Types of Morphisms in Categories

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00UT

Contents

1 Monomorphisms

1.1 Foundations 00UV

Let *C* be a category.

Definition 1.1.1.1. A morphism $m: A \to \emptyset$ **B** Lof C is a **monomorphism** if for every commutative diagram of the form

$$C \xrightarrow{f} A \xrightarrow{m} B,$$

we have f = g.

Example 1.1.1.2. Let $f: A \to B$ be a function of the following conditions are equivalent:

- 1. The function f is injective.
- 2. The function f is a monomorphism in Sets.

Proof. Suppose that f is a monomorphism and consider the following diagram:

$$\{*\} \xrightarrow{[u]} A \xrightarrow{f} B,$$

where [x] and [y] are the morphisms picking the elements x and y of A. Then f(x) = f(y) iff $f \circ [x] = f \circ [y]$, implying [x] = [y], and hence x = y. Therefore f is injective.

¹That is, with $m \circ f = m \circ g$.

Conversely, suppose that f is injective. Proceeding by contrapositive, we claim that given a pair of maps $g,h\colon C\rightrightarrows A$ such that $g\neq h$, then $f\circ g\neq f\circ h$. Indeed, as g and h are different maps, there exists must exist at least one element $x\in C$ such that $g(x)\neq h(x)$. But then we have $f(g(x))\neq f(h(x))$, as f is injective. Thus $f\circ g\neq f\circ h$, and we are done.

Proposition 1.1.1.3. Let C be a category with pullbacks and $f: A \to B$ be a morphism of C.

- 1. Characterisations. The following conditions are equivalent:
 - (a) The morphism f is a monomorphism.

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(b) For each $X \in \text{Obj}(C)$, the map of sets

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$$f_* : \operatorname{Hom}_{\operatorname{Sets}}(X, A) \to \operatorname{Hom}_{\operatorname{Sets}}(X, B)$$

is injective.

(c) The kernel pair of f is trivial, i.e. we have

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$$A \times_B A \cong A, \qquad A \xrightarrow{\operatorname{id}_A} A \\ \downarrow f \\ A \xrightarrow{f} B.$$

2. Monomorphisms vs.Injective Maps. Let

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- *C* be a concrete category;
- $\overline{\kappa}$: $C \to \mathsf{Sets}$ be the forgetful functor from C to Sets ;
- $f: A \rightarrow B$ be a morphism of C.

If 忘 preserves pullbacks, then the following conditions are equivalent:

- (a) The morphism f is a monomorphism.
- (b) The morphism *f* is injective.
- 3. Stability Properties. The class of all monorally hisms of C is stable under the following operations:
 - (a) Composition. If f and q are monomorphisms, then so is $q \circ f$.²

 $^{^2}$ Conversely, if $g\circ f$ is a monomorphism, then so is f .

(b) Pullbacks. Let

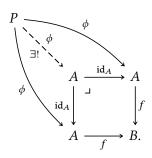
$$\begin{array}{ccc}
A \times_C B & \longrightarrow & B \\
\downarrow^{m'} & & \downarrow^{m} \\
A & \longrightarrow & C
\end{array}$$

be a diagram in C. If m is a monomorphism in C, then so is m'.

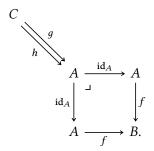
4. Morphisms From the Terminal Object Are Monomorphisms. If C has a terminal object \mathbb{F}_C , then every morphism of C from \mathbb{F}_C is a monomorphism.

Proof. ??, *Characterisations*: The equivalence between ???? is clear. We claim that ???? are equivalent:

1. $?? \implies ??$: Suppose that f is a monomorphism. Then A satisfies the universal property of the pullback:

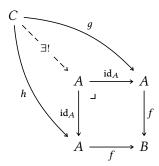


2. $?? \Longrightarrow ??:$ Suppose that $A \cong A \times_B A$ and let $g, h: C \rightrightarrows A$ be a pair of morphisms. Consider the diagram



The universal property of the pullback says that there exists a unique morphism

 $C \rightarrow A$ making the diagram



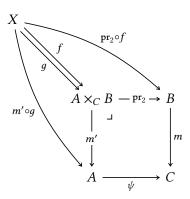
commute, which implies g = h. Therefore, f is a monomorphism.

??, Monomorphisms vs. Injective Maps: Assume that f is injective. As the forgetful functor from C to Sets is faithful, we see that **??** together with **??** imply that f is a monomorphism. Conversely, assume that f is a monomorphism. As F preserves pullbacks, it also preserves kernel pairs. By **??**, we see that F preserves monomorphisms. Thus F_f is a monomorphism, and hence is injective by **??**.

??, Stability Properties: Let $f, g: X \Rightarrow A \times_C B$ be two morphisms such that the diagram

$$X \xrightarrow{f} A \times_C B \xrightarrow{m'} A$$

commutes. It follows that the diagram



also commutes. From the universal property of the pullback, it follows that there must be precisely one morphism from X to $A \times_C B$ making the above diagram commute. Thus f = g and m' is a monomorphism.

??, Morphisms From the Terminal Object Are Monomorphisms: Clear.

1.2 Monomorphism Reflecting Functors

Definition 1.2.1.1. A functor $F: C \to \mathcal{D}$ we flects **monomorphisms** if, for each morphism f of C, whenever F_f is a monomorphism, so is f.

Proposition 1.2.1.2. Let $F: C \to \mathcal{D}$ be a **GONO** or. If F is faithful, then it reflects monomorphisms.

Proof. Let $f: A \to B$ be a morphism of C and suppose that $F_f: F_A \to F_B$ is a monomorphism. Let $g, h: B \rightrightarrows C$ be two morphisms of C such that $g \circ f = h \circ f$. As F is faithful, we must have

$$F_q \circ F_f = F_{q \circ f} = F_{h \circ f} = F_h \circ F_f$$
,

but as F_f is a monomorphism, it must be that $F_g = F_h$. Using the faithfulness of F again, we see that g = h. Therefore f is a monomorphism.

1.3 Split Monomo

Let *C* be a category.

Definition 1.3.1.1. A morphism $f: A \to \mathcal{B} \otimes f \mathcal{K}$ is a **split monomorphism**³ if there exists a morphism $g: B \to A$ of \mathcal{B} such that⁴

$$g \circ f = \mathrm{id}_A$$
.

Proposition 1.3.1.2. Let C be a category. 00VB

1. *Split Monomorphisms are Monomorphisms.* If *m* is a split monomorphism, then *m* is a monomorphism.

Proof. ??, *Split Monomorphisms are Monomorphisms*: Let $m: A \to B$ be a split monomorphism of C, let $e: B \to A$ be a morphism of C with

$$e \circ m = id_A$$

and let $f, g: C \Rightarrow A$ be two morphisms of C such that the diagram

$$C \xrightarrow{f} A \xrightarrow{m} B$$

³Further Terminology: Also called a **section**, or a **split monic** morphism.

⁴ Warning: There exist monomorphisms which are not split monomorphisms, e.g. $\mathbb{Z}/2 \hookrightarrow \mathbb{Z}/4$ in Ring.

commutes. Then we have

$$f = id_A \circ f$$

$$= (e \circ m) \circ f$$

$$= e \circ (m \circ f)$$

$$= e \circ (m \circ g)$$

$$= (e \circ m) \circ g$$

$$= id_A \circ g$$

$$= g,$$

showing m to be a monomorphism.

2 Epimorphism VD

2.1 Foundations **00VE**

Let *C* be a category.

Definition 2.1.1.1. A morphism $f: A \rightarrow \emptyset B/\overline{b} f C$ is an **epimorphism** if for every commutative diagram of the form

$$A \xrightarrow{f} B \xrightarrow{g} C$$
,

we have g = h.

Example 2.1.1.2. Let $f: A \to B$ be a function of the following conditions are equivalent:

- 1. The function f is injective.
- 2. The function f is an epimorphism in Sets.

Proof. Suppose that f is surjective and let $g, h \colon B \rightrightarrows C$ be morphisms such that $g \circ f = h \circ f$. Then for each $a \in A$, we have

$$g(f(a)) = h(f(a)),$$

but this implies that

$$g(b) = h(b)$$

for each $b \in B$, as f is surjective. Thus g = h and f is an epimorphism.

⁵That is, with $g \circ f = h \circ f$.

To prove the converse, we proceed by contrapositive. So suppose that f is not surjective and consider the diagram

$$A \xrightarrow{f} B \xrightarrow{g} C$$
,

where *h* is the map defined by h(b) = 0 for each $b \in B$ and *g* is the map defined by

$$g(b) = \begin{cases} 1 & \text{if } b \in \text{Im}(f), \\ 0 & \text{otherwise.} \end{cases}$$

Then $h \circ f = g \circ f$, as h(f(a)) = 1 = g(f(a)) for each $a \in A$. However, for any $b \in B \setminus \operatorname{Im}(f)$, we have

$$g(b) = 0 \neq 1 = h(b).$$

Therefore $g \neq h$ and f is not an epimorphism.

Proposition 2.1.1.3. Let C be a category. 00VH

1. Characterisations. Let C be a category will pullbacks and $f: A \to B$ be a morphism of C. The following conditions are equivalent:

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- (a) The morphism f is an epimorphism.
- (b) For each $X \in Obj(C)$, the map of sets

$$f^* : \operatorname{Hom}_{\mathsf{Sets}}(B, X) \to \operatorname{Hom}_{\mathsf{Sets}}(A, X)$$

is injective.

(c) The cokernel pair of f is trivial, i.e. we have

$$B \coprod_A B \cong B \qquad \begin{cases} B \longleftarrow B \\ \uparrow & \uparrow \\ B \longleftarrow A. \end{cases}$$

- 2. Epimorphisms vs. Surjective Maps. Let
 - *C* be a concrete category;
 - $\overline{\kappa}$: *C* → Sets be the forgetful functor from *C* to Sets;
 - $f: A \rightarrow B$ be a morphism of C.

If 忘 preserves pushouts, then the following conditions are equivalent:

- (a) The morphism f is a epimorphism.
- (b) The morphism f is surjective.
- 3. *Stability Properties.* The class of all epimor of *C* is stable under the following operations:
 - (a) *Composition.* If f and g are epimorphisms, then so is $g \circ f$.
 - (b) Pushouts. Let

$$\begin{array}{ccc}
A \coprod_{C} B & \longleftarrow & B \\
\downarrow e' & & & \uparrow e \\
A & \longrightarrow & C
\end{array}$$

be a diagram in C. If m is an epimorphism in C, then so is e'.

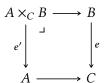
4. *Morphisms to the Initial Object Are Monomorphisms.* If C has an initial object then every morphism of C to \emptyset_C is a epimorphism.

Proof. This is dual to ??.

2.2 Regular Epim@phisms

Proposition 2.2.1.1. Let C be a category. 00VS

1. Stability Under Pullbacks. Consider the diagram



in C. If e is a regular epimorphism, then so is e'.

Proof. Epimorphisms Need Not Be Stable Under Pullback. : Regular Epimorphisms Are Stable Under Pullback. : □

2.3 Effective Epin hisms

Let *C* be a category.

Definition 2.3.1.1. An epimorphism $f: A \otimes B$ of C is **effective** if we have an isomorphism

$$B \cong CoEq(A \times_B A \Rightarrow A).$$

⁶Conversely, if $g \circ f$ is a epimorphism, then so is g.

2.4 Split Epimorphosms

Let *C* be a category.

Definition 2.4.1.1. A morphism $f: A \to \mathcal{B} \cap \mathcal{C}$ is a **retraction**⁷ if there is an arrow $g: B \to A$ such that $f \circ g = \mathrm{id}_B$.

Proposition 2.4.1.2. Let $f: A \to B$ be a modphism of C.

1. Every split epimorphism is an epimorphism.⁸

Proof. This is dual to ??.

Appendices

A Other Chapters

Sets	Category Theory
1. Sets	
2. Constructions With Sets	11. Categories
3. Pointed Sets	12. Types of Morphisms in Categories
4. Tensor Products of Pointed Sets	
5. Relations	13. Adjunctions and the Yoneda Lemma
6. Spans	
7. Posets	14. Constructions With Categories
Indexed and Fibred Sets	15. Kan Extensions
7. Indexed Sets	
8. Fibred Sets	Bicategories
9. Un/Straightening for Indexed and Fibred Sets	17. Bicategories

⁷Further Terminology: Also called a **split epimorphism**.

⁸ Warning: There are epimorphisms which are not split epimorphisms, however, e.g. $\mathbb{Z} \hookrightarrow \mathbb{Z}/2$.

18. Internal Adjunctions	31. Hypergroups
Internal Category Theory	32. Hypersemirings and Hyperrings
19. Internal Categories	33. Quantales
Cyclic Stuff	Near-Rings
20. The Cycle Category	34. Near-Semirings
Cubical Stuff	35. Near-Rings
21. The Cube Category	Real Analysis
Globular Stuff	36. Real Analysis in One Variable
22. The Globe Category	37. Real Analysis in Several Variables
Cellular Stuff	Measure Theory
23. The Cell Category	38. Measurable Spaces
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24. Monoids	Probability Theory
25. Constructions With Monoids	39. Probability Theory
Monoids With Zero	Stochastic Analysis
26. Monoids With Zero	40. Stochastic Processes, Martingales,
27. Constructions With Monoids With	and Brownian Motion
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29. Constructions With Groups	43. Topological and Smooth Manifolds
Hyper Algebra	Schemes
30. Hypermonoids	44. Schemes