Fibred Sets

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This chapter contains a 00S0 discussion of the un/straightening equivalence in the context of sets, as well as a general discussion of indexed and fibred sets. In particular, it contains:

1. A discussion of indexed sets (i.e. functors $K_{\mathsf{disc}} \to \mathsf{Sets}$ with K a set), constructions with them like dependent sums and dependent products, and their properties (**????**);
2. A discussion of fibred sets (i.e. maps of sets $X \to K$), constructions with them like dependent sums and dependent products, and their properties (**????**);
3. A discussion of the un/straightening equivalence for indexed and fibred sets (**??**).

Contents

1 Fibred Sets 00S1

1.1 Foundations 00S2

Let K be a set.

Definition 1.1.1.1. A K-**fibred set** is a pair 00S3 (X, ϕ) consisting of[1]

- *The Underlying Set.* A set X, called the **underlying set of** (X, ϕ);
- *The Fibration.* A map of sets $\phi \colon X \to K$.

[1] *Further Terminology:* The **fibre of** (X, ϕ) **over** $x \in K$ is the set $\phi^{-1}(x)$ (also written ϕ_x) defined by

$$\phi^{-1}(x) \overset{\text{def}}{=} \mathrm{pt} \times_{[x], K, \phi} X, \qquad \begin{tikzcd} \phi^{-1}(x) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "\phi"] \\ \mathrm{pt} \arrow[r, "{[x]}"'] & K. \end{tikzcd}$$

1.2 Morphisms of Fibred Sets 00S4

Definition 1.2.1.1. 00S5 A **morphism of K-fibred sets from (X,ϕ) to (Y,ψ)** is a function $f\colon X\to Y$ such that the diagram[2]

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\phi"'] & & Y \arrow[dl, "\psi"] \\ & K & \end{tikzcd}$$

commutes.

1.3 The Category of Fibred Sets Over a Fixed Base 00S6

Definition 1.3.1.1. 00S7 The **category of K-fibred sets** is the category $\mathsf{FibSets}(K)$ defined as the slice category $\mathsf{Sets}_{/K}$ of Sets over K:

$$\mathsf{FibSets}(K) \overset{\text{def}}{=} \mathsf{Sets}_{/K}.$$

Remark 1.3.1.2. 00S8 In detail $\mathsf{FibSets}(K)$ is the category where

- *Objects.* The objects of $\mathsf{FibSets}(K)$ are pairs (X,ϕ) consisting of
  - *The Fibred Set.* A set X;
  - *The Fibration.* A function $\phi\colon X\to K$;
- *Morphisms.* A morphism of $\mathsf{FibSets}(K)$ from (X,ϕ) to (Y,ψ) is a function $f\colon X\to$

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[2] *Further Terminology:* The **transport map associated to f at $x\in K$** is the function

$$f^*_x\colon \phi^{-1}(x)\to\psi^{-1}(x)$$

given by the dashed map in the diagram

$$\begin{tikzcd} \phi^{-1}(x) \arrow[rr] \arrow[dd] \arrow[dr, dashed] \arrow[dr, phantom, very near start, "\lrcorner"] & & X \arrow[dr, "f"] \arrow[dd, "\phi" near start] & \\ & \psi^{-1}(x) \arrow[rr] \arrow[dd] \arrow[dr, phantom, very near start, "\lrcorner"] & & Y \arrow[dd, "\psi"] \\ \mathrm{pt} \arrow[rr, "{[x]}" near start] \arrow[dr, equal] & & K \arrow[dr, equal] & \\ & \mathrm{pt} \arrow[rr, "{[x]}"'] & & K. \end{tikzcd}$$

Y making the diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\phi"'] & & Y \arrow[dl, "\psi"] \\ & X & \end{tikzcd}$$

commute;

- *Identities.* For each $(X, \phi) \in \mathsf{Obj}(\mathsf{FibSets}(K))$, the unit map
$$\mathbb{1}^{\mathsf{FibSets}(K)}_{(X,\phi)} \colon \mathrm{pt} \to \mathrm{Hom}_{\mathsf{FibSets}(K)}((X,\phi),(X,\phi))$$
of $\mathsf{FibSets}(K)$ at (X, ϕ) is given by
$$\mathrm{id}^{\mathsf{FibSets}(K)}_{(X,\phi)} \overset{\text{def}}{=} \mathrm{id}_X,$$
as witnessed by the commutativity of the diagram
$$\begin{tikzcd} X \arrow[rr, equal, "\mathrm{id}_X"] \arrow[dr, "\phi"'] & & X \arrow[dl, "\phi"] \\ & K & \end{tikzcd}$$
in Sets;

- *Composition.* For each $\mathbf{X} = (X, \phi)$, $\mathbf{Y} = (Y, \psi)$, $\mathbf{Z} = (Z, \chi) \in \mathsf{Obj}(\mathsf{FibSets}(K))$, the composition map
$$\circ^{\mathsf{FibSets}(K)}_{\mathbf{X},\mathbf{Y},\mathbf{Z}} \colon \mathrm{Hom}_{\mathsf{FibSets}(K)}(\mathbf{Y},\mathbf{Z}) \times \mathrm{Hom}_{\mathsf{FibSets}(K)}(\mathbf{X},\mathbf{Y}) \to \mathrm{Hom}_{\mathsf{FibSets}(K)}(\mathbf{X},\mathbf{Z})$$
of $\mathsf{FibSets}(K)$ at $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ is defined by
$$\circ^{\mathsf{FibSets}(K)}_{\mathbf{X},\mathbf{Y},\mathbf{Z}} \overset{\text{def}}{=} \circ^{\mathsf{Sets}}_{X,Y,Z},$$
as witnessed by the commutativity of the diagram
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "\phi"'] & Y \arrow[r, "g"] \arrow[d, "\psi"] & Z \arrow[dl, "\chi"] \\ & K & \end{tikzcd}$$
in Sets.

1.4 The Category of Fibred Sets

Definition 1.4.1.1. The **category of fibred sets** is the category $\mathsf{FibSets}$ defined as the Grothendieck construction of the functor $\mathsf{FibSets}\colon \mathsf{Sets}^{\mathsf{op}} \to \mathsf{Cats}$ of **??**:

$$\mathsf{FibSets} \overset{\text{def}}{=} \int^{\mathsf{Sets}} \mathsf{FibSets}.$$

Remark 1.4.1.2. In detail, the **category of fibred sets** is the category $\mathsf{FibSets}$ where

- *Objects.* The objects of $\mathsf{FibSets}$ are pairs $(K, (X, \phi_X))$ consisting of
  - *The Base Set.* A set K;
  - *The Fibred Set.* A K-fibred set $\phi_X\colon X \to K$;
- *Morphisms.* A morphism of $\mathsf{FibSets}$ from $(K, (X, \phi_X))$ to $(K', (Y, \phi_Y))$ is a pair (ϕ, f) consisting of
  - *The Base Map.* A map of sets $\phi\colon K \to K'$;
  - *The Morphism of Fibred Sets.* A morphism of K-fibred sets

$$f\colon (X, \phi_X) \to \phi_Y^*(Y), \qquad \begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\phi_X"'] & & Y \times_{K'} K \arrow[dl, "\mathrm{pr}_2"] \\ & K; & \end{tikzcd}$$

- *Identities.* For each $(K, X) \in \mathrm{Obj}(\mathsf{FibSets})$, the unit map

$$\mathbb{1}^{\mathsf{FibSets}}_{(K,X)}\colon \mathrm{pt} \to \mathsf{FibSets}((K, X), (K, X))$$

  of $\mathsf{FibSets}$ at (K, X) is defined by

$$\mathrm{id}^{\mathsf{FibSets}}_{(K,X)} \overset{\text{def}}{=} (\mathrm{id}_K, \sim),$$

  where $\sim$ is the isomorphism $X \to X \times_K K$ as in the diagram

$$\begin{tikzcd} X \arrow[rr, dashed, "\sim"] \arrow[dr, "\phi_X"'] & & X \times_K K \arrow[dl, "\mathrm{pr}_2"] \\ & K; & \end{tikzcd}$$

- *Composition.* For each $\mathbf{X} = (K, X), \mathbf{Y} = (K', Y), \mathbf{Z} = (K'', Z) \in \mathrm{Obj}(\mathsf{FibSets})$, the composition map

$$\circ^{\mathsf{FibSets}}_{\mathbf{X},\mathbf{Y},\mathbf{Z}}\colon \mathsf{FibSets}(\mathbf{Y}, \mathbf{Z}) \times \mathsf{FibSets}(\mathbf{X}, \mathbf{Y}) \to \mathsf{FibSets}(\mathbf{X}, \mathbf{Z})$$

of $\mathsf{FibSets}$ at $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ is defined by

$$g \circ^{\mathsf{FibSets}}_{\mathbf{X},\mathbf{Y},\mathbf{Z}} f \overset{\text{def}}{=} (g \times_{K'} \mathrm{id}_K) \circ f$$

as in the diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[dr, "\phi_X"'] & Y \times_{K'} K \arrow[r, "g \times_{K'} \mathrm{id}_K"] \arrow[d, "\mathrm{pr}_2"] & \overbrace{(Z \times_{K''} K') \times_{K'} K}^{\cong Z \times_{K''} K} \arrow[dl, "\mathrm{pr}_2"] \\
& K; &
\end{tikzcd}$$

for each $f \in \mathrm{Obj}(\mathsf{FibSets}(\mathbf{X}, \mathbf{Y}))$ and each $g \in \mathrm{Obj}(\mathsf{FibSets}(\mathbf{Y}, \mathbf{Z}))$.

2 Constructions With Fibred Sets 00SC

2.1 Change of Base 00SD

Let $f \colon K \to K'$ be a function and let (X, ϕ_X) be a K'-fibred set.

Definition 2.1.1.1. The **change of base of** 00SE (X, ϕ_X) **to** K is the K-fibred set $f^*(X)$ defined by

$$f^*(X) \overset{\text{def}}{=} (K \times_{K'} X, \mathrm{pr}_1), \qquad \begin{tikzcd}
f^*(X) \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "\phi_X"] \\
K \arrow[r, "f"'] & K'.
\end{tikzcd}$$

Proposition 2.1.1.2. The assignment $X \mapsto f^*(X)$ 00SF defines a functor

$$f^* \colon \mathsf{FibSets}(K') \to \mathsf{FibSets}(K),$$

where

- *Action on Objects.* For each $(X, \phi_X) \in \mathrm{Obj}(\mathsf{FibSets}(K'))$, we have

$$f^*(X, \phi_X) \overset{\text{def}}{=} f^*(X);$$

- *Action on Morphisms.* For each $(X, \phi_X), (Y, \phi_Y) \in \mathrm{Obj}(\mathsf{FibSets}(K'))$, the action on Hom-sets

$$f^*_{X,Y} \colon \mathrm{Hom}_{\mathsf{FibSets}(K')}(X,Y) \to \mathrm{Hom}_{\mathsf{FibSets}(K)}(f^*(X), f^*(Y))$$

of f^* at $((X,\phi_X),(Y,\phi_Y))$ is the map sending a morphism of K'-fibred sets

$$g \colon (X,\phi_X) \to (Y,\phi_Y)$$

to the morphism of K-fibred sets given by the dashed morphism in the diagram

$$\begin{tikzcd}
f^*(X) \arrow[rr] \arrow[dr, dashed] \arrow[dr, phantom, very near start, "\lrcorner"] \arrow[dd] & & X \arrow[dr, "g"] \arrow[dd, dashed, "\phi_X" near start] & \\
& f^*(Y) \arrow[rr] \arrow[dd] \arrow[dr, phantom, very near start, "\lrcorner"] & & Y \arrow[dd, "\phi_Y"] \\
K \arrow[rr, "f" near start] \arrow[dr, equal] & & K' \arrow[dr, equal] & \\
& K \arrow[rr, "f"'] & & K'.
\end{tikzcd}$$

Proof. Omitted. □

Proposition 2.1.1.3. The assignment $K \mapsto \mathsf{FibSets}(K)$ defines a functor

$$\mathsf{FibSets} \colon \mathsf{Sets}^{\mathsf{op}} \to \mathsf{Cats},$$

where

- *Action on Objects.* For each $K \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$[\mathsf{FibSets}](K) \overset{\text{def}}{=} \mathsf{FibSets}(K);$$

- *Action on Morphisms.* For each $K, K' \in \mathrm{Obj}(\mathsf{Sets})$, the action on Hom-sets

$$\mathsf{Sets}_{/(-)|K,K'} \colon \mathsf{Sets}^{\mathsf{op}}(K,K') \to \mathsf{Fun}(\mathsf{FibSets}(K), \mathsf{FibSets}(K'))$$

of $\mathsf{Sets}_{/(-)}$ at (K,K') is the map sending a map of sets $f \colon K \to K'$ to the functor

$$\mathsf{Sets}_{/f} \colon \mathsf{FibSets}(K') \to \mathsf{FibSets}(K)$$

defined by

$$\mathsf{Sets}_{/f} \overset{\text{def}}{=} f^*.$$

Proof. Omitted. □

2.2 Dependent Sums 00SH

Let $f\colon K \to K'$ be a function and let (X, ϕ_X) be a K-fibred set.

Definition 2.2.1.1. The **dependent sum**[3] **of** 00SJ (X, ϕ_X) is the K'-fibred set $\Sigma_f(X)$[4] defined by

$$\begin{aligned}\Sigma_f(X) &\stackrel{\text{def}}{=} (\Sigma_f(X), \Sigma_f(\phi_X))\\ &\stackrel{\text{def}}{=} (X, f \circ \phi_X).\end{aligned}$$

Proposition 2.2.1.2. Let $f\colon K \to K'$ be a function. 00SK

1. *Functoriality.* The assignment $X \mapsto \Sigma_f(X)$ 00SL defines a functor

   $$\Sigma_f\colon \mathsf{FibSets}(K) \to \mathsf{FibSets}(K'),$$

   where

   - *Action on Objects.* For each $(X, \phi_X) \in \mathrm{Obj}(\mathsf{FibSets}(K))$, we have

     $$\Sigma_f(X, \phi_X) \stackrel{\text{def}}{=} (\Sigma_f(X), \Sigma_f(\phi_X));$$

   - *Action on Morphisms.* For each $(X, \phi_X), (Y, \phi_Y) \in \mathrm{Obj}(\mathsf{FibSets}(K))$, the action on Hom-sets

     $$\Sigma_{f|X,Y}\colon \mathrm{Hom}_{\mathsf{FibSets}(K)}(X, Y) \to \mathrm{Hom}_{\mathsf{FibSets}(K)}(\Sigma_f(X), \Sigma_f(Y))$$

     of Σ_f at $((X, \phi_X), (Y, \phi_Y))$ is the map sending a morphism of K-fibred sets

     $$g\colon (X, \phi_X) \to (Y, \phi_Y)$$

     to the morphism of K'-fibred sets defined by

     $$\Sigma_f(g) \stackrel{\text{def}}{=} g.$$

2. *Interaction With Fibres.* We have a bijection of sets 00SM

   $$\Sigma_f(\phi_X)^{-1}(k') \cong \coprod_{k \in f^{-1}(k')} \phi_X^{-1}(k)$$

   for each $k' \in K'$.

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[3]The name "dependent sum" comes from the fact that the fibre $\Sigma_f(\phi_X)^{-1}(x)$ of $\Sigma_f(X)$ at $x \in K'$ is given by

$$\Sigma_f(\phi_X)^{-1}(x) \cong \coprod_{y \in f^{-1}(x)} \phi_X^{-1}(y);$$

see **??** of **??**.

[4]*Further Notation:* Also written $f_*(X)$.

Proof. **??**, *Functoriality*: Omitted.
??, *Interaction With Fibres*: Indeed, we have

$$\begin{aligned}\Sigma_f(\phi_X)^{-1}(k') &\stackrel{\text{def}}{=} \mathrm{pt} \times_{[k'],K',f\circ\phi_X} X\\ &\cong \{x \in X \mid f(\phi_X(x)) = k'\}\\ &\cong \coprod_{k\in f^{-1}(k')} \{x \in X \mid \phi_X(x) = k\}\\ &\cong \coprod_{k\in f^{-1}(k')} \phi_X^{-1}(k)\end{aligned}$$

for each $k' \in K'$. □

2.3 Dependent Products 00SN

Let $f\colon K \to K'$ be a function and let (X, ϕ_X) be a K-fibred set.

Definition 2.3.1.1. 00SP The **dependent product**[5] **of** (X, ϕ_X) is the K'-fibred set $\Pi_f(X)$[6] consisting of[7]

- *The Underlying Set.* The set $\Pi_f(X)$ defined by

$$\Pi_f(X) \stackrel{\text{def}}{=} \coprod_{k'\in K'} \prod_{k\in f^{-1}(k')} \phi_X^{-1}(k);$$

- *The Fibration.* The map of sets

$$\Pi_f(\phi_X)\colon \Pi_f(X) \to K'$$

  defined by sending an element of

$$\Pi_f(X) \stackrel{\text{def}}{=} \coprod_{k'\in K'} \prod_{k\in f^{-1}(k')} \phi_X^{-1}(k)$$

  to its index k' in K'.

Example 2.3.1.2. 00SQ Here are some examples of dependent products of sets.

[5]The name "dependent product" comes from the fact that the fibre $\Pi_f(\phi_X)^{-1}(k')$ of $\Pi_f(X)$ at $k' \in K'$ is given by

$$\Pi_f(\phi_X)^{-1}(k') \cong \prod_{k\in f^{-1}(k')} \phi_X^{-1}(k);$$

see **??** of **??**.

[6]*Further Notation:* Also written $f_!(X)$.

[7]We can also define dependent products via the internal **Hom** in $\mathsf{FibSets}(K')$; see **??** of **??**.

1. *Spaces of Sections.* Let $K = X$, $K' = \mathrm{pt}$, let $\phi \colon E \longrightarrow X$ be a map of sets, and write $!_X \colon X \longrightarrow \mathrm{pt}$ for the terminal map from X to pt. We have a bijection of sets

$$\begin{aligned}\Pi_{!_X}((E,\phi)) &\cong \Gamma_X(\phi)\\ &\overset{\text{def}}{=} \{h \in \mathsf{Sets}(X,E) \mid \phi \circ h = \mathrm{id}_X\}.\end{aligned}$$

2. *Function Spaces.* Let $K = K' = \mathrm{pt}$ and write $!_X \colon X \longrightarrow \mathrm{pt}$ and $!_Y \colon Y \longrightarrow \mathrm{pt}$ for the terminal maps from X and Y to pt. We have a bijection of sets

$$\mathsf{Sets}(X,Y) \cong \Pi_{!_X}(!_X^*(Y,!_Y)).$$

Proof. **??**, *Spaces of Sections*: Indeed, we have

$$\begin{aligned}\Pi_{!_X}((E,\phi)) &\overset{\text{def}}{=} \coprod_{\star\in\mathrm{pt}} \prod_{k\in !_X^{-1}(\star)} \phi_X^{-1}(k)\\ &= \prod_{x\in X} \phi_X^{-1}(x)\\ &\cong \{h \in \mathsf{Sets}(X,E) \mid \phi_X \circ h = \mathrm{id}_X\}\\ &\overset{\text{def}}{=} \Gamma_X(\phi).\end{aligned}$$

??, *Function Spaces*: Indeed, we have

$$\begin{aligned}\Pi_{!_X}(!_X^*(Y,!_Y)) &\overset{\text{def}}{=} \Pi_{!_X}(X \times_{!_X,\mathrm{pt},!_Y} Y)\\ &\overset{\text{def}}{=} \coprod_{\star\in\mathrm{pt}} \prod_{x\in !_X^{-1}(\star)} \mathrm{pr}_1^{-1}(x)\\ &= \prod_{x\in X} Y\\ &\cong \mathsf{Sets}(X,Y).\end{aligned}$$

This finishes the proof. □

Proposition 2.3.1.3. Let $f \colon K \longrightarrow K'$ be a function.

1. *Functoriality.* The assignment $X \mapsto \Pi_f(X)$ defines a functor

$$\Pi_f \colon \mathsf{FibSets}(K) \longrightarrow \mathsf{FibSets}(K'),$$

where

- *Action on Objects.* For each $(X,\phi_X) \in \mathrm{Obj}(\mathsf{FibSets}(K))$, we have

$$\Pi_f(X,\phi_X) \overset{\text{def}}{=} \Pi_f(X);$$

- *Action on Morphisms.* For each $(X, \phi_X), (Y, \phi_Y) \in \mathsf{Obj}(\mathsf{FibSets}(K))$, the action on Hom-sets

$$\Pi_{f|X,Y} \colon \mathrm{Hom}_{\mathsf{FibSets}(K)}(X, Y) \to \mathrm{Hom}_{\mathsf{FibSets}(K')}(\Pi_f(X), \Pi_f(Y))$$

of Π_f at $((X, \phi_X), (Y, \phi_Y))$ is the map sending a morphism of K-fibred sets

$$\xi \colon (X, \phi_X) \to (Y, \phi_Y), \qquad \begin{tikzcd} X \arrow[rr, "\xi"] \arrow[dr, "\phi_X"'] & & Y \arrow[dl, "\phi_Y"] \\ & K & \end{tikzcd}$$

to the morphism

$$\Pi_f(\xi) \colon (\Pi_f(X), \Pi_f(\phi_X)) \to (\Pi_f(Y), \Pi_f(\phi_Y)) \qquad \begin{tikzcd} \Pi_f(X) \arrow[rr, "\Pi_f(\xi)"] \arrow[dr, "\Pi_f(\phi_X)"'] & & \Pi_f(Y) \arrow[dl, "\Pi_f(\phi_Y)"] \\ & K & \end{tikzcd}$$

of K'-fibred sets given by[8]

$$[\Pi_f(\xi)]((x_k)_{k \in f^{-1}(k')}) \stackrel{\text{def}}{=} (\xi(x_k))_{k \in f^{-1}(k')}$$

for each $(x_k)_{k \in f^{-1}(k')} \in \Pi_f(X)$.

2. *Interaction With Fibres.* We have a bijection of sets 00SV

$$\Pi_f(\phi_X)^{-1}(k') \cong \prod_{k \in f^{-1}(k')} \phi_X^{-1}(k);$$

for each $k' \in K'$.

3. *Construction Using the Internal Hom.* We have 00SW

$$\Pi_f(X, \phi_X) = (K' \times_{\mathbf{Hom}_{\mathsf{FibSets}(K')}((K,f),(K,f))} \mathbf{Hom}_{\mathsf{FibSets}(K')}((K, f), (X, f \circ \phi_X)), \mathrm{pr}_1),$$

[8] Note that we indeed have $\xi(x_k) \in \phi_Y^{-1}(k)$, since

$$\begin{aligned} \phi_Y(\xi(x_k)) &= [\phi_Y \circ \xi](x_k) \\ &= \phi_X(x_k) \\ &= k, \end{aligned}$$

where we have used that ξ is a morphism of K-fibred sets for the second equality.

forming a pullback diagram

$$\begin{tikzcd}
\Pi_f(X,\phi_X) \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Hom}_{\mathsf{FibSets}(K')}((K,f),(X,f\circ\phi_X)) \arrow[d, "(\phi_X)_*"] \\
K' \arrow[r, "I"'] & \mathbf{Hom}_{\mathsf{FibSets}(K')}((K,f),(K,f)),
\end{tikzcd}$$

where the bottom map is given by

$$I(k') \stackrel{\text{def}}{=} \mathrm{id}_{f^{-1}(k')}$$

for each $k' \in K'$ and where $\mathbf{Hom}_{\mathsf{FibSets}(K')}$ denotes the internal Hom of $\mathsf{FibSets}(K')$ of **??**.

4. *Internal Homs via Dependent Products.* We have 00SX

$$\mathbf{Hom}_{\mathsf{FibSets}(K)}(X,Y) \cong \Pi_{\phi_X}(\phi_X^*(Y)).$$

Proof. **??**, *Functoriality*: Omitted.
??, *Interaction With Fibres*: Clear.
??, *Construction Using the Internal Hom*: Using the explicit formula for pullbacks of sets given in Constructions With Sets, Definition 1.3.1.1, we see that the pullback

$$K' \times_{\mathbf{Hom}_{\mathsf{FibSets}(K')}((K,f),(K,f))} \mathbf{Hom}_{\mathsf{FibSets}(K')}((K,f),(X,f\circ\phi_X))$$

is given by

$$\left\{(k',h) \in \coprod_{k'\in K'} \mathsf{Sets}(f^{-1}(k'),\phi_X^{-1}(f^{-1}(k'))) \;\middle|\; \phi_X \circ h = \mathrm{id}_{f^{-1}(k')}\right\},$$

which is isomorphic to

$$\coprod_{k'\in K'} \left\{h \in \mathsf{Sets}(f^{-1}(k'),\phi_X^{-1}(f^{-1}(k'))) \;\middle|\; \phi_X \circ h = \mathrm{id}_{f^{-1}(k')}\right\}.$$

We claim that

$$\left\{h \in \mathsf{Sets}(f^{-1}(k'),\phi_X^{-1}(f^{-1}(k'))) \;\middle|\; \phi_X \circ h = \mathrm{id}_{f^{-1}(k')}\right\} \cong \prod_{k\in f^{-1}(k')} \phi_X^{-1}(k),$$

so that the pullback is indeed given by $\Pi_f(X)$. There are two cases:

1. If $f^{-1}(k') = \emptyset$, then there is only one map from $f^{-1}(k')$ to $\phi_X^{-1}(f^{-1}(k'))$ (the inclusion), so $\mathsf{Sets}(f^{-1}(k'), \phi_X^{-1}(f^{-1}(k'))) \cong \mathrm{pt}$. Since products indexed by the empty set are isomorphic to pt, the isomorphism follows.

2. Otherwise, by the condition $\phi_X \circ h = \mathrm{id}_{f^{-1}(k')}$, it follows that, for each $k \in f^{-1}(k')$, we must have
$$\phi_X(h(k)) = k,$$
and thus $h(k) \in \phi_X^{-1}(k)$. Therefore, a map from $f^{-1}(k')$ to $\phi_X^{-1}(f^{-1}(k'))$ consists of a choice of an element from $\phi_X^{-1}(k)$ for each $k \in f^{-1}(k')$, which is precisely given by an element of the product $\prod_{k \in f^{-1}(k')} \phi_X^{-1}(k)$, showing the bijection to be true.

??, *Internal Homs via Dependent Products*: Indeed we have

$$\begin{aligned}
\Pi_{\phi_X}(\phi_X^*(Y)) &\stackrel{\text{def}}{=} \Pi_{\phi_X}(X \times_K Y) \\
&\stackrel{\text{def}}{=} \coprod_{k \in K} \prod_{x \in \phi_X^{-1}(k)} \mathrm{pr}_1^{-1}(x) \\
&\cong \coprod_{k \in K} \prod_{x \in \phi_X^{-1}(k)} \{y \in Y \mid \phi_X(x) = \phi_Y(y)\} \\
&\cong \coprod_{k \in K} \prod_{x \in \phi_X^{-1}(k)} \{y \in Y \mid k = \phi_Y(y)\} \\
&\cong \coprod_{k \in K} \prod_{x \in \phi_X^{-1}(k)} \phi_Y^{-1}(k) \\
&\cong \coprod_{k \in K} \mathsf{Sets}(\phi_X^{-1}(k), \phi_Y^{-1}(k)) \\
&\stackrel{\text{def}}{=} \mathbf{Hom}_{\mathsf{FibSets}(K)}(X, Y).
\end{aligned}$$

This finishes the proof. □

2.4 Internal Homs 00SY

Let K be a set and let (X, ϕ_X) and (Y, ϕ_Y) be K-fibred sets.

Definition 2.4.1.1. The **internal Hom of fibred sets from** (X, ϕ_X) **to** (Y, ϕ_Y) is the fibred set $\mathbf{Hom}_{\mathsf{FibSets}(K)}(X, Y)$ consisting of

- *The Underlying Set.* The set $\mathbf{Hom}_{\mathsf{FibSets}(K)}(X, Y)$ defined by
$$\mathbf{Hom}_{\mathsf{FibSets}(K)}(X, Y) \stackrel{\text{def}}{=} \coprod_{k \in K} \mathsf{Sets}(\phi_X^{-1}(k), \phi_Y^{-1}(k));$$

- *The Fibration.* The map of sets[9]

$$\phi_{\mathbf{Hom}_{\mathsf{FibSets}(K)}(X,Y)}\colon \underbrace{\mathbf{Hom}_{\mathsf{FibSets}(K)}(X,Y)}_{\coprod_{k\in K}\mathsf{Sets}(\phi_X^{-1}(k),\phi_Y^{-1}(k))} \to K$$

defined by sending a map $f\colon \phi_X^{-1}(k) \to \phi_Y^{-1}(k)$ to its index $k \in K$.

Proof. Omitted. □

Proposition 2.4.1.2. 00T0 Let K be a set and let (X, ϕ_X) and (Y, ϕ_Y) be K-fibred sets.

1. *Functoriality.* 00T1 Let (X, ϕ_X) and (Y, ϕ_Y) be K-fibred sets.
   (a) The assignment $X \mapsto \mathbf{Hom}_{\mathsf{FibSets}(K)}(X, Y)$ defines a functor
   $$\mathbf{Hom}_{\mathsf{FibSets}(K)}(X, -)\colon \mathsf{FibSets}(K) \to \mathsf{FibSets}(K).$$
   (b) The assignment $Y \mapsto \mathbf{Hom}_{\mathsf{FibSets}(K)}(X, Y)$ defines a functor
   $$\mathbf{Hom}_{\mathsf{FibSets}(K)}(-, Y)\colon \mathsf{FibSets}(K)^{\mathsf{op}} \to \mathsf{FibSets}(K).$$
   (c) The assignment $(X, Y) \mapsto \mathbf{Hom}_{\mathsf{FibSets}(K)}(X, Y)$ defines a functor
   $$\mathbf{Hom}_{\mathsf{FibSets}(K)}(-_1, -_2)\colon \mathsf{FibSets}(K)^{\mathsf{op}} \times \mathsf{FibSets}(K) \to \mathsf{FibSets}(K).$$
2. *Internal Homs via Dependent Products.* We have 00T2
   $$\mathbf{Hom}_{\mathsf{FibSets}(K)}(X, Y) \cong \Pi_{\phi_X}(\phi_X^*(Y)).$$

Proof. **??**, *Functoriality*: Omitted.
??, *Internal Homs via Dependent Products*: This was proved in **??** of **??**. □

2.5 Adjointness for Fibred Sets 00T3

Let $f\colon K \to K'$ be a map of sets.

---

[9] The fibres of the internal **Hom** of $\mathsf{FibSets}(K)$ are precisely the sets $\mathsf{Sets}(\phi_X^{-1}(k), \phi_Y^{-1}(k))$, i.e. we have

$$\phi_{\mathbf{Hom}_{\mathsf{FibSets}(K)}(X,Y)|k} \cong \mathsf{Sets}(\phi_X^{-1}(k), \phi_Y^{-1}(k))$$

for each $k \in K$.

Proposition 2.5.1.1. We have a triple adjunction 02T4

$$
(\Sigma_f \dashv f^* \dashv \Pi_f)\colon\quad
\begin{tikzcd}
\mathsf{FibSets}(K) \arrow[r, bend left, "\Sigma_f"] \arrow[r, bend right, "\Pi_f"'] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, phantom, shift right=3, "\perp"] & \mathsf{FibSets}(K') \arrow[l, "f^*" description]
\end{tikzcd}.
$$

We offer two proofs. The first uses the corresponding adjunction for indexed sets (Indexed Sets, Proposition 4.5.1.1) and the un/straightening equivalence together with its compatibility with dependent sums and products to "transfer" the adjunction to fibred sets, while the second is a direct proof.

Proof. The Adjunction $\Sigma_f \dashv f^*$: The adjunction

$$
(\Sigma_f \dashv f^*)\colon\quad
\begin{tikzcd}
\mathsf{ISets}(K) \arrow[r, bend left, "\Sigma_f"] \arrow[r, phantom, "\perp"] & \mathsf{ISets}(K') \arrow[l, bend left, "f^*"]
\end{tikzcd}
$$

of Indexed Sets, Proposition 4.5.1.1 gives a unit and counit of the form

$$
\begin{aligned}
\eta &\colon \mathrm{id}_{\mathsf{ISets}(K)} \Longrightarrow \Sigma_f \circ f^*,\\
\epsilon &\colon f^* \circ \Sigma_f \Longrightarrow \mathrm{id}_{\mathsf{ISets}(K')}.
\end{aligned}
$$

With these in hand, we construct natural transformations

$$
\begin{aligned}
\eta' &\colon \mathrm{id}_{\mathsf{FibSets}(K)} \Longrightarrow \Sigma_f \circ f^*,\\
\epsilon' &\colon f^* \circ \Sigma_f \Longrightarrow \mathrm{id}_{\mathsf{FibSets}(K')}
\end{aligned}
$$

as follows:

1. *The Unit.* We define $\eta'\colon \mathrm{id}_{\mathsf{FibSets}(K)} \Longrightarrow \Sigma_f \circ f^*$ as the pasting of the diagram

$$
\begin{tikzcd}
 & & \mathsf{FibSets}(K') \arrow[drr, "f^*"] & & \\
\mathsf{FibSets}(K) \arrow[urr, "\Sigma_f"] & (1) & \mathsf{ISets}(K') \arrow[u, "\mathrm{St}_{K'}"] \arrow[dr, "f^*"] & (2) & \mathsf{FibSets}(K) \\
(3) & \mathsf{ISets}(K) \arrow[ul, "\mathrm{St}_K"] \arrow[ur, "\Sigma_f"] \arrow[rr, "\mathrm{id}_{\mathsf{ISets}(K)}" description] \arrow[r, phantom, shift left=4, "\Uparrow\eta"] & & \mathsf{ISets}(K) \arrow[ur, "\mathrm{St}_K"'] & (4) \\
\mathsf{FibSets}(K) \arrow[uu, "\mathrm{id}_{\mathsf{FibSets}(K)}"] \arrow[ur, "\mathrm{Un}_K"] \arrow[rrrr, "\mathrm{id}_{\mathsf{FibSets}(K)}"'] & & (5) & & \mathsf{FibSets}(K), \arrow[uu, "\mathrm{id}_{\mathsf{FibSets}(K)}"'] \arrow[ul, "\mathrm{Un}_K"']
\end{tikzcd}
$$

where:

(a) Subdiagram (1) commutes by Un/Straightening for Indexed and Fibred Sets, **??** of **??**.

(b) Subdiagram (2) commutes by Un/Straightening for Indexed and Fibred Sets, **??** of **??**.

(c) Subdiagram (3) commutes by Un/Straightening for Indexed and Fibred Sets, **??**.

(d) Subdiagram (4) commutes by Un/Straightening for Indexed and Fibred Sets, **??**.

(e) Subdiagram (5) commutes by unitality of composition.

2. *The Counit.* We define $\epsilon' : f^* \circ \Sigma_f \Longrightarrow \mathrm{id}_{\mathsf{FibSets}(K')}$ as the pasting of the diagram

$$\begin{tikzcd}
\mathsf{FibSets}(K') \arrow[rrrr, "\mathrm{id}_{\mathsf{FibSets}(K')}"] \arrow[dr, "\mathrm{Un}_K"] \arrow[dd, "\mathrm{id}_{\mathsf{FibSets}(K')}"'] & & & & \mathsf{FibSets}(K') \arrow[dl, "\mathrm{Un}_K"'] \arrow[dd, "\mathrm{id}_{\mathsf{FibSets}(K')}"] \\
(2) & \mathsf{ISets}(K') \arrow[rr, "\mathrm{id}_{\mathsf{ISets}(K)}"] \arrow[dl, "\mathrm{St}_{K'}"'] \arrow[dr, "f^*"] & (1) & \mathsf{ISets}(K') \arrow[dr, "\mathrm{St}_{K'}"] & (3) \\
\mathsf{FibSets}(K') \arrow[drr, "f^*"'] & (4) & \mathsf{ISets}(K) \arrow[ur, "\Sigma_f"] \arrow[d, "\mathrm{St}_K"] \arrow[u, phantom, "\Uparrow \epsilon"] & (5) & \mathsf{FibSets}(K') \\
& & \mathsf{FibSets}(K) \arrow[urr, "\Sigma_f"'] & &
\end{tikzcd}$$

where:

(a) Subdiagram (1) commutes by unitality of composition.

(b) Subdiagram (2) commutes by Un/Straightening for Indexed and Fibred Sets, **??**.

(c) Subdiagram (3) commutes by Un/Straightening for Indexed and Fibred Sets, **??**.

(d) Subdiagram (4) commutes by Un/Straightening for Indexed and Fibred Sets, **??** of **??**.

(e) Subdiagram (5) commutes by Un/Straightening for Indexed and Fibred Sets, **??** of **??**.

Next, we prove the left triangle identity,

$$\begin{tikzcd}
& \mathsf{FibSets}(K') \arrow[rr, "\mathrm{id}_{\mathsf{FibSets}(K')}"] \arrow[dr, "f^*"] \arrow[d, phantom, "\Uparrow\eta"] & \arrow[d, phantom, "\Uparrow\epsilon"] & \mathsf{FibSets}(K') \\
\mathsf{FibSets}(K) \arrow[ur, "\Sigma_f"] \arrow[rr, "\mathrm{id}_{\mathsf{FibSets}(K)}"'] & {} & \mathsf{FibSets}(K) \arrow[ur, "\Sigma_f"'] &
\end{tikzcd}
=
\begin{tikzcd}
& \mathsf{FibSets}(K') \arrow[rr, "\mathrm{id}_{\mathsf{FibSets}(K')}"] & & \mathsf{FibSets}(K'), \\
\mathsf{FibSets}(K) \arrow[ur, "\Sigma_f"] \arrow[rr, "\mathrm{id}_{\mathsf{FibSets}(K)}"'] & \arrow[r, phantom, "\Leftarrow \mathrm{id}_{\Sigma_f}"] & \mathsf{FibSets}(K) \arrow[ur, "\Sigma_f"'] &
\end{tikzcd}$$

whose left side in our case looks like this:

$$\begin{tikzcd}
& & \mathsf{FibSets}(K') \arrow[rrr, "\mathrm{id}_{\mathsf{FibSets}(K')}"] \arrow[drr, "f^*"] & & & \mathsf{FibSets}(K') \arrow[rrrr, "\mathrm{id}_{\mathsf{FibSets}(K')}"] \arrow[dd, "\mathrm{id}_{\mathsf{FibSets}(K')}"'] \arrow[dr, "\mathrm{Un}_K"] & & & & \mathsf{FibSets}(K') \arrow[dl, "\mathrm{Un}_K"'] \arrow[dd, "\mathrm{id}_{\mathsf{FibSets}(K')}"] \\
\mathsf{FibSets}(K) \arrow[urr, "\Sigma_f"] & & \mathsf{ISets}(K') \arrow[u, "\mathrm{St}_{K'}"] \arrow[dr, "f^*"] \arrow[d, phantom, "\Uparrow\eta"] & & \mathsf{FibSets}(K) & & \mathsf{ISets}(K') \arrow[rr, "\mathrm{id}_{\mathsf{ISets}(K)}"] \arrow[dl, "\mathrm{St}_{K'}"'] \arrow[dr, "f^*"] & \arrow[d, phantom, "\Uparrow\epsilon"] & \mathsf{ISets}(K') \arrow[dr, "\mathrm{St}_{K'}"] & \\
& \mathsf{ISets}(K) \arrow[ul, "\mathrm{St}_K"] \arrow[ur, "\Sigma_f"] \arrow[rr, "\mathrm{id}_{\mathsf{ISets}(K)}"] & {} & \mathsf{ISets}(K) \arrow[ur, "\mathrm{St}_K"] & & \mathsf{FibSets}(K') \arrow[drr, "f^*"'] & & \mathsf{ISets}(K) \arrow[ur, "\Sigma_f"] \arrow[d, "\mathrm{St}_K"] & & \mathsf{FibSets}(K'). \\
\mathsf{FibSets}(K) \arrow[uu, "\mathrm{id}_{\mathsf{FibSets}(K)}"] \arrow[ur, "\mathrm{Un}_K"] \arrow[rrrr, "\mathrm{id}_{\mathsf{FibSets}(K)}"'] & & & & \mathsf{FibSets}(K) \arrow[ul, "\mathrm{Un}_K"'] \arrow[uu, "\mathrm{id}_{\mathsf{FibSets}(K)}"'] \arrow[rrr, "\mathrm{id}_{\mathsf{FibSets}(K)}"'] & & & \mathsf{FibSets}(K) \arrow[urr, "\Sigma_f"'] & &
\end{tikzcd}$$

It can be rearranged into

$$\begin{tikzcd}
& & \mathsf{FibSets}(K') \arrow[rrr, "\mathrm{id}_{\mathsf{FibSets}(K')}"] & & & \mathsf{FibSets}(K') \arrow[rrrr, "\mathrm{id}_{\mathsf{FibSets}(K')}"] \arrow[d, phantom, "(1)"] & & & & \mathsf{FibSets}(K') \arrow[dl, "\mathrm{Un}_K"'] \arrow[dd, "\mathrm{id}_{\mathsf{FibSets}(K')}"] \\
\mathsf{FibSets}(K) \arrow[urr, "\Sigma_f"] & & \mathsf{ISets}(K') \arrow[u, "\mathrm{St}_{K'}"] \arrow[rrrr, "\mathrm{id}_{\mathsf{ISets}(K')}"] \arrow[dr, "f^*"] \arrow[d, phantom, "\Uparrow\eta"] & & \arrow[d, phantom, "(2)"] & & \mathsf{ISets}(K') \arrow[rr, "\mathrm{id}_{\mathsf{ISets}(K)}"] \arrow[dr, "f^*"'] & \arrow[d, phantom, "\Uparrow\epsilon"] & \mathsf{ISets}(K') \arrow[dr, "\mathrm{St}_{K'}"] & \\
& \mathsf{ISets}(K) \arrow[ul, "\mathrm{St}_K"] \arrow[ur, "\Sigma_f"] \arrow[rr, "\mathrm{id}_{\mathsf{ISets}(K)}"] & {} & \mathsf{ISets}(K) \arrow[rrrr, "\mathrm{id}_{\mathsf{ISets}(K')}"] \arrow[d, phantom, "(3)"] & & & & \mathsf{ISets}(K) \arrow[ur, "\Sigma_f"] \arrow[d, "\mathrm{St}_K"] & & \mathsf{FibSets}(K'), \\
\mathsf{FibSets}(K) \arrow[uu, "\mathrm{id}_{\mathsf{FibSets}(K)}"] \arrow[ur, "\mathrm{Un}_K"] \arrow[rrrr, "\mathrm{id}_{\mathsf{FibSets}(K)}"'] & & & & \mathsf{FibSets}(K) \arrow[rrr, "\mathrm{id}_{\mathsf{FibSets}(K)}"'] & & & \mathsf{FibSets}(K) \arrow[urr, "\Sigma_f"'] & &
\end{tikzcd}$$

where:

1. Subdiagram (1) commutes by Un/Straightening for Indexed and Fibred Sets, **??**.
2. Subdiagram (2) commutes by unitality of composition.
3. Subdiagram (3) commutes by Un/Straightening for Indexed and Fibred Sets, **??**.

And then, it can be rearranged into

$$\begin{tikzcd}
& & \mathsf{FibSets}(K') \arrow[rrrrrrr, "\mathrm{id}_{\mathsf{FibSets}(K')}"] & & & & & & & \mathsf{FibSets}(K') \arrow[dl, "\mathrm{Un}_K"'] \arrow[dd, "\mathrm{id}_{\mathsf{FibSets}(K')}"] \\
\mathsf{FibSets}(K) \arrow[urr, "\Sigma_f"] & & \mathsf{ISets}(K') \arrow[u, "\mathrm{St}_{K'}"] \arrow[rrrrrr, "\mathrm{id}_{\mathsf{ISets}(K')}"] \arrow[drrrrr, "f^*"] \arrow[d, phantom, "\eta\Uparrow"] & & & & & \arrow[d, phantom, "\Uparrow\epsilon"] & \mathsf{ISets}(K') \arrow[dr, "\mathrm{St}_{K'}"] & \\
& \mathsf{ISets}(K) \arrow[ul, "\mathrm{St}_K"] \arrow[ur, "\Sigma_f"] \arrow[rrrrrr, "\mathrm{id}_{\mathsf{ISets}(K)}"] & {} & & & & & \mathsf{ISets}(K) \arrow[ur, "\Sigma_f"] \arrow[d, "\mathrm{St}_K"] & & \mathsf{FibSets}(K'), \\
\mathsf{FibSets}(K) \arrow[uu, "\mathrm{id}_{\mathsf{FibSets}(K)}"] \arrow[ur, "\mathrm{Un}_K"] \arrow[rrrrrrr, "\mathrm{id}_{\mathsf{FibSets}(K)}"'] & & & & & & & \mathsf{FibSets}(K) \arrow[urr, "\Sigma_f"'] & &
\end{tikzcd}$$

which by the left triangle identity for (η, ϵ), becomes

$$\begin{tikzcd}
 & & \mathsf{FibSets}(K') \arrow[rrrr, "\mathrm{id}_{\mathsf{FibSets}(K')}"] & & & & \mathsf{FibSets}(K') \arrow[dl, "\mathrm{Un}_K"'] \arrow[dd, "\mathrm{id}_{\mathsf{FibSets}(K')}"] \\
\mathsf{FibSets}(K) \arrow[urr, "\Sigma_f"] & & \mathsf{ISets}(K') \arrow[u, "\mathrm{St}_{K'}"] \arrow[rrr, "\mathrm{id}_{\mathsf{ISets}(K')}"] \arrow[drr, phantom, "\mathrm{id}"] & & & \mathsf{ISets}(K') \arrow[dr, "\mathrm{St}_{K'}"] & \\
 & \mathsf{ISets}(K) \arrow[ul, "\mathrm{St}_K"] \arrow[ur, "\Sigma_f"'] \arrow[rrr, "\mathrm{id}_{\mathsf{ISets}(K)}"] & & & \mathsf{ISets}(K) \arrow[ur, "\Sigma_f"] \arrow[d, "\mathrm{St}_K"] & & \mathsf{FibSets}(K'), \\
\mathsf{FibSets}(K) \arrow[uu, "\mathrm{id}_{\mathsf{FibSets}(K)}"] \arrow[ur, "\mathrm{Un}_K"] \arrow[rrrr, "\mathrm{id}_{\mathsf{FibSets}(K)}"'] & & & & \mathsf{FibSets}(K) \arrow[urr, "\Sigma_f"'] & &
\end{tikzcd}$$

finishing the proof of the left triangle identity. The proof of the right triangle identity is similar, and is thus omitted.

The Adjunction $f^* \dashv \Pi_f$: This proof is similar to the proof of the adjunction $\Sigma_f \dashv f^*$, and is thus omitted. $\square$

We proceed to the direct proof of **??**.

Proof. The Adjunction $\Sigma_f \dashv f^*$: We claim there's a bijection

$$\mathrm{Hom}_{\mathsf{FibSets}(K')}(\Sigma_f(X), Y) \cong \mathrm{Hom}_{\mathsf{FibSets}(K)}(X, f^*(Y)),$$

natural in $(X, \phi_X) \in \mathrm{Obj}(\mathsf{FibSets}(K))$ and $(Y, \phi_Y) \in \mathrm{Obj}(\mathsf{FibSets}(K'))$:

- *Map I.* We define a map

$$\Phi_{X,Y} : \mathrm{Hom}_{\mathsf{FibSets}(K')}(\Sigma_f(X), Y) \to \mathrm{Hom}_{\mathsf{FibSets}(K)}(X, f^*(Y)),$$

  by sending a morphism

$$\xi : \Sigma_f(X) \to Y, \qquad \begin{tikzcd} X \arrow[rr, "\xi"] \arrow[d, "\phi_X"'] & & Y \arrow[ddl, "\phi_Y"] \\ K \arrow[dr, "f"'] & & \\ & K' & \end{tikzcd}$$

  of K'-fibred sets to the morphism

$$\xi^\dagger : X \to f^*(Y), \qquad \begin{tikzcd} X \arrow[rr, "\xi^\dagger"] \arrow[dr, "\phi_X"'] & & K \times_{K'} Y \arrow[dl, "\mathrm{pr}_1"] \\ & K' & \end{tikzcd}$$

of K-fibred sets given by the dashed morphism in the diagram

$$\begin{tikzcd}
X \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left, "\xi"] \arrow[ddr, bend right, "\phi_X"'] & & \\
& K \times_{K'} Y \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "\phi_Y"] \\
& K \arrow[r, "f"'] & K'.
\end{tikzcd}$$

- *Map II.* We define a map

$$\Psi_{X,Y} \colon \mathrm{Hom}_{\mathsf{FibSets}(K)}(X, f^*(Y)) \to \mathrm{Hom}_{\mathsf{FibSets}(K')}(\Sigma_f(X), Y),$$

given by sending a map

$$\xi \colon X \to f^*(Y), \qquad \begin{tikzcd} X \arrow[rr, "\xi"] \arrow[dr, "\phi_X"'] & & K \times_{K'} Y \arrow[dl, "\mathrm{pr}_1"] \\ & K' & \end{tikzcd}$$

of K'-fibred sets to the map

$$\xi^\dagger \colon \Sigma_f(X) \to Y, \qquad \begin{tikzcd} X \arrow[rr, "\xi^\dagger"] \arrow[d, "\phi_X"'] & & Y \arrow[ddl, "\phi_Y"] \\ K \arrow[dr, "f"'] & & \\ & K' & \end{tikzcd}$$

of K-fibred sets given by

$$\xi^\dagger \overset{\text{def}}{=} \mathrm{pr}_2 \circ \xi,$$

where we indeed have

$$\begin{aligned}
\phi_Y \circ (\mathrm{pr}_2 \circ \xi) &= (\phi_Y \circ \mathrm{pr}_2) \circ \xi \\
&= (f \circ \mathrm{pr}_1) \circ \xi && \text{(by the pullback square of } K \times_{K'} Y) \\
&= f \circ (\mathrm{pr}_1 \circ \xi) \\
&= f \circ \phi_X. && \text{(since } \xi \text{ is a morphism of } K'\text{-fibred sets)}
\end{aligned}$$

- *Naturality I.* We need to show that, given a morphism
$$\alpha\colon (X,\phi_X) \longrightarrow (X',\phi_{X'})$$
of K-fibred sets, the diagram
$$\begin{tikzcd}
\mathrm{Hom}_{\mathsf{FibSets}(K')}(\Sigma_f(X'),Y) \arrow[r, "\Phi_{X',Y}"] \arrow[d, "\Sigma_f(\alpha)^*"'] & \mathrm{Hom}_{\mathsf{FibSets}(K)}(X',f^*(Y)), \arrow[d, "\alpha^*"] \\
\mathrm{Hom}_{\mathsf{FibSets}(K')}(\Sigma_f(X),Y) \arrow[r, "\Phi_{X,Y}"'] & \mathrm{Hom}_{\mathsf{FibSets}(K)}(X,f^*(Y))
\end{tikzcd}$$
commutes. Indeed, given a morphism
$$\xi\colon \Sigma_f(X') \longrightarrow Y, \qquad \begin{tikzcd}
X' \arrow[rr, "\xi"] \arrow[dr, "\phi_{X'}"'] & & Y \arrow[ddl, "\phi_Y"] \\
& K \arrow[d, "f"'] & \\
& K' &
\end{tikzcd}$$
of K'-fibred-sets, the map $\Phi_{X',Y}(\xi)\circ\alpha$ is the composition, coloured in vermillion, of the dashed arrow with α in the diagram
$$\begin{tikzcd}
X \arrow[dr, "\alpha"] \arrow[ddrr, dashed, bend left, "\exists!"] \arrow[ddrrr, bend left, "\xi\circ\alpha"] \arrow[dddrr, bend right, "\phi_{X'}\circ\alpha"'] & & & \\
& X' \arrow[dr, dashed, "\exists!"] \arrow[drr, "\xi"] \arrow[ddr, "\phi_{X'}"'] & & \\
& & K\times_{K'}Y \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "\phi_Y"] \\
& & K \arrow[r, "f"'] & K',
\end{tikzcd}$$
while $\Phi_{X,Y}(\xi\circ\Sigma_f(\alpha))$ is given by the dashed arrow, coloured in blue. Since both the blue arrow and the vermillion arrow make the outer pullback diagram commute, it follows from the universal property of the pullback that they must be equal, i.e. that
$$\Phi_{X',Y}(\xi)\circ\alpha = \Phi_{X,Y}(\xi\circ\Sigma_f(\alpha)),$$
showing that the naturality diagram above indeed commutes.

- *Naturality II.* We need to show that, given a morphism

$$\beta\colon (Y,\phi_Y) \longrightarrow (Y',\phi_{Y'})$$

of K-fibred sets, the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathsf{FibSets}(K')}(\Sigma_f(X),Y) \arrow[r, "\Phi_{X,Y}"] \arrow[d, "\beta_*"'] & \mathrm{Hom}_{\mathsf{FibSets}(K)}(X,f^*(Y)), \arrow[d, "f^*(\beta)_*"] \\
\mathrm{Hom}_{\mathsf{FibSets}(K')}(\Sigma_f(X),Y') \arrow[r, "\Phi_{X,Y'}"'] & \mathrm{Hom}_{\mathsf{FibSets}(K)}(X,f^*(Y'))
\end{tikzcd}$$

commutes. Indeed, given a morphism

$$\xi\colon \Sigma_f(X') \longrightarrow Y, \qquad \begin{tikzcd}
X' \arrow[rr, "\xi"] \arrow[dr, "\phi_{X'}"'] & & Y \arrow[ddl, "\phi_Y"] \\
& K \arrow[d, "f"'] & \\
& K' &
\end{tikzcd}$$

of K'-fibred-sets, the map $f^*(\beta)\circ\Phi_{X,Y}(\xi)$ is the composition, coloured in vermillion, of the dashed arrow from X to $K\times_{K'} Y$ with the dashed arrow from $K\times_{K'} Y$ to $K\times_{K'} Y'$ in the diagram

$$\begin{tikzcd}
X \arrow[drrr, bend left, "\xi"] \arrow[dr, dashed, color=orange, "\exists!"] \arrow[ddrr, dashed, bend left, color=blue, "\exists!"] \arrow[dddr, bend right, "\phi_X"'] & & & & \\
& K\times_{K'} Y \arrow[rr] \arrow[dd] \arrow[dr, dashed, color=orange] \arrow[dr, phantom, very near start, "\lrcorner"] & & Y \arrow[dr, "\beta"] \arrow[dd, "\phi_Y"] & \\
& & K\times_{K'} Y' \arrow[rr] \arrow[dd] \arrow[dr, phantom, very near start, "\lrcorner"] & & Y' \arrow[dd, "\phi_{Y'}"] \\
& K \arrow[rr, "f"] \arrow[dr, equal] & & K \arrow[dr, equal] & \\
& & K \arrow[rr, "f"'] & & K',
\end{tikzcd}$$

while $\Phi_{X,Y'}(\beta\circ\xi)$ is given by the dashed arrow from X to $K\times_{K'} Y'$, coloured in blue. Since both the blue arrow and the vermillion arrow make the outer pullback diagram for $K \times_{K'} Y'$ commute, it follows from the universal property of the pullback that they must be equal, i.e. that

$$f^*(\beta) \circ \Phi_{X,Y}(\xi) = \Phi_{X,Y'}(\beta \circ \xi),$$

showing that the naturality diagram above indeed commutes.

- *Invertibility I.* We claim that

$$\Psi_{X,Y} \circ \Phi_{X,Y} = \mathrm{id}_{\mathrm{Hom}_{\mathsf{FibSets}(K')}(\Sigma_f(X),Y)}.$$

Indeed, $\Phi_{X,Y}$ sends a map

$$\xi \colon \Sigma_f(X) \longrightarrow Y, \qquad \begin{tikzcd} X \arrow[rr, "\xi"] \arrow[dr, "\phi_X"'] & & Y \arrow[ddl, "\phi_Y"] \\ & K \arrow[d, "f"'] & \\ & K' & \end{tikzcd}$$

of K'-fibred sets to the dashed morphism in the diagram

$$\begin{tikzcd} X \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left, "\xi"] \arrow[ddr, bend right, "\phi_X"'] & & \\ & K \times_{K'} Y \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "\phi_Y"] \\ & K \arrow[r, "f"'] & K', \end{tikzcd}$$

and $\Psi_{X,Y}$ then postcomposes that map with pr_2, which, by the commutativity of the diagram above, is ξ again, showing the claimed equality to be true.

- *Invertibility II.* We claim that

$$\Phi_{X,Y} \circ \Psi_{X,Y} = \mathrm{id}_{\mathrm{Hom}_{\mathsf{FibSets}(K)}(X,f^*(Y))}.$$

Indeed, $\Psi_{X,Y}$ sends a map

$$\xi \colon X \longrightarrow f^*(Y), \qquad \begin{tikzcd} X \arrow[rr, "\xi"] \arrow[dr, "\phi_X"'] & & K \times_{K'} Y \arrow[dl, "\mathrm{pr}_1"] \\ & K' & \end{tikzcd}$$

of K'-fibred sets to $\mathrm{pr}_2 \circ \xi$, which is then sent by $\Phi_{X,Y}$ to the dashed morphism in the diagram

$$\begin{tikzcd}
X \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left, "\mathrm{pr}_2\circ\xi"] \arrow[ddr, bend right, "\phi_X"'] & & \\
& K \times_{K'} Y \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "\phi_Y"] \\
& K \arrow[r, "f"'] & K',
\end{tikzcd}$$

which, by the commutativity of the subdiagram marked with (†), is given by ξ again, showing the claimed equality to be true.

The Adjunction $f^* \dashv \Pi_f$: We claim there's a bijection

$$\mathrm{Hom}_{\mathsf{FibSets}(K)}(f^*(X), Y), \cong \mathrm{Hom}_{\mathsf{FibSets}(K')}(X, \Pi_f(Y))$$

natural in $(X, \phi_X) \in \mathsf{Obj}(\mathsf{FibSets}(K'))$ and $(Y, \phi_Y) \in \mathsf{Obj}(\mathsf{FibSets}(K))$:

1. *Map I.* We define a map

$$\Phi_{X,Y} : \mathrm{Hom}_{\mathsf{FibSets}(K)}(f^*(X), Y) \to \mathrm{Hom}_{\mathsf{FibSets}(K')}(X, \Pi_f(Y))$$

defined as follows. Given a morphism

$$\xi : f^*(X) \to Y, \qquad \begin{tikzcd} K \times_{K'} X \arrow[rr, "\xi"] \arrow[dr, "\mathrm{pr}_1"'] & & Y \arrow[dl, "\phi_Y"] \\ & K & \end{tikzcd}$$

of K-fibred sets, where

$$\begin{aligned} f^*(X) &\overset{\text{def}}{=} K \times_{K'} X \\ &\overset{\text{def}}{=} \{(k, x) \in K \times X \mid f(k) = \phi_X(x)\}, \end{aligned}$$

we construct a morphism

$$\xi^\dagger : X \to \Pi_f(Y), \qquad \begin{tikzcd} X \arrow[rr, "\xi^\dagger"] \arrow[dr, "\phi_X"'] & & \Pi_f(Y) \arrow[dl, "\Pi_f(\phi_Y)"] \\ & K' & \end{tikzcd}$$

of K'-fibred sets, where

$$\Pi_f(Y) \stackrel{\text{def}}{=} \coprod_{k' \in K'} \prod_{k \in f^{-1}(k')} \phi_Y^{-1}(k),$$

by defining

$$\xi^\dagger(x) \stackrel{\text{def}}{=} (\xi(k,x))_{k \in f^{-1}(\phi_X(x))}$$

for each $x \in X$. There are two things to be checked here:

- We have $\xi(k,x) \in \phi_Y^{-1}(\phi_X(x))$ since $\phi_Y(\xi(k,x)) = \phi_X(x)$ as ξ is a morphism of K-fibred sets.
- The map $\xi^\dagger$ is indeed a morphism of K'-fibred sets, i.e. we have

  $$\Pi_f(\phi_Y) \circ \xi^\dagger = \phi_X,$$

  since

  $$[\Pi_f(\phi_Y)]\Big((\xi(k,x))_{k \in f^{-1}(\phi_X(x))}\Big) = \phi_X(x)$$

  for each $x \in X$.

2. *Map II.* We define a map

$$\Psi_{X,Y} \colon \mathrm{Hom}_{\mathsf{FibSets}(K')}(X, \Pi_f(Y)) \to \mathrm{Hom}_{\mathsf{FibSets}(K)}(f^*(X), Y)$$

as follows. Given a morphism

$$\xi \colon X \to \Pi_f(Y), \qquad \begin{tikzcd} X \arrow[rr, "\xi"] \arrow[dr, "\phi_X"'] & & \Pi_f(Y) \arrow[dl, "\Pi_f(\phi_Y)"] \\ & K' & \end{tikzcd}$$

of K'-fibred sets, where

$$\Pi_f(Y) \stackrel{\text{def}}{=} \coprod_{k' \in K'} \prod_{k \in f^{-1}(k')} \phi_Y^{-1}(k),$$

we construct a morphism

$$\xi^\dagger \colon f^*(X) \to Y, \qquad \begin{tikzcd} K \times_{K'} X \arrow[rr, "\xi^\dagger"] \arrow[dr, "\mathrm{pr}_1"'] & & Y \arrow[dl, "\phi_Y"] \\ & K & \end{tikzcd}$$

of K-fibred sets, where

$$\begin{aligned} f^*(X) &\stackrel{\text{def}}{=} K \times_{K'} X \\ &\stackrel{\text{def}}{=} \{(k,x) \in K \times X \mid f(k) = \phi_X(x)\}, \end{aligned}$$

by defining

$$\xi^\dagger(k,x) \stackrel{\text{def}}{=} \xi(x)_k$$

for each $(k,x) \in f^*(X)$, where $\xi(x)_k$ is the kth component of $\xi(x) = (y_k)_{k \in f^{-1}(k')}$. We also need to check that $\xi^\dagger$ is a morphism of K-fibred sets, i.e. that

$$\phi_Y \circ \xi^\dagger = \mathrm{pr}_1,$$

or

$$\phi_Y(\xi^\dagger(k,x)) = k,$$

for each $(k,x) \in f^*(X)$, which is clear, since $\xi^\dagger(k,x) \in \phi_Y^{-1}(k)$ by definition.

3. *Naturality I.* We need to show that, given a morphism

$$\alpha \colon (X, \phi_X) \to (X', \phi_{X'})$$

of K'-fibred sets, the diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathsf{FibSets}(K')}(f^*(X'), Y) \arrow[r, "\Phi_{X',Y}"] \arrow[d, "f^*(\alpha)^*"'] & \mathrm{Hom}_{\mathsf{FibSets}(K)}(X', \Pi_f(Y)) \arrow[d, "\alpha^*"] \\ \mathrm{Hom}_{\mathsf{FibSets}(K')}(f^*(X), Y) \arrow[r, "\Phi_{X,Y}"'] & \mathrm{Hom}_{\mathsf{FibSets}(K)}(X, \Pi_f(Y)) \end{tikzcd}$$

commutes. Indeed, given a morphism $\xi \colon f^*(X') \to Y$ of K'-fibred sets, we have

$$\begin{aligned} [[\Phi_{X,Y} \circ f^*(\alpha)](\xi)](x) &\stackrel{\text{def}}{=} [\Phi_{X,Y}(\xi \circ f^*(\alpha))](x) \\ &\stackrel{\text{def}}{=} ([\xi \circ f^*(\alpha)](k,x))_{k \in f^{-1}(\phi_X(x))} \\ &\stackrel{\text{def}}{=} (\xi(k, \alpha(x)))_{k \in f^{-1}(\phi_X(x))} \\ &\stackrel{\text{def}}{=} \alpha^*((\xi(k,x))_{k \in f^{-1}(\phi_X(x))}) \\ &\stackrel{\text{def}}{=} \alpha^*(\xi^\dagger(x)) \\ &\stackrel{\text{def}}{=} [[\alpha^* \circ \Phi_{X,Y}](\xi)](x) \end{aligned}$$

for each $x \in X$.

4. *Naturality II.* We need to show that, given a morphism

$$\beta\colon (Y,\phi_Y) \to (Y',\phi_{Y'})$$

of K-fibred sets, the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathsf{FibSets}(K')}(f^*(X),Y) \arrow[r, "\Phi_{X,Y}"] \arrow[d, "\beta_*"'] & \mathrm{Hom}_{\mathsf{FibSets}(K)}(X,\Pi_f(Y)) \arrow[d, "\Pi_f(\beta)_*"] \\
\mathrm{Hom}_{\mathsf{FibSets}(K')}(f^*(X),Y') \arrow[r, "\Phi_{X,Y'}"'] & \mathrm{Hom}_{\mathsf{FibSets}(K)}(X,\Pi_f(Y'))
\end{tikzcd}$$

commutes. Indeed, given a morphism $\xi\colon X \to \Pi_f(Y)$ of K-fibred sets, we have

$$\begin{aligned}
[[\Phi_{X,Y'} \circ \beta_*](\xi)](x) &\overset{\text{def}}{=} [\Phi_{X,Y'}(\beta\circ\xi)](x) \\
&\overset{\text{def}}{=} [\Phi_{X,Y'}(\beta\circ\xi)](x) \\
&\overset{\text{def}}{=} ([\beta\circ\xi](k,x))_{k\in f^{-1}(\phi_X(x))} \\
&\overset{\text{def}}{=} (\beta(\xi(k,x)))_{k\in f^{-1}(\phi_X(x))} \\
&\overset{\text{def}}{=} \Pi_f(\beta)_*((\xi(k,x))_{k\in f^{-1}(\phi_X(x))}) \\
&\overset{\text{def}}{=} [\Pi_f(\beta)_* \circ \xi^{\dagger}](x) \\
&\overset{\text{def}}{=} [\Pi_f(\beta)_* \circ \Phi_{X,Y'}(\xi)](x)
\end{aligned}$$

for each $x \in X$.

5. *Invertibility I.* We claim that

$$\Psi_{X,Y} \circ \Phi_{X,Y} = \mathrm{id}_{\mathrm{Hom}_{\mathsf{FibSets}(K)}(f^*(X),Y)}.$$

Indeed, given a morphism $\xi\colon f^*(X') \to Y$ of K'-fibred sets, we have

$$\begin{aligned}
[[\Psi_{X,Y} \circ \Phi_{X,Y}](\xi)](k,x) &\overset{\text{def}}{=} [\Psi_{X,Y}(\Phi_{X,Y}(\xi))](k,x) \\
&\overset{\text{def}}{=} ([\Phi_{X,Y}(\xi)](x))_k \\
&\overset{\text{def}}{=} ((\xi(k_1,x))_{k_1\in f^{-1}(\phi_X(x))})_k \\
&\overset{\text{def}}{=} \xi(k,x)
\end{aligned}$$

for each $(k,x) \in f^*(X)$, and thus the stated equality follows.

6. *Invertibility II.* We claim that

$$\Phi_{X,Y} \circ \Psi_{X,Y} = \mathrm{id}_{\mathrm{Hom}_{\mathsf{FibSets}(K')}(X,\Pi_f(Y))}.$$

Indeed, given a morphism $\xi\colon X \longrightarrow \Pi_f(Y)$ of K-fibred sets, write

$$\xi(x) = (y_k)_{k\in f^{-1}(k'_x)}.$$

We then have

$$\begin{aligned}[][\Phi_{X,Y}\circ\Psi_{X,Y}](\xi)](x) &\overset{\text{def}}{=} [\Phi_{X,Y}(\Psi_{X,Y}(\xi))](x)\\ &\overset{\text{def}}{=} ([\Psi_{X,Y}(\xi)](k_1,x))_{k_1\in f^{-1}(\phi_X(x))}\\ &\overset{\text{def}}{=} ((\xi(x))_{k_1})_{k_1\in f^{-1}(\phi_X(x))}\\ &\overset{\text{def}}{=} (((y_k)_{k\in f^{-1}(k'_x)})_{k_1})_{k_1\in f^{-1}(\phi_X(x))}\\ &\overset{\text{def}}{=} (y_{k_1})_{k_1\in f^{-1}(\phi_X(x))}\\ &= (y_{k_1})_{k_1\in f^{-1}(k'_x)}\\ &= (y_k)_{k\in f^{-1}(k'_x)}\\ &\overset{\text{def}}{=} \xi(x)\end{aligned}$$

for each $x \in X$, where the equality $\phi_X(x) = k'_x$ follows from the fact that ξ is a morphism of K'-fibred sets. Thus the stated equality follows.

This finishes the proof. □

Appendices

A Other Chapters