

Sets

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0000 This chapter (will eventually) contain material on axiomatic set theory, as well as a couple other things.

Contents

1	The Enrichment of Sets in Classical Truth Values	1
1.1	(-2) -Categories	1
1.2	(-1) -Categories	1
1.3	0-Categories	3
1.4	Tables of Analogies Between Set Theory and Category Theory	4
A	Other Chapters	6

0001 1 The Enrichment of Sets in Classical Truth Values

0002 1.1 (-2) -Categories

0003 DEFINITION 1.1.1 ► (-2) -CATEGORIES

A (-2) -category is the “necessarily true” truth value.^{1,2,3}

¹Thus, there is only one (-2) -category.

²A $(-n)$ -category for $n = 3, 4, \dots$ is also the “necessarily true” truth value, coinciding with a (-2) -category.

³For motivation, see [BS10, p. 13].

0004 1.2 (-1) -Categories

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DEFINITION 1.2.1 ► (-1) -CATEGORIES

A (-1) -**category** is a classical truth value.

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REMARK 1.2.2 ► MOTIVATION FOR (-1) -CATEGORIES

¹ (-1) -categories should be thought of as being “categories enriched in (-2) -categories”, having a collection of objects and, for each pair of objects, a Hom-object $\text{Hom}(x, y)$ that is a (-2) -category (i.e. trivial).

Therefore, a (-1) -category C is either ([BS10, pp. 33–34]):

1. *Empty*, having no objects;
2. *Contractible*, having a collection of objects $\{a, b, c, \dots\}$, but with $\text{Hom}_C(a, b)$ being a (-2) -category (i.e. trivial) for all $a, b \in \text{Obj}(C)$, forcing all objects of C to be uniquely isomorphic to each other.

As such, there are only two (-1) -categories, up to equivalence:

- The (-1) -category false (the empty one);
- The (-1) -category true (the contractible one).

¹For more motivation, see [BS10, p. 13].

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DEFINITION 1.2.3 ► THE POSET OF TRUTH VALUES

The **poset of truth values**¹ is the poset $(\{\text{true}, \text{false}\}, \leq)$ ² consisting of

- *The Underlying Set.* The set $\{\text{true}, \text{false}\}$ whose elements are the truth values true and false;
- *The Partial Order.* The partial order

$$\leq: \{\text{true}, \text{false}\} \times \{\text{true}, \text{false}\} \rightarrow \{\text{true}, \text{false}\}$$

on $\{\text{true}, \text{false}\}$ defined by³

$$\begin{aligned} \text{false} &\leq \text{false} \stackrel{\text{def}}{=} \text{true}, \\ \text{true} &\leq \text{false} \stackrel{\text{def}}{=} \text{false}, \\ \text{false} &\leq \text{true} \stackrel{\text{def}}{=} \text{true}, \\ \text{true} &\leq \text{true} \stackrel{\text{def}}{=} \text{true}. \end{aligned}$$

¹*Further Terminology:* Also called the **poset of (-1) -categories**.

²*Further Notation:* Also written $\{t, f\}$.

³This partial order coincides with logical implication.

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PROPOSITION 1.2.4 ► CARTESIAN CLOSEDNESS OF THE POSET OF TRUTH VALUES

The poset of truth values $\{t, f\}$ is Cartesian closed with product given by¹

$$t \times t = t,$$

$$t \times f = f,$$

$$f \times t = f,$$

$$f \times f = f,$$

and internal Hom $\mathbf{Hom}_{\{t,f\}}$ given by the partial order of $\{t, f\}$, i.e. by

$$\mathbf{Hom}_{\{t,f\}}(t, t) = t,$$

$$\mathbf{Hom}_{\{t,f\}}(t, f) = f,$$

$$\mathbf{Hom}_{\{t,f\}}(f, t) = t,$$

$$\mathbf{Hom}_{\{t,f\}}(f, f) = t.$$

¹Note that \times coincides with the “and” operator, while $\mathbf{Hom}_{\{t,f\}}$ coincides with the logical implication operator.

PROOF 1.2.5 ► PROOF OF PROPOSITION 1.2.4

Omitted.

0009 **1.3 0-Categories**

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DEFINITION 1.3.1 ► 0-CATEGORIES

A **0-category** is a poset.¹

¹*Motivation:* A 0-category is precisely a category enriched in the poset of (-1) -categories.

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DEFINITION 1.3.2 ► 0-GROUPOIDS

A **0-groupoid** is a 0-category in which every morphism is invertible.¹

¹That is, a set.

000C

1.4 Tables of Analogies Between Set Theory and Category Theory

Here we record some analogies between notions in set theory and category theory. Note that the analogies relating to presheaves relate equally well to copresheaves, as the opposite X^{op} of a set X is just X again.

Basics:

SET THEORY	CATEGORY THEORY
Enrichment in $\{\text{true}, \text{false}\}$	Enrichment in Sets
Set X	Category C
Element $x \in X$	Object $X \in \text{Obj}(C)$
Function	Functor
Function $X \rightarrow \{\text{true}, \text{false}\}$	Functor $C \rightarrow \text{Sets}$
Function $X \rightarrow \{\text{true}, \text{false}\}$	Presheaf $C^{\text{op}} \rightarrow \text{Sets}$

Powersets and categories of presheaves:

SET THEORY	CATEGORY THEORY
Powerset $\mathcal{P}(X)$	Presheaf category $\mathbf{PSh}(C)$
Characteristic function $\chi_{\{x\}}$	Representable presheaf h_X
Characteristic embedding $\chi_{(-)} : X \hookrightarrow \mathcal{P}(X)$	Yoneda embedding $\mathcal{J} : C^{\text{op}} \hookrightarrow \mathbf{PSh}(C)$
Characteristic relation $\chi_X(-_1, -_2)$	Hom profunctor $\text{Hom}_C(-_1, -_2)$
The Yoneda lemma for sets $\text{Hom}_{\mathcal{P}(X)}(\chi_x, \chi_U) = \chi_U(x)$	The Yoneda lemma for categories $\text{Nat}(h_X, \mathcal{F}) \cong \mathcal{F}(X)$
The characteristic embedding is fully faithful, $\text{Hom}_{\mathcal{P}(X)}(\chi_x, \chi_y) = \chi_X(x, y)$	The Yoneda embedding is fully faithful, $\text{Nat}(h_X, h_Y) \cong \text{Hom}_C(X, Y)$
Subsets are unions of their elements $U = \bigcup_{x \in U} \{x\}$ or $\chi_U = \text{colim}_{\chi_x \in \text{Sets}(U, \{\text{t}, \text{f}\})} (\chi_x)$	Presheaves are colimits of representables, $\mathcal{F} \cong \text{colim}_{h_X \in \int_C \mathcal{F}} (h_X)$

Categories of elements:

SET THEORY	CATEGORY THEORY
Assignment $U \mapsto \chi_U$	Assignment $\mathcal{F} \mapsto \int_C \mathcal{F}$ (the category of elements)
Assignment $U \mapsto \chi_U$ giving an isomorphism $\mathcal{P}(X) \cong \text{Sets}(X, \{\text{t}, \text{f}\})$	Assignment $\mathcal{F} \mapsto \int_C \mathcal{F}$ giving an equivalence $\mathbf{PSh}(C) \stackrel{\text{eq.}}{\cong} \mathbf{DFib}(C)$

Functions between powersets and functors between presheaf categories:

SET THEORY	CATEGORY THEORY
Direct image function $f_*: \mathcal{P}(X) \rightarrow \mathcal{P}(Y)$	Inverse image functor $f^{-1}: \text{PSh}(\mathcal{C}) \rightarrow \text{PSh}(\mathcal{D})$
Inverse image function $f^{-1}: \mathcal{P}(Y) \rightarrow \mathcal{P}(X)$	Direct image functor $f_*: \text{PSh}(\mathcal{D}) \rightarrow \text{PSh}(\mathcal{C})$
Direct image with compact support function $f_!: \mathcal{P}(X) \rightarrow \mathcal{P}(Y)$	Direct image with compact support functor $f_!: \text{PSh}(\mathcal{C}) \rightarrow \text{PSh}(\mathcal{D})$

Relations and profunctors:

SET THEORY	CATEGORY THEORY
Relation $R: X \times Y \rightarrow \{t, f\}$	Profunctor $\mathfrak{p}: \mathcal{D}^{\text{op}} \times \mathcal{C} \rightarrow \text{Sets}$
Relation $R: X \rightarrow \mathcal{P}(Y)$	Profunctor $\mathfrak{p}: \mathcal{C} \rightarrow \text{PSh}(\mathcal{D})$
Relation as a cocontinuous morphism of posets $R: (\mathcal{P}(X), \subset) \rightarrow (\mathcal{P}(Y), \subset)$	Profunctor as a colimit-preserving functor $\mathfrak{p}: \text{PSh}(\mathcal{C}) \rightarrow \text{PSh}(\mathcal{D})$

Appendices

A Other Chapters

Set Theory

1. Sets
2. Constructions With Sets
3. Pointed Sets
4. Tensor Products of Pointed Sets
5. Indexed and Fibred Sets
6. Relations
7. Spans
8. Posets

Category Theory

9. Categories
10. Constructions With Categories
11. Kan Extensions

Bicategories

12. Bicategories
13. Internal Adjunctions

Internal Category Theory

14. Internal Categories

Cyclic Stuff

15. The Cycle Category

Cubical Stuff

16. The Cube Category

Globular Stuff

17. The Globe Category

Cellular Stuff

18. The Cell Category

Monoids

19. Monoids

20. Constructions With Monoids

Monoids With Zero

21. Monoids With Zero

22. Constructions With Monoids With Zero

Groups

23. Groups

24. Constructions With Groups

Hyper Algebra

25. Hypermonoids

26. Hypergroups

27. Hypersemirings and Hyperrings

28. Quantales

Near-Rings

29. Near-Semirings

30. Near-Rings

Real Analysis

31. Real Analysis in One Variable

32. Real Analysis in Several Variables

Measure Theory

33. Measurable Spaces

34. Measures and Integration

Probability Theory

34. Probability Theory

Stochastic Analysis

35. Stochastic Processes, Martingales, and Brownian Motion

36. Itô Calculus

37. Stochastic Differential Equations

Differential Geometry

38. Topological and Smooth Manifolds

Schemes

39. Schemes