Spans

December 3, 2023

00QB This chapter contains some material about spans. Notably, we discuss and explore:

1. The basic definitions around spans (Section 1);
2. The relation between spans and functions (Proposition 7.1.1);
3. The relation between spans and relations (Propositions 7.2.4 and 7.3.1 and Remark 7.5.1).
4. "Hyperpointed sets" (**??**). I don't know why I wrote this...

TODO:

1. internal adjoint equivalences in **Rel**
2. internal adjoint equivalences in Span
3. 2-categorical limits in **Rel**;
4. morphism of internal adjunctions in **Rel**;
5. morphism of internal adjunctions in Span;
6. morphism of co/monads in Span;
7. What is $\mathrm{Adj}(\mathrm{Span}(A, B))$?
8. monoids, comonoids, pseudomonoids, etc. in Span.
9. write down the dumb intuition about spans inducing morphisms $\mathrm{Sets}(S, A) \to \mathrm{Sets}(S, B)$ instead of $\mathcal{P}(A) \to \mathcal{P}(B)$ from the similarity between
$$S \to A \times B$$
and
$$A \times B \to \{\mathrm{t}, \mathrm{f}\}.$$
This intuition is justified by taking $A = \mathrm{pt}$ or $B = \mathrm{pt}$.

10. What about using the direct image with compact support in $g\big(f^{-1}(a)\big)$?
11. Monads in Span | develop this in the level of morphisms too
12. Comonads in Span are spans whose legs are equal | develop this in the level of morphisms too
13. Does Span have an internal **Hom**?
14. Examples of spans
15. Functional and total spans
16. closed symmetric monoidal category of spans
17. double category of relations
18. collage of a span
19. equivalence spans?
20. functoriality of powersets for spans
21. Is Span a closed bicategory?
22. skew monoidal structure on Span(A, B)
23. Adjunctions in Span
24. Isomorphisms in Span
25. Equivalences in Span
26. Interaction between the above notions in Span vs.in **Rel** via the comparison functors

Contents

00QC 1 Spans

00QD 1.1 The Walking Span

00QE **Definition 1.1.1 ► The Walking Span**

The **walking span** is the category Λ that looks like this:

$$\begin{tikzcd}[row sep=small, column sep=small] & {[0]} \arrow[loop above] \arrow[dl] \arrow[dr] & \\ {[-1]} \arrow[loop left] & & {[1]} \arrow[loop right] \end{tikzcd}$$

00QF 1.2 Spans

Let A and B be sets.

00QG **Definition 1.2.1 ► Spans**

A **span from A to B**[1] is a functor $F\colon \Lambda \longrightarrow \mathsf{Sets}$ such that

$$\begin{aligned} F([-1]) &= A, \\ F([1]) &= B. \end{aligned}$$

[1] *Further Terminology:* Also called a **roof from A to B** or a **correspondence from A to B**.

00QH **Remark 1.2.2 ► Unwinding Definition 1.2.1**

In detail, a **span from A to B** is a triple (S, f, g) consisting of[1,2]

- *The Underlying Set.* A set S, called the **underlying set of** (S, f, g);
- *The Legs.* A pair of functions $f\colon S \longrightarrow A$ and $g\colon S \longrightarrow B$.

[1] *Picture:*

$$\begin{tikzcd}[row sep=small, column sep=small] & S \arrow[dl, "f"'] \arrow[dr, "g"] & \\ A & & B. \end{tikzcd}$$

[2] Every span (S, f, g) from A to B determines in particular a relation $R\colon A \nrightarrow B$ via

$$R \overset{\text{def}}{=} \{(f(a), g(a)) \mid a \in A\},$$

i.e. where $R(a) = g\big(f^{-1}(a)\big)$ for each $a \in A$; see Proposition 7.2.4.

00QJ 1.3 Morphisms of Spans

00QK **Definition 1.3.1 ► Morphisms of Spans**

A **morphism of spans** (R, f_1, g_1) **to** (S, f_2, g_2)[1] is a natural transformation $(R, f_1, g_1) \Longrightarrow (S, f_2, g_2)$.

[1] *Further Terminology:* Also called a **morphism of roofs from** (R, f_1, g_1) **to** (S, f_2, g_2) or a **morphism of correspondences from** (R, f_1, g_1) **to** (S, f_2, g_2).

00QL **Remark 1.3.2 ► Unwinding Definition 1.3.1**

In detail, a **morphism of spans from** (R, f_1, g_1) **to** (S, f_2, g_2) is a function $\phi\colon R \longrightarrow S$ making the diagram[1]

$$\begin{tikzcd}
 & R \arrow[dl, "f_1"'] \arrow[dr, "g_1"] \arrow[rrr, "\phi"] & & & S \arrow[dl, "f_2"'] \arrow[dr, "g_2"] & \\
A \arrow[rrr, equal] & & B \arrow[rrr, equal] & A & & B.
\end{tikzcd}$$

commute.

[1] *Alternative Picture:*

$$\begin{tikzcd}
 & R \arrow[dl, "f_1"'] \arrow[dr, "g_1"] \arrow[dd, "\phi"] & \\
A & & B. \\
 & S \arrow[ul, "f_2"] \arrow[ur, "g_2"'] &
\end{tikzcd}$$

00QM 1.4 Functional Spans

Let $\lambda = \left(A \xleftarrow{f} S \xrightarrow{g} B\right)$ be a span. A morphism of spans from id_A to $\lambda \diamond \lambda^\dagger$ is a morphism

$$s\colon A \longrightarrow S \times_B S$$

making the diagram

$$\begin{tikzcd} & A \arrow[dl, equal, "\mathrm{id}_A"'] \arrow[dr, equal, "\mathrm{id}_A"] \arrow[d, "s"] & \\ A & {} \arrow[d] & A \\ & S \times_B S \arrow[ul, "f \circ \mathrm{pr}'_1"] \arrow[ur, "f \circ \mathrm{pr}'_2"'] & \end{tikzcd}$$

commute, where $S \times_B S$ is the pullback

$$S \times_B S \cong \{(s, t) \in S \times S \mid g(s) = g(t)\} \qquad \begin{tikzcd} S \times_B S \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & S \arrow[d, "g"] \\ S \arrow[r, "g"'] & B \end{tikzcd}$$

of S with itself along g.

00QN
1.5 Total Spans

00QP
2 Categories of Spans

00QQ
2.1 Categories of Spans

Let A and B be sets.

00QR
Definition 2.1.1 ► The Category of Spans From A to B

The **category of spans from A to B** is the category $\mathrm{Span}(A, B)$ defined by

$$\mathrm{Span}(A, B) \overset{\mathrm{def}}{=} \mathrm{Fun}(\Lambda, \mathrm{Sets}) \underset{\mathrm{ev}_{[-1]}, \mathrm{Sets}, [A]}{\times} \mathrm{pt} \underset{[B], \mathrm{Sets}, \mathrm{ev}_{[1]}}{\times} \mathrm{Fun}(\Lambda, \mathrm{Sets}),$$

as in the diagram

$$\begin{tikzcd}[column sep=small]
 & & \mathsf{Span}(A,B) \arrow[dl] \arrow[dr] \arrow[dd, phantom, very near start, "\lrcorner"] & & \\
 & \mathsf{Fun}(\Lambda,\mathsf{Sets}) \underset{\mathsf{Sets}}{\times} \mathsf{pt} \arrow[dl] \arrow[dr] \arrow[dd, phantom, very near start, "\lrcorner"] & & \mathsf{pt} \underset{\mathsf{Sets}}{\times} \mathsf{Fun}(\Lambda,\mathsf{Sets}) \arrow[dl] \arrow[dr] \arrow[dd, phantom, very near start, "\lrcorner"] & \\
\mathsf{Fun}(\Lambda,\mathsf{Sets}) \arrow[dr, "\mathsf{ev}_{[-1]}"'] & & \mathsf{pt} \arrow[dl, "{[A]}"] \arrow[dr, "{[B]}"'] & & \mathsf{Fun}(\Lambda,\mathsf{Sets}). \arrow[dl, "\mathsf{ev}_{[1]}"] \\
 & \mathsf{Sets} & & \mathsf{Sets} &
\end{tikzcd}$$

00QS **Remark 2.1.2 ► Unwinding Definition 2.1.1**

In detail, the **category of spans from A to B** is the category $\mathsf{Span}(A,B)$ where

- *Objects.* The objects of $\mathsf{Span}(A,B)$ are spans from A to B;
- *Morphisms.* The morphism of $\mathsf{Span}(A,B)$ are morphisms of spans;
- *Identities.* The unit map

  $$\mathbb{1}^{\mathsf{Span}(A,B)}_{(S,f,g)} \colon \mathsf{pt} \longrightarrow \mathsf{Hom}_{\mathsf{Span}(A,B)}((S,f,g),(S,f,g))$$

  of $\mathsf{Span}(A,B)$ at (S,f,g) is defined by[1]

  $$\mathrm{id}^{\mathsf{Span}(A,B)}_{(S,f,g)} \overset{\mathrm{def}}{=} \mathrm{id}_S;$$

- *Composition.* The composition map

  $$\circ^{\mathsf{Span}(A,B)}_{R,S,T} \colon \mathsf{Hom}_{\mathsf{Span}(A,B)}(S,T) \times \mathsf{Hom}_{\mathsf{Span}(A,B)}(R,S) \longrightarrow \mathsf{Hom}_{\mathsf{Span}(A,B)}(R,T)$$

  of $\mathsf{Span}(A,B)$ at $((R,f_1,g_1),(S,f_2,g_2),(T,f_3,g_3))$ is defined by[2]

  $$\psi \circ^{\mathsf{Span}(A,B)}_{R,S,T} \phi \overset{\mathrm{def}}{=} \psi \circ \phi.$$

[1] *Picture:*

$$\begin{tikzcd}
& S \arrow[ld, "f"'] \arrow[rd, "g"] \arrow[rrr, "\mathrm{id}_S"] & & & S \arrow[ld, "f"'] \arrow[rd, "g"] & \\
A \arrow[rrr, equal] & & B \arrow[rrr, equal] & A & & B.
\end{tikzcd}$$

[2] *Picture:*

$$\begin{tikzcd}
& R \arrow[ld, "f_1"'] \arrow[rd, "g_1"] \arrow[rrr, "\phi"] & & & S \arrow[ld, "f_2"'] \arrow[rd, "g_2"] \arrow[rrr, "\psi"] & & & T \arrow[ld, "f_3"'] \arrow[rd, "g_3"] & \\
A \arrow[rrr, equal] & & B \arrow[rrr, equal] & A \arrow[rrr, equal] & & B \arrow[rrr, equal] & A & & B.
\end{tikzcd}$$

00QT 2.2 The Bicategory of Spans

00QU **Definition 2.2.1 ► The Bicategory of Spans**

The **bicategory of spans** is the bicategory Span where

- *Objects.* The objects of Span are sets;
- Hom-*Categories.* For each $A, B \in \mathrm{Obj}(\mathsf{Span})$, we have

$$\mathrm{Hom}_{\mathsf{Span}}(A, B) \stackrel{\text{def}}{=} \mathsf{Span}(A, B);$$

- *Identities.* For each $A \in \mathrm{Obj}(\mathsf{Span})$, the unit functor

$$\mathbb{1}^{\mathsf{Span}}_A \colon \mathrm{pt} \to \mathsf{Span}(A, A)$$

of Span at A is the functor picking the span $(A, \mathrm{id}_A, \mathrm{id}_A)$:

$$\begin{tikzcd} & A \arrow[dl, "\mathrm{id}_A"'] \arrow[dr, "\mathrm{id}_A"] & \\ A & & A. \end{tikzcd}$$

- *Composition.* For each $A, B, C \in \mathrm{Obj}(\mathsf{Span})$, the composition bifunctor

$$\circ^{\mathsf{Span}}_{A,B,C} \colon \mathsf{Span}(B, C) \times \mathsf{Span}(A, B) \to \mathsf{Span}(A, C)$$

of Span at (A, B, C) is the bifunctor where

 – *Action on Objects.* The composition of two spans

$$\begin{tikzcd} & R \arrow[dl, "f_1"'] \arrow[dr, "g_1"] & \\ A & & B \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & S \arrow[dl, "f_2"'] \arrow[dr, "g_2"] & \\ B & & C \end{tikzcd}$$

 is the span $(R \times_B S, f_1 \circ \mathrm{pr}_1, g_2 \circ \mathrm{pr}_2)$, constructed as in the diagram

$$\begin{tikzcd} & & R \times_B S \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] \arrow[ddll, bend right, "f_1 \circ \mathrm{pr}_1"'] \arrow[ddrr, bend left, "g_2 \circ \mathrm{pr}_2"] & & \\ & R \arrow[dl, "f_1"'] \arrow[dr, "g_1"] & & S \arrow[dl, "f_2"'] \arrow[dr, "g_2"] & \\ A & & B & & C. \end{tikzcd}$$

 – *Action on Morphisms.* The horizontal composition of 2-morphisms is defined via functoriality of pullbacks: given morphisms of spans

$$\begin{tikzcd} & R \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[dd, "\phi"] & \\ A & & B \\ & R' \arrow[ul, "f'"] \arrow[ur, "g'"'] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & S \arrow[dl, "h"'] \arrow[dr, "k"] \arrow[dd, "\psi"] & \\ B & & C, \\ & S' \arrow[ul, "h'"] \arrow[ur, "k'"'] & \end{tikzcd}$$

their horizontal composition is the morphism of spans

$$\begin{tikzcd}
& R \times_B S \arrow[dl, "f \circ \mathrm{pr}_1"'] \arrow[dr, "k \circ \mathrm{pr}_2"] \arrow[dd, dashed, "\exists !"] & \\
A & & C, \\
& R' \times_B S' \arrow[ul, "h' \circ \mathrm{pr}'_1"] \arrow[ur, "k' \circ \mathrm{pr}'_2"'] &
\end{tikzcd}$$

constructed as in the diagram

$$\begin{tikzcd}
& & R \times_B S \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] \arrow[ddll, bend right, "f \circ \mathrm{pr}_1"'] \arrow[ddrr, bend left, "k \circ \mathrm{pr}_2"] \arrow[dddd, dashed, "\exists !"'] & & \\
& R \arrow[dl, "f"'] \arrow[dr, "g"] & & S \arrow[dl, "h"'] \arrow[dr, "k"] & \\
A & & B & & C; \\
& R' \arrow[ul, "f'"] \arrow[ur, "g'"'] & & S' \arrow[ul, "h'"] \arrow[ur, "k'"'] & \\
& & R' \times_B S' \arrow[ul, "\mathrm{pr}'_1"] \arrow[ur, "\mathrm{pr}'_2"'] \arrow[uull, bend left, "f' \circ \mathrm{pr}'_1"] \arrow[uurr, bend right, "k' \circ \mathrm{pr}'_2"'] & &
\end{tikzcd}$$

- *Associators and Unitors.* The associator and unitors are defined using the universal property of the pullback.

00QV 2.3 The Monoidal Bicategory of Spans

00QW 2.4 The Double Category of Spans

00QX **Definition 2.4.1 ► The Double Category of Spans**

The **double category of spans** is the double category $\mathsf{Span}^{\mathsf{dbl}}$ where

- *Objects.* The objects of $\mathsf{Span}^{\mathsf{dbl}}$ are sets;
- *Vertical Morphisms.* The vertical morphisms of $\mathsf{Span}^{\mathsf{dbl}}$ are functions $f\colon A \longrightarrow B$;
- *Horizontal Morphisms.* The horizontal morphisms of $\mathsf{Span}^{\mathsf{dbl}}$ are spans $(S, \phi, \psi)\colon A \nrightarrow X$;
- *2-Morphisms.* A 2-cell

$$\begin{tikzcd}
A \arrow[r, "{(R,\phi_R,\psi_R)}", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & B \arrow[d, "g"] \\
X \arrow[r, "{(S,\phi_S,\psi_S)}"', "\shortmid"{marking}] & Y
\end{tikzcd}$$

of $\mathsf{Span}^{\mathsf{dbl}}$ is a morphism of spans from the span

$$\begin{tikzcd}
 & & R \arrow[dll, "\phi_R"'] \arrow[dr, "\psi_R"] & & \\
 & & & B \arrow[dr, "g"] & \\
A & & & & Y
\end{tikzcd}$$

to the span

$$\begin{tikzcd}
 & & A \times_X S \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner", very near start, rotate=-45] & & \\
 & A \arrow[dl, "f"'] \arrow[dr, "f"] & & S \arrow[dl, "\phi_S"'] \arrow[dr, "\psi_S"] & \\
X & & X & & Y;
\end{tikzcd}$$

- *Horizontal Identities.* The horizontal unit functor

$$\mathbb{1}^{\mathsf{Span}^{\mathsf{dbl}}} \colon \left(\mathsf{Span}^{\mathsf{dbl}}\right)_0 \to \left(\mathsf{Span}^{\mathsf{dbl}}\right)_1$$

of $\mathsf{Span}^{\mathsf{dbl}}$ is the functor where

  - *Action on Objects.* For each $A \in \mathrm{Obj}\left(\left(\mathsf{Span}^{\mathsf{dbl}}\right)_0\right)$, we have

$$\mathbb{1}_A \overset{\text{def}}{=} (A, \mathrm{id}_A, \mathrm{id}_A),$$

    as in the diagram

$$\begin{tikzcd} & A \arrow[dl, equals, "\mathrm{id}_A"'] \arrow[dr, equals, "\mathrm{id}_A"] & \\ A & & A; \end{tikzcd}$$

  - *Action on Morphisms.* For each vertical morphism $f \colon A \to B$ of $\mathsf{Span}^{\mathsf{dbl}}$, i.e. each map of sets f from A to B, the identity 2-morphism

$$\begin{tikzcd} A \arrow[r, "\mathbb{1}_A", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow \mathbb{1}_f"] & A \arrow[d, "f"] \\ B \arrow[r, "\mathbb{1}_B"', "\shortmid"{marking}] & B \end{tikzcd}$$

    of f is the morphism of spans from

$$\begin{tikzcd} & & A \arrow[dll, equals, "\mathrm{id}_A"'] \arrow[dr, equals, "\mathrm{id}_A"] & & \\ & & & A \arrow[dr, "f"] & \\ A & & & & B \end{tikzcd}$$

to

$$\begin{tikzcd}
 & & A \times_B B \arrow[dl] \arrow[dr] \arrow[d, phantom, very near start, "\lrcorner" rotate=-45] & & \\
 & A \arrow[dl, "f"'] \arrow[dr, "f"] & & B \arrow[dl, equal, "\mathrm{id}_B"'] \arrow[dr, equal, "\mathrm{id}_B"] & \\
B & & B & & B
\end{tikzcd}$$

given by the isomorphism $A \xrightarrow{\cong} A \times_B B$;

- *Vertical Identities.* For each $A \in \mathrm{Obj}\left(\mathsf{Span}^{\mathsf{dbl}}\right)$, we have

$$\mathrm{id}_A^{\mathsf{Span}^{\mathsf{dbl}}} \stackrel{\text{def}}{=} \mathrm{id}_A;$$

- *Identity 2-Morphisms.* For each horizontal morphism $R\colon A \nrightarrow B$ of $\mathsf{Span}^{\mathsf{dbl}}$, the identity 2-morphism

$$\begin{tikzcd}
A \arrow[r, "S", "\shortmid" marking] \arrow[d, "\mathrm{id}_A"'] \arrow[dr, phantom, "\Downarrow \mathrm{id}_S"] & B \arrow[d, "\mathrm{id}_B"] \\
A \arrow[r, "S"', "\shortmid" marking] & B
\end{tikzcd}$$

of R is the morphism of spans from

$$\begin{tikzcd}
 & S \arrow[dl, "\phi_S"'] \arrow[dr, "\psi_S"] & & \\
 & & B \arrow[dr, equal, "\mathrm{id}_B"] & \\
A & & & B
\end{tikzcd}$$

to

$$
\begin{tikzcd}
 & & A \times_A S \arrow[dl] \arrow[dr] \arrow[d, phantom, "\lrcorner" rotate=-45] & & \\
 & A \arrow[dl, equal, "\mathrm{id}_A"'] \arrow[dr, equal, "\mathrm{id}_A"] & & S \arrow[dl, "\phi_S"'] \arrow[dr, "\psi_S"] & \\
A & & A & & B
\end{tikzcd}
$$

given by the isomorphism $S \xrightarrow{\cong} A \times_A S$;

- *Horizontal Composition.* The horizontal composition functor

$$\odot^{\mathsf{Span}^{\mathsf{dbl}}} \colon \left(\mathsf{Span}^{\mathsf{dbl}}\right)_1 \times_{\left(\mathsf{Span}^{\mathsf{dbl}}\right)_0} \left(\mathsf{Span}^{\mathsf{dbl}}\right)_1 \to \left(\mathsf{Span}^{\mathsf{dbl}}\right)_1$$

of $\mathsf{Span}^{\mathsf{dbl}}$ is the functor where

  - *Action on Objects.* For each composable pair

$$A \overset{(R,\phi_R,\psi_R)}{\nrightarrow} B \overset{(S,\phi_S,\psi_S)}{\nrightarrow} C$$

of horizontal morphisms of $\mathsf{Span}^{\mathsf{dbl}}$, we have

$$(S, \phi_S, \psi_S) \odot (R, \phi_R, \psi_R) \stackrel{\text{def}}{=} S \circ^{\mathsf{Span}}_{A,B,C} R,$$

where $S \circ^{\mathsf{Span}}_{A,B,C} R$ is the composition of (R, ϕ_R, ψ_R) and (S, ϕ_S, ψ_S) defined as in Definition 2.2.1;

  - *Action on Morphisms.* For each horizontally composable pair

$$
\begin{tikzcd}
A \arrow[r, "{(R,\phi_R,\psi_R)}", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & B \arrow[d, "g"] \\
X \arrow[r, "{(T,\phi_T,\psi_T)}"', "\shortmid"{marking}] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r, "{(S,\phi_S,\psi_S)}", "\shortmid"{marking}] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\beta"] & C \arrow[d, "h"] \\
Y \arrow[r, "{(U,\phi_U,\psi_U)}"', "\shortmid"{marking}] & Z
\end{tikzcd}
$$

of 2-morphisms of $\mathsf{Span}^{\mathsf{dbl}}$,

- *Vertical Composition of 1-Morphisms.* For each composable pair $A \xrightarrow{F} B \xrightarrow{G} C$ of vertical morphisms of $\mathsf{Span}^{\mathsf{dbl}}$, i.e. maps of sets, we have

$$g \circ^{\mathsf{Span}^{\mathsf{dbl}}} f \overset{\text{def}}{=} g \circ f;$$

- *Vertical Composition of 2-Morphisms.* For each vertically composable pair

$$\begin{tikzcd} A \arrow[r, "{(R,\phi_R,\psi_R)}", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & X \arrow[d, "g"] \\ B \arrow[r, "{(S,\phi_S,\psi_S)}"', "\shortmid"{marking}] & Y \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "{(S,\phi_S,\psi_S)}", "\shortmid"{marking}] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow\beta"] & Y \arrow[d, "k"] \\ C \arrow[r, "{(T,\phi_T,\psi_T)}"', "\shortmid"{marking}] & Z \end{tikzcd}$$

of 2-morphisms of $\mathsf{Span}^{\mathsf{dbl}}$,

- *Associators and Unitors.* The associator and unitors of $\mathsf{Span}^{\mathsf{dbl}}$ are defined using the universal property of the pullback.

00QY 2.5 Properties of The Bicategory of Spans

00QZ **Proposition 2.5.1 ► Properties of the Bicategory of Spans**

Let $\lambda = \left(A \overset{f}{\leftarrow} S \overset{g}{\rightarrow} B\right)$ be a span.

00R0 1. *Self-Duality.*

00R1 2. *Isomorphisms in* Span.

00R2 3. *Equivalences in* Span.

00R3 4. *Adjunctions in* Span. Let A and B be sets.[1]

00R4 (a) We have a natural bijection

$$\left\{\begin{array}{c}\text{Adjunctions in Span}\\ \text{from } A \text{ to } B\end{array}\right\} \cong \left\{\begin{array}{c}\text{Spans } A \overset{f}{\leftarrow} S \overset{g}{\rightarrow} B\\ \text{from } A \text{ to } B \text{ with}\\ f \text{ an isomorphism}\end{array}\right\}.$$

00R5 (b) We have an equivalence of categories

$$\mathsf{MapSpan}(A,B) \overset{\text{eq.}}{\cong} \mathsf{Sets}(A,B)_{\mathsf{disc}},$$

where $\mathsf{MapSpan}(A,B)$ is the full subcategory of $\mathsf{Span}(A,B)$ spanned by the spans $A \xleftarrow{f} S \xrightarrow{g} B$ from A to B with f an isomorphism.

00R6 (c) We have a biequivalence of bicategories

$$\mathsf{MapSpan} \overset{\text{eq.}}{\cong} \mathsf{Sets}_{\mathsf{bidisc}},$$

where $\mathsf{MapSpan}$ is the sub-bicategory of Span whose Hom-categories are given by $\mathsf{MapSpan}(A,B)$.

00R7 5. *Monads in* Span.

00R8 6. *Comonads in* Span.

00R9 7. *Monomorphisms in* Span.

00RA 8. *Epimorphisms in* Span.

00RB 9. *Existence of Right Kan Extensions.*

00RC 10. *Existence of Right Kan Lifts.*

00RD 11. *Closedness.*

[1] In the literature (e.g. [**ref**]),...are called maps and denoted by $\mathsf{MapSpan}(A,B)$

PROOF 2.5.2 ► PROOF OF PROPOSITION 2.5.1

Item 1: Self-Duality

Item 2: Isomorphisms in Span

Item 3: Equivalences in Span

Item 4: Adjunctions in Span

We first prove Item 4a.

We proceed step by step:

1. *From Adjunctions in* Span *to Functions.* An adjunction in Span from A to B consists of a pair of spans

$$\begin{tikzcd} & R \arrow[dl, "f"'] \arrow[dr, "g"] & \\ A & & B \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & S' \arrow[dl, "h"'] \arrow[dr, "k"] & \\ B & & A \end{tikzcd}$$

together with maps

$$\begin{tikzcd} & A \arrow[dl, equal, "\mathrm{id}_A"'] \arrow[dr, equal, "\mathrm{id}_A"] \arrow[dd, "\phi"] & \\ A & & A \\ & S \times_B S' \arrow[ul, "f \circ \mathrm{pr}'_1"] \arrow[ur, "k \circ \mathrm{pr}'_2"'] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & S' \times_A S \arrow[dl, "h \circ \mathrm{pr}_1"'] \arrow[dr, "g \circ \mathrm{pr}_2"] \arrow[dd, "\psi"] & \\ B \arrow[dr, equal, "\mathrm{id}_B"'] & & B. \arrow[dl, equal, "\mathrm{id}_B"] \\ & B & \end{tikzcd}$$

We claim that these conditions

2. *From Functions to Adjunctions in* ***Rel****.*

3. *Invertibility: From Functions to Adjunctions Back to Functions.*

4. *Invertibility: From Adjunctions to Functions Back to Adjunctions.*

We now proceed to the proof of Item 4b. For this, we will construct a functor

$$F\colon \mathsf{Sets}(A, B)_{\mathsf{disc}} \longrightarrow \mathsf{MapSpan}(A, B)$$

and prove it to be essentially surjective and fully faithful, and thus an equivalence by Categories, **??** of **??**. Indeed, given a map $f\colon A \longrightarrow B$, let $F(f)$ be the representable span associated to f of Definition 5.1.1, and let F send the unique (identity) morphism from f to itself to the identity morphism of $F(f)$ in $\mathsf{MapSpan}(A, B)$. We now prove that F is fully faithful and essentially surjective:

1. *F Is Fully Faithful:* Given maps $f, g \colon A \rightrightarrows B$, we need to show that

$$\mathrm{Hom}_{\mathsf{MapSpan}(A,B)}(F(f), F(g)) = \begin{cases} \mathrm{pt} & \text{if } f = g, \\ \emptyset & \text{otherwise.} \end{cases}$$

Indeed, a morphism from $F(f)$ to $F(g)$ takes the form

$$\begin{tikzcd} & A \arrow[dl, equal] \arrow[dr, "f"] \arrow[dd, "\phi"] & \\ A & & B. \\ & A \arrow[ul, equal] \arrow[ur, "g"'] & \end{tikzcd}$$

From the relations $\mathrm{id}_A = \mathrm{id}_A \circ \phi$ and $f = g \circ \phi$, we see that $\phi = \mathrm{id}_A$, and thus from the relation $f = g \circ \phi$ there is such a morphism iff $f = g$.

2. *F Is Essentially Surjective:* Let λ be a span of the form

$$\begin{tikzcd} & S \arrow[dl, dashed, "\phi"', "\sim"] \arrow[dr, "f"] & \\ A & & B. \end{tikzcd}$$

we claim that $\lambda \cong F\left(f \circ \phi^{-1}\right)$. Indeed, we have morphisms

$$\begin{tikzcd} & S \arrow[dl, dashed, "\phi"', "\sim"] \arrow[dr, "f"] \arrow[dd, "\phi"] & \\ A & & B \\ & A \arrow[ul, equal] \arrow[ur, "f\circ\phi^{-1}"'] & \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} & A \arrow[dl, equal] \arrow[dr, "f\circ\phi^{-1}"] \arrow[dd, "\phi^{-1}"] & \\ A & & B, \\ & S \arrow[ul, dashed, "\phi", "\sim"'] \arrow[ur, "f"'] & \end{tikzcd}$$

inverse to each other in $\mathsf{MapSpan}(A, B)$, and thus $\lambda \cong F\left(f \circ \phi^{-1}\right)$.

Finally, we prove Item 4c.

Item 5: Monads in Span

Item 6: Comonads in Span

Item 7: Monomorphisms in Span

Item 8: Epimorphisms in Span

Item 9: Existence of Right Kan Extensions

Item 10: Existence of Right Kan Lifts

Item 11: Closedness

00RE # 3 Limits of Spans

00RF # 4 Colimits of Spans

00RG # 5 Constructions With Spans

00RH ## 5.1 Representable Spans

00RJ **Definition 5.1.1 ▶ Representable Spans**

Let $f\colon A \longrightarrow B$ be a function.

- The **representable span associated to** f is the span

$$\begin{tikzcd} & A \arrow[dl, equal, "\mathrm{id}_A"'] \arrow[dr, "f"] & \\ A & & B \end{tikzcd}$$

from A to B.

- The **corepresentable span associated to** f is the span

$$\begin{tikzcd} & A \arrow[dl, "f"'] \arrow[dr, equal, "\mathrm{id}_A"] & \\ B & & A \end{tikzcd}$$

from B to A.

00RK 5.2 Composition of Spans

00RL **Definition 5.2.1 ► Composition of Spans**

The **composition** of two spans

$$\begin{tikzcd} & R \arrow[dl, "f_1"'] \arrow[dr, "g_1"] & \\ A & & B \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & S \arrow[dl, "f_2"'] \arrow[dr, "g_2"] & \\ B & & C \end{tikzcd}$$

is the span $(R \times_B S, f_1 \circ \mathrm{pr}_1, g_2 \circ \mathrm{pr}_2)$, constructed as in the diagram

$$\begin{tikzcd} & & R \times_B S \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] \arrow[dd, phantom, very near start, "\lrcorner" rotate=-45] \arrow[ddll, bend right, "f_1 \circ \mathrm{pr}_1"'] \arrow[ddrr, bend left, "g_2 \circ \mathrm{pr}_2"] & & \\ & R \arrow[dl, "f_1"'] \arrow[dr, "g_1"] & & S \arrow[dl, "f_2"'] \arrow[dr, "g_2"] & \\ A & & B & & C. \end{tikzcd}$$

00RM 5.3 Horizontal Composition of Morphisms of Spans

00RN

Definition 5.3.1 ► Horizontal Composition of Morphisms of Spans

The **horizontal composition** of a pair of 2-morphisms of spans

$$\begin{tikzcd} & R \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[d] & \\ A & \phi & B \\ & R' \arrow[ul, "f'"] \arrow[ur, "g'"'] \arrow[from=u, to=d] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & S \arrow[dl, "h"'] \arrow[dr, "k"] \arrow[d] & \\ B & \psi & C, \\ & S' \arrow[ul, "h'"] \arrow[ur, "k'"'] & \end{tikzcd}$$

is the morphism of spans

$$\begin{tikzcd} & R \times_B S \arrow[dl, "f \circ \mathrm{pr}_1"'] \arrow[dr, "k \circ \mathrm{pr}_2"] \arrow[dd, dashed, "\exists !"] & \\ A & & C, \\ & R' \times_B S' \arrow[ul, "h' \circ \mathrm{pr}'_1"] \arrow[ur, "k' \circ \mathrm{pr}'_2"'] & \end{tikzcd}$$

constructed as in the diagram

$$
\begin{tikzcd}
 & & R \times_B S \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] \arrow[dddd, dashed, "\exists!"'] \arrow[ddll, bend right, "f \circ \mathrm{pr}_1"'] \arrow[ddrr, bend left, "k \circ \mathrm{pr}_2"] & & \\
 & R \arrow[dl, "f"'] \arrow[dr, "g"] & & S \arrow[dl, "h"'] \arrow[dr, "k"] & \\
A & & B & & C; \\
 & R' \arrow[ul, "f'"] \arrow[ur, "g'"'] & & S' \arrow[ul, "h'"] \arrow[ur, "k'"'] & \\
 & & R' \times_B S' \arrow[ul, "\mathrm{pr}'_1"] \arrow[ur, "\mathrm{pr}'_2"'] \arrow[uull, bend left, "f' \circ \mathrm{pr}'_1"] \arrow[uurr, bend right, "k' \circ \mathrm{pr}'_2"'] & &
\end{tikzcd}
$$

00RP **5.4 Properties of Composition of Spans**

00RQ **Proposition 5.4.1 ► Properties of Composition of Spans**

Let $\lambda = \left(A \overset{f}{\leftarrow} S \overset{g}{\rightarrow} B\right)$ be a span.

00RR 1. *Functoriality.*

Proof 5.4.2 ► Proof of Proposition 5.4.1

00RS **5.5 The Inverse of a Span**

00RT # 6 Functoriality of Spans

00RU 6.1 Direct Images

00RV 6.2 Functoriality of Spans on Powersets

00RW 7 Comparison of Spans to Functions and Relations

00RX 7.1 Comparison to Functions

00RY **Proposition 7.1.1 ► Comparison of Spans to Functions**

We have a pseudofunctor

$$\iota\colon \mathsf{Sets}_{\mathsf{bidisc}} \longrightarrow \mathsf{Span}$$

from $\mathsf{Sets}_{\mathsf{bidisc}}$ to Span where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathsf{Sets}_{\mathsf{bidisc}})$, we have

$$\iota(A) \overset{\mathrm{def}}{=} A;$$

- *Action on* Hom*-Categories.* For each $A, B \in \mathrm{Obj}(\mathsf{Sets}_{\mathsf{bidisc}})$, the action on Hom-categories

$$\iota_{A,B}\colon \mathsf{Sets}(A,B)_{\mathsf{disc}} \longrightarrow \mathsf{Span}(A,B)$$

of ι at (A, B) is the functor defined on objects by sending a function $f\colon A \to B$ to the span

$$\begin{tikzcd} & A \arrow[dl, equal, "\mathrm{id}_A"'] \arrow[dr, "f"] & \\ A & & B \end{tikzcd}$$

from A to B.

- *Strict Unity Constraints.* For each $A \in \mathrm{Obj}(\mathsf{Sets}_{\mathsf{bidisc}})$, the strict unity constraint

$$\iota^0_A\colon \mathrm{id}_{\iota(A)} \Longrightarrow \iota(\mathrm{id}_A)$$

of ι at A is given by the identity morphism of spans

$$\begin{tikzcd} & A \arrow[dl, equal, "\mathrm{id}_A"'] \arrow[dr, equal, "\mathrm{id}_A"] \arrow[dd, equal, "\mathrm{id}"] & \\ A & & A, \\ & A \arrow[ul, equal, "\mathrm{id}_A"] \arrow[ur, equal, "\mathrm{id}_A"'] & \end{tikzcd}$$

as indeed $\mathrm{id}_{\iota(A)} = \iota(\mathrm{id}_A)$;

- *Pseudofunctoriality Constraints.* For each $A, B, C \in \mathrm{Obj}(\mathsf{Sets}_{\mathsf{bidisc}})$, each $f \in \mathrm{Hom}_{\mathsf{Sets}_{\mathsf{bidisc}}}(A, B)$, and each $g \in \mathrm{Hom}_{\mathsf{Sets}_{\mathsf{bidisc}}}(B, C)$, the pseudofunctoriality constraint

$$\iota^2_{g,f} \colon \iota(g) \circ \iota(f) \Longrightarrow \iota(g \circ f)$$

of ι at (f, g) is the morphism of spans from the span

$$\begin{tikzcd} & & A \times_B B \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] \arrow[ddll, bend right, "\mathrm{id}_A \circ \mathrm{pr}_1"'] \arrow[ddrr, bend left, "g \circ \mathrm{pr}_2"] & & \\ & A \arrow[dl, equal, "\mathrm{id}_A"'] \arrow[dr, "f"] & & B \arrow[dl, equal, "\mathrm{id}_B"'] \arrow[dr, "g"] & \\ A & & B & & C. \end{tikzcd}$$

to the span

$$\begin{tikzcd} & A \arrow[dl, equal, "\mathrm{id}_A"'] \arrow[dr, "g \circ f"] & \\ A & & C \end{tikzcd}$$

given by the isomorphism $A \times_B B \cong A$.

PROOF 7.1.2 ► PROOF OF PROPOSITION 7.1.1

Omitted. ∎

00RZ
7.2 Comparison to Relations: From Span to Rel

00S0
7.2.1 Relations Associated to Spans

Let $\lambda = \left(A \overset{f}{\leftarrow} S \overset{g}{\rightarrow} B\right)$ be a span.

00S1
Definition 7.2.1 ► The Relation Associated to a Span

The **relation associated to** λ is the relation

$$S(\lambda)\colon A \nrightarrow B$$

from A to B defined as follows:

- Viewing relations from A to B as functions $A \times B \to \{\text{true}, \text{false}\}$, we define

$$\iota_{A,B}(S)^a_b \overset{\text{def}}{=} \begin{cases} \text{true} & \text{if there exists } x \in S \text{ such} \\ & \text{that } a = f(x) \text{ and } b = g(x), \\ \text{false} & \text{otherwise} \end{cases}$$

  for each $(a, b) \in A \times B$.
- Viewing relations from A to B as functions $A \to \mathcal{P}(B)$, we define

$$[\iota_{A,B}(S)](a) \overset{\text{def}}{=} g(f^{-1}(a))$$

  for each $a \in A$.
- Viewing relations from A to B as subsets of $A \times B$, we define

$$\iota_{A,B}(S) \overset{\text{def}}{=} \{(f(x), g(x)) \mid x \in S\}.$$

00S2
Proposition 7.2.2 ► Properties of Relations Associated to Spans

Let $\lambda = \left(A \overset{f}{\leftarrow} S \overset{g}{\rightarrow} B\right)$ be a span.

00S3
1. *Interaction With Identities.*

00S4
2. *Interaction With Composition.*

00S5 3. *Interaction With Inverses.*

PROOF 7.2.3 ▶ PROOF OF PROPOSITION 7.2.2

00S6 7.2.2 The Comparison Functor from Span to Rel

00S7 PROPOSITION 7.2.4 ▶ COMPARISON OF SPANS TO RELATIONS I

We have a pseudofunctor

$$\iota\colon \mathsf{Span} \longrightarrow \mathbf{Rel}$$

from Span to **Rel** where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathsf{Span})$, we have

$$\iota(A) \overset{\text{def}}{=} A;$$

- *Action on* Hom*-Categories.* For each $A, B \in \mathrm{Obj}(\mathsf{Span})$, the action on Hom-categories

$$\iota_{A,B}\colon \mathsf{Span}(A,B) \longrightarrow \mathbf{Rel}(A,B)$$

  of ι at (A, B) is the functor where

  – *Action on Objects.* Given a span

$$\begin{tikzcd} & S \arrow[dl, "f"'] \arrow[dr, "g"] & \\ A & & B \end{tikzcd}$$

  from A to B, the image

$$\iota_{A,B}(S)\colon A \nrightarrow B$$

  of S by ι is the relation from A to B defined as follows:

- ⁎ Viewing relations as functions $A \times B \longrightarrow \{\text{true}, \text{false}\}$, we define

$$\iota_{A,B}(S)^a_b \overset{\text{def}}{=} \begin{cases} \text{true} & \text{if there exists } x \in S \\ & \text{such that } a = f(x) \\ & \text{and } b = g(x), \\ \text{false} & \text{otherwise} \end{cases}$$

  for each $(a, b) \in A \times B$;
- ⁎ Viewing relations as functions $A \longrightarrow \mathcal{P}(B)$, we define

$$[\iota_{A,B}(S)](a) \overset{\text{def}}{=} g(f^{-1}(a))$$

  for each $a \in A$;
- ⁎ Viewing relations as subsets of $A \times B$, we define

$$\iota_{A,B}(S) \overset{\text{def}}{=} \{(f(x), g(x)) \mid x \in S\}.$$

– *Action on Morphisms.* Given a morphism of spans

$$\begin{tikzcd} & R \arrow[dl, "f_R"'] \arrow[dr, "g_R"] \arrow[dd, "\phi"] & \\ A & & B, \\ & S \arrow[ul, "f_S"] \arrow[ur, "g_S"'] & \end{tikzcd}$$

we have a corresponding inclusion of relations

$$\iota_{A,B}(\phi) \colon \iota_{A,B}(R) \subset \iota_{A,B}(S),$$

since we have $a \sim_{\iota_{A,B}(R)} b$ iff there exists $x \in R$ such that $a = f_R(x)$ and $b = g_R(x)$, in which case we then have

$$\begin{aligned} a &= f_R(x) \\ &= f_S(\phi(x)), \\ b &= g_R(x) \\ &= g_S(\phi(x)), \end{aligned}$$

so that $a \sim_{\iota_{A,B}(S)} b$, and thus $\iota_{A,B}(R) \subset \iota_{A,B}(S)$.

Proof 7.2.5 ▶ Proof of Proposition 7.2.4

Omitted.

00S8 7.3 Comparison to Relations: From Rel to Span

00S9 **Proposition 7.3.1 ▶ Comparison of Spans to Relations II**

We have a lax functor

$$\left(\iota,\iota^2,\iota^0\right)\colon \mathbf{Rel} \longrightarrow \mathsf{Span}$$

from **Rel** to Span where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathsf{Span})$, we have

$$\iota(A) \stackrel{\text{def}}{=} A;$$

- *Action on* Hom*-Categories.* For each $A, B \in \mathrm{Obj}(\mathsf{Span})$, the action on Hom-categories

$$\iota_{A,B}\colon \mathbf{Rel}(A,B) \longrightarrow \mathsf{Span}(A,B)$$

  of ι at (A,B) is the functor where

  – *Action on Objects.* Given a relation $R\colon A \nrightarrow B$ from A to B, we define a span

$$\iota_{A,B}(R)\colon A \nrightarrow B$$

    from A to B by

$$\iota_{A,B}(R) \stackrel{\text{def}}{=} (R, \restriction \mathrm{pr}_1 R, \restriction \mathrm{pr}_2 R),$$

    where $R \subset A \times B$ and $\restriction \mathrm{pr}_1 R$ and $\restriction \mathrm{pr}_2 R$ are the restriction of the projections

$$\mathrm{pr}_1\colon A \times B \longrightarrow A,$$
$$\mathrm{pr}_2\colon A \times B \longrightarrow B$$

    to R;

- *Action on Morphisms.* Given an inclusion $\phi\colon R \subset S$ of relations, we have a corresponding morphism of spans

$$\iota_{A,B}(\phi)\colon \iota_{A,B}(R) \to \iota_{A,B}(S)$$

as in the diagram

$$\begin{tikzcd}
 & R \arrow[dl, "\upharpoonright\mathrm{pr}_1 R"'] \arrow[dr, "\upharpoonright\mathrm{pr}_2 R"] \arrow[dd, hook] & \\
A & & B. \\
 & S \arrow[ul, "\upharpoonright\mathrm{pr}_1 S"] \arrow[ur, "\upharpoonright\mathrm{pr}_2 S"'] &
\end{tikzcd}$$

- *The Lax Functoriality Constraints.* The lax functoriality constraint

$$\iota^2_{R,S}\colon \iota(S) \circ \iota(R) \Longrightarrow \iota(S \diamond R)$$

of ι at (R,S) is given by the morphism of spans from

$$\begin{tikzcd}
 & & R \times_B S \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] \arrow[dd, phantom, very near start, "\lrcorner" rotate=-45] \arrow[ddll, bend right, "\upharpoonright\mathrm{pr}_1 R \circ \mathrm{pr}_1"'] \arrow[ddrr, bend left, "\upharpoonright\mathrm{pr}_2 S \circ \mathrm{pr}_2"] & & \\
 & R \arrow[dl, "\upharpoonright\mathrm{pr}_1 R"'] \arrow[dr, "\upharpoonright\mathrm{pr}_2 R"] & & S \arrow[dl, "\upharpoonright\mathrm{pr}_1 S"'] \arrow[dr, "\upharpoonright\mathrm{pr}_2 S"] & \\
A & & B & & C
\end{tikzcd}$$

to

$$\begin{tikzcd}
 & S \diamond R \arrow[dl, "\upharpoonright\mathrm{pr}_1 S\diamond R"'] \arrow[dr, "\upharpoonright\mathrm{pr}_2 S\diamond R"] & \\
A & & C
\end{tikzcd}$$

given by the natural inclusion $R \times_B S \hookrightarrow S \diamond R$, since we have

$$R \times_B S = \{((a_R, b_R), (b_S, c_S)) \in R \times S \mid b_R = b_S\};$$

$$S \diamond R = \left\{ (a,c) \in A \times C \;\middle|\; \begin{array}{l} \text{there exists some } b \in B \text{ such} \\ \text{that } (a,b) \in R \text{ and } (b,c) \in S \end{array} \right\};$$

- *The Lax Unity Constraints.* The lax unity constraint[1]

$$\iota^0_A\colon \underbrace{\mathrm{id}_{\iota(A)}}_{(A,\mathrm{id}_A,\mathrm{id}_A)} \Longrightarrow \underbrace{\iota(\chi_A)}_{(\Delta_A,\ulcorner\mathrm{pr}_1\Delta_A,\ulcorner\mathrm{pr}_2\Delta_A)}$$

of ι at A is given by the diagonal morphism of A, as in the diagram

$$\begin{tikzcd} & A \arrow[dl, equal, "\mathrm{id}_A"'] \arrow[dr, equal, "\mathrm{id}_A"] \arrow[dd, "\delta_A"] & \\ A & & A. \\ & \Delta_A \arrow[ul, "\ulcorner\mathrm{pr}_1\Delta_A"] \arrow[ur, "\ulcorner\mathrm{pr}_2\Delta_A"'] & \end{tikzcd}$$

[1]Which is in fact strong, as δ_A is an isomorphism.

Proof 7.3.2 ► Proof of Proposition 7.2.4

Omitted.

00SA

7.4 Comparison to Relations: The Wehrheim–Woodward Construction

00SB

7.5 Comparison to Multirelations

00SC

Remark 7.5.1 ► Interaction With Multirelations

The pseudofunctor of Proposition 7.2.4 and the lax functor of Proposition 7.3.1 fail to be equivalences of bicategories. This happens essentially because a span $(S,f,g)\colon A \nrightarrow B$ from A to B may relate elements $a \in A$ and $b \in B$ by more than one element, e.g. there could be $s \neq s' \in S$ such that $a = f(s) = f(s')$ and $b = g(s) = g(s')$.

Thus, in a sense, spans may be thought of as "relations with multiplicity". And indeed, if instead of considering relations from A to B, i.e. functions

$$R\colon A \times B \longrightarrow \{\text{true}, \text{false}\}$$

from $A \times B$ to $\{\text{true}, \text{false}\} \cong \{0,1\}$, we consider functions

$$R\colon A \times B \to \mathbb{N} \cup \{\infty\}$$

from $A \times B$ to $\mathbb{N} \cup \{\infty\}$, then we obtain the notion of a **multirelation from A to B**, and these turn out to assemble together with sets into a bicategory MRel that is biequivalent to Span; see [**some-algebraic-laws-for-spans-and-their-connections-with-multirelations**].

00SD 7.6 Comparison to Relations via Double Categories

00SE **Remark 7.6.1 ▶ Interaction With Double Categories and Adjointness**

There are double functors between the double categories $\mathrm{Rel}^{\mathrm{dbl}}$ and $\mathrm{Span}^{\mathrm{dbl}}$ analogous to the functors of Propositions 7.2.4 and 7.3.1, assembling moreover into a strict-lax adjunction of double functors; see [**higher-dimensional-categories**].

Appendices

A Other Chapters