Indexed and Fibred Sets

December 11, 2023

This chapter contains a discussion of the un/straightening equivalence in the context of sets, as well as a general discussion of indexed and fibred sets. In particular, it contains:

1. A discussion of indexed sets (i.e. functors $K_{\mathsf{disc}} \to \mathsf{Sets}$ with K a set), constructions with them like dependent sums and dependent products, and their properties (Sections 1 and 2);
2. A discussion of fibred sets (i.e. maps of sets $X \to K$), constructions with them like dependent sums and dependent products, and their properties (Sections 3 and 4);
3. A discussion of the un/straightening equivalence for indexed and fibred sets (Section 5).

Contents

1 Indexed Sets

1.1 Foundations

Let K be a set.

Definition 1.1.1.1. A **K-indexed set** is a functor $X \colon K_{\mathsf{disc}} \to \mathsf{Sets}$.

Remark 1.1.1.2. By Categories, **??**, a **K-indexed set** consists of a K-indexed collection

$$X^{\dagger} \colon K \to \mathrm{Obj}(\mathsf{Sets}),$$

of sets, assigning a set $X^{\dagger}_{x} \stackrel{\text{def}}{=} X_x$ to each element x of K.

1.2 Morphisms of Indexed Sets

Let $X \colon K_{\mathsf{disc}} \to \mathsf{Sets}$ and $Y \colon K_{\mathsf{disc}} \to \mathsf{Sets}$ be indexed sets.

Definition 1.2.1.1. A **morphism of K-indexed sets from X to Y**[1] is a natural transformation

$$f\colon X \Longrightarrow Y, \qquad \begin{tikzcd} K_{\mathsf{disc}} \arrow[r, bend left, "X"{name=U}] \arrow[r, bend right, "Y"'{name=D}] & \mathsf{Sets} \arrow[from=U, to=D, Rightarrow, "f"'] \end{tikzcd}$$

from X to Y.

Remark 1.2.1.2. In detail, a **morphism of K-indexed sets** consists of a K-indexed collection

$$\{f_x\colon X_x \to Y_x\}_{x\in K}$$

of maps of sets.

1.3 The Category of Sets Indexed by a Fixed Set

Let K be a set.

Definition 1.3.1.1. The **category of K-indexed sets** is the category $\mathsf{ISets}(K)$ defined by

$$\mathsf{ISets}(K) \overset{\text{def}}{=} \mathsf{Fun}\big(K_{\mathsf{disc}}, \mathsf{Sets}\big).$$

Remark 1.3.1.2. In detail, the **category of K-indexed sets** is the category $\mathsf{ISets}(K)$ where

- *Objects.* The objects of $\mathsf{ISets}(K)$ are K-indexed sets as in Definition 1.1.1.1;
- *Morphisms.* The morphisms of $\mathsf{ISets}(K)$ are morphisms of K-indexed sets as in Definition 1.2.1.1;
- *Identities.* For each $X \in \mathrm{Obj}(\mathsf{ISets}(K))$, the unit map

  $$\mathbb{1}^{\mathsf{ISets}(K)}_X\colon \mathrm{pt} \to \mathrm{Hom}_{\mathsf{ISets}(K)}(X, X)$$

  of $\mathsf{ISets}(K)$ at X is defined by

  $$\mathrm{id}^{\mathsf{ISets}(K)}_X \overset{\text{def}}{=} \{\mathrm{id}_{X_x}\}_{x\in K};$$

- *Composition.* For each $X, Y, Z \in \mathrm{Obj}(\mathsf{ISets}(K))$, the composition map

  $$\circ^{\mathsf{ISets}(K)}_{X,Y,Z}\colon \mathrm{Hom}_{\mathsf{ISets}(K)}(Y, Z) \times \mathrm{Hom}_{\mathsf{ISets}(K)}(X, Y) \to \mathrm{Hom}_{\mathsf{ISets}(K)}(X, Z)$$

  of $\mathsf{ISets}(K)$ at (X, Y, Z) is defined by

  $$\{g_x\}_{x\in K} \circ^{\mathsf{ISets}(K)}_{X,Y,Z} \{f_x\}_{x\in K} \overset{\text{def}}{=} \{g_x \circ f_x\}_{x\in K}.$$

[1] *Further Terminology:* Also called a **K-indexed map of sets from X to Y**.

1.4 The Category of Indexed Sets

Definition 1.4.1.1. The **category of indexed sets** is the category ISets defined as the Grothendieck construction of the functor $\mathsf{ISets}\colon \mathsf{Sets}^{\mathsf{op}} \to \mathsf{Cats}$ of Proposition 2.1.1.4:

$$\mathsf{ISets} \overset{\text{def}}{=} \int^{\mathsf{Sets}} \mathsf{ISets}.$$

Remark 1.4.1.2. In detail, the **category of indexed sets** is the category ISets where

- *Objects.* The objects of ISets are pairs (K, X) consisting of
  - *The Indexing Set.* A set K;
  - *The Indexed Set.* A K-indexed set $X\colon K_{\mathsf{disc}} \to \mathsf{Sets}$;
- *Morphisms.* A morphism of ISets from (K, X) to (K', Y) is a pair (ϕ, f) consisting of
  - *The Reindexing Map.* A map of sets $\phi\colon K \to K'$;
  - *The Morphism of Indexed Sets.* A morphism of K-indexed sets $f\colon X \to \phi_*(Y)$ as in the diagram

$$f\colon X \to \phi_*(Y), \qquad \begin{tikzcd} K_{\mathsf{disc}} \arrow[rr, "\phi"] \arrow[dr, "X"'] & \arrow[d, phantom, "\overset{f}{\Rightarrow}"] & K'_{\mathsf{disc}} \arrow[dl, "Y"] \\ & \mathsf{Sets}; & \end{tikzcd}$$

- *Identities.* For each $(K, X) \in \mathrm{Obj}(\mathsf{ISets})$, the unit map

$$\mathbb{1}^{\mathsf{ISets}}_{(K,X)}\colon \mathrm{pt} \to \mathsf{ISets}((K, X), (K, X))$$

  of ISets at (K, X) is defined by

$$\mathrm{id}^{\mathsf{ISets}}_{(K,X)} \overset{\text{def}}{=} (\mathrm{id}_K, \mathrm{id}_X).$$

- *Composition.* For each $\mathbf{X} = (K, X)$, $\mathbf{Y} = (K', Y)$, $\mathbf{Z} = (K'', Z) \in \mathrm{Obj}(\mathsf{ISets})$, the composition map

$$\circ^{\mathsf{ISets}}_{\mathbf{X},\mathbf{Y},\mathbf{Z}}\colon \mathsf{ISets}(\mathbf{Y}, \mathbf{Z}) \times \mathsf{ISets}(\mathbf{X}, \mathbf{Y}) \to \mathsf{ISets}(\mathbf{X}, \mathbf{Z})$$

  of ISets at $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ is defined by

$$(\psi, g) \circ (\phi, f) \overset{\text{def}}{=} (\psi \circ \phi, (g \circ \mathrm{id}_\phi) \circ f),$$

as in the diagram

$$
\begin{tikzcd}
K_{\mathsf{disc}} \arrow[r, "\phi"] \arrow[dr, "X"'] \arrow[dr, phantom, shift left=3, "\overset{f}{\Nearrow}"] & K'_{\mathsf{disc}} \arrow[r, "\psi"] \arrow[d, "Y"] \arrow[r, phantom, shift right=4, "\overset{g}{\Nearrow}"] & K''_{\mathsf{disc}} \arrow[dl, "Z"] \\
& \mathsf{Sets}; &
\end{tikzcd}
$$

for each $(\phi, f) \in \mathsf{ISets}(\mathbf{X}, \mathbf{Y})$ and each $(\psi, g) \in \mathsf{ISets}(\mathbf{Y}, \mathbf{Z})$.

2 Constructions With Indexed Sets

2.1 Change of Indexing

Let $\phi\colon K \to K'$ be a function and let X be a K'-indexed set.

Definition 2.1.1.1. The **change of indexing of** X **to** K is the K-indexed set $\phi^*(X)$ defined by

$$\phi^*(X) \overset{\text{def}}{=} X \circ \phi_{\mathsf{disc}}.$$

Remark 2.1.1.2. In detail, the **change of indexing of** X **to** K is the K-indexed set $\phi^*(X)$ defined by

$$\phi^*(X)_x \overset{\text{def}}{=} X_{\phi(x)}$$

for each $x \in K$.

Proposition 2.1.1.3. The assignment $X \mapsto \phi^*(X)$ defines a functor

$$\phi^*\colon \mathsf{ISets}(K') \to \mathsf{ISets}(K),$$

where

- *Action on Objects.* For each $X \in \mathrm{Obj}(\mathsf{ISets}(K'))$, we have

$$[\phi^*](X) \overset{\text{def}}{=} \phi^*(X);$$

- *Action on Morphisms.* For each $X, Y \in \mathrm{Obj}(\mathsf{ISets}(K'))$, the action on Hom-sets

$$\phi^*_{X,Y}\colon \mathrm{Hom}_{\mathsf{ISets}(K')}(X, Y) \to \mathrm{Hom}_{\mathsf{ISets}(K)}(\phi^*(X), \phi^*(Y))$$

of ϕ^* at (X, Y) is the map sending a morphism of K'-indexed sets

$$f = \{f_x \colon X_x \to Y_x\}_{x \in K'}$$

from X to Y to the morphism of K-indexed sets defined by

$$\phi^*(f) \overset{\text{def}}{=} \left\{f_{\phi(x)} \colon X_{\phi(x)} \to Y_{\phi(x)}\right\}_{x \in K}.$$

Proof. Omitted. □

Proposition 2.1.1.4. The assignment $K \mapsto \mathsf{ISets}(K)$ defines a functor

$$\mathsf{ISets} \colon \mathsf{Sets}^{\mathsf{op}} \to \mathsf{Cats},$$

where

- *Action on Objects.* For each $K \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$[\mathsf{ISets}](K) \overset{\text{def}}{=} \mathsf{ISets}(K);$$

- *Action on Morphisms.* For each $K, K' \in \mathrm{Obj}(\mathsf{Sets})$, the action on Hom-sets

$$\mathsf{ISets}_{K,K'} \colon \mathsf{Sets}^{\mathsf{op}}(K, K') \to \mathsf{Fun}(\mathsf{ISets}(K), \mathsf{ISets}(K'))$$

  of ISets at (K, K') is the map defined by

$$\mathsf{ISets}_{K,K'}(\phi) \overset{\text{def}}{=} \phi^*$$

  for each $\phi \in \mathsf{Sets}^{\mathsf{op}}(K, K')$.

Proof. Omitted. □

2.2 Dependent Sums

Let $\phi \colon K \to K'$ be a function and let X be a K-indexed set.

Definition 2.2.1.1. The **dependent sum of** X is the K'-indexed set $\Sigma_\phi(X)$[2] defined by

$$\Sigma_\phi(X) \overset{\text{def}}{=} \mathrm{Lan}_\phi(X),$$

and hence given by

$$\Sigma_\phi(X)_x \cong \coprod_{y \in \phi^{-1}(x)} X_y$$

for each $x \in K'$.

[2] *Further Notation:* Also written $\phi_*(X)$.

Proposition 2.2.1.2. The assignment $X \mapsto \Sigma_\phi(X)$ defines a functor

$$\Sigma_\phi \colon \mathsf{ISets}(K) \to \mathsf{ISets}(K'),$$

where

- *Action on Objects.* For each $X \in \mathrm{Obj}(\mathsf{ISets}(K))$, we have

$$[\Sigma_\phi](X) \stackrel{\text{def}}{=} \Sigma_\phi(X);$$

- *Action on Morphisms.* For each $X, Y \in \mathrm{Obj}(\mathsf{ISets}(K))$, the action on Hom-sets

$$\Sigma_{\phi|X,Y} \colon \mathrm{Hom}_{\mathsf{ISets}(K)}(X, Y) \to \mathrm{Hom}_{\mathsf{ISets}(K')}(\Sigma_\phi(X), \Sigma_\phi(Y))$$

  of Σ_ϕ at (X, Y) is the map sending a morphism of K-indexed sets

$$f \colon X \to Y$$

  to the morphism of K'-indexed sets defined by

$$\begin{aligned} \Sigma_\phi(f) &\stackrel{\text{def}}{=} \mathrm{Lan}_\phi(f); \\ &\cong \coprod_{y \in \phi^{-1}(X)} f_y. \end{aligned}$$

Proof. Omitted. □

2.3 Dependent Products

Let $\phi \colon K \to K'$ be a function and let X be a K-indexed set.

Definition 2.3.1.1. The **dependent product of** X is the K'-indexed set $\Pi_\phi(X)$[3] defined by

$$\Pi_\phi(X) \stackrel{\text{def}}{=} \mathrm{Ran}_\phi(X),$$

and hence given by

$$\Pi_\phi(X)_x \cong \prod_{y \in \phi^{-1}(x)} X_y$$

for each $x \in K'$.

[3] *Further Notation:* Also written $\phi_!(X)$.

Proposition 2.3.1.2. The assignment $X \mapsto \Pi_\phi(X)$ defines a functor

$$\Pi_\phi \colon \mathsf{ISets}(K) \to \mathsf{ISets}(K'),$$

where

- *Action on Objects.* For each $X \in \mathrm{Obj}(\mathsf{ISets}(K))$, we have
$$[\Pi_\phi](X) \overset{\text{def}}{=} \Pi_\phi(X);$$
- *Action on Morphisms.* For each $X, Y \in \mathrm{Obj}(\mathsf{ISets}(K))$, the action on Hom-sets
$$\Pi_{\phi|X,Y} \colon \mathrm{Hom}_{\mathsf{ISets}(K)}(X, Y) \to \mathrm{Hom}_{\mathsf{ISets}(K')}(\Pi_\phi(X), \Pi_\phi(Y))$$
of Π_ϕ at (X, Y) is the map sending a morphism of K-indexed sets
$$f \colon X \to Y$$
to the morphism of K'-indexed sets defined by
$$\begin{aligned} \Pi_\phi(f) &\overset{\text{def}}{=} \mathrm{Ran}_\phi(f); \\ &\cong \prod_{y \in \phi^{-1}(x)} f_y. \end{aligned}$$

Proof. Omitted. □

2.4 Internal Homs

Let K be a set and let X and Y be K-indexed sets.

Definition 2.4.1.1. The **internal Hom of indexed sets from X to Y** is the indexed set $\mathbf{Hom}_{\mathsf{ISets}(K)}(X, Y)$ defined by

$$\mathbf{Hom}_{\mathsf{ISets}(K)}(X, Y) \overset{\text{def}}{=} \mathsf{Sets}(X_x, Y_x)$$

for each $x \in K$.

2.5 Adjointness of Indexed Sets

Let $\phi \colon K \to K'$ be a map of sets.

Proposition 2.5.1.1. We have a triple adjunction

$$(\Sigma_\phi \dashv \phi^* \dashv \Pi_\phi)\colon \quad \begin{tikzcd} \mathsf{ISets}(K) \arrow[r, bend left, "\Sigma_\phi"] \arrow[r, bend right, "\Pi_\phi"'] & \mathsf{ISets}(K') \arrow[l, "\phi^*" description] \end{tikzcd}.$$

Proof. This follows from Kan Extensions, **??** of **??**. □

3 Fibred Sets

3.1 Foundations

Let K be a set.

Definition 3.1.1.1. A **K-fibred set** is a pair (X, ϕ) consisting of[4]

- *The Underlying Set.* A set X, called the **underlying set of** (X, ϕ);
- *The Fibration.* A map of sets $\phi\colon X \to K$.

3.2 Morphisms of Fibred Sets

Definition 3.2.1.1. A **morphism of K-fibred sets from** (X, ϕ) **to** (Y, ψ) is a function $f\colon X \to Y$ such that the diagram[5]

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "\phi"'] & & Y \arrow[dl, "\psi"] \\
& K &
\end{tikzcd}$$

commutes.

[4] *Further Terminology:* The **fibre of** (X, ϕ) **over** $x \in K$ is the set $\phi^{-1}(x)$ (also written ϕ_x) defined by

$$\phi^{-1}(x) \overset{\text{def}}{=} \mathrm{pt} \times_{[x], K, \phi} X, \qquad \begin{tikzcd}
\phi^{-1}(x) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "\phi"] \\
\mathrm{pt} \arrow[r, "{[x]}"'] & K.
\end{tikzcd}$$

[5] *Further Terminology:* The **transport map associated to f at $x \in K$** is the function

$$f^*_x\colon \phi^{-1}(x) \to \psi^{-1}(x)$$

given by the dashed map in the diagram

$$\begin{tikzcd}
\phi^{-1}(x) \arrow[rr] \arrow[dr, dashed] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & & X \arrow[dr, "f"] \arrow[dd, "\phi" near start] & \\
& \psi^{-1}(x) \arrow[rr] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & & Y \arrow[dd, "\psi"] \\
\mathrm{pt} \arrow[rr, "{[x]}" near start] \arrow[dr, equal] & & K \arrow[dr, equal] & \\
& \mathrm{pt} \arrow[rr, "{[x]}"'] & & K.
\end{tikzcd}$$

3.3 The Category of Fibred Sets Over a Fixed Base

Definition 3.3.1.1. The **category of K-fibred sets** is the category $\mathsf{FibSets}(K)$ defined as the slice category $\mathsf{Sets}_{/K}$ of Sets over K:

$$\mathsf{FibSets}(K) \overset{\text{def}}{=} \mathsf{Sets}_{/K}.$$

Remark 3.3.1.2. In detail $\mathsf{FibSets}(K)$ is the category where

- *Objects.* The objects of $\mathsf{FibSets}(K)$ are pairs (X, ϕ) consisting of
  - *The Fibred Set.* A set X;
  - *The Fibration.* A function $\phi\colon X \to K$;
- *Morphisms.* A morphism of $\mathsf{FibSets}(K)$ from (X, ϕ) to (Y, ψ) is a function $f\colon X \to Y$ making the diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\phi"'] & & Y \arrow[dl, "\psi"] \\ & X & \end{tikzcd}$$

  commute;
- *Identities.* For each $(X, \phi) \in \mathrm{Obj}(\mathsf{FibSets}(K))$, the unit map

$$\mathbb{1}^{\mathsf{FibSets}(K)}_{(X,\phi)}\colon \mathrm{pt} \to \mathrm{Hom}_{\mathsf{FibSets}(K)}((X, \phi), (X, \phi))$$

  of $\mathsf{FibSets}(K)$ at (X, ϕ) is given by

$$\mathrm{id}^{\mathsf{FibSets}(K)}_{(X,\phi)} \overset{\text{def}}{=} \mathrm{id}_X,$$

  as witnessed by the commutativity of the diagram

$$\begin{tikzcd} X \arrow[rr, equal, "\mathrm{id}_X"] \arrow[dr, "\phi"'] & & X \arrow[dl, "\phi"] \\ & K & \end{tikzcd}$$

  in Sets;
- *Composition.* For each $\mathbf{X} = (X, \phi)$, $\mathbf{Y} = (Y, \psi)$, $\mathbf{Z} = (Z, \chi) \in$

$\mathrm{Obj}(\mathsf{FibSets}(K))$, the composition map

$$\circ^{\mathsf{FibSets}(K)}_{\mathbf{X},\mathbf{Y},\mathbf{Z}} \colon \mathrm{Hom}_{\mathsf{FibSets}(K)}(\mathbf{Y},\mathbf{Z}) \times \mathrm{Hom}_{\mathsf{FibSets}(K)}(\mathbf{X},\mathbf{Y}) \to \mathrm{Hom}_{\mathsf{FibSets}(K)}(\mathbf{X},\mathbf{Z})$$

of $\mathsf{FibSets}(K)$ at $(\mathbf{X},\mathbf{Y},\mathbf{Z})$ is defined by

$$\circ^{\mathsf{FibSets}(K)}_{\mathbf{X},\mathbf{Y},\mathbf{Z}} \overset{\mathrm{def}}{=} \circ^{\mathsf{Sets}}_{X,Y,Z},$$

as witnessed by the commutativity of the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "\phi"'] & Y \arrow[r, "g"] \arrow[d, "\psi"] & Z \arrow[dl, "\chi"] \\ & K & \end{tikzcd}$$

in Sets.

3.4 The Category of Fibred Sets

Definition 3.4.1.1. The **category of fibred sets** is the category $\mathsf{FibSets}$ defined as the Grothendieck construction of the functor $\mathsf{FibSets} \colon \mathsf{Sets}^{\mathsf{op}} \to \mathsf{Cats}$ of Proposition 4.1.1.3:

$$\mathsf{FibSets} \overset{\mathrm{def}}{=} \int^{\mathsf{Sets}} \mathsf{FibSets}.$$

Remark 3.4.1.2. In detail, the **category of fibred sets** is the category $\mathsf{FibSets}$ where

- *Objects.* The objects of $\mathsf{FibSets}$ are pairs $(K,(X,\phi_X))$ consisting of
  - *The Base Set.* A set K;
  - *The Fibred Set.* A K-fibred set $\phi_X \colon X \to K$;
- *Morphisms.* A morphism of $\mathsf{FibSets}$ from $(K,(X,\phi_X))$ to $(K',(Y,\phi_Y))$ is a pair (ϕ,f) consisting of
  - *The Base Map.* A map of sets $\phi \colon K \to K'$;
  - *The Morphism of Fibred Sets.* A morphism of K-fibred sets

$$f \colon (X,\phi_X) \to \phi_Y^*(Y), \qquad \begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\phi_X"'] & & Y \times_{K'} K \arrow[dl, "\mathrm{pr}_2"] \\ & K; & \end{tikzcd}$$

- *Identities.* For each $(K, X) \in \mathrm{Obj}(\mathsf{FibSets})$, the unit map

  $$\mathbb{1}^{\mathsf{FibSets}}_{(K,X)} \colon \mathrm{pt} \to \mathsf{FibSets}((K, X), (K, X))$$

  of $\mathsf{FibSets}$ at (K, X) is defined by

  $$\mathrm{id}^{\mathsf{FibSets}}_{(K,X)} \overset{\mathrm{def}}{=} (\mathrm{id}_K, \sim),$$

  where $\sim$ is the isomorphism $X \to X \times_K K$ as in the diagram

  $$\begin{tikzcd} X \arrow[rr, dashed, "\sim"] \arrow[dr, "\phi_X"'] & & X \times_K K \arrow[dl, "\mathrm{pr}_2"] \\ & K; & \end{tikzcd}$$

- *Composition.* For each $\mathbf{X} = (K, X), \mathbf{Y} = (K', Y), \mathbf{Z} = (K'', Z) \in \mathrm{Obj}(\mathsf{FibSets})$, the composition map

  $$\circ^{\mathsf{FibSets}}_{\mathbf{X},\mathbf{Y},\mathbf{Z}} \colon \mathsf{FibSets}(\mathbf{Y}, \mathbf{Z}) \times \mathsf{FibSets}(\mathbf{X}, \mathbf{Y}) \to \mathsf{FibSets}(\mathbf{X}, \mathbf{Z})$$

  of $\mathsf{FibSets}$ at $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ is defined by

  $$g \circ^{\mathsf{FibSets}}_{\mathbf{X},\mathbf{Y},\mathbf{Z}} f \overset{\mathrm{def}}{=} (g \times_{K'} \mathrm{id}_K) \circ f$$

  as in the diagram

  $$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "\phi_X"'] & Y \times_{K'} K \arrow[r, "g \times_{K'} \mathrm{id}_K"] \arrow[d, "\mathrm{pr}_2"] & \overbrace{(Z \times_{K''} K') \times_{K'} K}^{\cong Z \times_{K''} K} \arrow[dl, "\mathrm{pr}_2"] \\ & K; & \end{tikzcd}$$

  for each $f \in \mathsf{FibSets}(\mathbf{X}, \mathbf{Y})$ and each $g \in \mathsf{FibSets}(\mathbf{Y}, \mathbf{Z})$.

4 Constructions With Fibred Sets

4.1 Change of Base

Let $f \colon K \to K'$ be a function and let (X, ϕ_X) be a K'-fibred set.

Definition 4.1.1.1. The **change of base of** (X, ϕ_X) **to** K is the K-fibred set $f^*(X)$ defined by

$$f^*(X) \overset{\text{def}}{=} (K \times_{K'} X, \mathrm{pr}_1), \qquad \begin{tikzcd} f^*(X) \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] \arrow[dr, phantom, very near start, "\lrcorner"] & X \arrow[d, "\phi_X"] \\ K \arrow[r, "f"'] & K'. \end{tikzcd}$$

Proposition 4.1.1.2. The assignment $X \mapsto f^*(X)$ defines a functor

$$f^* \colon \mathsf{FibSets}(K') \to \mathsf{FibSets}(K),$$

where

- *Action on Objects.* For each $(X, \phi_X) \in \mathrm{Obj}(\mathsf{FibSets}(K'))$, we have
$$f^*(X, \phi_X) \overset{\text{def}}{=} f^*(X);$$
- *Action on Morphisms.* For each $(X, \phi_X), (Y, \phi_Y) \in \mathrm{Obj}(\mathsf{FibSets}(K'))$, the action on Hom-sets
$$f^*_{X,Y} \colon \mathrm{Hom}_{\mathsf{FibSets}(K')}(X, Y) \to \mathrm{Hom}_{\mathsf{FibSets}(K)}(f^*(X), f^*(Y))$$
of f^* at $((X, \phi_X), (Y, \phi_Y))$ is the map sending a morphism of K'-fibred sets
$$g \colon (X, \phi_X) \to (Y, \phi_Y)$$
to the morphism of K-fibred sets given by the dashed morphism in the diagram
$$\begin{tikzcd} f^*(X) \arrow[rr] \arrow[dd] \arrow[dr, dashed] & & X \arrow[dr, "g"] \arrow[dd, "\phi_X" near start] & \\ & f^*(Y) \arrow[rr] \arrow[dd] & & Y \arrow[dd, "\phi_Y"] \\ K \arrow[rr, "f" near start] \arrow[dr, equal] & & K' \arrow[dr, equal] & \\ & K \arrow[rr, "f"'] & & K'. \end{tikzcd}$$

Proof. Omitted. □

Proposition 4.1.1.3. The assignment $K \mapsto \mathsf{FibSets}(K)$ defines a functor

$$\mathsf{FibSets} \colon \mathsf{Sets}^{\mathsf{op}} \to \mathsf{Cats},$$

where

- *Action on Objects.* For each $K \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$[\mathsf{FibSets}](K) \stackrel{\text{def}}{=} \mathsf{FibSets}(K);$$

- *Action on Morphisms.* For each $K, K' \in \mathrm{Obj}(\mathsf{Sets})$, the action on Hom-sets

$$\mathsf{Sets}_{/(-)|K,K'} \colon \mathsf{Sets}^{\mathsf{op}}(K, K') \to \mathsf{Fun}(\mathsf{FibSets}(K), \mathsf{FibSets}(K'))$$

of $\mathsf{Sets}_{/(-)}$ at (K, K') is the map sending a map of sets $f \colon K \to K'$ to the functor

$$\mathsf{Sets}_{/f} \colon \mathsf{FibSets}(K') \to \mathsf{FibSets}(K)$$

defined by

$$\mathsf{Sets}_{/f} \stackrel{\text{def}}{=} f^*.$$

Proof. Omitted. □

4.2 Dependent Sums

Let $f \colon K \to K'$ be a function and let (X, ϕ_X) be a K-fibred set.

Definition 4.2.1.1. The **dependent sum**[6] **of** (X, ϕ_X) is the K'-fibred set $\Sigma_f(X)$[7] defined by

$$\begin{aligned} \Sigma_f(X) &\stackrel{\text{def}}{=} (\Sigma_f(X), \Sigma_f(\phi_X)) \\ &\stackrel{\text{def}}{=} (X, f \circ \phi_X). \end{aligned}$$

Proposition 4.2.1.2. Let $f \colon K \to K'$ be a function.

1. *Functoriality.* The assignment $X \mapsto \Sigma_f(X)$ defines a functor

$$\Sigma_f \colon \mathsf{FibSets}(K) \to \mathsf{FibSets}(K'),$$

where

[6]The name "dependent sum" comes from the fact that the fibre $\Sigma_f(\phi_X)^{-1}(x)$ of $\Sigma_f(X)$ at $x \in K'$ is given by

$$\Sigma_f(\phi_X)^{-1}(x) \cong \coprod_{y \in f^{-1}(x)} \phi_X^{-1}(y);$$

see Item 2 of Proposition 4.2.1.2.

[7]*Further Notation:* Also written $f_*(X)$.

- *Action on Objects.* For each $(X, \phi_X) \in \mathrm{Obj}(\mathsf{FibSets}(K))$, we have

$$\Sigma_f(X, \phi_X) \stackrel{\text{def}}{=} (\Sigma_f(X), \Sigma_f(\phi_X));$$

- *Action on Morphisms.* For each $(X, \phi_X), (Y, \phi_Y) \in \mathrm{Obj}(\mathsf{FibSets}(K))$, the action on Hom-sets

$$\Sigma_{f|X,Y} \colon \mathrm{Hom}_{\mathsf{FibSets}(K)}(X, Y) \to \mathrm{Hom}_{\mathsf{FibSets}(K)}(\Sigma_f(X), \Sigma_f(Y))$$

of Σ_f at $((X, \phi_X), (Y, \phi_Y))$ is the map sending a morphism of K-fibred sets

$$g \colon (X, \phi_X) \to (Y, \phi_Y)$$

to the morphism of K'-fibred sets defined by

$$\Sigma_f(g) \stackrel{\text{def}}{=} g.$$

2. *Interaction With Fibres.* We have a bijection of sets

$$\Sigma_f(\phi_X)^{-1}(x) \cong \coprod_{y \in f^{-1}(x)} \phi_X^{-1}(y)$$

for each $x \in K'$.

Proof. *Item 1, Functoriality*: Omitted.
Item 2, Interaction With Fibres: Indeed, we have

$$\begin{aligned}
\Sigma_f(\phi_X)^{-1}(x) &\stackrel{\text{def}}{=} \mathrm{pt} \times_{[x], K', f \circ \phi_X} X \\
&\cong \{(a, y) \in X \times K \mid f(\phi_X(a)) = x\} \\
&\cong \coprod_{y \in f^{-1}(x)} \phi_X^{-1}(y)
\end{aligned}$$

for each $x \in K'$. □

4.3 Dependent Products

Let $f \colon K \to K'$ be a function and let (X, ϕ_X) be a K-fibred set.

Definition 4.3.1.1. The **dependent product**[8] **of** (X, ϕ_X) is the K'-fibred

[8] The name “dependent product” comes from the fact that the fibre $\Pi_f(\phi_X)^{-1}(x)$ of

set $\Pi_f(X)$[9] consisting of[10]

- *The Underlying Set.* The set $\Pi_f(X)$ defined by

$$\begin{aligned}\Pi_f(X) &\overset{\text{def}}{=} \coprod_{k'\in K'} \Gamma^{\phi_X}_{f^{-1}(k')}\Big(\phi_X^{-1}\Big(f^{-1}(k')\Big)\Big)\\ &\overset{\text{def}}{=} \left\{(k',h)\in\coprod_{k'\in K'}\mathsf{Sets}\Big(f^{-1}(k'),\phi_X^{-1}\Big(f^{-1}(k')\Big)\Big)\ \middle|\ \phi_X\circ h=\mathrm{id}_{f^{-1}(k')}\right\}\\ &\cong \coprod_{k'\in K'}\left\{h\in\mathsf{Sets}\Big(f^{-1}(k'),\phi_X^{-1}\Big(f^{-1}(k')\Big)\Big)\ \middle|\ \phi_X\circ h=\mathrm{id}_{f^{-1}(k')}\right\}\\ &\cong \coprod_{k'\in K'}\prod_{k\in f^{-1}(k')}\phi_X^{-1}(k);\end{aligned}$$

- *The Fibration.* The map of sets

$$\Pi_f(\phi_X)\colon \Pi_f(X)\to K'$$

defined by sending an element of

$$\Pi_f(X)\cong\coprod_{k'\in K'}\prod_{k\in f^{-1}(k')}\phi_X^{-1}(k)$$

to its index k' in K'.

Proof. The first bijection in the definition of $\Pi_f(X)$ is clear, so it only remains to show the bijection

$$\left\{h\in\mathsf{Sets}\Big(f^{-1}(k'),\phi_X^{-1}\Big(f^{-1}(k')\Big)\Big)\ \middle|\ \phi_X\circ h=\mathrm{id}_{f^{-1}(k')}\right\}\cong\prod_{k\in f^{-1}(k')}\phi_X^{-1}(k).$$

There are two cases:

1. If $f^{-1}(k')=\emptyset$, then there is only one map from $f^{-1}(k')$ to $\phi_X^{-1}(f^{-1}(k'))$ (the inclusion), so $\mathsf{Sets}\Big(f^{-1}(k'),\phi_X^{-1}(f^{-1}(k'))\Big)\cong\mathrm{pt}$. Since products indexed by the empty set are isomorphic to pt, the isomorphism follows.

$\Pi_f(X)$ at $x\in K'$ is given by

$$\Pi_f(\phi_X)^{-1}(x)\cong\prod_{y\in f^{-1}(x)}\phi_X^{-1}(y);$$

see Item 2 of Proposition 4.3.1.3.

[9] *Further Notation:* Also written $f_!(X)$.

[10] We can also define dependent products via the internal **Hom** in $\mathsf{FibSets}(K')$; see Item 3

2. Otherwise, by the condition $\phi_X \circ h = \mathrm{id}_{f^{-1}(k')}$, it follows that, for each $k \in f^{-1}(k')$, we must have
$$\phi_X(h(k)) = k,$$
and thus $h(k) \in \phi_X^{-1}(k)$. Therefore, a map from $f^{-1}(k')$ to $\phi_X^{-1}(f^{-1}(k'))$ consists of a choice of an element from $\phi_X^{-1}(k)$ for each $k \in f^{-1}(k')$, which is precisely given by an element of the product $\prod_{k \in f^{-1}(k')} \phi_X^{-1}(k)$, showing the bijection to be true.

This finishes the proof. □

Example 4.3.1.2. Here are some examples of dependent products of sets.

1. *Spaces of Sections.* Let $K = X$, $K' = \mathrm{pt}$, let $\phi\colon E \to X$ be a map of sets, and write $!_X\colon X \to \mathrm{pt}$ for the terminal map from X to pt. We have a bijection of sets
$$\begin{aligned}\Pi_{!_X}((E,\phi)) &\cong \Gamma_X(\phi)\\ &\cong \{h \in \mathsf{Sets}(X,E) \mid \phi \circ h = \mathrm{id}_X\}.\end{aligned}$$

2. *Function Spaces.* Let $K = K' = \mathrm{pt}$. We have a bijection of sets
$$\mathsf{Sets}(X,Y) \cong \Pi_{!_X}(!_X^*(Y)).$$

Proposition 4.3.1.3. Let $f\colon K \to K'$ be a function.

1. *Functoriality.* The assignment $X \mapsto \Pi_f(X)$ defines a functor
$$\Pi_f\colon \mathsf{FibSets}(K) \to \mathsf{FibSets}(K'),$$
where
   - *Action on Objects.* For each $(X,\phi_X) \in \mathrm{Obj}(\mathsf{FibSets}(K))$, we have
$$\Pi_f(X,\phi_X) \overset{\mathrm{def}}{=} \Pi_f(X);$$
   - *Action on Morphisms.* For each $(X,\phi_X),(Y,\phi_Y) \in \mathrm{Obj}(\mathsf{FibSets}(K))$, the action on Hom-sets
$$\Pi_{f|X,Y}\colon \mathrm{Hom}_{\mathsf{FibSets}(K)}(X,Y) \to \mathrm{Hom}_{\mathsf{FibSets}(K')}(\Pi_f(X),\Pi_f(Y))$$

of Proposition 4.3.1.3.

of Π_f at $((X,\phi_X),(Y,\phi_Y))$ is the map sending a morphism of K-fibred sets

$$\xi\colon (X,\phi_X) \to (Y,\phi_Y)$$

to the morphism of K'-fibred sets from

$$\Pi_f(X) \overset{\text{def}}{=} \left\{(x,h) \in \textstyle\coprod_{x\in K'} \mathsf{Sets}\Big(f^{-1}(x), \phi_X^{-1}(f^{-1}(x))\Big) \;\middle|\; \phi_X \circ h = \mathrm{id}_{f^{-1}(x)}\right\};$$

to

$$\Pi_f(Y) \overset{\text{def}}{=} \left\{(x,h) \in \textstyle\coprod_{x\in K'} \mathsf{Sets}\Big(f^{-1}(x), \phi_Y^{-1}(f^{-1}(x))\Big) \;\middle|\; \phi_Y \circ h = \mathrm{id}_{f^{-1}(x)}\right\};$$

induced by the composition

$$\begin{aligned}
\mathsf{Sets}\Big(f^{-1}(x), \phi_X^{-1}\Big(f^{-1}(x)\Big)\Big) &= \mathsf{Sets}\Big(f^{-1}(x), [\phi_Y \circ \xi]^{-1}\Big(f^{-1}(x)\Big)\Big)\\
&= \mathsf{Sets}\Big(f^{-1}(x), \xi^{-1}\Big(\phi_Y^{-1}\Big(f^{-1}(x)\Big)\Big)\Big)\\
&\xrightarrow{\xi_*} \mathsf{Sets}\Big(f^{-1}(x), \xi\Big(\xi^{-1}\Big(\phi_Y^{-1}\Big(f^{-1}(x)\Big)\Big)\Big)\Big)\\
&\xrightarrow{\iota_*} \mathsf{Sets}\Big(f^{-1}(x), \phi_Y^{-1}\Big(f^{-1}(x)\Big)\Big),
\end{aligned}$$

where $\iota\colon \xi\Big(\xi^{-1}\Big(\phi_Y^{-1}(f^{-1}(x))\Big)\Big) \hookrightarrow \phi_Y^{-1}(f^{-1}(x))$ is the canonical inclusion, and thus given on elements by

$$[\Pi_f(\xi)](k',h) = (k', \xi \circ h),$$

for each $(k',h) \in \Pi_f(X)$.[11]

2. *Interaction With Fibres.* We have a bijection of sets

$$\Pi_f(\phi_X)^{-1}(k') \cong \prod_{k\in f^{-1}(k')} \phi_X^{-1}(k)$$

for each $k' \in K'$.

[11] Note that the section condition is satisfied: given $(x,h) \in \Pi_f(X)$, we have

$$\begin{aligned}
\phi_Y \circ [\Pi_f(\xi)](h) &\overset{\text{def}}{=} \phi_Y \circ (\xi \circ h)\\
&= (\phi_Y \circ \xi) \circ h\\
&= \phi_X \circ h\\
&= \mathrm{id}_{f^{-1}(x)}.
\end{aligned}$$

3. *Construction Using the Internal Hom.* We have

$$\Pi_f(X,\phi_X) = \left(K' \times_{\mathbf{Hom}_{\mathsf{FibSets}(K')}((K,f),(K,f))} \mathbf{Hom}_{\mathsf{FibSets}(K')}((K,f),(X,f\circ\phi_X)), \mathrm{pr}_1\right),$$

forming a pullback diagram

$$\begin{tikzcd}
\Pi_f(X,\phi_X) \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Hom}_{\mathsf{FibSets}(K')}((K,f),(X,f\circ\phi_X)) \arrow[d, "(\phi_X)_*"] \\
K' \arrow[r, "I"'] & \mathbf{Hom}_{\mathsf{FibSets}(K')}((K,f),(K,f)),
\end{tikzcd}$$

where the bottom map is given by

$$I(k') \overset{\mathrm{def}}{=} \mathrm{id}_{f^{-1}(k')}$$

for each $k' \in K'$ and where $\mathbf{Hom}_{\mathsf{FibSets}(K')}$ denotes the internal Hom of $\mathsf{FibSets}(K')$ of Definition 4.4.1.1.

Proof. Item 1, Functoriality: Omitted.
Item 2, Interaction With Fibres: Indeed, we have

$$\begin{aligned}
\Pi_f(\phi_X)^{-1}(k') &\overset{\mathrm{def}}{=} \{(k,h) \in \Pi_f(X) \mid [\Pi_f(\phi_X)](h) = k'\} \\
&\overset{\mathrm{def}}{=} \{(k,h) \in \Pi_f(X) \mid k = k'\} \\
&\cong \left\{h \in \mathsf{Sets}\left(f^{-1}(k'), \phi_X^{-1}\left(f^{-1}(k')\right)\right) \;\middle|\; \phi_X \circ h = \mathrm{id}_{f^{-1}(k')}\right\} \\
&\cong \prod_{k \in f^{-1}(k')} \phi_X^{-1}(k)
\end{aligned}$$

for each $k' \in K'$, where the last bijection is proved in the proof of Definition 4.3.1.1.
Item 3, Construction Using the Internal Hom: Omitted. □

4.4 Internal Homs

Let K be a set and let (X,ϕ_X) and (Y,ϕ_Y) be K-fibred sets.

Definition 4.4.1.1. The **internal Hom of fibred sets from** (X,ϕ_X) **to** (Y,ϕ_Y) is the fibred set $\mathbf{Hom}_{\mathsf{FibSets}(K)}(X,Y)$ consisting of

- *The Underlying Set.* The set $\mathbf{Hom}_{\mathsf{FibSets}(K)}(X,Y)$ defined by

$$\mathbf{Hom}_{\mathsf{FibSets}(K)}(X,Y) \overset{\mathrm{def}}{=} \coprod_{x\in K} \mathsf{Sets}\left(\phi_X^{-1}(x), \phi_Y^{-1}(x)\right);$$

- *The Fibration.* The map of sets[12]

$$\phi_{\mathbf{Hom}_{\mathsf{FibSets}(K)}(X,Y)} \colon \underbrace{\mathbf{Hom}_{\mathsf{FibSets}(K)}(X,Y)}_{\coprod_{x\in K}\mathsf{Sets}\left(\phi_X^{-1}(x),\phi_Y^{-1}(x)\right)} \to K$$

defined by sending a map $f\colon \phi_X^{-1}(x) \to \phi_Y^{-1}(x)$ to its index $x \in K$.

4.5 Adjointness for Fibred Sets

Let $f\colon K \to K'$ be a map of sets.

Proposition 4.5.1.1. We have a triple adjunction

$$(\Sigma_f \dashv f^* \dashv \Pi_f)\colon \quad \begin{tikzcd} \mathsf{FibSets}(K) \arrow[r, bend left, "\Sigma_f"] \arrow[r, bend right, "\Pi_f"'] & \mathsf{FibSets}(K') \arrow[l, "f^*" description] \end{tikzcd}.$$

Proof. We offer two proofs. The first uses the corresponding adjunction for indexed sets (Proposition 2.5.1.1) and the un/straightening equivalence together with its compatibility with dependent sums and products to "transfer" the adjunction to fibred sets, while the second is a direct one.

Transferring the Adjunction From Indexed Sets Part I: The Adjunction $\Sigma_f \dashv f^*$: The adjunction

$$(\Sigma_f \dashv f^*)\colon \quad \begin{tikzcd} \mathsf{ISets}(K) \arrow[r, bend left, "\Sigma_f"] \arrow[r, phantom, "\perp"] & \mathsf{ISets}(K') \arrow[l, bend left, "f^*"] \end{tikzcd}$$

of Proposition 2.5.1.1 gives a unit and counit of the form

$$\begin{aligned} \eta\colon \mathrm{id}_{\mathsf{ISets}(K)} &\Longrightarrow \Sigma_f \circ f^*, \\ \epsilon\colon f^* \circ \Sigma_f &\Longrightarrow \mathrm{id}_{\mathsf{ISets}(K')}. \end{aligned}$$

[12]The fibres of the internal **Hom** of $\mathsf{FibSets}(K)$ are precisely the sets $\mathsf{Sets}\left(\phi_X^{-1}(x),\phi_Y^{-1}(x)\right)$, i.e. we have

$$\phi_{\mathbf{Hom}_{\mathsf{FibSets}(K)}(X,Y)|x} \cong \mathsf{Sets}\left(\phi_X^{-1}(x),\phi_Y^{-1}(x)\right)$$

for each $x \in K$.

With these in hand, we construct natural transformations

$$\eta' \colon \mathrm{id}_{\mathsf{FibSets}(K)} \Longrightarrow \Sigma_f \circ f^*,$$
$$\epsilon' \colon f^* \circ \Sigma_f \Longrightarrow \mathrm{id}_{\mathsf{FibSets}(K')}$$

as follows:

1. *The Unit.* We define $\eta' \colon \mathrm{id}_{\mathsf{FibSets}(K)} \Longrightarrow \Sigma_f \circ f^*$ as the pasting of the diagram

$$\begin{tikzcd}
 & & \mathsf{FibSets}(K') \arrow[drr, "f^*"] & & \\
\mathsf{FibSets}(K) \arrow[urr, "\Sigma_f"] \arrow[rr, phantom, "(1)"] & & \mathsf{ISets}(K') \arrow[u, "\mathrm{St}_{K'}"] \arrow[dr, "f^*"] \arrow[rr, phantom, "(2)"] & & \mathsf{FibSets}(K) \\
 & \mathsf{ISets}(K) \arrow[ul, "\mathrm{St}_K"] \arrow[ur, "\Sigma_f"] \arrow[rr, "\mathrm{id}_{\mathsf{ISets}(K)}"] \arrow[l, phantom, "(3)"] & {} \arrow[u, phantom, "\Uparrow\eta"] \arrow[d, phantom, "(5)"] & \mathsf{ISets}(K) \arrow[ur, "\mathrm{St}_K"] \arrow[r, phantom, "(4)"] & {} \\
\mathsf{FibSets}(K) \arrow[uu, "\mathrm{id}_{\mathsf{FibSets}(K)}"] \arrow[ur, "\mathrm{Un}_K"] \arrow[rrrr, "\mathrm{id}_{\mathsf{FibSets}(K)}"'] & & {} & & \mathsf{FibSets}(K), \arrow[ul, "\mathrm{Un}_K"] \arrow[uu, "\mathrm{id}_{\mathsf{FibSets}(K)}"']
\end{tikzcd}$$

   where:

   (a) Subdiagram (1) commutes by Item 3 of Proposition 5.1.1.2.
   (b) Subdiagram (2) commutes by Item 2 of Proposition 5.1.1.2.
   (c) Subdiagram (3) commutes by Theorem 5.3.1.1.
   (d) Subdiagram (4) commutes by Theorem 5.3.1.1.
   (e) Subdiagram (5) commutes by unitality of composition.

2. *The Counit.* We define $\epsilon' \colon f^* \circ \Sigma_f \Longrightarrow \mathrm{id}_{\mathsf{FibSets}(K')}$ as the pasting of

the diagram

$$\begin{tikzcd}
\mathsf{FibSets}(K') \arrow[rrrr, "\mathrm{id}_{\mathsf{FibSets}(K')}"] \arrow[dr, "\mathrm{Un}_K"] \arrow[dd, "\mathrm{id}_{\mathsf{FibSets}(K')}"'] & & \arrow[d, phantom, "(1)"] & & \mathsf{FibSets}(K') \arrow[dl, "\mathrm{Un}_K"'] \arrow[dd, "\mathrm{id}_{\mathsf{FibSets}(K')}"] \\
\arrow[r, phantom, "(2)"] & \mathsf{ISets}(K') \arrow[rr, "\mathrm{id}_{\mathsf{ISets}(K)}"] \arrow[dl, "\mathrm{St}_{K'}"'] \arrow[dr, "f^*"] & & \mathsf{ISets}(K') \arrow[dr, "\mathrm{St}_{K'}"] \arrow[r, phantom, "(3)"] & \\
\mathsf{FibSets}(K') \arrow[drr, "f^*"'] \arrow[r, phantom, "(4)"] & & \mathsf{ISets}(K) \arrow[ur, "\Sigma_f"'] \arrow[u, phantom, "\Uparrow\epsilon"] \arrow[d, "\mathrm{St}_K"] \arrow[r, phantom, "(5)"] & & \mathsf{FibSets}(K') \\
& & \mathsf{FibSets}(K) \arrow[urr, "\Sigma_f"'] & &
\end{tikzcd}$$

where:

(a) Subdiagram (1) commutes by unitality of composition.
(b) Subdiagram (2) commutes by Theorem 5.3.1.1.
(c) Subdiagram (3) commutes by Theorem 5.3.1.1.
(d) Subdiagram (4) commutes by Item 3 of Proposition 5.1.1.2.
(e) Subdiagram (5) commutes by Item 2 of Proposition 5.1.1.2.

Next, we prove the left triangle identity,

$$\begin{tikzcd}
& \mathsf{FibSets}(K') \arrow[rr, "\mathrm{id}_{\mathsf{FibSets}(K')}"] \arrow[dr, "f^*"] & & \mathsf{FibSets}(K') \\
\mathsf{FibSets}(K) \arrow[ur, "\Sigma_f"] \arrow[rr, "\mathrm{id}_{\mathsf{FibSets}(K)}"'] \arrow[r, phantom, "\Uparrow\eta"] & & \mathsf{FibSets}(K) \arrow[ur, "\Sigma_f"'] \arrow[u, phantom, "\Uparrow\epsilon"] &
\end{tikzcd}
=
\begin{tikzcd}
& \mathsf{FibSets}(K') \arrow[rr, "\mathrm{id}_{\mathsf{FibSets}(K')}"] & & \mathsf{FibSets}(K'), \\
\mathsf{FibSets}(K) \arrow[ur, "\Sigma_f"] \arrow[rr, "\mathrm{id}_{\mathsf{FibSets}(K)}"'] & & \mathsf{FibSets}(K) \arrow[ur, "\Sigma_f"'] \arrow[ul, phantom, "\Leftarrow \mathrm{id}_{\Sigma_f}"] &
\end{tikzcd}$$

whose left side in our case looks like this:

$$\begin{tikzcd}
& & \mathsf{FibSets}(K') \arrow[rrr, "\mathrm{id}_{\mathsf{FibSets}(K')}"] \arrow[drr, "f^*"] & & & \mathsf{FibSets}(K') \arrow[rrrr, "\mathrm{id}_{\mathsf{FibSets}(K')}"] \arrow[dd, "\mathrm{id}_{\mathsf{FibSets}(K')}"'] \arrow[dr, "\mathrm{Un}_K"] & & & & \mathsf{FibSets}(K') \arrow[dl, "\mathrm{Un}_K"'] \arrow[dd, "\mathrm{id}_{\mathsf{FibSets}(K')}"] \\
\mathsf{FibSets}(K) \arrow[urr, "\Sigma_f"] & & \mathsf{ISets}(K') \arrow[u, "\mathrm{St}_{K'}"] \arrow[dr, "f^*"] & & \mathsf{FibSets}(K) & & \mathsf{ISets}(K') \arrow[rr, "\mathrm{id}_{\mathsf{ISets}(K)}"] \arrow[dl, "\mathrm{St}_{K'}"'] \arrow[dr, "f^*"] & & \mathsf{ISets}(K') \arrow[dr, "\mathrm{St}_{K'}"] & \\
& \mathsf{ISets}(K) \arrow[ul, "\mathrm{St}_K"] \arrow[ur, "\Sigma_f"] \arrow[rr, "\mathrm{id}_{\mathsf{ISets}(K)}"] \arrow[from=3-2, to=2-4, phantom, "\Uparrow\eta"] & & \mathsf{ISets}(K) \arrow[ur, "\mathrm{St}_K"] & & \mathsf{FibSets}(K') \arrow[drr, "f^*"'] & & \mathsf{ISets}(K) \arrow[ur, "\Sigma_f"'] \arrow[u, phantom, "\Uparrow\epsilon"] \arrow[d, "\mathrm{St}_K"] & & \mathsf{FibSets}(K'). \\
\mathsf{FibSets}(K) \arrow[uu, "\mathrm{id}_{\mathsf{FibSets}(K)}"] \arrow[ur, "\mathrm{Un}_K"] \arrow[rrrr, "\mathrm{id}_{\mathsf{FibSets}(K)}"'] & & & & \mathsf{FibSets}(K) \arrow[ul, "\mathrm{Un}_K"'] \arrow[uu, "\mathrm{id}_{\mathsf{FibSets}(K)}"'] \arrow[rrr, "\mathrm{id}_{\mathsf{FibSets}(K)}"'] & & & \mathsf{FibSets}(K) \arrow[urr, "\Sigma_f"'] & &
\end{tikzcd}$$

It can be rearranged into

$$
\begin{tikzcd}
&& \mathsf{FibSets}(K') \arrow[rrr, "\mathrm{id}_{\mathsf{FibSets}(K')}"] &&& \mathsf{FibSets}(K') \arrow[rrrr, "\mathrm{id}_{\mathsf{FibSets}(K')}"] \arrow[d, phantom, "(1)"] &&&& \mathsf{FibSets}(K') \arrow[dl, "\mathrm{Un}_K"'] \arrow[dd, "\mathrm{id}_{\mathsf{FibSets}(K')}"] \\
\mathsf{FibSets}(K) \arrow[urr, "\Sigma_f"] && \mathsf{ISets}(K') \arrow[u, "\mathrm{St}_{K'}"] \arrow[rrrr, "\mathrm{id}_{\mathsf{ISets}(K')}"] \arrow[dr, "f^*"] \arrow[d, phantom, "\Uparrow\eta"] &&& {} \arrow[d, phantom, "(2)"] & \mathsf{ISets}(K') \arrow[rr, "\mathrm{id}_{\mathsf{ISets}(K)}"] \arrow[dr, "f^*"'] & {} \arrow[d, phantom, "\Uparrow\epsilon"] & \mathsf{ISets}(K') \arrow[dr, "\mathrm{St}_{K'}"] & \\
& \mathsf{ISets}(K) \arrow[ul, "\mathrm{St}_K"'] \arrow[ur, "\Sigma_f"] \arrow[r, "\mathrm{id}_{\mathsf{ISets}(K)}"] & {} & \mathsf{ISets}(K) \arrow[rrrr, "\mathrm{id}_{\mathsf{ISets}(K')}"] & {} \arrow[d, phantom, "(3)"] &&& \mathsf{ISets}(K) \arrow[ur, "\Sigma_f"'] \arrow[d, "\mathrm{St}_K"] && \mathsf{FibSets}(K'), \\
\mathsf{FibSets}(K) \arrow[uu, "\mathrm{id}_{\mathsf{FibSets}(K)}"] \arrow[ur, "\mathrm{Un}_K"] \arrow[rrrr, "\mathrm{id}_{\mathsf{FibSets}(K)}"'] &&&& \mathsf{FibSets}(K) \arrow[rrr, "\mathrm{id}_{\mathsf{FibSets}(K)}"'] &&& \mathsf{FibSets}(K) \arrow[urr, "\Sigma_f"'] &&
\end{tikzcd}
$$

where:

1. Subdiagram (1) commutes by Theorem 5.3.1.1.
2. Subdiagram (2) commutes by unitality of composition.
3. Subdiagram (3) commutes by Theorem 5.3.1.1.

And then, it can be rearranged into

$$
\begin{tikzcd}
&& \mathsf{FibSets}(K') \arrow[rrr, "\mathrm{id}_{\mathsf{FibSets}(K')}"] &&& \mathsf{FibSets}(K') \arrow[dl, "\mathrm{Un}_K"'] \arrow[dd, "\mathrm{id}_{\mathsf{FibSets}(K')}"] \\
\mathsf{FibSets}(K) \arrow[urr, "\Sigma_f"] && \mathsf{ISets}(K') \arrow[u, "\mathrm{St}_{K'}"] \arrow[rr, "\mathrm{id}_{\mathsf{ISets}(K)}"] \arrow[drr, "f^*"] \arrow[d, phantom, "\eta\Uparrow"] & {} \arrow[d, phantom, "\Uparrow\epsilon"] & \mathsf{ISets}(K') \arrow[dr, "\mathrm{St}_{K'}"] & \\
& \mathsf{ISets}(K) \arrow[ul, "\mathrm{St}_K"'] \arrow[ur, "\Sigma_f"] \arrow[rr, "\mathrm{id}_{\mathsf{ISets}(K)}"] & {} & \mathsf{ISets}(K) \arrow[ur, "\Sigma_f"'] \arrow[d, "\mathrm{St}_K"] && \mathsf{FibSets}(K'), \\
\mathsf{FibSets}(K) \arrow[uu, "\mathrm{id}_{\mathsf{FibSets}(K)}"] \arrow[ur, "\mathrm{Un}_K"] \arrow[rrr, "\mathrm{id}_{\mathsf{FibSets}(K)}"'] &&& \mathsf{FibSets}(K) \arrow[urr, "\Sigma_f"'] &&
\end{tikzcd}
$$

which by the left triangle identity for (η, ϵ), becomes

$$
\begin{tikzcd}
&& \mathsf{FibSets}(K') \arrow[rrr, "\mathrm{id}_{\mathsf{FibSets}(K')}"] &&& \mathsf{FibSets}(K') \arrow[dl, "\mathrm{Un}_K"'] \arrow[dd, "\mathrm{id}_{\mathsf{FibSets}(K')}"] \\
\mathsf{FibSets}(K) \arrow[urr, "\Sigma_f"] && \mathsf{ISets}(K') \arrow[u, "\mathrm{St}_{K'}"] \arrow[rr, "\mathrm{id}_{\mathsf{ISets}(K)}"] & {} \arrow[d, phantom, "\mathsf{id}"] & \mathsf{ISets}(K') \arrow[dr, "\mathrm{St}_{K'}"] & \\
& \mathsf{ISets}(K) \arrow[ul, "\mathrm{St}_K"'] \arrow[ur, "\Sigma_f"] \arrow[rr, "\mathrm{id}_{\mathsf{ISets}(K)}"] && \mathsf{ISets}(K) \arrow[ur, "\Sigma_f"'] \arrow[d, "\mathrm{St}_K"] && \mathsf{FibSets}(K'), \\
\mathsf{FibSets}(K) \arrow[uu, "\mathrm{id}_{\mathsf{FibSets}(K)}"] \arrow[ur, "\mathrm{Un}_K"] \arrow[rrr, "\mathrm{id}_{\mathsf{FibSets}(K)}"'] &&& \mathsf{FibSets}(K) \arrow[urr, "\Sigma_f"'] &&
\end{tikzcd}
$$

finishing the proof of the left triangle identity. The proof of the right triangle identity is similar, and is thus omitted.

Transferring the Adjunction From Indexed Sets Part II: The Adjunction $f^* \dashv \Pi_f$: This proof is similar to the proof of the adjunction $\Sigma_f \dashv f^*$, and is thus omitted.

Direct Proof Part I: The Adjunction $\Sigma_f \dashv f^*$*:* We claim there's a bijection

$$\mathrm{Hom}_{\mathsf{FibSets}(K')}(\Sigma_f(X), Y) \cong \mathrm{Hom}_{\mathsf{FibSets}(K)}(X, f_*(Y)),$$

natural in $(X, \phi_X) \in \mathsf{FibSets}(K)$ and $(Y, \phi_Y) \in \mathsf{FibSets}(K')$:

- *Map I.* We define a map

  $$\Phi_{X,Y} \colon \mathrm{Hom}_{\mathsf{FibSets}(K')}(\Sigma_f(X), Y) \to \mathrm{Hom}_{\mathsf{FibSets}(K)}(X, f_*(Y)),$$

  by sending a morphism

  $$\xi \colon \Sigma_f(X) \to Y, \qquad \begin{tikzcd} X \arrow[rr, "\xi"] \arrow[dr, "\phi_X"'] & & Y \arrow[ddl, "\phi_Y"] \\ & K \arrow[d, "f"'] & \\ & K' & \end{tikzcd}$$

  of K'-fibred sets to the morphism

  $$\xi^\dagger \colon X \to f^*(Y), \qquad \begin{tikzcd} X \arrow[rr, "\xi^\dagger"] \arrow[dr, "\phi_X"'] & & K \times_{K'} Y \arrow[dl, "\mathrm{pr}_1"] \\ & K' & \end{tikzcd}$$

  of K-fibred sets given by the dashed morphism in the diagram

  $$\begin{tikzcd} X \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left, "\xi"] \arrow[ddr, bend right, "\phi_X"'] & & \\ & K \times_{K'} Y \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "\phi_Y"] \\ & K \arrow[r, "f"'] & K'. \end{tikzcd}$$

- *Map II.* We define a map

  $$\Psi_{X,Y} \colon \mathrm{Hom}_{\mathsf{FibSets}(K)}(X, f_*(Y)) \to \mathrm{Hom}_{\mathsf{FibSets}(K')}(\Sigma_f(X), Y),$$

given by sending a map

$$\xi\colon X \to f^*(Y), \qquad \begin{tikzcd} X \arrow[rr, "\xi"] \arrow[dr, "\phi_X"'] & & K \times_{K'} Y \arrow[dl, "\mathrm{pr}_1"] \\ & K' & \end{tikzcd}$$

of K'-fibred sets to the map

$$\xi^\dagger\colon \Sigma_f(X) \to Y, \qquad \begin{tikzcd} X \arrow[rr, "\xi^\dagger"] \arrow[d, "\phi_X"'] & & Y \arrow[ddl, "\phi_Y"] \\ K \arrow[dr, "f"'] & & \\ & K' & \end{tikzcd}$$

of K-fibred sets given by

$$\xi^\dagger \overset{\text{def}}{=} \mathrm{pr}_2 \circ \xi,$$

where we indeed have

$$\begin{aligned} \phi_Y \circ (\mathrm{pr}_2 \circ \xi) &= (\phi_Y \circ \mathrm{pr}_2) \circ \xi \\ &= (f \circ \mathrm{pr}_1) \circ \xi && \text{(by the pullback square of } K \times_{K'} Y\text{)} \\ &= f \circ (\mathrm{pr}_1 \circ \xi) \\ &= f \circ \phi_X. && \text{(since } \xi \text{ is a morphism of } K'\text{-fibred sets)} \end{aligned}$$

- *Naturality I.* We need to show that, given a morphism

$$\alpha\colon (X, \phi_X) \to \left(X', \phi_{X'}\right)$$

of K-fibred sets, the diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathsf{FibSets}(K')}(\Sigma_f(X'), Y) \arrow[r, "\Phi_{X',Y}"] \arrow[d, "\Sigma_f(\alpha)^*"'] & \mathrm{Hom}_{\mathsf{FibSets}(K)}(X', f_*(Y)), \arrow[d, "\alpha^*"] \\ \mathrm{Hom}_{\mathsf{FibSets}(K')}(\Sigma_f(X), Y) \arrow[r, "\Phi_{X,Y}"'] & \mathrm{Hom}_{\mathsf{FibSets}(K)}(X, f_*(Y)), \end{tikzcd}$$

commutes. Indeed, given a morphism

$$\xi\colon \Sigma_f(X') \to Y, \qquad \begin{tikzcd} X' \arrow[rr, "\xi"] \arrow[d, "\phi_{X'}"'] & & Y \arrow[ddl, "\phi_Y"] \\ K \arrow[dr, "f"'] & & \\ & K' & \end{tikzcd}$$

of K'-fibred-sets, the map $\Phi_{X',Y}(\xi) \circ \alpha$ is the composition, coloured in vermillion, of the dashed arrow with α in the diagram

$$\begin{tikzcd}
X \arrow[dr, "\alpha"] \arrow[ddrr, dashed, "\exists!"] \arrow[ddrrr, bend left, "\xi\circ\alpha"] \arrow[dddrr, bend right, "\phi_{X'}\circ\alpha"'] & & & \\
& X' \arrow[dr, dashed, "\exists!"] \arrow[drr, "\xi"] \arrow[ddr, bend right, "\phi_{X'}"'] & & \\
& & K \times_{K'} Y \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "\phi_Y"] \\
& & K \arrow[r, "f"'] & K',
\end{tikzcd}$$

while $\Phi_{X,Y}(\xi \circ \Sigma_f(\alpha))$ is given by the dashed arrow, coloured in blue. Since both the blue arrow and the vermillion arrow make the outer pullback diagram commute, it follows from the universal property of the pullback that they must be equal, i.e. that

$$\Phi_{X',Y}(\xi) \circ \alpha = \Phi_{X,Y}(\xi \circ \Sigma_f(\alpha)),$$

showing that the naturality diagram above indeed commutes.

- *Naturality II.* We need to show that, given a morphism

$$\beta\colon (Y, \phi_Y) \to (Y', \phi_{Y'})$$

of K-fibred sets, the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathsf{FibSets}(K')}(\Sigma_f(X), Y) \arrow[r, "\Phi_{X,Y}"] \arrow[d, "\beta_*"'] & \mathrm{Hom}_{\mathsf{FibSets}(K)}(X, f_*(Y)), \arrow[d, "f_*(\beta)_*"] \\
\mathrm{Hom}_{\mathsf{FibSets}(K')}(\Sigma_f(X), Y') \arrow[r, "\Phi_{X,Y'}"'] & \mathrm{Hom}_{\mathsf{FibSets}(K)}(X, f_*(Y')),
\end{tikzcd}$$

commutes. Indeed, given a morphism

$$\xi\colon \Sigma_f(X') \to Y, \qquad \begin{tikzcd}
X' \arrow[rr, "\xi"] \arrow[dr, "\phi_{X'}"'] & & Y \arrow[ddl, "\phi_Y"] \\
& K \arrow[d, "f"'] & \\
& K' &
\end{tikzcd}$$

of K'-fibred-sets, the map $f_*(\beta) \circ \Phi_{X,Y}(\xi)$ is the composition, coloured in vermillion, of the dashed arrow from X to $K \times_{K'} Y$ with the dashed arrow from $K \times_{K'} Y$ to $K \times_{K'} Y'$ in the diagram

$$
\begin{tikzcd}
X \arrow[dr, dashed, "\exists!"] \arrow[ddrr, dashed, "\exists!"] \arrow[drrr, bend left, "\xi"] \arrow[dddr, bend right, "\phi_X"'] & & & & \\
& K \times_{K'} Y \arrow[rr] \arrow[dd] \arrow[dr, dashed] \arrow[dr, phantom, "\lrcorner", very near start] & & Y \arrow[dr, "\beta"] \arrow[dd, "\phi_Y"] & \\
& & K \times_{K'} Y' \arrow[rr] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & & Y' \arrow[dd, "\phi_{Y'}"] \\
& K \arrow[rr, "f"] \arrow[dr, equal] & & K \arrow[dr, equal] & \\
& & K \arrow[rr, "f"'] & & K',
\end{tikzcd}
$$

while $\Phi_{X,Y'}(\beta \circ \xi)$ is given by the dashed arrow from X to $K \times_{K'} Y'$, coloured in blue. Since both the blue arrow and the vermillion arrow make the outer pullback diagram for $K \times_{K'} Y'$ commute, it follows from the universal property of the pullback that they must be equal, i.e. that

$$f_*(\beta) \circ \Phi_{X,Y}(\xi) = \Phi_{X,Y'}(\beta \circ \xi),$$

showing that the naturality diagram above indeed commutes.

- *Invertibility I.* We claim that

$$\Psi_{X,Y} \circ \Phi_{X,Y} = \mathrm{id}_{\mathrm{Hom}_{\mathsf{FibSets}(K')}(\Sigma_f(X),Y)}.$$

Indeed, $\Phi_{X,Y}$ sends a map

$$
\xi \colon \Sigma_f(X) \to Y, \qquad
\begin{tikzcd}
X \arrow[rrr, "\xi"] \arrow[dr, "\phi_X"'] & & & Y \arrow[ddl, "\phi_Y"] \\
& K \arrow[dr, "f"'] & & \\
& & K' &
\end{tikzcd}
$$

of K'-fibred sets to the dashed morphism in the diagram

$$\begin{tikzcd} X \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left, "\xi"] \arrow[ddr, bend right, "\phi_X"'] & & \\ & K \times_{K'} Y \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "\phi_Y"] \\ & K \arrow[r, "f"'] & K', \end{tikzcd}$$

and $\Psi_{X,Y}$ then postcomposes that map with pr_2, which, by the commutativity of the diagram above, is ξ again, showing the claimed equality to be true.

- *Invertibility II.* We claim that

$$\Phi_{X,Y} \circ \Psi_{X,Y} = \mathrm{id}_{\mathrm{Hom}_{\mathsf{FibSets}(K)}(X, f^*(Y))}.$$

Indeed, $\Psi_{X,Y}$ sends a map

$$\xi \colon X \to f^*(Y), \qquad \begin{tikzcd} X \arrow[rr, "\xi"] \arrow[dr, "\phi_X"'] & & K \times_{K'} Y \arrow[dl, "\mathrm{pr}_1"] \\ & K' & \end{tikzcd}$$

of K'-fibred sets to $\mathrm{pr}_2 \circ \xi$, which is then sent by $\Phi_{X,Y}$ to the dashed morphism in the diagram

$$\begin{tikzcd} X \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left, "\mathrm{pr}_2 \circ \xi"] \arrow[ddr, bend right, "\phi_X"'] \arrow[drr, phantom, "(\dagger)"] & & \\ & K \times_{K'} Y \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "\phi_Y"] \\ & K \arrow[r, "f"'] & K', \end{tikzcd}$$

which, by the commutativity of the subdiagram marked with $(\dagger)$, is given by ξ again, showing the claimed equality to be true.

Direct Proof Part II: The Adjunction $f^* \dashv \Pi_f$: We claim there's a bijection

$$\mathrm{Hom}_{\mathsf{FibSets}(K)}(f_*(X), Y), \cong \mathrm{Hom}_{\mathsf{FibSets}(K')}(X, \Pi_f(Y))$$

natural in $(X, \phi_X) \in \mathsf{FibSets}(K')$ and $(Y, \phi_Y) \in \mathsf{FibSets}(K)$:

1. *Map I.* We define a map

   $$\Phi_{X,Y} \colon \mathrm{Hom}_{\mathsf{FibSets}(K)}(f_*(X), Y) \to \mathrm{Hom}_{\mathsf{FibSets}(K')}(X, \Pi_f(Y))$$

   defined as follows. Given a morphism

   $$\xi \colon f^*(X) \to Y, \qquad \begin{tikzcd} K \times_{K'} X \arrow[rr, "\xi"] \arrow[dr, "\mathrm{pr}_1"'] & & Y \arrow[dl, "\phi_Y"] \\ & K & \end{tikzcd}$$

   of K-fibred sets, where

   $$\begin{aligned} f^*(X) &\stackrel{\text{def}}{=} K \times_{K'} X \\ &\stackrel{\text{def}}{=} \{(k, x) \in K \times X \mid f(k) = \phi_X(x)\}, \end{aligned}$$

   we construct a morphism

   $$\xi^\dagger \colon X \to \Pi_f(Y), \qquad \begin{tikzcd} X \arrow[rr, "\xi^\dagger"] \arrow[dr, "\phi_X"'] & & \Pi_f(Y) \arrow[dl, "\Pi_f(\phi_Y)"] \\ & K' & \end{tikzcd}$$

   of K'-fibred sets, where

   $$\Pi_f(Y) \stackrel{\text{def}}{=} \left\{ (k', h) \in \coprod_{k' \in K'} \mathsf{Sets}\Big(f^{-1}(k'), \phi_Y^{-1}\Big(f^{-1}(k')\Big)\Big) \;\middle|\; \phi_Y \circ h = \mathrm{id}_{f^{-1}(k')} \right\},$$

   by defining

   $$\xi^\dagger(x) \stackrel{\text{def}}{=} (\phi_X(x), h_{x,\xi})$$

   for each $x \in X$, where

   $$h_{x,\xi} \colon f^{-1}(\phi_X(x)) \to \phi_Y^{-1}\Big(f^{-1}(\phi_X(x))\Big)$$

is the morphism from $f^{-1}(\phi_X(x)) \subset K$ to $\phi_Y^{-1}(f^{-1}(\phi_X(x))) \subset Y$ given by
$$h_{x,\xi}(k) \overset{\text{def}}{=} \xi(k,x)$$
for each $k \in f^{-1}(\phi_X(x))$. Notice that the diagram

$$\begin{tikzcd} X \arrow[rr, "\xi^\dagger"] \arrow[dr, "\phi_X"'] & & \Pi_f(Y) \arrow[dl, "\Pi_f(\phi_Y)"] \\ & K' & \end{tikzcd}$$

indeed commutes since we have

$$\begin{aligned} [\Pi_f(\phi_Y) \circ \xi^\dagger](x) &\overset{\text{def}}{=} [\Pi_f(\phi_Y)]\Big(\xi^\dagger(x)\Big) \\ &\overset{\text{def}}{=} [\Pi_f(\phi_Y)](\phi_X(x), h_{x,\xi}) \\ &\overset{\text{def}}{=} \phi_X(x) \end{aligned}$$

for each $x \in X$.

2. *Map II.* We define a map
$$\Psi_{X,Y} \colon \mathrm{Hom}_{\mathsf{FibSets}(K')}(X, \Pi_f(Y)) \to \mathrm{Hom}_{\mathsf{FibSets}(K)}(f_*(X), Y)$$
given by sending a morphism

3. *Naturality I.*

4. *Naturality II.*

5. *Invertibility I.*

6. *Invertibility II.*

This finishes the proof. □

5 Un/Straightening for Indexed and Fibred Sets

5.1 Straightening for Fibred Sets

Let K be a set and let (X, ϕ) be a K-fibred set.

Definition 5.1.1.1. The **straightening of** (X, ϕ) is the K-indexed set

$$\mathrm{St}_K(X, \phi) \colon K_{\mathsf{disc}} \to \mathsf{Sets}$$

defined by

$$\mathrm{St}_K(X, \phi)_x \stackrel{\text{def}}{=} \phi^{-1}(x)$$

for each $x \in K$.

Proposition 5.1.1.2. Let K be a set.

1. *Functoriality.* The assignment $(X, \phi) \mapsto \mathrm{St}_K(X, \phi)$ defines a functor

$$\mathrm{St}_K \colon \mathsf{FibSets}(K) \to \mathsf{ISets}(K)$$

- *Action on Objects.* For each $(X, \phi) \in \mathrm{Obj}(\mathsf{FibSets}(K))$, we have

$$[\mathrm{St}_K](X, \phi) \stackrel{\text{def}}{=} \mathrm{St}_K(X, \phi);$$

- *Action on Morphisms.* For each $(X, \phi), (Y, \psi) \in \mathrm{Obj}(\mathsf{FibSets}(K))$, the action on Hom-sets

$$\mathrm{St}_{K|X,Y} \colon \mathrm{Hom}_{\mathsf{FibSets}(K)}(X, Y) \to \mathrm{Hom}_{\mathsf{ISets}(K)}(\mathrm{St}_K(X), \mathrm{St}_K(Y))$$

of St_K at (X, Y) is given by sending a morphism

$$f \colon (X, \phi) \to (Y, \psi)$$

of K-fibred sets to the morphism

$$\mathrm{St}_K(f) \colon \mathrm{St}_K(X, \phi) \to \mathrm{St}_K(Y, \psi)$$

of K-indexed sets defined by

$$\mathrm{St}_K(f) \stackrel{\text{def}}{=} \{f^*_x\}_{x \in K},$$

where f^*_x is the transport map associated to f at $x \in K$ of Definition 3.2.1.1.

2. *Interaction With Change of Base/Indexing.* Let $f \colon K \to K'$ be a map of sets. The diagram

$$\begin{tikzcd}
\mathsf{FibSets}(K') \arrow[r, "f^*"] \arrow[d, "\mathrm{St}_{K'}"'] & \mathsf{FibSets}(K) \arrow[d, "\mathrm{St}_K"] \\
\mathsf{ISets}(K') \arrow[r, "f^*"'] & \mathsf{ISets}(K)
\end{tikzcd}$$

commutes.

3. *Interaction With Dependent Sums.* Let $f\colon K \to K'$ be a map of sets. The diagram

$$\begin{tikzcd}
\mathsf{FibSets}(K) \arrow[r, "\Sigma_f"] \arrow[d, "\mathrm{St}_K"'] & \mathsf{FibSets}(K') \arrow[d, "\mathrm{St}_{K'}"] \\
\mathsf{ISets}(K) \arrow[r, "\Sigma_f"'] & \mathsf{ISets}(K')
\end{tikzcd}$$

commutes.

4. *Interaction With Dependent Products.* Let $f\colon K \to K'$ be a map of sets. The diagram

$$\begin{tikzcd}
\mathsf{Sets}_{/K} \arrow[r, "\Pi_f"] \arrow[d, "\mathrm{St}_K"'] & \mathsf{FibSets}(K') \arrow[d, "\mathrm{St}_{K'}"] \\
\mathsf{ISets}(K) \arrow[r, "\Pi_f"'] & \mathsf{ISets}(K')
\end{tikzcd}$$

commutes.

Proof. *Item 1*, *Functoriality*: Omitted.
Item 2, *Interaction With Change of Base/Indexing*: Indeed, we have

$$\begin{aligned}
\mathrm{St}_K(f^*(X,\phi))_x &\stackrel{\text{def}}{=} \mathrm{St}_K(K \times_{K'} X)_x \\
&\stackrel{\text{def}}{=} \left(\mathrm{pr}_1^{K\times_{K'}X}\right)^{-1}(x) \\
&= \left\{(k,y) \in K \times_{K'} X \;\middle|\; \mathrm{pr}_1^{K\times_{K'}X}(k,y) = x\right\} \\
&= \{(k,y) \in K \times_{K'} X \mid k = x\} \\
&= \{(k,y) \in K \times X \mid k = x \text{ and } f(k) = \phi(y)\} \\
&\cong \{y \in X \mid \phi(y) = f(x)\} \\
&= \phi^{-1}(f(x)) \\
&\stackrel{\text{def}}{=} f^*\left(\phi^{-1}(x)\right) \\
&\stackrel{\text{def}}{=} f^*(\mathrm{St}_{K'}(X,\phi)_x)
\end{aligned}$$

for each $(X,\phi) \in \mathrm{Obj}(\mathsf{FibSets}(K'))$ and each $x \in K$, and similarly for morphisms.

Item 3, Interaction With Dependent Sums: Indeed, we have

$$\begin{aligned}\mathrm{St}_{K'}(\Sigma_f(X,\phi))_x &\overset{\text{def}}{=} \Sigma_f(\phi)^{-1}(x)\\ &\cong \coprod_{\substack{y\in X\\ f(y)=x}} \phi^{-1}(y)\\ &\cong \Sigma_f\Big(\phi^{-1}(x)\Big)\\ &\overset{\text{def}}{=} \Sigma_f(\mathrm{St}_K(X,\phi)_x)\end{aligned}$$

for each $(X,\phi) \in \mathrm{Obj}(\mathsf{FibSets}(K))$ and each $x \in K'$, where we have used Item 2 of Proposition 4.2.1.2 for the first bijection, and similarly for morphisms.

Item 4, Interaction With Dependent Products: Indeed, we have

$$\begin{aligned}\mathrm{St}_{K'}(\Pi_f(X,\phi))_x &\overset{\text{def}}{=} \Pi_f(\phi)^{-1}(x)\\ &\cong \prod_{\substack{y\in X\\ f(y)=x}} \phi^{-1}(y)\\ &\cong \Pi_f\Big(\phi^{-1}(x)\Big)\\ &\overset{\text{def}}{=} \Pi_f(\mathrm{St}_K(X,\phi)_x)\end{aligned}$$

for each $(X,\phi) \in \mathrm{Obj}(\mathsf{FibSets}(K))$ and each $x \in K'$, where we have used Item 2 of Proposition 4.3.1.3 for the first bijection, and similarly for morphisms. □

5.2 Unstraightening for Indexed Sets

Let K be a set and let X be a K-indexed set.

Definition 5.2.1.1. The **unstraightening of** X is the K-fibred set

$$\phi_{\mathrm{Un}_K}\colon \mathrm{Un}_K(X) \to K$$

consisting of

- *The Underlying Set.* The set $\mathrm{Un}_K(X)$ defined by

$$\mathrm{Un}_K(X) \overset{\text{def}}{=} \coprod_{x\in K} X_x;$$

- *The Fibration.* The map of sets

$$\phi_{\mathrm{Un}_K}\colon \mathrm{Un}_K(X) \to K$$

  defined by sending an element of $\coprod_{x\in K} X_x$ to its index in K.

Proposition 5.2.1.2. Let K be a set.

1. *Functoriality.* The assignment $X \mapsto \mathrm{Un}_K(X)$ defines a functor
$$\mathrm{Un}_K \colon \mathsf{ISets}(K) \to \mathsf{FibSets}(K)$$
   - *Action on Objects.* For each $X \in \mathrm{Obj}(\mathsf{ISets}(K))$, we have
$$[\mathrm{Un}_K](X) \stackrel{\text{def}}{=} \mathrm{Un}_K(X);$$
   - *Action on Morphisms.* For each $X, Y \in \mathrm{Obj}(\mathsf{ISets}(K))$, the action on Hom-sets
$$\mathrm{Un}_{K|X,Y} \colon \mathrm{Hom}_{\mathsf{ISets}(K)}(X, Y) \to \mathrm{Hom}_{\mathsf{FibSets}(K)}(\mathrm{Un}_K(X), \mathrm{Un}_K(Y))$$
   of Un_K at (X, Y) is defined by
$$\mathrm{Un}_{K|X,Y}(f) \stackrel{\text{def}}{=} \coprod_{x \in K} f_x^*.$$
2. *Interaction With Fibres.* We have a bijection of sets
$$\phi_{\mathrm{Un}_K}^{-1}(x) \cong X_x$$
for each $x \in K$.
3. *As a Pullback.* We have a bijection of sets
$$\mathrm{Un}_K(X) \cong K_{\mathsf{disc}} \times_{\mathsf{Sets}} \mathsf{Sets}_*, \qquad \begin{tikzcd} \mathrm{Un}_K(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Sets}_* \arrow[d, "忘"] \\ K_{\mathsf{disc}} \arrow[r, "X"'] & \mathsf{Sets}. \end{tikzcd}$$
4. *As a Colimit.* We have a bijection of sets
$$\mathrm{Un}_K(X) \cong \mathrm{colim}(X).$$
5. *Interaction With Change of Indexing/Base.* Let $f \colon K \to K'$ be a map of sets. The diagram
$$\begin{tikzcd} \mathsf{ISets}(K') \arrow[r, "f^*"] \arrow[d, "\mathrm{Un}_{K'}"'] & \mathsf{ISets}(K) \arrow[d, "\mathrm{Un}_K"] \\ \mathsf{FibSets}(K') \arrow[r, "f^*"'] & \mathsf{FibSets}(K) \end{tikzcd}$$
commutes.

6. *Interaction With Dependent Sums.* Let $f\colon K \to K'$ be a map of sets. The diagram
$$\begin{tikzcd} \mathsf{ISets}(K) \arrow[r, "\Sigma_f"] \arrow[d, "\mathrm{Un}_K"'] & \mathsf{ISets}(K') \arrow[d, "\mathrm{Un}_{K'}"] \\ \mathsf{FibSets}(K) \arrow[r, "\Sigma_f"'] & \mathsf{FibSets}(K') \end{tikzcd}$$
commutes.

7. *Interaction With Dependent Products.* Let $f\colon K \to K'$ be a map of sets. The diagram
$$\begin{tikzcd} \mathsf{ISets}(K) \arrow[r, "\Pi_f"] \arrow[d, "\mathrm{Un}_K"'] & \mathsf{ISets}(K') \arrow[d, "\mathrm{Un}_{K'}"] \\ \mathsf{FibSets}(K) \arrow[r, "\Pi_f"'] & \mathsf{FibSets}(K') \end{tikzcd}$$
commutes.

Proof. *Item 1, Functoriality*: Omitted.
Item 2, Interaction With Fibres: Omitted.
Item 3, As a Pullback: Omitted.
Item 4, As a Colimit: Clear.
Item 5, Interaction With Change of Indexing/Base: Indeed, we have
$$\begin{aligned} \mathrm{Un}_K(f^*(X)) &\overset{\text{def}}{=} \mathrm{Un}_K(X \circ f) \\ &\overset{\text{def}}{=} \coprod_{x \in K} X_{f(x)} \\ &\cong \left\{ (x,(y,a)) \in K \times \coprod_{y \in K'} X_y \;\middle|\; f(x) = y \right\} \\ &\cong K \times_{K'} \coprod_{y \in K'} X_y \\ &\overset{\text{def}}{=} K \times_{K'} \mathrm{Un}_{K'}(X) \\ &\overset{\text{def}}{=} f^*(\mathrm{Un}_{K'}(X)) \end{aligned}$$
for each $X \in \mathrm{Obj}(\mathsf{ISets}(K'))$. Similarly, it can be shown that we also have $\mathrm{Un}_K(f^*(\phi)) = f^*(\mathrm{Un}_{K'}(\phi))$ and that $\mathrm{Un}_K \circ f^* = f^* \circ \mathrm{Un}_{K'}$ also holds on morphisms.

Item 6, Interaction With Dependent Sums: Indeed, we have

$$\begin{aligned}
\mathrm{Un}_{K'}(\Sigma_f(X)) &\stackrel{\text{def}}{=} \coprod_{x\in K'} \Sigma_f(X)_x \\
&\cong \coprod_{x\in K'} \coprod_{y\in f^{-1}(x)} X_y \\
&\cong \coprod_{y\in K} X_y \\
&\cong \mathrm{Un}_K(X) \\
&\stackrel{\text{def}}{=} \Sigma_f(\mathrm{Un}_K(X))
\end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{ISets}(K))$, where we have used Item 2 of Proposition 4.2.1.2 for the first bijection. Similarly, it can be shown that we also have $\mathrm{Un}_{K'}(\Sigma_f(\phi)) = \Sigma_f(\phi_{\mathrm{Un}_K})$ and that $\mathrm{Un}_{K'} \circ \Sigma_f = \Sigma_f \circ \mathrm{Un}_K$ also holds on morphisms.
Item 7, Interaction With Dependent Products: Indeed, we have

$$\begin{aligned}
\mathrm{Un}_{K'}(\Pi_f(X)) &\stackrel{\text{def}}{=} \coprod_{x\in K'} \Pi_f(X)_x \\
&\cong \coprod_{x\in K'} \prod_{y\in f^{-1}(x)} X_y \\
&\cong \left\{ (x,h) \in \coprod_{x\in K'} \mathsf{Sets}\big(f^{-1}(x), \phi^{-1}_{\mathrm{Un}_K}(f^{-1}(x))\big) \;\middle|\; \phi\circ h = \mathrm{id}_{f^{-1}(x)} \right\} \\
&\stackrel{\text{def}}{=} \Pi_f\left(\coprod_{y\in K} X_y\right) \\
&\stackrel{\text{def}}{=} \Pi_f(\mathrm{Un}_K(X))
\end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{ISets}(K))$, where we have used Item 2 of Proposition 4.3.1.3 for the first bijection. Similarly, it can be shown that we also have $\mathrm{Un}_{K'}(\Pi_f(\phi)) = \Pi_f(\phi_{\mathrm{Un}_K})$ and that $\mathrm{Un}_{K'} \circ \Pi_f = \Pi_f \circ \mathrm{Un}_K$ also holds on morphisms. □

5.3 The Un/Straightening Equivalence

Theorem 5.3.1.1. We have an isomorphism of categories

$$(\mathrm{St}_K \dashv \mathrm{Un}_K)\colon\quad \begin{tikzcd} \mathsf{FibSets}(K) \arrow[r, bend left, "\mathrm{St}_K"] \arrow[r, phantom, "\perp"] & \mathsf{ISets}(K) \arrow[l, bend left, "\mathrm{Un}_K"] \end{tikzcd}.$$

Proof. Omitted. □

6 Miscellany

6.1 Other Kinds of Un/Straightening

Remark 6.1.1.1. There are also other kinds of un/straightening for sets, where Sets is replaced by $\mathbf{Rel}$ or Span:

- *Un/Straightening With* $\mathbf{Rel}$*, I.* We have an isomorphism of sets

  $$\mathrm{Rel}(A, B) \cong \mathsf{Sets}(B \times A, \{\mathsf{true}, \mathsf{false}\}).$$

  by the definition of a relation from A to B, Relations, Definition 1.1.1.1.

- *Un/Straightening With* $\mathbf{Rel}$*, II.* We have an equivalence of categories

  $$\mathsf{LaxFun}(K_{\mathsf{disc}}, \mathbf{Rel}) \overset{\text{eq.}}{\cong} \mathsf{Cats}^{\mathsf{fth}}_{/K_{\mathsf{disc}}},$$

  where $\mathsf{Cats}^{\mathsf{fth}}_{/K_{\mathsf{disc}}}$ is the full subcategory of $\mathsf{Cats}_{/K_{\mathsf{disc}}}$ spanned by the faithful functors; see [Nie04, Theorem 3.1].

- *Un/Straightening With* Span*, I.* For each $A, B \in \mathrm{Obj}(\mathsf{Sets})$, we have a morphism of sets

  $$\mathsf{Span}(A, B) \to \mathsf{Sets}(A \times B, \mathbb{N} \cup \{\infty\})$$

  which assemble into an equivalence of categories between $\mathsf{Span}(\mathsf{Sets})$ and the category MRel of "multirelations"; see Spans, Remark 7.5.1.1.

- *Un/Straightening With* Span*, II.* We have an equivalence of categories

  $$\mathsf{LaxFun}(K_{\mathsf{disc}}, \mathsf{Span}) \overset{\text{eq.}}{\cong} \mathsf{Cats}_{/K_{\mathsf{disc}}};$$

  see [nLa23, Section 3].

Appendices

A Other Chapters

Probability Theory

Stochastic Analysis

Differential Geometry

Schemes