Internal Adjunctions

December 3, 2023

Create tags:

1. https://www.google.com/search?q=mate+of+an+adjunction
2. Moreover, by uniqueness of adjoints (Internal Adjunctions, Item 2 of Proposition 1.2.1.4), this implies also that $S = f^{-1}$.
3. define bicategory $\mathsf{Adj}(\mathcal{C})$
4. walking monad
5. proposition: 2-functors preserve unitors and associators
6. https://ncatlab.org/nlab/show/2-category+of+adjunctions. Is there a 3-category too?
7. https://ncatlab.org/nlab/show/free+monad
8. https://ncatlab.org/nlab/show/CatAdj
9. https://ncatlab.org/nlab/show/Adj
10. $\mathsf{Adj}(\mathsf{Adj}(\mathcal{C}))$

Contents

1 Internal Adjunctions

1.1 The Walking Adjunction

Definition 1.1.1.1. The **walking adjunction** is the bicategory Adj freely generated by[1]

- *Objects.* A pair of objects A and B;
- *Morphisms.* A pair of morphisms

$$\begin{aligned} L &\colon A \to B, \\ R &\colon B \to A; \end{aligned}$$

- 2-*Morphisms.* A pair of 2-morphisms

$$\begin{aligned} \eta &\colon \mathrm{id}_A \to L \circ R, \\ \epsilon &\colon R \circ L \to \mathrm{id}_B; \end{aligned}$$

[1] See [SS86] for an explicit description of the 2-category (as opposed to a bicategory) version of Adj in terms of finite ordinals, similar to the description of the 2-category version of the walking monad (**??**) as a subcategory of Δ.

subject to the equalities

$$
\begin{tikzcd}
 & B \arrow[rr, "\mathrm{id}_B"] \arrow[dr, "R"] \arrow[d, phantom, "\Uparrow\eta"] & & B \\
A \arrow[ur, "L"] \arrow[rr, "\mathrm{id}_A"'] & {} & A \arrow[ur, "L"'] \arrow[u, phantom, "\Uparrow\epsilon"] &
\end{tikzcd}
=
\begin{tikzcd}
 & B \arrow[rr, "\mathrm{id}_B"] & & B \\
A \arrow[ur, "L"] \arrow[rr, "\mathrm{id}_A"'] & & A \arrow[ur, "L"'] \arrow[ul, Rightarrow, "\mathrm{id}_L"] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & A \arrow[rr, "\mathrm{id}_A"] \arrow[dr, "L"] \arrow[d, phantom, "\Downarrow\epsilon"] & & A \\
B \arrow[ur, "R"] \arrow[rr, "\mathrm{id}_B"'] & {} & B \arrow[ur, "R"'] \arrow[u, phantom, "\Downarrow\eta"] &
\end{tikzcd}
=
\begin{tikzcd}
 & A \arrow[rr, "\mathrm{id}_A"] \arrow[dr, Rightarrow, "\mathrm{id}_R"] & & A \\
B \arrow[ur, "R"] \arrow[rr, "\mathrm{id}_B"'] & & B \arrow[ur, "R"'] &
\end{tikzcd}
$$

of pasting diagrams, which are equivalent to the following conditions:

1. *The Left Triangle Identity.* The diagram

$$
\begin{tikzcd}
L \circ \mathrm{id}_A \arrow[r, Rightarrow, "\mathrm{id}_L \circ \eta"] \arrow[ddrr, Rightarrow, "\rho_L^{\mathsf{Adj}}"'] & L \circ (R \circ L) \arrow[r, Rightarrow, "(\alpha^{\mathsf{Adj}}_{L,R,L})^{-1}"] & (L \circ R) \circ L \arrow[d, Rightarrow, "\epsilon \circ \mathrm{id}_L"] \\
 & & \mathrm{id}_B \circ L \arrow[d, Rightarrow, "\lambda_L^{\mathsf{Adj}}"] \\
 & & L
\end{tikzcd}
$$

   commutes.

2. *The Right Triangle Identity.* The diagram

$$
\begin{tikzcd}
\mathrm{id}_A \circ R \arrow[r, Rightarrow, "\eta \circ \mathrm{id}_R"] \arrow[ddrr, Rightarrow, "\lambda_R^{\mathsf{Adj}}"'] & (R \circ L) \circ R \arrow[r, Rightarrow, "\alpha^{\mathsf{Adj}}_{R,L,R}"] & R \circ (L \circ R) \arrow[d, Rightarrow, "\mathrm{id}_R \circ \epsilon"] \\
 & & R \circ \mathrm{id}_B \arrow[d, Rightarrow, "\rho_R^{\mathsf{Adj}}"] \\
 & & R.
\end{tikzcd}
$$

1.2 Internal Adjunctions

Let $\mathcal{C}$ be a bicategory.

Definition 1.2.1.1. An **internal adjunction in $\mathcal{C}$**[2,3] is a 2-functor $\mathsf{Adj} \to \mathcal{C}$.

Remark 1.2.1.2. In detail, an **internal adjunction in $\mathcal{C}$** consists of

- *Objects.* A pair of objects A and B of $\mathcal{C}$;
- *Morphisms.* A pair of morphisms

$$\begin{aligned} L\colon A &\to B, \\ R\colon B &\to A \end{aligned}$$

  of $\mathcal{C}$;
- 2-*Morphisms.* A pair of 2-morphisms

$$\begin{aligned} \eta\colon \mathrm{id}_A &\to L \circ R, \\ \epsilon\colon R \circ L &\to \mathrm{id}_B \end{aligned}$$

  of $\mathcal{C}$;

subject to the equalities

$$\begin{tikzcd} & B \arrow[r, "\mathrm{id}_B"] \arrow[dr, "R"] & B \\ A \arrow[ur, "L"] \arrow[r, "\mathrm{id}_A"'] \arrow[ur, phantom, shift left=6, "\Uparrow\eta"] & A \arrow[ur, "L"'] \arrow[u, phantom, shift right=6, "\Uparrow\epsilon"] & \end{tikzcd} = \begin{tikzcd} & B \arrow[r, "\mathrm{id}_B"] & B \\ A \arrow[ur, "L"] \arrow[r, "\mathrm{id}_A"'] & A \arrow[ur, "L"'] \arrow[u, phantom, "\Leftarrow \mathrm{id}_L" description] & \end{tikzcd}$$

$$\begin{tikzcd} & A \arrow[r, "\mathrm{id}_A"] \arrow[dr, "L"] & A \\ B \arrow[ur, "R"] \arrow[r, "\mathrm{id}_B"'] \arrow[ur, phantom, shift left=6, "\Downarrow\epsilon"] & B \arrow[ur, "R"'] \arrow[u, phantom, shift right=6, "\Downarrow\eta"] & \end{tikzcd} = \begin{tikzcd} & A \arrow[r, "\mathrm{id}_A"] \arrow[d, phantom, "\Rightarrow \mathrm{id}_R" description] & A \\ B \arrow[ur, "R"] \arrow[r, "\mathrm{id}_B"'] & B \arrow[ur, "R"'] & \end{tikzcd}$$

[2] *Further Terminology:* Also called an **adjunction internal to $\mathcal{C}$**.

[3] *Further Terminology:* In this situation, we also call (g, f) an **adjoint pair**, f the **left adjoint** of the pair, g the **right adjoint** of the pair, η the **unit** of the adjunction, and ϵ

of pasting diagrams in $\mathcal{C}$, which are equivalent to the following conditions:[4]

1. *The Left Triangle Identity.* The diagram

$$\begin{tikzcd}
L \circ \mathrm{id}_A \arrow[r, Rightarrow, "\mathrm{id}_L \circ \eta"] \arrow[rrdd, Rightarrow, "\rho^{\mathcal{C}}_L"'] & L \circ (R \circ L) \arrow[r, Rightarrow, "(\alpha^{\mathcal{C}}_{L,R,L})^{-1}"] & (L \circ R) \circ L \arrow[d, Rightarrow, "\epsilon \circ \mathrm{id}_L"] \\
 & & \mathrm{id}_B \circ L \arrow[d, Rightarrow, "\lambda^{\mathcal{C}}_L"] \\
 & & L
\end{tikzcd}$$

   commutes.

2. *The Right Triangle Identity.* The diagram

$$\begin{tikzcd}
\mathrm{id}_A \circ R \arrow[r, Rightarrow, "\eta \circ \mathrm{id}_R"] \arrow[rrdd, Rightarrow, "\lambda^{\mathcal{C}}_R"'] & (R \circ L) \circ R \arrow[r, Rightarrow, "\alpha^{\mathcal{C}}_{R,L,R}"] & R \circ (L \circ R) \arrow[d, Rightarrow, "\mathrm{id}_R \circ \epsilon"] \\
 & & R \circ \mathrm{id}_B \arrow[d, Rightarrow, "\rho^{\mathcal{C}}_R"] \\
 & & R.
\end{tikzcd}$$

Example 1.2.1.3. Here are some examples of internal adjunctions.

1. *Internal Adjunctions in* Cats_2*.* The internal adjunctions in the 2-category Cats_2 of categories, functors, and natural transformations are precisely the adjunctions of Categories, **??**.

the **counit** of the adjunction.

[4]When $\mathcal{C}$ is a 2-category, these diagrams take the following form:

$$\begin{tikzcd}
L \arrow[r, Rightarrow, "\mathrm{id}_L \circ \eta"] \arrow[dr, Rightarrow, "\mathrm{id}_L"'] & L \circ R \circ L \arrow[d, Rightarrow, "\epsilon \circ \mathrm{id}_L"] \\
 & L
\end{tikzcd}
\qquad
\begin{tikzcd}
R \arrow[r, Rightarrow, "\mathrm{id}_R \circ \eta"] \arrow[dr, Rightarrow, "\mathrm{id}_L"'] & R \circ L \circ R \arrow[d, Rightarrow, "\epsilon \circ \mathrm{id}_R"] \\
 & R.
\end{tikzcd}$$

2. *Internal Adjunctions in* $\mathbf{Rel}$*.* The internal adjunctions in $\mathbf{Rel}$ are precisely the relations of the form $\mathrm{Gr}(f) \dashv f^{-1}$ with f a function; see Relations, Item 4 of Proposition 2.5.1.1.

3. *Internal Adjunctions in* Span*.* The internal adjunctions in Span are precisely the spans of the form

$$\begin{tikzcd} & S \arrow[dl, "\phi"'] \arrow[dr, "g"] & \\ A & & B \end{tikzcd}$$

with ϕ an isomorphism; see Spans, Item 4 of Proposition 2.5.1.1.

Proposition 1.2.1.4. Let $\mathcal{C}$ be a bicategory.

1. *Duality.* Let (f, g, η, ϵ) be an internal adjunction in $\mathcal{C}$.
   (a) The quadruple (g, f, η, ϵ) is an internal adjunction in $\mathcal{C}^{\mathsf{op}}$.
   (b) The quadruple (g, f, ϵ, η) is an internal adjunction in $\mathcal{C}^{\mathsf{co}}$.
   (c) The quadruple (f, g, η, ϵ) is an internal adjunction in $\mathcal{C}^{\mathsf{coop}}$.

2. *Uniqueness of Adjoints.* Let (f, g, η, ϵ) and $(f, g', \eta', \epsilon')$ be internal adjunctions in $\mathcal{C}$. We have a canonical isomorphism[5]

$$g \xRightarrow{(\lambda^{\mathcal{C}}_{g})^{-1}} \mathrm{id}_A \circ g \xRightarrow{\eta' \circ \mathrm{id}_g} (g' \circ f) \circ g \xRightarrow{\alpha^{\mathcal{C}}_{g',f,g}} g' \circ (f \circ g) \xRightarrow{\mathrm{id}_{g'} \circ \epsilon} g' \circ \mathrm{id}_B \xRightarrow{(\rho^{\mathcal{C}}_{g'})^{-1}} g'$$

with inverse

$$g' \xRightarrow{(\lambda^{\mathcal{C}}_{g'})^{-1}} \mathrm{id}_B \circ g' \xRightarrow{\eta \circ \mathrm{id}_{g'}} (g \circ f) \circ g' \xRightarrow{\alpha^{\mathcal{C}}_{g',f,g}} g \circ (f \circ g') \xRightarrow{\mathrm{id}_g \circ \epsilon'} g \circ \mathrm{id}_B \xRightarrow{(\lambda^{\mathcal{C}}_{g})^{-1}} g.$$

3. *Carrying Internal Adjunctions Through Pseudofunctors.* Let $F \colon \mathcal{C} \longrightarrow \mathcal{D}$ be a pseudofunctor and (f, g, η, ϵ) be an internal adjunction in $\mathcal{C}$. There is an induced internal adjunction[6]

$$(F(f), F(g), \overline{\eta}, \overline{\epsilon})$$

in $\mathcal{D}$, where:

[5] *Slogan:* Left adjoints are unique up to canonical isomorphism. Dually, so are right adjoints.

[6] *Warning:* Lax or oplax functors which are not pseudofunctors need not preserve

(a) The unit

$$\overline{\eta}\colon \mathrm{id}_{F(A)} \Longrightarrow F(g) \circ F(f)$$

is the composition

$$\mathrm{id}_{F(A)} \xRightarrow{F_A} F(\mathrm{id}_A) \xRightarrow{F(\eta)} F(g \circ f) \xRightarrow{F^{-1}_{g,f}} F(g) \circ F(f).$$

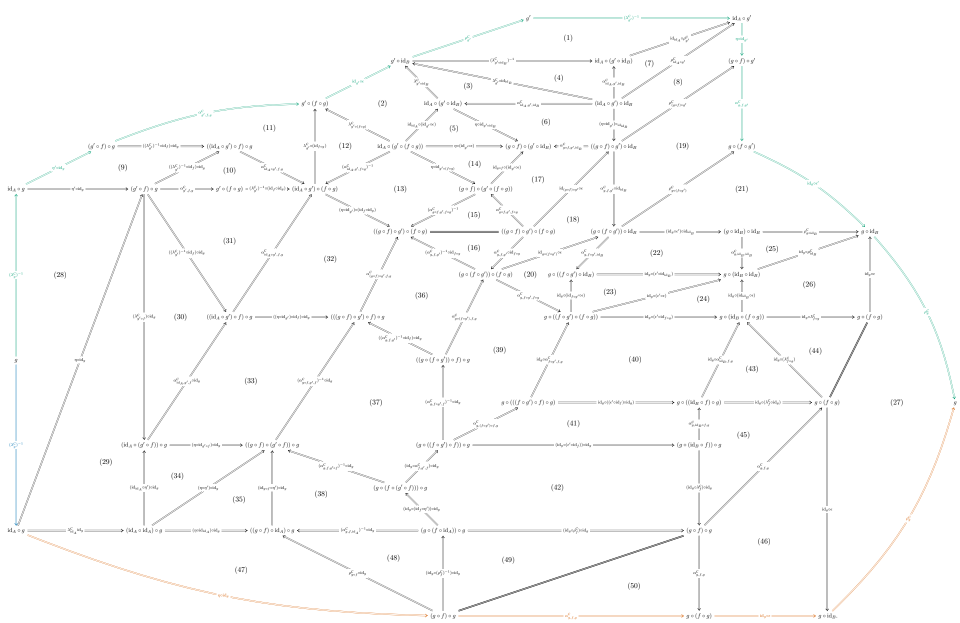
(b) The counit

$$\overline{\epsilon}\colon F(f) \circ F(g) \Longrightarrow \mathrm{id}_{F(B)}$$

is the composition

$$F(f) \circ F(g) \xRightarrow{F_{f,g}} F(f \circ g) \xRightarrow{F(\epsilon)} F(\mathrm{id}_B) \xRightarrow{F_B} \mathrm{id}_{F(B)}.$$

Proof. Item 1, Duality: Omitted.[7]
Item 2, Uniqueness of Adjoints: [8]Consider the diagram (if you *really* want to consider it I fear you will need to zoom in)



In this diagram:

internal adjunctions.

[7] *Reference:* [JY21, Exercise 6.6.2].

[8] *Reference:* [JY21, Lemma 6.1.6].

1. The morphisms in green are the composition $g \overset{\cong}{\Longrightarrow} g' \overset{\cong}{\Longrightarrow} g$;

2. The morphisms in red are equal to $\lambda_g^{\mathcal{C}}$ by the right triangle identity for (f, g, η, ϵ). Hence the composition of the morphism in blue with the morphisms in red is the identity;

3. Subdiagrams (1), (2), (10), (11), (29), (31), and (43) commute by the naturality of the left unitor of $\mathcal{C}$ and its inverse;

4. Subdiagrams (8), (19), and (21) commute by the naturality of the right unitor of $\mathcal{C}$ and its inverse;

5. Subdiagrams (6), (13), (17), (18), (20), (22), (32), (33), (36), (38), (40), (41), and (45) commute by the naturality of the associator of $\mathcal{C}$ and its inverse;

6. Subdiagrams (37), (39), and (42) commute by the pentagon identity for $\mathcal{C}$;

7. Subdiagrams (3), (4), (7), (12), (25), (30), and (48) commute by Bicategories, **??** of **??**;

8. Subdiagrams (5), (14), (23), (24), (34), and (35) commute by middle-four exchange;

9. Subdiagrams (9), (15), (16), (27), (28), (44), (46), (49), and (50) commute trivially;

10. Subdiagram (26) commutes by Bicategories, **????** of **??**;

11. Subdiagram (47) commutes by Bicategories, **??** of **??** and the naturality of the left unitor of right unitor of $\mathcal{C}$.

Hence $g \cong g'$.
Item 3, Carrying Internal Adjunctions Through Pseudofunctors: [9]We claim that the left and right triangle identities for $(F(f), F(g), \overline{\eta}, \overline{\epsilon})$ hold:

1. The left triangle identity for $(F(f), F(g), \overline{\eta}, \overline{\epsilon})$ is the condition that the

[9]*Reference:* [JY21, Proposition 6.1.7].

boundary diagram of the diagram (you may need to zoom in)

$$\begin{tikzcd}
F(f)\circ \mathrm{id}_{F(A)} \arrow[rrr, Rightarrow, "\mathrm{id}_{F(f)}\circ\overline{\eta}"] \arrow[dr, Rightarrow, "\mathrm{id}_{F(f)}\circ F_A"] \arrow[dd, Rightarrow, "\rho^{\mathcal{D}}_{F(f)}"'] & & & F(f)\circ(F(g)\circ F(f)) \arrow[dr, Rightarrow, "(\alpha^{\mathcal{D}}_{F(f),F(g),F(f)})^{-1}"] & & \\
& F(f)\circ F(\mathrm{id}_A) \arrow[r, Rightarrow, "\mathrm{id}_{F(f)}\circ F(\eta)"] \arrow[dd, Rightarrow, "F_{f,\mathrm{id}_A}"] \arrow[ur, phantom, "(1)"] \arrow[ddr, phantom, "(3)"] & F(f)\circ F(g\circ f) \arrow[ur, Rightarrow, "\mathrm{id}_{F(f)}\circ F^{-1}_{g,f}"] \arrow[dd, Rightarrow, "F_{f,g\circ f}"] \arrow[ddrr, phantom, "(4)"] & & (F(f)\circ F(g))\circ F(f) \arrow[dd, Rightarrow, "F_{f,g}\circ\mathrm{id}_{F(f)}"] \arrow[dddddr, Rightarrow, bend left, "\overline{\epsilon}\circ\mathrm{id}_{F(f)}"] & \\
F(f) \arrow[dr, Rightarrow, "(F(\rho^{\mathcal{C}}_f))^{-1}"'] \arrow[ur, phantom, "(2)"] & & & & & \\
& F(f\circ\mathrm{id}_A) \arrow[r, Rightarrow, "F(\mathrm{id}_f\circ\eta)"] \arrow[ddddrrr, Rightarrow, bend right, "F(\rho^{\mathcal{C}}_f)"'] & F(f\circ(g\circ f)) \arrow[dr, Rightarrow, "F((\alpha^{\mathcal{C}}_{f,g,f})^{-1})"] \arrow[ddd, phantom, "(5)"] & & F(f\circ g)\circ F(f) \arrow[dl, Rightarrow, "F_{f\circ g,f}"'] \arrow[dd, Rightarrow, "F(\epsilon)\circ\mathrm{id}_{F(f)}"] \arrow[dr, phantom, "(7)"] & \\
& & & F((f\circ g)\circ f) \arrow[dd, Rightarrow, "F(\epsilon\circ\mathrm{id}_f)"] \arrow[dr, phantom, "(6)"] & & \\
& & & & F(\mathrm{id}_B)\circ F(f) \arrow[dl, Rightarrow, "F_{\mathrm{id}_B,f}"'] \arrow[dr, Rightarrow, "F^{-1}_B\circ\mathrm{id}_{F(f)}"] & \\
& & & F(\mathrm{id}_B\circ f) \arrow[dr, Rightarrow, "F(\lambda^{\mathcal{C}}_f)"'] \arrow[rr, phantom, "(8)"] & & \mathrm{id}_{F(B)}\circ F(f) \arrow[dl, Rightarrow, "\lambda^{\mathcal{D}}_{F(f)}"] \\
& & & & F(f). &
\end{tikzcd}$$

commutes. Since

(a) Subdiagrams (1) and (7) commute by applying middle-four exchange twice,

(b) Subdiagrams (2) and (8) commute by the left and right lax unity conditions for F,

(c) Subdiagrams (3) and (6) commute by the naturality of the lax functoriality constraints of F,

(d) Subdiagram (4) commutes by the lax associativity condition for F, and

(e) Subdiagram (5) commutes by the left triangle identity for (f, g, η, ϵ),

so does the boundary diagram.

2. The right triangle identity for $(F(f), F(g), \overline{\eta}, \overline{\epsilon})$ is the condition that

the boundary diagram of the diagram (you may need to zoom in)

$$
\begin{tikzcd}
\mathrm{id}_{F(A)} \circ F(g) \arrow[rrrr, Rightarrow, "\overline{\eta}\circ\mathrm{id}_{F(g)}"] \arrow[dr, Rightarrow, "F_A\circ\mathrm{id}_{F(g)}"] \arrow[dd, Rightarrow, "\lambda^{\mathcal{D}}_{F(g)}"'] & & \arrow[d, phantom, "(1)"] & & (F(g)\circ F(f))\circ F(g) \arrow[dr, Rightarrow, "\alpha^{\mathcal{D}}_{F(g),F(f),F(g)}"] & & \\
\arrow[dr, phantom, "(2)"] & F(\mathrm{id}_A)\circ F(g) \arrow[rr, Rightarrow, "F(\eta)\circ\mathrm{id}_{F(g)}"] \arrow[dd, Rightarrow, "F_{\mathrm{id}_A,g}"] & \arrow[dd, phantom, "(3)"] & F(g\circ f)\circ F(g) \arrow[ur, Rightarrow, "F^{-1}_{g,f}\circ\mathrm{id}_{F(g)}"] \arrow[dd, Rightarrow, "F_{g\circ f,g}"] & \arrow[dd, phantom, "(4)"] & F(g)\circ(F(f)\circ F(g)) \arrow[dd, Rightarrow, "\mathrm{id}_{F(g)}\circ F_{f,g}"] \arrow[dddddr, Rightarrow, bend left, "\mathrm{id}_{F(g)}\circ\overline{\epsilon}"] & \\
F(g) \arrow[dr, Rightarrow, "(F(\lambda^{\mathcal{C}}_g))^{-1}"'] & & & & & & \\
& F(\mathrm{id}_A\circ g) \arrow[rr, Rightarrow, "F(\eta\circ\mathrm{id}_g)"] \arrow[ddddrrrr, Rightarrow, bend right, "F(\lambda^{\mathcal{C}}_g)"'] & & F((g\circ f)\circ g) \arrow[dr, Rightarrow, "F(\alpha^{\mathcal{C}}_{g,f,g})"] & & F(g)\circ F(f\circ g) \arrow[dl, Rightarrow, "F_{g,f\circ g}"'] \arrow[dd, Rightarrow, "\mathrm{id}_{F(g)}\circ F(\epsilon)"] \arrow[dr, phantom, "(7)"] & \\
& & & \arrow[d, phantom, "(5)"] & F(g\circ(f\circ g)) \arrow[dd, Rightarrow, "F(\mathrm{id}_g\circ\epsilon)"] \arrow[dr, phantom, "(6)"] & & \\
& & & & & F(g)\circ F(\mathrm{id}_B) \arrow[dl, Rightarrow, "F_{g,\mathrm{id}_B}"'] \arrow[dr, Rightarrow, "\mathrm{id}_{F(g)}\circ F_B^{-1}"] \arrow[dd, phantom, "(8)"] & \\
& & & & F(g\circ\mathrm{id}_B) \arrow[dr, Rightarrow, "F(\rho^{\mathcal{C}}_g)"'] & & F(g)\circ\mathrm{id}_{F(B)} \arrow[dl, Rightarrow, "\rho^{\mathcal{D}}_{F(g)}"] \\
& & & & & F(g). &
\end{tikzcd}
$$

commutes. Since

(a) Subdiagrams (1) and (7) commute by applying middle-four exchange twice,

(b) Subdiagrams (2) and (8) commute by the left and right lax unity conditions for F,

(c) Subdiagrams (3) and (6) commute by the naturality of the lax functoriality constraints of F,

(d) Subdiagram (4) commutes by the lax associativity condition for F, and

(e) Subdiagram (5) commutes by the right triangle identity for (f, g, η, ϵ),

so does the boundary diagram.

This finishes the proof. □

1.3 Internal Adjoint Equivalences

Let $\mathcal{C}$ be a bicategory.

Definition 1.3.1.1. An internal adjunction (f, g, η, ϵ) in $\mathcal{C}$ is an **internal adjoint equivalence** if η and ϵ are isomorphisms in $\mathcal{C}$.

Example 1.3.1.2. Here are some examples of internal adjoint equivalences.

1. *Internal Adjoint Equivalences in* Cats_2*.* The internal adjoint equivalences in the 2-category Cats_2 of categories, functors, and natural transformations are precisely the adjoint equivalences of Categories, **??**.[10]

2. *Internal Adjoint Equivalences in* Mod*.* The internal adjoint equivalences in Mod are precisely the invertible R-modules; see **??**.[11]

3. *Internal Adjoint Equivalences in* $\mathsf{PseudoFun}(\mathcal{C}, \mathcal{D})$*.* The internal adjoint equivalences in $\mathsf{PseudoFun}(\mathcal{C}, \mathcal{D})$ are precisely the invertible strong transformations; see **??**.[12]

4. *Internal Adjoint Equivalences in* $\mathbf{Rel}$*.* The internal adjoint equivalences in $\mathbf{Rel}$ are precisely the relations of the form $\mathrm{Gr}(f) \dashv f^{-1}$ with f an isomorphism; see **??**.

5. *Internal Adjoint Equivalences in* Span*.* The internal adjoint equivalences in Span are precisely the spans of the form $A \xleftarrow{\phi} S \xrightarrow{\psi} B$ with ϕ and ψ isomorphisms; see **??**.

Proposition 1.3.1.3. Let $\mathcal{C}$ be a bicategory.

1. *Carrying Internal Adjoint Equivalences Through Pseudofunctors.* Let $F \colon \mathcal{C} \longrightarrow \mathcal{D}$ be a pseudofunctor and (f, g, η, ϵ) be an internal adjunction in $\mathcal{C}$. If (f, g, η, ϵ) is an internal adjoint equivalence in $\mathcal{C}$, then the induced internal adjunction
$$(F(f), F(g), \overline{\eta}, \overline{\epsilon})$$
in $\mathcal{D}$ of Item 3 of Proposition 1.2.1.4 is an internal adjoint equivalence as well.

2. *Internal Adjunctions Always Refine to Internal Adjoint Equivalences.* Let (f, g, η, ϵ) be an internal adjunction in $\mathcal{C}$. If f is an equivalence,

[10] *Reference:* [JY21, Examples 6.2.5].
[11] *Reference:* [JY21, Examples 6.2.6].
[12] *Reference:* [JY21, Examples 6.2.7].

then there exist 2-morphisms

$$\overline{\eta}\colon \mathrm{id}_A \Longrightarrow g \circ f$$
$$\overline{\epsilon}\colon f \circ g \Longrightarrow \mathrm{id}_B$$

of $\mathcal{C}$ such that $(f, g, \overline{\eta}, \overline{\epsilon})$ is an internal adjoint equivalence.

Proof. *Item 1, Carrying Internal Adjoint Equivalences Through Pseudofunctors*: See [JY21, Proposition 6.2.3].
Item 2, Internal Adjunctions Always Refine to Internal Adjoint Equivalences: See [JY21, Proposition 6.2.4]. □

1.4 Mates

Let $\mathcal{C}$ be a bicategory, let (f, g, η, ϵ) and $(f', g', \eta', \epsilon')$ be adjunctions, and let h and k be morphisms of $\mathcal{C}$ as in the diagram

$$\begin{tikzcd}
A \arrow[r, bend left, "f"] \arrow[r, phantom, "\perp"] \arrow[d, "h"'] & B \arrow[l, bend left, "g"] \arrow[d, "k"] \\
C \arrow[r, bend left, "f'"] \arrow[r, phantom, "\perp"] & D. \arrow[l, bend left, "g'"]
\end{tikzcd}$$

Definition 1.4.1.1. The **mates** of a pair of 2-morphisms

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "h"'] & B \arrow[d, "k"] \\
C \arrow[r, "f'"'] \arrow[ur, phantom, "\Nearrow\omega"] & D
\end{tikzcd}
\qquad
\begin{aligned}
\omega &\colon f' \circ h \Longrightarrow k \circ f,\\
\nu &\colon h \circ g \Longrightarrow g' \circ k
\end{aligned}
\qquad
\begin{tikzcd}
A \arrow[d, "h"'] \arrow[dr, phantom, "\Searrow\nu"] & B \arrow[l, "g"'] \arrow[d, "k"] \\
C & D \arrow[l, "g'"]
\end{tikzcd}$$

are the 2-morphisms

$$\begin{tikzcd}
A \arrow[d, "h"'] \arrow[dr, phantom, "\Searrow\omega^\dagger"] & B \arrow[l, "g"'] \arrow[d, "k"] \\
C & D \arrow[l, "g'"]
\end{tikzcd}
\qquad
\begin{aligned}
\omega^\dagger &\colon h \circ g \Longrightarrow g' \circ k,\\
\nu^\dagger &\colon f' \circ h \Longrightarrow k \circ f
\end{aligned}
\qquad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "h"'] & B \arrow[d, "k"] \\
C \arrow[r, "f'"'] \arrow[ur, phantom, "\Nearrow\nu^\dagger"] & D
\end{tikzcd}$$

defined as the pastings of the diagrams[13]

$$
\begin{tikzcd}
B \arrow[rrr, bend left, "k"] \arrow[dr, "g"] \arrow[drr, bend left, "\mathrm{id}_B"] \arrow[ddd, bend right, "g"'] & & & D \arrow[ddd, bend left, "g'"] \\
& A \arrow[r, "f"] \arrow[d, "h"] \arrow[u, Rightarrow, "\epsilon"] & B \arrow[d, "k"] \arrow[ur, Rightarrow, "\rho^{C}_{h\circ g}"'] & \\
& C \arrow[r, "f'"] \arrow[ur, Rightarrow, "\omega"] \arrow[drr, bend right, "\mathrm{id}_C"] & D \arrow[dr, "g'"] & \\
A \arrow[ur, Rightarrow, "(\lambda^{C}_{h\circ g})^{-1}"] \arrow[rrr, bend right, "h"'] & & \mathrm{id}_C \arrow[u, Rightarrow, "\eta'"] & C.
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[rrr, bend left, "f"] \arrow[dr, "f"] \arrow[ddr, bend right, "\mathrm{id}_A"] \arrow[ddd, bend right, "h"'] & & & B \arrow[ddd, bend left, "k"] \\
& B \arrow[r, "k"] \arrow[d, "g"] & D \arrow[d, "g'"] \arrow[ur, Rightarrow, "\lambda^{C}_{h\circ g}"] \arrow[ddr, bend left, "\mathrm{id}_D"] & \\
& A \arrow[r, "h"] \arrow[u, phantom, "\overset{\eta}{\Rightarrow}"] \arrow[ur, Rightarrow, "\nu"] & C \arrow[dr, "f'"] \arrow[r, phantom, "\overset{\epsilon'}{\Rightarrow}"] & \\
C \arrow[ur, Rightarrow, "(\rho^{C}_{h\circ g})^{-1}"'] \arrow[rrr, bend right, "f'"'] & & & D.
\end{tikzcd}
$$

Proposition 1.4.1.2. Let $\omega\colon f' \circ h \Longrightarrow k \circ f$ and $\nu\colon h \circ g \Longrightarrow g' \circ k$ be 2-morphisms.

1. *The Mate Correspondence.* The map

$$
\begin{aligned}
(-)^{\dagger}\colon \mathrm{Hom}_{\mathsf{Hom}_{C}(A,C)}(f' \circ h, k \circ f) &\longrightarrow \mathrm{Hom}_{\mathsf{Hom}_{C}(B,D)}(h \circ g, g' \circ k) \\
\omega &\longmapsto \omega^{\dagger}
\end{aligned}
$$

   is a bijection.

Proof. *Item 1, The Mate Correspondence*: Here we give a proof for 2-categories (which indirectly proves also the general case by Bicategories, **??**). A proof for general bicategories can be found in [JY21, Lemma 6.1.13].

[13] If C is a 2-category, these pasting diagrams become the following:

$$
\begin{tikzcd}
B \arrow[r, "g"] \arrow[d, "k"'] & A \arrow[d, "h"] \arrow[dl, Rightarrow, "\omega^{\dagger}"'] \\
D \arrow[r, "g'"'] & C
\end{tikzcd}
=
\begin{tikzcd}
B \arrow[r, "g"] \arrow[dr, "\mathrm{id}_B"'] & A \arrow[r, "h"] \arrow[d, "f"] \arrow[dl, Rightarrow, "\epsilon"'] & C \arrow[d, "f'"] \arrow[dl, Rightarrow, "\omega"'] \arrow[dr, "\mathrm{id}_C"] & \\
& B \arrow[r, "k"'] & D \arrow[r, "g'"'] & C.
\end{tikzcd}
$$

$$
\begin{tikzcd}
C \arrow[r, "f'"] & D \\
A \arrow[u, "h"] \arrow[r, "f"'] & B \arrow[u, "k"'] \arrow[from=1-1, Rightarrow, "\nu^{\dagger}"]
\end{tikzcd}
=
\begin{tikzcd}
& A \arrow[r, "h"] \arrow[dr, Rightarrow, "\nu"] & C \arrow[r, "f'"] \arrow[d, leftrightarrow, "g'"] & D \\
A \arrow[ur, "\mathrm{id}_A"] \arrow[r, "f"'] & B \arrow[u, "g"] \arrow[r, "k"'] & D. \arrow[ur, "\mathrm{id}_D"'] &
\end{tikzcd}
$$

Let

$$\nu\colon h\circ g \Longrightarrow g'\circ k \qquad \begin{tikzcd} A \arrow[d, "h"'] \arrow[dr, phantom, "\overset{\nu}{\searrow}"] & B \arrow[l, "g"'] \arrow[d, "k"] \\ C & D \arrow[l, "g'"] \end{tikzcd}$$

be a 2-morphism of $\mathcal{C}$. The mate $\nu^\dagger$ of ν is then given by

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h"'] & B \arrow[d, "k"] \\ C \arrow[r, "f'"'] \arrow[ur, phantom, "\overset{\nu^\dagger}{\nearrow}"] & D \end{tikzcd} = \begin{tikzcd} & A \arrow[dl, "\mathrm{id}_A"'] \arrow[d, "f"] \arrow[dl, phantom, shift left=3, "\overset{\eta}{\searrow}"] \\ A \arrow[d, "h"'] \arrow[dr, phantom, "\overset{\nu}{\searrow}"] & B \arrow[l, "g"'] \arrow[d, "k"] \\ C \arrow[d, "f'"'] \arrow[dr, phantom, shift right=3, "\overset{\epsilon'}{\searrow}"] & D \arrow[l, "g'"'] \arrow[dl, "\mathrm{id}_D"] \\ D, & \end{tikzcd}$$

and the mate of $\nu^\dagger$ is the 2-morphism $\left(\nu^\dagger\right)^\dagger\colon f'\circ h \Longrightarrow k\circ f$ given by

$$\begin{tikzcd} A \arrow[d, "h"'] \arrow[dr, phantom, "\overset{(\nu^\dagger)^\dagger}{\searrow}"] & B \arrow[l, "g"'] \arrow[d, "k"] \\ C & D. \arrow[l, "g'"] \end{tikzcd} = \begin{tikzcd} & & A \arrow[dl, "\mathrm{id}_A"'] \arrow[d, "f"] \arrow[dl, phantom, shift left=3, "\overset{\eta}{\searrow}"] & B \arrow[l, "g"'] \arrow[dl, "\mathrm{id}_B"] \arrow[dl, phantom, shift right=3, "\overset{\epsilon}{\searrow}"] \\ & A \arrow[d, "h"'] \arrow[dr, phantom, "\overset{\nu}{\searrow}"] & B \arrow[l, "g"'] \arrow[d, "k"] & \\ & C \arrow[dl, "\mathrm{id}_C"'] \arrow[d, "f'"] \arrow[dl, phantom, shift left=3, "\overset{\eta'}{\searrow}"] & D \arrow[l, "g'"'] \arrow[dl, "\mathrm{id}_D"] \arrow[dl, phantom, shift right=3, "\overset{\epsilon'}{\searrow}"] & \\ C & D \arrow[l, "g'"] & & \end{tikzcd} = \begin{tikzcd} & & A \arrow[dl, "\mathrm{id}_A"'] \arrow[d, phantom, "\Downarrow \mathrm{id}_g"] & B \arrow[l, "g"'] \arrow[dl, "\mathrm{id}_B"] \\ & A \arrow[d, "h"'] \arrow[dr, phantom, "\overset{\nu}{\searrow}"] & B \arrow[l, "g"'] \arrow[d, "k"] & \\ & C \arrow[dl, "\mathrm{id}_C"'] \arrow[d, phantom, "\Downarrow \mathrm{id}_{g'}"] & D \arrow[l, "g'"'] \arrow[dl, "\mathrm{id}_D"] & \\ C & D \arrow[l, "g'"] & & \end{tikzcd} = \begin{tikzcd} A \arrow[d, "h"'] \arrow[dr, phantom, "\overset{\nu}{\searrow}"] & B \arrow[l, "g"'] \arrow[d, "k"] \\ C & D. \arrow[l, "g'"] \end{tikzcd}$$

Similarly, $(\omega)^{\dagger\dagger} = \omega$. $\square$

2 Morphisms of Internal Adjunctions

2.1 Lax Morphisms of Internal Adjunctions

Let $\mathcal{C}$ be a bicategory and let $(A, B, F, G, \eta, \epsilon)$ and $(A', B', F', G', \eta', \epsilon')$ be internal adjunctions in $\mathcal{C}$.

Definition 2.1.1.1. A **lax morphism of internal adjunctions** from $(A, B, F, G, \eta, \epsilon)$ to $(A', B', F', G', \eta', \epsilon')$ is a lax transformation between these viewed as 2-functors from the walking adjunction.

Remark 2.1.1.2. In detail, a **lax morphism of internal adjunctions** from $(A, B, F, G, \eta, \epsilon)$ to $(A', B', F', G', \eta', \epsilon')$ consists of

- 1-*Morphisms.* A pair of 1-morphisms

$$
\begin{aligned}
\phi &\colon A \to A', \\
\psi &\colon B \to B'
\end{aligned}
$$

  of $\mathcal{C}$;

- 2-*Morphisms.* A pair of 2-morphisms

$$
\begin{tikzcd}
A \arrow[r, "F"] \arrow[d, "\phi"'] & B \arrow[d, "\psi"] \\
A' \arrow[r, "F'"'] \arrow[ur, phantom, "\overset{\alpha}{\Nearrow}"] & B'
\end{tikzcd}
\qquad
\begin{aligned}
\alpha &\colon F' \circ \phi \Rightarrow \psi \circ F, \\
\beta &\colon G' \circ \phi \Rightarrow \psi \circ G
\end{aligned}
\qquad
\begin{tikzcd}
A \arrow[d, "\phi"'] & B \arrow[l, "G"'] \arrow[d, "\psi"] \arrow[dl, phantom, "\overset{\beta}{\Nwarrow}"] \\
A' & B' \arrow[l, "G'"]
\end{tikzcd};
$$

  of $\mathcal{C}$;

satisfying the following conditions:

1. *Compatibility With Units.* We have an equality

$$
\begin{tikzcd}
 & B \arrow[dr, bend left, "G"] & \\
A \arrow[ur, bend left, "F"] \arrow[rr, "\mathrm{id}_A" description] \arrow[d, "\phi"'] \arrow[drr, "\phi" description] \arrow[r, phantom, "\lambda^{\mathcal{C}}_{\phi}\Nearrow"'] & \arrow[u, phantom, "\Uparrow\eta"] \arrow[r, phantom, "\Nearrow\rho^{\mathcal{C},-1}_{\phi}"'] & A \arrow[d, "\phi"] \\
A' \arrow[rr, bend right, "\mathrm{id}_{A'}"'] & & A'
\end{tikzcd}
=
\begin{tikzcd}
 & B \arrow[dr, bend left, "G"] \arrow[dd, "\psi"] & \\
A \arrow[ur, bend left, "F"] \arrow[ddd, "\phi"'] \arrow[dr, phantom, "\overset{\alpha}{\Nearrow}"] & & A \arrow[ddd, "\phi"] \arrow[dl, phantom, "\overset{\beta}{\Nearrow}"] \\
 & B' \arrow[ddr, "G'"] & \\
 & \arrow[u, phantom, "\Uparrow\eta'"] & \\
A' \arrow[uur, "F'"] \arrow[rr, bend right, "\mathrm{id}_{A'}"'] & & A'
\end{tikzcd}
$$

  of pasting diagrams in $\mathcal{C}$;

2. *Compatibility With Counits.* We have an equality

$$
\begin{tikzcd}
B \arrow[rr, bend left, "\mathrm{id}_B"] \arrow[dr, "G"'] \arrow[dd, "\psi"'] & & B \arrow[dd, "\psi"] \\
& A \arrow[ur, "F"'] \arrow[dd, "\phi"] \arrow[u, phantom, "\Uparrow\epsilon"] & \\
B' \arrow[dr, bend right, "G'"'] \arrow[ur, Rightarrow, "\beta"] & & B' \\
& A' \arrow[ur, bend right, "F'"'] \arrow[uuur, Rightarrow, "\alpha"'] &
\end{tikzcd}
=
\begin{tikzcd}
B \arrow[rr, bend left, "\mathrm{id}_B"] \arrow[dr, dash] \arrow[dd, "\psi"'] & & B \arrow[dd, "\psi"] \\
& \psi \arrow[dr] \arrow[ur, Rightarrow, "\rho^{C,-1}_{\psi}"] & \\
B' \arrow[rr, "\mathrm{id}_{B'}"] \arrow[dr, bend right, "G'"'] \arrow[ur, Rightarrow, "\lambda^{C}_{\psi}"] & & B' \\
& A' \arrow[ur, bend right, "F'"'] \arrow[u, phantom, "\epsilon'\Uparrow"] &
\end{tikzcd}
$$

of pasting diagrams in C.

2.2 Oplax Morphisms of Internal Adjunctions

Let C be a bicategory and let $(A, B, F, G, \eta, \epsilon)$ and $(A', B', F', G', \eta', \epsilon')$ be internal adjunctions in C.

Definition 2.2.1.1. An **oplax morphism of internal adjunctions** from $(A, B, F, G, \eta, \epsilon)$ to $(A', B', F', G', \eta', \epsilon')$ is an oplax transformation between these viewed as 2-functors from the walking adjunction.

Remark 2.2.1.2. In detail, an **oplax morphism of internal adjunctions** from $(A, B, F, G, \eta, \epsilon)$ to $(A', B', F', G', \eta', \epsilon')$ consists of

- 1-*Morphisms.* A pair of 1-morphisms

$$
\begin{aligned}
\phi &\colon A \to A', \\
\psi &\colon B \to B'
\end{aligned}
$$

  of C;

- 2-*Morphisms.* A pair of 2-morphisms

$$
\begin{tikzcd}
A \arrow[r, "F"] \arrow[d, "\phi"'] & B \arrow[d, "\psi"] \arrow[dl, Rightarrow, "\alpha"'] \\
A' \arrow[r, "F'"'] & B'
\end{tikzcd}
\qquad
\begin{aligned}
\alpha &\colon \psi \circ F \Rightarrow F' \circ \phi, \\
\beta &\colon \psi \circ G \Rightarrow G' \circ \phi
\end{aligned}
\qquad
\begin{tikzcd}
A \arrow[d, "\phi"'] \arrow[dr, Rightarrow, "\beta"] & B \arrow[l, "G"'] \arrow[d, "\psi"] \\
A' & B' \arrow[l, "G'"]
\end{tikzcd};
$$

  of C;

satisfying the following conditions:

1. *Compatibility With Units.* We have an equality

$$
\begin{tikzcd}
 & A \arrow[dr, bend left, "F"] \arrow[d, phantom, "\Downarrow\epsilon"] & \\
B \arrow[ur, bend left, "G"] \arrow[rr, "\mathrm{id}_B"] \arrow[dd, "\psi"'] \arrow[ddrr, "\psi" description] & & B \arrow[dd, "\psi"] \arrow[dl, phantom, "\overset{\rho^{C}_{\phi}}{\Swarrow}"] \\
 & \arrow[dl, phantom, "\overset{\lambda^{C,-1}_{\phi}}{\Swarrow}"] & \\
B' \arrow[rr, bend right, "\mathrm{id}_{B'}"'] & & B'
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & A \arrow[dr, bend left, "F"] \arrow[dd, "\phi"] & \\
B \arrow[ur, bend left, "G"] \arrow[dd, "\psi"'] \arrow[r, phantom, "\overset{\beta}{\Swarrow}"] & & B \arrow[dd, "\psi"] \arrow[l, phantom, "\overset{\alpha}{\Swarrow}"] \\
 & A' \arrow[dr, "F'"] \arrow[d, phantom, "\Downarrow\epsilon'"] & \\
B' \arrow[ur, "G'"] \arrow[rr, bend right, "\mathrm{id}_{B'}"'] & & B'
\end{tikzcd}
$$

   of pasting diagrams in $\mathcal{C}$;

2. *Compatibility With Counits.* We have an equality

$$
\begin{tikzcd}
A \arrow[rr, bend left, "\mathrm{id}_A"] \arrow[dr, "F"] \arrow[dd, "\phi"'] & \arrow[d, phantom, "\Downarrow\eta"] & A \arrow[dd, "\phi"] \\
 & B \arrow[ur, "G"] \arrow[dd, "\psi"] \arrow[dl, phantom, "\overset{\alpha}{\Swarrow}"] \arrow[r, phantom, "\overset{\beta}{\Swarrow}"] & \\
A' \arrow[dr, "F'"'] & & A' \\
 & B' \arrow[ur, "G'"'] &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A \arrow[rr, bend left, "\mathrm{id}_A"] \arrow[ddrr, "\phi" description] \arrow[dd, "\phi"'] & & A \arrow[dd, "\phi"] \arrow[dl, phantom, "\overset{\rho^{C}_{\psi}}{\Swarrow}"] \\
 & \arrow[dl, phantom, "\overset{\lambda^{C,-1}_{\psi}}{\Swarrow}"] & \\
A' \arrow[rr, "\mathrm{id}_{A'}"] \arrow[dr, "F'"'] & \arrow[d, phantom, "\Downarrow\eta'"] & A' \\
 & B' \arrow[ur, "G'"'] &
\end{tikzcd}
$$

   of pasting diagrams in $\mathcal{C}$.

2.3 Strong Morphisms of Internal Adjunctions

Let $\mathcal{C}$ be a bicategory and let $(A, B, F, G, \eta, \epsilon)$ and $(A', B', F', G', \eta', \epsilon')$ be internal adjunctions in $\mathcal{C}$.

Definition 2.3.1.1. A **strong morphism of internal adjunctions** from $(A, B, F, G, \eta, \epsilon)$ to $(A', B', F', G', \eta', \epsilon')$ is a strong transformation between these viewed as 2-functors from the walking adjunction.

Remark 2.3.1.2. In detail, a **strong morphism of internal adjunctions** from $(A, B, F, G, \eta, \epsilon)$ to $(A', B', F', G', \eta', \epsilon')$ is equivalently:

1. A lax morphism of internal adjunctions as in Remark 2.1.1.2 whose 2-morphisms are invertible.

2. An oplax morphism of internal adjunctions as in Remark 2.2.1.2 whose 2-morphisms are invertible.

2.4 Strict Morphisms of Internal Adjunctions

Let $\mathcal{C}$ be a bicategory and let $(A, B, F, G, \eta, \epsilon)$ and $(A', B', F', G', \eta', \epsilon')$ be internal adjunctions in $\mathcal{C}$.

Definition 2.4.1.1. A **strict morphism of internal adjunctions** from $(A, B, F, G, \eta, \epsilon)$ to $(A', B', F', G', \eta', \epsilon')$ is a strict transformation between these viewed as 2-functors from the walking adjunction.

Remark 2.4.1.2. In detail, a **strict morphism of internal adjunctions** from $(A, B, F, G, \eta, \epsilon)$ to $(A', B', F', G', \eta', \epsilon')$ is equivalently:

1. A lax morphism of internal adjunctions as in Remark 2.1.1.2 whose 2-morphisms are identities.

2. An oplax morphism of internal adjunctions as in Remark 2.2.1.2 whose 2-morphisms are identities.

3 2-Morphisms Between Morphisms of Internal Adjunctions

3.1 2-Morphisms Between Lax Morphisms of Internal Adjunctions

Let $\mathcal{C}$ be a bicategory, let $(A, B, F, G, \eta, \epsilon)$ and $(A', B', F', G', \eta', \epsilon')$ be internal adjunctions in $\mathcal{C}$, and let $(\phi_1, \psi_1, \alpha_1, \beta_1)$ and $(\phi_2, \psi_2, \alpha_2, \beta_2)$ be lax morphisms of internal adjunctions from $(A, B, F, G, \eta, \epsilon)$ to $(A', B', F', G', \eta', \epsilon')$.

Definition 3.1.1.1. A **2-morphism from** $(\phi_1, \psi_1, \alpha_1, \beta_1)$ **to** $(\phi_2, \psi_2, \alpha_2, \beta_2)$ is a modification between these viewed as lax transformations.

Remark 3.1.1.2. In detail, a **2-morphism from** $(\phi_1, \psi_1, \alpha_1, \beta_1)$ **to** $(\phi_2, \psi_2, \alpha_2, \beta_2)$ consist of 2-morphisms

$$\Gamma\colon \phi_1 \Rightarrow \phi_2$$
$$\Sigma\colon \psi_1 \Rightarrow \psi_2$$

of $\mathcal{C}$ such that we have equalities

$$
\begin{tikzcd}
A \arrow[r, "F"] \arrow[d, bend right, "\phi_1"'] \arrow[d, bend left, "\phi_2"] \arrow[d, phantom, "\overset{\Gamma}{\Rightarrow}"] & B \arrow[d, bend left, "\psi_2"] \\
A' \arrow[r, "F'"'] \arrow[ur, phantom, "\alpha_2\ \Nearrow"] & B'
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, "F"] \arrow[d, bend right, "\phi_1"'] & B \arrow[d, bend right, "\psi_1"'] \arrow[d, bend left, "\psi_2"] \arrow[d, phantom, "\overset{\Sigma}{\Rightarrow}"] \\
A' \arrow[r, "F'"'] \arrow[ur, phantom, "\alpha_1\ \Nearrow"] & B'
\end{tikzcd}
$$

$$
\begin{tikzcd}
B \arrow[r, "G"] \arrow[d, bend right, "\psi_1"'] \arrow[d, bend left, "\psi_2"] \arrow[d, phantom, "\overset{\Sigma}{\Rightarrow}"] & A \arrow[d, bend left, "\phi_2"] \\
B' \arrow[r, "G'"'] \arrow[ur, phantom, "\beta_2\ \Nearrow"] & A'
\end{tikzcd}
=
\begin{tikzcd}
B \arrow[r, "G"] \arrow[d, bend right, "\psi_1"'] & A \arrow[d, bend right, "\phi_1"'] \arrow[d, bend left, "\phi_2"] \arrow[d, phantom, "\overset{\Gamma}{\Rightarrow}"] \\
B' \arrow[r, "G'"'] \arrow[ur, phantom, "\beta_1\ \Nearrow"] & A'
\end{tikzcd}
$$

of pasting diagrams in $\mathcal{C}$.

3.2 2-Morphisms Between Oplax Morphisms of Internal Adjunctions

Let $\mathcal{C}$ be a bicategory, let $(A, B, F, G, \eta, \epsilon)$ and $(A', B', F', G', \eta', \epsilon')$ be internal adjunctions in $\mathcal{C}$, and let $(\phi_1, \psi_1, \alpha_1, \beta_1)$ and $(\phi_2, \psi_2, \alpha_2, \beta_2)$ be oplax morphisms of internal adjunctions from $(A, B, F, G, \eta, \epsilon)$ to $(A', B', F', G', \eta', \epsilon')$.

Definition 3.2.1.1. A **2-morphism from** $(\phi_1, \psi_1, \alpha_1, \beta_1)$ **to** $(\phi_2, \psi_2, \alpha_2, \beta_2)$ is a modification between these viewed as oplax transformations.

Remark 3.2.1.2. In detail, a **2-morphism from** $(\phi_1, \psi_1, \alpha_1, \beta_1)$ **to** $(\phi_2, \psi_2, \alpha_2, \beta_2)$ consist of 2-morphisms

$$\Gamma \colon \phi_1 \Rightarrow \phi_2$$
$$\Sigma \colon \psi_1 \Rightarrow \psi_2$$

of C such that we have equalities

$$
\begin{tikzcd}
A \arrow[r, "F"] \arrow[d, bend right, "\phi_2"'] \arrow[d, bend left, "\phi_1"] \arrow[d, phantom, "\overset{\Gamma}{\Leftarrow}"] \arrow[dr, phantom, "\swarrow \alpha_1"] & B \arrow[d, bend left, "\psi_1"] \\
A' \arrow[r, "F'"'] & B'
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, "F"] \arrow[d, bend right, "\phi_2"'] \arrow[dr, phantom, "\swarrow \alpha_2"] & B \arrow[d, bend right, "\psi_2"'] \arrow[d, bend left, "\psi_1"] \arrow[d, phantom, "\overset{\Sigma}{\Leftarrow}"] \\
A' \arrow[r, "F'"'] & B'
\end{tikzcd}
$$

$$
\begin{tikzcd}
B \arrow[r, "G"] \arrow[d, bend right, "\psi_2"'] \arrow[d, bend left, "\psi_1"] \arrow[d, phantom, "\overset{\Sigma}{\Leftarrow}"] \arrow[dr, phantom, "\swarrow \beta_1"] & A \arrow[d, bend left, "\phi_1"] \\
B' \arrow[r, "G'"'] & A'
\end{tikzcd}
=
\begin{tikzcd}
B \arrow[r, "G"] \arrow[d, bend right, "\psi_2"'] \arrow[dr, phantom, "\swarrow \beta_2"] & A \arrow[d, bend right, "\phi_2"'] \arrow[d, bend left, "\phi_1"] \arrow[d, phantom, "\overset{\Gamma}{\Leftarrow}"] \\
B' \arrow[r, "G'"'] & A'
\end{tikzcd}
$$

of pasting diagrams in C.

3.3 2-Morphisms Between Strong Morphisms of Internal Adjunctions

Let C be a bicategory, let $(A, B, F, G, \eta, \epsilon)$ and $(A', B', F', G', \eta', \epsilon')$ be internal adjunctions in C, and let $(\phi_1, \psi_1, \alpha_1, \beta_1)$ and $(\phi_2, \psi_2, \alpha_2, \beta_2)$ be strong morphisms of internal adjunctions from $(A, B, F, G, \eta, \epsilon)$ to $(A', B', F', G', \eta', \epsilon')$.

Definition 3.3.1.1. A **2-morphism from** $(\phi_1, \psi_1, \alpha_1, \beta_1)$ **to** $(\phi_2, \psi_2, \alpha_2, \beta_2)$ is a modification between these viewed as strong transformations.

Remark 3.3.1.2. In detail, a **2-morphism from** $(\phi_1, \psi_1, \alpha_1, \beta_1)$ **to** $(\phi_2, \psi_2, \alpha_2, \beta_2)$ is equivalently:

- A 2-morphism (Γ, Σ) from $(\phi_1, \psi_1, \alpha_1, \beta_1)$ to $(\phi_2, \psi_2, \alpha_2, \beta_2)$ viewed as lax transformations as in Remark 3.1.1.2.
- A 2-morphism (Γ, Σ) from $(\phi_1, \psi_1, \alpha_1, \beta_1)$ to $(\phi_2, \psi_2, \alpha_2, \beta_2)$ viewed as oplax transformations as in Remark 3.2.1.2.

3.4 2-Morphisms Between Strict Morphisms of Internal Adjunctions

Let C be a bicategory, let $(A, B, F, G, \eta, \epsilon)$ and $(A', B', F', G', \eta', \epsilon')$ be internal adjunctions in C, and let $(\phi_1, \psi_1, \alpha_1, \beta_1)$ and $(\phi_2, \psi_2, \alpha_2, \beta_2)$ be lax morphisms of internal adjunctions from $(A, B, F, G, \eta, \epsilon)$ to $(A', B', F', G', \eta', \epsilon')$.

Definition 3.4.1.1. A **2-morphism from** $(\phi_1, \psi_1, \alpha_1, \beta_1)$ **to** $(\phi_2, \psi_2, \alpha_2, \beta_2)$ is a modification between these viewed as strict transformations.

Remark 3.4.1.2. In detail, a **2-morphism from** $(\phi_1, \psi_1, \alpha_1, \beta_1)$ **to** $(\phi_2, \psi_2, \alpha_2, \beta_2)$ is equivalently:

- A 2-morphism (Γ, Σ) from $(\phi_1, \psi_1, \alpha_1, \beta_1)$ to $(\phi_2, \psi_2, \alpha_2, \beta_2)$ viewed as lax transformations as in Remark 3.1.1.2.
- A 2-morphism (Γ, Σ) from $(\phi_1, \psi_1, \alpha_1, \beta_1)$ to $(\phi_2, \psi_2, \alpha_2, \beta_2)$ viewed as oplax transformations as in Remark 3.2.1.2.
- A 2-morphism (Γ, Σ) from $(\phi_1, \psi_1, \alpha_1, \beta_1)$ to $(\phi_2, \psi_2, \alpha_2, \beta_2)$ viewed as strong transformations as in Remark 3.3.1.2.

4 Bicategories of Internal Adjunctions in a Bicategory

Appendices

A Other Chapters