Constructions With Sets

December 3, 2023

This chapter contains some material relating to constructions with sets. Notably, it contains:

1. Explicit descriptions of the major types of co/limits in Sets, including in particular pushouts and coequalisers (see Definitions 2.3.1.1 and 2.4.1.1 and Remarks 2.3.1.2 and 2.4.1.2);

2. A discussion of powersets as decategorifications of categories of presheaves (Remarks 4.1.1.2 and 4.2.1.2);

3. A lengthy discussion of the adjoint triple

$$f_* \dashv f^{-1} \dashv f_! \colon \mathcal{P}(A) \overset{\rightleftarrows}{\to} \mathcal{P}(B)$$

of functors (morphisms of posets) between $\mathcal{P}(A)$ and $\mathcal{P}(B)$ induced by a map of sets $f\colon A \to B$, along with a discussion of the properties of f_*, f^{-1}, and $f_!$.

Contents

1 Limits of Sets

1.1 Products of Families of Sets

Let $\{A_i\}_{i \in I}$ be a family of sets.

Definition 1.1.1.1. The **product**[1] **of** $\{A_i\}_{i \in I}$ is the set $\prod_{i \in I} A_i$ defined by

$$\prod_{i \in I} A_i \stackrel{\text{def}}{=} \left\{ f \in \mathsf{Sets}\left(I, \bigcup_{i \in I} A_i\right) \,\middle|\, \begin{array}{l} \text{for each } i \in I \text{, we} \\ \text{have } f(i) \in A_i \end{array} \right\}.$$

1.2 Binary Products of Sets

Let A and B be sets.

[1] *Further Terminology:* Also called the **Cartesian product of** $\{A_i\}_{i \in I}$.

Definition 1.2.1.1. The **product**[2] **of** A **and** B is the set $A \times B$ defined by

$$\begin{aligned} A \times B &\stackrel{\text{def}}{=} \prod_{z \in \{A,B\}} z \\ &\stackrel{\text{def}}{=} \{f \in \mathsf{Sets}(\{0,1\}, A \cup B) \mid \text{we have } f(0) \in A \text{ and } f(1) \in B\} \\ &\cong \{\{\{a\},\{a,b\}\} \in \mathcal{P}(\mathcal{P}(A \cup B)) \mid \text{we have } a \in A \text{ and } b \in B\}. \end{aligned}$$

Proposition 1.2.1.2. Let A, B, C, and X be sets.

1. *Functoriality.* The assignments $A, B, (A,B) \mapsto A \times B$ define functors

$$\begin{aligned} A \times -_2 &\colon \mathsf{Sets} \to \mathsf{Sets}, \\ -_1 \times B &\colon \mathsf{Sets} \to \mathsf{Sets}, \\ -_1 \times -_2 &\colon \mathsf{Sets} \times \mathsf{Sets} \to \mathsf{Sets}, \end{aligned}$$

   where $-_1 \times -_2$ is the functor where

   - *Action on Objects.* For each $(A,B) \in \mathrm{Obj}(\mathsf{Sets} \times \mathsf{Sets})$, we have

$$[-_1 \times -_2](A,B) \stackrel{\text{def}}{=} A \times B;$$

   - *Action on Morphisms.* For each $(A,B), (X,Y) \in \mathrm{Obj}(\mathsf{Sets})$, the action on Hom-sets

$$\times_{(A,B),(X,Y)} \colon \mathsf{Sets}(A,X) \times \mathsf{Sets}(B,Y) \to \mathsf{Sets}(A \times B, X \times Y)$$

     of $\times$ at $((A,B),(X,Y))$ is defined by sending (f,g) to the function

$$f \times g \colon A \times B \to X \times Y$$

     defined by

$$[f \times g](a,b) \stackrel{\text{def}}{=} (f(a), g(b))$$

     for each $(a,b) \in A \times B$;

   and where $A \times -$ and $- \times B$ are the partial functors of $-_1 \times -_2$ at $A, B \in \mathrm{Obj}(\mathsf{Sets})$.

[2] *Further Terminology:* Also called the **Cartesian product of** A **and** B or the **binary Cartesian product of** A **and** B, for emphasis.
This can also be thought of as the $(\mathbb{E}_{-1}, \mathbb{E}_{-1})$**-tensor product of** A **and** B.

2. *Adjointness.* We have adjunctions

$$(A \times - \dashv \mathsf{Sets}(A,-))\colon\quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "A\times -"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}, \arrow[l, bend left, "\mathsf{Sets}(A,-)"] \end{tikzcd}$$

$$(- \times B \dashv \mathsf{Sets}(B,-))\colon\quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "-\times B"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}, \arrow[l, bend left, "\mathsf{Sets}(B,-)"] \end{tikzcd}$$

   witnessed by bijections

$$\begin{aligned} \mathsf{Sets}(A \times B, C) &\cong \mathsf{Sets}(A, \mathsf{Sets}(B, C)),\\ \mathsf{Sets}(A \times B, C) &\cong \mathsf{Sets}(B, \mathsf{Sets}(A, C)), \end{aligned}$$

   natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

3. *Associativity.* We have an isomorphism of sets

$$(A \times B) \times C \cong A \times (B \times C),$$

   natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

4. *Unitality.* We have isomorphisms of sets

$$\begin{aligned} \mathrm{pt} \times A &\cong A,\\ A \times \mathrm{pt} &\cong A, \end{aligned}$$

   natural in $A \in \mathrm{Obj}(\mathsf{Sets})$.

5. *Commutativity.* We have an isomorphism of sets

$$A \times B \cong B \times A,$$

   natural in $A, B \in \mathrm{Obj}(\mathsf{Sets})$.

6. *Annihilation With the Empty Set.* We have isomorphisms of sets

$$\begin{aligned} A \times \emptyset &\cong \emptyset,\\ \emptyset \times A &\cong \emptyset, \end{aligned}$$

   natural in $A \in \mathrm{Obj}(\mathsf{Sets})$.

7. *Distributivity Over Unions.* We have isomorphisms of sets
$$A \times (B \cup C) = (A \times B) \cup (A \times C),$$
$$(A \cup B) \times C = (A \times C) \cup (B \times C),$$
natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

8. *Distributivity Over Intersections.* We have isomorphisms of sets
$$A \times (B \cap C) = (A \times B) \cap (A \times C),$$
$$(A \cap B) \times C = (A \times C) \cap (B \times C),$$
natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

9. *Distributivity Over Differences.* We have isomorphisms of sets
$$A \times (B \setminus C) = (A \times B) \setminus (A \times C),$$
$$(A \setminus B) \times C = (A \times C) \setminus (B \times C),$$
natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

10. *Distributivity Over Symmetric Differences.* We have isomorphisms of sets
$$A \times (B \bigtriangleup C) = (A \times B) \bigtriangleup (A \times C),$$
$$(A \bigtriangleup B) \times C = (A \times C) \bigtriangleup (B \times C),$$
natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

11. *Symmetric Monoidality.* The triple $(\mathsf{Sets}, \times, \mathrm{pt})$ is a symmetric monoidal category.

12. *Symmetric Bimonoidality.* The quintuple $(\mathsf{Sets}, \coprod, \emptyset, \times, \mathrm{pt})$ is a symmetric bimonoidal category.

Proof. *Item 1, Functoriality*: Omitted.
Item 2, Adjointness: Omitted.
Item 3, Associativity: Clear.
Item 4, Unitality: Clear.
Item 5, Commutativity: Clear.
Item 6, Annihilation With the Empty Set: Clear.
Item 7, Distributivity Over Unions: Omitted.
Item 8, Distributivity Over Intersections: Omitted.
Item 9, Distributivity Over Differences: Omitted.
Item 10, Distributivity Over Symmetric Differences: Omitted.
Item 11, Symmetric Monoidality: Omitted.
Item 12, Symmetric Bimonoidality: Omitted. □

1.3 Pullbacks

Let A, B, and C be sets and let $f\colon A\to C$ and $g\colon B\to C$ be functions.

Definition 1.3.1.1. The **pullback of** A **and** B **over** C **along** f **and** g[3] is the set $A\times_C B$ defined by

$$A\times_C B\overset{\mathrm{def}}{=}\{(a,b)\in A\times B\mid f(a)=g(b)\}.$$

Example 1.3.1.2. Here are some examples of pullbacks of sets.

1. *Unions via Intersections.* Let $A,B\subset X$. We have a bijection of sets

$$A\cap B\cong A\times_{A\cup B}B.$$

Proposition 1.3.1.3. Let A, B, C, and X be sets.

1. *Associativity.* We have an isomorphism of sets

$$(A\times_X B)\times_X C\cong A\times_X(B\times_X C),$$

natural in $A,B,C,X\in\mathrm{Obj}(\mathsf{Sets})$.

2. *Unitality.* We have isomorphisms of sets

$$X\times_X A\cong A,$$
$$A\times_X X\cong A,$$

natural in $A,X\in\mathrm{Obj}(\mathsf{Sets})$.

3. *Commutativity.* We have an isomorphism of sets

$$A\times_X B\cong B\times_X A,$$

natural in $A,B,X\in\mathrm{Obj}(\mathsf{Sets})$.

4. *Annihilation With the Empty Set.* We have isomorphisms of sets

$$A\times_X\emptyset\cong\emptyset,$$
$$\emptyset\times_X A\cong\emptyset,$$

natural in $A,X\in\mathrm{Obj}(\mathsf{Sets})$.

5. *Symmetric Monoidality.* The triple $(\mathsf{Sets},\times_X,X)$ is a symmetric monoidal category.

Proof. Item 1, *Associativity*: Clear.
Item 2, *Unitality*: Clear.
Item 3, *Commutativity*: Clear.
Item 4, *Annihilation With the Empty Set*: Clear.
Item 5, *Symmetric Monoidality*: Omitted. □

[3] *Further Terminology:* Also called the **fibre product of** A **and** B **over** C **along** f **and** g.

1.4 Equalisers

Let A and B be sets and let $f, g\colon A \rightrightarrows B$ be functions.

Definition 1.4.1.1. The **equaliser of** f **and** g is the set $\mathrm{Eq}(f, g)$ defined by

$$\mathrm{Eq}(f, g) \stackrel{\text{def}}{=} \{a \in A \mid f(a) = g(a)\}.$$

Proposition 1.4.1.2. Let A, B, and C be sets.

1. *Associativity.* We have an isomorphism of sets[4]

$$\underbrace{\mathrm{Eq}\big(f \circ \mathrm{eq}(g,h), g \circ \mathrm{eq}(g,h)\big)}_{=\mathrm{Eq}(f\circ\mathrm{eq}(g,h),h\circ\mathrm{eq}(g,h))} \cong \mathrm{Eq}(f,g,h) \cong \underbrace{\mathrm{Eq}\big(f \circ \mathrm{eq}(f,g), h \circ \mathrm{eq}(f,g)\big)}_{=\mathrm{Eq}(g\circ\mathrm{eq}(f,g),h\circ\mathrm{eq}(f,g))},$$

where $\mathrm{Eq}(f,g,h)$ is the limit of the diagram

$$\begin{tikzcd} A \arrow[r, shift left=2, "f"] \arrow[r, "g" description] \arrow[r, shift right=2, "h"'] & B \end{tikzcd}$$

in Sets.

[4]That is: the following constructions give the same result:

1. Take the equaliser of (f,g,h), i.e. the limit of the diagram

$$\begin{tikzcd} A \arrow[r, shift left=2, "f"] \arrow[r, "g" description] \arrow[r, shift right=2, "h"'] & B \end{tikzcd}$$

in Sets.

2. First take the equaliser of f and g, forming a diagram

$$\mathrm{Eq}(f,g) \overset{\mathrm{eq}(f,g)}{\hookrightarrow} A \underset{g}{\overset{f}{\rightrightarrows}} B$$

and then take the equaliser of the composition

$$\mathrm{Eq}(f,g) \overset{\mathrm{eq}(f,g)}{\hookrightarrow} A \underset{h}{\overset{f}{\rightrightarrows}} B,$$

obtaining a subset

$$\mathrm{Eq}\big(f \circ \mathrm{eq}(f,g), h \circ \mathrm{eq}(f,g)\big) = \mathrm{Eq}\big(g \circ \mathrm{eq}(f,g), h \circ \mathrm{eq}(f,g)\big)$$

of $\mathrm{Eq}(f,g)$.

3. First take the equaliser of g and h, forming a diagram

$$\mathrm{Eq}(g,h) \overset{\mathrm{eq}(g,h)}{\hookrightarrow} A \underset{h}{\overset{g}{\rightrightarrows}} B$$

and then take the equaliser of the composition

$$\mathrm{Eq}(g,h) \overset{\mathrm{eq}(g,h)}{\hookrightarrow} A \underset{g}{\overset{f}{\rightrightarrows}} B,$$

obtaining a subset

$$\mathrm{Eq}\big(f \circ \mathrm{eq}(g,h), g \circ \mathrm{eq}(g,h)\big) = \mathrm{Eq}\big(f \circ \mathrm{eq}(g,h), h \circ \mathrm{eq}(g,h)\big)$$

of $\mathrm{Eq}(g,h)$.

4. *Unitality.* We have an isomorphism of sets
$$\mathrm{Eq}(f, f) \cong A.$$

5. *Commutativity.* We have an isomorphism of sets
$$\mathrm{Eq}(f, g) \cong \mathrm{Eq}(g, f).$$

6. *Interaction With Composition.* Let
$$A \underset{g}{\overset{f}{\rightrightarrows}} B \underset{k}{\overset{h}{\rightrightarrows}} C$$
be functions. We have an inclusion of sets
$$\mathrm{Eq}\big(h \circ f \circ \mathrm{eq}(f, g), k \circ g \circ \mathrm{eq}(f, g)\big) \subset \mathrm{Eq}(h \circ f, k \circ g),$$
where $\mathrm{Eq}\big(h \circ f \circ \mathrm{eq}(f, g), k \circ g \circ \mathrm{eq}(f, g)\big)$ is the equaliser of the composition
$$\mathrm{Eq}(f, g) \overset{\mathrm{eq}(f,g)}{\hookrightarrow} A \underset{g}{\overset{f}{\rightrightarrows}} B \underset{k}{\overset{h}{\rightrightarrows}} C.$$

Proof. Item 1, *Associativity*: Clear.
Item 4, *Unitality*: Clear.
Item 5, *Commutativity*: Clear.
Item 6, *Interaction With Composition*: Omitted. □

2 Colimits of Sets

2.1 Coproducts of Families of Sets

Let $\{A_i\}_{i\in I}$ be a family of sets.

Definition 2.1.1.1. The **disjoint union of the family** $\{A_i\}_{i\in I}$ is the set $\coprod_{i\in I} A_i$ defined by
$$\coprod_{i\in I} A_i \overset{\mathrm{def}}{=} \bigcup_{i\in I} \left\{ (x, i) \in \left(\bigcup_{i\in I} A_i \right) \times I \,\middle|\, x \in A_i \right\}.$$

2.2 Binary Coproducts

Let A and B be sets.

Definition 2.2.1.1. The **coproduct**[5] **of** A **and** B is the set $A \coprod B$ defined by

$$\begin{aligned} A \coprod B &\overset{\text{def}}{=} \coprod_{z \in \{A,B\}} z \\ &\overset{\text{def}}{=} \{(a,0) \mid a \in A\} \cup \{(b,1) \mid b \in B\}. \end{aligned}$$

Proposition 2.2.1.2. Let A, B, C, and X be sets.

1. *Functoriality.* The assignment $A, B, (A,B) \mapsto A \coprod B$ defines functors

$$\begin{aligned} A \coprod -_2 &\colon \mathsf{Sets} \to \mathsf{Sets}, \\ -_1 \coprod B &\colon \mathsf{Sets} \to \mathsf{Sets}, \\ -_1 \coprod -_2 &\colon \mathsf{Sets} \times \mathsf{Sets} \to \mathsf{Sets}, \end{aligned}$$

where $-_1 \coprod -_2$ is the functor where

- *Action on Objects.* For each $(A,B) \in \mathrm{Obj}(\mathsf{Sets} \times \mathsf{Sets})$, we have

$$[-_1 \coprod -_2](A,B) \overset{\text{def}}{=} A \coprod B;$$

- *Action on Morphisms.* For each $(A,B), (X,Y) \in \mathrm{Obj}(\mathsf{Sets})$, the action on Hom-sets

$$\coprod_{(A,B),(X,Y)} \colon \mathsf{Sets}(A,X) \times \mathsf{Sets}(B,Y) \to \mathsf{Sets}(A \coprod B, X \coprod Y)$$

of $\coprod$ at $((A,B),(X,Y))$ is defined by sending (f,g) to the function

$$f \coprod g \colon A \coprod B \to X \coprod Y$$

defined by

$$[f \coprod g](x) \overset{\text{def}}{=} \begin{cases} f(x) & \text{if } x \in A, \\ g(x) & \text{if } x \in B \end{cases}$$

for each $x \in A \coprod B$;

and where $A \coprod -$ and $- \coprod B$ are the partial functors of $-_1 \coprod -_2$ at $A, B \in \mathrm{Obj}(\mathsf{Sets})$.

[5] *Further Terminology:* Also called the **disjoint union of** A **and** B, or the **binary disjoint union of** A **and** B, for emphasis.

2. *Associativity.* We have an isomorphism of sets

$$(A \coprod B) \coprod C \cong A \coprod (B \coprod C),$$

natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

3. *Unitality.* We have isomorphisms of sets

$$A \coprod \emptyset \cong A,$$
$$\emptyset \coprod A \cong A,$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$.

4. *Commutativity.* We have an isomorphism of sets

$$A \coprod B \cong B \coprod A,$$

natural in $A, B \in \mathrm{Obj}(\mathsf{Sets})$.

5. *Symmetric Monoidality.* The triple $(\mathsf{Sets}, \coprod, \emptyset)$ is a symmetric monoidal category.

Proof. *Item 1, Functoriality*: Omitted.
Item 2, Associativity: Clear.
Item 3, Unitality: Clear.
Item 4, Commutativity: Clear.
Item 5, Symmetric Monoidality: Omitted. □

2.3 Pushouts

Let A, B, and C be sets and let $f\colon C \to A$ and $g\colon C \to B$ be functions.

Definition 2.3.1.1. The **pushout of A and B over C along f and g**[6] is the set $A \coprod_C B$ defined by

$$A \coprod_C B \overset{\text{def}}{=} A \coprod B/{\sim_C},$$

where $\sim_C$ is the equivalence relation on $A \coprod B$ generated by $f(c) \sim_C g(c)$.

Remark 2.3.1.2. In detail, the relation $\sim$ of Definition 2.3.1.1 is given by declaring $a \sim b$ iff one of the following conditions is satisfied:

- We have $a, b \in A$ and $a = b$;
- We have $a, b \in B$ and $a = b$;

[6] *Further Terminology:* Also called the **fibre coproduct of A and B over C along f and g**.

- There exist $x_1, \ldots, x_n \in A \coprod B$ such that $a \sim' x_1 \sim' \cdots \sim' x_n \sim' b$, where we declare $x \sim' y$ if one of the following conditions is satisfied:

  1. There exists $c \in C$ such that $x = f(c)$ and $y = g(c)$.
  2. There exists $c \in C$ such that $x = g(c)$ and $y = f(c)$.

  That is: we require the following condition to be satisfied:

  ($\star$) There exist $x_1, \ldots, x_n \in A \coprod B$ satisfying the following conditions:
    1. There exists $c_0 \in C$ satisfying one of the following conditions:
       (a) We have $a = f(c_0)$ and $x_1 = g(c_0)$.
       (b) We have $a = g(c_0)$ and $x_1 = f(c_0)$.
    2. For each $1 \leq i \leq n-1$, there exists $c_i \in C$ satisfying one of the following conditions:
       (a) We have $x_i = f(c_i)$ and $x_{i+1} = g(c_i)$.
       (b) We have $x_i = g(c_i)$ and $x_{i+1} = f(c_i)$.
    3. There exists $c_n \in C$ satisfying one of the following conditions:
       (a) We have $x_n = f(c_n)$ and $b = g(c_n)$.
       (b) We have $x_n = g(c_n)$ and $b = f(c_n)$.

Example 2.3.1.3. Here are some examples of pushouts of sets.

1. *Wedge Sums of Pointed Sets.* The wedge sum of two pointed sets of **??** is an example of a pushout of sets.

2. *Intersections via Unions.* Let $A, B \subset X$. We have a bijection of sets

$$A \cup B \cong A \coprod_{A \cap B} B.$$

Proposition 2.3.1.4. Let A, B, C, and X be sets.

1. *Associativity.* We have an isomorphism of sets

$$(A \coprod_X B) \coprod_X C \cong A \coprod_X (B \coprod_X C),$$

   natural in $A, B, C, X \in \mathrm{Obj}(\mathsf{Sets})$.

2. *Unitality.* We have isomorphisms of sets

$$\emptyset \coprod_X A \cong A,$$
$$A \coprod_X \emptyset \cong A,$$

   natural in $A, X \in \mathrm{Obj}(\mathsf{Sets})$.

3. *Commutativity.* We have an isomorphism of sets

$$A \coprod_X B \cong B \coprod_X A,$$

natural in $A, B, X \in \mathrm{Obj}(\mathsf{Sets})$.

4. *Annihilation With the Empty Set.* We have isomorphisms of sets

$$A \coprod_X \emptyset \cong \emptyset,$$
$$\emptyset \coprod_X A \cong \emptyset,$$

natural in $A, X \in \mathrm{Obj}(\mathsf{Sets})$.

5. *Symmetric Monoidality.* The triple $(\mathsf{Sets}, \coprod_X, \emptyset)$ is a symmetric monoidal category.

Proof. *Item 1, Associativity*: Clear.
Item 2, Unitality: Clear.
Item 3, Commutativity: Clear.
Item 4, Annihilation With the Empty Set: Clear.
Item 5, Symmetric Monoidality: Omitted. □

2.4 Coequalisers

Let A and B be sets and let $f, g \colon A \rightrightarrows B$ be functions.

Definition 2.4.1.1. The **coequaliser of** f **and** g is the set $\mathrm{CoEq}(f, g)$ defined by

$$\mathrm{CoEq}(f, g) \stackrel{\text{def}}{=} B/{\sim},$$

where $\sim$ is the equivalence relation on B generated by $f(a) \sim g(a)$.

Remark 2.4.1.2. In detail, the relation $\sim$ of Definition 2.4.1.1 is given by declaring $a \sim b$ iff one of the following conditions is satisfied:

- We have $a = b$;
- There exist $x_1, \ldots, x_n \in B$ such that $a \sim' x_1 \sim' \cdots \sim' x_n \sim' b$, where we declare $x \sim' y$ if one of the following conditions is satisfied:
  1. There exists $z \in A$ such that $x = f(z)$ and $y = g(z)$.
  2. There exists $z \in A$ such that $x = g(z)$ and $y = f(z)$.

  That is: we require the following condition to be satisfied:

(★) There exist $x_1, \ldots, x_n \in B$ satisfying the following conditions:

1. There exists $z_0 \in A$ satisfying one of the following conditions:
   (a) We have $a = f(z_0)$ and $x_1 = g(z_0)$.
   (b) We have $a = g(z_0)$ and $x_1 = f(z_0)$.
2. For each $1 \leq i \leq n-1$, there exists $z_i \in A$ satisfying one of the following conditions:
   (a) We have $x_i = f(z_i)$ and $x_{i+1} = g(z_i)$.
   (b) We have $x_i = g(z_i)$ and $x_{i+1} = f(z_i)$.
3. There exists $z_n \in A$ satisfying one of the following conditions:
   (a) We have $x_n = f(z_n)$ and $b = g(z_n)$.
   (b) We have $x_n = g(z_n)$ and $b = f(z_n)$.

Example 2.4.1.3. Here are some examples of coequalisers of sets.

1. *Quotients by Equivalence Relations.* Let R be an equivalence relation on a set X. We have a bijection of sets

$$X/{\sim_R} \cong \mathrm{CoEq}\left(R \hookrightarrow X \times X \underset{\mathrm{pr}_2}{\overset{\mathrm{pr}_1}{\rightrightarrows}} X\right).$$

Proposition 2.4.1.4. Let A, B, and C be sets.

1. *Associativity.* We have an isomorphism of sets[7]

$$\underbrace{\mathrm{CoEq}(\mathrm{coeq}(f,g)\circ f, \mathrm{coeq}(f,g)\circ h)}_{=\mathrm{CoEq}(\mathrm{coeq}(f,g)\circ g, \mathrm{coeq}(f,g)\circ h)} \cong \mathrm{CoEq}(f,g,h) \cong \underbrace{\mathrm{CoEq}(\mathrm{coeq}(g,h)\circ f, \mathrm{coeq}(g,h)\circ g)}_{=\mathrm{CoEq}(\mathrm{coeq}(g,h)\circ f, \mathrm{coeq}(g,h)\circ h)},$$

---

[7] That is: the following constructions give the same result:

1. Take the coequaliser of (f, g, h), i.e. the colimit of the diagram

$$A \overset{f}{\underset{h}{\overset{\xrightarrow{\quad g \quad}}{\rightrightarrows}}} B$$

in Sets.

2. First take the coequaliser of f and g, forming a diagram

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \overset{\mathrm{coeq}(f,g)}{\twoheadrightarrow} \mathrm{CoEq}(f,g)$$

and then take the coequaliser of the composition

$$A \underset{h}{\overset{f}{\rightrightarrows}} B \overset{\mathrm{coeq}(f,g)}{\twoheadrightarrow} \mathrm{CoEq}(f,g),$$

where $\mathrm{CoEq}(f, g, h)$ is the colimit of the diagram

$$A \underset{h}{\overset{f}{\underset{\rightarrow g\rightarrow}{\rightrightarrows}}} B$$

in Sets.

4. *Unitality.* We have an isomorphism of sets

$$\mathrm{CoEq}(f, f) \cong B.$$

5. *Commutativity.* We have an isomorphism of sets

$$\mathrm{CoEq}(f, g) \cong \mathrm{CoEq}(g, f).$$

6. *Interaction With Composition.* Let

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \underset{k}{\overset{h}{\rightrightarrows}} C$$

be functions. We have a surjection

$$\mathrm{CoEq}(h \circ f, k \circ g) \twoheadrightarrow \mathrm{CoEq}\big(\mathrm{coeq}(h, k) \circ h \circ f, \mathrm{coeq}(h, k) \circ k \circ g\big)$$

exhibiting $\mathrm{CoEq}\big(\mathrm{coeq}(h, k) \circ h \circ f, \mathrm{coeq}(h, k) \circ k \circ g\big)$ as a quotient of $\mathrm{CoEq}(h \circ f, k \circ g)$ by the relation generated by declaring $h(y) \sim k(y)$ for each $y \in B$.

---

obtaining a quotient

$$\mathrm{CoEq}\big(\mathrm{coeq}(f, g) \circ f, \mathrm{coeq}(f, g) \circ h\big) = \mathrm{CoEq}\big(\mathrm{coeq}(f, g) \circ g, \mathrm{coeq}(f, g) \circ h\big)$$

of $\mathrm{CoEq}(f, g)$

3. First take the coequaliser of g and h, forming a diagram

$$A \underset{h}{\overset{g}{\rightrightarrows}} B \overset{\mathrm{coeq}(g,h)}{\twoheadrightarrow} \mathrm{CoEq}(g, h)$$

and then take the coequaliser of the composition

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \overset{\mathrm{coeq}(g,h)}{\twoheadrightarrow} \mathrm{CoEq}(g, h),$$

obtaining a quotient

$$\mathrm{CoEq}\big(\mathrm{coeq}(g, h) \circ f, \mathrm{coeq}(g, h) \circ g\big) = \mathrm{CoEq}\big(\mathrm{coeq}(g, h) \circ f, \mathrm{coeq}(g, h) \circ h\big)$$

of $\mathrm{CoEq}(g, h)$.

Proof. *Item 1*, *Associativity*: Omitted.
Item 4, *Unitality*: Clear.
Item 5, *Commutativity*: Clear.
Item 6, *Interaction With Composition*: Omitted. □

3 Operations With Sets

3.1 The Empty Set

Definition 3.1.1.1. The **empty set** is the set $\emptyset$ defined by

$$\emptyset \overset{\text{def}}{=} \{x \in X \mid x \neq x\},$$

where A is the set in the set existence axiom, **??** of **??**.

3.2 Singleton Sets

Let X be a set.

Definition 3.2.1.1. The **singleton set containing** X is the set $\{X\}$ defined by

$$\{X\} \overset{\text{def}}{=} \{X, X\},$$

where $\{X, X\}$ is the pairing of X with itself (Definition 3.3.1.1).

3.3 Pairings of Sets

Let X and Y be sets.

Definition 3.3.1.1. The **pairing of** X **and** Y is the set $\{X, Y\}$ defined by

$$\{X, Y\} \overset{\text{def}}{=} \{x \in A \mid x = X \text{ or } x = Y\},$$

where A is the set in the axiom of pairing, **??** of **??**.

3.4 Unions of Families

Let $\{A_i\}_{i \in I}$ be a family of sets.

Definition 3.4.1.1. The **union of the family** $\{A_i\}_{i \in I}$ is the set $\bigcup_{i \in I} A_i$ defined by

$$\bigcup_{i \in I} A_i \overset{\text{def}}{=} \{x \in F \mid \text{there exists some } i \in I \text{ such that } x \in A_i\},$$

where F is the set in the axiom of union, **??** of **??**.

3.5 Binary Unions

Let A and B be sets.

Definition 3.5.1.1. The **union**[8] **of** A **and** B is the set $A \cup B$ defined by

$$A \cup B \overset{\text{def}}{=} \bigcup_{z \in \{A,B\}} z.$$

Proposition 3.5.1.2. Let X be a set.

1. *Functoriality.* The assignments $U, V, (U, V) \mapsto U \cup V$ define functors

$$\begin{aligned} U \cup -\colon (\mathcal{P}(X), \subset) &\to (\mathcal{P}(X), \subset), \\ - \cup V\colon (\mathcal{P}(X), \subset) &\to (\mathcal{P}(X), \subset), \\ -_1 \cup -_2\colon (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \subset) &\to (\mathcal{P}(X), \subset), \end{aligned}$$

where $-_1 \cup -_2$ is the functor where

- *Action on Objects.* For each $(U, V) \in \mathcal{P}(X) \times \mathcal{P}(X)$, we have

$$[-_1 \cup -_2](U, V) \overset{\text{def}}{=} U \cup V;$$

- *Action on Morphisms.* For each pair of morphisms

$$\begin{aligned} \iota_U &\colon U \hookrightarrow U', \\ \iota_V &\colon V \hookrightarrow V' \end{aligned}$$

of $\mathcal{P}(X) \times \mathcal{P}(X)$, the image

$$\iota_U \cup \iota_V \colon U \cup V \hookrightarrow U' \cup V'$$

of (ι_U, ι_V) by $\cup$ is the inclusion

$$U \cup V \subset U' \cup V'$$

i.e. where we have

($\star$) If $U \subset U'$ and $V \subset V'$, then $U \cup V \subset U' \cup V'$;

and where $U \cup -$ and $- \cup V$ are the partial functors of $-_1 \cup -_2$ at $U, V \in \mathcal{P}(X)$.

[8] *Further Terminology:* Also called the **binary union of** A **and** B, for emphasis.

2. *Via Intersections and Symmetric Differences.* We have an equality of sets
$$U \cup V = (U \mathbin{\triangle} V) \mathbin{\triangle} (U \cap V)$$
for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

3. *Associativity.* We have an equality of sets
$$(U \cup V) \cup W = U \cup (V \cup W)$$
for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

4. *Unitality.* We have equalities of sets
$$\begin{aligned} U \cup \emptyset &= U, \\ \emptyset \cup U &= U \end{aligned}$$
for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

5. *Commutativity.* We have an equality of sets
$$U \cup V = V \cup U$$
for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

6. *Idempotency.* We have an equality of sets
$$U \cup U = U$$
for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

7. *Distributivity Over Intersections.* We have equalities of sets
$$\begin{aligned} U \cup (V \cap W) &= (U \cup V) \cap (U \cup W), \\ (U \cap V) \cup W &= (U \cup W) \cap (V \cup W) \end{aligned}$$
for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

8. *Interaction With Powersets and Semirings.* The quintuple $(\mathcal{P}(X), \cup, \cap, \emptyset, X)$ is an idempotent commutative semiring.

Proof. Item 1, *Functoriality*: Omitted.
Item 2, *Via Intersections and Symmetric Differences*: Omitted.
Item 3, *Associativity*: Clear.
Item 4, *Unitality*: Clear.
Item 5, *Commutativity*: Clear.
Item 6, *Idempotency*: Clear.
Item 7, *Distributivity Over Intersections*: Omitted.
Item 8, *Interaction With Powersets and Semirings*: This follows from Items 3 to 6 and Items 3 to 5, 7 and 8 of Proposition 3.7.1.2. □

3.6 Intersections of Families

Let $\mathcal{F}$ be a family of sets.

Definition 3.6.1.1. The **intersection of a family** $\mathcal{F}$ **of sets** is the set $\bigcap_{X\in\mathcal{F}} X$ defined by

$$\bigcap_{X\in\mathcal{F}} X \stackrel{\text{def}}{=} \left\{ z \in \bigcup_{X\in\mathcal{F}} X \,\middle|\, \text{for each } X \in \mathcal{F}\text{, we have } z \in X \right\}.$$

3.7 Binary Intersections

Let X and Y be sets.

Definition 3.7.1.1. The **intersection**[9] **of** X **and** Y is the set $X \cap Y$ defined by

$$X \cap Y \stackrel{\text{def}}{=} \bigcap_{z\in\{X,Y\}} z.$$

Proposition 3.7.1.2. Let X be a set.

1. *Functoriality.* The assignments $U, V, (U, V) \mapsto U \cap V$ define functors

$$\begin{aligned} U \cap -\colon (\mathcal{P}(X), \subset) &\to (\mathcal{P}(X), \subset), \\ - \cap V\colon (\mathcal{P}(X), \subset) &\to (\mathcal{P}(X), \subset), \\ -_1 \cap -_2\colon (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \subset) &\to (\mathcal{P}(X), \subset), \end{aligned}$$

where $-_1 \cap -_2$ is the functor where

- *Action on Objects.* For each $(U, V) \in \mathcal{P}(X) \times \mathcal{P}(X)$, we have

$$[-_1 \cap -_2](U, V) \stackrel{\text{def}}{=} U \cap V;$$

- *Action on Morphisms.* For each pair of morphisms

$$\begin{aligned} \iota_U &\colon U \hookrightarrow U', \\ \iota_V &\colon V \hookrightarrow V' \end{aligned}$$

of $\mathcal{P}(X) \times \mathcal{P}(X)$, the image

$$\iota_U \cap \iota_V \colon U \cap V \hookrightarrow U' \cap V'$$

of (ι_U, ι_V) by $\cap$ is the inclusion

$$U \cap V \subset U' \cap V'$$

i.e. where we have

[9] *Further Terminology:* Also called the **binary intersection of** X **and** Y, for emphasis.

(★) If $U \subset U'$ and $V \subset V'$, then $U \cap V \subset U' \cap V'$;

and where $U \cap -$ and $- \cap V$ are the partial functors of $-_1 \cap -_2$ at $U, V \in \mathcal{P}(X)$.

2. *Adjointness.* We have adjunctions

$$\big(U \cap - \dashv \mathbf{Hom}_{\mathcal{P}(X)}(U, -)\big)\colon \quad \begin{tikzcd} \mathcal{P}(X) \arrow[r, bend left, "U\cap-"] \arrow[r, phantom, "\perp"] & \mathcal{P}(X), \arrow[l, bend left, "\mathbf{Hom}_{\mathcal{P}(X)}(U,-)"] \end{tikzcd}$$

$$\big(- \cap V \dashv \mathbf{Hom}_{\mathcal{P}(X)}(V, -)\big)\colon \quad \begin{tikzcd} \mathcal{P}(X) \arrow[r, bend left, "-\cap V"] \arrow[r, phantom, "\perp"] & \mathcal{P}(X), \arrow[l, bend left, "\mathbf{Hom}_{\mathcal{P}(X)}(V,-)"] \end{tikzcd}$$

where

$$\mathbf{Hom}_{\mathcal{P}(X)}(-_1, -_2)\colon \mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \to \mathcal{P}(X)$$

is the bifunctor defined by[10]

$$\mathbf{Hom}_{\mathcal{P}(X)}(U, V) \stackrel{\text{def}}{=} (X \setminus U) \cup V$$

witnessed by bijections

$$\begin{aligned} \mathrm{Hom}_{\mathcal{P}(X)}(U \cap V, W) &\cong \mathrm{Hom}_{\mathcal{P}(X)}\big(U, \mathbf{Hom}_{\mathcal{P}(X)}(V, W)\big), \\ \mathrm{Hom}_{\mathcal{P}(X)}(U \cap V, W) &\cong \mathrm{Hom}_{\mathcal{P}(X)}\big(V, \mathbf{Hom}_{\mathcal{P}(X)}(U, W)\big), \end{aligned}$$

natural in $U, V, W \in \mathcal{P}(X)$, i.e. where:

(a) The following conditions are equivalent:

  i. We have $U \cap V \subset W$.

  ii. We have $U \subset \mathbf{Hom}_{\mathcal{P}(X)}(V, W)$.

  iii. We have $U \subset (X \setminus V) \cup W$.

(b) The following conditions are equivalent:

  i. We have $V \cap U \subset W$.

  ii. We have $V \subset \mathbf{Hom}_{\mathcal{P}(X)}(U, W)$.

  iii. We have $V \subset (X \setminus U) \cup W$.

10 *Intuition:* Since intersections are the products in $\mathcal{P}(X)$, the left adjoint $\mathbf{Hom}_{\mathcal{P}(X)}(U, V)$ works as a function type $U \to V$.

Now, under the Curry–Howard correspondence, the function type $U \to V$ corresponds to implication $U \implies V$, which is logically equivalent to the statement $\neg U \vee V$, which in turn corresponds to the set $U^{\mathsf{c}} \vee V \stackrel{\text{def}}{=} (X \setminus U) \cup V$.

3. *Associativity.* We have an equality of sets

$$(U \cap V) \cap W = U \cap (V \cap W)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

4. *Unitality.* Let X be a set and let $U \in \mathcal{P}(X)$. We have equalities of sets

$$\begin{aligned} X \cap U &= U, \\ U \cap X &= U \end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

5. *Commutativity.* We have an equality of sets

$$U \cap V = V \cap U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

6. *Idempotency.* We have an equality of sets

$$U \cap U = U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

7. *Distributivity Over Unions.* We have equalities of sets

$$\begin{aligned} U \cap (V \cup W) &= (U \cap V) \cup (U \cap W), \\ (U \cup V) \cap W &= (U \cap W) \cup (V \cap W) \end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

8. *Annihilation With the Empty Set.* We have an equality of sets

$$\begin{aligned} \emptyset \cap X &= \emptyset, \\ X \cap \emptyset &= \emptyset \end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

9. *Interaction With Powersets and Monoids With Zero.* The quadruple $((\mathcal{P}(X), \emptyset), \cap, X)$ is a commutative monoid with zero.

10. *Interaction With Powersets and Semirings.* The quintuple $(\mathcal{P}(X), \cup, \cap, \emptyset, X)$ is an idempotent commutative semiring.

Proof. *Item 1, Functoriality*: Omitted.
Item 2, Adjointness: See [MSE 267469].
Item 3, Associativity: Clear.
Item 4, Unitality: Clear.
Item 5, Commutativity: Clear.
Item 6, Idempotency: Clear.
Item 7, Distributivity Over Unions: Omitted.
Item 8, Annihilation With the Empty Set: Clear.
Item 9, Interaction With Powersets and Monoids With Zero: This follows from Items 3 to 5 and 8.
Item 10, Interaction With Powersets and Semirings: This follows from Items 3 to 6 and Items 3 to 5, 7 and 8 of Proposition 3.7.1.2. □

3.8 Differences

Let X and Y be sets.

Definition 3.8.1.1. The **difference of** X **and** Y is the set $X \setminus Y$ defined by

$$X \setminus Y \stackrel{\text{def}}{=} \{a \in X \mid a \notin Y\}.$$

Proposition 3.8.1.2. Let X be a set.

1. *Functoriality.* The assignments $U, V, (U, V) \mapsto U \cap V$ define functors

$$\begin{aligned} U \setminus -&\colon (\mathcal{P}(X), \supset) \to (\mathcal{P}(X), \subset), \\ - \setminus V&\colon (\mathcal{P}(X), \subset) \to (\mathcal{P}(X), \subset), \\ -_1 \setminus -_2&\colon (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \supset) \to (\mathcal{P}(X), \subset), \end{aligned}$$

where $-_1 \setminus -_2$ is the functor where

- *Action on Objects.* For each $(U, V) \in \mathcal{P}(X) \times \mathcal{P}(X)$, we have

$$[-_1 \setminus -_2](U, V) \stackrel{\text{def}}{=} U \setminus V;$$

- *Action on Morphisms.* For each pair of morphisms

$$\begin{aligned} \iota_A &\colon A \hookrightarrow B, \\ \iota_U &\colon U \hookrightarrow V \end{aligned}$$

of $\mathcal{P}(X) \times \mathcal{P}(X)$, the image

$$\iota_U \setminus \iota_V \colon A \setminus V \hookrightarrow B \setminus U$$

of (ι_U, ι_V) by $\setminus$ is the inclusion

$$A \setminus V \subset B \setminus U$$

i.e. where we have

($\star$) If $A \subset B$ and $U \subset V$, then $A \setminus V \subset B \setminus U$;

and where $U \setminus -$ and $- \setminus V$ are the partial functors of $-_1 \setminus -_2$ at $U, V \in \mathcal{P}(X)$.

2. *De Morgan's Laws.* We have equalities of sets

$$\begin{aligned} X \setminus (U \cup V) &= (X \setminus U) \cap (X \setminus V), \\ X \setminus (U \cap V) &= (X \setminus U) \cup (X \setminus V) \end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

3. *Interaction With Unions I.* We have equalities of sets

$$(U \setminus V) \cup W = (U \cup W) \setminus (V \setminus W)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

4. *Interaction With Unions II.* We have equalities of sets

$$(U \cup V) \setminus W = (U \setminus W) \cup (V \setminus W)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

5. *Interaction With Intersections.* We have equalities of sets

$$\begin{aligned} (U \setminus V) \cap W &= (U \cap W) \setminus V \\ &= U \cap (W \setminus V) \end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

6. *Triple Differences.* We have

$$U \setminus (V \setminus W) = (U \cap W) \cup (U \setminus V)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

7. *Left Annihilation.* We have

$$\emptyset \setminus U = \emptyset$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

8. *Right Unitality.* We have
$$U \setminus \emptyset = U$$
for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

9. *Invertibility.* We have
$$U \setminus U = \emptyset$$
for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

10. *Interaction With Containment.* The following conditions are equivalent:
    (a) We have $V \setminus U \subset W$.
    (b) We have $V \setminus W \subset U$.

Proof. Item 1, *Functoriality*: Omitted.
Item 2, *De Morgan's Laws*: Omitted.
Item 3, *Interaction With Unions I*: Omitted.
Item 4, *Interaction With Unions II*: Omitted.
Item 5, *Interaction With Intersections*: Omitted.
Item 6, *Triple Differences*: Omitted.
Item 7, *Left Annihilation*: Clear.
Item 8, *Right Unitality*: Clear.
Item 9, *Invertibility*: Clear.
Item 10, *Interaction With Containment*: Omitted. □

3.9 Complements

Let X be a set and let $U \in \mathcal{P}(X)$.

Definition 3.9.1.1. The **complement of** U is the set U^{c} defined by

$$\begin{aligned} U^{\mathsf{c}} &\stackrel{\text{def}}{=} X \setminus U \\ &\stackrel{\text{def}}{=} \{a \in X \mid a \notin U\}. \end{aligned}$$

Proposition 3.9.1.2. Let X be a set.

1. *Functoriality.* The assignment $U \mapsto U^{\mathsf{c}}$ defines a functor
$$(-)^{\mathsf{c}} \colon \mathcal{P}(X)^{\mathsf{op}} \to \mathcal{P}(X),$$
where

- *Action on Objects.* For each $U \in \mathcal{P}(X)$, we have

$$[(-)^{\mathsf{c}}](U) \stackrel{\text{def}}{=} U^{\mathsf{c}};$$

- *Action on Morphisms.* For each morphism $\iota_U : U \hookrightarrow V$ of $\mathcal{P}(X)$, the image

$$\iota_U^{\mathsf{c}} : V^{\mathsf{c}} \hookrightarrow U^{\mathsf{c}}$$

of ι_U by $(-)^{\mathsf{c}}$ is the inclusion

$$V^{\mathsf{c}} \subset U^{\mathsf{c}}$$

i.e. where we have

($\star$) If $U \subset V$, then $V^{\mathsf{c}} \subset U^{\mathsf{c}}$.

2. *De Morgan's Laws.* We have equalities of sets

$$\begin{aligned}(U \cup V)^{\mathsf{c}} &= U^{\mathsf{c}} \cap V^{\mathsf{c}},\\ (U \cap V)^{\mathsf{c}} &= U^{\mathsf{c}} \cup V^{\mathsf{c}}\end{aligned}$$

for each $X \in \text{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

3. *Involutority.* We have

$$(U^{\mathsf{c}})^{\mathsf{c}} = U$$

for each $X \in \text{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

Proof. *Item 1*, *Functoriality*: Clear.
Item 2, *De Morgan's Laws*: Omitted.
Item 3, *Involutority*: Clear. □

3.10 Symmetric Differences

Let A and B be sets.

Definition 3.10.1.1. The **symmetric difference of** A **and** B is the set $A \triangle B$ defined by

$$A \triangle B \stackrel{\text{def}}{=} (A \setminus B) \cup (B \setminus A).$$

Proposition 3.10.1.2. Let X be a set.

1. *Lack of Functoriality.* The assignment $(U, V) \mapsto U \triangle V$ ***does not*** define a functor

$$-_1 \triangle -_2 : (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \subset) \to (\mathcal{P}(X), \subset).$$

2. *Via Unions and Intersections.* We have[11]

$$U \mathbin{\triangle} V = (U \cup V) \setminus (U \cap V)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

3. *Associativity.* We have[12]

$$(U \mathbin{\triangle} V) \mathbin{\triangle} W = U \mathbin{\triangle} (V \mathbin{\triangle} W)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

4. *Unitality.* We have

$$\begin{aligned} U \mathbin{\triangle} \emptyset &= U, \\ \emptyset \mathbin{\triangle} U &= U \end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

5. *Invertibility.* We have

$$U \mathbin{\triangle} U = \emptyset$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

6. *Commutativity.* We have

$$U \mathbin{\triangle} V = V \mathbin{\triangle} U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

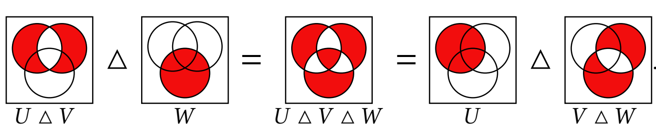
7. *"Transitivity".* We have

$$(U \mathbin{\triangle} V) \mathbin{\triangle} (V \mathbin{\triangle} W) = U \mathbin{\triangle} W$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

[11] *Illustration:*

[12] *Illustration:*



8. *The Triangle Inequality for Symmetric Differences.* We have

$$U \mathbin{\triangle} W \subset U \mathbin{\triangle} V \cup V \mathbin{\triangle} W$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

9. *Distributivity Over Intersections.* We have

$$\begin{aligned} U \cap (V \mathbin{\triangle} W) &= (U \cap V) \mathbin{\triangle} (U \cap W), \\ (U \mathbin{\triangle} V) \cap W &= (U \cap W) \mathbin{\triangle} (V \cap W) \end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

10. *Interaction With Indicator Functions.* We have

$$\chi_{U \triangle V} \equiv \chi_U + \chi_V \mod 2$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

11. *Bijectivity.* Given $A, B \subset \mathcal{P}(X)$, the maps

$$\begin{aligned} A \mathbin{\triangle} -&\colon \mathcal{P}(X) \to \mathcal{P}(X), \\ - \mathbin{\triangle} B&\colon \mathcal{P}(X) \to \mathcal{P}(X) \end{aligned}$$

are bijections with inverses given by

$$\begin{aligned} (A \mathbin{\triangle} -)^{-1} &= - \cup (A \cap -), \\ (- \mathbin{\triangle} B)^{-1} &= - \cup (B \cap -). \end{aligned}$$

Moreover, the map

$$C \mapsto C \mathbin{\triangle} (A \mathbin{\triangle} B)$$

is a bijection of $\mathcal{P}(X)$ onto itself sending A to B and B to A.

12. *Interaction With Powersets and Groups I.* The quadruple $\left(\mathcal{P}(X), \triangle, \emptyset, \mathrm{id}_{\mathcal{P}(X)}\right)$ is

an abelian group.[13,14,15]

13. *Interaction With Powersets and Groups II.* Every element of $\mathcal{P}(X)$ has order 2 with respect to $\triangle$, and thus $\mathcal{P}(X)$ is a *Boolean group* (i.e. an abelian 2-group).

14. *Interaction With Powersets and Vector Spaces I.* The pair $\left(\mathcal{P}(X), \alpha_{\mathcal{P}(X)}\right)$ consisting of

    - The group $\mathcal{P}(X)$ of Item 12;
    - The map $\alpha_{\mathcal{P}(X)} \colon \mathbb{F}_2 \times \mathcal{P}(X) \to \mathcal{P}(X)$ defined by

    $$\begin{aligned} 0 \cdot U &\stackrel{\text{def}}{=} \emptyset, \\ 1 \cdot U &\stackrel{\text{def}}{=} U; \end{aligned}$$

    is an $\mathbb{F}_2$-vector space.

15. *Interaction With Powersets and Vector Spaces II.* If X is finite, then:

    (a) The set of singletons sets on the elements of X forms a basis for the $\mathbb{F}_2$-vector space $\left(\mathcal{P}(X), \alpha_{\mathcal{P}(X)}\right)$ of Item 14.

    (b) We have
    $$\dim(\mathcal{P}(X)) = \#\mathcal{P}(X).$$

16. *Interaction With Powersets and Rings.* The quintuple $(\mathcal{P}(X), \triangle, \cap, \emptyset, X)$ is a commutative ring.[16]

[13] *Example:* When $X = \emptyset$, we have an isomorphism of groups between $\mathcal{P}(\emptyset)$ and the trivial group:
$$\left(\mathcal{P}(\emptyset), \triangle, \emptyset, \mathrm{id}_{\mathcal{P}(\emptyset)}\right) \cong \mathrm{pt}.$$

[14] *Example:* When $X = \mathrm{pt}$, we have an isomorphism of groups between $\mathcal{P}(\mathrm{pt})$ and $\mathbb{Z}_{/2}$:
$$\left(\mathcal{P}(\mathrm{pt}), \triangle, \emptyset, \mathrm{id}_{\mathcal{P}(\mathrm{pt})}\right) \cong \mathbb{Z}_{/2}.$$

[15] *Example:* When $X = \{0, 1\}$, we have an isomorphism of groups between $\mathcal{P}(\{0, 1\})$ and $\mathbb{Z}_{/2} \times \mathbb{Z}_{/2}$:
$$\left(\mathcal{P}(\{0, 1\}), \triangle, \emptyset, \mathrm{id}_{\mathcal{P}(\{0,1\})}\right) \cong \mathbb{Z}_{/2} \times \mathbb{Z}_{/2}.$$

[16] *Warning:* The analogous statement replacing intersections by unions (i.e. that the quintuple $(\mathcal{P}(X), \triangle, \cup, \emptyset, X)$ is a ring) is false, however. See [Pro23b] for a proof.

Proof. *Item 1, Lack of Functoriality*: Omitted.
Item 2, Via Unions and Intersections: Omitted.
Item 3, Associativity: Omitted.
Item 4, Unitality: Clear.
Item 5, Invertibility: Clear.
Item 6, Commutativity: Clear.
Item 7, "Transitivity": We have

$$\begin{aligned}(U \bigtriangleup V) \bigtriangleup (V \bigtriangleup W) &= U \bigtriangleup (V \bigtriangleup (V \bigtriangleup W)) && \text{(by Item 3)}\\ &= U \bigtriangleup ((V \bigtriangleup V) \bigtriangleup W) && \text{(by Item 3)}\\ &= U \bigtriangleup (\emptyset \bigtriangleup W) && \text{(by Item 5)}\\ &= U \bigtriangleup W && \text{(by Item 4)}\end{aligned}$$

Item 8, The Triangle Inequality for Symmetric Differences: This follows from Items 2 and 7.
Item 9, Distributivity Over Intersections: Omitted.
Item 10, Interaction With Indicator Functions: Clear.
Item 11, Bijectivity: Clear.
Item 12, Interaction With Powersets and Groups I: This follows from Items 3 to 6.
Item 13, Interaction With Powersets and Groups II: This follows from Item 5.
Item 14, Interaction With Powersets and Vector Spaces I: Clear.
Item 15, Interaction With Powersets and Vector Spaces II: Omitted.
Item 16, Interaction With Powersets and Rings: This follows from Items 9 and 12 and Items 8 and 9 of Proposition 3.7.1.2.[17] □

3.11 Ordered Pairs

Let A and B be sets.

Definition 3.11.1.1. The **ordered pair associated to** A **and** B is the set (A, B) defined by

$$(A, B) \overset{\text{def}}{=} \{\{A\}, \{A, B\}\}.$$

Proposition 3.11.1.2. Let A and B be sets.

1. *Uniqueness.* Let A, B, C, and D be sets. The following conditions are equivalent:
   (a) We have $(A, B) = (C, D)$.
   (b) We have $A = C$ and $B = D$.

Proof. *Item 1, Uniqueness*: See [Cie97, Theorem 1.2.3]. □

[17] *Reference:* [Pro23a].

4 Powersets

4.1 Characteristic Functions

Let X be a set.

Definition 4.1.1.1. Let $U \subset X$ and let $x \in X$.

1. The **characteristic function of** U[18] is the function[19]

$$\chi_U : X \to \{\mathsf{t}, \mathsf{f}\}$$

defined by

$$\chi_U(x) \stackrel{\text{def}}{=} \begin{cases} \mathsf{true} & \text{if } x \in U, \\ \mathsf{false} & \text{if } x \notin U \end{cases}$$

for each $x \in X$.

2. The **characteristic function of** x is the function[20]

$$\chi_x : X \to \{\mathsf{t}, \mathsf{f}\}$$

defined by

$$\chi_x \stackrel{\text{def}}{=} \chi_{\{x\}},$$

i.e. by

$$\chi_x(y) \stackrel{\text{def}}{=} \begin{cases} \mathsf{true} & \text{if } x = y, \\ \mathsf{false} & \text{if } x \neq y \end{cases}$$

for each $y \in X$.

3. The **characteristic relation on** X[21] is the relation[22]

$$\chi_X(-_1, -_2) : X \times X \to \{\mathsf{t}, \mathsf{f}\}$$

on X defined by[23]

$$\chi_X(x, y) \stackrel{\text{def}}{=} \begin{cases} \mathsf{true} & \text{if } x = y, \\ \mathsf{false} & \text{if } x \neq y \end{cases}$$

for each $x, y \in X$.

---

[18] *Further Terminology:* Also called the **indicator function of** U.
[19] *Further Notation:* Also written $\chi_X(U, -)$ or $\chi_X(-, U)$.
[20] *Further Notation:* Also written χ_x, $\chi_X(x, -)$, or $\chi_X(-, x)$.
[21] *Further Terminology:* Also called the **identity relation on** X.
[22] *Further Notation:* Also written χ^{-1}_{-2}, or $\sim_{\mathrm{id}}$ in the context of relations.
[23] As a subset of $X \times X$, the relation χ_X corresponds to the diagonal $\Delta_X \subset X \times X$ of X.

4. The **characteristic embedding**[24] **of** X **into** $\mathcal{P}(X)$ is the function

$$\chi_{(-)} \colon X \hookrightarrow \mathcal{P}(X)$$

defined by

$$\chi_{(-)}(x) \stackrel{\text{def}}{=} \chi_x$$

for each $x \in X$.

Remark 4.1.1.2. The definitions in Definition 4.1.1.1 are decategorifications of co/presheaves, representable co/presheaves, Hom profunctors, and the Yoneda embedding:[25]

1. A function

$$f \colon X \to \{\mathsf{t}, \mathsf{f}\}$$

is a decategorification of a presheaf

$$\mathcal{F} \colon C^{\mathsf{op}} \to \mathsf{Sets},$$

with the characteristic functions χ_U of the subsets of X being the primordial examples (and, in fact, all examples) of these.

[24] The name "characteristic *embedding*" comes from the fact that there is an analogue of fully faithfulness for $\chi_{(-)}$: given a set X, we have

$$\mathrm{Hom}_{\mathcal{P}(X)}(\chi_x, \chi_y) = \chi_X(x, y),$$

for each $x, y \in X$.

[25] These statements can be made precise by using the embeddings

$$(-)_{\mathsf{disc}} \colon \mathsf{Sets} \hookrightarrow \mathsf{Cats},$$
$$(-)_{\mathsf{disc}} \colon \{\mathsf{t}, \mathsf{f}\}_{\mathsf{disc}} \hookrightarrow \mathsf{Sets}$$

of sets into categories and of classical truth values into sets.

For instance, in this approach the characteristic function

$$\chi_x \colon X \to \{\mathsf{t}, \mathsf{f}\}$$

of an element x of X, defined by

$$\chi_x(y) \stackrel{\text{def}}{=} \begin{cases} \mathsf{true} & \text{if } x = y, \\ \mathsf{false} & \text{if } x \neq y \end{cases}$$

for each $y \in X$, is recovered as the representable presheaf

$$\mathrm{Hom}_{X_{\mathsf{disc}}}(-, x) \colon X_{\mathsf{disc}} \to \mathsf{Sets}$$

of the corresponding object x of X_{disc}, defined on objects by

$$\mathrm{Hom}_{X_{\mathsf{disc}}}(y, x) \stackrel{\text{def}}{=} \begin{cases} \mathrm{pt} & \text{if } x = y, \\ \emptyset & \text{if } x \neq y \end{cases}$$

for each $y \in \mathrm{Obj}(X_{\mathsf{disc}})$.

2. The characteristic function
$$\chi_x \colon X \to \{\mathsf{t}, \mathsf{f}\}$$
of an *element* x of X is a decategorification of the representable presheaf
$$h_X \colon \mathcal{C}^{\mathsf{op}} \to \mathsf{Sets}$$
of an *object* x of a category $\mathcal{C}$.

3. The characteristic relation
$$\chi_X(-_1, -_2) \colon X \times X \to \{\mathsf{t}, \mathsf{f}\}$$
of X is a decategorification of the Hom profunctor
$$\mathrm{Hom}_{\mathcal{C}}(-_1, -_2) \colon \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathsf{Sets}$$
of a category $\mathcal{C}$.

4. The characteristic embedding
$$\chi_{(-)} \colon X \hookrightarrow \mathcal{P}(X)$$
of X into $\mathcal{P}(X)$ is a decategorification of the Yoneda embedding
$$よ \colon \mathcal{C}^{\mathsf{op}} \hookrightarrow \mathsf{PSh}(\mathcal{C})$$
of a category $\mathcal{C}$ into $\mathsf{PSh}(\mathcal{C})$.

5. There is also a direct parallel between unions and colimits:
   - An element of $\mathcal{P}(X)$ is a union of elements of X, viewed as one-point subsets $\{x\} \in \mathcal{P}(A)$;
   - An object of $\mathsf{PSh}(\mathcal{C})$ is a colimit of objects of $\mathcal{C}$, viewed as representable presheaves $h_X \in \mathrm{Obj}(\mathsf{PSh}(\mathcal{C}))$.

Proposition 4.1.1.3. Let $f \colon A \to B$ be a function. We have an inclusion
$$\chi_B \circ (f \times f) \subset \chi_A, \qquad \begin{tikzcd} A \times A \arrow[r, "\chi_A(-_1,-_2)"] \arrow[d, "f\times f"'] \arrow[dr, phantom, "\subset"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ B \times B \arrow[r, "\chi_B(-_1,-_2)"'] & \{\mathsf{true}, \mathsf{false}\}. \end{tikzcd}$$

Proof. The inclusion $\chi_B(f(a), f(b)) \subset \chi_A(a, b)$ is equivalent to the statement "if $a = b$, then $f(a) = f(b)$", which is true. □

Proposition 4.1.1.4. Let X be a set and let $U \subset X$ be a subset of X. We have

$$\mathrm{Hom}_{\mathcal{P}(X)}(\chi_x, \chi_U) = \chi_U(x)$$

for each $x \in X$, giving an equality of functions

$$\mathrm{Hom}_{\mathcal{P}(X)}\big(\chi_{(-)}, \chi_U\big) = \chi_U.$$

Proof. Clear. □

Corollary 4.1.1.5. The characteristic embedding is fully faithful, i.e., we have

$$\mathrm{Hom}_{\mathcal{P}(X)}\big(\chi_x, \chi_y\big) = \chi_X(x, y)$$

for each $x, y \in X$.

Proof. This follows from Proposition 4.1.1.4. □

4.2 Powersets

Let X be a set.

Definition 4.2.1.1. The **powerset of** X is the set $\mathcal{P}(X)$ defined by

$$\mathcal{P}(X) \stackrel{\text{def}}{=} \{U \in P \mid U \subset X\},$$

where P is the set in the axiom of powerset, **??** of **??**.

Remark 4.2.1.2. The powerset of a set is a decategorification of the category of presheaves of a category: while[26]

[26] This parallel is based on the following comparison:

- A category is enriched over the category

$$\mathsf{Sets} \stackrel{\text{def}}{=} \mathsf{Cats}_0$$

  of sets (i.e. "0-categories"), with presheaves taking values on it;

- A set is enriched over the set

$$\{\mathsf{t}, \mathsf{f}\} \stackrel{\text{def}}{=} \mathsf{Cats}_{-1}$$

  of classical truth values (i.e. "(-1)-categories"), with characteristic functions taking values on it.

- The powerset of a set X is equivalently (Item 6 of Proposition 4.2.1.3) the set

$$\mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\})$$

of functions from X to the set $\{\mathsf{t}, \mathsf{f}\}$ of classical truth values;

- The category of presheaves on a category $\mathcal{C}$ is the category

$$\mathsf{Fun}(\mathcal{C}^{\mathsf{op}}, \mathsf{Sets})$$

of functors from $\mathcal{C}^{\mathsf{op}}$ to the category Sets of sets.

Proposition 4.2.1.3. Let X be a set.

1. *Functoriality.* The assignment $X \mapsto \mathcal{P}(X)$ defines functors

$$\begin{aligned} \mathcal{P}_* &\colon \mathsf{Sets} \to \mathsf{Sets}, \\ \mathcal{P}^{-1} &\colon \mathsf{Sets}^{\mathsf{op}} \to \mathsf{Sets}, \\ \mathcal{P}_! &\colon \mathsf{Sets} \to \mathsf{Sets} \end{aligned}$$

where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$\begin{aligned} \mathcal{P}_*(A) &\stackrel{\text{def}}{=} \mathcal{P}(A), \\ \mathcal{P}^{-1}(A) &\stackrel{\text{def}}{=} \mathcal{P}(A), \\ \mathcal{P}_!(A) &\stackrel{\text{def}}{=} \mathcal{P}(A); \end{aligned}$$

- *Action on Morphisms.* For each morphism $f \colon A \to B$ of Sets, the images

$$\begin{aligned} \mathcal{P}_*(f) &\colon \mathcal{P}(A) \to \mathcal{P}(B), \\ \mathcal{P}^{-1}(f) &\colon \mathcal{P}(B) \to \mathcal{P}(A), \\ \mathcal{P}_!(f) &\colon \mathcal{P}(A) \to \mathcal{P}(B) \end{aligned}$$

of f by $\mathcal{P}_*$, $\mathcal{P}^{-1}$, and $\mathcal{P}_!$ are defined by

$$\begin{aligned} \mathcal{P}_*(f) &\stackrel{\text{def}}{=} f_*, \\ \mathcal{P}^{-1}(f) &\stackrel{\text{def}}{=} f^{-1}, \\ \mathcal{P}_!(f) &\stackrel{\text{def}}{=} f_!, \end{aligned}$$

as in Definitions 4.3.1.1, 4.4.1.1 and 4.5.1.1.

2. *Adjointness I.* We have an adjunction

$$\left(\mathcal{P}^{-1} \dashv \mathcal{P}^{-1,\mathsf{op}}\right)\colon \quad \begin{tikzcd} \mathsf{Sets}^{\mathsf{op}} \arrow[r, bend left, "\mathcal{P}^{-1}"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}, \arrow[l, bend left, "\mathcal{P}^{-1,\mathsf{op}}"] \end{tikzcd}$$

witnessed by a bijection

$$\underbrace{\mathsf{Sets}^{\mathsf{op}}(\mathcal{P}(X), Y)}_{\stackrel{\text{def}}{=} \mathsf{Sets}(Y, \mathcal{P}(X))} \cong \mathsf{Sets}(X, \mathcal{P}(Y)),$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets})$ and $Y \in \mathrm{Obj}\left(\mathsf{Sets}^{\mathsf{op}}\right)$.

3. *Adjointness II.* We have an adjunction

$$(\mathrm{Gr} \dashv \mathcal{P}_*)\colon \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "\mathrm{Gr}"] \arrow[r, phantom, "\perp"] & \mathrm{Rel}, \arrow[l, bend left, "\mathcal{P}_*"] \end{tikzcd}$$

witnessed by a bijection of sets

$$\mathrm{Rel}(\mathrm{Gr}(A), B) \cong \mathsf{Sets}(A, \mathcal{P}(B))$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$ and $B \in \mathrm{Obj}(\mathrm{Rel})$, where Gr is the graph functor of Relations, Item 1 of Proposition 3.1.1.2.

4. *Symmetric Strong Monoidality With Respect to Coproducts.* The powerset functor $\mathcal{P}_*$ of Item 1 has a symmetric strong monoidal structure

$$\left(\mathcal{P}_*, \mathcal{P}_*^{\coprod}, \mathcal{P}_{*|\mathbb{1}}^{\coprod}\right)\colon (\mathsf{Sets}, \textstyle\coprod, \emptyset) \longrightarrow (\mathsf{Sets}, \times, \mathrm{pt})$$

being equipped with isomorphisms

$$\mathcal{P}_{*|X,Y}^{\coprod}\colon \mathcal{P}(X) \times \mathcal{P}(Y) \xrightarrow{\cong} \mathcal{P}(X \textstyle\coprod Y),$$
$$\mathcal{P}_{*|\mathbb{1}}^{\coprod}\colon \mathrm{pt} \xrightarrow{=} \mathcal{P}(\emptyset),$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

5. *Symmetric Lax Monoidality With Respect to Products.* The powerset functor $\mathcal{P}_*$ of Item 1 has a symmetric lax monoidal structure

$$\left(\mathcal{P}_*, \mathcal{P}_*^{\otimes}, \mathcal{P}_{*|\mathbb{1}}^{\otimes}\right)\colon (\mathsf{Sets}, \times, \mathrm{pt}) \longrightarrow (\mathsf{Sets}, \times, \mathrm{pt})$$

being equipped with isomorphisms

$$\mathcal{P}^{\otimes}_{*|X,Y} \colon \mathcal{P}(X) \times \mathcal{P}(Y) \to \mathcal{P}(X \times Y),$$
$$\mathcal{P}^{\otimes}_{*|\mathbb{1}} \colon \mathrm{pt} \xrightarrow{=} \mathcal{P}(\emptyset),$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$, where $\mathcal{P}^{\otimes}_{*|X,Y}$ is given by

$$\mathcal{P}^{\otimes}_{*|X,Y}(U, V) \overset{\mathrm{def}}{=} U \times V$$

for each $(U, V) \in \mathcal{P}(X) \times \mathcal{P}(Y)$.

6. *Powersets as Sets of Functions.* The assignment $U \mapsto \chi_U$ defines a bijection[27]

$$\chi_{(-)} \colon \mathcal{P}(X) \xrightarrow{\cong} \mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\}),$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets})$.

7. *Powersets as Sets of Relations.* We have bijections

$$\mathcal{P}(X) \cong \mathrm{Rel}(\mathrm{pt}, X),$$
$$\mathcal{P}(X) \cong \mathrm{Rel}(X, \mathrm{pt}),$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets})$.

8. *As a Free Cocompletion: Universal Property.* The pair $(\mathcal{P}(X), \chi_{(-)})$ consisting of

   - The powerset $\mathcal{P}(X)$ of X;
   - The characteristic embedding $\chi_{(-)} \colon X \hookrightarrow \mathcal{P}(X)$ of X into $\mathcal{P}(X)$;

   satisfies the following universal property:

   (★) Given another pair (Y, f) consisting of
   - A cocomplete poset $(Y, \preceq)$;
   - A function $f \colon X \to Y$;

   there exists a unique cocontinuous morphism of posets $(\mathcal{P}(X), \subset) \xrightarrow{\exists!}$

[27] This bijection is a decategorified form of the equivalence

$$\mathsf{PSh}(\mathcal{C}) \overset{\text{eq.}}{\cong} \mathsf{DFib}(\mathcal{C})$$

of Fibred Categories, **??** of **??**, with $\chi_{(-)}$ being a decategorified version of the category of elements construction of Fibred Categories, **??**.

See also **??** of **??**.

$(Y, \preceq)$ making the diagram

$$\begin{tikzcd} & \mathcal{P}(X) \arrow[d, dashed, "\exists!"] \\ X \arrow[ur, "\chi_X"] \arrow[r, "f"'] & Y \end{tikzcd}$$

commute.

9. *As a Free Cocompletion: Adjointness.* We have an adjunction[28]

$$(\chi_{(-)} \dashv 忘) \colon \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "\chi_{(-)}"] \arrow[r, phantom, "\perp"] & \mathsf{Pos}^{\mathsf{cocomp.}} \arrow[l, bend left, "忘"] \end{tikzcd},$$

witnessed by a bijection

$$\mathsf{Pos}^{\mathsf{cocomp.}}((\mathcal{P}(X), \subset), (Y, \preceq)) \cong \mathsf{Sets}(X, Y),$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets})$ and $(Y, \preceq) \in \mathrm{Obj}(\mathsf{Pos})$, where

- We have a natural map

  $$\chi_X^* \colon \mathsf{Pos}^{\mathsf{cocomp.}}((\mathcal{P}(X), \subset), (Y, \preceq)) \longrightarrow \mathsf{Sets}(X, Y)$$

  defined by

  $$\chi_X^*(f) \stackrel{\text{def}}{=} f \circ \chi_X,$$

  i.e. by sending a cocontinuous morphism of posets $f \colon \mathcal{P}(X) \longrightarrow Y$ to the composition

  $$X \overset{\chi_X}{\hookrightarrow} \mathcal{P}(X) \xrightarrow{f} Y;$$

- We have a natural map

  $$\mathrm{Lan}_{\chi_X} \colon \mathsf{Sets}(X, Y) \longrightarrow \mathsf{Pos}^{\mathsf{cocomp.}}((\mathcal{P}(X), \subset), (Y, \preceq))$$

  computed by

  $$\begin{aligned} [\mathrm{Lan}_{\chi_X}(f)](U) &\cong \int^{x \in X} \chi_{\mathcal{P}(X)}(\chi_x, U) \odot f(x) \\ &\cong \int^{x \in X} \chi_U(x) \odot f(x) && \text{(by Proposition 4.1.1.4)} \\ &\cong \bigvee_{x \in X} (\chi_U(x) \odot f(x)) \end{aligned}$$

  for each $U \in \mathcal{P}(X)$, where:

[28] In this sense, $\mathcal{P}(A)$ is the free cocompletion of A. (Note that, despite its name, however, this is not an

– $\bigvee$ is the join in $(Y, \preceq)$;
– We have

$$\begin{aligned}\mathsf{true} \odot f(x) &\overset{\text{def}}{=} f(x),\\ \mathsf{false} \odot f(x) &\overset{\text{def}}{=} \varnothing_Y,\end{aligned}$$

where $\varnothing_Y$ is the minimal element of $(Y, \preceq)$.

Proof. *Item 1, Functoriality*: This follows from Items 3 and 4 of Proposition 4.3.1.4, Items 3 and 4 of Proposition 4.4.1.4, and Items 3 and 4 of Proposition 4.5.1.6.
Item 2, Adjointness I: Omitted.
Item 3, Adjointness II: Omitted.
Item 4, Symmetric Strong Monoidality With Respect to Coproducts: Omitted.
Item 5, Symmetric Lax Monoidality With Respect to Products: Omitted.
Item 6, Powersets as Sets of Functions: Omitted.
Item 7, Powersets as Sets of Relations: Omitted.
Item 8, As a Free Cocompletion: Universal Property: This is a rephrasing of **??**.
Item 9, As a Free Cocompletion: Adjointness: Omitted. □

4.3 Direct Images

Let A and B be sets and let $f: A \to B$ be a function.

Definition 4.3.1.1. The **direct image function associated to** f is the function[29]

$$f_*: \mathcal{P}(A) \to \mathcal{P}(B)$$

defined by[30,31]

$$\begin{aligned} f_*(U) &\overset{\text{def}}{=} f(U) \\ &\overset{\text{def}}{=} \left\{ b \in B \,\middle|\, \begin{array}{l}\text{there exists some } a \in U\\ \text{such that } b = f(a)\end{array} \right\} \\ &= \{ f(a) \in B \mid a \in U \} \end{aligned}$$

idempotent operation, as we have $\mathcal{P}(\mathcal{P}(A)) \neq \mathcal{P}(A)$.)

[29] *Further Notation:* Also written $\exists_f : \mathcal{P}(A) \to \mathcal{P}(B)$. This notation comes from the fact that the following statements are equivalent, where $b \in B$ and $U \in \mathcal{P}(A)$:

- We have $b \in \exists_f(U)$.
- There exists some $a \in U$ such that $f(a) = b$.

[30] *Further Terminology:* The set $f(U)$ is called the **direct image of** U **by** f.
[31] We also have

$$f_*(U) = B \setminus f_!(A \setminus U);$$

for each $U \in \mathcal{P}(A)$.

Remark 4.3.1.2. Identifying subsets of A with functions from A to $\{\mathsf{true}, \mathsf{false}\}$ via Item 6 of Proposition 4.2.1.3, we see that the direct image function associated to f is equivalently the function

$$f_* \colon \mathcal{P}(A) \to \mathcal{P}(B)$$

defined by

$$\begin{aligned} f_*(\chi_U) &\stackrel{\text{def}}{=} \mathrm{Lan}_f(\chi_U) \\ &= \operatorname{colim}\left(\left(f \overset{\rightarrow}{\times} \underline{(-_1)}\right) \overset{\mathrm{pr}}{\twoheadrightarrow} A \xrightarrow{\chi_U} \{\mathsf{t}, \mathsf{f}\}\right) \\ &= \operatorname*{colim}_{\substack{a \in A \\ f(a) = -_1}} (\chi_U(a)) \\ &= \bigvee_{\substack{a \in A \\ f(a) = -_1}} (\chi_U(a)). \end{aligned}$$

So, in other words, we have

$$\begin{aligned} [f_*(\chi_U)](b) &= \bigvee_{\substack{a \in A \\ f(a) = b}} (\chi_U(a)) \\ &= \begin{cases} \mathsf{true} & \text{if there exists some } a \in A \text{ such} \\ & \text{that } f(a) = b \text{ and } a \in U, \\ \mathsf{false} & \text{otherwise} \end{cases} \\ &= \begin{cases} \mathsf{true} & \text{if there exists some } a \in U \\ & \text{such that } f(a) = b, \\ \mathsf{false} & \text{otherwise} \end{cases} \end{aligned}$$

for each $b \in B$.

Proposition 4.3.1.3. Let $f \colon A \to B$ be a function.

1. *Functoriality.* The assignment $U \mapsto f_*(U)$ defines a functor

$$f_* \colon (\mathcal{P}(A), \subset) \to (\mathcal{P}(B), \subset)$$

where

see Item 7 of Proposition 4.3.1.3.

- *Action on Objects.* For each $U \in \mathcal{P}(A)$, we have

$$[f_*](U) \stackrel{\text{def}}{=} f_*(U);$$

- *Action on Morphisms.* For each $U, V \in \mathcal{P}(A)$:

  ($\star$) If $U \subset V$, then $f_*(U) \subset f_*(V)$.

2. *Triple Adjointness.* We have a triple adjunction

$$\left(f_* \dashv f^{-1} \dashv f_!\right)\colon \quad \begin{tikzcd} \mathcal{P}(A) \arrow[r, bend left=50, "f_*"] \arrow[r, bend right=50, "f_!"'] & \mathcal{P}(B), \arrow[l, "f^{-1}"' description] \end{tikzcd}$$

witnessed by bijections of sets

$$\begin{aligned} \mathrm{Hom}_{\mathcal{P}(B)}(f_*(U), V) &\cong \mathrm{Hom}_{\mathcal{P}(A)}\left(U, f^{-1}(V)\right), \\ \mathrm{Hom}_{\mathcal{P}(A)}\left(f^{-1}(U), V\right) &\cong \mathrm{Hom}_{\mathcal{P}(A)}(U, f_!(V)), \end{aligned}$$

natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$ and (respectively) $V \in \mathcal{P}(A)$ and $U \in \mathcal{P}(B)$, i.e. where:

(a) The following conditions are equivalent:
   i. We have $f_*(U) \subset V$.
   ii. We have $U \subset f^{-1}(V)$.

(b) The following conditions are equivalent:
   i. We have $f^{-1}(U) \subset V$.
   ii. We have $U \subset f_!(V)$.

3. *Preservation of Colimits.* We have an equality of sets

$$f_*\left(\bigcup_{i \in I} U_i\right) = \bigcup_{i \in I} f_*(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have equalities

$$\begin{aligned} f_*(U) \cup f_*(V) &= f_*(U \cup V), \\ f_*(\emptyset) &= \emptyset, \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

4. *Oplax Preservation of Limits.* We have an inclusion of sets

$$f_*\left(\bigcap_{i\in I} U_i\right) \subset \bigcap_{i\in I} f_*(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have inclusions

$$\begin{gathered} f_*(U \cap V) \subset f_*(U) \cap f_*(V), \\ f_*(A) \subset B, \end{gathered}$$

natural in $U, V \in \mathcal{P}(A)$.

5. *Symmetric Strict Monoidality With Respect to Unions.* The direct image function of Item 1 has a symmetric strict monoidal structure

$$\left(f_*, f_*^{\otimes}, f_{*|\mathbb{1}}^{\otimes}\right) \colon (\mathcal{P}(A), \cup, \emptyset) \longrightarrow (\mathcal{P}(B), \cup, \emptyset),$$

being equipped with equalities

$$\begin{gathered} f_{*|U,V}^{\otimes} \colon f_*(U) \cup f_*(V) \xrightarrow{=} f_*(U \cup V), \\ f_{*|\mathbb{1}}^{\otimes} \colon \emptyset \xrightarrow{=} \emptyset, \end{gathered}$$

natural in $U, V \in \mathcal{P}(A)$.

6. *Symmetric Oplax Monoidality With Respect to Intersections.* The direct image function of Item 1 has a symmetric oplax monoidal structure

$$\left(f_*, f_*^{\otimes}, f_{*|\mathbb{1}}^{\otimes}\right) \colon (\mathcal{P}(A), \cap, A) \longrightarrow (\mathcal{P}(B), \cap, B),$$

being equipped with inclusions

$$\begin{gathered} f_{*|U,V}^{\otimes} \colon f_*(U \cap V) \hookrightarrow f_*(U) \cap f_*(V), \\ f_{*|\mathbb{1}}^{\otimes} \colon f_*(A) \hookrightarrow B, \end{gathered}$$

natural in $U, V \in \mathcal{P}(A)$.

7. *Relation to Direct Images With Compact Support.* We have

$$f_*(U) = B \setminus f_!(A \setminus U)$$

for each $U \in \mathcal{P}(A)$.

Proof. *Item 1*, *Functoriality*: Clear.
Item 2, *Triple Adjointness*: This follows from Kan Extensions, **??** of **??**.
Item 3, *Preservation of Colimits*: This follows from Item 2 and Categories, **??** of **??**.
Item 4, *Oplax Preservation of Limits*: Omitted.
Item 5, *Symmetric Strict Monoidality With Respect to Unions*: This follows from Item 3.
Item 6, *Symmetric Oplax Monoidality With Respect to Intersections*: This follows from **??**.
Item 7, *Relation to Direct Images With Compact Support*: Applying **??** of **??** to $A \setminus U$, we have

$$\begin{aligned} f_!(A \setminus U) &= B \setminus f_*(A \setminus (A \setminus U)) \\ &= B \setminus f_*(U). \end{aligned}$$

Taking complements, we then obtain

$$\begin{aligned} f_*(U) &= B \setminus (B \setminus f_*(U)), \\ &= B \setminus f_!(A \setminus U), \end{aligned}$$

which finishes the proof. □

Proposition 4.3.1.4. Let $f\colon A \to B$ be a function.

1. *Functionality I.* The assignment $f \mapsto f_*$ defines a function

$$(-)_{*|A,B}\colon \mathsf{Sets}(A, B) \to \mathsf{Sets}(\mathcal{P}(A), \mathcal{P}(B)).$$

2. *Functionality II.* The assignment $f \mapsto f_*$ defines a function

$$(-)_{*|A,B}\colon \mathsf{Sets}(A, B) \to \mathsf{Pos}((\mathcal{P}(A), \subset), (\mathcal{P}(B), \subset)).$$

3. *Interaction With Identities.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$(\mathrm{id}_A)_* = \mathrm{id}_{\mathcal{P}(A)};$$

4. *Interaction With Composition.* For each pair of composable functions $f\colon A \to B$ and $g\colon B \to C$, we have

$$(g \circ f)_* = g_* \circ f_*, \qquad \begin{tikzcd} \mathcal{P}(A) \arrow[r, "f_*"] \arrow[dr, "(g\circ f)_*"'] & \mathcal{P}(B) \arrow[d, "g_*"] \\ & \mathcal{P}(C). \end{tikzcd}$$

Proof. *Item 1*, *Functionality I*: Clear.
Item 2, *Functionality II*: Clear.
Item 3, *Interaction With Identities*: This follows from Kan Extensions, **??** of **??**.
Item 4, *Interaction With Composition*: This follows from Kan Extensions, **??** of **??**. □

4.4 Inverse Images

Let A and B be sets and let $f: A \to B$ be a function.

Definition 4.4.1.1. The **inverse image function associated to** f is the function[32]

$$f^{-1}: \mathcal{P}(B) \to \mathcal{P}(A)$$

defined by[33]

$$f^{-1}(V) \stackrel{\text{def}}{=} \{a \in A \mid \text{we have } f(a) \in V\}$$

for each $V \in \mathcal{P}(B)$.

Remark 4.4.1.2. Identifying subsets of B with functions from B to $\{\mathsf{true}, \mathsf{false}\}$ via Item 6 of Proposition 4.2.1.3, we see that the inverse image function associated to f is equivalently the function

$$f^*: \mathcal{P}(B) \to \mathcal{P}(A)$$

defined by

$$f^*(\chi_V) \stackrel{\text{def}}{=} \chi_V \circ f$$

for each $\chi_V \in \mathcal{P}(B)$, where $\chi_V \circ f$ is the composition

$$A \xrightarrow{f} B \xrightarrow{\chi_V} \{\mathsf{true}, \mathsf{false}\}$$

in Sets.

Proposition 4.4.1.3. Let $f: A \to B$ be a function.

1. *Functoriality.* The assignment $V \mapsto f^{-1}(V)$ defines a functor

$$f^{-1}: (\mathcal{P}(B), \subset) \to (\mathcal{P}(A), \subset)$$

   where

   - *Action on Objects.* For each $V \in \mathcal{P}(B)$, we have

   $$[f^{-1}](V) \stackrel{\text{def}}{=} f^{-1}(V);$$

   - *Action on Morphisms.* For each $U, V \in \mathcal{P}(B)$:
     ($\star$) If $U \subset V$, then $f^{-1}(U) \subset f^{-1}(V)$.

[32] *Further Notation:* Also written $f^*: \mathcal{P}(B) \to \mathcal{P}(A)$.
[33] *Further Terminology:* The set $f^{-1}(V)$ is called the **inverse image of** V **by** f.

2. *Triple Adjointness.* We have a triple adjunction

$$(f_* \dashv f^{-1} \dashv f_!)\colon\quad \begin{tikzcd} \mathcal{P}(A) \arrow[r, bend left=45, "f_*"] \arrow[r, bend right=45, "f_!"'] & \mathcal{P}(B), \arrow[l, "f^{-1}"'] \end{tikzcd}$$

witnessed by bijections of sets

$$\begin{aligned} \mathrm{Hom}_{\mathcal{P}(B)}(f_*(U), V) &\cong \mathrm{Hom}_{\mathcal{P}(A)}\left(U, f^{-1}(V)\right),\\ \mathrm{Hom}_{\mathcal{P}(A)}\left(f^{-1}(U), V\right) &\cong \mathrm{Hom}_{\mathcal{P}(A)}(U, f_!(V)), \end{aligned}$$

natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$ and (respectively) $V \in \mathcal{P}(A)$ and $U \in \mathcal{P}(B)$, i.e. where:

(a) The following conditions are equivalent:
   i. We have $f_*(U) \subset V$;
   ii. We have $U \subset f^{-1}(V)$;

(b) The following conditions are equivalent:
   i. We have $f^{-1}(U) \subset V$.
   ii. We have $U \subset f_!(V)$.

3. *Preservation of Colimits.* We have an equality of sets

$$f^{-1}\left(\bigcup_{i\in I} U_i\right) = \bigcup_{i\in I} f^{-1}(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(B)^{\times I}$. In particular, we have equalities

$$\begin{aligned} f^{-1}(U) \cup f^{-1}(V) &= f^{-1}(U \cup V),\\ f^{-1}(\emptyset) &= \emptyset, \end{aligned}$$

natural in $U, V \in \mathcal{P}(B)$.

4. *Preservation of Limits.* We have an equality of sets

$$f^{-1}\left(\bigcap_{i\in I} U_i\right) = \bigcap_{i\in I} f^{-1}(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(B)^{\times I}$. In particular, we have equalities

$$\begin{aligned} f^{-1}(U) \cap f^{-1}(V) &= f^{-1}(U \cap V),\\ f^{-1}(B) &= A, \end{aligned}$$

natural in $U, V \in \mathcal{P}(B)$.

5. *Symmetric Strict Monoidality With Respect to Unions.* The inverse image function of Item 1 has a symmetric strict monoidal structure

$$\left(f^{-1}, f^{-1,\otimes}, f_{\mathbb{1}}^{-1,\otimes}\right) \colon (\mathcal{P}(B), \cup, \emptyset) \to (\mathcal{P}(A), \cup, \emptyset),$$

being equipped with equalities

$$f_{U,V}^{-1,\otimes} \colon f^{-1}(U) \cup f^{-1}(V) \xrightarrow{=} f^{-1}(U \cup V),$$
$$f_{\mathbb{1}}^{-1,\otimes} \colon \emptyset \xrightarrow{=} f^{-1}(\emptyset),$$

natural in $U, V \in \mathcal{P}(B)$.

6. *Symmetric Strict Monoidality With Respect to Intersections.* The inverse image function of Item 1 has a symmetric strict monoidal structure

$$\left(f^{-1}, f^{-1,\otimes}, f_{\mathbb{1}}^{-1,\otimes}\right) \colon (\mathcal{P}(B), \cap, B) \to (\mathcal{P}(A), \cap, A),$$

being equipped with equalities

$$f_{U,V}^{-1,\otimes} \colon f^{-1}(U) \cap f^{-1}(V) \xrightarrow{=} f^{-1}(U \cap V),$$
$$f_{\mathbb{1}}^{-1,\otimes} \colon A \xrightarrow{=} f^{-1}(B),$$

natural in $U, V \in \mathcal{P}(B)$.

Proof. *Item 1, Functoriality*: Clear.
Item 2, Triple Adjointness: This follows from Kan Extensions, **??** of **??**.
Item 3, Preservation of Colimits: This follows from Item 2 and Categories, **??** of **??**.
Item 4, Preservation of Limits: This follows from Item 2 and Categories, **??** of **??**.
Item 5, Symmetric Strict Monoidality With Respect to Unions: This follows from Item 3.
Item 6, Symmetric Strict Monoidality With Respect to Intersections: This follows from Item 4. □

Proposition 4.4.1.4. Let $f \colon A \to B$ be a function.

1. *Functionality I.* The assignment $f \mapsto f^{-1}$ defines a function

$$(-)_{A,B}^{-1} \colon \mathsf{Sets}(A, B) \to \mathsf{Sets}(\mathcal{P}(B), \mathcal{P}(A)).$$

2. *Functionality II.* The assignment $f \mapsto f^{-1}$ defines a function

$$(-)_{A,B}^{-1} \colon \mathsf{Sets}(A, B) \to \mathsf{Pos}((\mathcal{P}(B), \subset), (\mathcal{P}(A), \subset)).$$

3. *Interaction With Identities.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have
$$\mathrm{id}_A^{-1} = \mathrm{id}_{\mathcal{P}(A)};$$

4. *Interaction With Composition.* For each pair of composable functions $f: A \to B$ and $g: B \to C$, we have
$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}, \qquad \begin{tikzcd} \mathcal{P}(C) \arrow[r, "g^{-1}"] \arrow[dr, "(g\circ f)^{-1}"'] & \mathcal{P}(B) \arrow[d, "f^{-1}"] \\ & \mathcal{P}(A). \end{tikzcd}$$

Proof. Item 1, *Functionality I*: Clear.
Item 2, *Functionality II*: Clear.
Item 3, *Interaction With Identities*: This follows from Categories, **??** of **??**.
Item 4, *Interaction With Composition*: This follows from Categories, **??** of **??**. □

4.5 Direct Images With Compact Support

Let A and B be sets and let $f: A \to B$ be a function.

Definition 4.5.1.1. The **direct image with compact support function associated to** f is the function[34]
$$f_!: \mathcal{P}(A) \to \mathcal{P}(B)$$
defined by[35,36]
$$\begin{aligned} f_!(U) &\overset{\text{def}}{=} \left\{ b \in B \;\middle|\; \begin{array}{l} \text{for each } a \in A, \text{ if we have} \\ f(a) = b, \text{ then } a \in U \end{array} \right\} \\ &= \left\{ b \in B \;\middle|\; \text{we have } f^{-1}(b) \subset U \right\} \end{aligned}$$
for each $U \in \mathcal{P}(A)$.

[34] *Further Notation:* Also written $\forall_f: \mathcal{P}(A) \to \mathcal{P}(B)$. This notation comes from the fact that the following statements are equivalent, where $b \in B$ and $U \in \mathcal{P}(A)$:

- We have $b \in \forall_f(U)$.
- For each $a \in A$, if $b = f(a)$, then $a \in U$.

[35] *Further Terminology:* The set $f_!(U)$ is called the **direct image with compact support of** U **by** f.

[36] We also have
$$f_!(U) = B \setminus f_*(A \setminus U);$$

Remark 4.5.1.2. Identifying subsets of A with functions from A to $\{\mathsf{true}, \mathsf{false}\}$ via Item 6 of Proposition 4.2.1.3, we see that the direct image with compact support function associated to f is equivalently the function

$$f_! \colon \mathcal{P}(A) \to \mathcal{P}(B)$$

defined by

$$\begin{aligned} f_!(\chi_U) &\overset{\text{def}}{=} \mathrm{Ran}_f(\chi_U) \\ &= \lim\left(\left(\underline{(-_1)} \overrightarrow{\times} f\right) \overset{\mathrm{pr}}{\twoheadrightarrow} A \xrightarrow{\chi_U} \{\mathsf{true}, \mathsf{false}\}\right) \\ &= \lim_{\substack{a \in A \\ f(a) = -_1}} (\chi_U(a)) \\ &= \bigwedge_{\substack{a \in A \\ f(a) = -_1}} (\chi_U(a)). \end{aligned}$$

So, in other words, we have

$$\begin{aligned} [f_!(\chi_U)](b) &= \bigwedge_{\substack{a \in A \\ f(a) = b}} (\chi_U(a)) \\ &= \begin{cases} \mathsf{true} & \text{if, for each } a \in A \text{ such that} \\ & f(a) = b, \text{ we have } a \in U, \\ \mathsf{false} & \text{otherwise} \end{cases} \\ &= \begin{cases} \mathsf{true} & \text{if } f^{-1}(b) \subset U \\ \mathsf{false} & \text{otherwise} \end{cases} \end{aligned}$$

for each $b \in B$.

---

see Item 7 of Proposition 4.5.1.5.

Definition 4.5.1.3. Let U be a subset of A.[37,38]

1. The **image part of the direct image with compact support** $f_!(U)$ **of** U is the set $f_{!,\mathrm{im}}(U)$ defined by

$$\begin{aligned} f_{!,\mathrm{im}}(U) &\overset{\mathrm{def}}{=} f_!(U) \cap \mathrm{Im}(f) \\ &= \left\{ b \in B \,\middle|\, \begin{array}{l} \text{we have } f^{-1}(b) \subset U \\ \text{and } f^{-1}(b) \neq \emptyset \end{array} \right\}. \end{aligned}$$

2. The **complement part of the direct image with compact support** $f_!(U)$ **of** U is the set $f_{!,\mathrm{cp}}(U)$ defined by

$$\begin{aligned} f_{!,\mathrm{cp}}(U) &\overset{\mathrm{def}}{=} f_!(U) \cap (B \setminus \mathrm{Im}(f)) \\ &= B \setminus \mathrm{Im}(f) \\ &= \left\{ b \in B \,\middle|\, \begin{array}{l} \text{we have } f^{-1}(b) \subset U \\ \text{and } f^{-1}(b) = \emptyset \end{array} \right\} \\ &= \left\{ b \in B \,\middle|\, f^{-1}(b) = \emptyset \right\}. \end{aligned}$$

Example 4.5.1.4. Here are some examples of direct images with compact support.

1. *The Multiplication by Two Map on the Natural Numbers.* Consider the function $f\colon \mathbb{N} \to \mathbb{N}$ given by

$$f(n) \overset{\mathrm{def}}{=} 2n$$

[37] Note that we have

$$f_!(U) = f_{!,\mathrm{im}}(U) \cup f_{!,\mathrm{cp}}(U),$$

as

$$\begin{aligned} f_!(U) &= f_!(U) \cap B \\ &= f_!(U) \cap (\mathrm{Im}(f) \cup (B \setminus \mathrm{Im}(f))) \\ &= (f_!(U) \cap \mathrm{Im}(f)) \cup (f_!(U) \cap (B \setminus \mathrm{Im}(f))) \\ &\overset{\mathrm{def}}{=} f_{!,\mathrm{im}}(U) \cup f_{!,\mathrm{cp}}(U). \end{aligned}$$

[38] In terms of the meet computation of $f_!(U)$ of Remark 4.5.1.2, namely

$$f_!(\chi_U) = \bigwedge_{\substack{a \in A \\ f(a) = -_1}} (\chi_U(a)),$$

we see that $f_{!,\mathrm{im}}$ corresponds to meets indexed over nonempty sets, while $f_{!,\mathrm{cp}}$ corresponds to meets indexed over the empty set.

for each $n \in \mathbb{N}$. Since f is injective, we have

$$\begin{aligned} f_{!,\mathrm{im}}(U) &= f_*(U) \\ f_{!,\mathrm{cp}}(U) &= \{\text{odd natural numbers}\} \end{aligned}$$

for any $U \subset \mathbb{N}$.

2. *Parabolas.* Consider the function $f\colon \mathbb{R} \to \mathbb{R}$ given by

$$f(x) \overset{\mathrm{def}}{=} x^2$$

for each $x \in \mathbb{R}$. We have

$$f_{!,\mathrm{cp}}(U) = \mathbb{R}_{<0}$$

for any $U \subset \mathbb{R}$. Moreover, since $f^{-1}(x) = \left\{-\sqrt{x}, \sqrt{x}\right\}$, we have e.g.:

$$\begin{aligned} f_{!,\mathrm{im}}([0,1]) &= \{0\}, \\ f_{!,\mathrm{im}}([-1,1]) &= [0,1], \\ f_{!,\mathrm{im}}([1,2]) &= \emptyset, \\ f_{!,\mathrm{im}}([-2,-1] \cup [1,2]) &= [1,4]. \end{aligned}$$

3. *Circles.* Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x,y) \overset{\mathrm{def}}{=} x^2 + y^2$$

for each $(x,y) \in \mathbb{R}^2$. We have

$$f_{!,\mathrm{cp}}(U) = \mathbb{R}_{<0}$$

for any $U \subset \mathbb{R}^2$, and since

$$f^{-1}(r) = \begin{cases} \text{a circle of radius } r \text{ about the origin} & \text{if } r > 0, \\ \{(0,0)\} & \text{if } r = 0, \\ \emptyset & \text{if } r < 0, \end{cases}$$

we have e.g.:

$$\begin{aligned} f_{!,\mathrm{im}}([-1,1] \times [-1,1]) &= [0,1], \\ f_{!,\mathrm{im}}(([-1,1] \times [-1,1]) \setminus [-1,1] \times \{0\}) &= \emptyset. \end{aligned}$$

Proposition 4.5.1.5. Let $f\colon A \to B$ be a function.

1. *Functoriality.* The assignment $U \mapsto f_!(U)$ defines a functor
$$f_! \colon (\mathcal{P}(A), \subset) \to (\mathcal{P}(B), \subset)$$
where
   - *Action on Objects.* For each $U \in \mathcal{P}(A)$, we have
   $$[f_!](U) \stackrel{\text{def}}{=} f_!(U);$$
   - *Action on Morphisms.* For each $U, V \in \mathcal{P}(A)$:
   ($\star$) If $U \subset V$, then $f_!(U) \subset f_!(V)$.

2. *Triple Adjointness.* We have a triple adjunction
$$\left(f_* \dashv f^{-1} \dashv f_!\right)\colon \quad \begin{tikzcd} \mathcal{P}(A) \arrow[r, bend left, "f_*"] \arrow[r, bend right, "f_!"'] & \mathcal{P}(B), \arrow[l, "f^{-1}"'] \end{tikzcd}$$
witnessed by bijections of sets
$$\begin{aligned} \mathrm{Hom}_{\mathcal{P}(B)}(f_*(U), V) &\cong \mathrm{Hom}_{\mathcal{P}(A)}\left(U, f^{-1}(V)\right), \\ \mathrm{Hom}_{\mathcal{P}(A)}\left(f^{-1}(U), V\right) &\cong \mathrm{Hom}_{\mathcal{P}(A)}(U, f_!(V)), \end{aligned}$$
natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$ and (respectively) $V \in \mathcal{P}(A)$ and $U \in \mathcal{P}(B)$, i.e. where:
   (a) The following conditions are equivalent:
      i. We have $f_*(U) \subset V$;
      ii. We have $U \subset f^{-1}(V)$;
   (b) The following conditions are equivalent:
      i. We have $f^{-1}(U) \subset V$.
      ii. We have $U \subset f_!(V)$.

3. *Lax Preservation of Colimits.* We have an inclusion of sets
$$\bigcup_{i \in I} f_!(U_i) \subset f_!\left(\bigcup_{i \in I} U_i\right),$$
natural in $\{U_i\}_{i \in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have inclusions
$$\begin{aligned} f_!(U) \cup f_!(V) &\hookrightarrow f_!(U \cup V), \\ \emptyset &\hookrightarrow f_!(\emptyset), \end{aligned}$$
natural in $U, V \in \mathcal{P}(A)$.

4. *Preservation of Limits.* We have an equality of sets

$$f_!\left(\bigcap_{i\in I} U_i\right) = \bigcap_{i\in I} f_!(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have equalities

$$\begin{aligned} f^{-1}(U \cap V) &= f_!(U) \cap f^{-1}(V), \\ f_!(A) &= B, \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

5. *Symmetric Lax Monoidality With Respect to Unions.* The direct image with compact support function of Item 1 has a symmetric lax monoidal structure

$$\left(f_!, f_!^{\otimes}, f_{!|\mathbb{1}}^{\otimes}\right) \colon (\mathcal{P}(A), \cup, \emptyset) \longrightarrow (\mathcal{P}(B), \cup, \emptyset),$$

being equipped with inclusions

$$\begin{aligned} f_{!|U,V}^{\otimes} &\colon f_!(U) \cup f_!(V) \hookrightarrow f_!(U \cup V), \\ f_{!|\mathbb{1}}^{\otimes} &\colon \emptyset \hookrightarrow f_!(\emptyset), \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

6. *Symmetric Strict Monoidality With Respect to Intersections.* The direct image function of Item 1 has a symmetric strict monoidal structure

$$\left(f_!, f_!^{\otimes}, f_{!|\mathbb{1}}^{\otimes}\right) \colon (\mathcal{P}(A), \cap, A) \longrightarrow (\mathcal{P}(B), \cap, B),$$

being equipped with equalities

$$\begin{aligned} f_{!|U,V}^{\otimes} &\colon f_!(U \cap V) \xrightarrow{=} f_!(U) \cap f_!(V), \\ f_{!|\mathbb{1}}^{\otimes} &\colon f_!(A) \xrightarrow{=} B, \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

7. *Relation to Direct Images.* We have

$$f_!(U) = B \setminus f_*(A \setminus U)$$

for each $U \in \mathcal{P}(A)$.

8. *Interaction With Injections.* If f is injective, then we have

$$\begin{aligned} f_{!,\mathrm{im}}(U) &= f_*(U), \\ f_{!,\mathrm{cp}}(U) &= B \setminus \mathrm{Im}(f), \\ f_!(U) &= f_{!,\mathrm{im}}(U) \cup f_{!,\mathrm{cp}}(U) \\ &= f_*(U) \cup (B \setminus \mathrm{Im}(f)) \end{aligned}$$

for each $U \in \mathcal{P}(A)$.

9. *Interaction With Surjections.* If f is surjective, then we have

$$\begin{aligned} f_{!,\mathrm{im}}(U) &\subset f_*(U), \\ f_{!,\mathrm{cp}}(U) &= \emptyset, \\ f_!(U) &\subset f_*(U) \end{aligned}$$

for each $U \in \mathcal{P}(A)$.

Proof. *Item 1, Functoriality*: Clear.
Item 2, Triple Adjointness: This follows from Kan Extensions, **??** of **??**.
Item 3, Lax Preservation of Colimits: Omitted.
Item 4, Preservation of Limits: Omitted. This follows from Item 2 and Categories, **??** of **??**.
Item 5, Symmetric Lax Monoidality With Respect to Unions: This follows from **??**.
Item 6, Symmetric Strict Monoidality With Respect to Intersections: This follows from Item 4.
Item 7, Relation to Direct Images: We claim that $f_!(U) = B \setminus f_*(A \setminus U)$.

- *The First Implication.* We claim that

  $$f_!(U) \subset B \setminus f_*(A \setminus U).$$

  Let $b \in f_!(U)$. We need to show that $b \notin f_*(A \setminus U)$, i.e. that there is no $a \in A \setminus U$ such that $f(a) = b$.

  This is indeed the case, as otherwise we would have $a \in f^{-1}(b)$ and $a \notin U$, contradicting $f^{-1}(b) \subset U$ (which holds since $b \in f_!(U)$).

  Thus $b \in B \setminus f_*(A \setminus U)$.

- *The Second Implication.* We claim that

  $$B \setminus f_*(A \setminus U) \subset f_!(U).$$

  Let $b \in B \setminus f_*(A \setminus U)$. We need to show that $b \in f_!(U)$, i.e. that $f^{-1}(b) \subset U$.

  Since $b \notin f_*(A \setminus U)$, there exists no $a \in A \setminus U$ such that $b = f(a)$, and hence $f^{-1}(b) \subset U$.

  Thus $b \in f_!(U)$.

This finishes the proof of Item 7.
Item 8, *Interaction With Injections*: Clear.
Item 9, *Interaction With Surjections*: Clear. □

Proposition 4.5.1.6. Let $f\colon A \longrightarrow B$ be a function.

1. *Functionality I.* The assignment $f \mapsto f_!$ defines a function
$$(-)_{!|A,B}\colon \mathsf{Sets}(A,B) \longrightarrow \mathsf{Sets}(\mathcal{P}(A),\mathcal{P}(B)).$$

2. *Functionality II.* The assignment $f \mapsto f_!$ defines a function
$$(-)_{!|A,B}\colon \mathsf{Sets}(A,B) \longrightarrow \mathsf{Pos}((\mathcal{P}(A),\subset),(\mathcal{P}(B),\subset)).$$

3. *Interaction With Identities.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have
$$(\mathrm{id}_A)_! = \mathrm{id}_{\mathcal{P}(A)};$$

4. *Interaction With Composition.* For each pair of composable functions $f\colon A \longrightarrow B$ and $g\colon B \longrightarrow C$, we have
$$(g \circ f)_! = g_! \circ f_!, \qquad \begin{tikzcd} \mathcal{P}(A) \arrow[r, "f_!"] \arrow[dr, "(g\circ f)_!"'] & \mathcal{P}(B) \arrow[d, "g_!"] \\ & \mathcal{P}(C). \end{tikzcd}$$

Proof. *Item 1*, *Functionality I*: Clear.
Item 2, *Functionality II*: Clear.
Item 3, *Interaction With Identities*: This follows from Kan Extensions, **??** of **??**.
Item 4, *Interaction With Composition*: This follows from Kan Extensions, **??** of **??**. □

Appendices

A Other Chapters

Probability Theory

Stochastic Analysis

Differential Geometry

Schemes