

Constructions With Sets

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000D This chapter contains some material relating to constructions with sets. Notably, it contains:

1. Explicit descriptions of the major types of co/limits in **Sets**, including in particular pushouts and coequalisers (see **Definitions 2.3.1.1** and **2.4.1.1** and **Remarks 2.3.1.2** and **2.4.1.2**);
2. A discussion of powersets as decategorifications of categories of presheaves (**Remarks 4.1.1.2** and **4.2.1.2**);
3. A lengthy discussion of the adjoint triple

$$f_* \dashv f^{-1} \dashv f_! : \mathcal{P}(A) \xrightarrow{\cong} \mathcal{P}(B)$$

of functors (morphisms of posets) between $\mathcal{P}(A)$ and $\mathcal{P}(B)$ induced by a map of sets $f : A \rightarrow B$, along with a discussion of the properties of f_* , f^{-1} , and $f_!$.

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000E 1 Limits of Sets

000F 1.1 Products of Families of Sets

Let $\{A_i\}_{i \in I}$ be a family of sets.

000G **Definition 1.1.1.1.** The **product**¹ of $\{A_i\}_{i \in I}$ is the set $\prod_{i \in I} A_i$ defined by

$$\prod_{i \in I} A_i \stackrel{\text{def}}{=} \left\{ f \in \text{Sets} \left(I, \bigcup_{i \in I} A_i \right) \mid \begin{array}{l} \text{for each } i \in I, \text{ we} \\ \text{have } f(i) \in A_i \end{array} \right\}.$$

000H 1.2 Binary Products of Sets

Let A and B be sets.

¹*Further Terminology:* Also called the **Cartesian product** of $\{A_i\}_{i \in I}$.

000J Definition 1.2.1.1. The **product**² of A and B is the set $A \times B$ defined by

$$\begin{aligned} A \times B &\stackrel{\text{def}}{=} \prod_{z \in \{A, B\}} z \\ &\stackrel{\text{def}}{=} \{f \in \mathbf{Sets}(\{0, 1\}, A \cup B) \mid \text{we have } f(0) \in A \text{ and } f(1) \in B\} \\ &\cong \{\{\{a\}, \{a, b\}\} \in \mathcal{P}(\mathcal{P}(A \cup B)) \mid \text{we have } a \in A \text{ and } b \in B\}. \end{aligned}$$

000K Proposition 1.2.1.2. Let A , B , C , and X be sets.

000L 1. *Functoriality.* The assignments $A, B, (A, B) \mapsto A \times B$ define functors

$$\begin{aligned} A \times -_2 &: \mathbf{Sets} \rightarrow \mathbf{Sets}, \\ -_1 \times B &: \mathbf{Sets} \rightarrow \mathbf{Sets}, \\ -_1 \times -_2 &: \mathbf{Sets} \times \mathbf{Sets} \rightarrow \mathbf{Sets}, \end{aligned}$$

where $-_1 \times -_2$ is the functor where

- *Action on Objects.* For each $(A, B) \in \text{Obj}(\mathbf{Sets} \times \mathbf{Sets})$, we have

$$[-_1 \times -_2](A, B) \stackrel{\text{def}}{=} A \times B;$$

- *Action on Morphisms.* For each $(A, B), (X, Y) \in \text{Obj}(\mathbf{Sets})$, the action on Hom-sets

$$\times_{(A, B), (X, Y)}: \mathbf{Sets}(A, X) \times \mathbf{Sets}(B, Y) \rightarrow \mathbf{Sets}(A \times B, X \times Y)$$

of \times at $((A, B), (X, Y))$ is defined by sending (f, g) to the function

$$f \times g: A \times B \rightarrow X \times Y$$

defined by

$$[f \times g](a, b) \stackrel{\text{def}}{=} (f(a), g(b))$$

for each $(a, b) \in A \times B$;

and where $A \times -$ and $- \times B$ are the partial functors of $-_1 \times -_2$ at $A, B \in \text{Obj}(\mathbf{Sets})$.

²*Further Terminology:* Also called the **Cartesian product of A and B** or the **binary Cartesian product of A and B** , for emphasis.

This can also be thought of as the $(\mathbb{E}_{-1}, \mathbb{E}_{-1})$ -**tensor product of A and B** .

000M 2. *Adjointness.* We have adjunctions

$$(A \times - \dashv \text{Sets}(A, -)): \text{Sets} \begin{array}{c} \xrightarrow{A \times -} \\ \perp \\ \xleftarrow{\text{Sets}(A, -)} \end{array} \text{Sets},$$

$$(- \times B \dashv \text{Sets}(B, -)): \text{Sets} \begin{array}{c} \xrightarrow{- \times B} \\ \perp \\ \xleftarrow{\text{Sets}(B, -)} \end{array} \text{Sets},$$

witnessed by bijections

$$\begin{aligned} \text{Sets}(A \times B, C) &\cong \text{Sets}(A, \text{Sets}(B, C)), \\ \text{Sets}(A \times B, C) &\cong \text{Sets}(B, \text{Sets}(A, C)), \end{aligned}$$

natural in $A, B, C \in \text{Obj}(\text{Sets})$.

000N 3. *Associativity.* We have an isomorphism of sets

$$(A \times B) \times C \cong A \times (B \times C),$$

natural in $A, B, C \in \text{Obj}(\text{Sets})$.

000P 4. *Unitality.* We have isomorphisms of sets

$$\begin{aligned} \text{pt} \times A &\cong A, \\ A \times \text{pt} &\cong A, \end{aligned}$$

natural in $A \in \text{Obj}(\text{Sets})$.

000Q 5. *Commutativity.* We have an isomorphism of sets

$$A \times B \cong B \times A,$$

natural in $A, B \in \text{Obj}(\text{Sets})$.

000R 6. *Annihilation With the Empty Set.* We have isomorphisms of sets

$$\begin{aligned} A \times \emptyset &\cong \emptyset, \\ \emptyset \times A &\cong \emptyset, \end{aligned}$$

natural in $A \in \text{Obj}(\text{Sets})$.

000S 7. *Distributivity Over Unions.* We have isomorphisms of sets

$$\begin{aligned} A \times (B \cup C) &= (A \times B) \cup (A \times C), \\ (A \cup B) \times C &= (A \times C) \cup (B \times C), \end{aligned}$$

natural in $A, B, C \in \text{Obj}(\text{Sets})$.

000T 8. *Distributivity Over Intersections.* We have isomorphisms of sets

$$\begin{aligned} A \times (B \cap C) &= (A \times B) \cap (A \times C), \\ (A \cap B) \times C &= (A \times C) \cap (B \times C), \end{aligned}$$

natural in $A, B, C \in \text{Obj}(\text{Sets})$.

000U 9. *Distributivity Over Differences.* We have isomorphisms of sets

$$\begin{aligned} A \times (B \setminus C) &= (A \times B) \setminus (A \times C), \\ (A \setminus B) \times C &= (A \times C) \setminus (B \times C), \end{aligned}$$

natural in $A, B, C \in \text{Obj}(\text{Sets})$.

000V 10. *Distributivity Over Symmetric Differences.* We have isomorphisms of sets

$$\begin{aligned} A \times (B \triangle C) &= (A \times B) \triangle (A \times C), \\ (A \triangle B) \times C &= (A \times C) \triangle (B \times C), \end{aligned}$$

natural in $A, B, C \in \text{Obj}(\text{Sets})$.

000W 11. *Symmetric Monoidality.* The triple $(\text{Sets}, \times, \text{pt})$ is a symmetric monoidal category.

000X 12. *Symmetric Bimonoidality.* The quintuple $(\text{Sets}, \coprod, \emptyset, \times, \text{pt})$ is a symmetric bimonoidal category.

Proof. **Item 1, Functoriality:** Omitted.

Item 2, Adjointness: Omitted.

Item 3, Associativity: Clear.

Item 4, Unitality: Clear.

Item 5, Commutativity: Clear.

Item 6, Annihilation With the Empty Set: Clear.

Item 7, Distributivity Over Unions: Omitted.

Item 8, Distributivity Over Intersections: Omitted.

Item 9, Distributivity Over Differences: Omitted.

Item 10, Distributivity Over Symmetric Differences: Omitted.

Item 11, Symmetric Monoidality: Omitted.

Item 12, Symmetric Bimonoidality: Omitted. □

000Y 1.3 Pullbacks

Let A , B , and C be sets and let $f: A \rightarrow C$ and $g: B \rightarrow C$ be functions.

000Z Definition 1.3.1.1. The **pullback of A and B over C along f and g** ³ is the set $A \times_C B$ defined by

$$A \times_C B \stackrel{\text{def}}{=} \{(a, b) \in A \times B \mid f(a) = g(b)\}.$$

0010 Example 1.3.1.2. Here are some examples of pullbacks of sets.

0011 1. *Unions via Intersections.* Let $A, B \subset X$. We have a bijection of sets

$$A \cap B \cong A \times_{A \cup B} B.$$

0012 Proposition 1.3.1.3. Let A , B , C , and X be sets.

0013 1. *Associativity.* We have an isomorphism of sets

$$(A \times_X B) \times_X C \cong A \times_X (B \times_X C),$$

natural in $A, B, C, X \in \text{Obj}(\text{Sets})$.

0014 2. *Unitality.* We have isomorphisms of sets

$$X \times_X A \cong A,$$

$$A \times_X X \cong A,$$

natural in $A, X \in \text{Obj}(\text{Sets})$.

0015 3. *Commutativity.* We have an isomorphism of sets

$$A \times_X B \cong B \times_X A,$$

natural in $A, B, X \in \text{Obj}(\text{Sets})$.

0016 4. *Annihilation With the Empty Set.* We have isomorphisms of sets

$$A \times_X \emptyset \cong \emptyset,$$

$$\emptyset \times_X A \cong \emptyset,$$

natural in $A, X \in \text{Obj}(\text{Sets})$.

³*Further Terminology:* Also called the **fibre product of A and B over C along f**

- 0017 5. *Symmetric Monoidality.* The triple $(\mathbf{Sets}, \times_X, X)$ is a symmetric monoidal category.

Proof. Item 1, Associativity: Clear.

Item 2, Unitality: Clear.

Item 3, Commutativity: Clear.

Item 4, Annihilation With the Empty Set: Clear.

Item 5, Symmetric Monoidality: Omitted. \square

0018 1.4 Equalisers

Let A and B be sets and let $f, g: A \rightrightarrows B$ be functions.

- 0019 **Definition 1.4.1.1.** The **equaliser of f and g** is the set $\text{Eq}(f, g)$ defined by

$$\text{Eq}(f, g) \stackrel{\text{def}}{=} \{a \in A \mid f(a) = g(a)\}.$$

- 001A **Proposition 1.4.1.2.** Let A , B , and C be sets.

- 001B 1. *Associativity.* We have an isomorphism of sets⁴

$$\underbrace{\text{Eq}(f \circ \text{eq}(g, h), g \circ \text{eq}(g, h))}_{=\text{Eq}(f \circ \text{eq}(g, h), h \circ \text{eq}(g, h))} \cong \text{Eq}(f, g, h) \cong \underbrace{\text{Eq}(f \circ \text{eq}(f, g), h \circ \text{eq}(f, g))}_{=\text{Eq}(g \circ \text{eq}(f, g), h \circ \text{eq}(f, g))},$$

and g .

⁴That is: the following constructions give the same result:

1. Take the equaliser of (f, g, h) , i.e. the limit of the diagram

$$A \begin{array}{c} \xrightarrow{f} \\ \xrightarrow[-g]{} \\ \xrightarrow{h} \end{array} B$$

in \mathbf{Sets} .

2. First take the equaliser of f and g , forming a diagram

$$\text{Eq}(f, g) \xrightarrow{\text{eq}(f, g)} A \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} B$$

and then take the equaliser of the composition

$$\text{Eq}(f, g) \xrightarrow{\text{eq}(f, g)} A \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{h} \end{array} B,$$

obtaining a subset

$$\text{Eq}(f \circ \text{eq}(f, g), h \circ \text{eq}(f, g)) = \text{Eq}(g \circ \text{eq}(f, g), h \circ \text{eq}(f, g))$$

of $\text{Eq}(f, g)$.

where $\text{Eq}(f, g, h)$ is the limit of the diagram

$$A \begin{array}{c} \xrightarrow{f} \\ \xrightarrow[-g]{} \\ \xrightarrow{h} \end{array} B$$

in **Sets**.

- 001C** 4. *Unitality*. We have an isomorphism of sets

$$\text{Eq}(f, f) \cong A.$$

- 001D** 5. *Commutativity*. We have an isomorphism of sets

$$\text{Eq}(f, g) \cong \text{Eq}(g, f).$$

- 001E** 6. *Interaction With Composition*. Let

$$A \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} B \begin{array}{c} \xrightarrow{h} \\ \xrightarrow{k} \end{array} C$$

be functions. We have an inclusion of sets

$$\text{Eq}(h \circ f \circ \text{eq}(f, g), k \circ g \circ \text{eq}(f, g)) \subset \text{Eq}(h \circ f, k \circ g),$$

where $\text{Eq}(h \circ f \circ \text{eq}(f, g), k \circ g \circ \text{eq}(f, g))$ is the equaliser of the composition

$$\text{Eq}(f, g) \xrightarrow{\text{eq}(f, g)} A \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} B \begin{array}{c} \xrightarrow{h} \\ \xrightarrow{k} \end{array} C.$$

Proof. Item 1, Associativity: Clear.

Item 4, Unitality: Clear.

Item 5, Commutativity: Clear.

Item 6, Interaction With Composition: Omitted. □

-
3. First take the equaliser of g and h , forming a diagram

$$\text{Eq}(g, h) \xrightarrow{\text{eq}(g, h)} A \begin{array}{c} \xrightarrow{g} \\ \xrightarrow{h} \end{array} B$$

and then take the equaliser of the composition

$$\text{Eq}(g, h) \xrightarrow{\text{eq}(g, h)} A \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} B,$$

001F 2 Colimits of Sets

001G 2.1 Coproducts of Families of Sets

Let $\{A_i\}_{i \in I}$ be a family of sets.

001H **Definition 2.1.1.1.** The **disjoint union of the family** $\{A_i\}_{i \in I}$ is the set $\coprod_{i \in I} A_i$ defined by

$$\coprod_{i \in I} A_i \stackrel{\text{def}}{=} \bigcup_{i \in I} \left\{ (x, i) \in \left(\bigcup_{i \in I} A_i \right) \times I \mid x \in A_i \right\}.$$

001J 2.2 Binary Coproducts

Let A and B be sets.

001K **Definition 2.2.1.1.** The **coproduct**⁵ of A and B is the set $A \coprod B$ defined by

$$\begin{aligned} A \coprod B &\stackrel{\text{def}}{=} \coprod_{z \in \{A, B\}} z \\ &\stackrel{\text{def}}{=} \{(a, 0) \mid a \in A\} \cup \{(b, 1) \mid b \in B\}. \end{aligned}$$

001L **Proposition 2.2.1.2.** Let A , B , C , and X be sets.

001M 1. *Functoriality.* The assignment $A, B, (A, B) \mapsto A \coprod B$ defines functors

$$\begin{aligned} A \coprod -_2 &: \mathbf{Sets} \rightarrow \mathbf{Sets}, \\ -_1 \coprod B &: \mathbf{Sets} \rightarrow \mathbf{Sets}, \\ -_1 \coprod -_2 &: \mathbf{Sets} \times \mathbf{Sets} \rightarrow \mathbf{Sets}, \end{aligned}$$

where $-_1 \coprod -_2$ is the functor where

- *Action on Objects.* For each $(A, B) \in \mathbf{Obj}(\mathbf{Sets} \times \mathbf{Sets})$, we have

$$[-_1 \coprod -_2](A, B) \stackrel{\text{def}}{=} A \coprod B;$$

obtaining a subset

$$\text{Eq}(f \circ \text{eq}(g, h), g \circ \text{eq}(g, h)) = \text{Eq}(f \circ \text{eq}(g, h), h \circ \text{eq}(g, h))$$

of $\text{Eq}(g, h)$.

⁵*Further Terminology:* Also called the **disjoint union of A and B** , or the **binary**

- *Action on Morphisms.* For each $(A, B), (X, Y) \in \text{Obj}(\text{Sets})$, the action on Hom-sets

$$\coprod_{(A,B),(X,Y)} : \text{Sets}(A, X) \times \text{Sets}(B, Y) \rightarrow \text{Sets}(A \coprod B, X \coprod Y)$$

of \coprod at $((A, B), (X, Y))$ is defined by sending (f, g) to the function

$$f \coprod g : A \coprod B \rightarrow X \coprod Y$$

defined by

$$[f \coprod g](x) \stackrel{\text{def}}{=} \begin{cases} f(x) & \text{if } x \in A, \\ g(x) & \text{if } x \in B \end{cases}$$

for each $x \in A \coprod B$;

and where $A \coprod -$ and $- \coprod B$ are the partial functors of $-_1 \coprod -_2$ at $A, B \in \text{Obj}(\text{Sets})$.

- 001N** 2. *Associativity.* We have an isomorphism of sets

$$(A \coprod B) \coprod C \cong A \coprod (B \coprod C),$$

natural in $A, B, C \in \text{Obj}(\text{Sets})$.

- 001P** 3. *Unitality.* We have isomorphisms of sets

$$\begin{aligned} A \coprod \emptyset &\cong A, \\ \emptyset \coprod A &\cong A, \end{aligned}$$

natural in $A \in \text{Obj}(\text{Sets})$.

- 001Q** 4. *Commutativity.* We have an isomorphism of sets

$$A \coprod B \cong B \coprod A,$$

natural in $A, B \in \text{Obj}(\text{Sets})$.

- 001R** 5. *Symmetric Monoidality.* The triple $(\text{Sets}, \coprod, \emptyset)$ is a symmetric monoidal category.

Proof. **Item 1, Functoriality:** Omitted.

Item 2, Associativity: Clear.

Item 3, Unitality: Clear.

Item 4, Commutativity: Clear.

Item 5, Symmetric Monoidality: Omitted. □

001S 2.3 Pushouts

Let A , B , and C be sets and let $f: C \rightarrow A$ and $g: C \rightarrow B$ be functions.

001T Definition 2.3.1.1. The **pushout of A and B over C along f and g** ⁶ is the set $A \amalg_C B$ defined by

$$A \amalg_C B \stackrel{\text{def}}{=} A \amalg B / \sim_C,$$

where \sim_C is the equivalence relation on $A \amalg B$ generated by $f(c) \sim_C g(c)$.

001U Remark 2.3.1.2. In detail, the relation \sim of **Definition 2.3.1.1** is given by declaring $a \sim b$ iff one of the following conditions is satisfied:

- We have $a, b \in A$ and $a = b$;
- We have $a, b \in B$ and $a = b$;
- There exist $x_1, \dots, x_n \in A \amalg B$ such that $a \sim' x_1 \sim' \dots \sim' x_n \sim' b$, where we declare $x \sim' y$ if one of the following conditions is satisfied:
 1. There exists $c \in C$ such that $x = f(c)$ and $y = g(c)$.
 2. There exists $c \in C$ such that $x = g(c)$ and $y = f(c)$.

That is: we require the following condition to be satisfied:

- (★) There exist $x_1, \dots, x_n \in A \amalg B$ satisfying the following conditions:
 1. There exists $c_0 \in C$ satisfying one of the following conditions:
 - (a) We have $a = f(c_0)$ and $x_1 = g(c_0)$.
 - (b) We have $a = g(c_0)$ and $x_1 = f(c_0)$.
 2. For each $1 \leq i \leq n-1$, there exists $c_i \in C$ satisfying one of the following conditions:
 - (a) We have $x_i = f(c_i)$ and $x_{i+1} = g(c_i)$.
 - (b) We have $x_i = g(c_i)$ and $x_{i+1} = f(c_i)$.
 3. There exists $c_n \in C$ satisfying one of the following conditions:
 - (a) We have $x_n = f(c_n)$ and $b = g(c_n)$.
 - (b) We have $x_n = g(c_n)$ and $b = f(c_n)$.

001V Example 2.3.1.3. Here are some examples of pushouts of sets.

disjoint union of A and B , for emphasis.

⁶ *Further Terminology:* Also called the **fibre coproduct of A and B over C along f and g** .

001W 1. *Wedge Sums of Pointed Sets.* The wedge sum of two pointed sets of ?? is an example of a pushout of sets.

001X 2. *Intersections via Unions.* Let $A, B \subset X$. We have a bijection of sets

$$A \cup B \cong A \coprod_{A \cap B} B.$$

001Y **Proposition 2.3.1.4.** Let A, B, C , and X be sets.

001Z 1. *Associativity.* We have an isomorphism of sets

$$(A \coprod_X B) \coprod_X C \cong A \coprod_X (B \coprod_X C),$$

natural in $A, B, C, X \in \text{Obj}(\text{Sets})$.

0020 2. *Unitality.* We have isomorphisms of sets

$$\begin{aligned} \emptyset \coprod_X A &\cong A, \\ A \coprod_X \emptyset &\cong A, \end{aligned}$$

natural in $A, X \in \text{Obj}(\text{Sets})$.

0021 3. *Commutativity.* We have an isomorphism of sets

$$A \coprod_X B \cong B \coprod_X A,$$

natural in $A, B, X \in \text{Obj}(\text{Sets})$.

0022 4. *Annihilation With the Empty Set.* We have isomorphisms of sets

$$\begin{aligned} A \coprod_X \emptyset &\cong \emptyset, \\ \emptyset \coprod_X A &\cong \emptyset, \end{aligned}$$

natural in $A, X \in \text{Obj}(\text{Sets})$.

0023 5. *Symmetric Monoidality.* The triple $(\text{Sets}, \coprod_X, \emptyset)$ is a symmetric monoidal category.

Proof. **Item 1, Associativity:** Clear.

Item 2, Unitality: Clear.

Item 3, Commutativity: Clear.

Item 4, Annihilation With the Empty Set: Clear.

Item 5, Symmetric Monoidality: Omitted. □

0024 2.4 Coequalisers

Let A and B be sets and let $f, g: A \rightrightarrows B$ be functions.

0025 **Definition 2.4.1.1.** The **coequaliser of f and g** is the set $\text{CoEq}(f, g)$ defined by

$$\text{CoEq}(f, g) \stackrel{\text{def}}{=} B / \sim,$$

where \sim is the equivalence relation on B generated by $f(a) \sim g(a)$.

0026 **Remark 2.4.1.2.** In detail, the relation \sim of **Definition 2.4.1.1** is given by declaring $a \sim b$ iff one of the following conditions is satisfied:

- We have $a = b$;
- There exist $x_1, \dots, x_n \in B$ such that $a \sim' x_1 \sim' \dots \sim' x_n \sim' b$, where we declare $x \sim' y$ if one of the following conditions is satisfied:
 1. There exists $z \in A$ such that $x = f(z)$ and $y = g(z)$.
 2. There exists $z \in A$ such that $x = g(z)$ and $y = f(z)$.

That is: we require the following condition to be satisfied:

- (\star) There exist $x_1, \dots, x_n \in B$ satisfying the following conditions:
 1. There exists $z_0 \in A$ satisfying one of the following conditions:
 - (a) We have $a = f(z_0)$ and $x_1 = g(z_0)$.
 - (b) We have $a = g(z_0)$ and $x_1 = f(z_0)$.
 2. For each $1 \leq i \leq n-1$, there exists $z_i \in A$ satisfying one of the following conditions:
 - (a) We have $x_i = f(z_i)$ and $x_{i+1} = g(z_i)$.
 - (b) We have $x_i = g(z_i)$ and $x_{i+1} = f(z_i)$.
 3. There exists $z_n \in A$ satisfying one of the following conditions:
 - (a) We have $x_n = f(z_n)$ and $b = g(z_n)$.
 - (b) We have $x_n = g(z_n)$ and $b = f(z_n)$.

0027 **Example 2.4.1.3.** Here are some examples of coequalisers of sets.

0028 1. *Quotients by Equivalence Relations.* Let R be an equivalence relation on a set X . We have a bijection of sets

$$X / \sim_R \cong \text{CoEq} \left(R \hookrightarrow X \times X \begin{array}{c} \xrightarrow{\text{pr}_1} \\ \xrightarrow{\text{pr}_2} \end{array} X \right).$$

0029 Proposition 2.4.1.4. Let A , B , and C be sets.

002A 1. *Associativity.* We have an isomorphism of sets⁷

$$\underbrace{\text{CoEq}(\text{coeq}(f, g) \circ f, \text{coeq}(f, g) \circ h)}_{=\text{CoEq}(\text{coeq}(f, g) \circ g, \text{coeq}(f, g) \circ h)} \cong \text{CoEq}(f, g, h) \cong \underbrace{\text{CoEq}(\text{coeq}(g, h) \circ f, \text{coeq}(g, h) \circ g)}_{=\text{CoEq}(\text{coeq}(g, h) \circ f, \text{coeq}(g, h) \circ h)},$$

where $\text{CoEq}(f, g, h)$ is the colimit of the diagram

$$A \begin{array}{c} \xrightarrow{f} \\ \xrightarrow[-g]{\rightarrow} \\ \xrightarrow{h} \end{array} B$$

in **Sets**.

⁷That is: the following constructions give the same result:

1. Take the coequaliser of (f, g, h) , i.e. the colimit of the diagram

$$A \begin{array}{c} \xrightarrow{f} \\ \xrightarrow[-g]{\rightarrow} \\ \xrightarrow{h} \end{array} B$$

in **Sets**.

2. First take the coequaliser of f and g , forming a diagram

$$A \begin{array}{c} \xrightarrow{f} \\ \xrightarrow[-g]{\rightarrow} \end{array} B \xrightarrow{\text{coeq}(f, g)} \text{CoEq}(f, g)$$

and then take the coequaliser of the composition

$$A \begin{array}{c} \xrightarrow{f} \\ \xrightarrow[-h]{\rightarrow} \end{array} B \xrightarrow{\text{coeq}(f, g)} \text{CoEq}(f, g),$$

obtaining a quotient

$$\text{CoEq}(\text{coeq}(f, g) \circ f, \text{coeq}(f, g) \circ h) = \text{CoEq}(\text{coeq}(f, g) \circ g, \text{coeq}(f, g) \circ h)$$

of $\text{CoEq}(f, g)$

3. First take the coequaliser of g and h , forming a diagram

$$A \begin{array}{c} \xrightarrow{g} \\ \xrightarrow[-h]{\rightarrow} \end{array} B \xrightarrow{\text{coeq}(g, h)} \text{CoEq}(g, h)$$

and then take the coequaliser of the composition

$$A \begin{array}{c} \xrightarrow{f} \\ \xrightarrow[-g]{\rightarrow} \end{array} B \xrightarrow{\text{coeq}(g, h)} \text{CoEq}(g, h),$$

obtaining a quotient

$$\text{CoEq}(\text{coeq}(g, h) \circ f, \text{coeq}(g, h) \circ g) = \text{CoEq}(\text{coeq}(g, h) \circ f, \text{coeq}(g, h) \circ h)$$

of $\text{CoEq}(g, h)$.

002B 4. *Unitality.* We have an isomorphism of sets

$$\text{CoEq}(f, f) \cong B.$$

002C 5. *Commutativity.* We have an isomorphism of sets

$$\text{CoEq}(f, g) \cong \text{CoEq}(g, f).$$

002D 6. *Interaction With Composition.* Let

$$A \begin{matrix} \xrightarrow{f} \\ \xrightarrow{g} \end{matrix} B \begin{matrix} \xrightarrow{h} \\ \xrightarrow{k} \end{matrix} C$$

be functions. We have a surjection

$$\text{CoEq}(h \circ f, k \circ g) \twoheadrightarrow \text{CoEq}(\text{coeq}(h, k) \circ h \circ f, \text{coeq}(h, k) \circ k \circ g)$$

exhibiting $\text{CoEq}(\text{coeq}(h, k) \circ h \circ f, \text{coeq}(h, k) \circ k \circ g)$ as a quotient of $\text{CoEq}(h \circ f, k \circ g)$ by the relation generated by declaring $h(y) \sim k(y)$ for each $y \in B$.

Proof. *Item 1, Associativity:* Omitted.

Item 4, Unitality: Clear.

Item 5, Commutativity: Clear.

Item 6, Interaction With Composition: Omitted. □

002E 3 Operations With Sets

002F 3.1 The Empty Set

002G **Definition 3.1.1.1.** The **empty set** is the set \emptyset defined by

$$\emptyset \stackrel{\text{def}}{=} \{x \in X \mid x \neq x\},$$

where A is the set in the set existence axiom, ?? of ??.

002H 3.2 Singleton Sets

Let X be a set.

002J **Definition 3.2.1.1.** The **singleton set containing** X is the set $\{X\}$ defined by

$$\{X\} \stackrel{\text{def}}{=} \{X, X\},$$

where $\{X, X\}$ is the pairing of X with itself (**Definition 3.3.1.1**).

002K 3.3 Pairings of Sets

Let X and Y be sets.

002L Definition 3.3.1.1. The **pairing of X and Y** is the set $\{X, Y\}$ defined by

$$\{X, Y\} \stackrel{\text{def}}{=} \{x \in A \mid x = X \text{ or } x = Y\},$$

where A is the set in the axiom of pairing, ?? of ??.

002M 3.4 Unions of Families

Let $\{A_i\}_{i \in I}$ be a family of sets.

002N Definition 3.4.1.1. The **union of the family $\{A_i\}_{i \in I}$** is the set $\bigcup_{i \in I} A_i$ defined by

$$\bigcup_{i \in I} A_i \stackrel{\text{def}}{=} \{x \in F \mid \text{there exists some } i \in I \text{ such that } x \in A_i\},$$

where F is the set in the axiom of union, ?? of ??.

002P 3.5 Binary Unions

Let A and B be sets.

002Q Definition 3.5.1.1. The **union⁸ of A and B** is the set $A \cup B$ defined by

$$A \cup B \stackrel{\text{def}}{=} \bigcup_{z \in \{A, B\}} z.$$

002R Proposition 3.5.1.2. Let X be a set.

002S 1. *Functoriality.* The assignments $U, V, (U, V) \mapsto U \cup V$ define functors

$$\begin{aligned} U \cup - &: (\mathcal{P}(X), \subset) \rightarrow (\mathcal{P}(X), \subset), \\ - \cup V &: (\mathcal{P}(X), \subset) \rightarrow (\mathcal{P}(X), \subset), \\ -_1 \cup -_2 &: (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \subset) \rightarrow (\mathcal{P}(X), \subset), \end{aligned}$$

where $-_1 \cup -_2$ is the functor where

- *Action on Objects.* For each $(U, V) \in \mathcal{P}(X) \times \mathcal{P}(X)$, we have

$$[-_1 \cup -_2](U, V) \stackrel{\text{def}}{=} U \cup V;$$

⁸*Further Terminology:* Also called the **binary union of A and B** , for emphasis.

- *Action on Morphisms.* For each pair of morphisms

$$\begin{aligned}\iota_U &: U \hookrightarrow U', \\ \iota_V &: V \hookrightarrow V'\end{aligned}$$

of $\mathcal{P}(X) \times \mathcal{P}(X)$, the image

$$\iota_U \cup \iota_V : U \cup V \hookrightarrow U' \cup V'$$

of (ι_U, ι_V) by \cup is the inclusion

$$U \cup V \subset U' \cup V'$$

i.e. where we have

$$(\star) \text{ If } U \subset U' \text{ and } V \subset V', \text{ then } U \cup V \subset U' \cup V';$$

and where $U \cup -$ and $- \cup V$ are the partial functors of $-_1 \cup -_2$ at $U, V \in \mathcal{P}(X)$.

- 002T** 2. *Via Intersections and Symmetric Differences.* We have an equality of sets

$$U \cup V = (U \triangle V) \triangle (U \cap V)$$

for each $X \in \text{Obj}(\mathbf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

- 002U** 3. *Associativity.* We have an equality of sets

$$(U \cup V) \cup W = U \cup (V \cup W)$$

for each $X \in \text{Obj}(\mathbf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

- 002V** 4. *Unitality.* We have equalities of sets

$$\begin{aligned}U \cup \emptyset &= U, \\ \emptyset \cup U &= U\end{aligned}$$

for each $X \in \text{Obj}(\mathbf{Sets})$ and each $U \in \mathcal{P}(X)$.

- 002W** 5. *Commutativity.* We have an equality of sets

$$U \cup V = V \cup U$$

for each $X \in \text{Obj}(\mathbf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

002X 6. *Idempotency.* We have an equality of sets

$$U \cup U = U$$

for each $X \in \text{Obj}(\mathbf{Sets})$ and each $U \in \mathcal{P}(X)$.

002Y 7. *Distributivity Over Intersections.* We have equalities of sets

$$\begin{aligned} U \cup (V \cap W) &= (U \cup V) \cap (U \cup W), \\ (U \cap V) \cup W &= (U \cup W) \cap (V \cup W) \end{aligned}$$

for each $X \in \text{Obj}(\mathbf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

002Z 8. *Interaction With Powersets and Semirings.* The quintuple $(\mathcal{P}(X), \cup, \cap, \emptyset, X)$ is an idempotent commutative semiring.

Proof. *Item 1, Functoriality:* Omitted.

Item 2, Via Intersections and Symmetric Differences: Omitted.

Item 3, Associativity: Clear.

Item 4, Unitality: Clear.

Item 5, Commutativity: Clear.

Item 6, Idempotency: Clear.

Item 7, Distributivity Over Intersections: Omitted.

Item 8, Interaction With Powersets and Semirings: This follows from *Items 3 to 6* and *Items 3 to 5, 7 and 8* of *Proposition 3.7.1.2*. \square

0030 3.6 Intersections of Families

Let \mathcal{F} be a family of sets.

0031 **Definition 3.6.1.1.** The **intersection of a family \mathcal{F} of sets** is the set $\bigcap_{X \in \mathcal{F}} X$ defined by

$$\bigcap_{X \in \mathcal{F}} X \stackrel{\text{def}}{=} \left\{ z \in \bigcup_{X \in \mathcal{F}} X \mid \text{for each } X \in \mathcal{F}, \text{ we have } z \in X \right\}.$$

0032 3.7 Binary Intersections

Let X and Y be sets.

0033 Definition 3.7.1.1. The **intersection**⁹ of X and Y is the set $X \cap Y$ defined by

$$X \cap Y \stackrel{\text{def}}{=} \bigcap_{z \in \{X, Y\}} z.$$

0034 Proposition 3.7.1.2. Let X be a set.

0035 1. *Functoriality.* The assignments $U, V, (U, V) \mapsto U \cap V$ define functors

$$\begin{aligned} U \cap - &: (\mathcal{P}(X), \subset) \rightarrow (\mathcal{P}(X), \subset), \\ - \cap V &: (\mathcal{P}(X), \subset) \rightarrow (\mathcal{P}(X), \subset), \\ -_1 \cap -_2 &: (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \subset) \rightarrow (\mathcal{P}(X), \subset), \end{aligned}$$

where $-_1 \cap -_2$ is the functor where

- *Action on Objects.* For each $(U, V) \in \mathcal{P}(X) \times \mathcal{P}(X)$, we have

$$[-_1 \cap -_2](U, V) \stackrel{\text{def}}{=} U \cap V;$$

- *Action on Morphisms.* For each pair of morphisms

$$\begin{aligned} \iota_U &: U \hookrightarrow U', \\ \iota_V &: V \hookrightarrow V' \end{aligned}$$

of $\mathcal{P}(X) \times \mathcal{P}(X)$, the image

$$\iota_U \cap \iota_V : U \cap V \hookrightarrow U' \cap V'$$

of (ι_U, ι_V) by \cap is the inclusion

$$U \cap V \subset U' \cap V'$$

i.e. where we have

$$(\star) \text{ If } U \subset U' \text{ and } V \subset V', \text{ then } U \cap V \subset U' \cap V';$$

and where $U \cap -$ and $- \cap V$ are the partial functors of $-_1 \cap -_2$ at $U, V \in \mathcal{P}(X)$.

⁹ *Further Terminology:* Also called the **binary intersection of X and Y** , for emphasis.

0036 2. *Adjointness.* We have adjunctions

$$\begin{aligned} (U \cap - \dashv \mathbf{Hom}_{\mathcal{P}(X)}(U, -)) : \mathcal{P}(X) &\begin{array}{c} \xrightarrow{U \cap -} \\ \perp \\ \xleftarrow{\mathbf{Hom}_{\mathcal{P}(X)}(U, -)} \end{array} \mathcal{P}(X), \\ (- \cap V \dashv \mathbf{Hom}_{\mathcal{P}(X)}(V, -)) : \mathcal{P}(X) &\begin{array}{c} \xrightarrow{- \cap V} \\ \perp \\ \xleftarrow{\mathbf{Hom}_{\mathcal{P}(X)}(V, -)} \end{array} \mathcal{P}(X), \end{aligned}$$

where

$$\mathbf{Hom}_{\mathcal{P}(X)}(-_1, -_2) : \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X) \rightarrow \mathcal{P}(X)$$

is the bifunctor defined by¹⁰

$$\mathbf{Hom}_{\mathcal{P}(X)}(U, V) \stackrel{\text{def}}{=} (X \setminus U) \cup V$$

witnessed by bijections

$$\begin{aligned} \text{Hom}_{\mathcal{P}(X)}(U \cap V, W) &\cong \text{Hom}_{\mathcal{P}(X)}(U, \mathbf{Hom}_{\mathcal{P}(X)}(V, W)), \\ \text{Hom}_{\mathcal{P}(X)}(U \cap V, W) &\cong \text{Hom}_{\mathcal{P}(X)}(V, \mathbf{Hom}_{\mathcal{P}(X)}(U, W)), \end{aligned}$$

natural in $U, V, W \in \mathcal{P}(X)$, i.e. where:

(a) The following conditions are equivalent:

- i. We have $U \cap V \subset W$.
- ii. We have $U \subset \mathbf{Hom}_{\mathcal{P}(X)}(V, W)$.
- iii. We have $U \subset (X \setminus V) \cup W$.

(b) The following conditions are equivalent:

- i. We have $V \cap U \subset W$.
- ii. We have $V \subset \mathbf{Hom}_{\mathcal{P}(X)}(U, W)$.
- iii. We have $V \subset (X \setminus U) \cup W$.

0037 3. *Associativity.* We have an equality of sets

$$(U \cap V) \cap W = U \cap (V \cap W)$$

for each $X \in \text{Obj}(\mathbf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

¹⁰ *Intuition:* Since intersections are the products in $\mathcal{P}(X)$, the left adjoint $\mathbf{Hom}_{\mathcal{P}(X)}(U, V)$ works as a function type $U \rightarrow V$.

- 0038 4. *Unitality.* Let X be a set and let $U \in \mathcal{P}(X)$. We have equalities of sets

$$\begin{aligned} X \cap U &= U, \\ U \cap X &= U \end{aligned}$$

for each $X \in \text{Obj}(\mathbf{Sets})$ and each $U \in \mathcal{P}(X)$.

- 0039 5. *Commutativity.* We have an equality of sets

$$U \cap V = V \cap U$$

for each $X \in \text{Obj}(\mathbf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

- 003A 6. *Idempotency.* We have an equality of sets

$$U \cap U = U$$

for each $X \in \text{Obj}(\mathbf{Sets})$ and each $U \in \mathcal{P}(X)$.

- 003B 7. *Distributivity Over Unions.* We have equalities of sets

$$\begin{aligned} U \cap (V \cup W) &= (U \cap V) \cup (U \cap W), \\ (U \cup V) \cap W &= (U \cap W) \cup (V \cap W) \end{aligned}$$

for each $X \in \text{Obj}(\mathbf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

- 003C 8. *Annihilation With the Empty Set.* We have an equality of sets

$$\begin{aligned} \emptyset \cap X &= \emptyset, \\ X \cap \emptyset &= \emptyset \end{aligned}$$

for each $X \in \text{Obj}(\mathbf{Sets})$ and each $U \in \mathcal{P}(X)$.

- 003D 9. *Interaction With Powersets and Monoids With Zero.* The quadruple $((\mathcal{P}(X), \emptyset), \cap, X)$ is a commutative monoid with zero.

- 003E 10. *Interaction With Powersets and Semirings.* The quintuple $(\mathcal{P}(X), \cup, \cap, \emptyset, X)$ is an idempotent commutative semiring.

Now, under the Curry–Howard correspondence, the function type $U \rightarrow V$ corresponds to implication $U \Rightarrow V$, which is logically equivalent to the statement $\neg U \vee V$, which in turn corresponds to the set $U^c \vee V \stackrel{\text{def}}{=} (X \setminus U) \cup V$.

Proof. Item 1, Functoriality: Omitted.

Item 2, Adjointness: See [MSE 267469].

Item 3, Associativity: Clear.

Item 4, Unitality: Clear.

Item 5, Commutativity: Clear.

Item 6, Idempotency: Clear.

Item 7, Distributivity Over Unions: Omitted.

Item 8, Annihilation With the Empty Set: Clear.

Item 9, Interaction With Powersets and Monoids With Zero: This follows from **Items 3 to 5** and **8**.

Item 10, Interaction With Powersets and Semirings: This follows from **Items 3 to 6** and **Items 3 to 5, 7** and **8** of **Proposition 3.7.1.2**. \square

003F 3.8 Differences

Let X and Y be sets.

003G Definition 3.8.1.1. The **difference of X and Y** is the set $X \setminus Y$ defined by

$$X \setminus Y \stackrel{\text{def}}{=} \{a \in X \mid a \notin Y\}.$$

003H Proposition 3.8.1.2. Let X be a set.

003J 1. *Functoriality.* The assignments $U, V, (U, V) \mapsto U \cap V$ define functors

$$\begin{aligned} U \setminus - &: (\mathcal{P}(X), \supset) \rightarrow (\mathcal{P}(X), \subset), \\ - \setminus V &: (\mathcal{P}(X), \subset) \rightarrow (\mathcal{P}(X), \subset), \\ -_1 \setminus -_2 &: (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \supset) \rightarrow (\mathcal{P}(X), \subset), \end{aligned}$$

where $-_1 \setminus -_2$ is the functor where

- *Action on Objects.* For each $(U, V) \in \mathcal{P}(X) \times \mathcal{P}(X)$, we have

$$[-_1 \setminus -_2](U, V) \stackrel{\text{def}}{=} U \setminus V;$$

- *Action on Morphisms.* For each pair of morphisms

$$\begin{aligned} \iota_A &: A \hookrightarrow B, \\ \iota_U &: U \hookrightarrow V \end{aligned}$$

of $\mathcal{P}(X) \times \mathcal{P}(X)$, the image

$$\iota_U \setminus \iota_V: A \setminus V \hookrightarrow B \setminus U$$

of (ι_U, ι_V) by \setminus is the inclusion

$$A \setminus V \subset B \setminus U$$

i.e. where we have

$$(\star) \text{ If } A \subset B \text{ and } U \subset V, \text{ then } A \setminus V \subset B \setminus U;$$

and where $U \setminus -$ and $- \setminus V$ are the partial functors of $-_1 \setminus -_2$ at $U, V \in \mathcal{P}(X)$.

003K 2. *De Morgan's Laws.* We have equalities of sets

$$\begin{aligned} X \setminus (U \cup V) &= (X \setminus U) \cap (X \setminus V), \\ X \setminus (U \cap V) &= (X \setminus U) \cup (X \setminus V) \end{aligned}$$

for each $X \in \text{Obj}(\mathbf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

003L 3. *Interaction With Unions I.* We have equalities of sets

$$(U \setminus V) \cup W = (U \cup W) \setminus (V \setminus W)$$

for each $X \in \text{Obj}(\mathbf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

003M 4. *Interaction With Unions II.* We have equalities of sets

$$(U \cup V) \setminus W = (U \setminus W) \cup (V \setminus W)$$

for each $X \in \text{Obj}(\mathbf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

003N 5. *Interaction With Intersections.* We have equalities of sets

$$\begin{aligned} (U \setminus V) \cap W &= (U \cap W) \setminus V \\ &= U \cap (W \setminus V) \end{aligned}$$

for each $X \in \text{Obj}(\mathbf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

003P 6. *Triple Differences.* We have

$$U \setminus (V \setminus W) = (U \cap W) \cup (U \setminus V)$$

for each $X \in \text{Obj}(\mathbf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

003Q 7. *Left Annihilation.* We have

$$\emptyset \setminus U = \emptyset$$

for each $X \in \text{Obj}(\mathbf{Sets})$ and each $U \in \mathcal{P}(X)$.

003R 8. *Right Unitality*. We have

$$U \setminus \emptyset = U$$

for each $X \in \text{Obj}(\mathbf{Sets})$ and each $U \in \mathcal{P}(X)$.

003S 9. *Invertibility*. We have

$$U \setminus U = \emptyset$$

for each $X \in \text{Obj}(\mathbf{Sets})$ and each $U \in \mathcal{P}(X)$.

003T 10. *Interaction With Containment*. The following conditions are equivalent:

- (a) We have $V \setminus U \subset W$.
- (b) We have $V \setminus W \subset U$.

Proof. *Item 1, Functoriality*: Omitted.

Item 2, De Morgan's Laws: Omitted.

Item 3, Interaction With Unions I: Omitted.

Item 4, Interaction With Unions II: Omitted.

Item 5, Interaction With Intersections: Omitted.

Item 6, Triple Differences: Omitted.

Item 7, Left Annihilation: Clear.

Item 8, Right Unitality: Clear.

Item 9, Invertibility: Clear.

Item 10, Interaction With Containment: Omitted. □

003U 3.9 Complements

Let X be a set and let $U \in \mathcal{P}(X)$.

003V **Definition 3.9.1.1.** The **complement of U** is the set U^c defined by

$$\begin{aligned} U^c &\stackrel{\text{def}}{=} X \setminus U \\ &\stackrel{\text{def}}{=} \{a \in X \mid a \notin U\}. \end{aligned}$$

003W **Proposition 3.9.1.2.** Let X be a set.

003X 1. *Functoriality*. The assignment $U \mapsto U^c$ defines a functor

$$(-)^c: \mathcal{P}(X)^{\text{op}} \rightarrow \mathcal{P}(X),$$

where

- *Action on Objects.* For each $U \in \mathcal{P}(X)$, we have

$$[(-)^c](U) \stackrel{\text{def}}{=} U^c;$$

- *Action on Morphisms.* For each morphism $\iota_U: U \hookrightarrow V$ of $\mathcal{P}(X)$, the image

$$\iota_U^c: V^c \hookrightarrow U^c$$

of ι_U by $(-)^c$ is the inclusion

$$V^c \subset U^c$$

i.e. where we have

(\star) If $U \subset V$, then $V^c \subset U^c$.

- 003Y 2. *De Morgan's Laws.* We have equalities of sets

$$(U \cup V)^c = U^c \cap V^c,$$

$$(U \cap V)^c = U^c \cup V^c$$

for each $X \in \text{Obj}(\mathbf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

- 003Z 3. *Involutority.* We have

$$(U^c)^c = U$$

for each $X \in \text{Obj}(\mathbf{Sets})$ and each $U \in \mathcal{P}(X)$.

Proof. Item 1, Functoriality: Clear.

Item 2, De Morgan's Laws: Omitted.

Item 3, Involutority: Clear. □

0040 3.10 Symmetric Differences

Let A and B be sets.

- 0041 **Definition 3.10.1.1.** The **symmetric difference of A and B** is the set $A \triangle B$ defined by

$$A \triangle B \stackrel{\text{def}}{=} (A \setminus B) \cup (B \setminus A).$$

- 0042 **Proposition 3.10.1.2.** Let X be a set.

- 0043 1. *Lack of Functoriality.* The assignment $(U, V) \mapsto U \triangle V$ **does not** define a functor

$$-_1 \triangle -_2: (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \subset) \rightarrow (\mathcal{P}(X), \subset).$$

0044 2. *Via Unions and Intersections.* We have¹¹

$$U \triangle V = (U \cup V) \setminus (U \cap V)$$

for each $X \in \text{Obj}(\text{Sets})$ and each $U, V \in \mathcal{P}(X)$.

0045 3. *Associativity.* We have¹²

$$(U \triangle V) \triangle W = U \triangle (V \triangle W)$$

for each $X \in \text{Obj}(\text{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

0046 4. *Unitality.* We have

$$\begin{aligned} U \triangle \emptyset &= U, \\ \emptyset \triangle U &= U \end{aligned}$$

for each $X \in \text{Obj}(\text{Sets})$ and each $U \in \mathcal{P}(X)$.

0047 5. *Invertibility.* We have

$$U \triangle U = \emptyset$$

for each $X \in \text{Obj}(\text{Sets})$ and each $U \in \mathcal{P}(X)$.

0048 6. *Commutativity.* We have

$$U \triangle V = V \triangle U$$

for each $X \in \text{Obj}(\text{Sets})$ and each $U, V \in \mathcal{P}(X)$.

0049 7. *“Transitivity”.* We have

$$(U \triangle V) \triangle (V \triangle W) = U \triangle W$$

for each $X \in \text{Obj}(\text{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

¹¹*Illustration:*

$$\begin{array}{|c|} \hline \text{Venn diagram of } U \triangle V \\ \hline \end{array} = \begin{array}{|c|} \hline \text{Venn diagram of } U \cup V \\ \hline \end{array} \setminus \begin{array}{|c|} \hline \text{Venn diagram of } U \cap V \\ \hline \end{array}.$$

¹²*Illustration:*

$$\begin{array}{|c|} \hline \text{Venn diagram of } U \triangle V \\ \hline \end{array} \triangle \begin{array}{|c|} \hline \text{Venn diagram of } W \\ \hline \end{array} = \begin{array}{|c|} \hline \text{Venn diagram of } U \triangle V \triangle W \\ \hline \end{array} = \begin{array}{|c|} \hline \text{Venn diagram of } U \\ \hline \end{array} \triangle \begin{array}{|c|} \hline \text{Venn diagram of } V \triangle W \\ \hline \end{array}.$$

004A 8. *The Triangle Inequality for Symmetric Differences.* We have

$$U \triangle W \subset U \triangle V \cup V \triangle W$$

for each $X \in \text{Obj}(\mathbf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

004B 9. *Distributivity Over Intersections.* We have

$$\begin{aligned} U \cap (V \triangle W) &= (U \cap V) \triangle (U \cap W), \\ (U \triangle V) \cap W &= (U \cap W) \triangle (V \cap W) \end{aligned}$$

for each $X \in \text{Obj}(\mathbf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

004C 10. *Interaction With Indicator Functions.* We have

$$\chi_{U \triangle V} \equiv \chi_U + \chi_V \pmod{2}$$

for each $X \in \text{Obj}(\mathbf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

004D 11. *Bijectivity.* Given $A, B \subset \mathcal{P}(X)$, the maps

$$\begin{aligned} A \triangle -: \mathcal{P}(X) &\rightarrow \mathcal{P}(X), \\ - \triangle B: \mathcal{P}(X) &\rightarrow \mathcal{P}(X) \end{aligned}$$

are bijections with inverses given by

$$\begin{aligned} (A \triangle -)^{-1} &= - \cup (A \cap -), \\ (- \triangle B)^{-1} &= - \cup (B \cap -). \end{aligned}$$

Moreover, the map

$$C \mapsto C \triangle (A \triangle B)$$

is a bijection of $\mathcal{P}(X)$ onto itself sending A to B and B to A .

004E 12. *Interaction With Powersets and Groups I.* The quadruple $(\mathcal{P}(X), \triangle, \emptyset, \text{id}_{\mathcal{P}(X)})$

is an abelian group.^{13,14,15}

004F 13. *Interaction With Powersets and Groups II.* Every element of $\mathcal{P}(X)$ has order 2 with respect to \triangle , and thus $\mathcal{P}(X)$ is a *Boolean group* (i.e. an abelian 2-group).

004G 14. *Interaction With Powersets and Vector Spaces I.* The pair $(\mathcal{P}(X), \alpha_{\mathcal{P}(X)})$ consisting of

- The group $\mathcal{P}(X)$ of **Item 12**;
- The map $\alpha_{\mathcal{P}(X)}: \mathbb{F}_2 \times \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ defined by

$$\begin{aligned} 0 \cdot U &\stackrel{\text{def}}{=} \emptyset, \\ 1 \cdot U &\stackrel{\text{def}}{=} U; \end{aligned}$$

is an \mathbb{F}_2 -vector space.

004H 15. *Interaction With Powersets and Vector Spaces II.* If X is finite, then:

- (a) The set of singletons sets on the elements of X forms a basis for the \mathbb{F}_2 -vector space $(\mathcal{P}(X), \alpha_{\mathcal{P}(X)})$ of **Item 14**.
- (b) We have

$$\dim(\mathcal{P}(X)) = \#\mathcal{P}(X).$$

004J 16. *Interaction With Powersets and Rings.* The quintuple $(\mathcal{P}(X), \triangle, \cap, \emptyset, X)$ is a commutative ring.¹⁶

¹³*Example:* When $X = \emptyset$, we have an isomorphism of groups between $\mathcal{P}(\emptyset)$ and the trivial group:


$$(\mathcal{P}(\emptyset), \triangle, \emptyset, \text{id}_{\mathcal{P}(\emptyset)}) \cong \text{pt}.$$

¹⁴*Example:* When $X = \text{pt}$, we have an isomorphism of groups between $\mathcal{P}(\text{pt})$ and $\mathbb{Z}/2$:

$$(\mathcal{P}(\text{pt}), \triangle, \emptyset, \text{id}_{\mathcal{P}(\text{pt})}) \cong \mathbb{Z}/2.$$

¹⁵*Example:* When $X = \{0, 1\}$, we have an isomorphism of groups between $\mathcal{P}(\{0, 1\})$ and $\mathbb{Z}/2 \times \mathbb{Z}/2$:

$$(\mathcal{P}(\{0, 1\}), \triangle, \emptyset, \text{id}_{\mathcal{P}(\{0, 1\})}) \cong \mathbb{Z}/2 \times \mathbb{Z}/2.$$

¹⁶ *Warning:* The analogous statement replacing intersections by unions (i.e. that the quintuple $(\mathcal{P}(X), \triangle, \cup, \emptyset, X)$ is a ring) is false, however. See [Pro23b] for a proof.

Proof. Item 1, Lack of Functoriality: Omitted.

Item 2, Via Unions and Intersections: Omitted.

Item 3, Associativity: Omitted.

Item 4, Unitality: Clear.

Item 5, Invertibility: Clear.

Item 6, Commutativity: Clear.

Item 7, “Transitivity”: We have

$$\begin{aligned}
 (U \triangle V) \triangle (V \triangle W) &= U \triangle (V \triangle (V \triangle W)) && \text{(by Item 3)} \\
 &= U \triangle ((V \triangle V) \triangle W) && \text{(by Item 3)} \\
 &= U \triangle (\emptyset \triangle W) && \text{(by Item 5)} \\
 &= U \triangle W && \text{(by Item 4)}
 \end{aligned}$$

Item 8, The Triangle Inequality for Symmetric Differences: This follows from Items 2 and 7.

Item 9, Distributivity Over Intersections: Omitted.

Item 10, Interaction With Indicator Functions: Clear.

Item 11, Bijectivity: Clear.

Item 12, Interaction With Powersets and Groups I: This follows from Items 3 to 6.

Item 13, Interaction With Powersets and Groups II: This follows from Item 5.

Item 14, Interaction With Powersets and Vector Spaces I: Clear.

Item 15, Interaction With Powersets and Vector Spaces II: Omitted.

Item 16, Interaction With Powersets and Rings: This follows from Items 9 and 12 and Items 8 and 9 of Proposition 3.7.1.2.¹⁷ \square

004K 3.11 Ordered Pairs

Let A and B be sets.

004L Definition 3.11.1.1. The ordered pair associated to A and B is the set (A, B) defined by

$$(A, B) \stackrel{\text{def}}{=} \{\{A\}, \{A, B\}\}.$$

004M Proposition 3.11.1.2. Let A and B be sets.

004N 1. *Uniqueness.* Let A , B , C , and D be sets. The following conditions are equivalent:

¹⁷Reference: [Pro23a].

- (a) We have $(A, B) = (C, D)$.
 (b) We have $A = C$ and $B = D$.

Proof. **Item 1, Uniqueness:** See [Cie97, Theorem 1.2.3]. □

004P 4 Powersets

004Q 4.1 Characteristic Functions

Let X be a set.

004R Definition 4.1.1.1. Let $U \subset X$ and let $x \in X$.

004S 1. The **characteristic function of U** ¹⁸ is the function¹⁹

$$\chi_U: X \rightarrow \{\mathbf{t}, \mathbf{f}\}$$

defined by

$$\chi_U(x) \stackrel{\text{def}}{=} \begin{cases} \mathbf{true} & \text{if } x \in U, \\ \mathbf{false} & \text{if } x \notin U \end{cases}$$

for each $x \in X$.

004T 2. The **characteristic function of x** is the function²⁰

$$\chi_x: X \rightarrow \{\mathbf{t}, \mathbf{f}\}$$

defined by

$$\chi_x \stackrel{\text{def}}{=} \chi_{\{x\}},$$

i.e. by

$$\chi_x(y) \stackrel{\text{def}}{=} \begin{cases} \mathbf{true} & \text{if } x = y, \\ \mathbf{false} & \text{if } x \neq y \end{cases}$$

for each $y \in X$.

¹⁸ *Further Terminology:* Also called the **indicator function of U** .

¹⁹ *Further Notation:* Also written $\chi_X(U, -)$ or $\chi_X(-, U)$.

²⁰ *Further Notation:* Also written χ_x , $\chi_X(x, -)$, or $\chi_X(-, x)$.

- 004U 3. The **characteristic relation on X** ²¹ is the relation²²

$$\chi_X(-1, -2): X \times X \rightarrow \{\mathbf{t}, \mathbf{f}\}$$

on X defined by²³

$$\chi_X(x, y) \stackrel{\text{def}}{=} \begin{cases} \text{true} & \text{if } x = y, \\ \text{false} & \text{if } x \neq y \end{cases}$$

for each $x, y \in X$.

- 004V 4. The **characteristic embedding**²⁴ of X into $\mathcal{P}(X)$ is the function

$$\chi_{(-)}: X \hookrightarrow \mathcal{P}(X)$$

defined by

$$\chi_{(-)}(x) \stackrel{\text{def}}{=} \chi_x$$

for each $x \in X$.

- 004W **Remark 4.1.1.2.** The definitions in [Definition 4.1.1.1](#) are decategorifications of co/presheaves, representable co/presheaves, Hom profunctors, and the Yoneda embedding.²⁵

²¹*Further Terminology:* Also called the **identity relation on X** .

²²*Further Notation:* Also written χ_{-2}^{-1} , or \sim_{id} in the context of relations.

²³As a subset of $X \times X$, the relation χ_X corresponds to the diagonal $\Delta_X \subset X \times X$ of X .

²⁴The name “characteristic *embedding*” comes from the fact that there is an analogue of fully faithfulness for $\chi_{(-)}$: given a set X , we have

$$\text{Hom}_{\mathcal{P}(X)}(\chi_x, \chi_y) = \chi_X(x, y),$$

for each $x, y \in X$.

²⁵These statements can be made precise by using the embeddings

$$\begin{aligned} (-)_{\text{disc}}: \mathbf{Sets} &\hookrightarrow \mathbf{Cats}, \\ (-)_{\text{disc}}: \{\mathbf{t}, \mathbf{f}\}_{\text{disc}} &\hookrightarrow \mathbf{Sets} \end{aligned}$$

of sets into categories and of classical truth values into sets.

For instance, in this approach the characteristic function

$$\chi_x: X \rightarrow \{\mathbf{t}, \mathbf{f}\}$$

of an element x of X , defined by

$$\chi_x(y) \stackrel{\text{def}}{=} \begin{cases} \text{true} & \text{if } x = y, \\ \text{false} & \text{if } x \neq y \end{cases}$$

1. A function

$$f: X \rightarrow \{\mathbf{t}, \mathbf{f}\}$$

is a decategorification of a presheaf

$$\mathcal{F}: C^{\text{op}} \rightarrow \mathbf{Sets},$$

with the characteristic functions χ_U of the subsets of X being the primordial examples (and, in fact, all examples) of these.

2. The characteristic function

$$\chi_x: X \rightarrow \{\mathbf{t}, \mathbf{f}\}$$

of an *element* x of X is a decategorification of the representable presheaf

$$h_X: C^{\text{op}} \rightarrow \mathbf{Sets}$$

of an *object* x of a category C .

3. The characteristic relation

$$\chi_X(-, -): X \times X \rightarrow \{\mathbf{t}, \mathbf{f}\}$$

of X is a decategorification of the Hom profunctor

$$\text{Hom}_C(-, -): C^{\text{op}} \times C \rightarrow \mathbf{Sets}$$

of a category C .

4. The characteristic embedding

$$\chi_{(-)}: X \hookrightarrow \mathcal{P}(X)$$

of X into $\mathcal{P}(X)$ is a decategorification of the Yoneda embedding

$$\mathcal{Y}: C^{\text{op}} \hookrightarrow \mathbf{PSh}(C)$$

of a category C into $\mathbf{PSh}(C)$.

for each $y \in X$, is recovered as the representable presheaf

$$\text{Hom}_{X_{\text{disc}}}(-, x): X_{\text{disc}} \rightarrow \mathbf{Sets}$$

5. There is also a direct parallel between unions and colimits:

- An element of $\mathcal{P}(X)$ is a union of elements of X , viewed as one-point subsets $\{x\} \in \mathcal{P}(A)$;
- An object of $\mathbf{PSh}(C)$ is a colimit of objects of C , viewed as representable presheaves $h_X \in \mathbf{Obj}(\mathbf{PSh}(C))$.

004X Proposition 4.1.1.3. Let $f: A \rightarrow B$ be a function. We have an inclusion

$$\begin{array}{ccc} A \times A & \xrightarrow{\chi_A(-1, -2)} & \{\text{true}, \text{false}\} \\ \chi_B \circ (f \times f) \subset \chi_A, \quad f \times f \downarrow & \subset & \downarrow \text{id}_{\{\text{true}, \text{false}\}} \\ B \times B & \xrightarrow{\chi_B(-1, -2)} & \{\text{true}, \text{false}\}. \end{array}$$

Proof. The inclusion $\chi_B(f(a), f(b)) \subset \chi_A(a, b)$ is equivalent to the statement “if $a = b$, then $f(a) = f(b)$ ”, which is true. \square

004Y Proposition 4.1.1.4. Let X be a set and let $U \subset X$ be a subset of X . We have

$$\text{Hom}_{\mathcal{P}(X)}(\chi_x, \chi_U) = \chi_U(x)$$

for each $x \in X$, giving an equality of functions

$$\text{Hom}_{\mathcal{P}(X)}(\chi_{(-)}, \chi_U) = \chi_U.$$

Proof. Clear. \square

004Z Corollary 4.1.1.5. The characteristic embedding is fully faithful, i.e., we have

$$\text{Hom}_{\mathcal{P}(X)}(\chi_x, \chi_y) = \chi_X(x, y)$$

for each $x, y \in X$.

Proof. This follows from **Proposition 4.1.1.4**. \square

of the corresponding object x of X_{disc} , defined on objects by

$$\text{Hom}_{X_{\text{disc}}}(y, x) \stackrel{\text{def}}{=} \begin{cases} \text{pt} & \text{if } x = y, \\ \emptyset & \text{if } x \neq y \end{cases}$$

for each $y \in \mathbf{Obj}(X_{\text{disc}})$.

0050 4.2 Powersets

Let X be a set.

0051 **Definition 4.2.1.1.** The **powerset of X** is the set $\mathcal{P}(X)$ defined by

$$\mathcal{P}(X) \stackrel{\text{def}}{=} \{U \in P \mid U \subset X\},$$

where P is the set in the axiom of powerset, ?? of ??.

0052 **Remark 4.2.1.2.** The powerset of a set is a decategorification of the category of presheaves of a category: while²⁶

- The powerset of a set X is equivalently (Item 6 of Proposition 4.2.1.3) the set

$$\mathbf{Sets}(X, \{\mathbf{t}, \mathbf{f}\})$$

of functions from X to the set $\{\mathbf{t}, \mathbf{f}\}$ of classical truth values;

- The category of presheaves on a category C is the category

$$\mathbf{Fun}(C^{\text{op}}, \mathbf{Sets})$$

of functors from C^{op} to the category \mathbf{Sets} of sets.

0053 **Proposition 4.2.1.3.** Let X be a set.

0054 1. *Functoriality.* The assignment $X \mapsto \mathcal{P}(X)$ defines functors

$$\begin{aligned} \mathcal{P}_* &: \mathbf{Sets} \rightarrow \mathbf{Sets}, \\ \mathcal{P}^{-1} &: \mathbf{Sets}^{\text{op}} \rightarrow \mathbf{Sets}, \\ \mathcal{P}_! &: \mathbf{Sets} \rightarrow \mathbf{Sets} \end{aligned}$$

where

²⁶This parallel is based on the following comparison:

- A category is enriched over the category

$$\mathbf{Sets} \stackrel{\text{def}}{=} \mathbf{Cats}_0$$

of sets (i.e. “0-categories”), with presheaves taking values on it;

- A set is enriched over the set

$$\{\mathbf{t}, \mathbf{f}\} \stackrel{\text{def}}{=} \mathbf{Cats}_{-1}$$

of classical truth values (i.e. “(−1)-categories”), with characteristic functions taking values on it.

- *Action on Objects.* For each $A \in \text{Obj}(\text{Sets})$, we have

$$\begin{aligned}\mathcal{P}_*(A) &\stackrel{\text{def}}{=} \mathcal{P}(A), \\ \mathcal{P}^{-1}(A) &\stackrel{\text{def}}{=} \mathcal{P}(A), \\ \mathcal{P}_!(A) &\stackrel{\text{def}}{=} \mathcal{P}(A);\end{aligned}$$

- *Action on Morphisms.* For each morphism $f: A \rightarrow B$ of Sets , the images

$$\begin{aligned}\mathcal{P}_*(f): \mathcal{P}(A) &\rightarrow \mathcal{P}(B), \\ \mathcal{P}^{-1}(f): \mathcal{P}(B) &\rightarrow \mathcal{P}(A), \\ \mathcal{P}_!(f): \mathcal{P}(A) &\rightarrow \mathcal{P}(B)\end{aligned}$$

of f by \mathcal{P}_* , \mathcal{P}^{-1} , and $\mathcal{P}_!$ are defined by

$$\begin{aligned}\mathcal{P}_*(f) &\stackrel{\text{def}}{=} f_*, \\ \mathcal{P}^{-1}(f) &\stackrel{\text{def}}{=} f^{-1}, \\ \mathcal{P}_!(f) &\stackrel{\text{def}}{=} f_!,\end{aligned}$$

as in [Definitions 4.3.1.1](#), [4.4.1.1](#) and [4.5.1.1](#).

- 0055** 2. *Adjointness I.* We have an adjunction

$$(\mathcal{P}^{-1} \dashv \mathcal{P}^{-1, \text{op}}): \text{Sets}^{\text{op}} \begin{array}{c} \xrightarrow{\mathcal{P}^{-1}} \\ \perp \\ \xleftarrow{\mathcal{P}^{-1, \text{op}}} \end{array} \text{Sets},$$

witnessed by a bijection

$$\underbrace{\text{Sets}^{\text{op}}(\mathcal{P}(X), Y)}_{\stackrel{\text{def}}{=} \text{Sets}(Y, \mathcal{P}(X))} \cong \text{Sets}(X, \mathcal{P}(Y)),$$

natural in $X \in \text{Obj}(\text{Sets})$ and $Y \in \text{Obj}(\text{Sets}^{\text{op}})$.

- 0056** 3. *Adjointness II.* We have an adjunction

$$(\text{Gr} \dashv \mathcal{P}_*): \text{Sets} \begin{array}{c} \xrightarrow{\text{Gr}} \\ \perp \\ \xleftarrow{\mathcal{P}_*} \end{array} \text{Rel},$$

witnessed by a bijection of sets

$$\text{Rel}(\text{Gr}(A), B) \cong \text{Sets}(A, \mathcal{P}(B))$$

natural in $A \in \text{Obj}(\text{Sets})$ and $B \in \text{Obj}(\text{Rel})$, where Gr is the graph functor of [Relations](#), [Item 1](#) of [Proposition 3.1.1.2](#).

- 0057** 4. *Symmetric Strong Monoidality With Respect to Coproducts.* The powerset functor \mathcal{P}_* of **Item 1** has a symmetric strong monoidal structure

$$\left(\mathcal{P}_*, \mathcal{P}_*^{\coprod}, \mathcal{P}_{*\|\kern-0.08em\|}^{\coprod} \right) : (\mathbf{Sets}, \coprod, \emptyset) \rightarrow (\mathbf{Sets}, \times, \text{pt})$$

being equipped with isomorphisms

$$\begin{aligned} \mathcal{P}_{*|X,Y}^{\coprod} : \mathcal{P}(X) \times \mathcal{P}(Y) &\xrightarrow{\cong} \mathcal{P}(X \coprod Y), \\ \mathcal{P}_{*\|\kern-0.08em\|}^{\coprod} : \text{pt} &\xrightarrow{=} \mathcal{P}(\emptyset), \end{aligned}$$

natural in $X, Y \in \text{Obj}(\mathbf{Sets})$.

- 0058** 5. *Symmetric Lax Monoidality With Respect to Products.* The powerset functor \mathcal{P}_* of **Item 1** has a symmetric lax monoidal structure

$$\left(\mathcal{P}_*, \mathcal{P}_*^{\otimes}, \mathcal{P}_{*\|\kern-0.08em\|}^{\otimes} \right) : (\mathbf{Sets}, \times, \text{pt}) \rightarrow (\mathbf{Sets}, \times, \text{pt})$$

being equipped with isomorphisms

$$\begin{aligned} \mathcal{P}_{*|X,Y}^{\otimes} : \mathcal{P}(X) \times \mathcal{P}(Y) &\rightarrow \mathcal{P}(X \times Y), \\ \mathcal{P}_{*\|\kern-0.08em\|}^{\otimes} : \text{pt} &\xrightarrow{=} \mathcal{P}(\emptyset), \end{aligned}$$

natural in $X, Y \in \text{Obj}(\mathbf{Sets})$, where $\mathcal{P}_{*|X,Y}^{\otimes}$ is given by

$$\mathcal{P}_{*|X,Y}^{\otimes}(U, V) \stackrel{\text{def}}{=} U \times V$$

for each $(U, V) \in \mathcal{P}(X) \times \mathcal{P}(Y)$.

- 0059** 6. *Powersets as Sets of Functions.* The assignment $U \mapsto \chi_U$ defines a bijection²⁷

$$\chi_{(-)} : \mathcal{P}(X) \xrightarrow{\cong} \mathbf{Sets}(X, \{\text{t}, \text{f}\}),$$

natural in $X \in \text{Obj}(\mathbf{Sets})$.

²⁷This bijection is a decategorified form of the equivalence

$$\mathbf{PSh}(C) \stackrel{\text{eq.}}{\cong} \mathbf{DFib}(C)$$

of Fibred Categories, ?? of ??, with $\chi_{(-)}$ being a decategorified version of the category of elements construction of Fibred Categories, ??.

See also ?? of ??.

005A 7. *Powersets as Sets of Relations.* We have bijections

$$\begin{aligned}\mathcal{P}(X) &\cong \text{Rel}(\text{pt}, X), \\ \mathcal{P}(X) &\cong \text{Rel}(X, \text{pt}),\end{aligned}$$

natural in $X \in \text{Obj}(\mathbf{Sets})$.

005B 8. *As a Free Cocompletion: Universal Property.* The pair $(\mathcal{P}(X), \chi_{(-)})$ consisting of

- The powerset $\mathcal{P}(X)$ of X ;
- The characteristic embedding $\chi_{(-)}: X \hookrightarrow \mathcal{P}(X)$ of X into $\mathcal{P}(X)$;

satisfies the following universal property:

- (\star) Given another pair (Y, f) consisting of
- A cocomplete poset (Y, \preceq) ;
 - A function $f: X \rightarrow Y$;

there exists a unique cocontinuous morphism of posets $(\mathcal{P}(X), \subset) \xrightarrow{\exists!} (Y, \preceq)$ making the diagram

$$\begin{array}{ccc} & & \mathcal{P}(X) \\ & \nearrow \chi_X & \downarrow \exists! \\ X & \xrightarrow{f} & Y\end{array}$$

commute.

005C 9. *As a Free Cocompletion: Adjointness.* We have an adjunction²⁸

$$(\chi_{(-)} \dashv \text{忘}): \mathbf{Sets} \begin{array}{c} \xrightarrow{\chi_{(-)}} \\ \perp \\ \xleftarrow{\text{忘}} \end{array} \mathbf{Pos}^{\text{cocomp.}},$$

witnessed by a bijection

$$\mathbf{Pos}^{\text{cocomp.}}((\mathcal{P}(X), \subset), (Y, \preceq)) \cong \mathbf{Sets}(X, Y),$$

natural in $X \in \text{Obj}(\mathbf{Sets})$ and $(Y, \preceq) \in \text{Obj}(\mathbf{Pos})$, where

²⁸In this sense, $\mathcal{P}(A)$ is the free cocompletion of A . (Note that, despite its name, however,

- We have a natural map

$$\chi_X^* : \mathbf{Pos}^{\text{cocomp.}}((\mathcal{P}(X), \subset), (Y, \preceq)) \rightarrow \mathbf{Sets}(X, Y)$$

defined by

$$\chi_X^*(f) \stackrel{\text{def}}{=} f \circ \chi_X,$$

i.e. by sending a cocontinuous morphism of posets $f : \mathcal{P}(X) \rightarrow Y$ to the composition

$$X \xrightarrow{\chi_X} \mathcal{P}(X) \xrightarrow{f} Y;$$

- We have a natural map

$$\text{Lan}_{\chi_X} : \mathbf{Sets}(X, Y) \rightarrow \mathbf{Pos}^{\text{cocomp.}}((\mathcal{P}(X), \subset), (Y, \preceq))$$

computed by

$$\begin{aligned} [\text{Lan}_{\chi_X}(f)](U) &\cong \int^{x \in X} \chi_{\mathcal{P}(X)}(\chi_x, U) \odot f(x) \\ &\cong \int^{x \in X} \chi_U(x) \odot f(x) && \text{(by Proposition 4.1.1.4)} \\ &\cong \bigvee_{x \in X} (\chi_U(x) \odot f(x)) \end{aligned}$$

for each $U \in \mathcal{P}(X)$, where:

- \bigvee is the join in (Y, \preceq) ;
- We have

$$\begin{aligned} \text{true} \odot f(x) &\stackrel{\text{def}}{=} f(x), \\ \text{false} \odot f(x) &\stackrel{\text{def}}{=} \emptyset_Y, \end{aligned}$$

where \emptyset_Y is the minimal element of (Y, \preceq) .

Proof. Item 1, Functoriality: This follows from Items 3 and 4 of Proposition 4.3.1.4, Items 3 and 4 of Proposition 4.4.1.4, and Items 3 and 4 of Proposition 4.5.1.6.

Item 2, Adjointness I: Omitted.

Item 3, Adjointness II: Omitted.

Item 4, Symmetric Strong Monoidality With Respect to Coproducts: Omitted.

Item 5, Symmetric Lax Monoidality With Respect to Products: Omitted.

Item 6, Powersets as Sets of Functions: Omitted.

Item 7, Powersets as Sets of Relations: Omitted.

Item 8, As a Free Cocompletion: Universal Property: This is a rephrasing of ??.

Item 9, As a Free Cocompletion: Adjointness: Omitted. \square

005D 4.3 Direct Images

Let A and B be sets and let $f: A \rightarrow B$ be a function.

005E Definition 4.3.1.1. The **direct image function** associated to f is the function²⁹

$$f_*: \mathcal{P}(A) \rightarrow \mathcal{P}(B)$$

defined by^{30,31}

$$\begin{aligned} f_*(U) &\stackrel{\text{def}}{=} f(U) \\ &\stackrel{\text{def}}{=} \left\{ b \in B \mid \begin{array}{l} \text{there exists some } a \in \\ U \text{ such that } b = f(a) \end{array} \right\} \\ &= \{ f(a) \in B \mid a \in U \} \end{aligned}$$

for each $U \in \mathcal{P}(A)$.

005F Remark 4.3.1.2. Identifying subsets of A with functions from A to $\{\text{true}, \text{false}\}$ via *Item 6* of *Proposition 4.2.1.3*, we see that the direct image function associated to f is equivalently the function

$$f_*: \mathcal{P}(A) \rightarrow \mathcal{P}(B)$$

this is not an idempotent operation, as we have $\mathcal{P}(\mathcal{P}(A)) \neq \mathcal{P}(A)$.)

²⁹*Further Notation:* Also written $\exists_f: \mathcal{P}(A) \rightarrow \mathcal{P}(B)$. This notation comes from the fact that the following statements are equivalent, where $b \in B$ and $U \in \mathcal{P}(A)$:

- We have $b \in \exists_f(U)$.
- There exists some $a \in U$ such that $f(a) = b$.

³⁰*Further Terminology:* The set $f(U)$ is called the **direct image of U by f** .

³¹We also have

$$f_*(U) = B \setminus f_!(A \setminus U);$$

see *Item 7* of *Proposition 4.3.1.3*.

defined by

$$\begin{aligned}
 f_*(\chi_U) &\stackrel{\text{def}}{=} \text{Lan}_f(\chi_U) \\
 &= \text{colim} \left(\left(f \times \overrightarrow{(-1)} \right) \xrightarrow{\text{pr}} A \xrightarrow{\chi_U} \{\text{t}, \text{f}\} \right) \\
 &= \text{colim}_{\substack{a \in A \\ f(a) = -1}} (\chi_U(a)) \\
 &= \bigvee_{\substack{a \in A \\ f(a) = -1}} (\chi_U(a)).
 \end{aligned}$$

So, in other words, we have

$$\begin{aligned}
 [f_*(\chi_U)](b) &= \bigvee_{\substack{a \in A \\ f(a) = b}} (\chi_U(a)) \\
 &= \begin{cases} \text{true} & \text{if there exists some } a \in A \text{ such} \\ & \text{that } f(a) = b \text{ and } a \in U, \\ \text{false} & \text{otherwise} \end{cases} \\
 &= \begin{cases} \text{true} & \text{if there exists some } a \in U \\ & \text{such that } f(a) = b, \\ \text{false} & \text{otherwise} \end{cases}
 \end{aligned}$$

for each $b \in B$.

005G Proposition 4.3.1.3. Let $f: A \rightarrow B$ be a function.

005H 1. *Functoriality.* The assignment $U \mapsto f_*(U)$ defines a functor

$$f_*: (\mathcal{P}(A), \subset) \rightarrow (\mathcal{P}(B), \subset)$$

where

- *Action on Objects.* For each $U \in \mathcal{P}(A)$, we have

$$[f_*](U) \stackrel{\text{def}}{=} f_*(U);$$

- *Action on Morphisms.* For each $U, V \in \mathcal{P}(A)$:
 (★) If $U \subset V$, then $f_*(U) \subset f_*(V)$.

005J 2. *Triple Adjointness.* We have a triple adjunction

$$(f_* \dashv f^{-1} \dashv f_!): \mathcal{P}(A) \begin{array}{c} \xrightarrow{f_*} \\ \perp \\ \xleftarrow{f^{-1}} \\ \perp \\ \xrightarrow{f_!} \end{array} \mathcal{P}(B),$$

witnessed by bijections of sets

$$\begin{aligned} \mathrm{Hom}_{\mathcal{P}(B)}(f_*(U), V) &\cong \mathrm{Hom}_{\mathcal{P}(A)}(U, f^{-1}(V)), \\ \mathrm{Hom}_{\mathcal{P}(A)}(f^{-1}(U), V) &\cong \mathrm{Hom}_{\mathcal{P}(A)}(U, f_!(V)), \end{aligned}$$

natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$ and (respectively) $V \in \mathcal{P}(A)$ and $U \in \mathcal{P}(B)$, i.e. where:

(a) The following conditions are equivalent:

- i. We have $f_*(U) \subset V$.
- ii. We have $U \subset f^{-1}(V)$.

(b) The following conditions are equivalent:

- i. We have $f^{-1}(U) \subset V$.
- ii. We have $U \subset f_!(V)$.

005K 3. *Preservation of Colimits.* We have an equality of sets

$$f_*\left(\bigcup_{i \in I} U_i\right) = \bigcup_{i \in I} f_*(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have equalities

$$\begin{aligned} f_*(U) \cup f_*(V) &= f_*(U \cup V), \\ f_*(\emptyset) &= \emptyset, \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

005L 4. *Oplax Preservation of Limits.* We have an inclusion of sets

$$f_*\left(\bigcap_{i \in I} U_i\right) \subset \bigcap_{i \in I} f_*(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have inclusions

$$\begin{aligned} f_*(U \cap V) &\subset f_*(U) \cap f_*(V), \\ f_*(A) &\subset B, \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

- 005M** 5. *Symmetric Strict Monoidality With Respect to Unions.* The direct image function of **Item 1** has a symmetric strict monoidal structure

$$(f_*, f_*^\otimes, f_{*|\mathbb{K}}^\otimes): (\mathcal{P}(A), \cup, \emptyset) \rightarrow (\mathcal{P}(B), \cup, \emptyset),$$

being equipped with equalities

$$\begin{aligned} f_{*|U,V}^\otimes: f_*(U) \cup f_*(V) &\xrightarrow{=} f_*(U \cup V), \\ f_{*|\mathbb{K}}^\otimes: \emptyset &\xrightarrow{=} \emptyset, \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

- 005N** 6. *Symmetric Oplax Monoidality With Respect to Intersections.* The direct image function of **Item 1** has a symmetric oplax monoidal structure

$$(f_*, f_*^\otimes, f_{*|\mathbb{K}}^\otimes): (\mathcal{P}(A), \cap, A) \rightarrow (\mathcal{P}(B), \cap, B),$$

being equipped with inclusions

$$\begin{aligned} f_{*|U,V}^\otimes: f_*(U \cap V) &\hookrightarrow f_*(U) \cap f_*(V), \\ f_{*|\mathbb{K}}^\otimes: f_*(A) &\hookrightarrow B, \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

- 005P** 7. *Relation to Direct Images With Compact Support.* We have

$$f_*(U) = B \setminus f_!(A \setminus U)$$

for each $U \in \mathcal{P}(A)$.

Proof. Item 1, Functoriality: Clear.

Item 2, Triple Adjointness: This follows from **Kan Extensions**, ?? of ??.

Item 3, Preservation of Colimits: This follows from **Item 2** and **Categories**, ?? of ??.

Item 4, Oplax Preservation of Limits: Omitted.

Item 5, Symmetric Strict Monoidality With Respect to Unions: This follows from **Item 3**.

Item 6, Symmetric Oplax Monoidality With Respect to Intersections: This follows from ??.

Item 7, Relation to Direct Images With Compact Support: Applying ?? of ?? to $A \setminus U$, we have

$$\begin{aligned} f_!(A \setminus U) &= B \setminus f_*(A \setminus (A \setminus U)) \\ &= B \setminus f_*(U). \end{aligned}$$

Taking complements, we then obtain

$$\begin{aligned} f_*(U) &= B \setminus (B \setminus f_*(U)), \\ &= B \setminus f_!(A \setminus U), \end{aligned}$$

which finishes the proof. \square

005Q Proposition 4.3.1.4. Let $f: A \rightarrow B$ be a function.

005R 1. *Functionality I.* The assignment $f \mapsto f_*$ defines a function

$$(-)_{*|A,B}: \mathbf{Sets}(A, B) \rightarrow \mathbf{Sets}(\mathcal{P}(A), \mathcal{P}(B)).$$

005S 2. *Functionality II.* The assignment $f \mapsto f_*$ defines a function

$$(-)_{*|A,B}: \mathbf{Sets}(A, B) \rightarrow \mathbf{Pos}((\mathcal{P}(A), \subset), (\mathcal{P}(B), \subset)).$$

005T 3. *Interaction With Identities.* For each $A \in \mathbf{Obj}(\mathbf{Sets})$, we have

$$(\mathrm{id}_A)_* = \mathrm{id}_{\mathcal{P}(A)};$$

005U 4. *Interaction With Composition.* For each pair of composable functions $f: A \rightarrow B$ and $g: B \rightarrow C$, we have

$$(g \circ f)_* = g_* \circ f_*,$$

$$\begin{array}{ccc} \mathcal{P}(A) & \xrightarrow{f_*} & \mathcal{P}(B) \\ & \searrow (g \circ f)_* & \downarrow g_* \\ & & \mathcal{P}(C). \end{array}$$

Proof. **Item 1, Functionality I:** Clear.

Item 2, Functionality II: Clear.

Item 3, Interaction With Identities: This follows from **Kan Extensions**, ?? of ??.

Item 4, Interaction With Composition: This follows from **Kan Extensions**, ?? of ??.

\square

005V 4.4 Inverse Images

Let A and B be sets and let $f: A \rightarrow B$ be a function.

005W Definition 4.4.1.1. The **inverse image function associated to f** is the function³²

$$f^{-1}: \mathcal{P}(B) \rightarrow \mathcal{P}(A)$$

defined by³³

$$f^{-1}(V) \stackrel{\text{def}}{=} \{a \in A \mid \text{we have } f(a) \in V\}$$

for each $V \in \mathcal{P}(B)$.

005X Remark 4.4.1.2. Identifying subsets of B with functions from B to $\{\text{true}, \text{false}\}$ via **Item 6** of **Proposition 4.2.1.3**, we see that the inverse image function associated to f is equivalently the function

$$f^*: \mathcal{P}(B) \rightarrow \mathcal{P}(A)$$

defined by

$$f^*(\chi_V) \stackrel{\text{def}}{=} \chi_V \circ f$$

for each $\chi_V \in \mathcal{P}(B)$, where $\chi_V \circ f$ is the composition

$$A \xrightarrow{f} B \xrightarrow{\chi_V} \{\text{true}, \text{false}\}$$

in Sets.

005Y Proposition 4.4.1.3. Let $f: A \rightarrow B$ be a function.

005Z 1. *Functoriality.* The assignment $V \mapsto f^{-1}(V)$ defines a functor

$$f^{-1}: (\mathcal{P}(B), \subset) \rightarrow (\mathcal{P}(A), \subset)$$

where

- *Action on Objects.* For each $V \in \mathcal{P}(B)$, we have

$$[f^{-1}](V) \stackrel{\text{def}}{=} f^{-1}(V);$$

- *Action on Morphisms.* For each $U, V \in \mathcal{P}(B)$:

$$(\star) \text{ If } U \subset V, \text{ then } f^{-1}(U) \subset f^{-1}(V).$$

³²*Further Notation:* Also written $f^*: \mathcal{P}(B) \rightarrow \mathcal{P}(A)$.

³³*Further Terminology:* The set $f^{-1}(V)$ is called the **inverse image of V by f** .

0060 2. *Triple Adjointness.* We have a triple adjunction

$$(f_* \dashv f^{-1} \dashv f_!): \mathcal{P}(A) \begin{matrix} \xrightarrow{f_*} \\ \perp \\ \xleftarrow{f^{-1}} \\ \perp \\ \xrightarrow{f_!} \end{matrix} \mathcal{P}(B),$$

witnessed by bijections of sets

$$\begin{aligned} \mathrm{Hom}_{\mathcal{P}(B)}(f_*(U), V) &\cong \mathrm{Hom}_{\mathcal{P}(A)}(U, f^{-1}(V)), \\ \mathrm{Hom}_{\mathcal{P}(A)}(f^{-1}(U), V) &\cong \mathrm{Hom}_{\mathcal{P}(A)}(U, f_!(V)), \end{aligned}$$

natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$ and (respectively) $V \in \mathcal{P}(A)$ and $U \in \mathcal{P}(B)$, i.e. where:

(a) The following conditions are equivalent:

- i. We have $f_*(U) \subset V$;
- ii. We have $U \subset f^{-1}(V)$;

(b) The following conditions are equivalent:

- i. We have $f^{-1}(U) \subset V$.
- ii. We have $U \subset f_!(V)$.

0061 3. *Preservation of Colimits.* We have an equality of sets

$$f^{-1}\left(\bigcup_{i \in I} U_i\right) = \bigcup_{i \in I} f^{-1}(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(B)^{\times I}$. In particular, we have equalities

$$\begin{aligned} f^{-1}(U) \cup f^{-1}(V) &= f^{-1}(U \cup V), \\ f^{-1}(\emptyset) &= \emptyset, \end{aligned}$$

natural in $U, V \in \mathcal{P}(B)$.

0062 4. *Preservation of Limits.* We have an equality of sets

$$f^{-1}\left(\bigcap_{i \in I} U_i\right) = \bigcap_{i \in I} f^{-1}(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(B)^{\times I}$. In particular, we have equalities

$$\begin{aligned} f^{-1}(U) \cap f^{-1}(V) &= f^{-1}(U \cap V), \\ f^{-1}(B) &= A, \end{aligned}$$

natural in $U, V \in \mathcal{P}(B)$.

- 0063** 5. *Symmetric Strict Monoidality With Respect to Unions.* The inverse image function of **Item 1** has a symmetric strict monoidal structure

$$(f^{-1}, f^{-1, \otimes}, f_{\mathbb{K}}^{-1, \otimes}) : (\mathcal{P}(B), \cup, \emptyset) \rightarrow (\mathcal{P}(A), \cup, \emptyset),$$

being equipped with equalities

$$\begin{aligned} f_{U, V}^{-1, \otimes} : f^{-1}(U) \cup f^{-1}(V) &\xrightarrow{=} f^{-1}(U \cup V), \\ f_{\mathbb{K}}^{-1, \otimes} : \emptyset &\xrightarrow{=} f^{-1}(\emptyset), \end{aligned}$$

natural in $U, V \in \mathcal{P}(B)$.

- 0064** 6. *Symmetric Strict Monoidality With Respect to Intersections.* The inverse image function of **Item 1** has a symmetric strict monoidal structure

$$(f^{-1}, f^{-1, \otimes}, f_{\mathbb{K}}^{-1, \otimes}) : (\mathcal{P}(B), \cap, B) \rightarrow (\mathcal{P}(A), \cap, A),$$

being equipped with equalities

$$\begin{aligned} f_{U, V}^{-1, \otimes} : f^{-1}(U) \cap f^{-1}(V) &\xrightarrow{=} f^{-1}(U \cap V), \\ f_{\mathbb{K}}^{-1, \otimes} : A &\xrightarrow{=} f^{-1}(B), \end{aligned}$$

natural in $U, V \in \mathcal{P}(B)$.

Proof. Item 1, Functoriality: Clear.

Item 2, Triple Adjointness: This follows from **Kan Extensions**, ?? of ??.

Item 3, Preservation of Colimits: This follows from **Item 2** and **Categories**, ?? of ??.

Item 4, Preservation of Limits: This follows from **Item 2** and **Categories**, ?? of ??.

Item 5, Symmetric Strict Monoidality With Respect to Unions: This follows from **Item 3**.

Item 6, Symmetric Strict Monoidality With Respect to Intersections: This follows from **Item 4**. \square

0065 Proposition 4.4.1.4. Let $f: A \rightarrow B$ be a function.

0066 1. *Functionality I.* The assignment $f \mapsto f^{-1}$ defines a function

$$(-)_{A,B}^{-1}: \mathbf{Sets}(A, B) \rightarrow \mathbf{Sets}(\mathcal{P}(B), \mathcal{P}(A)).$$

0067 2. *Functionality II.* The assignment $f \mapsto f^{-1}$ defines a function

$$(-)_{A,B}^{-1}: \mathbf{Sets}(A, B) \rightarrow \mathbf{Pos}((\mathcal{P}(B), \subset), (\mathcal{P}(A), \subset)).$$

0068 3. *Interaction With Identities.* For each $A \in \mathbf{Obj}(\mathbf{Sets})$, we have

$$\mathrm{id}_A^{-1} = \mathrm{id}_{\mathcal{P}(A)};$$

0069 4. *Interaction With Composition.* For each pair of composable functions $f: A \rightarrow B$ and $g: B \rightarrow C$, we have

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}, \quad \begin{array}{ccc} \mathcal{P}(C) & \xrightarrow{g^{-1}} & \mathcal{P}(B) \\ & \searrow (g \circ f)^{-1} & \downarrow f^{-1} \\ & & \mathcal{P}(A). \end{array}$$

Proof. **Item 1, Functionality I:** Clear.

Item 2, Functionality II: Clear.

Item 3, Interaction With Identities: This follows from **Categories**, ?? of ??.

Item 4, Interaction With Composition: This follows from **Categories**, ?? of ??. \square

006A 4.5 Direct Images With Compact Support

Let A and B be sets and let $f: A \rightarrow B$ be a function.

006B Definition 4.5.1.1. The **direct image with compact support function associated to f** is the function³⁴

$$f_!: \mathcal{P}(A) \rightarrow \mathcal{P}(B)$$

³⁴*Further Notation:* Also written $\forall_f: \mathcal{P}(A) \rightarrow \mathcal{P}(B)$. This notation comes from the fact that the following statements are equivalent, where $b \in B$ and $U \in \mathcal{P}(A)$:

- We have $b \in \forall_f(U)$.

defined by^{35,36}

$$\begin{aligned} f_!(U) &\stackrel{\text{def}}{=} \left\{ b \in B \mid \begin{array}{l} \text{for each } a \in A, \text{ if we have} \\ f(a) = b, \text{ then } a \in U \end{array} \right\} \\ &= \{ b \in B \mid \text{we have } f^{-1}(b) \subset U \} \end{aligned}$$

for each $U \in \mathcal{P}(A)$.

006C Remark 4.5.1.2. Identifying subsets of A with functions from A to $\{\text{true}, \text{false}\}$ via **Item 6** of **Proposition 4.2.1.3**, we see that the direct image with compact support function associated to f is equivalently the function

$$f_! : \mathcal{P}(A) \rightarrow \mathcal{P}(B)$$

defined by

$$\begin{aligned} f_!(\chi_U) &\stackrel{\text{def}}{=} \text{Ran}_f(\chi_U) \\ &= \lim \left(\left(\underline{(-1)} \xrightarrow{\times} f \right) \xrightarrow{\text{pr}} A \xrightarrow{\chi_U} \{\text{true}, \text{false}\} \right) \\ &= \lim_{\substack{a \in A \\ f(a) = -1}} (\chi_U(a)) \\ &= \bigwedge_{\substack{a \in A \\ f(a) = -1}} (\chi_U(a)). \end{aligned}$$

So, in other words, we have

$$\begin{aligned} [f_!(\chi_U)](b) &= \bigwedge_{\substack{a \in A \\ f(a) = b}} (\chi_U(a)) \\ &= \begin{cases} \text{true} & \text{if, for each } a \in A \text{ such that} \\ & f(a) = b, \text{ we have } a \in U, \\ \text{false} & \text{otherwise} \end{cases} \\ &= \begin{cases} \text{true} & \text{if } f^{-1}(b) \subset U \\ \text{false} & \text{otherwise} \end{cases} \end{aligned}$$

for each $b \in B$.

-
- For each $a \in A$, if $b = f(a)$, then $a \in U$.

³⁵*Further Terminology:* The set $f_!(U)$ is called the **direct image with compact support of U by f** .

³⁶We also have

$$f_!(U) = B \setminus f_*(A \setminus U);$$

006D Definition 4.5.1.3. Let U be a subset of A .^{37,38}

1. The **image part of the direct image with compact support** $f_!(U)$ of U is the set $f_{!,\text{im}}(U)$ defined by

$$\begin{aligned} f_{!,\text{im}}(U) &\stackrel{\text{def}}{=} f_!(U) \cap \text{Im}(f) \\ &= \left\{ b \in B \mid \begin{array}{l} \text{we have } f^{-1}(b) \subset U \\ \text{and } f^{-1}(b) \neq \emptyset \end{array} \right\}. \end{aligned}$$

2. The **complement part of the direct image with compact support** $f_!(U)$ of U is the set $f_{!,\text{cp}}(U)$ defined by

$$\begin{aligned} f_{!,\text{cp}}(U) &\stackrel{\text{def}}{=} f_!(U) \cap (B \setminus \text{Im}(f)) \\ &= B \setminus \text{Im}(f) \\ &= \left\{ b \in B \mid \begin{array}{l} \text{we have } f^{-1}(b) \subset U \\ \text{and } f^{-1}(b) = \emptyset \end{array} \right\} \\ &= \{ b \in B \mid f^{-1}(b) = \emptyset \}. \end{aligned}$$

006E Example 4.5.1.4. Here are some examples of direct images with compact support.

1. *The Multiplication by Two Map on the Natural Numbers.* Consider the function $f: \mathbb{N} \rightarrow \mathbb{N}$ given by

$$f(n) \stackrel{\text{def}}{=} 2n$$

see **Item 7 of Proposition 4.5.1.5.**

³⁷Note that we have

$$f_!(U) = f_{!,\text{im}}(U) \cup f_{!,\text{cp}}(U),$$

as

$$\begin{aligned} f_!(U) &= f_!(U) \cap B \\ &= f_!(U) \cap (\text{Im}(f) \cup (B \setminus \text{Im}(f))) \\ &= (f_!(U) \cap \text{Im}(f)) \cup (f_!(U) \cap (B \setminus \text{Im}(f))) \\ &\stackrel{\text{def}}{=} f_{!,\text{im}}(U) \cup f_{!,\text{cp}}(U). \end{aligned}$$

³⁸In terms of the meet computation of $f_!(U)$ of **Remark 4.5.1.2**, namely

$$f_!(\chi_U) = \bigwedge_{\substack{a \in A \\ f(a) = -1}} (\chi_U(a)),$$

we see that $f_{!,\text{im}}$ corresponds to meets indexed over nonempty sets, while $f_{!,\text{cp}}$ corresponds

for each $n \in \mathbb{N}$. Since f is injective, we have

$$\begin{aligned} f_{!,\text{im}}(U) &= f_*(U) \\ f_{!,\text{cp}}(U) &= \{\text{odd natural numbers}\} \end{aligned}$$

for any $U \subset \mathbb{N}$.

2. *Parabolas.* Consider the function $f: \mathbb{R} \rightarrow \mathbb{R}$ given by

$$f(x) \stackrel{\text{def}}{=} x^2$$

for each $x \in \mathbb{R}$. We have

$$f_{!,\text{cp}}(U) = \mathbb{R}_{<0}$$

for any $U \subset \mathbb{R}$. Moreover, since $f^{-1}(x) = \{-\sqrt{x}, \sqrt{x}\}$, we have e.g.:

$$\begin{aligned} f_{!,\text{im}}([0, 1]) &= \{0\}, \\ f_{!,\text{im}}([-1, 1]) &= [0, 1], \\ f_{!,\text{im}}([1, 2]) &= \emptyset, \\ f_{!,\text{im}}([-2, -1] \cup [1, 2]) &= [1, 4]. \end{aligned}$$

3. *Circles.* Consider the function $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ given by

$$f(x, y) \stackrel{\text{def}}{=} x^2 + y^2$$

for each $(x, y) \in \mathbb{R}^2$. We have

$$f_{!,\text{cp}}(U) = \mathbb{R}_{<0}$$

for any $U \subset \mathbb{R}^2$, and since

$$f^{-1}(r) = \begin{cases} \text{a circle of radius } r \text{ about the origin} & \text{if } r > 0, \\ \{(0, 0)\} & \text{if } r = 0, \\ \emptyset & \text{if } r < 0, \end{cases}$$

we have e.g.:

$$\begin{aligned} f_{!,\text{im}}([-1, 1] \times [-1, 1]) &= [0, 1], \\ f_{!,\text{im}}([-1, 1] \times [-1, 1] \setminus [-1, 1] \times \{0\}) &= \emptyset. \end{aligned}$$

006F Proposition 4.5.1.5. Let $f: A \rightarrow B$ be a function.

006G 1. *Functoriality.* The assignment $U \mapsto f_!(U)$ defines a functor

$$f_!: (\mathcal{P}(A), \subset) \rightarrow (\mathcal{P}(B), \subset)$$

where

- *Action on Objects.* For each $U \in \mathcal{P}(A)$, we have

$$[f_!](U) \stackrel{\text{def}}{=} f_!(U);$$

- *Action on Morphisms.* For each $U, V \in \mathcal{P}(A)$:
 (★) If $U \subset V$, then $f_!(U) \subset f_!(V)$.

006H 2. *Triple Adjointness.* We have a triple adjunction

$$(f_* \dashv f^{-1} \dashv f_!): \mathcal{P}(A) \begin{array}{c} \xrightarrow{f_*} \\ \perp \\ \xleftarrow{f^{-1}} \\ \perp \\ \xrightarrow{f_!} \end{array} \mathcal{P}(B),$$

witnessed by bijections of sets

$$\begin{aligned} \text{Hom}_{\mathcal{P}(B)}(f_*(U), V) &\cong \text{Hom}_{\mathcal{P}(A)}(U, f^{-1}(V)), \\ \text{Hom}_{\mathcal{P}(A)}(f^{-1}(U), V) &\cong \text{Hom}_{\mathcal{P}(A)}(U, f_!(V)), \end{aligned}$$

natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$ and (respectively) $V \in \mathcal{P}(A)$ and $U \in \mathcal{P}(B)$, i.e. where:

- (a) The following conditions are equivalent:
 - i. We have $f_*(U) \subset V$;
 - ii. We have $U \subset f^{-1}(V)$;
- (b) The following conditions are equivalent:
 - i. We have $f^{-1}(U) \subset V$.
 - ii. We have $U \subset f_!(V)$.

to meets indexed over the empty set.

006J 3. *Lax Preservation of Colimits.* We have an inclusion of sets

$$\bigcup_{i \in I} f_!(U_i) \subset f_!\left(\bigcup_{i \in I} U_i\right),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have inclusions

$$\begin{aligned} f_!(U) \cup f_!(V) &\hookrightarrow f_!(U \cup V), \\ \emptyset &\hookrightarrow f_!(\emptyset), \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

006K 4. *Preservation of Limits.* We have an equality of sets

$$f_!\left(\bigcap_{i \in I} U_i\right) = \bigcap_{i \in I} f_!(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have equalities

$$\begin{aligned} f^{-1}(U \cap V) &= f^{-1}(U) \cap f^{-1}(V), \\ f_!(A) &= B, \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

006L 5. *Symmetric Lax Monoidality With Respect to Unions.* The direct image with compact support function of **Item 1** has a symmetric lax monoidal structure

$$\left(f_!, f_!^{\otimes}, f_{!|\mu}^{\otimes}\right): (\mathcal{P}(A), \cup, \emptyset) \rightarrow (\mathcal{P}(B), \cup, \emptyset),$$

being equipped with inclusions

$$\begin{aligned} f_{!|U,V}^{\otimes}: f_!(U) \cup f_!(V) &\hookrightarrow f_!(U \cup V), \\ f_{!|\mu}^{\otimes}: \emptyset &\hookrightarrow f_!(\emptyset), \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

006M 6. *Symmetric Strict Monoidality With Respect to Intersections.* The direct image function of **Item 1** has a symmetric strict monoidal structure

$$\left(f_!, f_!^{\otimes}, f_{!|\mu}^{\otimes}\right): (\mathcal{P}(A), \cap, A) \rightarrow (\mathcal{P}(B), \cap, B),$$

being equipped with equalities

$$\begin{aligned} f_{!|U,V}^\otimes &: f_!(U \cap V) \xrightarrow{=} f_!(U) \cap f_!(V), \\ f_{!|\mathbb{K}}^\otimes &: f_!(A) \xrightarrow{=} B, \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

006N 7. *Relation to Direct Images.* We have

$$f_!(U) = B \setminus f_*(A \setminus U)$$

for each $U \in \mathcal{P}(A)$.

006P 8. *Interaction With Injections.* If f is injective, then we have

$$\begin{aligned} f_{!,\text{im}}(U) &= f_*(U), \\ f_{!,\text{cp}}(U) &= B \setminus \text{Im}(f), \\ f_!(U) &= f_{!,\text{im}}(U) \cup f_{!,\text{cp}}(U) \\ &= f_*(U) \cup (B \setminus \text{Im}(f)) \end{aligned}$$

for each $U \in \mathcal{P}(A)$.

006Q 9. *Interaction With Surjections.* If f is surjective, then we have

$$\begin{aligned} f_{!,\text{im}}(U) &\subset f_*(U), \\ f_{!,\text{cp}}(U) &= \emptyset, \\ f_!(U) &\subset f_*(U) \end{aligned}$$

for each $U \in \mathcal{P}(A)$.

Proof. **Item 1, Functoriality:** Clear.

Item 2, Triple Adjointness: This follows from **Kan Extensions**, ?? of ??.

Item 3, Lax Preservation of Colimits: Omitted.

Item 4, Preservation of Limits: Omitted. This follows from **Item 2** and **Categories**, ?? of ??.

Item 5, Symmetric Lax Monoidality With Respect to Unions: This follows from ??.

Item 6, Symmetric Strict Monoidality With Respect to Intersections: This follows from **Item 4**.

Item 7, Relation to Direct Images: We claim that $f_!(U) = B \setminus f_*(A \setminus U)$.

- *The First Implication.* We claim that

$$f_!(U) \subset B \setminus f_*(A \setminus U).$$

Let $b \in f_!(U)$. We need to show that $b \notin f_*(A \setminus U)$, i.e. that there is no $a \in A \setminus U$ such that $f(a) = b$.

This is indeed the case, as otherwise we would have $a \in f^{-1}(b)$ and $a \notin U$, contradicting $f^{-1}(b) \subset U$ (which holds since $b \in f_!(U)$).

Thus $b \in B \setminus f_*(A \setminus U)$.

- *The Second Implication.* We claim that

$$B \setminus f_*(A \setminus U) \subset f_!(U).$$

Let $b \in B \setminus f_*(A \setminus U)$. We need to show that $b \in f_!(U)$, i.e. that $f^{-1}(b) \subset U$.

Since $b \notin f_*(A \setminus U)$, there exists no $a \in A \setminus U$ such that $b = f(a)$, and hence $f^{-1}(b) \subset U$.

Thus $b \in f_!(U)$.

This finishes the proof of **Item 7**.

Item 8, Interaction With Injections: Clear.

Item 9, Interaction With Surjections: Clear. □

006R Proposition 4.5.1.6. Let $f: A \rightarrow B$ be a function.

- 006S** 1. *Functionality I.* The assignment $f \mapsto f_!$ defines a function

$$(-)_{!|A,B}: \mathbf{Sets}(A, B) \rightarrow \mathbf{Sets}(\mathcal{P}(A), \mathcal{P}(B)).$$

- 006T** 2. *Functionality II.* The assignment $f \mapsto f_!$ defines a function

$$(-)_{!|A,B}: \mathbf{Sets}(A, B) \rightarrow \mathbf{Pos}((\mathcal{P}(A), \subset), (\mathcal{P}(B), \subset)).$$

- 006U** 3. *Interaction With Identities.* For each $A \in \mathbf{Obj}(\mathbf{Sets})$, we have

$$(\mathrm{id}_A)_! = \mathrm{id}_{\mathcal{P}(A)};$$

- 006V** 4. *Interaction With Composition.* For each pair of composable functions $f: A \rightarrow B$ and $g: B \rightarrow C$, we have

$$(g \circ f)_! = g_! \circ f_!,$$

$$\begin{array}{ccc} \mathcal{P}(A) & \xrightarrow{f_!} & \mathcal{P}(B) \\ & \searrow (g \circ f)_! & \downarrow g_! \\ & & \mathcal{P}(C). \end{array}$$

Proof. Item 1, Functionality I: Clear.

Item 2, Functionality II: Clear.

Item 3, Interaction With Identities: This follows from [Kan Extensions](#), ?? of ??.

Item 4, Interaction With Composition: This follows from [Kan Extensions](#), ?? of ??.

□

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