# The Gigantic Mess Project

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Index of Set Theory 450

# Part 1

# Sets

CHAPTER 1

# Sets

This chapter (will eventually) contain material on axiomatic set theory, as well as a couple other things.

## 1.1. The Enrichment of Sets in Classical Truth Values

### 1.1.1. $(-2)$-Categories.

**Definition 1.1.1.1.1.** A $(-2)$**-category** is the "necessarily true" truth value.[1,2,3]

### 1.1.2. $(-1)$-Categories.

**Definition 1.1.2.1.1.** A $(-1)$**-category** is a classical truth value.

**Remark 1.1.2.1.2.** [4]$(-1)$-categories should be thought of as being "categories enriched in $(-2)$-categories", having a collection of objects and, for each pair of objects, a Hom-object $\mathrm{Hom}(x, y)$ that is a $(-2)$-category (i.e. trivial).
Therefore, a $(-1)$-category $\mathcal{C}$ is either ([BS10, pp. 33–34]):

(1) *Empty*, having no objects;
(2) *Contractible*, having a collection of objects $\{a, b, c, \ldots\}$, but with $\mathrm{Hom}_{\mathcal{C}}(a, b)$ being a $(-2)$-category (i.e. trivial) for all $a, b \in \mathrm{Obj}(\mathcal{C})$, forcing all objects of $\mathcal{C}$ to be uniquely isomorphic to each other.

As such, there are only two $(-1)$-categories, up to equivalence:

- The $(-1)$-category $\mathsf{false}$ (the empty one);
- The $(-1)$-category $\mathsf{true}$ (the contractible one).

**Definition 1.1.2.1.3.** The **poset of truth values**[5] is the poset $(\{\mathsf{true}, \mathsf{false}\}, \preceq)$[6] consisting of

- *The Underlying Set.* The set $\{\mathsf{true}, \mathsf{false}\}$ whose elements are the truth values $\mathsf{true}$ and $\mathsf{false}$;
- *The Partial Order.* The partial order

$$\preceq : \{\mathsf{true}, \mathsf{false}\} \times \{\mathsf{true}, \mathsf{false}\} \to \{\mathsf{true}, \mathsf{false}\}$$

1. Thus, there is only one $(-2)$-category.
2. A $(-n)$-category for $n = 3, 4, \ldots$ is also the "necessarily true" truth value, coinciding with a $(-2)$-category.
3. For motivation, see [BS10, p. 13].
4. For more motivation, see [BS10, p. 13].
5. *Further Terminology:* Also called the **poset of** $(-1)$**-categories**.
6. *Further Notation:* Also written $\{\mathsf{t}, \mathsf{f}\}$.

on $\{\mathsf{true}, \mathsf{false}\}$ defined by[7]

$$
\begin{aligned}
\mathsf{false} \preceq \mathsf{false} &\overset{\text{def}}{=} \mathsf{true},\\
\mathsf{true} \preceq \mathsf{false} &\overset{\text{def}}{=} \mathsf{false},\\
\mathsf{false} \preceq \mathsf{true} &\overset{\text{def}}{=} \mathsf{true},\\
\mathsf{true} \preceq \mathsf{true} &\overset{\text{def}}{=} \mathsf{true}.
\end{aligned}
$$

**Proposition 1.1.2.1.4.** The poset of truth values $\{\mathsf{t}, \mathsf{f}\}$ is Cartesian closed with product given by[8]

$$
\begin{aligned}
\mathsf{t} \times \mathsf{t} &= \mathsf{t},\\
\mathsf{t} \times \mathsf{f} &= \mathsf{f},\\
\mathsf{f} \times \mathsf{t} &= \mathsf{f},\\
\mathsf{f} \times \mathsf{f} &= \mathsf{f},
\end{aligned}
$$

and internal Hom $\mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}$ given by the partial order of $\{\mathsf{t}, \mathsf{f}\}$, i.e. by

$$
\begin{aligned}
\mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t}, \mathsf{t}) &= \mathsf{t},\\
\mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t}, \mathsf{f}) &= \mathsf{f},\\
\mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{t}) &= \mathsf{t},\\
\mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{f}) &= \mathsf{t}.
\end{aligned}
$$

**Proof.** *Existence of Products*: We claim that the products $\mathsf{t} \times \mathsf{t}$, $\mathsf{t} \times \mathsf{f}$, $\mathsf{f} \times \mathsf{t}$, and $\mathsf{f} \times \mathsf{f}$ satisfy the universal property of the product in $\{\mathsf{t}, \mathsf{f}\}$. Indeed, consider the diagrams

$$
\begin{tikzcd}
 & P_1 \arrow[dl, bend right, "p^1_1"'] \arrow[d, dashed, "\exists!"] \arrow[dr, bend left, "p^1_2"] & \\
\mathsf{t} & \underbrace{\mathsf{t} \times \mathsf{t}}_{=\mathsf{t}} \arrow[l, two heads, "\mathrm{pr}_1"] \arrow[r, two heads, "\mathrm{pr}_2"'] & \mathsf{t}
\end{tikzcd}
\quad
\begin{tikzcd}
 & P_2 \arrow[dl, bend right, "p^2_1"'] \arrow[d, dashed, "\exists!"] \arrow[dr, bend left, "p^2_2"] & \\
\mathsf{t} & \underbrace{\mathsf{t} \times \mathsf{f}}_{=\mathsf{f}} \arrow[l, two heads, "\mathrm{pr}_1"] \arrow[r, two heads, "\mathrm{pr}_2"'] & \mathsf{f}
\end{tikzcd}
\quad
\begin{tikzcd}
 & P_3 \arrow[dl, bend right, "p^3_1"'] \arrow[d, dashed, "\exists!"] \arrow[dr, bend left, "p^3_2"] & \\
\mathsf{f} & \underbrace{\mathsf{f} \times \mathsf{t}}_{=\mathsf{f}} \arrow[l, two heads, "\mathrm{pr}_1"] \arrow[r, two heads, "\mathrm{pr}_2"'] & \mathsf{t}
\end{tikzcd}
\quad
\begin{tikzcd}
 & P_4 \arrow[dl, bend right, "p^4_1"'] \arrow[d, dashed, "\exists!"] \arrow[dr, bend left, "p^4_2"] & \\
\mathsf{f} & \underbrace{\mathsf{f} \times \mathsf{f}}_{=\mathsf{f}} \arrow[l, two heads, "\mathrm{pr}_1"] \arrow[r, two heads, "\mathrm{pr}_2"'] & \mathsf{f}.
\end{tikzcd}
$$

Here:

(1) If $P_1 = \mathsf{t}$, then $p^1_1 = p^1_2 = \mathrm{id}_{\mathsf{t}}$, and there's indeed a unique morphism from $P_1$ to $\mathsf{t}$ making the diagram commute, namely $\mathrm{id}_{\mathsf{t}}$;
(2) If $P_1 = \mathsf{f}$, then $p^1_1 = p^1_2$ are given by the unique morphism from $\mathsf{f}$ to $\mathsf{t}$, and there's indeed a unique morphism from $P_1$ to $\mathsf{t}$ making the diagram commute, namely the unique morphism from $\mathsf{f}$ to $\mathsf{t}$;
(3) If $P_2 = \mathsf{t}$, then there is no morphism $p^2_2$.
(4) If $P_2 = \mathsf{f}$, then $p^2_1$ is the unique morphism from $\mathsf{f}$ to $\mathsf{t}$ while $p^2_2 = \mathrm{id}_{\mathsf{f}}$, and there's indeed a unique morphism from $P_2$ to $\mathsf{f}$ making the diagram commute, namely $\mathrm{id}_{\mathsf{f}}$;
(5) The proof for $P_3$ is similar to the one for $P_2$;
(6) If $P_4 = \mathsf{t}$, then there is no morphism $p^4_1$ or $p^4_2$.
(7) If $P_4 = \mathsf{f}$, then $p^4_1 = p^4_2 = \mathrm{id}_{\mathsf{f}}$, and there's indeed a unique morphism from $P_4$ to $\mathsf{f}$ making the diagram commute, namely $\mathrm{id}_{\mathsf{f}}$.

[7] This partial order coincides with logical implication.

[8] Note that $\times$ coincides with the "and" operator, while $\mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}$ coincides with the logical implication operator.

*Cartesian Closedness*: We claim there's a bijection

$$\mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(A \times B, C) \cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\Big(A, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(B, C)\Big)$$

natural in $A, B, C \in \{\mathsf{t}, \mathsf{f}\}$. Indeed:

- For $(A, B, C) = (\mathsf{t}, \mathsf{t}, \mathsf{t})$, we have
$$\begin{aligned}\mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t} \times \mathsf{t}, \mathsf{t}) &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t}, \mathsf{t})\\ &= \{\mathrm{id}_{\mathsf{true}}\}\\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t}, \mathsf{t})\\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\Big(\mathsf{t}, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t}, \mathsf{t})\Big).\end{aligned}$$
- For $(A, B, C) = (\mathsf{t}, \mathsf{t}, \mathsf{f})$, we have
$$\begin{aligned}\mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t} \times \mathsf{t}, \mathsf{f}) &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t}, \mathsf{f})\\ &= \emptyset\\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t}, \mathsf{f})\\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\Big(\mathsf{t}, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t}, \mathsf{f})\Big).\end{aligned}$$
- For $(A, B, C) = (\mathsf{t}, \mathsf{f}, \mathsf{t})$, we have
$$\begin{aligned}\mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t} \times \mathsf{f}, \mathsf{t}) &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{t})\\ &\cong \mathrm{pt}\\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{t})\\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\Big(\mathsf{f}, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{t})\Big).\end{aligned}$$
- For $(A, B, C) = (\mathsf{t}, \mathsf{f}, \mathsf{f})$, we have
$$\begin{aligned}\mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t} \times \mathsf{f}, \mathsf{f}) &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{f})\\ &\cong \{\mathrm{id}_{\mathsf{false}}\}\\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{f})\\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\Big(\mathsf{t}, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{f})\Big).\end{aligned}$$
- For $(A, B, C) = (\mathsf{f}, \mathsf{t}, \mathsf{t})$, we have
$$\begin{aligned}\mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f} \times \mathsf{t}, \mathsf{t}) &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{t})\\ &\cong \mathrm{pt}\\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{t})\\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\Big(\mathsf{f}, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t}, \mathsf{t})\Big).\end{aligned}$$
- For $(A, B, C) = (\mathsf{f}, \mathsf{t}, \mathsf{f})$, we have
$$\begin{aligned}\mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f} \times \mathsf{t}, \mathsf{f}) &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{f})\\ &\cong \{\mathrm{id}_{\mathsf{false}}\}\\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{f})\\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\Big(\mathsf{f}, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t}, \mathsf{f})\Big).\end{aligned}$$

- For $(A, B, C) = (\mathsf{f}, \mathsf{f}, \mathsf{t})$, we have
$$\begin{aligned}\mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f} \times \mathsf{f}, \mathsf{t}) &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{t}) \\ &\cong \mathrm{pt} \\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{t}) \\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\Big(\mathsf{f}, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{t})\Big).\end{aligned}$$
- For $(A, B, C) = (\mathsf{f}, \mathsf{f}, \mathsf{f})$, we have
$$\begin{aligned}\mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f} \times \mathsf{f}, \mathsf{f}) &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{f}) \\ &= \{\mathrm{id}_{\mathsf{false}}\} \\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{f}) \\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\Big(\mathsf{f}, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{f})\Big).\end{aligned}$$

The proof of naturality is omitted. □

### 1.1.3. 0-Categories.

**Definition 1.1.3.1.1.** A **0-category** is a poset.[9]

**Definition 1.1.3.1.2.** A **0-groupoid** is a 0-category in which every morphism is invertible.[10]

**1.1.4. Tables of Analogies Between Set Theory and Category Theory.** Here we record some analogies between notions in set theory and category theory. Note that the analogies relating to presheaves relate equally well to copresheaves, as the opposite $X^{\mathsf{op}}$ of a set $X$ is just $X$ again.
Basics:

| Set Theory | Category Theory |
|---|---|
| Enrichment in $\{\mathsf{true}, \mathsf{false}\}$ | Enrichment in $\mathsf{Sets}$ |
| Set $X$ | Category $\mathcal{C}$ |
| Element $x \in X$ | Object $X \in \mathrm{Obj}(\mathcal{C})$ |
| Function | Functor |
| Function $X \to \{\mathsf{true}, \mathsf{false}\}$ | Functor $\mathcal{C} \to \mathsf{Sets}$ |
| Function $X \to \{\mathsf{true}, \mathsf{false}\}$ | Presheaf $\mathcal{C}^{\mathsf{op}} \to \mathsf{Sets}$ |

Powersets and categories of presheaves:

[9] *Motivation:* A 0-category is precisely a category enriched in the poset of $(-1)$-categories.
[10] That is, a *set*.

| Set Theory | Category Theory |
|---|---|
| Powerset $\mathcal{P}(X)$ | Presheaf category $\mathsf{PSh}(\mathcal{C})$ |
| Characteristic function $\chi_{\{x\}}$ | Representable presheaf $h_X$ |
| Characteristic embedding<br>$\chi_{(-)}\colon X \hookrightarrow \mathcal{P}(X)$ | Yoneda embedding<br>$\text{よ}\colon \mathcal{C}^{\mathsf{op}} \hookrightarrow \mathsf{PSh}(\mathcal{C})$ |
| Characteristic relation $\chi_X(-_1, -_2)$ | Hom profunctor $\mathrm{Hom}_{\mathcal{C}}(-_1, -_2)$ |
| The Yoneda lemma for sets<br>$\mathrm{Hom}_{\mathcal{P}(X)}(\chi_x, \chi_U) = \chi_U(x)$ | The Yoneda lemma for categories<br>$\mathrm{Nat}(h_X, \mathcal{F}) \cong \mathcal{F}(X)$ |
| The characteristic<br>embedding is fully faithful,<br>$\mathrm{Hom}_{\mathcal{P}(X)}(\chi_x, \chi_y) = \chi_X(x, y)$ | The Yoneda<br>embedding is fully faithful,<br>$\mathrm{Nat}(h_X, h_Y) \cong \mathrm{Hom}_{\mathcal{C}}(X, Y)$ |
| Subsets are unions of their elements<br>$U = \bigcup_{x \in U} \{x\}$<br>or<br>$\chi_U = \operatorname*{colim}_{\chi_x \in \mathsf{Sets}(U, \{\mathsf{t}, \mathsf{f}\})} (\chi_x)$ | Presheaves are<br>colimits of representables,<br>$\mathcal{F} \cong \operatorname*{colim}_{h_X \in \int_{\mathcal{C}} \mathcal{F}} (h_X)$ |

Categories of elements:

| Set Theory | Category Theory |
|---|---|
| Assignment $U \mapsto \chi_U$ | Assignment $\mathcal{F} \mapsto \int_{\mathcal{C}} \mathcal{F}$<br>(the category of elements) |
| Assignment $U \mapsto \chi_U$<br>giving an isomorphism<br>$\mathcal{P}(X) \cong \mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\})$ | Assignment $\mathcal{F} \mapsto \int_{\mathcal{C}} \mathcal{F}$<br>giving an equivalence<br>$\mathsf{PSh}(\mathcal{C}) \overset{\text{eq.}}{\cong} \mathsf{DFib}(\mathcal{C})$ |

Functions between powersets and functors between presheaf categories:

| Set Theory | Category Theory |
|---|---|
| Direct image function<br>$f_*\colon \mathcal{P}(X) \to \mathcal{P}(Y)$ | Inverse image functor<br>$f^{-1}\colon \mathsf{PSh}(\mathcal{C}) \to \mathsf{PSh}(\mathcal{D})$ |
| Inverse image function<br>$f^{-1}\colon \mathcal{P}(Y) \to \mathcal{P}(X)$ | Direct image functor<br>$f_*\colon \mathsf{PSh}(\mathcal{D}) \to \mathsf{PSh}(\mathcal{C})$ |
| Direct image with<br>compact support function<br>$f_!\colon \mathcal{P}(X) \to \mathcal{P}(Y)$ | Direct image with<br>compact support functor<br>$f_!\colon \mathsf{PSh}(\mathcal{C}) \to \mathsf{PSh}(\mathcal{D})$ |

Relations and profunctors:

| Set Theory | Category Theory |
|---|---|
| Relation $R\colon X \times Y \to \{\mathsf{t}, \mathsf{f}\}$ | Profunctor $\mathfrak{p}\colon \mathcal{D}^{\mathsf{op}} \times \mathcal{C} \to \mathsf{Sets}$ |
| Relation $R\colon X \to \mathcal{P}(Y)$ | Profunctor $\mathfrak{p}\colon \mathcal{C} \to \mathsf{PSh}(\mathcal{D})$ |
| Relation as a cocontinuous morphism of posets $R\colon (\mathcal{P}(X), \subset) \to (\mathcal{P}(Y), \subset)$ | Profunctor as a colimit-preserving functor $\mathfrak{p}\colon \mathsf{PSh}(\mathcal{C}) \to \mathsf{PSh}(\mathcal{D})$ |

# Appendices

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(3) Pointed Sets
(4) Tensor Products of Pointed Sets
(5) Relations
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(45) Schemes

## 1.2. Other Chapters

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(45) Schemes

CHAPTER 2

# Constructions With Sets

This chapter contains some material relating to constructions with sets. Notably, it contains:

(1) Explicit descriptions of the major types of co/limits in $\mathsf{Sets}$, including in particular pushouts and coequalisers (see Definitions 2.2.3.1.1 and 2.2.4.1.1 and Remarks 2.2.3.1.2 and 2.2.4.1.2);
(2) A discussion of powersets as decategorifications of categories of presheaves (Remarks 2.4.1.1.2 and 2.4.3.1.2);
(3) A lengthy discussion of the adjoint triple

$$f_* \dashv f^{-1} \dashv f_! \colon \mathcal{P}(A) \overset{\rightleftarrows}{\rightarrow} \mathcal{P}(B)$$

of functors (morphisms of posets) between $\mathcal{P}(A)$ and $\mathcal{P}(B)$ induced by a map of sets $f\colon A \to B$, along with a discussion of the properties of $f_*$, $f^{-1}$, and $f_!$.

## 2.1. Limits of Sets

**2.1.1. Products of Families of Sets.** Let $\{A_i\}_{i\in I}$ be a family of sets.

**Definition 2.1.1.1.1.** The **product**[1] **of** $\{A_i\}_{i\in I}$ is the pair $\left(\prod_{i\in I} A_i, \{\mathrm{pr}_i\}_{i\in I}\right)$ consisting of

- *The Limit.* The set $\prod_{i\in I} A_i$ defined by[2]

$$\prod_{i\in I} A_i \overset{\text{def}}{=} \left\{ f \in \mathsf{Sets}\left(I, \bigcup_{i\in I} A_i\right) \;\middle|\; \begin{array}{l}\text{for each } i \in I, \text{ we}\\ \text{have } f(i) \in A_i\end{array} \right\}.$$

- *The Cone.* The collection

$$\left\{ \mathrm{pr}_i \colon \prod_{i\in I} A_i \to A_i \right\}_{i\in I}$$

of maps given by

$$\mathrm{pr}_i(f) \overset{\text{def}}{=} f(i)$$

for each $f \in \prod_{i\in I} A_i$ and each $i \in I$.

---

[1] *Further Terminology:* Also called the **Cartesian product of** $\{A_i\}_{i\in I}$.

[2] Less formally, $\prod_{i\in I} A_i$ is the set whose elements are $I$-indexed collections $(a_i)_{i\in I}$ with $a_i \in A_i$ for each $i \in I$.

**Proof.** We claim that $\prod_{i\in I} A_i$ is the categorical product of $\{A_i\}_{i\in I}$ in $\mathsf{Sets}$. Indeed, suppose we have, for each $i \in I$, a diagram of the form

$$\begin{tikzcd} P \arrow[dr, "p_i"] & \\ \prod_{i\in I} A_i \arrow[r, "\mathrm{pr}_i"'] & A_i \end{tikzcd}$$

in $\mathsf{Sets}$. Then there exists a unique map $\phi\colon P \to \prod_{i\in I} A_i$, uniquely determined by the condition $\mathrm{pr}_i \circ \phi = p_i$ for each $i \in I$, being necessarily given by

$$\phi(x) = (p_i(x))_{i\in I}$$

for each $x \in P$. □

**Proposition 2.1.1.1.2.** Let $\{A_i\}_{i\in I}$ be a family of sets.

(1) *Functoriality.* The assignment $\{A_i\}_{i\in I} \mapsto \prod_{i\in I} A_i$ defines a functor

$$\prod_{i\in I}\colon \mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}) \to \mathsf{Sets}$$

where

- *Action on Objects.* For each $(A_i)_{i\in I} \in \mathrm{Obj}(\mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}))$, we have

$$\left[\prod_{i\in I}\right]((A_i)_{i\in I}) \overset{\text{def}}{=} \prod_{i\in I} A_i$$

- *Action on Morphisms.* For each $(A_i)_{i\in I}, (B_i)_{i\in I} \in \mathrm{Obj}(\mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}))$, the action on Hom-sets

$$\left(\prod_{i\in I}\right)_{(A_i)_{i\in I},(B_i)_{i\in I}} : \mathrm{Nat}\big((A_i)_{i\in I}, (B_i)_{i\in I}\big) \to \mathsf{Sets}\left(\prod_{i\in I} A_i, \prod_{i\in I} B_i\right)$$

of $\prod_{i\in I}$ at $\big((A_i)_{i\in I}, (B_i)_{i\in I}\big)$ is defined by sending a map

$$\{f_i\colon A_i \to B_i\}_{i\in I}$$

in $\mathrm{Nat}\big((A_i)_{i\in I}, (B_i)_{i\in I}\big)$ to the map of sets

$$\prod_{i\in I} f_i\colon \prod_{i\in I} A_i \to \prod_{i\in I} B_i$$

defined by

$$\left[\prod_{i\in I} f_i\right]((a_i)_{i\in I}) \overset{\text{def}}{=} (f_i(a_i))_{i\in I}$$

for each $(a_i)_{i\in I} \in \prod_{i\in I} A_i$.

**Proof.** *Item 1, Functoriality*: Clear. □

**2.1.2. Binary Products of Sets.** Let $A$ and $B$ be sets.

**Definition 2.1.2.1.1.** The **product**[3] **of $A$ and $B$** is the pair $(A \times B, \{\mathrm{pr}_1, \mathrm{pr}_2\})$ consisting of

[3] *Further Terminology:* Also called the **Cartesian product of $A$ and $B$** or the **binary**

- *The Limit.* The set $A \times B$ defined by[4]

$$\begin{aligned} A \times B &\overset{\text{def}}{=} \prod_{z \in \{A,B\}} z \\ &\overset{\text{def}}{=} \{f \in \mathsf{Sets}(\{0,1\}, A \cup B) \mid \text{we have } f(0) \in A \text{ and } f(1) \in B\} \\ &\cong \{\{\{a\},\{a,b\}\} \in \mathcal{P}(\mathcal{P}(A \cup B)) \mid \text{we have } a \in A \text{ and } b \in B\}. \end{aligned}$$

- *The Cone.* The maps

$$\begin{aligned} \mathrm{pr}_1 &\colon A \times B \to A, \\ \mathrm{pr}_2 &\colon A \times B \to B \end{aligned}$$

  defined by

$$\begin{aligned} \mathrm{pr}_1(a,b) &\overset{\text{def}}{=} a, \\ \mathrm{pr}_2(a,b) &\overset{\text{def}}{=} b \end{aligned}$$

  for each $(a,b) \in A \times B$.

**Proof.** We claim that $A \times B$ is the categorical product of $A$ and $B$ in $\mathsf{Sets}$. Indeed, suppose we have a diagram of the form

$$\begin{tikzcd} & P \arrow[dl, bend right, "p_1"'] \arrow[dr, bend left, "p_2"] & \\ A & A \times B \arrow[l, "\mathrm{pr}_1"] \arrow[r, "\mathrm{pr}_2"'] & B \end{tikzcd}$$

in $\mathsf{Sets}$. Then there exists a unique map $\phi\colon P \to A \times B$, uniquely determined by the conditions

$$\begin{aligned} \mathrm{pr}_1 \circ \phi &= p_1, \\ \mathrm{pr}_2 \circ \phi &= p_2, \end{aligned}$$

being necessarily given by

$$\phi(x) = (p_1(x), p_2(x))$$

for each $x \in P$. □

**Proposition 2.1.2.1.2.** Let $A$, $B$, $C$, and $X$ be sets.

(1) *Functoriality.* The assignments $A, B, (A,B) \mapsto A \times B$ define functors

$$\begin{aligned} A \times -_2 &\colon \mathsf{Sets} \to \mathsf{Sets}, \\ -_1 \times B &\colon \mathsf{Sets} \to \mathsf{Sets}, \\ -_1 \times -_2 &\colon \mathsf{Sets} \times \mathsf{Sets} \to \mathsf{Sets}, \end{aligned}$$

where $-_1 \times -_2$ is the functor where

- *Action on Objects.* For each $(A,B) \in \mathrm{Obj}(\mathsf{Sets} \times \mathsf{Sets})$, we have

$$[-_1 \times -_2](A,B) \overset{\text{def}}{=} A \times B;$$

---

**Cartesian product of $A$ and $B$**, for emphasis.
This can also be thought of as the $(\mathbb{E}_{-1}, \mathbb{E}_{-1})$**-tensor product of $A$ and $B$**.
[4]In other words, $A \times B$ is the set whose elements are ordered pairs $(a,b)$ with $a \in A$ and $b \in B$ as in Definition 2.3.4.1.1

- *Action on Morphisms.* For each $(A, B), (X, Y) \in \mathrm{Obj}(\mathsf{Sets})$, the action on Hom-sets

$$\times_{(A,B),(X,Y)} \colon \mathsf{Sets}(A, X) \times \mathsf{Sets}(B, Y) \to \mathsf{Sets}(A \times B, X \times Y)$$

  of $\times$ at $((A, B), (X, Y))$ is defined by sending $(f, g)$ to the function

$$f \times g \colon A \times B \to X \times Y$$

  defined by

$$[f \times g](a, b) \stackrel{\text{def}}{=} (f(a), g(b))$$

  for each $(a, b) \in A \times B$;

and where $A \times -$ and $- \times B$ are the partial functors of $-_1 \times -_2$ at $A, B \in \mathrm{Obj}(\mathsf{Sets})$.

(2) *Adjointness.* We have adjunctions

$$(A \times - \dashv \mathsf{Sets}(A, -))\colon\ \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "A\times -"] \arrow[r, phantom, "\perp"] & \mathsf{Sets} \arrow[l, bend left, "\mathsf{Sets}(A,-)"] \end{tikzcd},$$

$$(- \times B \dashv \mathsf{Sets}(B, -))\colon\ \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "-\times B"] \arrow[r, phantom, "\perp"] & \mathsf{Sets} \arrow[l, bend left, "\mathsf{Sets}(B,-)"] \end{tikzcd},$$

witnessed by bijections

$$\mathsf{Sets}(A \times B, C) \cong \mathsf{Sets}(A, \mathsf{Sets}(B, C)),$$
$$\mathsf{Sets}(A \times B, C) \cong \mathsf{Sets}(B, \mathsf{Sets}(A, C)),$$

natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

(3) *Associativity.* We have an isomorphism of sets

$$(A \times B) \times C \cong A \times (B \times C),$$

natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

(4) *Unitality.* We have isomorphisms of sets

$$\mathrm{pt} \times A \cong A,$$
$$A \times \mathrm{pt} \cong A,$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$.

(5) *Commutativity.* We have an isomorphism of sets

$$A \times B \cong B \times A,$$

natural in $A, B \in \mathrm{Obj}(\mathsf{Sets})$.

(6) *Annihilation With the Empty Set.* We have isomorphisms of sets

$$A \times \emptyset \cong \emptyset,$$
$$\emptyset \times A \cong \emptyset,$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$.

(7) *Distributivity Over Unions.* We have isomorphisms of sets

$$A \times (B \cup C) = (A \times B) \cup (A \times C),$$
$$(A \cup B) \times C = (A \times C) \cup (B \times C).$$

(8) *Distributivity Over Intersections.* We have isomorphisms of sets

$$A \times (B \cap C) = (A \times B) \cap (A \times C),$$
$$(A \cap B) \times C = (A \times C) \cap (B \times C).$$

(9) *Middle-Four Exchange with Respect to Intersections.* We have an isomorphism of sets

$$(A \times B) \cap (C \times D) \cong (A \cap B) \times (C \cap D).$$

(10) *Distributivity Over Differences.* We have isomorphisms of sets

$$A \times (B \setminus C) = (A \times B) \setminus (A \times C),$$
$$(A \setminus B) \times C = (A \times C) \setminus (B \times C),$$

natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

(11) *Distributivity Over Symmetric Differences.* We have isomorphisms of sets

$$A \times (B \triangle C) = (A \times B) \triangle (A \times C),$$
$$(A \triangle B) \times C = (A \times C) \triangle (B \times C),$$

natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

(12) *Symmetric Monoidality.* The triple $(\mathsf{Sets}, \times, \mathrm{pt})$ is a symmetric monoidal category.

(13) *Symmetric Bimonoidality.* The quintuple $(\mathsf{Sets}, \coprod, \emptyset, \times, \mathrm{pt})$ is a symmetric bimonoidal category.

**Proof.** *Item 1, Functoriality*: This follows by applying associativity and unitality componentwise.

*Item 2, Adjointness*: We prove only that there's an adjunction $X \times - \dashv \mathrm{Hom}_{\mathsf{Sets}}(-, Z)$, witnessed by a bijection

$$\mathrm{Hom}_{\mathsf{Sets}}(X \times Y, Z) \cong \mathrm{Hom}_{\mathsf{Sets}}(X, \mathrm{Hom}_{\mathsf{Sets}}(Y, Z)),$$

natural in $Y, Z \in \mathrm{Obj}(\mathsf{Sets})$, as the proof of the existence of the adjunction $- \times Y \dashv \mathrm{Hom}_{\mathsf{Sets}}(-, Z)$ follows almost exactly in the same way.[5]

- *Map I.* We define a map

  $$\Phi_{Y,Z} \colon \mathrm{Hom}_{\mathsf{Sets}}(X \times Y, Z) \to \mathrm{Hom}_{\mathsf{Sets}}(X, \mathrm{Hom}_{\mathsf{Sets}}(Y, Z)),$$

  by sending a morphism $\xi \colon X \times Y \to Z$ to the morphism

  $$\xi^{\dagger} \colon X \to \mathrm{Hom}_{\mathsf{Sets}(Y,Z)}$$

  defined by

  $$\xi^{\dagger}(x) \overset{\mathrm{def}}{=} \xi_x$$

  for each $x \in X$, where $\xi_x \colon Y \to Z$ is the map defined by

  $$\xi_x(y) \overset{\mathrm{def}}{=} \xi(x, y)$$

  for each $y \in Y$.

- *Map II.* We define a map

  $$\Psi_{Y,Z} \colon \mathrm{Hom}_{\mathsf{Sets}}(X, \mathrm{Hom}_{\mathsf{Sets}}(Y, Z)), \to \mathrm{Hom}_{\mathsf{Sets}}(X \times Y, Z)$$

[5] Here we sometimes denote a map $f \colon X \to Y$ by $[x \mapsto f(x)]$, similar to the lambda notation $\lambda x.f(x)$.

given by sending a map $\xi\colon X \to \mathrm{Hom}_{\mathsf{Sets}}(Y, Z)$ to the map

$$\xi^\dagger\colon X \times Y \to Z$$

defined by

$$\xi^\dagger(x, y) \overset{\text{def}}{=} [\xi(x)](y)$$

for each $(x, y) \in X \times Y$.

- *Naturality I.* We need to show that, given a function $g\colon Y \to Y'$, the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathsf{Sets}}(X \times Y', Z) \arrow[r, "\Phi_{Y',Z}"] \arrow[d, "\mathrm{id}_X \times g^*"'] & \mathrm{Hom}_{\mathsf{Sets}}(X, \mathrm{Hom}_{\mathsf{Sets}}(Y', Z)), \arrow[d, "(g^*)_*"] \\
\mathrm{Hom}_{\mathsf{Sets}}(X \times Y, Z) \arrow[r, "\Phi_{Y,Z}"'] & \mathrm{Hom}_{\mathsf{Sets}}(X, \mathrm{Hom}_{\mathsf{Sets}}(Y, Z)),
\end{tikzcd}$$

commutes. Indeed, given a morphism $\xi\colon X' \times Y \to Z$, we have

$$\begin{aligned}
[\Phi_{Y,Z} \circ (g^* \times \mathrm{id}_Y)](\xi) &\overset{\text{def}}{=} (\xi(-_1, g(-_2)))^\dagger \\
&\overset{\text{def}}{=} \xi_{-_1}(g(-_2)) \\
&\overset{\text{def}}{=} (g_*)^*(\xi_{-_1}(-_2)) \\
&\overset{\text{def}}{=} (g_*)^*\Big(\xi^\dagger\Big) \\
&\overset{\text{def}}{=} [(g_*)^* \circ \Phi_{Y',Z}](\xi).
\end{aligned}$$

- *Naturality II.* We need to show that, given a function $h\colon Z \to Z'$, the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathsf{Sets}}(X \times Y, Z) \arrow[r, "\Phi_{Y,Z}"] \arrow[d, "h_*"'] & \mathrm{Hom}_{\mathsf{Sets}}(X, \mathrm{Hom}_{\mathsf{Sets}}(Y, Z)), \arrow[d, "(h_*)_*"] \\
\mathrm{Hom}_{\mathsf{Sets}}(X \times Y, Z') \arrow[r, "\Phi_{Y,Z'}"'] & \mathrm{Hom}_{\mathsf{Sets}}(X, \mathrm{Hom}_{\mathsf{Sets}}(Y, Z')),
\end{tikzcd}$$

commutes. Indeed, given a morphism $\xi\colon X \times Y \to Z$, we have

$$\begin{aligned}
[\Phi_{Y,Z} \circ h_*](\xi) &\overset{\text{def}}{=} (h(\xi(-_1, -_2)))^\dagger \\
&\overset{\text{def}}{=} [x \mapsto [y \mapsto h(\xi(x, y))]] \\
&\overset{\text{def}}{=} [x \mapsto h_*([y \mapsto \xi(x, y)])] \\
&\overset{\text{def}}{=} [x \mapsto h_*\Big(\xi^\dagger(x)\Big)] \\
&\overset{\text{def}}{=} h_*\Big(\xi^\dagger\Big) \\
&\overset{\text{def}}{=} [(h_*)_* \circ \Phi_{Y,Z}](\xi).
\end{aligned}$$

- *Invertibility I.* We claim that

$$\Psi_{X,Y} \circ \Phi_{X,Y} = \mathrm{id}_{\mathrm{Hom}_{\mathsf{Sets}}(X \times Y, Z)}.$$

Indeed, given a morphism $\xi\colon X\times Y\to Z$, we have

$$\begin{aligned}[\Psi_{X,Y}\circ\Phi_{X,Y}](\xi) &\stackrel{\text{def}}{=} \Psi_{X,Y}(\Phi_{X,Y}(\xi))\\ &\stackrel{\text{def}}{=} \Psi_{X,Y}([x\mapsto[y\mapsto\xi(x,y)]])\\ &\stackrel{\text{def}}{=} [(x,y)\mapsto \mathrm{ev}_x([x\mapsto \mathrm{ev}_y([y\mapsto\xi(x,y)]])]\\ &\stackrel{\text{def}}{=} [(x,y)\mapsto \mathrm{ev}_x([x\mapsto\xi(x,y)])]\\ &\stackrel{\text{def}}{=} [(x,y)\mapsto\xi(x,y)]\\ &\stackrel{\text{def}}{=} \xi.\end{aligned}$$

- *Invertibility II.* We claim that

$$\Phi_{X,Y}\circ\Psi_{X,Y}=\mathrm{id}_{\mathrm{Hom}_{\mathsf{Sets}}(X,\mathrm{Hom}_{\mathsf{Sets}}(Y,Z))}.$$

Indeed, given a morphism $\xi\colon X\to\mathrm{Hom}_{\mathsf{Sets}}(Y,Z)$, we have

$$\begin{aligned}[\Phi_{X,Y}\circ\Psi_{X,Y}](\xi) &\stackrel{\text{def}}{=} \Phi_{X,Y}(\Psi_{X,Y}(\xi))\\ &\stackrel{\text{def}}{=} \Phi_{X,Y}(\Psi_{X,Y}([x\mapsto\xi(x)]))\\ &\stackrel{\text{def}}{=} \Phi_{X,Y}([(x,y)\mapsto \mathrm{ev}_x([x\mapsto \mathrm{ev}_y(\xi(x))])])\\ &\stackrel{\text{def}}{=} \Phi_{X,Y}([(x,y)\mapsto\xi(x,y)])\\ &\stackrel{\text{def}}{=} [x\mapsto[y\mapsto\xi(x,y)]]\\ &\stackrel{\text{def}}{=} \xi.\end{aligned}$$

*Item 3, Associativity*: See [Pro24a].
*Item 4, Unitality*: Clear.
*Item 5, Commutativity*: See [Pro24b].
*Item 6, Annihilation With the Empty Set*: See [Pro24f].
*Item 7, Distributivity Over Unions*: See [Pro24e].
*Item 8, Distributivity Over Intersections*: See [Pro24g, Corollary 1].
*Item 9, Middle-Four Exchange With Respect to Intersections*: See [Pro24g, Corollary 1].
*Item 10, Distributivity Over Differences*: See [Pro24c].
*Item 11, Distributivity Over Symmetric Differences*: See [Pro24d].
*Item 12, Symmetric Monoidality*: See [`MO 382264`].
*Item 13, Symmetric Bimonoidality*: Omitted. □

**2.1.3. Pullbacks.** Let $A$, $B$, and $C$ be sets and let $f\colon A\to C$ and $g\colon B\to C$ be functions.

**Definition 2.1.3.1.1.** The **pullback of $A$ and $B$ over $C$ along $f$ and $g$**[6] is the pair[7] $(A\times_C B,\{\mathrm{pr}_1,\mathrm{pr}_2\})$ consisting of

- *The Limit.* The set $A\times_C B$ defined by

$$A\times_C B\stackrel{\text{def}}{=}\{(a,b)\in A\times B\mid f(a)=g(b)\}.$$

- *The Cone.* The maps

$$\begin{aligned}\mathrm{pr}_1&\colon A\times_C B\to A,\\ \mathrm{pr}_2&\colon A\times_C B\to B\end{aligned}$$

[6] *Further Terminology:* Also called the **fibre product of $A$ and $B$ over $C$ along $f$ and $g$**.
[7] *Further Notation:* Also written $A\times_{f,C,g}B$.

defined by

$$\begin{aligned}\mathrm{pr}_1(a,b) &\overset{\mathrm{def}}{=} a,\\ \mathrm{pr}_2(a,b) &\overset{\mathrm{def}}{=} b\end{aligned}$$

for each $(a,b) \in A \times_C B$.

**Proof.** We claim that $A \times_C B$ is the categorical pullback of $A$ and $B$ over $C$ with respect to $(f,g)$ in $\mathsf{Sets}$. First we need to check that the relevant pullback diagram commutes, i.e. that we have

$$f \circ \mathrm{pr}_1 = g \circ \mathrm{pr}_2, \qquad \begin{tikzcd} A \times_C B \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C. \end{tikzcd}$$

Indeed, given $(a,b) \in A \times_C B$, we have

$$\begin{aligned}[f \circ \mathrm{pr}_1](a,b) &= f(\mathrm{pr}_1(a,b))\\ &= f(a)\\ &= g(b)\\ &= g(\mathrm{pr}_2(a,b))\\ &= [g \circ \mathrm{pr}_2](a,b),\end{aligned}$$

where $f(a) = g(b)$ since $(a,b) \in A \times_C B$. Next, we prove that $A \times_C B$ satisfies the universal property of the pullback. Suppose we have a diagram of the form

$$\begin{tikzcd} P \arrow[drr, bend left, "p_2"] \arrow[ddr, bend right, "p_1"'] & & \\ & A \times_C B \arrow[r, dashed, "\mathrm{pr}_2"] \arrow[d, dashed, "\mathrm{pr}_1"] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "g"] \\ & A \arrow[r, "f"'] & C \end{tikzcd}$$

in $\mathsf{Sets}$. Then there exists a unique map $\phi\colon P \to A \times_C B$, uniquely determined by the conditions

$$\begin{aligned}\mathrm{pr}_1 \circ \phi &= p_1,\\ \mathrm{pr}_2 \circ \phi &= p_2,\end{aligned}$$

being necessarily given by

$$\phi(x) = (p_1(x), p_2(x))$$

for each $x \in P$, where we note that $(p_1(x), p_2(x)) \in A \times B$ indeed lies in $A \times_C B$ by the condition

$$f \circ p_1 = g \circ p_2,$$

which gives

$$f(p_1(x)) = g(p_2(x))$$

for each $x \in P$, so that $(p_1(x), p_2(x)) \in A \times_C B$. $\square$

**Example 2.1.3.1.2.** Here are some examples of pullbacks of sets.

(1) *Unions via Intersections.* Let $A, B \subset X$. We have a bijection of sets

$$A \cap B \cong A \times_{A \cup B} B, \qquad \begin{tikzcd} A \cap B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "\iota_B"] \\ A \arrow[r, "\iota_A"'] & A \cup B. \end{tikzcd}$$

**Proof.** *Item 1, Unions via Intersections*: Indeed, we have

$$\begin{aligned} A \times_{A \cup B} B &\cong \{(x, y) \in A \times B \mid x = y\} \\ &\cong A \cap B. \end{aligned}$$

This finishes the proof. $\square$

**Proposition 2.1.3.1.3.** Let $A$, $B$, $C$, and $X$ be sets.

(1) *Functoriality.* The assignment $(A, B, C, f, g) \mapsto A \times_{f,C,g} B$ defines a functor

$$-_1 \times_{-_3} -_1 \colon \mathsf{Fun}(\mathcal{P}, \mathsf{Sets}) \to \mathsf{Sets},$$

where $\mathcal{P}$ is the category that looks like this:

$$\begin{tikzcd} & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet. \end{tikzcd}$$

In particular, the action on morphisms of $-_1 \times_{-_3} -_1$ is given by sending a morphism

$$\begin{tikzcd} A \times_C B \arrow[rr] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & & B \arrow[dd, "g"' near start] \arrow[dr, "\psi"] & \\ & A' \times_{C'} B' \arrow[rr] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & & B' \arrow[dd, "g'"] \\ A \arrow[rr, "f" near start] \arrow[dr, "\phi"'] & & C \arrow[dr, "\chi"] & \\ & A' \arrow[rr, "f'"'] & & C' \end{tikzcd}$$

in $\mathsf{Fun}(\mathcal{P}, \mathsf{Sets})$ to the map $\xi \colon A \times_C B \xrightarrow{\exists!} A' \times_{C'} B'$ given by

$$\xi(a, b) \stackrel{\text{def}}{=} (\phi(a), \psi(b))$$

for each $(a,b) \in A \times_C B$, which is the unique map making the diagram

$$\begin{tikzcd}
A \times_C B \arrow[rr] \arrow[dd] \arrow[dr, dashed] \arrow[dr, phantom, "\lrcorner", very near start] & & B \arrow[dd, "g" near start] \arrow[dr, "\psi"] & \\
& A' \times_{C'} B' \arrow[rr] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & & B' \arrow[dd, "g'"] \\
A \arrow[rr, "f" near start] \arrow[dr, "\phi"'] & & C \arrow[dr, "\chi"] & \\
& A' \arrow[rr, "f'"'] & & C'
\end{tikzcd}$$

commute.

(2) *Associativity.* Given a diagram

$$\begin{tikzcd}
A \arrow[dr, "f"'] & & B \arrow[dl, "g"] \arrow[dr, "h"'] & & C \arrow[dl, "k"] \\
& X & & Y &
\end{tikzcd}$$

in $\mathsf{Sets}$, we have isomorphisms

$$(A \times_X B) \times_Y C \cong (A \times_X B) \times_B (B \times_Y C) \cong A \times_X (B \times_Y C),$$

where these pullbacks are built as in the diagrams

$$\begin{tikzcd}[column sep=tiny]
& & (A \times_X B) \times_Y C \arrow[dl] \arrow[drr] \arrow[d, phantom, "\vee"] & & \\
& A \times_X B \arrow[dl] \arrow[dr] \arrow[d, phantom, "\vee"] & {} & & \\
A \arrow[dr, "f"'] & {} & B \arrow[dl, "g"] \arrow[dr, "h"'] & & C, \arrow[dl, "k"] \\
& X & & Y &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=tiny]
& & (A \times_X B) \times_B (B \times_Y C) \arrow[dl] \arrow[dr] \arrow[d, phantom, "\vee"] & & \\
& A \times_X B \arrow[dl] \arrow[dr] \arrow[d, phantom, "\vee"] & {} & B \times_Y C \arrow[dl] \arrow[dr] \arrow[d, phantom, "\vee"] & \\
A \arrow[dr, "f"'] & {} & B \arrow[dl, "g"] \arrow[dr, "h"'] & {} & C, \arrow[dl, "k"] \\
& X & & Y &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=tiny]
& & A \times_X (B \times_Y C) \arrow[dll] \arrow[dr] \arrow[d, phantom, "\vee"] & & \\
& & {} & B \times_Y C \arrow[dl] \arrow[dr] \arrow[d, phantom, "\vee"] & \\
A \arrow[dr, "f"'] & & B \arrow[dl, "g"] \arrow[dr, "h"'] & {} & C. \arrow[dl, "k"] \\
& X & & Y &
\end{tikzcd}$$

(3) *Unitality.* We have isomorphisms of sets

$$\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\
X \arrow[r, equal] & X
\end{tikzcd}
\qquad
\begin{aligned}
X \times_X A &\cong A, \\
A \times_X X &\cong A,
\end{aligned}
\qquad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, equal] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, equal] \\
X \arrow[r, "f"'] & X.
\end{tikzcd}$$

(4) *Commutativity.* We have an isomorphism of sets

$$\begin{tikzcd}
A \times_X B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "g"] \\
A \arrow[r, "f"'] & X,
\end{tikzcd}
\qquad
A \times_X B \cong B \times_X A
\qquad
\begin{tikzcd}
B \times_X A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\
B \arrow[r, "g"'] & X.
\end{tikzcd}$$

(5) *Annihilation With the Empty Set.* We have isomorphisms of sets

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \emptyset \arrow[d] \\ A \arrow[r, "f"'] & X, \end{tikzcd} \qquad \begin{aligned} A \times_X \emptyset &\cong \emptyset, \\ \emptyset \times_X A &\cong \emptyset, \end{aligned} \qquad \begin{tikzcd} \emptyset \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\ \emptyset \arrow[r] & X. \end{tikzcd}$$

(6) *Interaction With Products.* We have

$$A \times_{\mathrm{pt}} B \cong A \times B, \qquad \begin{tikzcd} A \times B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "!_B"] \\ A \arrow[r, "!_A"'] & \mathrm{pt}. \end{tikzcd}$$

(7) *Symmetric Monoidality.* The triple $(\mathsf{Sets}, \times_X, X)$ is a symmetric monoidal category.

**Proof.** *Item 1, Functoriality*: This is a special case of functoriality of co/limits, **??** of **??**, with the explicit expression for $\xi$ following from the commutativity of the cube pullback diagram.
*Item 2, Associativity*: Indeed, we have

$$\begin{aligned}
(A \times_X B) \times_Y C &\cong \{((a,b),c) \in (A \times_X B) \times C \mid h(b) = k(c)\} \\
&\cong \{((a,b),c) \in (A \times B) \times C \mid f(a) = g(b) \text{ and } h(b) = k(c)\} \\
&\cong \{(a,(b,c)) \in A \times (B \times C) \mid f(a) = g(b) \text{ and } h(b) = k(c)\} \\
&\cong \{(a,(b,c)) \in A \times (B \times_Y C) \mid f(a) = g(b)\} \\
&\cong A \times_X (B \times_Y C)
\end{aligned}$$

and

$$\begin{aligned}
(A \times_X B) \times_B (B \times_Y C) &\cong \left\{ \left((a,b),\left(b',c\right)\right) \in (A \times_X B) \times (B \times_Y C) \;\middle|\; b = b' \right\} \\
&\cong \left\{ \left((a,b),\left(b',c\right)\right) \in (A \times B) \times (B \times C) \;\middle|\; \begin{array}{l} f(a) = g(b),\ b = b', \\ \text{and } h\left(b'\right) = k(c) \end{array} \right\} \\
&\cong \left\{ \left(a,\left(b,\left(b',c\right)\right)\right) \in A \times (B \times (B \times C)) \;\middle|\; \begin{array}{l} f(a) = g(b),\ b = b', \\ \text{and } h\left(b'\right) = k(c) \end{array} \right\} \\
&\cong \left\{ \left(a,\left(\left(b,b'\right),c\right)\right) \in A \times ((B \times B) \times C) \;\middle|\; \begin{array}{l} f(a) = g(b),\ b = b', \\ \text{and } h\left(b'\right) = k(c) \end{array} \right\} \\
&\cong \left\{ \left(a,\left(\left(b,b'\right),c\right)\right) \in A \times ((B \times_B B) \times C) \;\middle|\; \begin{array}{l} f(a) = g(b) \text{ and} \\ h\left(b'\right) = k(c) \end{array} \right\} \\
&\cong \{(a,(b,c)) \in A \times (B \times C) \mid f(a) = g(b) \text{ and } h(b) = k(c)\} \\
&\cong A \times_X (B \times_Y C),
\end{aligned}$$

where we have used Item 3 for the isomorphism $B \times_B B \cong B$.
*Item 3, Unitality*: Indeed, we have

$$\begin{aligned}
X \times_X A &\cong \{(x,a) \in X \times A \mid f(a) = x\}, \\
A \times_X X &\cong \{(a,x) \in X \times A \mid f(a) = x\},
\end{aligned}$$

which are isomorphic to $A$ via the maps $(x,a) \mapsto a$ and $(a,x) \mapsto a$.
*Item 4, Commutativity*: Clear.

*Item 5, Annihilation With the Empty Set*: Clear.
*Item 6, Interaction With Products*: Clear.
*Item 7, Symmetric Monoidality*: Omitted. □

**2.1.4. Equalisers.** Let $A$ and $B$ be sets and let $f, g\colon A \rightrightarrows B$ be functions.

**Definition 2.1.4.1.1.** The **equaliser of $f$ and $g$** is the pair $(\mathrm{Eq}(f,g), \mathrm{eq}(f,g))$ consisting of

- *The Limit.* The set $\mathrm{Eq}(f,g)$ defined by
$$\mathrm{Eq}(f,g) \overset{\mathrm{def}}{=} \{a \in A \mid f(a) = g(a)\}.$$
- *The Cone.* The inclusion map
$$\mathrm{eq}(f,g)\colon \mathrm{Eq}(f,g) \hookrightarrow A.$$

**Proof.** We claim that $\mathrm{Eq}(f,g)$ is the categorical equaliser of $f$ and $g$ in $\mathsf{Sets}$. First we need to check that the relevant equaliser diagram commutes, i.e. that we have
$$f \circ \mathrm{eq}(f,g) = g \circ \mathrm{eq}(f,g),$$
which indeed holds by the definition of the set $\mathrm{Eq}(f,g)$. Next, we prove that $\mathrm{Eq}(f,g)$ satisfies the universal property of the equaliser. Suppose we have a diagram of the form
$$\begin{tikzcd}
\mathrm{Eq}(f,g) \arrow[r, hook, "\mathrm{eq}(f,g)"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\
E \arrow[ur, "e"'] & &
\end{tikzcd}$$
in $\mathsf{Sets}$. Then there exists a unique map $\phi\colon E \to \mathrm{Eq}(f,g)$, uniquely determined by the condition
$$\mathrm{eq}(f,g) \circ \phi = e$$
being necessarily given by
$$\phi(x) = e(x)$$
for each $x \in E$, where we note that $e(x) \in A$ indeed lies in $\mathrm{Eq}(f,g)$ by the condition
$$f \circ e = g \circ e,$$
which gives
$$f(e(x)) = g(e(x))$$
for each $x \in E$, so that $e(x) \in \mathrm{Eq}(f,g)$. □

**Proposition 2.1.4.1.2.** Let $A$, $B$, and $C$ be sets.

(1) *Associativity.* We have an isomorphism of sets[8]
$$\underbrace{\mathrm{Eq}(f \circ \mathrm{eq}(g,h), g \circ \mathrm{eq}(g,h))}_{=\mathrm{Eq}(f\circ\mathrm{eq}(g,h), h\circ\mathrm{eq}(g,h))} \cong \mathrm{Eq}(f,g,h) \cong \underbrace{\mathrm{Eq}(f \circ \mathrm{eq}(f,g), h \circ \mathrm{eq}(f,g))}_{=\mathrm{Eq}(g\circ\mathrm{eq}(f,g), h\circ\mathrm{eq}(f,g))},$$

[8]That is, the following three ways of forming "the" equaliser of $(f,g,h)$ agree:

(1) Take the equaliser of $(f,g,h)$, i.e. the limit of the diagram
$$\begin{tikzcd} A \arrow[r, shift left=2, "f"] \arrow[r, "g" description] \arrow[r, shift right=2, "h"'] & B \end{tikzcd}$$
in $\mathsf{Sets}$.

where $\mathrm{Eq}(f,g,h)$ is the limit of the diagram

$$\begin{tikzcd} A \arrow[r, shift left=2, "f"] \arrow[r, "g" description] \arrow[r, shift right=2, "h"'] & B \end{tikzcd}$$

in $\mathsf{Sets}$, being explicitly given by

$$\mathrm{Eq}(f,g,h) \cong \{a \in A \mid f(a) = g(a) = h(a)\}.$$

(4) *Unitality.* We have an isomorphism of sets

$$\mathrm{Eq}(f,f) \cong A.$$

(5) *Commutativity.* We have an isomorphism of sets

$$\mathrm{Eq}(f,g) \cong \mathrm{Eq}(g,f).$$

(6) *Interaction With Composition.* Let

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \underset{k}{\overset{h}{\rightrightarrows}} C$$

be functions. We have an inclusion of sets

$$\mathrm{Eq}(h \circ f \circ \mathrm{eq}(f,g), k \circ g \circ \mathrm{eq}(f,g)) \subset \mathrm{Eq}(h \circ f, k \circ g),$$

where $\mathrm{Eq}(h \circ f \circ \mathrm{eq}(f,g), k \circ g \circ \mathrm{eq}(f,g))$ is the equaliser of the composition

$$\mathrm{Eq}(f,g) \overset{\mathrm{eq}(f,g)}{\hookrightarrow} A \underset{g}{\overset{f}{\rightrightarrows}} B \underset{k}{\overset{h}{\rightrightarrows}} C.$$

**Proof.** *Item 1, Associativity*: We first prove that $\mathrm{Eq}(f,g,h)$ is indeed given by

$$\mathrm{Eq}(f,g,h) \cong \{a \in A \mid f(a) = g(a) = h(a)\}.$$

---

(2) First take the equaliser of $f$ and $g$, forming a diagram

$$\mathrm{Eq}(f,g) \overset{\mathrm{eq}(f,g)}{\hookrightarrow} A \underset{g}{\overset{f}{\rightrightarrows}} B$$

and then take the equaliser of the composition

$$\mathrm{Eq}(f,g) \overset{\mathrm{eq}(f,g)}{\hookrightarrow} A \underset{h}{\overset{f}{\rightrightarrows}} B,$$

obtaining a subset

$$\mathrm{Eq}(f \circ \mathrm{eq}(f,g), h \circ \mathrm{eq}(f,g)) = \mathrm{Eq}(g \circ \mathrm{eq}(f,g), h \circ \mathrm{eq}(f,g))$$

of $\mathrm{Eq}(f,g)$.

(3) First take the equaliser of $g$ and $h$, forming a diagram

$$\mathrm{Eq}(g,h) \overset{\mathrm{eq}(g,h)}{\hookrightarrow} A \underset{h}{\overset{g}{\rightrightarrows}} B$$

and then take the equaliser of the composition

$$\mathrm{Eq}(g,h) \overset{\mathrm{eq}(g,h)}{\hookrightarrow} A \underset{g}{\overset{f}{\rightrightarrows}} B,$$

obtaining a subset

$$\mathrm{Eq}(f \circ \mathrm{eq}(g,h), g \circ \mathrm{eq}(g,h)) = \mathrm{Eq}(f \circ \mathrm{eq}(g,h), h \circ \mathrm{eq}(g,h))$$

of $\mathrm{Eq}(g,h)$.

Indeed, suppose we have a diagram of the form

$$\begin{tikzcd} \mathrm{Eq}(f,g,h) \arrow[r, hook, "\mathrm{eq}(f,g,h)"] & A \arrow[r, shift left=2, "f"] \arrow[r, "g"] \arrow[r, shift right=2, "h"'] & B \\ E \arrow[ur, "e"'] & & \end{tikzcd}$$

in $\mathsf{Sets}$. Then there exists a unique map $\phi\colon E \to \mathrm{Eq}(f,g,h)$, uniquely determined by the condition

$$\mathrm{eq}(f,g) \circ \phi = e$$

being necessarily given by

$$\phi(x) = e(x)$$

for each $x \in E$, where we note that $e(x) \in A$ indeed lies in $\mathrm{Eq}(f,g,h)$ by the condition

$$f \circ e = g \circ e = h \circ e,$$

which gives

$$f(e(x)) = g(e(x)) = h(e(x))$$

for each $x \in E$, so that $e(x) \in \mathrm{Eq}(f,g,h)$.
We now check the equalities

$$\mathrm{Eq}(f \circ \mathrm{eq}(g,h), g \circ \mathrm{eq}(g,h)) \cong \mathrm{Eq}(f,g,h) \cong \mathrm{Eq}(f \circ \mathrm{eq}(f,g), h \circ \mathrm{eq}(f,g)).$$

Indeed, we have

$$\begin{aligned} \mathrm{Eq}(f \circ \mathrm{eq}(g,h), g \circ \mathrm{eq}(g,h)) &\cong \{x \in \mathrm{Eq}(g,h) \mid [f \circ \mathrm{eq}(g,h)](a) = [g \circ \mathrm{eq}(g,h)](a)\} \\ &\cong \{x \in \mathrm{Eq}(g,h) \mid f(a) = g(a)\} \\ &\cong \{x \in A \mid f(a) = g(a) \text{ and } g(a) = h(a)\} \\ &\cong \{x \in A \mid f(a) = g(a) = h(a)\} \\ &\cong \mathrm{Eq}(f,g,h). \end{aligned}$$

Similarly, we have

$$\begin{aligned} \mathrm{Eq}(f \circ \mathrm{eq}(f,g), h \circ \mathrm{eq}(f,g)) &\cong \{x \in \mathrm{Eq}(f,g) \mid [f \circ \mathrm{eq}(f,g)](a) = [h \circ \mathrm{eq}(f,g)](a)\} \\ &\cong \{x \in \mathrm{Eq}(f,g) \mid f(a) = h(a)\} \\ &\cong \{x \in A \mid f(a) = h(a) \text{ and } f(a) = g(a)\} \\ &\cong \{x \in A \mid f(a) = g(a) = h(a)\} \\ &\cong \mathrm{Eq}(f,g,h). \end{aligned}$$

*Item 4*, *Unitality*: Clear.
*Item 5*, *Commutativity*: Clear.
*Item 6*, *Interaction With Composition*: Indeed, we have

$$\begin{aligned} \mathrm{Eq}(h \circ f \circ \mathrm{eq}(f,g), k \circ g \circ \mathrm{eq}(f,g)) &\cong \{a \in \mathrm{Eq}(f,g) \mid h(f(a)) = k(g(a))\} \\ &\cong \{a \in A \mid f(a) = g(a) \text{ and } h(f(a)) = k(g(a))\}. \end{aligned}$$

and

$$\mathrm{Eq}(h \circ f, k \circ g) \cong \{a \in A \mid h(f(a)) = k(g(a))\},$$

and thus there's an inclusion from $\mathrm{Eq}(h \circ f \circ \mathrm{eq}(f,g), k \circ g \circ \mathrm{eq}(f,g))$ to $\mathrm{Eq}(h \circ f, k \circ g)$. □

## 2.2. Colimits of Sets

**2.2.1. Coproducts of Families of Sets.** Let $\{A_i\}_{i \in I}$ be a family of sets.

**Definition 2.2.1.1.1.** The **disjoint union of the family** $\{A_i\}_{i\in I}$ is the pair $(\coprod_{i\in I} A_i, \{\mathrm{inj}_i\}_{i\in I})$ consisting of

- *The Colimit.* The set $\coprod_{i\in I} A_i$ defined by
$$\coprod_{i\in I} A_i \stackrel{\text{def}}{=} \left\{(i,x) \in I \times \left(\bigcup_{i\in I} A_i\right) \;\middle|\; x \in A_i\right\}.$$
- *The Cocone.* The collection
$$\left\{\mathrm{inj}_i\colon A_i \to \coprod_{i\in I} A_i\right\}_{i\in I}$$
of maps given by
$$\mathrm{inj}_i(x) \stackrel{\text{def}}{=} (i,x)$$
for each $x \in A_i$ and each $i \in I$.

**Proof.** We claim that $\coprod_{i\in I} A_i$ is the categorical coproduct of $\{A_i\}_{i\in I}$ in $\mathsf{Sets}$. Indeed, suppose we have, for each $i \in I$, a diagram of the form

$$\begin{tikzcd} & C \\ A_i \arrow[ur, "\iota_i"] \arrow[r, "\mathrm{inj}_i"'] & \coprod_{i\in I} A_i \end{tikzcd}$$

in $\mathsf{Sets}$. Then there exists a unique map $\phi\colon \coprod_{i\in I} A_i \to C$, uniquely determined by the condition $\phi \circ \mathrm{inj}_i = \iota_i$ for each $i \in I$, being necessarily given by
$$\phi(i,x) = \iota_i(x)$$
for each $(i,x) \in \coprod_{i\in I} A_i$. $\square$

**Proposition 2.2.1.1.2.** Let $\{A_i\}_{i\in I}$ be a family of sets.

(1) *Functoriality.* The assignment $\{A_i\}_{i\in I} \mapsto \coprod_{i\in I} A_i$ defines a functor
$$\coprod_{i\in I}\colon \mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}) \to \mathsf{Sets}$$
where
- *Action on Objects.* For each $(A_i)_{i\in I} \in \mathrm{Obj}(\mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}))$, we have
$$\left[\coprod_{i\in I}\right]((A_i)_{i\in I}) \stackrel{\text{def}}{=} \coprod_{i\in I} A_i$$
- *Action on Morphisms.* For each $(A_i)_{i\in I}, (B_i)_{i\in I} \in \mathrm{Obj}(\mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}))$, the action on Hom-sets
$$\left(\coprod_{i\in I}\right)_{(A_i)_{i\in I},(B_i)_{i\in I}}\colon \mathrm{Nat}\big((A_i)_{i\in I}, (B_i)_{i\in I}\big) \to \mathsf{Sets}\left(\coprod_{i\in I} A_i, \coprod_{i\in I} B_i\right)$$
of $\coprod_{i\in I}$ at $\big((A_i)_{i\in I}, (B_i)_{i\in I}\big)$ is defined by sending a map
$$\{f_i\colon A_i \to B_i\}_{i\in I}$$

in $\mathrm{Nat}\big((A_i)_{i\in I}, (B_i)_{i\in I}\big)$ to the map of sets

$$\coprod_{i\in I} f_i \colon \coprod_{i\in I} A_i \to \coprod_{i\in I} B_i$$

defined by

$$\left[\coprod_{i\in I} f_i\right](i,a) \stackrel{\text{def}}{=} f_i(a)$$

for each $(i,a) \in \coprod_{i\in I} A_i$.

**Proof.** *Item 1, Functoriality*: Clear. □

**2.2.2. Binary Coproducts.** Let $A$ and $B$ be sets.

**Definition 2.2.2.1.1.** The **coproduct**[9] **of** $A$ **and** $B$ is the pair $(A \coprod B, \{\mathrm{inj}_1, \mathrm{inj}_2\})$ consisting of

- *The Colimit.* The set $A \coprod B$ defined by

$$\begin{aligned} A \coprod B &\stackrel{\text{def}}{=} \coprod_{z\in\{A,B\}} z \\ &\cong \{(0,a) \mid a \in A\} \cup \{(1,b) \mid b \in B\}. \end{aligned}$$

- *The Cocone.* The maps

$$\begin{aligned} \mathrm{inj}_1 &\colon A \to A \coprod B, \\ \mathrm{inj}_2 &\colon B \to A \coprod B, \end{aligned}$$

  given by

$$\begin{aligned} \mathrm{inj}_1(a) &\stackrel{\text{def}}{=} (0,a), \\ \mathrm{inj}_2(b) &\stackrel{\text{def}}{=} (1,b), \end{aligned}$$

  for each $a \in A$ and each $b \in B$.

**Proof.** We claim that $A \coprod B$ is the categorical coproduct of $A$ and $B$ in $\mathsf{Sets}$. Indeed, suppose we have a diagram of the form

$$\begin{tikzcd} & C & \\ A \arrow[ur, bend left, "i_A"] \arrow[r, "\mathrm{inj}_A"'] & A \coprod B & B \arrow[ul, bend right, "i_B"'] \arrow[l, "\mathrm{inj}_B"] \end{tikzcd}$$

in $\mathsf{Sets}$. Then there exists a unique map $\phi\colon A \coprod B \to C$, uniquely determined by the conditions

$$\begin{aligned} \phi \circ \mathrm{inj}_A &= i_A, \\ \phi \circ \mathrm{inj}_B &= i_B, \end{aligned}$$

being necessarily given by

$$\phi(x) = \begin{cases} i_A(x) & \text{if } x \in A, \\ i_B(x) & \text{if } x \in B \end{cases}$$

for each $x \in C$. □

[9] *Further Terminology:* Also called the **disjoint union of** $A$ **and** $B$, or the **binary disjoint union of** $A$ **and** $B$, for emphasis.

**Proposition 2.2.2.1.2.** Let $A$, $B$, $C$, and $X$ be sets.

(1) *Functoriality.* The assignment $A, B, (A, B) \mapsto A \coprod B$ defines functors

$$A \coprod -_2 \colon \mathsf{Sets} \to \mathsf{Sets},$$
$$-_1 \coprod B \colon \mathsf{Sets} \to \mathsf{Sets},$$
$$-_1 \coprod -_2 \colon \mathsf{Sets} \times \mathsf{Sets} \to \mathsf{Sets},$$

where $-_1 \coprod -_2$ is the functor where

- *Action on Objects.* For each $(A, B) \in \mathrm{Obj}(\mathsf{Sets} \times \mathsf{Sets})$, we have
$$[-_1 \coprod -_2](A, B) \stackrel{\text{def}}{=} A \coprod B;$$
- *Action on Morphisms.* For each $(A, B), (X, Y) \in \mathrm{Obj}(\mathsf{Sets})$, the action on Hom-sets
$$\coprod_{(A,B),(X,Y)} \colon \mathsf{Sets}(A, X) \times \mathsf{Sets}(B, Y) \to \mathsf{Sets}(A \coprod B, X \coprod Y)$$
of $\coprod$ at $((A, B), (X, Y))$ is defined by sending $(f, g)$ to the function
$$f \coprod g \colon A \coprod B \to X \coprod Y$$
defined by
$$[f \coprod g](x) \stackrel{\text{def}}{=} \begin{cases} f(x) & \text{if } x \in A, \\ g(x) & \text{if } x \in B \end{cases}$$
for each $x \in A \coprod B$;

and where $A \coprod -$ and $- \coprod B$ are the partial functors of $-_1 \coprod -_2$ at $A, B \in \mathrm{Obj}(\mathsf{Sets})$.

(2) *Associativity.* We have an isomorphism of sets
$$(A \coprod B) \coprod C \cong A \coprod (B \coprod C),$$
natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

(3) *Unitality.* We have isomorphisms of sets
$$A \coprod \emptyset \cong A,$$
$$\emptyset \coprod A \cong A,$$
natural in $A \in \mathrm{Obj}(\mathsf{Sets})$.

(4) *Commutativity.* We have an isomorphism of sets
$$A \coprod B \cong B \coprod A,$$
natural in $A, B \in \mathrm{Obj}(\mathsf{Sets})$.

(5) *Symmetric Monoidality.* The triple $(\mathsf{Sets}, \coprod, \emptyset)$ is a symmetric monoidal category.

**Proof.** *Item 1, Functoriality*: Clear.
*Item 2, Associativity*: Clear.
*Item 3, Unitality*: Clear.
*Item 4, Commutativity*: Clear.
*Item 5, Symmetric Monoidality*: Omitted. □

**2.2.3. Pushouts.** Let $A$, $B$, and $C$ be sets and let $f \colon C \to A$ and $g \colon C \to B$ be functions.

**Definition 2.2.3.1.1.** The **pushout of $A$ and $B$ over $C$ along $f$ and $g$**[10] is the pair $(A \coprod_C B, \{\mathrm{inj}_1, \mathrm{inj}2\})$ consisting of

- *The Colimit.* The set $A \coprod_C B$ defined by
$$A \coprod_C B \overset{\text{def}}{=} A \coprod B/{\sim_C},$$
where $\sim_C$ is the equivalence relation on $A \coprod B$ generated by $(0, f(c)) \sim_C (1, g(c))$.
- *The Cocone.* The maps
$$\mathrm{inj}_1 \colon A \to A \coprod_C B,$$
$$\mathrm{inj}_2 \colon B \to A \coprod_C B$$
given by
$$\mathrm{inj}_1(a) \overset{\text{def}}{=} [(0, a)]$$
$$\mathrm{inj}_2(b) \overset{\text{def}}{=} [(1, b)]$$
for each $a \in A$ and each $b \in B$.

**Proof.** We claim that $A \coprod_C B$ is the categorical pushout of $A$ and $B$ over $C$ with respect to $(f, g)$ in $\mathsf{Sets}$. First we need to check that the relevant pushout diagram commutes, i.e. that we have

$$\mathrm{inj}_1 \circ f = \mathrm{inj}_2 \circ g, \qquad \begin{tikzcd} A \coprod_C B & B \arrow[l, "\mathrm{inj}_2"'] \\ A \arrow[u, "\mathrm{inj}_1"] & C. \arrow[l, "f"] \arrow[u, "g"'] \end{tikzcd}$$

Indeed, given $c \in C$, we have

$$\begin{aligned} [\mathrm{inj}_1 \circ f](c) &= \mathrm{inj}_1(f(c)) \\ &= [(0, f(c))] \\ &= [(1, g(c))] \\ &= \mathrm{inj}_2(g(c)) \\ &= [\mathrm{inj}_2 \circ g](c), \end{aligned}$$

where $[(0, f(c))] = [(1, g(c))]$ by the definition of the relation $\sim$ on $B$. Next, we prove that $A \coprod_C B$ satisfies the universal property of the pushout. Suppose we have a diagram of the form

$$\begin{tikzcd} P & & \\ & A \coprod_C B \arrow[dr, phantom, "\ulcorner"] & B \arrow[l, dashed, "\mathrm{inj}_2"'] \arrow[ull, bend right, "i_2"'] \\ & A \arrow[u, dashed, "\mathrm{inj}_1"] \arrow[uul, bend left, "i_1"] & C \arrow[l, "f"] \arrow[u, "g"'] \end{tikzcd}$$

[10] *Further Terminology:* Also called the **fibre coproduct of $A$ and $B$ over $C$ along $f$ and $g$**.

in $\mathsf{Sets}$. Then there exists a unique map $\phi\colon A\coprod_C B \to P$, uniquely determined by the conditions

$$\begin{aligned}\phi \circ \mathrm{inj}_1 &= i_1,\\ \phi \circ \mathrm{inj}_2 &= i_2,\end{aligned}$$

being necessarily given by

$$\phi(x) = \begin{cases} i_1(a) & \text{if } x = [(0,a)],\\ i_2(b) & \text{if } x = [(1,b)]\end{cases}$$

for each $x \in A \coprod_C B$, where the well-definedness of $\phi$ is guaranteed by the equality $i_1 \circ f = i_2 \circ g$ and the definition of the relation $\sim$ on $A \coprod B$ as follows.

(1) *Case 1:* Suppose we have $x = [(0,a)] = [(0,a')]$ for some $a, a' \in A$. Then, by Remark 2.2.3.1.2, we have a sequence
$$(0,a) \sim' x_1 \sim' \cdots \sim' x_n \sim' (0,a').$$

(2) *Case 2:* Suppose we have $x = [(1,b)] = [(1,b')]$ for some $b, b' \in B$. Then, by Remark 2.2.3.1.2, we have a sequence
$$(1,b) \sim' x_1 \sim' \cdots \sim' x_n \sim' (1,b').$$

(3) *Case 3:* Suppose we have $x = [(0,a)] = [(1,b)]$ for some $a \in A$ and $b \in B$. Then, by Remark 2.2.3.1.2, we have a sequence
$$(0,a) \sim' x_1 \sim' \cdots \sim' x_n \sim' (1,b).$$

In all these cases, we declare $x \sim' y$ iff there exists some $c \in C$ such that $x = (0, f(c))$ and $y = (1, g(c))$ or $x = (1, g(c))$ and $y = (0, f(c))$. Then, the equality $i_1 \circ f = i_2 \circ g$ gives

$$\begin{aligned}\phi([x]) &= \phi([(0,f(c))])\\ &\overset{\text{def}}{=} i_1(f(c))\\ &= i_2(g(c))\\ &\overset{\text{def}}{=} \phi([(1,g(c))])\\ &= \phi([y]),\end{aligned}$$

with the case where $x = (1, g(c))$ and $y = (0, f(c))$ similarly giving $\phi([x]) = \phi([y])$. Thus, if $x \sim' y$, then $\phi([x]) = \phi([y])$. Applying this equality pairwise to the sequences

$$\begin{aligned}(0,a) \sim' x_1 \sim' \cdots \sim' x_n \sim' (0,a'),\\ (1,b) \sim' x_1 \sim' \cdots \sim' x_n \sim' (1,b'),\\ (0,a) \sim' x_1 \sim' \cdots \sim' x_n \sim' (1,b)\end{aligned}$$

gives

$$\begin{aligned}\phi([(0,a)]) &= \phi([(0,a')]),\\ \phi([(1,b)]) &= \phi([(1,b')]),\\ \phi([(0,a)]) &= \phi([(1,b)]),\end{aligned}$$

showing $\phi$ to be well-defined. $\square$

**Remark 2.2.3.1.2.** In detail, by **??**, the relation $\sim$ of Definition 2.2.3.1.1 is given by declaring $a \sim b$ iff one of the following conditions is satisfied:

- We have $a, b \in A$ and $a = b$;
- We have $a, b \in B$ and $a = b$;
- There exist $x_1, \ldots, x_n \in A \coprod B$ such that $a \sim' x_1 \sim' \cdots \sim' x_n \sim' b$, where we declare $x \sim' y$ if one of the following conditions is satisfied:
  (1) There exists $c \in C$ such that $x = (0, f(c))$ and $y = (1, g(c))$.
  (2) There exists $c \in C$ such that $x = (1, g(c))$ and $y = (0, f(c))$.

  That is: we require the following condition to be satisfied:
  ($\star$) There exist $x_1, \ldots, x_n \in A \coprod B$ satisfying the following conditions:
    (1) There exists $c_0 \in C$ satisfying one of the following conditions:
      (a) We have $a = f(c_0)$ and $x_1 = g(c_0)$.
      (b) We have $a = g(c_0)$ and $x_1 = f(c_0)$.
    (2) For each $1 \leq i \leq n-1$, there exists $c_i \in C$ satisfying one of the following conditions:
      (a) We have $x_i = f(c_i)$ and $x_{i+1} = g(c_i)$.
      (b) We have $x_i = g(c_i)$ and $x_{i+1} = f(c_i)$.
    (3) There exists $c_n \in C$ satisfying one of the following conditions:
      (a) We have $x_n = f(c_n)$ and $b = g(c_n)$.
      (b) We have $x_n = g(c_n)$ and $b = f(c_n)$.

**Example 2.2.3.1.3.** Here are some examples of pushouts of sets.

(1) *Wedge Sums of Pointed Sets.* The wedge sum of two pointed sets of **??** is an example of a pushout of sets.
(2) *Intersections via Unions.* Let $A, B \subset X$. We have a bijection of sets

$$A \cup B \cong A \textstyle\coprod_{A \cap B} B, \qquad \begin{tikzcd} A \cup B & B \arrow[l, hook'] \\ A \arrow[u, hook] \arrow[ur, phantom, very near end, "\ulcorner"] & A \cap B. \arrow[l, hook'] \arrow[u, hook] \end{tikzcd}$$

**Proof.** *Item 1, Wedge Sums of Pointed Sets*: Follows by definition.
*Item 2, Intersections via Unions*: Indeed, $A \coprod_{A \cap B} B$ is the quotient of $A \coprod B$ by the equivalence relation obtained by declaring $(0, a) \sim (1, b)$ iff $a = b \in A \cap B$, which is in bijection with $A \cup B$ via the map with $[(0, a)] \mapsto a$ and $[(1, b)] \mapsto b$. $\square$

**Proposition 2.2.3.1.4.** Let $A$, $B$, $C$, and $X$ be sets.

(1) *Functoriality.* The assignment $(A, B, C, f, g) \mapsto A \coprod_{f,C,g} B$ defines a functor

$$-_1 \textstyle\coprod_{-_3} -_1 \colon \mathsf{Fun}(\mathcal{P}, \mathsf{Sets}) \to \mathsf{Sets},$$

where $\mathcal{P}$ is the category that looks like this:

$$\begin{tikzcd} & \bullet \\ \bullet & \bullet. \arrow[l] \arrow[u] \end{tikzcd}$$

In particular, the action on morphisms of $-_1 \coprod_{-_3} -_1$ is given by sending a morphism

$$\begin{tikzcd}
A \coprod_C B \arrow[dr, phantom, "\ulcorner", very near start] & & B \arrow[ll] \arrow[dr, "\psi"] & \\
& A' \coprod_{C'} B' \arrow[dr, phantom, "\ulcorner", very near start] & & B' \arrow[ll] \\
A \arrow[uu] \arrow[dr, "\phi"'] & & C \arrow[ll, "f"'] \arrow[uu, "g"] \arrow[dr, "\chi"] & \\
& A' \arrow[uu] & & C' \arrow[ll, "f'"] \arrow[uu, "g'"']
\end{tikzcd}$$

in $\mathsf{Fun}(\mathcal{P}, \mathsf{Sets})$ to the map $\xi\colon A \coprod_C B \xrightarrow{\exists!} A' \coprod_{C'} B'$ given by

$$\xi(x) \overset{\text{def}}{=} \begin{cases} \phi(a) & \text{if } x = [(0,a)], \\ \psi(b) & \text{if } x = [(1,b)] \end{cases}$$

for each $x \in A \coprod_C B$, which is the unique map making the diagram

$$\begin{tikzcd}
A \coprod_C B \arrow[dr, phantom, "\ulcorner", very near start] \arrow[dr, dashed] & & B \arrow[ll] \arrow[dr, "\psi"] & \\
& A' \coprod_{C'} B' \arrow[dr, phantom, "\ulcorner", very near start] & & B' \arrow[ll] \\
A \arrow[uu] \arrow[dr, "\phi"'] & & C \arrow[ll, "f"'] \arrow[uu, "g"] \arrow[dr, "\chi"] & \\
& A' \arrow[uu] & & C' \arrow[ll, "f'"] \arrow[uu, "g'"']
\end{tikzcd}$$

commute.

(2) *Associativity.* Given a diagram

$$\begin{tikzcd}
A & & B & & C \\
& X \arrow[ul, "f"] \arrow[ur, "g"'] & & Y \arrow[ul, "h"] \arrow[ur, "k"'] &
\end{tikzcd}$$

in $\mathsf{Sets}$, we have isomorphisms

$$(A \coprod_X B) \coprod_Y C \cong (A \coprod_X B) \coprod_B (B \coprod_Y C) \cong A \coprod_X (B \coprod_Y C),$$

where these pullbacks are built as in the diagrams

$$\begin{tikzcd}[column sep=tiny]
& & (A \coprod_X B) \coprod_Y C & & \\
& A \coprod_X B \arrow[ur] \arrow[ur, phantom, "\wedge", near end] & & & \\
A \arrow[ur] \arrow[ur, phantom, "\wedge", shift left=3ex] & & B \arrow[ul] \arrow[uu, phantom, "\wedge"] & & C, \arrow[uull] \\
& X \arrow[ul, "f"] \arrow[ur, "g"'] & & Y \arrow[ul, "h"] \arrow[ur, "k"'] &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=tiny]
& & (A \coprod_X B) \coprod_B (B \coprod_Y C) & & \\
& A \coprod_X B \arrow[ur] & & B \coprod_Y C \arrow[ul] & \\
A \arrow[ur] & & B \arrow[ul] \arrow[ur] & & C, \arrow[ul] \\
& X \arrow[ul, "f"] \arrow[ur, "g"'] & & Y \arrow[ul, "h"] \arrow[ur, "k"'] &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=tiny]
& & A \coprod_X (B \coprod_Y C) & & \\
& & & B \coprod_Y C \arrow[ul] & \\
A \arrow[uurr] & & B \arrow[ur] & & C. \arrow[ul] \\
& X \arrow[ul, "f"] \arrow[ur, "g"'] & & Y \arrow[ul, "h"] \arrow[ur, "k"'] &
\end{tikzcd}$$

(3) *Unitality.* We have isomorphisms of sets

$$\begin{tikzcd} A \arrow[r, equal] & A \\ X \arrow[u, "f"] \arrow[r, equal] \arrow[ur, phantom, "\ulcorner", very near end] & X \arrow[u, "f"'] \end{tikzcd} \qquad \begin{aligned} X \textstyle\coprod_X A &\cong A, \\ A \textstyle\coprod_X X &\cong A, \end{aligned} \qquad \begin{tikzcd} A & X \arrow[l, "f"'] \\ X \arrow[u, equal] \arrow[ur, phantom, "\ulcorner", very near end] & X. \arrow[l, "f"] \arrow[u, equal] \end{tikzcd}$$

(4) *Commutativity.* We have an isomorphism of sets

$$\begin{tikzcd} A \coprod_X B & B \arrow[l] \\ A \arrow[u] \arrow[ur, phantom, "\ulcorner", very near end] & X, \arrow[l, "f"] \arrow[u, "g"'] \end{tikzcd} \qquad A \textstyle\coprod_X B \cong B \textstyle\coprod_X A \qquad \begin{tikzcd} B \coprod_X A & A \arrow[l] \\ B \arrow[u] \arrow[ur, phantom, "\ulcorner", very near end] & X. \arrow[l, "g"] \arrow[u, "f"'] \end{tikzcd}$$

(5) *Interaction With Coproducts.* We have

$$A \textstyle\coprod_{\emptyset} B \cong A \coprod B, \qquad \begin{tikzcd} A \coprod B & B \arrow[l] \\ A \arrow[u] \arrow[ur, phantom, "\ulcorner", very near end] & \emptyset. \arrow[l, hook, "\iota_A"] \arrow[u, hook, "\iota_B"'] \end{tikzcd}$$

(6) *Symmetric Monoidality.* The triple $(\mathsf{Sets}, \coprod_X, \emptyset)$ is a symmetric monoidal category.

**Proof.** *Item 1, Functoriality*: This is a special case of functoriality of co/limits, **??** of **??**, with the explicit expression for $\xi$ following from the commutativity of the cube pushout diagram.
*Item 2, Associativity*: Omitted.
*Item 3, Unitality*: Omitted.
*Item 4, Commutativity*: Clear.
*Item 5, Interaction With Coproducts*: Clear.
*Item 6, Symmetric Monoidality*: Omitted. □

**2.2.4. Coequalisers.** Let $A$ and $B$ be sets and let $f, g\colon A \rightrightarrows B$ be functions.

**Definition 2.2.4.1.1.** The **coequaliser of** $f$ **and** $g$ is the pair $(\mathrm{CoEq}(f,g), \mathrm{coeq}(f,g))$ consisting of

- *The Colimit.* The set $\mathrm{CoEq}(f,g)$ defined by
$$\mathrm{CoEq}(f,g) \overset{\text{def}}{=} B/{\sim},$$
where $\sim$ is the equivalence relation on $B$ generated by $f(a) \sim g(a)$.
- *The Cocone.* The map
$$\mathrm{coeq}(f,g)\colon B \to \mathrm{CoEq}(f,g)$$
given by the quotient map $\pi\colon B \twoheadrightarrow B/{\sim}$ with respect to the equivalence relation generated by $f(a) \sim g(a)$.

**Proof.** We claim that $\mathrm{CoEq}(f,g)$ is the categorical coequaliser of $f$ and $g$ in $\mathsf{Sets}$. First we need to check that the relevant coequaliser diagram commutes, i.e. that we have
$$\mathrm{coeq}(f,g) \circ f = \mathrm{coeq}(f,g) \circ g.$$

Indeed, we have

$$\begin{aligned}
[\mathrm{coeq}(f,g)\circ f](a) &\stackrel{\text{def}}{=} [\mathrm{coeq}(f,g)](f(a))\\
&\stackrel{\text{def}}{=} [f(a)]\\
&= [g(a)]\\
&\stackrel{\text{def}}{=} [\mathrm{coeq}(f,g)](g(a))\\
&\stackrel{\text{def}}{=} [\mathrm{coeq}(f,g)\circ g](a)
\end{aligned}$$

for each $a \in A$. Next, we prove that $\mathrm{CoEq}(f,g)$ satisfies the universal property of the coequaliser. Suppose we have a diagram of the form

$$\begin{tikzcd}
A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, hook, two heads, "\mathrm{coeq}(f,g)"] \arrow[dr, "c"'] & \mathrm{CoEq}(f,g)\\
& & C
\end{tikzcd}$$

in $\mathsf{Sets}$. Then, since $c(f(a)) = c(g(a))$ for each $a \in A$, it follows from **????** of **??** that there exists a unique map $\mathrm{CoEq}(f,g) \xrightarrow{\exists!} C$ making the diagram

$$\begin{tikzcd}
A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, hook, two heads, "\mathrm{coeq}(f,g)"] \arrow[dr, "c"'] & \mathrm{CoEq}(f,g) \arrow[d, dashed, "\exists!"]\\
& & C
\end{tikzcd}$$

commutes. □

**Remark 2.2.4.1.2.** In detail, by **??**, the relation $\sim$ of Definition 2.2.4.1.1 is given by declaring $a \sim b$ iff one of the following conditions is satisfied:

- We have $a = b$;
- There exist $x_1, \ldots, x_n \in B$ such that $a \sim' x_1 \sim' \cdots \sim' x_n \sim' b$, where we declare $x \sim' y$ if one of the following conditions is satisfied:
  (1) There exists $z \in A$ such that $x = f(z)$ and $y = g(z)$.
  (2) There exists $z \in A$ such that $x = g(z)$ and $y = f(z)$.
  That is: we require the following condition to be satisfied:
  ($\star$) There exist $x_1, \ldots, x_n \in B$ satisfying the following conditions:
    (1) There exists $z_0 \in A$ satisfying one of the following conditions:
      (a) We have $a = f(z_0)$ and $x_1 = g(z_0)$.
      (b) We have $a = g(z_0)$ and $x_1 = f(z_0)$.
    (2) For each $1 \leq i \leq n-1$, there exists $z_i \in A$ satisfying one of the following conditions:
      (a) We have $x_i = f(z_i)$ and $x_{i+1} = g(z_i)$.
      (b) We have $x_i = g(z_i)$ and $x_{i+1} = f(z_i)$.
    (3) There exists $z_n \in A$ satisfying one of the following conditions:
      (a) We have $x_n = f(z_n)$ and $b = g(z_n)$.
      (b) We have $x_n = g(z_n)$ and $b = f(z_n)$.

**Example 2.2.4.1.3.** Here are some examples of coequalisers of sets.

(1) *Quotients by Equivalence Relations.* Let $R$ be an equivalence relation on a set $X$. We have a bijection of sets

$$X/\sim_R \cong \mathrm{CoEq}\left(R \hookrightarrow X \times X \underset{\mathrm{pr}_2}{\overset{\mathrm{pr}_1}{\rightrightarrows}} X\right).$$

**Proof.** *Item 1, Quotients by Equivalence Relations*: See [Pro24z]. ☐

**Proposition 2.2.4.1.4.** Let $A$, $B$, and $C$ be sets.

(1) *Associativity.* We have an isomorphism of sets[11]

$$\underbrace{\mathrm{CoEq}(\mathrm{coeq}(f,g)\circ f, \mathrm{coeq}(f,g)\circ h)}_{=\mathrm{CoEq}(\mathrm{coeq}(f,g)\circ g, \mathrm{coeq}(f,g)\circ h)} \cong \mathrm{CoEq}(f,g,h) \cong \underbrace{\mathrm{CoEq}(\mathrm{coeq}(g,h)\circ f, \mathrm{coeq}(g,h)\circ g)}_{=\mathrm{CoEq}(\mathrm{coeq}(g,h)\circ f, \mathrm{coeq}(g,h)\circ h)},$$

where $\mathrm{CoEq}(f,g,h)$ is the colimit of the diagram

$$A \overset{\xrightarrow{f}}{\underset{\xrightarrow[h]{}}{\xrightarrow{g}}} B$$

in $\mathsf{Sets}$.

(4) *Unitality.* We have an isomorphism of sets

$$\mathrm{CoEq}(f,f) \cong B.$$

(5) *Commutativity.* We have an isomorphism of sets

$$\mathrm{CoEq}(f,g) \cong \mathrm{CoEq}(g,f).$$

---

[11] That is, the following three ways of forming "the" coequaliser of $(f,g,h)$ agree:

(1) Take the coequaliser of $(f,g,h)$, i.e. the colimit of the diagram

$$A \overset{\xrightarrow{f}}{\underset{\xrightarrow[h]{}}{\xrightarrow{g}}} B$$

in $\mathsf{Sets}$.

(2) First take the coequaliser of $f$ and $g$, forming a diagram

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \overset{\mathrm{coeq}(f,g)}{\twoheadrightarrow} \mathrm{CoEq}(f,g)$$

and then take the coequaliser of the composition

$$A \underset{h}{\overset{f}{\rightrightarrows}} B \overset{\mathrm{coeq}(f,g)}{\twoheadrightarrow} \mathrm{CoEq}(f,g),$$

obtaining a quotient

$$\mathrm{CoEq}(\mathrm{coeq}(f,g)\circ f, \mathrm{coeq}(f,g)\circ h) = \mathrm{CoEq}(\mathrm{coeq}(f,g)\circ g, \mathrm{coeq}(f,g)\circ h)$$

of $\mathrm{CoEq}(f,g)$

(3) First take the coequaliser of $g$ and $h$, forming a diagram

$$A \underset{h}{\overset{g}{\rightrightarrows}} B \overset{\mathrm{coeq}(g,h)}{\twoheadrightarrow} \mathrm{CoEq}(g,h)$$

and then take the coequaliser of the composition

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \overset{\mathrm{coeq}(g,h)}{\twoheadrightarrow} \mathrm{CoEq}(g,h),$$

obtaining a quotient

$$\mathrm{CoEq}(\mathrm{coeq}(g,h)\circ f, \mathrm{coeq}(g,h)\circ g) = \mathrm{CoEq}(\mathrm{coeq}(g,h)\circ f, \mathrm{coeq}(g,h)\circ h)$$

of $\mathrm{CoEq}(g,h)$.

(6) *Interaction With Composition.* Let

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \underset{k}{\overset{h}{\rightrightarrows}} C$$

be functions. We have a surjection

$$\mathrm{CoEq}(h \circ f, k \circ g) \twoheadrightarrow \mathrm{CoEq}(\mathrm{coeq}(h,k) \circ h \circ f, \mathrm{coeq}(h,k) \circ k \circ g)$$

exhibiting $\mathrm{CoEq}(\mathrm{coeq}(h,k) \circ h \circ f, \mathrm{coeq}(h,k) \circ k \circ g)$ as a quotient of $\mathrm{CoEq}(h \circ f, k \circ g)$ by the relation generated by declaring $h(y) \sim k(y)$ for each $y \in B$.

**Proof.** *Item 1, Associativity*: Omitted.
*Item 4, Unitality*: Clear.
*Item 5, Commutativity*: Clear.
*Item 6, Interaction With Composition*: Omitted. □

## 2.3. Operations With Sets

### 2.3.1. The Empty Set.

**Definition 2.3.1.1.1.** The **empty set** is the set $\emptyset$ defined by

$$\emptyset \overset{\text{def}}{=} \{x \in X \mid x \neq x\},$$

where $A$ is the set in the set existence axiom, **??** of **??**.

### 2.3.2. Singleton Sets. Let $X$ be a set.

**Definition 2.3.2.1.1.** The **singleton set containing** $X$ is the set $\{X\}$ defined by

$$\{X\} \overset{\text{def}}{=} \{X, X\},$$

where $\{X, X\}$ is the pairing of $X$ with itself (Definition 2.3.3.1.1).

### 2.3.3. Pairings of Sets. Let $X$ and $Y$ be sets.

**Definition 2.3.3.1.1.** The **pairing of** $X$ **and** $Y$ is the set $\{X, Y\}$ defined by

$$\{X, Y\} \overset{\text{def}}{=} \{x \in A \mid x = X \text{ or } x = Y\},$$

where $A$ is the set in the axiom of pairing, **??** of **??**.

### 2.3.4. Ordered Pairs. Let $A$ and $B$ be sets.

**Definition 2.3.4.1.1.** The **ordered pair associated to** $A$ **and** $B$ is the set $(A, B)$ defined by

$$(A, B) \overset{\text{def}}{=} \{\{A\}, \{A, B\}\}.$$

**Proposition 2.3.4.1.2.** Let $A$ and $B$ be sets.

(1) *Uniqueness.* Let $A$, $B$, $C$, and $D$ be sets. The following conditions are equivalent:
  (a) We have $(A, B) = (C, D)$.
  (b) We have $A = C$ and $B = D$.

**Proof.** *Item 1, Uniqueness*: See [Cie97, Theorem 1.2.3]. □

### 2.3.5. Unions of Families. Let $\{A_i\}_{i \in I}$ be a family of sets.

**Definition 2.3.5.1.1.** The **union of the family** $\{A_i\}_{i\in I}$ is the set $\bigcup_{i\in I} A_i$ defined by

$$\bigcup_{i\in I} A_i \overset{\text{def}}{=} \{x \in F \mid \text{there exists some } i \in I \text{ such that } x \in A_i\},$$

where $F$ is the set in the axiom of union, **??** of **??**.

**2.3.6. Binary Unions.** Let $A$ and $B$ be sets.

**Definition 2.3.6.1.1.** The **union**[12] **of** $A$ **and** $B$ is the set $A \cup B$ defined by

$$A \cup B \overset{\text{def}}{=} \bigcup_{z\in\{A,B\}} z.$$

**Proposition 2.3.6.1.2.** Let $X$ be a set.

(1) *Functoriality.* The assignments $U, V, (U, V) \mapsto U \cup V$ define functors

$$\begin{aligned} U \cup -&\colon (\mathcal{P}(X), \subset) \to (\mathcal{P}(X), \subset), \\ - \cup V&\colon (\mathcal{P}(X), \subset) \to (\mathcal{P}(X), \subset), \\ -_1 \cup -_2&\colon (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \subset) \to (\mathcal{P}(X), \subset), \end{aligned}$$

where $-_1 \cup -_2$ is the functor where

- *Action on Objects.* For each $(U, V) \in \mathcal{P}(X) \times \mathcal{P}(X)$, we have
$$[-_1 \cup -_2](U, V) \overset{\text{def}}{=} U \cup V;$$
- *Action on Morphisms.* For each pair of morphisms
$$\begin{aligned} \iota_U &\colon U \hookrightarrow U', \\ \iota_V &\colon V \hookrightarrow V' \end{aligned}$$
of $\mathcal{P}(X) \times \mathcal{P}(X)$, the image
$$\iota_U \cup \iota_V \colon U \cup V \hookrightarrow U' \cup V'$$
of $(\iota_U, \iota_V)$ by $\cup$ is the inclusion
$$U \cup V \subset U' \cup V'$$
i.e. where we have
  ($\star$) If $U \subset U'$ and $V \subset V'$, then $U \cup V \subset U' \cup V'$;

and where $U \cup -$ and $- \cup V$ are the partial functors of $-_1 \cup -_2$ at $U, V \in \mathcal{P}(X)$.

(2) *Via Intersections and Symmetric Differences.* We have an equality of sets

$$U \cup V = (U \bigtriangleup V) \bigtriangleup (U \cap V)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

(3) *Associativity.* We have an equality of sets

$$(U \cup V) \cup W = U \cup (V \cup W)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

(4) *Unitality.* We have equalities of sets

$$\begin{aligned} U \cup \emptyset &= U, \\ \emptyset \cup U &= U \end{aligned}$$

[12] *Further Terminology:* Also called the **binary union of** $A$ **and** $B$, for emphasis.

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

(5) *Commutativity.* We have an equality of sets

$$U \cup V = V \cup U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

(6) *Idempotency.* We have an equality of sets

$$U \cup U = U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

(7) *Distributivity Over Intersections.* We have equalities of sets

$$U \cup (V \cap W) = (U \cup V) \cap (U \cup W),$$
$$(U \cap V) \cup W = (U \cup W) \cap (V \cup W)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

(8) *Interaction With Characteristic Functions I.* We have

$$\chi_{U \cup V} = \max(\chi_U, \chi_V)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

(9) *Interaction With Characteristic Functions II.* We have

$$\chi_{U \cup V} = \chi_U + \chi_V - \chi_{U \cap V}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

(10) *Interaction With Powersets and Semirings.* The quintuple $(\mathcal{P}(X), \cup, \cap, \emptyset, X)$ is an idempotent commutative semiring.

**Proof.** *Item 1, Functoriality*: See [Pro24an].
*Item 2, Via Intersections and Symmetric Differences*: See [Pro24ay].
*Item 3, Associativity*: See [Pro24ba].
*Item 4, Unitality*: This follows from [Pro24bd] and Item 5.
*Item 5, Commutativity*: See [Pro24bb].
*Item 6, Idempotency*: See [Pro24am].
*Item 7, Distributivity Over Intersections*: See [Pro24az].
*Item 8, Interaction With Characteristic Functions I*: See [Pro24k].
*Item 9, Interaction With Characteristic Functions II*: See [Pro24k].
*Item 10, Interaction With Powersets and Semirings*: This follows from Items 3 to 6 and Items 3 to 5, 7 and 8 of Proposition 2.3.8.1.2. □

**2.3.7. Intersections of Families.** Let $\mathcal{F}$ be a family of sets.

**Definition 2.3.7.1.1.** The **intersection of a family $\mathcal{F}$ of sets** is the set $\bigcap_{X \in \mathcal{F}} X$ defined by

$$\bigcap_{X \in \mathcal{F}} X \overset{\text{def}}{=} \left\{ z \in \bigcup_{X \in \mathcal{F}} X \;\middle|\; \text{for each } X \in \mathcal{F}\text{, we have } z \in X \right\}.$$

**2.3.8. Binary Intersections.** Let $X$ and $Y$ be sets.

**Definition 2.3.8.1.1.** The **intersection**[13] **of $X$ and $Y$** is the set $X \cap Y$ defined by

$$X \cap Y \overset{\text{def}}{=} \bigcap_{z \in \{X, Y\}} z.$$

[13] *Further Terminology:* Also called the **binary intersection of $X$ and $Y$**, for emphasis.

**Proposition 2.3.8.1.2.** Let $X$ be a set.

(1) *Functoriality.* The assignments $U, V, (U, V) \mapsto U \cap V$ define functors

$$\begin{aligned} U \cap - &\colon (\mathcal{P}(X), \subset) \to (\mathcal{P}(X), \subset), \\ - \cap V &\colon (\mathcal{P}(X), \subset) \to (\mathcal{P}(X), \subset), \\ -_1 \cap -_2 &\colon (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \subset) \to (\mathcal{P}(X), \subset), \end{aligned}$$

where $-_1 \cap -_2$ is the functor where

- *Action on Objects.* For each $(U, V) \in \mathcal{P}(X) \times \mathcal{P}(X)$, we have
$$[-_1 \cap -_2](U, V) \overset{\text{def}}{=} U \cap V;$$
- *Action on Morphisms.* For each pair of morphisms
$$\begin{aligned} \iota_U &\colon U \hookrightarrow U', \\ \iota_V &\colon V \hookrightarrow V' \end{aligned}$$
of $\mathcal{P}(X) \times \mathcal{P}(X)$, the image
$$\iota_U \cap \iota_V \colon U \cap V \hookrightarrow U' \cap V'$$
of $(\iota_U, \iota_V)$ by $\cap$ is the inclusion
$$U \cap V \subset U' \cap V'$$
i.e. where we have
  ($\star$) If $U \subset U'$ and $V \subset V'$, then $U \cap V \subset U' \cap V'$;

and where $U \cap -$ and $- \cap V$ are the partial functors of $-_1 \cap -_2$ at $U, V \in \mathcal{P}(X)$.

(2) *Adjointness.* We have adjunctions

$$\Big(U \cap - \dashv \mathbf{Hom}_{\mathcal{P}(X)}(U, -)\Big)\colon \quad \begin{tikzcd} \mathcal{P}(X) \arrow[r, bend left, "U\cap-"] \arrow[r, phantom, "\perp"] & \mathcal{P}(X), \arrow[l, bend left, "\mathbf{Hom}_{\mathcal{P}(X)}(U,-)"] \end{tikzcd}$$

$$\Big(- \cap V \dashv \mathbf{Hom}_{\mathcal{P}(X)}(V, -)\Big)\colon \quad \begin{tikzcd} \mathcal{P}(X) \arrow[r, bend left, "-\cap V"] \arrow[r, phantom, "\perp"] & \mathcal{P}(X), \arrow[l, bend left, "\mathbf{Hom}_{\mathcal{P}(X)}(V,-)"] \end{tikzcd}$$

where

$$\mathbf{Hom}_{\mathcal{P}(X)}(-_1, -_2) \colon \mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \to \mathcal{P}(X)$$

is the bifunctor defined by[14]

$$\mathbf{Hom}_{\mathcal{P}(X)}(U, V) \overset{\text{def}}{=} (X \setminus U) \cup V$$

witnessed by bijections

$$\begin{aligned} \mathrm{Hom}_{\mathcal{P}(X)}(U \cap V, W) &\cong \mathrm{Hom}_{\mathcal{P}(X)}\Big(U, \mathbf{Hom}_{\mathcal{P}(X)}(V, W)\Big), \\ \mathrm{Hom}_{\mathcal{P}(X)}(U \cap V, W) &\cong \mathrm{Hom}_{\mathcal{P}(X)}\Big(V, \mathbf{Hom}_{\mathcal{P}(X)}(U, W)\Big), \end{aligned}$$

natural in $U, V, W \in \mathcal{P}(X)$, i.e. where:

[14] *Intuition:* Since intersections are the products in $\mathcal{P}(X)$, the left adjoint $\mathbf{Hom}_{\mathcal{P}(X)}(U, V)$ works as a function type $U \to V$.
Now, under the Curry–Howard correspondence, the function type $U \to V$ corresponds to implication $U \implies V$, which is logically equivalent to the statement $\neg U \vee V$, which in turn corresponds to the set $U^{\mathsf{c}} \vee V \overset{\text{def}}{=} (X \setminus U) \cup V$.

(a) The following conditions are equivalent:
   (i) We have $U \cap V \subset W$.
   (ii) We have $U \subset \mathbf{Hom}_{\mathcal{P}(X)}(V, W)$.
   (iii) We have $U \subset (X \setminus V) \cup W$.
(b) The following conditions are equivalent:
   (i) We have $V \cap U \subset W$.
   (ii) We have $V \subset \mathbf{Hom}_{\mathcal{P}(X)}(U, W)$.
   (iii) We have $V \subset (X \setminus U) \cup W$.

(3) *Associativity.* We have an equality of sets

$$(U \cap V) \cap W = U \cap (V \cap W)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

(4) *Unitality.* Let $X$ be a set and let $U \in \mathcal{P}(X)$. We have equalities of sets

$$X \cap U = U,$$
$$U \cap X = U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

(5) *Commutativity.* We have an equality of sets

$$U \cap V = V \cap U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

(6) *Idempotency.* We have an equality of sets

$$U \cap U = U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

(7) *Distributivity Over Unions.* We have equalities of sets

$$U \cap (V \cup W) = (U \cap V) \cup (U \cap W),$$
$$(U \cup V) \cap W = (U \cap W) \cup (V \cap W)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

(8) *Annihilation With the Empty Set.* We have an equality of sets

$$\emptyset \cap X = \emptyset,$$
$$X \cap \emptyset = \emptyset$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

(9) *Interaction With Characteristic Functions I.* We have

$$\chi_{U \cap V} = \chi_U \chi_V$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

(10) *Interaction With Characteristic Functions II.* We have

$$\chi_{U \cap V} = \min(\chi_U, \chi_V)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

(11) *Interaction With Powersets and Monoids With Zero.* The quadruple $((\mathcal{P}(X), \emptyset), \cap, X)$ is a commutative monoid with zero.

(12) *Interaction With Powersets and Semirings.* The quintuple $(\mathcal{P}(X), \cup, \cap, \emptyset, X)$ is an idempotent commutative semiring.

**Proof.** *Item 1, Functoriality*: See [Pro24al].

*Item 2, Adjointness*: See [MSE 267469].
*Item 3, Associativity*: See [Pro24t].
*Item 4, Unitality*: This follows from [Pro24x] and Item 5.
*Item 5, Commutativity*: See [Pro24u].
*Item 6, Idempotency*: See [Pro24ak].
*Item 7, Distributivity Over Unions*: See [Pro24aj].
*Item 8, Annihilation With the Empty Set*: This follows from [Pro24v] and Item 5.
*Item 9, Interaction With Characteristic Functions I*: See [Pro24h].
*Item 10, Interaction With Characteristic Functions II*: See [Pro24h].
*Item 11, Interaction With Powersets and Monoids With Zero*: This follows from Items 3 to 5 and 8.
*Item 12, Interaction With Powersets and Semirings*: This follows from Items 3 to 6 and Items 3 to 5, 7 and 8 of Proposition 2.3.8.1.2. □

**2.3.9. Differences.** Let $X$ and $Y$ be sets.

**Definition 2.3.9.1.1.** The **difference of $X$ and $Y$** is the set $X \setminus Y$ defined by

$$X \setminus Y \overset{\text{def}}{=} \{a \in X \mid a \notin Y\}.$$

**Proposition 2.3.9.1.2.** Let $X$ be a set.

(1) *Functoriality.* The assignments $U, V, (U, V) \mapsto U \cap V$ define functors

$$U \setminus -\colon (\mathcal{P}(X), \supset) \to (\mathcal{P}(X), \subset),$$
$$- \setminus V\colon (\mathcal{P}(X), \subset) \to (\mathcal{P}(X), \subset),$$
$$-_1 \setminus -_2\colon (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \supset) \to (\mathcal{P}(X), \subset),$$

where $-_1 \setminus -_2$ is the functor where

- *Action on Objects.* For each $(U, V) \in \mathcal{P}(X) \times \mathcal{P}(X)$, we have
$$[-_1 \setminus -_2](U, V) \overset{\text{def}}{=} U \setminus V;$$
- *Action on Morphisms.* For each pair of morphisms
$$\iota_A\colon A \hookrightarrow B,$$
$$\iota_U\colon U \hookrightarrow V$$
of $\mathcal{P}(X) \times \mathcal{P}(X)$, the image
$$\iota_U \setminus \iota_V\colon A \setminus V \hookrightarrow B \setminus U$$
of $(\iota_U, \iota_V)$ by $\setminus$ is the inclusion
$$A \setminus V \subset B \setminus U$$
i.e. where we have
  ($\star$) If $A \subset B$ and $U \subset V$, then $A \setminus V \subset B \setminus U$;

and where $U \setminus -$ and $- \setminus V$ are the partial functors of $-_1 \setminus -_2$ at $U, V \in \mathcal{P}(X)$.

(2) *De Morgan's Laws.* We have equalities of sets

$$X \setminus (U \cup V) = (X \setminus U) \cap (X \setminus V),$$
$$X \setminus (U \cap V) = (X \setminus U) \cup (X \setminus V)$$

for each $X \in \mathsf{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

(3) *Interaction With Unions I.* We have equalities of sets
$$U \setminus (V \cup W) = (U \setminus V) \cap (U \setminus W)$$
for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

(4) *Interaction With Unions II.* We have equalities of sets
$$(U \setminus V) \cup W = (U \cup W) \setminus (V \setminus W)$$
for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

(5) *Interaction With Unions III.* We have equalities of sets
$$\begin{aligned} U \setminus (V \cup W) &= (U \cup W) \setminus (V \cup W) \\ &= (U \setminus V) \setminus W \\ &= (U \setminus W) \setminus V \end{aligned}$$
for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

(6) *Interaction With Unions IV.* We have equalities of sets
$$(U \cup V) \setminus W = (U \setminus W) \cup (V \setminus W)$$
for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

(7) *Interaction With Intersections.* We have equalities of sets
$$\begin{aligned} (U \setminus V) \cap W &= (U \cap W) \setminus V \\ &= U \cap (W \setminus V) \end{aligned}$$
for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

(8) *Interaction With Complements.* We have an equality of sets
$$U \setminus V = U \cap V^{\mathsf{c}}$$
for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

(9) *Interaction With Symmetric Differences.* We have an equality of sets
$$U \setminus V = U \bigtriangleup (U \cap V)$$
for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

(10) *Triple Differences.* We have
$$U \setminus (V \setminus W) = (U \cap W) \cup (U \setminus V)$$
for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

(11) *Left Annihilation.* We have
$$\emptyset \setminus U = \emptyset$$
for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

(12) *Right Unitality.* We have
$$U \setminus \emptyset = U$$
for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

(13) *Invertibility.* We have
$$U \setminus U = \emptyset$$
for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

(14) *Interaction With Containment.* The following conditions are equivalent:

(a) We have $V \setminus U \subset W$.

(b) We have $V \setminus W \subset U$.

(15) *Interaction With Characteristic Functions.* We have

$$\chi_{U \setminus V} = \chi_U - \chi_{U \cap V}$$

for each $X \in \text{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

**Proof.** *Item 1, Functoriality*: See [Pro24ad] and [Pro24ah].
*Item 2, De Morgan's Laws*: See [Pro24p].
*Item 3, Interaction With Unions I*: See [Pro24q].
*Item 4, Interaction With Unions II*: Omitted.
*Item 5, Interaction With Unions III*: See [Pro24ai].
*Item 6, Interaction With Unions IV*: See [Pro24ac].
*Item 7, Interaction With Intersections*: See [Pro24w].
*Item 8, Interaction With Complements*: See [Pro24aa].
*Item 9, Interaction With Symmetric Differences*: See [Pro24ab].
*Item 10, Triple Differences*: See [Pro24ag].
*Item 11, Left Annihilation*: Clear.
*Item 12, Right Unitality*: See [Pro24ae].
*Item 13, Invertibility*: See [Pro24af].
*Item 14, Interaction With Containment*: Omitted.
*Item 15, Interaction With Characteristic Functions*: See [Pro24i]. □

**2.3.10. Complements.** Let $X$ be a set and let $U \in \mathcal{P}(X)$.

**Definition 2.3.10.1.1.** The **complement of** $U$ is the set $U^{\mathsf{c}}$ defined by

$$\begin{aligned} U^{\mathsf{c}} &\overset{\text{def}}{=} X \setminus U \\ &\overset{\text{def}}{=} \{a \in X \mid a \notin U\}. \end{aligned}$$

**Proposition 2.3.10.1.2.** Let $X$ be a set.

(1) *Functoriality.* The assignment $U \mapsto U^{\mathsf{c}}$ defines a functor

$$(-)^{\mathsf{c}} \colon \mathcal{P}(X)^{\mathsf{op}} \to \mathcal{P}(X),$$

where

- *Action on Objects.* For each $U \in \mathcal{P}(X)$, we have

$$[(-)^{\mathsf{c}}](U) \overset{\text{def}}{=} U^{\mathsf{c}};$$

- *Action on Morphisms.* For each morphism $\iota_U \colon U \hookrightarrow V$ of $\mathcal{P}(X)$, the image

$$\iota_U^{\mathsf{c}} \colon V^{\mathsf{c}} \hookrightarrow U^{\mathsf{c}}$$

of $\iota_U$ by $(-)^{\mathsf{c}}$ is the inclusion

$$V^{\mathsf{c}} \subset U^{\mathsf{c}}$$

i.e. where we have

($\star$) If $U \subset V$, then $V^{\mathsf{c}} \subset U^{\mathsf{c}}$.

(2) *De Morgan's Laws.* We have equalities of sets

$$(U \cup V)^{\mathsf{c}} = U^{\mathsf{c}} \cap V^{\mathsf{c}},$$
$$(U \cap V)^{\mathsf{c}} = U^{\mathsf{c}} \cup V^{\mathsf{c}}$$

for each $X \in \text{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

(3) *Involutority.* We have

$$(U^{\mathsf{c}})^{\mathsf{c}} = U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

(4) *Interaction With Characteristic Functions.* We have

$$\chi_{U^{\mathsf{c}}} = 1 - \chi_U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

**Proof.** *Item 1*, *Functoriality*: This follows from Item 1 of Proposition 2.3.9.1.2.
*Item 2*, *De Morgan's Laws*: See [Pro24p].
*Item 3*, *Involutority*: See [Pro24l].
*Item 4*, *Interaction With Characteristic Functions*: Clear. □

**2.3.11. Symmetric Differences.** Let $A$ and $B$ be sets.

**Definition 2.3.11.1.1.** The **symmetric difference of** $A$ **and** $B$ is the set $A \triangle B$ defined by

$$A \triangle B \overset{\text{def}}{=} (A \setminus B) \cup (B \setminus A).$$

**Proposition 2.3.11.1.2.** Let $X$ be a set.

(1) *Lack of Functoriality.* The assignment $(U, V) \mapsto U \triangle V$ ***need not*** define functors

$$U \triangle -_2 \colon (\mathcal{P}(X), \subset) \to (\mathcal{P}(X), \subset),$$
$$-_1 \triangle V \colon (\mathcal{P}(X), \subset) \to (\mathcal{P}(X), \subset),$$
$$-_1 \triangle -_2 \colon (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \subset) \to (\mathcal{P}(X), \subset).$$

(2) *Via Unions and Intersections.* We have[15]

$$U \triangle V = (U \cup V) \setminus (U \cap V)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

(3) *Associativity.* We have[16]

$$(U \triangle V) \triangle W = U \triangle (V \triangle W)$$

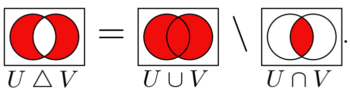
for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

(4) *Commutativity.* We have

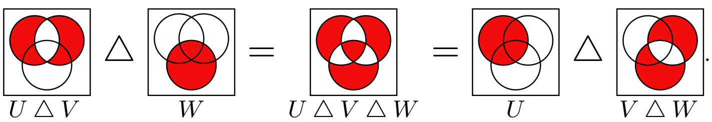
$$U \triangle V = V \triangle U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

[15] *Illustration:*




[16] *Illustration:*



(5) *Unitality.* We have

$$U \bigtriangleup \emptyset = U,$$
$$\emptyset \bigtriangleup U = U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

(6) *Invertibility.* We have

$$U \bigtriangleup U = \emptyset$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

(7) *Interaction With Unions.* We have

$$(U \bigtriangleup V) \cup (V \bigtriangleup T) = (U \cup V \cup W) \setminus (U \cap V \cap W)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

(8) *Interaction With Complements I.* We have

$$U \bigtriangleup U^{\mathsf{c}} = X$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

(9) *Interaction With Complements II.* We have

$$U \bigtriangleup X = U^{\mathsf{c}},$$
$$X \bigtriangleup U = U^{\mathsf{c}}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

(10) *Interaction With Complements III.* We have

$$U^{\mathsf{c}} \bigtriangleup V^{\mathsf{c}} = U \bigtriangleup V$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

(11) *"Transitivity".* We have

$$(U \bigtriangleup V) \bigtriangleup (V \bigtriangleup W) = U \bigtriangleup W$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

(12) *The Triangle Inequality for Symmetric Differences.* We have

$$U \bigtriangleup W \subset U \bigtriangleup V \cup V \bigtriangleup W$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

(13) *Distributivity Over Intersections.* We have

$$U \cap (V \bigtriangleup W) = (U \cap V) \bigtriangleup (U \cap W),$$
$$(U \bigtriangleup V) \cap W = (U \cap W) \bigtriangleup (V \cap W)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

(14) *Interaction With Characteristic Functions.* We have

$$\chi_{U \bigtriangleup V} = \chi_U + \chi_V - 2\chi_{U \cap V}$$

and thus, in particular, we have

$$\chi_{U \bigtriangleup V} \equiv \chi_U + \chi_V \mod 2$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

(15) *Bijectivity.* Given $A, B \subset \mathcal{P}(X)$, the maps

$$A \bigtriangleup - \colon \mathcal{P}(X) \to \mathcal{P}(X),$$
$$- \bigtriangleup B \colon \mathcal{P}(X) \to \mathcal{P}(X)$$

are bijections with inverses given by

$$(A \bigtriangleup -)^{-1} = - \cup (A \cap -),$$
$$(- \bigtriangleup B)^{-1} = - \cup (B \cap -).$$

Moreover, the map

$$C \mapsto C \bigtriangleup (A \bigtriangleup B)$$

is a bijection of $\mathcal{P}(X)$ onto itself sending $A$ to $B$ and $B$ to $A$.

(16) *Interaction With Powersets and Groups.* Let $X$ be a set.
   (a) The quadruple $\left(\mathcal{P}(X), \bigtriangleup, \emptyset, \mathrm{id}_{\mathcal{P}(X)}\right)$ is an abelian group.[17]
   (b) Every element of $\mathcal{P}(X)$ has order 2 with respect to $\bigtriangleup$, and thus $\mathcal{P}(X)$ is a *Boolean group* (i.e. an abelian 2-group).

(4) *Interaction With Powersets and Vector Spaces I.* The pair $\left(\mathcal{P}(X), \alpha_{\mathcal{P}(X)}\right)$ consisting of
   - The group $\mathcal{P}(X)$ of **??**;
   - The map $\alpha_{\mathcal{P}(X)} \colon \mathbb{F}_2 \times \mathcal{P}(X) \to \mathcal{P}(X)$ defined by

$$0 \cdot U \stackrel{\text{def}}{=} \emptyset,$$
$$1 \cdot U \stackrel{\text{def}}{=} U;$$

   is an $\mathbb{F}_2$-vector space.

(5) *Interaction With Powersets and Vector Spaces II.* If $X$ is finite, then:
   (a) The set of singletons sets on the elements of $X$ forms a basis for the $\mathbb{F}_2$-vector space $\left(\mathcal{P}(X), \alpha_{\mathcal{P}(X)}\right)$ of Item 4.
   (b) We have

$$\dim(\mathcal{P}(X)) = \#\mathcal{P}(X).$$

(6) *Interaction With Powersets and Rings.* The quintuple $(\mathcal{P}(X), \bigtriangleup, \cap, \emptyset, X)$ is a commutative ring.[18]

**Proof.** *Item 1, Lack of Functoriality*: Omitted.
*Item 2, Via Unions and Intersections*: See [Pro24r].
*Item 3, Associativity*: See [Pro24ao].
*Item 4, Commutativity*: See [Pro24ap].

[17] Here are some examples:

(1) When $X = \emptyset$, we have an isomorphism of groups between $\mathcal{P}(\emptyset)$ and the trivial group:

$$\left(\mathcal{P}(\emptyset), \bigtriangleup, \emptyset, \mathrm{id}_{\mathcal{P}(\emptyset)}\right) \cong \mathrm{pt}.$$

(2) When $X = \mathrm{pt}$, we have an isomorphism of groups between $\mathcal{P}(\mathrm{pt})$ and $\mathbb{Z}_{/2}$:

$$\left(\mathcal{P}(\mathrm{pt}), \bigtriangleup, \emptyset, \mathrm{id}_{\mathcal{P}(\mathrm{pt})}\right) \cong \mathbb{Z}_{/2}.$$

(3) When $X = \{0, 1\}$, we have an isomorphism of groups between $\mathcal{P}(\{0, 1\})$ and $\mathbb{Z}_{/2} \times \mathbb{Z}_{/2}$:

$$\left(\mathcal{P}(\{0, 1\}), \bigtriangleup, \emptyset, \mathrm{id}_{\mathcal{P}(\{0,1\})}\right) \cong \mathbb{Z}_{/2} \times \mathbb{Z}_{/2}.$$

[18] *Warning:* The analogous statement replacing intersections by unions (i.e. that the quintuple $(\mathcal{P}(X), \bigtriangleup, \cup, \emptyset, X)$ is a ring) is false, however. See [Pro24aw] for a proof.
END TEXTDBEND

*Item 5, Unitality*: This follows from Item 4 and [Pro24at].
*Item 6, Invertibility*: See [Pro24av].
*Item 7, Interaction With Unions*: See [Pro24bc].
*Item 8, Interaction With Complements I*: See [Pro24as].
*Item 9, Interaction With Complements II*: This follows from Item 4 and [Pro24ax].
*Item 10, Interaction With Complements III*: See [Pro24aq].
*Item 11, "Transitivity"*: We have

$$\begin{aligned}
(U \triangle V) \triangle (V \triangle W) &= U \triangle (V \triangle (V \triangle W)) && \text{(by Item 3)}\\
&= U \triangle ((V \triangle V) \triangle W) && \text{(by Item 3)}\\
&= U \triangle (\emptyset \triangle W) && \text{(by Item 6)}\\
&= U \triangle W && \text{(by Item 5)}
\end{aligned}$$

*Item 12, The Triangle Inequality for Symmetric Differences*: This follows from Items 2 and 11.
*Item 13, Distributivity Over Intersections*: See [Pro24s].
*Item 14, Interaction With Characteristic Functions*: See [Pro24j].
*Item 15, Bijectivity*: Clear.
*Item 16, Interaction With Powersets and Groups*: Item 16a follows from[19] Items 3 to 6, while Item 3b follows from Item 6.
*Item 4, Interaction With Powersets and Vector Spaces I*: Clear.
*Item 5, Interaction With Powersets and Vector Spaces II*: Omitted.
*Item 6, Interaction With Powersets and Rings*: This follows from Items 8 and 11 of Proposition 2.3.8.1.2 and Items 13 and 16.[20] □

## 2.4. Powersets

### 2.4.1. Characteristic Functions.

Let $X$ be a set.

**Definition 2.4.1.1.1.** Let $U \subset X$ and let $x \in X$.

(1) The **characteristic function of** $U$[21] is the function[22]

$$\chi_U \colon X \to \{\mathsf{t}, \mathsf{f}\}$$

defined by

$$\chi_U(x) \overset{\text{def}}{=} \begin{cases} \mathsf{true} & \text{if } x \in U, \\ \mathsf{false} & \text{if } x \notin U \end{cases}$$

for each $x \in X$.

(2) The **characteristic function of** $x$ is the function[23]

$$\chi_x \colon X \to \{\mathsf{t}, \mathsf{f}\}$$

defined by

$$\chi_x \overset{\text{def}}{=} \chi_{\{x\}},$$

[19] Reference: [Pro24ar].
[20] *Reference:* [Pro24au].
[21] *Further Terminology:* Also called the **indicator function of** $U$.
[22] *Further Notation:* Also written $\chi_X(U, -)$ or $\chi_X(-, U)$.
[23] *Further Notation:* Also written $\chi_x$, $\chi_X(x, -)$, or $\chi_X(-, x)$.

i.e. by

$$\chi_x(y) \overset{\text{def}}{=} \begin{cases} \mathsf{true} & \text{if } x = y, \\ \mathsf{false} & \text{if } x \neq y \end{cases}$$

for each $y \in X$.

(3) The **characteristic relation on** $X$[24] is the relation[25]

$$\chi_X(-_1, -_2) \colon X \times X \to \{\mathsf{t}, \mathsf{f}\}$$

on $X$ defined by[26]

$$\chi_X(x, y) \overset{\text{def}}{=} \begin{cases} \mathsf{true} & \text{if } x = y, \\ \mathsf{false} & \text{if } x \neq y \end{cases}$$

for each $x, y \in X$.

(4) The **characteristic embedding**[27] **of** $X$ **into** $\mathcal{P}(X)$ is the function

$$\chi_{(-)} \colon X \hookrightarrow \mathcal{P}(X)$$

defined by

$$\chi_{(-)}(x) \overset{\text{def}}{=} \chi_x$$

for each $x \in X$.

**Remark 2.4.1.1.2.** The definitions in Definition 2.4.1.1.1 are decategorifications of co/presheaves, representable co/presheaves, Hom profunctors, and the Yoneda embedding:[28]

(1) A function

$$f \colon X \to \{\mathsf{t}, \mathsf{f}\}$$

---

[24] *Further Terminology:* Also called the **identity relation on** $X$.

[25] *Further Notation:* Also written $\chi^{-1}_{-2}$, or $\sim_{\text{id}}$ in the context of relations.

[26] As a subset of $X \times X$, the relation $\chi_X$ corresponds to the diagonal $\Delta_X \subset X \times X$ of $X$.

[27] The name "characteristic *embedding*" comes from the fact that there is an analogue of fully faithfulness for $\chi_{(-)}$: given a set $X$, we have

$$\mathrm{Hom}_{\mathcal{P}(X)}(\chi_x, \chi_y) = \chi_X(x, y),$$

for each $x, y \in X$.

[28] These statements can be made precise by using the embeddings

$$(-)_{\mathsf{disc}} \colon \mathsf{Sets} \hookrightarrow \mathsf{Cats},$$
$$(-)_{\mathsf{disc}} \colon \{\mathsf{t}, \mathsf{f}\}_{\mathsf{disc}} \hookrightarrow \mathsf{Sets}$$

of sets into categories and of classical truth values into sets.
For instance, in this approach the characteristic function

$$\chi_x \colon X \to \{\mathsf{t}, \mathsf{f}\}$$

of an element $x$ of $X$, defined by

$$\chi_x(y) \overset{\text{def}}{=} \begin{cases} \mathsf{true} & \text{if } x = y, \\ \mathsf{false} & \text{if } x \neq y \end{cases}$$

for each $y \in X$, is recovered as the representable presheaf

$$\mathrm{Hom}_{X_{\mathsf{disc}}}(-, x) \colon X_{\mathsf{disc}} \to \mathsf{Sets}$$

of the corresponding object $x$ of $X_{\mathsf{disc}}$, defined on objects by

$$\mathrm{Hom}_{X_{\mathsf{disc}}}(y, x) \overset{\text{def}}{=} \begin{cases} \mathrm{pt} & \text{if } x = y, \\ \emptyset & \text{if } x \neq y \end{cases}$$

for each $y \in \mathrm{Obj}(X_{\mathsf{disc}})$.

is a decategorification of a presheaf

$$\mathcal{F}\colon \mathcal{C}^{\mathsf{op}} \to \mathsf{Sets},$$

with the characteristic functions $\chi_U$ of the subsets of $X$ being the primordial examples (and, in fact, all examples) of these.

(2) The characteristic function

$$\chi_x\colon X \to \{\mathsf{t},\mathsf{f}\}$$

of an *element* $x$ of $X$ is a decategorification of the representable presheaf

$$h_X\colon \mathcal{C}^{\mathsf{op}} \to \mathsf{Sets}$$

of an *object* $x$ of a category $\mathcal{C}$.

(3) The characteristic relation

$$\chi_X(-_1,-_2)\colon X \times X \to \{\mathsf{t},\mathsf{f}\}$$

of $X$ is a decategorification of the Hom profunctor

$$\mathrm{Hom}_{\mathcal{C}}(-_1,-_2)\colon \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathsf{Sets}$$

of a category $\mathcal{C}$.

(4) The characteristic embedding

$$\chi_{(-)}\colon X \hookrightarrow \mathcal{P}(X)$$

of $X$ into $\mathcal{P}(X)$ is a decategorification of the Yoneda embedding

$$よ\colon \mathcal{C}^{\mathsf{op}} \hookrightarrow \mathsf{PSh}(\mathcal{C})$$

of a category $\mathcal{C}$ into $\mathsf{PSh}(\mathcal{C})$.

(5) There is also a direct parallel between unions and colimits:
- An element of $\mathcal{P}(X)$ is a union of elements of $X$, viewed as one-point subsets $\{x\} \in \mathcal{P}(A)$;
- An object of $\mathsf{PSh}(\mathcal{C})$ is a colimit of objects of $\mathcal{C}$, viewed as representable presheaves $h_X \in \mathrm{Obj}(\mathsf{PSh}(\mathcal{C}))$.

**Proposition 2.4.1.1.3.** Let $X$ be a set.

(1) *The Inclusion of Characteristic Relations Associated to a Function.* Let $f\colon A \to B$ be a function. We have an inclusion[29]

$$\chi_B \circ (f \times f) \subset \chi_A, \qquad \begin{tikzcd} A \times A \arrow[rr, "f \times f"] \arrow[dr, "\chi_A"'] \arrow[rr, phantom, shift right=4ex, "\supset"] & & B \times B \arrow[dl, "\chi_B"] \\ & \{\mathsf{t},\mathsf{f}\}. & \end{tikzcd}$$

(2) *Interaction With Unions I.* We have

$$\chi_{U \cup V} = \max(\chi_U, \chi_V)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

(3) *Interaction With Unions II.* We have

$$\chi_{U \cup V} = \chi_U + \chi_V - \chi_{U \cap V}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

---

[29]This is the 0-categorical version of **??**.

(4) *Interaction With Intersections I.* We have

$$\chi_{U\cap V} = \chi_U \chi_V$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

(5) *Interaction With Intersections II.* We have

$$\chi_{U\cap V} = \min(\chi_U, \chi_V)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

(6) *Interaction With Differences.* We have

$$\chi_{U\setminus V} = \chi_U - \chi_{U\cap V}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

(7) *Interaction With Complements.* We have

$$\chi_{U^{\mathsf{c}}} = 1 - \chi_U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

(8) *Interaction With Symmetric Differences.* We have

$$\chi_{U\triangle V} = \chi_U + \chi_V - 2\chi_{U\cap V}$$

and thus, in particular, we have

$$\chi_{U\triangle V} \equiv \chi_U + \chi_V \mod 2$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

**Proof.** *Item 1, The Inclusion of Characteristic Relations Associated to a Function*: The inclusion $\chi_B(f(a), f(b)) \subset \chi_A(a, b)$ is equivalent to the statement "if $a = b$, then $f(a) = f(b)$", which is true.

*Item 2, Interaction With Unions I*: This is a repetition of Item 8 of Proposition 2.3.6.1.2 and is proved there.

*Item 3, Interaction With Unions II*: This is a repetition of Item 9 of Proposition 2.3.6.1.2 and is proved there.

*Item 4, Interaction With Intersections I*: This is a repetition of Item 9 of Proposition 2.3.8.1.2 and is proved there.

*Item 5, Interaction With Intersections II*: This is a repetition of Item 10 of Proposition 2.3.8.1.2 and is proved there.

*Item 6, Interaction With Differences*: This is a repetition of Item 15 of Proposition 2.3.9.1.2 and is proved there.

*Item 7, Interaction With Complements*: This is a repetition of Item 4 of Proposition 2.3.10.1.2 and is proved there.

*Item 8, Interaction With Symmetric Differences*: This is a repetition of Item 14 of Proposition 2.3.11.1.2 and is proved there. □

**2.4.2. The Yoneda Lemma for Sets.** Let $X$ be a set and let $U \subset X$ be a subset of $X$.

**Proposition 2.4.2.1.1.** We have

$$\chi_{\mathcal{P}(X)}(\chi_x, \chi_U) = \chi_U(x)$$

for each $x \in X$, giving an equality of functions

$$\chi_{\mathcal{P}(X)}\Big(\chi_{(-)}, \chi_U\Big) = \chi_U.$$

**Proof.** Clear. □

**Corollary 2.4.2.1.2.** The characteristic embedding is fully faithful, i.e., we have

$$\chi_{\mathcal{P}(X)}(\chi_x, \chi_y) = \chi_X(x, y)$$

for each $x, y \in X$.

**Proof.** This follows from Proposition 2.4.2.1.1. □

**2.4.3. Powersets.** Let $X$ be a set.

**Definition 2.4.3.1.1.** The **powerset of** $X$ is the set $\mathcal{P}(X)$ defined by

$$\mathcal{P}(X) \overset{\text{def}}{=} \{U \in P \mid U \subset X\},$$

where $P$ is the set in the axiom of powerset, **??** of **??**.

**Remark 2.4.3.1.2.** The powerset of a set is a decategorification of the category of presheaves of a category: while[30]

- The powerset of a set $X$ is equivalently (Item 6 of Proposition 2.4.3.1.3) the set
$$\mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\})$$
of functions from $X$ to the set $\{\mathsf{t}, \mathsf{f}\}$ of classical truth values;
- The category of presheaves on a category $\mathcal{C}$ is the category
$$\mathsf{Fun}(\mathcal{C}^{\mathsf{op}}, \mathsf{Sets})$$
of functors from $\mathcal{C}^{\mathsf{op}}$ to the category $\mathsf{Sets}$ of sets.

**Proposition 2.4.3.1.3.** Let $X$ be a set.

(1) *Functoriality.* The assignment $X \mapsto \mathcal{P}(X)$ defines functors

$$\mathcal{P}_* \colon \mathsf{Sets} \to \mathsf{Sets},$$
$$\mathcal{P}^{-1} \colon \mathsf{Sets}^{\mathsf{op}} \to \mathsf{Sets},$$
$$\mathcal{P}_! \colon \mathsf{Sets} \to \mathsf{Sets}$$

where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have
$$\mathcal{P}_*(A) \overset{\text{def}}{=} \mathcal{P}(A),$$
$$\mathcal{P}^{-1}(A) \overset{\text{def}}{=} \mathcal{P}(A),$$
$$\mathcal{P}_!(A) \overset{\text{def}}{=} \mathcal{P}(A);$$
- *Action on Morphisms.* For each morphism $f \colon A \to B$ of $\mathsf{Sets}$, the images
$$\mathcal{P}_*(f) \colon \mathcal{P}(A) \to \mathcal{P}(B),$$
$$\mathcal{P}^{-1}(f) \colon \mathcal{P}(B) \to \mathcal{P}(A),$$
$$\mathcal{P}_!(f) \colon \mathcal{P}(A) \to \mathcal{P}(B)$$

---

[30] This parallel is based on the following comparison:

- A category is enriched over the category
$$\mathsf{Sets} \overset{\text{def}}{=} \mathsf{Cats}_0$$
of sets (i.e. “0-categories”), with presheaves taking values on it;
- A set is enriched over the set
$$\{\mathsf{t}, \mathsf{f}\} \overset{\text{def}}{=} \mathsf{Cats}_{-1}$$

of $f$ by $\mathcal{P}_*$, $\mathcal{P}^{-1}$, and $\mathcal{P}_!$ are defined by

$$\mathcal{P}_*(f) \stackrel{\text{def}}{=} f_*,$$
$$\mathcal{P}^{-1}(f) \stackrel{\text{def}}{=} f^{-1},$$
$$\mathcal{P}_!(f) \stackrel{\text{def}}{=} f_!,$$

as in Definitions 2.4.4.1.1, 2.4.5.1.1 and 2.4.6.1.1.

(2) *Adjointness I.* We have an adjunction

$$\left(\mathcal{P}^{-1} \dashv \mathcal{P}^{-1,\mathsf{op}}\right)\colon \quad \begin{tikzcd} \mathsf{Sets}^{\mathsf{op}} \arrow[r, bend left, "\mathcal{P}^{-1}"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}, \arrow[l, bend left, "\mathcal{P}^{-1,\mathsf{op}}"] \end{tikzcd}$$

witnessed by a bijection

$$\underbrace{\mathsf{Sets}^{\mathsf{op}}(\mathcal{P}(X), Y)}_{\stackrel{\text{def}}{=} \mathsf{Sets}(Y, \mathcal{P}(X))} \cong \mathsf{Sets}(X, \mathcal{P}(Y)),$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets})$ and $Y \in \mathrm{Obj}(\mathsf{Sets}^{\mathsf{op}})$.

(3) *Adjointness II.* We have an adjunction

$$(\mathrm{Gr} \dashv \mathcal{P}_*)\colon \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "\mathrm{Gr}"] \arrow[r, phantom, "\perp"] & \mathrm{Rel}, \arrow[l, bend left, "\mathcal{P}_*"] \end{tikzcd}$$

witnessed by a bijection of sets

$$\mathrm{Rel}(\mathrm{Gr}(A), B) \cong \mathsf{Sets}(A, \mathcal{P}(B))$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$ and $B \in \mathrm{Obj}(\mathrm{Rel})$, where Gr is the graph functor of **??** of **??**.

(4) *Symmetric Strong Monoidality With Respect to Coproducts.* The powerset functor $\mathcal{P}_*$ of Item 1 has a symmetric strong monoidal structure

$$\left(\mathcal{P}_*, \mathcal{P}_*^{\coprod}, \mathcal{P}_{*|\mathbb{1}}^{\coprod}\right)\colon (\mathsf{Sets}, \textstyle\coprod, \emptyset) \to (\mathsf{Sets}, \times, \mathrm{pt})$$

being equipped with isomorphisms

$$\mathcal{P}_{*|X,Y}^{\coprod}\colon \mathcal{P}(X) \times \mathcal{P}(Y) \xrightarrow{\cong} \mathcal{P}(X \textstyle\coprod Y),$$
$$\mathcal{P}_{*|\mathbb{1}}^{\coprod}\colon \mathrm{pt} \xrightarrow{\equiv} \mathcal{P}(\emptyset),$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

(5) *Symmetric Lax Monoidality With Respect to Products.* The powerset functor $\mathcal{P}_*$ of Item 1 has a symmetric lax monoidal structure

$$\left(\mathcal{P}_*, \mathcal{P}_*^{\otimes}, \mathcal{P}_{*|\mathbb{1}}^{\otimes}\right)\colon (\mathsf{Sets}, \times, \mathrm{pt}) \to (\mathsf{Sets}, \times, \mathrm{pt})$$

being equipped with isomorphisms

$$\mathcal{P}_{*|X,Y}^{\otimes}\colon \mathcal{P}(X) \times \mathcal{P}(Y) \to \mathcal{P}(X \times Y),$$
$$\mathcal{P}_{*|\mathbb{1}}^{\otimes}\colon \mathrm{pt} \xrightarrow{\equiv} \mathcal{P}(\emptyset),$$

---

of classical truth values (i.e. "$(-1)$-categories"), with characteristic functions taking values on it.

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$, where $\mathcal{P}^{\otimes}_{*|X,Y}$ is given by

$$\mathcal{P}^{\otimes}_{*|X,Y}(U, V) \stackrel{\text{def}}{=} U \times V$$

for each $(U, V) \in \mathcal{P}(X) \times \mathcal{P}(Y)$.

(6) *Powersets as Sets of Functions.* The assignment $U \mapsto \chi_U$ defines a bijection[31]

$$\chi_{(-)} \colon \mathcal{P}(X) \xrightarrow{\cong} \mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\}),$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets})$.

(7) *Powersets as Sets of Relations.* We have bijections

$$\mathcal{P}(X) \cong \mathrm{Rel}(\mathrm{pt}, X),$$
$$\mathcal{P}(X) \cong \mathrm{Rel}(X, \mathrm{pt}),$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets})$.

(8) *As a Free Cocompletion: Universal Property.* The pair $\left(\mathcal{P}(X), \chi_{(-)}\right)$ consisting of
- The powerset $\mathcal{P}(X)$ of $X$;
- The characteristic embedding $\chi_{(-)} \colon X \hookrightarrow \mathcal{P}(X)$ of $X$ into $\mathcal{P}(X)$;

satisfies the following universal property:

($\star$) Given another pair $(Y, f)$ consisting of
  - A cocomplete poset $(Y, \preceq)$;
  - A function $f \colon X \to Y$;

  there exists a unique cocontinuous morphism of posets $(\mathcal{P}(X), \subset) \xrightarrow{\exists!} (Y, \preceq)$ making the diagram

$$\begin{tikzcd} & \mathcal{P}(X) \arrow[d, dashed, "\exists!"] \\ X \arrow[ur, "\chi_X"] \arrow[r, "f"'] & Y \end{tikzcd}$$

  commute.

(9) *As a Free Cocompletion: Adjointness.* We have an adjunction[32]

$$\left(\chi_{(-)} \dashv 忘\right) \colon \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "\chi_{(-)}"] \arrow[r, phantom, "\perp"] & \mathsf{Pos}^{\mathsf{cocomp.}} \arrow[l, bend left, "忘"] \end{tikzcd},$$

witnessed by a bijection

$$\mathsf{Pos}^{\mathsf{cocomp.}}((\mathcal{P}(X), \subset), (Y, \preceq)) \cong \mathsf{Sets}(X, Y),$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets})$ and $(Y, \preceq) \in \mathrm{Obj}(\mathsf{Pos})$, where

[31]This bijection is a decategorified form of the equivalence

$$\mathsf{PSh}(\mathcal{C}) \stackrel{\text{eq.}}{\cong} \mathsf{DFib}(\mathcal{C})$$

of **??** of **??**, with $\chi_{(-)}$ being a decategorified version of the category of elements construction of **??**.
See also **??** of **??**.

[32]In this sense, $\mathcal{P}(A)$ is the free cocompletion of $A$. (Note that, despite its name, however, this is not an idempotent operation, as we have $\mathcal{P}(\mathcal{P}(A)) \neq \mathcal{P}(A)$.)

- We have a natural map
  $\chi_X^*\colon \mathsf{Pos}^{\mathsf{cocomp.}}((\mathcal{P}(X),\subset),(Y,\preceq)) \to \mathsf{Sets}(X,Y)$
  defined by
  $$\chi_X^*(f) \overset{\text{def}}{=} f \circ \chi_X,$$
  i.e. by sending a cocontinuous morphism of posets $f\colon \mathcal{P}(X) \to Y$ to the composition
  $$X \overset{\chi_X}{\hookrightarrow} \mathcal{P}(X) \xrightarrow{f} Y;$$
- We have a natural map
  $\mathrm{Lan}_{\chi_X}\colon \mathsf{Sets}(X,Y) \to \mathsf{Pos}^{\mathsf{cocomp.}}((\mathcal{P}(X),\subset),(Y,\preceq))$
  computed by
  $$\begin{aligned}[\mathrm{Lan}_{\chi_X}(f)](U) &\cong \int^{x\in X} \chi_{\mathcal{P}(X)}(\chi_x,U)\odot f(x)\\ \text{(by Proposition 2.4.2.1.1)}\qquad &\cong \int^{x\in X} \chi_U(x)\odot f(x)\\ &\cong \bigvee_{x\in X}(\chi_U(x)\odot f(x))\end{aligned}$$
  for each $U \in \mathcal{P}(X)$, where:
  - $\bigvee$ is the join in $(Y,\preceq)$;
  - We have
    $$\begin{aligned}\mathsf{true}\odot f(x) &\overset{\text{def}}{=} f(x),\\ \mathsf{false}\odot f(x) &\overset{\text{def}}{=} \varnothing_Y,\end{aligned}$$
    where $\varnothing_Y$ is the minimal element of $(Y,\preceq)$.

**Proof.** *Item 1, Functoriality*: This follows from Items 3 and 4 of Proposition 2.4.4.1.4, Items 3 and 4 of Proposition 2.4.5.1.4, and Items 3 and 4 of Proposition 2.4.6.1.6.
*Item 2, Adjointness I*: Omitted.
*Item 3, Adjointness II*: We have

$$\begin{aligned}\mathrm{Rel}(\mathrm{Gr}(A),B) &= \mathcal{P}(A\times B)\\ \text{(by Item 6)}\qquad &= \mathsf{Sets}(A\times B,\{\mathsf{t},\mathsf{f}\})\\ \text{(by Item 2 of Proposition 2.1.2.1.2)}\qquad &= \mathsf{Sets}(A,\mathsf{Sets}(B,\{\mathsf{t},\mathsf{f}\}))\\ \text{(by Item 6)}\qquad &= \mathsf{Sets}(A,\mathcal{P}(B))\end{aligned}$$

with all bijections natural in $A$ and $B$.
*Item 4, Symmetric Strong Monoidality With Respect to Coproducts*: Omitted.
*Item 5, Symmetric Lax Monoidality With Respect to Products*: Omitted.
*Item 6, Powersets as Sets of Functions*: Omitted.
*Item 7, Powersets as Sets of Relations*: Omitted.
*Item 8, As a Free Cocompletion: Universal Property*: This is a rephrasing of **??**.
*Item 9, As a Free Cocompletion: Adjointness*: Omitted. □

**2.4.4. Direct Images.** Let $A$ and $B$ be sets and let $f\colon A \to B$ be a function.

**Definition 2.4.4.1.1.** The **direct image function associated to** $f$ is the function[33]

$$f_* \colon \mathcal{P}(A) \to \mathcal{P}(B)$$

defined by[34,35]

$$\begin{aligned} f_*(U) &\stackrel{\text{def}}{=} f(U) \\ &\stackrel{\text{def}}{=} \left\{ b \in B \;\middle|\; \begin{array}{l} \text{there exists some } a \in \\ U \text{ such that } b = f(a) \end{array} \right\} \\ &= \{ f(a) \in B \mid a \in U \} \end{aligned}$$

for each $U \in \mathcal{P}(A)$.

**Remark 2.4.4.1.2.** Identifying subsets of $A$ with functions from $A$ to $\{\mathsf{true}, \mathsf{false}\}$ via Item 6 of Proposition 2.4.3.1.3, we see that the direct image function associated to $f$ is equivalently the function

$$f_* \colon \mathcal{P}(A) \to \mathcal{P}(B)$$

defined by

$$\begin{aligned} f_*(\chi_U) &\stackrel{\text{def}}{=} \mathrm{Lan}_f(\chi_U) \\ &= \operatorname{colim}\left(\left(f \overset{\rightarrow}{\times} \underline{(-_1)}\right) \overset{\mathrm{pr}}{\twoheadrightarrow} A \xrightarrow{\chi_U} \{\mathsf{t}, \mathsf{f}\}\right) \\ &= \operatorname*{colim}_{\substack{a \in A \\ f(a) = -_1}} (\chi_U(a)) \\ &= \bigvee_{\substack{a \in A \\ f(a) = -_1}} (\chi_U(a)). \end{aligned}$$

So, in other words, we have

$$\begin{aligned} [f_*(\chi_U)](b) &= \bigvee_{\substack{a \in A \\ f(a) = b}} (\chi_U(a)) \\ &= \begin{cases} \mathsf{true} & \text{if there exists some } a \in A \text{ such} \\ & \text{that } f(a) = b \text{ and } a \in U, \\ \mathsf{false} & \text{otherwise} \end{cases} \\ &= \begin{cases} \mathsf{true} & \text{if there exists some } a \in U \\ & \text{such that } f(a) = b, \\ \mathsf{false} & \text{otherwise} \end{cases} \end{aligned}$$

for each $b \in B$.

---

[33] *Further Notation:* Also written $\exists_f \colon \mathcal{P}(A) \to \mathcal{P}(B)$. This notation comes from the fact that the following statements are equivalent, where $b \in B$ and $U \in \mathcal{P}(A)$:

- We have $b \in \exists_f(U)$.
- There exists some $a \in U$ such that $f(a) = b$.

[34] *Further Terminology:* The set $f(U)$ is called the **direct image of $U$ by $f$**.

[35] We also have

$$f_*(U) = B \setminus f_!(A \setminus U);$$

see Item 7 of Proposition 2.4.4.1.3.

**Proposition 2.4.4.1.3.** Let $f\colon A \to B$ be a function.

(1) *Functoriality.* The assignment $U \mapsto f_*(U)$ defines a functor

$$f_*\colon (\mathcal{P}(A), \subset) \to (\mathcal{P}(B), \subset)$$

where

- *Action on Objects.* For each $U \in \mathcal{P}(A)$, we have

$$[f_*](U) \stackrel{\text{def}}{=} f_*(U);$$

- *Action on Morphisms.* For each $U, V \in \mathcal{P}(A)$:
  - ($\star$) If $U \subset V$, then $f_*(U) \subset f_*(V)$.

(2) *Triple Adjointness.* We have a triple adjunction

$$\left(f_* \dashv f^{-1} \dashv f_!\right)\colon \quad \begin{tikzcd} \mathcal{P}(A) \arrow[r, bend left, "f_*"] \arrow[r, bend right, "f_!"'] & \mathcal{P}(B), \arrow[l, "f^{-1}" description] \end{tikzcd}$$

witnessed by bijections of sets

$$\mathrm{Hom}_{\mathcal{P}(B)}(f_*(U), V) \cong \mathrm{Hom}_{\mathcal{P}(A)}\left(U, f^{-1}(V)\right),$$
$$\mathrm{Hom}_{\mathcal{P}(A)}\left(f^{-1}(U), V\right) \cong \mathrm{Hom}_{\mathcal{P}(A)}(U, f_!(V)),$$

natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$ and (respectively) $V \in \mathcal{P}(A)$ and $U \in \mathcal{P}(B)$, i.e. where:

(a) The following conditions are equivalent:
   (i) We have $f_*(U) \subset V$.
   (ii) We have $U \subset f^{-1}(V)$.

(b) The following conditions are equivalent:
   (i) We have $f^{-1}(U) \subset V$.
   (ii) We have $U \subset f_!(V)$.

(3) *Preservation of Colimits.* We have an equality of sets

$$f_*\left(\bigcup_{i\in I} U_i\right) = \bigcup_{i\in I} f_*(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have equalities

$$f_*(U) \cup f_*(V) = f_*(U \cup V),$$
$$f_*(\emptyset) = \emptyset,$$

natural in $U, V \in \mathcal{P}(A)$.

(4) *Oplax Preservation of Limits.* We have an inclusion of sets

$$f_*\left(\bigcap_{i\in I} U_i\right) \subset \bigcap_{i\in I} f_*(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have inclusions

$$f_*(U \cap V) \subset f_*(U) \cap f_*(V),$$
$$f_*(A) \subset B,$$

natural in $U, V \in \mathcal{P}(A)$.

(5) *Symmetric Strict Monoidality With Respect to Unions.* The direct image function of Item 1 has a symmetric strict monoidal structure

$$\left(f_*, f_*^{\otimes}, f_{*|\mathbb{1}}^{\otimes}\right): (\mathcal{P}(A), \cup, \emptyset) \to (\mathcal{P}(B), \cup, \emptyset),$$

being equipped with equalities

$$f_{*|U,V}^{\otimes}: f_*(U) \cup f_*(V) \xrightarrow{=} f_*(U \cup V),$$
$$f_{*|\mathbb{1}}^{\otimes}: \emptyset \xrightarrow{=} \emptyset,$$

natural in $U, V \in \mathcal{P}(A)$.

(6) *Symmetric Oplax Monoidality With Respect to Intersections.* The direct image function of Item 1 has a symmetric oplax monoidal structure

$$\left(f_*, f_*^{\otimes}, f_{*|\mathbb{1}}^{\otimes}\right): (\mathcal{P}(A), \cap, A) \to (\mathcal{P}(B), \cap, B),$$

being equipped with inclusions

$$f_{*|U,V}^{\otimes}: f_*(U \cap V) \hookrightarrow f_*(U) \cap f_*(V),$$
$$f_{*|\mathbb{1}}^{\otimes}: f_*(A) \hookrightarrow B,$$

natural in $U, V \in \mathcal{P}(A)$.

(7) *Relation to Direct Images With Compact Support.* We have

$$f_*(U) = B \setminus f_!(A \setminus U)$$

for each $U \in \mathcal{P}(A)$.

**Proof.** *Item 1, Functoriality*: Clear.
*Item 2, Triple Adjointness*: This follows from **??** of **??**.
*Item 3, Preservation of Colimits*: This follows from Item 2 and **??** of **??**.
*Item 4, Oplax Preservation of Limits*: Omitted.
*Item 5, Symmetric Strict Monoidality With Respect to Unions*: This follows from Item 3.
*Item 6, Symmetric Oplax Monoidality With Respect to Intersections*: This follows from **??**.
*Item 7, Relation to Direct Images With Compact Support*: Applying **??** of **??** to $A \setminus U$, we have

$$\begin{aligned} f_!(A \setminus U) &= B \setminus f_*(A \setminus (A \setminus U)) \\ &= B \setminus f_*(U). \end{aligned}$$

Taking complements, we then obtain

$$\begin{aligned} f_*(U) &= B \setminus (B \setminus f_*(U)), \\ &= B \setminus f_!(A \setminus U), \end{aligned}$$

which finishes the proof. □

**Proposition 2.4.4.1.4.** Let $f: A \to B$ be a function.

(1) *Functionality I.* The assignment $f \mapsto f_*$ defines a function

$$(-)_{*|A,B}: \mathsf{Sets}(A, B) \to \mathsf{Sets}(\mathcal{P}(A), \mathcal{P}(B)).$$

(2) *Functionality II.* The assignment $f \mapsto f_*$ defines a function

$$(-)_{*|A,B}: \mathsf{Sets}(A, B) \to \mathsf{Pos}((\mathcal{P}(A), \subset), (\mathcal{P}(B), \subset)).$$

(3) *Interaction With Identities.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$(\mathrm{id}_A)_* = \mathrm{id}_{\mathcal{P}(A)};$$

(4) *Interaction With Composition.* For each pair of composable functions $f: A \to B$ and $g: B \to C$, we have

$$(g \circ f)_* = g_* \circ f_*, \qquad \begin{tikzcd} \mathcal{P}(A) \arrow[r, "f_*"] \arrow[dr, "(g\circ f)_*"'] & \mathcal{P}(B) \arrow[d, "g_*"] \\ & \mathcal{P}(C). \end{tikzcd}$$

**Proof.** *Item 1, Functionality I*: Clear.
*Item 2, Functionality II*: Clear.
*Item 3, Interaction With Identities*: This follows from **??** of **??**.
*Item 4, Interaction With Composition*: This follows from **??** of **??**. □

**2.4.5. Inverse Images.** Let $A$ and $B$ be sets and let $f: A \to B$ be a function.

**Definition 2.4.5.1.1.** The **inverse image function associated to** $f$ is the function[36]

$$f^{-1}: \mathcal{P}(B) \to \mathcal{P}(A)$$

defined by[37]

$$f^{-1}(V) \stackrel{\text{def}}{=} \{a \in A \mid \text{we have } f(a) \in V\}$$

for each $V \in \mathcal{P}(B)$.

**Remark 2.4.5.1.2.** Identifying subsets of $B$ with functions from $B$ to $\{\mathsf{true}, \mathsf{false}\}$ via Item 6 of Proposition 2.4.3.1.3, we see that the inverse image function associated to $f$ is equivalently the function

$$f^*: \mathcal{P}(B) \to \mathcal{P}(A)$$

defined by

$$f^*(\chi_V) \stackrel{\text{def}}{=} \chi_V \circ f$$

for each $\chi_V \in \mathcal{P}(B)$, where $\chi_V \circ f$ is the composition

$$A \xrightarrow{f} B \xrightarrow{\chi_V} \{\mathsf{true}, \mathsf{false}\}$$

in $\mathsf{Sets}$.

**Proposition 2.4.5.1.3.** Let $f: A \to B$ be a function.

(1) *Functoriality.* The assignment $V \mapsto f^{-1}(V)$ defines a functor

$$f^{-1}: (\mathcal{P}(B), \subset) \to (\mathcal{P}(A), \subset)$$

where

- *Action on Objects.* For each $V \in \mathcal{P}(B)$, we have

$$[f^{-1}](V) \stackrel{\text{def}}{=} f^{-1}(V);$$

- *Action on Morphisms.* For each $U, V \in \mathcal{P}(B)$:
  ($\star$) If $U \subset V$, then $f^{-1}(U) \subset f^{-1}(V)$.

[36] *Further Notation:* Also written $f^*: \mathcal{P}(B) \to \mathcal{P}(A)$.
[37] *Further Terminology:* The set $f^{-1}(V)$ is called the **inverse image of** $V$ **by** $f$.

(2) *Triple Adjointness.* We have a triple adjunction

$$\left(f_* \dashv f^{-1} \dashv f_!\right)\colon \quad \begin{tikzcd} \mathcal{P}(A) \arrow[r, bend left, "f_*"] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, phantom, shift right=2, "\perp"] \arrow[r, bend right, "f_!"'] & \mathcal{P}(B), \arrow[l, "f^{-1}" description] \end{tikzcd}$$

witnessed by bijections of sets

$$\mathrm{Hom}_{\mathcal{P}(B)}(f_*(U), V) \cong \mathrm{Hom}_{\mathcal{P}(A)}\left(U, f^{-1}(V)\right),$$
$$\mathrm{Hom}_{\mathcal{P}(A)}\left(f^{-1}(U), V\right) \cong \mathrm{Hom}_{\mathcal{P}(A)}(U, f_!(V)),$$

natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$ and (respectively) $V \in \mathcal{P}(A)$ and $U \in \mathcal{P}(B)$, i.e. where:

(a) The following conditions are equivalent:
  (i) We have $f_*(U) \subset V$;
  (ii) We have $U \subset f^{-1}(V)$;

(b) The following conditions are equivalent:
  (i) We have $f^{-1}(U) \subset V$.
  (ii) We have $U \subset f_!(V)$.

(3) *Preservation of Colimits.* We have an equality of sets

$$f^{-1}\left(\bigcup_{i\in I} U_i\right) = \bigcup_{i\in I} f^{-1}(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(B)^{\times I}$. In particular, we have equalities

$$f^{-1}(U) \cup f^{-1}(V) = f^{-1}(U \cup V),$$
$$f^{-1}(\emptyset) = \emptyset,$$

natural in $U, V \in \mathcal{P}(B)$.

(4) *Preservation of Limits.* We have an equality of sets

$$f^{-1}\left(\bigcap_{i\in I} U_i\right) = \bigcap_{i\in I} f^{-1}(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(B)^{\times I}$. In particular, we have equalities

$$f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V),$$
$$f^{-1}(B) = A,$$

natural in $U, V \in \mathcal{P}(B)$.

(5) *Symmetric Strict Monoidality With Respect to Unions.* The inverse image function of Item 1 has a symmetric strict monoidal structure

$$\left(f^{-1}, f^{-1,\otimes}, f^{-1,\otimes}_{\mathbb{1}}\right)\colon (\mathcal{P}(B), \cup, \emptyset) \to (\mathcal{P}(A), \cup, \emptyset),$$

being equipped with equalities

$$f^{-1,\otimes}_{U,V}\colon f^{-1}(U) \cup f^{-1}(V) \xrightarrow{=} f^{-1}(U \cup V),$$
$$f^{-1,\otimes}_{\mathbb{1}}\colon \emptyset \xrightarrow{=} f^{-1}(\emptyset),$$

natural in $U, V \in \mathcal{P}(B)$.

(6) *Symmetric Strict Monoidality With Respect to Intersections.* The inverse image function of Item 1 has a symmetric strict monoidal structure

$$\left(f^{-1}, f^{-1,\otimes}, f^{-1,\otimes}_{\mathbb{1}}\right) : (\mathcal{P}(B), \cap, B) \to (\mathcal{P}(A), \cap, A),$$

being equipped with equalities

$$f^{-1,\otimes}_{U,V} : f^{-1}(U) \cap f^{-1}(V) \xrightarrow{=} f^{-1}(U \cap V),$$
$$f^{-1,\otimes}_{\mathbb{1}} : A \xrightarrow{=} f^{-1}(B),$$

natural in $U, V \in \mathcal{P}(B)$.

**Proof.** *Item 1, Functoriality*: Clear.
*Item 2, Triple Adjointness*: This follows from **??** of **??**.
*Item 3, Preservation of Colimits*: This follows from Item 2 and **??** of **??**.
*Item 4, Preservation of Limits*: This follows from Item 2 and **??** of **??**.
*Item 5, Symmetric Strict Monoidality With Respect to Unions*: This follows from Item 3.
*Item 6, Symmetric Strict Monoidality With Respect to Intersections*: This follows from Item 4. □

**Proposition 2.4.5.1.4.** Let $f : A \to B$ be a function.

(1) *Functionality I.* The assignment $f \mapsto f^{-1}$ defines a function

$$(-)^{-1}_{A,B} : \mathsf{Sets}(A, B) \to \mathsf{Sets}(\mathcal{P}(B), \mathcal{P}(A)).$$

(2) *Functionality II.* The assignment $f \mapsto f^{-1}$ defines a function

$$(-)^{-1}_{A,B} : \mathsf{Sets}(A, B) \to \mathsf{Pos}((\mathcal{P}(B), \subset), (\mathcal{P}(A), \subset)).$$

(3) *Interaction With Identities.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$\mathrm{id}_A^{-1} = \mathrm{id}_{\mathcal{P}(A)};$$

(4) *Interaction With Composition.* For each pair of composable functions $f : A \to B$ and $g : B \to C$, we have

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}, \qquad \begin{tikzcd} \mathcal{P}(C) \arrow[r, "g^{-1}"] \arrow[dr, "(g\circ f)^{-1}"'] & \mathcal{P}(B) \arrow[d, "f^{-1}"] \\ & \mathcal{P}(A). \end{tikzcd}$$

**Proof.** *Item 1, Functionality I*: Clear.
*Item 2, Functionality II*: Clear.
*Item 3, Interaction With Identities*: This follows from **??** of **??**.
*Item 4, Interaction With Composition*: This follows from **??** of **??**. □

**2.4.6. Direct Images With Compact Support.** Let $A$ and $B$ be sets and let $f : A \to B$ be a function.

**Definition 2.4.6.1.1.** The **direct image with compact support function associated to $f$** is the function[38]

$$f_!\colon \mathcal{P}(A) \to \mathcal{P}(B)$$

defined by[39,40]

$$\begin{aligned} f_!(U) &\overset{\text{def}}{=} \left\{ b \in B \;\middle|\; \begin{array}{l} \text{for each } a \in A\text{, if we have} \\ f(a) = b\text{, then } a \in U \end{array} \right\} \\ &= \left\{ b \in B \;\middle|\; \text{we have } f^{-1}(b) \subset U \right\} \end{aligned}$$

for each $U \in \mathcal{P}(A)$.

**Remark 2.4.6.1.2.** Identifying subsets of $A$ with functions from $A$ to $\{\mathsf{true}, \mathsf{false}\}$ via Item 6 of Proposition 2.4.3.1.3, we see that the direct image with compact support function associated to $f$ is equivalently the function

$$f_!\colon \mathcal{P}(A) \to \mathcal{P}(B)$$

defined by

$$\begin{aligned} f_!(\chi_U) &\overset{\text{def}}{=} \mathrm{Ran}_f(\chi_U) \\ &= \lim\left(\left(\underline{(-_1)} \overset{\rightarrow}{\times} f\right) \overset{\mathrm{pr}}{\twoheadrightarrow} A \xrightarrow{\chi_U} \{\mathsf{true}, \mathsf{false}\}\right) \\ &= \lim_{\substack{a \in A \\ f(a) = -_1}} (\chi_U(a)) \\ &= \bigwedge_{\substack{a \in A \\ f(a) = -_1}} (\chi_U(a)). \end{aligned}$$

So, in other words, we have

$$\begin{aligned} [f_!(\chi_U)](b) &= \bigwedge_{\substack{a \in A \\ f(a) = b}} (\chi_U(a)) \\ &= \begin{cases} \mathsf{true} & \text{if, for each } a \in A \text{ such that} \\ & f(a) = b\text{, we have } a \in U, \\ \mathsf{false} & \text{otherwise} \end{cases} \\ &= \begin{cases} \mathsf{true} & \text{if } f^{-1}(b) \subset U \\ \mathsf{false} & \text{otherwise} \end{cases} \end{aligned}$$

for each $b \in B$.

---

[38] *Further Notation:* Also written $\forall_f\colon \mathcal{P}(A) \to \mathcal{P}(B)$. This notation comes from the fact that the following statements are equivalent, where $b \in B$ and $U \in \mathcal{P}(A)$:

- We have $b \in \forall_f(U)$.
- For each $a \in A$, if $b = f(a)$, then $a \in U$.

[39] *Further Terminology:* The set $f_!(U)$ is called the **direct image with compact support of $U$ by $f$**.

[40] We also have

$$f_!(U) = B \setminus f_*(A \setminus U);$$

see Item 7 of Proposition 2.4.6.1.5.

**Definition 2.4.6.1.3.** Let $U$ be a subset of $A$.[41,42]

(1) The **image part of the direct image with compact support $f_!(U)$ of $U$** is the set $f_{!,\mathrm{im}}(U)$ defined by

$$\begin{aligned} f_{!,\mathrm{im}}(U) &\stackrel{\mathrm{def}}{=} f_!(U) \cap \mathrm{Im}(f) \\ &= \left\{ b \in B \,\middle|\, \begin{array}{l} \text{we have } f^{-1}(b) \subset \\ U \text{ and } f^{-1}(b) \neq \emptyset \end{array} \right\}. \end{aligned}$$

(2) The **complement part of the direct image with compact support $f_!(U)$ of $U$** is the set $f_{!,\mathrm{cp}}(U)$ defined by

$$\begin{aligned} f_{!,\mathrm{cp}}(U) &\stackrel{\mathrm{def}}{=} f_!(U) \cap (B \setminus \mathrm{Im}(f)) \\ &= B \setminus \mathrm{Im}(f) \\ &= \left\{ b \in B \,\middle|\, \begin{array}{l} \text{we have } f^{-1}(b) \subset \\ U \text{ and } f^{-1}(b) = \emptyset \end{array} \right\} \\ &= \left\{ b \in B \,\middle|\, f^{-1}(b) = \emptyset \right\}. \end{aligned}$$

**Example 2.4.6.1.4.** Here are some examples of direct images with compact support.

(1) *The Multiplication by Two Map on the Natural Numbers.* Consider the function $f\colon \mathbb{N} \to \mathbb{N}$ given by

$$f(n) \stackrel{\mathrm{def}}{=} 2n$$

for each $n \in \mathbb{N}$. Since $f$ is injective, we have

$$\begin{aligned} f_{!,\mathrm{im}}(U) &= f_*(U) \\ f_{!,\mathrm{cp}}(U) &= \{\text{odd natural numbers}\} \end{aligned}$$

for any $U \subset \mathbb{N}$.

(2) *Parabolas.* Consider the function $f\colon \mathbb{R} \to \mathbb{R}$ given by

$$f(x) \stackrel{\mathrm{def}}{=} x^2$$

for each $x \in \mathbb{R}$. We have

$$f_{!,\mathrm{cp}}(U) = \mathbb{R}_{<0}$$

---

[41] Note that we have

$$f_!(U) = f_{!,\mathrm{im}}(U) \cup f_{!,\mathrm{cp}}(U),$$

as

$$\begin{aligned} f_!(U) &= f_!(U) \cap B \\ &= f_!(U) \cap (\mathrm{Im}(f) \cup (B \setminus \mathrm{Im}(f))) \\ &= (f_!(U) \cap \mathrm{Im}(f)) \cup (f_!(U) \cap (B \setminus \mathrm{Im}(f))) \\ &\stackrel{\mathrm{def}}{=} f_{!,\mathrm{im}}(U) \cup f_{!,\mathrm{cp}}(U). \end{aligned}$$

[42] In terms of the meet computation of $f_!(U)$ of Remark 2.4.6.1.2, namely

$$f_!(\chi_U) = \bigwedge_{\substack{a \in A \\ f(a) = -1}} (\chi_U(a)),$$

we see that $f_{!,\mathrm{im}}$ corresponds to meets indexed over nonempty sets, while $f_{!,\mathrm{cp}}$ corresponds to meets indexed over the empty set.

for any $U \subset \mathbb{R}$. Moreover, since $f^{-1}(x) = \{-\sqrt{x}, \sqrt{x}\}$, we have e.g.:

$$\begin{aligned} f_{!,\mathrm{im}}([0,1]) &= \{0\}, \\ f_{!,\mathrm{im}}([-1,1]) &= [0,1], \\ f_{!,\mathrm{im}}([1,2]) &= \emptyset, \\ f_{!,\mathrm{im}}([-2,-1] \cup [1,2]) &= [1,4]. \end{aligned}$$

(3) *Circles.* Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x,y) \overset{\mathrm{def}}{=} x^2 + y^2$$

for each $(x,y) \in \mathbb{R}^2$. We have

$$f_{!,\mathrm{cp}}(U) = \mathbb{R}_{<0}$$

for any $U \subset \mathbb{R}^2$, and since

$$f^{-1}(r) = \begin{cases} \text{a circle of radius } r \text{ about the origin} & \text{if } r > 0, \\ \{(0,0)\} & \text{if } r = 0, \\ \emptyset & \text{if } r < 0, \end{cases}$$

we have e.g.:

$$\begin{aligned} f_{!,\mathrm{im}}([-1,1] \times [-1,1]) &= [0,1], \\ f_{!,\mathrm{im}}(([-1,1] \times [-1,1]) \setminus [-1,1] \times \{0\}) &= \emptyset. \end{aligned}$$

**Proposition 2.4.6.1.5.** Let $f\colon A \to B$ be a function.

(1) *Functoriality.* The assignment $U \mapsto f_!(U)$ defines a functor

$$f_!\colon (\mathcal{P}(A), \subset) \to (\mathcal{P}(B), \subset)$$

where
- *Action on Objects.* For each $U \in \mathcal{P}(A)$, we have

$$[f_!](U) \overset{\mathrm{def}}{=} f_!(U);$$

- *Action on Morphisms.* For each $U, V \in \mathcal{P}(A)$:
  - ($\star$) If $U \subset V$, then $f_!(U) \subset f_!(V)$.

(2) *Triple Adjointness.* We have a triple adjunction

$$\Big(f_* \dashv f^{-1} \dashv f_!\Big)\colon \quad \begin{tikzcd} \mathcal{P}(A) \arrow[r, bend left=45, "f_*"] \arrow[r, bend right=45, "f_!"'] & \mathcal{P}(B) \arrow[l, "f^{-1}"' description] \end{tikzcd},$$

witnessed by bijections of sets

$$\begin{aligned} \mathrm{Hom}_{\mathcal{P}(B)}(f_*(U), V) &\cong \mathrm{Hom}_{\mathcal{P}(A)}\Big(U, f^{-1}(V)\Big), \\ \mathrm{Hom}_{\mathcal{P}(A)}\Big(f^{-1}(U), V\Big) &\cong \mathrm{Hom}_{\mathcal{P}(A)}(U, f_!(V)), \end{aligned}$$

natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$ and (respectively) $V \in \mathcal{P}(A)$ and $U \in \mathcal{P}(B)$, i.e. where:

(a) The following conditions are equivalent:
   (i) We have $f_*(U) \subset V$;
   (ii) We have $U \subset f^{-1}(V)$;

(b) The following conditions are equivalent:

(i) We have $f^{-1}(U) \subset V$.
(ii) We have $U \subset f_!(V)$.

(3) *Lax Preservation of Colimits.* We have an inclusion of sets

$$\bigcup_{i\in I} f_!(U_i) \subset f_!\left(\bigcup_{i\in I} U_i\right),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have inclusions

$$f_!(U) \cup f_!(V) \hookrightarrow f_!(U \cup V),$$
$$\emptyset \hookrightarrow f_!(\emptyset),$$

natural in $U, V \in \mathcal{P}(A)$.

(4) *Preservation of Limits.* We have an equality of sets

$$f_!\left(\bigcap_{i\in I} U_i\right) = \bigcap_{i\in I} f_!(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have equalities

$$f^{-1}(U \cap V) = f_!(U) \cap f^{-1}(V),$$
$$f_!(A) = B,$$

natural in $U, V \in \mathcal{P}(A)$.

(5) *Symmetric Lax Monoidality With Respect to Unions.* The direct image with compact support function of Item 1 has a symmetric lax monoidal structure

$$\left(f_!, f_!^{\otimes}, f_{!|\mathbb{1}}^{\otimes}\right) \colon (\mathcal{P}(A), \cup, \emptyset) \to (\mathcal{P}(B), \cup, \emptyset),$$

being equipped with inclusions

$$f_{!|U,V}^{\otimes} \colon f_!(U) \cup f_!(V) \hookrightarrow f_!(U \cup V),$$
$$f_{!|\mathbb{1}}^{\otimes} \colon \emptyset \hookrightarrow f_!(\emptyset),$$

natural in $U, V \in \mathcal{P}(A)$.

(6) *Symmetric Strict Monoidality With Respect to Intersections.* The direct image function of Item 1 has a symmetric strict monoidal structure

$$\left(f_!, f_!^{\otimes}, f_{!|\mathbb{1}}^{\otimes}\right) \colon (\mathcal{P}(A), \cap, A) \to (\mathcal{P}(B), \cap, B),$$

being equipped with equalities

$$f_{!|U,V}^{\otimes} \colon f_!(U \cap V) \xrightarrow{=} f_!(U) \cap f_!(V),$$
$$f_{!|\mathbb{1}}^{\otimes} \colon f_!(A) \xrightarrow{=} B,$$

natural in $U, V \in \mathcal{P}(A)$.

(7) *Relation to Direct Images.* We have

$$f_!(U) = B \setminus f_*(A \setminus U)$$

for each $U \in \mathcal{P}(A)$.

(8) *Interaction With Injections.* If $f$ is injective, then we have

$$\begin{aligned} f_{!,\mathrm{im}}(U) &= f_*(U), \\ f_{!,\mathrm{cp}}(U) &= B \setminus \mathrm{Im}(f), \\ f_!(U) &= f_{!,\mathrm{im}}(U) \cup f_{!,\mathrm{cp}}(U) \\ &= f_*(U) \cup (B \setminus \mathrm{Im}(f)) \end{aligned}$$

for each $U \in \mathcal{P}(A)$.

(9) *Interaction With Surjections.* If $f$ is surjective, then we have

$$\begin{aligned} f_{!,\mathrm{im}}(U) &\subset f_*(U), \\ f_{!,\mathrm{cp}}(U) &= \emptyset, \\ f_!(U) &\subset f_*(U) \end{aligned}$$

for each $U \in \mathcal{P}(A)$.

**Proof.** *Item 1, Functoriality*: Clear.
*Item 2, Triple Adjointness*: This follows from **??** of **??**.
*Item 3, Lax Preservation of Colimits*: Omitted.
*Item 4, Preservation of Limits*: Omitted. This follows from Item 2 and **??** of **??**.
*Item 5, Symmetric Lax Monoidality With Respect to Unions*: This follows from **??**.
*Item 6, Symmetric Strict Monoidality With Respect to Intersections*: This follows from Item 4.
*Item 7, Relation to Direct Images*: We claim that $f_!(U) = B \setminus f_*(A \setminus U)$.

- *The First Implication.* We claim that
$$f_!(U) \subset B \setminus f_*(A \setminus U).$$
Let $b \in f_!(U)$. We need to show that $b \notin f_*(A \setminus U)$, i.e. that there is no $a \in A \setminus U$ such that $f(a) = b$.
  This is indeed the case, as otherwise we would have $a \in f^{-1}(b)$ and $a \notin U$, contradicting $f^{-1}(b) \subset U$ (which holds since $b \in f_!(U)$).
  Thus $b \in B \setminus f_*(A \setminus U)$.
- *The Second Implication.* We claim that
$$B \setminus f_*(A \setminus U) \subset f_!(U).$$
Let $b \in B \setminus f_*(A \setminus U)$. We need to show that $b \in f_!(U)$, i.e. that $f^{-1}(b) \subset U$.
  Since $b \notin f_*(A \setminus U)$, there exists no $a \in A \setminus U$ such that $b = f(a)$, and hence $f^{-1}(b) \subset U$.
  Thus $b \in f_!(U)$.

This finishes the proof of Item 7.
*Item 8, Interaction With Injections*: Clear.
*Item 9, Interaction With Surjections*: Clear. □

**Proposition 2.4.6.1.6.** Let $f\colon A \to B$ be a function.

(1) *Functionality I.* The assignment $f \mapsto f_!$ defines a function

$$(-)_{!|A,B}\colon \mathsf{Sets}(A, B) \to \mathsf{Sets}(\mathcal{P}(A), \mathcal{P}(B)).$$

(2) *Functionality II.* The assignment $f \mapsto f_!$ defines a function

$$(-)_{!|A,B} \colon \mathsf{Sets}(A,B) \to \mathsf{Pos}((\mathcal{P}(A),\subset),(\mathcal{P}(B),\subset)).$$

(3) *Interaction With Identities.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$(\mathrm{id}_A)_! = \mathrm{id}_{\mathcal{P}(A)};$$

(4) *Interaction With Composition.* For each pair of composable functions $f \colon A \to B$ and $g \colon B \to C$, we have

$$(g \circ f)_! = g_! \circ f_!, \qquad \begin{tikzcd} \mathcal{P}(A) \arrow[r, "f_!"] \arrow[dr, "(g\circ f)_!"'] & \mathcal{P}(B) \arrow[d, "g_!"] \\ & \mathcal{P}(C). \end{tikzcd}$$

**Proof.** *Item 1*, *Functionality I*: Clear.
*Item 2*, *Functionality II*: Clear.
*Item 3*, *Interaction With Identities*: This follows from **??** of **??**.
*Item 4*, *Interaction With Composition*: This follows from **??** of **??**. □

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CHAPTER 3

# Pointed Sets

This chapter contains some foundational material on pointed sets.

## 3.1. Pointed Sets

### 3.1.1. Foundations.

**Definition 3.1.1.1.1.** A **pointed set**[1] is equivalently

- An $\mathbb{E}_0$-monoid in $(\mathrm{N}_\bullet(\mathsf{Sets}), \mathrm{pt})$;
- A pointed object in $(\mathsf{Sets}, \mathrm{pt})$.

**Remark 3.1.1.1.2.** In detail, a **pointed set** is a pair $(X, x_0)$ consisting of

- *The Underlying Set.* A set $X$, called the **underlying set of** $(X, x_0)$;
- *The Basepoint.* A morphism
$$[x_0]\colon \mathrm{pt} \to X$$
in $\mathsf{Sets}$, determining an element $x_0 \in X$, called the **basepoint of** $X$.

**Example 3.1.1.1.3.** The 0**-sphere**[2] is the pointed set $(S^0, 0)$[3] consisting of

- *The Underlying Set.* The set $S^0$ defined by
$$S^0 \stackrel{\text{def}}{=} \{0, 1\};$$
- *The Basepoint.* The element 0 of $S^0$.

**Example 3.1.1.1.4.** The **trivial pointed set** is the pointed set $(\mathrm{pt}, \star)$ consisting of

- *The Underlying Set.* The punctual set $\mathrm{pt} \stackrel{\text{def}}{=} \{\star\}$;
- *The Basepoint.* The element $\star$ of pt.

**Example 3.1.1.1.5.** The **underlying pointed set** of a semimodule $(M, \alpha_M)$ is the pointed set $(M, 0_M)$.

**Example 3.1.1.1.6.** The **underlying pointed set** of a module $(M, \alpha_M)$ is the pointed set $(M, 0_M)$.

### 3.1.2. Morphisms of Pointed Sets.

**Definition 3.1.2.1.1.** A **morphism of pointed sets**[4] is equivalently

- A morphism of $\mathbb{E}_0$-monoids in $(\mathrm{N}_\bullet(\mathsf{Sets}), \mathrm{pt})$.

---

[1] *Further Terminology:* Also called an $\mathbb{F}_1$**-module**.

[2] *Further Terminology:* Also called the **underlying pointed set of the field with one element**.

[3] *Further Notation:* Also denoted $(\mathbb{F}_1, 0)$.

[4] *Further Terminology:* Also called a **pointed function** or a **morphism of** $\mathbb{F}_1$**-modules**.

- A morphism of pointed objects in $(\mathsf{Sets}, \mathrm{pt})$.

**Remark 3.1.2.1.2.** In detail, a **morphism of pointed sets** $f\colon (X, x_0) \to (Y, y_0)$ is a morphism of sets $f\colon X \to Y$ such that the diagram

$$\begin{tikzcd} & \mathrm{pt} \arrow[dl, "{[x_0]}"'] \arrow[dr, "{[y_0]}"] & \\ X \arrow[rr, "f"'] & & Y \end{tikzcd}$$

commutes, i.e. such that

$$f(x_0) = y_0.$$

### 3.1.3. The Category of Pointed Sets.

**Definition 3.1.3.1.1.** The **category of pointed sets** is the category $\mathsf{Sets}_*$ defined equivalently as

- The homotopy category of the $\infty$-category $\mathsf{Mon}_{\mathbb{E}_0}(\mathrm{N}_\bullet(\mathsf{Sets}), \mathrm{pt})$ of **??**;
- The category $\mathsf{Sets}_*$ of **??**.

**Remark 3.1.3.1.2.** In detail, the **category of pointed sets** is the category $\mathsf{Sets}_*$ where

- *Objects.* The objects of $\mathsf{Sets}_*$ are pointed sets;
- *Morphisms.* The morphisms of $\mathsf{Sets}_*$ are morphisms of pointed sets;
- *Identities.* For each $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, the unit map

  $$\mathbb{1}^{\mathsf{Sets}_*}_{(X,x_0)}\colon \mathrm{pt} \to \mathsf{Sets}_*((X, x_0), (X, x_0))$$

  of $\mathsf{Sets}_*$ at $(X, x_0)$ is defined by[5]

  $$\mathrm{id}^{\mathsf{Sets}_*}_{(X,x_0)} \overset{\mathrm{def}}{=} \mathrm{id}_X;$$

- *Composition.* For each $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, the composition map

$$\circ^{\mathsf{Sets}_*}_{(X,x_0),(Y,y_0),(Z,z_0)}\colon \mathsf{Sets}_*((Y, y_0), (Z, z_0)) \times \mathsf{Sets}_*((X, x_0), (Y, y_0)) \to \mathsf{Sets}_*((X, x_0), (Z, z_0))$$

  of $\mathsf{Sets}_*$ at $((X, x_0), (Y, y_0), (Z, z_0))$ is defined by[6]

  $$g \circ^{\mathsf{Sets}_*}_{(X,x_0),(Y,y_0),(Z,z_0)} f \overset{\mathrm{def}}{=} g \circ f.$$

### 3.1.4. Elementary Properties of Pointed Sets.

[5]Note that $\mathrm{id}_X$ is indeed a morphism of pointed sets, as we have $\mathrm{id}_X(x_0) = x_0$.

[6]Note that the composition of two morphisms of pointed sets is indeed a morphism of pointed sets, as we have

$$\begin{aligned} g(f(x_0)) &= g(y_0) \\ &= z_0, \end{aligned}$$

or

$$\begin{tikzcd} & \mathrm{pt} \arrow[dl, "{[x_0]}"'] \arrow[d, "{[y_0]}"] \arrow[dr, "{[z_0]}"] & \\ X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z \end{tikzcd}$$

in terms of diagrams.

**Proposition 3.1.4.1.1.** Let $(X, x_0)$ be a pointed set.

(1) *Completeness.* The category $\mathsf{Sets}_*$ of pointed sets and morphisms between them is complete, having in particular products (Definition 3.2.1.1.1), pullbacks (Definition 3.2.3.1.1), and equalisers (Definition 3.2.2.1.1).

(2) *Cocompleteness.* The category $\mathsf{Sets}_*$ of pointed sets and morphisms between them is cocomplete, having in particular coproducts (Definition 3.3.1.1.1), pushouts (Definition 3.3.2.1.1), and coequalisers (Definition 3.3.3.1.1).

(3) *Failure To Be Cartesian Closed.* The category $\mathsf{Sets}_*$ is not Cartesian closed.

(4) *Relation to Partial Functions.* We have an equivalence of categories[7]

$$\mathsf{Sets}_* \overset{\text{eq.}}{\cong} \mathsf{Sets}^{\text{part.}}$$

between the category of pointed sets and pointed functions between them and the category of sets and partial functions between them.

**Proof.** *Item 1, Completeness*: Omitted.
*Item 2, Cocompleteness*: Omitted.
*Item 3, Failure To Be Cartesian Closed*: See [**MSE2855868**].
*Item 4, Relation to Partial Functions*: Omitted. □

## 3.2. Limits of Pointed Sets

**3.2.1. Products.** Let $(X, x_0)$ and $(Y, y_0)$ be pointed sets.

**Definition 3.2.1.1.1.** The **product of** $(X, x_0)$ **and** $(Y, y_0)$ is the pointed set $(X \times Y, (x_0, y_0))$.

**3.2.2. Equalisers.** Let $f, g \colon (X, x_0) \rightrightarrows (Y, y_0)$ be morphisms of pointed sets.

**Definition 3.2.2.1.1.** The **equaliser of** $(f, g)$ is the pointed set $(\mathrm{Eq}_*(f, g), x_0)$ consisting of

- *The Underlying Set.* The set $\mathrm{Eq}_*(f, g)$ defined by
$$\mathrm{Eq}_*(f, g) \stackrel{\text{def}}{=} \{x \in X \mid f(x) = y_0 = g(x)\};$$
- *The Basepoint.* The element $x_0$ of $\mathrm{Eq}_*(f, g)$.

**3.2.3. Pullbacks.** Let $(X, x_0)$, $(Y, y_0)$, and $(Z, z_0)$ be pointed sets and let $f \colon (X, x_0) \to (Z, z_0)$ and $g \colon (Y, y_0) \to (Z, z_0)$ be morphisms of pointed sets.

**Definition 3.2.3.1.1.** The **pullback of** $(X, x_0)$ **and** $(Y, y_0)$ **over** $(Z, z_0)$ **along** $(f, g)$ is the pointed set $\big((X, x_0) \times_{(z, z_0)} (Y, y_0), p_0\big)$ consisting of

- *The Underlying Set.* The set $(X, x_0) \times_{(z, z_0)} (Y, y_0)$ defined by
$$(X, x_0) \times_{(z, z_0)} (Y, y_0) \stackrel{\text{def}}{=} \{(x, y) \in X \times Y \mid f(x) = z_0 = g(y)\};$$
- *The Basepoint.* The element $(x_0, y_0)$ of $(X, x_0) \times_{(z, z_0)} (Y, y_0)$.

[7] *Warning:* This is not an isomorphism of categories, only an equivalence.
END TEXTDBEND

## 3.3. Colimits of Pointed Sets

**3.3.1. Coproducts.** Let $(X, x_0)$ and $(Y, y_0)$ be pointed sets.

**Definition 3.3.1.1.1.** The **coproduct of** $(X, x_0)$ **and** $(Y, y_0)$ is their wedge sum $(X \vee Y, p_0)$ of Definition 3.4.3.1.1.

**3.3.2. Pushouts.** Let $(X, x_0)$, $(Y, y_0)$, and $(Z, z_0)$ be pointed sets and let $f\colon (Z, z_0) \to (X, x_0)$ and $g\colon (Z, z_0) \to (Y, y_0)$ be morphisms of pointed sets.

**Definition 3.3.2.1.1.** The **pushout of** $(X, x_0)$ **and** $(Y, y_0)$ **over** $(Z, z_0)$ **along** $(f, g)$ is the pointed set $\left(X \coprod_{f,Z,g} Y, p_0\right)$, where $p_0 = [x_0] = [y_0]$.

**3.3.3. Coequalisers.** Let $f, g\colon (X, x_0) \rightrightarrows (Y, y_0)$ be morphisms of pointed sets.

**Definition 3.3.3.1.1.** The **coequaliser of** $(f, g)$ is the pointed set $(\mathrm{CoEq}(f, g), x_0)$.

## 3.4. Constructions With Pointed Sets

**3.4.1. Internal Homs.** Let $(X, x_0)$ and $(Y, y_0)$ be pointed sets.

**Definition 3.4.1.1.1.** The **pointed set of morphisms of pointed sets from** $(X, x_0)$ **to** $(Y, y_0)$ is the pointed set $\mathbf{Sets}_*(X, Y)$ consisting of

- *The Underlying Set.* The set $\mathbf{Sets}_*((X, x_0), (Y, y_0))$ of morphisms of pointed sets from $(X, x_0)$ to $(Y, y_0)$;
- *The Basepoint.* The element
$$\Delta_{y_0}\colon (X, x_0) \to (Y, y_0)$$
of $\mathbf{Sets}_*((X, x_0), (Y, y_0))$.

**3.4.2. Free Pointed Sets.** Let $X$ be a set.

**Definition 3.4.2.1.1.** The **free pointed set on** $X$ is the pointed set $X^+$ consisting of

- *The Underlying Set.* The set $X^+$ defined by
$$X^+ \overset{\text{def}}{=} X \coprod \mathrm{pt};$$
- *The Basepoint.* The element $\star$ of $X^+$.

**Proposition 3.4.2.1.2.** Let $X$ be a set.

(1) *Functoriality.* The assignment $X \mapsto X^+$ defines a functor
$$(-)^+\colon \mathsf{Sets} \to \mathsf{Sets}_*,$$
where
- *Action on Objects.* For each $X \in \mathrm{Obj}(\mathsf{Sets})$, we have
$$[(-)^+](X) \overset{\text{def}}{=} X_+,$$
where $X_+$ is the pointed set of Definition 3.4.2.1.1;
- *Action on Morphisms.* For each morphism $f\colon X \to Y$ of $\mathsf{Sets}$, the image
$$f_+\colon X_+ \to Y_+$$
of $f$ by $(-)^+$ is the map of pointed sets defined by
$$f^+(x) \overset{\text{def}}{=} \begin{cases} f(x) & \text{if } x \in X, \\ \star & \text{if } x = \star. \end{cases}$$

(2) *Adjointness.* We have an adjunction

$$\left((-)^{+} \dashv 忘\right)\colon \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "(-)^{+}"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_{*}, \arrow[l, bend left, "忘"] \end{tikzcd}$$

witnessed by a bijection of sets

$$\mathsf{Sets}_{*}((X_{+}, \star), (Y, y_0)) \cong \mathsf{Sets}(X, Y),$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets})$ and $(Y, y_0) \in \mathrm{Obj}(\mathsf{Sets}_{*})$.

(3) *Symmetric Strong Monoidality With Respect to Wedge Sums.* The free pointed set functor of Item 1 has a symmetric strong monoidal structure

$$\left((-)^{+}, (-)^{+,\coprod}, (-)^{+,\coprod}_{\mathbb{1}}\right)\colon (\mathsf{Sets}, \coprod, \emptyset) \to (\mathsf{Sets}_{*}, \vee, \mathrm{pt}),$$

being equipped with isomorphisms of pointed sets

$$(-)^{+,\coprod}_{X,Y}\colon X^{+} \vee Y^{+} \xrightarrow{\cong} (X \textstyle\coprod Y)^{+},$$

$$(-)^{+,\coprod}_{\mathbb{1}}\colon \mathrm{pt} \xrightarrow{\cong} \emptyset^{+},$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

(4) *Symmetric Strong Monoidality With Respect to Smash Products.* The free pointed set functor of Item 1 has a symmetric strong monoidal structure

$$\left((-)^{+}, (-)^{+,\times}, (-)^{+,\times}_{\mathbb{1}}\right)\colon (\mathsf{Sets}, \times, \mathrm{pt}) \to \left(\mathsf{Sets}_{*}, \wedge, S^0\right),$$

being equipped with isomorphisms of pointed sets

$$(-)^{+,\times}_{X,Y}\colon X^{+} \wedge Y^{+} \xrightarrow{\cong} (X \times Y)^{+},$$

$$(-)^{+,\times}_{\mathbb{1}}\colon S^0 \xrightarrow{\cong} \mathrm{pt}^{+},$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

**Proof.** *Item 1, Functoriality*: Clear.
*Item 2, Adjointness*: Clear.
*Item 3, Symmetric Strong Monoidality With Respect to Wedge Sums*: Omitted.
*Item 4, Symmetric Strong Monoidality With Respect to Smash Products*: Omitted. □

**3.4.3. Wedge Sums of Pointed Sets.** Let $(X, x_0)$ and $(Y, y_0)$ be pointed sets.

**Definition 3.4.3.1.1.** The **wedge sum of $X$ and $Y$** is the pointed set $(X \vee Y, p_0)$ consisting of

- *The Underlying Set.* The set $X \vee Y$ defined by[8]

$$\begin{aligned}(X \vee Y, p_0) &\overset{\text{def}}{=} (X, x_0) \coprod (Y, y_0)\\ &\cong (X \textstyle\coprod_{\mathrm{pt}} Y, p_0)\\ &\cong (X \textstyle\coprod Y/\sim, p_0),\end{aligned}\qquad \begin{tikzcd} X \vee Y \arrow[dr, phantom, "\ulcorner", very near start] & Y \arrow[l] \\ X \arrow[u] & \mathrm{pt}, \arrow[l, hook, "{[x_0]}"] \arrow[u, hook, "{[y_0]}"'] \end{tikzcd}$$

  where $\sim$ is the equivalence relation on $X \coprod Y$ given by $x_0 \sim y_0$;
- *The Basepoint.* The element $p_0$ of $X \vee Y$ defined by

$$\begin{aligned}p_0 &\overset{\text{def}}{=} [x_0]\\ &= [y_0].\end{aligned}$$

**Proposition 3.4.3.1.2.** Let $(X, x_0)$ and $(Y, y_0)$ be pointed sets.

(1) *Functoriality.* The assignments $(X, x_0), (Y, y_0), ((X, x_0), (Y, y_0)) \mapsto (X \vee Y, p_0)$ define functors

$$\begin{aligned}X \vee - &\colon \mathsf{Sets}_* \to \mathsf{Sets}_*,\\ - \vee Y &\colon \mathsf{Sets}_* \to \mathsf{Sets}_*,\\ -_1 \vee -_2 &\colon \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*.\end{aligned}$$

(2) *Associativity.* We have an isomorphism of pointed sets

$$(X \vee Y) \vee Z \cong X \vee (Y \vee Z),$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathsf{Sets}_*$.

(3) *Unitality.* We have isomorphisms of pointed sets

$$\begin{aligned}\mathrm{pt} \vee X &\cong X,\\ X \vee \mathrm{pt} &\cong X,\end{aligned}$$

natural in $(X, x_0) \in \mathsf{Sets}_*$.

(4) *Commutativity.* We have an isomorphism of pointed sets

$$X \vee Y \cong Y \vee X,$$

natural in $(X, x_0), (Y, y_0) \in \mathsf{Sets}_*$.

(5) *Symmetric Monoidality.* The triple $(\mathsf{Sets}_*, \vee, \mathrm{pt})$ is a symmetric monoidal category.

(6) *Symmetric Strong Monoidality With Respect to Free Pointed Sets.* The free pointed set functor of Item 1 of Proposition 3.4.2.1.2 has a symmetric strong monoidal structure

$$\left((-)^+, (-)^{+,\coprod}, (-)^{+,\coprod}_{\mathbb{1}}\right) \colon (\mathsf{Sets}, \textstyle\coprod, \emptyset) \to (\mathsf{Sets}_*, \vee, \mathrm{pt}),$$

being equipped with isomorphisms of pointed sets

$$\begin{aligned}(-)^{+,\coprod}_{X,Y} &\colon X^+ \vee Y^+ \xrightarrow{\cong} (X \textstyle\coprod Y)^+,\\ (-)^{+,\coprod}_{\mathbb{1}} &\colon \mathrm{pt} \xrightarrow{\cong} \emptyset^+,\end{aligned}$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

[8] Here $(X, x_0) \coprod (Y, y_0)$ is the coproduct of $(X, x_0)$ and $(Y, y_0)$ in $\mathsf{Sets}_*$.

(7) *The Fold Map.* We have a natural transformation

$$\nabla \colon \vee \circ \Delta^{\mathsf{Cats}}_{\mathsf{Sets}_*} \Longrightarrow \mathrm{id}_{\mathsf{Sets}_*}, \qquad \begin{tikzcd} & \mathsf{Sets}_* \times \mathsf{Sets}_* \arrow[dr, "\vee"] \arrow[d, phantom, "\Downarrow \nabla"] & \\ \mathsf{Sets}_* \arrow[ur, "\Delta^{\mathsf{Cats}}_{\mathsf{Sets}_*}"] \arrow[rr, bend right, "\mathrm{id}_{\mathsf{Sets}_*}"'] & & \mathsf{Sets}_*, \end{tikzcd}$$

called the **fold map**, whose component

$$\nabla_X \colon X \vee X \to X$$

at $X$ is given by the composition

$$\begin{aligned} X &\xrightarrow{\Delta_X} X \times X \\ &\twoheadrightarrow X \times X/\sim \\ &\overset{\mathrm{def}}{=\!=} X \vee X. \end{aligned}$$

**Proof.** *Item 1, Functoriality*: Omitted.
*Item 2, Associativity*: Omitted.
*Item 3, Unitality*: Omitted.
*Item 4, Commutativity*: Omitted.
*Item 5, Symmetric Monoidality*: Omitted.
*Item 6, Symmetric Strong Monoidality With Respect to Free Pointed Sets*: Omitted.
*Item 7, The Fold Map*: Omitted. □

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CHAPTER 4

# Tensor Products of Pointed Sets

This chapter contains some material on tensor products of pointed sets.

## 4.1. Bilinear Morphisms of Pointed Sets

**4.1.1. Left Bilinear Morphisms of Pointed Sets.** Let $(X, x_0)$, $(Y, y_0)$, and $(Z, z_0)$ be pointed sets.

**Definition 4.1.1.1.1.** A **left bilinear morphism of pointed sets from** $(X \times Y, (x_0, y_0))$ **to** $(Z, z_0)$ is a map of sets

$$f \colon (X \times Y, (x_0, y_0)) \to (Z, z_0)$$

satisfying the following condition:[1,2]

($\star$) *Left Unital Bilinearity.* The diagram

$$\begin{tikzcd}
 & \mathrm{pt} \times \mathrm{pt} \arrow[dr, dashed, "\sim"] & \\
\mathrm{pt} \times Y \arrow[ur, "\mathrm{id}_{\mathrm{pt}} \times \epsilon_Y"] \arrow[d, "{[x_0] \times \mathrm{id}_Y}"'] & & \mathrm{pt} \arrow[d, "{[z_0]}"] \\
X \times Y \arrow[rr, "f"'] & & Z
\end{tikzcd}$$

commutes, i.e. for each $y \in Y$, we have

$$f(x_0, y) = z_0.$$

**Definition 4.1.1.1.2.** The **set of left bilinear morphisms of pointed sets from** $(X \times Y, (x_0, y_0))$ **to** $(Z, z_0)$ is the set $\mathrm{Hom}^{\otimes, \mathrm{L}}_{\mathsf{Sets}_*}(X \times Y, Z)$ defined by

$$\mathrm{Hom}^{\otimes, \mathrm{L}}_{\mathsf{Sets}_*}(X \times Y, Z) \overset{\mathrm{def}}{=} \{f \in \mathsf{Sets}_*(A \times B, C) \mid f \text{ is left bilinear}\}.$$

**4.1.2. Right Bilinear Morphisms of Pointed Sets.** Let $(X, x_0)$, $(Y, y_0)$, and $(Z, z_0)$ be pointed sets.

**Definition 4.1.2.1.1.** A **right bilinear morphism of pointed sets from** $(X \times Y, (x_0, y_0))$ **to** $(Z, z_0)$ is a map of sets

$$f \colon (X \times Y, (x_0, y_0)) \to (Z, z_0)$$

[1] *Slogan:* $f$ is left bilinear if it preserves basepoints in its first argument.

[2] Succinctly, $f$ is bilinear if we have

$$f(x_0, y) = z_0$$

for each $y \in Y$.

satisfying the following condition:[3,4]

(⋆) *Right Unital Bilinearity.* The diagram

$$\begin{tikzcd} & \mathrm{pt} \times \mathrm{pt} \arrow[dr, dashed, "\sim"] & \\ X \times \mathrm{pt} \arrow[ur, "\epsilon_X \times \mathrm{id}_{\mathrm{pt}}"] \arrow[d, "\mathrm{id}_X \times [y_0]"'] & & \mathrm{pt} \arrow[d, "[z_0]"] \\ X \times Y \arrow[rr, "f"'] & & Z \end{tikzcd}$$

commutes, i.e. for each $x \in X$, we have

$$f(x, y_0) = z_0.$$

**Definition 4.1.2.1.2.** The **set of right bilinear morphisms of pointed sets from** $(X \times Y, (x_0, y_0))$ **to** $(Z, z_0)$ is the set $\mathrm{Hom}^{\otimes,\mathrm{R}}_{\mathsf{Sets}_*}(X \times Y, Z)$ defined by

$$\mathrm{Hom}^{\otimes,\mathrm{R}}_{\mathsf{Sets}_*}(X \times Y, Z) \overset{\mathrm{def}}{=} \{f \in \mathsf{Sets}_*(A \times B, C) \mid f \text{ is right bilinear}\}.$$

**4.1.3. Bilinear Morphisms of Pointed Sets.** Let $(X, x_0)$, $(Y, y_0)$, and $(Z, z_0)$ be pointed sets.

**Definition 4.1.3.1.1.** A **bilinear morphism of pointed sets from** $(X \times Y, (x_0, y_0))$ **to** $(Z, z_0)$ is a map of sets

$$f : (X \times Y, (x_0, y_0)) \to (Z, z_0)$$

that is both left bilinear and right bilinear.

**Remark 4.1.3.1.2.** In detail, a **bilinear morphism of pointed sets from** $(X \times Y, (x_0, y_0))$ **to** $(Z, z_0)$ is a map of sets

$$f : (X \times Y, (x_0, y_0)) \to (Z, z_0)$$

satisfying the following conditions:[5,6]

---

[3] *Slogan:* $f$ is right bilinear if it preserves basepoints in its second argument.

[4] Succinctly, $f$ is bilinear if we have

$$f(x, y_0) = z_0$$

for each $x \in X$.

[5] *Slogan:* $f$ is bilinear if it preserves basepoints in each argument.

[6] Succinctly, $f$ is bilinear if we have

$$f(x_0, y) = z_0,$$
$$f(x, y_0) = z_0$$

for each $x \in X$ and each $y \in Y$.

(1) *Left Unital Bilinearity.* The diagram

$$\begin{tikzcd}
 & \mathrm{pt}\times\mathrm{pt} \arrow[dr, dashed, "\sim"] & \\
\mathrm{pt}\times Y \arrow[ur, "\mathrm{id}_{\mathrm{pt}}\times\epsilon_Y"] \arrow[d, "{[x_0]\times\mathrm{id}_Y}"'] & & \mathrm{pt} \arrow[d, "{[z_0]}"] \\
X\times Y \arrow[rr, "f"'] & & Z
\end{tikzcd}$$

commutes, i.e. for each $y \in Y$, we have

$$f(x_0, y) = z_0.$$

(2) *Right Unital Bilinearity.* The diagram

$$\begin{tikzcd}
 & \mathrm{pt}\times\mathrm{pt} \arrow[dr, dashed, "\sim"] & \\
X\times\mathrm{pt} \arrow[ur, "\epsilon_X\times\mathrm{id}_{\mathrm{pt}}"] \arrow[d, "{\mathrm{id}_X\times[y_0]}"'] & & \mathrm{pt} \arrow[d, "{[z_0]}"] \\
X\times Y \arrow[rr, "f"'] & & Z
\end{tikzcd}$$

commutes, i.e. for each $x \in X$, we have

$$f(x, y_0) = z_0.$$

**Definition 4.1.3.1.3.** The **set of bilinear morphisms of pointed sets from** $(X \times Y, (x_0, y_0))$ **to** $(Z, z_0)$ is the set $\mathrm{Hom}^{\otimes}_{\mathsf{Sets}_*}(X \times Y, Z)$ defined by

$$\mathrm{Hom}^{\otimes}_{\mathsf{Sets}_*}(X \times Y, Z) \stackrel{\text{def}}{=} \{f \in \mathsf{Sets}_*(A \times B, C) \mid f \text{ is bilinear}\}.$$

## 4.2. Tensors and Cotensors of Pointed Sets by Sets

**4.2.1. Tensors of Pointed Sets by Sets.** Let $(X, x_0)$ be a pointed set and let $A$ be a set.

**Definition 4.2.1.1.1.** The **tensor of** $(X, x_0)$ **by** $A$ is the pointed set $A \odot (X, x_0)$ satisfying the following universal property:

(**UP**) We have a bijection

$$\mathsf{Sets}_*(A \odot X, K) \cong \mathsf{Sets}(A, \mathbf{\mathsf{Sets}}_*(X, K)),$$

natural in $(K, k_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

**Remark 4.2.1.1.2.** The tensor of $(X, x_0)$ by $A$ satisfies the following universal property:

$$\mathsf{Sets}_*(A \odot X, K) \cong \mathsf{Sets}^{\otimes}_{\mathbb{E}_0}(A \times X, K),$$

where $\mathsf{Sets}^{\otimes}_{\mathbb{E}_0}(A \times X, K)$ is the set defined by

$$\mathsf{Sets}^{\otimes}_{\mathbb{E}_0}(A \times X, K) \stackrel{\text{def}}{=} \left\{ f \in \mathsf{Sets}(A \times X, K) \;\middle|\; \begin{array}{l} \text{for each } a \in A, \text{ we} \\ \text{have } f(a, x_0) = k_0 \end{array} \right\}.$$

**Construction 4.2.1.1.3.** Concretely, the **tensor of** $(X, x_0)$ **by** $A$ is the pointed set $A \odot (X, x_0)$ consisting of

- *The Underlying Set.* The set $A \odot X$ given by
$$A \odot X \cong \bigvee_{a \in A} (X, x_0);$$
- *The Basepoint.* The point $[x_0]$ of $\bigvee_{a \in A}(X, x_0)$.

**4.2.2. Cotensors of Pointed Sets by Sets.** Let $(X, x_0)$ be a pointed set and let $A$ be a set.

**Definition 4.2.2.1.1.** The **cotensor of** $(X, x_0)$ **by** $A$ is the pointed set $A \pitchfork (X, x_0)$ satisfying the following universal property:

(**UP**) We have a bijection
$$\mathsf{Sets}_*(K, A \pitchfork X) \cong \mathsf{Sets}(A, \mathbf{Sets}_*(K, X)),$$
natural in $(K, k_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

**Remark 4.2.2.1.2.** The cotensor of $(X, x_0)$ by $A$ satisfies the following universal property:
$$\mathsf{Sets}_*(K, A \pitchfork X) \cong \mathsf{Sets}^{\otimes}_{\mathbb{E}_0}(A \times K, X),$$
where $\mathsf{Sets}^{\otimes}_{\mathbb{E}_0}(A \times K, X)$ is the set defined by
$$\mathsf{Sets}^{\otimes}_{\mathbb{E}_0}(A \times K, X) \overset{\text{def}}{=} \left\{ f \in \mathsf{Sets}(A \times K, X) \;\middle|\; \begin{array}{l} \text{for each } a \in A, \text{ we} \\ \text{have } f(a, k_0) = x_0 \end{array} \right\}.$$

**Construction 4.2.2.1.3.** Concretely, the **cotensor of** $(X, x_0)$ **by** $A$ is the pointed set $A \pitchfork (X, x_0)$ consisting of

- *The Underlying Set.* The set $A \pitchfork X$ given by
$$A \pitchfork X \cong \bigwedge_{a \in A} (X, x_0);$$
- *The Basepoint.* The point $[(x_0, x_0, x_0, \ldots)]$ of $\bigwedge_{a \in A}(X, x_0)$.

## 4.3. The Left Tensor Product of Pointed Sets

**4.3.1. Foundations.** Let $(X, x_0)$ and $(Y, y_0)$ be pointed sets.

**Definition 4.3.1.1.1.** The **left tensor product of pointed sets** is the functor
$$\lhd_{\mathsf{Sets}_*} \colon \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*$$
defined as the composition
$$\mathsf{Sets}_* \times \mathsf{Sets}_* \xrightarrow{\mathrm{id} \times 忘} \mathsf{Sets}_* \times \mathsf{Sets} \xrightarrow{\beta^{\mathsf{Cats}_2}_{\mathsf{Sets}_*, \mathsf{Sets}}} \mathsf{Sets} \times \mathsf{Sets}_* \xrightarrow{\odot} \mathsf{Sets}_*.$$

**Remark 4.3.1.1.2.** The left tensor product of pointed sets satisfies the following universal property:[7]
$$\mathsf{Sets}_*(X \lhd_{\mathsf{Sets}_*} Y, Z) \cong \mathrm{Hom}^{\otimes, \mathrm{L}}_{\mathsf{Sets}_*}(X \times Y, Z).$$

[7]Namely, a pointed map $f \colon X \lhd_{\mathsf{Sets}_*} Y \to Z$ is the same as a map $f^\dagger \colon X \times Y \to Z$ such that
$$f^\dagger(x_0, y) = z_0$$
for each $y \in Y$.

**Remark 4.3.1.1.3.** In detail, the **left tensor product of** $(X, x_0)$ **and** $(Y, y_0)$ is the pointed set $(X \lhd_{\mathsf{Sets}_*} Y, [x_0])$ consisting of[8]

- *The Underlying Set.* The set $X \lhd_{\mathsf{Sets}_*} Y$ defined by
$$\begin{aligned} X \lhd_{\mathsf{Sets}_*} Y &\overset{\text{def}}{=} |Y| \odot X \\ &\cong \bigvee_{y \in Y} (X, x_0); \end{aligned}$$
- *The Underlying Basepoint.* The point $[x_0]$ of $\bigvee_{y \in Y}(X, x_0)$.

**Proposition 4.3.1.1.4.** Let $(X, x_0)$ and $(Y, y_0)$ be pointed sets.

(1) *Functoriality.* The assignments $X, Y, (X, Y) \mapsto X \lhd_{\mathsf{Sets}_*} Y$ define functors
$$\begin{aligned} X \lhd_{\mathsf{Sets}_*} - &\colon \mathsf{Sets}_* \to \mathsf{Sets}_*, \\ - \lhd_{\mathsf{Sets}_*} Y &\colon \mathsf{Sets}_* \to \mathsf{Sets}_*, \\ -_1 \lhd_{\mathsf{Sets}_*} -_2 &\colon \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*. \end{aligned}$$

**Proof.** *Item 1, Functoriality*: Omitted. □

### 4.3.2. The Skew Associator.

**Definition 4.3.2.1.1.** The **skew associator of the left tensor product of pointed sets** is the natural isomorphism
$$\alpha^{\mathsf{Sets}_*, \lhd} \colon \lhd_{\mathsf{Sets}_*} \circ (\lhd_{\mathsf{Sets}_*} \times \mathrm{id}_{\mathsf{Sets}_*}) \overset{\cong}{\Longrightarrow} \lhd_{\mathsf{Sets}_*} \circ (\mathrm{id}_{\mathsf{Sets}_*} \times \lhd_{\mathsf{Sets}_*}),$$
whose component
$$\alpha^{\mathsf{Sets}_*, \lhd}_{X,Y,Z} \colon (X \lhd_{\mathsf{Sets}_*} Y) \lhd_{\mathsf{Sets}_*} Z \xrightarrow{\cong} X \lhd_{\mathsf{Sets}_*} (Y \lhd_{\mathsf{Sets}_*} Z)$$

[8] *Further Notation:* We write $x \lhd_{\mathsf{Sets}_*} y$ for the image of $(x, y)$ under the map
$$X \times Y \to \underbrace{X \lhd_{\mathsf{Sets}_*} Y}_{\cong \bigvee_{y \in Y} (X, x_0)}.$$
sending $(x, y)$ to the element $x \in X$ in the $y$th copy of $X$ in $\bigvee_{y \in Y}(X, x_0)$. Note that we have
$$x_0 \lhd_{\mathsf{Sets}_*} y = x_0 \lhd_{\mathsf{Sets}_*} y',$$
for each $y, y' \in Y$.

at $(X, Y, Z)$ is given by the composition[9]

$$\begin{aligned}(X \lhd_{\mathsf{Sets}_*} Y) \lhd_{\mathsf{Sets}_*} Z &\overset{\text{def}}{=} |Z| \odot (X \lhd_{\mathsf{Sets}_*} Y)\\ &\overset{\text{def}}{=} |Z| \odot (|Y| \odot X)\\ &\cong \bigvee_{z\in Z} (|Y| \odot X, [x_0])\\ &\overset{\text{def}}{=} \bigvee_{z\in Z}\left(\bigvee_{y\in Y}(X, x_0)\right)\\ &\cong \bigvee_{(z,y)\in\bigvee_{z\in Z}(Y,y_0)} (X, x_0)\\ &\overset{\text{def}}{=} \bigvee_{(z,y)\in|Z|\odot Y} (X, x_0)\\ &\cong ||Z| \odot Y| \odot X\\ &\overset{\text{def}}{=} |Y \lhd_{\mathsf{Sets}_*} Z| \odot X\\ &\overset{\text{def}}{=} X \lhd_{\mathsf{Sets}_*} (Y \lhd_{\mathsf{Sets}_*} Z),\end{aligned}$$

where the isomorphism

$$\bigvee_{z\in Z}\left(\bigvee_{y\in Y}(X, x_0)\right) \cong \bigvee_{(y,z)\in\bigvee_{z\in Z}(Y,y_0)} (X, x_0)$$

is given by $[(z,(y,x))] \mapsto [((z,y),x)]$.

### 4.3.3. The Skew Left Unitor.

**Definition 4.3.3.1.1.** The **skew left unitor of the left tensor product of pointed sets** is the natural transformation

$$\lambda^{\mathsf{Sets}_*,\lhd}\colon \lhd_{\mathsf{Sets}_*} \circ \left(\mathbb{1}^{\mathsf{Sets}_*} \times \mathrm{id}_{\mathsf{Sets}_*}\right) \Longrightarrow \mathrm{id}_{\mathsf{Sets}_*},$$

whose component

$$\lambda^{\mathsf{Sets}_*,\lhd}_X\colon S^0 \lhd_{\mathsf{Sets}_*} X \to X$$

at $X$ is given by the composition[10]

$$\begin{aligned}S^0 \lhd_{\mathsf{Sets}_*} X &\cong |X| \odot S^0\\ &\cong \bigvee_{x\in X} S^0\\ &\to X\end{aligned}$$

[9] In other words, $\alpha^{\mathsf{Sets}_*,\lhd}_{X,Y,Z}$ acts on elements as

$$\alpha^{\mathsf{Sets}_*,\lhd}_{X,Y,Z}((x \lhd_{\mathsf{Sets}_*} y) \lhd_{\mathsf{Sets}_*} z) \overset{\text{def}}{=} x \lhd_{\mathsf{Sets}_*} (y \lhd_{\mathsf{Sets}_*} z)$$

for each $(x \lhd_{\mathsf{Sets}_*} y) \lhd_{\mathsf{Sets}_*} z \in (X \lhd_{\mathsf{Sets}_*} Y) \lhd_{\mathsf{Sets}_*} Z$.

[10] In other words, $\lambda^{\mathsf{Sets}_*,\lhd}_X$ acts on elements as

$$\begin{aligned}\lambda^{\mathsf{Sets}_*,\lhd}_X(x \lhd_{\mathsf{Sets}_*} 0) &\overset{\text{def}}{=} x,\\ \lambda^{\mathsf{Sets}_*,\lhd}_X(x \lhd_{\mathsf{Sets}_*} 1) &\overset{\text{def}}{=} x,\end{aligned}$$

for each $x \in X$.

where $\bigvee_{x\in X} S^0 \to X$ is the map given by

$$\begin{aligned}(x,0) &\mapsto x,\\(x,1) &\mapsto x.\end{aligned}$$

#### 4.3.4. The Skew Right Unitor.

**Definition 4.3.4.1.1.** The **skew right unitor of the left tensor product of pointed sets** is the natural transformation

$$\rho^{\mathsf{Sets}_*,\lhd}\colon \mathrm{id}_{\mathsf{Sets}_*} \Longrightarrow \lhd_{\mathsf{Sets}_*} \circ \left(\mathrm{id}_{\mathsf{Sets}_*} \times \mathbb{1}^{\mathsf{Sets}_*}\right),$$

whose component

$$\rho^{\mathsf{Sets}_*,\lhd}_X\colon X \to X \lhd_{\mathsf{Sets}_*} S^0$$

at $X$ is given by the composition[11]

$$\begin{aligned}X &\to X \vee X\\&\cong |S^0| \odot X\\&\cong S^0 \lhd_{\mathsf{Sets}_*} X,\end{aligned}$$

where $X \to X \vee X$ is the map sending $X$ to the first factor of $X$ in $X \vee X$.

#### 4.3.5. The Left-Skew Monoidal Category Structure on Pointed Sets.

**Proposition 4.3.5.1.1.** The category $\mathsf{Sets}_*$ admits a left-skew monoidal category structure consisting of[12]

- *The Skew Monoidal Product.* The left tensor product functor

  $$\lhd_{\mathsf{Sets}_*}\colon \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*,$$

  of Proposition 4.3.1.1.4;
- *The Skew Monoidal Unit.* The functor

  $$\mathbb{1}^{\mathsf{Sets}_*}\colon \mathsf{pt} \to \mathsf{Sets}_*$$

  defined by

  $$\mathbb{1}_{\mathsf{Sets}_*} \overset{\mathrm{def}}{=} S^0;$$
- *The Skew Associators.* The natural isomorphism

  $$\alpha^{\mathsf{Sets}_*,\lhd}\colon \lhd_{\mathsf{Sets}_*} \circ (\lhd_{\mathsf{Sets}_*} \times \mathrm{id}_{\mathsf{Sets}_*}) \overset{\cong}{\Longrightarrow} \lhd_{\mathsf{Sets}_*} \circ (\mathrm{id}_{\mathsf{Sets}_*} \times \lhd_{\mathsf{Sets}_*}),$$

  of Definition 4.3.2.1.1;
- *The Skew Left Unitors.* The natural transformation

  $$\lambda^{\mathsf{Sets}_*,\lhd}\colon \lhd_{\mathsf{Sets}_*} \circ \left(\mathbb{1}^{\mathsf{Sets}_*} \times \mathrm{id}_{\mathsf{Sets}_*}\right) \Longrightarrow \mathrm{id}_{\mathsf{Sets}_*},$$

  of Definition 4.3.3.1.1;
- *The Skew Right Unitors.* The natural transformation

  $$\rho^{\mathsf{Sets}_*,\lhd}\colon \mathrm{id}_{\mathsf{Sets}_*} \Longrightarrow \lhd_{\mathsf{Sets}_*} \circ \left(\mathrm{id}_{\mathsf{Sets}_*} \times \mathbb{1}^{\mathsf{Sets}_*}\right),$$

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[11] In other words, $\rho^{\mathsf{Sets}_*,\lhd}_X$ acts on elements as

$$\rho^{\mathsf{Sets}_*,\lhd}_X(x) \overset{\mathrm{def}}{=} x \lhd_{\mathsf{Sets}_*} 0$$

for each $x \in X$.

[12] Note in particular that, differently from general left-skew monoidal categories, the skew associator of $\left(\mathsf{Sets}_*, \lhd_{\mathsf{Sets}_*}, S^0\right)$ is a natural isomorphism.

of Definition 4.3.4.1.1.

**Proof.** Omitted. □

## 4.4. The Right Tensor Product of Pointed Sets

**4.4.1. Foundations.** Let $(X, x_0)$ and $(Y, y_0)$ be pointed sets.

**Definition 4.4.1.1.1.** The **right tensor product of pointed sets** is the functor

$$\rhd_{\mathsf{Sets}_*} \colon \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*$$

defined as the composition

$$\mathsf{Sets}_* \times \mathsf{Sets}_* \xrightarrow{忘 \times \mathrm{id}} \mathsf{Sets} \times \mathsf{Sets}_* \xrightarrow{\odot} \mathsf{Sets}_*.$$

**Remark 4.4.1.1.2.** The right tensor product of pointed sets satisfies the following universal property:[13]

$$\mathsf{Sets}_*(X \rhd_{\mathsf{Sets}_*} Y, Z) \cong \mathrm{Hom}^{\otimes,\mathrm{R}}_{\mathsf{Sets}_*}(X \times Y, Z).$$

**Remark 4.4.1.1.3.** In detail, the **right tensor product of** $(X, x_0)$ **and** $(Y, y_0)$ is the pointed set $(X \rhd_{\mathsf{Sets}_*} Y, [y_0])$ consisting of [14]

- *The Underlying Set.* The set $X \rhd_{\mathsf{Sets}_*} Y$ defined by
$$\begin{aligned} X \rhd_{\mathsf{Sets}_*} Y &\overset{\text{def}}{=} |X| \odot Y \\ &\cong \bigvee_{x \in X} (Y, y_0); \end{aligned}$$
- *The Underlying Basepoint.* The point $[y_0]$ of $\bigvee_{x \in X}(Y, y_0)$.

**Proposition 4.4.1.1.4.** Let $(X, x_0)$ and $(Y, y_0)$ be pointed sets.

(1) *Functoriality.* The assignments $X, Y, (X, Y) \mapsto X \rhd_{\mathsf{Sets}_*} Y$ define functors

$$\begin{aligned} X \rhd_{\mathsf{Sets}_*} - &\colon \mathsf{Sets}_* \to \mathsf{Sets}_*, \\ - \rhd_{\mathsf{Sets}_*} Y &\colon \mathsf{Sets}_* \to \mathsf{Sets}_*, \\ -_1 \rhd_{\mathsf{Sets}_*} -_2 &\colon \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*. \end{aligned}$$

**Proof.** *Item 1, Functoriality*: Omitted. □

**4.4.2. The Skew Associator.**

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[13] Namely, a pointed map $f \colon X \lhd_{\mathsf{Sets}_*} Y \to Z$ is the same as a map $f^\dagger \colon X \times Y \to Z$ such that

$$f^\dagger(x, y_0) = z_0$$

for each $y \in Y$.

[14] *Further Notation:* We write $x \rhd_{\mathsf{Sets}_*} y$ for the image of $(x, y)$ under the map

$$X \times Y \to \underbrace{X \rhd_{\mathsf{Sets}_*} Y}_{\cong \bigvee_{x \in X}(Y, y_0)}.$$

sending $(x, y)$ to the element $y \in Y$ in the $x$th copy of $Y$ in $\bigvee_{x \in X}(Y, y_0)$. Note that we have

$$x \rhd_{\mathsf{Sets}_*} y_0 = x' \rhd_{\mathsf{Sets}_*} y_0,$$

for each $x, x' \in X$.

**Definition 4.4.2.1.1.** The **skew associator of the right tensor product of pointed sets** is the natural isomorphism

$$\alpha^{\mathsf{Sets}_*,\triangleright}\colon \triangleright_{\mathsf{Sets}_*}\circ(\mathrm{id}_{\mathsf{Sets}_*}\times\triangleright_{\mathsf{Sets}_*})\overset{\cong}{\Longrightarrow}\triangleright_{\mathsf{Sets}_*}\circ(\triangleright_{\mathsf{Sets}_*}\times\mathrm{id}_{\mathsf{Sets}_*}),$$

whose component

$$\alpha^{\mathsf{Sets}_*,\triangleright}_{X,Y,Z}\colon X\triangleright_{\mathsf{Sets}_*}(Y\triangleright_{\mathsf{Sets}_*}Z)\xrightarrow{\cong}(X\triangleright_{\mathsf{Sets}_*}Y)\triangleright_{\mathsf{Sets}_*}Z$$

at $(X,Y,Z)$ is given by the composition[15]

$$\begin{aligned}
X\triangleright_{\mathsf{Sets}_*}(Y\triangleright_{\mathsf{Sets}_*}Z) &\overset{\text{def}}{=} |X|\odot(Y\triangleright_{\mathsf{Sets}_*}Z)\\
&\overset{\text{def}}{=} |X|\odot(|Y|\odot Z)\\
&\cong |X|\odot\left(\bigvee_{y\in Y}(Z,z_0)\right)\\
&\cong \bigvee_{x\in X}\left(\bigvee_{y\in Y}(Z,z_0)\right)\\
&\cong \bigvee_{(x,y)\in\bigvee_{x\in X}(Y,y_0)}(Z,z_0)\\
&\cong \left|\bigvee_{x\in X}(Y,y_0)\right|\odot Z\\
&\overset{\text{def}}{=} |X\odot Y|\odot Z\\
&\overset{\text{def}}{=} |X\triangleright_{\mathsf{Sets}_*}Y|\odot Z\\
&\overset{\text{def}}{=} (X\triangleright_{\mathsf{Sets}_*}Y)\triangleright_{\mathsf{Sets}_*}Z
\end{aligned}$$

where the isomorphism

$$\bigvee_{x\in X}\left(\bigvee_{y\in Y}(Z,z_0)\right)\cong\bigvee_{(x,y)\in\bigvee_{x\in X}(Y,y_0)}(Z,z_0)$$

is given by $[(x,(y,z))]\mapsto[((x,y),z)]$.

### 4.4.3. The Skew Left Unitor.

**Definition 4.4.3.1.1.** The **skew left unitor of the right tensor product of pointed sets** is the natural transformation

$$\lambda^{\mathsf{Sets}_*,\triangleright}\colon \mathrm{id}_{\mathsf{Sets}_*}\Longrightarrow\triangleright_{\mathsf{Sets}_*}\circ\left(\mathbb{1}^{\mathsf{Sets}_*}\times\mathrm{id}_{\mathsf{Sets}_*}\right),$$

whose component

$$\lambda^{\mathsf{Sets}_*,\triangleright}_X\colon X\to S^0\triangleright_{\mathsf{Sets}_*}X$$

[15] In other words, $\alpha^{\mathsf{Sets}_*,\triangleright}_{X,Y,Z}$ acts on elements as

$$\alpha^{\mathsf{Sets}_*,\triangleright}_{X,Y,Z}(x\triangleright_{\mathsf{Sets}_*}(y\triangleright_{\mathsf{Sets}_*}z))\overset{\text{def}}{=}(x\triangleright_{\mathsf{Sets}_*}y)\triangleright_{\mathsf{Sets}_*}z$$

for each $x\triangleright_{\mathsf{Sets}_*}(y\triangleright_{\mathsf{Sets}_*}z)\in X\triangleright_{\mathsf{Sets}_*}(Y\triangleright_{\mathsf{Sets}_*}Z)$.

at $X$ is given by the composition[16]

$$\begin{aligned} X &\to X \vee X \\ &\cong |S^0| \odot X \\ &\cong S^0 \rhd_{\mathsf{Sets}_*} X, \end{aligned}$$

where $X \to X \vee X$ is the map sending $X$ to the first factor of $X$ in $X \vee X$.

### 4.4.4. The Skew Right Unitor.

**Definition 4.4.4.1.1.** The **skew right unitor of the right tensor product of pointed sets** is the natural transformation

$$\rho^{\mathsf{Sets}_*,\rhd} \colon \rhd_{\mathsf{Sets}_*} \circ \left(\mathrm{id}_{\mathsf{Sets}_*} \times \not\Bbbk^{\mathsf{Sets}_*}\right) \Longrightarrow \mathrm{id}_{\mathsf{Sets}_*},$$

whose component[17]

$$\rho^{\mathsf{Sets}_*,\rhd}_X \colon X \rhd_{\mathsf{Sets}_*} S^0 \to X$$

at $X$ is given by the composition

$$\begin{aligned} X \rhd_{\mathsf{Sets}_*} S^0 &\cong |X| \odot S^0 \\ &\cong \bigvee_{x\in X} S^0 \\ &\to X \end{aligned}$$

where $\bigvee_{x\in X} S^0 \to X$ is the map given by

$$\begin{aligned} (x,0) &\mapsto x, \\ (x,1) &\mapsto x. \end{aligned}$$

### 4.4.5. The Right-Skew Monoidal Category Structure on Pointed Sets.

**Proposition 4.4.5.1.1.** The category $\mathsf{Sets}_*$ admits a right-skew monoidal category structure consisting of[18]

- *The Skew Monoidal Product.* The right tensor product functor

  $$\rhd_{\mathsf{Sets}_*} \colon \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*,$$

  of Item 1;
- *The Skew Monoidal Unit.* The functor

  $$\not\Bbbk^{\mathsf{Sets}_*} \colon \mathsf{pt} \to \mathsf{Sets}_*$$

  defined by

  $$\not\Bbbk_{\mathsf{Sets}_*} \overset{\text{def}}{=} S^0;$$

[16] In other words, $\lambda^{\mathsf{Sets}_*,\rhd}_X$ acts on elements as

$$\lambda^{\mathsf{Sets}_*,\rhd}_X(x) \overset{\text{def}}{=} 0 \rhd_{\mathsf{Sets}_*} x$$

for each $x \in X$.

[17] In other words, $\rho^{\mathsf{Sets}_*,\rhd}_X$ acts on elements as

$$\begin{aligned} \rho^{\mathsf{Sets}_*,\rhd}_X(x \rhd_{\mathsf{Sets}_*} 0) &\overset{\text{def}}{=} x, \\ \rho^{\mathsf{Sets}_*,\rhd}_X(x \rhd_{\mathsf{Sets}_*} 1) &\overset{\text{def}}{=} x \end{aligned}$$

for each $x \in X$.

[18] Note in particular that, differently from general right-skew monoidal categories, the skew associator of $\left(\mathsf{Sets}_*, \rhd_{\mathsf{Sets}_*}, S^0\right)$ is a natural isomorphism.

- *The Skew Associators.* The natural isomorphism
$$\alpha^{\mathsf{Sets}_*,\triangleright}\colon \triangleright_{\mathsf{Sets}_*}\circ(\triangleright_{\mathsf{Sets}_*}\times\mathrm{id}_{\mathsf{Sets}_*})\overset{\cong}{\Longrightarrow}\triangleright_{\mathsf{Sets}_*}\circ(\mathrm{id}_{\mathsf{Sets}_*}\times\triangleright_{\mathsf{Sets}_*}),$$
of Definition 4.4.2.1.1;
- *The Skew Left Unitors.* The natural transformation
$$\lambda^{\mathsf{Sets}_*,\triangleright}\colon \mathrm{id}_{\mathsf{Sets}_*}\Longrightarrow\triangleright_{\mathsf{Sets}_*}\circ\left(\nVdash^{\mathsf{Sets}_*}\times\mathrm{id}_{\mathsf{Sets}_*}\right),$$
of Definition 4.3.3.1.1;
- *The Skew Right Unitors.* The natural transformation
$$\rho^{\mathsf{Sets}_*,\triangleright}\colon \triangleright_{\mathsf{Sets}_*}\circ\left(\mathrm{id}_{\mathsf{Sets}_*}\times\nVdash^{\mathsf{Sets}_*}\right)\Longrightarrow\mathrm{id}_{\mathsf{Sets}_*},$$
of Definition 4.3.4.1.1.

**Proof.** Omitted. □

## 4.5. Smash Products of Pointed Sets

**4.5.1. Foundations.** Let $(X,x_0)$ and $(Y,y_0)$ be pointed sets.

**Definition 4.5.1.1.1.** The **smash product of** $(X,x_0)$ **and** $(Y,y_0)$[19] is the pointed set $X\wedge Y$[20] such that we have a bijection
$$\mathsf{Sets}_*(X\wedge Y,Z)\cong\mathrm{Hom}^{\otimes}_{\mathsf{Sets}_*}(X\times Y,Z),$$
natural in $(X,x_0),(Y,y_0),(Z,z_0)\in\mathrm{Obj}(\mathsf{Sets}_*)$.

**Remark 4.5.1.1.2.** In detail, the **smash product of** $(X,x_0)$ **and** $(Y,y_0)$ is the pair $((X\wedge Y,[(x_0,y_0)]),\iota)$ consisting of
- A pointed set $(X\wedge Y,[(x_0,y_0)])$;
- A bilinear morphism of pointed sets $\iota\colon(X\times Y,(x_0,y_0))\to X\wedge Y$;

satisfying the following universal property:

**(UP)** Given another such pair $((Z,z_0),f)$ consisting of
  - A pointed set $(Z,z_0)$;
  - A bilinear morphism of pointed sets $f\colon(X\times Y,(x_0,y_0))\to X\wedge Y$;

  there exists a unique morphism of pointed sets $X\wedge Y\xrightarrow{\exists!}Z$ making the diagram
$$\begin{tikzcd} & X\wedge Y \arrow[d, dashed, "\exists!"] \\ X\times Y \arrow[ur, "\iota"] \arrow[r, "f"'] & Z \end{tikzcd}$$
  commute.

[19] *Further Terminology:* Also called the **tensor product of** $\mathbb{F}_1$**-modules of** $(X,x_0)$ **and** $(Y,y_0)$ or the **tensor product of** $(X,x_0)$ **and** $(Y,y_0)$ **over** $\mathbb{F}_1$.
[20] *Further Notation:* Also written $X\otimes_{\mathbb{F}_1}Y$.

**Construction 4.5.1.1.3.** Concretely, the **smash product of** $(X, x_0)$ **and** $(Y, y_0)$ is the pointed set $(X \wedge Y, [(x_0, y_0)])$ consisting of[21]

- *The Underlying Set.* The set $X \wedge Y$ defined by

$$\begin{aligned} X \wedge Y &\cong \mathrm{pt} \coprod_{X \vee Y} (X \times Y) \\ &\stackrel{\text{def}}{=} \frac{X \times Y}{X \vee Y} \\ &\cong X \times Y/\sim, \end{aligned} \qquad \begin{tikzcd} X \wedge Y \arrow[dr, phantom, "\ulcorner", very near start] & X \times Y \arrow[l] \\ \mathrm{pt} \arrow[u] & X \vee Y, \arrow[l, "!"] \arrow[u, hook] \end{tikzcd}$$

  where $\sim$ is the equivalence relation of $X \times Y$ obtained by declaring $(x, y) \sim (x', y')$ iff $(x, y), (x', y') \in X \vee Y$, i.e. by declaring

$$\begin{aligned} (x_0, y) &\sim (x_0, y'), \\ (x, y_0) &\sim (x', y_0) \end{aligned}$$

  for all $x \in X$ and all $y \in Y$;
- *The Basepoint.* The element $[(x_0, y_0)]$ of $X \wedge Y$ given by the equivalence class of $(x_0, y_0)$ under the equivalence relation $\sim$ on $X \times Y$.

**Proof.** Clear. □

**Example 4.5.1.1.4.** Here are some examples of smash products of pointed sets.

(1) *Smashing With $S^0$.* For any pointed set $X$, we have isomorphisms of pointed sets

$$\begin{aligned} S^0 \wedge X &\cong X, \\ X \wedge S^0 &\cong X. \end{aligned}$$

**Proposition 4.5.1.1.5.** Let $(X, x_0)$ and $(Y, y_0)$ be pointed sets.

(1) *Functoriality.* The assignments $(X, x_0), (Y, y_0), ((X, x_0), (Y, y_0)) \mapsto X \wedge Y$ define functors

$$\begin{aligned} X \wedge - &\colon \mathsf{Sets}_* \to \mathsf{Sets}_*, \\ - \wedge Y &\colon \mathsf{Sets}_* \to \mathsf{Sets}_*, \\ -_1 \wedge -_2 &\colon \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*. \end{aligned}$$

(2) *Adjointness.* We have adjunctions

$$(X \wedge - \dashv \mathbf{Sets}_*(X, -))\colon \quad \begin{tikzcd} \mathsf{Sets}_* \arrow[r, bend left, "X \wedge -"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_*, \arrow[l, bend left, "\mathbf{Sets}_*(X,-)"] \end{tikzcd}$$

$$(- \wedge Y \dashv \mathbf{Sets}_*(Y, -))\colon \quad \begin{tikzcd} \mathsf{Sets}_* \arrow[r, bend left, "- \wedge Y"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_*, \arrow[l, bend left, "\mathbf{Sets}_*(Y,-)"] \end{tikzcd}$$

[21] *Further Notation:* We write $x \wedge y$ for the image of $(x, y)$ under the quotient map

$$X \times Y \twoheadrightarrow \underbrace{\frac{X \times Y}{X \vee Y}}_{\stackrel{\text{def}}{=} X \wedge Y}$$

witnessed by bijections

$$\begin{aligned}\mathsf{Sets}_*(X \wedge Y, Z) &\cong \mathsf{Sets}_*(X, \mathbf{Sets}_*(Y, Z)),\\ \mathsf{Sets}_*(X \wedge Y, Z) &\cong \mathsf{Sets}_*(X, \mathbf{Sets}_*(A, Z)),\end{aligned}$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, which internalise to isomorphisms of pointed sets

$$\begin{aligned}\mathbf{Sets}_*(X \wedge Y, Z) &\cong \mathbf{Sets}_*(X, \mathbf{Sets}_*(Y, Z)),\\ \mathbf{Sets}_*(X \wedge Y, Z) &\cong \mathbf{Sets}_*(X, \mathbf{Sets}_*(A, Z)),\end{aligned}$$

again natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

(3) *Closed Symmetric Monoidality.* The quadruple $(\mathsf{Sets}_*, \wedge, S^0, \mathbf{Sets}_*)$ is a closed symmetric monoidal category.

(4) *Morphisms From the Monoidal Unit.* We have a bijection of sets[22]

$$\mathsf{Sets}_*\Big(S^0, X\Big) \cong X,$$

natural in $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, internalising also to an isomorphism of pointed sets

$$\mathbf{Sets}_*\Big(S^0, X\Big) \cong (X, x_0),$$

again natural in $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

(5) *Symmetric Strong Monoidality With Respect to Free Pointed Sets.* The free pointed set functor of **??** of **??** has a symmetric strong monoidal structure

$$\Big((-)^+, (-)^{+,\times}, (-)^{+,\times}_{\mathbb{1}}\Big) \colon (\mathsf{Sets}, \times, \mathrm{pt}) \to \Big(\mathsf{Sets}_*, \wedge, S^0\Big),$$

being equipped with isomorphisms of pointed sets

$$\begin{aligned}(-)^{+,\times}_{X,Y} \colon X^+ \wedge Y^+ &\xrightarrow{\cong} (X \times Y)^+,\\ (-)^{+,\times}_{\mathbb{1}} \colon S^0 &\xrightarrow{\cong} \mathrm{pt}^+,\end{aligned}$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

(6) *Distributivity Over Wedge Sums.* We have isomorphisms of pointed sets

$$\begin{aligned}X \wedge (Y \vee Z) &\cong (X \wedge Y) \vee (X \wedge Z),\\ (X \vee Y) \wedge Z &\cong (X \wedge Z) \vee (Y \wedge Z),\end{aligned}$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

(7) *Universal Property I.* The symmetric monoidal structure on the category $\mathsf{Sets}_*$ is uniquely determined by the following requirements:

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Note that we have

$$\begin{aligned}x \wedge y_0 &= x' \wedge y_0,\\ x_0 \wedge y &= x_0 \wedge y'\end{aligned}$$

for each $x, x' \in X$ and each $y, y' \in Y$.

[22]In other words, the forgetful functor

$$忘\colon \mathsf{Sets}_* \to \mathsf{Sets}$$

defined on objects by sending a pointed set to its underlying set is corepresentable by $S^0$.

(a) *Two-Sided Preservation of Colimits.* The smash product

$$\wedge\colon \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*$$

of $\mathsf{Sets}_*$ preserves colimits separately in each variable.

(b) *The Unit Object Is $S^0$.* We have $\mathbb{1}_{\mathsf{Sets}_*} = S^0$.

(8) *Universal Property II.* The symmetric monoidal structure on the category $\mathsf{Sets}_*$ is the unique symmetric monoidal structure on $\mathsf{Sets}_*$ such that the free pointed set functor

$$(-)^+\colon \mathsf{Sets} \to \mathsf{Sets}_*$$

admits a symmetric monoidal structure.

(9) *Existence of Monoidal Diagonals.* The triple $(\mathsf{Sets}_*, \wedge, S^0)$ is a monoidal category with diagonals:

(a) *Monoidal Diagonals.* The natural transformation

$$\Delta\colon \mathrm{id}_{\mathsf{Sets}_*} \Longrightarrow \wedge \circ \Delta^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}, \qquad \begin{tikzcd} \mathsf{Sets}_* \arrow[rr, bend left, "\mathrm{id}_{\mathsf{Sets}_*}"] \arrow[dr, "\Delta^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}"'] \arrow[rr, phantom, "\Downarrow\Delta"] & & \mathsf{Sets}_* \\ & \mathsf{Sets}_* \times \mathsf{Sets}_*, \arrow[ur, "\wedge"'] & \end{tikzcd}$$

whose component

$$\Delta_X\colon (X, x_0) \to (X \wedge X, [(x_0, x_0)])$$

at $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$ is given by the composition

$$\begin{aligned} (X, x_0) &\xrightarrow{\Delta_X} (X \times X, (x_0, x_0)) \\ &\twoheadrightarrow \left(\tfrac{X\times X}{X \vee X}, [(x_0, x_0)]\right) \\ &\overset{\mathrm{def}}{=\!=} (X \wedge X, [(x_0, x_0)]) \end{aligned}$$

in $\mathsf{Sets}_*$, is a monoidal natural transformation:

(i) *Naturality.* For each morphism $f\colon X \to Y$ of pointed sets, the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\Delta_X"'] & Y \arrow[d, "\Delta_Y"] \\ X \wedge X \arrow[r, "f \wedge f"'] & Y \wedge Y \end{tikzcd}$$

commutes.

(ii) *Compatibility With Strong Monoidality Constraints.* For each $(X, x_0), (Y, y_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, the diagram

$$\begin{tikzcd} X \wedge Y \arrow[r, "\Delta_X \wedge \Delta_Y"] \arrow[d, equal] & (X \wedge X) \wedge (Y \wedge Y) \arrow[d, dashed, "\wr"] \\ X \wedge Y \arrow[r, "\Delta_{X \wedge Y}"'] & (X \wedge Y) \wedge (X \wedge Y) \end{tikzcd}$$

commutes.

(iii) *Compatibility With Strong Unitality Constraints.* The diagram

$$\begin{tikzcd} S^0 \arrow[d, equal] \arrow[dr, "(\lambda^{\mathsf{Sets}_*}_{S^0})^{-1}=(\rho^{\mathsf{Sets}_*}_{S^0})^{-1}"] & \\ S^0 \arrow[r, "\Delta_{S^0}"'] & S^0 \wedge S^0 \end{tikzcd}$$

commutes.

(b) *The Diagonal of the Unit.* The component

$$\Delta^{\mathsf{Sets}_*}_{S^0} \colon S^0 \xrightarrow{\cong} S^0 \wedge S^0$$

of the diagonal natural transformation of $\mathsf{Sets}_*$ at $S^0$ is an isomorphism.

(10) *Comonoids in* $\mathsf{Sets}_*$. The symmetric monoidal functor

$$\left((-)^+, (-)^{+,\times}, (-)^{+,\times}_{\mathbb{1}}\right) \colon (\mathsf{Sets}, \times, \mathrm{pt}) \to \left(\mathsf{Sets}_*, \wedge, S^0\right),$$

of **??** of **??** lifts to an equivalence of categories

$$\begin{aligned} \mathsf{CoMon}\left(\mathsf{Sets}_*, \wedge, S^0\right) &\overset{\text{eq.}}{\cong} \mathsf{CoMon}(\mathsf{Sets}, \times, \mathrm{pt}) \\ &\cong \mathsf{Sets}. \end{aligned}$$

**Proof.** *Item 1, Functoriality*: Omitted.
*Item 2, Adjointness*: Omitted.
*Item 3, Closed Symmetric Monoidality*: Omitted.
*Item 4, Morphisms From the Monoidal Unit*: Omitted.
*Item 5, Symmetric Strong Monoidality With Respect to Free Pointed Sets*: Omitted.
*Item 6, Distributivity Over Wedge Sums*: This follows from Item 3, **??** of **??**, and the fact that $\vee$ is the coproduct in $\mathsf{Sets}_*$.
*Item 7, Universal Property I*: Omitted.
*Item 8, Universal Property II*: See [GGN15, Theorem 5.1].
*Item 9, Existence of Monoidal Diagonals*: Omitted.
*Item 10, Comonoids in* $\mathsf{Sets}_*$: See [PS19, Lemma 2.4]. □

# Appendices

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CHAPTER 5

# Relations

This chapter contains some material about relations and constructions with them. Notably, we discuss and explore:

(1) The definition of relations (Section 5.1.1).
(2) How relations may be viewed as decategorification of profunctors (Section 5.1.2).
(3) The various kind of categories that relations form, namely:
   (a) A category (Section 5.2.1),
   (b) A monoidal category (Section 5.2.2),
   (c) A 2-category (Section 5.2.3), and
   (d) A double category (Section 5.2.4).
(4) The various categorical properties of the 2-category of relations, including (Section 5.2.5):
   (a) The self-duality of $\mathsf{Rel}$ and $\mathbf{Rel}$ (Items 1 and 2 of Proposition 5.2.5.1.1);
   (b) Identifications of equivalences and isomorphisms in $\mathbf{Rel}$ with bijections (Item 3 of Proposition 5.2.5.1.1);
   (c) Identifications of adjunctions in $\mathbf{Rel}$ with functions (Item 4 of Proposition 5.2.5.1.1);
   (d) Identifications of monads in $\mathbf{Rel}$ with preorders (Item 5 of Proposition 5.2.5.1.1);
   (e) Identifications of comonads in $\mathbf{Rel}$ with subsets (Item 6 of Proposition 5.2.5.1.1);
   (f) Characterisations of monomorphisms in $\mathsf{Rel}$ (Item 7 of Proposition 5.2.5.1.1);
   (g) Characterisations of epimorphisms in $\mathsf{Rel}$ (Item 8 of Proposition 5.2.5.1.1);
   (h) The partial co/completeness of $\mathsf{Rel}$ (Item 10 of Proposition 5.2.5.1.1);
   (i) The existence of right Kan extensions and right Kan lifts in $\mathsf{Rel}$ (Items 11 and 12 of Proposition 5.2.5.1.1);
   (j) The closedness of $\mathbf{Rel}$ (Item 13 of Proposition 5.2.5.1.1).
(5) The various kinds of constructions involving relations, such as graphs, domains, ranges, unions, intersections, products, inverse relations, composition of relations, and collages (Section 5.3).
(6) Equivalence relations (Section 5.4) and quotient sets (Section 5.4.5).
(7) The adjoint pairs

$$R_* \dashv R_{-1} \colon \mathcal{P}(A) \rightleftarrows \mathcal{P}(B),$$
$$R^{-1} \dashv R_! \colon \mathcal{P}(B) \rightleftarrows \mathcal{P}(A)$$

of functors (morphisms of posets) between $\mathcal{P}(A)$ and $\mathcal{P}(B)$ induced by a relation $R\colon A \nrightarrow B$, as well as the properties of $R_*$, $R_{-1}$, $R^{-1}$, and $R_!$ (Section 5.5).

Here we also note that:

(a) These two pairs of adjoint functors are the counterpart for relations of the adjoint triple $f_* \dashv f^{-1} \dashv f_!$ induced by a function $f\colon A \to B$ studied in ??;

(b) We have $R_{-1} = R^{-1}$ iff $R$ is total and functional (Item 8 of Proposition 5.5.2.1.3).

(c) As a consequence of the previous item, when $R$ comes from a function $f$ the pair of adjunctions

$$R_* \dashv R_{-1} = R^{-1} \dashv R_!$$

reduces to the triple adjunction

$$f_* \dashv f^{-1} \dashv f_!$$

from before.

(d) The pairs $R_* \dashv R_{-1}$ and $R^{-1} \dashv R_!$ later make an appearance in the context of continuous, open, and closed relations between topological spaces (??).

(8) A notion of *relative preorder* that is to that of a preorder as relative monads are to monads, extending the identifications of monads in $\mathsf{Rel}$ with preorders of Item 5 of Proposition 5.2.5.1.1 to “relative monads in $\mathsf{Rel}$”.

## 5.1. Relations

**5.1.1. Foundations.** Let $A$ and $B$ be sets.

**Definition 5.1.1.1.1.** A **relation** $R\colon A \nrightarrow B$ **from** $A$ **to** $B$[1,2] is a subset $R$ of $A \times B$.[3]

**Definition 5.1.1.1.2.** Let $A$ and $B$ be sets.

(1) The **set of relations from** $A$ **to** $B$ is the set $\mathrm{Rel}(A, B)$ defined by

$$\mathrm{Rel}(A, B) \overset{\text{def}}{=} \{\text{Relations from } A \text{ to } B\}.$$

(2) The **poset of relations from** $A$ **to** $B$ is the poset

$$\mathbf{Rel}(A, B) \overset{\text{def}}{=} (\mathrm{Rel}(A, B), \subset)$$

consisting of

- *The Underlying Set.* The set $\mathrm{Rel}(A, B)$ of Item 1;
- *The Partial Order.* The partial order

$$\subset\colon \mathrm{Rel}(A, B) \times \mathrm{Rel}(A, B) \to \{\mathsf{true}, \mathsf{false}\}$$

on $\mathrm{Rel}(A, B)$ given by inclusion of relations.

---

[1] *Further Terminology:* Also called a **multivalued function from** $A$ **to** $B$, a **relation over** $A$ **and** $B$, **relation on** $A$ **and** $B$, a **binary relation over** $A$ **and** $B$, or a **binary relation on** $A$ **and** $B$.

[2] *Further Terminology:* When $A = B$, we also call $R \subset A \times A$ a **relation on** $A$.

[3] *Further Notation:* Given elements $a \in A$ and $b \in B$, we write $a \sim_R b$ to mean $(a, b) \in R$.

**Remark 5.1.1.1.3.** A relation from $A$ to $B$ is equivalently:[4]

(1) A subset of $A \times B$;
(2) A function from $A \times B$ to $\{\mathsf{true}, \mathsf{false}\}$;
(3) A function from $A$ to $\mathcal{P}(B)$;
(4) A function from $B$ to $\mathcal{P}(A)$;
(5) A cocontinuous morphism of posets from $(\mathcal{P}(A), \subset)$ to $(\mathcal{P}(B), \subset)$.

That is: we have bijections of sets

$$\begin{aligned}\mathrm{Rel}(A, B) &\overset{\mathrm{def}}{=} \mathcal{P}(A \times B),\\ &\cong \mathsf{Sets}(A \times B, \{\mathsf{true}, \mathsf{false}\}),\\ &\cong \mathsf{Sets}(A, \mathcal{P}(B)),\\ &\cong \mathsf{Sets}(B, \mathcal{P}(A)),\\ &\cong \mathrm{Hom}^{\mathrm{cocont}}_{\mathsf{Pos}}(\mathcal{P}(A), \mathcal{P}(B)),\end{aligned}$$

natural in $A, B \in \mathrm{Obj}(\mathsf{Sets})$.

**Proof.** We claim that Items 1 to 5 are indeed equivalent:

- *Item 1* $\iff$ *Item 2:* This is a special case of **??** of **??**.
- *Item 2* $\iff$ *Item 3:* This is an instance of currying, following from the bijections

  $$\begin{aligned}\mathsf{Sets}(A \times B, \{\mathsf{true}, \mathsf{false}\}) &\cong \mathsf{Sets}(A, \mathsf{Sets}(B, \{\mathsf{true}, \mathsf{false}\}))\\ &\cong \mathsf{Sets}(A, \mathcal{P}(B)),\end{aligned}$$

  where the last bijection is from **??** of **??**.
- *Item 2* $\iff$ *Item 4:* This is also an instance of currying, following from the bijections

  $$\begin{aligned}\mathsf{Sets}(A \times B, \{\mathsf{true}, \mathsf{false}\}) &\cong \mathsf{Sets}(B, \mathsf{Sets}(B, \{\mathsf{true}, \mathsf{false}\}))\\ &\cong \mathsf{Sets}(B, \mathcal{P}(A)),\end{aligned}$$

  where again the last bijection is from **??** of **??**.
- *Item 2* $\iff$ *Item 5:* This follows from the universal property of the powerset $\mathcal{P}(X)$ of a set $X$ as the free cocompletion of $X$ via the characteristic embedding

  $$\chi_X : X \hookrightarrow \mathcal{P}(X)$$

  of $X$ into $\mathcal{P}(X)$ (**??** of **??**).

  In particular, the bijection

  $$\mathrm{Rel}(A, B) \cong \mathrm{Hom}^{\mathrm{cocont}}_{\mathsf{Pos}}(\mathcal{P}(A), \mathcal{P}(B))$$

  is given by taking a relation $R \colon A \nrightarrow B$, passing to its associated function $f \colon A \to \mathcal{P}(B)$ from $A$ to $B$ and then extending $f$ from $A$ to all of $\mathcal{P}(A)$ by taking its left Kan extension along $\chi_X$.

  This coincides with the direct image function $f_* \colon \mathcal{P}(A) \to \mathcal{P}(B)$ of **??**.

This finishes the proof. □

[4] *Intuition:* In particular, we may think of a relation $R \colon A \to \mathcal{P}(B)$ from $A$ to $B$ as a multivalued function from $A$ to $B$ (including the possibility of a given $a \in A$ having no value at all).

**Proposition 5.1.1.1.4.** Let $A$ and $B$ be sets.

(1) *End Formula for The Poset of Relations.* Let $R, S\colon A \nrightarrow B$ be relations. We have

$$\mathrm{Hom}_{\mathbf{Rel}(A,B)}(R,S) \cong \int_{a\in A}\int_{b\in B} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R^a_b, S^a_b).$$

**Proof.** *Item 1, End Formula for The Poset of Relations*: Unwinding the expression inside the end on the right hand side, we have

$$\int_{a\in A}\int_{b\in B} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R^a_b, S^a_b) \cong \begin{cases} \mathrm{pt} & \text{if for each } (a,b)\in A\times B, \\ & \text{if } a\sim_R b, \text{ then } a\sim_S b, \\ \emptyset & \text{otherwise.} \end{cases}$$

On the left hand-side, we have

$$\mathrm{Hom}_{\mathbf{Rel}(A,B)}(R,S) \cong \begin{cases} \mathrm{pt} & \text{if } R\subset S, \\ \emptyset & \text{otherwise.} \end{cases}$$

It is then clear that the conditions for each set to evaluate to pt are equivalent, implying that those two sets are isomorphic, finishing the proof. □

### 5.1.2. Relations as Decategorifications of Profunctors.

**Remark 5.1.2.1.1.** The notion of a relation is a decategorification of that of a profunctor:

(1) A profunctor from a category $\mathcal{C}$ to a category $\mathcal{D}$ is a functor

$$\mathfrak{p}\colon \mathcal{D}^{\mathsf{op}} \times \mathcal{C} \to \mathsf{Sets}.$$

(2) A relation on sets $A$ and $B$ is a function

$$R\colon A\times B \to \{\mathsf{true}, \mathsf{false}\}.$$

Here we notice that:

- The opposite $X^{\mathsf{op}}$ of a set $X$ is itself, as $(-)^{\mathsf{op}}\colon \mathsf{Cats} \to \mathsf{Cats}$ restricts to the identity endofunctor on $\mathsf{Sets}$;
- The values that profunctors and relations take are directly related in relation to decategorification:
  - A category is enriched over the category
  
    $$\mathsf{Sets} \overset{\text{def}}{=} \mathsf{Cats}_0$$
    
    of sets, with profunctors taking values on it;
  - A set is enriched over the set
  
    $$\{\mathsf{true}, \mathsf{false}\} \overset{\text{def}}{=} \mathsf{Cats}_{-1}$$
    
    of classical truth values, with relations taking values on it;

**Remark 5.1.2.1.2.** Extending Remark 5.1.2.1.1, the equivalent definitions of relations in Remark 5.1.1.1.3 are also related to the corresponding ones for profunctors (**??**), which state that a profunctor $\mathfrak{p}\colon \mathcal{C} \nrightarrow \mathcal{D}$ is equivalently:

(1) A functor $\mathfrak{p}\colon \mathcal{D}^{\mathsf{op}} \times \mathcal{C} \to \mathsf{Sets}$;
(2) A functor $\mathfrak{p}\colon \mathcal{C} \to \mathsf{PSh}(\mathcal{D})$;
(3) A functor $\mathfrak{p}\colon \mathcal{D}^{\mathsf{op}} \to \mathsf{Fun}(\mathcal{C}, \mathsf{Sets})$;
(4) A colimit-preserving functor $\mathfrak{p}\colon \mathsf{PSh}(\mathcal{C}) \to \mathsf{PSh}(\mathcal{D})$.

Indeed:

- The equivalence between Items 1 and 2 (and also that between Items 1 and 3, which is proved analogously) is an instance of currying, both for profunctors as well as for relations, using the isomorphisms

$$\begin{aligned}\mathsf{Sets}(A \times B, \{\mathsf{true}, \mathsf{false}\}) &\cong \mathsf{Sets}(A, \mathsf{Sets}(B, \{\mathsf{true}, \mathsf{false}\}))\\ &\cong \mathsf{Sets}(A, \mathcal{P}(B)),\\ \mathsf{Fun}(\mathcal{D}^{\mathsf{op}} \times \mathcal{D}, \mathsf{Sets}) &\cong \mathsf{Fun}(\mathcal{C}, \mathsf{Fun}(\mathcal{D}^{\mathsf{op}}, \mathsf{Sets}))\\ &\cong \mathsf{Fun}(\mathcal{C}, \mathsf{PSh}(\mathcal{D})).\end{aligned}$$

- The equivalence between Items 1 and 3 follows from the universal properties of:
  - The powerset $\mathcal{P}(X)$ of a set $X$ as the free cocompletion of $X$ via the characteristic embedding
  $$\chi_{(-)}\colon X \hookrightarrow \mathcal{P}(X)$$
  of $X$ into $\mathcal{P}(X)$ (**??** of **??**);
  - The category $\mathsf{PSh}(\mathcal{C})$ of presheaves on a category $\mathcal{C}$ as the free cocompletion of $\mathcal{C}$ via the Yoneda embedding
  $$よ\colon \mathcal{C} \hookrightarrow \mathsf{PSh}(\mathcal{C})$$
  of $\mathcal{C}$ into $\mathsf{PSh}(\mathcal{C})$ (**??** of **??**).

### 5.1.3. Examples of Relations.

**Example 5.1.3.1.1.** The **trivial relation on $A$ and $B$** is the relation $\sim_{\mathrm{triv}}$ defined by[5,6,7]

$$\sim_{\mathrm{triv}} \overset{\mathrm{def}}{=} A \times A.$$

**Example 5.1.3.1.2.** The **cotrivial relation on $A$ and $B$** is the relation $\sim_{\mathrm{cotriv}}$ defined by[8,9,10]

$$\sim_{\mathrm{cotriv}} \overset{\mathrm{def}}{=} \emptyset.$$

[5]This is the unique relation $R$ on $A$ and $B$ such that we have $a \sim_R b$ for all $a \in A$ and all $b \in B$.

[6]As a function from $A \times A$ to $\{\mathsf{true}, \mathsf{false}\}$, the relation $\sim_{\mathrm{triv}}$ is the constant function

$$\Delta_{\mathsf{true}}\colon A \times B \to \{\mathsf{true}, \mathsf{false}\}$$

from $A \times B$ to $\{\mathsf{true}, \mathsf{false}\}$ taking value $\mathsf{true}$.

[7]As a function from $A$ to $\mathcal{P}(B)$, the relation $\sim_{\mathrm{triv}}$ is the function

$$\Delta_{\mathsf{true}}\colon A \to \mathcal{P}(B)$$

defined by

$$\Delta_{\mathsf{true}}(a) \overset{\mathrm{def}}{=} B$$

for each $a \in A$.

[8]This is the unique relation $R$ on $A$ and $B$ such that we have $a \sim_R b$ for no $a \in A$ and no $b \in B$.

[9]As a function from $A \times B$ to $\{\mathsf{true}, \mathsf{false}\}$, the relation $\sim_{\mathrm{cotriv}}$ is the constant function

$$\Delta_{\mathsf{false}}\colon A \times B \to \{\mathsf{true}, \mathsf{false}\}$$

from $A \times B$ to $\{\mathsf{true}, \mathsf{false}\}$ taking value $\mathsf{false}$.

[10]As a function from $A$ to $\mathcal{P}(A)$, the relation $\sim_{\mathrm{cotriv}}$ is the function

$$\Delta_{\mathsf{false}}\colon A \to \mathcal{P}(A)$$

**Example 5.1.3.1.3.** The characteristic relation on $A$ of **??** of **??** is another example of a relation. It is in fact the unique relation on $A$ making the following conditions equivalent, for each $a, b \in A$:

(1) We have $a \sim_{\mathrm{id}} b$.
(2) We have $a = b$.

**Example 5.1.3.1.4.** Square roots are examples of relations:

(1) *Square Roots in $\mathbb{R}$.* The assignment $x \mapsto \sqrt{x}$ defines a relation
$$\sqrt{-}\colon \mathbb{R} \to \mathcal{P}(\mathbb{R})$$
from $\mathbb{R}$ to itself, being explicitly given by
$$\sqrt{x} \overset{\mathrm{def}}{=} \begin{cases} 0 & \text{if } x = 0, \\ \left\{-\sqrt{|x|}, \sqrt{|x|}\right\} & \text{if } x \neq 0. \end{cases}$$
(2) *Square Roots in $\mathbb{Q}$.* Square roots in $\mathbb{Q}$ are similar to square roots in $\mathbb{R}$, though now additionally it may also occur that $\sqrt{-}\colon \mathbb{Q} \to \mathcal{P}(\mathbb{Q})$ sends a rational number $x$ (e.g. 2) to the empty set (since $\sqrt{2} \notin \mathbb{Q}$).

**Example 5.1.3.1.5.** The complex logarithm defines a relation
$$\log\colon \mathbb{C} \to \mathcal{P}(\mathbb{C})$$
from $\mathbb{C}$ to itself, where we have
$$\log(a+bi) \overset{\mathrm{def}}{=} \left\{\log\left(\sqrt{a^2+b^2}\right) + i\arg(a+bi) + (2\pi i)k \;\middle|\; k \in \mathbb{Z}\right\}$$
for each $a + bi \in \mathbb{C}$.

**Example 5.1.3.1.6.** See [**wikipedia:multivalued-functions**] for more examples of relations, such as antiderivation, inverse trigonometric functions, and inverse hyperbolic functions.

**5.1.4. Functional Relations.** Let $A$ and $B$ be sets.

**Definition 5.1.4.1.1.** A relation $R\colon A \nrightarrow B$ is **functional** if, for each $a \in A$, the set $R(a)$ is either empty or a singleton.

**Proposition 5.1.4.1.2.** Let $R\colon A \nrightarrow B$ be a relation.

(1) *Characterisations.* The following conditions are equivalent:
  (a) The relation $R$ is functional.
  (b) We have $R \diamond R^\dagger \subset \chi_B$.

**Proof.** *Item 1, Characterisations*: We claim that Items 1a and 1b are indeed equivalent:

- *Item 1a $\Longrightarrow$ Item 1b:* Let $(b, b') \in B \times B$. We need to show that
$$[R \diamond R^\dagger](b, b') \preceq_{\{\mathsf{t},\mathsf{f}\}} \chi_B(b, b'),$$
i.e. that if there exists some $a \in A$ such that $b \sim_{R^\dagger} a$ and $a \sim_R b'$, then $b = b'$. But since $b \sim_{R^\dagger} a$ is the same as $a \sim_R b$, we have both $a \sim_R b$ and $a \sim_R b'$ at the same time, which implies $b = b'$ since $R$ is functional.

---

defined by
$$\Delta_{\mathsf{true}}(a) \overset{\mathrm{def}}{=} \emptyset$$

- *Item 1b* $\implies$ *Item 1a:* Suppose that we have $a \sim_R b$ and $a \sim_R b'$ for $b, b' \in B$. We claim that $b = b'$:
  (1) Since $a \sim_R b$, we have $b \sim_{R^\dagger} a$.
  (2) Since $R \diamond R^\dagger \subset \chi_B$, we have
  $$[R \diamond R^\dagger](b, b') \preceq_{\{\mathsf{t},\mathsf{f}\}} \chi_B(b, b'),$$
  and since $b \sim_{R^\dagger} a$ and $a \sim_R b'$, it follows that $[R \diamond R^\dagger](b, b') = \mathsf{true}$, and thus $\chi_B(b, b') = \mathsf{true}$ as well, i.e. $b = b'$.

This finishes the proof. □

**5.1.5. Total Relations.** Let $A$ and $B$ be sets.

**Definition 5.1.5.1.1.** A relation $R \colon A \nrightarrow B$ is **total** if, for each $a \in A$, we have $R(a) \neq \emptyset$.

**Proposition 5.1.5.1.2.** Let $R \colon A \nrightarrow B$ be a relation.

(1) *Characterisations.* The following conditions are equivalent:
  (a) The relation $R$ is total.
  (b) We have $\chi_A \subset R^\dagger \diamond R$.

**Proof.** *Item 1, Characterisations*: We claim that Items 1a and 1b are indeed equivalent:

- *Item 1a* $\implies$ *Item 1b:* We have to show that, for each $(a, a') \in A$, we have
  $$\chi_A(a, a') \preceq_{\{\mathsf{t},\mathsf{f}\}} [R^\dagger \diamond R](a, a'),$$
  i.e. that if $a = a'$, then there exists some $b \in B$ such that $a \sim_R b$ and $b \sim_{R^\dagger} a'$ (i.e. $a \sim_R b$ again), which follows from the totality of $R$.
- *Item 1b* $\implies$ *Item 1a:* Given $a \in A$, since $\chi_A \subset R^\dagger \diamond R$, we must have
  $$\{a\} \subset [R^\dagger \diamond R](a),$$
  implying that there must exist some $b \in B$ such that $a \sim_R b$ and $b \sim_{R^\dagger} a$ (i.e. $a \sim_R b$) and thus $R(a) \neq \emptyset$, as $b \in R(a)$.

This finishes the proof. □

## 5.2. Categories of Relations

### 5.2.1. The Category of Relations.

**Definition 5.2.1.1.1.** The **category of relations** is the category $\mathsf{Rel}$ where

- *Objects.* The objects of $\mathsf{Rel}$ are sets;
- *Morphisms.* For each $A, B \in \mathrm{Obj}(\mathsf{Sets})$, we have
  $$\mathsf{Rel}(A, B) \stackrel{\text{def}}{=} \mathrm{Rel}(A, B);$$
- *Identities.* For each $A \in \mathrm{Obj}(\mathsf{Rel})$, the unit map
  $$\mathbb{1}^{\mathsf{Rel}}_A \colon \mathrm{pt} \to \mathrm{Rel}(A, A)$$
  of $\mathsf{Rel}$ at $A$ is defined by
  $$\mathrm{id}^{\mathsf{Rel}}_A \stackrel{\text{def}}{=} \chi_A(-_1, -_2),$$
  where $\chi_A(-_1, -_2)$ is the characteristic relation of $A$ of **??** of **??**;

- *Composition.* For each $A, B, C \in \mathrm{Obj}(\mathsf{Rel})$, the composition map
$$\circ^{\mathsf{Rel}}_{A,B,C} \colon \mathrm{Rel}(B, C) \times \mathrm{Rel}(A, B) \to \mathrm{Rel}(A, C)$$
of $\mathsf{Rel}$ at $(A, B, C)$ is defined by
$$S \circ^{\mathsf{Rel}}_{A,B,C} R \stackrel{\text{def}}{=} S \diamond R$$
for each $(S, R) \in \mathrm{Rel}(B, C) \times \mathrm{Rel}(A, B)$, where $S \diamond R$ is the composition of $S$ and $R$ of Definition 5.3.12.1.1.

### 5.2.2. The Closed Symmetric Monoidal Category of Relations.

5.2.2.1. *The Monoidal Product.*

**Definition 5.2.2.1.1.** The **monoidal product of** $\mathsf{Rel}$ is the functor
$$\times \colon \mathsf{Rel} \times \mathsf{Rel} \to \mathsf{Rel}$$
where

- *Action on Objects.* We have
$$\times(A, B) \stackrel{\text{def}}{=} A \times B,$$
where $A \times B$ is the Cartesian product of sets of **??**;
- *Action on Morphisms.* For each $(A, C), (B, D) \in \mathrm{Obj}(\mathsf{Rel} \times \mathsf{Rel})$, the action on morphisms
$$\times_{(A,C),(B,D)} \colon \mathrm{Rel}(A, B) \times \mathrm{Rel}(C, D) \to \mathrm{Rel}(A \times C, B \times D)$$
of $\times$ is given by sending a pair of morphisms $(R, S)$ of the form
$$R \colon A \nrightarrow B,$$
$$S \colon C \nrightarrow D$$
to the relation
$$R \times S \colon A \times C \nrightarrow B \times D$$
of Definition 5.3.9.1.1.

5.2.2.2. *The Monoidal Unit.*

**Definition 5.2.2.2.1.** The **monoidal unit of** $\mathsf{Rel}$ is the functor
$$\mathbb{1}^{\mathsf{Rel}} \colon \mathrm{pt} \to \mathsf{Rel}$$
picking the set
$$\mathbb{1}_{\mathsf{Rel}} \stackrel{\text{def}}{=} \mathrm{pt}$$
of $\mathsf{Rel}$.

5.2.2.3. *The Associator.*

**Definition 5.2.2.3.1.** The **associator of** $\mathsf{Rel}$ is the natural isomorphism

$$\alpha^{\mathsf{Rel}} \colon \times \circ ((\times) \times \mathsf{id}) \stackrel{\cong}{\Longrightarrow} \times \circ (\mathsf{id} \times (\times)), \qquad \begin{tikzcd} \mathsf{Rel} \times \mathsf{Rel} \times \mathsf{Rel} \arrow[r, "\mathsf{id} \times (\times)"] \arrow[d, "(\times) \times \mathsf{id}"'] & \mathsf{Rel} \times \mathsf{Rel} \arrow[d, "\times"] \\ \mathsf{Rel} \times \mathsf{Rel} \arrow[r, "\times"'] \arrow[ur, phantom, "\alpha^{\mathsf{Rel}}"] & \mathsf{Rel}, \end{tikzcd}$$

for each $a \in A$.

whose component

$$\alpha^{\mathsf{Rel}}_{A,B,C}\colon (A\times B)\times C \mathrel{+\!\!\!\to} A\times(B\times C)$$

at $(A,B,C)$ is defined by declaring

$$((a,b),c)\sim_{\alpha^{\mathsf{Rel}}_{A,B,C}} (a',(b',c'))$$

iff $a=a'$, $b=b'$, and $c=c'$.

5.2.2.4. *The Left Unitor.*

**Definition 5.2.2.4.1.** The **left unitor of** $\mathsf{Rel}$ is the natural isomorphism

$$\lambda^{\mathsf{Rel}}\colon \times\circ(\mathbb{1}^{\mathsf{Rel}}\times\mathsf{id}) \overset{\cong}{\Longrightarrow} \boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathsf{Rel}}, \qquad \begin{tikzcd} \mathsf{pt}\times\mathsf{Rel} \arrow[r, "\mathbb{1}^{\mathsf{Rel}}\times\mathsf{id}"] \arrow[dr, dashed, bend right, "\boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathsf{Rel}}"'] & \mathsf{Rel}\times\mathsf{Rel} \arrow[d, "\times"] \arrow[dl, phantom, "\Swarrow\lambda^{\mathsf{Rel}}"] \\ & \mathsf{Rel}, \end{tikzcd}$$

whose component

$$\lambda^{\mathsf{Rel}}_{A}\colon \mathbb{1}_{\mathsf{Rel}}\times A \mathrel{+\!\!\!\to} A$$

at $A$ is defined by declaring

$$(\star,a)\sim_{\lambda^{\mathsf{Rel}}_{A}} b$$

iff $a=b$.

5.2.2.5. *The Right Unitor.*

**Definition 5.2.2.5.1.** The **right unitor of** $\mathsf{Rel}$ is the natural isomorphism

$$\rho^{\mathsf{Rel}}\colon \times\circ(\mathsf{id}\times\mathbb{1}^{\mathsf{Rel}}) \overset{\cong}{\Longrightarrow} \boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathsf{Rel}}, \qquad \begin{tikzcd} \mathsf{Rel}\times\mathsf{pt} \arrow[r, "\mathsf{id}\times\mathbb{1}^{\mathsf{Rel}}"] \arrow[dr, dashed, bend right, "\boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathsf{Rel}}"'] & \mathsf{Rel}\times\mathsf{Rel} \arrow[d, "\times"] \arrow[dl, phantom, "\Swarrow\rho^{\mathsf{Rel}}"] \\ & \mathsf{Rel}, \end{tikzcd}$$

whose component

$$\rho^{\mathsf{Rel}}_{A}\colon A\times\mathbb{1}_{\mathsf{Rel}} \mathrel{+\!\!\!\to} A$$

at $A$ is defined by declaring

$$(a,\star)\sim_{\rho^{\mathsf{Rel}}_{A}} b$$

iff $a=b$.

5.2.2.6. *The Symmetry.*

**Definition 5.2.2.6.1.** The **symmetry of** $\mathsf{Rel}$ is the natural isomorphism

$$\sigma^{\mathsf{Rel}} \colon \times \Longrightarrow \times \circ \boldsymbol{\sigma}^{\mathsf{Cats}_2}_{\mathsf{Rel},\mathsf{Rel}}, \qquad \begin{tikzcd} \mathsf{Rel} \times \mathsf{Rel} \arrow[rr, "\times"] \arrow[dr, bend right, "\boldsymbol{\sigma}^{\mathsf{Cats}_2}_{\mathsf{Rel},\mathsf{Rel}}"'] & {} \arrow[d, phantom, "\overset{\Vert}{\Downarrow \sigma^{\mathsf{Rel}}}"] & \mathsf{Rel}, \\ & \mathsf{Rel} \times \mathsf{Rel} \arrow[ur, bend right, "\times"'] & \end{tikzcd}$$

whose component

$$\sigma^{\mathsf{Rel}}_{A,B} \colon A \times B \to B \times A$$

at $(A, B)$ is defined by declaring

$$(a, b) \sim_{\sigma^{\mathsf{Rel}}_{A,B}} (b', a')$$

iff $a = a'$ and $b = b'$.

5.2.2.7. *The Internal Hom.*

**Definition 5.2.2.7.1.** The **internal Hom of** $\mathsf{Rel}$ is the functor

$$\mathbf{Hom}_{\mathsf{Rel}} \colon \mathsf{Rel}^{\mathsf{op}} \times \mathsf{Rel} \to \mathsf{Rel}$$

defined by

$$\mathbf{Hom}_{\mathsf{Rel}}(A, B) \overset{\text{def}}{=} A \times B$$

for each $A, B \in \mathrm{Obj}(\mathsf{Rel})$.

**Proposition 5.2.2.7.2.** Let $A, B, C \in \mathrm{Obj}(\mathsf{Rel})$.

(1) *Via Self-Duality.* The internal Hom $\mathbf{Hom}_{\mathsf{Rel}}$ of $\mathsf{Rel}$ is given by the composition

$$\mathsf{Rel}^{\mathsf{op}} \times \mathsf{Rel} \xrightarrow{\cong} \mathsf{Rel} \times \mathsf{Rel} \xrightarrow{\times} \mathsf{Rel},$$

where the self-duality equivalence $\mathsf{Rel}^{\mathsf{op}} \cong \mathsf{Rel}$ comes from Item 1 of Proposition 5.2.5.1.1.

(2) *Adjointness.* We have adjunctions

$$(A \times - \dashv \mathbf{Hom}_{\mathsf{Rel}}(A, -)) \colon \quad \begin{tikzcd} \mathsf{Rel} \arrow[r, bend left, "A \times -"] \arrow[r, phantom, "\perp"] & \mathsf{Rel}, \arrow[l, bend left, "\mathbf{Hom}_{\mathsf{Rel}}(A,-)"] \end{tikzcd}$$

$$(- \times B \dashv \mathbf{Hom}_{\mathsf{Rel}}(B, -)) \colon \quad \begin{tikzcd} \mathsf{Rel} \arrow[r, bend left, "- \times B"] \arrow[r, phantom, "\perp"] & \mathsf{Rel}, \arrow[l, bend left, "\mathbf{Hom}_{\mathsf{Rel}}(B,-)"] \end{tikzcd}$$

witnessed by bijections

$$\begin{aligned} \mathrm{Rel}(A \times B, C) &\cong \mathrm{Rel}(A, \mathbf{Hom}_{\mathrm{Rel}}(B, C)) \\ &\overset{\text{def}}{=} \mathrm{Rel}(A, B \times C), \\ \mathrm{Rel}(A \times B, C) &\cong \mathrm{Rel}(B, \mathbf{Hom}_{\mathrm{Rel}}(A, C)) \\ &\overset{\text{def}}{=} \mathrm{Rel}(B, A \times C), \end{aligned}$$

natural in $A, B, C \in \mathrm{Obj}(\mathsf{Rel})$.

**Proof.** *Item 1, Via Self-Duality*: Omitted.

*Item 2, Adjointness*: Indeed, we have

$$\begin{aligned}\mathrm{Rel}(A\times B,C) &\stackrel{\text{def}}{=} \mathsf{Sets}(A\times B\times C,\{\mathsf{true},\mathsf{false}\})\\ &\stackrel{\text{def}}{=} \mathrm{Rel}(A,B\times C)\\ &\stackrel{\text{def}}{=} \mathrm{Rel}(A,\mathbf{Hom}_{\mathrm{Rel}}(B,C)),\end{aligned}$$

and similarly for the bijection $\mathrm{Rel}(A\times B,C)\cong\mathrm{Rel}(B,\mathbf{Hom}_{\mathrm{Rel}}(A,C))$. □

5.2.2.8. *The Closed Symmetric Monoidal Category of Relations.*

**Definition 5.2.2.8.1.** The **closed symmetric monoidal category of relations** is the closed symmetric monoidal category

$$\left(\mathsf{Rel},\times,\mathbb{1}_{\mathsf{Rel}},\alpha^{\mathsf{Rel}},\lambda^{\mathsf{Rel}},\rho^{\mathsf{Rel}},\sigma^{\mathsf{Rel}},\mathbf{Hom}_{\mathsf{Rel}}\right)$$

consisting of

- *The Underlying Category.* The category $\mathsf{Rel}$ of sets and relations of Definition 5.2.1.1.1;
- *The Monoidal Product.* The functor
$$\times\colon \mathsf{Rel}\times\mathsf{Rel}\to\mathsf{Rel}$$
of Definition 5.2.2.1.1;
- *The Monoidal Unit.* The functor $\mathbb{1}^{\mathsf{Rel}}$ of Definition 5.2.2.2.1;
- *The Associator.* The natural isomorphism $\alpha^{\mathsf{Rel}}$ of Definition 5.2.2.3.1;
- *The Left Unitor.* The natural isomorphism $\lambda^{\mathsf{Rel}}$ of Definition 5.2.2.4.1;
- *The Right Unitor.* The natural isomorphism $\rho^{\mathsf{Rel}}$ of Definition 5.2.2.5.1;
- *The Symmetry.* The natural isomorphism $\sigma^{\mathsf{Rel}}$ of Definition 5.2.2.6.1;
- *The Internal Hom.* The functor
$$\mathbf{Hom}_{\mathsf{Rel}}\colon \mathsf{Rel}^{\mathsf{op}}\times\mathsf{Rel}\to\mathsf{Rel}$$
of Definition 5.2.2.7.1.

### 5.2.3. The 2-Category of Relations.

**Definition 5.2.3.1.1.** The **2-category of relations** is the locally posetal 2-category $\mathbf{Rel}$ where

- *Objects.* The objects of $\mathbf{Rel}$ are sets;
- ***Hom**-Objects.* For each $A,B\in\mathrm{Obj}(\mathsf{Sets})$, we have
$$\begin{aligned}\mathrm{Hom}_{\mathbf{Rel}}(A,B) &\stackrel{\text{def}}{=} \mathbf{Rel}(A,B)\\ &\stackrel{\text{def}}{=} (\mathrm{Rel}(A,B),\subset);\end{aligned}$$
- *Identities.* For each $A\in\mathrm{Obj}(\mathbf{Rel})$, the unit map
$$\mathbb{1}^{\mathbf{Rel}}_A\colon \mathrm{pt}\to\mathbf{Rel}(A,A)$$
of $\mathbf{Rel}$ at $A$ is defined by
$$\mathrm{id}^{\mathbf{Rel}}_A \stackrel{\text{def}}{=} \chi_A(-_1,-_2),$$
where $\chi_A(-_1,-_2)$ is the characteristic relation of $A$ of **??** of **??**;
- *Composition.* For each $A,B,C\in\mathrm{Obj}(\mathbf{Rel})$, the composition map[11]
$$\circ^{\mathbf{Rel}}_{A,B,C}\colon \mathbf{Rel}(B,C)\times\mathbf{Rel}(A,B)\to\mathbf{Rel}(A,C)$$

[11] Note that this is indeed a morphism of posets: given relations $R_1,R_2\in\mathbf{Rel}(A,B)$ and

of **Rel** at $(A, B, C)$ is defined by

$$S \circ^{\mathsf{Rel}}_{A,B,C} R \overset{\text{def}}{=} S \diamond R$$

for each $(S, R) \in \mathsf{Rel}(B, C) \times \mathsf{Rel}(A, B)$, where $S \diamond R$ is the composition of $S$ and $R$ of Definition 5.3.12.1.1.

### 5.2.4. The Double Category of Relations.

5.2.4.1. *The Double Category of Relations.*

**Definition 5.2.4.1.1.** The **double category of relations** is the locally posetal double category $\mathsf{Rel}^{\mathsf{dbl}}$ where

- *Objects.* The objects of $\mathsf{Rel}^{\mathsf{dbl}}$ are sets;
- *Vertical Morphisms.* The vertical morphisms of $\mathsf{Rel}^{\mathsf{dbl}}$ are maps of sets $f : A \to B$;
- *Horizontal Morphisms.* The horizontal morphisms of $\mathsf{Rel}^{\mathsf{dbl}}$ are relations $R : A \nrightarrow X$;
- 2-*Morphisms.* A 2-cell

$$\begin{tikzcd} A \arrow[r, "R", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & B \arrow[d, "g"] \\ X \arrow[r, "S"', "\shortmid"{marking}] & Y \end{tikzcd}$$

of $\mathsf{Rel}^{\mathsf{dbl}}$ is either non-existent or an inclusion of relations of the form

$$R \subset S \circ (f \times g), \qquad \begin{tikzcd} A \times B \arrow[r, "R"] \arrow[d, "f \times g"'] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ X \times Y \arrow[r, "S"'] & \{\mathsf{true}, \mathsf{false}\}; \end{tikzcd}$$

- *Horizontal Identities.* The horizontal unit functor of $\mathsf{Rel}^{\mathsf{dbl}}$ is the functor of Definition 5.2.4.2.1;
- *Vertical Identities.* For each $A \in \mathrm{Obj}\left(\mathsf{Rel}^{\mathsf{dbl}}\right)$, we have

$$\mathrm{id}^{\mathsf{Rel}^{\mathsf{dbl}}}_A \overset{\text{def}}{=} \mathrm{id}_A;$$

- *Identity* 2-*Morphisms.* For each horizontal morphism $R : A \nrightarrow B$ of $\mathsf{Rel}^{\mathsf{dbl}}$, the identity 2-morphism

$$\begin{tikzcd} A \arrow[r, "R", "\shortmid"{marking}] \arrow[d, "\mathrm{id}_A"'] \arrow[dr, phantom, "\Downarrow\mathrm{id}_R"] & B \arrow[d, "\mathrm{id}_B"] \\ A \arrow[r, "R"', "\shortmid"{marking}] & B \end{tikzcd}$$

---

$S_1, S_2 \in \mathbf{Rel}(B, C)$ such that

$$R_1 \subset R_2,$$
$$S_1 \subset S_2,$$

of $R$ is the identity inclusion

$$R \subset R, \qquad \begin{tikzcd} B \times A \arrow[r, "R"] \arrow[d, "\mathrm{id}_B \times \mathrm{id}_A"'] \arrow[dr, phantom, "\subset"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ B \times A \arrow[r, "R"'] & \{\mathsf{true}, \mathsf{false}\}; \end{tikzcd}$$

- *Horizontal Composition.* The horizontal composition functor of $\mathsf{Rel}^{\mathsf{dbl}}$ is the functor of Definition 5.2.4.3.1;
- *Vertical Composition of* 1*-Morphisms.* For each composable pair $A \xrightarrow{F} B \xrightarrow{G} C$ of vertical morphisms of $\mathsf{Rel}^{\mathsf{dbl}}$, i.e. maps of sets, we have
$$g \circ^{\mathsf{Rel}^{\mathsf{dbl}}} f \stackrel{\text{def}}{=} g \circ f;$$
- *Vertical Composition of* 2*-Morphisms.* The vertical composition of 2-morphisms in $\mathsf{Rel}^{\mathsf{dbl}}$ is defined as in Definition 5.2.4.4.1;
- *Associators.* The associators of $\mathsf{Rel}^{\mathsf{dbl}}$ is defined as in Definition 5.2.4.5.1;
- *Left Unitors.* The left unitors of $\mathsf{Rel}^{\mathsf{dbl}}$ is defined as in Definition 5.2.4.6.1;
- *Right Unitors.* The right unitors of $\mathsf{Rel}^{\mathsf{dbl}}$ is defined as in Definition 5.2.4.7.1.

5.2.4.2. *Horizontal Identities.*

**Definition 5.2.4.2.1.** The **horizontal unit functor** of $\mathsf{Rel}^{\mathsf{dbl}}$ is the functor

$$\not\Vdash^{\mathsf{Rel}^{\mathsf{dbl}}} \colon \mathsf{Rel}^{\mathsf{dbl}}_0 \to \mathsf{Rel}^{\mathsf{dbl}}_1$$

of $\mathsf{Rel}^{\mathsf{dbl}}$ is the functor where

- *Action on Objects.* For each $A \in \mathrm{Obj}\left(\mathsf{Rel}^{\mathsf{dbl}}_0\right)$, we have
$$\not\Vdash_A \stackrel{\text{def}}{=} \chi_A(-_1, -_2);$$
- *Action on Morphisms.* For each vertical morphism $f \colon A \to B$ of $\mathsf{Rel}^{\mathsf{dbl}}$, i.e. each map of sets $f$ from $A$ to $B$, the identity 2-morphism
$$\begin{tikzcd} A \arrow[r, "\not\Vdash_A", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\overset{\Vert}{\underset{\Downarrow}{\not\Vdash_f}}"] & A \arrow[d, "f"] \\ B \arrow[r, "\not\Vdash_B"', "\shortmid"{marking}] & B \end{tikzcd}$$
of $f$ is the inclusion
$$\chi_B \circ (f \times f) \subset \chi_A, \qquad \begin{tikzcd} A \times A \arrow[r, "\chi_A(-_1,-_2)"] \arrow[d, "f \times f"'] \arrow[dr, phantom, "\subset"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ B \times B \arrow[r, "\chi_B(-_1,-_2)"'] & \{\mathsf{true}, \mathsf{false}\} \end{tikzcd}$$
of **??** of **??**.

---

we have also $S_1 \diamond R_1 \subset S_2 \diamond R_2$.

5.2.4.3. *Horizontal Composition.*

**Definition 5.2.4.3.1.** The **horizontal composition functor** of $\mathsf{Rel}^{\mathsf{dbl}}$ is the functor

$$\odot^{\mathsf{Rel}^{\mathsf{dbl}}} \colon \mathsf{Rel}_1^{\mathsf{dbl}} \underset{\mathsf{Rel}_0^{\mathsf{dbl}}}{\times} \mathsf{Rel}_1^{\mathsf{dbl}} \to \mathsf{Rel}_1^{\mathsf{dbl}}$$

of $\mathsf{Rel}^{\mathsf{dbl}}$ is the functor where

- *Action on Objects.* For each composable pair $A \overset{R}{\nrightarrow} B \overset{S}{\nrightarrow} C$ of horizontal morphisms of $\mathsf{Rel}^{\mathsf{dbl}}$, we have

  $$S \odot R \overset{\text{def}}{=} S \diamond R,$$

  where $S \diamond R$ is the composition of $R$ and $S$ of Definition 5.3.12.1.1;
- *Action on Morphisms.* For each horizontally composable pair

  $$\begin{tikzcd} A \arrow[r, "R", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & B \arrow[d, "g"] \\ X \arrow[r, "T"', "\shortmid"{marking}] & Y \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "S", "\shortmid"{marking}] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\beta"] & C \arrow[d, "h"] \\ Y \arrow[r, "U"', "\shortmid"{marking}] & Z \end{tikzcd}$$

  of 2-morphisms of $\mathsf{Rel}^{\mathsf{dbl}}$, i.e. for each pair

  $$\begin{tikzcd} A \times B \arrow[r, "R"] \arrow[d, "f \times g"'] \arrow[dr, phantom, "\circlearrowright"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ X \times Y \arrow[r, "T"'] & \{\mathsf{true}, \mathsf{false}\} \end{tikzcd} \qquad \begin{tikzcd} B \times C \arrow[r, "S"] \arrow[d, "g \times h"'] \arrow[dr, phantom, "\circlearrowright"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ Y \times Z \arrow[r, "U"'] & \{\mathsf{true}, \mathsf{false}\} \end{tikzcd}$$

  of inclusions of relations, the horizontal composition

  $$\begin{tikzcd} A \arrow[r, "S \odot R", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\beta\odot\alpha"] & C \arrow[d, "h"] \\ X \arrow[r, "U \odot T"', "\shortmid"{marking}] & Z \end{tikzcd}$$

  of $\alpha$ and $\beta$ is the inclusion of relations[12]

  $$(U \diamond T) \circ (f \times h) \subset (S \diamond R) \qquad \begin{tikzcd} A \times C \arrow[r, "S \diamond R"] \arrow[d, "f \times h"'] \arrow[dr, phantom, "\circlearrowright"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ X \times Z \arrow[r, "U \diamond T"'] & \{\mathsf{true}, \mathsf{false}\}. \end{tikzcd}$$

---

[12]This is justified by noting that, given $(a, c) \in A \times C$, the statement

- We have $a \sim_{(U \diamond T) \circ (f \times h)} c$, i.e. $f(a) \sim_{U \diamond T} h(c)$, i.e. there exists some $y \in Y$ such that:
  1. We have $f(a) \sim_T y$;
  2. We have $y \sim_U h(c)$;

is implied by the statement

- We have $a \sim_{S \diamond R} c$, i.e. there exists some $b \in B$ such that:
  1. We have $a \sim_R b$;

5.2.4.4. *Vertical Composition of 2-Morphisms.*

**Definition 5.2.4.4.1.** The **vertical composition** in $\mathsf{Rel}^{\mathsf{dbl}}$ is defined as follows: for each vertically composable pair

$$\begin{tikzcd} A \arrow[r, "R", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & X \arrow[d, "g"] \\ B \arrow[r, "S"', "\shortmid"{marking}] & Y \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "S", "\shortmid"{marking}] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow\beta"] & Y \arrow[d, "k"] \\ C \arrow[r, "T"', "\shortmid"{marking}] & Z \end{tikzcd}$$

of 2-morphisms of $\mathsf{Rel}^{\mathsf{dbl}}$, i.e. for each each pair

$$\begin{tikzcd} A \times X \arrow[r, "R"] \arrow[d, "f\times g"'] \arrow[dr, phantom, "\subset"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ B \times Y \arrow[r, "S"'] & \{\mathsf{true}, \mathsf{false}\} \end{tikzcd} \qquad \begin{tikzcd} B \times Y \arrow[r, "S"] \arrow[d, "h\times k"'] \arrow[dr, phantom, "\subset"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ C \times Z \arrow[r, "T"'] & \{\mathsf{true}, \mathsf{false}\} \end{tikzcd}$$

of inclusions of relations, we define the vertical composition

$$\begin{tikzcd} A \arrow[r, "R", "\shortmid"{marking}] \arrow[d, "h\circ f"'] \arrow[dr, phantom, "\Downarrow\beta\circ\alpha"] & X \arrow[d, "k\circ g"] \\ C \arrow[r, "T"', "\shortmid"{marking}] & Z \end{tikzcd}$$

of $\alpha$ and $\beta$ as the inclusion of relations

$$T \circ [(h \circ f) \times (k \circ g)] \subset R, \qquad \begin{tikzcd} A \times X \arrow[r, "R"] \arrow[d, "(h\circ f)\times(k\circ g)"'] \arrow[dr, phantom, "\subset"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ C \times Z \arrow[r, "T"'] & \{\mathsf{true}, \mathsf{false}\} \end{tikzcd}$$

given by the pasting of inclusions[13]

$$\begin{tikzcd} A \times X \arrow[r, "R"] \arrow[d, "f\times g"'] \arrow[dr, phantom, "\subset"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ B \times Y \arrow[r, dashed, "S"] \arrow[d, "h\times k"'] \arrow[dr, phantom, "\subset"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ C \times Z \arrow[r, "T"'] & \{\mathsf{true}, \mathsf{false}\}. \end{tikzcd}$$

---

(2) We have $b \sim_S c$;

since:

- If $a \sim_R b$, then $f(a) \sim_T g(b)$, as $T \circ (f \times g) \subset R$;
- If $b \sim_S c$, then $g(b) \sim_U h(c)$, as $U \circ (g \times h) \subset S$;

[13]This is justified by noting that, given $(a, x) \in A \times X$, the statement

5.2.4.5. *The Associators.*

**Definition 5.2.4.5.1.** For each composable triple $A \overset{R}{\relbar\joinrel\mapstochar\joinrel\rightarrow} B \overset{S}{\relbar\joinrel\mapstochar\joinrel\rightarrow} C \overset{T}{\relbar\joinrel\mapstochar\joinrel\rightarrow} D$ of horizontal morphisms of $\mathsf{Rel}^{\mathsf{dbl}}$, the component

$$\alpha^{\mathsf{Rel}^{\mathsf{dbl}}}_{T,S,R} \colon (T \odot S) \odot R \overset{\cong}{\Longrightarrow} T \odot (S \odot R), \qquad \begin{tikzcd} A \arrow[r, "R", "\shortmid"{marking}] \arrow[d, "\mathrm{id}_A"'] & B \arrow[r, "S", "\shortmid"{marking}] \arrow[d, phantom, "\alpha^{\mathsf{Rel}^{\mathsf{dbl}}}_{T,S,R}\Downarrow"] & C \arrow[r, "T", "\shortmid"{marking}] & D \arrow[d, "\mathrm{id}_D"] \\ A \arrow[r, "R"', "\shortmid"{marking}] & B \arrow[r, "S"', "\shortmid"{marking}] & C \arrow[r, "T"', "\shortmid"{marking}] & D \end{tikzcd}$$

of the associator of $\mathsf{Rel}^{\mathsf{dbl}}$ at $(R, S, T)$ is the identity inclusion[14]

$$(T \diamond S) \diamond R = T \diamond (S \diamond R) \qquad \begin{tikzcd} A \times B \arrow[r, "(T\diamond S)\diamond R"] \arrow[d, equal] \arrow[dr, phantom, "\mathbin{/\!\!/}"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ A \times B \arrow[r, "T\diamond(S\diamond R)"'] & \{\mathsf{true}, \mathsf{false}\}. \end{tikzcd}$$

5.2.4.6. *The Left Unitors.*

**Definition 5.2.4.6.1.** For each horizontal morphism $R \colon A \relbar\joinrel\mapstochar\joinrel\rightarrow B$ of $\mathsf{Rel}^{\mathsf{dbl}}$, the component

$$\lambda^{\mathsf{Rel}^{\mathsf{dbl}}}_{R} \colon \chi_B \odot R \overset{\cong}{\Longrightarrow} R, \qquad \begin{tikzcd} A \arrow[r, "R", "\shortmid"{marking}] \arrow[d, "\mathrm{id}_A"'] & B \arrow[r, "\chi_B", "\shortmid"{marking}] \arrow[d, phantom, "\lambda^{\mathsf{Rel}^{\mathsf{dbl}}}_{R}\Downarrow"] & B \arrow[d, "\mathrm{id}_B"] \\ A \arrow[rr, "R"', "\shortmid"{marking}] & & B \end{tikzcd}$$

of the left unitor of $\mathsf{Rel}^{\mathsf{dbl}}$ at $R$ is the identity inclusion[15]

$$R = \chi_B \diamond R, \qquad \begin{tikzcd} A \times B \arrow[r, "\chi_B\diamond R"] \arrow[d, equal] \arrow[dr, phantom, "\mathbin{/\!\!/}"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ A \times B \arrow[r, "R"'] & \{\mathsf{true}, \mathsf{false}\}. \end{tikzcd}$$

5.2.4.7. *The Right Unitors.*

- We have $h(f(a)) \sim_T k(g(x))$;

is implied by the statement

- We have $a \sim_R x$;

since

- If $a \sim_R x$, then $f(a) \sim_S g(x)$, as $S \circ (f \times g) \subset R$;
- If $b \sim_S y$, then $h(b) \sim_T k(y)$, as $T \circ (h \times k) \subset S$, and thus, in particular:
  - If $f(a) \sim_S g(x)$, then $h(f(a)) \sim_T k(g(x))$;

[14]This is justified by Item 2 of Proposition 5.3.12.1.3.
[15]This is justified by Item 3 of Proposition 5.3.12.1.3.

**Definition 5.2.4.7.1.** For each horizontal morphism $R\colon A \nrightarrow B$ of $\mathsf{Rel}^{\mathsf{dbl}}$, the component

$$\rho^{\mathsf{Rel}^{\mathsf{dbl}}}_R \colon R \odot \nVdash_A \overset{\cong}{\Longrightarrow} R, \qquad \begin{tikzcd} A \arrow[r, "\nVdash_A", "|"] \arrow[d, "\mathrm{id}_A"'] & A \arrow[r, "R", "|"] \arrow[d, phantom, "\rho^{\mathsf{Rel}^{\mathsf{dbl}}}_R \Downarrow"] & B \arrow[d, "\mathrm{id}_B"] \\ A \arrow[rr, "R"', "|"] & & B \end{tikzcd}$$

of the right unitor of $\mathsf{Rel}^{\mathsf{dbl}}$ at $R$ is the identity inclusion[16]

$$R = R \diamond \chi_A, \qquad \begin{tikzcd} A \times B \arrow[r, "R\diamond\chi_A"] \arrow[d, equal] \arrow[dr, phantom, "\mathbin{/\!\!/}"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ A \times B \arrow[r, "R"'] & \{\mathsf{true}, \mathsf{false}\}. \end{tikzcd}$$

### 5.2.5. Properties of the Category of Relations.

**Proposition 5.2.5.1.1.** Let $A$ and $B$ be sets.

(1) *Self-Duality I.* The category Rel is self-dual, i.e. we have an equivalence
$$\mathrm{Rel}^{\mathsf{op}} \overset{\text{eq.}}{\cong} \mathrm{Rel}$$
of categories.

(2) *Self-Duality II.* The bicategory $\mathbf{Rel}$ is self-dual, i.e. we have a biequivalence
$$\mathbf{Rel}^{\mathsf{op}} \overset{\text{eq.}}{\cong} \mathbf{Rel}$$
of bicategories.

(3) *Equivalences and Isomorphisms in Rel.* Let $R\colon A \nrightarrow B$ be a relation from $A$ to $B$. The following conditions are equivalent:

  (a) The relation $R\colon A \nrightarrow B$ is an equivalence in $\mathbf{Rel}$, i.e. there exists a relation $R^{-1}\colon B \nrightarrow A$ from $B$ to $A$ together with isomorphisms
  $$R^{-1} \diamond R \cong \chi_A,$$
  $$R \diamond R^{-1} \cong \chi_B.$$

  (b) The relation $R\colon A \nrightarrow B$ is an isomorphism in Rel, i.e. there exists a relation $R^{-1}\colon B \nrightarrow A$ from $B$ to $A$ such that we have
  $$R^{-1} \diamond R = \chi_A,$$
  $$R \diamond R^{-1} = \chi_B.$$

  (c) There exists a bijection $f\colon A \xrightarrow{\cong} B$ with $R = \mathrm{Gr}(f)$.

(4) *Adjunctions in* $\mathbf{Rel}$. We have a natural bijection
$$\left\{\begin{matrix}\text{Adjunctions in } \mathbf{Rel} \\ \text{from } A \text{ to } B\end{matrix}\right\} \cong \left\{\begin{matrix}\text{Functions} \\ \text{from } A \text{ to } B\end{matrix}\right\},$$

[16]This is justified by Item 3 of Proposition 5.3.12.1.3.

with every adjunction in **Rel** being of the form $\mathrm{Gr}(f) \dashv f^{-1}$ for some function $f$.

(5) *Monads in* **Rel**. We have a natural bijection[17]

$$\left\{\begin{matrix}\text{Monads in} \\ \textbf{Rel} \text{ on } A\end{matrix}\right\} \cong \{\text{Preorders on } A\}.$$

(6) *Comonads in* **Rel**. We have a natural bijection

$$\left\{\begin{matrix}\text{Comonads in} \\ \textbf{Rel} \text{ on } A\end{matrix}\right\} \cong \{\text{Subsets of } A\}.$$

(7) *Characterisations of Monomorphisms in* $\mathsf{Rel}$. Let $R\colon A \nrightarrow B$ be a relation. The following conditions are equivalent:
   (a) The relation $R$ is a monomorphism in $\mathsf{Rel}$.
   (b) The direct image function

$$R_*\colon \mathcal{P}(A) \to \mathcal{P}(B)$$

   associated to $R$ is injective.
   (c) The direct image with compact support function

$$R_!\colon \mathcal{P}(A) \to \mathcal{P}(B)$$

   associated to $R$ is injective.

   Moreover, if $R$ is a monomorphism, then it satisfies the following condition, and the converse holds if $R$ is total:
   ($\star$) For each $a, a' \in A$, if there exists some $b \in B$ such that $a \sim_R b$ and $a' \sim_R b$, then $a = a'$.

(8) *Epimorphisms in* $\mathsf{Rel}$. Let $R\colon A \nrightarrow B$ be a relation. The following conditions are equivalent:
   (a) The relation $R$ is an epimorphism in $\mathsf{Rel}$.
   (b) The weak inverse image function

$$R^{-1}\colon \mathcal{P}(B) \to \mathcal{P}(A)$$

   associated to $R$ is injective.
   (c) The strong inverse image function

$$R_{-1}\colon \mathcal{P}(B) \to \mathcal{P}(A)$$

   associated to $R$ is injective.
   (d) The function $R\colon A \to \mathcal{P}(B)$ is "surjective on singletons":
      ($\star$) For each $b \in B$, there exists some $a \in A$ such that $R(a) = \{b\}$.

(9) *As a Kleisli Category.* We have an isomorphism of categories

$$\mathrm{Rel} \cong \mathsf{FreeAlg}_{\mathcal{P}},$$

   where $\mathcal{P}$ is the powerset monad of **??**.

(10) *Co/Completeness (Or Lack Thereof).* The category Rel is not co/complete, but admits some co/limits:
   (a) *Zero Objects.* The category Rel has a zero object, the empty set $\emptyset$.

[17] See also Section 5.6 for an extension of this correspondence to "relative monads on **Rel**".

(b) *Co/Products.* The category Rel has co/products, both given by disjoint union of sets.
(c) *Lack of Co/Equalisers.* The category Rel does not have co/equalisers.
(d) *Limits of Graphs of Functions.* The category Rel has limits whose arrows are all graphs of functions.
(e) *Colimits of Graphs of Functions.* The category Rel has colimits whose arrows are all graphs of functions, and these agree with the corresponding limits in $\mathsf{Sets}$.

(11) *Existence of Right Kan Extensions.* The right Kan extension

$$\mathrm{Ran}_R\colon \mathrm{Rel}(A,X)\to\mathrm{Rel}(B,X)$$

along a relation $R\colon A \nrightarrow B$ in $\mathbf{\mathsf{Rel}}$ exists and is given by

$$\mathrm{Ran}_R(S)\stackrel{\mathrm{def}}{=}\int_{a\in A}\mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R^a_{-1},S^a_{-2})$$

for each $S\in\mathrm{Rel}(A,X)$, so that the following conditions are equivalent:
(a) We have $b\sim_{\mathrm{Ran}_R(S)} x$.
(b) For each $a\in A$, if $a\sim_R b$, then $a\sim_S x$.

(12) *Existence of Right Kan Lifts.* The right Kan lift

$$\mathrm{Rift}_R\colon \mathrm{Rel}(X,B)\to\mathrm{Rel}(X,A)$$

along a relation $R\colon A \nrightarrow B$ in $\mathbf{\mathsf{Rel}}$ exists and is given by

$$\mathrm{Rift}_R(S)\stackrel{\mathrm{def}}{=}\int_{b\in B}\mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^{-2}_b,S^{-1}_b\right)$$

for each $S\in\mathrm{Rel}(X,B)$, so that the following conditions are equivalent:
(a) We have $x\sim_{\mathrm{Rift}_R(S)} a$.
(b) For each $b\in B$, if $a\sim_R b$, then $x\sim_S b$.

(13) *Closedness.* The bicategory $\mathbf{\mathsf{Rel}}$ is a closed bicategory, there being, for each $R\colon A\nrightarrow B$ and set $X$, a pair of adjunctions

$$(R^*\dashv\mathrm{Ran}_R)\colon\quad \begin{tikzcd}\mathrm{Rel}(B,X) \arrow[r, bend left, "R^*"] \arrow[r, phantom, "\perp"] & \mathrm{Rel}(A,X), \arrow[l, bend left, "\mathrm{Ran}_R"]\end{tikzcd}$$

$$(R_*\dashv\mathrm{Rift}_R)\colon\quad \begin{tikzcd}\mathrm{Rel}(X,A) \arrow[r, bend left, "R_*"] \arrow[r, phantom, "\perp"] & \mathrm{Rel}(X,B), \arrow[l, bend left, "\mathrm{Rift}_R"]\end{tikzcd}$$

witnessed by bijections

$$\mathbf{Rel}(S\diamond R,T)\cong\mathbf{Rel}(S,\mathrm{Ran}_R(T)),$$
$$\mathbf{Rel}(R\diamond U,V)\cong\mathbf{Rel}(U,\mathrm{Rift}_R(V)),$$

natural in $S\in\mathrm{Rel}(B,X)$, $T\in\mathrm{Rel}(A,X)$, $U\in\mathrm{Rel}(X,A)$, and $V\in\mathrm{Rel}(X,B)$.

**Proof.** *Item 1, Self-Duality I*: Omitted.
*Item 2, Self-Duality II*: Omitted.

*Item 3, Equivalences and Isomorphisms in Rel*: We claim that Items 3a to 3c are indeed equivalent:

- *Item 3a $\iff$ Item 3b:* This follows from the fact that **Rel** is locally posetal, so that natural isomorphisms and equalities of 1-cells in **Rel** coincide.
- *Item 3b $\implies$ Item 3c:* The equalities in Item 3b imply $R \dashv R^{-1}$, and thus by Item 4, there exists a function $f_R \colon A \to B$ associated to $R$, where, for each $a \in A$, the image $f_R(a)$ of $a$ by $f_R$ is the unique element of $R(a)$, which implies $R = \mathrm{Gr}(f_R)$ in particular. Furthermore, we have $R^{-1} = f_R^{-1}$ (as in Definition 5.3.2.1.1). The conditions from Item 3b then become the following:
$$f_R^{-1} \diamond f_R = \chi_A,$$
$$f_R \diamond f_R^{-1} = \chi_B.$$
All that is left is to show then is that $f_R$ is a bijection:
  - *The Function $f_R$ Is Injective.* Let $a, b \in A$ and suppose that $f_R(a) = f_R(b)$. Since $a \sim_R f_R(a)$ and $f_R(a) = f_R(b) \sim_{R^{-1}} b$, the condition $f_R^{-1} \diamond f_R = \chi_A$ implies that $a = b$, showing $f_R$ to be injective.
  - *The Function $f_R$ Is Surjective.* Let $b \in B$. Applying the condition $f_R \diamond f_R^{-1} = \chi_B$ to $(b, b)$, it follows that there exists some $a \in A$ such that $f_R^{-1}(b) = a$ and $f_R(a) = b$. This shows $f_R$ to be surjective.
- *Item 3c $\implies$ Item 3b:* By Item 2, we have an adjunction $\mathrm{Gr}(f) \dashv f^{-1}$, giving inclusions
$$\chi_A \subset f^{-1} \diamond \mathrm{Gr}(f),$$
$$\mathrm{Gr}(f) \diamond f^{-1} \subset \chi_B.$$
We claim the reverse inclusions are also true:
  - *$f^{-1} \diamond Gr(f) \subset \chi_A$:* This is equivalent to the statement that if $f(a) = b$ and $f^{-1}(b) = a'$, then $a = a'$, which follows from the injectivity of $f$.
  - *$\chi_B \subset Gr(f) \diamond f^{-1}$:* This is equivalent to the statement that given $b \in B$ there exists some $a \in A$ such that $f^{-1}(b) = a$ and $f(a) = b$, which follows from the surjectivity of $f$.

*Item 4, Adjunctions in **Rel***: We proceed step by step:

(1) *From Adjunctions in **Rel** to Functions.* An adjunction in **Rel** from $A$ to $B$ consists of a pair of relations
$$R \colon A \nrightarrow B,$$
$$S \colon B \nrightarrow A,$$
together with inclusions
$$\chi_A \subset S \diamond R,$$
$$R \diamond S \subset \chi_B.$$
We claim that these conditions imply that $R$ is total and functional, i.e. that $R(a)$ is a singleton for each $a \in A$:

(a) $R(a)$ *Has an Element.* Given $a \in A$, since $\chi_A \subset S \diamond R$, we must have $\{a\} \subset S(R(a))$, implying that there exists some $b \in B$ such that $a \sim_R b$ and $b \sim_S a$, and thus $R(a) \neq \emptyset$, as $b \in R(a)$.

(b) $R(a)$ *Has No More Than One Element.* Suppose that we have $a \sim_R b$ and $a \sim_R b'$ for $b, b' \in B$. We claim that $b = b'$:

(i) Since $\chi_A \subset S \diamond R$, there exists some $k \in B$ such that $a \sim_R k$ and $k \sim_S a$.

(ii) Since $R \diamond S \subset \chi_B$, if $b'' \sim_S a'$ and $a' \sim_R b'''$, then $b'' = b'''$.

(iii) Applying the above to $b'' = k$, $b''' = b$, and $a' = a$, since $k \sim_S a$ and $a \sim_R b'$, we have $k = b$.

(iv) Similarly $k = b'$.

(v) Thus $b = b'$.

Together, the above two items show $R(a)$ to be a singleton, being thus given by $\mathrm{Gr}(f)$ for some function $f \colon A \to B$, which gives a map

$$\left\{\begin{matrix}\text{Adjunctions in } \textbf{Rel} \\ \text{from } A \text{ to } B\end{matrix}\right\} \to \left\{\begin{matrix}\text{Functions} \\ \text{from } A \text{ to } B\end{matrix}\right\}.$$

Moreover, by uniqueness of adjoints (**??** of **??**), this implies also that $S = f^{-1}$.

(2) *From Functions to Adjunctions in* **Rel**. By Item 2 of Proposition 5.3.1.1.2, every function $f \colon A \to B$ gives rise to an adjunction $\mathrm{Gr}(f) \dashv f^{-1}$ in Rel, giving a map

$$\left\{\begin{matrix}\text{Functions} \\ \text{from } A \text{ to } B\end{matrix}\right\} \to \left\{\begin{matrix}\text{Adjunctions in } \textbf{Rel} \\ \text{from } A \text{ to } B\end{matrix}\right\}.$$

(3) *Invertibility: From Functions to Adjunctions Back to Functions.* We need to show that starting with a function $f \colon A \to B$, passing to $\mathrm{Gr}(f) \dashv f^{-1}$, and then passing again to a function gives $f$ again. This is clear however, since we have $a \sim_{\mathrm{Gr}(f)} b$ iff $f(a) = b$.

(4) *Invertibility: From Adjunctions to Functions Back to Adjunctions.* We need to show that, given an adjunction $R \dashv S$ in **Rel** giving rise to a function $f_{R,S} \colon A \to B$, we have

$$\begin{aligned}\mathrm{Gr}(f_{R,S}) &= R, \\ f^{-1}_{R,S} &= S.\end{aligned}$$

We check these explicitly:

- *Gr*$(f_{R,S}) = R$. We have

$$\begin{aligned}\mathrm{Gr}(f_{R,S}) &\overset{\text{def}}{=} \{(a, f_{R,S}(a)) \in A \times B \mid a \in A\} \\ &\overset{\text{def}}{=} \{(a, R(a)) \in A \times B \mid a \in A\} \\ &= R.\end{aligned}$$

- $f^{-1}_{R,S} = S$. We first claim that, given $a \in A$ and $b \in B$, the following conditions are equivalent:
  - We have $a \sim_R b$.
  - We have $b \sim_S a$.

  Indeed:

- *If $a \sim_R b$, then $b \sim_S a$:* Since $\chi_A \subset S \diamond R$, there exists $k \in B$ such that $a \sim_R k$ and $k \sim_S a$, but since $a \sim_R b$ and $R$ is functional, we have $k = b$ and thus $b \sim_S a$.
- *If $b \sim_S a$, then $a \sim_R b$:* First note that since $R$ is total we have $a \sim_R b'$ for some $b' \in B$. Now, since $R \diamond S \subset \chi_B$, $b \sim_S a$, and $a \sim_R b'$, we have $b = b'$, and thus $a \sim_R b$.

Having show this, we now have

$$\begin{aligned} f^{-1}_{R,S}(b) &\overset{\text{def}}{=} \{a \in A \mid f_{R,S}(a) = b\} \\ &\overset{\text{def}}{=} \{a \in A \mid a \sim_R b\} \\ &= \{a \in A \mid b \sim_S a\} \\ &\overset{\text{def}}{=} S(b). \end{aligned}$$

for each $b \in B$, showing $f^{-1}_{R,S} = S$.

This finishes the proof.

*Item 5, Monads in* **Rel**: A monad in **Rel** on $A$ consists of a relation $R \colon A \nrightarrow A$ together with maps

$$\begin{aligned} \mu_R &\colon R \diamond R \subset R, \\ \eta_R &\colon \chi_A \subset R \end{aligned}$$

making the diagrams

$$\begin{tikzcd} \chi_A \diamond R \arrow[r, "\eta_R \diamond \mathrm{id}_R", "\shortmid" marking] \arrow[dr, equal, "\lambda^{\mathbf{Rel}(A,B)}_R"'] & R \diamond R \arrow[d, "\mu_R", "\shortmid" marking] \\ & R \end{tikzcd} \qquad \begin{tikzcd} & R \diamond (R \diamond R) \arrow[dl, equal, "\alpha^{\mathbf{Rel}(A,B)}_{R,R,R}"'] \arrow[dr, "\mathrm{id}_R \diamond \mu_R", "\shortmid" marking] & \\ (R \diamond R) \diamond R \arrow[d, "\mu_R \diamond \mathrm{id}_R"', "\shortmid" marking] & & R \diamond R \arrow[d, "\mu_R", "\shortmid" marking] \\ R \diamond R \arrow[rr, "\mu_R"', "\shortmid" marking] & & R \end{tikzcd} \qquad \begin{tikzcd} R \diamond \chi_A \arrow[r, "\mathrm{id}_R \diamond \eta_R", "\shortmid" marking] \arrow[dr, equal, "\rho^{\mathbf{Rel}(A,B)}_R"'] & R \diamond R \arrow[d, "\mu_R", "\shortmid" marking] \\ & R \end{tikzcd}$$

commute. However, since all morphisms involved are inclusions, the commutativity of the above diagrams is automatic, and hence all that is left is the data of the two maps $\mu_R$ and $\eta_R$, which correspond respectively to the following conditions:

(1) For each $a, b, c \in A$, if $a \sim_R b$ and $b \sim_R c$, then $a \sim_R c$.
(2) For each $a \in A$, we have $a \sim_R a$.

These are exactly the requirements for $R$ to be a preorder (**??**). Conversely any preorder $\preceq$ gives rise to a pair of maps $\mu_\preceq$ and $\eta_\preceq$, forming a monad on $A$.

*Item 6, Comonads in* **Rel**: A comonad in **Rel** on $A$ consists of a relation $R \colon A \nrightarrow A$ together with maps

$$\begin{aligned} \Delta_R &\colon R \subset R \diamond R, \\ \epsilon_R &\colon R \subset \chi_A \end{aligned}$$

making the diagrams

$$
\begin{tikzcd}
R \arrow[r, "\Delta_R", "|" marking] \arrow[dr, equals, "\lambda^{\mathbf{Rel}(A,B),-1}_R"'] & R \diamond R \arrow[d, "\epsilon_R \diamond \mathrm{id}_R", "|" marking] \\
 & \chi_A \diamond R
\end{tikzcd}
\qquad
\begin{tikzcd}
 & R \diamond R \arrow[dr, "\mathrm{id}_R \diamond \Delta_R", "|" marking] & \\
R \arrow[ur, "\Delta_R", "|" marking] \arrow[d, "\Delta_R"', "|" marking] & & R \diamond (R \diamond R) \arrow[d, equals, "\alpha^{\mathbf{Rel}(A,B),-1}_{R,R,R}"] \\
R \diamond R \arrow[rr, "\Delta_R \diamond \mathrm{id}_R"', "|" marking] & & (R \diamond R) \diamond R
\end{tikzcd}
\qquad
\begin{tikzcd}
R \arrow[r, "\Delta_R", "|" marking] \arrow[dr, equals, "\rho^{\mathbf{Rel}(A,B),-1}_R"'] & R \diamond R \arrow[d, "\mathrm{id}_R \diamond \epsilon_R", "|" marking] \\
 & R \diamond \chi_A
\end{tikzcd}
$$

commute. However, since all morphisms involved are inclusions, the commutativity of the above diagrams is automatic, and hence all that is left is the data of the two maps $\Delta_R$ and $\epsilon_R$, which correspond respectively to the following conditions:

(1) For each $a, b \in A$, if $a \sim_R b$, then there exists some $k \in A$ such that $a \sim_R k$ and $k \sim_R b$.
(2) For each $a, b \in A$, if $a \sim_R b$, then $a = b$.

Taking $k = b$ in the first condition above shows it to be trivially satisfied, while the second condition implies $R \subset \Delta_A$, i.e. $R$ must be a subset of $A$. Conversely, any subset $U$ of $A$ satisfies $U \subset \Delta_A$, defining a comonad as above.

*Item 7, Monomorphisms in* $\mathsf{Rel}$: Firstly note that Items 7b and 7c are equivalent by Item 7 of Proposition 5.5.1.1.3. We then claim that Items 7a and 7b are also equivalent:

- *Item 7a* $\implies$ *Item 7b:* Let $U, V \in \mathcal{P}(A)$ and consider the diagram

$$\mathrm{pt} \underset{V}{\overset{U}{\rightrightarrows}} A \overset{R}{\longrightarrow} B.$$

By Remark 5.5.1.1.2, we have

$$R_*(U) = R \diamond U,$$
$$R_*(V) = R \diamond V.$$

Now, if $R \diamond U = R \diamond V$, i.e. $R_*(U) = R_*(V)$, then $U = V$ since $R$ is assumed to be a monomorphism, showing $R_*$ to be injective.

- *Item 7b* $\implies$ *Item 7a:* Conversely, suppose that $R_*$ is injective, consider the diagram

$$K \underset{T}{\overset{S}{\rightrightarrows}} A \overset{R}{\longrightarrow} B,$$

and suppose that $R \diamond S = R \diamond T$. Note that, since $R_*$ is injective, given a diagram of the form

$$\mathrm{pt} \underset{V}{\overset{U}{\rightrightarrows}} A \overset{R}{\longrightarrow} B,$$

if $R_*(U) = R \diamond U = R \diamond V = R_*(V)$, then $U = V$. In particular, for each $k \in K$, we may consider the diagram

$$\mathrm{pt} \xrightarrow{[k]} K \underset{T}{\overset{S}{\rightrightarrows}} A \xrightarrow{R} B,$$

for which we have $R \diamond S \diamond [k] = R \diamond T \diamond [k]$, implying that we have

$$S(k) = S \diamond [k] = T \diamond [k] = T(k)$$

for each $k \in K$, implying $S = T$, and thus $R$ is a monomorphism.

We can also prove this in a more abstract way, following [MSE 350788]:

- *Item 7a* $\Longrightarrow$ *Item 7b:* Assume that $R$ is a monomorphism.
  - We first notice that the functor $\mathrm{Rel}(\mathrm{pt}, -)\colon \mathrm{Rel} \to \mathsf{Sets}$ maps $R$ to $R_*$ by Remark 5.5.1.1.2.
  - Since $\mathrm{Rel}(\mathrm{pt}, -)$ preserves all limits by **??** of **??**, it follows by **??** of **??** that $\mathrm{Rel}(\mathrm{pt}, -)$ also preserves monomorphisms.
  - Since $R$ is a monomorphism and $\mathrm{Rel}(\mathrm{pt}, -)$ maps $R$ to $R_*$, it follows that $R_*$ is also a monomorphism.
  - Since the monomorphisms in $\mathsf{Sets}$ are precisely the injections (**??** of **??**), it follows that $R_*$ is injective.
- *Item 7b* $\Longrightarrow$ *Item 7a:* Assume that $R_*$ is injective.
  - We first notice that the functor $\mathrm{Rel}(\mathrm{pt}, -)\colon \mathrm{Rel} \to \mathsf{Sets}$ maps $R$ to $R_*$ by Remark 5.5.1.1.2.
  - Since the monomorphisms in $\mathsf{Sets}$ are precisely the injections (**??** of **??**), it follows that $R_*$ is a monomorphism.
  - Since $\mathrm{Rel}(\mathrm{pt}, -)$ is faithful, it follows by **??** of **??** that $\mathrm{Rel}(\mathrm{pt}, -)$ reflects monomorphisms.
  - Since $R_*$ is a monomorphism and $\mathrm{Rel}(\mathrm{pt}, -)$ maps $R$ to $R_*$, it follows that $R$ is also a monomorphism.

Finally, we prove the second part of the statement. Assume that $R$ is a monomorphism, let $a, a' \in A$ such that $a \sim_R b$ and $a' \sim_R b$ for some $b \in B$, and consider the diagram

$$\mathrm{pt} \underset{[a']}{\overset{[a]}{\rightrightarrows}} A \xrightarrow{R} B.$$

Since $\star \sim_{[a]} a$ and $a \sim_R b$, we have $\star \sim_{R \diamond [a]} b$. Similarly, $\star \sim_{R \diamond [a']} b$. Thus $R \diamond [a] = R \diamond [a']$, and since $R$ is a monomorphism, we have $[a] = [a']$, i.e. $a = a'$.

Conversely, assume the condition

($\star$) For each $a, a' \in A$, if there exists some $b \in B$ such that $a \sim_R b$ and $a' \sim_R b$, then $a = a'$,

consider the diagram

$$K \underset{T}{\overset{S}{\rightrightarrows}} A \xrightarrow{R} B,$$

and let $(k, a) \in S$. Since $R$ is total and $a \in A$, there exists some $b \in B$ such that $a \sim_R b$. In this case, we have $k \sim_{R \diamond S} b$, and since $R \diamond S = R \diamond T$, we have also $k \sim_{R \diamond T} b$. Thus there must exist some $a' \in A$ such that $k \sim_T a'$

and $a' \sim_R b$. However, since $a, a' \sim_R b$, we must have $a = a'$, and thus $(k, a) \in T$ as well.
A similar argument shows that if $(k, a) \in T$, then $(k, a) \in S$, and thus $S = T$ and $R$ is a monomorphism.
*Item 8, Epimorphisms in* Rel: Firstly note that Items 8b and 8c are equivalent by Item 7 of Proposition 5.5.2.1.3. We then claim that Items 8a and 8b are also equivalent:

- *Item 8a* $\Longrightarrow$ *Item 8b:* Let $U, V \in \mathcal{P}(A)$ and consider the diagram
$$A \overset{R}{\longrightarrow\!\!\!\!\!\!+\;\;} B \underset{V}{\overset{U}{\rightrightarrows}} \mathrm{pt}.$$
By Remark 5.5.1.1.2, we have
$$\begin{aligned} R^{-1}(U) &= U \diamond R, \\ R^{-1}(V) &= V \diamond R. \end{aligned}$$
Now, if $U \diamond R = V \diamond R$, i.e. $R^{-1}(U) = R^{-1}(V)$, then $U = V$ since $R$ is assumed to be an epimorphism, showing $R^{-1}$ to be injective.
- *Item 8b* $\Longrightarrow$ *Item 8a:* Conversely, suppose that $R^{-1}$ is injective, consider the diagram
$$A \overset{R}{\longrightarrow\!\!\!\!\!\!+\;\;} B \underset{T}{\overset{S}{\rightrightarrows}} K,$$
and suppose that $S \diamond R = T \diamond R$. Note that, since $R^{-1}$ is injective, given a diagram of the form
$$A \overset{R}{\longrightarrow\!\!\!\!\!\!+\;\;} B \underset{V}{\overset{U}{\rightrightarrows}} \mathrm{pt},$$
if $R^{-1}(U) = U \diamond R = V \diamond R = R^{-1}(V)$, then $U = V$. In particular, for each $k \in K$, we may consider the diagram
$$A \overset{R}{\longrightarrow\!\!\!\!\!\!+\;\;} B \underset{T}{\overset{S}{\rightrightarrows}} K \overset{[k]}{\longrightarrow\!\!\!\!\!\!+\;\;} \mathrm{pt},$$
for which we have $[k] \diamond S \diamond R = [k] \diamond T \diamond R$, implying that we have
$$S^{-1}(k) = [k] \diamond S = [k] \diamond T = T^{-1}(k)$$
for each $k \in K$, implying $S = T$, and thus $R$ is an epimorphism.

We can also prove this in a more abstract way, following [MSE 350788]:

- *Item 8a* $\Longrightarrow$ *Item 8b:* Assume that $R$ is an epimorphism.
  - We first notice that the functor $\mathrm{Rel}(-, \mathrm{pt})\colon \mathrm{Rel}^{\mathsf{op}} \to \mathsf{Sets}$ maps $R$ to $R^{-1}$ by Remark 5.5.3.1.2.
  - Since $\mathrm{Rel}(-, \mathrm{pt})$ preserves limits by **??** of **??**, it follows by **??** of **??** that $\mathrm{Rel}(-, \mathrm{pt})$ also preserves monomorphisms.
  - That is: $\mathrm{Rel}(-, \mathrm{pt})$ sends monomorphisms in $\mathrm{Rel}^{\mathsf{op}}$ to monomorphisms in $\mathsf{Sets}$.
  - The monomorphisms $\mathrm{Rel}^{\mathsf{op}}$ are precisely the epimorphisms in Rel by **??** of **??**.

  - Since $R$ is an epimorphism and $\mathrm{Rel}(-, \mathrm{pt})$ maps $R$ to $R^{-1}$, it follows that $R^{-1}$ is a monomorphism.
  - Since the monomorphisms in $\mathsf{Sets}$ are precisely the injections (**??** of **??**), it follows that $R^{-1}$ is injective.
- *Item 8b* $\Longrightarrow$ *Item 8a:* Assume that $R^{-1}$ is injective.
  - We first notice that the functor $\mathrm{Rel}(-, \mathrm{pt}) \colon \mathrm{Rel}^{\mathsf{op}} \to \mathsf{Sets}$ maps $R$ to $R^{-1}$ by Remark 5.5.3.1.2.
  - Since the monomorphisms in $\mathsf{Sets}$ are precisely the injections (**??** of **??**), it follows that $R^{-1}$ is a monomorphism.
  - Since $\mathrm{Rel}(-, \mathrm{pt})$ is faithful, it follows by **??** of **??** that $\mathrm{Rel}(, \mathrm{pt})$ reflects monomorphisms.
  - That is: $\mathrm{Rel}(-, \mathrm{pt})$ reflects monomorphisms in $\mathsf{Sets}$ to monomorphisms in $\mathrm{Rel}^{\mathsf{op}}$.
  - The monomorphisms $\mathrm{Rel}^{\mathsf{op}}$ are precisely the epimorphisms in Rel by **??** of **??**.
  - Since $R^{-1}$ is a monomorphism and $\mathrm{Rel}(-, \mathrm{pt})$ maps $R$ to $R^{-1}$, it follows that $R$ is an epimorphism.

Finally, we claim that Items 8b and 8d are also equivalent, following [MO 350788]:

- *Item 8b* $\Longrightarrow$ *Item 8d:* Since $B \setminus \{b\} \subset B$ and $R^{-1}$ is injective, we have $R^{-1}(B \setminus \{b\}) \subsetneq R^{-1}(B)$. So taking some $a \in R^{-1}(B) \setminus R^{-1}(B \setminus \{b\})$ we get an element of $A$ such that $R(a) = \{b\}$.
- *Item 8d* $\Longrightarrow$ *Item 8b:* Let $U, V \subset B$ with $U \neq V$. Without loss of generality, we can assume $U \setminus V \neq \emptyset$; otherwise just swap $U$ and $V$. Let then $b \in U \setminus V$. By assumption, there exists an $a \in A$ with $R(a) = \{b\}$. Then $a \in R^{-1}(U)$ but $a \notin R^{-1}(V)$, and thus $R^{-1}(U) \neq R^{-1}(V)$, showing $R^{-1}$ to be injective.

*Item 9, As a Kleisli Category*: Omitted.
*Item 10, Co/Completeness (Or Lack Thereof)*: Omitted.
*Item 11, Existence of Right Kan Extensions*: We have

$$
\begin{aligned}
\mathrm{Hom}_{\mathbf{Rel}(A,X)}(S \diamond R, T) &\cong \int_{a \in A} \int_{x \in X} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}((S \diamond R)^a_x, T^a_x) \\
&\cong \int_{a \in A} \int_{x \in X} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(\left(\int^{b \in B} S^b_x \times R^a_b\right), T^a_x\right) \\
&\cong \int_{a \in A} \int_{x \in X} \int_{b \in B} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(S^b_x \times R^a_b, T^a_x) \\
&\cong \int_{a \in A} \int_{x \in X} \int_{b \in B} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(S^b_x, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R^a_b, T^a_x)) \\
&\cong \int_{b \in B} \int_{x \in X} \int_{a \in A} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(S^b_x, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R^a_b, T^a_x)) \\
&\cong \int_{b \in B} \int_{x \in X} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(S^b_x, \int_{a \in A} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R^a_b, T^a_x)\right) \\
&\cong \mathrm{Hom}_{\mathbf{Rel}(B,X)}\left(S, \int_{a \in A} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R^a_{-_1}, T^a_{-_2})\right)
\end{aligned}
$$

naturally in each $S \in \mathbf{Rel}(B, X)$ and each $T \in \mathbf{Rel}(A, X)$, showing that

$$\int_{a\in A} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R^{a}_{-_1}, T^{a}_{-_2})$$

is right adjoint to the precomposition functor $-\diamond R$, being thus the right Kan extension along $R$. Here we have used the following results, respectively (i.e. for each $\cong$ sign):

(1) Item 1 of Proposition 5.1.1.1.4;
(2) Definition 5.3.12.1.1;
(3) **??** of **??**;
(4) **??**;
(5) **??** of **??**;
(6) **??** of **??**;
(7) Item 1 of Proposition 5.1.1.1.4.

*Item 12, Existence of Right Kan Lifts*: We have

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{Rel}(X,B)}(R\diamond S, T) &\cong \int_{x\in X}\int_{b\in B} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}((R\diamond S)^{x}_{b}, T^{x}_{b})\\
&\cong \int_{x\in X}\int_{b\in B} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(\left(\int^{a\in A} R^{a}_{b}\times S^{x}_{a}\right), T^{x}_{b}\right)\\
&\cong \int_{x\in X}\int_{b\in B}\int_{a\in A} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R^{a}_{b}\times S^{x}_{a}, T^{x}_{b})\\
&\cong \int_{x\in X}\int_{b\in B}\int_{a\in A} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(S^{x}_{a}, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R^{a}_{b}, T^{x}_{b}))\\
&\cong \int_{x\in X}\int_{a\in A}\int_{b\in B} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(S^{x}_{a}, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R^{a}_{b}, T^{x}_{b}))\\
&\cong \int_{x\in X}\int_{a\in A} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(S^{x}_{a}, \int_{b\in B}\mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R^{a}_{b}, T^{x}_{b})\right)\\
&\cong \mathrm{Hom}_{\mathbf{Rel}(X,A)}\left(S, \int_{b\in B}\mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R^{-_2}_{b}, T^{-_1}_{b})\right)
\end{aligned}$$

naturally in each $S \in \mathbf{Rel}(X, A)$ and each $T \in \mathbf{Rel}(X, B)$, showing that

$$\int_{b\in B} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^{-_2}_{b}, T^{-_1}_{b}\right)$$

is right adjoint to the postcomposition functor $R\diamond -$, being thus the right Kan lift along $R$. Here we have used the following results, respectively (i.e. for each $\cong$ sign):

(1) Item 1 of Proposition 5.1.1.1.4;
(2) Definition 5.3.12.1.1;
(3) **??** of **??**;
(4) **??**;
(5) **??** of **??**;
(6) **??** of **??**;
(7) Item 1 of Proposition 5.1.1.1.4.

*Item 13, Closedness*: This has been proved as part of the proof of Items 11 and 12. □

## 5.3. Constructions With Relations

**5.3.1. The Graph of a Function.** Let $f\colon A \to B$ be a function.

**Definition 5.3.1.1.1.** The **graph of** $f$ is the relation $\mathrm{Gr}(f)\colon A \nrightarrow B$ defined as follows:[18]

- Viewing relations from $A$ to $B$ as subsets of $A \times B$, we define
$$\mathrm{Gr}(f) \overset{\text{def}}{=} \{(a, f(a)) \in A \times B \mid a \in A\};$$
- Viewing relations from $A$ to $B$ as functions $A \times B \to \{\mathsf{true}, \mathsf{false}\}$, we define
$$[\mathrm{Gr}(f)](a, b) \overset{\text{def}}{=} \begin{cases} \mathsf{true} & \text{if } b = f(a), \\ \mathsf{false} & \text{otherwise} \end{cases}$$
for each $(a, b) \in A \times B$;
- Viewing relations from $A$ to $B$ as functions $A \to \mathcal{P}(B)$, we define
$$[\mathrm{Gr}(f)](a) \overset{\text{def}}{=} \{f(a)\}$$
for each $a \in A$, i.e. we define $\mathrm{Gr}(f)$ as the composition
$$A \xrightarrow{f} B \xrightarrow{\chi_B} \mathcal{P}(B).$$

**Proposition 5.3.1.1.2.** Let $f\colon A \to B$ be a function.

(1) *Functoriality.* The assignment $A \mapsto \mathrm{Gr}(A)$ defines a functor
$$\mathrm{Gr}\colon \mathsf{Sets} \to \mathrm{Rel}$$
where
- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have
$$\mathrm{Gr}(A) \overset{\text{def}}{=} A;$$
- *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathsf{Sets})$, the action on Hom-sets
$$\mathrm{Gr}_{A,B}\colon \mathsf{Sets}(A, B) \to \underbrace{\mathrm{Rel}(\mathrm{Gr}(A), \mathrm{Gr}(B))}_{\overset{\text{def}}{=}\mathrm{Rel}(A,B)}$$
of Gr at $(A, B)$ is defined by
$$\mathrm{Gr}_{A,B}(f) \overset{\text{def}}{=} \mathrm{Gr}(f),$$
where $\mathrm{Gr}(f)$ is the graph of $f$ as in Definition 5.3.1.1.1.

In particular:
- *Preservation of Identities.* We have
$$\mathrm{Gr}(\mathrm{id}_A) = \chi_A$$
for each $A \in \mathrm{Obj}(\mathsf{Sets})$.
- *Preservation of Composition.* We have
$$\mathrm{Gr}(g \circ f) = \mathrm{Gr}(g) \diamond \mathrm{Gr}(f)$$
for each pair of functions $f\colon A \to B$ and $g\colon B \to C$.

[18] *Further Notation:* We write $\mathrm{Gr}(A)$ for $\mathrm{Gr}(\mathrm{id}_A)$, and call it the **graph** of $A$.

(2) *Adjointness Inside* **Rel**. We have an adjunction

$$\left(\mathrm{Gr}(f) \dashv f^{-1}\right)\colon \quad \begin{tikzcd} A \arrow[r, bend left, "\mathrm{Gr}(f)"] \arrow[r, phantom, "\perp"] & B \arrow[l, bend left, "f^{-1}"] \end{tikzcd}$$

in **Rel**, where $f^{-1}$ is the inverse of $f$ of Definition 5.3.2.1.1.

(3) *Adjointness.* We have an adjunction

$$(\mathrm{Gr} \dashv \mathcal{P}_*)\colon \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "\mathrm{Gr}"] \arrow[r, phantom, "\perp"] & \mathrm{Rel}, \arrow[l, bend left, "\mathcal{P}_*"] \end{tikzcd}$$

witnessed by a bijection of sets

$$\mathrm{Rel}(\mathrm{Gr}(A), B) \cong \mathsf{Sets}(A, \mathcal{P}(B))$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$ and $B \in \mathrm{Obj}(\mathrm{Rel})$.

(4) *Interaction With Inverses.* We have

$$\mathrm{Gr}(f)^\dagger = f^{-1},$$
$$\left(f^{-1}\right)^\dagger = \mathrm{Gr}(f).$$

(5) *Cocontinuity.* The functor $\mathrm{Gr}\colon \mathsf{Sets} \to \mathrm{Rel}$ of Item 1 preserves colimits.

(6) *Characterisations.* Let $R\colon A \nrightarrow B$ be a relation. The following conditions are equivalent:
  (a) There exists a function $f\colon A \to B$ such that $R = \mathrm{Gr}(f)$.
  (b) The relation $R$ is total and functional.
  (c) The weak and strong inverse images of $R$ agree, i.e. we have $R^{-1} = R_{-1}$.
  (d) The relation $R$ has a right adjoint $R^\dagger$ in Rel.

**Proof.** *Item 1, Functoriality*: Clear.

*Item 2, Adjointness Inside* **Rel**: We need to check that there are inclusions

$$\chi_A \subset f^{-1} \diamond \mathrm{Gr}(f),$$
$$\mathrm{Gr}(f) \diamond f^{-1} \subset \chi_B.$$

These correspond respectively to the following conditions:

(1) For each $a \in A$, there exists some $b \in B$ such that $a \sim_{\mathrm{Gr}(f)} b$ and $b \sim_{f^{-1}} a$.

(2) For each $a, b \in A$, if $a \sim_{\mathrm{Gr}(f)} b$ and $b \sim_{f^{-1}} a$, then $a = b$.

In other words, the first condition states that the image of any $a \in A$ by $f$ is nonempty, whereas the second condition states that $f$ is not multivalued. As $f$ is a function, both of these statements are true, and we are done.

*Item 3, Adjointness*: The stated bijection follows from Remark 5.1.1.1.3, with naturality being clear.

*Item 4, Interaction With Inverses*: Clear.

*Item 5, Cocontinuity*: Omitted.

*Item 6, Characterisations*: We claim that Items 6a to 6d are indeed equivalent:

- *Item 6a* $\iff$ *Item 6b*. This is shown in the proof of Item 4 of Proposition 5.2.5.1.1.

- *Item 6b* $\implies$ *Item 6c*. If $R$ is total and functional, then, for each $a \in A$, the set $R(a)$ is a singleton, implying that

$$\begin{aligned} R^{-1}(V) &\stackrel{\text{def}}{=} \{a \in A \mid R(a) \cap V \neq \emptyset\}, \\ R_{-1}(V) &\stackrel{\text{def}}{=} \{a \in A \mid R(a) \subset V\} \end{aligned}$$

  are equal for all $V \in \mathcal{P}(B)$, as the conditions $R(a) \cap V \neq \emptyset$ and $R(a) \subset V$ are equivalent when $R(a)$ is a singleton.
- *Item 6c* $\implies$ *Item 6b*. We claim that $R$ is indeed total and functional:
  - *Totality*. If we had $R(a) = \emptyset$ for some $a \in A$, then we would have $a \in R_{-1}(\emptyset)$, so that $R_{-1}(\emptyset) \neq \emptyset$. But since $R^{-1}(\emptyset) = \emptyset$, this would imply $R_{-1}(\emptyset) \neq R^{-1}(\emptyset)$, a contradiction. Thus $R(a) \neq \emptyset$ for all $a \in A$ and $R$ is total.
  - *Functionality*. If $R^{-1} = R_{-1}$, then we have

$$\begin{aligned} \{a\} &= R^{-1}(\{b\}) \\ &= R_{-1}(\{b\}) \end{aligned}$$

    for each $b \in R(a)$ and each $a \in A$, and thus $R(a) \subset \{b\}$. But since $R$ is total, we must have $R(a) = \{b\}$, and thus we see that $R$ is functional.
- *Item 6a* $\iff$ *Item 6d*. This follows from Item 4 of Proposition 5.2.5.1.1.

This finishes the proof. □

**5.3.2. The Inverse of a Function.** Let $f \colon A \to B$ be a function.

**Definition 5.3.2.1.1.** The **inverse of** $f$ is the relation $f^{-1} \colon B \nrightarrow A$ defined as follows:

- Viewing relations from $B$ to $A$ as subsets of $B \times A$, we define

$$f^{-1} \stackrel{\text{def}}{=} \left\{ \left(b, f^{-1}(b)\right) \in B \times A \;\middle|\; a \in A \right\},$$

  where

$$f^{-1}(b) \stackrel{\text{def}}{=} \{a \in A \mid f(a) = b\}$$

  for each $b \in B$.
- Viewing relations from $B$ to $A$ as functions $B \times A \to \{\mathsf{true}, \mathsf{false}\}$, we define

$$f^{-1}(b, a) \stackrel{\text{def}}{=} \begin{cases} \mathsf{true} & \text{if there exists } a \in A \text{ with } f(a) = b, \\ \mathsf{false} & \text{otherwise} \end{cases}$$

  for each $(b, a) \in B \times A$;
- Viewing relations from $B$ to $A$ as functions $B \to \mathcal{P}(A)$, we define

$$f^{-1}(b) \stackrel{\text{def}}{=} \{a \in A \mid f(a) = b\}$$

  for each $b \in B$.

**Proposition 5.3.2.1.2.** Let $f \colon A \to B$ be a function.

(1) *Functoriality*. The assignment $A \mapsto A$, $f \mapsto f^{-1}$ defines a functor

$$(-)^{-1} \colon \mathsf{Sets} \to \mathrm{Rel}$$

where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have
$$\left[(-)^{-1}\right](A) \stackrel{\text{def}}{=} A;$$
- *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathsf{Sets})$, the action on Hom-sets
$$(-)^{-1}_{A,B} \colon \mathsf{Sets}(A, B) \to \mathrm{Rel}(A, B)$$
of $(-)^{-1}$ at $(A, B)$ is defined by
$$(-)^{-1}_{A,B}(f) \stackrel{\text{def}}{=} \left[(-)^{-1}\right](f),$$
where $f^{-1}$ is the inverse of $f$ as in Definition 5.3.2.1.1.

In particular:
- *Preservation of Identities.* We have
$$\mathrm{id}_A^{-1} = \chi_A$$
for each $A \in \mathrm{Obj}(\mathsf{Sets})$.
- *Preservation of Composition.* We have
$$(g \circ f)^{-1} = g^{-1} \diamond f^{-1}$$
for pair of functions $f \colon A \to B$ and $g \colon B \to C$.

(2) *Adjointness Inside* **Rel**. We have an adjunction
$$\left(\mathrm{Gr}(f) \dashv f^{-1}\right)\colon \quad \begin{tikzcd} A \arrow[r, bend left, "\mathrm{Gr}(f)", "\shortmid"{marking}] \arrow[r, phantom, "\perp"] & B \arrow[l, bend left, "f^{-1}", "\shortmid"{marking}] \end{tikzcd}$$
in **Rel**.

(3) *Interaction With Inverses of Relations.* We have
$$\left(f^{-1}\right)^{\dagger} = \mathrm{Gr}(f),$$
$$\mathrm{Gr}(f)^{\dagger} = f^{-1}.$$

**Proof.** *Item 1, Functoriality*: Clear.
*Item 2, Adjointness Inside* **Rel**: This is proved in Item 2 of Proposition 5.3.1.1.2.
*Item 3, Interaction With Inverses of Relations*: Clear. □

**5.3.3. Representable Relations.** Let $A$ and $B$ be sets.

**Definition 5.3.3.1.1.** Let $f \colon A \to B$ and $g \colon B \to A$ be functions.[19]

---

[19]More generally, given functions
$$f \colon A \to C,$$
$$g \colon B \to D$$
and a relation $B \nrightarrow D$, we may consider the composite relation
$$A \times B \xrightarrow{f \times g} C \times D \xrightarrow{R} \{\mathsf{true}, \mathsf{false}\},$$
for which we have $a \sim_{R \circ (f \times g)} b$ iff $f(a) \sim_R g(b)$.

(1) The **representable relation associated to** $f$ is the relation $\chi_f \colon A \nrightarrow B$ defined as the composition

$$A \times B \xrightarrow{f \times \mathrm{id}_B} B \times B \xrightarrow{\chi_B} \{\mathsf{true}, \mathsf{false}\},$$

i.e. given by declaring $a \sim_{\chi_f} b$ iff $f(a) = b$.

(2) The **corepresentable relation associated to** $g$ is the relation $\chi^g \colon B \nrightarrow A$ defined as the composition

$$B \times A \xrightarrow{g \times \mathrm{id}_A} A \times A \xrightarrow{\chi_A} \{\mathsf{true}, \mathsf{false}\},$$

i.e. given by declaring $b \sim_{\chi^g} a$ iff $g(b) = a$.

**5.3.4. The Domain and Range of a Relation.** Let $A$ and $B$ be sets.

**Definition 5.3.4.1.1.** Let $R \subset A \times B$ be a relation.[20,21]

(1) The **domain of** $R$ is the subset $\mathrm{dom}(R)$ of $A$ defined by

$$\mathrm{dom}(R) \overset{\text{def}}{=} \left\{ a \in A \,\middle|\, \begin{array}{l} \text{there exists some } b \in B \\ \text{such that } a \sim_R b \end{array} \right\}.$$

(2) The **range of** $R$ is the subset $\mathrm{range}(R)$ of $B$ defined by

$$\mathrm{range}(R) \overset{\text{def}}{=} \left\{ b \in B \,\middle|\, \begin{array}{l} \text{there exists some } a \in A \\ \text{such that } a \sim_R b \end{array} \right\}.$$

**5.3.5. Binary Unions of Relations.** Let $A$ and $B$ be sets and let $R$ and $S$ be relations from $A$ to $B$.

**Definition 5.3.5.1.1.** The **union of** $R$ **and** $S$[22] is the relation $R \cup S$ from $A$ to $B$ defined as follows:

[20]Following **??**, we may compute the (characteristic functions associated to the) domain and range of a relation using the following colimit formulas:

$$\begin{aligned} \chi_{\mathrm{dom}(R)}(a) &\cong \operatorname*{colim}_{b \in B}(R^a_b) \qquad (a \in A) \\ &\cong \bigvee_{b \in B} R^a_b, \\ \chi_{\mathrm{range}(R)}(b) &\cong \operatorname*{colim}_{a \in A}(R^a_b) \qquad (b \in B) \\ &\cong \bigvee_{a \in A} R^a_b, \end{aligned}$$

where the join $\bigvee$ is taken in the poset $(\{\mathsf{true}, \mathsf{false}\}, \preceq)$ of **??**.

[21]Viewing $R$ as a function $R \colon A \to \mathcal{P}(B)$, we have

$$\begin{aligned} \mathrm{dom}(R) &\cong \operatorname*{colim}_{y \in Y}(R(y)) \\ &\cong \bigcup_{y \in Y} R(y), \\ \mathrm{range}(R) &\cong \operatorname*{colim}_{x \in X}(R(x)) \\ &\cong \bigcup_{x \in X} R(x), \end{aligned}$$

[22]*Further Terminology:* Also called the **binary union of** $R$ **and** $S$, for emphasis.

- Viewing relations from $A$ to $B$ as subsets of $A \times B$, we define[23]
  $$R \cup S \overset{\text{def}}{=} \{(a,b) \in B \times A \mid \text{we have } a \sim_R b \text{ or } a \sim_S b\}.$$
- Viewing relations from $A$ to $B$ as functions $A \to \mathcal{P}(B)$, we define
  $$[R \cup S](a) \overset{\text{def}}{=} R(a) \cup S(a)$$
  for each $a \in A$.

**Proposition 5.3.5.1.2.** Let $R$, $S$, $R_1$, and $R_2$ be relations from $A$ to $B$, and let $S_1$ and $S_2$ be relations from $B$ to $C$.

(1) *Interaction With Inverses.* We have
$$(R \cup S)^\dagger = R^\dagger \cup S^\dagger.$$
(2) *Interaction With Composition.* We have
$$(S_1 \diamond R_1) \cup (S_2 \diamond R_2) \overset{\text{poss.}}{\neq} (S_1 \cup S_2) \diamond (R_1 \cup R_2).$$

**Proof.** *Item 1, Interaction With Inverses*: Clear.
*Item 2, Interaction With Composition*: Unwinding the definitions, we see that:

(1) The condition for $(S_1 \diamond R_1) \cup (S_2 \diamond R_2)$ is:
   (a) There exists some $b \in B$ such that:
      (i) $a \sim_{R_1} b$ and $b \sim_{S_1} c$;
   or
      (i) $a \sim_{R_2} b$ and $b \sim_{S_2} c$;
(3) The condition for $(S_1 \cup S_2) \diamond (R_1 \cup R_2)$ is:
   (a) There exists some $b \in B$ such that:
      (i) $a \sim_{R_1} b$ or $a \sim_{R_2} b$;
   and
      (i) $b \sim_{S_1} c$ or $b \sim_{S_2} c$.

These two conditions may fail to agree (counterexample omitted), and thus the two resulting relations on $A \times C$ may differ. □

**5.3.6. Unions of Families of Relations.** Let $A$ and $B$ be sets and let $\{R_i\}_{i \in I}$ be a family of relations from $A$ to $B$.

**Definition 5.3.6.1.1.** The **union of the family** $\{R_i\}_{i \in I}$ is the relation $\bigcup_{i \in I} R_i$ from $A$ to $B$ defined as follows:

- Viewing relations from $A$ to $B$ as subsets of $A \times B$, we define[24]
  $$\bigcup_{i \in I} R_i \overset{\text{def}}{=} \left\{ (a,b) \in (A \times B)^{\times I} \,\middle|\, \begin{array}{l} \text{there exists some } i \in I \\ \text{such that } a \sim_{R_i} b \end{array} \right\}.$$
- Viewing relations from $A$ to $B$ as functions $A \to \mathcal{P}(B)$, we define
  $$\left[\bigcup_{i \in I} R_i\right](a) \overset{\text{def}}{=} \bigcup_{i \in I} R_i(a)$$
  for each $a \in A$.

[23] This is the same as the union of $R$ and $S$ as subsets of $A \times B$.
[24] This is the same as the union of $\{R_i\}_{i \in I}$ as a collection of subsets of $A \times B$.

**Proposition 5.3.6.1.2.** Let $A$ and $B$ be sets and let $\{R_i\}_{i\in I}$ be a family of relations from $A$ to $B$.

(1) *Interaction With Inverses.* We have

$$\left(\bigcup_{i\in I} R_i\right)^{\dagger} = \bigcup_{i\in I} R_i^{\dagger}.$$

**Proof.** *Item 1, Interaction With Inverses*: Clear. □

**5.3.7. Binary Intersections of Relations.** Let $A$ and $B$ be sets and let $R$ and $S$ be relations from $A$ to $B$.

**Definition 5.3.7.1.1.** The **intersection of** $R$ **and** $S$[25] is the relation $R\cap S$ from $A$ to $B$ defined as follows:

- Viewing relations from $A$ to $B$ as subsets of $A\times B$, we define[26]

  $R\cap S \overset{\text{def}}{=} \{(a,b)\in B\times A \mid \text{we have } a\sim_R b \text{ and } a\sim_S b\}$.

- Viewing relations from $A$ to $B$ as functions $A\to\mathcal{P}(B)$, we define

  $$[R\cap S](a) \overset{\text{def}}{=} R(a)\cap S(a)$$

  for each $a\in A$.

**Proposition 5.3.7.1.2.** Let $R$, $S$, $R_1$, and $R_2$ be relations from $A$ to $B$, and let $S_1$ and $S_2$ be relations from $B$ to $C$.

(1) *Interaction With Inverses.* We have

$$(R\cap S)^{\dagger} = R^{\dagger}\cap S^{\dagger}.$$

(2) *Interaction With Composition.* We have

$$(S_1\diamond R_1)\cap(S_2\diamond R_2) = (S_1\cap S_2)\diamond(R_1\cap R_2).$$

**Proof.** *Item 1, Interaction With Inverses*: Clear.
*Item 2, Interaction With Composition*: Unwinding the definitions, we see that:

(1) The condition for $(S_1\diamond R_1)\cap(S_2\diamond R_2)$ is:
   (a) There exists some $b\in B$ such that:
      (i) $a\sim_{R_1} b$ and $b\sim_{S_1} c$;
   and
      (i) $a\sim_{R_2} b$ and $b\sim_{S_2} c$;
(3) The condition for $(S_1\cap S_2)\diamond(R_1\cap R_2)$ is:
   (a) There exists some $b\in B$ such that:
      (i) $a\sim_{R_1} b$ and $a\sim_{R_2} b$;
   and
      (i) $b\sim_{S_1} c$ and $b\sim_{S_2} c$.

These two conditions agree, and thus so do the two resulting relations on $A\times C$. □

**5.3.8. Intersections of Families of Relations.** Let $A$ and $B$ be sets and let $\{R_i\}_{i\in I}$ be a family of relations from $A$ to $B$.

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[25] *Further Terminology:* Also called the **binary intersection of** $R$ **and** $S$, for emphasis.
[26] This is the same as the intersection of $R$ and $S$ as subsets of $A\times B$.

**Definition 5.3.8.1.1.** The **intersection of the family** $\{R_i\}_{i\in I}$ is the relation $\bigcup_{i\in I} R_i$ defined as follows:

- Viewing relations from $A$ to $B$ as subsets of $A\times B$, we define[27]

$$\bigcup_{i\in I} R_i \overset{\text{def}}{=} \left\{ (a,b)\in (A\times B)^{\times I} \;\middle|\; \begin{array}{l}\text{for each } i\in I,\\ \text{we have } a\sim_{R_i} b\end{array} \right\}.$$

- Viewing relations from $A$ to $B$ as functions $A\to\mathcal{P}(B)$, we define

$$\left[\bigcap_{i\in I} R_i\right](a) \overset{\text{def}}{=} \bigcap_{i\in I} R_i(a)$$

  for each $a\in A$.

**Proposition 5.3.8.1.2.** Let $A$ and $B$ be sets and let $\{R_i\}_{i\in I}$ be a family of relations from $A$ to $B$.

(1) *Interaction With Inverses.* We have

$$\left(\bigcap_{i\in I} R_i\right)^{\dagger} = \bigcap_{i\in I} R_i^{\dagger}.$$

**Proof.** *Item 1, Interaction With Inverses*: Clear. □

**5.3.9. Binary Products of Relations.** Let $A$, $B$, $X$, and $Y$ be sets, let $R\colon A \mathrel{+\!\!\!\!\!\rightarrow} B$ be a relation from $A$ to $B$, and let $S\colon X \mathrel{+\!\!\!\!\!\rightarrow} Y$ be a relation from $X$ to $Y$.

**Definition 5.3.9.1.1.** The **product of $R$ and $S$**[28] is the relation $R\times S$ from $A\times X$ to $B\times Y$ defined as follows:

- Viewing relations from $A\times X$ to $B\times Y$ as subsets of $(A\times X)\times(B\times Y)$, we define $R\times S$ as the Cartesian product of $R$ and $S$ as subsets of $A\times X$ and $B\times Y$;[29]
- Viewing relations from $A\times X$ to $B\times Y$ as functions $A\times X\to\mathcal{P}(B\times Y)$, we define $R\times S$ as the composition

$$A\times X \xrightarrow{R\times S} \mathcal{P}(B)\times\mathcal{P}(Y) \overset{\mathcal{P}^{\otimes}_{B,Y}}{\hookrightarrow} \mathcal{P}(B\times Y)$$

  in $\mathsf{Sets}$, i.e. by

$$[R\times S](a,x) \overset{\text{def}}{=} R(a)\times S(x)$$

  for each $(a,x)\in A\times X$.

**Proposition 5.3.9.1.2.** Let $A$, $B$, $X$, and $Y$ be sets.

(1) *Interaction With Inverses.* Let

$$R\colon A \mathrel{+\!\!\!\!\!\rightarrow} A,$$
$$S\colon X \mathrel{+\!\!\!\!\!\rightarrow} X$$

  We have

$$(R\times S)^{\dagger} = R^{\dagger}\times S^{\dagger}.$$

[27]This is the same as the intersection of $\{R_i\}_{i\in I}$ as a collection of subsets of $A\times B$.

[28]*Further Terminology:* Also called the **binary product of $R$ and $S$**, for emphasis.

[29]That is, $R\times S$ is the relation given by declaring $(a,x)\sim_{R\times S}(b,y)$ iff $a\sim_R b$ and $x\sim_S y$.

(2) *Interaction With Composition.* Let

$$\begin{aligned} R_1 &\colon A \nrightarrow B, \\ S_1 &\colon B \nrightarrow C, \\ R_2 &\colon X \nrightarrow Y, \\ S_2 &\colon Y \nrightarrow Z \end{aligned}$$

be relations. We have

$$(S_1 \diamond R_1) \times (S_2 \diamond R_2) = (S_1 \times S_2) \diamond (R_1 \times R_2).$$

**Proof.** *Item 1, Interaction With Inverses*: Unwinding the definitions, we see that:

(1) We have $(a, x) \sim_{(R\times S)^\dagger} (b, y)$ iff :
  - We have $(b, y) \sim_{R\times S} (a, x)$, i.e. iff :
    - We have $b \sim_R a$;
    - We have $y \sim_S x$;

(2) We have $(a, x) \sim_{R^\dagger \times S^\dagger} (b, y)$ iff :
  - We have $a \sim_{R^\dagger} b$ and $x \sim_{S^\dagger} y$, i.e. iff :
    - We have $b \sim_R a$;
    - We have $y \sim_S x$.

These two conditions agree, and thus the two resulting relations on $A \times X$ are equal.

*Item 2, Interaction With Composition*: Unwinding the definitions, we see that:

(1) We have $(a, x) \sim_{(S_1 \diamond R_1)\times(S_2 \diamond R_2)} (c, z)$ iff :
  (a) We have $a \sim_{S_1 \diamond R_1} c$ and $x \sim_{S_2 \diamond R_2} z$, i.e. iff :
    (i) There exists some $b \in B$ such that $a \sim_{R_1} b$ and $b \sim_{S_1} c$;
    (ii) There exists some $y \in Y$ such that $x \sim_{R_2} y$ and $y \sim_{S_2} z$;

(2) We have $(a, x) \sim_{(S_1 \times S_2)\diamond(R_1 \times R_2)} (c, z)$ iff :
  (a) There exists some $(b, y) \in B \times Y$ such that $(a, x) \sim_{R_1 \times R_2} (b, y)$ and $(b, y) \sim_{S_1 \times S_2} (c, z)$, i.e. such that:
    (i) We have $a \sim_{R_1} b$ and $x \sim_{R_2} y$;
    (ii) We have $b \sim_{S_1} c$ and $y \sim_{S_2} z$.

These two conditions agree, and thus the two resulting relations from $A \times X$ to $C \times Z$ are equal. □

**5.3.10. Products of Families of Relations.** Let $\{A_i\}_{i\in I}$ and $\{B_i\}_{i\in I}$ be families of sets, and let $\{R_i \colon A_i \nrightarrow B_i\}_{i\in I}$ be a family of relations.

**Definition 5.3.10.1.1.** The **product of the family** $\{R_i\}_{i\in I}$ is the relation $\prod_{i\in I} R_i$ from $\prod_{i\in I} A_i$ to $\prod_{i\in I} B_i$ defined as follows:

- Viewing relations as subsets, we define $\prod_{i\in I} R_i$ as its product as a family of sets, i.e. we have

$$\prod_{i\in I} R_i \overset{\text{def}}{=} \left\{ (a_i, b_i)_{i\in I} \in \prod_{i\in I} (A_i \times B_i) \;\middle|\; \begin{array}{l} \text{for each } i \in I, \\ \text{we have } a_i \sim_{R_i} b_i \end{array} \right\}.$$

- Viewing relations as functions to powersets, we define

$$\left[\prod_{i\in I} R_i\right]((a_i)_{i\in I}) \overset{\text{def}}{=} \prod_{i\in I} R_i(a_i)$$

  for each $(a_i)_{i\in I} \in \prod_{i\in I} R_i$.

**5.3.11. The Inverse of a Relation.** Let $A$, $B$, and $C$ be sets and let $R \subset A \times B$ be a relation.

**Definition 5.3.11.1.1.** The **inverse of** $R$[30] is the relation $R^\dagger$ defined as follows:

- Viewing relations as subsets, we define

$$R^\dagger \overset{\text{def}}{=} \{(b,a) \in B \times A \mid \text{we have } b \sim_R a\}.$$

- Viewing relations as functions $A \times B \to \{\mathsf{true}, \mathsf{false}\}$, we define

$$[R^\dagger]_a^b \overset{\text{def}}{=} R_b^a$$

  for each $(b,a) \in B \times A$.
- Viewing relations as functions $A \to \mathcal{P}(B)$, we define

$$\begin{aligned}[R^\dagger](b) &\overset{\text{def}}{=} R^\dagger(\{b\})\\ &\overset{\text{def}}{=} \{a \in A \mid b \in R(a)\}\end{aligned}$$

  for each $b \in B$, where $R^\dagger(\{b\})$ is the fibre of $R$ over $\{b\}$.

**Example 5.3.11.1.2.** Here are some examples of inverses of relations.

(1) *Less Than Equal Signs.* We have $(\leq)^\dagger = \geq$.
(2) *Greater Than Equal Signs.* Dually to **??**, we have $(\geq)^\dagger = \leq$.
(3) *Functions.* Let $f\colon A \to B$ be a function. We have

$$\begin{aligned}\mathrm{Gr}(f)^\dagger &= f^{-1},\\ \left(f^{-1}\right)^\dagger &= \mathrm{Gr}(f).\end{aligned}$$

**Proposition 5.3.11.1.3.** Let $R\colon A \nrightarrow B$ and $S\colon B \nrightarrow C$ be relations.

(1) *Interaction With Ranges and Domains.* We have

$$\begin{aligned}\mathrm{dom}\left(R^\dagger\right) &= \mathrm{range}(R),\\ \mathrm{range}\left(R^\dagger\right) &= \mathrm{dom}(R).\end{aligned}$$

(2) *Interaction With Composition I.* We have

$$(S \diamond R)^\dagger = R^\dagger \diamond S^\dagger.$$

(3) *Interaction With Composition II.* We have

$$\begin{aligned}\chi_B(-_1, -_2) &\subset R \diamond R^\dagger,\\ \chi_A(-_1, -_2) &\subset R^\dagger \diamond R.\end{aligned}$$

(4) *Invertibility.* We have

$$\left(R^\dagger\right)^\dagger = R.$$

[30] *Further Terminology:* Also called the **opposite of** $R$, the **transpose of** $R$, or the

(5) *Identity.* We have

$$\chi_A^\dagger(-_1,-_2) = \chi_A(-_1,-_2).$$

**Proof.** *Item 1, Interaction With Ranges and Domains*: Clear.
*Item 2, Interaction With Composition I*: Clear.
*Item 3, Interaction With Composition II*: Clear.
*Item 4, Invertibility*: Clear.
*Item 5, Identity*: Clear. □

**5.3.12. Composition of Relations.** Let $A$, $B$, and $C$ be sets and let $R \subset A \times B$ and $S \subset B \times C$ be relations.

**Definition 5.3.12.1.1.** The **composition of** $R$ **and** $S$ is the relation $S \diamond R$ defined as follows:

- Viewing relations from $A$ to $C$ as subsets of $A \times C$, we define

$$S \diamond R \stackrel{\text{def}}{=} \left\{ (a,c) \in A \times C \;\middle|\; \begin{array}{l} \text{there exists some } b \in B \text{ such} \\ \text{that } a \sim_R b \text{ and } b \sim_S c \end{array} \right\}.$$

- Viewing relations as functions $A \times B \to \{\mathsf{true}, \mathsf{false}\}$, we define

$$\begin{aligned} (S \diamond R)^{-_1}_{-_2} &\stackrel{\text{def}}{=} \int^{y \in B} S^{-_1}_{y} \times R^{y}_{-_2} \\ &= \bigvee_{y \in B} S^{-_1}_{y} \times R^{y}_{-_2}, \end{aligned}$$

  where the join $\bigvee$ is taken in the poset $(\{\mathsf{true}, \mathsf{false}\}, \preceq)$ of **??**.
- Viewing relations as functions $A \to \mathcal{P}(B)$, we define

$$S \diamond R \stackrel{\text{def}}{=} \mathrm{Lan}_{\chi_B}(S) \circ R, \qquad \begin{tikzcd} & B \arrow[r, "S"] \arrow[d, "\chi_B"'] \arrow[dr, phantom, "\Downarrow"] & \mathcal{P}(C), \\ A \arrow[r, "R"'] & \mathcal{P}(B) \arrow[ur, "\mathrm{Lan}_{\chi_B}(S)"'] & \end{tikzcd}$$

  where $\mathrm{Lan}_{\chi_B}(S)$ is computed by the formula

$$\begin{aligned} [\mathrm{Lan}_{\chi_B}(S)](V) &\cong \int^{y \in B} \chi_{\mathcal{P}(B)}(\chi_y, V) \odot S_y \\ &\cong \int^{y \in B} \chi_V(y) \odot S_y \\ &\cong \bigcup_{y \in B} \chi_V(y) \odot S_y \\ &\cong \bigcup_{y \in V} S_y \end{aligned}$$

---

**converse of** $R$.

for each $V \in \mathcal{P}(B)$. In other words, $S \diamond R$ is defined by[31]

$$\begin{aligned}[S \diamond R](a) &\overset{\text{def}}{=} S(R(a)) \\ &\overset{\text{def}}{=} \bigcup_{x \in R(a)} S(x).\end{aligned}$$

for each $a \in A$.

**Example 5.3.12.1.2.** Here are some examples of composition of relations.

(1) *Composing Less/Greater Than Equal With Greater/Less Than Equal Signs.* We have

$$\begin{aligned}\leq \diamond \geq &= \sim_{\text{triv}}, \\ \geq \diamond \leq &= \sim_{\text{triv}} .\end{aligned}$$

(2) *Composing Less/Greater Than Equal Signs With Less/Greater Than Equal Signs.* We have

$$\begin{aligned}\leq \diamond \leq &= \leq, \\ \geq \diamond \geq &= \geq.\end{aligned}$$

**Proposition 5.3.12.1.3.** Let $R\colon A \nrightarrow B$, $S\colon B \nrightarrow C$, and $T\colon C \nrightarrow D$ be relations.

(1) *Interaction With Ranges and Domains.* We have

$$\begin{aligned}\operatorname{dom}(S \diamond R) &\subset \operatorname{dom}(R), \\ \operatorname{range}(S \diamond R) &\subset \operatorname{range}(S).\end{aligned}$$

(2) *Associativity.* We have

$$(T \diamond S) \diamond R = T \diamond (S \diamond R).$$

(3) *Unitality.* We have

$$\begin{aligned}\chi_B \diamond R &= R, \\ R \diamond \chi_A &= R.\end{aligned}$$

(4) *Interaction With Inverses.* We have

$$(S \diamond R)^\dagger = R^\dagger \diamond S^\dagger.$$

(5) *Interaction With Composition.* We have

$$\begin{aligned}\chi_B(-_1, -_2) &\subset R \diamond R^\dagger, \\ \chi_A(-_1, -_2) &\subset R^\dagger \diamond R.\end{aligned}$$

**Proof.** *Item 1, Interaction With Ranges and Domains*: Clear.

[31]That is: the relation $R$ may send $a \in A$ to a number of elements $\{b_i\}_{i \in I}$ in $B$, and then the relation $S$ may send the image of each of the $b_i$'s to a number of elements $\{S(b_i)\}_{i \in I} = \left\{\{c_{j_i}\}_{j_i \in J_i}\right\}_{i \in I}$ in $C$.

*Item 2, Associativity*: Indeed, we have

$$
\begin{aligned}
(T \diamond S) \diamond R &\overset{\text{def}}{=} \left( \int^{y \in C} T^{-1}_{x} \times S^{x}_{-_2} \right) \diamond R \\
&\overset{\text{def}}{=} \int^{x \in B} \left( \int^{y \in C} T^{-1}_{x} \times S^{x}_{y} \right) \diamond R^{y}_{-_2} \\
&= \int^{x \in B} \int^{y \in C} \left( T^{-1}_{x} \times S^{x}_{y} \right) \diamond R^{y}_{-_2} \\
&= \int^{y \in C} \int^{x \in B} \left( T^{-1}_{x} \times S^{x}_{y} \right) \diamond R^{y}_{-_2} \\
&= \int^{y \in C} \int^{x \in B} T^{-1}_{x} \times \left( S^{x}_{y} \diamond R^{y}_{-_2} \right) \\
&= \int^{x \in B} T^{-1}_{x} \times \left( \int^{y \in C} S^{x}_{y} \diamond R^{y}_{-_2} \right) \\
&\overset{\text{def}}{=} \int^{x \in B} T^{-1}_{x} \times (S \diamond R)^{x}_{-_2} \\
&\overset{\text{def}}{=} T \diamond (S \diamond R).
\end{aligned}
$$

In the language of relations, given $a \in A$ and $d \in D$, the stated equality witnesses the equivalence of the following two statements:

(1) We have $a \sim_{(T \diamond S) \diamond R} d$, i.e. there exists some $b \in B$ such that:
   (a) We have $a \sim_R b$;
   (b) We have $b \sim_{T \diamond S} d$, i.e. there exists some $c \in C$ such that:
      (i) We have $b \sim_S c$;
      (ii) We have $c \sim_T d$;
(2) We have $a \sim_{T \diamond (S \diamond R)} d$, i.e. there exists some $c \in C$ such that:
   (a) We have $a \sim_{S \diamond R} c$, i.e. there exists some $b \in B$ such that:
      (i) We have $a \sim_R b$;
      (ii) We have $b \sim_S c$;
   (b) We have $c \sim_T d$;

both of which are equivalent to the statement

- There exist $b \in B$ and $c \in C$ such that $a \sim_R b \sim_S c \sim_T d$.

*Item 3, Unitality*: Indeed, we have

$$
\begin{aligned}
\chi_B \diamond R &\overset{\text{def}}{=} \int^{x \in B} (\chi_B)^{-1}_{x} \times R^{x}_{-_2} \\
&= \bigvee_{x \in B} (\chi_B)^{-1}_{x} \times R^{x}_{-_2} \\
&= \bigvee_{\substack{x \in B \\ x = -_1}} R^{x}_{-_2} \\
&= R^{-_1}_{-_2},
\end{aligned}
$$

and

$$\begin{aligned} R \diamond \chi_A &\overset{\text{def}}{=} \int^{x \in A} R^{-1}_{x} \times (\chi_A)^{x}_{-2} \\ &= \bigvee_{x \in B} R^{-1}_{x} \times (\chi_A)^{x}_{-2} \\ &= \bigvee_{\substack{x \in B \\ x = -2}} R^{-1}_{x} \\ &= R^{-1}_{-2}. \end{aligned}$$

In the language of relations, given $a \in A$ and $b \in B$:

- The equality
$$\chi_B \diamond R = R$$
witnesses the equivalence of the following two statements:
  (1) We have $a \sim_b B$.
  (2) There exists some $b' \in B$ such that:
    (a) We have $a \sim_R b'$
    (b) We have $b' \sim_{\chi_B} b$, i.e. $b' = b$.
- The equality
$$R \diamond \chi_A = R$$
witnesses the equivalence of the following two statements:
  (1) There exists some $a' \in A$ such that:
    (a) We have $a \sim_{\chi_B} a'$, i.e. $a = a'$.
    (b) We have $a' \sim_R b$
  (2) We have $a \sim_b B$.

*Item 4, Interaction With Inverses*: Clear.
*Item 5, Interaction With Composition*: Clear. □

**5.3.13. The Collage of a Relation.** Let $A$ and $B$ be sets and let $R\colon A \nrightarrow B$ be a relation from $A$ to $B$.

**Definition 5.3.13.1.1.** The **collage of** $R$[32] is the poset $\mathbf{Coll}(R) \overset{\text{def}}{=} \left(\mathrm{Coll}(R), \preceq_{\mathbf{Coll}(R)}\right)$ consisting of

- *The Underlying Set.* The set $\mathrm{Coll}(R)$ defined by
$$\mathrm{Coll}(R) \overset{\text{def}}{=} A \coprod B.$$
- *The Partial Order.* The partial order
$$\preceq_{\mathbf{Coll}(R)} \colon \mathrm{Coll}(R) \times \mathrm{Coll}(R) \to \{\mathsf{true}, \mathsf{false}\}$$
on $\mathrm{Coll}(R)$ defined by
$$\preceq(a,b) \overset{\text{def}}{=} \begin{cases} \mathsf{true} & \text{if } a = b \text{ or } a \sim_R b, \\ \mathsf{false} & \text{otherwise.} \end{cases}$$

**Proposition 5.3.13.1.2.** Let $A$ and $B$ be sets and let $R\colon A \nrightarrow B$ be a relation from $A$ to $B$.

[32] *Further Terminology:* Also called the **cograph of** $R$.

(1) *Functoriality I.* The assignment $R \mapsto \mathbf{Coll}(R)$ defines a functor[33]

$$\mathbf{Coll}\colon \mathbf{Rel}(A,B) \to \mathsf{Pos}_{/\Delta^1}(A,B),$$

where

- *Action on Objects.* For each $R \in \mathrm{Obj}(\mathbf{Rel}(A,B))$, we have

$$[\mathbf{Coll}](R) \stackrel{\text{def}}{=} (\mathbf{Coll}(R), \phi_R)$$

for each $R \in \mathbf{Rel}(A,B)$, where
  - The poset $\mathbf{Coll}(R)$ is the collage of $R$ of Definition 5.3.13.1.1;
  - The morphism $\phi_R\colon \mathbf{Coll}(R) \to \Delta^1$ is given by

$$\phi_R(x) \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if } x \in A, \\ 1 & \text{if } x \in B \end{cases}$$

  for each $x \in \mathbf{Coll}(R)$;
- *Action on Morphisms.* For each $R, S \in \mathrm{Obj}(\mathbf{Rel}(A,B))$, the action on Hom-sets

$$\mathbf{Coll}_{R,S}\colon \mathrm{Hom}_{\mathbf{Rel}(A,B)}(R,S) \to \mathsf{Pos}(\mathbf{Coll}(R), \mathbf{Coll}(S))$$

of $\mathbf{Coll}$ at $(R,S)$ is given by sending an inclusion

$$\iota\colon R \subset S$$

to the morphism

$$\mathbf{Coll}(\iota)\colon \mathbf{Coll}(R) \to \mathbf{Coll}(S)$$

of posets over $\Delta^1$ defined by

$$[\mathbf{Coll}(\iota)](x) \stackrel{\text{def}}{=} x$$

for each $x \in \mathbf{Coll}(R)$.[34]

(2) *Equivalence.* The functor of Item 1 is an equivalence of categories.

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[33]Here $\mathsf{Pos}_{/\Delta^1}(A,B)$ is the category defined as the pullback

$$\mathsf{Pos}_{/\Delta^1}(A,B) \stackrel{\text{def}}{=} \mathsf{pt} \underset{[A],\mathsf{Pos},\mathrm{fib}_0}{\times} \mathsf{Pos}_{/\Delta^1} \underset{\mathrm{fib}_1,\mathsf{Pos},[B]}{\times} \mathsf{pt},$$

as in the diagram

$$\begin{tikzcd}
 & & \mathsf{Pos}_{/\Delta^1}(A,B) \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner"] & & \\
 & \mathsf{Pos}_{/\Delta^1} \underset{\mathsf{Pos}}{\times} \mathsf{pt} \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner"] & & \mathsf{pt} \underset{\mathsf{Pos}}{\times} \mathsf{Pos}_{/\Delta^1} \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner"] & \\
\mathsf{pt} \arrow[dr, "{[A]}"'] & & \mathsf{Pos}_{/\Delta^1} \arrow[dl, "\mathrm{fib}_{[0]}"] \arrow[dr, "\mathrm{fib}_{[1]}"'] & & \mathsf{pt}. \arrow[dl, "{[B]}"] \\
 & \mathsf{Pos} & & \mathsf{Pos} &
\end{tikzcd}$$

Explicitly, an object of $\mathsf{Pos}_{/\Delta^1}(A,B)$ is a pair $(X, \phi_X)$ consisting of

- A poset $X$;
- A morphism $\phi_X\colon X \to \Delta^1$;

such that $\phi_X^{-1}(0) = A$ and $\phi_X^{-1}(0) = B$, with morphisms between such objects being morphisms of posets over $\Delta^1$.

[34]Note that this is indeed a morphism of posets: if $x \preceq_{\mathbf{Coll}(R)} y$, then $x = y$ or $x \sim_R y$, so we have either $x = y$ or $x \sim_S y$ (as $R \subset S$), and thus $x \preceq_{\mathbf{Coll}(S)} y$.

**Proof.** *Item 1, Functoriality*: Clear.
*Item 2, Equivalence*: Omitted. □

## 5.4. Equivalence Relations

### 5.4.1. Reflexive Relations.

5.4.1.1. *Foundations.* Let $A$ be a set.

**Definition 5.4.1.1.1.** A **reflexive relation** is equivalently:[35]

- An $\mathbb{E}_0$-monoid in $(\mathrm{N}_\bullet(\mathbf{Rel}(A,A)), \chi_A)$;
- A pointed object in $(\mathbf{Rel}(A,A), \chi_A)$.

**Remark 5.4.1.1.2.** In detail, a relation $R$ on $A$ is **reflexive** if we have an inclusion

$$\eta_R \colon \chi_A \subset R$$

of relations in $\mathbf{Rel}(A,A)$, i.e. if, for each $a \in A$, we have $a \sim_R a$.

**Definition 5.4.1.1.3.** Let $A$ be a set.

(1) The **set of reflexive relations on** $A$ is the subset $\mathrm{Rel}^{\mathsf{refl}}(A,A)$ of $\mathrm{Rel}(A,A)$ spanned by the reflexive relations.
(2) The **poset of relations on** $A$ is is the subposet $\mathbf{Rel}^{\mathsf{refl}}(A,A)$ of $\mathbf{Rel}(A,A)$ spanned by the reflexive relations.

**Proposition 5.4.1.1.4.** Let $R$ and $S$ be relations on $A$.

(1) *Interaction With Inverses.* If $R$ is reflexive, then so is $R^\dagger$.
(2) *Interaction With Composition.* If $R$ and $S$ are reflexive, then so is $S \diamond R$.

**Proof.** *Item 1, Interaction With Inverses*: Clear.
*Item 2, Interaction With Composition*: Clear. □

5.4.1.2. *The Reflexive Closure of a Relation.* Let $R$ be a relation on $A$.

**Definition 5.4.1.2.1.** The **reflexive closure** of $\sim_R$ is the relation $\sim_R^{\mathrm{refl}}$[36] satisfying the following universal property:[37]

($\star$) Given another reflexive relation $\sim_S$ on $A$ such that $R \subset S$, there exists an inclusion $\sim_R^{\mathrm{refl}} \subset \sim_S$.

**Construction 5.4.1.2.2.** Concretely, $\sim_R^{\mathrm{refl}}$ is the free pointed object on $R$ in $(\mathbf{Rel}(A,A), \chi_A)$[38], being given by

$$\begin{aligned} R^{\mathrm{refl}} &\overset{\mathrm{def}}{=} R \coprod^{\mathbf{Rel}(A,A)} \Delta_A \\ &= R \cup \Delta_A \\ &= \{(a,b) \in A \times A \mid \text{we have } a \sim_R b \text{ or } a = b\}. \end{aligned}$$

**Proof.** Clear. □

**Proposition 5.4.1.2.3.** Let $R$ be a relation on $A$.

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[35] Note that since $\mathbf{Rel}(A,A)$ is posetal, reflexivity is a property of a relation, rather than extra structure.
[36] *Further Notation:* Also written $R^{\mathrm{refl}}$.
[37] *Slogan:* The reflexive closure of $R$ is the smallest reflexive relation containing $R$.
[38] Or, equivalently, the free $\mathbb{E}_0$-monoid on $R$ in $(\mathrm{N}_\bullet(\mathbf{Rel}(A,A)), \chi_A)$.

(1) *Adjointness.* We have an adjunction

$$\Big((-)^{\mathrm{refl}} \dashv 忘\Big)\colon \quad \begin{tikzcd} \mathbf{Rel}(A,A) \arrow[r, bend left, "(-)^{\mathrm{refl}}"] \arrow[r, phantom, "\perp"] & \mathbf{Rel}^{\mathsf{refl}}(A,A), \arrow[l, bend left, "忘"] \end{tikzcd}$$

witnessed by a bijection of sets

$$\mathbf{Rel}^{\mathsf{refl}}\Big(R^{\mathrm{refl}}, S\Big) \cong \mathbf{Rel}(R,S),$$

natural in $R \in \mathrm{Obj}\Big(\mathbf{Rel}^{\mathsf{refl}}(A,A)\Big)$ and $S \in \mathrm{Obj}(\mathbf{Rel}(A,A))$.

(2) *The Reflexive Closure of a Reflexive Relation.* If $R$ is reflexive, then $R^{\mathrm{refl}} = R$.

(3) *Idempotency.* We have

$$\Big(R^{\mathrm{refl}}\Big)^{\mathrm{refl}} = R^{\mathrm{refl}}.$$

(4) *Interaction With Inverses.* We have

$$\Big(R^{\dagger}\Big)^{\mathrm{refl}} = \Big(R^{\mathrm{refl}}\Big)^{\dagger}, \qquad \begin{tikzcd} \mathrm{Rel}(A,A) \arrow[r, "(-)^{\mathrm{refl}}"] \arrow[d, "(-)^{\dagger}"'] & \mathrm{Rel}(A,A) \arrow[d, "(-)^{\dagger}"] \\ \mathrm{Rel}(A,A) \arrow[r, "(-)^{\mathrm{refl}}"'] & \mathrm{Rel}(A,A). \end{tikzcd}$$

(5) *Interaction With Composition.* We have

$$(S \diamond R)^{\mathrm{refl}} = S^{\mathrm{refl}} \diamond R^{\mathrm{refl}}, \qquad \begin{tikzcd} \mathrm{Rel}(A,A) \times \mathrm{Rel}(A,A) \arrow[r, "\diamond"] \arrow[d, "(-)^{\mathrm{refl}} \times (-)^{\mathrm{refl}}"'] & \mathrm{Rel}(A,A) \arrow[d, "(-)^{\mathrm{refl}}"] \\ \mathrm{Rel}(A,A) \times \mathrm{Rel}(A,A) \arrow[r, "\diamond"'] & \mathrm{Rel}(A,A). \end{tikzcd}$$

**Proof.** *Item 1, Adjointness*: This is a rephrasing of the universal property of the reflexive closure of a relation, stated in Definition 5.4.1.2.1.
*Item 2, The Reflexive Closure of a Reflexive Relation*: Clear.
*Item 3, Idempotency*: This follows from Item 2.
*Item 4, Interaction With Inverses*: Clear.
*Item 5, Interaction With Composition*: This follows from Item 2 of Proposition 5.4.1.1.4. □

### 5.4.2. Symmetric Relations.

5.4.2.1. *Foundations.* Let $A$ be a set.

**Definition 5.4.2.1.1.** A relation $R$ on $A$ is **symmetric** if, for each $a, b \in A$, the following conditions are equivalent:[39]

(1) We have $a \sim_R b$.
(2) We have $b \sim_R a$.

**Definition 5.4.2.1.2.** Let $A$ be a set.

[39]That is, $R$ is symmetric if $R^{\dagger} = R$.

(1) The **set of symmetric relations on** $A$ is the subset $\mathrm{Rel}^{\mathsf{symm}}(A, A)$ of $\mathrm{Rel}(A, A)$ spanned by the symmetric relations.
(2) The **poset of relations on** $A$ is is the subposet $\mathbf{Rel}^{\mathsf{symm}}(A, A)$ of $\mathbf{Rel}(A, A)$ spanned by the symmetric relations.

**Proposition 5.4.2.1.3.** Let $R$ and $S$ be relations on $A$.
(1) *Interaction With Inverses.* If $R$ is symmetric, then so is $R^{\dagger}$.
(2) *Interaction With Composition.* If $R$ and $S$ are symmetric, then so is $S \diamond R$.

**Proof.** *Item 1, Interaction With Inverses*: Clear.
*Item 2, Interaction With Composition*: Clear. □

5.4.2.2. *The Symmetric Closure of a Relation.* Let $R$ be a relation on $A$.

**Definition 5.4.2.2.1.** The **symmetric closure** of $\sim_R$ is the relation $\sim^{\mathrm{symm}}_R$[40] satisfying the following universal property:[41]
($\star$) Given another symmetric relation $\sim_S$ on $A$ such that $R \subset S$, there exists an inclusion $\sim^{\mathrm{symm}}_R \subset \sim_S$.

**Construction 5.4.2.2.2.** Concretely, $\sim^{\mathrm{symm}}_R$ is the symmetric relation on $A$ defined by

$$\begin{aligned} R^{\mathrm{symm}} &\overset{\mathrm{def}}{=} R \cup R^{\dagger} \\ &= \{(a, b) \in A \times A \mid \text{we have } a \sim_R b \text{ or } b \sim_R a\}. \end{aligned}$$

**Proof.** Clear. □

**Proposition 5.4.2.2.3.** Let $R$ be a relation on $A$.
(1) *Adjointness.* We have an adjunction

$$((-)^{\mathrm{symm}} \dashv 忘) \colon \quad \begin{tikzcd} \mathbf{Rel}(A, A) \arrow[r, bend left, "(-)^{\mathrm{symm}}"] \arrow[r, phantom, "\perp"] & \mathbf{Rel}^{\mathsf{symm}}(A, A), \arrow[l, bend left, "忘"] \end{tikzcd}$$

witnessed by a bijection of sets

$$\mathbf{Rel}^{\mathsf{symm}}(R^{\mathrm{symm}}, S) \cong \mathbf{Rel}(R, S),$$

natural in $R \in \mathrm{Obj}(\mathbf{Rel}^{\mathsf{symm}}(A, A))$ and $S \in \mathrm{Obj}(\mathbf{Rel}(A, A))$.
(2) *The Symmetric Closure of a Symmetric Relation.* If $R$ is symmetric, then $R^{\mathrm{symm}} = R$.
(3) *Idempotency.* We have

$$(R^{\mathrm{symm}})^{\mathrm{symm}} = R^{\mathrm{symm}}.$$

(4) *Interaction With Inverses.* We have

$$\left(R^{\dagger}\right)^{\mathrm{symm}} = \left(R^{\mathrm{symm}}\right)^{\dagger}, \qquad \begin{tikzcd} \mathrm{Rel}(A, A) \arrow[r, "(-)^{\mathrm{symm}}"] \arrow[d, "(-)^{\dagger}"'] & \mathrm{Rel}(A, A) \arrow[d, "(-)^{\dagger}"] \\ \mathrm{Rel}(A, A) \arrow[r, "(-)^{\mathrm{symm}}"'] & \mathrm{Rel}(A, A). \end{tikzcd}$$

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[40] *Further Notation:* Also written $R^{\mathrm{symm}}$.
[41] *Slogan:* The symmetric closure of $R$ is the smallest symmetric relation containing $R$.

(5) *Interaction With Composition.* We have

$$(S \diamond R)^{\mathrm{symm}} = S^{\mathrm{symm}} \diamond R^{\mathrm{symm}}, \qquad \begin{tikzcd} \mathrm{Rel}(A,A) \times \mathrm{Rel}(A,A) \arrow[r, "\diamond"] \arrow[d, "(-)^{\mathrm{symm}} \times (-)^{\mathrm{symm}}"'] & \mathrm{Rel}(A,A) \arrow[d, "(-)^{\mathrm{symm}}"] \\ \mathrm{Rel}(A,A) \times \mathrm{Rel}(A,A) \arrow[r, "\diamond"'] & \mathrm{Rel}(A,A). \end{tikzcd}$$

**Proof.** *Item 1, Adjointness*: This is a rephrasing of the universal property of the symmetric closure of a relation, stated in Definition 5.4.2.2.1.
*Item 2, The Symmetric Closure of a Symmetric Relation*: Clear.
*Item 3, Idempotency*: This follows from Item 2.
*Item 4, Interaction With Inverses*: Clear.
*Item 5, Interaction With Composition*: This follows from Item 2 of Proposition 5.4.2.1.3. □

### 5.4.3. Transitive Relations.

5.4.3.1. *Foundations.* Let $A$ be a set.

**Definition 5.4.3.1.1.** A **transitive relation** is equivalently:[42]

- A non-unital $\mathbb{E}_1$-monoid in $(\mathrm{N}_\bullet(\mathbf{Rel}(A,A)), \diamond)$;
- A non-unital monoid in $(\mathbf{Rel}(A,A), \diamond)$.

**Remark 5.4.3.1.2.** In detail, a relation $R$ on $A$ is **transitive** if we have an inclusion

$$\mu_R \colon R \diamond R \subset R$$

of relations in $\mathbf{Rel}(A,A)$, i.e. if, for each $a, c \in A$, the following condition is satisfied:

($\star$) If there exists some $b \in A$ such that $a \sim_R b$ and $b \sim_R c$, then $a \sim_R c$.

**Definition 5.4.3.1.3.** Let $A$ be a set.

(1) The **set of transitive relations from $A$ to $B$** is the subset $\mathrm{Rel}^{\mathrm{trans}}(A)$ of $\mathrm{Rel}(A,A)$ spanned by the transitive relations.
(2) The **poset of relations from $A$ to $B$** is is the subposet $\mathbf{Rel}^{\mathsf{trans}}(A)$ of $\mathbf{Rel}(A,A)$ spanned by the transitive relations.

**Proposition 5.4.3.1.4.** Let $R$ and $S$ be relations on $A$.

(1) *Interaction With Inverses.* If $R$ is transitive, then so is $R^\dagger$.
(2) *Interaction With Composition.* If $R$ and $S$ are transitive, then $S \diamond R$ ***may fail to be transitive***.

**Proof.** *Item 1, Interaction With Inverses*: Clear.
*Item 2, Interaction With Composition*: See [**MSE2096272**].[43] □

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[42]Note that since $\mathbf{Rel}(A,A)$ is posetal, transitivity is a property of a relation, rather than extra structure.

[43]*Intuition:* Transitivity for $R$ and $S$ fails to imply that of $S \diamond R$ because the composition operation for relations intertwines $R$ and $S$ in an incompatible way:

(1) If $a \sim_{S \diamond R} c$ and $c \sim_{S \diamond r} e$, then:
  (a) There is some $b \in A$ such that:
    (i) $a \sim_R b$;

5.4.3.2. *The Transitive Closure of a Relation.* Let $R$ be a relation on $A$.

**Definition 5.4.3.2.1.** The **transitive closure** of $\sim_R$ is the relation $\sim_R^{\mathrm{trans}}$[44] satisfying the following universal property:[45]

- $(\star)$ Given another transitive relation $\sim_S$ on $A$ such that $R \subset S$, there exists an inclusion $\sim_R^{\mathrm{trans}} \subset \sim_S$.

**Construction 5.4.3.2.2.** Concretely, $\sim_R^{\mathrm{trans}}$ is the free non-unital monoid on $R$ in $(\mathbf{Rel}(A, A), \diamond)$[46], being given by

$$\begin{aligned} R^{\mathrm{trans}} &\overset{\mathrm{def}}{=} \coprod_{n=1}^{\infty} R^{\diamond n} \\ &\overset{\mathrm{def}}{=} \bigcup_{n=1}^{\infty} R^{\diamond n} \\ &\overset{\mathrm{def}}{=} \left\{ (a, b) \in A \times B \;\middle|\; \begin{array}{l} \text{there exists some } (x_1, \ldots, x_n) \in R^{\times n} \\ \text{such that } a \sim_R x_1 \sim_R \cdots \sim_R x_n \sim_R b \end{array} \right\}. \end{aligned}$$

**Proof.** Clear. □

**Proposition 5.4.3.2.3.** Let $R$ be a relation on $A$.

(1) *Adjointness.* We have an adjunction

$$\Big((-)^{\mathrm{trans}} \dashv 忘\Big)\colon \quad \begin{tikzcd} \mathbf{Rel}(A, A) \arrow[r, bend left, "(-)^{\mathrm{trans}}"] \arrow[r, phantom, "\perp"] & \mathbf{Rel}^{\mathsf{trans}}(A, A), \arrow[l, bend left, "忘"] \end{tikzcd}$$

witnessed by a bijection of sets

$$\mathbf{Rel}^{\mathsf{trans}}\Big(R^{\mathrm{trans}}, S\Big) \cong \mathbf{Rel}(R, S),$$

natural in $R \in \mathrm{Obj}(\mathbf{Rel}^{\mathsf{trans}}(A, A))$ and $S \in \mathrm{Obj}(\mathbf{Rel}(A, B))$.

(2) *The Transitive Closure of a Transitive Relation.* If $R$ is transitive, then $R^{\mathrm{trans}} = R$.

(3) *Idempotency.* We have

$$\Big(R^{\mathrm{trans}}\Big)^{\mathrm{trans}} = R^{\mathrm{trans}}.$$

(4) *Interaction With Inverses.* We have

$$\Big(R^{\dagger}\Big)^{\mathrm{trans}} = \Big(R^{\mathrm{trans}}\Big)^{\dagger}, \qquad \begin{tikzcd} \mathrm{Rel}(A, A) \arrow[r, "(-)^{\mathrm{trans}}"] \arrow[d, "(-)^{\dagger}"'] & \mathrm{Rel}(A, A) \arrow[d, "(-)^{\dagger}"] \\ \mathrm{Rel}(A, A) \arrow[r, "(-)^{\mathrm{trans}}"'] & \mathrm{Rel}(A, A). \end{tikzcd}$$

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(ii) $b \sim_S c$;

(b) There is some $d \in A$ such that:

(i) $c \sim_R d$;

(ii) $d \sim_S e$.

[44] *Further Notation:* Also written $R^{\mathrm{trans}}$.

[45] *Slogan:* The transitive closure of $R$ is the smallest transitive relation containing $R$.

[46] Or, equivalently, the free non-unital $\mathbb{E}_1$-monoid on $R$ in $(\mathrm{N}_{\bullet}(\mathbf{Rel}(A, A)), \diamond)$.

(5) *Interaction With Composition.* We have

$$(S \diamond R)^{\mathrm{trans}} \overset{\text{poss.}}{\neq} S^{\mathrm{trans}} \diamond R^{\mathrm{trans}}, \qquad \begin{tikzcd} \mathrm{Rel}(A,A) \times \mathrm{Rel}(A,A) \arrow[r, "\diamond"] \arrow[d, "(-)^{\mathrm{trans}} \times (-)^{\mathrm{trans}}"'] \arrow[dr, phantom, "\times"] & \mathrm{Rel}(A,A) \arrow[d, "(-)^{\mathrm{trans}}"] \\ \mathrm{Rel}(A,A) \times \mathrm{Rel}(A,A) \arrow[r, "\diamond"'] & \mathrm{Rel}(A,A). \end{tikzcd}$$

**Proof.** *Item 1, Adjointness*: This is a rephrasing of the universal property of the transitive closure of a relation, stated in Definition 5.4.3.2.1.
*Item 2, The Transitive Closure of a Transitive Relation*: Clear.
*Item 3, Idempotency*: This follows from Item 2.
*Item 4, Interaction With Inverses*: We have

$$\begin{aligned} \left(R^{\dagger}\right)^{\mathrm{trans}} &= \bigcup_{n=1}^{\infty} \left(R^{\dagger}\right)^{\diamond n} && \text{(by Construction 5.4.3.2.2)} \\ &= \bigcup_{n=1}^{\infty} (R^{\diamond n})^{\dagger} && \text{(by Item 4 of Proposition 5.3.12.1.3)} \\ &= \left(\bigcup_{n=1}^{\infty} R^{\diamond n}\right)^{\dagger} && \text{(by Item 1 of Proposition 5.3.6.1.2)} \\ &= \left(R^{\mathrm{trans}}\right)^{\dagger}. && \text{(by Construction 5.4.3.2.2)} \end{aligned}$$

*Item 5, Interaction With Composition*: This follows from Item 2 of Proposition 5.4.3.1.4. □

### 5.4.4. Equivalence Relations.

5.4.4.1. *Foundations.* Let $A$ be a set.

**Definition 5.4.4.1.1.** A relation $R$ is an **equivalence relation** if it is reflexive, symmetric, and transitive.[47]

**Example 5.4.4.1.2.** The **kernel of a function** $f: A \to B$ is the equivalence $\sim_{\mathrm{Ker}(f)}$ on $A$ obtained by declaring $a \sim_{\mathrm{Ker}(f)} b$ iff $f(a) = f(b)$.[48]

**Definition 5.4.4.1.3.** Let $A$ and $B$ be sets.

(1) The **set of equivalence relations from** $A$ **to** $B$ is the subset $\mathrm{Rel}^{\mathrm{eq}}(A,B)$ of $\mathrm{Rel}(A,B)$ spanned by the equivalence relations.
(2) The **poset of relations from** $A$ **to** $B$ is is the subposet $\mathbf{Rel}^{\mathrm{eq}}(A,B)$ of $\mathbf{Rel}(A,B)$ spanned by the equivalence relations.

5.4.4.2. *The Equivalence Closure of a Relation.* Let $R$ be a relation on $A$.

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[47] *Further Terminology:* If instead $R$ is just symmetric and transitive, then it is called a **partial equivalence relation**.
[48] The kernel $\mathrm{Ker}(f): A \nrightarrow A$ of $f$ is the monad induced by the adjunction $\mathrm{Gr}(f) \dashv f^{-1}: A \rightleftarrows B$ in **Rel** of Item 2 of Proposition 5.3.1.1.2.

**Definition 5.4.4.2.1.** The **equivalence closure**[49] of $\sim_R$ is the relation $\sim_R^{\mathrm{eq}}$[50] satisfying the following universal property:[51]

($\star$) Given another equivalence relation $\sim_S$ on $A$ such that $R \subset S$, there exists an inclusion $\sim_R^{\mathrm{eq}} \subset \sim_S$.

**Construction 5.4.4.2.2.** Concretely, $\sim_R^{\mathrm{eq}}$ is the equivalence relation on $A$ defined by

$$\begin{aligned}
R^{\mathrm{eq}} &\overset{\mathrm{def}}{=} \left(\left(R^{\mathrm{refl}}\right)^{\mathrm{symm}}\right)^{\mathrm{trans}} \\
&= \left((R^{\mathrm{symm}})^{\mathrm{trans}}\right)^{\mathrm{refl}} \\
&= \left\{ (a,b) \in A \times B \;\middle|\; \begin{array}{l} \text{there exists } (x_1,\ldots,x_n) \in R^{\times n} \text{ satisfying at} \\ \text{least one of the following conditions:} \\ \quad 1.\ \text{The following conditions are satisfied:} \\ \qquad \text{(a) We have } a \sim_R x_1 \text{ or } x_1 \sim_R a; \\ \qquad \text{(b) We have } x_i \sim_R x_{i+1} \text{ or } x_{i+1} \sim_R x_i \\ \qquad\quad \text{for each } 1 \leq i \leq n-1; \\ \qquad \text{(c) We have } b \sim_R x_n \text{ or } x_n \sim_R b; \\ \quad 2.\ \text{We have } a = b. \end{array} \right\}.
\end{aligned}$$

**Proof.** From the universal properties of the reflexive, symmetric, and transitive closures of a relation (Definitions 5.4.1.2.1, 5.4.2.2.1 and 5.4.3.2.1), we see that it suffices to prove that:

(1) The symmetric closure of a reflexive relation is still reflexive;
(2) The transitive closure of a symmetric relation is still symmetric;

which are both clear. □

**Proposition 5.4.4.2.3.** Let $R$ be a relation on $A$.

(1) *Adjointness.* We have an adjunction

$$\left((-)^{\mathrm{eq}} \dashv 忘\right)\colon \quad \begin{tikzcd} \mathbf{Rel}(A,B) \arrow[r, bend left, "(-)^{\mathrm{eq}}"] \arrow[r, phantom, "\perp"] & \mathbf{Rel}^{\mathrm{eq}}(A,B), \arrow[l, bend left, "忘"] \end{tikzcd}$$

witnessed by a bijection of sets

$$\mathbf{Rel}^{\mathrm{eq}}(R^{\mathrm{eq}}, S) \cong \mathbf{Rel}(R, S),$$

natural in $R \in \mathrm{Obj}(\mathbf{Rel}^{\mathrm{eq}}(A,B))$ and $S \in \mathrm{Obj}(\mathbf{Rel}(A,B))$.

(2) *The Equivalence Closure of an Equivalence Relation.* If $R$ is an equivalence relation, then $R^{\mathrm{eq}} = R$.

(3) *Idempotency.* We have

$$(R^{\mathrm{eq}})^{\mathrm{eq}} = R^{\mathrm{eq}}.$$

[49] *Further Terminology:* Also called the **equivalence relation associated to** $\sim_R$.
[50] *Further Notation:* Also written $R^{\mathrm{eq}}$.
[51] *Slogan:* The equivalence closure of $R$ is the smallest equivalence relation containing $R$.

**Proof.** *Item 1, Adjointness*: This is a rephrasing of the universal property of the equivalence closure of a relation, stated in Definition 5.4.4.2.1.
*Item 2, The Equivalence Closure of an Equivalence Relation*: Clear.
*Item 3, Idempotency*: This follows from Item 2. ☐

### 5.4.5. Quotients by Equivalence Relations.

5.4.5.1. *Equivalence Classes.* Let $A$ be a set, let $R$ be a relation on $A$, and let $a \in A$.

**Definition 5.4.5.1.1.** The **equivalence class associated to** $a$ is the set $[a]$ defined by

$$\begin{aligned} [a] &\stackrel{\text{def}}{=} \{x \in X \mid x \sim_R a\} \\ &= \{x \in X \mid a \sim_R x\}. \qquad \text{(since } R \text{ is symmetric)} \end{aligned}$$

5.4.5.2. *Quotients of Sets by Equivalence Relations.* Let $A$ be a set and let $R$ be a relation on $A$.

**Definition 5.4.5.2.1.** The **quotient of** $X$ **by** $R$ is the set $X/\!\sim_R$ defined by

$$X/\!\sim_R \stackrel{\text{def}}{=} \{[a] \in \mathcal{P}(X) \mid a \in X\}.$$

**Remark 5.4.5.2.2.** The reason we define quotient sets for equivalence relations only is that each of the properties of being an equivalence relation—reflexivity, symmetry, and transitivity—ensures that the equivalences classes $[a]$ of $X$ under $R$ are well-behaved:

- *Reflexivity.* If $R$ is reflexive, then, for each $a \in X$, we have $a \in [a]$.
- *Symmetry.* The equivalence class $[a]$ of an element $a$ of $X$ is defined by
$$[a] \stackrel{\text{def}}{=} \{x \in X \mid x \sim_R a\},$$
but we could equally well define
$$[a]' \stackrel{\text{def}}{=} \{x \in X \mid a \sim_R x\}$$
instead. This is not a problem when $R$ is symmetric, as we then have $[a] = [a]'$.[52]
- *Transitivity.* If $R$ is transitive, then $[a]$ and $[b]$ are disjoint iff $a \nsim_R b$, and equal otherwise.

**Proposition 5.4.5.2.3.** Let $f\colon X \to Y$ be a function and let $R$ be a relation on $X$.

(1) *As a Coequaliser.* We have an isomorphism of sets
$$X/\!\sim_R^{\text{eq}} \cong \operatorname{CoEq}\left(R \hookrightarrow X \times X \underset{\text{pr}_2}{\overset{\text{pr}_1}{\rightrightarrows}} X\right),$$
where $\sim_R^{\text{eq}}$ is the equivalence relation generated by $\sim_R$.

[52]When categorifying equivalence relations, one finds that $[a]$ and $[a]'$ correspond to presheaves and copresheaves; see **??**.

(2) *As a Pushout.* We have an isomorphism of sets[53]

$$X/{\sim}^{\mathrm{eq}}_R \cong X \coprod_{\mathrm{Eq}(\mathrm{pr}_1,\mathrm{pr}_2)} X, \qquad \begin{tikzcd} X/{\sim}^{\mathrm{eq}}_R \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[l] \\ X \arrow[u] & \mathrm{Eq}(\mathrm{pr}_1,\mathrm{pr}_2). \arrow[l] \arrow[u] \end{tikzcd}$$

where ${\sim}^{\mathrm{eq}}_R$ is the equivalence relation generated by ${\sim}_R$.

(3) *The First Isomorphism Theorem for Sets.* We have an isomorphism of sets[54,55]

$$X/{\sim}_{\mathrm{Ker}(f)} \cong \mathrm{Im}(f).$$

(4) *Descending Functions to Quotient Sets, I.* Let $R$ be an equivalence relation on $X$. The following conditions are equivalent:

(a) There exists a map

$$\overline{f}\colon X/{\sim}_R \to Y$$

making the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, two heads, "q"'] & Y \\ X/{\sim}_R \arrow[ur, dotted, "\exists", "\overline{f}"'] & \end{tikzcd}$$

commute.

(b) For each $x, y \in X$, if $x \sim_R y$, then $f(x) = f(y)$.

(5) *Descending Functions to Quotient Sets, II.* Let $R$ be an equivalence relation on $X$. If the conditions of Item 4 hold, then $\overline{f}$ is the *unique*

[53]Dually, we also have an isomorphism of sets

$$\mathrm{Eq}(\mathrm{pr}_1,\mathrm{pr}_2) \cong X \times_{X/{\sim}^{\mathrm{eq}}_R} X, \qquad \begin{tikzcd} \mathrm{Eq}(\mathrm{pr}_1,\mathrm{pr}_2) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d] \\ X \arrow[r] & X/{\sim}^{\mathrm{eq}}_R. \end{tikzcd}$$

[54]*Further Terminology:* The set $X/{\sim}_{\mathrm{Ker}(f)}$ is often called the **coimage of** $f$, and denoted by $\mathrm{Coim}(f)$.

[55]In a sense this is a result relating the monad in **Rel** induced by $f$ with the comonad in **Rel** induced by $f$, as the kernel and image

$$\mathrm{Ker}(f)\colon X \nrightarrow X,$$
$$\mathrm{Im}(f) \subset Y$$

of $f$ are respectively the induced monads and comonads of the adjunction

$$\left(\mathrm{Gr}(f) \dashv f^{-1}\right)\colon \quad \begin{tikzcd} A \arrow[r, bend left, "\mathrm{Gr}(f)"] \arrow[r, phantom, "\perp"] & B \arrow[l, bend left, "f^{-1}"] \end{tikzcd}$$

of Item 2 of Proposition 5.3.1.1.2.

map making the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "q"'] & Y \\ X/\sim_R \arrow[ur, dashed, "\exists!", "\overline{f}"'] & \end{tikzcd}$$

commute.

(6) *Descending Functions to Quotient Sets, III.* Let $R$ be an equivalence relation on $X$. If the conditions of Item 4 hold, then the following conditions are equivalent:
   (a) The map $\overline{f}$ is an injection.
   (b) For each $x, y \in X$, we have $x \sim_R y$ iff $f(x) = f(y)$.

(7) *Descending Functions to Quotient Sets, IV.* Let $R$ be an equivalence relation on $X$. If the conditions of Item 4 hold, then the following conditions are equivalent:
   (a) The map $f\colon X \to Y$ is surjective.
   (b) The map $\overline{f}\colon X/\sim_R \to Y$ is surjective.

(8) *Descending Functions to Quotient Sets, V.* Let $R$ be a relation on $X$ and let $\sim_R^{\mathrm{eq}}$ be the equivalence relation associated to $R$. The following conditions are equivalent:
   (a) The map $f$ satisfies the equivalent conditions of Item 4:
      - There exists a map
      $$\overline{f}\colon X/\sim_R^{\mathrm{eq}} \to Y$$
      making the diagram
      $$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "q"'] & Y \\ X/\sim_R^{\mathrm{eq}} \arrow[ur, dotted, "\exists", "\overline{f}"'] & \end{tikzcd}$$
      commute.
      - For each $x, y \in X$, if $x \sim_R^{\mathrm{eq}} y$, then $f(x) = f(y)$.
   (b) For each $x, y \in X$, if $x \sim_R y$, then $f(x) = f(y)$.

**Proof.** *Item 1, As a Coequaliser*: Omitted.
*Item 2, As a Pushout*: Omitted.
*Item 3, The First Isomorphism Theorem for Sets*: Clear.
*Item 4, Descending Functions to Quotient Sets, I*: See [Pro24o].
*Item 5, Descending Functions to Quotient Sets, II*: See [Pro24y].
*Item 6, Descending Functions to Quotient Sets, III*: See [Pro24n].
*Item 7, Descending Functions to Quotient Sets, IV*: See [Pro24m].
*Item 8, Descending Functions to Quotient Sets, V*: The implication Item 8a $\Longrightarrow$ Item 8b is clear.
Conversely, suppose that, for each $x, y \in X$, if $x \sim_R y$, then $f(x) = f(y)$. Spelling out the definition of the equivalence closure of $R$, we see that the condition $x \sim_R^{\mathrm{eq}} y$ unwinds to the following:

($\star$) There exist $(x_1, \ldots, x_n) \in R^{\times n}$ satisfying at least one of the following conditions:
   (1) The following conditions are satisfied:
      (a) We have $x \sim_R x_1$ or $x_1 \sim_R x$;
      (b) We have $x_i \sim_R x_{i+1}$ or $x_{i+1} \sim_R x_i$ for each $1 \leq i \leq n-1$;
      (c) We have $y \sim_R x_n$ or $x_n \sim_R y$;
   (2) We have $x = y$.

Now, if $x = y$, then $f(x) = f(y)$ trivially; otherwise, we have

$$\begin{aligned} f(x) &= f(x_1), \\ f(x_1) &= f(x_2), \\ &\ \ \vdots \\ f(x_{n-1}) &= f(x_n), \\ f(x_n) &= f(y), \end{aligned}$$

and $f(x) = f(y)$, as we wanted to show. □

## 5.5. Functoriality of Powersets

**5.5.1. Direct Images.** Let $A$ and $B$ be sets and let $R\colon A \nrightarrow B$ be a relation.

**Definition 5.5.1.1.1.** The **direct image function associated to $R$** is the function[56]

$$R_*\colon \mathcal{P}(A) \to \mathcal{P}(B)$$

defined by[57,58]

$$\begin{aligned} R_*(U) &\overset{\text{def}}{=} R(U) \\ &\overset{\text{def}}{=} \bigcup_{a \in U} R(a) \\ &= \left\{ b \in B \,\middle|\, \begin{aligned} &\text{there exists some } a \in U \\ &\text{such that } b \in R(a) \end{aligned} \right\} \end{aligned}$$

for each $U \in \mathcal{P}(A)$.

**Remark 5.5.1.1.2.** Identifying subsets of $A$ with relations from pt to $A$ via **??** of **??**, we see that the direct image function associated to $R$ is equivalently the function

$$R_*\colon \underbrace{\mathcal{P}(A)}_{\cong \mathrm{Rel}(\mathrm{pt}, A)} \to \underbrace{\mathcal{P}(B)}_{\cong \mathrm{Rel}(\mathrm{pt}, B)}$$

defined by

$$R_*(U) \overset{\text{def}}{=} R \diamond U$$

[56] *Further Notation:* Also written $\exists_R\colon \mathcal{P}(A) \to \mathcal{P}(B)$. This notation comes from the fact that the following statements are equivalent, where $b \in B$ and $U \in \mathcal{P}(A)$:

- We have $b \in \exists_R(U)$.
- There exists some $a \in U$ such that $b \in f(a)$.

[57] *Further Terminology:* The set $R(U)$ is called the **direct image of $U$ by $R$**.

[58] We also have

$$R_*(U) = B \setminus R_!(A \setminus U);$$

see Item 7 of Proposition 5.5.1.1.3.

for each $U \in \mathcal{P}(A)$, where $R \diamond U$ is the composition

$$\mathrm{pt} \overset{U}{\nrightarrow} A \overset{R}{\nrightarrow} B.$$

**Proposition 5.5.1.1.3.** Let $R \colon A \nrightarrow B$ be a relation.

(1) *Functoriality.* The assignment $U \mapsto R_*(U)$ defines a functor

$$R_* \colon (\mathcal{P}(A), \subset) \to (\mathcal{P}(B), \subset)$$

where

- *Action on Objects.* For each $U \in \mathcal{P}(A)$, we have

$$[R_*](U) \overset{\text{def}}{=} R_*(U);$$

- *Action on Morphisms.* For each $U, V \in \mathcal{P}(A)$:
  - If $U \subset V$, then $R_*(U) \subset R_*(V)$.

(2) *Adjointness.* We have an adjunction

$$(R_* \dashv R_{-1}) \colon \quad \begin{tikzcd} \mathcal{P}(A) \arrow[r, bend left, "R_*"] \arrow[r, phantom, "\perp"] & \mathcal{P}(B), \arrow[l, bend left, "R_{-1}"] \end{tikzcd}$$

witnessed by a bijections of sets

$$\mathrm{Hom}_{\mathcal{P}(A)}(R_*(U), V) \cong \mathrm{Hom}_{\mathcal{P}(A)}(U, R_{-1}(V)),$$

natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$, i.e. such that:

($\star$) The following conditions are equivalent:
  - We have $R_*(U) \subset V$;
  - We have $U \subset R_{-1}(V)$.

(3) *Preservation of Colimits.* We have an equality of sets

$$R_*\left(\bigcup_{i \in I} U_i\right) = \bigcup_{i \in I} R_*(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have equalities

$$R_*(U) \cup R_*(V) = R_*(U \cup V),$$
$$R_*(\emptyset) = \emptyset,$$

natural in $U, V \in \mathcal{P}(A)$.

(4) *Oplax Preservation of Limits.* We have an inclusion of sets

$$R_*\left(\bigcap_{i \in I} U_i\right) \subset \bigcap_{i \in I} R_*(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have inclusions

$$R_*(U \cap V) \subset R_*(U) \cap R_*(V),$$
$$R_*(A) \subset B,$$

natural in $U, V \in \mathcal{P}(A)$.

(5) *Symmetric Strict Monoidality With Respect to Unions.* The direct image function of Item 1 has a symmetric strict monoidal structure

$$\left(R_*, R_*^{\otimes}, R_{*|\mathbb{1}}^{\otimes}\right) \colon (\mathcal{P}(A), \cup, \emptyset) \to (\mathcal{P}(B), \cup, \emptyset),$$

being equipped with equalities

$$R^{\otimes}_{*|U,V}\colon R_*(U)\cup R_*(V)\xrightarrow{=}R_*(U\cup V),$$
$$R^{\otimes}_{*|\mathbb{1}}\colon \emptyset\xrightarrow{=}\emptyset,$$

natural in $U,V\in\mathcal{P}(A)$.

(6) *Symmetric Oplax Monoidality With Respect to Intersections.* The direct image function of Item 1 has a symmetric oplax monoidal structure

$$\left(R_*,R^{\otimes}_*,R^{\otimes}_{*|\mathbb{1}}\right)\colon(\mathcal{P}(A),\cap,A)\to(\mathcal{P}(B),\cap,B),$$

being equipped with inclusions

$$R^{\otimes}_{*|U,V}\colon R_*(U\cap V)\subset R_*(U)\cap R_*(V),$$
$$R^{\otimes}_{*|\mathbb{1}}\colon R_*(A)\subset B,$$

natural in $U,V\in\mathcal{P}(A)$.

(7) *Relation to Direct Images With Compact Support.* We have

$$R_*(U)=B\setminus R_!(A\setminus U)$$

for each $U\in\mathcal{P}(A)$.

**Proof.** *Item 1, Functoriality*: Clear.
*Item 2, Adjointness*: This follows from **??** of **??**.
*Item 3, Preservation of Colimits*: This follows from Item 2 and **??** of **??**.
*Item 4, Oplax Preservation of Limits*: Omitted.
*Item 5, Symmetric Strict Monoidality With Respect to Unions*: This follows from Item 3.
*Item 6, Symmetric Oplax Monoidality With Respect to Intersections*: This follows from Item 4.
*Item 7, Relation to Direct Images With Compact Support*: The proof proceeds in the same way as in the case of functions (**??** of **??**): applying Item 7 of Proposition 5.5.4.1.3 to $A\setminus U$, we have

$$\begin{aligned}R_!(A\setminus U)&=B\setminus R_*(A\setminus(A\setminus U))\\&=B\setminus R_*(U).\end{aligned}$$

Taking complements, we then obtain

$$\begin{aligned}R_*(U)&=B\setminus(B\setminus R_*(U)),\\&=B\setminus R_!(A\setminus U),\end{aligned}$$

which finishes the proof. □

**Proposition 5.5.1.1.4.** Let $R\colon A\nrightarrow B$ be a relation.

(1) *Functionality I.* The assignment $R\mapsto R_*$ defines a function

$$(-)_*\colon \mathrm{Rel}(A,B)\to\mathsf{Sets}(\mathcal{P}(A),\mathcal{P}(B)).$$

(2) *Functionality II.* The assignment $R\mapsto R_*$ defines a function

$$(-)_*\colon \mathrm{Rel}(A,B)\to\mathsf{Pos}((\mathcal{P}(A),\subset),(\mathcal{P}(B),\subset)).$$

(3) *Interaction With Identities.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have[59]

$$(\chi_A)_* = \mathrm{id}_{\mathcal{P}(A)};$$

(4) *Interaction With Composition.* For each pair of composable relations $R\colon A \nrightarrow B$ and $S\colon B \nrightarrow C$, we have[60]

$$(S \diamond R)_* = S_* \circ R_*, \qquad \begin{tikzcd} \mathcal{P}(A) \arrow[r, "R_*"] \arrow[dr, "(S\diamond R)_*"'] & \mathcal{P}(B) \arrow[d, "S_*"] \\ & \mathcal{P}(C). \end{tikzcd}$$

**Proof.** *Item 1, Functionality I*: Clear.
*Item 2, Functionality II*: Clear.
*Item 3, Interaction With Identities*: Indeed, we have

$$\begin{aligned} (\chi_A)_*(U) &\stackrel{\text{def}}{=} \bigcup_{a \in U} \chi_A(a) \\ &\stackrel{\text{def}}{=} \bigcup_{a \in U} \{a\} \\ &= U \\ &\stackrel{\text{def}}{=} \mathrm{id}_{\mathcal{P}(A)}(U) \end{aligned}$$

for each $U \in \mathcal{P}(A)$. Thus $(\chi_A)_* = \mathrm{id}_{\mathcal{P}(A)}$.
*Item 4, Interaction With Composition*: Indeed, we have

$$\begin{aligned} (S \diamond R)_*(U) &\stackrel{\text{def}}{=} \bigcup_{a \in U} [S \diamond R](a) \\ &\stackrel{\text{def}}{=} \bigcup_{a \in U} S(R(a)) \\ &\stackrel{\text{def}}{=} \bigcup_{a \in U} S_*(R(a)) \\ &= S_*\left(\bigcup_{a \in U} R(a)\right) \\ &\stackrel{\text{def}}{=} S_*(R_*(U)) \\ &\stackrel{\text{def}}{=} [S_* \circ R_*](U) \end{aligned}$$

for each $U \in \mathcal{P}(A)$, where we used Item 3 of Proposition 5.5.1.1.3. Thus $(S \diamond R)_* = S_* \circ R_*$. □

[59]That is, the postcomposition function

$$(\chi_A)_*\colon \mathrm{Rel}(\mathrm{pt}, A) \to \mathrm{Rel}(\mathrm{pt}, A)$$

is equal to $\mathrm{id}_{\mathrm{Rel}(\mathrm{pt},A)}$.
[60]That is, we have

$$(S \diamond R)_* = S_* \circ R_*, \qquad \begin{tikzcd} \mathrm{Rel}(\mathrm{pt}, A) \arrow[r, "R_*"] \arrow[dr, "(S\diamond R)_*"'] & \mathrm{Rel}(\mathrm{pt}, B) \arrow[d, "S_*"] \\ & \mathrm{Rel}(\mathrm{pt}, C). \end{tikzcd}$$

**5.5.2. Strong Inverse Images.** Let $A$ and $B$ be sets and let $R\colon A \nrightarrow B$ be a relation.

**Definition 5.5.2.1.1.** The **strong inverse image function associated to** $R$ is the function

$$R_{-1}\colon \mathcal{P}(B) \to \mathcal{P}(A)$$

defined by[61]

$$R_{-1}(V) \overset{\text{def}}{=} \{a \in A \mid R(a) \subset V\}$$

for each $V \in \mathcal{P}(B)$.

**Remark 5.5.2.1.2.** Identifying subsets of $B$ with relations from pt to $B$ via **??** of **??**, we see that the inverse image function associated to $R$ is equivalently the function

$$R_{-1}\colon \underbrace{\mathcal{P}(B)}_{\cong \mathrm{Rel}(\mathrm{pt},B)} \to \underbrace{\mathcal{P}(A)}_{\cong \mathrm{Rel}(\mathrm{pt},A)}$$

defined by

$$R_{-1}(V) \overset{\text{def}}{=} \mathrm{Rift}_R(V), \qquad \begin{tikzcd} & A \arrow[d, "R", "+"' marking] \arrow[dl, phantom, "\Downarrow"] \\ \mathrm{pt} \arrow[ur, dashed, "\mathrm{Rift}_R(V)", "\times" marking] \arrow[r, "V"', "+" marking] & B, \end{tikzcd}$$

and being explicitly computed by

$$\begin{aligned} R_{-1}(V) &\overset{\text{def}}{=} \mathrm{Rift}_R(V) \\ &\cong \int_{x \in B} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R^x_{-1}, V^x_{-2}), \end{aligned}$$

where we have used Item 12 of Proposition 5.2.5.1.1.

[61] *Further Terminology:* The set $R_{-1}(V)$ is called the **strong inverse image of $V$ by $R$**.

**Proof.** We have

$$
\begin{aligned}
\mathrm{Rift}_R(V) &\cong \int_{x\in B} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\big(R^x_{-_1}, V^x_{-_2}\big)\\
&= \left\{ a\in A \;\middle|\; \int_{x\in B} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R^x_a, V^x_\star) = \mathsf{true} \right\}\\
&= \left\{ a\in A \;\middle|\; \begin{array}{l}
\text{for each } x\in B, \text{ at least one of the}\\
\text{following conditions hold:}\\[6pt]
\quad 1.\ \text{We have } R^x_a = \mathsf{false}\\
\quad 2.\ \text{The following conditions hold:}\\[6pt]
\qquad \text{(a) We have } R^x_a = \mathsf{true}\\
\qquad \text{(b) We have } V^x_\star = \mathsf{true}
\end{array} \right\}\\
&= \left\{ a\in A \;\middle|\; \begin{array}{l}
\text{for each } x\in B, \text{ at least one of the}\\
\text{following conditions hold:}\\[6pt]
\quad 1.\ \text{We have } x\notin R(a)\\
\quad 2.\ \text{The following conditions hold:}\\[6pt]
\qquad \text{(a) We have } x\in R(a)\\
\qquad \text{(b) We have } x\in V
\end{array} \right\}\\
&= \{a\in A \mid \text{for each } x\in R(a), \text{ we have } x\in V\}\\
&= \{a\in A \mid R(a)\subset V\}\\
&\overset{\text{def}}{=} R_{-1}(V).
\end{aligned}
$$

This finishes the proof. □

**Proposition 5.5.2.1.3.** Let $R\colon A \nrightarrow B$ be a relation.

(1) *Functoriality.* The assignment $V \mapsto R_{-1}(V)$ defines a functor

$$R_{-1}\colon (\mathcal{P}(B),\subset) \to (\mathcal{P}(A),\subset)$$

where

- *Action on Objects.* For each $V\in\mathcal{P}(B)$, we have
$$[R_{-1}](V) \overset{\text{def}}{=} R_{-1}(V);$$
- *Action on Morphisms.* For each $U,V\in\mathcal{P}(B)$:
  - If $U\subset V$, then $R_{-1}(U)\subset R_{-1}(V)$.

(2) *Adjointness.* We have an adjunction

$$
(R_* \dashv R_{-1})\colon\quad \begin{tikzcd} \mathcal{P}(A) \arrow[r, bend left, "R_*"] \arrow[r, phantom, "\perp"] & \mathcal{P}(B), \arrow[l, bend left, "R_{-1}"] \end{tikzcd}
$$

witnessed by a bijections of sets

$$\mathrm{Hom}_{\mathcal{P}(A)}(R_*(U),V) \cong \mathrm{Hom}_{\mathcal{P}(A)}(U, R_{-1}(V)),$$

natural in $U\in\mathcal{P}(A)$ and $V\in\mathcal{P}(B)$, i.e. such that:

($\star$) The following conditions are equivalent:
  - We have $R_*(U)\subset V$;

– We have $U \subset R_{-1}(V)$.

(3) *Lax Preservation of Colimits.* We have an inclusion of sets

$$\bigcup_{i\in I} R_{-1}(U_i) \subset R_{-1}\left(\bigcup_{i\in I} U_i\right),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(B)^{\times I}$. In particular, we have inclusions

$$R_{-1}(U) \cup R_{-1}(V) \subset R_{-1}(U \cup V),$$
$$\emptyset \subset R_{-1}(\emptyset),$$

natural in $U, V \in \mathcal{P}(B)$.

(4) *Preservation of Limits.* We have an equality of sets

$$R_{-1}\left(\bigcap_{i\in I} U_i\right) = \bigcap_{i\in I} R_{-1}(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(B)^{\times I}$. In particular, we have equalities

$$R_{-1}(U \cap V) = R_{-1}(U) \cap R_{-1}(V),$$
$$R_{-1}(B) = B,$$

natural in $U, V \in \mathcal{P}(B)$.

(5) *Symmetric Lax Monoidality With Respect to Unions.* The direct image with compact support function of Item 1 has a symmetric lax monoidal structure

$$\left(R_{-1}, R_{-1}^{\otimes}, R_{-1|\mathbb{1}}^{\otimes}\right) : (\mathcal{P}(A), \cup, \emptyset) \to (\mathcal{P}(B), \cup, \emptyset),$$

being equipped with inclusions

$$R_{-1|U,V}^{\otimes} : R_{-1}(U) \cup R_{-1}(V) \subset R_{-1}(U \cup V),$$
$$R_{-1|\mathbb{1}}^{\otimes} : \emptyset \subset R_{-1}(\emptyset),$$

natural in $U, V \in \mathcal{P}(B)$.

(6) *Symmetric Strict Monoidality With Respect to Intersections.* The direct image function of Item 1 has a symmetric strict monoidal structure

$$\left(R_{-1}, R_{-1}^{\otimes}, R_{-1|\mathbb{1}}^{\otimes}\right) : (\mathcal{P}(A), \cap, A) \to (\mathcal{P}(B), \cap, B),$$

being equipped with equalities

$$R_{-1|U,V}^{\otimes} : R_{-1}(U \cap V) \xrightarrow{=} R_{-1}(U) \cap R_{-1}(V),$$
$$R_{-1|\mathbb{1}}^{\otimes} : R_{-1}(A) \xrightarrow{=} B,$$

natural in $U, V \in \mathcal{P}(B)$.

(7) *Interaction With Weak Inverse Images I.* We have

$$R_{-1}(V) = A \setminus R^{-1}(B \setminus V)$$

for each $V \in \mathcal{P}(B)$.

(8) *Interaction With Weak Inverse Images II.* Let $R \colon A \nrightarrow B$ be a relation from $A$ to $B$.

(a) If $R$ is a total relation, then we have an inclusion of sets

$$R_{-1}(V) \subset R^{-1}(V)$$

natural in $V \in \mathcal{P}(B)$.

(b) If $R$ is total and functional, then the above inclusion is in fact an equality.

(c) Conversely, if we have $R_{-1} = R^{-1}$, then $R$ is total and functional.

**Proof.** *Item 1, Functoriality*: Clear.
*Item 2, Adjointness*: This follows from **??** of **??**.
*Item 3, Lax Preservation of Colimits*: Omitted.
*Item 4, Preservation of Limits*: This follows from Item 2 and **??** of **??**.
*Item 5, Symmetric Lax Monoidality With Respect to Unions*: This follows from Item 3.
*Item 6, Symmetric Strict Monoidality With Respect to Intersections*: This follows from Item 4.
*Item 7, Interaction With Weak Inverse Images I*: We claim we have an equality

$$R_{-1}(B \setminus V) = A \setminus R^{-1}(V).$$

Indeed, we have

$$\begin{aligned} R_{-1}(B \setminus V) &= \{a \in A \mid R(a) \subset B \setminus V\}, \\ A \setminus R^{-1}(V) &= \{a \in A \mid R(a) \cap V = \emptyset\}. \end{aligned}$$

Taking $V = B \setminus V$ then implies the original statement.
*Item 8, Interaction With Weak Inverse Images II*: Item 8a is clear, while Items 8b and 8c follow from Item 6 of Proposition 5.3.1.1.2. □

**Proposition 5.5.2.1.4.** Let $R\colon A \nrightarrow B$ be a relation.

(1) *Functionality I.* The assignment $R \mapsto R_{-1}$ defines a function

$$(-)_{-1}\colon \mathsf{Sets}(A, B) \to \mathsf{Sets}(\mathcal{P}(A), \mathcal{P}(B)).$$

(2) *Functionality II.* The assignment $R \mapsto R_{-1}$ defines a function

$$(-)_{-1}\colon \mathsf{Sets}(A, B) \to \mathsf{Pos}((\mathcal{P}(A), \subset), (\mathcal{P}(B), \subset)).$$

(3) *Interaction With Identities.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$(\mathrm{id}_A)_{-1} = \mathrm{id}_{\mathcal{P}(A)};$$

(4) *Interaction With Composition.* For each pair of composable relations $R\colon A \nrightarrow B$ and $S\colon B \nrightarrow C$, we have

$$(S \diamond R)_{-1} = R_{-1} \circ S_{-1}, \qquad \begin{tikzcd} \mathcal{P}(C) \arrow[r, "S_{-1}"] \arrow[dr, "(S\diamond R)_{-1}"'] & \mathcal{P}(B) \arrow[d, "R_{-1}"] \\ & \mathcal{P}(A). \end{tikzcd}$$

**Proof.** *Item 1, Functionality I*: Clear.
*Item 2, Functionality II*: Clear.
*Item 3, Interaction With Identities*: Indeed, we have

$$\begin{aligned} (\chi_A)_{-1}(U) &\overset{\text{def}}{=} \{a \in A \mid \chi_A(a) \subset U\} \\ &\overset{\text{def}}{=} \{a \in A \mid \{a\} \subset U\} \\ &= U \end{aligned}$$

for each $U \in \mathcal{P}(A)$. Thus $(\chi_A)_{-1} = \mathrm{id}_{\mathcal{P}(A)}$.

*Item 4, Interaction With Composition*: Indeed, we have

$$\begin{aligned}
(S \diamond R)_{-1}(U) &\stackrel{\text{def}}{=} \{a \in A \mid [S \diamond R](a) \subset U\}\\
&\stackrel{\text{def}}{=} \{a \in A \mid S(R(a)) \subset U\}\\
&\stackrel{\text{def}}{=} \{a \in A \mid S_*(R(a)) \subset U\}\\
&= \{a \in A \mid R(a) \subset S_{-1}(U)\}\\
&\stackrel{\text{def}}{=} R_{-1}(S_{-1}(U))\\
&\stackrel{\text{def}}{=} [R_{-1} \circ S_{-1}](U)
\end{aligned}$$

for each $U \in \mathcal{P}(C)$, where we used Item 2 of Proposition 5.5.2.1.3, which implies that the conditions

- We have $S_*(R(a)) \subset U$;
- We have $R(a) \subset S_{-1}(U)$;

are equivalent. Thus $(S \diamond R)_{-1} = R_{-1} \circ S_{-1}$. □

**5.5.3. Weak Inverse Images.** Let $A$ and $B$ be sets and let $R \colon A \nrightarrow B$ be a relation.

**Definition 5.5.3.1.1.** The **weak inverse image function associated to** $R$[62] is the function

$$R^{-1} \colon \mathcal{P}(B) \to \mathcal{P}(A)$$

defined by[63]

$$R^{-1}(V) \stackrel{\text{def}}{=} \{a \in A \mid R(a) \cap V \neq \emptyset\}$$

for each $V \in \mathcal{P}(B)$.

**Remark 5.5.3.1.2.** Identifying subsets of $B$ with relations from $B$ to pt via **??** of **??**, we see that the weak inverse image function associated to $R$ is equivalently the function

$$R^{-1} \colon \underbrace{\mathcal{P}(B)}_{\cong \mathrm{Rel}(B,\mathrm{pt})} \to \underbrace{\mathcal{P}(A)}_{\cong \mathrm{Rel}(A,\mathrm{pt})}$$

defined by

$$R^{-1}(V) \stackrel{\text{def}}{=} V \diamond R$$

for each $V \in \mathcal{P}(A)$, where $R \diamond V$ is the composition

$$A \overset{R}{\nrightarrow} B \overset{V}{\nrightarrow} \mathrm{pt}.$$

Explicitly, we have

$$\begin{aligned}
R^{-1}(V) &\stackrel{\text{def}}{=} V \diamond R\\
&\stackrel{\text{def}}{=} \int^{x \in B} V^{-1}_x \times R^x_{-2}.
\end{aligned}$$

[62] *Further Terminology:* Also called simply the **inverse image function associated to** $R$.

[63] *Further Terminology:* The set $R^{-1}(V)$ is called the **weak inverse image of** $V$ **by** $R$ or simply the **inverse image of** $V$ **by** $R$.

**Proof.** We have

$$
\begin{aligned}
V \diamond R &\overset{\text{def}}{=} \int^{x \in B} V_x^{-1} \times R^x_{-2} \\
&= \left\{ a \in A \;\middle|\; \int^{x \in B} V_x^{\star} \times R^x_a = \mathsf{true} \right\} \\
&= \left\{ a \in A \;\middle|\; \begin{array}{l} \text{there exists } x \in B \text{ such that the} \\ \text{following conditions hold:} \\ \quad 1.\ \text{We have } V_x^{\star} = \mathsf{true} \\ \quad 2.\ \text{We have } R^x_a = \mathsf{true} \end{array} \right\} \\
&= \left\{ a \in A \;\middle|\; \begin{array}{l} \text{there exists } x \in B \text{ such that the} \\ \text{following conditions hold:} \\ \quad 1.\ \text{We have } x \in V \\ \quad 2.\ \text{We have } x \in R(a) \end{array} \right\} \\
&= \{ a \in A \mid \text{there exists } x \in V \text{ such that } x \in R(a) \} \\
&= \{ a \in A \mid R(a) \cap V \neq \emptyset \} \\
&\overset{\text{def}}{=} R^{-1}(V)
\end{aligned}
$$

This finishes the proof. □

**Proposition 5.5.3.1.3.** Let $R \colon A \nrightarrow B$ be a relation.

(1) *Functoriality.* The assignment $V \mapsto R^{-1}(V)$ defines a functor

$$R^{-1} \colon (\mathcal{P}(B), \subset) \to (\mathcal{P}(A), \subset)$$

where

- *Action on Objects.* For each $V \in \mathcal{P}(B)$, we have
$$[R^{-1}](V) \overset{\text{def}}{=} R^{-1}(V);$$
- *Action on Morphisms.* For each $U, V \in \mathcal{P}(B)$:
  - If $U \subset V$, then $R^{-1}(U) \subset R^{-1}(V)$.

(2) *Adjointness.* We have an adjunction

$$\left(R^{-1} \dashv R_!\right)\colon \quad \begin{tikzcd} \mathcal{P}(B) \arrow[r, bend left, "R^{-1}"] \arrow[r, phantom, "\perp"] & \mathcal{P}(A) \arrow[l, bend left, "R_!"] \end{tikzcd},$$

witnessed by a bijections of sets

$$\mathrm{Hom}_{\mathcal{P}(A)}\left(R^{-1}(U), V\right) \cong \mathrm{Hom}_{\mathcal{P}(A)}(U, R_!(V)),$$

natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$, i.e. such that:

($\star$) The following conditions are equivalent:
  - We have $R^{-1}(U) \subset V$;
  - We have $U \subset R_!(V)$.

(3) *Preservation of Colimits.* We have an equality of sets
$$R^{-1}\left(\bigcup_{i\in I} U_i\right) = \bigcup_{i\in I} R^{-1}(U_i),$$
natural in $\{U_i\}_{i\in I} \in \mathcal{P}(B)^{\times I}$. In particular, we have equalities
$$\begin{aligned} R^{-1}(U) \cup R^{-1}(V) &= R^{-1}(U \cup V), \\ R^{-1}(\emptyset) &= \emptyset, \end{aligned}$$
natural in $U, V \in \mathcal{P}(B)$.

(4) *Oplax Preservation of Limits.* We have an inclusion of sets
$$R^{-1}\left(\bigcap_{i\in I} U_i\right) \subset \bigcap_{i\in I} R^{-1}(U_i),$$
natural in $\{U_i\}_{i\in I} \in \mathcal{P}(B)^{\times I}$. In particular, we have inclusions
$$\begin{aligned} R^{-1}(U \cap V) &\subset R^{-1}(U) \cap R^{-1}(V), \\ R^{-1}(A) &\subset B, \end{aligned}$$
natural in $U, V \in \mathcal{P}(B)$.

(5) *Symmetric Strict Monoidality With Respect to Unions.* The direct image function of Item 1 has a symmetric strict monoidal structure
$$\left(R^{-1}, R^{-1,\otimes}, R^{-1,\otimes}_{\mathbb{1}}\right) \colon (\mathcal{P}(A), \cup, \emptyset) \to (\mathcal{P}(B), \cup, \emptyset),$$
being equipped with equalities
$$\begin{aligned} R^{-1,\otimes}_{U,V} \colon R^{-1}(U) \cup R^{-1}(V) &\xrightarrow{=} R^{-1}(U \cup V), \\ R^{-1,\otimes}_{\mathbb{1}} \colon \emptyset &\xrightarrow{=} \emptyset, \end{aligned}$$
natural in $U, V \in \mathcal{P}(B)$.

(6) *Symmetric Oplax Monoidality With Respect to Intersections.* The direct image function of Item 1 has a symmetric oplax monoidal structure
$$\left(R^{-1}, R^{-1,\otimes}, R^{-1,\otimes}_{\mathbb{1}}\right) \colon (\mathcal{P}(A), \cap, A) \to (\mathcal{P}(B), \cap, B),$$
being equipped with inclusions
$$\begin{aligned} R^{-1,\otimes}_{U,V} \colon R^{-1}(U \cap V) &\subset R^{-1}(U) \cap R^{-1}(V), \\ R^{-1,\otimes}_{\mathbb{1}} \colon R^{-1}(A) &\subset B, \end{aligned}$$
natural in $U, V \in \mathcal{P}(B)$.

(7) *Interaction With Strong Inverse Images I.* We have
$$R^{-1}(V) = A \setminus R_{-1}(B \setminus V)$$
for each $V \in \mathcal{P}(B)$.

(8) *Interaction With Strong Inverse Images II.* Let $R \colon A \nrightarrow B$ be a relation from $A$ to $B$.
   (a) If $R$ is a total relation, then we have an inclusion of sets
   $$R_{-1}(V) \subset R^{-1}(V)$$
   natural in $V \in \mathcal{P}(B)$.

(b) If $R$ is total and functional, then the above inclusion is in fact an equality.

(c) Conversely, if we have $R_{-1} = R^{-1}$, then $R$ is total and functional.

**Proof.** *Item 1, Functoriality*: Clear.
*Item 2, Adjointness*: This follows from **??** of **??**.
*Item 3, Preservation of Colimits*: This follows from Item 2 and **??** of **??**.
*Item 4, Oplax Preservation of Limits*: Omitted.
*Item 5, Symmetric Strict Monoidality With Respect to Unions*: This follows from Item 3.
*Item 6, Symmetric Oplax Monoidality With Respect to Intersections*: This follows from Item 4.
*Item 7, Interaction With Strong Inverse Images I*: This follows from Item 7 of Proposition 5.5.2.1.3.
*Item 8, Interaction With Strong Inverse Images II*: This was proved in Item 8 of Proposition 5.5.2.1.3. □

**Proposition 5.5.3.1.4.** Let $R\colon A \nrightarrow B$ be a relation.

(1) *Functionality I.* The assignment $R \mapsto R^{-1}$ defines a function

$$(-)^{-1}\colon \mathrm{Rel}(A, B) \to \mathsf{Sets}(\mathcal{P}(A), \mathcal{P}(B)).$$

(2) *Functionality II.* The assignment $R \mapsto R^{-1}$ defines a function

$$(-)^{-1}\colon \mathrm{Rel}(A, B) \to \mathsf{Pos}((\mathcal{P}(A), \subset), (\mathcal{P}(B), \subset)).$$

(3) *Interaction With Identities.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have[64]

$$(\chi_A)^{-1} = \mathrm{id}_{\mathcal{P}(A)};$$

(4) *Interaction With Composition.* For each pair of composable relations $R\colon A \nrightarrow B$ and $S\colon B \nrightarrow C$, we have[65]

$$(S \diamond R)^{-1} = R^{-1} \circ S^{-1}, \qquad \begin{tikzcd} \mathcal{P}(C) \arrow[r, "S^{-1}"] \arrow[dr, "(S\diamond R)^{-1}"'] & \mathcal{P}(B) \arrow[d, "R^{-1}"] \\ & \mathcal{P}(A). \end{tikzcd}$$

**Proof.** *Item 1, Functionality I*: Clear.
*Item 2, Functionality II*: Clear.
*Item 3, Interaction With Identities*: This follows from **??** of **??**.

[64]That is, the postcomposition

$$(\chi_A)^{-1}\colon \mathrm{Rel}(\mathrm{pt}, A) \to \mathrm{Rel}(\mathrm{pt}, A)$$

is equal to $\mathrm{id}_{\mathrm{Rel}(\mathrm{pt},A)}$.

[65]That is, we have

$$(S \diamond R)^{-1} = R^{-1} \circ S^{-1}, \qquad \begin{tikzcd} \mathrm{Rel}(\mathrm{pt}, C) \arrow[r, "R^{-1}"] \arrow[dr, "(S\diamond R)^{-1}"'] & \mathrm{Rel}(\mathrm{pt}, B) \arrow[d, "S^{-1}"] \\ & \mathrm{Rel}(\mathrm{pt}, A). \end{tikzcd}$$

*Item 4*, *Interaction With Composition*: This follows from **??** of **??**. □

**5.5.4. Direct Images With Compact Support.** Let $A$ and $B$ be sets and let $R\colon A \nrightarrow B$ be a relation.

**Definition 5.5.4.1.1.** The **direct image with compact support function associated to $R$** is the function[66]

$$R_!\colon \mathcal{P}(A) \to \mathcal{P}(B)$$

defined by[67,68]

$$\begin{aligned} R_!(U) &\overset{\text{def}}{=} \left\{ b \in B \;\middle|\; \begin{array}{l} \text{for each } a \in A, \text{ if we have} \\ b \in R(a), \text{ then } a \in U \end{array} \right\} \\ &= \left\{ b \in B \;\middle|\; R^{-1}(b) \subset U \right\} \end{aligned}$$

for each $U \in \mathcal{P}(A)$.

**Remark 5.5.4.1.2.** Identifying subsets of $B$ with relations from pt to $B$ via **??** of **??**, we see that the direct image with compact support function associated to $R$ is equivalently the function

$$R_!\colon \underbrace{\mathcal{P}(A)}_{\cong \mathrm{Rel}(A,\mathrm{pt})} \to \underbrace{\mathcal{P}(B)}_{\cong \mathrm{Rel}(B,\mathrm{pt})}$$

defined by

$$R_!(U) \overset{\text{def}}{=} \mathrm{Ran}_R(U), \qquad \begin{tikzcd} & B \arrow[d, dashed, "\mathrm{Ran}_R(U)", "|" marking] \\ A \arrow[ur, "R", "/" marking] \arrow[r, "U"', "|" marking] & \mathrm{pt}, \end{tikzcd}$$

being explicitly computed by

$$\begin{aligned} R^*(U) &\overset{\text{def}}{=} \mathrm{Ran}_R(U) \\ &\cong \int_{a \in A} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R_a^{-2}, U_a^{-1}), \end{aligned}$$

where we have used Item 11 of Proposition 5.2.5.1.1.

---

[66] *Further Notation:* Also written $\forall_R\colon \mathcal{P}(A) \to \mathcal{P}(B)$. This notation comes from the fact that the following statements are equivalent, where $b \in B$ and $U \in \mathcal{P}(A)$:

- We have $b \in \forall_R(U)$.
- For each $a \in A$, if $b \in R(a)$, then $a \in U$.

[67] *Further Terminology:* The set $R_!(U)$ is called the **direct image with compact support of $U$ by $R$**.

[68] We also have

$$R_!(U) = B \setminus R_*(A \setminus U);$$

see Item 7 of Proposition 5.5.4.1.3.

**Proof.** We have

$$
\begin{aligned}
\mathrm{Ran}_R(V) &\cong \int_{a\in A} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^{-_2}_a, U^{-_1}_a\right)\\
&= \left\{ b\in B \;\middle|\; \int_{a\in A} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^b_a, U^\star_a\right) = \mathsf{true} \right\}\\
&= \left\{ b\in B \;\middle|\; \begin{array}{l} \text{for each } a\in A\text{, at least one of the}\\ \text{following conditions hold:}\\ \quad 1.\ \text{We have } R^b_a = \mathsf{false}\\ \quad 2.\ \text{The following conditions hold:}\\ \qquad \text{(a) We have } R^b_a = \mathsf{true}\\ \qquad \text{(b) We have } U^\star_a = \mathsf{true} \end{array} \right\}\\
&= \left\{ b\in B \;\middle|\; \begin{array}{l} \text{for each } a\in A\text{, at least one of the}\\ \text{following conditions hold:}\\ \quad 1.\ \text{We have } b\notin R(A)\\ \quad 2.\ \text{The following conditions hold:}\\ \qquad \text{(a) We have } b\in R(a)\\ \qquad \text{(b) We have } a\in U \end{array} \right\}\\
&= \left\{ b\in B \;\middle|\; \begin{array}{l} \text{for each } a\in A\text{, if we have}\\ b\in R(a)\text{, then } a\in U \end{array} \right\}\\
&= \left\{ b\in B \;\middle|\; R^{-1}(b)\subset U \right\}\\
&\overset{\text{def}}{=} R^{-1}(U).
\end{aligned}
$$

This finishes the proof. □

**Proposition 5.5.4.1.3.** Let $R\colon A \nrightarrow B$ be a relation.

(1) *Functoriality.* The assignment $U \mapsto R_!(U)$ defines a functor

$$R_!\colon (\mathcal{P}(A), \subset) \to (\mathcal{P}(B), \subset)$$

where

- *Action on Objects.* For each $U\in\mathcal{P}(A)$, we have
$$[R_!](U) \overset{\text{def}}{=} R_!(U);$$
- *Action on Morphisms.* For each $U, V\in\mathcal{P}(A)$:
  - If $U\subset V$, then $R_!(U)\subset R_!(V)$.

(2) *Adjointness.* We have an adjunction

$$
\left(R^{-1} \dashv R_!\right)\colon\quad
\begin{tikzcd}
\mathcal{P}(B) \arrow[r, bend left, "R^{-1}"] \arrow[r, phantom, "\perp"] & \mathcal{P}(A), \arrow[l, bend left, "R_!"]
\end{tikzcd}
$$

witnessed by a bijections of sets

$$\mathrm{Hom}_{\mathcal{P}(A)}\left(R^{-1}(U), V\right) \cong \mathrm{Hom}_{\mathcal{P}(A)}(U, R_!(V)),$$

natural in $U\in\mathcal{P}(A)$ and $V\in\mathcal{P}(B)$, i.e. such that:

(⋆) The following conditions are equivalent:
- We have $R^{-1}(U) \subset V$;
- We have $U \subset R_!(V)$.

(3) *Lax Preservation of Colimits.* We have an inclusion of sets
$$\bigcup_{i\in I} R_!(U_i) \subset R_!\left(\bigcup_{i\in I} U_i\right),$$
natural in $\{U_i\}_{i\in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have inclusions
$$\begin{aligned} R_!(U) \cup R_!(V) &\subset R_!(U \cup V), \\ \emptyset &\subset R_!(\emptyset), \end{aligned}$$
natural in $U, V \in \mathcal{P}(A)$.

(4) *Preservation of Limits.* We have an equality of sets
$$R_!\left(\bigcap_{i\in I} U_i\right) = \bigcap_{i\in I} R_!(U_i),$$
natural in $\{U_i\}_{i\in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have equalities
$$\begin{aligned} R_!(U \cap V) &= R_!(U) \cap R_!(V), \\ R_!(A) &= B, \end{aligned}$$
natural in $U, V \in \mathcal{P}(A)$.

(5) *Symmetric Lax Monoidality With Respect to Unions.* The direct image with compact support function of Item 1 has a symmetric lax monoidal structure
$$\left(R_!, R_!^{\otimes}, R_{!|\mathbb{1}}^{\otimes}\right) : (\mathcal{P}(A), \cup, \emptyset) \to (\mathcal{P}(B), \cup, \emptyset),$$
being equipped with inclusions
$$\begin{aligned} R_{!|U,V}^{\otimes} &: R_!(U) \cup R_!(V) \subset R_!(U \cup V), \\ R_{!|\mathbb{1}}^{\otimes} &: \emptyset \subset R_!(\emptyset), \end{aligned}$$
natural in $U, V \in \mathcal{P}(A)$.

(6) *Symmetric Strict Monoidality With Respect to Intersections.* The direct image function of Item 1 has a symmetric strict monoidal structure
$$\left(R_!, R_!^{\otimes}, R_{!|\mathbb{1}}^{\otimes}\right) : (\mathcal{P}(A), \cap, A) \to (\mathcal{P}(B), \cap, B),$$
being equipped with equalities
$$\begin{aligned} R_{!|U,V}^{\otimes} &: R_!(U \cap V) \xrightarrow{=} R_!(U) \cap R_!(V), \\ R_{!|\mathbb{1}}^{\otimes} &: R_!(A) \xrightarrow{=} B, \end{aligned}$$
natural in $U, V \in \mathcal{P}(A)$.

(7) *Relation to Direct Images.* We have
$$R_!(U) = B \setminus R_*(A \setminus U)$$
for each $U \in \mathcal{P}(A)$.

**Proof.** *Item 1, Functoriality*: Clear.
*Item 2, Adjointness*: This follows from **??** of **??**.

*Item 3, Lax Preservation of Colimits*: Omitted.
*Item 4, Preservation of Limits*: This follows from Item 2 and **??** of **??**.
*Item 5, Symmetric Lax Monoidality With Respect to Unions*: This follows from Item 3.
*Item 6, Symmetric Strict Monoidality With Respect to Intersections*: This follows from Item 4.
*Item 7, Relation to Direct Images*: This follows from Item 7 of Proposition 5.5.1.1.3. Alternatively, we may prove it directly as follows, with the proof proceeding in the same way as in the case of functions (**??** of **??**). We claim that $R_!(U) = B \setminus R_*(A \setminus U)$:

- *The First Implication.* We claim that
$$R_!(U) \subset B \setminus R_*(A \setminus U).$$
Let $b \in R_!(U)$. We need to show that $b \notin R_*(A \setminus U)$, i.e. that there is no $a \in A \setminus U$ such that $b \in R(a)$.
This is indeed the case, as otherwise we would have $a \in R^{-1}(b)$ and $a \notin U$, contradicting $R^{-1}(b) \subset U$ (which holds since $b \in R_!(U)$).
Thus $b \in B \setminus R_*(A \setminus U)$.
- *The Second Implication.* We claim that
$$B \setminus R_*(A \setminus U) \subset R_!(U).$$
Let $b \in B \setminus R_*(A \setminus U)$. We need to show that $b \in R_!(U)$, i.e. that $R^{-1}(b) \subset U$.
Since $b \notin R_*(A \setminus U)$, there exists no $a \in A \setminus U$ such that $b \in R(a)$, and hence $R^{-1}(b) \subset U$.
Thus $b \in R_!(U)$.

This finishes the proof. □

**Proposition 5.5.4.1.4.** Let $R\colon A \nrightarrow B$ be a relation.

(1) *Functionality I.* The assignment $R \mapsto R_!$ defines a function
$$(-)_!\colon \mathsf{Sets}(A, B) \to \mathsf{Sets}(\mathcal{P}(A), \mathcal{P}(B)).$$
(2) *Functionality II.* The assignment $R \mapsto R_!$ defines a function
$$(-)_!\colon \mathsf{Sets}(A, B) \to \mathrm{Hom}_{\mathsf{Pos}}((\mathcal{P}(A), \subset), (\mathcal{P}(B), \subset)).$$
(3) *Interaction With Identities.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have
$$(\mathrm{id}_A)_! = \mathrm{id}_{\mathcal{P}(A)};$$
(4) *Interaction With Composition.* For each pair of composable relations $R\colon A \nrightarrow B$ and $S\colon B \nrightarrow C$, we have
$$(S \diamond R)_! = S_! \circ R_!, \qquad \begin{tikzcd} \mathcal{P}(A) \arrow[r, "R_!"] \arrow[dr, "(S\diamond R)_!"'] & \mathcal{P}(B) \arrow[d, "S_!"] \\ & \mathcal{P}(C). \end{tikzcd}$$

**Proof.** *Item 1, Functionality I*: Clear.
*Item 2, Functionality II*: Clear.

*Item 3, Interaction With Identities*: Indeed, we have

$$\begin{aligned}
(\chi_A)_!(U) &\stackrel{\text{def}}{=} \left\{a \in A \,\middle|\, \chi_A^{-1}(a) \subset U\right\}\\
&\stackrel{\text{def}}{=} \{a \in A \mid \{a\} \subset U\}\\
&= U
\end{aligned}$$

for each $U \in \mathcal{P}(A)$. Thus $(\chi_A)_! = \mathrm{id}_{\mathcal{P}(A)}$.
*Item 4, Interaction With Composition*: Indeed, we have

$$\begin{aligned}
(S \diamond R)_!(U) &\stackrel{\text{def}}{=} \left\{c \in C \,\middle|\, [S \diamond R]^{-1}(c) \subset U\right\}\\
&\stackrel{\text{def}}{=} \left\{c \in C \,\middle|\, S^{-1}\Big(R^{-1}(c)\Big) \subset U\right\}\\
&= \left\{c \in C \,\middle|\, R^{-1}(c) \subset S_!(U)\right\}\\
&\stackrel{\text{def}}{=} R_!(S_!(U))\\
&\stackrel{\text{def}}{=} [R_! \circ S_!](U)
\end{aligned}$$

for each $U \in \mathcal{P}(C)$, where we used Item 2 of Proposition 5.5.4.1.3, which implies that the conditions

- We have $S^{-1}(R^{-1}(c)) \subset U$;
- We have $R^{-1}(c) \subset S_!(U)$;

are equivalent. Thus $(S \diamond R)_! = S_! \circ R_!$. □

### 5.5.5. Functoriality of Powersets.

**Proposition 5.5.5.1.1.** The assignment $X \mapsto \mathcal{P}(X)$ defines functors[69]

$$\begin{aligned}
\mathcal{P}_* &\colon \mathrm{Rel} \to \mathsf{Sets},\\
\mathcal{P}_{-1} &\colon \mathrm{Rel}^{\mathsf{op}} \to \mathsf{Sets},\\
\mathcal{P}^{-1} &\colon \mathrm{Rel}^{\mathsf{op}} \to \mathsf{Sets},\\
\mathcal{P}_! &\colon \mathrm{Rel} \to \mathsf{Sets}
\end{aligned}$$

where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathrm{Rel})$, we have
$$\begin{aligned}
\mathcal{P}_*(A) &\stackrel{\text{def}}{=} \mathcal{P}(A),\\
\mathcal{P}_{-1}(A) &\stackrel{\text{def}}{=} \mathcal{P}(A),\\
\mathcal{P}^{-1}(A) &\stackrel{\text{def}}{=} \mathcal{P}(A),\\
\mathcal{P}_!(A) &\stackrel{\text{def}}{=} \mathcal{P}(A);
\end{aligned}$$
- *Action on Morphisms.* For each morphism $R \colon A \nrightarrow B$ of Rel, the images
$$\begin{aligned}
\mathcal{P}_*(R) &\colon \mathcal{P}(A) \to \mathcal{P}(B),\\
\mathcal{P}_{-1}(R) &\colon \mathcal{P}(B) \to \mathcal{P}(A),\\
\mathcal{P}^{-1}(R) &\colon \mathcal{P}(B) \to \mathcal{P}(A),\\
\mathcal{P}_!(R) &\colon \mathcal{P}(A) \to \mathcal{P}(B)
\end{aligned}$$

[69] The functor $\mathcal{P}_* \colon \mathrm{Rel} \to \mathsf{Sets}$ admits a left adjoint; see Item 3 of Proposition 5.3.1.1.2.

of $R$ by $\mathcal{P}_*$, $\mathcal{P}_{-1}$, $\mathcal{P}^{-1}$, and $\mathcal{P}_!$ are defined by

$$\begin{aligned}\mathcal{P}_*(R) &\overset{\text{def}}{=} R_*,\\ \mathcal{P}_{-1}(R) &\overset{\text{def}}{=} R_{-1},\\ \mathcal{P}^{-1}(R) &\overset{\text{def}}{=} R^{-1},\\ \mathcal{P}_!(R) &\overset{\text{def}}{=} R_!,\end{aligned}$$

as in Definitions 5.5.1.1.1, 5.5.2.1.1, 5.5.3.1.1 and 5.5.4.1.1.

**Proof.** This follows from Items 3 and 4 of Proposition 5.5.1.1.4, Items 3 and 4 of Proposition 5.5.2.1.4, Items 3 and 4 of Proposition 5.5.3.1.4, and Items 3 and 4 of Proposition 5.5.4.1.4. □

**5.5.6. Functoriality of Powersets: Relations on Powersets.** Let $A$ and $B$ be sets and let $R\colon A \mathrel{\mapstochar\mkern-5mu\rightarrow} B$ be a relation.

**Definition 5.5.6.1.1.** The **relation on powersets associated to** $R$ is the relation

$$\mathcal{P}(R)\colon \mathcal{P}(A) \mathrel{\mapstochar\mkern-5mu\rightarrow} \mathcal{P}(B)$$

defined by[70]

$$\mathcal{P}(R)^V_U \overset{\text{def}}{=} \mathbf{Rel}(\chi_{\text{pt}}, V \diamond R \diamond U)$$

for each $U \in \mathcal{P}(A)$ and each $V \in \mathcal{P}(B)$.

**Remark 5.5.6.1.2.** In detail, we have $U \sim_{\mathcal{P}(R)} V$ iff the following equivalent conditions hold:

- We have $\chi_{\text{pt}} \subset V \diamond R \diamond U$.
- We have $(V \diamond R \diamond U)^\star_\star = \mathsf{true}$, i.e. we have
$$\int^{a\in A}\int^{b\in B} V^\star_b \times R^b_a \times U^a_\star = \mathsf{true}.$$
- There exists some $a \in A$ and some $b \in B$ such that:
  - We have $U^a_\star = \mathsf{true}$;
  - We have $R^b_a = \mathsf{true}$;
  - We have $V^\star_b = \mathsf{true}$.
- There exists some $a \in A$ and some $b \in B$ such that:
  - We have $a \in U$;
  - We have $a \sim_R b$;
  - We have $b \in V$.

**Proposition 5.5.6.1.3.** The assignment $R \mapsto \mathcal{P}(R)$ defines a functor

$$\mathcal{P}\colon \mathrm{Rel} \to \mathrm{Rel}.$$

**Proof.** Omitted. □

## 5.6. Relative Preorders

**5.6.1. The Left Skew Monoidal Structure on Rel$(A, B)$.** Let $A$ and $B$ be sets and let $J\colon A \mathrel{\mapstochar\mkern-5mu\rightarrow} B$ be a relation.

[70]Illustration:

$$\begin{tikzcd} \mathrm{pt} \arrow[r, "U"'] \arrow[rrr, bend left, "\chi_{\mathrm{pt}}"] & A \arrow[r, "R"'] & B \arrow[r, "V"'] & \mathrm{pt}. \end{tikzcd}$$

5.6.1.1. *The Left Skew Monoidal Product.*

**Definition 5.6.1.1.1.** The **left $J$-skew monoidal product of $\mathbf{Rel}(A,B)$** is the functor

$$\triangleleft_J \colon \mathbf{Rel}(A,B)\times\mathbf{Rel}(A,B)\to\mathbf{Rel}(A,B)$$

where

- *Action on Objects.* For each $R,S\in\mathrm{Obj}(\mathbf{Rel}(A,B))$, we have

$$S\triangleleft_J R\overset{\text{def}}{=}S\diamond\mathrm{Rift}_J(R),\qquad \begin{tikzcd} & A \arrow[r, "S", "\shortmid"{marking}] \arrow[d, "J", "\shortmid"{marking}] \arrow[dl, phantom, "\Leftarrow"] & B; \\ A \arrow[ur, dashed, "\mathrm{Rift}_J(R)", "\shortmid"{marking}] \arrow[r, "R"', "\shortmid"{marking}] & B \end{tikzcd}$$

- *Action on Morphisms.* For each $R,S,R',S'\in\mathrm{Obj}(\mathbf{Rel}(A,B))$, the action on Hom-sets

$$(\triangleleft_J)_{(G,F),(G',F')}\colon \mathrm{Hom}_{\mathbf{Rel}(A,B)}\big(S,S'\big)\times\mathrm{Hom}_{\mathbf{Rel}(A,B)}\big(R,R'\big)\to\mathrm{Hom}_{\mathbf{Rel}(A,B)}\big(S\triangleleft_J R,S'\triangleleft_J R'\big)$$

  of $\triangleleft_J$ at $((R,S),(R',S'))$ is defined by[71]

$$\beta\triangleleft_J\alpha\overset{\text{def}}{=}\beta\diamond\mathrm{Rift}_J(\alpha),\qquad \begin{tikzcd} & A \arrow[r, bend left, "S", "\shortmid"{marking}] \arrow[r, bend right, "S'"', "\shortmid"{marking}] \arrow[r, phantom, "\beta\Downarrow"] \arrow[d, "J", "\shortmid"{marking}] \arrow[dl, phantom, "\Leftarrow"] & B \\ A \arrow[ur, dashed, bend left, "\mathrm{Rift}_J(R)"] \arrow[ur, dashed, "\mathrm{Rift}_J(R')"'] \arrow[r, "R", "\shortmid"{marking}] \arrow[r, bend right, "R'"', "\shortmid"{marking}] & B \end{tikzcd}$$

  for each $\beta\in\mathrm{Hom}_{\mathbf{Rel}(A,B)}(S,S')$ and each $\alpha\in\mathrm{Hom}_{\mathbf{Rel}(A,B)}(R,R')$.

5.6.1.2. *The Left Skew Monoidal Unit.*

**Definition 5.6.1.2.1.** The **left $J$-skew monoidal unit of $\mathbf{Rel}(A,B)$** is the functor

$$\mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleleft}\colon \mathsf{pt}\to\mathbf{Rel}(A,B)$$

picking the object

$$\mathbb{1}^{\triangleleft}_{\mathbf{Rel}(A,B)}\overset{\text{def}}{=}J$$

of $\mathbf{Rel}(A,B)$.

5.6.1.3. *The Left Skew Associators.*

**Definition 5.6.1.3.1.** The **left $J$-skew associator of $\mathbf{Rel}(A,B)$** is the natural transformation

$$\alpha^{\mathbf{Rel}(A,B),\triangleleft}\colon \triangleleft_J\circ(\triangleleft_J\times\mathsf{id})\Longrightarrow\triangleleft_J\circ(\mathsf{id}\times\triangleleft_J),$$

[71]Since $\mathbf{Rel}(A,B)$ is posetal, this is to say that if $S\subset S'$ and $R\subset R'$, then $S\triangleleft_J R\subset S'\triangleleft_J R'$.

whose component

$$\alpha^{\mathbf{Rel}(A,B),\triangleleft}_{T,S,R}\colon \underbrace{(T \triangleleft_J S) \triangleleft_J R}_{\stackrel{\text{def}}{=} T\diamond\mathrm{Rift}_J(S)\diamond\mathrm{Rift}_J(R)} \hookrightarrow \underbrace{T \triangleleft_J (S \triangleleft_J R)}_{\stackrel{\text{def}}{=} T\diamond\mathrm{Rift}_J(S\diamond\mathrm{Rift}_J(R))}$$

at $(T, S, R)$ is given by

$$\alpha^{\mathbf{Rel}(A,B),\triangleleft}_{T,S,R} \stackrel{\text{def}}{=} \mathrm{id}_T \diamond \gamma,$$

where

$$\gamma\colon \mathrm{Rift}_J(S) \diamond \mathrm{Rift}_J(R) \hookrightarrow \mathrm{Rift}_J(S \diamond \mathrm{Rift}_J(R))$$

is the inclusion adjunct to the inclusion

$$\epsilon_S \star \mathrm{id}_{\mathrm{Rift}_J(R)}\colon \underbrace{J \diamond \mathrm{Rift}_J(S) \diamond \mathrm{Rift}_J(R)}_{\stackrel{\text{def}}{=} J_*(\mathrm{Rift}_J(S)\diamond\mathrm{Rift}_J(R))} \hookrightarrow S \diamond \mathrm{Rift}_J(R)$$

under the adjunction $J_* \dashv \mathrm{Rift}_J$, where $\epsilon\colon J \diamond \mathrm{Rift}_J \Longrightarrow \mathrm{id}_{\mathbf{Rel}(A,B)}$ is the counit of the adjunction $J_* \dashv \mathrm{Rift}_J$.

5.6.1.4. *The Left Skew Left Unitors.*

**Definition 5.6.1.4.1.** The **left $J$-skew left unitor of $\mathbf{Rel}(A, B)$** is the natural transformation

$$\lambda^{\mathbf{Rel}(A,B),\triangleleft}\colon \triangleleft_J \circ \left(\not\Vdash^{\mathbf{Rel}(A,B)}_{\triangleleft} \times \mathsf{id}\right) \Longrightarrow \mathsf{id},$$

whose component

$$\lambda^{\mathbf{Rel}(A,B),\triangleleft}_R\colon \underbrace{J \triangleleft_J R}_{\stackrel{\text{def}}{=} J\diamond\mathrm{Rift}_J(R)} \hookrightarrow R$$

at $R$ is given by

$$\lambda^{\mathbf{Rel}(A,B),\triangleleft}_R \stackrel{\text{def}}{=} \epsilon_R,$$

where $\epsilon\colon J \diamond \mathrm{Rift}_J \Longrightarrow \mathrm{id}_{\mathbf{Rel}(A,B)}$ is the counit of the adjunction $J_* \dashv \mathrm{Rift}_J$.

5.6.1.5. *The Left Skew Right Unitors.*

**Definition 5.6.1.5.1.** The **left $J$-skew right unitor of $\mathbf{Rel}(A, B)$** is the natural transformation

$$\rho^{\mathbf{Rel}(A,B),\triangleleft}\colon \mathsf{id} \Longrightarrow \triangleleft_J \circ \left(\mathsf{id} \times \not\Vdash^{\mathbf{Rel}(A,B)}_{\triangleleft}\right),$$

whose component

$$\rho^{\mathbf{Rel}(A,B),\triangleleft}_R\colon R \hookrightarrow \underbrace{R \triangleleft_J J}_{\stackrel{\text{def}}{=} R\diamond\mathrm{Rift}_J(J)}$$

at $R$ is given by

$$\rho^{\mathbf{Rel}(A,B),\triangleleft}_R \stackrel{\text{def}}{=} \mathrm{id}_R \star \sigma,$$

where $\sigma\colon \mathrm{id}_A \Longrightarrow \mathrm{Rift}_J(J)$ is the universal transformation included in the data of the right Kan lift $\mathrm{Rift}_J(J)$.

5.6.1.6. *The Left Skew Monoidal Structure on* ***Rel***$(A, B)$.

**Definition 5.6.1.6.1.** The **left $J$-skew monoidal category of relations from $A$ to $B$** is the left skew monoidal category

$$\left(\mathbf{Rel}(A, B), \triangleleft_J, \not\Vdash^{\mathbf{Rel}(A,B)}_{\triangleleft}, \alpha^{\mathbf{Rel}(A,B),\triangleleft}, \lambda^{\mathbf{Rel}(A,B),\triangleleft}, \rho^{\mathbf{Rel}(A,B),\triangleleft}\right)$$

consisting of

- *The Underlying Category.* The posetal category associated to the poset $\mathbf{Rel}(A, B)$ of relations from $A$ to $B$ of Item 2 of Definition 5.1.1.1.2;
- *The Skew Monoidal Product.* The functor $\triangleleft_J$ of Definition 5.6.1.1.1;
- *The Skew Monoidal Unit.* The functor $\mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleleft}$ of Definition 5.6.1.2.1;
- *The Skew Associators.* The natural transformation $\alpha^{\mathbf{Rel}(A,B),\triangleleft}$ of Definition 5.6.1.3.1;
- *The Skew Left Unitors.* The natural transformation $\lambda^{\mathbf{Rel}(A,B),\triangleleft}$ of Definition 5.6.1.4.1;
- *The Skew Right Unitors.* The natural transformation $\rho^{\mathbf{Rel}(A,B),\triangleleft}$ of Definition 5.6.1.5.1.

**5.6.2. Left Relative Preorders.** Let $A$ and $B$ be sets and let $J \colon A \nrightarrow B$ be a relation.

**Definition 5.6.2.1.1.** A **left $J$-relative preorder from $A$ to $B$** is equivalently:

- An $\mathbb{E}_1$-skew monoid in $(\mathrm{N}_\bullet(\mathbf{Rel}(A, B)), \triangleleft_J, J)$;
- A skew monoid in $(\mathbf{Rel}(A, B), \triangleleft_J, J)$.

**Remark 5.6.2.1.2.** In detail, a **left $J$-relative preorder** $(R, \mu_R, \eta_R)$ **from $A$ to $B$** consists of

- *The Underlying Relation.* A relation
$$R \colon A \nrightarrow B,$$
called the **underlying relation of** $(R, \mu_R, \eta_R)$;
- *The Multiplication Inclusion.* An inclusion of relations
$$\mu_R \colon R \triangleleft_J R \subset R,$$
called the **multiplication** of $(R, \mu_R, \eta_R)$;
- *The Unit Inclusion.* An inclusion of relations
$$\eta_R \colon J \subset R,$$
called the **unit** of $(R, \mu_R, \eta_R)$.

**Remark 5.6.2.1.3.** In other words, a **left $J$-relative preorder from $A$ to $B$** is a relation $R \colon A \nrightarrow B$ from $A$ to $B$ satisfying the following conditions:

(1) *$J$-Transitivity.* For each $a \in A$ and each $c \in B$, we have
$$a \sim_{R \diamond \mathrm{Rift}_J(R)} c$$
i.e. the following condition is satisfied:[72]

[72] *Illustration:* If we have

$$\begin{tikzcd}
 & \vdots & & \\
 & x & & \\
a \arrow[uur, squiggly, "R"] \arrow[ur, squiggly, "R"] \arrow[r, squiggly, "R"] \arrow[dr, squiggly, "R"'] \arrow[ddr, squiggly, "R"'] & x' & b \arrow[uul, squiggly, "J"'] \arrow[ul, squiggly, "J"'] \arrow[l, squiggly, "J"'] \arrow[dl, squiggly, "J"] \arrow[ddl, squiggly, "J"] \arrow[r, squiggly, "R"] & c \\
 & x'' & & \\
 & \vdots & &
\end{tikzcd}$$

- (★) If there exists some $b \in A$ such that:
  - We have $a \sim_{\mathrm{Rift}_J(R)} b$, i.e. for each $x \in B$, if $b \sim_J x$, then $a \sim_R x$;[73]
  - We have $b \sim_R c$;

  then $a \sim_R c$.

(2) *$J$-Unitality.* For each $a \in A$ and each $b \in B$, the following condition is satisfied:
- (★) If $a \sim_J b$, then $a \sim_R b$.

**5.6.3. The Right Skew Monoidal Structure on $\mathbf{Rel}(A, B)$.** Let $A$ and $B$ be sets and let $J\colon A \nrightarrow B$ be a relation.

5.6.3.1. *The Right Skew Monoidal Product.*

**Definition 5.6.3.1.1.** The **right $J$-skew monoidal product of $\mathbf{Rel}(A, B)$** is the functor

$$\triangleright_J \colon \mathbf{Rel}(A, B) \times \mathbf{Rel}(A, B) \to \mathbf{Rel}(A, B)$$

where

- *Action on Objects.* For each $R, S \in \mathrm{Obj}(\mathbf{Rel}(A, B))$, we have

$$S \triangleright_J R \overset{\mathrm{def}}{=} \mathrm{Ran}_J(S) \diamond R, \qquad \begin{tikzcd} A \arrow[r, "R", "\shortmid" marking] & B \arrow[r, dashed, "\mathrm{Ran}_J(S)", "\shortmid" marking] & B; \\ & A \arrow[u, "J", "\shortmid" marking] \arrow[ur, "S"'] \arrow[u, phantom, shift left=4, "\Leftarrow"] & \end{tikzcd}$$

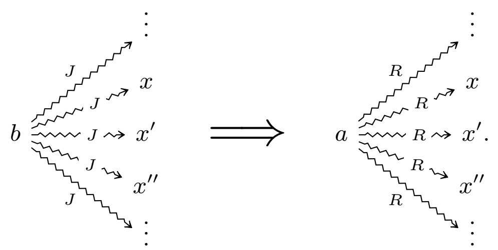
- *Action on Morphisms.* For each $R, S, R', S' \in \mathrm{Obj}(\mathbf{Rel}(A, B))$, the action on Hom-sets

$$(\triangleright_J)_{(S,R),(S',R')} \colon \mathrm{Hom}_{\mathbf{Rel}(A,B)}\big(S, S'\big) \times \mathrm{Hom}_{\mathbf{Rel}(A,B)}\big(R, R'\big) \to \mathrm{Hom}_{\mathbf{Rel}(A,B)}\big(S \triangleright_J R, S' \triangleright_J R'\big)$$

---

then $a \sim_R c$.

[73]Illustration:



of $\triangleright_J$ at $((S,R),(S',R'))$ is defined by[74]

$$\beta \triangleright_J \alpha \overset{\text{def}}{=} \mathrm{Ran}_J(\beta) \diamond \alpha, \qquad \begin{tikzcd} A \arrow[r, bend left, "R'"] \arrow[r, bend right, "R"'] \arrow[r, phantom, "\alpha\Uparrow"] & B \arrow[rr, dashed, bend left, "\mathrm{Ran}_J(S')"] \arrow[rr, dashed, "\mathrm{Ran}_J(S)"'] \arrow[rr, phantom, bend left=15, "\mathrm{Ran}_J(\beta)\Uparrow"] & & B \\ & A \arrow[u, "J"] \arrow[urr, "S'"] \arrow[urr, bend right, "S"'] \arrow[urr, phantom, bend right=15, "\Uparrow\beta"] & & \end{tikzcd}$$

for each $\beta \in \mathrm{Hom}_{\mathbf{Rel}(A,B)}(S,S')$ and each $\alpha \in \mathrm{Hom}_{\mathbf{Rel}(A,B)}(R,R')$.

5.6.3.2. *The Right Skew Monoidal Unit.*

**Definition 5.6.3.2.1.** The **right $J$-skew monoidal unit of $\mathbf{Rel}(A,B)$** is the functor

$$\mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleright} \colon \mathsf{pt} \to \mathbf{Rel}(A,B)$$

picking the object

$$\mathbb{1}^{\triangleright}_{\mathbf{Rel}(A,B)} \overset{\text{def}}{=} J$$

of $\mathbf{Rel}(A,B)$.

5.6.3.3. *The Right Skew Associators.*

**Definition 5.6.3.3.1.** The **right $J$-skew associator of $\mathbf{Rel}(A,B)$** is the natural transformation

$$\alpha^{\mathbf{Rel}(A,B),\triangleright} \colon \triangleright_J \circ (\mathsf{id} \times \triangleright_J) \Longrightarrow \triangleright_J \circ (\triangleright_J \times \mathsf{id}),$$

whose component

$$\alpha^{\mathbf{Rel}(A,B),\triangleright}_{T,S,R} \colon \underbrace{T \triangleright_J (S \triangleright_J R)}_{\overset{\text{def}}{=}\mathrm{Ran}_J(T)\diamond(\mathrm{Ran}_J(S)\diamond R)} \hookrightarrow \underbrace{(T \triangleright_J S) \triangleright_J R}_{\overset{\text{def}}{=}\mathrm{Ran}_J(\mathrm{Ran}_J(T)\diamond S)\diamond R}$$

at $(T,S,R)$ is given by

$$\alpha^{\mathbf{Rel}(A,B),\triangleright}_{T,S,R} \overset{\text{def}}{=} \gamma \diamond \mathrm{id}_R,$$

where

$$\gamma \colon \mathrm{Ran}_J(T) \diamond \mathrm{Ran}_J(S) \hookrightarrow \mathrm{Ran}_J(\mathrm{Ran}_J(T) \diamond S)$$

is the inclusion adjunct to the inclusion

$$\mathrm{id}_{\mathrm{Ran}_J(T)} \diamond \epsilon_S \colon \underbrace{\mathrm{Ran}_J(T) \diamond \mathrm{Ran}_J(S) \diamond J}_{\overset{\text{def}}{=} J^*(\mathrm{Ran}_J(T)\diamond\mathrm{Ran}_J(S))} \hookrightarrow \mathrm{Ran}_J(T) \diamond S$$

under the adjunction $J^* \dashv \mathrm{Ran}_J$, where $\epsilon \colon \mathrm{Ran}_J \diamond J \Longrightarrow \mathrm{id}_{\mathbf{Rel}(A,B)}$ is the counit of the adjunction $J^* \dashv \mathrm{Ran}_J$.

5.6.3.4. *The Right Skew Left Unitors.*

**Definition 5.6.3.4.1.** The **right $J$-skew left unitor of $\mathbf{Rel}(A,B)$** is the natural transformation

$$\lambda^{\mathbf{Rel}(A,B),\triangleright} \colon \mathsf{id} \Longrightarrow \triangleright_J \circ \left(\mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleright} \times \mathsf{id}\right),$$

[74]Since $\mathbf{Rel}(A,B)$ is posetal, this is to say that if $S \subset S'$ and $R \subset R'$, then $S \triangleright_J R \subset S' \triangleright_J R'$.

whose component

$$\lambda^{\mathbf{Rel}(A,B),\triangleright}_{R} : R \hookrightarrow \underbrace{J \triangleright_J R}_{\overset{\text{def}}{=}\mathrm{Ran}_J(J)\diamond R}$$

at $R$ is given by

$$\lambda^{\mathbf{Rel}(A,B),\triangleright}_{R} \overset{\text{def}}{=} \sigma \diamond \mathrm{id}_R,$$

where $\sigma\colon \mathrm{id}_B \Longrightarrow \mathrm{Ran}_J(J)$ is the universal transformation included in the data of the right Kan extension $\mathrm{Ran}_J(J)$.

5.6.3.5. *The Right Skew Right Unitors.*

**Definition 5.6.3.5.1.** The **right $J$-skew right unitor of $\mathbf{Rel}(A,B)$** is the natural transformation

$$\rho^{\mathbf{Rel}(A,B),\triangleright} : \triangleright_J \circ \left(\mathsf{id} \times \Bbbk^{\mathbf{Rel}(A,B)}_{\triangleright}\right) \Longrightarrow \mathsf{id},$$

whose component

$$\rho^{\mathbf{Rel}(A,B),\triangleright}_{S} : \underbrace{S \triangleright_J J}_{\overset{\text{def}}{=}\mathrm{Ran}_J(S)\diamond J} \hookrightarrow S$$

at $S$ is given by

$$\rho^{\mathbf{Rel}(A,B),\triangleright}_{S} \overset{\text{def}}{=} \epsilon_R,$$

where $\epsilon\colon \mathrm{Ran}_J \diamond J \Longrightarrow \mathrm{id}_{\mathbf{Rel}(A,B)}$ is the counit of the adjunction $J^* \dashv \mathrm{Ran}_J$.

5.6.3.6. *The Right Skew Monoidal Structure on **Rel**$(A,B)$.*

**Definition 5.6.3.6.1.** The **right $J$-skew monoidal category of functors from $A$ to $B$** is the right skew monoidal category

$$\left(\mathbf{Rel}(A,B), \triangleright_J, \Bbbk^{\mathbf{Rel}(A,B)}_{\triangleright}, \alpha^{\mathbf{Rel}(A,B),\triangleright}, \lambda^{\mathbf{Rel}(A,B),\triangleright}, \rho^{\mathbf{Rel}(A,B),\triangleright}\right)$$

consisting of

- *The Underlying Category.* The posetal category associated to the poset $\mathbf{Rel}(A,B)$ of relations from $A$ to $B$ of Item 2 of Definition 5.1.1.1.2;
- *The Skew Monoidal Product.* The functor $\triangleright_J$ of Definition 5.6.3.1.1;
- *The Skew Monoidal Unit.* The functor $\Bbbk^{\mathbf{Rel}(A,B)}_{\triangleright}$ of Definition 5.6.3.2.1;
- *The Skew Associators.* The natural transformation $\alpha^{\mathbf{Rel}(A,B),\triangleright}$ of Definition 5.6.3.3.1;
- *The Skew Left Unitors.* The natural transformation $\lambda^{\mathbf{Rel}(A,B),\triangleright}$ of Definition 5.6.3.4.1;
- *The Skew Right Unitors.* The natural transformation $\rho^{\mathbf{Rel}(A,B),\triangleright}$ of Definition 5.6.3.5.1.

**5.6.4. Right Relative Preorders.** Let $A$ and $B$ be sets and let $J\colon A \nrightarrow B$ be a relation.

**Definition 5.6.4.1.1.** A **right $J$-relative preorder from $A$ to $B$** is equivalently:

- An $\mathbb{E}_1$-skew monoid in $(\mathrm{N}_\bullet(\mathbf{Rel}(A,B)), \triangleright_J, J)$;
- A skew monoid in $(\mathbf{Rel}(A,B), \triangleright_J, J)$.

**Remark 5.6.4.1.2.** In detail, a **right $J$-relative preorder** $(R, \mu_R, \eta_R)$ **from $A$ to $B$** consists of

- *The Underlying Relation.* A relation
$$R\colon A \nrightarrow B,$$
called the **underlying relation of** $(R, \mu_R, \eta_R)$;
- *The Multiplication Inclusion.* An inclusion of relations
$$\mu_R\colon R \mathbin{\triangleright_J} R \subset R,$$
called the **multiplication** of $(R, \mu_R, \eta_R)$;
- *The Unit Inclusion.* An inclusion of relations
$$\eta_R\colon J \subset R,$$
called the **unit** of $(R, \mu_R, \eta_R)$.

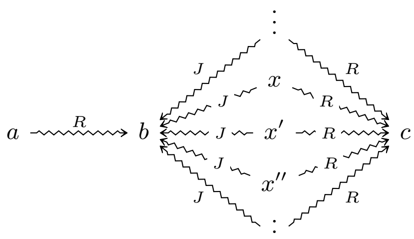
**Remark 5.6.4.1.3.** In other words, a **right $J$-relative preorder from $A$ to $B$** is a relation $R\colon A \nrightarrow B$ from $A$ to $B$ satisfying the following conditions:

(1) *$J$-Transitivity.* For each $a \in A$ and each $c \in B$, we have
$$a \sim_{\mathrm{Ran}_J(R) \diamond R} c,$$
i.e. the following condition is satisfied:[75]
 (⋆) If there exists some $b \in B$ such that:
  – We have $a \sim_R b$;
  – We have $b \sim_{\mathrm{Ran}_J(R)} c$, i.e. for each $x \in A$, if $x \sim_J b$, then $x \sim_R c$;[76]
 then $a \sim_R c$.

(2) *$J$-Unitality.* For each $a \in A$ and each $b \in B$, the following condition is satisfied:
 (⋆) If $a \sim_J b$, then $a \sim_R b$.

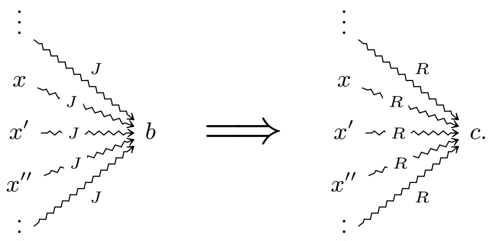
# Appendices

[75] *Illustration:* If we have




then $a \sim_R c$.

[76] Illustration:



## 5.A. Other Chapters

**Sets**

(1) Sets
(2) Constructions With Sets
(3) Pointed Sets
(4) Tensor Products of Pointed Sets
(5) Relations
(6) Spans
(7) Posets

**Indexed and Fibred Sets**

(7) Indexed Sets
(8) Fibred Sets
(9) Un/Straightening for Indexed and Fibred Sets

**Category Theory**

(11) Categories
(12) Types of Morphisms in Categories
(13) Adjunctions and the Yoneda Lemma
(14) Constructions With Categories
(15) Profunctors
(16) Cartesian Closed Categories
(17) Kan Extensions

**Bicategories**

(18) Bicategories
(19) Internal Adjunctions

**Internal Category Theory**

(20) Internal Categories

**Cyclic Stuff**

(21) The Cycle Category

**Cubical Stuff**

(22) The Cube Category

**Globular Stuff**

(23) The Globe Category

**Cellular Stuff**

(24) The Cell Category

**Monoids**

(25) Monoids
(26) Constructions With Monoids

**Monoids With Zero**

(27) Monoids With Zero
(28) Constructions With Monoids With Zero

**Groups**

(29) Groups
(30) Constructions With Groups

**Hyper Algebra**

(31) Hypermonoids
(32) Hypergroups
(33) Hypersemirings and Hyperrings
(34) Quantales

**Near-Rings**

(35) Near-Semirings
(36) Near-Rings

**Real Analysis**

(37) Real Analysis in One Variable
(38) Real Analysis in Several Variables

**Measure Theory**

(39) Measurable Spaces
(40) Measures and Integration

**Probability Theory**

(40) Probability Theory

**Stochastic Analysis**

(41) Stochastic Processes, Martingales, and Brownian Motion
(42) Itô Calculus
(43) Stochastic Differential Equations

**Differential Geometry**

(44) Topological and Smooth Manifolds

**Schemes**

(45) Schemes

## 5.2. Other Chapters

Sets

(1) Sets
(2) Constructions With Sets
(3) Pointed Sets
(4) Tensor Products of Pointed Sets
(5) Relations
(6) Spans
(7) Posets

Indexed and Fibred Sets

(7) Indexed Sets
(8) Fibred Sets
(9) Un/Straightening for Indexed and Fibred Sets

Category Theory

(11) Categories
(12) Types of Morphisms in Categories
(13) Adjunctions and the Yoneda Lemma
(14) Constructions With Categories
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(18) Bicategories
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(20) Internal Categories

Cyclic Stuff

(21) The Cycle Category

Cubical Stuff

(22) The Cube Category

Globular Stuff

(23) The Globe Category

Cellular Stuff

(24) The Cell Category

Monoids

(25) Monoids
(26) Constructions With Monoids

Monoids With Zero

(27) Monoids With Zero
(28) Constructions With Monoids With Zero

Groups

(29) Groups
(30) Constructions With Groups

Hyper Algebra

(31) Hypermonoids
(32) Hypergroups
(33) Hypersemirings and Hyperrings
(34) Quantales

Near-Rings

(35) Near-Semirings
(36) Near-Rings

Real Analysis

(37) Real Analysis in One Variable
(38) Real Analysis in Several Variables

Measure Theory

(39) Measurable Spaces
(40) Measures and Integration

Probability Theory

(40) Probability Theory

Stochastic Analysis

(41) Stochastic Processes, Martingales, and Brownian Motion
(42) Itô Calculus
(43) Stochastic Differential Equations

Differential Geometry

(44) Topological and Smooth Manifolds

Schemes

(45) Schemes

CHAPTER 6

# Spans

This chapter contains some material about spans. Notably, we discuss and explore:

(1) The basic definitions around spans (Section 6.1);
(2) The relation between spans and functions (Proposition 6.8.1.1.1);
(3) The relation between spans and relations (Propositions 6.8.2.2.1 and 6.8.3.1.1 and Remark 6.8.5.1.1).
(4) "Hyperpointed sets" (**??**). I don't know why I wrote this…

TODO:

(1) `https://www.sciencedirect.com/science/article/pii/0022404994900094?ref=pdf_download&fr=RR-2&rr=834107b75c906aa4`
(2) `https://arxiv.org/abs/1605.08100`
(3) `https://arxiv.org/abs/1603.08181`
(4) `https://arxiv.org/abs/1601.02307`
(5) `https://arxiv.org/abs/1507.01460`
(6) `https://arxiv.org/abs/1506.08870`
(7) `https://arxiv.org/abs/1505.00048`
(8) `https://arxiv.org/abs/1501.07592`
(9) `https://arxiv.org/abs/1501.04664`
(10) `https://arxiv.org/abs/1501.00792`
(11) `https://arxiv.org/abs/1412.6560`
(12) `https://arxiv.org/abs/1412.0212`
(13) `https://arxiv.org/abs/1409.0837`
(14) `https://arxiv.org/abs/1408.5220`
(15) `https://arxiv.org/abs/1308.6548`
(16) `https://arxiv.org/abs/1304.0219`
(17) `https://arxiv.org/abs/1210.8192`
(18) `https://arxiv.org/abs/1210.1433`
(19) `https://arxiv.org/abs/1201.3789`
(20) `https://arxiv.org/abs/1112.0560`
(21) `https://arxiv.org/abs/1109.1598`
(22) `https://arxiv.org/abs/1101.4594`
(23) `https://arxiv.org/abs/1012.6001`
(24) `https://arxiv.org/abs/1011.3243`
(25) `https://arxiv.org/abs/0910.2996`
(26) `https://arxiv.org/abs/0810.2361`
(27) `https://arxiv.org/abs/0803.2429`
(28) `https://arxiv.org/abs/0712.2525`
(29) `https://arxiv.org/abs/0706.1286`
(30) `https://arxiv.org/abs/math/0611930`

(31) https://arxiv.org/abs/2311.15342
(32) https://arxiv.org/abs/2310.19428
(33) https://arxiv.org/abs/2309.08084
(34) https://arxiv.org/abs/2308.01662
(35) https://arxiv.org/abs/2301.11860
(36) https://arxiv.org/abs/2301.01199
(37) https://arxiv.org/abs/2212.09060
(38) https://arxiv.org/abs/2208.07183
(39) https://arxiv.org/abs/2205.06892
(40) https://arxiv.org/abs/2203.16179
(41) https://arxiv.org/abs/2201.09551
(42) https://arxiv.org/abs/2112.04599
(43) https://arxiv.org/abs/2111.10968
(44) https://arxiv.org/abs/2107.07621
(45) https://arxiv.org/abs/2106.14743
(46) https://arxiv.org/abs/2105.14654
(47) https://arxiv.org/abs/2102.08051
(48) https://arxiv.org/abs/2102.04386
(49) https://arxiv.org/abs/2101.06734
(50) https://arxiv.org/abs/2011.11042
(51) https://arxiv.org/abs/2010.15722
(52) https://arxiv.org/abs/2006.10375
(53) https://arxiv.org/abs/2006.10375
(54) https://arxiv.org/abs/2005.10496
(55) https://arxiv.org/abs/2003.11541
(56) https://arxiv.org/abs/2002.10334
(57) https://arxiv.org/abs/1909.00069
(58) https://arxiv.org/abs/1907.02695
(59) https://arxiv.org/abs/1905.06671
(60) define a relational span
(61) consider giving Ran and Rift their dedicated sections on the relations chapter, perhaps together with the other sections on co/limits
(62) https://arxiv.org/abs/1710.02742
(63) https://arxiv.org/search/math?searchtype=author&query=Walker,+Charles
(64) https://arxiv.org/abs/1706.09575
(65) https://arxiv.org/abs/1710.01465
(66) fibred categories: https://arxiv.org/abs/1806.02376
(67) https://arxiv.org/abs/1806.10477v2
(68) double categorical limits in $\mathsf{Rel}^{\mathsf{dbl}}$
(69) double categorical limits in $\mathsf{Span}^{\mathsf{dbl}}$
(70) internal adjoint equivalences in $\mathbf{Rel}$
(71) internal adjoint equivalences in $\mathsf{Span}$
(72) 2-categorical limits in $\mathbf{Rel}$;
(73) morphism of internal adjunctions in $\mathbf{Rel}$;
(74) morphism of internal adjunctions in $\mathsf{Span}$;
(75) morphism of co/monads in $\mathsf{Span}$;
(76) What is $\mathsf{Adj}(\mathsf{Span}(A, B))$?
(77) monoids, comonoids, pseudomonoids, etc. in $\mathsf{Span}$.

(78) write down the dumb intuition about spans inducing morphisms $\mathsf{Sets}(S, A) \to \mathsf{Sets}(S, B)$ instead of $\mathcal{P}(A) \to \mathcal{P}(B)$ from the similarity between

$$S \to A \times B$$

and

$$A \times B \to \{\mathsf{t}, \mathsf{f}\}.$$

This intuition is justified by taking $A = \mathrm{pt}$ or $B = \mathrm{pt}$.

(79) What about using the direct image with compact support in $g(f^{-1}(a))$?

(80) Monads in $\mathsf{Span}$ | develop this in the level of morphisms too

(81) Comonads in $\mathsf{Span}$ are spans whose legs are equal | develop this in the level of morphisms too

(82) Does $\mathsf{Span}$ have an internal **Hom**?

(83) Examples of spans

(84) Functional and total spans

(85) closed symmetric monoidal category of spans

(86) double category of relations

(87) collage of a span

(88) equivalence spans?

(89) functoriality of powersets for spans

(90) Is $\mathsf{Span}$ a closed bicategory?

(91) skew monoidal structure on $\mathsf{Span}(A, B)$

(92) Adjunctions in $\mathsf{Span}$

(93) Isomorphisms in $\mathsf{Span}$

(94) Equivalences in $\mathsf{Span}$

(95) Interaction between the above notions in $\mathsf{Span}$ vs.in $\mathbf{Rel}$ via the comparison functors

(96) $\mathrm{Hom}_{\mathcal{C}}(S, A) \times \mathrm{Hom}_{\mathcal{C}}(f^*(S), A)$.

(97) Proof of non-existence of left Kan extensions/lifts in $\mathbf{Rel}$ (when do these exist btw?)

(98) description of unitors and associators of span

(99) add intuition for spans as relations with multiple witnesses:

   (a) Given a span $A \xleftarrow{f} S \xrightarrow{g} B$, we have a functor

   $$\mathrm{St}(S) \colon (A \times B)_{\mathsf{disc}} \to \mathsf{Sets}$$

   given by

   $$\begin{aligned}[\mathrm{St}(S)](a, b) &\stackrel{\text{def}}{=} \mathrm{St}(S)_{ab} \\ &\stackrel{\text{def}}{=} \{s \in S \mid f(s) = a \text{ and } g(s) = b\}.\end{aligned}$$

   (b) Given a functor

   $$S \colon (A \times B)_{\mathsf{disc}} \to \mathsf{Sets},$$

   we have a map of sets

   $$\mathrm{Un}(S) \colon \coprod_{(a,b) \in A \times B} S(a, b) \to A \times B,$$

   determining a span from $A$ to $B$.

   (c) How do these interact with left/right Kan extensions/lifts?

   (d) Un/straightening for spans of categories: assignment $(a, b) \mapsto \mathrm{Wit}_S(a, b)$.

(e) Fix the TODO below

## 6.1. Spans

### 6.1.1. The Walking Span.

**Definition 6.1.1.1.1.** The **walking span** is the category $\Lambda$ that looks like this:

$$\begin{tikzcd}
 & {[0]} \arrow[loop above] \arrow[dl] \arrow[dr] & \\
{[-1]} \arrow[loop left] & & {[1]} \arrow[loop right]
\end{tikzcd}$$

**6.1.2. Spans.** Let $A$ and $B$ be sets.

**Definition 6.1.2.1.1.** A **span from $A$ to $B$**[1] is a functor $F\colon \Lambda \to \mathsf{Sets}$ such that

$$\begin{aligned} F([-1]) &= A, \\ F([1]) &= B. \end{aligned}$$

**Remark 6.1.2.1.2.** In detail, a **span from $A$ to $B$** is a triple $(S, f, g)$ consisting of[2,3]

- *The Underlying Set.* A set $S$, called the **underlying set of** $(S, f, g)$;
- *The Legs.* A pair of functions $f\colon S \to A$ and $g\colon S \to B$.

**Remark 6.1.2.1.3.** A span

$$\begin{tikzcd}
 & S \arrow[dl, "f"'] \arrow[dr, "g"] & \\
A & & B.
\end{tikzcd}$$

from $A$ to $B$ may be thought of as a relation from $A$ to $B$ which can relate an element $a \in A$ to an element $b \in B$ in multiple ways via $f$ and $g$, with the "set of witnesses of $a \sim_S b$" being given by

$$\mathrm{Wit}_S(a, b) \overset{\mathrm{def}}{=} \{s \in S \mid a = f(s) \text{ and } g(s) = b\}.$$

This analogy is made precise by Remark 6.8.5.1.1 and Section 6.7.

### 6.1.3. Morphisms of Spans.

[1] *Further Terminology:* Also called a **roof from $A$ to $B$** or a **correspondence from $A$ to $B$**.

[2] *Picture:*

$$\begin{tikzcd}
 & S \arrow[dl, "f"'] \arrow[dr, "g"] & \\
A & & B.
\end{tikzcd}$$

[3] Every span $(S, f, g)$ from $A$ to $B$ determines in particular a relation $R\colon A \nrightarrow B$ via

$$R \overset{\mathrm{def}}{=} \{(f(a), g(a)) \mid a \in A\},$$

i.e. where $R(a) = g\big(f^{-1}(a)\big)$ for each $a \in A$; see Proposition 6.8.2.2.1.

**Definition 6.1.3.1.1.** A **morphism of spans from** $(R, f_1, g_1)$ **to** $(S, f_2, g_2)$[4] is a natural transformation $(R, f_1, g_1) \Longrightarrow (S, f_2, g_2)$.

**Remark 6.1.3.1.2.** In detail, a **morphism of spans from** $(R, f_1, g_1)$ **to** $(S, f_2, g_2)$ is a function $\phi\colon R \to S$ making the diagram[5]

$$\begin{tikzcd}
 & R \arrow[dl, "f_1"'] \arrow[dr, "g_1"] \arrow[rrr, "\phi"] & & & S \arrow[dl, "f_2"'] \arrow[dr, "g_2"] & \\
A \arrow[rrr, equal] & & B \arrow[rrr, equal] & A & & B.
\end{tikzcd}$$

commute.

### 6.1.4. Functional Spans.

Let $\lambda = \left(A \xleftarrow{f} S \xrightarrow{g} B\right)$ be a span.

**Definition 6.1.4.1.1.** The span $\lambda$ is **functional** if $f$ the following equivalent conditions are satisfied:

(1) The associated relation $g \circ f^{-1}$ of $\lambda$ is functional.
(2) For each $s, t \in S$, if $f(s) = f(t)$, then $g(s) = g(t)$.
(3) this "$f$-relative injectivity" condition is the same as being a monomorphism/monoid/whatever in nice category | maybe this is the same as being a skew monoid in $\mathsf{Span}(A, B)$ or something?

(1) a[6]

[4] *Further Terminology:* Also called a **morphism of roofs from** $(R, f_1, g_1)$ **to** $(S, f_2, g_2)$ or a **morphism of correspondences from** $(R, f_1, g_1)$ **to** $(S, f_2, g_2)$.

[5] *Alternative Picture:*

$$\begin{tikzcd}
 & R \arrow[dl, "f_1"'] \arrow[dr, "g_1"] \arrow[dd, "\phi"] & \\
A & & B. \\
 & S \arrow[ul, "f_2"] \arrow[ur, "g_2"'] &
\end{tikzcd}$$

[6] Here we could perhaps also use the direct image with compact support $g_!$ of $g$ (see **??**) instead of the usual direct image, although the expression for $g_!\left(f^{-1}(a)\right)$ seems a bit weird. It can also actually be given as a right Kan extension (**??** of **??**):

$$\begin{aligned}
g_!\left(f^{-1}(a)\right) &= g_!(\{s \in S \mid f(s) = a\}) \\
&= \left\{b \in B \;\middle|\; g^{-1}(b) \subset \{s \in S \mid f(s) = a\}\right\} \\
&= \{b \in B \mid \text{for each } s \in S, \text{ if } g(s) = b, \text{ then } f(s) = a\} \\
&= [\mathrm{Ran}_g^\dagger(f)](a)
\end{aligned}$$

as in the diagram

$$\mathrm{Ran}_g^\dagger(f)\colon A \nrightarrow B \qquad \begin{tikzcd}
 & B \arrow[d, dashed, "\mathrm{Ran}_g(f)"] \arrow[dl, Rightarrow, shorten=2mm] \\
S \arrow[ur, "g", "\shortmid"{marking}] \arrow[r, "f"', "\shortmid"{marking}] & A.
\end{tikzcd}$$

**Definition 6.1.4.1.2.** The span $\lambda$ is **total** if $f$ is surjective.

Let $\lambda = \left(A \overset{f}{\leftarrow} S \overset{g}{\rightarrow} B\right)$ be a span. A morphism of spans from $\mathrm{id}_A$ to $\lambda \diamond \lambda^\dagger$ is a morphism

$$s\colon A \to S \times_B S$$

making the diagram

$$\begin{tikzcd}
 & A \arrow[dl, equal, "\mathrm{id}_A"'] \arrow[dr, equal, "\mathrm{id}_A"] \arrow[dd, "s"] & \\
A & & A \\
 & S \times_B S \arrow[ul, "f \circ \mathrm{pr}'_1"] \arrow[ur, "f \circ \mathrm{pr}'_2"'] &
\end{tikzcd}$$

commute, where $S \times_B S$ is the pullback

$$S \times_B S \cong \{(s,t) \in S \times S \mid g(s) = g(t)\} \qquad \begin{tikzcd}
S \times_B S \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & S \arrow[d, "g"] \\
S \arrow[r, "g"'] & B
\end{tikzcd}$$

of $S$ with itself along $g$. In particular, $\mathrm{pr}_1 \circ s$ and $\mathrm{pr}_2 \circ s$ are both left-inverses/retractions for $f$, i.e. we have

$$(\mathrm{pr}_1 \circ s) \circ f \cong \mathrm{id}_A,$$
$$(\mathrm{pr}_2 \circ s) \circ f \cong \mathrm{id}_A.$$

Thus, by **??** of **??**, $f$ is injective if $A \neq \emptyset$.

### 6.1.5. Total Spans.

## 6.2. Categories of Spans

### 6.2.1. The Category of Spans Between Two Sets.

Let $A$ and $B$ be sets.

**Definition 6.2.1.1.1.** The **category of spans from $A$ to $B$** is the category $\mathsf{Span}(A,B)$ defined by

$$\mathsf{Span}(A,B) \overset{\text{def}}{=} \mathsf{Fun}(\Lambda, \mathsf{Sets}) \underset{\mathrm{ev}_{[-1]}, \mathsf{Sets}, [A]}{\times} \mathsf{pt} \underset{[B], \mathsf{Sets}, \mathrm{ev}_{[1]}}{\times} \mathsf{Fun}(\Lambda, \mathsf{Sets}),$$

as in the diagram

$$\begin{tikzcd}[column sep=small]
 & & & \mathsf{Span}(A,B) \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & & & \\
 & & \mathsf{Fun}(\Lambda,\mathsf{Sets}) \underset{\mathsf{Sets}}{\times} \mathsf{pt} \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & & \mathsf{pt} \underset{\mathsf{Sets}}{\times} \mathsf{Fun}(\Lambda,\mathsf{Sets}) \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & & \\
 & \mathsf{Fun}(\Lambda,\mathsf{Sets}) \arrow[dr, "\mathrm{ev}_{[-1]}"'] & & \mathsf{pt} \arrow[dl, "{[A]}"] \arrow[dr, "{[B]}"'] & & \mathsf{Fun}(\Lambda,\mathsf{Sets}). \arrow[dl, "\mathrm{ev}_{[1]}"] & \\
 & & \mathsf{Sets} & & \mathsf{Sets} & &
\end{tikzcd}$$

**Remark 6.2.1.1.2.** In detail, the **category of spans from** $A$ **to** $B$ is the category $\mathsf{Span}(A,B)$ where

- *Objects.* The objects of $\mathsf{Span}(A,B)$ are spans from $A$ to $B$;
- *Morphisms.* The morphism of $\mathsf{Span}(A,B)$ are morphisms of spans;
- *Identities.* The unit map
$$\mathbb{1}^{\mathsf{Span}(A,B)}_{(S,f,g)} \colon \mathrm{pt} \to \mathrm{Hom}_{\mathsf{Span}(A,B)}((S,f,g),(S,f,g))$$
of $\mathsf{Span}(A,B)$ at $(S,f,g)$ is defined by[7]
$$\mathrm{id}^{\mathsf{Span}(A,B)}_{(S,f,g)} \overset{\mathrm{def}}{=} \mathrm{id}_S;$$
- *Composition.* The composition map
$$\circ^{\mathsf{Span}(A,B)}_{R,S,T} \colon \mathrm{Hom}_{\mathsf{Span}(A,B)}(S,T) \times \mathrm{Hom}_{\mathsf{Span}(A,B)}(R,S) \to \mathrm{Hom}_{\mathsf{Span}(A,B)}(R,T)$$
of $\mathsf{Span}(A,B)$ at $((R,f_1,g_1),(S,f_2,g_2),(T,f_3,g_3))$ is defined by[8]
$$\psi \circ^{\mathsf{Span}(A,B)}_{R,S,T} \phi \overset{\mathrm{def}}{=} \psi \circ \phi.$$

[7] *Picture:*

$$\begin{tikzcd}
 & S \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[rr, "\mathrm{id}_S"] & & S \arrow[dl, "f"'] \arrow[dr, "g"] & \\
A \arrow[r, equal] & A & B \arrow[r, equal] & B. &
\end{tikzcd}$$

[8] *Picture:*

$$\begin{tikzcd}
 & R \arrow[dl, "f_1"'] \arrow[dr, "g_1"] \arrow[rr, "\phi"] & & S \arrow[dl, "f_2"'] \arrow[dr, "g_2"] \arrow[rr, "\psi"] & & T \arrow[dl, "f_3"'] \arrow[dr, "g_3"] & \\
A \arrow[r, equal] & A \arrow[r, equal] & A & B \arrow[r, equal] & B \arrow[r, equal] & B. &
\end{tikzcd}$$

**Proposition 6.2.1.1.3.** Let $A$ and $B$ be sets.

(1) *As a Pullback.* We have an isomorphism of categories

$$\mathsf{Span}(A,B) \cong \mathsf{Sets}_{/A} \times_{\mathsf{Sets}} \mathsf{Sets}_{/B}, \qquad \begin{tikzcd} \mathsf{Span}(A,B) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Sets}_{/B} \arrow[d, "忘"] \\ \mathsf{Sets}_{/A} \arrow[r, "忘"'] & \mathsf{Sets}. \end{tikzcd}$$

**Proof.** *Item 1, As a Pullback*: In detail, the pullback $\mathsf{Sets}_{/A} \times_{\mathsf{Sets}} \mathsf{Sets}_{/B}$ is the category where

- *Objects.* The objects of $\mathsf{Sets}_{/A} \times_{\mathsf{Sets}} \mathsf{Sets}_{/B}$ consist of pairs $((S,f),(S',g))$ of objects of $\mathsf{Sets}$ consisting of
  - A pair $(S,f)$ in $\mathrm{Obj}\big(\mathsf{Sets}_{/A}\big)$ consisting of a set $S$ and a map $f\colon S \to A$;
  - A pair $(S',g)$ in $\mathrm{Obj}\big(\mathsf{Sets}_{/B}\big)$ consisting of a set $S'$ and a map $g\colon S \to B$;

  such that
  $$\underbrace{忘(S,f)}_{\overset{\mathrm{def}}{=} S} = \underbrace{忘(S',g)}_{\overset{\mathrm{def}}{=} S'}.$$
  Thus the objects of $\mathsf{Sets}_{/A} \times_{\mathsf{Sets}} \mathsf{Sets}_{/B}$ are the same as spans from $A$ to $B$.
- *Morphisms.* A morphism of $\mathsf{Sets}_{/A} \times_{\mathsf{Sets}} \mathsf{Sets}_{/B}$ from $(S,f,g)$ to $(S',f',g')$ consists of a pair of morphisms
  $$\phi\colon S \to S'$$
  $$\psi\colon S \to S'$$
  such that the diagrams
  $$\begin{tikzcd} S \arrow[rr, "\phi"] \arrow[dr, "f"'] & & S' \arrow[dl, "f'"] \\ & A & \end{tikzcd} \qquad \begin{tikzcd} S \arrow[rr, "\psi"] \arrow[dr, "g"'] & & S' \arrow[dl, "g'"] \\ & B & \end{tikzcd}$$
  such that
  $$\underbrace{忘(\phi)}_{\overset{\mathrm{def}}{=} \phi} = \underbrace{忘(\psi)}_{\overset{\mathrm{def}}{=} \psi}.$$
  Thus the morphisms of $\mathsf{Sets}_{/A} \times_{\mathsf{Sets}} \mathsf{Sets}_{/B}$ are also the same as morphisms of spans from $(S,f,g)$ to $(S,f',g')$.
- *Identities and Composition.* The identities and composition of $\mathsf{Sets}_{/A} \times_{\mathsf{Sets}} \mathsf{Sets}_{/B}$ are also the same as those in $\mathsf{Span}(A,B)$.

This finishes the proof. □

### 6.2.2. The Bicategory of Spans.

**Definition 6.2.2.1.1.** The **bicategory of spans** is the bicategory $\mathsf{Span}$ where

- *Objects.* The objects of $\mathsf{Span}$ are sets;

- $\mathsf{Hom}$-*Categories.* For each $A, B \in \mathrm{Obj}(\mathsf{Span})$, we have
$$\mathsf{Hom}_{\mathsf{Span}}(A, B) \overset{\mathrm{def}}{=} \mathsf{Span}(A, B);$$
- *Identities.* For each $A \in \mathrm{Obj}(\mathsf{Span})$, the unit functor
$$\mathbb{1}^{\mathsf{Span}}_{A} \colon \mathsf{pt} \to \mathsf{Span}(A, A)$$
of $\mathsf{Span}$ at $A$ is the functor picking the span $(A, \mathrm{id}_A, \mathrm{id}_A)$:
$$\begin{tikzcd} & A \arrow[dl, "\mathrm{id}_A"'] \arrow[dr, "\mathrm{id}_A"] & \\ A & & A. \end{tikzcd}$$
- *Composition.* For each $A, B, C \in \mathrm{Obj}(\mathsf{Span})$, the composition bifunctor
$$\circ^{\mathsf{Span}}_{A,B,C} \colon \mathsf{Span}(B, C) \times \mathsf{Span}(A, B) \to \mathsf{Span}(A, C)$$
of $\mathsf{Span}$ at $(A, B, C)$ is the bifunctor where
  - *Action on Objects.* The composition of two spans
$$\begin{tikzcd} & R \arrow[dl, "f_1"'] \arrow[dr, "g_1"] & \\ A & & B \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & S \arrow[dl, "f_2"'] \arrow[dr, "g_2"] & \\ B & & C \end{tikzcd}$$
is the span $(R \times_B S, f_1 \circ \mathrm{pr}_1, g_2 \circ \mathrm{pr}_2)$, constructed as in the diagram
$$\begin{tikzcd} & & R \times_B S \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] \arrow[ddll, bend right, "f_1 \circ \mathrm{pr}_1"'] \arrow[ddrr, bend left, "g_2 \circ \mathrm{pr}_2"] & & \\ & R \arrow[dl, "f_1"'] \arrow[dr, "g_1"] & & S \arrow[dl, "f_2"'] \arrow[dr, "g_2"] & \\ A & & B & & C. \end{tikzcd}$$
  - *Action on Morphisms.* The horizontal composition of 2-morphisms is defined via functoriality of pullbacks: given morphisms of spans
$$\begin{tikzcd} & R \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[dd, "\phi"] & \\ A & & B \\ & R' \arrow[ul, "f'"] \arrow[ur, "g'"'] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & S \arrow[dl, "h"'] \arrow[dr, "k"] \arrow[dd, "\psi"] & \\ B & & C, \\ & S' \arrow[ul, "h'"] \arrow[ur, "k'"'] & \end{tikzcd}$$

their horizontal composition is the morphism of spans

$$\begin{tikzcd}
 & R \times_B S \arrow[dl, "f \circ \mathrm{pr}_1"'] \arrow[dr, "k \circ \mathrm{pr}_2"] \arrow[dd, dashed, "\exists!"] & \\
A & & C, \\
 & R' \times_B S' \arrow[ul, "h' \circ \mathrm{pr}'_1"] \arrow[ur, "k' \circ \mathrm{pr}'_2"'] &
\end{tikzcd}$$

constructed as in the diagram

$$\begin{tikzcd}
 & & R \times_B S \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] \arrow[ddll, bend right, "f \circ \mathrm{pr}_1"'] \arrow[ddrr, bend left, "k \circ \mathrm{pr}_2"] \arrow[dddd, dashed, "\exists!"'] \arrow[dd, phantom, very near start, "\lrcorner", rotate=-45] & & \\
 & R \arrow[dl, "f"'] \arrow[dr, "g"] & & S \arrow[dl, "h"'] \arrow[dr, "k"] & \\
A & & B & & C; \\
 & R' \arrow[ul, "f'"] \arrow[ur, "g'"'] & & S' \arrow[ul, "h'"] \arrow[ur, "k'"'] & \\
 & & R' \times_B S' \arrow[ul, "\mathrm{pr}'_1"] \arrow[ur, "\mathrm{pr}'_2"'] \arrow[uull, bend left, "f' \circ \mathrm{pr}'_1"] \arrow[uurr, bend right, "k' \circ \mathrm{pr}'_2"'] \arrow[uu, phantom, very near start, "\urcorner", rotate=-45] & &
\end{tikzcd}$$

- *Associators and Unitors.* The associator and unitors are defined using the universal property of the pullback.

### 6.2.3. The Monoidal Bicategory of Spans.

### 6.2.4. The Double Category of Spans.

**Definition 6.2.4.1.1.** The **double category of spans** is the double category $\mathsf{Span}^{\mathsf{dbl}}$ where

- *Objects.* The objects of $\mathsf{Span}^{\mathsf{dbl}}$ are sets;
- *Vertical Morphisms.* The vertical morphisms of $\mathsf{Span}^{\mathsf{dbl}}$ are functions $f : A \to B$;
- *Horizontal Morphisms.* The horizontal morphisms of $\mathsf{Span}^{\mathsf{dbl}}$ are spans $(S, \phi, \psi) : A \nrightarrow X$;

- 2-*Morphisms.* A 2-cell

$$
\begin{tikzcd}
A \arrow[r, "{(R,\phi_R,\psi_R)}", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\overset{\|}{\underset{\Downarrow}{\alpha}}"] & B \arrow[d, "g"] \\
X \arrow[r, "{(S,\phi_S,\psi_S)}"', "\shortmid"{marking}] & Y
\end{tikzcd}
$$

  of $\mathsf{Span}^{\mathsf{dbl}}$ is a morphism of spans from the span

$$
\begin{tikzcd}
 & R \arrow[dl, "\phi_R"'] \arrow[dr, "\psi_R"] & \\
 & & B \arrow[d, "g"] \\
A & & Y
\end{tikzcd}
$$

  to the span

$$
\begin{tikzcd}
 & & A \times_X S \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner", very near start] & & \\
 & A \arrow[dl, "f"'] \arrow[dr, "f"] & & S \arrow[dl, "\phi_S"'] \arrow[dr, "\psi_S"] & \\
X & & X & & Y;
\end{tikzcd}
$$

- *Horizontal Identities.* The horizontal unit functor

$$\nVdash^{\mathsf{Span}^{\mathsf{dbl}}} \colon \left(\mathsf{Span}^{\mathsf{dbl}}\right)_0 \to \left(\mathsf{Span}^{\mathsf{dbl}}\right)_1$$

  of $\mathsf{Span}^{\mathsf{dbl}}$ is the functor where
  - *Action on Objects.* For each $A \in \mathrm{Obj}\Big(\left(\mathsf{Span}^{\mathsf{dbl}}\right)_0\Big)$, we have

    $$\nVdash_A \overset{\mathrm{def}}{=} (A, \mathrm{id}_A, \mathrm{id}_A),$$

    as in the diagram

$$
\begin{tikzcd}
 & A \arrow[dl, equal, "\mathrm{id}_A"'] \arrow[dr, equal, "\mathrm{id}_A"] & \\
A & & A;
\end{tikzcd}
$$

  - *Action on Morphisms.* For each vertical morphism $f \colon A \to B$ of $\mathsf{Span}^{\mathsf{dbl}}$, i.e. each map of sets $f$ from $A$ to $B$, the identity 2-morphism

$$
\begin{tikzcd}
A \arrow[r, "\nVdash_A", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\overset{\|}{\underset{\Downarrow}{\nVdash_f}}"] & A \arrow[d, "f"] \\
B \arrow[r, "\nVdash_B"', "\shortmid"{marking}] & B
\end{tikzcd}
$$

of $f$ is the morphism of spans from

$$
\begin{tikzcd}
 & & A \arrow[dll, equal, "\mathrm{id}_A"'] \arrow[dr, equal, "\mathrm{id}_A"] & & \\
 & & & A \arrow[dr, "f"] & \\
A & & & & B
\end{tikzcd}
$$

to

$$
\begin{tikzcd}
 & & A \times_B B \arrow[dl] \arrow[dr] \arrow[d, phantom, "\lrcorner" rotate=-45] & & \\
 & A \arrow[dl, "f"'] \arrow[dr, "f"] & {} & B \arrow[dl, equal, "\mathrm{id}_B"'] \arrow[dr, equal, "\mathrm{id}_B"] & \\
B & & B & & B
\end{tikzcd}
$$

given by the isomorphism $A \xrightarrow{\cong} A \times_B B$;

- *Vertical Identities.* For each $A \in \mathrm{Obj}\left(\mathsf{Span}^{\mathsf{dbl}}\right)$, we have

$$
\mathrm{id}_A^{\mathsf{Span}^{\mathsf{dbl}}} \overset{\text{def}}{=} \mathrm{id}_A;
$$

- *Identity 2-Morphisms.* For each horizontal morphism $R\colon A \nrightarrow B$ of $\mathsf{Span}^{\mathsf{dbl}}$, the identity 2-morphism

$$
\begin{tikzcd}
A \arrow[r, "S", "|" marking] \arrow[d, "\mathrm{id}_A"'] \arrow[dr, phantom, "\overset{\|}{\underset{\Downarrow}{\mathrm{id}_S}}"] & B \arrow[d, "\mathrm{id}_B"] \\
A \arrow[r, "S"', "|" marking] & B
\end{tikzcd}
$$

of $R$ is the morphism of spans from

$$
\begin{tikzcd}
 & & S \arrow[dll, "\phi_S"'] \arrow[dr, "\psi_S"] & & \\
 & & & B \arrow[dr, equal, "\mathrm{id}_B"] & \\
A & & & & B
\end{tikzcd}
$$

to

$$
\begin{tikzcd}
 & & A \times_A S \arrow[dl] \arrow[dr] \arrow[d, phantom, "\lrcorner" rotate=-45] & & \\
 & A \arrow[dl, equal, "\mathrm{id}_A"'] \arrow[dr, equal, "\mathrm{id}_A"] & {} & S \arrow[dl, "\phi_S"'] \arrow[dr, "\psi_S"] & \\
A & & A & & B
\end{tikzcd}
$$

given by the isomorphism $S \xrightarrow{\cong} A \times_A S$;

- *Horizontal Composition.* The horizontal composition functor

  $$\odot^{\mathsf{Span}^{\mathsf{dbl}}} \colon \left(\mathsf{Span}^{\mathsf{dbl}}\right)_1 \times_{\left(\mathsf{Span}^{\mathsf{dbl}}\right)_0} \left(\mathsf{Span}^{\mathsf{dbl}}\right)_1 \to \left(\mathsf{Span}^{\mathsf{dbl}}\right)_1$$

  of $\mathsf{Span}^{\mathsf{dbl}}$ is the functor where
  - *Action on Objects.* For each composable pair

    $$A \overset{(R,\phi_R,\psi_R)}{\nrightarrow} B \overset{(S,\phi_S,\psi_S)}{\nrightarrow} C$$

    of horizontal morphisms of $\mathsf{Span}^{\mathsf{dbl}}$, we have

    $$(S,\phi_S,\psi_S) \odot (R,\phi_R,\psi_R) \overset{\text{def}}{=} S \circ^{\mathsf{Span}}_{A,B,C} R,$$

    where $S \circ^{\mathsf{Span}}_{A,B,C} R$ is the composition of $(R,\phi_R,\psi_R)$ and $(S,\phi_S,\psi_S)$ defined as in Definition 6.2.2.1.1;
  - *Action on Morphisms.* For each horizontally composable pair

    $$\begin{tikzcd} A \arrow[r, "{(R,\phi_R,\psi_R)}", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & B \arrow[d, "g"] \\ X \arrow[r, "{(T,\phi_T,\psi_T)}"', "|" marking] & Y \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "{(S,\phi_S,\psi_S)}", "|" marking] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\beta"] & C \arrow[d, "h"] \\ Y \arrow[r, "{(U,\phi_U,\psi_U)}"', "|" marking] & Z \end{tikzcd}$$

    of 2-morphisms of $\mathsf{Span}^{\mathsf{dbl}}$,
- *Vertical Composition of 1-Morphisms.* For each composable pair $A \xrightarrow{F} B \xrightarrow{G} C$ of vertical morphisms of $\mathsf{Span}^{\mathsf{dbl}}$, i.e. maps of sets, we have

  $$g \circ^{\mathsf{Span}^{\mathsf{dbl}}} f \overset{\text{def}}{=} g \circ f;$$
- *Vertical Composition of 2-Morphisms.* For each vertically composable pair

  $$\begin{tikzcd} A \arrow[r, "{(R,\phi_R,\psi_R)}", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & X \arrow[d, "g"] \\ B \arrow[r, "{(S,\phi_S,\psi_S)}"', "|" marking] & Y \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "{(S,\phi_S,\psi_S)}", "|" marking] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow\beta"] & Y \arrow[d, "k"] \\ C \arrow[r, "{(T,\phi_T,\psi_T)}"', "|" marking] & Z \end{tikzcd}$$

  of 2-morphisms of $\mathsf{Span}^{\mathsf{dbl}}$,
- *Associators and Unitors.* The associator and unitors of $\mathsf{Span}^{\mathsf{dbl}}$ are defined using the universal property of the pullback.

### 6.2.5. Properties of The Bicategory of Spans.

**Proposition 6.2.5.1.1.** Let $\lambda = (A \overset{f}{\leftarrow} S \overset{g}{\to} B)$ be a span.

(1) *Self-Duality.*
(2) *Isomorphisms in* $\mathsf{Span}$.
(3) *Equivalences in* $\mathsf{Span}$.
(4) *Adjunctions in* $\mathsf{Span}$. Let $A$ and $B$ be sets.[9]

[9] In the literature (e.g. [**ref**]),…are called maps and denoted by $\mathsf{MapSpan}(A,B)$

(a) We have a natural bijection

$$\left\{\begin{array}{c}\text{Adjunctions in } \mathsf{Span}\\ \text{from } A \text{ to } B\end{array}\right\} \cong \left\{\begin{array}{c}\text{Spans } A \overset{f}{\leftarrow} S \overset{g}{\to} B\\ \text{from } A \text{ to } B \text{ with}\\ f \text{ an isomorphism}\end{array}\right\}.$$

(b) We have an equivalence of categories

$$\mathsf{MapSpan}(A, B) \overset{\text{eq.}}{\cong} \mathsf{Sets}(A, B)_{\mathsf{disc}},$$

where $\mathsf{MapSpan}(A, B)$ is the full subcategory of $\mathsf{Span}(A, B)$ spanned by the spans $A \overset{f}{\leftarrow} S \overset{g}{\to} B$ from $A$ to $B$ with $f$ an isomorphism.

(c) We have a biequivalence of bicategories

$$\mathsf{MapSpan} \overset{\text{eq.}}{\cong} \mathsf{Sets}_{\mathsf{bidisc}},$$

where $\mathsf{MapSpan}$ is the sub-bicategory of $\mathsf{Span}$ whose $\mathsf{Hom}$-categories are given by $\mathsf{MapSpan}(A, B)$.

(5) *Monads in* $\mathsf{Span}$*.*
(6) *Comonads in* $\mathsf{Span}$*.*
(7) *Monomorphisms in* $\mathsf{Span}$*.*
(8) *Epimorphisms in* $\mathsf{Span}$*.*
(9) *Existence of Right Kan Extensions.*
(10) *Existence of Right Kan Lifts.*
(11) *Closedness.*

**Proof.** *Item 1, Self-Duality*:
*Item 2, Isomorphisms in* $\mathsf{Span}$:
*Item 3, Equivalences in* $\mathsf{Span}$:
*Item 4, Adjunctions in* $\mathsf{Span}$: We first prove Item 4a.
We proceed step by step:

(1) *From Adjunctions in* $\mathsf{Span}$ *to Functions.* An adjunction in $\mathsf{Span}$ from $A$ to $B$ consists of a pair of spans

$$\begin{tikzcd} & R \arrow[dl, "f"'] \arrow[dr, "g"] & \\ A & & B \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & S' \arrow[dl, "h"'] \arrow[dr, "k"] & \\ B & & A \end{tikzcd}$$

together with maps

$$\begin{tikzcd} & A \arrow[dl, equals, "\mathrm{id}_A"'] \arrow[dr, equals, "\mathrm{id}_A"] \arrow[dd, "\phi"] & \\ A & & A \\ & S \times_B S' \arrow[ul, "f \circ \mathrm{pr}'_1"] \arrow[ur, "k \circ \mathrm{pr}'_2"'] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & S' \times_A S \arrow[dl, "h \circ \mathrm{pr}_1"'] \arrow[dr, "g \circ \mathrm{pr}_2"] \arrow[dd, "\psi"] & \\ B \arrow[dr, equals, "\mathrm{id}_B"'] & & B. \arrow[dl, equals, "\mathrm{id}_B"] \\ & B & \end{tikzcd}$$

We claim that these conditions

(2) *From Functions to Adjunctions in* $\mathbf{Rel}$*.*
(3) *Invertibility: From Functions to Adjunctions Back to Functions.*
(4) *Invertibility: From Adjunctions to Functions Back to Adjunctions.*

We now proceed to the proof of Item 4b. For this, we will construct a functor

$$F\colon \mathsf{Sets}(A,B)_{\mathsf{disc}} \to \mathsf{MapSpan}(A,B)$$

and prove it to be essentially surjective and fully faithful, and thus an equivalence by **??** of **??**. Indeed, given a map $f\colon A\to B$, let $F(f)$ be the representable span associated to $f$ of Definition 6.5.1.1.1, and let $F$ send the unique (identity) morphism from $f$ to itself to the identity morphism of $F(f)$ in $\mathsf{MapSpan}(A,B)$. We now prove that $F$ is fully faithful and essentially surjective:

1. *$F$ Is Fully Faithful:* Given maps $f,g\colon A\rightrightarrows B$, we need to show that
$$\mathrm{Hom}_{\mathsf{MapSpan}(A,B)}(F(f),F(g)) = \begin{cases} \mathrm{pt} & \text{if } f=g,\\ \emptyset & \text{otherwise.}\end{cases}$$
Indeed, a morphism from $F(f)$ to $F(g)$ takes the form
$$\begin{tikzcd} & A \arrow[dl, equal] \arrow[dd, "\phi"] \arrow[dr, "f"] & \\ A \arrow[dr, equal] & & B. \\ & A \arrow[ur, "g"'] & \end{tikzcd}$$
From the relations $\mathrm{id}_A = \mathrm{id}_A\circ\phi$ and $f = g\circ\phi$, we see that $\phi=\mathrm{id}_A$, and thus from the relation $f=g\circ\phi$ there is such a morphism iff $f=g$.
2. *$F$ Is Essentially Surjective:* Let $\lambda$ be a span of the form
$$\begin{tikzcd} & S \arrow[dl, dashed, "\phi"', "\sim"] \arrow[dr, "f"] & \\ A & & B. \end{tikzcd}$$
we claim that $\lambda\cong F(f\circ\phi^{-1})$. Indeed, we have morphisms
$$\begin{tikzcd} & S \arrow[dl, dashed, "\phi"', "\sim"] \arrow[dd, "\phi"] \arrow[dr, "f"] & \\ A \arrow[dr, equal] & & B \\ & A \arrow[ur, "f\circ\phi^{-1}"'] & \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} & A \arrow[dl, equal] \arrow[dd, "\phi^{-1}"] \arrow[dr, "f\circ\phi^{-1}"] & \\ A & & B, \\ & S \arrow[ul, dashed, "\phi", "\sim"'] \arrow[ur, "f"'] & \end{tikzcd}$$
inverse to each other in $\mathsf{MapSpan}(A,B)$, and thus $\lambda\cong F(f\circ\phi^{-1})$.

Finally, we prove Item 4c.

*Item 5, Monads in* $\mathsf{Span}$:

*Item 6, Comonads in* $\mathsf{Span}$:

*Item 7, Monomorphisms in* $\mathsf{Span}$:

*Item 8, Epimorphisms in* $\mathsf{Span}$:

*Item 9, Existence of Right Kan Extensions*:

*Item 10, Existence of Right Kan Lifts*:

*Item 11, Closedness*:

□

## 6.3. Limits of Spans

### 6.3.1. tmp2.

$$\mathrm{Hom}_{\mathrm{Rel}(A,X)}(\mathrm{Lan}_S(R),T)\cong\mathrm{Hom}_{\mathrm{Rel}(B,X)}(R,T\diamond S)$$

(1) $\mathrm{Lan}_S(R)\subset T$, i.e. if $a\sim_{\mathrm{Lan}_S(R)}x$, then $a\sim_T x$.
(2) $R\subset T\diamond S$, i.e. if $b\sim_R x$, then there exists some $a\in A$ such that $a\sim_S b$ and $b\sim_T x$.

### 6.3.2. tmp.

$$\begin{tikzcd} & S \arrow[dl, "f"'] \arrow[dr, "g"] & \\ A & & B \end{tikzcd}$$

$$\begin{tikzcd} & S' \arrow[dl, "\phi_{S'}"'] \arrow[dr, "\psi_{S'}"] & & & & S\times_B S' \arrow[dl, "f\circ\mathrm{pr}_1"'] \arrow[dr, "\psi_{S'}\circ\mathrm{pr}_2"] & \\ B & & X & \mapsto & A & & X \end{tikzcd}$$

$$\begin{tikzcd} & S'' \arrow[dl, "\phi_{S''}"'] \arrow[dr, "\psi_{S''}"] & & & & ? \arrow[dl, "?"'] \arrow[dr, "?"] & \\ A & & X & \mapsto & B & & X \end{tikzcd}$$

$$\mathrm{Hom}_{\mathsf{Span}(A,X)}(S\times_B S',K)\cong\mathrm{Hom}_{\mathsf{Span}(B,X)}(S',R(K))$$

$$\begin{tikzcd} & S\times_B S' \arrow[dl, "f\circ\mathrm{pr}_1"'] \arrow[dr, "\psi_{S'}\circ\mathrm{pr}_2"] \arrow[dd, "\xi"] & & & & S' \arrow[dl, "\phi_{S'}"'] \arrow[dr, "\psi_{S'}"] \arrow[dd, "\xi^\dagger"] & \\ A & & X & & B & & X \\ & S'' \arrow[ul, "\phi_{S''}"] \arrow[ur, "\psi_{S''}"'] & & & & \Pi_g(S'') \arrow[ul, "?"] \arrow[ur, "?"'] & \end{tikzcd}$$

### 6.3.3. Left Kan Extensions.

Let $\lambda=\left(A\overset{f}{\leftarrow}S\overset{g}{\rightarrow}B\right)$ be a span.

**Proposition 6.3.3.1.1.** The left Kan extension

$$\mathrm{Lan}_\lambda\colon\mathsf{Span}(A,X)\to\mathsf{Span}(B,X)$$

along $\lambda$ in $\mathsf{Span}$ exists and is the functor given on objects by sending a span $\lambda'$ in $\mathsf{Span}(A,X)$ as in

$$\begin{tikzcd} & S' \arrow[dl, "\phi"'] \arrow[dr, "\psi"] & \\ A & & X \end{tikzcd}$$

to the span

$$\mathrm{Lan}_\lambda(\lambda')\overset{\mathrm{def}}{=}(\mathrm{Lan}_\lambda(S'),\mathrm{Lan}_\lambda(\phi),\mathrm{Lan}_\lambda(\psi)),$$

in $\mathsf{Span}(B,X)$ where

- The set $\mathrm{Lan}_\lambda(S')$ is given by

$$\begin{aligned}\mathrm{Lan}_\lambda(S') &\stackrel{\text{def}}{=} \Sigma_g(S')\\ &\stackrel{\text{def}}{=} S'\end{aligned}$$

where $\Sigma_g(S')$ is the dependent sum of $\phi\colon S' \to A$ along $g$ of **??**;
- The map $\mathrm{Lan}_\lambda(\phi)\colon \mathrm{Lan}_\lambda(S') \to B$ is given by $\Sigma_g(\phi)$;
- The map $\mathrm{Lan}_\lambda(\psi)\colon \mathrm{Lan}_\lambda(S') \to X$ is given by $\psi$.

**Proof.** □

**6.3.4. Right Kan Extensions.** Let $\lambda = \left(A \xleftarrow{f} S \xrightarrow{g} B\right)$ be a span.

**Proposition 6.3.4.1.1.** The right Kan extension

$$\mathrm{Ran}_\lambda\colon \mathsf{Span}(A,X) \to \mathsf{Span}(B,X)$$

along $\lambda$ in $\mathsf{Span}$ exists and is the functor given on objects by sending a span $\lambda'$ in $\mathsf{Span}(A,X)$ as in

$$\begin{tikzcd} & S' \arrow[dl, "\phi_{S'}"'] \arrow[dr, "\psi_{S'}"] & \\ A & & X \end{tikzcd}$$

to the span

$$\mathrm{Ran}_\lambda(\lambda') \stackrel{\text{def}}{=} (\mathrm{Ran}_\lambda(S'), \mathrm{Ran}_\lambda(\phi_{S'}), \mathrm{Ran}_\lambda(\psi_{S'})),$$

in $\mathsf{Span}(B,X)$ where
- The set $\mathrm{Ran}_\lambda(S')$ is given by

$$\mathrm{Ran}_\lambda(S') \stackrel{\text{def}}{=} \coprod_{b\in B} \prod_{s\in g^{-1}(b)} \phi_{S'}^{-1}(f(s));$$

- The map $\mathrm{Ran}_\lambda(\phi_{S'})\colon \mathrm{Ran}_\lambda(S') \to B$ is given by

$$[\mathrm{Ran}_\lambda(\phi_{S'})](b, (s'_s)_{s\in g^{-1}(b)}) \stackrel{\text{def}}{=} b;$$

for each $\left(b, (s'_s)_{s\in g^{-1}(b)}\right) \in \mathrm{Ran}_\lambda(S')$;
- The map $\mathrm{Ran}_\lambda(\psi_{S'})\colon \mathrm{Ran}_\lambda(S') \to X$ is given by

$$[\mathrm{Ran}_\lambda(\psi_{S'})]\left(b, (s'_s)_{s\in g^{-1}(b)}\right) \stackrel{\text{def}}{=} \psi_{S'}(s'_i)$$

for each $\left(b, (s'_s)_{s\in g^{-1}(b)}\right) \in \mathrm{Ran}_\lambda(S')$, where the $i$ in $s'_i$ denotes any $s \in g^{-1}(b)$, as we have $\psi_{S'}(s'_i) = \psi_{S'}\left(s'_j\right)$ for all $s \in g^{-1}(b)$.[10]

**Proof.** □

**6.3.5. Right Kan Lifts.** (Although right Kan lifts aren't really limits, this is probably the most appropriate to place this section.)

Let $\lambda = \left(A \xleftarrow{f} S \xrightarrow{g} B\right)$ be a span.

**Proposition 6.3.5.1.1.** The right Kan lift

$$\mathrm{Rift}_\lambda\colon \mathsf{Span}(X,B) \to \mathsf{Span}(X,A)$$

[10]Indeed

along $\lambda$ in $\mathsf{Span}$ exists and is the functor given on objects by sending a span $\lambda'$ in $\mathsf{Span}(X, B)$ as in

$$\begin{tikzcd}
 & S' \arrow[dl, "\phi"'] \arrow[dr, "\psi"] & \\
X & & B
\end{tikzcd}$$

to the span

$$\mathrm{Rift}_\lambda(\lambda') \stackrel{\mathrm{def}}{=} (\mathrm{Rift}_\lambda(S'), \mathrm{Rift}_\lambda(\phi), \mathrm{Rift}_\lambda(\psi)),$$

in $\mathsf{Span}(X, A)$ where

- The set $\mathrm{Rift}_\lambda(S')$ is given by
$$\mathrm{Rift}_\lambda(S') \stackrel{\mathrm{def}}{=} \Pi_f(S'),$$
where $\Pi_f(S')$ is the dependent product of $\psi \colon S' \to A$ along $f$ of **??**;
- The map $\mathrm{Rift}_\lambda(\phi) \colon \mathrm{Rift}_\lambda(S') \to X$ is given by $\phi$;
- The map $\mathrm{Rift}_\lambda(\psi) \colon \mathrm{Rift}_\lambda(S') \to A$ is given by $\Pi_f(\psi)$.

**Proof.** □

## 6.4. Colimits of Spans

## 6.5. Constructions With Spans

### 6.5.1. Representable Spans.

**Definition 6.5.1.1.1.** Let $f \colon A \to B$ be a function.

- The **representable span associated to** $f$ is the span
$$\begin{tikzcd}
 & A \arrow[dl, equal, "\mathrm{id}_A"'] \arrow[dr, "f"] & \\
A & & B
\end{tikzcd}$$
from $A$ to $B$.
- The **corepresentable span associated to** $f$ is the span
$$\begin{tikzcd}
 & A \arrow[dl, "f"'] \arrow[dr, equal, "\mathrm{id}_A"] & \\
B & & A
\end{tikzcd}$$
from $B$ to $A$.

### 6.5.2. Composition of Spans.

**Definition 6.5.2.1.1.** The **composition** of two spans

$$\begin{tikzcd}
 & R \arrow[dl, "f_1"'] \arrow[dr, "g_1"] & \\
A & & B
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
 & S \arrow[dl, "f_2"'] \arrow[dr, "g_2"] & \\
B & & C
\end{tikzcd}$$

is the span $(R \times_B S, f_1 \circ \mathrm{pr}_1, g_2 \circ \mathrm{pr}_2)$, constructed as in the diagram

$$\begin{tikzcd}
 & & R \times_B S \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] \arrow[ddll, bend right, "f_1 \circ \mathrm{pr}_1"'] \arrow[ddrr, bend left, "g_2 \circ \mathrm{pr}_2"] & & \\
 & R \arrow[dl, "f_1"'] \arrow[dr, "g_1"] & & S \arrow[dl, "f_2"'] \arrow[dr, "g_2"] & \\
A & & B & & C.
\end{tikzcd}$$

### 6.5.3. Horizontal Composition of Morphisms of Spans.

**Definition 6.5.3.1.1.** The **horizontal composition** of a pair of 2-morphisms of spans

$$\begin{tikzcd}
 & R \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[dd, "\phi" description] & \\
A & & B \\
 & R' \arrow[ul, "f'"] \arrow[ur, "g'"'] &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
 & S \arrow[dl, "h"'] \arrow[dr, "k"] \arrow[dd, "\psi" description] & \\
B & & C, \\
 & S' \arrow[ul, "h'"] \arrow[ur, "k'"'] &
\end{tikzcd}$$

is the morphism of spans

$$\begin{tikzcd}
 & R \times_B S \arrow[dl, "f \circ \mathrm{pr}_1"'] \arrow[dr, "k \circ \mathrm{pr}_2"] \arrow[dd, dashed, "\exists!" description] & \\
A & & C, \\
 & R' \times_B S' \arrow[ul, "h' \circ \mathrm{pr}'_1"] \arrow[ur, "k' \circ \mathrm{pr}'_2"'] &
\end{tikzcd}$$

constructed as in the diagram

$$\begin{tikzcd}
 & & R \times_B S \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] \arrow[dll, bend right, "f \circ \mathrm{pr}_1"'] \arrow[drr, bend left, "k \circ \mathrm{pr}_2"] \arrow[dd, phantom, "\lrcorner"] \arrow[dddd, dashed, "\exists!"'] & & \\
 & R \arrow[dl, "f"'] \arrow[dr, "g"] & & S \arrow[dl, "h"'] \arrow[dr, "k"] & \\
A & & B & & C; \\
 & R' \arrow[ul, "f'"] \arrow[ur, "g'"'] & & S' \arrow[ul, "h'"'] \arrow[ur, "k'"] & \\
 & & R' \times_B S' \arrow[ul, "\mathrm{pr}'_1"] \arrow[ur, "\mathrm{pr}'_2"'] \arrow[ull, bend left, "f' \circ \mathrm{pr}'_1"] \arrow[urr, bend right, "k' \circ \mathrm{pr}'_2"'] \arrow[uu, phantom, "\urcorner"] & &
\end{tikzcd}$$

### 6.5.4. Properties of Composition of Spans.

**Proposition 6.5.4.1.1.** Let $\lambda = \left(A \overset{f}{\leftarrow} S \overset{g}{\rightarrow} B\right)$ be a span.

(1) *Functoriality.*

**Proof.** □

### 6.5.5. The Inverse of a Span.

## 6.6. Functoriality of Spans

### 6.6.1. Direct Images.

### 6.6.2. Functoriality of Spans on Powersets.

## 6.7. Un/Straightening for Spans

**6.7.1. Straightening for Spans.** Let $A$ and $B$ be sets and let $(S, f, g)$ be a span from $A$ to $B$.

**Definition 6.7.1.1.1.** The **straightening of** $(S, f, g)$ is the $(A \times B)$-indexed set

$$\mathrm{St}_{A,B}(S) \colon (A \times B)_{\mathsf{disc}} \to \mathsf{Sets}$$

defined as the straightening of $S$, viewed as an $(A \times B)$-fibred set, as in **??**.

**Remark 6.7.1.1.2.** In detail, $\mathrm{St}_{A,B}(S)$ is the $(A \times B)$-indexed set defined by[11]

$$\begin{aligned}[\mathrm{St}_{A,B}(S)](a,b) &\stackrel{\text{def}}{=} \mathrm{Wit}_S(a,b)\\ &\stackrel{\text{def}}{=} S_{ab}\\ &\stackrel{\text{def}}{=} \{s \in S \mid f(s) = a \text{ and } g(s) = b\}.\end{aligned}$$

**Proposition 6.7.1.1.3.** Let $A$ and $B$ be sets and let $(S, f, g)$ be a span.

(1) *Functoriality.* The assignment $(S, f, g) \mapsto \mathrm{St}_{A,B}(S)$ defines a functor

$$\mathrm{St}_{A,B} \colon \mathsf{Span}(A, B) \to \mathsf{ISets}(A \times B)$$

- *Action on Objects.* For each $(S, f, g) \in \mathrm{Obj}(\mathsf{Span}(A, B))$, we have
$$[\mathrm{St}_{A,B}](S, f, g) \stackrel{\text{def}}{=} \mathrm{St}_{A,B}(S);$$
- *Action on Morphisms.* For each $(S_1, f_1, g_1), (S_2, f_2, g_2) \in \mathrm{Obj}(\mathsf{Span}(A, B))$, the action on Hom-sets
$$\mathrm{St}_{A,B|S_1,S_2} \colon \mathrm{Hom}_{\mathsf{Span}(A,B)}(S_1, S_2) \to \mathrm{Hom}_{\mathsf{ISets}(A\times B)}(\mathrm{St}_{A,B}(S_1), \mathrm{St}_{A,B}(S_2))$$
of $\mathrm{St}_{A,B}$ at $(S_1, S_2)$ is given by sending a morphism
$$\phi \colon (S_1, f_1, g_1) \to (S_2, f_2, g_2)$$
of spans from $A$ to $B$ to the morphism
$$\mathrm{St}_{A,B}(\phi) \colon \mathrm{St}_{A,B}(S_1) \to \mathrm{St}_{A,B}(S_2)$$
of $(A \times B)$-indexed sets defined by
$$\mathrm{St}_{A,B}(\phi) \stackrel{\text{def}}{=} \{\phi^*_{ab}\}_{(a,b)\in A\times B},$$
where $\phi^*_{ab}$ is the transport map associated to $\phi$ at $(a, b) \in A \times B$ of **??**.

**Proof.** *Item 1, Functoriality*: This is the special case of **??** of **??** where $K = A \times B$. □

**6.7.2. Unstraightening for Spans.** Let $A$ and $B$ be sets and let $S \colon (A \times B)_{\mathsf{disc}} \to \mathsf{Sets}$ be an $(A \times B)$-indexed set.

**Definition 6.7.2.1.1.** The **unstraightening of** $S$ is the span

$$\begin{tikzcd} & \mathrm{Un}_{A,B}(S) \arrow[dl, "f_{\mathrm{Un}_{A,B}(S)}"'] \arrow[dr, "g_{\mathrm{Un}_{A,B}(S)}"] & \\ A & & B \end{tikzcd}$$

from $A$ to $B$ where

$$\mathrm{Un}_{A,B}(S) \stackrel{\text{def}}{=} \coprod_{(a,b)\in A\times B} S(a, b)$$

and where the maps $f_{\mathrm{Un}_{A,B}(S)}$ and $g_{\mathrm{Un}_{A,B}(S)}$ are given by

$$\begin{aligned} f_{\mathrm{Un}_{A,B}(S)}((a, b), s) &\stackrel{\text{def}}{=} a,\\ g_{\mathrm{Un}_{A,B}(S)}((a, b), s) &\stackrel{\text{def}}{=} b \end{aligned}$$

[11]Here we may think of $\mathrm{Wit}_S(a, b)$ as the "set of witnesses in $S$ that $a \sim b$ holds"; see Remark 6.1.2.1.3.

for each $((a,b),s) \in \mathrm{Un}_{A,B}(S)$.

**Proposition 6.7.2.1.2.** Let $A$ and $B$ be sets.

(1) *Functoriality.* The assignment $S \mapsto \mathrm{Un}_{A,B}(S)$ defines a functor

$$\mathrm{Un}_{A,B} \colon \mathsf{ISets}(A \times B) \to \mathsf{Span}(A, B)$$

- *Action on Objects.* For each $S \in \mathrm{Obj}(\mathsf{ISets}(A \times B))$, we have
$$[\mathrm{Un}_{A,B}](S) \stackrel{\text{def}}{=} \mathrm{Un}_{A,B}(S);$$
- *Action on Morphisms.* For each $S, S' \in \mathrm{Obj}(\mathsf{ISets}(A \times B))$, the action on Hom-sets
$$\mathrm{Un}_{A,B|S,S'} \colon \mathrm{Hom}_{\mathsf{ISets}(A\times B)}(S, S') \to \mathrm{Hom}_{\mathsf{Span}(A,B)}(\mathrm{Un}_{A,B}(S), \mathrm{Un}_{A,B}(S'))$$
of $\mathrm{Un}_{A,B}$ at $(S, S')$ is defined by
$$\mathrm{Un}_{A,B|S,S'}(f) \stackrel{\text{def}}{=} \coprod_{(a,b)\in A\times B} f_{ab}.$$

(2) *Interaction With Fibres.* Viewing the legs of $\mathrm{Un}_{A,B}(S)$ as a morphism $(f,g) \colon \mathrm{Un}_{A,B}(S) \to A \times B$, we have a bijection of sets

$$(f,g)^{-1}_{\mathrm{Un}_{A,B}(S)}(a,b) \cong S(a,b)$$

for each $(a,b) \in A \times B$.

(3) *As a Pullback.* We have a bijection of sets

$$\mathrm{Un}_{A,B}(S) \cong (A \times B)_{\mathsf{disc}} \times_{\mathsf{Sets}} \mathsf{Sets}_*, \qquad \begin{tikzcd} \mathrm{Un}_{A,B}(S) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Sets}_* \arrow[d, "忘"] \\ (A \times B)_{\mathsf{disc}} \arrow[r, "S"'] & \mathsf{Sets}. \end{tikzcd}$$

(4) *As a Colimit.* We have a bijection of sets

$$\mathrm{Un}_{A,B}(S) \cong \mathrm{colim}(S).$$

**Proof.** *Item 1, Functoriality*: This is the special case of **??** of **??** where $K = A \times B$.
*Item 2, Interaction With Fibres*: This is the special case of **??** of **??** where $K = A \times B$.
*Item 3, As a Pullback*: This is the special case of **??** of **??** where $K = A \times B$.
*Item 4, As a Colimit*: This is the special case of **??** of **??** where $K = A \times B$. □

### 6.7.3. The Un/Straightening Equivalence for Spans.

**Theorem 6.7.3.1.1.** We have an isomorphism of categories

$$(\mathrm{St}_{A,B} \dashv \mathrm{Un}_{A,B}) \colon \quad \begin{tikzcd} \mathsf{Span}(A,B) \arrow[r, bend left, "\mathrm{St}_{A,B}"] \arrow[r, phantom, "\perp"] & \mathsf{ISets}(A \times B). \arrow[l, bend left, "\mathrm{Un}_{A,B}"] \end{tikzcd}$$

**Proof.** This is the special case of **??** where $K = A \times B$. □

## 6.8. Comparison of Spans to Functions and Relations

### 6.8.1. Comparison to Functions.

**Proposition 6.8.1.1.1.** We have a pseudofunctor

$$\iota\colon \mathsf{Sets}_{\mathsf{bidisc}} \to \mathsf{Span}$$

from $\mathsf{Sets}_{\mathsf{bidisc}}$ to $\mathsf{Span}$ where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathsf{Sets}_{\mathsf{bidisc}})$, we have

  $$\iota(A) \stackrel{\text{def}}{=} A;$$

- *Action on* $\mathsf{Hom}$*-Categories.* For each $A, B \in \mathrm{Obj}(\mathsf{Sets}_{\mathsf{bidisc}})$, the action on $\mathsf{Hom}$-categories

  $$\iota_{A,B}\colon \mathsf{Sets}(A,B)_{\mathsf{disc}} \to \mathsf{Span}(A,B)$$

  of $\iota$ at $(A,B)$ is the functor defined on objects by sending a function $f\colon A \to B$ to the span

  $$\begin{tikzcd} & A \arrow[dl, equal, "\mathrm{id}_A"'] \arrow[dr, "f"] & \\ A & & B \end{tikzcd}$$

  from $A$ to $B$.
- *Strict Unity Constraints.* For each $A \in \mathrm{Obj}(\mathsf{Sets}_{\mathsf{bidisc}})$, the strict unity constraint

  $$\iota^0_A\colon \mathrm{id}_{\iota(A)} \Longrightarrow \iota(\mathrm{id}_A)$$

  of $\iota$ at $A$ is given by the identity morphism of spans

  $$\begin{tikzcd} & A \arrow[dl, equal, "\mathrm{id}_A"'] \arrow[dr, equal, "\mathrm{id}_A"] \arrow[dd, equal, "\mathrm{id}" description] & \\ A & & A, \\ & A \arrow[ul, equal, "\mathrm{id}_A"] \arrow[ur, equal, "\mathrm{id}_A"'] & \end{tikzcd}$$

  as indeed $\mathrm{id}_{\iota(A)} = \iota(\mathrm{id}_A)$;
- *Pseudofunctoriality Constraints.* For each $A, B, C \in \mathrm{Obj}(\mathsf{Sets}_{\mathsf{bidisc}})$, each $f \in \mathsf{Hom}_{\mathsf{Sets}_{\mathsf{bidisc}}}(A,B)$, and each $g \in \mathsf{Hom}_{\mathsf{Sets}_{\mathsf{bidisc}}}(B,C)$, the pseudofunctoriality constraint

  $$\iota^2_{g,f}\colon \iota(g) \circ \iota(f) \Longrightarrow \iota(g \circ f)$$

  of $\iota$ at $(f,g)$ is the morphism of spans from the span

  $$\begin{tikzcd} & & A \times_B B \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] \arrow[ddll, bend right, "\mathrm{id}_A \circ \mathrm{pr}_1"'] \arrow[ddrr, bend left, "g \circ \mathrm{pr}_2"] & & \\ & A \arrow[dl, equal, "\mathrm{id}_A"'] \arrow[dr, "f"] & & B \arrow[dl, equal, "\mathrm{id}_B"'] \arrow[dr, "g"] & \\ A & & B & & C. \end{tikzcd}$$

to the span

$$\begin{tikzcd}
& A \arrow[dl, equal, "\mathrm{id}_A"'] \arrow[dr, "g\circ f"] & \\
A & & C
\end{tikzcd}$$

given by the isomorphism $A \times_B B \cong A$.

**Proof.** Omitted. □

### 6.8.2. Comparison to Relations: From Span to Rel.

6.8.2.1. *Relations Associated to Spans.* Let $\lambda = \left(A \xleftarrow{f} S \xrightarrow{g} B\right)$ be a span.

**Definition 6.8.2.1.1.** The **relation associated to** $\lambda$ is the relation

$$S(\lambda)\colon A \nrightarrow B$$

from $A$ to $B$ defined as follows:

- Viewing relations from $A$ to $B$ as functions $A \times B \to \{\mathsf{true}, \mathsf{false}\}$, we define
$$\iota_{A,B}(S)^a_b \overset{\text{def}}{=} \begin{cases} \mathsf{true} & \text{if there exists } x \in S \text{ such} \\ & \text{that } a = f(x) \text{ and } b = g(x), \\ \mathsf{false} & \text{otherwise} \end{cases}$$
for each $(a, b) \in A \times B$.
- Viewing relations from $A$ to $B$ as functions $A \to \mathcal{P}(B)$, we define
$$[\iota_{A,B}(S)](a) \overset{\text{def}}{=} g(f^{-1}(a))$$
for each $a \in A$.
- Viewing relations from $A$ to $B$ as subsets of $A \times B$, we define
$$\iota_{A,B}(S) \overset{\text{def}}{=} \{(f(x), g(x)) \mid x \in S\}.$$

**Proposition 6.8.2.1.2.** Let $\lambda = \left(A \xleftarrow{f} S \xrightarrow{g} B\right)$ be a span.

(1) *Interaction With Identities.*
(2) *Interaction With Composition.*
(3) *Interaction With Inverses.*

**Proof.** □

6.8.2.2. *The Comparison Functor from* Span *to* **Rel**.

**Proposition 6.8.2.2.1.** We have a pseudofunctor

$$\iota\colon \mathsf{Span} \to \mathbf{Rel}$$

from Span to **Rel** where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathsf{Span})$, we have
$$\iota(A) \overset{\text{def}}{=} A;$$
- *Action on* Hom*-Categories.* For each $A, B \in \mathrm{Obj}(\mathsf{Span})$, the action on Hom-categories
$$\iota_{A,B}\colon \mathsf{Span}(A, B) \to \mathbf{Rel}(A, B)$$

of $\iota$ at $(A, B)$ is the functor where

- *Action on Objects.* Given a span

$$\begin{tikzcd} & S \arrow[dl, "f"'] \arrow[dr, "g"] & \\ A & & B \end{tikzcd}$$

from $A$ to $B$, the image

$$\iota_{A,B}(S) \colon A \nrightarrow B$$

of $S$ by $\iota$ is the relation from $A$ to $B$ defined as follows:

  * Viewing relations as functions $A \times B \to \{\mathsf{true}, \mathsf{false}\}$, we define

$$\iota_{A,B}(S)^a_b \overset{\text{def}}{=} \begin{cases} \mathsf{true} & \text{if there exists } x \in S \\ & \text{such that } a = f(x) \\ & \text{and } b = g(x), \\ \mathsf{false} & \text{otherwise} \end{cases}$$

  for each $(a, b) \in A \times B$;

  * Viewing relations as functions $A \to \mathcal{P}(B)$, we define

$$[\iota_{A,B}(S)](a) \overset{\text{def}}{=} g(f^{-1}(a))$$

  for each $a \in A$;

  * Viewing relations as subsets of $A \times B$, we define

$$\iota_{A,B}(S) \overset{\text{def}}{=} \{(f(x), g(x)) \mid x \in S\}.$$

- *Action on Morphisms.* Given a morphism of spans

$$\begin{tikzcd} & R \arrow[dl, "f_R"'] \arrow[dr, "g_R"] \arrow[dd, "\phi"] & \\ A & & B, \\ & S \arrow[ul, "f_S"] \arrow[ur, "g_S"'] & \end{tikzcd}$$

we have a corresponding inclusion of relations

$$\iota_{A,B}(\phi) \colon \iota_{A,B}(R) \subset \iota_{A,B}(S),$$

since we have $a \sim_{\iota_{A,B}(R)} b$ iff there exists $x \in R$ such that $a = f_R(x)$ and $b = g_R(x)$, in which case we then have

$$\begin{aligned} a &= f_R(x) \\ &= f_S(\phi(x)), \\ b &= g_R(x) \\ &= g_S(\phi(x)), \end{aligned}$$

so that $a \sim_{\iota_{A,B}(S)} b$, and thus $\iota_{A,B}(R) \subset \iota_{A,B}(S)$.

**Proof.** Omitted. □

**6.8.3. Comparison to Relations: From $\mathsf{Rel}$ to $\mathsf{Span}$.**

**Proposition 6.8.3.1.1.** We have a lax functor

$$\left(\iota, \iota^2, \iota^0\right) \colon \mathsf{\textbf{Rel}} \to \mathsf{Span}$$

from $\mathsf{\textbf{Rel}}$ to $\mathsf{Span}$ where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathsf{Span})$, we have
$$\iota(A) \overset{\text{def}}{=} A;$$
- *Action on* $\mathsf{Hom}$*-Categories.* For each $A, B \in \mathrm{Obj}(\mathsf{Span})$, the action on $\mathsf{Hom}$-categories
$$\iota_{A,B} \colon \mathbf{Rel}(A, B) \to \mathsf{Span}(A, B)$$
of $\iota$ at $(A, B)$ is the functor where
  - *Action on Objects.* Given a relation $R \colon A \nrightarrow B$ from $A$ to $B$, we define a span
  $$\iota_{A,B}(R) \colon A \nrightarrow B$$
  from $A$ to $B$ by
  $$\iota_{A,B}(R) \overset{\text{def}}{=} (R, \mathrm{pr}_1|_R, \mathrm{pr}_2|_R),$$
  where $R \subset A \times B$ and $\mathrm{pr}_1|_R$ and $\mathrm{pr}_2|_R$ are the restriction of the projections
  $$\begin{aligned} \mathrm{pr}_1 &\colon A \times B \to A, \\ \mathrm{pr}_2 &\colon A \times B \to B \end{aligned}$$
  to $R$;
  - *Action on Morphisms.* Given an inclusion $\phi \colon R \subset S$ of relations, we have a corresponding morphism of spans
  $$\iota_{A,B}(\phi) \colon \iota_{A,B}(R) \to \iota_{A,B}(S)$$
  as in the diagram
  $$\begin{tikzcd} & R \arrow[dl, "\mathrm{pr}_1|_R"'] \arrow[dr, "\mathrm{pr}_2|_R"] \arrow[dd, hook] & \\ A & & B. \\ & S \arrow[ul, "\mathrm{pr}_1|_S"] \arrow[ur, "\mathrm{pr}_2|_S"'] & \end{tikzcd}$$
- *The Lax Functoriality Constraints.* The lax functoriality constraint
$$\iota^2_{R,S} \colon \iota(S) \circ \iota(R) \Longrightarrow \iota(S \diamond R)$$
of $\iota$ at $(R, S)$ is given by the morphism of spans from
$$\begin{tikzcd} & & R \times_B S \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] \arrow[d, phantom, "\lrcorner" rotate=-45] \arrow[ddll, bend right, "\mathrm{pr}_1|_R \circ \mathrm{pr}_1"'] \arrow[ddrr, bend left, "\mathrm{pr}_2|_S \circ \mathrm{pr}_2"] & & \\ & R \arrow[dl, "\mathrm{pr}_1|_R"'] \arrow[dr, "\mathrm{pr}_2|_R"] & & S \arrow[dl, "\mathrm{pr}_1|_S"'] \arrow[dr, "\mathrm{pr}_2|_S"] & \\ A & & B & & C \end{tikzcd}$$

to

$$\begin{tikzcd}
 & S \diamond R \arrow[dl, "\mathrm{pr}_1|_{S\diamond R}"'] \arrow[dr, "\mathrm{pr}_2|_{S\diamond R}"] & \\
A & & C
\end{tikzcd}$$

given by the natural inclusion $R \times_B S \hookrightarrow S \diamond R$, since we have

$$R \times_B S = \{((a_R, b_R), (b_S, c_S)) \in R \times S \mid b_R = b_S\};$$

$$S \diamond R = \left\{(a, c) \in A \times C \;\middle|\; \begin{array}{l}\text{there exists some } b \in B \text{ such}\\ \text{that } (a, b) \in R \text{ and } (b, c) \in S\end{array}\right\};$$

- *The Lax Unity Constraints.* The lax unity constraint[12]

$$\iota^0_A \colon \underbrace{\mathrm{id}_{\iota(A)}}_{(A,\mathrm{id}_A,\mathrm{id}_A)} \Longrightarrow \underbrace{\iota(\chi_A)}_{\left(\Delta_A,\mathrm{pr}_1|_{\Delta_A},\mathrm{pr}_2|_{\Delta_A}\right)}$$

of $\iota$ at $A$ is given by the diagonal morphism of $A$, as in the diagram

$$\begin{tikzcd}
 & A \arrow[dl, equal, "\mathrm{id}_A"'] \arrow[dr, equal, "\mathrm{id}_A"] \arrow[dd, "\delta_A"] & \\
A & & A. \\
 & \Delta_A \arrow[ul, "\mathrm{pr}_1|_{\Delta_A}"] \arrow[ur, "\mathrm{pr}_2|_{\Delta_A}"'] &
\end{tikzcd}$$

**Proof.** Omitted. □

### 6.8.4. Comparison to Relations: The Wehrheim–Woodward Construction.

### 6.8.5. Comparison to Multirelations.

**Remark 6.8.5.1.1.** The pseudofunctor of Proposition 6.8.2.2.1 and the lax functor of Proposition 6.8.3.1.1 fail to be equivalences of bicategories. This happens essentially because a span $(S, f, g) \colon A \nrightarrow B$ from $A$ to $B$ may relate elements $a \in A$ and $b \in B$ by more than one element, e.g. there could be $s \neq s' \in S$ such that $a = f(s) = f(s')$ and $b = g(s) = g(s')$.
Thus, in a sense, spans may be thought of as "relations with multiplicity". And indeed, if instead of considering relations from $A$ to $B$, i.e. functions

$$R \colon A \times B \to \{\mathsf{true}, \mathsf{false}\}$$

from $A \times B$ to $\{\mathsf{true}, \mathsf{false}\} \cong \{0, 1\}$, we consider functions

$$R \colon A \times B \to \mathbb{N} \cup \{\infty\}$$

from $A \times B$ to $\mathbb{N} \cup \{\infty\}$, then we obtain the notion of a **multirelation from $A$ to $B$**, and these turn out to assemble together with sets into a bicategory $\mathsf{MRel}$ that is biequivalent to $\mathsf{Span}$; see [**some-algebraic-laws-for-spans-and-their-connections-with-m**

### 6.8.6. Comparison to Relations via Double Categories.

**Remark 6.8.6.1.1.** There are double functors between the double categories $\mathsf{Rel}^{\mathsf{dbl}}$ and $\mathsf{Span}^{\mathsf{dbl}}$ analogous to the functors of Propositions 6.8.2.2.1 and 6.8.3.1.1, assembling moreover into a strict-lax adjunction of double functors; see [**higher-dimensional-categories**].

[12] Which is in fact strong, as $\delta_A$ is an isomorphism.

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CHAPTER 7

# Posets

Rename this to "Preorders, Partial Orders, and Posets".

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### 7.1.1. Section.

7.1.1.1. *Section.*

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# Part 2

# Indexed and Fibred Sets

CHAPTER 8

# Indexed Sets

This chapter contains a discussion of indexed sets, the set-theoretical counterpart to indexed categories. In particular, here we explore:

(1) Indexed sets, i.e. functors $K_{\mathsf{disc}} \to \mathsf{Sets}$ with $K$ a set;
(2) The limits and colimits in the category of $K$-indexed sets;
(3) Constructions with indexed sets like dependent sums, dependent products, and internal Homs.

## 8.1. Indexed Sets

**8.1.1. Foundations.** Let $K$ be a set.

**Definition 8.1.1.1.1.** A **$K$-indexed set** is a functor $X\colon K_{\mathsf{disc}} \to \mathsf{Sets}$.

**Remark 8.1.1.1.2.** By **??**, a **$K$-indexed set** consists of a $K$-indexed collection

$$X^{\dagger}\colon K \to \mathrm{Obj}(\mathsf{Sets}),$$

of sets, assigning a set $X^{\dagger}_{x} \overset{\text{def}}{=} X_x$ to each element $x$ of $K$.

**8.1.2. Morphisms of Indexed Sets.** Let $X\colon K_{\mathsf{disc}} \to \mathsf{Sets}$ and $Y\colon K_{\mathsf{disc}} \to \mathsf{Sets}$ be indexed sets.

**Definition 8.1.2.1.1.** A **morphism of $K$-indexed sets from $X$ to $Y$**[1] is a natural transformation

$$f\colon X \Longrightarrow Y, \qquad \begin{tikzcd} K_{\mathsf{disc}} \arrow[r, bend left, "X"{name=U}] \arrow[r, bend right, "Y"'{name=D}] & \mathsf{Sets} \arrow[from=U, to=D, Rightarrow, "f"'] \end{tikzcd}$$

from $X$ to $Y$.

**Remark 8.1.2.1.2.** In detail, a **morphism of $K$-indexed sets** consists of a $K$-indexed collection

$$\{f_x\colon X_x \to Y_x\}_{x\in K}$$

of maps of sets.

**8.1.3. The Category of Sets Indexed by a Fixed Set.** Let $K$ be a set.

**Definition 8.1.3.1.1.** The **category of $K$-indexed sets** is the category $\mathsf{ISets}(K)$ defined by

$$\mathsf{ISets}(K) \overset{\text{def}}{=} \mathsf{Fun}\Big(K_{\mathsf{disc}}, \mathsf{Sets}\Big).$$

**Remark 8.1.3.1.2.** In detail, the **category of $K$-indexed sets** is the category $\mathsf{ISets}(K)$ where

[1] *Further Terminology:* Also called a **$K$-indexed map of sets from $X$ to $Y$**.

- *Objects.* The objects of $\mathsf{ISets}(K)$ are $K$-indexed sets as in Definition 8.1.1.1.1;
- *Morphisms.* The morphisms of $\mathsf{ISets}(K)$ are morphisms of $K$-indexed sets as in Definition 8.1.2.1.1;
- *Identities.* For each $X \in \mathrm{Obj}(\mathsf{ISets}(K))$, the unit map

$$\mathbb{1}^{\mathsf{ISets}(K)}_X : \mathrm{pt} \to \mathrm{Hom}_{\mathsf{ISets}(K)}(X, X)$$

  of $\mathsf{ISets}(K)$ at $X$ is defined by

$$\mathrm{id}^{\mathsf{ISets}(K)}_X \overset{\text{def}}{=} \{\mathrm{id}_{X_x}\}_{x \in K};$$

- *Composition.* For each $X, Y, Z \in \mathrm{Obj}(\mathsf{ISets}(K))$, the composition map

$$\circ^{\mathsf{ISets}(K)}_{X,Y,Z} : \mathrm{Hom}_{\mathsf{ISets}(K)}(Y, Z) \times \mathrm{Hom}_{\mathsf{ISets}(K)}(X, Y) \to \mathrm{Hom}_{\mathsf{ISets}(K)}(X, Z)$$

  of $\mathsf{ISets}(K)$ at $(X, Y, Z)$ is defined by

$$\{g_x\}_{x \in K} \circ^{\mathsf{ISets}(K)}_{X,Y,Z} \{f_x\}_{x \in K} \overset{\text{def}}{=} \{g_x \circ f_x\}_{x \in K}.$$

### 8.1.4. The Category of Indexed Sets.

**Definition 8.1.4.1.1.** The **category of indexed sets** is the category $\mathsf{ISets}$ defined as the Grothendieck construction of the functor $\mathsf{ISets} \colon \mathsf{Sets}^{\mathsf{op}} \to \mathsf{Cats}$ of Proposition 8.4.1.1.4:

$$\mathsf{ISets} \overset{\text{def}}{=} \int^{\mathsf{Sets}} \mathsf{ISets}.$$

**Remark 8.1.4.1.2.** In detail, the **category of indexed sets** is the category $\mathsf{ISets}$ where

- *Objects.* The objects of $\mathsf{ISets}$ are pairs $(K, X)$ consisting of
  - *The Indexing Set.* A set $K$;
  - *The Indexed Set.* A $K$-indexed set $X \colon K_{\mathsf{disc}} \to \mathsf{Sets}$;
- *Morphisms.* A morphism of $\mathsf{ISets}$ from $(K, X)$ to $(K', Y)$ is a pair $(\phi, f)$ consisting of
  - *The Reindexing Map.* A map of sets $\phi \colon K \to K'$;
  - *The Morphism of Indexed Sets.* A morphism of $K$-indexed sets $f \colon X \to \phi_*(Y)$ as in the diagram

$$f \colon X \to \phi_*(Y), \qquad \begin{tikzcd} K_{\mathsf{disc}} \arrow[rr, "\phi"] \arrow[dr, "X"'] & \arrow[d, phantom, "\overset{f}{\Rightarrow}"] & K'_{\mathsf{disc}} \arrow[dl, "Y"] \\ & \mathsf{Sets}; & \end{tikzcd}$$

- *Identities.* For each $(K, X) \in \mathrm{Obj}(\mathsf{ISets})$, the unit map

$$\mathbb{1}^{\mathsf{ISets}}_{(K,X)} : \mathrm{pt} \to \mathsf{ISets}((K, X), (K, X))$$

  of $\mathsf{ISets}$ at $(K, X)$ is defined by

$$\mathrm{id}^{\mathsf{ISets}}_{(K,X)} \overset{\text{def}}{=} (\mathrm{id}_K, \mathrm{id}_X).$$

- *Composition.* For each $\mathbf{X} = (K, X)$, $\mathbf{Y} = (K', Y)$, $\mathbf{Z} = (K'', Z) \in \mathrm{Obj}(\mathsf{ISets})$, the composition map

$$\circ^{\mathsf{ISets}}_{\mathbf{X},\mathbf{Y},\mathbf{Z}} : \mathsf{ISets}(\mathbf{Y}, \mathbf{Z}) \times \mathsf{ISets}(\mathbf{X}, \mathbf{Y}) \to \mathsf{ISets}(\mathbf{X}, \mathbf{Z})$$

of $\mathsf{ISets}$ at $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ is defined by

$$(\psi, g) \circ (\phi, f) \overset{\text{def}}{=} (\psi \circ \phi, (g \star \mathrm{id}_{\phi}) \circ f),$$

as in the diagram

$$\begin{tikzcd}
K_{\mathsf{disc}} \arrow[r, "\phi"] \arrow[dr, "X"'] \arrow[r, phantom, shift right=4ex, "\overset{f}{\Nearrow}"] & K'_{\mathsf{disc}} \arrow[r, "\psi"] \arrow[d, "Y"] \arrow[r, phantom, shift right=4ex, "\overset{g}{\Nearrow}"] & K''_{\mathsf{disc}} \arrow[dl, "Z"] \\
& \mathsf{Sets} &
\end{tikzcd}$$

for each $(\phi, f) \in \mathsf{ISets}(\mathbf{X}, \mathbf{Y})$ and each $(\psi, g) \in \mathsf{ISets}(\mathbf{Y}, \mathbf{Z})$.

## 8.2. Limits of Indexed Sets

**8.2.1. Products of $K$-Indexed Sets.** Let $X : K_{\mathsf{disc}} \to \mathsf{Sets}$ and $Y : K_{\mathsf{disc}} \to \mathsf{Sets}$ be indexed sets.

**Definition 8.2.1.1.1.** The **product of $X$ and $Y$** is the $K$-indexed set $X \times Y : K_{\mathsf{disc}} \to \mathsf{Sets}$ defined by

$$(X \times Y)_k \overset{\text{def}}{=} X_k \times Y_k$$

for each $k \in K$.

**Proof.** That this agrees with the categorical product in $\mathsf{ISets}(K)$ follows from **??** of **??**. ☐

**8.2.2. Pullbacks of $K$-Indexed Sets.** Let $X, Y, Z : K_{\mathsf{disc}} \to \mathsf{Sets}$ be $K$-indexed sets and let $f : X \to Z$ and $g : Y \to Z$ be morphisms of $K$-indexed sets.

**Definition 8.2.2.1.1.** The **pullback of $X$ and $Y$ over $Z$** is the $K$-indexed set $X \times_Z Y : K_{\mathsf{disc}} \to \mathsf{Sets}$ defined by

$$(X \times_Z Y)_k \overset{\text{def}}{=} X_k \times_{Z_k} Y_k$$

for each $k \in K$.

**Proof.** That this agrees with the categorical pullback in $\mathsf{ISets}(K)$ follows from **??** of **??**. ☐

**8.2.3. Equalisers of $K$-Indexed Sets.** Let $X, Y : K_{\mathsf{disc}} \to \mathsf{Sets}$ be $K$-indexed sets and let $f, g : X \rightrightarrows Y$ be morphisms of $K$-indexed sets.

**Definition 8.2.3.1.1.** The **equaliser of $f$ and $g$** is the $K$-indexed set $\mathrm{Eq}(f, g) : K_{\mathsf{disc}} \to \mathsf{Sets}$ defined by

$$(\mathrm{Eq}(f, g))_k \overset{\text{def}}{=} \mathrm{Eq}(f_k, g_k)$$

for each $k \in K$.

**Proof.** That this agrees with the categorical equaliser in $\mathsf{ISets}(K)$ follows from **??** of **??**. ☐

**8.2.4. Products in $\mathsf{ISets}$.** Let $X : K_{\mathsf{disc}} \to \mathsf{Sets}$ and $Y : K'_{\mathsf{disc}} \to \mathsf{Sets}$ be indexed sets.

**Definition 8.2.4.1.1.** The **product of $X$ and $Y$** is the $(K \times K')$-indexed set

$$X \times Y : (K \times K')_{\mathsf{disc}} \to \mathsf{Sets}$$

defined by

$$(X \times Y)_{(k,k')} \overset{\text{def}}{=} X_k \times Y_{k'}$$

for each $(k, k') \in K \times K'$.

**Proof.** We claim that this agrees with the categorical product in $\mathsf{ISets}$. □

**8.2.5. Pullbacks in $\mathsf{ISets}$.** Let $X : K_{\mathsf{disc}} \to \mathsf{Sets}$ be a $K$-indexed set, let $Y : K'_{\mathsf{disc}} \to \mathsf{Sets}$ be a $K'$-indexed set, let $Z : K''_{\mathsf{disc}} \to \mathsf{Sets}$ be a $K''$-indexed set, and let $(\phi, f): X \to Z$ and $(\psi, g): Y \to Z$ be morphisms of indexed sets (as in Remark 8.1.4.1.2).

**Definition 8.2.5.1.1.** The **pullback of $X$ and $Y$ over $Z$** is the $(K \times_{K''} K)$-indexed set

$$X \times_Z Y : (K \times_{K''} K)_{\mathsf{disc}} \to \mathsf{Sets}$$

defined by

$$\begin{aligned}(X \times_Z Y)_{(k,k')} &\overset{\text{def}}{=} X_k \times_{Z_{\phi(k)}} Y_{k'}\\ &\overset{\text{def}}{=} X_k \times_{Z_{\psi(k)}} Y_{k'}\end{aligned}$$

for each $(k, k') \in K \times_{K''} K'$.

**Proof.** We claim that this agrees with the categorical pullback in $\mathsf{ISets}$. □

**8.2.6. Equalisers in $\mathsf{ISets}$.** Let $X : K_{\mathsf{disc}} \to \mathsf{Sets}$ be a $K$-indexed set, let $Y : K'_{\mathsf{disc}} \to \mathsf{Sets}$ be a $K'$-indexed set, and let $(\phi, f), (\psi, g): X \to Y$ be morphisms of indexed sets (as in Remark 8.1.4.1.2).

**Definition 8.2.6.1.1.** The **equaliser of $(\phi, f)$ and $(\psi, g)$** is the $\mathrm{Eq}(\phi, \psi)$-indexed set $\mathrm{Eq}(f, g) : \mathrm{Eq}(\phi, \psi) \to \mathsf{Sets}$ defined by

$$(\mathrm{Eq}(f, g))_k \overset{\text{def}}{=} \mathrm{Eq}(f_k, g_k)$$

for each $k \in \mathrm{Eq}(\phi, \psi)$.

**Proof.** We claim that this agrees with the categorical equaliser in $\mathsf{ISets}$. □

## 8.3. Colimits of Indexed Sets

**8.3.1. Coproducts of $K$-Indexed Sets.** Let $X : K_{\mathsf{disc}} \to \mathsf{Sets}$ and $Y : K_{\mathsf{disc}} \to \mathsf{Sets}$ be indexed sets.

**Definition 8.3.1.1.1.** The **coproduct** of $X$ and $Y$ is the $K$-k-indexed set $X \coprod Y : K_{\mathsf{disc}} \to \mathsf{Sets}$ defined by

$$(X \coprod Y)_k \overset{\text{def}}{=} X_k \coprod Y_k$$

for each $k \in K$.

**Proof.** That this agrees with the categorical coproduct in $\mathsf{ISets}(K)$ follows from **??** of **??**. □

**8.3.2. Pushouts of $K$-Indexed Sets.** Let $X, Y, Z : K_{\mathsf{disc}} \to \mathsf{Sets}$ be $K$-indexed sets and let $f : Z \to X$ and $g : Z \to Y$ be morphisms of $K$-indexed sets.

**Definition 8.3.2.1.1.** The **pushout** of $X$ and $Y$ is the $K$-indexed set $X \coprod_Z Y \colon K_{\mathsf{disc}} \to \mathsf{Sets}$ defined by

$$(X \coprod_Z Y)_k \stackrel{\text{def}}{=} X_k \coprod_{Z_k} Y_k$$

for each $k \in K$.

**Proof.** That this agrees with the categorical pushout in $\mathsf{ISets}(K)$ follows from **??** of **??**. □

**8.3.3. Coequalisers of $K$-Indexed Sets.** Let $X, Y \colon K_{\mathsf{disc}} \to \mathsf{Sets}$ be $K$-indexed sets and let $f, g \colon X \rightrightarrows Y$ be morphisms of $K$-indexed sets.

**Definition 8.3.3.1.1.** The **coequaliser** of $X$ and $Y$ is the $K$-indexed set $\mathrm{CoEq}(f, g) \colon K_{\mathsf{disc}} \to \mathsf{Sets}$ defined by

$$(\mathrm{CoEq}(f, g))_k \stackrel{\text{def}}{=} \mathrm{CoEq}(f_k, g_k)$$

for each $k \in K$.

**Proof.** That this agrees with the categorical coequaliser in $\mathsf{ISets}(K)$ follows from **??** of **??**. □

## 8.4. Constructions With Indexed Sets

**8.4.1. Change of Indexing.** Let $\phi \colon K \to K'$ be a function and let $X$ be a $K'$-indexed set.

**Definition 8.4.1.1.1.** The **change of indexing of $X$ to $K$** is the $K$-indexed set $\phi^*(X)$ defined by

$$\phi^*(X) \stackrel{\text{def}}{=} X \circ \phi_{\mathsf{disc}}.$$

**Remark 8.4.1.1.2.** In detail, the **change of indexing of $X$ to $K$** is the $K$-indexed set $\phi^*(X)$ defined by

$$\phi^*(X)_x \stackrel{\text{def}}{=} X_{\phi(x)}$$

for each $x \in K$.

**Proposition 8.4.1.1.3.** The assignment $X \mapsto \phi^*(X)$ defines a functor

$$\phi^* \colon \mathsf{ISets}(K') \to \mathsf{ISets}(K),$$

where

- *Action on Objects.* For each $X \in \mathrm{Obj}(\mathsf{ISets}(K'))$, we have
  $$[\phi^*](X) \stackrel{\text{def}}{=} \phi^*(X);$$
- *Action on Morphisms.* For each $X, Y \in \mathrm{Obj}(\mathsf{ISets}(K'))$, the action on Hom-sets
  $$\phi^*_{X,Y} \colon \mathrm{Hom}_{\mathsf{ISets}(K')}(X, Y) \to \mathrm{Hom}_{\mathsf{ISets}(K)}(\phi^*(X), \phi^*(Y))$$
  of $\phi^*$ at $(X, Y)$ is the map sending a morphism of $K'$-indexed sets
  $$f = \{f_x \colon X_x \to Y_x\}_{x \in K'}$$
  from $X$ to $Y$ to the morphism of $K$-indexed sets defined by
  $$\phi^*(f) \stackrel{\text{def}}{=} \left\{f_{\phi(x)} \colon X_{\phi(x)} \to Y_{\phi(x)}\right\}_{x \in K}.$$

**Proof.** Omitted. □

**Proposition 8.4.1.1.4.** The assignment $K \mapsto \mathsf{ISets}(K)$ defines a functor

$$\mathsf{ISets}\colon \mathsf{Sets}^{\mathsf{op}} \to \mathsf{Cats},$$

where

- *Action on Objects.* For each $K \in \mathrm{Obj}(\mathsf{Sets})$, we have
$$[\mathsf{ISets}](K) \stackrel{\text{def}}{=} \mathsf{ISets}(K);$$
- *Action on Morphisms.* For each $K, K' \in \mathrm{Obj}(\mathsf{Sets})$, the action on Hom-sets
$$\mathsf{ISets}_{K,K'}\colon \mathsf{Sets}^{\mathsf{op}}(K, K') \to \mathsf{Fun}(\mathsf{ISets}(K), \mathsf{ISets}(K'))$$
of $\mathsf{ISets}$ at $(K, K')$ is the map defined by
$$\mathsf{ISets}_{K,K'}(\phi) \stackrel{\text{def}}{=} \phi^*$$
for each $\phi \in \mathsf{Sets}^{\mathsf{op}}(K, K')$.

**Proof.** Omitted. □

**8.4.2. Dependent Sums.** Let $\phi\colon K \to K'$ be a function and let $X$ be a $K$-indexed set.

**Definition 8.4.2.1.1.** The **dependent sum of** $X$ is the $K'$-indexed set $\Sigma_\phi(X)$[2] defined by

$$\Sigma_\phi(X) \stackrel{\text{def}}{=} \mathrm{Lan}_\phi(X),$$

and hence given by

$$\Sigma_\phi(X)_x \cong \coprod_{y \in \phi^{-1}(x)} X_y$$

for each $x \in K'$.

**Proposition 8.4.2.1.2.** The assignment $X \mapsto \Sigma_\phi(X)$ defines a functor

$$\Sigma_\phi\colon \mathsf{ISets}(K) \to \mathsf{ISets}(K'),$$

where

- *Action on Objects.* For each $X \in \mathrm{Obj}(\mathsf{ISets}(K))$, we have
$$[\Sigma_\phi](X) \stackrel{\text{def}}{=} \Sigma_\phi(X);$$
- *Action on Morphisms.* For each $X, Y \in \mathrm{Obj}(\mathsf{ISets}(K))$, the action on Hom-sets
$$\Sigma_{\phi|X,Y}\colon \mathrm{Hom}_{\mathsf{ISets}(K)}(X, Y) \to \mathrm{Hom}_{\mathsf{ISets}(K')}(\Sigma_\phi(X), \Sigma_\phi(Y))$$
of $\Sigma_\phi$ at $(X, Y)$ is the map sending a morphism of $K$-indexed sets
$$f\colon X \to Y$$
to the morphism of $K'$-indexed sets defined by
$$\begin{aligned}\Sigma_\phi(f) &\stackrel{\text{def}}{=} \mathrm{Lan}_\phi(f);\\ &\cong \coprod_{y \in \phi^{-1}(X)} f_y.\end{aligned}$$

**Proof.** Omitted. □

[2] *Further Notation:* Also written $\phi_*(X)$.

**8.4.3. Dependent Products.** Let $\phi\colon K \to K'$ be a function and let $X$ be a $K$-indexed set.

**Definition 8.4.3.1.1.** The **dependent product of** $X$ is the $K'$-indexed set $\Pi_\phi(X)$[3] defined by

$$\Pi_\phi(X) \overset{\text{def}}{=} \mathrm{Ran}_\phi(X),$$

and hence given by

$$\Pi_\phi(X)_x \cong \prod_{y\in\phi^{-1}(x)} X_y$$

for each $x \in K'$.

**Proposition 8.4.3.1.2.** The assignment $X \mapsto \Pi_\phi(X)$ defines a functor

$$\Pi_\phi\colon \mathsf{ISets}(K) \to \mathsf{ISets}(K'),$$

where

- *Action on Objects.* For each $X \in \mathrm{Obj}(\mathsf{ISets}(K))$, we have
$$[\Pi_\phi](X) \overset{\text{def}}{=} \Pi_\phi(X);$$
- *Action on Morphisms.* For each $X, Y \in \mathrm{Obj}(\mathsf{ISets}(K))$, the action on Hom-sets
$$\Pi_{\phi|X,Y}\colon \mathrm{Hom}_{\mathsf{ISets}(K)}(X,Y) \to \mathrm{Hom}_{\mathsf{ISets}(K')}(\Pi_\phi(X),\Pi_\phi(Y))$$
of $\Pi_\phi$ at $(X,Y)$ is the map sending a morphism of $K$-indexed sets
$$f\colon X \to Y$$
to the morphism of $K'$-indexed sets defined by
$$\begin{aligned}\Pi_\phi(f) &\overset{\text{def}}{=} \mathrm{Ran}_\phi(f);\\ &\cong \prod_{y\in\phi^{-1}(x)} f_y.\end{aligned}$$

**Proof.** Omitted. □

**8.4.4. Internal Homs.** Let $K$ be a set and let $X$ and $Y$ be $K$-indexed sets.

**Definition 8.4.4.1.1.** The **internal Hom of indexed sets from** $X$ **to** $Y$ is the indexed set $\mathbf{Hom}_{\mathsf{ISets}(K)}(X,Y)$ defined by

$$\mathbf{Hom}_{\mathsf{ISets}(K)}(X,Y) \overset{\text{def}}{=} \mathsf{Sets}(X_x, Y_x)$$

for each $x \in K$.

**8.4.5. Adjointness of Indexed Sets.** Let $\phi\colon K \to K'$ be a map of sets.

**Proposition 8.4.5.1.1.** We have a triple adjunction

$$(\Sigma_\phi \dashv \phi^* \dashv \Pi_\phi)\colon\quad \begin{tikzcd} \mathsf{ISets}(K) \arrow[r, bend left, "\Sigma_\phi"] \arrow[r, bend right, "\Pi_\phi"'] & \mathsf{ISets}(K'). \arrow[l, "\phi^*"'] \end{tikzcd}$$

**Proof.** This follows from **??** of **??**. □

[3] *Further Notation:* Also written $\phi_!(X)$.

# Appendices

## 8.A. Other Chapters


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(2) Constructions With Sets
(3) Pointed Sets
(4) Tensor Products of Pointed Sets
(5) Relations
(6) Spans
(7) Posets

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(7) Indexed Sets
(8) Fibred Sets
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CHAPTER 9

# Fibred Sets

This chapter contains a discussion of the un/straightening equivalence in the context of sets, as well as a general discussion of indexed and fibred sets. In particular, it contains:

(1) A discussion of indexed sets (i.e. functors $K_{\mathsf{disc}} \to \mathsf{Sets}$ with $K$ a set), constructions with them like dependent sums and dependent products, and their properties (**????**);
(2) A discussion of fibred sets (i.e. maps of sets $X \to K$), constructions with them like dependent sums and dependent products, and their properties (Sections 9.1 and 9.2);
(3) A discussion of the un/straightening equivalence for indexed and fibred sets (**??**).

## 9.1. Fibred Sets

**9.1.1. Foundations.** Let $K$ be a set.

**Definition 9.1.1.1.1.** A **$K$-fibred set** is a pair $(X, \phi)$ consisting of[1]

- *The Underlying Set.* A set $X$, called the **underlying set of** $(X, \phi)$;
- *The Fibration.* A map of sets $\phi\colon X \to K$.

**9.1.2. Morphisms of Fibred Sets.**

**Definition 9.1.2.1.1.** A **morphism of $K$-fibred sets from** $(X, \phi)$ **to** $(Y, \psi)$ is a function $f\colon X \to Y$ such that the diagram[2]

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\phi"'] & & Y \arrow[dl, "\psi"] \\ & K & \end{tikzcd}$$

commutes.

**9.1.3. The Category of Fibred Sets Over a Fixed Base.**

[1] *Further Terminology:* The **fibre of** $(X, \phi)$ **over** $x \in K$ is the set $\phi^{-1}(x)$ (also written $\phi_x$) defined by

$$\phi^{-1}(x) \overset{\text{def}}{=} \mathrm{pt} \times_{[x], K, \phi} X, \qquad \begin{tikzcd} \phi^{-1}(x) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "\phi"] \\ \mathrm{pt} \arrow[r, "{[x]}"'] & K. \end{tikzcd}$$

[2] *Further Terminology:* The **transport map associated to $f$ at** $x \in K$ is the function

$$f^*_x\colon \phi^{-1}(x) \to \psi^{-1}(x)$$

**Definition 9.1.3.1.1.** The **category of $K$-fibred sets** is the category $\mathsf{FibSets}(K)$ defined as the slice category $\mathsf{Sets}_{/K}$ of $\mathsf{Sets}$ over $K$:

$$\mathsf{FibSets}(K) \stackrel{\text{def}}{=} \mathsf{Sets}_{/K}.$$

**Remark 9.1.3.1.2.** In detail $\mathsf{FibSets}(K)$ is the category where

- *Objects.* The objects of $\mathsf{FibSets}(K)$ are pairs $(X, \phi)$ consisting of
  - *The Fibred Set.* A set $X$;
  - *The Fibration.* A function $\phi\colon X \to K$;
- *Morphisms.* A morphism of $\mathsf{FibSets}(K)$ from $(X, \phi)$ to $(Y, \psi)$ is a function $f\colon X \to Y$ making the diagram

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "\phi"'] & & Y \arrow[dl, "\psi"] \\
& X &
\end{tikzcd}$$

  commute;
- *Identities.* For each $(X, \phi) \in \mathrm{Obj}(\mathsf{FibSets}(K))$, the unit map

$$\mathbb{1}^{\mathsf{FibSets}(K)}_{(X,\phi)}\colon \mathrm{pt} \to \mathrm{Hom}_{\mathsf{FibSets}(K)}((X, \phi), (X, \phi))$$

  of $\mathsf{FibSets}(K)$ at $(X, \phi)$ is given by

$$\mathrm{id}^{\mathsf{FibSets}(K)}_{(X,\phi)} \stackrel{\text{def}}{=} \mathrm{id}_X,$$

  as witnessed by the commutativity of the diagram

$$\begin{tikzcd}
X \arrow[rr, equal, "\mathrm{id}_X"] \arrow[dr, "\phi"'] & & X \arrow[dl, "\phi"] \\
& K &
\end{tikzcd}$$

  in $\mathsf{Sets}$;
- *Composition.* For each $\mathbf{X} = (X, \phi)$, $\mathbf{Y} = (Y, \psi)$, $\mathbf{Z} = (Z, \chi) \in \mathrm{Obj}(\mathsf{FibSets}(K))$, the composition map

$$\circ^{\mathsf{FibSets}(K)}_{\mathbf{X},\mathbf{Y},\mathbf{Z}}\colon \mathrm{Hom}_{\mathsf{FibSets}(K)}(\mathbf{Y}, \mathbf{Z}) \times \mathrm{Hom}_{\mathsf{FibSets}(K)}(\mathbf{X}, \mathbf{Y}) \to \mathrm{Hom}_{\mathsf{FibSets}(K)}(\mathbf{X}, \mathbf{Z})$$

  of $\mathsf{FibSets}(K)$ at $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ is defined by

$$\circ^{\mathsf{FibSets}(K)}_{\mathbf{X},\mathbf{Y},\mathbf{Z}} \stackrel{\text{def}}{=} \circ^{\mathsf{Sets}}_{X,Y,Z},$$

---

given by the dashed map in the diagram

$$\begin{tikzcd}
\phi^{-1}(x) \arrow[rr] \arrow[dr, dashed] \arrow[dd] \arrow[dr, phantom, very near start, "\lrcorner"] & & X \arrow[dr, "f"] \arrow[dd, "\phi" near start] & \\
& \psi^{-1}(x) \arrow[rr] \arrow[dd] \arrow[dr, phantom, very near start, "\lrcorner"] & & Y \arrow[dd, "\psi"] \\
\mathrm{pt} \arrow[rr, "{[x]}" near start] \arrow[dr, equal] & & K \arrow[dr, equal] & \\
& \mathrm{pt} \arrow[rr, "{[x]}"'] & & K.
\end{tikzcd}$$

as witnessed by the commutativity of the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "\phi"'] & Y \arrow[r, "g"] \arrow[d, "\psi"] & Z \arrow[dl, "\chi"] \\ & K & \end{tikzcd}$$

in $\mathsf{Sets}$.

### 9.1.4. The Category of Fibred Sets.

**Definition 9.1.4.1.1.** The **category of fibred sets** is the category $\mathsf{FibSets}$ defined as the Grothendieck construction of the functor $\mathsf{FibSets}\colon \mathsf{Sets}^{\mathsf{op}} \to \mathsf{Cats}$ of Proposition 9.2.1.1.3:

$$\mathsf{FibSets} \overset{\text{def}}{=} \int^{\mathsf{Sets}} \mathsf{FibSets}.$$

**Remark 9.1.4.1.2.** In detail, the **category of fibred sets** is the category $\mathsf{FibSets}$ where

- *Objects.* The objects of $\mathsf{FibSets}$ are pairs $(K, (X, \phi_X))$ consisting of
  - *The Base Set.* A set $K$;
  - *The Fibred Set.* A $K$-fibred set $\phi_X\colon X \to K$;
- *Morphisms.* A morphism of $\mathsf{FibSets}$ from $(K, (X, \phi_X))$ to $(K', (Y, \phi_Y))$ is a pair $(\phi, f)$ consisting of
  - *The Base Map.* A map of sets $\phi\colon K \to K'$;
  - *The Morphism of Fibred Sets.* A morphism of $K$-fibred sets

$$f\colon (X, \phi_X) \to \phi_Y^*(Y), \qquad \begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "\phi_X"'] & Y \times_{K'} K \arrow[d, "\mathrm{pr}_2"] \\ & K; \end{tikzcd}$$

- *Identities.* For each $(K, X) \in \mathrm{Obj}(\mathsf{FibSets})$, the unit map

$$\mathbb{1}^{\mathsf{FibSets}}_{(K,X)}\colon \mathrm{pt} \to \mathsf{FibSets}((K, X), (K, X))$$

  of $\mathsf{FibSets}$ at $(K, X)$ is defined by

$$\mathrm{id}^{\mathsf{FibSets}}_{(K,X)} \overset{\text{def}}{=} (\mathrm{id}_K, \sim),$$

  where $\sim$ is the isomorphism $X \to X \times_K K$ as in the diagram

$$\begin{tikzcd} X \arrow[r, dashed, "\sim"] \arrow[dr, "\phi_X"'] & X \times_K K \arrow[d, "\mathrm{pr}_2"] \\ & K; \end{tikzcd}$$

- *Composition.* For each $\mathbf{X} = (K, X), \mathbf{Y} = (K', Y), \mathbf{Z} = (K'', Z) \in \mathrm{Obj}(\mathsf{FibSets})$, the composition map

$$\circ^{\mathsf{FibSets}}_{\mathbf{X},\mathbf{Y},\mathbf{Z}}\colon \mathsf{FibSets}(\mathbf{Y}, \mathbf{Z}) \times \mathsf{FibSets}(\mathbf{X}, \mathbf{Y}) \to \mathsf{FibSets}(\mathbf{X}, \mathbf{Z})$$

  of $\mathsf{FibSets}$ at $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ is defined by

$$g \circ^{\mathsf{FibSets}}_{\mathbf{X},\mathbf{Y},\mathbf{Z}} f \overset{\text{def}}{=} (g \times_{K'} \mathrm{id}_K) \circ f$$

as in the diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[dr, "\phi_X"'] & Y \times_{K'} K \arrow[r, "g \times_{K'} \mathrm{id}_K"] \arrow[d, "\mathrm{pr}_2"] & \overbrace{(Z \times_{K''} K') \times_{K'} K}^{\cong Z \times_{K''} K} \arrow[dl, "\mathrm{pr}_2"] \\
& K; &
\end{tikzcd}$$

for each $f \in \mathrm{Obj}(\mathsf{FibSets}(\mathbf{X}, \mathbf{Y}))$ and each $g \in \mathrm{Obj}(\mathsf{FibSets}(\mathbf{Y}, \mathbf{Z}))$.

## 9.2. Constructions With Fibred Sets

**9.2.1. Change of Base.** Let $f \colon K \to K'$ be a function and let $(X, \phi_X)$ be a $K'$-fibred set.

**Definition 9.2.1.1.1.** The **change of base of** $(X, \phi_X)$ **to** $K$ is the $K$-fibred set $f^*(X)$ defined by

$$f^*(X) \overset{\text{def}}{=} (K \times_{K'} X, \mathrm{pr}_1), \qquad \begin{tikzcd}
f^*(X) \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "\phi_X"] \\
K \arrow[r, "f"'] & K'.
\end{tikzcd}$$

**Proposition 9.2.1.1.2.** The assignment $X \mapsto f^*(X)$ defines a functor

$$f^* \colon \mathsf{FibSets}(K') \to \mathsf{FibSets}(K),$$

where

- *Action on Objects.* For each $(X, \phi_X) \in \mathrm{Obj}(\mathsf{FibSets}(K'))$, we have
$$f^*(X, \phi_X) \overset{\text{def}}{=} f^*(X);$$
- *Action on Morphisms.* For each $(X, \phi_X), (Y, \phi_Y) \in \mathrm{Obj}(\mathsf{FibSets}(K'))$, the action on Hom-sets
$$f^*_{X,Y} \colon \mathrm{Hom}_{\mathsf{FibSets}(K')}(X, Y) \to \mathrm{Hom}_{\mathsf{FibSets}(K)}(f^*(X), f^*(Y))$$
of $f^*$ at $((X, \phi_X), (Y, \phi_Y))$ is the map sending a morphism of $K'$-fibred sets
$$g \colon (X, \phi_X) \to (Y, \phi_Y)$$
to the morphism of $K$-fibred sets given by the dashed morphism in the diagram
$$\begin{tikzcd}
f^*(X) \arrow[rr] \arrow[dd] \arrow[dr, dashed] \arrow[dr, phantom, "\lrcorner", very near start] & & X \arrow[dr, "g"] \arrow[dd, "\phi_X" near start] & \\
& f^*(Y) \arrow[rr, crossing over] \arrow[dr, phantom, "\lrcorner", very near start] & & Y \arrow[dd, "\phi_Y"] \\
K \arrow[rr, "f" near start] \arrow[dr, equal] & & K' \arrow[dr, equal] & \\
& K \arrow[rr, "f"'] \arrow[from=uu, crossing over] & & K'.
\end{tikzcd}$$

**Proof.** Omitted. □

**Proposition 9.2.1.1.3.** The assignment $K \mapsto \mathsf{FibSets}(K)$ defines a functor

$$\mathsf{FibSets}\colon \mathsf{Sets}^{\mathsf{op}} \to \mathsf{Cats},$$

where

- *Action on Objects.* For each $K \in \mathrm{Obj}(\mathsf{Sets})$, we have
$$[\mathsf{FibSets}](K) \overset{\text{def}}{=} \mathsf{FibSets}(K);$$
- *Action on Morphisms.* For each $K, K' \in \mathrm{Obj}(\mathsf{Sets})$, the action on Hom-sets
$$\mathsf{Sets}_{/(-)|K,K'}\colon \mathsf{Sets}^{\mathsf{op}}(K, K') \to \mathsf{Fun}(\mathsf{FibSets}(K), \mathsf{FibSets}(K'))$$
of $\mathsf{Sets}_{/(-)}$ at $(K, K')$ is the map sending a map of sets $f\colon K \to K'$ to the functor
$$\mathsf{Sets}_{/f}\colon \mathsf{FibSets}(K') \to \mathsf{FibSets}(K)$$
defined by
$$\mathsf{Sets}_{/f} \overset{\text{def}}{=} f^*.$$

**Proof.** Omitted. □

**9.2.2. Dependent Sums.** Let $f\colon K \to K'$ be a function and let $(X, \phi_X)$ be a $K$-fibred set.

**Definition 9.2.2.1.1.** The **dependent sum**[3] **of** $(X, \phi_X)$ is the $K'$-fibred set $\Sigma_f(X)$[4] defined by

$$\begin{aligned}\Sigma_f(X) &\overset{\text{def}}{=} (\Sigma_f(X), \Sigma_f(\phi_X))\\ &\overset{\text{def}}{=} (X, f \circ \phi_X).\end{aligned}$$

**Proposition 9.2.2.1.2.** Let $f\colon K \to K'$ be a function.

(1) *Functoriality.* The assignment $X \mapsto \Sigma_f(X)$ defines a functor
$$\Sigma_f\colon \mathsf{FibSets}(K) \to \mathsf{FibSets}(K'),$$
where
  - *Action on Objects.* For each $(X, \phi_X) \in \mathrm{Obj}(\mathsf{FibSets}(K))$, we have
$$\Sigma_f(X, \phi_X) \overset{\text{def}}{=} (\Sigma_f(X), \Sigma_f(\phi_X));$$
  - *Action on Morphisms.* For each $(X, \phi_X), (Y, \phi_Y) \in \mathrm{Obj}(\mathsf{FibSets}(K))$, the action on Hom-sets
$$\Sigma_{f|X,Y}\colon \mathrm{Hom}_{\mathsf{FibSets}(K)}(X, Y) \to \mathrm{Hom}_{\mathsf{FibSets}(K)}(\Sigma_f(X), \Sigma_f(Y))$$
of $\Sigma_f$ at $((X, \phi_X), (Y, \phi_Y))$ is the map sending a morphism of $K$-fibred sets
$$g\colon (X, \phi_X) \to (Y, \phi_Y)$$

[3]The name "dependent sum" comes from the fact that the fibre $\Sigma_f(\phi_X)^{-1}(x)$ of $\Sigma_f(X)$ at $x \in K'$ is given by
$$\Sigma_f(\phi_X)^{-1}(x) \cong \coprod_{y \in f^{-1}(x)} \phi_X^{-1}(y);$$
see Item 2 of Proposition 9.2.2.1.2.

[4]*Further Notation:* Also written $f_*(X)$.

to the morphism of $K'$-fibred sets defined by

$$\Sigma_f(g) \stackrel{\text{def}}{=} g.$$

(2) *Interaction With Fibres.* We have a bijection of sets

$$\Sigma_f(\phi_X)^{-1}(k') \cong \coprod_{k \in f^{-1}(k')} \phi_X^{-1}(k)$$

for each $k' \in K'$.

**Proof.** *Item 1, Functoriality*: Omitted.
*Item 2, Interaction With Fibres*: Indeed, we have

$$\begin{aligned}\Sigma_f(\phi_X)^{-1}(k') &\stackrel{\text{def}}{=} \mathrm{pt} \times_{[k'],K',f\circ\phi_X} X \\ &\cong \{x \in X \mid f(\phi_X(x)) = k'\} \\ &\cong \coprod_{k \in f^{-1}(k')} \{x \in X \mid \phi_X(x) = k\} \\ &\cong \coprod_{k \in f^{-1}(k')} \phi_X^{-1}(k)\end{aligned}$$

for each $k' \in K'$. □

**9.2.3. Dependent Products.** Let $f \colon K \to K'$ be a function and let $(X, \phi_X)$ be a $K$-fibred set.

**Definition 9.2.3.1.1.** The **dependent product**[5] **of** $(X, \phi_X)$ is the $K'$-fibred set $\Pi_f(X)$[6] consisting of[7]

- *The Underlying Set.* The set $\Pi_f(X)$ defined by
$$\Pi_f(X) \stackrel{\text{def}}{=} \coprod_{k' \in K'} \prod_{k \in f^{-1}(k')} \phi_X^{-1}(k);$$
- *The Fibration.* The map of sets
$$\Pi_f(\phi_X) \colon \Pi_f(X) \to K'$$
defined by sending an element of
$$\Pi_f(X) \stackrel{\text{def}}{=} \coprod_{k' \in K'} \prod_{k \in f^{-1}(k')} \phi_X^{-1}(k)$$
to its index $k'$ in $K'$.

**Example 9.2.3.1.2.** Here are some examples of dependent products of sets.

(1) *Spaces of Sections.* Let $K = X$, $K' = \mathrm{pt}$, let $\phi \colon E \to X$ be a map of sets, and write $!_X \colon X \to \mathrm{pt}$ for the terminal map from $X$ to pt.

[5] The name "dependent product" comes from the fact that the fibre $\Pi_f(\phi_X)^{-1}(k')$ of $\Pi_f(X)$ at $k' \in K'$ is given by

$$\Pi_f(\phi_X)^{-1}(k') \cong \prod_{k \in f^{-1}(k')} \phi_X^{-1}(k);$$

see Item 2 of Proposition 9.2.3.1.3.

[6] *Further Notation:* Also written $f_!(X)$.

[7] We can also define dependent products via the internal **Hom** in $\mathsf{FibSets}(K')$; see Item 3 of Proposition 9.2.3.1.3.

We have a bijection of sets

$$\begin{aligned}\Pi_{!_X}((E,\phi)) &\cong \Gamma_X(\phi)\\ &\stackrel{\text{def}}{=} \{h \in \mathsf{Sets}(X,E) \mid \phi\circ h = \mathrm{id}_X\}.\end{aligned}$$

(2) *Function Spaces.* Let $K = K' = \mathrm{pt}$ and write $!_X \colon X \to \mathrm{pt}$ and $!_Y \colon Y \to \mathrm{pt}$ for the terminal maps from $X$ and $Y$ to pt. We have a bijection of sets

$$\mathsf{Sets}(X,Y) \cong \Pi_{!_X}(!_X^*(Y,!_Y)).$$

**Proof.** *Item 1, Spaces of Sections*: Indeed, we have

$$\begin{aligned}\Pi_{!_X}((E,\phi)) &\stackrel{\text{def}}{=} \coprod_{\star\in\mathrm{pt}}\prod_{k\in !_X^{-1}(\star)} \phi_X^{-1}(k)\\ &= \prod_{x\in X}\phi_X^{-1}(x)\\ &\cong \{h \in \mathsf{Sets}(X,E) \mid \phi_X\circ h = \mathrm{id}_X\}\\ &\stackrel{\text{def}}{=} \Gamma_X(\phi).\end{aligned}$$

*Item 2, Function Spaces*: Indeed, we have

$$\begin{aligned}\Pi_{!_X}(!_X^*(Y,!_Y)) &\stackrel{\text{def}}{=} \Pi_{!_X}(X\times_{!_X,\mathrm{pt},!_Y} Y)\\ &\stackrel{\text{def}}{=} \coprod_{\star\in\mathrm{pt}}\prod_{x\in !_X^{-1}(\star)} \mathrm{pr}_1^{-1}(x)\\ &= \prod_{x\in X} Y\\ &\cong \mathsf{Sets}(X,Y).\end{aligned}$$

This finishes the proof. □

**Proposition 9.2.3.1.3.** Let $f\colon K \to K'$ be a function.

(1) *Functoriality.* The assignment $X \mapsto \Pi_f(X)$ defines a functor

$$\Pi_f\colon \mathsf{FibSets}(K) \to \mathsf{FibSets}(K'),$$

where

- *Action on Objects.* For each $(X,\phi_X) \in \mathrm{Obj}(\mathsf{FibSets}(K))$, we have
$$\Pi_f(X,\phi_X) \stackrel{\text{def}}{=} \Pi_f(X);$$
- *Action on Morphisms.* For each $(X,\phi_X),(Y,\phi_Y) \in \mathrm{Obj}(\mathsf{FibSets}(K))$, the action on Hom-sets
$$\Pi_{f|X,Y}\colon \mathrm{Hom}_{\mathsf{FibSets}(K)}(X,Y) \to \mathrm{Hom}_{\mathsf{FibSets}(K')}(\Pi_f(X),\Pi_f(Y))$$
of $\Pi_f$ at $((X,\phi_X),(Y,\phi_Y))$ is the map sending a morphism of $K$-fibred sets
$$\xi\colon (X,\phi_X) \to (Y,\phi_Y),\qquad \begin{tikzcd} X \arrow[rr, "\xi"] \arrow[dr, "\phi_X"'] & & Y \arrow[dl, "\phi_Y"] \\ & K & \end{tikzcd}$$

to the morphism

$$\Pi_f(\xi)\colon (\Pi_f(X), \Pi_f(\phi_X)) \to (\Pi_f(Y), \Pi_f(\phi_Y)) \qquad \begin{tikzcd} \Pi_f(X) \arrow[rr, "\Pi_f(\xi)"] \arrow[dr, "\Pi_f(\phi_X)"'] & & \Pi_f(Y) \arrow[dl, "\Pi_f(\phi_Y)"] \\ & K & \end{tikzcd}$$

of $K'$-fibred sets given by[8]

$$[\Pi_f(\xi)]((x_k)_{k \in f^{-1}(k')}) \stackrel{\text{def}}{=} (\xi(x_k))_{k \in f^{-1}(k')}$$

for each $(x_k)_{k \in f^{-1}(k')} \in \Pi_f(X)$.

(2) *Interaction With Fibres.* We have a bijection of sets

$$\Pi_f(\phi_X)^{-1}(k') \cong \prod_{k \in f^{-1}(k')} \phi_X^{-1}(k);$$

for each $k' \in K'$.

(3) *Construction Using the Internal Hom.* We have

$$\Pi_f(X, \phi_X) = \left(K' \times_{\mathbf{Hom}_{\mathsf{FibSets}(K')}((K,f),(K,f))} \mathbf{Hom}_{\mathsf{FibSets}(K')}((K,f),(X, f \circ \phi_X)), \mathrm{pr}_1\right),$$

forming a pullback diagram

$$\begin{tikzcd} \Pi_f(X, \phi_X) \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Hom}_{\mathsf{FibSets}(K')}((K,f),(X, f \circ \phi_X)) \arrow[d, "(\phi_X)_*"] \\ K' \arrow[r, "I"'] & \mathbf{Hom}_{\mathsf{FibSets}(K')}((K,f),(K,f)), \end{tikzcd}$$

where the bottom map is given by

$$I(k') \stackrel{\text{def}}{=} \mathrm{id}_{f^{-1}(k')}$$

for each $k' \in K'$ and where $\mathbf{Hom}_{\mathsf{FibSets}(K')}$ denotes the internal Hom of $\mathsf{FibSets}(K')$ of Definition 9.2.4.1.1.

(4) *Internal Homs via Dependent Products.* We have

$$\mathbf{Hom}_{\mathsf{FibSets}(K)}(X, Y) \cong \Pi_{\phi_X}(\phi_X^*(Y)).$$

**Proof.** *Item 1, Functoriality*: Omitted.
*Item 2, Interaction With Fibres*: Clear.
*Item 3, Construction Using the Internal Hom*: Using the explicit formula for pullbacks of sets given in **??**, we see that the pullback

$$K' \times_{\mathbf{Hom}_{\mathsf{FibSets}(K')}((K,f),(K,f))} \mathbf{Hom}_{\mathsf{FibSets}(K')}((K,f),(X, f \circ \phi_X))$$

[8]Note that we indeed have $\xi(x_k) \in \phi_Y^{-1}(k)$, since

$$\begin{aligned} \phi_Y(\xi(x_k)) &= [\phi_Y \circ \xi](x_k) \\ &= \phi_X(x_k) \\ &= k, \end{aligned}$$

where we have used that $\xi$ is a morphism of $K$-fibred sets for the second equality.

is given by

$$\left\{(k',h)\in\coprod_{k'\in K'}\mathsf{Sets}\Big(f^{-1}(k'),\phi_X^{-1}\Big(f^{-1}(k')\Big)\Big)\;\middle|\;\phi_X\circ h=\mathrm{id}_{f^{-1}(k')}\right\},$$

which is isomorphic to

$$\coprod_{k'\in K'}\left\{h\in\mathsf{Sets}\Big(f^{-1}(k'),\phi_X^{-1}\Big(f^{-1}(k')\Big)\Big)\;\middle|\;\phi_X\circ h=\mathrm{id}_{f^{-1}(k')}\right\}.$$

We claim that

$$\left\{h\in\mathsf{Sets}\Big(f^{-1}(k'),\phi_X^{-1}\Big(f^{-1}(k')\Big)\Big)\;\middle|\;\phi_X\circ h=\mathrm{id}_{f^{-1}(k')}\right\}\cong\prod_{k\in f^{-1}(k')}\phi_X^{-1}(k),$$

so that the pullback is indeed given by $\Pi_f(X)$. There are two cases:

(1) If $f^{-1}(k')=\emptyset$, then there is only one map from $f^{-1}(k')$ to $\phi_X^{-1}(f^{-1}(k'))$ (the inclusion), so $\mathsf{Sets}\Big(f^{-1}(k'),\phi_X^{-1}(f^{-1}(k'))\Big)\cong\mathrm{pt}$. Since products indexed by the empty set are isomorphic to pt, the isomorphism follows.

(2) Otherwise, by the condition $\phi_X\circ h=\mathrm{id}_{f^{-1}(k')}$, it follows that, for each $k\in f^{-1}(k')$, we must have

$$\phi_X(h(k))=k,$$

and thus $h(k)\in\phi_X^{-1}(k)$. Therefore, a map from $f^{-1}(k')$ to $\phi_X^{-1}(f^{-1}(k'))$ consists of a choice of an element from $\phi_X^{-1}(k)$ for each $k\in f^{-1}(k')$, which is precisely given by an element of the product $\prod_{k\in f^{-1}(k')}\phi_X^{-1}(k)$, showing the bijection to be true.

*Item 4, Internal Homs via Dependent Products*: Indeed we have

$$\begin{aligned}\Pi_{\phi_X}(\phi_X^*(Y))&\overset{\text{def}}{=}\Pi_{\phi_X}(X\times_K Y)\\&\overset{\text{def}}{=}\coprod_{k\in K}\prod_{x\in\phi_X^{-1}(k)}\mathrm{pr}_1^{-1}(x)\\&\cong\coprod_{k\in K}\prod_{x\in\phi_X^{-1}(k)}\{y\in Y\mid\phi_X(x)=\phi_Y(y)\}\\&\cong\coprod_{k\in K}\prod_{x\in\phi_X^{-1}(k)}\{y\in Y\mid k=\phi_Y(y)\}\\&\cong\coprod_{k\in K}\prod_{x\in\phi_X^{-1}(k)}\phi_Y^{-1}(k)\\&\cong\coprod_{k\in K}\mathsf{Sets}\Big(\phi_X^{-1}(k),\phi_Y^{-1}(k)\Big)\\&\overset{\text{def}}{=}\mathbf{Hom}_{\mathsf{FibSets}(K)}(X,Y).\end{aligned}$$

This finishes the proof. □

**9.2.4. Internal Homs.** Let $K$ be a set and let $(X,\phi_X)$ and $(Y,\phi_Y)$ be $K$-fibred sets.

**Definition 9.2.4.1.1.** The **internal Hom of fibred sets from** $(X,\phi_X)$ **to** $(Y,\phi_Y)$ is the fibred set $\mathbf{Hom}_{\mathsf{FibSets}(K)}(X,Y)$ consisting of

- *The Underlying Set.* The set $\mathbf{Hom}_{\mathsf{FibSets}(K)}(X,Y)$ defined by
$$\mathbf{Hom}_{\mathsf{FibSets}(K)}(X,Y) \stackrel{\text{def}}{=} \coprod_{k\in K} \mathsf{Sets}\Big(\phi_X^{-1}(k), \phi_Y^{-1}(k)\Big);$$
- *The Fibration.* The map of sets[9]
$$\phi_{\mathbf{Hom}_{\mathsf{FibSets}(K)}(X,Y)}\colon \underbrace{\mathbf{Hom}_{\mathsf{FibSets}(K)}(X,Y)}_{\coprod_{k\in K}\mathsf{Sets}\left(\phi_X^{-1}(k),\phi_Y^{-1}(k)\right)} \to K$$
defined by sending a map $f\colon \phi_X^{-1}(k) \to \phi_Y^{-1}(k)$ to its index $k \in K$.

**Proof.** Omitted. □

**Proposition 9.2.4.1.2.** Let $K$ be a set and let $(X,\phi_X)$ and $(Y,\phi_Y)$ be $K$-fibred sets.

(1) *Functoriality.* Let $(X,\phi_X)$ and $(Y,\phi_Y)$ be $K$-fibred sets.
   (a) The assignment $X \mapsto \mathbf{Hom}_{\mathsf{FibSets}(K)}(X,Y)$ defines a functor
   $$\mathbf{Hom}_{\mathsf{FibSets}(K)}(X,-)\colon \mathsf{FibSets}(K) \to \mathsf{FibSets}(K).$$
   (b) The assignment $Y \mapsto \mathbf{Hom}_{\mathsf{FibSets}(K)}(X,Y)$ defines a functor
   $$\mathbf{Hom}_{\mathsf{FibSets}(K)}(-,Y)\colon \mathsf{FibSets}(K)^{\mathsf{op}} \to \mathsf{FibSets}(K).$$
   (c) The assignment $(X,Y) \mapsto \mathbf{Hom}_{\mathsf{FibSets}(K)}(X,Y)$ defines a functor
$$\mathbf{Hom}_{\mathsf{FibSets}(K)}(-_1,-_2)\colon \mathsf{FibSets}(K)^{\mathsf{op}} \times \mathsf{FibSets}(K) \to \mathsf{FibSets}(K).$$
(2) *Internal Homs via Dependent Products.* We have
$$\mathbf{Hom}_{\mathsf{FibSets}(K)}(X,Y) \cong \Pi_{\phi_X}(\phi_X^*(Y)).$$

**Proof.** *Item 1, Functoriality*: Omitted.
*Item 2, Internal Homs via Dependent Products*: This was proved in Item 4 of Proposition 9.2.3.1.3. □

**9.2.5. Adjointness for Fibred Sets.** Let $f\colon K \to K'$ be a map of sets.

**Proposition 9.2.5.1.1.** We have a triple adjunction

$$(\Sigma_f \dashv f^* \dashv \Pi_f)\colon\quad \begin{tikzcd} \mathsf{FibSets}(K) \arrow[r, bend left, "\Sigma_f"] \arrow[r, bend right, "\Pi_f"'] & \mathsf{FibSets}(K') \arrow[l, "f^*"'] \end{tikzcd}.$$

We offer two proofs. The first uses the corresponding adjunction for indexed sets (**??**) and the un/straightening equivalence together with its compatibility with dependent sums and products to "transfer" the adjunction to fibred sets, while the second is a direct proof.

[9]The fibres of the internal $\mathbf{Hom}$ of $\mathsf{FibSets}(K)$ are precisely the sets $\mathsf{Sets}\big(\phi_X^{-1}(k),\phi_Y^{-1}(k)\big)$, i.e. we have
$$\phi_{\mathbf{Hom}_{\mathsf{FibSets}(K)}(X,Y)|k} \cong \mathsf{Sets}\big(\phi_X^{-1}(k),\phi_Y^{-1}(k)\big)$$
for each $k \in K$.

**Proof.** *The Adjunction* $\Sigma_f \dashv f^*$: The adjunction

$$
(\Sigma_f \dashv f^*)\colon\quad
\begin{tikzcd}
\mathsf{ISets}(K) \arrow[r, bend left, "\Sigma_f"] \arrow[r, phantom, "\perp"] & \mathsf{ISets}(K') \arrow[l, bend left, "f^*"]
\end{tikzcd}
$$

of **??** gives a unit and counit of the form

$$
\begin{aligned}
\eta &\colon \mathrm{id}_{\mathsf{ISets}(K)} \Longrightarrow \Sigma_f \circ f^*,\\
\epsilon &\colon f^* \circ \Sigma_f \Longrightarrow \mathrm{id}_{\mathsf{ISets}(K')}.
\end{aligned}
$$

With these in hand, we construct natural transformations

$$
\begin{aligned}
\eta' &\colon \mathrm{id}_{\mathsf{FibSets}(K)} \Longrightarrow \Sigma_f \circ f^*,\\
\epsilon' &\colon f^* \circ \Sigma_f \Longrightarrow \mathrm{id}_{\mathsf{FibSets}(K')}
\end{aligned}
$$

as follows:

(1) *The Unit.* We define $\eta'\colon \mathrm{id}_{\mathsf{FibSets}(K)} \Longrightarrow \Sigma_f \circ f^*$ as the pasting of the diagram

$$
\begin{tikzcd}
 & & \mathsf{FibSets}(K') \arrow[drr, "f^*"] & & \\
\mathsf{FibSets}(K) \arrow[urr, "\Sigma_f"] \arrow[r, phantom, "(1)"] & {} & \mathsf{ISets}(K') \arrow[u, "\mathrm{St}_{K'}"] \arrow[dr, "f^*"] \arrow[r, phantom, "(2)"] & {} & \mathsf{FibSets}(K) \\
 \arrow[r, phantom, "(3)"] & \mathsf{ISets}(K) \arrow[ul, "\mathrm{St}_K"] \arrow[ur, "\Sigma_f"] \arrow[rr, "\mathrm{id}_{\mathsf{ISets}(K)}"] \arrow[r, phantom, shift left=4, "\Uparrow\eta"'] & & \mathsf{ISets}(K) \arrow[ur, "\mathrm{St}_K"] \arrow[r, phantom, "(4)"] & {} \\
\mathsf{FibSets}(K) \arrow[uu, "\mathrm{id}_{\mathsf{FibSets}(K)}"] \arrow[ur, "\mathrm{Un}_K"] \arrow[rrrr, "\mathrm{id}_{\mathsf{FibSets}(K)}"'] \arrow[rrrr, phantom, shift left=3, "(5)"] & & & & \mathsf{FibSets}(K), \arrow[ul, "\mathrm{Un}_K"'] \arrow[uu, "\mathrm{id}_{\mathsf{FibSets}(K)}"']
\end{tikzcd}
$$

where:

(a) Subdiagram (1) commutes by **??** of **??**.
(b) Subdiagram (2) commutes by **??** of **??**.
(c) Subdiagram (3) commutes by **??**.
(d) Subdiagram (4) commutes by **??**.
(e) Subdiagram (5) commutes by unitality of composition.

(2) *The Counit.* We define $\epsilon' \colon f^* \circ \Sigma_f \Longrightarrow \mathrm{id}_{\mathsf{FibSets}(K')}$ as the pasting of the diagram

$$
\begin{tikzcd}
\mathsf{FibSets}(K') \arrow[rrrr, "\mathrm{id}_{\mathsf{FibSets}(K')}"] \arrow[dd, "\mathrm{id}_{\mathsf{FibSets}(K')}"'] \arrow[dr, "\mathrm{Un}_K"] & & (1) & & \mathsf{FibSets}(K') \arrow[dd, "\mathrm{id}_{\mathsf{FibSets}(K')}"] \arrow[dl, "\mathrm{Un}_K"'] \\
(2) & \mathsf{ISets}(K') \arrow[rr, "\mathrm{id}_{\mathsf{ISets}(K)}"] \arrow[dl, "\mathrm{St}_{K'}"'] \arrow[dr, "f^*"] & \arrow[d, phantom, "\Uparrow\epsilon"] & \mathsf{ISets}(K') \arrow[dr, "\mathrm{St}_{K'}"] & (3) \\
\mathsf{FibSets}(K') \arrow[drr, "f^*"'] & (4) & \mathsf{ISets}(K) \arrow[ur, "\Sigma_f"] \arrow[d, "\mathrm{St}_K"] & (5) & \mathsf{FibSets}(K') \\
& & \mathsf{FibSets}(K) \arrow[urr, "\Sigma_f"'] & &
\end{tikzcd}
$$

where:
(a) Subdiagram (1) commutes by unitality of composition.
(b) Subdiagram (2) commutes by **??**.
(c) Subdiagram (3) commutes by **??**.
(d) Subdiagram (4) commutes by **??** of **??**.
(e) Subdiagram (5) commutes by **??** of **??**.

Next, we prove the left triangle identity,

$$
\begin{tikzcd}
& \mathsf{FibSets}(K') \arrow[rr, "\mathrm{id}_{\mathsf{FibSets}(K')}"] \arrow[dr, "f^*"] & \arrow[d, phantom, "\Uparrow\epsilon"] & \mathsf{FibSets}(K') \\
\mathsf{FibSets}(K) \arrow[ur, "\Sigma_f"] \arrow[rr, "\mathrm{id}_{\mathsf{FibSets}(K)}"'] & \arrow[u, phantom, "\Uparrow\eta"] & \mathsf{FibSets}(K) \arrow[ur, "\Sigma_f"'] &
\end{tikzcd}
=
\begin{tikzcd}
& \mathsf{FibSets}(K') \arrow[rr, "\mathrm{id}_{\mathsf{FibSets}(K')}"] & & \mathsf{FibSets}(K'), \\
\mathsf{FibSets}(K) \arrow[ur, "\Sigma_f"] \arrow[rr, "\mathrm{id}_{\mathsf{FibSets}(K)}"'] & & \mathsf{FibSets}(K) \arrow[ur, "\Sigma_f"'] \arrow[ul, phantom, "\Leftarrow\mathrm{id}_{\Sigma_f}"] &
\end{tikzcd}
$$

whose left side in our case looks like this:

$$
\begin{tikzcd}[column sep=small]
& & \mathsf{FibSets}(K') \arrow[rrr, "\mathrm{id}_{\mathsf{FibSets}(K')}"] \arrow[drr, "f^*"] & & & \mathsf{FibSets}(K') \arrow[rrrr, "\mathrm{id}_{\mathsf{FibSets}(K')}"] \arrow[dd, "\mathrm{id}_{\mathsf{FibSets}(K')}"'] \arrow[dr, "\mathrm{Un}_K"] & & & & \mathsf{FibSets}(K') \arrow[dd, "\mathrm{id}_{\mathsf{FibSets}(K')}"] \arrow[dl, "\mathrm{Un}_K"'] \\
\mathsf{FibSets}(K) \arrow[urr, "\Sigma_f"] & & \mathsf{ISets}(K') \arrow[u, "\mathrm{St}_{K'}"] \arrow[dr, "f^*"] & & \mathsf{FibSets}(K) & & \mathsf{ISets}(K') \arrow[rr, "\mathrm{id}_{\mathsf{ISets}(K)}"] \arrow[dl, "\mathrm{St}_{K'}"'] \arrow[dr, "f^*"] & \arrow[d, phantom, "\Uparrow\epsilon"] & \mathsf{ISets}(K') \arrow[dr, "\mathrm{St}_{K'}"] & \\
& \mathsf{ISets}(K) \arrow[ul, "\mathrm{St}_K"] \arrow[ur, "\Sigma_f"] \arrow[rr, "\mathrm{id}_{\mathsf{ISets}(K)}"] & \arrow[u, phantom, "\Uparrow\eta"] & \mathsf{ISets}(K) \arrow[ur, "\mathrm{St}_K"] & & \mathsf{FibSets}(K') \arrow[drr, "f^*"'] & & \mathsf{ISets}(K) \arrow[ur, "\Sigma_f"] \arrow[d, "\mathrm{St}_K"] & & \mathsf{FibSets}(K'). \\
\mathsf{FibSets}(K) \arrow[uu, "\mathrm{id}_{\mathsf{FibSets}(K)}"] \arrow[ur, "\mathrm{Un}_K"] \arrow[rrrr, "\mathrm{id}_{\mathsf{FibSets}(K)}"'] & & & & \mathsf{FibSets}(K) \arrow[uu, "\mathrm{id}_{\mathsf{FibSets}(K)}"'] \arrow[ul, "\mathrm{Un}_K"'] \arrow[rrr, "\mathrm{id}_{\mathsf{FibSets}(K)}"'] & & & \mathsf{FibSets}(K) \arrow[urr, "\Sigma_f"'] & &
\end{tikzcd}
$$

It can be rearranged into

$$
\begin{tikzcd}[column sep=small]
& & \mathsf{FibSets}(K') \arrow[rrr, "\mathrm{id}_{\mathsf{FibSets}(K')}"] & & & \mathsf{FibSets}(K') \arrow[rrrr, "\mathrm{id}_{\mathsf{FibSets}(K')}"] & & & & \mathsf{FibSets}(K') \arrow[dd, "\mathrm{id}_{\mathsf{FibSets}(K')}"] \arrow[dl, "\mathrm{Un}_K"'] \\
\mathsf{FibSets}(K) \arrow[urr, "\Sigma_f"] & & \mathsf{ISets}(K') \arrow[u, "\mathrm{St}_{K'}"] \arrow[dr, "f^*"] \arrow[rrrr, "\mathrm{id}_{\mathsf{ISets}(K')}"] & & & (1) & \mathsf{ISets}(K') \arrow[rr, "\mathrm{id}_{\mathsf{ISets}(K)}"] \arrow[dr, "f^*"] & \arrow[d, phantom, "\Uparrow\epsilon"] & \mathsf{ISets}(K') \arrow[dr, "\mathrm{St}_{K'}"] & \\
& \mathsf{ISets}(K) \arrow[ul, "\mathrm{St}_K"] \arrow[ur, "\Sigma_f"] \arrow[rr, "\mathrm{id}_{\mathsf{ISets}(K)}"] & \arrow[u, phantom, "\Uparrow\eta"] & \mathsf{ISets}(K) \arrow[rrrr, "\mathrm{id}_{\mathsf{ISets}(K')}"] & & (2) & & \mathsf{ISets}(K) \arrow[ur, "\Sigma_f"] \arrow[d, "\mathrm{St}_K"] & & \mathsf{FibSets}(K'), \\
\mathsf{FibSets}(K) \arrow[uu, "\mathrm{id}_{\mathsf{FibSets}(K)}"] \arrow[ur, "\mathrm{Un}_K"] \arrow[rrrr, "\mathrm{id}_{\mathsf{FibSets}(K)}"'] & & & & \mathsf{FibSets}(K) \arrow[u, phantom, "(3)"] \arrow[rrr, "\mathrm{id}_{\mathsf{FibSets}(K)}"'] & & & \mathsf{FibSets}(K) \arrow[urr, "\Sigma_f"'] & &
\end{tikzcd}
$$

where:
(1) Subdiagram (1) commutes by **??**.
(2) Subdiagram (2) commutes by unitality of composition.

(3) Subdiagram (3) commutes by **??**.

And then, it can be rearranged into

$$
\begin{tikzcd}
 & & \mathsf{FibSets}(K') \arrow[rrr, "\mathrm{id}_{\mathsf{FibSets}(K')}"] & & & \mathsf{FibSets}(K') \arrow[dl, "\mathrm{Un}_K"'] \arrow[dd, "\mathrm{id}_{\mathsf{FibSets}(K')}"] \\
\mathsf{FibSets}(K) \arrow[urr, "\Sigma_f"] & & \mathsf{ISets}(K') \arrow[u, "\mathrm{St}_{K'}"] \arrow[rr, "\mathrm{id}_{\mathsf{ISets}(K)}"] \arrow[dr, "f^*"] \arrow[d, phantom, "\Uparrow\eta"] & \arrow[d, phantom, "\Uparrow\epsilon"] & \mathsf{ISets}(K') \arrow[dr, "\mathrm{St}_{K'}"] & \\
 & \mathsf{ISets}(K) \arrow[ul, "\mathrm{St}_K"] \arrow[ur, "\Sigma_f"] \arrow[rr, "\mathrm{id}_{\mathsf{ISets}(K)}"] & & \mathsf{ISets}(K) \arrow[ur, "\Sigma_f"] \arrow[d, "\mathrm{St}_K"] & & \mathsf{FibSets}(K'), \\
\mathsf{FibSets}(K) \arrow[uu, "\mathrm{id}_{\mathsf{FibSets}(K)}"] \arrow[ur, "\mathrm{Un}_K"] \arrow[rrr, "\mathrm{id}_{\mathsf{FibSets}(K)}"'] & & & \mathsf{FibSets}(K) \arrow[urr, "\Sigma_f"'] & &
\end{tikzcd}
$$

which by the left triangle identity for $(\eta, \epsilon)$, becomes

$$
\begin{tikzcd}
 & & \mathsf{FibSets}(K') \arrow[rrr, "\mathrm{id}_{\mathsf{FibSets}(K')}"] & & & \mathsf{FibSets}(K') \arrow[dl, "\mathrm{Un}_K"'] \arrow[dd, "\mathrm{id}_{\mathsf{FibSets}(K')}"] \\
\mathsf{FibSets}(K) \arrow[urr, "\Sigma_f"] & & \mathsf{ISets}(K') \arrow[u, "\mathrm{St}_{K'}"] \arrow[rr, "\mathrm{id}_{\mathsf{ISets}(K)}"] \arrow[dr, phantom, "\mathsf{id}"] & & \mathsf{ISets}(K') \arrow[dr, "\mathrm{St}_{K'}"] & \\
 & \mathsf{ISets}(K) \arrow[ul, "\mathrm{St}_K"] \arrow[ur, "\Sigma_f"] \arrow[rr, "\mathrm{id}_{\mathsf{ISets}(K)}"] & & \mathsf{ISets}(K) \arrow[ur, "\Sigma_f"] \arrow[d, "\mathrm{St}_K"] & & \mathsf{FibSets}(K'), \\
\mathsf{FibSets}(K) \arrow[uu, "\mathrm{id}_{\mathsf{FibSets}(K)}"] \arrow[ur, "\mathrm{Un}_K"] \arrow[rrr, "\mathrm{id}_{\mathsf{FibSets}(K)}"'] & & & \mathsf{FibSets}(K) \arrow[urr, "\Sigma_f"'] & &
\end{tikzcd}
$$

finishing the proof of the left triangle identity. The proof of the right triangle identity is similar, and is thus omitted.

*The Adjunction* $f^* \dashv \Pi_f$: This proof is similar to the proof of the adjunction $\Sigma_f \dashv f^*$, and is thus omitted. □

We proceed to the direct proof of Proposition 9.2.5.1.1.

**Proof.** *The Adjunction* $\Sigma_f \dashv f^*$: We claim there's a bijection

$$\mathrm{Hom}_{\mathsf{FibSets}(K')}(\Sigma_f(X), Y) \cong \mathrm{Hom}_{\mathsf{FibSets}(K)}(X, f^*(Y)),$$

natural in $(X, \phi_X) \in \mathrm{Obj}(\mathsf{FibSets}(K))$ and $(Y, \phi_Y) \in \mathrm{Obj}(\mathsf{FibSets}(K'))$:

- *Map I.* We define a map

  $$\Phi_{X,Y} \colon \mathrm{Hom}_{\mathsf{FibSets}(K')}(\Sigma_f(X), Y) \to \mathrm{Hom}_{\mathsf{FibSets}(K)}(X, f^*(Y)),$$

  by sending a morphism

  $$\xi \colon \Sigma_f(X) \to Y, \qquad \begin{tikzcd} X \arrow[rr, "\xi"] \arrow[dr, "\phi_X"'] & & Y \arrow[ddl, "\phi_Y"] \\ & K \arrow[d, "f"'] & \\ & K' & \end{tikzcd}$$

  of $K'$-fibred sets to the morphism

  $$\xi^\dagger \colon X \to f^*(Y), \qquad \begin{tikzcd} X \arrow[rr, "\xi^\dagger"] \arrow[dr, "\phi_X"'] & & K \times_{K'} Y \arrow[dl, "\mathrm{pr}_1"] \\ & K' & \end{tikzcd}$$

of $K$-fibred sets given by the dashed morphism in the diagram

$$\begin{tikzcd} X \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left, "\xi"] \arrow[ddr, bend right, "\phi_X"'] & & \\ & K \times_{K'} Y \arrow[r, dashed, "\mathrm{pr}_2"] \arrow[d, dashed, "\mathrm{pr}_1"] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "\phi_Y"] \\ & K \arrow[r, "f"'] & K'. \end{tikzcd}$$

- *Map II.* We define a map

$$\Psi_{X,Y} \colon \mathrm{Hom}_{\mathsf{FibSets}(K)}(X, f^*(Y)) \to \mathrm{Hom}_{\mathsf{FibSets}(K')}(\Sigma_f(X), Y),$$

given by sending a map

$$\xi \colon X \to f^*(Y), \qquad \begin{tikzcd}[column sep=small] X \arrow[rr, "\xi"] \arrow[dr, "\phi_X"'] & & K \times_{K'} Y \arrow[dl, "\mathrm{pr}_1"] \\ & K' & \end{tikzcd}$$

of $K'$-fibred sets to the map

$$\xi^\dagger \colon \Sigma_f(X) \to Y, \qquad \begin{tikzcd}[column sep=small] X \arrow[rr, "\xi^\dagger"] \arrow[d, "\phi_X"'] & & Y \arrow[ddl, "\phi_Y"] \\ K \arrow[dr, "f"'] & & \\ & K' & \end{tikzcd}$$

of $K$-fibred sets given by

$$\xi^\dagger \overset{\text{def}}{=} \mathrm{pr}_2 \circ \xi,$$

where we indeed have

$$\begin{aligned} \phi_Y \circ (\mathrm{pr}_2 \circ \xi) &= (\phi_Y \circ \mathrm{pr}_2) \circ \xi \\ &= (f \circ \mathrm{pr}_1) \circ \xi && \text{(by the pullback square of } K \times_{K'} Y) \\ &= f \circ (\mathrm{pr}_1 \circ \xi) \\ &= f \circ \phi_X. && \text{(since } \xi \text{ is a morphism of } K'\text{-fibred sets)} \end{aligned}$$

- *Naturality I.* We need to show that, given a morphism

$$\alpha \colon (X, \phi_X) \to (X', \phi_{X'})$$

of $K$-fibred sets, the diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathsf{FibSets}(K')}(\Sigma_f(X'), Y) \arrow[r, "\Phi_{X',Y}"] \arrow[d, "\Sigma_f(\alpha)^*"'] & \mathrm{Hom}_{\mathsf{FibSets}(K)}(X', f^*(Y)), \arrow[d, "\alpha^*"] \\ \mathrm{Hom}_{\mathsf{FibSets}(K')}(\Sigma_f(X), Y) \arrow[r, "\Phi_{X,Y}"'] & \mathrm{Hom}_{\mathsf{FibSets}(K)}(X, f^*(Y)) \end{tikzcd}$$

commutes. Indeed, given a morphism

$$\xi\colon \Sigma_f(X') \to Y, \qquad \begin{tikzcd} X' \arrow[rr, "\xi"] \arrow[dr, "\phi_{X'}"'] & & Y \arrow[ddl, "\phi_Y"] \\ & K \arrow[d, "f"'] & \\ & K' & \end{tikzcd}$$

of $K'$-fibred-sets, the map $\Phi_{X',Y}(\xi) \circ \alpha$ is the composition, coloured in vermillion, of the dashed arrow with $\alpha$ in the diagram

$$\begin{tikzcd}
X \arrow[dr, "\alpha"] \arrow[ddrr, dashed, "\exists!"] \arrow[ddrrr, bend left, "\xi\circ\alpha"] \arrow[dddrr, bend right, "\phi_{X'}\circ\alpha"'] & & & \\
& X' \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left, "\xi"] \arrow[ddr, bend right, "\phi_{X'}"'] & & \\
& & K \times_{K'} Y \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "\phi_Y"] \\
& & K \arrow[r, "f"'] & K',
\end{tikzcd}$$

while $\Phi_{X,Y}(\xi \circ \Sigma_f(\alpha))$ is given by the dashed arrow, coloured in blue. Since both the blue arrow and the vermillion arrow make the outer pullback diagram commute, it follows from the universal property of the pullback that they must be equal, i.e. that

$$\Phi_{X',Y}(\xi) \circ \alpha = \Phi_{X,Y}(\xi \circ \Sigma_f(\alpha)),$$

showing that the naturality diagram above indeed commutes.

- *Naturality II.* We need to show that, given a morphism

$$\beta\colon (Y, \phi_Y) \to (Y', \phi_{Y'})$$

of $K$-fibred sets, the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathsf{FibSets}(K')}(\Sigma_f(X), Y) \arrow[r, "\Phi_{X,Y}"] \arrow[d, "\beta_*"'] & \mathrm{Hom}_{\mathsf{FibSets}(K)}(X, f^*(Y)), \arrow[d, "f^*(\beta)_*"] \\
\mathrm{Hom}_{\mathsf{FibSets}(K')}(\Sigma_f(X), Y') \arrow[r, "\Phi_{X,Y'}"'] & \mathrm{Hom}_{\mathsf{FibSets}(K)}(X, f^*(Y'))
\end{tikzcd}$$

commutes. Indeed, given a morphism

$$\xi\colon \Sigma_f(X') \to Y, \qquad \begin{tikzcd} X' \arrow[rr, "\xi"] \arrow[dr, "\phi_{X'}"'] & & Y \arrow[ddl, "\phi_Y"] \\ & K \arrow[d, "f"'] & \\ & K' & \end{tikzcd}$$

of $K'$-fibred-sets, the map $f^*(\beta) \circ \Phi_{X,Y}(\xi)$ is the composition, coloured in vermillion, of the dashed arrow from $X$ to $K \times_{K'} Y$

with the dashed arrow from $K \times_{K'} Y$ to $K \times_{K'} Y'$ in the diagram

$$\begin{tikzcd}
X \arrow[dr, dashed, "\exists!"] \arrow[drr, dashed, "\exists!"] \arrow[drrr, bend left, "\xi"] \arrow[dddr, bend right, "\phi_X"'] & & & & \\
& K \times_{K'} Y \arrow[rr] \arrow[dr, dashed] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & & Y \arrow[dr, "\beta"] \arrow[dd, "\phi_Y"] & \\
& & K \times_{K'} Y' \arrow[rr] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & & Y' \arrow[dd, "\phi_{Y'}"] \\
& K \arrow[rr, "f"] \arrow[dr, equal] & & K \arrow[dr, equal] & \\
& & K \arrow[rr, "f"'] & & K',
\end{tikzcd}$$

while $\Phi_{X,Y'}(\beta \circ \xi)$ is given by the dashed arrow from $X$ to $K \times_{K'} Y'$, coloured in blue. Since both the blue arrow and the vermillion arrow make the outer pullback diagram for $K \times_{K'} Y'$ commute, it follows from the universal property of the pullback that they must be equal, i.e. that

$$f^*(\beta) \circ \Phi_{X,Y}(\xi) = \Phi_{X,Y'}(\beta \circ \xi),$$

showing that the naturality diagram above indeed commutes.

- *Invertibility I.* We claim that

$$\Psi_{X,Y} \circ \Phi_{X,Y} = \mathrm{id}_{\mathrm{Hom}_{\mathsf{FibSets}(K')}\left(\Sigma_f(X),Y\right)}.$$

Indeed, $\Phi_{X,Y}$ sends a map

$$\xi \colon \Sigma_f(X) \to Y, \qquad \begin{tikzcd}
X \arrow[rr, "\xi"] \arrow[dr, "\phi_X"'] & & Y \arrow[ddl, "\phi_Y"] \\
& K \arrow[d, "f"'] & \\
& K' &
\end{tikzcd}$$

of $K'$-fibred sets to the dashed morphism in the diagram

$$\begin{tikzcd}
X \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left, "\xi"] \arrow[ddr, bend right, "\phi_X"'] & & \\
& K \times_{K'} Y \arrow[r, dashed, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "\phi_Y"] \\
& K \arrow[r, "f"'] & K',
\end{tikzcd}$$

and $\Psi_{X,Y}$ then postcomposes that map with $\mathrm{pr}_2$, which, by the commutativity of the diagram above, is $\xi$ again, showing the claimed equality to be true.

- *Invertibility II.* We claim that
$$\Phi_{X,Y} \circ \Psi_{X,Y} = \mathrm{id}_{\mathrm{Hom}_{\mathsf{FibSets}(K)}(X, f^*(Y))}.$$
Indeed, $\Psi_{X,Y}$ sends a map
$$\xi \colon X \to f^*(Y), \qquad \begin{tikzcd} X \arrow[rr, "\xi"] \arrow[dr, "\phi_X"'] & & K \times_{K'} Y \arrow[dl, "\mathrm{pr}_1"] \\ & K' & \end{tikzcd}$$
of $K'$-fibred sets to $\mathrm{pr}_2 \circ \xi$, which is then sent by $\Phi_{X,Y}$ to the dashed morphism in the diagram
$$\begin{tikzcd} X \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left, "\mathrm{pr}_2 \circ \xi"] \arrow[ddr, bend right, "\phi_X"'] \arrow[dr, phantom, shift left=6, "(\dagger)"] & & \\ & K \times_{K'} Y \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] \arrow[dr, phantom, very near start, "\lrcorner"] & Y \arrow[d, "\phi_Y"] \\ & K \arrow[r, "f"'] & K', \end{tikzcd}$$
which, by the commutativity of the subdiagram marked with $(\dagger)$, is given by $\xi$ again, showing the claimed equality to be true.

*The Adjunction $f^* \dashv \Pi_f$:* We claim there's a bijection
$$\mathrm{Hom}_{\mathsf{FibSets}(K)}(f^*(X), Y), \cong \mathrm{Hom}_{\mathsf{FibSets}(K')}(X, \Pi_f(Y))$$
natural in $(X, \phi_X) \in \mathrm{Obj}(\mathsf{FibSets}(K'))$ and $(Y, \phi_Y) \in \mathrm{Obj}(\mathsf{FibSets}(K))$:

(1) *Map I.* We define a map
$$\Phi_{X,Y} \colon \mathrm{Hom}_{\mathsf{FibSets}(K)}(f^*(X), Y) \to \mathrm{Hom}_{\mathsf{FibSets}(K')}(X, \Pi_f(Y))$$
defined as follows. Given a morphism
$$\xi \colon f^*(X) \to Y, \qquad \begin{tikzcd} K \times_{K'} X \arrow[rr, "\xi"] \arrow[dr, "\mathrm{pr}_1"'] & & Y \arrow[dl, "\phi_Y"] \\ & K & \end{tikzcd}$$
of $K$-fibred sets, where
$$\begin{aligned} f^*(X) &\overset{\text{def}}{=} K \times_{K'} X \\ &\overset{\text{def}}{=} \{(k, x) \in K \times X \mid f(k) = \phi_X(x)\}, \end{aligned}$$

we construct a morphism

$$\xi^\dagger\colon X \to \Pi_f(Y),\qquad \begin{tikzcd} X \arrow[rr, "\xi^\dagger"] \arrow[dr, "\phi_X"'] & & \Pi_f(Y) \arrow[dl, "\Pi_f(\phi_Y)"] \\ & K' & \end{tikzcd}$$

of $K'$-fibred sets, where

$$\Pi_f(Y) \stackrel{\text{def}}{=} \coprod_{k'\in K'} \prod_{k\in f^{-1}(k')} \phi_Y^{-1}(k),$$

by defining

$$\xi^\dagger(x) \stackrel{\text{def}}{=} (\xi(k,x))_{k\in f^{-1}(\phi_X(x))}$$

for each $x \in X$. There are two things to be checked here:

- We have $\xi(k,x) \in \phi_Y^{-1}(\phi_X(x))$ since $\phi_Y(\xi(k,x)) = \phi_X(x)$ as $\xi$ is a morphism of $K$-fibred sets.
- The map $\xi^\dagger$ is indeed a morphism of $K'$-fibred sets, i.e. we have

  $$\Pi_f(\phi_Y) \circ \xi^\dagger = \phi_X,$$

  since

  $$[\Pi_f(\phi_Y)]\Big((\xi(k,x))_{k\in f^{-1}(\phi_X(x))}\Big) = \phi_X(x)$$

  for each $x \in X$.

(2) *Map II.* We define a map

$$\Psi_{X,Y}\colon \mathrm{Hom}_{\mathsf{FibSets}(K')}(X, \Pi_f(Y)) \to \mathrm{Hom}_{\mathsf{FibSets}(K)}(f^*(X), Y)$$

as follows. Given a morphism

$$\xi\colon X \to \Pi_f(Y),\qquad \begin{tikzcd} X \arrow[rr, "\xi"] \arrow[dr, "\phi_X"'] & & \Pi_f(Y) \arrow[dl, "\Pi_f(\phi_Y)"] \\ & K' & \end{tikzcd}$$

of $K'$-fibred sets, where

$$\Pi_f(Y) \stackrel{\text{def}}{=} \coprod_{k'\in K'} \prod_{k\in f^{-1}(k')} \phi_Y^{-1}(k),$$

we construct a morphism

$$\xi^\dagger\colon f^*(X) \to Y,\qquad \begin{tikzcd} K \times_{K'} X \arrow[rr, "\xi^\dagger"] \arrow[dr, "\mathrm{pr}_1"'] & & Y \arrow[dl, "\phi_Y"] \\ & K & \end{tikzcd}$$

of $K$-fibred sets, where

$$\begin{aligned} f^*(X) &\stackrel{\text{def}}{=} K \times_{K'} X \\ &\stackrel{\text{def}}{=} \{(k,x) \in K \times X \mid f(k) = \phi_X(x)\}, \end{aligned}$$

by defining

$$\xi^\dagger(k,x) \stackrel{\text{def}}{=} \xi(x)_k$$

for each $(k,x) \in f^*(X)$, where $\xi(x)_k$ is the $k$th component of $\xi(x) = (y_k)_{k\in f^{-1}(k')}$. We also need to check that $\xi^\dagger$ is a morphism of $K$-fibred sets, i.e. that

$$\phi_Y \circ \xi^\dagger = \mathrm{pr}_1,$$

or

$$\phi_Y\Big(\xi^\dagger(k,x)\Big) = k,$$

for each $(k,x) \in f^*(X)$, which is clear, since $\xi^\dagger(k,x) \in \phi_Y^{-1}(k)$ by definition.

(3) *Naturality I.* We need to show that, given a morphism

$$\alpha\colon (X,\phi_X) \to (X',\phi_{X'})$$

of $K'$-fibred sets, the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathsf{FibSets}(K')}(f^*(X'),Y) \arrow[r, "\Phi_{X',Y}"] \arrow[d, "f^*(\alpha)^*"'] & \mathrm{Hom}_{\mathsf{FibSets}(K)}(X',\Pi_f(Y)) \arrow[d, "\alpha^*"] \\
\mathrm{Hom}_{\mathsf{FibSets}(K')}(f^*(X),Y) \arrow[r, "\Phi_{X,Y}"'] & \mathrm{Hom}_{\mathsf{FibSets}(K)}(X,\Pi_f(Y))
\end{tikzcd}$$

commutes. Indeed, given a morphism $\xi\colon f^*(X') \to Y$ of $K'$-fibred sets, we have

$$\begin{aligned}
[[\Phi_{X,Y} \circ f^*(\alpha)](\xi)](x) &\overset{\text{def}}{=} [\Phi_{X,Y}(\xi \circ f^*(\alpha))](x) \\
&\overset{\text{def}}{=} ([\xi \circ f^*(\alpha)](k,x))_{k \in f^{-1}(\phi_X(x))} \\
&\overset{\text{def}}{=} (\xi(k,\alpha(x)))_{k \in f^{-1}(\phi_X(x))} \\
&\overset{\text{def}}{=} \alpha^*\Big((\xi(k,x))_{k \in f^{-1}(\phi_X(x))}\Big) \\
&\overset{\text{def}}{=} \alpha^*\Big(\xi^\dagger(x)\Big) \\
&\overset{\text{def}}{=} [[\alpha^* \circ \Phi_{X,Y}](\xi)](x)
\end{aligned}$$

for each $x \in X$.

(4) *Naturality II.* We need to show that, given a morphism

$$\beta\colon (Y,\phi_Y) \to (Y',\phi_{Y'})$$

of $K$-fibred sets, the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathsf{FibSets}(K')}(f^*(X),Y) \arrow[r, "\Phi_{X,Y}"] \arrow[d, "\beta_*"'] & \mathrm{Hom}_{\mathsf{FibSets}(K)}(X,\Pi_f(Y)) \arrow[d, "\Pi_f(\beta)_*"] \\
\mathrm{Hom}_{\mathsf{FibSets}(K')}(f^*(X),Y') \arrow[r, "\Phi_{X,Y'}"'] & \mathrm{Hom}_{\mathsf{FibSets}(K)}(X,\Pi_f(Y'))
\end{tikzcd}$$

commutes. Indeed, given a morphism $\xi\colon X \to \Pi_f(Y)$ of $K$-fibred sets, we have

$$\begin{aligned}
[[\Phi_{X,Y'} \circ \beta_*](\xi)](x) &\stackrel{\text{def}}{=} [\Phi_{X,Y'}(\beta \circ \xi)](x)\\
&\stackrel{\text{def}}{=} [\Phi_{X,Y'}(\beta \circ \xi)](x)\\
&\stackrel{\text{def}}{=} ([\beta \circ \xi](k,x))_{k\in f^{-1}(\phi_X(x))}\\
&\stackrel{\text{def}}{=} (\beta(\xi(k,x)))_{k\in f^{-1}(\phi_X(x))}\\
&\stackrel{\text{def}}{=} \Pi_f(\beta)_*\Big((\xi(k,x))_{k\in f^{-1}(\phi_X(x))}\Big)\\
&\stackrel{\text{def}}{=} [\Pi_f(\beta)_* \circ \xi^\dagger](x)\\
&\stackrel{\text{def}}{=} [\Pi_f(\beta)_* \circ \Phi_{X,Y'}(\xi)](x)
\end{aligned}$$

for each $x \in X$.

(5) *Invertibility I.* We claim that

$$\Psi_{X,Y} \circ \Phi_{X,Y} = \mathrm{id}_{\mathrm{Hom}_{\mathsf{FibSets}(K)}(f^*(X),Y)}.$$

Indeed, given a morphism $\xi\colon f^*(X') \to Y$ of $K'$-fibred sets, we have

$$\begin{aligned}
[[\Psi_{X,Y} \circ \Phi_{X,Y}](\xi)](k,x) &\stackrel{\text{def}}{=} [\Psi_{X,Y}(\Phi_{X,Y}(\xi))](k,x)\\
&\stackrel{\text{def}}{=} ([\Phi_{X,Y}(\xi)](x))_k\\
&\stackrel{\text{def}}{=} \Big((\xi(k_1,x))_{k_1\in f^{-1}(\phi_X(x))}\Big)_k\\
&\stackrel{\text{def}}{=} \xi(k,x)
\end{aligned}$$

for each $(k,x) \in f^*(X)$, and thus the stated equality follows.

(6) *Invertibility II.* We claim that

$$\Phi_{X,Y} \circ \Psi_{X,Y} = \mathrm{id}_{\mathrm{Hom}_{\mathsf{FibSets}(K')}(X,\Pi_f(Y))}.$$

Indeed, given a morphism $\xi\colon X \to \Pi_f(Y)$ of $K$-fibred sets, write

$$\xi(x) = (y_k)_{k\in f^{-1}(k'_x)}.$$

We then have

$$\begin{aligned}
[[\Phi_{X,Y} \circ \Psi_{X,Y}](\xi)](x) &\stackrel{\text{def}}{=} [\Phi_{X,Y}(\Psi_{X,Y}(\xi))](x)\\
&\stackrel{\text{def}}{=} ([\Psi_{X,Y}(\xi)](k_1,x))_{k_1\in f^{-1}(\phi_X(x))}\\
&\stackrel{\text{def}}{=} \Big((\xi(x))_{k_1}\Big)_{k_1\in f^{-1}(\phi_X(x))}\\
&\stackrel{\text{def}}{=} \bigg(\Big((y_k)_{k\in f^{-1}(k'_x)}\Big)_{k_1}\bigg)_{k_1\in f^{-1}(\phi_X(x))}\\
&\stackrel{\text{def}}{=} (y_{k_1})_{k_1\in f^{-1}(\phi_X(x))}\\
&= (y_{k_1})_{k_1\in f^{-1}(k'_x)}\\
&= (y_k)_{k\in f^{-1}(k'_x)}\\
&\stackrel{\text{def}}{=} \xi(x)
\end{aligned}$$

for each $x \in X$, where the equality $\phi_X(x) = k'_x$ follows from the fact that $\xi$ is a morphism of $K'$-fibred sets. Thus the stated equality follows.

This finishes the proof. □

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CHAPTER 10

# Un/Straightening for Indexed and Fibred Sets

This chapter contains a discussion of the un/straightening equivalence in the context of sets, as well as a general discussion of indexed and fibred sets. In particular, it contains:

(1) A discussion of indexed sets (i.e. functors $K_{\mathsf{disc}} \to \mathsf{Sets}$ with $K$ a set), constructions with them like dependent sums and dependent products, and their properties (**??**);
(2) A discussion of fibred sets (i.e. maps of sets $X \to K$), constructions with them like dependent sums and dependent products, and their properties (**??**);
(3) A discussion of the un/straightening equivalence for indexed and fibred sets (Section 10.1).

## 10.1. Un/Straightening for Indexed and Fibred Sets

**10.1.1. Straightening for Fibred Sets.** Let $K$ be a set and let $(X, \phi)$ be a $K$-fibred set.

**Definition 10.1.1.1.1.** The **straightening of** $(X, \phi)$ is the $K$-indexed set

$$\mathrm{St}_K(X, \phi) \colon K_{\mathsf{disc}} \to \mathsf{Sets}$$

defined by

$$\mathrm{St}_K(X, \phi)_x \stackrel{\mathrm{def}}{=} \phi^{-1}(x)$$

for each $x \in K$.

**Proposition 10.1.1.1.2.** Let $K$ be a set.

(1) *Functoriality.* The assignment $(X, \phi) \mapsto \mathrm{St}_K(X, \phi)$ defines a functor

$$\mathrm{St}_K \colon \mathsf{FibSets}(K) \to \mathsf{ISets}(K)$$

- *Action on Objects.* For each $(X, \phi) \in \mathrm{Obj}(\mathsf{FibSets}(K))$, we have

$$[\mathrm{St}_K](X, \phi) \stackrel{\mathrm{def}}{=} \mathrm{St}_K(X, \phi);$$

- *Action on Morphisms.* For each $(X, \phi), (Y, \psi) \in \mathrm{Obj}(\mathsf{FibSets}(K))$, the action on Hom-sets

$$\mathrm{St}_{K|X,Y} \colon \mathrm{Hom}_{\mathsf{FibSets}(K)}(X, Y) \to \mathrm{Hom}_{\mathsf{ISets}(K)}(\mathrm{St}_K(X), \mathrm{St}_K(Y))$$

of $\mathrm{St}_K$ at $(X, Y)$ is given by sending a morphism

$$f \colon (X, \phi) \to (Y, \psi)$$

of $K$-fibred sets to the morphism

$$\mathrm{St}_K(f) \colon \mathrm{St}_K(X, \phi) \to \mathrm{St}_K(Y, \psi)$$

of $K$-indexed sets defined by

$$\mathrm{St}_K(f) \stackrel{\mathrm{def}}{=} \{f_x^*\}_{x\in K},$$

where $f_x^*$ is the transport map associated to $f$ at $x \in K$ of **??**.

(2) *Interaction With Change of Base/Indexing.* Let $f\colon K \to K'$ be a map of sets. The diagram

$$\begin{tikzcd} \mathsf{FibSets}(K') \arrow[r, "f^*"] \arrow[d, "\mathrm{St}_{K'}"'] & \mathsf{FibSets}(K) \arrow[d, "\mathrm{St}_K"] \\ \mathsf{ISets}(K') \arrow[r, "f^*"'] & \mathsf{ISets}(K) \end{tikzcd}$$

commutes.

(3) *Interaction With Dependent Sums.* Let $f\colon K \to K'$ be a map of sets. The diagram

$$\begin{tikzcd} \mathsf{FibSets}(K) \arrow[r, "\Sigma_f"] \arrow[d, "\mathrm{St}_K"'] & \mathsf{FibSets}(K') \arrow[d, "\mathrm{St}_{K'}"] \\ \mathsf{ISets}(K) \arrow[r, "\Sigma_f"'] & \mathsf{ISets}(K') \end{tikzcd}$$

commutes.

(4) *Interaction With Dependent Products.* Let $f\colon K \to K'$ be a map of sets. The diagram

$$\begin{tikzcd} \mathsf{Sets}_{/K} \arrow[r, "\Pi_f"] \arrow[d, "\mathrm{St}_K"'] & \mathsf{FibSets}(K') \arrow[d, "\mathrm{St}_{K'}"] \\ \mathsf{ISets}(K) \arrow[r, "\Pi_f"'] & \mathsf{ISets}(K') \end{tikzcd}$$

commutes.

**Proof.** *Item 1*, *Functoriality*: Omitted.
*Item 2*, *Interaction With Change of Base/Indexing*: Indeed, we have

$$\begin{aligned} \mathrm{St}_K(f^*(X,\phi))_x &\stackrel{\mathrm{def}}{=} \mathrm{St}_K(K \times_{K'} X)_x \\ &\stackrel{\mathrm{def}}{=} \left(\mathrm{pr}_1^{K\times_{K'}X}\right)^{-1}(x) \\ &= \left\{(k,y) \in K \times_{K'} X \;\middle|\; \mathrm{pr}_1^{K\times_{K'}X}(k,y) = x\right\} \\ &= \{(k,y) \in K \times_{K'} X \mid k = x\} \\ &= \{(k,y) \in K \times X \mid k = x \text{ and } f(k) = \phi(y)\} \\ &\cong \{y \in X \mid \phi(y) = f(x)\} \\ &= \phi^{-1}(f(x)) \\ &\stackrel{\mathrm{def}}{=} f^*\left(\phi^{-1}(x)\right) \\ &\stackrel{\mathrm{def}}{=} f^*(\mathrm{St}_{K'}(X,\phi)_x) \end{aligned}$$

for each $(X,\phi) \in \mathrm{Obj}(\mathsf{FibSets}(K'))$ and each $x \in K$, and similarly for morphisms.

*Item 3, Interaction With Dependent Sums*: Indeed, we have

$$\begin{aligned}\mathrm{St}_{K'}(\Sigma_f(X,\phi))_x &\stackrel{\text{def}}{=} \Sigma_f(\phi)^{-1}(x)\\ &\cong \coprod_{\substack{y\in X\\ f(y)=x}} \phi^{-1}(y)\\ &\cong \Sigma_f\Big(\phi^{-1}(x)\Big)\\ &\stackrel{\text{def}}{=} \Sigma_f(\mathrm{St}_K(X,\phi)_x)\end{aligned}$$

for each $(X,\phi) \in \mathrm{Obj}(\mathsf{FibSets}(K))$ and each $x \in K'$, where we have used **??** of **??** for the first bijection, and similarly for morphisms.

*Item 4, Interaction With Dependent Products*: Indeed, we have

$$\begin{aligned}\mathrm{St}_{K'}(\Pi_f(X,\phi))_x &\stackrel{\text{def}}{=} \Pi_f(\phi)^{-1}(x)\\ &\cong \prod_{\substack{y\in X\\ f(y)=x}} \phi^{-1}(y)\\ &\cong \Pi_f\Big(\phi^{-1}(x)\Big)\\ &\stackrel{\text{def}}{=} \Pi_f(\mathrm{St}_K(X,\phi)_x)\end{aligned}$$

for each $(X,\phi) \in \mathrm{Obj}(\mathsf{FibSets}(K))$ and each $x \in K'$, where we have used **??** of **??** for the first bijection, and similarly for morphisms. □

**10.1.2. Unstraightening for Indexed Sets.** Let $K$ be a set and let $X$ be a $K$-indexed set.

**Definition 10.1.2.1.1.** The **unstraightening of** $X$ is the $K$-fibred set

$$\phi_{\mathrm{Un}_K} \colon \mathrm{Un}_K(X) \to K$$

consisting of

- *The Underlying Set.* The set $\mathrm{Un}_K(X)$ defined by
$$\mathrm{Un}_K(X) \stackrel{\text{def}}{=} \coprod_{x\in K} X_x;$$
- *The Fibration.* The map of sets
$$\phi_{\mathrm{Un}_K} \colon \mathrm{Un}_K(X) \to K$$
defined by sending an element of $\coprod_{x\in K} X_x$ to its index in $K$.

**Proposition 10.1.2.1.2.** Let $K$ be a set.

(1) *Functoriality.* The assignment $X \mapsto \mathrm{Un}_K(X)$ defines a functor
$$\mathrm{Un}_K \colon \mathsf{ISets}(K) \to \mathsf{FibSets}(K)$$
- *Action on Objects.* For each $X \in \mathrm{Obj}(\mathsf{ISets}(K))$, we have
$$[\mathrm{Un}_K](X) \stackrel{\text{def}}{=} \mathrm{Un}_K(X);$$
- *Action on Morphisms.* For each $X, Y \in \mathrm{Obj}(\mathsf{ISets}(K))$, the action on Hom-sets

$$\mathrm{Un}_{K|X,Y} \colon \mathrm{Hom}_{\mathsf{ISets}(K)}(X,Y) \to \mathrm{Hom}_{\mathsf{FibSets}(K)}(\mathrm{Un}_K(X),\mathrm{Un}_K(Y))$$

of $\mathrm{Un}_K$ at $(X, Y)$ is defined by

$$\mathrm{Un}_{K|X,Y}(f) \stackrel{\text{def}}{=} \coprod_{x \in K} f_x^*.$$

(2) *Interaction With Fibres.* We have a bijection of sets

$$\phi_{\mathrm{Un}_K}^{-1}(x) \cong X_x$$

for each $x \in K$.

(3) *As a Pullback.* We have a bijection of sets

$$\mathrm{Un}_K(X) \cong K_{\mathsf{disc}} \times_{\mathsf{Sets}} \mathsf{Sets}_*, \qquad \begin{tikzcd} \mathrm{Un}_K(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Sets}_* \arrow[d, "忘"] \\ K_{\mathsf{disc}} \arrow[r, "X"'] & \mathsf{Sets}. \end{tikzcd}$$

(4) *As a Colimit.* We have a bijection of sets

$$\mathrm{Un}_K(X) \cong \mathrm{colim}(X).$$

(5) *Interaction With Change of Indexing/Base.* Let $f: K \to K'$ be a map of sets. The diagram

$$\begin{tikzcd} \mathsf{ISets}(K') \arrow[r, "f^*"] \arrow[d, "\mathrm{Un}_{K'}"'] & \mathsf{ISets}(K) \arrow[d, "\mathrm{Un}_K"] \\ \mathsf{FibSets}(K') \arrow[r, "f^*"'] & \mathsf{FibSets}(K) \end{tikzcd}$$

commutes.

(6) *Interaction With Dependent Sums.* Let $f: K \to K'$ be a map of sets. The diagram

$$\begin{tikzcd} \mathsf{ISets}(K) \arrow[r, "\Sigma_f"] \arrow[d, "\mathrm{Un}_K"'] & \mathsf{ISets}(K') \arrow[d, "\mathrm{Un}_{K'}"] \\ \mathsf{FibSets}(K) \arrow[r, "\Sigma_f"'] & \mathsf{FibSets}(K') \end{tikzcd}$$

commutes.

(7) *Interaction With Dependent Products.* Let $f: K \to K'$ be a map of sets. The diagram

$$\begin{tikzcd} \mathsf{ISets}(K) \arrow[r, "\Pi_f"] \arrow[d, "\mathrm{Un}_K"'] & \mathsf{ISets}(K') \arrow[d, "\mathrm{Un}_{K'}"] \\ \mathsf{FibSets}(K) \arrow[r, "\Pi_f"'] & \mathsf{FibSets}(K') \end{tikzcd}$$

commutes.

**Proof.** *Item 1*, *Functoriality*: Omitted.
*Item 2*, *Interaction With Fibres*: Omitted.
*Item 3*, *As a Pullback*: Omitted.

*Item 4, As a Colimit*: Clear.
*Item 5, Interaction With Change of Indexing/Base*: Indeed, we have

$$\begin{aligned}
\mathrm{Un}_K(f^*(X)) &\overset{\text{def}}{=} \mathrm{Un}_K(X \circ f)\\
&\overset{\text{def}}{=} \coprod_{x\in K} X_{f(x)}\\
&\cong \left\{(x,(y,a)) \in K \times \coprod_{y\in K'} X_y \;\middle|\; f(x) = y\right\}\\
&\cong K \times_{K'} \coprod_{y\in K'} X_y\\
&\overset{\text{def}}{=} K \times_{K'} \mathrm{Un}_{K'}(X)\\
&\overset{\text{def}}{=} f^*(\mathrm{Un}_{K'}(X))
\end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{ISets}(K'))$. Similarly, it can be shown that we also have $\mathrm{Un}_K(f^*(\phi)) = f^*(\mathrm{Un}_{K'}(\phi))$ and that $\mathrm{Un}_K \circ f^* = f^* \circ \mathrm{Un}_{K'}$ also holds on morphisms.
*Item 6, Interaction With Dependent Sums*: Indeed, we have

$$\begin{aligned}
\mathrm{Un}_{K'}(\Sigma_f(X)) &\overset{\text{def}}{=} \coprod_{x\in K'} \Sigma_f(X)_x\\
&\cong \coprod_{x\in K'} \coprod_{y\in f^{-1}(x)} X_y\\
&\cong \coprod_{y\in K} X_y\\
&\cong \mathrm{Un}_K(X)\\
&\overset{\text{def}}{=} \Sigma_f(\mathrm{Un}_K(X))
\end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{ISets}(K))$, where we have used **??** of **??** for the first bijection. Similarly, it can be shown that we also have $\mathrm{Un}_{K'}(\Sigma_f(\phi)) = \Sigma_f(\phi_{\mathrm{Un}_K})$ and that $\mathrm{Un}_{K'} \circ \Sigma_f = \Sigma_f \circ \mathrm{Un}_K$ also holds on morphisms.
*Item 7, Interaction With Dependent Products*: Indeed, we have

$$\begin{aligned}
\mathrm{Un}_{K'}(\Pi_f(X)) &\overset{\text{def}}{=} \coprod_{x\in K'} \Pi_f(X)_x\\
&\cong \coprod_{x\in K'} \prod_{y\in f^{-1}(x)} X_y\\
&\cong \left\{(x,h) \in \coprod_{x\in K'} \mathsf{Sets}\big(f^{-1}(x), \phi^{-1}_{\mathrm{Un}_K}\big(f^{-1}(x)\big)\big) \;\middle|\; \phi \circ h = \mathrm{id}_{f^{-1}(x)}\right\}\\
&\overset{\text{def}}{=} \Pi_f\left(\coprod_{y\in K} X_y\right)\\
&\overset{\text{def}}{=} \Pi_f(\mathrm{Un}_K(X))
\end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{ISets}(K))$, where we have used **??** of **??** for the first bijection. Similarly, it can be shown that we also have $\mathrm{Un}_{K'}(\Pi_f(\phi)) = \Pi_f(\phi_{\mathrm{Un}_K})$ and that $\mathrm{Un}_{K'} \circ \Pi_f = \Pi_f \circ \mathrm{Un}_K$ also holds on morphisms. □

#### 10.1.3. The Un/Straightening Equivalence.

**Theorem 10.1.3.1.1.** We have an isomorphism of categories

$$(\mathrm{St}_K \dashv \mathrm{Un}_K)\colon \quad \mathsf{FibSets}(K) \underset{\mathrm{Un}_K}{\overset{\mathrm{St}_K}{\underset{\perp}{\rightleftarrows}}} \mathsf{ISets}(K).$$

**Proof.** Omitted. □

## 10.2. Miscellany

### 10.2.1. Other Kinds of Un/Straightening.

**Remark 10.2.1.1.1.** There are also other kinds of un/straightening for sets, where $\mathsf{Sets}$ is replaced by $\mathbf{Rel}$ or $\mathsf{Span}$:

- *Un/Straightening With* $\mathbf{Rel}$*, I.* We have an isomorphism of sets
$$\mathrm{Rel}(A, B) \cong \mathsf{Sets}(B \times A, \{\mathsf{true}, \mathsf{false}\}).$$
by the definition of a relation from $A$ to $B$, **??**.
- *Un/Straightening With* $\mathbf{Rel}$*, II.* We have an equivalence of categories
$$\mathsf{LaxFun}(K_{\mathsf{disc}}, \mathbf{Rel}) \overset{\text{eq.}}{\cong} \mathsf{Cats}^{\mathsf{fth}}_{/K_{\mathsf{disc}}},$$
where $\mathsf{Cats}^{\mathsf{fth}}_{/K_{\mathsf{disc}}}$ is the full subcategory of $\mathsf{Cats}_{/K_{\mathsf{disc}}}$ spanned by the faithful functors; see [Nie04, Theorem 3.1].
- *Un/Straightening With* $\mathsf{Span}$*, I.* For each $A, B \in \mathrm{Obj}(\mathsf{Sets})$, we have a morphism of sets
$$\mathsf{Span}(A, B) \to \mathsf{Sets}(A \times B, \mathbb{N} \cup \{\infty\})$$
which assemble into an equivalence of categories between $\mathsf{Span}(\mathsf{Sets})$ and the category $\mathsf{MRel}$ of "multirelations"; see **??**.
- *Un/Straightening With* $\mathsf{Span}$*, II.* We have an equivalence of categories
$$\mathsf{LaxFun}(K_{\mathsf{disc}}, \mathsf{Span}) \overset{\text{eq.}}{\cong} \mathsf{Cats}_{/K_{\mathsf{disc}}};$$
see [nLa24a, Section 3].

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# Part 3

# Category Theory

CHAPTER 11

# Categories

Create tags (see [MSE 350788] for some of these):

(1) define bicategory $\mathsf{Adj}(\mathcal{C})$
(2) internal **Hom** in categories of co/Cartesian fibrations
(3) https://mathoverflow.net/questions/460146/universal-property-of-isbell-duality
(4) http://www.tac.mta.ca/tac/volumes/36/12/36-12abs.html
(5) justify adjunctions via homs
(6) https://ncatlab.org/nlab/show/enrichment+versus+internalisation
(7) https://arxiv.org/pdf/2004.08964.pdf

## 11.1. Categories

### 11.1.1. Foundations.

**Definition 11.1.1.1.1.** A **category** $(\mathcal{C}, \circ^{\mathcal{C}}, \mathbb{1}^{\mathcal{C}})$ consists of[1,2]

- *Objects.* A class $\mathrm{Obj}(\mathcal{C})$ of **objects**;
- *Morphisms.* For each $A, B \in \mathrm{Obj}(\mathcal{C})$, a class $\mathrm{Hom}_{\mathcal{C}}(A, B)$, called the **class of morphisms of $\mathcal{C}$ from $A$ to $B$**;
- *Identities.* For each $A \in \mathrm{Obj}(\mathcal{C})$, a map of sets
$$\mathbb{1}^{\mathcal{C}}_A \colon \mathrm{pt} \to \mathrm{Hom}_{\mathcal{C}}(A, A),$$
called the **unit map of $\mathcal{C}$ at $A$**, determining a morphism
$$\mathrm{id}_A \colon A \to A$$
of $\mathcal{C}$, called the **identity morphism of $A$**;
- *Composition.* For each $A, B, C \in \mathrm{Obj}(\mathcal{C})$, a map of sets
$$\circ^{\mathcal{C}}_{A,B,C} \colon \mathrm{Hom}_{\mathcal{C}}(B, C) \times \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{C}}(A, C),$$
called the **composition map of $\mathcal{C}$ at $(A, B, C)$**;

such that the following conditions are satisfied:

[1] *Further Notation:* We also write $\mathcal{C}(A, B)$ for $\mathrm{Hom}_{\mathcal{C}}(A, B)$.
[2] *Further Notation:* We write $\mathrm{Mor}(\mathcal{C})$ for the class of all morphisms of $\mathcal{C}$.

(1) *Associativity.* The diagram

$$\begin{tikzcd}
& & \mathrm{Hom}_{\mathcal{C}}(C,D)\times(\mathrm{Hom}_{\mathcal{C}}(B,C)\times\mathrm{Hom}_{\mathcal{C}}(A,B)) \arrow[drr, "\mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(C,D)}\times\circ^{\mathcal{C}}_{A,B,C}"] & & \\
(\mathrm{Hom}_{\mathcal{C}}(C,D)\times\mathrm{Hom}_{\mathcal{C}}(B,C))\times\mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[urr, dashed, "\alpha^{\mathsf{Sets}}_{\mathrm{Hom}_{\mathcal{C}}(C,D),\mathrm{Hom}_{\mathcal{C}}(B,C),\mathrm{Hom}_{\mathcal{C}}(A,B)}", "\sim"'] \arrow[dr, "\circ^{\mathcal{C}}_{B,C,D}\times\mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(A,B)}"'] & & & & \mathrm{Hom}_{\mathcal{C}}(C,D)\times\mathrm{Hom}_{\mathcal{C}}(A,C) \arrow[dl, "\circ^{\mathcal{C}}_{A,C,D}"] \\
& \mathrm{Hom}_{\mathcal{C}}(B,D)\times\mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[rr, "\circ^{\mathcal{C}}_{A,B,D}"'] & & \mathrm{Hom}_{\mathcal{C}}(A,D) &
\end{tikzcd}$$

commutes, i.e. for each composable triple $(f, g, h)$ of morphisms of $\mathcal{C}$, we have

$$(f \circ g) \circ h = f \circ (g \circ h).$$

(2) *Left Unitality.* The diagram

$$\begin{tikzcd}
\mathrm{pt}\times\mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[d, "\mathbb{1}^{\mathcal{C}}_B\times\mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(A,B)}"'] \arrow[dr, dashed, "\lambda^{\mathsf{Sets}}_{\mathrm{Hom}_{\mathcal{C}}(A,B)}", "\sim"'] & \\
\mathrm{Hom}_{\mathcal{C}}(B,B)\times\mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[r, "\circ^{\mathcal{C}}_{A,B,B}"'] & \mathrm{Hom}_{\mathcal{C}}(A,B)
\end{tikzcd}$$

commutes, i.e. for each morphism $f\colon A \to B$ of $\mathcal{C}$, we have

$$\mathrm{id}_B \circ f = f.$$

(3) *Right Unitality.* The diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(A,B)\times\mathrm{pt} \arrow[d, "\mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(A,B)}\times\mathbb{1}^{\mathcal{C}}_A"'] \arrow[dr, dashed, "\rho^{\mathsf{Sets}}_{\mathrm{Hom}_{\mathcal{C}}(A,B)}", "\sim"'] & \\
\mathrm{Hom}_{\mathcal{C}}(A,B)\times\mathrm{Hom}_{\mathcal{C}}(A,A) \arrow[r, "\circ^{\mathcal{C}}_{A,A,B}"'] & \mathrm{Hom}_{\mathcal{C}}(A,B)
\end{tikzcd}$$

commutes, i.e. for each morphism $f\colon A \to B$ of $\mathcal{C}$, we have

$$f \circ \mathrm{id}_A = f.$$

**Definition 11.1.1.1.2.** Let $\kappa$ be a regular cardinal. A category $\mathcal{C}$ is

(1) **Locally small** if, for each $A, B \in \mathrm{Obj}(\mathcal{C})$, the class $\mathrm{Hom}_{\mathcal{C}}(A, B)$ is a set.

(2) **Locally essentially small** if, for each $A, B \in \mathrm{Obj}(\mathcal{C})$, the class

$$\mathrm{Hom}_{\mathcal{C}}(A, B)/\{\text{isomorphisms}\}$$

is a set.

(3) **Small** if $\mathcal{C}$ is locally small and $\mathrm{Obj}(\mathcal{C})$ is a set.
(4) **$\kappa$-Small** if $\mathcal{C}$ is locally small, $\mathrm{Obj}(\mathcal{C})$ is a set, and we have $\#\mathrm{Obj}(\mathcal{C}) < \kappa$.

### 11.1.2. Examples of Categories.

**Example 11.1.2.1.1.** The **punctual category**[3] is the category $\mathsf{pt}$ where

- *Objects.* We have
$$\mathrm{Obj}(\mathsf{pt}) \stackrel{\text{def}}{=} \{\star\};$$
- *Morphisms.* The unique Hom-set of $\mathsf{pt}$ is defined by
$$\mathrm{Hom}_{\mathsf{pt}}(\star, \star) \stackrel{\text{def}}{=} \{\mathrm{id}_\star\};$$
- *Identities.* The unit map
$$\mathbb{1}^{\mathsf{pt}}_{\star} \colon \mathrm{pt} \to \mathrm{Hom}_{\mathsf{pt}}(\star, \star)$$
of $\mathsf{pt}$ at $\star$ is defined by
$$\mathrm{id}^{\mathsf{pt}}_{\star} \stackrel{\text{def}}{=} \mathrm{id}_\star;$$
- *Composition.* The composition map
$$\circ^{\mathsf{pt}}_{\star,\star,\star} \colon \mathrm{Hom}_{\mathsf{pt}}(\star, \star) \times \mathrm{Hom}_{\mathsf{pt}}(\star, \star) \to \mathrm{Hom}_{\mathsf{pt}}(\star, \star)$$
of $\mathsf{pt}$ at $(\star, \star, \star)$ is given by the bijection $\mathrm{pt} \times \mathrm{pt} \cong \mathrm{pt}$.

**Example 11.1.2.1.2.** We have an isomorphism of categories[4]

$$\mathsf{Mon} \cong \mathsf{pt} \underset{\mathsf{Sets}}{\times} \mathsf{Cats}, \qquad \begin{tikzcd} \mathsf{Mon} \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathsf{Cats} \arrow[d, "\mathrm{Obj}"] \\ \mathsf{pt} \arrow[r, "{[\mathrm{pt}]}"'] & \mathsf{Sets} \end{tikzcd}$$

via the delooping functor $\mathsf{B} \colon \mathsf{Mon} \to \mathsf{Cats}$ of **??** of **??**.

**Proof.** Omitted. $\square$

**Example 11.1.2.1.3.** The **empty category** is the category $\emptyset_{\mathsf{cat}}$ where

- *Objects.* We have
$$\mathrm{Obj}(\emptyset_{\mathsf{cat}}) \stackrel{\text{def}}{=} \emptyset;$$
- *Morphisms.* We have
$$\mathrm{Mor}(\emptyset_{\mathsf{cat}}) \stackrel{\text{def}}{=} \emptyset;$$

---

[3] *Further Terminology:* Also called the **singleton category**.

[4] This can be enhanced to an isomorphism of 2-categories

$$\mathsf{Mon}_{2-\mathsf{disc}} \cong \mathsf{pt}_{\mathsf{bi}} \underset{\mathsf{Sets}_{2-\mathsf{disc}}}{\times} \mathsf{Cats}_{2,*}, \qquad \begin{tikzcd} \mathsf{Mon}_{2-\mathsf{disc}} \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathsf{Cats}_{2,*} \arrow[d, "\mathrm{Obj}"] \\ \mathsf{pt}_{\mathsf{bi}} \arrow[r, "{[\mathrm{pt}]}"'] & \mathsf{Sets}_{2-\mathsf{disc}} \end{tikzcd}$$

between the discrete 2-category $\mathsf{Mon}_{2-\mathsf{disc}}$ on $\mathsf{Mon}$ and the 2-category of pointed categories with one object.

- *Identities and Composition.* Having no objects, $\emptyset_{\mathsf{cat}}$ has no unit nor composition maps.

**Example 11.1.2.1.4.** The ***n*th ordinal category** is the category $\ltimes$ where[5]

- *Objects.* We have
$$\mathrm{Obj}(\ltimes) \overset{\mathrm{def}}{=} \{[0], \ldots, [n]\};$$
- *Morphisms.* For each $[i], [j] \in \mathrm{Obj}(\ltimes)$, we have
$$\mathrm{Hom}_{\ltimes}([i],[j]) \overset{\mathrm{def}}{=} \begin{cases} \left\{\mathrm{id}_{[i]}\right\} & \text{if } [i] = [j], \\ \{[i] \to [j]\} & \text{if } [j] < [i], \\ \emptyset & \text{if } [j] > [i]; \end{cases}$$
- *Identities.* For each $[i] \in \mathrm{Obj}(\ltimes)$, the unit map
$$\nVdash^{\ltimes}_{[i]} : \mathrm{pt} \to \mathrm{Hom}_{\ltimes}([i],[i])$$
of $\ltimes$ at $[i]$ is defined by
$$\mathrm{id}^{\ltimes}_{[i]} \overset{\mathrm{def}}{=} \mathrm{id}_{[i]};$$
- *Composition.* For each $[i], [j], [k] \in \mathrm{Obj}(\ltimes)$, the composition map
$$\circ^{\ltimes}_{[i],[j],[k]} : \mathrm{Hom}_{\ltimes}([j],[k]) \times \mathrm{Hom}_{\ltimes}([i],[j]) \to \mathrm{Hom}_{\ltimes}([i],[k])$$
of $\ltimes$ at $([i],[j],[k])$ is defined by
$$\begin{aligned} \mathrm{id}_{[i]} \circ \mathrm{id}_{[i]} &= \mathrm{id}_{[i]}, \\ ([j] \to [k]) \circ ([i] \to [j]) &= ([i] \to [k]). \end{aligned}$$

**Example 11.1.2.1.5.** Here we list all the other categories that appear throughout this work.

- The category $\mathsf{Sets}_*$ of pointed sets of **??**.
- The category $\mathsf{Rel}$ of sets and relations of **??**.
- The category $\mathsf{Span}(A, B)$ of spans from a set $A$ to a set $B$ of **??**.
- The category $\mathsf{ISets}(K)$ of $K$-indexed sets of **??**.
- The category $\mathsf{ISets}$ of indexed sets of **??**.

[5] In other words, $\ltimes$ is the category associated to the poset
$$[0] \to [1] \to \cdots \to [n-1] \to [n].$$
The category $\ltimes$ for $n \geq 2$ may also be defined in terms of $\nvdash$ and joins: we have isomorphisms of categories
$$\begin{aligned} \nVdash &\cong \nvdash \star \nvdash, \\ \nvDash &\cong \nVdash \star \nvdash \\ &\cong (\nvdash \star \nvdash) \star \nvdash, \\ \nVDash &\cong \nvDash \star \nvdash \\ &\cong (\nVdash \star \nvdash) \star \nvdash \\ &\cong ((\nvdash \star \nvdash) \star \nvdash) \star \nvdash, \\ \ntrianglerighteq &\cong \nVDash \star \nvdash \\ &\cong (\nvDash \star \nvdash) \star \nvdash \\ &\cong ((\nVdash \star \nvdash) \star \nvdash) \star \nvdash \\ &\cong (((\nvdash \star \nvdash) \star \nvdash) \star \nvdash) \star \nvdash, \end{aligned}$$
and so on.

- The category $\mathsf{FibSets}(K)$ of $K$-fibred sets of **??**.
- The category $\mathsf{FibSets}$ of fibred sets of **??**.

**11.1.3. Subcategories.** Let $\mathcal{C}$ be a category.

**Definition 11.1.3.1.1.** A **subcategory** of $\mathcal{C}$ is a category $\mathcal{A}$ satisfying the following conditions:

(1) *Objects.* We have $\mathrm{Obj}(\mathcal{A}) \subset \mathrm{Obj}(\mathcal{C})$.
(2) *Morphisms.* For each $A, B \in \mathrm{Obj}(\mathcal{A})$, we have
$$\mathrm{Hom}_{\mathcal{A}}(A, B) \subset \mathrm{Hom}_{\mathcal{C}}(A, B).$$
(3) *Identities.* For each $A \in \mathrm{Obj}(\mathcal{A})$, we have
$$\mathbb{1}^{\mathcal{A}}_{A} = \mathbb{1}^{\mathcal{C}}_{A}.$$
(4) *Composition.* For each $A, B, C \in \mathrm{Obj}(\mathcal{A})$, we have
$$\circ^{\mathcal{A}}_{A,B,C} = \circ^{\mathcal{C}}_{A,B,C}.$$

**Definition 11.1.3.1.2.** A subcategory $\mathcal{A}$ of $\mathcal{C}$ is **full** if the canonical inclusion functor $\mathcal{A} \to \mathcal{C}$ is full, i.e. if, for each $A, B \in \mathrm{Obj}(\mathcal{A})$, the inclusion
$$\iota_{A,B} \colon \mathrm{Hom}_{\mathcal{A}}(A, B) \hookrightarrow \mathrm{Hom}_{\mathcal{C}}(A, B)$$
is surjective (and thus bijective).

**Definition 11.1.3.1.3.** A subcategory $\mathcal{A}$ of a category $\mathcal{C}$ is **strictly full** if it satisfies the following conditions:

(1) *Fullness.* The subcategory $\mathcal{A}$ is full.
(2) *Closedness Under Isomorphisms.* The class $\mathrm{Obj}(\mathcal{A})$ is closed under isomorphisms.[6]

**Definition 11.1.3.1.4.** A subcategory $\mathcal{A}$ of $\mathcal{C}$ is **wide**[7] if $\mathrm{Obj}(\mathcal{A}) = \mathrm{Obj}(\mathcal{C})$.

**11.1.4. Skeletons of Categories.**

**Definition 11.1.4.1.1.** A[8] **skeleton** of a category $\mathcal{C}$ is a full subcategory $\mathsf{Sk}(\mathcal{C})$ with one object from each isomorphism class of objects of $\mathcal{C}$.

**Definition 11.1.4.1.2.** A category $\mathcal{C}$ is **skeletal** if $\mathcal{C} \cong \mathsf{Sk}(\mathcal{C})$.[9]

**Proposition 11.1.4.1.3.** Let $\mathcal{C}$ be a category.

(1) *Existence.* Assuming the axiom of choice, $\mathsf{Sk}(\mathcal{C})$ always exists.
(2) *Pseudofunctoriality.* The assignment $\mathcal{C} \mapsto \mathsf{Sk}(\mathcal{C})$ defines a pseudofunctor
$$\mathsf{Sk} \colon \mathsf{Cats}_2 \to \mathsf{Cats}_2.$$
(3) *Uniqueness Up to Equivalence.* Any two skeletons of $\mathcal{C}$ are equivalent.
(4) *Inclusions of Skeletons Are Equivalences.* The inclusion
$$\iota_{\mathcal{C}} \colon \mathsf{Sk}(\mathcal{C}) \hookrightarrow \mathcal{C}$$
of a skeleton of $\mathcal{C}$ into $\mathcal{C}$ is an equivalence of categories.

[6]That is, given $A \in \mathrm{Obj}(\mathcal{A})$ and $C \in \mathrm{Obj}(\mathcal{C})$, if $C \cong A$, then $C \in \mathrm{Obj}(\mathcal{A})$.

[7]*Further Terminology:* Also called **lluf**.

[8]Due to Item 3 of Proposition 11.1.4.1.3, we often refer to any such full subcategory $\mathsf{Sk}(\mathcal{C})$ of $\mathcal{C}$ as *the* skeleton of $\mathcal{C}$.

[9]That is, $\mathcal{C}$ is **skeletal** if isomorphic objects of $\mathcal{C}$ are equal.

**Proof.** *Item 1, Existence*: See [nLab23, Section "Existence of Skeletons of Categories"].
*Item 2, Pseudofunctoriality*: See [nLab23, Section "Skeletons as an Endo-Pseudofunctor on $\mathfrak{Cat}$"].
*Item 3, Uniqueness Up to Equivalence*: Clear.
*Item 4, Inclusions of Skeletons Are Equivalences*: Clear. □

**11.1.5. Precomposition and Postcomposition.** Let $\mathcal{C}$ be a category and let $A, B, C \in \mathrm{Obj}(\mathcal{C})$.

**Definition 11.1.5.1.1.** Let $f\colon A \to B$ and $g\colon B \to C$ be morphisms of $\mathcal{C}$.

- The **precomposition function associated to** $f$ is the function
$$f^*\colon \mathrm{Hom}_{\mathcal{C}}(B, C) \to \mathrm{Hom}_{\mathcal{C}}(A, C)$$
defined by
$$f^*(\phi) \overset{\text{def}}{=} \phi \circ f$$
for each $\phi \in \mathrm{Hom}_{\mathcal{C}}(B, C)$.
- The **postcomposition function associated to** $g$ is the function
$$g_*\colon \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{C}}(A, C)$$
defined by
$$g_*(\phi) \overset{\text{def}}{=} g \circ \phi$$
for each $\phi \in \mathrm{Hom}_{\mathcal{C}}(A, B)$.

**Proposition 11.1.5.1.2.** Let $A, B, C, D \in \mathrm{Obj}(\mathcal{C})$ and let $f\colon A \to B$ and $g\colon B \to C$ be morphisms of $\mathcal{C}$.

(1) *Interaction Between Precomposition and Postcomposition.* We have
$$g_* \circ f^* = f^* \circ g_*, \qquad \begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(B, C) \arrow[r, "g_*"] \arrow[d, "f^*"'] & \mathrm{Hom}_{\mathcal{C}}(B, D) \arrow[d, "f^*"] \\ \mathrm{Hom}_{\mathcal{C}}(A, C) \arrow[r, "g_*"'] & \mathrm{Hom}_{\mathcal{C}}(A, D). \end{tikzcd}$$

(2) *Interaction With Composition I.* We have
$$(g \circ f)^* = f^* \circ g^*, \qquad \begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(X, A) \arrow[r, "f_*"] \arrow[dr, "(g\circ f)_*"'] & \mathrm{Hom}_{\mathcal{C}}(X, B) \arrow[d, "g_*"] \\ & \mathrm{Hom}_{\mathcal{C}}(X, C), \end{tikzcd}$$
$$(g \circ f)_* = g_* \circ f_*, \qquad \begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(C, X) \arrow[r, "g^*"] \arrow[dr, "(g\circ f)^*"'] & \mathrm{Hom}_{\mathcal{C}}(B, X) \arrow[d, "f^*"] \\ & \mathrm{Hom}_{\mathcal{C}}(A, X). \end{tikzcd}$$

(3) *Interaction With Composition II.* We have

$$\begin{tikzcd} \mathrm{pt} \arrow[r, "{[f]}"] \arrow[dr, "{[g\circ f]}"'] & \mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[d, "g_*"] \\ & \mathrm{Hom}_{\mathcal{C}}(A,C) \end{tikzcd} \qquad \begin{aligned} [g\circ f] &= g_* \circ [f], \\ [g\circ f] &= f^* \circ [g], \end{aligned} \qquad \begin{tikzcd} \mathrm{pt} \arrow[r, "{[g]}"] \arrow[dr, "{[g\circ f]}"'] & \mathrm{Hom}_{\mathcal{C}}(B,C) \arrow[d, "f^*"] \\ & \mathrm{Hom}_{\mathcal{C}}(A,C). \end{tikzcd}$$

(4) *Interaction With Composition III.* We have

$$f^* \circ \circ^{\mathcal{C}}_{A,B,C} = \circ^{\mathcal{C}}_{X,B,C} \circ (f^* \times \mathsf{id}), \qquad \begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(B,C) \times \mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[r, "\circ^{\mathcal{C}}_{A,B,C}"] \arrow[d, "\mathsf{id}\times f^*"'] & \mathrm{Hom}_{\mathcal{C}}(A,C) \arrow[d, "f^*"] \\ \mathrm{Hom}_{\mathcal{C}}(B,C) \times \mathrm{Hom}_{\mathcal{C}}(X,B) \arrow[r, "\circ^{\mathcal{C}}_{X,B,C}"'] & \mathrm{Hom}_{\mathcal{C}}(X,C), \end{tikzcd}$$

$$g_* \circ \circ^{\mathcal{C}}_{A,B,C} = \circ^{\mathcal{C}}_{A,B,D} \circ (\mathsf{id} \times g_*), \qquad \begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(B,C) \times \mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[r, "\circ^{\mathcal{C}}_{A,B,C}"] \arrow[d, "g_*\times\mathsf{id}"'] & \mathrm{Hom}_{\mathcal{C}}(A,C) \arrow[d, "g^*"] \\ \mathrm{Hom}_{\mathcal{C}}(B,D) \times \mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[r, "\circ^{\mathcal{C}}_{A,B,D}"'] & \mathrm{Hom}_{\mathcal{C}}(A,D). \end{tikzcd}$$

(5) *Interaction With Identities.* We have

$$\begin{aligned} (\mathrm{id}_A)^* &= \mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(A,B)}, \\ (\mathrm{id}_B)_* &= \mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(A,B)}. \end{aligned}$$

**Proof.** *Item 1, Interaction Between Precomposition and Postcomposition*: Clear.
*Item 2, Interaction With Composition I*: Clear.
*Item 3, Interaction With Composition II*: Clear.
*Item 4, Interaction With Composition III*: Clear.
*Item 5, Interaction With Identities*: Clear. □

## 11.2. The Quadruple Adjunction With Sets

**11.2.1. Statement.** Let $\mathcal{C}$ be a category.

**Proposition 11.2.1.1.1.** We have a quadruple adjunction

$$(\pi_0 \dashv (-)_{\mathsf{disc}} \dashv \mathrm{Obj} \dashv (-)_{\mathsf{indisc}}) \colon \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left=20, "(-)_{\mathsf{disc}}"] \arrow[r, bend right=60, "(-)_{\mathsf{indisc}}"'] & \mathsf{Cats}, \arrow[l, bend right=60, "\pi_0"'] \arrow[l, bend left=20, "\mathrm{Obj}"] \end{tikzcd}$$

witnessed by bijections of sets

$$\begin{aligned} \mathrm{Hom}_{\mathsf{Sets}}(\pi_0(\mathcal{C}), X) &\cong \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, X_{\mathsf{disc}}), \\ \mathrm{Hom}_{\mathsf{Cats}}(X_{\mathsf{disc}}, \mathcal{C}) &\cong \mathrm{Hom}_{\mathsf{Sets}}(X, \mathrm{Obj}(\mathcal{C})), \\ \mathrm{Hom}_{\mathsf{Sets}}(\mathrm{Obj}(\mathcal{C}), X) &\cong \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, X_{\mathsf{indisc}}), \end{aligned}$$

natural in $\mathcal{C} \in \mathrm{Obj}(\mathsf{Cats})$ and $X \in \mathrm{Obj}(\mathsf{Sets})$, where

- The functor
$$\pi_0\colon \mathsf{Cats} \to \mathsf{Sets},$$
the **connected components functor**, is the functor sending a category to its set of connected components of Definition 11.2.3.1.1.
- The functor
$$(-)_{\mathsf{disc}}\colon \mathsf{Sets} \to \mathsf{Cats},$$
the **discrete category functor**, is the functor sending a set to its associated discrete category of Definition 11.2.5.1.1.
- The functor
$$\mathrm{Obj}\colon \mathsf{Cats} \to \mathsf{Sets},$$
the **object functor**, is the functor sending a category to its set of objects.
- The functor
$$(-)_{\mathsf{indisc}}\colon \mathsf{Sets} \to \mathsf{Cats},$$
the **indiscrete category functor**, is the functor sending a set to its associated indiscrete category of Definition 11.2.6.1.1.

**Proof.** Omitted. □

**11.2.2. Connected Components of Categories.** Let $\mathcal{C}$ be a category.

**Definition 11.2.2.1.1.** A **connected component** of $\mathcal{C}$ is a full subcategory $\mathcal{I}$ of $\mathcal{C}$ satisfying the following conditions:[10]

(1) *Non-Emptiness.* We have $\mathrm{Obj}(\mathcal{I}) \neq \emptyset$.

(2) *Connectedness.* There exists a zigzag of arrows between any two objects of $\mathcal{I}$.

**11.2.3. Sets of Connected Components of Categories.** Let $\mathcal{C}$ be a category.

**Definition 11.2.3.1.1.** The **set of connected components of** $\mathcal{C}$ is the set $\pi_0(\mathcal{C})$ whose elements are the connected components of $\mathcal{C}$.

**Proposition 11.2.3.1.2.** Let $\mathcal{C}$ be a category.

(1) *Functoriality.* The assignment $\mathcal{C} \mapsto \pi_0(\mathcal{C})$ defines a functor
$$\pi_0\colon \mathsf{Cats} \to \mathsf{Sets}.$$

(2) *Adjointness.* We have a quadruple adjunction
$$(\pi_0 \dashv (-)_{\mathsf{disc}} \dashv \mathrm{Obj} \dashv (-)_{\mathsf{indisc}})\colon \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left=20, "(-)_{\mathsf{disc}}" description] \arrow[r, bend right=60, "(-)_{\mathsf{indisc}}"'] & \mathsf{Cats}. \arrow[l, bend right=60, "\pi_0"'] \arrow[l, bend left=20, "\mathrm{Obj}" description] \end{tikzcd}$$

[10] In other words, a **connected component** of $\mathcal{C}$ is an element of the set $\mathrm{Obj}(\mathcal{C})/\sim$ with $\sim$ the equivalence relation generated by the relation $\sim'$ obtained by declaring $A \sim' B$ iff there exists a morphism of $\mathcal{C}$ from $A$ to $B$.

(3) *Interaction With Groupoids.* If $\mathcal{C}$ is a groupoid, then we have an isomorphism of categories

$$\pi_0(\mathcal{C}) \cong \mathrm{K}(\mathcal{C}),$$

where $\mathrm{K}(\mathcal{C})$ is the set of isomorphism classes of $\mathcal{C}$ of **??**.

(4) *Preservation of Colimits.* The functor $\pi_0$ of Item 1 preserves colimits. In particular, we have bijections of sets

$$\begin{aligned}\pi_0(\mathcal{C} \textstyle\coprod \mathcal{D}) &\cong \pi_0(\mathcal{C}) \textstyle\coprod \pi_0(\mathcal{D}),\\ \pi_0(\mathcal{C} \textstyle\coprod_{\mathcal{E}} \mathcal{D}) &\cong \pi_0(\mathcal{C}) \textstyle\coprod_{\pi_0(\mathcal{E})} \pi_0(\mathcal{D}),\\ \pi_0\left(\mathrm{CoEq}\left(\mathcal{C} \underset{G}{\overset{F}{\rightrightarrows}} \mathcal{D}\right)\right) &\cong \mathrm{CoEq}\left(\pi_0(\mathcal{C}) \underset{\pi_0(G)}{\overset{\pi_0(F)}{\rightrightarrows}} \pi_0(\mathcal{D})\right),\end{aligned}$$

natural in $\mathcal{C}, \mathcal{D}, \mathcal{E} \in \mathrm{Obj}(\mathsf{Cats})$.

(5) *Symmetric Strong Monoidality With Respect to Coproducts.* The connected components functor of Item 1 has a symmetric strong monoidal structure

$$\left(\pi_0, \pi_0^{\coprod}, \pi_{0|\mathbb{1}}^{\coprod}\right) \colon (\mathsf{Cats}, \textstyle\coprod, \emptyset_{\mathsf{cat}}) \to (\mathsf{Sets}, \textstyle\coprod, \emptyset),$$

being equipped with isomorphisms

$$\begin{aligned}\pi_{0|\mathcal{C},\mathcal{D}}^{\coprod} &\colon \pi_0(\mathcal{C}) \textstyle\coprod \pi_0(\mathcal{D}) \xrightarrow{\cong} \pi_0(\mathcal{C} \textstyle\coprod \mathcal{D}),\\ \pi_{0|\mathbb{1}}^{\coprod} &\colon \emptyset \xrightarrow{\cong} \pi_0(\emptyset_{\mathsf{cat}}),\end{aligned}$$

natural in $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$.

(6) *Symmetric Strong Monoidality With Respect to Products.* The connected components functor of Item 1 has a symmetric strong monoidal structure

$$\left(\pi_0, \pi_0^{\otimes}, \pi_{0|\mathbb{1}}^{\otimes}\right) \colon (\mathsf{Cats}, \times, \mathsf{pt}) \to (\mathsf{Sets}, \times, \mathrm{pt}),$$

being equipped with isomorphisms

$$\begin{aligned}\pi_{0|\mathcal{C},\mathcal{D}}^{\otimes} &\colon \pi_0(\mathcal{C}) \times \pi_0(\mathcal{D}) \xrightarrow{\cong} \pi_0(\mathcal{C} \times \mathcal{D}),\\ \pi_{0|\mathbb{1}}^{\otimes} &\colon \mathrm{pt} \xrightarrow{\cong} \pi_0(\mathsf{pt}),\end{aligned}$$

natural in $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$.

**Proof.** *Item 1, Functoriality*: Clear.
*Item 2, Adjointness*: This is proved in Proposition 11.2.1.1.1.
*Item 3, Interaction With Groupoids*: Clear.
*Item 4, Preservation of Colimits*: This follows from Item 2 and **??** of **??**.
*Item 5, Symmetric Strong Monoidality With Respect to Coproducts*: Omitted.
*Item 6, Symmetric Strong Monoidality With Respect to Products*: Omitted. □

### 11.2.4. Connected Categories.

**Definition 11.2.4.1.1.** A category $\mathcal{C}$ is **connected** if $\pi_0(\mathcal{C}) \cong \mathrm{pt}$.[11,12]

[11] *Further Terminology:* A category is **disconnected** if it is not connected.

[12] *Example:* A groupoid is connected iff any two of its objects are isomorphic.

**11.2.5. Discrete Categories.** Let $X$ be a set.

**Definition 11.2.5.1.1.** The **discrete category on a set** $X$ is the category $X_{\mathsf{disc}}$ where

- *Objects.* We have
$$\mathrm{Obj}(X_{\mathsf{disc}}) \stackrel{\text{def}}{=} X;$$
- *Morphisms.* For each $A, B \in \mathrm{Obj}(X_{\mathsf{disc}})$, we have
$$\mathrm{Hom}_{X_{\mathsf{disc}}}(A, B) \stackrel{\text{def}}{=} \begin{cases} \mathrm{id}_A & \text{if } A = B, \\ \emptyset & \text{if } A \neq B; \end{cases}$$
- *Identities.* For each $A \in \mathrm{Obj}(X_{\mathsf{disc}})$, the unit map
$$\mathbb{1}^{X_{\mathsf{disc}}}_{A} \colon \mathrm{pt} \to \mathrm{Hom}_{X_{\mathsf{disc}}}(A, A)$$
of $X_{\mathsf{disc}}$ at $A$ is defined by
$$\mathrm{id}^{X_{\mathsf{disc}}}_{A} \stackrel{\text{def}}{=} \mathrm{id}_A;$$
- *Composition.* For each $A, B, C \in \mathrm{Obj}(X_{\mathsf{disc}})$, the composition map
$$\circ^{X_{\mathsf{disc}}}_{A,B,C} \colon \mathrm{Hom}_{X_{\mathsf{disc}}}(B, C) \times \mathrm{Hom}_{X_{\mathsf{disc}}}(A, B) \to \mathrm{Hom}_{X_{\mathsf{disc}}}(A, C)$$
of $X_{\mathsf{disc}}$ at $(A, B, C)$ is defined by
$$\mathrm{id}_A \circ \mathrm{id}_A \stackrel{\text{def}}{=} \mathrm{id}_A.$$

**Proposition 11.2.5.1.2.** Let $X$ be a set.

(1) *Functoriality.* The assignment $X \mapsto X_{\mathsf{disc}}$ defines a functor
$$(-)_{\mathsf{disc}} \colon \mathsf{Sets} \to \mathsf{Cats}.$$
(2) *Adjointness.* We have a quadruple adjunction
$$(\pi_0 \dashv (-)_{\mathsf{disc}} \dashv \mathrm{Obj} \dashv (-)_{\mathsf{indisc}}) \colon \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left=20, "(-)_{\mathsf{disc}}"] \arrow[r, bend right=60, "(-)_{\mathsf{indisc}}"'] & \mathsf{Cats}. \arrow[l, bend right=60, "\pi_0"'] \arrow[l, bend left=20, "\mathrm{Obj}"] \end{tikzcd}$$
(3) *Symmetric Strong Monoidality With Respect to Coproducts.* The functor of Item 1 has a symmetric strong monoidal structure
$$\left((-)_{\mathsf{disc}}, (-)^{\coprod}_{\mathsf{disc}}, (-)^{\coprod}_{\mathsf{disc}|\mathbb{1}}\right) \colon (\mathsf{Sets}, \textstyle\coprod, \emptyset) \to (\mathsf{Cats}, \textstyle\coprod, \emptyset_{\mathsf{cat}}),$$
being equipped with isomorphisms
$$(-)^{\coprod}_{\mathsf{disc}|X,Y} \colon X_{\mathsf{disc}} \textstyle\coprod Y_{\mathsf{disc}} \xrightarrow{\cong} (X \textstyle\coprod Y)_{\mathsf{disc}},$$
$$(-)^{\coprod}_{\mathsf{disc}|\mathbb{1}} \colon \emptyset_{\mathsf{cat}} \xrightarrow{\cong} \emptyset_{\mathsf{disc}},$$
natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.
(4) *Symmetric Strong Monoidality With Respect to Products.* The functor of Item 1 has a symmetric strong monoidal structure
$$\left((-)_{\mathsf{disc}}, (-)^{\otimes}_{\mathsf{disc}}, (-)^{\otimes}_{\mathsf{disc}|\mathbb{1}}\right) \colon (\mathsf{Sets}, \times, \mathrm{pt}) \to (\mathsf{Cats}, \times, \mathsf{pt}),$$

being equipped with isomorphisms

$$(-)^{\otimes}_{\mathsf{disc}|X,Y}\colon X_{\mathsf{disc}} \times Y_{\mathsf{disc}} \xrightarrow{\cong} (X \times Y)_{\mathsf{disc}},$$

$$(-)^{\otimes}_{\mathsf{disc}|\mathbb{1}}\colon \mathsf{pt} \xrightarrow{\cong} \mathrm{pt}_{\mathsf{disc}},$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

**Proof.** *Item 1, Functoriality*: Clear.
*Item 2, Adjointness*: This is proved in Proposition 11.2.1.1.1.
*Item 3, Symmetric Strong Monoidality With Respect to Coproducts*: Omitted.
*Item 4, Symmetric Strong Monoidality With Respect to Products*: Omitted. □

### 11.2.6. Indiscrete Categories.

**Definition 11.2.6.1.1.** The **indiscrete category on a set** $X$[13] is the category $X_{\mathsf{indisc}}$ where

- *Objects.* We have
$$\mathrm{Obj}(X_{\mathsf{indisc}}) \stackrel{\text{def}}{=} X;$$
- *Morphisms.* For each $A, B \in \mathrm{Obj}(X_{\mathsf{indisc}})$, we have
$$\mathrm{Hom}_{X_{\mathsf{disc}}}(A, B) \stackrel{\text{def}}{=} \{[A] \to [B]\};$$
- *Identities.* For each $A \in \mathrm{Obj}(X_{\mathsf{indisc}})$, the unit map
$$\mathbb{1}^{X_{\mathsf{indisc}}}_{A}\colon \mathrm{pt} \to \mathrm{Hom}_{X_{\mathsf{indisc}}}(A, A)$$
of $X_{\mathsf{indisc}}$ at $A$ is defined by
$$\mathrm{id}^{X_{\mathsf{indisc}}}_{A} \stackrel{\text{def}}{=} \{[A] \to [A]\};$$
- *Composition.* For each $A, B, C \in \mathrm{Obj}(X_{\mathsf{indisc}})$, the composition map
$$\circ^{X_{\mathsf{indisc}}}_{A,B,C}\colon \mathrm{Hom}_{X_{\mathsf{indisc}}}(B, C) \times \mathrm{Hom}_{X_{\mathsf{indisc}}}(A, B) \to \mathrm{Hom}_{X_{\mathsf{indisc}}}(A, C)$$
of $X_{\mathsf{disc}}$ at $(A, B, C)$ is defined by
$$([B] \to [C]) \circ ([A] \to [B]) \stackrel{\text{def}}{=} ([A] \to [C]).$$

**Proposition 11.2.6.1.2.** Let $X$ be a set.

(1) *Functoriality.* The assignment $X \mapsto X_{\mathsf{indisc}}$ defines a functor
$$(-)_{\mathsf{indisc}}\colon \mathsf{Sets} \to \mathsf{Cats}.$$

(2) *Adjointness.* We have a quadruple adjunction

$$(\pi_0 \dashv (-)_{\mathsf{disc}} \dashv \mathrm{Obj} \dashv (-)_{\mathsf{indisc}})\colon\quad \begin{tikzcd} \mathsf{Sets} \arrow[rr, bend left=50, "(-)_{\mathsf{disc}}"' description] \arrow[rr, bend right=80, "(-)_{\mathsf{indisc}}"'] & & \mathsf{Cats}. \arrow[ll, bend right=80, "\pi_0"'] \arrow[ll, bend left=50, "\mathrm{Obj}" description] \end{tikzcd}$$

---

[13] *Further Terminology:* Also called the **chaotic category on** $X$.

(3) *Symmetric Strong Monoidality With Respect to Products.* The functor of Item 1 has a symmetric strong monoidal structure

$$\left((-)_{\mathsf{indisc}}, (-)^{\otimes}_{\mathsf{indisc}}, (-)^{\otimes}_{\mathsf{indisc}|\mathbb{1}}\right) \colon (\mathsf{Sets}, \times, \mathrm{pt}) \to (\mathsf{Cats}, \times, \mathsf{pt}),$$

being equipped with isomorphisms

$$(-)^{\otimes}_{\mathsf{indisc}|X,Y} \colon X_{\mathsf{indisc}} \times Y_{\mathsf{indisc}} \xrightarrow{\cong} (X \times Y)_{\mathsf{indisc}},$$

$$(-)^{\otimes}_{\mathsf{indisc}|\mathbb{1}} \colon \mathsf{pt} \xrightarrow{\cong} \mathrm{pt}_{\mathsf{indisc}},$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

**Proof.** *Item 1, Functoriality*: Clear.
*Item 2, Adjointness*: This is proved in Proposition 11.2.1.1.1.
*Item 3, Symmetric Strong Monoidality With Respect to Products*: Omitted. □

## 11.3. Groupoids

**11.3.1. Foundations.** Let $\mathcal{C}$ be a category.

**Definition 11.3.1.1.1.** A morphism $f \colon A \to B$ of $\mathcal{C}$ is an **isomorphism** if there exists a morphism $f^{-1} \colon B \to A$ of $\mathcal{C}$ such that

$$f \circ f^{-1} = \mathrm{id}_B,$$
$$f^{-1} \circ f = \mathrm{id}_A.$$

**Definition 11.3.1.1.2.** A **groupoid** is a category in which every morphism is an isomorphism.

**11.3.2. The Groupoid Completion of a Category.** Let $\mathcal{C}$ be a category.

**Definition 11.3.2.1.1.** The **groupoid completion of $\mathcal{C}$**[14] is the pair $(\mathrm{K}_0(\mathcal{C}), \iota_{\mathcal{C}})$ consisting of

- A groupoid $\mathrm{K}_0(\mathcal{C})$;
- A functor $\iota_{\mathcal{C}} \colon \mathcal{C} \to \mathrm{K}_0(\mathcal{C})$;

satisfying the following universal property:[15]

(UP) Given another such pair $(\mathcal{G}, i)$, there exists a unique functor $\mathrm{K}_0(\mathcal{C}) \xrightarrow{\exists!} \mathcal{G}$ making the diagram

$$\begin{tikzcd} & \mathrm{K}_0(\mathcal{C}) \arrow[d, dashed, "\exists!"] \\ \mathcal{C} \arrow[ur, "\iota_{\mathcal{C}}"] \arrow[r, "i"'] & \mathcal{G} \end{tikzcd}$$

commute.

**Proposition 11.3.2.1.2.** Let $\mathcal{C}$ be a category.

(1) *Functoriality.* The assignment $\mathcal{C} \mapsto \mathrm{K}_0(\mathcal{C})$ defines a functor

$$\mathrm{K}_0 \colon \mathsf{Cats} \to \mathsf{Grpd}.$$

[14] *Further Terminology:* Also called the **Grothendieck groupoid of $\mathcal{C}$** or the **Grothendieck groupoid completion of $\mathcal{C}$**.

[15] See Item 5 of Proposition 11.3.2.1.2 for an explicit construction.

(2) *2-Functoriality.* The assignment $\mathcal{C} \mapsto \mathrm{K}_0(\mathcal{C})$ defines a 2-functor

$$\mathrm{K}_0 \colon \mathsf{Cats}_2 \to \mathsf{Grpd}_2.$$

(3) *Adjointness.* We have an adjunction

$$(\mathrm{K}_0 \dashv \iota)\colon \quad \begin{tikzcd} \mathsf{Cats} \arrow[r, shift left, "\mathrm{K}_0"] \arrow[r, phantom, "\perp"] & \mathsf{Grpd}, \arrow[l, shift left, hook, "\iota"] \end{tikzcd}$$

witnessed by a bijection of sets

$$\mathrm{Hom}_{\mathsf{Grpd}}(\mathrm{K}_0(\mathcal{C}), \mathcal{G}) \cong \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, \mathcal{G}),$$

natural in $\mathcal{C} \in \mathrm{Obj}(\mathsf{Cats})$ and $\mathcal{G} \in \mathrm{Obj}(\mathsf{Grpd})$, forming, together with the functor $\mathsf{Core}$ of Item 1 of Proposition 11.3.3.1.3, a triple adjunction

$$(\mathrm{K}_0 \dashv \iota \dashv \mathsf{Core})\colon \quad \begin{tikzcd} \mathsf{Cats} \arrow[r, bend left, "\mathrm{K}_0"] \arrow[r, bend right, "\mathsf{Core}"'] & \mathsf{Grpd}, \arrow[l, hook', "\iota" description] \end{tikzcd}$$

witnessed by bijections of sets

$$\begin{aligned} \mathrm{Hom}_{\mathsf{Grpd}}(\mathrm{K}_0(\mathcal{C}), \mathcal{G}) &\cong \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, \mathcal{G}), \\ \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{G}, \mathcal{D}) &\cong \mathrm{Hom}_{\mathsf{Grpd}}(\mathcal{G}, \mathsf{Core}(\mathcal{D})), \end{aligned}$$

natural in $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$ and $\mathcal{G} \in \mathrm{Obj}(\mathsf{Grpd})$.

(4) *2-Adjointness.* We have a 2-adjunction

$$(\mathrm{K}_0 \dashv \iota)\colon \quad \begin{tikzcd} \mathsf{Cats} \arrow[r, shift left, "\mathrm{K}_0"] \arrow[r, phantom, "\perp_2"] & \mathsf{Grpd}, \arrow[l, shift left, hook, "\iota"] \end{tikzcd}$$

witnessed by an isomorphism of categories

$$\mathsf{Fun}(\mathrm{K}_0(\mathcal{C}), \mathcal{G}) \cong \mathsf{Fun}(\mathcal{C}, \mathcal{G}),$$

natural in $\mathcal{C} \in \mathrm{Obj}(\mathsf{Cats})$ and $\mathcal{G} \in \mathrm{Obj}(\mathsf{Grpd})$, forming, together with the 2-functor $\mathsf{Core}$ of Item 2 of Proposition 11.3.3.1.3, a triple 2-adjunction

$$(\mathrm{K}_0 \dashv \iota \dashv \mathsf{Core})\colon \quad \begin{tikzcd} \mathsf{Cats} \arrow[r, bend left, "\mathrm{K}_0"] \arrow[r, bend right, "\mathsf{Core}"'] & \mathsf{Grpd}, \arrow[l, hook', "\iota" description] \end{tikzcd}$$

witnessed by isomorphisms of categories

$$\begin{aligned} \mathsf{Fun}(\mathrm{K}_0(\mathcal{C}), \mathcal{G}) &\cong \mathsf{Fun}(\mathcal{C}, \mathcal{G}), \\ \mathsf{Fun}(\mathcal{G}, \mathcal{D}) &\cong \mathsf{Fun}(\mathcal{G}, \mathsf{Core}(\mathcal{D})), \end{aligned}$$

natural in $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$ and $\mathcal{G} \in \mathrm{Obj}(\mathsf{Grpd})$.

(5) *Interaction With Classifying Spaces.* We have an isomorphism of groupoids

$$\mathrm{K}_0(\mathcal{C}) \cong \Pi_{\leq 1}(|\mathrm{N}_\bullet(\mathcal{C})|),$$

natural in $\mathcal{C} \in \mathrm{Obj}(\mathsf{Cats})$; i.e. the diagram

$$\begin{tikzcd}
\mathsf{Cats} \arrow[r, "\mathrm{K}_0"] \arrow[d, "\mathrm{N}_\bullet"'] & \mathsf{Grp} \arrow[d, phantom, "" ] \\
\mathsf{sSets} \arrow[r, "|-|"'] \arrow[ur, phantom, "\wr"] & \mathsf{Top} \arrow[u, "\Pi_{\leq 1}"']
\end{tikzcd}$$

commutes up to natural isomorphism.

(6) *Symmetric Strong Monoidality With Respect to Coproducts.* The groupoid completion functor of Item 1 has a symmetric strong monoidal structure

$$\left(\mathrm{K}_0, \mathrm{K}_0^{\coprod}, \mathrm{K}_{0|\mathbb{1}}^{\coprod}\right) : (\mathsf{Cats}, \coprod, \emptyset_{\mathsf{cat}}) \to (\mathsf{Grpd}, \coprod, \emptyset_{\mathsf{cat}})$$

being equipped with isomorphisms

$$\mathrm{K}_{0|\mathcal{C},\mathcal{D}}^{\coprod} : \mathrm{K}_0(\mathcal{C}) \coprod \mathrm{K}_0(\mathcal{D}) \xrightarrow{\cong} \mathrm{K}_0(\mathcal{C} \coprod \mathcal{D}),$$

$$\mathrm{K}_{0|\mathbb{1}}^{\coprod} : \emptyset_{\mathsf{cat}} \xrightarrow{\cong} \mathrm{K}_0(\emptyset_{\mathsf{cat}}),$$

natural in $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$.

(7) *Symmetric Strong Monoidality With Respect to Products.* The groupoid completion functor of Item 1 has a symmetric strong monoidal structure

$$\left(\mathrm{K}_0, \mathrm{K}_0^{\times}, \mathrm{K}_{0|\mathbb{1}}^{\times}\right) : (\mathsf{Cats}, \times, \mathrm{pt}) \to (\mathsf{Grpd}, \times, \mathrm{pt})$$

being equipped with isomorphisms

$$\mathrm{K}_{0|\mathcal{C},\mathcal{D}}^{\times} : \mathrm{K}_0(\mathcal{C}) \times \mathrm{K}_0(\mathcal{D}) \xrightarrow{\cong} \mathrm{K}_0(\mathcal{C} \times \mathcal{D}),$$

$$\mathrm{K}_{0|\mathbb{1}}^{\times} : \mathrm{pt} \xrightarrow{\cong} \mathrm{K}_0(\mathrm{pt}),$$

natural in $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$.

**Proof.** *Item 1, Functoriality*: Omitted.
*Item 2, 2-Functoriality*: Omitted.
*Item 3, Adjointness*: Omitted.
*Item 4, 2-Adjointness*: Omitted.
*Item 5, Interaction With Classifying Spaces*: See Corollary 18.33 of `https://web.ma.utexas.edu/users/dafr/M392C-2012/Notes/lecture18.pdf`.
*Item 6, Symmetric Strong Monoidality With Respect to Coproducts*: Omitted.
*Item 7, Symmetric Strong Monoidality With Respect to Products*: Omitted. □

### 11.3.3. The Core of a Category. 
Let $\mathcal{C}$ be a category.

**Definition 11.3.3.1.1.** The **core** of $\mathcal{C}$ is the pair $(\mathsf{Core}(\mathcal{C}), \iota_{\mathcal{C}})$[16] consisting of

(1) A groupoid $\mathsf{Core}(\mathcal{C})$;
(2) A functor $\iota_{\mathcal{C}} : \mathsf{Core}(\mathcal{C}) \hookrightarrow \mathcal{C}$;

satisfying the following universal property:

[16] *Further Notation:* Also written $\mathcal{C}^{\simeq}$.

(UP) Given another such pair $(\mathcal{G}, i)$, there exists a unique functor $\mathcal{G} \xrightarrow{\exists!} \mathsf{Core}(\mathcal{C})$ making the diagram

$$\begin{tikzcd}
& \mathsf{Core}(\mathcal{C}) \arrow[d, hook, "\iota_{\mathcal{C}}"] \\
\mathcal{G} \arrow[ur, dashed, "\exists!"] \arrow[r, "i"'] & \mathcal{C}
\end{tikzcd}$$

commute.

**Construction 11.3.3.1.2.** The core of $\mathcal{C}$ is the wide subcategory of $\mathcal{C}$ spanned by the isomorphisms of $\mathcal{C}$, i.e. the category $\mathsf{Core}(\mathcal{C})$ where[17]

(1) *Objects.* We have

$$\mathrm{Obj}(\mathsf{Core}(\mathcal{C})) \stackrel{\text{def}}{=} \mathrm{Obj}(\mathcal{C});$$

(2) *Morphisms.* The morphisms of $\mathsf{Core}(\mathcal{C})$ are the isomorphisms of $\mathcal{C}$.

**Proof.** This follows from the fact that functors preserve isomorphisms. □

**Proposition 11.3.3.1.3.** Let $\mathcal{C}$ be a category.

(1) *Functoriality.* The assignment $\mathcal{C} \mapsto \mathsf{Core}(\mathcal{C})$ defines a functor

$$\mathsf{Core}\colon \mathsf{Cats} \to \mathsf{Grpd}.$$

(2) *2-Functoriality.* The assignment $\mathcal{C} \mapsto \mathsf{Core}(\mathcal{C})$ defines a 2-functor

$$\mathsf{Core}\colon \mathsf{Cats}_2 \to \mathsf{Grpd}_2.$$

(3) *Adjointness.* We have an adjunction

$$(\iota \dashv \mathsf{Core})\colon \quad \begin{tikzcd} \mathsf{Grpd} \arrow[r, hook, bend left, "\iota"] \arrow[r, phantom, "\perp"] & \mathsf{Cats}, \arrow[l, bend left, "\mathsf{Core}"] \end{tikzcd}$$

witnessed by a bijection of sets

$$\mathrm{Hom}_{\mathsf{Cats}}(\mathcal{G}, \mathcal{D}) \cong \mathrm{Hom}_{\mathsf{Grpd}}(\mathcal{G}, \mathsf{Core}(\mathcal{D})),$$

natural in $\mathcal{G} \in \mathrm{Obj}(\mathsf{Grpd})$ and $\mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$, forming, together with the functor $\mathrm{K}_0$ of Item 1 of Proposition 11.3.2.1.2, a triple adjunction

$$(\mathrm{K}_0 \dashv \iota \dashv \mathsf{Core})\colon \quad \begin{tikzcd} \mathsf{Cats} \arrow[r, bend left=50, "\mathrm{K}_0"] \arrow[r, bend right=50, phantom, ""] & \mathsf{Grpd}, \arrow[l, hook', "\iota" description] \end{tikzcd}$$

witnessed by bijections of sets

$$\mathrm{Hom}_{\mathsf{Grpd}}(\mathrm{K}_0(\mathcal{C}), \mathcal{G}) \cong \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, \mathcal{G}),$$
$$\mathrm{Hom}_{\mathsf{Cats}}(\mathcal{G}, \mathcal{D}) \cong \mathrm{Hom}_{\mathsf{Grpd}}(\mathcal{G}, \mathsf{Core}(\mathcal{D})),$$

natural in $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$ and $\mathcal{G} \in \mathrm{Obj}(\mathsf{Grpd})$.

[17] *Slogan:* The groupoid $\mathsf{Core}(\mathcal{C})$ is the maximal subgroupoid of $\mathcal{C}$.

(4) *2-Adjointness.* We have an adjunction

$$(\iota \dashv \mathsf{Core})\colon \quad \begin{tikzcd} \mathsf{Grpd} \arrow[r, bend left, "\iota"] \arrow[r, phantom, "\perp_2"] & \mathsf{Cats}, \arrow[l, bend left, "\mathsf{Core}"] \end{tikzcd}$$

witnessed by an isomorphism of categories

$$\mathsf{Fun}(\mathcal{G}, \mathcal{D}) \cong \mathsf{Fun}(\mathcal{G}, \mathsf{Core}(\mathcal{D})),$$

natural in $\mathcal{G} \in \mathrm{Obj}(\mathsf{Grpd})$ and $\mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$, forming, together with the 2-functor $\mathrm{K}_0$ of Item 2 of Proposition 11.3.2.1.2, a triple 2-adjunction

$$(\mathrm{K}_0 \dashv \iota \dashv \mathsf{Core})\colon \quad \begin{tikzcd} \mathsf{Cats} \arrow[r, bend left=50, "\mathrm{K}_0"] \arrow[r, bend right=50, "\mathsf{Core}"'] & \mathsf{Grpd}, \arrow[l, "\iota" description] \end{tikzcd}$$

witnessed by isomorphisms of categories

$$\mathsf{Fun}(\mathrm{K}_0(\mathcal{C}), \mathcal{G}) \cong \mathsf{Fun}(\mathcal{C}, \mathcal{G}),$$
$$\mathsf{Fun}(\mathcal{G}, \mathcal{D}) \cong \mathsf{Fun}(\mathcal{G}, \mathsf{Core}(\mathcal{D})),$$

natural in $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$ and $\mathcal{G} \in \mathrm{Obj}(\mathsf{Grpd})$.

(5) *Symmetric Strong Monoidality With Respect to Products.* The core functor of Item 1 has a symmetric strong monoidal structure

$$\left(\mathsf{Core}, \mathsf{Core}^{\times}, \mathsf{Core}^{\times}_{\mathbb{1}}\right) \colon (\mathsf{Cats}, \times, \mathsf{pt}) \to (\mathsf{Grpd}, \times, \mathsf{pt})$$

being equipped with isomorphisms

$$\mathsf{Core}^{\times}_{\mathcal{C},\mathcal{D}} \colon \mathsf{Core}(\mathcal{C}) \times \mathsf{Core}(\mathcal{D}) \xrightarrow{\cong} \mathsf{Core}(\mathcal{C} \times \mathcal{D}),$$
$$\mathsf{Core}^{\times}_{\mathbb{1}} \colon \mathsf{pt} \xrightarrow{\cong} \mathsf{Core}(\mathsf{pt}),$$

natural in $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$.

(6) *Symmetric Strong Monoidality With Respect to Coproducts.* The core functor of Item 1 has a symmetric strong monoidal structure

$$\left(\mathsf{Core}, \mathsf{Core}^{\coprod}, \mathsf{Core}^{\coprod}_{\mathbb{1}}\right) \colon (\mathsf{Cats}, \coprod, \emptyset_{\mathsf{cat}}) \to (\mathsf{Grpd}, \coprod, \emptyset_{\mathsf{cat}})$$

being equipped with isomorphisms

$$\mathsf{Core}^{\coprod}_{\mathcal{C},\mathcal{D}} \colon \mathsf{Core}(\mathcal{C}) \coprod \mathsf{Core}(\mathcal{D}) \xrightarrow{\cong} \mathsf{Core}(\mathcal{C} \coprod \mathcal{D}),$$
$$\mathsf{Core}^{\coprod}_{\mathbb{1}} \colon \emptyset_{\mathsf{cat}} \xrightarrow{\cong} \mathsf{Core}(\emptyset_{\mathsf{cat}}),$$

natural in $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$.

**Proof.** *Item 1, Functoriality*: Omitted.
*Item 2, 2-Functoriality*: Omitted.
*Item 3, Adjointness*: The adjunction $(\mathrm{K}_0 \dashv \iota)$ follows from the universal property of the Gabriel–Zisman localisation of a category with respect to a class of morphisms (**??**), while the adjunction $(\iota \dashv \mathsf{Core})$ is a reformulation of the universal property of the core of a category (Definition 11.3.3.1.1).[18]

[18] *Reference:* [Rie17, Example 4.1.15]

*Item 4, 2-Adjointness*: Omitted.
*Item 5, Symmetric Strong Monoidality With Respect to Products*: Omitted.
*Item 6, Symmetric Strong Monoidality With Respect to Coproducts*: Omitted.
□

## 11.4. Functors

**11.4.1. Foundations.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

**Definition 11.4.1.1.1.** A **functor** $F\colon \mathcal{C} \to \mathcal{D}$ **from** $\mathcal{C}$ **to** $\mathcal{D}$[19] consists of[20]

(1) *Action on Objects.* A map of sets
$$F\colon \mathrm{Obj}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{D}),$$
called the **action on objects of** $F$;

(2) *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathcal{C})$, a map
$$F_{A,B}\colon \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F_A, F_B),$$
called the **action on morphisms of** $F$ **at** $(A, B)$[21];

satisfying the following conditions:

(1) *Preservation of Identities.* For each $A \in \mathrm{Obj}(\mathcal{C})$, the diagram
$$\begin{tikzcd}
\mathrm{pt} \arrow[d, "\mathbb{1}^{\mathcal{C}}_{A}"'] \arrow[dr, "\mathbb{1}^{\mathcal{D}}_{F_A}"] & \\
\mathrm{Hom}_{\mathcal{C}}(A, A) \arrow[r, "F_{A,A}"'] & \mathrm{Hom}_{\mathcal{D}}(F_A, F_A)
\end{tikzcd}$$
commutes, i.e. we have
$$F(\mathrm{id}_A) = \mathrm{id}_{F_A}.$$

(2) *Preservation of Composition.* For each $A, B, C \in \mathrm{Obj}(\mathcal{C})$, the diagram
$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(B, C) \times \mathrm{Hom}_{\mathcal{C}}(A, B) \arrow[r, "\circ^{\mathcal{C}}_{A,B,C}"] \arrow[d, "F_{B,C} \times F_{A,B}"'] & \mathrm{Hom}_{\mathcal{C}}(A, C) \arrow[d, "F_{A,C}"] \\
\mathrm{Hom}_{\mathcal{D}}(F_B, F_C) \times \mathrm{Hom}_{\mathcal{D}}(F_A, F_B) \arrow[r, "\circ^{\mathcal{D}}_{F_A,F_B,F_C}"'] & \mathrm{Hom}_{\mathcal{D}}(F_A, F_C)
\end{tikzcd}$$
commutes, i.e. for each composable pair $(g, f)$ of morphisms of $\mathcal{C}$, we have
$$F(g \circ f) = F(g) \circ F(f).$$

**Example 11.4.1.1.2.** The **identity functor** of a category $\mathcal{C}$ is the functor $\mathrm{id}_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{C}$ where

[19] *Further Terminology:* Also called a **covariant functor**.
[20] *Further Notation:* Given functors $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{C}^{\mathsf{op}} \to \mathcal{D}$, we will sometimes write $F_A$ for $F(A)$ (resp. $G^A$ for $G(A)$) and $F_f$ for $F(f)$ (resp. $G^f$ for $G(f)$). This has been called Einstein notation in the literature.
[21] *Further Terminology:* Also called **action on Hom-sets of** $F$ **at** $(A, B)$.

(1) *Action on Objects.* For each $A \in \mathrm{Obj}(\mathcal{C})$, we have
$$\mathrm{id}_{\mathcal{C}}(A) \overset{\mathrm{def}}{=} A;$$
(2) *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathcal{C})$, the action on morphisms map
$$(\mathrm{id}_{\mathcal{C}})_{A,B} \colon \mathrm{Hom}_{\mathcal{C}}(A,B) \to \underbrace{\mathrm{Hom}_{\mathcal{C}}(\mathrm{id}_{\mathcal{C}}(A), \mathrm{id}_{\mathcal{C}}(B))}_{\overset{\mathrm{def}}{=}\mathrm{Hom}_{\mathcal{C}}(A,B)}$$
of $\mathrm{id}_{\mathcal{C}}$ at $(A, B)$ is defined by
$$(\mathrm{id}_{\mathcal{C}})_{A,B} \overset{\mathrm{def}}{=} \mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(A,B)}.$$

**Proof.** *Preservation of Identities*: We have $\mathrm{id}_{\mathcal{C}}(\mathrm{id}_A) \overset{\mathrm{def}}{=} \mathrm{id}_A$ for each $A \in \mathrm{Obj}(\mathcal{C})$ by definition.

*Preservation of Compositions*: For each composable pair $A \xrightarrow{f} B \xrightarrow{g} B$ of morphisms of $\mathcal{C}$, we have
$$\begin{aligned}\mathrm{id}_{\mathcal{C}}(g \circ f) &\overset{\mathrm{def}}{=} g \circ f \\ &\overset{\mathrm{def}}{=} \mathrm{id}_{\mathcal{C}}(g) \circ \mathrm{id}_{\mathcal{C}}(f).\end{aligned}$$
This finishes the proof. □

**Definition 11.4.1.1.3.** The **composition** of two functors $F \colon \mathcal{C} \to \mathcal{D}$ and $G \colon \mathcal{D} \to \mathcal{E}$ is the functor $G \circ F$ where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathcal{C})$, we have
$$[G \circ F](A) \overset{\mathrm{def}}{=} G(F(A));$$
- *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathcal{C})$, the action on morphisms map
$$(G \circ F)_{A,B} \colon \mathrm{Hom}_{\mathcal{C}}(A,B) \to \mathrm{Hom}_{\mathcal{E}}(G_{F_A}, G_{F_B})$$
of $G \circ F$ at $(A, B)$ is defined by
$$[G \circ F](f) \overset{\mathrm{def}}{=} G(F(f)).$$

**Proof.** *Preservation of Identities*: For each $A \in \mathrm{Obj}(\mathcal{C})$, we have
$$\begin{aligned} &\text{(functoriality of } F\text{)} & G_{F_{\mathrm{id}_A}} &= G_{\mathrm{id}_{F_A}} \\ &\text{(functoriality of } G\text{)} & &= \mathrm{id}_{G_{F_A}}.\end{aligned}$$
*Preservation of Composition*: For each composable pair $(g, f)$ of morphisms of $\mathcal{C}$, we have
$$\begin{aligned} &\text{(functoriality of } F\text{)} & G_{F_{g\circ f}} &= G_{F_g \circ F_f} \\ &\text{(functoriality of } G\text{)} & &= G_{F_g} \circ G_{F_f}.\end{aligned}$$
This finishes the proof. □

**Proposition 11.4.1.1.4.** Let $F \colon \mathcal{C} \to \mathcal{D}$ be a functor.

(1) *Preservation of Isomorphisms.* If $f$ is an isomorphism in $\mathcal{C}$, then $F(f)$ is an isomorphism in $\mathcal{D}$.[22]

[22]When the converse holds, we call $F$ *conservative*, see Definition 11.4.6.1.1.

**Proof.** *Item 1, Preservation of Isomorphisms*: Indeed, we have

$$\begin{aligned} F(f)^{-1} \circ F(f) &= F\Big(f^{-1} \circ f\Big) \\ &= F(\mathrm{id}_A) \\ &= \mathrm{id}_{F(A)} \end{aligned}$$

and

$$\begin{aligned} F(f) \circ F(f)^{-1} &= F\Big(f \circ f^{-1}\Big) \\ &= F(\mathrm{id}_A) \\ &= \mathrm{id}_{F(A)}, \end{aligned}$$

showing $F(f)$ to be an isomorphism. □

**11.4.2. Faithful Functors.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

**Definition 11.4.2.1.1.** A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **faithful** if, for each $A, B \in \mathrm{Obj}(\mathcal{C})$, the action on morphisms map

$$F_{A,B} \colon \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F_A, F_B)$$

of $F$ at $(A, B)$ is injective.

**Proposition 11.4.2.1.2.** Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor

(1) *Characterisations.* The following conditions are equivalent:
   (a) The functor $F\colon \mathcal{C} \to \mathcal{D}$ is faithful.
   (b) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor
   $$F^* \colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$
   is faithful.
   (c) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor
   $$F_* \colon \mathsf{Fun}(\mathcal{X}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{X}, \mathcal{D})$$
   is faithful.

**Proof.** *Item 1, Characterisations*: Omitted. □

**11.4.3. Full Functors.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

**Definition 11.4.3.1.1.** A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **full** if, for each $A, B \in \mathrm{Obj}(\mathcal{C})$, the action on morphisms map

$$F_{A,B} \colon \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F_A, F_B)$$

of $F$ at $(A, B)$ is surjective.

**Proposition 11.4.3.1.2.** Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor

(1) *Characterisations.* The following conditions are equivalent:
   (a) The functor $F\colon \mathcal{C} \to \mathcal{D}$ is full.
   (b) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor
   $$F^* \colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$
   is full.
   (c) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor
   $$F_* \colon \mathsf{Fun}(\mathcal{X}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{X}, \mathcal{D})$$
   is full.

**Proof.** *Item 1, Characterisations*: Omitted. □

**11.4.4. Fully Faithful Functors.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

**Definition 11.4.4.1.1.** A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **fully faithful** if $F$ is full and faithful, i.e. if, for each $A, B \in \mathrm{Obj}(\mathcal{C})$, the action on morphisms map

$$F_{A,B}\colon \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F_A, F_B)$$

of $F$ at $(A, B)$ is bijective.

**Proposition 11.4.4.1.2.** Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor

(1) *Characterisations.* The following conditions are equivalent:
  (a) The functor $F\colon \mathcal{C} \to \mathcal{D}$ is fully faithful.
  (b) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor

$$F^*\colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

  is fully faithful.
  (c) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor

$$F_*\colon \mathsf{Fun}(\mathcal{X}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{X}, \mathcal{D})$$

  is fully faithful.

(2) *Conservativity.* If $F$ is fully faithful, then $F$ is conservative.

**Proof.** *Item 1, Characterisations*: Omitted.
*Item 2, Conservativity*: This is proved in Item 2 of Proposition 11.4.6.1.2. □

**11.4.5. Essentially Surjective Functors.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

**Definition 11.4.5.1.1.** A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **essentially surjective** if, for each $D \in \mathrm{Obj}(\mathcal{D})$, there exists some object $A$ of $\mathcal{C}$ with $F(A) \cong D$.

**11.4.6. Conservative Functors.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

**Definition 11.4.6.1.1.** A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **conservative** if it satisfies the following condition:

($\star$) For each $f \in \mathrm{Mor}(\mathcal{C})$, if $F(f)$ is an isomorphism in $\mathcal{D}$, then $f$ is an isomorphism in $\mathcal{C}$.

**Proposition 11.4.6.1.2.** Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor.

(1) *Characterisations.* The following conditions are equivalent:
  (a) The functor $F$ is conservative.
  (b) For each $f \in \mathrm{Mor}(\mathcal{C})$, the morphism $F(f)$ is an isomorphism in $\mathcal{D}$ iff $f$ is an isomorphism in $\mathcal{C}$.

(2) *Interaction With Fully Faithfulness.* Every fully faithful functor is conservative.

**Proof.** *Item 1, Characterisations*: This follows from Item 1 of Proposition 11.4.1.1.4.
*Item 2, Interaction With Fully Faithfulness*: Let $F\colon \mathcal{C} \to \mathcal{D}$ be a fully faithful functor, let $f\colon A \to B$ be a morphism of $\mathcal{C}$, and suppose that $F_f$ is

an isomorphism. We have

$$\begin{aligned} F(\mathrm{id}_B) &= \mathrm{id}_{F(B)} \\ &= F(f) \circ F(f)^{-1} \\ &= F\Big(f \circ f^{-1}\Big). \end{aligned}$$

Similarly, $F(\mathrm{id}_A) = F(f^{-1} \circ f)$. But since $F$ is fully faithful, we must have

$$\begin{aligned} f \circ f^{-1} &= \mathrm{id}_B, \\ f^{-1} \circ f &= \mathrm{id}_A, \end{aligned}$$

showing $f$ to be an isomorphism. Thus $F$ is conservative. □

### 11.4.7. Equivalences of Categories.

**Definition 11.4.7.1.1.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

- An **equivalence of categories** between $\mathcal{C}$ and $\mathcal{D}$ consists of a pair of functors

$$\begin{aligned} F &: \mathcal{C} \to \mathcal{D}, \\ G &: \mathcal{D} \to \mathcal{C} \end{aligned}$$

  together with natural isomorphisms

$$\begin{aligned} \eta &: \mathrm{id}_{\mathcal{C}} \overset{\cong}{\Longrightarrow} G \circ F, \\ \epsilon &: F \circ G \overset{\cong}{\Longrightarrow} \mathrm{id}_{\mathcal{D}}. \end{aligned}$$

- An **adjoint equivalence of categories** between $\mathcal{C}$ and $\mathcal{D}$ is an equivalence $(F, G, \eta, \epsilon)$ between $\mathcal{C}$ and $\mathcal{D}$ which is also an adjunction.

**Proposition 11.4.7.1.2.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor.

(1) *Characterisations.* If $\mathcal{C}$ and $\mathcal{D}$ are small[23], then the following conditions are equivalent:[24]
  (a) The functor $F$ is an equivalence of categories.
  (b) The functor $F$ is fully faithful and essentially surjective.
  (c) The induced functor

$$F|_{\mathsf{Sk}(\mathcal{C})} : \mathsf{Sk}(\mathcal{C}) \to \mathsf{Sk}(\mathcal{D})$$

  is an *isomorphism* of categories.

(2) *Two-Out-of-Three.* Let

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "G \circ F"] \arrow[dr, "F"'] & & \mathcal{E} \\ & \mathcal{D} \arrow[ur, "G"'] & \end{tikzcd}$$

  be a diagram in $\mathsf{Cats}$. If two out of the three functors among $F$, $G$, and $G \circ F$ are equivalences of categories, then so is the third.

[23]Otherwise there will be size issues. One can also work with large categories and universes, or require $F$ to be *constructively* essentially surjective; see [**MSE1465107**].
[24]In ZFC, the equivalence between Item 1a and Item 1b is equivalent to the axiom of choice; see [MO 119454].
In Univalent Foundations, this is true without requiring neither the axiom of choice nor the law of the excluded middle.

(3) *Stability Under Composition.* Let

$$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D} \underset{G'}{\overset{F'}{\rightleftarrows}} \mathcal{E}$$

be a diagram in $\mathsf{Cats}$. If $(F, G)$ and $(F', G')$ are equivalences of categories, then so is their composite $(F' \circ F, G' \circ G)$.

(4) *Equivalences vs.Adjoint Equivalences.* Every equivalence of categories can be promoted to an adjoint equivalence.[25]

(5) *Interaction With Groupoids.* If $\mathcal{C}$ and $\mathcal{D}$ are groupoids, then the following conditions are equivalent:
   (a) The functor $F$ is an equivalence of groupoids.
   (b) The following conditions are satisfied:
      (i) The functor $F$ induces a bijection

      $$\pi_0(F) \colon \pi_0(\mathcal{C}) \to \pi_0(\mathcal{D})$$

      of sets.
      (ii) For each $A \in \mathrm{Obj}(\mathcal{C})$, the induced map

      $$F_{x,x} \colon \mathrm{Aut}_{\mathcal{C}}(A) \to \mathrm{Aut}_{\mathcal{D}}(F_A)$$

      is an isomorphism of groups.

**Proof.** *Item 1, Characterisations*: We claim that Items 1a to 1c are indeed equivalent:

(1) *Item 1a $\Longrightarrow$ Item 1b.* Clear.

(2) *Item 1b $\Longrightarrow$ Item 1a.* Since $F$ is essentially surjective and $\mathcal{C}$ and $\mathcal{D}$ are small, we can choose, using the axiom of choice, for each $B \in \mathrm{Obj}(\mathcal{D})$, an object $j_B$ of $\mathcal{C}$ and an isomorphism $i_B \colon B \to F_{j_B}$ of $\mathcal{D}$.

   Since $F$ is fully faithful, we can extend the assignment $B \mapsto j_B$ to a *unique* functor $j \colon \mathcal{D} \to \mathcal{C}$ such that the isomorphisms $i_B \colon B \to F_{j_B}$ assemble into a natural isomorphism $\eta \colon \mathrm{id}_{\mathcal{D}} \overset{\cong}{\Longrightarrow} F \circ j$, with a similar natural isomorphism $\epsilon \colon \mathrm{id}_{\mathcal{C}} \overset{\cong}{\Longrightarrow} j \circ F$. Hence $F$ is an equivalence.

(3) *Item 1a $\Longrightarrow$ Item 1c.* This follows from **??**.

*Item 2, Two-Out-of-Three*: Omitted.
*Item 3, Stability Under Composition*: Clear.
*Item 4, Equivalences vs.Adjoint Equivalences*: See [Rie17, Proposition 4.4.5].
*Item 5, Interaction With Groupoids*: See [nLa24b, Proposition 4.4]. □

### 11.4.8. Isomorphisms of Categories.

**Definition 11.4.8.1.1.** An **isomorphism of categories** is a pair of functors

$$F \colon \mathcal{C} \to \mathcal{D},$$
$$G \colon \mathcal{D} \to \mathcal{C}$$

[25]More precisely, we can promote an equivalence of categories $(F, G, \eta, \epsilon)$ to adjoint equivalences $(F, G, \eta', \epsilon)$ and $(F, G, \eta, \epsilon')$.

such that we have

$$G \circ F = \mathrm{id}_{\mathcal{C}},$$

$$F \circ G = \mathrm{id}_{\mathcal{D}}.$$

**Example 11.4.8.1.2.** Categories can be equivalent but non-isomorphic. For example, the category consisting of two isomorphic objects is equivalent to $\mathsf{pt}$, but not isomorphic to it.

**Proposition 11.4.8.1.3.** Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor.

(1) *Characterisations.* If $\mathcal{C}$ and $\mathcal{D}$ are small, then the following conditions are equivalent:
 (a) The functor $F$ is an isomorphism of categories.
 (b) The functor $F$ is fully faithful and a bijection on objects.

**Proof.** *Item 1, Characterisations*: Omitted, but similar to Item 1 of Proposition 11.4.7.1.2. □

### 11.4.9. The Natural Transformation Associated to a Functor.

**Definition 11.4.9.1.1.** Every functor $F\colon \mathcal{C} \to \mathcal{D}$ defines a natural transformation[26]

$$F^{\dagger}\colon \mathrm{Hom}_{\mathcal{C}} \Longrightarrow \mathrm{Hom}_{\mathcal{D}} \circ (F^{\mathsf{op}} \times F), \qquad \begin{tikzcd} \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \arrow[rr, "F^{\mathsf{op}} \times F"] \arrow[dr, "\mathrm{Hom}_{\mathcal{C}}"'] & \arrow[d, phantom, "\overset{F^{\dagger}}{\Longrightarrow}"] & \mathcal{D}^{\mathsf{op}} \times \mathcal{D} \arrow[dl, "\mathrm{Hom}_{\mathcal{D}}"] \\ & \mathsf{Sets}, & \end{tikzcd}$$

called the **natural transformation associated to** $F$, consisting of the collection

$$\left\{F^{\dagger}_{A,B}\colon \mathrm{Hom}_{\mathcal{C}}(A,B) \to \mathrm{Hom}_{\mathcal{D}}(F_A, F_B)\right\}_{(A,B)\in\mathrm{Obj}(\mathcal{C}^{\mathsf{op}}\times\mathcal{C})}$$

with

$$F^{\dagger}_{A,B} \overset{\mathrm{def}}{=} F_{A,B}.$$

**Proof.** The naturality condition for $F^{\dagger}$ is the requirement that for each morphism

$$(\phi,\psi)\colon (X,Y) \to (A,B)$$

of $\mathcal{C}^{\mathsf{op}} \times \mathcal{C}$, the diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(X,Y) \arrow[r, "\phi^* \circ \psi_* = \psi_* \circ \phi^*"] \arrow[d, "F_{X,Y}"'] & \mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[d, "F_{A,B}"] \\ \mathrm{Hom}_{\mathcal{D}}(F_X, F_Y) \arrow[r, "F(\phi)^* \circ F(\psi)_* = F(\psi)_* \circ F(\phi)^*"'] & \mathrm{Hom}_{\mathcal{D}}(F_A, F_B), \end{tikzcd}$$

[26]This is the 1-categorical version of ?? of ??.

acting on elements as

$$\begin{tikzcd}
f \arrow[r, mapsto] \arrow[d, mapsto] & \psi \circ f \circ \phi \arrow[d, mapsto] \\
F(f) \arrow[r, mapsto] & F(\psi) \circ F(f) \circ F(\psi) = F(\psi \circ f \circ \phi)
\end{tikzcd}$$

commutes, which follows from the functoriality of $F$. □

**Proposition 11.4.9.1.2.** Let $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{E}$ be functors.

(1) *Interaction With Natural Isomorphisms.* The following conditions are equivalent:
  (a) The natural transformation $F^{\dagger}\colon \mathrm{Hom}_{\mathcal{C}} \Longrightarrow \mathrm{Hom}_{\mathcal{D}} \circ (F^{\mathsf{op}} \times F)$ associated to $F$ is a natural isomorphism.
  (b) The functor $F$ is fully faithful.

(2) *Interaction With Composition.* We have an equality of pasting diagrams

$$\begin{tikzcd}
\mathcal{C}^{\mathsf{op}} \times \mathcal{C} \arrow[r, "F^{\mathsf{op}} \times F"] \arrow[dr, "\mathrm{Hom}_{\mathcal{C}}"'] & \mathcal{D}^{\mathsf{op}} \times \mathcal{D} \arrow[r, "G^{\mathsf{op}} \times G"] \arrow[d, "\mathrm{Hom}_{\mathcal{D}}"] \arrow[dl, phantom, "\Rightarrow F^{\dagger}"] & \mathcal{E}^{\mathsf{op}} \times \mathcal{E} \arrow[dl, "\mathrm{Hom}_{\mathcal{E}}"] \arrow[dll, phantom, "\Rightarrow G^{\dagger}"] \\
 & \mathsf{Sets} &
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{C}^{\mathsf{op}} \times \mathcal{C} \arrow[rr, "(G \circ F)^{\mathsf{op}} \times (G \circ F)"] \arrow[dr, "\mathrm{Hom}_{\mathcal{C}}"'] & \arrow[d, phantom, "\Rightarrow (G \circ F)^{\dagger}"] & \mathcal{E}^{\mathsf{op}} \times \mathcal{E}, \arrow[dl, "\mathrm{Hom}_{\mathcal{E}}"] \\
 & \mathsf{Sets} &
\end{tikzcd}$$

in $\mathsf{Cats}_2$, i.e. we have

$$(G \circ F)^{\dagger} = \left(G^{\dagger} \star \mathrm{id}_{F^{\mathsf{op}} \times F}\right) \circ F^{\dagger}.$$

(3) *Interaction With Identities.* We have

$$\mathrm{id}_{\mathcal{C}}^{\dagger} = \mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(-_1, -_2)},$$

i.e. the natural transformation associated to $\mathrm{id}_{\mathcal{C}}$ is the identity natural transformation of the functor $\mathrm{Hom}_{\mathcal{C}}(-_1, -_2)$.

**Proof.** *Item 1*, *Interaction With Natural Isomorphisms*: Clear.
*Item 2*, *Interaction With Composition*: Clear.
*Item 3*, *Interaction With Identities*: Clear. □

## 11.5. Natural Transformations

**11.5.1. Foundations.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories and $F, G\colon \mathcal{C} \rightrightarrows \mathcal{D}$ be functors.

**Definition 11.5.1.1.1.** A **transformation**[27,28] $\alpha\colon F \overset{\text{unnat}}{\Longrightarrow} G$ **from** $F$ **to** $G$ is a collection

$$\{\alpha_A\colon F(A) \to G(A)\}_{A \in \mathrm{Obj}(\mathcal{C})}$$

of morphisms of $\mathcal{D}$.

[27] *Further Terminology:* Also called an **unnatural transformation** for emphasis.
[28] *Further Notation:* We write $\mathrm{UnNat}(F, G)$ for the set of unnatural transformations from $F$ to $G$.

**Definition 11.5.1.1.2.** A **natural transformation**[29] $\alpha\colon F \Longrightarrow G$ **from** $F$ **to** $G$ is a transformation

$$\{\alpha_A\colon F(A) \to G(A)\}_{A\in\mathrm{Obj}(\mathcal{C})}$$

from $F$ to $G$ such that, for each morphism $f\colon A \to B$ of $\mathcal{C}$, the diagram

$$\begin{tikzcd}
F(A) \arrow[r, "F(f)"] \arrow[d, "\alpha_A"'] & F(B) \arrow[d, "\alpha_B"] \\
G(A) \arrow[r, "G(f)"'] & G(B)
\end{tikzcd}$$

commutes.[30,31]

**Example 11.5.1.1.3.** The **identity natural transformation** $\mathrm{id}_F\colon F \Longrightarrow F$ **of** $F$ is the natural transformation consisting of the collection

$$\left\{\mathrm{id}_{F(A)}\colon F(A) \to F(A)\right\}_{A\in\mathrm{Obj}(\mathcal{C})}.$$

**Proof.** The naturality condition for $\mathrm{id}_F$ is the requirement that, for each morphism $f\colon A \to B$ of $\mathcal{C}$, the diagram

$$\begin{tikzcd}
F(A) \arrow[r, "F(f)"] \arrow[d, "\mathrm{id}_{F(A)}"'] & F(B) \arrow[d, "\mathrm{id}_{F(B)}"] \\
F(A) \arrow[r, "F(f)"'] & F(B)
\end{tikzcd}$$

commutes, which follows from unitality of the composition of $\mathcal{C}$. □

**Definition 11.5.1.1.4.** Two natural transformations $\alpha, \beta\colon F \Longrightarrow G$ are **equal** if we have

$$\alpha_A = \beta_A$$

for each $A \in \mathrm{Obj}(\mathcal{C})$.

### 11.5.2. Vertical Composition of Natural Transformations.

**Definition 11.5.2.1.1.** The **vertical composition** of two natural transformations $\alpha\colon F \Longrightarrow G$ and $\beta\colon G \Longrightarrow H$ as in the diagram

$$\begin{tikzcd}[row sep=large]
\mathcal{C} \arrow[r, bend left=60, "F", ""{name=U, below}] \arrow[r, "G" description, ""{name=M}] \arrow[r, bend right=60, "H"', ""{name=D, above}] & \mathcal{D}
\arrow[Rightarrow, from=U, to=M, "\alpha"']
\arrow[Rightarrow, from=M, to=D, "\beta"']
\end{tikzcd}$$

---

[29]Pictured in diagrams as

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "F", ""{name=U, below}] \arrow[r, bend right, "G"', ""{name=D, above}] & \mathcal{D}.
\arrow[Rightarrow, from=U, to=D, "\alpha"']
\end{tikzcd}$$

[30]*Further Terminology:* The morphism $\alpha_A\colon F_A \to G_A$ is called the **component of** $\alpha$ **at** $A$.

[31]*Further Notation:* We write $\mathrm{Nat}(F, G)$ for the set of natural transformations from $F$ to $G$.

is the natural transformation $\beta \circ \alpha \colon F \Longrightarrow H$ consisting of the collection

$$\{(\beta \circ \alpha)_A \colon F(A) \to H(A)\}_{A \in \mathrm{Obj}(\mathcal{C})}$$

with

$$(\beta \circ \alpha)_A \stackrel{\mathrm{def}}{=} \beta_A \circ \alpha_A$$

for each $A \in \mathrm{Obj}(\mathcal{C})$.

**Proof.** The naturality condition for $\beta \circ \alpha$ is the requirement that the boundary of the diagram

$$\begin{tikzcd}
F(A) \arrow[r, "F(f)"] \arrow[d, "\alpha_A"'] \arrow[dr, phantom, "(1)"] & F(B) \arrow[d, "\alpha_B"] \\
G(A) \arrow[r, "G(f)"] \arrow[d, "\beta_A"'] \arrow[dr, phantom, "(2)"] & G(B) \arrow[d, "\beta_B"] \\
H(A) \arrow[r, "H(f)"'] & H(B)
\end{tikzcd}$$

commutes. Since

(1) Subdiagram (1) commutes by the naturality of $\alpha$;
(2) Subdiagram (2) commutes by the naturality of $\beta$;

so does the boundary diagram. Hence $\beta \circ \alpha$ is a natural transformation. $\square$

**Proposition 11.5.2.1.2.** Let $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ be categories.

(1) *Functionality.* The assignment $(\beta, \alpha) \mapsto \beta \circ \alpha$ defines a function
$$\circ_{F,G,H} \colon \mathrm{Nat}(G, H) \times \mathrm{Nat}(F, G) \to \mathrm{Nat}(F, H).$$

(2) *Associativity.* Let $F, G, H, K \colon \mathcal{C} \overset{\rightrightarrows}{\rightrightarrows} \mathcal{D}$ be functors. The diagram

$$\begin{tikzcd}
& \mathrm{Nat}(H,K) \times (\mathrm{Nat}(G,H) \times \mathrm{Nat}(F,G)) \arrow[dr, "\mathrm{id}_{\mathrm{Nat}(H,K)} \times \circ_{F,G,H}"] & \\
(\mathrm{Nat}(H,K) \times \mathrm{Nat}(G,H)) \times \mathrm{Nat}(F,G) \arrow[ur, dashed, "\alpha^{\mathsf{Sets}}_{\mathrm{Nat}(H,K),\mathrm{Nat}(G,H),\mathrm{Nat}(F,G)}", "\sim"'] \arrow[d, "\circ_{G,H,K} \times \mathrm{id}_{\mathrm{Nat}(F,G)}"'] & & \mathrm{Nat}(H,K) \times \mathrm{Nat}(F,H) \arrow[d, "\circ_{F,H,K}"] \\
\mathrm{Nat}(G,K) \times \mathrm{Nat}(F,G) \arrow[rr, "\circ_{F,G,K}"'] & & \mathrm{Nat}(F,K)
\end{tikzcd}$$

commutes, i.e. given natural transformations

$$\begin{aligned}\alpha\colon F &\Longrightarrow G,\\ \beta\colon G &\Longrightarrow H,\\ \gamma\colon H &\Longrightarrow K,\end{aligned}$$

we have

$$(\gamma \circ \beta) \circ \alpha = \gamma \circ (\beta \circ \alpha).$$

(3) *Unitality.* Let $F, G\colon \mathcal{C} \rightrightarrows \mathcal{D}$ be functors.

(a) *Left Unitality.* The diagram

$$\begin{tikzcd}
\mathrm{pt} \times \mathrm{Nat}(F,G) \arrow[d, "{[\mathrm{id}_G] \times \mathrm{id}_{\mathrm{Nat}(F,G)}}"'] \arrow[dr, dashed, "\lambda^{\mathsf{Sets}}_{\mathrm{Nat}(F,G)}", "\sim"'] & \\
\mathrm{Nat}(G,G) \times \mathrm{Nat}(F,G) \arrow[r, "\circ_{F,G,G}"'] & \mathrm{Nat}(F,G)
\end{tikzcd}$$

commutes, i.e. given a natural transformation $\alpha\colon F \Longrightarrow G$, we have

$$\mathrm{id}_G \circ \alpha = \alpha.$$

(b) *Right Unitality.* The diagram

$$\begin{tikzcd}
\mathrm{Nat}(F,G) \times \mathrm{pt} \arrow[d, "{\mathrm{id}_{\mathrm{Nat}(F,G)} \times [\mathrm{id}_F]}"'] \arrow[dr, dashed, "\rho^{\mathsf{Sets}}_{\mathrm{Nat}(F,G)}", "\sim"'] & \\
\mathrm{Nat}(F,G) \times \mathrm{Nat}(F,F) \arrow[r, "\circ^{\mathcal{C}}_{F,F,G}"'] & \mathrm{Nat}(F,G)
\end{tikzcd}$$

commutes, i.e. given a natural transformation $\alpha\colon F \Longrightarrow G$, we have

$$\alpha \circ \mathrm{id}_F = \alpha.$$

(4) *Middle Four Exchange.* Let $F_1, F_2, F_3\colon \mathcal{C} \to \mathcal{D}$ and $G_1, G_2, G_3\colon \mathcal{D} \to \mathcal{E}$ be functors. The diagram

$$\begin{tikzcd}
(\mathrm{Nat}(G_2,G_3) \times \mathrm{Nat}(G_1,G_2)) \times (\mathrm{Nat}(F_2,F_3) \times \mathrm{Nat}(F_1,F_2)) \arrow[rr, dashed, leftrightarrow, "\mu_4", "\sim"'] \arrow[d, "\circ_{G_1,G_2,G_3} \times \circ_{F_1,F_2,F_3}"'] & & (\mathrm{Nat}(G_2,G_3) \times \mathrm{Nat}(F_2,F_3)) \times (\mathrm{Nat}(G_1,G_2) \times \mathrm{Nat}(F_1,F_2)) \arrow[d, "\star_{F_2,F_3,G_2,G_3} \times \star_{F_1,F_2,G_1,G_2}"] \\
\mathrm{Nat}(G_1,G_3) \times \mathrm{Nat}(F_1,F_3) \arrow[dr, "\star_{F_1,F_3,G_1,G_3}"'] & & \mathrm{Nat}(G_2 \circ F_2, G_3 \circ F_3) \times \mathrm{Nat}(G_1 \circ F_1, G_2 \circ F_2) \arrow[dl, "\circ_{G_1 \circ F_1, G_2 \circ F_2, G_3 \circ F_3}"] \\
& \mathrm{Nat}(G_1 \circ F_1, G_3 \circ F_3) &
\end{tikzcd}$$

commutes, i.e. given a diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=60, "F_1"{name=U}] \arrow[r, "F_2"{name=M, description}] \arrow[r, bend right=60, "F_3"'{name=L}] & \mathcal{D} \arrow[r, bend left=60, "G_1"{name=U2}] \arrow[r, "G_2"{name=M2, description}] \arrow[r, bend right=60, "G_3"'{name=L2}] & \mathcal{E}
\arrow[from=U, to=M, Rightarrow, "\alpha"']
\arrow[from=M, to=L, Rightarrow, "\alpha'"']
\arrow[from=U2, to=M2, Rightarrow, "\beta"']
\arrow[from=M2, to=L2, Rightarrow, "\beta'"']
\end{tikzcd}$$

in $\mathsf{Cats}_2$, we have

$$(\beta' \star \alpha') \circ (\beta \star \alpha) = (\beta' \circ \beta) \star (\alpha' \circ \alpha).$$

**Proof.** *Item 1, Functionality*: Clear.
*Item 2, Associativity*: Indeed, we have

$$\begin{aligned}((\gamma \circ \beta) \circ \alpha)_A &= (\gamma_A \circ \beta_A) \circ \alpha_A \\ &= \gamma_A \circ (\beta_A \circ \alpha_A) \\ &= (\gamma \circ (\beta \circ \alpha))_A\end{aligned}$$

for each $A \in \mathrm{Obj}(\mathcal{C})$, showing the desired equality.
*Item 3, Unitality*: We have

$$\begin{aligned}(\mathrm{id}_G \circ \alpha)_A &= \mathrm{id}_G \circ \alpha_A \\ &= \alpha_A, \\ (\alpha \circ \mathrm{id}_F)_A &= \alpha_A \circ \mathrm{id}_F \\ &= \alpha_A\end{aligned}$$

for each $A \in \mathrm{Obj}(\mathcal{C})$, showing the desired equality.
*Item 4, Middle Four Exchange*: This is proved in Item 4 of Proposition 11.5.3.1.2. ☐

### 11.5.3. Horizontal Composition of Natural Transformations.

**Definition 11.5.3.1.1.** The **horizontal composition**[32,33] of two natural transformations $\alpha\colon F \Longrightarrow G$ and $\beta\colon H \Longrightarrow K$ as in the diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{D} \arrow[r, bend left, "H"] \arrow[r, bend right, "K"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{E} \end{tikzcd}$$

of $\alpha$ and $\beta$ is the natural transformation

$$\beta \star \alpha\colon (H \circ F) \Longrightarrow (K \circ G),$$

as in the diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "H \circ F"] \arrow[r, bend right, "K \circ G"'] \arrow[r, phantom, "\Downarrow \beta\star\alpha"] & \mathcal{E}, \end{tikzcd}$$

consisting of the collection

$$\{(\beta \star \alpha)_A\colon H(F(A)) \to K(G(A))\}_{A \in \mathrm{Obj}(\mathcal{C})},$$

---

[32] *Further Terminology:* Also called the **Godement product** of $\alpha$ and $\beta$.
[33] Horizontal composition forms a map

$$\star_{(F,H),(G,K)}\colon \mathrm{Nat}(H,K) \times \mathrm{Nat}(F,G) \to \mathrm{Nat}(H \circ F, K \circ G).$$

of morphisms of $\mathcal{E}$ with

$$(\beta \star \alpha)_A \stackrel{\text{def}}{=} \beta_{G(A)} \circ H(\alpha_A) = K(\alpha_A) \circ \beta_{F(A)}, \qquad \begin{tikzcd} H(F(A)) \arrow[r, "H(\alpha_A)"] \arrow[d, "\beta_{F(A)}"'] & H(G(A)) \arrow[d, "\beta_{G(A)}"] \\ K(F(A)) \arrow[r, "K(\alpha_A)"'] & K(G(A)). \end{tikzcd}$$

**Proof.** First, we claim that we indeed have

$$\beta_{G(A)} \circ H(\alpha_A) = K(\alpha_A) \circ \beta_{F(A)}, \qquad \begin{tikzcd} H(F(A)) \arrow[r, "H(\alpha_A)"] \arrow[d, "\beta_{F(A)}"'] & H(G(A)) \arrow[d, "\beta_{G(A)}"] \\ K(F(A)) \arrow[r, "K(\alpha_A)"'] & K(G(A)). \end{tikzcd}$$

This is, however, simply the naturality square for $\beta$ applied to the morphism $\alpha_A \colon F(A) \to G(A)$. Next, we check the naturality condition for $\beta \star \alpha$, which is the requirement that the boundary of the diagram

$$\begin{tikzcd} H(F(A)) \arrow[r, "H(F(f))"] \arrow[d, "H(\alpha_A)"'] \arrow[dr, phantom, "(1)"] & H(F(B)) \arrow[d, "H(\alpha_B)"] \\ H(G(A)) \arrow[r, "H(G(f))" description] \arrow[d, "\beta_{G(A)}"'] \arrow[dr, phantom, "(2)"] & H(G(B)) \arrow[d, "\beta_{G(B)}"] \\ K(G(A)) \arrow[r, "K(G(f))"'] & K(G(B)) \end{tikzcd}$$

commutes. Since

(1) Subdiagram (1) commutes by the naturality of $\alpha$;
(2) Subdiagram (2) commutes by the naturality of $\beta$;

so does the boundary diagram. Hence $\beta \circ \alpha$ is a natural transformation.[34] □

**Proposition 11.5.3.1.2.** Let $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ be categories.

(1) *Functionality.* The assignment $(\beta, \alpha) \mapsto \beta \star \alpha$ defines a function

$$\star_{(F,G),(H,K)} \colon \mathrm{Nat}(H, K) \times \mathrm{Nat}(F, G) \to \mathrm{Nat}(H \circ F, K \circ G).$$

(2) *Associativity.* Let

$$\mathcal{C} \underset{G_1}{\overset{F_1}{\rightrightarrows}} \mathcal{D} \underset{G_2}{\overset{F_2}{\rightrightarrows}} \mathcal{E} \underset{G_3}{\overset{F_3}{\rightrightarrows}} \mathcal{F}$$

[34] *Reference:* [Bor94, Proposition 1.3.4].

be a diagram in $\mathsf{Cats}_2$. The diagram

$$\begin{tikzcd}
\mathrm{Nat}(F_3,G_3)\times\mathrm{Nat}(F_2,G_2)\times\mathrm{Nat}(F_1,G_1) \arrow[r, "\star_{(F_2,G_2),(F_3,G_3)}\times\mathsf{id}"] \arrow[d, "\mathsf{id}\times\star_{(F_1,G_1),(F_2,G_2)}"'] & \mathrm{Nat}(F_3\circ F_2, G_3\circ G_2)\times\mathrm{Nat}(F_1,G_1) \arrow[d, "\star_{(F_3\circ F_2),(G_3\circ G_2,F_1,G_1)}"] \\
\mathrm{Nat}(F_3,G_3)\times\mathrm{Nat}(F_2\circ F_1, G_2\circ G_1) \arrow[r, "\star_{(F_2\circ F_1),(G_2\circ G_1,F_3,G_3)}"'] & \mathrm{Nat}(F_3\circ F_2\circ F_1, G_3\circ G_2\circ G_1)
\end{tikzcd}$$

commutes, i.e. given natural transformations

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "F_1"{name=A}] \arrow[r, bend right, "G_1"'{name=B}] \arrow[from=A, to=B, phantom, "\alpha\Downarrow"] & \mathcal{D} \arrow[r, bend left, "F_2"{name=C}] \arrow[r, bend right, "G_2"'{name=D}] \arrow[from=C, to=D, phantom, "\beta\Downarrow"] & \mathcal{E} \arrow[r, bend left, "F_3"{name=E}] \arrow[r, bend right, "G_3"'{name=F}] \arrow[from=E, to=F, phantom, "\gamma\Downarrow"] & \mathcal{F},
\end{tikzcd}$$

we have

$$(\gamma\star\beta)\star\alpha = \gamma\star(\beta\star\alpha).$$

(3) *Interaction With Identities.* Let $F\colon\mathcal{C}\to\mathcal{D}$ and $G\colon\mathcal{D}\to\mathcal{E}$ be functors. The diagram

$$\begin{tikzcd}
\mathrm{pt}\times\mathrm{pt} \arrow[r, "{[\mathrm{id}_G]\times[\mathrm{id}_F]}"] & \mathrm{Nat}(G,G)\times\mathrm{Nat}(F,F) \arrow[d, "\star_{(F,F),(G,G)}"] \\
\mathrm{pt} \arrow[u, dashed, "\wr"] \arrow[r, "{[\mathrm{id}_{G\circ F}]}"'] & \mathrm{Nat}(G\circ F, G\circ F)
\end{tikzcd}$$

commutes, i.e. we have

$$\mathrm{id}_G\star\mathrm{id}_F = \mathrm{id}_{G\circ F}.$$

(4) *Middle Four Exchange.* Let $F_1,F_2,F_3\colon\mathcal{C}\to\mathcal{D}$ and $G_1,G_2,G_3\colon\mathcal{D}\to\mathcal{E}$ be functors. The diagram

$$\begin{tikzcd}
(\mathrm{Nat}(G_2,G_3)\times\mathrm{Nat}(G_1,G_2))\times(\mathrm{Nat}(F_2,F_3)\times\mathrm{Nat}(F_1,F_2)) \arrow[rr, dashed, leftrightarrow, "\mu_4", "\sim"'] \arrow[d, "\circ_{G_1,G_2,G_3}\times\circ_{F_1,F_2,F_3}"'] & & (\mathrm{Nat}(G_2,G_3)\times\mathrm{Nat}(F_2,F_3))\times(\mathrm{Nat}(G_1,G_2)\times\mathrm{Nat}(F_1,F_2)) \arrow[d, "\star_{F_2,F_3,G_2,G_3}\times\star_{F_1,F_2,G_1,G_2}"] \\
\mathrm{Nat}(G_1,G_3)\times\mathrm{Nat}(F_1,F_3) \arrow[dr, "\star_{F_1,F_3,G_1,G_3}"'] & & \mathrm{Nat}(G_2\circ F_2, G_3\circ F_3)\times\mathrm{Nat}(G_1\circ F_1, G_2\circ F_2) \arrow[dl, "\circ_{G_1\circ F_1,G_2\circ F_2,G_3\circ F_3}"] \\
& \mathrm{Nat}(G_1\circ F_1, G_3\circ F_3) &
\end{tikzcd}$$

commutes, i.e. given a diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=60, "F_1"{name=A}] \arrow[r, "F_2"{name=B, description}] \arrow[r, bend right=60, "F_3"'{name=C}] \arrow[from=A, to=B, phantom, "\alpha\Downarrow"] \arrow[from=B, to=C, phantom, "\alpha'\Downarrow"] & \mathcal{D} \arrow[r, bend left=60, "G_1"{name=D}] \arrow[r, "G_2"{name=E, description}] \arrow[r, bend right=60, "G_3"'{name=F}] \arrow[from=D, to=E, phantom, "\beta\Downarrow"] \arrow[from=E, to=F, phantom, "\beta'\Downarrow"] & \mathcal{E}
\end{tikzcd}$$

in $\mathsf{Cats}_2$, we have

$$(\beta'\star\alpha')\circ(\beta\star\alpha) = (\beta'\circ\beta)\star(\alpha'\circ\alpha).$$

**Proof.** *Item 1, Functionality*: Clear.
*Item 2, Associativity*: Omitted.

*Item 3, Interaction With Identities*: We have

$$\begin{aligned}(\mathrm{id}_G \star \mathrm{id}_F)_A &\overset{\mathrm{def}}{=} (\mathrm{id}_G)_{F_A} \circ G_{(\mathrm{id}_F)_A}\\ &\overset{\mathrm{def}}{=} \mathrm{id}_{G_{F_A}} \circ G_{\mathrm{id}_{F_A}}\\ &= \mathrm{id}_{G_{F_A}} \circ \mathrm{id}_{G_{F_A}}\\ &= \mathrm{id}_{G_{F_A}}\\ &\overset{\mathrm{def}}{=} (\mathrm{id}_{G\circ F})_A\end{aligned}$$

for each $A \in \mathrm{Obj}(\mathcal{C})$, showing the desired equality.
*Item 4, Middle Four Exchange*: Let $A \in \mathrm{Obj}(\mathcal{C})$ and consider the diagram

$$\begin{tikzcd}
 & & G_{F''_A} \arrow[dr, "\beta_{F''_A}"] & & \\
G_{F_A} \arrow[r, "G_{\alpha_A}"] & G_{F'_A} \arrow[ur, "G_{\alpha'_A}"] \arrow[dr, "\beta_{F'_A}"'] \arrow[rr, phantom, "(1)"] & & G'_{F''_A} \arrow[r, "\beta'_{F''_A}"] & G''_{F''_A}. \\
 & & G'_{F'_A} \arrow[ur, "G'_{\alpha'_A}"'] & &
\end{tikzcd}$$

The top composition is $((\beta' \circ \beta) \star (\alpha' \circ \alpha))_A$ and the bottom composition is $((\beta' \star \alpha') \circ (\beta \star \alpha))_A$. Since Subdiagram (1) commutes, they are equal. □

### 11.5.4. Properties of Natural Transformations.

**Proposition 11.5.4.1.1.** Let $F, G\colon \mathcal{C} \rightrightarrows \mathcal{D}$ be functors. The following data are equivalent:[35]

(1) A natural transformation $\alpha\colon F \Longrightarrow G$.
(2) A functor $[\alpha]\colon \mathcal{C} \to \mathcal{D}^{\Vdash}$ filling the diagram

$$\begin{tikzcd}
 & \mathcal{D} \\
\mathcal{C} \arrow[ur, "F"] \arrow[r, "{[\alpha]}"] \arrow[dr, "G"'] & \mathcal{D}^{\Vdash}. \arrow[u, "\mathrm{ev}_0"'] \arrow[d, "\mathrm{ev}_1"] \\
 & \mathcal{D}
\end{tikzcd}$$

(3) A functor $[\alpha]\colon \mathcal{C} \times \Vdash \to \mathcal{D}$ filling the diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[dr, "F"] & \\
\mathcal{C} \times \Vdash \arrow[u, "\mathrm{ev}_0"] \arrow[d, "\mathrm{ev}_1"'] \arrow[r, "{[\alpha]}"] & \mathcal{D}. \\
\mathcal{C} \arrow[ur, "G"'] &
\end{tikzcd}$$

[35]Taken from [MO 64365].

**Proof.** *From Item 1 to Item 2 and Back*: We may identify $\mathcal{D}^{\nVdash}$ with $\mathsf{Arr}(\mathcal{D})$. Given a natural transformation $\alpha\colon F \Longrightarrow G$, we have a functor

$$
\begin{aligned}
[\alpha]\colon \mathcal{C} &\longrightarrow \mathcal{D}^{\nVdash} \\
A &\longmapsto \alpha_A \\
(f\colon A \to B) &\longmapsto \left(\begin{tikzcd} F_A \arrow[r, "F_f"] \arrow[d, "\alpha_A"'] & F_B \arrow[d, "\alpha_B"] \\ G_A \arrow[r, "G_f"'] & G_B \end{tikzcd}\right)
\end{aligned}
$$

making the diagram in Item 2 commute. Conversely, every such functor gives rise to a natural transformation from $F$ to $G$, and these constructions are inverse to each other.

*From Item 2 to Item 3 and Back*: This follows from Item 3 of Proposition 11.6.1.1.2. $\square$

### 11.5.5. Natural Isomorphisms.

**Definition 11.5.5.1.1.** A natural transformation $\alpha\colon F \Longrightarrow G$ between functors $F, G\colon \mathcal{C} \to \mathcal{D}$ between categories $\mathcal{C}$ and $\mathcal{D}$ is a **natural isomorphism** if there exists a natural transformation $\alpha^{-1}\colon G \Longrightarrow F$ such that

$$
\begin{aligned}
\alpha^{-1} \circ \alpha &= \mathrm{id}_F, \\
\alpha \circ \alpha^{-1} &= \mathrm{id}_G.
\end{aligned}
$$

**Proposition 11.5.5.1.2.** Let $\alpha\colon F \Longrightarrow G$ be a natural transformation.

(1) *Characterisations.* The following conditions are equivalent:
   (a) The natural transformation $\alpha$ is a natural isomorphism.
   (b) For each $A \in \mathrm{Obj}(\mathcal{C})$, the morphism $\alpha_A\colon F_A \to G_A$ is an isomorphism.

(2) *Componentwise Inverses of Natural Transformations Assemble Into Natural Transformations.* Let $\alpha^{-1}\colon G \Longrightarrow F$ be a transformation such that, for each $A \in \mathrm{Obj}(\mathcal{C})$, we have

$$
\begin{aligned}
\alpha_A^{-1} \circ \alpha_A &= \mathrm{id}_{F(A)}, \\
\alpha_A \circ \alpha_A^{-1} &= \mathrm{id}_{G(A)}.
\end{aligned}
$$

Then $\alpha^{-1}$ is a natural transformation.

**Proof.** *Item 1, Characterisations*: The implication Item 1a $\Longrightarrow$ Item 1b is clear, whereas the implication Item 1b $\Longrightarrow$ Item 1a follows from Item 2.

*Item 2, Componentwise Inverses of Natural Transformations Assemble Into Natural Transformations*: The naturality condition for $\alpha^{-1}$ corresponds to the commutativity of the diagram

$$
\begin{tikzcd}
G(A) \arrow[r, "G(f)"] \arrow[d, "\alpha_A^{-1}"'] & G(B) \arrow[d, "\alpha_B^{-1}"] \\
F(A) \arrow[r, "F(f)"'] & F(B)
\end{tikzcd}
$$

for each $A, B \in \mathrm{Obj}(\mathcal{C})$ and each $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$. Considering the diagram

$$\begin{tikzcd}
G(A) \arrow[r, "G(f)"] \arrow[d, "\alpha_A^{-1}"'] \arrow[dr, phantom, "(1)"] & G(B) \arrow[d, "\alpha_B^{-1}"] \\
F(A) \arrow[r, "F(f)"] \arrow[d, "\alpha_A"'] \arrow[dr, phantom, "(2)"] & F(B) \arrow[d, "\alpha_B"] \\
G(A) \arrow[r, "G(f)"'] & G(B),
\end{tikzcd}$$

where the boundary diagram as well as Subdiagram (2) commute, we have

$$\begin{aligned}
G(f) &= G(f) \circ \mathrm{id}_{G(A)} \\
&= G(f) \circ \alpha_A \circ \alpha_A^{-1} \\
&= \alpha_B \circ F(f) \circ \alpha_A^{-1}.
\end{aligned}$$

Postcomposing both sides with $\alpha_B^{-1}$, we get

$$\begin{aligned}
\alpha_B^{-1} \circ G(f) &= \alpha_B^{-1} \circ \alpha_B \circ F(f) \circ \alpha_A^{-1} \\
&= \mathrm{id}_{F(B)} \circ F(f) \circ \alpha_A^{-1} \\
&= F(f) \circ \alpha_A^{-1},
\end{aligned}$$

which is the naturality condition we wanted to show. Thus $\alpha^{-1}$ is a natural transformation. □

## 11.6. Categories of Categories

**11.6.1. Functor Categories.** Let $\mathcal{C}$ be a category and $\mathcal{D}$ be a small category.

**Definition 11.6.1.1.1.** The **category of functors from $\mathcal{C}$ to $\mathcal{D}$**[36] is the category $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$[37] where

- *Objects.* The objects of $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$ are functors from $\mathcal{C}$ to $\mathcal{D}$;
- *Morphisms.* For each $F, G \in \mathrm{Obj}(\mathsf{Fun}(\mathcal{C}, \mathcal{D}))$, we have
$$\mathrm{Hom}_{\mathsf{Fun}(\mathcal{C},\mathcal{D})}(F, G) \stackrel{\text{def}}{=} \mathrm{Nat}(F, G);$$
- *Identities.* For each $F \in \mathrm{Obj}(\mathsf{Fun}(\mathcal{C}, \mathcal{D}))$, the unit map
$$\mathbb{1}_F^{\mathsf{Fun}(\mathcal{C},\mathcal{D})} \colon \mathrm{pt} \to \mathrm{Nat}(F, F)$$
of $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$ at $F$ is given by
$$\mathrm{id}_F^{\mathsf{Fun}(\mathcal{C},\mathcal{D})} \stackrel{\text{def}}{=} \mathrm{id}_F,$$
where $\mathrm{id}_F \colon F \Longrightarrow F$ is the identity natural transformation of $F$ of Example 11.5.1.1.3;

[36] *Further Terminology:* Also called the **functor category** $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$.
[37] *Further Notation:* Also written $\mathcal{D}^{\mathcal{C}}$ and $[\mathcal{C}, \mathcal{D}]$.

- *Composition.* For each $F, G, H \in \mathrm{Obj}(\mathsf{Fun}(\mathcal{C}, \mathcal{D}))$, the composition map
$$\circ^{\mathsf{Fun}(\mathcal{C},\mathcal{D})}_{F,G,H} \colon \mathrm{Nat}(G, H) \times \mathrm{Nat}(F, G) \to \mathrm{Nat}(F, H)$$
of $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$ at $(F, G, H)$ is given by
$$\beta \circ^{\mathsf{Fun}(\mathcal{C},\mathcal{D})}_{F,G,H} \alpha \overset{\text{def}}{=} \beta \circ \alpha,$$
where $\beta \circ \alpha$ is the vertical composition of $\alpha$ and $\beta$ of Item 1 of Proposition 11.5.2.1.2.

**Proposition 11.6.1.1.2.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories and let $F \colon \mathcal{C} \to \mathcal{D}$ be a functor.

(1) *Functoriality.* The assignments $\mathcal{C}, \mathcal{D}, (\mathcal{C}, \mathcal{D}) \mapsto \mathsf{Fun}(\mathcal{C}, \mathcal{D})$ define functors
$$\begin{aligned} \mathsf{Fun}(\mathcal{C}, -_2) &\colon \mathsf{Cats} \to \mathsf{Cats}, \\ \mathsf{Fun}(-_1, \mathcal{D}) &\colon \mathsf{Cats}^{\mathsf{op}} \to \mathsf{Cats}, \\ \mathsf{Fun}(-_1, -_2) &\colon \mathsf{Cats}^{\mathsf{op}} \times \mathsf{Cats} \to \mathsf{Cats}. \end{aligned}$$

(2) *2-Functoriality.* The assignments $\mathcal{C}, \mathcal{D}, (\mathcal{C}, \mathcal{D}) \mapsto \mathsf{Fun}(\mathcal{C}, \mathcal{D})$ define 2-functors
$$\begin{aligned} \mathsf{Fun}(\mathcal{C}, -_2) &\colon \mathsf{Cats}_2 \to \mathsf{Cats}_2, \\ \mathsf{Fun}(-_1, \mathcal{D}) &\colon \mathsf{Cats}_2^{\mathsf{op}} \to \mathsf{Cats}_2, \\ \mathsf{Fun}(-_1, -_2) &\colon \mathsf{Cats}_2^{\mathsf{op}} \times \mathsf{Cats}_2 \to \mathsf{Cats}_2. \end{aligned}$$

(3) *Adjointness.* We have adjunctions
$$(\mathcal{C} \times - \dashv \mathsf{Fun}(\mathcal{C}, -)) \colon \quad \begin{tikzcd} \mathsf{Cats} \arrow[r, bend left, "\mathcal{C}\times-"] \arrow[r, phantom, "\perp"] & \mathsf{Cats}, \arrow[l, bend left, "\mathsf{Fun}(\mathcal{C},-)"] \end{tikzcd}$$
$$(- \times \mathcal{D} \dashv \mathsf{Fun}(\mathcal{D}, -)) \colon \quad \begin{tikzcd} \mathsf{Cats} \arrow[r, bend left, "-\times\mathcal{D}"] \arrow[r, phantom, "\perp"] & \mathsf{Cats}, \arrow[l, bend left, "\mathsf{Fun}(\mathcal{D},-)"] \end{tikzcd}$$
witnessed by bijections of sets
$$\begin{aligned} \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) &\cong \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{D}, \mathsf{Fun}(\mathcal{C}, \mathcal{E})), \\ \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) &\cong \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, \mathsf{Fun}(\mathcal{D}, \mathcal{E})), \end{aligned}$$
natural in $\mathcal{C}, \mathcal{D}, \mathcal{E} \in \mathrm{Obj}(\mathsf{Cats})$.

(4) *2-Adjointness.* We have 2-adjunctions
$$(\mathcal{C} \times - \dashv \mathsf{Fun}(\mathcal{C}, -)) \colon \quad \begin{tikzcd} \mathsf{Cats}_2 \arrow[r, bend left, "\mathcal{C}\times-"] \arrow[r, phantom, "\perp_2"] & \mathsf{Cats}_2, \arrow[l, bend left, "\mathsf{Fun}(\mathcal{C},-)"] \end{tikzcd}$$
$$(- \times \mathcal{D} \dashv \mathsf{Fun}(\mathcal{D}, -)) \colon \quad \begin{tikzcd} \mathsf{Cats}_2 \arrow[r, bend left, "-\times\mathcal{D}"] \arrow[r, phantom, "\perp_2"] & \mathsf{Cats}_2, \arrow[l, bend left, "\mathsf{Fun}(\mathcal{D},-)"] \end{tikzcd}$$

witnessed by isomorphisms of categories

$$\mathsf{Fun}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) \cong \mathsf{Fun}(\mathcal{D}, \mathsf{Fun}(\mathcal{C}, \mathcal{E})),$$
$$\mathsf{Fun}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) \cong \mathsf{Fun}(\mathcal{C}, \mathsf{Fun}(\mathcal{D}, \mathcal{E})),$$

natural in $\mathcal{C}, \mathcal{D}, \mathcal{E} \in \mathrm{Obj}(\mathsf{Cats}_2)$.

(5) *Trivial Functor Categories.* We have a canonical isomorphism of categories

$$\mathsf{Fun}(\mathsf{pt}, \mathcal{C}) \cong \mathcal{C},$$

natural in $\mathcal{C} \in \mathrm{Obj}(\mathsf{Cats})$.

(6) *Objectwise Computation of Co/Limits.* Let

$$D \colon \mathcal{I} \to \mathsf{Fun}(\mathcal{C}, \mathcal{D})$$

be a diagram in $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$. We have isomorphisms

$$\lim(D)_A \cong \lim_{i \in I}(D_i(A)),$$
$$\operatorname{colim}(D)_A \cong \operatorname*{colim}_{i \in I}(D_i(A)),$$

naturally in $A \in \mathrm{Obj}(\mathcal{C})$.

(7) *Bicompleteness.* If $\mathcal{E}$ is co/complete, then so is $\mathsf{Fun}(\mathcal{C}, \mathcal{E})$.

(8) *Abelianness.* If $\mathcal{E}$ is abelian, then so is $\mathsf{Fun}(\mathcal{C}, \mathcal{E})$.

(9) *Monomorphisms and Epimorphisms.* Let $\alpha \colon F \Longrightarrow G$ be a morphism of $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$. The following conditions are equivalent:

(a) The natural transformation

$$\alpha \colon F \Longrightarrow G$$

is a monomorphism (resp. epimorphism) in $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$.

(b) For each $A \in \mathrm{Obj}(\mathcal{C})$, the morphism

$$\alpha_A \colon F_A \to G_A$$

is a monomorphism (resp. epimorphism) in $\mathcal{D}$.

**Proof.** *Item 1, Functoriality*: Omitted.
*Item 2, 2-Functoriality*: Omitted.
*Item 3, Adjointness*: Omitted.
*Item 4, 2-Adjointness*: Omitted.
*Item 5, Trivial Functor Categories*: Omitted.
*Item 6, Objectwise Computation of Co/Limits*: Omitted.
*Item 7, Bicompleteness*: This follows from **??**.
*Item 8, Abelianness*: Omitted.
*Item 9, Monomorphisms and Epimorphisms*: Omitted. □

### 11.6.2. The Category of Categories and Functors.

**Definition 11.6.2.1.1.** The **category of (small) categories and functors** is the category $\mathsf{Cats}$ where

- *Objects.* The objects of $\mathsf{Cats}$ are small categories;
- *Morphisms.* For each $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$, we have

$$\mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, \mathcal{D}) \stackrel{\text{def}}{=} \mathrm{Obj}(\mathsf{Fun}(\mathcal{C}, \mathcal{D}));$$

- *Identities.* For each $\mathcal{C} \in \mathrm{Obj}(\mathsf{Cats})$, the unit map

$$\mathbb{1}^{\mathsf{Cats}}_{\mathcal{C}} \colon \mathrm{pt} \to \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, \mathcal{C})$$

of $\mathsf{Cats}$ at $\mathcal{C}$ is defined by

$$\mathrm{id}_{\mathcal{C}}^{\mathsf{Cats}} \stackrel{\text{def}}{=} \mathrm{id}_{\mathcal{C}},$$

where $\mathrm{id}_{\mathcal{C}} \colon \mathcal{C} \to \mathcal{C}$ is the identity functor of $\mathcal{C}$ of Example 11.4.1.1.2;

- *Composition.* For each $\mathcal{C}, \mathcal{D}, \mathcal{E} \in \mathrm{Obj}(\mathsf{Cats})$, the composition map

  $$\circ_{\mathcal{C},\mathcal{D},\mathcal{E}}^{\mathsf{Cats}} \colon \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{D}, \mathcal{E}) \times \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, \mathcal{D}) \to \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, \mathcal{E})$$

  of $\mathsf{Cats}$ at $(\mathcal{C}, \mathcal{D}, \mathcal{E})$ is given by

  $$G \circ_{\mathcal{C},\mathcal{D},\mathcal{E}}^{\mathsf{Cats}} F \stackrel{\text{def}}{=} G \circ F,$$

  where $G \circ F \colon \mathcal{C} \to \mathcal{E}$ is the composition of $F$ and $G$ of Definition 11.4.1.1.3.

**Proposition 11.6.2.1.2.** Let $\mathcal{C}$ be a category.

(1) *Co/Completeness.* The category $\mathsf{Cats}$ is complete and cocomplete.
(2) *Cartesian Monoidal Structure.* The quadruple $(\mathsf{Cats}, \times, \mathsf{pt}, \mathsf{Fun})$ is a Cartesian closed monoidal category.

**Proof.** *Item 1, Co/Completeness*: This follows from
*Item 2, Cartesian Monoidal Structure*: Omitted. □

### 11.6.3. The 2-Category of Categories, Functors, and Natural Transformations.

**Definition 11.6.3.1.1.** The **2-category of (small) categories, functors, and natural transformations** is the 2-category $\mathsf{Cats}_2$ where

- *Objects.* The objects of $\mathsf{Cats}_2$ are small categories;
- $\mathsf{Hom}$-*Categories.* For each $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats}_2)$, we have

  $$\mathsf{Hom}_{\mathsf{Cats}_2}(\mathcal{C}, \mathcal{D}) \stackrel{\text{def}}{=} \mathsf{Fun}(\mathcal{C}, \mathcal{D});$$

- *Identities.* For each $\mathcal{C} \in \mathrm{Obj}(\mathsf{Cats}_2)$, the unit functor

  $$\mathbb{1}_{\mathcal{C}}^{\mathsf{Cats}_2} \colon \mathsf{pt} \to \mathsf{Fun}(\mathcal{C}, \mathcal{C})$$

  of $\mathsf{Cats}_2$ at $\mathcal{C}$ is the functor picking the identity functor $\mathrm{id}_{\mathcal{C}} \colon \mathcal{C} \to \mathcal{C}$ of $\mathcal{C}$;
- *Composition.* For each $\mathcal{C}, \mathcal{D}, \mathcal{E} \in \mathrm{Obj}(\mathsf{Cats}_2)$, the composition bifunctor

  $$\circ_{\mathcal{C},\mathcal{D},\mathcal{E}}^{\mathsf{Cats}_2} \colon \mathsf{Hom}_{\mathsf{Cats}_2}(\mathcal{D}, \mathcal{E}) \times \mathsf{Hom}_{\mathsf{Cats}_2}(\mathcal{C}, \mathcal{D}) \to \mathsf{Hom}_{\mathsf{Cats}_2}(\mathcal{C}, \mathcal{E})$$

  of $\mathsf{Cats}_2$ at $(\mathcal{C}, \mathcal{D}, \mathcal{E})$ is the functor where
  - *Action on Objects.* For each object $(G, F) \in \mathrm{Obj}(\mathsf{Hom}_{\mathsf{Cats}_2}(\mathcal{D}, \mathcal{E}) \times \mathsf{Hom}_{\mathsf{Cats}_2}(\mathcal{C}, \mathcal{D}))$, we have

    $$\circ_{\mathcal{C},\mathcal{D},\mathcal{E}}^{\mathsf{Cats}_2}(G, F) \stackrel{\text{def}}{=} G \circ F;$$

  - *Action on Morphisms.* For each morphism $(\beta, \alpha) \colon (K, H) \Longrightarrow (G, F)$ of $\mathsf{Hom}_{\mathsf{Cats}_2}(\mathcal{D}, \mathcal{E}) \times \mathsf{Hom}_{\mathsf{Cats}_2}(\mathcal{C}, \mathcal{D})$, we have

    $$\circ_{\mathcal{C},\mathcal{D},\mathcal{E}}^{\mathsf{Cats}_2}(\beta, \alpha) \stackrel{\text{def}}{=} \beta \star \alpha,$$

    where $\beta \star \alpha$ is the horizontal composition of $\alpha$ and $\beta$ of Definition 11.5.3.1.1.

**Proposition 11.6.3.1.2.** Let $\mathcal{C}$ be a category.

(1) *2-Categorical Co/Completeness.* The 2-category $\mathsf{Cats}_2$ is complete and cocomplete as a 2-category, having all 2-categorical and bicategorical co/limits.

**Proof.** *Item 1, Co/Completeness*: This follows from □

### 11.6.4. The Category of Groupoids.

**Definition 11.6.4.1.1.** The **category of (small) groupoids** is the full subcategory $\mathsf{Grpd}$ of $\mathsf{Cats}$ spanned by the groupoids.

### 11.6.5. The 2-Category of Groupoids.

**Definition 11.6.5.1.1.** The **2-category of (small) groupoids** is the full sub-2-category $\mathsf{Grpd}_2$ of $\mathsf{Cats}_2$ spanned by the groupoids.

## 11.7. Miscellany

### 11.7.1. Concrete Categories.

**Definition 11.7.1.1.1.** A category $\mathcal{C}$ is **concrete** if there exists a faithful functor $F\colon \mathcal{C} \to \mathsf{Sets}$.

### 11.7.2. Balanced Categories.

**Definition 11.7.2.1.1.** A category is **balanced** if every morphism which is both a monomorphism and an epimorphism is an isomorphism.

**11.7.3. Monoid Actions on Objects of Categories.** Let $A$ be a monoid, let $\mathcal{C}$ be a category, and let $X \in \mathrm{Obj}(\mathcal{C})$.

**Definition 11.7.3.1.1.** An **$A$-action on $X$** is a functor $\lambda\colon \mathsf{B}A \to \mathcal{C}$ with $\lambda(\star) = X$.

**Remark 11.7.3.1.2.** In detail, an **$A$-action on $X$** is an $A$-action on $\mathsf{End}_{\mathcal{C}}(X)$, consisting of a morphism

$$\lambda\colon A \to \underbrace{\mathsf{End}_{\mathcal{C}}(X)}_{\overset{\text{def}}{=}\mathrm{Hom}_{\mathcal{C}}(X,X)}$$

satisfying the following conditions:

(1) *Preservation of Identities.* We have
$$\lambda_{1_A} = \mathrm{id}_X.$$

(2) *Preservation of Composition.* For each $a, b \in A$, we have
$$\lambda_b \circ \lambda_a = \lambda_{ab}, \qquad \begin{tikzcd} X \arrow[r, "\lambda_a"] \arrow[dr, "\lambda_{ab}"'] & X \arrow[d, "\lambda_b"] \\ & X. \end{tikzcd}$$

**11.7.4. Group Actions on Objects of Categories.** Let $G$ be a group, let $\mathcal{C}$ be a category, and let $X \in \mathrm{Obj}(\mathcal{C})$.

**Definition 11.7.4.1.1.** A **$G$-action on $X$** is a functor $\lambda\colon \mathsf{B}G \to \mathcal{C}$ with $\lambda(\star) = X$.

**Remark 11.7.4.1.2.** In detail, a ***G*-action on** $X$ is a $G$-action on $\mathsf{Aut}_C(X)$, consisting of a morphism

$$\lambda\colon G \to \underbrace{\mathsf{End}_C(X)}_{\overset{\text{def}}{=}\mathrm{Hom}_C(X,X)}$$

satisfying the following conditions:

(1) *Preservation of Identities.* We have

$$\lambda_{1_A} = \mathrm{id}_X.$$

(2) *Preservation of Composition.* For each $a, b \in A$, we have

$$\lambda_b \circ \lambda_a = \lambda_{ab}, \qquad \begin{tikzcd} X \arrow[r, "\lambda_a"] \arrow[dr, "\lambda_{ab}"'] & X \arrow[d, "\lambda_b"] \\ & X. \end{tikzcd}$$

# Appendices

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(4) Tensor Products of Pointed Sets
(5) Relations
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## 11.2. Other Chapters


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Schemes

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CHAPTER 12

# Types of Morphisms in Categories

Create tags (see [MSE 350788] for some of these):

(1) **??**
(2) **??**
(3) **??**
(4) **??**
(5) write material on sections and retractions

## 12.1. Monomorphisms

**12.1.1. Foundations.** Let $\mathcal{C}$ be a category.

**Definition 12.1.1.1.1.** A morphism $m\colon A \to B$ of $\mathcal{C}$ is a **monomorphism** if, for each diagram of the form

$$m \circ f = m \circ g, \qquad X \underset{g}{\overset{f}{\rightrightarrows}} A \xrightarrow{m} B$$

in $\mathsf{Cats}$, we have $f = g$.

**Example 12.1.1.1.2.** Here are some examples of monomorphisms.

(1) *Monomorphisms in* $\mathsf{Sets}$. The monomorphisms in $\mathsf{Sets}$ are precisely the injections.

**Proof.** *Item 1, Monomorphisms in* $\mathsf{Sets}$: Let $f\colon A \to B$ be a morphism in $\mathsf{Sets}$. Suppose that $f$ is a monomorphism and consider the diagram

$$\{*\} \underset{[y]}{\overset{[x]}{\rightrightarrows}} A \xrightarrow{f} B,$$

where $[x]$ and $[y]$ are the morphisms picking the elements $x$ and $y$ of $A$. If $f(x) = f(y)$, then $f \circ [x] = f \circ [y]$, and thus $[x] = [y]$ since $f$ is a monomorphism. Hence $x = y$ and we see that $f$ is injective.
Conversely, suppose that $f$ is injective. Proceeding by contrapositive, we claim that given a pair of maps $g, h\colon X \rightrightarrows A$ such that $g \neq h$, then $f \circ g \neq f \circ h$. Indeed, as $g$ and $h$ are different maps, there must exist at least one element $x \in X$ such that $g(x) \neq h(x)$. But then we have $f(g(x)) \neq f(h(x))$, since $f$ is injective. Thus $f \circ g \neq f \circ h$, and we are done, having showed that $f$ is a monomorphism. □

**Proposition 12.1.1.1.3.** Let $f\colon A \to B$ be a morphism of $\mathcal{C}$.

(1) *Characterisations.* The following conditions are equivalent:
  (a) The morphism $f$ is a monomorphism.
  (b) For each $X \in \mathrm{Obj}(\mathcal{C})$, the map of sets

$$f_*\colon \mathrm{Hom}_{\mathsf{Sets}}(X, A) \to \mathrm{Hom}_{\mathsf{Sets}}(X, B)$$

is injective.

(c) The kernel pair of $f$ is trivial, i.e. we have

$$A \times_B A \cong A, \qquad \begin{tikzcd} A \arrow[r, "\mathrm{id}_A"] \arrow[d, "\mathrm{id}_A"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\ A \arrow[r, "f"'] & B. \end{tikzcd}$$

(2) *Duality.* The following conditions are equivalent:
   (a) The morphism $f\colon A \to B$ is a monomorphism in $\mathcal{C}$.
   (b) The morphism $f^\dagger\colon B \to A$ is an epimorphism in $\mathcal{C}^{\mathsf{op}}$.

(3) *Monomorphisms vs.Injective Maps.* Let
   - $\mathcal{C}$ be a concrete category as in **??**;
   - $忘_{\mathcal{C}}\colon \mathcal{C} \to \mathsf{Sets}$ be the forgetful functor from $\mathcal{C}$ to $\mathsf{Sets}$;
   - $f\colon A \to B$ be a morphism of $\mathcal{C}$.

   If $忘_{\mathcal{C}}$ preserves pullbacks, then the following conditions are equivalent:
   (a) The morphism $f$ is a monomorphism.
   (b) The morphism $忘(f)_{\mathcal{C}}$ is injective.

(4) *Stability Properties.* The class of all monomorphisms of $\mathcal{C}$ is stable under the following operations:
   (a) *Composition.* If $f$ and $g$ are monomorphisms, then so is $g \circ f$.[1]
   (b) *Pullbacks.* Let

$$\begin{tikzcd} A \times_C B \arrow[r] \arrow[d, "m'"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "m"] \\ A \arrow[r] & C \end{tikzcd}$$

   be a diagram in $\mathcal{C}$. If $m$ is a monomorphism in $\mathcal{C}$, then so is $m'$.

(5) *Morphisms From the Terminal Object Are Monomorphisms.* If $\mathcal{C}$ has a terminal object $\not\Vdash_{\mathcal{C}}$, then every morphism of $\mathcal{C}$ from $\not\Vdash_{\mathcal{C}}$ is a monomorphism.

**Proof.** *Item 1*, *Characterisations*: The equivalence between Items 1a and 1b is clear. We claim that Items 1a and 1c are equivalent:

(1) *Item 1a* $\Longrightarrow$ *Item 1c*: Suppose that $f$ is a monomorphism. Then $A$ satisfies the universal property of the pullback:

$$\begin{tikzcd} P \arrow[dr, dashed, "\phi", "\exists!"'] \arrow[drr, bend left, "\phi"] \arrow[ddr, bend right, "\phi"'] & & \\ & A \arrow[r, "\mathrm{id}_A"] \arrow[d, "\mathrm{id}_A"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\ & A \arrow[r, "f"'] & B. \end{tikzcd}$$

[1] Conversely, if $g \circ f$ is a monomorphism, then so is $f$.

(2) *Item 1c* $\implies$ *Item 1a*: Suppose that $A \cong A \times_B A$ and let $g, h \colon C \rightrightarrows A$ be a pair of morphisms. Consider the diagram

$$
\begin{tikzcd}
C \arrow[dr, shift left, "g"] \arrow[dr, shift right, "h"'] & & \\
& A \arrow[r, "\mathrm{id}_A"] \arrow[d, "\mathrm{id}_A"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\
& A \arrow[r, "f"'] & B.
\end{tikzcd}
$$

The universal property of the pullback says that there exists a unique morphism $C \to A$ making the diagram

$$
\begin{tikzcd}
C \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "h"'] & & \\
& A \arrow[r, "\mathrm{id}_A"] \arrow[d, "\mathrm{id}_A"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\
& A \arrow[r, "f"'] & B
\end{tikzcd}
$$

commute, which implies $g = h$. Therefore, $f$ is a monomorphism.

*Item 3*, *Monomorphisms vs.Injective Maps*: Assume that $f$ is injective. As the forgetful functor from $\mathcal{C}$ to $\mathsf{Sets}$ is faithful, we see that Proposition 12.1.2.1.2 together with **??** imply that $f$ is a monomorphism.

Conversely, assume that $f$ is a monomorphism. As $F$ preserves pullbacks, it also preserves kernel pairs. By **??**, we see that $F$ preserves monomorphisms. Thus $F_f$ is a monomorphism, and hence is injective by **??**.

*Item 4*, *Stability Properties*: Let $f, g \colon X \rightrightarrows A \times_C B$ be two morphisms such that the diagram

$$
\begin{tikzcd}
X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & A \times_C B \arrow[r, "m'"] & A
\end{tikzcd}
$$

commutes. It follows that the diagram

$$
\begin{tikzcd}
X \arrow[dr, shift left, "f"] \arrow[dr, shift right, "g"'] \arrow[drr, bend left, "\mathrm{pr}_2 \circ f"] \arrow[ddr, bend right, "m' \circ g"'] & & \\
& A \times_C B \arrow[r, dashed, "\mathrm{pr}_2"] \arrow[d, "m'"] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "m"] \\
& A \arrow[r, "\psi"'] & C
\end{tikzcd}
$$

also commutes. From the universal property of the pullback, it follows that there must be precisely one morphism from $X$ to $A \times_C B$ making the above diagram commute. Thus $f = g$ and $m'$ is a monomorphism.
*Item 5, Morphisms From the Terminal Object Are Monomorphisms*: Clear. □

### 12.1.2. Monomorphism-Reflecting Functors.

**Definition 12.1.2.1.1.** A functor $F\colon \mathcal{C} \to \mathcal{D}$ **reflects monomorphisms** if, for each morphism $f$ of $\mathcal{C}$, whenever $F_f$ is a monomorphism, so is $f$.

**Proposition 12.1.2.1.2.** Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor. If $F$ is faithful, then it reflects monomorphisms.

**Proof.** Let $f\colon A \to B$ be a morphism of $\mathcal{C}$ and suppose that $F_f\colon F_A \to F_B$ is a monomorphism. Let $g, h\colon B \rightrightarrows C$ be two morphisms of $\mathcal{C}$ such that $g \circ f = h \circ f$. As $F$ is faithful, we must have

$$F_g \circ F_f = F_{g \circ f} = F_{h \circ f} = F_h \circ F_f,$$

but as $F_f$ is a monomorphism, it must be that $F_g = F_h$. Using the faithfulness of $F$ again, we see that $g = h$. Therefore $f$ is a monomorphism. □

**12.1.3. Split Monomorphisms.** Let $\mathcal{C}$ be a category.

**Definition 12.1.3.1.1.** A morphism $f\colon A \to B$ of $\mathcal{C}$ is a **split monomorphism**[2] if there exists a morphism $g\colon B \to A$ of $\mathcal{B}$ such that[3]

$$g \circ f = \mathrm{id}_A.$$

**Proposition 12.1.3.1.2.** Let $\mathcal{C}$ be a category.

(1) *Split Monomorphisms are Monomorphisms.* If $m$ is a split monomorphism, then $m$ is a monomorphism.

**Proof.** *Item 1, Split Monomorphisms are Monomorphisms*: Let $m\colon A \to B$ be a split monomorphism of $\mathcal{C}$, let $e\colon B \to A$ be a morphism of $\mathcal{C}$ with

$$e \circ m = \mathrm{id}_A,$$

and let $f, g\colon C \rightrightarrows A$ be two morphisms of $\mathcal{C}$ such that the diagram

$$\begin{tikzcd} C \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & A \arrow[r, "m"] & B \end{tikzcd}$$

commutes. Then we have

$$\begin{aligned} f &= \mathrm{id}_A \circ f \\ &= (e \circ m) \circ f \\ &= e \circ (m \circ f) \\ &= e \circ (m \circ g) \\ &= (e \circ m) \circ g \\ &= \mathrm{id}_A \circ g \\ &= g, \end{aligned}$$

[2] *Further Terminology:* Also called a **section**, or a **split monic** morphism.

[3] *Warning:* There exist monomorphisms which are not split monomorphisms, e.g.

showing $m$ to be a monomorphism. $\square$

## 12.2. Epimorphisms

**12.2.1. Foundations.** Let $\mathcal{C}$ be a category.

**Definition 12.2.1.1.1.** A morphism $f\colon A \to B$ of $\mathcal{C}$ is an **epimorphism** if for every commutative[4] diagram of the form

$$A \xrightarrow{f} B \underset{h}{\overset{g}{\rightrightarrows}} C,$$

we have $g = h$.

**Example 12.2.1.1.2.** Let $f\colon A \to B$ be a function. The following conditions are equivalent:

(1) The function $f$ is injective.
(2) The function $f$ is an epimorphism in $\mathsf{Sets}$.

**Proof.** Suppose that $f$ is surjective and let $g, h\colon B \rightrightarrows C$ be morphisms such that $g \circ f = h \circ f$. Then for each $a \in A$, we have

$$g(f(a)) = h(f(a)),$$

but this implies that

$$g(b) = h(b)$$

for each $b \in B$, as $f$ is surjective. Thus $g = h$ and $f$ is an epimorphism.
To prove the converse, we proceed by contrapositive. So suppose that $f$ is not surjective and consider the diagram

$$A \xrightarrow{f} B \underset{h}{\overset{g}{\rightrightarrows}} C,$$

where $h$ is the map defined by $h(b) = 0$ for each $b \in B$ and $g$ is the map defined by

$$g(b) = \begin{cases} 1 & \text{if } b \in \mathrm{Im}(f), \\ 0 & \text{otherwise.} \end{cases}$$

Then $h \circ f = g \circ f$, as $h(f(a)) = 1 = g(f(a))$ for each $a \in A$. However, for any $b \in B \setminus \mathrm{Im}(f)$, we have

$$g(b) = 0 \neq 1 = h(b).$$

Therefore $g \neq h$ and $f$ is not an epimorphism. $\square$

**Proposition 12.2.1.1.3.** Let $\mathcal{C}$ be a category.

(1) *Characterisations.* Let $\mathcal{C}$ be a category with pullbacks and $f\colon A \to B$ be a morphism of $\mathcal{C}$. The following conditions are equivalent:
  (a) The morphism $f$ is an epimorphism.
  (b) For each $X \in \mathrm{Obj}(\mathcal{C})$, the map of sets

$$f^*\colon \mathrm{Hom}_{\mathsf{Sets}}(B, X) \to \mathrm{Hom}_{\mathsf{Sets}}(A, X)$$

  is injective.

---

$\mathbb{Z}/2 \hookrightarrow \mathbb{Z}/4$ in $\mathsf{Ring}$.
[4]That is, with $g \circ f = h \circ f$.

(c) The cokernel pair of $f$ is trivial, i.e. we have

$$B \coprod_A B \cong B \qquad \begin{tikzcd} B \arrow[dr, phantom, "\ulcorner", very near start] & B \arrow[l] \\ B \arrow[u] & A. \arrow[l, "f"] \arrow[u, "f"'] \end{tikzcd}$$

(2) *Epimorphisms vs.Surjective Maps.* Let
- $\mathcal{C}$ be a concrete category;
- $忘\colon \mathcal{C} \to \mathsf{Sets}$ be the forgetful functor from $\mathcal{C}$ to $\mathsf{Sets}$;
- $f\colon A \to B$ be a morphism of $\mathcal{C}$.

If $忘$ preserves pushouts, then the following conditions are equivalent:

(a) The morphism $f$ is a epimorphism.
(b) The morphism $f$ is surjective.

(3) *Stability Properties.* The class of all epimorphisms of $\mathcal{C}$ is stable under the following operations:

(a) *Composition.* If $f$ and $g$ are epimorphisms, then so is $g \circ f$.[5]
(b) *Pushouts.* Let

$$\begin{tikzcd} A \coprod_C B \arrow[dr, phantom, "\ulcorner", very near start] & B \arrow[l] \\ A \arrow[u, "e'"] \arrow[r] & C \arrow[u, "e"'] \end{tikzcd}$$

be a diagram in $\mathcal{C}$. If $m$ is an epimorphism in $\mathcal{C}$, then so is $e'$.

(4) *Morphisms to the Initial Object Are Monomorphisms.* If $\mathcal{C}$ has an initial object $\emptyset_{\mathcal{C}}$, then every morphism of $\mathcal{C}$ to $\emptyset_{\mathcal{C}}$ is a epimorphism.

**Proof.** This is dual to Proposition 12.1.1.1.3. □

### 12.2.2. Regular Epimorphisms.

**Proposition 12.2.2.1.1.** Let $\mathcal{C}$ be a category.

(1) *Stability Under Pullbacks.* Consider the diagram

$$\begin{tikzcd} A \times_C B \arrow[r] \arrow[d, "e'"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "e"] \\ A \arrow[r] & C \end{tikzcd}$$

in $\mathcal{C}$. If $e$ is a regular epimorphism, then so is $e'$.

**Proof.** *Epimorphisms Need Not Be Stable Under Pullback.* : *Regular Epimorphisms Are Stable Under Pullback.* : □

### 12.2.3. Effective Epimorphisms.

Let $\mathcal{C}$ be a category.

**Definition 12.2.3.1.1.** An epimorphism $f\colon A \to B$ of $\mathcal{C}$ is **effective** if we have an isomorphism

$$B \cong \mathrm{CoEq}(A \times_B A \rightrightarrows A).$$

[5]Conversely, if $g \circ f$ is a epimorphism, then so is $g$.

**12.2.4. Split Epimorphisms.** Let $\mathcal{C}$ be a category.

**Definition 12.2.4.1.1.** A morphism $f\colon A \to B$ of $\mathcal{C}$ is a **retraction**[6] if there is an arrow $g\colon B \to A$ such that $f \circ g = \mathrm{id}_B$.

**Proposition 12.2.4.1.2.** Let $f\colon A \to B$ be a morphism of $\mathcal{C}$.

(1) Every split epimorphism is an epimorphism.[7]

**Proof.** This is dual to ??. □

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[6] *Further Terminology:* Also called a **split epimorphism**.

[7] *Warning:* There are epimorphisms which are not split epimorphisms, however, e.g. $\mathbb{Z} \hookrightarrow \mathbb{Z}/2$.

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CHAPTER 13

# Adjunctions and the Yoneda Lemma

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CHAPTER 15

# Profunctors

## 15.1. Profunctors

**15.1.1. Foundations.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

**Definition 15.1.1.1.1.** A **profunctor**[1] $\mathfrak{p}\colon \mathcal{C} \nrightarrow \mathcal{D}$ from $\mathcal{C}$ to $\mathcal{D}$ is a functor $\mathfrak{p}\colon \mathcal{D}^{\mathsf{op}} \times \mathcal{C} \to \mathsf{Sets}$.

**Remark 15.1.1.1.2.** Equivalently, we may define a profunctor from $\mathcal{C}$ to $\mathcal{D}$ as:

(1) A functor $\mathfrak{p}\colon \mathcal{D}^{\mathsf{op}} \times \mathcal{C} \to \mathsf{Sets}$;
(2) A functor $\mathfrak{p}\colon \mathcal{C} \to \mathsf{PSh}(\mathcal{D})$;
(3) A functor $\mathfrak{p}\colon \mathcal{D}^{\mathsf{op}} \to \mathsf{Fun}(\mathcal{C}, \mathsf{Sets})$;
(4) A cocontinuous functor $\mathfrak{p}\colon \mathsf{PSh}(\mathcal{C}) \to \mathsf{PSh}(\mathcal{D})$;

That is, we have isomorphisms of categories

$$\begin{aligned}\mathsf{Prof}(\mathcal{C}, \mathcal{D}) &\cong \mathsf{Fun}(\mathcal{C}, \mathsf{PSh}(\mathcal{D})),\\ &\cong \mathsf{Fun}(\mathcal{D}^{\mathsf{op}}, \mathsf{CoPSh}(\mathcal{C})),\\ &\cong \mathsf{CoContFun}(\mathsf{PSh}(\mathcal{C}), \mathsf{PSh}(\mathcal{D})),\end{aligned}$$

natural in $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$.

**Proof.** We claim that Items 1 to 4 are indeed equivalent:

- The equivalence between Items 1 and 2 is an instance of currying, following from the isomorphisms of categories

$$\mathsf{Fun}(\mathcal{D}^{\mathsf{op}} \times \mathcal{C}, \mathsf{Sets}) \cong \mathsf{Fun}(\mathcal{C}, \mathsf{Fun}(\mathcal{D}^{\mathsf{op}}, \mathsf{Sets})) \qquad \stackrel{\text{def}}{=} \mathsf{Fun}(\mathcal{C}, \mathsf{PSh}(\mathcal{D})).$$

- The equivalence between Items 1 and 3 is also an instance of currying, following from the isomorphisms of categories

$$\mathsf{Fun}(\mathcal{D}^{\mathsf{op}} \times \mathcal{C}, \mathsf{Sets}) \cong \mathsf{Fun}(\mathcal{D}^{\mathsf{op}}, \mathsf{Fun}(\mathcal{C}, \mathsf{Sets})) \qquad \stackrel{\text{def}}{=} \mathsf{Fun}(\mathcal{D}^{\mathsf{op}}, \mathsf{Fun}(\mathcal{C}, \mathsf{Sets})).$$

- The equivalence between Items 1 and 4 follows from the universal property of the category $\mathsf{PSh}(\mathcal{C})$ of presheaves on $\mathcal{C}$ as the free cocompletion of $\mathcal{C}$ via the Yoneda embedding

$$よ\colon \mathcal{C}^{\mathsf{op}} \hookrightarrow \mathsf{PSh}(\mathcal{C})$$

of $\mathcal{C}$ into $\mathsf{PSh}(\mathcal{C})$ (**??** of **??**).

This finishes the proof. $\square$

## 15.2. Operations With Profunctors

**15.2.1. The Domain and Range of a Profunctor.**

[1] *Further Terminology:* Also called a **distributor**, a **bimodule**, a **correspondence**, or a **relator**.

**Definition 15.2.1.1.1.** Let $\mathfrak{p}\colon \mathcal{C} \nrightarrow \mathcal{D}$ be a profunctor.[2]

(1) The **domain of** $\mathfrak{p}$ is the presheaf $\mathrm{dom}(\mathfrak{p})\colon \mathcal{D}^{\mathsf{op}} \to \mathsf{Sets}$ on $\mathcal{D}$ defined by
$$\mathrm{dom}(\mathfrak{p})^{-} \overset{\text{def}}{=} \operatorname*{colim}_{B\in\mathcal{D}}\left(\mathfrak{p}^{-}_{B}\right).$$

(2) The **range of** $\mathfrak{p}$ is the copresheaf $\mathrm{range}(\mathfrak{p})\colon \mathcal{C} \to \mathsf{Sets}$ on $\mathcal{C}$ defined by
$$\mathrm{range}(\mathfrak{p})_{-} \overset{\text{def}}{=} \operatorname*{colim}_{A\in\mathcal{D}}\left(\mathfrak{p}^{A}_{-}\right).$$

**15.2.2. Composition of Profunctors.** Let $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ be categories and let $\mathfrak{p}\colon \mathcal{C} \nrightarrow \mathcal{D}$ and $\mathfrak{q}\colon \mathcal{D} \nrightarrow \mathcal{E}$ be profunctors.

**Definition 15.2.2.1.1.** The **composition of** $\mathfrak{p}$ **and** $\mathfrak{q}$ is the profunctor $\mathfrak{q} \diamond \mathfrak{p}\colon \mathcal{C} \nrightarrow \mathcal{E}$ defined by[3]
$$(\mathfrak{q} \diamond \mathfrak{p})^{-1}_{-2} \overset{\text{def}}{=} \int^{B\in\mathcal{D}} \mathfrak{q}^{-1}_{B} \times \mathfrak{p}^{B}_{-2}.$$

**15.2.3. Representable Profunctors.**

**Definition 15.2.3.1.1.** The **representable profunctor associated to a functor** $F\colon \mathcal{C} \to \mathcal{D}$ is the profunctor $\widehat{F}^{*}\colon \mathcal{C} \nrightarrow \mathcal{D}$ defined as the adjunct of the composition
$$\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{\text{よ}} \mathsf{PSh}(\mathcal{D})$$
under the adjunction
$$\mathsf{Fun}(\mathcal{D}^{\mathsf{op}} \times \mathcal{C}, \mathsf{Sets}) \cong \mathsf{Fun}(\mathcal{C}, \mathsf{PSh}(\mathcal{D}))$$
of **??** of **??**.[4]

[2]In other words, the domain and range of $\mathfrak{p}$ are the functors
$$\mathrm{dom}(\mathfrak{p})\colon \mathcal{D}^{\mathsf{op}} \to \mathsf{Sets},$$
$$\mathrm{range}(\mathfrak{p})\colon \mathcal{C} \to \mathsf{Sets}$$
defined by
$$\begin{tikzcd} \mathcal{D}^{\mathsf{op}} \arrow[r, "\mathfrak{p}^{\dagger}"] \arrow[dr, "\mathrm{dom}(\mathfrak{p})"'] & \mathsf{PSh}(\mathcal{D}) \arrow[d, "\mathrm{colim}"] \\ & \mathsf{Sets}, \end{tikzcd} \qquad \begin{aligned} \mathrm{dom}(\mathfrak{p}) &\overset{\text{def}}{=} \mathrm{colim} \circ \mathfrak{p}^{\dagger}, \\ \mathrm{range}(\mathfrak{p}) &\overset{\text{def}}{=} \mathrm{colim} \circ \mathfrak{p}^{\ddagger}, \end{aligned} \qquad \begin{tikzcd} \mathcal{C} \arrow[r, "\mathfrak{p}^{\ddagger}"] \arrow[dr, "\mathrm{range}(\mathfrak{p})"'] & \mathsf{Fun}(\mathcal{C}, \mathsf{Sets}) \arrow[d, "\mathrm{colim}"] \\ & \mathsf{Sets}. \end{tikzcd}$$

[3]Alternatively, we may define $\mathfrak{q} \diamond \mathfrak{p}$ (using the equivalent definition of Item 2 of Remark 15.1.1.1.2) by
$$(\mathfrak{q} \diamond \mathfrak{p})^{\dagger} \overset{\text{def}}{=} \mathrm{Lan}_{\text{よ}}(\mathfrak{p}^{\dagger}) \circ \mathfrak{q}^{\dagger}, \qquad \begin{tikzcd} & \mathcal{D} \arrow[r, "\mathfrak{p}^{\dagger}"] \arrow[d, hook, "\text{よ}"'] & \mathsf{PSh}(\mathcal{C}). \\ \mathcal{E} \arrow[r, "\mathfrak{q}^{\dagger}"'] & \mathsf{PSh}(\mathcal{D}) \arrow[ur, bend right, "\mathrm{Lan}_{\text{よ}}(\mathfrak{p}^{\dagger})"'] & \end{tikzcd}$$

[4]That is, we have
$$\widehat{F}^{*} \overset{\text{def}}{=} \mathrm{Hom}_{\mathcal{D}}(-_1, F_{-2}).$$

**Definition 15.2.3.1.2.** A profunctor is **representable** if it is isomorphic to a representable profunctor.

**Definition 15.2.3.1.3.** The **corepresentable**[5] **profunctor associated to a functor** $F\colon \mathcal{C} \to \mathcal{D}$ is the profunctor $\widehat{F}_*\colon \mathcal{D} \nrightarrow \mathcal{C}$ defined as the adjunct of the composition

$$\mathcal{C}^{\mathsf{op}} \xrightarrow{F^{\mathsf{op}}} \mathcal{D}^{\mathsf{op}} \xrightarrow{\text{よ}} \mathsf{CoPSh}(\mathcal{D})$$

under the adjunction

$$\mathsf{Fun}(\mathcal{C}^{\mathsf{op}} \times \mathcal{D}, \mathsf{Sets}) \cong \mathsf{Fun}(\mathcal{C}^{\mathsf{op}}, \mathsf{CoPSh}(\mathcal{D}))$$

of **??** of **??**.[6]

**Definition 15.2.3.1.4.** A profunctor is **corepresentable** if it is isomorphic to a corepresentable profunctor.

**15.2.4. Collages.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

**Definition 15.2.4.1.1.** The **collage** of a profunctor $\mathfrak{p}\colon \mathcal{C} \nrightarrow \mathcal{D}$ is the category $\mathsf{Coll}(\mathfrak{p})$[7] where[8]

- *Objects.* We have
$$\mathrm{Obj}(\mathsf{Coll}(\mathfrak{p})) \overset{\text{def}}{=} \mathrm{Obj}(\mathcal{C}) \coprod \mathrm{Obj}(\mathcal{D});$$
- *Morphisms.* For each $A, B \in \mathrm{Obj}(\mathsf{Coll}(\mathfrak{p}))$, we have
$$\mathrm{Hom}_{\mathsf{Coll}(\mathfrak{p})}(A,B) \overset{\text{def}}{=} \begin{cases} \mathrm{Hom}_{\mathcal{C}}(A,B) & \text{if } A,B \in \mathrm{Obj}(\mathcal{C}), \\ \mathrm{Hom}_{\mathcal{D}}(A,B) & \text{if } A,B \in \mathrm{Obj}(\mathcal{D}), \\ \mathfrak{p}(A,B) & \text{if } A \in \mathrm{Obj}(\mathcal{C}) \text{ and } B \in \mathrm{Obj}(\mathcal{D}), \\ \emptyset & \text{if } A \in \mathrm{Obj}(\mathcal{D}) \text{ and } B \in \mathrm{Obj}(\mathcal{C}); \end{cases}$$
- *Identities.* For each $A \in \mathrm{Obj}(\mathsf{Coll}(\mathfrak{p}))$, the unit map
$$\mathbb{1}^{\mathsf{Coll}(\mathfrak{p})}_A\colon \mathrm{pt} \to \mathrm{Hom}_{\mathsf{Coll}(\mathfrak{p})}(A,A)$$

---

[5]Some authors call both $\widehat{F}^*$ and $\widehat{F}_*$ the **representable profunctors associated to** $F$.

[6]That is:
$$\widehat{F}_* \overset{\text{def}}{=} \mathrm{Hom}_{\mathcal{D}}(F_{-1}, -_2,).$$

[7]*Further Notation:* Also written $\mathcal{C} \star^{\mathfrak{p}} \mathcal{D}$, notably in [**HigherToposTheory**].

[8]We also have a functor $\phi\colon \mathsf{Coll}(\mathfrak{p}) \to \mathbb{1}$ where

- *Actions on Objects.* For each $A \in \mathrm{Obj}(\mathsf{Coll}(\mathfrak{p}))$, we have
$$\phi_A \overset{\text{def}}{=} \begin{cases} [0] & \text{if } A \in \mathrm{Obj}(\mathcal{C}), \\ [1] & \text{if } A \in \mathrm{Obj}(\mathcal{D}). \end{cases}$$
- *Actions on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathsf{Coll}(\mathfrak{p}))$, the action on morphisms
$$\phi_{A,B}\colon \mathrm{Hom}_{\mathsf{Coll}(\mathfrak{p})}(A,B) \to \mathrm{Hom}_{\mathsf{Coll}(\mathfrak{p})}(\phi_A, \phi_B)$$
of $\phi$ at $(A,B)$ is given by
$$\phi_{A,B}(f) \overset{\text{def}}{=} \begin{cases} \mathrm{id}_{[0]} & \text{if } A,B \in \mathrm{Obj}(\mathcal{C}), \\ \mathrm{id}_{[1]} & \text{if } A,B \in \mathrm{Obj}(\mathcal{D}), \\ [0] \to [1] & \text{if } A \in \mathrm{Obj}(\mathcal{C}) \text{ and } B \in \mathrm{Obj}(\mathcal{D}). \end{cases}$$
If $A \in \mathrm{Obj}(\mathcal{D})$ and $B \in \mathrm{Obj}(\mathcal{C})$, we have $\phi_{A,B} \overset{\text{def}}{=} \mathrm{id}_{\emptyset}$.

of $\mathsf{Coll}(\mathfrak{p})$ at $A$ is defined by

$$\mathrm{id}_A \overset{\text{def}}{=} \begin{cases} \mathrm{id}^{\mathcal{C}}_A & \text{if } A \in \mathrm{Obj}(\mathcal{C}), \\ \mathrm{id}^{\mathcal{D}}_A & \text{if } A \in \mathrm{Obj}(\mathcal{D}); \end{cases}$$

- *Composition.* For each $A, B, C \in \mathrm{Obj}(\mathsf{Coll}(\mathfrak{p}))$, the composition map

$$\circ^{\mathsf{Coll}(\mathfrak{p})}_{A,B,C} \colon \mathrm{Hom}_{\mathsf{Coll}(\mathfrak{p})}(B, C) \times \mathrm{Hom}_{\mathsf{Coll}(\mathfrak{p})}(A, B) \to \mathrm{Hom}_{\mathsf{Coll}(\mathfrak{p})}(A, C)$$

of $\mathsf{Coll}(\mathfrak{p})$ at $(A, B, C)$ is defined by[9]

$$\circ^{\mathsf{Coll}(\mathfrak{p})}_{A,B,C} \overset{\text{def}}{=} \begin{cases} \circ^{\mathcal{C}}_{A,B,C} & \text{if } A, B, C \in \mathrm{Obj}(\mathcal{C}), \\ \mathfrak{p}^{A,B}_C & \text{if } A, B \in \mathrm{Obj}(\mathcal{C}) \text{ and } C \in \mathrm{Obj}(\mathcal{D}), \\ \iota & \text{if } A, C \in \mathrm{Obj}(\mathcal{C}) \text{ and } B \in \mathrm{Obj}(\mathcal{D}), \\ \iota & \text{if } B, C \in \mathrm{Obj}(\mathcal{C}) \text{ and } A \in \mathrm{Obj}(\mathcal{D}), \\ \mathfrak{p}^{A}_{B,C} & \text{if } A \in \mathrm{Obj}(\mathcal{C}) \text{ and } B, C \in \mathrm{Obj}(\mathcal{D}), \\ \iota & \text{if } B \in \mathrm{Obj}(\mathcal{C}) \text{ and } A, C \in \mathrm{Obj}(\mathcal{D}), \\ \iota & \text{if } C \in \mathrm{Obj}(\mathcal{C}) \text{ and } A, B \in \mathrm{Obj}(\mathcal{D}), \\ \circ^{\mathcal{D}}_{A,B,C} & \text{if } A, B, C \in \mathrm{Obj}(\mathcal{D}). \end{cases}$$

**Example 15.2.4.1.2.** If $\mathfrak{p}$ is the constant functor $\Delta_{\mathrm{pt}} \colon \mathcal{D}^{\mathsf{op}} \times \mathcal{C} \to \mathsf{Sets}$ with value pt, then $\mathsf{Coll}(\mathfrak{p})$ is the join $\mathcal{C} \star \mathcal{D}$ of $\mathcal{C}$ and $\mathcal{D}$ of **??**.

**Proposition 15.2.4.1.3.** Let $\mathfrak{p} \colon \mathcal{C} \nrightarrow \mathcal{D}$ be a profunctor.

(1) *Functoriality.* The assignment $\mathfrak{p} \mapsto \mathsf{Coll}(\mathfrak{p})$ defines a functor[10]

$$\mathsf{Coll}_{\mathcal{C},\mathcal{D}} \colon \mathsf{Prof}(\mathcal{C}, \mathcal{D}) \to \mathsf{Cats}_{/\nVdash}(\mathcal{C}, \mathcal{D}),$$

where

[9]Here the maps $\mathfrak{p}^{A,B}_C$ and $\mathfrak{p}^{A}_{B,C}$ are the maps

$$\mathfrak{p}^{A,B}_C \colon \mathfrak{p}^{B}_C \times \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathfrak{p}^{A}_C,$$
$$\mathfrak{p}^{A}_{B,C} \colon \mathrm{Hom}_{\mathcal{D}}(B, C) \times \mathfrak{p}^{A}_B \to \mathfrak{p}^{A}_C$$

coming from the profunctor structure of $\mathfrak{p}$ and the $\iota$'s are inclusions of the empty set into the appropriate Hom sets.

[10]Here $\mathsf{Cats}_{/\nVdash}(\mathcal{C}, \mathcal{D})$ is the category defined as the pullback

$$\mathsf{Cats}_{/\nVdash}(\mathcal{C}, \mathcal{D}) \overset{\text{def}}{=} \mathsf{pt} \underset{[\mathcal{C}],\mathsf{Cats},\mathrm{fib}_0}{\times} \mathsf{Cats}_{/\nVdash} \underset{\mathrm{fib}_1,\mathsf{Cats},[\mathcal{D}]}{\times} \mathsf{pt},$$

as in the diagram

$$\begin{tikzcd}
 & & \mathsf{Cats}_{/\nVdash}(\mathcal{C}, \mathcal{D}) \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45] & & \\
 & \mathsf{Cats}_{/\nVdash} \underset{\mathsf{Cats}}{\times} \mathsf{pt} \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45] & & \mathsf{pt} \underset{\mathsf{Cats}}{\times} \mathsf{Cats}_{/\nVdash} \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45] & \\
\mathsf{pt} \arrow[dr, "{[\mathcal{C}]}"'] & & \mathsf{Cats}_{/\nVdash} \arrow[dl, "\mathrm{fib}_{[0]}"] \arrow[dr, "\mathrm{fib}_{[1]}"'] & & \mathsf{pt}. \arrow[dl, "{[\mathcal{D}]}"] \\
 & \mathsf{Cats} & & \mathsf{Cats} &
\end{tikzcd}$$

- *Action on Objects.* For each $\mathfrak{p} \in \mathrm{Obj}(\mathsf{Prof}(\mathcal{C}, \mathcal{D}))$, we have
$$[\mathsf{Coll}](\mathfrak{p}) \stackrel{\text{def}}{=} \mathsf{Coll}(\mathfrak{p});$$
- *Action on Morphisms.* For each $\mathfrak{p}, \mathfrak{q} \in \mathrm{Obj}(\mathsf{Prof}(\mathcal{C}, \mathcal{D}))$, the action on Hom-sets
$$\mathsf{Coll}_{\mathfrak{p},\mathfrak{q}} \colon \mathrm{Nat}(\mathfrak{p}, \mathfrak{q}) \to \mathsf{Fun}_{/\Bbbk}(\mathsf{Coll}(\mathfrak{p}), \mathsf{Coll}(\mathfrak{q}))$$
of $\mathsf{Coll}$ at $(\mathfrak{p}, \mathfrak{q})$ is the function sending a natural transformation $\alpha \colon \mathfrak{p} \Longrightarrow \mathfrak{q}$ to the functor
$$\mathsf{Coll}(\alpha) \colon \mathsf{Coll}(\mathfrak{p}) \to \mathsf{Coll}(\mathfrak{q})$$
over $\Bbbk$ where
  - *Action on Objects.* For each $X \in \mathrm{Obj}(\mathsf{Coll}(\mathfrak{p}))$, we have
$$[\mathsf{Coll}(\alpha)](X) \stackrel{\text{def}}{=} X;$$
  - *Action on Morphisms.* For each $X, Y \in \mathrm{Obj}(\mathsf{Coll}(\mathfrak{p}))$, the action on Hom-sets
$$\mathsf{Coll}(\alpha)_{X,Y} \colon \mathrm{Hom}_{\mathsf{Coll}(\mathfrak{p})}(X, Y) \to \underbrace{\mathrm{Hom}_{\mathsf{Coll}(\mathfrak{q})}([\mathsf{Coll}(\alpha)](X), [\mathsf{Coll}(\alpha)](Y))}_{\stackrel{\text{def}}{=} \mathrm{Hom}_{\mathsf{Coll}(\mathfrak{q})}(X,Y)}$$
of $\mathsf{Coll}(\alpha)$ at $(X, Y)$ is defined as follows:
    * If $X, Y \in \mathrm{Obj}(\mathcal{C})$ or $X, Y \in \mathrm{Obj}(\mathcal{D})$, then we have
$$\mathsf{Coll}(\alpha)_{X,Y}(f) \stackrel{\text{def}}{=} f$$
for each $f \in \mathrm{Hom}_{\mathsf{Coll}(\mathfrak{p})}(X, Y)$.
    * If $X \in \mathrm{Obj}(\mathcal{C})$ and $Y \in \mathrm{Obj}(\mathcal{D})$, then
$$\mathsf{Coll}(\alpha)_{X,Y} \colon \underbrace{\mathrm{Hom}_{\mathsf{Coll}(\mathfrak{p})}(X, Y)}_{\stackrel{\text{def}}{=} \mathfrak{p}^X_Y} \to \underbrace{\mathrm{Hom}_{\mathsf{Coll}(\mathfrak{q})}(X, Y)}_{\stackrel{\text{def}}{=} \mathfrak{q}^X_Y}$$
is defined by
$$\mathsf{Coll}(\alpha)_{X,Y}(f) \stackrel{\text{def}}{=} \alpha^X_Y;$$
    * If $Y \in \mathrm{Obj}(\mathcal{C})$ and $X \in \mathrm{Obj}(\mathcal{D})$, then we have
$$\mathsf{Coll}(\alpha)_{X,Y}(f) \stackrel{\text{def}}{=} \mathrm{id}_{\emptyset}.$$

(2) *Collages as Lax Colimits.* We have an isomorphism of categories
$$\mathsf{Coll}(\mathfrak{p}) \cong \mathsf{colim}^{\mathsf{lax}}(\mathfrak{p}),$$
functorial in $\mathfrak{p}$, where the above lax colimit is taken in the bicategory $\mathsf{Prof}$.

(3) *Profunctors vs. Collages.* We have an equivalence of categories
$$(\mathsf{Coll} \dashv \Gamma)\colon \quad \mathsf{Prof}(\mathcal{C}, \mathcal{D}) \underset{\Gamma}{\overset{\mathsf{Coll}}{\rightleftarrows}} \mathsf{Cats}_{/\Bbbk}, \quad (\perp_{\mathsf{eq}})$$
where $\Gamma \colon \mathsf{Cats}_{/\Bbbk} \to \mathsf{Prof}(\mathcal{C}, \mathcal{D})$ is the functor sending a functor $\mathcal{E} \to \Bbbk$ to the profunctor
$$\Gamma(\mathfrak{p}) \colon \mathcal{C} \nrightarrow \mathcal{D}$$
given on objects by
$$\Gamma(\mathfrak{p})^A_B \stackrel{\text{def}}{=} \mathrm{Hom}_{\mathcal{E}}(A, B)$$

for each $A, B \in \mathrm{Obj}(\mathcal{E})$.

**Proof.** *Item 1, Functoriality*: Omitted.
*Item 2, Collages as Lax Colimits*: See [**collages-as-lax-colimits**].
*Item 3, Profunctors vs. Collages*: See [**joyal:distributors-and-barrels**]. □

## 15.3. Categories of Profunctors

### 15.3.1. The Bicategory of Profunctors.

**Definition 15.3.1.1.1.** The **bicategory of profunctors** is the bicategory $\mathsf{Prof}$ where[11]

(1) *Objects.* The objects of $\mathsf{Prof}$ are categories;
(2) *1-Morphisms.* The 1-morphisms of $\mathsf{Prof}$ are profunctors;
(3) *2-Morphisms.* The 2-morphisms of $\mathsf{Prof}$ are natural transformations between profunctors;
(4) *Identities.* For each $\mathcal{C} \in \mathrm{Obj}(\mathsf{Prof})$, we have

$$\mathrm{id}_{\mathcal{C}}^{\mathsf{Prof}} \stackrel{\mathrm{def}}{=} \mathrm{Hom}_{\mathcal{C}}(-,-);$$

(5) *Composition.* For each $\mathcal{C}, \mathcal{D}, \mathcal{E} \in \mathrm{Obj}(\mathsf{Prof})$, the composition bifunctor

$$\diamond \colon \mathsf{Prof}(\mathcal{D}, \mathcal{E}) \times \mathsf{Prof}(\mathcal{C}, \mathcal{D}) \to \mathsf{Prof}(\mathcal{C}, \mathcal{E})$$

is defined on objects by sending profunctors $\mathfrak{p} \colon \mathcal{C} \nrightarrow \mathcal{D}$ and $\mathfrak{q} \colon \mathcal{D} \nrightarrow \mathcal{E}$ to the profunctor $\mathfrak{q} \diamond \mathfrak{p}$ of Definition 15.2.2.1.1.

**Proof.** See Definition 15.3.1.1.1. □

### 15.3.2. Properties of $\mathsf{Prof}$.

**Proposition 15.3.2.1.1.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

(1) *Self-Duality.* The bicategory $\mathsf{Prof}$ is self-dual: we have a biequivalence of bicategories

$$(-)^{\mathsf{op}} \colon \mathsf{Prof} \xrightarrow{\cong} \mathsf{Prof}^{\mathsf{op}}$$

where

- *Action on Objects.* The functor $(-)^{\mathsf{op}}$ sends categories to their opposites;
- *Action on 1-Morphisms.* The functor $(-)^{\mathsf{op}}$ sends profunctors to itself under the identification

$$\begin{aligned}\mathsf{Prof}(\mathcal{C}, \mathcal{D}) &\stackrel{\mathrm{def}}{=} \mathsf{Fun}(\mathcal{D}^{\mathsf{op}} \times \mathcal{C}, \mathsf{Sets}),\\ &\cong \mathsf{Fun}(\mathcal{C} \times \mathcal{D}^{\mathsf{op}}, \mathsf{Sets}),\\ &\stackrel{\mathrm{def}}{=} \mathsf{Prof}(\mathcal{D}^{\mathsf{op}}, \mathcal{C}^{\mathsf{op}});\end{aligned}$$

- *Action on 2-Morphisms.* The functor $(-)^{\mathsf{op}}$ sends natural transformations between profunctors to themselves.

[11] The bicategory $\mathsf{Prof}$ admits a nice strictification to a 2-category: it is biequivalent to the sub-bicategory of $\mathsf{Cats}$ spanned by the presheaf categories, cocontinuous functors between them, and natural transformation between these.

(2) *Relation to* $\mathsf{Cats}$. The co/representable profunctor constructions of Definitions 15.2.3.1.1 and 15.2.3.1.3 define embeddings of bicategories

$$\mathsf{Cats}^{\mathsf{op}} \hookrightarrow \mathsf{Prof},$$
$$\mathsf{Cats}^{\mathsf{co}} \hookrightarrow \mathsf{Prof}.$$

(3) *Equivalences in* $\mathsf{Prof}$ *and Cauchy Completions.* Every category is equivalent to its Cauchy completion in $\mathsf{Prof}$.
(4) *Equivalences in* $\mathsf{Prof}$. The following conditions are equivalent:
 (a) The categories $\mathcal{C}$ and $\mathcal{D}$ are equivalent in $\mathsf{Prof}$.
 (b) The categories $\mathsf{PSh}(\mathcal{C})$ and $\mathsf{PSh}(\mathcal{D})$ are equivalent in $\mathsf{Cats}_2$.
 (c) The Cauchy completions of $\mathcal{C}$ and $\mathcal{D}$ are equivalent in $\mathsf{Cats}_2$.
(5) *Adjunctions in* $\mathsf{Prof}$. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. The following data are equivalent:
 (a) An adjunction in $\mathsf{Prof}$ from $\mathcal{C}$ to $\mathcal{D}$.
 (b) A functor from $\mathcal{C}$ to the Cauchy completion $\overline{\mathcal{D}}$ of $\mathcal{D}$.
 (c) A semifunctor from $\mathcal{C}$ to $\mathcal{D}$.
(6) *As a Kleisli Bicategory.* We have a biequivalence of bicategories

$$\mathsf{Prof} \cong \mathsf{FreePsAlg}_{\mathsf{PSh}},$$

where $\mathsf{PSh}$ is the presheaf category relative pseudomonad of [**relative-pseudomonads-kleisl**
(7) *Closedness.* The bicategory $\mathsf{Prof}$ is a closed bicategory, where given a profunctor $\mathfrak{p}\colon \mathcal{C} \nrightarrow \mathcal{D}$ and a category $\mathcal{X}$:
 - *Right Kan Extensions.* The right adjoint

$$\mathrm{Ran}_{\mathfrak{p}}\colon \mathrm{Rel}(\mathcal{C},\mathcal{X}) \to \mathrm{Rel}(\mathcal{D},\mathcal{X})$$

 to the precomposition functor $\mathfrak{p}^*\colon \mathrm{Rel}(\mathcal{D},\mathcal{X}) \to \mathrm{Rel}(\mathcal{C},\mathcal{X})$ is given by

$$\mathrm{Ran}_{\mathfrak{p}}(\mathfrak{q}) \overset{\text{def}}{=} \int_{A\in\mathcal{C}} \mathsf{Sets}\Big(\mathfrak{p}_A^{-2}, \mathfrak{q}_A^{-1}\Big)$$

 for each $\mathfrak{q} \in \mathrm{Rel}(\mathcal{C},\mathcal{X})$.
 - *Right Kan Lifts.* The right adjoint to the postcomposition functor

$$\mathrm{Rift}_{\mathfrak{p}}\colon \mathrm{Rel}(\mathcal{X},\mathcal{D}) \to \mathrm{Rel}(\mathcal{X},\mathcal{C})$$

 to the postcomposition functor $\mathfrak{p}_*\colon \mathrm{Rel}(\mathcal{X},\mathcal{C}) \to \mathrm{Rel}(\mathcal{X},\mathcal{D})$ is given by

$$\mathrm{Rift}_{\mathfrak{p}}(\mathfrak{q}) \overset{\text{def}}{=} \int_{B\in\mathcal{D}} \mathsf{Sets}\Big(\mathfrak{p}^B_{-1}, \mathfrak{q}^B_{-2}\Big)$$

 for each $\mathfrak{q} \in \mathrm{Rel}(\mathcal{X},\mathcal{D})$.
(8) *Un/Straightening for Profunctors: Two-Sided Discrete Fibrations.* We have an equivalence of categories

$$\mathsf{Prof}(\mathcal{C},\mathcal{D}) \cong \mathsf{DFib}(\mathcal{C},\mathcal{D}).$$

**Proof.** *Item 1, Self-Duality*: See [**loregian2020coend**].
*Item 2, Relation to* $\mathsf{Cats}$: See [**loregian2020coend**].
*Item 3, Equivalences in* $\mathsf{Prof}$ *and Cauchy Completions*: See [**borceux-2**].
*Item 4, Equivalences in* $\mathsf{Prof}$: See [**borceux-2**].
*Item 5, Adjunctions in* $\mathsf{Prof}$: Omitted.

*Item 6, As a Kleisli Bicategory*: See [**relative-pseudomonads-kleisli-bicategories-and-substituti**
*Item 7, Closedness*: Omitted.
*Item 8, Un/Straightening for Profunctors: Two-Sided Discrete Fibrations*: See [**riehl:two-sided-discrete-fibrations**] □

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CHAPTER 16

# Cartesian Closed Categories

Create tags (see [MSE 350788] for some of these):

(1) define bicategory $\mathsf{Adj}(\mathcal{C})$
(2) https://www.google.com/search?q=category+of+categories+is+not+locally+cartesian+closed
(3) https://math.stackexchange.com/questions/2864916/are-there-important-locally-cartesian-closed-categories-that-actually-are-not-ca
(4) Cats is not locally Cartesian closed: $f^*$ does have a left adjoint (the proof for fibred sets seems to apply for any category with pullbacks), but does not have a right adjoint, as proved in example 1.7 of https://sinhp.github.io/files/CT/notes_on_lcccs.pdf
(5) internal **Hom** in categories of co/Cartesian fibrations
(6) https://mathoverflow.net/questions/460146/universal-property-of-isbell-duality
(7) http://www.tac.mta.ca/tac/volumes/36/12/36-12abs.html
(8) Cartesian closed categories and locally Cartesian closed categories
    (a) https://ncatlab.org/nlab/show/locally+cartesian+closed+functor
    (b) https://ncatlab.org/nlab/show/cartesian+closed+functor
    (c) https://ncatlab.org/nlab/show/locally+cartesian+closed+category
    (d) https://ncatlab.org/nlab/show/Frobenius+reciprocity
(9) https://math.stackexchange.com/questions/3657046/the-inverse-of-a-natural-isomorphism-is-a-natural-isomorphism to justify adjunctions via homs
(10) https://ncatlab.org/nlab/show/enrichment+versus+internalisation
(11) https://mathoverflow.net/questions/382239/proof-that-a-cartesian-category-is-monoidal
(12) https://arxiv.org/pdf/2004.08964.pdf

Create tags:

(1) https://www.google.com/search?q=category+of+categories+is+not+locally+cartesian+closed
(2) https://math.stackexchange.com/questions/2864916/are-there-important-locally-cartesian-closed-categories-that-actually-are-not-ca
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but does not have a right adjoint, as proved in example 1.7 of https://sinhp.github.io/files/CT/notes_on_lcccs.pdf

(4) Cartesian closed categories and locally Cartesian closed categories
   (a) https://ncatlab.org/nlab/show/locally+cartesian+closed+functor
   (b) https://ncatlab.org/nlab/show/cartesian+closed+functor
   (c) https://ncatlab.org/nlab/show/locally+cartesian+closed+category
   (d) https://ncatlab.org/nlab/show/Frobenius+reciprocity

(5) https://mathoverflow.net/questions/382239/proof-that-a-cartesian-category-is-monoidal

### 16.1. Cartesian Closed Categories

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CHAPTER 17

# Kan Extensions

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# Part 4

# Bicategories

CHAPTER 18

# Bicategories

Create tags and TODO:

(1) spans in bicategories: add Proposition 7 here: https://arxiv.org/abs/1903.03890
(2) add fact: internal adjunctions in $\mathsf{PseudoFun}(\mathcal{C}, \mathcal{D})$ are precisely the invertible strong transformations as in [JY21, Example 6.2.7]. What are the internal adjunctions?

## 18.1. Monomorphisms in Bicategories

### 18.1.1. Faithful Monomorphisms.

Let $\mathcal{C}$ be a bicategory.

**Definition 18.1.1.1.1.** A 1-morphism $f\colon A \to B$ is a **faithful monomorphism in $\mathcal{C}$** if the following equivalent conditions are satisfied:

(1) For each $X \in \mathrm{Obj}(\mathcal{C})$, the functor
$$f_*\colon \mathsf{Hom}_{\mathcal{C}}(X, A) \to \mathsf{Hom}_{\mathcal{C}}(X, B)$$
given by postcomposition by $f$ is faithful.

(2) Given a diagram in $\mathcal{C}$ of the form
$$\begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & A \arrow[r, "f"] & B, \end{tikzcd}$$
if we have $\mathrm{id}_f \star \alpha = \mathrm{id}_f \star \beta$, then $\alpha = \beta$.

**Example 18.1.1.1.2.** Here are some examples of faithful monomorphisms.

(1) *Full Monomorphisms in* $\mathsf{Cats}_2$.
(2) *Full Monomorphisms in* $\mathbf{Rel}$.
(3) *Full Monomorphisms in* $\mathsf{Span}$.

### 18.1.2. Full Monomorphisms.

Let $\mathcal{C}$ be a bicategory.

**Definition 18.1.2.1.1.** A 1-morphism $f\colon A \to B$ is a **full monomorphism in $\mathcal{C}$** if the following equivalent conditions are satisfied:

(1) For each $X \in \mathrm{Obj}(\mathcal{C})$, the functor
$$f_*\colon \mathsf{Hom}_{\mathcal{C}}(X, A) \to \mathsf{Hom}_{\mathcal{C}}(X, B)$$
given by postcomposition by $f$ is full.

(2) For each $X \in \mathrm{Obj}(\mathcal{C})$ and each 2-morphism
$$\gamma\colon f \circ \phi \Longrightarrow f \circ \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"] \arrow[r, bend right, "f\circ\psi"'] \arrow[r, phantom, "\gamma\Downarrow"] & B \end{tikzcd}$$

of $C$, there exists a 2-morphism $\alpha\colon \phi \Longrightarrow \psi$ of $C$ such that we have an equality

$$\begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & A \arrow[r, "f"] & B \end{tikzcd} = \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"] \arrow[r, bend right, "f\circ\psi"'] \arrow[r, phantom, "\gamma\Downarrow"] & B \end{tikzcd}$$

of pasting diagrams in $C$, i.e. such that we have

$$\gamma = \mathrm{id}_f \star \alpha.$$

**Example 18.1.2.1.2.** Here are some examples of full monomorphisms.

(1) *Full Monomorphisms in* $\mathsf{Cats}_2$*.*
(2) *Full Monomorphisms in* $\mathbf{Rel}$*.*
(3) *Full Monomorphisms in* $\mathsf{Span}$*.*

**18.1.3. Fully Faithful Monomorphisms.** Let $C$ be a bicategory.

**Definition 18.1.3.1.1.** A 1-morphism $f\colon A \to B$ is a **fully faithful monomorphism in** $C$ if the following equivalent conditions are satisfied:

(1) The 1-morphism $f$ is fully and faithful.
(2) For each $X \in \mathrm{Obj}(C)$, the functor

$$f_*\colon \mathsf{Hom}_C(X, A) \to \mathsf{Hom}_C(X, B)$$

given by postcomposition by $f$ is fully faithful.
(3) The conditions in Item 1 of Definition 18.1.1.1.1 and Item 1 of Definition 18.1.2.1.1 hold.

**Example 18.1.3.1.2.** Here are some examples of fully faithful monomorphisms.

(1) *Fully Faithful Monomorphisms in* $\mathsf{Cats}_2$*.*
(2) *Fully Faithful Monomorphisms in* $\mathbf{Rel}$*.*
(3) *Fully Faithful Monomorphisms in* $\mathsf{Span}$*.*

**18.1.4. Strict Monomorphisms.** Let $C$ be a bicategory.

**Definition 18.1.4.1.1.** A 1-morphism $f\colon A \to B$ is a **strict monomorphism in** $C$ if the following equivalent conditions are satisfied:

(1) For each $X \in \mathrm{Obj}(C)$, the action on objects

$$f_*\colon \mathrm{Obj}(\mathsf{Hom}_C(X, A)) \to \mathrm{Obj}(\mathsf{Hom}_C(X, B))$$

of the functor

$$f_*\colon \mathsf{Hom}_C(X, A) \to \mathsf{Hom}_C(X, B)$$

given by postcomposition by $f$ is injective.
(2) For each diagram in $C$ of the form

$$X \underset{\psi}{\overset{\phi}{\rightrightarrows}} A \xrightarrow{f} B,$$

if $f \circ \phi = f \circ \psi$, then $\phi = \psi$.

**Example 18.1.4.1.2.** Here are some examples of strict monomorphisms.

(1) *Strict Monomorphisms in* $\mathsf{Cats}_2$*.*
(2) *Strict Monomorphisms in* $\mathbf{Rel}$*.*
(3) *Strict Monomorphisms in* $\mathsf{Span}$*.*

## 18.2. Epimorphisms in Bicategories

**18.2.1. Faithful Epimorphisms.** Let $C$ be a bicategory.

**Definition 18.2.1.1.1.** A 1-morphism $f\colon A \to B$ is a **faithful epimorphism in $C$** if the following equivalent conditions are satisfied:

(1) For each $X \in \mathrm{Obj}(C)$, the functor
$$f^*\colon \mathsf{Hom}_C(B,X) \to \mathsf{Hom}_C(A,X)$$
given by precomposition by $f$ is faithful.

(2) Given a diagram in $C$ of the form
$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & X, \end{tikzcd}$$
if we have $\alpha \star \mathrm{id}_f = \beta \star \mathrm{id}_f$, then $\alpha = \beta$.

**Example 18.2.1.1.2.** Here are some examples of faithful epimorphisms.

(1) *Full Epimorphisms in* $\mathsf{Cats}_2$.
(2) *Full Epimorphisms in* $\mathbf{Rel}$.
(3) *Full Epimorphisms in* $\mathsf{Span}$.

**18.2.2. Full Epimorphisms.** Let $C$ be a bicategory.

**Definition 18.2.2.1.1.** A 1-morphism $f\colon A \to B$ is a **full epimorphism in $C$** if the following equivalent conditions are satisfied:

(1) For each $X \in \mathrm{Obj}(C)$, the functor
$$f^*\colon \mathsf{Hom}_C(B,X) \to \mathsf{Hom}_C(A,X)$$
given by precomposition by $f$ is full.

(2) For each $X \in \mathrm{Obj}(C)$ and each 2-morphism
$$\gamma\colon \phi \circ f \Longrightarrow \psi \circ f, \qquad \begin{tikzcd} X \arrow[r, bend left, "\phi\circ f"] \arrow[r, bend right, "\psi\circ f"'] \arrow[r, phantom, "\gamma\Downarrow"] & B \end{tikzcd}$$
of $C$, there exists a 2-morphism $\alpha\colon \phi \Longrightarrow \psi$ of $C$ such that we have an equality
$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & X \end{tikzcd} \quad = \quad \begin{tikzcd} A \arrow[r, bend left, "\phi\circ f"] \arrow[r, bend right, "\psi\circ f"'] \arrow[r, phantom, "\gamma\Downarrow"] & X \end{tikzcd}$$
of pasting diagrams in $C$, i.e. such that we have
$$\gamma = \alpha \star \mathrm{id}_f.$$

**Example 18.2.2.1.2.** Here are some examples of full epimorphisms.

(1) *Full Epimorphisms in* $\mathsf{Cats}_2$.
(2) *Full Epimorphisms in* $\mathbf{Rel}$.
(3) *Full Epimorphisms in* $\mathsf{Span}$.

**18.2.3. Fully Faithful Epimorphisms.** Let $C$ be a bicategory.

**Definition 18.2.3.1.1.** A 1-morphism $f\colon A \to B$ is a **fully faithful epimorphism in $C$** if the following equivalent conditions are satisfied:

(1) The 1-morphism $f$ is fully and faithful.
(2) For each $X \in \mathrm{Obj}(C)$, the functor
$$f^*\colon \mathsf{Hom}_C(B, X) \to \mathsf{Hom}_C(A, X)$$
given by precomposition by $f$ is fully faithful.
(3) The conditions in Item 1 of Definition 18.2.1.1.1 and Item 1 of Definition 18.2.2.1.1 hold.

**Example 18.2.3.1.2.** Here are some examples of fully faithful epimorphisms.

(1) *Fully Faithful Epimorphisms in* $\mathsf{Cats}_2$.
(2) *Fully Faithful Epimorphisms in* **Rel**.
(3) *Fully Faithful Epimorphisms in* $\mathsf{Span}$.

**18.2.4. Strict Epimorphisms.** Let $C$ be a bicategory.

**Definition 18.2.4.1.1.** A 1-morphism $f\colon A \to B$ is a **strict epimorphism in $C$** if the following equivalent conditions are satisfied:

(1) For each $X \in \mathrm{Obj}(C)$, the action on objects
$$f^*\colon \mathrm{Obj}(\mathsf{Hom}_C(B, X)) \to \mathrm{Obj}(\mathsf{Hom}_C(A, X))$$
of the functor
$$f^*\colon \mathsf{Hom}_C(B, X) \to \mathsf{Hom}_C(A, X)$$
given by precomposition by $f$ is injective.
(2) For each diagram in $C$ of the form
$$A \xrightarrow{f} B \overset{\phi}{\underset{\psi}{\rightrightarrows}} X,$$
if $\phi \circ f = \psi \circ f$, then $\phi = \psi$.

**Example 18.2.4.1.2.** Here are some examples of strict epimorphisms.

(1) *Strict Epimorphisms in* $\mathsf{Cats}_2$.
(2) *Strict Epimorphisms in* **Rel**.
(3) *Strict Epimorphisms in* $\mathsf{Span}$.

## 18.3. bicategories of spans

**Proposition 18.3.0.1.1.** Let $A$ and $B$ be objects of $C$.

(1) *As a Pullback.* We have an isomorphism of categories
$$\mathsf{Span}_C(A, B) \cong C_{/A} \times_C C_{/B}, \qquad \begin{tikzcd} \mathsf{Span}(A,B) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C_{/B} \arrow[d, "\text{忘}"] \\ C_{/A} \arrow[r, "\text{忘}"'] & C. \end{tikzcd}$$

**Proof.** Item 1, *As a Pullback*: In detail, the pullback $C_{/A} \times_C C_{/B}$ is the category where

- *Objects.* The objects of $C_{/A} \times_C C_{/B}$ consist of pairs $((S, f), (S', g))$ of objects of $C$ consisting of

  - A pair $(S,f)$ in $\mathrm{Obj}\left(\mathcal{C}_{/A}\right)$ consisting of an object $S$ of $\mathcal{C}$ and a morphism $f\colon S\to A$ of $\mathcal{C}$;
  - A pair $(S',g)$ in $\mathrm{Obj}\left(\mathcal{C}_{/B}\right)$ consisting of an object $S'$ of $\mathcal{C}$ and a morphism $g\colon S\to B$ of $\mathcal{C}$;

  such that

  $$\underbrace{忘(S,f)}_{\overset{\text{def}}{=}S} = \underbrace{忘(S',g)}_{\overset{\text{def}}{=}S'}.$$

  Thus the objects of $\mathcal{C}_{/A}\times_{\mathcal{C}}\mathcal{C}_{/B}$ are the same as spans in $\mathcal{C}$ from $A$ to $B$.
- *Morphisms.* A morphism of $\mathcal{C}_{/A}\times_{\mathcal{C}}\mathcal{C}_{/B}$ from $(S,f,g)$ to $(S',f',g')$ consists of a pair of morphisms

  $$\phi\colon S\to S'$$
  $$\psi\colon S\to S'$$

  such that the diagrams

  $$\begin{tikzcd} S \arrow[rr, "\phi"] \arrow[dr, "f"'] & & S' \arrow[dl, "f'"] \\ & A & \end{tikzcd} \qquad \begin{tikzcd} S \arrow[rr, "\psi"] \arrow[dr, "g"'] & & S' \arrow[dl, "g'"] \\ & B & \end{tikzcd}$$

  such that

  $$\underbrace{忘(\phi)}_{\overset{\text{def}}{=}\phi} = \underbrace{忘(\psi)}_{\overset{\text{def}}{=}\psi}.$$

  Thus the morphisms of $\mathcal{C}_{/A}\times_{\mathcal{C}}\mathcal{C}_{/B}$ are also the same as morphisms of spans in $\mathcal{C}$ from $(S,f,g)$ to $(S,f',g')$.
- *Identities and Composition.* The identities and composition of $\mathcal{C}_{/A}\times_{\mathcal{C}}\mathcal{C}_{/B}$ are also the same as those in $\mathsf{Span}_{\mathcal{C}}(A,B)$.

This finishes the proof. □

# Appendices

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(3) Pointed Sets
(4) Tensor Products of Pointed Sets
(5) Relations
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(7) Posets

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(25) Monoids
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(27) Monoids With Zero
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(45) Schemes

## 18.2. Other Chapters

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(1) Sets
(2) Constructions With Sets
(3) Pointed Sets
(4) Tensor Products of Pointed Sets
(5) Relations
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- (25) Monoids
- (26) Constructions With Monoids

Monoids With Zero
- (27) Monoids With Zero
- (28) Constructions With Monoids With Zero

Groups
- (29) Groups
- (30) Constructions With Groups

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- (31) Hypermonoids
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- (33) Hypersemirings and Hyperrings
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- (36) Near-Rings

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- (37) Real Analysis in One Variable
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- (39) Measurable Spaces
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- (40) Probability Theory

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- (41) Stochastic Processes, Martingales, and Brownian Motion
- (42) Itô Calculus
- (43) Stochastic Differential Equations

Differential Geometry
- (44) Topological and Smooth Manifolds

Schemes
- (45) Schemes

CHAPTER 19

# Internal Adjunctions

Create tags:

(1) https://www.google.com/search?q=mate+of+an+adjunction
(2) Moreover, by uniqueness of adjoints (Item 2 of Proposition 19.1.2.1.4), this implies also that $S = f^{-1}$.
(3) define bicategory $\mathsf{Adj}(\mathcal{C})$
(4) walking monad
(5) proposition: 2-functors preserve unitors and associators
(6) https://ncatlab.org/nlab/show/2-category+of+adjunctions. Is there a 3-category too?
(7) https://ncatlab.org/nlab/show/free+monad
(8) https://ncatlab.org/nlab/show/CatAdj
(9) https://ncatlab.org/nlab/show/Adj
(10) $\mathsf{Adj}(\mathsf{Adj}(\mathcal{C}))$

## 19.1. Internal Adjunctions

### 19.1.1. The Walking Adjunction.

**Definition 19.1.1.1.1.** The **walking adjunction** is the bicategory $\mathsf{Adj}$ freely generated by[1]

- *Objects.* A pair of objects $A$ and $B$;
- *Morphisms.* A pair of morphisms

$$L\colon A \to B,$$
$$R\colon B \to A;$$

- 2-*Morphisms.* A pair of 2-morphisms

$$\eta\colon \mathrm{id}_A \to R \circ L,$$
$$\epsilon\colon L \circ R \to \mathrm{id}_B;$$

[1] See [SS86] for an explicit description of the 2-category (as opposed to a bicategory) version of $\mathsf{Adj}$ in terms of finite ordinals, similar to the description of the 2-category version of the walking monad (**??**) as a subcategory of $\Delta$.

subject to the equalities

$$
\begin{tikzcd}
& B \arrow[r, "\mathrm{id}_B"] \arrow[dr, "R"] \arrow[d, phantom, "\Uparrow\eta"] & B \arrow[d, phantom, "\Uparrow\epsilon"] \\
A \arrow[ur, "L"] \arrow[r, "\mathrm{id}_A"'] & A \arrow[ur, "L"'] & B
\end{tikzcd}
=
\begin{tikzcd}
& B \arrow[r, "\mathrm{id}_B"] & B \\
A \arrow[ur, "L"] \arrow[r, "\mathrm{id}_A"'] & A \arrow[ul, Rightarrow, "\mathrm{id}_L"] \arrow[ur, "L"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
& A \arrow[r, "\mathrm{id}_A"] \arrow[dr, "L"] \arrow[d, phantom, "\Downarrow\epsilon"] & A \arrow[d, phantom, "\Downarrow\eta"] \\
B \arrow[ur, "R"] \arrow[r, "\mathrm{id}_B"'] & B \arrow[ur, "R"'] & 
\end{tikzcd}
=
\begin{tikzcd}
& A \arrow[r, "\mathrm{id}_A"] \arrow[dr, Rightarrow, "\mathrm{id}_R"] & A \\
B \arrow[ur, "R"] \arrow[r, "\mathrm{id}_B"'] & B \arrow[ur, "R"'] &
\end{tikzcd}
$$

of pasting diagrams, which are equivalent to the following conditions:

(1) *The Left Triangle Identity.* The diagram

$$
\begin{tikzcd}
L \circ \mathrm{id}_A \arrow[r, Rightarrow, "\mathrm{id}_L \star \eta"] \arrow[ddrr, Rightarrow, "\rho^{\mathsf{Adj}}_L"'] & L \circ (R \circ L) \arrow[r, Rightarrow, "(\alpha^{\mathsf{Adj}}_{L,R,L})^{-1}"] & (L \circ R) \circ L \arrow[d, Rightarrow, "\epsilon \star \mathrm{id}_L"] \\
& & \mathrm{id}_B \circ L \arrow[d, Rightarrow, "\lambda^{\mathsf{Adj}}_L"] \\
& & L
\end{tikzcd}
$$

commutes.

(2) *The Right Triangle Identity.* The diagram

$$
\begin{tikzcd}
\mathrm{id}_A \circ R \arrow[r, Rightarrow, "\eta \star \mathrm{id}_R"] \arrow[ddrr, Rightarrow, "\lambda^{\mathsf{Adj}}_R"'] & (R \circ L) \circ R \arrow[r, Rightarrow, "\alpha^{\mathsf{Adj}}_{R,L,R}"] & R \circ (L \circ R) \arrow[d, Rightarrow, "\mathrm{id}_R \star \epsilon"] \\
& & R \circ \mathrm{id}_B \arrow[d, Rightarrow, "\rho^{\mathsf{Adj}}_R"] \\
& & R.
\end{tikzcd}
$$

**19.1.2. Internal Adjunctions.** Let $C$ be a bicategory.

**Definition 19.1.2.1.1.** An **internal adjunction in $C$**[2,3] is a 2-functor $\mathsf{Adj} \to C$.

**Remark 19.1.2.1.2.** In detail, an **internal adjunction in $C$** consists of

---

[2] *Further Terminology:* Also called an **adjunction internal to $C$**.

[3] *Further Terminology:* In this situation, we also call $(g, f)$ an **adjoint pair**, $f$ the **left adjoint** of the pair, $g$ the **right adjoint** of the pair, $\eta$ the **unit** of the adjunction, and $\epsilon$ the **counit** of the adjunction.

- *Objects.* A pair of objects $A$ and $B$ of $\mathcal{C}$;
- *Morphisms.* A pair of morphisms

$$L\colon A \to B,$$
$$R\colon B \to A$$

  of $\mathcal{C}$;
- 2-*Morphisms.* A pair of 2-morphisms

$$\eta\colon \mathrm{id}_A \to R \circ L,$$
$$\epsilon\colon L \circ R \to \mathrm{id}_B$$

  of $\mathcal{C}$;

subject to the equalities

$$\begin{tikzcd}
& B \arrow[rr, "\mathrm{id}_B"] \arrow[dr, "R"] \arrow[d, phantom, "\Uparrow\eta"] & \arrow[d, phantom, "\Uparrow\epsilon"] & B \\
A \arrow[ur, "L"] \arrow[rr, "\mathrm{id}_A"'] & {} & A \arrow[ur, "L"'] &
\end{tikzcd}
=
\begin{tikzcd}
& B \arrow[rr, "\mathrm{id}_B"] & & B \\
A \arrow[ur, "L"] \arrow[rr, "\mathrm{id}_A"'] & & A \arrow[ul, Rightarrow, "\mathrm{id}_L"] \arrow[ur, "L"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
& A \arrow[rr, "\mathrm{id}_A"] \arrow[dr, "L"] \arrow[d, phantom, "\Downarrow\epsilon"] & \arrow[d, phantom, "\Downarrow\eta"] & A \\
B \arrow[ur, "R"] \arrow[rr, "\mathrm{id}_B"'] & {} & B \arrow[ur, "R"'] &
\end{tikzcd}
=
\begin{tikzcd}
& A \arrow[rr, "\mathrm{id}_A"] \arrow[dr, Rightarrow, "\mathrm{id}_R"] & & A \\
B \arrow[ur, "R"] \arrow[rr, "\mathrm{id}_B"'] & & B \arrow[ur, "R"'] &
\end{tikzcd}$$

of pasting diagrams in $\mathcal{C}$, which are equivalent to the following conditions:[4]

(1) *The Left Triangle Identity.* The diagram

$$\begin{tikzcd}
L \circ \mathrm{id}_A \arrow[r, Rightarrow, "\mathrm{id}_L \star \eta"] \arrow[ddrr, Rightarrow, "\rho^{\mathcal{C}}_L"'] & L \circ (R \circ L) \arrow[r, Rightarrow, "(\alpha^{\mathcal{C}}_{L,R,L})^{-1}"] & (L \circ R) \circ L \arrow[d, Rightarrow, "\epsilon \star \mathrm{id}_L"] \\
& & \mathrm{id}_B \circ L \arrow[d, Rightarrow, "\lambda^{\mathcal{C}}_L"] \\
& & L
\end{tikzcd}$$

commutes.

---

[4]When $\mathcal{C}$ is a 2-category, these diagrams take the following form:

$$\begin{tikzcd}
L \arrow[r, Rightarrow, "\mathrm{id}_L \star \eta"] \arrow[dr, Rightarrow, "\mathrm{id}_L"'] & L \circ R \circ L \arrow[d, Rightarrow, "\epsilon \star \mathrm{id}_L"] \\
& L
\end{tikzcd}
\qquad
\begin{tikzcd}
R \arrow[r, Rightarrow, "\mathrm{id}_R \star \eta"] \arrow[dr, Rightarrow, "\mathrm{id}_L"'] & R \circ L \circ R \arrow[d, Rightarrow, "\epsilon \star \mathrm{id}_R"] \\
& R.
\end{tikzcd}$$

(2) *The Right Triangle Identity.* The diagram

$$\begin{tikzcd}
\mathrm{id}_A \circ R \arrow[r, Rightarrow, "\eta \star \mathrm{id}_R"] \arrow[ddrr, Rightarrow, "\lambda^{\mathcal{C}}_R"'] & (R \circ L) \circ R \arrow[r, Rightarrow, "\alpha^{\mathcal{C}}_{R,L,R}"] & R \circ (L \circ R) \arrow[d, Rightarrow, "\mathrm{id}_R \star \epsilon"] \\
 & & R \circ \mathrm{id}_B \arrow[d, Rightarrow, "\rho^{\mathcal{C}}_R"] \\
 & & R.
\end{tikzcd}$$

**Example 19.1.2.1.3.** Here are some examples of internal adjunctions.

(1) *Internal Adjunctions in* $\mathsf{Cats}_2$*.* The internal adjunctions in the 2-category $\mathsf{Cats}_2$ of categories, functors, and natural transformations are precisely the adjunctions of **??**.

(2) *Internal Adjunctions in* **Rel**. The internal adjunctions in **Rel** are precisely the relations of the form $\mathrm{Gr}(f) \dashv f^{-1}$ with $f$ a function; see **??** of **??**.

(3) *Internal Adjunctions in* $\mathsf{Span}$*.* The internal adjunctions in $\mathsf{Span}$ are precisely the spans of the form

$$\begin{tikzcd}
 & S \arrow[dl, "\phi"'] \arrow[dr, "g"] & \\
A & & B
\end{tikzcd}$$

with $\phi$ an isomorphism; see **??** of **??**.

**Proposition 19.1.2.1.4.** Let $\mathcal{C}$ be a bicategory.

(1) *Duality.* Let $(f, g, \eta, \epsilon)$ be an internal adjunction in $\mathcal{C}$.
  (a) The quadruple $(g, f, \eta, \epsilon)$ is an internal adjunction in $\mathcal{C}^{\mathsf{op}}$.
  (b) The quadruple $(g, f, \epsilon, \eta)$ is an internal adjunction in $\mathcal{C}^{\mathsf{co}}$.
  (c) The quadruple $(f, g, \eta, \epsilon)$ is an internal adjunction in $\mathcal{C}^{\mathsf{coop}}$.

(2) *Uniqueness of Adjoints.* Let $(f, g, \eta, \epsilon)$ and $(f, g', \eta', \epsilon')$ be internal adjunctions in $\mathcal{C}$. We have a canonical isomorphism[5]

$$g \xRightarrow{(\lambda^{\mathcal{C}}_g)^{-1}} \mathrm{id}_A \circ g \xRightarrow{\eta' \star \mathrm{id}_g} (g' \circ f) \circ g \xRightarrow{\alpha^{\mathcal{C}}_{g',f,g}} g' \circ (f \circ g) \xRightarrow{\mathrm{id}_{g'} \star \epsilon} g' \circ \mathrm{id}_B \xRightarrow{(\rho^{\mathcal{C}}_{g'})^{-1}} g'$$

with inverse

$$g' \xRightarrow{(\lambda^{\mathcal{C}}_{g'})^{-1}} \mathrm{id}_B \circ g' \xRightarrow{\eta \star \mathrm{id}_{g'}} (g \circ f) \circ g' \xRightarrow{\alpha^{\mathcal{C}}_{g',f,g}} g \circ (f \circ g') \xRightarrow{\mathrm{id}_g \star \epsilon'} g \circ \mathrm{id}_B \xRightarrow{(\lambda^{\mathcal{C}}_g)^{-1}} g.$$

(3) *Carrying Internal Adjunctions Through Pseudofunctors.* Let $F \colon \mathcal{C} \longrightarrow \mathcal{D}$ be a pseudofunctor and $(f, g, \eta, \epsilon)$ be an internal adjunction in $\mathcal{C}$. There is an induced internal adjunction[6]

$$(F(f), F(g), \overline{\eta}, \overline{\epsilon})$$

[5] *Slogan:* Left adjoints are unique up to canonical isomorphism. Dually, so are right adjoints.

[6] *Warning:* Lax or oplax functors which are not pseudofunctors need not preserve internal adjunctions.

in $\mathcal{D}$, where:

(a) The unit

$$\overline{\eta}\colon \mathrm{id}_{F(A)} \Longrightarrow F(g) \circ F(f)$$

is the composition

$$\mathrm{id}_{F(A)} \xRightarrow{F_A} F(\mathrm{id}_A) \xRightarrow{F(\eta)} F(g \circ f) \xRightarrow{F^{-1}_{g,f}} F(g) \circ F(f).$$

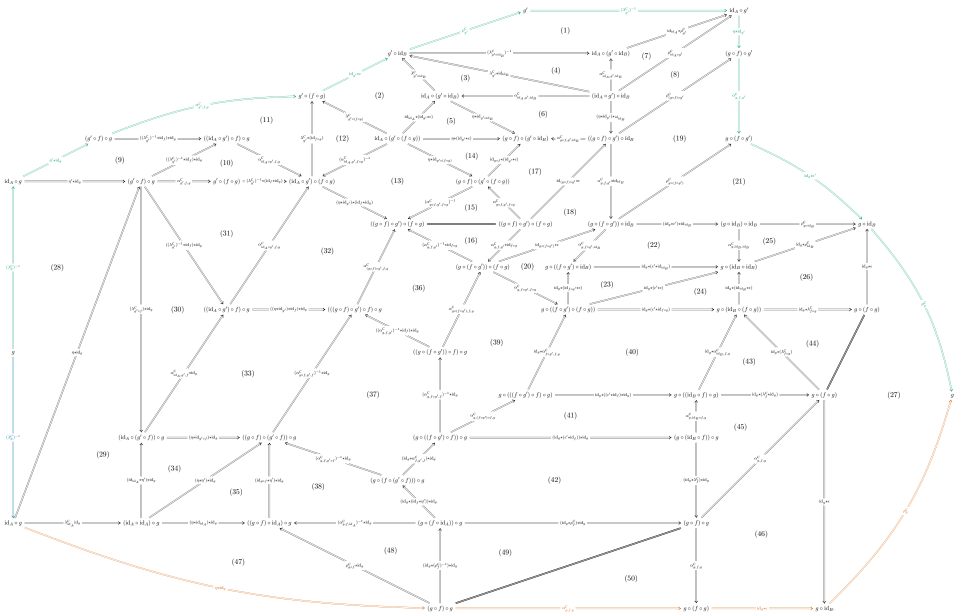
(b) The counit

$$\overline{\epsilon}\colon F(f) \circ F(g) \Longrightarrow \mathrm{id}_{F(B)}$$

is the composition

$$F(f) \circ F(g) \xRightarrow{F_{f,g}} F(f \circ g) \xRightarrow{F(\epsilon)} F(\mathrm{id}_B) \xRightarrow{F_B} \mathrm{id}_{F(B)}.$$

**Proof.** *Item 1, Duality*: Omitted.[7]
*Item 2, Uniqueness of Adjoints*: [8]Consider the diagram (if you *really* want to consider it I fear you will need to zoom in)



In this diagram:

(1) The morphisms in green are the composition $g \xRightarrow{\cong} g' \xRightarrow{\cong} g$;
(2) The morphisms in red are equal to $\lambda^{\mathcal{C}}_g$ by the right triangle identity for $(f, g, \eta, \epsilon)$. Hence the composition of the morphism in blue with the morphisms in red is the identity;
(3) Subdiagrams (1), (2), (10), (11), (29), (31), and (43) commute by the naturality of the left unitor of $\mathcal{C}$ and its inverse;
(4) Subdiagrams (8), (19), and (21) commute by the naturality of the right unitor of $\mathcal{C}$ and its inverse;

[7] *Reference:* [JY21, Exercise 6.6.2].
[8] *Reference:* [JY21, Lemma 6.1.6].

(5) Subdiagrams (6), (13), (17), (18), (20), (22), (32), (33), (36), (38), (40), (41), and (45) commute by the naturality of the associator of $\mathcal{C}$ and its inverse;
(6) Subdiagrams (37), (39), and (42) commute by the pentagon identity for $\mathcal{C}$;
(7) Subdiagrams (3), (4), (7), (12), (25), (30), and (48) commute by **??** of **??**;
(8) Subdiagrams (5), (14), (23), (24), (34), and (35) commute by middle-four exchange;
(9) Subdiagrams (9), (15), (16), (27), (28), (44), (46), (49), and (50) commute trivially;
(10) Subdiagram (26) commutes by **????** of **??**;
(11) Subdiagram (47) commutes by **??** of **??** and the naturality of the left unitor of right unitor of $\mathcal{C}$.

Hence $g \cong g'$.

*Item 3, Carrying Internal Adjunctions Through Pseudofunctors*: [9]We claim that the left and right triangle identities for $(F(f), F(g), \overline{\eta}, \overline{\epsilon})$ hold:

(1) The left triangle identity for $(F(f), F(g), \overline{\eta}, \overline{\epsilon})$ is the condition that the boundary diagram of the diagram (you may need to zoom in)

$$
\begin{tikzcd}
F(f)\circ \mathrm{id}_{F(A)} \arrow[rrr, Rightarrow, "\mathrm{id}_{F(f)}\star\overline{\eta}"] \arrow[dr, Rightarrow, "\mathrm{id}_{F(f)}\star F_A"] \arrow[dd, Rightarrow, "\rho^{\mathcal{D}}_{F(f)}"'] & \arrow[dr, phantom, "(1)"] & & F(f)\circ(F(g)\circ F(f)) \arrow[dr, Rightarrow, "(\alpha^{\mathcal{D}}_{F(f),F(g),F(f)})^{-1}"] & & \\
\arrow[dr, phantom, "(2)"] & F(f)\circ F(\mathrm{id}_A) \arrow[r, Rightarrow, "\mathrm{id}_{F(f)}\star F(\eta)"] \arrow[dd, Rightarrow, "F_{f,\mathrm{id}_A}"] & F(f)\circ F(g\circ f) \arrow[ur, Rightarrow, "\mathrm{id}_{F(f)}\star F^{-1}_{g,f}"] \arrow[dd, Rightarrow, "F_{f,g\circ f}"] & & (F(f)\circ F(g))\circ F(f) \arrow[dd, Rightarrow, "F_{f,g}\star\mathrm{id}_{F(f)}"] \arrow[dddddr, Rightarrow, bend left, "\overline{\epsilon}\star\mathrm{id}_{F(f)}"] & \\
F(f) \arrow[dr, Rightarrow, "(F(\rho^{\mathcal{C}}_f))^{-1}"'] & \arrow[r, phantom, "(3)"] & \arrow[rr, phantom, "(4)"] & & & \\
& F(f\circ\mathrm{id}_A) \arrow[r, Rightarrow, "F(\mathrm{id}_f\star\eta)"] \arrow[ddddrrr, Rightarrow, bend right, "F(\rho^{\mathcal{C}}_f)"'] & F(f\circ(g\circ f)) \arrow[dr, Rightarrow, "F((\alpha^{\mathcal{C}}_{f,g,f})^{-1})"'] & & F(f\circ g)\circ F(f) \arrow[dl, Rightarrow, "F_{f\circ g,f}"] \arrow[dd, Rightarrow, "F(\epsilon)\star\mathrm{id}_{F(f)}"] \arrow[dr, phantom, "(7)"] & \\
& & \arrow[d, phantom, "(5)"] & F((f\circ g)\circ f) \arrow[dd, Rightarrow, "F(\epsilon\star\mathrm{id}_f)"] \arrow[dr, phantom, "(6)"] & & \\
& & & & F(\mathrm{id}_B)\circ F(f) \arrow[dl, Rightarrow, "F_{\mathrm{id}_B,f}"'] \arrow[dr, Rightarrow, "F_B^{-1}\star\mathrm{id}_{F(f)}"] & \\
& & & F(\mathrm{id}_B\circ f) \arrow[dr, Rightarrow, "F(\lambda^{\mathcal{C}}_f)"'] \arrow[rr, phantom, "(8)"] & & \mathrm{id}_{F(B)}\circ F(f) \arrow[dl, Rightarrow, "\lambda^{\mathcal{D}}_{F(f)}"] \\
& & & & F(f). &
\end{tikzcd}
$$

commutes. Since

(a) Subdiagrams (1) and (7) commute by applying middle-four exchange twice,
(b) Subdiagrams (2) and (8) commute by the left and right lax unity conditions for $F$,
(c) Subdiagrams (3) and (6) commute by the naturality of the lax functoriality constraints of $F$,

[9] *Reference:* [JY21, Proposition 6.1.7].

(d) Subdiagram (4) commutes by the lax associativity condition for $F$, and
(e) Subdiagram (5) commutes by the left triangle identity for $(f, g, \eta, \epsilon)$,

so does the boundary diagram.

(2) The right triangle identity for $(F(f), F(g), \overline{\eta}, \overline{\epsilon})$ is the condition that the boundary diagram of the diagram (you may need to zoom in)

$$\begin{tikzcd}[arrows=Rightarrow]
\mathrm{id}_{F(A)} \circ F(g) \arrow[rrr, "\overline{\eta} \star \mathrm{id}_{F(g)}"] \arrow[dr, "F_A \star \mathrm{id}_{F(g)}"] \arrow[dd, "\lambda^{\mathcal{D}}_{F(g)}"'] & & (1) & (F(g) \circ F(f)) \circ F(g) \arrow[dr, "\alpha^{\mathcal{D}}_{F(g),F(f),F(g)}"] & & \\
(2) & F(\mathrm{id}_A) \circ F(g) \arrow[r, "F(\eta) \star \mathrm{id}_{F(g)}"] \arrow[dd, "F_{\mathrm{id}_A, g}"] & F(g \circ f) \circ F(g) \arrow[ur, "F^{-1}_{g,f} \star \mathrm{id}_{F(g)}"] \arrow[dd, "F_{g \circ f, g}"] & & F(g) \circ (F(f) \circ F(g)) \arrow[dd, "\mathrm{id}_{F(g)} \star F_{f,g}"] \arrow[ddddr, bend left, "\mathrm{id}_{F(g)} \star \overline{\epsilon}"] & \\
F(g) \arrow[dr, "(F(\lambda^{\mathcal{C}}_g))^{-1}"'] & & (3) & (4) & & \\
& F(\mathrm{id}_A \circ g) \arrow[r, "F(\eta \star \mathrm{id}_g)"] \arrow[ddddrrr, bend right, "F(\lambda^{\mathcal{C}}_g)"'] & F((g \circ f) \circ g) \arrow[dr, "F(\alpha^{\mathcal{C}}_{g,f,g})"'] & & F(g) \circ F(f \circ g) \arrow[dl, "F_{g, f \circ g}"'] \arrow[dd, "\mathrm{id}_{F(g)} \star F(\epsilon)"] & (7) \\
& & & F(g \circ (f \circ g)) \arrow[dd, "F(\mathrm{id}_g \star \epsilon)"] & & \\
& & (5) & (6) & F(g) \circ F(\mathrm{id}_B) \arrow[dl, "F_{g, \mathrm{id}_B}"'] \arrow[dr, "\mathrm{id}_{F(g)} \star F_B^{-1}"] & \\
& & & F(g \circ \mathrm{id}_B) \arrow[dr, "F(\rho^{\mathcal{C}}_g)"'] & (8) & F(g) \circ \mathrm{id}_{F(B)} \arrow[dl, "\rho^{\mathcal{D}}_{F(g)}"] \\
& & & & F(g). &
\end{tikzcd}$$

commutes. Since

(a) Subdiagrams (1) and (7) commute by applying middle-four exchange twice,
(b) Subdiagrams (2) and (8) commute by the left and right lax unity conditions for $F$,
(c) Subdiagrams (3) and (6) commute by the naturality of the lax functoriality constraints of $F$,
(d) Subdiagram (4) commutes by the lax associativity condition for $F$, and
(e) Subdiagram (5) commutes by the right triangle identity for $(f, g, \eta, \epsilon)$,

so does the boundary diagram.

This finishes the proof. □

### 19.1.3. Internal Adjoint Equivalences.

Let $\mathcal{C}$ be a bicategory.

**Definition 19.1.3.1.1.** An internal adjunction $(f, g, \eta, \epsilon)$ in $\mathcal{C}$ is an **internal adjoint equivalence** if $\eta$ and $\epsilon$ are isomorphisms in $\mathcal{C}$.

**Example 19.1.3.1.2.** Here are some examples of internal adjoint equivalences.

(1) *Internal Adjoint Equivalences in* $\mathsf{Cats}_2$*.* The internal adjoint equivalences in the 2-category $\mathsf{Cats}_2$ of categories, functors, and natural transformations are precisely the adjoint equivalences of **??**.[10]
(2) *Internal Adjoint Equivalences in* $\mathsf{Mod}$*.* The internal adjoint equivalences in $\mathsf{Mod}$ are precisely the invertible $R$-modules; see **??**.[11]
(3) *Internal Adjoint Equivalences in* $\mathsf{PseudoFun}(\mathcal{C}, \mathcal{D})$*.* The internal adjoint equivalences in $\mathsf{PseudoFun}(\mathcal{C}, \mathcal{D})$ are precisely the invertible strong transformations; see **??**.[12]
(4) *Internal Adjoint Equivalences in* $\mathbf{Rel}$*.* The internal adjoint equivalences in $\mathbf{Rel}$ are precisely the relations of the form $\mathrm{Gr}(f) \dashv f^{-1}$ with $f$ an isomorphism; see **??**.
(5) *Internal Adjoint Equivalences in* $\mathsf{Span}$*.* The internal adjoint equivalences in $\mathsf{Span}$ are precisely the spans of the form $A \xleftarrow{\phi} S \xrightarrow{\psi} B$ with $\phi$ and $\psi$ isomorphisms; see **??**.

**Proposition 19.1.3.1.3.** Let $\mathcal{C}$ be a bicategory.

(1) *Carrying Internal Adjoint Equivalences Through Pseudofunctors.* Let $F \colon \mathcal{C} \longrightarrow \mathcal{D}$ be a pseudofunctor and $(f, g, \eta, \epsilon)$ be an internal adjunction in $\mathcal{C}$. If $(f, g, \eta, \epsilon)$ is an internal adjoint equivalence in $\mathcal{C}$, then the induced internal adjunction
$$(F(f), F(g), \overline{\eta}, \overline{\epsilon})$$
in $\mathcal{D}$ of Item 3 of Proposition 19.1.2.1.4 is an internal adjoint equivalence as well.
(2) *Internal Adjunctions Always Refine to Internal Adjoint Equivalences.* Let $(f, g, \eta, \epsilon)$ be an internal adjunction in $\mathcal{C}$. If $f$ is an equivalence, then there exist 2-morphisms
$$\begin{aligned} \overline{\eta} &\colon \mathrm{id}_A \Longrightarrow g \circ f \\ \overline{\epsilon} &\colon f \circ g \Longrightarrow \mathrm{id}_B \end{aligned}$$
of $\mathcal{C}$ such that $(f, g, \overline{\eta}, \overline{\epsilon})$ is an internal adjoint equivalence.

**Proof.** *Item 1, Carrying Internal Adjoint Equivalences Through Pseudofunctors*: See [JY21, Proposition 6.2.3].
*Item 2, Internal Adjunctions Always Refine to Internal Adjoint Equivalences*: See [JY21, Proposition 6.2.4]. □

**19.1.4. Mates.** Let $\mathcal{C}$ be a bicategory, let $(f, g, \eta, \epsilon)$ and $(f', g', \eta', \epsilon')$ be adjunctions, and let $h$ and $k$ be morphisms of $\mathcal{C}$ as in the diagram
$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, phantom, "\perp"] \arrow[d, "h"'] & B \arrow[l, bend left, "g"] \arrow[d, "k"] \\ C \arrow[r, bend left, "f'"] \arrow[r, phantom, "\perp"] & D. \arrow[l, bend left, "g'"] \end{tikzcd}$$

[10] *Reference:* [JY21, Examples 6.2.5].
[11] *Reference:* [JY21, Examples 6.2.6].
[12] *Reference:* [JY21, Examples 6.2.7].

**Definition 19.1.4.1.1.** The **mates** of a pair of 2-morphisms

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "h"'] & B \arrow[d, "k"] \\
C \arrow[r, "f'"'] \arrow[ur, phantom, "\overset{\omega}{\Nearrow}"] & D
\end{tikzcd}
\qquad
\begin{aligned}
\omega &\colon f'\circ h \Longrightarrow k\circ f,\\
\nu &\colon h\circ g \Longrightarrow g'\circ k
\end{aligned}
\qquad
\begin{tikzcd}
A \arrow[d, "h"'] \arrow[dr, phantom, "\overset{\nu}{\Searrow}"] & B \arrow[l, "g"'] \arrow[d, "k"] \\
C & D \arrow[l, "g'"]
\end{tikzcd}
$$

are the 2-morphisms

$$
\begin{tikzcd}
A \arrow[d, "h"'] \arrow[dr, phantom, "\overset{\omega^\dagger}{\Searrow}"] & B \arrow[l, "g"'] \arrow[d, "k"] \\
C & D \arrow[l, "g'"]
\end{tikzcd}
\qquad
\begin{aligned}
\omega^\dagger &\colon h\circ g \Longrightarrow g'\circ k,\\
\nu^\dagger &\colon f'\circ h \Longrightarrow k\circ f
\end{aligned}
\qquad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "h"'] & B \arrow[d, "k"] \\
C \arrow[r, "f'"'] \arrow[ur, phantom, "\overset{\nu^\dagger}{\Nearrow}"] & D
\end{tikzcd}
$$

defined as the pastings of the diagrams[13]

$$
\begin{tikzcd}
B \arrow[rrr, bend left=10, "k"] \arrow[dr, "g"] \arrow[drr, bend left, "\mathrm{id}_B"] \arrow[ddd, bend right, "g"'] & & & D \arrow[ddd, bend left, "g'"] \\
& A \arrow[r, "f"] \arrow[d, "h"] \arrow[u, phantom, "\overset{\epsilon}{\Uparrow}"] & B \arrow[d, "k"] \arrow[ur, phantom, "\overset{\rho^C_{h\circ g}}{\Nearrow}"] & \\
& C \arrow[r, "f'"'] \arrow[ur, phantom, "\overset{\omega}{\Nearrow}"] \arrow[drr, bend right, "\mathrm{id}_C"'] & D \arrow[dr, "g'"] \arrow[d, phantom, "\overset{\eta'}{\Uparrow}"] & \\
A \arrow[rrr, bend right=10, "h"'] \arrow[ur, phantom, "\overset{(\lambda^C_{h\circ g})^{-1}}{\Nearrow}"] & & & C.
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[rrr, bend left=10, "f"] \arrow[dr, "f"] \arrow[ddr, bend right, "\mathrm{id}_A"'] \arrow[ddd, bend right, "h"'] & & & B \arrow[ddd, bend left, "k"] \\
& B \arrow[r, "k"] \arrow[d, "g"] \arrow[l, phantom, "\overset{\eta}{\Rightarrow}"] & D \arrow[d, "g'"] \arrow[ddr, bend left, "\mathrm{id}_D"] \arrow[ur, phantom, "\overset{\lambda^C_{h\circ g}}{\Nearrow}"] & \\
& A \arrow[r, "h"'] \arrow[ur, phantom, "\overset{\nu}{\Nearrow}"] & C \arrow[dr, "f'"] \arrow[r, phantom, "\overset{\epsilon'}{\Rightarrow}"] & \\
C \arrow[rrr, bend right=10, "f'"'] \arrow[ur, phantom, "\overset{(\rho^C_{h\circ g})^{-1}}{\Nearrow}"] & & & D.
\end{tikzcd}
$$

**Proposition 19.1.4.1.2.** Let $\omega\colon f'\circ h \Longrightarrow k\circ f$ and $\nu\colon h\circ g \Longrightarrow g'\circ k$ be 2-morphisms.

(1) *The Mate Correspondence.* The map

$$
\begin{aligned}
(-)^\dagger\colon \mathrm{Hom}_{\mathsf{Hom}_C(A,C)}(f'\circ h, k\circ f) &\longrightarrow \mathrm{Hom}_{\mathsf{Hom}_C(B,D)}(h\circ g, g'\circ k)\\
\omega &\longmapsto \omega^\dagger
\end{aligned}
$$

---

[13] If $C$ is a 2-category, these pasting diagrams become the following:

$$
\begin{tikzcd}
B \arrow[r, "g"] \arrow[d, "k"'] & A \arrow[d, "h"] \\
D \arrow[r, "g'"'] \arrow[ur, phantom, "\overset{\omega^\dagger}{\Swarrow}"] & C
\end{tikzcd}
=
\begin{tikzcd}
B \arrow[r, "g"] \arrow[dr, "\mathrm{id}_B"'] & A \arrow[r, "h"] \arrow[d, "f"] \arrow[dl, phantom, "\overset{\epsilon}{\Swarrow}"] & C \arrow[d, "f'"] \arrow[dr, "\mathrm{id}_C"] \arrow[dl, phantom, "\overset{\omega}{\Swarrow}"] & \\
& B \arrow[r, "k"'] & D \arrow[r, "g'"'] \arrow[ur, phantom, "\overset{\eta'}{\Swarrow}"] & C.
\end{tikzcd}
$$

$$
\begin{tikzcd}
C \arrow[r, "f'"] \arrow[dr, phantom, "\overset{\nu^\dagger}{\Searrow}"] & D \\
A \arrow[u, "h"] \arrow[r, "f"'] & B \arrow[u, "k"']
\end{tikzcd}
=
\begin{tikzcd}
& A \arrow[r, "h"] \arrow[dr, phantom, "\overset{\nu}{\Searrow}"] & C \arrow[r, "f'"] \arrow[dr, phantom, "\overset{\epsilon'}{\Searrow}"] & D \\
A \arrow[ur, "\mathrm{id}_A"] \arrow[r, "f"'] \arrow[ur, phantom, "\overset{\eta}{\Searrow}"] & B \arrow[u, "g"] \arrow[r, "k"'] & D. \arrow[u, "g'"'] \arrow[ur, "\mathrm{id}_D"'] &
\end{tikzcd}
$$

is a bijection.

**Proof.** *Item 1, The Mate Correspondence*: Here we give a proof for 2-categories (which indirectly proves also the general case by **??**). A proof for general bicategories can be found in [JY21, Lemma 6.1.13].
Let

$$\nu\colon h\circ g \Longrightarrow g'\circ k \qquad \begin{tikzcd} A \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow\nu"] & B \arrow[l, "g"'] \arrow[d, "k"] \\ C & D \arrow[l, "g'"] \end{tikzcd}$$

be a 2-morphism of $C$. The mate $\nu^\dagger$ of $\nu$ is then given by

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h"'] \arrow[dr, phantom, "\Uparrow\nu^\dagger"] & B \arrow[d, "k"] \\ C \arrow[r, "f'"'] & D \end{tikzcd} = \begin{tikzcd} & A \arrow[dl, "\mathrm{id}_A"'] \arrow[d, "f"] \arrow[dl, phantom, shift left=3, "\Downarrow\eta"] \\ A \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow\nu"] & B \arrow[l, "g"'] \arrow[d, "k"] \\ C \arrow[d, "f'"'] \arrow[dr, phantom, shift left=3, "\Downarrow\epsilon'"] & D \arrow[l, "g'"'] \arrow[dl, "\mathrm{id}_D"] \\ D, & \end{tikzcd}$$

and the mate of $\nu^\dagger$ is the 2-morphism $\left(\nu^\dagger\right)^\dagger\colon f'\circ h \Longrightarrow k\circ f$ given by

$$\begin{tikzcd} A \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow(\nu^\dagger)^\dagger"] & B \arrow[l, "g"'] \arrow[d, "k"] \\ C & D. \arrow[l, "g'"] \end{tikzcd} = \begin{tikzcd} & & A \arrow[dl, "\mathrm{id}_A"'] \arrow[d, "f"] \arrow[dl, phantom, shift left=3, "\Downarrow\eta"] \arrow[r, phantom, shift right=3, "\Downarrow\epsilon"] & B \arrow[l, "g"'] \arrow[dl, "\mathrm{id}_B"] \\ & A \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow\nu"] & B \arrow[l, "g"'] \arrow[d, "k"] & \\ & C \arrow[dl, "\mathrm{id}_C"'] \arrow[d, "f'"] \arrow[dl, phantom, shift left=3, "\Downarrow\eta'"] \arrow[r, phantom, shift right=3, "\Downarrow\epsilon'"] & D \arrow[l, "g'"'] \arrow[dl, "\mathrm{id}_D"] & \\ C & D \arrow[l, "g'"] & & \end{tikzcd} = \begin{tikzcd} & & A \arrow[dl, "\mathrm{id}_A"'] \arrow[d, phantom, "\Downarrow\mathrm{id}_g"] & B \arrow[l, "g"'] \arrow[dl, "\mathrm{id}_B"] \\ & A \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow\nu"] & B \arrow[l, "g"'] \arrow[d, "k"] & \\ & C \arrow[dl, "\mathrm{id}_C"'] \arrow[d, phantom, "\Downarrow\mathrm{id}_{g'}"] & D \arrow[l, "g'"'] \arrow[dl, "\mathrm{id}_D"] & \\ C & D \arrow[l, "g'"] & & \end{tikzcd} = \begin{tikzcd} A \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow\nu"] & B \arrow[l, "g"'] \arrow[d, "k"] \\ C & D. \arrow[l, "g'"] \end{tikzcd}$$

Similarly, $(\omega)^{\dagger\dagger} = \omega$. ☐

## 19.2. Morphisms of Internal Adjunctions

**19.2.1. Lax Morphisms of Internal Adjunctions.** Let $C$ be a bicategory and let $(A, B, F, G, \eta, \epsilon)$ and $(A', B', F', G', \eta', \epsilon')$ be internal adjunctions in $C$.

**Definition 19.2.1.1.1.** A **lax morphism of internal adjunctions** from $(A, B, F, G, \eta, \epsilon)$ to $(A', B', F', G', \eta', \epsilon')$ is a lax transformation between these viewed as 2-functors from the walking adjunction.

**Remark 19.2.1.1.2.** In detail, a **lax morphism of internal adjunctions** from $(A, B, F, G, \eta, \epsilon)$ to $(A', B', F', G', \eta', \epsilon')$ consists of

- 1-*Morphisms.* A pair of 1-morphisms
$$\phi \colon A \to A',$$
$$\psi \colon B \to B'$$
of $C$;
- 2-*Morphisms.* A pair of 2-morphisms
$$\begin{tikzcd} A \arrow[r, "F"] \arrow[d, "\phi"'] & B \arrow[d, "\psi"] \\ A' \arrow[r, "F'"'] \arrow[ur, phantom, "\overset{\alpha}{\Nearrow}"] & B' \end{tikzcd} \qquad \begin{aligned} \alpha &\colon F' \circ \phi \Rightarrow \psi \circ F, \\ \beta &\colon G' \circ \phi \Rightarrow \psi \circ G \end{aligned} \qquad \begin{tikzcd} A \arrow[d, "\phi"'] & B \arrow[l, "G"'] \arrow[d, "\psi"] \\ A' & B' \arrow[l, "G'"] \arrow[ul, phantom, "\overset{\beta}{\Nwarrow}"] \end{tikzcd};$$
of $C$;

satisfying the following conditions:

(1) *Compatibility With Units.* We have an equality
$$\begin{tikzcd} & B \arrow[dr, bend left, "G"] \arrow[d, phantom, "\Uparrow\eta"] & \\ A \arrow[ur, bend left, "F"] \arrow[rr, "\mathrm{id}_A"] \arrow[d, "\phi"'] \arrow[drr, "\phi"] & & A \arrow[d, "\phi"] \\ A' \arrow[rr, bend right, "\mathrm{id}_{A'}"'] \arrow[r, phantom, "\overset{\lambda^C_\phi}{\Nearrow}"] & \phantom{x} \arrow[r, phantom, "\overset{\rho^{C,-1}_\phi}{\Nearrow}"] & A' \end{tikzcd} = \begin{tikzcd} & B \arrow[dr, bend left, "G"] \arrow[d, "\psi"] & \\ A \arrow[ur, bend left, "F"] \arrow[d, "\phi"'] \arrow[r, phantom, "\overset{\alpha}{\Nearrow}"] & B' \arrow[dr, "G'"] \arrow[r, phantom, "\overset{\beta}{\Nearrow}"] \arrow[d, phantom, "\overset{\Uparrow\eta'}{\Vert}"] & A \arrow[d, "\phi"] \\ A' \arrow[ur, "F'"] \arrow[rr, bend right, "\mathrm{id}_{A'}"'] & & A' \end{tikzcd}$$
of pasting diagrams in $C$;

(2) *Compatibility With Counits.* We have an equality
$$\begin{tikzcd} B \arrow[rr, bend left, "\mathrm{id}_B"] \arrow[dr, "G"] \arrow[d, "\psi"'] & \Uparrow\epsilon & B \arrow[d, "\psi"] \\ B' \arrow[dr, bend right, "G'"'] \arrow[r, phantom, "\overset{\beta}{\Nearrow}"] & A \arrow[ur, "F"] \arrow[d, "\phi"] \arrow[r, phantom, "\overset{\alpha}{\Nearrow}"] & B' \\ & A' \arrow[ur, bend right, "F'"'] & \end{tikzcd} = \begin{tikzcd} B \arrow[rr, bend left, "\mathrm{id}_B"] \arrow[d, "\psi"'] \arrow[drr, "\psi"] & \overset{\rho^{C,-1}_\psi}{\Nearrow} & B \arrow[d, "\psi"] \\ B' \arrow[rr, "\mathrm{id}_{B'}"] \arrow[dr, bend right, "G'"'] \arrow[r, phantom, "\overset{\lambda^C_\psi}{\Nearrow}"] & \epsilon'\Uparrow & B' \\ & A' \arrow[ur, bend right, "F'"'] & \end{tikzcd}$$
of pasting diagrams in $C$.

**19.2.2. Oplax Morphisms of Internal Adjunctions.** Let $C$ be a bicategory and let $(A, B, F, G, \eta, \epsilon)$ and $(A', B', F', G', \eta', \epsilon')$ be internal adjunctions in $C$.

**Definition 19.2.2.1.1.** An **oplax morphism of internal adjunctions** from $(A, B, F, G, \eta, \epsilon)$ to $(A', B', F', G', \eta', \epsilon')$ is an oplax transformation between these viewed as 2-functors from the walking adjunction.

**Remark 19.2.2.1.2.** In detail, an **oplax morphism of internal adjunctions** from $(A, B, F, G, \eta, \epsilon)$ to $(A', B', F', G', \eta', \epsilon')$ consists of

- 1-*Morphisms.* A pair of 1-morphisms
$$\begin{aligned}\phi &\colon A \to A',\\ \psi &\colon B \to B'\end{aligned}$$
of $C$;
- 2-*Morphisms.* A pair of 2-morphisms
$$\begin{tikzcd} A \arrow[r, "F"] \arrow[d, "\phi"'] & B \arrow[d, "\psi"] \arrow[dl, Rightarrow, "\alpha"'] \\ A' \arrow[r, "F'"'] & B' \end{tikzcd} \qquad \begin{aligned}\alpha &\colon \psi \circ F \Rightarrow F' \circ \phi,\\ \beta &\colon \psi \circ G \Rightarrow G' \circ \phi\end{aligned} \qquad \begin{tikzcd} A \arrow[d, "\phi"'] \arrow[dr, Rightarrow, "\beta"] & B \arrow[l, "G"'] \arrow[d, "\psi"] \\ A' & B' \arrow[l, "G'"] \end{tikzcd};$$
of $C$;

satisfying the following conditions:

(1) *Compatibility With Units.* We have an equality
$$\begin{tikzcd} & A \arrow[dr, bend left, "F"] \arrow[d, phantom, "\epsilon\Downarrow"] & \\ B \arrow[ur, bend left, "G"] \arrow[rr, "\mathrm{id}_B"] \arrow[d, "\psi"'] \arrow[drr, "\psi"] & {} & B \arrow[d, "\psi"] \arrow[dll, Rightarrow, "\rho^{C}_{\phi}"'] \\ B' \arrow[rr, bend right, "\mathrm{id}_{B'}"'] \arrow[r, phantom, "\lambda^{C,-1}_{\phi}"] & {} & B' \end{tikzcd} = \begin{tikzcd} & A \arrow[dr, bend left, "F"] \arrow[dd, "\phi"] \arrow[ddl, Rightarrow, "\beta"] & \\ B \arrow[ur, bend left, "G"] \arrow[dd, "\psi"'] & & B \arrow[dd, "\psi"] \arrow[dl, Rightarrow, "\alpha"'] \\ & A' \arrow[dr, "F'"] \arrow[d, phantom, "\epsilon'\Downarrow"] & \\ B' \arrow[ur, "G'"] \arrow[rr, bend right, "\mathrm{id}_{B'}"'] & {} & B' \end{tikzcd}$$
of pasting diagrams in $C$;

(2) *Compatibility With Counits.* We have an equality
$$\begin{tikzcd} A \arrow[rr, bend left, "\mathrm{id}_A"] \arrow[dr, "F"] \arrow[dd, "\phi"'] & \Downarrow\eta & A \arrow[dd, "\phi"] \arrow[ddl, Rightarrow, "\beta"] \\ & B \arrow[ur, "G"] \arrow[dd, "\psi"] \arrow[dl, Rightarrow, "\alpha"'] & \\ A' \arrow[dr, "F'"'] & & A' \\ & B' \arrow[ur, "G'"'] & \end{tikzcd} = \begin{tikzcd} A \arrow[rr, bend left, "\mathrm{id}_A"] \arrow[d, "\phi"'] \arrow[drr, "\phi"] & {} & A \arrow[d, "\phi"] \arrow[dll, Rightarrow, "\rho^{C}_{\psi}"'] \\ A' \arrow[rr, "\mathrm{id}_{A'}"] \arrow[dr, bend right, "F'"'] & \eta'\Downarrow & A' \\ & B' \arrow[ur, bend right, "G'"'] & \end{tikzcd}$$
(with $\lambda^{C,-1}_{\psi}$ in the left region of the right-hand diagram) of pasting diagrams in $C$.

**19.2.3. Strong Morphisms of Internal Adjunctions.** Let $C$ be a bicategory and let $(A, B, F, G, \eta, \epsilon)$ and $(A', B', F', G', \eta', \epsilon')$ be internal adjunctions in $C$.

**Definition 19.2.3.1.1.** A **strong morphism of internal adjunctions** from $(A, B, F, G, \eta, \epsilon)$ to $(A', B', F', G', \eta', \epsilon')$ is a strong transformation between these viewed as 2-functors from the walking adjunction.

**Remark 19.2.3.1.2.** In detail, a **strong morphism of internal adjunctions** from $(A, B, F, G, \eta, \epsilon)$ to $(A', B', F', G', \eta', \epsilon')$ is equivalently:

(1) A lax morphism of internal adjunctions as in Remark 19.2.1.1.2 whose 2-morphisms are invertible.
(2) An oplax morphism of internal adjunctions as in Remark 19.2.2.1.2 whose 2-morphisms are invertible.

**19.2.4. Strict Morphisms of Internal Adjunctions.** Let $C$ be a bicategory and let $(A, B, F, G, \eta, \epsilon)$ and $(A', B', F', G', \eta', \epsilon')$ be internal adjunctions in $C$.

**Definition 19.2.4.1.1.** A **strict morphism of internal adjunctions** from $(A, B, F, G, \eta, \epsilon)$ to $(A', B', F', G', \eta', \epsilon')$ is a strict transformation between these viewed as 2-functors from the walking adjunction.

**Remark 19.2.4.1.2.** In detail, a **strict morphism of internal adjunctions** from $(A, B, F, G, \eta, \epsilon)$ to $(A', B', F', G', \eta', \epsilon')$ is equivalently:

(1) A lax morphism of internal adjunctions as in Remark 19.2.1.1.2 whose 2-morphisms are identities.
(2) An oplax morphism of internal adjunctions as in Remark 19.2.2.1.2 whose 2-morphisms are identities.

## 19.3. 2-Morphisms Between Morphisms of Internal Adjunctions

**19.3.1. 2-Morphisms Between Lax Morphisms of Internal Adjunctions.** Let $C$ be a bicategory, let $(A, B, F, G, \eta, \epsilon)$ and $(A', B', F', G', \eta', \epsilon')$ be internal adjunctions in $C$, and let $(\phi_1, \psi_1, \alpha_1, \beta_1)$ and $(\phi_2, \psi_2, \alpha_2, \beta_2)$ be lax morphisms of internal adjunctions from $(A, B, F, G, \eta, \epsilon)$ to $(A', B', F', G', \eta', \epsilon')$.

**Definition 19.3.1.1.1.** A **2-morphism from** $(\phi_1, \psi_1, \alpha_1, \beta_1)$ **to** $(\phi_2, \psi_2, \alpha_2, \beta_2)$ is a modification between these viewed as lax transformations.

**Remark 19.3.1.1.2.** In detail, a **2-morphism from** $(\phi_1, \psi_1, \alpha_1, \beta_1)$ **to** $(\phi_2, \psi_2, \alpha_2, \beta_2)$ consist of 2-morphisms

$$\Gamma\colon \phi_1 \Rightarrow \phi_2$$
$$\Sigma\colon \psi_1 \Rightarrow \psi_2$$

of $\mathcal{C}$ such that we have equalities

$$
\begin{tikzcd}
A \arrow[r, "F"] \arrow[d, bend right=60, "\phi_1"'] \arrow[d, bend left=60, "\phi_2"] \arrow[d, phantom, "\overset{\Gamma}{\Rightarrow}"] & B \arrow[d, bend left, "\psi_2"] \\
A' \arrow[r, "F'"'] \arrow[ur, phantom, "\overset{\alpha_2}{\Nearrow}"] & B'
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, "F"] \arrow[d, bend right, "\phi_1"'] & B \arrow[d, bend right=60, "\psi_1"'] \arrow[d, bend left=60, "\psi_2"] \arrow[d, phantom, "\overset{\Sigma}{\Rightarrow}"] \\
A' \arrow[r, "F'"'] \arrow[ur, phantom, "\overset{\alpha_1}{\Nearrow}"] & B'
\end{tikzcd}
$$

$$
\begin{tikzcd}
B \arrow[r, "G"] \arrow[d, bend right=60, "\psi_1"'] \arrow[d, bend left=60, "\psi_2"] \arrow[d, phantom, "\overset{\Sigma}{\Rightarrow}"] & A \arrow[d, bend left, "\phi_2"] \\
B' \arrow[r, "G'"'] \arrow[ur, phantom, "\overset{\beta_2}{\Nearrow}"] & A'
\end{tikzcd}
=
\begin{tikzcd}
B \arrow[r, "G"] \arrow[d, bend right, "\psi_1"'] & A \arrow[d, bend right=60, "\phi_1"'] \arrow[d, bend left=60, "\phi_2"] \arrow[d, phantom, "\overset{\Gamma}{\Rightarrow}"] \\
B' \arrow[r, "G'"'] \arrow[ur, phantom, "\overset{\beta_1}{\Nearrow}"] & A'
\end{tikzcd}
$$

of pasting diagrams in $\mathcal{C}$.

**19.3.2. 2-Morphisms Between Oplax Morphisms of Internal Adjunctions.** Let $\mathcal{C}$ be a bicategory, let $(A, B, F, G, \eta, \epsilon)$ and $(A', B', F', G', \eta', \epsilon')$ be internal adjunctions in $\mathcal{C}$, and let $(\phi_1, \psi_1, \alpha_1, \beta_1)$ and $(\phi_2, \psi_2, \alpha_2, \beta_2)$ be oplax morphisms of internal adjunctions from $(A, B, F, G, \eta, \epsilon)$ to $(A', B', F', G', \eta', \epsilon')$.

**Definition 19.3.2.1.1.** A **2-morphism from** $(\phi_1, \psi_1, \alpha_1, \beta_1)$ **to** $(\phi_2, \psi_2, \alpha_2, \beta_2)$ is a modification between these viewed as oplax transformations.

**Remark 19.3.2.1.2.** In detail, a **2-morphism from** $(\phi_1, \psi_1, \alpha_1, \beta_1)$ **to** $(\phi_2, \psi_2, \alpha_2, \beta_2)$ consist of 2-morphisms

$$
\Gamma\colon \phi_1 \Rightarrow \phi_2
$$
$$
\Sigma\colon \psi_1 \Rightarrow \psi_2
$$

of $\mathcal{C}$ such that we have equalities

$$
\begin{tikzcd}
A \arrow[r, "F"] \arrow[d, bend right=60, "\phi_2"'] \arrow[d, bend left=60, "\phi_1"] \arrow[d, phantom, "\overset{\Gamma}{\Leftarrow}"] & B \arrow[d, bend left, "\psi_1"] \arrow[dl, phantom, "\overset{\alpha_1}{\Swarrow}"] \\
A' \arrow[r, "F'"'] & B'
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, "F"] \arrow[d, bend right, "\phi_2"'] & B \arrow[d, bend right=60, "\psi_2"'] \arrow[d, bend left=60, "\psi_1"] \arrow[d, phantom, "\overset{\Sigma}{\Leftarrow}"] \arrow[dl, phantom, "\overset{\alpha_2}{\Swarrow}"] \\
A' \arrow[r, "F'"'] & B'
\end{tikzcd}
$$

$$
\begin{tikzcd}
B \arrow[r, "G"] \arrow[d, bend right=60, "\psi_2"'] \arrow[d, bend left=60, "\psi_1"] \arrow[d, phantom, "\overset{\Sigma}{\Leftarrow}"] & A \arrow[d, bend left, "\phi_1"] \arrow[dl, phantom, "\overset{\beta_1}{\Swarrow}"] \\
B' \arrow[r, "G'"'] & A'
\end{tikzcd}
=
\begin{tikzcd}
B \arrow[r, "G"] \arrow[d, bend right, "\psi_2"'] & A \arrow[d, bend right=60, "\phi_2"'] \arrow[d, bend left=60, "\phi_1"] \arrow[d, phantom, "\overset{\Gamma}{\Leftarrow}"] \arrow[dl, phantom, "\overset{\beta_2}{\Swarrow}"] \\
B' \arrow[r, "G'"'] & A'
\end{tikzcd}
$$

of pasting diagrams in $\mathcal{C}$.

**19.3.3. 2-Morphisms Between Strong Morphisms of Internal Adjunctions.** Let $\mathcal{C}$ be a bicategory, let $(A, B, F, G, \eta, \epsilon)$ and $(A', B', F', G', \eta', \epsilon')$ be internal adjunctions in $\mathcal{C}$, and let $(\phi_1, \psi_1, \alpha_1, \beta_1)$ and $(\phi_2, \psi_2, \alpha_2, \beta_2)$ be strong morphisms of internal adjunctions from $(A, B, F, G, \eta, \epsilon)$ to $(A', B', F', G', \eta', \epsilon')$.

**Definition 19.3.3.1.1.** A **2-morphism from** $(\phi_1, \psi_1, \alpha_1, \beta_1)$ **to** $(\phi_2, \psi_2, \alpha_2, \beta_2)$ is a modification between these viewed as strong transformations.

**Remark 19.3.3.1.2.** In detail, a **2-morphism from** $(\phi_1, \psi_1, \alpha_1, \beta_1)$ **to** $(\phi_2, \psi_2, \alpha_2, \beta_2)$ is equivalently:

- A 2-morphism $(\Gamma, \Sigma)$ from $(\phi_1, \psi_1, \alpha_1, \beta_1)$ to $(\phi_2, \psi_2, \alpha_2, \beta_2)$ viewed as lax transformations as in Remark 19.3.1.1.2.
- A 2-morphism $(\Gamma, \Sigma)$ from $(\phi_1, \psi_1, \alpha_1, \beta_1)$ to $(\phi_2, \psi_2, \alpha_2, \beta_2)$ viewed as oplax transformations as in Remark 19.3.2.1.2.

**19.3.4. 2-Morphisms Between Strict Morphisms of Internal Adjunctions.** Let $\mathcal{C}$ be a bicategory, let $(A, B, F, G, \eta, \epsilon)$ and $(A', B', F', G', \eta', \epsilon')$ be internal adjunctions in $\mathcal{C}$, and let $(\phi_1, \psi_1, \alpha_1, \beta_1)$ and $(\phi_2, \psi_2, \alpha_2, \beta_2)$ be lax morphisms of internal adjunctions from $(A, B, F, G, \eta, \epsilon)$ to $(A', B', F', G', \eta', \epsilon')$.

**Definition 19.3.4.1.1.** A **2-morphism from** $(\phi_1, \psi_1, \alpha_1, \beta_1)$ **to** $(\phi_2, \psi_2, \alpha_2, \beta_2)$ is a modification between these viewed as strict transformations.

**Remark 19.3.4.1.2.** In detail, a **2-morphism from** $(\phi_1, \psi_1, \alpha_1, \beta_1)$ **to** $(\phi_2, \psi_2, \alpha_2, \beta_2)$ is equivalently:

- A 2-morphism $(\Gamma, \Sigma)$ from $(\phi_1, \psi_1, \alpha_1, \beta_1)$ to $(\phi_2, \psi_2, \alpha_2, \beta_2)$ viewed as lax transformations as in Remark 19.3.1.1.2.
- A 2-morphism $(\Gamma, \Sigma)$ from $(\phi_1, \psi_1, \alpha_1, \beta_1)$ to $(\phi_2, \psi_2, \alpha_2, \beta_2)$ viewed as oplax transformations as in Remark 19.3.2.1.2.
- A 2-morphism $(\Gamma, \Sigma)$ from $(\phi_1, \psi_1, \alpha_1, \beta_1)$ to $(\phi_2, \psi_2, \alpha_2, \beta_2)$ viewed as strong transformations as in Remark 19.3.3.1.2.

## 19.4. Bicategories of Internal Adjunctions in a Bicategory

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# Part 5

# Internal Category Theory

CHAPTER 20

# Internal Categories

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# Cyclic Stuff

CHAPTER 21

# The Cycle Category

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(44) Topological and Smooth Manifolds

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(2) Constructions With Sets
(3) Pointed Sets
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(7) Indexed Sets
(8) Fibred Sets
(9) Un/Straightening for Indexed and Fibred Sets

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(16) Cartesian Closed Categories
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(18) Bicategories
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(20) Internal Categories

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(23) The Globe Category

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(24) The Cell Category

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(25) Monoids
(26) Constructions With Monoids

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(27) Monoids With Zero
(28) Constructions With Monoids With Zero

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(29) Groups
(30) Constructions With Groups

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(35) Near-Semirings
(36) Near-Rings

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(37) Real Analysis in One Variable
(38) Real Analysis in Several Variables

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(39) Measurable Spaces

(40) Measures and Integration

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(40) Probability Theory

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(41) Stochastic Processes, Martingales, and Brownian Motion
(42) Itô Calculus
(43) Stochastic Differential Equations

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(44) Topological and Smooth Manifolds

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(45) Schemes

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(27) Monoids With Zero
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(37) Real Analysis in One Variable
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# Globular Stuff

CHAPTER 23

# The Globe Category

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(7) Indexed Sets
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(9) Un/Straightening for Indexed and Fibred Sets

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(27) Monoids With Zero
(28) Constructions With Monoids With Zero

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(31) Hypermonoids
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(35) Near-Semirings
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(37) Real Analysis in One Variable
(38) Real Analysis in Several Variables

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(39) Measurable Spaces

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(40) Probability Theory

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(41) Stochastic Processes, Martingales, and Brownian Motion
(42) Itô Calculus
(43) Stochastic Differential Equations

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(45) Schemes

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# Cellular Stuff

CHAPTER 24

# The Cell Category

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(7) Indexed Sets
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(27) Monoids With Zero
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(37) Real Analysis in One Variable
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(40) Probability Theory

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(41) Stochastic Processes, Martingales, and Brownian Motion
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(45) Schemes

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CHAPTER 25

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CHAPTER 26

# Constructions With Monoids

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(43) Stochastic Differential Equations

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# Monoids With Zero

CHAPTER 27

# Monoids With Zero

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CHAPTER 28

# Constructions With Monoids With Zero

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# Groups

CHAPTER 29

# Groups

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# Differential Geometry

CHAPTER 45

# Topological and Smooth Manifolds

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# Part 20

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CHAPTER 46

# Schemes

## 46.1. Introduction

In this document we define schemes. A basic reference is [**EGA**].

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# Part 21

# Secret Part

CHAPTER 47

# To Do List

This chapter contains some material about relations and constructions with them. Notably, we discuss and explore:

## 47.1. Notes to Self

### 47.1.1. Things To Ask On MO/Zulip.

**Remark 47.1.1.1.1.** Here is a list of things to be asked on MO/Zulip.

(1) What are
   (a) Cartesian bicategories
   (b) Double categories of relations (https://arxiv.org/abs/2107.07621)
   (c) Categories of relations
   (d) Allegories
   (e) 1-Category equipped with relations (https://ncatlab.org/nlab/show/1-category+equipped+with+relations)

   good for? What have these notions been developed for, why are they important, and what have they lead to?

### 47.1.2. Things To Explore/Add.

**Remark 47.1.2.1.1.** Here is a list of things to be explored.

(1) https://mathoverflow.net/a/461814
(2) there's some cool stuff in https://arxiv.org/abs/2312.00990, e.g. on cofunctors.
(3) internal adjunctions in $\mathsf{Mod}$ as in [JY21, Section 6.3]; see [JY21, Example 6.2.6].
(4) write the "profunctors" equivalent of the relations chapter
(5) change $\chi_B$ notation throughout the notes
(6) maybe note that skew monoidal structures on $\mathbf{Rel}(A, B)$ satisfy coherence trivially since the 2-morphisms are inclusions
(7) reconsider notation $\mathsf{FreeAlg}_{\mathcal{P}}$ in Relations
(8) Constructions With Sets: Isbell duality for powersets
(9) Categories: comma category notation as in https://mathoverflow.net/questions/455630
(10) Universal property of the bicategory of spans, https://ncatlab.org/nlab/show/span
(11) Codensity monad $\mathrm{Ran}_J(J)$ of a relation (What about $\mathrm{Rift}_J(J)$?)
(12) Relative *co*monads in $\mathsf{Rel}$.
(13) Write proper sections on straightening for lax functors from sets to Rel or Span (displayed sets) when I study the corresponding notions for categories

(14) Write about cospans.
(15) CoCartesian fibration classifying $\mathsf{Fun}(F, G)$, https://mathoverflow.net/questions/457533/cocartesian-fibration-classifying-mathrmfunf-g
(16) Constructions With Sets: functoriality of limits/colimits, like functoriality of pullbacks
(17) https://ncatlab.org/nlab/show/adjoint+lifting+theorem
(18) https://ncatlab.org/nlab/show/Gabriel%E2%80%93Ulmer+duality

### 47.1.3. Random Cool Papers.

**Remark 47.1.3.1.1.** Here are some random cool papers that appeared on arXiv and that I want to check eventually.

(1) A Derived Geometric Approach to Propagation of Solution Singularities for Non-linear PDEs I: Foundations
(2) The Fundamental Theorem of Calculus point-free, with applications to exponentials and logarithms

### 47.1.4. Omitted Proofs To Add.

Не так благотворна истина, как зловредна ее видимость.

*Даниил Данковский*

Truth does not do as much good in the world as the appearance of truth does evil.

*Daniil Dankovsky*

There's a very large number of omitted proofs throughout these notes; here I list some of the ones that I really want to add to the notes at some point.

**Remark 47.1.4.1.1.** Here is a list of omitted proofs that I really want to eventually write up or add a reference to.

- **??** of **??**
- **??** of **??**
- **??** of **??**
- **??** of **??**
- **??** of **??**
- **??** of **??**
- **??** of **??**

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