

Relations

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00D0 This chapter contains some material about relations and constructions with them. Notably, we discuss and explore:

1. The definition of relations (Section 1.1).
2. How relations may be viewed as decategorification of profunctors (Section 1.2).
3. The various kind of categories that relations form, namely:
 - (a) A category (Section 2.1),
 - (b) A monoidal category (Section 2.2),
 - (c) A 2-category (Section 2.3), and
 - (d) A double category (Section 2.4).
4. The various categorical properties of the 2-category of relations, including (Section 2.5):
 - (a) The self-duality of \mathbf{Rel} and \mathbf{Rel} (Items 1 and 2 of Proposition 2.5.1);
 - (b) Identifications of equivalences and isomorphisms in \mathbf{Rel} with bijections (Item 3 of Proposition 2.5.1);
 - (c) Identifications of adjunctions in \mathbf{Rel} with functions (Item 4 of Proposition 2.5.1);
 - (d) Identifications of monads in \mathbf{Rel} with preorders (Item 5 of Proposition 2.5.1);
 - (e) Identifications of comonads in \mathbf{Rel} with subsets (Item 6 of Proposition 2.5.1);
 - (f) Characterisations of monomorphisms in \mathbf{Rel} (Item 7 of Proposition 2.5.1);
 - (g) Characterisations of epimorphisms in \mathbf{Rel} (Item 8 of Proposition 2.5.1);
 - (h) The partial co/completeness of \mathbf{Rel} (Item 10 of Proposition 2.5.1);
 - (i) The existence of right Kan extensions and right Kan lifts in \mathbf{Rel} (Items 11 and 12 of Proposition 2.5.1);

- (j) The closedness of **Rel** (Item 13 of Proposition 2.5.1).
- 5. The various kinds of constructions involving relations, such as graphs, domains, ranges, unions, intersections, products, inverse relations, composition of relations, and collages (Section 3).
- 6. Equivalence relations (Section 4) and quotient sets (Section 4.5).
- 7. The adjoint pairs

$$\begin{aligned} R_* \dashv R_{-1} &: \mathcal{P}(A) \rightleftarrows \mathcal{P}(B), \\ R^{-1} \dashv R_! &: \mathcal{P}(B) \rightleftarrows \mathcal{P}(A) \end{aligned}$$

of functors (morphisms of posets) between $\mathcal{P}(A)$ and $\mathcal{P}(B)$ induced by a relation $R: A \rightarrowtail B$, as well as the properties of R_* , R_{-1} , R^{-1} , and $R_!$ (Section 5).

Here we also note that:

- (a) These two pairs of adjoint functors are the counterpart for relations of the adjoint triple $f_* \dashv f^{-1} \dashv f_!$ induced by a function $f: A \rightarrow B$ studied in [Constructions With Sets, Section 4](#);
- (b) We have $R_{-1} = R^{-1}$ iff R is total and functional (Item 8 of Proposition 5.2.4).
- (c) As a consequence of the previous item, when R comes from a function f the pair of adjunctions

$$R_* \dashv R_{-1} = R^{-1} \dashv R_!$$

reduces to the triple adjunction

$$f_* \dashv f^{-1} \dashv f_!$$

from before.

- (d) The pairs $R_* \dashv R_{-1}$ and $R^{-1} \dashv R_!$ later make an appearance in the context of continuous, open, and closed relations between topological spaces (Topological Spaces, ??).
- 8. A notion of *relative preorder* that is to that of a preorder as relative monads are to monads, extending the identifications of monads in **Rel** with preorders of Item 5 of Proposition 2.5.1 to “relative monads in **Rel**”.

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00D1 1 Relations

00D2 1.1 Foundations

Let A and B be sets.

00D3 DEFINITION 1.1.1 ► RELATIONS

A **relation** $R: A \rightarrow B$ **from** A **to** B ^{1,2} is a subset R of $A \times B$.³

¹*Further Terminology:* Also called a **multivalued function from A to B** , a **relation over A and B** , **relation on A and B** , a **binary relation over A and B** , or a **binary relation on A and B** .

²*Further Terminology:* When $A = B$, we also call $R \subset A \times A$ a **relation on A** .

³*Further Notation:* Given elements $a \in A$ and $b \in B$, we write $a \sim_R b$ to mean $(a, b) \in R$.

00D4 DEFINITION 1.1.2 ► THE PO/SET OF RELATIONS OVER TWO SETS

Let A and B be sets.

- 00D5 1. The **set of relations from A to B** is the set $\mathbf{Rel}(A, B)$ defined by

$$\mathbf{Rel}(A, B) \stackrel{\text{def}}{=} \{\text{Relations from } A \text{ to } B\}.$$

00D6

2. The **poset of relations from A to B** is the poset

$$\mathbf{Rel}(A, B) \stackrel{\text{def}}{=} (\text{Rel}(A, B), \subset)$$

consisting of

- *The Underlying Set.* The set $\text{Rel}(A, B)$ of **Item 1**;
- *The Partial Order.* The partial order

$$\subset : \text{Rel}(A, B) \times \text{Rel}(A, B) \rightarrow \{\text{true}, \text{false}\}$$

on $\text{Rel}(A, B)$ given by inclusion of relations.

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REMARK 1.1.3 ► EQUIVALENT DEFINITIONS OF RELATIONS

A relation from A to B is equivalently:¹

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1. A subset of $A \times B$;

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2. A function from $A \times B$ to $\{\text{true}, \text{false}\}$;

00DA

3. A function from A to $\mathcal{P}(B)$;

00DB

4. A function from B to $\mathcal{P}(A)$;

00DC

5. A cocontinuous morphism of posets from $(\mathcal{P}(A), \subset)$ to $(\mathcal{P}(B), \subset)$.

That is: we have bijections of sets

$$\begin{aligned} \text{Rel}(A, B) &\stackrel{\text{def}}{=} \mathcal{P}(A \times B), \\ &\cong \text{Sets}(A \times B, \{\text{true}, \text{false}\}), \\ &\cong \text{Sets}(A, \mathcal{P}(B)), \\ &\cong \text{Sets}(B, \mathcal{P}(A)), \\ &\cong \text{Hom}_{\mathbf{Pos}}^{\text{cocont}}(\mathcal{P}(A), \mathcal{P}(B)), \end{aligned}$$

natural in $A, B \in \text{Obj}(\text{Sets})$.

¹*Intuition:* In particular, we may think of a relation $R: A \rightarrow \mathcal{P}(B)$ from A to B as a multivalued function from A to B (including the possibility of a given $a \in A$ having no value at all).

PROOF 1.1.4 ► PROOF OF REMARK 1.1.3

We claim that **Items 1 to 5** are indeed equivalent:

- **Item 1** \iff **Item 2**: This is a special case of **Constructions With Sets, Item 6** of **Proposition 4.2.3**.

- **Item 2** \iff **Item 3**: This is an instance of currying, following from the bijections

$$\begin{aligned} \text{Sets}(A \times B, \{\text{true}, \text{false}\}) &\cong \text{Sets}(A, \text{Sets}(B, \{\text{true}, \text{false}\})) \\ &\cong \text{Sets}(A, \mathcal{P}(B)), \end{aligned}$$

where the last bijection is from **Constructions With Sets, Item 6** of **Proposition 4.2.3**.

- **Item 2** \iff **Item 4**: This is also an instance of currying, following from the bijections

$$\begin{aligned} \text{Sets}(A \times B, \{\text{true}, \text{false}\}) &\cong \text{Sets}(B, \text{Sets}(A, \{\text{true}, \text{false}\})) \\ &\cong \text{Sets}(B, \mathcal{P}(A)), \end{aligned}$$

where again the last bijection is from **Constructions With Sets, Item 6** of **Proposition 4.2.3**.

- **Item 2** \iff **Item 5**: This follows from the universal property of the powerset $\mathcal{P}(X)$ of a set X as the free cocompletion of X via the characteristic embedding

$$\chi_X: X \hookrightarrow \mathcal{P}(X)$$


of X into $\mathcal{P}(X)$ (**Constructions With Sets, Item 9** of **Proposition 4.2.3**).

In particular, the bijection

$$\text{Rel}(A, B) \cong \text{Hom}_{\text{Pos}}^{\text{cocont}}(\mathcal{P}(A), \mathcal{P}(B))$$

is given by taking a relation $R: A \rightarrowtail B$, passing to its associated function $f: A \rightarrow \mathcal{P}(B)$ from A to B and then extending f from A to all of $\mathcal{P}(A)$ by taking its left Kan extension along χ_X .

This coincides with the direct image function $f_*: \mathcal{P}(A) \rightarrow \mathcal{P}(B)$ of **Constructions With Sets, Definition 4.3.1**.

This finishes the proof. 

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PROPOSITION 1.1.5 ► PROPERTIES OF RELATIONS

Let A and B be sets.

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1. *End Formula for The Poset of Relations.* Let $R, S: A \rightarrow B$ be relations. We have

$$\mathrm{Hom}_{\mathbf{Rel}(A,B)}(R, S) \cong \int_{a \in A} \int_{b \in B} \mathrm{Hom}_{\{\mathbf{t}, \mathbf{f}\}}(R_b^a, S_b^a).$$


PROOF 1.1.6 ► PROOF OF PROPOSITION 1.1.5**Item 1: End Formula for The Poset of Relations**

Unwinding the expression inside the end on the right hand side, we have

$$\int_{a \in A} \int_{b \in B} \mathrm{Hom}_{\{\mathbf{t}, \mathbf{f}\}}(R_b^a, S_b^a) \cong \begin{cases} \text{pt} & \text{if for each } (a, b) \in A \times B, \\ & \text{if } a \sim_R b, \text{ then } a \sim_S b, \\ \emptyset & \text{otherwise.} \end{cases}$$

On the left hand-side, we have

$$\mathrm{Hom}_{\mathbf{Rel}(A,B)}(R, S) \cong \begin{cases} \text{pt} & \text{if } R \subset S, \\ \emptyset & \text{otherwise.} \end{cases}$$

It is then clear that the conditions for each set to evaluate to pt are equivalent, implying that those two sets are isomorphic, finishing the proof. 

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1.2 Relations as Decategorifications of Profunctors

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REMARK 1.2.1 ► RELATIONS AS DECATEGORIFICATIONS OF PROFUNCTORS I

The notion of a relation is a decategorification of that of a profunctor:

1. A profunctor from a category \mathcal{C} to a category \mathcal{D} is a functor

$$\mathbf{p}: \mathcal{D}^{\mathrm{op}} \times \mathcal{C} \rightarrow \mathbf{Sets}.$$

2. A relation on sets A and B is a function

$$R: A \times B \rightarrow \{\text{true}, \text{false}\}.$$

Here we notice that:

- The opposite X^{op} of a set X is itself, as $(-)^{\text{op}}: \text{Cats} \rightarrow \text{Cats}$ restricts to the identity endofunctor on Sets ;
- The values that profunctors and relations take are directly related in relation to decategorification:

- A category is enriched over the category

$$\text{Sets} \stackrel{\text{def}}{=} \text{Cats}_0$$

of sets, with profunctors taking values on it;

- A set is enriched over the set

$$\{\text{true}, \text{false}\} \stackrel{\text{def}}{=} \text{Cats}_{-1}$$

of classical truth values, with relations taking values on it;

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REMARK 1.2.2 ► RELATIONS AS DECATEGORIFICATIONS OF PROFUNCTORS II

Extending [Remark 1.2.1](#), the equivalent definitions of relations in [Remark 1.1.3](#) are also related to the corresponding ones for profunctors ([Categories](#), ??), which state that a profunctor $\mathfrak{p}: C \nrightarrow D$ is equivalently:

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1. A functor $\mathfrak{p}: D^{\text{op}} \times C \rightarrow \text{Sets}$;

00DK

2. A functor $\mathfrak{p}: C \rightarrow \text{PSh}(D)$;

00DL

3. A functor $\mathfrak{p}: D^{\text{op}} \rightarrow \text{Fun}(C, \text{Sets})$;

00DM

4. A colimit-preserving functor $\mathfrak{p}: \text{PSh}(C) \rightarrow \text{PSh}(D)$.

Indeed:

- The equivalence between [Items 1](#) and [2](#) (and also that between [Items 1](#) and [3](#), which is proved analogously) is an instance of currying, both for profunctors

as well as for relations, using the isomorphisms

$$\begin{aligned}\text{Sets}(A \times B, \{\text{true}, \text{false}\}) &\cong \text{Sets}(A, \text{Sets}(B, \{\text{true}, \text{false}\})) \\ &\cong \text{Sets}(A, \mathcal{P}(B)), \\ \text{Fun}(\mathcal{D}^{\text{op}} \times \mathcal{D}, \text{Sets}) &\cong \text{Fun}(C, \text{Fun}(\mathcal{D}^{\text{op}}, \text{Sets})) \\ &\cong \text{Fun}(C, \text{PSh}(\mathcal{D})).\end{aligned}$$

- The equivalence between **Items 1** and **3** follows from the universal properties of:

- The powerset $\mathcal{P}(X)$ of a set X as the free cocompletion of X via the characteristic embedding

$$\chi(-) : X \hookrightarrow \mathcal{P}(X)$$

of X into $\mathcal{P}(X)$ (**Constructions With Sets**, **Item 9** of **Proposition 4.2.3**);

- The category $\text{PSh}(C)$ of presheaves on a category C as the free cocompletion of C via the Yoneda embedding

$$\mathcal{Y} : C \hookrightarrow \text{PSh}(C)$$

of C into $\text{PSh}(C)$ (**Categories**, ?? of ??).

00DN 1.3 Examples of Relations

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EXAMPLE 1.3.1 ► THE TRIVIAL RELATION

The **trivial relation on A and B** is the relation \sim_{triv} defined by^{1,2,3}

$$\sim_{\text{triv}} \stackrel{\text{def}}{=} A \times A.$$

¹This is the unique relation R on A and B such that we have $a \sim_R b$ for all $a \in A$ and all $b \in B$.

²As a function from $A \times A$ to $\{\text{true}, \text{false}\}$, the relation \sim_{triv} is the constant function

$$\Delta_{\text{true}} : A \times B \rightarrow \{\text{true}, \text{false}\}$$

from $A \times B$ to $\{\text{true}, \text{false}\}$ taking value **true**.

³As a function from A to $\mathcal{P}(B)$, the relation \sim_{triv} is the function

$$\Delta_{\text{true}} : A \rightarrow \mathcal{P}(B)$$

defined by

$$\Delta_{\text{true}}(a) \stackrel{\text{def}}{=} B$$

for each $a \in A$.

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EXAMPLE 1.3.2 ► THE COTRIVIAL RELATION

The **cotrivial relation** on A and B is the relation \sim_{cotriv} defined by^{1,2,3}

$$\sim_{\text{cotriv}} \stackrel{\text{def}}{=} \emptyset.$$

¹This is the unique relation R on A and B such that we have $a \sim_R b$ for no $a \in A$ and no $b \in B$.

²As a function from $A \times B$ to $\{\text{true}, \text{false}\}$, the relation \sim_{cotriv} is the constant function

$$\Delta_{\text{false}} : A \times B \rightarrow \{\text{true}, \text{false}\}$$

from $A \times B$ to $\{\text{true}, \text{false}\}$ taking value false.

³As a function from A to $\mathcal{P}(A)$, the relation \sim_{cotriv} is the function

$$\Delta_{\text{false}} : A \rightarrow \mathcal{P}(A)$$

defined by

$$\Delta_{\text{true}}(a) \stackrel{\text{def}}{=} \emptyset$$

for each $a \in A$.

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EXAMPLE 1.3.3 ► THE CHARACTERISTIC RELATION OF A SET

The characteristic relation on A of [Constructions With Sets, Item 3](#) of [Definition 4.1.1](#) is another example of a relation. It is in fact the unique relation on A making the following conditions equivalent, for each $a, b \in A$:

1. We have $a \sim_{\text{id}} b$.
2. We have $a = b$.

00DS

EXAMPLE 1.3.4 ► SQUARE ROOTS

Square roots are examples of relations:

1. *Square Roots in \mathbb{R}* . The assignment $x \mapsto \sqrt{x}$ defines a relation

$$\sqrt{-} : \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R})$$

from \mathbb{R} to itself, being explicitly given by

$$\sqrt{x} \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if } x = 0, \\ \{-\sqrt{|x|}, \sqrt{|x|}\} & \text{if } x \neq 0. \end{cases}$$

2. *Square Roots in \mathbb{Q} .* Square roots in \mathbb{Q} are similar to square roots in \mathbb{R} , though now additionally it may also occur that $\sqrt{\cdot}: \mathbb{Q} \rightarrow \mathcal{P}(\mathbb{Q})$ sends a rational number x (e.g. 2) to the empty set (since $\sqrt{2} \notin \mathbb{Q}$).

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EXAMPLE 1.3.5 ► COMPLEX LOGARITHMS

The complex logarithm defines a relation

$$\log: \mathbb{C} \rightarrow \mathcal{P}(\mathbb{C})$$

from \mathbb{C} to itself, where we have

$$\log(a + bi) \stackrel{\text{def}}{=} \left\{ \log\left(\sqrt{a^2 + b^2}\right) + i \arg(a + bi) + (2\pi i)k \mid k \in \mathbb{Z} \right\}$$

for each $a + bi \in \mathbb{C}$.

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EXAMPLE 1.3.6 ► MORE EXAMPLES OF RELATIONS

See [\[wikipedia:multivalued-functions\]](#) for more examples of relations, such as antiderivation, inverse trigonometric functions, and inverse hyperbolic functions.

00DV 1.4 Functional Relations

Let A and B be sets.

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DEFINITION 1.4.1 ► FUNCTIONAL RELATIONS

A relation $R: A \rightarrow B$ is **functional** if, for each $a \in A$, the set $R(a)$ is either empty or a singleton.

00DX

PROPOSITION 1.4.2 ► PROPERTIES OF FUNCTIONAL RELATIONS

Let $R: A \rightarrow B$ be a relation.

00DY

1. *Characterisations.* The following conditions are equivalent:

00DZ

(a) The relation R is functional.

00E0

(b) We have $R \diamond R^\dagger \subset \chi_B$.

PROOF 1.4.3 ► PROOF OF PROPOSITION 1.4.2**Item 1: Characterisations**

We claim that **Items 1a** and **1b** are indeed equivalent:

• **Item 1a** \implies **Item 1b**: Let $(b, b') \in B \times B$. We need to show that

$$[R \diamond R^\dagger](b, b') \leq_{\{t, f\}} \chi_B(b, b'),$$

i.e. that if there exists some $a \in A$ such that $b \sim_{R^\dagger} a$ and $a \sim_R b'$, then $b = b'$. But since $b \sim_{R^\dagger} a$ is the same as $a \sim_R b$, we have both $a \sim_R b$ and $a \sim_R b'$ at the same time, which implies $b = b'$ since R is functional.


• **Item 1b** \implies **Item 1a**: Suppose that we have $a \sim_R b$ and $a \sim_R b'$ for $b, b' \in B$. We claim that $b = b'$:

1. Since $a \sim_R b$, we have $b \sim_{R^\dagger} a$.

2. Since $R \diamond R^\dagger \subset \chi_B$, we have

$$[R \diamond R^\dagger](b, b') \leq_{\{t, f\}} \chi_B(b, b'),$$

and since $b \sim_{R^\dagger} a$ and $a \sim_R b'$, it follows that $[R \diamond R^\dagger](b, b') = \text{true}$, and thus $\chi_B(b, b') = \text{true}$ as well, i.e. $b = b'$.

This finishes the proof. 

00E1 1.5 Total Relations

Let A and B be sets.

00E2 DEFINITION 1.5.1 ► TOTAL RELATIONS

A relation $R: A \rightarrow B$ is **total** if, for each $a \in A$, we have $R(a) \neq \emptyset$.

00E3 PROPOSITION 1.5.2 ► PROPERTIES OF TOTAL RELATIONS

Let $R: A \rightarrow B$ be a relation.

00E4 1. *Characterisations.* The following conditions are equivalent:

00E5 (a) The relation R is total.

00E6 (b) We have $\chi_A \subset R^\dagger \diamond R$.

PROOF 1.5.3 ► PROOF OF PROPOSITION 1.5.2

Item 1: Characterisations

We claim that **Items 1a** and **1b** are indeed equivalent:

• **Item 1a** \implies **Item 1b**: We have to show that, for each $(a, a') \in A$, we have


$$\chi_A(a, a') \leq_{\{t, f\}} [R^\dagger \diamond R](a, a'),$$

i.e. that if $a = a'$, then there exists some $b \in B$ such that $a \sim_R b$ and $b \sim_{R^\dagger} a'$ (i.e. $a \sim_R b$ again), which follows from the totality of R .

• **Item 1b** \implies **Item 1a**: Given $a \in A$, since $\chi_A \subset R^\dagger \diamond R$, we must have

$$\{a\} \subset [R^\dagger \diamond R](a),$$

implying that there must exist some $b \in B$ such that $a \sim_R b$ and $b \sim_{R^\dagger} a$ (i.e. $a \sim_R b$) and thus $R(a) \neq \emptyset$, as $b \in R(a)$.

This finishes the proof. 

00E7 2 Categories of Relations

00E8 2.1 The Category of Relations

00E9

DEFINITION 2.1.1 ► THE CATEGORY OF RELATIONS

The **category of relations** is the category Rel where

- *Objects.* The objects of Rel are sets;
- *Morphisms.* For each $A, B \in \text{Obj}(\text{Sets})$, we have

$$\text{Rel}(A, B) \stackrel{\text{def}}{=} \text{Rel}(A, B);$$

- *Identities.* For each $A \in \text{Obj}(\text{Rel})$, the unit map

$$\mathbb{K}_A^{\text{Rel}} : \text{pt} \rightarrow \text{Rel}(A, A)$$

of Rel at A is defined by

$$\text{id}_A^{\text{Rel}} \stackrel{\text{def}}{=} \chi_A(-_1, -_2),$$

where $\chi_A(-_1, -_2)$ is the characteristic relation of A of **Constructions With Sets, Item 3 of Definition 4.1.1**;

- *Composition.* For each $A, B, C \in \text{Obj}(\text{Rel})$, the composition map

$$\circ_{A,B,C}^{\text{Rel}} : \text{Rel}(B, C) \times \text{Rel}(A, B) \rightarrow \text{Rel}(A, C)$$

of Rel at (A, B, C) is defined by

$$S \circ_{A,B,C}^{\text{Rel}} R \stackrel{\text{def}}{=} S \diamond R$$

for each $(S, R) \in \text{Rel}(B, C) \times \text{Rel}(A, B)$, where $S \diamond R$ is the composition of S and R of **Definition 3.12.1**.

00EA 2.2 The Closed Symmetric Monoidal Category of Relations**2.2.1 The Monoidal Product**

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DEFINITION 2.2.1 ► THE MONOIDAL PRODUCT OF REL

The **monoidal product** of Rel is the functor

$$\times : \text{Rel} \times \text{Rel} \rightarrow \text{Rel}$$

where

- *Action on Objects.* We have

$$\times(A, B) \stackrel{\text{def}}{=} A \times B,$$

where $A \times B$ is the Cartesian product of sets of [Constructions With Sets, Definition 1.2.1](#);

- *Action on Morphisms.* For each $(A, C), (B, D) \in \text{Obj}(\text{Rel} \times \text{Rel})$, the action on morphisms

$$\times_{(A,C),(B,D)} : \text{Rel}(A, B) \times \text{Rel}(C, D) \rightarrow \text{Rel}(A \times C, B \times D)$$

of \times is given by sending a pair of morphisms (R, S) of the form

$$\begin{aligned} R &: A \rightarrowtail B, \\ S &: C \rightarrowtail D \end{aligned}$$

to the relation

$$R \times S : A \times C \rightarrowtail B \times D$$

of [Definition 3.9.1](#).

2.2.2 The Monoidal Unit

00EE

DEFINITION 2.2.2 ► THE MONOIDAL UNIT OF Rel

The **monoidal unit** of Rel is the functor

$$\mathbb{K}^{\text{Rel}} : \text{pt} \rightarrow \text{Rel}$$

picking the set

$$\mathbb{K}_{\text{Rel}} \stackrel{\text{def}}{=} \text{pt}$$

of Rel.

2.2.3 The Associator

00EG

DEFINITION 2.2.3 ► THE ASSOCIATOR OF Rel

The **associator** of Rel is the natural isomorphism

$$\alpha^{\text{Rel}} : \times \circ ((\times) \times \text{id}) \xrightarrow{\cong} \times \circ (\text{id} \times (\times)),$$

whose component

$$\alpha_{A,B,C}^{\text{Rel}} : (A \times B) \times C \rightarrowtail A \times (B \times C)$$

at (A, B, C) is defined by declaring

$$((a, b), c) \sim_{\alpha_{A,B,C}^{\text{Rel}}} (a', (b', c'))$$

iff $a = a'$, $b = b'$, and $c = c'$.

2.2.4 The Left Unitor

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DEFINITION 2.2.4 ► THE LEFT UNITOR OF Rel

The **left unitor** of Rel is the natural isomorphism

$$\lambda^{\text{Rel}} : \times \circ (\text{pt}^{\text{Rel}} \times \text{id}) \xrightarrow{\cong} \lambda_{\text{Rel}}^{\text{Cats}_2},$$

whose component

$$\lambda_A^{\text{Rel}} : \text{pt}^{\text{Rel}} \times A \rightarrowtail A$$

at A is defined by declaring

$$(\star, a) \sim_{\lambda_A^{\text{Rel}}} b$$

iff $a = b$.

2.2.5 The Right Unitor

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DEFINITION 2.2.5 ► THE RIGHT UNITOR OF Rel

The **right unitor** of Rel is the natural isomorphism

$$\rho^{\text{Rel}}: \times \circ (\text{id} \times \mathbb{K}^{\text{Rel}}) \xRightarrow{\cong} \rho_{\text{Rel}}^{\text{Cats}_2},$$

whose component

$$\rho_A^{\text{Rel}}: A \times \mathbb{K}^{\text{Rel}} \rightarrow A$$

at A is defined by declaring

$$(a, \star) \sim_{\rho_A^{\text{Rel}}} b$$

iff $a = b$.

2.2.6 The Symmetry

00EN

DEFINITION 2.2.6 ► THE SYMMETRY OF Rel

The **symmetry** of Rel is the natural isomorphism

$$\sigma^{\text{Rel}} : \times \Rightarrow \times \circ \sigma_{\text{Rel}, \text{Rel}}^{\text{Cats}_2}$$

whose component

$$\sigma_{A,B}^{\text{Rel}} : A \times B \rightarrow B \times A$$

at (A, B) is defined by declaring

$$(a, b) \sim_{\sigma_{A,B}^{\text{Rel}}} (b', a')$$

iff $a = a'$ and $b = b'$.

2.2.7 The Internal Hom

00EQ

DEFINITION 2.2.7 ► THE INTERNAL HOM OF Rel

The **internal Hom** of Rel is the functor

$$\mathbf{Hom}_{\text{Rel}} : \text{Rel}^{\text{op}} \times \text{Rel} \rightarrow \text{Rel}$$

defined by

$$\mathbf{Hom}_{\text{Rel}}(A, B) \stackrel{\text{def}}{=} A \times B$$

for each $A, B \in \text{Obj}(\text{Rel})$.

00ER

PROPOSITION 2.2.8 ► PROPERTIES OF THE INTERNAL HOM OF Rel

Let $A, B, C \in \text{Obj}(\text{Rel})$.

00ES

1. *Via Self-Duality.* The internal Hom $\mathbf{Hom}_{\text{Rel}}$ of Rel is given by the composition

$$\text{Rel}^{\text{op}} \times \text{Rel} \xrightarrow{\cong} \text{Rel} \times \text{Rel} \xrightarrow{\times} \text{Rel},$$

where the self-duality equivalence $\text{Rel}^{\text{op}} \cong \text{Rel}$ comes from [Item 1 of Propo-](#)

sition 2.5.1.

00ET

2. *Adjointness.* We have adjunctions

$$(A \times - \dashv \mathbf{Hom}_{\mathbf{Rel}}(A, -)) : \mathbf{Rel} \begin{array}{c} \xrightarrow{A \times -} \\ \perp \\ \xleftarrow{\mathbf{Hom}_{\mathbf{Rel}}(A, -)} \end{array} \mathbf{Rel},$$

$$(- \times B \dashv \mathbf{Hom}_{\mathbf{Rel}}(B, -)) : \mathbf{Rel} \begin{array}{c} \xrightarrow{- \times B} \\ \perp \\ \xleftarrow{\mathbf{Hom}_{\mathbf{Rel}}(B, -)} \end{array} \mathbf{Rel},$$

witnessed by bijections

$$\begin{aligned} \mathbf{Rel}(A \times B, C) &\cong \mathbf{Rel}(A, \mathbf{Hom}_{\mathbf{Rel}}(B, C)) \\ &\stackrel{\text{def}}{=} \mathbf{Rel}(A, B \times C), \\ \mathbf{Rel}(A \times B, C) &\cong \mathbf{Rel}(B, \mathbf{Hom}_{\mathbf{Rel}}(A, C)) \\ &\stackrel{\text{def}}{=} \mathbf{Rel}(B, A \times C), \end{aligned}$$

natural in $A, B, C \in \mathbf{Obj}(\mathbf{Rel})$.

PROOF 2.2.9 ► PROOF OF PROPOSITION 2.2.8


Item 1: Via Self-Duality

Omitted.

Item 2: Adjointness

Indeed, we have

$$\begin{aligned} \mathbf{Rel}(A \times B, C) &\stackrel{\text{def}}{=} \mathbf{Sets}(A \times B \times C, \{\text{true}, \text{false}\}) \\ &\stackrel{\text{def}}{=} \mathbf{Rel}(A, B \times C) \\ &\stackrel{\text{def}}{=} \mathbf{Rel}(A, \mathbf{Hom}_{\mathbf{Rel}}(B, C)), \end{aligned}$$

and similarly for the bijection $\mathbf{Rel}(A \times B, C) \cong \mathbf{Rel}(B, \mathbf{Hom}_{\mathbf{Rel}}(A, C))$. 

2.2.8 The Closed Symmetric Monoidal Category of Relations

00EV

DEFINITION 2.2.10 ► THE CLOSED SYMMETRIC MONOIDAL CATEGORY OF RELATIONS

The **closed symmetric monoidal category of relations** is the closed symmetric monoidal category

$$\left(\mathbf{Rel}, \times, \mathbb{K}_{\mathbf{Rel}}, \alpha^{\mathbf{Rel}}, \lambda^{\mathbf{Rel}}, \rho^{\mathbf{Rel}}, \sigma^{\mathbf{Rel}}, \mathbf{Hom}_{\mathbf{Rel}} \right)$$

consisting of

- *The Underlying Category.* The category \mathbf{Rel} of sets and relations of [Definition 2.1.1](#);
- *The Monoidal Product.* The functor

$$\times: \mathbf{Rel} \times \mathbf{Rel} \rightarrow \mathbf{Rel}$$

of [Definition 2.2.1](#);

- *The Monoidal Unit.* The functor $\mathbb{K}^{\mathbf{Rel}}$ of [Definition 2.2.2](#);
- *The Associator.* The natural isomorphism $\alpha^{\mathbf{Rel}}$ of [Definition 2.2.3](#);
- *The Left Unitor.* The natural isomorphism $\lambda^{\mathbf{Rel}}$ of [Definition 2.2.4](#);
- *The Right Unitor.* The natural isomorphism $\rho^{\mathbf{Rel}}$ of [Definition 2.2.5](#);
- *The Symmetry.* The natural isomorphism $\sigma^{\mathbf{Rel}}$ of [Definition 2.2.6](#);
- *The Internal Hom.* The functor

$$\mathbf{Hom}_{\mathbf{Rel}}: \mathbf{Rel}^{\text{op}} \times \mathbf{Rel} \rightarrow \mathbf{Rel}$$

of [Definition 2.2.7](#).

00EW 2.3 The 2-Category of Relations

00EX

DEFINITION 2.3.1 ► THE 2-CATEGORY OF RELATIONS

The **2-category of relations** is the locally posetal 2-category \mathbf{Rel} where

- *Objects.* The objects of \mathbf{Rel} are sets;

- **Hom-Objects.** For each $A, B \in \text{Obj}(\mathbf{Sets})$, we have

$$\begin{aligned} \text{Hom}_{\mathbf{Rel}}(A, B) &\stackrel{\text{def}}{=} \mathbf{Rel}(A, B) \\ &\stackrel{\text{def}}{=} (\text{Rel}(A, B), \subset); \end{aligned}$$

- **Identities.** For each $A \in \text{Obj}(\mathbf{Rel})$, the unit map

$$\mathbb{K}_A^{\mathbf{Rel}}: \text{pt} \rightarrow \mathbf{Rel}(A, A)$$

of \mathbf{Rel} at A is defined by

$$\text{id}_A^{\mathbf{Rel}} \stackrel{\text{def}}{=} \chi_A(-1, -2),$$

where $\chi_A(-1, -2)$ is the characteristic relation of A of **Constructions With Sets, Item 3 of Definition 4.1.1**;

- **Composition.** For each $A, B, C \in \text{Obj}(\mathbf{Rel})$, the composition map¹

$$\circ_{A,B,C}^{\mathbf{Rel}}: \mathbf{Rel}(B, C) \times \mathbf{Rel}(A, B) \rightarrow \mathbf{Rel}(A, C)$$

of \mathbf{Rel} at (A, B, C) is defined by

$$S \circ_{A,B,C}^{\mathbf{Rel}} R \stackrel{\text{def}}{=} S \diamond R$$

for each $(S, R) \in \mathbf{Rel}(B, C) \times \mathbf{Rel}(A, B)$, where $S \diamond R$ is the composition of S and R of **Definition 3.12.1**.

¹Note that this is indeed a morphism of posets: given relations $R_1, R_2 \in \mathbf{Rel}(A, B)$ and $S_1, S_2 \in \mathbf{Rel}(B, C)$ such that

$$\begin{aligned} R_1 &\subset R_2, \\ S_1 &\subset S_2, \end{aligned}$$

we have also $S_1 \diamond R_1 \subset S_2 \diamond R_2$.

00EY 2.4 The Double Category of Relations

2.4.1 The Double Category of Relations

00F0

DEFINITION 2.4.1 ► THE DOUBLE CATEGORY OF RELATIONS

The **double category of relations** is the locally posetal double category Rel^{dbl} where

- *Objects.* The objects of Rel^{dbl} are sets;
- *Vertical Morphisms.* The vertical morphisms of Rel^{dbl} are maps of sets $f: A \rightarrow B$;
- *Horizontal Morphisms.* The horizontal morphisms of Rel^{dbl} are relations $R: A \rightarrowtail B$;
- *2-Morphisms.* A 2-cell

$$\begin{array}{ccc} A & \xrightarrow{R} & B \\ f \downarrow & \Downarrow \alpha & \downarrow g \\ X & \xrightarrow{S} & Y \end{array}$$

of Rel^{dbl} is either non-existent or an inclusion of relations of the form

$$\begin{array}{ccccc} A \times B & \xrightarrow{R} & \{\text{true}, \text{false}\} \\ f \times g \downarrow & \subset & \downarrow \text{id}_{\{\text{true}, \text{false}\}} \\ X \times Y & \xrightarrow{S} & \{\text{true}, \text{false}\}; \end{array}$$

- *Horizontal Identities.* The horizontal unit functor of Rel^{dbl} is the functor of [Definition 2.4.2](#);
- *Vertical Identities.* For each $A \in \text{Obj}(\text{Rel}^{\text{dbl}})$, we have

$$\text{id}_A^{\text{Rel}^{\text{dbl}}} \stackrel{\text{def}}{=} \text{id}_A;$$

- *Identity 2-Morphisms.* For each horizontal morphism $R: A \rightarrowtail B$ of Rel^{dbl} , the identity 2-morphism

$$\begin{array}{ccc}
 A & \xrightarrow{R} & B \\
 \text{id}_A \downarrow & \Downarrow \text{id}_R & \downarrow \text{id}_B \\
 A & \xrightarrow{R} & B
 \end{array}$$

of R is the identity inclusion

$$\begin{array}{ccc}
 B \times A & \xrightarrow{R} & \{\text{true}, \text{false}\} \\
 \text{id}_B \times \text{id}_A \downarrow & \subset & \downarrow \text{id}_{\{\text{true}, \text{false}\}} \\
 B \times A & \xrightarrow{R} & \{\text{true}, \text{false}\};
 \end{array}$$

- *Horizontal Composition.* The horizontal composition functor of Rel^{dbl} is the functor of [Definition 2.4.3](#);
- *Vertical Composition of 1-Morphisms.* For each composable pair $A \xrightarrow{F} B \xrightarrow{G} C$ of vertical morphisms of Rel^{dbl} , i.e. maps of sets, we have

$$g \circ^{\text{Rel}^{\text{dbl}}} f \stackrel{\text{def}}{=} g \circ f;$$

- *Vertical Composition of 2-Morphisms.* The vertical composition of 2-morphisms in Rel^{dbl} is defined as in [Definition 2.4.4](#);
- *Associators.* The associators of Rel^{dbl} is defined as in [Definition 2.4.5](#);
- *Left Unitors.* The left unitors of Rel^{dbl} is defined as in [Definition 2.4.6](#);
- *Right Unitors.* The right unitors of Rel^{dbl} is defined as in [Definition 2.4.7](#).

2.4.2 Horizontal Identities

00F2

DEFINITION 2.4.2 ► THE HORIZONTAL IDENTITIES OF Rel^{dbl}

The **horizontal unit functor** of Rel^{dbl} is the functor

$$\mathbb{K}^{\text{Rel}^{\text{dbl}}} : \text{Rel}_0^{\text{dbl}} \rightarrow \text{Rel}_1^{\text{dbl}}$$

of Rel^{dbl} is the functor where

- *Action on Objects.* For each $A \in \text{Obj}(\text{Rel}_0^{\text{dbl}})$, we have

$$\mathbb{K}_A \stackrel{\text{def}}{=} \chi_A(-1, -2);$$

- *Action on Morphisms.* For each vertical morphism $f: A \rightarrow B$ of Rel^{dbl} , i.e. each map of sets f from A to B , the identity 2-morphism

$$\begin{array}{ccc} A & \xrightarrow{\mathbb{K}_A} & A \\ f \downarrow & \Downarrow \mathbb{K}_f & \downarrow f \\ B & \xrightarrow{\mathbb{K}_B} & B \end{array}$$

of f is the inclusion

$$\begin{array}{ccccc} A \times A & \xrightarrow{\chi_A(-1, -2)} & \{\text{true}, \text{false}\} \\ \downarrow f \times f & \subset & \downarrow \text{id}_{\{\text{true}, \text{false}\}} \\ B \times B & \xrightarrow{\chi_B(-1, -2)} & \{\text{true}, \text{false}\} \end{array}$$

of **Constructions With Sets, Proposition 4.1.3**.

2.4.3 Horizontal Composition

00F4

DEFINITION 2.4.3 ► THE HORIZONTAL COMPOSITION OF Rel^{dbl}

The **horizontal composition functor** of Rel^{dbl} is the functor

$$\odot^{\text{Rel}^{\text{dbl}}} : \text{Rel}_1^{\text{dbl}} \times_{\text{Rel}_0^{\text{dbl}}} \text{Rel}_1^{\text{dbl}} \rightarrow \text{Rel}_1^{\text{dbl}}$$

of Rel^{dbl} is the functor where

- *Action on Objects.* For each composable pair $A \xrightarrow{R} B \xrightarrow{S} C$ of horizontal morphisms of Rel^{dbl} , we have

$$S \odot R \stackrel{\text{def}}{=} S \diamond R,$$

where $S \diamond R$ is the composition of R and S of [Definition 3.12.1](#);

- *Action on Morphisms.* For each horizontally composable pair

$$\begin{array}{ccc} A & \xrightarrow{R} & B \\ f \downarrow & \Downarrow \alpha & \downarrow g \\ X & \xrightarrow{T} & Y \end{array} \quad \begin{array}{ccc} B & \xrightarrow{S} & C \\ g \downarrow & \Downarrow \beta & \downarrow h \\ Y & \xrightarrow{U} & Z \end{array}$$

of 2-morphisms of Rel^{dbl} , i.e. for each pair

$$\begin{array}{ccc} A \times B & \xrightarrow{R} & \{\text{true}, \text{false}\} \\ f \times g \downarrow & \subset & \downarrow \text{id}_{\{\text{true}, \text{false}\}} \\ X \times Y & \xrightarrow{T} & \{\text{true}, \text{false}\} \end{array} \quad \begin{array}{ccc} B \times C & \xrightarrow{S} & \{\text{true}, \text{false}\} \\ g \times h \downarrow & \subset & \downarrow \text{id}_{\{\text{true}, \text{false}\}} \\ Y \times Z & \xrightarrow{U} & \{\text{true}, \text{false}\} \end{array}$$

of inclusions of relations, the horizontal composition

$$\begin{array}{ccc} A & \xrightarrow{S \odot R} & C \\ f \downarrow & \Downarrow \beta \odot \alpha & \downarrow h \\ X & \xrightarrow{U \odot T} & Z \end{array}$$

of α and β is the inclusion of relations¹

$$(U \diamond T) \circ (f \times h) \subset (S \diamond R)$$

$$\begin{array}{ccc} A \times C & \xrightarrow{S \diamond R} & \{\text{true}, \text{false}\} \\ f \times h \downarrow & \subset & \downarrow \text{id}_{\{\text{true}, \text{false}\}} \\ X \times Z & \xrightarrow{U \diamond T} & \{\text{true}, \text{false}\}. \end{array}$$

¹This is justified by noting that, given $(a, c) \in A \times C$, the statement

– We have $a \sim_{(U \diamond T) \circ (f \times h)} c$, i.e. $f(a) \sim_{U \diamond T} h(c)$, i.e. there exists some $y \in Y$ such that:

1. We have $f(a) \sim_T y$;
2. We have $y \sim_U h(c)$;

is implied by the statement

– We have $a \sim_{S \diamond R} c$, i.e. there exists some $b \in B$ such that:

1. We have $a \sim_R b$;
2. We have $b \sim_S c$;

since:

- If $a \sim_R b$, then $f(a) \sim_T g(b)$, as $T \circ (f \times g) \subset R$;
- If $b \sim_S c$, then $g(b) \sim_U h(c)$, as $U \circ (g \times h) \subset S$;

2.4.4 Vertical Composition of 2-Morphisms

00F6

DEFINITION 2.4.4 ► THE VERTICAL COMPOSITION OF 2-MORPHISMS IN Rel^{dbl}

The **vertical composition** in Rel^{dbl} is defined as follows: for each vertically composable pair

$$\begin{array}{ccc} A & \xrightarrow{R} & X \\ f \downarrow & \Downarrow \alpha & \downarrow g \\ B & \xrightarrow{S} & Y \end{array} \quad \begin{array}{ccc} B & \xrightarrow{S} & Y \\ h \downarrow & \Downarrow \beta & \downarrow k \\ C & \xrightarrow{T} & Z \end{array}$$

of 2-morphisms of Rel^{dbl} , i.e. for each pair

$$\begin{array}{ccc} A \times X & \xrightarrow{R} & \{\text{true}, \text{false}\} \\ f \times g \downarrow & \subset & \downarrow \text{id}_{\{\text{true}, \text{false}\}} \\ B \times Y & \xrightarrow{S} & \{\text{true}, \text{false}\} \end{array} \quad \begin{array}{ccc} B \times Y & \xrightarrow{S} & \{\text{true}, \text{false}\} \\ h \times k \downarrow & \subset & \downarrow \text{id}_{\{\text{true}, \text{false}\}} \\ C \times Z & \xrightarrow{T} & \{\text{true}, \text{false}\} \end{array}$$

of inclusions of relations, we define the vertical composition

$$\begin{array}{ccc} A & \xrightarrow{R} & X \\ h \circ f \downarrow & \beta \circ \alpha \Downarrow & \downarrow k \circ g \\ C & \xrightarrow{T} & Z \end{array}$$

of α and β as the inclusion of relations

$$T \circ [(h \circ f) \times (k \circ g)] \subset R, \quad \begin{array}{ccc} A \times X & \xrightarrow{R} & \{\text{true}, \text{false}\} \\ (h \circ f) \times (k \circ g) \downarrow & \subset & \downarrow \text{id}_{\{\text{true}, \text{false}\}} \\ C \times Z & \xrightarrow{T} & \{\text{true}, \text{false}\} \end{array}$$

given by the pasting of inclusions¹

$$\begin{array}{ccc} A \times X & \xrightarrow{R} & \{\text{true}, \text{false}\} \\ f \times g \downarrow & \subset & \downarrow \text{id}_{\{\text{true}, \text{false}\}} \\ B \times Y & \xrightarrow{S} & \{\text{true}, \text{false}\} \\ h \times k \downarrow & \subset & \downarrow \text{id}_{\{\text{true}, \text{false}\}} \\ C \times Z & \xrightarrow{T} & \{\text{true}, \text{false}\}. \end{array}$$

¹This is justified by noting that, given $(a, x) \in A \times X$, the statement

· We have $h(f(a)) \sim_T k(g(x))$;

is implied by the statement

- We have $a \sim_R x$;

since

- If $a \sim_R x$, then $f(a) \sim_S g(x)$, as $S \circ (f \times g) \subset R$;
- If $b \sim_S y$, then $h(b) \sim_T k(y)$, as $T \circ (h \times k) \subset S$, and thus, in particular:
 - If $f(a) \sim_S g(x)$, then $h(f(a)) \sim_T k(g(x))$;

2.4.5 The Associators

00F8

DEFINITION 2.4.5 ► THE ASSOCIATORS OF Rel^{dbl}

For each composable triple $A \xrightarrow{R} B \xrightarrow{S} C \xrightarrow{T} D$ of horizontal morphisms of Rel^{dbl} , the component

$$\alpha_{T,S,R}^{\text{Rel}^{\text{dbl}}} : (T \odot S) \odot R \xrightarrow{\cong} T \odot (S \odot R),$$

of the associator of Rel^{dbl} at (R, S, T) is the identity inclusion¹

$$(T \odot S) \odot R = T \odot (S \odot R)$$

¹This is justified by Item 2 of Proposition 3.12.3.

2.4.6 The Left Unitors

00FA

DEFINITION 2.4.6 ► THE LEFT UNITORS OF Rel^{dbl}

For each horizontal morphism $R: A \rightarrowtail B$ of Rel^{dbl} , the component

$$\lambda_R^{\text{Rel}^{\text{dbl}}} : \mathbb{K}_B \odot R \xRightarrow{\cong} R,$$

$$\begin{array}{ccccc} A & \xrightarrow{\quad R \quad} & B & \xrightarrow{\quad \mathbb{K}_B \quad} & B \\ \text{id}_A \downarrow & & \lambda_R^{\text{Rel}^{\text{dbl}}} \Downarrow & & \downarrow \text{id}_B \\ A & \xrightarrow{\quad R \quad} & B & & \end{array}$$

of the left unitor of Rel^{dbl} at R is the identity inclusion¹

$$R = \chi_B \diamond R,$$

$$\begin{array}{ccc} A \times B & \xrightarrow{\chi_B \diamond R} & \{\text{true}, \text{false}\} \\ \parallel & \cong & \downarrow \text{id}_{\{\text{true}, \text{false}\}} \\ A \times B & \xrightarrow{\quad R \quad} & \{\text{true}, \text{false}\}. \end{array}$$

¹This is justified by Item 3 of Proposition 3.12.3.

2.4.7 The Right Unitors

00FC

DEFINITION 2.4.7 ► THE RIGHT UNITORS OF Rel^{dbl}

For each horizontal morphism $R: A \rightarrowtail B$ of Rel^{dbl} , the component

$$\rho_R^{\text{Rel}^{\text{dbl}}} : R \odot \mathbb{K}_A \xRightarrow{\cong} R,$$

$$\begin{array}{ccccc} A & \xrightarrow{\quad \mathbb{K}_A \quad} & A & \xrightarrow{\quad R \quad} & B \\ \text{id}_A \downarrow & & \rho_R^{\text{Rel}^{\text{dbl}}} \Downarrow & & \downarrow \text{id}_B \\ A & \xrightarrow{\quad R \quad} & B & & \end{array}$$

of the right unitor of $\mathbf{Rel}^{\text{dbl}}$ at R is the identity inclusion¹

$$R = R \diamond \chi_A, \quad \begin{array}{ccc} A \times B & \xrightarrow{R \diamond \chi_A} & \{\text{true}, \text{false}\} \\ \parallel & \cong & \downarrow \text{id}_{\{\text{true}, \text{false}\}} \\ A \times B & \xrightarrow{R} & \{\text{true}, \text{false}\}. \end{array}$$

¹This is justified by Item 3 of Proposition 3.12.3.

00FD 2.5 Properties of the Category of Relations

00FE PROPOSITION 2.5.1 ► PROPERTIES OF THE CATEGORY OF RELATIONS

Let A and B be sets.

- 00FF 1. *Self-Duality I.* The category \mathbf{Rel} is self-dual, i.e. we have an equivalence

$$\mathbf{Rel}^{\text{op}} \stackrel{\text{eq.}}{\cong} \mathbf{Rel}$$

of categories.

- 00FG 2. *Self-Duality II.* The bicategory \mathbf{Rel} is self-dual, i.e. we have a biequivalence

$$\mathbf{Rel}^{\text{op}} \stackrel{\text{eq.}}{\cong} \mathbf{Rel}$$

of bicategories.

- 00FH 3. *Equivalences and Isomorphisms in \mathbf{Rel} .* Let $R: A \rightarrowtail B$ be a relation from A to B . The following conditions are equivalent:

- 00FJ (a) The relation $R: A \rightarrowtail B$ is an equivalence in \mathbf{Rel} , i.e. there exists a relation $R^{-1}: B \rightarrowtail A$ from B to A together with isomorphisms

$$R^{-1} \diamond R \cong \chi_A,$$

$$R \diamond R^{-1} \cong \chi_B.$$

- 00FK (b) The relation $R: A \rightarrowtail B$ is an isomorphism in \mathbf{Rel} , i.e. there exists a relation $R^{-1}: B \rightarrowtail A$ from B to A such that we have

$$R^{-1} \diamond R = \chi_A,$$

$$R \diamond R^{-1} = \chi_B.$$

00FL

(c) There exists a bijection $f: A \xrightarrow{\cong} B$ with $R = \text{Gr}(f)$.

00FM

4. *Adjunctions in **Rel***. We have a natural bijection

$$\left\{ \begin{array}{c} \text{Adjunctions in } \mathbf{Rel} \\ \text{from } A \text{ to } B \end{array} \right\} \cong \left\{ \begin{array}{c} \text{Functions} \\ \text{from } A \text{ to } B \end{array} \right\},$$

with every adjunction in **Rel** being of the form $\text{Gr}(f) \dashv f^{-1}$ for some function f .

00FN

5. *Monads in **Rel***. We have a natural bijection¹

$$\left\{ \begin{array}{c} \text{Monads in} \\ \mathbf{Rel} \text{ on } A \end{array} \right\} \cong \{\text{Preorders on } A\}.$$

00FP

6. *Comonads in **Rel***. We have a natural bijection

$$\left\{ \begin{array}{c} \text{Comonads in} \\ \mathbf{Rel} \text{ on } A \end{array} \right\} \cong \{\text{Subsets of } A\}.$$

00FQ

7. *Characterisations of Monomorphisms in Rel*. Let $R: A \rightarrowtail B$ be a relation. The following conditions are equivalent:

00FR

(a) The relation R is a monomorphism in **Rel**.

00FS

(b) The direct image function

$$R_*: \mathcal{P}(A) \rightarrow \mathcal{P}(B)$$

associated to R is injective.

00FT

(c) The direct image with compact support function

$$R_!: \mathcal{P}(A) \rightarrow \mathcal{P}(B)$$

associated to R is injective.

Moreover, if R is a monomorphism, then it satisfies the following condition, and the converse holds if R is total:

(★) For each $a, a' \in A$, if there exists some $b \in B$ such that $a \sim_R b$ and $a' \sim_R b$, then $a = a'$.

00FU

8. *Epimorphisms in Rel.* Let $R: A \rightarrow B$ be a relation. The following conditions are equivalent:

00FV

(a) The relation R is an epimorphism in Rel.

00FW

(b) The weak inverse image function

$$R^{-1}: \mathcal{P}(B) \rightarrow \mathcal{P}(A)$$

associated to R is injective.

00FX

(c) The strong inverse image function

$$R_{-1}: \mathcal{P}(B) \rightarrow \mathcal{P}(A)$$

associated to R is injective.

00FY

(d) The function $R: A \rightarrow \mathcal{P}(B)$ is “surjective on singletons”:

(★) For each $b \in B$, there exists some $a \in A$ such that $R(a) = \{b\}$.

00FZ

9. *As a Kleisli Category.* We have an isomorphism of categories

$$\text{Rel} \cong \text{FreeAlg}_{\mathcal{P}},$$

where \mathcal{P} is the powerset monad of Monads, ??.

00G0

10. *Co/Completeness (Or Lack Thereof).* The category Rel is not co/complete, but admits some co/limits:

(a) *Zero Objects.* The category Rel has a zero object, the empty set \emptyset .

(b) *Co/Products.* The category Rel has co/products, both given by disjoint union of sets.

(c) *Lack of Co/Equalisers.* The category Rel does not have co/equalisers.

(d) *Limits of Graphs of Functions.* The category Rel has limits whose arrows are all graphs of functions.

(e) *Colimits of Graphs of Functions.* The category Rel has colimits whose arrows are all graphs of functions, and these agree with the corresponding limits in Sets.

00G1

11. *Existence of Right Kan Extensions.* The right Kan extension

$$\text{Ran}_R: \text{Rel}(A, X) \rightarrow \text{Rel}(B, X)$$

along a relation $R: A \rightarrowtail B$ exists and is given by

$$\text{Ran}_R(S) \stackrel{\text{def}}{=} \int_{a \in A} \mathbf{Hom}_{\{t, f\}}(R_{-1}^a, S_{-2}^a)$$

for each $S \in \text{Rel}(A, X)$, so that the following conditions are equivalent:

- (a) We have $b \sim_{\text{Ran}_R(S)} x$.
- (b) For each $a \in A$, if $a \sim_R b$, then $a \sim_S x$.

00G2

12. *Existence of Right Kan Lifts.* The right Kan lift

$$\text{Rift}_R: \text{Rel}(X, B) \rightarrow \text{Rel}(X, A)$$

along a relation $R: A \rightarrowtail B$ exists and is given by

$$\text{Rift}_R(S) \stackrel{\text{def}}{=} \int_{b \in B} \mathbf{Hom}_{\{t, f\}}(R_b^{-2}, S_b^{-1})$$

for each $S \in \text{Rel}(X, B)$, so that the following conditions are equivalent:

- (a) We have $x \sim_{\text{Rift}_R(S)} a$.
- (b) For each $b \in B$, if $a \sim_R b$, then $x \sim_S b$.

00G3

13. *Closedness.* The bicategory **Rel** is a closed bicategory, there being, for each $R: A \rightarrowtail B$ and set X , a pair of adjunctions

$$\begin{array}{ccc} (R^* \dashv \text{Ran}_R): & \text{Rel}(B, X) & \begin{array}{c} \xrightarrow{R^*} \\ \perp \\ \xleftarrow{\text{Ran}_R} \end{array} & \text{Rel}(A, X), \\ (R_* \dashv \text{Rift}_R): & \text{Rel}(X, A) & \begin{array}{c} \xrightarrow{R_*} \\ \perp \\ \xleftarrow{\text{Rift}_R} \end{array} & \text{Rel}(X, B), \end{array}$$

witnessed by bijections

$$\mathbf{Rel}(S \diamond R, T) \cong \mathbf{Rel}(S, \text{Ran}_R(T)),$$

$$\mathbf{Rel}(R \diamond U, V) \cong \mathbf{Rel}(U, \text{Rift}_R(V)),$$

natural in $S \in \text{Rel}(B, X)$, $T \in \text{Rel}(A, X)$, $U \in \text{Rel}(X, A)$, and $V \in \text{Rel}(X, B)$.

¹See also [Section 6](#) for an extension of this correspondence to “relative monads on **Rel**”.

PROOF 2.5.2 ► PROOF OF PROPOSITION 2.5.1

Item 1: Self-Duality I

Omitted.

Item 2: Self-Duality II

Omitted.

Item 3: Equivalences and Isomorphisms in Rel

We claim that **Items 3a to 3c** are indeed equivalent:

- **Item 3a** \iff **Item 3b**: This follows from the fact that **Rel** is locally posetal, so that natural isomorphisms and equalities of 1-cells in **Rel** coincide.
- **Item 3b** \implies **Item 3c**: The equalities in **Item 3b** imply $R \dashv R^{-1}$, and thus by **Item 4**, there exists a function $f_R: A \rightarrow B$ associated to R , where, for each $a \in A$, the image $f_R(a)$ of a by f_R is the unique element of $R(a)$, which implies $R = \text{Gr}(f_R)$ in particular. Furthermore, we have $R^{-1} = f_R^{-1}$ (as in **Definition 3.2.1**). The conditions from **Item 3b** then become the following:

$$\begin{aligned} f_R^{-1} \diamond f_R &= \chi_A, \\ f_R \diamond f_R^{-1} &= \chi_B. \end{aligned}$$

All that is left is to show then is that f_R is a bijection:

- *The Function f_R Is Injective.* Let $a, b \in A$ and suppose that $f_R(a) = f_R(b)$. Since $a \sim_R f_R(a)$ and $f_R(a) = f_R(b) \sim_{R^{-1}} b$, the condition $f_R^{-1} \diamond f_R = \chi_A$ implies that $a = b$, showing f_R to be injective.
- *The Function f_R Is Surjective.* Let $b \in B$. Applying the condition $f_R \diamond f_R^{-1} = \chi_B$ to (b, b) , it follows that there exists some $a \in A$ such that $f_R^{-1}(b) = a$ and $f_R(a) = b$. This shows f_R to be surjective.
- **Item 3c** \implies **Item 3b**: By **Item 2**, we have an adjunction $\text{Gr}(f) \dashv f^{-1}$, giving inclusions

$$\begin{aligned} \chi_A &\subset f^{-1} \diamond \text{Gr}(f), \\ \text{Gr}(f) \diamond f^{-1} &\subset \chi_B. \end{aligned}$$

We claim the reverse inclusions are also true:

- $f^{-1} \diamond Gr(f) \subset \chi_A$: This is equivalent to the statement that if $f(a) = b$ and $f^{-1}(b) = a'$, then $a = a'$, which follows from the injectivity of f .
- $\chi_B \subset Gr(f) \diamond f^{-1}$: This is equivalent to the statement that given $b \in B$ there exists some $a \in A$ such that $f^{-1}(b) = a$ and $f(a) = b$, which follows from the surjectivity of f .

Item 4: Adjunctions in **Rel**

We proceed step by step:

1. *From Adjunctions in **Rel** to Functions.* An adjunction in **Rel** from A to B consists of a pair of relations

$$\begin{aligned} R &: A \rightarrowtail B, \\ S &: B \rightarrowtail A, \end{aligned}$$

together with inclusions

$$\begin{aligned} \chi_A &\subset S \diamond R, \\ R \diamond S &\subset \chi_B. \end{aligned}$$

We claim that these conditions imply that R is total and functional, i.e. that $R(a)$ is a singleton for each $a \in A$:

- (a) *$R(a)$ Has an Element.* Given $a \in A$, since $\chi_A \subset S \diamond R$, we must have $\{a\} \subset S(R(a))$, implying that there exists some $b \in B$ such that $a \sim_R b$ and $b \sim_S a$, and thus $R(a) \neq \emptyset$, as $b \in R(a)$.
- (b) *$R(a)$ Has No More Than One Element.* Suppose that we have $a \sim_R b$ and $a \sim_R b'$ for $b, b' \in B$. We claim that $b = b'$:
 - i. Since $\chi_A \subset S \diamond R$, there exists some $k \in B$ such that $a \sim_R k$ and $k \sim_S a$.
 - ii. Since $R \diamond S \subset \chi_B$, if $b'' \sim_S a'$ and $a' \sim_R b'''$, then $b'' = b'''$.
 - iii. Applying the above to $b'' = k, b''' = b$, and $a' = a$, since $k \sim_S a$ and $a \sim_R b'$, we have $k = b$.
 - iv. Similarly $k = b'$.
 - v. Thus $b = b'$.

Together, the above two items show $R(a)$ to be a singleton, being thus given by $\text{Gr}(f)$ for some function $f: A \rightarrow B$, which gives a map

$$\left\{ \begin{array}{c} \text{Adjunctions in } \mathbf{Rel} \\ \text{from } A \text{ to } B \end{array} \right\} \rightarrow \left\{ \begin{array}{c} \text{Functions} \\ \text{from } A \text{ to } B \end{array} \right\}.$$

Moreover, by uniqueness of adjoints (**Internal Adjunctions, Item 2 of Proposition 1.2.4**), this implies also that $S = f^{-1}$.

2. *From Functions to Adjunctions in **Rel**.* By **Item 2 of Proposition 3.1.2**, every function $f: A \rightarrow B$ gives rise to an adjunction $\text{Gr}(f) \dashv f^{-1}$ in **Rel**, giving a map

$$\left\{ \begin{array}{c} \text{Functions} \\ \text{from } A \text{ to } B \end{array} \right\} \rightarrow \left\{ \begin{array}{c} \text{Adjunctions in } \mathbf{Rel} \\ \text{from } A \text{ to } B \end{array} \right\}.$$

3. *Invertibility: From Functions to Adjunctions Back to Functions.* We need to show that starting with a function $f: A \rightarrow B$, passing to $\text{Gr}(f) \dashv f^{-1}$, and then passing again to a function gives f again. This is clear however, since we have $a \sim_{\text{Gr}(f)} b$ iff $f(a) = b$.
4. *Invertibility: From Adjunctions to Functions Back to Adjunctions.* We need to show that, given an adjunction $R \dashv S$ in **Rel** giving rise to a function $f_{R,S}: A \rightarrow B$, we have

$$\begin{aligned} \text{Gr}(f_{R,S}) &= R, \\ f_{R,S}^{-1} &= S. \end{aligned}$$

We check these explicitly:

- $\text{Gr}(f_{R,S}) = R$. We have

$$\begin{aligned} \text{Gr}(f_{R,S}) &\stackrel{\text{def}}{=} \{(a, f_{R,S}(a)) \in A \times B \mid a \in A\} \\ &\stackrel{\text{def}}{=} \{(a, R(a)) \in A \times B \mid a \in A\} \\ &= R. \end{aligned}$$

- $f_{R,S}^{-1} = S$. We first claim that, given $a \in A$ and $b \in B$, the following conditions are equivalent:

- We have $a \sim_R b$.
- We have $b \sim_S a$.

Indeed:

- If $a \sim_R b$, then $b \sim_S a$: Since $\chi_A \subset S \diamond R$, there exists $k \in B$ such that $a \sim_R k$ and $k \sim_S a$, but since $a \sim_R b$ and R is functional, we have $k = b$ and thus $b \sim_S a$.
- If $b \sim_S a$, then $a \sim_R b$: First note that since R is total we have $a \sim_R b'$ for some $b' \in B$. Now, since $R \diamond S \subset \chi_B$, $b \sim_S a$, and $a \sim_R b'$, we have $b = b'$, and thus $a \sim_R b$.

Having show this, we now have

$$\begin{aligned} f_{R,S}^{-1}(b) &\stackrel{\text{def}}{=} \{a \in A \mid f_{R,S}(a) = b\} \\ &\stackrel{\text{def}}{=} \{a \in A \mid a \sim_R b\} \\ &= \{a \in A \mid b \sim_S a\} \\ &\stackrel{\text{def}}{=} S(b). \end{aligned}$$

for each $b \in B$, showing $f_{R,S}^{-1} = S$.

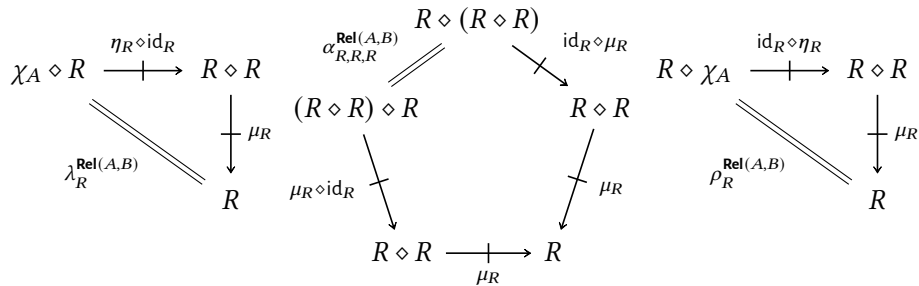
This finishes the proof.

Item 5: Monads in **Rel**

A monad in **Rel** on A consists of a relation $R: A \rightarrowtail A$ together with maps

$$\begin{aligned} \mu_R: R \diamond R &\subset R, \\ \eta_R: \chi_A &\subset R \end{aligned}$$

making the diagrams



commute. However, since all morphisms involved are inclusions, the commutativity of the above diagrams is automatic, and hence all that is left is the data of the two maps μ_R and η_R , which correspond respectively to the following conditions:

1. For each $a, b, c \in A$, if $a \sim_R b$ and $b \sim_R c$, then $a \sim_R c$.
2. For each $a \in A$, we have $a \sim_R a$.

These are exactly the requirements for R to be a preorder (**Posets**, ??). Conversely any preorder \leq gives rise to a pair of maps μ_\leq and η_\leq , forming a monad on A .

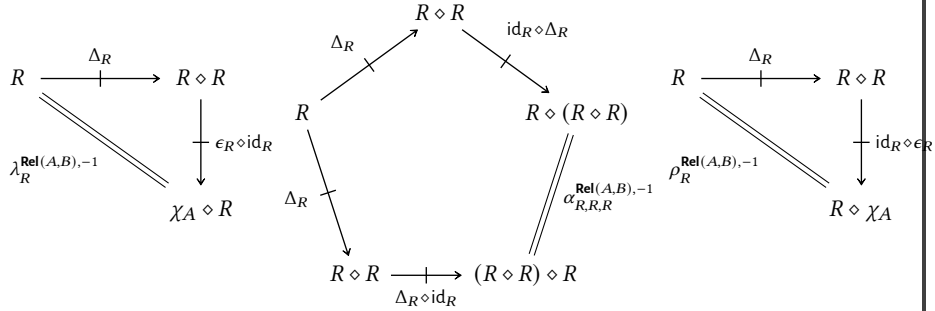
Item 6: Comonads in **Rel**

A comonad in **Rel** on A consists of a relation $R: A \multimap A$ together with maps

$$\Delta_R: R \subset R \diamond R,$$

$$\epsilon_R: R \subset \chi_A$$

making the diagrams



commute. However, since all morphisms involved are inclusions, the commutativity of the above diagrams is automatic, and hence all that is left is the data of the two maps Δ_R and ϵ_R , which correspond respectively to the following conditions:

1. For each $a, b \in A$, if $a \sim_R b$, then there exists some $k \in A$ such that $a \sim_R k$ and $k \sim_R b$.
2. For each $a, b \in A$, if $a \sim_R b$, then $a = b$.

Taking $k = b$ in the first condition above shows it to be trivially satisfied, while the second condition implies $R \subset \Delta_A$, i.e. R must be a subset of A . Conversely, any subset U of A satisfies $U \subset \Delta_A$, defining a comonad as above.

Item 7: Monomorphisms in Rel

Firstly note that **Items 7b** and **7c** are equivalent by **Item 7** of **Proposition 5.1.3**. We then claim that **Items 7a** and **7b** are also equivalent:

- **Item 7a** \implies **Item 7b**: Let $U, V \in \mathcal{P}(A)$ and consider the diagram

$$\text{pt} \begin{array}{c} \xrightarrow{U} \\ \text{---} \\ \xrightarrow{V} \end{array} A \xrightarrow{R} B.$$

By **Remark 5.1.2**, we have

$$R_*(U) = R \diamond U,$$

$$R_*(V) = R \diamond V.$$

Now, if $R \diamond U = R \diamond V$, i.e. $R_*(U) = R_*(V)$, then $U = V$ since R is assumed to be a monomorphism, showing R_* to be injective.

- **Item 7b** \implies **Item 7a**: Conversely, suppose that R_* is injective, consider the diagram

$$K \begin{array}{c} \xrightarrow{S} \\ \text{---} \\ \xrightarrow{T} \end{array} A \xrightarrow{R} B,$$

and suppose that $R \diamond S = R \diamond T$. Note that, since R_* is injective, given a diagram of the form

$$\text{pt} \begin{array}{c} \xrightarrow{U} \\ \text{---} \\ \xrightarrow{V} \end{array} A \xrightarrow{R} B,$$

if $R_*(U) = R \diamond U = R \diamond V = R_*(V)$, then $U = V$. In particular, for each $k \in K$, we may consider the diagram

$$\text{pt} \xrightarrow{[k]} K \begin{array}{c} \xrightarrow{S} \\ \text{---} \\ \xrightarrow{T} \end{array} A \xrightarrow{R} B,$$

for which we have $R \diamond S \diamond [k] = R \diamond T \diamond [k]$, implying that we have

$$S(k) = S \diamond [k] = T \diamond [k] = T(k)$$

for each $k \in K$, implying $S = T$, and thus R is a monomorphism.

We can also prove this in a more abstract way, following [MSE 350788]:

- *Item 7a* \implies *Item 7b*: Assume that R is a monomorphism.
 - We first notice that the functor $\text{Rel}(\text{pt}, -) : \text{Rel} \rightarrow \text{Sets}$ maps R to R_* by Remark 5.1.2.
 - Since $\text{Rel}(\text{pt}, -)$ preserves all limits by Limits and Colimits, ?? of ??, it follows by Categories, ?? of ?? that $\text{Rel}(\text{pt}, -)$ also preserves monomorphisms.
 - Since R is a monomorphism and $\text{Rel}(\text{pt}, -)$ maps R to R_* , it follows that R_* is also a monomorphism.
 - Since the monomorphisms in Sets are precisely the injections (Categories, ?? of ??), it follows that R_* is injective.
- *Item 7b* \implies *Item 7a*: Assume that R_* is injective.
 - We first notice that the functor $\text{Rel}(\text{pt}, -) : \text{Rel} \rightarrow \text{Sets}$ maps R to R_* by Remark 5.1.2.
 - Since the monomorphisms in Sets are precisely the injections (Categories, ?? of ??), it follows that R_* is a monomorphism.
 - Since $\text{Rel}(\text{pt}, -)$ is faithful, it follows by Categories, ?? of ?? that $\text{Rel}(\text{pt}, -)$ reflects monomorphisms.
 - Since R_* is a monomorphism and $\text{Rel}(\text{pt}, -)$ maps R to R_* , it follows that R is also a monomorphism.

Finally, we prove the second part of the statement. Assume that R is a monomorphism, let $a, a' \in A$ such that $a \sim_R b$ and $a' \sim_R b$ for some $b \in B$, and consider the diagram

$$\begin{array}{ccc} & [a] & \\ \text{pt} & \begin{array}{c} \xrightarrow{\quad} \\ \text{---} \\ \xrightarrow{\quad} \end{array} & A \xrightarrow{R} B \\ & [a'] & \end{array}$$

Since $\star \sim_{[a]} a$ and $a \sim_R b$, we have $\star \sim_{R \diamond [a]} b$. Similarly, $\star \sim_{R \diamond [a']} b$. Thus $R \diamond [a] = R \diamond [a']$, and since R is a monomorphism, we have $[a] = [a']$, i.e. $a = a'$.

Conversely, assume the condition

- (\star) For each $a, a' \in A$, if there exists some $b \in B$ such that $a \sim_R b$ and $a' \sim_R b$, then $a = a'$,

consider the diagram

$$K \begin{array}{c} \xrightarrow{S} \\ \text{---} \\ \xrightarrow{T} \end{array} A \xrightarrow{R} B,$$

and let $(k, a) \in S$. Since R is total and $a \in A$, there exists some $b \in B$ such that $a \sim_R b$. In this case, we have $k \sim_{R \diamond S} b$, and since $R \diamond S = R \diamond T$, we have also $k \sim_{R \diamond T} b$. Thus there must exist some $a' \in A$ such that $k \sim_T a'$ and $a' \sim_R b$. However, since $a, a' \sim_R b$, we must have $a = a'$, and thus $(k, a) \in T$ as well.

A similar argument shows that if $(k, a) \in T$, then $(k, a) \in S$, and thus $S = T$ and R is a monomorphism.

Item 8: Epimorphisms in Rel

Firstly note that **Items 8b** and **8c** are equivalent by **Item 7** of **Proposition 5.2.4**. We then claim that **Items 8a** and **8b** are also equivalent:

- **Item 8a** \implies **Item 8b**: Let $U, V \in \mathcal{P}(A)$ and consider the diagram

$$A \xrightarrow{R} B \begin{array}{c} \xrightarrow{U} \\ \text{---} \\ \xrightarrow{V} \end{array} \text{pt.}$$

By **Remark 5.1.2**, we have

$$R^{-1}(U) = U \diamond R,$$

$$R^{-1}(V) = V \diamond R.$$

Now, if $U \diamond R = V \diamond R$, i.e. $R^{-1}(U) = R^{-1}(V)$, then $U = V$ since R is assumed to be an epimorphism, showing R^{-1} to be injective.

- **Item 8b** \implies **Item 8a**: Conversely, suppose that R^{-1} is injective, consider the diagram

$$A \xrightarrow{R} B \begin{array}{c} \xrightarrow{S} \\ \text{---} \\ \xrightarrow{T} \end{array} K,$$

and suppose that $S \diamond R = T \diamond R$. Note that, since R^{-1} is injective, given a diagram of the form

$$A \xrightarrow{R} B \begin{array}{c} \xrightarrow{U} \\ \text{---} \\ \xrightarrow{V} \end{array} \text{pt,}$$

if $R^{-1}(U) = U \diamond R = V \diamond R = R^{-1}(V)$, then $U = V$. In particular, for each $k \in K$, we may consider the diagram

$$A \xrightarrow{R} B \begin{array}{c} \xrightarrow{S} \\ \xleftarrow{T} \end{array} K \xrightarrow{[k]} \text{pt},$$

for which we have $[k] \diamond S \diamond R = [k] \diamond T \diamond R$, implying that we have

$$S^{-1}(k) = [k] \diamond S = [k] \diamond T = T^{-1}(k)$$

for each $k \in K$, implying $S = T$, and thus R is an epimorphism.

We can also prove this in a more abstract way, following [MSE 350788]:

- *Item 8a* \implies *Item 8b*: Assume that R is an epimorphism.
 - We first notice that the functor $\text{Rel}(-, \text{pt}) : \text{Rel}^{\text{op}} \rightarrow \text{Sets}$ maps R to R^{-1} by Remark 5.3.2.
 - Since $\text{Rel}(-, \text{pt})$ preserves limits by Limits and Colimits, ?? of ??, it follows by Categories, ?? of ?? that $\text{Rel}(-, \text{pt})$ also preserves monomorphisms.
 - That is: $\text{Rel}(-, \text{pt})$ sends monomorphisms in Rel^{op} to monomorphisms in Sets.
 - The monomorphisms Rel^{op} are precisely the epimorphisms in Rel by Categories, ?? of ??.
 - Since R is an epimorphism and $\text{Rel}(-, \text{pt})$ maps R to R^{-1} , it follows that R^{-1} is a monomorphism.
 - Since the monomorphisms in Sets are precisely the injections (Categories, ?? of ??), it follows that R^{-1} is injective.
- *Item 8b* \implies *Item 8a*: Assume that R^{-1} is injective.
 - We first notice that the functor $\text{Rel}(-, \text{pt}) : \text{Rel}^{\text{op}} \rightarrow \text{Sets}$ maps R to R^{-1} by Remark 5.3.2.
 - Since the monomorphisms in Sets are precisely the injections (Categories, ?? of ??), it follows that R^{-1} is a monomorphism.

- Since $\text{Rel}(-, \text{pt})$ is faithful, it follows by **Categories**, ?? of ?? that $\text{Rel}(-, \text{pt})$ reflects monomorphisms.
- That is: $\text{Rel}(-, \text{pt})$ reflects monomorphisms in **Sets** to monomorphisms in Rel^{op} .
- The monomorphisms Rel^{op} are precisely the epimorphisms in **Rel** by **Categories**, ?? of ??.
- Since R^{-1} is a monomorphism and $\text{Rel}(-, \text{pt})$ maps R to R^{-1} , it follows that R is an epimorphism.

Finally, we claim that **Items 8b** and **8d** are also equivalent, following [MO 350788]:

- **Item 8b** \implies **Item 8d**: Since $B \setminus \{b\} \subset B$ and R^{-1} is injective, we have $R^{-1}(B \setminus \{b\}) \subsetneq R^{-1}(B)$. So taking some $a \in R^{-1}(B) \setminus R^{-1}(B \setminus \{b\})$ we get an element of A such that $R(a) = \{b\}$.
- **Item 8d** \implies **Item 8b**: Let $U, V \subset B$ with $U \neq V$. Without loss of generality, we can assume $U \setminus V \neq \emptyset$; otherwise just swap U and V . Let then $b \in U \setminus V$. By assumption, there exists an $a \in A$ with $R(a) = \{b\}$. Then $a \in R^{-1}(U)$ but $a \notin R^{-1}(V)$, and thus $R^{-1}(U) \neq R^{-1}(V)$, showing R^{-1} to be injective.

Item 9: As a Kleisli Category

Omitted.

Item 10: Co/Completeness (Or Lack Thereof)

Omitted.

Item 11: Existence of Right Kan Extensions

We have

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{Rel}(A,X)}(S \diamond R, T) &\cong \int_{a \in A} \int_{x \in X} \mathbf{Hom}_{\{t,f\}}((S \diamond R)_x^a, T_x^a) \\
&\cong \int_{a \in A} \int_{x \in X} \mathbf{Hom}_{\{t,f\}}\left(\left(\int_{b \in B} S_x^b \times R_b^a\right), T_x^a\right) \\
&\cong \int_{a \in A} \int_{x \in X} \int_{b \in B} \mathbf{Hom}_{\{t,f\}}(S_x^b \times R_b^a, T_x^a) \\
&\cong \int_{a \in A} \int_{x \in X} \int_{b \in B} \mathbf{Hom}_{\{t,f\}}(S_x^b, \mathbf{Hom}_{\{t,f\}}(R_b^a, T_x^a)) \\
&\cong \int_{b \in B} \int_{x \in X} \int_{a \in A} \mathbf{Hom}_{\{t,f\}}(S_x^b, \mathbf{Hom}_{\{t,f\}}(R_b^a, T_x^a)) \\
&\cong \int_{b \in B} \int_{x \in X} \mathbf{Hom}_{\{t,f\}}\left(S_x^b, \int_{a \in A} \mathbf{Hom}_{\{t,f\}}(R_b^a, T_x^a)\right) \\
&\cong \mathrm{Hom}_{\mathbf{Rel}(B,X)}\left(S, \int_{a \in A} \mathbf{Hom}_{\{t,f\}}(R_{-1}^a, T_{-2}^a)\right)
\end{aligned}$$

naturally in each $S \in \mathbf{Rel}(B, X)$ and each $T \in \mathbf{Rel}(A, X)$, showing that

$$\int_{a \in A} \mathbf{Hom}_{\{t,f\}}(R_{-1}^a, T_{-2}^a)$$

is right adjoint to the precomposition functor $- \diamond R$, being thus the right Kan extension along R . Here we have used the following results, respectively (i.e. for each \cong sign):

1. Item 1 of Proposition 1.1.5;
2. Definition 3.12.1;
3. Diagonal Category Theory, ?? of ??;
4. Sets, Proposition 1.2.4;
5. Diagonal Category Theory, ?? of ??;
6. Diagonal Category Theory, ?? of ??;
7. Item 1 of Proposition 1.1.5.

Item 12: Existence of Right Kan Lifts

We have

$$\begin{aligned}
 \text{Hom}_{\mathbf{Rel}(X,B)}(R \diamond S, T) &\cong \int_{x \in X} \int_{b \in B} \mathbf{Hom}_{\{t,f\}}((R \diamond S)_b^x, T_b^x) \\
 &\cong \int_{x \in X} \int_{b \in B} \mathbf{Hom}_{\{t,f\}}\left(\left(\int_{a \in A} R_b^a \times S_a^x\right), T_b^x\right) \\
 &\cong \int_{x \in X} \int_{b \in B} \int_{a \in A} \mathbf{Hom}_{\{t,f\}}(R_b^a \times S_a^x, T_b^x) \\
 &\cong \int_{x \in X} \int_{b \in B} \int_{a \in A} \mathbf{Hom}_{\{t,f\}}(S_a^x, \mathbf{Hom}_{\{t,f\}}(R_b^a, T_b^x)) \\
 &\cong \int_{x \in X} \int_{a \in A} \int_{b \in B} \mathbf{Hom}_{\{t,f\}}(S_a^x, \mathbf{Hom}_{\{t,f\}}(R_b^a, T_b^x)) \\
 &\cong \int_{x \in X} \int_{a \in A} \mathbf{Hom}_{\{t,f\}}\left(S_a^x, \int_{b \in B} \mathbf{Hom}_{\{t,f\}}(R_b^a, T_b^x)\right) \\
 &\cong \text{Hom}_{\mathbf{Rel}(X,A)}\left(S, \int_{b \in B} \mathbf{Hom}_{\{t,f\}}(R_b^{-2}, T_b^{-1})\right)
 \end{aligned}$$

naturally in each $S \in \mathbf{Rel}(X, A)$ and each $T \in \mathbf{Rel}(X, B)$, showing that

$$\int_{b \in B} \mathbf{Hom}_{\{t,f\}}(R_b^{-2}, T_b^{-1})$$

is right adjoint to the postcomposition functor $R \diamond -$, being thus the right Kan lift along R . Here we have used the following results, respectively (i.e. for each \cong sign):

1. Item 1 of Proposition 1.1.5;
2. Definition 3.12.1;
3. Diagonal Category Theory, ?? of ??;
4. Sets, Proposition 1.2.4;
5. Diagonal Category Theory, ?? of ??;
6. Diagonal Category Theory, ?? of ??;
7. Item 1 of Proposition 1.1.5.

Item 13: Closedness

This has been proved as part of the proof of Items 11 and 12.



00G4 3 Constructions With Relations

00G5 3.1 The Graph of a Function

Let $f: A \rightarrow B$ be a function.

00G6 DEFINITION 3.1.1 ► THE GRAPH OF A FUNCTION

The **graph of f** is the relation $\text{Gr}(f): A \rightarrow B$ defined as follows:¹

- Viewing relations from A to B as subsets of $A \times B$, we define

$$\text{Gr}(f) \stackrel{\text{def}}{=} \{(a, f(a)) \in A \times B \mid a \in A\};$$

- Viewing relations from A to B as functions $A \times B \rightarrow \{\text{true}, \text{false}\}$, we define

$$[\text{Gr}(f)](a, b) \stackrel{\text{def}}{=} \begin{cases} \text{true} & \text{if } b = f(a), \\ \text{false} & \text{otherwise} \end{cases}$$

for each $(a, b) \in A \times B$;

- Viewing relations from A to B as functions $A \rightarrow \mathcal{P}(B)$, we define

$$[\text{Gr}(f)](a) \stackrel{\text{def}}{=} \{f(a)\}$$

for each $a \in A$, i.e. we define $\text{Gr}(f)$ as the composition

$$A \xrightarrow{f} B \xrightarrow{\chi_B} \mathcal{P}(B).$$

¹Further Notation: We write $\text{Gr}(A)$ for $\text{Gr}(\text{id}_A)$, and call it the **graph** of A .

00G7 PROPOSITION 3.1.2 ► PROPERTIES OF GRAPHS OF FUNCTIONS

Let $f: A \rightarrow B$ be a function.

- 00G8 1. *Functoriality.* The assignment $A \mapsto \text{Gr}(A)$ defines a functor

$$\text{Gr}: \text{Sets} \rightarrow \text{Rel}$$

where

- *Action on Objects.* For each $A \in \text{Obj}(\text{Sets})$, we have

$$\text{Gr}(A) \stackrel{\text{def}}{=} A;$$

- *Action on Morphisms.* For each $A, B \in \text{Obj}(\text{Sets})$, the action on Hom-sets

$$\text{Gr}_{A,B}: \text{Sets}(A, B) \rightarrow \underbrace{\text{Rel}(\text{Gr}(A), \text{Gr}(B))}_{\stackrel{\text{def}}{=} \text{Rel}(A, B)}$$

of Gr at (A, B) is defined by

$$\text{Gr}_{A,B}(f) \stackrel{\text{def}}{=} \text{Gr}(f),$$

where $\text{Gr}(f)$ is the graph of f as in [Definition 3.1.1](#).

In particular:

- *Preservation of Identities.* We have

$$\text{Gr}(\text{id}_A) = \chi_A$$

for each $A \in \text{Obj}(\text{Sets})$.

- *Preservation of Composition.* We have

$$\text{Gr}(g \circ f) = \text{Gr}(g) \diamond \text{Gr}(f)$$

for each pair of functions $f: A \rightarrow B$ and $g: B \rightarrow C$.

00G9

2. *Adjointness Inside **Rel**.* We have an adjunction

$$(\text{Gr}(f) \dashv f^{-1}): A \begin{array}{c} \xrightarrow{\text{Gr}(f)} \\ \dashv \\ \xleftarrow{f^{-1}} \end{array} B$$

in **Rel**, where f^{-1} is the inverse of f of [Definition 3.2.1](#).

00GA

3. *Adjointness.* We have an adjunction

$$(Gr \dashv \mathcal{P}_*) : \text{Sets} \begin{array}{c} \xrightarrow{Gr} \\ \perp \\ \xleftarrow{\mathcal{P}_*} \end{array} \text{Rel},$$

witnessed by a bijection of sets

$$\text{Rel}(Gr(A), B) \cong \text{Sets}(A, \mathcal{P}(B))$$

natural in $A \in \text{Obj}(\text{Sets})$ and $B \in \text{Obj}(\text{Rel})$.

00GB

4. *Interaction With Inverses.* We have

$$\begin{aligned} Gr(f)^\dagger &= f^{-1}, \\ (f^{-1})^\dagger &= Gr(f). \end{aligned}$$

00GC

5. *Cocontinuity.* The functor $Gr : \text{Sets} \rightarrow \text{Rel}$ of **Item 1** preserves colimits.

00GD

6. *Characterisations.* Let $R : A \rightarrowtail B$ be a relation. The following conditions are equivalent:

00GE

(a) There exists a function $f : A \rightarrow B$ such that $R = Gr(f)$.

00GF

(b) The relation R is total and functional.

00GG

(c) The weak and strong inverse images of R agree, i.e. we have $R^{-1} = R_{-1}$.

00GH

(d) The relation R has a right adjoint R^\dagger in Rel .

PROOF 3.1.3 ► PROOF OF PROPOSITION 3.1.2

Item 1: Functoriality

Clear.

Item 2: Adjointness Inside **Rel**

We need to check that there are inclusions

$$\begin{aligned}\chi_A &\subset f^{-1} \diamond \text{Gr}(f), \\ \text{Gr}(f) \diamond f^{-1} &\subset \chi_B.\end{aligned}$$

These correspond respectively to the following conditions:

1. For each $a \in A$, there exists some $b \in B$ such that $a \sim_{\text{Gr}(f)} b$ and $b \sim_{f^{-1}} a$.
2. For each $a, b \in A$, if $a \sim_{\text{Gr}(f)} b$ and $b \sim_{f^{-1}} a$, then $a = b$.

In other words, the first condition states that the image of any $a \in A$ by f is nonempty, whereas the second condition states that f is not multivalued. As f is a function, both of these statements are true, and we are done.

Item 3: Adjointness

The stated bijection follows from [Remark 1.1.3](#), with naturality being clear.

Item 4: Interaction With Inverses

Clear.

Item 5: Cocontinuity

Omitted.

Item 6: Characterisations

We claim that [Items 6a to 6d](#) are indeed equivalent:

- [Item 6a](#) \iff [Item 6b](#). This is shown in the proof of [Item 4](#) of [Proposition 2.5.1](#).
- [Item 6b](#) \implies [Item 6c](#). If R is total and functional, then, for each $a \in A$, the set $R(a)$ is a singleton, implying that

$$\begin{aligned}R^{-1}(V) &\stackrel{\text{def}}{=} \{a \in A \mid R(a) \cap V \neq \emptyset\}, \\ R_{-1}(V) &\stackrel{\text{def}}{=} \{a \in A \mid R(a) \subset V\}\end{aligned}$$

are equal for all $V \in \mathcal{P}(B)$, as the conditions $R(a) \cap V \neq \emptyset$ and $R(a) \subset V$ are equivalent when $R(a)$ is a singleton.


- [Item 6c](#) \implies [Item 6b](#). We claim that R is indeed total and functional:
 - *Totality*. If we had $R(a) = \emptyset$ for some $a \in A$, then we would have $a \in R_{-1}(\emptyset)$, so that $R_{-1}(\emptyset) \neq \emptyset$. But since $R^{-1}(\emptyset) = \emptyset$, this would imply $R_{-1}(\emptyset) \neq R^{-1}(\emptyset)$, a contradiction. Thus $R(a) \neq \emptyset$ for all $a \in A$ and R is total.

– *Functionality.* If $R^{-1} = R_{-1}$, then we have

$$\begin{aligned}\{a\} &= R^{-1}(\{b\}) \\ &= R_{-1}(\{b\})\end{aligned}$$

for each $b \in R(a)$ and each $a \in A$, and thus $R(a) \subset \{b\}$. But since R is total, we must have $R(a) = \{b\}$, and thus we see that R is functional.

· *Item 6a* \iff *Item 6d*. This follows from *Item 4* of [Proposition 2.5.1](#).

This finishes the proof. 

00GJ 3.2 The Inverse of a Function

Let $f: A \rightarrow B$ be a function.

00GK DEFINITION 3.2.1 ► THE INVERSE OF A FUNCTION

The **inverse of f** is the relation $f^{-1}: B \rightarrow A$ defined as follows:

· Viewing relations from B to A as subsets of $B \times A$, we define

$$f^{-1} \stackrel{\text{def}}{=} \{(b, f^{-1}(b)) \in B \times A \mid a \in A\},$$

where

$$f^{-1}(b) \stackrel{\text{def}}{=} \{a \in A \mid f(a) = b\}$$

for each $b \in B$.

· Viewing relations from B to A as functions $B \times A \rightarrow \{\text{true}, \text{false}\}$, we define

$$f^{-1}(b, a) \stackrel{\text{def}}{=} \begin{cases} \text{true} & \text{if there exists } a \in A \text{ with } f(a) = b, \\ \text{false} & \text{otherwise} \end{cases}$$

for each $(b, a) \in B \times A$;

· Viewing relations from B to A as functions $B \rightarrow \mathcal{P}(A)$, we define

$$f^{-1}(b) \stackrel{\text{def}}{=} \{a \in A \mid f(a) = b\}$$

for each $b \in B$.

00GL

PROPOSITION 3.2.2 ► PROPERTIES OF INVERSES OF FUNCTIONS

Let $f: A \rightarrow B$ be a function.

00GM

1. *Functoriality.* The assignment $A \mapsto A, f \mapsto f^{-1}$ defines a functor

$$(-)^{-1}: \mathbf{Sets} \rightarrow \mathbf{Rel}$$

where

- *Action on Objects.* For each $A \in \mathbf{Obj}(\mathbf{Sets})$, we have

$$[(-)^{-1}](A) \stackrel{\text{def}}{=} A;$$

- *Action on Morphisms.* For each $A, B \in \mathbf{Obj}(\mathbf{Sets})$, the action on Hom-sets

$$(-)^{-1}_{A,B}: \mathbf{Sets}(A, B) \rightarrow \mathbf{Rel}(A, B)$$

of $(-)^{-1}$ at (A, B) is defined by

$$(-)^{-1}_{A,B}(f) \stackrel{\text{def}}{=} [(-)^{-1}](f),$$

where f^{-1} is the inverse of f as in **Definition 3.2.1**.

In particular:

- *Preservation of Identities.* We have

$$\text{id}_A^{-1} = \chi_A$$

for each $A \in \mathbf{Obj}(\mathbf{Sets})$.

- *Preservation of Composition.* We have

$$(g \circ f)^{-1} = g^{-1} \diamond f^{-1}$$

for pair of functions $f: A \rightarrow B$ and $g: B \rightarrow C$.

00GN

2. *Adjointness Inside **Rel**.* We have an adjunction

$$(\text{Gr}(f) \dashv f^{-1}): A \begin{array}{c} \xrightarrow{\text{Gr}(f)} \\ \perp \\ \xleftarrow{f^{-1}} \end{array} B$$

in **Rel**.

00GP

3. *Interaction With Inverses of Relations.* We have

$$(f^{-1})^{\dagger} = \text{Gr}(f),$$

$$\text{Gr}(f)^{\dagger} = f^{-1}.$$

PROOF 3.2.3 ► PROOF OF PROPOSITION 3.2.2

Item 1: Functoriality

Clear.

Item 2: Adjointness Inside **Rel**

This is proved in **Item 2** of **Proposition 3.1.2**.

Item 3: Interaction With Inverses of Relations

Clear. 

00GQ 3.3 Representable Relations

Let A and B be sets.

00GR

DEFINITION 3.3.1 ► REPRESENTABLE RELATIONS

Let $f: A \rightarrow B$ and $g: B \rightarrow A$ be functions.¹

1. The **representable relation associated to f** is the relation $\chi_f: A \rightarrowtail B$ defined as the composition

$$A \times B \xrightarrow{f \times \text{id}_B} B \times B \xrightarrow{\chi_B} \{\text{true}, \text{false}\},$$

i.e. given by declaring $a \sim_{\chi_f} b$ iff $f(a) = b$.

2. The **corepresentable relation associated to g** is the relation $\chi^g: B \rightarrowtail A$ defined as the composition

$$B \times A \xrightarrow{g \times \text{id}_A} A \times A \xrightarrow{\chi_A} \{\text{true}, \text{false}\},$$

i.e. given by declaring $b \sim_{\chi^g} a$ iff $g(b) = a$.

¹More generally, given functions

$$f: A \rightarrow C,$$

$$g: B \rightarrow D$$

and a relation $B \rightarrow D$, we may consider the composite relation

$$A \times B \xrightarrow{f \times g} C \times D \xrightarrow{R} \{\text{true}, \text{false}\},$$

for which we have $a \sim_{R \circ (f \times g)} b$ iff $f(a) \sim_R g(b)$.

00GS 3.4 The Domain and Range of a Relation

Let A and B be sets.

00GT

DEFINITION 3.4.1 ► THE DOMAIN AND RANGE OF A RELATION

Let $R \subset A \times B$ be a relation.^{1,2}

1. The **domain of R** is the subset $\text{dom}(R)$ of A defined by

$$\text{dom}(R) \stackrel{\text{def}}{=} \left\{ a \in A \mid \begin{array}{l} \text{there exists some } b \in B \\ \text{such that } a \sim_R b \end{array} \right\}.$$

2. The **range of R** is the subset $\text{range}(R)$ of B defined by

$$\text{range}(R) \stackrel{\text{def}}{=} \left\{ b \in B \mid \begin{array}{l} \text{there exists some } a \in A \\ \text{such that } a \sim_R b \end{array} \right\}.$$

¹Following [Categories](#), ??, we may compute the (characteristic functions associated to the) domain and range of a relation using the following colimit formulas:

$$\begin{aligned} \chi_{\text{dom}(R)}(a) &\cong \text{colim}_{b \in B} (R_b^a) & (a \in A) \\ &\cong \bigvee_{b \in B} R_b^a, \\ \chi_{\text{range}(R)}(b) &\cong \text{colim}_{a \in A} (R_b^a) & (b \in B) \\ &\cong \bigvee_{a \in A} R_b^a, \end{aligned}$$

where the join \bigvee is taken in the poset $(\{\text{true}, \text{false}\}, \leq)$ of [Constructions With Sets](#), Definition 1.2.3.

²Viewing R as a function $R: A \rightarrow \mathcal{P}(B)$, we have

$$\begin{aligned}\text{dom}(R) &\cong \text{colim}_{y \in Y} (R(y)) \\ &\cong \bigcup_{y \in Y} R(y), \\ \text{range}(R) &\cong \text{colim}_{x \in X} (R(x)) \\ &\cong \bigcup_{x \in X} R(x),\end{aligned}$$

00GU 3.5 Binary Unions of Relations

Let A and B be sets and let R and S be relations from A to B .

00GV DEFINITION 3.5.1 ► BINARY UNIONS OF RELATIONS

The **union of R and S** ¹ is the relation $R \cup S$ from A to B defined as follows:

- Viewing relations from A to B as subsets of $A \times B$, we define²

$$R \cup S \stackrel{\text{def}}{=} \{(a, b) \in B \times A \mid \text{we have } a \sim_R b \text{ or } a \sim_S b\}.$$

- Viewing relations from A to B as functions $A \rightarrow \mathcal{P}(B)$, we define

$$[R \cup S](a) \stackrel{\text{def}}{=} R(a) \cup S(a)$$

for each $a \in A$.

¹Further Terminology: Also called the **binary union of R and S** , for emphasis.

²This is the same as the union of R and S as subsets of $A \times B$.

00GW PROPOSITION 3.5.2 ► PROPERTIES OF BINARY UNIONS OF RELATIONS

Let R, S, R_1 , and R_2 be relations from A to B , and let S_1 and S_2 be relations from B to C .

- 00HF 1. *Interaction With Inverses.* We have

$$(R \cup S)^\dagger = R^\dagger \cup S^\dagger.$$

00HG

2. *Interaction With Composition.* We have

$$(S_1 \diamond R_1) \cup (S_2 \diamond R_2) \stackrel{\text{poss.}}{\neq} (S_1 \cup S_2) \diamond (R_1 \cup R_2).$$

PROOF 3.5.3 ► PROOF OF PROPOSITION 3.5.2

Item 1: Interaction With Inverses

Clear.

Item 2: Interaction With Composition

Unwinding the definitions, we see that:

1. The condition for $(S_1 \diamond R_1) \cup (S_2 \diamond R_2)$ is:

(a) There exists some $b \in B$ such that:

i. $a \sim_{R_1} b$ and $b \sim_{S_1} c$;

or

i. $a \sim_{R_2} b$ and $b \sim_{S_2} c$;


3. The condition for $(S_1 \cup S_2) \diamond (R_1 \cup R_2)$ is:

(a) There exists some $b \in B$ such that:

i. $a \sim_{R_1} b$ or $a \sim_{R_2} b$;

and

i. $b \sim_{S_1} c$ or $b \sim_{S_2} c$.

These two conditions may fail to agree (counterexample omitted), and thus the two resulting relations on $A \times C$ may differ. 

00GZ 3.6 Unions of Families of Relations

Let A and B be sets and let $\{R_i\}_{i \in I}$ be a family of relations from A to B .

00H0

DEFINITION 3.6.1 ► THE UNION OF A FAMILY OF RELATIONS

The **union of the family** $\{R_i\}_{i \in I}$ is the relation $\bigcup_{i \in I} R_i$ from A to B defined as follows:

- Viewing relations from A to B as subsets of $A \times B$, we define¹

$$\bigcup_{i \in I} R_i \stackrel{\text{def}}{=} \left\{ (a, b) \in (A \times B)^{\times I} \mid \begin{array}{l} \text{there exists some } i \in I \\ \text{such that } a \sim_{R_i} b \end{array} \right\}.$$

- Viewing relations from A to B as functions $A \rightarrow \mathcal{P}(B)$, we define

$$\left[\bigcup_{i \in I} R_i \right] (a) \stackrel{\text{def}}{=} \bigcup_{i \in I} R_i(a)$$

for each $a \in A$.

¹This is the same as the union of $\{R_i\}_{i \in I}$ as a collection of subsets of $A \times B$.

00H1

PROPOSITION 3.6.2 ► PROPERTIES OF UNIONS OF FAMILIES OF RELATIONS

Let A and B be sets and let $\{R_i\}_{i \in I}$ be a family of relations from A to B .

00H2

1. *Interaction With Inverses.* We have

$$\left(\bigcup_{i \in I} R_i \right)^{\dagger} = \bigcup_{i \in I} R_i^{\dagger}.$$

PROOF 3.6.3 ► PROOF OF PROPOSITION 3.6.2

Item 1: Interaction With Inverses

Clear. 

00H3 3.7 Binary Intersections of Relations

Let A and B be sets and let R and S be relations from A to B .

00H4

DEFINITION 3.7.1 ► BINARY INTERSECTIONS OF RELATIONS

The **intersection of R and S** ¹ is the relation $R \cap S$ from A to B defined as follows:

- Viewing relations from A to B as subsets of $A \times B$, we define²

$$R \cap S \stackrel{\text{def}}{=} \{(a, b) \in A \times B \mid \text{we have } a \sim_R b \text{ and } a \sim_S b\}.$$

- Viewing relations from A to B as functions $A \rightarrow \mathcal{P}(B)$, we define

$$[R \cap S](a) \stackrel{\text{def}}{=} R(a) \cap S(a)$$

for each $a \in A$.

¹Further Terminology: Also called the **binary intersection of R and S** , for emphasis.

²This is the same as the intersection of R and S as subsets of $A \times B$.

00H5

PROPOSITION 3.7.2 ► PROPERTIES OF BINARY INTERSECTIONS OF RELATIONS

Let R, S, R_1 , and R_2 be relations from A to B , and let S_1 and S_2 be relations from B to C .

00H6

1. *Interaction With Inverses.* We have

$$(R \cap S)^\dagger = R^\dagger \cap S^\dagger.$$

00H7

2. *Interaction With Composition.* We have

$$(S_1 \diamond R_1) \cap (S_2 \diamond R_2) = (S_1 \cap S_2) \diamond (R_1 \cap R_2).$$

PROOF 3.7.3 ► PROOF OF PROPOSITION 3.7.2

Item 1: Interaction With Inverses

Clear.

Item 2: Interaction With Composition

Unwinding the definitions, we see that:

1. The condition for $(S_1 \diamond R_1) \cap (S_2 \diamond R_2)$ is:

- (a) There exists some $b \in B$ such that:

i. $a \sim_{R_1} b$ and $b \sim_{S_1} c$;

and

i. $a \sim_{R_2} b$ and $b \sim_{S_2} c$;


3. The condition for $(S_1 \cap S_2) \diamond (R_1 \cap R_2)$ is:

(a) There exists some $b \in B$ such that:

i. $a \sim_{R_1} b$ and $a \sim_{R_2} b$;

and

i. $b \sim_{S_1} c$ and $b \sim_{S_2} c$.

These two conditions agree, and thus so do the two resulting relations on $A \times C$. 

00H8 3.8 Intersections of Families of Relations

Let A and B be sets and let $\{R_i\}_{i \in I}$ be a family of relations from A to B .

DEFINITION 3.8.1 ► THE INTERSECTION OF A FAMILY OF RELATIONS

The **intersection of the family** $\{R_i\}_{i \in I}$ is the relation $\bigcup_{i \in I} R_i$ defined as follows:

- Viewing relations from A to B as subsets of $A \times B$, we define¹

$$\bigcup_{i \in I} R_i \stackrel{\text{def}}{=} \left\{ (a, b) \in (A \times B)^{\times I} \left| \begin{array}{l} \text{for each } i \in I, \\ \text{we have } a \sim_{R_i} b \end{array} \right. \right\}.$$

- Viewing relations from A to B as functions $A \rightarrow \mathcal{P}(B)$, we define

$$\left[\bigcap_{i \in I} R_i \right] (a) \stackrel{\text{def}}{=} \bigcap_{i \in I} R_i(a)$$

for each $a \in A$.

¹This is the same as the intersection of $\{R_i\}_{i \in I}$ as a collection of subsets of $A \times B$.

00HA

PROPOSITION 3.8.2 ► PROPERTIES OF INTERSECTIONS OF FAMILIES OF RELATIONS

Let A and B be sets and let $\{R_i\}_{i \in I}$ be a family of relations from A to B .

00HB

1. *Interaction With Inverses.* We have

$$\left(\bigcap_{i \in I} R_i \right)^\dagger = \bigcap_{i \in I} R_i^\dagger.$$

PROOF 3.8.3 ► PROOF OF PROPOSITION 3.8.2

Item 1: Interaction With Inverses

Clear.



00HC

3.9 Binary Products of Relations

Let A, B, X , and Y be sets, let $R: A \rightarrow B$ be a relation from A to B , and let $S: X \rightarrow Y$ be a relation from X to Y .

00HD

DEFINITION 3.9.1 ► BINARY PRODUCTS OF RELATIONS

The **product of R and S** ¹ is the relation $R \times S$ from $A \times X$ to $B \times Y$ defined as follows:

- Viewing relations from $A \times X$ to $B \times Y$ as subsets of $(A \times X) \times (B \times Y)$, we define $R \times S$ as the Cartesian product of R and S as subsets of $A \times X$ and $B \times Y$;²
- Viewing relations from $A \times X$ to $B \times Y$ as functions $A \times X \rightarrow \mathcal{P}(B \times Y)$, we define $R \times S$ as the composition

$$A \times X \xrightarrow{R \times S} \mathcal{P}(B) \times \mathcal{P}(Y) \xrightarrow{\mathcal{P}_{B,Y}^\otimes} \mathcal{P}(B \times Y)$$

in Sets, i.e. by

$$[R \times S](a, x) \stackrel{\text{def}}{=} R(a) \times S(x)$$

for each $(a, x) \in A \times X$.

¹Further Terminology: Also called the **binary product of R and S** , for emphasis. That is, $R \times S$ is the relation given by declaring $(a, x) \sim_{R \times S} (b, y)$ iff $a \sim_R b$ and $x \sim_S y$.

00HE

PROPOSITION 3.9.2 ► PROPERTIES OF BINARY PRODUCTS OF RELATIONS

Let A, B, X , and Y be sets.

00HF

1. *Interaction With Inverses.* Let

$$R: A \rightarrowtail A,$$

$$S: X \rightarrowtail X$$

We have

$$(R \times S)^\dagger = R^\dagger \times S^\dagger.$$

00HG

2. *Interaction With Composition.* Let

$$R_1: A \rightarrowtail B,$$

$$S_1: B \rightarrowtail C,$$

$$R_2: X \rightarrowtail Y,$$

$$S_2: Y \rightarrowtail Z$$

be relations. We have

$$(S_1 \diamond R_1) \times (S_2 \diamond R_2) = (S_1 \times S_2) \diamond (R_1 \times R_2).$$

PROOF 3.9.3 ► PROOF OF PROPOSITION 3.5.2**Item 1: Interaction With Inverses**

Unwinding the definitions, we see that:

1. We have $(a, x) \sim_{(R \times S)^\dagger} (b, y)$ iff:
 - We have $(b, y) \sim_{R \times S} (a, x)$, i.e. iff:
 - We have $b \sim_R a$;
 - We have $y \sim_S x$;
2. We have $(a, x) \sim_{R^\dagger \times S^\dagger} (b, y)$ iff:
 - We have $a \sim_{R^\dagger} b$ and $x \sim_{S^\dagger} y$, i.e. iff:
 - We have $b \sim_R a$;


– We have $y \sim_S x$.

These two conditions agree, and thus the two resulting relations on $A \times X$ are equal.

Item 2: Interaction With Composition

Unwinding the definitions, we see that:

1. We have $(a, x) \sim_{(S_1 \diamond R_1) \times (S_2 \diamond R_2)} (c, z)$ iff:
 - (a) We have $a \sim_{S_1 \diamond R_1} c$ and $x \sim_{S_2 \diamond R_2} z$, i.e. iff:
 - i. There exists some $b \in B$ such that $a \sim_{R_1} b$ and $b \sim_{S_1} c$;
 - ii. There exists some $y \in Y$ such that $x \sim_{R_2} y$ and $y \sim_{S_2} z$;
2. We have $(a, x) \sim_{(S_1 \times S_2) \diamond (R_1 \times R_2)} (c, z)$ iff:
 - (a) There exists some $(b, y) \in B \times Y$ such that $(a, x) \sim_{R_1 \times R_2} (b, y)$ and $(b, y) \sim_{S_1 \times S_2} (c, z)$, i.e. such that:
 - i. We have $a \sim_{R_1} b$ and $x \sim_{R_2} y$;
 - ii. We have $b \sim_{S_1} c$ and $y \sim_{S_2} z$.

These two conditions agree, and thus the two resulting relations from $A \times X$ to $C \times Z$ are equal. 

00HH 3.10 Products of Families of Relations

Let $\{A_i\}_{i \in I}$ and $\{B_i\}_{i \in I}$ be families of sets, and let $\{R_i: A_i \rightarrow B_i\}_{i \in I}$ be a family of relations.

00HJ

DEFINITION 3.10.1 ► THE PRODUCT OF A FAMILY OF RELATIONS

The **product of the family** $\{R_i\}_{i \in I}$ is the relation $\prod_{i \in I} R_i$ from $\prod_{i \in I} A_i$ to $\prod_{i \in I} B_i$ defined as follows:

- Viewing relations as subsets, we define $\prod_{i \in I} R_i$ as its product as a family of sets, i.e. we have

$$\prod_{i \in I} R_i \stackrel{\text{def}}{=} \left\{ (a_i, b_i)_{i \in I} \in \prod_{i \in I} (A_i \times B_i) \mid \begin{array}{l} \text{for each } i \in I, \\ \text{we have } a_i \sim_{R_i} b_i \end{array} \right\}.$$

- Viewing relations as functions to powersets, we define

$$\left[\prod_{i \in I} R_i \right] ((a_i)_{i \in I}) \stackrel{\text{def}}{=} \prod_{i \in I} R_i(a_i)$$

for each $(a_i)_{i \in I} \in \prod_{i \in I} R_i$.

00HK 3.11 The Inverse of a Relation

Let A, B , and C be sets and let $R \subset A \times B$ be a relation.

00HL DEFINITION 3.11.1 ► THE INVERSE OF A RELATION

The **inverse of R** ¹ is the relation R^\dagger defined as follows:

- Viewing relations as subsets, we define

$$R^\dagger \stackrel{\text{def}}{=} \{(b, a) \in B \times A \mid \text{we have } b \sim_R a\}.$$

- Viewing relations as functions $A \times B \rightarrow \{\text{true}, \text{false}\}$, we define

$$[R^\dagger]_a^b \stackrel{\text{def}}{=} R_b^a$$

for each $(b, a) \in B \times A$.

- Viewing relations as functions $A \rightarrow \mathcal{P}(B)$, we define

$$\begin{aligned} [R^\dagger](b) &\stackrel{\text{def}}{=} R^\dagger(\{b\}) \\ &\stackrel{\text{def}}{=} \{a \in A \mid b \in R(a)\} \end{aligned}$$

for each $b \in B$, where $R^\dagger(\{b\})$ is the fibre of R over $\{b\}$.

¹Further Terminology: Also called the **opposite of R** , the **transpose of R** , or the **converse of R** .

00HM EXAMPLE 3.11.2 ► EXAMPLES OF INVERSES OF RELATIONS

Here are some examples of inverses of relations.

1. *Less Than Equal Signs.* We have $(\leq)^\dagger = \geq$.

00HP 2. *Greater Than Equal Signs.* Dually to ??, we have $(\geq)^\dagger = \leq$.

00HQ 3. *Functions.* Let $f: A \rightarrow B$ be a function. We have

$$\begin{aligned}\text{Gr}(f)^\dagger &= f^{-1}, \\ (f^{-1})^\dagger &= \text{Gr}(f).\end{aligned}$$

00HR **PROPOSITION 3.11.3 ► PROPERTIES OF INVERSES OF RELATIONS**

Let $R: A \rightarrowtail B$ and $S: B \rightarrowtail C$ be relations.

00HS 1. *Interaction With Ranges and Domains.* We have

$$\begin{aligned}\text{dom}(R^\dagger) &= \text{range}(R), \\ \text{range}(R^\dagger) &= \text{dom}(R).\end{aligned}$$

00HT 2. *Interaction With Composition I.* We have

$$(S \diamond R)^\dagger = R^\dagger \diamond S^\dagger.$$

00HU 3. *Interaction With Composition II.* We have

$$\begin{aligned}\chi_B(-1, -2) &\subset R \diamond R^\dagger, \\ \chi_A(-1, -2) &\subset R^\dagger \diamond R.\end{aligned}$$

00HV 4. *Invertibility.* We have

$$(R^\dagger)^\dagger = R.$$

00HW 5. *Identity.* We have

$$\chi_A^\dagger(-1, -2) = \chi_A(-1, -2).$$

PROOF 3.11.4 ► PROOF OF PROPOSITION 3.11.3

Item 1: Interaction With Ranges and Domains

Clear.

Item 2: Interaction With Composition I

Clear.

Item 3: Interaction With Composition II

Clear.

Item 4: Invertibility

Clear.

Item 5: Identity

Clear.

**00HX 3.12 Composition of Relations**

Let A , B , and C be sets and let $R \subset A \times B$ and $S \subset B \times C$ be relations.

00HY DEFINITION 3.12.1 ► COMPOSITION OF RELATIONS

The **composition of R and S** is the relation $S \diamond R$ defined as follows:

- Viewing relations from A to C as subsets of $A \times C$, we define

$$S \diamond R \stackrel{\text{def}}{=} \left\{ (a, c) \in A \times C \mid \begin{array}{l} \text{there exists some } b \in B \text{ such} \\ \text{that } a \sim_R b \text{ and } b \sim_S c \end{array} \right\}.$$

- Viewing relations as functions $A \times B \rightarrow \{\text{true}, \text{false}\}$, we define

$$\begin{aligned} (S \diamond R)_{-2}^{-1} &\stackrel{\text{def}}{=} \int^{y \in B} S_y^{-1} \times R_{-2}^y \\ &= \bigvee_{y \in B} S_y^{-1} \times R_{-2}^y, \end{aligned}$$

where the join \bigvee is taken in the poset $(\{\text{true}, \text{false}\}, \leq)$ of **Sets, Definition 1.2.3**.

· Viewing relations as functions $A \rightarrow \mathcal{P}(B)$, we define

$$S \diamond R \stackrel{\text{def}}{=} \text{Lan}_{\chi_B}(S) \circ R,$$

where $\text{Lan}_{\chi_B}(S)$ is computed by the formula

$$\begin{aligned} [\text{Lan}_{\chi_B}(S)](V) &\cong \int^{y \in B} \chi_{\mathcal{P}(B)}(\chi_y, V) \odot S_y \\ &\cong \int^{y \in B} \chi_V(y) \odot S_y \\ &\cong \bigcup_{y \in B} \chi_V(y) \odot S_y \\ &\cong \bigcup_{y \in V} S_y \end{aligned}$$

for each $V \in \mathcal{P}(B)$. In other words, $S \diamond R$ is defined by¹

$$\begin{aligned} [S \diamond R](a) &\stackrel{\text{def}}{=} S(R(a)) \\ &\stackrel{\text{def}}{=} \bigcup_{x \in R(a)} S(x). \end{aligned}$$

for each $a \in A$.

¹That is: the relation R may send $a \in A$ to a number of elements $\{b_i\}_{i \in I}$ in B , and then the relation S may send the image of each of the b_i 's to a number of elements $\{S(b_i)\}_{i \in I} = \left\{ \{c_{j_i}\}_{j_i \in J_i} \right\}_{i \in I}$ in C .

00HZ

EXAMPLE 3.12.2 ► EXAMPLES OF COMPOSITION OF RELATIONS

Here are some examples of composition of relations.

1. *Composing Less/Greater Than Equal With Greater/Less Than Equal Signs.* We have

$$\begin{aligned}\leq \diamond \geq &= \sim_{\text{triv}}, \\ \geq \diamond \leq &= \sim_{\text{triv}}.\end{aligned}$$

2. *Composing Less/Greater Than Equal Signs With Less/Greater Than Equal Signs.* We have

$$\begin{aligned}\leq \diamond \leq &= \leq, \\ \geq \diamond \geq &= \geq.\end{aligned}$$

00J0

PROPOSITION 3.12.3 ► PROPERTIES OF COMPOSITION OF RELATIONS

Let $R: A \rightarrow B$, $S: B \rightarrow C$, and $T: C \rightarrow D$ be relations.

00J1

1. *Interaction With Ranges and Domains.* We have

$$\begin{aligned}\text{dom}(S \diamond R) &\subset \text{dom}(R), \\ \text{range}(S \diamond R) &\subset \text{range}(S).\end{aligned}$$

00J2

2. *Associativity.* We have

$$(T \diamond S) \diamond R = T \diamond (S \diamond R).$$

00J3

3. *Unitality.* We have

$$\begin{aligned}\chi_B \diamond R &= R, \\ R \diamond \chi_A &= R.\end{aligned}$$

00J4

4. *Interaction With Inverses.* We have

$$(S \diamond R)^\dagger = R^\dagger \diamond S^\dagger.$$

00J5

5. *Interaction With Composition.* We have

$$\begin{aligned}\chi_B(-_1, -_2) &\subset R \diamond R^\dagger, \\ \chi_A(-_1, -_2) &\subset R^\dagger \diamond R.\end{aligned}$$

PROOF 3.12.4 ► PROOF OF PROPOSITION 3.12.3

Item 1: Interaction With Ranges and Domains

Clear.

Item 2: Associativity

Indeed, we have

$$\begin{aligned}
 (T \diamond S) \diamond R &\stackrel{\text{def}}{=} \left(\int^{y \in C} T_x^{-1} \times S_{-2}^x \right) \diamond R \\
 &\stackrel{\text{def}}{=} \int^{x \in B} \left(\int^{y \in C} T_x^{-1} \times S_y^x \right) \diamond R_{-2}^y \\
 &= \int^{x \in B} \int^{y \in C} \left(T_x^{-1} \times S_y^x \right) \diamond R_{-2}^y \\
 &= \int^{y \in C} \int^{x \in B} \left(T_x^{-1} \times S_y^x \right) \diamond R_{-2}^y \\
 &= \int^{y \in C} \int^{x \in B} T_x^{-1} \times \left(S_y^x \diamond R_{-2}^y \right) \\
 &= \int^{x \in B} T_x^{-1} \times \left(\int^{y \in C} S_y^x \diamond R_{-2}^y \right) \\
 &\stackrel{\text{def}}{=} \int^{x \in B} T_x^{-1} \times (S \diamond R)_{-2}^x \\
 &\stackrel{\text{def}}{=} T \diamond (S \diamond R).
 \end{aligned}$$

In the language of relations, given $a \in A$ and $d \in D$, the stated equality witnesses the equivalence of the following two statements:

1. We have $a \sim_{(T \diamond S) \diamond R} d$, i.e. there exists some $b \in B$ such that:
 - (a) We have $a \sim_R b$;
 - (b) We have $b \sim_{T \diamond S} d$, i.e. there exists some $c \in C$ such that:
 - i. We have $b \sim_S c$;
 - ii. We have $c \sim_T d$;
2. We have $a \sim_{T \diamond (S \diamond R)} d$, i.e. there exists some $c \in C$ such that:
 - (a) We have $a \sim_{S \diamond R} c$, i.e. there exists some $b \in B$ such that:

- i. We have $a \sim_R b$;
- ii. We have $b \sim_S c$;
- (b) We have $c \sim_T d$;

both of which are equivalent to the statement

- There exist $b \in B$ and $c \in C$ such that $a \sim_R b \sim_S c \sim_T d$.

Item 3: Unitality

Indeed, we have

$$\begin{aligned}
 \chi_B \diamond R &\stackrel{\text{def}}{=} \int^{x \in B} (\chi_B)_x^{-1} \times R_{-2}^x \\
 &= \bigvee_{x \in B} (\chi_B)_x^{-1} \times R_{-2}^x \\
 &= \bigvee_{\substack{x \in B \\ x = -1}} R_{-2}^x \\
 &= R_{-2}^{-1},
 \end{aligned}$$

and

$$\begin{aligned}
 R \diamond \chi_A &\stackrel{\text{def}}{=} \int^{x \in A} R_x^{-1} \times (\chi_A)_{-2}^x \\
 &= \bigvee_{x \in B} R_x^{-1} \times (\chi_A)_{-2}^x \\
 &= \bigvee_{\substack{x \in B \\ x = -2}} R_x^{-1} \\
 &= R_{-2}^{-1}.
 \end{aligned}$$

In the language of relations, given $a \in A$ and $b \in B$:

- The equality

$$\chi_B \diamond R = R$$

witnesses the equivalence of the following two statements:

1. We have $a \sim_b B$.

2. There exists some $b' \in B$ such that:

- (a) We have $a \sim_R b'$
- (b) We have $b' \sim_{\chi_B} b$, i.e. $b' = b$.

· The equality

$$R \diamond \chi_A = R$$

witnesses the equivalence of the following two statements:

1. There exists some $a' \in A$ such that:

- (a) We have $a \sim_{\chi_B} a'$, i.e. $a = a'$.
- (b) We have $a' \sim_R b$

2. We have $a \sim_b B$.

Item 4: Interaction With Inverses

Clear.

Item 5: Interaction With Composition

Clear.



00J6 3.13 The Collage of a Relation

Let A and B be sets and let $R: A \rightarrow B$ be a relation from A to B .

00J7 DEFINITION 3.13.1 ► THE COLLAGE OF A RELATION

The **collage of R** ¹ is the poset $\mathbf{Coll}(R) \stackrel{\text{def}}{=} (\text{Coll}(R), \leq_{\mathbf{Coll}(R)})$ consisting of

· *The Underlying Set.* The set $\text{Coll}(R)$ defined by

$$\text{Coll}(R) \stackrel{\text{def}}{=} A \coprod B.$$

· *The Partial Order.* The partial order

$$\leq_{\mathbf{Coll}(R)}: \text{Coll}(R) \times \text{Coll}(R) \rightarrow \{\text{true}, \text{false}\}$$

on $\text{Coll}(R)$ defined by

$$\leq(a, b) \stackrel{\text{def}}{=} \begin{cases} \text{true} & \text{if } a = b \text{ or } a \sim_R b, \\ \text{false} & \text{otherwise.} \end{cases}$$

¹Further Terminology: Also called the **cograph** of R .

00J8

PROPOSITION 3.13.2 ► PROPERTIES OF COLLAGES OF RELATIONS

Let A and B be sets and let $R: A \rightarrow B$ be a relation from A to B .

00J9

1. *Functoriality I.* The assignment $R \mapsto \mathbf{Coll}(R)$ defines a functor¹

$$\mathbf{Coll}: \mathbf{Rel}(A, B) \rightarrow \mathbf{Pos}_{/\Delta^1}(A, B),$$

where

- *Action on Objects.* For each $R \in \mathbf{Obj}(\mathbf{Rel}(A, B))$, we have

$$[\mathbf{Coll}](R) \stackrel{\text{def}}{=} (\mathbf{Coll}(R), \phi_R)$$

for each $R \in \mathbf{Rel}(A, B)$, where

- The poset $\mathbf{Coll}(R)$ is the collage of R of [Definition 3.13.1](#);
- The morphism $\phi_R: \mathbf{Coll}(R) \rightarrow \Delta^1$ is given by

$$\phi_R(x) \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if } x \in A, \\ 1 & \text{if } x \in B \end{cases}$$

for each $x \in \mathbf{Coll}(R)$;

- *Action on Morphisms.* For each $R, S \in \mathbf{Obj}(\mathbf{Rel}(A, B))$, the action on Hom-sets

$$\mathbf{Coll}_{R,S}: \mathbf{Hom}_{\mathbf{Rel}(A,B)}(R, S) \rightarrow \mathbf{Pos}(\mathbf{Coll}(R), \mathbf{Coll}(S))$$

of \mathbf{Coll} at (R, S) is given by sending an inclusion

$$\iota: R \subset S$$

to the morphism

$$\mathbf{Coll}(\iota): \mathbf{Coll}(R) \rightarrow \mathbf{Coll}(S)$$

of posets over Δ^1 defined by

$$[\mathbf{Coll}(\iota)](x) \stackrel{\text{def}}{=} x$$

for each $x \in \mathbf{Coll}(R)$.²

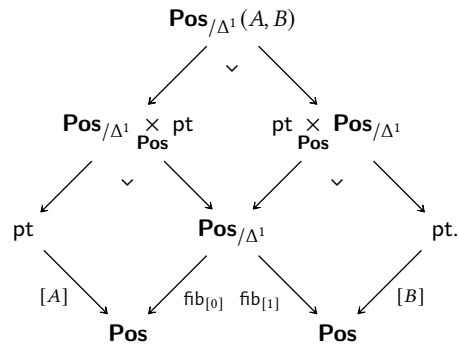
00JA

2. *Equivalence.* The functor of **Item 1** is an equivalence of categories.

¹Here $\text{Pos}_{/\Delta^1}(A, B)$ is the category defined as the pullback

$$\text{Pos}_{/\Delta^1}(A, B) \stackrel{\text{def}}{=} \text{pt}_{[A], \text{Pos}, \text{fib}_0} \times_{\text{Pos}} \text{Pos}_{/\Delta^1} \times_{\text{fib}_1, \text{Pos}, [B]} \text{pt},$$

as in the diagram



Explicitly, an object of $\text{Pos}_{/\Delta^1}(A, B)$ is a pair (X, ϕ_X) consisting of

- A poset X ;
- A morphism $\phi_X: X \rightarrow \Delta^1$;

such that $\phi_X^{-1}(0) = A$ and $\phi_X^{-1}(1) = B$, with morphisms between such objects being morphisms of posets over Δ^1 .

²Note that this is indeed a morphism of posets: if $x \leq_{\text{Coll}(R)} y$, then $x = y$ or $x \sim_R y$, so we have either $x = y$ or $x \sim_S y$ (as $R \subset S$), and thus $x \leq_{\text{Coll}(S)} y$.

PROOF 3.13.3 ► PROOF OF PROPOSITION 3.13.2

Item 1: Functoriality

Clear.

Item 2: Equivalence

Omitted.



00JB 4 Equivalence Relations

00JC 4.1 Reflexive Relations

4.1.1 Foundations

Let A be a set.

00JE

DEFINITION 4.1.1 ► REFLEXIVE RELATIONS

A **reflexive relation** is equivalently:¹

- An \mathbb{E}_0 -monoid in $(N_\bullet(\mathbf{Rel}(A, A)), \chi_A)$;
- A pointed object in $(\mathbf{Rel}(A, A), \chi_A)$.

¹Note that since $\mathbf{Rel}(A, A)$ is posetal, reflexivity is a property of a relation, rather than extra structure.

00JF

REMARK 4.1.2 ► UNWINDING DEFINITION 4.1.1

In detail, a relation R on A is **reflexive** if we have an inclusion

$$\eta_R: \chi_A \subset R$$

of relations in $\mathbf{Rel}(A, A)$, i.e. if, for each $a \in A$, we have $a \sim_R a$.

00JG

DEFINITION 4.1.3 ► THE PO/SET OF REFLEXIVE RELATIONS ON A SET

Let A be a set.

00JH

1. The **set of reflexive relations on A** is the subset $\mathbf{Rel}^{\text{refl}}(A, A)$ of $\mathbf{Rel}(A, A)$ spanned by the reflexive relations.

00JJ

2. The **poset of relations on A** is the subposet $\mathbf{Rel}^{\text{refl}}(A, A)$ of $\mathbf{Rel}(A, A)$ spanned by the reflexive relations.

00JK

PROPOSITION 4.1.4 ► PROPERTIES OF REFLEXIVE RELATIONS

Let R and S be relations on A .

00JL

1. *Interaction With Inverses.* If R is reflexive, then so is R^\dagger .

00JM

2. *Interaction With Composition.* If R and S are reflexive, then so is $S \diamond R$.

PROOF 4.1.5 ► PROOF OF PROPOSITION 4.1.4

Item 1: Interaction With Inverses

Clear.

Item 2: Interaction With Composition

Clear.



4.1.2 The Reflexive Closure of a Relation

Let R be a relation on A .

00JP

DEFINITION 4.1.6 ► THE REFLEXIVE CLOSURE OF A RELATION

The **reflexive closure** of \sim_R is the relation \sim_R^{refl} ¹ satisfying the following universal property:²

- (★) Given another reflexive relation \sim_S on A such that $R \subset S$, there exists an inclusion $\sim_R^{\text{refl}} \subset \sim_S$.

¹Further Notation: Also written R^{refl} .

²Slogan: The reflexive closure of R is the smallest reflexive relation containing R .

00JQ

CONSTRUCTION 4.1.7 ► THE REFLEXIVE CLOSURE OF A RELATION

Concretely, \sim_R^{refl} is the free pointed object on R in $(\mathbf{Rel}(A, A), \chi_A)$ ¹, being given by

$$\begin{aligned} R^{\text{refl}} &\stackrel{\text{def}}{=} R \coprod^{\mathbf{Rel}(A, A)} \Delta_A \\ &= R \cup \Delta_A \\ &= \{(a, b) \in A \times A \mid \text{we have } a \sim_R b \text{ or } a = b\}. \end{aligned}$$

¹Or, equivalently, the free \mathbb{E}_0 -monoid on R in $(\mathbf{N}_\bullet(\mathbf{Rel}(A, A)), \chi_A)$.

PROOF 4.1.8 ► PROOF OF CONSTRUCTION 4.1.7

Clear.



00JR

PROPOSITION 4.1.9 ► PROPERTIES OF THE REFLEXIVE CLOSURE OF A RELATION

Let R be a relation on A .

00JS

1. *Adjointness.* We have an adjunction

$$\left((-)^{\text{refl}} \dashv \overset{\circlearrowleft}{\text{忘}} \right): \mathbf{Rel}(A, A) \overset{(-)^{\text{refl}}}{\underset{\overset{\circlearrowleft}{\text{忘}}}{\rightleftarrows}} \mathbf{Rel}^{\text{refl}}(A, A),$$

witnessed by a bijection of sets

$$\mathbf{Rel}^{\text{refl}}(R^{\text{refl}}, S) \cong \mathbf{Rel}(R, S),$$

natural in $R \in \text{Obj}(\mathbf{Rel}^{\text{refl}}(A, A))$ and $S \in \text{Obj}(\mathbf{Rel}(A, A))$.

00JT

2. *The Reflexive Closure of a Reflexive Relation.* If R is reflexive, then $R^{\text{refl}} = R$.

00JU

3. *Idempotency.* We have

$$(R^{\text{refl}})^{\text{refl}} = R^{\text{refl}}.$$

00JV

4. *Interaction With Inverses.* We have

$$\left(R^{\dagger} \right)^{\text{refl}} = \left(R^{\text{refl}} \right)^{\dagger}, \quad \begin{array}{ccc} \mathbf{Rel}(A, A) & \xrightarrow{(-)^{\text{refl}}} & \mathbf{Rel}(A, A) \\ (-)^{\dagger} \downarrow & & \downarrow (-)^{\dagger} \\ \mathbf{Rel}(A, A) & \xrightarrow{(-)^{\text{refl}}} & \mathbf{Rel}(A, A). \end{array}$$

00JW

5. *Interaction With Composition.* We have

$$(S \diamond R)^{\text{refl}} = S^{\text{refl}} \diamond R^{\text{refl}}, \quad \begin{array}{ccc} \mathbf{Rel}(A, A) \times \mathbf{Rel}(A, A) & \xrightarrow{\diamond} & \mathbf{Rel}(A, A) \\ (-)^{\text{refl}} \times (-)^{\text{refl}} \downarrow & & \downarrow (-)^{\text{refl}} \\ \mathbf{Rel}(A, A) \times \mathbf{Rel}(A, A) & \xrightarrow{\diamond} & \mathbf{Rel}(A, A). \end{array}$$

PROOF 4.1.10 ► PROOF OF PROPOSITION 4.1.9

Item 1: Adjointness

This is a rephrasing of the universal property of the reflexive closure of a relation, stated in [Definition 4.1.6](#).

Item 2: The Reflexive Closure of a Reflexive Relation

Clear.

Item 3: Idempotency

This follows from [Item 2](#).

Item 4: Interaction With Inverses

Clear.

Item 5: Interaction With Composition

This follows from [Item 2](#) of [Proposition 4.1.4](#). 

00JX 4.2 Symmetric Relations

4.2.1 Foundations

Let A be a set.

00JZ DEFINITION 4.2.1 ► SYMMETRIC RELATIONS

A relation R on A is **symmetric** if, for each $a, b \in A$, the following conditions are equivalent:¹

1. We have $a \sim_R b$.
2. We have $b \sim_R a$.

¹That is, R is symmetric if $R^\dagger = R$.

00K0 DEFINITION 4.2.2 ► THE PO/SET OF SYMMETRIC RELATIONS ON A SET

Let A be a set.

1. The **set of symmetric relations on A** is the subset $\text{Rel}^{\text{symm}}(A, A)$ of $\text{Rel}(A, A)$ spanned by the symmetric relations.

00K1

00K2

2. The **poset of relations on A** is the subposet $\mathbf{Rel}^{\text{symm}}(A, A)$ of $\mathbf{Rel}(A, A)$ spanned by the symmetric relations.

00K3

PROPOSITION 4.2.3 ► PROPERTIES OF SYMMETRIC RELATIONS

Let R and S be relations on A .

00K4

1. *Interaction With Inverses.* If R is symmetric, then so is R^\dagger .

00K5

2. *Interaction With Composition.* If R and S are symmetric, then so is $S \diamond R$.

PROOF 4.2.4 ► PROOF OF PROPOSITION 4.2.3

Item 1: Interaction With Inverses

Clear.

Item 2: Interaction With Composition

Clear.



4.2.2 The Symmetric Closure of a Relation

Let R be a relation on A .

00K7

DEFINITION 4.2.5 ► THE SYMMETRIC CLOSURE OF A RELATION

The **symmetric closure** of \sim_R is the relation \sim_R^{symm} ¹ satisfying the following universal property:²

- (★) Given another symmetric relation \sim_S on A such that $R \subset S$, there exists an inclusion $\sim_R^{\text{symm}} \subset \sim_S$.

¹Further Notation: Also written R^{symm} .

²Slogan: The symmetric closure of R is the smallest symmetric relation containing R .

00K8

CONSTRUCTION 4.2.6 ► THE SYMMETRIC CLOSURE OF A RELATION

Concretely, \sim_R^{symm} is the symmetric relation on A defined by

$$\begin{aligned} R^{\text{symm}} &\stackrel{\text{def}}{=} R \cup R^\dagger \\ &= \{(a, b) \in A \times A \mid \text{we have } a \sim_R b \text{ or } b \sim_R a\}. \end{aligned}$$

PROOF 4.2.7 ► PROOF OF CONSTRUCTION 4.2.6

Clear. 

00K9

PROPOSITION 4.2.8 ► PROPERTIES OF THE SYMMETRIC CLOSURE OF A RELATION

Let R be a relation on A .

00KA

1. *Adjointness.* We have an adjunction

$$((-)^{\text{symm}} \dashv \text{忘}): \mathbf{Rel}(A, A) \begin{matrix} \xrightarrow{(-)^{\text{symm}}} \\ \perp \\ \xleftarrow{\text{忘}} \end{matrix} \mathbf{Rel}^{\text{symm}}(A, A),$$

witnessed by a bijection of sets

$$\mathbf{Rel}^{\text{symm}}(R^{\text{symm}}, S) \cong \mathbf{Rel}(R, S),$$

natural in $R \in \text{Obj}(\mathbf{Rel}^{\text{symm}}(A, A))$ and $S \in \text{Obj}(\mathbf{Rel}(A, A))$.

00KB

2. *The Symmetric Closure of a Symmetric Relation.* If R is symmetric, then $R^{\text{symm}} = R$.

00KC

3. *Idempotency.* We have

$$(R^{\text{symm}})^{\text{symm}} = R^{\text{symm}}.$$

00KD

4. *Interaction With Inverses.* We have

$$\begin{array}{ccc} \mathbf{Rel}(A, A) & \xrightarrow{(-)^{\text{symm}}} & \mathbf{Rel}(A, A) \\ (-)^\dagger \downarrow & & \downarrow (-)^\dagger \\ \mathbf{Rel}(A, A) & \xrightarrow{(-)^{\text{symm}}} & \mathbf{Rel}(A, A) \end{array}$$

$$(R^\dagger)^{\text{symm}} = (R^{\text{symm}})^\dagger,$$

00KE

5. *Interaction With Composition.* We have

$$\begin{array}{ccc}
 & \text{Rel}(A, A) \times \text{Rel}(A, A) \xrightarrow{\diamond} \text{Rel}(A, A) & \\
 (S \diamond R)^{\text{symm}} = S^{\text{symm}} \diamond R^{\text{symm}}, & \begin{array}{c} \downarrow (-)^{\text{symm}} \times (-)^{\text{symm}} \\ \text{Rel}(A, A) \times \text{Rel}(A, A) \xrightarrow{\diamond} \text{Rel}(A, A). \end{array} & \begin{array}{c} \downarrow (-)^{\text{symm}} \\ \text{Rel}(A, A) \end{array}
 \end{array}$$

PROOF 4.2.9 ► PROOF OF PROPOSITION 4.2.8

Item 1: Adjointness

This is a rephrasing of the universal property of the symmetric closure of a relation, stated in [Definition 4.2.5](#).

Item 2: The Symmetric Closure of a Symmetric Relation

Clear.

Item 3: Idempotency

This follows from [Item 2](#).

Item 4: Interaction With Inverses

Clear.

Item 5: Interaction With Composition

This follows from [Item 2](#) of [Proposition 4.2.3](#). 

00KF 4.3 Transitive Relations

4.3.1 Foundations

Let A be a set.

00KH

DEFINITION 4.3.1 ► TRANSITIVE RELATIONS

A **transitive relation** is equivalently:¹

- A non-unital \mathbb{B}_1 -monoid in $(\mathbf{N}_\bullet(\mathbf{Rel}(A, A)), \diamond)$;
- A non-unital monoid in $(\mathbf{Rel}(A, A), \diamond)$.

¹Note that since $\mathbf{Rel}(A, A)$ is posetal, transitivity is a property of a relation, rather than extra structure.

00KJ

REMARK 4.3.2 ► UNWINDING DEFINITION 4.3.1

In detail, a relation R on A is **transitive** if we have an inclusion

$$\mu_R: R \diamond R \subset R$$

of relations in $\mathbf{Rel}(A, A)$, i.e. if, for each $a, c \in A$, the following condition is satisfied:

(★) If there exists some $b \in A$ such that $a \sim_R b$ and $b \sim_R c$, then $a \sim_R c$.

00KK

DEFINITION 4.3.3 ► THE PO/SET OF TRANSITIVE RELATIONS ON A SET

Let A be a set.

00KL

1. The **set of transitive relations from A to B** is the subset $\mathbf{Rel}^{\text{trans}}(A)$ of $\mathbf{Rel}(A, A)$ spanned by the transitive relations.

00KM

2. The **poset of relations from A to B** is the subposet $\mathbf{Rel}^{\text{trans}}(A)$ of $\mathbf{Rel}(A, A)$ spanned by the transitive relations.

00KN

PROPOSITION 4.3.4 ► PROPERTIES OF TRANSITIVE RELATIONS

Let R and S be relations on A .

00KP

1. *Interaction With Inverses.* If R is transitive, then so is R^\dagger .

00KQ

2. *Interaction With Composition.* If R and S are transitive, then $S \diamond R$ **may fail to be transitive**.

PROOF 4.3.5 ► PROOF OF PROPOSITION 4.3.4

Item 1: Interaction With Inverses

Clear.

Item 2: Interaction With Composition

See [MSE2096272].¹



¹*Intuition:* Transitivity for R and S fails to imply that of $S \diamond R$ because the composition operation for relations intertwines R and S in an incompatible way:

1. If $a \sim_{S \diamond R} c$ and $c \sim_{S \diamond R} e$, then:
 - (a) There is some $b \in A$ such that:
 - i. $a \sim_R b$;
 - ii. $b \sim_S c$;
 - (b) There is some $d \in A$ such that:
 - i. $c \sim_R d$;
 - ii. $d \sim_S e$.

4.3.2 The Transitive Closure of a Relation

Let R be a relation on A .

00KS

DEFINITION 4.3.6 ► THE TRANSITIVE CLOSURE OF A RELATION

The **transitive closure** of \sim_R is the relation \sim_R^{trans} ¹ satisfying the following universal property:²

- (★) Given another transitive relation \sim_S on A such that $R \subset S$, there exists an inclusion $\sim_R^{\text{trans}} \subset \sim_S$.

¹*Further Notation:* Also written R^{trans} .

²*Slogan:* The transitive closure of R is the smallest transitive relation containing R .

00KT

CONSTRUCTION 4.3.7 ► THE TRANSITIVE CLOSURE OF A RELATION

Concretely, \sim_R^{trans} is the free non-unital monoid on R in $(\mathbf{Rel}(A, A), \diamond)$ ¹, being given by

$$\begin{aligned} R^{\text{trans}} &\stackrel{\text{def}}{=} \coprod_{n=1}^{\infty} R^{\diamond n} \\ &\stackrel{\text{def}}{=} \bigcup_{n=1}^{\infty} R^{\diamond n} \\ &\stackrel{\text{def}}{=} \left\{ (a, b) \in A \times B \mid \begin{array}{l} \text{there exists some } (x_1, \dots, x_n) \in R^{\times n} \\ \text{such that } a \sim_R x_1 \sim_R \dots \sim_R x_n \sim_R b \end{array} \right\}. \end{aligned}$$

¹Or, equivalently, the free non-unital \mathbb{E}_1 -monoid on R in $(\mathbf{N}_\bullet(\mathbf{Rel}(A, A)), \diamond)$.

PROOF 4.3.8 ► PROOF OF CONSTRUCTION 4.3.7

Clear.



00KU

PROPOSITION 4.3.9 ► PROPERTIES OF THE TRANSITIVE CLOSURE OF A RELATION

Let R be a relation on A .

00KV

1. *Adjointness.* We have an adjunction

$$((-)^{\text{trans}} \dashv \overset{\circ}{\text{忘}}): \mathbf{Rel}(A, A) \begin{matrix} \xrightarrow{(-)^{\text{trans}}} \\ \perp \\ \xleftarrow{\overset{\circ}{\text{忘}}} \end{matrix} \mathbf{Rel}^{\text{trans}}(A, A),$$

witnessed by a bijection of sets

$$\mathbf{Rel}^{\text{trans}}(R^{\text{trans}}, S) \cong \mathbf{Rel}(R, S),$$

natural in $R \in \text{Obj}(\mathbf{Rel}^{\text{trans}}(A, A))$ and $S \in \text{Obj}(\mathbf{Rel}(A, B))$.

00KW

2. *The Transitive Closure of a Transitive Relation.* If R is transitive, then $R^{\text{trans}} = R$.

00KX

3. *Idempotency.* We have

$$(R^{\text{trans}})^{\text{trans}} = R^{\text{trans}}.$$

00KY

4. *Interaction With Inverses.* We have

$$\begin{array}{ccc} \mathbf{Rel}(A, A) & \xrightarrow{(-)^{\text{trans}}} & \mathbf{Rel}(A, A) \\ \downarrow (-)^{\dagger} & & \downarrow (-)^{\dagger} \\ \mathbf{Rel}(A, A) & \xrightarrow{(-)^{\text{trans}}} & \mathbf{Rel}(A, A) \end{array}$$

$$(R^{\dagger})^{\text{trans}} = (R^{\text{trans}})^{\dagger},$$

00KZ

5. *Interaction With Composition.* We have

$$\begin{array}{ccc} \mathbf{Rel}(A, A) \times \mathbf{Rel}(A, A) & \xrightarrow{\diamond} & \mathbf{Rel}(A, A) \\ \downarrow (-)^{\text{trans}} \times (-)^{\text{trans}} & \text{X} & \downarrow (-)^{\text{trans}} \\ \mathbf{Rel}(A, A) \times \mathbf{Rel}(A, A) & \xrightarrow{\diamond} & \mathbf{Rel}(A, A) \end{array}$$

$$(S \diamond R)^{\text{trans}} \stackrel{\text{poss.}}{\neq} S^{\text{trans}} \diamond R^{\text{trans}},$$

PROOF 4.3.10 ► PROOF OF PROPOSITION 4.3.9

Item 1: Adjointness

This is a rephrasing of the universal property of the transitive closure of a relation, stated in [Definition 4.3.6](#).

Item 2: The Transitive Closure of a Transitive Relation

Clear.

Item 3: Idempotency

This follows from [Item 2](#).

Item 4: Interaction With Inverses

We have

$$\begin{aligned}
 (R^\dagger)^{\text{trans}} &= \bigcup_{n=1}^{\infty} (R^\dagger)^{\diamond n} && \text{(by Construction 4.3.7)} \\
 &= \bigcup_{n=1}^{\infty} (R^{\diamond n})^\dagger && \text{(by Item 4 of Proposition 3.12.3)} \\
 &= \left(\bigcup_{n=1}^{\infty} R^{\diamond n} \right)^\dagger && \text{(by Item 1 of Proposition 3.6.2)} \\
 &= (R^{\text{trans}})^\dagger. && \text{(by Construction 4.3.7)}
 \end{aligned}$$

Item 5: Interaction With Composition

This follows from [Item 2](#) of [Proposition 4.3.4](#). 

00L0 4.4 Equivalence Relations

4.4.1 Foundations

Let A be a set.

00L2 DEFINITION 4.4.1 ► EQUIVALENCE RELATIONS

A relation R is an **equivalence relation** if it is reflexive, symmetric, and transitive.¹

¹*Further Terminology:* If instead R is just symmetric and transitive, then it is called a **partial equivalence relation**.

00L3

EXAMPLE 4.4.2 ► THE KERNEL OF A FUNCTION

The **kernel of a function** $f: A \rightarrow B$ is the equivalence $\sim_{\text{Ker}(f)}$ on A obtained by declaring $a \sim_{\text{Ker}(f)} b$ iff $f(a) = f(b)$.¹

¹The kernel $\text{Ker}(f): A \dashv\rightarrow A$ of f is the monad induced by the adjunction $\text{Gr}(f) \dashv f^{-1}: A \rightleftarrows B$ in **Rel** of Item 2 of Proposition 3.1.2.

00L4

DEFINITION 4.4.3 ► THE PO/SET OF EQUIVALENCE RELATIONS ON A SET

Let A and B be sets.

00L5

1. The **set of equivalence relations from A to B** is the subset $\text{Rel}^{\text{eq}}(A, B)$ of $\text{Rel}(A, B)$ spanned by the equivalence relations.

00L6

2. The **poset of relations from A to B** is the subposet **Rel**^{eq}(A, B) of **Rel**(A, B) spanned by the equivalence relations.

4.4.2 The Equivalence Closure of a Relation

Let R be a relation on A .

00L8

DEFINITION 4.4.4 ► THE EQUIVALENCE CLOSURE OF A RELATION

The **equivalence closure**¹ of \sim_R is the relation \sim_R^{eq} ² satisfying the following universal property:³

- (★) Given another equivalence relation \sim_S on A such that $R \subset S$, there exists an inclusion $\sim_R^{\text{eq}} \subset \sim_S$.

¹Further Terminology: Also called the **equivalence relation associated to \sim_R** .

²Further Notation: Also written R^{eq} .

³Slogan: The equivalence closure of R is the smallest equivalence relation containing R .

00L9

CONSTRUCTION 4.4.5 ► THE EQUIVALENCE CLOSURE OF A RELATION

Concretely, \sim_R^{eq} is the equivalence relation on A defined by

$$\begin{aligned}
 R^{\text{eq}} &\stackrel{\text{def}}{=} \left(\left(R^{\text{refl}} \right)^{\text{symm}} \right)^{\text{trans}} \\
 &= \left(\left(R^{\text{symm}} \right)^{\text{trans}} \right)^{\text{refl}} \\
 &= \left\{ (a, b) \in A \times B \mid \begin{array}{l} \text{there exists } (x_1, \dots, x_n) \in R^{\times n} \text{ satisfying at} \\ \text{least one of the following conditions:} \\ \\ 1. \text{ The following conditions are satisfied:} \\ \quad (a) \text{ We have } a \sim_R x_1 \text{ or } x_1 \sim_R a; \\ \quad (b) \text{ We have } x_i \sim_R x_{i+1} \text{ or } x_{i+1} \sim_R x_i \\ \quad \quad \text{for each } 1 \leq i \leq n-1; \\ \quad (c) \text{ We have } b \sim_R x_n \text{ or } x_n \sim_R b; \\ \\ 2. \text{ We have } a = b. \end{array} \right\}.
 \end{aligned}$$

PROOF 4.4.6 ► PROOF OF CONSTRUCTION 4.4.5


From the universal properties of the reflexive, symmetric, and transitive closures of a relation (**Definitions 4.1.6, 4.2.5 and 4.3.6**), we see that it suffices to prove that:

00LA

1. The symmetric closure of a reflexive relation is still reflexive;

00LB

2. The transitive closure of a symmetric relation is still symmetric;

which are both clear. 

00LC

PROPOSITION 4.4.7 ► PROPERTIES OF EQUIVALENCE RELATIONS

Let R be a relation on A .

00LD

1. *Adjointness.* We have an adjunction

$$\left((-)^{\text{eq}} \dashv \text{忘} \right): \mathbf{Rel}(A, B) \begin{array}{c} \xrightarrow{(-)^{\text{eq}}} \\ \perp \\ \xleftarrow{\text{忘}} \end{array} \mathbf{Rel}^{\text{eq}}(A, B),$$

witnessed by a bijection of sets

$$\mathbf{Rel}^{\text{eq}}(R^{\text{eq}}, S) \cong \mathbf{Rel}(R, S),$$

natural in $R \in \text{Obj}(\mathbf{Rel}^{\text{eq}}(A, B))$ and $S \in \text{Obj}(\mathbf{Rel}(A, B))$.

00LE

2. *The Equivalence Closure of an Equivalence Relation.* If R is an equivalence relation, then $R^{\text{eq}} = R$.

00LF

3. *Idempotency.* We have

$$(R^{\text{eq}})^{\text{eq}} = R^{\text{eq}}.$$

PROOF 4.4.8 ► PROOF OF PROPOSITION 4.4.7


Item 1: Adjointness

This is a rephrasing of the universal property of the equivalence closure of a relation, stated in [Definition 4.4.4](#).

Item 2: The Equivalence Closure of an Equivalence Relation

Clear.

Item 3: Idempotency

This follows from [Item 2](#). 

00LG 4.5 Quotients by Equivalence Relations

4.5.1 Equivalence Classes

Let A be a set, let R be a relation on A , and let $a \in A$.

00LJ

DEFINITION 4.5.1 ► EQUIVALENCE CLASSES

The **equivalence class associated to** a is the set $[a]$ defined by

$$\begin{aligned} [a] &\stackrel{\text{def}}{=} \{x \in X \mid x \sim_R a\} \\ &= \{x \in X \mid a \sim_R x\}. \end{aligned} \quad (\text{since } R \text{ is symmetric})$$

4.5.2 Quotients of Sets by Equivalence Relations

Let A be a set and let R be a relation on A .

00LL

DEFINITION 4.5.2 ► QUOTIENTS OF SETS BY EQUIVALENCE RELATIONS

The **quotient of X by R** is the set X/\sim_R defined by

$$X/\sim_R \stackrel{\text{def}}{=} \{[a] \in \mathcal{P}(X) \mid a \in X\}.$$

00LM

REMARK 4.5.3 ► WHY USE “EQUIVALENCE” RELATIONS FOR QUOTIENT SETS

The reason we define quotient sets for equivalence relations only is that each of the properties of being an equivalence relation—reflexivity, symmetry, and transitivity—ensures that the equivalence classes $[a]$ of X under R are well-behaved:

- *Reflexivity.* If R is reflexive, then, for each $a \in X$, we have $a \in [a]$.
- *Symmetry.* The equivalence class $[a]$ of an element a of X is defined by

$$[a] \stackrel{\text{def}}{=} \{x \in X \mid x \sim_R a\},$$

but we could equally well define

$$[a]' \stackrel{\text{def}}{=} \{x \in X \mid a \sim_R x\}$$

instead. This is not a problem when R is symmetric, as we then have $[a] = [a]'$.¹

- *Transitivity.* If R is transitive, then $[a]$ and $[b]$ are disjoint iff $a \not\sim_R b$, and equal otherwise.

¹When categorifying equivalence relations, one finds that $[a]$ and $[a]'$ correspond to presheaves and copresheaves; see [Constructions With Categories](#), ??.

00LN

PROPOSITION 4.5.4 ► PROPERTIES OF QUOTIENT SETS

Let $f: X \rightarrow Y$ be a function and let R be a relation on X .

00LP

1. *As a Coequaliser.* We have an isomorphism of sets

$$X/\sim_R^{\text{eq}} \cong \text{CoEq} \left(R \hookrightarrow X \times X \begin{array}{c} \xrightarrow{\text{pr}_1} \\ \xrightarrow{\text{pr}_2} \end{array} X \right),$$

where \sim_R^{eq} is the equivalence relation generated by \sim_R .

00LQ

2. *As a Pushout.* We have an isomorphism of sets¹

$$X/\sim_R^{\text{eq}} \cong X \amalg_{\text{Eq}(pr_1, pr_2)} X,$$

$$\begin{array}{ccc} X/\sim_R^{\text{eq}} & \longleftarrow & X \\ \uparrow & \ulcorner & \uparrow \\ X & \longleftarrow & \text{Eq}(pr_1, pr_2). \end{array}$$

where \sim_R^{eq} is the equivalence relation generated by \sim_R .

00LR

3. *The First Isomorphism Theorem for Sets.* We have an isomorphism of sets^{2,3}

$$X/\sim_{\text{Ker}(f)} \cong \text{Im}(f).$$

00LS

4. *Descending Functions to Quotient Sets, I.* Let R be an equivalence relation on X . The following conditions are equivalent:

- (a) There exists a map

$$\bar{f}: X/\sim_R \rightarrow Y$$

making the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ q \downarrow & \searrow \bar{f} & \nearrow \exists \\ X/\sim_R & & \end{array}$$

commute.

- (b) For each $x, y \in X$, if $x \sim_R y$, then $f(x) = f(y)$.

00LT

5. *Descending Functions to Quotient Sets, II.* Let R be an equivalence relation on X . If the conditions of **Item 4** hold, then \bar{f} is the *unique* map making the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ q \downarrow & \searrow \bar{f} & \nearrow \exists! \\ X/\sim_R & & \end{array}$$

commute.

00LU

6. *Descending Functions to Quotient Sets, III.* Let R be an equivalence relation on X . If the conditions of **Item 4** hold, then the following conditions are equivalent:

- (a) The map \bar{f} is an injection.
- (b) For each $x, y \in X$, we have $x \sim_R y$ iff $f(x) = f(y)$.

00LV

7. *Descending Functions to Quotient Sets, IV.* Let R be an equivalence relation on X . If the conditions of **Item 4** hold, then the following conditions are equivalent:

- (a) The map $f: X \rightarrow Y$ is surjective.
- (b) The map $\bar{f}: X/\sim_R \rightarrow Y$ is surjective.

00LW

8. *Descending Functions to Quotient Sets, V.* Let R be a relation on X and let \sim_R^{eq} be the equivalence relation associated to R . The following conditions are equivalent:

00LX

- (a) The map f satisfies the equivalent conditions of **Item 4**:
 - There exists a map

$$\bar{f}: X/\sim_R^{\text{eq}} \rightarrow Y$$

making the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ q \downarrow & \exists & \nearrow \bar{f} \\ X/\sim_R^{\text{eq}} & & \end{array}$$

commute.

- For each $x, y \in X$, if $x \sim_R^{\text{eq}} y$, then $f(x) = f(y)$.

00LY

- (b) For each $x, y \in X$, if $x \sim_R y$, then $f(x) = f(y)$.

¹Dually, we also have an isomorphism of sets

$$\text{Eq}(pr_1, pr_2) \cong X \times_{X/\sim_R^{\text{eq}}} X,$$

$$\begin{array}{ccc} \text{Eq}(pr_1, pr_2) & \longrightarrow & X \\ \downarrow & \lrcorner & \downarrow \\ X & \longrightarrow & X/\sim_R^{\text{eq}} \end{array}$$

²*Further Terminology:* The set $X/\sim_{\text{Ker}(f)}$ is often called the **coimage of f** , and denoted by $\text{Coim}(f)$.

³In a sense this is a result relating the monad in **Rel** induced by f with the comonad in **Rel** induced by f , as the kernel and image

$$\begin{aligned} \text{Ker}(f) &: X \rightrightarrows X, \\ \text{Im}(f) &\subset Y \end{aligned}$$

of f are respectively the induced monads and comonads of the adjunction

$$\left(\text{Gr}(f) \dashv f^{-1} \right): \begin{array}{ccc} & \text{Gr}(f) & \\ & \downarrow & \\ A & \begin{array}{c} \dashv \\ \vdash \end{array} & B \\ & \uparrow & \\ & f^{-1} & \end{array}$$

of Item 2 of Proposition 3.1.2.

PROOF 4.5.5 ► PROOF OF PROPOSITION 4.5.4

Item 1: As a Coequaliser

Omitted.

Item 2: As a Pushout

Omitted.

Item 3: The First Isomorphism Theorem for Sets

Clear.

Item 4: Descending Functions to Quotient Sets, I

See [Pro23c].

Item 5: Descending Functions to Quotient Sets, II

See [Pro23d].

Item 6: Descending Functions to Quotient Sets, III

See [Pro23b].

Item 7: Descending Functions to Quotient Sets, IV

See [Pro23a].

Item 8: Descending Functions to Quotient Sets, V


The implication **Item 8a** \implies **Item 8b** is clear.

Conversely, suppose that, for each $x, y \in X$, if $x \sim_R y$, then $f(x) = f(y)$. Spelling out the definition of the equivalence closure of R , we see that the condition $x \sim_R^{\text{eq}} y$ unwinds to the following:

- (★) There exist $(x_1, \dots, x_n) \in R^{\times n}$ satisfying at least one of the following conditions:
 1. The following conditions are satisfied:
 - (a) We have $x \sim_R x_1$ or $x_1 \sim_R x$;
 - (b) We have $x_i \sim_R x_{i+1}$ or $x_{i+1} \sim_R x_i$ for each $1 \leq i \leq n-1$;
 - (c) We have $y \sim_R x_n$ or $x_n \sim_R y$;
 2. We have $x = y$.

Now, if $x = y$, then $f(x) = f(y)$ trivially; otherwise, we have

$$\begin{aligned}
 f(x) &= f(x_1), \\
 f(x_1) &= f(x_2), \\
 &\vdots \\
 f(x_{n-1}) &= f(x_n), \\
 f(x_n) &= f(y),
 \end{aligned}$$

and $f(x) = f(y)$, as we wanted to show. 

00LZ 5 Functoriality of Powersets

00M0 5.1 Direct Images

Let A and B be sets and let $R: A \rightarrow B$ be a relation.

00M1

DEFINITION 5.1.1 ► DIRECT IMAGES

The **direct image function associated to R** is the function¹

$$R_* : \mathcal{P}(A) \rightarrow \mathcal{P}(B)$$

defined by^{2,3}

$$\begin{aligned} R_*(U) &\stackrel{\text{def}}{=} R(U) \\ &\stackrel{\text{def}}{=} \bigcup_{a \in U} R(a) \\ &= \left\{ b \in B \mid \begin{array}{l} \text{there exists some } a \in U \\ \text{such that } b \in R(a) \end{array} \right\} \end{aligned}$$

for each $U \in \mathcal{P}(A)$.

¹*Further Notation:* Also written $\exists_R : \mathcal{P}(A) \rightarrow \mathcal{P}(B)$. This notation comes from the fact that the following statements are equivalent, where $b \in B$ and $U \in \mathcal{P}(A)$:

- We have $b \in \exists_R(U)$.
- There exists some $a \in U$ such that $b \in f(a)$.

²*Further Terminology:* The set $R(U)$ is called the **direct image of U by R** .

³We also have

$$R_*(U) = B \setminus R_!(A \setminus U);$$

see Item 7 of Proposition 5.1.3.

00M2

REMARK 5.1.2 ► UNWINDING DEFINITION 5.1.1

Identifying subsets of A with relations from pt to A via **Constructions With Sets**, Item 7 of Proposition 4.2.3, we see that the direct image function associated to R is equivalently the function

$$R_* : \underbrace{\mathcal{P}(A)}_{\cong \text{Rel}(\text{pt}, A)} \rightarrow \underbrace{\mathcal{P}(B)}_{\cong \text{Rel}(\text{pt}, B)}$$

defined by

$$R_*(U) \stackrel{\text{def}}{=} R \diamond U$$

for each $U \in \mathcal{P}(A)$, where $R \diamond U$ is the composition

$$\text{pt} \xrightarrow{U} A \xrightarrow{R} B.$$

00M3

PROPOSITION 5.1.3 ► PROPERTIES OF DIRECT IMAGE FUNCTIONS

Let $R: A \rightarrowtail B$ be a relation.

00M4

1. *Functoriality.* The assignment $U \mapsto R_*(U)$ defines a functor

$$R_*: (\mathcal{P}(A), \subset) \rightarrow (\mathcal{P}(B), \subset)$$

where

- *Action on Objects.* For each $U \in \mathcal{P}(A)$, we have

$$[R_*](U) \stackrel{\text{def}}{=} R_*(U);$$

- *Action on Morphisms.* For each $U, V \in \mathcal{P}(A)$:

- If $U \subset V$, then $R_*(U) \subset R_*(V)$.

00M5

2. *Adjointness.* We have an adjunction

$$(R_* \dashv R_{-1}): \mathcal{P}(A) \begin{matrix} \xrightarrow{R_*} \\ \perp \\ \xleftarrow{R_{-1}} \end{matrix} \mathcal{P}(B),$$

witnessed by a bijections of sets

$$\text{Hom}_{\mathcal{P}(A)}(R_*(U), V) \cong \text{Hom}_{\mathcal{P}(A)}(U, R_{-1}(V)),$$

natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$, i.e. such that:

(★) The following conditions are equivalent:

- We have $R_*(U) \subset V$;
- We have $U \subset R_{-1}(V)$.

00M6

3. *Preservation of Colimits.* We have an equality of sets

$$R_*\left(\bigcup_{i \in I} U_i\right) = \bigcup_{i \in I} R_*(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have equalities

$$R_*(U) \cup R_*(V) = R_*(U \cup V),$$

$$R_*(\emptyset) = \emptyset,$$

natural in $U, V \in \mathcal{P}(A)$.

00M7

4. *Oplax Preservation of Limits.* We have an inclusion of sets

$$R_*\left(\bigcap_{i \in I} U_i\right) \subset \bigcap_{i \in I} R_*(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have inclusions

$$\begin{aligned} R_*(U \cap V) &\subset R_*(U) \cap R_*(V), \\ R_*(A) &\subset B, \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

00M8

5. *Symmetric Strict Monoidality With Respect to Unions.* The direct image function of **Item 1** has a symmetric strict monoidal structure

$$\left(R_*, R_*^\otimes, R_{*|\#}^\otimes\right): (\mathcal{P}(A), \cup, \emptyset) \rightarrow (\mathcal{P}(B), \cup, \emptyset),$$

being equipped with equalities

$$\begin{aligned} R_{*|U,V}^\otimes: R_*(U) \cup R_*(V) &\xrightarrow{=} R_*(U \cup V), \\ R_{*|\#}^\otimes: \emptyset &\xrightarrow{=} \emptyset, \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

00M9

6. *Symmetric Oplax Monoidality With Respect to Intersections.* The direct image function of **Item 1** has a symmetric oplax monoidal structure

$$\left(R_*, R_*^\otimes, R_{*|\#}^\otimes\right): (\mathcal{P}(A), \cap, A) \rightarrow (\mathcal{P}(B), \cap, B),$$

being equipped with inclusions

$$\begin{aligned} R_{*|U,V}^\otimes: R_*(U \cap V) &\subset R_*(U) \cap R_*(V), \\ R_{*|\#}^\otimes: R_*(A) &\subset B, \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

00MA

7. *Relation to Direct Images With Compact Support.* We have

$$R_*(U) = B \setminus R_!(A \setminus U)$$

for each $U \in \mathcal{P}(A)$.

PROOF 5.1.4 ► PROOF OF PROPOSITION 5.1.3

Item 1: Functoriality

Clear.

Item 2: Adjointness

This follows from [Kan Extensions](#), ?? of ??.

Item 3: Preservation of Colimits

This follows from [Item 2](#) and [Categories](#), ?? of ??.

Item 4: Oplax Preservation of Limits

Omitted.

Item 5: Symmetric Strict Monoidality With Respect to Unions

This follows from [Item 3](#).

Item 6: Symmetric Oplax Monoidality With Respect to Intersections

This follows from [Item 4](#).


Item 7: Relation to Direct Images With Compact Support

The proof proceeds in the same way as in the case of functions ([Constructions With Sets](#), [Item 7](#) of [Proposition 4.3.3](#)): applying [Item 7](#) of [Proposition 5.4.4](#) to $A \setminus U$, we have

$$\begin{aligned} R_!(A \setminus U) &= B \setminus R_*(A \setminus (A \setminus U)) \\ &= B \setminus R_*(U). \end{aligned}$$

Taking complements, we then obtain

$$\begin{aligned} R_*(U) &= B \setminus (B \setminus R_*(U)), \\ &= B \setminus R_!(A \setminus U), \end{aligned}$$

which finishes the proof. 

00MB

PROPOSITION 5.1.5 ► PROPERTIES OF THE DIRECT IMAGE FUNCTION OPERATION

Let $R: A \rightarrow B$ be a relation.

00MC

1. *Functionality I.* The assignment $R \mapsto R_*$ defines a function

$$(-)_* : \text{Rel}(A, B) \rightarrow \text{Sets}(\mathcal{P}(A), \mathcal{P}(B)).$$

00MD

2. *Functionality II.* The assignment $R \mapsto R_*$ defines a function

$$(-)_* : \text{Rel}(A, B) \rightarrow \text{Pos}((\mathcal{P}(A), \subset), (\mathcal{P}(B), \subset)).$$

00ME

3. *Interaction With Identities.* For each $A \in \text{Obj}(\text{Sets})$, we have¹

$$(\chi_A)_* = \text{id}_{\mathcal{P}(A)};$$

00MF

4. *Interaction With Composition.* For each pair of composable relations $R: A \rightarrowtail B$ and $S: B \rightarrowtail C$, we have²

$$(S \diamond R)_* = S_* \circ R_*,$$

$$\begin{array}{ccc} \mathcal{P}(A) & \xrightarrow{R_*} & \mathcal{P}(B) \\ & \searrow (S \diamond R)_* & \downarrow S_* \\ & & \mathcal{P}(C). \end{array}$$

¹That is, the postcomposition function

$$(\chi_A)_* : \text{Rel}(\text{pt}, A) \rightarrow \text{Rel}(\text{pt}, A)$$

is equal to $\text{id}_{\text{Rel}(\text{pt}, A)}$.

²That is, we have

$$(S \diamond R)_* = S_* \circ R_*,$$

$$\begin{array}{ccc} \text{Rel}(\text{pt}, A) & \xrightarrow{R_*} & \text{Rel}(\text{pt}, B) \\ & \searrow (S \diamond R)_* & \downarrow S_* \\ & & \text{Rel}(\text{pt}, C). \end{array}$$

PROOF 5.1.6 ► PROOF OF PROPOSITION 5.1.5

Item 1: Functionality I

Clear.

Item 2: Functionality II

Clear.

Item 3: Interaction With Identities

Indeed, we have

$$\begin{aligned}
 (\chi_A)_*(U) &\stackrel{\text{def}}{=} \bigcup_{a \in U} \chi_A(a) \\
 &\stackrel{\text{def}}{=} \bigcup_{a \in U} \{a\} \\
 &= U \\
 &\stackrel{\text{def}}{=} \text{id}_{\mathcal{P}(A)}(U)
 \end{aligned}$$

for each $U \in \mathcal{P}(A)$. Thus $(\chi_A)_* = \text{id}_{\mathcal{P}(A)}$.

Item 4: Interaction With Composition

Indeed, we have

$$\begin{aligned}
 (S \diamond R)_*(U) &\stackrel{\text{def}}{=} \bigcup_{a \in U} [S \diamond R](a) \\
 &\stackrel{\text{def}}{=} \bigcup_{a \in U} S(R(a)) \\
 &\stackrel{\text{def}}{=} \bigcup_{a \in U} S_*(R(a)) \\
 &= S_* \left(\bigcup_{a \in U} R(a) \right) \\
 &\stackrel{\text{def}}{=} S_*(R_*(U)) \\
 &\stackrel{\text{def}}{=} [S_* \circ R_*](U)
 \end{aligned}$$

for each $U \in \mathcal{P}(A)$, where we used [Item 3](#) of [Proposition 5.1.3](#). Thus $(S \diamond R)_* = S_* \circ R_*$. 

00MG 5.2 Strong Inverse Images

Let A and B be sets and let $R: A \rightarrow B$ be a relation.

00MH

DEFINITION 5.2.1 ► STRONG INVERSE IMAGES

The **strong inverse image function associated to R** is the function

$$R_{-1}: \mathcal{P}(B) \rightarrow \mathcal{P}(A)$$

defined by¹

$$R_{-1}(V) \stackrel{\text{def}}{=} \{a \in A \mid R(a) \subset V\}$$

for each $V \in \mathcal{P}(B)$.

¹Further Terminology: The set $R_{-1}(V)$ is called the **strong inverse image of V by R** .

00MJ

REMARK 5.2.2 ► UNWINDING DEFINITION 5.2.1

Identifying subsets of B with relations from pt to B via **Constructions With Sets, Item 7** of **Proposition 4.2.3**, we see that the inverse image function associated to R is equivalently the function

$$R_{-1}: \underbrace{\mathcal{P}(B)}_{\cong \text{Rel}(\text{pt}, B)} \rightarrow \underbrace{\mathcal{P}(A)}_{\cong \text{Rel}(\text{pt}, A)}$$

defined by

$$R_{-1}(V) \stackrel{\text{def}}{=} \text{Rift}_R(V),$$

and being explicitly computed by


$$\begin{aligned} R_{-1}(V) &\stackrel{\text{def}}{=} \text{Rift}_R(V) \\ &\cong \int_{x \in B} \text{Hom}_{\{\text{t}, \text{f}\}}(R_{-1}^x, V_{-2}^x), \end{aligned}$$

where we have used **Item 12** of **Proposition 2.5.1**.

PROOF 5.2.3 ► PROOF OF REMARK 5.2.2

We have

$$\begin{aligned}
 \text{Rift}_R(V) &\cong \int_{x \in B} \text{Hom}_{\{t, f\}}(R_{-1}^x, V_{-2}^x) \\
 &= \left\{ a \in A \mid \int_{x \in B} \text{Hom}_{\{t, f\}}(R_a^x, V_{\star}^x) = \text{true} \right\} \\
 &= \left\{ a \in A \mid \begin{array}{l} \text{for each } x \in B, \text{ at least one of the} \\ \text{following conditions hold:} \\ \begin{array}{l} 1. \text{ We have } R_a^x = \text{false} \\ 2. \text{ The following conditions hold:} \\ \begin{array}{l} (a) \text{ We have } R_a^x = \text{true} \\ (b) \text{ We have } V_{\star}^x = \text{true} \end{array} \end{array} \end{array} \right\} \\
 &= \left\{ a \in A \mid \begin{array}{l} \text{for each } x \in B, \text{ at least one of the} \\ \text{following conditions hold:} \\ \begin{array}{l} 1. \text{ We have } x \notin R(a) \\ 2. \text{ The following conditions hold:} \\ \begin{array}{l} (a) \text{ We have } x \in R(a) \\ (b) \text{ We have } x \in V \end{array} \end{array} \end{array} \right\} \\
 &= \{a \in A \mid \text{for each } x \in R(a), \text{ we have } x \in V\} \\
 &= \{a \in A \mid R(a) \subset V\} \\
 &\stackrel{\text{def}}{=} R_{-1}(V).
 \end{aligned}$$

This finishes the proof. 

00MK

PROPOSITION 5.2.4 ► PROPERTIES OF STRONG INVERSE IMAGES

Let $R: A \dashrightarrow B$ be a relation.

00ML

1. *Functoriality.* The assignment $V \mapsto R_{-1}(V)$ defines a functor

$$R_{-1}: (\mathcal{P}(B), \subset) \rightarrow (\mathcal{P}(A), \subset)$$

where

- *Action on Objects.* For each $V \in \mathcal{P}(B)$, we have

$$[R_{-1}](V) \stackrel{\text{def}}{=} R_{-1}(V);$$

- *Action on Morphisms.* For each $U, V \in \mathcal{P}(B)$:
 - If $U \subset V$, then $R_{-1}(U) \subset R_{-1}(V)$.

00MM

2. *Adjointness.* We have an adjunction

$$(R_* \dashv R_{-1}) : \mathcal{P}(A) \begin{matrix} \xrightarrow{R_*} \\ \perp \\ \xleftarrow{R_{-1}} \end{matrix} \mathcal{P}(B),$$

witnessed by a bijections of sets

$$\text{Hom}_{\mathcal{P}(A)}(R_*(U), V) \cong \text{Hom}_{\mathcal{P}(A)}(U, R_{-1}(V)),$$

natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$, i.e. such that:

- (★) The following conditions are equivalent:

- We have $R_*(U) \subset V$;
- We have $U \subset R_{-1}(V)$.

00MN

3. *Lax Preservation of Colimits.* We have an inclusion of sets

$$\bigcup_{i \in I} R_{-1}(U_i) \subset R_{-1}\left(\bigcup_{i \in I} U_i\right),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(B)^{\times I}$. In particular, we have inclusions

$$\begin{aligned} R_{-1}(U) \cup R_{-1}(V) &\subset R_{-1}(U \cup V), \\ \emptyset &\subset R_{-1}(\emptyset), \end{aligned}$$

natural in $U, V \in \mathcal{P}(B)$.

00MP

4. *Preservation of Limits.* We have an equality of sets

$$R_{-1}\left(\bigcap_{i \in I} U_i\right) = \bigcap_{i \in I} R_{-1}(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(B)^{\times I}$. In particular, we have equalities

$$\begin{aligned} R_{-1}(U \cap V) &= R_{-1}(U) \cap R_{-1}(V), \\ R_{-1}(B) &= B, \end{aligned}$$

natural in $U, V \in \mathcal{P}(B)$.

00MQ

5. *Symmetric Lax Monoidality With Respect to Unions.* The direct image with compact support function of **Item 1** has a symmetric lax monoidal structure

$$\left(R_{-1}, R_{-1}^{\otimes}, R_{-1|*}^{\otimes} \right): (\mathcal{P}(A), \cup, \emptyset) \rightarrow (\mathcal{P}(B), \cup, \emptyset),$$

being equipped with inclusions

$$\begin{aligned} R_{-1|U,V}^{\otimes}: R_{-1}(U) \cup R_{-1}(V) &\subset R_{-1}(U \cup V), \\ R_{-1|*}^{\otimes}: \emptyset &\subset R_{-1}(\emptyset), \end{aligned}$$

natural in $U, V \in \mathcal{P}(B)$.

00MR

6. *Symmetric Strict Monoidality With Respect to Intersections.* The direct image function of **Item 1** has a symmetric strict monoidal structure

$$\left(R_{-1}, R_{-1}^{\otimes}, R_{-1|*}^{\otimes} \right): (\mathcal{P}(A), \cap, A) \rightarrow (\mathcal{P}(B), \cap, B),$$

being equipped with equalities

$$\begin{aligned} R_{-1|U,V}^{\otimes}: R_{-1}(U \cap V) &\xrightarrow{=} R_{-1}(U) \cap R_{-1}(V), \\ R_{-1|*}^{\otimes}: R_{-1}(A) &\xrightarrow{=} B, \end{aligned}$$

natural in $U, V \in \mathcal{P}(B)$.

00MS

7. *Interaction With Weak Inverse Images I.* We have

$$R_{-1}(V) = A \setminus R^{-1}(B \setminus V)$$

for each $V \in \mathcal{P}(B)$.

00MT

8. *Interaction With Weak Inverse Images II.* Let $R: A \rightarrowtail B$ be a relation from A to B .

00MU

(a) If R is a total relation, then we have an inclusion of sets

$$R_{-1}(V) \subset R^{-1}(V)$$

natural in $V \in \mathcal{P}(B)$.

00MV

(b) If R is total and functional, then the above inclusion is in fact an equality.

00MW

(c) Conversely, if we have $R_{-1} = R^{-1}$, then R is total and functional.

PROOF 5.2.5 ► PROOF OF PROPOSITION 5.2.4

Item 1: Functoriality

Clear.

Item 2: Adjointness

This follows from [Kan Extensions](#), ?? of ??.

Item 3: Lax Preservation of Colimits

Omitted.

Item 4: Preservation of Limits

This follows from [Item 2](#) and [Categories](#), ?? of ??.

Item 5: Symmetric Lax Monoidality With Respect to Unions

This follows from [Item 3](#).

Item 6: Symmetric Strict Monoidality With Respect to Intersections

This follows from [Item 4](#).

Item 7: Interaction With Weak Inverse Images I

We claim we have an equality

$$R_{-1}(B \setminus V) = A \setminus R^{-1}(V).$$

Indeed, we have

$$R_{-1}(B \setminus V) = \{a \in A \mid R(a) \subset B \setminus V\},$$

$$A \setminus R^{-1}(V) = \{a \in A \mid R(a) \cap V = \emptyset\}.$$

Taking $V = B \setminus V$ then implies the original statement.

Item 8: Interaction With Weak Inverse Images II

Item 8a is clear, while Items 8b and 8c follow from Item 6 of Proposition 3.1.2. 

00MX

PROPOSITION 5.2.6 ► PROPERTIES OF THE STRONG INVERSE IMAGE FUNCTION OPERATION

Let $R: A \rightarrowtail B$ be a relation.

00MY

1. *Functionality I.* The assignment $R \mapsto R_{-1}$ defines a function

$$(-)_{-1}: \text{Sets}(A, B) \rightarrow \text{Sets}(\mathcal{P}(A), \mathcal{P}(B)).$$

00MZ

2. *Functionality II.* The assignment $R \mapsto R_{-1}$ defines a function

$$(-)_{-1}: \text{Sets}(A, B) \rightarrow \text{Pos}((\mathcal{P}(A), \subset), (\mathcal{P}(B), \subset)).$$

00N0

3. *Interaction With Identities.* For each $A \in \text{Obj}(\text{Sets})$, we have

$$(\text{id}_A)_{-1} = \text{id}_{\mathcal{P}(A)};$$

00N1

4. *Interaction With Composition.* For each pair of composable relations $R: A \rightarrowtail B$ and $S: B \rightarrowtail C$, we have

$$(S \diamond R)_{-1} = R_{-1} \circ S_{-1},$$

$$\begin{array}{ccc} \mathcal{P}(C) & \xrightarrow{S_{-1}} & \mathcal{P}(B) \\ & \searrow (S \diamond R)_{-1} & \downarrow R_{-1} \\ & & \mathcal{P}(A). \end{array}$$

PROOF 5.2.7 ► PROOF OF PROPOSITION 5.2.6

Item 1: Functionality I

Clear.

Item 2: Functionality II

Clear.

Item 3: Interaction With Identities

Indeed, we have

$$\begin{aligned} (\chi_A)_{-1}(U) &\stackrel{\text{def}}{=} \{a \in A \mid \chi_A(a) \subset U\} \\ &\stackrel{\text{def}}{=} \{a \in A \mid \{a\} \subset U\} \\ &= U \end{aligned}$$

for each $U \in \mathcal{P}(A)$. Thus $(\chi_A)_{-1} = \text{id}_{\mathcal{P}(A)}$.


Item 4: Interaction With Composition

Indeed, we have

$$\begin{aligned} (S \diamond R)_{-1}(U) &\stackrel{\text{def}}{=} \{a \in A \mid [S \diamond R](a) \subset U\} \\ &\stackrel{\text{def}}{=} \{a \in A \mid S(R(a)) \subset U\} \\ &\stackrel{\text{def}}{=} \{a \in A \mid S_*(R(a)) \subset U\} \\ &= \{a \in A \mid R(a) \subset S_{-1}(U)\} \\ &\stackrel{\text{def}}{=} R_{-1}(S_{-1}(U)) \\ &\stackrel{\text{def}}{=} [R_{-1} \circ S_{-1}](U) \end{aligned}$$

for each $U \in \mathcal{P}(C)$, where we used [Item 2](#) of [Proposition 5.2.4](#), which implies that the conditions

- We have $S_*(R(a)) \subset U$;
- We have $R(a) \subset S_{-1}(U)$;

are equivalent. Thus $(S \diamond R)_{-1} = R_{-1} \circ S_{-1}$. 

00N2 5.3 Weak Inverse Images

Let A and B be sets and let $R: A \rightarrow B$ be a relation.

00N3

DEFINITION 5.3.1 ► WEAK INVERSE IMAGES

The **weak inverse image function associated to R** is the function

$$R^{-1}: \mathcal{P}(B) \rightarrow \mathcal{P}(A)$$

defined by²

$$R^{-1}(V) \stackrel{\text{def}}{=} \{a \in A \mid R(a) \cap V \neq \emptyset\}$$

for each $V \in \mathcal{P}(B)$.

¹Further Terminology: Also called simply the **inverse image function associated to R** .

²Further Terminology: The set $R^{-1}(V)$ is called the **weak inverse image of V by R** or simply the **inverse image of V by R** .

00N4

REMARK 5.3.2 ► UNWINDING DEFINITION 5.3.1

Identifying subsets of B with relations from B to pt via **Constructions With Sets, Item 7** of **Proposition 4.2.3**, we see that the weak inverse image function associated to R is equivalently the function

$$R^{-1}: \underbrace{\mathcal{P}(B)}_{\cong \text{Rel}(B, \text{pt})} \rightarrow \underbrace{\mathcal{P}(A)}_{\cong \text{Rel}(A, \text{pt})}$$

defined by

$$R^{-1}(V) \stackrel{\text{def}}{=} V \diamond R$$

for each $V \in \mathcal{P}(A)$, where $R \diamond V$ is the composition

$$A \xrightarrow{R} B \xrightarrow{V} \text{pt}.$$


Explicitly, we have

$$\begin{aligned} R^{-1}(V) &\stackrel{\text{def}}{=} V \diamond R \\ &\stackrel{\text{def}}{=} \int^{x \in B} V_x^{-1} \times R_{-2}^x. \end{aligned}$$

PROOF 5.3.3 ► PROOF OF REMARK 5.3.2

We have

$$\begin{aligned}
 V \diamond R &\stackrel{\text{def}}{=} \int^{x \in B} V_x^{-1} \times R_{-2}^x \\
 &= \left\{ a \in A \mid \int^{x \in B} V_x^\star \times R_a^x = \text{true} \right\} \\
 &= \left\{ a \in A \mid \begin{array}{l} \text{there exists } x \in B \text{ such that the} \\ \text{following conditions hold:} \\ 1. \text{ We have } V_x^\star = \text{true} \\ 2. \text{ We have } R_a^x = \text{true} \end{array} \right\} \\
 &= \left\{ a \in A \mid \begin{array}{l} \text{there exists } x \in B \text{ such that the} \\ \text{following conditions hold:} \\ 1. \text{ We have } x \in V \\ 2. \text{ We have } x \in R(a) \end{array} \right\} \\
 &= \{ a \in A \mid \text{there exists } x \in V \text{ such that } x \in R(a) \} \\
 &= \{ a \in A \mid R(a) \cap V \neq \emptyset \} \\
 &\stackrel{\text{def}}{=} R^{-1}(V)
 \end{aligned}$$

This finishes the proof. 

00N5

PROPOSITION 5.3.4 ► PROPERTIES OF WEAK INVERSE IMAGE FUNCTIONS

Let $R: A \dashv B$ be a relation.

00N6

1. *Functoriality.* The assignment $V \mapsto R^{-1}(V)$ defines a functor

$$R^{-1}: (\mathcal{P}(B), \subset) \rightarrow (\mathcal{P}(A), \subset)$$

where

- *Action on Objects.* For each $V \in \mathcal{P}(B)$, we have

$$[R^{-1}](V) \stackrel{\text{def}}{=} R^{-1}(V);$$

• *Action on Morphisms.* For each $U, V \in \mathcal{P}(B)$:

- If $U \subset V$, then $R^{-1}(U) \subset R^{-1}(V)$.

00N7

2. *Adjointness.* We have an adjunction

$$(R^{-1} \dashv R_!): \mathcal{P}(B) \begin{matrix} \xrightarrow{R^{-1}} \\ \perp \\ \xleftarrow{R_!} \end{matrix} \mathcal{P}(A),$$

witnessed by a bijections of sets

$$\mathrm{Hom}_{\mathcal{P}(A)}(R^{-1}(U), V) \cong \mathrm{Hom}_{\mathcal{P}(A)}(U, R_!(V)),$$

natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$, i.e. such that:

(★) The following conditions are equivalent:

- We have $R^{-1}(U) \subset V$;
- We have $U \subset R_!(V)$.

00N8

3. *Preservation of Colimits.* We have an equality of sets

$$R^{-1}\left(\bigcup_{i \in I} U_i\right) = \bigcup_{i \in I} R^{-1}(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(B)^{\times I}$. In particular, we have equalities

$$\begin{aligned} R^{-1}(U) \cup R^{-1}(V) &= R^{-1}(U \cup V), \\ R^{-1}(\emptyset) &= \emptyset, \end{aligned}$$

natural in $U, V \in \mathcal{P}(B)$.

00N9

4. *Oplax Preservation of Limits.* We have an inclusion of sets

$$R^{-1}\left(\bigcap_{i \in I} U_i\right) \subset \bigcap_{i \in I} R^{-1}(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(B)^{\times I}$. In particular, we have inclusions

$$\begin{aligned} R^{-1}(U \cap V) &\subset R^{-1}(U) \cap R^{-1}(V), \\ R^{-1}(A) &\subset B, \end{aligned}$$

natural in $U, V \in \mathcal{P}(B)$.

00NA

5. *Symmetric Strict Monoidality With Respect to Unions.* The direct image function of **Item 1** has a symmetric strict monoidal structure

$$\left(R^{-1}, R^{-1, \otimes}, R_{\#}^{-1, \otimes}\right): (\mathcal{P}(A), \cup, \emptyset) \rightarrow (\mathcal{P}(B), \cup, \emptyset),$$

being equipped with equalities

$$R_{U,V}^{-1, \otimes}: R^{-1}(U) \cup R^{-1}(V) \xrightarrow{=} R^{-1}(U \cup V),$$

$$R_{\#}^{-1, \otimes}: \emptyset \xrightarrow{=} \emptyset,$$

natural in $U, V \in \mathcal{P}(B)$.

00NB

6. *Symmetric Oplax Monoidality With Respect to Intersections.* The direct image function of **Item 1** has a symmetric oplax monoidal structure

$$\left(R^{-1}, R^{-1, \otimes}, R_{\#}^{-1, \otimes}\right): (\mathcal{P}(A), \cap, A) \rightarrow (\mathcal{P}(B), \cap, B),$$

being equipped with inclusions

$$R_{U,V}^{-1, \otimes}: R^{-1}(U \cap V) \subset R^{-1}(U) \cap R^{-1}(V),$$

$$R_{\#}^{-1, \otimes}: R^{-1}(A) \subset B,$$

natural in $U, V \in \mathcal{P}(B)$.

00NC

7. *Interaction With Strong Inverse Images I.* We have

$$R^{-1}(V) = A \setminus R_{-1}(B \setminus V)$$

for each $V \in \mathcal{P}(B)$.

00ND

8. *Interaction With Strong Inverse Images II.* Let $R: A \rightarrowtail B$ be a relation from A to B .

00NE

- (a) If R is a total relation, then we have an inclusion of sets

$$R_{-1}(V) \subset R^{-1}(V)$$

natural in $V \in \mathcal{P}(B)$.

00NF

- (b) If R is total and functional, then the above inclusion is in fact an equality.

00NG

- (c) Conversely, if we have $R_{-1} = R^{-1}$, then R is total and functional.

PROOF 5.3.5 ► PROOF OF PROPOSITION 5.3.4

Item 1: Functoriality

Clear.

Item 2: Adjointness

This follows from [Kan Extensions](#), ?? of ??.

Item 3: Preservation of Colimits

This follows from [Item 2](#) and [Categories](#), ?? of ??.

Item 4: Oplax Preservation of Limits

Omitted.

Item 5: Symmetric Strict Monoidality With Respect to Unions

This follows from [Item 3](#).

Item 6: Symmetric Oplax Monoidality With Respect to Intersections

This follows from [Item 4](#).

Item 7: Interaction With Strong Inverse Images I

This follows from [Item 7](#) of [Proposition 5.2.4](#).

Item 8: Interaction With Strong Inverse Images II

This was proved in [Item 8](#) of [Proposition 5.2.4](#). 

00NH

PROPOSITION 5.3.6 ► PROPERTIES OF THE WEAK INVERSE IMAGE FUNCTION OPERATION

Let $R: A \dashv B$ be a relation.

00NJ

- 1.
- Functionality I*
- . The assignment
- $R \mapsto R^{-1}$
- defines a function

$$(-)^{-1}: \text{Rel}(A, B) \rightarrow \text{Sets}(\mathcal{P}(A), \mathcal{P}(B)).$$

00NK

- 2.
- Functionality II*
- . The assignment
- $R \mapsto R^{-1}$
- defines a function

$$(-)^{-1}: \text{Rel}(A, B) \rightarrow \text{Pos}((\mathcal{P}(A), \subset), (\mathcal{P}(B), \subset)).$$

00NL

- 3.
- Interaction With Identities*
- . For each
- $A \in \text{Obj}(\text{Sets})$
- , we have
- ¹

$$(\chi_A)^{-1} = \text{id}_{\mathcal{P}(A)};$$

00NM

4. *Interaction With Composition.* For each pair of composable relations $R: A \rightarrowtail B$ and $S: B \rightarrowtail C$, we have²

$$(S \diamond R)^{-1} = R^{-1} \circ S^{-1},$$

$$\begin{array}{ccc} \mathcal{P}(C) & \xrightarrow{S^{-1}} & \mathcal{P}(B) \\ & \searrow (S \diamond R)^{-1} & \downarrow R^{-1} \\ & & \mathcal{P}(A). \end{array}$$

¹That is, the postcomposition

$$(\chi_A)^{-1}: \text{Rel}(\text{pt}, A) \rightarrow \text{Rel}(\text{pt}, A)$$

is equal to $\text{id}_{\text{Rel}(\text{pt}, A)}$.

²That is, we have

$$(S \diamond R)^{-1} = R^{-1} \circ S^{-1},$$

$$\begin{array}{ccc} \text{Rel}(\text{pt}, C) & \xrightarrow{R^{-1}} & \text{Rel}(\text{pt}, B) \\ & \searrow (S \diamond R)^{-1} & \downarrow S^{-1} \\ & & \text{Rel}(\text{pt}, A). \end{array}$$

PROOF 5.3.7 ► PROOF OF PROPOSITION 5.3.6

Item 1: Functionality I

Clear.

Item 2: Functionality II

Clear.

Item 3: Interaction With Identities

This follows from **Categories**, ?? of ??.

Item 4: Interaction With Composition

This follows from **Categories**, ?? of ??.



00NN 5.4 Direct Images With Compact Support

Let A and B be sets and let $R: A \rightarrowtail B$ be a relation.

00NP

DEFINITION 5.4.1 ► DIRECT IMAGES WITH COMPACT SUPPORT

The **direct image with compact support function associated to R** is the function¹

$$R_! : \mathcal{P}(A) \rightarrow \mathcal{P}(B)$$

defined by^{2,3}

$$\begin{aligned} R_!(U) &\stackrel{\text{def}}{=} \left\{ b \in B \mid \begin{array}{l} \text{for each } a \in A, \text{ if we have} \\ b \in R(a), \text{ then } a \in U \end{array} \right\} \\ &= \{ b \in B \mid R^{-1}(b) \subset U \} \end{aligned}$$

for each $U \in \mathcal{P}(A)$.

¹*Further Notation:* Also written $\forall_R : \mathcal{P}(A) \rightarrow \mathcal{P}(B)$. This notation comes from the fact that the following statements are equivalent, where $b \in B$ and $U \in \mathcal{P}(A)$:

- We have $b \in \forall_R(U)$.
- For each $a \in A$, if $b \in R(a)$, then $a \in U$.

²*Further Terminology:* The set $R_!(U)$ is called the **direct image with compact support of U by R** .

³We also have

$$R_!(U) = B \setminus R_*(A \setminus U);$$

see Item 7 of Proposition 5.4.4.

00NQ

REMARK 5.4.2 ► UNWINDING DEFINITION 5.4.1

Identifying subsets of B with relations from pt to B via **Constructions With Sets, Item 7 of Proposition 4.2.3**, we see that the direct image with compact support function associated to R is equivalently the function

$$R_! : \underbrace{\mathcal{P}(A)}_{\cong \text{Rel}(A, \text{pt})} \rightarrow \underbrace{\mathcal{P}(B)}_{\cong \text{Rel}(B, \text{pt})}$$

defined by

$$R_!(U) \stackrel{\text{def}}{=} \text{Ran}_R(U),$$

being explicitly computed by


$$\begin{aligned} R^*(U) &\stackrel{\text{def}}{=} \text{Ran}_R(U) \\ &\cong \int_{a \in A} \text{Hom}_{\{t,f\}}(R_a^{-2}, U_a^{-1}), \end{aligned}$$

where we have used [Item 11](#) of [Proposition 2.5.1](#).

PROOF 5.4.3 ► PROOF OF REMARK 5.4.2

We have

$$\begin{aligned} \text{Ran}_R(V) &\cong \int_{a \in A} \text{Hom}_{\{t,f\}}(R_a^{-2}, U_a^{-1}) \\ &= \left\{ b \in B \mid \int_{a \in A} \text{Hom}_{\{t,f\}}(R_a^b, U_a^\star) = \text{true} \right\} \\ &= \left\{ b \in B \mid \begin{array}{l} \text{for each } a \in A, \text{ at least one of the} \\ \text{following conditions hold:} \\ \begin{array}{l} 1. \text{ We have } R_a^b = \text{false} \\ 2. \text{ The following conditions hold:} \\ \begin{array}{l} (a) \text{ We have } R_a^b = \text{true} \\ (b) \text{ We have } U_a^\star = \text{true} \end{array} \end{array} \end{array} \right\} \\ &= \left\{ b \in B \mid \begin{array}{l} \text{for each } a \in A, \text{ at least one of the} \\ \text{following conditions hold:} \\ \begin{array}{l} 1. \text{ We have } b \notin R(a) \\ 2. \text{ The following conditions hold:} \\ \begin{array}{l} (a) \text{ We have } b \in R(a) \\ (b) \text{ We have } a \in U \end{array} \end{array} \end{array} \right\} \\ &= \left\{ b \in B \mid \begin{array}{l} \text{for each } a \in A, \text{ if we have} \\ b \in R(a), \text{ then } a \in U \end{array} \right\} \\ &= \{ b \in B \mid R^{-1}(b) \subset U \} \\ &\stackrel{\text{def}}{=} R^{-1}(U). \end{aligned}$$

This finishes the proof. 

00NR

PROPOSITION 5.4.4 ► PROPERTIES OF DIRECT IMAGES WITH COMPACT SUPPORT

Let $R: A \rightarrow B$ be a relation.

00NS

1. *Functoriality.* The assignment $U \mapsto R_!(U)$ defines a functor

$$R_!: (\mathcal{P}(A), \subset) \rightarrow (\mathcal{P}(B), \subset)$$

where

- *Action on Objects.* For each $U \in \mathcal{P}(A)$, we have

$$[R_!](U) \stackrel{\text{def}}{=} R_!(U);$$

- *Action on Morphisms.* For each $U, V \in \mathcal{P}(A)$:
 - If $U \subset V$, then $R_!(U) \subset R_!(V)$.

00NT

2. *Adjointness.* We have an adjunction

$$(R^{-1} \dashv R_!): \mathcal{P}(B) \begin{matrix} \xrightarrow{R^{-1}} \\ \perp \\ \xleftarrow{R_!} \end{matrix} \mathcal{P}(A),$$

witnessed by a bijections of sets

$$\text{Hom}_{\mathcal{P}(A)}(R^{-1}(U), V) \cong \text{Hom}_{\mathcal{P}(A)}(U, R_!(V)),$$

natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$, i.e. such that:

(★) The following conditions are equivalent:

- We have $R^{-1}(U) \subset V$;
- We have $U \subset R_!(V)$.

00NU

3. *Lax Preservation of Colimits.* We have an inclusion of sets

$$\bigcup_{i \in I} R_!(U_i) \subset R_!\left(\bigcup_{i \in I} U_i\right),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have inclusions

$$\begin{aligned} R_!(U) \cup R_!(V) &\subset R_!(U \cup V), \\ \emptyset &\subset R_!(\emptyset), \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

00NV

4. *Preservation of Limits.* We have an equality of sets

$$R_!\left(\bigcap_{i \in I} U_i\right) = \bigcap_{i \in I} R_!(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have equalities

$$\begin{aligned} R_!(U \cap V) &= R_!(U) \cap R_!(V), \\ R_!(A) &= B, \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

00NW

5. *Symmetric Lax Monoidality With Respect to Unions.* The direct image with compact support function of **Item 1** has a symmetric lax monoidal structure

$$(R_!, R_!^\otimes, R_{!|_{\mathcal{P}}}^\otimes) : (\mathcal{P}(A), \cup, \emptyset) \rightarrow (\mathcal{P}(B), \cup, \emptyset),$$

being equipped with inclusions

$$\begin{aligned} R_{!|_{U,V}}^\otimes : R_!(U) \cup R_!(V) &\subset R_!(U \cup V), \\ R_{!|_{\mathcal{P}}}^\otimes : \emptyset &\subset R_!(\emptyset), \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

00NX

6. *Symmetric Strict Monoidality With Respect to Intersections.* The direct image function of **Item 1** has a symmetric strict monoidal structure

$$(R_!, R_!^\otimes, R_{!|_{\mathcal{P}}}^\otimes) : (\mathcal{P}(A), \cap, A) \rightarrow (\mathcal{P}(B), \cap, B),$$

being equipped with equalities

$$\begin{aligned} R_{!|_{U,V}}^\otimes : R_!(U \cap V) &\xrightarrow{=} R_!(U) \cap R_!(V), \\ R_{!|_{\mathcal{P}}}^\otimes : R_!(A) &\xrightarrow{=} B, \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

00NY

7. *Relation to Direct Images.* We have

$$R_!(U) = B \setminus R_*(A \setminus U)$$

for each $U \in \mathcal{P}(A)$.

PROOF 5.4.5 ► PROOF OF PROPOSITION 5.4.4

Item 1: Functoriality

Clear.

Item 2: Adjointness

This follows from **Kan Extensions**, ?? of ??.

Item 3: Lax Preservation of Colimits

Omitted.

Item 4: Preservation of Limits

This follows from **Item 2** and **Categories**, ?? of ??.

Item 5: Symmetric Lax Monoidality With Respect to Unions

This follows from **Item 3**.

Item 6: Symmetric Strict Monoidality With Respect to Intersections

This follows from **Item 4**.

Item 7: Relation to Direct Images

This follows from **Item 7** of **Proposition 5.1.3**. Alternatively, we may prove it directly as follows, with the proof proceeding in the same way as in the case of functions (**Constructions With Sets**, **Item 7** of **Proposition 4.5.5**).

We claim that $R_!(U) = B \setminus R_*(A \setminus U)$:

• *The First Implication.* We claim that

$$R_!(U) \subset B \setminus R_*(A \setminus U).$$

Let $b \in R_!(U)$. We need to show that $b \notin R_*(A \setminus U)$, i.e. that there is no $a \in A \setminus U$ such that $b \in R(a)$.

This is indeed the case, as otherwise we would have $a \in R^{-1}(b)$ and $a \notin U$, contradicting $R^{-1}(b) \subset U$ (which holds since $b \in R_!(U)$).

Thus $b \in B \setminus R_*(A \setminus U)$.


• *The Second Implication.* We claim that

$$B \setminus R_*(A \setminus U) \subset R_!(U).$$

Let $b \in B \setminus R_*(A \setminus U)$. We need to show that $b \in R_!(U)$, i.e. that $R^{-1}(b) \subset U$.

Since $b \notin R_*(A \setminus U)$, there exists no $a \in A \setminus U$ such that $b \in R(a)$, and hence $R^{-1}(b) \subset U$.

Thus $b \in R_!(U)$.

This finishes the proof. 

PROPOSITION 5.4.6 ► PROPERTIES OF THE DIRECT IMAGE WITH COMPACT SUPPORT FUNCTION OPERATION

Let $R: A \rightarrowtail B$ be a relation.

1. *Functionality I.* The assignment $R \mapsto R_!$ defines a function

$$(-)_!: \text{Sets}(A, B) \rightarrow \text{Sets}(\mathcal{P}(A), \mathcal{P}(B)).$$

2. *Functionality II.* The assignment $R \mapsto R_!$ defines a function

$$(-)_!: \text{Sets}(A, B) \rightarrow \text{Hom}_{\text{Pos}}((\mathcal{P}(A), \subset), (\mathcal{P}(B), \subset)).$$

3. *Interaction With Identities.* For each $A \in \text{Obj}(\text{Sets})$, we have

$$(\text{id}_A)_! = \text{id}_{\mathcal{P}(A)};$$

4. *Interaction With Composition.* For each pair of composable relations $R: A \rightarrowtail B$ and $S: B \rightarrowtail C$, we have

$$(S \diamond R)_! = S_! \circ R_!,$$

$$\begin{array}{ccc} \mathcal{P}(A) & \xrightarrow{R_!} & \mathcal{P}(B) \\ & \searrow (S \diamond R)_! & \downarrow S_! \\ & & \mathcal{P}(C). \end{array}$$

PROOF 5.4.7 ► PROOF OF PROPOSITION 5.4.6

Item 1: Functionality I

Clear.

Item 2: Functionality II

Clear.

Item 3: Interaction With Identities

Indeed, we have

$$\begin{aligned} (\chi_A)_!(U) &\stackrel{\text{def}}{=} \{a \in A \mid \chi_A^{-1}(a) \subset U\} \\ &\stackrel{\text{def}}{=} \{a \in A \mid \{a\} \subset U\} \\ &= U \end{aligned}$$

for each $U \in \mathcal{P}(A)$. Thus $(\chi_A)_! = \text{id}_{\mathcal{P}(A)}$.

Item 4: Interaction With Composition

Indeed, we have

$$\begin{aligned} (S \diamond R)_!(U) &\stackrel{\text{def}}{=} \{c \in C \mid [S \diamond R]^{-1}(c) \subset U\} \\ &\stackrel{\text{def}}{=} \{c \in C \mid S^{-1}(R^{-1}(c)) \subset U\} \\ &= \{c \in C \mid R^{-1}(c) \subset S_!(U)\} \\ &\stackrel{\text{def}}{=} R_!(S_!(U)) \\ &\stackrel{\text{def}}{=} [R_! \circ S_!](U) \end{aligned}$$

for each $U \in \mathcal{P}(C)$, where we used **Item 2** of **Proposition 5.4.4**, which implies that the conditions

- We have $S^{-1}(R^{-1}(c)) \subset U$;
- We have $R^{-1}(c) \subset S_!(U)$;

are equivalent. Thus $(S \diamond R)_! = S_! \circ R_!$.



00P5

PROPOSITION 5.5.1 ► FUNCTORIALITY OF POWERSETS I

The assignment $X \mapsto \mathcal{P}(X)$ defines functors¹

$$\begin{aligned}\mathcal{P}_* &: \text{Rel} \rightarrow \text{Sets}, \\ \mathcal{P}_{-1} &: \text{Rel}^{\text{op}} \rightarrow \text{Sets}, \\ \mathcal{P}^{-1} &: \text{Rel}^{\text{op}} \rightarrow \text{Sets}, \\ \mathcal{P}_! &: \text{Rel} \rightarrow \text{Sets}\end{aligned}$$

where

- *Action on Objects.* For each $A \in \text{Obj}(\text{Rel})$, we have

$$\begin{aligned}\mathcal{P}_*(A) &\stackrel{\text{def}}{=} \mathcal{P}(A), \\ \mathcal{P}_{-1}(A) &\stackrel{\text{def}}{=} \mathcal{P}(A), \\ \mathcal{P}^{-1}(A) &\stackrel{\text{def}}{=} \mathcal{P}(A), \\ \mathcal{P}_!(A) &\stackrel{\text{def}}{=} \mathcal{P}(A);\end{aligned}$$

- *Action on Morphisms.* For each morphism $R: A \rightarrowtail B$ of Rel , the images

$$\begin{aligned}\mathcal{P}_*(R) &: \mathcal{P}(A) \rightarrow \mathcal{P}(B), \\ \mathcal{P}_{-1}(R) &: \mathcal{P}(B) \rightarrow \mathcal{P}(A), \\ \mathcal{P}^{-1}(R) &: \mathcal{P}(B) \rightarrow \mathcal{P}(A), \\ \mathcal{P}_!(R) &: \mathcal{P}(A) \rightarrow \mathcal{P}(B)\end{aligned}$$

of R by \mathcal{P}_* , \mathcal{P}_{-1} , \mathcal{P}^{-1} , and $\mathcal{P}_!$ are defined by

$$\begin{aligned}\mathcal{P}_*(R) &\stackrel{\text{def}}{=} R_*, \\ \mathcal{P}_{-1}(R) &\stackrel{\text{def}}{=} R_{-1}, \\ \mathcal{P}^{-1}(R) &\stackrel{\text{def}}{=} R^{-1}, \\ \mathcal{P}_!(R) &\stackrel{\text{def}}{=} R_!,\end{aligned}$$

as in [Definitions 5.1.1, 5.2.1, 5.3.1](#) and [5.4.1](#).

¹The functor $\mathcal{P}_*: \text{Rel} \rightarrow \text{Sets}$ admits a left adjoint; see [Item 3](#) of [Proposition 3.1.2](#).

PROOF 5.5.2 ► PROOF OF PROPOSITION 5.5.1

This follows from **Items 3 and 4** of **Proposition 5.1.5**, **Items 3 and 4** of **Proposition 5.2.6**, **Items 3 and 4** of **Proposition 5.3.6**, and **Items 3 and 4** of **Proposition 5.4.6**.



00P6 5.6 Functoriality of Powersets: Relations on Powersets

Let A and B be sets and let $R: A \rightarrow B$ be a relation.

00P7 DEFINITION 5.6.1 ► THE RELATION ON POWERSETS ASSOCIATED TO A RELATION

The **relation on powersets associated to R** is the relation

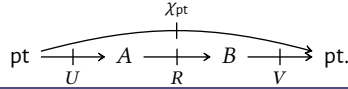
$$\mathcal{P}(R): \mathcal{P}(A) \rightarrow \mathcal{P}(B)$$

defined by¹

$$\mathcal{P}(R)_U^V \stackrel{\text{def}}{=} \mathbf{Rel}(\chi_{\text{pt}}, V \diamond R \diamond U)$$

for each $U \in \mathcal{P}(A)$ and each $V \in \mathcal{P}(B)$.

¹Illustration:



00P8 REMARK 5.6.2 ► UNWINDING DEFINITION 5.6.1

In detail, we have $U \sim_{\mathcal{P}(R)} V$ iff the following equivalent conditions hold:

- We have $\chi_{\text{pt}} \subset V \diamond R \diamond U$.
- We have $(V \diamond R \diamond U)_{\star}^{\star} = \text{true}$, i.e. we have

$$\int^{a \in A} \int^{b \in B} V_b^{\star} \times R_a^b \times U_a^{\star} = \text{true}.$$

- There exists some $a \in A$ and some $b \in B$ such that:
 - We have $U_a^{\star} = \text{true}$;
 - We have $R_a^b = \text{true}$;
 - We have $V_b^{\star} = \text{true}$.

· There exists some $a \in A$ and some $b \in B$ such that:

- We have $a \in U$;
- We have $a \sim_R b$;
- We have $b \in V$.


00P9

PROPOSITION 5.6.3 ► FUNCTORIALITY OF POWERSETS II

The assignment $R \mapsto \mathcal{P}(R)$ defines a functor

$$\mathcal{P}: \mathbf{Rel} \rightarrow \mathbf{Rel}.$$

PROOF 5.6.4 ► PROOF OF PROPOSITION 5.6.3

Omitted. 

00PA 6 Relative Preorders

00PB 6.1 The Left Skew Monoidal Structure on $\mathbf{Rel}(A, B)$

Let A and B be sets and let $J: A \rightarrowtail B$ be a relation.

6.1.1 The Left Skew Monoidal Product

00PD

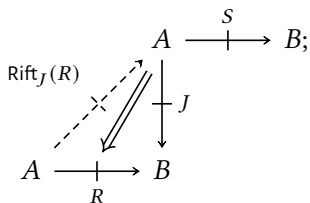
DEFINITION 6.1.1 ► THE LEFT J -SKEW MONOIDAL PRODUCT OF $\mathbf{REL}(A, B)$

The **left J -skew monoidal product of $\mathbf{Rel}(A, B)$** is the functor

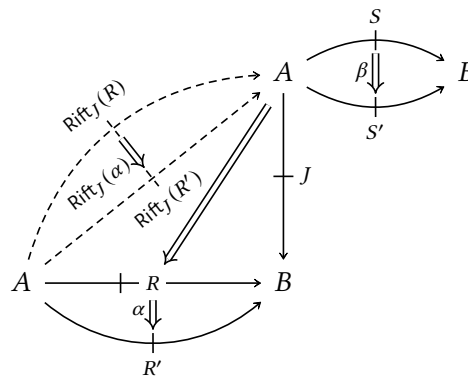
$$\triangleleft_J: \mathbf{Rel}(A, B) \times \mathbf{Rel}(A, B) \rightarrow \mathbf{Rel}(A, B)$$

where

· *Action on Objects.* For each $R, S \in \mathbf{Obj}(\mathbf{Rel}(A, B))$, we have

$$S \triangleleft_J R \stackrel{\text{def}}{=} S \diamond \text{Rift}_J(R),$$


of \triangleleft_J at $((R, S), (R', S'))$ is defined by¹



¹Since $\mathbf{Rel}(A, B)$ is posetal, this is to say that if $S \subset S'$ and $R \subset R'$, then $S \triangleleft_J R \subset S' \triangleleft_J R'$.

6.1.3 The Left Skew Associators

00PH

DEFINITION 6.1.3 ► THE LEFT J -SKEW ASSOCIATOR OF $\mathbf{Rel}(A, B)$

The **left J -skew associator of $\mathbf{Rel}(A, B)$** is the natural transformation

$$\alpha^{\mathbf{Rel}(A,B),\triangleleft} : \triangleleft_J \circ (\triangleleft_J \times \text{id}) \Longrightarrow \triangleleft_J \circ (\text{id} \times \triangleleft_J),$$

whose component

$$\alpha_{T,S,R}^{\mathbf{Rel}(A,B),\triangleleft} : \underbrace{(T \triangleleft_J S) \triangleleft_J R}_{\stackrel{\text{def}}{=} T \diamond \text{Rift}_J(S) \diamond \text{Rift}_J(R)} \hookrightarrow \underbrace{T \triangleleft_J (S \triangleleft_J R)}_{\stackrel{\text{def}}{=} T \diamond \text{Rift}_J(S \diamond \text{Rift}_J(R))}$$

at (T, S, R) is given by

$$\alpha_{T,S,R}^{\mathbf{Rel}(A,B),\triangleleft} \stackrel{\text{def}}{=} \text{id}_T \diamond \gamma,$$

where

$$\gamma : \text{Rift}_J(S) \diamond \text{Rift}_J(R) \hookrightarrow \text{Rift}_J(S \diamond \text{Rift}_J(R))$$

is the inclusion adjunct to the inclusion

$$\epsilon_S \circ \text{id}_{\text{Rift}_J(R)} : \underbrace{J \diamond \text{Rift}_J(S) \diamond \text{Rift}_J(R)}_{\stackrel{\text{def}}{=} J_* (\text{Rift}_J(S) \diamond \text{Rift}_J(R))} \hookrightarrow S \diamond \text{Rift}_J(R)$$

under the adjunction $J_* \dashv \text{Rift}_J$, where $\epsilon : J \diamond \text{Rift}_J \Longrightarrow \text{id}_{\mathbf{Rel}(A,B)}$ is the counit of the adjunction $J_* \dashv \text{Rift}_J$.

6.1.4 The Left Skew Left Unitors

00PK

DEFINITION 6.1.4 ► THE LEFT J -SKEW LEFT UNITOR OF $\mathbf{Rel}(A, B)$

The **left J -skew left unitor of $\mathbf{Rel}(A, B)$** is the natural transformation

$$\lambda^{\mathbf{Rel}(A,B),\triangleleft} : \triangleleft_J \circ \left(\mathbb{K}_{\triangleleft}^{\mathbf{Rel}(A,B)} \times \text{id} \right) \Longrightarrow \text{id},$$

whose component

$$\lambda_R^{\mathbf{Rel}(A,B),\triangleleft} : \underbrace{J \triangleleft_J R}_{\stackrel{\text{def}}{=} J \diamond \text{Rift}_J(R)} \hookrightarrow R$$

at R is given by

$$\lambda_R^{\mathbf{Rel}(A,B),\triangleleft} \stackrel{\text{def}}{=} \epsilon_R,$$

where $\epsilon: J \diamond \text{Rift}_J \Rightarrow \text{id}_{\mathbf{Rel}(A,B)}$ is the counit of the adjunction $J_* \dashv \text{Rift}_J$.

6.1.5 The Left Skew Right Unitors

00PM

DEFINITION 6.1.5 ► THE LEFT J -SKEW RIGHT UNITOR OF $\mathbf{Rel}(A, B)$

The **left J -skew right unitor of $\mathbf{Rel}(A, B)$** is the natural transformation

$$\rho^{\mathbf{Rel}(A,B),\triangleleft}: \text{id} \Rightarrow \triangleleft_J \circ (\text{id} \times \mathbb{K}_{\triangleleft}^{\mathbf{Rel}(A,B)}),$$

whose component

$$\rho_R^{\mathbf{Rel}(A,B),\triangleleft}: R \hookrightarrow \underbrace{R \triangleleft_J J}_{\stackrel{\text{def}}{=} R \diamond \text{Rift}_J(J)}$$

at R is given by

$$\rho_R^{\mathbf{Rel}(A,B),\triangleleft} \stackrel{\text{def}}{=} \text{id}_R \circ \sigma,$$

where $\sigma: \text{id}_A \Rightarrow \text{Rift}_J(J)$ is the universal transformation included in the data of the right Kan lift $\text{Rift}_J(J)$.

6.1.6 The Left Skew Monoidal Structure on $\mathbf{Rel}(A, B)$

00PP

DEFINITION 6.1.6 ► THE LEFT J -SKEW MONOIDAL STRUCTURE ON $\mathbf{Rel}(A, B)$

The **left J -skew monoidal category of relations from A to B** is the left skew monoidal category

$$\left(\mathbf{Rel}(A, B), \triangleleft_J, \mathbb{K}_{\triangleleft}^{\mathbf{Rel}(A,B)}, \alpha^{\mathbf{Rel}(A,B),\triangleleft}, \lambda^{\mathbf{Rel}(A,B),\triangleleft}, \rho^{\mathbf{Rel}(A,B),\triangleleft} \right)$$

consisting of

- *The Underlying Category.* The posetal category associated to the poset $\mathbf{Rel}(A, B)$ of relations from A to B of [Item 2](#) of [Definition 1.1.2](#);
- *The Skew Monoidal Product.* The functor \triangleleft_J of [Definition 6.1.1](#);

- *The Skew Monoidal Unit.* The functor $\mathbb{K}_{\triangleleft}^{\mathbf{Rel}(A,B)}$ of Definition 6.1.2;
- *The Skew Associators.* The natural transformation $\alpha^{\mathbf{Rel}(A,B),\triangleleft}$ of Definition 6.1.3;
- *The Skew Left Unitors.* The natural transformation $\lambda^{\mathbf{Rel}(A,B),\triangleleft}$ of Definition 6.1.4;
- *The Skew Right Unitors.* The natural transformation $\rho^{\mathbf{Rel}(A,B),\triangleleft}$ of Definition 6.1.5.

00PQ 6.2 Left Relative Preorders

Let A and B be sets and let $J: A \rightarrowtail B$ be a relation.

00PR DEFINITION 6.2.1 ► LEFT J -RELATIVE PREORDERS

A **left J -relative preorder from A to B** is equivalently:

- An \mathbb{E}_1 -skew monoid in $(\mathbf{N}_\bullet(\mathbf{Rel}(A,B)), \triangleleft_J, J)$;
- A skew monoid in $(\mathbf{Rel}(A,B), \triangleleft_J, J)$.

00PS REMARK 6.2.2 ► UNWINDING DEFINITION 6.2.1, I

In detail, a **left J -relative preorder (R, μ_R, η_R) from A to B** consists of

- *The Underlying Relation.* A relation

$$R: A \rightarrowtail B,$$
 called the **underlying relation of (R, μ_R, η_R)** ;
- *The Multiplication Inclusion.* An inclusion of relations

$$\mu_R: R \triangleleft_J R \subset R,$$
 called the **multiplication of (R, μ_R, η_R)** ;
- *The Unit Inclusion.* An inclusion of relations

$$\eta_R: J \subset R,$$

called the **unit** of (R, μ_R, η_R) .

00PT

REMARK 6.2.3 ► UNWINDING DEFINITION 6.2.1, II

In other words, a **left J -relative preorder from A to B** is a relation $R: A \rightarrow B$ from A to B satisfying the following conditions:

1. *J-Transitivity*. For each $a \in A$ and each $c \in B$, we have

$$a \sim_{R \diamond \text{Rift}_J(R)} c$$

i.e. the following condition is satisfied:¹

(★) If there exists some $b \in A$ such that:

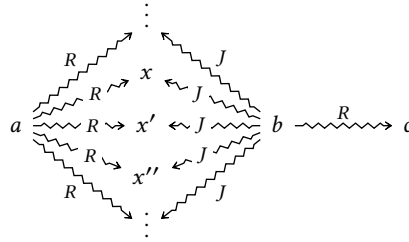
- We have $a \sim_{\text{Rift}_J(R)} b$, i.e. for each $x \in B$, if $b \sim_J x$, then $a \sim_R x$;²
- We have $b \sim_R c$;

then $a \sim_R c$.

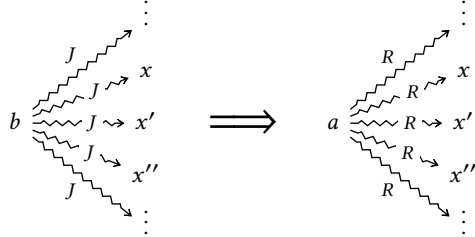
2. *J-Unitality*. For each $a \in A$ and each $b \in B$, the following condition is satisfied:

(★) If $a \sim_J b$, then $a \sim_R b$.

¹Illustration: If we have



then $a \sim_R c$.

²Illustration:**00PU 6.3 The Right Skew Monoidal Structure on $\mathbf{Rel}(A, B)$**

Let A and B be sets and let $J: A \rightarrow B$ be a relation.

6.3.1 The Right Skew Monoidal Product**00PW****DEFINITION 6.3.1 ► THE RIGHT J -SKEW MONOIDAL PRODUCT OF $\mathbf{Rel}(A, B)$**

The **right J -skew monoidal product of $\mathbf{Rel}(A, B)$** is the functor

$$\triangleright_J: \mathbf{Rel}(A, B) \times \mathbf{Rel}(A, B) \rightarrow \mathbf{Rel}(A, B)$$

where

- *Action on Objects.* For each $R, S \in \text{Obj}(\mathbf{Rel}(A, B))$, we have

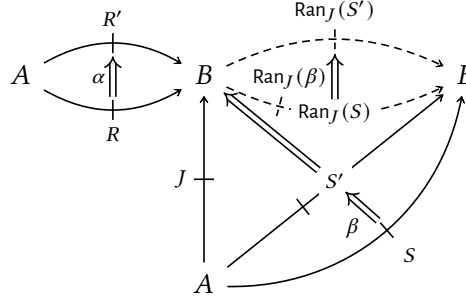
$$S \triangleright_J R \stackrel{\text{def}}{=} \text{Ran}_J(S) \diamond R,$$

- *Action on Morphisms.* For each $R, S, R', S' \in \text{Obj}(\mathbf{Rel}(A, B))$, the action on Hom-sets

$$(\triangleright_J)_{(S,R),(S',R')} : \text{Hom}_{\mathbf{Rel}(A,B)}(S, S') \times \text{Hom}_{\mathbf{Rel}(A,B)}(R, R') \rightarrow \text{Hom}_{\mathbf{Rel}(A,B)}(S \triangleright_J R, S' \triangleright_J R')$$

of \triangleright_J at $((S, R), (S', R'))$ is defined by¹

$$\beta \triangleright_J \alpha \stackrel{\text{def}}{=} \text{Ran}_J(\beta) \diamond \alpha,$$



for each $\beta \in \text{Hom}_{\mathbf{Rel}(A, B)}(S, S')$ and each $\alpha \in \text{Hom}_{\mathbf{Rel}(A, B)}(R, R')$.

¹Since $\mathbf{Rel}(A, B)$ is posetal, this is to say that if $S \subset S'$ and $R \subset R'$, then $S \triangleright_J R \subset S' \triangleright_J R'$.

6.3.2 The Right Skew Monoidal Unit

00PY

DEFINITION 6.3.2 ► THE RIGHT J -SKEW MONOIDAL UNIT OF $\mathbf{Rel}(A, B)$

The **right J -skew monoidal unit** of $\mathbf{Rel}(A, B)$ is the functor

$$\mathbb{K}_{\triangleright}^{\mathbf{Rel}(A, B)} : \text{pt} \rightarrow \mathbf{Rel}(A, B)$$

picking the object

$$\mathbb{K}_{\triangleright}^{\mathbf{Rel}(A, B)} \stackrel{\text{def}}{=} J$$

of $\mathbf{Rel}(A, B)$.

6.3.3 The Right Skew Associators

00Q0

DEFINITION 6.3.3 ► THE RIGHT J -SKEW ASSOCIATOR OF $\mathbf{Rel}(A, B)$

The **right J -skew associator** of $\mathbf{Rel}(A, B)$ is the natural transformation

$$\alpha^{\mathbf{Rel}(A, B), \triangleright} : \triangleright_J \circ (\text{id} \times \triangleright_J) \Longrightarrow \triangleright_J \circ (\triangleright_J \times \text{id}),$$

whose component

$$\alpha_{T,S,R}^{\mathbf{Rel}(A,B),\triangleright} : \underbrace{T \triangleright_J (S \triangleright_J R)}_{\stackrel{\text{def}}{=} \text{Ran}_J(T) \diamond (\text{Ran}_J(S) \diamond R)} \hookrightarrow \underbrace{(T \triangleright_J S) \triangleright_J R}_{\stackrel{\text{def}}{=} \text{Ran}_J(\text{Ran}_J(T) \diamond S) \diamond R}$$

at (T, S, R) is given by

$$\alpha_{T,S,R}^{\mathbf{Rel}(A,B),\triangleright} \stackrel{\text{def}}{=} \gamma \diamond \text{id}_R,$$

where

$$\gamma : \text{Ran}_J(T) \diamond \text{Ran}_J(S) \hookrightarrow \text{Ran}_J(\text{Ran}_J(T) \diamond S)$$

is the inclusion adjunct to the inclusion

$$\text{id}_{\text{Ran}_J(T)} \diamond \epsilon_S : \underbrace{\text{Ran}_J(T) \diamond \text{Ran}_J(S) \diamond J}_{\stackrel{\text{def}}{=} J^*(\text{Ran}_J(T) \diamond \text{Ran}_J(S))} \hookrightarrow \text{Ran}_J(T) \diamond S$$

under the adjunction $J^* \dashv \text{Ran}_J$, where $\epsilon : \text{Ran}_J \diamond J \Rightarrow \text{id}_{\mathbf{Rel}(A,B)}$ is the counit of the adjunction $J^* \dashv \text{Ran}_J$.

6.3.4 The Right Skew Left Unitors

00Q2

DEFINITION 6.3.4 ► THE RIGHT J -SKEW LEFT UNITOR OF $\mathbf{Rel}(A, B)$

The **right J -skew left unitor** of $\mathbf{Rel}(A, B)$ is the natural transformation

$$\lambda^{\mathbf{Rel}(A,B),\triangleright} : \text{id} \Rightarrow \triangleright_J \circ \left(\kappa_{\triangleright}^{\mathbf{Rel}(A,B)} \times \text{id} \right),$$

whose component

$$\lambda_R^{\mathbf{Rel}(A,B),\triangleright} : R \hookrightarrow \underbrace{J \triangleright_J R}_{\stackrel{\text{def}}{=} \text{Ran}_J(J) \diamond R}$$

at R is given by

$$\lambda_R^{\mathbf{Rel}(A,B),\triangleright} \stackrel{\text{def}}{=} \sigma \diamond \text{id}_R,$$

where $\sigma : \text{id}_B \Rightarrow \text{Ran}_J(J)$ is the universal transformation included in the data of the right Kan extension $\text{Ran}_J(J)$.

6.3.5 The Right Skew Right Unitors

00Q4

DEFINITION 6.3.5 ► THE RIGHT J -SKEW RIGHT UNITOR OF $\mathbf{Rel}(A, B)$

The **right J -skew right unitor** of $\mathbf{Rel}(A, B)$ is the natural transformation

$$\rho^{\mathbf{Rel}(A,B),\triangleright} : \triangleright_J \circ (\mathrm{id} \times \mathbb{K}_{\triangleright}^{\mathbf{Rel}(A,B)}) \Rightarrow \mathrm{id},$$

whose component

$$\rho_S^{\mathbf{Rel}(A,B),\triangleright} : \underbrace{S \triangleright_J J}_{\stackrel{\text{def}}{=} \mathrm{Ran}_J(S) \diamond J} \hookrightarrow S$$

at S is given by

$$\rho_S^{\mathbf{Rel}(A,B),\triangleright} \stackrel{\text{def}}{=} \epsilon_R,$$

where $\epsilon : \mathrm{Ran}_J \diamond J \Rightarrow \mathrm{id}_{\mathbf{Rel}(A,B)}$ is the counit of the adjunction $J^* \dashv \mathrm{Ran}_J$.

6.3.6 The Right Skew Monoidal Structure on $\mathbf{Rel}(A, B)$

00Q6

DEFINITION 6.3.6 ► THE RIGHT J -SKEW MONOIDAL STRUCTURE ON $\mathbf{Rel}(A, B)$

The **right J -skew monoidal category of functors from A to B** is the right skew monoidal category

$$\left(\mathbf{Rel}(A, B), \triangleright_J, \mathbb{K}_{\triangleright}^{\mathbf{Rel}(A,B)}, \alpha^{\mathbf{Rel}(A,B),\triangleright}, \lambda^{\mathbf{Rel}(A,B),\triangleright}, \rho^{\mathbf{Rel}(A,B),\triangleright} \right)$$

consisting of

- *The Underlying Category.* The posetal category associated to the poset $\mathbf{Rel}(A, B)$ of relations from A to B of [Item 2 of Definition 1.1.2](#);
- *The Skew Monoidal Product.* The functor \triangleright_J of [Definition 6.3.1](#);
- *The Skew Monoidal Unit.* The functor $\mathbb{K}_{\triangleright}^{\mathbf{Rel}(A,B)}$ of [Definition 6.3.2](#);
- *The Skew Associators.* The natural transformation $\alpha^{\mathbf{Rel}(A,B),\triangleright}$ of [Definition 6.3.3](#);
- *The Skew Left Unitors.* The natural transformation $\lambda^{\mathbf{Rel}(A,B),\triangleright}$ of [Definition 6.3.4](#);
- *The Skew Right Unitors.* The natural transformation $\rho^{\mathbf{Rel}(A,B),\triangleright}$ of [Definition 6.3.5](#).

00Q7 6.4 Right Relative Preorders

Let A and B be sets and let $J: A \rightarrowtail B$ be a relation.

00Q8 DEFINITION 6.4.1 ► RIGHT J -RELATIVE PREORDERS

A **right J -relative preorder from A to B** is equivalently:

- An \mathbb{E}_1 -skew monoid in $(\mathbf{N}_\bullet(\mathbf{Rel}(A, B)), \triangleright_J, J)$;
- A skew monoid in $(\mathbf{Rel}(A, B), \triangleright_J, J)$.

00Q9 REMARK 6.4.2 ► UNWINDING DEFINITION 6.4.1, I

In detail, a **right J -relative preorder (R, μ_R, η_R) from A to B** consists of

- *The Underlying Relation.* A relation

$$R: A \rightarrowtail B,$$

called the **underlying relation of (R, μ_R, η_R)** ;

- *The Multiplication Inclusion.* An inclusion of relations

$$\mu_R: R \triangleright_J R \subset R,$$

called the **multiplication of (R, μ_R, η_R)** ;

- *The Unit Inclusion.* An inclusion of relations

$$\eta_R: J \subset R,$$

called the **unit of (R, μ_R, η_R)** .

00QA REMARK 6.4.3 ► UNWINDING DEFINITION 6.4.1, II

In other words, a **right J -relative preorder from A to B** is a relation $R: A \rightarrowtail B$ from A to B satisfying the following conditions:

1. *J -Transitivity.* For each $a \in A$ and each $c \in B$, we have

$$a \sim_{\text{Ran}_J(R) \diamond R} c,$$

i.e. the following condition is satisfied:¹

(★) If there exists some $b \in B$ such that:

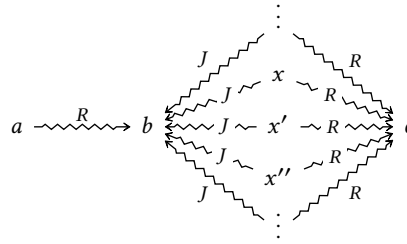
- We have $a \sim_R b$;
- We have $b \sim_{\text{Ran}_J(R)} c$, i.e. for each $x \in A$, if $x \sim_J b$, then $x \sim_R c$;²

then $a \sim_R c$.

2. *J*-Unitality. For each $a \in A$ and each $b \in B$, the following condition is satisfied:

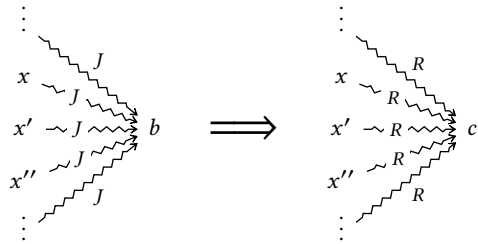
(★) If $a \sim_J b$, then $a \sim_R b$.

¹Illustration: If we have



then $a \sim_R c$.

²Illustration:



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A Other Chapters

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