2: Adjointness

Clear.

Item 3: Symmetric Strong Monoidality With Respect to Wedge Sums

Omitted.

Item 4: Symmetric Strong Monoidality With Respect to Smash Products

Omitted.

3.4.3 Wedge Sums of Pointed Sets

Let (X,x_0) and (Y,y_0) be pointed sets.

Definition 3.4.3.1 ▶ Wedge Sums of Pointed Sets

The **wedge sum of** X **and** Y is the pointed set $(X\vee Y,p_0)$ consisting of

- *The Underlying Set.* The set $X\vee Y$ defined by[1]

$$\begin{aligned}(X\vee Y,p_0)&\overset{\text{def}}{=}(X,x_0)\textstyle\coprod(Y,y_0)\\&\cong\left(X\textstyle\coprod_{\mathrm{pt}}Y,p_0\right)\\&\cong\left(X\textstyle\coprod Y/\sim,p_0\right),\end{aligned}$$

$$\begin{tikzcd} X\vee Y \arrow[dr, phantom, "\ulcorner", very near start] & Y \arrow[l] \\ X \arrow[u] & \mathrm{pt}, \arrow[l, hook', "{[x_0]}"] \arrow[u, hook', "{[y_0]}"'] \end{tikzcd}$$

where $\sim$ is the equivalence relation on $X\coprod Y$ given by $x_0\sim y_0$;

- *The Basepoint*. The element p_0 of $X \vee Y$ defined by

$$\begin{aligned} p_0 &\overset{\text{def}}{=} [x_0] \\ &= [y_0]. \end{aligned}$$

[1]Here $(X, x_0) \coprod (Y, y_0)$ is the coproduct of (X, x_0) and (Y, y_0) in Sets_*.

PROPOSITION 3.4.3.2 ► PROPERTIES OF WEDGE SUMS OF POINTED SETS

Let (X, x_0) and (Y, y_0) be pointed sets.

1. *Functoriality*. The assignments $(X, x_0), (Y, y_0), ((X, x_0), (Y, y_0)) \mapsto (X \vee Y, p_0)$ define functors

$$\begin{aligned} X \vee -\colon \mathsf{Sets}_* &\to \mathsf{Sets}_*, \\ - \vee Y\colon \mathsf{Sets}_* &\to \mathsf{Sets}_*, \\ -_1 \vee -_2\colon \mathsf{Sets}_* \times \mathsf{Sets}_* &\to \mathsf{Sets}_*. \end{aligned}$$

2. *Associativity*. We have an isomorphism of pointed sets

$$(X \vee Y) \vee Z \cong X \vee (Y \vee Z),$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathsf{Sets}_*$.

3. *Unitality*. We have isomorphisms of pointed sets

$$\begin{aligned} \mathrm{pt} \vee X &\cong X, \\ X \vee \mathrm{pt} &\cong X, \end{aligned}$$

natural in $(X, x_0) \in \mathsf{Sets}_*$.

4. *Commutativity*. We have an isomorphism of pointed sets

$$X \vee Y \cong Y \vee X,$$

natural in $(X, x_0), (Y, y_0) \in \mathsf{Sets}_*$.

5. *Symmetric Monoidality*. The triple $(\mathsf{Sets}_*, \vee, \mathrm{pt})$ is a symmetric monoidal category.

6. *Symmetric Strong Monoidality With Respect to Free Pointed Sets*. The free pointed set functor of Item 1 of Proposition 3.4.2.2 has a symmetric strong monoidal structure

$$\left((-)^+, (-)^{+,\coprod}, (-)^{+,\coprod}_{\mathbb{1}}\right)\colon (\mathsf{Sets}, \coprod, \emptyset) \to (\mathsf{Sets}_*, \vee, \mathrm{pt}),$$

being equipped with isomorphisms of pointed sets

$$(-)^{+,\coprod}_{X,Y} \colon X^+ \vee Y^+ \xrightarrow{\cong} (X \coprod Y)^+,$$
$$(-)^{+,\coprod}_{\mathbb{1}} \colon \mathrm{pt} \xrightarrow{\cong} \emptyset^+,$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

7. *The Fold Map.* We have a natural transformation

$$\nabla \colon \vee \circ \Delta^{\mathsf{Cats}}_{\mathsf{Sets}_*} \Longrightarrow \mathrm{id}_{\mathsf{Sets}_*}, \qquad \begin{tikzcd} & \mathsf{Sets}_* \times \mathsf{Sets}_* \arrow[dr, "\vee"] \arrow[d, phantom, "\Downarrow\nabla"] & \\ \mathsf{Sets}_* \arrow[ur, "\Delta^{\mathsf{Cats}}_{\mathsf{Sets}_*}"] \arrow[rr, bend right, "\mathrm{id}_{\mathsf{Sets}_*}"'] & & \mathsf{Sets}_*, \end{tikzcd}$$

called the **fold map**, whose component

$$\nabla_X \colon X \vee X \to X$$

at X is given by the composition

$$\begin{aligned} X &\xrightarrow{\Delta_X} X \times X \\ &\twoheadrightarrow X \times X/\sim \\ &\overset{\text{def}}{=\!=} X \vee X. \end{aligned}$$

Proof 3.4.3.3 ► Proof of Proposition 3.4.3.2

Item 1: Functoriality

Omitted.

Item 2: Associativity

Omitted.

Item 3: Unitality

Omitted.

Item 4: Commutativity

Omitted.

Item 5: Symmetric Monoidality

Omitted.

Item 6: Symmetric Strong Monoidality With Respect to Free Pointed Sets

Omitted.

Item 7: The Fold Map

Omitted.

Appendices

3.A Other Chapters

Chapter 4

Tensor Products of Pointed Sets

This chapter contains some material on tensor products of pointed sets.

Contents

4.1 Bilinear Morphisms of Pointed Sets

4.1.1 Left Bilinear Morphisms of Pointed Sets

Let (X, x_0), (Y, y_0), and (Z, z_0) be pointed sets.

Definition 4.1.1.1 ► Left Bilinear Morphisms of Pointed Sets

A **left bilinear morphism of pointed sets from** $(X \times Y, (x_0, y_0))$ **to** (Z, z_0) is a map of sets

$$f\colon (X \times Y, (x_0, y_0)) \to (Z, z_0)$$

satisfying the following condition:[1,2]

(★) *Left Unital Bilinearity.* The diagram

$$\begin{tikzcd}
 & \mathrm{pt} \times \mathrm{pt} \arrow[dr, dashed, "\sim"] & \\
\mathrm{pt} \times Y \arrow[ur, "\mathrm{id}_{\mathrm{pt}} \times \epsilon_Y"] \arrow[d, "{[x_0]} \times \mathrm{id}_Y"'] & & \mathrm{pt} \arrow[d, "{[z_0]}"] \\
X \times Y \arrow[rr, "f"'] & & Z
\end{tikzcd}$$

commutes, i.e. for each $y \in Y$, we have

$$f(x_0, y) = z_0.$$

[1] *Slogan:* f is left bilinear if it preserves basepoints in its first argument.

[2] Succinctly, f is bilinear if we have

$$f(x_0, y) = z_0$$

for each $y \in Y$.

Definition 4.1.1.2 ► The Set of Left Bilinear Morphisms of Pointed Sets

The **set of left bilinear morphisms of pointed sets from** $(X \times Y, (x_0, y_0))$ **to** (Z, z_0) is the set $\mathrm{Hom}^{\otimes, \mathrm{L}}_{\mathsf{Sets}_*}(X \times Y, Z)$ defined by

$$\mathrm{Hom}^{\otimes, \mathrm{L}}_{\mathsf{Sets}_*}(X \times Y, Z) \overset{\mathrm{def}}{=} \{f \in \mathsf{Sets}_*(A \times B, C) \mid f \text{ is left bilinear}\}.$$

4.1.2 Right Bilinear Morphisms of Pointed Sets

Let (X, x_0), (Y, y_0), and (Z, z_0) be pointed sets.

Definition 4.1.2.1 ► Right Bilinear Morphisms of pointed sets

A **right bilinear morphism of pointed sets from** $(X \times Y, (x_0, y_0))$ **to** (Z, z_0) is a map of sets

$$f\colon (X \times Y, (x_0, y_0)) \to (Z, z_0)$$

satisfying the following condition:[1,2]

(★) *Right Unital Bilinearity.* The diagram

$$\begin{tikzcd}
 & \mathrm{pt} \times \mathrm{pt} \arrow[dr, dashed, "\sim"] & \\
X \times \mathrm{pt} \arrow[ur, "\epsilon_X \times \mathrm{id}_{\mathrm{pt}}"] \arrow[d, "\mathrm{id}_X \times [y_0]"'] & & \mathrm{pt} \arrow[d, "{[z_0]}"] \\
X \times Y \arrow[rr, "f"'] & & Z
\end{tikzcd}$$

commutes, i.e. for each $x \in X$, we have

$$f(x, y_0) = z_0.$$

[1] *Slogan:* f is right bilinear if it preserves basepoints in its second argument.
[2] Succinctly, f is bilinear if we have

$$f(x, y_0) = z_0$$

for each $x \in X$.

Definition 4.1.2.2 ► The Set of Right Bilinear Morphisms of Pointed Sets

The **set of right bilinear morphisms of pointed sets from** $(X \times Y, (x_0, y_0))$ **to** (Z, z_0) is the set $\mathrm{Hom}^{\otimes,\mathrm{R}}_{\mathsf{Sets}_*}(X \times Y, Z)$ defined by

$$\mathrm{Hom}^{\otimes,\mathrm{R}}_{\mathsf{Sets}_*}(X \times Y, Z) \overset{\text{def}}{=} \{f \in \mathsf{Sets}_*(A \times B, C) \mid f \text{ is right bilinear}\}.$$

4.1.3 Bilinear Morphisms of Pointed Sets

Let (X, x_0), (Y, y_0), and (Z, z_0) be pointed sets.

Definition 4.1.3.1 ► Bilinear Morphisms of pointed sets

A **bilinear morphism of pointed sets from** $(X \times Y, (x_0, y_0))$ **to** (Z, z_0) is a map of sets

$$f\colon (X \times Y, (x_0, y_0)) \to (Z, z_0)$$

that is both left bilinear and right bilinear.

Remark 4.1.3.2 ▶ Unwinding Definition 4.1.3.1

In detail, a **bilinear morphism of pointed sets from** $(X \times Y, (x_0, y_0))$ **to** (Z, z_0) is a map of sets

$$f\colon (X \times Y, (x_0, y_0)) \to (Z, z_0)$$

satisfying the following conditions:[1,2]

1. *Left Unital Bilinearity.* The diagram

$$\begin{tikzcd}
 & \mathrm{pt} \times \mathrm{pt} \arrow[dr, dashed, "\sim"] & \\
\mathrm{pt} \times Y \arrow[ur, "\mathrm{id}_{\mathrm{pt}} \times \epsilon_Y"] \arrow[d, "{[x_0] \times \mathrm{id}_Y}"'] & & \mathrm{pt} \arrow[d, "{[z_0]}"] \\
X \times Y \arrow[rr, "f"'] & & Z
\end{tikzcd}$$

   commutes, i.e. for each $y \in Y$, we have

$$f(x_0, y) = z_0.$$

2. *Right Unital Bilinearity.* The diagram

$$\begin{tikzcd}
 & \mathrm{pt} \times \mathrm{pt} \arrow[dr, dashed, "\sim"] & \\
X \times \mathrm{pt} \arrow[ur, "\epsilon_X \times \mathrm{id}_{\mathrm{pt}}"] \arrow[d, "{\mathrm{id}_X \times [y_0]}"'] & & \mathrm{pt} \arrow[d, "{[z_0]}"] \\
X \times Y \arrow[rr, "f"'] & & Z
\end{tikzcd}$$

   commutes, i.e. for each $x \in X$, we have

$$f(x, y_0) = z_0.$$

[1] *Slogan:* f is bilinear if it preserves basepoints in each argument.

[2] Succinctly, f is bilinear if we have

$$f(x_0, y) = z_0,$$
$$f(x, y_0) = z_0$$

for each $x \in X$ and each $y \in Y$.

Definition 4.1.3.3 ► The Set of Bilinear Morphisms of Pointed Sets

The **set of bilinear morphisms of pointed sets from** $(X \times Y, (x_0, y_0))$ **to** (Z, z_0) is the set $\mathrm{Hom}^{\otimes}_{\mathsf{Sets}_*}(X \times Y, Z)$ defined by

$$\mathrm{Hom}^{\otimes}_{\mathsf{Sets}_*}(X \times Y, Z) \stackrel{\text{def}}{=} \{f \in \mathsf{Sets}_*(A \times B, C) \mid f \text{ is bilinear}\}.$$

4.2 Tensors and Cotensors of Pointed Sets by Sets

4.2.1 Tensors of Pointed Sets by Sets

Let (X, x_0) be a pointed set and let A be a set.

Definition 4.2.1.1 ► Tensors of Pointed Sets by Sets

The **tensor of** (X, x_0) **by** A is the pointed set $A \odot (X, x_0)$ satisfying the following universal property:

(UP) We have a bijection

$$\mathsf{Sets}_*(A \odot X, K) \cong \mathsf{Sets}(A, \mathbf{Sets}_*(X, K)),$$

natural in $(K, k_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

Remark 4.2.1.2 ► Unwinding Definition 4.2.1.1

The tensor of (X, x_0) by A satisfies the following universal property:

$$\mathsf{Sets}_*(A \odot X, K) \cong \mathsf{Sets}^{\otimes}_{\mathbb{E}_0}(A \times X, K),$$

where $\mathsf{Sets}^{\otimes}_{\mathbb{E}_0}(A \times X, K)$ is the set defined by

$$\mathsf{Sets}^{\otimes}_{\mathbb{E}_0}(A \times X, K) \stackrel{\text{def}}{=} \left\{ f \in \mathsf{Sets}(A \times X, K) \,\middle|\, \begin{array}{l} \text{for each } a \in A, \text{ we have} \\ f(a, x_0) = k_0 \end{array} \right\}.$$

Construction 4.2.1.3 ► Construction of Tensors of Pointed Sets by Sets

Concretely, the **tensor of** (X, x_0) **by** A is the pointed set $A \odot (X, x_0)$ consisting of

- *The Underlying Set.* The set $A \odot X$ given by
$$A \odot X \cong \bigvee_{a \in A} (X, x_0);$$
- *The Basepoint.* The point $[x_0]$ of $\bigvee_{a \in A}(X, x_0)$.

4.2.2 Cotensors of Pointed Sets by Sets

Let (X, x_0) be a pointed set and let A be a set.

Definition 4.2.2.1 ► Cotensors of Pointed Sets by Sets

The **cotensor of** (X, x_0) **by** A is the pointed set $A \pitchfork (X, x_0)$ satisfying the following universal property:

(UP) We have a bijection

$$\mathsf{Sets}_*(K, A \pitchfork X) \cong \mathsf{Sets}(A, \mathbf{Sets}_*(K, X)),$$

natural in $(K, k_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

Remark 4.2.2.2 ► Unwinding Definition 4.2.2.1

The cotensor of (X, x_0) by A satisfies the following universal property:

$$\mathsf{Sets}_*(K, A \pitchfork X) \cong \mathsf{Sets}^{\otimes}_{\mathbb{E}_0}(A \times K, X),$$

where $\mathsf{Sets}^{\otimes}_{\mathbb{E}_0}(A \times K, X)$ is the set defined by

$$\mathsf{Sets}^{\otimes}_{\mathbb{E}_0}(A \times K, X) \stackrel{\text{def}}{=} \left\{ f \in \mathsf{Sets}(A \times K, X) \,\middle|\, \begin{array}{l}\text{for each } a \in A \text{, we have}\\ f(a, k_0) = x_0\end{array} \right\}.$$

Construction 4.2.2.3 ► Construction of Cotensors of Pointed Sets by Sets

Concretely, the **cotensor of** (X, x_0) **by** A is the pointed set $A \pitchfork (X, x_0)$ consisting of

- *The Underlying Set.* The set $A \pitchfork X$ given by

$$A \pitchfork X \cong \bigwedge_{a \in A} (X, x_0);$$

- *The Basepoint.* The point $[(x_0, x_0, x_0, \ldots)]$ of $\bigwedge_{a \in A}(X, x_0)$.

4.3 The Left Tensor Product of Pointed Sets

4.3.1 Foundations

Let (X, x_0) and (Y, y_0) be pointed sets.

Definition 4.3.1.1 ▶ The Left Tensor Product of Pointed Sets

The **left tensor product of pointed sets** is the functor

$$\lhd_{\mathsf{Sets}_*} \colon \mathsf{Sets}_* \times \mathsf{Sets}_* \longrightarrow \mathsf{Sets}_*$$

defined as the composition

$$\mathsf{Sets}_* \times \mathsf{Sets}_* \xrightarrow{\mathrm{id}\times 忘} \mathsf{Sets}_* \times \mathsf{Sets} \xrightarrow{\beta^{\mathsf{Cats}_2}_{\mathsf{Sets}_*,\mathsf{Sets}}} \mathsf{Sets} \times \mathsf{Sets}_* \xrightarrow{\odot} \mathsf{Sets}_*.$$

Remark 4.3.1.2 ▶ Unwinding Definition 4.3.1.1, I: Universal Property

The left tensor product of pointed sets satisfies the following universal property:[1]

$$\mathsf{Sets}_*(X \lhd_{\mathsf{Sets}_*} Y, Z) \cong \mathrm{Hom}^{\otimes,\mathrm{L}}_{\mathsf{Sets}_*}(X \times Y, Z).$$

[1] Namely, a pointed map $f \colon X \lhd_{\mathsf{Sets}_*} Y \to Z$ is the same as a map $f^\dagger \colon X \times Y \to Z$ such that

$$f^\dagger(x_0, y) = z_0$$

for each $y \in Y$.

Remark 4.3.1.3 ▶ Unwinding Definition 4.3.1.1, II: Explicit Description

In detail, the **left tensor product of** (X, x_0) **and** (Y, y_0) is the pointed set $(X \lhd_{\mathsf{Sets}_*} Y, [x_0])$ consisting of[1]

- *The Underlying Set.* The set $X \lhd_{\mathsf{Sets}_*} Y$ defined by

$$\begin{aligned} X \lhd_{\mathsf{Sets}_*} Y &\overset{\mathrm{def}}{=} |Y| \odot X \\ &\cong \bigvee_{y \in Y} (X, x_0); \end{aligned}$$

- *The Underlying Basepoint.* The point $[x_0]$ of $\bigvee_{y \in Y}(X, x_0)$.

[1] *Further Notation:* We write $x \lhd_{\mathsf{Sets}_*} y$ for the image of (x, y) under the map

$$X \times Y \to \underbrace{X \lhd_{\mathsf{Sets}_*} Y}_{\cong \bigvee_{y\in Y}(X,x_0)}.$$

sending (x, y) to the element $x \in X$ in the yth copy of X in $\bigvee_{y \in Y}(X, x_0)$. Note that we have

$$x_0 \lhd_{\mathsf{Sets}_*} y = x_0 \lhd_{\mathsf{Sets}_*} y',$$

for each $y, y' \in Y$.

Proposition 4.3.1.4 ▶ Properties of Left Tensor Products of Pointed Sets

Let (X, x_0) and (Y, y_0) be pointed sets.

1. *Functoriality.* The assignments $X, Y, (X, Y) \mapsto X \lhd_{\mathsf{Sets}_*} Y$ define functors

$$
\begin{aligned}
X \lhd_{\mathsf{Sets}_*} - &\colon \mathsf{Sets}_* \to \mathsf{Sets}_*,\\
- \lhd_{\mathsf{Sets}_*} Y &\colon \mathsf{Sets}_* \to \mathsf{Sets}_*,\\
-_1 \lhd_{\mathsf{Sets}_*} -_2 &\colon \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*.
\end{aligned}
$$

Proof 4.3.1.5 ▶ Proof of Proposition 4.3.1.4

Item 1: Functoriality

Omitted.

4.3.2 The Skew Associator

Definition 4.3.2.1 ▶ The Skew Associator of $\lhd_{\mathsf{Sets}_*}$

The **skew associator of the left tensor product of pointed sets** is the natural isomorphism

$$\alpha^{\mathsf{Sets}_*, \lhd} \colon \lhd_{\mathsf{Sets}_*} \circ \left(\lhd_{\mathsf{Sets}_*} \times \mathrm{id}_{\mathsf{Sets}_*}\right) \stackrel{\cong}{\Longrightarrow} \lhd_{\mathsf{Sets}_*} \circ \left(\mathrm{id}_{\mathsf{Sets}_*} \times \lhd_{\mathsf{Sets}_*}\right),$$

whose component

$$\alpha^{\mathsf{Sets}_*, \lhd}_{X,Y,Z} \colon \left(X \lhd_{\mathsf{Sets}_*} Y\right) \lhd_{\mathsf{Sets}_*} Z \xrightarrow{\cong} X \lhd_{\mathsf{Sets}_*} \left(Y \lhd_{\mathsf{Sets}_*} Z\right)$$

at (X, Y, Z) is given by the composition[1]

$$\begin{aligned}
(X \lhd_{\mathsf{Sets}_*} Y) \lhd_{\mathsf{Sets}_*} Z &\stackrel{\text{def}}{=} |Z| \odot (X \lhd_{\mathsf{Sets}_*} Y) \\
&\stackrel{\text{def}}{=} |Z| \odot (|Y| \odot X) \\
&\cong \bigvee_{z \in Z} (|Y| \odot X, [x_0]) \\
&\stackrel{\text{def}}{=} \bigvee_{z \in Z} \left(\bigvee_{y \in Y} (X, x_0) \right) \\
&\cong \bigvee_{(z,y) \in \bigvee_{z \in Z}(Y, y_0)} (X, x_0) \\
&\stackrel{\text{def}}{=} \bigvee_{(z,y) \in |Z| \odot Y} (X, x_0) \\
&\cong ||Z| \odot Y| \odot X \\
&\stackrel{\text{def}}{=} |Y \lhd_{\mathsf{Sets}_*} Z| \odot X \\
&\stackrel{\text{def}}{=} X \lhd_{\mathsf{Sets}_*} (Y \lhd_{\mathsf{Sets}_*} Z),
\end{aligned}$$

where the isomorphism

$$\bigvee_{z \in Z} \left(\bigvee_{y \in Y} (X, x_0) \right) \cong \bigvee_{(y,z) \in \bigvee_{z \in Z}(Y, y_0)} (X, x_0)$$

is given by $[(z, (y, x))] \mapsto [((z, y), x)]$.

[1] In other words, $\alpha^{\mathsf{Sets}_*, \lhd}_{X,Y,Z}$ acts on elements as

$$\alpha^{\mathsf{Sets}_*, \lhd}_{X,Y,Z}((x \lhd_{\mathsf{Sets}_*} y) \lhd_{\mathsf{Sets}_*} z) \stackrel{\text{def}}{=} x \lhd_{\mathsf{Sets}_*} (y \lhd_{\mathsf{Sets}_*} z)$$

for each $(x \lhd_{\mathsf{Sets}_*} y) \lhd_{\mathsf{Sets}_*} z \in (X \lhd_{\mathsf{Sets}_*} Y) \lhd_{\mathsf{Sets}_*} Z$.

4.3.3 The Skew Left Unitor

Definition 4.3.3.1 ▶ The Skew Left Unitor of $\lhd_{\mathsf{Sets}_*}$

The **skew left unitor of the left tensor product of pointed sets** is the natural transformation

$$\lambda^{\mathsf{Sets}_*, \lhd} \colon \lhd_{\mathsf{Sets}_*} \circ \left(\mathbb{1}^{\mathsf{Sets}_*} \times \mathrm{id}_{\mathsf{Sets}_*} \right) \Longrightarrow \mathrm{id}_{\mathsf{Sets}_*},$$

whose component

$$\lambda^{\mathsf{Sets}_*, \lhd}_X \colon S^0 \lhd_{\mathsf{Sets}_*} X \to X$$

at X is given by the composition[1]

$$\begin{aligned} S^0 \lhd_{\mathsf{Sets}_*} X &\cong |X| \odot S^0 \\ &\cong \bigvee_{x\in X} S^0 \\ &\to X \end{aligned}$$

where $\bigvee_{x\in X} S^0 \to X$ is the map given by

$$\begin{aligned} (x,0) &\mapsto x, \\ (x,1) &\mapsto x. \end{aligned}$$

[1]In other words, $\lambda^{\mathsf{Sets}_*,\lhd}_X$ acts on elements as

$$\begin{aligned} \lambda^{\mathsf{Sets}_*,\lhd}_X(x \lhd_{\mathsf{Sets}_*} 0) &\overset{\text{def}}{=} x, \\ \lambda^{\mathsf{Sets}_*,\lhd}_X(x \lhd_{\mathsf{Sets}_*} 1) &\overset{\text{def}}{=} x, \end{aligned}$$

for each $x \in X$.

4.3.4 The Skew Right Unitor

Definition 4.3.4.1 ▶ The Skew Right Unitor of $\lhd_{\mathsf{Sets}_*}$

The **skew right unitor of the left tensor product of pointed sets** is the natural transformation

$$\rho^{\mathsf{Sets}_*,\lhd}\colon \mathrm{id}_{\mathsf{Sets}_*} \Longrightarrow \lhd_{\mathsf{Sets}_*} \circ \left(\mathrm{id}_{\mathsf{Sets}_*} \times \not{\mathbb{F}}^{\mathsf{Sets}_*}\right),$$

whose component

$$\rho^{\mathsf{Sets}_*,\lhd}_X\colon X \to X \lhd_{\mathsf{Sets}_*} S^0$$

at X is given by the composition[1]

$$\begin{aligned} X &\to X \vee X \\ &\cong |S^0| \odot X \\ &\cong S^0 \lhd_{\mathsf{Sets}_*} X, \end{aligned}$$

where $X \to X \vee X$ is the map sending X to the first factor of X in $X \vee X$.

[1]In other words, $\rho^{\mathsf{Sets}_*,\lhd}_X$ acts on elements as

$$\rho^{\mathsf{Sets}_*,\lhd}_X(x) \overset{\text{def}}{=} x \lhd_{\mathsf{Sets}_*} 0$$

for each $x \in X$.

4.3.5 The Left-Skew Monoidal Category Structure on Pointed Sets

Proposition 4.3.5.1 ▶ The Left-Skew Monoidal Category Structure on Pointed Sets

The category Sets_* admits a left-skew monoidal category structure consisting of[1]

- *The Skew Monoidal Product.* The left tensor product functor

$$\lhd_{\mathsf{Sets}_*} \colon \mathsf{Sets}_* \times \mathsf{Sets}_* \longrightarrow \mathsf{Sets}_*,$$

  of Proposition 4.3.1.4;

- *The Skew Monoidal Unit.* The functor

$$\mathbb{1}^{\mathsf{Sets}_*} \colon \mathrm{pt} \longrightarrow \mathsf{Sets}_*$$

  defined by

$$\mathbb{1}_{\mathsf{Sets}_*} \overset{\mathrm{def}}{=} S^0;$$

- *The Skew Associators.* The natural isomorphism

$$\alpha^{\mathsf{Sets}_*,\lhd} \colon \lhd_{\mathsf{Sets}_*} \circ \left(\lhd_{\mathsf{Sets}_*} \times \mathrm{id}_{\mathsf{Sets}_*}\right) \overset{\cong}{\Longrightarrow} \lhd_{\mathsf{Sets}_*} \circ \left(\mathrm{id}_{\mathsf{Sets}_*} \times \lhd_{\mathsf{Sets}_*}\right),$$

  of Definition 4.3.2.1;

- *The Skew Left Unitors.* The natural transformation

$$\lambda^{\mathsf{Sets}_*,\lhd} \colon \lhd_{\mathsf{Sets}_*} \circ \left(\mathbb{1}^{\mathsf{Sets}_*} \times \mathrm{id}_{\mathsf{Sets}_*}\right) \Longrightarrow \mathrm{id}_{\mathsf{Sets}_*},$$

  of Definition 4.3.3.1;

- *The Skew Right Unitors.* The natural transformation

$$\rho^{\mathsf{Sets}_*,\lhd} \colon \mathrm{id}_{\mathsf{Sets}_*} \Longrightarrow \lhd_{\mathsf{Sets}_*} \circ \left(\mathrm{id}_{\mathsf{Sets}_*} \times \mathbb{1}^{\mathsf{Sets}_*}\right),$$

  of Definition 4.3.4.1.

[1] Note in particular that, differently from general left-skew monoidal categories, the skew associator of $(\mathsf{Sets}_*, \lhd_{\mathsf{Sets}_*}, S^0)$ is a natural isomorphism.

Proof 4.3.5.2 ▶ Proof of Proposition 4.3.5.1

Omitted. □

4.4 The Right Tensor Product of Pointed Sets

4.4.1 Foundations

Let (X, x_0) and (Y, y_0) be pointed sets.

The **right tensor product of pointed sets** is the functor

$$\rhd_{\mathsf{Sets}_*} \colon \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*$$

defined as the composition

$$\mathsf{Sets}_* \times \mathsf{Sets}_* \xrightarrow{忘\times \mathrm{id}} \mathsf{Sets} \times \mathsf{Sets}_* \xrightarrow{\odot} \mathsf{Sets}_*.$$

Remark 4.4.1.2 ► Unwinding Definition 4.4.1.1, I: Universal Property

The right tensor product of pointed sets satisfies the following universal property:[1]

$$\mathsf{Sets}_*(X \rhd_{\mathsf{Sets}_*} Y, Z) \cong \mathrm{Hom}^{\otimes,\mathrm{R}}_{\mathsf{Sets}_*}(X \times Y, Z).$$

[1] Namely, a pointed map $f \colon X \lhd_{\mathsf{Sets}_*} Y \to Z$ is the same as a map $f^{\dagger} \colon X \times Y \to Z$ such that

$$f^{\dagger}(x, y_0) = z_0$$

for each $y \in Y$.

Remark 4.4.1.3 ► Unwinding Definition 4.4.1.1, II: Explicit Description

In detail, the **right tensor product of** (X, x_0) **and** (Y, y_0) is the pointed set $(X \rhd_{\mathsf{Sets}_*} Y, [y_0])$ consisting of[1]

- *The Underlying Set.* The set $X \rhd_{\mathsf{Sets}_*} Y$ defined by

$$\begin{aligned} X \rhd_{\mathsf{Sets}_*} Y &\stackrel{\text{def}}{=} |X| \odot Y \\ &\cong \bigvee_{x \in X} (Y, y_0); \end{aligned}$$

- *The Underlying Basepoint.* The point $[y_0]$ of $\bigvee_{x \in X}(Y, y_0)$.

[1] *Further Notation:* We write $x \rhd_{\mathsf{Sets}_*} y$ for the image of (x, y) under the map

$$X \times Y \to \underbrace{X \rhd_{\mathsf{Sets}_*} Y}_{\cong \bigvee_{x \in X}(Y, y_0)}.$$

sending (x, y) to the element $y \in Y$ in the xth copy of Y in $\bigvee_{x \in X}(Y, y_0)$. Note that we have

$$x \rhd_{\mathsf{Sets}_*} y_0 = x' \rhd_{\mathsf{Sets}_*} y_0,$$

for each $x, x' \in X$.

Proposition 4.4.1.4 ► Properties of Right Tensor Products of Pointed Sets

Let (X, x_0) and (Y, y_0) be pointed sets.

1. *Functoriality.* The assignments $X, Y, (X, Y) \mapsto X \rhd_{\mathsf{Sets}_*} Y$ define functors

$$\begin{aligned} X \rhd_{\mathsf{Sets}_*} - &\colon \mathsf{Sets}_* \to \mathsf{Sets}_*, \\ - \rhd_{\mathsf{Sets}_*} Y &\colon \mathsf{Sets}_* \to \mathsf{Sets}_*, \\ -_1 \rhd_{\mathsf{Sets}_*} -_2 &\colon \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*. \end{aligned}$$

Proof 4.4.1.5 ► Proof of Proposition 4.4.1.4

Item 1: Functoriality

Omitted. □

4.4.2 The Skew Associator

Definition 4.4.2.1 ► The Skew Associator of $\rhd_{\mathsf{Sets}_*}$

The **skew associator of the right tensor product of pointed sets** is the natural isomorphism

$$\alpha^{\mathsf{Sets}_*, \rhd} \colon \rhd_{\mathsf{Sets}_*} \circ \left(\mathrm{id}_{\mathsf{Sets}_*} \times \rhd_{\mathsf{Sets}_*}\right) \overset{\cong}{\Longrightarrow} \rhd_{\mathsf{Sets}_*} \circ \left(\rhd_{\mathsf{Sets}_*} \times \mathrm{id}_{\mathsf{Sets}_*}\right),$$

whose component

$$\alpha^{\mathsf{Sets}_*, \rhd}_{X,Y,Z} \colon X \rhd_{\mathsf{Sets}_*} \left(Y \rhd_{\mathsf{Sets}_*} Z\right) \xrightarrow{\cong} \left(X \rhd_{\mathsf{Sets}_*} Y\right) \rhd_{\mathsf{Sets}_*} Z$$

at (X, Y, Z) is given by the composition[1]

$$\begin{aligned}
X \rhd_{\mathsf{Sets}_*} \left(Y \rhd_{\mathsf{Sets}_*} Z\right) &\overset{\text{def}}{=} |X| \odot \left(Y \rhd_{\mathsf{Sets}_*} Z\right) \\
&\overset{\text{def}}{=} |X| \odot (|Y| \odot Z) \\
&\cong |X| \odot \left(\bigvee_{y \in Y} (Z, z_0)\right) \\
&\cong \bigvee_{x \in X}\left(\bigvee_{y \in Y} (Z, z_0)\right) \\
&\cong \bigvee_{(x,y) \in \bigvee_{x \in X}(Y, y_0)} (Z, z_0) \\
&\cong \left|\bigvee_{x \in X} (Y, y_0)\right| \odot Z \\
&\overset{\text{def}}{=} |X \odot Y| \odot Z \\
&\overset{\text{def}}{=} |X \rhd_{\mathsf{Sets}_*} Y| \odot Z \\
&\overset{\text{def}}{=} \left(X \rhd_{\mathsf{Sets}_*} Y\right) \rhd_{\mathsf{Sets}_*} Z
\end{aligned}$$

where the isomorphism

$$\bigvee_{x \in X}\left(\bigvee_{y \in Y} (Z, z_0)\right) \cong \bigvee_{(x,y) \in \bigvee_{x \in X}(Y, y_0)} (Z, z_0)$$

is given by $[(x, (y, z))] \mapsto [((x, y), z)]$.

[1] In other words, $\alpha^{\mathsf{Sets}_*, \rhd}_{X,Y,Z}$ acts on elements as

$$\alpha^{\mathsf{Sets}_*, \rhd}_{X,Y,Z}\left(x \rhd_{\mathsf{Sets}_*} \left(y \rhd_{\mathsf{Sets}_*} z\right)\right) \overset{\text{def}}{=} \left(x \rhd_{\mathsf{Sets}_*} y\right) \rhd_{\mathsf{Sets}_*} z$$

for each $x \rhd_{\mathsf{Sets}_*} \left(y \rhd_{\mathsf{Sets}_*} z\right) \in X \rhd_{\mathsf{Sets}_*} \left(Y \rhd_{\mathsf{Sets}_*} Z\right)$.

4.4.3 The Skew Left Unitor

Definition 4.4.3.1 ► The Skew Left Unitor of $\rhd_{\mathsf{Sets}_*}$

The **skew left unitor of the right tensor product of pointed sets** is the natural transformation

$$\lambda^{\mathsf{Sets}_*, \rhd} \colon \mathrm{id}_{\mathsf{Sets}_*} \Longrightarrow \rhd_{\mathsf{Sets}_*} \circ \left(\mathbb{1}^{\mathsf{Sets}_*} \times \mathrm{id}_{\mathsf{Sets}_*}\right),$$

whose component

$$\lambda_X^{\mathsf{Sets}_*,\triangleright} \colon X \to S^0 \triangleright_{\mathsf{Sets}_*} X$$

at X is given by the composition[1]

$$\begin{aligned} X &\to X \vee X \\ &\cong |S^0| \odot X \\ &\cong S^0 \triangleright_{\mathsf{Sets}_*} X, \end{aligned}$$

where $X \to X \vee X$ is the map sending X to the first factor of X in $X \vee X$.

[1] In other words, $\lambda_X^{\mathsf{Sets}_*,\triangleright}$ acts on elements as

$$\lambda_X^{\mathsf{Sets}_*,\triangleright}(x) \overset{\text{def}}{=} 0 \triangleright_{\mathsf{Sets}_*} x$$

for each $x \in X$.

4.4.4 The Skew Right Unitor

Definition 4.4.4.1 ▶ The Skew Right Unitor of $\triangleright_{\mathsf{Sets}_*}$

The **skew right unitor of the right tensor product of pointed sets** is the natural transformation

$$\rho^{\mathsf{Sets}_*,\triangleright} \colon \triangleright_{\mathsf{Sets}_*} \circ \left(\mathrm{id}_{\mathsf{Sets}_*} \times \mathbb{1}^{\mathsf{Sets}_*}\right) \Longrightarrow \mathrm{id}_{\mathsf{Sets}_*},$$

whose component[1]

$$\rho_X^{\mathsf{Sets}_*,\triangleright} \colon X \triangleright_{\mathsf{Sets}_*} S^0 \to X$$

at X is given by the composition

$$\begin{aligned} X \triangleright_{\mathsf{Sets}_*} S^0 &\cong |X| \odot S^0 \\ &\cong \bigvee_{x \in X} S^0 \\ &\to X \end{aligned}$$

where $\bigvee_{x\in X} S^0 \to X$ is the map given by

$$\begin{aligned} (x, 0) &\mapsto x, \\ (x, 1) &\mapsto x. \end{aligned}$$

[1] In other words, $\rho_X^{\mathsf{Sets}_*,\triangleright}$ acts on elements as

$$\begin{aligned} \rho_X^{\mathsf{Sets}_*,\triangleright}(x \triangleright_{\mathsf{Sets}_*} 0) &\overset{\text{def}}{=} x, \\ \rho_X^{\mathsf{Sets}_*,\triangleright}(x \triangleright_{\mathsf{Sets}_*} 1) &\overset{\text{def}}{=} x \end{aligned}$$

for each $x \in X$.

4.4.5 The Right-Skew Monoidal Category Structure on Pointed Sets

Proposition 4.4.5.1 ▶ The Right-Skew Monoidal Category Structure on Pointed Sets

The category Sets_* admits a right-skew monoidal category structure consisting of[1]

- *The Skew Monoidal Product.* The right tensor product functor

$$\triangleright_{\mathsf{Sets}_*} \colon \mathsf{Sets}_* \times \mathsf{Sets}_* \longrightarrow \mathsf{Sets}_*,$$

  of Item 1;

- *The Skew Monoidal Unit.* The functor

$$\mathbb{1}^{\mathsf{Sets}_*} \colon \mathrm{pt} \longrightarrow \mathsf{Sets}_*$$

  defined by

$$\mathbb{1}_{\mathsf{Sets}_*} \stackrel{\text{def}}{=} S^0;$$

- *The Skew Associators.* The natural isomorphism

$$\alpha^{\mathsf{Sets}_*,\triangleright} \colon \triangleright_{\mathsf{Sets}_*} \circ (\triangleright_{\mathsf{Sets}_*} \times \mathrm{id}_{\mathsf{Sets}_*}) \stackrel{\cong}{\Longrightarrow} \triangleright_{\mathsf{Sets}_*} \circ (\mathrm{id}_{\mathsf{Sets}_*} \times \triangleright_{\mathsf{Sets}_*}),$$

  of Definition 4.4.2.1;

- *The Skew Left Unitors.* The natural transformation

$$\lambda^{\mathsf{Sets}_*,\triangleright} \colon \mathrm{id}_{\mathsf{Sets}_*} \Longrightarrow \triangleright_{\mathsf{Sets}_*} \circ \left(\mathbb{1}^{\mathsf{Sets}_*} \times \mathrm{id}_{\mathsf{Sets}_*}\right),$$

  of Definition 4.3.3.1;

- *The Skew Right Unitors.* The natural transformation

$$\rho^{\mathsf{Sets}_*,\triangleright} \colon \triangleright_{\mathsf{Sets}_*} \circ \left(\mathrm{id}_{\mathsf{Sets}_*} \times \mathbb{1}^{\mathsf{Sets}_*}\right) \Longrightarrow \mathrm{id}_{\mathsf{Sets}_*},$$

  of Definition 4.3.4.1.

[1] Note in particular that, differently from general right-skew monoidal categories, the skew associator of $(\mathsf{Sets}_*, \triangleright_{\mathsf{Sets}_*}, S^0)$ is a natural isomorphism.

Proof 4.4.5.2 ▶ Proof of Proposition 4.3.5.1

Omitted. □

4.5 Smash Products of Pointed Sets

4.5.1 Foundations

Let (X, x_0) and (Y, y_0) be pointed sets.

The **smash product of** (X, x_0) **and** (Y, y_0)[1] is the pointed set $X \wedge Y$[2] such that we have a bijection

$$\mathsf{Sets}_*(X \wedge Y, Z) \cong \mathrm{Hom}^{\otimes}_{\mathsf{Sets}_*}(X \times Y, Z),$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

[1] *Further Terminology:* Also called the **tensor product of** $\mathbb{F}_1$**-modules of** (X, x_0) **and** (Y, y_0) or the **tensor product of** (X, x_0) **and** (Y, y_0) **over** $\mathbb{F}_1$.

[2] *Further Notation:* Also written $X \otimes_{\mathbb{F}_1} Y$.

Remark 4.5.1.2 ▶ Unwinding Definition 4.5.1.1

In detail, the **smash product of** (X, x_0) **and** (Y, y_0) is the pair $((X \wedge Y, [(x_0, y_0)]), \iota)$ consisting of

- A pointed set $(X \wedge Y, [(x_0, y_0)])$;
- A bilinear morphism of pointed sets $\iota\colon (X \times Y, (x_0, y_0)) \to X \wedge Y$;

satisfying the following universal property:

(UP) Given another such pair $((Z, z_0), f)$ consisting of

  - A pointed set (Z, z_0);
  - A bilinear morphism of pointed sets $f\colon (X \times Y, (x_0, y_0)) \to X \wedge Y$;

  there exists a unique morphism of pointed sets $X \wedge Y \xrightarrow{\exists!} Z$ making the diagram

$$\begin{tikzcd} & X \wedge Y \arrow[d, dashed, "\exists!"] \\ X \times Y \arrow[ur, "\iota"] \arrow[r, "f"'] & Z \end{tikzcd}$$

  commute.

Construction 4.5.1.3 ▶ Smash Products of Pointed Sets

Concretely, the **smash product of** (X, x_0) **and** (Y, y_0) is the pointed set $(X \wedge Y, [(x_0, y_0)])$ consisting of[1]

- *The Underlying Set.* The set $X \wedge Y$ defined by

$$\begin{aligned} X \wedge Y &\cong \mathrm{pt} \coprod_{X \vee Y} (X \times Y) \\ &\overset{\text{def}}{=} \frac{X \times Y}{X \vee Y} \\ &\cong X \times Y/\sim, \end{aligned} \qquad \begin{tikzcd} X \wedge Y \arrow[dr, phantom, "\ulcorner", very near start] & X \times Y \arrow[l] \\ \mathrm{pt} \arrow[u] & X \vee Y, \arrow[l, "!"] \arrow[u, hook] \end{tikzcd}$$

where $\sim$ is the equivalence relation of $X \times Y$ obtained by declaring $(x, y) \sim (x', y')$ iff $(x, y), (x', y') \in X \vee Y$, i.e. by declaring

$$\begin{aligned} (x_0, y) &\sim (x_0, y'), \\ (x, y_0) &\sim (x', y_0) \end{aligned}$$

for all $x \in X$ and all $y \in Y$;

- *The Basepoint.* The element $[(x_0, y_0)]$ of $X \wedge Y$ given by the equivalence class of (x_0, y_0) under the equivalence relation $\sim$ on $X \times Y$.

---

[1] *Further Notation:* We write $x \wedge y$ for the image of (x, y) under the quotient map

$$X \times Y \twoheadrightarrow \underbrace{\frac{X \times Y}{X \vee Y}}_{\overset{\text{def}}{=} X \wedge Y}$$

Note that we have

$$\begin{aligned} x \wedge y_0 &= x' \wedge y_0, \\ x_0 \wedge y &= x_0 \wedge y' \end{aligned}$$

for each $x, x' \in X$ and each $y, y' \in Y$.

PROOF 4.5.1.4 ► PROOF OF ??

Clear.

EXAMPLE 4.5.1.5 ► EXAMPLES OF SMASH PRODUCTS OF POINTED SETS

Here are some examples of smash products of pointed sets.

1. *Smashing With S^0.* For any pointed set X, we have isomorphisms of pointed sets

$$\begin{aligned} S^0 \wedge X &\cong X, \\ X \wedge S^0 &\cong X. \end{aligned}$$

Proposition 4.5.1.6 ▶ Properties of Smash Products of Pointed Sets

Let (X, x_0) and (Y, y_0) be pointed sets.

1. *Functoriality.* The assignments $(X, x_0), (Y, y_0), ((X, x_0), (Y, y_0)) \mapsto X \wedge Y$ define functors

$$\begin{aligned} X \wedge - \colon \mathsf{Sets}_* &\to \mathsf{Sets}_*, \\ - \wedge Y \colon \mathsf{Sets}_* &\to \mathsf{Sets}_*, \\ -_1 \wedge -_2 \colon \mathsf{Sets}_* \times \mathsf{Sets}_* &\to \mathsf{Sets}_*. \end{aligned}$$

2. *Adjointness.* We have adjunctions

$$(X \wedge - \dashv \mathbf{Sets}_*(X, -)) \colon \quad \begin{tikzcd} \mathsf{Sets}_* \arrow[r, bend left, "X\wedge -"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_*, \arrow[l, bend left, "\mathbf{Sets}_*(X,-)"] \end{tikzcd}$$

$$(- \wedge Y \dashv \mathbf{Sets}_*(Y, -)) \colon \quad \begin{tikzcd} \mathsf{Sets}_* \arrow[r, bend left, "-\wedge Y"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_*, \arrow[l, bend left, "\mathbf{Sets}_*(Y,-)"] \end{tikzcd}$$

witnessed by bijections

$$\begin{aligned} \mathsf{Sets}_*(X \wedge Y, Z) &\cong \mathsf{Sets}_*(X, \mathbf{Sets}_*(Y, Z)), \\ \mathsf{Sets}_*(X \wedge Y, Z) &\cong \mathsf{Sets}_*(X, \mathbf{Sets}_*(A, Z)), \end{aligned}$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, which internalise to isomorphisms of pointed sets

$$\begin{aligned} \mathbf{Sets}_*(X \wedge Y, Z) &\cong \mathbf{Sets}_*(X, \mathbf{Sets}_*(Y, Z)), \\ \mathbf{Sets}_*(X \wedge Y, Z) &\cong \mathbf{Sets}_*(X, \mathbf{Sets}_*(A, Z)), \end{aligned}$$

again natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

3. *Closed Symmetric Monoidality.* The quadruple $(\mathsf{Sets}_*, \wedge, S^0, \mathbf{Sets}_*)$ is a closed symmetric monoidal category.

4. *Morphisms From the Monoidal Unit.* We have a bijection of sets[1]

$$\mathsf{Sets}_*(S^0, X) \cong X,$$

natural in $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, internalising also to an isomorphism of pointed sets

$$\mathbf{Sets}_*(S^0, X) \cong (X, x_0),$$

again natural in $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

5. *Symmetric Strong Monoidality With Respect to Free Pointed Sets.* The free pointed set functor of **??** of **??** has a symmetric strong monoidal structure

$$\left((-)^{+}, (-)^{+,\times}, (-)^{+,\times}_{\mathbb{1}}\right) \colon (\mathsf{Sets}, \times, \mathrm{pt}) \to \left(\mathsf{Sets}_{*}, \wedge, S^{0}\right),$$

being equipped with isomorphisms of pointed sets

$$(-)^{+,\times}_{X,Y} \colon X^{+} \wedge Y^{+} \xrightarrow{\cong} (X \times Y)^{+},$$
$$(-)^{+,\times}_{\mathbb{1}} \colon S^{0} \xrightarrow{\cong} \mathrm{pt}^{+},$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

6. *Distributivity Over Wedge Sums.* We have isomorphisms of pointed sets

$$X \wedge (Y \vee Z) \cong (X \wedge Y) \vee (X \wedge Z),$$
$$(X \vee Y) \wedge Z \cong (X \wedge Z) \vee (Y \wedge Z),$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_{*})$.

7. *Universal Property I.* The symmetric monoidal structure on the category Sets_{*} is uniquely determined by the following requirements:

   (a) *Two-Sided Preservation of Colimits.* The smash product

   $$\wedge \colon \mathsf{Sets}_{*} \times \mathsf{Sets}_{*} \to \mathsf{Sets}_{*}$$

   of Sets_{*} preserves colimits separately in each variable.

   (b) *The Unit Object Is S^0.* We have $\mathbb{1}_{\mathsf{Sets}_{*}} = S^{0}$.

8. *Universal Property II.* The symmetric monoidal structure on the category Sets_{*} is the unique symmetric monoidal structure on Sets_{*} such that the free pointed set functor

$$(-)^{+} \colon \mathsf{Sets} \to \mathsf{Sets}_{*}$$

admits a symmetric monoidal structure.

9. *Existence of Monoidal Diagonals.* The triple $\left(\mathsf{Sets}_{*}, \wedge, S^{0}\right)$ is a monoidal category with diagonals:

   (a) *Monoidal Diagonals.* The natural transformation

$$\Delta \colon \mathrm{id}_{\mathsf{Sets}_{*}} \Longrightarrow \wedge \circ \Delta^{\mathsf{Cats}_2}_{\mathsf{Sets}_{*}}, \qquad \begin{tikzcd} \mathsf{Sets}_{*} \arrow[rr, "\mathrm{id}_{\mathsf{Sets}_{*}}"] \arrow[dr, "\Delta^{\mathsf{Cats}_2}_{\mathsf{Sets}_{*}}"'] & {} \arrow[d, phantom, "\Downarrow \Delta"] & \mathsf{Sets}_{*} \\ & \mathsf{Sets}_{*} \times \mathsf{Sets}_{*}, \arrow[ur, "\wedge"'] & \end{tikzcd}$$

whose component

$$\Delta_X \colon (X, x_0) \to (X \wedge X, [(x_0, x_0)])$$

at $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$ is given by the composition

$$\begin{aligned}
(X, x_0) &\xrightarrow{\Delta_X} (X \times X, (x_0, x_0)) \\
&\twoheadrightarrow \left(\tfrac{X \times X}{X \vee X}, [(x_0, x_0)]\right) \\
&\overset{\text{def}}{=\!=} (X \wedge X, [(x_0, x_0)])
\end{aligned}$$

in Sets_*, is a monoidal natural transformation:

i. *Naturality.* For each morphism $f \colon X \to Y$ of pointed sets, the diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\Delta_X"'] & Y \arrow[d, "\Delta_Y"] \\
X \wedge X \arrow[r, "f \wedge f"'] & Y \wedge Y
\end{tikzcd}$$

commutes.

ii. *Compatibility With Strong Monoidality Constraints.* For each $(X, x_0), (Y, y_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, the diagram

$$\begin{tikzcd}
X \wedge Y \arrow[r, "\Delta_X \wedge \Delta_Y"] \arrow[d, equal] & (X \wedge X) \wedge (Y \wedge Y) \arrow[d, dashed, "\wr"] \\
X \wedge Y \arrow[r, "\Delta_{X \wedge Y}"'] & (X \wedge Y) \wedge (X \wedge Y)
\end{tikzcd}$$

commutes.

iii. *Compatibility With Strong Unitality Constraints.* The diagram

$$\begin{tikzcd}
S^0 \arrow[d, equal] \arrow[dr, "(\lambda^{\mathsf{Sets}_*}_{S^0})^{-1} = (\rho^{\mathsf{Sets}_*}_{S^0})^{-1}"] & \\
S^0 \arrow[r, "\Delta_{S^0}"'] & S^0 \wedge S^0
\end{tikzcd}$$

commutes.

(b) *The Diagonal of the Unit.* The component

$$\Delta^{\mathsf{Sets}_*}_{S^0}\colon S^0 \xrightarrow{\cong} S^0 \wedge S^0$$

of the diagonal natural transformation of Sets_* at S^0 is an isomorphism.

10. *Comonoids in* Sets_*. The symmetric monoidal functor

$$\left((-)^+, (-)^{+,\times}, (-)^{+,\times}_{\mathbb{1}}\right)\colon (\mathsf{Sets}, \times, \mathrm{pt}) \longrightarrow \left(\mathsf{Sets}_*, \wedge, S^0\right),$$

of **??** of **??** lifts to an equivalence of categories

$$\begin{aligned}\mathsf{CoMon}\left(\mathsf{Sets}_*, \wedge, S^0\right) &\overset{\text{eq.}}{\cong} \mathsf{CoMon}(\mathsf{Sets}, \times, \mathrm{pt})\\ &\cong \mathsf{Sets}.\end{aligned}$$

[1] In other words, the forgetful functor

$$忘\colon \mathsf{Sets}_* \longrightarrow \mathsf{Sets}$$

defined on objects by sending a pointed set to its underlying set is corepresentable by S^0.

Proof 4.5.1.7 ► Proof of Proposition 4.5.1.6

Item 1: Functoriality

Omitted.

Item 2: Adjointness

Omitted.

Item 3: Closed Symmetric Monoidality

Omitted.

Item 4: Morphisms From the Monoidal Unit

Omitted.

Item 5: Symmetric Strong Monoidality With Respect to Free Pointed Sets

Omitted.

Item 6: Distributivity Over Wedge Sums

This follows from Item 3, **??** of **??**, and the fact that $\vee$ is the coproduct in Sets_*.

Item 7: Universal Property I

Omitted.

Item 8: Universal Property II

See [GGN15, Theorem 5.1].

Item 9: Existence of Monoidal Diagonals

Omitted.

Item 10: Comonoids in Sets_*

See [PS19, Lemma 2.4].

Appendices

4.A Other Chapters

Chapter 5

Relations

This chapter contains some material about relations and constructions with them. Notably, we discuss and explore:

1. The definition of relations (Section 5.1.1).
2. How relations may be viewed as decategorification of profunctors (Section 5.1.2).
3. The various kind of categories that relations form, namely:
   (a) A category (Section 5.2.1),
   (b) A monoidal category (Section 5.2.2),
   (c) A 2-category (Section 5.2.3), and
   (d) A double category (Section 5.2.4).
4. The various categorical properties of the 2-category of relations, including (Section 5.2.5):
   (a) The self-duality of Rel and **Rel** (Items 1 and 2 of Proposition 5.2.5.1);
   (b) Identifications of equivalences and isomorphisms in **Rel** with bijections (Item 3 of Proposition 5.2.5.1);
   (c) Identifications of adjunctions in **Rel** with functions (Item 4 of Proposition 5.2.5.1);
   (d) Identifications of monads in **Rel** with preorders (Item 5 of Proposition 5.2.5.1);
   (e) Identifications of comonads in **Rel** with subsets (Item 6 of Proposition 5.2.5.1);
   (f) Characterisations of monomorphisms in Rel (Item 7 of Proposition 5.2.5.1);
   (g) Characterisations of epimorphisms in Rel (Item 8 of Proposition 5.2.5.1);
   (h) The partial co/completeness of Rel (Item 10 of Proposition 5.2.5.1);
   (i) The existence of right Kan extensions and right Kan lifts in Rel (Items 11 and 12 of Proposition 5.2.5.1);
   (j) The closedness of **Rel** (Item 13 of Proposition 5.2.5.1).

5. The various kinds of constructions involving relations, such as graphs, domains, ranges, unions, intersections, products, inverse relations, composition of relations, and collages (Section 5.3).

6. Equivalence relations (Section 5.4) and quotient sets (Section 5.4.5).

7. The adjoint pairs

$$R_* \dashv R_{-1} \colon \mathcal{P}(A) \rightleftarrows \mathcal{P}(B),$$
$$R^{-1} \dashv R_! \colon \mathcal{P}(B) \rightleftarrows \mathcal{P}(A)$$

of functors (morphisms of posets) between $\mathcal{P}(A)$ and $\mathcal{P}(B)$ induced by a relation $R \colon A \nrightarrow B$, as well as the properties of R_*, R_{-1}, R^{-1}, and $R_!$ (Section 5.5).

Here we also note that:

(a) These two pairs of adjoint functors are the counterpart for relations of the adjoint triple $f_* \dashv f^{-1} \dashv f_!$ induced by a function $f \colon A \to B$ studied in **??**;

(b) We have $R_{-1} = R^{-1}$ iff R is total and functional (Item 8 of Proposition 5.5.2.4).

(c) As a consequence of the previous item, when R comes from a function f the pair of adjunctions

$$R_* \dashv R_{-1} = R^{-1} \dashv R_!$$

reduces to the triple adjunction

$$f_* \dashv f^{-1} \dashv f_!$$

from before.

(d) The pairs $R_* \dashv R_{-1}$ and $R^{-1} \dashv R_!$ later make an appearance in the context of continuous, open, and closed relations between topological spaces (**??**).

8. A notion of *relative preorder* that is to that of a preorder as relative monads are to monads, extending the identifications of monads in Rel with preorders of Item 5 of Proposition 5.2.5.1 to "relative monads in Rel".

Contents

5.1 Relations

5.1.1 Foundations

Let A and B be sets.

Definition 5.1.1.1 ► Relations

A **relation** $R\colon A \nrightarrow B$ **from** A **to** B[1,2] is a subset R of $A \times B$.[3]

[1] *Further Terminology:* Also called a **multivalued function from** A **to** B, a **relation over** A **and** B, **relation on** A **and** B, a **binary relation over** A **and** B, or a **binary relation on** A **and** B.

[2] *Further Terminology:* When $A = B$, we also call $R \subset A \times A$ a **relation on** A.

[3] *Further Notation:* Given elements $a \in A$ and $b \in B$, we write $a \sim_R b$ to mean $(a, b) \in R$.

Definition 5.1.1.2 ► The Po/Set of Relations Over Two Sets

Let A and B be sets.

1. The **set of relations from** A **to** B is the set $\mathrm{Rel}(A, B)$ defined by
$$\mathrm{Rel}(A, B) \overset{\text{def}}{=} \{\text{Relations from } A \text{ to } B\}.$$

2. The **poset of relations from** A **to** B is the poset
$$\mathbf{Rel}(A, B) \overset{\text{def}}{=} (\mathrm{Rel}(A, B), \subset)$$
consisting of
   - *The Underlying Set.* The set $\mathrm{Rel}(A, B)$ of Item 1;
   - *The Partial Order.* The partial order
   $$\subset\colon \mathrm{Rel}(A, B) \times \mathrm{Rel}(A, B) \to \{\text{true}, \text{false}\}$$
   on $\mathrm{Rel}(A, B)$ given by inclusion of relations.

Remark 5.1.1.3 ► Equivalent Definitions of Relations

A relation from A to B is equivalently:[1]

1. A subset of $A \times B$;
2. A function from $A \times B$ to $\{\text{true}, \text{false}\}$;
3. A function from A to $\mathcal{P}(B)$;
4. A function from B to $\mathcal{P}(A)$;

5. A cocontinuous morphism of posets from $(\mathcal{P}(A), \subset)$ to $(\mathcal{P}(B), \subset)$.

That is: we have bijections of sets

$$\begin{aligned}\mathsf{Rel}(A, B) &\stackrel{\text{def}}{=} \mathcal{P}(A \times B),\\ &\cong \mathsf{Sets}(A \times B, \{\mathsf{true}, \mathsf{false}\}),\\ &\cong \mathsf{Sets}(A, \mathcal{P}(B)),\\ &\cong \mathsf{Sets}(B, \mathcal{P}(A)),\\ &\cong \mathrm{Hom}^{\mathrm{cocont}}_{\mathsf{Pos}}(\mathcal{P}(A), \mathcal{P}(B)),\end{aligned}$$

natural in $A, B \in \mathrm{Obj}(\mathsf{Sets})$.

[1] *Intuition:* In particular, we may think of a relation $R\colon A \to \mathcal{P}(B)$ from A to B as a multivalued function from A to B (including the possibility of a given $a \in A$ having no value at all).

Proof 5.1.1.4 ▶ Proof of Remark 5.1.1.3

We claim that Items 1 to 5 are indeed equivalent:

- *Item 1* $\iff$ *Item 2*: This is a special case of **??** of **??**.
- *Item 2* $\iff$ *Item 3*: This is an instance of currying, following from the bijections

  $$\begin{aligned}\mathsf{Sets}(A \times B, \{\mathsf{true}, \mathsf{false}\}) &\cong \mathsf{Sets}(A, \mathsf{Sets}(B, \{\mathsf{true}, \mathsf{false}\}))\\ &\cong \mathsf{Sets}(A, \mathcal{P}(B)),\end{aligned}$$

  where the last bijection is from **??** of **??**.
- *Item 2* $\iff$ *Item 4*: This is also an instance of currying, following from the bijections

  $$\begin{aligned}\mathsf{Sets}(A \times B, \{\mathsf{true}, \mathsf{false}\}) &\cong \mathsf{Sets}(B, \mathsf{Sets}(B, \{\mathsf{true}, \mathsf{false}\}))\\ &\cong \mathsf{Sets}(B, \mathcal{P}(A)),\end{aligned}$$

  where again the last bijection is from **??** of **??**.
- *Item 2* $\iff$ *Item 5*: This follows from the universal property of the powerset $\mathcal{P}(X)$ of a set X as the free cocompletion of X via the characteristic embedding

  $$\chi_X \colon X \hookrightarrow \mathcal{P}(X)$$

  of X into $\mathcal{P}(X)$ (**??** of **??**).

In particular, the bijection

$$\mathsf{Rel}(A, B) \cong \mathsf{Hom}^{\mathsf{cocont}}_{\mathsf{Pos}}(\mathcal{P}(A), \mathcal{P}(B))$$

is given by taking a relation $R\colon A \nrightarrow B$, passing to its associated function $f\colon A \to \mathcal{P}(B)$ from A to B and then extending f from A to all of $\mathcal{P}(A)$ by taking its left Kan extension along χ_X.

This coincides with the direct image function $f_*\colon \mathcal{P}(A) \to \mathcal{P}(B)$ of **??**.

This finishes the proof. □

Proposition 5.1.1.5 ► Properties of Relations

Let A and B be sets.

1. *End Formula for The Poset of Relations.* Let $R, S\colon A \nrightarrow B$ be relations. We have

$$\mathsf{Hom}_{\mathbf{Rel}(A,B)}(R, S) \cong \int_{a\in A}\int_{b\in B} \mathsf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R^a_b, S^a_b).$$

Proof 5.1.1.6 ► Proof of Proposition 5.1.1.5

Item 1: End Formula for The Poset of Relations

Unwinding the expression inside the end on the right hand side, we have

$$\int_{a\in A}\int_{b\in B} \mathsf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R^a_b, S^a_b) \cong \begin{cases} \mathrm{pt} & \text{if for each } (a,b) \in A \times B, \\ & \text{if } a \sim_R b, \text{ then } a \sim_S b, \\ \emptyset & \text{otherwise.} \end{cases}$$

On the left hand-side, we have

$$\mathsf{Hom}_{\mathbf{Rel}(A,B)}(R, S) \cong \begin{cases} \mathrm{pt} & \text{if } R \subset S, \\ \emptyset & \text{otherwise.} \end{cases}$$

It is then clear that the conditions for each set to evaluate to pt are equivalent, implying that those two sets are isomorphic, finishing the proof. □

5.1.2 Relations as Decategorifications of Profunctors

Remark 5.1.2.1 ▶ Relations as Decategorifications of Profunctors I

The notion of a relation is a decategorification of that of a profunctor:

1. A profunctor from a category $\mathcal{C}$ to a category $\mathcal{D}$ is a functor

$$\mathfrak{p}\colon \mathcal{D}^{\mathsf{op}} \times \mathcal{C} \to \mathsf{Sets}.$$

2. A relation on sets A and B is a function

$$R\colon A \times B \to \{\mathsf{true}, \mathsf{false}\}.$$

Here we notice that:

- The opposite X^{op} of a set X is itself, as $(-)^{\mathsf{op}}\colon \mathsf{Cats} \to \mathsf{Cats}$ restricts to the identity endofunctor on Sets;
- The values that profunctors and relations take are directly related in relation to decategorification:
  - A category is enriched over the category

    $$\mathsf{Sets} \overset{\mathrm{def}}{=} \mathsf{Cats}_0$$

    of sets, with profunctors taking values on it;
  - A set is enriched over the set

    $$\{\mathsf{true}, \mathsf{false}\} \overset{\mathrm{def}}{=} \mathsf{Cats}_{-1}$$

    of classical truth values, with relations taking values on it;

Remark 5.1.2.2 ▶ Relations as Decategorifications of Profunctors II

Extending Remark 5.1.2.1, the equivalent definitions of relations in Remark 5.1.1.3 are also related to the corresponding ones for profunctors (**??**), which state that a profunctor $\mathfrak{p}\colon \mathcal{C} \nrightarrow \mathcal{D}$ is equivalently:

1. A functor $\mathfrak{p}\colon \mathcal{D}^{\mathsf{op}} \times \mathcal{C} \to \mathsf{Sets}$;
2. A functor $\mathfrak{p}\colon \mathcal{C} \to \mathsf{PSh}(\mathcal{D})$;
3. A functor $\mathfrak{p}\colon \mathcal{D}^{\mathsf{op}} \to \mathsf{Fun}(\mathcal{C}, \mathsf{Sets})$;
4. A colimit-preserving functor $\mathfrak{p}\colon \mathsf{PSh}(\mathcal{C}) \to \mathsf{PSh}(\mathcal{D})$.

Indeed:

- The equivalence between Items 1 and 2 (and also that between Items 1 and 3, which is proved analogously) is an instance of currying, both for profunctors as well as for relations, using the isomorphisms

$$\begin{aligned}\mathsf{Sets}(A \times B, \{\mathsf{true}, \mathsf{false}\}) &\cong \mathsf{Sets}(A, \mathsf{Sets}(B, \{\mathsf{true}, \mathsf{false}\}))\\ &\cong \mathsf{Sets}(A, \mathcal{P}(B)),\\ \mathsf{Fun}(\mathcal{D}^{\mathsf{op}} \times \mathcal{D}, \mathsf{Sets}) &\cong \mathsf{Fun}(\mathcal{C}, \mathsf{Fun}(\mathcal{D}^{\mathsf{op}}, \mathsf{Sets}))\\ &\cong \mathsf{Fun}(\mathcal{C}, \mathsf{PSh}(\mathcal{D})).\end{aligned}$$

- The equivalence between Items 1 and 3 follows from the universal properties of:
  - The powerset $\mathcal{P}(X)$ of a set X as the free cocompletion of X via the characteristic embedding

    $$\chi_{(-)} \colon X \hookrightarrow \mathcal{P}(X)$$

    of X into $\mathcal{P}(X)$ (**??** of **??**);
  - The category $\mathsf{PSh}(\mathcal{C})$ of presheaves on a category $\mathcal{C}$ as the free cocompletion of $\mathcal{C}$ via the Yoneda embedding

    $$よ \colon \mathcal{C} \hookrightarrow \mathsf{PSh}(\mathcal{C})$$

    of $\mathcal{C}$ into $\mathsf{PSh}(\mathcal{C})$ (**??** of **??**).

5.1.3 Examples of Relations

EXAMPLE 5.1.3.1 ▶ THE TRIVIAL RELATION

The **trivial relation on A and B** is the relation $\sim_{\mathrm{triv}}$ defined by[1,2,3]

$$\sim_{\mathrm{triv}} \overset{\mathrm{def}}{=} A \times A.$$

[1]This is the unique relation R on A and B such that we have $a \sim_R b$ for all $a \in A$ and all $b \in B$.

[2]As a function from $A \times A$ to $\{\mathsf{true}, \mathsf{false}\}$, the relation $\sim_{\mathrm{triv}}$ is the constant function

$$\Delta_{\mathsf{true}} \colon A \times B \to \{\mathsf{true}, \mathsf{false}\}$$

from $A \times B$ to $\{\mathsf{true}, \mathsf{false}\}$ taking value true.

[3]As a function from A to $\mathcal{P}(B)$, the relation $\sim_{\mathrm{triv}}$ is the function

$$\Delta_{\mathsf{true}} \colon A \to \mathcal{P}(B)$$

defined by

$$\Delta_{\mathsf{true}}(a) \overset{\mathrm{def}}{=} B$$

for each $a \in A$.

Example 5.1.3.2 ▶ The Cotrivial Relation

The **cotrivial relation on A and B** is the relation $\sim_{\text{cotriv}}$ defined by[1,2,3]

$$\sim_{\text{cotriv}} \overset{\text{def}}{=} \emptyset.$$

[1]This is the unique relation R on A and B such that we have $a \sim_R b$ for no $a \in A$ and no $b \in B$.

[2]As a function from $A \times B$ to $\{\text{true}, \text{false}\}$, the relation $\sim_{\text{cotriv}}$ is the constant function

$$\Delta_{\text{false}} \colon A \times B \to \{\text{true}, \text{false}\}$$

from $A \times B$ to $\{\text{true}, \text{false}\}$ taking value false.

[3]As a function from A to $\mathcal{P}(A)$, the relation $\sim_{\text{cotriv}}$ is the function

$$\Delta_{\text{false}} \colon A \to \mathcal{P}(A)$$

defined by

$$\Delta_{\text{true}}(a) \overset{\text{def}}{=} \emptyset$$

for each $a \in A$.

Example 5.1.3.3 ▶ The Characteristic Relation of a Set

The characteristic relation on A of **??** of **??** is another example of a relation. It is in fact the unique relation on A making the following conditions equivalent, for each $a, b \in A$:

1. We have $a \sim_{\text{id}} b$.
2. We have $a = b$.

Example 5.1.3.4 ▶ Square Roots

Square roots are examples of relations:

1. *Square Roots in $\mathbb{R}$*. The assignment $x \mapsto \sqrt{x}$ defines a relation

   $$\sqrt{-} \colon \mathbb{R} \to \mathcal{P}(\mathbb{R})$$

   from $\mathbb{R}$ to itself, being explicitly given by

   $$\sqrt{x} \overset{\text{def}}{=} \begin{cases} 0 & \text{if } x = 0, \\ \left\{-\sqrt{|x|}, \sqrt{|x|}\right\} & \text{if } x \neq 0. \end{cases}$$

2. *Square Roots in $\mathbb{Q}$*. Square roots in $\mathbb{Q}$ are similar to square roots in $\mathbb{R}$, though now additionally it may also occur that $\sqrt{-} \colon \mathbb{Q} \to \mathcal{P}(\mathbb{Q})$ sends a rational number x (e.g. 2) to the empty set (since $\sqrt{2} \notin \mathbb{Q}$).

Example 5.1.3.5 ▶ Complex Logarithms

The complex logarithm defines a relation

$$\log\colon \mathbb{C} \to \mathcal{P}(\mathbb{C})$$

from $\mathbb{C}$ to itself, where we have

$$\log(a+bi) \overset{\text{def}}{=} \left\{\log\left(\sqrt{a^2+b^2}\right) + i\arg(a+bi) + (2\pi i)k \,\middle|\, k \in \mathbb{Z}\right\}$$

for each $a + bi \in \mathbb{C}$.

Example 5.1.3.6 ▶ More Examples of Relations

See [**wikipedia:multivalued-functions**] for more examples of relations, such as antiderivation, inverse trigonometric functions, and inverse hyperbolic functions.

5.1.4 Functional Relations

Let A and B be sets.

Definition 5.1.4.1 ▶ Functional Relations

A relation $R\colon A \nrightarrow B$ is **functional** if, for each $a \in A$, the set $R(a)$ is either empty or a singleton.

Proposition 5.1.4.2 ▶ Properties of Functional Relations

Let $R\colon A \nrightarrow B$ be a relation.

1. *Characterisations.* The following conditions are equivalent:
   (a) The relation R is functional.
   (b) We have $R \diamond R^{\dagger} \subset \chi_B$.

Proof 5.1.4.3 ▶ Proof of Proposition 5.1.4.2

Item 1: Characterisations

We claim that Items 1a and 1b are indeed equivalent:

- *Item 1a* $\implies$ *Item 1b*: Let $(b, b') \in B \times B$. We need to show that

$$[R \diamond R^{\dagger}](b,b') \preceq_{\{\mathsf{t},\mathsf{f}\}} \chi_B(b,b'),$$

i.e. that if there exists some $a \in A$ such that $b \sim_{R^\dagger} a$ and $a \sim_R b'$, then $b = b'$. But since $b \sim_{R^\dagger} a$ is the same as $a \sim_R b$, we have both $a \sim_R b$ and $a \sim_R b'$ at the same time, which implies $b = b'$ since R is functional.

- *Item 1b* $\implies$ *Item 1a*: Suppose that we have $a \sim_R b$ and $a \sim_R b'$ for $b, b' \in B$. We claim that $b = b'$:

  1. Since $a \sim_R b$, we have $b \sim_{R^\dagger} a$.
  2. Since $R \diamond R^\dagger \subset \chi_B$, we have

  $$[R \diamond R^\dagger](b, b') \preceq_{\{\mathsf{t},\mathsf{f}\}} \chi_B(b, b'),$$

  and since $b \sim_{R^\dagger} a$ and $a \sim_R b'$, it follows that $[R \diamond R^\dagger](b, b') = \mathsf{true}$, and thus $\chi_B(b, b') = \mathsf{true}$ as well, i.e. $b = b'$.

This finishes the proof. ∎

5.1.5 Total Relations

Let A and B be sets.

Definition 5.1.5.1 ► Total Relations

A relation $R \colon A \nrightarrow B$ is **total** if, for each $a \in A$, we have $R(a) \neq \emptyset$.

Proposition 5.1.5.2 ► Properties of Total Relations

Let $R \colon A \nrightarrow B$ be a relation.

1. *Characterisations.* The following conditions are equivalent:

   (a) The relation R is total.

   (b) We have $\chi_A \subset R^\dagger \diamond R$.

Proof 5.1.5.3 ► Proof of Proposition 5.1.5.2

Item 1: Characterisations

We claim that Items 1a and 1b are indeed equivalent:

- *Item 1a* $\implies$ *Item 1b*: We have to show that, for each $(a, a') \in A$, we have

  $$\chi_A(a, a') \preceq_{\{\mathsf{t},\mathsf{f}\}} [R^\dagger \diamond R](a, a'),$$

  i.e. that if $a = a'$, then there exists some $b \in B$ such that $a \sim_R b$ and

$b \sim_{R^\dagger} a'$ (i.e. $a \sim_R b$ again), which follows from the totality of R.

- *Item 1b* $\Longrightarrow$ *Item 1a*: Given $a \in A$, since $\chi_A \subset R^\dagger \diamond R$, we must have

$$\{a\} \subset [R^\dagger \diamond R](a),$$

implying that there must exist some $b \in B$ such that $a \sim_R b$ and $b \sim_{R^\dagger} a$ (i.e. $a \sim_R b$) and thus $R(a) \neq \emptyset$, as $b \in R(a)$.

This finishes the proof. □

5.2 Categories of Relations

5.2.1 The Category of Relations

Definition 5.2.1.1 ▶ The Category of Relations

The **category of relations** is the category Rel where

- *Objects.* The objects of Rel are sets;
- *Morphisms.* For each $A, B \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$\mathrm{Rel}(A, B) \overset{\mathrm{def}}{=} \mathrm{Rel}(A, B);$$

- *Identities.* For each $A \in \mathrm{Obj}(\mathsf{Rel})$, the unit map

$$\Bbbk^{\mathsf{Rel}}_A \colon \mathrm{pt} \longrightarrow \mathrm{Rel}(A, A)$$

of Rel at A is defined by

$$\mathrm{id}^{\mathsf{Rel}}_A \overset{\mathrm{def}}{=} \chi_A(-_1, -_2),$$

where $\chi_A(-_1, -_2)$ is the characteristic relation of A of **??** of **??**;

- *Composition.* For each $A, B, C \in \mathrm{Obj}(\mathsf{Rel})$, the composition map

$$\circ^{\mathsf{Rel}}_{A,B,C} \colon \mathrm{Rel}(B, C) \times \mathrm{Rel}(A, B) \longrightarrow \mathrm{Rel}(A, C)$$

of Rel at (A, B, C) is defined by

$$S \circ^{\mathsf{Rel}}_{A,B,C} R \overset{\mathrm{def}}{=} S \diamond R$$

for each $(S, R) \in \mathrm{Rel}(B, C) \times \mathrm{Rel}(A, B)$, where $S \diamond R$ is the composition of S and R of Definition 5.3.12.1.

5.2.2 The Closed Symmetric Monoidal Category of Relations

5.2.2.1 The Monoidal Product

Definition 5.2.2.1 ▶ The Monoidal Product of Rel

The **monoidal product of** Rel is the functor

$$\times \colon \mathsf{Rel} \times \mathsf{Rel} \longrightarrow \mathsf{Rel}$$

where

- *Action on Objects.* We have

$$\times(A, B) \stackrel{\text{def}}{=} A \times B,$$

  where $A \times B$ is the Cartesian product of sets of **??**;

- *Action on Morphisms.* For each $(A, C), (B, D) \in \mathrm{Obj}(\mathsf{Rel} \times \mathsf{Rel})$, the action on morphisms

$$\times_{(A,C),(B,D)} \colon \mathsf{Rel}(A, B) \times \mathsf{Rel}(C, D) \longrightarrow \mathsf{Rel}(A \times C, B \times D)$$

  of $\times$ is given by sending a pair of morphisms (R, S) of the form

$$R \colon A \nrightarrow B,$$
$$S \colon C \nrightarrow D$$

  to the relation

$$R \times S \colon A \times C \nrightarrow B \times D$$

  of Definition 5.3.9.1.

5.2.2.2 The Monoidal Unit

Definition 5.2.2.2 ▶ The Monoidal Unit of Rel

The **monoidal unit of** Rel is the functor

$$\mathbb{1}^{\mathsf{Rel}} \colon \mathrm{pt} \longrightarrow \mathsf{Rel}$$

picking the set

$$\mathbb{1}_{\mathsf{Rel}} \stackrel{\text{def}}{=} \mathrm{pt}$$

of Rel.

5.2.2.3 The Associator

Definition 5.2.2.3 ► The Associator of Rel

The **associator of** Rel is the natural isomorphism

$$\alpha^{\mathsf{Rel}}\colon \times\circ((\times)\times\mathrm{id}) \overset{\cong}{\Longrightarrow} \times\circ(\mathrm{id}\times(\times)), \qquad \begin{tikzcd} \mathsf{Rel}\times\mathsf{Rel}\times\mathsf{Rel} \arrow[r, "\mathrm{id}\times(\times)"] \arrow[d, "(\times)\times\mathrm{id}"'] & \mathsf{Rel}\times\mathsf{Rel} \arrow[d, "\times"] \\ \mathsf{Rel}\times\mathsf{Rel} \arrow[r, "\times"'] \arrow[ur, phantom, "\Nearrow \alpha^{\mathsf{Rel}}"] & \mathsf{Rel}, \end{tikzcd}$$

whose component

$$\alpha^{\mathsf{Rel}}_{A,B,C}\colon (A\times B)\times C \nrightarrow A\times(B\times C)$$

at (A,B,C) is defined by declaring

$$((a,b),c) \sim_{\alpha^{\mathsf{Rel}}_{A,B,C}} (a',(b',c'))$$

iff $a=a'$, $b=b'$, and $c=c'$.

5.2.2.4 The Left Unitor

Definition 5.2.2.4 ► The Left Unitor of Rel

The **left unitor of** Rel is the natural isomorphism

$$\lambda^{\mathsf{Rel}}\colon \times\circ(\mathbb{1}^{\mathsf{Rel}}\times\mathrm{id}) \overset{\cong}{\Longrightarrow} \lambda^{\mathsf{Cats}_2}_{\mathsf{Rel}}, \qquad \begin{tikzcd} \mathrm{pt}\times\mathsf{Rel} \arrow[r, "\mathbb{1}^{\mathsf{Rel}}\times\mathrm{id}"] \arrow[dr, dashed, bend right, "\lambda^{\mathsf{Cats}_2}_{\mathsf{Rel}}"'] & \mathsf{Rel}\times\mathsf{Rel} \arrow[d, "\times"] \arrow[dl, phantom, "\Swarrow \lambda^{\mathsf{Rel}}"] \\ & \mathsf{Rel}, \end{tikzcd}$$

whose component

$$\lambda^{\mathsf{Rel}}_A\colon \mathbb{1}_{\mathsf{Rel}}\times A \nrightarrow A$$

at A is defined by declaring

$$(\star,a) \sim_{\lambda^{\mathsf{Rel}}_A} b$$

iff $a=b$.

5.2.2.5 The Right Unitor

DEFINITION 5.2.2.5 ► THE RIGHT UNITOR OF Rel

The **right unitor of** Rel is the natural isomorphism

$$\rho^{\mathsf{Rel}} \colon \times \circ (\mathrm{id} \times \mathbb{1}^{\mathsf{Rel}}) \overset{\cong}{\Longrightarrow} \boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathsf{Rel}}, \qquad \begin{tikzcd} \mathsf{Rel} \times \mathrm{pt} \arrow[r, "\mathrm{id} \times \mathbb{1}^{\mathsf{Rel}}"] \arrow[dr, dashed, bend right, "\boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathsf{Rel}}"'] & \mathsf{Rel} \times \mathsf{Rel} \arrow[d, "\times"] \arrow[dl, phantom, "\overset{\rho^{\mathsf{Rel}}}{\Swarrow}"] \\ & \mathsf{Rel}, \end{tikzcd}$$

whose component

$$\rho^{\mathsf{Rel}}_A \colon A \times \mathbb{1}_{\mathsf{Rel}} \nrightarrow A$$

at A is defined by declaring

$$(a, \star) \sim_{\rho^{\mathsf{Rel}}_A} b$$

iff $a = b$.

5.2.2.6 The Symmetry

DEFINITION 5.2.2.6 ► THE SYMMETRY OF Rel

The **symmetry of** Rel is the natural isomorphism

$$\sigma^{\mathsf{Rel}} \colon \times \Longrightarrow \times \circ \boldsymbol{\sigma}^{\mathsf{Cats}_2}_{\mathsf{Rel},\mathsf{Rel}}, \qquad \begin{tikzcd} \mathsf{Rel} \times \mathsf{Rel} \arrow[r, "\times"] \arrow[dr, bend right, "\boldsymbol{\sigma}^{\mathsf{Cats}_2}_{\mathsf{Rel},\mathsf{Rel}}"'] & \mathsf{Rel}, \\ & \mathsf{Rel} \times \mathsf{Rel} \arrow[u, bend right, "\times"'] \arrow[u, phantom, "\overset{\sigma^{\mathsf{Rel}}}{\Downarrow}"] \end{tikzcd}$$

whose component

$$\sigma^{\mathsf{Rel}}_{A,B} \colon A \times B \longrightarrow B \times A$$

at (A, B) is defined by declaring

$$(a, b) \sim_{\sigma^{\mathsf{Rel}}_{A,B}} (b', a')$$

iff $a = a'$ and $b = b'$.

5.2.2.7 The Internal Hom

Definition 5.2.2.7 ▶ The Internal Hom of Rel

The **internal Hom of** Rel is the functor

$$\mathbf{Hom}_{\mathsf{Rel}} \colon \mathsf{Rel}^{\mathsf{op}} \times \mathsf{Rel} \to \mathsf{Rel}$$

defined by

$$\mathbf{Hom}_{\mathsf{Rel}}(A, B) \stackrel{\text{def}}{=} A \times B$$

for each $A, B \in \mathrm{Obj}(\mathsf{Rel})$.

Proposition 5.2.2.8 ▶ Properties of the Internal Hom of Rel

Let $A, B, C \in \mathrm{Obj}(\mathsf{Rel})$.

1. *Via Self-Duality.* The internal Hom $\mathbf{Hom}_{\mathsf{Rel}}$ of Rel is given by the composition

$$\mathsf{Rel}^{\mathsf{op}} \times \mathsf{Rel} \xrightarrow{\cong} \mathsf{Rel} \times \mathsf{Rel} \xrightarrow{\times} \mathsf{Rel},$$

   where the self-duality equivalence $\mathsf{Rel}^{\mathsf{op}} \cong \mathsf{Rel}$ comes from Item 1 of Proposition 5.2.5.1.

2. *Adjointness.* We have adjunctions

$$(A \times - \dashv \mathbf{Hom}_{\mathsf{Rel}}(A, -)) \colon \quad \begin{tikzcd} \mathsf{Rel} \arrow[r, bend left, "A\times -"] \arrow[r, phantom, "\perp"] & \mathsf{Rel}, \arrow[l, bend left, "\mathbf{Hom}_{\mathsf{Rel}}(A,-)"] \end{tikzcd}$$

$$(- \times B \dashv \mathbf{Hom}_{\mathsf{Rel}}(B, -)) \colon \quad \begin{tikzcd} \mathsf{Rel} \arrow[r, bend left, "-\times B"] \arrow[r, phantom, "\perp"] & \mathsf{Rel}, \arrow[l, bend left, "\mathbf{Hom}_{\mathsf{Rel}}(B,-)"] \end{tikzcd}$$

   witnessed by bijections

$$\begin{aligned} \mathsf{Rel}(A \times B, C) &\cong \mathsf{Rel}(A, \mathbf{Hom}_{\mathsf{Rel}}(B, C)) \\ &\stackrel{\text{def}}{=} \mathsf{Rel}(A, B \times C), \\ \mathsf{Rel}(A \times B, C) &\cong \mathsf{Rel}(B, \mathbf{Hom}_{\mathsf{Rel}}(A, C)) \\ &\stackrel{\text{def}}{=} \mathsf{Rel}(B, A \times C), \end{aligned}$$

   natural in $A, B, C \in \mathrm{Obj}(\mathsf{Rel})$.

Proof 5.2.2.9 ► Proof of Proposition 5.2.2.8

Item 1: Via Self-Duality

Omitted.

Item 2: Adjointness

Indeed, we have

$$\begin{aligned}\mathsf{Rel}(A \times B, C) &\stackrel{\text{def}}{=} \mathsf{Sets}(A \times B \times C, \{\mathsf{true}, \mathsf{false}\})\\ &\stackrel{\text{def}}{=} \mathsf{Rel}(A, B \times C)\\ &\stackrel{\text{def}}{=} \mathsf{Rel}(A, \mathbf{Hom}_{\mathsf{Rel}}(B, C)),\end{aligned}$$

and similarly for the bijection $\mathsf{Rel}(A \times B, C) \cong \mathsf{Rel}(B, \mathbf{Hom}_{\mathsf{Rel}}(A, C))$. □

5.2.2.8 The Closed Symmetric Monoidal Category of Relations

Definition 5.2.2.10 ► The Closed Symmetric Monoidal Category of Relations

The **closed symmetric monoidal category of relations** is the closed symmetric monoidal category

$$\left(\mathsf{Rel}, \times, \mathbb{1}_{\mathsf{Rel}}, \alpha^{\mathsf{Rel}}, \lambda^{\mathsf{Rel}}, \rho^{\mathsf{Rel}}, \sigma^{\mathsf{Rel}}, \mathbf{Hom}_{\mathsf{Rel}}\right)$$

consisting of

- *The Underlying Category.* The category Rel of sets and relations of Definition 5.2.1.1;
- *The Monoidal Product.* The functor

  $$\times\colon \mathsf{Rel} \times \mathsf{Rel} \to \mathsf{Rel}$$

  of Definition 5.2.2.1;
- *The Monoidal Unit.* The functor $\mathbb{1}^{\mathsf{Rel}}$ of Definition 5.2.2.2;
- *The Associator.* The natural isomorphism α^{Rel} of Definition 5.2.2.3;
- *The Left Unitor.* The natural isomorphism λ^{Rel} of Definition 5.2.2.4;
- *The Right Unitor.* The natural isomorphism ρ^{Rel} of Definition 5.2.2.5;
- *The Symmetry.* The natural isomorphism σ^{Rel} of Definition 5.2.2.6;
- *The Internal Hom.* The functor

  $$\mathbf{Hom}_{\mathsf{Rel}}\colon \mathsf{Rel}^{\mathsf{op}} \times \mathsf{Rel} \to \mathsf{Rel}$$

  of Definition 5.2.2.7.

5.2.3 The 2-Category of Relations

Definition 5.2.3.1 ► The 2-Category of Relations

The 2-**category of relations** is the locally posetal 2-category **Rel** where

- *Objects.* The objects of **Rel** are sets;
- ***Hom**-Objects.* For each $A, B \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$\begin{aligned}\mathrm{Hom}_{\mathbf{Rel}}(A, B) &\stackrel{\text{def}}{=} \mathbf{Rel}(A, B)\\ &\stackrel{\text{def}}{=} (\mathrm{Rel}(A, B), \subset);\end{aligned}$$

- *Identities.* For each $A \in \mathrm{Obj}(\mathbf{Rel})$, the unit map

$$\mathbb{1}^{\mathbf{Rel}}_{A} \colon \mathrm{pt} \to \mathbf{Rel}(A, A)$$

  of **Rel** at A is defined by

$$\mathrm{id}^{\mathbf{Rel}}_{A} \stackrel{\text{def}}{=} \chi_A(-_1, -_2),$$

  where $\chi_A(-_1, -_2)$ is the characteristic relation of A of **??** of **??**;

- *Composition.* For each $A, B, C \in \mathrm{Obj}(\mathbf{Rel})$, the composition map[1]

$$\circ^{\mathbf{Rel}}_{A,B,C} \colon \mathbf{Rel}(B, C) \times \mathbf{Rel}(A, B) \to \mathbf{Rel}(A, C)$$

  of **Rel** at (A, B, C) is defined by

$$S \circ^{\mathbf{Rel}}_{A,B,C} R \stackrel{\text{def}}{=} S \diamond R$$

  for each $(S, R) \in \mathbf{Rel}(B, C) \times \mathbf{Rel}(A, B)$, where $S \diamond R$ is the composition of S and R of Definition 5.3.12.1.

[1] Note that this is indeed a morphism of posets: given relations $R_1, R_2 \in \mathbf{Rel}(A, B)$ and $S_1, S_2 \in \mathbf{Rel}(B, C)$ such that

$$\begin{aligned}R_1 &\subset R_2,\\ S_1 &\subset S_2,\end{aligned}$$

we have also $S_1 \diamond R_1 \subset S_2 \diamond R_2$.

5.2.4 The Double Category of Relations

5.2.4.1 The Double Category of Relations

Definition 5.2.4.1 ► The Double Category of Relations

The **double category of relations** is the locally posetal double category $\mathsf{Rel}^{\mathsf{dbl}}$ where

- *Objects.* The objects of $\mathsf{Rel}^{\mathsf{dbl}}$ are sets;
- *Vertical Morphisms.* The vertical morphisms of $\mathsf{Rel}^{\mathsf{dbl}}$ are maps of sets $f\colon A \to B$;
- *Horizontal Morphisms.* The horizontal morphisms of $\mathsf{Rel}^{\mathsf{dbl}}$ are relations $R\colon A \nrightarrow X$;
- *2-Morphisms.* A 2-cell

$$\begin{tikzcd} A \arrow[r, "R", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & B \arrow[d, "g"] \\ X \arrow[r, "S"', "\shortmid"{marking}] & Y \end{tikzcd}$$

  of $\mathsf{Rel}^{\mathsf{dbl}}$ is either non-existent or an inclusion of relations of the form

$$R \subset S \circ (f \times g), \qquad \begin{tikzcd} A \times B \arrow[r, "R"] \arrow[d, "f\times g"'] \arrow[dr, phantom, "\subset"] & \{\mathrm{true}, \mathrm{false}\} \arrow[d, "\mathrm{id}_{\{\mathrm{true},\mathrm{false}\}}"] \\ X \times Y \arrow[r, "S"'] & \{\mathrm{true}, \mathrm{false}\}; \end{tikzcd}$$

- *Horizontal Identities.* The horizontal unit functor of $\mathsf{Rel}^{\mathsf{dbl}}$ is the functor of Definition 5.2.4.2;
- *Vertical Identities.* For each $A \in \mathrm{Obj}\left(\mathsf{Rel}^{\mathsf{dbl}}\right)$, we have

$$\mathrm{id}_A^{\mathsf{Rel}^{\mathsf{dbl}}} \overset{\mathrm{def}}{=} \mathrm{id}_A;$$

- *Identity 2-Morphisms.* For each horizontal morphism $R\colon A \nrightarrow B$ of $\mathsf{Rel}^{\mathsf{dbl}}$, the identity 2-morphism

$$\begin{tikzcd} A \arrow[r, "R", "\shortmid"{marking}] \arrow[d, "\mathrm{id}_A"'] \arrow[dr, phantom, "\Downarrow\mathrm{id}_R"] & B \arrow[d, "\mathrm{id}_B"] \\ A \arrow[r, "R"', "\shortmid"{marking}] & B \end{tikzcd}$$

of R is the identity inclusion

$$
R \subset R, \qquad
\begin{tikzcd}
B \times A \arrow[r, "R"] \arrow[d, "\mathrm{id}_B \times \mathrm{id}_A"'] \arrow[dr, phantom, "\subset"] & \{\mathrm{true}, \mathrm{false}\} \arrow[d, "\mathrm{id}_{\{\mathrm{true},\mathrm{false}\}}"] \\
B \times A \arrow[r, "R"'] & \{\mathrm{true}, \mathrm{false}\};
\end{tikzcd}
$$

- *Horizontal Composition.* The horizontal composition functor of $\mathsf{Rel}^{\mathsf{dbl}}$ is the functor of Definition 5.2.4.3;
- *Vertical Composition of 1-Morphisms.* For each composable pair $A \xrightarrow{F} B \xrightarrow{G} C$ of vertical morphisms of $\mathsf{Rel}^{\mathsf{dbl}}$, i.e. maps of sets, we have

$$g \circ^{\mathsf{Rel}^{\mathsf{dbl}}} f \stackrel{\text{def}}{=} g \circ f;$$

- *Vertical Composition of 2-Morphisms.* The vertical composition of 2-morphisms in $\mathsf{Rel}^{\mathsf{dbl}}$ is defined as in Definition 5.2.4.4;
- *Associators.* The associators of $\mathsf{Rel}^{\mathsf{dbl}}$ is defined as in Definition 5.2.4.5;
- *Left Unitors.* The left unitors of $\mathsf{Rel}^{\mathsf{dbl}}$ is defined as in Definition 5.2.4.6;
- *Right Unitors.* The right unitors of $\mathsf{Rel}^{\mathsf{dbl}}$ is defined as in Definition 5.2.4.7.

5.2.4.2 Horizontal Identities

Definition 5.2.4.2 ▶ The Horizontal Identities of $\mathsf{Rel}^{\mathsf{dbl}}$

The **horizontal unit functor** of $\mathsf{Rel}^{\mathsf{dbl}}$ is the functor

$$\mathbb{1}^{\mathsf{Rel}^{\mathsf{dbl}}} \colon \mathsf{Rel}^{\mathsf{dbl}}_0 \to \mathsf{Rel}^{\mathsf{dbl}}_1$$

of $\mathsf{Rel}^{\mathsf{dbl}}$ is the functor where

- *Action on Objects.* For each $A \in \mathrm{Obj}\left(\mathsf{Rel}^{\mathsf{dbl}}_0\right)$, we have

$$\mathbb{1}_A \stackrel{\text{def}}{=} \chi_A(-_1, -_2);$$

- *Action on Morphisms.* For each vertical morphism $f \colon A \to B$ of $\mathsf{Rel}^{\mathsf{dbl}}$, i.e.

each map of sets f from A to B, the identity 2-morphism

$$\begin{tikzcd} A \arrow[r, "\chi_A", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow \chi_f"] & A \arrow[d, "f"] \\ B \arrow[r, "\chi_B"', "\shortmid"{marking}] & B \end{tikzcd}$$

of f is the inclusion

$$\chi_B \circ (f \times f) \subset \chi_A, \qquad \begin{tikzcd} A \times A \arrow[r, "\chi_A(-_1,-_2)"] \arrow[d, "f \times f"'] \arrow[dr, phantom, "\subset"] & \{\text{true}, \text{false}\} \arrow[d, "\mathrm{id}_{\{\text{true},\text{false}\}}"] \\ B \times B \arrow[r, "\chi_B(-_1,-_2)"'] & \{\text{true}, \text{false}\} \end{tikzcd}$$

of **??** of **??**.

5.2.4.3 Horizontal Composition

Definition 5.2.4.3 ▶ The Horizontal Composition of $\mathsf{Rel}^{\mathsf{dbl}}$

The **horizontal composition functor** of $\mathsf{Rel}^{\mathsf{dbl}}$ is the functor

$$\odot^{\mathsf{Rel}^{\mathsf{dbl}}} \colon \mathsf{Rel}^{\mathsf{dbl}}_1 \underset{\mathsf{Rel}^{\mathsf{dbl}}_0}{\times} \mathsf{Rel}^{\mathsf{dbl}}_1 \to \mathsf{Rel}^{\mathsf{dbl}}_1$$

of $\mathsf{Rel}^{\mathsf{dbl}}$ is the functor where

- *Action on Objects.* For each composable pair $A \overset{R}{\nrightarrow} B \overset{S}{\nrightarrow} C$ of horizontal morphisms of $\mathsf{Rel}^{\mathsf{dbl}}$, we have

$$S \odot R \overset{\text{def}}{=} S \diamond R,$$

where $S \diamond R$ is the composition of R and S of Definition 5.3.12.1;

- *Action on Morphisms.* For each horizontally composable pair

$$\begin{tikzcd} A \arrow[r, "R", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow \alpha"] & B \arrow[d, "g"] \\ X \arrow[r, "T"', "\shortmid"{marking}] & Y \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "S", "\shortmid"{marking}] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow \beta"] & C \arrow[d, "h"] \\ Y \arrow[r, "U"', "\shortmid"{marking}] & Z \end{tikzcd}$$

of 2-morphisms of $\mathsf{Rel}^{\mathsf{dbl}}$, i.e. for each pair

$$
\begin{tikzcd}
A \times B \arrow[r, "R"] \arrow[d, "f \times g"'] \arrow[dr, phantom, "\subset"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\
X \times Y \arrow[r, "T"'] & \{\mathsf{true}, \mathsf{false}\}
\end{tikzcd}
\qquad
\begin{tikzcd}
B \times C \arrow[r, "S"] \arrow[d, "g \times h"'] \arrow[dr, phantom, "\subset"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\
Y \times Z \arrow[r, "U"'] & \{\mathsf{true}, \mathsf{false}\}
\end{tikzcd}
$$

of inclusions of relations, the horizontal composition

$$
\begin{tikzcd}
A \arrow[r, "S \odot R", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\substack{\| \\ \beta \odot \alpha \\ \Downarrow}"] & C \arrow[d, "h"] \\
X \arrow[r, "U \odot T"', "|" marking] & Z
\end{tikzcd}
$$

of α and β is the inclusion of relations[1]

$$
(U \diamond T) \circ (f \times h) \subset (S \diamond R) \qquad
\begin{tikzcd}
A \times C \arrow[r, "S \diamond R"] \arrow[d, "f \times h"'] \arrow[dr, phantom, "\subset"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\
X \times Z \arrow[r, "U \diamond T"'] & \{\mathsf{true}, \mathsf{false}\}.
\end{tikzcd}
$$

---

[1] This is justified by noting that, given $(a, c) \in A \times C$, the statement

- We have $a \sim_{(U \diamond T) \circ (f \times h)} c$, i.e. $f(a) \sim_{U \diamond T} h(c)$, i.e. there exists some $y \in Y$ such that:
    1. We have $f(a) \sim_T y$;
    2. We have $y \sim_U h(c)$;

is implied by the statement

- We have $a \sim_{S \diamond R} c$, i.e. there exists some $b \in B$ such that:
    1. We have $a \sim_R b$;
    2. We have $b \sim_S c$;

since:

- If $a \sim_R b$, then $f(a) \sim_T g(b)$, as $T \circ (f \times g) \subset R$;
- If $b \sim_S c$, then $g(b) \sim_U h(c)$, as $U \circ (g \times h) \subset S$;

5.2.4.4 Vertical Composition of 2-Morphisms

Definition 5.2.4.4 ► The Vertical Composition of 2-Morphisms in $\mathsf{Rel}^{\mathsf{dbl}}$

The **vertical composition** in $\mathsf{Rel}^{\mathsf{dbl}}$ is defined as follows: for each vertically composable pair

$$\begin{tikzcd} A \arrow[r, "R", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & X \arrow[d, "g"] \\ B \arrow[r, "S"', "\shortmid"{marking}] & Y \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "S", "\shortmid"{marking}] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow\beta"] & Y \arrow[d, "k"] \\ C \arrow[r, "T"', "\shortmid"{marking}] & Z \end{tikzcd}$$

of 2-morphisms of $\mathsf{Rel}^{\mathsf{dbl}}$, i.e. for each each pair

$$\begin{tikzcd} A \times X \arrow[r, "R"] \arrow[d, "f \times g"'] \arrow[dr, phantom, "\subset"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true}, \mathsf{false}\}}"] \\ B \times Y \arrow[r, "S"'] & \{\mathsf{true}, \mathsf{false}\} \end{tikzcd} \qquad \begin{tikzcd} B \times Y \arrow[r, "S"] \arrow[d, "h \times k"'] \arrow[dr, phantom, "\subset"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true}, \mathsf{false}\}}"] \\ C \times Z \arrow[r, "T"'] & \{\mathsf{true}, \mathsf{false}\} \end{tikzcd}$$

of inclusions of relations, we define the vertical composition

$$\begin{tikzcd} A \arrow[r, "R", "\shortmid"{marking}] \arrow[d, "h \circ f"'] \arrow[dr, phantom, "\Downarrow\beta\circ\alpha"] & X \arrow[d, "k \circ g"] \\ C \arrow[r, "T"', "\shortmid"{marking}] & Z \end{tikzcd}$$

of α and β as the inclusion of relations

$$T \circ [(h \circ f) \times (k \circ g)] \subset R, \qquad \begin{tikzcd} A \times X \arrow[r, "R"] \arrow[d, "(h\circ f) \times (k \circ g)"'] \arrow[dr, phantom, "\subset"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true}, \mathsf{false}\}}"] \\ C \times Z \arrow[r, "T"'] & \{\mathsf{true}, \mathsf{false}\} \end{tikzcd}$$

given by the pasting of inclusions[1]

$$
\begin{tikzcd}
A \times X \arrow[r, "R"] \arrow[d, "f\times g"'] \arrow[dr, phantom, "\subset"] & \{\mathrm{true}, \mathrm{false}\} \arrow[d, "\mathrm{id}_{\{\mathrm{true},\mathrm{false}\}}"] \\
B \times Y \arrow[r, "S"] \arrow[d, "h\times k"'] \arrow[dr, phantom, "\subset"] & \{\mathrm{true}, \mathrm{false}\} \arrow[d, "\mathrm{id}_{\{\mathrm{true},\mathrm{false}\}}"] \\
C \times Z \arrow[r, "T"'] & \{\mathrm{true}, \mathrm{false}\}.
\end{tikzcd}
$$

[1]This is justified by noting that, given $(a, x) \in A \times X$, the statement

- We have $h(f(a)) \sim_T k(g(x))$;

is implied by the statement

- We have $a \sim_R x$;

since

- If $a \sim_R x$, then $f(a) \sim_S g(x)$, as $S \circ (f \times g) \subset R$;
- If $b \sim_S y$, then $h(b) \sim_T k(y)$, as $T \circ (h \times k) \subset S$, and thus, in particular:
  - If $f(a) \sim_S g(x)$, then $h(f(a)) \sim_T k(g(x))$;

5.2.4.5 The Associators

Definition 5.2.4.5 ▶ The Associators of $\mathsf{Rel}^{\mathsf{dbl}}$

For each composable triple $A \overset{R}{\nrightarrow} B \overset{S}{\nrightarrow} C \overset{T}{\nrightarrow} D$ of horizontal morphisms of $\mathsf{Rel}^{\mathsf{dbl}}$, the component

$$
\alpha^{\mathsf{Rel}^{\mathsf{dbl}}}_{T,S,R} \colon (T \odot S) \odot R \overset{\cong}{\Longrightarrow} T \odot (S \odot R), \qquad
\begin{tikzcd}
A \arrow[r, "R", "\shortmid"{marking}] \arrow[d, "\mathrm{id}_A"'] & B \arrow[r, "S", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow \alpha^{\mathsf{Rel}^{\mathsf{dbl}}}_{T,S,R}"] & C \arrow[r, "T", "\shortmid"{marking}] & D \arrow[d, "\mathrm{id}_D"] \\
A \arrow[r, "R"', "\shortmid"{marking}] & B \arrow[r, "S"', "\shortmid"{marking}] & C \arrow[r, "T"', "\shortmid"{marking}] & D
\end{tikzcd}
$$

of the associator of $\mathsf{Rel}^{\mathsf{dbl}}$ at (R, S, T) is the identity inclusion[1]

$$
(T \diamond S) \diamond R = T \diamond (S \diamond R) \qquad
\begin{tikzcd}
A \times B \arrow[r, "(T\diamond S)\diamond R"] \arrow[d, equal] \arrow[dr, phantom, "\mathbin{/\!/}"] & \{\mathrm{true}, \mathrm{false}\} \arrow[d, "\mathrm{id}_{\{\mathrm{true},\mathrm{false}\}}"] \\
A \times B \arrow[r, "T\diamond(S\diamond R)"'] & \{\mathrm{true}, \mathrm{false}\}.
\end{tikzcd}
$$

[1]This is justified by Item 2 of Proposition 5.3.12.3.

5.2.4.6 The Left Unitors

Definition 5.2.4.6 ► The Left Unitors of $\mathsf{Rel}^{\mathsf{dbl}}$

For each horizontal morphism $R\colon A \nrightarrow B$ of $\mathsf{Rel}^{\mathsf{dbl}}$, the component

$$\lambda^{\mathsf{Rel}^{\mathsf{dbl}}}_R \colon \chi_B \odot R \overset{\cong}{\Longrightarrow} R, \qquad \begin{tikzcd} A \arrow[r, "R", "\shortmid"{marking}] \arrow[d, "\mathrm{id}_A"'] & B \arrow[r, "\chi_B", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow \lambda^{\mathsf{Rel}^{\mathsf{dbl}}}_R"] & B \arrow[d, "\mathrm{id}_B"] \\ A \arrow[rr, "R"', "\shortmid"{marking}] & & B \end{tikzcd}$$

of the left unitor of $\mathsf{Rel}^{\mathsf{dbl}}$ at R is the identity inclusion[1]

$$R = \chi_B \diamond R, \qquad \begin{tikzcd} A \times B \arrow[r, "\chi_B \diamond R"] \arrow[d, equal] \arrow[dr, phantom, "\sslash"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ A \times B \arrow[r, "R"'] & \{\mathsf{true}, \mathsf{false}\}. \end{tikzcd}$$

[1]This is justified by Item 3 of Proposition 5.3.12.3.

5.2.4.7 The Right Unitors

Definition 5.2.4.7 ► The Right Unitors of $\mathsf{Rel}^{\mathsf{dbl}}$

For each horizontal morphism $R\colon A \nrightarrow B$ of $\mathsf{Rel}^{\mathsf{dbl}}$, the component

$$\rho^{\mathsf{Rel}^{\mathsf{dbl}}}_R \colon R \odot \chi_A \overset{\cong}{\Longrightarrow} R, \qquad \begin{tikzcd} A \arrow[r, "\chi_A", "\shortmid"{marking}] \arrow[d, "\mathrm{id}_A"'] & A \arrow[r, "R", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow \rho^{\mathsf{Rel}^{\mathsf{dbl}}}_R"] & B \arrow[d, "\mathrm{id}_B"] \\ A \arrow[rr, "R"', "\shortmid"{marking}] & & B \end{tikzcd}$$

of the right unitor of $\mathrm{Rel}^{\mathrm{dbl}}$ at R is the identity inclusion[1]

$$R = R \diamond \chi_A, \qquad \begin{tikzcd} A \times B \arrow[r, "R \diamond \chi_A"] \arrow[d, equal] \arrow[dr, phantom, "\swarrow"] & \{\mathrm{true}, \mathrm{false}\} \arrow[d, "\mathrm{id}_{\{\mathrm{true},\mathrm{false}\}}"] \\ A \times B \arrow[r, "R"'] & \{\mathrm{true}, \mathrm{false}\}. \end{tikzcd}$$

[1] This is justified by Item 3 of Proposition 5.3.12.3.

5.2.5 Properties of the Category of Relations

Proposition 5.2.5.1 ▶ Properties of the Category of Relations

Let A and B be sets.

1. *Self-Duality I.* The category Rel is self-dual, i.e. we have an equivalence

$$\mathrm{Rel}^{\mathrm{op}} \overset{\mathrm{eq.}}{\cong} \mathrm{Rel}$$

   of categories.

2. *Self-Duality II.* The bicategory **Rel** is self-dual, i.e. we have a biequivalence

$$\mathbf{Rel}^{\mathrm{op}} \overset{\mathrm{eq.}}{\cong} \mathbf{Rel}$$

   of bicategories.

3. *Equivalences and Isomorphisms in Rel.* Let $R \colon A \nrightarrow B$ be a relation from A to B. The following conditions are equivalent:

   (a) The relation $R \colon A \nrightarrow B$ is an equivalence in **Rel**, i.e. there exists a relation $R^{-1} \colon B \nrightarrow A$ from B to A together with isomorphisms

$$R^{-1} \diamond R \cong \chi_A,$$
$$R \diamond R^{-1} \cong \chi_B.$$

   (b) The relation $R \colon A \nrightarrow B$ is an isomorphism in Rel, i.e. there exists a relation $R^{-1} \colon B \nrightarrow A$ from B to A such that we have

$$R^{-1} \diamond R = \chi_A,$$
$$R \diamond R^{-1} = \chi_B.$$

   (c) There exists a bijection $f \colon A \xrightarrow{\cong} B$ with $R = \mathrm{Gr}(f)$.

4. *Adjunctions in* **Rel**. We have a natural bijection

$$\left\{\begin{matrix}\text{Adjunctions in }\mathbf{Rel}\\ \text{from } A \text{ to } B\end{matrix}\right\} \cong \left\{\begin{matrix}\text{Functions}\\ \text{from } A \text{ to } B\end{matrix}\right\},$$

with every adjunction in **Rel** being of the form $\mathrm{Gr}(f) \dashv f^{-1}$ for some function f.

5. *Monads in* **Rel**. We have a natural bijection[1]

$$\left\{\begin{matrix}\text{Monads in}\\ \mathbf{Rel}\text{ on } A\end{matrix}\right\} \cong \{\text{Preorders on } A\}.$$

6. *Comonads in* **Rel**. We have a natural bijection

$$\left\{\begin{matrix}\text{Comonads in}\\ \mathbf{Rel}\text{ on } A\end{matrix}\right\} \cong \{\text{Subsets of } A\}.$$

7. *Characterisations of Monomorphisms in* Rel. Let $R \colon A \nrightarrow B$ be a relation. The following conditions are equivalent:

   (a) The relation R is a monomorphism in Rel.

   (b) The direct image function

   $$R_* \colon \mathcal{P}(A) \to \mathcal{P}(B)$$

   associated to R is injective.

   (c) The direct image with compact support function

   $$R_! \colon \mathcal{P}(A) \to \mathcal{P}(B)$$

   associated to R is injective.

   Moreover, if R is a monomorphism, then it satisfies the following condition, and the converse holds if R is total:

   $(\star)$ For each $a, a' \in A$, if there exists some $b \in B$ such that $a \sim_R b$ and $a' \sim_R b$, then $a = a'$.

8. *Epimorphisms in* Rel. Let $R \colon A \nrightarrow B$ be a relation. The following conditions are equivalent:

   (a) The relation R is an epimorphism in Rel.

(b) The weak inverse image function

$$R^{-1}\colon \mathcal{P}(B) \to \mathcal{P}(A)$$

associated to R is injective.

(c) The strong inverse image function

$$R_{-1}\colon \mathcal{P}(B) \to \mathcal{P}(A)$$

associated to R is injective.

(d) The function $R\colon A \to \mathcal{P}(B)$ is "surjective on singletons":

($\star$) For each $b \in B$, there exists some $a \in A$ such that $R(a) = \{b\}$.

9. *As a Kleisli Category.* We have an isomorphism of categories

$$\mathsf{Rel} \cong \mathsf{FreeAlg}_{\mathcal{P}},$$

where $\mathcal{P}$ is the powerset monad of **??**.

10. *Co/Completeness (Or Lack Thereof).* The category Rel is not co/complete, but admits some co/limits:

(a) *Zero Objects.* The category Rel has a zero object, the empty set $\emptyset$.

(b) *Co/Products.* The category Rel has co/products, both given by disjoint union of sets.

(c) *Lack of Co/Equalisers.* The category Rel does not have co/equalisers.

(d) *Limits of Graphs of Functions.* The category Rel has limits whose arrows are all graphs of functions.

(e) *Colimits of Graphs of Functions.* The category Rel has colimits whose arrows are all graphs of functions, and these agree with the corresponding limits in Sets.

11. *Existence of Right Kan Extensions.* The right Kan extension

$$\mathrm{Ran}_R\colon \mathsf{Rel}(A, X) \to \mathsf{Rel}(B, X)$$

along a relation $R\colon A \nrightarrow B$ in **Rel** exists and is given by

$$\mathrm{Ran}_R(S) \stackrel{\text{def}}{=} \int_{a\in A} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^a_{-_1}, S^a_{-_2}\right)$$

for each $S \in \mathsf{Rel}(A, X)$, so that the following conditions are equivalent:

(a) We have $b \sim_{\mathrm{Ran}_R(S)} x$.

(b) For each $a \in A$, if $a \sim_R b$, then $a \sim_S x$.

12. *Existence of Right Kan Lifts.* The right Kan lift

$$\mathrm{Rift}_R \colon \mathrm{Rel}(X, B) \longrightarrow \mathrm{Rel}(X, A)$$

along a relation $R \colon A \nrightarrow B$ in **Rel** exists and is given by

$$\mathrm{Rift}_R(S) \stackrel{\text{def}}{=} \int_{b \in B} \mathbf{Hom}_{\{\mathrm{t},\mathrm{f}\}}\left(R_b^{-2}, S_b^{-1}\right)$$

for each $S \in \mathrm{Rel}(X, B)$, so that the following conditions are equivalent:

(a) We have $x \sim_{\mathrm{Rift}_R(S)} a$.

(b) For each $b \in B$, if $a \sim_R b$, then $x \sim_S b$.

13. *Closedness.* The bicategory **Rel** is a closed bicategory, there being, for each $R \colon A \nrightarrow B$ and set X, a pair of adjunctions

$$(R^* \dashv \mathrm{Ran}_R) \colon \quad \begin{tikzcd} \mathrm{Rel}(B, X) \arrow[r, bend left, "R^*"] \arrow[r, phantom, "\perp"] & \mathrm{Rel}(A, X), \arrow[l, bend left, "\mathrm{Ran}_R"] \end{tikzcd}$$

$$(R_* \dashv \mathrm{Rift}_R) \colon \quad \begin{tikzcd} \mathrm{Rel}(X, A) \arrow[r, bend left, "R_*"] \arrow[r, phantom, "\perp"] & \mathrm{Rel}(X, B), \arrow[l, bend left, "\mathrm{Rift}_R"] \end{tikzcd}$$

witnessed by bijections

$$\mathbf{Rel}(S \diamond R, T) \cong \mathbf{Rel}(S, \mathrm{Ran}_R(T)),$$
$$\mathbf{Rel}(R \diamond U, V) \cong \mathbf{Rel}(U, \mathrm{Rift}_R(V)),$$

natural in $S \in \mathrm{Rel}(B, X)$, $T \in \mathrm{Rel}(A, X)$, $U \in \mathrm{Rel}(X, A)$, and $V \in \mathrm{Rel}(X, B)$.

[1] See also Section 5.6 for an extension of this correspondence to "relative monads on **Rel**".

Proof 5.2.5.2 ► Proof of Proposition 5.2.5.1

Item 1: Self-Duality I

Omitted.

Item 2: Self-Duality II

Omitted.

Item 3: Equivalences and Isomorphisms in Rel

We claim that Items 3a to 3c are indeed equivalent:

- *Item 3a* $\iff$ *Item 3b*: This follows from the fact that **Rel** is locally posetal, so that natural isomorphisms and equalities of 1-cells in **Rel** coincide.
- *Item 3b* $\implies$ *Item 3c*: The equalities in Item 3b imply $R \dashv R^{-1}$, and thus by Item 4, there exists a function $f_R \colon A \longrightarrow B$ associated to R, where, for each $a \in A$, the image $f_R(a)$ of a by f_R is the unique element of $R(a)$, which implies $R = \mathrm{Gr}(f_R)$ in particular. Furthermore, we have $R^{-1} = f_R^{-1}$ (as in Definition 5.3.2.1). The conditions from Item 3b then become the following:

  $$\begin{aligned} f_R^{-1} \diamond f_R &= \chi_A, \\ f_R \diamond f_R^{-1} &= \chi_B. \end{aligned}$$

  All that is left is to show then is that f_R is a bijection:

  - *The Function f_R Is Injective.* Let $a, b \in A$ and suppose that $f_R(a) = f_R(b)$. Since $a \sim_R f_R(a)$ and $f_R(a) = f_R(b) \sim_{R^{-1}} b$, the condition $f_R^{-1} \diamond f_R = \chi_A$ implies that $a = b$, showing f_R to be injective.
  - *The Function f_R Is Surjective.* Let $b \in B$. Applying the condition $f_R \diamond f_R^{-1} = \chi_B$ to (b, b), it follows that there exists some $a \in A$ such that $f_R^{-1}(b) = a$ and $f_R(a) = b$. This shows f_R to be surjective.
- *Item 3c* $\implies$ *Item 3b*: By Item 2, we have an adjunction $\mathrm{Gr}(f) \dashv f^{-1}$, giving inclusions

  $$\begin{aligned} \chi_A &\subset f^{-1} \diamond \mathrm{Gr}(f), \\ \mathrm{Gr}(f) \diamond f^{-1} &\subset \chi_B. \end{aligned}$$

  We claim the reverse inclusions are also true:

  - $f^{-1} \diamond Gr(f) \subset \chi_A$: This is equivalent to the statement that if $f(a) = b$ and $f^{-1}(b) = a'$, then $a = a'$, which follows from the injectivity of f.
  - $\chi_B \subset Gr(f) \diamond f^{-1}$: This is equivalent to the statement that given $b \in B$ there exists some $a \in A$ such that $f^{-1}(b) = a$ and $f(a) = b$, which follows from the surjectivity of f.

Item 4: Adjunctions in Rel

We proceed step by step:

1. *From Adjunctions in* **Rel** *to Functions.* An adjunction in **Rel** from A to B consists of a pair of relations

$$\begin{aligned} R &\colon A \nrightarrow B, \\ S &\colon B \nrightarrow A, \end{aligned}$$

together with inclusions

$$\begin{aligned} \chi_A &\subset S \diamond R, \\ R \diamond S &\subset \chi_B. \end{aligned}$$

We claim that these conditions imply that R is total and functional, i.e. that $R(a)$ is a singleton for each $a \in A$:

   (a) $R(a)$ *Has an Element.* Given $a \in A$, since $\chi_A \subset S \diamond R$, we must have $\{a\} \subset S(R(a))$, implying that there exists some $b \in B$ such that $a \sim_R b$ and $b \sim_S a$, and thus $R(a) \neq \emptyset$, as $b \in R(a)$.

   (b) $R(a)$ *Has No More Than One Element.* Suppose that we have $a \sim_R b$ and $a \sim_R b'$ for $b, b' \in B$. We claim that $b = b'$:

      i. Since $\chi_A \subset S \diamond R$, there exists some $k \in B$ such that $a \sim_R k$ and $k \sim_S a$.

      ii. Since $R \diamond S \subset \chi_B$, if $b'' \sim_S a'$ and $a' \sim_R b'''$, then $b'' = b'''$.

      iii. Applying the above to $b'' = k$, $b''' = b$, and $a' = a$, since $k \sim_S a$ and $a \sim_R b'$, we have $k = b$.

      iv. Similarly $k = b'$.

      v. Thus $b = b'$.

Together, the above two items show $R(a)$ to be a singleton, being thus given by $\mathrm{Gr}(f)$ for some function $f \colon A \to B$, which gives a map

$$\left\{\begin{matrix}\text{Adjunctions in } \mathbf{Rel} \\ \text{from } A \text{ to } B\end{matrix}\right\} \to \left\{\begin{matrix}\text{Functions} \\ \text{from } A \text{ to } B\end{matrix}\right\}.$$

Moreover, by uniqueness of adjoints (**??** of **??**), this implies also that $S = f^{-1}$.

2. *From Functions to Adjunctions in* **Rel**. By Item 2 of Proposition 5.3.1.2, every function $f \colon A \to B$ gives rise to an adjunction $\mathrm{Gr}(f) \dashv f^{-1}$ in Rel, giving a map

$$\left\{\begin{matrix}\text{Functions} \\ \text{from } A \text{ to } B\end{matrix}\right\} \to \left\{\begin{matrix}\text{Adjunctions in } \mathbf{Rel} \\ \text{from } A \text{ to } B\end{matrix}\right\}.$$

3. *Invertibility: From Functions to Adjunctions Back to Functions.* We need to show that starting with a function $f\colon A \to B$, passing to $\mathrm{Gr}(f) \dashv f^{-1}$, and then passing again to a function gives f again. This is clear however, since we have $a \sim_{\mathrm{Gr}(f)} b$ iff $f(a) = b$.

4. *Invertibility: From Adjunctions to Functions Back to Adjunctions.* We need to show that, given an adjunction $R \dashv S$ in **Rel** giving rise to a function $f_{R,S}\colon A \to B$, we have

$$\begin{aligned} \mathrm{Gr}\big(f_{R,S}\big) &= R, \\ f_{R,S}^{-1} &= S. \end{aligned}$$

We check these explicitly:

· *$\mathrm{Gr}\big(f_{R,S}\big) = R$.* We have

$$\begin{aligned} \mathrm{Gr}\big(f_{R,S}\big) &\overset{\text{def}}{=} \big\{\big(a, f_{R,S}(a)\big) \in A \times B \,\big|\, a \in A\big\} \\ &\overset{\text{def}}{=} \{(a, R(a)) \in A \times B \mid a \in A\} \\ &= R. \end{aligned}$$

· *$f_{R,S}^{-1} = S$.* We first claim that, given $a \in A$ and $b \in B$, the following conditions are equivalent:

- We have $a \sim_R b$.
- We have $b \sim_S a$.

Indeed:

- *If $a \sim_R b$, then $b \sim_S a$:* Since $\chi_A \subset S \diamond R$, there exists $k \in B$ such that $a \sim_R k$ and $k \sim_S a$, but since $a \sim_R b$ and R is functional, we have $k = b$ and thus $b \sim_S a$.
- *If $b \sim_S a$, then $a \sim_R b$:* First note that since R is total we have $a \sim_R b'$ for some $b' \in B$. Now, since $R \diamond S \subset \chi_B$, $b \sim_S a$, and $a \sim_R b'$, we have $b = b'$, and thus $a \sim_R b$.

Having show this, we now have

$$\begin{aligned} f_{R,S}^{-1}(b) &\overset{\text{def}}{=} \big\{a \in A \,\big|\, f_{R,S}(a) = b\big\} \\ &\overset{\text{def}}{=} \{a \in A \mid a \sim_R b\} \\ &= \{a \in A \mid b \sim_S a\} \\ &\overset{\text{def}}{=} S(b). \end{aligned}$$

for each $b \in B$, showing $f_{R,S}^{-1} = S$.

This finishes the proof.

Item 5: Monads in Rel

A monad in **Rel** on A consists of a relation $R\colon A \nrightarrow A$ together with maps

$$\mu_R\colon R \diamond R \subset R,$$
$$\eta_R\colon \chi_A \subset R$$

making the diagrams

$$\begin{tikzcd}
\chi_A \diamond R \arrow[r, "\eta_R\diamond\mathrm{id}_R", "\shortmid"{marking}] \arrow[dr, equals, "\lambda^{\mathbf{Rel}(A,B)}_R"'] & R \diamond R \arrow[d, "\mu_R", "\shortmid"{marking}] \\
& R
\end{tikzcd}
\qquad
\begin{tikzcd}
& & R \diamond (R \diamond R) \arrow[drr, "\mathrm{id}_R\diamond\mu_R", "\shortmid"{marking}] & & \\
(R \diamond R) \diamond R \arrow[urr, equals, "\alpha^{\mathbf{Rel}(A,B)}_{R,R,R}"] \arrow[dr, "\mu_R\diamond\mathrm{id}_R"', "\shortmid"{marking}] & & & & R \diamond R \arrow[dl, "\mu_R", "\shortmid"{marking}] \\
& R \diamond R \arrow[rr, "\mu_R"', "\shortmid"{marking}] & & R &
\end{tikzcd}
\qquad
\begin{tikzcd}
R \diamond \chi_A \arrow[r, "\mathrm{id}_R\diamond\eta_R", "\shortmid"{marking}] \arrow[dr, equals, "\rho^{\mathbf{Rel}(A,B)}_R"'] & R \diamond R \arrow[d, "\mu_R", "\shortmid"{marking}] \\
& R
\end{tikzcd}$$

commute. However, since all morphisms involved are inclusions, the commutativity of the above diagrams is automatic, and hence all that is left is the data of the two maps μ_R and η_R, which correspond respectively to the following conditions:

1. For each $a, b, c \in A$, if $a \sim_R b$ and $b \sim_R c$, then $a \sim_R c$.
2. For each $a \in A$, we have $a \sim_R a$.

These are exactly the requirements for R to be a preorder (**??**). Conversely any preorder $\preceq$ gives rise to a pair of maps $\mu_\preceq$ and $\eta_\preceq$, forming a monad on A.

Item 6: Comonads in Rel

A comonad in **Rel** on A consists of a relation $R\colon A \nrightarrow A$ together with maps

$$\Delta_R\colon R \subset R \diamond R,$$
$$\epsilon_R\colon R \subset \chi_A$$

making the diagrams

$$\begin{tikzcd}
R \arrow[r, "\Delta_R", "\shortmid"{marking}] \arrow[dr, equals, "\lambda^{\mathbf{Rel}(A,B),-1}_R"'] & R \diamond R \arrow[d, "\epsilon_R\diamond\mathrm{id}_R", "\shortmid"{marking}] \\
& \chi_A \diamond R
\end{tikzcd}
\qquad
\begin{tikzcd}
& & R \diamond R \arrow[drr, "\mathrm{id}_R\diamond\Delta_R", "\shortmid"{marking}] & & \\
R \arrow[urr, "\Delta_R", "\shortmid"{marking}] \arrow[dr, "\Delta_R"', "\shortmid"{marking}] & & & & R \diamond (R \diamond R) \arrow[dl, equals, "\alpha^{\mathbf{Rel}(A,B),-1}_{R,R,R}"] \\
& R \diamond R \arrow[rr, "\Delta_R\diamond\mathrm{id}_R"', "\shortmid"{marking}] & & (R \diamond R) \diamond R &
\end{tikzcd}
\qquad
\begin{tikzcd}
R \arrow[r, "\Delta_R", "\shortmid"{marking}] \arrow[dr, equals, "\rho^{\mathbf{Rel}(A,B),-1}_R"'] & R \diamond R \arrow[d, "\mathrm{id}_R\diamond\epsilon_R", "\shortmid"{marking}] \\
& R \diamond \chi_A
\end{tikzcd}$$

commute. However, since all morphisms involved are inclusions, the commutativity of the above diagrams is automatic, and hence all that is left is the data of the two maps Δ_R and ϵ_R, which correspond respectively to the following conditions:

1. For each $a, b \in A$, if $a \sim_R b$, then there exists some $k \in A$ such that $a \sim_R k$ and $k \sim_R b$.

2. For each $a, b \in A$, if $a \sim_R b$, then $a = b$.

Taking $k = b$ in the first condition above shows it to be trivially satisfied, while the second condition implies $R \subset \Delta_A$, i.e. R must be a subset of A. Conversely, any subset U of A satisfies $U \subset \Delta_A$, defining a comonad as above.

Item 7: Monomorphisms in Rel

Firstly note that Items 7b and 7c are equivalent by Item 7 of Proposition 5.5.1.3. We then claim that Items 7a and 7b are also equivalent:

- *Item 7a* $\Longrightarrow$ *Item 7b*: Let $U, V \in \mathcal{P}(A)$ and consider the diagram

$$\mathrm{pt} \underset{V}{\overset{U}{\rightrightarrows}} A \xrightarrow{R} B.$$

  By Remark 5.5.1.2, we have

$$\begin{aligned} R_*(U) &= R \diamond U, \\ R_*(V) &= R \diamond V. \end{aligned}$$

  Now, if $R \diamond U = R \diamond V$, i.e. $R_*(U) = R_*(V)$, then $U = V$ since R is assumed to be a monomorphism, showing R_* to be injective.

- *Item 7b* $\Longrightarrow$ *Item 7a*: Conversely, suppose that R_* is injective, consider the diagram

$$K \underset{T}{\overset{S}{\rightrightarrows}} A \xrightarrow{R} B,$$

  and suppose that $R \diamond S = R \diamond T$. Note that, since R_* is injective, given a diagram of the form

$$\mathrm{pt} \underset{V}{\overset{U}{\rightrightarrows}} A \xrightarrow{R} B,$$

  if $R_*(U) = R \diamond U = R \diamond V = R_*(V)$, then $U = V$. In particular, for each $k \in K$, we may consider the diagram

$$\mathrm{pt} \xrightarrow{[k]} K \underset{T}{\overset{S}{\rightrightarrows}} A \xrightarrow{R} B,$$

for which we have $R \diamond S \diamond [k] = R \diamond T \diamond [k]$, implying that we have

$$S(k) = S \diamond [k] = T \diamond [k] = T(k)$$

for each $k \in K$, implying $S = T$, and thus R is a monomorphism.

We can also prove this in a more abstract way, following [MSE 350788]:

- *Item 7a* $\implies$ *Item 7b*: Assume that R is a monomorphism.
  - We first notice that the functor $\mathsf{Rel}(\mathrm{pt}, -) \colon \mathsf{Rel} \to \mathsf{Sets}$ maps R to R_* by Remark 5.5.1.2.
  - Since $\mathsf{Rel}(\mathrm{pt}, -)$ preserves all limits by **??** of **??**, it follows by **??** of **??** that $\mathsf{Rel}(\mathrm{pt}, -)$ also preserves monomorphisms.
  - Since R is a monomorphism and $\mathsf{Rel}(\mathrm{pt}, -)$ maps R to R_*, it follows that R_* is also a monomorphism.
  - Since the monomorphisms in Sets are precisely the injections (**??** of **??**), it follows that R_* is injective.
- *Item 7b* $\implies$ *Item 7a*: Assume that R_* is injective.
  - We first notice that the functor $\mathsf{Rel}(\mathrm{pt}, -) \colon \mathsf{Rel} \to \mathsf{Sets}$ maps R to R_* by Remark 5.5.1.2.
  - Since the monomorphisms in Sets are precisely the injections (**??** of **??**), it follows that R_* is a monomorphism.
  - Since $\mathsf{Rel}(\mathrm{pt}, -)$ is faithful, it follows by **??** of **??** that $\mathsf{Rel}(\mathrm{pt}, -)$ reflects monomorphisms.
  - Since R_* is a monomorphism and $\mathsf{Rel}(\mathrm{pt}, -)$ maps R to R_*, it follows that R is also a monomorphism.

Finally, we prove the second part of the statement. Assume that R is a monomorphism, let $a, a' \in A$ such that $a \sim_R b$ and $a' \sim_R b$ for some $b \in B$, and consider the diagram

$$\mathrm{pt} \underset{[a']}{\overset{[a]}{\rightrightarrows}} A \overset{R}{\nrightarrow} B.$$

Since $\star \sim_{[a]} a$ and $a \sim_R b$, we have $\star \sim_{R \diamond [a]} b$. Similarly, $\star \sim_{R \diamond [a']} b$. Thus $R \diamond [a] = R \diamond [a']$, and since R is a monomorphism, we have $[a] = [a']$, i.e. $a = a'$.

Conversely, assume the condition

($\star$) For each $a, a' \in A$, if there exists some $b \in B$ such that $a \sim_R b$ and $a' \sim_R b$, then $a = a'$, consider the diagram

$$K \underset{T}{\overset{S}{\rightrightarrows}} A \overset{R}{\nrightarrow} B,$$

and let $(k, a) \in S$. Since R is total and $a \in A$, there exists some $b \in B$ such that $a \sim_R b$. In this case, we have $k \sim_{R \diamond S} b$, and since $R \diamond S = R \diamond T$, we have also $k \sim_{R \diamond T} b$. Thus there must exist some $a' \in A$ such that $k \sim_T a'$ and $a' \sim_R b$. However, since $a, a' \sim_R b$, we must have $a = a'$, and thus $(k, a) \in T$ as well.

A similar argument shows that if $(k, a) \in T$, then $(k, a) \in S$, and thus $S = T$ and R is a monomorphism.

Item 8: Epimorphisms in Rel

Firstly note that Items 8b and 8c are equivalent by Item 7 of Proposition 5.5.2.4. We then claim that Items 8a and 8b are also equivalent:

- *Item 8a* $\implies$ *Item 8b*: Let $U, V \in \mathcal{P}(A)$ and consider the diagram

  $$A \overset{R}{\nrightarrow} B \underset{V}{\overset{U}{\rightrightarrows}} \mathrm{pt}.$$

  By Remark 5.5.1.2, we have

  $$R^{-1}(U) = U \diamond R,$$
  $$R^{-1}(V) = V \diamond R.$$

  Now, if $U \diamond R = V \diamond R$, i.e. $R^{-1}(U) = R^{-1}(V)$, then $U = V$ since R is assumed to be an epimorphism, showing R^{-1} to be injective.

- *Item 8b* $\implies$ *Item 8a*: Conversely, suppose that R^{-1} is injective, consider the diagram

  $$A \overset{R}{\nrightarrow} B \underset{T}{\overset{S}{\rightrightarrows}} K,$$

  and suppose that $S \diamond R = T \diamond R$. Note that, since R^{-1} is injective, given a diagram of the form

  $$A \overset{R}{\nrightarrow} B \underset{V}{\overset{U}{\rightrightarrows}} \mathrm{pt},$$

  if $R^{-1}(U) = U \diamond R = V \diamond R = R^{-1}(V)$, then $U = V$. In particular, for each $k \in K$, we may consider the diagram

  $$A \overset{R}{\nrightarrow} B \underset{T}{\overset{S}{\rightrightarrows}} K \overset{[k]}{\nrightarrow} \mathrm{pt},$$

for which we have $[k] \diamond S \diamond R = [k] \diamond T \diamond R$, implying that we have

$$S^{-1}(k) = [k] \diamond S = [k] \diamond T = T^{-1}(k)$$

for each $k \in K$, implying $S = T$, and thus R is an epimorphism.

We can also prove this in a more abstract way, following [MSE 350788]:

- *Item 8a* $\implies$ *Item 8b*: Assume that R is an epimorphism.
  - We first notice that the functor $\mathsf{Rel}(-, \mathrm{pt}) \colon \mathsf{Rel}^{\mathsf{op}} \to \mathsf{Sets}$ maps R to R^{-1} by Remark 5.5.3.2.
  - Since $\mathsf{Rel}(-, \mathrm{pt})$ preserves limits by **??** of **??**, it follows by **??** of **??** that $\mathsf{Rel}(-, \mathrm{pt})$ also preserves monomorphisms.
  - That is: $\mathsf{Rel}(-, \mathrm{pt})$ sends monomorphisms in $\mathsf{Rel}^{\mathsf{op}}$ to monomorphisms in Sets.
  - The monomorphisms $\mathsf{Rel}^{\mathsf{op}}$ are precisely the epimorphisms in Rel by **??** of **??**.
  - Since R is an epimorphism and $\mathsf{Rel}(-, \mathrm{pt})$ maps R to R^{-1}, it follows that R^{-1} is a monomorphism.
  - Since the monomorphisms in Sets are precisely the injections (**??** of **??**), it follows that R^{-1} is injective.
- *Item 8b* $\implies$ *Item 8a*: Assume that R^{-1} is injective.
  - We first notice that the functor $\mathsf{Rel}(-, \mathrm{pt}) \colon \mathsf{Rel}^{\mathsf{op}} \to \mathsf{Sets}$ maps R to R^{-1} by Remark 5.5.3.2.
  - Since the monomorphisms in Sets are precisely the injections (**??** of **??**), it follows that R^{-1} is a monomorphism.
  - Since $\mathsf{Rel}(-, \mathrm{pt})$ is faithful, it follows by **??** of **??** that $\mathsf{Rel}(, \mathrm{pt})$ reflects monomorphisms.
  - That is: $\mathsf{Rel}(-, \mathrm{pt})$ reflects monomorphisms in Sets to monomorphisms in $\mathsf{Rel}^{\mathsf{op}}$.
  - The monomorphisms $\mathsf{Rel}^{\mathsf{op}}$ are precisely the epimorphisms in Rel by **??** of **??**.
  - Since R^{-1} is a monomorphism and $\mathsf{Rel}(-, \mathrm{pt})$ maps R to R^{-1}, it follows that R is an epimorphism.

Finally, we claim that Items 8b and 8d are also equivalent, following [MO 350788]:

- *Item 8b* $\implies$ *Item 8d*: Since $B \setminus \{b\} \subset B$ and R^{-1} is injective, we have $R^{-1}(B \setminus \{b\}) \subsetneq R^{-1}(B)$. So taking some $a \in R^{-1}(B) \setminus R^{-1}(B \setminus \{b\})$ we get an element of A such that $R(a) = \{b\}$.

- *Item 8d* $\implies$ *Item 8b*: Let $U, V \subset B$ with $U \neq V$. Without loss of generality, we can assume $U \setminus V \neq \emptyset$; otherwise just swap U and V. Let then $b \in U \setminus V$. By assumption, there exists an $a \in A$ with $R(a) = \{b\}$. Then $a \in R^{-1}(U)$ but $a \notin R^{-1}(V)$, and thus $R^{-1}(U) \neq R^{-1}(V)$, showing R^{-1} to be injective.

Item 9: As a Kleisli Category

Omitted.

Item 10: Co/Completeness (Or Lack Thereof)

Omitted.

Item 11: Existence of Right Kan Extensions

We have

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{Rel}(A,X)}(S \diamond R, T) &\cong \int_{a \in A} \int_{x \in X} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left((S \diamond R)^a_x, T^a_x\right) \\
&\cong \int_{a \in A} \int_{x \in X} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(\left(\int^{b \in B} S^b_x \times R^a_b\right), T^a_x\right) \\
&\cong \int_{a \in A} \int_{x \in X} \int_{b \in B} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(S^b_x \times R^a_b, T^a_x\right) \\
&\cong \int_{a \in A} \int_{x \in X} \int_{b \in B} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(S^b_x, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^a_b, T^a_x\right)\right) \\
&\cong \int_{b \in B} \int_{x \in X} \int_{a \in A} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(S^b_x, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^a_b, T^a_x\right)\right) \\
&\cong \int_{b \in B} \int_{x \in X} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(S^b_x, \int_{a \in A} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^a_b, T^a_x\right)\right) \\
&\cong \mathrm{Hom}_{\mathbf{Rel}(B,X)}\left(S, \int_{a \in A} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^a_{-_1}, T^a_{-_2}\right)\right)
\end{aligned}$$

naturally in each $S \in \mathbf{Rel}(B, X)$ and each $T \in \mathbf{Rel}(A, X)$, showing that

$$\int_{a \in A} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^a_{-_1}, T^a_{-_2}\right)$$

is right adjoint to the precomposition functor $- \diamond R$, being thus the right Kan extension along R. Here we have used the following results, respectively (i.e. for each $\cong$ sign):

1. Item 1 of Proposition 5.1.1.5;
2. Definition 5.3.12.1;

3. ?? of ??;

4. ??;

5. ?? of ??;

6. ?? of ??;

7. Item 1 of Proposition 5.1.1.5.

Item 12: Existence of Right Kan Lifts

We have

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{Rel}(X,B)}(R \diamond S, T) &\cong \int_{x\in X}\int_{b\in B} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left((R\diamond S)^x_b, T^x_b\right)\\
&\cong \int_{x\in X}\int_{b\in B} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(\left(\int^{a\in A} R^a_b \times S^x_a\right), T^x_b\right)\\
&\cong \int_{x\in X}\int_{b\in B}\int_{a\in A} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^a_b \times S^x_a, T^x_b\right)\\
&\cong \int_{x\in X}\int_{b\in B}\int_{a\in A} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(S^x_a, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^a_b, T^x_b\right)\right)\\
&\cong \int_{x\in X}\int_{a\in A}\int_{b\in B} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(S^x_a, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^a_b, T^x_b\right)\right)\\
&\cong \int_{x\in X}\int_{a\in A} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(S^x_a, \int_{b\in B}\mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^a_b, T^x_b\right)\right)\\
&\cong \mathrm{Hom}_{\mathbf{Rel}(X,A)}\left(S, \int_{b\in B}\mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^{-_2}_b, T^{-_1}_b\right)\right)
\end{aligned}$$

naturally in each $S \in \mathbf{Rel}(X, A)$ and each $T \in \mathbf{Rel}(X, B)$, showing that

$$\int_{b\in B}\mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^{-_2}_b, T^{-_1}_b\right)$$

is right adjoint to the postcomposition functor $R \diamond -$, being thus the right Kan lift along R. Here we have used the following results, respectively (i.e. for each $\cong$ sign):

1. Item 1 of Proposition 5.1.1.5;

2. Definition 5.3.12.1;

3. ?? of ??;

4. ??;

5. ?? of ??;

6. ?? of ??;

7. Item 1 of Proposition 5.1.1.5.

Item 13: Closedness

This has been proved as part of the proof of Items 11 and 12.

5.3 Constructions With Relations

5.3.1 The Graph of a Function

Let $f \colon A \to B$ be a function.

Definition 5.3.1.1 ▶ The Graph of a Function

The **graph of** f is the relation $\mathrm{Gr}(f) \colon A \nrightarrow B$ defined as follows:[1]

- Viewing relations from A to B as subsets of $A \times B$, we define

$$\mathrm{Gr}(f) \overset{\text{def}}{=} \{(a, f(a)) \in A \times B \mid a \in A\};$$

- Viewing relations from A to B as functions $A \times B \to \{\text{true}, \text{false}\}$, we define

$$[\mathrm{Gr}(f)](a, b) \overset{\text{def}}{=} \begin{cases} \text{true} & \text{if } b = f(a), \\ \text{false} & \text{otherwise} \end{cases}$$

for each $(a, b) \in A \times B$;

- Viewing relations from A to B as functions $A \to \mathcal{P}(B)$, we define

$$[\mathrm{Gr}(f)](a) \overset{\text{def}}{=} \{f(a)\}$$

for each $a \in A$, i.e. we define $\mathrm{Gr}(f)$ as the composition

$$A \xrightarrow{f} B \xrightarrow{\chi_B} \mathcal{P}(B).$$

[1] *Further Notation:* We write $\mathrm{Gr}(A)$ for $\mathrm{Gr}(\mathrm{id}_A)$, and call it the **graph** of A.

Proposition 5.3.1.2 ▶ Properties of Graphs of Functions

Let $f \colon A \to B$ be a function.

1. *Functoriality.* The assignment $A \mapsto \mathrm{Gr}(A)$ defines a functor

$$\mathrm{Gr} \colon \mathsf{Sets} \longrightarrow \mathrm{Rel}$$

where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$\mathrm{Gr}(A) \stackrel{\mathrm{def}}{=} A;$$

- *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathsf{Sets})$, the action on Hom-sets

$$\mathrm{Gr}_{A,B} \colon \mathsf{Sets}(A, B) \longrightarrow \underbrace{\mathrm{Rel}(\mathrm{Gr}(A), \mathrm{Gr}(B))}_{\stackrel{\mathrm{def}}{=} \mathrm{Rel}(A,B)}$$

  of Gr at (A, B) is defined by

$$\mathrm{Gr}_{A,B}(f) \stackrel{\mathrm{def}}{=} \mathrm{Gr}(f),$$

  where $\mathrm{Gr}(f)$ is the graph of f as in Definition 5.3.1.1.

In particular:

- *Preservation of Identities.* We have

$$\mathrm{Gr}(\mathrm{id}_A) = \chi_A$$

  for each $A \in \mathrm{Obj}(\mathsf{Sets})$.
- *Preservation of Composition.* We have

$$\mathrm{Gr}(g \circ f) = \mathrm{Gr}(g) \diamond \mathrm{Gr}(f)$$

  for each pair of functions $f \colon A \longrightarrow B$ and $g \colon B \longrightarrow C$.

2. *Adjointness Inside* **Rel**. We have an adjunction

$$\left(\mathrm{Gr}(f) \dashv f^{-1}\right)\colon \quad \begin{tikzcd} A \arrow[r, bend left, "\mathrm{Gr}(f)"] \arrow[r, phantom, "\perp"] & B \arrow[l, bend left, "f^{-1}"] \end{tikzcd}$$

in **Rel**, where f^{-1} is the inverse of f of Definition 5.3.2.1.

3. *Adjointness*. We have an adjunction

$$(\mathrm{Gr} \dashv \mathcal{P}_*)\colon \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "\mathrm{Gr}"] \arrow[r, phantom, "\perp"] & \mathsf{Rel}, \arrow[l, bend left, "\mathcal{P}_*"] \end{tikzcd}$$

witnessed by a bijection of sets

$$\mathsf{Rel}(\mathrm{Gr}(A), B) \cong \mathsf{Sets}(A, \mathcal{P}(B))$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$ and $B \in \mathrm{Obj}(\mathsf{Rel})$.

4. *Interaction With Inverses*. We have

$$\begin{aligned} \mathrm{Gr}(f)^\dagger &= f^{-1}, \\ (f^{-1})^\dagger &= \mathrm{Gr}(f). \end{aligned}$$

5. *Cocontinuity*. The functor $\mathrm{Gr}\colon \mathsf{Sets} \to \mathsf{Rel}$ of Item 1 preserves colimits.

6. *Characterisations*. Let $R\colon A \nrightarrow B$ be a relation. The following conditions are equivalent:

   (a) There exists a function $f\colon A \to B$ such that $R = \mathrm{Gr}(f)$.

   (b) The relation R is total and functional.

   (c) The weak and strong inverse images of R agree, i.e. we have $R^{-1} = R_{-1}$.

   (d) The relation R has a right adjoint $R^\dagger$ in Rel.

Proof 5.3.1.3 ▶ Proof of Proposition 5.3.1.2

Item 1: Functoriality

Clear.

Item 2: Adjointness Inside Rel

We need to check that there are inclusions

$$\begin{aligned} \chi_A &\subset f^{-1} \diamond \mathrm{Gr}(f), \\ \mathrm{Gr}(f) \diamond f^{-1} &\subset \chi_B. \end{aligned}$$

These correspond respectively to the following conditions:

1. For each $a \in A$, there exists some $b \in B$ such that $a \sim_{\mathrm{Gr}(f)} b$ and $b \sim_{f^{-1}} a$.

2. For each $a, b \in A$, if $a \sim_{\mathrm{Gr}(f)} b$ and $b \sim_{f^{-1}} a$, then $a = b$.

In other words, the first condition states that the image of any $a \in A$ by f is nonempty, whereas the second condition states that f is not multivalued. As f is a function, both of these statements are true, and we are done.

Item 3: Adjointness

The stated bijection follows from Remark 5.1.1.3, with naturality being clear.

Item 4: Interaction With Inverses

Clear.

Item 5: Cocontinuity

Omitted.

Item 6: Characterisations

We claim that Items 6a to 6d are indeed equivalent:

- *Item 6a* $\iff$ *Item 6b*. This is shown in the proof of Item 4 of Proposition 5.2.5.1.
- *Item 6b* $\implies$ *Item 6c*. If R is total and functional, then, for each $a \in A$, the set $R(a)$ is a singleton, implying that

$$\begin{aligned} R^{-1}(V) &\overset{\text{def}}{=} \{a \in A \mid R(a) \cap V \neq \emptyset\}, \\ R_{-1}(V) &\overset{\text{def}}{=} \{a \in A \mid R(a) \subset V\} \end{aligned}$$

  are equal for all $V \in \mathcal{P}(B)$, as the conditions $R(a) \cap V \neq \emptyset$ and $R(a) \subset V$ are equivalent when $R(a)$ is a singleton.
- *Item 6c* $\implies$ *Item 6b*. We claim that R is indeed total and functional:
  - *Totality.* If we had $R(a) = \emptyset$ for some $a \in A$, then we would have $a \in R_{-1}(\emptyset)$, so that $R_{-1}(\emptyset) \neq \emptyset$. But since $R^{-1}(\emptyset) = \emptyset$, this would imply $R_{-1}(\emptyset) \neq R^{-1}(\emptyset)$, a contradiction. Thus $R(a) \neq \emptyset$ for all $a \in A$ and R is total.
  - *Functionality.* If $R^{-1} = R_{-1}$, then we have

$$\begin{aligned} \{a\} &= R^{-1}(\{b\}) \\ &= R_{-1}(\{b\}) \end{aligned}$$

    for each $b \in R(a)$ and each $a \in A$, and thus $R(a) \subset \{b\}$. But since R is total, we must have $R(a) = \{b\}$, and thus we see that R is functional.
- *Item 6a* $\iff$ *Item 6d*. This follows from Item 4 of Proposition 5.2.5.1.

This finishes the proof. ∎

5.3.2 The Inverse of a Function

Let $f: A \to B$ be a function.

Definition 5.3.2.1 ▶ The Inverse of a Function

The **inverse of** f is the relation $f^{-1}: B \nrightarrow A$ defined as follows:

- Viewing relations from B to A as subsets of $B \times A$, we define
$$f^{-1} \overset{\text{def}}{=} \left\{ (b, f^{-1}(b)) \in B \times A \,\middle|\, a \in A \right\},$$
where
$$f^{-1}(b) \overset{\text{def}}{=} \{a \in A \mid f(a) = b\}$$
for each $b \in B$.
- Viewing relations from B to A as functions $B \times A \to \{\text{true}, \text{false}\}$, we define
$$f^{-1}(b, a) \overset{\text{def}}{=} \begin{cases} \text{true} & \text{if there exists } a \in A \text{ with } f(a) = b, \\ \text{false} & \text{otherwise} \end{cases}$$
for each $(b, a) \in B \times A$;
- Viewing relations from B to A as functions $B \to \mathcal{P}(A)$, we define
$$f^{-1}(b) \overset{\text{def}}{=} \{a \in A \mid f(a) = b\}$$
for each $b \in B$.

Proposition 5.3.2.2 ▶ Properties of Inverses of Functions

Let $f: A \to B$ be a function.

1. *Functoriality.* The assignment $A \mapsto A, f \mapsto f^{-1}$ defines a functor
$$(-)^{-1}: \mathsf{Sets} \to \mathsf{Rel}$$
where
   - *Action on Objects.* For each $A \in \text{Obj}(\mathsf{Sets})$, we have
$$\left[(-)^{-1}\right](A) \overset{\text{def}}{=} A;$$
   - *Action on Morphisms.* For each $A, B \in \text{Obj}(\mathsf{Sets})$, the action on Hom-sets
$$(-)^{-1}_{A,B}: \mathsf{Sets}(A, B) \to \mathsf{Rel}(A, B)$$

of $(-)^{-1}$ at (A, B) is defined by

$$(-)^{-1}_{A,B}(f) \stackrel{\text{def}}{=} \left[(-)^{-1}\right](f),$$

where f^{-1} is the inverse of f as in Definition 5.3.2.1.

In particular:

- *Preservation of Identities.* We have

$$\mathrm{id}_A^{-1} = \chi_A$$

for each $A \in \mathrm{Obj}(\mathsf{Sets})$.

- *Preservation of Composition.* We have

$$(g \circ f)^{-1} = g^{-1} \diamond f^{-1}$$

for pair of functions $f \colon A \to B$ and $g \colon B \to C$.

2. *Adjointness Inside* **Rel**. We have an adjunction

$$(\mathrm{Gr}(f) \dashv f^{-1})\colon \quad \begin{tikzcd} A \arrow[r, bend left, "\mathrm{Gr}(f)"] \arrow[r, phantom, "\perp"] & B \arrow[l, bend left, "f^{-1}"] \end{tikzcd}$$

in **Rel**.

3. *Interaction With Inverses of Relations.* We have

$$\begin{aligned}(f^{-1})^{\dagger} &= \mathrm{Gr}(f),\\ \mathrm{Gr}(f)^{\dagger} &= f^{-1}.\end{aligned}$$

Proof 5.3.2.3 ► Proof of Proposition 5.3.2.2

Item 1: Functoriality

Clear.

Item 2: Adjointness Inside Rel

This is proved in Item 2 of Proposition 5.3.1.2.

Item 3: Interaction With Inverses of Relations

Clear. ∎

5.3.3 Representable Relations

Let A and B be sets.

Definition 5.3.3.1 ▶ Representable Relations

Let $f\colon A \to B$ and $g\colon B \to A$ be functions.[1]

1. The **representable relation associated to** f is the relation $\chi_f\colon A \nrightarrow B$ defined as the composition

$$A \times B \xrightarrow{f\times\mathrm{id}_B} B \times B \xrightarrow{\chi_B} \{\text{true}, \text{false}\},$$

i.e. given by declaring $a \sim_{\chi_f} b$ iff $f(a) = b$.

2. The **corepresentable relation associated to** g is the relation $\chi^g\colon B \nrightarrow A$ defined as the composition

$$B \times A \xrightarrow{g\times\mathrm{id}_A} A \times A \xrightarrow{\chi_A} \{\text{true}, \text{false}\},$$

i.e. given by declaring $b \sim_{\chi^g} a$ iff $g(b) = a$.

[1] More generally, given functions

$$\begin{aligned} f&\colon A \to C, \\ g&\colon B \to D \end{aligned}$$

and a relation $B \nrightarrow D$, we may consider the composite relation

$$A \times B \xrightarrow{f\times g} C \times D \xrightarrow{R} \{\text{true}, \text{false}\},$$

for which we have $a \sim_{R\circ(f\times g)} b$ iff $f(a) \sim_R g(b)$.

5.3.4 The Domain and Range of a Relation

Let A and B be sets.

Definition 5.3.4.1 ▶ The Domain and Range of a Relation

Let $R \subset A \times B$ be a relation.[1,2]

1. The **domain of** R is the subset $\mathrm{dom}(R)$ of A defined by

$$\mathrm{dom}(R) \overset{\text{def}}{=} \left\{ a \in A \;\middle|\; \begin{array}{l} \text{there exists some } b \in B \\ \text{such that } a \sim_R b \end{array} \right\}.$$

2. The **range of** R is the subset $\mathrm{range}(R)$ of B defined by

$$\mathrm{range}(R) \overset{\text{def}}{=} \left\{ b \in B \;\middle|\; \begin{array}{l} \text{there exists some } a \in A \\ \text{such that } a \sim_R b \end{array} \right\}.$$

[1] Following ??, we may compute the (characteristic functions associated to the) domain and range of a relation using the following colimit formulas:

$$
\begin{aligned}
\chi_{\mathrm{dom}(R)}(a) &\cong \operatorname*{colim}_{b\in B}\left(R^a_b\right) \qquad (a \in A)\\
&\cong \bigvee_{b\in B} R^a_b,\\
\chi_{\mathrm{range}(R)}(b) &\cong \operatorname*{colim}_{a\in A}\left(R^a_b\right) \qquad (b \in B)\\
&\cong \bigvee_{a\in A} R^a_b,
\end{aligned}
$$

where the join $\bigvee$ is taken in the poset $(\{\mathrm{true}, \mathrm{false}\}, \preceq)$ of **??**.

[2] Viewing R as a function $R\colon A \to \mathcal{P}(B)$, we have

$$
\begin{aligned}
\mathrm{dom}(R) &\cong \operatorname*{colim}_{y\in Y}(R(y))\\
&\cong \bigcup_{y\in Y} R(y),\\
\mathrm{range}(R) &\cong \operatorname*{colim}_{x\in X}(R(x))\\
&\cong \bigcup_{x\in X} R(x),
\end{aligned}
$$

5.3.5 Binary Unions of Relations

Let A and B be sets and let R and S be relations from A to B.

Definition 5.3.5.1 ► Binary Unions of Relations

The **union of R and S**[1] is the relation $R \cup S$ from A to B defined as follows:

- Viewing relations from A to B as subsets of $A \times B$, we define[2]

$$R \cup S \stackrel{\text{def}}{=} \{(a, b) \in B \times A \mid \text{we have } a \sim_R b \text{ or } a \sim_S b\}.$$

- Viewing relations from A to B as functions $A \to \mathcal{P}(B)$, we define

$$[R \cup S](a) \stackrel{\text{def}}{=} R(a) \cup S(a)$$

for each $a \in A$.

[1] *Further Terminology:* Also called the **binary union of R and S**, for emphasis.

[2] This is the same as the union of R and S as subsets of $A \times B$.

Proposition 5.3.5.2 ► Properties of Binary Unions of Relations

Let R, S, R_1, and R_2 be relations from A to B, and let S_1 and S_2 be relations from B to C.

1. *Interaction With Inverses.* We have
$$(R \cup S)^{\dagger} = R^{\dagger} \cup S^{\dagger}.$$

2. *Interaction With Composition.* We have
$$(S_1 \diamond R_1) \cup (S_2 \diamond R_2) \overset{\text{poss.}}{\neq} (S_1 \cup S_2) \diamond (R_1 \cup R_2).$$

Proof 5.3.5.3 ► Proof of Proposition 5.3.5.2

Item 1: Interaction With Inverses

Clear.

Item 2: Interaction With Composition

Unwinding the definitions, we see that:

1. The condition for $(S_1 \diamond R_1) \cup (S_2 \diamond R_2)$ is:

   (a) There exists some $b \in B$ such that:

      i. $a \sim_{R_1} b$ and $b \sim_{S_1} c$;

   or

      i. $a \sim_{R_2} b$ and $b \sim_{S_2} c$;

3. The condition for $(S_1 \cup S_2) \diamond (R_1 \cup R_2)$ is:

   (a) There exists some $b \in B$ such that:

      i. $a \sim_{R_1} b$ or $a \sim_{R_2} b$;

   and

      i. $b \sim_{S_1} c$ or $b \sim_{S_2} c$.

These two conditions may fail to agree (counterexample omitted), and thus the two resulting relations on $A \times C$ may differ. □

5.3.6 Unions of Families of Relations

Let A and B be sets and let $\{R_i\}_{i \in I}$ be a family of relations from A to B.

Definition 5.3.6.1 ► The Union of a Family of Relations

The **union of the family** $\{R_i\}_{i\in I}$ is the relation $\bigcup_{i\in I} R_i$ from A to B defined as follows:

- Viewing relations from A to B as subsets of $A \times B$, we define[1]

$$\bigcup_{i\in I} R_i \stackrel{\text{def}}{=} \left\{ (a,b) \in (A\times B)^{\times I} \;\middle|\; \begin{array}{l}\text{there exists some } i \in I\\ \text{such that } a \sim_{R_i} b\end{array} \right\}.$$

- Viewing relations from A to B as functions $A \to \mathcal{P}(B)$, we define

$$\left[\bigcup_{i\in I} R_i\right](a) \stackrel{\text{def}}{=} \bigcup_{i\in I} R_i(a)$$

  for each $a \in A$.

[1]This is the same as the union of $\{R_i\}_{i\in I}$ as a collection of subsets of $A \times B$.

Proposition 5.3.6.2 ► Properties of Unions of Families of Relations

Let A and B be sets and let $\{R_i\}_{i\in I}$ be a family of relations from A to B.

1. *Interaction With Inverses.* We have

$$\left(\bigcup_{i\in I} R_i\right)^{\dagger} = \bigcup_{i\in I} R_i^{\dagger}.$$

Proof 5.3.6.3 ► Proof of Proposition 5.3.6.2

Item 1: Interaction With Inverses

Clear.

5.3.7 Binary Intersections of Relations

Let A and B be sets and let R and S be relations from A to B.

Definition 5.3.7.1 ► Binary Intersections of Relations

The **intersection of R and S**[1] is the relation $R \cap S$ from A to B defined as follows:

- Viewing relations from A to B as subsets of $A \times B$, we define[2]

$$R \cap S \stackrel{\text{def}}{=} \{(a,b) \in B \times A \mid \text{we have } a \sim_R b \text{ and } a \sim_S b\}.$$

- Viewing relations from A to B as functions $A \longrightarrow \mathcal{P}(B)$, we define

$$[R \cap S](a) \stackrel{\text{def}}{=} R(a) \cap S(a)$$

for each $a \in A$.

[1] *Further Terminology:* Also called the **binary intersection of R and S**, for emphasis.
[2] This is the same as the intersection of R and S as subsets of $A \times B$.

Proposition 5.3.7.2 ► Properties of Binary Intersections of Relations

Let R, S, R_1, and R_2 be relations from A to B, and let S_1 and S_2 be relations from B to C.

1. *Interaction With Inverses.* We have

$$(R \cap S)^{\dagger} = R^{\dagger} \cap S^{\dagger}.$$

2. *Interaction With Composition.* We have

$$(S_1 \diamond R_1) \cap (S_2 \diamond R_2) = (S_1 \cap S_2) \diamond (R_1 \cap R_2).$$

Proof 5.3.7.3 ► Proof of Proposition 5.3.7.2

Item 1: Interaction With Inverses

Clear.

Item 2: Interaction With Composition

Unwinding the definitions, we see that:

1. The condition for $(S_1 \diamond R_1) \cap (S_2 \diamond R_2)$ is:
   (a) There exists some $b \in B$ such that:
      i. $a \sim_{R_1} b$ and $b \sim_{S_1} c$;
   and
      i. $a \sim_{R_2} b$ and $b \sim_{S_2} c$;
3. The condition for $(S_1 \cap S_2) \diamond (R_1 \cap R_2)$ is:
   (a) There exists some $b \in B$ such that:
      i. $a \sim_{R_1} b$ and $a \sim_{R_2} b$;
   and
      i. $b \sim_{S_1} c$ and $b \sim_{S_2} c$.

These two conditions agree, and thus so do the two resulting relations on $A \times C$. ∎

5.3.8 Intersections of Families of Relations

Let A and B be sets and let $\{R_i\}_{i\in I}$ be a family of relations from A to B.

Definition 5.3.8.1 ► The Intersection of a Family of Relations

The **intersection of the family** $\{R_i\}_{i\in I}$ is the relation $\bigcup_{i\in I} R_i$ defined as follows:

- Viewing relations from A to B as subsets of $A \times B$, we define[1]

$$\bigcup_{i\in I} R_i \overset{\text{def}}{=} \left\{ (a,b) \in (A \times B)^{\times I} \,\middle|\, \begin{array}{l} \text{for each } i \in I, \\ \text{we have } a \sim_{R_i} b \end{array} \right\}.$$

- Viewing relations from A to B as functions $A \longrightarrow \mathcal{P}(B)$, we define

$$\left[\bigcap_{i\in I} R_i\right](a) \overset{\text{def}}{=} \bigcap_{i\in I} R_i(a)$$

  for each $a \in A$.

[1] This is the same as the intersection of $\{R_i\}_{i\in I}$ as a collection of subsets of $A \times B$.

Proposition 5.3.8.2 ► Properties of Intersections of Families of Relations

Let A and B be sets and let $\{R_i\}_{i\in I}$ be a family of relations from A to B.

1. *Interaction With Inverses.* We have

$$\left(\bigcap_{i\in I} R_i\right)^{\dagger} = \bigcap_{i\in I} R_i^{\dagger}.$$

Proof 5.3.8.3 ► Proof of Proposition 5.3.8.2

Item 1: Interaction With Inverses

Clear.

5.3.9 Binary Products of Relations

Let A, B, X, and Y be sets, let $R \colon A \nrightarrow B$ be a relation from A to B, and let $S \colon X \nrightarrow Y$ be a relation from X to Y.

Definition 5.3.9.1 ► Binary Products of Relations

The **product of R and S**[1] is the relation $R \times S$ from $A \times X$ to $B \times Y$ defined as follows:

- Viewing relations from $A \times X$ to $B \times Y$ as subsets of $(A \times X) \times (B \times Y)$, we define $R \times S$ as the Cartesian product of R and S as subsets of $A \times X$ and $B \times Y$;[2]

- Viewing relations from $A \times X$ to $B \times Y$ as functions $A \times X \to \mathcal{P}(B \times Y)$, we define $R \times S$ as the composition

$$A \times X \xrightarrow{R \times S} \mathcal{P}(B) \times \mathcal{P}(Y) \overset{\mathcal{P}^{\otimes}_{B,Y}}{\hookrightarrow} \mathcal{P}(B \times Y)$$

  in Sets, i.e. by

$$[R \times S](a, x) \overset{\text{def}}{=} R(a) \times S(x)$$

  for each $(a, x) \in A \times X$.

[1] *Further Terminology:* Also called the **binary product of R and S**, for emphasis.

[2] That is, $R \times S$ is the relation given by declaring $(a, x) \sim_{R \times S} (b, y)$ iff $a \sim_R b$ and $x \sim_S y$.

Proposition 5.3.9.2 ► Properties of Binary Products of Relations

Let A, B, X, and Y be sets.

1. *Interaction With Inverses.* Let

$$\begin{aligned} R &\colon A \nrightarrow A, \\ S &\colon X \nrightarrow X \end{aligned}$$

   We have

$$(R \times S)^{\dagger} = R^{\dagger} \times S^{\dagger}.$$

2. *Interaction With Composition.* Let

$$\begin{aligned} R_1 &\colon A \nrightarrow B, \\ S_1 &\colon B \nrightarrow C, \\ R_2 &\colon X \nrightarrow Y, \\ S_2 &\colon Y \nrightarrow Z \end{aligned}$$

   be relations. We have

$$(S_1 \diamond R_1) \times (S_2 \diamond R_2) = (S_1 \times S_2) \diamond (R_1 \times R_2).$$

PROOF 5.3.9.3 ► PROOF OF PROPOSITION 5.3.5.2

Item 1: Interaction With Inverses

Unwinding the definitions, we see that:

1. We have $(a, x) \sim_{(R\times S)^\dagger} (b, y)$ iff:
   - We have $(b, y) \sim_{R\times S} (a, x)$, i.e. iff:
     - We have $b \sim_R a$;
     - We have $y \sim_S x$;
2. We have $(a, x) \sim_{R^\dagger \times S^\dagger} (b, y)$ iff:
   - We have $a \sim_{R^\dagger} b$ and $x \sim_{S^\dagger} y$, i.e. iff:
     - We have $b \sim_R a$;
     - We have $y \sim_S x$.

These two conditions agree, and thus the two resulting relations on $A \times X$ are equal.

Item 2: Interaction With Composition

Unwinding the definitions, we see that:

1. We have $(a, x) \sim_{(S_1 \diamond R_1)\times(S_2 \diamond R_2)} (c, z)$ iff:
   (a) We have $a \sim_{S_1 \diamond R_1} c$ and $x \sim_{S_2 \diamond R_2} z$, i.e. iff:
      i. There exists some $b \in B$ such that $a \sim_{R_1} b$ and $b \sim_{S_1} c$;
      ii. There exists some $y \in Y$ such that $x \sim_{R_2} y$ and $y \sim_{S_2} z$;
2. We have $(a, x) \sim_{(S_1\times S_2)\diamond(R_1\times R_2)} (c, z)$ iff:
   (a) There exists some $(b, y) \in B \times Y$ such that $(a, x) \sim_{R_1\times R_2} (b, y)$ and $(b, y) \sim_{S_1\times S_2} (c, z)$, i.e. such that:
      i. We have $a \sim_{R_1} b$ and $x \sim_{R_2} y$;
      ii. We have $b \sim_{S_1} c$ and $y \sim_{S_2} z$.

These two conditions agree, and thus the two resulting relations from $A \times X$ to $C \times Z$ are equal.

5.3.10 Products of Families of Relations

Let $\{A_i\}_{i\in I}$ and $\{B_i\}_{i\in I}$ be families of sets, and let $\{R_i : A_i \nrightarrow B_i\}_{i\in I}$ be a family of relations.

Definition 5.3.10.1 ► The Product of a Family of Relations

The **product of the family** $\{R_i\}_{i\in I}$ is the relation $\prod_{i\in I} R_i$ from $\prod_{i\in I} A_i$ to $\prod_{i\in I} B_i$ defined as follows:

- Viewing relations as subsets, we define $\prod_{i\in I} R_i$ as its product as a family of sets, i.e. we have

$$\prod_{i\in I} R_i \stackrel{\text{def}}{=} \left\{ (a_i, b_i)_{i\in I} \in \prod_{i\in I} (A_i \times B_i) \;\middle|\; \begin{array}{l} \text{for each } i \in I, \\ \text{we have } a_i \sim_{R_i} b_i \end{array} \right\}.$$

- Viewing relations as functions to powersets, we define

$$\left[\prod_{i\in I} R_i\right]((a_i)_{i\in I}) \stackrel{\text{def}}{=} \prod_{i\in I} R_i(a_i)$$

  for each $(a_i)_{i\in I} \in \prod_{i\in I} R_i$.

5.3.11 The Inverse of a Relation

Let A, B, and C be sets and let $R \subset A \times B$ be a relation.

Definition 5.3.11.1 ► The Inverse of a Relation

The **inverse of** R[1] is the relation $R^\dagger$ defined as follows:

- Viewing relations as subsets, we define

$$R^\dagger \stackrel{\text{def}}{=} \{(b, a) \in B \times A \mid \text{we have } b \sim_R a\}.$$

- Viewing relations as functions $A \times B \to \{\text{true}, \text{false}\}$, we define

$$[R^\dagger]^b_a \stackrel{\text{def}}{=} R^a_b$$

  for each $(b, a) \in B \times A$.

- Viewing relations as functions $A \to \mathcal{P}(B)$, we define

$$\begin{aligned} [R^\dagger](b) &\stackrel{\text{def}}{=} R^\dagger(\{b\}) \\ &\stackrel{\text{def}}{=} \{a \in A \mid b \in R(a)\} \end{aligned}$$

  for each $b \in B$, where $R^\dagger(\{b\})$ is the fibre of R over $\{b\}$.

[1] *Further Terminology:* Also called the **opposite of** R, the **transpose of** R, or the **converse of** R.

Example 5.3.11.2 ▶ Examples of Inverses of Relations

Here are some examples of inverses of relations.

1. *Less Than Equal Signs.* We have $(\leq)^{\dagger} = \geq$.
2. *Greater Than Equal Signs.* Dually to **??**, we have $(\geq)^{\dagger} = \leq$.
3. *Functions.* Let $f \colon A \to B$ be a function. We have

$$\begin{aligned} \mathrm{Gr}(f)^{\dagger} &= f^{-1}, \\ \left(f^{-1}\right)^{\dagger} &= \mathrm{Gr}(f). \end{aligned}$$

Proposition 5.3.11.3 ▶ Properties of Inverses of Relations

Let $R \colon A \nrightarrow B$ and $S \colon B \nrightarrow C$ be relations.

1. *Interaction With Ranges and Domains.* We have

$$\begin{aligned} \mathrm{dom}\left(R^{\dagger}\right) &= \mathrm{range}(R), \\ \mathrm{range}\left(R^{\dagger}\right) &= \mathrm{dom}(R). \end{aligned}$$

2. *Interaction With Composition I.* We have

$$(S \diamond R)^{\dagger} = R^{\dagger} \diamond S^{\dagger}.$$

3. *Interaction With Composition II.* We have

$$\begin{aligned} \chi_B(-_1, -_2) &\subset R \diamond R^{\dagger}, \\ \chi_A(-_1, -_2) &\subset R^{\dagger} \diamond R. \end{aligned}$$

4. *Invertibility.* We have

$$\left(R^{\dagger}\right)^{\dagger} = R.$$

5. *Identity.* We have

$$\chi_A^{\dagger}(-_1, -_2) = \chi_A(-_1, -_2).$$

Proof 5.3.11.4 ▶ Proof of Proposition 5.3.11.3

Item 1: Interaction With Ranges and Domains

Clear.

Item 2: Interaction With Composition I

Clear.

Item 3: Interaction With Composition II

Clear.

Item 4: Invertibility

Clear.

Item 5: Identity

Clear.

5.3.12 Composition of Relations

Let A, B, and C be sets and let $R \subset A \times B$ and $S \subset B \times C$ be relations.

Definition 5.3.12.1 ▶ Composition of Relations

The **composition of** R **and** S is the relation $S \diamond R$ defined as follows:

- Viewing relations from A to C as subsets of $A \times C$, we define

$$S \diamond R \overset{\text{def}}{=} \left\{ (a, c) \in A \times C \;\middle|\; \begin{array}{l} \text{there exists some } b \in B \text{ such} \\ \text{that } a \sim_R b \text{ and } b \sim_S c \end{array} \right\}.$$

- Viewing relations as functions $A \times B \to \{\text{true}, \text{false}\}$, we define

$$\begin{aligned} (S \diamond R)^{-1}_{-2} &\overset{\text{def}}{=} \int^{y \in B} S^{-1}_{y} \times R^{y}_{-2} \\ &= \bigvee_{y \in B} S^{-1}_{y} \times R^{y}_{-2}, \end{aligned}$$

  where the join $\bigvee$ is taken in the poset $(\{\text{true}, \text{false}\}, \preceq)$ of **??**.

- Viewing relations as functions $A \to \mathcal{P}(B)$, we define

$$S \diamond R \overset{\text{def}}{=} \mathrm{Lan}_{\chi_B}(S) \circ R, \qquad \begin{tikzcd} & B \arrow[r, "S"] \arrow[d, hook, "\chi_B"'] & \mathcal{P}(C), \\ A \arrow[r, "R"'] & \mathcal{P}(B) \arrow[ur, "\mathrm{Lan}_{\chi_B}(S)"'] \arrow[u, phantom, shift left=3, "\Leftarrow"] & \end{tikzcd}$$

where $\mathrm{Lan}_{\chi_B}(S)$ is computed by the formula

$$\begin{aligned}
[\mathrm{Lan}_{\chi_B}(S)](V) &\cong \int^{y\in B} \chi_{\mathcal{P}(B)}\big(\chi_y, V\big) \odot S_y \\
&\cong \int^{y\in B} \chi_V(y) \odot S_y \\
&\cong \bigcup_{y\in B} \chi_V(y) \odot S_y \\
&\cong \bigcup_{y\in V} S_y
\end{aligned}$$

for each $V \in \mathcal{P}(B)$. In other words, $S \diamond R$ is defined by[1]

$$\begin{aligned}
[S \diamond R](a) &\stackrel{\text{def}}{=} S(R(a)) \\
&\stackrel{\text{def}}{=} \bigcup_{x\in R(a)} S(x).
\end{aligned}$$

for each $a \in A$.

[1]That is: the relation R may send $a \in A$ to a number of elements $\{b_i\}_{i\in I}$ in B, and then the relation S may send the image of each of the b_i's to a number of elements $\{S(b_i)\}_{i\in I} = \left\{\{c_{j_i}\}_{j_i\in J_i}\right\}_{i\in I}$ in C.

Example 5.3.12.2 ▶ Examples of Composition of Relations

Here are some examples of composition of relations.

1. *Composing Less/Greater Than Equal With Greater/Less Than Equal Signs.* We have

$$\begin{aligned}
\leq \diamond \geq &= \sim_{\text{triv}}, \\
\geq \diamond \leq &= \sim_{\text{triv}} .
\end{aligned}$$

2. *Composing Less/Greater Than Equal Signs With Less/Greater Than Equal Signs.* We have

$$\begin{aligned}
\leq \diamond \leq &= \leq, \\
\geq \diamond \geq &= \geq.
\end{aligned}$$

PROPOSITION 5.3.12.3 ► PROPERTIES OF COMPOSITION OF RELATIONS

Let $R\colon A \rightarrow\!\!\!\!\!+ B$, $S\colon B \rightarrow\!\!\!\!\!+ C$, and $T\colon C \rightarrow\!\!\!\!\!+ D$ be relations.

1. *Interaction With Ranges and Domains.* We have

$$\begin{aligned}\mathrm{dom}(S \diamond R) &\subset \mathrm{dom}(R),\\ \mathrm{range}(S \diamond R) &\subset \mathrm{range}(S).\end{aligned}$$

2. *Associativity.* We have

$$(T \diamond S) \diamond R = T \diamond (S \diamond R).$$

3. *Unitality.* We have

$$\begin{aligned}\chi_B \diamond R &= R,\\ R \diamond \chi_A &= R.\end{aligned}$$

4. *Interaction With Inverses.* We have

$$(S \diamond R)^{\dagger} = R^{\dagger} \diamond S^{\dagger}.$$

5. *Interaction With Composition.* We have

$$\begin{aligned}\chi_B(-_1, -_2) &\subset R \diamond R^{\dagger},\\ \chi_A(-_1, -_2) &\subset R^{\dagger} \diamond R.\end{aligned}$$

PROOF 5.3.12.4 ► PROOF OF PROPOSITION 5.3.12.3

Item 1: Interaction With Ranges and Domains

Clear.

Item 2: Associativity

Indeed, we have

$$\begin{aligned}
(T \diamond S) \diamond R &\overset{\text{def}}{=} \left(\int^{y \in C} T^{-1}_{x} \times S^{x}_{-_2}\right) \diamond R \\
&\overset{\text{def}}{=} \int^{x \in B} \left(\int^{y \in C} T^{-1}_{x} \times S^{x}_{y}\right) \diamond R^{y}_{-_2} \\
&= \int^{x \in B} \int^{y \in C} \left(T^{-1}_{x} \times S^{x}_{y}\right) \diamond R^{y}_{-_2} \\
&= \int^{y \in C} \int^{x \in B} \left(T^{-1}_{x} \times S^{x}_{y}\right) \diamond R^{y}_{-_2} \\
&= \int^{y \in C} \int^{x \in B} T^{-1}_{x} \times \left(S^{x}_{y} \diamond R^{y}_{-_2}\right) \\
&= \int^{x \in B} T^{-1}_{x} \times \left(\int^{y \in C} S^{x}_{y} \diamond R^{y}_{-_2}\right) \\
&\overset{\text{def}}{=} \int^{x \in B} T^{-1}_{x} \times (S \diamond R)^{x}_{-_2} \\
&\overset{\text{def}}{=} T \diamond (S \diamond R).
\end{aligned}$$

In the language of relations, given $a \in A$ and $d \in D$, the stated equality witnesses the equivalence of the following two statements:

1. We have $a \sim_{(T \diamond S) \diamond R} d$, i.e. there exists some $b \in B$ such that:
   (a) We have $a \sim_R b$;
   (b) We have $b \sim_{T \diamond S} d$, i.e. there exists some $c \in C$ such that:
      i. We have $b \sim_S c$;
      ii. We have $c \sim_T d$;
2. We have $a \sim_{T \diamond (S \diamond R)} d$, i.e. there exists some $c \in C$ such that:
   (a) We have $a \sim_{S \diamond R} c$, i.e. there exists some $b \in B$ such that:
      i. We have $a \sim_R b$;
      ii. We have $b \sim_S c$;
   (b) We have $c \sim_T d$;

both of which are equivalent to the statement

- There exist $b \in B$ and $c \in C$ such that $a \sim_R b \sim_S c \sim_T d$.

Item 3: Unitality

Indeed, we have

$$\begin{aligned}\chi_B \diamond R &\stackrel{\text{def}}{=} \int^{x\in B} (\chi_B)^{-1}_{x} \times R^{x}_{-2}\\ &= \bigvee_{x\in B} (\chi_B)^{-1}_{x} \times R^{x}_{-2}\\ &= \bigvee_{\substack{x\in B\\ x=-1}} R^{x}_{-2}\\ &= R^{-1}_{-2},\end{aligned}$$

and

$$\begin{aligned}R \diamond \chi_A &\stackrel{\text{def}}{=} \int^{x\in A} R^{-1}_{x} \times (\chi_A)^{x}_{-2}\\ &= \bigvee_{x\in B} R^{-1}_{x} \times (\chi_A)^{x}_{-2}\\ &= \bigvee_{\substack{x\in B\\ x=-2}} R^{-1}_{x}\\ &= R^{-1}_{-2}.\end{aligned}$$

In the language of relations, given $a \in A$ and $b \in B$:

- The equality

$$\chi_B \diamond R = R$$

witnesses the equivalence of the following two statements:

1. We have $a \sim_b B$.
2. There exists some $b' \in B$ such that:
   (a) We have $a \sim_R b'$
   (b) We have $b' \sim_{\chi_B} b$, i.e. $b' = b$.

- The equality

$$R \diamond \chi_A = R$$

witnesses the equivalence of the following two statements:

1. There exists some $a' \in A$ such that:
   (a) We have $a \sim_{\chi_B} a'$, i.e. $a = a'$.
   (b) We have $a' \sim_R b$

2. We have $a \sim_b B$.

Item 4: Interaction With Inverses

Clear.

Item 5: Interaction With Composition

Clear.

5.3.13 The Collage of a Relation

Let A and B be sets and let $R\colon A \nrightarrow B$ be a relation from A to B.

Definition 5.3.13.1 ► The Collage of a Relation

The **collage of** R[1] is the poset $\mathbf{Coll}(R) \overset{\text{def}}{=} (\mathrm{Coll}(R), \preceq_{\mathbf{Coll}(R)})$ consisting of

- *The Underlying Set.* The set $\mathrm{Coll}(R)$ defined by

$$\mathrm{Coll}(R) \overset{\text{def}}{=} A \coprod B.$$

- *The Partial Order.* The partial order

$$\preceq_{\mathbf{Coll}(R)} : \mathrm{Coll}(R) \times \mathrm{Coll}(R) \to \{\mathrm{true}, \mathrm{false}\}$$

on $\mathrm{Coll}(R)$ defined by

$$\preceq(a, b) \overset{\text{def}}{=} \begin{cases} \mathrm{true} & \text{if } a = b \text{ or } a \sim_R b, \\ \mathrm{false} & \text{otherwise.} \end{cases}$$

[1] *Further Terminology:* Also called the **cograph of** R.

Proposition 5.3.13.2 ► Properties of Collages of Relations

Let A and B be sets and let $R\colon A \nrightarrow B$ be a relation from A to B.

1. *Functoriality I.* The assignment $R \mapsto \mathbf{Coll}(R)$ defines a functor[1]

$$\mathbf{Coll}\colon \mathbf{Rel}(A, B) \to \mathrm{Pos}_{/\Delta^1}(A, B),$$

where

- *Action on Objects.* For each $R \in \mathrm{Obj}(\mathbf{Rel}(A, B))$, we have

$$[\mathbf{Coll}](R) \overset{\text{def}}{=} (\mathbf{Coll}(R), \phi_R)$$

for each $R \in \mathbf{Rel}(A, B)$, where

- The poset $\mathbf{Coll}(R)$ is the collage of R of Definition 5.3.13.1;
- The morphism $\phi_R \colon \mathbf{Coll}(R) \to \Delta^1$ is given by

$$\phi_R(x) \overset{\text{def}}{=} \begin{cases} 0 & \text{if } x \in A, \\ 1 & \text{if } x \in B \end{cases}$$

for each $x \in \mathbf{Coll}(R)$;

- *Action on Morphisms.* For each $R, S \in \mathrm{Obj}(\mathbf{Rel}(A, B))$, the action on Hom-sets

$$\mathbf{Coll}_{R,S} \colon \mathrm{Hom}_{\mathbf{Rel}(A,B)}(R, S) \to \mathrm{Pos}(\mathbf{Coll}(R), \mathbf{Coll}(S))$$

of $\mathbf{Coll}$ at (R, S) is given by sending an inclusion

$$\iota \colon R \subset S$$

to the morphism

$$\mathbf{Coll}(\iota) \colon \mathbf{Coll}(R) \to \mathbf{Coll}(S)$$

of posets over Δ^1 defined by

$$[\mathbf{Coll}(\iota)](x) \overset{\text{def}}{=} x$$

for each $x \in \mathbf{Coll}(R)$.[2]

2. *Equivalence.* The functor of Item 1 is an equivalence of categories.

---

[1]Here $\mathrm{Pos}_{/\Delta^1}(A, B)$ is the category defined as the pullback

$$\mathrm{Pos}_{/\Delta^1}(A, B) \overset{\text{def}}{=} \mathrm{pt} \underset{[A],\mathsf{Pos},\mathrm{fib}_0}{\times} \mathrm{Pos}_{/\Delta^1} \underset{\mathrm{fib}_1,\mathsf{Pos},[B]}{\times} \mathrm{pt},$$

as in the diagram

$$\begin{tikzcd}
 & & \mathbf{Pos}_{/\Delta^1}(A, B) \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & & \\
 & \mathbf{Pos}_{/\Delta^1} \underset{\mathbf{Pos}}{\times} \mathrm{pt} \arrow[dl] \arrow[dr] & & \mathrm{pt} \underset{\mathbf{Pos}}{\times} \mathbf{Pos}_{/\Delta^1} \arrow[dl] \arrow[dr] & \\
\mathrm{pt} \arrow[dr, "{[A]}"'] & & \mathbf{Pos}_{/\Delta^1} \arrow[dl, "\mathrm{fib}_{[0]}"] \arrow[dr, "\mathrm{fib}_{[1]}"'] & & \mathrm{pt}. \arrow[dl, "{[B]}"] \\
 & \mathbf{Pos} & & \mathbf{Pos} &
\end{tikzcd}$$

Explicitly, an object of $\mathrm{Pos}_{/\Delta^1}(A, B)$ is a pair (X, ϕ_X) consisting of

- A poset X;
- A morphism $\phi_X \colon X \to \Delta^1$;

such that $\phi_X^{-1}(0) = A$ and $\phi_X^{-1}(0) = B$, with morphisms between such objects being morphisms of posets over Δ^1.

[2]Note that this is indeed a morphism of posets: if $x \preceq_{\mathbf{Coll}(R)} y$, then $x = y$ or $x \sim_R y$, so we have either $x = y$ or $x \sim_S y$ (as $R \subset S$), and thus $x \preceq_{\mathbf{Coll}(S)} y$.

PROOF 5.3.13.3 ▶ PROOF OF PROPOSITION 5.3.13.2

Item 1: Functoriality

Clear.

Item 2: Equivalence

Omitted.

5.4 Equivalence Relations

5.4.1 Reflexive Relations

5.4.1.1 Foundations

Let A be a set.

DEFINITION 5.4.1.1 ▶ REFLEXIVE RELATIONS

A **reflexive relation** is equivalently:[1]

- An $\mathbb{E}_0$-monoid in $(\mathsf{N}_\bullet(\mathbf{Rel}(A, A)), \chi_A)$;
- A pointed object in $(\mathbf{Rel}(A, A), \chi_A)$.

[1] Note that since $\mathbf{Rel}(A, A)$ is posetal, reflexivity is a property of a relation, rather than extra structure.

REMARK 5.4.1.2 ▶ UNWINDING DEFINITION 5.4.1.1

In detail, a relation R on A is **reflexive** if we have an inclusion

$$\eta_R \colon \chi_A \subset R$$

of relations in $\mathbf{Rel}(A, A)$, i.e. if, for each $a \in A$, we have $a \sim_R a$.

DEFINITION 5.4.1.3 ▶ THE PO/SET OF REFLEXIVE RELATIONS ON A SET

Let A be a set.

1. The **set of reflexive relations on** A is the subset $\mathrm{Rel}^{\mathrm{refl}}(A, A)$ of $\mathrm{Rel}(A, A)$ spanned by the reflexive relations.
2. The **poset of relations on** A is is the subposet $\mathbf{Rel}^{\mathrm{refl}}(A, A)$ of $\mathbf{Rel}(A, A)$ spanned by the reflexive relations.

Proposition 5.4.1.4 ► Properties of Reflexive Relations

Let R and S be relations on A.

1. *Interaction With Inverses.* If R is reflexive, then so is $R^{\dagger}$.
2. *Interaction With Composition.* If R and S are reflexive, then so is $S \diamond R$.

Proof 5.4.1.5 ► Proof of Proposition 5.4.1.4

Item 1: Interaction With Inverses

Clear.

Item 2: Interaction With Composition

Clear.

5.4.1.2 The Reflexive Closure of a Relation

Let R be a relation on A.

Definition 5.4.1.6 ► The Reflexive Closure of a Relation

The **reflexive closure** of $\sim_R$ is the relation $\sim_R^{\mathrm{refl}}$[1] satisfying the following universal property:[2]

($\star$) Given another reflexive relation $\sim_S$ on A such that $R \subset S$, there exists an inclusion $\sim_R^{\mathrm{refl}} \subset \sim_S$.

[1] *Further Notation:* Also written R^{refl}.
[2] *Slogan:* The reflexive closure of R is the smallest reflexive relation containing R.

Construction 5.4.1.7 ► The Reflexive Closure of a Relation

Concretely, $\sim_R^{\mathrm{refl}}$ is the free pointed object on R in $(\mathbf{Rel}(A, A), \chi_A)$[1], being given by

$$\begin{aligned} R^{\mathrm{refl}} &\stackrel{\mathrm{def}}{=} R \coprod^{\mathbf{Rel}(A,A)} \Delta_A \\ &= R \cup \Delta_A \\ &= \{(a, b) \in A \times A \mid \text{we have } a \sim_R b \text{ or } a = b\}. \end{aligned}$$

[1] Or, equivalently, the free $\mathbb{E}_0$-monoid on R in $(\mathsf{N}_\bullet(\mathbf{Rel}(A, A)), \chi_A)$.

Proof 5.4.1.8 ► Proof of ??

Clear.

Proposition 5.4.1.9 ▶ Properties of the Reflexive Closure of a Relation

Let R be a relation on A.

1. *Adjointness.* We have an adjunction
$$\left((-)^{\mathrm{refl}} \dashv 忘\right)\colon\quad \begin{tikzcd} \mathbf{Rel}(A,A) \arrow[r, bend left, "(-)^{\mathrm{refl}}"] \arrow[r, phantom, "\perp"] & \mathbf{Rel}^{\mathrm{refl}}(A,A), \arrow[l, bend left, "忘"] \end{tikzcd}$$
witnessed by a bijection of sets
$$\mathbf{Rel}^{\mathrm{refl}}\left(R^{\mathrm{refl}}, S\right) \cong \mathbf{Rel}(R,S),$$
natural in $R \in \mathrm{Obj}\left(\mathbf{Rel}^{\mathrm{refl}}(A,A)\right)$ and $S \in \mathrm{Obj}(\mathbf{Rel}(A,A))$.

2. *The Reflexive Closure of a Reflexive Relation.* If R is reflexive, then $R^{\mathrm{refl}} = R$.

3. *Idempotency.* We have
$$\left(R^{\mathrm{refl}}\right)^{\mathrm{refl}} = R^{\mathrm{refl}}.$$

4. *Interaction With Inverses.* We have
$$\left(R^{\dagger}\right)^{\mathrm{refl}} = \left(R^{\mathrm{refl}}\right)^{\dagger}, \qquad \begin{tikzcd} \mathrm{Rel}(A,A) \arrow[r, "(-)^{\mathrm{refl}}"] \arrow[d, "(-)^{\dagger}"'] & \mathrm{Rel}(A,A) \arrow[d, "(-)^{\dagger}"] \\ \mathrm{Rel}(A,A) \arrow[r, "(-)^{\mathrm{refl}}"'] & \mathrm{Rel}(A,A). \end{tikzcd}$$

5. *Interaction With Composition.* We have
$$(S \diamond R)^{\mathrm{refl}} = S^{\mathrm{refl}} \diamond R^{\mathrm{refl}}, \qquad \begin{tikzcd} \mathrm{Rel}(A,A) \times \mathrm{Rel}(A,A) \arrow[r, "\diamond"] \arrow[d, "(-)^{\mathrm{refl}} \times (-)^{\mathrm{refl}}"'] & \mathrm{Rel}(A,A) \arrow[d, "(-)^{\mathrm{refl}}"] \\ \mathrm{Rel}(A,A) \times \mathrm{Rel}(A,A) \arrow[r, "\diamond"'] & \mathrm{Rel}(A,A). \end{tikzcd}$$

Proof 5.4.1.10 ► Proof of Proposition 5.4.1.9

Item 1: Adjointness

This is a rephrasing of the universal property of the reflexive closure of a relation, stated in Definition 5.4.1.6.

Item 2: The Reflexive Closure of a Reflexive Relation

Clear.

Item 3: Idempotency

This follows from Item 2.

Item 4: Interaction With Inverses

Clear.

Item 5: Interaction With Composition

This follows from Item 2 of Proposition 5.4.1.4.

5.4.2 Symmetric Relations

5.4.2.1 Foundations

Let A be a set.

Definition 5.4.2.1 ► Symmetric Relations

A relation R on A is **symmetric** if, for each $a, b \in A$, the following conditions are equivalent:[1]

1. We have $a \sim_R b$.
2. We have $b \sim_R a$.

[1] That is, R is symmetric if $R^\dagger = R$.

Definition 5.4.2.2 ► The Po/Set of Symmetric Relations on a Set

Let A be a set.

1. The **set of symmetric relations on** A is the subset $\mathrm{Rel}^{\mathrm{symm}}(A, A)$ of $\mathrm{Rel}(A, A)$ spanned by the symmetric relations.
2. The **poset of relations on** A is is the subposet $\mathbf{Rel}^{\mathrm{symm}}(A, A)$ of $\mathbf{Rel}(A, A)$ spanned by the symmetric relations.

Proposition 5.4.2.3 ▶ Properties of Symmetric Relations

Let R and S be relations on A.

1. *Interaction With Inverses.* If R is symmetric, then so is $R^{\dagger}$.
2. *Interaction With Composition.* If R and S are symmetric, then so is $S \diamond R$.

Proof 5.4.2.4 ▶ Proof of Proposition 5.4.2.3

Item 1: Interaction With Inverses

Clear.

Item 2: Interaction With Composition

Clear.

5.4.2.2 The Symmetric Closure of a Relation

Let R be a relation on A.

Definition 5.4.2.5 ▶ The Symmetric Closure of a Relation

The **symmetric closure** of $\sim_R$ is the relation $\sim_R^{\mathrm{symm}}$[1] satisfying the following universal property:[2]

(★) Given another symmetric relation $\sim_S$ on A such that $R \subset S$, there exists an inclusion $\sim_R^{\mathrm{symm}} \subset \sim_S$.

[1] *Further Notation:* Also written R^{symm}.

[2] *Slogan:* The symmetric closure of R is the smallest symmetric relation containing R.

Construction 5.4.2.6 ▶ The Symmetric Closure of a Relation

Concretely, $\sim_R^{\mathrm{symm}}$ is the symmetric relation on A defined by

$$\begin{aligned} R^{\mathrm{symm}} &\overset{\mathrm{def}}{=} R \cup R^{\dagger} \\ &= \{(a,b) \in A \times A \mid \text{we have } a \sim_R b \text{ or } b \sim_R a\}. \end{aligned}$$

Proof 5.4.2.7 ▶ Proof of ??

Clear.

Proposition 5.4.2.8 ▶ Properties of the Symmetric Closure of a Relation

Let R be a relation on A.

1. *Adjointness.* We have an adjunction
$$\left((-)^{\mathsf{symm}} \dashv 忘\right)\colon \quad \begin{tikzcd} \mathbf{Rel}(A,A) \arrow[r, bend left, "(-)^{\mathsf{symm}}"] \arrow[r, phantom, "\perp"] & \mathbf{Rel}^{\mathsf{symm}}(A,A), \arrow[l, bend left, "忘"] \end{tikzcd}$$
witnessed by a bijection of sets
$$\mathbf{Rel}^{\mathsf{symm}}(R^{\mathsf{symm}},S) \cong \mathbf{Rel}(R,S),$$
natural in $R \in \mathrm{Obj}(\mathbf{Rel}^{\mathsf{symm}}(A,A))$ and $S \in \mathrm{Obj}(\mathbf{Rel}(A,A))$.

2. *The Symmetric Closure of a Symmetric Relation.* If R is symmetric, then $R^{\mathsf{symm}} = R$.

3. *Idempotency.* We have
$$(R^{\mathsf{symm}})^{\mathsf{symm}} = R^{\mathsf{symm}}.$$

4. *Interaction With Inverses.* We have
$$\left(R^{\dagger}\right)^{\mathsf{symm}} = \left(R^{\mathsf{symm}}\right)^{\dagger}, \qquad \begin{tikzcd} \mathrm{Rel}(A,A) \arrow[r, "(-)^{\mathsf{symm}}"] \arrow[d, "(-)^{\dagger}"'] & \mathrm{Rel}(A,A) \arrow[d, "(-)^{\dagger}"] \\ \mathrm{Rel}(A,A) \arrow[r, "(-)^{\mathsf{symm}}"'] & \mathrm{Rel}(A,A). \end{tikzcd}$$

5. *Interaction With Composition.* We have
$$(S \diamond R)^{\mathsf{symm}} = S^{\mathsf{symm}} \diamond R^{\mathsf{symm}}, \qquad \begin{tikzcd} \mathrm{Rel}(A,A) \times \mathrm{Rel}(A,A) \arrow[r, "\diamond"] \arrow[d, "(-)^{\mathsf{symm}} \times (-)^{\mathsf{symm}}"'] & \mathrm{Rel}(A,A) \arrow[d, "(-)^{\mathsf{symm}}"] \\ \mathrm{Rel}(A,A) \times \mathrm{Rel}(A,A) \arrow[r, "\diamond"'] & \mathrm{Rel}(A,A). \end{tikzcd}$$

Proof 5.4.2.9 ▶ Proof of Proposition 5.4.2.8

Item 1: Adjointness

This is a rephrasing of the universal property of the symmetric closure of a relation, stated in Definition 5.4.2.5.

Item 2: The Symmetric Closure of a Symmetric Relation

Clear.

Item 3: Idempotency

This follows from Item 2.

Item 4: Interaction With Inverses

Clear.

Item 5: Interaction With Composition

This follows from Item 2 of Proposition 5.4.2.3.

5.4.3 Transitive Relations

5.4.3.1 Foundations

Let A be a set.

Definition 5.4.3.1 ► Transitive Relations

A **transitive relation** is equivalently:[1]

- A non-unital $\mathbb{E}_1$-monoid in $(\mathrm{N}_\bullet(\mathbf{Rel}(A, A)), \diamond)$;
- A non-unital monoid in $(\mathbf{Rel}(A, A), \diamond)$.

[1]Note that since $\mathbf{Rel}(A, A)$ is posetal, transitivity is a property of a relation, rather than extra structure.

Remark 5.4.3.2 ► Unwinding Definition 5.4.3.1

In detail, a relation R on A is **transitive** if we have an inclusion

$$\mu_R \colon R \diamond R \subset R$$

of relations in $\mathbf{Rel}(A, A)$, i.e. if, for each $a, c \in A$, the following condition is satisfied:

($\star$) If there exists some $b \in A$ such that $a \sim_R b$ and $b \sim_R c$, then $a \sim_R c$.

Definition 5.4.3.3 ► The Po/Set of Transitive Relations on a Set

Let A be a set.

1. The **set of transitive relations from A to B** is the subset $\mathrm{Rel}^{\mathrm{trans}}(A)$ of

$\mathsf{Rel}(A, A)$ spanned by the transitive relations.

2. The **poset of relations from A to B** is is the subposet $\mathbf{Rel}^{\mathsf{trans}}(A)$ of $\mathbf{Rel}(A, A)$ spanned by the transitive relations.

PROPOSITION 5.4.3.4 ► PROPERTIES OF TRANSITIVE RELATIONS

Let R and S be relations on A.

1. *Interaction With Inverses.* If R is transitive, then so is $R^{\dagger}$.

2. *Interaction With Composition.* If R and S are transitive, then $S \diamond R$ ***may fail to be transitive***.

PROOF 5.4.3.5 ► PROOF OF PROPOSITION 5.4.3.4

Item 1: Interaction With Inverses

Clear.

Item 2: Interaction With Composition

See [**MSE2096272**].[1] ☐

[1] *Intuition:* Transitivity for R and S fails to imply that of $S \diamond R$ because the composition operation for relations intertwines R and S in an incompatible way:

1. If $a \sim_{S\diamond R} c$ and $c \sim_{S\diamond r} e$, then:
   (a) There is some $b \in A$ such that:
      i. $a \sim_R b$;
      ii. $b \sim_S c$;
   (b) There is some $d \in A$ such that:
      i. $c \sim_R d$;
      ii. $d \sim_S e$.

5.4.3.2 The Transitive Closure of a Relation

Let R be a relation on A.

DEFINITION 5.4.3.6 ► THE TRANSITIVE CLOSURE OF A RELATION

The **transitive closure** of $\sim_R$ is the relation $\sim_R^{\mathrm{trans}}$[1] satisfying the following universal property:[2]

($\star$) Given another transitive relation $\sim_S$ on A such that $R \subset S$, there exists an inclusion $\sim_R^{\mathrm{trans}} \subset \sim_S$.

[1] *Further Notation:* Also written R^{trans}.
[2] *Slogan:* The transitive closure of R is the smallest transitive relation containing R.

Construction 5.4.3.7 ▶ The Transitive Closure of a Relation

Concretely, $\sim_R^{\mathrm{trans}}$ is the free non-unital monoid on R in $(\mathbf{Rel}(A, A), \diamond)$[1], being given by

$$\begin{aligned} R^{\mathrm{trans}} &\overset{\mathrm{def}}{=} \coprod_{n=1}^{\infty} R^{\diamond n} \\ &\overset{\mathrm{def}}{=} \bigcup_{n=1}^{\infty} R^{\diamond n} \\ &\overset{\mathrm{def}}{=} \left\{ (a, b) \in A \times B \;\middle|\; \begin{array}{l} \text{there exists some } (x_1, \ldots, x_n) \in R^{\times n} \\ \text{such that } a \sim_R x_1 \sim_R \cdots \sim_R x_n \sim_R b \end{array} \right\}. \end{aligned}$$

[1] Or, equivalently, the free non-unital $\mathbb{E}_1$-monoid on R in $(\mathsf{N}_\bullet(\mathbf{Rel}(A, A)), \diamond)$.

Proof 5.4.3.8 ▶ Proof of ??

Clear. ∎

Proposition 5.4.3.9 ▶ Properties of the Transitive Closure of a Relation

Let R be a relation on A.

1. *Adjointness.* We have an adjunction

$$\left((-)^{\mathrm{trans}} \dashv 忘\right)\colon \quad \begin{tikzcd} \mathbf{Rel}(A, A) \arrow[r, bend left, "(-)^{\mathrm{trans}}"] \arrow[r, phantom, "\perp"] & \mathbf{Rel}^{\mathrm{trans}}(A, A), \arrow[l, bend left, "忘"] \end{tikzcd}$$

   witnessed by a bijection of sets

$$\mathbf{Rel}^{\mathrm{trans}}(R^{\mathrm{trans}}, S) \cong \mathbf{Rel}(R, S),$$

   natural in $R \in \mathrm{Obj}(\mathbf{Rel}^{\mathrm{trans}}(A, A))$ and $S \in \mathrm{Obj}(\mathbf{Rel}(A, B))$.

2. *The Transitive Closure of a Transitive Relation.* If R is transitive, then $R^{\mathrm{trans}} = R$.

3. *Idempotency.* We have

$$\left(R^{\mathrm{trans}}\right)^{\mathrm{trans}} = R^{\mathrm{trans}}.$$

4. *Interaction With Inverses.* We have

$$\left(R^{\dagger}\right)^{\mathrm{trans}} = \left(R^{\mathrm{trans}}\right)^{\dagger}, \qquad \begin{tikzcd} \mathrm{Rel}(A,A) \arrow[r, "(-)^{\mathrm{trans}}"] \arrow[d, "(-)^{\dagger}"'] & \mathrm{Rel}(A,A) \arrow[d, "(-)^{\dagger}"] \\ \mathrm{Rel}(A,A) \arrow[r, "(-)^{\mathrm{trans}}"'] & \mathrm{Rel}(A,A). \end{tikzcd}$$

5. *Interaction With Composition.* We have

$$(S \diamond R)^{\mathrm{trans}} \overset{\text{poss.}}{\neq} S^{\mathrm{trans}} \diamond R^{\mathrm{trans}}, \qquad \begin{tikzcd} \mathrm{Rel}(A,A) \times \mathrm{Rel}(A,A) \arrow[r, "\diamond"] \arrow[d, "(-)^{\mathrm{trans}} \times (-)^{\mathrm{trans}}"'] \arrow[dr, phantom, "\times"] & \mathrm{Rel}(A,A) \arrow[d, "(-)^{\mathrm{trans}}"] \\ \mathrm{Rel}(A,A) \times \mathrm{Rel}(A,A) \arrow[r, "\diamond"'] & \mathrm{Rel}(A,A). \end{tikzcd}$$

PROOF 5.4.3.10 ► PROOF OF PROPOSITION 5.4.3.9

Item 1: Adjointness

This is a rephrasing of the universal property of the transitive closure of a relation, stated in Definition 5.4.3.6.

Item 2: The Transitive Closure of a Transitive Relation

Clear.

Item 3: Idempotency

This follows from Item 2.

Item 4: Interaction With Inverses

We have

$$\begin{aligned} \left(R^{\dagger}\right)^{\mathrm{trans}} &= \bigcup_{n=1}^{\infty} \left(R^{\dagger}\right)^{\diamond n} && \text{(by ??)} \\ &= \bigcup_{n=1}^{\infty} \left(R^{\diamond n}\right)^{\dagger} && \text{(by Item 4 of Proposition 5.3.12.3)} \\ &= \left(\bigcup_{n=1}^{\infty} R^{\diamond n}\right)^{\dagger} && \text{(by Item 1 of Proposition 5.3.6.2)} \\ &= \left(R^{\mathrm{trans}}\right)^{\dagger}. && \text{(by ??)} \end{aligned}$$

Item 5: Interaction With Composition

This follows from Item 2 of Proposition 5.4.3.4. □

5.4.4 Equivalence Relations

5.4.4.1 Foundations

Let A be a set.

Definition 5.4.4.1 ▶ Equivalence Relations

A relation R is an **equivalence relation** if it is reflexive, symmetric, and transitive.[1]

[1] *Further Terminology:* If instead R is just symmetric and transitive, then it is called a **partial equivalence relation**.

Example 5.4.4.2 ▶ The Kernel of a Function

The **kernel of a function** $f: A \to B$ is the equivalence $\sim_{\mathrm{Ker}(f)}$ on A obtained by declaring $a \sim_{\mathrm{Ker}(f)} b$ iff $f(a) = f(b)$.[1]

[1] The kernel $\mathrm{Ker}(f): A \nrightarrow A$ of f is the monad induced by the adjunction $\mathrm{Gr}(f) \dashv f^{-1}: A \rightleftarrows B$ in **Rel** of Item 2 of Proposition 5.3.1.2.

Definition 5.4.4.3 ▶ The Po/Set of Equivalence Relations on a Set

Let A and B be sets.

1. The **set of equivalence relations from A to B** is the subset $\mathrm{Rel}^{\mathrm{eq}}(A, B)$ of $\mathrm{Rel}(A, B)$ spanned by the equivalence relations.
2. The **poset of relations from A to B** is is the subposet $\mathbf{Rel}^{\mathrm{eq}}(A, B)$ of $\mathbf{Rel}(A, B)$ spanned by the equivalence relations.

5.4.4.2 The Equivalence Closure of a Relation

Let R be a relation on A.

Definition 5.4.4.4 ▶ The Equivalence Closure of a Relation

The **equivalence closure**[1] of $\sim_R$ is the relation $\sim_R^{\mathrm{eq}}$[2] satisfying the following universal property:[3]

(★) Given another equivalence relation $\sim_S$ on A such that $R \subset S$, there exists an inclusion $\sim_R^{\mathrm{eq}} \subset \sim_S$.

[1] *Further Terminology:* Also called the **equivalence relation associated to** $\sim_R$.

[2] *Further Notation:* Also written R^{eq}.
[3] *Slogan:* The equivalence closure of R is the smallest equivalence relation containing R.

Construction 5.4.4.5 ▶ The Equivalence Closure of a Relation

Concretely, $\sim_R^{\mathrm{eq}}$ is the equivalence relation on A defined by

$$\begin{aligned} R^{\mathrm{eq}} &\overset{\mathrm{def}}{=} \left(\left(R^{\mathrm{refl}}\right)^{\mathrm{symm}}\right)^{\mathrm{trans}} \\ &= \left(\left(R^{\mathrm{symm}}\right)^{\mathrm{trans}}\right)^{\mathrm{refl}} \\ &= \left\{ (a,b) \in A \times B \,\middle|\, \begin{array}{l} \text{there exists } (x_1,\ldots,x_n) \in R^{\times n} \text{ satisfying at} \\ \text{least one of the following conditions:} \\ \quad 1.\ \text{The following conditions are satisfied:} \\ \qquad \text{(a) We have } a \sim_R x_1 \text{ or } x_1 \sim_R a; \\ \qquad \text{(b) We have } x_i \sim_R x_{i+1} \text{ or } x_{i+1} \sim_R x_i \\ \qquad\quad \text{for each } 1 \leq i \leq n-1; \\ \qquad \text{(c) We have } b \sim_R x_n \text{ or } x_n \sim_R b; \\ \quad 2.\ \text{We have } a = b. \end{array} \right\}. \end{aligned}$$

Proof 5.4.4.6 ▶ Proof of ??

From the universal properties of the reflexive, symmetric, and transitive closures of a relation (Definitions 5.4.1.6, 5.4.2.5 and 5.4.3.6), we see that it suffices to prove that:

1. The symmetric closure of a reflexive relation is still reflexive;
2. The transitive closure of a symmetric relation is still symmetric;

which are both clear. □

Proposition 5.4.4.7 ▶ Properties of Equivalence Relations

Let R be a relation on A.

1. *Adjointness.* We have an adjunction

$$\left((-)^{\mathrm{eq}} \dashv 忘\right)\colon\quad \mathbf{Rel}(A,B) \underset{忘}{\overset{(-)^{\mathrm{eq}}}{\underset{\perp}{\rightleftarrows}}} \mathbf{Rel}^{\mathrm{eq}}(A,B),$$

witnessed by a bijection of sets

$$\mathbf{Rel}^{\mathrm{eq}}(R^{\mathrm{eq}}, S) \cong \mathbf{Rel}(R, S),$$

natural in $R \in \mathrm{Obj}(\mathbf{Rel}^{\mathrm{eq}}(A, B))$ and $S \in \mathrm{Obj}(\mathbf{Rel}(A, B))$.

2. *The Equivalence Closure of an Equivalence Relation.* If R is an equivalence relation, then $R^{\mathrm{eq}} = R$.

3. *Idempotency.* We have

$$(R^{\mathrm{eq}})^{\mathrm{eq}} = R^{\mathrm{eq}}.$$

Proof 5.4.4.8 ► Proof of Proposition 5.4.4.7

Item 1: Adjointness

This is a rephrasing of the universal property of the equivalence closure of a relation, stated in Definition 5.4.4.4.

Item 2: The Equivalence Closure of an Equivalence Relation

Clear.

Item 3: Idempotency

This follows from Item 2.

5.4.5 Quotients by Equivalence Relations

5.4.5.1 Equivalence Classes

Let A be a set, let R be a relation on A, and let $a \in A$.

Definition 5.4.5.1 ► Equivalence Classes

The **equivalence class associated to** a is the set $[a]$ defined by

$$\begin{aligned} [a] &\stackrel{\mathrm{def}}{=} \{x \in X \mid x \sim_R a\} \\ &= \{x \in X \mid a \sim_R x\}. && \text{(since } R \text{ is symmetric)} \end{aligned}$$

5.4.5.2 Quotients of Sets by Equivalence Relations

Let A be a set and let R be a relation on A.

Definition 5.4.5.2 ► Quotients of Sets by Equivalence Relations

The **quotient of** X **by** R is the set $X/\sim_R$ defined by

$$X/\sim_R \overset{\text{def}}{=} \{[a] \in \mathcal{P}(X) \mid a \in X\}.$$

Remark 5.4.5.3 ► Why Use "Equivalence" Relations for Quotient Sets

The reason we define quotient sets for equivalence relations only is that each of the properties of being an equivalence relation—reflexivity, symmetry, and transitivity—ensures that the equivalences classes $[a]$ of X under R are well-behaved:

- *Reflexivity.* If R is reflexive, then, for each $a \in X$, we have $a \in [a]$.
- *Symmetry.* The equivalence class $[a]$ of an element a of X is defined by

  $$[a] \overset{\text{def}}{=} \{x \in X \mid x \sim_R a\},$$

  but we could equally well define

  $$[a]' \overset{\text{def}}{=} \{x \in X \mid a \sim_R x\}$$

  instead. This is not a problem when R is symmetric, as we then have $[a] = [a]'$.[1]
- *Transitivity.* If R is transitive, then $[a]$ and $[b]$ are disjoint iff $a \nsim_R b$, and equal otherwise.

[1]When categorifying equivalence relations, one finds that $[a]$ and $[a]'$ correspond to presheaves and copresheaves; see **??**.

Proposition 5.4.5.4 ► Properties of Quotient Sets

Let $f\colon X \to Y$ be a function and let R be a relation on X.

1. *As a Coequaliser.* We have an isomorphism of sets

   $$X/\sim_R^{\text{eq}} \cong \operatorname{CoEq}\left(R \hookrightarrow X \times X \underset{\text{pr}_2}{\overset{\text{pr}_1}{\rightrightarrows}} X\right),$$

   where $\sim_R^{\text{eq}}$ is the equivalence relation generated by $\sim_R$.

2. *As a Pushout.* We have an isomorphism of sets[1]

$$X/{\sim_R^{\mathrm{eq}}} \cong X \textstyle\coprod_{\mathrm{Eq}(\mathrm{pr}_1,\mathrm{pr}_2)} X, \qquad \begin{tikzcd} X/{\sim_R^{\mathrm{eq}}} \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[l] \\ X \arrow[u] & \mathrm{Eq}(\mathrm{pr}_1, \mathrm{pr}_2). \arrow[l] \arrow[u] \end{tikzcd}$$

where $\sim_R^{\mathrm{eq}}$ is the equivalence relation generated by $\sim_R$.

3. *The First Isomorphism Theorem for Sets.* We have an isomorphism of sets[2,3]

$$X/{\sim_{\mathrm{Ker}(f)}} \cong \mathrm{Im}(f).$$

4. *Descending Functions to Quotient Sets, I.* Let R be an equivalence relation on X. The following conditions are equivalent:

   (a) There exists a map

   $$\bar{f} \colon X/{\sim_R} \to Y$$

   making the diagram

   $$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "q"'] & Y \\ X/{\sim_R} \arrow[ur, dotted, "\exists", "\bar{f}"'] \end{tikzcd}$$

   commute.

   (b) For each $x, y \in X$, if $x \sim_R y$, then $f(x) = f(y)$.

5. *Descending Functions to Quotient Sets, II.* Let R be an equivalence relation on X. If the conditions of Item 4 hold, then $\bar{f}$ is the *unique* map making the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "q"'] & Y \\ X/{\sim_R} \arrow[ur, dashed, "\exists!", "\bar{f}"'] \end{tikzcd}$$

commute.

6. *Descending Functions to Quotient Sets, III.* Let R be an equivalence relation on X. If the conditions of Item 4 hold, then the following conditions are equivalent:

   (a) The map $\overline{f}$ is an injection.

   (b) For each $x, y \in X$, we have $x \sim_R y$ iff $f(x) = f(y)$.

7. *Descending Functions to Quotient Sets, IV.* Let R be an equivalence relation on X. If the conditions of Item 4 hold, then the following conditions are equivalent:

   (a) The map $f: X \to Y$ is surjective.

   (b) The map $\overline{f}: X/{\sim_R} \to Y$ is surjective.

8. *Descending Functions to Quotient Sets, V.* Let R be a relation on X and let $\sim_R^{\mathrm{eq}}$ be the equivalence relation associated to R. The following conditions are equivalent:

   (a) The map f satisfies the equivalent conditions of Item 4:

      - There exists a map
$$\overline{f}: X/{\sim_R^{\mathrm{eq}}} \to Y$$
        making the diagram
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "q"'] & Y \\ X/{\sim_R^{\mathrm{eq}}} \arrow[ur, dotted, "\exists", "\overline{f}"'] & \end{tikzcd}$$
        commute.

      - For each $x, y \in X$, if $x \sim_R^{\mathrm{eq}} y$, then $f(x) = f(y)$.

   (b) For each $x, y \in X$, if $x \sim_R y$, then $f(x) = f(y)$.

[1] Dually, we also have an isomorphism of sets

$$\mathrm{Eq}(\mathrm{pr}_1, \mathrm{pr}_2) \cong X \times_{X/{\sim_R^{\mathrm{eq}}}} X, \qquad \begin{tikzcd} \mathrm{Eq}(\mathrm{pr}_1, \mathrm{pr}_2) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d] \\ X \arrow[r] & X/{\sim_R^{\mathrm{eq}}}. \end{tikzcd}$$

[2] *Further Terminology:* The set $X/{\sim_{\mathrm{Ker}(f)}}$ is often called the **coimage of** f, and denoted by $\mathrm{Coim}(f)$.

[3] In a sense this is a result relating the monad in **Rel** induced by f with the comonad in **Rel** induced by f, as the kernel and image

$$\mathrm{Ker}(f)\colon X \nrightarrow X,$$
$$\mathrm{Im}(f) \subset Y$$

of f are respectively the induced monads and comonads of the adjunction

$$\left(\mathrm{Gr}(f) \dashv f^{-1}\right)\colon\ \begin{tikzcd} A \arrow[r, bend left, "\mathrm{Gr}(f)"] \arrow[r, phantom, "\perp"] & B \arrow[l, bend left, "f^{-1}"] \end{tikzcd}$$

of Item 2 of Proposition 5.3.1.2.

PROOF 5.4.5.5 ► PROOF OF PROPOSITION 5.4.5.4

Item 1: As a Coequaliser

Omitted.

Item 2: As a Pushout

Omitted.

Item 3: The First Isomorphism Theorem for Sets

Clear.

Item 4: Descending Functions to Quotient Sets, I

See [Pro24o].

Item 5: Descending Functions to Quotient Sets, II

See [Pro24y].

Item 6: Descending Functions to Quotient Sets, III

See [Pro24n].

Item 7: Descending Functions to Quotient Sets, IV

See [Pro24m].

Item 8: Descending Functions to Quotient Sets, V

The implication Item 8a $\implies$ Item 8b is clear.

Conversely, suppose that, for each $x, y \in X$, if $x \sim_R y$, then $f(x) = f(y)$. Spelling out the definition of the equivalence closure of R, we see that the condition $x \sim_R^{\mathrm{eq}} y$ unwinds to the following:

($\star$) There exist $(x_1, \ldots, x_n) \in R^{\times n}$ satisfying at least one of the following conditions:

1. The following conditions are satisfied:
   (a) We have $x \sim_R x_1$ or $x_1 \sim_R x$;
   (b) We have $x_i \sim_R x_{i+1}$ or $x_{i+1} \sim_R x_i$ for each $1 \leq i \leq n-1$;
   (c) We have $y \sim_R x_n$ or $x_n \sim_R y$;
2. We have $x = y$.

Now, if $x = y$, then $f(x) = f(y)$ trivially; otherwise, we have

$$\begin{aligned} f(x) &= f(x_1), \\ f(x_1) &= f(x_2), \\ &\vdots \\ f(x_{n-1}) &= f(x_n), \\ f(x_n) &= f(y), \end{aligned}$$

and $f(x) = f(y)$, as we wanted to show.

5.5 Functoriality of Powersets

5.5.1 Direct Images

Let A and B be sets and let $R\colon A \nrightarrow B$ be a relation.

Definition 5.5.1.1 ► Direct Images

The **direct image function associated to** R is the function[1]

$$R_* \colon \mathcal{P}(A) \to \mathcal{P}(B)$$

defined by[2,3]

$$\begin{aligned} R_*(U) &\overset{\text{def}}{=} R(U) \\ &\overset{\text{def}}{=} \bigcup_{a \in U} R(a) \\ &= \left\{ b \in B \;\middle|\; \begin{array}{l} \text{there exists some } a \in U \\ \text{such that } b \in R(a) \end{array} \right\} \end{aligned}$$

for each $U \in \mathcal{P}(A)$.

[1] *Further Notation:* Also written $\exists_R \colon \mathcal{P}(A) \to \mathcal{P}(B)$. This notation comes from the fact that the following statements are equivalent, where $b \in B$ and $U \in \mathcal{P}(A)$:

- We have $b \in \exists_R(U)$.
- There exists some $a \in U$ such that $b \in f(a)$.

[2] *Further Terminology:* The set $R(U)$ is called the **direct image of** U **by** R.

[3] We also have

$$R_*(U) = B \setminus R_!(A \setminus U);$$

see Item 7 of Proposition 5.5.1.3.

Remark 5.5.1.2 ▶ Unwinding Definition 5.5.1.1

Identifying subsets of A with relations from pt to A via **??** of **??**, we see that the direct image function associated to R is equivalently the function

$$R_* \colon \underbrace{\mathcal{P}(A)}_{\cong \mathrm{Rel}(\mathrm{pt}, A)} \to \underbrace{\mathcal{P}(B)}_{\cong \mathrm{Rel}(\mathrm{pt}, B)}$$

defined by

$$R_*(U) \stackrel{\text{def}}{=} R \diamond U$$

for each $U \in \mathcal{P}(A)$, where $R \diamond U$ is the composition

$$\mathrm{pt} \overset{U}{\nrightarrow} A \overset{R}{\nrightarrow} B.$$

Proposition 5.5.1.3 ▶ Properties of Direct Image Functions

Let $R \colon A \nrightarrow B$ be a relation.

1. *Functoriality.* The assignment $U \mapsto R_*(U)$ defines a functor

$$R_* \colon (\mathcal{P}(A), \subset) \to (\mathcal{P}(B), \subset)$$

where

- *Action on Objects.* For each $U \in \mathcal{P}(A)$, we have

$$[R_*](U) \stackrel{\text{def}}{=} R_*(U);$$

- *Action on Morphisms.* For each $U, V \in \mathcal{P}(A)$:
  - If $U \subset V$, then $R_*(U) \subset R_*(V)$.

2. *Adjointness.* We have an adjunction

$$(R_* \dashv R_{-1}) \colon \begin{tikzcd} \mathcal{P}(A) \arrow[r, bend left, "R_*"] \arrow[r, phantom, "\perp"] & \mathcal{P}(B), \arrow[l, bend left, "R_{-1}"] \end{tikzcd}$$

witnessed by a bijections of sets

$$\mathrm{Hom}_{\mathcal{P}(A)}(R_*(U), V) \cong \mathrm{Hom}_{\mathcal{P}(A)}(U, R_{-1}(V)),$$

natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$, i.e. such that:

($\star$) The following conditions are equivalent:
- We have $R_*(U) \subset V$;
- We have $U \subset R_{-1}(V)$.

3. *Preservation of Colimits.* We have an equality of sets

$$R_*\left(\bigcup_{i\in I} U_i\right) = \bigcup_{i\in I} R_*(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have equalities

$$R_*(U) \cup R_*(V) = R_*(U \cup V),$$
$$R_*(\emptyset) = \emptyset,$$

natural in $U, V \in \mathcal{P}(A)$.

4. *Oplax Preservation of Limits.* We have an inclusion of sets

$$R_*\left(\bigcap_{i\in I} U_i\right) \subset \bigcap_{i\in I} R_*(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have inclusions

$$R_*(U \cap V) \subset R_*(U) \cap R_*(V),$$
$$R_*(A) \subset B,$$

natural in $U, V \in \mathcal{P}(A)$.

5. *Symmetric Strict Monoidality With Respect to Unions.* The direct image function of Item 1 has a symmetric strict monoidal structure

$$\left(R_*, R_*^{\otimes}, R_{*|\mathbb{1}}^{\otimes}\right) : (\mathcal{P}(A), \cup, \emptyset) \to (\mathcal{P}(B), \cup, \emptyset),$$

being equipped with equalities

$$R_{*|U,V}^{\otimes} : R_*(U) \cup R_*(V) \xrightarrow{=} R_*(U \cup V),$$
$$R_{*|\mathbb{1}}^{\otimes} : \emptyset \xrightarrow{=} \emptyset,$$

natural in $U, V \in \mathcal{P}(A)$.

6. *Symmetric Oplax Monoidality With Respect to Intersections.* The direct image function of Item 1 has a symmetric oplax monoidal structure

$$\left(R_*, R_*^{\otimes}, R_{*|\mathbb{1}}^{\otimes}\right) \colon (\mathcal{P}(A), \cap, A) \to (\mathcal{P}(B), \cap, B),$$

being equipped with inclusions

$$R_{*|U,V}^{\otimes} \colon R_*(U \cap V) \subset R_*(U) \cap R_*(V),$$
$$R_{*|\mathbb{1}}^{\otimes} \colon R_*(A) \subset B,$$

natural in $U, V \in \mathcal{P}(A)$.

7. *Relation to Direct Images With Compact Support.* We have

$$R_*(U) = B \setminus R_!(A \setminus U)$$

for each $U \in \mathcal{P}(A)$.

Proof 5.5.1.4 ▶ Proof of Proposition 5.5.1.3

Item 1: Functoriality

Clear.

Item 2: Adjointness

This follows from **??** of **??**.

Item 3: Preservation of Colimits

This follows from Item 2 and **??** of **??**.

Item 4: Oplax Preservation of Limits

Omitted.

Item 5: Symmetric Strict Monoidality With Respect to Unions

This follows from Item 3.

Item 6: Symmetric Oplax Monoidality With Respect to Intersections

This follows from Item 4.

Item 7: Relation to Direct Images With Compact Support

The proof proceeds in the same way as in the case of functions (**??** of **??**): applying Item 7 of Proposition 5.5.4.4 to $A \setminus U$, we have

$$\begin{aligned} R_!(A \setminus U) &= B \setminus R_*(A \setminus (A \setminus U)) \\ &= B \setminus R_*(U). \end{aligned}$$

Taking complements, we then obtain

$$\begin{aligned} R_*(U) &= B \setminus (B \setminus R_*(U)), \\ &= B \setminus R_!(A \setminus U), \end{aligned}$$

which finishes the proof. □

Proposition 5.5.1.5 ▶ Properties of the Direct Image Function Operation

Let $R\colon A \nrightarrow B$ be a relation.

1. *Functionality I*. The assignment $R \mapsto R_*$ defines a function

$$(-)_*\colon \mathsf{Rel}(A, B) \to \mathsf{Sets}(\mathcal{P}(A), \mathcal{P}(B)).$$

2. *Functionality II*. The assignment $R \mapsto R_*$ defines a function

$$(-)_*\colon \mathsf{Rel}(A, B) \to \mathsf{Pos}((\mathcal{P}(A), \subset), (\mathcal{P}(B), \subset)).$$

3. *Interaction With Identities*. For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have[1]

$$(\chi_A)_* = \mathrm{id}_{\mathcal{P}(A)};$$

4. *Interaction With Composition*. For each pair of composable relations $R\colon A \nrightarrow B$ and $S\colon B \nrightarrow C$, we have[2]

$$(S \diamond R)_* = S_* \circ R_*, \qquad \begin{tikzcd} \mathcal{P}(A) \arrow[r, "R_*"] \arrow[dr, "(S\diamond R)_*"'] & \mathcal{P}(B) \arrow[d, "S_*"] \\ & \mathcal{P}(C). \end{tikzcd}$$

[1]That is, the postcomposition function

$$(\chi_A)_*\colon \mathsf{Rel}(\mathrm{pt}, A) \to \mathsf{Rel}(\mathrm{pt}, A)$$

is equal to $\mathrm{id}_{\mathsf{Rel}(\mathrm{pt},A)}$.

[2]That is, we have

$$(S \diamond R)_* = S_* \circ R_*, \qquad \begin{tikzcd} \mathsf{Rel}(\mathrm{pt}, A) \arrow[r, "R_*"] \arrow[dr, "(S\diamond R)_*"'] & \mathsf{Rel}(\mathrm{pt}, B) \arrow[d, "S_*"] \\ & \mathsf{Rel}(\mathrm{pt}, C). \end{tikzcd}$$

Proof 5.5.1.6 ▶ Proof of Proposition 5.5.1.5

Item 1: Functoriality I

Clear.

Item 2: Functoriality II

Clear.

Item 3: Interaction With Identities

Indeed, we have

$$
\begin{aligned}
(\chi_A)_*(U) &\overset{\text{def}}{=} \bigcup_{a \in U} \chi_A(a) \\
&\overset{\text{def}}{=} \bigcup_{a \in U} \{a\} \\
&= U \\
&\overset{\text{def}}{=} \mathrm{id}_{\mathcal{P}(A)}(U)
\end{aligned}
$$

for each $U \in \mathcal{P}(A)$. Thus $(\chi_A)_* = \mathrm{id}_{\mathcal{P}(A)}$.

Item 4: Interaction With Composition

Indeed, we have

$$
\begin{aligned}
(S \diamond R)_*(U) &\overset{\text{def}}{=} \bigcup_{a \in U} [S \diamond R](a) \\
&\overset{\text{def}}{=} \bigcup_{a \in U} S(R(a)) \\
&\overset{\text{def}}{=} \bigcup_{a \in U} S_*(R(a)) \\
&= S_*\left(\bigcup_{a \in U} R(a)\right) \\
&\overset{\text{def}}{=} S_*(R_*(U)) \\
&\overset{\text{def}}{=} [S_* \circ R_*](U)
\end{aligned}
$$

for each $U \in \mathcal{P}(A)$, where we used Item 3 of Proposition 5.5.1.3. Thus $(S \diamond R)_* = S_* \circ R_*$.

5.5.2 Strong Inverse Images

Let A and B be sets and let $R \colon A \nrightarrow B$ be a relation.

Definition 5.5.2.1 ▶ Strong Inverse Images

The **strong inverse image function associated to** R is the function

$$R_{-1} \colon \mathcal{P}(B) \to \mathcal{P}(A)$$

defined by[1]

$$R_{-1}(V) \overset{\text{def}}{=} \{a \in A \mid R(a) \subset V\}$$

for each $V \in \mathcal{P}(B)$.

[1] *Further Terminology:* The set $R_{-1}(V)$ is called the **strong inverse image of** V **by** R.

Remark 5.5.2.2 ▶ Unwinding Definition 5.5.2.1

Identifying subsets of B with relations from pt to B via **??** of **??**, we see that the inverse image function associated to R is equivalently the function

$$R_{-1} \colon \underbrace{\mathcal{P}(B)}_{\cong \mathrm{Rel}(\mathrm{pt}, B)} \to \underbrace{\mathcal{P}(A)}_{\cong \mathrm{Rel}(\mathrm{pt}, A)}$$

defined by

$$R_{-1}(V) \overset{\text{def}}{=} \mathrm{Rift}_R(V), \qquad \begin{tikzcd} & A \arrow[d, "R", "\mid" marking] \\ \mathrm{pt} \arrow[ur, dashed, "\mathrm{Rift}_R(V)", "\mid" marking] \arrow[r, "V"', "\mid" marking] \arrow[ur, phantom, shift right=2ex, "\Downarrow"] & B, \end{tikzcd}$$

and being explicitly computed by

$$\begin{aligned} R_{-1}(V) &\overset{\text{def}}{=} \mathrm{Rift}_R(V) \\ &\cong \int_{x \in B} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^x_{-1}, V^x_{-2}\right), \end{aligned}$$

where we have used Item 12 of Proposition 5.2.5.1.

Proof 5.5.2.3 ▶ Proof of Remark 5.5.2.2

We have

$$
\begin{aligned}
\mathrm{Rift}_R(V) &\cong \int_{x\in B} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^x_{-_1}, V^x_{-_2}\right)\\
&= \left\{a\in A \,\middle|\, \int_{x\in B} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^x_a, V^x_\star\right) = \mathsf{true}\right\}\\
&= \left\{a \in A \,\middle|\, \begin{array}{l}\text{for each } x\in B\text{, at least one of the}\\ \text{following conditions hold:}\\ \quad 1.\ \text{We have } R^x_a = \mathsf{false}\\ \quad 2.\ \text{The following conditions hold:}\\ \qquad \text{(a) We have } R^x_a = \mathsf{true}\\ \qquad \text{(b) We have } V^x_\star = \mathsf{true}\end{array}\right\}\\
&= \left\{a \in A \,\middle|\, \begin{array}{l}\text{for each } x\in B\text{, at least one of the}\\ \text{following conditions hold:}\\ \quad 1.\ \text{We have } x\notin R(a)\\ \quad 2.\ \text{The following conditions hold:}\\ \qquad \text{(a) We have } x\in R(a)\\ \qquad \text{(b) We have } x\in V\end{array}\right\}\\
&= \{a\in A \mid \text{for each } x\in R(a)\text{, we have } x\in V\}\\
&= \{a\in A \mid R(a)\subset V\}\\
&\overset{\text{def}}{=} R_{-1}(V).
\end{aligned}
$$

This finishes the proof. □

Proposition 5.5.2.4 ▶ Properties of Strong Inverse Images

Let $R\colon A \nrightarrow B$ be a relation.

1. *Functoriality.* The assignment $V \mapsto R_{-1}(V)$ defines a functor

$$R_{-1}\colon (\mathcal{P}(B), \subset) \to (\mathcal{P}(A), \subset)$$

where

- *Action on Objects.* For each $V\in\mathcal{P}(B)$, we have

$$[R_{-1}](V) \overset{\text{def}}{=} R_{-1}(V);$$

- *Action on Morphisms.* For each $U, V \in \mathcal{P}(B)$:
  - If $U \subset V$, then $R_{-1}(U) \subset R_{-1}(V)$.

2. *Adjointness.* We have an adjunction

$$
\begin{tikzcd}
(R_* \dashv R_{-1})\colon \quad \mathcal{P}(A) \arrow[r, bend left, "R_*"] \arrow[r, phantom, "\perp"] & \mathcal{P}(B), \arrow[l, bend left, "R_{-1}"]
\end{tikzcd}
$$

witnessed by a bijections of sets

$$\mathrm{Hom}_{\mathcal{P}(A)}(R_*(U), V) \cong \mathrm{Hom}_{\mathcal{P}(A)}(U, R_{-1}(V)),$$

natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$, i.e. such that:

($\star$) The following conditions are equivalent:
  - We have $R_*(U) \subset V$;
  - We have $U \subset R_{-1}(V)$.

3. *Lax Preservation of Colimits.* We have an inclusion of sets

$$\bigcup_{i \in I} R_{-1}(U_i) \subset R_{-1}\left(\bigcup_{i \in I} U_i\right),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(B)^{\times I}$. In particular, we have inclusions

$$
\begin{aligned}
R_{-1}(U) \cup R_{-1}(V) &\subset R_{-1}(U \cup V), \\
\emptyset &\subset R_{-1}(\emptyset),
\end{aligned}
$$

natural in $U, V \in \mathcal{P}(B)$.

4. *Preservation of Limits.* We have an equality of sets

$$R_{-1}\left(\bigcap_{i \in I} U_i\right) = \bigcap_{i \in I} R_{-1}(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(B)^{\times I}$. In particular, we have equalities

$$
\begin{aligned}
R_{-1}(U \cap V) &= R_{-1}(U) \cap R_{-1}(V), \\
R_{-1}(B) &= B,
\end{aligned}
$$

natural in $U, V \in \mathcal{P}(B)$.

5. *Symmetric Lax Monoidality With Respect to Unions.* The direct image with compact support function of Item 1 has a symmetric lax monoidal structure

$$\left(R_{-1}, R_{-1}^{\otimes}, R_{-1|\mathbb{1}}^{\otimes}\right)\colon (\mathcal{P}(A), \cup, \emptyset) \longrightarrow (\mathcal{P}(B), \cup, \emptyset),$$

being equipped with inclusions

$$R_{-1|U,V}^{\otimes}\colon R_{-1}(U) \cup R_{-1}(V) \subset R_{-1}(U \cup V),$$
$$R_{-1|\mathbb{1}}^{\otimes}\colon \emptyset \subset R_{-1}(\emptyset),$$

natural in $U, V \in \mathcal{P}(B)$.

6. *Symmetric Strict Monoidality With Respect to Intersections.* The direct image function of Item 1 has a symmetric strict monoidal structure

$$\left(R_{-1}, R_{-1}^{\otimes}, R_{-1|\mathbb{1}}^{\otimes}\right)\colon (\mathcal{P}(A), \cap, A) \longrightarrow (\mathcal{P}(B), \cap, B),$$

being equipped with equalities

$$R_{-1|U,V}^{\otimes}\colon R_{-1}(U \cap V) \xrightarrow{=} R_{-1}(U) \cap R_{-1}(V),$$
$$R_{-1|\mathbb{1}}^{\otimes}\colon R_{-1}(A) \xrightarrow{=} B,$$

natural in $U, V \in \mathcal{P}(B)$.

7. *Interaction With Weak Inverse Images I.* We have

$$R_{-1}(V) = A \setminus R^{-1}(B \setminus V)$$

for each $V \in \mathcal{P}(B)$.

8. *Interaction With Weak Inverse Images II.* Let $R\colon A \nrightarrow B$ be a relation from A to B.

   (a) If R is a total relation, then we have an inclusion of sets

   $$R_{-1}(V) \subset R^{-1}(V)$$

   natural in $V \in \mathcal{P}(B)$.

   (b) If R is total and functional, then the above inclusion is in fact an equality.

   (c) Conversely, if we have $R_{-1} = R^{-1}$, then R is total and functional.

Proof 5.5.2.5 ► Proof of Proposition 5.5.2.4

Item 1: Functoriality

Clear.

Item 2: Adjointness

This follows from **??** of **??**.

Item 3: Lax Preservation of Colimits

Omitted.

Item 4: Preservation of Limits

This follows from Item 2 and **??** of **??**.

Item 5: Symmetric Lax Monoidality With Respect to Unions

This follows from Item 3.

Item 6: Symmetric Strict Monoidality With Respect to Intersections

This follows from Item 4.

Item 7: Interaction With Weak Inverse Images I

We claim we have an equality

$$R_{-1}(B \setminus V) = A \setminus R^{-1}(V).$$

Indeed, we have

$$\begin{aligned} R_{-1}(B \setminus V) &= \{a \in A \mid R(a) \subset B \setminus V\}, \\ A \setminus R^{-1}(V) &= \{a \in A \mid R(a) \cap V = \emptyset\}. \end{aligned}$$

Taking $V = B \setminus V$ then implies the original statement.

Item 8: Interaction With Weak Inverse Images II

Item 8a is clear, while Items 8b and 8c follow from Item 6 of Proposition 5.3.1.2. □

Proposition 5.5.2.6 ► Properties of the Strong Inverse Image Function Operation

Let $R\colon A \nrightarrow B$ be a relation.

1. *Functionality I.* The assignment $R \mapsto R_{-1}$ defines a function

$$(-)_{-1}\colon \mathsf{Sets}(A, B) \to \mathsf{Sets}(\mathcal{P}(A), \mathcal{P}(B)).$$

2. *Functionality II.* The assignment $R \mapsto R_{-1}$ defines a function

$$(-)_{-1}\colon \mathsf{Sets}(A, B) \to \mathsf{Pos}((\mathcal{P}(A), \subset), (\mathcal{P}(B), \subset)).$$

3. *Interaction With Identities.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$(\mathrm{id}_A)_{-1} = \mathrm{id}_{\mathcal{P}(A)};$$

4. *Interaction With Composition.* For each pair of composable relations $R\colon A \nrightarrow B$ and $S\colon B \nrightarrow C$, we have

$$(S \diamond R)_{-1} = R_{-1} \circ S_{-1}, \qquad \begin{tikzcd} \mathcal{P}(C) \arrow[r, "S_{-1}"] \arrow[dr, "(S\diamond R)_{-1}"'] & \mathcal{P}(B) \arrow[d, "R_{-1}"] \\ & \mathcal{P}(A). \end{tikzcd}$$

Proof 5.5.2.7 ▶ Proof of Proposition 5.5.2.6

Item 1: Functionality I

Clear.

Item 2: Functionality II

Clear.

Item 3: Interaction With Identities

Indeed, we have

$$\begin{aligned} (\chi_A)_{-1}(U) &\overset{\text{def}}{=} \{a \in A \mid \chi_A(a) \subset U\} \\ &\overset{\text{def}}{=} \{a \in A \mid \{a\} \subset U\} \\ &= U \end{aligned}$$

for each $U \in \mathcal{P}(A)$. Thus $(\chi_A)_{-1} = \mathrm{id}_{\mathcal{P}(A)}$.

Item 4: Interaction With Composition

Indeed, we have

$$\begin{aligned} (S \diamond R)_{-1}(U) &\overset{\text{def}}{=} \{a \in A \mid [S \diamond R](a) \subset U\} \\ &\overset{\text{def}}{=} \{a \in A \mid S(R(a)) \subset U\} \\ &\overset{\text{def}}{=} \{a \in A \mid S_*(R(a)) \subset U\} \\ &= \{a \in A \mid R(a) \subset S_{-1}(U)\} \\ &\overset{\text{def}}{=} R_{-1}(S_{-1}(U)) \\ &\overset{\text{def}}{=} [R_{-1} \circ S_{-1}](U) \end{aligned}$$

for each $U \in \mathcal{P}(C)$, where we used Item 2 of Proposition 5.5.2.4, which implies that the conditions

- We have $S_*(R(a)) \subset U$;
- We have $R(a) \subset S_{-1}(U)$;

are equivalent. Thus $(S \diamond R)_{-1} = R_{-1} \circ S_{-1}$.

5.5.3 Weak Inverse Images

Let A and B be sets and let $R\colon A \nrightarrow B$ be a relation.

Definition 5.5.3.1 ▶ Weak Inverse Images

The **weak inverse image function associated to** R[1] is the function

$$R^{-1}\colon \mathcal{P}(B) \to \mathcal{P}(A)$$

defined by[2]

$$R^{-1}(V) \stackrel{\text{def}}{=} \{a \in A \mid R(a) \cap V \neq \emptyset\}$$

for each $V \in \mathcal{P}(B)$.

[1] *Further Terminology:* Also called simply the **inverse image function associated to** R.

[2] *Further Terminology:* The set $R^{-1}(V)$ is called the **weak inverse image of** V **by** R or simply the **inverse image of** V **by** R.

Remark 5.5.3.2 ▶ Unwinding Definition 5.5.3.1

Identifying subsets of B with relations from B to pt via **??** of **??**, we see that the weak inverse image function associated to R is equivalently the function

$$R^{-1}\colon \underbrace{\mathcal{P}(B)}_{\cong \mathrm{Rel}(B,\mathrm{pt})} \to \underbrace{\mathcal{P}(A)}_{\cong \mathrm{Rel}(A,\mathrm{pt})}$$

defined by

$$R^{-1}(V) \stackrel{\text{def}}{=} V \diamond R$$

for each $V \in \mathcal{P}(A)$, where $R \diamond V$ is the composition

$$A \overset{R}{\nrightarrow} B \overset{V}{\nrightarrow} \mathrm{pt}.$$

Explicitly, we have

$$\begin{aligned} R^{-1}(V) &\stackrel{\text{def}}{=} V \diamond R \\ &\stackrel{\text{def}}{=} \int^{x \in B} V^{-1}_{x} \times R^{x}_{-_2}. \end{aligned}$$

Proof 5.5.3.3 ► Proof of Remark 5.5.3.2

We have

$$\begin{aligned}
V \diamond R &\overset{\text{def}}{=} \int^{x\in B} V^{-1}_{x} \times R^{x}_{-_2}\\
&= \left\{ a \in A \,\middle|\, \int^{x\in B} V^{\star}_{x} \times R^{x}_{a} = \text{true} \right\}\\
&= \left\{ a \in A \,\middle|\, \begin{array}{l}\text{there exists } x \in B \text{ such that the}\\ \text{following conditions hold:}\\ \quad 1.\ \text{We have } V^{\star}_{x} = \text{true}\\ \quad 2.\ \text{We have } R^{x}_{a} = \text{true}\end{array} \right\}\\
&= \left\{ a \in A \,\middle|\, \begin{array}{l}\text{there exists } x \in B \text{ such that the}\\ \text{following conditions hold:}\\ \quad 1.\ \text{We have } x \in V\\ \quad 2.\ \text{We have } x \in R(a)\end{array} \right\}\\
&= \{a \in A \mid \text{there exists } x \in V \text{ such that } x \in R(a)\}\\
&= \{a \in A \mid R(a) \cap V \neq \emptyset\}\\
&\overset{\text{def}}{=} R^{-1}(V)
\end{aligned}$$

This finishes the proof.

Proposition 5.5.3.4 ► Properties of Weak Inverse Image Functions

Let $R \colon A \nrightarrow B$ be a relation.

1. *Functoriality.* The assignment $V \mapsto R^{-1}(V)$ defines a functor

$$R^{-1} \colon (\mathcal{P}(B), \subset) \to (\mathcal{P}(A), \subset)$$

where

- *Action on Objects.* For each $V \in \mathcal{P}(B)$, we have

$$[R^{-1}](V) \overset{\text{def}}{=} R^{-1}(V);$$

- *Action on Morphisms.* For each $U, V \in \mathcal{P}(B)$:
  - If $U \subset V$, then $R^{-1}(U) \subset R^{-1}(V)$.

2. *Adjointness.* We have an adjunction

$$
(R^{-1} \dashv R_!)\colon \quad \begin{tikzcd} \mathcal{P}(B) \arrow[r, bend left, "R^{-1}"] \arrow[r, phantom, "\perp"] & \mathcal{P}(A), \arrow[l, bend left, "R_!"] \end{tikzcd}
$$

witnessed by a bijections of sets

$$
\mathrm{Hom}_{\mathcal{P}(A)}\left(R^{-1}(U), V\right) \cong \mathrm{Hom}_{\mathcal{P}(A)}(U, R_!(V)),
$$

natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$, i.e. such that:

(★) The following conditions are equivalent:
   - We have $R^{-1}(U) \subset V$;
   - We have $U \subset R_!(V)$.

3. *Preservation of Colimits.* We have an equality of sets

$$
R^{-1}\left(\bigcup_{i \in I} U_i\right) = \bigcup_{i \in I} R^{-1}(U_i),
$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(B)^{\times I}$. In particular, we have equalities

$$
\begin{aligned}
R^{-1}(U) \cup R^{-1}(V) &= R^{-1}(U \cup V),\\
R^{-1}(\emptyset) &= \emptyset,
\end{aligned}
$$

natural in $U, V \in \mathcal{P}(B)$.

4. *Oplax Preservation of Limits.* We have an inclusion of sets

$$
R^{-1}\left(\bigcap_{i \in I} U_i\right) \subset \bigcap_{i \in I} R^{-1}(U_i),
$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(B)^{\times I}$. In particular, we have inclusions

$$
\begin{aligned}
R^{-1}(U \cap V) &\subset R^{-1}(U) \cap R^{-1}(V),\\
R^{-1}(A) &\subset B,
\end{aligned}
$$

natural in $U, V \in \mathcal{P}(B)$.

5. *Symmetric Strict Monoidality With Respect to Unions.* The direct image function of Item 1 has a symmetric strict monoidal structure

$$
\left(R^{-1}, R^{-1,\otimes}, R_{\mathbb{1}}^{-1,\otimes}\right)\colon (\mathcal{P}(A), \cup, \emptyset) \to (\mathcal{P}(B), \cup, \emptyset),
$$

being equipped with equalities

$$R^{-1,\otimes}_{U,V}\colon R^{-1}(U)\cup R^{-1}(V)\xrightarrow{=}R^{-1}(U\cup V),$$
$$R^{-1,\otimes}_{\mathbb{1}}\colon \emptyset\xrightarrow{=}\emptyset,$$

natural in $U,V\in\mathcal{P}(B)$.

6. *Symmetric Oplax Monoidality With Respect to Intersections.* The direct image function of Item 1 has a symmetric oplax monoidal structure

$$\left(R^{-1},R^{-1,\otimes},R^{-1,\otimes}_{\mathbb{1}}\right)\colon(\mathcal{P}(A),\cap,A)\to(\mathcal{P}(B),\cap,B),$$

being equipped with inclusions

$$R^{-1,\otimes}_{U,V}\colon R^{-1}(U\cap V)\subset R^{-1}(U)\cap R^{-1}(V),$$
$$R^{-1,\otimes}_{\mathbb{1}}\colon R^{-1}(A)\subset B,$$

natural in $U,V\in\mathcal{P}(B)$.

7. *Interaction With Strong Inverse Images I.* We have

$$R^{-1}(V)=A\setminus R_{-1}(B\setminus V)$$

for each $V\in\mathcal{P}(B)$.

8. *Interaction With Strong Inverse Images II.* Let $R\colon A\nrightarrow B$ be a relation from A to B.

   (a) If R is a total relation, then we have an inclusion of sets

   $$R_{-1}(V)\subset R^{-1}(V)$$

   natural in $V\in\mathcal{P}(B)$.

   (b) If R is total and functional, then the above inclusion is in fact an equality.

   (c) Conversely, if we have $R_{-1}=R^{-1}$, then R is total and functional.

Proof 5.5.3.5 ► Proof of Proposition 5.5.3.4

Item 1: Functoriality

Clear.

Item 2: Adjointness

This follows from ?? of ??.

Item 3: Preservation of Colimits

This follows from Item 2 and ?? of ??.

Item 4: Oplax Preservation of Limits

Omitted.

Item 5: Symmetric Strict Monoidality With Respect to Unions

This follows from Item 3.

Item 6: Symmetric Oplax Monoidality With Respect to Intersections

This follows from Item 4.

Item 7: Interaction With Strong Inverse Images I

This follows from Item 7 of Proposition 5.5.2.4.

Item 8: Interaction With Strong Inverse Images II

This was proved in Item 8 of Proposition 5.5.2.4.

Proposition 5.5.3.6 ► Properties of the Weak Inverse Image Function Operation

Let $R\colon A \nrightarrow B$ be a relation.

1. *Functionality I.* The assignment $R \mapsto R^{-1}$ defines a function

$$(-)^{-1}\colon \mathrm{Rel}(A,B) \to \mathrm{Sets}(\mathcal{P}(A),\mathcal{P}(B)).$$

2. *Functionality II.* The assignment $R \mapsto R^{-1}$ defines a function

$$(-)^{-1}\colon \mathrm{Rel}(A,B) \to \mathrm{Pos}((\mathcal{P}(A),\subset),(\mathcal{P}(B),\subset)).$$

3. *Interaction With Identities.* For each $A \in \mathrm{Obj}(\mathrm{Sets})$, we have[1]

$$(\chi_A)^{-1} = \mathrm{id}_{\mathcal{P}(A)};$$

4. *Interaction With Composition.* For each pair of composable relations $R\colon A \nrightarrow B$ and $S\colon B \nrightarrow C$, we have[2]

$$(S \diamond R)^{-1} = R^{-1} \circ S^{-1}, \qquad \begin{tikzcd} \mathcal{P}(C) \arrow[r, "S^{-1}"] \arrow[dr, "(S\diamond R)^{-1}"'] & \mathcal{P}(B) \arrow[d, "R^{-1}"] \\ & \mathcal{P}(A). \end{tikzcd}$$

[1]That is, the postcomposition

$$(\chi_A)^{-1}\colon \mathrm{Rel}(\mathrm{pt}, A) \to \mathrm{Rel}(\mathrm{pt}, A)$$

is equal to $\mathrm{id}_{\mathrm{Rel}(\mathrm{pt}, A)}$.

[2]That is, we have

$$(S \diamond R)^{-1} = R^{-1} \circ S^{-1}, \qquad \begin{tikzcd} \mathrm{Rel}(\mathrm{pt}, C) \arrow[r, "R^{-1}"] \arrow[dr, "(S\diamond R)^{-1}"'] & \mathrm{Rel}(\mathrm{pt}, B) \arrow[d, "S^{-1}"] \\ & \mathrm{Rel}(\mathrm{pt}, A). \end{tikzcd}$$

Proof 5.5.3.7 ► Proof of Proposition 5.5.3.6

Item 1: Functoriality I

Clear.

Item 2: Functoriality II

Clear.

Item 3: Interaction With Identities

This follows from **??** of **??**.

Item 4: Interaction With Composition

This follows from **??** of **??**.

5.5.4 Direct Images With Compact Support

Let A and B be sets and let $R\colon A \nrightarrow B$ be a relation.

Definition 5.5.4.1 ► Direct Images With Compact Support

The **direct image with compact support function associated to** R is the function[1]

$$R_!\colon \mathcal{P}(A) \to \mathcal{P}(B)$$

defined by[2,3]

$$\begin{aligned} R_!(U) &\stackrel{\text{def}}{=} \left\{ b \in B \,\middle|\, \begin{array}{l} \text{for each } a \in A \text{, if we have} \\ b \in R(a) \text{, then } a \in U \end{array} \right\} \\ &= \left\{ b \in B \,\middle|\, R^{-1}(b) \subset U \right\} \end{aligned}$$

for each $U \in \mathcal{P}(A)$.

[1]*Further Notation:* Also written $\forall_R\colon \mathcal{P}(A) \to \mathcal{P}(B)$. This notation comes from the fact that the following statements are equivalent, where $b \in B$ and $U \in \mathcal{P}(A)$:

- We have $b \in \forall_R(U)$.
- For each $a \in A$, if $b \in R(a)$, then $a \in U$.

[2]*Further Terminology:* The set $R_!(U)$ is called the **direct image with compact support of U by R**.
[3]We also have

$$R_!(U) = B \setminus R_*(A \setminus U);$$

see Item 7 of Proposition 5.5.4.4.

Remark 5.5.4.2 ▶ Unwinding Definition 5.5.4.1

Identifying subsets of B with relations from pt to B via **??** of **??**, we see that the direct image with compact support function associated to R is equivalently the function

$$R_! \colon \underbrace{\mathcal{P}(A)}_{\cong \mathrm{Rel}(A,\mathrm{pt})} \to \underbrace{\mathcal{P}(B)}_{\cong \mathrm{Rel}(B,\mathrm{pt})}$$

defined by

$$R_!(U) \stackrel{\text{def}}{=} \mathrm{Ran}_R(U), \qquad \begin{tikzcd} & B \arrow[d, dashed, "\mathrm{Ran}_R(U)"] \\ A \arrow[ur, "R", "\shortmid" marking] \arrow[r, "U"', "\shortmid" marking] & \mathrm{pt}, \end{tikzcd}$$

being explicitly computed by

$$\begin{aligned} R^*(U) &\stackrel{\text{def}}{=} \mathrm{Ran}_R(U) \\ &\cong \int_{a \in A} \mathrm{Hom}_{\{\mathrm{t},\mathrm{f}\}}\left(R_a^{-2}, U_a^{-1}\right), \end{aligned}$$

where we have used Item 11 of Proposition 5.2.5.1.

Proof 5.5.4.3 ► Proof of Remark 5.5.4.2

We have

$$
\begin{aligned}
\mathrm{Ran}_R(V) &\cong \int_{a\in A} \mathrm{Hom}_{\{\mathrm{t,f}\}}\left(R_a^{-_2}, U_a^{-_1}\right) \\
&= \left\{ b \in B \;\middle|\; \int_{a\in A} \mathrm{Hom}_{\{\mathrm{t,f}\}}\left(R_a^b, U_a^\star\right) = \mathrm{true} \right\} \\
&= \left\{ b \in B \;\middle|\; \begin{array}{l} \text{for each } a \in A\text{, at least one of the} \\ \text{following conditions hold:} \\ \quad 1.\ \text{We have } R_a^b = \mathrm{false} \\ \quad 2.\ \text{The following conditions hold:} \\ \qquad \text{(a) We have } R_a^b = \mathrm{true} \\ \qquad \text{(b) We have } U_a^\star = \mathrm{true} \end{array} \right\} \\
&= \left\{ b \in B \;\middle|\; \begin{array}{l} \text{for each } a \in A\text{, at least one of the} \\ \text{following conditions hold:} \\ \quad 1.\ \text{We have } b \notin R(A) \\ \quad 2.\ \text{The following conditions hold:} \\ \qquad \text{(a) We have } b \in R(a) \\ \qquad \text{(b) We have } a \in U \end{array} \right\} \\
&= \left\{ b \in B \;\middle|\; \begin{array}{l} \text{for each } a \in A\text{, if we have} \\ b \in R(a)\text{, then } a \in U \end{array} \right\} \\
&= \left\{ b \in B \;\middle|\; R^{-1}(b) \subset U \right\} \\
&\overset{\mathrm{def}}{=} R^{-1}(U).
\end{aligned}
$$

This finishes the proof. □

Proposition 5.5.4.4 ► Properties of Direct Images With Compact Support

Let $R\colon A \nrightarrow B$ be a relation.

1. *Functoriality.* The assignment $U \mapsto R_!(U)$ defines a functor

$$R_!\colon (\mathcal{P}(A), \subset) \to (\mathcal{P}(B), \subset)$$

   where

- *Action on Objects.* For each $U \in \mathcal{P}(A)$, we have

$$[R_!](U) \stackrel{\text{def}}{=} R_!(U);$$

- *Action on Morphisms.* For each $U, V \in \mathcal{P}(A)$:
  - If $U \subset V$, then $R_!(U) \subset R_!(V)$.

2. *Adjointness.* We have an adjunction

$$(R^{-1} \dashv R_!)\colon \quad \begin{tikzcd} \mathcal{P}(B) \arrow[r, bend left, "R^{-1}"] \arrow[r, phantom, "\perp"] & \mathcal{P}(A), \arrow[l, bend left, "R_!"] \end{tikzcd}$$

witnessed by a bijections of sets

$$\mathrm{Hom}_{\mathcal{P}(A)}\left(R^{-1}(U), V\right) \cong \mathrm{Hom}_{\mathcal{P}(A)}(U, R_!(V)),$$

natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$, i.e. such that:

(★) The following conditions are equivalent:
  - We have $R^{-1}(U) \subset V$;
  - We have $U \subset R_!(V)$.

3. *Lax Preservation of Colimits.* We have an inclusion of sets

$$\bigcup_{i \in I} R_!(U_i) \subset R_!\left(\bigcup_{i \in I} U_i\right),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have inclusions

$$R_!(U) \cup R_!(V) \subset R_!(U \cup V),$$
$$\emptyset \subset R_!(\emptyset),$$

natural in $U, V \in \mathcal{P}(A)$.

4. *Preservation of Limits.* We have an equality of sets

$$R_!\left(\bigcap_{i \in I} U_i\right) = \bigcap_{i \in I} R_!(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have equalities

$$R_!(U \cap V) = R_!(U) \cap R_!(V),$$
$$R_!(A) = B,$$

natural in $U, V \in \mathcal{P}(A)$.

5. *Symmetric Lax Monoidality With Respect to Unions.* The direct image with compact support function of Item 1 has a symmetric lax monoidal structure

$$\left(R_!, R_!^{\otimes}, R_{!|\mathbb{1}}^{\otimes}\right): (\mathcal{P}(A), \cup, \emptyset) \to (\mathcal{P}(B), \cup, \emptyset),$$

being equipped with inclusions

$$R_{!|U,V}^{\otimes}: R_!(U) \cup R_!(V) \subset R_!(U \cup V),$$
$$R_{!|\mathbb{1}}^{\otimes}: \emptyset \subset R_!(\emptyset),$$

natural in $U, V \in \mathcal{P}(A)$.

6. *Symmetric Strict Monoidality With Respect to Intersections.* The direct image function of Item 1 has a symmetric strict monoidal structure

$$\left(R_!, R_!^{\otimes}, R_{!|\mathbb{1}}^{\otimes}\right): (\mathcal{P}(A), \cap, A) \to (\mathcal{P}(B), \cap, B),$$

being equipped with equalities

$$R_{!|U,V}^{\otimes}: R_!(U \cap V) \xrightarrow{=} R_!(U) \cap R_!(V),$$
$$R_{!|\mathbb{1}}^{\otimes}: R_!(A) \xrightarrow{=} B,$$

natural in $U, V \in \mathcal{P}(A)$.

7. *Relation to Direct Images.* We have

$$R_!(U) = B \setminus R_*(A \setminus U)$$

for each $U \in \mathcal{P}(A)$.

Proof 5.5.4.5 ► Proof of Proposition 5.5.4.4

Item 1: Functoriality

Clear.

Item 2: Adjointness

This follows from **??** of **??**.

Item 3: Lax Preservation of Colimits

Omitted.

Item 4: Preservation of Limits

This follows from Item 2 and **??** of **??**.

Item 5: Symmetric Lax Monoidality With Respect to Unions

This follows from Item 3.

Item 6: Symmetric Strict Monoidality With Respect to Intersections

This follows from Item 4.

Item 7: Relation to Direct Images

This follows from Item 7 of Proposition 5.5.1.3. Alternatively, we may prove it directly as follows, with the proof proceeding in the same way as in the case of functions (**??** of **??**).

We claim that $R_!(U) = B \setminus R_*(A \setminus U)$:

- *The First Implication.* We claim that

$$R_!(U) \subset B \setminus R_*(A \setminus U).$$

  Let $b \in R_!(U)$. We need to show that $b \notin R_*(A \setminus U)$, i.e. that there is no $a \in A \setminus U$ such that $b \in R(a)$.

  This is indeed the case, as otherwise we would have $a \in R^{-1}(b)$ and $a \notin U$, contradicting $R^{-1}(b) \subset U$ (which holds since $b \in R_!(U)$).

  Thus $b \in B \setminus R_*(A \setminus U)$.

- *The Second Implication.* We claim that

$$B \setminus R_*(A \setminus U) \subset R_!(U).$$

  Let $b \in B \setminus R_*(A \setminus U)$. We need to show that $b \in R_!(U)$, i.e. that $R^{-1}(b) \subset U$.

  Since $b \notin R_*(A \setminus U)$, there exists no $a \in A \setminus U$ such that $b \in R(a)$, and hence $R^{-1}(b) \subset U$.

  Thus $b \in R_!(U)$.

This finishes the proof.

Proposition 5.5.4.6 ► Properties of the Direct Image With Compact Support Function Operation

Let $R\colon A \nrightarrow B$ be a relation.

1. *Functionality I.* The assignment $R \mapsto R_!$ defines a function

$$(-)_!\colon \mathsf{Sets}(A, B) \to \mathsf{Sets}(\mathcal{P}(A), \mathcal{P}(B)).$$

2. *Functionality II.* The assignment $R \mapsto R_!$ defines a function

$$(-)_! \colon \mathsf{Sets}(A, B) \to \mathrm{Hom}_{\mathsf{Pos}}((\mathcal{P}(A), \subset), (\mathcal{P}(B), \subset)).$$

3. *Interaction With Identities.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$(\mathrm{id}_A)_! = \mathrm{id}_{\mathcal{P}(A)};$$

4. *Interaction With Composition.* For each pair of composable relations $R \colon A \nrightarrow B$ and $S \colon B \nrightarrow C$, we have

$$(S \diamond R)_! = S_! \circ R_!, \qquad \begin{tikzcd} \mathcal{P}(A) \arrow[r, "R_!"] \arrow[dr, "(S \diamond R)_!"'] & \mathcal{P}(B) \arrow[d, "S_!"] \\ & \mathcal{P}(C). \end{tikzcd}$$

Proof 5.5.4.7 ► Proof of Proposition 5.5.4.6

Item 1: Functionality I

Clear.

Item 2: Functionality II

Clear.

Item 3: Interaction With Identities

Indeed, we have

$$\begin{aligned} (\chi_A)_!(U) &\overset{\text{def}}{=} \left\{ a \in A \,\middle|\, \chi_A^{-1}(a) \subset U \right\} \\ &\overset{\text{def}}{=} \{ a \in A \mid \{a\} \subset U \} \\ &= U \end{aligned}$$

for each $U \in \mathcal{P}(A)$. Thus $(\chi_A)_! = \mathrm{id}_{\mathcal{P}(A)}$.

Item 4: Interaction With Composition

Indeed, we have

$$\begin{aligned} (S \diamond R)_!(U) &\overset{\text{def}}{=} \left\{ c \in C \,\middle|\, [S \diamond R]^{-1}(c) \subset U \right\} \\ &\overset{\text{def}}{=} \left\{ c \in C \,\middle|\, S^{-1}\big(R^{-1}(c)\big) \subset U \right\} \\ &= \left\{ c \in C \,\middle|\, R^{-1}(c) \subset S_!(U) \right\} \\ &\overset{\text{def}}{=} R_!(S_!(U)) \\ &\overset{\text{def}}{=} [R_! \circ S_!](U) \end{aligned}$$

for each $U \in \mathcal{P}(C)$, where we used Item 2 of Proposition 5.5.4.4, which implies that the conditions

- We have $S^{-1}\big(R^{-1}(c)\big) \subset U$;
- We have $R^{-1}(c) \subset S_!(U)$;

are equivalent. Thus $(S \diamond R)_! = S_! \circ R_!$.

5.5.5 Functoriality of Powersets

Proposition 5.5.5.1 ▶ Functoriality of Powersets I

The assignment $X \mapsto \mathcal{P}(X)$ defines functors[1]

$$\begin{aligned}\mathcal{P}_*&\colon \mathsf{Rel} \longrightarrow \mathsf{Sets},\\ \mathcal{P}_{-1}&\colon \mathsf{Rel}^{\mathsf{op}} \longrightarrow \mathsf{Sets},\\ \mathcal{P}^{-1}&\colon \mathsf{Rel}^{\mathsf{op}} \longrightarrow \mathsf{Sets},\\ \mathcal{P}_!&\colon \mathsf{Rel} \longrightarrow \mathsf{Sets}\end{aligned}$$

where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathsf{Rel})$, we have

$$\begin{aligned}\mathcal{P}_*(A) &\stackrel{\text{def}}{=} \mathcal{P}(A),\\ \mathcal{P}_{-1}(A) &\stackrel{\text{def}}{=} \mathcal{P}(A),\\ \mathcal{P}^{-1}(A) &\stackrel{\text{def}}{=} \mathcal{P}(A),\\ \mathcal{P}_!(A) &\stackrel{\text{def}}{=} \mathcal{P}(A);\end{aligned}$$

- *Action on Morphisms.* For each morphism $R\colon A \nrightarrow B$ of Rel, the images

$$\begin{aligned}\mathcal{P}_*(R)&\colon \mathcal{P}(A) \longrightarrow \mathcal{P}(B),\\ \mathcal{P}_{-1}(R)&\colon \mathcal{P}(B) \longrightarrow \mathcal{P}(A),\\ \mathcal{P}^{-1}(R)&\colon \mathcal{P}(B) \longrightarrow \mathcal{P}(A),\\ \mathcal{P}_!(R)&\colon \mathcal{P}(A) \longrightarrow \mathcal{P}(B)\end{aligned}$$

  of R by $\mathcal{P}_*$, $\mathcal{P}_{-1}$, $\mathcal{P}^{-1}$, and $\mathcal{P}_!$ are defined by

$$\begin{aligned}\mathcal{P}_*(R) &\stackrel{\text{def}}{=} R_*,\\ \mathcal{P}_{-1}(R) &\stackrel{\text{def}}{=} R_{-1},\\ \mathcal{P}^{-1}(R) &\stackrel{\text{def}}{=} R^{-1},\\ \mathcal{P}_!(R) &\stackrel{\text{def}}{=} R_!,\end{aligned}$$

as in Definitions 5.5.1.1, 5.5.2.1, 5.5.3.1 and 5.5.4.1.

[1]The functor $\mathcal{P}_* \colon \mathsf{Rel} \to \mathsf{Sets}$ admits a left adjoint; see Item 3 of Proposition 5.3.1.2.

PROOF 5.5.5.2 ▶ PROOF OF PROPOSITION 5.5.5.1

This follows from Items 3 and 4 of Proposition 5.5.1.5, Items 3 and 4 of Proposition 5.5.2.6, Items 3 and 4 of Proposition 5.5.3.6, and Items 3 and 4 of Proposition 5.5.4.6.

5.5.6 Functoriality of Powersets: Relations on Powersets

Let A and B be sets and let $R \colon A \nrightarrow B$ be a relation.

DEFINITION 5.5.6.1 ▶ THE RELATION ON POWERSETS ASSOCIATED TO A RELATION

The **relation on powersets associated to** R is the relation

$$\mathcal{P}(R) \colon \mathcal{P}(A) \nrightarrow \mathcal{P}(B)$$

defined by[1]

$$\mathcal{P}(R)^V_U \overset{\text{def}}{=} \mathbf{Rel}(\chi_{\text{pt}}, V \diamond R \diamond U)$$

for each $U \in \mathcal{P}(A)$ and each $V \in \mathcal{P}(B)$.

[1]Illustration:

$$\begin{tikzcd}
\text{pt} \arrow[r, "U"', "\shortmid"{marking}] \arrow[rrr, bend left, "\chi_{\text{pt}}", "\shortmid"{marking}] & A \arrow[r, "R"', "\shortmid"{marking}] & B \arrow[r, "V"', "\shortmid"{marking}] & \text{pt}.
\end{tikzcd}$$

REMARK 5.5.6.2 ▶ UNWINDING DEFINITION 5.5.6.1

In detail, we have $U \sim_{\mathcal{P}(R)} V$ iff the following equivalent conditions hold:

- We have $\chi_{\text{pt}} \subset V \diamond R \diamond U$.
- We have $(V \diamond R \diamond U)^{\star}_{\star} = \text{true}$, i.e. we have

$$\int^{a \in A} \int^{b \in B} V^{\star}_{b} \times R^{b}_{a} \times U^{a}_{\star} = \text{true}.$$

- There exists some $a \in A$ and some $b \in B$ such that:
  - We have $U^{a}_{\star} = \text{true}$;
  - We have $R^{b}_{a} = \text{true}$;
  - We have $V^{\star}_{b} = \text{true}$.

- There exists some $a \in A$ and some $b \in B$ such that:
  - We have $a \in U$;
  - We have $a \sim_R b$;
  - We have $b \in V$.

Proposition 5.5.6.3 ► Functoriality of Powersets II

The assignment $R \mapsto \mathcal{P}(R)$ defines a functor

$$\mathcal{P}\colon \mathsf{Rel} \longrightarrow \mathsf{Rel}.$$

Proof 5.5.6.4 ► Proof of Proposition 5.5.6.3

Omitted.

5.6 Relative Preorders

5.6.1 The Left Skew Monoidal Structure on $\mathbf{Rel}(A, B)$

Let A and B be sets and let $J\colon A \nrightarrow B$ be a relation.

5.6.1.1 The Left Skew Monoidal Product

Definition 5.6.1.1 ► The Left J-Skew Monoidal Product of $\mathrm{Rel}(A, B)$

The **left J-skew monoidal product of $\mathbf{Rel}(A, B)$** is the functor

$$\lhd_J\colon \mathbf{Rel}(A, B) \times \mathbf{Rel}(A, B) \longrightarrow \mathbf{Rel}(A, B)$$

where

- *Action on Objects.* For each $R, S \in \mathrm{Obj}(\mathbf{Rel}(A, B))$, we have

$$S \lhd_J R \stackrel{\text{def}}{=} S \diamond \mathrm{Rift}_J(R), \qquad \begin{tikzcd} & A \arrow[r, "S", "\mapstochar" marking] \arrow[d, "J", "\mapstochar" marking] \arrow[dl, phantom, "\Swarrow"] & B; \\ A \arrow[r, "R"', "\mapstochar" marking] \arrow[ur, dashed, "\mathrm{Rift}_J(R)", "\mapstochar" marking] & B & \end{tikzcd}$$

- *Action on Morphisms.* For each $R, S, R', S' \in \mathrm{Obj}(\mathbf{Rel}(A, B))$, the action on Hom-sets

$$(\lhd_J)_{(G,F),(G',F')}\colon \mathrm{Hom}_{\mathbf{Rel}(A,B)}(S, S') \times \mathrm{Hom}_{\mathbf{Rel}(A,B)}(R, R') \to \mathrm{Hom}_{\mathbf{Rel}(A,B)}(S \lhd_J R, S' \lhd_J R')$$

of $\lhd_J$ at $((R,S),(R',S'))$ is defined by[1]

$$\beta \lhd_J \alpha \overset{\text{def}}{=} \beta \diamond \mathrm{Rift}_J(\alpha),$$

$$\begin{tikzcd}
 & A \arrow[r, bend left, "S"] \arrow[r, bend right, "S'"'] \arrow[r, phantom, "\beta\Downarrow"] \arrow[d, "J"] & B \\
A \arrow[ur, dashed, bend left, "\mathrm{Rift}_J(R)"] \arrow[ur, dashed, "\mathrm{Rift}_J(R')"'] \arrow[ur, phantom, "\mathrm{Rift}_J(\alpha)\Downarrow"] \arrow[r, "R"] \arrow[r, bend right, "R'"'] \arrow[r, phantom, "\alpha\Downarrow"'] & B &
\end{tikzcd}$$

for each $\beta \in \mathrm{Hom}_{\mathbf{Rel}(A,B)}(S,S')$ and each $\alpha \in \mathrm{Hom}_{\mathbf{Rel}(A,B)}(R,R')$.

[1] Since $\mathbf{Rel}(A,B)$ is posetal, this is to say that if $S \subset S'$ and $R \subset R'$, then $S \lhd_J R \subset S' \lhd_J R'$.

5.6.1.2 The Left Skew Monoidal Unit

Definition 5.6.1.2 ▶ The Left J-Skew Monoidal Unit of Rel(A,B)

The **left J-skew monoidal unit of Rel(A,B)** is the functor

$$\mathbb{1}^{\mathbf{Rel}(A,B)}_{\lhd} \colon \mathrm{pt} \longrightarrow \mathbf{Rel}(A,B)$$

picking the object

$$\mathbb{1}^{\lhd}_{\mathbf{Rel}(A,B)} \overset{\text{def}}{=} J$$

of $\mathbf{Rel}(A,B)$.

5.6.1.3 The Left Skew Associators

Definition 5.6.1.3 ▶ The Left J-Skew Associator of Rel(A,B)

The **left J-skew associator of Rel(A,B)** is the natural transformation

$$\alpha^{\mathbf{Rel}(A,B),\lhd} \colon \lhd_J \circ (\lhd_J \times \mathrm{id}) \Longrightarrow \lhd_J \circ (\mathrm{id} \times \lhd_J),$$

whose component

$$\alpha^{\mathbf{Rel}(A,B),\lhd}_{T,S,R} \colon \underbrace{(T \lhd_J S) \lhd_J R}_{\overset{\text{def}}{=} T \diamond \mathrm{Rift}_J(S) \diamond \mathrm{Rift}_J(R)} \hookrightarrow \underbrace{T \lhd_J (S \lhd_J R)}_{\overset{\text{def}}{=} T \diamond \mathrm{Rift}_J(S \diamond \mathrm{Rift}_J(R))}$$

at (T, S, R) is given by

$$\alpha^{\mathbf{Rel}(A,B),\triangleleft}_{T,S,R} \stackrel{\text{def}}{=} \mathrm{id}_T \diamond \gamma,$$

where

$$\gamma\colon \mathrm{Rift}_J(S) \diamond \mathrm{Rift}_J(R) \hookrightarrow \mathrm{Rift}_J\big(S \diamond \mathrm{Rift}_J(R)\big)$$

is the inclusion adjunct to the inclusion

$$\epsilon_S \star \mathrm{id}_{\mathrm{Rift}_J(R)}\colon \underbrace{J \diamond \mathrm{Rift}_J(S) \diamond \mathrm{Rift}_J(R)}_{\stackrel{\text{def}}{=} J_*(\mathrm{Rift}_J(S)\diamond\mathrm{Rift}_J(R))} \hookrightarrow S \diamond \mathrm{Rift}_J(R)$$

under the adjunction $J_* \dashv \mathrm{Rift}_J$, where $\epsilon\colon J \diamond \mathrm{Rift}_J \Longrightarrow \mathrm{id}_{\mathbf{Rel}(A,B)}$ is the counit of the adjunction $J_* \dashv \mathrm{Rift}_J$.

5.6.1.4 The Left Skew Left Unitors

Definition 5.6.1.4 ▶ The Left J-Skew Left Unitor of $\mathbf{Rel}(A,B)$

The **left J-skew left unitor of $\mathbf{Rel}(A,B)$** is the natural transformation

$$\lambda^{\mathbf{Rel}(A,B),\triangleleft}\colon \triangleleft_J \circ \left(\mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleleft} \times \mathrm{id}\right) \Longrightarrow \mathrm{id},$$

whose component

$$\lambda^{\mathbf{Rel}(A,B),\triangleleft}_R\colon \underbrace{J \triangleleft_J R}_{\stackrel{\text{def}}{=} J\diamond\mathrm{Rift}_J(R)} \hookrightarrow R$$

at R is given by

$$\lambda^{\mathbf{Rel}(A,B),\triangleleft}_R \stackrel{\text{def}}{=} \epsilon_R,$$

where $\epsilon\colon J \diamond \mathrm{Rift}_J \Longrightarrow \mathrm{id}_{\mathbf{Rel}(A,B)}$ is the counit of the adjunction $J_* \dashv \mathrm{Rift}_J$.

5.6.1.5 The Left Skew Right Unitors

Definition 5.6.1.5 ▶ The Left J-Skew Right Unitor of $\mathbf{Rel}(A,B)$

The **left J-skew right unitor of $\mathbf{Rel}(A,B)$** is the natural transformation

$$\rho^{\mathbf{Rel}(A,B),\triangleleft}\colon \mathrm{id} \Longrightarrow \triangleleft_J \circ \left(\mathrm{id} \times \mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleleft}\right),$$

whose component

$$\rho^{\mathbf{Rel}(A,B),\triangleleft}_R\colon R \hookrightarrow \underbrace{R \triangleleft_J J}_{\stackrel{\text{def}}{=} R\diamond\mathrm{Rift}_J(J)}$$

at R is given by

$$\rho_R^{\mathbf{Rel}(A,B),\lhd} \overset{\text{def}}{=} \mathrm{id}_R \star \sigma,$$

where $\sigma\colon \mathrm{id}_A \Longrightarrow \mathrm{Rift}_J(J)$ is the universal transformation included in the data of the right Kan lift $\mathrm{Rift}_J(J)$.

5.6.1.6 The Left Skew Monoidal Structure on $\mathbf{Rel}(A, B)$

Definition 5.6.1.6 ▶ The Left J-Skew Monoidal Structure on $\mathbf{Rel}(A, B)$

The **left J-skew monoidal category of relations from A to B** is the left skew monoidal category

$$\left(\mathbf{Rel}(A, B), \lhd_J, \not\Vdash_{\lhd}^{\mathbf{Rel}(A,B)}, \alpha^{\mathbf{Rel}(A,B),\lhd}, \lambda^{\mathbf{Rel}(A,B),\lhd}, \rho^{\mathbf{Rel}(A,B),\lhd}\right)$$

consisting of

- *The Underlying Category.* The posetal category associated to the poset $\mathbf{Rel}(A, B)$ of relations from A to B of Item 2 of Definition 5.1.1.2;
- *The Skew Monoidal Product.* The functor $\lhd_J$ of Definition 5.6.1.1;
- *The Skew Monoidal Unit.* The functor $\not\Vdash_{\lhd}^{\mathbf{Rel}(A,B)}$ of Definition 5.6.1.2;
- *The Skew Associators.* The natural transformation $\alpha^{\mathbf{Rel}(A,B),\lhd}$ of Definition 5.6.1.3;
- *The Skew Left Unitors.* The natural transformation $\lambda^{\mathbf{Rel}(A,B),\lhd}$ of Definition 5.6.1.4;
- *The Skew Right Unitors.* The natural transformation $\rho^{\mathbf{Rel}(A,B),\lhd}$ of Definition 5.6.1.5.

5.6.2 Left Relative Preorders

Let A and B be sets and let $J\colon A \nrightarrow B$ be a relation.

Definition 5.6.2.1 ▶ Left J-Relative Preorders

A **left J-relative preorder from A to B** is equivalently:

- An $\mathbb{E}_1$-skew monoid in $\left(\mathrm{N}_{\bullet}(\mathbf{Rel}(A, B)), \lhd_J, J\right)$;
- A skew monoid in $\left(\mathbf{Rel}(A, B), \lhd_J, J\right)$.

Remark 5.6.2.2 ▶ Unwinding Definition 5.6.2.1, I

In detail, a **left J-relative preorder** (R, μ_R, η_R) **from** A **to** B consists of

- *The Underlying Relation.* A relation

$$R\colon A \nrightarrow B,$$

called the **underlying relation of** (R, μ_R, η_R);

- *The Multiplication Inclusion.* An inclusion of relations

$$\mu_R\colon R \lhd_J R \subset R,$$

called the **multiplication** of (R, μ_R, η_R);

- *The Unit Inclusion.* An inclusion of relations

$$\eta_R\colon J \subset R,$$

called the **unit** of (R, μ_R, η_R).

Remark 5.6.2.3 ▶ Unwinding Definition 5.6.2.1, II

In other words, a **left J-relative preorder from** A **to** B is a relation $R\colon A \nrightarrow B$ from A to B satisfying the following conditions:

1. *J-Transitivity.* For each $a \in A$ and each $c \in B$, we have

$$a \sim_{R \diamond \mathrm{Rift}_J(R)} c$$

i.e. the following condition is satisfied:[1]

(★) If there exists some $b \in A$ such that:

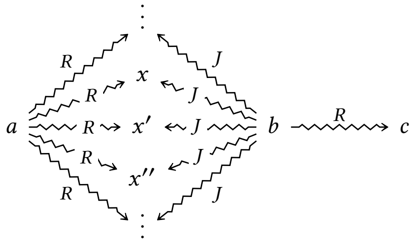
- We have $a \sim_{\mathrm{Rift}_J(R)} b$, i.e. for each $x \in B$, if $b \sim_J x$, then $a \sim_R x$;[2]
- We have $b \sim_R c$;

then $a \sim_R c$.

2. *J-Unitality.* For each $a \in A$ and each $b \in B$, the following condition is satisfied:

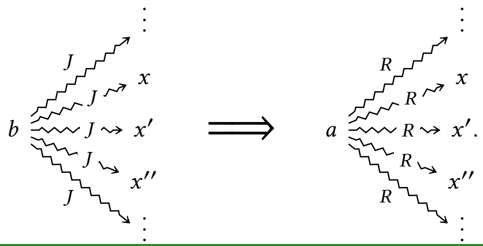
(★) If $a \sim_J b$, then $a \sim_R b$.

[1] *Illustration:* If we have



then $a \sim_R c$.

[2] Illustration:



5.6.3 The Right Skew Monoidal Structure on $\mathbf{Rel}(A, B)$

Let A and B be sets and let $J \colon A \nrightarrow B$ be a relation.

5.6.3.1 The Right Skew Monoidal Product

Definition 5.6.3.1 ► The Right J-Skew Monoidal Product of $\mathrm{Rel}(A, B)$

The **right J-skew monoidal product of $\mathbf{Rel}(A, B)$** is the functor

$$\triangleright_J \colon \mathbf{Rel}(A, B) \times \mathbf{Rel}(A, B) \longrightarrow \mathbf{Rel}(A, B)$$

where

- *Action on Objects.* For each $R, S \in \mathrm{Obj}(\mathbf{Rel}(A, B))$, we have

$$S \triangleright_J R \overset{\text{def}}{=} \mathrm{Ran}_J(S) \diamond R, \qquad \begin{tikzcd} A \arrow[r, "R", "\shortmid"{marking}] & B \arrow[r, dashed, "\mathrm{Ran}_J(S)", "\shortmid"{marking}] & B; \\ & A \arrow[u, "J", "\shortmid"{marking}] \arrow[ur, "S"', "\shortmid"{marking}] \arrow[u, phantom, shift left=4, "\Nwarrow"] & \end{tikzcd}$$

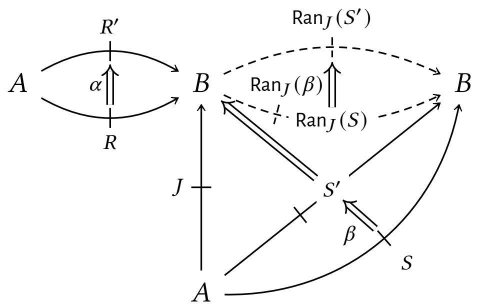
- *Action on Morphisms.* For each $R, S, R', S' \in \mathrm{Obj}(\mathbf{Rel}(A, B))$, the action on

Hom-sets

$$(\rhd_J)_{(S,R),(S',R')} : \mathrm{Hom}_{\mathbf{Rel}(A,B)}(S,S') \times \mathrm{Hom}_{\mathbf{Rel}(A,B)}(R,R') \to \mathrm{Hom}_{\mathbf{Rel}(A,B)}(S \rhd_J R, S' \rhd_J R')$$

of $\rhd_J$ at $((S,R),(S',R'))$ is defined by[1]

$$\beta \rhd_J \alpha \overset{\text{def}}{=} \mathrm{Ran}_J(\beta) \diamond \alpha,$$



for each $\beta \in \mathrm{Hom}_{\mathbf{Rel}(A,B)}(S,S')$ and each $\alpha \in \mathrm{Hom}_{\mathbf{Rel}(A,B)}(R,R')$.

[1]Since $\mathbf{Rel}(A,B)$ is posetal, this is to say that if $S \subset S'$ and $R \subset R'$, then $S \rhd_J R \subset S' \rhd_J R'$.

5.6.3.2 The Right Skew Monoidal Unit

Definition 5.6.3.2 ▶ The Right J-Skew Monoidal Unit of $\mathbf{Rel}(A,B)$

The **right J-skew monoidal unit of $\mathbf{Rel}(A,B)$** is the functor

$$\mathbb{1}^{\mathbf{Rel}(A,B)}_{\rhd} : \mathrm{pt} \longrightarrow \mathbf{Rel}(A,B)$$

picking the object

$$\mathbb{1}^{\rhd}_{\mathbf{Rel}(A,B)} \overset{\text{def}}{=} J$$

of $\mathbf{Rel}(A,B)$.

5.6.3.3 The Right Skew Associators

Definition 5.6.3.3 ▶ The Right J-Skew Associator of $\mathbf{Rel}(A,B)$

The **right J-skew associator of $\mathbf{Rel}(A,B)$** is the natural transformation

$$\alpha^{\mathbf{Rel}(A,B),\rhd} : \rhd_J \circ (\mathrm{id} \times \rhd_J) \Longrightarrow \rhd_J \circ (\rhd_J \times \mathrm{id}),$$

whose component

$$\alpha^{\mathbf{Rel}(A,B),\rhd}_{T,S,R} : \underbrace{T \rhd_J (S \rhd_J R)}_{\overset{\text{def}}{=} \mathrm{Ran}_J(T) \diamond (\mathrm{Ran}_J(S) \diamond R)} \hookrightarrow \underbrace{(T \rhd_J S) \rhd_J R}_{\overset{\text{def}}{=} \mathrm{Ran}_J(\mathrm{Ran}_J(T) \diamond S) \diamond R}$$

at (T, S, R) is given by

$$\alpha^{\mathbf{Rel}(A,B),\triangleright}_{T,S,R} \overset{\text{def}}{=} \gamma \diamond \mathrm{id}_R,$$

where

$$\gamma\colon \mathrm{Ran}_J(T) \diamond \mathrm{Ran}_J(S) \hookrightarrow \mathrm{Ran}_J\big(\mathrm{Ran}_J(T) \diamond S\big)$$

is the inclusion adjunct to the inclusion

$$\mathrm{id}_{\mathrm{Ran}_J(T)} \diamond \epsilon_S\colon \underbrace{\mathrm{Ran}_J(T) \diamond \mathrm{Ran}_J(S) \diamond J}_{\overset{\text{def}}{=} J^*(\mathrm{Ran}_J(T) \diamond \mathrm{Ran}_J(S))} \hookrightarrow \mathrm{Ran}_J(T) \diamond S$$

under the adjunction $J^* \dashv \mathrm{Ran}_J$, where $\epsilon\colon \mathrm{Ran}_J \diamond J \Longrightarrow \mathrm{id}_{\mathbf{Rel}(A,B)}$ is the counit of the adjunction $J^* \dashv \mathrm{Ran}_J$.

5.6.3.4 The Right Skew Left Unitors

Definition 5.6.3.4 ► The Right J-Skew Left Unitor of Rel(A, B)

The **right J-skew left unitor of Rel(A, B)** is the natural transformation

$$\lambda^{\mathbf{Rel}(A,B),\triangleright}\colon \mathrm{id} \Longrightarrow \triangleright_J \circ \left(\mathbb{K}^{\mathbf{Rel}(A,B)}_{\triangleright} \times \mathrm{id}\right),$$

whose component

$$\lambda^{\mathbf{Rel}(A,B),\triangleright}_R\colon R \hookrightarrow \underbrace{J \triangleright_J R}_{\overset{\text{def}}{=} \mathrm{Ran}_J(J) \diamond R}$$

at R is given by

$$\lambda^{\mathbf{Rel}(A,B),\triangleright}_R \overset{\text{def}}{=} \sigma \diamond \mathrm{id}_R,$$

where $\sigma\colon \mathrm{id}_B \Longrightarrow \mathrm{Ran}_J(J)$ is the universal transformation included in the data of the right Kan extension $\mathrm{Ran}_J(J)$.

5.6.3.5 The Right Skew Right Unitors

Definition 5.6.3.5 ► The Right J-Skew Right Unitor of Rel(A, B)

The **right J-skew right unitor of Rel(A, B)** is the natural transformation

$$\rho^{\mathbf{Rel}(A,B),\triangleright}\colon \triangleright_J \circ \left(\mathrm{id} \times \mathbb{K}^{\mathbf{Rel}(A,B)}_{\triangleright}\right) \Longrightarrow \mathrm{id},$$

whose component

$$\rho^{\mathbf{Rel}(A,B),\triangleright}_{S} \colon \underbrace{S \triangleright_J J}_{\overset{\text{def}}{=} \mathrm{Ran}_J(S) \diamond J} \hookrightarrow S$$

at S is given by

$$\rho^{\mathbf{Rel}(A,B),\triangleright}_{S} \overset{\text{def}}{=} \epsilon_R,$$

where $\epsilon \colon \mathrm{Ran}_J \diamond J \Longrightarrow \mathrm{id}_{\mathbf{Rel}(A,B)}$ is the counit of the adjunction $J^* \dashv \mathrm{Ran}_J$.

5.6.3.6 The Right Skew Monoidal Structure on $\mathbf{Rel}(A, B)$

Definition 5.6.3.6 ► The Right J-Skew Monoidal Structure on $\mathrm{Rel}(A, B)$

The **right J-skew monoidal category of functors from A to B** is the right skew monoidal category

$$\left(\mathbf{Rel}(A, B), \triangleright_J, \mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleright}, \alpha^{\mathbf{Rel}(A,B),\triangleright}, \lambda^{\mathbf{Rel}(A,B),\triangleright}, \rho^{\mathbf{Rel}(A,B),\triangleright}\right)$$

consisting of

- *The Underlying Category.* The posetal category associated to the poset $\mathbf{Rel}(A, B)$ of relations from A to B of Item 2 of Definition 5.1.1.2;
- *The Skew Monoidal Product.* The functor $\triangleright_J$ of Definition 5.6.3.1;
- *The Skew Monoidal Unit.* The functor $\mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleright}$ of Definition 5.6.3.2;
- *The Skew Associators.* The natural transformation $\alpha^{\mathbf{Rel}(A,B),\triangleright}$ of Definition 5.6.3.3;
- *The Skew Left Unitors.* The natural transformation $\lambda^{\mathbf{Rel}(A,B),\triangleright}$ of Definition 5.6.3.4;
- *The Skew Right Unitors.* The natural transformation $\rho^{\mathbf{Rel}(A,B),\triangleright}$ of Definition 5.6.3.5.

5.6.4 Right Relative Preorders

Let A and B be sets and let $J \colon A \nrightarrow B$ be a relation.

Definition 5.6.4.1 ▶ Right J-Relative Preorders

A **right J-relative preorder from A to B** is equivalently:

- An $\mathbb{E}_1$-skew monoid in $\left(\mathsf{N}_{\bullet}(\mathbf{Rel}(A,B)), \triangleright_J, J\right)$;
- A skew monoid in $\left(\mathbf{Rel}(A,B), \triangleright_J, J\right)$.

Remark 5.6.4.2 ▶ Unwinding Definition 5.6.4.1, I

In detail, a **right J-relative preorder** (R, μ_R, η_R) **from** A **to** B consists of

- *The Underlying Relation.* A relation

$$R\colon A \nrightarrow B,$$

called the **underlying relation of** (R, μ_R, η_R);

- *The Multiplication Inclusion.* An inclusion of relations

$$\mu_R\colon R \triangleright_J R \subset R,$$

called the **multiplication** of (R, μ_R, η_R);

- *The Unit Inclusion.* An inclusion of relations

$$\eta_R\colon J \subset R,$$

called the **unit** of (R, μ_R, η_R).

Remark 5.6.4.3 ▶ Unwinding Definition 5.6.4.1, II

In other words, a **right J-relative preorder from A to B** is a relation $R\colon A \nrightarrow B$ from A to B satisfying the following conditions:

1. *J-Transitivity.* For each $a \in A$ and each $c \in B$, we have

$$a \sim_{\mathrm{Ran}_J(R) \diamond R} c,$$

i.e. the following condition is satisfied:[1]

(★) If there exists some $b \in B$ such that:

- We have $a \sim_R b$;
- We have $b \sim_{\mathrm{Ran}_J(R)} c$, i.e. for each $x \in A$, if $x \sim_J b$, then $x \sim_R c$;[2]

then $a \sim_R c$.

2. *J-Unitality.* For each $a \in A$ and each $b \in B$, the following condition is satisfied:

   (★) If $a \sim_J b$, then $a \sim_R b$.

[1] *Illustration:* If we have

$$
\begin{tikzcd}
 & & \vdots \arrow[ddl, squiggly, "J"'] \arrow[ddr, squiggly, "R"] & \\
 & & x \arrow[dl, squiggly, "J"'] \arrow[dr, squiggly, "R"] & \\
a \arrow[r, squiggly, "R"] & b & x' \arrow[l, squiggly, "J"'] \arrow[r, squiggly, "R"] & c \\
 & & x'' \arrow[ul, squiggly, "J"] \arrow[ur, squiggly, "R"'] & \\
 & & \vdots \arrow[uul, squiggly, "J"] \arrow[uur, squiggly, "R"'] &
\end{tikzcd}
$$

then $a \sim_R c$.

[2] Illustration:

$$
\begin{tikzcd}
\vdots \arrow[ddr, squiggly, "J"] & \\
x \arrow[dr, squiggly, "J"] & \\
x' \arrow[r, squiggly, "J"] & b \\
x'' \arrow[ur, squiggly, "J"'] & \\
\vdots \arrow[uur, squiggly, "J"'] &
\end{tikzcd}
\Longrightarrow
\begin{tikzcd}
\vdots \arrow[ddr, squiggly, "R"] & \\
x \arrow[dr, squiggly, "R"] & \\
x' \arrow[r, squiggly, "R"] & c. \\
x'' \arrow[ur, squiggly, "R"'] & \\
\vdots \arrow[uur, squiggly, "R"'] &
\end{tikzcd}
$$

Appendices

5.A Other Chapters

Chapter 6

Spans

This chapter contains some material about spans. Notably, we discuss and explore:

1. The basic definitions around spans (Section 6.1);
2. The relation between spans and functions (Proposition 6.8.1.1);
3. The relation between spans and relations (Propositions 6.8.2.4 and 6.8.3.1 and Remark 6.8.5.1).
4. “Hyperpointed sets” (**??**). I don’t know why I wrote this...

TODO:

1. https://www.sciencedirect.com/science/article/pii/0022404994900094?ref=pdf_download&fr=RR-2&rr=834107b75c906aa4
2. https://arxiv.org/abs/1605.08100
3. https://arxiv.org/abs/1603.08181
4. https://arxiv.org/abs/1601.02307
5. https://arxiv.org/abs/1507.01460
6. https://arxiv.org/abs/1506.08870
7. https://arxiv.org/abs/1505.00048
8. https://arxiv.org/abs/1501.07592
9. https://arxiv.org/abs/1501.04664
10. https://arxiv.org/abs/1501.00792
11. https://arxiv.org/abs/1412.6560
12. https://arxiv.org/abs/1412.0212

13. https://arxiv.org/abs/1409.0837
14. https://arxiv.org/abs/1408.5220
15. https://arxiv.org/abs/1308.6548
16. https://arxiv.org/abs/1304.0219
17. https://arxiv.org/abs/1210.8192
18. https://arxiv.org/abs/1210.1433
19. https://arxiv.org/abs/1201.3789
20. https://arxiv.org/abs/1112.0560
21. https://arxiv.org/abs/1109.1598
22. https://arxiv.org/abs/1101.4594
23. https://arxiv.org/abs/1012.6001
24. https://arxiv.org/abs/1011.3243
25. https://arxiv.org/abs/0910.2996
26. https://arxiv.org/abs/0810.2361
27. https://arxiv.org/abs/0803.2429
28. https://arxiv.org/abs/0712.2525
29. https://arxiv.org/abs/0706.1286
30. https://arxiv.org/abs/math/0611930
31. https://arxiv.org/abs/2311.15342
32. https://arxiv.org/abs/2310.19428
33. https://arxiv.org/abs/2309.08084
34. https://arxiv.org/abs/2308.01662
35. https://arxiv.org/abs/2301.11860
36. https://arxiv.org/abs/2301.01199
37. https://arxiv.org/abs/2212.09060
38. https://arxiv.org/abs/2208.07183
39. https://arxiv.org/abs/2205.06892

40. https://arxiv.org/abs/2203.16179
41. https://arxiv.org/abs/2201.09551
42. https://arxiv.org/abs/2112.04599
43. https://arxiv.org/abs/2111.10968
44. https://arxiv.org/abs/2107.07621
45. https://arxiv.org/abs/2106.14743
46. https://arxiv.org/abs/2105.14654
47. https://arxiv.org/abs/2102.08051
48. https://arxiv.org/abs/2102.04386
49. https://arxiv.org/abs/2101.06734
50. https://arxiv.org/abs/2011.11042
51. https://arxiv.org/abs/2010.15722
52. https://arxiv.org/abs/2006.10375
53. https://arxiv.org/abs/2006.10375
54. https://arxiv.org/abs/2005.10496
55. https://arxiv.org/abs/2003.11541
56. https://arxiv.org/abs/2002.10334
57. https://arxiv.org/abs/1909.00069
58. https://arxiv.org/abs/1907.02695
59. https://arxiv.org/abs/1905.06671
60. define a relational span
61. consider giving Ran and Rift their dedicated sections on the relations chapter, perhaps together with the other sections on co/limits
62. https://arxiv.org/abs/1710.02742
63. https://arxiv.org/search/math?searchtype=author&query=Walker,+Charles
64. https://arxiv.org/abs/1706.09575
65. https://arxiv.org/abs/1710.01465

66. fibred categories: https://arxiv.org/abs/1806.02376
67. https://arxiv.org/abs/1806.10477v2
68. double categorical limits in $\mathsf{Rel}^{\mathsf{dbl}}$
69. double categorical limits in $\mathsf{Span}^{\mathsf{dbl}}$
70. internal adjoint equivalences in **Rel**
71. internal adjoint equivalences in Span
72. 2-categorical limits in **Rel**;
73. morphism of internal adjunctions in **Rel**;
74. morphism of internal adjunctions in Span;
75. morphism of co/monads in Span;
76. What is $\mathrm{Adj}(\mathrm{Span}(A, B))$?
77. monoids, comonoids, pseudomonoids, etc. in Span.
78. write down the dumb intuition about spans inducing morphisms $\mathrm{Sets}(S, A) \to \mathrm{Sets}(S, B)$ instead of $\mathcal{P}(A) \to \mathcal{P}(B)$ from the similarity between

    $$S \to A \times B$$

    and

    $$A \times B \to \{\mathrm{t}, \mathrm{f}\}.$$

    This intuition is justified by taking $A = \mathrm{pt}$ or $B = \mathrm{pt}$.
79. What about using the direct image with compact support in $g\big(f^{-1}(a)\big)$?
80. Monads in Span | develop this in the level of morphisms too
81. Comonads in Span are spans whose legs are equal | develop this in the level of morphisms too
82. Does Span have an internal **Hom**?
83. Examples of spans
84. Functional and total spans
85. closed symmetric monoidal category of spans
86. double category of relations
87. collage of a span

88. equivalence spans?
89. functoriality of powersets for spans
90. Is Span a closed bicategory?
91. skew monoidal structure on Span(A, B)
92. Adjunctions in Span
93. Isomorphisms in Span
94. Equivalences in Span
95. Interaction between the above notions in Span vs.in **Rel** via the comparison functors
96. $\mathrm{Hom}_C(S, A) \times \mathrm{Hom}_C(f^*(S), A)$.
97. Proof of non-existence of left Kan extensions/lifts in **Rel** (when do these exist btw?)
98. description of unitors and associators of span
99. add intuition for spans as relations with multiple witnesses:
    (a) Given a span $A \xleftarrow{f} S \xrightarrow{g} B$, we have a functor
    $$\mathsf{St}(S)\colon (A \times B)_{\mathsf{disc}} \to \mathsf{Sets}$$
    given by
    $$\begin{aligned}[\mathsf{St}(S)](a, b) &\overset{\text{def}}{=} \mathsf{St}(S)_{ab}\\ &\overset{\text{def}}{=} \{s \in S \mid f(s) = a \text{ and } g(s) = b\}.\end{aligned}$$
    (b) Given a functor
    $$S\colon (A \times B)_{\mathsf{disc}} \to \mathsf{Sets},$$
    we have a map of sets
    $$\mathsf{Un}(S)\colon \coprod_{(a,b)\in A\times B} S(a, b) \to A \times B,$$
    determining a span from A to B.
    (c) How do these interact with left/right Kan extensions/lifts?
    (d) Un/straightening for spans of categories: assignment $(a, b) \mapsto \mathsf{Wit}_S(a, b)$.
    (e) Fix the TODO below

Contents

6.1 Spans

6.1.1 The Walking Span

Definition 6.1.1.1 ► The Walking Span

The **walking span** is the category Λ that looks like this:

$$\begin{tikzcd} & {[0]} \arrow[loop above] \arrow[dl] \arrow[dr] & \\ {[-1]} \arrow[loop left] & & {[1]} \arrow[loop right] \end{tikzcd}$$

6.1.2 Spans

Let A and B be sets.

Definition 6.1.2.1 ► Spans

A **span from** A **to** B[1] is a functor $F\colon \Lambda \to \mathsf{Sets}$ such that

$$\begin{aligned} F([-1]) &= A, \\ F([1]) &= B. \end{aligned}$$

[1] *Further Terminology:* Also called a **roof from** A **to** B or a **correspondence from** A **to** B.

Remark 6.1.2.2 ► Unwinding Definition 6.1.2.1

In detail, a **span from** A **to** B is a triple (S, f, g) consisting of[1,2]

- *The Underlying Set.* A set S, called the **underlying set of** (S, f, g);
- *The Legs.* A pair of functions $f\colon S \to A$ and $g\colon S \to B$.

[1] *Picture:*

$$\begin{tikzcd} & S \arrow[dl, "f"'] \arrow[dr, "g"] & \\ A & & B. \end{tikzcd}$$

[2]Every span (S, f, g) from A to B determines in particular a relation $R\colon A \nrightarrow B$ via

$$R \overset{\text{def}}{=} \{(f(a), g(a)) \mid a \in A\},$$

i.e. where $R(a) = g(f^{-1}(a))$ for each $a \in A$; see Proposition 6.8.2.4.

Remark 6.1.2.3 ▶ Spans as Relations With Multiple Witnesses

A span

$$\begin{tikzcd} & S \arrow[dl, "f"'] \arrow[dr, "g"] & \\ A & & B. \end{tikzcd}$$

from A to B may be thought of as a relation from A to B which can relate an element $a \in A$ to an element $b \in B$ in multiple ways via f and g, with the "set of witnesses of $a \sim_S b$" being given by

$$\mathrm{Wit}_S(a, b) \overset{\text{def}}{=} \{s \in S \mid a = f(s) \text{ and } g(s) = b\}.$$

This analogy is made precise by Remark 6.8.5.1 and Section 6.7.

6.1.3 Morphisms of Spans

Definition 6.1.3.1 ▶ Morphisms of Spans

A **morphism of spans from** (R, f_1, g_1) **to** (S, f_2, g_2)[1] is a natural transformation $(R, f_1, g_1) \Longrightarrow (S, f_2, g_2)$.

[1]*Further Terminology:* Also called a **morphism of roofs from** (R, f_1, g_1) **to** (S, f_2, g_2) or a **morphism of correspondences from** (R, f_1, g_1) **to** (S, f_2, g_2).

Remark 6.1.3.2 ▶ Unwinding Definition 6.1.3.1

In detail, a **morphism of spans from** (R, f_1, g_1) **to** (S, f_2, g_2) is a function $\phi\colon R \to S$ making the diagram[1]

$$\begin{tikzcd} & R \arrow[dl, "f_1"'] \arrow[dr, "g_1"] \arrow[rrr, "\phi"] & & & S \arrow[dl, "f_2"'] \arrow[dr, "g_2"] & \\ A \arrow[rrr, equal] & & B \arrow[rrr, equal] & A & & B. \end{tikzcd}$$

commute.

[1] *Alternative Picture:*

$$\begin{tikzcd} & R \arrow[dl, "f_1"'] \arrow[d, "\phi"] \arrow[dr, "g_1"] & \\ A & S \arrow[l, "f_2"] \arrow[r, "g_2"'] & B. \end{tikzcd}$$

6.1.4 Functional Spans

Let $\lambda = \left(A \xleftarrow{f} S \xrightarrow{g} B\right)$ be a span.

Definition 6.1.4.1 ► Functional Spans

The span λ is **functional** if f the following equivalent conditions are satisfied:

1. The associated relation $g \circ f^{-1}$ of λ is functional.
2. For each $s, t \in S$, if $f(s) = f(t)$, then $g(s) = g(t)$.
3. this "f-relative injectivity" condition is the same as being a monomorphism/monoid/whatever in nice category | maybe this is the same as being a skew monoid in $\mathsf{Span}(A, B)$ or something?

1. a[1]

[1] Here we could perhaps also use the direct image with compact support $g_!$ of g (see **??**) instead of the usual direct image, although the expression for $g_!(f^{-1}(a))$ seems a bit weird. It can also actually be given as a right Kan extension (**??** of **??**):

$$\begin{aligned} g_!\left(f^{-1}(a)\right) &= g_!(\{s \in S \mid f(s) = a\}) \\ &= \left\{b \in B \,\middle|\, g^{-1}(b) \subset \{s \in S \mid f(s) = a\}\right\} \\ &= \{b \in B \mid \text{for each } s \in S, \text{if } g(s) = b, \text{then } f(s) = a\} \\ &= [\mathrm{Ran}^{\dagger}_{g}(f)](a) \end{aligned}$$

as in the diagram

$$\mathrm{Ran}^{\dagger}_{g}(f) \colon A \nrightarrow B \qquad \begin{tikzcd} & B \arrow[d, dashed, "\mathrm{Ran}_g(f)"] \\ S \arrow[ur, "g"] \arrow[r, "f"'] & A. \end{tikzcd}$$

Definition 6.1.4.2 ▶ Total Spans

The span λ is **total** if f is surjective.

Let $\lambda = \left(A \overset{f}{\leftarrow} S \overset{g}{\rightarrow} B\right)$ be a span. A morphism of spans from id_A to $\lambda \diamond \lambda^\dagger$ is a morphism

$$s\colon A \to S \times_B S$$

making the diagram

$$\begin{tikzcd}
 & A \arrow[dl, equal, "\mathrm{id}_A"'] \arrow[dr, equal, "\mathrm{id}_A"] \arrow[dd, "s"] & \\
A & & A \\
 & S \times_B S \arrow[ul, "f\circ\mathrm{pr}'_1"] \arrow[ur, "f\circ\mathrm{pr}'_2"'] &
\end{tikzcd}$$

commute, where $S \times_B S$ is the pullback

$$S \times_B S \cong \{(s,t) \in S \times S \mid g(s) = g(t)\} \qquad \begin{tikzcd}
S \times_B S \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & S \arrow[d, "g"] \\
S \arrow[r, "g"'] & B
\end{tikzcd}$$

of S with itself along g. In particular, $\mathrm{pr}_1 \circ s$ and $\mathrm{pr}_2 \circ s$ are both left-inverses/retractions for f, i.e. we have

$$\begin{aligned}
(\mathrm{pr}_1 \circ s) \circ f &\cong \mathrm{id}_A,\\
(\mathrm{pr}_2 \circ s) \circ f &\cong \mathrm{id}_A.
\end{aligned}$$

Thus, by **??** of **??**, f is injective if $A \neq \emptyset$.

6.1.5 Total Spans

6.2 Categories of Spans

6.2.1 The Category of Spans Between Two Sets

Let A and B be sets.

Definition 6.2.1.1 ► The Category of Spans From A to B

The **category of spans from A to B** is the category $\mathsf{Span}(A,B)$ defined by

$$\mathsf{Span}(A,B) \stackrel{\text{def}}{=} \mathsf{Fun}(\Lambda,\mathsf{Sets}) \underset{\mathrm{ev}_{[-1]},\mathsf{Sets},[A]}{\times} \mathrm{pt} \underset{[B],\mathsf{Sets},\mathrm{ev}_{[1]}}{\times} \mathsf{Fun}(\Lambda,\mathsf{Sets}),$$

as in the diagram

$$\begin{tikzcd}
 & & & \mathsf{Span}(A,B) \arrow[dl] \arrow[dr] \arrow[dd, phantom, very near start, "\lrcorner", rotate=-45] & & & \\
 & & \mathsf{Fun}(\Lambda,\mathsf{Sets}) \underset{\mathsf{Sets}}{\times} \mathrm{pt} \arrow[dl] \arrow[dr] \arrow[dd, phantom, very near start, "\lrcorner", rotate=-45] & & \mathrm{pt} \underset{\mathsf{Sets}}{\times} \mathsf{Fun}(\Lambda,\mathsf{Sets}) \arrow[dl] \arrow[dr] \arrow[dd, phantom, very near start, "\lrcorner", rotate=-45] & & \\
 & \mathsf{Fun}(\Lambda,\mathsf{Sets}) \arrow[dr, "\mathrm{ev}_{[-1]}"'] & & \mathrm{pt} \arrow[dl, "{[A]}"] \arrow[dr, "{[B]}"'] & & \mathsf{Fun}(\Lambda,\mathsf{Sets}). \arrow[dl, "\mathrm{ev}_{[1]}"] & \\
 & & \mathsf{Sets} & & \mathsf{Sets} & &
\end{tikzcd}$$

Remark 6.2.1.2 ► Unwinding Definition 6.2.1.1

In detail, the **category of spans from A to B** is the category $\mathsf{Span}(A,B)$ where

- *Objects.* The objects of $\mathsf{Span}(A,B)$ are spans from A to B;
- *Morphisms.* The morphism of $\mathsf{Span}(A,B)$ are morphisms of spans;
- *Identities.* The unit map

  $$\mathbb{1}^{\mathsf{Span}(A,B)}_{(S,f,g)} \colon \mathrm{pt} \to \mathrm{Hom}_{\mathsf{Span}(A,B)}((S,f,g),(S,f,g))$$

  of $\mathsf{Span}(A,B)$ at (S,f,g) is defined by[1]

  $$\mathrm{id}^{\mathsf{Span}(A,B)}_{(S,f,g)} \stackrel{\text{def}}{=} \mathrm{id}_S;$$

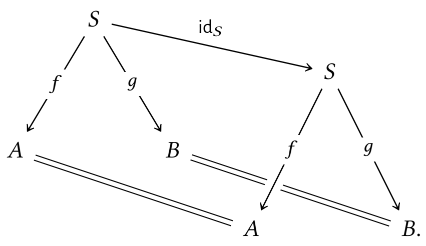
- *Composition.* The composition map

  $$\circ^{\mathsf{Span}(A,B)}_{R,S,T} \colon \mathrm{Hom}_{\mathsf{Span}(A,B)}(S,T) \times \mathrm{Hom}_{\mathsf{Span}(A,B)}(R,S) \to \mathrm{Hom}_{\mathsf{Span}(A,B)}(R,T)$$

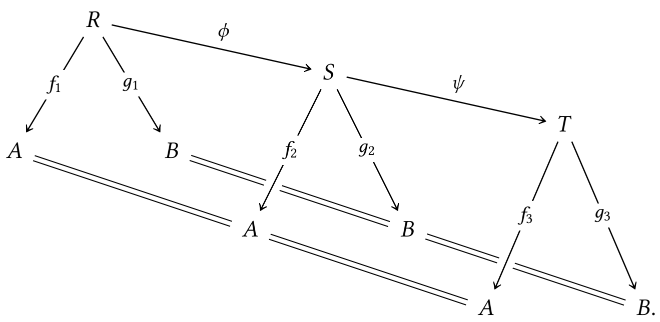
  of $\mathsf{Span}(A,B)$ at $((R,f_1,g_1),(S,f_2,g_2),(T,f_3,g_3))$ is defined by[2]

  $$\psi \circ^{\mathsf{Span}(A,B)}_{R,S,T} \phi \stackrel{\text{def}}{=} \psi \circ \phi.$$

[1] *Picture:*



[2] *Picture:*



Proposition 6.2.1.3 ▶ Properties of the Category of Spans Between Two Sets

Let A and B be sets.

1. *As a Pullback.* We have an isomorphism of categories

$$\mathsf{Span}(A, B) \cong \mathsf{Sets}_{/A} \times_{\mathsf{Sets}} \mathsf{Sets}_{/B}, \qquad \begin{tikzcd} \mathsf{Span}(A,B) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Sets}_{/B} \arrow[d, "忘"] \\ \mathsf{Sets}_{/A} \arrow[r, "忘"'] & \mathsf{Sets}. \end{tikzcd}$$

Proof 6.2.1.4 ▶ Proof of Proposition 6.2.1.3

Item 1: As a Pullback

In detail, the pullback $\mathsf{Sets}_{/A} \times_{\mathsf{Sets}} \mathsf{Sets}_{/B}$ is the category where

- *Objects.* The objects of $\mathsf{Sets}_{/A} \times_{\mathsf{Sets}} \mathsf{Sets}_{/B}$ consist of pairs $((S, f), (S', g))$ of objects of Sets consisting of
  - A pair (S, f) in $\mathrm{Obj}(\mathsf{Sets}_{/A})$ consisting of a set S and a map $f: S \to A$;
  - A pair (S', g) in $\mathrm{Obj}(\mathsf{Sets}_{/B})$ consisting of a set S' and a map $g: S \to$

B;

such that

$$\underbrace{\text{忘}(S, f)}_{\stackrel{\text{def}}{=} S} = \underbrace{\text{忘}(S', g)}_{\stackrel{\text{def}}{=} S'}.$$

Thus the objects of $\mathsf{Sets}_{/A} \times_{\mathsf{Sets}} \mathsf{Sets}_{/B}$ are the same as spans from A to B.

- *Morphisms.* A morphism of $\mathsf{Sets}_{/A} \times_{\mathsf{Sets}} \mathsf{Sets}_{/B}$ from (S, f, g) to (S', f', g') consists of a pair of morphisms

$$\begin{aligned} \phi &: S \to S' \\ \psi &: S \to S' \end{aligned}$$

such that the diagrams

$$\begin{tikzcd} S \arrow[rr, "\phi"] \arrow[dr, "f"'] & & S' \arrow[dl, "f'"] \\ & A & \end{tikzcd} \qquad \begin{tikzcd} S \arrow[rr, "\psi"] \arrow[dr, "g"'] & & S' \arrow[dl, "g'"] \\ & B & \end{tikzcd}$$

such that

$$\underbrace{\text{忘}(\phi)}_{\stackrel{\text{def}}{=} \phi} = \underbrace{\text{忘}(\psi)}_{\stackrel{\text{def}}{=} \psi}.$$

Thus the morphisms of $\mathsf{Sets}_{/A} \times_{\mathsf{Sets}} \mathsf{Sets}_{/B}$ are also the same as morphisms of spans from (S, f, g) to (S, f', g').

- *Identities and Composition.* The identities and composition of $\mathsf{Sets}_{/A} \times_{\mathsf{Sets}} \mathsf{Sets}_{/B}$ are also the same as those in $\mathsf{Span}(A, B)$.

This finishes the proof.

6.2.2 The Bicategory of Spans

Definition 6.2.2.1 ► The Bicategory of Spans

The **bicategory of spans** is the bicategory Span where

- *Objects.* The objects of Span are sets;
- Hom-*Categories.* For each $A, B \in \mathrm{Obj}(\mathsf{Span})$, we have

$$\mathsf{Hom}_{\mathsf{Span}}(A, B) \stackrel{\text{def}}{=} \mathsf{Span}(A, B);$$

- *Identities.* For each $A \in \mathrm{Obj}(\mathsf{Span})$, the unit functor

$$\mathbb{1}^{\mathsf{Span}}_A \colon \mathrm{pt} \to \mathsf{Span}(A, A)$$

of Span at A is the functor picking the span $(A, \mathrm{id}_A, \mathrm{id}_A)$:

$$\begin{tikzcd}
 & A \arrow[dl, "\mathrm{id}_A"'] \arrow[dr, "\mathrm{id}_A"] & \\
A & & A.
\end{tikzcd}$$

- *Composition.* For each $A, B, C \in \mathrm{Obj}(\mathsf{Span})$, the composition bifunctor

$$\circ^{\mathsf{Span}}_{A,B,C} \colon \mathsf{Span}(B, C) \times \mathsf{Span}(A, B) \to \mathsf{Span}(A, C)$$

of Span at (A, B, C) is the bifunctor where

  - *Action on Objects.* The composition of two spans

$$\begin{tikzcd}
 & R \arrow[dl, "f_1"'] \arrow[dr, "g_1"] & & & S \arrow[dl, "f_2"'] \arrow[dr, "g_2"] & \\
A & & B & B & & C
\end{tikzcd}
\quad \text{and} \quad$$

is the span $(R \times_B S, f_1 \circ \mathrm{pr}_1, g_2 \circ \mathrm{pr}_2)$, constructed as in the diagram

$$\begin{tikzcd}
 & & R \times_B S \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] \arrow[dd, phantom, very near start, "\lrcorner" rotate=-45] \arrow[ddll, bend right, "f_1 \circ \mathrm{pr}_1"'] \arrow[ddrr, bend left, "g_2 \circ \mathrm{pr}_2"] & & \\
 & R \arrow[dl, "f_1"'] \arrow[dr, "g_1"] & & S \arrow[dl, "f_2"'] \arrow[dr, "g_2"] & \\
A & & B & & C.
\end{tikzcd}$$

  - *Action on Morphisms.* The horizontal composition of 2-morphisms is defined via functoriality of pullbacks: given morphisms of spans

$$\begin{tikzcd}
 & R \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[dd, "\phi"] & \\
A & & B \\
 & R' \arrow[ul, "f'"] \arrow[ur, "g'"'] &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
 & S \arrow[dl, "h"'] \arrow[dr, "k"] \arrow[dd, "\psi"] & \\
B & & C, \\
 & S' \arrow[ul, "h'"] \arrow[ur, "k'"'] &
\end{tikzcd}$$

their horizontal composition is the morphism of spans

$$
\begin{tikzcd}
 & R \times_B S \arrow[dl, "f \circ \mathrm{pr}_1"'] \arrow[dr, "k \circ \mathrm{pr}_2"] \arrow[dd, dashed, "\exists!"] & \\
A & & C, \\
 & R' \times_B S' \arrow[ul, "h' \circ \mathrm{pr}'_1"] \arrow[ur, "k' \circ \mathrm{pr}'_2"'] &
\end{tikzcd}
$$

constructed as in the diagram

$$
\begin{tikzcd}
 & & R \times_B S \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] \arrow[dll, bend right, "f \circ \mathrm{pr}_1"'] \arrow[drr, bend left, "k \circ \mathrm{pr}_2"] \arrow[dddd, dashed, bend right=10, "\exists!"'] \arrow[dd, phantom, "\lrcorner", very near start] & & \\
 & R \arrow[dl, "f"'] \arrow[dr, "g"] & & S \arrow[dl, "h"'] \arrow[dr, "k"] & \\
A & & B & & C; \\
 & R' \arrow[ul, "f'"] \arrow[ur, "g'"'] & & S' \arrow[ul, "h'"] \arrow[ur, "k'"'] & \\
 & & R' \times_B S' \arrow[ul, "\mathrm{pr}'_1"] \arrow[ur, "\mathrm{pr}'_2"'] \arrow[ull, bend left, "f' \circ \mathrm{pr}'_1"] \arrow[urr, bend right, "k' \circ \mathrm{pr}'_2"'] \arrow[uu, phantom, "\urcorner", very near start] & &
\end{tikzcd}
$$

- *Associators and Unitors.* The associator and unitors are defined using the universal property of the pullback.

6.2.3 The Monoidal Bicategory of Spans

6.2.4 The Double Category of Spans

Definition 6.2.4.1 ► The Double Category of Spans

The **double category of spans** is the double category $\mathsf{Span}^{\mathsf{dbl}}$ where

- *Objects.* The objects of $\mathsf{Span}^{\mathsf{dbl}}$ are sets;
- *Vertical Morphisms.* The vertical morphisms of $\mathsf{Span}^{\mathsf{dbl}}$ are functions $f\colon A \to B$;
- *Horizontal Morphisms.* The horizontal morphisms of $\mathsf{Span}^{\mathsf{dbl}}$ are spans $(S, \phi, \psi)\colon A \nrightarrow X$;
- *2-Morphisms.* A 2-cell

$$\begin{tikzcd} A \arrow[r, "{(R,\phi_R,\psi_R)}", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & B \arrow[d, "g"] \\ X \arrow[r, "{(S,\phi_S,\psi_S)}"', "\shortmid"{marking}] & Y \end{tikzcd}$$

of $\mathsf{Span}^{\mathsf{dbl}}$ is a morphism of spans from the span

$$\begin{tikzcd} & & R \arrow[dll, "\phi_R"'] \arrow[d, "\psi_R"] \\ A & & B \arrow[d, "g"] \\ & & Y \end{tikzcd}$$

to the span

$$\begin{tikzcd} & & A \times_X S \arrow[dl] \arrow[dr] & & \\ & A \arrow[dl, "f"'] \arrow[dr, "f"] & & S \arrow[dl, "\phi_S"'] \arrow[dr, "\psi_S"] & \\ X & & X & & Y; \end{tikzcd}$$

- *Horizontal Identities.* The horizontal unit functor

$$\mathbb{1}^{\mathsf{Span}^{\mathsf{dbl}}}\colon \left(\mathsf{Span}^{\mathsf{dbl}}\right)_0 \to \left(\mathsf{Span}^{\mathsf{dbl}}\right)_1$$

of $\mathsf{Span}^{\mathsf{dbl}}$ is the functor where

– *Action on Objects.* For each $A \in \mathrm{Obj}\Big(\big(\mathsf{Span}^{\mathsf{dbl}}\big)_0\Big)$, we have

$$\mathbb{1}_A \overset{\mathrm{def}}{=} (A, \mathrm{id}_A, \mathrm{id}_A),$$

as in the diagram

$$\begin{tikzcd} & A \arrow[dl, equal, "\mathrm{id}_A"'] \arrow[dr, equal, "\mathrm{id}_A"] & \\ A & & A; \end{tikzcd}$$

– *Action on Morphisms.* For each vertical morphism $f\colon A \to B$ of $\mathsf{Span}^{\mathsf{dbl}}$, i.e. each map of sets f from A to B, the identity 2-morphism

$$\begin{tikzcd} A \arrow[r, "\mathbb{1}_A", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\overset{\Vert}{\underset{\Downarrow}{\mathbb{1}_f}}"] & A \arrow[d, "f"] \\ B \arrow[r, "\mathbb{1}_B"', "\shortmid"{marking}] & B \end{tikzcd}$$

of f is the morphism of spans from

$$\begin{tikzcd} & & A \arrow[dll, equal, "\mathrm{id}_A"'] \arrow[dr, equal, "\mathrm{id}_A"] & & \\ & & & A \arrow[dr, "f"] & \\ A & & & & B \end{tikzcd}$$

to

$$\begin{tikzcd} & & A \times_B B \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner"{rotate=-45}, very near start] & & \\ & A \arrow[dl, "f"'] \arrow[dr, "f"] & & B \arrow[dl, equal, "\mathrm{id}_B"'] \arrow[dr, equal, "\mathrm{id}_B"] & \\ B & & B & & B \end{tikzcd}$$

given by the isomorphism $A \xrightarrow{\cong} A \times_B B$;

- *Vertical Identities.* For each $A \in \mathrm{Obj}\big(\mathsf{Span}^{\mathsf{dbl}}\big)$, we have

$$\mathrm{id}_A^{\mathsf{Span}^{\mathsf{dbl}}} \overset{\mathrm{def}}{=} \mathrm{id}_A;$$

- *Identity 2-Morphisms.* For each horizontal morphism $R\colon A \nrightarrow B$ of $\mathsf{Span}^{\mathsf{dbl}}$, the identity 2-morphism

$$
\begin{tikzcd}
A \arrow[r, "S", "|"{marking}] \arrow[d, "\mathrm{id}_A"'] \arrow[dr, phantom, "\overset{\|}{\mathrm{id}_S}\Downarrow"] & B \arrow[d, "\mathrm{id}_B"] \\
A \arrow[r, "S"', "|"{marking}] & B
\end{tikzcd}
$$

of R is the morphism of spans from

$$
\begin{tikzcd}
 & & S \arrow[dll, "\phi_S"'] \arrow[dr, "\psi_S"] & & \\
 & & & B \arrow[dr, equal, "\mathrm{id}_B"] & \\
A & & & & B
\end{tikzcd}
$$

to

$$
\begin{tikzcd}
 & & A \times_A S \arrow[dl] \arrow[dr] \arrow[dd, phantom, very near start, "\lrcorner"{rotate=-45}] & & \\
 & A \arrow[dl, equal, "\mathrm{id}_A"'] \arrow[dr, equal, "\mathrm{id}_A"] & & S \arrow[dl, "\phi_S"'] \arrow[dr, "\psi_S"] & \\
A & & A & & B
\end{tikzcd}
$$

given by the isomorphism $S \xrightarrow{\cong} A \times_A S$;

- *Horizontal Composition.* The horizontal composition functor

$$
\odot^{\mathsf{Span}^{\mathsf{dbl}}} \colon \left(\mathsf{Span}^{\mathsf{dbl}}\right)_1 \times_{\left(\mathsf{Span}^{\mathsf{dbl}}\right)_0} \left(\mathsf{Span}^{\mathsf{dbl}}\right)_1 \to \left(\mathsf{Span}^{\mathsf{dbl}}\right)_1
$$

of $\mathsf{Span}^{\mathsf{dbl}}$ is the functor where

  - *Action on Objects.* For each composable pair

$$
A \overset{(R,\phi_R,\psi_R)}{\nrightarrow} B \overset{(S,\phi_S,\psi_S)}{\nrightarrow} C
$$

  of horizontal morphisms of $\mathsf{Span}^{\mathsf{dbl}}$, we have

$$
(S, \phi_S, \psi_S) \odot (R, \phi_R, \psi_R) \overset{\text{def}}{=} S \circ^{\mathsf{Span}}_{A,B,C} R,
$$

  where $S \circ^{\mathsf{Span}}_{A,B,C} R$ is the composition of (R, ϕ_R, ψ_R) and (S, ϕ_S, ψ_S) defined as in Definition 6.2.2.1;

– *Action on Morphisms.* For each horizontally composable pair

$$
\begin{tikzcd}
A \arrow[r, "{(R,\phi_R,\psi_R)}", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & B \arrow[d, "g"] \\
X \arrow[r, "{(T,\phi_T,\psi_T)}"', "\shortmid"{marking}] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r, "{(S,\phi_S,\psi_S)}", "\shortmid"{marking}] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\beta"] & C \arrow[d, "h"] \\
Y \arrow[r, "{(U,\phi_U,\psi_U)}"', "\shortmid"{marking}] & Z
\end{tikzcd}
$$

of 2-morphisms of $\mathsf{Span}^{\mathsf{dbl}}$,

- *Vertical Composition of 1-Morphisms.* For each composable pair $A \xrightarrow{F} B \xrightarrow{G} C$ of vertical morphisms of $\mathsf{Span}^{\mathsf{dbl}}$, i.e. maps of sets, we have

$$g \circ^{\mathsf{Span}^{\mathsf{dbl}}} f \stackrel{\text{def}}{=} g \circ f;$$

- *Vertical Composition of 2-Morphisms.* For each vertically composable pair

$$
\begin{tikzcd}
A \arrow[r, "{(R,\phi_R,\psi_R)}", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & X \arrow[d, "g"] \\
B \arrow[r, "{(S,\phi_S,\psi_S)}"', "\shortmid"{marking}] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r, "{(S,\phi_S,\psi_S)}", "\shortmid"{marking}] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow\beta"] & Y \arrow[d, "k"] \\
C \arrow[r, "{(T,\phi_T,\psi_T)}"', "\shortmid"{marking}] & Z
\end{tikzcd}
$$

of 2-morphisms of $\mathsf{Span}^{\mathsf{dbl}}$,

- *Associators and Unitors.* The associator and unitors of $\mathsf{Span}^{\mathsf{dbl}}$ are defined using the universal property of the pullback.

6.2.5 Properties of The Bicategory of Spans

Proposition 6.2.5.1 ► Properties of the Bicategory of Spans

Let $\lambda = \left(A \xleftarrow{f} S \xrightarrow{g} B\right)$ be a span.

1. *Self-Duality.*
2. *Isomorphisms in* Span.
3. *Equivalences in* Span.
4. *Adjunctions in* Span. Let A and B be sets.[1]

(a) We have a natural bijection

$$\left\{\begin{matrix}\text{Adjunctions in Span}\\ \text{from } A \text{ to } B\end{matrix}\right\} \cong \left\{\begin{matrix}\text{Spans } A \xleftarrow{f} S \xrightarrow{g} B\\ \text{from } A \text{ to } B \text{ with}\\ f \text{ an isomorphism}\end{matrix}\right\}.$$

(b) We have an equivalence of categories

$$\mathsf{MapSpan}(A,B) \overset{\text{eq.}}{\cong} \mathsf{Sets}(A,B)_{\mathsf{disc}},$$

where $\mathsf{MapSpan}(A,B)$ is the full subcategory of $\mathsf{Span}(A,B)$ spanned by the spans $A \xleftarrow{f} S \xrightarrow{g} B$ from A to B with f an isomorphism.

(c) We have a biequivalence of bicategories

$$\mathsf{MapSpan} \overset{\text{eq.}}{\cong} \mathsf{Sets}_{\mathsf{bidisc}},$$

where MapSpan is the sub-bicategory of Span whose Hom-categories are given by $\mathsf{MapSpan}(A,B)$.

5. *Monads in* Span.

6. *Comonads in* Span.

7. *Monomorphisms in* Span.

8. *Epimorphisms in* Span.

9. *Existence of Right Kan Extensions.*

10. *Existence of Right Kan Lifts.*

11. *Closedness.*

[1] In the literature (e.g. [**ref**]), ...are called maps and denoted by $\mathsf{MapSpan}(A,B)$

Proof 6.2.5.2 ► Proof of Proposition 6.2.5.1

Item 1: Self-Duality

Item 2: Isomorphisms in Span

Item 3: Equivalences in Span

Item 4: Adjunctions in Span

We first prove Item 4a.

We proceed step by step:

1. *From Adjunctions in* Span *to Functions.* An adjunction in Span from A to B consists of a pair of spans

$$\begin{tikzcd} & R \arrow[dl, "f"'] \arrow[dr, "g"] & \\ A & & B \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & S' \arrow[dl, "h"'] \arrow[dr, "k"] & \\ B & & A \end{tikzcd}$$

together with maps

$$\begin{tikzcd} & A \arrow[dl, equal, "\mathrm{id}_A"'] \arrow[dr, equal, "\mathrm{id}_A"] \arrow[dd, "\phi"] & \\ A & & A \\ & S \times_B S' \arrow[ul, "f\circ\mathrm{pr}'_1"] \arrow[ur, "k\circ\mathrm{pr}'_2"'] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & S' \times_A S \arrow[dl, "h\circ\mathrm{pr}_1"'] \arrow[dr, "g\circ\mathrm{pr}_2"] \arrow[dd, "\psi"] & \\ B & & B. \\ & B \arrow[ul, equal, "\mathrm{id}_B"] \arrow[ur, equal, "\mathrm{id}_B"'] & \end{tikzcd}$$

We claim that these conditions

2. *From Functions to Adjunctions in* **Rel**.

3. *Invertibility: From Functions to Adjunctions Back to Functions.*

4. *Invertibility: From Adjunctions to Functions Back to Adjunctions.*

We now proceed to the proof of Item 4b. For this, we will construct a functor

$$F\colon \mathsf{Sets}(A, B)_{\mathsf{disc}} \to \mathsf{MapSpan}(A, B)$$

and prove it to be essentially surjective and fully faithful, and thus an equivalence by **??** of **??**. Indeed, given a map $f\colon A \to B$, let $F(f)$ be the representable span associated to f of Definition 6.5.1.1, and let F send the unique (identity) morphism from f to itself to the identity morphism of $F(f)$ in $\mathsf{MapSpan}(A, B)$. We now prove that F is fully faithful and essentially surjective:

1. *F Is Fully Faithful:* Given maps $f, g\colon A \rightrightarrows B$, we need to show that

$$\mathrm{Hom}_{\mathsf{MapSpan}(A,B)}(F(f), F(g)) = \begin{cases} \mathrm{pt} & \text{if } f = g, \\ \emptyset & \text{otherwise.} \end{cases}$$

   Indeed, a morphism from $F(f)$ to $F(g)$ takes the form

$$\begin{tikzcd} & A \arrow[dl, equal] \arrow[dr, "f"] \arrow[dd, "\phi"] & \\ A & & B. \\ & A \arrow[ul, equal] \arrow[ur, "g"'] & \end{tikzcd}$$

   From the relations $\mathrm{id}_A = \mathrm{id}_A \circ \phi$ and $f = g \circ \phi$, we see that $\phi = \mathrm{id}_A$, and thus from the relation $f = g \circ \phi$ there is such a morphism iff $f = g$.

2. *F Is Essentially Surjective:* Let λ be a span of the form

$$\begin{tikzcd} & S \arrow[dl, dashed, "\phi"', "\sim"] \arrow[dr, "f"] & \\ A & & B. \end{tikzcd}$$

   we claim that $\lambda \cong F(f \circ \phi^{-1})$. Indeed, we have morphisms

$$\begin{tikzcd} & S \arrow[dl, dashed, "\phi"', "\sim"] \arrow[dr, "f"] \arrow[dd, "\phi"] & \\ A & & B \\ & A \arrow[ul, equal] \arrow[ur, "f\circ\phi^{-1}"'] & \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} & A \arrow[dl, equal] \arrow[dr, "f\circ\phi^{-1}"] \arrow[dd, "\phi^{-1}"] & \\ A & & B, \\ & S \arrow[ul, dashed, "\phi", "\sim"'] \arrow[ur, "f"'] & \end{tikzcd}$$

   inverse to each other in $\mathsf{MapSpan}(A, B)$, and thus $\lambda \cong F(f \circ \phi^{-1})$.

Finally, we prove Item 4c.

Item 5: Monads in Span

Item 6: Comonads in Span

Item 7: Monomorphisms in Span

Item 8: Epimorphisms in Span

Item 9: Existence of Right Kan Extensions

Item 10: Existence of Right Kan Lifts

Item 11: Closedness

6.3 Limits of Spans

6.3.1 tmp2

$$\mathrm{Hom}_{\mathsf{Rel}(A,X)}(\mathrm{Lan}_S(R), T) \cong \mathrm{Hom}_{\mathsf{Rel}(B,X)}(R, T \diamond S)$$

1. $\mathrm{Lan}_S(R) \subset T$, i.e. if $a \sim_{\mathrm{Lan}_S(R)} x$, then $a \sim_T x$.
2. $R \subset T \diamond S$, i.e. if $b \sim_R x$, then there exists some $a \in A$ such that $a \sim_S b$ and $b \sim_T x$.

6.3.2 tmp

$$\begin{tikzcd}
 & S \arrow[dl, "f"'] \arrow[dr, "g"] & \\
A & & B
\end{tikzcd}$$

$$\begin{tikzcd}
 & S' \arrow[dl, "\phi_{S'}"'] \arrow[dr, "\psi_{S'}"] & & & & S \times_B S' \arrow[dl, "f \circ \mathrm{pr}_1"'] \arrow[dr, "\psi_{S'} \circ \mathrm{pr}_2"] & \\
B & & X & \mapsto & A & & X
\end{tikzcd}$$

$$\begin{tikzcd}
 & S'' \arrow[dl, "\phi_{S''}"'] \arrow[dr, "\psi_{S''}"] & & & & ? \arrow[dl, "?"'] \arrow[dr, "?"] & \\
A & & X & \mapsto & B & & X
\end{tikzcd}$$

$$\mathrm{Hom}_{\mathsf{Span}(A,X)}(S \times_B S', K) \cong \mathrm{Hom}_{\mathsf{Span}(B,X)}(S', R(K))$$

$$\begin{tikzcd}
 & S \times_B S' \arrow[dl, "f \circ \mathrm{pr}_1"'] \arrow[dr, "\psi_{S'} \circ \mathrm{pr}_2"] \arrow[dd, "\xi"] & \\
A & & X \\
 & S'' \arrow[ul, "\phi_{S''}"] \arrow[ur, "\psi_{S''}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & S' \arrow[dl, "\phi_{S'}"'] \arrow[dr, "\psi_{S'}"] \arrow[dd, "\xi^\dagger"] & \\
B & & X \\
 & \Pi_g(S'') \arrow[ul, "?"] \arrow[ur, "?"'] &
\end{tikzcd}$$

6.3.3 Left Kan Extensions

Let $\lambda = \left(A \xleftarrow{f} S \xrightarrow{g} B\right)$ be a span.

Proposition 6.3.3.1 ► Left Kan Extensions In Span

The left Kan extension

$$\mathrm{Lan}_\lambda \colon \mathsf{Span}(A, X) \longrightarrow \mathsf{Span}(B, X)$$

along λ in Span exists and is the functor given on objects by sending a span λ' in $\mathsf{Span}(A, X)$ as in

$$\begin{tikzcd} & S' \arrow[dl, "\phi"'] \arrow[dr, "\psi"] & \\ A & & X \end{tikzcd}$$

to the span

$$\mathrm{Lan}_\lambda(\lambda') \stackrel{\text{def}}{=} (\mathrm{Lan}_\lambda(S'), \mathrm{Lan}_\lambda(\phi), \mathrm{Lan}_\lambda(\psi)),$$

in $\mathsf{Span}(B, X)$ where

- The set $\mathrm{Lan}_\lambda(S')$ is given by

  $$\begin{aligned} \mathrm{Lan}_\lambda(S') &\stackrel{\text{def}}{=} \Sigma_g(S') \\ &\stackrel{\text{def}}{=} S' \end{aligned}$$

  where $\Sigma_g(S')$ is the dependent sum of $\phi \colon S' \longrightarrow A$ along g of **??**;
- The map $\mathrm{Lan}_\lambda(\phi) \colon \mathrm{Lan}_\lambda(S') \longrightarrow B$ is given by $\Sigma_g(\phi)$;
- The map $\mathrm{Lan}_\lambda(\psi) \colon \mathrm{Lan}_\lambda(S') \longrightarrow X$ is given by ψ.

Proof 6.3.3.2 ► Proof of Proposition 6.3.4.1

6.3.4 Right Kan Extensions

Let $\lambda = \left(A \xleftarrow{f} S \xrightarrow{g} B\right)$ be a span.

Proposition 6.3.4.1 ► Right Kan Extensions In Span

The right Kan extension

$$\mathrm{Ran}_\lambda\colon \mathsf{Span}(A, X) \longrightarrow \mathsf{Span}(B, X)$$

along λ in Span exists and is the functor given on objects by sending a span λ' in $\mathsf{Span}(A, X)$ as in

$$\begin{tikzcd} & S' \arrow[dl, "\phi_{S'}"'] \arrow[dr, "\psi_{S'}"] & \\ A & & X \end{tikzcd}$$

to the span

$$\mathrm{Ran}_\lambda(\lambda') \overset{\text{def}}{=} (\mathrm{Ran}_\lambda(S'), \mathrm{Ran}_\lambda(\phi_{S'}), \mathrm{Ran}_\lambda(\psi_{S'})),$$

in $\mathsf{Span}(B, X)$ where

- The set $\mathrm{Ran}_\lambda(S')$ is given by

$$\mathrm{Ran}_\lambda(S') \overset{\text{def}}{=} \coprod_{b\in B} \prod_{s\in g^{-1}(b)} \phi_{S'}^{-1}(f(s));$$

- The map $\mathrm{Ran}_\lambda(\phi_{S'})\colon \mathrm{Ran}_\lambda(S') \longrightarrow B$ is given by

$$[\mathrm{Ran}_\lambda(\phi_{S'})](b, (s'_s)_{s\in g^{-1}(b)}) \overset{\text{def}}{=} b;$$

for each $\left(b, (s'_s)_{s\in g^{-1}(b)}\right) \in \mathrm{Ran}_\lambda(S')$;

- The map $\mathrm{Ran}_\lambda(\psi_{S'})\colon \mathrm{Ran}_\lambda(S') \longrightarrow X$ is given by

$$[\mathrm{Ran}_\lambda(\psi_{S'})]\left(b, (s'_s)_{s\in g^{-1}(b)}\right) \overset{\text{def}}{=} \psi_{S'}(s'_i)$$

for each $\left(b, (s'_s)_{s\in g^{-1}(b)}\right) \in \mathrm{Ran}_\lambda(S')$, where the i in s'_i denotes any $s \in g^{-1}(b)$, as we have $\psi_{S'}(s'_i) = \psi_{S'}\left(s'_j\right)$ for all $s \in g^{-1}(b)$.[1]

[1] Indeed

Proof 6.3.4.2 ► Proof of Proposition 6.3.4.1

6.3.5 Right Kan Lifts

(Although right Kan lifts aren't really limits, this is probably the most appropriate to place this section.)

Let $\lambda = \left(A \xleftarrow{f} S \xrightarrow{g} B\right)$ be a span.

Proposition 6.3.5.1 ▶ Right Kan Lifts in Span

The right Kan lift

$$\mathrm{Rift}_\lambda \colon \mathsf{Span}(X, B) \to \mathsf{Span}(X, A)$$

along λ in Span exists and is the functor given on objects by sending a span λ' in $\mathsf{Span}(X, B)$ as in

$$\begin{tikzcd} & S' \arrow[dl, "\phi"'] \arrow[dr, "\psi"] & \\ X & & B \end{tikzcd}$$

to the span

$$\mathrm{Rift}_\lambda(\lambda') \overset{\text{def}}{=} (\mathrm{Rift}_\lambda(S'), \mathrm{Rift}_\lambda(\phi), \mathrm{Rift}_\lambda(\psi)),$$

in $\mathsf{Span}(X, A)$ where

- The set $\mathrm{Rift}_\lambda(S')$ is given by

$$\mathrm{Rift}_\lambda(S') \overset{\text{def}}{=} \Pi_f(S'),$$

where $\Pi_f(S')$ is the dependent product of $\psi \colon S' \to A$ along f of **??**;
- The map $\mathrm{Rift}_\lambda(\phi) \colon \mathrm{Rift}_\lambda(S') \to X$ is given by ϕ;
- The map $\mathrm{Rift}_\lambda(\psi) \colon \mathrm{Rift}_\lambda(S') \to A$ is given by $\Pi_f(\psi)$.

Proof 6.3.5.2 ▶ Proof of Proposition 6.3.5.1

6.4 Colimits of Spans

6.5 Constructions With Spans

6.5.1 Representable Spans

Definition 6.5.1.1 ▶ Representable Spans

Let $f \colon A \to B$ be a function.

- The **representable span associated to** f is the span

$$\begin{tikzcd} & A \arrow[dl, equal, "\mathrm{id}_A"'] \arrow[dr, "f"] & \\ A & & B \end{tikzcd}$$

from A to B.

- The **corepresentable span associated to** f is the span

$$\begin{tikzcd} & A \arrow[dl, "f"'] \arrow[dr, equal, "\mathrm{id}_A"] & \\ B & & A \end{tikzcd}$$

from B to A.

6.5.2 Composition of Spans

Definition 6.5.2.1 ▶ Composition of Spans

The **composition** of two spans

$$\begin{tikzcd} & R \arrow[dl, "f_1"'] \arrow[dr, "g_1"] & \\ A & & B \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & S \arrow[dl, "f_2"'] \arrow[dr, "g_2"] & \\ B & & C \end{tikzcd}$$

is the span $(R \times_B S, f_1 \circ \mathrm{pr}_1, g_2 \circ \mathrm{pr}_2)$, constructed as in the diagram

$$\begin{tikzcd} & & R \times_B S \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] \arrow[ddll, bend right, "f_1 \circ \mathrm{pr}_1"'] \arrow[ddrr, bend left, "g_2 \circ \mathrm{pr}_2"] & & \\ & R \arrow[dl, "f_1"'] \arrow[dr, "g_1"] & & S \arrow[dl, "f_2"'] \arrow[dr, "g_2"] & \\ A & & B & & C. \end{tikzcd}$$

6.5.3 Horizontal Composition of Morphisms of Spans

Definition 6.5.3.1 ► Horizontal Composition of Morphisms of Spans

The **horizontal composition** of a pair of 2-morphisms of spans

$$\begin{tikzcd} & R \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[dd, "\phi"] & \\ A & & B \\ & R' \arrow[ul, "f'"] \arrow[ur, "g'"'] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & S \arrow[dl, "h"'] \arrow[dr, "k"] \arrow[dd, "\psi"] & \\ B & & C, \\ & S' \arrow[ul, "h'"] \arrow[ur, "k'"'] & \end{tikzcd}$$

is the morphism of spans

$$\begin{tikzcd} & R \times_B S \arrow[dl, "f \circ \mathrm{pr}_1"'] \arrow[dr, "k \circ \mathrm{pr}_2"] \arrow[dd, dashed, "\exists!"] & \\ A & & C, \\ & R' \times_B S' \arrow[ul, "h' \circ \mathrm{pr}'_1"] \arrow[ur, "k' \circ \mathrm{pr}'_2"'] & \end{tikzcd}$$

constructed as in the diagram

$$\begin{tikzcd} & & R \times_B S \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] \arrow[dddd, dashed, "\exists!"'] \arrow[ddll, bend right, "f \circ \mathrm{pr}_1"'] \arrow[ddrr, bend left, "k \circ \mathrm{pr}_2"] & & \\ & R \arrow[dl, "f"'] \arrow[dr, "g"] & & S \arrow[dl, "h"'] \arrow[dr, "k"] & \\ A & & B & & C; \\ & R' \arrow[ul, "f'"] \arrow[ur, "g'"'] & & S' \arrow[ul, "h'"] \arrow[ur, "k'"'] & \\ & & R' \times_B S' \arrow[ul, "\mathrm{pr}'_1"] \arrow[ur, "\mathrm{pr}'_2"'] \arrow[uull, bend left, "f' \circ \mathrm{pr}'_1"] \arrow[uurr, bend right, "k' \circ \mathrm{pr}'_2"'] & & \end{tikzcd}$$

6.5.4 Properties of Composition of Spans

Let $\lambda = \left(A \overset{f}{\leftarrow} S \overset{g}{\rightarrow} B\right)$ be a span.

1. *Functoriality.*

Proof 6.5.4.2 ▶ Proof of Proposition 6.5.4.1

6.5.5 The Inverse of a Span

6.6 Functoriality of Spans

6.6.1 Direct Images

6.6.2 Functoriality of Spans on Powersets

6.7 Un/Straightening for Spans

6.7.1 Straightening for Spans

Let A and B be sets and let (S, f, g) be a span from A to B.

Definition 6.7.1.1 ▶ The Straightening of a Span

The **straightening of** (S, f, g) is the $(A \times B)$-indexed set

$$\mathrm{St}_{A,B}(S)\colon (A \times B)_{\mathrm{disc}} \to \mathsf{Sets}$$

defined as the straightening of S, viewed as an $(A \times B)$-fibred set, as in **??**.

Remark 6.7.1.2 ▶ Unwinding Definition 6.7.1.1

In detail, $\mathrm{St}_{A,B}(S)$ is the $(A \times B)$-indexed set defined by[1]

$$\begin{aligned}[\mathrm{St}_{A,B}(S)](a,b) &\overset{\mathrm{def}}{=} \mathrm{Wit}_S(a,b)\\ &\overset{\mathrm{def}}{=} S_{ab}\\ &\overset{\mathrm{def}}{=} \{s \in S \mid f(s) = a \text{ and } g(s) = b\}.\end{aligned}$$

[1] Here we may think of $\mathrm{Wit}_S(a,b)$ as the "set of witnesses in S that $a \sim b$ holds"; see Remark 6.1.2.3.

Proposition 6.7.1.3 ▶ Properties of Straightening for Spans

Let A and B be sets and let (S, f, g) be a span.

1. *Functoriality.* The assignment $(S, f, g) \mapsto \mathrm{St}_{A,B}(S)$ defines a functor

$$\mathrm{St}_{A,B} \colon \mathsf{Span}(A, B) \to \mathsf{ISets}(A \times B)$$

   - *Action on Objects.* For each $(S, f, g) \in \mathrm{Obj}(\mathsf{Span}(A, B))$, we have

$$[\mathrm{St}_{A,B}](S, f, g) \overset{\text{def}}{=} \mathrm{St}_{A,B}(S);$$

   - *Action on Morphisms.* For each $(S_1, f_1, g_1), (S_2, f_2, g_2) \in \mathrm{Obj}(\mathsf{Span}(A, B))$, the action on Hom-sets

$$\mathrm{St}_{A,B|S_1,S_2} \colon \mathrm{Hom}_{\mathsf{Span}(A,B)}(S_1, S_2) \to \mathrm{Hom}_{\mathsf{ISets}(A\times B)}(\mathrm{St}_{A,B}(S_1), \mathrm{St}_{A,B}(S_2))$$

     of $\mathrm{St}_{A,B}$ at (S_1, S_2) is given by sending a morphism

$$\phi \colon (S_1, f_1, g_1) \to (S_2, f_2, g_2)$$

     of spans from A to B to the morphism

$$\mathrm{St}_{A,B}(\phi) \colon \mathrm{St}_{A,B}(S_1) \to \mathrm{St}_{A,B}(S_2)$$

     of $(A \times B)$-indexed sets defined by

$$\mathrm{St}_{A,B}(\phi) \overset{\text{def}}{=} \{\phi^*_{ab}\}_{(a,b)\in A\times B},$$

     where ϕ^*_{ab} is the transport map associated to ϕ at $(a, b) \in A \times B$ of **??**.

Proof 6.7.1.4 ▶ Proof of Proposition 6.7.1.3

Item 1: Functoriality

This is the special case of **??** of **??** where $K = A \times B$.

6.7.2 Unstraightening for Spans

Let A and B be sets and let $S \colon (A \times B)_{\mathsf{disc}} \to \mathsf{Sets}$ be an $(A \times B)$-indexed set.

Definition 6.7.2.1 ► Unstraightening for Spans

The **unstraightening of** S is the span

$$\begin{tikzcd} & \mathsf{Un}_{A,B}(S) \arrow[dl, "f_{\mathsf{Un}_{A,B}(S)}"'] \arrow[dr, "g_{\mathsf{Un}_{A,B}(S)}"] & \\ A & & B \end{tikzcd}$$

from A to B where

$$\mathsf{Un}_{A,B}(S) \overset{\text{def}}{=} \coprod_{(a,b)\in A\times B} S(a,b)$$

and where the maps $f_{\mathsf{Un}_{A,B}(S)}$ and $g_{\mathsf{Un}_{A,B}(S)}$ are given by

$$\begin{aligned} f_{\mathsf{Un}_{A,B}(S)}((a,b),s) &\overset{\text{def}}{=} a, \\ g_{\mathsf{Un}_{A,B}(S)}((a,b),s) &\overset{\text{def}}{=} b \end{aligned}$$

for each $((a,b),s) \in \mathsf{Un}_{A,B}(S)$.

Proposition 6.7.2.2 ► Properties of Unstraightening for Spans

Let A and B be sets.

1. *Functoriality.* The assignment $S \mapsto \mathsf{Un}_{A,B}(S)$ defines a functor

$$\mathsf{Un}_{A,B} \colon \mathsf{ISets}(A\times B) \to \mathsf{Span}(A,B)$$

   - *Action on Objects.* For each $S \in \mathrm{Obj}(\mathsf{ISets}(A\times B))$, we have

$$[\mathsf{Un}_{A,B}](S) \overset{\text{def}}{=} \mathsf{Un}_{A,B}(S);$$

   - *Action on Morphisms.* For each $S, S' \in \mathrm{Obj}(\mathsf{ISets}(A\times B))$, the action on Hom-sets

$$\mathsf{Un}_{A,B|S,S'} \colon \mathrm{Hom}_{\mathsf{ISets}(A\times B)}(S,S') \to \mathrm{Hom}_{\mathsf{Span}(A,B)}\big(\mathsf{Un}_{A,B}(S), \mathsf{Un}_{A,B}(S')\big)$$

   of $\mathsf{Un}_{A,B}$ at (S,S') is defined by

$$\mathsf{Un}_{A,B|S,S'}(f) \overset{\text{def}}{=} \coprod_{(a,b)\in A\times B} f_{ab}.$$

2. *Interaction With Fibres.* Viewing the legs of $\mathsf{Un}_{A,B}(S)$ as a morphism $(f,g)\colon \mathsf{Un}_{A,B}(S) \to A\times B$, we have a bijection of sets

$$(f,g)^{-1}_{\mathsf{Un}_{A,B}(S)}(a,b) \cong S(a,b)$$

   for each $(a,b) \in A\times B$.

3. *As a Pullback.* We have a bijection of sets

$$\mathsf{Un}_{A,B}(S) \cong (A \times B)_{\mathsf{disc}} \times_{\mathsf{Sets}} \mathsf{Sets}_*, \qquad \begin{tikzcd} \mathsf{Un}_{A,B}(S) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Sets}_* \arrow[d, "忘"] \\ (A \times B)_{\mathsf{disc}} \arrow[r, "S"'] & \mathsf{Sets}. \end{tikzcd}$$

4. *As a Colimit.* We have a bijection of sets

$$\mathsf{Un}_{A,B}(S) \cong \mathsf{colim}(S).$$

Proof 6.7.2.3 ▶ Proof of Proposition 6.7.2.2

Item 1: Functoriality

This is the special case of **??** of **??** where $K = A \times B$.

Item 2: Interaction With Fibres

This is the special case of **??** of **??** where $K = A \times B$.

Item 3: As a Pullback

This is the special case of **??** of **??** where $K = A \times B$.

Item 4: As a Colimit

This is the special case of **??** of **??** where $K = A \times B$.

6.7.3 The Un/Straightening Equivalence for Spans

Theorem 6.7.3.1 ▶ Un/Straightening for Spans

We have an isomorphism of categories

$$(\mathsf{St}_{A,B} \dashv \mathsf{Un}_{A,B}) : \quad \begin{tikzcd} \mathsf{Span}(A, B) \arrow[r, bend left, "\mathsf{St}_{A,B}"] \arrow[r, phantom, "\perp"] & \mathsf{ISets}(A \times B). \arrow[l, bend left, "\mathsf{Un}_{A,B}"] \end{tikzcd}$$

Proof 6.7.3.2 ▶ Proof of Theorem 6.7.3.1

This is the special case of **??** where $K = A \times B$.

6.8 Comparison of Spans to Functions and Relations

6.8.1 Comparison to Functions

Proposition 6.8.1.1 ▶ Comparison of Spans to Functions

We have a pseudofunctor

$$\iota\colon \mathsf{Sets}_{\mathsf{bidisc}} \longrightarrow \mathsf{Span}$$

from $\mathsf{Sets}_{\mathsf{bidisc}}$ to Span where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathsf{Sets}_{\mathsf{bidisc}})$, we have

$$\iota(A) \overset{\mathrm{def}}{=} A;$$

- *Action on* Hom*-Categories.* For each $A, B \in \mathrm{Obj}(\mathsf{Sets}_{\mathsf{bidisc}})$, the action on Hom-categories

$$\iota_{A,B}\colon \mathsf{Sets}(A,B)_{\mathsf{disc}} \longrightarrow \mathsf{Span}(A,B)$$

of ι at (A,B) is the functor defined on objects by sending a function $f\colon A \to B$ to the span

$$\begin{tikzcd} & A \arrow[dl, equal, "\mathrm{id}_A"'] \arrow[dr, "f"] & \\ A & & B \end{tikzcd}$$

from A to B.

- *Strict Unity Constraints.* For each $A \in \mathrm{Obj}(\mathsf{Sets}_{\mathsf{bidisc}})$, the strict unity constraint

$$\iota^0_A\colon \mathrm{id}_{\iota(A)} \Longrightarrow \iota(\mathrm{id}_A)$$

of ι at A is given by the identity morphism of spans

$$\begin{tikzcd} & A \arrow[dl, equal, "\mathrm{id}_A"'] \arrow[dr, equal, "\mathrm{id}_A"] \arrow[dd, equal, "\mathrm{id}"] & \\ A & & A, \\ & A \arrow[ul, equal, "\mathrm{id}_A"] \arrow[ur, equal, "\mathrm{id}_A"'] & \end{tikzcd}$$

as indeed $\mathrm{id}_{\iota(A)} = \iota(\mathrm{id}_A)$;

- *Pseudofunctoriality Constraints.* For each $A, B, C \in \mathrm{Obj}(\mathsf{Sets}_{\mathsf{bidisc}})$, each $f \in \mathrm{Hom}_{\mathsf{Sets}_{\mathsf{bidisc}}}(A,B)$, and each $g \in \mathrm{Hom}_{\mathsf{Sets}_{\mathsf{bidisc}}}(B,C)$, the pseudofunctoriality constraint

$$\iota^2_{g,f}\colon \iota(g) \circ \iota(f) \Longrightarrow \iota(g \circ f)$$

of ι at (f, g) is the morphism of spans from the span

$$\begin{tikzcd}
 & & A \times_B B \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] \arrow[dd, phantom, very near start, "\lrcorner" rotate=-45] \arrow[ddll, bend right, "\mathrm{id}_A \circ \mathrm{pr}_1"'] \arrow[ddrr, bend left, "g \circ \mathrm{pr}_2"] & & \\
 & A \arrow[dl, equal, "\mathrm{id}_A"'] \arrow[dr, "f"] & & B \arrow[dl, equal, "\mathrm{id}_B"'] \arrow[dr, "g"] & \\
A & & B & & C.
\end{tikzcd}$$

to the span

$$\begin{tikzcd}
 & A \arrow[dl, equal, "\mathrm{id}_A"'] \arrow[dr, "g \circ f"] & \\
A & & C
\end{tikzcd}$$

given by the isomorphism $A \times_B B \cong A$.

PROOF 6.8.1.2 ▶ PROOF OF PROPOSITION 6.8.1.1

Omitted.

6.8.2 Comparison to Relations: From Span to Rel

6.8.2.1 Relations Associated to Spans

Let $\lambda = \left(A \overset{f}{\leftarrow} S \overset{g}{\rightarrow} B\right)$ be a span.

DEFINITION 6.8.2.1 ▶ THE RELATION ASSOCIATED TO A SPAN

The **relation associated to** λ is the relation

$$S(\lambda) \colon A \nrightarrow B$$

from A to B defined as follows:

- Viewing relations from A to B as functions $A \times B \to \{\text{true}, \text{false}\}$, we define

$$\iota_{A,B}(S)^a_b \overset{\text{def}}{=} \begin{cases} \text{true} & \text{if there exists } x \in S \text{ such that } a = f(x) \text{ and } b = g(x), \\ \text{false} & \text{otherwise} \end{cases}$$

  for each $(a, b) \in A \times B$.

- Viewing relations from A to B as functions $A \longrightarrow \mathcal{P}(B)$, we define

$$[\iota_{A,B}(S)](a) \stackrel{\text{def}}{=} g(f^{-1}(a))$$

for each $a \in A$.

- Viewing relations from A to B as subsets of $A \times B$, we define

$$\iota_{A,B}(S) \stackrel{\text{def}}{=} \{(f(x), g(x)) \mid x \in S\}.$$

Proposition 6.8.2.2 ▶ Properties of Relations Associated to Spans

Let $\lambda = \left(A \overset{f}{\leftarrow} S \overset{g}{\rightarrow} B\right)$ be a span.

1. *Interaction With Identities.*
2. *Interaction With Composition.*
3. *Interaction With Inverses.*

Proof 6.8.2.3 ▶ Proof of Proposition 6.8.2.2

6.8.2.2 The Comparison Functor from Span to Rel

Proposition 6.8.2.4 ▶ Comparison of Spans to Relations I

We have a pseudofunctor

$$\iota\colon \mathsf{Span} \longrightarrow \mathbf{Rel}$$

from Span to **Rel** where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathsf{Span})$, we have

$$\iota(A) \stackrel{\text{def}}{=} A;$$

- *Action on* Hom*-Categories.* For each $A, B \in \mathrm{Obj}(\mathsf{Span})$, the action on Hom-categories

$$\iota_{A,B}\colon \mathsf{Span}(A, B) \longrightarrow \mathbf{Rel}(A, B)$$

of ι at (A, B) is the functor where

- *Action on Objects.* Given a span

$$\begin{tikzcd} & S \arrow[dl, "f"'] \arrow[dr, "g"] & \\ A & & B \end{tikzcd}$$

from A to B, the image

$$\iota_{A,B}(S)\colon A \nrightarrow B$$

of S by ι is the relation from A to B defined as follows:

  * Viewing relations as functions $A \times B \to \{\mathrm{true}, \mathrm{false}\}$, we define

$$\iota_{A,B}(S)^a_b \overset{\text{def}}{=} \begin{cases} \mathrm{true} & \text{if there exists } x \in S \\ & \text{such that } a = f(x) \\ & \text{and } b = g(x), \\ \mathrm{false} & \text{otherwise} \end{cases}$$

  for each $(a, b) \in A \times B$;

  * Viewing relations as functions $A \to \mathcal{P}(B)$, we define

$$[\iota_{A,B}(S)](a) \overset{\text{def}}{=} g(f^{-1}(a))$$

  for each $a \in A$;

  * Viewing relations as subsets of $A \times B$, we define

$$\iota_{A,B}(S) \overset{\text{def}}{=} \{(f(x), g(x)) \mid x \in S\}.$$

- *Action on Morphisms.* Given a morphism of spans

$$\begin{tikzcd} & R \arrow[dl, "f_R"'] \arrow[dr, "g_R"] \arrow[dd, "\phi"] & \\ A & & B, \\ & S \arrow[ul, "f_S"] \arrow[ur, "g_S"'] & \end{tikzcd}$$

we have a corresponding inclusion of relations

$$\iota_{A,B}(\phi)\colon \iota_{A,B}(R) \subset \iota_{A,B}(S),$$

since we have $a \sim_{\iota_{A,B}(R)} b$ iff there exists $x \in R$ such that $a = f_R(x)$ and $b = g_R(x)$, in which case we then have

$$\begin{aligned} a &= f_R(x) \\ &= f_S(\phi(x)), \\ b &= g_R(x) \\ &= g_S(\phi(x)), \end{aligned}$$

so that $a \sim_{\iota_{A,B}(S)} b$, and thus $\iota_{A,B}(R) \subset \iota_{A,B}(S)$.

Proof 6.8.2.5 ▶ Proof of Proposition 6.8.2.4

Omitted.

6.8.3 Comparison to Relations: From Rel to Span

Proposition 6.8.3.1 ▶ Comparison of Spans to Relations II

We have a lax functor

$$\left(\iota, \iota^2, \iota^0\right) : \mathbf{Rel} \to \mathsf{Span}$$

from **Rel** to Span where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathsf{Span})$, we have

  $$\iota(A) \stackrel{\text{def}}{=} A;$$

- *Action on* Hom*-Categories.* For each $A, B \in \mathrm{Obj}(\mathsf{Span})$, the action on Hom-categories

  $$\iota_{A,B} : \mathbf{Rel}(A, B) \to \mathsf{Span}(A, B)$$

  of ι at (A, B) is the functor where

  – *Action on Objects.* Given a relation $R : A \nrightarrow B$ from A to B, we define a span

    $$\iota_{A,B}(R) : A \nrightarrow B$$

    from A to B by

    $$\iota_{A,B}(R) \stackrel{\text{def}}{=} (R, \upharpoonright \mathrm{pr}_1 R, \upharpoonright \mathrm{pr}_2 R),$$

    where $R \subset A \times B$ and $\upharpoonright \mathrm{pr}_1 R$ and $\upharpoonright \mathrm{pr}_2 R$ are the restriction of the projections

    $$\begin{aligned} \mathrm{pr}_1 &: A \times B \to A, \\ \mathrm{pr}_2 &: A \times B \to B \end{aligned}$$

to R;

– *Action on Morphisms.* Given an inclusion $\phi\colon R \subset S$ of relations, we have a corresponding morphism of spans

$$\iota_{A,B}(\phi)\colon \iota_{A,B}(R) \longrightarrow \iota_{A,B}(S)$$

as in the diagram

$$\begin{tikzcd}
 & R \arrow[dl, "\restriction \mathrm{pr}_1 R"'] \arrow[dr, "\restriction \mathrm{pr}_2 R"] \arrow[dd, hook] & \\
A & & B. \\
 & S \arrow[ul, "\restriction \mathrm{pr}_1 S"] \arrow[ur, "\restriction \mathrm{pr}_2 S"'] &
\end{tikzcd}$$

· *The Lax Functoriality Constraints.* The lax functoriality constraint

$$\iota^2_{R,S}\colon \iota(S) \circ \iota(R) \Longrightarrow \iota(S \diamond R)$$

of ι at (R, S) is given by the morphism of spans from

$$\begin{tikzcd}
 & & R \times_B S \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] \arrow[ddll, bend right, "\restriction \mathrm{pr}_1 R \circ \mathrm{pr}_1"'] \arrow[ddrr, bend left, "\restriction \mathrm{pr}_2 S \circ \mathrm{pr}_2"] & & \\
 & R \arrow[dl, "\restriction \mathrm{pr}_1 R"'] \arrow[dr, "\restriction \mathrm{pr}_2 R"] & & S \arrow[dl, "\restriction \mathrm{pr}_1 S"'] \arrow[dr, "\restriction \mathrm{pr}_2 S"] & \\
A & & B & & C
\end{tikzcd}$$

to

$$\begin{tikzcd}
 & S \diamond R \arrow[dl, "\restriction \mathrm{pr}_1 S \diamond R"'] \arrow[dr, "\restriction \mathrm{pr}_2 S \diamond R"] & \\
A & & C
\end{tikzcd}$$

given by the natural inclusion $R \times_B S \hookrightarrow S \diamond R$, since we have

$$R \times_B S = \{((a_R, b_R), (b_S, c_S)) \in R \times S \mid b_R = b_S\};$$

$$S \diamond R = \left\{ (a, c) \in A \times C \;\middle|\; \begin{array}{l} \text{there exists some } b \in B \text{ such} \\ \text{that } (a, b) \in R \text{ and } (b, c) \in S \end{array} \right\};$$

- *The Lax Unity Constraints.* The lax unity constraint[1]

$$\iota^0_A\colon \underbrace{\mathrm{id}_{\iota(A)}}_{(A,\mathrm{id}_A,\mathrm{id}_A)} \Longrightarrow \underbrace{\iota(\chi_A)}_{(\Delta_A,\restriction\mathrm{pr}_1\Delta_A,\restriction\mathrm{pr}_2\Delta_A)}$$

  of ι at A is given by the diagonal morphism of A, as in the diagram

$$\begin{tikzcd} & A \arrow[dl, equal, "\mathrm{id}_A"'] \arrow[dr, equal, "\mathrm{id}_A"] \arrow[dd, "\delta_A"] & \\ A & & A. \\ & \Delta_A \arrow[ul, "\restriction\mathrm{pr}_1\Delta_A"] \arrow[ur, "\restriction\mathrm{pr}_2\Delta_A"'] & \end{tikzcd}$$

[1] Which is in fact strong, as δ_A is an isomorphism.

Proof 6.8.3.2 ► Proof of Proposition 6.8.2.4

Omitted.

6.8.4 Comparison to Relations: The Wehrheim–Woodward Construction

6.8.5 Comparison to Multirelations

Remark 6.8.5.1 ► Interaction With Multirelations

The pseudofunctor of Proposition 6.8.2.4 and the lax functor of Proposition 6.8.3.1 fail to be equivalences of bicategories. This happens essentially because a span $(S,f,g)\colon A \nrightarrow B$ from A to B may relate elements $a \in A$ and $b \in B$ by more than one element, e.g. there could be $s \neq s' \in S$ such that $a = f(s) = f(s')$ and $b = g(s) = g(s')$.

Thus, in a sense, spans may be thought of as “relations with multiplicity”. And indeed, if instead of considering relations from A to B, i.e. functions

$$R\colon A\times B \longrightarrow \{\mathrm{true},\mathrm{false}\}$$

from $A\times B$ to $\{\mathrm{true},\mathrm{false}\} \cong \{0,1\}$, we consider functions

$$R\colon A\times B \longrightarrow \mathbb{N}\cup\{\infty\}$$

from $A \times B$ to $\mathbb{N} \cup \{\infty\}$, then we obtain the notion of a **multirelation from A to B**, and these turn out to assemble together with sets into a bicategory MRel that is biequivalent to Span; see [**some-algebraic-laws-for-spans-and-their-connections-with-multirelations**].

6.8.6 Comparison to Relations via Double Categories

Remark 6.8.6.1 ▶ Interaction With Double Categories and Adjointness

There are double functors between the double categories $\mathrm{Rel}^{\mathrm{dbl}}$ and $\mathrm{Span}^{\mathrm{dbl}}$ analogous to the functors of Propositions 6.8.2.4 and 6.8.3.1, assembling moreover into a strict-lax adjunction of double functors; see [**higher-dimensional-categories**].

Appendices

6.A Other Chapters

Chapter 7

Posets

Rename this to "Preorders, Partial Orders, and Posets".

Contents

7.1 Section

7.1.1 Section

7.1.1.1 Section

Appendices

7.A Other Chapters

Sets

1. Sets
2. Constructions With Sets
3. Pointed Sets
4. Tensor Products of Pointed Sets
5. Relations
6. Spans
7. Posets

Indexed and Fibred Sets

7. Indexed Sets
8. Fibred Sets
9. Un/Straightening for Indexed and Fibred Sets

Category Theory

11. Categories
12. Types of Morphisms in Categories

Part II

Indexed and Fibred Sets

Chapter 8

Indexed Sets

This chapter contains a discussion of indexed sets, the set-theoretical counterpart to indexed categories. In particular, here we explore:

1. Indexed sets, i.e. functors $K_{\mathsf{disc}} \to \mathsf{Sets}$ with K a set;
2. The limits and colimits in the category of K-indexed sets;
3. Constructions with indexed sets like dependent sums, dependent products, and internal Homs.

Contents

8.1 Indexed Sets

8.1.1 Foundations

Let K be a set.

Definition 8.1.1.1 ▶ Indexed Sets

A K**-indexed set** is a functor $X: K_{\mathsf{disc}} \to \mathsf{Sets}$.

Remark 8.1.1.2 ▶ Unwinding Definition 8.1.1.1

By **??**, a K**-indexed set** consists of a K-indexed collection

$$X^{\dagger}: K \to \mathrm{Obj}(\mathsf{Sets}),$$

of sets, assigning a set $X^{\dagger}_x \stackrel{\text{def}}{=} X_x$ to each element x of K.

8.1.2 Morphisms of Indexed Sets

Let $X: K_{\mathsf{disc}} \to \mathsf{Sets}$ and $Y: K_{\mathsf{disc}} \to \mathsf{Sets}$ be indexed sets.

Definition 8.1.2.1 ▶ Morphisms of Indexed Sets

A **morphism of** K**-indexed sets from** X **to** Y[1] is a natural transformation

$$f: X \Longrightarrow Y, \qquad \begin{tikzcd} K_{\mathsf{disc}} \arrow[r, bend left, "X"] \arrow[r, bend right, "Y"'] \arrow[r, phantom, "f\Downarrow"] & \mathsf{Sets} \end{tikzcd}$$

from X to Y.

[1] *Further Terminology:* Also called a K**-indexed map of sets from** X **to** Y.

Remark 8.1.2.2 ▶ Unwinding Definition 8.1.2.1

In detail, a **morphism of** K**-indexed sets** consists of a K-indexed collection

$$\{f_x: X_x \to Y_x\}_{x \in K}$$

of maps of sets.

8.1.3 The Category of Sets Indexed by a Fixed Set

Let K be a set.

Definition 8.1.3.1 ▶ The Category of K-Indexed Sets

The **category of K-indexed sets** is the category $\mathsf{ISets}(K)$ defined by

$$\mathsf{ISets}(K) \stackrel{\text{def}}{=} \mathsf{Fun}\big(K_{\mathsf{disc}}, \mathsf{Sets}\big).$$

Remark 8.1.3.2 ▶ Unwinding Definition 8.1.3.1

In detail, the **category of K-indexed sets** is the category $\mathsf{ISets}(K)$ where

- *Objects.* The objects of $\mathsf{ISets}(K)$ are K-indexed sets as in Definition 8.1.1.1;
- *Morphisms.* The morphisms of $\mathsf{ISets}(K)$ are morphisms of K-indexed sets as in Definition 8.1.2.1;
- *Identities.* For each $X \in \mathrm{Obj}(\mathsf{ISets}(K))$, the unit map

$$\mathbb{1}_X^{\mathsf{ISets}(K)} \colon \mathrm{pt} \longrightarrow \mathrm{Hom}_{\mathsf{ISets}(K)}(X, X)$$

  of $\mathsf{ISets}(K)$ at X is defined by

$$\mathrm{id}_X^{\mathsf{ISets}(K)} \stackrel{\text{def}}{=} \{\mathrm{id}_{X_x}\}_{x \in K};$$

- *Composition.* For each $X, Y, Z \in \mathrm{Obj}(\mathsf{ISets}(K))$, the composition map

$$\circ_{X,Y,Z}^{\mathsf{ISets}(K)} \colon \mathrm{Hom}_{\mathsf{ISets}(K)}(Y, Z) \times \mathrm{Hom}_{\mathsf{ISets}(K)}(X, Y) \longrightarrow \mathrm{Hom}_{\mathsf{ISets}(K)}(X, Z)$$

  of $\mathsf{ISets}(K)$ at (X, Y, Z) is defined by

$$\{g_x\}_{x \in K} \circ_{X,Y,Z}^{\mathsf{ISets}(K)} \{f_x\}_{x \in K} \stackrel{\text{def}}{=} \{g_x \circ f_x\}_{x \in K}.$$

8.1.4 The Category of Indexed Sets

Definition 8.1.4.1 ► The Category of Indexed Sets

The **category of indexed sets** is the category ISets defined as the Grothendieck construction of the functor $\mathsf{ISets}\colon \mathsf{Sets}^{\mathsf{op}} \to \mathsf{Cats}$ of Proposition 8.4.1.5:

$$\mathsf{ISets} \stackrel{\text{def}}{=} \int^{\mathsf{Sets}} \mathsf{ISets}.$$

Remark 8.1.4.2 ► Unwinding Definition 8.1.4.1

In detail, the **category of indexed sets** is the category ISets where

- *Objects.* The objects of ISets are pairs (K, X) consisting of
  - *The Indexing Set.* A set K;
  - *The Indexed Set.* A K-indexed set $X\colon K_{\mathsf{disc}} \to \mathsf{Sets}$;
- *Morphisms.* A morphism of ISets from (K, X) to (K', Y) is a pair (ϕ, f) consisting of
  - *The Reindexing Map.* A map of sets $\phi\colon K \to K'$;
  - *The Morphism of Indexed Sets.* A morphism of K-indexed sets $f\colon X \to \phi_*(Y)$ as in the diagram

$$f\colon X \to \phi_*(Y), \qquad \begin{tikzcd} K_{\mathsf{disc}} \arrow[rr, "\phi"] \arrow[dr, "X"'] & \arrow[d, phantom, "\overset{f}{\Rightarrow}"] & K'_{\mathsf{disc}} \arrow[dl, "Y"] \\ & \mathsf{Sets}; & \end{tikzcd}$$

- *Identities.* For each $(K, X) \in \mathrm{Obj}(\mathsf{ISets})$, the unit map

$$\mathbb{1}^{\mathsf{ISets}}_{(K,X)}\colon \mathrm{pt} \to \mathsf{ISets}((K, X), (K, X))$$

  of ISets at (K, X) is defined by

$$\mathrm{id}^{\mathsf{ISets}}_{(K,X)} \stackrel{\text{def}}{=} (\mathrm{id}_K, \mathrm{id}_X).$$

- *Composition.* For each $\mathbf{X} = (K, X)$, $\mathbf{Y} = (K', Y)$, $\mathbf{Z} = (K'', Z) \in \mathrm{Obj}(\mathsf{ISets})$, the composition map

$$\circ^{\mathsf{ISets}}_{\mathbf{X},\mathbf{Y},\mathbf{Z}}\colon \mathsf{ISets}(\mathbf{Y}, \mathbf{Z}) \times \mathsf{ISets}(\mathbf{X}, \mathbf{Y}) \to \mathsf{ISets}(\mathbf{X}, \mathbf{Z})$$

  of ISets at $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ is defined by

$$(\psi, g) \circ (\phi, f) \stackrel{\text{def}}{=} (\psi \circ \phi, (g \star \mathrm{id}_\phi) \circ f),$$

as in the diagram

$$\begin{tikzcd}
K_{\mathrm{disc}} \arrow[r, "\phi"] \arrow[dr, "X"'] \arrow[dr, phantom, shift left=4, "\overset{f}{\Nearrow}"] & K'_{\mathrm{disc}} \arrow[r, "\psi"] \arrow[d, "Y"] \arrow[dr, phantom, shift left=1, "\overset{g}{\Nearrow}"] & K''_{\mathrm{disc}} \arrow[dl, "Z"] \\
& \mathsf{Sets} &
\end{tikzcd}$$

for each $(\phi, f) \in \mathsf{ISets}(\mathbf{X}, \mathbf{Y})$ and each $(\psi, g) \in \mathsf{ISets}(\mathbf{Y}, \mathbf{Z})$.

8.2 Limits of Indexed Sets

8.2.1 Products of K-Indexed Sets

Let $X : K_{\mathrm{disc}} \to \mathsf{Sets}$ and $Y : K_{\mathrm{disc}} \to \mathsf{Sets}$ be indexed sets.

Definition 8.2.1.1 ► Products of K-Indexed Sets

The **product of X and Y** is the K-indexed set $X \times Y : K_{\mathrm{disc}} \to \mathsf{Sets}$ defined by

$$(X \times Y)_k \overset{\mathrm{def}}{=} X_k \times Y_k$$

for each $k \in K$.

Proof 8.2.1.2 ► Proof of Definition 8.2.1.1

That this agrees with the categorical product in $\mathsf{ISets}(K)$ follows from **??** of **??**.

8.2.2 Pullbacks of K-Indexed Sets

Let $X, Y, Z : K_{\mathrm{disc}} \to \mathsf{Sets}$ be K-indexed sets and let $f : X \to Z$ and $g : Y \to Z$ be morphisms of K-indexed sets.

Definition 8.2.2.1 ► Pullbacks of K-Indexed Sets

The **pullback of X and Y over Z** is the K-indexed set $X \times_Z Y : K_{\mathrm{disc}} \to \mathsf{Sets}$ defined by

$$(X \times_Z Y)_k \overset{\mathrm{def}}{=} X_k \times_{Z_k} Y_k$$

for each $k \in K$.

Proof 8.2.2.2 ▶ Proof of Definition 8.2.2.1

That this agrees with the categorical pullback in $\mathsf{ISets}(K)$ follows from **??** of **??**. □

8.2.3 Equalisers of K-Indexed Sets

Let $X, Y \colon K_{\mathsf{disc}} \to \mathsf{Sets}$ be K-indexed sets and let $f, g \colon X \rightrightarrows Y$ be morphisms of K-indexed sets.

Definition 8.2.3.1 ▶ Equalisers of K-Indexed Sets

The **equaliser of f and g** is the K-indexed set $\mathrm{Eq}(f, g) \colon K_{\mathsf{disc}} \to \mathsf{Sets}$ defined by

$$(\mathrm{Eq}(f, g))_k \overset{\text{def}}{=} \mathrm{Eq}(f_k, g_k)$$

for each $k \in K$.

Proof 8.2.3.2 ▶ Proof of Definition 8.2.3.1

That this agrees with the categorical equaliser in $\mathsf{ISets}(K)$ follows from **??** of **??**. □

8.2.4 Products in ISets

Let $X \colon K_{\mathsf{disc}} \to \mathsf{Sets}$ and $Y \colon K'_{\mathsf{disc}} \to \mathsf{Sets}$ be indexed sets.

Definition 8.2.4.1 ▶ Products of Indexed Sets

The **product of X and Y** is the $(K \times K')$-indexed set

$$X \times Y \colon (K \times K')_{\mathsf{disc}} \to \mathsf{Sets}$$

defined by

$$(X \times Y)_{(k,k')} \overset{\text{def}}{=} X_k \times Y_{k'}$$

for each $(k, k') \in K \times K'$.

Proof 8.2.4.2 ▶ Proof of Definition 8.2.4.1

We claim that this agrees with the categorical product in ISets. □

8.2.5 Pullbacks in ISets

Let $X \colon K_{\mathsf{disc}} \to \mathsf{Sets}$ be a K-indexed set, let $Y \colon K'_{\mathsf{disc}} \to \mathsf{Sets}$ be a K'-indexed set, let $Z \colon K''_{\mathsf{disc}} \to \mathsf{Sets}$ be a K''-indexed set, and let $(\phi, f) \colon X \to Z$ and $(\psi, g) \colon Y \to Z$ be morphisms of indexed sets (as in Remark 8.1.4.2).

Definition 8.2.5.1 ▶ Pullbacks of Indexed Sets

The **pullback of X and Y over Z** is the $(K \times_{K''} K)$-indexed set

$$X \times_Z Y \colon (K \times_{K''} K)_{\mathsf{disc}} \to \mathsf{Sets}$$

defined by

$$\begin{aligned}(X \times_Z Y)_{(k,k')} &\overset{\text{def}}{=} X_k \times_{Z_{\phi(k)}} Y_{k'} \\ &\overset{\text{def}}{=} X_k \times_{Z_{\psi(k)}} Y_{k'}\end{aligned}$$

for each $(k, k') \in K \times_{K''} K'$.

Proof 8.2.5.2 ▶ Proof of Definition 8.2.2.1

We claim that this agrees with the categorical pullback in ISets.

8.2.6 Equalisers in ISets

Let $X \colon K_{\mathsf{disc}} \to \mathsf{Sets}$ be a K-indexed set, let $Y \colon K'_{\mathsf{disc}} \to \mathsf{Sets}$ be a K'-indexed set, and let $(\phi, f), (\psi, g) \colon X \to Y$ be morphisms of indexed sets (as in Remark 8.1.4.2).

Definition 8.2.6.1 ▶ Equalisers of Indexed Sets

The **equaliser of** (ϕ, f) **and** (ψ, g) is the $\mathsf{Eq}(\phi, \psi)$-indexed set $\mathsf{Eq}(f, g) \colon \mathsf{Eq}(\phi, \psi) \to \mathsf{Sets}$ defined by

$$(\mathsf{Eq}(f, g))_k \overset{\text{def}}{=} \mathsf{Eq}(f_k, g_k)$$

for each $k \in \mathsf{Eq}(\phi, \psi)$.

Proof 8.2.6.2 ▶ Proof of Definition 8.2.6.1

We claim that this agrees with the categorical equaliser in ISets.

8.3 Colimits of Indexed Sets

8.3.1 Coproducts of K-Indexed Sets

Let $X \colon K_{\mathsf{disc}} \to \mathsf{Sets}$ and $Y \colon K_{\mathsf{disc}} \to \mathsf{Sets}$ be indexed sets.

Definition 8.3.1.1 ► Coproducts of Indexed Sets

The **coproduct** of X and Y is the K-k-indexed set $X \coprod Y \colon K_{\mathsf{disc}} \to \mathsf{Sets}$ defined by

$$(X \coprod Y)_k \stackrel{\text{def}}{=} X_k \coprod Y_k$$

for each $k \in K$.

Proof 8.3.1.2 ► Proof of Definition 8.3.1.1

That this agrees with the categorical coproduct in $\mathsf{ISets}(K)$ follows from **??** of **??**. ∎

8.3.2 Pushouts of K-Indexed Sets

Let $X, Y, Z \colon K_{\mathsf{disc}} \to \mathsf{Sets}$ be K-indexed sets and let $f \colon Z \to X$ and $g \colon Z \to Y$ be morphisms of K-indexed sets.

Definition 8.3.2.1 ► Pushouts of K-Indexed Sets

The **pushout** of X and Y is the K-indexed set $X \coprod_Z Y \colon K_{\mathsf{disc}} \to \mathsf{Sets}$ defined by

$$(X \coprod_Z Y)_k \stackrel{\text{def}}{=} X_k \coprod_{Z_k} Y_k$$

for each $k \in K$.

Proof 8.3.2.2 ► Proof of Definition 8.3.2.1

That this agrees with the categorical pushout in $\mathsf{ISets}(K)$ follows from **??** of **??**. ∎

8.3.3 Coequalisers of K-Indexed Sets

Let $X, Y \colon K_{\mathsf{disc}} \to \mathsf{Sets}$ be K-indexed sets and let $f, g \colon X \rightrightarrows Y$ be morphisms of K-indexed sets.

Definition 8.3.3.1 ► Coequalisers of K-Indexed Sets

The **coequaliser** of X and Y is the K-indexed set $\mathrm{CoEq}(f, g) \colon K_{\mathsf{disc}} \to \mathsf{Sets}$ defined by

$$(\mathrm{CoEq}(f, g))_k \stackrel{\text{def}}{=} \mathrm{CoEq}(f_k, g_k)$$

for each $k \in K$.

Proof 8.3.3.2 ► Proof of Definition 8.3.3.1

That this agrees with the categorical coequaliser in $\mathsf{ISets}(K)$ follows from **??** of **??**.

8.4 Constructions With Indexed Sets

8.4.1 Change of Indexing

Let $\phi\colon K \to K'$ be a function and let X be a K'-indexed set.

Definition 8.4.1.1 ► Change of Indexing of Indexed Sets

The **change of indexing of** X **to** K is the K-indexed set $\phi^*(X)$ defined by

$$\phi^*(X) \overset{\text{def}}{=} X \circ \phi_{\mathsf{disc}}.$$

Remark 8.4.1.2 ► Unwinding Definition 8.4.1.1

In detail, the **change of indexing of** X **to** K is the K-indexed set $\phi^*(X)$ defined by

$$\phi^*(X)_x \overset{\text{def}}{=} X_{\phi(x)}$$

for each $x \in K$.

Proposition 8.4.1.3 ► Functoriality of Change of Indexing

The assignment $X \mapsto \phi^*(X)$ defines a functor

$$\phi^*\colon \mathsf{ISets}(K') \to \mathsf{ISets}(K),$$

where

- *Action on Objects.* For each $X \in \mathrm{Obj}(\mathsf{ISets}(K'))$, we have

$$[\phi^*](X) \overset{\text{def}}{=} \phi^*(X);$$

- *Action on Morphisms.* For each $X, Y \in \mathrm{Obj}(\mathsf{ISets}(K'))$, the action on Hom-sets

$$\phi^*_{X,Y}\colon \mathrm{Hom}_{\mathsf{ISets}(K')}(X,Y) \to \mathrm{Hom}_{\mathsf{ISets}(K)}(\phi^*(X),\phi^*(Y))$$

of ϕ^* at (X,Y) is the map sending a morphism of K'-indexed sets

$$f = \{f_x\colon X_x \to Y_x\}_{x\in K'}$$

from X to Y to the morphism of K-indexed sets defined by

$$\phi^*(f) \overset{\text{def}}{=} \left\{f_{\phi(x)} \colon X_{\phi(x)} \to Y_{\phi(x)}\right\}_{x \in K}.$$

Proof 8.4.1.4 ▶ Proof of Proposition 8.4.1.3

Omitted. ∎

Proposition 8.4.1.5 ▶ Functoriality of Categories of K-Indexed Sets

The assignment $K \mapsto \mathsf{ISets}(K)$ defines a functor

$$\mathsf{ISets} \colon \mathsf{Sets}^{\mathsf{op}} \to \mathsf{Cats},$$

where

- *Action on Objects.* For each $K \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$[\mathsf{ISets}](K) \overset{\text{def}}{=} \mathsf{ISets}(K);$$

- *Action on Morphisms.* For each $K, K' \in \mathrm{Obj}(\mathsf{Sets})$, the action on Hom-sets

$$\mathsf{ISets}_{K,K'} \colon \mathsf{Sets}^{\mathsf{op}}(K, K') \to \mathsf{Fun}(\mathsf{ISets}(K), \mathsf{ISets}(K'))$$

of ISets at (K, K') is the map defined by

$$\mathsf{ISets}_{K,K'}(\phi) \overset{\text{def}}{=} \phi^*$$

for each $\phi \in \mathsf{Sets}^{\mathsf{op}}(K, K')$.

Proof 8.4.1.6 ▶ Proof of Proposition 8.4.1.5

Omitted. ∎

8.4.2 Dependent Sums

Let $\phi \colon K \to K'$ be a function and let X be a K-indexed set.

Definition 8.4.2.1 ▶ Dependent Sums of Indexed Sets

The **dependent sum of** X is the K'-indexed set $\Sigma_\phi(X)$[1] defined by

$$\Sigma_\phi(X) \overset{\text{def}}{=} \mathrm{Lan}_\phi(X),$$

and hence given by

$$\Sigma_\phi(X)_x \cong \coprod_{y \in \phi^{-1}(x)} X_y$$

for each $x \in K'$.

[1] *Further Notation:* Also written $\phi_*(X)$.

Proposition 8.4.2.2 ▶ Functoriality of Dependent Sums

The assignment $X \mapsto \Sigma_\phi(X)$ defines a functor

$$\Sigma_\phi \colon \mathsf{ISets}(K) \to \mathsf{ISets}(K'),$$

where

- *Action on Objects.* For each $X \in \mathrm{Obj}(\mathsf{ISets}(K))$, we have

$$[\Sigma_\phi](X) \overset{\text{def}}{=} \Sigma_\phi(X);$$

- *Action on Morphisms.* For each $X, Y \in \mathrm{Obj}(\mathsf{ISets}(K))$, the action on Hom-sets

$$\Sigma_{\phi|X,Y} \colon \mathrm{Hom}_{\mathsf{ISets}(K)}(X, Y) \to \mathrm{Hom}_{\mathsf{ISets}(K')}\big(\Sigma_\phi(X), \Sigma_\phi(Y)\big)$$

  of Σ_ϕ at (X, Y) is the map sending a morphism of K-indexed sets

$$f \colon X \to Y$$

  to the morphism of K'-indexed sets defined by

$$\begin{aligned} \Sigma_\phi(f) &\overset{\text{def}}{=} \mathrm{Lan}_\phi(f); \\ &\cong \coprod_{y \in \phi^{-1}(X)} f_y. \end{aligned}$$

Proof 8.4.2.3 ▶ Proof of Proposition 8.4.2.2

Omitted. ∎

8.4.3 Dependent Products

Let $\phi \colon K \to K'$ be a function and let X be a K-indexed set.

Definition 8.4.3.1 ▶ Dependent Products of Indexed Sets

The **dependent product of** X is the K'-indexed set $\Pi_\phi(X)$[1] defined by

$$\Pi_\phi(X) \stackrel{\text{def}}{=} \mathrm{Ran}_\phi(X),$$

and hence given by

$$\Pi_\phi(X)_x \cong \prod_{y\in\phi^{-1}(x)} X_y$$

for each $x \in K'$.

[1] *Further Notation:* Also written $\phi_!(X)$.

Proposition 8.4.3.2 ▶ Functoriality of Dependent Products

The assignment $X \mapsto \Pi_\phi(X)$ defines a functor

$$\Pi_\phi \colon \mathsf{ISets}(K) \to \mathsf{ISets}(K'),$$

where

- *Action on Objects.* For each $X \in \mathrm{Obj}(\mathsf{ISets}(K))$, we have

  $$[\Pi_\phi](X) \stackrel{\text{def}}{=} \Pi_\phi(X);$$

- *Action on Morphisms.* For each $X, Y \in \mathrm{Obj}(\mathsf{ISets}(K))$, the action on Hom-sets

  $$\Pi_{\phi|X,Y} \colon \mathrm{Hom}_{\mathsf{ISets}(K)}(X,Y) \to \mathrm{Hom}_{\mathsf{ISets}(K')}(\Pi_\phi(X), \Pi_\phi(Y))$$

  of Π_ϕ at (X, Y) is the map sending a morphism of K-indexed sets

  $$f \colon X \to Y$$

  to the morphism of K'-indexed sets defined by

  $$\begin{aligned}\Pi_\phi(f) &\stackrel{\text{def}}{=} \mathrm{Ran}_\phi(f);\\ &\cong \prod_{y\in\phi^{-1}(x)} f_y.\end{aligned}$$

Proof 8.4.3.3 ▶ Proof of Proposition 8.4.3.2

Omitted. □

8.4.4 Internal Homs

Let K be a set and let X and Y be K-indexed sets.

DEFINITION 8.4.4.1 ▶ INTERNAL HOM OF INDEXED SETS

The **internal Hom of indexed sets from** X **to** Y is the indexed set $\mathbf{Hom}_{\mathsf{ISets}(K)}(X, Y)$ defined by

$$\mathbf{Hom}_{\mathsf{ISets}(K)}(X, Y) \overset{\text{def}}{=} \mathsf{Sets}(X_x, Y_x)$$

for each $x \in K$.

8.4.5 Adjointness of Indexed Sets

Let $\phi\colon K \to K'$ be a map of sets.

PROPOSITION 8.4.5.1 ▶ ADJOINTNESS OF INDEXED SETS

We have a triple adjunction

$$(\Sigma_\phi \dashv \phi^* \dashv \Pi_\phi)\colon\quad \begin{tikzcd} \mathsf{ISets}(K) \arrow[r, bend left=50, "\Sigma_\phi"] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, phantom, shift right=2, "\perp"] \arrow[r, bend right=50, "\Pi_\phi"'] & \mathsf{ISets}(K') \arrow[l, "\phi^*"' description] \end{tikzcd}.$$

PROOF 8.4.5.2 ▶ PROOF OF PROPOSITION 8.4.5.1

This follows from **??** of **??**.

Appendices

8.A Other Chapters

Sets

Indexed and Fibred Sets

Category Theory

Chapter 9

Fibred Sets

This chapter contains a discussion of the un/straightening equivalence in the context of sets, as well as a general discussion of indexed and fibred sets. In particular, it contains:

1. A discussion of indexed sets (i.e. functors $K_{\mathsf{disc}} \to \mathsf{Sets}$ with K a set), constructions with them like dependent sums and dependent products, and their properties (**????**);

2. A discussion of fibred sets (i.e. maps of sets $X \to K$), constructions with them like dependent sums and dependent products, and their properties (Sections 9.1 and 9.2);

3. A discussion of the un/straightening equivalence for indexed and fibred sets (**??**).

Contents

9.1 Fibred Sets

9.1.1 Foundations

Let K be a set.

Definition 9.1.1.1 ▶ Fibred Sets

A K-**fibred set** is a pair (X, ϕ) consisting of[1]

- *The Underlying Set.* A set X, called the **underlying set of** (X, ϕ);
- *The Fibration.* A map of sets $\phi\colon X \to K$.

[1] *Further Terminology:* The **fibre of** (X, ϕ) **over** $x \in K$ is the set $\phi^{-1}(x)$ (also written ϕ_x) defined by

$$\phi^{-1}(x) \overset{\text{def}}{=} \mathrm{pt} \times_{[x], K, \phi} X, \qquad \begin{tikzcd} \phi^{-1}(x) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "\phi"] \\ \mathrm{pt} \arrow[r, "{[x]}"'] & K. \end{tikzcd}$$

9.1.2 Morphisms of Fibred Sets

Definition 9.1.2.1 ▶ Morphisms of Fibred Sets

A **morphism of** K**-fibred sets from** (X, ϕ) **to** (Y, ψ) is a function $f\colon X \to Y$ such that the diagram[1]

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\phi"'] & & Y \arrow[dl, "\psi"] \\ & K & \end{tikzcd}$$

commutes.

[1] *Further Terminology:* The **transport map associated to** f **at** $x \in K$ is the function

$$f_x^*\colon \phi^{-1}(x) \to \psi^{-1}(x)$$

given by the dashed map in the diagram

$$\begin{tikzcd} \phi^{-1}(x) \arrow[rr] \arrow[dd] \arrow[dr, dashed] \arrow[dr, phantom, "\lrcorner", very near start] & & X \arrow[dd, "\phi", near start] \arrow[dr, "f"] & \\ & \psi^{-1}(x) \arrow[rr] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & & Y \arrow[dd, "\psi"] \\ \mathrm{pt} \arrow[rr, "{[x]}", near start] \arrow[dr, equals] & & K \arrow[dr, equals] & \\ & \mathrm{pt} \arrow[rr, "{[x]}"'] & & K. \end{tikzcd}$$

9.1.3 The Category of Fibred Sets Over a Fixed Base

Definition 9.1.3.1 ▶ The Category of K-Fibred Sets

The **category of K-fibred sets** is the category $\mathsf{FibSets}(K)$ defined as the slice category $\mathsf{Sets}_{/K}$ of Sets over K:

$$\mathsf{FibSets}(K) \stackrel{\text{def}}{=} \mathsf{Sets}_{/K}.$$

Remark 9.1.3.2 ▶ Unwinding Definition 9.1.3.1

In detail $\mathsf{FibSets}(K)$ is the category where

- *Objects.* The objects of $\mathsf{FibSets}(K)$ are pairs (X, ϕ) consisting of
  - *The Fibred Set.* A set X;
  - *The Fibration.* A function $\phi \colon X \to K$;
- *Morphisms.* A morphism of $\mathsf{FibSets}(K)$ from (X, ϕ) to (Y, ψ) is a function $f \colon X \to Y$ making the diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\phi"'] & & Y \arrow[dl, "\psi"] \\ & X & \end{tikzcd}$$

  commute;
- *Identities.* For each $(X, \phi) \in \mathrm{Obj}(\mathsf{FibSets}(K))$, the unit map

$$\mathbb{1}^{\mathsf{FibSets}(K)}_{(X,\phi)} \colon \mathrm{pt} \to \mathrm{Hom}_{\mathsf{FibSets}(K)}((X, \phi), (X, \phi))$$

  of $\mathsf{FibSets}(K)$ at (X, ϕ) is given by

$$\mathrm{id}^{\mathsf{FibSets}(K)}_{(X,\phi)} \stackrel{\text{def}}{=} \mathrm{id}_X,$$

  as witnessed by the commutativity of the diagram

$$\begin{tikzcd} X \arrow[rr, equal, "\mathrm{id}_X"] \arrow[dr, "\phi"'] & & X \arrow[dl, "\phi"] \\ & K & \end{tikzcd}$$

  in Sets;

- *Composition*. For each $\mathbf{X} = (X, \phi)$, $\mathbf{Y} = (Y, \psi)$, $\mathbf{Z} = (Z, \chi) \in \mathrm{Obj}(\mathsf{FibSets}(K))$, the composition map

$$\circ^{\mathsf{FibSets}(K)}_{\mathbf{X},\mathbf{Y},\mathbf{Z}} \colon \mathrm{Hom}_{\mathsf{FibSets}(K)}(\mathbf{Y}, \mathbf{Z}) \times \mathrm{Hom}_{\mathsf{FibSets}(K)}(\mathbf{X}, \mathbf{Y}) \to \mathrm{Hom}_{\mathsf{FibSets}(K)}(\mathbf{X}, \mathbf{Z})$$

  of $\mathsf{FibSets}(K)$ at $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ is defined by

$$\circ^{\mathsf{FibSets}(K)}_{\mathbf{X},\mathbf{Y},\mathbf{Z}} \overset{\text{def}}{=} \circ^{\mathsf{Sets}}_{X,Y,Z},$$

  as witnessed by the commutativity of the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "\phi"'] & Y \arrow[r, "g"] \arrow[d, "\psi"] & Z \arrow[dl, "\chi"] \\ & K & \end{tikzcd}$$

  in Sets.

9.1.4 The Category of Fibred Sets

Definition 9.1.4.1 ▶ The Category of Fibred Sets

The **category of fibred sets** is the category $\mathsf{FibSets}$ defined as the Grothendieck construction of the functor $\mathsf{FibSets} \colon \mathsf{Sets}^{\mathsf{op}} \to \mathsf{Cats}$ of Proposition 9.2.1.4:

$$\mathsf{FibSets} \overset{\text{def}}{=} \int^{\mathsf{Sets}} \mathsf{FibSets}.$$

Remark 9.1.4.2 ▶ Unwinding Definition 9.1.4.1

In detail, the **category of fibred sets** is the category $\mathsf{FibSets}$ where

- *Objects*. The objects of $\mathsf{FibSets}$ are pairs $(K, (X, \phi_X))$ consisting of
  - *The Base Set*. A set K;
  - *The Fibred Set*. A K-fibred set $\phi_X \colon X \to K$;
- *Morphisms*. A morphism of $\mathsf{FibSets}$ from $(K, (X, \phi_X))$ to $(K', (Y, \phi_Y))$ is a pair (ϕ, f) consisting of
  - *The Base Map*. A map of sets $\phi \colon K \to K'$;

- *The Morphism of Fibred Sets.* A morphism of K-fibred sets

$$f\colon (X,\phi_X) \to \phi_Y^*(Y), \qquad \begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\phi_X"'] & & Y \times_{K'} K \arrow[dl, "\mathrm{pr}_2"] \\ & K; & \end{tikzcd}$$

· *Identities.* For each $(K,X) \in \mathrm{Obj}(\mathsf{FibSets})$, the unit map

$$\mathbb{1}^{\mathsf{FibSets}}_{(K,X)}\colon \mathrm{pt} \to \mathsf{FibSets}((K,X),(K,X))$$

of FibSets at (K,X) is defined by

$$\mathrm{id}^{\mathsf{FibSets}}_{(K,X)} \overset{\mathrm{def}}{=} (\mathrm{id}_K, \sim),$$

where $\sim$ is the isomorphism $X \to X \times_K K$ as in the diagram

$$\begin{tikzcd} X \arrow[rr, dashed, "\sim"] \arrow[dr, "\phi_X"'] & & X \times_K K \arrow[dl, "\mathrm{pr}_2"] \\ & K; & \end{tikzcd}$$

· *Composition.* For each $\mathbf{X} = (K,X), \mathbf{Y} = (K',Y), \mathbf{Z} = (K'',Z) \in \mathrm{Obj}(\mathsf{FibSets})$, the composition map

$$\circ^{\mathsf{FibSets}}_{\mathbf{X},\mathbf{Y},\mathbf{Z}}\colon \mathsf{FibSets}(\mathbf{Y},\mathbf{Z}) \times \mathsf{FibSets}(\mathbf{X},\mathbf{Y}) \to \mathsf{FibSets}(\mathbf{X},\mathbf{Z})$$

of FibSets at $(\mathbf{X},\mathbf{Y},\mathbf{Z})$ is defined by

$$g \circ^{\mathsf{FibSets}}_{\mathbf{X},\mathbf{Y},\mathbf{Z}} f \overset{\mathrm{def}}{=} (g \times_{K'} \mathrm{id}_K) \circ f$$

as in the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "\phi_X"'] & Y \times_{K'} K \arrow[r, "g \times_{K'} \mathrm{id}_K"] \arrow[d, "\mathrm{pr}_2"] & \overbrace{(Z \times_{K''} K') \times_{K'} K}^{\cong Z \times_{K''} K} \arrow[dl, "\mathrm{pr}_2"] \\ & K; & \end{tikzcd}$$

for each $f \in \mathrm{Obj}(\mathsf{FibSets}(\mathbf{X},\mathbf{Y}))$ and each $g \in \mathrm{Obj}(\mathsf{FibSets}(\mathbf{Y},\mathbf{Z}))$.

9.2 Constructions With Fibred Sets

9.2.1 Change of Base

Let $f \colon K \to K'$ be a function and let (X, ϕ_X) be a K'-fibred set.

Definition 9.2.1.1 ► Change of Base for Fibred Sets

The **change of base of** (X, ϕ_X) **to** K is the K-fibred set $f^*(X)$ defined by

$$f^*(X) \overset{\text{def}}{=} (K \times_{K'} X, \mathrm{pr}_1), \qquad \begin{tikzcd} f^*(X) \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "\phi_X"] \\ K \arrow[r, "f"'] & K'. \end{tikzcd}$$

Proposition 9.2.1.2 ► Functoriality of Change of Base

The assignment $X \mapsto f^*(X)$ defines a functor

$$f^* \colon \mathsf{FibSets}(K') \to \mathsf{FibSets}(K),$$

where

- *Action on Objects.* For each $(X, \phi_X) \in \mathrm{Obj}(\mathsf{FibSets}(K'))$, we have
$$f^*(X, \phi_X) \overset{\text{def}}{=} f^*(X);$$
- *Action on Morphisms.* For each $(X, \phi_X), (Y, \phi_Y) \in \mathrm{Obj}(\mathsf{FibSets}(K'))$, the action on Hom-sets
$$f^*_{X,Y} \colon \mathrm{Hom}_{\mathsf{FibSets}(K')}(X, Y) \to \mathrm{Hom}_{\mathsf{FibSets}(K)}(f^*(X), f^*(Y))$$
of f^* at $((X, \phi_X), (Y, \phi_Y))$ is the map sending a morphism of K'-fibred sets
$$g \colon (X, \phi_X) \to (Y, \phi_Y)$$
to the morphism of K-fibred sets given by the dashed morphism in the diagram
$$\begin{tikzcd} f^*(X) \arrow[rr] \arrow[dr, dashed] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & & X \arrow[dr, "g"] \arrow[dd, "\phi_X" near start] & \\ & f^*(Y) \arrow[rr] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & & Y \arrow[dd, "\phi_Y"] \\ K \arrow[rr, "f" near start] \arrow[dr, equal] & & K' \arrow[dr, equal] & \\ & K \arrow[rr, "f"'] & & K'. \end{tikzcd}$$

Proof 9.2.1.3 ▶ Proof of Proposition 9.2.1.2

Omitted. □

Proposition 9.2.1.4 ▶ Functoriality of Categories of K-Fibred Sets

The assignment $K \mapsto \mathsf{FibSets}(K)$ defines a functor

$$\mathsf{FibSets}\colon \mathsf{Sets}^{\mathsf{op}} \longrightarrow \mathsf{Cats},$$

where

- *Action on Objects.* For each $K \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$[\mathsf{FibSets}](K) \stackrel{\text{def}}{=} \mathsf{FibSets}(K);$$

- *Action on Morphisms.* For each $K, K' \in \mathrm{Obj}(\mathsf{Sets})$, the action on Hom-sets

$$\mathsf{Sets}_{/(-)|K,K'}\colon \mathsf{Sets}^{\mathsf{op}}(K, K') \longrightarrow \mathsf{Fun}(\mathsf{FibSets}(K), \mathsf{FibSets}(K'))$$

  of $\mathsf{Sets}_{/(-)}$ at (K, K') is the map sending a map of sets $f\colon K \longrightarrow K'$ to the functor

$$\mathsf{Sets}_{/f}\colon \mathsf{FibSets}(K') \longrightarrow \mathsf{FibSets}(K)$$

  defined by

$$\mathsf{Sets}_{/f} \stackrel{\text{def}}{=} f^*.$$

Proof 9.2.1.5 ▶ Proof of Proposition 9.2.1.4

Omitted. □

9.2.2 Dependent Sums

Let $f\colon K \longrightarrow K'$ be a function and let (X, ϕ_X) be a K-fibred set.

Definition 9.2.2.1 ▶ Dependent Sums for Fibred Sets

The **dependent sum**[1] **of** (X, ϕ_X) is the K'-fibred set $\Sigma_f(X)$[2] defined by

$$\begin{aligned}\Sigma_f(X) &\stackrel{\text{def}}{=} (\Sigma_f(X), \Sigma_f(\phi_X))\\ &\stackrel{\text{def}}{=} (X, f \circ \phi_X).\end{aligned}$$

[1] The name "dependent sum" comes from the fact that the fibre $\Sigma_f(\phi_X)^{-1}(x)$ of $\Sigma_f(X)$ at $x \in K'$ is given by

$$\Sigma_f(\phi_X)^{-1}(x) \cong \coprod_{y \in f^{-1}(x)} \phi_X^{-1}(y);$$

see Item 2 of Proposition 9.2.2.2.

[2] *Further Notation:* Also written $f_*(X)$.

Proposition 9.2.2.2 ► Properties of Dependent Sums of Fibred Sets

Let $f\colon K \to K'$ be a function.

1. *Functoriality.* The assignment $X \mapsto \Sigma_f(X)$ defines a functor

$$\Sigma_f\colon \mathsf{FibSets}(K) \to \mathsf{FibSets}(K'),$$

where

- *Action on Objects.* For each $(X, \phi_X) \in \mathrm{Obj}(\mathsf{FibSets}(K))$, we have

$$\Sigma_f(X, \phi_X) \stackrel{\text{def}}{=} (\Sigma_f(X), \Sigma_f(\phi_X));$$

- *Action on Morphisms.* For each $(X, \phi_X), (Y, \phi_Y) \in \mathrm{Obj}(\mathsf{FibSets}(K))$, the action on Hom-sets

$$\Sigma_{f|X,Y}\colon \mathrm{Hom}_{\mathsf{FibSets}(K)}(X, Y) \to \mathrm{Hom}_{\mathsf{FibSets}(K)}(\Sigma_f(X), \Sigma_f(Y))$$

of Σ_f at $((X, \phi_X), (Y, \phi_Y))$ is the map sending a morphism of K-fibred sets

$$g\colon (X, \phi_X) \to (Y, \phi_Y)$$

to the morphism of K'-fibred sets defined by

$$\Sigma_f(g) \stackrel{\text{def}}{=} g.$$

2. *Interaction With Fibres.* We have a bijection of sets

$$\Sigma_f(\phi_X)^{-1}(k') \cong \coprod_{k \in f^{-1}(k')} \phi_X^{-1}(k)$$

for each $k' \in K'$.

Proof 9.2.2.3 ▶ Proof of Proposition 9.2.2.2

Item 1: Functoriality

Omitted.

Item 2: Interaction With Fibres

Indeed, we have

$$\begin{aligned}\Sigma_f(\phi_X)^{-1}(k') &\overset{\text{def}}{=} \mathrm{pt} \times_{[k'],K',f\circ\phi_X} X\\ &\cong \{x \in X \mid f(\phi_X(x)) = k'\}\\ &\cong \coprod_{k\in f^{-1}(k')} \{x \in X \mid \phi_X(x) = k\}\\ &\cong \coprod_{k\in f^{-1}(k')} \phi_X^{-1}(k)\end{aligned}$$

for each $k' \in K'$. □

9.2.3 Dependent Products

Let $f : K \to K'$ be a function and let (X, ϕ_X) be a K-fibred set.

Definition 9.2.3.1 ▶ Dependent Products of Fibred Sets

The **dependent product**[1] **of** (X, ϕ_X) is the K'-fibred set $\Pi_f(X)$[2] consisting of[3]

- *The Underlying Set.* The set $\Pi_f(X)$ defined by
$$\Pi_f(X) \overset{\text{def}}{=} \coprod_{k'\in K'} \prod_{k\in f^{-1}(k')} \phi_X^{-1}(k);$$
- *The Fibration.* The map of sets
$$\Pi_f(\phi_X) : \Pi_f(X) \to K'$$
defined by sending an element of
$$\Pi_f(X) \overset{\text{def}}{=} \coprod_{k'\in K'} \prod_{k\in f^{-1}(k')} \phi_X^{-1}(k)$$
to its index k' in K'.

[1] The name "dependent product" comes from the fact that the fibre $\Pi_f(\phi_X)^{-1}(k')$ of $\Pi_f(X)$ at $k' \in K'$ is given by
$$\Pi_f(\phi_X)^{-1}(k') \cong \prod_{k\in f^{-1}(k')} \phi_X^{-1}(k);$$

see Item 2 of Proposition 9.2.3.4.

[2]*Further Notation:* Also written $f_!(X)$.

[3]We can also define dependent products via the internal **Hom** in $\mathsf{FibSets}(K')$; see Item 3 of Proposition 9.2.3.4.

Example 9.2.3.2 ▶ Examples of Dependent Products of Sets

Here are some examples of dependent products of sets.

1. *Spaces of Sections.* Let $K = X$, $K' = \mathrm{pt}$, let $\phi\colon E \longrightarrow X$ be a map of sets, and write $!_X\colon X \longrightarrow \mathrm{pt}$ for the terminal map from X to pt. We have a bijection of sets

$$\begin{aligned}\Pi_{!_X}((E,\phi)) &\cong \Gamma_X(\phi)\\ &\overset{\text{def}}{=} \{h \in \mathsf{Sets}(X,E) \mid \phi \circ h = \mathrm{id}_X\}.\end{aligned}$$

2. *Function Spaces.* Let $K = K' = \mathrm{pt}$ and write $!_X\colon X \longrightarrow \mathrm{pt}$ and $!_Y\colon Y \longrightarrow \mathrm{pt}$ for the terminal maps from X and Y to pt. We have a bijection of sets

$$\mathsf{Sets}(X,Y) \cong \Pi_{!_X}\big(!_X^*(Y,!_Y)\big).$$

Proof 9.2.3.3 ▶ Proof of Example 9.2.3.2

Item 1: Spaces of Sections

Indeed, we have

$$\begin{aligned}\Pi_{!_X}((E,\phi)) &\overset{\text{def}}{=} \coprod_{\star\in\mathrm{pt}} \prod_{k\in !_X^{-1}(\star)} \phi_X^{-1}(k)\\ &= \prod_{x\in X} \phi_X^{-1}(x)\\ &\cong \{h \in \mathsf{Sets}(X,E) \mid \phi_X \circ h = \mathrm{id}_X\}\\ &\overset{\text{def}}{=} \Gamma_X(\phi).\end{aligned}$$

Item 2: Function Spaces

Indeed, we have

$$\begin{aligned}\Pi_{!_X}\left(!_X^*(Y,!_Y)\right) &\overset{\text{def}}{=} \Pi_{!_X}\left(X \times_{!_X,\mathrm{pt},!_Y} Y\right)\\ &\overset{\text{def}}{=} \coprod_{\star\in\mathrm{pt}} \prod_{x\in !_X^{-1}(\star)} \mathrm{pr}_1^{-1}(x)\\ &= \prod_{x\in X} Y\\ &\cong \mathsf{Sets}(X,Y).\end{aligned}$$

This finishes the proof.

PROPOSITION 9.2.3.4 ► PROPERTIES OF DEPENDENT PRODUCTS OF FIBRED SETS

Let $f\colon K \to K'$ be a function.

1. *Functoriality.* The assignment $X \mapsto \Pi_f(X)$ defines a functor

$$\Pi_f\colon \mathsf{FibSets}(K) \to \mathsf{FibSets}(K'),$$

where

- *Action on Objects.* For each $(X,\phi_X) \in \mathrm{Obj}(\mathsf{FibSets}(K))$, we have

$$\Pi_f(X,\phi_X) \overset{\text{def}}{=} \Pi_f(X);$$

- *Action on Morphisms.* For each $(X,\phi_X), (Y,\phi_Y) \in \mathrm{Obj}(\mathsf{FibSets}(K))$, the action on Hom-sets

$$\Pi_{f|X,Y}\colon \mathrm{Hom}_{\mathsf{FibSets}(K)}(X,Y) \to \mathrm{Hom}_{\mathsf{FibSets}(K')}\left(\Pi_f(X),\Pi_f(Y)\right)$$

of Π_f at $((X,\phi_X),(Y,\phi_Y))$ is the map sending a morphism of K-fibred sets

$$\xi\colon (X,\phi_X) \to (Y,\phi_Y), \qquad \begin{tikzcd} X \arrow[rr, "\xi"] \arrow[dr, "\phi_X"'] & & Y \arrow[dl, "\phi_Y"] \\ & K & \end{tikzcd}$$

to the morphism

$$\Pi_f(\xi)\colon \left(\Pi_f(X),\Pi_f(\phi_X)\right) \to \left(\Pi_f(Y),\Pi_f(\phi_Y)\right) \qquad \begin{tikzcd} \Pi_f(X) \arrow[rr, "\Pi_f(\xi)"] \arrow[dr, "\Pi_f(\phi_X)"'] & & \Pi_f(Y) \arrow[dl, "\Pi_f(\phi_Y)"] \\ & K & \end{tikzcd}$$

of K'-fibred sets given by[1]

$$[\Pi_f(\xi)]((x_k)_{k\in f^{-1}(k')}) \overset{\text{def}}{=} (\xi(x_k))_{k\in f^{-1}(k')}$$

for each $(x_k)_{k\in f^{-1}(k')} \in \Pi_f(X)$.

2. *Interaction With Fibres.* We have a bijection of sets

$$\Pi_f(\phi_X)^{-1}(k') \cong \prod_{k\in f^{-1}(k')} \phi_X^{-1}(k);$$

for each $k' \in K'$.

3. *Construction Using the Internal Hom.* We have

$$\Pi_f(X, \phi_X) = \left(K' \times_{\mathbf{Hom}_{\mathsf{FibSets}(K')}((K,f),(K,f))} \mathbf{Hom}_{\mathsf{FibSets}(K')}((K,f),(X, f\circ\phi_X)), \mathrm{pr}_1\right),$$

forming a pullback diagram

$$\begin{tikzcd}
\Pi_f(X,\phi_X) \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Hom}_{\mathsf{FibSets}(K')}((K,f),(X,f\circ\phi_X)) \arrow[d, "(\phi_X)_*"] \\
K' \arrow[r, "I"'] & \mathbf{Hom}_{\mathsf{FibSets}(K')}((K,f),(K,f)),
\end{tikzcd}$$

where the bottom map is given by

$$I(k') \overset{\text{def}}{=} \mathrm{id}_{f^{-1}(k')}$$

for each $k' \in K'$ and where $\mathbf{Hom}_{\mathsf{FibSets}(K')}$ denotes the internal Hom of $\mathsf{FibSets}(K')$ of Definition 9.2.4.1.

4. *Internal Homs via Dependent Products.* We have

$$\mathbf{Hom}_{\mathsf{FibSets}(K)}(X, Y) \cong \Pi_{\phi_X}\left(\phi_X^*(Y)\right).$$

[1]Note that we indeed have $\xi(x_k) \in \phi_Y^{-1}(k)$, since

$$\begin{aligned}
\phi_Y(\xi(x_k)) &= [\phi_Y \circ \xi](x_k) \\
&= \phi_X(x_k) \\
&= k,
\end{aligned}$$

where we have used that ξ is a morphism of K-fibred sets for the second equality.

Proof 9.2.3.5 ► Proof of Proposition 9.2.3.4

Item 1: Functoriality

Omitted.

Item 2: Interaction With Fibres

Clear.

Item 3: Construction Using the Internal Hom

Using the explicit formula for pullbacks of sets given in **??**, we see that the pullback

$$K' \times_{\mathbf{Hom}_{\mathsf{FibSets}(K')}((K,f),(K,f))} \mathbf{Hom}_{\mathsf{FibSets}(K')}((K,f),(X,f\circ\phi_X))$$

is given by

$$\left\{(k',h) \in \coprod_{k'\in K'} \mathsf{Sets}\big(f^{-1}(k'), \phi_X^{-1}\big(f^{-1}(k')\big)\big) \,\middle|\, \phi_X \circ h = \mathrm{id}_{f^{-1}(k')}\right\},$$

which is isomorphic to

$$\coprod_{k'\in K'} \big\{h \in \mathsf{Sets}\big(f^{-1}(k'), \phi_X^{-1}\big(f^{-1}(k')\big)\big) \,\big|\, \phi_X \circ h = \mathrm{id}_{f^{-1}(k')}\big\}.$$

We claim that

$$\big\{h \in \mathsf{Sets}\big(f^{-1}(k'), \phi_X^{-1}\big(f^{-1}(k')\big)\big) \,\big|\, \phi_X \circ h = \mathrm{id}_{f^{-1}(k')}\big\} \cong \prod_{k\in f^{-1}(k')} \phi_X^{-1}(k),$$

so that the pullback is indeed given by $\Pi_f(X)$. There are two cases:

1. If $f^{-1}(k') = \emptyset$, then there is only one map from $f^{-1}(k')$ to $\phi_X^{-1}\big(f^{-1}(k')\big)$ (the inclusion), so $\mathsf{Sets}\big(f^{-1}(k'), \phi_X^{-1}\big(f^{-1}(k')\big)\big) \cong \mathrm{pt}$. Since products indexed by the empty set are isomorphic to pt, the isomorphism follows.

2. Otherwise, by the condition $\phi_X \circ h = \mathrm{id}_{f^{-1}(k')}$, it follows that, for each $k \in f^{-1}(k')$, we must have

$$\phi_X(h(k)) = k,$$

and thus $h(k) \in \phi_X^{-1}(k)$. Therefore, a map from $f^{-1}(k')$ to $\phi_X^{-1}\big(f^{-1}(k')\big)$ consists of a choice of an element from $\phi_X^{-1}(k)$ for each $k \in f^{-1}(k')$, which is precisely given by an element of the product $\prod_{k\in f^{-1}(k')} \phi_X^{-1}(k)$, showing the bijection to be true.

Item 4: Internal Homs via Dependent Products

Indeed we have

$$\begin{aligned}
\Pi_{\phi_X}(\phi_X^*(Y)) &\stackrel{\text{def}}{=} \Pi_{\phi_X}(X \times_K Y)\\
&\stackrel{\text{def}}{=} \coprod_{k\in K} \prod_{x\in\phi_X^{-1}(k)} \mathrm{pr}_1^{-1}(x)\\
&\cong \coprod_{k\in K} \prod_{x\in\phi_X^{-1}(k)} \{y\in Y \mid \phi_X(x) = \phi_Y(y)\}\\
&\cong \coprod_{k\in K} \prod_{x\in\phi_X^{-1}(k)} \{y\in Y \mid k = \phi_Y(y)\}\\
&\cong \coprod_{k\in K} \prod_{x\in\phi_X^{-1}(k)} \phi_Y^{-1}(k)\\
&\cong \coprod_{k\in K} \mathsf{Sets}\big(\phi_X^{-1}(k), \phi_Y^{-1}(k)\big)\\
&\stackrel{\text{def}}{=} \mathbf{Hom}_{\mathsf{FibSets}(K)}(X, Y).
\end{aligned}$$

This finishes the proof.

9.2.4 Internal Homs

Let K be a set and let (X, ϕ_X) and (Y, ϕ_Y) be K-fibred sets.

Definition 9.2.4.1 ► Internal Hom of Fibred Sets

The **internal Hom of fibred sets from** (X, ϕ_X) **to** (Y, ϕ_Y) is the fibred set $\mathbf{Hom}_{\mathsf{FibSets}(K)}(X, Y)$ consisting of

- *The Underlying Set.* The set $\mathbf{Hom}_{\mathsf{FibSets}(K)}(X, Y)$ defined by

$$\mathbf{Hom}_{\mathsf{FibSets}(K)}(X, Y) \stackrel{\text{def}}{=} \coprod_{k\in K} \mathsf{Sets}\big(\phi_X^{-1}(k), \phi_Y^{-1}(k)\big);$$

- *The Fibration.* The map of sets[1]

$$\phi_{\mathbf{Hom}_{\mathsf{FibSets}(K)}(X,Y)} \colon \underbrace{\mathbf{Hom}_{\mathsf{FibSets}(K)}(X, Y)}_{\coprod_{k\in K} \mathsf{Sets}(\phi_X^{-1}(k), \phi_Y^{-1}(k))} \to K$$

  defined by sending a map $f \colon \phi_X^{-1}(k) \to \phi_Y^{-1}(k)$ to its index $k \in K$.

[1]The fibres of the internal **Hom** of $\mathsf{FibSets}(K)$ are precisely the sets $\mathsf{Sets}\left(\phi_X^{-1}(k), \phi_Y^{-1}(k)\right)$, i.e. we have

$$\phi_{\mathbf{Hom}_{\mathsf{FibSets}(K)}(X,Y)|k} \cong \mathsf{Sets}\left(\phi_X^{-1}(k), \phi_Y^{-1}(k)\right)$$

for each $k \in K$.

Proof 9.2.4.2 ▶ Proof of Definition 9.2.4.1

Omitted.

Proposition 9.2.4.3 ▶ Properties of Internal Homs of Fibred Sets

Let K be a set and let (X, ϕ_X) and (Y, ϕ_Y) be K-fibred sets.

1. *Functoriality.* Let (X, ϕ_X) and (Y, ϕ_Y) be K-fibred sets.

   (a) The assignment $X \mapsto \mathbf{Hom}_{\mathsf{FibSets}(K)}(X, Y)$ defines a functor

   $$\mathbf{Hom}_{\mathsf{FibSets}(K)}(X, -) \colon \mathsf{FibSets}(K) \to \mathsf{FibSets}(K).$$

   (b) The assignment $Y \mapsto \mathbf{Hom}_{\mathsf{FibSets}(K)}(X, Y)$ defines a functor

   $$\mathbf{Hom}_{\mathsf{FibSets}(K)}(-, Y) \colon \mathsf{FibSets}(K)^{\mathsf{op}} \to \mathsf{FibSets}(K).$$

   (c) The assignment $(X, Y) \mapsto \mathbf{Hom}_{\mathsf{FibSets}(K)}(X, Y)$ defines a functor

   $$\mathbf{Hom}_{\mathsf{FibSets}(K)}(-_1, -_2) \colon \mathsf{FibSets}(K)^{\mathsf{op}} \times \mathsf{FibSets}(K) \to \mathsf{FibSets}(K).$$

2. *Internal Homs via Dependent Products.* We have

   $$\mathbf{Hom}_{\mathsf{FibSets}(K)}(X, Y) \cong \Pi_{\phi_X}(\phi_X^*(Y)).$$

Proof 9.2.4.4 ▶ Proof of Proposition 9.2.4.3

Item 1: Functoriality

Omitted.

Item 2: Internal Homs via Dependent Products

This was proved in Item 4 of Proposition 9.2.3.4.

9.2.5 Adjointness for Fibred Sets

Let $f \colon K \to K'$ be a map of sets.

Proposition 9.2.5.1 ► Adjointness for Fibred Sets

We have a triple adjunction

$$(\Sigma_f \dashv f^* \dashv \Pi_f)\colon \quad \begin{tikzcd} \mathsf{FibSets}(K) \arrow[r, bend left, "\Sigma_f"] \arrow[r, bend right, "\Pi_f"'] & \mathsf{FibSets}(K') \arrow[l, "f^*"' description] \end{tikzcd}.$$

We offer two proofs. The first uses the corresponding adjunction for indexed sets (**??**) and the un/straightening equivalence together with its compatibility with dependent sums and products to "transfer" the adjunction to fibred sets, while the second is a direct proof.

Proof 9.2.5.2 ► First Proof of Proposition 9.2.5.1

The Adjunction $\Sigma_f \dashv f^*$

The adjunction

$$(\Sigma_f \dashv f^*)\colon \quad \begin{tikzcd} \mathsf{ISets}(K) \arrow[r, bend left, "\Sigma_f"] \arrow[r, phantom, "\perp"] & \mathsf{ISets}(K') \arrow[l, bend left, "f^*"] \end{tikzcd}$$

of **??** gives a unit and counit of the form

$$\begin{aligned} \eta &\colon \mathrm{id}_{\mathsf{ISets}(K)} \Longrightarrow \Sigma_f \circ f^*, \\ \epsilon &\colon f^* \circ \Sigma_f \Longrightarrow \mathrm{id}_{\mathsf{ISets}(K')}. \end{aligned}$$

With these in hand, we construct natural transformations

$$\begin{aligned} \eta' &\colon \mathrm{id}_{\mathsf{FibSets}(K)} \Longrightarrow \Sigma_f \circ f^*, \\ \epsilon' &\colon f^* \circ \Sigma_f \Longrightarrow \mathrm{id}_{\mathsf{FibSets}(K')} \end{aligned}$$

as follows:

1. *The Unit.* We define $\eta'\colon \mathrm{id}_{\mathsf{FibSets}(K)} \Longrightarrow \Sigma_f \circ f^*$ as the pasting of the

diagram

$$
\begin{tikzcd}
 & & \mathsf{FibSets}(K') \arrow[drr, "f^*"] & & \\
\mathsf{FibSets}(K) \arrow[urr, "\Sigma_f"] \arrow[r, phantom, "(1)"] & & \mathsf{ISets}(K') \arrow[u, "\mathsf{St}_{K'}"] \arrow[dr, "f^*"] \arrow[d, phantom, "\Uparrow\eta"] \arrow[r, phantom, "(2)"] & & \mathsf{FibSets}(K) \\
 & \mathsf{ISets}(K) \arrow[ul, "\mathsf{St}_K"] \arrow[ur, "\Sigma_f"] \arrow[rr, "\mathrm{id}_{\mathsf{ISets}(K)}"] \arrow[l, phantom, "(3)"] & {} \arrow[d, phantom, "(5)"] & \mathsf{ISets}(K) \arrow[ur, "\mathsf{St}_K"] \arrow[r, phantom, "(4)"] & {} \\
\mathsf{FibSets}(K) \arrow[uu, "\mathrm{id}_{\mathsf{FibSets}(K)}"] \arrow[ur, "\mathsf{Un}_K"] \arrow[rrrr, "\mathrm{id}_{\mathsf{FibSets}(K)}"'] & & {} & & \mathsf{FibSets}(K), \arrow[uu, "\mathrm{id}_{\mathsf{FibSets}(K)}"'] \arrow[ul, "\mathsf{Un}_K"']
\end{tikzcd}
$$

where:

(a) Subdiagram (1) commutes by **??** of **??**.

(b) Subdiagram (2) commutes by **??** of **??**.

(c) Subdiagram (3) commutes by **??**.

(d) Subdiagram (4) commutes by **??**.

(e) Subdiagram (5) commutes by unitality of composition.

2. *The Counit.* We define $\epsilon' \colon f^* \circ \Sigma_f \Longrightarrow \mathrm{id}_{\mathsf{FibSets}(K')}$ as the pasting of the diagram

$$
\begin{tikzcd}
\mathsf{FibSets}(K') \arrow[rrrr, "\mathrm{id}_{\mathsf{FibSets}(K')}"] \arrow[dr, "\mathsf{Un}_K"] \arrow[dd, "\mathrm{id}_{\mathsf{FibSets}(K')}"'] & & {} \arrow[d, phantom, "(1)"] & & \mathsf{FibSets}(K') \arrow[dl, "\mathsf{Un}_K"'] \arrow[dd, "\mathrm{id}_{\mathsf{FibSets}(K')}"] \\
{} \arrow[r, phantom, "(2)"] & \mathsf{ISets}(K') \arrow[rr, "\mathrm{id}_{\mathsf{ISets}(K)}"] \arrow[dl, "\mathsf{St}_{K'}"'] \arrow[dr, "f^*"] & {} & \mathsf{ISets}(K') \arrow[dr, "\mathsf{St}_{K'}"] \arrow[r, phantom, "(3)"] & {} \\
\mathsf{FibSets}(K') \arrow[drr, "f^*"'] \arrow[r, phantom, "(4)"] & & \mathsf{ISets}(K) \arrow[ur, "\Sigma_f"'] \arrow[d, "\mathsf{St}_K"] \arrow[u, phantom, "\Uparrow\epsilon"] \arrow[r, phantom, "(5)"] & & \mathsf{FibSets}(K') \\
 & & \mathsf{FibSets}(K) \arrow[urr, "\Sigma_f"'] & &
\end{tikzcd}
$$

where:

(a) Subdiagram (1) commutes by unitality of composition.

(b) Subdiagram (2) commutes by **??**.

(c) Subdiagram (3) commutes by **??**.

(d) Subdiagram (4) commutes by **??** of **??**.

(e) Subdiagram (5) commutes by **??** of **??**.

Next, we prove the left triangle identity,

$$
\begin{tikzcd}
& \mathsf{FibSets}(K') \arrow[rr, "\mathrm{id}_{\mathsf{FibSets}(K')}"] \arrow[dr, "f^*"] & & \mathsf{FibSets}(K') \\
\mathsf{FibSets}(K) \arrow[ur, "\Sigma_f"] \arrow[rr, "\mathrm{id}_{\mathsf{FibSets}(K)}"'] \arrow[r, phantom, "\Uparrow\eta"] & & \mathsf{FibSets}(K) \arrow[ur, "\Sigma_f"'] \arrow[u, phantom, "\Uparrow\epsilon"] &
\end{tikzcd}
=
\begin{tikzcd}
& \mathsf{FibSets}(K') \arrow[rr, "\mathrm{id}_{\mathsf{FibSets}(K')}"] & & \mathsf{FibSets}(K'), \\
\mathsf{FibSets}(K) \arrow[ur, "\Sigma_f"] \arrow[rr, "\mathrm{id}_{\mathsf{FibSets}(K)}"'] & & \mathsf{FibSets}(K) \arrow[ur, "\Sigma_f"'] \arrow[ul, phantom, "\Leftarrow\mathrm{id}_{\Sigma_f}"] &
\end{tikzcd}
$$

whose left side in our case looks like this:

$$
\begin{tikzcd}
& & \mathsf{FibSets}(K') \arrow[rrr, "\mathrm{id}_{\mathsf{FibSets}(K')}"] \arrow[drr, "f^*"] & & & \mathsf{FibSets}(K') \arrow[rrrr, "\mathrm{id}_{\mathsf{FibSets}(K')}"] \arrow[dd, "\mathrm{id}_{\mathsf{FibSets}(K')}"'] \arrow[dr, "\mathsf{Un}_K"] & & & & \mathsf{FibSets}(K') \arrow[dl, "\mathsf{Un}_K"'] \arrow[dd, "\mathrm{id}_{\mathsf{FibSets}(K')}"] \\
\mathsf{FibSets}(K) \arrow[urr, "\Sigma_f"] & & \mathsf{ISets}(K') \arrow[u, "\mathsf{St}_{K'}"] \arrow[dr, "f^*"] \arrow[d, phantom, "\Uparrow\eta"] & & \mathsf{FibSets}(K) & & \mathsf{ISets}(K') \arrow[rr, "\mathrm{id}_{\mathsf{ISets}(K)}"] \arrow[dl, "\mathsf{St}_{K'}"'] \arrow[dr, "f^*"] & & \mathsf{ISets}(K') \arrow[dr, "\mathsf{St}_{K'}"] & \\
& \mathsf{ISets}(K) \arrow[ul, "\mathsf{St}_K"] \arrow[ur, "\Sigma_f"] \arrow[rr, "\mathrm{id}_{\mathsf{ISets}(K)}"] & & \mathsf{ISets}(K) \arrow[ur, "\mathsf{St}_K"] & & \mathsf{FibSets}(K') \arrow[drr, "f^*"'] & & \mathsf{ISets}(K) \arrow[ur, "\Sigma_f"] \arrow[u, phantom, "\Uparrow\epsilon"] \arrow[d, "\mathsf{St}_K"] & & \mathsf{FibSets}(K'). \\
\mathsf{FibSets}(K) \arrow[uu, "\mathrm{id}_{\mathsf{FibSets}(K)}"] \arrow[ur, "\mathsf{Un}_K"] \arrow[rrrr, "\mathrm{id}_{\mathsf{FibSets}(K)}"'] & & & & \mathsf{FibSets}(K) \arrow[ul, "\mathsf{Un}_K"'] \arrow[uu, "\mathrm{id}_{\mathsf{FibSets}(K)}"'] \arrow[rrr, "\mathrm{id}_{\mathsf{FibSets}(K)}"'] & & & \mathsf{FibSets}(K) \arrow[urr, "\Sigma_f"'] & &
\end{tikzcd}
$$

It can be rearranged into

$$
\begin{tikzcd}
& & \mathsf{FibSets}(K') \arrow[rrr, "\mathrm{id}_{\mathsf{FibSets}(K')}"] & & & \mathsf{FibSets}(K') \arrow[rrrr, "\mathrm{id}_{\mathsf{FibSets}(K')}"] \arrow[d, phantom, "(1)"] & & & & \mathsf{FibSets}(K') \arrow[dl, "\mathsf{Un}_K"'] \arrow[dd, "\mathrm{id}_{\mathsf{FibSets}(K')}"] \\
\mathsf{FibSets}(K) \arrow[urr, "\Sigma_f"] & & \mathsf{ISets}(K') \arrow[u, "\mathsf{St}_{K'}"] \arrow[dr, "f^*"] \arrow[d, phantom, "\Uparrow\eta"] \arrow[rrrr, "\mathrm{id}_{\mathsf{ISets}(K')}"] & & & \arrow[d, phantom, "(2)"] & \mathsf{ISets}(K') \arrow[rr, "\mathrm{id}_{\mathsf{ISets}(K)}"] \arrow[dr, "f^*"] & & \mathsf{ISets}(K') \arrow[dr, "\mathsf{St}_{K'}"] & \\
& \mathsf{ISets}(K) \arrow[ul, "\mathsf{St}_K"] \arrow[ur, "\Sigma_f"] \arrow[rr, "\mathrm{id}_{\mathsf{ISets}(K)}"] & & \mathsf{ISets}(K) \arrow[rrrr, "\mathrm{id}_{\mathsf{ISets}(K')}"] \arrow[dr, phantom, "(3)"] & & & & \mathsf{ISets}(K) \arrow[ur, "\Sigma_f"] \arrow[u, phantom, "\Uparrow\epsilon"] \arrow[d, "\mathsf{St}_K"] & & \mathsf{FibSets}(K'), \\
\mathsf{FibSets}(K) \arrow[uu, "\mathrm{id}_{\mathsf{FibSets}(K)}"] \arrow[ur, "\mathsf{Un}_K"] \arrow[rrrr, "\mathrm{id}_{\mathsf{FibSets}(K)}"'] & & & & \mathsf{FibSets}(K) \arrow[rrr, "\mathrm{id}_{\mathsf{FibSets}(K)}"'] & & & \mathsf{FibSets}(K) \arrow[urr, "\Sigma_f"'] & &
\end{tikzcd}
$$

where:

1. Subdiagram (1) commutes by **??**.
2. Subdiagram (2) commutes by unitality of composition.
3. Subdiagram (3) commutes by **??**.

And then, it can be rearranged into

$$\begin{tikzcd}
& & \mathsf{FibSets}(K') \arrow[rrrr, "\mathrm{id}_{\mathsf{FibSets}(K')}"] & & & & \mathsf{FibSets}(K') \arrow[dl, "\mathsf{Un}_K"'] \arrow[dd, "\mathrm{id}_{\mathsf{FibSets}(K')}"] \\
\mathsf{FibSets}(K) \arrow[urr, "\Sigma_f"] & & \mathsf{ISets}(K') \arrow[u, "\mathsf{St}_{K'}"] \arrow[rrr, "\mathrm{id}_{\mathsf{ISets}(K)}"] \arrow[drr, "f^*"] & & & \mathsf{ISets}(K') \arrow[dr, "\mathsf{St}_{K'}"] & \\
& \mathsf{ISets}(K) \arrow[ul, "\mathsf{St}_K"'] \arrow[ur, "\Sigma_f"] \arrow[rrr, "\mathrm{id}_{\mathsf{ISets}(K)}"] & & & \mathsf{ISets}(K) \arrow[ur, "\Sigma_f"] \arrow[d, "\mathsf{St}_K"] & & \mathsf{FibSets}(K'), \\
\mathsf{FibSets}(K) \arrow[uu, "\mathrm{id}_{\mathsf{FibSets}(K)}"] \arrow[ur, "\mathsf{Un}_K"] \arrow[rrrr, "\mathrm{id}_{\mathsf{FibSets}(K)}"'] & & & & \mathsf{FibSets}(K) \arrow[urr, "\Sigma_f"'] & &
\arrow[from=3-3, to=2-3, phantom, "\Uparrow\eta"]
\arrow[from=3-5, to=2-5, phantom, "\Uparrow\epsilon"]
\end{tikzcd}$$

which by the left triangle identity for (η, ϵ), becomes

$$\begin{tikzcd}
& & \mathsf{FibSets}(K') \arrow[rrrr, "\mathrm{id}_{\mathsf{FibSets}(K')}"] & & & & \mathsf{FibSets}(K') \arrow[dl, "\mathsf{Un}_K"'] \arrow[dd, "\mathrm{id}_{\mathsf{FibSets}(K')}"] \\
\mathsf{FibSets}(K) \arrow[urr, "\Sigma_f"] & & \mathsf{ISets}(K') \arrow[u, "\mathsf{St}_{K'}"] \arrow[rrr, "\mathrm{id}_{\mathsf{ISets}(K)}"] \arrow[drr, phantom, "\mathrm{id}"] & & & \mathsf{ISets}(K') \arrow[dr, "\mathsf{St}_{K'}"] & \\
& \mathsf{ISets}(K) \arrow[ul, "\mathsf{St}_K"'] \arrow[ur, "\Sigma_f"] \arrow[rrr, "\mathrm{id}_{\mathsf{ISets}(K)}"] & & & \mathsf{ISets}(K) \arrow[ur, "\Sigma_f"] \arrow[d, "\mathsf{St}_K"] & & \mathsf{FibSets}(K'), \\
\mathsf{FibSets}(K) \arrow[uu, "\mathrm{id}_{\mathsf{FibSets}(K)}"] \arrow[ur, "\mathsf{Un}_K"] \arrow[rrrr, "\mathrm{id}_{\mathsf{FibSets}(K)}"'] & & & & \mathsf{FibSets}(K) \arrow[urr, "\Sigma_f"'] & &
\end{tikzcd}$$

finishing the proof of the left triangle identity. The proof of the right triangle identity is similar, and is thus omitted.

The Adjunction $f^* \dashv \Pi_f$

This proof is similar to the proof of the adjunction $\Sigma_f \dashv f^*$, and is thus omitted. □

We proceed to the direct proof of Proposition 9.2.5.1.

PROOF 9.2.5.3 ▶ SECOND PROOF OF PROPOSITION 9.2.5.1

The Adjunction $\Sigma_f \dashv f^*$

We claim there's a bijection

$$\mathrm{Hom}_{\mathsf{FibSets}(K')}(\Sigma_f(X), Y) \cong \mathrm{Hom}_{\mathsf{FibSets}(K)}(X, f^*(Y)),$$

natural in $(X, \phi_X) \in \mathrm{Obj}(\mathsf{FibSets}(K))$ and $(Y, \phi_Y) \in \mathrm{Obj}(\mathsf{FibSets}(K'))$:

- *Map I.* We define a map

$$\Phi_{X,Y} \colon \mathrm{Hom}_{\mathsf{FibSets}(K')}(\Sigma_f(X), Y) \to \mathrm{Hom}_{\mathsf{FibSets}(K)}(X, f^*(Y)),$$

by sending a morphism

$$\xi\colon \Sigma_f(X) \to Y, \qquad \begin{tikzcd} X \arrow[rr, "\xi"] \arrow[dr, "\phi_X"'] & & Y \arrow[ddl, "\phi_Y"] \\ & K \arrow[d, "f"'] & \\ & K' & \end{tikzcd}$$

of K'-fibred sets to the morphism

$$\xi^\dagger\colon X \to f^*(Y), \qquad \begin{tikzcd} X \arrow[rr, "\xi^\dagger"] \arrow[dr, "\phi_X"'] & & K \times_{K'} Y \arrow[dl, "\mathrm{pr}_1"] \\ & K' & \end{tikzcd}$$

of K-fibred sets given by the dashed morphism in the diagram

$$\begin{tikzcd} X \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left, "\xi"] \arrow[ddr, bend right, "\phi_X"'] & & \\ & K \times_{K'} Y \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "\phi_Y"] \\ & K \arrow[r, "f"'] & K'. \end{tikzcd}$$

- *Map II.* We define a map

  $$\Psi_{X,Y}\colon \mathrm{Hom}_{\mathsf{FibSets}(K)}(X, f^*(Y)) \to \mathrm{Hom}_{\mathsf{FibSets}(K')}\big(\Sigma_f(X), Y\big),$$

  given by sending a map

  $$\xi\colon X \to f^*(Y), \qquad \begin{tikzcd} X \arrow[rr, "\xi"] \arrow[dr, "\phi_X"'] & & K \times_{K'} Y \arrow[dl, "\mathrm{pr}_1"] \\ & K' & \end{tikzcd}$$

  of K'-fibred sets to the map

  $$\xi^\dagger\colon \Sigma_f(X) \to Y, \qquad \begin{tikzcd} X \arrow[rr, "\xi^\dagger"] \arrow[dr, "\phi_X"'] & & Y \arrow[ddl, "\phi_Y"] \\ & K \arrow[d, "f"'] & \\ & K' & \end{tikzcd}$$

of K-fibred sets given by

$$\xi^\dagger \stackrel{\text{def}}{=} \mathrm{pr}_2 \circ \xi,$$

where we indeed have

$$\begin{aligned}
\phi_Y \circ (\mathrm{pr}_2 \circ \xi) &= (\phi_Y \circ \mathrm{pr}_2) \circ \xi \\
&= (f \circ \mathrm{pr}_1) \circ \xi && \text{(by the pullback square of } K \times_{K'} Y\text{)} \\
&= f \circ (\mathrm{pr}_1 \circ \xi) \\
&= f \circ \phi_X. && \text{(since } \xi \text{ is a morphism of } K'\text{-fibred sets)}
\end{aligned}$$

- *Naturality I*. We need to show that, given a morphism

$$\alpha \colon (X, \phi_X) \to (X', \phi_{X'})$$

of K-fibred sets, the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathsf{FibSets}(K')}(\Sigma_f(X'), Y) \arrow[r, "\Phi_{X',Y}"] \arrow[d, "\Sigma_f(\alpha)^*"'] & \mathrm{Hom}_{\mathsf{FibSets}(K)}(X', f^*(Y)), \arrow[d, "\alpha^*"] \\
\mathrm{Hom}_{\mathsf{FibSets}(K')}(\Sigma_f(X), Y) \arrow[r, "\Phi_{X,Y}"'] & \mathrm{Hom}_{\mathsf{FibSets}(K)}(X, f^*(Y))
\end{tikzcd}$$

commutes. Indeed, given a morphism

$$\xi \colon \Sigma_f(X') \to Y, \qquad \begin{tikzcd}
X' \arrow[rr, "\xi"] \arrow[dr, "\phi_{X'}"'] & & Y \arrow[ddl, "\phi_Y"] \\
& K \arrow[d, "f"'] & \\
& K' &
\end{tikzcd}$$

of K'-fibred-sets, the map $\Phi_{X',Y}(\xi) \circ \alpha$ is the composition, coloured in vermillion, of the dashed arrow with α in the diagram

$$\begin{tikzcd}
X \arrow[dr, dashed, "\alpha"] \arrow[ddrr, dashed, "\exists!"] \arrow[ddrrr, bend left, "\xi\circ\alpha"] \arrow[dddrr, bend right, "\phi_{X'}\circ\alpha"'] & & & \\
& X' \arrow[drr, bend left, "\xi"] \arrow[dr, dashed, "\exists!"] \arrow[ddr, bend right, "\phi_{X'}"'] & & \\
& & K \times_{K'} Y \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "\phi_Y"] \\
& & K \arrow[r, "f"'] & K',
\end{tikzcd}$$

while $\Phi_{X,Y}(\xi \circ \Sigma_f(\alpha))$ is given by the dashed arrow, coloured in blue. Since both the blue arrow and the vermillion arrow make the outer pullback diagram commute, it follows from the universal property of the pullback that they must be equal, i.e. that

$$\Phi_{X',Y}(\xi) \circ \alpha = \Phi_{X,Y}(\xi \circ \Sigma_f(\alpha)),$$

showing that the naturality diagram above indeed commutes.

- *Naturality II*. We need to show that, given a morphism

$$\beta \colon (Y, \phi_Y) \to (Y', \phi_{Y'})$$

of K-fibred sets, the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathsf{FibSets}(K')}(\Sigma_f(X), Y) \arrow[r, "\Phi_{X,Y}"] \arrow[d, "\beta_*"'] & \mathrm{Hom}_{\mathsf{FibSets}(K)}(X, f^*(Y)), \arrow[d, "f^*(\beta)_*"] \\
\mathrm{Hom}_{\mathsf{FibSets}(K')}(\Sigma_f(X), Y') \arrow[r, "\Phi_{X,Y'}"'] & \mathrm{Hom}_{\mathsf{FibSets}(K)}(X, f^*(Y'))
\end{tikzcd}$$

commutes. Indeed, given a morphism

$$\xi \colon \Sigma_f(X') \to Y, \qquad \begin{tikzcd}
X' \arrow[rr, "\xi"] \arrow[dr, "\phi_{X'}"'] & & Y \arrow[ddl, "\phi_Y"] \\
& K \arrow[d, "f"'] & \\
& K' &
\end{tikzcd}$$

of K'-fibred-sets, the map $f^*(\beta) \circ \Phi_{X,Y}(\xi)$ is the composition, coloured in vermillion, of the dashed arrow from X to $K \times_{K'} Y$ with the dashed arrow from $K \times_{K'} Y$ to $K \times_{K'} Y'$ in the diagram

$$
\begin{tikzcd}
X \arrow[dr, dashed, "\exists!"] \arrow[ddrr, dashed, "\exists!"] \arrow[drrr, bend left, "\xi"] \arrow[dddr, bend right, "\phi_X"'] \\
& K \times_{K'} Y \arrow[rr] \arrow[dd] \arrow[dr, dashed] \arrow[drr, phantom, "\lrcorner"] & & Y \arrow[dr, "\beta"] \arrow[dd, "\phi_Y"] \\
& & K \times_{K'} Y' \arrow[rr] \arrow[dd] \arrow[drr, phantom, "\lrcorner"] & & Y' \arrow[dd, "\phi_{Y'}"] \\
& K \arrow[rr, "f"] \arrow[dr, equal] & & K \arrow[dr, equal] \\
& & K \arrow[rr, "f"'] & & K',
\end{tikzcd}
$$

while $\Phi_{X,Y'}(\beta \circ \xi)$ is given by the dashed arrow from X to $K \times_{K'} Y'$, coloured in blue. Since both the blue arrow and the vermillion arrow make the outer pullback diagram for $K \times_{K'} Y'$ commute, it follows from the universal property of the pullback that they must be equal, i.e. that

$$f^*(\beta) \circ \Phi_{X,Y}(\xi) = \Phi_{X,Y'}(\beta \circ \xi),$$

showing that the naturality diagram above indeed commutes.

- *Invertibility I.* We claim that

$$\Psi_{X,Y} \circ \Phi_{X,Y} = \mathrm{id}_{\mathrm{Hom}_{\mathsf{FibSets}(K')}(\Sigma_f(X),Y)}.$$

Indeed, $\Phi_{X,Y}$ sends a map

$$
\xi \colon \Sigma_f(X) \to Y, \qquad
\begin{tikzcd}
X \arrow[rr, "\xi"] \arrow[dr, "\phi_X"'] & & Y \arrow[ddl, "\phi_Y"] \\
& K \arrow[d, "f"'] \\
& K'
\end{tikzcd}
$$

of K'-fibred sets to the dashed morphism in the diagram

$$\begin{tikzcd}
X \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left, "\xi"] \arrow[ddr, bend right, "\phi_X"'] & & \\
& K \times_{K'} Y \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "\phi_Y"] \\
& K \arrow[r, "f"'] & K',
\end{tikzcd}$$

and $\Psi_{X,Y}$ then postcomposes that map with pr_2, which, by the commutativity of the diagram above, is ξ again, showing the claimed equality to be true.

- *Invertibility II.* We claim that

$$\Phi_{X,Y} \circ \Psi_{X,Y} = \mathrm{id}_{\mathrm{Hom}_{\mathsf{FibSets}(K)}(X, f^*(Y))}.$$

Indeed, $\Psi_{X,Y}$ sends a map

$$\xi \colon X \to f^*(Y), \qquad \begin{tikzcd}
X \arrow[rr, "\xi"] \arrow[dr, "\phi_X"'] & & K \times_{K'} Y \arrow[dl, "\mathrm{pr}_1"] \\
& K' &
\end{tikzcd}$$

of K'-fibred sets to $\mathrm{pr}_2 \circ \xi$, which is then sent by $\Phi_{X,Y}$ to the dashed morphism in the diagram

$$\begin{tikzcd}
X \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left, "\mathrm{pr}_2 \circ \xi"] \arrow[ddr, bend right, "\phi_X"'] & (\dagger) & \\
& K \times_{K'} Y \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "\phi_Y"] \\
& K \arrow[r, "f"'] & K',
\end{tikzcd}$$

which, by the commutativity of the subdiagram marked with $(\dagger)$, is given by ξ again, showing the claimed equality to be true.

The Adjunction $f^* \dashv \Pi_f$

We claim there's a bijection

$$\mathrm{Hom}_{\mathsf{FibSets}(K)}(f^*(X), Y), \cong \mathrm{Hom}_{\mathsf{FibSets}(K')}\big(X, \Pi_f(Y)\big)$$

natural in $(X, \phi_X) \in \mathrm{Obj}(\mathsf{FibSets}(K'))$ and $(Y, \phi_Y) \in \mathrm{Obj}(\mathsf{FibSets}(K))$:

1. *Map I.* We define a map

   $$\Phi_{X,Y} \colon \mathrm{Hom}_{\mathsf{FibSets}(K)}(f^*(X), Y) \to \mathrm{Hom}_{\mathsf{FibSets}(K')}\big(X, \Pi_f(Y)\big)$$

   defined as follows. Given a morphism

   $$\xi \colon f^*(X) \to Y, \qquad \begin{tikzcd} K \times_{K'} X \arrow[rr, "\xi"] \arrow[dr, "\mathrm{pr}_1"'] & & Y \arrow[dl, "\phi_Y"] \\ & K & \end{tikzcd}$$

   of K-fibred sets, where

   $$\begin{aligned} f^*(X) &\overset{\text{def}}{=} K \times_{K'} X \\ &\overset{\text{def}}{=} \{(k, x) \in K \times X \mid f(k) = \phi_X(x)\}, \end{aligned}$$

   we construct a morphism

   $$\xi^\dagger \colon X \to \Pi_f(Y), \qquad \begin{tikzcd} X \arrow[rr, "\xi^\dagger"] \arrow[dr, "\phi_X"'] & & \Pi_f(Y) \arrow[dl, "\Pi_f(\phi_Y)"] \\ & K' & \end{tikzcd}$$

   of K'-fibred sets, where

   $$\Pi_f(Y) \overset{\text{def}}{=} \coprod_{k' \in K'} \prod_{k \in f^{-1}(k')} \phi_Y^{-1}(k),$$

   by defining

   $$\xi^\dagger(x) \overset{\text{def}}{=} (\xi(k, x))_{k \in f^{-1}(\phi_X(x))}$$

   for each $x \in X$. There are two things to be checked here:

   - We have $\xi(k, x) \in \phi_Y^{-1}(\phi_X(x))$ since $\phi_Y(\xi(k, x)) = \phi_X(x)$ as ξ is a morphism of K-fibred sets.

- The map $\xi^\dagger$ is indeed a morphism of K'-fibred sets, i.e. we have

  $$\Pi_f(\phi_Y) \circ \xi^\dagger = \phi_X,$$

  since

  $$[\Pi_f(\phi_Y)]\Big((\xi(k,x))_{k\in f^{-1}(\phi_X(x))}\Big) = \phi_X(x)$$

  for each $x \in X$.

2. *Map II*. We define a map

$$\Psi_{X,Y} \colon \mathrm{Hom}_{\mathsf{FibSets}(K')}\big(X, \Pi_f(Y)\big) \to \mathrm{Hom}_{\mathsf{FibSets}(K)}(f^*(X), Y)$$

as follows. Given a morphism

$$\xi \colon X \to \Pi_f(Y), \qquad \begin{tikzcd} X \arrow[rr, "\xi"] \arrow[dr, "\phi_X"'] & & \Pi_f(Y) \arrow[dl, "\Pi_f(\phi_Y)"] \\ & K' & \end{tikzcd}$$

of K'-fibred sets, where

$$\Pi_f(Y) \overset{\text{def}}{=} \coprod_{k'\in K'} \prod_{k\in f^{-1}(k')} \phi_Y^{-1}(k),$$

we construct a morphism

$$\xi^\dagger \colon f^*(X) \to Y, \qquad \begin{tikzcd} K \times_{K'} X \arrow[rr, "\xi^\dagger"] \arrow[dr, "\mathrm{pr}_1"'] & & Y \arrow[dl, "\phi_Y"] \\ & K & \end{tikzcd}$$

of K-fibred sets, where

$$\begin{aligned} f^*(X) &\overset{\text{def}}{=} K \times_{K'} X \\ &\overset{\text{def}}{=} \{(k,x) \in K \times X \mid f(k) = \phi_X(x)\}, \end{aligned}$$

by defining

$$\xi^\dagger(k,x) \overset{\text{def}}{=} \xi(x)_k$$

for each $(k,x) \in f^*(X)$, where $\xi(x)_k$ is the kth component of $\xi(x) = (y_k)_{k\in f^{-1}(k')}$. We also need to check that $\xi^\dagger$ is a morphism of K-fibred sets, i.e. that

$$\phi_Y \circ \xi^\dagger = \mathrm{pr}_1,$$

or

$$\phi_Y\Big(\xi^\dagger(k,x)\Big) = k,$$

for each $(k,x) \in f^*(X)$, which is clear, since $\xi^\dagger(k,x) \in \phi_Y^{-1}(k)$ by definition.

3. *Naturality I.* We need to show that, given a morphism

$$\alpha\colon (X,\phi_X) \to (X',\phi_{X'})$$

of K'-fibred sets, the diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathsf{FibSets}(K')}(f^*(X'),Y) \arrow[r, "\Phi_{X',Y}"] \arrow[d, "f^*(\alpha)^*"'] & \mathrm{Hom}_{\mathsf{FibSets}(K)}(X',\Pi_f(Y)) \arrow[d, "\alpha^*"] \\ \mathrm{Hom}_{\mathsf{FibSets}(K')}(f^*(X),Y) \arrow[r, "\Phi_{X,Y}"'] & \mathrm{Hom}_{\mathsf{FibSets}(K)}(X,\Pi_f(Y)) \end{tikzcd}$$

commutes. Indeed, given a morphism $\xi\colon f^*(X') \to Y$ of K'-fibred sets, we have

$$\begin{aligned} [[\Phi_{X,Y} \circ f^*(\alpha)](\xi)](x) &\overset{\text{def}}{=} [\Phi_{X,Y}(\xi \circ f^*(\alpha))](x) \\ &\overset{\text{def}}{=} ([\xi \circ f^*(\alpha)](k,x))_{k\in f^{-1}(\phi_X(x))} \\ &\overset{\text{def}}{=} (\xi(k,\alpha(x)))_{k\in f^{-1}(\phi_X(x))} \\ &\overset{\text{def}}{=} \alpha^*\Big((\xi(k,x))_{k\in f^{-1}(\phi_X(x))}\Big) \\ &\overset{\text{def}}{=} \alpha^*\Big(\xi^\dagger(x)\Big) \\ &\overset{\text{def}}{=} [[\alpha^* \circ \Phi_{X,Y}](\xi)](x) \end{aligned}$$

for each $x \in X$.

4. *Naturality II.* We need to show that, given a morphism

$$\beta\colon (Y,\phi_Y) \to (Y',\phi_{Y'})$$

of K-fibred sets, the diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathsf{FibSets}(K')}(f^*(X),Y) \arrow[r, "\Phi_{X,Y}"] \arrow[d, "\beta_*"'] & \mathrm{Hom}_{\mathsf{FibSets}(K)}(X,\Pi_f(Y)) \arrow[d, "\Pi_f(\beta)_*"] \\ \mathrm{Hom}_{\mathsf{FibSets}(K')}(f^*(X),Y') \arrow[r, "\Phi_{X,Y'}"'] & \mathrm{Hom}_{\mathsf{FibSets}(K)}(X,\Pi_f(Y')) \end{tikzcd}$$

commutes. Indeed, given a morphism $\xi\colon X \to \Pi_f(Y)$ of K-fibred sets, we have

$$\begin{aligned}
[[\Phi_{X,Y'} \circ \beta_*](\xi)](x) &\overset{\text{def}}{=} [\Phi_{X,Y'}(\beta \circ \xi)](x)\\
&\overset{\text{def}}{=} [\Phi_{X,Y'}(\beta \circ \xi)](x)\\
&\overset{\text{def}}{=} ([\beta \circ \xi](k,x))_{k\in f^{-1}(\phi_X(x))}\\
&\overset{\text{def}}{=} (\beta(\xi(k,x)))_{k\in f^{-1}(\phi_X(x))}\\
&\overset{\text{def}}{=} \Pi_f(\beta)_*\Big((\xi(k,x))_{k\in f^{-1}(\phi_X(x))}\Big)\\
&\overset{\text{def}}{=} [\Pi_f(\beta)_* \circ \xi^\dagger](x)\\
&\overset{\text{def}}{=} [\Pi_f(\beta)_* \circ \Phi_{X,Y'}(\xi)](x)
\end{aligned}$$

for each $x \in X$.

5. *Invertibility I.* We claim that

$$\Psi_{X,Y} \circ \Phi_{X,Y} = \mathrm{id}_{\mathrm{Hom}_{\mathsf{FibSets}(K)}(f^*(X),Y)}.$$

Indeed, given a morphism $\xi\colon f^*(X') \to Y$ of K'-fibred sets, we have

$$\begin{aligned}
[[\Psi_{X,Y} \circ \Phi_{X,Y}](\xi)](k,x) &\overset{\text{def}}{=} [\Psi_{X,Y}(\Phi_{X,Y}(\xi))](k,x)\\
&\overset{\text{def}}{=} ([\Phi_{X,Y}(\xi)](x))_k\\
&\overset{\text{def}}{=} \Big((\xi(k_1,x))_{k_1\in f^{-1}(\phi_X(x))}\Big)_k\\
&\overset{\text{def}}{=} \xi(k,x)
\end{aligned}$$

for each $(k,x) \in f^*(X)$, and thus the stated equality follows.

6. *Invertibility II.* We claim that

$$\Phi_{X,Y} \circ \Psi_{X,Y} = \mathrm{id}_{\mathrm{Hom}_{\mathsf{FibSets}(K')}(X,\Pi_f(Y))}.$$

Indeed, given a morphism $\xi\colon X \to \Pi_f(Y)$ of K-fibred sets, write

$$\xi(x) = (y_k)_{k\in f^{-1}(k'_x)}.$$

We then have

$$
\begin{aligned}
[[\Phi_{X,Y} \circ \Psi_{X,Y}](\xi)](x) &\overset{\text{def}}{=} [\Phi_{X,Y}(\Psi_{X,Y}(\xi))](x) \\
&\overset{\text{def}}{=} ([\Psi_{X,Y}(\xi)](k_1, x))_{k_1 \in f^{-1}(\phi_X(x))} \\
&\overset{\text{def}}{=} \left((\xi(x))_{k_1}\right)_{k_1 \in f^{-1}(\phi_X(x))} \\
&\overset{\text{def}}{=} \left(\left((y_k)_{k \in f^{-1}(k'_x)}\right)_{k_1}\right)_{k_1 \in f^{-1}(\phi_X(x))} \\
&\overset{\text{def}}{=} (y_{k_1})_{k_1 \in f^{-1}(\phi_X(x))} \\
&= (y_{k_1})_{k_1 \in f^{-1}(k'_x)} \\
&= (y_k)_{k \in f^{-1}(k'_x)} \\
&\overset{\text{def}}{=} \xi(x)
\end{aligned}
$$

for each $x \in X$, where the equality $\phi_X(x) = k'_x$ follows from the fact that ξ is a morphism of K'-fibred sets. Thus the stated equality follows.

This finishes the proof. ∎

Appendices

9.A Other Chapters

Chapter 10

Un/Straightening for Indexed and Fibred Sets

This chapter contains a discussion of the un/straightening equivalence in the context of sets, as well as a general discussion of indexed and fibred sets. In particular, it contains:

1. A discussion of indexed sets (i.e. functors $K_{\mathsf{disc}} \to \mathsf{Sets}$ with K a set), constructions with them like dependent sums and dependent products, and their properties (**????**);

2. A discussion of fibred sets (i.e. maps of sets $X \to K$), constructions with them like dependent sums and dependent products, and their properties (**????**);

3. A discussion of the un/straightening equivalence for indexed and fibred sets (Section 10.1).

Contents

10.1 Un/Straightening for Indexed and Fibred Sets

10.1.1 Straightening for Fibred Sets

Let K be a set and let (X, ϕ) be a K-fibred set.

Definition 10.1.1.1 ▶ The Straightening of a Fibred Set

The **straightening of** (X, ϕ) is the K-indexed set

$$\mathsf{St}_K(X, \phi)\colon K_{\mathsf{disc}} \to \mathsf{Sets}$$

defined by

$$\mathsf{St}_K(X, \phi)_x \overset{\text{def}}{=} \phi^{-1}(x)$$

for each $x \in K$.

Proposition 10.1.1.2 ▶ Properties of Straightening for Fibred Sets

Let K be a set.

1. *Functoriality.* The assignment $(X, \phi) \mapsto \mathsf{St}_K(X, \phi)$ defines a functor

$$\mathsf{St}_K\colon \mathsf{FibSets}(K) \to \mathsf{ISets}(K)$$

   - *Action on Objects.* For each $(X, \phi) \in \mathrm{Obj}(\mathsf{FibSets}(K))$, we have

$$[\mathsf{St}_K](X, \phi) \overset{\text{def}}{=} \mathsf{St}_K(X, \phi);$$

   - *Action on Morphisms.* For each $(X, \phi), (Y, \psi) \in \mathrm{Obj}(\mathsf{FibSets}(K))$, the action on Hom-sets

$$\mathsf{St}_{K|X,Y}\colon \mathrm{Hom}_{\mathsf{FibSets}(K)}(X, Y) \to \mathrm{Hom}_{\mathsf{ISets}(K)}(\mathsf{St}_K(X), \mathsf{St}_K(Y))$$

   of St_K at (X, Y) is given by sending a morphism

$$f\colon (X, \phi) \to (Y, \psi)$$

   of K-fibred sets to the morphism

$$\mathsf{St}_K(f)\colon \mathsf{St}_K(X, \phi) \to \mathsf{St}_K(Y, \psi)$$

   of K-indexed sets defined by

$$\mathsf{St}_K(f) \overset{\text{def}}{=} \{f_x^*\}_{x \in K},$$

   where f_x^* is the transport map associated to f at $x \in K$ of **??**.

2. *Interaction With Change of Base/Indexing.* Let $f\colon K \to K'$ be a map of sets. The diagram

$$\begin{tikzcd} \mathsf{FibSets}(K') \arrow[r, "f^*"] \arrow[d, "\mathsf{St}_{K'}"'] & \mathsf{FibSets}(K) \arrow[d, "\mathsf{St}_K"] \\ \mathsf{ISets}(K') \arrow[r, "f^*"'] & \mathsf{ISets}(K) \end{tikzcd}$$

   commutes.

3. *Interaction With Dependent Sums.* Let $f\colon K \to K'$ be a map of sets. The diagram

$$\begin{tikzcd}
\mathsf{FibSets}(K) \arrow[r, "\Sigma_f"] \arrow[d, "\mathsf{St}_K"'] & \mathsf{FibSets}(K') \arrow[d, "\mathsf{St}_{K'}"] \\
\mathsf{ISets}(K) \arrow[r, "\Sigma_f"'] & \mathsf{ISets}(K')
\end{tikzcd}$$

commutes.

4. *Interaction With Dependent Products.* Let $f\colon K \to K'$ be a map of sets. The diagram

$$\begin{tikzcd}
\mathsf{Sets}_{/K} \arrow[r, "\Pi_f"] \arrow[d, "\mathsf{St}_K"'] & \mathsf{FibSets}(K') \arrow[d, "\mathsf{St}_{K'}"] \\
\mathsf{ISets}(K) \arrow[r, "\Pi_f"'] & \mathsf{ISets}(K')
\end{tikzcd}$$

commutes.

Proof 10.1.1.3 ► Proof of Proposition 10.1.1.2

Item 1: Functoriality

Omitted.

Item 2: Interaction With Change of Base/Indexing

Indeed, we have

$$\begin{aligned}
\mathsf{St}_K(f^*(X,\phi))_x &\stackrel{\text{def}}{=} \mathsf{St}_K(K \times_{K'} X)_x \\
&\stackrel{\text{def}}{=} \left(\mathrm{pr}_1^{K\times_{K'}X}\right)^{-1}(x) \\
&= \left\{(k,y) \in K \times_{K'} X \,\middle|\, \mathrm{pr}_1^{K\times_{K'}X}(k,y) = x\right\} \\
&= \{(k,y) \in K \times_{K'} X \mid k = x\} \\
&= \{(k,y) \in K \times X \mid k = x \text{ and } f(k) = \phi(y)\} \\
&\cong \{y \in X \mid \phi(y) = f(x)\} \\
&= \phi^{-1}(f(x)) \\
&\stackrel{\text{def}}{=} f^*(\phi^{-1}(x)) \\
&\stackrel{\text{def}}{=} f^*(\mathsf{St}_{K'}(X,\phi)_x)
\end{aligned}$$

for each $(X,\phi) \in \mathrm{Obj}(\mathsf{FibSets}(K'))$ and each $x \in K$, and similarly for morphisms.

Item 3: Interaction With Dependent Sums

Indeed, we have

$$\begin{aligned}\mathrm{St}_{K'}\left(\Sigma_f(X,\phi)\right)_x &\overset{\text{def}}{=} \Sigma_f(\phi)^{-1}(x)\\ &\cong \coprod_{\substack{y\in X\\ f(y)=x}} \phi^{-1}(y)\\ &\cong \Sigma_f\left(\phi^{-1}(x)\right)\\ &\overset{\text{def}}{=} \Sigma_f\left(\mathrm{St}_K(X,\phi)_x\right)\end{aligned}$$

for each $(X,\phi)\in\mathrm{Obj}(\mathsf{FibSets}(K))$ and each $x\in K'$, where we have used **??** of **??** for the first bijection, and similarly for morphisms.

Item 4: Interaction With Dependent Products

Indeed, we have

$$\begin{aligned}\mathrm{St}_{K'}\left(\Pi_f(X,\phi)\right)_x &\overset{\text{def}}{=} \Pi_f(\phi)^{-1}(x)\\ &\cong \prod_{\substack{y\in X\\ f(y)=x}} \phi^{-1}(y)\\ &\cong \Pi_f\left(\phi^{-1}(x)\right)\\ &\overset{\text{def}}{=} \Pi_f\left(\mathrm{St}_K(X,\phi)_x\right)\end{aligned}$$

for each $(X,\phi)\in\mathrm{Obj}(\mathsf{FibSets}(K))$ and each $x\in K'$, where we have used **??** of **??** for the first bijection, and similarly for morphisms.

10.1.2 Unstraightening for Indexed Sets

Let K be a set and let X be a K-indexed set.

Definition 10.1.2.1 ► The Unstraightening of an Indexed Set

The **unstraightening of** X is the K-fibred set

$$\phi_{\mathrm{Un}_K}\colon \mathrm{Un}_K(X)\longrightarrow K$$

consisting of

- *The Underlying Set.* The set $\mathrm{Un}_K(X)$ defined by

$$\mathrm{Un}_K(X)\overset{\text{def}}{=}\coprod_{x\in K} X_x;$$

- *The Fibration*. The map of sets

$$\phi_{\mathsf{Un}_K} : \mathsf{Un}_K(X) \to K$$

defined by sending an element of $\coprod_{x \in K} X_x$ to its index in K.

Proposition 10.1.2.2 ▶ Properties of Unstraightening for Indexed Sets

Let K be a set.

1. *Functoriality*. The assignment $X \mapsto \mathsf{Un}_K(X)$ defines a functor

$$\mathsf{Un}_K : \mathsf{ISets}(K) \to \mathsf{FibSets}(K)$$

- *Action on Objects*. For each $X \in \mathrm{Obj}(\mathsf{ISets}(K))$, we have

$$[\mathsf{Un}_K](X) \stackrel{\text{def}}{=} \mathsf{Un}_K(X);$$

- *Action on Morphisms*. For each $X, Y \in \mathrm{Obj}(\mathsf{ISets}(K))$, the action on Hom-sets

$$\mathsf{Un}_{K|X,Y} : \mathrm{Hom}_{\mathsf{ISets}(K)}(X, Y) \to \mathrm{Hom}_{\mathsf{FibSets}(K)}(\mathsf{Un}_K(X), \mathsf{Un}_K(Y))$$

of Un_K at (X, Y) is defined by

$$\mathsf{Un}_{K|X,Y}(f) \stackrel{\text{def}}{=} \coprod_{x \in K} f_x^*.$$

2. *Interaction With Fibres*. We have a bijection of sets

$$\phi_{\mathsf{Un}_K}^{-1}(x) \cong X_x$$

for each $x \in K$.

3. *As a Pullback*. We have a bijection of sets

$$\mathsf{Un}_K(X) \cong K_{\mathsf{disc}} \times_{\mathsf{Sets}} \mathsf{Sets}_*, \qquad \begin{tikzcd} \mathsf{Un}_K(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Sets}_* \arrow[d, "忘"] \\ K_{\mathsf{disc}} \arrow[r, "X"'] & \mathsf{Sets}. \end{tikzcd}$$

4. *As a Colimit*. We have a bijection of sets

$$\mathsf{Un}_K(X) \cong \mathrm{colim}(X).$$

5. *Interaction With Change of Indexing/Base.* Let $f\colon K \to K'$ be a map of sets. The diagram

$$\begin{tikzcd} \mathsf{ISets}(K') \arrow[r, "f^*"] \arrow[d, "\mathsf{Un}_{K'}"'] & \mathsf{ISets}(K) \arrow[d, "\mathsf{Un}_{K}"] \\ \mathsf{FibSets}(K') \arrow[r, "f^*"'] & \mathsf{FibSets}(K) \end{tikzcd}$$

commutes.

6. *Interaction With Dependent Sums.* Let $f\colon K \to K'$ be a map of sets. The diagram

$$\begin{tikzcd} \mathsf{ISets}(K) \arrow[r, "\Sigma_f"] \arrow[d, "\mathsf{Un}_{K}"'] & \mathsf{ISets}(K') \arrow[d, "\mathsf{Un}_{K'}"] \\ \mathsf{FibSets}(K) \arrow[r, "\Sigma_f"'] & \mathsf{FibSets}(K') \end{tikzcd}$$

commutes.

7. *Interaction With Dependent Products.* Let $f\colon K \to K'$ be a map of sets. The diagram

$$\begin{tikzcd} \mathsf{ISets}(K) \arrow[r, "\Pi_f"] \arrow[d, "\mathsf{Un}_{K}"'] & \mathsf{ISets}(K') \arrow[d, "\mathsf{Un}_{K'}"] \\ \mathsf{FibSets}(K) \arrow[r, "\Pi_f"'] & \mathsf{FibSets}(K') \end{tikzcd}$$

commutes.

Proof 10.1.2.3 ► Proof of Proposition 10.1.2.2

Item 1: Functoriality

Omitted.

Item 2: Interaction With Fibres

Omitted.

Item 3: As a Pullback

Omitted.

Item 4: As a Colimit

Clear.

Item 5: Interaction With Change of Indexing/Base

Indeed, we have

$$\begin{aligned}\mathsf{Un}_K(f^*(X)) &\overset{\text{def}}{=} \mathsf{Un}_K(X \circ f)\\ &\overset{\text{def}}{=} \coprod_{x\in K} X_{f(x)}\\ &\cong \left\{(x,(y,a)) \in K \times \coprod_{y\in K'} X_y \,\middle|\, f(x) = y\right\}\\ &\cong K \times_{K'} \coprod_{y\in K'} X_y\\ &\overset{\text{def}}{=} K \times_{K'} \mathsf{Un}_{K'}(X)\\ &\overset{\text{def}}{=} f^*(\mathsf{Un}_{K'}(X))\end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{ISets}(K'))$. Similarly, it can be shown that we also have $\mathsf{Un}_K(f^*(\phi)) = f^*(\mathsf{Un}_{K'}(\phi))$ and that $\mathsf{Un}_K \circ f^* = f^* \circ \mathsf{Un}_{K'}$ also holds on morphisms.

Item 6: Interaction With Dependent Sums

Indeed, we have

$$\begin{aligned}\mathsf{Un}_{K'}\left(\Sigma_f(X)\right) &\overset{\text{def}}{=} \coprod_{x\in K'} \Sigma_f(X)_x\\ &\cong \coprod_{x\in K'} \coprod_{y\in f^{-1}(x)} X_y\\ &\cong \coprod_{y\in K} X_y\\ &\cong \mathsf{Un}_K(X)\\ &\overset{\text{def}}{=} \Sigma_f(\mathsf{Un}_K(X))\end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{ISets}(K))$, where we have used **??** of **??** for the first bijection. Similarly, it can be shown that we also have $\mathsf{Un}_{K'}\left(\Sigma_f(\phi)\right) = \Sigma_f\left(\phi_{\mathsf{Un}_K}\right)$ and that $\mathsf{Un}_{K'} \circ \Sigma_f = \Sigma_f \circ \mathsf{Un}_K$ also holds on morphisms.

Item 7: Interaction With Dependent Products

Indeed, we have

$$
\begin{aligned}
\mathsf{Un}_{K'}(\Pi_f(X)) &\stackrel{\text{def}}{=} \coprod_{x\in K'} \Pi_f(X)_x\\
&\cong \coprod_{x\in K'} \prod_{y\in f^{-1}(x)} X_y\\
&\cong \left\{ (x,h) \in \coprod_{x\in K'} \mathsf{Sets}\Big(f^{-1}(x), \phi^{-1}_{\mathsf{Un}_K}(f^{-1}(x))\Big) \,\middle|\, \phi \circ h = \mathrm{id}_{f^{-1}(x)} \right\}\\
&\stackrel{\text{def}}{=} \Pi_f\left(\coprod_{y\in K} X_y\right)\\
&\stackrel{\text{def}}{=} \Pi_f(\mathsf{Un}_K(X))
\end{aligned}
$$

for each $X \in \mathsf{Obj}(\mathsf{ISets}(K))$, where we have used **??** of **??** for the first bijection. Similarly, it can be shown that we also have $\mathsf{Un}_{K'}(\Pi_f(\phi)) = \Pi_f(\phi_{\mathsf{Un}_K})$ and that $\mathsf{Un}_{K'} \circ \Pi_f = \Pi_f \circ \mathsf{Un}_K$ also holds on morphisms. ▭

10.1.3 The Un/Straightening Equivalence

Theorem 10.1.3.1 ► Un/Straightening for Indexed and Fibred Sets

We have an isomorphism of categories

$$
(\mathsf{St}_K \dashv \mathsf{Un}_K)\colon \quad
\begin{tikzcd}
\mathsf{FibSets}(K) \arrow[r, bend left, "\mathsf{St}_K"] \arrow[r, phantom, "\perp"] & \mathsf{ISets}(K). \arrow[l, bend left, "\mathsf{Un}_K"]
\end{tikzcd}
$$

Proof 10.1.3.2 ► Proof of Theorem 10.1.3.1

Omitted. ▭

10.2 Miscellany

10.2.1 Other Kinds of Un/Straightening

Remark 10.2.1.1 ► Other Kinds of Un/Straightening

There are also other kinds of un/straightening for sets, where Sets is replaced by **Rel** or Span:

- *Un/Straightening With* **Rel**, *I.* We have an isomorphism of sets

$$\mathrm{Rel}(A, B) \cong \mathrm{Sets}(B \times A, \{\mathrm{true}, \mathrm{false}\}).$$

by the definition of a relation from A to B, ??.

- *Un/Straightening With* **Rel**, *II.* We have an equivalence of categories

$$\mathrm{LaxFun}(K_{\mathrm{disc}}, \mathbf{Rel}) \overset{\text{eq.}}{\cong} \mathrm{Cats}^{\mathrm{fth}}_{/K_{\mathrm{disc}}},$$

where $\mathrm{Cats}^{\mathrm{fth}}_{/K_{\mathrm{disc}}}$ is the full subcategory of $\mathrm{Cats}_{/K_{\mathrm{disc}}}$ spanned by the faithful functors; see [Nie04, Theorem 3.1].

- *Un/Straightening With* Span, *I.* For each $A, B \in \mathrm{Obj}(\mathrm{Sets})$, we have a morphism of sets

$$\mathrm{Span}(A, B) \longrightarrow \mathrm{Sets}(A \times B, \mathbb{N} \cup \{\infty\})$$

which assemble into an equivalence of categories between Span(Sets) and the category MRel of "multirelations"; see ??.

- *Un/Straightening With* Span, *II.* We have an equivalence of categories

$$\mathrm{LaxFun}(K_{\mathrm{disc}}, \mathrm{Span}) \overset{\text{eq.}}{\cong} \mathrm{Cats}_{/K_{\mathrm{disc}}};$$

see [nLa24a, Section 3].

Appendices

10.A Other Chapters

Part III

Category Theory

Chapter 11

Categories

Create tags (see [MSE 350788] for some of these):

1. define bicategory $\mathsf{Adj}(\mathcal{C})$
2. internal **Hom** in categories of co/Cartesian fibrations
3. https://mathoverflow.net/questions/460146/universal-property-of-isbell-duality
4. http://www.tac.mta.ca/tac/volumes/36/12/36-12abs.html
5. justify adjunctions via homs
6. https://ncatlab.org/nlab/show/enrichment+versus+internalisation
7. https://arxiv.org/pdf/2004.08964.pdf

Contents

11.1 Categories

11.1.1 Foundations

Definition 11.1.1.1 ► Categories

A **category** $\left(\mathcal{C},\circ^{\mathcal{C}},\mathbb{1}^{\mathcal{C}}\right)$ consists of[1,2]

- *Objects.* A class $\mathrm{Obj}(\mathcal{C})$ of **objects**;
- *Morphisms.* For each $A,B\in\mathrm{Obj}(\mathcal{C})$, a class $\mathrm{Hom}_{\mathcal{C}}(A,B)$, called the **class of morphisms of $\mathcal{C}$ from A to B**;
- *Identities.* For each $A\in\mathrm{Obj}(\mathcal{C})$, a map of sets

  $$\mathbb{1}^{\mathcal{C}}_{A}\colon \mathrm{pt}\longrightarrow \mathrm{Hom}_{\mathcal{C}}(A,A),$$

  called the **unit map of $\mathcal{C}$ at A**, determining a morphism

  $$\mathrm{id}_{A}\colon A\longrightarrow A$$

  of $\mathcal{C}$, called the **identity morphism of A**;
- *Composition.* For each $A,B,C\in\mathrm{Obj}(\mathcal{C})$, a map of sets

  $$\circ^{\mathcal{C}}_{A,B,C}\colon \mathrm{Hom}_{\mathcal{C}}(B,C)\times\mathrm{Hom}_{\mathcal{C}}(A,B)\longrightarrow \mathrm{Hom}_{\mathcal{C}}(A,C),$$

  called the **composition map of $\mathcal{C}$ at (A,B,C)**;

such that the following conditions are satisfied:

1. *Associativity.* The diagram

$$\begin{tikzcd}
 & \mathrm{Hom}_{\mathcal{C}}(C,D)\times(\mathrm{Hom}_{\mathcal{C}}(B,C)\times\mathrm{Hom}_{\mathcal{C}}(A,B)) \arrow[dr, "\mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(C,D)}\times\circ^{\mathcal{C}}_{A,B,C}"] & \\
(\mathrm{Hom}_{\mathcal{C}}(C,D)\times\mathrm{Hom}_{\mathcal{C}}(B,C))\times\mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[ur, dashed, "\alpha^{\mathsf{Sets}}_{\mathrm{Hom}_{\mathcal{C}}(C,D),\mathrm{Hom}_{\mathcal{C}}(B,C),\mathrm{Hom}_{\mathcal{C}}(A,B)}", "\sim"'] \arrow[d, "\circ^{\mathcal{C}}_{B,C,D}\times\mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(A,B)}"'] & & \mathrm{Hom}_{\mathcal{C}}(C,D)\times\mathrm{Hom}_{\mathcal{C}}(A,C) \arrow[d, "\circ^{\mathcal{C}}_{A,C,D}"] \\
\mathrm{Hom}_{\mathcal{C}}(B,D)\times\mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[rr, "\circ^{\mathcal{C}}_{A,B,D}"'] & & \mathrm{Hom}_{\mathcal{C}}(A,D)
\end{tikzcd}$$

   commutes, i.e. for each composable triple (f,g,h) of morphisms of $\mathcal{C}$, we have

   $$(f\circ g)\circ h=f\circ(g\circ h).$$

2. *Left Unitality.* The diagram

$$\begin{tikzcd}
\mathrm{pt} \times \mathrm{Hom}_{\mathcal{C}}(A, B) \arrow[d, "\Bbbk_B^{\mathcal{C}} \times \mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(A,B)}"'] \arrow[drr, dashed, "\lambda^{\mathsf{Sets}}_{\mathrm{Hom}_{\mathcal{C}}(A,B)}", "\sim"'] & & \\
\mathrm{Hom}_{\mathcal{C}}(B, B) \times \mathrm{Hom}_{\mathcal{C}}(A, B) \arrow[rr, "\circ^{\mathcal{C}}_{A,B,B}"'] & & \mathrm{Hom}_{\mathcal{C}}(A, B)
\end{tikzcd}$$

commutes, i.e. for each morphism $f : A \longrightarrow B$ of $\mathcal{C}$, we have

$$\mathrm{id}_B \circ f = f.$$

3. *Right Unitality.* The diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(A, B) \times \mathrm{pt} \arrow[d, "\mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(A,B)} \times \Bbbk_A^{\mathcal{C}}"'] \arrow[drr, dashed, "\rho^{\mathsf{Sets}}_{\mathrm{Hom}_{\mathcal{C}}(A,B)}", "\sim"'] & & \\
\mathrm{Hom}_{\mathcal{C}}(A, B) \times \mathrm{Hom}_{\mathcal{C}}(A, A) \arrow[rr, "\circ^{\mathcal{C}}_{A,A,B}"'] & & \mathrm{Hom}_{\mathcal{C}}(A, B)
\end{tikzcd}$$

commutes, i.e. for each morphism $f : A \longrightarrow B$ of $\mathcal{C}$, we have

$$f \circ \mathrm{id}_A = f.$$

[1] *Further Notation:* We also write $\mathcal{C}(A, B)$ for $\mathrm{Hom}_{\mathcal{C}}(A, B)$.
[2] *Further Notation:* We write $\mathrm{Mor}(\mathcal{C})$ for the class of all morphisms of $\mathcal{C}$.

Definition 11.1.1.2 ► Size Conditions on Categories

Let κ be a regular cardinal. A category $\mathcal{C}$ is

1. **Locally small** if, for each $A, B \in \mathrm{Obj}(\mathcal{C})$, the class $\mathrm{Hom}_{\mathcal{C}}(A, B)$ is a set.
2. **Locally essentially small** if, for each $A, B \in \mathrm{Obj}(\mathcal{C})$, the class

$$\mathrm{Hom}_{\mathcal{C}}(A, B)/\{\text{isomorphisms}\}$$

is a set.

3. **Small** if $\mathcal{C}$ is locally small and $\mathrm{Obj}(\mathcal{C})$ is a set.
4. **κ-Small** if $\mathcal{C}$ is locally small, $\mathrm{Obj}(\mathcal{C})$ is a set, and we have $\#\mathrm{Obj}(\mathcal{C}) < \kappa$.

11.1.2 Examples of Categories

Example 11.1.2.1 ▶ The Punctual Category

The **punctual category**[1] is the category pt where

- *Objects*. We have
$$\mathrm{Obj}(\mathrm{pt}) \overset{\text{def}}{=} \{\star\};$$
- *Morphisms*. The unique Hom-set of pt is defined by
$$\mathrm{Hom}_{\mathrm{pt}}(\star,\star) \overset{\text{def}}{=} \{\mathrm{id}_\star\};$$
- *Identities*. The unit map
$$\mathbb{1}^{\mathrm{pt}}_{\star} \colon \mathrm{pt} \longrightarrow \mathrm{Hom}_{\mathrm{pt}}(\star,\star)$$
of pt at $\star$ is defined by
$$\mathrm{id}^{\mathrm{pt}}_\star \overset{\text{def}}{=} \mathrm{id}_\star;$$
- *Composition*. The composition map
$$\circ^{\mathrm{pt}}_{\star,\star,\star} \colon \mathrm{Hom}_{\mathrm{pt}}(\star,\star) \times \mathrm{Hom}_{\mathrm{pt}}(\star,\star) \longrightarrow \mathrm{Hom}_{\mathrm{pt}}(\star,\star)$$
of pt at $(\star,\star,\star)$ is given by the bijection $\mathrm{pt} \times \mathrm{pt} \cong \mathrm{pt}$.

[1] *Further Terminology:* Also called the **singleton category**.

Example 11.1.2.2 ▶ Monoids as One-Object Categories

We have an isomorphism of categories[1]

$$\mathrm{Mon} \cong \mathrm{pt} \underset{\mathrm{Sets}}{\times} \mathrm{Cats}, \qquad \begin{tikzcd} \mathrm{Mon} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Cats} \arrow[d, "\mathrm{Obj}"] \\ \mathrm{pt} \arrow[r, "{[\mathrm{pt}]}"'] & \mathrm{Sets} \end{tikzcd}$$

via the delooping functor $\mathbf{B} \colon \mathrm{Mon} \longrightarrow \mathrm{Cats}$ of **??** of **??**.

[1] This can be enhanced to an isomorphism of 2-categories

$$\mathrm{Mon}_{2\text{-disc}} \cong \mathrm{pt}_{\mathrm{bi}} \underset{\mathrm{Sets}_{2\text{-disc}}}{\times} \mathrm{Cats}_{2,*}, \qquad \begin{tikzcd} \mathrm{Mon}_{2\text{-disc}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Cats}_{2,*} \arrow[d, "\mathrm{Obj}"] \\ \mathrm{pt}_{\mathrm{bi}} \arrow[r, "{[\mathrm{pt}]}"'] & \mathrm{Sets}_{2\text{-disc}} \end{tikzcd}$$

between the discrete 2-category $\mathrm{Mon}_{2\text{-disc}}$ on Mon and the 2-category of pointed categories with one object.

PROOF 11.1.2.3 ► PROOF OF EXAMPLE 11.1.2.2

Omitted. ∎

EXAMPLE 11.1.2.4 ► THE EMPTY CATEGORY

The **empty category** is the category $\emptyset_{\mathsf{cat}}$ where

- *Objects*. We have
$$\mathrm{Obj}(\emptyset_{\mathsf{cat}}) \overset{\text{def}}{=} \emptyset;$$
- *Morphisms*. We have
$$\mathrm{Mor}(\emptyset_{\mathsf{cat}}) \overset{\text{def}}{=} \emptyset;$$
- *Identities and Composition*. Having no objects, $\emptyset_{\mathsf{cat}}$ has no unit nor composition maps.

EXAMPLE 11.1.2.5 ► ORDINAL CATEGORIES

The **nth ordinal category** is the category $\ltimes$ where[1]

- *Objects*. We have
$$\mathrm{Obj}(\ltimes) \overset{\text{def}}{=} \{[0], \ldots, [n]\};$$
- *Morphisms*. For each $[i], [j] \in \mathrm{Obj}(\ltimes)$, we have
$$\mathrm{Hom}_{\ltimes}([i], [j]) \overset{\text{def}}{=} \begin{cases} \{\mathrm{id}_{[i]}\} & \text{if } [i] = [j], \\ \{[i] \to [j]\} & \text{if } [j] < [i], \\ \emptyset & \text{if } [j] > [i]; \end{cases}$$
- *Identities*. For each $[i] \in \mathrm{Obj}(\ltimes)$, the unit map
$$\mathbb{1}^{\ltimes}_{[i]} \colon \mathrm{pt} \to \mathrm{Hom}_{\ltimes}([i], [i])$$
of $\ltimes$ at $[i]$ is defined by
$$\mathrm{id}^{\ltimes}_{[i]} \overset{\text{def}}{=} \mathrm{id}_{[i]};$$
- *Composition*. For each $[i], [j], [k] \in \mathrm{Obj}(\ltimes)$, the composition map
$$\circ^{\ltimes}_{[i],[j],[k]} \colon \mathrm{Hom}_{\ltimes}([j], [k]) \times \mathrm{Hom}_{\ltimes}([i], [j]) \to \mathrm{Hom}_{\ltimes}([i], [k])$$
of $\ltimes$ at $([i], [j], [k])$ is defined by
$$\begin{aligned} \mathrm{id}_{[i]} \circ \mathrm{id}_{[i]} &= \mathrm{id}_{[i]}, \\ ([j] \to [k]) \circ ([i] \to [j]) &= ([i] \to [k]). \end{aligned}$$

[1]In other words, $\ltimes$ is the category associated to the poset

$$[0] \to [1] \to \cdots \to [n-1] \to [n].$$

The category $\ltimes$ for $n \geq 2$ may also be defined in terms of $\nvdash$ and joins: we have isomorphisms of categories

$$\begin{aligned}
\nVdash &\cong \nvdash \star \nvdash, \\
\nvDash &\cong \nVdash \star \nvdash \\
&\cong (\nvdash \star \nvdash) \star \nvdash, \\
\nVDash &\cong \nvDash \star \nvdash \\
&\cong (\nVdash \star \nvdash) \star \nvdash \\
&\cong ((\nvdash \star \nvdash) \star \nvdash) \star \nvdash, \\
\ntrianglerighteq &\cong \nVDash \star \nvdash \\
&\cong (\nvDash \star \nvdash) \star \nvdash \\
&\cong ((\nVdash \star \nvdash) \star \nvdash) \star \nvdash \\
&\cong (((\nvdash \star \nvdash) \star \nvdash) \star \nvdash) \star \nvdash,
\end{aligned}$$

and so on.

EXAMPLE 11.1.2.6 ► MORE EXAMPLES OF CATEGORIES

Here we list all the other categories that appear throughout this work.

- The category Sets_* of pointed sets of **??**.
- The category Rel of sets and relations of **??**.
- The category $\mathsf{Span}(A, B)$ of spans from a set A to a set B of **??**.
- The category $\mathsf{ISets}(K)$ of K-indexed sets of **??**.
- The category ISets of indexed sets of **??**.
- The category $\mathsf{FibSets}(K)$ of K-fibred sets of **??**.
- The category $\mathsf{FibSets}$ of fibred sets of **??**.

11.1.3 Subcategories

Let $\mathcal{C}$ be a category.

DEFINITION 11.1.3.1 ► SUBCATEGORIES

A **subcategory** of $\mathcal{C}$ is a category $\mathcal{A}$ satisfying the following conditions:

1. *Objects.* We have $\mathrm{Obj}(\mathcal{A}) \subset \mathrm{Obj}(\mathcal{C})$.

2. *Morphisms.* For each $A, B \in \mathrm{Obj}(\mathcal{A})$, we have

$$\mathrm{Hom}_{\mathcal{A}}(A, B) \subset \mathrm{Hom}_{C}(A, B).$$

3. *Identities.* For each $A \in \mathrm{Obj}(\mathcal{A})$, we have

$$\mathbb{1}^{\mathcal{A}}_{A} = \mathbb{1}^{C}_{A}.$$

4. *Composition.* For each $A, B, C \in \mathrm{Obj}(\mathcal{A})$, we have

$$\circ^{\mathcal{A}}_{A,B,C} = \circ^{C}_{A,B,C}.$$

Definition 11.1.3.2 ▶ Full Subcategories

A subcategory $\mathcal{A}$ of C is **full** if the canonical inclusion functor $\mathcal{A} \to C$ is full, i.e. if, for each $A, B \in \mathrm{Obj}(\mathcal{A})$, the inclusion

$$\iota_{A,B} \colon \mathrm{Hom}_{\mathcal{A}}(A, B) \hookrightarrow \mathrm{Hom}_{C}(A, B)$$

is surjective (and thus bijective).

Definition 11.1.3.3 ▶ Strictly Full Subcategories

A subcategory $\mathcal{A}$ of a category C is **strictly full** if it satisfies the following conditions:

1. *Fullness.* The subcategory $\mathcal{A}$ is full.

2. *Closedness Under Isomorphisms.* The class $\mathrm{Obj}(\mathcal{A})$ is closed under isomorphisms.[1]

[1] That is, given $A \in \mathrm{Obj}(\mathcal{A})$ and $C \in \mathrm{Obj}(C)$, if $C \cong A$, then $C \in \mathrm{Obj}(\mathcal{A})$.

Definition 11.1.3.4 ▶ Wide Subcategories

A subcategory $\mathcal{A}$ of C is **wide**[1] if $\mathrm{Obj}(\mathcal{A}) = \mathrm{Obj}(C)$.

[1] *Further Terminology:* Also called **lluf**.

11.1.4 Skeletons of Categories

Definition 11.1.4.1 ► Skeletons of Categories

A[1] **skeleton** of a category $\mathcal{C}$ is a full subcategory $\mathsf{Sk}(\mathcal{C})$ with one object from each isomorphism class of objects of $\mathcal{C}$.

[1] Due to Item 3 of Proposition 11.1.4.3, we often refer to any such full subcategory $\mathsf{Sk}(\mathcal{C})$ of $\mathcal{C}$ as *the* skeleton of $\mathcal{C}$.

Definition 11.1.4.2 ► Skeletal Categories

A category $\mathcal{C}$ is **skeletal** if $\mathcal{C} \cong \mathsf{Sk}(\mathcal{C})$.[1]

[1] That is, $\mathcal{C}$ is **skeletal** if isomorphic objects of $\mathcal{C}$ are equal.

Proposition 11.1.4.3 ► Properties of Skeletons of Categories

Let $\mathcal{C}$ be a category.

1. *Existence.* Assuming the axiom of choice, $\mathsf{Sk}(\mathcal{C})$ always exists.
2. *Pseudofunctoriality.* The assignment $\mathcal{C} \mapsto \mathsf{Sk}(\mathcal{C})$ defines a pseudofunctor

$$\mathsf{Sk}\colon \mathsf{Cats}_2 \longrightarrow \mathsf{Cats}_2.$$

3. *Uniqueness Up to Equivalence.* Any two skeletons of $\mathcal{C}$ are equivalent.
4. *Inclusions of Skeletons Are Equivalences.* The inclusion

$$\iota_{\mathcal{C}}\colon \mathsf{Sk}(\mathcal{C}) \hookrightarrow \mathcal{C}$$

of a skeleton of $\mathcal{C}$ into $\mathcal{C}$ is an equivalence of categories.

Proof 11.1.4.4 ► Proof of Proposition 11.1.4.3

Item 1: Existence

See [nLab23, Section "Existence of Skeletons of Categories"].

Item 2: Pseudofunctoriality

See [nLab23, Section "Skeletons as an Endo-Pseudofunctor on $\mathfrak{Cat}$"].

Item 3: Uniqueness Up to Equivalence

Clear.

Item 4: Inclusions of Skeletons Are Equivalences

Clear.

11.1.5 Precomposition and Postcomposition

Let $\mathcal{C}$ be a category and let $A, B, C \in \mathrm{Obj}(\mathcal{C})$.

Let $f\colon A \to B$ and $g\colon B \to C$ be morphisms of $\mathcal{C}$.

- The **precomposition function associated to** f is the function

$$f^*\colon \mathrm{Hom}_{\mathcal{C}}(B, C) \to \mathrm{Hom}_{\mathcal{C}}(A, C)$$

  defined by

$$f^*(\phi) \overset{\text{def}}{=} \phi \circ f$$

  for each $\phi \in \mathrm{Hom}_{\mathcal{C}}(B, C)$.

- The **postcomposition function associated to** g is the function

$$g_*\colon \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{C}}(A, C)$$

  defined by

$$g_*(\phi) \overset{\text{def}}{=} g \circ \phi$$

  for each $\phi \in \mathrm{Hom}_{\mathcal{C}}(A, B)$.

Proposition 11.1.5.2 ▶ Properties of Pre/Postcomposition

Let $A, B, C, D \in \mathrm{Obj}(\mathcal{C})$ and let $f\colon A \to B$ and $g\colon B \to C$ be morphisms of $\mathcal{C}$.

1. *Interaction Between Precomposition and Postcomposition.* We have

$$g_* \circ f^* = f^* \circ g_*, \qquad \begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(B, C) \arrow[r, "g_*"] \arrow[d, "f^*"'] & \mathrm{Hom}_{\mathcal{C}}(B, D) \arrow[d, "f^*"] \\ \mathrm{Hom}_{\mathcal{C}}(A, C) \arrow[r, "g_*"'] & \mathrm{Hom}_{\mathcal{C}}(A, D). \end{tikzcd}$$

2. *Interaction With Composition I.* We have

$$(g \circ f)^* = f^* \circ g^*, \qquad \begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(X, A) \arrow[r, "f_*"] \arrow[dr, "(g\circ f)_*"'] & \mathrm{Hom}_{\mathcal{C}}(X, B) \arrow[d, "g_*"] \\ & \mathrm{Hom}_{\mathcal{C}}(X, C), \end{tikzcd}$$

$$(g \circ f)_* = g_* \circ f_*, \qquad \begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(C, X) \arrow[r, "g^*"] \arrow[dr, "(g\circ f)^*"'] & \mathrm{Hom}_{\mathcal{C}}(B, X) \arrow[d, "f^*"] \\ & \mathrm{Hom}_{\mathcal{C}}(A, X). \end{tikzcd}$$

3. *Interaction With Composition II.* We have

$$
\begin{tikzcd}
\mathrm{pt} \arrow[r, "{[f]}"] \arrow[dr, "{[g\circ f]}"'] & \mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[d, "g_*"] \\
 & \mathrm{Hom}_{\mathcal{C}}(A,C)
\end{tikzcd}
\qquad
\begin{aligned}
[g\circ f] &= g_* \circ [f],\\
[g\circ f] &= f^* \circ [g],
\end{aligned}
\qquad
\begin{tikzcd}
\mathrm{pt} \arrow[r, "{[g]}"] \arrow[dr, "{[g\circ f]}"'] & \mathrm{Hom}_{\mathcal{C}}(B,C) \arrow[d, "f^*"] \\
 & \mathrm{Hom}_{\mathcal{C}}(A,C).
\end{tikzcd}
$$

4. *Interaction With Composition III.* We have

$$
f^* \circ \circ^{\mathcal{C}}_{A,B,C} = \circ^{\mathcal{C}}_{X,B,C} \circ (f^* \times \mathrm{id}),
\qquad
\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(B,C)\times\mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[r, "\circ^{\mathcal{C}}_{A,B,C}"] \arrow[d, "\mathrm{id}\times f^*"'] & \mathrm{Hom}_{\mathcal{C}}(A,C) \arrow[d, "f^*"] \\
\mathrm{Hom}_{\mathcal{C}}(B,C)\times\mathrm{Hom}_{\mathcal{C}}(X,B) \arrow[r, "\circ^{\mathcal{C}}_{X,B,C}"'] & \mathrm{Hom}_{\mathcal{C}}(X,C),
\end{tikzcd}
$$

$$
g_* \circ \circ^{\mathcal{C}}_{A,B,C} = \circ^{\mathcal{C}}_{A,B,D} \circ (\mathrm{id} \times g_*),
\qquad
\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(B,C)\times\mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[r, "\circ^{\mathcal{C}}_{A,B,C}"] \arrow[d, "g_*\times\mathrm{id}"'] & \mathrm{Hom}_{\mathcal{C}}(A,C) \arrow[d, "g^*"] \\
\mathrm{Hom}_{\mathcal{C}}(B,D)\times\mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[r, "\circ^{\mathcal{C}}_{A,B,D}"'] & \mathrm{Hom}_{\mathcal{C}}(A,D).
\end{tikzcd}
$$

5. *Interaction With Identities.* We have

$$
\begin{aligned}
(\mathrm{id}_A)^* &= \mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(A,B)},\\
(\mathrm{id}_B)_* &= \mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(A,B)}.
\end{aligned}
$$

Proof 11.1.5.3 ► Proof of Proposition 11.1.5.2

Item 1: Interaction Between Precomposition and Postcomposition

Clear.

Item 2: Interaction With Composition I

Clear.

Item 3: Interaction With Composition II

Clear.

Item 4: Interaction With Composition III

Clear.

Item 5: Interaction With Identities

Clear.

11.2 The Quadruple Adjunction With Sets

11.2.1 Statement

Let $\mathcal{C}$ be a category.

Proposition 11.2.1.1 ► The Quadruple Adjunction Between Sets and Cats

We have a quadruple adjunction

$$(\pi_0 \dashv (-)_{\mathsf{disc}} \dashv \mathsf{Obj} \dashv (-)_{\mathsf{indisc}})\colon \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left=20, "(-)_{\mathsf{disc}}"] \arrow[r, bend right=70, "(-)_{\mathsf{indisc}}"'] & \mathsf{Cats}, \arrow[l, bend right=70, "\pi_0"'] \arrow[l, bend left=20, "\mathsf{Obj}"] \end{tikzcd}$$

witnessed by bijections of sets

$$\begin{aligned} \mathrm{Hom}_{\mathsf{Sets}}(\pi_0(\mathcal{C}), X) &\cong \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, X_{\mathsf{disc}}), \\ \mathrm{Hom}_{\mathsf{Cats}}(X_{\mathsf{disc}}, \mathcal{C}) &\cong \mathrm{Hom}_{\mathsf{Sets}}(X, \mathsf{Obj}(\mathcal{C})), \\ \mathrm{Hom}_{\mathsf{Sets}}(\mathsf{Obj}(\mathcal{C}), X) &\cong \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, X_{\mathsf{indisc}}), \end{aligned}$$

natural in $\mathcal{C} \in \mathsf{Obj}(\mathsf{Cats})$ and $X \in \mathsf{Obj}(\mathsf{Sets})$, where

- The functor
  $$\pi_0\colon \mathsf{Cats} \longrightarrow \mathsf{Sets},$$
  the **connected components functor**, is the functor sending a category to its set of connected components of Definition 11.2.3.1.

- The functor
  $$(-)_{\mathsf{disc}}\colon \mathsf{Sets} \longrightarrow \mathsf{Cats},$$
  the **discrete category functor**, is the functor sending a set to its associated discrete category of Definition 11.2.5.1.

- The functor
  $$\mathsf{Obj}\colon \mathsf{Cats} \longrightarrow \mathsf{Sets},$$
  the **object functor**, is the functor sending a category to its set of objects.

- The functor

$$(-)_{\text{indisc}} \colon \mathsf{Sets} \longrightarrow \mathsf{Cats},$$

the **indiscrete category functor**, is the functor sending a set to its associated indiscrete category of Definition 11.2.6.1.

Proof 11.2.1.2 ► Proof of Proposition 11.2.1.1

Omitted. ∎

11.2.2 Connected Components of Categories

Let $\mathcal{C}$ be a category.

Definition 11.2.2.1 ► Connected Components of Categories

A **connected component** of $\mathcal{C}$ is a full subcategory $\mathcal{I}$ of $\mathcal{C}$ satisfying the following conditions:[1]

1. *Non-Emptiness.* We have $\text{Obj}(\mathcal{I}) \neq \emptyset$.
2. *Connectedness.* There exists a zigzag of arrows between any two objects of $\mathcal{I}$.

[1] In other words, a **connected component** of $\mathcal{C}$ is an element of the set $\text{Obj}(\mathcal{C})/\sim$ with $\sim$ the equivalence relation generated by the relation $\sim'$ obtained by declaring $A \sim' B$ iff there exists a morphism of $\mathcal{C}$ from A to B.

11.2.3 Sets of Connected Components of Categories

Let $\mathcal{C}$ be a category.

Definition 11.2.3.1 ► Sets of Connected Components of Categories

The **set of connected components of** $\mathcal{C}$ is the set $\pi_0(\mathcal{C})$ whose elements are the connected components of $\mathcal{C}$.

Proposition 11.2.3.2 ► Properties of Sets of Connected Components

Let $\mathcal{C}$ be a category.

1. *Functoriality.* The assignment $\mathcal{C} \mapsto \pi_0(\mathcal{C})$ defines a functor

$$\pi_0 \colon \mathsf{Cats} \longrightarrow \mathsf{Sets}.$$

2. *Adjointness.* We have a quadruple adjunction

$$(\pi_0 \dashv (-)_{\mathsf{disc}} \dashv \mathsf{Obj} \dashv (-)_{\mathsf{indisc}})\colon \quad \begin{tikzcd}[column sep=huge] \mathsf{Sets} \arrow[r, bend left=15, "(-)_{\mathsf{disc}}" description] \arrow[r, bend right=60, "(-)_{\mathsf{indisc}}"'] \arrow[r, phantom, shift left=4, "\perp"] \arrow[r, phantom, "\perp"] \arrow[r, phantom, shift right=4, "\perp"] & \mathsf{Cats}. \arrow[l, bend right=60, "\pi_0"'] \arrow[l, bend left=15, "\mathsf{Obj}" description] \end{tikzcd}$$

3. *Interaction With Groupoids.* If $\mathcal{C}$ is a groupoid, then we have an isomorphism of categories

$$\pi_0(\mathcal{C}) \cong \mathsf{K}(\mathcal{C}),$$

where $\mathsf{K}(\mathcal{C})$ is the set of isomorphism classes of $\mathcal{C}$ of **??**.

4. *Preservation of Colimits.* The functor π_0 of Item 1 preserves colimits. In particular, we have bijections of sets

$$\begin{aligned} \pi_0(\mathcal{C} \textstyle\coprod \mathcal{D}) &\cong \pi_0(\mathcal{C}) \textstyle\coprod \pi_0(\mathcal{D}),\\ \pi_0(\mathcal{C} \textstyle\coprod_{\mathcal{E}} \mathcal{D}) &\cong \pi_0(\mathcal{C}) \textstyle\coprod_{\pi_0(\mathcal{E})} \pi_0(\mathcal{D}),\\ \pi_0\left(\mathsf{CoEq}\left(\mathcal{C} \underset{G}{\overset{F}{\rightrightarrows}} \mathcal{D}\right)\right) &\cong \mathsf{CoEq}\left(\pi_0(\mathcal{C}) \underset{\pi_0(G)}{\overset{\pi_0(F)}{\rightrightarrows}} \pi_0(\mathcal{D})\right), \end{aligned}$$

natural in $\mathcal{C}, \mathcal{D}, \mathcal{E} \in \mathsf{Obj}(\mathsf{Cats})$.

5. *Symmetric Strong Monoidality With Respect to Coproducts.* The connected components functor of Item 1 has a symmetric strong monoidal structure

$$\left(\pi_0, \pi_0^{\coprod}, \pi_{0|\mathbb{1}}^{\coprod}\right)\colon (\mathsf{Cats}, \textstyle\coprod, \emptyset_{\mathsf{cat}}) \longrightarrow (\mathsf{Sets}, \textstyle\coprod, \emptyset),$$

being equipped with isomorphisms

$$\begin{gathered} \pi_{0|\mathcal{C},\mathcal{D}}^{\coprod}\colon \pi_0(\mathcal{C}) \textstyle\coprod \pi_0(\mathcal{D}) \xrightarrow{\cong} \pi_0(\mathcal{C} \textstyle\coprod \mathcal{D}),\\ \pi_{0|\mathbb{1}}^{\coprod}\colon \emptyset \xrightarrow{\cong} \pi_0(\emptyset_{\mathsf{cat}}), \end{gathered}$$

natural in $\mathcal{C}, \mathcal{D} \in \mathsf{Obj}(\mathsf{Cats})$.

6. *Symmetric Strong Monoidality With Respect to Products.* The connected components functor of Item 1 has a symmetric strong monoidal structure

$$\left(\pi_0, \pi_0^{\otimes}, \pi_{0|\mathbb{1}}^{\otimes}\right)\colon (\mathsf{Cats}, \times, \mathsf{pt}) \longrightarrow (\mathsf{Sets}, \times, \mathsf{pt}),$$

being equipped with isomorphisms

$$\pi^{\otimes}_{0|C,\mathcal{D}} \colon \pi_0(C) \times \pi_0(\mathcal{D}) \xrightarrow{\cong} \pi_0(C \times \mathcal{D}),$$
$$\pi^{\otimes}_{0|\mathbb{1}} \colon \mathrm{pt} \xrightarrow{\cong} \pi_0(\mathrm{pt}),$$

natural in $C, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$.

Proof 11.2.3.3 ► Proof of Proposition 11.2.3.2

Item 1: Functoriality

Clear.

Item 2: Adjointness

This is proved in Proposition 11.2.1.1.

Item 3: Interaction With Groupoids

Clear.

Item 4: Preservation of Colimits

This follows from Item 2 and ?? of ??.

Item 5: Symmetric Strong Monoidality With Respect to Coproducts

Omitted.

Item 6: Symmetric Strong Monoidality With Respect to Products

Omitted.

11.2.4 Connected Categories

Definition 11.2.4.1 ► Connected Categories

A category C is **connected** if $\pi_0(C) \cong \mathrm{pt}$.[1,2]

[1] *Further Terminology:* A category is **disconnected** if it is not connected.
[2] *Example:* A groupoid is connected iff any two of its objects are isomorphic.

11.2.5 Discrete Categories

Let X be a set.

Definition 11.2.5.1 ► The Discrete Category on a Set

The **discrete category on a set** X is the category X_{disc} where

- *Objects.* We have
$$\mathrm{Obj}(X_{\mathsf{disc}}) \overset{\mathrm{def}}{=} X;$$
- *Morphisms.* For each $A, B \in \mathrm{Obj}(X_{\mathsf{disc}})$, we have
$$\mathrm{Hom}_{X_{\mathsf{disc}}}(A, B) \overset{\mathrm{def}}{=} \begin{cases} \mathrm{id}_A & \text{if } A = B, \\ \emptyset & \text{if } A \neq B; \end{cases}$$
- *Identities.* For each $A \in \mathrm{Obj}(X_{\mathsf{disc}})$, the unit map
$$\mathbb{1}^{X_{\mathsf{disc}}}_A \colon \mathrm{pt} \longrightarrow \mathrm{Hom}_{X_{\mathsf{disc}}}(A, A)$$
of X_{disc} at A is defined by
$$\mathrm{id}^{X_{\mathsf{disc}}}_A \overset{\mathrm{def}}{=} \mathrm{id}_A;$$
- *Composition.* For each $A, B, C \in \mathrm{Obj}(X_{\mathsf{disc}})$, the composition map
$$\circ^{X_{\mathsf{disc}}}_{A,B,C} \colon \mathrm{Hom}_{X_{\mathsf{disc}}}(B, C) \times \mathrm{Hom}_{X_{\mathsf{disc}}}(A, B) \longrightarrow \mathrm{Hom}_{X_{\mathsf{disc}}}(A, C)$$
of X_{disc} at (A, B, C) is defined by
$$\mathrm{id}_A \circ \mathrm{id}_A \overset{\mathrm{def}}{=} \mathrm{id}_A.$$

Proposition 11.2.5.2 ► Properties of Discrete Categories on Sets

Let X be a set.

1. *Functoriality.* The assignment $X \mapsto X_{\mathsf{disc}}$ defines a functor
$$(-)_{\mathsf{disc}} \colon \mathsf{Sets} \longrightarrow \mathsf{Cats}.$$
2. *Adjointness.* We have a quadruple adjunction
$$(\pi_0 \dashv (-)_{\mathsf{disc}} \dashv \mathrm{Obj} \dashv (-)_{\mathsf{indisc}}) \colon \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left=20, "(-)_{\mathsf{disc}}"] \arrow[r, bend right=60, "(-)_{\mathsf{indisc}}"'] & \mathsf{Cats}. \arrow[l, bend right=60, "\pi_0"'] \arrow[l, bend left=20, "\mathrm{Obj}"] \end{tikzcd}$$

3. *Symmetric Strong Monoidality With Respect to Coproducts.* The functor of Item 1 has a symmetric strong monoidal structure

$$\left((-)_{\mathsf{disc}}, (-)^{\coprod}_{\mathsf{disc}}, (-)^{\coprod}_{\mathsf{disc}|\mathbb{1}}\right) : (\mathsf{Sets}, \coprod, \emptyset) \to (\mathsf{Cats}, \coprod, \emptyset_{\mathsf{cat}}),$$

being equipped with isomorphisms

$$(-)^{\coprod}_{\mathsf{disc}|X,Y} : X_{\mathsf{disc}} \coprod Y_{\mathsf{disc}} \xrightarrow{\cong} (X \coprod Y)_{\mathsf{disc}},$$
$$(-)^{\coprod}_{\mathsf{disc}|\mathbb{1}} : \emptyset_{\mathsf{cat}} \xrightarrow{\cong} \emptyset_{\mathsf{disc}},$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

4. *Symmetric Strong Monoidality With Respect to Products.* The functor of Item 1 has a symmetric strong monoidal structure

$$\left((-)_{\mathsf{disc}}, (-)^{\otimes}_{\mathsf{disc}}, (-)^{\otimes}_{\mathsf{disc}|\mathbb{1}}\right) : (\mathsf{Sets}, \times, \mathrm{pt}) \to (\mathsf{Cats}, \times, \mathrm{pt}),$$

being equipped with isomorphisms

$$(-)^{\otimes}_{\mathsf{disc}|X,Y} : X_{\mathsf{disc}} \times Y_{\mathsf{disc}} \xrightarrow{\cong} (X \times Y)_{\mathsf{disc}},$$
$$(-)^{\otimes}_{\mathsf{disc}|\mathbb{1}} : \mathrm{pt} \xrightarrow{\cong} \mathrm{pt}_{\mathsf{disc}},$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

Proof 11.2.5.3 ▶ Proof of Proposition 11.2.5.2

Item 1: Functoriality

Clear.

Item 2: Adjointness

This is proved in Proposition 11.2.1.1.

Item 3: Symmetric Strong Monoidality With Respect to Coproducts

Omitted.

Item 4: Symmetric Strong Monoidality With Respect to Products

Omitted.

11.2.6 Indiscrete Categories

Definition 11.2.6.1 ► The Indiscrete category on a Set

The **indiscrete category on a set** X[1] is the category X_{indisc} where

- *Objects*. We have
$$\mathrm{Obj}(X_{\mathsf{indisc}}) \stackrel{\mathrm{def}}{=} X;$$
- *Morphisms*. For each $A, B \in \mathrm{Obj}(X_{\mathsf{indisc}})$, we have
$$\mathrm{Hom}_{X_{\mathsf{disc}}}(A, B) \stackrel{\mathrm{def}}{=} \{[A] \longrightarrow [B]\};$$
- *Identities*. For each $A \in \mathrm{Obj}(X_{\mathsf{indisc}})$, the unit map
$$\mathbb{1}^{X_{\mathsf{indisc}}}_{A} \colon \mathrm{pt} \longrightarrow \mathrm{Hom}_{X_{\mathsf{indisc}}}(A, A)$$
of X_{indisc} at A is defined by
$$\mathrm{id}^{X_{\mathsf{indisc}}}_{A} \stackrel{\mathrm{def}}{=} \{[A] \longrightarrow [A]\};$$
- *Composition*. For each $A, B, C \in \mathrm{Obj}(X_{\mathsf{indisc}})$, the composition map
$$\circ^{X_{\mathsf{indisc}}}_{A,B,C} \colon \mathrm{Hom}_{X_{\mathsf{indisc}}}(B, C) \times \mathrm{Hom}_{X_{\mathsf{indisc}}}(A, B) \longrightarrow \mathrm{Hom}_{X_{\mathsf{indisc}}}(A, C)$$
of X_{disc} at (A, B, C) is defined by
$$([B] \longrightarrow [C]) \circ ([A] \longrightarrow [B]) \stackrel{\mathrm{def}}{=} ([A] \longrightarrow [C]).$$

[1] *Further Terminology:* Also called the **chaotic category on** X.

Proposition 11.2.6.2 ► Properties of Indiscrete Categories on Sets

Let X be a set.

1. *Functoriality*. The assignment $X \mapsto X_{\mathsf{indisc}}$ defines a functor
$$(-)_{\mathsf{indisc}} \colon \mathsf{Sets} \longrightarrow \mathsf{Cats}.$$
2. *Adjointness*. We have a quadruple adjunction
$$(\pi_0 \dashv (-)_{\mathsf{disc}} \dashv \mathrm{Obj} \dashv (-)_{\mathsf{indisc}})\colon \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left=20, "(-)_{\mathsf{disc}}" description] \arrow[r, bend right=60, "(-)_{\mathsf{indisc}}"'] & \mathsf{Cats}. \arrow[l, bend right=60, "\pi_0"'] \arrow[l, bend left=20, "\mathrm{Obj}" description] \end{tikzcd}$$

3. *Symmetric Strong Monoidality With Respect to Products.* The functor of Item 1 has a symmetric strong monoidal structure

$$\left((-)_{\mathsf{indisc}}, (-)^{\otimes}_{\mathsf{indisc}}, (-)^{\otimes}_{\mathsf{indisc}|\mathbb{1}}\right) : (\mathsf{Sets}, \times, \mathrm{pt}) \to (\mathsf{Cats}, \times, \mathrm{pt}),$$

being equipped with isomorphisms

$$(-)^{\otimes}_{\mathsf{indisc}|X,Y} : X_{\mathsf{indisc}} \times Y_{\mathsf{indisc}} \xrightarrow{\cong} (X \times Y)_{\mathsf{indisc}},$$
$$(-)^{\otimes}_{\mathsf{indisc}|\mathbb{1}} : \mathrm{pt} \xrightarrow{\cong} \mathrm{pt}_{\mathsf{indisc}},$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

Proof 11.2.6.3 ▶ Proof of Proposition 11.2.6.2

Item 1: Functoriality

Clear.

Item 2: Adjointness

This is proved in Proposition 11.2.1.1.

Item 3: Symmetric Strong Monoidality With Respect to Products

Omitted.

11.3 Groupoids

11.3.1 Foundations

Let $\mathcal{C}$ be a category.

Definition 11.3.1.1 ▶ Isomorphisms

A morphism $f : A \to B$ of $\mathcal{C}$ is an **isomorphism** if there exists a morphism $f^{-1} : B \to A$ of $\mathcal{C}$ such that

$$f \circ f^{-1} = \mathrm{id}_B,$$
$$f^{-1} \circ f = \mathrm{id}_A.$$

Definition 11.3.1.2 ▶ Groupoids

A **groupoid** is a category in which every morphism is an isomorphism.

11.3.2 The Groupoid Completion of a Category

Let $\mathcal{C}$ be a category.

Definition 11.3.2.1 ▶ The Groupoid Completion of a Category

The **groupoid completion of** $\mathcal{C}$[1] is the pair $(\mathsf{K}_0(\mathcal{C}), \iota_{\mathcal{C}})$ consisting of

- A groupoid $\mathsf{K}_0(\mathcal{C})$;
- A functor $\iota_{\mathcal{C}} \colon \mathcal{C} \to \mathsf{K}_0(\mathcal{C})$;

satisfying the following universal property:[2]

(UP) Given another such pair $(\mathcal{G}, i)$, there exists a unique functor $\mathsf{K}_0(\mathcal{C}) \xrightarrow{\exists!} \mathcal{G}$ making the diagram

$$\begin{tikzcd} & \mathsf{K}_0(\mathcal{C}) \arrow[d, dashed, "\exists!"] \\ \mathcal{C} \arrow[ur, "\iota_{\mathcal{C}}"] \arrow[r, "i"'] & \mathcal{G} \end{tikzcd}$$

commute.

[1] *Further Terminology:* Also called the **Grothendieck groupoid of** $\mathcal{C}$ or the **Grothendieck groupoid completion of** $\mathcal{C}$.

[2] See Item 5 of Proposition 11.3.2.2 for an explicit construction.

Proposition 11.3.2.2 ▶ Properties of Groupoid Completion

Let $\mathcal{C}$ be a category.

1. *Functoriality.* The assignment $\mathcal{C} \mapsto \mathsf{K}_0(\mathcal{C})$ defines a functor

$$\mathsf{K}_0 \colon \mathsf{Cats} \to \mathsf{Grpd}.$$

2. *2-Functoriality.* The assignment $\mathcal{C} \mapsto \mathsf{K}_0(\mathcal{C})$ defines a 2-functor

$$\mathsf{K}_0 \colon \mathsf{Cats}_2 \to \mathsf{Grpd}_2.$$

3. *Adjointness.* We have an adjunction

$$(\mathsf{K}_0 \dashv \iota)\colon \quad \begin{tikzcd} \mathsf{Cats} \arrow[r, bend left, "\mathsf{K}_0"] \arrow[r, phantom, "\perp"] & \mathsf{Grpd}, \arrow[l, bend left, "\iota"] \end{tikzcd}$$

witnessed by a bijection of sets

$$\mathrm{Hom}_{\mathsf{Grpd}}(\mathsf{K}_0(\mathcal{C}), \mathcal{G}) \cong \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, \mathcal{G}),$$

natural in $\mathcal{C} \in \mathrm{Obj}(\mathsf{Cats})$ and $\mathcal{G} \in \mathrm{Obj}(\mathsf{Grpd})$, forming, together with the functor Core of Item 1 of Proposition 11.3.3.4, a triple adjunction

$$(\mathrm{K}_0 \dashv \iota \dashv \mathrm{Core})\colon\quad \begin{tikzcd} \mathsf{Cats} \arrow[r, bend left=45, "\mathrm{K}_0"] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, bend right=45, "\mathrm{Core}"'] \arrow[r, phantom, shift right=3, "\perp"] & \mathsf{Grpd} \arrow[l, "\iota" description] \end{tikzcd},$$

witnessed by bijections of sets

$$\begin{aligned} \mathrm{Hom}_{\mathsf{Grpd}}(\mathrm{K}_0(\mathcal{C}), \mathcal{G}) &\cong \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, \mathcal{G}), \\ \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{G}, \mathcal{D}) &\cong \mathrm{Hom}_{\mathsf{Grpd}}(\mathcal{G}, \mathrm{Core}(\mathcal{D})), \end{aligned}$$

natural in $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$ and $\mathcal{G} \in \mathrm{Obj}(\mathsf{Grpd})$.

4. *2-Adjointness.* We have a 2-adjunction

$$(\mathrm{K}_0 \dashv \iota)\colon\quad \begin{tikzcd} \mathsf{Cats} \arrow[r, shift left, "\mathrm{K}_0"] \arrow[r, phantom, "\perp_2"] & \mathsf{Grpd} \arrow[l, shift left, "\iota"] \end{tikzcd},$$

witnessed by an isomorphism of categories

$$\mathsf{Fun}(\mathrm{K}_0(\mathcal{C}), \mathcal{G}) \cong \mathsf{Fun}(\mathcal{C}, \mathcal{G}),$$

natural in $\mathcal{C} \in \mathrm{Obj}(\mathsf{Cats})$ and $\mathcal{G} \in \mathrm{Obj}(\mathsf{Grpd})$, forming, together with the 2-functor Core of Item 2 of Proposition 11.3.3.4, a triple 2-adjunction

$$(\mathrm{K}_0 \dashv \iota \dashv \mathrm{Core})\colon\quad \begin{tikzcd} \mathsf{Cats} \arrow[r, bend left=45, "\mathrm{K}_0"] \arrow[r, phantom, shift left=3, "\perp_2"] \arrow[r, bend right=45, "\mathrm{Core}"'] \arrow[r, phantom, shift right=3, "\perp_2"] & \mathsf{Grpd} \arrow[l, "\iota" description] \end{tikzcd},$$

witnessed by isomorphisms of categories

$$\begin{aligned} \mathsf{Fun}(\mathrm{K}_0(\mathcal{C}), \mathcal{G}) &\cong \mathsf{Fun}(\mathcal{C}, \mathcal{G}), \\ \mathsf{Fun}(\mathcal{G}, \mathcal{D}) &\cong \mathsf{Fun}(\mathcal{G}, \mathrm{Core}(\mathcal{D})), \end{aligned}$$

natural in $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$ and $\mathcal{G} \in \mathrm{Obj}(\mathsf{Grpd})$.

5. *Interaction With Classifying Spaces.* We have an isomorphism of groupoids

$$\mathrm{K}_0(\mathcal{C}) \cong \Pi_{\leq 1}(|\mathrm{N}_\bullet(\mathcal{C})|),$$

natural in $\mathcal{C} \in \mathrm{Obj}(\mathsf{Cats})$; i.e. the diagram

$$\begin{tikzcd}
\mathsf{Cats} \arrow[r, "\mathrm{K}_0"] \arrow[d, "\mathrm{N}_\bullet"'] \arrow[dr, phantom, "\wr"] & \mathsf{Grp} \\
\mathsf{sSets} \arrow[r, "|-|"'] & \mathsf{Top} \arrow[u, "\Pi_{\leq 1}"']
\end{tikzcd}$$

commutes up to natural isomorphism.

6. *Symmetric Strong Monoidality With Respect to Coproducts.* The groupoid completion functor of Item 1 has a symmetric strong monoidal structure

$$\left(\mathrm{K}_0, \mathrm{K}_0^{\coprod}, \mathrm{K}_{0|\mathbb{1}}^{\coprod}\right) : \left(\mathsf{Cats}, \coprod, \emptyset_{\mathsf{cat}}\right) \to \left(\mathsf{Grpd}, \coprod, \emptyset_{\mathsf{cat}}\right)$$

being equipped with isomorphisms

$$\mathrm{K}_{0|\mathcal{C},\mathcal{D}}^{\coprod} : \mathrm{K}_0(\mathcal{C}) \coprod \mathrm{K}_0(\mathcal{D}) \xrightarrow{\cong} \mathrm{K}_0(\mathcal{C} \coprod \mathcal{D}),$$
$$\mathrm{K}_{0|\mathbb{1}}^{\coprod} : \emptyset_{\mathsf{cat}} \xrightarrow{\cong} \mathrm{K}_0(\emptyset_{\mathsf{cat}}),$$

natural in $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$.

7. *Symmetric Strong Monoidality With Respect to Products.* The groupoid completion functor of Item 1 has a symmetric strong monoidal structure

$$\left(\mathrm{K}_0, \mathrm{K}_0^{\times}, \mathrm{K}_{0|\mathbb{1}}^{\times}\right) : (\mathsf{Cats}, \times, \mathrm{pt}) \to (\mathsf{Grpd}, \times, \mathrm{pt})$$

being equipped with isomorphisms

$$\mathrm{K}_{0|\mathcal{C},\mathcal{D}}^{\times} : \mathrm{K}_0(\mathcal{C}) \times \mathrm{K}_0(\mathcal{D}) \xrightarrow{\cong} \mathrm{K}_0(\mathcal{C} \times \mathcal{D}),$$
$$\mathrm{K}_{0|\mathbb{1}}^{\times} : \mathrm{pt} \xrightarrow{\cong} \mathrm{K}_0(\mathrm{pt}),$$

natural in $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$.

Proof 11.3.2.3 ▶ Proof of Proposition 11.3.2.2

Item 1: Functoriality

Omitted.

Item 2: 2-Functoriality

Omitted.

Item 3: Adjointness

Omitted.

Item 4: 2-Adjointness

Omitted.

Item 5: Interaction With Classifying Spaces

See Corollary 18.33 of https://web.ma.utexas.edu/users/dafr/M392C-2012/Notes/lecture18.pdf.

Item 6: Symmetric Strong Monoidality With Respect to Coproducts

Omitted.

Item 7: Symmetric Strong Monoidality With Respect to Products

Omitted.

11.3.3 The Core of a Category

Let $\mathcal{C}$ be a category.

Definition 11.3.3.1 ► The Core of a Category

The **core** of $\mathcal{C}$ is the pair $(\mathrm{Core}(\mathcal{C}), \iota_{\mathcal{C}})$[1] consisting of

1. A groupoid $\mathrm{Core}(\mathcal{C})$;
2. A functor $\iota_{\mathcal{C}} \colon \mathrm{Core}(\mathcal{C}) \hookrightarrow \mathcal{C}$;

satisfying the following universal property:

(UP) Given another such pair $(\mathcal{G}, i)$, there exists a unique functor $\mathcal{G} \xrightarrow{\exists!} \mathrm{Core}(\mathcal{C})$ making the diagram

$$\begin{tikzcd} & \mathrm{Core}(\mathcal{C}) \arrow[d, "\iota_{\mathcal{C}}"] \\ \mathcal{G} \arrow[ur, dashed, "\exists!"] \arrow[r, "i"'] & \mathcal{C} \end{tikzcd}$$

commute.

[1] *Further Notation:* Also written $\mathcal{C}^{\simeq}$.

Construction 11.3.3.2 ▶ Construction of the Core of a Category

The core of $\mathcal{C}$ is the wide subcategory of $\mathcal{C}$ spanned by the isomorphisms of $\mathcal{C}$, i.e. the category $\mathrm{Core}(\mathcal{C})$ where[1]

1. *Objects.* We have
$$\mathrm{Obj}(\mathrm{Core}(\mathcal{C})) \stackrel{\text{def}}{=} \mathrm{Obj}(\mathcal{C});$$
2. *Morphisms.* The morphisms of $\mathrm{Core}(\mathcal{C})$ are the isomorphisms of $\mathcal{C}$.

[1] *Slogan:* The groupoid $\mathrm{Core}(\mathcal{C})$ is the maximal subgroupoid of $\mathcal{C}$.

Proof 11.3.3.3 ▶ Proof of ??

This follows from the fact that functors preserve isomorphisms. ∎

Proposition 11.3.3.4 ▶ Properties of the Core of a Category

Let $\mathcal{C}$ be a category.

1. *Functoriality.* The assignment $\mathcal{C} \mapsto \mathrm{Core}(\mathcal{C})$ defines a functor
$$\mathrm{Core}\colon \mathsf{Cats} \longrightarrow \mathsf{Grpd}.$$
2. *2-Functoriality.* The assignment $\mathcal{C} \mapsto \mathrm{Core}(\mathcal{C})$ defines a 2-functor
$$\mathrm{Core}\colon \mathsf{Cats}_2 \longrightarrow \mathsf{Grpd}_2.$$
3. *Adjointness.* We have an adjunction
$$(\iota \dashv \mathrm{Core})\colon \quad \begin{tikzcd} \mathsf{Grpd} \arrow[r, bend left, hook, "\iota"] \arrow[r, phantom, "\perp"] & \mathsf{Cats}, \arrow[l, bend left, "\mathrm{Core}"] \end{tikzcd}$$
witnessed by a bijection of sets
$$\mathrm{Hom}_{\mathsf{Cats}}(\mathcal{G}, \mathcal{D}) \cong \mathrm{Hom}_{\mathsf{Grpd}}(\mathcal{G}, \mathrm{Core}(\mathcal{D})),$$
natural in $\mathcal{G} \in \mathrm{Obj}(\mathsf{Grpd})$ and $\mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$, forming, together with the functor K_0 of Item 1 of Proposition 11.3.2.2, a triple adjunction
$$(\mathrm{K}_0 \dashv \iota \dashv \mathrm{Core})\colon \quad \begin{tikzcd} \mathsf{Cats} \arrow[r, bend left, "\mathrm{K}_0"] \arrow[r, bend right, "\mathrm{Core}"'] & \mathsf{Grpd}, \arrow[l, hook', "\iota" description] \end{tikzcd}$$

witnessed by bijections of sets

$$\begin{aligned}\mathrm{Hom}_{\mathsf{Grpd}}(\mathrm{K}_0(\mathcal{C}),\mathcal{G}) &\cong \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C},\mathcal{G}),\\ \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{G},\mathcal{D}) &\cong \mathrm{Hom}_{\mathsf{Grpd}}(\mathcal{G},\mathsf{Core}(\mathcal{D})),\end{aligned}$$

natural in $\mathcal{C},\mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$ and $\mathcal{G} \in \mathrm{Obj}(\mathsf{Grpd})$.

4. *2-Adjointness.* We have an adjunction

$$(\iota \dashv \mathsf{Core})\colon\quad \begin{tikzcd}\mathsf{Grpd} \arrow[r, bend left, hook, "\iota"] \arrow[r, phantom, "\perp_2"] & \mathsf{Cats} \arrow[l, bend left, "\mathsf{Core}"]\end{tikzcd},$$

witnessed by an isomorphism of categories

$$\mathsf{Fun}(\mathcal{G},\mathcal{D}) \cong \mathsf{Fun}(\mathcal{G},\mathsf{Core}(\mathcal{D})),$$

natural in $\mathcal{G} \in \mathrm{Obj}(\mathsf{Grpd})$ and $\mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$, forming, together with the 2-functor K_0 of Item 2 of Proposition 11.3.2.2, a triple 2-adjunction

$$(\mathrm{K}_0 \dashv \iota \dashv \mathsf{Core})\colon\quad \begin{tikzcd}\mathsf{Cats} \arrow[r, bend left=50, "\mathrm{K}_0"] \arrow[r, bend right=50, "\mathsf{Core}"'] & \mathsf{Grpd} \arrow[l, hook', "\iota" description]\end{tikzcd},$$

witnessed by isomorphisms of categories

$$\begin{aligned}\mathsf{Fun}(\mathrm{K}_0(\mathcal{C}),\mathcal{G}) &\cong \mathsf{Fun}(\mathcal{C},\mathcal{G}),\\ \mathsf{Fun}(\mathcal{G},\mathcal{D}) &\cong \mathsf{Fun}(\mathcal{G},\mathsf{Core}(\mathcal{D})),\end{aligned}$$

natural in $\mathcal{C},\mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$ and $\mathcal{G} \in \mathrm{Obj}(\mathsf{Grpd})$.

5. *Symmetric Strong Monoidality With Respect to Products.* The core functor of Item 1 has a symmetric strong monoidal structure

$$(\mathsf{Core},\mathsf{Core}^{\times},\mathsf{Core}^{\times}_{\mathbb{1}})\colon (\mathsf{Cats},\times,\mathrm{pt}) \longrightarrow (\mathsf{Grpd},\times,\mathrm{pt})$$

being equipped with isomorphisms

$$\begin{aligned}\mathsf{Core}^{\times}_{\mathcal{C},\mathcal{D}}\colon \mathsf{Core}(\mathcal{C})\times\mathsf{Core}(\mathcal{D}) &\xrightarrow{\cong} \mathsf{Core}(\mathcal{C}\times\mathcal{D}),\\ \mathsf{Core}^{\times}_{\mathbb{1}}\colon \mathrm{pt} &\xrightarrow{\cong} \mathsf{Core}(\mathrm{pt}),\end{aligned}$$

natural in $\mathcal{C},\mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$.

6. *Symmetric Strong Monoidality With Respect to Coproducts.* The core functor of Item 1 has a symmetric strong monoidal structure

$$\left(\mathsf{Core}, \mathsf{Core}^{\coprod}, \mathsf{Core}^{\coprod}_{\mathbb{1}}\right) \colon (\mathsf{Cats}, \coprod, \emptyset_{\mathsf{cat}}) \to (\mathsf{Grpd}, \coprod, \emptyset_{\mathsf{cat}})$$

being equipped with isomorphisms

$$\mathsf{Core}^{\coprod}_{\mathcal{C},\mathcal{D}} \colon \mathsf{Core}(\mathcal{C}) \coprod \mathsf{Core}(\mathcal{D}) \xrightarrow{\cong} \mathsf{Core}(\mathcal{C} \coprod \mathcal{D}),$$
$$\mathsf{Core}^{\coprod}_{\mathbb{1}} \colon \emptyset_{\mathsf{cat}} \xrightarrow{\cong} \mathsf{Core}(\emptyset_{\mathsf{cat}}),$$

natural in $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$.

Proof 11.3.3.5 ▶ Proof of Proposition 11.3.3.4

Item 1: Functoriality

Omitted.

Item 2: 2-Functoriality

Omitted.

Item 3: Adjointness

The adjunction $(\mathrm{K}_0 \dashv \iota)$ follows from the universal property of the Gabriel–Zisman localisation of a category with respect to a class of morphisms (**??**), while the adjunction $(\iota \dashv \mathsf{Core})$ is a reformulation of the universal property of the core of a category (Definition 11.3.3.1).[1]

Item 4: 2-Adjointness

Omitted.

Item 5: Symmetric Strong Monoidality With Respect to Products

Omitted.

Item 6: Symmetric Strong Monoidality With Respect to Coproducts

Omitted. □

[1] *Reference:* [Rie17, Example 4.1.15]

11.4 Functors

11.4.1 Foundations

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 11.4.1.1 ▶ Functors

A **functor** $F\colon \mathcal{C} \to \mathcal{D}$ **from** $\mathcal{C}$ **to** $\mathcal{D}$[1] consists of[2]

1. *Action on Objects.* A map of sets

$$F\colon \mathrm{Obj}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{D}),$$

called the **action on objects of** F;

2. *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathcal{C})$, a map

$$F_{A,B}\colon \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F_A, F_B),$$

called the **action on morphisms of** F **at** (A, B)[3];

satisfying the following conditions:

1. *Preservation of Identities.* For each $A \in \mathrm{Obj}(\mathcal{C})$, the diagram

$$\begin{tikzcd}
\mathrm{pt} \arrow[d, "\mathbb{1}^{\mathcal{C}}_{A}"'] \arrow[dr, "\mathbb{1}^{\mathcal{D}}_{F_A}"] & \\
\mathrm{Hom}_{\mathcal{C}}(A, A) \arrow[r, "F_{A,A}"'] & \mathrm{Hom}_{\mathcal{D}}(F_A, F_A)
\end{tikzcd}$$

commutes, i.e. we have

$$F(\mathrm{id}_A) = \mathrm{id}_{F_A}.$$

2. *Preservation of Composition.* For each $A, B, C \in \mathrm{Obj}(\mathcal{C})$, the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(B, C) \times \mathrm{Hom}_{\mathcal{C}}(A, B) \arrow[r, "\circ^{\mathcal{C}}_{A,B,C}"] \arrow[d, "F_{B,C} \times F_{A,B}"'] & \mathrm{Hom}_{\mathcal{C}}(A, C) \arrow[d, "F_{A,C}"] \\
\mathrm{Hom}_{\mathcal{D}}(F_B, F_C) \times \mathrm{Hom}_{\mathcal{D}}(F_A, F_B) \arrow[r, "\circ^{\mathcal{D}}_{F_A,F_B,F_C}"'] & \mathrm{Hom}_{\mathcal{D}}(F_A, F_C)
\end{tikzcd}$$

commutes, i.e. for each composable pair (g, f) of morphisms of $\mathcal{C}$, we have

$$F(g \circ f) = F(g) \circ F(f).$$

---

[1] *Further Terminology:* Also called a **covariant functor**.

[2] *Further Notation:* Given functors $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{C}^{\mathsf{op}} \to \mathcal{D}$, we will sometimes write F_A for $F(A)$ (resp. G^A for $G(A)$) and F_f for $F(f)$ (resp. G^f for $G(f)$). This has been called Einstein notation in the literature.

[3] *Further Terminology:* Also called **action on Hom-sets of** F **at** (A, B).

Example 11.4.1.2 ▶ Identity Functors

The **identity functor** of a category $\mathcal{C}$ is the functor $\mathrm{id}_{\mathcal{C}} \colon \mathcal{C} \to \mathcal{C}$ where

1. *Action on Objects.* For each $A \in \mathrm{Obj}(\mathcal{C})$, we have

$$\mathrm{id}_{\mathcal{C}}(A) \stackrel{\text{def}}{=} A;$$

2. *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathcal{C})$, the action on morphisms map

$$(\mathrm{id}_{\mathcal{C}})_{A,B} \colon \mathrm{Hom}_{\mathcal{C}}(A, B) \to \underbrace{\mathrm{Hom}_{\mathcal{C}}(\mathrm{id}_{\mathcal{C}}(A), \mathrm{id}_{\mathcal{C}}(B))}_{\stackrel{\text{def}}{=}\mathrm{Hom}_{\mathcal{C}}(A,B)}$$

of $\mathrm{id}_{\mathcal{C}}$ at (A, B) is defined by

$$(\mathrm{id}_{\mathcal{C}})_{A,B} \stackrel{\text{def}}{=} \mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(A,B)}.$$

Proof 11.4.1.3 ▶ Proof of Example 11.4.1.2

Preservation of Identities

We have $\mathrm{id}_{\mathcal{C}}(\mathrm{id}_A) \stackrel{\text{def}}{=} \mathrm{id}_A$ for each $A \in \mathrm{Obj}(\mathcal{C})$ by definition.

Preservation of Compositions

For each composable pair $A \xrightarrow{f} B \xrightarrow{g} B$ of morphisms of $\mathcal{C}$, we have

$$\begin{aligned} \mathrm{id}_{\mathcal{C}}(g \circ f) &\stackrel{\text{def}}{=} g \circ f \\ &\stackrel{\text{def}}{=} \mathrm{id}_{\mathcal{C}}(g) \circ \mathrm{id}_{\mathcal{C}}(f). \end{aligned}$$

This finishes the proof. ∎

Definition 11.4.1.4 ▶ Composition of Functors

The **composition** of two functors $F \colon \mathcal{C} \to \mathcal{D}$ and $G \colon \mathcal{D} \to \mathcal{E}$ is the functor $G \circ F$ where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathcal{C})$, we have

$$[G \circ F](A) \stackrel{\text{def}}{=} G(F(A));$$

- *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathcal{C})$, the action on morphisms map

$$(G \circ F)_{A,B} \colon \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{E}}(G_{F_A}, G_{F_B})$$

of $G \circ F$ at (A, B) is defined by

$$[G \circ F](f) \stackrel{\text{def}}{=} G(F(f)).$$

PROOF 11.4.1.5 ► PROOF OF DEFINITION 11.4.1.4

Preservation of Identities

For each $A \in \text{Obj}(\mathcal{C})$, we have

$$\begin{aligned} G_{F_{\text{id}_A}} &= G_{\text{id}_{F_A}} && \text{(functoriality of } F\text{)} \\ &= \text{id}_{G_{F_A}}. && \text{(functoriality of } G\text{)} \end{aligned}$$

Preservation of Composition

For each composable pair (g, f) of morphisms of $\mathcal{C}$, we have

$$\begin{aligned} G_{F_{g \circ f}} &= G_{F_g \circ F_f} && \text{(functoriality of } F\text{)} \\ &= G_{F_g} \circ G_{F_f}. && \text{(functoriality of } G\text{)} \end{aligned}$$

This finishes the proof.

PROPOSITION 11.4.1.6 ► ELEMENTARY PROPERTIES OF FUNCTORS

Let $F: \mathcal{C} \to \mathcal{D}$ be a functor.

1. *Preservation of Isomorphisms.* If f is an isomorphism in $\mathcal{C}$, then $F(f)$ is an isomorphism in $\mathcal{D}$.[1]

[1] When the converse holds, we call F *conservative*, see Definition 11.4.6.1.

PROOF 11.4.1.7 ► PROOF OF PROPOSITION 11.4.1.6

Item 1: Preservation of Isomorphisms

Indeed, we have

$$\begin{aligned} F(f)^{-1} \circ F(f) &= F\left(f^{-1} \circ f\right) \\ &= F(\text{id}_A) \\ &= \text{id}_{F(A)} \end{aligned}$$

and

$$\begin{aligned} F(f) \circ F(f)^{-1} &= F(f \circ f^{-1}) \\ &= F(\mathrm{id}_A) \\ &= \mathrm{id}_{F(A)}, \end{aligned}$$

showing $F(f)$ to be an isomorphism.

11.4.2 Faithful Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 11.4.2.1 ► Faithful Functors

A functor $F \colon \mathcal{C} \to \mathcal{D}$ is **faithful** if, for each $A, B \in \mathrm{Obj}(\mathcal{C})$, the action on morphisms map

$$F_{A,B} \colon \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F_A, F_B)$$

of F at (A, B) is injective.

Proposition 11.4.2.2 ► Properties of Faithful Functors

Let $F \colon \mathcal{C} \to \mathcal{D}$ be a functor

1. *Characterisations.* The following conditions are equivalent:

   (a) The functor $F \colon \mathcal{C} \to \mathcal{D}$ is faithful.

   (b) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor

   $$F^* \colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

   is faithful.

   (c) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor

   $$F_* \colon \mathsf{Fun}(\mathcal{X}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{X}, \mathcal{D})$$

   is faithful.

Proof 11.4.2.3 ► Proof of Proposition 11.4.2.2

Item 1: Characterisations

Omitted.

11.4.3 Full Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 11.4.3.1 ▶ Full Functors

A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **full** if, for each $A, B \in \mathrm{Obj}(\mathcal{C})$, the action on morphisms map

$$F_{A,B}\colon \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F_A, F_B)$$

of F at (A, B) is surjective.

Proposition 11.4.3.2 ▶ Properties of Full Functors

Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor

1. *Characterisations.* The following conditions are equivalent:
   (a) The functor $F\colon \mathcal{C} \to \mathcal{D}$ is full.
   (b) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor
   $$F^*\colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$
   is full.
   (c) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor
   $$F_*\colon \mathsf{Fun}(\mathcal{X}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{X}, \mathcal{D})$$
   is full.

Proof 11.4.3.3 ▶ Proof of Proposition 11.4.3.2

Item 1: Characterisations

Omitted. □

11.4.4 Fully Faithful Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 11.4.4.1 ▶ Fully Faithful Functors

A functor $F: \mathcal{C} \to \mathcal{D}$ is **fully faithful** if F is full and faithful, i.e. if, for each $A, B \in \mathrm{Obj}(\mathcal{C})$, the action on morphisms map

$$F_{A,B}: \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F_A, F_B)$$

of F at (A, B) is bijective.

Proposition 11.4.4.2 ▶ Properties of Fully Faithful Functors

Let $F: \mathcal{C} \to \mathcal{D}$ be a functor

1. *Characterisations.* The following conditions are equivalent:
   (a) The functor $F: \mathcal{C} \to \mathcal{D}$ is fully faithful.
   (b) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor
   $$F^*: \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$
   is fully faithful.
   (c) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor
   $$F_*: \mathsf{Fun}(\mathcal{X}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{X}, \mathcal{D})$$
   is fully faithful.
2. *Conservativity.* If F is fully faithful, then F is conservative.

Proof 11.4.4.3 ▶ Proof of Proposition 11.4.4.2

Item 1: Characterisations

Omitted.

Item 2: Conservativity

This is proved in Item 2 of Proposition 11.4.6.2.

11.4.5 Essentially Surjective Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 11.4.5.1 ▶ Essentially Surjective Functors

A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **essentially surjective** if, for each $D \in \mathrm{Obj}(\mathcal{D})$, there exists some object A of $\mathcal{C}$ with $F(A) \cong D$.

11.4.6 Conservative Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 11.4.6.1 ▶ Conservative Functors

A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **conservative** if it satisfies the following condition:

($\star$) For each $f \in \mathrm{Mor}(\mathcal{C})$, if $F(f)$ is an isomorphism in $\mathcal{D}$, then f is an isomorphism in $\mathcal{C}$.

Proposition 11.4.6.2 ▶ Properties of Conservative Functors

Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor.

1. *Characterisations.* The following conditions are equivalent:
   (a) The functor F is conservative.
   (b) For each $f \in \mathrm{Mor}(\mathcal{C})$, the morphism $F(f)$ is an isomorphism in $\mathcal{D}$ iff f is an isomorphism in $\mathcal{C}$.
2. *Interaction With Fully Faithfulness.* Every fully faithful functor is conservative.

Proof 11.4.6.3 ▶ Proof of Proposition 11.4.6.2

Item 1: Characterisations

This follows from Item 1 of Proposition 11.4.1.6.

Item 2: Interaction With Fully Faithfulness

Let $F\colon \mathcal{C} \to \mathcal{D}$ be a fully faithful functor, let $f: A \to B$ be a morphism of $\mathcal{C}$, and suppose that F_f is an isomorphism. We have

$$\begin{aligned} F(\mathrm{id}_B) &= \mathrm{id}_{F(B)} \\ &= F(f) \circ F(f)^{-1} \\ &= F\left(f \circ f^{-1}\right). \end{aligned}$$

Similarly, $F(\mathrm{id}_A) = F(f^{-1} \circ f)$. But since F is fully faithful, we must have

$$\begin{aligned} f \circ f^{-1} &= \mathrm{id}_B, \\ f^{-1} \circ f &= \mathrm{id}_A, \end{aligned}$$

showing f to be an isomorphism. Thus F is conservative.

11.4.7 Equivalences of Categories

Definition 11.4.7.1 ► Equivalences of Categories

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

- An **equivalence of categories** between $\mathcal{C}$ and $\mathcal{D}$ consists of a pair of functors

$$\begin{aligned} F &: \mathcal{C} \to \mathcal{D}, \\ G &: \mathcal{D} \to \mathcal{C} \end{aligned}$$

  together with natural isomorphisms

$$\begin{aligned} \eta &: \mathrm{id}_{\mathcal{C}} \overset{\cong}{\Longrightarrow} G \circ F, \\ \epsilon &: F \circ G \overset{\cong}{\Longrightarrow} \mathrm{id}_{\mathcal{D}}. \end{aligned}$$

- An **adjoint equivalence of categories** between $\mathcal{C}$ and $\mathcal{D}$ is an equivalence (F, G, η, ϵ) between $\mathcal{C}$ and $\mathcal{D}$ which is also an adjunction.

Proposition 11.4.7.2 ► Properties of Equivalences of Categories

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor.

1. *Characterisations.* If $\mathcal{C}$ and $\mathcal{D}$ are small[1], then the following conditions are equivalent:[2]

   (a) The functor F is an equivalence of categories.

   (b) The functor F is fully faithful and essentially surjective.

   (c) The induced functor

$$\upharpoonright F\mathrm{Sk}(\mathcal{C}) : \mathrm{Sk}(\mathcal{C}) \to \mathrm{Sk}(\mathcal{D})$$

   is an *isomorphism* of categories.

2. *Two-Out-of-Three.* Let

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "G\circ F"] \arrow[dr, "F"'] & & \mathcal{E} \\ & \mathcal{D} \arrow[ur, "G"'] & \end{tikzcd}$$

be a diagram in Cats. If two out of the three functors among F, G, and $G \circ F$ are equivalences of categories, then so is the third.

3. *Stability Under Composition.* Let

$$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D} \underset{G'}{\overset{F'}{\rightleftarrows}} \mathcal{E}$$

be a diagram in Cats. If (F, G) and (F', G') are equivalences of categories, then so is their composite $(F' \circ F, G' \circ G)$.

4. *Equivalences vs.Adjoint Equivalences.* Every equivalence of categories can be promoted to an adjoint equivalence.[3]

5. *Interaction With Groupoids.* If $\mathcal{C}$ and $\mathcal{D}$ are groupoids, then the following conditions are equivalent:

   (a) The functor F is an equivalence of groupoids.

   (b) The following conditions are satisfied:

      i. The functor F induces a bijection

      $$\pi_0(F)\colon \pi_0(\mathcal{C}) \to \pi_0(\mathcal{D})$$

      of sets.

      ii. For each $A \in \mathrm{Obj}(\mathcal{C})$, the induced map

      $$F_{x,x}\colon \mathrm{Aut}_{\mathcal{C}}(A) \to \mathrm{Aut}_{\mathcal{D}}(F_A)$$

      is an isomorphism of groups.

[1] Otherwise there will be size issues. One can also work with large categories and universes, or require F to be *constructively* essentially surjective; see [**MSE1465107**].

[2] In ZFC, the equivalence between Item 1a and Item 1b is equivalent to the axiom of choice; see [MO 119454].

In Univalent Foundations, this is true without requiring neither the axiom of choice nor the law of the excluded middle.

[3] More precisely, we can promote an equivalence of categories (F, G, η, ϵ) to adjoint equivalences (F, G, η', ϵ) and (F, G, η, ϵ').

Proof 11.4.7.3 ► Proof of Proposition 11.4.7.2

Item 1: Characterisations

We claim that Items 1a to 1c are indeed equivalent:

1. *Item 1a* $\implies$ *Item 1b*. Clear.

2. *Item 1b* $\implies$ *Item 1a*. Since F is essentially surjective and $\mathcal{C}$ and $\mathcal{D}$ are small, we can choose, using the axiom of choice, for each $B \in \mathrm{Obj}(\mathcal{D})$, an object j_B of $\mathcal{C}$ and an isomorphism $i_B \colon B \to F_{j_B}$ of $\mathcal{D}$.

   Since F is fully faithful, we can extend the assignment $B \mapsto j_B$ to a *unique* functor $j \colon \mathcal{D} \to \mathcal{C}$ such that the isomorphisms $i_B \colon B \to F_{j_B}$ assemble into a natural isomorphism $\eta \colon \mathrm{id}_{\mathcal{D}} \overset{\cong}{\Longrightarrow} F \circ j$, with a similar natural isomorphism $\epsilon \colon \mathrm{id}_{\mathcal{C}} \overset{\cong}{\Longrightarrow} j \circ F$. Hence F is an equivalence.

3. *Item 1a* $\implies$ *Item 1c*. This follows from **??**.

Item 2: Two-Out-of-Three

Omitted.

Item 3: Stability Under Composition

Clear.

Item 4: Equivalences vs.Adjoint Equivalences

See [Rie17, Proposition 4.4.5].

Item 5: Interaction With Groupoids

See [nLa24b, Proposition 4.4].

11.4.8 Isomorphisms of Categories

Definition 11.4.8.1 ► Isomorphisms of Categories

An **isomorphism of categories** is a pair of functors

$$F \colon \mathcal{C} \to \mathcal{D},$$
$$G \colon \mathcal{D} \to \mathcal{C}$$

such that we have

$$G \circ F = \mathrm{id}_{\mathcal{C}},$$

$$F \circ G = \mathrm{id}_{\mathcal{D}}.$$

Example 11.4.8.2 ▶ Equivalent But Non-Isomorphic Categories

Categories can be equivalent but non-isomorphic. For example, the category consisting of two isomorphic objects is equivalent to pt, but not isomorphic to it.

Proposition 11.4.8.3 ▶ Properties of Isomorphisms of Categories

Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor.

1. *Characterisations.* If $\mathcal{C}$ and $\mathcal{D}$ are small, then the following conditions are equivalent:

   (a) The functor F is an isomorphism of categories.

   (b) The functor F is fully faithful and a bijection on objects.

Proof 11.4.8.4 ▶ Proof of Proposition 11.4.8.3

Item 1: Characterisations

Omitted, but similar to Item 1 of Proposition 11.4.7.2.

11.4.9 The Natural Transformation Associated to a Functor

Definition 11.4.9.1 ▶ The Natural Transformation Associated to a Functor

Every functor $F\colon \mathcal{C} \to \mathcal{D}$ defines a natural transformation[1]

$$F^{\dagger}\colon \mathrm{Hom}_{\mathcal{C}} \Longrightarrow \mathrm{Hom}_{\mathcal{D}} \circ (F^{\mathsf{op}} \times F), \qquad \begin{tikzcd} \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \arrow[rr, "F^{\mathsf{op}} \times F"] \arrow[dr, "\mathrm{Hom}_{\mathcal{C}}"'] & \arrow[d, phantom, "\overset{F^{\dagger}}{\Nearrow}"] & \mathcal{D}^{\mathsf{op}} \times \mathcal{D} \arrow[dl, "\mathrm{Hom}_{\mathcal{D}}"] \\ & \mathsf{Sets}, & \end{tikzcd}$$

called the **natural transformation associated to** F, consisting of the collection

$$\left\{F^{\dagger}_{A,B}\colon \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F_A, F_B)\right\}_{(A,B)\in\mathrm{Obj}(\mathcal{C}^{\mathsf{op}}\times\mathcal{C})}$$

with

$$F^{\dagger}_{A,B} \stackrel{\text{def}}{=} F_{A,B}.$$

[1]This is the 1-categorical version of **??** of **??**.

Proof 11.4.9.2 ▶ Proof of Definition 11.4.9.1

The naturality condition for $F^{\dagger}$ is the requirement that for each morphism

$$(\phi, \psi) \colon (X, Y) \to (A, B)$$

of $\mathcal{C}^{\mathsf{op}} \times \mathcal{C}$, the diagram

$$\begin{tikzcd}
\mathsf{Hom}_{\mathcal{C}}(X, Y) \arrow[rr, "\phi^{*}\circ\psi_{*}=\psi_{*}\circ\phi^{*}"] \arrow[d, "F_{X,Y}"'] & & \mathsf{Hom}_{\mathcal{C}}(A, B) \arrow[d, "F_{A,B}"] \\
\mathsf{Hom}_{\mathcal{D}}(F_X, F_Y) \arrow[rr, "F(\phi)^{*}\circ F(\psi)_{*}=F(\psi)_{*}\circ F(\phi)^{*}"'] & & \mathsf{Hom}_{\mathcal{D}}(F_A, F_B),
\end{tikzcd}$$

acting on elements as

$$\begin{tikzcd}
f \arrow[r, mapsto] \arrow[d, mapsto] & \psi \circ f \circ \phi \arrow[d, mapsto] \\
F(f) \arrow[r, mapsto] & F(\psi) \circ F(f) \circ F(\psi) = F(\psi \circ f \circ \phi)
\end{tikzcd}$$

commutes, which follows from the functoriality of F. ∎

Proposition 11.4.9.3 ▶ Properties of Natural Transformations Associated to Functors

Let $F \colon \mathcal{C} \to \mathcal{D}$ and $G \colon \mathcal{D} \to \mathcal{E}$ be functors.

1. *Interaction With Natural Isomorphisms.* The following conditions are equivalent:

   (a) The natural transformation $F^{\dagger} \colon \mathsf{Hom}_{\mathcal{C}} \Longrightarrow \mathsf{Hom}_{\mathcal{D}} \circ (F^{\mathsf{op}} \times F)$ associated to F is a natural isomorphism.

   (b) The functor F is fully faithful.

2. *Interaction With Composition.* We have an equality of pasting diagrams

$$\begin{tikzcd}
\mathcal{C}^{\mathsf{op}}\times\mathcal{C} \arrow[r, "F^{\mathsf{op}}\times F"] \arrow[dr, "\mathsf{Hom}_{\mathcal{C}}"'] \arrow[r, phantom, shift right=5ex, "\Nearrow F^{\dagger}"] & \mathcal{D}^{\mathsf{op}}\times\mathcal{D} \arrow[r, "G^{\mathsf{op}}\times G"] \arrow[d, "\mathsf{Hom}_{\mathcal{D}}"] \arrow[r, phantom, shift right=5ex, "\Nearrow G^{\dagger}"] & \mathcal{E}^{\mathsf{op}}\times\mathcal{E} \arrow[dl, "\mathsf{Hom}_{\mathcal{E}}"] \\
& \mathsf{Sets} &
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{C}^{\mathsf{op}}\times\mathcal{C} \arrow[rr, "(G\circ F)^{\mathsf{op}}\times(G\circ F)"] \arrow[dr, "\mathsf{Hom}_{\mathcal{C}}"'] \arrow[rr, phantom, shift right=5ex, "\Nearrow (G\circ F)^{\dagger}"] & & \mathcal{E}^{\mathsf{op}}\times\mathcal{E}, \arrow[dl, "\mathsf{Hom}_{\mathcal{E}}"] \\
& \mathsf{Sets} &
\end{tikzcd}$$

in Cats_2, i.e. we have

$$(G \circ F)^{\dagger} = \left(G^{\dagger} \star \mathrm{id}_{F^{\mathrm{op}} \times F}\right) \circ F^{\dagger}.$$

3. *Interaction With Identities.* We have

$$\mathrm{id}_{C}^{\dagger} = \mathrm{id}_{\mathrm{Hom}_{C}(-_1,-_2)},$$

i.e. the natural transformation associated to id_C is the identity natural transformation of the functor $\mathrm{Hom}_C(-_1, -_2)$.

Proof 11.4.9.4 ▶ Proof of Proposition 11.4.9.3

Item 1: Interaction With Natural Isomorphisms

Clear.

Item 2: Interaction With Composition

Clear.

Item 3: Interaction With Identities

Clear.

11.5 Natural Transformations

11.5.1 Foundations

Let C and $\mathcal{D}$ be categories and $F, G\colon C \rightrightarrows \mathcal{D}$ be functors.

Definition 11.5.1.1 ▶ Transformations

A **transformation**[1,2] $\alpha\colon F \overset{\text{unnat}}{\Longrightarrow} G$ **from** F **to** G is a collection

$$\{\alpha_A\colon F(A) \to G(A)\}_{A \in \mathrm{Obj}(C)}$$

of morphisms of $\mathcal{D}$.

[1] *Further Terminology:* Also called an **unnatural transformation** for emphasis.
[2] *Further Notation:* We write $\mathrm{UnNat}(F, G)$ for the set of unnatural transformations from F to G.

Definition 11.5.1.2 ▶ Natural Transformations

A **natural transformation**[1] $\alpha\colon F \Longrightarrow G$ **from** F **to** G is a transformation

$$\{\alpha_A\colon F(A) \to G(A)\}_{A \in \mathrm{Obj}(C)}$$

from F to G such that, for each morphism $f\colon A \to B$ of $\mathcal{C}$, the diagram

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\alpha_A"'] & F(B) \arrow[d, "\alpha_B"] \\ G(A) \arrow[r, "G(f)"'] & G(B) \end{tikzcd}$$

commutes.[2,3]

[1] Pictured in diagrams as

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{D}. \end{tikzcd}$$

[2] *Further Terminology:* The morphism $\alpha_A\colon F_A \to G_A$ is called the **component of** α **at** A.
[3] *Further Notation:* We write $\mathrm{Nat}(F, G)$ for the set of natural transformations from F to G.

Example 11.5.1.3 ▶ Identity Natural Transformations

The **identity natural transformation** $\mathrm{id}_F\colon F \Longrightarrow F$ **of** F is the natural transformation consisting of the collection

$$\{\mathrm{id}_{F(A)}\colon F(A) \to F(A)\}_{A\in\mathrm{Obj}(\mathcal{C})}.$$

Proof 11.5.1.4 ▶ Proof of Example 11.5.1.3

The naturality condition for id_F is the requirement that, for each morphism $f\colon A \to B$ of $\mathcal{C}$, the diagram

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\mathrm{id}_{F(A)}"'] & F(B) \arrow[d, "\mathrm{id}_{F(B)}"] \\ F(A) \arrow[r, "F(f)"'] & F(B) \end{tikzcd}$$

commutes, which follows from unitality of the composition of $\mathcal{C}$.

Definition 11.5.1.5 ▶ Equality of Natural Transformations

Two natural transformations $\alpha, \beta\colon F \Longrightarrow G$ are **equal** if we have

$$\alpha_A = \beta_A$$

for each $A \in \mathrm{Obj}(\mathcal{C})$.

11.5.2 Vertical Composition of Natural Transformations

The **vertical composition** of two natural transformations $\alpha\colon F \implies G$ and $\beta\colon G \implies H$ as in the diagram

$$\begin{tikzcd} \mathcal{C} \arrow[rr, bend left, "F"] \arrow[rr, "G" description] \arrow[rr, bend right, "H"'] & & \mathcal{D} \end{tikzcd}$$

with $\alpha\Downarrow$ between F and G, and $\beta\Downarrow$ between G and H,

is the natural transformation $\beta \circ \alpha\colon F \implies H$ consisting of the collection

$$\left\{(\beta \circ \alpha)_A \colon F(A) \to H(A)\right\}_{A \in \mathrm{Obj}(\mathcal{C})}$$

with

$$(\beta \circ \alpha)_A \stackrel{\text{def}}{=} \beta_A \circ \alpha_A$$

for each $A \in \mathrm{Obj}(\mathcal{C})$.

PROOF 11.5.2.2 ▶ PROOF OF DEFINITION 11.5.2.1

The naturality condition for $\beta \circ \alpha$ is the requirement that the boundary of the diagram

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\alpha_A"'] \arrow[dr, phantom, "(1)"] & F(B) \arrow[d, "\alpha_B"] \\ G(A) \arrow[r, "G(f)" description] \arrow[d, "\beta_A"'] \arrow[dr, phantom, "(2)"] & G(B) \arrow[d, "\beta_B"] \\ H(A) \arrow[r, "H(f)"'] & H(B) \end{tikzcd}$$

commutes. Since

1. Subdiagram (1) commutes by the naturality of α;
2. Subdiagram (2) commutes by the naturality of β;

so does the boundary diagram. Hence $\beta \circ \alpha$ is a natural transformation. ∎

Proposition 11.5.2.3 ▶ Properties of Vertical Composition of Natural Transformations

Let $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ be categories.

1. *Functionality.* The assignment $(\beta, \alpha) \mapsto \beta \circ \alpha$ defines a function

$$\circ_{F,G,H} \colon \mathrm{Nat}(G, H) \times \mathrm{Nat}(F, G) \to \mathrm{Nat}(F, H).$$

2. *Associativity.* Let $F, G, H, K \colon \mathcal{C} \rightrightarrows \mathcal{D}$ be functors. The diagram

$$\begin{tikzcd} & \mathrm{Nat}(H,K) \times (\mathrm{Nat}(G,H) \times \mathrm{Nat}(F,G)) \arrow[dr, "\mathrm{id}_{\mathrm{Nat}(H,K)} \times \circ_{F,G,H}"] & \\ (\mathrm{Nat}(H,K) \times \mathrm{Nat}(G,H)) \times \mathrm{Nat}(F,G) \arrow[ur, dashed, "\alpha^{\mathsf{Sets}}_{\mathrm{Nat}(H,K),\mathrm{Nat}(G,H),\mathrm{Nat}(F,G)}", "\sim"'] \arrow[d, "\circ_{G,H,K} \times \mathrm{id}_{\mathrm{Nat}(F,G)}"'] & & \mathrm{Nat}(H,K) \times \mathrm{Nat}(F,H) \arrow[d, "\circ_{F,H,K}"] \\ \mathrm{Nat}(G,K) \times \mathrm{Nat}(F,G) \arrow[rr, "\circ_{F,G,K}"'] & & \mathrm{Nat}(F,K) \end{tikzcd}$$

commutes, i.e. given natural transformations

$$\begin{aligned} \alpha &\colon F \Longrightarrow G, \\ \beta &\colon G \Longrightarrow H, \\ \gamma &\colon H \Longrightarrow K, \end{aligned}$$

we have

$$(\gamma \circ \beta) \circ \alpha = \gamma \circ (\beta \circ \alpha).$$

3. *Unitality.* Let $F, G \colon \mathcal{C} \rightrightarrows \mathcal{D}$ be functors.

   (a) *Left Unitality.* The diagram

$$\begin{tikzcd} \mathrm{pt} \times \mathrm{Nat}(F,G) \arrow[d, "{[\mathrm{id}_G] \times \mathrm{id}_{\mathrm{Nat}(F,G)}}"'] \arrow[dr, dashed, "\lambda^{\mathsf{Sets}}_{\mathrm{Nat}(F,G)}", "\sim"'] & \\ \mathrm{Nat}(G,G) \times \mathrm{Nat}(F,G) \arrow[r, "\circ_{F,G,G}"'] & \mathrm{Nat}(F,G) \end{tikzcd}$$

commutes, i.e. given a natural transformation $\alpha\colon F \Longrightarrow G$, we have

$$\mathrm{id}_G \circ \alpha = \alpha.$$

(b) *Right Unitality.* The diagram

$$\begin{tikzcd}
\mathsf{Nat}(F,G) \times \mathrm{pt} \arrow[d, "\mathrm{id}_{\mathsf{Nat}(F,G)} \times [\mathrm{id}_F]"'] \arrow[drr, dashed, "\rho^{\mathsf{Sets}}_{\mathsf{Nat}(F,G)}", "\sim"'] & & \\
\mathsf{Nat}(F,G) \times \mathsf{Nat}(F,F) \arrow[rr, "\circ^{\mathcal{C}}_{F,F,G}"'] & & \mathsf{Nat}(F,G)
\end{tikzcd}$$

commutes, i.e. given a natural transformation $\alpha\colon F \Longrightarrow G$, we have

$$\alpha \circ \mathrm{id}_F = \alpha.$$

4. *Middle Four Exchange.* Let $F_1, F_2, F_3\colon \mathcal{C} \to \mathcal{D}$ and $G_1, G_2, G_3\colon \mathcal{D} \to \mathcal{E}$ be functors. The diagram

$$\begin{tikzcd}
(\mathsf{Nat}(G_2,G_3) \times \mathsf{Nat}(G_1,G_2)) \times (\mathsf{Nat}(F_2,F_3) \times \mathsf{Nat}(F_1,F_2)) \arrow[rr, dashed, leftrightarrow, "\mu_4", "\sim"'] \arrow[d, "\circ_{G_1,G_2,G_3} \times \circ_{F_1,F_2,F_3}"'] & & (\mathsf{Nat}(G_2,G_3) \times \mathsf{Nat}(F_2,F_3)) \times (\mathsf{Nat}(G_1,G_2) \times \mathsf{Nat}(F_1,F_2)) \arrow[d, "\star_{F_2,F_3,G_2,G_3} \times \star_{F_1,F_2,G_1,G_2}"] \\
\mathsf{Nat}(G_1,G_3) \times \mathsf{Nat}(F_1,F_3) \arrow[dr, "\star_{F_1,F_3,G_1,G_3}"'] & & \mathsf{Nat}(G_2 \circ F_2, G_3 \circ F_3) \times \mathsf{Nat}(G_1 \circ F_1, G_2 \circ F_2) \arrow[dl, "\circ_{G_1 \circ F_1, G_2 \circ F_2, G_3 \circ F_3}"] \\
& \mathsf{Nat}(G_1 \circ F_1, G_3 \circ F_3) &
\end{tikzcd}$$

commutes, i.e. given a diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=60, "F_1"{name=U1}] \arrow[r, "F_2"{name=M1, description}] \arrow[r, bend right=60, "F_3"'{name=D1}] & \mathcal{D} \arrow[r, bend left=60, "G_1"{name=U2}] \arrow[r, "G_2"{name=M2, description}] \arrow[r, bend right=60, "G_3"'{name=D2}] & \mathcal{E}
\arrow[Rightarrow, from=U1, to=M1, "\alpha"']
\arrow[Rightarrow, from=M1, to=D1, "\alpha'"']
\arrow[Rightarrow, from=U2, to=M2, "\beta"']
\arrow[Rightarrow, from=M2, to=D2, "\beta'"']
\end{tikzcd}$$

in Cats_2, we have

$$(\beta' \star \alpha') \circ (\beta \star \alpha) = (\beta' \circ \beta) \star (\alpha' \circ \alpha).$$

Proof 11.5.2.4 ▶ Proof of Proposition 11.5.2.3

Item 1: Functionality

Clear.

Item 2: Associativity

Indeed, we have

$$\begin{aligned}((\gamma \circ \beta) \circ \alpha)_A &= (\gamma_A \circ \beta_A) \circ \alpha_A \\ &= \gamma_A \circ (\beta_A \circ \alpha_A) \\ &= (\gamma \circ (\beta \circ \alpha))_A\end{aligned}$$

for each $A \in \mathrm{Obj}(\mathcal{C})$, showing the desired equality.

Item 3: Unitality

We have

$$\begin{aligned}(\mathrm{id}_G \circ \alpha)_A &= \mathrm{id}_G \circ \alpha_A \\ &= \alpha_A, \\ (\alpha \circ \mathrm{id}_F)_A &= \alpha_A \circ \mathrm{id}_F \\ &= \alpha_A\end{aligned}$$

for each $A \in \mathrm{Obj}(\mathcal{C})$, showing the desired equality.

Item 4: Middle Four Exchange

This is proved in Item 4 of Proposition 11.5.3.3.

11.5.3 Horizontal Composition of Natural Transformations

Definition 11.5.3.1 ▶ Horizontal Composition of Natural Transformations

The **horizontal composition**[1,2] of two natural transformations $\alpha\colon F \Longrightarrow G$ and $\beta\colon H \Longrightarrow K$ as in the diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{D} \arrow[r, bend left, "H"] \arrow[r, bend right, "K"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{E} \end{tikzcd}$$

of α and β is the natural transformation

$$\beta \star \alpha\colon (H \circ F) \Longrightarrow (K \circ G),$$

as in the diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "H\circ F"] \arrow[r, bend right, "K\circ G"'] \arrow[r, phantom, "\Downarrow \beta\star\alpha"] & \mathcal{E}, \end{tikzcd}$$

consisting of the collection

$$\left\{(\beta \star \alpha)_A : H(F(A)) \to K(G(A))\right\}_{A\in\mathrm{Obj}(\mathcal{C})},$$

of morphisms of $\mathcal{E}$ with

$$\begin{aligned}(\beta \star \alpha)_A &\overset{\text{def}}{=} \beta_{G(A)} \circ H(\alpha_A)\\ &= K(\alpha_A) \circ \beta_{F(A)},\end{aligned} \qquad \begin{tikzcd} H(F(A)) \arrow[r, "H(\alpha_A)"] \arrow[d, "\beta_{F(A)}"'] & H(G(A)) \arrow[d, "\beta_{G(A)}"] \\ K(F(A)) \arrow[r, "K(\alpha_A)"'] & K(G(A)). \end{tikzcd}$$

[1] *Further Terminology:* Also called the **Godement product** of α and β.

[2] Horizontal composition forms a map

$$\star_{(F,H),(G,K)} : \mathrm{Nat}(H,K) \times \mathrm{Nat}(F,G) \to \mathrm{Nat}(H \circ F, K \circ G).$$

PROOF 11.5.3.2 ► PROOF OF DEFINITION 11.5.3.1

First, we claim that we indeed have

$$\beta_{G(A)} \circ H(\alpha_A) = K(\alpha_A) \circ \beta_{F(A)}, \qquad \begin{tikzcd} H(F(A)) \arrow[r, "H(\alpha_A)"] \arrow[d, "\beta_{F(A)}"'] & H(G(A)) \arrow[d, "\beta_{G(A)}"] \\ K(F(A)) \arrow[r, "K(\alpha_A)"'] & K(G(A)). \end{tikzcd}$$

This is, however, simply the naturality square for β applied to the morphism $\alpha_A : F(A) \to G(A)$. Next, we check the naturality condition for $\beta \star \alpha$, which is the requirement that the boundary of the diagram

$$\begin{tikzcd}
H(F(A)) \arrow[r, "H(F(f))"] \arrow[d, "H(\alpha_A)"'] \arrow[dr, phantom, "(1)"] & H(F(B)) \arrow[d, "H(\alpha_B)"] \\
H(G(A)) \arrow[r, "H(G(f))"] \arrow[d, "\beta_{G(A)}"'] \arrow[dr, phantom, "(2)"] & H(G(B)) \arrow[d, "\beta_{G(B)}"] \\
K(G(A)) \arrow[r, "K(G(f))"'] & K(G(B))
\end{tikzcd}$$

commutes. Since

1. Subdiagram (1) commutes by the naturality of α;
2. Subdiagram (2) commutes by the naturality of β;

so does the boundary diagram. Hence $\beta \circ \alpha$ is a natural transformation.[1]

[1] *Reference:* [Bor94, Proposition 1.3.4].

Proposition 11.5.3.3 ▶ Properties of Horizontal Composition of Natural Transformations

Let $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ be categories.

1. *Functionality.* The assignment $(\beta, \alpha) \mapsto \beta \star \alpha$ defines a function

$$\star_{(F,G),(H,K)} \colon \mathsf{Nat}(H, K) \times \mathsf{Nat}(F, G) \to \mathsf{Nat}(H \circ F, K \circ G).$$

2. *Associativity.* Let

$$\mathcal{C} \underset{G_1}{\overset{F_1}{\rightrightarrows}} \mathcal{D} \underset{G_2}{\overset{F_2}{\rightrightarrows}} \mathcal{E} \underset{G_3}{\overset{F_3}{\rightrightarrows}} \mathcal{F}$$

be a diagram in Cats_2. The diagram

$$\begin{tikzcd}
\mathsf{Nat}(F_3, G_3) \times \mathsf{Nat}(F_2, G_2) \times \mathsf{Nat}(F_1, G_1) \arrow[r, "\star_{(F_2,G_2),(F_3,G_3)} \times \mathrm{id}"] \arrow[d, "\mathrm{id} \times \star_{(F_1,G_1),(F_2,G_2)}"'] & \mathsf{Nat}(F_3 \circ F_2, G_3 \circ G_2) \times \mathsf{Nat}(F_1, G_1) \arrow[d, "\star_{(F_3 \circ F_2),(G_3 \circ G_2, F_1, G_1)}"] \\
\mathsf{Nat}(F_3, G_3) \times \mathsf{Nat}(F_2 \circ F_1, G_2 \circ G_1) \arrow[r, "\star_{(F_2 \circ F_1),(G_2 \circ G_1, F_3, G_3)}"'] & \mathsf{Nat}(F_3 \circ F_2 \circ F_1, G_3 \circ G_2 \circ G_1)
\end{tikzcd}$$

commutes, i.e. given natural transformations

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F_1"] \arrow[r, bend right, "G_1"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{D} \arrow[r, bend left, "F_2"] \arrow[r, bend right, "G_2"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{E} \arrow[r, bend left, "F_3"] \arrow[r, bend right, "G_3"'] \arrow[r, phantom, "\gamma\Downarrow"] & \mathcal{F}, \end{tikzcd}$$

we have

$$(\gamma \star \beta) \star \alpha = \gamma \star (\beta \star \alpha).$$

3. *Interaction With Identities.* Let $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{E}$ be functors. The diagram

$$\begin{tikzcd} \mathrm{pt} \times \mathrm{pt} \arrow[r, "{[\mathrm{id}_G] \times [\mathrm{id}_F]}"] & \mathrm{Nat}(G, G) \times \mathrm{Nat}(F, F) \arrow[d, "\star_{(F,F),(G,G)}"] \\ \mathrm{pt} \arrow[u, dashed, "\wr"] \arrow[r, "{[\mathrm{id}_{G \circ F}]}"'] & \mathrm{Nat}(G \circ F, G \circ F) \end{tikzcd}$$

commutes, i.e. we have

$$\mathrm{id}_G \star \mathrm{id}_F = \mathrm{id}_{G \circ F}.$$

4. *Middle Four Exchange.* Let $F_1, F_2, F_3\colon \mathcal{C} \to \mathcal{D}$ and $G_1, G_2, G_3\colon \mathcal{D} \to \mathcal{E}$ be functors. The diagram

$$\begin{tikzcd} (\mathrm{Nat}(G_2, G_3) \times \mathrm{Nat}(G_1, G_2)) \times (\mathrm{Nat}(F_2, F_3) \times \mathrm{Nat}(F_1, F_2)) \arrow[rr, dashed, leftrightarrow, "\mu_4", "\sim"'] \arrow[d, "\circ_{G_1,G_2,G_3} \times \circ_{F_1,F_2,F_3}"'] & & (\mathrm{Nat}(G_2, G_3) \times \mathrm{Nat}(F_2, F_3)) \times (\mathrm{Nat}(G_1, G_2) \times \mathrm{Nat}(F_1, F_2)) \arrow[d, "\star_{F_2,F_3,G_2,G_3} \times \star_{F_1,F_2,G_1,G_2}"] \\ \mathrm{Nat}(G_1, G_3) \times \mathrm{Nat}(F_1, F_3) \arrow[dr, "\star_{F_1,F_3,G_1,G_3}"'] & & \mathrm{Nat}(G_2 \circ F_2, G_3 \circ F_3) \times \mathrm{Nat}(G_1 \circ F_1, G_2 \circ F_2) \arrow[dl, "\circ_{G_1 \circ F_1, G_2 \circ F_2, G_3 \circ F_3}"] \\ & \mathrm{Nat}(G_1 \circ F_1, G_3 \circ F_3) & \end{tikzcd}$$

commutes, i.e. given a diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=50, "F_1"] \arrow[r, "F_2" description] \arrow[r, bend right=50, "F_3"'] \arrow[r, phantom, shift left=3, "\alpha\Downarrow"] \arrow[r, phantom, shift right=3, "\alpha'\Downarrow"] & \mathcal{D} \arrow[r, bend left=50, "G_1"] \arrow[r, "G_2" description] \arrow[r, bend right=50, "G_3"'] \arrow[r, phantom, shift left=3, "\beta\Downarrow"] \arrow[r, phantom, shift right=3, "\beta'\Downarrow"] & \mathcal{E} \end{tikzcd}$$

in Cats_2, we have

$$(\beta' \star \alpha') \circ (\beta \star \alpha) = (\beta' \circ \beta) \star (\alpha' \circ \alpha).$$

Proof 11.5.3.4 ► Proof of Proposition 11.5.3.3

Item 1: Functionality

Clear.

Item 2: Associativity

Omitted.

Item 3: Interaction With Identities

We have

$$\begin{aligned}(\mathrm{id}_G \star \mathrm{id}_F)_A &\overset{\text{def}}{=} (\mathrm{id}_G)_{F_A} \circ G_{(\mathrm{id}_F)_A}\\ &\overset{\text{def}}{=} \mathrm{id}_{G_{F_A}} \circ G_{\mathrm{id}_{F_A}}\\ &= \mathrm{id}_{G_{F_A}} \circ \mathrm{id}_{G_{F_A}}\\ &= \mathrm{id}_{G_{F_A}}\\ &\overset{\text{def}}{=} (\mathrm{id}_{G\circ F})_A\end{aligned}$$

for each $A \in \mathrm{Obj}(\mathcal{C})$, showing the desired equality.

Item 4: Middle Four Exchange

Let $A \in \mathrm{Obj}(\mathcal{C})$ and consider the diagram

$$\begin{tikzcd}
 & & G_{F''_A} \arrow[dr, "\beta_{F''_A}"] & & \\
G_{F_A} \arrow[r, "G_{\alpha_A}"] & G_{F'_A} \arrow[ur, "G_{\alpha'_A}"] \arrow[dr, "\beta_{F'_A}"'] & (1) & G''_{F''_A} \arrow[r, "\beta'_{F''_A}"] & G''_{F''_A}. \\
 & & G'_{F'_A} \arrow[ur, "G'_{\alpha'_A}"'] & &
\end{tikzcd}$$

The top composition is $((\beta' \circ \beta) \star (\alpha' \circ \alpha))_A$ and the bottom composition is $((\beta' \star \alpha') \circ (\beta \star \alpha))_A$. Since Subdiagram (1) commutes, they are equal.

11.5.4 Properties of Natural Transformations

Proposition 11.5.4.1 ► Natural Transformations as Categorical Homotopies

Let $F, G \colon \mathcal{C} \rightrightarrows \mathcal{D}$ be functors. The following data are equivalent:[1]

1. A natural transformation $\alpha \colon F \Longrightarrow G$.

2. A functor $[\alpha]\colon \mathcal{C} \to \mathcal{D}^{\nVdash}$ filling the diagram

$$\begin{tikzcd}
 & \mathcal{D} \\
\mathcal{C} \arrow[ur, "F"] \arrow[dr, "G"'] \arrow[r, "{[\alpha]}"] & \mathcal{D}^{\nVdash}. \arrow[u, "\mathrm{ev}_0"'] \arrow[d, "\mathrm{ev}_1"] \\
 & \mathcal{D}
\end{tikzcd}$$

3. A functor $[\alpha]\colon \mathcal{C} \times \nVdash \to \mathcal{D}$ filling the diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[dr, "F"] & \\
\mathcal{C} \times \nVdash \arrow[u, "\mathrm{ev}_0"] \arrow[d, "\mathrm{ev}_1"'] \arrow[r, "{[\alpha]}"] & \mathcal{D}. \\
\mathcal{C} \arrow[ur, "G"'] &
\end{tikzcd}$$

[1]Taken from [MO 64365].

PROOF 11.5.4.2 ▶ PROOF OF PROPOSITION 11.5.4.1

From Item 1 to Item 2 and Back

We may identify $\mathcal{D}^{\nVdash}$ with $\mathrm{Arr}(\mathcal{D})$. Given a natural transformation $\alpha\colon F \Longrightarrow G$, we have a functor

$$\begin{aligned}
[\alpha]\colon \mathcal{C} &\longrightarrow \mathcal{D}^{\nVdash} \\
A &\longmapsto \alpha_A \\
(f\colon A \to B) &\longmapsto \left(\begin{tikzcd} F_A \arrow[r, "F_f"] \arrow[d, "\alpha_A"'] & F_B \arrow[d, "\alpha_B"] \\ G_A \arrow[r, "G_f"'] & G_B \end{tikzcd}\right)
\end{aligned}$$

making the diagram in Item 2 commute. Conversely, every such functor gives rise to a natural transformation from F to G, and these constructions are inverse to each other.

From Item 2 to Item 3 and Back

This follows from Item 3 of Proposition 11.6.1.2.

11.5.5 Natural Isomorphisms

Definition 11.5.5.1 ► Natural Isomorphisms

A natural transformation $\alpha \colon F \Longrightarrow G$ between functors $F, G \colon \mathcal{C} \to \mathcal{D}$ between categories $\mathcal{C}$ and $\mathcal{D}$ is a **natural isomorphism** if there exists a natural transformation $\alpha^{-1} \colon G \Longrightarrow F$ such that

$$\alpha^{-1} \circ \alpha = \mathrm{id}_F,$$
$$\alpha \circ \alpha^{-1} = \mathrm{id}_G.$$

Proposition 11.5.5.2 ► Properties of Natural Isomorphisms

Let $\alpha \colon F \Longrightarrow G$ be a natural transformation.

1. *Characterisations.* The following conditions are equivalent:

   (a) The natural transformation α is a natural isomorphism.

   (b) For each $A \in \mathrm{Obj}(\mathcal{C})$, the morphism $\alpha_A \colon F_A \to G_A$ is an isomorphism.

2. *Componentwise Inverses of Natural Transformations Assemble Into Natural Transformations.* Let $\alpha^{-1} \colon G \Longrightarrow F$ be a transformation such that, for each $A \in \mathrm{Obj}(\mathcal{C})$, we have

$$\alpha_A^{-1} \circ \alpha_A = \mathrm{id}_{F(A)},$$
$$\alpha_A \circ \alpha_A^{-1} = \mathrm{id}_{G(A)}.$$

   Then α^{-1} is a natural transformation.

Proof 11.5.5.3 ► Proof of Proposition 11.5.5.2

Item 1: Characterisations

The implication Item 1a $\Longrightarrow$ Item 1b is clear, whereas the implication Item 1b $\Longrightarrow$ Item 1a follows from Item 2.

Item 2: Componentwise Inverses of Natural Transformations Assemble Into Nat

The naturality condition for α^{-1} corresponds to the commutativity of the diagram

$$\begin{tikzcd} G(A) \arrow[r, "G(f)"] \arrow[d, "\alpha_A^{-1}"'] & G(B) \arrow[d, "\alpha_B^{-1}"] \\ F(A) \arrow[r, "F(f)"'] & F(B) \end{tikzcd}$$

for each $A, B \in \mathrm{Obj}(\mathcal{C})$ and each $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$. Considering the diagram

$$\begin{tikzcd} G(A) \arrow[r, "G(f)"] \arrow[d, "\alpha_A^{-1}"'] \arrow[dr, phantom, "(1)"] & G(B) \arrow[d, "\alpha_B^{-1}"] \\ F(A) \arrow[r, "F(f)" description] \arrow[d, "\alpha_A"'] \arrow[dr, phantom, "(2)"] & F(B) \arrow[d, "\alpha_B"] \\ G(A) \arrow[r, "G(f)"'] & G(B), \end{tikzcd}$$

where the boundary diagram as well as Subdiagram (2) commute, we have

$$\begin{aligned} G(f) &= G(f) \circ \mathrm{id}_{G(A)} \\ &= G(f) \circ \alpha_A \circ \alpha_A^{-1} \\ &= \alpha_B \circ F(f) \circ \alpha_A^{-1}. \end{aligned}$$

Postcomposing both sides with α_B^{-1}, we get

$$\begin{aligned} \alpha_B^{-1} \circ G(f) &= \alpha_B^{-1} \circ \alpha_B \circ F(f) \circ \alpha_A^{-1} \\ &= \mathrm{id}_{F(B)} \circ F(f) \circ \alpha_A^{-1} \\ &= F(f) \circ \alpha_A^{-1}, \end{aligned}$$

which is the naturality condition we wanted to show. Thus α^{-1} is a natural transformation.

11.6 Categories of Categories

11.6.1 Functor Categories

Let $\mathcal{C}$ be a category and $\mathcal{D}$ be a small category.

Definition 11.6.1.1 ► Functor Categories

The **category of functors from $\mathcal{C}$ to $\mathcal{D}$**[1] is the category $\mathsf{Fun}(\mathcal{C},\mathcal{D})$[2] where

- *Objects.* The objects of $\mathsf{Fun}(\mathcal{C},\mathcal{D})$ are functors from $\mathcal{C}$ to $\mathcal{D}$;
- *Morphisms.* For each $F, G \in \mathrm{Obj}(\mathsf{Fun}(\mathcal{C},\mathcal{D}))$, we have

$$\mathrm{Hom}_{\mathsf{Fun}(\mathcal{C},\mathcal{D})}(F,G) \stackrel{\text{def}}{=} \mathrm{Nat}(F,G);$$

- *Identities.* For each $F \in \mathrm{Obj}(\mathsf{Fun}(\mathcal{C},\mathcal{D}))$, the unit map

$$\mathbb{1}^{\mathsf{Fun}(\mathcal{C},\mathcal{D})}_F \colon \mathrm{pt} \longrightarrow \mathrm{Nat}(F,F)$$

  of $\mathsf{Fun}(\mathcal{C},\mathcal{D})$ at F is given by

$$\mathrm{id}^{\mathsf{Fun}(\mathcal{C},\mathcal{D})}_F \stackrel{\text{def}}{=} \mathrm{id}_F,$$

  where $\mathrm{id}_F \colon F \Longrightarrow F$ is the identity natural transformation of F of Example 11.5.1.3;

- *Composition.* For each $F, G, H \in \mathrm{Obj}(\mathsf{Fun}(\mathcal{C},\mathcal{D}))$, the composition map

$$\circ^{\mathsf{Fun}(\mathcal{C},\mathcal{D})}_{F,G,H} \colon \mathrm{Nat}(G,H) \times \mathrm{Nat}(F,G) \longrightarrow \mathrm{Nat}(F,H)$$

  of $\mathsf{Fun}(\mathcal{C},\mathcal{D})$ at (F,G,H) is given by

$$\beta \circ^{\mathsf{Fun}(\mathcal{C},\mathcal{D})}_{F,G,H} \alpha \stackrel{\text{def}}{=} \beta \circ \alpha,$$

  where $\beta \circ \alpha$ is the vertical composition of α and β of Item 1 of Proposition 11.5.2.3.

[1] *Further Terminology:* Also called the **functor category** $\mathsf{Fun}(\mathcal{C},\mathcal{D})$.
[2] *Further Notation:* Also written $\mathcal{D}^{\mathcal{C}}$ and $[\mathcal{C},\mathcal{D}]$.

Proposition 11.6.1.2 ► Properties of Functor Categories

Let $\mathcal{C}$ and $\mathcal{D}$ be categories and let $F \colon \mathcal{C} \to \mathcal{D}$ be a functor.

1. *Functoriality.* The assignments $\mathcal{C}, \mathcal{D}, (\mathcal{C},\mathcal{D}) \mapsto \mathsf{Fun}(\mathcal{C},\mathcal{D})$ define functors

$$\begin{aligned}
\mathsf{Fun}(\mathcal{C},-_2) &\colon \mathsf{Cats} \longrightarrow \mathsf{Cats},\\
\mathsf{Fun}(-_1,\mathcal{D}) &\colon \mathsf{Cats}^{\mathsf{op}} \longrightarrow \mathsf{Cats},\\
\mathsf{Fun}(-_1,-_2) &\colon \mathsf{Cats}^{\mathsf{op}} \times \mathsf{Cats} \longrightarrow \mathsf{Cats}.
\end{aligned}$$

2. *2-Functoriality.* The assignments $\mathcal{C}, \mathcal{D}, (\mathcal{C}, \mathcal{D}) \mapsto \mathsf{Fun}(\mathcal{C}, \mathcal{D})$ define 2-functors

$$\begin{aligned}\mathsf{Fun}(\mathcal{C}, -_2) &\colon \mathsf{Cats}_2 \to \mathsf{Cats}_2,\\ \mathsf{Fun}(-_1, \mathcal{D}) &\colon \mathsf{Cats}_2^{\mathsf{op}} \to \mathsf{Cats}_2,\\ \mathsf{Fun}(-_1, -_2) &\colon \mathsf{Cats}_2^{\mathsf{op}} \times \mathsf{Cats}_2 \to \mathsf{Cats}_2.\end{aligned}$$

3. *Adjointness.* We have adjunctions

$$(\mathcal{C} \times - \dashv \mathsf{Fun}(\mathcal{C}, -))\colon \quad \begin{tikzcd} \mathsf{Cats} \arrow[r, bend left, "\mathcal{C}\times-"] \arrow[r, phantom, "\perp"] & \mathsf{Cats}, \arrow[l, bend left, "\mathsf{Fun}(\mathcal{C},-)"] \end{tikzcd}$$

$$(- \times \mathcal{D} \dashv \mathsf{Fun}(\mathcal{D}, -))\colon \quad \begin{tikzcd} \mathsf{Cats} \arrow[r, bend left, "-\times\mathcal{D}"] \arrow[r, phantom, "\perp"] & \mathsf{Cats}, \arrow[l, bend left, "\mathsf{Fun}(\mathcal{D},-)"] \end{tikzcd}$$

witnessed by bijections of sets

$$\begin{aligned}\mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) &\cong \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{D}, \mathsf{Fun}(\mathcal{C}, \mathcal{E})),\\ \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) &\cong \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, \mathsf{Fun}(\mathcal{D}, \mathcal{E})),\end{aligned}$$

natural in $\mathcal{C}, \mathcal{D}, \mathcal{E} \in \mathrm{Obj}(\mathsf{Cats})$.

4. *2-Adjointness.* We have 2-adjunctions

$$(\mathcal{C} \times - \dashv \mathsf{Fun}(\mathcal{C}, -))\colon \quad \begin{tikzcd} \mathsf{Cats}_2 \arrow[r, bend left, "\mathcal{C}\times-"] \arrow[r, phantom, "\perp_2"] & \mathsf{Cats}_2, \arrow[l, bend left, "\mathsf{Fun}(\mathcal{C},-)"] \end{tikzcd}$$

$$(- \times \mathcal{D} \dashv \mathsf{Fun}(\mathcal{D}, -))\colon \quad \begin{tikzcd} \mathsf{Cats}_2 \arrow[r, bend left, "-\times\mathcal{D}"] \arrow[r, phantom, "\perp_2"] & \mathsf{Cats}_2, \arrow[l, bend left, "\mathsf{Fun}(\mathcal{D},-)"] \end{tikzcd}$$

witnessed by isomorphisms of categories

$$\begin{aligned}\mathsf{Fun}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) &\cong \mathsf{Fun}(\mathcal{D}, \mathsf{Fun}(\mathcal{C}, \mathcal{E})),\\ \mathsf{Fun}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) &\cong \mathsf{Fun}(\mathcal{C}, \mathsf{Fun}(\mathcal{D}, \mathcal{E})),\end{aligned}$$

natural in $\mathcal{C}, \mathcal{D}, \mathcal{E} \in \mathrm{Obj}(\mathsf{Cats}_2)$.

5. *Trivial Functor Categories.* We have a canonical isomorphism of categories

$$\mathsf{Fun}(\mathrm{pt}, \mathcal{C}) \cong \mathcal{C},$$

natural in $\mathcal{C} \in \mathrm{Obj}(\mathsf{Cats})$.

6. *Objectwise Computation of Co/Limits.* Let

$$D\colon \mathcal{I} \to \mathsf{Fun}(\mathcal{C}, \mathcal{D})$$

be a diagram in $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$. We have isomorphisms

$$\begin{aligned} \lim(D)_A &\cong \lim_{i\in\mathcal{I}}(D_i(A)), \\ \operatorname{colim}(D)_A &\cong \operatorname*{colim}_{i\in\mathcal{I}}(D_i(A)), \end{aligned}$$

naturally in $A \in \mathrm{Obj}(\mathcal{C})$.

7. *Bicompleteness.* If $\mathcal{E}$ is co/complete, then so is $\mathsf{Fun}(\mathcal{C}, \mathcal{E})$.

8. *Abelianness.* If $\mathcal{E}$ is abelian, then so is $\mathsf{Fun}(\mathcal{C}, \mathcal{E})$.

9. *Monomorphisms and Epimorphisms.* Let $\alpha\colon F \Longrightarrow G$ be a morphism of $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$. The following conditions are equivalent:

   (a) The natural transformation

   $$\alpha\colon F \Longrightarrow G$$

   is a monomorphism (resp. epimorphism) in $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$.

   (b) For each $A \in \mathrm{Obj}(\mathcal{C})$, the morphism

   $$\alpha_A\colon F_A \to G_A$$

   is a monomorphism (resp. epimorphism) in $\mathcal{D}$.

Proof 11.6.1.3 ▶ Proof of Proposition 11.6.1.2

Item 1: Functoriality

Omitted.

Item 2: 2-Functoriality

Omitted.

Item 3: Adjointness

Omitted.

Item 4: 2-Adjointness

Omitted.

Item 5: Trivial Functor Categories

Omitted.

Item 6: Objectwise Computation of Co/Limits

Omitted.

Item 7: Bicompleteness

This follows from **??**.

Item 8: Abelianness

Omitted.

Item 9: Monomorphisms and Epimorphisms

Omitted.

11.6.2 The Category of Categories and Functors

Definition 11.6.2.1 ▶ The Category of Categories and Functors

The **category of** (**small**) **categories and functors** is the category Cats where

- *Objects.* The objects of Cats are small categories;
- *Morphisms.* For each $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$, we have

  $$\mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, \mathcal{D}) \stackrel{\text{def}}{=} \mathrm{Obj}(\mathsf{Fun}(\mathcal{C}, \mathcal{D}));$$

- *Identities.* For each $\mathcal{C} \in \mathrm{Obj}(\mathsf{Cats})$, the unit map

  $$\mathbb{1}^{\mathsf{Cats}}_{\mathcal{C}} \colon \mathrm{pt} \to \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, \mathcal{C})$$

  of Cats at $\mathcal{C}$ is defined by

  $$\mathrm{id}^{\mathsf{Cats}}_{\mathcal{C}} \stackrel{\text{def}}{=} \mathrm{id}_{\mathcal{C}},$$

  where $\mathrm{id}_{\mathcal{C}} \colon \mathcal{C} \to \mathcal{C}$ is the identity functor of $\mathcal{C}$ of Example 11.4.1.2;
- *Composition.* For each $\mathcal{C}, \mathcal{D}, \mathcal{E} \in \mathrm{Obj}(\mathsf{Cats})$, the composition map

  $$\circ^{\mathsf{Cats}}_{\mathcal{C},\mathcal{D},\mathcal{E}} \colon \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{D}, \mathcal{E}) \times \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, \mathcal{D}) \to \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, \mathcal{E})$$

  of Cats at $(\mathcal{C}, \mathcal{D}, \mathcal{E})$ is given by

  $$G \circ^{\mathsf{Cats}}_{\mathcal{C},\mathcal{D},\mathcal{E}} F \stackrel{\text{def}}{=} G \circ F,$$

  where $G \circ F \colon \mathcal{C} \to \mathcal{E}$ is the composition of F and G of Definition 11.4.1.4.

Proposition 11.6.2.2 ► Properties of the Category Cats

Let $\mathcal{C}$ be a category.

1. *Co/Completeness.* The category Cats is complete and cocomplete.

2. *Cartesian Monoidal Structure.* The quadruple $(\mathsf{Cats}, \times, \mathrm{pt}, \mathsf{Fun})$ is a Cartesian closed monoidal category.

Proof 11.6.2.3 ► Proof of Proposition 11.6.2.2

Item 1: Co/Completeness

This follows from

Item 2: Cartesian Monoidal Structure

Omitted. □

11.6.3 The 2-Category of Categories, Functors, and Natural Transformations

Definition 11.6.3.1 ► The 2-Category of Categories

The 2-**category of** (**small**) **categories, functors, and natural transformations** is the 2-category Cats_2 where

- *Objects.* The objects of Cats_2 are small categories;
- Hom-*Categories.* For each $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats}_2)$, we have

$$\mathrm{Hom}_{\mathsf{Cats}_2}(\mathcal{C}, \mathcal{D}) \stackrel{\mathrm{def}}{=} \mathsf{Fun}(\mathcal{C}, \mathcal{D});$$

- *Identities.* For each $\mathcal{C} \in \mathrm{Obj}(\mathsf{Cats}_2)$, the unit functor

$$\mathbb{1}^{\mathsf{Cats}_2}_{\mathcal{C}} \colon \mathrm{pt} \to \mathsf{Fun}(\mathcal{C}, \mathcal{C})$$

  of Cats_2 at $\mathcal{C}$ is the functor picking the identity functor $\mathrm{id}_{\mathcal{C}} \colon \mathcal{C} \to \mathcal{C}$ of $\mathcal{C}$;
- *Composition.* For each $\mathcal{C}, \mathcal{D}, \mathcal{E} \in \mathrm{Obj}(\mathsf{Cats}_2)$, the composition bifunctor

$$\circ^{\mathsf{Cats}_2}_{\mathcal{C},\mathcal{D},\mathcal{E}} \colon \mathrm{Hom}_{\mathsf{Cats}_2}(\mathcal{D}, \mathcal{E}) \times \mathrm{Hom}_{\mathsf{Cats}_2}(\mathcal{C}, \mathcal{D}) \to \mathrm{Hom}_{\mathsf{Cats}_2}(\mathcal{C}, \mathcal{E})$$

  of Cats_2 at $(\mathcal{C}, \mathcal{D}, \mathcal{E})$ is the functor where
  - *Action on Objects.* For each object $(G, F) \in \mathrm{Obj}(\mathrm{Hom}_{\mathsf{Cats}_2}(\mathcal{D}, \mathcal{E}) \times \mathrm{Hom}_{\mathsf{Cats}_2}(\mathcal{C}, \mathcal{D}))$, we have

$$\circ^{\mathsf{Cats}_2}_{\mathcal{C},\mathcal{D},\mathcal{E}}(G, F) \stackrel{\mathrm{def}}{=} G \circ F;$$

– *Action on Morphisms.* For each morphism $(\beta, \alpha)\colon (K, H) \Longrightarrow (G, F)$ of $\mathrm{Hom}_{\mathsf{Cats}_2}(\mathcal{D}, \mathcal{E}) \times \mathrm{Hom}_{\mathsf{Cats}_2}(\mathcal{C}, \mathcal{D})$, we have

$$\circ^{\mathsf{Cats}_2}_{\mathcal{C},\mathcal{D},\mathcal{E}}(\beta, \alpha) \stackrel{\text{def}}{=} \beta \star \alpha,$$

where $\beta \star \alpha$ is the horizontal composition of α and β of Definition 11.5.3.1.

Proposition 11.6.3.2 ▶ Properties of the 2-Category Cats_2

Let $\mathcal{C}$ be a category.

1. *2-Categorical Co/Completeness.* The 2-category Cats_2 is complete and cocomplete as a 2-category, having all 2-categorical and bicategorical co/limits.

Proof 11.6.3.3 ▶ Proof of Proposition 11.6.3.2

Item 1: Co/Completeness

This follows from

11.6.4 The Category of Groupoids

Definition 11.6.4.1 ▶ The Category of Small Groupoids

The **category of** (**small**) **groupoids** is the full subcategory Grpd of Cats spanned by the groupoids.

11.6.5 The 2-Category of Groupoids

Definition 11.6.5.1 ▶ The 2-Category of Small Groupoids

The 2-**category of** (**small**) **groupoids** is the full sub-2-category Grpd_2 of Cats_2 spanned by the groupoids.

11.7 Miscellany

11.7.1 Concrete Categories

Definition 11.7.1.1 ▶ Concrete Categories

A category $\mathcal{C}$ is **concrete** if there exists a faithful functor $F\colon \mathcal{C} \to \mathsf{Sets}$.

11.7.2 Balanced Categories

Definition 11.7.2.1 ▶ Balanced Categories

A category is **balanced** if every morphism which is both a monomorphism and an epimorphism is an isomorphism.

11.7.3 Monoid Actions on Objects of Categories

Let A be a monoid, let $\mathcal{C}$ be a category, and let $X \in \mathrm{Obj}(\mathcal{C})$.

Definition 11.7.3.1 ▶ Monoid Actions on Objects of Categories

An A-**action on** X is a functor $\lambda\colon \mathsf{B}A \to \mathcal{C}$ with $\lambda(\star) = X$.

Remark 11.7.3.2 ▶ Unwinding Definition 11.7.3.1

In detail, an A-**action on** X is an A-action on $\mathrm{End}_{\mathcal{C}}(X)$, consisting of a morphism

$$\lambda\colon A \to \underbrace{\mathrm{End}_{\mathcal{C}}(X)}_{\overset{\text{def}}{=}\mathrm{Hom}_{\mathcal{C}}(X,X)}$$

satisfying the following conditions:

1. *Preservation of Identities.* We have

$$\lambda_{1_A} = \mathrm{id}_X.$$

2. *Preservation of Composition.* For each $a, b \in A$, we have

$$\lambda_b \circ \lambda_a = \lambda_{ab}, \qquad \begin{tikzcd} X \arrow[r, "\lambda_a"] \arrow[dr, "\lambda_{ab}"'] & X \arrow[d, "\lambda_b"] \\ & X. \end{tikzcd}$$

11.7.4 Group Actions on Objects of Categories

Let G be a group, let $\mathcal{C}$ be a category, and let $X \in \mathrm{Obj}(\mathcal{C})$.

Definition 11.7.4.1 ► Group Actions on Objects of Categories

A G-**action on** X is a functor $\lambda\colon \mathsf{B}G \to \mathcal{C}$ with $\lambda(\star) = X$.

Remark 11.7.4.2 ► Unwinding Definition 11.7.4.1

In detail, a G-**action on** X is a G-action on $\mathrm{Aut}_{\mathcal{C}}(X)$, consisting of a morphism

$$\lambda\colon G \to \underbrace{\mathrm{End}_{\mathcal{C}}(X)}_{\stackrel{\text{def}}{=}\mathrm{Hom}_{\mathcal{C}}(X,X)}$$

satisfying the following conditions:

1. *Preservation of Identities.* We have
$$\lambda_{1_A} = \mathrm{id}_X.$$
2. *Preservation of Composition.* For each $a, b \in A$, we have
$$\lambda_b \circ \lambda_a = \lambda_{ab}, \qquad \begin{tikzcd} X \arrow[r, "\lambda_a"] \arrow[dr, "\lambda_{ab}"'] & X \arrow[d, "\lambda_b"] \\ & X. \end{tikzcd}$$

Appendices

11.A Other Chapters

Chapter 12

Types of Morphisms in Categories

Create tags (see [MSE 350788] for some of these):

1. ??
2. ??
3. ??
4. ??
5. write material on sections and retractions

Contents

12.1 Monomorphisms

12.1.1 Foundations

Let $\mathcal{C}$ be a category.

Definition 12.1.1.1 ► Monomorphisms

A morphism $m\colon A \longrightarrow B$ of C is a **monomorphism** if, for each diagram of the form

$$m \circ f = m \circ g, \qquad X \overset{f}{\underset{g}{\rightrightarrows}} A \xrightarrow{m} B$$

in Cats, we have $f = g$.

Example 12.1.1.2 ► Examples of Monomorphisms

Here are some examples of monomorphisms.

1. *Monomorphisms in* Sets. The monomorphisms in Sets are precisely the injections.

Proof 12.1.1.3 ► Proof of Example 12.1.1.2

Item 1: Monomorphisms in Sets

Let $f\colon A \longrightarrow B$ be a morphism in Sets. Suppose that f is a monomorphism and consider the diagram

$$\{*\} \overset{[x]}{\underset{[y]}{\rightrightarrows}} A \xrightarrow{f} B,$$

where $[x]$ and $[y]$ are the morphisms picking the elements x and y of A. If $f(x) = f(y)$, then $f \circ [x] = f \circ [y]$, and thus $[x] = [y]$ since f is a monomorphism. Hence $x = y$ and we see that f is injective.

Conversely, suppose that f is injective. Proceeding by contrapositive, we claim that given a pair of maps $g, h\colon X \rightrightarrows A$ such that $g \neq h$, then $f \circ g \neq f \circ h$. Indeed, as g and h are different maps, there must exist at least one element $x \in X$ such that $g(x) \neq h(x)$. But then we have $f(g(x)) \neq f(h(x))$, since f is injective. Thus $f \circ g \neq f \circ h$, and we are done, having showed that f is a monomorphism. □

Proposition 12.1.1.4 ► Properties of Monomorphisms

Let $f\colon A \longrightarrow B$ be a morphism of C.

1. *Characterisations.* The following conditions are equivalent:
   (a) The morphism f is a monomorphism.
   (b) For each $X \in \mathrm{Obj}(C)$, the map of sets

   $$f_*\colon \mathrm{Hom}_{\mathsf{Sets}}(X, A) \longrightarrow \mathrm{Hom}_{\mathsf{Sets}}(X, B)$$

   is injective.

(c) The kernel pair of f is trivial, i.e. we have

$$A \times_B A \cong A, \qquad \begin{tikzcd} A \arrow[r, "\mathrm{id}_A"] \arrow[d, "\mathrm{id}_A"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\ A \arrow[r, "f"'] & B. \end{tikzcd}$$

2. *Duality.* The following conditions are equivalent:

   (a) The morphism $f\colon A \to B$ is a monomorphism in $\mathcal{C}$.

   (b) The morphism $f^{\dagger}\colon B \to A$ is an epimorphism in $\mathcal{C}^{\mathrm{op}}$.

3. *Monomorphisms vs. Injective Maps.* Let

   · $\mathcal{C}$ be a concrete category as in **??**;

   · $忘_{\mathcal{C}}\colon \mathcal{C} \to \mathsf{Sets}$ be the forgetful functor from $\mathcal{C}$ to Sets;

   · $f\colon A \to B$ be a morphism of $\mathcal{C}$.

   If $忘_{\mathcal{C}}$ preserves pullbacks, then the following conditions are equivalent:

   (a) The morphism f is a monomorphism.

   (b) The morphism $忘(f)_{\mathcal{C}}$ is injective.

4. *Stability Properties.* The class of all monomorphisms of $\mathcal{C}$ is stable under the following operations:

   (a) *Composition.* If f and g are monomorphisms, then so is $g \circ f$.[1]

   (b) *Pullbacks.* Let

$$\begin{tikzcd} A \times_C B \arrow[r] \arrow[d, "m'"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "m"] \\ A \arrow[r] & C \end{tikzcd}$$

   be a diagram in $\mathcal{C}$. If m is a monomorphism in $\mathcal{C}$, then so is m'.

5. *Morphisms From the Terminal Object Are Monomorphisms.* If $\mathcal{C}$ has a terminal object $\mathbb{1}_{\mathcal{C}}$, then every morphism of $\mathcal{C}$ from $\mathbb{1}_{\mathcal{C}}$ is a monomorphism.

[1] Conversely, if $g \circ f$ is a monomorphism, then so is f.

PROOF 12.1.1.5 ► PROOF OF PROPOSITION 12.1.1.4

Item 1: Characterisations

The equivalence between Items 1a and 1b is clear. We claim that Items 1a and 1c are equivalent:

1. *Item 1a* $\Longrightarrow$ *Item 1c*: Suppose that f is a monomorphism. Then A satisfies the universal property of the pullback:

$$\begin{tikzcd}
P \arrow[dr, dashed, "\phi", "\exists!"'] \arrow[drr, bend left, "\phi"] \arrow[ddr, bend right, "\phi"'] & & \\
& A \arrow[r, "\mathrm{id}_A"] \arrow[d, "\mathrm{id}_A"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\
& A \arrow[r, "f"'] & B.
\end{tikzcd}$$

2. *Item 1c* $\Longrightarrow$ *Item 1a*: Suppose that $A \cong A \times_B A$ and let $g, h \colon C \rightrightarrows A$ be a pair of morphisms. Consider the diagram

$$\begin{tikzcd}
C \arrow[dr, shift left, "g"] \arrow[dr, shift right, "h"'] & & \\
& A \arrow[r, "\mathrm{id}_A"] \arrow[d, "\mathrm{id}_A"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\
& A \arrow[r, "f"'] & B.
\end{tikzcd}$$

The universal property of the pullback says that there exists a unique morphism $C \to A$ making the diagram

$$\begin{tikzcd}
C \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "h"'] & & \\
& A \arrow[r, "\mathrm{id}_A"] \arrow[d, "\mathrm{id}_A"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\
& A \arrow[r, "f"'] & B
\end{tikzcd}$$

commute, which implies $g = h$. Therefore, f is a monomorphism.

Item 3: Monomorphisms vs. Injective Maps

Assume that f is injective. As the forgetful functor from $\mathcal{C}$ to Sets is faithful, we see that Proposition 12.1.2.2 together with **??** imply that f is a monomorphism.

Conversely, assume that f is a monomorphism. As F preserves pullbacks, it also preserves kernel pairs. By **??**, we see that F preserves monomorphisms. Thus F_f is a monomorphism, and hence is injective by **??**.

Item 4: Stability Properties

Let $f, g\colon X \rightrightarrows A \times_C B$ be two morphisms such that the diagram

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & A \times_C B \arrow[r, "m'"] & A \end{tikzcd}$$

commutes. It follows that the diagram

$$\begin{tikzcd} X \arrow[dr, shift left, "f"] \arrow[dr, shift right, "g"'] \arrow[drr, bend left, "\mathrm{pr}_2 \circ f"] \arrow[ddr, bend right, "m' \circ g"'] & & \\ & A \times_C B \arrow[r, "\mathrm{pr}_2"] \arrow[d, "m'"] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "m"] \\ & A \arrow[r, "\psi"'] & C \end{tikzcd}$$

also commutes. From the universal property of the pullback, it follows that there must be precisely one morphism from X to $A \times_C B$ making the above diagram commute. Thus $f = g$ and m' is a monomorphism.

Item 5: Morphisms From the Terminal Object Are Monomorphisms

Clear. □

12.1.2 Monomorphism-Reflecting Functors

Definition 12.1.2.1 ► Monomorphism-Reflecting Functors

A functor $F\colon \mathcal{C} \to \mathcal{D}$ **reflects monomorphisms** if, for each morphism f of $\mathcal{C}$, whenever F_f is a monomorphism, so is f.

Proposition 12.1.2.2 ▶ Faithful Functors Reflect Monomorphisms

Let $F \colon \mathcal{C} \to \mathcal{D}$ be a functor. If F is faithful, then it reflects monomorphisms.

Proof 12.1.2.3 ▶ Proof of Proposition 12.1.2.2

Let $f \colon A \to B$ be a morphism of $\mathcal{C}$ and suppose that $F_f \colon F_A \to F_B$ is a monomorphism. Let $g, h \colon B \rightrightarrows C$ be two morphisms of $\mathcal{C}$ such that $g \circ f = h \circ f$. As F is faithful, we must have

$$F_g \circ F_f = F_{g \circ f} = F_{h \circ f} = F_h \circ F_f,$$

but as F_f is a monomorphism, it must be that $F_g = F_h$. Using the faithfulness of F again, we see that $g = h$. Therefore f is a monomorphism. ∎

12.1.3 Split Monomorphisms

Let $\mathcal{C}$ be a category.

Definition 12.1.3.1 ▶ Split Monomorphisms

A morphism $f \colon A \to B$ of $\mathcal{C}$ is a **split monomorphism**[1] if there exists a morphism $g \colon B \to A$ of $\mathcal{B}$ such that[2]

$$g \circ f = \mathrm{id}_A.$$

[1] *Further Terminology:* Also called a **section**, or a **split monic** morphism.

[2] *Warning:* There exist monomorphisms which are not split monomorphisms, e.g. $\mathbb{Z}/2 \hookrightarrow \mathbb{Z}/4$ in Ring.

Proposition 12.1.3.2 ▶ Properties of Split Monomorphisms

Let $\mathcal{C}$ be a category.

1. *Split Monomorphisms are Monomorphisms.* If m is a split monomorphism, then m is a monomorphism.

Proof 12.1.3.3 ▶ Proof of Proposition 12.1.3.2

Item 1: Split Monomorphisms are Monomorphisms

Let $m \colon A \to B$ be a split monomorphism of $\mathcal{C}$, let $e \colon B \to A$ be a morphism of $\mathcal{C}$ with

$$e \circ m = \mathrm{id}_A,$$

and let $f, g \colon C \rightrightarrows A$ be two morphisms of $\mathcal{C}$ such that the diagram

$$C \overset{f}{\underset{g}{\rightrightarrows}} A \xrightarrow{m} B$$

commutes. Then we have

$$\begin{aligned} f &= \mathrm{id}_A \circ f \\ &= (e \circ m) \circ f \\ &= e \circ (m \circ f) \\ &= e \circ (m \circ g) \\ &= (e \circ m) \circ g \\ &= \mathrm{id}_A \circ g \\ &= g, \end{aligned}$$

showing m to be a monomorphism. ∎

12.2 Epimorphisms

12.2.1 Foundations

Let $\mathcal{C}$ be a category.

Definition 12.2.1.1 ► Epimorphisms

A morphism $f \colon A \to B$ of $\mathcal{C}$ is an **epimorphism** if for every commutative[1] diagram of the form

$$A \xrightarrow{f} B \overset{g}{\underset{h}{\rightrightarrows}} C,$$

we have $g = h$.

[1] That is, with $g \circ f = h \circ f$.

Example 12.2.1.2 ► Epimorphisms in Sets

Let $f \colon A \to B$ be a function. The following conditions are equivalent:

1. The function f is injective.
2. The function f is an epimorphism in Sets.

Proof 12.2.1.3 ► Proof of Example 12.2.1.2

Suppose that f is surjective and let $g, h \colon B \rightrightarrows C$ be morphisms such that $g \circ f = h \circ f$. Then for each $a \in A$, we have

$$g(f(a)) = h(f(a)),$$

but this implies that

$$g(b) = h(b)$$

for each $b \in B$, as f is surjective. Thus $g = h$ and f is an epimorphism.

To prove the converse, we proceed by contrapositive. So suppose that f is not surjective and consider the diagram

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & C, \end{tikzcd}$$

where h is the map defined by $h(b) = 0$ for each $b \in B$ and g is the map defined by

$$g(b) = \begin{cases} 1 & \text{if } b \in \mathrm{Im}(f), \\ 0 & \text{otherwise.} \end{cases}$$

Then $h \circ f = g \circ f$, as $h(f(a)) = 1 = g(f(a))$ for each $a \in A$. However, for any $b \in B \setminus \mathrm{Im}(f)$, we have

$$g(b) = 0 \neq 1 = h(b).$$

Therefore $g \neq h$ and f is not an epimorphism. ∎

Proposition 12.2.1.4 ► Properties of Epimorphisms

Let $\mathcal{C}$ be a category.

1. *Characterisations.* Let $\mathcal{C}$ be a category with pullbacks and $f \colon A \to B$ be a morphism of $\mathcal{C}$. The following conditions are equivalent:

   (a) The morphism f is an epimorphism.

   (b) For each $X \in \mathrm{Obj}(\mathcal{C})$, the map of sets

   $$f^* \colon \mathrm{Hom}_{\mathsf{Sets}}(B, X) \to \mathrm{Hom}_{\mathsf{Sets}}(A, X)$$

   is injective.

(c) The cokernel pair of f is trivial, i.e. we have

$$B \coprod_A B \cong B \qquad \begin{tikzcd} B & B \arrow[l] \\ B \arrow[u] \arrow[ur, phantom, "\ulcorner"] & A. \arrow[l, "f"] \arrow[u, "f"'] \end{tikzcd}$$

2. *Epimorphisms vs. Surjective Maps.* Let

- $\mathcal{C}$ be a concrete category;
- 忘: $\mathcal{C} \to \mathsf{Sets}$ be the forgetful functor from $\mathcal{C}$ to Sets;
- $f\colon A \to B$ be a morphism of $\mathcal{C}$.

If 忘 preserves pushouts, then the following conditions are equivalent:

(a) The morphism f is a epimorphism.

(b) The morphism f is surjective.

3. *Stability Properties.* The class of all epimorphisms of $\mathcal{C}$ is stable under the following operations:

(a) *Composition.* If f and g are epimorphisms, then so is $g \circ f$.[1]

(b) *Pushouts.* Let

$$\begin{tikzcd} A \coprod_C B & B \arrow[l] \\ A \arrow[u, "e'"] \arrow[r] \arrow[ur, phantom, "\ulcorner"] & C \arrow[u, "e"'] \end{tikzcd}$$

be a diagram in $\mathcal{C}$. If m is an epimorphism in $\mathcal{C}$, then so is e'.

4. *Morphisms to the Initial Object Are Monomorphisms.* If $\mathcal{C}$ has an initial object $\emptyset_{\mathcal{C}}$, then every morphism of $\mathcal{C}$ to $\emptyset_{\mathcal{C}}$ is a epimorphism.

[1] Conversely, if $g \circ f$ is a epimorphism, then so is g.

Proof 12.2.1.5 ▶ Proof of Proposition 12.2.1.4

This is dual to Proposition 12.1.1.4.

12.2.2 Regular Epimorphisms

PROPOSITION 12.2.2.1 ▶ PROPERTIES OF REGULAR EPIMORPHISMS

Let $\mathcal{C}$ be a category.

1. *Stability Under Pullbacks.* Consider the diagram

$$\begin{tikzcd} A \times_C B \arrow[r] \arrow[d, "e'"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "e"] \\ A \arrow[r] & C \end{tikzcd}$$

in $\mathcal{C}$. If e is a regular epimorphism, then so is e'.

PROOF 12.2.2.2 ▶ PROOF OF PROPOSITION 12.2.2.1

Epimorphisms Need Not Be Stable Under Pullback.

Regular Epimorphisms Are Stable Under Pullback.

□

12.2.3 Effective Epimorphisms

Let $\mathcal{C}$ be a category.

DEFINITION 12.2.3.1 ▶ EFFECTIVE EPIMORPHISMS

An epimorphism $f: A \to B$ of $\mathcal{C}$ is **effective** if we have an isomorphism

$$B \cong \mathrm{CoEq}(A \times_B A \rightrightarrows A).$$

12.2.4 Split Epimorphisms

Let $\mathcal{C}$ be a category.

DEFINITION 12.2.4.1 ▶ RETRACTIONS

A morphism $f: A \to B$ of $\mathcal{C}$ is a **retraction**[1] if there is an arrow $g: B \to A$ such that $f \circ g = \mathrm{id}_B$.

[1] *Further Terminology:* Also called a **split epimorphism**.

Proposition 12.2.4.2 ▶ Properties of Split Epimorphisms

Let $f\colon A \to B$ be a morphism of C.

1. Every split epimorphism is an epimorphism.[1]

[1] *Warning:* There are epimorphisms which are not split epimorphisms, however, e.g. $\mathbb{Z} \hookrightarrow \mathbb{Z}/2$.

Proof 12.2.4.3 ▶ Proof of Proposition 12.2.4.2

This is dual to **??**.

Appendices

12.A Other Chapters

Chapter 13

Adjunctions and the Yoneda Lemma

Chapter 14

Constructions With Categories

Contents

Appendices

14.A Other Chapters

Sets

1. Sets
2. Constructions With Sets
3. Pointed Sets
4. Tensor Products of Pointed Sets
5. Relations
6. Spans
7. Posets

Indexed and Fibred Sets

7. Indexed Sets
8. Fibred Sets
9. Un/Straightening for Indexed and Fibred Sets

Category Theory

11. Categories
12. Types of Morphisms in Categories
13. Adjunctions and the Yoneda Lemma
14. Constructions With Categories
15. Profunctors
16. Cartesian Closed Categories
17. Kan Extensions

Bicategories

18. Bicategories
19. Internal Adjunctions

Internal Category Theory

20. Internal Categories

Chapter 15

Profunctors

Contents

15.1 Profunctors

15.1.1 Foundations

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 15.1.1.1 ► Profunctors

A **profunctor**[1] $\mathfrak{p}\colon \mathcal{C} \nrightarrow \mathcal{D}$ from $\mathcal{C}$ to $\mathcal{D}$ is a functor $\mathfrak{p}\colon \mathcal{D}^{\mathsf{op}} \times \mathcal{C} \to \mathsf{Sets}$.

[1] *Further Terminology:* Also called a **distributor**, a **bimodule**, a **correspondence**, or a **relator**.

Remark 15.1.1.2 ▶ Equivalent Definitions of Profunctors

Equivalently, we may define a profunctor from $\mathcal{C}$ to $\mathcal{D}$ as:

1. A functor $\mathfrak{p}\colon \mathcal{D}^{\mathsf{op}} \times \mathcal{C} \to \mathsf{Sets}$;
2. A functor $\mathfrak{p}\colon \mathcal{C} \to \mathsf{PSh}(\mathcal{D})$;
3. A functor $\mathfrak{p}\colon \mathcal{D}^{\mathsf{op}} \to \mathsf{Fun}(\mathcal{C}, \mathsf{Sets})$;
4. A cocontinuous functor $\mathfrak{p}\colon \mathsf{PSh}(\mathcal{C}) \to \mathsf{PSh}(\mathcal{D})$;

That is, we have isomorphisms of categories

$$\begin{aligned}\mathsf{Prof}(\mathcal{C}, \mathcal{D}) &\cong \mathsf{Fun}(\mathcal{C}, \mathsf{PSh}(\mathcal{D})),\\ &\cong \mathsf{Fun}(\mathcal{D}^{\mathsf{op}}, \mathsf{CoPSh}(\mathcal{C})),\\ &\cong \mathsf{CoContFun}(\mathsf{PSh}(\mathcal{C}), \mathsf{PSh}(\mathcal{D})),\end{aligned}$$

natural in $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$.

Proof 15.1.1.3 ▶ Proof of Remark 15.1.1.2

We claim that Items 1 to 4 are indeed equivalent:

- The equivalence between Items 1 and 2 is an instance of currying, following from the isomorphisms of categories

$$\mathsf{Fun}(\mathcal{D}^{\mathsf{op}} \times \mathcal{C}, \mathsf{Sets}) \cong \mathsf{Fun}(\mathcal{C}, \mathsf{Fun}(\mathcal{D}^{\mathsf{op}}, \mathsf{Sets})) \quad \overset{\text{def}}{=} \mathsf{Fun}(\mathcal{C}, \mathsf{PSh}(\mathcal{D})).$$

- The equivalence between Items 1 and 3 is also an instance of currying, following from the isomorphisms of categories

$$\mathsf{Fun}(\mathcal{D}^{\mathsf{op}} \times \mathcal{C}, \mathsf{Sets}) \cong \mathsf{Fun}(\mathcal{D}^{\mathsf{op}}, \mathsf{Fun}(\mathcal{C}, \mathsf{Sets})) \quad \overset{\text{def}}{=} \mathsf{Fun}(\mathcal{D}^{\mathsf{op}}, \mathsf{Fun}(\mathcal{C}, \mathsf{Sets})).$$

- The equivalence between Items 1 and 4 follows from the universal property of the category $\mathsf{PSh}(\mathcal{C})$ of presheaves on $\mathcal{C}$ as the free cocompletion of $\mathcal{C}$ via the Yoneda embedding

$$\text{よ}\colon \mathcal{C}^{\mathsf{op}} \hookrightarrow \mathsf{PSh}(\mathcal{C})$$

  of $\mathcal{C}$ into $\mathsf{PSh}(\mathcal{C})$ (**??** of **??**).

This finishes the proof. ∎

15.2 Operations With Profunctors

15.2.1 The Domain and Range of a Profunctor

Let $\mathfrak{p}\colon \mathcal{C} \nrightarrow \mathcal{D}$ be a profunctor.[1]

1. The **domain of** $\mathfrak{p}$ is the presheaf $\mathrm{dom}(\mathfrak{p})\colon \mathcal{D}^{\mathsf{op}} \to \mathsf{Sets}$ on $\mathcal{D}$ defined by

$$\mathrm{dom}(\mathfrak{p})^{-} \overset{\mathrm{def}}{=} \operatorname*{colim}_{B\in\mathcal{D}}\left(\mathfrak{p}_{B}^{-}\right).$$

2. The **range of** $\mathfrak{p}$ is the copresheaf $\mathrm{range}(\mathfrak{p})\colon \mathcal{C} \to \mathsf{Sets}$ on $\mathcal{C}$ defined by

$$\mathrm{range}(\mathfrak{p})_{-} \overset{\mathrm{def}}{=} \operatorname*{colim}_{A\in\mathcal{D}}\left(\mathfrak{p}_{-}^{A}\right).$$

[1] In other words, the domain and range of $\mathfrak{p}$ are the functors

$$\begin{aligned} \mathrm{dom}(\mathfrak{p})&\colon \mathcal{D}^{\mathsf{op}} \to \mathsf{Sets}, \\ \mathrm{range}(\mathfrak{p})&\colon \mathcal{C} \to \mathsf{Sets} \end{aligned}$$

defined by

$$\begin{tikzcd} \mathcal{D}^{\mathsf{op}} \arrow[r, "\mathfrak{p}^{\dagger}"] \arrow[dr, "\mathrm{dom}(\mathfrak{p})"'] & \mathsf{PSh}(\mathcal{D}) \arrow[d, "\mathrm{colim}"] \\ & \mathsf{Sets}, \end{tikzcd} \qquad \begin{aligned} \mathrm{dom}(\mathfrak{p}) &\overset{\mathrm{def}}{=} \mathrm{colim} \circ \mathfrak{p}^{\dagger}, \\ \mathrm{range}(\mathfrak{p}) &\overset{\mathrm{def}}{=} \mathrm{colim} \circ \mathfrak{p}^{\ddagger}, \end{aligned} \qquad \begin{tikzcd} \mathcal{C} \arrow[r, "\mathfrak{p}^{\ddagger}"] \arrow[dr, "\mathrm{range}(\mathfrak{p})"'] & \mathsf{Fun}(\mathcal{C}, \mathsf{Sets}) \arrow[d, "\mathrm{colim}"] \\ & \mathsf{Sets}. \end{tikzcd}$$

15.2.2 Composition of Profunctors

Let $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ be categories and let $\mathfrak{p}\colon \mathcal{C} \nrightarrow \mathcal{D}$ and $\mathfrak{q}\colon \mathcal{D} \nrightarrow \mathcal{E}$ be profunctors.

Definition 15.2.2.1 ▶ Composition of Profunctors

The **composition of** $\mathfrak{p}$ **and** $\mathfrak{q}$ is the profunctor $\mathfrak{q} \diamond \mathfrak{p}\colon \mathcal{C} \nrightarrow \mathcal{E}$ defined by[1]

$$(\mathfrak{q} \diamond \mathfrak{p})_{-_2}^{-_1} \overset{\mathrm{def}}{=} \int^{B\in\mathcal{D}} \mathfrak{q}_{B}^{-_1} \times \mathfrak{p}_{-_2}^{B}.$$

[1] Alternatively, we may define $\mathfrak{q} \diamond \mathfrak{p}$ (using the equivalent definition of Item 2 of Remark 15.1.1.2) by

$$(\mathfrak{q} \diamond \mathfrak{p})^{\dagger} \overset{\mathrm{def}}{=} \mathrm{Lan}_{よ}(\mathfrak{p}^{\dagger}) \circ \mathfrak{q}^{\dagger}, \qquad \begin{tikzcd} & \mathcal{D} \arrow[r, "\mathfrak{p}^{\dagger}"] \arrow[d, hook, "よ"'] & \mathsf{PSh}(\mathcal{C}). \\ \mathcal{E} \arrow[r, "\mathfrak{q}^{\dagger}"'] & \mathsf{PSh}(\mathcal{D}) \arrow[ur, bend right, "\mathrm{Lan}_{よ}(\mathfrak{p}^{\dagger})"'] & \end{tikzcd}$$

15.2.3 Representable Profunctors

Definition 15.2.3.1 ► The Representable Profunctor Associated to a Functor

The **representable profunctor associated to a functor** $F\colon \mathcal{C} \to \mathcal{D}$ is the profunctor $\widehat{F}^*\colon \mathcal{C} \nrightarrow \mathcal{D}$ defined as the adjunct of the composition

$$\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{よ} \mathsf{PSh}(\mathcal{D})$$

under the adjunction

$$\mathsf{Fun}(\mathcal{D}^{\mathsf{op}} \times \mathcal{C}, \mathsf{Sets}) \cong \mathsf{Fun}(\mathcal{C}, \mathsf{PSh}(\mathcal{D}))$$

of **??** of **??**.[1]

[1]That is, we have

$$\widehat{F}^* \stackrel{\text{def}}{=} \mathsf{Hom}_{\mathcal{D}}(-_1, F_{-_2}).$$

Definition 15.2.3.2 ► Representable Profunctors

A profunctor is **representable** if it is isomorphic to a representable profunctor.

Definition 15.2.3.3 ► The Corepresentable Profunctor Associated to a Functor

The **corepresentable**[1] **profunctor associated to a functor** $F\colon \mathcal{C} \to \mathcal{D}$ is the profunctor $\widehat{F}_*\colon \mathcal{D} \nrightarrow \mathcal{C}$ defined as the adjunct of the composition

$$\mathcal{C}^{\mathsf{op}} \xrightarrow{F^{\mathsf{op}}} \mathcal{D}^{\mathsf{op}} \xrightarrow{よ} \mathsf{CoPSh}(\mathcal{D})$$

under the adjunction

$$\mathsf{Fun}(\mathcal{C}^{\mathsf{op}} \times \mathcal{D}, \mathsf{Sets}) \cong \mathsf{Fun}(\mathcal{C}^{\mathsf{op}}, \mathsf{CoPSh}(\mathcal{D}))$$

of **??** of **??**.[2]

[1]Some authors call both $\widehat{F}^*$ and $\widehat{F}_*$ the **representable profunctors associated to** F.

[2]That is:

$$\widehat{F}_* \stackrel{\text{def}}{=} \mathsf{Hom}_{\mathcal{D}}(F_{-_1}, -_2,).$$

Definition 15.2.3.4 ► Corepresentable Profunctors

A profunctor is **corepresentable** if it is isomorphic to a corepresentable profunctor.

15.2.4 Collages

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 15.2.4.1 ► The Collage of a Profunctor

The **collage** of a profunctor $\mathfrak{p}\colon \mathcal{C} \nrightarrow \mathcal{D}$ is the category $\mathsf{Coll}(\mathfrak{p})$[1] where[2]

- *Objects.* We have
$$\mathrm{Obj}(\mathsf{Coll}(\mathfrak{p})) \overset{\mathrm{def}}{=} \mathrm{Obj}(\mathcal{C}) \coprod \mathrm{Obj}(\mathcal{D});$$
- *Morphisms.* For each $A, B \in \mathrm{Obj}(\mathsf{Coll}(\mathfrak{p}))$, we have
$$\mathrm{Hom}_{\mathsf{Coll}(\mathfrak{p})}(A,B) \overset{\mathrm{def}}{=} \begin{cases} \mathrm{Hom}_{\mathcal{C}}(A,B) & \text{if } A,B \in \mathrm{Obj}(\mathcal{C}),\\ \mathrm{Hom}_{\mathcal{D}}(A,B) & \text{if } A,B \in \mathrm{Obj}(\mathcal{D}),\\ \mathfrak{p}(A,B) & \text{if } A \in \mathrm{Obj}(\mathcal{C}) \text{ and } B \in \mathrm{Obj}(\mathcal{D}),\\ \emptyset & \text{if } A \in \mathrm{Obj}(\mathcal{D}) \text{ and } B \in \mathrm{Obj}(\mathcal{C}); \end{cases}$$
- *Identities.* For each $A \in \mathrm{Obj}(\mathsf{Coll}(\mathfrak{p}))$, the unit map
$$\mathbb{K}^{\mathsf{Coll}(\mathfrak{p})}_{A}\colon \mathrm{pt} \longrightarrow \mathrm{Hom}_{\mathsf{Coll}(\mathfrak{p})}(A,A)$$
of $\mathsf{Coll}(\mathfrak{p})$ at A is defined by
$$\mathrm{id}_A \overset{\mathrm{def}}{=} \begin{cases} \mathrm{id}^{\mathcal{C}}_A & \text{if } A \in \mathrm{Obj}(\mathcal{C}),\\ \mathrm{id}^{\mathcal{D}}_A & \text{if } A \in \mathrm{Obj}(\mathcal{D}); \end{cases}$$
- *Composition.* For each $A, B, C \in \mathrm{Obj}(\mathsf{Coll}(\mathfrak{p}))$, the composition map
$$\circ^{\mathsf{Coll}(\mathfrak{p})}_{A,B,C}\colon \mathrm{Hom}_{\mathsf{Coll}(\mathfrak{p})}(B,C) \times \mathrm{Hom}_{\mathsf{Coll}(\mathfrak{p})}(A,B) \longrightarrow \mathrm{Hom}_{\mathsf{Coll}(\mathfrak{p})}(A,C)$$
of $\mathsf{Coll}(\mathfrak{p})$ at (A,B,C) is defined by[3]
$$\circ^{\mathsf{Coll}(\mathfrak{p})}_{A,B,C} \overset{\mathrm{def}}{=} \begin{cases} \circ^{\mathcal{C}}_{A,B,C} & \text{if } A,B,C \in \mathrm{Obj}(\mathcal{C}),\\ \mathfrak{p}^{A,B}_{C} & \text{if } A,B \in \mathrm{Obj}(\mathcal{C}) \text{ and } C \in \mathrm{Obj}(\mathcal{D}),\\ \iota & \text{if } A,C \in \mathrm{Obj}(\mathcal{C}) \text{ and } B \in \mathrm{Obj}(\mathcal{D}),\\ \iota & \text{if } B,C \in \mathrm{Obj}(\mathcal{C}) \text{ and } A \in \mathrm{Obj}(\mathcal{D}),\\ \mathfrak{p}^{A}_{B,C} & \text{if } A \in \mathrm{Obj}(\mathcal{C}) \text{ and } B,C \in \mathrm{Obj}(\mathcal{D}),\\ \iota & \text{if } B \in \mathrm{Obj}(\mathcal{C}) \text{ and } A,C \in \mathrm{Obj}(\mathcal{D}),\\ \iota & \text{if } C \in \mathrm{Obj}(\mathcal{C}) \text{ and } A,B \in \mathrm{Obj}(\mathcal{D}),\\ \circ^{\mathcal{D}}_{A,B,C} & \text{if } A,B,C \in \mathrm{Obj}(\mathcal{D}). \end{cases}$$

[1] *Further Notation:* Also written $\mathcal{C} \star^{\mathfrak{p}} \mathcal{D}$, notably in [**HigherToposTheory**].
[2] We also have a functor $\phi\colon \mathsf{Coll}(\mathfrak{p}) \longrightarrow \mathbb{K}$ where

- *Actions on Objects.* For each $A \in \mathrm{Obj}(\mathrm{Coll}(\mathfrak{p}))$, we have

$$\phi_A \overset{\mathrm{def}}{=} \begin{cases} [0] & \text{if } A \in \mathrm{Obj}(C), \\ [1] & \text{if } A \in \mathrm{Obj}(\mathcal{D}). \end{cases}$$

- *Actions on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathrm{Coll}(\mathfrak{p}))$, the action on morphisms

$$\phi_{A,B} \colon \mathrm{Hom}_{\mathrm{Coll}(\mathfrak{p})}(A, B) \to \mathrm{Hom}_{\mathrm{Coll}(\mathfrak{p})}(\phi_A, \phi_B)$$

of ϕ at (A, B) is given by

$$\phi_{A,B}(f) \overset{\mathrm{def}}{=} \begin{cases} \mathrm{id}_{[0]} & \text{if } A, B \in \mathrm{Obj}(C), \\ \mathrm{id}_{[1]} & \text{if } A, B \in \mathrm{Obj}(\mathcal{D}), \\ [0] \to [1] & \text{if } A \in \mathrm{Obj}(C) \text{ and } B \in \mathrm{Obj}(\mathcal{D}). \end{cases}$$

If $A \in \mathrm{Obj}(\mathcal{D})$ and $B \in \mathrm{Obj}(C)$, we have $\phi_{A,B} \overset{\mathrm{def}}{=} \mathrm{id}_{\emptyset}$.

[3]Here the maps $\mathfrak{p}_C^{A,B}$ and $\mathfrak{p}_{B,C}^A$ are the maps

$$\mathfrak{p}_C^{A,B} \colon \mathfrak{p}_C^B \times \mathrm{Hom}_C(A, B) \to \mathfrak{p}_C^A,$$
$$\mathfrak{p}_{B,C}^A \colon \mathrm{Hom}_{\mathcal{D}}(B, C) \times \mathfrak{p}_B^A \to \mathfrak{p}_C^A$$

coming from the profunctor structure of $\mathfrak{p}$ and the ι's are inclusions of the empty set into the appropriate Hom sets.

Example 15.2.4.2 ▶ The Collage of Δ_{pt} ([HigherToposTheory])

If $\mathfrak{p}$ is the constant functor $\Delta_{\mathrm{pt}} \colon \mathcal{D}^{\mathrm{op}} \times C \to \mathsf{Sets}$ with value pt, then $\mathrm{Coll}(\mathfrak{p})$ is the join $C \star \mathcal{D}$ of C and $\mathcal{D}$ of **??**.

Proposition 15.2.4.3 ▶ Properties of Collages

Let $\mathfrak{p} \colon C \nrightarrow \mathcal{D}$ be a profunctor.

1. *Functoriality.* The assignment $\mathfrak{p} \mapsto \mathrm{Coll}(\mathfrak{p})$ defines a functor[1]

$$\mathrm{Coll}_{C,\mathcal{D}} \colon \mathsf{Prof}(C, \mathcal{D}) \to \mathsf{Cats}_{/\mathbb{1}}(C, \mathcal{D}),$$

where

- *Action on Objects.* For each $\mathfrak{p} \in \mathrm{Obj}(\mathsf{Prof}(C, \mathcal{D}))$, we have

$$[\mathrm{Coll}](\mathfrak{p}) \overset{\mathrm{def}}{=} \mathrm{Coll}(\mathfrak{p});$$

- *Action on Morphisms.* For each $\mathfrak{p}, \mathfrak{q} \in \mathrm{Obj}(\mathsf{Prof}(C, \mathcal{D}))$, the action on Hom-sets

$$\mathrm{Coll}_{\mathfrak{p},\mathfrak{q}} \colon \mathrm{Nat}(\mathfrak{p}, \mathfrak{q}) \to \mathsf{Fun}_{/\mathbb{1}}(\mathrm{Coll}(\mathfrak{p}), \mathrm{Coll}(\mathfrak{q}))$$

of Coll at $(\mathfrak{p}, \mathfrak{q})$ is the function sending a natural transformation $\alpha \colon \mathfrak{p} \Longrightarrow \mathfrak{q}$ to the functor

$$\mathsf{Coll}(\alpha) \colon \mathsf{Coll}(\mathfrak{p}) \to \mathsf{Coll}(\mathfrak{q})$$

over $\not\Vdash$ where

- *Action on Objects.* For each $X \in \mathrm{Obj}(\mathsf{Coll}(\mathfrak{p}))$, we have

$$[\mathsf{Coll}(\alpha)](X) \stackrel{\text{def}}{=} X;$$

- *Action on Morphisms.* For each $X, Y \in \mathrm{Obj}(\mathsf{Coll}(\mathfrak{p}))$, the action on Hom-sets

$$\mathsf{Coll}(\alpha)_{X,Y} \colon \mathrm{Hom}_{\mathsf{Coll}(\mathfrak{p})}(X, Y) \to \underbrace{\mathrm{Hom}_{\mathsf{Coll}(\mathfrak{q})}([\mathsf{Coll}(\alpha)](X), [\mathsf{Coll}(\alpha)](Y))}_{\stackrel{\text{def}}{=}\mathrm{Hom}_{\mathsf{Coll}(\mathfrak{q})}(X,Y)}$$

of $\mathsf{Coll}(\alpha)$ at (X, Y) is defined as follows:

  * If $X, Y \in \mathrm{Obj}(\mathcal{C})$ or $X, Y \in \mathrm{Obj}(\mathcal{D})$, then we have

$$\mathsf{Coll}(\alpha)_{X,Y}(f) \stackrel{\text{def}}{=} f$$

  for each $f \in \mathrm{Hom}_{\mathsf{Coll}(\mathfrak{p})}(X, Y)$.

  * If $X \in \mathrm{Obj}(\mathcal{C})$ and $Y \in \mathrm{Obj}(\mathcal{D})$, then

$$\mathsf{Coll}(\alpha)_{X,Y} \colon \underbrace{\mathrm{Hom}_{\mathsf{Coll}(\mathfrak{p})}(X, Y)}_{\stackrel{\text{def}}{=}\mathfrak{p}^X_Y} \to \underbrace{\mathrm{Hom}_{\mathsf{Coll}(\mathfrak{q})}(X, Y)}_{\stackrel{\text{def}}{=}\mathfrak{q}^X_Y}$$

  is defined by

$$\mathsf{Coll}(\alpha)_{X,Y}(f) \stackrel{\text{def}}{=} \alpha^X_Y;$$

  * If $Y \in \mathrm{Obj}(\mathcal{C})$ and $X \in \mathrm{Obj}(\mathcal{D})$, then we have

$$\mathsf{Coll}(\alpha)_{X,Y}(f) \stackrel{\text{def}}{=} \mathrm{id}_{\emptyset}.$$

2. *Collages as Lax Colimits.* We have an isomorphism of categories

$$\mathsf{Coll}(\mathfrak{p}) \cong \mathrm{colim}^{\mathsf{lax}}(\mathfrak{p}),$$

functorial in $\mathfrak{p}$, where the above lax colimit is taken in the bicategory Prof.

3. *Profunctors vs.Collages.* We have an equivalence of categories

$$(\mathsf{Coll} \dashv \Gamma)\colon \quad \begin{tikzcd} \mathsf{Prof}(\mathcal{C}, \mathcal{D}) \arrow[r, bend left, "\mathsf{Coll}"] \arrow[r, phantom, "\perp_{\mathrm{eq}}"] & \mathsf{Cats}_{/\not\Vdash}, \arrow[l, bend left, "\Gamma"] \end{tikzcd}$$

where $\Gamma\colon \mathsf{Cats}_{/\nVdash} \longrightarrow \mathsf{Prof}(\mathcal{C},\mathcal{D})$ is the functor sending a functor $\mathcal{E} \longrightarrow \nVdash$ to the profunctor

$$\Gamma(\mathfrak{p})\colon \mathcal{C} \nrightarrow \mathcal{D}$$

given on objects by

$$\Gamma(\mathfrak{p})^{A}_{B} \overset{\text{def}}{=} \mathrm{Hom}_{\mathcal{E}}(A,B)$$

for each $A, B \in \mathrm{Obj}(\mathcal{E})$.

---

[1] Here $\mathsf{Cats}_{/\nVdash}(\mathcal{C},\mathcal{D})$ is the category defined as the pullback

$$\mathsf{Cats}_{/\nVdash}(\mathcal{C},\mathcal{D}) \overset{\text{def}}{=} \mathrm{pt} \underset{[\mathcal{C}],\mathsf{Cats},\mathrm{fib}_0}{\times} \mathsf{Cats}_{/\nVdash} \underset{\mathrm{fib}_1,\mathsf{Cats},[\mathcal{D}]}{\times} \mathrm{pt},$$

as in the diagram

$$\begin{tikzcd}
 & & \mathsf{Cats}_{/\nVdash}(\mathcal{C},\mathcal{D}) \arrow[dl] \arrow[dr] \arrow[dd, phantom, very near start, "\lrcorner" rotate=-45] & & \\
 & \mathsf{Cats}_{/\nVdash} \underset{\mathsf{Cats}}{\times} \mathrm{pt} \arrow[dl] \arrow[dr] \arrow[dd, phantom, very near start, "\lrcorner" rotate=-45] & & \mathrm{pt} \underset{\mathsf{Cats}}{\times} \mathsf{Cats}_{/\nVdash} \arrow[dl] \arrow[dr] \arrow[dd, phantom, very near start, "\lrcorner" rotate=-45] & \\
\mathrm{pt} \arrow[dr, "{[\mathcal{C}]}"'] & & \mathsf{Cats}_{/\nVdash} \arrow[dl, "\mathrm{fib}_{[0]}"] \arrow[dr, "\mathrm{fib}_{[1]}"'] & & \mathrm{pt}. \arrow[dl, "{[\mathcal{D}]}"] \\
 & \mathsf{Cats} & & \mathsf{Cats} &
\end{tikzcd}$$

Proof 15.2.4.4 ► Proof of Proposition 15.2.4.3

Item 1: Functoriality

Omitted.

Item 2: Collages as Lax Colimits

See [**collages-as-lax-colimits**].

Item 3: Profunctors vs.Collages

See [**joyal:distributors-and-barrels**]. □

15.3 Categories of Profunctors

15.3.1 The Bicategory of Profunctors

Definition 15.3.1.1 ▶ The Bicategory of Profunctors

The **bicategory of profunctors** is the bicategory Prof where[1]

1. *Objects.* The objects of Prof are categories;
2. *1-Morphisms.* The 1-morphisms of Prof are profunctors;
3. *2-Morphisms.* The 2-morphisms of Prof are natural transformations between profunctors;
4. *Identities.* For each $\mathcal{C} \in \mathrm{Obj}(\mathsf{Prof})$, we have

$$\mathrm{id}^{\mathsf{Prof}}_{\mathcal{C}} \overset{\text{def}}{=} \mathrm{Hom}_{\mathcal{C}}(-,-);$$

5. *Composition.* For each $\mathcal{C}, \mathcal{D}, \mathcal{E} \in \mathrm{Obj}(\mathsf{Prof})$, the composition bifunctor

$$\diamond \colon \mathsf{Prof}(\mathcal{D}, \mathcal{E}) \times \mathsf{Prof}(\mathcal{C}, \mathcal{D}) \to \mathsf{Prof}(\mathcal{C}, \mathcal{E})$$

is defined on objects by sending profunctors $\mathfrak{p} \colon \mathcal{C} \nrightarrow \mathcal{D}$ and $\mathfrak{q} \colon \mathcal{D} \nrightarrow \mathcal{E}$ to the profunctor $\mathfrak{q} \diamond \mathfrak{p}$ of Definition 15.2.2.1.

[1] The bicategory Prof admits a nice strictification to a 2-category: it is biequivalent to the sub-bicategory of Cats spanned by the presheaf categories, cocontinuous functors between them, and natural transformation between these.

Proof 15.3.1.2 ▶ Proof of Definition 15.3.1.1

See Definition 15.3.1.1. □

15.3.2 Properties of Prof

Proposition 15.3.2.1 ▶ Properties of the Bicategory of Profunctors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

1. *Self-Duality.* The bicategory Prof is self-dual: we have a biequivalence of bicategories

$$(-)^{\mathsf{op}} \colon \mathsf{Prof} \xrightarrow{\cong} \mathsf{Prof}^{\mathsf{op}}$$

where

- *Action on Objects.* The functor $(-)^{\mathsf{op}}$ sends categories to their opposites;
- *Action on 1-Morphisms.* The functor $(-)^{\mathsf{op}}$ sends profunctors to itself

under the identification

$$\begin{aligned}\mathsf{Prof}(\mathcal{C},\mathcal{D}) &\stackrel{\text{def}}{=} \mathsf{Fun}(\mathcal{D}^{\mathsf{op}}\times\mathcal{C},\mathsf{Sets}),\\ &\cong \mathsf{Fun}(\mathcal{C}\times\mathcal{D}^{\mathsf{op}},\mathsf{Sets}),\\ &\stackrel{\text{def}}{=} \mathsf{Prof}(\mathcal{D}^{\mathsf{op}},\mathcal{C}^{\mathsf{op}});\end{aligned}$$

- *Action on 2-Morphisms.* The functor $(-)^{\mathsf{op}}$ sends natural transformations between profunctors to themselves.

2. *Relation to* Cats. The co/representable profunctor constructions of Definitions 15.2.3.1 and 15.2.3.3 define embeddings of bicategories

$$\begin{aligned}\mathsf{Cats}^{\mathsf{op}} &\hookrightarrow \mathsf{Prof},\\ \mathsf{Cats}^{\mathsf{co}} &\hookrightarrow \mathsf{Prof}.\end{aligned}$$

3. *Equivalences in* Prof *and Cauchy Completions.* Every category is equivalent to its Cauchy completion in Prof.

4. *Equivalences in* Prof. The following conditions are equivalent:

   (a) The categories $\mathcal{C}$ and $\mathcal{D}$ are equivalent in Prof.

   (b) The categories $\mathsf{PSh}(\mathcal{C})$ and $\mathsf{PSh}(\mathcal{D})$ are equivalent in Cats_2.

   (c) The Cauchy completions of $\mathcal{C}$ and $\mathcal{D}$ are equivalent in Cats_2.

5. *Adjunctions in* Prof. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. The following data are equivalent:

   (a) An adjunction in Prof from $\mathcal{C}$ to $\mathcal{D}$.

   (b) A functor from $\mathcal{C}$ to the Cauchy completion $\overline{\mathcal{D}}$ of $\mathcal{D}$.

   (c) A semifunctor from $\mathcal{C}$ to $\mathcal{D}$.

6. *As a Kleisli Bicategory.* We have a biequivalence of bicategories

$$\mathsf{Prof} \cong \mathsf{FreePsAlg}_{\mathsf{PSh}},$$

where PSh is the presheaf category relative pseudomonad of [**relative-pseudomonads-kleisli-bicategories-and-substitution-monoidal-structures**].

7. *Closedness.* The bicategory Prof is a closed bicategory, where given a profunctor $\mathfrak{p}\colon \mathcal{C} \nrightarrow \mathcal{D}$ and a category $\mathcal{X}$:

- *Right Kan Extensions.* The right adjoint

$$\mathrm{Ran}_{\mathfrak{p}} \colon \mathrm{Rel}(\mathcal{C}, \mathcal{X}) \to \mathrm{Rel}(\mathcal{D}, \mathcal{X})$$

  to the precomposition functor $\mathfrak{p}^* \colon \mathrm{Rel}(\mathcal{D}, \mathcal{X}) \to \mathrm{Rel}(\mathcal{C}, \mathcal{X})$ is given by

$$\mathrm{Ran}_{\mathfrak{p}}(\mathfrak{q}) \overset{\text{def}}{=} \int_{A \in \mathcal{C}} \mathrm{Sets}(\mathfrak{p}^{-2}_{A}, \mathfrak{q}^{-1}_{A})$$

  for each $\mathfrak{q} \in \mathrm{Rel}(\mathcal{C}, \mathcal{X})$.
- *Right Kan Lifts.* The right adjoint to the postcomposition functor

$$\mathrm{Rift}_{\mathfrak{p}} \colon \mathrm{Rel}(\mathcal{X}, \mathcal{D}) \to \mathrm{Rel}(\mathcal{X}, \mathcal{C})$$

  to the postcomposition functor $\mathfrak{p}_* \colon \mathrm{Rel}(\mathcal{X}, \mathcal{C}) \to \mathrm{Rel}(\mathcal{X}, \mathcal{D})$ is given by

$$\mathrm{Rift}_{\mathfrak{p}}(\mathfrak{q}) \overset{\text{def}}{=} \int_{B \in \mathcal{D}} \mathrm{Sets}\Big(\mathfrak{p}^{B}_{-_1}, \mathfrak{q}^{B}_{-_2}\Big)$$

  for each $\mathfrak{q} \in \mathrm{Rel}(\mathcal{X}, \mathcal{D})$.

8. *Un/Straightening for Profunctors: Two-Sided Discrete Fibrations.* We have an equivalence of categories

$$\mathrm{Prof}(\mathcal{C}, \mathcal{D}) \cong \mathrm{DFib}(\mathcal{C}, \mathcal{D}).$$

PROOF 15.3.2.2 ▶ PROOF OF PROPOSITION 15.3.2.1

Item 1: Self-Duality

See [**loregian2020coend**].

Item 2: Relation to Cats

See [**loregian2020coend**].

Item 3: Equivalences in Prof and Cauchy Completions

See [**borceux-2**].

Item 4: Equivalences in Prof

See [**borceux-2**].

Item 5: Adjunctions in Prof

Omitted.

Item 6: As a Kleisli Bicategory

See [**relative-pseudomonads-kleisli-bicategories-and-substitution-monoidal-structures**].

Item 7: Closedness

Omitted.

Item 8: Un/Straightening for Profunctors: Two-Sided Discrete Fibrations

See [**riehl:two-sided-discrete-fibrations**]

Chapter 16

Cartesian Closed Categories

Create tags (see [MSE 350788] for some of these):

1. define bicategory $\mathsf{Adj}(C)$
2. https://www.google.com/search?q=category+of+categories+is+not+locally+cartesian+closed
3. https://math.stackexchange.com/questions/2864916/are-there-important-locally-cartesian-closed-categories-that-actually-are-not-ca
4. Cats is not locally Cartesian closed: f^* does have a left adjoint (the proof for fibred sets seems to apply for any category with pullbacks), but does not have a right adjoint, as proved in example 1.7 of https://sinhp.github.io/files/CT/notes_on_lcccs.pdf
5. internal **Hom** in categories of co/Cartesian fibrations
6. https://mathoverflow.net/questions/460146/universal-property-of-isbell-duality
7. http://www.tac.mta.ca/tac/volumes/36/12/36-12abs.html
8. Cartesian closed categories and locally Cartesian closed categories
   (a) https://ncatlab.org/nlab/show/locally+cartesian+closed+functor
   (b) https://ncatlab.org/nlab/show/cartesian+closed+functor
   (c) https://ncatlab.org/nlab/show/locally+cartesian+closed+category
   (d) https://ncatlab.org/nlab/show/Frobenius+reciprocity
9. https://math.stackexchange.com/questions/3657046/the-inver

se-of-a-natural-isomorphism-is-a-natural-isomorphism to justify adjunctions via homs

10. https://ncatlab.org/nlab/show/enrichment+versus+internalisation

11. https://mathoverflow.net/questions/382239/proof-that-a-cartesian-category-is-monoidal

12. https://arxiv.org/pdf/2004.08964.pdf

Create tags:

1. https://www.google.com/search?q=category+of+categories+is+not+locally+cartesian+closed

2. https://math.stackexchange.com/questions/2864916/are-there-important-locally-cartesian-closed-categories-that-actually-are-not-ca

3. Cats is not locally Cartesian closed: f^* does have a left adjoint (the proof for fibred sets seems to apply for any category with pullbacks), but does not have a right adjoint, as proved in example 1.7 of https://sinhp.github.io/files/CT/notes_on_lcccs.pdf

4. Cartesian closed categories and locally Cartesian closed categories
   (a) https://ncatlab.org/nlab/show/locally+cartesian+closed+functor
   (b) https://ncatlab.org/nlab/show/cartesian+closed+functor
   (c) https://ncatlab.org/nlab/show/locally+cartesian+closed+category
   (d) https://ncatlab.org/nlab/show/Frobenius+reciprocity

5. https://mathoverflow.net/questions/382239/proof-that-a-cartesian-category-is-monoidal

Contents

16.1 Cartesian Closed Categories

Appendices

16.A Other Chapters

Chapter 17

Kan Extensions

Contents

Appendices

17.A Other Chapters

Sets

1. Sets
2. Constructions With Sets
3. Pointed Sets
4. Tensor Products of Pointed Sets
5. Relations
6. Spans
7. Posets

Indexed and Fibred Sets

7. Indexed Sets
8. Fibred Sets
9. Un/Straightening for Indexed and Fibred Sets

Category Theory

11. Categories
12. Types of Morphisms in Categories
13. Adjunctions and the Yoneda Lemma
14. Constructions With Categories
15. Profunctors
16. Cartesian Closed Categories
17. Kan Extensions

Bicategories

18. Bicategories
19. Internal Adjunctions

Internal Category Theory

20. Internal Categories

Cyclic Stuff

Part IV

Bicategories

Chapter 18

Bicategories

Contents

Create tags and TODO:

1. spans in bicategories: add Proposition 7 here: https://arxiv.org/abs/1903.03890
2. add fact: internal adjunctions in $\mathsf{PseudoFun}(\mathcal{C},\mathcal{D})$ are precisely the invertible strong transformations as in [JY21, Example 6.2.7]. What are the internal adjunctions?

18.1 Monomorphisms in Bicategories

18.1.1 Faithful Monomorphisms

Let $\mathcal{C}$ be a bicategory.

Definition 18.1.1.1 ► Faithful Monomorphisms

A 1-morphism $f \colon A \to B$ is a **faithful monomorphism in C** if the following equivalent conditions are satisfied:

1. For each $X \in \mathrm{Obj}(C)$, the functor
$$f_* \colon \mathrm{Hom}_C(X, A) \to \mathrm{Hom}_C(X, B)$$
given by postcomposition by f is faithful.

2. Given a diagram in C of the form
$$\begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & A \arrow[r, "f"] & B, \end{tikzcd}$$
if we have $\mathrm{id}_f \star \alpha = \mathrm{id}_f \star \beta$, then $\alpha = \beta$.

Example 18.1.1.2 ► Examples of Faithful Monomorphisms

Here are some examples of faithful monomorphisms.

1. *Full Monomorphisms in* Cats_2.
2. *Full Monomorphisms in* **Rel**.
3. *Full Monomorphisms in* Span.

18.1.2 Full Monomorphisms

Let C be a bicategory.

Definition 18.1.2.1 ► Full Monomorphisms

A 1-morphism $f \colon A \to B$ is a **full monomorphism in C** if the following equivalent conditions are satisfied:

1. For each $X \in \mathrm{Obj}(C)$, the functor
$$f_* \colon \mathrm{Hom}_C(X, A) \to \mathrm{Hom}_C(X, B)$$
given by postcomposition by f is full.

2. For each $X \in \mathrm{Obj}(\mathcal{C})$ and each 2-morphism

$$\gamma\colon f\circ\phi \Longrightarrow f\circ\psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"] \arrow[r, bend right, "f\circ\psi"'] \arrow[r, phantom, "\gamma\Downarrow"] & B \end{tikzcd}$$

of $\mathcal{C}$, there exists a 2-morphism $\alpha\colon \phi \Longrightarrow \psi$ of $\mathcal{C}$ such that we have an equality

$$\begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & A \arrow[r, "f"] & B \end{tikzcd} = \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"] \arrow[r, bend right, "f\circ\psi"'] \arrow[r, phantom, "\gamma\Downarrow"] & B \end{tikzcd}$$

of pasting diagrams in $\mathcal{C}$, i.e. such that we have

$$\gamma = \mathrm{id}_f \star \alpha.$$

Example 18.1.2.2 ► Examples of Full Monomorphisms

Here are some examples of full monomorphisms.

1. *Full Monomorphisms in* Cats_2.
2. *Full Monomorphisms in* **Rel**.
3. *Full Monomorphisms in* Span.

18.1.3 Fully Faithful Monomorphisms

Let $\mathcal{C}$ be a bicategory.

Definition 18.1.3.1 ► Fully Faithful Monomorphisms

A 1-morphism $f\colon A \to B$ is a **fully faithful monomorphism in** $\mathcal{C}$ if the following equivalent conditions are satisfied:

1. The 1-morphism f is fully and faithful.
2. For each $X \in \mathrm{Obj}(\mathcal{C})$, the functor
$$f_*\colon \mathrm{Hom}_{\mathcal{C}}(X, A) \to \mathrm{Hom}_{\mathcal{C}}(X, B)$$
given by postcomposition by f is fully faithful.
3. The conditions in Item 1 of Definition 18.1.1.1 and Item 1 of Definition 18.1.2.1 hold.

Example 18.1.3.2 ▶ Examples of Fully Faithful Monomorphisms

Here are some examples of fully faithful monomorphisms.

1. *Fully Faithful Monomorphisms in* Cats_2.
2. *Fully Faithful Monomorphisms in* **Rel**.
3. *Fully Faithful Monomorphisms in* Span.

18.1.4 Strict Monomorphisms

Let C be a bicategory.

Definition 18.1.4.1 ▶ Strict Monomorphisms

A 1-morphism $f\colon A \to B$ is a **strict monomorphism in** C if the following equivalent conditions are satisfied:

1. For each $X \in \mathrm{Obj}(C)$, the action on objects
$$f_*\colon \mathrm{Obj}(\mathrm{Hom}_C(X,A)) \to \mathrm{Obj}(\mathrm{Hom}_C(X,B))$$
of the functor
$$f_*\colon \mathrm{Hom}_C(X,A) \to \mathrm{Hom}_C(X,B)$$
given by postcomposition by f is injective.
2. For each diagram in C of the form
$$X \underset{\psi}{\overset{\phi}{\rightrightarrows}} A \xrightarrow{f} B,$$
if $f \circ \phi = f \circ \psi$, then $\phi = \psi$.

Example 18.1.4.2 ▶ Examples of Strict Monomorphisms

Here are some examples of strict monomorphisms.

1. *Strict Monomorphisms in* Cats_2.
2. *Strict Monomorphisms in* **Rel**.
3. *Strict Monomorphisms in* Span.

18.2 Epimorphisms in Bicategories

18.2.1 Faithful Epimorphisms

Let C be a bicategory.

Definition 18.2.1.1 ► Faithful Epimorphisms

A 1-morphism $f\colon A \to B$ is a **faithful epimorphism in** C if the following equivalent conditions are satisfied:

1. For each $X \in \mathrm{Obj}(C)$, the functor
$$f^*\colon \mathrm{Hom}_C(B, X) \to \mathrm{Hom}_C(A, X)$$
given by precomposition by f is faithful.
2. Given a diagram in C of the form
$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & X, \end{tikzcd}$$
if we have $\alpha \star \mathrm{id}_f = \beta \star \mathrm{id}_f$, then $\alpha = \beta$.

Example 18.2.1.2 ► Examples of Faithful Epimorphisms

Here are some examples of faithful epimorphisms.

1. *Full Epimorphisms in* Cats_2.
2. *Full Epimorphisms in* **Rel**.
3. *Full Epimorphisms in* Span.

18.2.2 Full Epimorphisms

Let C be a bicategory.

Definition 18.2.2.1 ► Full Epimorphisms

A 1-morphism $f\colon A \to B$ is a **full epimorphism in** C if the following equivalent conditions are satisfied:

1. For each $X \in \mathrm{Obj}(\mathcal{C})$, the functor

$$f^* \colon \mathrm{Hom}_{\mathcal{C}}(B, X) \to \mathrm{Hom}_{\mathcal{C}}(A, X)$$

   given by precomposition by f is full.

2. For each $X \in \mathrm{Obj}(\mathcal{C})$ and each 2-morphism

$$\gamma \colon \phi \circ f \Longrightarrow \psi \circ f, \qquad \begin{tikzcd} X \arrow[r, bend left, "\phi\circ f"] \arrow[r, bend right, "\psi\circ f"'] \arrow[r, phantom, "\gamma\Downarrow"] & B \end{tikzcd}$$

   of $\mathcal{C}$, there exists a 2-morphism $\alpha \colon \phi \Longrightarrow \psi$ of $\mathcal{C}$ such that we have an equality

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & X \end{tikzcd} = \begin{tikzcd} A \arrow[r, bend left, "\phi\circ f"] \arrow[r, bend right, "\psi\circ f"'] \arrow[r, phantom, "\gamma\Downarrow"] & X \end{tikzcd}$$

   of pasting diagrams in $\mathcal{C}$, i.e. such that we have

$$\gamma = \alpha \star \mathrm{id}_f.$$

Example 18.2.2.2 ▶ Examples of Full Epimorphisms

Here are some examples of full epimorphisms.

1. *Full Epimorphisms in* Cats_2.
2. *Full Epimorphisms in* **Rel**.
3. *Full Epimorphisms in* Span.

18.2.3 Fully Faithful Epimorphisms

Let $\mathcal{C}$ be a bicategory.

Definition 18.2.3.1 ▶ Fully Faithful Epimorphisms

A 1-morphism $f \colon A \to B$ is a **fully faithful epimorphism in** $\mathcal{C}$ if the following equivalent conditions are satisfied:

1. The 1-morphism f is fully and faithful.

2. For each $X \in \mathrm{Obj}(\mathcal{C})$, the functor

$$f^* \colon \mathrm{Hom}_{\mathcal{C}}(B, X) \to \mathrm{Hom}_{\mathcal{C}}(A, X)$$

given by precomposition by f is fully faithful.

3. The conditions in Item 1 of Definition 18.2.1.1 and Item 1 of Definition 18.2.2.1 hold.

Example 18.2.3.2 ▶ Examples of Fully Faithful Epimorphisms

Here are some examples of fully faithful epimorphisms.

1. *Fully Faithful Epimorphisms in* Cats_2.
2. *Fully Faithful Epimorphisms in* **Rel**.
3. *Fully Faithful Epimorphisms in* Span.

18.2.4 Strict Epimorphisms

Let $\mathcal{C}$ be a bicategory.

Definition 18.2.4.1 ▶ Strict Epimorphisms

A 1-morphism $f \colon A \to B$ is a **strict epimorphism in** $\mathcal{C}$ if the following equivalent conditions are satisfied:

1. For each $X \in \mathrm{Obj}(\mathcal{C})$, the action on objects

$$f^* \colon \mathrm{Obj}(\mathrm{Hom}_{\mathcal{C}}(B, X)) \to \mathrm{Obj}(\mathrm{Hom}_{\mathcal{C}}(A, X))$$

of the functor

$$f^* \colon \mathrm{Hom}_{\mathcal{C}}(B, X) \to \mathrm{Hom}_{\mathcal{C}}(A, X)$$

given by precomposition by f is injective.

2. For each diagram in $\mathcal{C}$ of the form

$$A \xrightarrow{f} B \overset{\phi}{\underset{\psi}{\rightrightarrows}} X,$$

if $\phi \circ f = \psi \circ f$, then $\phi = \psi$.

Example 18.2.4.2 ▶ Examples of Strict Epimorphisms

Here are some examples of strict epimorphisms.

1. *Strict Epimorphisms in* Cats_2.
2. *Strict Epimorphisms in* **Rel**.
3. *Strict Epimorphisms in* Span.

18.3 bicategories of spans

Proposition 18.3.0.1 ▶ Properties of the Category of Spans Between Two Objects

Let A and B be objects of C.

1. *As a Pullback.* We have an isomorphism of categories

$$\mathsf{Span}_C(A,B) \cong C_{/A} \times_C C_{/B},\qquad \begin{tikzcd} \mathsf{Span}(A,B) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C_{/B} \arrow[d, "忘"] \\ C_{/A} \arrow[r, "忘"'] & C. \end{tikzcd}$$

Proof 18.3.0.2 ▶ Proof of Proposition 18.3.0.1

Item 1: As a Pullback

In detail, the pullback $C_{/A} \times_C C_{/B}$ is the category where

- *Objects.* The objects of $C_{/A} \times_C C_{/B}$ consist of pairs $((S,f),(S',g))$ of objects of C consisting of
  - A pair (S,f) in $\mathrm{Obj}(C_{/A})$ consisting of an object S of C and a morphism $f\colon S \to A$ of C;
  - A pair (S',g) in $\mathrm{Obj}(C_{/B})$ consisting of an object S' of C and a morphism $g\colon S \to B$ of C;

  such that

  $$\underbrace{忘(S,f)}_{\overset{\text{def}}{=}S} = \underbrace{忘(S',g)}_{\overset{\text{def}}{=}S'}.$$

  Thus the objects of $C_{/A} \times_C C_{/B}$ are the same as spans in C from A to B.

- *Morphisms.* A morphism of $\mathcal{C}_{/A} \times_{\mathcal{C}} \mathcal{C}_{/B}$ from (S, f, g) to (S', f', g') consists of a pair of morphisms

$$\phi\colon S \to S'$$
$$\psi\colon S \to S'$$

such that the diagrams

$$\begin{tikzcd} S \arrow[rr, "\phi"] \arrow[dr, "f"'] & & S' \arrow[dl, "f'"] \\ & A & \end{tikzcd} \qquad \begin{tikzcd} S \arrow[rr, "\psi"] \arrow[dr, "g"'] & & S' \arrow[dl, "g'"] \\ & B & \end{tikzcd}$$

such that

$$\underbrace{忘(\phi)}_{\stackrel{\text{def}}{=}\phi} = \underbrace{忘(\psi)}_{\stackrel{\text{def}}{=}\psi}.$$

Thus the morphisms of $\mathcal{C}_{/A} \times_{\mathcal{C}} \mathcal{C}_{/B}$ are also the same as morphisms of spans in $\mathcal{C}$ from (S, f, g) to (S, f', g').

- *Identities and Composition.* The identities and composition of $\mathcal{C}_{/A} \times_{\mathcal{C}} \mathcal{C}_{/B}$ are also the same as those in $\mathsf{Span}_{\mathcal{C}}(A, B)$.

This finishes the proof. □

Appendices

18.A Other Chapters

Chapter 19

Internal Adjunctions

Create tags:

1. https://www.google.com/search?q=mate+of+an+adjunction
2. Moreover, by uniqueness of adjoints (Item 2 of Proposition 19.1.2.4), this implies also that $S = f^{-1}$.
3. define bicategory $\mathsf{Adj}(\mathcal{C})$
4. walking monad
5. proposition: 2-functors preserve unitors and associators
6. https://ncatlab.org/nlab/show/2-category+of+adjunctions. Is there a 3-category too?
7. https://ncatlab.org/nlab/show/free+monad
8. https://ncatlab.org/nlab/show/CatAdj
9. https://ncatlab.org/nlab/show/Adj
10. $\mathsf{Adj}(\mathsf{Adj}(\mathcal{C}))$

Contents

19.1 Internal Adjunctions

19.1.1 The Walking Adjunction

Definition 19.1.1.1 ▶ The Walking Adjunction

The **walking adjunction** is the bicategory Adj freely generated by[1]

- *Objects.* A pair of objects A and B;
- *Morphisms.* A pair of morphisms

$$\begin{aligned} L&\colon A \to B,\\ R&\colon B \to A;\end{aligned}$$

- 2-*Morphisms.* A pair of 2-morphisms

$$\begin{aligned} \eta&\colon \mathrm{id}_A \to R\circ L,\\ \epsilon&\colon L\circ R \to \mathrm{id}_B;\end{aligned}$$

subject to the equalities

$$
\begin{tikzcd}
& B \arrow[rr, "\mathrm{id}_B"] \arrow[dr, "R"] & {} \arrow[d, phantom, "\Uparrow\epsilon"] & B \\
A \arrow[ur, "L"] \arrow[rr, "\mathrm{id}_A"'] & {} \arrow[u, phantom, "\Uparrow\eta"] & A \arrow[ur, "L"'] &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
& B \arrow[rr, "\mathrm{id}_B"] & & B \\
A \arrow[ur, "L"] \arrow[rr, "\mathrm{id}_A"'] & & A \arrow[ur, "L"'] \arrow[ul, Rightarrow, "\mathrm{id}_L"] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
& A \arrow[rr, "\mathrm{id}_A"] \arrow[dr, "L"] & {} \arrow[d, phantom, "\Downarrow\eta"] & A \\
B \arrow[ur, "R"] \arrow[rr, "\mathrm{id}_B"'] & {} \arrow[u, phantom, "\Downarrow\epsilon"] & B \arrow[ur, "R"'] &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
& A \arrow[rr, "\mathrm{id}_A"] \arrow[dr, Rightarrow, "\mathrm{id}_R"] & & A \\
B \arrow[ur, "R"] \arrow[rr, "\mathrm{id}_B"'] & & B \arrow[ur, "R"'] &
\end{tikzcd}
$$

of pasting diagrams, which are equivalent to the following conditions:

1. *The Left Triangle Identity.* The diagram

$$
\begin{tikzcd}
L \circ \mathrm{id}_A \arrow[r, Rightarrow, "\mathrm{id}_L \star \eta"] \arrow[ddrr, Rightarrow, "\rho^{\mathrm{Adj}}_L"'] & L \circ (R \circ L) \arrow[r, Rightarrow, "(\alpha^{\mathrm{Adj}}_{L,R,L})^{-1}"] & (L \circ R) \circ L \arrow[d, Rightarrow, "\epsilon \star \mathrm{id}_L"] \\
& & \mathrm{id}_B \circ L \arrow[d, Rightarrow, "\lambda^{\mathrm{Adj}}_L"] \\
& & L
\end{tikzcd}
$$

   commutes.

2. *The Right Triangle Identity.* The diagram

$$
\begin{tikzcd}
\mathrm{id}_A \circ R \arrow[r, Rightarrow, "\eta \star \mathrm{id}_R"] \arrow[ddrr, Rightarrow, "\lambda^{\mathrm{Adj}}_R"'] & (R \circ L) \circ R \arrow[r, Rightarrow, "\alpha^{\mathrm{Adj}}_{R,L,R}"] & R \circ (L \circ R) \arrow[d, Rightarrow, "\mathrm{id}_R \star \epsilon"] \\
& & R \circ \mathrm{id}_B \arrow[d, Rightarrow, "\rho^{\mathrm{Adj}}_R"] \\
& & R.
\end{tikzcd}
$$

[1]See [SS86] for an explicit description of the 2-category (as opposed to a bicategory) version of Adj in terms of finite ordinals, similar to the description of the 2-category version of the walking monad (??) as a subcategory of Δ.

19.1.2 Internal Adjunctions

Let C be a bicategory.

Definition 19.1.2.1 ▶ Internal Adjunctions

An **internal adjunction in** C[1,2] is a 2-functor $\mathsf{Adj} \to C$.

[1] *Further Terminology:* Also called an **adjunction internal to** C.

[2] *Further Terminology:* In this situation, we also call (g, f) an **adjoint pair**, f the **left adjoint** of the pair, g the **right adjoint** of the pair, η the **unit** of the adjunction, and ϵ the **counit** of the adjunction.

Remark 19.1.2.2 ▶ Unwinding Definition 19.1.2.1

In detail, an **internal adjunction in** C consists of

- *Objects.* A pair of objects A and B of C;
- *Morphisms.* A pair of morphisms

$$\begin{aligned} L\colon A &\to B, \\ R\colon B &\to A \end{aligned}$$

  of C;
- *2-Morphisms.* A pair of 2-morphisms

$$\begin{aligned} \eta\colon \mathrm{id}_A &\to R \circ L, \\ \epsilon\colon L \circ R &\to \mathrm{id}_B \end{aligned}$$

  of C;

subject to the equalities

$$\begin{tikzcd} & B \arrow[rr, "\mathrm{id}_B"] \arrow[dr, "R"] \arrow[d, phantom, "\Uparrow\eta"] & \arrow[d, phantom, "\Uparrow\epsilon"] & B \\ A \arrow[ur, "L"] \arrow[rr, "\mathrm{id}_A"'] & {} & A \arrow[ur, "L"'] & \end{tikzcd} = \begin{tikzcd} & B \arrow[rr, "\mathrm{id}_B"] & & B \\ A \arrow[ur, "L"] \arrow[rr, "\mathrm{id}_A"'] & & A \arrow[ur, "L"'] \arrow[ul, phantom, "\Leftarrow \mathrm{id}_L"] & \end{tikzcd}$$

$$\begin{tikzcd} & A \arrow[rr, "\mathrm{id}_A"] \arrow[dr, "L"] \arrow[d, phantom, "\Downarrow\epsilon"] & \arrow[d, phantom, "\Downarrow\eta"] & A \\ B \arrow[ur, "R"] \arrow[rr, "\mathrm{id}_B"'] & {} & B \arrow[ur, "R"'] & \end{tikzcd} = \begin{tikzcd} & A \arrow[rr, "\mathrm{id}_A"] \arrow[dr, phantom, "\Rightarrow \mathrm{id}_R"] & & A \\ B \arrow[ur, "R"] \arrow[rr, "\mathrm{id}_B"'] & & B \arrow[ur, "R"'] & \end{tikzcd}$$

of pasting diagrams in C, which are equivalent to the following conditions:[1]

1. *The Left Triangle Identity.* The diagram

$$\begin{tikzcd}
L \circ \mathrm{id}_A \arrow[r, Rightarrow, "\mathrm{id}_L \star \eta"] \arrow[ddrr, Rightarrow, "\rho^C_L"'] & L \circ (R \circ L) \arrow[r, Rightarrow, "(\alpha^C_{L,R,L})^{-1}"] & (L \circ R) \circ L \arrow[d, Rightarrow, "\epsilon \star \mathrm{id}_L"] \\
& & \mathrm{id}_B \circ L \arrow[d, Rightarrow, "\lambda^C_L"] \\
& & L
\end{tikzcd}$$

commutes.

2. *The Right Triangle Identity.* The diagram

$$\begin{tikzcd}
\mathrm{id}_A \circ R \arrow[r, Rightarrow, "\eta \star \mathrm{id}_R"] \arrow[ddrr, Rightarrow, "\lambda^C_R"'] & (R \circ L) \circ R \arrow[r, Rightarrow, "\alpha^C_{R,L,R}"] & R \circ (L \circ R) \arrow[d, Rightarrow, "\mathrm{id}_R \star \epsilon"] \\
& & R \circ \mathrm{id}_B \arrow[d, Rightarrow, "\rho^C_R"] \\
& & R.
\end{tikzcd}$$

---

[1]When C is a 2-category, these diagrams take the following form:

$$\begin{tikzcd}
L \arrow[r, Rightarrow, "\mathrm{id}_L \star \eta"] \arrow[dr, Rightarrow, "\mathrm{id}_L"'] & L \circ R \circ L \arrow[d, Rightarrow, "\epsilon \star \mathrm{id}_L"] \\
& L
\end{tikzcd}
\qquad
\begin{tikzcd}
R \arrow[r, Rightarrow, "\mathrm{id}_R \star \eta"] \arrow[dr, Rightarrow, "\mathrm{id}_L"'] & R \circ L \circ R \arrow[d, Rightarrow, "\epsilon \star \mathrm{id}_R"] \\
& R.
\end{tikzcd}$$

Example 19.1.2.3 ▶ Examples of Internal Adjunctions

Here are some examples of internal adjunctions.

1. *Internal Adjunctions in* Cats_2. The internal adjunctions in the 2-category Cats_2 of categories, functors, and natural transformations are precisely the adjunctions of **??**.

2. *Internal Adjunctions in* **Rel**. The internal adjunctions in **Rel** are precisely the relations of the form $\mathrm{Gr}(f) \dashv f^{-1}$ with f a function; see **??** of **??**.

3. *Internal Adjunctions in* Span. The internal adjunctions in Span are precisely the spans of the form

$$\begin{tikzcd} & S \arrow[dl, "\phi"'] \arrow[dr, "g"] & \\ A & & B \end{tikzcd}$$

with ϕ an isomorphism; see **??** of **??**.

PROPOSITION 19.1.2.4 ▶ PROPERTIES OF INTERNAL ADJUNCTIONS

Let C be a bicategory.

1. *Duality.* Let (f, g, η, ϵ) be an internal adjunction in C.
   (a) The quadruple (g, f, η, ϵ) is an internal adjunction in C^{op}.
   (b) The quadruple (g, f, ϵ, η) is an internal adjunction in C^{co}.
   (c) The quadruple (f, g, η, ϵ) is an internal adjunction in C^{coop}.

2. *Uniqueness of Adjoints.* Let (f, g, η, ϵ) and $(f, g', \eta', \epsilon')$ be internal adjunctions in C. We have a canonical isomorphism[1]

$$g \xRightarrow{(\lambda^C_g)^{-1}} \mathrm{id}_A \circ g \xRightarrow{\eta' \star \mathrm{id}_g} (g' \circ f) \circ g \xRightarrow{\alpha^C_{g',f,g}} g' \circ (f \circ g) \xRightarrow{\mathrm{id}_{g'} \star \epsilon} g' \circ \mathrm{id}_B \xRightarrow{(\rho^C_{g'})^{-1}} g'$$

with inverse

$$g' \xRightarrow{(\lambda^C_{g'})^{-1}} \mathrm{id}_B \circ g' \xRightarrow{\eta \star \mathrm{id}_{g'}} (g \circ f) \circ g' \xRightarrow{\alpha^C_{g',f,g}} g \circ (f \circ g') \xRightarrow{\mathrm{id}_g \star \epsilon'} g \circ \mathrm{id}_B \xRightarrow{(\lambda^C_g)^{-1}} g.$$

3. *Carrying Internal Adjunctions Through Pseudofunctors.* Let $F \colon C \longrightarrow \mathcal{D}$ be a pseudofunctor and (f, g, η, ϵ) be an internal adjunction in C. There is an induced internal adjunction[2]

$$(F(f), F(g), \bar{\eta}, \bar{\epsilon})$$

in $\mathcal{D}$, where:

   (a) The unit
$$\bar{\eta} \colon \mathrm{id}_{F(A)} \Longrightarrow F(g) \circ F(f)$$
   is the composition
$$\mathrm{id}_{F(A)} \xRightarrow{F_A} F(\mathrm{id}_A) \xRightarrow{F(\eta)} F(g \circ f) \xRightarrow{F^{-1}_{g,f}} F(g) \circ F(f).$$

(b) The counit

$$\bar{\epsilon}\colon F(f) \circ F(g) \Longrightarrow \mathrm{id}_{F(B)}$$

is the composition

$$F(f) \circ F(g) \xRightarrow{F_{f,g}} F(f \circ g) \xRightarrow{F(\epsilon)} F(\mathrm{id}_B) \xRightarrow{F_B} \mathrm{id}_{F(B)}.$$

[1] *Slogan:* Left adjoints are unique up to canonical isomorphism. Dually, so are right adjoints.

[2] *Warning:* Lax or oplax functors which are not pseudofunctors need not preserve internal adjunctions.

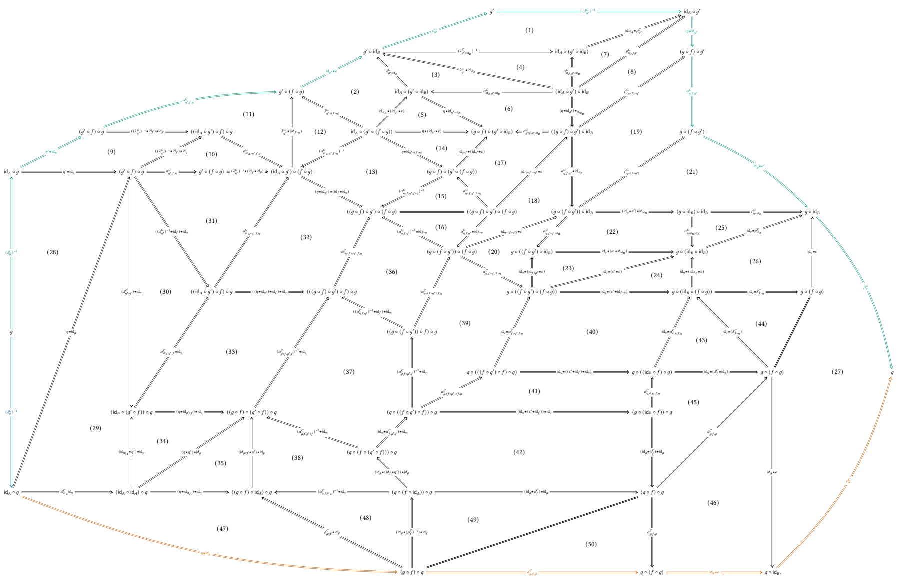
Proof 19.1.2.5 ▶ Proof of Proposition 19.1.2.4

Item 1: Duality

Omitted.[1]

Item 2: Uniqueness of Adjoints

[2]Consider the diagram (if you *really* want to consider it I fear you will need to zoom in)



In this diagram:

1. The morphisms in green are the composition $g \xRightarrow{\cong} g' \xRightarrow{\cong} g$;
2. The morphisms in red are equal to λ^{C}_{g} by the right triangle identity for

(f, g, η, ϵ). Hence the composition of the morphism in blue with the morphisms in red is the identity;

3. Subdiagrams (1), (2), (10), (11), (29), (31), and (43) commute by the naturality of the left unitor of C and its inverse;

4. Subdiagrams (8), (19), and (21) commute by the naturality of the right unitor of C and its inverse;

5. Subdiagrams (6), (13), (17), (18), (20), (22), (32), (33), (36), (38), (40), (41), and (45) commute by the naturality of the associator of C and its inverse;

6. Subdiagrams (37), (39), and (42) commute by the pentagon identity for C;

7. Subdiagrams (3), (4), (7), (12), (25), (30), and (48) commute by **??** of **??**;

8. Subdiagrams (5), (14), (23), (24), (34), and (35) commute by middle-four exchange;

9. Subdiagrams (9), (15), (16), (27), (28), (44), (46), (49), and (50) commute trivially;

10. Subdiagram (26) commutes by **????** of **??**;

11. Subdiagram (47) commutes by **??** of **??** and the naturality of the left unitor of right unitor of C.

Hence $g \cong g'$.

Item 3: Carrying Internal Adjunctions Through Pseudofunctors

[3]We claim that the left and right triangle identities for $(F(f), F(g), \overline{\eta}, \overline{\epsilon})$ hold:

1. The left triangle identity for $(F(f), F(g), \overline{\eta}, \overline{\epsilon})$ is the condition that the

boundary diagram of the diagram (you may need to zoom in)

$$\begin{tikzcd}
F(f)\circ \mathrm{id}_{F(A)} \arrow[rrr, Rightarrow, "\mathrm{id}_{F(f)}\star\overline{\eta}"] \arrow[dr, Rightarrow, "\mathrm{id}_{F(f)}\star F_A"] \arrow[dd, Rightarrow, "\rho^{\mathcal{D}}_{F(f)}"'] & & & F(f)\circ(F(g)\circ F(f)) \arrow[dr, Rightarrow, "(\alpha^{\mathcal{D}}_{F(f),F(g),F(f)})^{-1}"] & & \\
& F(f)\circ F(\mathrm{id}_A) \arrow[r, Rightarrow, "\mathrm{id}_{F(f)}\star F(\eta)"] \arrow[dd, Rightarrow, "F_{f,\mathrm{id}_A}"] \arrow[ur, phantom, "(1)"] \arrow[ddr, phantom, "(3)"] & F(f)\circ F(g\circ f) \arrow[ur, Rightarrow, "\mathrm{id}_{F(f)}\star F^{-1}_{g,f}"] \arrow[dd, Rightarrow, "F_{f,g\circ f}"] \arrow[ddrr, phantom, "(4)"] & & (F(f)\circ F(g))\circ F(f) \arrow[dd, Rightarrow, "F_{f,g}\star\mathrm{id}_{F(f)}"] \arrow[dddddr, Rightarrow, bend left, "\overline{\epsilon}\star\mathrm{id}_{F(f)}"] & \\
F(f) \arrow[dr, Rightarrow, "(F(\rho^{\mathcal{C}}_f))^{-1}"'] \arrow[ur, phantom, "(2)"] & & & & & \\
& F(f\circ\mathrm{id}_A) \arrow[r, Rightarrow, "F(\mathrm{id}_f\star\eta)"] \arrow[ddddrrr, Rightarrow, bend right, "F(\rho^{\mathcal{C}}_f)"'] & F(f\circ(g\circ f)) \arrow[dr, Rightarrow, "F((\alpha^{\mathcal{C}}_{f,g,f})^{-1})"] \arrow[ddd, phantom, "(5)"] & & F(f\circ g)\circ F(f) \arrow[dl, Rightarrow, "F_{f\circ g,f}"'] \arrow[dd, Rightarrow, "F(\epsilon)\star\mathrm{id}_{F(f)}"] \arrow[dr, phantom, "(7)"] & \\
& & & F((f\circ g)\circ f) \arrow[dd, Rightarrow, "F(\epsilon\star\mathrm{id}_f)"] \arrow[dr, phantom, "(6)"] & & \\
& & & & F(\mathrm{id}_B)\circ F(f) \arrow[dl, Rightarrow, "F_{\mathrm{id}_B,f}"'] \arrow[dr, Rightarrow, "F^{-1}_B\star\mathrm{id}_{F(f)}"] & \\
& & & F(\mathrm{id}_B\circ f) \arrow[dr, Rightarrow, "F(\lambda^{\mathcal{C}}_f)"] \arrow[rr, phantom, "(8)"] & & \mathrm{id}_{F(B)}\circ F(f) \arrow[dl, Rightarrow, "\lambda^{\mathcal{D}}_{F(f)}"] \\
& & & & F(f). &
\end{tikzcd}$$

commutes. Since

(a) Subdiagrams (1) and (7) commute by applying middle-four exchange twice,

(b) Subdiagrams (2) and (8) commute by the left and right lax unity conditions for F,

(c) Subdiagrams (3) and (6) commute by the naturality of the lax functoriality constraints of F,

(d) Subdiagram (4) commutes by the lax associativity condition for F, and

(e) Subdiagram (5) commutes by the left triangle identity for (f, g, η, ϵ),

so does the boundary diagram.

2. The right triangle identity for $(F(f), F(g), \overline{\eta}, \overline{\epsilon})$ is the condition that the

boundary diagram of the diagram (you may need to zoom in)

$$\begin{tikzcd}
\mathrm{id}_{F(A)} \circ F(g) \arrow[rrr, Rightarrow, "\bar{\eta}\star\mathrm{id}_{F(g)}"] \arrow[dd, Rightarrow, "\lambda^{\mathcal{D}}_{F(g)}"'] \arrow[dr, Rightarrow, "F_A\star\mathrm{id}_{F(g)}"] & {} \arrow[dr, phantom, "(1)"] & & (F(g) \circ F(f)) \circ F(g) \arrow[dr, Rightarrow, "\alpha^{\mathcal{D}}_{F(g),F(f),F(g)}"] & & \\
{} \arrow[dr, phantom, "(2)"] & F(\mathrm{id}_A) \circ F(g) \arrow[r, Rightarrow, "F(\eta)\star\mathrm{id}_{F(g)}"] \arrow[dd, Rightarrow, "F_{\mathrm{id}_A,g}"] & F(g \circ f) \circ F(g) \arrow[ur, Rightarrow, "F^{-1}_{g,f}\star\mathrm{id}_{F(g)}"] \arrow[dd, Rightarrow, "F_{g\circ f,g}"] & {} \arrow[dd, phantom, "(4)"] & F(g) \circ (F(f) \circ F(g)) \arrow[dd, Rightarrow, "\mathrm{id}_{F(g)}\star F_{f,g}"] \arrow[dddddr, Rightarrow, bend left, "\mathrm{id}_{F(g)}\star\bar{\epsilon}"] & \\
F(g) \arrow[dr, Rightarrow, "(F(\lambda^{\mathcal{C}}_g))^{-1}"'] & {} \arrow[r, phantom, "(3)"] & {} & & & \\
& F(\mathrm{id}_A \circ g) \arrow[r, Rightarrow, "F(\eta\star\mathrm{id}_g)"] \arrow[ddddrrr, Rightarrow, bend right, "F(\lambda^{\mathcal{C}}_g)"'] & F((g \circ f) \circ g) \arrow[dr, Rightarrow, "F(\alpha^{\mathcal{C}}_{g,f,g})"] & & F(g) \circ F(f \circ g) \arrow[dl, Rightarrow, "F_{g,f\circ g}"'] \arrow[dd, Rightarrow, "\mathrm{id}_{F(g)}\star F(\epsilon)"] \arrow[dr, phantom, "(7)"] & \\
& & {} \arrow[d, phantom, "(5)"] & F(g \circ (f \circ g)) \arrow[dd, Rightarrow, "F(\mathrm{id}_g\star\epsilon)"] \arrow[dr, phantom, "(6)"] & & {} \\
& & {} & & F(g) \circ F(\mathrm{id}_B) \arrow[dl, Rightarrow, "F_{g,\mathrm{id}_B}"'] \arrow[dr, Rightarrow, "\mathrm{id}_{F(g)}\star F_B^{-1}"] \arrow[dd, phantom, "(8)"] & \\
& & & F(g \circ \mathrm{id}_B) \arrow[dr, Rightarrow, "F(\rho^{\mathcal{C}}_g)"] & & F(g) \circ \mathrm{id}_{F(B)} \arrow[dl, Rightarrow, "\rho^{\mathcal{D}}_{F(g)}"] \\
& & & & F(g). &
\end{tikzcd}$$

commutes. Since

(a) Subdiagrams (1) and (7) commute by applying middle-four exchange twice,

(b) Subdiagrams (2) and (8) commute by the left and right lax unity conditions for F,

(c) Subdiagrams (3) and (6) commute by the naturality of the lax functoriality constraints of F,

(d) Subdiagram (4) commutes by the lax associativity condition for F, and

(e) Subdiagram (5) commutes by the right triangle identity for (f, g, η, ϵ),

so does the boundary diagram.

This finishes the proof.

[1] *Reference:* [JY21, Exercise 6.6.2].
[2] *Reference:* [JY21, Lemma 6.1.6].
[3] *Reference:* [JY21, Proposition 6.1.7].

19.1.3 Internal Adjoint Equivalences

Let C be a bicategory.

Definition 19.1.3.1 ► Internal Adjoint Equivalences

An internal adjunction (f, g, η, ϵ) in $\mathcal{C}$ is an **internal adjoint equivalence** if η and ϵ are isomorphisms in $\mathcal{C}$.

Example 19.1.3.2 ► Examples of Internal Adjoint Equivalences

Here are some examples of internal adjoint equivalences.

1. *Internal Adjoint Equivalences in* Cats_2. The internal adjoint equivalences in the 2-category Cats_2 of categories, functors, and natural transformations are precisely the adjoint equivalences of **??**.[1]

2. *Internal Adjoint Equivalences in* Mod. The internal adjoint equivalences in Mod are precisely the invertible R-modules; see **??**.[2]

3. *Internal Adjoint Equivalences in* $\mathsf{PseudoFun}(\mathcal{C}, \mathcal{D})$. The internal adjoint equivalences in $\mathsf{PseudoFun}(\mathcal{C}, \mathcal{D})$ are precisely the invertible strong transformations; see **??**.[3]

4. *Internal Adjoint Equivalences in* **Rel**. The internal adjoint equivalences in **Rel** are precisely the relations of the form $\mathrm{Gr}(f) \dashv f^{-1}$ with f an isomorphism; see **??**.

5. *Internal Adjoint Equivalences in* Span. The internal adjoint equivalences in Span are precisely the spans of the form $A \xleftarrow{\phi} S \xrightarrow{\psi} B$ with ϕ and ψ isomorphisms; see **??**.

---

[1] *Reference:* [JY21, Examples 6.2.5].
[2] *Reference:* [JY21, Examples 6.2.6].
[3] *Reference:* [JY21, Examples 6.2.7].

Proposition 19.1.3.3 ► Properties of Internal Adjoint Equivalences

Let $\mathcal{C}$ be a bicategory.

1. *Carrying Internal Adjoint Equivalences Through Pseudofunctors.* Let $F: \mathcal{C} \longrightarrow \mathcal{D}$ be a pseudofunctor and (f, g, η, ϵ) be an internal adjunction in $\mathcal{C}$. If (f, g, η, ϵ) is an internal adjoint equivalence in $\mathcal{C}$, then the induced internal adjunction
$$(F(f), F(g), \bar{\eta}, \bar{\epsilon})$$
in $\mathcal{D}$ of Item 3 of Proposition 19.1.2.4 is an internal adjoint equivalence as well.

2. *Internal Adjunctions Always Refine to Internal Adjoint Equivalences.* Let (f, g, η, ϵ) be an internal adjunction in $\mathcal{C}$. If f is an equivalence, then there

exist 2-morphisms

$$\bar{\eta}\colon \mathrm{id}_A \Longrightarrow g \circ f$$
$$\bar{\epsilon}\colon f \circ g \Longrightarrow \mathrm{id}_B$$

of C such that $(f, g, \bar{\eta}, \bar{\epsilon})$ is an internal adjoint equivalence.

PROOF 19.1.3.4 ► PROOF OF PROPOSITION 19.1.3.3

Item 1: Carrying Internal Adjoint Equivalences Through Pseudofunctors

See [JY21, Proposition 6.2.3].

Item 2: Internal Adjunctions Always Refine to Internal Adjoint Equivalences

See [JY21, Proposition 6.2.4].

19.1.4 Mates

Let C be a bicategory, let (f, g, η, ϵ) and $(f', g', \eta', \epsilon')$ be adjunctions, and let h and k be morphisms of C as in the diagram

$$\begin{tikzcd}
A \arrow[r, bend left, "f"] \arrow[r, phantom, "\perp"] \arrow[d, "h"'] & B \arrow[l, bend left, "g"] \arrow[d, "k"] \\
C \arrow[r, bend left, "f'"] \arrow[r, phantom, "\perp"] & D. \arrow[l, bend left, "g'"]
\end{tikzcd}$$

DEFINITION 19.1.4.1 ► MATES

The **mates** of a pair of 2-morphisms

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "h"'] & B \arrow[d, "k"] \\
C \arrow[r, "f'"'] \arrow[ur, phantom, "\stackrel{\omega}{\Nearrow}"] & D
\end{tikzcd}
\qquad
\begin{aligned}
\omega\colon f' \circ h &\Longrightarrow k \circ f, \\
\nu\colon h \circ g &\Longrightarrow g' \circ k
\end{aligned}
\qquad
\begin{tikzcd}
A \arrow[d, "h"'] \arrow[dr, phantom, "\stackrel{\nu}{\Searrow}"] & B \arrow[l, "g"'] \arrow[d, "k"] \\
C & D \arrow[l, "g'"]
\end{tikzcd}$$

are the 2-morphisms

$$
\begin{tikzcd}
A \arrow[d, "h"'] & B \arrow[l, "g"'] \arrow[d, "k"] \\
C & D \arrow[l, "g'"] \arrow[ul, phantom, "\Searrow\omega^{\dagger}"]
\end{tikzcd}
\qquad
\begin{aligned}
\omega^{\dagger}&\colon h\circ g \Longrightarrow g'\circ k,\\
\nu^{\dagger}&\colon f'\circ h \Longrightarrow k\circ f
\end{aligned}
\qquad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "h"'] & B \arrow[d, "k"] \\
C \arrow[r, "f'"'] \arrow[ur, phantom, "\Nearrow\nu^{\dagger}"] & D
\end{tikzcd}
$$

defined as the pastings of the diagrams[1]

$$
\begin{tikzcd}
B \arrow[rrr, bend left, "k"] \arrow[dr, "g"] \arrow[drr, bend left, "\mathrm{id}_B"] \arrow[ddd, bend right, "g"'] & & & D \arrow[ddd, bend left, "g'"] \\
& A \arrow[r, "f"] \arrow[d, "h"] \arrow[u, phantom, "\Uparrow\epsilon"] & B \arrow[d, "k"] \arrow[ur, phantom, "\Nearrow\rho^{C}_{h\circ g}"] & \\
& C \arrow[r, "f'"'] \arrow[drr, bend right, "\mathrm{id}_C"'] \arrow[dl, phantom, "\Nearrow(\lambda^{C}_{h\circ g})^{-1}"] \arrow[ur, phantom, "\Nearrow\omega"] & D \arrow[dr, "g'"] \arrow[d, phantom, "\Uparrow\eta'"] & \\
A \arrow[rrr, bend right, "h"'] & & & C.
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[rrr, bend left, "f"] \arrow[dr, "f"] \arrow[ddr, bend right, "\mathrm{id}_A"'] \arrow[ddd, bend right, "h"'] & & & B \arrow[ddd, bend left, "k"] \\
& B \arrow[r, "k"] \arrow[d, "g"] \arrow[l, phantom, "\Rightarrow\eta"] \arrow[urr, phantom, "\Nearrow\lambda^{C}_{h\circ g}"] & D \arrow[d, "g'"] \arrow[ddr, bend left, "\mathrm{id}_D"] & \\
& A \arrow[r, "h"'] \arrow[ur, phantom, "\Nearrow\nu"] \arrow[dl, phantom, "\Nearrow(\rho^{C}_{h\circ g})^{-1}"] & C \arrow[dr, "f'"] \arrow[r, phantom, "\Rightarrow\epsilon'"] & \\
C \arrow[rrr, bend right, "f'"'] & & & D.
\end{tikzcd}
$$

[1]If $\mathcal{C}$ is a 2-category, these pasting diagrams become the following:

$$
\begin{tikzcd}
B \arrow[r, "g"] \arrow[d, "k"'] & A \arrow[d, "h"] \\
D \arrow[r, "g'"'] \arrow[ur, phantom, "\Swarrow\omega^{\dagger}"] & C
\end{tikzcd}
=
\begin{tikzcd}
B \arrow[r, "g"] \arrow[dr, "\mathrm{id}_B"'] & A \arrow[r, "h"] \arrow[d, "f"] \arrow[dl, phantom, "\Swarrow\epsilon"] & C \arrow[d, "f'"] \arrow[dr, "\mathrm{id}_C"] \arrow[dl, phantom, "\Swarrow\omega"] & \\
& B \arrow[r, "k"'] & D \arrow[r, "g'"'] \arrow[ur, phantom, "\Swarrow\eta'"] & C.
\end{tikzcd}
$$

$$
\begin{tikzcd}
C \arrow[r, "f'"] & D \\
A \arrow[r, "f"'] \arrow[u, "h"] & B \arrow[u, "k"'] \arrow[ul, phantom, "\Searrow\nu^{\dagger}"]
\end{tikzcd}
=
\begin{tikzcd}
& A \arrow[r, "h"] & C \arrow[r, "f'"] \arrow[d, <->, "g'"] \arrow[dr, phantom, "\Searrow\epsilon'"] & D \\
A \arrow[r, "f"'] \arrow[ur, "\mathrm{id}_A"] & B \arrow[r, "k"'] \arrow[u, "g"'] \arrow[ul, phantom, "\Searrow\eta"] \arrow[ur, phantom, "\Searrow\nu"] & D. \arrow[ur, "\mathrm{id}_D"'] &
\end{tikzcd}
$$

Proposition 19.1.4.2 ▶ Properties of Mates

Let $\omega\colon f'\circ h \Longrightarrow k\circ f$ and $\nu\colon h\circ g \Longrightarrow g'\circ k$ be 2-morphisms.

1. *The Mate Correspondence.* The map

$$\begin{aligned}(-)^\dagger \colon \mathrm{Hom}_{\mathsf{Hom}_C(A,C)}(f' \circ h, k \circ f) &\longrightarrow \mathrm{Hom}_{\mathsf{Hom}_C(B,D)}(h \circ g, g' \circ k)\\ \omega &\longmapsto \omega^\dagger\end{aligned}$$

is a bijection.

PROOF 19.1.4.3 ► PROOF OF PROPOSITION 19.1.4.2

Item 1: The Mate Correspondence

Here we give a proof for 2-categories (which indirectly proves also the general case by **??**). A proof for general bicategories can be found in [JY21, Lemma 6.1.13].

Let

$$\nu \colon h \circ g \Longrightarrow g' \circ k \qquad \begin{tikzcd} A \arrow[d, "h"'] \arrow[dr, phantom, "\Searrow\nu"] & B \arrow[l, "g"'] \arrow[d, "k"] \\ C & D \arrow[l, "g'"] \end{tikzcd}$$

be a 2-morphism of C. The mate $\nu^\dagger$ of ν is then given by

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h"'] \arrow[dr, phantom, "\Nearrow\nu^\dagger"] & B \arrow[d, "k"] \\ C \arrow[r, "f'"'] & D \end{tikzcd} \quad = \quad \begin{tikzcd} & A \arrow[dl, "\mathrm{id}_A"'] \arrow[d, "f"] \arrow[dl, phantom, shift left=3, "\Searrow\eta"] \\ A \arrow[d, "h"'] \arrow[dr, phantom, "\Searrow\nu"] & B \arrow[l, "g"'] \arrow[d, "k"] \\ C \arrow[d, "f'"'] & D \arrow[l, "g'"'] \arrow[dl, "\mathrm{id}_D"] \arrow[dl, phantom, shift right=3, "\Searrow\epsilon'"] \\ D, & \end{tikzcd}$$

and the mate of $\nu^\dagger$ is the 2-morphism $\left(\nu^\dagger\right)^\dagger \colon f' \circ h \Longrightarrow k \circ f$ given by

$$
\begin{tikzcd}
A \arrow[d, "h"'] \arrow[dr, phantom, "\Searrow (\nu^\dagger)^\dagger"] & B \arrow[l, "g"'] \arrow[d, "k"] \\
C & D. \arrow[l, "g'"]
\end{tikzcd}
=
\begin{tikzcd}
 & & A \arrow[dl, "\mathrm{id}_A"'] \arrow[d, "f"] \arrow[dl, phantom, bend left, "\Searrow\eta"] & B \arrow[l, "g"'] \arrow[dl, "\mathrm{id}_B"] \arrow[dl, phantom, bend right, "\Searrow\epsilon"] \\
 & A \arrow[d, "h"'] \arrow[dr, phantom, "\Searrow\nu"] & B \arrow[l, "g"] \arrow[d, "k"] & \\
 & C \arrow[dl, "\mathrm{id}_C"'] \arrow[d, "f'"] \arrow[dl, phantom, bend left, "\Searrow\eta'"] & D \arrow[l, "g'"] \arrow[dl, "\mathrm{id}_D"] \arrow[dl, phantom, bend right, "\Searrow\epsilon'"] & \\
C & D \arrow[l, "g'"] & &
\end{tikzcd}
=
\begin{tikzcd}
 & & A \arrow[dl, "\mathrm{id}_A"'] \arrow[d, phantom, "\Downarrow \mathrm{id}_g"] & B \arrow[l, "g"'] \arrow[dl, "\mathrm{id}_B"] \\
 & A \arrow[d, "h"'] \arrow[dr, phantom, "\Searrow\nu"] & B \arrow[l, "g"] \arrow[d, "k"] & \\
 & C \arrow[dl, "\mathrm{id}_C"'] \arrow[d, phantom, "\Downarrow \mathrm{id}_{g'}"] & D \arrow[l, "g'"] \arrow[dl, "\mathrm{id}_D"] & \\
C & D \arrow[l, "g'"] & &
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[d, "h"'] \arrow[dr, phantom, "\Searrow \nu"] & B \arrow[l, "g"'] \arrow[d, "k"] \\
C & D. \arrow[l, "g'"]
\end{tikzcd}
$$

Similarly, $(\omega)^{\dagger\dagger} = \omega$.

19.2 Morphisms of Internal Adjunctions

19.2.1 Lax Morphisms of Internal Adjunctions

Let $\mathcal{C}$ be a bicategory and let $(A, B, F, G, \eta, \epsilon)$ and $(A', B', F', G', \eta', \epsilon')$ be internal adjunctions in $\mathcal{C}$.

Definition 19.2.1.1 ► Lax Morphisms of Internal Adjunctions

A **lax morphism of internal adjunctions** from $(A, B, F, G, \eta, \epsilon)$ to $(A', B', F', G', \eta', \epsilon')$ is a lax transformation between these viewed as 2-functors from the walking adjunction.

Remark 19.2.1.2 ► Unwinding Definition 19.2.1.1

In detail, a **lax morphism of internal adjunctions** from $(A, B, F, G, \eta, \epsilon)$ to $(A', B', F', G', \eta', \epsilon')$ consists of

- 1-*Morphisms*. A pair of 1-morphisms

$$
\begin{aligned}
\phi &\colon A \to A', \\
\psi &\colon B \to B'
\end{aligned}
$$

of $\mathcal{C}$;

- *2-Morphisms*. A pair of 2-morphisms

$$
\begin{tikzcd}
A \arrow[r, "F"] \arrow[d, "\phi"'] & B \arrow[d, "\psi"] \\
A' \arrow[r, "F'"'] \arrow[ur, phantom, "\Nearrow\alpha"] & B'
\end{tikzcd}
\qquad
\begin{aligned}
\alpha &\colon F' \circ \phi \Rightarrow \psi \circ F,\\
\beta &\colon G' \circ \phi \Rightarrow \psi \circ G
\end{aligned}
\qquad
\begin{tikzcd}
A \arrow[d, "\phi"'] & B \arrow[l, "G"'] \arrow[d, "\psi"] \arrow[dl, phantom, "\Nwarrow\beta"] \\
A' & B' \arrow[l, "G'"]
\end{tikzcd};
$$

  of C;

satisfying the following conditions:

1. *Compatibility With Units*. We have an equality

$$
\begin{tikzcd}
 & B \arrow[dr, bend left, "G"] \arrow[d, phantom, "\Uparrow\eta"] & \\
A \arrow[ur, bend left, "F"] \arrow[rr, "\mathrm{id}_A"] \arrow[d, "\phi"'] \arrow[drr, "\phi"] & {} & A \arrow[d, "\phi"] \\
A' \arrow[rr, bend right, "\mathrm{id}_{A'}"'] \arrow[ur, phantom, "\lambda^C_\phi"] & {} \arrow[ur, phantom, "\rho^{C,-1}_\phi"] & A'
\end{tikzcd}
=
\begin{tikzcd}
 & B \arrow[dr, bend left, "G"] \arrow[dd, "\psi"] & \\
A \arrow[ur, bend left, "F"] \arrow[dd, "\phi"'] \arrow[r, phantom, "\Nearrow\alpha"] & {} & A \arrow[dd, "\phi"] \arrow[l, phantom, "\Nearrow\beta"] \\
 & B' \arrow[dr, "G'"] & \\
A' \arrow[ur, "F'"] \arrow[rr, bend right, "\mathrm{id}_{A'}"'] & {} \arrow[u, phantom, "\Uparrow\eta'"] & A'
\end{tikzcd}
$$

   of pasting diagrams in C;

2. *Compatibility With Counits*. We have an equality

$$
\begin{tikzcd}
B \arrow[rr, bend left, "\mathrm{id}_B"] \arrow[dr, "G"] \arrow[dd, "\psi"'] & {} & B \arrow[dd, "\psi"] \\
 & A \arrow[ur, "F"] \arrow[dd, "\phi"] \arrow[u, phantom, "\Uparrow\epsilon"] & \\
B' \arrow[dr, "G'"'] \arrow[ur, phantom, "\Nearrow\beta"] & & B' \arrow[ul, phantom, "\Nearrow\alpha"] \\
 & A' \arrow[ur, "F'"'] &
\end{tikzcd}
=
\begin{tikzcd}
B \arrow[rr, bend left, "\mathrm{id}_B"] \arrow[d, "\psi"'] \arrow[drr, "\psi"] & {} & B \arrow[d, "\psi"] \\
B' \arrow[rr, "\mathrm{id}_{B'}"] \arrow[dr, "G'"'] \arrow[ur, phantom, "\lambda^C_\psi"] & {} \arrow[ur, phantom, "\rho^{C,-1}_\psi"] & B' \\
 & A' \arrow[ur, "F'"'] \arrow[u, phantom, "\Uparrow\epsilon'"] &
\end{tikzcd}
$$

   of pasting diagrams in C.

19.2.2 Oplax Morphisms of Internal Adjunctions

Let C be a bicategory and let $(A, B, F, G, \eta, \epsilon)$ and $(A', B', F', G', \eta', \epsilon')$ be internal adjunctions in C.

Definition 19.2.2.1 ▶ Oplax Morphisms of Internal Adjunctions

An **oplax morphism of internal adjunctions** from $(A, B, F, G, \eta, \epsilon)$ to $(A', B', F', G', \eta', \epsilon')$ is an oplax transformation between these viewed as 2-functors from the walking adjunction.

Remark 19.2.2.2 ▶ Unwinding Definition 19.2.2.1

In detail, an **oplax morphism of internal adjunctions** from $(A, B, F, G, \eta, \epsilon)$ to $(A', B', F', G', \eta', \epsilon')$ consists of

- 1-*Morphisms.* A pair of 1-morphisms

$$\begin{aligned}\phi &\colon A \to A',\\ \psi &\colon B \to B'\end{aligned}$$

  of C;

- 2-*Morphisms.* A pair of 2-morphisms

$$\begin{tikzcd} A \arrow[r, "F"] \arrow[d, "\phi"'] & B \arrow[d, "\psi"] \arrow[dl, phantom, "\swArrow\alpha"] \\ A' \arrow[r, "F'"'] & B' \end{tikzcd} \qquad \begin{aligned}\alpha &\colon \psi \circ F \Rightarrow F' \circ \phi,\\ \beta &\colon \psi \circ G \Rightarrow G' \circ \phi\end{aligned} \qquad \begin{tikzcd} A \arrow[d, "\phi"'] \arrow[dr, phantom, "\seArrow\beta"] & B \arrow[l, "G"'] \arrow[d, "\psi"] \\ A' & B' \arrow[l, "G'"] \end{tikzcd};$$

  of C;

satisfying the following conditions:

1. *Compatibility With Units.* We have an equality

$$\begin{tikzcd} & A \arrow[dr, bend left, "F"] \arrow[d, phantom, "\Downarrow\epsilon"] & \\ B \arrow[ur, bend left, "G"] \arrow[rr, "\mathrm{id}_B"] \arrow[d, "\psi"'] \arrow[drr, "\psi"] & {} & B \arrow[d, "\psi"] \arrow[dll] \arrow[dl, phantom, "\swArrow\rho^{C}_{\phi}"] \\ B' \arrow[rr, bend right, "\mathrm{id}_{B'}"'] \arrow[r, phantom, "\swArrow\lambda^{C,-1}_{\phi}"] & {} & B' \end{tikzcd} = \begin{tikzcd} & A \arrow[dr, bend left, "F"] \arrow[d, "\phi"] \arrow[dl, phantom, "\swArrow\beta"] & \\ B \arrow[ur, bend left, "G"] \arrow[d, "\psi"'] & A' \arrow[dr, "F'"] \arrow[d, phantom, "\Downarrow\epsilon'"] & B \arrow[d, "\psi"] \arrow[l, phantom, "\swArrow\alpha"] \\ B' \arrow[ur, "G'"] \arrow[rr, bend right, "\mathrm{id}_{B'}"'] & {} & B' \end{tikzcd}$$

   of pasting diagrams in C;

2. *Compatibility With Counits.* We have an equality

$$
\begin{tikzcd}
A \arrow[rr, bend left, "\mathrm{id}_A"] \arrow[dr, "F"'] \arrow[dd, "\phi"'] \arrow[rr, phantom, "\Downarrow\eta"] & & A \arrow[dd, "\phi"] \\
& B \arrow[ur, "G"'] \arrow[dd, "\psi"] \arrow[dl, phantom, "\swarrow\alpha"] \arrow[dr, phantom, "\swarrow\beta"] & \\
A' \arrow[dr, bend right, "F'"'] & & A' \\
& B' \arrow[ur, bend right, "G'"'] &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A \arrow[rr, bend left, "\mathrm{id}_A"] \arrow[ddrr, "\phi"] \arrow[dd, "\phi"'] & & A \arrow[dd, "\phi"] \arrow[dl, phantom, "\swarrow\rho^{C}_{\psi}"] \\
& \arrow[dl, phantom, "\swarrow\lambda^{C,-1}_{\psi}"] & \\
A' \arrow[rr, "\mathrm{id}_{A'}"] \arrow[dr, bend right, "F'"'] \arrow[rr, phantom, bend right=20, "\Downarrow\eta'"] & & A' \\
& B' \arrow[ur, bend right, "G'"'] &
\end{tikzcd}
$$

of pasting diagrams in C.

19.2.3 Strong Morphisms of Internal Adjunctions

Let C be a bicategory and let $(A, B, F, G, \eta, \epsilon)$ and $(A', B', F', G', \eta', \epsilon')$ be internal adjunctions in C.

Definition 19.2.3.1 ► Strong Morphisms of Internal Adjunctions

A **strong morphism of internal adjunctions** from $(A, B, F, G, \eta, \epsilon)$ to $(A', B', F', G', \eta', \epsilon')$ is a strong transformation between these viewed as 2-functors from the walking adjunction.

Remark 19.2.3.2 ► Unwinding Definition 19.2.3.1

In detail, a **strong morphism of internal adjunctions** from $(A, B, F, G, \eta, \epsilon)$ to $(A', B', F', G', \eta', \epsilon')$ is equivalently:

1. A lax morphism of internal adjunctions as in Remark 19.2.1.2 whose 2-morphisms are invertible.
2. An oplax morphism of internal adjunctions as in Remark 19.2.2.2 whose 2-morphisms are invertible.

19.2.4 Strict Morphisms of Internal Adjunctions

Let C be a bicategory and let $(A, B, F, G, \eta, \epsilon)$ and $(A', B', F', G', \eta', \epsilon')$ be internal adjunctions in C.

Definition 19.2.4.1 ▶ Strict Morphisms of Internal Adjunctions

A **strict morphism of internal adjunctions** from $(A, B, F, G, \eta, \epsilon)$ to $(A', B', F', G', \eta', \epsilon')$ is a strict transformation between these viewed as 2-functors from the walking adjunction.

Remark 19.2.4.2 ▶ Unwinding Definition 19.2.4.1

In detail, a **strict morphism of internal adjunctions** from $(A, B, F, G, \eta, \epsilon)$ to $(A', B', F', G', \eta', \epsilon')$ is equivalently:

1. A lax morphism of internal adjunctions as in Remark 19.2.1.2 whose 2-morphisms are identities.

2. An oplax morphism of internal adjunctions as in Remark 19.2.2.2 whose 2-morphisms are identities.

19.3 2-Morphisms Between Morphisms of Internal Adjunctions

19.3.1 2-Morphisms Between Lax Morphisms of Internal Adjunctions

Let C be a bicategory, let $(A, B, F, G, \eta, \epsilon)$ and $(A', B', F', G', \eta', \epsilon')$ be internal adjunctions in C, and let $(\phi_1, \psi_1, \alpha_1, \beta_1)$ and $(\phi_2, \psi_2, \alpha_2, \beta_2)$ be lax morphisms of internal adjunctions from $(A, B, F, G, \eta, \epsilon)$ to $(A', B', F', G', \eta', \epsilon')$.

Definition 19.3.1.1 ▶ 2-Morphisms Between Lax Morphisms of Internal Adjunctions

A **2-morphism from** $(\phi_1, \psi_1, \alpha_1, \beta_1)$ **to** $(\phi_2, \psi_2, \alpha_2, \beta_2)$ is a modification between these viewed as lax transformations.

Remark 19.3.1.2 ▶ Unwinding Definition 19.3.1.1

In detail, a **2-morphism from** $(\phi_1, \psi_1, \alpha_1, \beta_1)$ **to** $(\phi_2, \psi_2, \alpha_2, \beta_2)$ consist of 2-morphisms

$$\Gamma\colon \phi_1 \Rightarrow \phi_2$$
$$\Sigma\colon \psi_1 \Rightarrow \psi_2$$

of C such that we have equalities

$$\begin{tikzcd}
A \arrow[r, "F"] \arrow[d, bend right=50, "\phi_1"'] \arrow[d, bend left=50, "\phi_2"] \arrow[d, phantom, "\overset{\Gamma}{\Rightarrow}"] \arrow[dr, phantom, "\alpha_2 \Nearrow"] & B \arrow[d, bend left=30, "\psi_2"] \\
A' \arrow[r, "F'"'] & B'
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, "F"] \arrow[d, bend right=30, "\phi_1"'] \arrow[dr, phantom, "\alpha_1 \Nearrow"] & B \arrow[d, bend right=50, "\psi_1"'] \arrow[d, bend left=50, "\psi_2"] \arrow[d, phantom, "\overset{\Sigma}{\Rightarrow}"] \\
A' \arrow[r, "F'"'] & B'
\end{tikzcd}$$

$$\begin{tikzcd}
B \arrow[r, "G"] \arrow[d, bend right=50, "\psi_1"'] \arrow[d, bend left=50, "\psi_2"] \arrow[d, phantom, "\overset{\Sigma}{\Rightarrow}"] \arrow[dr, phantom, "\beta_2 \Nearrow"] & A \arrow[d, bend left=30, "\phi_2"] \\
B' \arrow[r, "G'"'] & A'
\end{tikzcd}
=
\begin{tikzcd}
B \arrow[r, "G"] \arrow[d, bend right=30, "\psi_1"'] \arrow[dr, phantom, "\beta_1 \Nearrow"] & A \arrow[d, bend right=50, "\phi_1"'] \arrow[d, bend left=50, "\phi_2"] \arrow[d, phantom, "\overset{\Gamma}{\Rightarrow}"] \\
B' \arrow[r, "G'"'] & A'
\end{tikzcd}$$

of pasting diagrams in C.

19.3.2 2-Morphisms Between Oplax Morphisms of Internal Adjunctions

Let C be a bicategory, let $(A, B, F, G, \eta, \epsilon)$ and $(A', B', F', G', \eta', \epsilon')$ be internal adjunctions in C, and let $(\phi_1, \psi_1, \alpha_1, \beta_1)$ and $(\phi_2, \psi_2, \alpha_2, \beta_2)$ be oplax morphisms of internal adjunctions from $(A, B, F, G, \eta, \epsilon)$ to $(A', B', F', G', \eta', \epsilon')$.

Definition 19.3.2.1 ► 2-Morphisms Between Oplax Morphisms of Internal Adjunctions

A **2-morphism from** $(\phi_1, \psi_1, \alpha_1, \beta_1)$ **to** $(\phi_2, \psi_2, \alpha_2, \beta_2)$ is a modification between these viewed as oplax transformations.

Remark 19.3.2.2 ► Unwinding Definition 19.3.2.1

In detail, a **2-morphism from** $(\phi_1, \psi_1, \alpha_1, \beta_1)$ **to** $(\phi_2, \psi_2, \alpha_2, \beta_2)$ consist of 2-morphisms

$$\Gamma \colon \phi_1 \Rightarrow \phi_2$$
$$\Sigma \colon \psi_1 \Rightarrow \psi_2$$

of C such that we have equalities

$$
\begin{tikzcd}
A \arrow[r, "F"] \arrow[d, bend right=50, "\phi_2"'] \arrow[d, bend left=50, "\phi_1"] \arrow[d, phantom, "\overset{\Gamma}{\Leftarrow}"] & B \arrow[d, "\psi_1"] \arrow[dl, phantom, "\alpha_1"] \\
A' \arrow[r, "F'"'] & B'
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, "F"] \arrow[d, "\phi_2"'] & B \arrow[d, bend right=50, "\psi_2"'] \arrow[d, bend left=50, "\psi_1"] \arrow[d, phantom, "\overset{\Sigma}{\Leftarrow}"] \arrow[dl, phantom, "\alpha_2"] \\
A' \arrow[r, "F'"'] & B'
\end{tikzcd}
$$

$$
\begin{tikzcd}
B \arrow[r, "G"] \arrow[d, bend right=50, "\psi_2"'] \arrow[d, bend left=50, "\psi_1"] \arrow[d, phantom, "\overset{\Sigma}{\Leftarrow}"] & A \arrow[d, "\phi_1"] \arrow[dl, phantom, "\beta_1"] \\
B' \arrow[r, "G'"'] & A'
\end{tikzcd}
=
\begin{tikzcd}
B \arrow[r, "G"] \arrow[d, "\psi_2"'] & A \arrow[d, bend right=50, "\phi_2"'] \arrow[d, bend left=50, "\phi_1"] \arrow[d, phantom, "\overset{\Gamma}{\Leftarrow}"] \arrow[dl, phantom, "\beta_2"] \\
B' \arrow[r, "G'"'] & A'
\end{tikzcd}
$$

of pasting diagrams in C.

19.3.3 2-Morphisms Between Strong Morphisms of Internal Adjunctions

Let C be a bicategory, let $(A, B, F, G, \eta, \epsilon)$ and $(A', B', F', G', \eta', \epsilon')$ be internal adjunctions in C, and let $(\phi_1, \psi_1, \alpha_1, \beta_1)$ and $(\phi_2, \psi_2, \alpha_2, \beta_2)$ be strong morphisms of internal adjunctions from $(A, B, F, G, \eta, \epsilon)$ to $(A', B', F', G', \eta', \epsilon')$.

Definition 19.3.3.1 ▶ 2-Morphisms Between Strong Morphisms of Internal Adjunctions

A **2-morphism from** $(\phi_1, \psi_1, \alpha_1, \beta_1)$ **to** $(\phi_2, \psi_2, \alpha_2, \beta_2)$ is a modification between these viewed as strong transformations.

Remark 19.3.3.2 ▶ Unwinding Definition 19.3.3.1

In detail, a **2-morphism from** $(\phi_1, \psi_1, \alpha_1, \beta_1)$ **to** $(\phi_2, \psi_2, \alpha_2, \beta_2)$ is equivalently:

- A 2-morphism (Γ, Σ) from $(\phi_1, \psi_1, \alpha_1, \beta_1)$ to $(\phi_2, \psi_2, \alpha_2, \beta_2)$ viewed as lax transformations as in Remark 19.3.1.2.
- A 2-morphism (Γ, Σ) from $(\phi_1, \psi_1, \alpha_1, \beta_1)$ to $(\phi_2, \psi_2, \alpha_2, \beta_2)$ viewed as oplax transformations as in Remark 19.3.2.2.

19.3.4 2-Morphisms Between Strict Morphisms of Internal Adjunctions

Let C be a bicategory, let $(A, B, F, G, \eta, \epsilon)$ and $(A', B', F', G', \eta', \epsilon')$ be internal adjunctions in C, and let $(\phi_1, \psi_1, \alpha_1, \beta_1)$ and $(\phi_2, \psi_2, \alpha_2, \beta_2)$ be lax morphisms of internal adjunctions from $(A, B, F, G, \eta, \epsilon)$ to $(A', B', F', G', \eta', \epsilon')$.

Definition 19.3.4.1 ▶ 2-Morphisms Between Strict Morphisms of Internal Adjunctions

A **2-morphism from** $(\phi_1, \psi_1, \alpha_1, \beta_1)$ **to** $(\phi_2, \psi_2, \alpha_2, \beta_2)$ is a modification between these viewed as strict transformations.

Remark 19.3.4.2 ▶ Unwinding Definition 19.3.4.1

In detail, a **2-morphism from** $(\phi_1, \psi_1, \alpha_1, \beta_1)$ **to** $(\phi_2, \psi_2, \alpha_2, \beta_2)$ is equivalently:

- A 2-morphism (Γ, Σ) from $(\phi_1, \psi_1, \alpha_1, \beta_1)$ to $(\phi_2, \psi_2, \alpha_2, \beta_2)$ viewed as lax transformations as in Remark 19.3.1.2.
- A 2-morphism (Γ, Σ) from $(\phi_1, \psi_1, \alpha_1, \beta_1)$ to $(\phi_2, \psi_2, \alpha_2, \beta_2)$ viewed as oplax transformations as in Remark 19.3.2.2.
- A 2-morphism (Γ, Σ) from $(\phi_1, \psi_1, \alpha_1, \beta_1)$ to $(\phi_2, \psi_2, \alpha_2, \beta_2)$ viewed as strong transformations as in Remark 19.3.3.2.

19.4 Bicategories of Internal Adjunctions in a Bicategory

Appendices

19.A Other Chapters

Part V

Internal Category Theory

Chapter 20

Internal Categories

Contents

Appendices

20.A Other Chapters

Part VI

Cyclic Stuff

Chapter 21

The Cycle Category

Contents

Appendices

21.A Other Chapters

Sets

1. Sets
2. Constructions With Sets
3. Pointed Sets
4. Tensor Products of Pointed Sets
5. Relations
6. Spans
7. Posets

Indexed and Fibred Sets

7. Indexed Sets
8. Fibred Sets
9. Un/Straightening for Indexed and Fibred Sets

Category Theory

11. Categories
12. Types of Morphisms in Categories
13. Adjunctions and the Yoneda Lemma
14. Constructions With Categories
15. Profunctors
16. Cartesian Closed Categories
17. Kan Extensions

Bicategories

18. Bicategories
19. Internal Adjunctions

Internal Category Theory

20. Internal Categories

Part VII

Cubical Stuff

Chapter 22

The Cube Category

Contents

Appendices

22.A Other Chapters

Sets

1. Sets
2. Constructions With Sets
3. Pointed Sets
4. Tensor Products of Pointed Sets
5. Relations
6. Spans
7. Posets

Indexed and Fibred Sets

7. Indexed Sets
8. Fibred Sets
9. Un/Straightening for Indexed and Fibred Sets

Category Theory

11. Categories
12. Types of Morphisms in Categories
13. Adjunctions and the Yoneda Lemma
14. Constructions With Categories
15. Profunctors
16. Cartesian Closed Categories
17. Kan Extensions

Bicategories

18. Bicategories
19. Internal Adjunctions

Internal Category Theory

20. Internal Categories

Part VIII

Globular Stuff

Chapter 23

The Globe Category

Contents

Appendices

23.A Other Chapters

Sets

1. Sets
2. Constructions With Sets
3. Pointed Sets
4. Tensor Products of Pointed Sets
5. Relations
6. Spans
7. Posets

Indexed and Fibred Sets

7. Indexed Sets
8. Fibred Sets
9. Un/Straightening for Indexed and Fibred Sets

Category Theory

11. Categories
12. Types of Morphisms in Categories
13. Adjunctions and the Yoneda Lemma
14. Constructions With Categories
15. Profunctors
16. Cartesian Closed Categories
17. Kan Extensions

Bicategories

18. Bicategories
19. Internal Adjunctions

Internal Category Theory

20. Internal Categories

Part IX

Cellular Stuff

Chapter 24

The Cell Category

Contents

Appendices

24.A Other Chapters

Sets

1. Sets
2. Constructions With Sets
3. Pointed Sets
4. Tensor Products of Pointed Sets
5. Relations
6. Spans
7. Posets

Indexed and Fibred Sets

7. Indexed Sets
8. Fibred Sets
9. Un/Straightening for Indexed and Fibred Sets

Category Theory

11. Categories
12. Types of Morphisms in Categories
13. Adjunctions and the Yoneda Lemma
14. Constructions With Categories
15. Profunctors
16. Cartesian Closed Categories
17. Kan Extensions

Bicategories

18. Bicategories
19. Internal Adjunctions

Internal Category Theory

20. Internal Categories

Part X

Monoids

Chapter 25

Monoids

Contents

Appendices

25.A Other Chapters

Chapter 26

Constructions With Monoids

Contents

Appendices

26.A Other Chapters

Part XI

Monoids With Zero

Chapter 27

Monoids With Zero

Contents

Appendices

27.A Other Chapters

Sets

1. Sets
2. Constructions With Sets
3. Pointed Sets
4. Tensor Products of Pointed Sets
5. Relations
6. Spans
7. Posets

Indexed and Fibred Sets

7. Indexed Sets
8. Fibred Sets
9. Un/Straightening for Indexed and Fibred Sets

Category Theory

11. Categories
12. Types of Morphisms in Categories
13. Adjunctions and the Yoneda Lemma
14. Constructions With Categories
15. Profunctors
16. Cartesian Closed Categories
17. Kan Extensions

Bicategories

18. Bicategories
19. Internal Adjunctions

Internal Category Theory

20. Internal Categories

Cyclic Stuff

Chapter 28

Constructions With Monoids With Zero

Contents

Appendices

28.A Other Chapters

Part XII

Groups

Chapter 29

Groups

Contents

Appendices

29.A Other Chapters

Sets

1. Sets
2. Constructions With Sets
3. Pointed Sets
4. Tensor Products of Pointed Sets
5. Relations
6. Spans
7. Posets

Indexed and Fibred Sets

7. Indexed Sets
8. Fibred Sets
9. Un/Straightening for Indexed and Fibred Sets

Category Theory

11. Categories
12. Types of Morphisms in Categories
13. Adjunctions and the Yoneda Lemma
14. Constructions With Categories
15. Profunctors
16. Cartesian Closed Categories
17. Kan Extensions

Bicategories

18. Bicategories
19. Internal Adjunctions

Internal Category Theory

20. Internal Categories

Chapter 30

Constructions With Groups

Contents

Appendices

30.A Other Chapters

Sets

1. Sets
2. Constructions With Sets
3. Pointed Sets
4. Tensor Products of Pointed Sets
5. Relations
6. Spans
7. Posets

Indexed and Fibred Sets

7. Indexed Sets
8. Fibred Sets
9. Un/Straightening for Indexed and Fibred Sets

Category Theory

11. Categories
12. Types of Morphisms in Categories
13. Adjunctions and the Yoneda Lemma
14. Constructions With Categories
15. Profunctors
16. Cartesian Closed Categories
17. Kan Extensions

Bicategories

18. Bicategories
19. Internal Adjunctions

Internal Category Theory

20. Internal Categories

Part XIII

Hyper Algebra

Chapter 31

Hypermonoids

Contents

Appendices

31.A Other Chapters

Sets

1. Sets
2. Constructions With Sets
3. Pointed Sets
4. Tensor Products of Pointed Sets
5. Relations
6. Spans
7. Posets

Indexed and Fibred Sets

7. Indexed Sets
8. Fibred Sets
9. Un/Straightening for Indexed and Fibred Sets

Category Theory

11. Categories
12. Types of Morphisms in Categories
13. Adjunctions and the Yoneda Lemma
14. Constructions With Categories
15. Profunctors
16. Cartesian Closed Categories
17. Kan Extensions

Bicategories

18. Bicategories
19. Internal Adjunctions

Internal Category Theory

20. Internal Categories

Chapter 32

Hypergroups

Contents

Appendices

32.A Other Chapters

Chapter 33

Hypersemirings and Hyperrings

Contents

Appendices

33.A Other Chapters

Sets

1. Sets
2. Constructions With Sets
3. Pointed Sets
4. Tensor Products of Pointed Sets
5. Relations
6. Spans
7. Posets

Indexed and Fibred Sets

7. Indexed Sets
8. Fibred Sets
9. Un/Straightening for Indexed and Fibred Sets

Category Theory

11. Categories
12. Types of Morphisms in Categories
13. Adjunctions and the Yoneda Lemma
14. Constructions With Categories
15. Profunctors
16. Cartesian Closed Categories
17. Kan Extensions

Bicategories

18. Bicategories
19. Internal Adjunctions

Internal Category Theory

20. Internal Categories

Chapter 34

Quantales

Contents

Appendices

34.A Other Chapters

Part XIV

Near-Rings

Chapter 35

Near-Semirings

Contents

Appendices

35.A Other Chapters

Chapter 36

Near-Rings

Contents

Appendices

36.A Other Chapters

Sets

1. Sets
2. Constructions With Sets
3. Pointed Sets
4. Tensor Products of Pointed Sets
5. Relations
6. Spans
7. Posets

Indexed and Fibred Sets

7. Indexed Sets
8. Fibred Sets
9. Un/Straightening for Indexed and Fibred Sets

Category Theory

11. Categories
12. Types of Morphisms in Categories
13. Adjunctions and the Yoneda Lemma
14. Constructions With Categories
15. Profunctors
16. Cartesian Closed Categories
17. Kan Extensions

Bicategories

18. Bicategories
19. Internal Adjunctions

Internal Category Theory

20. Internal Categories

Part XV

Real Analysis

Chapter 37

Real Analysis in One Variable

Contents

Appendices

37.A Other Chapters

Sets

1. Sets
2. Constructions With Sets
3. Pointed Sets
4. Tensor Products of Pointed Sets
5. Relations
6. Spans
7. Posets

Indexed and Fibred Sets

7. Indexed Sets
8. Fibred Sets
9. Un/Straightening for Indexed and Fibred Sets

Category Theory

11. Categories
12. Types of Morphisms in Categories
13. Adjunctions and the Yoneda Lemma
14. Constructions With Categories
15. Profunctors
16. Cartesian Closed Categories
17. Kan Extensions

Bicategories

18. Bicategories
19. Internal Adjunctions

Internal Category Theory

20. Internal Categories

Chapter 38

Real Analysis in Several Variables

Contents

Appendices

38.A Other Chapters

Sets

1. Sets
2. Constructions With Sets
3. Pointed Sets
4. Tensor Products of Pointed Sets
5. Relations
6. Spans
7. Posets

Indexed and Fibred Sets

7. Indexed Sets
8. Fibred Sets
9. Un/Straightening for Indexed and Fibred Sets

Category Theory

11. Categories
12. Types of Morphisms in Categories
13. Adjunctions and the Yoneda Lemma
14. Constructions With Categories
15. Profunctors
16. Cartesian Closed Categories
17. Kan Extensions

Bicategories

18. Bicategories
19. Internal Adjunctions

Internal Category Theory

20. Internal Categories

Part XVI

Measure Theory

Chapter 39

Measurable Spaces

Contents

Appendices

39.A Other Chapters

Sets

1. Sets
2. Constructions With Sets
3. Pointed Sets
4. Tensor Products of Pointed Sets
5. Relations
6. Spans
7. Posets

Indexed and Fibred Sets

7. Indexed Sets
8. Fibred Sets
9. Un/Straightening for Indexed and Fibred Sets

Category Theory

11. Categories
12. Types of Morphisms in Categories
13. Adjunctions and the Yoneda Lemma
14. Constructions With Categories
15. Profunctors
16. Cartesian Closed Categories
17. Kan Extensions

Bicategories

18. Bicategories
19. Internal Adjunctions

Internal Category Theory

20. Internal Categories

Chapter 40

Measures and Integration

Contents

Appendices

40.A Other Chapters

Sets

1. Sets
2. Constructions With Sets
3. Pointed Sets
4. Tensor Products of Pointed Sets
5. Relations
6. Spans
7. Posets

Indexed and Fibred Sets

7. Indexed Sets
8. Fibred Sets
9. Un/Straightening for Indexed and Fibred Sets

Category Theory

11. Categories
12. Types of Morphisms in Categories
13. Adjunctions and the Yoneda Lemma
14. Constructions With Categories
15. Profunctors
16. Cartesian Closed Categories
17. Kan Extensions

Bicategories

18. Bicategories
19. Internal Adjunctions

Internal Category Theory

20. Internal Categories

Part XVII

Probability Theory

Chapter 41

Probability Theory

Contents

Appendices

41.A Other Chapters

Sets

1. Sets
2. Constructions With Sets
3. Pointed Sets
4. Tensor Products of Pointed Sets
5. Relations
6. Spans
7. Posets

Indexed and Fibred Sets

7. Indexed Sets
8. Fibred Sets
9. Un/Straightening for Indexed and Fibred Sets

Category Theory

11. Categories
12. Types of Morphisms in Categories
13. Adjunctions and the Yoneda Lemma
14. Constructions With Categories
15. Profunctors
16. Cartesian Closed Categories
17. Kan Extensions

Bicategories

18. Bicategories
19. Internal Adjunctions

Internal Category Theory

20. Internal Categories

Part XVIII

Stochastic Analysis

Chapter 42

Stochastic Processes, Martingales, and Brownian Motion

Contents

Appendices

42.A Other Chapters

Sets

1. Sets
2. Constructions With Sets
3. Pointed Sets
4. Tensor Products of Pointed Sets
5. Relations
6. Spans
7. Posets

Indexed and Fibred Sets

7. Indexed Sets
8. Fibred Sets
9. Un/Straightening for Indexed and Fibred Sets

Category Theory

11. Categories
12. Types of Morphisms in Categories
13. Adjunctions and the Yoneda Lemma
14. Constructions With Categories
15. Profunctors
16. Cartesian Closed Categories
17. Kan Extensions

Bicategories

18. Bicategories
19. Internal Adjunctions

Chapter 43

Itô Calculus

Contents

Appendices

43.A Other Chapters

Sets

1. Sets
2. Constructions With Sets
3. Pointed Sets
4. Tensor Products of Pointed Sets
5. Relations
6. Spans
7. Posets

Indexed and Fibred Sets

7. Indexed Sets
8. Fibred Sets
9. Un/Straightening for Indexed and Fibred Sets

Category Theory

11. Categories
12. Types of Morphisms in Categories
13. Adjunctions and the Yoneda Lemma
14. Constructions With Categories
15. Profunctors
16. Cartesian Closed Categories
17. Kan Extensions

Bicategories

18. Bicategories
19. Internal Adjunctions

Internal Category Theory

20. Internal Categories

Cyclic Stuff

Chapter 44

Stochastic Differential Equations

Contents

Appendices

44.A Other Chapters

Sets

1. Sets
2. Constructions With Sets
3. Pointed Sets
4. Tensor Products of Pointed Sets
5. Relations
6. Spans
7. Posets

Indexed and Fibred Sets

7. Indexed Sets
8. Fibred Sets
9. Un/Straightening for Indexed and Fibred Sets

Category Theory

11. Categories
12. Types of Morphisms in Categories
13. Adjunctions and the Yoneda Lemma
14. Constructions With Categories
15. Profunctors
16. Cartesian Closed Categories
17. Kan Extensions

Bicategories

18. Bicategories
19. Internal Adjunctions

Internal Category Theory

20. Internal Categories

Part XIX

Differential Geometry

Chapter 45

Topological and Smooth Manifolds

Contents

Appendices

45.A Other Chapters

Sets

1. Sets
2. Constructions With Sets
3. Pointed Sets
4. Tensor Products of Pointed Sets
5. Relations
6. Spans
7. Posets

Indexed and Fibred Sets

7. Indexed Sets
8. Fibred Sets
9. Un/Straightening for Indexed and Fibred Sets

Category Theory

11. Categories
12. Types of Morphisms in Categories
13. Adjunctions and the Yoneda Lemma
14. Constructions With Categories
15. Profunctors
16. Cartesian Closed Categories
17. Kan Extensions

Bicategories

18. Bicategories
19. Internal Adjunctions

Internal Category Theory

20. Internal Categories

Part XX

Schemes

Chapter 46

Schemes

46.1 Introduction

In this document we define schemes. A basic reference is [**EGA**].

Part XXI

Secret Part

Chapter 47

To Do List

This chapter contains some material about relations and constructions with them. Notably, we discuss and explore:

Contents

47.1 Notes to Self

47.1.1 Things To Ask On MO/Zulip

Remark 47.1.1.1 ► Things To Ask On MO/Zulip

Here is a list of things to be asked on MO/Zulip.

1. What are
   - (a) Cartesian bicategories
   - (b) Double categories of relations (https://arxiv.org/abs/2107.07621)
   - (c) Categories of relations
   - (d) Allegories
   - (e) 1-Category equipped with relations (https://ncatlab.org/nlab/show/1-category+equipped+with+relations)

good for? What have these notions been developed for, why are they important, and what have they lead to?

47.1.2 Things To Explore/Add

Remark 47.1.2.1 ► Things To Explore/Add

Here is a list of things to be explored.

1. https://mathoverflow.net/a/461814
2. there's some cool stuff in https://arxiv.org/abs/2312.00990, e.g. on cofunctors.
3. internal adjunctions in Mod as in [JY21, Section 6.3]; see [JY21, Example 6.2.6].
4. write the "profunctors" equivalent of the relations chapter
5. change χ_B notation throughout the notes
6. maybe note that skew monoidal structures on $\mathbf{Rel}(A, B)$ satisfy coherence trivially since the 2-morphisms are inclusions
7. reconsider notation $\mathrm{FreeAlg}_{\mathcal{P}}$ in Relations
8. Constructions With Sets: Isbell duality for powersets
9. Categories: comma category notation as in https://mathoverflow.net/questions/455630
10. Universal property of the bicategory of spans, https://ncatlab.org/nlab/show/span
11. Codensity monad $\mathrm{Ran}_J(J)$ of a relation (What about $\mathrm{Rift}_J(J)$?)
12. Relative *co*monads in **Rel**.
13. Write proper sections on straightening for lax functors from sets to Rel or Span (displayed sets) when I study the corresponding notions for categories
14. Write about cospans.
15. CoCartesian fibration classifying $\mathrm{Fun}(F, G)$, https://mathoverflow.net/questions/457533/cocartesian-fibration-classifying-mathrmfunf-g

16. Constructions With Sets: functoriality of limits/colimits, like functoriality of pullbacks
17. https://ncatlab.org/nlab/show/adjoint+lifting+theorem
18. https://ncatlab.org/nlab/show/Gabriel%E2%80%93Ulmer+duality

47.1.3 Random Cool Papers

REMARK 47.1.3.1 ► RANDOM COOL PAPERS

Here are some random cool papers that appeared on arXiv and that I want to check eventually.

1. A Derived Geometric Approach to Propagation of Solution Singularities for Non-linear PDEs I: Foundations
2. The Fundamental Theorem of Calculus point-free, with applications to exponentials and logarithms

47.1.4 Omitted Proofs To Add

Не так благотворна истина, как зловредна ее видимость.

Даниил Данковский

Truth does not do as much good in the world as the appearance of truth does evil.

Daniil Dankovsky

There's a very large number of omitted proofs throughout these notes; here I list some of the ones that I really want to add to the notes at some point.

REMARK 47.1.4.1 ► OMITTED PROOFS TO ADD

Here is a list of omitted proofs that I really want to eventually write up or add a reference to.

- ?? of ??
- ?? of ??
- ?? of ??
- ?? of ??
- ?? of ??

- ?? of ??
- ?? of ??

Appendices

47.A Other Chapters

Hyper Algebra

Near-Rings

Real Analysis

Measure Theory

Probability Theory

Stochastic Analysis

Differential Geometry

Schemes

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Index of Notation

Index of Set Theory

R

S

U

Z