

Un/Straightening for Indexed and Fibred Sets

December 24, 2023

This chapter contains a discussion of the un/straightening equivalence in the context of sets, as well as a general discussion of indexed and fibred sets. In particular, it contains:

1. A discussion of indexed sets (i.e. functors $K_{\text{disc}} \rightarrow \text{Sets}$ with K a set), constructions with them like dependent sums and dependent products, and their properties (???)
2. A discussion of fibred sets (i.e. maps of sets $X \rightarrow K$), constructions with them like dependent sums and dependent products, and their properties (???)
3. A discussion of the un/straightening equivalence for indexed and fibred sets (??).

Contents

1 Un/Straightening for Indexed and Fibred Sets

1.1 Straightening for Fibred Sets

Let K be a set and let (X, ϕ) be a K -fibred set.

Definition 1.1.1.1. The **straightening of** (X, ϕ) is the K -indexed set

$$\text{St}_K(X, \phi) : K_{\text{disc}} \rightarrow \text{Sets}$$

defined by

$$\text{St}_K(X, \phi)_x \stackrel{\text{def}}{=} \phi^{-1}(x)$$

for each $x \in K$.

Proposition 1.1.1.2. Let K be a set.

1. *Functoriality.* The assignment $(X, \phi) \mapsto \text{St}_K(X, \phi)$ defines a functor

$$\text{St}_K : \text{FibSets}(K) \rightarrow \text{ISets}(K)$$

- *Action on Objects.* For each $(X, \phi) \in \text{Obj}(\text{FibSets}(K))$, we have

$$[\text{St}_K](X, \phi) \stackrel{\text{def}}{=} \text{St}_K(X, \phi);$$

- *Action on Morphisms.* For each $(X, \phi), (Y, \psi) \in \text{Obj}(\text{FibSets}(K))$, the action on Hom-sets

$$\text{St}_{K|X,Y} : \text{Hom}_{\text{FibSets}(K)}(X, Y) \rightarrow \text{Hom}_{\text{ISets}(K)}(\text{St}_K(X), \text{St}_K(Y))$$

of St_K at (X, Y) is given by sending a morphism

$$f : (X, \phi) \rightarrow (Y, \psi)$$

of K -fibred sets to the morphism

$$\text{St}_K(f) : \text{St}_K(X, \phi) \rightarrow \text{St}_K(Y, \psi)$$

of K -indexed sets defined by

$$\text{St}_K(f) \stackrel{\text{def}}{=} \{f_x^*\}_{x \in K},$$

where f_x^* is the transport map associated to f at $x \in K$ of ??.

2. *Interaction With Change of Base/Indexing.* Let $f : K \rightarrow K'$ be a map of sets. The diagram

$$\begin{array}{ccc} \text{FibSets}(K') & \xrightarrow{f^*} & \text{FibSets}(K) \\ \text{St}_{K'} \downarrow & & \downarrow \text{St}_K \\ \text{ISets}(K') & \xrightarrow{f^*} & \text{ISets}(K) \end{array}$$

commutes.

3. *Interaction With Dependent Sums.* Let $f : K \rightarrow K'$ be a map of sets. The diagram

$$\begin{array}{ccc} \text{FibSets}(K) & \xrightarrow{\Sigma_f} & \text{FibSets}(K') \\ \text{St}_K \downarrow & & \downarrow \text{St}_{K'} \\ \text{ISets}(K) & \xrightarrow{\Sigma_f} & \text{ISets}(K') \end{array}$$

commutes.

4. *Interaction With Dependent Products.* Let $f: K \rightarrow K'$ be a ~~map~~ **map** of sets. The diagram

$$\begin{array}{ccc} \mathbf{Sets}_{/K} & \xrightarrow{\Pi_f} & \mathbf{FibSets}(K') \\ \text{St}_K \downarrow & & \downarrow \text{St}_{K'} \\ \mathbf{ISets}(K) & \xrightarrow{\Pi_f} & \mathbf{ISets}(K') \end{array}$$

commutes.

Proof. **??**, *Functoriality*: Omitted.

??, *Interaction With Change of Base/Indexing*: Indeed, we have

$$\begin{aligned} \text{St}_K(f^*(X, \phi))_x &\stackrel{\text{def}}{=} \text{St}_K(K \times_{K'} X)_x \\ &\stackrel{\text{def}}{=} (\text{pr}_1^{K \times_{K'} X})^{-1}(x) \\ &= \{(k, y) \in K \times_{K'} X \mid \text{pr}_1^{K \times_{K'} X}(k, y) = x\} \\ &= \{(k, y) \in K \times_{K'} X \mid k = x\} \\ &= \{(k, y) \in K \times X \mid k = x \text{ and } f(k) = \phi(y)\} \\ &\cong \{y \in X \mid \phi(y) = f(x)\} \\ &= \phi^{-1}(f(x)) \\ &\stackrel{\text{def}}{=} f^*(\phi^{-1}(x)) \\ &\stackrel{\text{def}}{=} f^*(\text{St}_{K'}(X, \phi)_x) \end{aligned}$$

for each $(X, \phi) \in \text{Obj}(\mathbf{FibSets}(K'))$ and each $x \in K$, and similarly for morphisms.

??, *Interaction With Dependent Sums*: Indeed, we have

$$\begin{aligned} \text{St}_{K'}(\Sigma_f(X, \phi))_x &\stackrel{\text{def}}{=} \Sigma_f(\phi)^{-1}(x) \\ &\cong \coprod_{\substack{y \in X \\ f(y)=x}} \phi^{-1}(y) \\ &\cong \Sigma_f(\phi^{-1}(x)) \\ &\stackrel{\text{def}}{=} \Sigma_f(\text{St}_K(X, \phi)_x) \end{aligned}$$

for each $(X, \phi) \in \text{Obj}(\mathbf{FibSets}(K))$ and each $x \in K'$, where we have used **??** of **??** for the first bijection, and similarly for morphisms.

??, *Interaction With Dependent Products*: Indeed, we have

$$\begin{aligned}
 \mathrm{St}_{K'}(\Pi_f(X, \phi))_x &\stackrel{\mathrm{def}}{=} \Pi_f(\phi)^{-1}(x) \\
 &\cong \prod_{\substack{y \in X \\ f(y)=x}} \phi^{-1}(y) \\
 &\cong \Pi_f(\phi^{-1}(x)) \\
 &\stackrel{\mathrm{def}}{=} \Pi_f(\mathrm{St}_K(X, \phi)_x)
 \end{aligned}$$

for each $(X, \phi) \in \mathrm{Obj}(\mathrm{FibSets}(K))$ and each $x \in K'$, where we have used ?? of ?? for the first bijection, and similarly for morphisms. \square

1.2 Unstraightening for Indexed Sets

Let K be a set and let X be a K -indexed set.

Definition 1.2.1.1. The **unstraightening** Un_K is the K -fibred set

$$\phi_{\mathrm{Un}_K} : \mathrm{Un}_K(X) \rightarrow K$$

consisting of

- *The Underlying Set.* The set $\mathrm{Un}_K(X)$ defined by

$$\mathrm{Un}_K(X) \stackrel{\mathrm{def}}{=} \coprod_{x \in K} X_x;$$

- *The Fibration.* The map of sets

$$\phi_{\mathrm{Un}_K} : \mathrm{Un}_K(X) \rightarrow K$$

defined by sending an element of $\coprod_{x \in K} X_x$ to its index in K .

Proposition 1.2.1.2. Let K be a set. Un_K

1. *Functoriality.* The assignment $X \mapsto \mathrm{Un}_K(X)$ defines a functor

$$\mathrm{Un}_K : \mathrm{ISets}(K) \rightarrow \mathrm{FibSets}(K)$$

- *Action on Objects.* For each $X \in \mathrm{Obj}(\mathrm{ISets}(K))$, we have

$$[\mathrm{Un}_K](X) \stackrel{\mathrm{def}}{=} \mathrm{Un}_K(X);$$

- *Action on Morphisms.* For each $X, Y \in \text{Obj}(\text{ISets}(K))$, the action on Hom-sets

$$\text{Un}_K|_{X,Y} : \text{Hom}_{\text{ISets}(K)}(X, Y) \rightarrow \text{Hom}_{\text{FibSets}(K)}(\text{Un}_K(X), \text{Un}_K(Y))$$

of Un_K at (X, Y) is defined by

$$\text{Un}_K|_{X,Y}(f) \stackrel{\text{def}}{=} \prod_{x \in K} f_x^*.$$

2. *Interaction With Fibres.* We have a bijection of sets [9.4.1 J](#)

$$\phi_{\text{Un}_K}^{-1}(x) \cong X_x$$

for each $x \in K$.

3. *As a Pullback.* We have a bijection of sets [9.4.1 K](#)

$$\begin{array}{ccc} \text{Un}_K(X) & \rightarrow & \text{Sets}_* \\ \downarrow \lrcorner & & \downarrow \text{忘} \\ K_{\text{disc}} & \xrightarrow{X} & \text{Sets}. \end{array}$$

$\text{Un}_K(X) \cong K_{\text{disc}} \times_{\text{Sets}} \text{Sets}_*,$

4. *As a Colimit.* We have a bijection of sets [9.4.1 L](#)

$$\text{Un}_K(X) \cong \text{colim}(X).$$

5. *Interaction With Change of Indexing/Base.* Let $f : K \rightarrow K'$ be a map of sets. The diagram [9.4.1 F](#)

$$\begin{array}{ccc} \text{ISets}(K') & \xrightarrow{f^*} & \text{ISets}(K) \\ \text{Un}_{K'} \downarrow & & \downarrow \text{Un}_K \\ \text{FibSets}(K') & \xrightarrow{f^*} & \text{FibSets}(K) \end{array}$$

commutes.

6. *Interaction With Dependent Sums.* Let $f : K \rightarrow K'$ be a map of sets. The diagram [9.4.1 F](#)

$$\begin{array}{ccc} \text{ISets}(K) & \xrightarrow{\Sigma_f} & \text{ISets}(K') \\ \text{Un}_K \downarrow & & \downarrow \text{Un}_{K'} \\ \text{FibSets}(K) & \xrightarrow{\Sigma_f} & \text{FibSets}(K') \end{array}$$

commutes.

7. *Interaction With Dependent Products.* Let $f: K \rightarrow K'$ be a ~~map~~ ^{map} of sets. The diagram

$$\begin{array}{ccc} \mathbf{ISets}(K) & \xrightarrow{\Pi_f} & \mathbf{ISets}(K') \\ \mathbf{Un}_K \downarrow & & \downarrow \mathbf{Un}_{K'} \\ \mathbf{FibSets}(K) & \xrightarrow{\Pi_f} & \mathbf{FibSets}(K') \end{array}$$

commutes.

Proof. **??**, *Functoriality*: Omitted.

??, *Interaction With Fibres*: Omitted.

??, *As a Pullback*: Omitted.

??, *As a Colimit*: Clear.

??, *Interaction With Change of Indexing/Base*: Indeed, we have

$$\begin{aligned} \mathbf{Un}_K(f^*(X)) &\stackrel{\text{def}}{=} \mathbf{Un}_K(X \circ f) \\ &\stackrel{\text{def}}{=} \coprod_{x \in K} X_{f(x)} \\ &\cong \left\{ (x, (y, a)) \in K \times \coprod_{y \in K'} X_y \mid f(x) = y \right\} \\ &\cong K \times_{K'} \coprod_{y \in K'} X_y \\ &\stackrel{\text{def}}{=} K \times_{K'} \mathbf{Un}_{K'}(X) \\ &\stackrel{\text{def}}{=} f^*(\mathbf{Un}_{K'}(X)) \end{aligned}$$

for each $X \in \mathbf{Obj}(\mathbf{ISets}(K'))$. Similarly, it can be shown that we also have $\mathbf{Un}_K(f^*(\phi)) = f^*(\mathbf{Un}_{K'}(\phi))$ and that $\mathbf{Un}_K \circ f^* = f^* \circ \mathbf{Un}_{K'}$ also holds on morphisms.

??, *Interaction With Dependent Sums*: Indeed, we have

$$\begin{aligned} \mathbf{Un}_{K'}(\Sigma_f(X)) &\stackrel{\text{def}}{=} \coprod_{x \in K'} \Sigma_f(X)_x \\ &\cong \coprod_{x \in K'} \coprod_{y \in f^{-1}(x)} X_y \\ &\cong \coprod_{y \in K} X_y \\ &\cong \mathbf{Un}_K(X) \\ &\stackrel{\text{def}}{=} \Sigma_f(\mathbf{Un}_K(X)) \end{aligned}$$

for each $X \in \text{Obj}(\text{ISets}(K))$, where we have used ?? of ?? for the first bijection. Similarly, it can be shown that we also have $\text{Un}_{K'}(\Sigma_f(\phi)) = \Sigma_f(\phi_{\text{Un}_K})$ and that $\text{Un}_{K'} \circ \Sigma_f = \Sigma_f \circ \text{Un}_K$ also holds on morphisms.

??, *Interaction With Dependent Products*: Indeed, we have

$$\begin{aligned}
 \text{Un}_{K'}(\Pi_f(X)) &\stackrel{\text{def}}{=} \coprod_{x \in K'} \Pi_f(X)_x \\
 &\cong \coprod_{x \in K'} \prod_{y \in f^{-1}(x)} X_y \\
 &\cong \left\{ (x, h) \in \coprod_{x \in K'} \text{Sets}(f^{-1}(x), \phi_{\text{Un}_K}^{-1}(f^{-1}(x))) \mid \phi \circ h = \text{id}_{f^{-1}(x)} \right\} \\
 &\stackrel{\text{def}}{=} \Pi_f\left(\coprod_{y \in K} X_y\right) \\
 &\stackrel{\text{def}}{=} \Pi_f(\text{Un}_K(X))
 \end{aligned}$$

for each $X \in \text{Obj}(\text{ISets}(K))$, where we have used ?? of ?? for the first bijection. Similarly, it can be shown that we also have $\text{Un}_{K'}(\Pi_f(\phi)) = \Pi_f(\phi_{\text{Un}_K})$ and that $\text{Un}_{K'} \circ \Pi_f = \Pi_f \circ \text{Un}_K$ also holds on morphisms. \square

1.3 The Un/Straightening Equivalence

Theorem 1.3.1.1. We have an isomorphism of categories

$$(\text{St}_K \dashv \text{Un}_K): \text{FibSets}(K) \begin{array}{c} \xrightarrow{\text{St}_K} \\ \perp \\ \xleftarrow{\text{Un}_K} \end{array} \text{ISets}(K).$$

Proof. Omitted. \square

2 Miscellany

2.1 Other Kinds of Un/Straightening

Remark 2.1.1.1. There are also other kinds of un/straightening for sets, where Sets is replaced by **Rel** or Span:

- *Un/Straightening With Rel*, I. We have an isomorphism of sets

$$\text{Rel}(A, B) \cong \text{Sets}(B \times A, \{\text{true}, \text{false}\}).$$

by the definition of a relation from A to B , **Relations**, **Definition 1.1.1.1**.

- *Un/Straightening With Rel, II.* We have an equivalence of categories

$$\text{LaxFun}(K_{\text{disc}}, \mathbf{Rel}) \stackrel{\text{eq.}}{\cong} \text{Cats}_{/K_{\text{disc}}}^{\text{fth}},$$

where $\text{Cats}_{/K_{\text{disc}}}^{\text{fth}}$ is the full subcategory of $\text{Cats}_{/K_{\text{disc}}}$ spanned by the faithful functors; see [niefield:change-of-base-for-relational-variable-sets].

- *Un/Straightening With Span, I.* For each $A, B \in \text{Obj}(\text{Sets})$, we have a morphism of sets

$$\text{Span}(A, B) \rightarrow \text{Sets}(A \times B, \mathbb{N} \cup \{\infty\})$$

which assemble into an equivalence of categories between $\text{Span}(\text{Sets})$ and the category \mathbf{MRel} of “multirelations”; see [Spans](#), [Remark 7.5.1.1](#).

- *Un/Straightening With Span, II.* We have an equivalence of categories

$$\text{LaxFun}(K_{\text{disc}}, \text{Span}) \stackrel{\text{eq.}}{\cong} \text{Cats}_{/K_{\text{disc}}}:$$

see [nlab:displayed-category].

Appendices

A Other Chapters

Sets

1. [Sets](#)
2. [Constructions With Sets](#)
3. [Pointed Sets](#)
4. [Tensor Products of Pointed Sets](#)
5. [Relations](#)
6. [Spans](#)
7. [Posets](#)

Indexed and Fibred Sets

7. [Indexed Sets](#)

8. [Fibred Sets](#)

9. [Un/Straightening for Indexed and Fibred Sets](#)

Category Theory

11. [Categories](#)
12. [Types of Morphisms in Categories](#)
13. [Adjunctions and the Yoneda Lemma](#)
14. [Constructions With Categories](#)
15. [Kan Extensions](#)

Bicategories

17. [Bicategories](#)
18. [Internal Adjunctions](#)

Internal Category Theory

- 19. Internal Categories

Cyclic Stuff

- 20. The Cycle Category

Cubical Stuff

- 21. The Cube Category

Globular Stuff

- 22. The Globe Category

Cellular Stuff

- 23. The Cell Category

Monoids

- 24. Monoids
- 25. Constructions With Monoids

Monoids With Zero

- 26. Monoids With Zero
- 27. Constructions With Monoids With Zero

Groups

- 28. Groups
- 29. Constructions With Groups

Hyper Algebra

- 30. Hypermonoids
- 31. Hypergroups

- 32. Hypersemirings and Hyperrings

- 33. Quantaes

Near-Rings

- 34. Near-Semirings
- 35. Near-Rings

Real Analysis

- 36. Real Analysis in One Variable
- 37. Real Analysis in Several Variables

Measure Theory

- 38. Measurable Spaces
- 39. Measures and Integration

Probability Theory

- 39. Probability Theory

Stochastic Analysis

- 40. Stochastic Processes, Martingales, and Brownian Motion
- 41. Itô Calculus
- 42. Stochastic Differential Equations

Differential Geometry

- 43. Topological and Smooth Manifolds

Schemes

- 44. Schemes