

# Spans

December 3, 2023

00QB This chapter contains some material about spans. Notably, we discuss and explore:

1. The basic definitions around spans ([Section 1](#));
2. The relation between spans and functions ([Proposition 7.1.1.1](#));
3. The relation between spans and relations ([Propositions 7.2.2.1](#) and [7.3.1.1](#) and [Remark 7.5.1.1](#)).
4. “Hyperpointed sets” (??). I don’t know why I wrote this...

TODO:

1. internal adjoint equivalences in **Rel**
2. internal adjoint equivalences in **Span**
3. 2-categorical limits in **Rel**;
4. morphism of internal adjunctions in **Rel**;
5. morphism of internal adjunctions in **Span**;
6. morphism of co/monads in **Span**;
7. What is  $\text{Adj}(\text{Span}(A, B))$ ?
8. monoids, comonoids, pseudomonoids, etc. in **Span**.
9. write down the dumb intuition about spans inducing morphisms  $\text{Sets}(S, A) \rightarrow \text{Sets}(S, B)$  instead of  $\mathcal{P}(A) \rightarrow \mathcal{P}(B)$  from the similarity between

$$S \rightarrow A \times B$$

and

$$A \times B \rightarrow \{t, f\}.$$

This intuition is justified by taking  $A = \text{pt}$  or  $B = \text{pt}$ .

10. What about using the direct image with compact support in  $g(f^{-1}(a))$ ?
11. Monads in Span | develop this in the level of morphisms too
12. Comonads in Span are spans whose legs are equal | develop this in the level of morphisms too
13. Does Span have an internal **Hom**?
14. Examples of spans
15. Functional and total spans
16. closed symmetric monoidal category of spans
17. double category of relations
18. collage of a span
19. equivalence spans?
20. functoriality of powersets for spans
21. Is Span a closed bicategory?
22. skew monoidal structure on  $\text{Span}(A, B)$
23. Adjunctions in Span
24. Isomorphisms in Span
25. Equivalences in Span
26. Interaction between the above notions in Span vs. in **Rel** via the comparison functors

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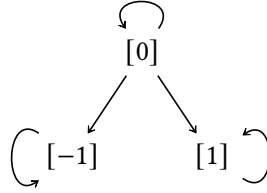
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## 00QC 1 Spans

### 00QD 1.1 The Walking Span

00QE **Definition 1.1.1.1.** The **walking span** is the category  $\Lambda$  that looks like this:



## 00QF 1.2 Spans

Let  $A$  and  $B$  be sets.

00QG **Definition 1.2.1.1.** A **span from  $A$  to  $B$** <sup>1</sup> is a functor  $F: \Lambda \rightarrow \text{Sets}$  such that

$$\begin{aligned} F([-1]) &= A, \\ F([1]) &= B. \end{aligned}$$

00QH **Remark 1.2.1.2.** In detail, a **span from  $A$  to  $B$**  is a triple  $(S, f, g)$  consisting of<sup>2,3</sup>

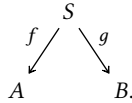
- *The Underlying Set.* A set  $S$ , called the **underlying set of  $(S, f, g)$** ;
- *The Legs.* A pair of functions  $f: S \rightarrow A$  and  $g: S \rightarrow B$ .

## 00QJ 1.3 Morphisms of Spans

00QK **Definition 1.3.1.1.** A **morphism of spans  $(R, f_1, g_1)$  to  $(S, f_2, g_2)$** <sup>4</sup> is a natural transformation  $(R, f_1, g_1) \Rightarrow (S, f_2, g_2)$ .

<sup>1</sup>Further Terminology: Also called a **roof from  $A$  to  $B$**  or a **correspondence from  $A$  to  $B$** .

<sup>2</sup>Picture:



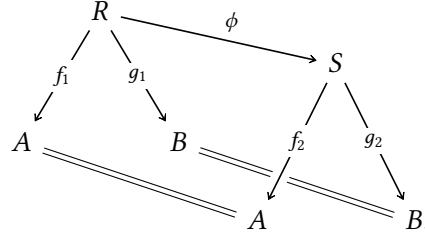
<sup>3</sup>Every span  $(S, f, g)$  from  $A$  to  $B$  determines in particular a relation  $R: A \nrightarrow B$  via

$$R \stackrel{\text{def}}{=} \{(f(a), g(a)) \mid a \in A\},$$

i.e. where  $R(a) = g(f^{-1}(a))$  for each  $a \in A$ ; see **Proposition 7.2.2.1**.

<sup>4</sup>Further Terminology: Also called a **morphism of roofs from  $(R, f_1, g_1)$  to  $(S, f_2, g_2)$**  or a **morphism of correspondences from  $(R, f_1, g_1)$  to  $(S, f_2, g_2)$** .

**00QL Remark 1.3.1.2.** In detail, a **morphism of spans from**  $(R, f_1, g_1)$  **to**  $(S, f_2, g_2)$  **is a** function  $\phi: R \rightarrow S$  **making the diagram**<sup>5</sup>



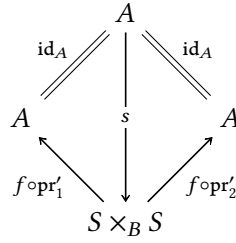
commute.

#### 00QM 1.4 Functional Spans

Let  $\lambda = \left( A \xleftarrow{f} S \xrightarrow{g} B \right)$  be a span. A morphism of spans from  $\text{id}_A$  to  $\lambda \diamond \lambda^\dagger$  is a morphism

$$s: A \rightarrow S \times_B S$$

making the diagram



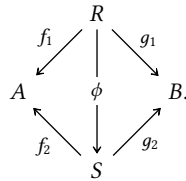
commute, where  $S \times_B S$  is the pullback

$$S \times_B S \cong \{(s, t) \in S \times S \mid g(s) = g(t)\}$$

$$\begin{array}{ccc} S \times_B S & \longrightarrow & S \\ \downarrow & \lrcorner & \downarrow g \\ S & \xrightarrow{g} & B \end{array}$$

of  $S$  with itself along  $g$ .

<sup>5</sup>Alternative Picture:



## 00QN 1.5 Total Spans

## 00QP 2 Categories of Spans

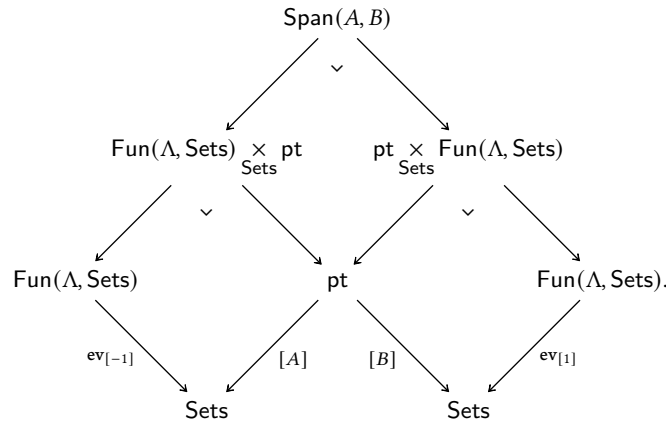
### 00QQ 2.1 Categories of Spans

Let  $A$  and  $B$  be sets.

**00QR Definition 2.1.1.1.** The **category of spans from  $A$  to  $B$**  is the category  $\text{Span}(A, B)$  defined by

$$\text{Span}(A, B) \stackrel{\text{def}}{=} \text{Fun}(\Lambda, \text{Sets}) \times_{\text{ev}_{[-1]}, \text{Sets}, [A]}^{\text{pt}} \times_{[B], \text{Sets}, \text{ev}_{[1]}}^{\text{pt}} \text{Fun}(\Lambda, \text{Sets}),$$

as in the diagram



**00QS Remark 2.1.1.2.** In detail, the **category of spans from  $A$  to  $B$**  is the category  $\text{Span}(A, B)$  where

- *Objects.* The objects of  $\text{Span}(A, B)$  are spans from  $A$  to  $B$ ;
- *Morphisms.* The morphism of  $\text{Span}(A, B)$  are morphisms of spans;
- *Identities.* The unit map

$$\mathbb{K}_{(S, f, g)}^{\text{Span}(A, B)} : \text{pt} \rightarrow \text{Hom}_{\text{Span}(A, B)}((S, f, g), (S, f, g))$$

of  $\text{Span}(A, B)$  at  $(S, f, g)$  is defined by<sup>6</sup>

$$\text{id}_{(S, f, g)}^{\text{Span}(A, B)} \stackrel{\text{def}}{=} \text{id}_S;$$

- *Composition.* The composition map

$$\circ_{R, S, T}^{\text{Span}(A, B)} : \text{Hom}_{\text{Span}(A, B)}(S, T) \times \text{Hom}_{\text{Span}(A, B)}(R, S) \rightarrow \text{Hom}_{\text{Span}(A, B)}(R, T)$$

of  $\text{Span}(A, B)$  at  $((R, f_1, g_1), (S, f_2, g_2), (T, f_3, g_3))$  is defined by<sup>7</sup>

$$\psi \circ_{R, S, T}^{\text{Span}(A, B)} \phi \stackrel{\text{def}}{=} \psi \circ \phi.$$

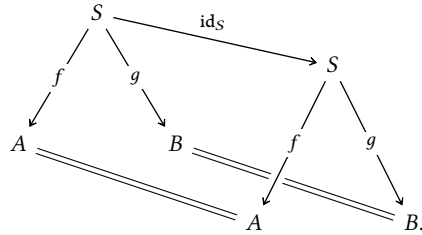
## 00QT 2.2 The Bicategory of Spans

00QU **Definition 2.2.1.1.** The **bicategory of spans** is the bicategory  $\text{Span}$  where

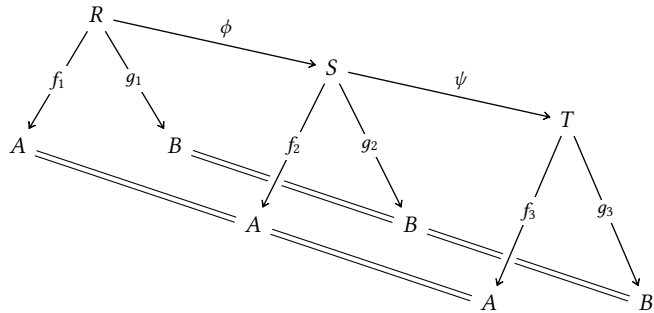
- *Objects.* The objects of  $\text{Span}$  are sets;
- *Hom-Categories.* For each  $A, B \in \text{Obj}(\text{Span})$ , we have

$$\text{Hom}_{\text{Span}}(A, B) \stackrel{\text{def}}{=} \text{Span}(A, B);$$

<sup>6</sup>Picture:



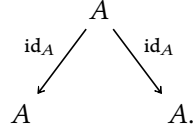
<sup>7</sup>Picture:



- *Identities.* For each  $A \in \text{Obj}(\text{Span})$ , the unit functor

$$\mathbb{K}_A^{\text{Span}} : \text{pt} \rightarrow \text{Span}(A, A)$$

of Span at  $A$  is the functor picking the span  $(A, \text{id}_A, \text{id}_A)$ :

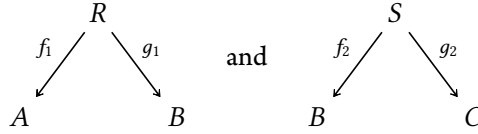


- *Composition.* For each  $A, B, C \in \text{Obj}(\text{Span})$ , the composition bifunctor

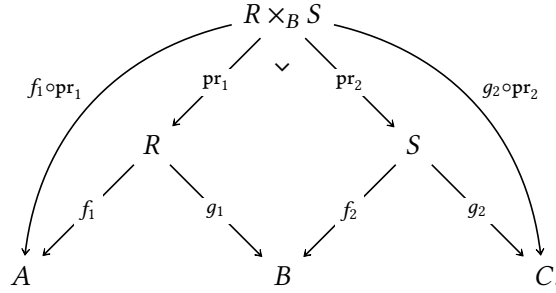
$$\circ_{A,B,C}^{\text{Span}} : \text{Span}(B, C) \times \text{Span}(A, B) \rightarrow \text{Span}(A, C)$$

of Span at  $(A, B, C)$  is the bifunctor where

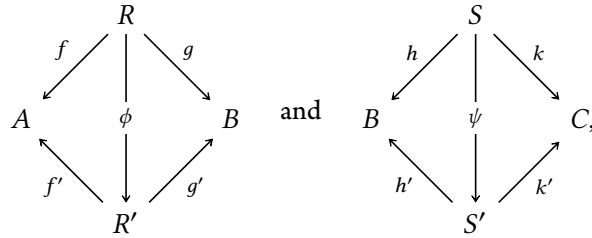
- *Action on Objects.* The composition of two spans



is the span  $(R \times_B S, f_1 \circ \text{pr}_1, g_2 \circ \text{pr}_2)$ , constructed as in the diagram

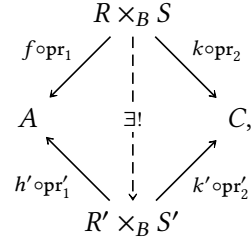


- *Action on Morphisms.* The horizontal composition of 2-morphisms is defined via functoriality of pullbacks: given morphisms of spans

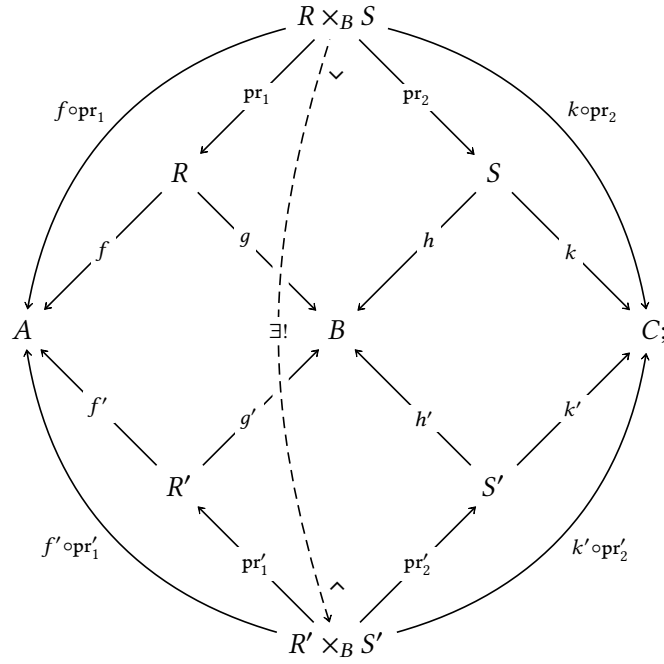




their horizontal composition is the morphism of spans



constructed as in the diagram



- *Associators and Unitors.* The associator and unitors are defined using the universal property of the pullback.

### 00QV 2.3 The Monoidal Bicategory of Spans

### 00QW 2.4 The Double Category of Spans

00QX **Definition 2.4.1.1.** The **double category of spans** is the double category  $\text{Span}^{\text{dbl}}$  where

- *Objects.* The objects of  $\text{Span}^{\text{dbl}}$  are sets;

- *Vertical Morphisms.* The vertical morphisms of  $\text{Span}^{\text{dbl}}$  are functions  $f: A \rightarrow B$ ;
- *Horizontal Morphisms.* The horizontal morphisms of  $\text{Span}^{\text{dbl}}$  are spans  $(S, \phi, \psi): A \rightarrowtail X$ ;
- *2-Morphisms.* A 2-cell

$$\begin{array}{ccc}
 A & \xrightarrow{(R, \phi_R, \psi_R)} & B \\
 f \downarrow & \Downarrow \alpha & \downarrow g \\
 X & \xrightarrow{(S, \phi_S, \psi_S)} & Y
 \end{array}$$

of  $\text{Span}^{\text{dbl}}$  is a morphism of spans from the span

$$\begin{array}{ccc}
 & R & \\
 \phi_R \swarrow & & \searrow \psi_R \\
 A & & B \\
 & & \searrow g \\
 & & Y
 \end{array}$$

to the span

$$\begin{array}{ccccc}
 & & A \times_X S & & \\
 & \swarrow & \downarrow \vee & \searrow & \\
 & A & & S & \\
 f \swarrow & & f \searrow & \phi_S \swarrow & \searrow \psi_S \\
 X & & X & & Y
 \end{array}$$

- *Horizontal Identities.* The horizontal unit functor

$$\mathbb{K}^{\text{Span}^{\text{dbl}}} : (\text{Span}^{\text{dbl}})_0 \rightarrow (\text{Span}^{\text{dbl}})_1$$

of  $\text{Span}^{\text{dbl}}$  is the functor where

- *Action on Objects.* For each  $A \in \text{Obj}((\text{Span}^{\text{dbl}})_0)$ , we have

$$\mathbb{K}_A \stackrel{\text{def}}{=} (A, \text{id}_A, \text{id}_A),$$

as in the diagram

$$\begin{array}{ccc} & A & \\ \text{id}_A \swarrow & & \searrow \text{id}_A \\ A & & A; \end{array}$$

- *Action on Morphisms.* For each vertical morphism  $f: A \rightarrow B$  of  $\text{Span}^{\text{dbl}}$ , i.e. each map of sets  $f$  from  $A$  to  $B$ , the identity 2-morphism

$$\begin{array}{ccccc} A & \xrightarrow{\text{id}_A} & A & & \\ f \downarrow & & \parallel & & \downarrow f \\ B & \xrightarrow{\text{id}_B} & B & & \end{array}$$

of  $f$  is the morphism of spans from

$$\begin{array}{ccc} & A & \\ \text{id}_A \swarrow & & \searrow \text{id}_A \\ A & & A \xrightarrow{f} B \end{array}$$

to

$$\begin{array}{ccccc} & A \times_B B & & & \\ & \swarrow \quad \searrow & & & \\ & A & & B & \\ f \swarrow & & \searrow f & & \parallel \text{id}_B \\ B & & B & & \parallel \text{id}_B \\ & & & & B \end{array}$$

given by the isomorphism  $A \xrightarrow{\cong} A \times_B B$ ;

- *Vertical Identities.* For each  $A \in \text{Obj}(\text{Span}^{\text{dbl}})$ , we have

$$\text{id}_A^{\text{Span}^{\text{dbl}}} \stackrel{\text{def}}{=} \text{id}_A;$$

- *Identity 2-Morphisms.* For each horizontal morphism  $R: A \rightarrowtail B$  of  $\text{Span}^{\text{dbl}}$ , the

identity 2-morphism

$$\begin{array}{ccc}
 A & \xrightarrow{S} & B \\
 \text{id}_A \downarrow & \Downarrow \text{id}_S & \downarrow \text{id}_B \\
 A & \xrightarrow{S} & B
 \end{array}$$

of  $R$  is the morphism of spans from

$$\begin{array}{ccc}
 & S & \\
 \phi_S \swarrow & & \searrow \psi_S \\
 A & & B \\
 & & \Downarrow \text{id}_B \\
 & & B
 \end{array}$$

to

$$\begin{array}{ccccc}
 & & A \times_A S & & \\
 & \swarrow & \downarrow \vee & \searrow & \\
 & A & & S & \\
 \text{id}_A \Downarrow & & \text{id}_A \Downarrow & \swarrow \phi_S & \searrow \psi_S \\
 A & & A & & B
 \end{array}$$

given by the isomorphism  $S \xrightarrow{\cong} A \times_A S$ ;

- *Horizontal Composition.* The horizontal composition functor

$$\odot^{\text{Span}^{\text{dbl}}} : (\text{Span}^{\text{dbl}})_1 \times_{(\text{Span}^{\text{dbl}})_0} (\text{Span}^{\text{dbl}})_1 \rightarrow (\text{Span}^{\text{dbl}})_1$$

of  $\text{Span}^{\text{dbl}}$  is the functor where

- *Action on Objects.* For each composable pair

$$\begin{array}{ccccc}
 & (R, \phi_R, \psi_R) & & (S, \phi_S, \psi_S) & \\
 A & \xrightarrow{\quad} & B & \xrightarrow{\quad} & C
 \end{array}$$

of horizontal morphisms of  $\text{Span}^{\text{dbl}}$ , we have

$$(S, \phi_S, \psi_S) \odot (R, \phi_R, \psi_R) \stackrel{\text{def}}{=} S \circ_{A,B,C}^{\text{Span}} R,$$

where  $S \circ_{A,B,C}^{\text{Span}} R$  is the composition of  $(R, \phi_R, \psi_R)$  and  $(S, \phi_S, \psi_S)$  defined as in [Definition 2.2.1.1](#);

- *Action on Morphisms.* For each horizontally composable pair

$$\begin{array}{ccc}
 A & \xrightarrow{(R, \phi_R, \psi_R)} & B \\
 f \downarrow & \Downarrow \alpha & \downarrow g \\
 X & \xrightarrow{(T, \phi_T, \psi_T)} & Y
 \end{array}
 \quad
 \begin{array}{ccc}
 B & \xrightarrow{(S, \phi_S, \psi_S)} & C \\
 g \downarrow & \Downarrow \beta & \downarrow h \\
 Y & \xrightarrow{(U, \phi_U, \psi_U)} & Z
 \end{array}$$

of 2-morphisms of  $\text{Span}^{\text{dbl}}$ ,

- *Vertical Composition of 1-Morphisms.* For each composable pair  $A \xrightarrow{F} B \xrightarrow{G} C$  of vertical morphisms of  $\text{Span}^{\text{dbl}}$ , i.e. maps of sets, we have

$$g \circ^{\text{Span}^{\text{dbl}}} f \stackrel{\text{def}}{=} g \circ f;$$

- *Vertical Composition of 2-Morphisms.* For each vertically composable pair

$$\begin{array}{ccc}
 A & \xrightarrow{(R, \phi_R, \psi_R)} & X \\
 f \downarrow & \Downarrow \alpha & \downarrow g \\
 B & \xrightarrow{(S, \phi_S, \psi_S)} & Y
 \end{array}
 \quad
 \begin{array}{ccc}
 B & \xrightarrow{(S, \phi_S, \psi_S)} & Y \\
 h \downarrow & \Downarrow \beta & \downarrow k \\
 C & \xrightarrow{(T, \phi_T, \psi_T)} & Z
 \end{array}$$

of 2-morphisms of  $\text{Span}^{\text{dbl}}$ ,

- *Associators and Unitors.* The associator and unitors of  $\text{Span}^{\text{dbl}}$  are defined using the universal property of the pullback.

## 00QY 2.5 Properties of The Bicategory of Spans

00QZ **Proposition 2.5.1.1.** Let  $\lambda = (A \xleftarrow{f} S \xrightarrow{g} B)$  be a span.

- 00R0 1. *Self-Duality.*
- 00R1 2. *Isomorphisms in Span.*
- 00R2 3. *Equivalences in Span.*
- 00R3 4. *Adjunctions in Span.* Let  $A$  and  $B$  be sets.<sup>8</sup>

<sup>8</sup>In the literature (e.g. [ref]), ... are called maps and denoted by  $\text{MapSpan}(A, B)$

00R4 (a) We have a natural bijection

$$\left\{ \begin{array}{c} \text{Adjunctions in Span} \\ \text{from } A \text{ to } B \end{array} \right\} \cong \left\{ \begin{array}{c} \text{Spans } A \xleftarrow{f} S \xrightarrow{g} B \\ \text{from } A \text{ to } B \text{ with} \\ f \text{ an isomorphism} \end{array} \right\}.$$

00R5 (b) We have an equivalence of categories

$$\text{MapSpan}(A, B) \stackrel{\text{eq.}}{\cong} \text{Sets}(A, B)_{\text{disc}},$$

where  $\text{MapSpan}(A, B)$  is the full subcategory of  $\text{Span}(A, B)$  spanned by the spans  $A \xleftarrow{f} S \xrightarrow{g} B$  from  $A$  to  $B$  with  $f$  an isomorphism.

00R6 (c) We have a biequivalence of bicategories

$$\text{MapSpan} \stackrel{\text{eq.}}{\cong} \text{Sets}_{\text{bidisc}},$$

where  $\text{MapSpan}$  is the sub-bicategory of  $\text{Span}$  whose Hom-categories are given by  $\text{MapSpan}(A, B)$ .

00R7 5. *Monads in Span.*

00R8 6. *Comonads in Span.*

00R9 7. *Monomorphisms in Span.*

00RA 8. *Epimorphisms in Span.*

00RB 9. *Existence of Right Kan Extensions.*

00RC 10. *Existence of Right Kan Lifts.*

00RD 11. *Closedness.*

*Proof. Item 1, Self-Duality:*

*Item 2, Isomorphisms in Span:*

*Item 3, Equivalences in Span:*

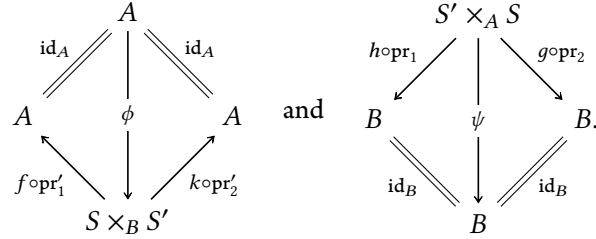
*Item 4, Adjunctions in Span: We first prove Item 4a.*

We proceed step by step:

1. *From Adjunctions in Span to Functions.* An adjunction in  $\text{Span}$  from  $A$  to  $B$  consists of a pair of spans

$$\begin{array}{ccc} & R & \\ f \swarrow & & \searrow g \\ A & & B \end{array} \quad \text{and} \quad \begin{array}{ccc} & S' & \\ h \swarrow & & \searrow k \\ B & & A \end{array}$$

together with maps



We claim that these conditions

2. *From Functions to Adjunctions in **Rel**.*
3. *Invertibility: From Functions to Adjunctions Back to Functions.*
4. *Invertibility: From Adjunctions to Functions Back to Adjunctions.*

We now proceed to the proof of **Item 4b**. For this, we will construct a functor

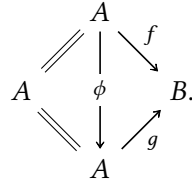
$$F: \text{Sets}(A, B)_{\text{disc}} \rightarrow \text{MapSpan}(A, B)$$

and prove it to be essentially surjective and fully faithful, and thus an equivalence by **Categories**, ?? of ??. Indeed, given a map  $f: A \rightarrow B$ , let  $F(f)$  be the representable span associated to  $f$  of **Definition 5.1.1.1**, and let  $F$  send the unique (identity) morphism from  $f$  to itself to the identity morphism of  $F(f)$  in  $\text{MapSpan}(A, B)$ . We now prove that  $F$  is fully faithful and essentially surjective:

1.  *$F$  Is Fully Faithful:* Given maps  $f, g: A \rightrightarrows B$ , we need to show that

$$\text{Hom}_{\text{MapSpan}(A, B)}(F(f), F(g)) = \begin{cases} \text{pt} & \text{if } f = g, \\ \emptyset & \text{otherwise.} \end{cases}$$

Indeed, a morphism from  $F(f)$  to  $F(g)$  takes the form



From the relations  $\text{id}_A = \text{id}_A \circ \phi$  and  $f = g \circ \phi$ , we see that  $\phi = \text{id}_A$ , and thus from the relation  $f = g \circ \phi$  there is such a morphism iff  $f = g$ .

2. *F Is Essentially Surjective*: Let  $\lambda$  be a span of the form

$$\begin{array}{ccc} & S & \\ \phi \swarrow & & \searrow f \\ A & & B. \end{array}$$

we claim that  $\lambda \cong F(f \circ \phi^{-1})$ . Indeed, we have morphisms

$$\begin{array}{ccc} & S & \\ \phi \swarrow & & \searrow f \\ A & & B \\ \parallel \swarrow & & \nearrow f \circ \phi^{-1} \\ & A & \end{array} \quad \text{and} \quad \begin{array}{ccc} & A & \\ \parallel \swarrow & & \searrow f \circ \phi^{-1} \\ A & & B \\ \nearrow \phi^{-1} & & \nwarrow f \\ & S & \end{array}$$

inverse to each other in  $\text{MapSpan}(A, B)$ , and thus  $\lambda \cong F(f \circ \phi^{-1})$ .

Finally, we prove **Item 4c**.

**Item 5**, *Monads in Span*:

**Item 6**, *Comonads in Span*:

**Item 7**, *Monomorphisms in Span*:

**Item 8**, *Epimorphisms in Span*:

**Item 9**, *Existence of Right Kan Extensions*:

**Item 10**, *Existence of Right Kan Lifts*:

**Item 11**, *Closedness*:

□

### 00RE 3 Limits of Spans

### 00RF 4 Colimits of Spans

### 00RG 5 Constructions With Spans

#### 00RH 5.1 Representable Spans

**00RJ Definition 5.1.1.1.** Let  $f: A \rightarrow B$  be a function.

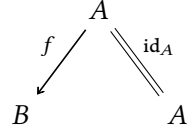
- The **representable span associated to  $f$**  is the span

$$\begin{array}{ccc} & A & \\ \text{id}_A \swarrow & & \searrow f \\ A & & B \end{array}$$

from  $A$  to  $B$ .



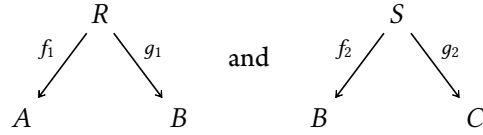
- The **corepresentable span associated to  $f$**  is the span



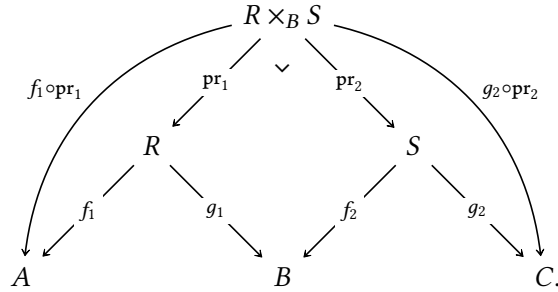
from  $B$  to  $A$ .

## 00RK 5.2 Composition of Spans

00RL **Definition 5.2.1.1.** The **composition** of two spans

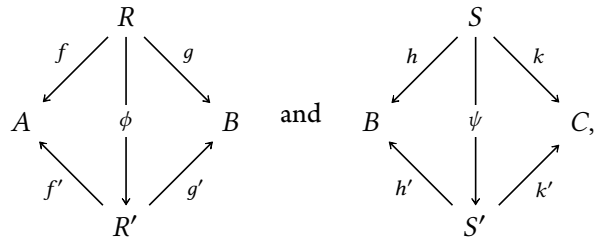


is the span  $(R \times_B S, f_1 \circ \text{pr}_1, g_2 \circ \text{pr}_2)$ , constructed as in the diagram

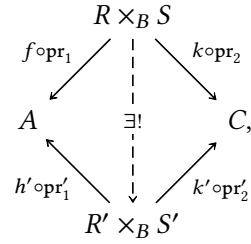


## 00RM 5.3 Horizontal Composition of Morphisms of Spans

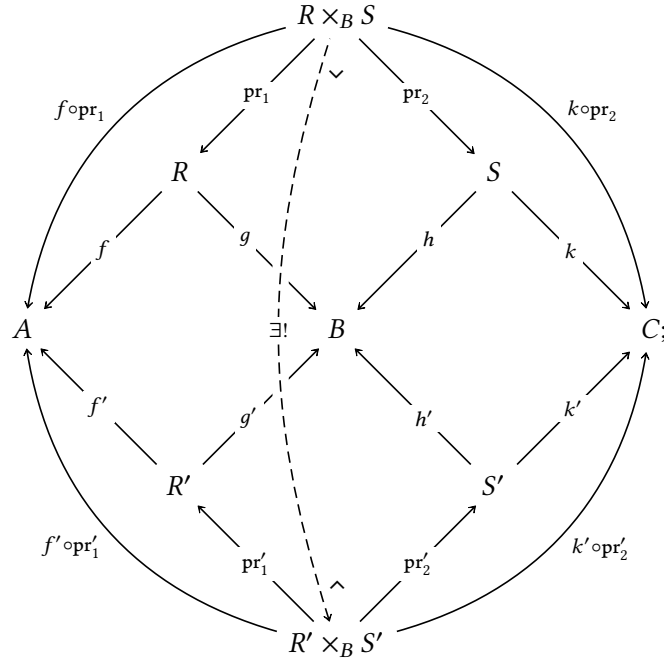
00RN **Definition 5.3.1.1.** The **horizontal composition** of a pair of 2-morphisms of spans



is the morphism of spans



constructed as in the diagram



#### 00RP 5.4 Properties of Composition of Spans

00RQ **Proposition 5.4.1.1.** Let  $\lambda = \left( A \xleftarrow{f} S \xrightarrow{g} B \right)$  be a span.

00RR 1. *Functoriality.*

*Proof.*

□

#### 00RS 5.5 The Inverse of a Span

## 00RT 6 Functoriality of Spans

### 00RU 6.1 Direct Images

### 00RV 6.2 Functoriality of Spans on Powersets

## 00RW 7 Comparison of Spans to Functions and Relations

### 00RX 7.1 Comparison to Functions

00RY **Proposition 7.1.1.1.** We have a pseudofunctor

$$\iota: \mathbf{Sets}_{\text{bidisc}} \rightarrow \mathbf{Span}$$

from  $\mathbf{Sets}_{\text{bidisc}}$  to  $\mathbf{Span}$  where

- *Action on Objects.* For each  $A \in \text{Obj}(\mathbf{Sets}_{\text{bidisc}})$ , we have

$$\iota(A) \stackrel{\text{def}}{=} A;$$

- *Action on Hom-Categories.* For each  $A, B \in \text{Obj}(\mathbf{Sets}_{\text{bidisc}})$ , the action on Hom-categories

$$\iota_{A,B}: \mathbf{Sets}(A, B)_{\text{disc}} \rightarrow \mathbf{Span}(A, B)$$

of  $\iota$  at  $(A, B)$  is the functor defined on objects by sending a function  $f: A \rightarrow B$  to the span

$$\begin{array}{ccc} & A & \\ \text{id}_A \swarrow & & \searrow f \\ A & & B \end{array}$$

from  $A$  to  $B$ .

- *Strict Unity Constraints.* For each  $A \in \text{Obj}(\mathbf{Sets}_{\text{bidisc}})$ , the strict unity constraint

$$\iota_A^0: \text{id}_{\iota(A)} \Longrightarrow \iota(\text{id}_A)$$

of  $\iota$  at  $A$  is given by the identity morphism of spans

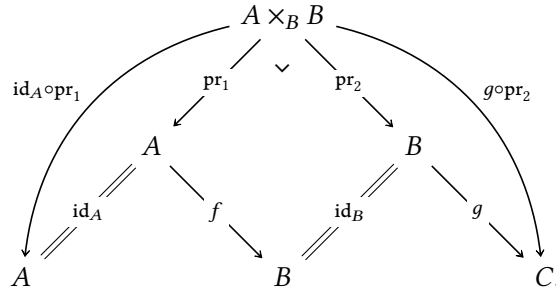
$$\begin{array}{ccccc} & & A & & \\ & \text{id}_A \swarrow & \parallel & \searrow \text{id}_A & \\ A & & \text{id} & & A, \\ & \swarrow \text{id}_A & \parallel & \searrow \text{id}_A & \\ & & A & & \end{array}$$

as indeed  $\text{id}_{\iota(A)} = \iota(\text{id}_A)$ ;

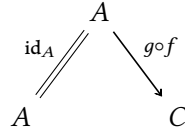
- *Pseudofunctionality Constraints.* For each  $A, B, C \in \text{Obj}(\text{Sets}_{\text{bidisc}})$ , each  $f \in \text{Hom}_{\text{Sets}_{\text{bidisc}}}(A, B)$ , and each  $g \in \text{Hom}_{\text{Sets}_{\text{bidisc}}}(B, C)$ , the pseudofunctionality constraint

$$\iota_{g,f}^2: \iota(g) \circ \iota(f) \implies \iota(g \circ f)$$

of  $\iota$  at  $(f, g)$  is the morphism of spans from the span



to the span



given by the isomorphism  $A \times_B B \cong A$ .

*Proof.* Omitted. □

## 7.2 Comparison to Relations: From Span to **Rel**

### 7.2.1 Relations Associated to Spans

Let  $\lambda = \left( A \xleftarrow{f} S \xrightarrow{g} B \right)$  be a span.

**Definition 7.2.1.1.** The **relation associated to  $\lambda$**  is the relation

$$S(\lambda): A \rightarrowtail B$$

from  $A$  to  $B$  defined as follows:

- Viewing relations from  $A$  to  $B$  as functions  $A \times B \rightarrow \{\text{true}, \text{false}\}$ , we define

$$\iota_{A,B}(S)_b^a \stackrel{\text{def}}{=} \begin{cases} \text{true} & \text{if there exists } x \in S \text{ such} \\ & \text{that } a = f(x) \text{ and } b = g(x), \\ \text{false} & \text{otherwise} \end{cases}$$

for each  $(a, b) \in A \times B$ .

- Viewing relations from  $A$  to  $B$  as functions  $A \rightarrow \mathcal{P}(B)$ , we define

$$[\iota_{A,B}(S)](a) \stackrel{\text{def}}{=} g(f^{-1}(a))$$

for each  $a \in A$ .

- Viewing relations from  $A$  to  $B$  as subsets of  $A \times B$ , we define

$$\iota_{A,B}(S) \stackrel{\text{def}}{=} \{(f(x), g(x)) \mid x \in S\}.$$

**00S2 Proposition 7.2.1.2.** Let  $\lambda = \left( A \xleftarrow{f} S \xrightarrow{g} B \right)$  be a span.

**00S3** 1. *Interaction With Identities.*

**00S4** 2. *Interaction With Composition.*

**00S5** 3. *Interaction With Inverses.*

*Proof.*

□

**00S6 7.2.2 The Comparison Functor from Span to Rel**

**00S7 Proposition 7.2.2.1.** We have a pseudofunctor

$$\iota: \mathbf{Span} \rightarrow \mathbf{Rel}$$

from **Span** to **Rel** where

- *Action on Objects.* For each  $A \in \mathbf{Obj}(\mathbf{Span})$ , we have

$$\iota(A) \stackrel{\text{def}}{=} A;$$

- *Action on Hom-Categories.* For each  $A, B \in \mathbf{Obj}(\mathbf{Span})$ , the action on Hom-categories

$$\iota_{A,B}: \mathbf{Span}(A, B) \rightarrow \mathbf{Rel}(A, B)$$

of  $\iota$  at  $(A, B)$  is the functor where

- *Action on Objects.* Given a span

$$\begin{array}{ccc} & S & \\ f \swarrow & & \searrow g \\ A & & B \end{array}$$

from  $A$  to  $B$ , the image

$$\iota_{A,B}(S): A \rightarrowtail B$$

of  $S$  by  $\iota$  is the relation from  $A$  to  $B$  defined as follows:

\* Viewing relations as functions  $A \times B \rightarrow \{\text{true}, \text{false}\}$ , we define

$$\iota_{A,B}(S)_b^a \stackrel{\text{def}}{=} \begin{cases} \text{true} & \text{if there exists } x \in S \\ & \text{such that } a = f(x) \\ & \text{and } b = g(x), \\ \text{false} & \text{otherwise} \end{cases}$$

for each  $(a, b) \in A \times B$ ;

\* Viewing relations as functions  $A \rightarrow \mathcal{P}(B)$ , we define

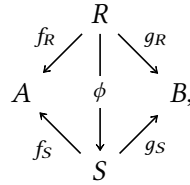
$$[\iota_{A,B}(S)](a) \stackrel{\text{def}}{=} g(f^{-1}(a))$$

for each  $a \in A$ ;

\* Viewing relations as subsets of  $A \times B$ , we define

$$\iota_{A,B}(S) \stackrel{\text{def}}{=} \{(f(x), g(x)) \mid x \in S\}.$$

– *Action on Morphisms.* Given a morphism of spans



we have a corresponding inclusion of relations

$$\iota_{A,B}(\phi): \iota_{A,B}(R) \subset \iota_{A,B}(S),$$

since we have  $a \sim_{\iota_{A,B}(R)} b$  iff there exists  $x \in R$  such that  $a = f_R(x)$  and  $b = g_R(x)$ , in which case we then have

$$\begin{aligned} a &= f_R(x) \\ &= f_S(\phi(x)), \\ b &= g_R(x) \\ &= g_S(\phi(x)), \end{aligned}$$

so that  $a \sim_{\iota_{A,B}(S)} b$ , and thus  $\iota_{A,B}(R) \subset \iota_{A,B}(S)$ .

*Proof.* Omitted. □

**00S8 7.3 Comparison to Relations: From Rel to Span****00S9 Proposition 7.3.1.1.** We have a lax functor

$$(\iota, \iota^2, \iota^0): \mathbf{Rel} \rightarrow \mathbf{Span}$$

from **Rel** to Span where

- *Action on Objects.* For each  $A \in \mathbf{Obj}(\mathbf{Span})$ , we have

$$\iota(A) \stackrel{\text{def}}{=} A;$$

- *Action on Hom-Categories.* For each  $A, B \in \mathbf{Obj}(\mathbf{Span})$ , the action on Hom-categories

$$\iota_{A,B}: \mathbf{Rel}(A, B) \rightarrow \mathbf{Span}(A, B)$$

of  $\iota$  at  $(A, B)$  is the functor where

- *Action on Objects.* Given a relation  $R: A \rightarrowtail B$  from  $A$  to  $B$ , we define a span

$$\iota_{A,B}(R): A \rightarrowtail B$$

from  $A$  to  $B$  by

$$\iota_{A,B}(R) \stackrel{\text{def}}{=} (R, \upharpoonright \text{pr}_1 R, \upharpoonright \text{pr}_2 R),$$

where  $R \subset A \times B$  and  $\upharpoonright \text{pr}_1 R$  and  $\upharpoonright \text{pr}_2 R$  are the restriction of the projections

$$\text{pr}_1: A \times B \rightarrow A,$$

$$\text{pr}_2: A \times B \rightarrow B$$

to  $R$ ;

- *Action on Morphisms.* Given an inclusion  $\phi: R \subset S$  of relations, we have a corresponding morphism of spans

$$\iota_{A,B}(\phi): \iota_{A,B}(R) \rightarrow \iota_{A,B}(S)$$

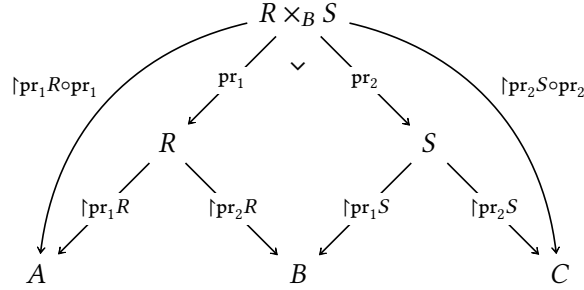
as in the diagram

$$\begin{array}{ccccc}
 & & R & & \\
 \upharpoonright \text{pr}_1 R & \swarrow & & \searrow & \upharpoonright \text{pr}_2 R \\
 A & & & & B \\
 \upharpoonright \text{pr}_1 S & \swarrow & \downarrow & \searrow & \upharpoonright \text{pr}_2 S \\
 & & S & & 
 \end{array}$$

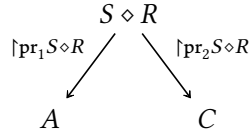
- *The Lax Functoriality Constraints.* The lax functoriality constraint

$$\iota_{R,S}^2: \iota(S) \circ \iota(R) \Longrightarrow \iota(S \diamond R)$$

of  $\iota$  at  $(R, S)$  is given by the morphism of spans from



to



given by the natural inclusion  $R \times_B S \hookrightarrow S \diamond R$ , since we have

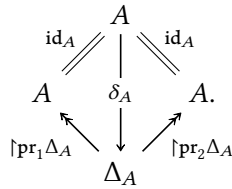
$$R \times_B S = \{((a_R, b_R), (b_S, c_S)) \in R \times S \mid b_R = b_S\};$$

$$S \diamond R = \left\{ (a, c) \in A \times C \mid \begin{array}{l} \text{there exists some } b \in B \text{ such that} \\ (a, b) \in R \text{ and } (b, c) \in S \end{array} \right\};$$

- *The Lax Unity Constraints.* The lax unity constraint<sup>9</sup>

$$\iota_A^0: \underbrace{\text{id}_{\iota(A)}}_{(A, \text{id}_A, \text{id}_A)} \Longrightarrow \underbrace{\iota(\chi_A)}_{(\Delta_A, \uparrow \text{pr}_1 \Delta_A, \uparrow \text{pr}_2 \Delta_A)}$$

of  $\iota$  at  $A$  is given by the diagonal morphism of  $A$ , as in the diagram



*Proof.* Omitted.

□

<sup>9</sup>Which is in fact strong, as  $\delta_A$  is an isomorphism.



**00SA 7.4 Comparison to Relations: The Wehrheim–Woodward Construction****00SB 7.5 Comparison to Multirelations**

**00SC Remark 7.5.1.1.** The pseudofunctor of [Proposition 7.2.2.1](#) and the lax functor of [Proposition 7.3.1.1](#) fail to be equivalences of bicategories. This happens essentially because a span  $(S, f, g): A \dashv B$  from  $A$  to  $B$  may relate elements  $a \in A$  and  $b \in B$  by more than one element, e.g. there could be  $s \neq s' \in S$  such that  $a = f(s) = f(s')$  and  $b = g(s) = g(s')$ .

Thus, in a sense, spans may be thought of as “relations with multiplicity”. And indeed, if instead of considering relations from  $A$  to  $B$ , i.e. functions

$$R: A \times B \rightarrow \{\text{true}, \text{false}\}$$

from  $A \times B$  to  $\{\text{true}, \text{false}\} \cong \{0, 1\}$ , we consider functions

$$R: A \times B \rightarrow \mathbb{N} \cup \{\infty\}$$

from  $A \times B$  to  $\mathbb{N} \cup \{\infty\}$ , then we obtain the notion of a **multirelation from  $A$  to  $B$** , and these turn out to assemble together with sets into a bicategory  $\mathbf{MRel}$  that is biequivalent to  $\mathbf{Span}$ ; see [\[some-algebraic-laws-for-spans-and-their-connections-with-multirelations\]](#).

**00SD 7.6 Comparison to Relations via Double Categories**

**00SE Remark 7.6.1.1.** There are double functors between the double categories  $\mathbf{Rel}^{\text{dbl}}$  and  $\mathbf{Span}^{\text{dbl}}$  analogous to the functors of [Propositions 7.2.2.1](#) and [7.3.1.1](#), assembling moreover into a strict-lax adjunction of double functors; see [\[higher-dimensional-categories\]](#).

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