Constructions With Sets

December 3, 2023

000D This chapter contains some material relating to constructions with sets. Notably, it contains:

1. Explicit descriptions of the major types of co/limits in Sets, including in particular pushouts and coequalisers (see Definitions 2.3.1 and 2.4.1 and Remarks 2.3.2 and 2.4.2);
2. A discussion of powersets as decategorifications of categories of presheaves (Remarks 4.1.2 and 4.2.2);
3. A lengthy discussion of the adjoint triple

$$f_* \dashv f^{-1} \dashv f_!\colon \mathcal{P}(A) \overset{\rightleftarrows}{\rightarrow} \mathcal{P}(B)$$

of functors (morphisms of posets) between $\mathcal{P}(A)$ and $\mathcal{P}(B)$ induced by a map of sets $f\colon A \longrightarrow B$, along with a discussion of the properties of f_*, f^{-1}, and $f_!$.

Contents

000E
1 Limits of Sets

000F
1.1 Products of Families of Sets

Let $\{A_i\}_{i\in I}$ be a family of sets.

000G
Definition 1.1.1 ▶ The Product of a Family of Sets

The **product**[1] **of** $\{A_i\}_{i\in I}$ is the set $\prod_{i\in I} A_i$ defined by

$$\prod_{i\in I} A_i \overset{\text{def}}{=} \left\{ f \in \mathsf{Sets}\left(I, \bigcup_{i\in I} A_i\right) \middle| \begin{array}{l}\text{for each } i \in I\text{, we}\\ \text{have } f(i) \in A_i\end{array} \right\}.$$

[1] *Further Terminology:* Also called the **Cartesian product of** $\{A_i\}_{i\in I}$.

000H
1.2 Binary Products of Sets

Let A and B be sets.

000J **Definition 1.2.1 ► Products of Sets**

The **product**[1] **of** A **and** B is the set $A \times B$ defined by

$$\begin{aligned} A \times B &\stackrel{\text{def}}{=} \prod_{z \in \{A,B\}} z \\ &\stackrel{\text{def}}{=} \{f \in \mathsf{Sets}(\{0,1\}, A \cup B) \mid \text{we have } f(0) \in A \text{ and } f(1) \in B\} \\ &\cong \{\{\{a\},\{a,b\}\} \in \mathcal{P}(\mathcal{P}(A \cup B)) \mid \text{we have } a \in A \text{ and } b \in B\}. \end{aligned}$$

[1] *Further Terminology:* Also called the **Cartesian product of** A **and** B or the **binary Cartesian product of** A **and** B, for emphasis.
This can also be thought of as the $(\mathbb{E}_{-1}, \mathbb{E}_{-1})$**-tensor product of** A **and** B.

000K **Proposition 1.2.2 ► Properties of Products of Sets**

Let $A, B, C,$ and X be sets.

000L 1. *Functoriality.* The assignments $A, B, (A, B) \mapsto A \times B$ define functors

$$\begin{aligned} A \times -_2 &: \mathsf{Sets} \to \mathsf{Sets}, \\ -_1 \times B &: \mathsf{Sets} \to \mathsf{Sets}, \\ -_1 \times -_2 &: \mathsf{Sets} \times \mathsf{Sets} \to \mathsf{Sets}, \end{aligned}$$

where $-_1 \times -_2$ is the functor where

- *Action on Objects.* For each $(A, B) \in \operatorname{Obj}(\mathsf{Sets} \times \mathsf{Sets})$, we have

$$[-_1 \times -_2](A, B) \stackrel{\text{def}}{=} A \times B;$$

- *Action on Morphisms.* For each $(A, B), (X, Y) \in \operatorname{Obj}(\mathsf{Sets})$, the action on Hom-sets

$$\times_{(A,B),(X,Y)} : \mathsf{Sets}(A, X) \times \mathsf{Sets}(B, Y) \to \mathsf{Sets}(A \times B, X \times Y)$$

of $\times$ at $((A, B), (X, Y))$ is defined by sending (f, g) to the function

$$f \times g : A \times B \to X \times Y$$

defined by

$$[f \times g](a, b) \stackrel{\text{def}}{=} (f(a), g(b))$$

for each $(a, b) \in A \times B$;

and where $A \times -$ and $- \times B$ are the partial functors of $-_1 \times -_2$ at $A, B \in \mathrm{Obj}(\mathsf{Sets})$.

000M 2. *Adjointness.* We have adjunctions

$$(A \times - \dashv \mathsf{Sets}(A, -)) \colon \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "A\times-"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}, \arrow[l, bend left, "\mathsf{Sets}(A,-)"] \end{tikzcd}$$

$$(- \times B \dashv \mathsf{Sets}(B, -)) \colon \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "-\times B"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}, \arrow[l, bend left, "\mathsf{Sets}(B,-)"] \end{tikzcd}$$

witnessed by bijections

$$\begin{aligned} \mathsf{Sets}(A \times B, C) &\cong \mathsf{Sets}(A, \mathsf{Sets}(B, C)), \\ \mathsf{Sets}(A \times B, C) &\cong \mathsf{Sets}(B, \mathsf{Sets}(A, C)), \end{aligned}$$

natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

000N 3. *Associativity.* We have an isomorphism of sets

$$(A \times B) \times C \cong A \times (B \times C),$$

natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

000P 4. *Unitality.* We have isomorphisms of sets

$$\begin{aligned} \mathrm{pt} \times A &\cong A, \\ A \times \mathrm{pt} &\cong A, \end{aligned}$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$.

000Q 5. *Commutativity.* We have an isomorphism of sets

$$A \times B \cong B \times A,$$

natural in $A, B \in \mathrm{Obj}(\mathsf{Sets})$.

000R 6. *Annihilation With the Empty Set.* We have isomorphisms of sets

$$\begin{aligned} A \times \emptyset &\cong \emptyset, \\ \emptyset \times A &\cong \emptyset, \end{aligned}$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$.

000S 7. *Distributivity Over Unions.* We have isomorphisms of sets

$$\begin{aligned} A \times (B \cup C) &= (A \times B) \cup (A \times C), \\ (A \cup B) \times C &= (A \times C) \cup (B \times C), \end{aligned}$$

natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

000T 8. *Distributivity Over Intersections.* We have isomorphisms of sets

$$\begin{aligned} A \times (B \cap C) &= (A \times B) \cap (A \times C), \\ (A \cap B) \times C &= (A \times C) \cap (B \times C), \end{aligned}$$

natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

000U 9. *Distributivity Over Differences.* We have isomorphisms of sets

$$\begin{aligned} A \times (B \setminus C) &= (A \times B) \setminus (A \times C), \\ (A \setminus B) \times C &= (A \times C) \setminus (B \times C), \end{aligned}$$

natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

000V 10. *Distributivity Over Symmetric Differences.* We have isomorphisms of sets

$$\begin{aligned} A \times (B \mathbin{\triangle} C) &= (A \times B) \mathbin{\triangle} (A \times C), \\ (A \mathbin{\triangle} B) \times C &= (A \times C) \mathbin{\triangle} (B \times C), \end{aligned}$$

natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

000W 11. *Symmetric Monoidality.* The triple $(\mathsf{Sets}, \times, \mathrm{pt})$ is a symmetric monoidal category.

000X 12. *Symmetric Bimonoidality.* The quintuple $(\mathsf{Sets}, \coprod, \emptyset, \times, \mathrm{pt})$ is a symmetric bimonoidal category.

Proof 1.2.3 ► Proof of Proposition 1.2.2

Item 1: Functoriality

Omitted.

Item 2: Adjointness

Omitted.

Item 3: Associativity

Clear.

Item 4: Unitality

Clear.

Item 5: Commutativity

Clear.

Item 6: Annihilation With the Empty Set

Clear.

Item 7: Distributivity Over Unions

Omitted.

Item 8: Distributivity Over Intersections

Omitted.

Item 9: Distributivity Over Differences

Omitted.

Item 10: Distributivity Over Symmetric Differences

Omitted.

Item 11: Symmetric Monoidality

Omitted.

Item 12: Symmetric Bimonoidality

Omitted. □

000Y 1.3 Pullbacks

Let A, B, and C be sets and let $f\colon A \to C$ and $g\colon B \to C$ be functions.

000Z **Definition 1.3.1 ► Pullbacks of Sets**

The **pullback of A and B over C along f and g**[1] is the set $A \times_C B$ defined by

$$A \times_C B \overset{\text{def}}{=} \{(a, b) \in A \times B \mid f(a) = g(b)\}.$$

[1] *Further Terminology:* Also called the **fibre product of A and B over C along f and g**.

0010 **Example 1.3.2 ► Examples of Pullbacks of Sets**

Here are some examples of pullbacks of sets.

0011 1. *Unions via Intersections.* Let $A, B \subset X$. We have a bijection of sets

$$A \cap B \cong A \times_{A \cup B} B.$$

0012 **Proposition 1.3.3 ► Properties of Pullbacks of Sets**

Let A, B, C, and X be sets.

0013 1. *Associativity.* We have an isomorphism of sets

$$(A \times_X B) \times_X C \cong A \times_X (B \times_X C),$$

natural in $A, B, C, X \in \mathrm{Obj}(\mathsf{Sets})$.

0014 2. *Unitality.* We have isomorphisms of sets

$$X \times_X A \cong A,$$
$$A \times_X X \cong A,$$

natural in $A, X \in \mathrm{Obj}(\mathsf{Sets})$.

0015 3. *Commutativity.* We have an isomorphism of sets

$$A \times_X B \cong B \times_X A,$$

natural in $A, B, X \in \mathrm{Obj}(\mathsf{Sets})$.

0016 4. *Annihilation With the Empty Set.* We have isomorphisms of sets

$$A \times_X \emptyset \cong \emptyset,$$
$$\emptyset \times_X A \cong \emptyset,$$

natural in $A, X \in \mathrm{Obj}(\mathsf{Sets})$.

0017 5. *Symmetric Monoidality.* The triple $(\mathsf{Sets}, \times_X, X)$ is a symmetric monoidal category.

Proof 1.3.4 ► Proof of Proposition 1.3.3

Item 1: Associativity

Clear.

Item 2: Unitality

Clear.

Item 3: Commutativity

Clear.

Item 4: Annihilation With the Empty Set

Clear.

Item 5: Symmetric Monoidality

Omitted.

0018 1.4 Equalisers

Let A and B be sets and let $f, g\colon A \rightrightarrows B$ be functions.

0019 **Definition 1.4.1 ► Equalisers of Sets**

The **equaliser of** f **and** g is the set $\mathrm{Eq}(f, g)$ defined by

$$\mathrm{Eq}(f, g) \stackrel{\text{def}}{=} \{a \in A \mid f(a) = g(a)\}.$$

001A **Proposition 1.4.2 ► Properties of Equalisers of Sets**

Let A, B, and C be sets.

001B 1. *Associativity.* We have an isomorphism of sets[1]

$$\underbrace{\mathrm{Eq}(f \circ \mathrm{eq}(g, h), g \circ \mathrm{eq}(g, h))}_{=\mathrm{Eq}(f\circ\mathrm{eq}(g,h),h\circ\mathrm{eq}(g,h))} \cong \mathrm{Eq}(f, g, h) \cong \underbrace{\mathrm{Eq}(f \circ \mathrm{eq}(f, g), h \circ \mathrm{eq}(f, g))}_{=\mathrm{Eq}(g\circ\mathrm{eq}(f,g),h\circ\mathrm{eq}(f,g))},$$

where $\mathrm{Eq}(f, g, h)$ is the limit of the diagram

$$\begin{tikzcd} A \arrow[r, shift left=2, "f"] \arrow[r, "g" description] \arrow[r, shift right=2, "h"'] & B \end{tikzcd}$$

in Sets.

001C 2. *Unitality.* We have an isomorphism of sets

$$\mathrm{Eq}(f, f) \cong A.$$

001D 3. *Commutativity.* We have an isomorphism of sets

$$\mathrm{Eq}(f, g) \cong \mathrm{Eq}(g, f).$$

001E 4. *Interaction With Composition.* Let

$$A \overset{f}{\underset{g}{\rightrightarrows}} B \overset{h}{\underset{k}{\rightrightarrows}} C$$

be functions. We have an inclusion of sets

$$\mathrm{Eq}(h \circ f \circ \mathrm{eq}(f, g), k \circ g \circ \mathrm{eq}(f, g)) \subset \mathrm{Eq}(h \circ f, k \circ g),$$

where $\mathrm{Eq}(h \circ f \circ \mathrm{eq}(f, g), k \circ g \circ \mathrm{eq}(f, g))$ is the equaliser of the composition

$$\mathrm{Eq}(f, g) \overset{\mathrm{eq}(f,g)}{\hookrightarrow} A \overset{f}{\underset{g}{\rightrightarrows}} B \overset{h}{\underset{k}{\rightrightarrows}} C.$$

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[1] That is: the following constructions give the same result:

(a) Take the equaliser of (f, g, h), i.e. the limit of the diagram

$$\begin{tikzcd} A \arrow[r, shift left=2, "f"] \arrow[r, "g" description] \arrow[r, shift right=2, "h"'] & B \end{tikzcd}$$

in Sets.

(b) First take the equaliser of f and g, forming a diagram

$$\mathrm{Eq}(f,g) \overset{\mathrm{eq}(f,g)}{\hookrightarrow} A \underset{g}{\overset{f}{\rightrightarrows}} B$$

and then take the equaliser of the composition

$$\mathrm{Eq}(f,g) \overset{\mathrm{eq}(f,g)}{\hookrightarrow} A \underset{h}{\overset{f}{\rightrightarrows}} B,$$

obtaining a subset

$$\mathrm{Eq}(f \circ \mathrm{eq}(f,g), h \circ \mathrm{eq}(f,g)) = \mathrm{Eq}(g \circ \mathrm{eq}(f,g), h \circ \mathrm{eq}(f,g))$$

of $\mathrm{Eq}(f,g)$.

(c) First take the equaliser of g and h, forming a diagram

$$\mathrm{Eq}(g,h) \overset{\mathrm{eq}(g,h)}{\hookrightarrow} A \underset{h}{\overset{g}{\rightrightarrows}} B$$

and then take the equaliser of the composition

$$\mathrm{Eq}(g,h) \overset{\mathrm{eq}(g,h)}{\hookrightarrow} A \underset{g}{\overset{f}{\rightrightarrows}} B,$$

obtaining a subset

$$\mathrm{Eq}(f \circ \mathrm{eq}(g,h), g \circ \mathrm{eq}(g,h)) = \mathrm{Eq}(f \circ \mathrm{eq}(g,h), h \circ \mathrm{eq}(g,h))$$

of $\mathrm{Eq}(g,h)$.

Proof 1.4.3 ► Proof of Proposition 1.4.2

Item 1: Associativity

Clear.

Item 2: Unitality

Clear.

Item 3: Commutativity

Clear.

Item 4: Interaction With Composition

Omitted.

001F 2 Colimits of Sets

001G 2.1 Coproducts of Families of Sets

Let $\{A_i\}_{i\in I}$ be a family of sets.

001H **Definition 2.1.1 ▶ Disjoint Unions of Families**

The **disjoint union of the family** $\{A_i\}_{i\in I}$ is the set $\coprod_{i\in I} A_i$ defined by

$$\coprod_{i\in I} A_i \overset{\text{def}}{=} \bigcup_{i\in I}\left\{(x,i)\in\left(\bigcup_{i\in I} A_i\right)\times I \;\middle|\; x\in A_i\right\}.$$

001J 2.2 Binary Coproducts

Let A and B be sets.

001K **Definition 2.2.1 ▶ Coproducts of Sets**

The **coproduct**[1] **of** A **and** B is the set $A\coprod B$ defined by

$$\begin{aligned} A\coprod B &\overset{\text{def}}{=} \coprod_{z\in\{A,B\}} z \\ &\overset{\text{def}}{=} \{(a,0)\mid a\in A\}\cup\{(b,1)\mid b\in B\}. \end{aligned}$$

[1] *Further Terminology:* Also called the **disjoint union of** A **and** B, or the **binary disjoint union of** A **and** B, for emphasis.

001L **Proposition 2.2.2 ▶ Properties of Coproducts of Sets**

Let $A, B, C,$ and X be sets.

001M 1. *Functoriality.* The assignment $A, B, (A,B)\mapsto A\coprod B$ defines functors

$$\begin{aligned} A\coprod -_2 &\colon \mathsf{Sets}\to\mathsf{Sets}, \\ -_1\coprod B &\colon \mathsf{Sets}\to\mathsf{Sets}, \\ -_1\coprod -_2 &\colon \mathsf{Sets}\times\mathsf{Sets}\to\mathsf{Sets}, \end{aligned}$$

where $-_1\coprod -_2$ is the functor where

- *Action on Objects.* For each $(A,B)\in\mathrm{Obj}(\mathsf{Sets}\times\mathsf{Sets})$, we have

$$[-_1\coprod -_2](A,B)\overset{\text{def}}{=} A\coprod B;$$

- *Action on Morphisms.* For each $(A, B), (X, Y) \in \mathrm{Obj}(\mathsf{Sets})$, the action on Hom-sets

$$\coprod_{(A,B),(X,Y)} \colon \mathsf{Sets}(A, X) \times \mathsf{Sets}(B, Y) \to \mathsf{Sets}(A \coprod B, X \coprod Y)$$

of $\coprod$ at $((A, B), (X, Y))$ is defined by sending (f, g) to the function

$$f \coprod g \colon A \coprod B \to X \coprod Y$$

defined by

$$[f \coprod g](x) \overset{\text{def}}{=} \begin{cases} f(x) & \text{if } x \in A, \\ g(x) & \text{if } x \in B \end{cases}$$

for each $x \in A \coprod B$;

and where $A \coprod -$ and $- \coprod B$ are the partial functors of $-_1 \coprod -_2$ at $A, B \in \mathrm{Obj}(\mathsf{Sets})$.

001N 2. *Associativity.* We have an isomorphism of sets

$$(A \coprod B) \coprod C \cong A \coprod (B \coprod C),$$

natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

001P 3. *Unitality.* We have isomorphisms of sets

$$\begin{aligned} A \coprod \emptyset &\cong A, \\ \emptyset \coprod A &\cong A, \end{aligned}$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$.

001Q 4. *Commutativity.* We have an isomorphism of sets

$$A \coprod B \cong B \coprod A,$$

natural in $A, B \in \mathrm{Obj}(\mathsf{Sets})$.

001R 5. *Symmetric Monoidality.* The triple $(\mathsf{Sets}, \coprod, \emptyset)$ is a symmetric monoidal category.

Proof 2.2.3 ▶ Proof of Proposition 2.2.2

Item 1: Functoriality

Omitted.

Item 2: Associativity

Clear.

Item 3: Unitality

Clear.

Item 4: Commutativity

Clear.

Item 5: Symmetric Monoidality

Omitted.

001S 2.3 Pushouts

Let A, B, and C be sets and let $f: C \to A$ and $g: C \to B$ be functions.

001T **Definition 2.3.1 ▶ Pushouts of Sets**

The **pushout of A and B over C along f and g**[1] is the set $A \coprod_C B$ defined by

$$A \coprod_C B \overset{\text{def}}{=} A \coprod B/{\sim_C},$$

where $\sim_C$ is the equivalence relation on $A \coprod B$ generated by $f(c) \sim_C g(c)$.

[1] *Further Terminology:* Also called the **fibre coproduct of A and B over C along f and g**.

001U **Remark 2.3.2 ▶ Unwinding Definition 2.3.1**

In detail, the relation $\sim$ of Definition 2.3.1 is given by declaring $a \sim b$ iff one of the following conditions is satisfied:

- We have $a, b \in A$ and $a = b$;
- We have $a, b \in B$ and $a = b$;
- There exist $x_1, \ldots, x_n \in A \coprod B$ such that $a \sim' x_1 \sim' \cdots \sim' x_n \sim' b$, where we declare $x \sim' y$ if one of the following conditions is satisfied:

1. There exists $c \in C$ such that $x = f(c)$ and $y = g(c)$.
2. There exists $c \in C$ such that $x = g(c)$ and $y = f(c)$.

That is: we require the following condition to be satisfied:

($\star$) There exist $x_1, \ldots, x_n \in A \coprod B$ satisfying the following conditions:

1. There exists $c_0 \in C$ satisfying one of the following conditions:
   (a) We have $a = f(c_0)$ and $x_1 = g(c_0)$.
   (b) We have $a = g(c_0)$ and $x_1 = f(c_0)$.
2. For each $1 \leq i \leq n-1$, there exists $c_i \in C$ satisfying one of the following conditions:
   (a) We have $x_i = f(c_i)$ and $x_{i+1} = g(c_i)$.
   (b) We have $x_i = g(c_i)$ and $x_{i+1} = f(c_i)$.
3. There exists $c_n \in C$ satisfying one of the following conditions:
   (a) We have $x_n = f(c_n)$ and $b = g(c_n)$.
   (b) We have $x_n = g(c_n)$ and $b = f(c_n)$.

001V **Example 2.3.3 ► Examples of Pushouts of Sets**

Here are some examples of pushouts of sets.

001W 1. *Wedge Sums of Pointed Sets.* The wedge sum of two pointed sets of **??** is an example of a pushout of sets.

001X 2. *Intersections via Unions.* Let $A, B \subset X$. We have a bijection of sets

$$A \cup B \cong A \coprod_{A \cap B} B.$$

001Y **Proposition 2.3.4 ► Properties of Pushouts of Sets**

Let A, B, C, and X be sets.

001Z 1. *Associativity.* We have an isomorphism of sets

$$(A \coprod_X B) \coprod_X C \cong A \coprod_X (B \coprod_X C),$$

natural in $A, B, C, X \in \mathrm{Obj}(\mathsf{Sets})$.

0020 2. *Unitality.* We have isomorphisms of sets

$$\emptyset \coprod_X A \cong A,$$
$$A \coprod_X \emptyset \cong A,$$

natural in $A, X \in \mathrm{Obj}(\mathsf{Sets})$.

0021 3. *Commutativity.* We have an isomorphism of sets

$$A \coprod_X B \cong B \coprod_X A,$$

natural in $A, B, X \in \mathrm{Obj}(\mathsf{Sets})$.

0022 4. *Annihilation With the Empty Set.* We have isomorphisms of sets

$$A \coprod_X \emptyset \cong \emptyset,$$
$$\emptyset \coprod_X A \cong \emptyset,$$

natural in $A, X \in \mathrm{Obj}(\mathsf{Sets})$.

0023 5. *Symmetric Monoidality.* The triple $(\mathsf{Sets}, \coprod_X, \emptyset)$ is a symmetric monoidal category.

PROOF 2.3.5 ► PROOF OF PROPOSITION 2.3.4

Item 1: Associativity

Clear.

Item 2: Unitality

Clear.

Item 3: Commutativity

Clear.

Item 4: Annihilation With the Empty Set

Clear.

Item 5: Symmetric Monoidality

Omitted.

0024 2.4 Coequalisers

Let A and B be sets and let $f, g\colon A \rightrightarrows B$ be functions.

0025 **Definition 2.4.1 ► Coequalisers of Sets**

The **coequaliser of f and g** is the set $\mathrm{CoEq}(f, g)$ defined by

$$\mathrm{CoEq}(f, g) \stackrel{\text{def}}{=} B/{\sim},$$

where $\sim$ is the equivalence relation on B generated by $f(a) \sim g(a)$.

0026 **Remark 2.4.2 ► Unwinding Definition 2.4.1**

In detail, the relation $\sim$ of Definition 2.4.1 is given by declaring $a \sim b$ iff one of the following conditions is satisfied:

- We have $a = b$;
- There exist $x_1, \ldots, x_n \in B$ such that $a \sim' x_1 \sim' \cdots \sim' x_n \sim' b$, where we declare $x \sim' y$ if one of the following conditions is satisfied:
  1. There exists $z \in A$ such that $x = f(z)$ and $y = g(z)$.
  2. There exists $z \in A$ such that $x = g(z)$ and $y = f(z)$.

  That is: we require the following condition to be satisfied:

  ($\star$) There exist $x_1, \ldots, x_n \in B$ satisfying the following conditions:
  1. There exists $z_0 \in A$ satisfying one of the following conditions:
     (a) We have $a = f(z_0)$ and $x_1 = g(z_0)$.
     (b) We have $a = g(z_0)$ and $x_1 = f(z_0)$.
  2. For each $1 \leq i \leq n-1$, there exists $z_i \in A$ satisfying one of the following conditions:
     (a) We have $x_i = f(z_i)$ and $x_{i+1} = g(z_i)$.
     (b) We have $x_i = g(z_i)$ and $x_{i+1} = f(z_i)$.
  3. There exists $z_n \in A$ satisfying one of the following conditions:
     (a) We have $x_n = f(z_n)$ and $b = g(z_n)$.
     (b) We have $x_n = g(z_n)$ and $b = f(z_n)$.

0027 **EXAMPLE 2.4.3 ▶ EXAMPLES OF COEQUALISERS OF SETS**

Here are some examples of coequalisers of sets.

0028 1. *Quotients by Equivalence Relations.* Let R be an equivalence relation on a set X. We have a bijection of sets

$$X/{\sim_R} \cong \mathrm{CoEq}\left(R \hookrightarrow X \times X \underset{\mathrm{pr}_2}{\overset{\mathrm{pr}_1}{\rightrightarrows}} X\right).$$

0029 **PROPOSITION 2.4.4 ▶ PROPERTIES OF COEQUALISERS OF SETS**

Let A, B, and C be sets.

002A 1. *Associativity.* We have an isomorphism of sets[1]

$$\underbrace{\mathrm{CoEq}(\mathrm{coeq}(f,g) \circ f, \mathrm{coeq}(f,g) \circ h)}_{=\mathrm{CoEq}(\mathrm{coeq}(f,g)\circ g, \mathrm{coeq}(f,g)\circ h)} \cong \mathrm{CoEq}(f,g,h) \cong \underbrace{\mathrm{CoEq}(\mathrm{coeq}(g,h) \circ f, \mathrm{coeq}(g,h) \circ g)}_{=\mathrm{CoEq}(\mathrm{coeq}(g,h)\circ f, \mathrm{coeq}(g,h)\circ h)},$$

where $\mathrm{CoEq}(f,g,h)$ is the colimit of the diagram

$$\begin{tikzcd} A \arrow[r, shift left=2, "f"] \arrow[r, "g" description] \arrow[r, shift right=2, "h"'] & B \end{tikzcd}$$

in Sets.

002B 2. *Unitality.* We have an isomorphism of sets

$$\mathrm{CoEq}(f,f) \cong B.$$

002C 3. *Commutativity.* We have an isomorphism of sets

$$\mathrm{CoEq}(f,g) \cong \mathrm{CoEq}(g,f).$$

002D 4. *Interaction With Composition.* Let

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \underset{k}{\overset{h}{\rightrightarrows}} C$$

be functions. We have a surjection

$$\mathrm{CoEq}(h \circ f, k \circ g) \twoheadrightarrow \mathrm{CoEq}(\mathrm{coeq}(h, k) \circ h \circ f, \mathrm{coeq}(h, k) \circ k \circ g)$$

exhibiting $\mathrm{CoEq}(\mathrm{coeq}(h, k) \circ h \circ f, \mathrm{coeq}(h, k) \circ k \circ g)$ as a quotient of $\mathrm{CoEq}(h \circ f, k \circ g)$ by the relation generated by declaring $h(y) \sim k(y)$ for each $y \in B$.

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[1] That is: the following constructions give the same result:

(a) Take the coequaliser of (f, g, h), i.e. the colimit of the diagram

$$\begin{tikzcd} A \arrow[r, shift left=2, "f"] \arrow[r, "g" description] \arrow[r, shift right=2, "h"'] & B \end{tikzcd}$$

in Sets.

(b) First take the coequaliser of f and g, forming a diagram

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \overset{\mathrm{coeq}(f,g)}{\twoheadrightarrow} \mathrm{CoEq}(f, g)$$

and then take the coequaliser of the composition

$$A \underset{h}{\overset{f}{\rightrightarrows}} B \overset{\mathrm{coeq}(f,g)}{\twoheadrightarrow} \mathrm{CoEq}(f, g),$$

obtaining a quotient

$$\mathrm{CoEq}(\mathrm{coeq}(f, g) \circ f, \mathrm{coeq}(f, g) \circ h) = \mathrm{CoEq}(\mathrm{coeq}(f, g) \circ g, \mathrm{coeq}(f, g) \circ h)$$

of $\mathrm{CoEq}(f, g)$

(c) First take the coequaliser of g and h, forming a diagram

$$A \underset{h}{\overset{g}{\rightrightarrows}} B \overset{\mathrm{coeq}(g,h)}{\twoheadrightarrow} \mathrm{CoEq}(g, h)$$

and then take the coequaliser of the composition

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \overset{\mathrm{coeq}(g,h)}{\twoheadrightarrow} \mathrm{CoEq}(g, h),$$

obtaining a quotient

$$\mathrm{CoEq}(\mathrm{coeq}(g, h) \circ f, \mathrm{coeq}(g, h) \circ g) = \mathrm{CoEq}(\mathrm{coeq}(g, h) \circ f, \mathrm{coeq}(g, h) \circ h)$$

of $\mathrm{CoEq}(g, h)$.

Proof 2.4.5 ► Proof of Proposition 2.4.4

Item 1: Associativity

Omitted.

Item 2: Unitality

Clear.

Item 3: Commutativity

Clear.

Item 4: Interaction With Composition

Omitted.

002E
3 Operations With Sets

002F
3.1 The Empty Set

002G **Definition 3.1.1 ► The Empty Set**

The **empty set** is the set $\emptyset$ defined by

$$\emptyset \overset{\text{def}}{=} \{x \in X \mid x \neq x\},$$

where A is the set in the set existence axiom, **??** of **??**.

002H
3.2 Singleton Sets

Let X be a set.

002J **Definition 3.2.1 ► Singleton Sets**

The **singleton set containing** X is the set $\{X\}$ defined by

$$\{X\} \overset{\text{def}}{=} \{X, X\},$$

where $\{X, X\}$ is the pairing of X with itself (Definition 3.3.1).

002K
3.3 Pairings of Sets

Let X and Y be sets.

002L **Definition 3.3.1 ▶ Pairings of Sets**

The **pairing of X and Y** is the set $\{X, Y\}$ defined by

$$\{X, Y\} \overset{\text{def}}{=} \{x \in A \mid x = X \text{ or } x = Y\},$$

where A is the set in the axiom of pairing, **??** of **??**.

002M
3.4 Unions of Families

Let $\{A_i\}_{i \in I}$ be a family of sets.

002N **Definition 3.4.1 ▶ Unions of Families**

The **union of the family $\{A_i\}_{i \in I}$** is the set $\bigcup_{i \in I} A_i$ defined by

$$\bigcup_{i \in I} A_i \overset{\text{def}}{=} \{x \in F \mid \text{there exists some } i \in I \text{ such that } x \in A_i\},$$

where F is the set in the axiom of union, **??** of **??**.

002P
3.5 Binary Unions

Let A and B be sets.

002Q **Definition 3.5.1 ▶ Binary Unions**

The **union**[1] **of A and B** is the set $A \cup B$ defined by

$$A \cup B \overset{\text{def}}{=} \bigcup_{z \in \{A, B\}} z.$$

[1] *Further Terminology:* Also called the **binary union of A and B**, for emphasis.

002R **Proposition 3.5.2 ▶ Properties of Binary Unions**

Let X be a set.

002S 1. *Functoriality.* The assignments $U, V, (U, V) \mapsto U \cup V$ define functors

$$\begin{aligned} U \cup -\colon (\mathcal{P}(X), \subset) &\to (\mathcal{P}(X), \subset),\\ - \cup V\colon (\mathcal{P}(X), \subset) &\to (\mathcal{P}(X), \subset),\\ -_1 \cup -_2\colon (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \subset) &\to (\mathcal{P}(X), \subset), \end{aligned}$$

where $-_1 \cup -_2$ is the functor where

- *Action on Objects.* For each $(U, V) \in \mathcal{P}(X) \times \mathcal{P}(X)$, we have

$$[-_1 \cup -_2](U, V) \stackrel{\text{def}}{=} U \cup V;$$

- *Action on Morphisms.* For each pair of morphisms

$$\begin{aligned} \iota_U &: U \hookrightarrow U',\\ \iota_V &: V \hookrightarrow V' \end{aligned}$$

of $\mathcal{P}(X) \times \mathcal{P}(X)$, the image

$$\iota_U \cup \iota_V \colon U \cup V \hookrightarrow U' \cup V'$$

of (ι_U, ι_V) by $\cup$ is the inclusion

$$U \cup V \subset U' \cup V'$$

i.e. where we have

($\star$) If $U \subset U'$ and $V \subset V'$, then $U \cup V \subset U' \cup V'$;

and where $U \cup -$ and $- \cup V$ are the partial functors of $-_1 \cup -_2$ at $U, V \in \mathcal{P}(X)$.

002T 2. *Via Intersections and Symmetric Differences.* We have an equality of sets

$$U \cup V = (U \mathbin{\triangle} V) \mathbin{\triangle} (U \cap V)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

002U 3. *Associativity.* We have an equality of sets

$$(U \cup V) \cup W = U \cup (V \cup W)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

002V 4. *Unitality.* We have equalities of sets

$$U \cup \emptyset = U,$$
$$\emptyset \cup U = U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

002W 5. *Commutativity.* We have an equality of sets

$$U \cup V = V \cup U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

002X 6. *Idempotency.* We have an equality of sets

$$U \cup U = U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

002Y 7. *Distributivity Over Intersections.* We have equalities of sets

$$U \cup (V \cap W) = (U \cup V) \cap (U \cup W),$$
$$(U \cap V) \cup W = (U \cup W) \cap (V \cup W)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

002Z 8. *Interaction With Powersets and Semirings.* The quintuple $(\mathcal{P}(X), \cup, \cap, \emptyset, X)$ is an idempotent commutative semiring.

PROOF 3.5.3 ► PROOF OF PROPOSITION 3.5.2

Item 1: Functoriality

Omitted.

Item 2: Via Intersections and Symmetric Differences

Omitted.

Item 3: Associativity

Clear.

Item 4: Unitality

Clear.

Item 5: Commutativity

Clear.

Item 6: Idempotency

Clear.

Item 7: Distributivity Over Intersections

Omitted.

Item 8: Interaction With Powersets and Semirings

This follows from Items 3 to 6 and Items 3 to 5, 7 and 8 of Proposition 3.7.2.

0030 3.6 Intersections of Families

Let $\mathcal{F}$ be a family of sets.

0031 **Definition 3.6.1 ► Intersections of Families**

The **intersection of a family $\mathcal{F}$ of sets** is the set $\bigcap_{X\in\mathcal{F}} X$ defined by

$$\bigcap_{X\in\mathcal{F}} X \overset{\text{def}}{=} \left\{ z \in \bigcup_{X\in\mathcal{F}} X \,\middle|\, \text{for each } X \in \mathcal{F}\text{, we have } z \in X \right\}.$$

0032 3.7 Binary Intersections

Let X and Y be sets.

0033 **Definition 3.7.1 ► Binary Intersections**

The **intersection**[1] **of X and Y** is the set $X \cap Y$ defined by

$$X \cap Y \overset{\text{def}}{=} \bigcap_{z\in\{X,Y\}} z.$$

[1] *Further Terminology:* Also called the **binary intersection of X and Y**, for emphasis.

0034 **PROPOSITION 3.7.2 ► PROPERTIES OF BINARY INTERSECTIONS**

Let X be a set.

0035 1. *Functoriality.* The assignments $U, V, (U, V) \mapsto U \cap V$ define functors

$$\begin{aligned} U \cap - &\colon (\mathcal{P}(X), \subset) \to (\mathcal{P}(X), \subset), \\ - \cap V &\colon (\mathcal{P}(X), \subset) \to (\mathcal{P}(X), \subset), \\ -_1 \cap -_2 &\colon (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \subset) \to (\mathcal{P}(X), \subset), \end{aligned}$$

where $-_1 \cap -_2$ is the functor where

- *Action on Objects.* For each $(U, V) \in \mathcal{P}(X) \times \mathcal{P}(X)$, we have

$$[-_1 \cap -_2](U, V) \stackrel{\text{def}}{=} U \cap V;$$

- *Action on Morphisms.* For each pair of morphisms

$$\begin{aligned} \iota_U &\colon U \hookrightarrow U', \\ \iota_V &\colon V \hookrightarrow V' \end{aligned}$$

of $\mathcal{P}(X) \times \mathcal{P}(X)$, the image

$$\iota_U \cap \iota_V \colon U \cap V \hookrightarrow U' \cap V'$$

of (ι_U, ι_V) by $\cap$ is the inclusion

$$U \cap V \subset U' \cap V'$$

i.e. where we have

(★) If $U \subset U'$ and $V \subset V'$, then $U \cap V \subset U' \cap V'$;

and where $U \cap -$ and $- \cap V$ are the partial functors of $-_1 \cap -_2$ at $U, V \in \mathcal{P}(X)$.

0036 2. *Adjointness.* We have adjunctions

$$\left(U \cap - \dashv \mathbf{Hom}_{\mathcal{P}(X)}(U, -)\right) \colon \quad \begin{tikzcd} \mathcal{P}(X) \arrow[r, bend left, "U \cap -"] \arrow[r, phantom, "\perp"] & \mathcal{P}(X) \arrow[l, bend left, "\mathbf{Hom}_{\mathcal{P}(X)}(U,-)"] \end{tikzcd},$$

$$\left(- \cap V \dashv \mathbf{Hom}_{\mathcal{P}(X)}(V, -)\right) \colon \quad \begin{tikzcd} \mathcal{P}(X) \arrow[r, bend left, "- \cap V"] \arrow[r, phantom, "\perp"] & \mathcal{P}(X) \arrow[l, bend left, "\mathbf{Hom}_{\mathcal{P}(X)}(V,-)"] \end{tikzcd},$$

where

$$\mathbf{Hom}_{\mathcal{P}(X)}(-_1,-_2)\colon \mathcal{P}(X)^{\mathsf{op}}\times\mathcal{P}(X)\to\mathcal{P}(X)$$

is the bifunctor defined by[1]

$$\mathbf{Hom}_{\mathcal{P}(X)}(U,V)\overset{\text{def}}{=}(X\setminus U)\cup V$$

witnessed by bijections

$$\begin{aligned}\mathrm{Hom}_{\mathcal{P}(X)}(U\cap V,W)&\cong\mathrm{Hom}_{\mathcal{P}(X)}\big(U,\mathbf{Hom}_{\mathcal{P}(X)}(V,W)\big),\\ \mathrm{Hom}_{\mathcal{P}(X)}(U\cap V,W)&\cong\mathrm{Hom}_{\mathcal{P}(X)}\big(V,\mathbf{Hom}_{\mathcal{P}(X)}(U,W)\big),\end{aligned}$$

natural in $U,V,W\in\mathcal{P}(X)$, i.e. where:

(a) The following conditions are equivalent:

  i. We have $U\cap V\subset W$.
  ii. We have $U\subset\mathbf{Hom}_{\mathcal{P}(X)}(V,W)$.
  iii. We have $U\subset(X\setminus V)\cup W$.

(b) The following conditions are equivalent:

  i. We have $V\cap U\subset W$.
  ii. We have $V\subset\mathbf{Hom}_{\mathcal{P}(X)}(U,W)$.
  iii. We have $V\subset(X\setminus U)\cup W$.

0037 3. *Associativity.* We have an equality of sets

$$(U\cap V)\cap W=U\cap(V\cap W)$$

for each $X\in\mathrm{Obj}(\mathsf{Sets})$ and each $U,V,W\in\mathcal{P}(X)$.

0038 4. *Unitality.* Let X be a set and let $U\in\mathcal{P}(X)$. We have equalities of sets

$$\begin{aligned}X\cap U&=U,\\ U\cap X&=U\end{aligned}$$

for each $X\in\mathrm{Obj}(\mathsf{Sets})$ and each $U\in\mathcal{P}(X)$.

0039 5. *Commutativity.* We have an equality of sets

$$U \cap V = V \cap U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

003A 6. *Idempotency.* We have an equality of sets

$$U \cap U = U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

003B 7. *Distributivity Over Unions.* We have equalities of sets

$$\begin{aligned} U \cap (V \cup W) &= (U \cap V) \cup (U \cap W), \\ (U \cup V) \cap W &= (U \cap W) \cup (V \cap W) \end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

003C 8. *Annihilation With the Empty Set.* We have an equality of sets

$$\begin{aligned} \emptyset \cap X &= \emptyset, \\ X \cap \emptyset &= \emptyset \end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

003D 9. *Interaction With Powersets and Monoids With Zero.* The quadruple $((\mathcal{P}(X), \emptyset), \cap, X)$ is a commutative monoid with zero.

003E 10. *Interaction With Powersets and Semirings.* The quintuple $(\mathcal{P}(X), \cup, \cap, \emptyset, X)$ is an idempotent commutative semiring.

[1] *Intuition:* Since intersections are the products in $\mathcal{P}(X)$, the left adjoint $\mathbf{Hom}_{\mathcal{P}(X)}(U, V)$ works as a function type $U \to V$.

Now, under the Curry–Howard correspondence, the function type $U \to V$ corresponds to implication $U \implies V$, which is logically equivalent to the statement $\neg U \vee V$, which in turn corresponds to the set $U^{\mathsf{c}} \vee V \overset{\text{def}}{=} (X \setminus U) \cup V$.

Proof 3.7.3 ► Proof of Proposition 3.7.2

Item 1: Functoriality

Omitted.

Item 2: Adjointness

See [MSE 267469].

Item 3: Associativity

Clear.

Item 4: Unitality

Clear.

Item 5: Commutativity

Clear.

Item 6: Idempotency

Clear.

Item 7: Distributivity Over Unions

Omitted.

Item 8: Annihilation With the Empty Set

Clear.

Item 9: Interaction With Powersets and Monoids With Zero

This follows from Items 3 to 5 and 8.

Item 10: Interaction With Powersets and Semirings

This follows from Items 3 to 6 and Items 3 to 5, 7 and 8 of Proposition 3.7.2.

003F
3.8 Differences

Let X and Y be sets.

003G
Definition 3.8.1 ► Differences

The **difference of** X **and** Y is the set $X \setminus Y$ defined by

$$X \setminus Y \overset{\text{def}}{=} \{a \in X \mid a \notin Y\}.$$

003H **Proposition 3.8.2 ▶ Properties of Differences**

Let X be a set.

003J 1. *Functoriality.* The assignments $U, V, (U, V) \mapsto U \cap V$ define functors

$$\begin{aligned} U \setminus -&\colon (\mathcal{P}(X), \supset) \to (\mathcal{P}(X), \subset), \\ - \setminus V&\colon (\mathcal{P}(X), \subset) \to (\mathcal{P}(X), \subset), \\ -_1 \setminus -_2&\colon (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \supset) \to (\mathcal{P}(X), \subset), \end{aligned}$$

where $-_1 \setminus -_2$ is the functor where

- *Action on Objects.* For each $(U, V) \in \mathcal{P}(X) \times \mathcal{P}(X)$, we have

$$[-_1 \setminus -_2](U, V) \stackrel{\text{def}}{=} U \setminus V;$$

- *Action on Morphisms.* For each pair of morphisms

$$\begin{aligned} \iota_A &\colon A \hookrightarrow B, \\ \iota_U &\colon U \hookrightarrow V \end{aligned}$$

of $\mathcal{P}(X) \times \mathcal{P}(X)$, the image

$$\iota_U \setminus \iota_V \colon A \setminus V \hookrightarrow B \setminus U$$

of (ι_U, ι_V) by $\setminus$ is the inclusion

$$A \setminus V \subset B \setminus U$$

i.e. where we have

($\star$) If $A \subset B$ and $U \subset V$, then $A \setminus V \subset B \setminus U$;

and where $U \setminus -$ and $- \setminus V$ are the partial functors of $-_1 \setminus -_2$ at $U, V \in \mathcal{P}(X)$.

003K 2. *De Morgan's Laws.* We have equalities of sets

$$\begin{aligned} X \setminus (U \cup V) &= (X \setminus U) \cap (X \setminus V), \\ X \setminus (U \cap V) &= (X \setminus U) \cup (X \setminus V) \end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

003L 3. *Interaction With Unions I.* We have equalities of sets

$$(U \setminus V) \cup W = (U \cup W) \setminus (V \setminus W)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

003M 4. *Interaction With Unions II.* We have equalities of sets

$$(U \cup V) \setminus W = (U \setminus W) \cup (V \setminus W)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

003N 5. *Interaction With Intersections.* We have equalities of sets

$$\begin{aligned}(U \setminus V) \cap W &= (U \cap W) \setminus V \\ &= U \cap (W \setminus V)\end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

003P 6. *Triple Differences.* We have

$$U \setminus (V \setminus W) = (U \cap W) \cup (U \setminus V)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

003Q 7. *Left Annihilation.* We have

$$\emptyset \setminus U = \emptyset$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

003R 8. *Right Unitality.* We have

$$U \setminus \emptyset = U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

003S 9. *Invertibility.* We have

$$U \setminus U = \emptyset$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

003T 10. *Interaction With Containment.* The following conditions are equivalent:

(a) We have $V \setminus U \subset W$.

(b) We have $V \setminus W \subset U$.

Proof 3.8.3 ► Proof of Proposition 3.8.2

Item 1: Functoriality

Omitted.

Item 2: De Morgan's Laws

Omitted.

Item 3: Interaction With Unions I

Omitted.

Item 4: Interaction With Unions II

Omitted.

Item 5: Interaction With Intersections

Omitted.

Item 6: Triple Differences

Omitted.

Item 7: Left Annihilation

Clear.

Item 8: Right Unitality

Clear.

Item 9: Invertibility

Clear.

Item 10: Interaction With Containment

Omitted. ∎

003U

3.9 Complements

Let X be a set and let $U \in \mathcal{P}(X)$.

003V

Definition 3.9.1 ► Complements

The **complement of** U is the set U^{c} defined by

$$\begin{aligned} U^{\mathsf{c}} &\overset{\text{def}}{=} X \setminus U \\ &\overset{\text{def}}{=} \{a \in X \mid a \notin U\}. \end{aligned}$$

003W **Proposition 3.9.2 ▶ Properties of Complements**

Let X be a set.

003X 1. *Functoriality.* The assignment $U \mapsto U^{\mathsf{c}}$ defines a functor

$$(-)^{\mathsf{c}} \colon \mathcal{P}(X)^{\mathsf{op}} \to \mathcal{P}(X),$$

where

- *Action on Objects.* For each $U \in \mathcal{P}(X)$, we have

$$[(-)^{\mathsf{c}}](U) \overset{\text{def}}{=} U^{\mathsf{c}};$$

- *Action on Morphisms.* For each morphism $\iota_U \colon U \hookrightarrow V$ of $\mathcal{P}(X)$, the image

$$\iota_U^{\mathsf{c}} \colon V^{\mathsf{c}} \hookrightarrow U^{\mathsf{c}}$$

of ι_U by $(-)^{\mathsf{c}}$ is the inclusion

$$V^{\mathsf{c}} \subset U^{\mathsf{c}}$$

i.e. where we have

($\star$) If $U \subset V$, then $V^{\mathsf{c}} \subset U^{\mathsf{c}}$.

003Y 2. *De Morgan's Laws.* We have equalities of sets

$$(U \cup V)^{\mathsf{c}} = U^{\mathsf{c}} \cap V^{\mathsf{c}},$$
$$(U \cap V)^{\mathsf{c}} = U^{\mathsf{c}} \cup V^{\mathsf{c}}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

003Z 3. *Involutority.* We have

$$(U^{\mathsf{c}})^{\mathsf{c}} = U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

Proof 3.9.3 ▶ Proof of Proposition 3.9.2

Item 1: Functoriality

Clear.

Item 2: De Morgan's Laws

Omitted.

Item 3: Involutority

Clear.

0040 3.10 Symmetric Differences

Let A and B be sets.

0041 **Definition 3.10.1 ▶ Symmetric Differences**

The **symmetric difference of A and B** is the set $A \vartriangle B$ defined by

$$A \vartriangle B \overset{\text{def}}{=} (A \setminus B) \cup (B \setminus A).$$

0042 **Proposition 3.10.2 ▶ Properties of Symmetric Differences**

Let X be a set.

0043 1. *Lack of Functoriality.* The assignment $(U, V) \mapsto U \vartriangle V$ ***does not*** define a functor

$$-_1 \vartriangle -_2 \colon (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \subset) \to (\mathcal{P}(X), \subset).$$

0044 2. *Via Unions and Intersections.* We have[1]

$$U \vartriangle V = (U \cup V) \setminus (U \cap V)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

0045 3. *Associativity.* We have[2]

$$(U \vartriangle V) \vartriangle W = U \vartriangle (V \vartriangle W)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

0046 4. *Unitality.* We have

$$U \triangle \emptyset = U,$$
$$\emptyset \triangle U = U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

0047 5. *Invertibility.* We have

$$U \triangle U = \emptyset$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

0048 6. *Commutativity.* We have

$$U \triangle V = V \triangle U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

0049 7. *"Transitivity".* We have

$$(U \triangle V) \triangle (V \triangle W) = U \triangle W$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

004A 8. *The Triangle Inequality for Symmetric Differences.* We have

$$U \triangle W \subset U \triangle V \cup V \triangle W$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

004B 9. *Distributivity Over Intersections.* We have

$$U \cap (V \triangle W) = (U \cap V) \triangle (U \cap W),$$
$$(U \triangle V) \cap W = (U \cap W) \triangle (V \cap W)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

004C 10. *Interaction With Indicator Functions.* We have

$$\chi_{U \triangle V} \equiv \chi_U + \chi_V \mod 2$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

004D 11. *Bijectivity.* Given $A, B \subset \mathcal{P}(X)$, the maps

$$A \mathbin{\triangle} -\colon \mathcal{P}(X) \to \mathcal{P}(X),$$
$$- \mathbin{\triangle} B\colon \mathcal{P}(X) \to \mathcal{P}(X)$$

are bijections with inverses given by

$$(A \mathbin{\triangle} -)^{-1} = - \cup (A \cap -),$$
$$(- \mathbin{\triangle} B)^{-1} = - \cup (B \cap -).$$

Moreover, the map

$$C \mapsto C \mathbin{\triangle} (A \mathbin{\triangle} B)$$

is a bijection of $\mathcal{P}(X)$ onto itself sending A to B and B to A.

004E 12. *Interaction With Powersets and Groups I.* The quadruple $\left(\mathcal{P}(X), \triangle, \emptyset, \mathrm{id}_{\mathcal{P}(X)}\right)$ is an abelian group.[3,4,5]

004F 13. *Interaction With Powersets and Groups II.* Every element of $\mathcal{P}(X)$ has order 2 with respect to $\triangle$, and thus $\mathcal{P}(X)$ is a *Boolean group* (i.e. an abelian 2-group).

004G 14. *Interaction With Powersets and Vector Spaces I.* The pair $\left(\mathcal{P}(X), \alpha_{\mathcal{P}(X)}\right)$ consisting of

- The group $\mathcal{P}(X)$ of Item 12;
- The map $\alpha_{\mathcal{P}(X)} \colon \mathbb{F}_2 \times \mathcal{P}(X) \to \mathcal{P}(X)$ defined by

$$0 \cdot U \stackrel{\text{def}}{=} \emptyset,$$
$$1 \cdot U \stackrel{\text{def}}{=} U;$$

is an $\mathbb{F}_2$-vector space.

004H 15. *Interaction With Powersets and Vector Spaces II.* If X is finite, then:

(a) The set of singletons sets on the elements of X forms a basis for the $\mathbb{F}_2$-vector space $\left(\mathcal{P}(X), \alpha_{\mathcal{P}(X)}\right)$ of Item 14.

(b) We have

$$\dim(\mathcal{P}(X)) = \#\mathcal{P}(X).$$

004J 16. *Interaction With Powersets and Rings.* The quintuple $(\mathcal{P}(X), \triangle, \cap, \emptyset, X)$ is a commutative ring.[6]

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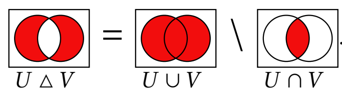
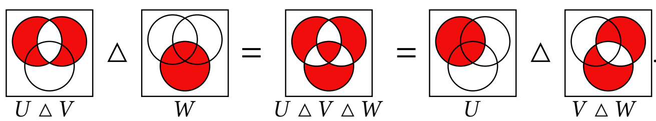
[1] *Illustration:*



[2] *Illustration:*



[3] *Example:* When $X = \emptyset$, we have an isomorphism of groups between $\mathcal{P}(\emptyset)$ and the trivial group:

$$\left(\mathcal{P}(\emptyset), \triangle, \emptyset, \mathrm{id}_{\mathcal{P}(\emptyset)}\right) \cong \mathrm{pt}.$$

[4] *Example:* When $X = \mathrm{pt}$, we have an isomorphism of groups between $\mathcal{P}(\mathrm{pt})$ and $\mathbb{Z}_{/2}$:

$$\left(\mathcal{P}(\mathrm{pt}), \triangle, \emptyset, \mathrm{id}_{\mathcal{P}(\mathrm{pt})}\right) \cong \mathbb{Z}_{/2}.$$

[5] *Example:* When $X = \{0, 1\}$, we have an isomorphism of groups between $\mathcal{P}(\{0, 1\})$ and $\mathbb{Z}_{/2} \times \mathbb{Z}_{/2}$:

$$\left(\mathcal{P}(\{0, 1\}), \triangle, \emptyset, \mathrm{id}_{\mathcal{P}(\{0,1\})}\right) \cong \mathbb{Z}_{/2} \times \mathbb{Z}_{/2}.$$

[6] *Warning:* The analogous statement replacing intersections by unions (i.e. that the quintuple $(\mathcal{P}(X), \triangle, \cup, \emptyset, X)$ is a ring) is false, however. See [Pro23b] for a proof.

Proof 3.10.3 ► Proof of Proposition 3.10.2

Item 1: Lack of Functoriality

Omitted.

Item 2: Via Unions and Intersections

Omitted.

Item 3: Associativity

Omitted.

Item 4: Unitality

Clear.

Item 5: Invertibility

Clear.

Item 6: Commutativity

Clear.

Item 7: "Transitivity"

We have

$$\begin{aligned}(U \triangle V) \triangle (V \triangle W) &= U \triangle (V \triangle (V \triangle W)) && \text{(by Item 3)}\\ &= U \triangle ((V \triangle V) \triangle W) && \text{(by Item 3)}\\ &= U \triangle (\emptyset \triangle W) && \text{(by Item 5)}\\ &= U \triangle W && \text{(by Item 4)}\end{aligned}$$

Item 8: The Triangle Inequality for Symmetric Differences

This follows from Items 2 and 7.

Item 9: Distributivity Over Intersections

Omitted.

Item 10: Interaction With Indicator Functions

Clear.

Item 11: Bijectivity

Clear.

Item 12: Interaction With Powersets and Groups I

This follows from Items 3 to 6.

Item 13: Interaction With Powersets and Groups II

This follows from Item 5.

Item 14: Interaction With Powersets and Vector Spaces I

Clear.

Item 15: Interaction With Powersets and Vector Spaces II

Omitted.

Item 16: Interaction With Powersets and Rings

This follows from Items 9 and 12 and Items 8 and 9 of Proposition 3.7.2.[1]

[1] *Reference:* [Pro23a].

004K 3.11 Ordered Pairs

Let A and B be sets.

004L **Definition 3.11.1 ► Ordered Pairs**

The **ordered pair associated to** A **and** B is the set (A, B) defined by

$$(A, B) \overset{\text{def}}{=} \{\{A\}, \{A, B\}\}.$$

004M **Proposition 3.11.2 ► Properties of Ordered Pairs**

Let A and B be sets.

004N 1. *Uniqueness.* Let $A, B, C,$ and D be sets. The following conditions are equivalent:
   (a) We have $(A, B) = (C, D)$.
   (b) We have $A = C$ and $B = D$.

Proof 3.11.3 ► Proof of Proposition 3.11.2

Item 1: Uniqueness

See [Cie97, Theorem 1.2.3].

004P 4 Powersets

004Q 4.1 Characteristic Functions

Let X be a set.

004R **Definition 4.1.1 ► Characteristic Functions**

Let $U \subset X$ and let $x \in X$.

004S 1. The **characteristic function of** U [1] is the function[2]

$$\chi_U : X \to \{\mathrm{t}, \mathrm{f}\}$$

defined by

$$\chi_U(x) \overset{\text{def}}{=} \begin{cases} \text{true} & \text{if } x \in U, \\ \text{false} & \text{if } x \notin U \end{cases}$$

for each $x \in X$.

004T 2. The **characteristic function of** x is the function [3]

$$\chi_x \colon X \to \{\mathsf{t}, \mathsf{f}\}$$

defined by

$$\chi_x \overset{\text{def}}{=} \chi_{\{x\}},$$

i.e. by

$$\chi_x(y) \overset{\text{def}}{=} \begin{cases} \text{true} & \text{if } x = y, \\ \text{false} & \text{if } x \neq y \end{cases}$$

for each $y \in X$.

004U 3. The **characteristic relation on** X [4] is the relation[5]

$$\chi_X(-_1, -_2) \colon X \times X \to \{\mathsf{t}, \mathsf{f}\}$$

on X defined by[6]

$$\chi_X(x, y) \overset{\text{def}}{=} \begin{cases} \text{true} & \text{if } x = y, \\ \text{false} & \text{if } x \neq y \end{cases}$$

for each $x, y \in X$.

004V 4. The **characteristic embedding** [7] **of** X **into** $\mathcal{P}(X)$ is the function

$$\chi_{(-)} \colon X \hookrightarrow \mathcal{P}(X)$$

defined by

$$\chi_{(-)}(x) \overset{\text{def}}{=} \chi_x$$

for each $x \in X$.

[1] *Further Terminology:* Also called the **indicator function of** U.
[2] *Further Notation:* Also written $\chi_X(U,-)$ or $\chi_X(-,U)$.
[3] *Further Notation:* Also written χ_x, $\chi_X(x,-)$, or $\chi_X(-,x)$.
[4] *Further Terminology:* Also called the **identity relation on** X.
[5] *Further Notation:* Also written $\chi_{-_2}^{-_1}$, or $\sim_{\mathrm{id}}$ in the context of relations.
[6] As a subset of $X \times X$, the relation χ_X corresponds to the diagonal $\Delta_X \subset X \times X$ of X.
[7] The name "characteristic *embedding*" comes from the fact that there is an analogue of fully faithfulness for $\chi_{(-)}$: given a set X, we have

$$\mathrm{Hom}_{\mathcal{P}(X)}(\chi_x, \chi_y) = \chi_X(x, y),$$

for each $x, y \in X$.

004W **Remark 4.1.2 ► Characteristic Functions as Decategorifications of Presheaves**

The definitions in Definition 4.1.1 are decategorifications of co/presheaves, representable co/presheaves, Hom profunctors, and the Yoneda embedding:[1]

1. A function
$$f: X \to \{\mathsf{t}, \mathsf{f}\}$$
is a decategorification of a presheaf
$$\mathcal{F}: \mathcal{C}^{\mathsf{op}} \to \mathsf{Sets},$$
with the characteristic functions χ_U of the subsets of X being the primordial examples (and, in fact, all examples) of these.

2. The characteristic function
$$\chi_x: X \to \{\mathsf{t}, \mathsf{f}\}$$
of an *element* x of X is a decategorification of the representable presheaf
$$h_X: \mathcal{C}^{\mathsf{op}} \to \mathsf{Sets}$$
of an *object* x of a category $\mathcal{C}$.

3. The characteristic relation
$$\chi_X(-_1, -_2): X \times X \to \{\mathsf{t}, \mathsf{f}\}$$
of X is a decategorification of the Hom profunctor
$$\mathrm{Hom}_{\mathcal{C}}(-_1, -_2): \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathsf{Sets}$$
of a category $\mathcal{C}$.

4. The characteristic embedding

$$\chi_{(-)}\colon X \hookrightarrow \mathcal{P}(X)$$

of X into $\mathcal{P}(X)$ is a decategorification of the Yoneda embedding

$$よ\colon \mathcal{C}^{\mathsf{op}} \hookrightarrow \mathsf{PSh}(\mathcal{C})$$

of a category $\mathcal{C}$ into $\mathsf{PSh}(\mathcal{C})$.

5. There is also a direct parallel between unions and colimits:

   - An element of $\mathcal{P}(X)$ is a union of elements of X, viewed as one-point subsets $\{x\} \in \mathcal{P}(A)$;
   - An object of $\mathsf{PSh}(\mathcal{C})$ is a colimit of objects of $\mathcal{C}$, viewed as representable presheaves $h_X \in \mathrm{Obj}(\mathsf{PSh}(\mathcal{C}))$.

[1]These statements can be made precise by using the embeddings

$$\begin{aligned}(-)_{\mathsf{disc}}&\colon \mathsf{Sets} \hookrightarrow \mathsf{Cats},\\ (-)_{\mathsf{disc}}&\colon \{\mathsf{t},\mathsf{f}\}_{\mathsf{disc}} \hookrightarrow \mathsf{Sets}\end{aligned}$$

of sets into categories and of classical truth values into sets.

For instance, in this approach the characteristic function

$$\chi_x\colon X \to \{\mathsf{t},\mathsf{f}\}$$

of an element x of X, defined by

$$\chi_x(y) \overset{\mathrm{def}}{=} \begin{cases}\mathsf{true} & \text{if } x = y,\\ \mathsf{false} & \text{if } x \neq y\end{cases}$$

for each $y \in X$, is recovered as the representable presheaf

$$\mathrm{Hom}_{X_{\mathsf{disc}}}(-,x)\colon X_{\mathsf{disc}} \to \mathsf{Sets}$$

of the corresponding object x of X_{disc}, defined on objects by

$$\mathrm{Hom}_{X_{\mathsf{disc}}}(y,x) \overset{\mathrm{def}}{=} \begin{cases}\mathrm{pt} & \text{if } x = y,\\ \emptyset & \text{if } x \neq y\end{cases}$$

for each $y \in \mathrm{Obj}(X_{\mathsf{disc}})$.

004X **Proposition 4.1.3 ▶ The Inclusion of Characteristic Relations Associated to a Function**

Let $f\colon A\to B$ be a function. We have an inclusion

$$\chi_B\circ(f\times f)\subset\chi_A,\qquad \begin{tikzcd} A\times A \arrow[r, "\chi_A(-_1,-_2)"] \arrow[d, "f\times f"'] \arrow[dr, phantom, "\subset"] & \{\mathrm{true},\mathrm{false}\} \arrow[d, "\mathrm{id}_{\{\mathrm{true},\mathrm{false}\}}"] \\ B\times B \arrow[r, "\chi_B(-_1,-_2)"'] & \{\mathrm{true},\mathrm{false}\}. \end{tikzcd}$$

Proof 4.1.4 ▶ Proof of Proposition 4.1.3

The inclusion $\chi_B(f(a),f(b))\subset\chi_A(a,b)$ is equivalent to the statement "if $a=b$, then $f(a)=f(b)$", which is true.

004Y **Proposition 4.1.5 ▶ The Yoneda Lemma for Sets**

Let X be a set and let $U\subset X$ be a subset of X. We have

$$\mathrm{Hom}_{\mathcal{P}(X)}(\chi_x,\chi_U)=\chi_U(x)$$

for each $x\in X$, giving an equality of functions

$$\mathrm{Hom}_{\mathcal{P}(X)}\big(\chi_{(-)},\chi_U\big)=\chi_U.$$

Proof 4.1.6 ▶ Proof of Proposition 4.1.5

Clear.

004Z **Corollary 4.1.7 ▶ The Characteristic Embedding Is Fully Faithful**

The characteristic embedding is fully faithful, i.e., we have

$$\mathrm{Hom}_{\mathcal{P}(X)}\big(\chi_x,\chi_y\big)=\chi_X(x,y)$$

for each $x,y\in X$.

PROOF 4.1.8 ▶ PROOF OF COROLLARY 4.1.7

This follows from Proposition 4.1.5.

0050 ## 4.2 Powersets

Let X be a set.

0051 **DEFINITION 4.2.1 ▶ POWERSETS**

The **powerset of** X is the set $\mathcal{P}(X)$ defined by

$$\mathcal{P}(X) \overset{\text{def}}{=} \{U \in P \mid U \subset X\},$$

where P is the set in the axiom of powerset, **??** of **??**.

0052 **REMARK 4.2.2 ▶ POWERSETS AS DECATEGORIFICATIONS OF CO/PRESHEAF CATEGORIES**

The powerset of a set is a decategorification of the category of presheaves of a category: while[1]

- The powerset of a set X is equivalently (Item 6 of Proposition 4.2.3) the set

  $$\mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\})$$

  of functions from X to the set $\{\mathsf{t}, \mathsf{f}\}$ of classical truth values;

- The category of presheaves on a category C is the category

  $$\mathsf{Fun}(C^{\mathsf{op}}, \mathsf{Sets})$$

  of functors from C^{op} to the category Sets of sets.

[1]This parallel is based on the following comparison:

- A category is enriched over the category

  $$\mathsf{Sets} \overset{\text{def}}{=} \mathsf{Cats}_0$$

  of sets (i.e. “0-categories”), with presheaves taking values on it;

- A set is enriched over the set

  $$\{\mathsf{t}, \mathsf{f}\} \overset{\text{def}}{=} \mathsf{Cats}_{-1}$$

  of classical truth values (i.e. “(-1)-categories”), with characteristic functions taking values on it.

0053 **PROPOSITION 4.2.3 ▶ PROPERTIES OF POWERSETS**

Let X be a set.

0054 1. *Functoriality.* The assignment $X \mapsto \mathcal{P}(X)$ defines functors

$$\begin{aligned}\mathcal{P}_*\colon \mathsf{Sets} &\to \mathsf{Sets},\\ \mathcal{P}^{-1}\colon \mathsf{Sets}^{\mathsf{op}} &\to \mathsf{Sets},\\ \mathcal{P}_!\colon \mathsf{Sets} &\to \mathsf{Sets}\end{aligned}$$

where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$\begin{aligned}\mathcal{P}_*(A) &\overset{\text{def}}{=} \mathcal{P}(A),\\ \mathcal{P}^{-1}(A) &\overset{\text{def}}{=} \mathcal{P}(A),\\ \mathcal{P}_!(A) &\overset{\text{def}}{=} \mathcal{P}(A);\end{aligned}$$

- *Action on Morphisms.* For each morphism $f\colon A \to B$ of Sets, the images

$$\begin{aligned}\mathcal{P}_*(f)\colon \mathcal{P}(A) &\to \mathcal{P}(B),\\ \mathcal{P}^{-1}(f)\colon \mathcal{P}(B) &\to \mathcal{P}(A),\\ \mathcal{P}_!(f)\colon \mathcal{P}(A) &\to \mathcal{P}(B)\end{aligned}$$

  of f by $\mathcal{P}_*$, $\mathcal{P}^{-1}$, and $\mathcal{P}_!$ are defined by

$$\begin{aligned}\mathcal{P}_*(f) &\overset{\text{def}}{=} f_*,\\ \mathcal{P}^{-1}(f) &\overset{\text{def}}{=} f^{-1},\\ \mathcal{P}_!(f) &\overset{\text{def}}{=} f_!,\end{aligned}$$

  as in Definitions 4.3.1, 4.4.1 and 4.5.1.

0055 2. *Adjointness I.* We have an adjunction

$$\left(\mathcal{P}^{-1} \dashv \mathcal{P}^{-1,\mathsf{op}}\right)\colon \quad \begin{tikzcd} \mathsf{Sets}^{\mathsf{op}} \arrow[r, bend left, "\mathcal{P}^{-1}"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}, \arrow[l, bend left, "\mathcal{P}^{-1,\mathsf{op}}"] \end{tikzcd}$$

witnessed by a bijection

$$\underbrace{\mathsf{Sets}^{\mathsf{op}}(\mathcal{P}(X), Y)}_{\overset{\text{def}}{=}\mathsf{Sets}(Y,\mathcal{P}(X))} \cong \mathsf{Sets}(X, \mathcal{P}(Y)),$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets})$ and $Y \in \mathrm{Obj}(\mathsf{Sets}^{\mathsf{op}})$.

0056 3. *Adjointness II.* We have an adjunction

$$(\mathsf{Gr} \dashv \mathcal{P}_*) : \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "\mathsf{Gr}"] \arrow[r, phantom, "\perp"] & \mathsf{Rel}, \arrow[l, bend left, "\mathcal{P}_*"] \end{tikzcd}$$

witnessed by a bijection of sets

$$\mathsf{Rel}(\mathsf{Gr}(A), B) \cong \mathsf{Sets}(A, \mathcal{P}(B))$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$ and $B \in \mathrm{Obj}(\mathsf{Rel})$, where Gr is the graph functor of Relations, Item 1 of Proposition 3.1.2.

0057 4. *Symmetric Strong Monoidality With Respect to Coproducts.* The powerset functor $\mathcal{P}_*$ of Item 1 has a symmetric strong monoidal structure

$$\left(\mathcal{P}_*, \mathcal{P}_*^{\coprod}, \mathcal{P}_{*|\mathbb{1}}^{\coprod}\right) : (\mathsf{Sets}, \coprod, \emptyset) \longrightarrow (\mathsf{Sets}, \times, \mathrm{pt})$$

being equipped with isomorphisms

$$\mathcal{P}_{*|X,Y}^{\coprod} : \mathcal{P}(X) \times \mathcal{P}(Y) \xrightarrow{\cong} \mathcal{P}(X \coprod Y),$$
$$\mathcal{P}_{*|\mathbb{1}}^{\coprod} : \mathrm{pt} \xrightarrow{=} \mathcal{P}(\emptyset),$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

0058 5. *Symmetric Lax Monoidality With Respect to Products.* The powerset functor $\mathcal{P}_*$ of Item 1 has a symmetric lax monoidal structure

$$\left(\mathcal{P}_*, \mathcal{P}_*^{\otimes}, \mathcal{P}_{*|\mathbb{1}}^{\otimes}\right) : (\mathsf{Sets}, \times, \mathrm{pt}) \longrightarrow (\mathsf{Sets}, \times, \mathrm{pt})$$

being equipped with isomorphisms

$$\mathcal{P}^{\otimes}_{*|X,Y} \colon \mathcal{P}(X) \times \mathcal{P}(Y) \to \mathcal{P}(X \times Y),$$
$$\mathcal{P}^{\otimes}_{*|\mathbb{1}} \colon \mathrm{pt} \xrightarrow{=} \mathcal{P}(\emptyset),$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$, where $\mathcal{P}^{\otimes}_{*|X,Y}$ is given by

$$\mathcal{P}^{\otimes}_{*|X,Y}(U, V) \stackrel{\text{def}}{=} U \times V$$

for each $(U, V) \in \mathcal{P}(X) \times \mathcal{P}(Y)$.

0059 6. *Powersets as Sets of Functions.* The assignment $U \mapsto \chi_U$ defines a bijection[1]

$$\chi_{(-)} \colon \mathcal{P}(X) \xrightarrow{\cong} \mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\}),$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets})$.

005A 7. *Powersets as Sets of Relations.* We have bijections

$$\mathcal{P}(X) \cong \mathrm{Rel}(\mathrm{pt}, X),$$
$$\mathcal{P}(X) \cong \mathrm{Rel}(X, \mathrm{pt}),$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets})$.

005B 8. *As a Free Cocompletion: Universal Property.* The pair $\big(\mathcal{P}(X), \chi_{(-)}\big)$ consisting of

- The powerset $\mathcal{P}(X)$ of X;
- The characteristic embedding $\chi_{(-)} \colon X \hookrightarrow \mathcal{P}(X)$ of X into $\mathcal{P}(X)$;

satisfies the following universal property:

(★) Given another pair (Y, f) consisting of
  - A cocomplete poset $(Y, \preceq)$;
  - A function $f \colon X \to Y$;

there exists a unique cocontinuous morphism of posets $(\mathcal{P}(X), \subset) \xrightarrow{\exists!} (Y, \preceq)$ making the diagram

$$\begin{tikzcd} & \mathcal{P}(X) \arrow[d, dashed, "\exists!"] \\ X \arrow[ur, "\chi_X"] \arrow[r, "f"'] & Y \end{tikzcd}$$

commute.

005C 9. *As a Free Cocompletion: Adjointness.* We have an adjunction[2]

$$(\chi_{(-)} \dashv 忘): \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "\chi_{(-)}"] \arrow[r, phantom, "\perp"] & \mathsf{Pos}^{\mathsf{cocomp.}} \arrow[l, bend left, "忘"] \end{tikzcd},$$

witnessed by a bijection

$$\mathsf{Pos}^{\mathsf{cocomp.}}((\mathcal{P}(X), \subset), (Y, \preceq)) \cong \mathsf{Sets}(X, Y),$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets})$ and $(Y, \preceq) \in \mathrm{Obj}(\mathsf{Pos})$, where

- We have a natural map

  $$\chi_X^* \colon \mathsf{Pos}^{\mathsf{cocomp.}}((\mathcal{P}(X), \subset), (Y, \preceq)) \to \mathsf{Sets}(X, Y)$$

  defined by

  $$\chi_X^*(f) \stackrel{\text{def}}{=} f \circ \chi_X,$$

  i.e. by sending a cocontinuous morphism of posets $f\colon \mathcal{P}(X) \to Y$ to the composition

  $$X \overset{\chi_X}{\hookrightarrow} \mathcal{P}(X) \xrightarrow{f} Y;$$

- We have a natural map

  $$\mathrm{Lan}_{\chi_X} \colon \mathsf{Sets}(X, Y) \to \mathsf{Pos}^{\mathsf{cocomp.}}((\mathcal{P}(X), \subset), (Y, \preceq))$$

computed by

$$
\begin{aligned}
[\mathrm{Lan}_{\chi_X}(f)](U) &\cong \int^{x\in X} \chi_{\mathcal{P}(X)}(\chi_x, U) \odot f(x) \\
&\cong \int^{x\in X} \chi_U(x) \odot f(x) && \text{(by Proposition 4.1.5)} \\
&\cong \bigvee_{x\in X} (\chi_U(x) \odot f(x))
\end{aligned}
$$

for each $U \in \mathcal{P}(X)$, where:

- $\bigvee$ is the join in $(Y, \preceq)$;
- We have

$$
\begin{aligned}
\mathsf{true} \odot f(x) &\stackrel{\text{def}}{=} f(x), \\
\mathsf{false} \odot f(x) &\stackrel{\text{def}}{=} \varnothing_Y,
\end{aligned}
$$

  where $\varnothing_Y$ is the minimal element of $(Y, \preceq)$.

[1]This bijection is a decategorified form of the equivalence

$$\mathsf{PSh}(C) \stackrel{\text{eq.}}{\cong} \mathsf{DFib}(C)$$

of Fibred Categories, **??** of **??**, with $\chi_{(-)}$ being a decategorified version of the category of elements construction of Fibred Categories, **??**.

See also **??** of **??**.

[2]In this sense, $\mathcal{P}(A)$ is the free cocompletion of A. (Note that, despite its name, however, this is not an idempotent operation, as we have $\mathcal{P}(\mathcal{P}(A)) \neq \mathcal{P}(A)$.)

Proof 4.2.4 ► Proof of Proposition 4.2.3

Item 1: Functoriality

This follows from Items 3 and 4 of Proposition 4.3.5, Items 3 and 4 of Proposition 4.4.5, and Items 3 and 4 of Proposition 4.5.7.

Item 2: Adjointness I

Omitted.

Item 3: Adjointness II

Omitted.

Item 4: Symmetric Strong Monoidality With Respect to Coproducts

Omitted.

Item 5: Symmetric Lax Monoidality With Respect to Products

Omitted.

Item 6: Powersets as Sets of Functions

Omitted.

Item 7: Powersets as Sets of Relations

Omitted.

Item 8: As a Free Cocompletion: Universal Property

This is a rephrasing of **??**.

Item 9: As a Free Cocompletion: Adjointness

Omitted.

005D 4.3 Direct Images

Let A and B be sets and let $f\colon A \longrightarrow B$ be a function.

005E **Definition 4.3.1 ▶ Direct Images**

The **direct image function associated to** f is the function[1]

$$f_*\colon \mathcal{P}(A) \to \mathcal{P}(B)$$

defined by[2,3]

$$\begin{aligned} f_*(U) &\overset{\text{def}}{=} f(U) \\ &\overset{\text{def}}{=} \left\{ b \in B \;\middle|\; \begin{array}{l}\text{there exists some } a \in \\ U \text{ such that } b = f(a)\end{array} \right\} \\ &= \{ f(a) \in B \mid a \in U \} \end{aligned}$$

for each $U \in \mathcal{P}(A)$.

[1] *Further Notation:* Also written $\exists_f\colon \mathcal{P}(A) \to \mathcal{P}(B)$. This notation comes from the fact that the following statements are equivalent, where $b \in B$ and $U \in \mathcal{P}(A)$:

- We have $b \in \exists_f(U)$.

- There exists some $a \in U$ such that $f(a) = b$.

[2] *Further Terminology:* The set $f(U)$ is called the **direct image of** U **by** f.
[3] We also have
$$f_*(U) = B \setminus f_!(A \setminus U);$$
see Item 7 of Proposition 4.3.3.

005F **Remark 4.3.2 ▶ Unwinding Definition 4.3.1**

Identifying subsets of A with functions from A to $\{\text{true}, \text{false}\}$ via Item 6 of Proposition 4.2.3, we see that the direct image function associated to f is equivalently the function
$$f_* \colon \mathcal{P}(A) \to \mathcal{P}(B)$$
defined by
$$\begin{aligned}
f_*(\chi_U) &\overset{\text{def}}{=} \mathrm{Lan}_f(\chi_U) \\
&= \operatorname{colim}\left(\left(f \overset{\rightarrow}{\times} \underline{(-_1)}\right) \overset{\text{pr}}{\twoheadrightarrow} A \xrightarrow{\chi_U} \{\mathsf{t}, \mathsf{f}\}\right) \\
&= \operatorname*{colim}_{\substack{a \in A \\ f(a) = -_1}} (\chi_U(a)) \\
&= \bigvee_{\substack{a \in A \\ f(a) = -_1}} (\chi_U(a)).
\end{aligned}$$
So, in other words, we have
$$\begin{aligned}
[f_*(\chi_U)](b) &= \bigvee_{\substack{a \in A \\ f(a) = b}} (\chi_U(a)) \\
&= \begin{cases} \text{true} & \text{if there exists some } a \in A \text{ such} \\ & \text{that } f(a) = b \text{ and } a \in U, \\ \text{false} & \text{otherwise} \end{cases} \\
&= \begin{cases} \text{true} & \text{if there exists some } a \in U \\ & \text{such that } f(a) = b, \\ \text{false} & \text{otherwise} \end{cases}
\end{aligned}$$
for each $b \in B$.

005G **Proposition 4.3.3 ▶ Properties of Direct Images I**

Let $f\colon A \to B$ be a function.

005H 1. *Functoriality.* The assignment $U \mapsto f_*(U)$ defines a functor

$$f_*\colon (\mathcal{P}(A), \subset) \to (\mathcal{P}(B), \subset)$$

where

- *Action on Objects.* For each $U \in \mathcal{P}(A)$, we have

$$[f_*](U) \overset{\text{def}}{=} f_*(U);$$

- *Action on Morphisms.* For each $U, V \in \mathcal{P}(A)$:

  (★) If $U \subset V$, then $f_*(U) \subset f_*(V)$.

005J 2. *Triple Adjointness.* We have a triple adjunction

$$(f_* \dashv f^{-1} \dashv f_!)\colon \quad \begin{tikzcd} \mathcal{P}(A) \arrow[r, bend left, "f_*"] \arrow[r, bend right, "f_!"'] & \mathcal{P}(B), \arrow[l, "f^{-1}"'] \end{tikzcd}$$

witnessed by bijections of sets

$$\begin{aligned} \mathrm{Hom}_{\mathcal{P}(B)}(f_*(U), V) &\cong \mathrm{Hom}_{\mathcal{P}(A)}\left(U, f^{-1}(V)\right), \\ \mathrm{Hom}_{\mathcal{P}(A)}\left(f^{-1}(U), V\right) &\cong \mathrm{Hom}_{\mathcal{P}(A)}(U, f_!(V)), \end{aligned}$$

natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$ and (respectively) $V \in \mathcal{P}(A)$ and $U \in \mathcal{P}(B)$, i.e. where:

(a) The following conditions are equivalent:

  i. We have $f_*(U) \subset V$.

  ii. We have $U \subset f^{-1}(V)$.

(b) The following conditions are equivalent:

  i. We have $f^{-1}(U) \subset V$.

  ii. We have $U \subset f_!(V)$.

005K 3. *Preservation of Colimits.* We have an equality of sets

$$f_*\left(\bigcup_{i\in I} U_i\right) = \bigcup_{i\in I} f_*(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have equalities

$$\begin{aligned} f_*(U) \cup f_*(V) &= f_*(U \cup V), \\ f_*(\emptyset) &= \emptyset, \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

005L 4. *Oplax Preservation of Limits.* We have an inclusion of sets

$$f_*\left(\bigcap_{i\in I} U_i\right) \subset \bigcap_{i\in I} f_*(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have inclusions

$$\begin{aligned} f_*(U \cap V) &\subset f_*(U) \cap f_*(V), \\ f_*(A) &\subset B, \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

005M 5. *Symmetric Strict Monoidality With Respect to Unions.* The direct image function of Item 1 has a symmetric strict monoidal structure

$$\left(f_*, f_*^{\otimes}, f_{*|\mathbb{1}}^{\otimes}\right) \colon (\mathcal{P}(A), \cup, \emptyset) \longrightarrow (\mathcal{P}(B), \cup, \emptyset),$$

being equipped with equalities

$$\begin{aligned} f_{*|U,V}^{\otimes} \colon f_*(U) \cup f_*(V) &\xrightarrow{=} f_*(U \cup V), \\ f_{*|\mathbb{1}}^{\otimes} \colon \emptyset &\xrightarrow{=} \emptyset, \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

005N 6. *Symmetric Oplax Monoidality With Respect to Intersections.* The direct image function of Item 1 has a symmetric oplax monoidal structure

$$\left(f_*, f_*^{\otimes}, f_{*|\mathbb{1}}^{\otimes}\right) \colon (\mathcal{P}(A), \cap, A) \to (\mathcal{P}(B), \cap, B),$$

being equipped with inclusions

$$f_{*|U,V}^{\otimes} \colon f_*(U \cap V) \hookrightarrow f_*(U) \cap f_*(V),$$
$$f_{*|\mathbb{1}}^{\otimes} \colon f_*(A) \hookrightarrow B,$$

natural in $U, V \in \mathcal{P}(A)$.

005P 7. *Relation to Direct Images With Compact Support.* We have

$$f_*(U) = B \setminus f_!(A \setminus U)$$

for each $U \in \mathcal{P}(A)$.

Proof 4.3.4 ► Proof of Proposition 4.3.3

Item 1: Functoriality

Clear.

Item 2: Triple Adjointness

This follows from Kan Extensions, ?? of ??.

Item 3: Preservation of Colimits

This follows from Item 2 and Categories, ?? of ??.

Item 4: Oplax Preservation of Limits

Omitted.

Item 5: Symmetric Strict Monoidality With Respect to Unions

This follows from Item 3.

Item 6: Symmetric Oplax Monoidality With Respect to Intersections

This follows from ??.

Item 7: Relation to Direct Images With Compact Support

Applying **??** of **??** to $A \setminus U$, we have

$$\begin{aligned} f_!(A \setminus U) &= B \setminus f_*(A \setminus (A \setminus U)) \\ &= B \setminus f_*(U). \end{aligned}$$

Taking complements, we then obtain

$$\begin{aligned} f_*(U) &= B \setminus (B \setminus f_*(U)), \\ &= B \setminus f_!(A \setminus U), \end{aligned}$$

which finishes the proof.

005Q **Proposition 4.3.5 ► Properties of Direct Images II**

Let $f \colon A \to B$ be a function.

005R 1. *Functionality I.* The assignment $f \mapsto f_*$ defines a function

$$(-)_{*|A,B} \colon \mathsf{Sets}(A, B) \to \mathsf{Sets}(\mathcal{P}(A), \mathcal{P}(B)).$$

005S 2. *Functionality II.* The assignment $f \mapsto f_*$ defines a function

$$(-)_{*|A,B} \colon \mathsf{Sets}(A, B) \to \mathsf{Pos}((\mathcal{P}(A), \subset), (\mathcal{P}(B), \subset)).$$

005T 3. *Interaction With Identities.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$(\mathrm{id}_A)_* = \mathrm{id}_{\mathcal{P}(A)};$$

005U 4. *Interaction With Composition.* For each pair of composable functions $f \colon A \to B$ and $g \colon B \to C$, we have

$$(g \circ f)_* = g_* \circ f_*, \qquad \begin{tikzcd} \mathcal{P}(A) \arrow[r, "f_*"] \arrow[dr, "(g\circ f)_*"'] & \mathcal{P}(B) \arrow[d, "g_*"] \\ & \mathcal{P}(C). \end{tikzcd}$$

Proof 4.3.6 ▶ Proof of Proposition 4.3.5

Item 1: Functionality I

Clear.

Item 2: Functionality II

Clear.

Item 3: Interaction With Identities

This follows from Kan Extensions, **??** of **??**.

Item 4: Interaction With Composition

This follows from Kan Extensions, **??** of **??**.

005V 4.4 Inverse Images

Let A and B be sets and let $f\colon A \longrightarrow B$ be a function.

005W **Definition 4.4.1 ▶ Inverse Images**

The **inverse image function associated to** f is the function[1]

$$f^{-1}\colon \mathcal{P}(B) \longrightarrow \mathcal{P}(A)$$

defined by[2]

$$f^{-1}(V) \overset{\text{def}}{=} \{a \in A \mid \text{we have } f(a) \in V\}$$

for each $V \in \mathcal{P}(B)$.

[1] *Further Notation:* Also written $f^{*}\colon \mathcal{P}(B) \to \mathcal{P}(A)$.
[2] *Further Terminology:* The set $f^{-1}(V)$ is called the **inverse image of** V **by** f.

005X **Remark 4.4.2 ▶ Unwinding Definition 4.4.1**

Identifying subsets of B with functions from B to $\{\text{true}, \text{false}\}$ via Item 6 of Proposition 4.2.3, we see that the inverse image function associated to f is equivalently the function

$$f^{*}\colon \mathcal{P}(B) \longrightarrow \mathcal{P}(A)$$

defined by

$$f^{*}(\chi_V) \overset{\text{def}}{=} \chi_V \circ f$$

for each $\chi_V \in \mathcal{P}(B)$, where $\chi_V \circ f$ is the composition

$$A \xrightarrow{f} B \xrightarrow{\chi_V} \{\text{true}, \text{false}\}$$

in Sets.

005Y **Proposition 4.4.3 ▶ Properties of Inverse Images I**

Let $f \colon A \to B$ be a function.

005Z 1. *Functoriality.* The assignment $V \mapsto f^{-1}(V)$ defines a functor

$$f^{-1} \colon (\mathcal{P}(B), \subset) \to (\mathcal{P}(A), \subset)$$

where

- *Action on Objects.* For each $V \in \mathcal{P}(B)$, we have

$$[f^{-1}](V) \overset{\text{def}}{=} f^{-1}(V);$$

- *Action on Morphisms.* For each $U, V \in \mathcal{P}(B)$:
  ($\star$) If $U \subset V$, then $f^{-1}(U) \subset f^{-1}(V)$.

0060 2. *Triple Adjointness.* We have a triple adjunction

$$\left(f_* \dashv f^{-1} \dashv f_!\right)\colon \quad \begin{tikzcd} \mathcal{P}(A) \arrow[r, bend left=45, "f_*"] \arrow[r, bend right=45, "f_!"'] & \mathcal{P}(B), \arrow[l, "f^{-1}"' description] \end{tikzcd}$$

witnessed by bijections of sets

$$\begin{aligned} \text{Hom}_{\mathcal{P}(B)}(f_*(U), V) &\cong \text{Hom}_{\mathcal{P}(A)}\left(U, f^{-1}(V)\right), \\ \text{Hom}_{\mathcal{P}(A)}\left(f^{-1}(U), V\right) &\cong \text{Hom}_{\mathcal{P}(A)}(U, f_!(V)), \end{aligned}$$

natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$ and (respectively) $V \in \mathcal{P}(A)$ and $U \in \mathcal{P}(B)$, i.e. where:

(a) The following conditions are equivalent:

  i. We have $f_*(U) \subset V$;

  ii. We have $U \subset f^{-1}(V)$;

(b) The following conditions are equivalent:

  i. We have $f^{-1}(U) \subset V$.

  ii. We have $U \subset f_!(V)$.

0061 3. *Preservation of Colimits.* We have an equality of sets

$$f^{-1}\left(\bigcup_{i \in I} U_i\right) = \bigcup_{i \in I} f^{-1}(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(B)^{\times I}$. In particular, we have equalities

$$\begin{aligned} f^{-1}(U) \cup f^{-1}(V) &= f^{-1}(U \cup V), \\ f^{-1}(\emptyset) &= \emptyset, \end{aligned}$$

natural in $U, V \in \mathcal{P}(B)$.

0062 4. *Preservation of Limits.* We have an equality of sets

$$f^{-1}\left(\bigcap_{i \in I} U_i\right) = \bigcap_{i \in I} f^{-1}(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(B)^{\times I}$. In particular, we have equalities

$$\begin{aligned} f^{-1}(U) \cap f^{-1}(V) &= f^{-1}(U \cap V), \\ f^{-1}(B) &= A, \end{aligned}$$

natural in $U, V \in \mathcal{P}(B)$.

0063 5. *Symmetric Strict Monoidality With Respect to Unions.* The inverse image function of Item 1 has a symmetric strict monoidal structure

$$\left(f^{-1}, f^{-1,\otimes}, f_{\not\Bbbk}^{-1,\otimes}\right) : (\mathcal{P}(B), \cup, \emptyset) \to (\mathcal{P}(A), \cup, \emptyset),$$

being equipped with equalities

$$f^{-1,\otimes}_{U,V}\colon f^{-1}(U)\cup f^{-1}(V)\xrightarrow{=}f^{-1}(U\cup V),$$
$$f^{-1,\otimes}_{\mathbb{1}}\colon \emptyset\xrightarrow{=}f^{-1}(\emptyset),$$

natural in $U,V\in\mathcal{P}(B)$.

0064 6. *Symmetric Strict Monoidality With Respect to Intersections.* The inverse image function of Item 1 has a symmetric strict monoidal structure

$$\left(f^{-1},f^{-1,\otimes},f^{-1,\otimes}_{\mathbb{1}}\right)\colon (\mathcal{P}(B),\cap,B)\longrightarrow(\mathcal{P}(A),\cap,A),$$

being equipped with equalities

$$f^{-1,\otimes}_{U,V}\colon f^{-1}(U)\cap f^{-1}(V)\xrightarrow{=}f^{-1}(U\cap V),$$
$$f^{-1,\otimes}_{\mathbb{1}}\colon A\xrightarrow{=}f^{-1}(B),$$

natural in $U,V\in\mathcal{P}(B)$.

Proof 4.4.4 ► Proof of Proposition 4.4.3

Item 1: Functoriality

Clear.

Item 2: Triple Adjointness

This follows from Kan Extensions, ?? of ??.

Item 3: Preservation of Colimits

This follows from Item 2 and Categories, ?? of ??.

Item 4: Preservation of Limits

This follows from Item 2 and Categories, ?? of ??.

Item 5: Symmetric Strict Monoidality With Respect to Unions

This follows from Item 3.

Item 6: Symmetric Strict Monoidality With Respect to Intersections

This follows from Item 4.

0065 **Proposition 4.4.5 ► Properties of Inverse Images II**

Let $f\colon A \to B$ be a function.

0066 1. *Functionality I.* The assignment $f \mapsto f^{-1}$ defines a function

$$(-)^{-1}_{A,B}\colon \mathsf{Sets}(A,B) \to \mathsf{Sets}(\mathcal{P}(B),\mathcal{P}(A)).$$

0067 2. *Functionality II.* The assignment $f \mapsto f^{-1}$ defines a function

$$(-)^{-1}_{A,B}\colon \mathsf{Sets}(A,B) \to \mathsf{Pos}((\mathcal{P}(B),\subset),(\mathcal{P}(A),\subset)).$$

0068 3. *Interaction With Identities.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$\mathrm{id}^{-1}_{A} = \mathrm{id}_{\mathcal{P}(A)};$$

0069 4. *Interaction With Composition.* For each pair of composable functions $f\colon A \to B$ and $g\colon B \to C$, we have

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}, \qquad \begin{tikzcd} \mathcal{P}(C) \arrow[r, "g^{-1}"] \arrow[dr, "(g\circ f)^{-1}"'] & \mathcal{P}(B) \arrow[d, "f^{-1}"] \\ & \mathcal{P}(A). \end{tikzcd}$$

Proof 4.4.6 ► Proof of Proposition 4.4.5

Item 1: Functionality I

Clear.

Item 2: Functionality II

Clear.

Item 3: Interaction With Identities

This follows from Categories, **??** of **??**.

Item 4: Interaction With Composition

This follows from Categories, **??** of **??**.

006A 4.5 Direct Images With Compact Support

Let A and B be sets and let $f\colon A \to B$ be a function.

006B **Definition 4.5.1 ▶ Direct Images With Compact Support**

The **direct image with compact support function associated to** f is the function[1]

$$f_!\colon \mathcal{P}(A) \to \mathcal{P}(B)$$

defined by[2,3]

$$\begin{aligned} f_!(U) &\overset{\text{def}}{=} \left\{ b \in B \,\middle|\, \begin{array}{l}\text{for each } a \in A, \text{ if we have}\\ f(a) = b, \text{ then } a \in U\end{array} \right\} \\ &= \left\{ b \in B \,\middle|\, \text{we have } f^{-1}(b) \subset U \right\} \end{aligned}$$

for each $U \in \mathcal{P}(A)$.

[1] *Further Notation:* Also written $\forall_f\colon \mathcal{P}(A) \to \mathcal{P}(B)$. This notation comes from the fact that the following statements are equivalent, where $b \in B$ and $U \in \mathcal{P}(A)$:

- We have $b \in \forall_f(U)$.
- For each $a \in A$, if $b = f(a)$, then $a \in U$.

[2] *Further Terminology:* The set $f_!(U)$ is called the **direct image with compact support of** U **by** f.

[3] We also have

$$f_!(U) = B \setminus f_*(A \setminus U);$$

see Item 7 of Proposition 4.5.5.

006C **Remark 4.5.2 ▶ Unwinding Definition 4.5.1**

Identifying subsets of A with functions from A to $\{\text{true}, \text{false}\}$ via Item 6 of Proposition 4.2.3, we see that the direct image with compact support function associated to f is equivalently the function

$$f_!\colon \mathcal{P}(A) \to \mathcal{P}(B)$$

defined by

$$
\begin{aligned}
f_!(\chi_U) &\overset{\text{def}}{=} \mathrm{Ran}_f(\chi_U)\\
&= \lim\left(\left(\underline{(-_1) \overset{\rightarrow}{\times} f}\right) \overset{\mathrm{pr}}{\twoheadrightarrow} A \xrightarrow{\chi_U} \{\mathrm{true}, \mathrm{false}\}\right)\\
&= \lim_{\substack{a\in A\\ f(a)=-_1}} (\chi_U(a))\\
&= \bigwedge_{\substack{a\in A\\ f(a)=-_1}} (\chi_U(a)).
\end{aligned}
$$

So, in other words, we have

$$
\begin{aligned}
[f_!(\chi_U)](b) &= \bigwedge_{\substack{a\in A\\ f(a)=b}} (\chi_U(a))\\
&= \begin{cases} \mathrm{true} & \text{if, for each } a \in A \text{ such that}\\ & f(a) = b, \text{ we have } a \in U,\\ \mathrm{false} & \text{otherwise} \end{cases}\\
&= \begin{cases} \mathrm{true} & \text{if } f^{-1}(b) \subset U\\ \mathrm{false} & \text{otherwise} \end{cases}
\end{aligned}
$$

for each $b \in B$.

006D **Definition 4.5.3 ► The Image and Complement Parts of $f_!$**

Let U be a subset of A.[1,2]

1. The **image part of the direct image with compact support $f_!(U)$ of U** is the set $f_{!,\mathrm{im}}(U)$ defined by

$$
\begin{aligned}
f_{!,\mathrm{im}}(U) &\overset{\text{def}}{=} f_!(U) \cap \mathrm{Im}(f)\\
&= \left\{ b \in B \;\middle|\; \begin{array}{l}\text{we have } f^{-1}(b) \subset U\\ \text{and } f^{-1}(b) \neq \emptyset\end{array} \right\}.
\end{aligned}
$$

2. The **complement part of the direct image with compact support $f_!(U)$ of**

U is the set $f_{!,\mathrm{cp}}(U)$ defined by

$$\begin{aligned} f_{!,\mathrm{cp}}(U) &\overset{\mathrm{def}}{=} f_!(U) \cap (B \setminus \mathrm{Im}(f)) \\ &= B \setminus \mathrm{Im}(f) \\ &= \left\{ b \in B \,\middle|\, \begin{array}{l}\text{we have } f^{-1}(b) \subset U \\ \text{and } f^{-1}(b) = \emptyset\end{array} \right\} \\ &= \left\{ b \in B \,\middle|\, f^{-1}(b) = \emptyset \right\}. \end{aligned}$$

[1]Note that we have

$$f_!(U) = f_{!,\mathrm{im}}(U) \cup f_{!,\mathrm{cp}}(U),$$

as

$$\begin{aligned} f_!(U) &= f_!(U) \cap B \\ &= f_!(U) \cap (\mathrm{Im}(f) \cup (B \setminus \mathrm{Im}(f))) \\ &= (f_!(U) \cap \mathrm{Im}(f)) \cup (f_!(U) \cap (B \setminus \mathrm{Im}(f))) \\ &\overset{\mathrm{def}}{=} f_{!,\mathrm{im}}(U) \cup f_{!,\mathrm{cp}}(U). \end{aligned}$$

[2]In terms of the meet computation of $f_!(U)$ of Remark 4.5.2, namely

$$f_!(\chi_U) = \bigwedge_{\substack{a \in A \\ f(a) = -_1}} (\chi_U(a)),$$

we see that $f_{!,\mathrm{im}}$ corresponds to meets indexed over nonempty sets, while $f_{!,\mathrm{cp}}$ corresponds to meets indexed over the empty set.

006E **Example 4.5.4 ► Examples of Direct Images With Compact Support**

Here are some examples of direct images with compact support.

1. *The Multiplication by Two Map on the Natural Numbers.* Consider the function $f \colon \mathbb{N} \longrightarrow \mathbb{N}$ given by
$$f(n) \overset{\mathrm{def}}{=} 2n$$
for each $n \in \mathbb{N}$. Since f is injective, we have
$$\begin{aligned} f_{!,\mathrm{im}}(U) &= f_*(U) \\ f_{!,\mathrm{cp}}(U) &= \{\text{odd natural numbers}\} \end{aligned}$$
for any $U \subset \mathbb{N}$.

2. *Parabolas.* Consider the function $f \colon \mathbb{R} \longrightarrow \mathbb{R}$ given by
$$f(x) \overset{\mathrm{def}}{=} x^2$$

for each $x \in \mathbb{R}$. We have

$$f_{!,\mathrm{cp}}(U) = \mathbb{R}_{<0}$$

for any $U \subset \mathbb{R}$. Moreover, since $f^{-1}(x) = \left\{-\sqrt{x}, \sqrt{x}\right\}$, we have e.g.:

$$\begin{aligned} f_{!,\mathrm{im}}([0,1]) &= \{0\}, \\ f_{!,\mathrm{im}}([-1,1]) &= [0,1], \\ f_{!,\mathrm{im}}([1,2]) &= \emptyset, \\ f_{!,\mathrm{im}}([-2,-1] \cup [1,2]) &= [1,4]. \end{aligned}$$

3. *Circles.* Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x,y) \overset{\mathrm{def}}{=} x^2 + y^2$$

for each $(x,y) \in \mathbb{R}^2$. We have

$$f_{!,\mathrm{cp}}(U) = \mathbb{R}_{<0}$$

for any $U \subset \mathbb{R}^2$, and since

$$f^{-1}(r) = \begin{cases} \text{a circle of radius } r \text{ about the origin} & \text{if } r > 0, \\ \{(0,0)\} & \text{if } r = 0, \\ \emptyset & \text{if } r < 0, \end{cases}$$

we have e.g.:

$$\begin{aligned} f_{!,\mathrm{im}}([-1,1] \times [-1,1]) &= [0,1], \\ f_{!,\mathrm{im}}(([-1,1] \times [-1,1]) \setminus [-1,1] \times \{0\}) &= \emptyset. \end{aligned}$$

006F **Proposition 4.5.5 ▶ Properties of Direct Images With Compact Support I**

Let $f\colon A \to B$ be a function.

006G 1. *Functoriality.* The assignment $U \mapsto f_!(U)$ defines a functor

$$f_!\colon (\mathcal{P}(A), \subset) \to (\mathcal{P}(B), \subset)$$

where

- *Action on Objects.* For each $U \in \mathcal{P}(A)$, we have

$$[f_!](U) \overset{\text{def}}{=} f_!(U);$$

- *Action on Morphisms.* For each $U, V \in \mathcal{P}(A)$:

  $(\star)$ If $U \subset V$, then $f_!(U) \subset f_!(V)$.

006H 2. *Triple Adjointness.* We have a triple adjunction

$$\left(f_* \dashv f^{-1} \dashv f_!\right)\colon\quad \begin{tikzcd} \mathcal{P}(A) \arrow[rr, bend left, "f_*"] \arrow[rr, bend right, "f_!"'] \arrow[rr, phantom, shift left=3, "\perp"] \arrow[rr, phantom, shift right=3, "\perp"] & & \mathcal{P}(B), \arrow[ll, "f^{-1}"'] \end{tikzcd}$$

witnessed by bijections of sets

$$\begin{aligned} \mathrm{Hom}_{\mathcal{P}(B)}(f_*(U), V) &\cong \mathrm{Hom}_{\mathcal{P}(A)}\left(U, f^{-1}(V)\right), \\ \mathrm{Hom}_{\mathcal{P}(A)}\left(f^{-1}(U), V\right) &\cong \mathrm{Hom}_{\mathcal{P}(A)}(U, f_!(V)), \end{aligned}$$

natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$ and (respectively) $V \in \mathcal{P}(A)$ and $U \in \mathcal{P}(B)$, i.e. where:

(a) The following conditions are equivalent:

   i. We have $f_*(U) \subset V$;

   ii. We have $U \subset f^{-1}(V)$;

(b) The following conditions are equivalent:

   i. We have $f^{-1}(U) \subset V$.

   ii. We have $U \subset f_!(V)$.

006J 3. *Lax Preservation of Colimits.* We have an inclusion of sets

$$\bigcup_{i \in I} f_!(U_i) \subset f_!\left(\bigcup_{i \in I} U_i\right),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have inclusions

$$f_!(U) \cup f_!(V) \hookrightarrow f_!(U \cup V),$$
$$\emptyset \hookrightarrow f_!(\emptyset),$$

natural in $U, V \in \mathcal{P}(A)$.

006K 4. *Preservation of Limits.* We have an equality of sets

$$f_!\left(\bigcap_{i\in I} U_i\right) = \bigcap_{i\in I} f_!(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have equalities

$$f^{-1}(U \cap V) = f_!(U) \cap f^{-1}(V),$$
$$f_!(A) = B,$$

natural in $U, V \in \mathcal{P}(A)$.

006L 5. *Symmetric Lax Monoidality With Respect to Unions.* The direct image with compact support function of Item 1 has a symmetric lax monoidal structure

$$\left(f_!, f_!^{\otimes}, f_{!|\mathbb{1}}^{\otimes}\right) \colon (\mathcal{P}(A), \cup, \emptyset) \longrightarrow (\mathcal{P}(B), \cup, \emptyset),$$

being equipped with inclusions

$$f_{!|U,V}^{\otimes} \colon f_!(U) \cup f_!(V) \hookrightarrow f_!(U \cup V),$$
$$f_{!|\mathbb{1}}^{\otimes} \colon \emptyset \hookrightarrow f_!(\emptyset),$$

natural in $U, V \in \mathcal{P}(A)$.

006M 6. *Symmetric Strict Monoidality With Respect to Intersections.* The direct image function of Item 1 has a symmetric strict monoidal structure

$$\left(f_!, f_!^{\otimes}, f_{!|\mathbb{1}}^{\otimes}\right) \colon (\mathcal{P}(A), \cap, A) \longrightarrow (\mathcal{P}(B), \cap, B),$$

being equipped with equalities

$$f_{!|U,V}^{\otimes} \colon f_!(U \cap V) \xrightarrow{=} f_!(U) \cap f_!(V),$$
$$f_{!|\mathbb{1}}^{\otimes} \colon f_!(A) \xrightarrow{=} B,$$

natural in $U, V \in \mathcal{P}(A)$.

006N 7. *Relation to Direct Images.* We have

$$f_!(U) = B \setminus f_*(A \setminus U)$$

for each $U \in \mathcal{P}(A)$.

006P 8. *Interaction With Injections.* If f is injective, then we have

$$\begin{aligned} f_{!,\mathrm{im}}(U) &= f_*(U), \\ f_{!,\mathrm{cp}}(U) &= B \setminus \mathrm{Im}(f), \\ f_!(U) &= f_{!,\mathrm{im}}(U) \cup f_{!,\mathrm{cp}}(U) \\ &= f_*(U) \cup (B \setminus \mathrm{Im}(f)) \end{aligned}$$

for each $U \in \mathcal{P}(A)$.

006Q 9. *Interaction With Surjections.* If f is surjective, then we have

$$\begin{aligned} f_{!,\mathrm{im}}(U) &\subset f_*(U), \\ f_{!,\mathrm{cp}}(U) &= \emptyset, \\ f_!(U) &\subset f_*(U) \end{aligned}$$

for each $U \in \mathcal{P}(A)$.

Proof 4.5.6 ► Proof of Proposition 4.5.5

Item 1: Functoriality

Clear.

Item 2: Triple Adjointness

This follows from Kan Extensions, **??** of **??**.

Item 3: Lax Preservation of Colimits

Omitted.

Item 4: Preservation of Limits

Omitted. This follows from Item 2 and Categories, **??** of **??**.

Item 5: Symmetric Lax Monoidality With Respect to Unions

This follows from **??**.

Item 6: Symmetric Strict Monoidality With Respect to Intersections

This follows from Item 4.

Item 7: Relation to Direct Images

We claim that $f_!(U) = B \setminus f_*(A \setminus U)$.

- *The First Implication.* We claim that

  $$f_!(U) \subset B \setminus f_*(A \setminus U).$$

  Let $b \in f_!(U)$. We need to show that $b \notin f_*(A \setminus U)$, i.e. that there is no $a \in A \setminus U$ such that $f(a) = b$.

  This is indeed the case, as otherwise we would have $a \in f^{-1}(b)$ and $a \notin U$, contradicting $f^{-1}(b) \subset U$ (which holds since $b \in f_!(U)$).

  Thus $b \in B \setminus f_*(A \setminus U)$.

- *The Second Implication.* We claim that

  $$B \setminus f_*(A \setminus U) \subset f_!(U).$$

  Let $b \in B \setminus f_*(A \setminus U)$. We need to show that $b \in f_!(U)$, i.e. that $f^{-1}(b) \subset U$.

  Since $b \notin f_*(A \setminus U)$, there exists no $a \in A \setminus U$ such that $b = f(a)$, and hence $f^{-1}(b) \subset U$.

  Thus $b \in f_!(U)$.

This finishes the proof of Item 7.

Item 8: Interaction With Injections

Clear.

Item 9: Interaction With Surjections

Clear.

006R **Proposition 4.5.7 ► Properties of Direct Images With Compact Support II**

Let $f\colon A \to B$ be a function.

006S 1. *Functionality I.* The assignment $f \mapsto f_!$ defines a function

$$(-)_{!|A,B}\colon \mathsf{Sets}(A,B) \to \mathsf{Sets}(\mathcal{P}(A),\mathcal{P}(B)).$$

006T 2. *Functionality II.* The assignment $f \mapsto f_!$ defines a function

$$(-)_{!|A,B}\colon \mathsf{Sets}(A,B) \to \mathsf{Pos}((\mathcal{P}(A),\subset),(\mathcal{P}(B),\subset)).$$

006U 3. *Interaction With Identities.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$(\mathrm{id}_A)_! = \mathrm{id}_{\mathcal{P}(A)};$$

006V 4. *Interaction With Composition.* For each pair of composable functions $f\colon A \to B$ and $g\colon B \to C$, we have

$$(g \circ f)_! = g_! \circ f_!, \qquad \begin{tikzcd} \mathcal{P}(A) \arrow[r, "f_!"] \arrow[dr, "(g\circ f)_!"'] & \mathcal{P}(B) \arrow[d, "g_!"] \\ & \mathcal{P}(C). \end{tikzcd}$$

Proof 4.5.8 ► Proof of Proposition 4.5.7

Item 1: Functionality I

Clear.

Item 2: Functionality II

Clear.

Item 3: Interaction With Identities

This follows from Kan Extensions, ?? of ??.

Item 4: Interaction With Composition

This follows from Kan Extensions, ?? of ??.

Appendices

A Other Chapters

Probability Theory

Stochastic Analysis

Differential Geometry

Schemes