Indexed Sets

December 12, 2023

This chapter contains a discussion of indexed sets, the set-theoretical counterpart to indexed categories. In particular, here we explore:

1. Indexed sets, i.e. functors $K_{\mathsf{disc}} \to \mathsf{Sets}$ with K a set;
2. The limits and colimits in the category of K-indexed sets;
3. Constructions with indexed sets like dependent sums, dependent products, and internal Homs.

Contents

1 Indexed Sets

1.1 Foundations

Let K be a set.

Definition 1.1.1.1. A **K-indexed set** is a functor $X \colon K_{\mathsf{disc}} \to \mathsf{Sets}$.

Remark 1.1.1.2. By Categories, **??**, a **K-indexed set** consists of a K-indexed collection

$$X^{\dagger} \colon K \to \mathrm{Obj}(\mathsf{Sets}),$$

of sets, assigning a set $X^{\dagger}_x \overset{\text{def}}{=} X_x$ to each element x of K.

1.2 Morphisms of Indexed Sets

Let $X \colon K_{\mathsf{disc}} \to \mathsf{Sets}$ and $Y \colon K_{\mathsf{disc}} \to \mathsf{Sets}$ be indexed sets.

Definition 1.2.1.1. A **morphism of K-indexed sets from X to Y**[1] is a natural transformation

$$f \colon X \Longrightarrow Y, \qquad \begin{tikzcd} K_{\mathsf{disc}} \arrow[r, bend left, "X"{name=U}] \arrow[r, bend right, "Y"'{name=D}] & \mathsf{Sets} \arrow[from=U, to=D, Rightarrow, "f"'] \end{tikzcd}$$

from X to Y.

Remark 1.2.1.2. In detail, a **morphism of K-indexed sets** consists of a K-indexed collection

$$\{f_x \colon X_x \to Y_x\}_{x \in K}$$

of maps of sets.

[1] *Further Terminology:* Also called a **K-indexed map of sets from X to Y**.

1.3 The Category of Sets Indexed by a Fixed Set

Let K be a set.

Definition 1.3.1.1. The **category of K-indexed sets** is the category $\mathsf{ISets}(K)$ defined by

$$\mathsf{ISets}(K) \stackrel{\text{def}}{=} \mathsf{Fun}\big(K_{\mathsf{disc}}, \mathsf{Sets}\big).$$

Remark 1.3.1.2. In detail, the **category of K-indexed sets** is the category $\mathsf{ISets}(K)$ where

- *Objects.* The objects of $\mathsf{ISets}(K)$ are K-indexed sets as in Definition 1.1.1.1;
- *Morphisms.* The morphisms of $\mathsf{ISets}(K)$ are morphisms of K-indexed sets as in Definition 1.2.1.1;
- *Identities.* For each $X \in \mathrm{Obj}(\mathsf{ISets}(K))$, the unit map

  $$\mathbb{1}^{\mathsf{ISets}(K)}_X : \mathrm{pt} \to \mathrm{Hom}_{\mathsf{ISets}(K)}(X, X)$$

  of $\mathsf{ISets}(K)$ at X is defined by

  $$\mathrm{id}^{\mathsf{ISets}(K)}_X \stackrel{\text{def}}{=} \{\mathrm{id}_{X_x}\}_{x \in K};$$

- *Composition.* For each $X, Y, Z \in \mathrm{Obj}(\mathsf{ISets}(K))$, the composition map

  $$\circ^{\mathsf{ISets}(K)}_{X,Y,Z} : \mathrm{Hom}_{\mathsf{ISets}(K)}(Y, Z) \times \mathrm{Hom}_{\mathsf{ISets}(K)}(X, Y) \to \mathrm{Hom}_{\mathsf{ISets}(K)}(X, Z)$$

  of $\mathsf{ISets}(K)$ at (X, Y, Z) is defined by

  $$\{g_x\}_{x \in K} \circ^{\mathsf{ISets}(K)}_{X,Y,Z} \{f_x\}_{x \in K} \stackrel{\text{def}}{=} \{g_x \circ f_x\}_{x \in K}.$$

1.4 The Category of Indexed Sets

Definition 1.4.1.1. The **category of indexed sets** is the category ISets defined as the Grothendieck construction of the functor $\mathsf{ISets} \colon \mathsf{Sets}^{\mathsf{op}} \to \mathsf{Cats}$ of Proposition 4.1.1.4:

$$\mathsf{ISets} \stackrel{\text{def}}{=} \int^{\mathsf{Sets}} \mathsf{ISets}.$$

Remark 1.4.1.2. In detail, the **category of indexed sets** is the category ISets where

- *Objects.* The objects of ISets are pairs (K, X) consisting of
  - *The Indexing Set.* A set K;
  - *The Indexed Set.* A K-indexed set $X\colon K_{\mathsf{disc}} \to \mathsf{Sets}$;
- *Morphisms.* A morphism of ISets from (K, X) to (K', Y) is a pair (ϕ, f) consisting of
  - *The Reindexing Map.* A map of sets $\phi\colon K \to K'$;
  - *The Morphism of Indexed Sets.* A morphism of K-indexed sets $f\colon X \to \phi_*(Y)$ as in the diagram

$$f\colon X \to \phi_*(Y), \qquad \begin{tikzcd} K_{\mathsf{disc}} \arrow[rr, "\phi"] \arrow[dr, "X"'] & \arrow[d, phantom, "\overset{f}{\Rightarrow}"] & K'_{\mathsf{disc}} \arrow[dl, "Y"] \\ & \mathsf{Sets}; & \end{tikzcd}$$

- *Identities.* For each $(K, X) \in \mathrm{Obj}(\mathsf{ISets})$, the unit map

$$\mathbb{1}^{\mathsf{ISets}}_{(K,X)}\colon \mathrm{pt} \to \mathsf{ISets}((K, X), (K, X))$$

  of ISets at (K, X) is defined by

$$\mathrm{id}^{\mathsf{ISets}}_{(K,X)} \overset{\mathrm{def}}{=} (\mathrm{id}_K, \mathrm{id}_X).$$

- *Composition.* For each $\mathbf{X} = (K, X)$, $\mathbf{Y} = (K', Y)$, $\mathbf{Z} = (K'', Z) \in \mathrm{Obj}(\mathsf{ISets})$, the composition map

$$\circ^{\mathsf{ISets}}_{\mathbf{X},\mathbf{Y},\mathbf{Z}}\colon \mathsf{ISets}(\mathbf{Y}, \mathbf{Z}) \times \mathsf{ISets}(\mathbf{X}, \mathbf{Y}) \to \mathsf{ISets}(\mathbf{X}, \mathbf{Z})$$

  of ISets at $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ is defined by

$$(\psi, g) \circ (\phi, f) \overset{\mathrm{def}}{=} (\psi \circ \phi, (g \circ \mathrm{id}_\phi) \circ f),$$

  as in the diagram

$$\begin{tikzcd} K_{\mathsf{disc}} \arrow[r, "\phi"] \arrow[dr, "X"'] \arrow[r, phantom, shift right=4, "\overset{f}{\Rightarrow}"] & K'_{\mathsf{disc}} \arrow[r, "\psi"] \arrow[d, "Y"] \arrow[r, phantom, shift right=4, "\overset{g}{\Rightarrow}"] & K''_{\mathsf{disc}} \arrow[dl, "Z"] \\ & \mathsf{Sets}; & \end{tikzcd}$$

  for each $(\phi, f) \in \mathsf{ISets}(\mathbf{X}, \mathbf{Y})$ and each $(\psi, g) \in \mathsf{ISets}(\mathbf{Y}, \mathbf{Z})$.

2 Limits of Indexed Sets

2.1 Products of K-Indexed Sets

Let $X \colon K_{\mathsf{disc}} \to \mathsf{Sets}$ and $Y \colon K_{\mathsf{disc}} \to \mathsf{Sets}$ be indexed sets.

Definition 2.1.1.1. The **product of** X **and** Y is the K-indexed set $X \times Y \colon K_{\mathsf{disc}} \to \mathsf{Sets}$ defined by

$$(X \times Y)_k \stackrel{\text{def}}{=} X_k \times Y_k$$

for each $k \in K$.

Proof. That this agrees with the categorical product in $\mathsf{ISets}(K)$ follows from Limits and Colimits, **??** of **??**. □

2.2 Pullbacks of K-Indexed Sets

Let $X, Y, Z \colon K_{\mathsf{disc}} \to \mathsf{Sets}$ be K-indexed sets and let $f \colon X \to Z$ and $g \colon Y \to Z$ be morphisms of K-indexed sets.

Definition 2.2.1.1. The **pullback of** X **and** Y **over** Z is the K-indexed set $X \times_Z Y \colon K_{\mathsf{disc}} \to \mathsf{Sets}$ defined by

$$(X \times_Z Y)_k \stackrel{\text{def}}{=} X_k \times_{Z_k} Y_k$$

for each $k \in K$.

Proof. That this agrees with the categorical pullback in $\mathsf{ISets}(K)$ follows from Limits and Colimits, **??** of **??**. □

2.3 Equalisers of K-Indexed Sets

Let $X, Y \colon K_{\mathsf{disc}} \to \mathsf{Sets}$ be K-indexed sets and let $f, g \colon X \rightrightarrows Y$ be morphisms of K-indexed sets.

Definition 2.3.1.1. The **equaliser of** f **and** g is the K-indexed set $\mathrm{Eq}(f,g) \colon K_{\mathsf{disc}} \to \mathsf{Sets}$ defined by

$$(\mathrm{Eq}(f,g))_k \stackrel{\text{def}}{=} \mathrm{Eq}(f_k, g_k)$$

for each $k \in K$.

Proof. That this agrees with the categorical equaliser in $\mathsf{ISets}(K)$ follows from Limits and Colimits, **??** of **??**. □

2.4 Products in ISets

Let $X\colon K_{\mathsf{disc}} \to \mathsf{Sets}$ and $Y\colon K'_{\mathsf{disc}} \to \mathsf{Sets}$ be indexed sets.

Definition 2.4.1.1. The **product of X and Y** is the $(K \times K')$-indexed set

$$X \times Y\colon (K \times K')_{\mathsf{disc}} \to \mathsf{Sets}$$

defined by

$$(X \times Y)_{(k,k')} \overset{\text{def}}{=} X_k \times Y_{k'}$$

for each $(k, k') \in K \times K'$.

Proof. We claim that this agrees with the categorical product in ISets. □

2.5 Pullbacks in ISets

Let $X\colon K_{\mathsf{disc}} \to \mathsf{Sets}$ be a K-indexed set, let $Y\colon K'_{\mathsf{disc}} \to \mathsf{Sets}$ be a K'-indexed set, let $Z\colon K''_{\mathsf{disc}} \to \mathsf{Sets}$ be a K''-indexed set, and let $(\phi, f)\colon X \to Z$ and $(\psi, g)\colon Y \to Z$ be morphisms of indexed sets (as in Remark 1.4.1.2).

Definition 2.5.1.1. The **pullback of X and Y over Z** is the $(K \times_{K''} K)$-indexed set

$$X \times_Z Y\colon (K \times_{K''} K)_{\mathsf{disc}} \to \mathsf{Sets}$$

defined by

$$\begin{aligned}(X \times_Z Y)_{(k,k')} &\overset{\text{def}}{=} X_k \times_{Z_{\phi(k)}} Y_{k'}\\ &\overset{\text{def}}{=} X_k \times_{Z_{\psi(k)}} Y_{k'}\end{aligned}$$

for each $(k, k') \in K \times_{K''} K'$.

Proof. We claim that this agrees with the categorical pullback in ISets. □

2.6 Equalisers in ISets

Let $X\colon K_{\mathsf{disc}} \to \mathsf{Sets}$ be a K-indexed set, let $Y\colon K'_{\mathsf{disc}} \to \mathsf{Sets}$ be a K'-indexed set, and let $(\phi, f), (\psi, g)\colon X \to Y$ be morphisms of indexed sets (as in Remark 1.4.1.2).

Definition 2.6.1.1. The **equaliser of (ϕ, f) and (ψ, g)** is the $\mathrm{Eq}(\phi, \psi)$-indexed set $\mathrm{Eq}(f, g)\colon \mathrm{Eq}(\phi, \psi) \to \mathsf{Sets}$ defined by

$$(\mathrm{Eq}(f, g))_k \overset{\text{def}}{=} \mathrm{Eq}(f_k, g_k)$$

for each $k \in \mathrm{Eq}(\phi, \psi)$.

Proof. We claim that this agrees with the categorical equaliser in ISets. □

3 Colimits of Indexed Sets

3.1 Coproducts of K-Indexed Sets

Let $X \colon K_{\mathsf{disc}} \to \mathsf{Sets}$ and $Y \colon K_{\mathsf{disc}} \to \mathsf{Sets}$ be indexed sets.

Definition 3.1.1.1. The **coproduct** of X and Y is the K-k-indexed set $X \coprod Y \colon K_{\mathsf{disc}} \to \mathsf{Sets}$ defined by

$$(X \coprod Y)_k \stackrel{\text{def}}{=} X_k \coprod Y_k$$

for each $k \in K$.

Proof. That this agrees with the categorical coproduct in $\mathsf{ISets}(K)$ follows from Limits and Colimits, **??** of **??**. □

3.2 Pushouts of K-Indexed Sets

Let $X, Y, Z \colon K_{\mathsf{disc}} \to \mathsf{Sets}$ be K-indexed sets and let $f \colon Z \to X$ and $g \colon Z \to Y$ be morphisms of K-indexed sets.

Definition 3.2.1.1. The **pushout** of X and Y is the K-indexed set $X \coprod_Z Y \colon K_{\mathsf{disc}} \to \mathsf{Sets}$ defined by

$$(X \coprod\nolimits_Z Y)_k \stackrel{\text{def}}{=} X_k \coprod\nolimits_{Z_k} Y_k$$

for each $k \in K$.

Proof. That this agrees with the categorical pushout in $\mathsf{ISets}(K)$ follows from Limits and Colimits, **??** of **??**. □

3.3 Coequalisers of K-Indexed Sets

Let $X, Y \colon K_{\mathsf{disc}} \to \mathsf{Sets}$ be K-indexed sets and let $f, g \colon X \rightrightarrows Y$ be morphisms of K-indexed sets.

Definition 3.3.1.1. The **coequaliser** of X and Y is the K-indexed set $\mathrm{CoEq}(f, g) \colon K_{\mathsf{disc}} \to \mathsf{Sets}$ defined by

$$(\mathrm{CoEq}(f, g))_k \stackrel{\text{def}}{=} \mathrm{CoEq}(f_k, g_k)$$

for each $k \in K$.

Proof. That this agrees with the categorical coequaliser in $\mathsf{ISets}(K)$ follows from Limits and Colimits, **??** of **??**. □

4 Constructions With Indexed Sets

4.1 Change of Indexing

Let $\phi\colon K \to K'$ be a function and let X be a K'-indexed set.

Definition 4.1.1.1. The **change of indexing of X to K** is the K-indexed set $\phi^*(X)$ defined by

$$\phi^*(X) \overset{\text{def}}{=} X \circ \phi_{\mathsf{disc}}.$$

Remark 4.1.1.2. In detail, the **change of indexing of X to K** is the K-indexed set $\phi^*(X)$ defined by

$$\phi^*(X)_x \overset{\text{def}}{=} X_{\phi(x)}$$

for each $x \in K$.

Proposition 4.1.1.3. The assignment $X \mapsto \phi^*(X)$ defines a functor

$$\phi^*\colon \mathsf{ISets}(K') \to \mathsf{ISets}(K),$$

where

- *Action on Objects.* For each $X \in \mathrm{Obj}(\mathsf{ISets}(K'))$, we have

$$[\phi^*](X) \overset{\text{def}}{=} \phi^*(X);$$

- *Action on Morphisms.* For each $X, Y \in \mathrm{Obj}(\mathsf{ISets}(K'))$, the action on Hom-sets

$$\phi^*_{X,Y}\colon \mathrm{Hom}_{\mathsf{ISets}(K')}(X, Y) \to \mathrm{Hom}_{\mathsf{ISets}(K)}(\phi^*(X), \phi^*(Y))$$

  of ϕ^* at (X, Y) is the map sending a morphism of K'-indexed sets

$$f = \{f_x\colon X_x \to Y_x\}_{x\in K'}$$

  from X to Y to the morphism of K-indexed sets defined by

$$\phi^*(f) \overset{\text{def}}{=} \left\{f_{\phi(x)}\colon X_{\phi(x)} \to Y_{\phi(x)}\right\}_{x\in K}.$$

Proof. Omitted. □

Proposition 4.1.1.4. The assignment $K \mapsto \mathsf{ISets}(K)$ defines a functor

$$\mathsf{ISets}\colon \mathsf{Sets}^{\mathsf{op}} \to \mathsf{Cats},$$

where

- *Action on Objects.* For each $K \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$[\mathsf{ISets}](K) \stackrel{\text{def}}{=} \mathsf{ISets}(K);$$

- *Action on Morphisms.* For each $K, K' \in \mathrm{Obj}(\mathsf{Sets})$, the action on Hom-sets

$$\mathsf{ISets}_{K,K'} \colon \mathsf{Sets}^{\mathsf{op}}(K, K') \to \mathsf{Fun}(\mathsf{ISets}(K), \mathsf{ISets}(K'))$$

  of ISets at (K, K') is the map defined by

$$\mathsf{ISets}_{K,K'}(\phi) \stackrel{\text{def}}{=} \phi^*$$

  for each $\phi \in \mathsf{Sets}^{\mathsf{op}}(K, K')$.

Proof. Omitted. □

4.2 Dependent Sums

Let $\phi \colon K \to K'$ be a function and let X be a K-indexed set.

Definition 4.2.1.1. The **dependent sum of** X is the K'-indexed set $\Sigma_\phi(X)$[2] defined by

$$\Sigma_\phi(X) \stackrel{\text{def}}{=} \mathrm{Lan}_\phi(X),$$

and hence given by

$$\Sigma_\phi(X)_x \cong \coprod_{y \in \phi^{-1}(x)} X_y$$

for each $x \in K'$.

Proposition 4.2.1.2. The assignment $X \mapsto \Sigma_\phi(X)$ defines a functor

$$\Sigma_\phi \colon \mathsf{ISets}(K) \to \mathsf{ISets}(K'),$$

where

- *Action on Objects.* For each $X \in \mathrm{Obj}(\mathsf{ISets}(K))$, we have

$$[\Sigma_\phi](X) \stackrel{\text{def}}{=} \Sigma_\phi(X);$$

- *Action on Morphisms.* For each $X, Y \in \mathrm{Obj}(\mathsf{ISets}(K))$, the action on

[2] *Further Notation:* Also written $\phi_*(X)$.

Hom-sets

$$\Sigma_{\phi|X,Y}: \mathrm{Hom}_{\mathsf{ISets}(K)}(X,Y) \to \mathrm{Hom}_{\mathsf{ISets}(K')}(\Sigma_\phi(X),\Sigma_\phi(Y))$$

of Σ_ϕ at (X,Y) is the map sending a morphism of K-indexed sets

$$f\colon X \to Y$$

to the morphism of K'-indexed sets defined by

$$\begin{aligned}\Sigma_\phi(f) &\stackrel{\text{def}}{=} \mathrm{Lan}_\phi(f);\\ &\cong \coprod_{y\in\phi^{-1}(X)} f_y.\end{aligned}$$

Proof. Omitted. □

4.3 Dependent Products

Let $\phi\colon K \to K'$ be a function and let X be a K-indexed set.

Definition 4.3.1.1. The **dependent product of** X is the K'-indexed set $\Pi_\phi(X)$[3] defined by

$$\Pi_\phi(X) \stackrel{\text{def}}{=} \mathrm{Ran}_\phi(X),$$

and hence given by

$$\Pi_\phi(X)_x \cong \prod_{y\in\phi^{-1}(x)} X_y$$

for each $x \in K'$.

Proposition 4.3.1.2. The assignment $X \mapsto \Pi_\phi(X)$ defines a functor

$$\Pi_\phi\colon \mathsf{ISets}(K) \to \mathsf{ISets}(K'),$$

where

- *Action on Objects.* For each $X \in \mathrm{Obj}(\mathsf{ISets}(K))$, we have

$$[\Pi_\phi](X) \stackrel{\text{def}}{=} \Pi_\phi(X);$$

- *Action on Morphisms.* For each $X, Y \in \mathrm{Obj}(\mathsf{ISets}(K))$, the action on

[3] *Further Notation:* Also written $\phi_!(X)$.

Hom-sets

$$\Pi_{\phi|X,Y} \colon \mathrm{Hom}_{\mathsf{ISets}(K)}(X,Y) \to \mathrm{Hom}_{\mathsf{ISets}(K')}(\Pi_\phi(X), \Pi_\phi(Y))$$

of Π_ϕ at (X,Y) is the map sending a morphism of K-indexed sets

$$f \colon X \to Y$$

to the morphism of K'-indexed sets defined by

$$\begin{aligned}\Pi_\phi(f) &\stackrel{\text{def}}{=} \mathrm{Ran}_\phi(f);\\ &\cong \prod_{y\in\phi^{-1}(x)} f_y.\end{aligned}$$

Proof. Omitted. □

4.4 Internal Homs

Let K be a set and let X and Y be K-indexed sets.

Definition 4.4.1.1. The **internal Hom of indexed sets from X to Y** is the indexed set $\mathbf{Hom}_{\mathsf{ISets}(K)}(X,Y)$ defined by

$$\mathbf{Hom}_{\mathsf{ISets}(K)}(X,Y) \stackrel{\text{def}}{=} \mathsf{Sets}(X_x, Y_x)$$

for each $x \in K$.

4.5 Adjointness of Indexed Sets

Let $\phi \colon K \to K'$ be a map of sets.

Proposition 4.5.1.1. We have a triple adjunction

$$(\Sigma_\phi \dashv \phi^* \dashv \Pi_\phi)\colon\quad \begin{tikzcd} \mathsf{ISets}(K) \arrow[r, bend left=45, "\Sigma_\phi"] \arrow[r, bend right=45, "\Pi_\phi"'] & \mathsf{ISets}(K') \arrow[l, "\phi^*" description] \end{tikzcd}.$$

Proof. This follows from Kan Extensions, **??** of **??**. □

Appendices

A Other Chapters

Probability Theory

Stochastic Analysis

Differential Geometry

Schemes