Un/Straightening for Indexed and Fibred Sets

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00T5 This chapter contains a discussion of the un/straightening equivalence in the context of sets, as well as a general discussion of indexed and fibred sets. In particular, it contains:

1. A discussion of indexed sets (i.e. functors $K_{\mathsf{disc}} \to \mathsf{Sets}$ with K a set), constructions with them like dependent sums and dependent products, and their properties (**????**);
2. A discussion of fibred sets (i.e. maps of sets $X \to K$), constructions with them like dependent sums and dependent products, and their properties (**????**);
3. A discussion of the un/straightening equivalence for indexed and fibred sets (**??**).

Contents

1 Un/Straightening for Indexed and Fibred Sets 00T6

1.1 Straightening for Fibred Sets 00T7

Let K be a set and let (X, ϕ) be a K-fibred set.

Definition 1.1.1.1. 00T8 The **straightening of** (X, ϕ) is the K-indexed set

$$\mathrm{St}_K(X, \phi) \colon K_{\mathsf{disc}} \to \mathsf{Sets}$$

defined by

$$\mathrm{St}_K(X, \phi)_x \stackrel{\text{def}}{=} \phi^{-1}(x)$$

for each $x \in K$.

Proposition 1.1.1.2. Let K be a set. 00T9

1. *Functoriality.* The assignment $(X, \phi) \mapsto \mathrm{St}_K(X, \phi)$ defines a functor 00TA

$$\mathrm{St}_K \colon \mathsf{FibSets}(K) \to \mathsf{ISets}(K)$$

   - *Action on Objects.* For each $(X, \phi) \in \mathrm{Obj}(\mathsf{FibSets}(K))$, we have

$$[\mathrm{St}_K](X, \phi) \stackrel{\text{def}}{=} \mathrm{St}_K(X, \phi);$$

   - *Action on Morphisms.* For each $(X, \phi), (Y, \psi) \in \mathrm{Obj}(\mathsf{FibSets}(K))$, the action on Hom-sets

$$\mathrm{St}_{K|X,Y} \colon \mathrm{Hom}_{\mathsf{FibSets}(K)}(X, Y) \to \mathrm{Hom}_{\mathsf{ISets}(K)}(\mathrm{St}_K(X), \mathrm{St}_K(Y))$$

   of St_K at (X, Y) is given by sending a morphism

$$f \colon (X, \phi) \to (Y, \psi)$$

   of K-fibred sets to the morphism

$$\mathrm{St}_K(f) \colon \mathrm{St}_K(X, \phi) \to \mathrm{St}_K(Y, \psi)$$

   of K-indexed sets defined by

$$\mathrm{St}_K(f) \stackrel{\text{def}}{=} \{f^*_x\}_{x \in K},$$

   where f^*_x is the transport map associated to f at $x \in K$ of **??**.

2. *Interaction With Change of Base/Indexing.* Let $f \colon K \to K'$ be a map of sets. The diagram 00TB

$$\begin{tikzcd}
\mathsf{FibSets}(K') \arrow[r, "f^*"] \arrow[d, "\mathrm{St}_{K'}"'] & \mathsf{FibSets}(K) \arrow[d, "\mathrm{St}_K"] \\
\mathsf{ISets}(K') \arrow[r, "f^*"'] & \mathsf{ISets}(K)
\end{tikzcd}$$

   commutes.

3. *Interaction With Dependent Sums.* Let $f \colon K \to K'$ be a map of sets. The diagram 00TC

$$\begin{tikzcd}
\mathsf{FibSets}(K) \arrow[r, "\Sigma_f"] \arrow[d, "\mathrm{St}_K"'] & \mathsf{FibSets}(K') \arrow[d, "\mathrm{St}_{K'}"] \\
\mathsf{ISets}(K) \arrow[r, "\Sigma_f"'] & \mathsf{ISets}(K')
\end{tikzcd}$$

   commutes.

4. *Interaction With Dependent Products.* Let $f\colon K \to K'$ be a map of sets. The diagram

$$\begin{tikzcd}
\mathsf{Sets}_{/K} \arrow[r, "\Pi_f"] \arrow[d, "\mathrm{St}_K"'] & \mathsf{FibSets}(K') \arrow[d, "\mathrm{St}_{K'}"] \\
\mathsf{ISets}(K) \arrow[r, "\Pi_f"'] & \mathsf{ISets}(K')
\end{tikzcd}$$

commutes.

Proof. **??**, *Functoriality*: Omitted.
??, *Interaction With Change of Base/Indexing*: Indeed, we have

$$\begin{aligned}
\mathrm{St}_K(f^*(X,\phi))_x &\stackrel{\text{def}}{=} \mathrm{St}_K(K \times_{K'} X)_x \\
&\stackrel{\text{def}}{=} (\mathrm{pr}_1^{K\times_{K'}X})^{-1}(x) \\
&= \left\{(k,y) \in K \times_{K'} X \;\middle|\; \mathrm{pr}_1^{K\times_{K'}X}(k,y) = x\right\} \\
&= \{(k,y) \in K \times_{K'} X \mid k = x\} \\
&= \{(k,y) \in K \times X \mid k = x \text{ and } f(k) = \phi(y)\} \\
&\cong \{y \in X \mid \phi(y) = f(x)\} \\
&= \phi^{-1}(f(x)) \\
&\stackrel{\text{def}}{=} f^*(\phi^{-1}(x)) \\
&\stackrel{\text{def}}{=} f^*(\mathrm{St}_{K'}(X,\phi)_x)
\end{aligned}$$

for each $(X,\phi) \in \mathrm{Obj}(\mathsf{FibSets}(K'))$ and each $x \in K$, and similarly for morphisms.
??, *Interaction With Dependent Sums*: Indeed, we have

$$\begin{aligned}
\mathrm{St}_{K'}(\Sigma_f(X,\phi))_x &\stackrel{\text{def}}{=} \Sigma_f(\phi)^{-1}(x) \\
&\cong \coprod_{\substack{y\in X\\ f(y)=x}} \phi^{-1}(y) \\
&\cong \Sigma_f(\phi^{-1}(x)) \\
&\stackrel{\text{def}}{=} \Sigma_f(\mathrm{St}_K(X,\phi)_x)
\end{aligned}$$

for each $(X,\phi) \in \mathrm{Obj}(\mathsf{FibSets}(K))$ and each $x \in K'$, where we have used **??** of **??** for the first bijection, and similarly for morphisms.

??, *Interaction With Dependent Products*: Indeed, we have

$$\begin{aligned}
\mathrm{St}_{K'}(\Pi_f(X,\phi))_x &\overset{\text{def}}{=} \Pi_f(\phi)^{-1}(x)\\
&\cong \prod_{\substack{y\in X\\ f(y)=x}} \phi^{-1}(y)\\
&\cong \Pi_f(\phi^{-1}(x))\\
&\overset{\text{def}}{=} \Pi_f(\mathrm{St}_K(X,\phi)_x)
\end{aligned}$$

for each $(X,\phi)\in \mathrm{Obj}(\mathsf{FibSets}(K))$ and each $x\in K'$, where we have used **??** of **??** for the first bijection, and similarly for morphisms. □

1.2 Unstraightening for Indexed Sets

00TE

Let K be a set and let X be a K-indexed set.

Definition 1.2.1.1. 00TF The **unstraightening of** X is the K-fibred set

$$\phi_{\mathrm{Un}_K}\colon \mathrm{Un}_K(X)\to K$$

consisting of

- *The Underlying Set.* The set $\mathrm{Un}_K(X)$ defined by

$$\mathrm{Un}_K(X) \overset{\text{def}}{=} \coprod_{x\in K} X_x;$$

- *The Fibration.* The map of sets

$$\phi_{\mathrm{Un}_K}\colon \mathrm{Un}_K(X)\to K$$

  defined by sending an element of $\coprod_{x\in K} X_x$ to its index in K.

Proposition 1.2.1.2. Let K be a set. 00TG

1. *Functoriality.* 00TH The assignment $X\mapsto \mathrm{Un}_K(X)$ defines a functor

$$\mathrm{Un}_K\colon \mathsf{ISets}(K)\to \mathsf{FibSets}(K)$$

   - *Action on Objects.* For each $X\in \mathrm{Obj}(\mathsf{ISets}(K))$, we have

$$[\mathrm{Un}_K](X) \overset{\text{def}}{=} \mathrm{Un}_K(X);$$

- *Action on Morphisms.* For each $X, Y \in \mathrm{Obj}(\mathsf{ISets}(K))$, the action on Hom-sets

$$\mathrm{Un}_{K|X,Y} \colon \mathrm{Hom}_{\mathsf{ISets}(K)}(X,Y) \to \mathrm{Hom}_{\mathsf{FibSets}(K)}(\mathrm{Un}_K(X), \mathrm{Un}_K(Y))$$

of Un_K at (X,Y) is defined by

$$\mathrm{Un}_{K|X,Y}(f) \overset{\text{def}}{=} \coprod_{x \in K} f_x^*.$$

2. *Interaction With Fibres.* We have a bijection of sets 00TJ

$$\phi^{-1}_{\mathrm{Un}_K}(x) \cong X_x$$

for each $x \in K$.

3. *As a Pullback.* We have a bijection of sets 00TK

$$\mathrm{Un}_K(X) \cong K_{\mathsf{disc}} \times_{\mathsf{Sets}} \mathsf{Sets}_*, \qquad \begin{tikzcd} \mathrm{Un}_K(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Sets}_* \arrow[d, "\text{忘}"] \\ K_{\mathsf{disc}} \arrow[r, "X"'] & \mathsf{Sets}. \end{tikzcd}$$

4. *As a Colimit.* We have a bijection of sets 00TL

$$\mathrm{Un}_K(X) \cong \mathrm{colim}(X).$$

5. *Interaction With Change of Indexing/Base.* Let $f \colon K \to K'$ be a map 00TM of sets. The diagram

$$\begin{tikzcd} \mathsf{ISets}(K') \arrow[r, "f^*"] \arrow[d, "\mathrm{Un}_{K'}"'] & \mathsf{ISets}(K) \arrow[d, "\mathrm{Un}_K"] \\ \mathsf{FibSets}(K') \arrow[r, "f^*"'] & \mathsf{FibSets}(K) \end{tikzcd}$$

commutes.

6. *Interaction With Dependent Sums.* Let $f \colon K \to K'$ be a map 00TN of sets. The diagram

$$\begin{tikzcd} \mathsf{ISets}(K) \arrow[r, "\Sigma_f"] \arrow[d, "\mathrm{Un}_K"'] & \mathsf{ISets}(K') \arrow[d, "\mathrm{Un}_{K'}"] \\ \mathsf{FibSets}(K) \arrow[r, "\Sigma_f"'] & \mathsf{FibSets}(K') \end{tikzcd}$$

commutes.

7. *Interaction With Dependent Products.* Let $f\colon K \to K'$ be a 00TP map of sets. The diagram

$$\begin{tikzcd}
\mathsf{ISets}(K) \arrow[r, "\Pi_f"] \arrow[d, "\mathrm{Un}_K"'] & \mathsf{ISets}(K') \arrow[d, "\mathrm{Un}_{K'}"] \\
\mathsf{FibSets}(K) \arrow[r, "\Pi_f"'] & \mathsf{FibSets}(K')
\end{tikzcd}$$

commutes.

Proof. **??**, *Functoriality*: Omitted.
??, *Interaction With Fibres*: Omitted.
??, *As a Pullback*: Omitted.
??, *As a Colimit*: Clear.
??, *Interaction With Change of Indexing/Base*: Indeed, we have

$$\begin{aligned}
\mathrm{Un}_K(f^*(X)) &\stackrel{\text{def}}{=} \mathrm{Un}_K(X \circ f) \\
&\stackrel{\text{def}}{=} \coprod_{x \in K} X_{f(x)} \\
&\cong \left\{ (x, (y, a)) \in K \times \coprod_{y \in K'} X_y \;\middle|\; f(x) = y \right\} \\
&\cong K \times_{K'} \coprod_{y \in K'} X_y \\
&\stackrel{\text{def}}{=} K \times_{K'} \mathrm{Un}_{K'}(X) \\
&\stackrel{\text{def}}{=} f^*(\mathrm{Un}_{K'}(X))
\end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{ISets}(K'))$. Similarly, it can be shown that we also have $\mathrm{Un}_K(f^*(\phi)) = f^*(\mathrm{Un}_{K'}(\phi))$ and that $\mathrm{Un}_K \circ f^* = f^* \circ \mathrm{Un}_{K'}$ also holds on morphisms.

??, *Interaction With Dependent Sums*: Indeed, we have

$$\begin{aligned}
\mathrm{Un}_{K'}(\Sigma_f(X)) &\overset{\text{def}}{=} \coprod_{x\in K'} \Sigma_f(X)_x \\
&\cong \coprod_{x\in K'} \coprod_{y\in f^{-1}(x)} X_y \\
&\cong \coprod_{y\in K} X_y \\
&\cong \mathrm{Un}_K(X) \\
&\overset{\text{def}}{=} \Sigma_f(\mathrm{Un}_K(X))
\end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{ISets}(K))$, where we have used **??** of **??** for the first bijection. Similarly, it can be shown that we also have $\mathrm{Un}_{K'}(\Sigma_f(\phi)) = \Sigma_f(\phi_{\mathrm{Un}_K})$ and that $\mathrm{Un}_{K'} \circ \Sigma_f = \Sigma_f \circ \mathrm{Un}_K$ also holds on morphisms.

??, *Interaction With Dependent Products*: Indeed, we have

$$\begin{aligned}
\mathrm{Un}_{K'}(\Pi_f(X)) &\overset{\text{def}}{=} \coprod_{x\in K'} \Pi_f(X)_x \\
&\cong \coprod_{x\in K'} \prod_{y\in f^{-1}(x)} X_y \\
&\cong \left\{ (x,h) \in \coprod_{x\in K'} \mathsf{Sets}(f^{-1}(x), \phi^{-1}_{\mathrm{Un}_K}(f^{-1}(x))) \;\middle|\; \phi \circ h = \mathrm{id}_{f^{-1}(x)} \right\} \\
&\overset{\text{def}}{=} \Pi_f\Big(\coprod_{y\in K} X_y\Big) \\
&\overset{\text{def}}{=} \Pi_f(\mathrm{Un}_K(X))
\end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{ISets}(K))$, where we have used **??** of **??** for the first bijection. Similarly, it can be shown that we also have $\mathrm{Un}_{K'}(\Pi_f(\phi)) = \Pi_f(\phi_{\mathrm{Un}_K})$ and that $\mathrm{Un}_{K'} \circ \Pi_f = \Pi_f \circ \mathrm{Un}_K$ also holds on morphisms. □

1.3 The Un/Straightening Equivalence 00TQ

Theorem 1.3.1.1. We have an isomorphism 00TR of categories

$$(\mathrm{St}_K \dashv \mathrm{Un}_K)\colon \quad \mathsf{FibSets}(K) \underset{\mathrm{Un}_K}{\overset{\mathrm{St}_K}{\rightleftarrows}} \mathsf{ISets}(K).$$

Proof. Omitted. □

2 Miscellany 00TS

2.1 Other Kinds of Un/Straightening 00TT

Remark 2.1.1.1. 00TU There are also other kinds of un/straightening for sets, where Sets is replaced by $\mathbf{Rel}$ or Span:

- *Un/Straightening With* $\mathbf{Rel}$*, I.* We have an isomorphism of sets

  $$\mathrm{Rel}(A, B) \cong \mathsf{Sets}(B \times A, \{\mathsf{true}, \mathsf{false}\}).$$

  by the definition of a relation from A to B, Relations, Definition 1.1.1.1.

- *Un/Straightening With* $\mathbf{Rel}$*, II.* We have an equivalence of categories

  $$\mathsf{LaxFun}(K_{\mathsf{disc}}, \mathbf{Rel}) \overset{\text{eq.}}{\cong} \mathsf{Cats}^{\mathsf{fth}}_{/K_{\mathsf{disc}}},$$

  where $\mathsf{Cats}^{\mathsf{fth}}_{/K_{\mathsf{disc}}}$ is the full subcategory of $\mathsf{Cats}_{/K_{\mathsf{disc}}}$ spanned by the faithful functors; see [**niefield:change-of-base-for-relational-variable-sets**].

- *Un/Straightening With* Span*, I.* For each $A, B \in \mathrm{Obj}(\mathsf{Sets})$, we have a morphism of sets

  $$\mathsf{Span}(A, B) \to \mathsf{Sets}(A \times B, \mathbb{N} \cup \{\infty\})$$

  which assemble into an equivalence of categories between $\mathsf{Span}(\mathsf{Sets})$ and the category MRel of "multirelations"; see Spans, Remark 7.5.1.1.

- *Un/Straightening With* Span*, II.* We have an equivalence of categories

  $$\mathsf{LaxFun}(K_{\mathsf{disc}}, \mathsf{Span}) \overset{\text{eq.}}{\cong} \mathsf{Cats}_{/K_{\mathsf{disc}}};$$

  see [**nlab:displayed-category**].

Appendices

A Other Chapters