

Un/Straightening for Indexed and Fibred Sets

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This chapter contains a discussion of the un/straightening equivalence in the context of sets, as well as a general discussion of indexed and fibred sets. In particular, it contains:

1. A discussion of indexed sets (i.e. functors $K_{\text{disc}} \rightarrow \mathbf{Sets}$ with K a set), constructions with them like dependent sums and dependent products, and their properties (????);
2. A discussion of fibred sets (i.e. maps of sets $X \rightarrow K$), constructions with them like dependent sums and dependent products, and their properties (????);
3. A discussion of the un/straightening equivalence for indexed and fibred sets ([Section 1](#)).

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1 Un/Straightening for Indexed and Fibred Sets

1.1 Straightening for Fibred Sets

Let K be a set and let (X, ϕ) be a K -fibred set.

Definition 1.1.1.1. The **straightening of** (X, ϕ) is the K -indexed set

$$\mathrm{St}_K(X, \phi): K_{\mathrm{disc}} \rightarrow \mathbf{Sets}$$

defined by

$$\mathrm{St}_K(X, \phi)_x \stackrel{\mathrm{def}}{=} \phi^{-1}(x)$$

for each $x \in K$.

Proposition 1.1.1.2. Let K be a set.

1. *Functoriality.* The assignment $(X, \phi) \mapsto \mathrm{St}_K(X, \phi)$ defines a functor

$$\mathrm{St}_K: \mathbf{FibSets}(K) \rightarrow \mathbf{ISets}(K)$$

- *Action on Objects.* For each $(X, \phi) \in \mathbf{Obj}(\mathbf{FibSets}(K))$, we have

$$[\mathrm{St}_K](X, \phi) \stackrel{\mathrm{def}}{=} \mathrm{St}_K(X, \phi);$$

- *Action on Morphisms.* For each $(X, \phi), (Y, \psi) \in \mathbf{Obj}(\mathbf{FibSets}(K))$, the action on Hom-sets

$$\mathrm{St}_{K|X,Y}: \mathbf{Hom}_{\mathbf{FibSets}(K)}(X, Y) \rightarrow \mathbf{Hom}_{\mathbf{ISets}(K)}(\mathrm{St}_K(X), \mathrm{St}_K(Y))$$

of St_K at (X, Y) is given by sending a morphism

$$f: (X, \phi) \rightarrow (Y, \psi)$$

of K -fibred sets to the morphism

$$\mathrm{St}_K(f): \mathrm{St}_K(X, \phi) \rightarrow \mathrm{St}_K(Y, \psi)$$

of K -indexed sets defined by

$$\mathrm{St}_K(f) \stackrel{\mathrm{def}}{=} \{f_x^*\}_{x \in K},$$

where f_x^* is the transport map associated to f at $x \in K$ of ??.

2. *Interaction With Change of Base/Indexing.* Let $f: K \rightarrow K'$ be a map

of sets. The diagram

$$\begin{array}{ccc} \text{FibSets}(K') & \xrightarrow{f^*} & \text{FibSets}(K) \\ \text{St}_{K'} \downarrow & & \downarrow \text{St}_K \\ \text{ISets}(K') & \xrightarrow{f^*} & \text{ISets}(K) \end{array}$$

commutes.

3. *Interaction With Dependent Sums.* Let $f: K \rightarrow K'$ be a map of sets. The diagram

$$\begin{array}{ccc} \text{FibSets}(K) & \xrightarrow{\Sigma_f} & \text{FibSets}(K') \\ \text{St}_K \downarrow & & \downarrow \text{St}_{K'} \\ \text{ISets}(K) & \xrightarrow{\Sigma_f} & \text{ISets}(K') \end{array}$$

commutes.

4. *Interaction With Dependent Products.* Let $f: K \rightarrow K'$ be a map of sets. The diagram

$$\begin{array}{ccc} \text{Sets}/_K & \xrightarrow{\Pi_f} & \text{FibSets}(K') \\ \text{St}_K \downarrow & & \downarrow \text{St}_{K'} \\ \text{ISets}(K) & \xrightarrow{\Pi_f} & \text{ISets}(K') \end{array}$$

commutes.

Proof. **Item 1, Functoriality:** Omitted.

Item 2, Interaction With Change of Base/Indexing: Indeed, we have

$$\begin{aligned}
\mathrm{St}_K(f^*(X, \phi))_x &\stackrel{\mathrm{def}}{=} \mathrm{St}_K(K \times_{K'} X)_x \\
&\stackrel{\mathrm{def}}{=} \left(\mathrm{pr}_1^{K \times_{K'} X} \right)^{-1}(x) \\
&= \left\{ (k, y) \in K \times_{K'} X \mid \mathrm{pr}_1^{K \times_{K'} X}(k, y) = x \right\} \\
&= \{ (k, y) \in K \times_{K'} X \mid k = x \} \\
&= \{ (k, y) \in K \times X \mid k = x \text{ and } f(k) = \phi(y) \} \\
&\cong \{ y \in X \mid \phi(y) = f(x) \} \\
&= \phi^{-1}(f(x)) \\
&\stackrel{\mathrm{def}}{=} f^*(\phi^{-1}(x)) \\
&\stackrel{\mathrm{def}}{=} f^*(\mathrm{St}_{K'}(X, \phi)_x)
\end{aligned}$$

for each $(X, \phi) \in \mathrm{Obj}(\mathrm{FibSets}(K'))$ and each $x \in K$, and similarly for morphisms.

Item 3, Interaction With Dependent Sums: Indeed, we have

$$\begin{aligned}
\mathrm{St}_{K'}(\Sigma_f(X, \phi))_x &\stackrel{\mathrm{def}}{=} \Sigma_f(\phi)^{-1}(x) \\
&\cong \coprod_{\substack{y \in X \\ f(y)=x}} \phi^{-1}(y) \\
&\cong \Sigma_f(\phi^{-1}(x)) \\
&\stackrel{\mathrm{def}}{=} \Sigma_f(\mathrm{St}_K(X, \phi)_x)
\end{aligned}$$

for each $(X, \phi) \in \mathrm{Obj}(\mathrm{FibSets}(K))$ and each $x \in K'$, where we have used ?? of ?? for the first bijection, and similarly for morphisms.

Item 4, Interaction With Dependent Products: Indeed, we have

$$\begin{aligned}
\mathrm{St}_{K'}(\Pi_f(X, \phi))_x &\stackrel{\mathrm{def}}{=} \Pi_f(\phi)^{-1}(x) \\
&\cong \prod_{\substack{y \in X \\ f(y)=x}} \phi^{-1}(y) \\
&\cong \Pi_f(\phi^{-1}(x)) \\
&\stackrel{\mathrm{def}}{=} \Pi_f(\mathrm{St}_K(X, \phi)_x)
\end{aligned}$$

for each $(X, \phi) \in \mathrm{Obj}(\mathrm{FibSets}(K))$ and each $x \in K'$, where we have used ?? of ?? for the first bijection, and similarly for morphisms. \square

1.2 Unstraightening for Indexed Sets

Let K be a set and let X be a K -indexed set.

Definition 1.2.1.1. The **unstraightening of X** is the K -fibred set

$$\phi_{\mathrm{Un}_K} : \mathrm{Un}_K(X) \rightarrow K$$

consisting of

- *The Underlying Set.* The set $\mathrm{Un}_K(X)$ defined by

$$\mathrm{Un}_K(X) \stackrel{\mathrm{def}}{=} \coprod_{x \in K} X_x;$$

- *The Fibration.* The map of sets

$$\phi_{\mathrm{Un}_K} : \mathrm{Un}_K(X) \rightarrow K$$

defined by sending an element of $\coprod_{x \in K} X_x$ to its index in K .

Proposition 1.2.1.2. Let K be a set.

1. *Functoriality.* The assignment $X \mapsto \mathrm{Un}_K(X)$ defines a functor

$$\mathrm{Un}_K : \mathbf{ISets}(K) \rightarrow \mathbf{FibSets}(K)$$

- *Action on Objects.* For each $X \in \mathbf{Obj}(\mathbf{ISets}(K))$, we have

$$[\mathrm{Un}_K](X) \stackrel{\mathrm{def}}{=} \mathrm{Un}_K(X);$$

- *Action on Morphisms.* For each $X, Y \in \mathbf{Obj}(\mathbf{ISets}(K))$, the action on Hom-sets

$$\mathrm{Un}_{K|X,Y} : \mathbf{Hom}_{\mathbf{ISets}(K)}(X, Y) \rightarrow \mathbf{Hom}_{\mathbf{FibSets}(K)}(\mathrm{Un}_K(X), \mathrm{Un}_K(Y))$$

of Un_K at (X, Y) is defined by

$$\mathrm{Un}_{K|X,Y}(f) \stackrel{\mathrm{def}}{=} \prod_{x \in K} f_x^*.$$

2. *Interaction With Fibres.* We have a bijection of sets

$$\phi_{\mathrm{Un}_K}^{-1}(x) \cong X_x$$

for each $x \in K$.

3. *As a Pullback.* We have a bijection of sets

$$\begin{array}{ccc} & \text{Un}_K(X) \rightarrow \text{Sets}_* & \\ & \downarrow \lrcorner & \downarrow \text{忘} \\ \text{Un}_K(X) \cong K_{\text{disc}} \times_{\text{Sets}} \text{Sets}_*, & K_{\text{disc}} & \xrightarrow{X} \text{Sets}. \end{array}$$

4. *As a Colimit.* We have a bijection of sets

$$\text{Un}_K(X) \cong \text{colim}(X).$$

5. *Interaction With Change of Indexing/Base.* Let $f: K \rightarrow K'$ be a map of sets. The diagram

$$\begin{array}{ccc} \text{ISets}(K') & \xrightarrow{f^*} & \text{ISets}(K) \\ \text{Un}_{K'} \downarrow & & \downarrow \text{Un}_K \\ \text{FibSets}(K') & \xrightarrow{f^*} & \text{FibSets}(K) \end{array}$$

commutes.

6. *Interaction With Dependent Sums.* Let $f: K \rightarrow K'$ be a map of sets. The diagram

$$\begin{array}{ccc} \text{ISets}(K) & \xrightarrow{\Sigma_f} & \text{ISets}(K') \\ \text{Un}_K \downarrow & & \downarrow \text{Un}_{K'} \\ \text{FibSets}(K) & \xrightarrow{\Sigma_f} & \text{FibSets}(K') \end{array}$$

commutes.

7. *Interaction With Dependent Products.* Let $f: K \rightarrow K'$ be a map of sets. The diagram

$$\begin{array}{ccc} \text{ISets}(K) & \xrightarrow{\Pi_f} & \text{ISets}(K') \\ \text{Un}_K \downarrow & & \downarrow \text{Un}_{K'} \\ \text{FibSets}(K) & \xrightarrow{\Pi_f} & \text{FibSets}(K') \end{array}$$

commutes.

Proof. Item 1, Functoriality: Omitted.

Item 2, Interaction With Fibres: Omitted.

Item 3, As a Pullback: Omitted.

Item 4, As a Colimit: Clear.

Item 5, Interaction With Change of Indexing/Base: Indeed, we have

$$\begin{aligned}
 \mathrm{Un}_K(f^*(X)) &\stackrel{\mathrm{def}}{=} \mathrm{Un}_K(X \circ f) \\
 &\stackrel{\mathrm{def}}{=} \coprod_{x \in K} X_{f(x)} \\
 &\cong \left\{ (x, (y, a)) \in K \times \coprod_{y \in K'} X_y \mid f(x) = y \right\} \\
 &\cong K \times_{K'} \coprod_{y \in K'} X_y \\
 &\stackrel{\mathrm{def}}{=} K \times_{K'} \mathrm{Un}_{K'}(X) \\
 &\stackrel{\mathrm{def}}{=} f^*(\mathrm{Un}_{K'}(X))
 \end{aligned}$$

for each $X \in \mathrm{Obj}(\mathbf{lSets}(K'))$. Similarly, it can be shown that we also have $\mathrm{Un}_K(f^*(\phi)) = f^*(\mathrm{Un}_{K'}(\phi))$ and that $\mathrm{Un}_K \circ f^* = f^* \circ \mathrm{Un}_{K'}$ also holds on morphisms.

Item 6, Interaction With Dependent Sums: Indeed, we have

$$\begin{aligned}
 \mathrm{Un}_{K'}(\Sigma_f(X)) &\stackrel{\mathrm{def}}{=} \coprod_{x \in K'} \Sigma_f(X)_x \\
 &\cong \coprod_{x \in K'} \coprod_{y \in f^{-1}(x)} X_y \\
 &\cong \coprod_{y \in K} X_y \\
 &\cong \mathrm{Un}_K(X) \\
 &\stackrel{\mathrm{def}}{=} \Sigma_f(\mathrm{Un}_K(X))
 \end{aligned}$$

for each $X \in \mathrm{Obj}(\mathbf{lSets}(K))$, where we have used ?? of ?? for the first bijection. Similarly, it can be shown that we also have $\mathrm{Un}_{K'}(\Sigma_f(\phi)) = \Sigma_f(\phi_{\mathrm{Un}_K})$ and that $\mathrm{Un}_{K'} \circ \Sigma_f = \Sigma_f \circ \mathrm{Un}_K$ also holds on morphisms.

Item 7, Interaction With Dependent Products: Indeed, we have

$$\begin{aligned}
\mathrm{Un}_{K'}(\Pi_f(X)) &\stackrel{\mathrm{def}}{=} \prod_{x \in K'} \Pi_f(X)_x \\
&\cong \prod_{x \in K'} \prod_{y \in f^{-1}(x)} X_y \\
&\cong \left\{ (x, h) \in \prod_{x \in K'} \mathrm{Sets}(f^{-1}(x), \phi_{\mathrm{Un}_K}^{-1}(f^{-1}(x))) \mid \phi \circ h = \mathrm{id}_{f^{-1}(x)} \right\} \\
&\stackrel{\mathrm{def}}{=} \Pi_f \left(\prod_{y \in K} X_y \right) \\
&\stackrel{\mathrm{def}}{=} \Pi_f(\mathrm{Un}_K(X))
\end{aligned}$$

for each $X \in \mathrm{Obj}(\mathbf{ISets}(K))$, where we have used ?? of ?? for the first bijection. Similarly, it can be shown that we also have $\mathrm{Un}_{K'}(\Pi_f(\phi)) = \Pi_f(\phi_{\mathrm{Un}_K})$ and that $\mathrm{Un}_{K'} \circ \Pi_f = \Pi_f \circ \mathrm{Un}_K$ also holds on morphisms. \square

1.3 The Un/Straightening Equivalence

Theorem 1.3.1.1. We have an isomorphism of categories

$$(\mathrm{St}_K \dashv \mathrm{Un}_K): \mathrm{FibSets}(K) \begin{array}{c} \xrightarrow{\mathrm{St}_K} \\ \perp \\ \xleftarrow{\mathrm{Un}_K} \end{array} \mathbf{ISets}(K).$$

Proof. Omitted. \square

2 Miscellany

2.1 Other Kinds of Un/Straightening

Remark 2.1.1.1. There are also other kinds of un/straightening for sets, where **Sets** is replaced by **Rel** or **Span**:

- *Un/Straightening With Rel, I.* We have an isomorphism of sets

$$\mathrm{Rel}(A, B) \cong \mathrm{Sets}(B \times A, \{\mathrm{true}, \mathrm{false}\}).$$

by the definition of a relation from A to B , **Relations**, **Definition 1.1.1.1**.

- *Un/Straightening With Rel, II.* We have an equivalence of categories

$$\mathrm{LaxFun}(K_{\mathrm{disc}}, \mathbf{Rel}) \stackrel{\mathrm{eq.}}{\cong} \mathrm{Cats}_{/K_{\mathrm{disc}}}^{\mathrm{fth}},$$

where $\mathbf{Cats}_{/K_{\text{disc}}}^{\text{fth}}$ is the full subcategory of $\mathbf{Cats}_{/K_{\text{disc}}}$ spanned by the faithful functors; see [Nie04, Theorem 3.1].

- *Un/Straightening With Span, I.* For each $A, B \in \text{Obj}(\mathbf{Sets})$, we have a morphism of sets

$$\text{Span}(A, B) \rightarrow \mathbf{Sets}(A \times B, \mathbb{N} \cup \{\infty\})$$

which assemble into an equivalence of categories between $\text{Span}(\mathbf{Sets})$ and the category \mathbf{MRel} of “multirelations”; see [Spans](#), [Remark 7.5.1.1](#).

- *Un/Straightening With Span, II.* We have an equivalence of categories

$$\text{LaxFun}(K_{\text{disc}}, \text{Span}) \stackrel{\text{eq.}}{\cong} \mathbf{Cats}_{/K_{\text{disc}}};$$

see [nLa23, Section 3].

Appendices

A Other Chapters

Set Theory

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2. [Constructions With Sets](#)
3. [Pointed Sets](#)
4. [Tensor Products of Pointed Sets](#)
5. Indexed and Fibred Sets
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Category Theory

9. [Categories](#)
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Bicategories

12. [Bicategories](#)
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Internal Category Theory

14. [Internal Categories](#)

Cyclic Stuff

15. [The Cycle Category](#)

Cubical Stuff

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Globular Stuff

17. [The Globe Category](#)

Cellular Stuff

18. The Cell Category

Monoids

19. Monoids
20. Constructions With Monoids

Monoids With Zero

21. Monoids With Zero
22. Constructions With Monoids With Zero

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23. Groups
24. Constructions With Groups

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25. Hypermonoids
26. Hypergroups
27. Hypersemirings and Hyperrings
28. Quantales

Near-Rings

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30. Near-Rings

Real Analysis

31. Real Analysis in One Variable
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33. Measurable Spaces
34. Measures and Integration

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34. Probability Theory

Stochastic Analysis

35. Stochastic Processes, Martingales, and Brownian Motion
36. Itô Calculus
37. Stochastic Differential Equations

Differential Geometry

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39. Schemes