Constructions With Sets

December 26, 2023

This chapter contains some material relating to constructions with sets. Notably, it contains:

1. Explicit descriptions of the major types of co/limits in Sets, including in particular pushouts and coequalisers (see Definitions 2.3.1.1 and 2.4.1.1 and Remarks 2.3.1.2 and 2.4.1.2);

2. A discussion of powersets as decategorifications of categories of presheaves (Remarks 4.1.1.2 and 4.2.1.2);

3. A lengthy discussion of the adjoint triple

$$f_* \dashv f^{-1} \dashv f_! \colon \mathcal{P}(A) \overset{\rightleftarrows}{\rightarrow} \mathcal{P}(B)$$

of functors (morphisms of posets) between $\mathcal{P}(A)$ and $\mathcal{P}(B)$ induced by a map of sets $f \colon A \to B$, along with a discussion of the properties of f_*, f^{-1}, and $f_!$.

Contents

1 Limits of Sets

1.1 Products of Families of Sets

Let $\{A_i\}_{i\in I}$ be a family of sets.

Definition 1.1.1.1. The **product**[1] **of** $\{A_i\}_{i\in I}$ is the pair $\left(\prod_{i\in I} A_i, \{\mathrm{pr}_i\}_{i\in I}\right)$ consisting of

- *The Limit.* The set $\prod_{i\in I} A_i$ defined by[2]

$$\prod_{i\in I} A_i \overset{\text{def}}{=} \left\{ f \in \mathsf{Sets}\left(I, \bigcup_{i\in I} A_i\right) \;\middle|\; \begin{array}{l}\text{for each } i \in I\text{, we}\\ \text{have } f(i) \in A_i\end{array} \right\}.$$

[1] *Further Terminology:* Also called the **Cartesian product of** $\{A_i\}_{i\in I}$.

[2] Less formally, $\prod_{i\in I} A_i$ is the set whose elements are I-indexed collections $(a_i)_{i\in I}$ with $a_i \in A_i$ for each $i \in I$.

- *The Cone.* The collection

$$\left\{\mathrm{pr}_i \colon \prod_{i\in I} A_i \to A_i\right\}_{i\in I}$$

of maps given by

$$\mathrm{pr}_i(f) \overset{\mathrm{def}}{=} f(i)$$

for each $f \in \prod_{i\in I} A_i$ and each $i \in I$.

Proof. We claim that $\prod_{i\in I} A_i$ is the categorical product of $\{A_i\}_{i\in I}$ in Sets. Indeed, suppose we have, for each $i \in I$, a diagram of the form

$$\begin{tikzcd} P \arrow[dr, "p_i"] & \\ \prod_{i\in I} A_i \arrow[r, "\mathrm{pr}_i"'] & A_i \end{tikzcd}$$

in Sets. Then there exists a unique map $\phi \colon P \to \prod_{i\in I} A_i$, uniquely determined by the condition $\mathrm{pr}_i \circ \phi = p_i$ for each $i \in I$, being necessarily given by

$$\phi(x) = (p_i(x))_{i\in I}$$

for each $x \in P$. $\square$

Proposition 1.1.1.2. Let $\{A_i\}_{i\in I}$ be a family of sets.

1. *Functoriality.* The assignment $\{A_i\}_{i\in I} \mapsto \prod_{i\in I} A_i$ defines a functor

$$\prod_{i\in I} \colon \mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}) \to \mathsf{Sets}$$

where

- *Action on Objects.* For each $(A_i)_{i\in I} \in \mathrm{Obj}(\mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}))$, we have

$$\left[\prod_{i\in I}\right]((A_i)_{i\in I}) \overset{\mathrm{def}}{=} \prod_{i\in I} A_i$$

- *Action on Morphisms.* For each $(A_i)_{i\in I}, (B_i)_{i\in I} \in \mathrm{Obj}(\mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}))$, the action on Hom-sets

$$\left(\prod_{i\in I}\right)_{(A_i)_{i\in I},(B_i)_{i\in I}} \colon \mathrm{Nat}\big((A_i)_{i\in I}, (B_i)_{i\in I}\big) \to \mathsf{Sets}\left(\prod_{i\in I} A_i, \prod_{i\in I} B_i\right)$$

of $\prod_{i\in I}$ at $\left((A_i)_{i\in I},(B_i)_{i\in I}\right)$ is defined by sending a map

$$\{f_i\colon A_i\to B_i\}_{i\in I}$$

in $\mathrm{Nat}\left((A_i)_{i\in I},(B_i)_{i\in I}\right)$ to the map of sets

$$\prod_{i\in I} f_i\colon \prod_{i\in I} A_i\to \prod_{i\in I} B_i$$

defined by

$$\left[\prod_{i\in I} f_i\right]((a_i)_{i\in I})\stackrel{\text{def}}{=}(f_i(a_i))_{i\in I}$$

for each $(a_i)_{i\in I}\in\prod_{i\in I}A_i$.

Proof. Item 1, Functoriality: Clear. □

1.2 Binary Products of Sets

Let A and B be sets.

Definition 1.2.1.1. The **product**[3] **of** A **and** B is the pair $(A\times B,\{\mathrm{pr}_1,\mathrm{pr}_2\})$ consisting of

- *The Limit.* The set $A\times B$ defined by[4]

$$\begin{aligned} A\times B &\stackrel{\text{def}}{=} \prod_{z\in\{A,B\}} z\\ &\stackrel{\text{def}}{=} \{f\in\mathsf{Sets}(\{0,1\},A\cup B)\mid \text{we have } f(0)\in A \text{ and } f(1)\in B\}\\ &\cong \{\{\{a\},\{a,b\}\}\in\mathcal{P}(\mathcal{P}(A\cup B))\mid \text{we have } a\in A \text{ and } b\in B\}.\end{aligned}$$

- *The Cone.* The maps

$$\begin{aligned}\mathrm{pr}_1&\colon A\times B\to A,\\ \mathrm{pr}_2&\colon A\times B\to B\end{aligned}$$

  defined by

$$\begin{aligned}\mathrm{pr}_1(a,b)&\stackrel{\text{def}}{=}a,\\ \mathrm{pr}_2(a,b)&\stackrel{\text{def}}{=}b\end{aligned}$$

  for each $(a,b)\in A\times B$.

[3] *Further Terminology:* Also called the **Cartesian product of** A **and** B or the **binary Cartesian product of** A **and** B, for emphasis.
This can also be thought of as the $(\mathbb{E}_{-1},\mathbb{E}_{-1})$**-tensor product of** A **and** B.

[4] Less formally, $A\times B$ is the set whose elements are pairs (a,b) with $a\in A$ and $b\in B$.

Proof. We claim that $A \times B$ is the categorical product of A and B in Sets. Indeed, suppose we have a diagram of the form

$$\begin{tikzcd} & P \arrow[dl, bend right, "p_1"'] \arrow[dr, bend left, "p_2"] & \\ A & A \times B \arrow[l, "\mathrm{pr}_1"] \arrow[r, "\mathrm{pr}_2"'] & B \end{tikzcd}$$

in Sets. Then there exists a unique map $\phi\colon P \to A \times B$, uniquely determined by the conditions

$$\begin{aligned} \mathrm{pr}_1 \circ \phi &= p_1, \\ \mathrm{pr}_2 \circ \phi &= p_2, \end{aligned}$$

being necessarily given by

$$\phi(x) = (p_1(x), p_2(x))$$

for each $x \in P$. □

Proposition 1.2.1.2. Let A, B, C, and X be sets.

1. *Functoriality.* The assignments $A, B, (A, B) \mapsto A \times B$ define functors

$$\begin{aligned} A \times -_2 &\colon \mathsf{Sets} \to \mathsf{Sets}, \\ -_1 \times B &\colon \mathsf{Sets} \to \mathsf{Sets}, \\ -_1 \times -_2 &\colon \mathsf{Sets} \times \mathsf{Sets} \to \mathsf{Sets}, \end{aligned}$$

   where $-_1 \times -_2$ is the functor where

   - *Action on Objects.* For each $(A, B) \in \mathrm{Obj}(\mathsf{Sets} \times \mathsf{Sets})$, we have

   $$[-_1 \times -_2](A, B) \stackrel{\text{def}}{=} A \times B;$$

   - *Action on Morphisms.* For each $(A, B), (X, Y) \in \mathrm{Obj}(\mathsf{Sets})$, the action on Hom-sets

   $$\times_{(A,B),(X,Y)} \colon \mathsf{Sets}(A, X) \times \mathsf{Sets}(B, Y) \to \mathsf{Sets}(A \times B, X \times Y)$$

     of $\times$ at $((A, B), (X, Y))$ is defined by sending (f, g) to the function

   $$f \times g \colon A \times B \to X \times Y$$

     defined by

   $$[f \times g](a, b) \stackrel{\text{def}}{=} (f(a), g(b))$$

     for each $(a, b) \in A \times B$;

and where $A \times -$ and $- \times B$ are the partial functors of $-_1 \times -_2$ at $A, B \in \mathrm{Obj}(\mathsf{Sets})$.

2. *Adjointness.* We have adjunctions

$$
\begin{tikzcd}
(A \times - \dashv \mathsf{Sets}(A,-))\colon & \mathsf{Sets} \arrow[r, bend left, "A\times -"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}, \arrow[l, bend left, "\mathsf{Sets}(A,-)"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
(- \times B \dashv \mathsf{Sets}(B,-))\colon & \mathsf{Sets} \arrow[r, bend left, "-\times B"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}, \arrow[l, bend left, "\mathsf{Sets}(B,-)"]
\end{tikzcd}
$$

witnessed by bijections

$$
\begin{aligned}
\mathsf{Sets}(A \times B, C) &\cong \mathsf{Sets}(A, \mathsf{Sets}(B, C)),\\
\mathsf{Sets}(A \times B, C) &\cong \mathsf{Sets}(B, \mathsf{Sets}(A, C)),
\end{aligned}
$$

natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

3. *Associativity.* We have an isomorphism of sets

$$(A \times B) \times C \cong A \times (B \times C),$$

natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

4. *Unitality.* We have isomorphisms of sets

$$
\begin{aligned}
\mathrm{pt} \times A &\cong A,\\
A \times \mathrm{pt} &\cong A,
\end{aligned}
$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$.

5. *Commutativity.* We have an isomorphism of sets

$$A \times B \cong B \times A,$$

natural in $A, B \in \mathrm{Obj}(\mathsf{Sets})$.

6. *Annihilation With the Empty Set.* We have isomorphisms of sets

$$
\begin{aligned}
A \times \emptyset &\cong \emptyset,\\
\emptyset \times A &\cong \emptyset,
\end{aligned}
$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$.

7. *Distributivity Over Unions.* We have isomorphisms of sets

$$A \times (B \cup C) = (A \times B) \cup (A \times C),$$
$$(A \cup B) \times C = (A \times C) \cup (B \times C).$$

8. *Distributivity Over Intersections.* We have isomorphisms of sets

$$A \times (B \cap C) = (A \times B) \cap (A \times C),$$
$$(A \cap B) \times C = (A \times C) \cap (B \times C).$$

9. *Middle-Four Exchange with Respect to Intersections.* We have an isomorphism of sets

$$(A \times B) \cap (C \times D) \cong (A \cap B) \times (C \cap D).$$

10. *Distributivity Over Differences.* We have isomorphisms of sets

$$A \times (B \setminus C) = (A \times B) \setminus (A \times C),$$
$$(A \setminus B) \times C = (A \times C) \setminus (B \times C),$$

natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

11. *Distributivity Over Symmetric Differences.* We have isomorphisms of sets

$$A \times (B \triangle C) = (A \times B) \triangle (A \times C),$$
$$(A \triangle B) \times C = (A \times C) \triangle (B \times C),$$

natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

12. *Symmetric Monoidality.* The triple $(\mathsf{Sets}, \times, \mathrm{pt})$ is a symmetric monoidal category.

13. *Symmetric Bimonoidality.* The quintuple $(\mathsf{Sets}, \coprod, \emptyset, \times, \mathrm{pt})$ is a symmetric bimonoidal category.

Proof. *Item 1*, *Functoriality*: This is clear, as associativity and unitality follow from applying associativity and unitality componentwise.
Item 2, *Adjointness*: We prove only that there's an adjunction $X \times - \dashv \mathrm{Hom}_{\mathsf{Sets}}(-, Z)$, witnessed by a bijection

$$\mathrm{Hom}_{\mathsf{Sets}}(X \times Y, Z) \cong \mathrm{Hom}_{\mathsf{Sets}}(X, \mathrm{Hom}_{\mathsf{Sets}}(Y, Z)),$$

natural in $Y, Z \in \mathrm{Obj}(\mathsf{Sets})$, as the proof of the existence of the adjunction $- \times Y \dashv \mathrm{Hom}_{\mathsf{Sets}}(-, Z)$ follows almost exactly in the same way.[5]

- *Map I.* We define a map
$$\Phi_{Y,Z} \colon \mathrm{Hom}_{\mathsf{Sets}}(X \times Y, Z) \to \mathrm{Hom}_{\mathsf{Sets}}(X, \mathrm{Hom}_{\mathsf{Sets}}(Y, Z)),$$
by sending a morphism $\xi \colon X \times Y \to Z$ to the morphism
$$\xi^{\dagger} \colon X \to \mathrm{Hom}_{\mathsf{Sets}(Y,Z)}$$
defined by
$$\xi^{\dagger}(x) \stackrel{\mathrm{def}}{=} \xi_x$$
for each $x \in X$, where $\xi_x \colon Y \to Z$ is the map defined by
$$\xi_x(y) \stackrel{\mathrm{def}}{=} \xi(x, y)$$
for each $y \in Y$.

- *Map II.* We define a map
$$\Psi_{Y,Z} \colon \mathrm{Hom}_{\mathsf{Sets}}(X, \mathrm{Hom}_{\mathsf{Sets}}(Y, Z)), \to \mathrm{Hom}_{\mathsf{Sets}}(X \times Y, Z)$$
given by sending a map $\xi \colon X \to \mathrm{Hom}_{\mathsf{Sets}}(Y, Z)$ to the map
$$\xi^{\dagger} \colon X \times Y \to Z$$
defined by
$$\xi^{\dagger}(x, y) \stackrel{\mathrm{def}}{=} [\xi(x)](y)$$
for each $(x, y) \in X \times Y$.

- *Naturality I.* We need to show that, given a function $g \colon Y \to Y'$, the diagram
$$\begin{tikzcd}
\mathrm{Hom}_{\mathsf{Sets}}(X \times Y', Z) \arrow[r, "\Phi_{Y',Z}"] \arrow[d, "\mathrm{id}_X \times g^*"'] & \mathrm{Hom}_{\mathsf{Sets}}(X, \mathrm{Hom}_{\mathsf{Sets}}(Y', Z)), \arrow[d, "(g^*)_*"] \\
\mathrm{Hom}_{\mathsf{Sets}}(X \times Y, Z) \arrow[r, "\Phi_{Y,Z}"'] & \mathrm{Hom}_{\mathsf{Sets}}(X, \mathrm{Hom}_{\mathsf{Sets}}(Y, Z)),
\end{tikzcd}$$

[5] Here we sometimes denote a map $f \colon X \to Y$ by $[x \mapsto f(x)]$, similar to the lambda notation $\lambda x.f(x)$.

commutes. Indeed, given a morphism $\xi\colon X' \times Y \to Z$, we have

$$\begin{aligned}
[\Phi_{Y,Z} \circ (g^* \times \mathrm{id}_Y)](\xi) &\overset{\text{def}}{=} (\xi(-_1, g(-_2)))^\dagger \\
&\overset{\text{def}}{=} \xi_{-_1}(g(-_2)) \\
&\overset{\text{def}}{=} (g_*)^*(\xi_{-_1}(-_2)) \\
&\overset{\text{def}}{=} (g_*)^*\Big(\xi^\dagger\Big) \\
&\overset{\text{def}}{=} [(g_*)^* \circ \Phi_{Y',Z}](\xi).
\end{aligned}$$

- *Naturality II.* We need to show that, given a function $h\colon Z \to Z'$, the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathsf{Sets}}(X \times Y, Z) \arrow[r, "\Phi_{Y,Z}"] \arrow[d, "h_*"'] & \mathrm{Hom}_{\mathsf{Sets}}(X, \mathrm{Hom}_{\mathsf{Sets}}(Y, Z)), \arrow[d, "(h_*)_*"] \\
\mathrm{Hom}_{\mathsf{Sets}}(X \times Y, Z') \arrow[r, "\Phi_{Y,Z'}"'] & \mathrm{Hom}_{\mathsf{Sets}}(X, \mathrm{Hom}_{\mathsf{Sets}}(Y, Z')),
\end{tikzcd}$$

commutes. Indeed, given a morphism $\xi\colon X \times Y \to Z$, we have

$$\begin{aligned}
[\Phi_{Y,Z} \circ h_*](\xi) &\overset{\text{def}}{=} (h(\xi(-_1, -_2)))^\dagger \\
&\overset{\text{def}}{=} [x \mapsto [y \mapsto h(\xi(x, y))]] \\
&\overset{\text{def}}{=} [x \mapsto h_*([y \mapsto \xi(x, y)])] \\
&\overset{\text{def}}{=} [x \mapsto h_*\Big(\xi^\dagger(x)\Big)] \\
&\overset{\text{def}}{=} h_*\Big(\xi^\dagger\Big) \\
&\overset{\text{def}}{=} [(h_*)_* \circ \Phi_{Y,Z}](\xi).
\end{aligned}$$

- *Invertibility I.* We claim that

$$\Psi_{X,Y} \circ \Phi_{X,Y} = \mathrm{id}_{\mathrm{Hom}_{\mathsf{Sets}}(X \times Y, Z)}.$$

Indeed, given a morphism $\xi\colon X \times Y \to Z$, we have

$$\begin{aligned}
[\Psi_{X,Y} \circ \Phi_{X,Y}](\xi) &\overset{\text{def}}{=} \Psi_{X,Y}(\Phi_{X,Y}(\xi)) \\
&\overset{\text{def}}{=} \Psi_{X,Y}([x \mapsto [y \mapsto \xi(x, y)]]) \\
&\overset{\text{def}}{=} [(x, y) \mapsto \mathrm{ev}_x([x \mapsto \mathrm{ev}_y([y \mapsto \xi(x, y)]])])] \\
&\overset{\text{def}}{=} [(x, y) \mapsto \mathrm{ev}_x([x \mapsto \xi(x, y)])] \\
&\overset{\text{def}}{=} [(x, y) \mapsto \xi(x, y)] \\
&\overset{\text{def}}{=} \xi.
\end{aligned}$$

- *Invertibility II.* We claim that

$$\Phi_{X,Y} \circ \Psi_{X,Y} = \mathrm{id}_{\mathrm{Hom}_{\mathsf{Sets}}(X,\mathrm{Hom}_{\mathsf{Sets}}(Y,Z))}.$$

  Indeed, given a morphism $\xi \colon X \to \mathrm{Hom}_{\mathsf{Sets}}(Y,Z)$, we have

$$\begin{aligned}[\Phi_{X,Y} \circ \Psi_{X,Y}](\xi) &\stackrel{\text{def}}{=} \Phi_{X,Y}(\Psi_{X,Y}(\xi)) \\ &\stackrel{\text{def}}{=} \Phi_{X,Y}(\Psi_{X,Y}([x \mapsto \xi(x)])) \\ &\stackrel{\text{def}}{=} \Phi_{X,Y}([(x,y) \mapsto \mathrm{ev}_x([x \mapsto \mathrm{ev}_y(\xi(x))])]) \\ &\stackrel{\text{def}}{=} \Phi_{X,Y}([(x,y) \mapsto \xi(x,y)]) \\ &\stackrel{\text{def}}{=} [x \mapsto [y \mapsto \xi(x,y)]] \\ &\stackrel{\text{def}}{=} \xi.\end{aligned}$$

Item 3, Associativity: Clear.
Item 4, Unitality: Clear.
Item 5, Commutativity: Clear.
Item 6, Annihilation With the Empty Set: Clear.
Item 7, Distributivity Over Unions: See [Pro24c].
Item 8, Distributivity Over Intersections: See [Pro24d, Corollary 1].
Item 9, Middle-Four Exchange With Respect to Intersections: See [Pro24d, Corollary 1].
Item 10, Distributivity Over Differences: See [Pro24a].
Item 11, Distributivity Over Symmetric Differences: See [Pro24b].
Item 12, Symmetric Monoidality: Omitted.
Item 13, Symmetric Bimonoidality: Omitted. □

1.3 Pullbacks

Let A, B, and C be sets and let $f\colon A \to C$ and $g\colon B \to C$ be functions.

Definition 1.3.1.1. The **pullback of A and B over C along f and g**[6] is the pair[7] $(A \times_C B, \{\mathrm{pr}_1, \mathrm{pr}_2\})$ consisting of

- *The Limit.* The set $A \times_C B$ defined by

$$A \times_C B \stackrel{\text{def}}{=} \{(a,b) \in A \times B \mid f(a) = g(b)\}.$$

[6] *Further Terminology:* Also called the **fibre product of A and B over C along f and g**.
[7] *Further Notation:* Also written $A \times_{f,C,g} B$.

- *The Cone.* The maps

$$\begin{aligned}\mathrm{pr}_1 &\colon A \times_C B \to A,\\ \mathrm{pr}_2 &\colon A \times_C B \to B\end{aligned}$$

  defined by

$$\begin{aligned}\mathrm{pr}_1(a,b) &\overset{\text{def}}{=} a,\\ \mathrm{pr}_2(a,b) &\overset{\text{def}}{=} b\end{aligned}$$

  for each $(a,b) \in A \times_C B$.

Proof. We claim that $A \times_C B$ is the categorical pullback of A and B over C with respect to (f,g) in Sets. First we need to check that the relevant pullback diagram commutes, i.e. that we have

$$f \circ \mathrm{pr}_1 = g \circ \mathrm{pr}_2, \qquad \begin{tikzcd} A \times_C B \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C. \end{tikzcd}$$

Indeed, given $(a,b) \in A \times_C B$, we have

$$\begin{aligned}[f \circ \mathrm{pr}_1](a,b) &= f(\mathrm{pr}_1(a,b))\\ &= f(a)\\ &= g(b)\\ &= g(\mathrm{pr}_2(a,b))\\ &= [g \circ \mathrm{pr}_2](a,b),\end{aligned}$$

where $f(a) = g(b)$ since $(a,b) \in A \times_C B$. Next, we prove that $A \times_C B$ satisfies the universal property of the pullback. Suppose we have a diagram of the form

$$\begin{tikzcd} P \arrow[drr, bend left, "p_2"] \arrow[ddr, bend right, "p_1"'] & & \\ & A \times_C B \arrow[r, dashed, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "g"] \\ & A \arrow[r, "f"'] & C \end{tikzcd}$$

in Sets. Then there exists a unique map $\phi\colon P \to A\times_C B$, uniquely determined by the conditions

$$\begin{aligned}\mathrm{pr}_1 \circ \phi &= p_1,\\ \mathrm{pr}_2 \circ \phi &= p_2,\end{aligned}$$

being necessarily given by

$$\phi(x) = (p_1(x), p_2(x))$$

for each $x \in P$, where we note that $(p_1(x), p_2(x)) \in A \times B$ indeed lies in $A \times_C B$ by the condition

$$f \circ p_1 = g \circ p_2,$$

which gives

$$f(p_1(x)) = g(p_2(x))$$

for each $x \in P$, so that $(p_1(x), p_2(x)) \in A \times_C B$. $\square$

Example 1.3.1.2. Here are some examples of pullbacks of sets.

1. *Unions via Intersections.* Let $A, B \subset X$. We have a bijection of sets

$$A \cap B \cong A \times_{A\cup B} B, \qquad \begin{tikzcd} A \cap B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "\iota_B"] \\ A \arrow[r, "\iota_A"'] & A \cup B. \end{tikzcd}$$

Proof. *Item 1, Unions via Intersections*: Indeed, we have

$$\begin{aligned}A \times_{A\cup B} B &\cong \{(x, y) \in A \times B \mid x = y\}\\ &\cong A \cap B.\end{aligned}$$

This finishes the proof. $\square$

Proposition 1.3.1.3. Let A, B, C, and X be sets.

1. *Functoriality.* The assignment $(A, B, C, f, g) \mapsto A \times_{f,C,g} B$ defines a functor

$$-_1 \times_{-_3} -_1 \colon \mathsf{Fun}(\mathcal{P}, \mathsf{Sets}) \to \mathsf{Sets},$$

where $\mathcal{P}$ is the category that looks like this:

$$
\begin{tikzcd}
 & \bullet \arrow[d] \\
\bullet \arrow[r] & \bullet.
\end{tikzcd}
$$

In particular, the action on morphisms of $-_1 \times_{-_3} -_1$ is given by sending a morphism

$$
\begin{tikzcd}
A \times_C B \arrow[rr] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & & B \arrow[dr, "\psi"] \arrow[dd, "g", near start] & \\
 & A' \times_{C'} B' \arrow[rr, crossing over] \arrow[dr, phantom, "\lrcorner", very near start] & & B' \arrow[dd, "g'"] \\
A \arrow[rr, "f", near start] \arrow[dr, "\phi"'] & & C \arrow[dr, "\chi"] & \\
 & A' \arrow[rr, "f'"'] \arrow[from=uu, crossing over] & & C'
\end{tikzcd}
$$

in $\mathsf{Fun}(\mathcal{P}, \mathsf{Sets})$ to the map $\xi \colon A \times_C B \xrightarrow{\exists!} A' \times_{C'} B'$ given by

$$
\xi(a, b) \overset{\text{def}}{=} (\phi(a), \psi(b))
$$

for each $(a, b) \in A \times_C B$, which is the unique map making the diagram

$$
\begin{tikzcd}
A \times_C B \arrow[rr] \arrow[dd] \arrow[dr, dashed] & & B \arrow[dr, "\psi"] \arrow[dd, "g", near start] & \\
 & A' \times_{C'} B' \arrow[rr, crossing over] \arrow[dr, phantom, "\lrcorner", very near start] & & B' \arrow[dd, "g'"] \\
A \arrow[rr, "f", near start] \arrow[dr, "\phi"'] & & C \arrow[dr, "\chi"] & \\
 & A' \arrow[rr, "f'"'] \arrow[from=uu, crossing over] & & C'
\end{tikzcd}
$$

commute.

2. *Associativity.* Given a diagram

$$
\begin{tikzcd}
A \arrow[dr, "f"'] & & B \arrow[dl, "g"] \arrow[dr, "h"'] & & C \arrow[dl, "k"] \\
 & X & & Y &
\end{tikzcd}
$$

in Sets, we have isomorphisms

$$(A \times_X B) \times_Y C \cong (A \times_X B) \times_B (B \times_Y C) \cong A \times_X (B \times_Y C),$$

where these pullbacks are built as in the diagrams

$$\begin{tikzcd}[column sep=small]
& & (A \times_X B) \times_Y C \arrow[dl] \arrow[ddrr] \arrow[d, phantom, very near start, "\lrcorner" {anchor=center, rotate=-45}] & & \\
& A \times_X B \arrow[dl] \arrow[dr] \arrow[d, phantom, very near start, "\lrcorner" {anchor=center, rotate=-45}] & & & \\
A \arrow[dr, "f"'] & & B \arrow[dl, "g"] \arrow[dr, "h"'] & & C, \arrow[dl, "k"] \\
& X & & Y &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=small]
& & (A \times_X B) \times_B (B \times_Y C) \arrow[dl] \arrow[dr] \arrow[d, phantom, very near start, "\lrcorner" {anchor=center, rotate=-45}] & & \\
& A \times_X B \arrow[dl] \arrow[dr] \arrow[d, phantom, very near start, "\lrcorner" {anchor=center, rotate=-45}] & & B \times_Y C \arrow[dl] \arrow[dr] \arrow[d, phantom, very near start, "\lrcorner" {anchor=center, rotate=-45}] & \\
A \arrow[dr, "f"'] & & B \arrow[dl, "g"] \arrow[dr, "h"'] & & C, \arrow[dl, "k"] \\
& X & & Y &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=small]
& & A \times_X (B \times_Y C) \arrow[ddll] \arrow[dr] \arrow[d, phantom, very near start, "\lrcorner" {anchor=center, rotate=-45}] & & \\
& & & B \times_Y C \arrow[dl] \arrow[dr] \arrow[d, phantom, very near start, "\lrcorner" {anchor=center, rotate=-45}] & \\
A \arrow[dr, "f"'] & & B \arrow[dl, "g"] \arrow[dr, "h"'] & & C. \arrow[dl, "k"] \\
& X & & Y &
\end{tikzcd}$$

3. *Unitality.* We have isomorphisms of sets

$$\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "f"'] \arrow[dr, phantom, very near start, "\lrcorner"] & A \arrow[d, "f"] \\
X \arrow[r, equal] & X
\end{tikzcd}
\qquad
\begin{aligned} X \times_X A &\cong A, \\ A \times_X X &\cong A, \end{aligned}
\qquad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, equal] \arrow[dr, phantom, very near start, "\lrcorner"] & X \arrow[d, equal] \\
X \arrow[r, "f"'] & X.
\end{tikzcd}$$

4. *Commutativity.* We have an isomorphism of sets

$$\begin{tikzcd}
A \times_X B \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & B \arrow[d, "g"] \\
A \arrow[r, "f"'] & X,
\end{tikzcd}
\qquad
A \times_X B \cong B \times_X A
\qquad
\begin{tikzcd}
B \times_X A \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & A \arrow[d, "f"] \\
B \arrow[r, "g"'] & X.
\end{tikzcd}$$

5. *Annihilation With the Empty Set.* We have isomorphisms of sets

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \emptyset \arrow[d] \\
A \arrow[r, "f"'] & X,
\end{tikzcd}
\qquad
\begin{aligned} A \times_X \emptyset &\cong \emptyset, \\ \emptyset \times_X A &\cong \emptyset, \end{aligned}
\qquad
\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & A \arrow[d, "f"] \\
\emptyset \arrow[r] & X.
\end{tikzcd}$$

6. *Interaction With Products.* We have

$$A \times_{\mathrm{pt}} B \cong A \times B,
\qquad
\begin{tikzcd}
A \times B \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & B \arrow[d, "!_B"] \\
A \arrow[r, "!_A"'] & \mathrm{pt}.
\end{tikzcd}$$

7. *Symmetric Monoidality.* The triple $(\mathsf{Sets}, \times_X, X)$ is a symmetric monoidal category.

Proof. *Item 1, Functoriality*: This is a special case of functoriality of co/limits, Limits and Colimits, **??** of **??**, with the explicit expression for ξ following from the commutativity of the cube pullback diagram.
Item 2, Associativity: Indeed, we have

$$\begin{aligned}(A \times_X B) \times_Y C &\cong \{((a,b),c) \in (A \times_X B) \times C \mid h(b) = k(c)\}\\ &\cong \{((a,b),c) \in (A \times B) \times C \mid f(a) = g(b) \text{ and } h(b) = k(c)\}\\ &\cong \{(a,(b,c)) \in A \times (B \times C) \mid f(a) = g(b) \text{ and } h(b) = k(c)\}\\ &\cong \{(a,(b,c)) \in A \times (B \times_Y C) \mid f(a) = g(b)\}\\ &\cong A \times_X (B \times_Y C)\end{aligned}$$

and

$$\begin{aligned}(A \times_X B) \times_B (B \times_Y C) &\cong \left\{\big((a,b),\big(b',c\big)\big) \in (A \times_X B) \times (B \times_Y C) \;\middle|\; b = b'\right\}\\ &\cong \left\{\big((a,b),\big(b',c\big)\big) \in (A \times B) \times (B \times C) \;\middle|\; \begin{array}{l} f(a) = g(b),\, b = b', \\ \text{and } h\big(b'\big) = k(c) \end{array}\right\}\\ &\cong \left\{\big(a,\big(b,\big(b',c\big)\big)\big) \in A \times (B \times (B \times C)) \;\middle|\; \begin{array}{l} f(a) = g(b),\, b = b', \\ \text{and } h\big(b'\big) = k(c) \end{array}\right\}\\ &\cong \left\{\big(a,\big(\big(b,b'\big),c\big)\big) \in A \times ((B \times B) \times C) \;\middle|\; \begin{array}{l} f(a) = g(b),\, b = b', \\ \text{and } h\big(b'\big) = k(c) \end{array}\right\}\\ &\cong \left\{\big(a,\big(\big(b,b'\big),c\big)\big) \in A \times ((B \times_B B) \times C) \;\middle|\; \begin{array}{l} f(a) = g(b) \text{ and} \\ h\big(b'\big) = k(c) \end{array}\right\}\\ &\cong \{(a,(b,c)) \in A \times (B \times C) \mid f(a) = g(b) \text{ and } h(b) = k(c)\}\\ &\cong A \times_X (B \times_Y C),\end{aligned}$$

where we have used Item 3 for the isomorphism $B \times_B B \cong B$.
Item 3, Unitality: Indeed, we have

$$\begin{aligned}X \times_X A &\cong \{(x,a) \in X \times A \mid f(a) = x\},\\ A \times_X X &\cong \{(a,x) \in X \times A \mid f(a) = x\},\end{aligned}$$

which are isomorphic to A via the maps $(x,a) \mapsto a$ and $(a,x) \mapsto a$.
Item 4, Commutativity: Clear.
Item 5, Annihilation With the Empty Set: Clear.
Item 6, Interaction With Products: Clear.
Item 7, Symmetric Monoidality: Omitted. □

1.4 Equalisers

Let A and B be sets and let $f, g\colon A \rightrightarrows B$ be functions.

Definition 1.4.1.1. The **equaliser of f and g** is the pair $(\mathrm{Eq}(f,g), \mathrm{eq}(f,g))$ consisting of

- *The Limit.* The set $\mathrm{Eq}(f,g)$ defined by

$$\mathrm{Eq}(f,g) \stackrel{\text{def}}{=} \{a \in A \mid f(a) = g(a)\}.$$

- *The Cone.* The inclusion map

$$\mathrm{eq}(f,g)\colon \mathrm{Eq}(f,g) \hookrightarrow A.$$

Proof. We claim that $\mathrm{Eq}(f,g)$ is the categorical equaliser of f and g in Sets. First we need to check that the relevant equaliser diagram commutes, i.e. that we have

$$f \circ \mathrm{eq}(f,g) = g \circ \mathrm{eq}(f,g),$$

which indeed holds by the definition of the set $\mathrm{Eq}(f,g)$. Next, we prove that $\mathrm{Eq}(f,g)$ satisfies the universal property of the equaliser. Suppose we have a diagram of the form

$$\begin{tikzcd}
\mathrm{Eq}(f,g) \arrow[r, hook, "\mathrm{eq}(f,g)"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\
E \arrow[ur, "e"'] & &
\end{tikzcd}$$

in Sets. Then there exists a unique map $\phi\colon E \to \mathrm{Eq}(f,g)$, uniquely determined by the condition

$$\mathrm{eq}(f,g) \circ \phi = e$$

being necessarily given by

$$\phi(x) = e(x)$$

for each $x \in E$, where we note that $e(x) \in A$ indeed lies in $\mathrm{Eq}(f,g)$ by the condition

$$f \circ e = g \circ e,$$

which gives

$$f(e(x)) = g(e(x))$$

for each $x \in E$, so that $e(x) \in \mathrm{Eq}(f,g)$. $\square$

Proposition 1.4.1.2. Let A, B, and C be sets.

1. *Associativity.* We have an isomorphism of sets[8]

$$\underbrace{\mathrm{Eq}(f\circ \mathrm{eq}(g,h), g\circ \mathrm{eq}(g,h))}_{=\mathrm{Eq}(f\circ \mathrm{eq}(g,h), h\circ \mathrm{eq}(g,h))} \cong \mathrm{Eq}(f,g,h) \cong \underbrace{\mathrm{Eq}(f\circ \mathrm{eq}(f,g), h\circ \mathrm{eq}(f,g))}_{=\mathrm{Eq}(g\circ \mathrm{eq}(f,g), h\circ \mathrm{eq}(f,g))},$$

where $\mathrm{Eq}(f,g,h)$ is the limit of the diagram

$$\begin{tikzcd} A \arrow[r, shift left=2, "f"] \arrow[r, dashed, "g" description] \arrow[r, shift right=2, "h"'] & B \end{tikzcd}$$

[8]That is, the following three ways of forming "the" equaliser of (f,g,h) agree:

1. Take the equaliser of (f,g,h), i.e. the limit of the diagram

$$\begin{tikzcd} A \arrow[r, shift left=2, "f"] \arrow[r, dashed, "g" description] \arrow[r, shift right=2, "h"'] & B \end{tikzcd}$$

in Sets.

2. First take the equaliser of f and g, forming a diagram

$$\mathrm{Eq}(f,g) \overset{\mathrm{eq}(f,g)}{\hookrightarrow} A \underset{g}{\overset{f}{\rightrightarrows}} B$$

and then take the equaliser of the composition

$$\mathrm{Eq}(f,g) \overset{\mathrm{eq}(f,g)}{\hookrightarrow} A \underset{h}{\overset{f}{\rightrightarrows}} B,$$

obtaining a subset

$$\mathrm{Eq}(f\circ \mathrm{eq}(f,g), h\circ \mathrm{eq}(f,g)) = \mathrm{Eq}(g\circ \mathrm{eq}(f,g), h\circ \mathrm{eq}(f,g))$$

of $\mathrm{Eq}(f,g)$.

3. First take the equaliser of g and h, forming a diagram

$$\mathrm{Eq}(g,h) \overset{\mathrm{eq}(g,h)}{\hookrightarrow} A \underset{h}{\overset{g}{\rightrightarrows}} B$$

and then take the equaliser of the composition

$$\mathrm{Eq}(g,h) \overset{\mathrm{eq}(g,h)}{\hookrightarrow} A \underset{g}{\overset{f}{\rightrightarrows}} B,$$

obtaining a subset

$$\mathrm{Eq}(f\circ \mathrm{eq}(g,h), g\circ \mathrm{eq}(g,h)) = \mathrm{Eq}(f\circ \mathrm{eq}(g,h), h\circ \mathrm{eq}(g,h))$$

of $\mathrm{Eq}(g,h)$.

in Sets, being explicitly given by

$$\mathrm{Eq}(f,g,h) \cong \{a \in A \mid f(a) = g(a) = h(a)\}.$$

4. *Unitality.* We have an isomorphism of sets

$$\mathrm{Eq}(f,f) \cong A.$$

5. *Commutativity.* We have an isomorphism of sets

$$\mathrm{Eq}(f,g) \cong \mathrm{Eq}(g,f).$$

6. *Interaction With Composition.* Let

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \underset{k}{\overset{h}{\rightrightarrows}} C$$

be functions. We have an inclusion of sets

$$\mathrm{Eq}(h \circ f \circ \mathrm{eq}(f,g), k \circ g \circ \mathrm{eq}(f,g)) \subset \mathrm{Eq}(h \circ f, k \circ g),$$

where $\mathrm{Eq}(h \circ f \circ \mathrm{eq}(f,g), k \circ g \circ \mathrm{eq}(f,g))$ is the equaliser of the composition

$$\mathrm{Eq}(f,g) \overset{\mathrm{eq}(f,g)}{\hookrightarrow} A \underset{g}{\overset{f}{\rightrightarrows}} B \underset{k}{\overset{h}{\rightrightarrows}} C.$$

Proof. *Item 1, Associativity*: We first prove that $\mathrm{Eq}(f,g,h)$ is indeed given by

$$\mathrm{Eq}(f,g,h) \cong \{a \in A \mid f(a) = g(a) = h(a)\}.$$

Indeed, suppose we have a diagram of the form

$$\begin{tikzcd}
\mathrm{Eq}(f,g,h) \arrow[r, hook, "\mathrm{eq}(f,g,h)"] & A \arrow[r, shift left=2, "f"] \arrow[r, "g" description] \arrow[r, shift right=2, "h"'] & B \\
E \arrow[ur, "e"'] & &
\end{tikzcd}$$

in Sets. Then there exists a unique map $\phi \colon E \to \mathrm{Eq}(f,g,h)$, uniquely determined by the condition

$$\mathrm{eq}(f,g) \circ \phi = e$$

being necessarily given by

$$\phi(x) = e(x)$$

for each $x \in E$, where we note that $e(x) \in A$ indeed lies in $\mathrm{Eq}(f, g, h)$ by the condition

$$f \circ e = g \circ e = h \circ e,$$

which gives

$$f(e(x)) = g(e(x)) = h(e(x))$$

for each $x \in E$, so that $e(x) \in \mathrm{Eq}(f, g, h)$.
We now check the equalities

$$\mathrm{Eq}(f \circ \mathrm{eq}(g,h), g \circ \mathrm{eq}(g,h)) \cong \mathrm{Eq}(f,g,h) \cong \mathrm{Eq}(f \circ \mathrm{eq}(f,g), h \circ \mathrm{eq}(f,g)).$$

Indeed, we have

$$\begin{aligned}
\mathrm{Eq}(f \circ \mathrm{eq}(g,h), g \circ \mathrm{eq}(g,h)) &\cong \{x \in \mathrm{Eq}(g,h) \mid [f \circ \mathrm{eq}(g,h)](a) = [g \circ \mathrm{eq}(g,h)](a)\} \\
&\cong \{x \in \mathrm{Eq}(g,h) \mid f(a) = g(a)\} \\
&\cong \{x \in A \mid f(a) = g(a) \text{ and } g(a) = h(a)\} \\
&\cong \{x \in A \mid f(a) = g(a) = h(a)\} \\
&\cong \mathrm{Eq}(f,g,h).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\mathrm{Eq}(f \circ \mathrm{eq}(f,g), h \circ \mathrm{eq}(f,g)) &\cong \{x \in \mathrm{Eq}(f,g) \mid [f \circ \mathrm{eq}(f,g)](a) = [h \circ \mathrm{eq}(f,g)](a)\} \\
&\cong \{x \in \mathrm{Eq}(f,g) \mid f(a) = h(a)\} \\
&\cong \{x \in A \mid f(a) = h(a) \text{ and } f(a) = g(a)\} \\
&\cong \{x \in A \mid f(a) = g(a) = h(a)\} \\
&\cong \mathrm{Eq}(f,g,h).
\end{aligned}$$

Item 4, Unitality: Clear.
Item 5, Commutativity: Clear.
Item 6, Interaction With Composition: Indeed, we have

$$\begin{aligned}
\mathrm{Eq}(h \circ f \circ \mathrm{eq}(f,g), k \circ g \circ \mathrm{eq}(f,g)) &\cong \{a \in \mathrm{Eq}(f,g) \mid h(f(a)) = k(g(a))\} \\
&\cong \{a \in A \mid f(a) = g(a) \text{ and } h(f(a)) = k(g(a))\}.
\end{aligned}$$

and

$$\mathrm{Eq}(h \circ f, k \circ g) \cong \{a \in A \mid h(f(a)) = k(g(a))\},$$

and thus there's an inclusion from $\mathrm{Eq}(h \circ f \circ \mathrm{eq}(f,g), k \circ g \circ \mathrm{eq}(f,g))$ to $\mathrm{Eq}(h \circ f, k \circ g)$. □

2 Colimits of Sets

2.1 Coproducts of Families of Sets

Let $\{A_i\}_{i\in I}$ be a family of sets.

Definition 2.1.1.1. The **disjoint union of the family** $\{A_i\}_{i\in I}$ is the pair $(\coprod_{i\in I} A_i, \{\mathrm{inj}_i\}_{i\in I})$ consisting of

- *The Colimit.* The set $\coprod_{i\in I} A_i$ defined by
$$\coprod_{i\in I} A_i \stackrel{\text{def}}{=} \left\{(i,x) \in I \times \left(\bigcup_{i\in I} A_i\right) \;\middle|\; x \in A_i\right\}.$$
- *The Cocone.* The collection
$$\left\{\mathrm{inj}_i\colon A_i \to \coprod_{i\in I} A_i\right\}_{i\in I}$$
of maps given by
$$\mathrm{inj}_i(x) \stackrel{\text{def}}{=} (i,x)$$
for each $x \in A_i$ and each $i \in I$.

Proof. We claim that $\coprod_{i\in I} A_i$ is the categorical coproduct of $\{A_i\}_{i\in I}$ in Sets. Indeed, suppose we have, for each $i \in I$, a diagram of the form

$$\begin{tikzcd} & C \\ A_i \arrow[ur, "i_i"] \arrow[r, "\mathrm{inj}_i"'] & \coprod_{i\in I} A_i \end{tikzcd}$$

in Sets. Then there exists a unique map $\phi\colon \coprod_{i\in I} A_i \to C$, uniquely determined by the condition $\phi \circ \mathrm{inj}_i = i_i$ for each $i \in I$, being necessarily given by
$$\phi(i,x) = i_i(x)$$
for each $(i,x) \in \coprod_{i\in I} A_i$. $\square$

Proposition 2.1.1.2. Let $\{A_i\}_{i\in I}$ be a family of sets.

1. *Functoriality.* The assignment $\{A_i\}_{i\in I} \mapsto \coprod_{i\in I} A_i$ defines a functor

$$\coprod_{i\in I}\colon \mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}) \to \mathsf{Sets}$$

where

- *Action on Objects.* For each $(A_i)_{i\in I} \in \mathrm{Obj}(\mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}))$, we have
$$\left[\coprod_{i\in I}\right]((A_i)_{i\in I}) \overset{\text{def}}{=} \coprod_{i\in I} A_i$$
- *Action on Morphisms.* For each $(A_i)_{i\in I}, (B_i)_{i\in I} \in \mathrm{Obj}(\mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}))$, the action on Hom-sets
$$\left(\coprod_{i\in I}\right)_{(A_i)_{i\in I},(B_i)_{i\in I}}\colon \mathrm{Nat}((A_i)_{i\in I}, (B_i)_{i\in I}) \to \mathsf{Sets}\left(\coprod_{i\in I} A_i, \coprod_{i\in I} B_i\right)$$
of $\coprod_{i\in I}$ at $((A_i)_{i\in I}, (B_i)_{i\in I})$ is defined by sending a map
$$\{f_i\colon A_i \to B_i\}_{i\in I}$$
in $\mathrm{Nat}((A_i)_{i\in I}, (B_i)_{i\in I})$ to the map of sets
$$\coprod_{i\in I} f_i\colon \coprod_{i\in I} A_i \to \coprod_{i\in I} B_i$$
defined by
$$\left[\coprod_{i\in I} f_i\right](i,a) \overset{\text{def}}{=} f_i(a)$$
for each $(i,a) \in \coprod_{i\in I} A_i$.

Proof. Item 1, *Functoriality*: Clear. □

2.2 Binary Coproducts

Let A and B be sets.

Definition 2.2.1.1. The **coproduct**[9] **of A and B** is the pair $(A \coprod B, \{\mathrm{inj}_1, \mathrm{inj}_2\})$ consisting of

[9] *Further Terminology:* Also called the **disjoint union of A and B**, or the **binary disjoint union of A and B**, for emphasis.

- *The Colimit.* The set $A \coprod B$ defined by

$$\begin{aligned} A \coprod B &\stackrel{\text{def}}{=} \coprod_{z \in \{A,B\}} z \\ &\cong \{(0,a) \mid a \in A\} \cup \{(1,b) \mid b \in B\}. \end{aligned}$$

- *The Cocone.* The maps

$$\begin{aligned} \mathrm{inj}_1 &\colon A \to A \coprod B, \\ \mathrm{inj}_2 &\colon B \to A \coprod B, \end{aligned}$$

  given by

$$\begin{aligned} \mathrm{inj}_1(a) &\stackrel{\text{def}}{=} (0,a), \\ \mathrm{inj}_2(b) &\stackrel{\text{def}}{=} (1,b), \end{aligned}$$

  for each $a \in A$ and each $b \in B$.

Proof. We claim that $A \coprod B$ is the categorical coproduct of A and B in Sets. Indeed, suppose we have a diagram of the form

$$\begin{tikzcd} & C & \\ A \arrow[ur, bend left, "i_A"] \arrow[r, "\mathrm{inj}_A"'] & A \coprod B & B \arrow[ul, bend right, "i_B"'] \arrow[l, "\mathrm{inj}_B"] \end{tikzcd}$$

in Sets. Then there exists a unique map $\phi \colon A \coprod B \to C$, uniquely determined by the conditions

$$\begin{aligned} \phi \circ \mathrm{inj}_A &= i_A, \\ \phi \circ \mathrm{inj}_B &= i_B, \end{aligned}$$

being necessarily given by

$$\phi(x) = \begin{cases} i_A(x) & \text{if } x \in A, \\ i_B(x) & \text{if } x \in B \end{cases}$$

for each $x \in C$. $\square$

Proposition 2.2.1.2. Let A, B, C, and X be sets.

1. *Functoriality.* The assignment $A, B, (A, B) \mapsto A \coprod B$ defines functors

$$\begin{aligned} A \textstyle\coprod -_2 &\colon \mathsf{Sets} \to \mathsf{Sets}, \\ -_1 \textstyle\coprod B &\colon \mathsf{Sets} \to \mathsf{Sets}, \\ -_1 \textstyle\coprod -_2 &\colon \mathsf{Sets} \times \mathsf{Sets} \to \mathsf{Sets}, \end{aligned}$$

   where $-_1 \coprod -_2$ is the functor where

   - *Action on Objects.* For each $(A, B) \in \mathrm{Obj}(\mathsf{Sets} \times \mathsf{Sets})$, we have

   $$[-_1 \textstyle\coprod -_2](A, B) \stackrel{\text{def}}{=} A \textstyle\coprod B;$$

   - *Action on Morphisms.* For each $(A, B), (X, Y) \in \mathrm{Obj}(\mathsf{Sets})$, the action on Hom-sets

   $$\textstyle\coprod_{(A,B),(X,Y)} \colon \mathsf{Sets}(A, X) \times \mathsf{Sets}(B, Y) \to \mathsf{Sets}(A \coprod B, X \coprod Y)$$

     of $\coprod$ at $((A, B), (X, Y))$ is defined by sending (f, g) to the function

   $$f \textstyle\coprod g \colon A \coprod B \to X \coprod Y$$

     defined by

   $$[f \textstyle\coprod g](x) \stackrel{\text{def}}{=} \begin{cases} f(x) & \text{if } x \in A, \\ g(x) & \text{if } x \in B \end{cases}$$

     for each $x \in A \coprod B$;

   and where $A \coprod -$ and $- \coprod B$ are the partial functors of $-_1 \coprod -_2$ at $A, B \in \mathrm{Obj}(\mathsf{Sets})$.

2. *Associativity.* We have an isomorphism of sets

$$(A \textstyle\coprod B) \coprod C \cong A \coprod (B \coprod C),$$

   natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

3. *Unitality.* We have isomorphisms of sets

$$\begin{aligned} A \textstyle\coprod \emptyset &\cong A, \\ \emptyset \textstyle\coprod A &\cong A, \end{aligned}$$

   natural in $A \in \mathrm{Obj}(\mathsf{Sets})$.

4. *Commutativity.* We have an isomorphism of sets

$$A \coprod B \cong B \coprod A,$$

natural in $A, B \in \mathrm{Obj}(\mathsf{Sets})$.

5. *Symmetric Monoidality.* The triple $(\mathsf{Sets}, \coprod, \emptyset)$ is a symmetric monoidal category.

Proof. *Item 1, Functoriality*: Clear.
Item 2, Associativity: Clear.
Item 3, Unitality: Clear.
Item 4, Commutativity: Clear.
Item 5, Symmetric Monoidality: Omitted. □

2.3 Pushouts

Let A, B, and C be sets and let $f\colon C \to A$ and $g\colon C \to B$ be functions.

Definition 2.3.1.1. The **pushout of A and B over C along f and g**[10] is the pair $(A \coprod_C B, \{\mathrm{inj}_1, \mathrm{inj2}\})$ consisting of

- *The Colimit.* The set $A \coprod_C B$ defined by

$$A \coprod_C B \overset{\mathrm{def}}{=} A \coprod B/{\sim_C},$$

where $\sim_C$ is the equivalence relation on $A \coprod B$ generated by $(0, f(c)) \sim_C (1, g(c))$.

- *The Cocone.* The maps

$$\mathrm{inj}_1\colon A \to A \coprod_C B,$$
$$\mathrm{inj}_2\colon B \to A \coprod_C B$$

given by

$$\mathrm{inj}_1(a) \overset{\mathrm{def}}{=} [(0, a)]$$
$$\mathrm{inj}_2(b) \overset{\mathrm{def}}{=} [(1, b)]$$

for each $a \in A$ and each $b \in B$.

[10] *Further Terminology:* Also called the **fibre coproduct of A and B over C along f and g**.

Proof. We claim that $A \coprod_C B$ is the categorical pushout of A and B over C with respect to (f, g) in Sets. First we need to check that the relevant pushout diagram commutes, i.e. that we have

$$\mathrm{inj}_1 \circ f = \mathrm{inj}_2 \circ g, \qquad \begin{tikzcd} A \coprod_C B & B \arrow[l, "\mathrm{inj}_2"'] \\ A \arrow[u, "\mathrm{inj}_1"] & C. \arrow[l, "f"] \arrow[u, "g"'] \end{tikzcd}$$

Indeed, given $c \in C$, we have

$$\begin{aligned} [\mathrm{inj}_1 \circ f](c) &= \mathrm{inj}_1(f(c)) \\ &= [(0, f(c))] \\ &= [(1, g(c))] \\ &= \mathrm{inj}_2(g(c)) \\ &= [\mathrm{inj}_2 \circ g](c), \end{aligned}$$

where $[(0, f(c))] = [(1, g(c))]$ by the definition of the relation $\sim$ on B. Next, we prove that $A \coprod_C B$ satisfies the universal property of the pushout. Suppose we have a diagram of the form

$$\begin{tikzcd} P & & \\ & A \coprod_C B \arrow[ul, dashed] & B \arrow[l, "\mathrm{inj}_2"'] \arrow[ull, bend right, "i_2"'] \\ & A \arrow[u, "\mathrm{inj}_1"] \arrow[uul, bend left, "i_1"] & C \arrow[l, "f"] \arrow[u, "g"'] \end{tikzcd}$$

in Sets. Then there exists a unique map $\phi\colon A \coprod_C B \to P$, uniquely determined by the conditions

$$\begin{aligned} \phi \circ \mathrm{inj}_1 &= i_1, \\ \phi \circ \mathrm{inj}_2 &= i_2, \end{aligned}$$

being necessarily given by

$$\phi(x) = \begin{cases} i_1(a) & \text{if } x = [(0, a)], \\ i_2(b) & \text{if } x = [(1, b)] \end{cases}$$

for each $x \in A \coprod_C B$, where the well-definedness of ϕ is guaranteed (through a somewhat involved but elementary argument; see [MSE 3774686]) by the equality $i_1 \circ f = i_2 \circ g$ and the definition of the relation $\sim$ on $A \coprod B$. □

Remark 2.3.1.2. In detail, by Relations, **??**, the relation $\sim$ of Definition 2.3.1.1 is given by declaring $a \sim b$ iff one of the following conditions is satisfied:

- We have $a, b \in A$ and $a = b$;
- We have $a, b \in B$ and $a = b$;
- There exist $x_1, \ldots, x_n \in A \coprod B$ such that $a \sim' x_1 \sim' \cdots \sim' x_n \sim' b$, where we declare $x \sim' y$ if one of the following conditions is satisfied:

  1. There exists $c \in C$ such that $x = f(c)$ and $y = g(c)$.
  2. There exists $c \in C$ such that $x = g(c)$ and $y = f(c)$.

  That is: we require the following condition to be satisfied:

  ($\star$) There exist $x_1, \ldots, x_n \in A \coprod B$ satisfying the following conditions:
    1. There exists $c_0 \in C$ satisfying one of the following conditions:
       (a) We have $a = f(c_0)$ and $x_1 = g(c_0)$.
       (b) We have $a = g(c_0)$ and $x_1 = f(c_0)$.
    2. For each $1 \leq i \leq n-1$, there exists $c_i \in C$ satisfying one of the following conditions:
       (a) We have $x_i = f(c_i)$ and $x_{i+1} = g(c_i)$.
       (b) We have $x_i = g(c_i)$ and $x_{i+1} = f(c_i)$.
    3. There exists $c_n \in C$ satisfying one of the following conditions:
       (a) We have $x_n = f(c_n)$ and $b = g(c_n)$.
       (b) We have $x_n = g(c_n)$ and $b = f(c_n)$.

Example 2.3.1.3. Here are some examples of pushouts of sets.

1. *Wedge Sums of Pointed Sets.* The wedge sum of two pointed sets of Pointed Sets, **??** is an example of a pushout of sets.

2. *Intersections via Unions.* Let $A, B \subset X$. We have a bijection of sets

$$A \cup B \cong A \coprod_{A \cap B} B, \qquad \begin{tikzcd} A \cup B \arrow[dr, phantom, "\ulcorner", very near start] & B \arrow[l, hook'] \\ A \arrow[u, hook] & A \cap B. \arrow[l, hook'] \arrow[u, hook] \end{tikzcd}$$

Proof. *Item 1, Wedge Sums of Pointed Sets*: Follows by definition.
Item 2, Intersections via Unions: Indeed, $A \coprod_{A \cap B} B$ is the quotient of $A \coprod B$ by the equivalence relation obtained by declaring $(0, a) \sim (1, b)$ iff $a = b \in A \cap B$, which is in bijection with $A \cup B$ via the map with $[(0, a)] \mapsto a$ and $[(1, b)] \mapsto b$. □

Proposition 2.3.1.4. Let A, B, C, and X be sets.

1. *Functoriality.* The assignment $(A, B, C, f, g) \mapsto A \coprod_{f,C,g} B$ defines a functor
$$-_1 \coprod_{-_3} -_1 \colon \mathsf{Fun}(\mathcal{P}, \mathsf{Sets}) \to \mathsf{Sets},$$
where $\mathcal{P}$ is the category that looks like this:
$$\begin{tikzcd} & \bullet \\ \bullet & \bullet. \arrow[l] \arrow[u] \end{tikzcd}$$
In particular, the action on morphisms of $-_1 \coprod_{-_3} -_1$ is given by sending a morphism
$$\begin{tikzcd} A \coprod_C B \arrow[dr, phantom, "\ulcorner", very near start] & & B \arrow[ll] \arrow[dr, "\psi"] & \\ & A' \coprod_{C'} B' \arrow[dr, phantom, "\ulcorner", very near start] & & B' \arrow[ll] \\ A \arrow[uu] \arrow[dr, "\phi"'] & & C \arrow[ll, "f"'] \arrow[uu, "g"] \arrow[dr, "\chi"] & \\ & A' \arrow[uu] & & C' \arrow[ll, "f'"] \arrow[uu, "g'"'] \end{tikzcd}$$
in $\mathsf{Fun}(\mathcal{P}, \mathsf{Sets})$ to the map $\xi \colon A \coprod_C B \xrightarrow{\exists!} A' \coprod_{C'} B'$ given by
$$\xi(x) \stackrel{\text{def}}{=} \begin{cases} \phi(a) & \text{if } x = [(0, a)], \\ \psi(b) & \text{if } x = [(1, b)] \end{cases}$$

for each $x \in A \coprod_C B$, which is the unique map making the diagram

$$
\begin{tikzcd}
A \coprod_C B \arrow[dr, dashed] & & B \arrow[ll] \arrow[dr, "\psi"] & \\
& A' \coprod_{C'} B' & & B' \arrow[ll] \\
A \arrow[uu] \arrow[dr, "\phi"'] & & C \arrow[ll, "f"'] \arrow[uu, "g"] \arrow[dr, "\chi"] & \\
& A' \arrow[uu] & & C' \arrow[ll, "f'"] \arrow[uu, "g'"']
\end{tikzcd}
$$

commute.

2. *Associativity.* Given a diagram

$$
\begin{tikzcd}
A & & B & & C \\
& X \arrow[ul, "f"] \arrow[ur, "g"'] & & Y \arrow[ul, "h"] \arrow[ur, "k"'] &
\end{tikzcd}
$$

in Sets, we have isomorphisms

$$(A \coprod_X B) \coprod_Y C \cong (A \coprod_X B) \coprod_B (B \coprod_Y C) \cong A \coprod_X (B \coprod_Y C),$$

where these pullbacks are built as in the diagrams

$$
\begin{tikzcd}
& & (A \coprod_X B) \coprod_Y C & & \\
& A \coprod_X B \arrow[ur] & & & \\
A \arrow[ur] & & B \arrow[ul] & & C, \arrow[uull] \\
& X \arrow[ul, "f"] \arrow[ur, "g"'] & & Y \arrow[ul, "h"] \arrow[ur, "k"'] &
\end{tikzcd}
\quad
\begin{tikzcd}
& & (A \coprod_X B) \coprod_B (B \coprod_Y C) & & \\
& A \coprod_X B \arrow[ur] & & B \coprod_Y C \arrow[ul] & \\
A \arrow[ur] & & B \arrow[ul] \arrow[ur] & & C, \arrow[ul] \\
& X \arrow[ul, "f"] \arrow[ur, "g"'] & & Y \arrow[ul, "h"] \arrow[ur, "k"'] &
\end{tikzcd}
\quad
\begin{tikzcd}
& & A \coprod_X (B \coprod_Y C) & & \\
& & & B \coprod_Y C \arrow[ul] & \\
A \arrow[uurr] & & B \arrow[ur] & & C. \arrow[ul] \\
& X \arrow[ul, "f"] \arrow[ur, "g"'] & & Y \arrow[ul, "h"] \arrow[ur, "k"'] &
\end{tikzcd}
$$

3. *Unitality.* We have isomorphisms of sets

$$
\begin{tikzcd}
A \arrow[r, equal] & A \\
X \arrow[u, "f"] \arrow[r, equal] & X \arrow[u, "f"']
\end{tikzcd}
\qquad
\begin{aligned}
X \coprod_X A &\cong A, \\
A \coprod_X X &\cong A,
\end{aligned}
\qquad
\begin{tikzcd}
A \arrow[d, equal] & X \arrow[l, "f"'] \arrow[d, equal] \\
X & X. \arrow[l, "f"]
\end{tikzcd}
$$

4. *Commutativity.* We have an isomorphism of sets

$$\begin{tikzcd} A \coprod_X B & B \arrow[l] \\ A \arrow[u] \arrow[ur, phantom, "\ulcorner", very near end] & X, \arrow[l, "f"] \arrow[u, "g"'] \end{tikzcd} \qquad A \coprod_X B \cong B \coprod_X A \qquad \begin{tikzcd} B \coprod_X A & A \arrow[l] \\ B \arrow[u] \arrow[ur, phantom, "\ulcorner", very near end] & X. \arrow[l, "g"] \arrow[u, "f"'] \end{tikzcd}$$

5. *Interaction With Coproducts.* We have

$$A \coprod_\emptyset B \cong A \coprod B, \qquad \begin{tikzcd} A \coprod B & B \arrow[l] \\ A \arrow[u] \arrow[ur, phantom, "\ulcorner", very near end] & \emptyset. \arrow[l, "\iota_A"] \arrow[u, "\iota_B"'] \end{tikzcd}$$

6. *Symmetric Monoidality.* The triple $(\mathsf{Sets}, \coprod_X, \emptyset)$ is a symmetric monoidal category.

Proof. *Item 1*, *Functoriality*: This is a special case of functoriality of co/limits, Limits and Colimits, **??** of **??**, with the explicit expression for ξ following from the commutativity of the cube pushout diagram.
Item 2, *Associativity*: Omitted.
Item 3, *Unitality*: Omitted.
Item 4, *Commutativity*: Clear.
Item 5, *Interaction With Coproducts*: Clear.
Item 6, *Symmetric Monoidality*: Omitted. □

2.4 Coequalisers

Let A and B be sets and let $f, g \colon A \rightrightarrows B$ be functions.

Definition 2.4.1.1. The **coequaliser of f and g** is the pair $(\mathrm{CoEq}(f,g), \mathrm{coeq}(f,g))$ consisting of

- *The Colimit.* The set $\mathrm{CoEq}(f,g)$ defined by

$$\mathrm{CoEq}(f,g) \stackrel{\text{def}}{=} B/{\sim},$$

  where $\sim$ is the equivalence relation on B generated by $f(a) \sim g(a)$.

- *The Cocone.* The map
$$\mathrm{coeq}(f,g)\colon B \to \mathrm{CoEq}(f,g)$$
given by the quotient map $\pi\colon B \twoheadrightarrow B/{\sim}$ with respect to the equivalence relation generated by $f(a) \sim g(a)$.

Proof. We claim that $\mathrm{CoEq}(f,g)$ is the categorical coequaliser of f and g in Sets. First we need to check that the relevant coequaliser diagram commutes, i.e. that we have
$$\mathrm{coeq}(f,g) \circ f = \mathrm{coeq}(f,g) \circ g.$$
Indeed, we have
$$\begin{aligned}[\mathrm{coeq}(f,g) \circ f](a) &\overset{\text{def}}{=} [\mathrm{coeq}(f,g)](f(a))\\ &\overset{\text{def}}{=} [f(a)]\\ &= [g(a)]\\ &\overset{\text{def}}{=} [\mathrm{coeq}(f,g)](g(a))\\ &\overset{\text{def}}{=} [\mathrm{coeq}(f,g) \circ g](a)\end{aligned}$$
for each $a \in A$. Next, we prove that $\mathrm{CoEq}(f,g)$ satisfies the universal property of the coequaliser. Suppose we have a diagram of the form
$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "\mathrm{coeq}(f,g)"] \arrow[dr, "c"'] & \mathrm{CoEq}(f,g) \\ & & C \end{tikzcd}$$
in Sets. Then, since $c(f(a)) = c(g(a))$ for each $a \in A$, it follows from Relations, **????** of **??** that there exists a unique map $\mathrm{CoEq}(f,g) \xrightarrow{\exists!} C$ making the diagram
$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "\mathrm{coeq}(f,g)"] \arrow[dr, "c"'] & \mathrm{CoEq}(f,g) \arrow[d, dashed, "\exists!"] \\ & & C \end{tikzcd}$$
commutes. □

Remark 2.4.1.2. In detail, by Relations, **??**, the relation $\sim$ of Definition 2.4.1.1 is given by declaring $a \sim b$ iff one of the following conditions is satisfied:

- We have $a = b$;
- There exist $x_1, \ldots, x_n \in B$ such that $a \sim' x_1 \sim' \cdots \sim' x_n \sim' b$, where we declare $x \sim' y$ if one of the following conditions is satisfied:
  1. There exists $z \in A$ such that $x = f(z)$ and $y = g(z)$.
  2. There exists $z \in A$ such that $x = g(z)$ and $y = f(z)$.

  That is: we require the following condition to be satisfied:

  ($\star$) There exist $x_1, \ldots, x_n \in B$ satisfying the following conditions:
    1. There exists $z_0 \in A$ satisfying one of the following conditions:
       (a) We have $a = f(z_0)$ and $x_1 = g(z_0)$.
       (b) We have $a = g(z_0)$ and $x_1 = f(z_0)$.
    2. For each $1 \leq i \leq n-1$, there exists $z_i \in A$ satisfying one of the following conditions:
       (a) We have $x_i = f(z_i)$ and $x_{i+1} = g(z_i)$.
       (b) We have $x_i = g(z_i)$ and $x_{i+1} = f(z_i)$.
    3. There exists $z_n \in A$ satisfying one of the following conditions:
       (a) We have $x_n = f(z_n)$ and $b = g(z_n)$.
       (b) We have $x_n = g(z_n)$ and $b = f(z_n)$.

Example 2.4.1.3. Here are some examples of coequalisers of sets.

1. *Quotients by Equivalence Relations.* Let R be an equivalence relation on a set X. We have a bijection of sets

$$X/{\sim_R} \cong \mathrm{CoEq}\left(R \hookrightarrow X \times X \underset{\mathrm{pr}_2}{\overset{\mathrm{pr}_1}{\rightrightarrows}} X\right).$$

Proof. *Item 1, Quotients by Equivalence Relations*: See [Pro24e]. □

Proposition 2.4.1.4. Let A, B, and C be sets.

1. *Associativity.* We have an isomorphism of sets[11]

$$\underbrace{\mathrm{CoEq}(\mathrm{coeq}(f,g)\circ f,\mathrm{coeq}(f,g)\circ h)}_{=\mathrm{CoEq}(\mathrm{coeq}(f,g)\circ g,\mathrm{coeq}(f,g)\circ h)} \cong \mathrm{CoEq}(f,g,h) \cong \underbrace{\mathrm{CoEq}(\mathrm{coeq}(g,h)\circ f,\mathrm{coeq}(g,h)\circ g)}_{=\mathrm{CoEq}(\mathrm{coeq}(g,h)\circ f,\mathrm{coeq}(g,h)\circ h)},$$

   where $\mathrm{CoEq}(f,g,h)$ is the colimit of the diagram

$$\begin{tikzcd} A \arrow[r, shift left=2, "f"] \arrow[r, "g" description] \arrow[r, shift right=2, "h"'] & B \end{tikzcd}$$

   in Sets.

[11] That is, the following three ways of forming "the" coequaliser of (f,g,h) agree:

1. Take the coequaliser of (f,g,h), i.e. the colimit of the diagram

$$\begin{tikzcd} A \arrow[r, shift left=2, "f"] \arrow[r, "g" description] \arrow[r, shift right=2, "h"'] & B \end{tikzcd}$$

   in Sets.
2. First take the coequaliser of f and g, forming a diagram

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \overset{\mathrm{coeq}(f,g)}{\twoheadrightarrow} \mathrm{CoEq}(f,g)$$

   and then take the coequaliser of the composition

$$A \underset{h}{\overset{f}{\rightrightarrows}} B \overset{\mathrm{coeq}(f,g)}{\twoheadrightarrow} \mathrm{CoEq}(f,g),$$

   obtaining a quotient

$$\mathrm{CoEq}(\mathrm{coeq}(f,g)\circ f,\mathrm{coeq}(f,g)\circ h) = \mathrm{CoEq}(\mathrm{coeq}(f,g)\circ g,\mathrm{coeq}(f,g)\circ h)$$

   of $\mathrm{CoEq}(f,g)$
3. First take the coequaliser of g and h, forming a diagram

$$A \underset{h}{\overset{g}{\rightrightarrows}} B \overset{\mathrm{coeq}(g,h)}{\twoheadrightarrow} \mathrm{CoEq}(g,h)$$

   and then take the coequaliser of the composition

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \overset{\mathrm{coeq}(g,h)}{\twoheadrightarrow} \mathrm{CoEq}(g,h),$$

   obtaining a quotient

$$\mathrm{CoEq}(\mathrm{coeq}(g,h)\circ f,\mathrm{coeq}(g,h)\circ g) = \mathrm{CoEq}(\mathrm{coeq}(g,h)\circ f,\mathrm{coeq}(g,h)\circ h)$$

   of $\mathrm{CoEq}(g,h)$.

4. *Unitality.* We have an isomorphism of sets
$$\mathrm{CoEq}(f, f) \cong B.$$

5. *Commutativity.* We have an isomorphism of sets
$$\mathrm{CoEq}(f, g) \cong \mathrm{CoEq}(g, f).$$

6. *Interaction With Composition.* Let
$$A \underset{g}{\overset{f}{\rightrightarrows}} B \underset{k}{\overset{h}{\rightrightarrows}} C$$
be functions. We have a surjection
$$\mathrm{CoEq}(h \circ f, k \circ g) \twoheadrightarrow \mathrm{CoEq}(\mathrm{coeq}(h, k) \circ h \circ f, \mathrm{coeq}(h, k) \circ k \circ g)$$
exhibiting $\mathrm{CoEq}(\mathrm{coeq}(h, k) \circ h \circ f, \mathrm{coeq}(h, k) \circ k \circ g)$ as a quotient of $\mathrm{CoEq}(h \circ f, k \circ g)$ by the relation generated by declaring $h(y) \sim k(y)$ for each $y \in B$.

Proof. *Item 1*, *Associativity*: Omitted.
Item 4, *Unitality*: Clear.
Item 5, *Commutativity*: Clear.
Item 6, *Interaction With Composition*: Omitted. □

3 Operations With Sets

3.1 The Empty Set

Definition 3.1.1.1. The **empty set** is the set $\emptyset$ defined by
$$\emptyset \overset{\mathrm{def}}{=} \{x \in X \mid x \neq x\},$$
where A is the set in the set existence axiom, **??** of **??**.

3.2 Singleton Sets

Let X be a set.

Definition 3.2.1.1. The **singleton set containing** X is the set $\{X\}$ defined by
$$\{X\} \overset{\mathrm{def}}{=} \{X, X\},$$
where $\{X, X\}$ is the pairing of X with itself (Definition 3.3.1.1).

3.3 Pairings of Sets

Let X and Y be sets.

Definition 3.3.1.1. The **pairing of** X **and** Y is the set $\{X, Y\}$ defined by

$$\{X, Y\} \stackrel{\text{def}}{=} \{x \in A \mid x = X \text{ or } x = Y\},$$

where A is the set in the axiom of pairing, **??** of **??**.

3.4 Unions of Families

Let $\{A_i\}_{i\in I}$ be a family of sets.

Definition 3.4.1.1. The **union of the family** $\{A_i\}_{i\in I}$ is the set $\bigcup_{i\in I} A_i$ defined by

$$\bigcup_{i\in I} A_i \stackrel{\text{def}}{=} \{x \in F \mid \text{there exists some } i \in I \text{ such that } x \in A_i\},$$

where F is the set in the axiom of union, **??** of **??**.

3.5 Binary Unions

Let A and B be sets.

Definition 3.5.1.1. The **union**[12] **of** A **and** B is the set $A \cup B$ defined by

$$A \cup B \stackrel{\text{def}}{=} \bigcup_{z\in\{A,B\}} z.$$

Proposition 3.5.1.2. Let X be a set.

1. *Functoriality.* The assignments $U, V, (U, V) \mapsto U \cup V$ define functors

$$\begin{aligned} U \cup - &: (\mathcal{P}(X), \subset) \to (\mathcal{P}(X), \subset), \\ - \cup V &: (\mathcal{P}(X), \subset) \to (\mathcal{P}(X), \subset), \\ -_1 \cup -_2 &: (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \subset) \to (\mathcal{P}(X), \subset), \end{aligned}$$

   where $-_1 \cup -_2$ is the functor where

   - *Action on Objects.* For each $(U, V) \in \mathcal{P}(X) \times \mathcal{P}(X)$, we have

$$[-_1 \cup -_2](U, V) \stackrel{\text{def}}{=} U \cup V;$$

[12] *Further Terminology:* Also called the **binary union of** A **and** B, for emphasis.

- *Action on Morphisms.* For each pair of morphisms

  $$\iota_U \colon U \hookrightarrow U',$$
  $$\iota_V \colon V \hookrightarrow V'$$

  of $\mathcal{P}(X) \times \mathcal{P}(X)$, the image

  $$\iota_U \cup \iota_V \colon U \cup V \hookrightarrow U' \cup V'$$

  of (ι_U, ι_V) by $\cup$ is the inclusion

  $$U \cup V \subset U' \cup V'$$

  i.e. where we have

  $(\star)$ If $U \subset U'$ and $V \subset V'$, then $U \cup V \subset U' \cup V'$;

and where $U \cup -$ and $- \cup V$ are the partial functors of $-_1 \cup -_2$ at $U, V \in \mathcal{P}(X)$.

2. *Via Intersections and Symmetric Differences.* We have an equality of sets
$$U \cup V = (U \bigtriangleup V) \bigtriangleup (U \cap V)$$
for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

3. *Associativity.* We have an equality of sets
$$(U \cup V) \cup W = U \cup (V \cup W)$$
for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

4. *Unitality.* We have equalities of sets
$$U \cup \emptyset = U,$$
$$\emptyset \cup U = U$$
for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

5. *Commutativity.* We have an equality of sets
$$U \cup V = V \cup U$$
for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

6. *Idempotency.* We have an equality of sets

$$U \cup U = U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

7. *Distributivity Over Intersections.* We have equalities of sets

$$\begin{aligned} U \cup (V \cap W) &= (U \cup V) \cap (U \cup W), \\ (U \cap V) \cup W &= (U \cup W) \cap (V \cup W) \end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

8. *Interaction With Powersets and Semirings.* The quintuple $(\mathcal{P}(X), \cup, \cap, \emptyset, X)$ is an idempotent commutative semiring.

Proof. *Item 1*, *Functoriality*: Omitted.
Item 2, *Via Intersections and Symmetric Differences*: Omitted.
Item 3, *Associativity*: Clear.
Item 4, *Unitality*: Clear.
Item 5, *Commutativity*: Clear.
Item 6, *Idempotency*: Clear.
Item 7, *Distributivity Over Intersections*: Omitted.
Item 8, *Interaction With Powersets and Semirings*: This follows from Items 3 to 6 and Items 3 to 5, 7 and 8 of Proposition 3.7.1.2. □

3.6 Intersections of Families

Let $\mathcal{F}$ be a family of sets.

Definition 3.6.1.1. The **intersection of a family $\mathcal{F}$ of sets** is the set $\bigcap_{X \in \mathcal{F}} X$ defined by

$$\bigcap_{X \in \mathcal{F}} X \stackrel{\text{def}}{=} \left\{ z \in \bigcup_{X \in \mathcal{F}} X \;\middle|\; \text{for each } X \in \mathcal{F}, \text{ we have } z \in X \right\}.$$

3.7 Binary Intersections

Let X and Y be sets.

Definition 3.7.1.1. The **intersection**[13] **of X and Y** is the set $X \cap Y$

[13] *Further Terminology:* Also called the **binary intersection of X and Y**, for emphasis.

defined by

$$X \cap Y \overset{\text{def}}{=} \bigcap_{z \in \{X,Y\}} z.$$

Proposition 3.7.1.2. Let X be a set.

1. *Functoriality.* The assignments $U, V, (U,V) \mapsto U \cap V$ define functors

$$\begin{aligned} U \cap -\colon (\mathcal{P}(X), \subset) &\to (\mathcal{P}(X), \subset), \\ - \cap V\colon (\mathcal{P}(X), \subset) &\to (\mathcal{P}(X), \subset), \\ -_1 \cap -_2\colon (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \subset) &\to (\mathcal{P}(X), \subset), \end{aligned}$$

where $-_1 \cap -_2$ is the functor where

- *Action on Objects.* For each $(U, V) \in \mathcal{P}(X) \times \mathcal{P}(X)$, we have

$$[-_1 \cap -_2](U, V) \overset{\text{def}}{=} U \cap V;$$

- *Action on Morphisms.* For each pair of morphisms

$$\begin{aligned} \iota_U\colon U &\hookrightarrow U', \\ \iota_V\colon V &\hookrightarrow V' \end{aligned}$$

of $\mathcal{P}(X) \times \mathcal{P}(X)$, the image

$$\iota_U \cap \iota_V\colon U \cap V \hookrightarrow U' \cap V'$$

of (ι_U, ι_V) by $\cap$ is the inclusion

$$U \cap V \subset U' \cap V'$$

i.e. where we have

($\star$) If $U \subset U'$ and $V \subset V'$, then $U \cap V \subset U' \cap V'$;

and where $U \cap -$ and $- \cap V$ are the partial functors of $-_1 \cap -_2$ at $U, V \in \mathcal{P}(X)$.

2. *Adjointness.* We have adjunctions

$$\Big(U \cap - \dashv \mathbf{Hom}_{\mathcal{P}(X)}(U, -)\Big)\colon \quad \begin{tikzcd} \mathcal{P}(X) \arrow[r, bend left, "U\cap-"] \arrow[r, phantom, "\perp"] & \mathcal{P}(X), \arrow[l, bend left, "\mathbf{Hom}_{\mathcal{P}(X)}(U,-)"] \end{tikzcd}$$

$$\Big(- \cap V \dashv \mathbf{Hom}_{\mathcal{P}(X)}(V, -)\Big)\colon \quad \begin{tikzcd} \mathcal{P}(X) \arrow[r, bend left, "-\cap V"] \arrow[r, phantom, "\perp"] & \mathcal{P}(X), \arrow[l, bend left, "\mathbf{Hom}_{\mathcal{P}(X)}(V,-)"] \end{tikzcd}$$

where

$$\mathbf{Hom}_{\mathcal{P}(X)}(-_1, -_2) \colon \mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \to \mathcal{P}(X)$$

is the bifunctor defined by[14]

$$\mathbf{Hom}_{\mathcal{P}(X)}(U, V) \stackrel{\text{def}}{=} (X \setminus U) \cup V$$

witnessed by bijections

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{P}(X)}(U \cap V, W) &\cong \mathrm{Hom}_{\mathcal{P}(X)}\Big(U, \mathbf{Hom}_{\mathcal{P}(X)}(V, W)\Big),\\
\mathrm{Hom}_{\mathcal{P}(X)}(U \cap V, W) &\cong \mathrm{Hom}_{\mathcal{P}(X)}\Big(V, \mathbf{Hom}_{\mathcal{P}(X)}(U, W)\Big),
\end{aligned}$$

natural in $U, V, W \in \mathcal{P}(X)$, i.e. where:

(a) The following conditions are equivalent:
   i. We have $U \cap V \subset W$.
   ii. We have $U \subset \mathbf{Hom}_{\mathcal{P}(X)}(V, W)$.
   iii. We have $U \subset (X \setminus V) \cup W$.

(b) The following conditions are equivalent:
   i. We have $V \cap U \subset W$.
   ii. We have $V \subset \mathbf{Hom}_{\mathcal{P}(X)}(U, W)$.
   iii. We have $V \subset (X \setminus U) \cup W$.

3. *Associativity.* We have an equality of sets

$$(U \cap V) \cap W = U \cap (V \cap W)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

4. *Unitality.* Let X be a set and let $U \in \mathcal{P}(X)$. We have equalities of sets

$$\begin{aligned}
X \cap U &= U,\\
U \cap X &= U
\end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

[14] *Intuition:* Since intersections are the products in $\mathcal{P}(X)$, the left adjoint $\mathbf{Hom}_{\mathcal{P}(X)}(U, V)$ works as a function type $U \to V$.

Now, under the Curry–Howard correspondence, the function type $U \to V$ corresponds to implication $U \implies V$, which is logically equivalent to the statement $\neg U \vee V$, which in

5. *Commutativity.* We have an equality of sets

$$U \cap V = V \cap U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

6. *Idempotency.* We have an equality of sets

$$U \cap U = U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

7. *Distributivity Over Unions.* We have equalities of sets

$$\begin{aligned} U \cap (V \cup W) &= (U \cap V) \cup (U \cap W), \\ (U \cup V) \cap W &= (U \cap W) \cup (V \cap W) \end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

8. *Annihilation With the Empty Set.* We have an equality of sets

$$\begin{aligned} \emptyset \cap X &= \emptyset, \\ X \cap \emptyset &= \emptyset \end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

9. *Interaction With Powersets and Monoids With Zero.* The quadruple $((\mathcal{P}(X), \emptyset), \cap, X)$ is a commutative monoid with zero.

10. *Interaction With Powersets and Semirings.* The quintuple $(\mathcal{P}(X), \cup, \cap, \emptyset, X)$ is an idempotent commutative semiring.

Proof. *Item 1*, *Functoriality*: Omitted.
Item 2, *Adjointness*: See [MSE 267469].
Item 3, *Associativity*: Clear.
Item 4, *Unitality*: Clear.
Item 5, *Commutativity*: Clear.
Item 6, *Idempotency*: Clear.
Item 7, *Distributivity Over Unions*: Omitted.
Item 8, *Annihilation With the Empty Set*: Clear.
Item 9, *Interaction With Powersets and Monoids With Zero*: This follows from Items 3 to 5 and 8.
Item 10, *Interaction With Powersets and Semirings*: This follows from Items 3 to 6 and Items 3 to 5, 7 and 8 of Proposition 3.7.1.2. □

3.8 Differences

Let X and Y be sets.

Definition 3.8.1.1. The **difference of X and Y** is the set $X \setminus Y$ defined by

$$X \setminus Y \overset{\text{def}}{=} \{a \in X \mid a \notin Y\}.$$

Proposition 3.8.1.2. Let X be a set.

1. *Functoriality.* The assignments $U, V, (U, V) \mapsto U \cap V$ define functors

$$\begin{aligned} U \setminus -\colon & (\mathcal{P}(X), \supset) \to (\mathcal{P}(X), \subset), \\ - \setminus V\colon & (\mathcal{P}(X), \subset) \to (\mathcal{P}(X), \subset), \\ -_1 \setminus -_2\colon & (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \supset) \to (\mathcal{P}(X), \subset), \end{aligned}$$

where $-_1 \setminus -_2$ is the functor where

- *Action on Objects.* For each $(U, V) \in \mathcal{P}(X) \times \mathcal{P}(X)$, we have

$$[-_1 \setminus -_2](U, V) \overset{\text{def}}{=} U \setminus V;$$

- *Action on Morphisms.* For each pair of morphisms

$$\begin{aligned} \iota_A &\colon A \hookrightarrow B, \\ \iota_U &\colon U \hookrightarrow V \end{aligned}$$

of $\mathcal{P}(X) \times \mathcal{P}(X)$, the image

$$\iota_U \setminus \iota_V \colon A \setminus V \hookrightarrow B \setminus U$$

of (ι_U, ι_V) by $\setminus$ is the inclusion

$$A \setminus V \subset B \setminus U$$

i.e. where we have

(★) If $A \subset B$ and $U \subset V$, then $A \setminus V \subset B \setminus U$;

and where $U \setminus -$ and $- \setminus V$ are the partial functors of $-_1 \setminus -_2$ at $U, V \in \mathcal{P}(X)$.

2. *De Morgan's Laws.* We have equalities of sets

$$\begin{aligned} X \setminus (U \cup V) &= (X \setminus U) \cap (X \setminus V), \\ X \setminus (U \cap V) &= (X \setminus U) \cup (X \setminus V) \end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

3. *Interaction With Unions I.* We have equalities of sets

$$(U \setminus V) \cup W = (U \cup W) \setminus (V \setminus W)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

4. *Interaction With Unions II.* We have equalities of sets

$$(U \cup V) \setminus W = (U \setminus W) \cup (V \setminus W)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

5. *Interaction With Intersections.* We have equalities of sets

$$\begin{aligned} (U \setminus V) \cap W &= (U \cap W) \setminus V \\ &= U \cap (W \setminus V) \end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

6. *Triple Differences.* We have

$$U \setminus (V \setminus W) = (U \cap W) \cup (U \setminus V)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

7. *Left Annihilation.* We have

$$\emptyset \setminus U = \emptyset$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

8. *Right Unitality.* We have

$$U \setminus \emptyset = U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

9. *Invertibility.* We have

$$U \setminus U = \emptyset$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

10. *Interaction With Containment.* The following conditions are equivalent:

    (a) We have $V \setminus U \subset W$.

    (b) We have $V \setminus W \subset U$.

Proof. Item 1, *Functoriality*: Omitted.
Item 2, *De Morgan's Laws*: Omitted.
Item 3, *Interaction With Unions I*: Omitted.
Item 4, *Interaction With Unions II*: Omitted.
Item 5, *Interaction With Intersections*: Omitted.
Item 6, *Triple Differences*: Omitted.
Item 7, *Left Annihilation*: Clear.
Item 8, *Right Unitality*: Clear.
Item 9, *Invertibility*: Clear.
Item 10, *Interaction With Containment*: Omitted. □

3.9 Complements

Let X be a set and let $U \in \mathcal{P}(X)$.

Definition 3.9.1.1. The **complement of** U is the set U^{c} defined by

$$\begin{aligned} U^{\mathsf{c}} &\overset{\text{def}}{=} X \setminus U \\ &\overset{\text{def}}{=} \{a \in X \mid a \notin U\}. \end{aligned}$$

Proposition 3.9.1.2. Let X be a set.

1. *Functoriality.* The assignment $U \mapsto U^{\mathsf{c}}$ defines a functor

$$(-)^{\mathsf{c}} \colon \mathcal{P}(X)^{\mathsf{op}} \to \mathcal{P}(X),$$

where

- *Action on Objects.* For each $U \in \mathcal{P}(X)$, we have

$$[(-)^{\mathsf{c}}](U) \overset{\text{def}}{=} U^{\mathsf{c}};$$

turn corresponds to the set $U^{\mathsf{c}} \vee V \overset{\text{def}}{=} (X \setminus U) \cup V$.

- *Action on Morphisms.* For each morphism $\iota_U \colon U \hookrightarrow V$ of $\mathcal{P}(X)$, the image
$$\iota_U^{\mathsf{c}} \colon V^{\mathsf{c}} \hookrightarrow U^{\mathsf{c}}$$
of ι_U by $(-)^{\mathsf{c}}$ is the inclusion
$$V^{\mathsf{c}} \subset U^{\mathsf{c}}$$
i.e. where we have
($\star$) If $U \subset V$, then $V^{\mathsf{c}} \subset U^{\mathsf{c}}$.

2. *De Morgan's Laws.* We have equalities of sets
$$(U \cup V)^{\mathsf{c}} = U^{\mathsf{c}} \cap V^{\mathsf{c}},$$
$$(U \cap V)^{\mathsf{c}} = U^{\mathsf{c}} \cup V^{\mathsf{c}}$$
for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

3. *Involutority.* We have
$$(U^{\mathsf{c}})^{\mathsf{c}} = U$$
for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

Proof. *Item 1, Functoriality*: Clear.
Item 2, De Morgan's Laws: Omitted.
Item 3, Involutority: Clear. □

3.10 Symmetric Differences

Let A and B be sets.

Definition 3.10.1.1. The **symmetric difference of A and B** is the set $A \triangle B$ defined by
$$A \triangle B \overset{\text{def}}{=} (A \setminus B) \cup (B \setminus A).$$

Proposition 3.10.1.2. Let X be a set.

1. *Lack of Functoriality.* The assignment $(U, V) \mapsto U \triangle V$ ***does not*** define a functor
$$-_1 \triangle -_2 \colon (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \subset) \to (\mathcal{P}(X), \subset).$$

2. *Via Unions and Intersections.* We have[15]

$$U \mathbin{\triangle} V = (U \cup V) \setminus (U \cap V)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

3. *Associativity.* We have[16]

$$(U \mathbin{\triangle} V) \mathbin{\triangle} W = U \mathbin{\triangle} (V \mathbin{\triangle} W)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

4. *Unitality.* We have

$$\begin{aligned} U \mathbin{\triangle} \emptyset &= U, \\ \emptyset \mathbin{\triangle} U &= U \end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

5. *Invertibility.* We have

$$U \mathbin{\triangle} U = \emptyset$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

6. *Commutativity.* We have

$$U \mathbin{\triangle} V = V \mathbin{\triangle} U$$

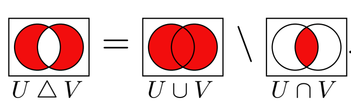
for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

7. *"Transitivity".* We have

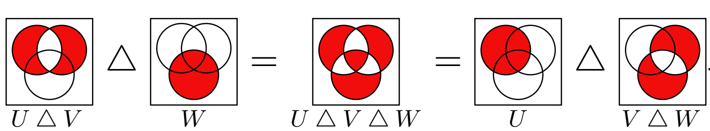
$$(U \mathbin{\triangle} V) \mathbin{\triangle} (V \mathbin{\triangle} W) = U \mathbin{\triangle} W$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

[15] *Illustration:*



[16] *Illustration:*



8. *The Triangle Inequality for Symmetric Differences.* We have

$$U \bigtriangleup W \subset U \bigtriangleup V \cup V \bigtriangleup W$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

9. *Distributivity Over Intersections.* We have

$$\begin{aligned} U \cap (V \bigtriangleup W) &= (U \cap V) \bigtriangleup (U \cap W), \\ (U \bigtriangleup V) \cap W &= (U \cap W) \bigtriangleup (V \cap W) \end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

10. *Interaction With Indicator Functions.* We have

$$\chi_{U \bigtriangleup V} \equiv \chi_U + \chi_V \mod 2$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

11. *Bijectivity.* Given $A, B \subset \mathcal{P}(X)$, the maps

$$\begin{aligned} A \bigtriangleup -&\colon \mathcal{P}(X) \to \mathcal{P}(X), \\ - \bigtriangleup B&\colon \mathcal{P}(X) \to \mathcal{P}(X) \end{aligned}$$

are bijections with inverses given by

$$\begin{aligned} (A \bigtriangleup -)^{-1} &= - \cup (A \cap -), \\ (- \bigtriangleup B)^{-1} &= - \cup (B \cap -). \end{aligned}$$

Moreover, the map

$$C \mapsto C \bigtriangleup (A \bigtriangleup B)$$

is a bijection of $\mathcal{P}(X)$ onto itself sending A to B and B to A.

12. *Interaction With Powersets and Groups I.* The quadruple $\left(\mathcal{P}(X), \bigtriangleup, \emptyset, \mathrm{id}_{\mathcal{P}(X)}\right)$

is an abelian group.[17,18,19]

13. *Interaction With Powersets and Groups II.* Every element of $\mathcal{P}(X)$ has order 2 with respect to $\triangle$, and thus $\mathcal{P}(X)$ is a *Boolean group* (i.e. an abelian 2-group).

14. *Interaction With Powersets and Vector Spaces I.* The pair $\left(\mathcal{P}(X), \alpha_{\mathcal{P}(X)}\right)$ consisting of

    - The group $\mathcal{P}(X)$ of Item 12;
    - The map $\alpha_{\mathcal{P}(X)} \colon \mathbb{F}_2 \times \mathcal{P}(X) \to \mathcal{P}(X)$ defined by

    $$\begin{aligned} 0 \cdot U &\overset{\text{def}}{=} \emptyset, \\ 1 \cdot U &\overset{\text{def}}{=} U; \end{aligned}$$

    is an $\mathbb{F}_2$-vector space.

15. *Interaction With Powersets and Vector Spaces II.* If X is finite, then:

    (a) The set of singletons sets on the elements of X forms a basis for the $\mathbb{F}_2$-vector space $\left(\mathcal{P}(X), \alpha_{\mathcal{P}(X)}\right)$ of Item 14.

    (b) We have
    $$\dim(\mathcal{P}(X)) = \#\mathcal{P}(X).$$

16. *Interaction With Powersets and Rings.* The quintuple $(\mathcal{P}(X), \triangle, \cap, \emptyset, X)$ is a commutative ring.[20]

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[17] *Example:* When $X = \emptyset$, we have an isomorphism of groups between $\mathcal{P}(\emptyset)$ and the trivial group:
$$\left(\mathcal{P}(\emptyset), \triangle, \emptyset, \mathrm{id}_{\mathcal{P}(\emptyset)}\right) \cong \mathrm{pt}.$$

[18] *Example:* When $X = \mathrm{pt}$, we have an isomorphism of groups between $\mathcal{P}(\mathrm{pt})$ and $\mathbb{Z}_{/2}$:
$$\left(\mathcal{P}(\mathrm{pt}), \triangle, \emptyset, \mathrm{id}_{\mathcal{P}(\mathrm{pt})}\right) \cong \mathbb{Z}_{/2}.$$

[19] *Example:* When $X = \{0,1\}$, we have an isomorphism of groups between $\mathcal{P}(\{0,1\})$ and $\mathbb{Z}_{/2} \times \mathbb{Z}_{/2}$:
$$\left(\mathcal{P}(\{0,1\}), \triangle, \emptyset, \mathrm{id}_{\mathcal{P}(\{0,1\})}\right) \cong \mathbb{Z}_{/2} \times \mathbb{Z}_{/2}.$$

[20] *Warning:* The analogous statement replacing intersections by unions (i.e. that the quintuple $(\mathcal{P}(X), \triangle, \cup, \emptyset, X)$ is a ring) is false, however. See [Pro24g] for a proof.

Proof. *Item 1, Lack of Functoriality*: Omitted.
Item 2, Via Unions and Intersections: Omitted.
Item 3, Associativity: Omitted.
Item 4, Unitality: Clear.
Item 5, Invertibility: Clear.
Item 6, Commutativity: Clear.
Item 7, "Transitivity": We have

$$\begin{aligned} (U \triangle V) \triangle (V \triangle W) &= U \triangle (V \triangle (V \triangle W)) && \text{(by Item 3)} \\ &= U \triangle ((V \triangle V) \triangle W) && \text{(by Item 3)} \\ &= U \triangle (\emptyset \triangle W) && \text{(by Item 5)} \\ &= U \triangle W && \text{(by Item 4)} \end{aligned}$$

Item 8, The Triangle Inequality for Symmetric Differences: This follows from Items 2 and 7.
Item 9, Distributivity Over Intersections: Omitted.
Item 10, Interaction With Indicator Functions: Clear.
Item 11, Bijectivity: Clear.
Item 12, Interaction With Powersets and Groups I: This follows from Items 3 to 6.
Item 13, Interaction With Powersets and Groups II: This follows from Item 5.
Item 14, Interaction With Powersets and Vector Spaces I: Clear.
Item 15, Interaction With Powersets and Vector Spaces II: Omitted.
Item 16, Interaction With Powersets and Rings: This follows from Items 9 and 12 and Items 8 and 9 of Proposition 3.7.1.2.[21] □

3.11 Ordered Pairs

Let A and B be sets.

Definition 3.11.1.1. The **ordered pair associated to A and B** is the set (A, B) defined by

$$(A, B) \stackrel{\text{def}}{=} \{\{A\}, \{A, B\}\}.$$

Proposition 3.11.1.2. Let A and B be sets.

1. *Uniqueness.* Let A, B, C, and D be sets. The following conditions are equivalent:

[21] *Reference:* [Pro24f].

(a) We have $(A, B) = (C, D)$.

(b) We have $A = C$ and $B = D$.

Proof. *Item 1*, *Uniqueness*: See [Cie97, Theorem 1.2.3]. □

4 Powersets

4.1 Characteristic Functions

Let X be a set.

Definition 4.1.1.1. Let $U \subset X$ and let $x \in X$.

1. The **characteristic function of** U[22] is the function[23]

$$\chi_U \colon X \to \{\mathsf{t}, \mathsf{f}\}$$

defined by

$$\chi_U(x) \overset{\text{def}}{=} \begin{cases} \mathsf{true} & \text{if } x \in U, \\ \mathsf{false} & \text{if } x \notin U \end{cases}$$

for each $x \in X$.

2. The **characteristic function of** x is the function[24]

$$\chi_x \colon X \to \{\mathsf{t}, \mathsf{f}\}$$

defined by

$$\chi_x \overset{\text{def}}{=} \chi_{\{x\}},$$

i.e. by

$$\chi_x(y) \overset{\text{def}}{=} \begin{cases} \mathsf{true} & \text{if } x = y, \\ \mathsf{false} & \text{if } x \neq y \end{cases}$$

for each $y \in X$.

[22] *Further Terminology:* Also called the **indicator function of** U.

[23] *Further Notation:* Also written $\chi_X(U, -)$ or $\chi_X(-, U)$.

[24] *Further Notation:* Also written χ_x, $\chi_X(x, -)$, or $\chi_X(-, x)$.

3. The **characteristic relation on** X[25] is the relation[26]

$$\chi_X(-_1,-_2)\colon X\times X\to\{\mathsf{t},\mathsf{f}\}$$

on X defined by[27]

$$\chi_X(x,y)\stackrel{\text{def}}{=}\begin{cases}\mathsf{true} & \text{if } x=y,\\ \mathsf{false} & \text{if } x\neq y\end{cases}$$

for each $x,y\in X$.

4. The **characteristic embedding**[28] **of** X **into** $\mathcal{P}(X)$ is the function

$$\chi_{(-)}\colon X\hookrightarrow\mathcal{P}(X)$$

defined by

$$\chi_{(-)}(x)\stackrel{\text{def}}{=}\chi_x$$

for each $x\in X$.

Remark 4.1.1.2. The definitions in Definition 4.1.1.1 are decategorifications of co/presheaves, representable co/presheaves, Hom profunctors, and the Yoneda embedding:[29]

[25]*Further Terminology:* Also called the **identity relation on** X.

[26]*Further Notation:* Also written χ^{-1}_{-2}, or $\sim_{\mathrm{id}}$ in the context of relations.

[27]As a subset of $X\times X$, the relation χ_X corresponds to the diagonal $\Delta_X\subset X\times X$ of X.

[28]The name "characteristic *embedding*" comes from the fact that there is an analogue of fully faithfulness for $\chi_{(-)}$: given a set X, we have

$$\mathrm{Hom}_{\mathcal{P}(X)}(\chi_x,\chi_y)=\chi_X(x,y),$$

for each $x,y\in X$.

[29]These statements can be made precise by using the embeddings

$$(-)_{\mathsf{disc}}\colon\mathsf{Sets}\hookrightarrow\mathsf{Cats},$$
$$(-)_{\mathsf{disc}}\colon\{\mathsf{t},\mathsf{f}\}_{\mathsf{disc}}\hookrightarrow\mathsf{Sets}$$

of sets into categories and of classical truth values into sets.

For instance, in this approach the characteristic function

$$\chi_x\colon X\to\{\mathsf{t},\mathsf{f}\}$$

of an element x of X, defined by

$$\chi_x(y)\stackrel{\text{def}}{=}\begin{cases}\mathsf{true} & \text{if } x=y,\\ \mathsf{false} & \text{if } x\neq y\end{cases}$$

1. A function
$$f\colon X \to \{\mathsf{t},\mathsf{f}\}$$
is a decategorification of a presheaf
$$\mathcal{F}\colon \mathcal{C}^{\mathsf{op}} \to \mathsf{Sets},$$
with the characteristic functions χ_U of the subsets of X being the primordial examples (and, in fact, all examples) of these.

2. The characteristic function
$$\chi_x\colon X \to \{\mathsf{t},\mathsf{f}\}$$
of an *element* x of X is a decategorification of the representable presheaf
$$h_X\colon \mathcal{C}^{\mathsf{op}} \to \mathsf{Sets}$$
of an *object* x of a category $\mathcal{C}$.

3. The characteristic relation
$$\chi_X(-_1,-_2)\colon X \times X \to \{\mathsf{t},\mathsf{f}\}$$
of X is a decategorification of the Hom profunctor
$$\mathrm{Hom}_{\mathcal{C}}(-_1,-_2)\colon \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathsf{Sets}$$
of a category $\mathcal{C}$.

4. The characteristic embedding
$$\chi_{(-)}\colon X \hookrightarrow \mathcal{P}(X)$$
of X into $\mathcal{P}(X)$ is a decategorification of the Yoneda embedding
$$よ\colon \mathcal{C}^{\mathsf{op}} \hookrightarrow \mathsf{PSh}(\mathcal{C})$$
of a category $\mathcal{C}$ into $\mathsf{PSh}(\mathcal{C})$.

for each $y \in X$, is recovered as the representable presheaf
$$\mathrm{Hom}_{X_{\mathsf{disc}}}(-,x)\colon X_{\mathsf{disc}} \to \mathsf{Sets}$$

5. There is also a direct parallel between unions and colimits:
    - An element of $\mathcal{P}(X)$ is a union of elements of X, viewed as one-point subsets $\{x\} \in \mathcal{P}(A)$;
    - An object of $\mathsf{PSh}(\mathcal{C})$ is a colimit of objects of $\mathcal{C}$, viewed as representable presheaves $h_X \in \mathrm{Obj}(\mathsf{PSh}(\mathcal{C}))$.

Proposition 4.1.1.3. Let $f\colon A \to B$ be a function. We have an inclusion

$$\chi_B \circ (f \times f) \subset \chi_A, \qquad \begin{tikzcd} A \times A \arrow[r, "\chi_A(-_1,-_2)"] \arrow[d, "f\times f"'] \arrow[dr, phantom, "\subset"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ B \times B \arrow[r, "\chi_B(-_1,-_2)"'] & \{\mathsf{true}, \mathsf{false}\}. \end{tikzcd}$$

Proof. The inclusion $\chi_B(f(a), f(b)) \subset \chi_A(a, b)$ is equivalent to the statement "if $a = b$, then $f(a) = f(b)$", which is true. □

Proposition 4.1.1.4. Let X be a set and let $U \subset X$ be a subset of X. We have

$$\mathrm{Hom}_{\mathcal{P}(X)}(\chi_x, \chi_U) = \chi_U(x)$$

for each $x \in X$, giving an equality of functions

$$\mathrm{Hom}_{\mathcal{P}(X)}\Big(\chi_{(-)}, \chi_U\Big) = \chi_U.$$

Proof. Clear. □

Corollary 4.1.1.5. The characteristic embedding is fully faithful, i.e., we have

$$\mathrm{Hom}_{\mathcal{P}(X)}(\chi_x, \chi_y) = \chi_X(x, y)$$

for each $x, y \in X$.

Proof. This follows from Proposition 4.1.1.4. □

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of the corresponding object x of X_{disc}, defined on objects by

$$\mathrm{Hom}_{X_{\mathsf{disc}}}(y, x) \overset{\mathrm{def}}{=} \begin{cases} \mathrm{pt} & \text{if } x = y, \\ \emptyset & \text{if } x \neq y \end{cases}$$

for each $y \in \mathrm{Obj}(X_{\mathsf{disc}})$.

4.2 Powersets

Let X be a set.

Definition 4.2.1.1. The **powerset of** X is the set $\mathcal{P}(X)$ defined by

$$\mathcal{P}(X) \overset{\text{def}}{=} \{U \in P \mid U \subset X\},$$

where P is the set in the axiom of powerset, **??** of **??**.

Remark 4.2.1.2. The powerset of a set is a decategorification of the category of presheaves of a category: while[30]

- The powerset of a set X is equivalently (Item 6 of Proposition 4.2.1.3) the set

  $$\mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\})$$

  of functions from X to the set $\{\mathsf{t}, \mathsf{f}\}$ of classical truth values;
- The category of presheaves on a category $\mathcal{C}$ is the category

  $$\mathsf{Fun}(\mathcal{C}^{\mathsf{op}}, \mathsf{Sets})$$

  of functors from $\mathcal{C}^{\mathsf{op}}$ to the category Sets of sets.

Proposition 4.2.1.3. Let X be a set.

1. *Functoriality.* The assignment $X \mapsto \mathcal{P}(X)$ defines functors

   $$\begin{aligned} \mathcal{P}_* &\colon \mathsf{Sets} \to \mathsf{Sets}, \\ \mathcal{P}^{-1} &\colon \mathsf{Sets}^{\mathsf{op}} \to \mathsf{Sets}, \\ \mathcal{P}_! &\colon \mathsf{Sets} \to \mathsf{Sets} \end{aligned}$$

   where

---

[30] This parallel is based on the following comparison:

- A category is enriched over the category

  $$\mathsf{Sets} \overset{\text{def}}{=} \mathsf{Cats}_0$$

  of sets (i.e. "0-categories"), with presheaves taking values on it;
- A set is enriched over the set

  $$\{\mathsf{t}, \mathsf{f}\} \overset{\text{def}}{=} \mathsf{Cats}_{-1}$$

  of classical truth values (i.e. "(-1)-categories"), with characteristic functions taking values on it.

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$\begin{aligned}\mathcal{P}_*(A) &\overset{\text{def}}{=} \mathcal{P}(A),\\ \mathcal{P}^{-1}(A) &\overset{\text{def}}{=} \mathcal{P}(A),\\ \mathcal{P}_!(A) &\overset{\text{def}}{=} \mathcal{P}(A);\end{aligned}$$

- *Action on Morphisms.* For each morphism $f\colon A \to B$ of Sets, the images

$$\begin{aligned}\mathcal{P}_*(f)&\colon \mathcal{P}(A) \to \mathcal{P}(B),\\ \mathcal{P}^{-1}(f)&\colon \mathcal{P}(B) \to \mathcal{P}(A),\\ \mathcal{P}_!(f)&\colon \mathcal{P}(A) \to \mathcal{P}(B)\end{aligned}$$

  of f by $\mathcal{P}_*$, $\mathcal{P}^{-1}$, and $\mathcal{P}_!$ are defined by

$$\begin{aligned}\mathcal{P}_*(f) &\overset{\text{def}}{=} f_*,\\ \mathcal{P}^{-1}(f) &\overset{\text{def}}{=} f^{-1},\\ \mathcal{P}_!(f) &\overset{\text{def}}{=} f_!,\end{aligned}$$

  as in Definitions 4.3.1.1, 4.4.1.1 and 4.5.1.1.

2. *Adjointness I.* We have an adjunction

$$\left(\mathcal{P}^{-1} \dashv \mathcal{P}^{-1,\mathsf{op}}\right)\colon\quad \begin{tikzcd} \mathsf{Sets}^{\mathsf{op}} \arrow[r, bend left, "\mathcal{P}^{-1}"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}, \arrow[l, bend left, "\mathcal{P}^{-1,\mathsf{op}}"] \end{tikzcd}$$

   witnessed by a bijection

$$\underbrace{\mathsf{Sets}^{\mathsf{op}}(\mathcal{P}(X), Y)}_{\overset{\text{def}}{=}\mathsf{Sets}(Y,\mathcal{P}(X))} \cong \mathsf{Sets}(X, \mathcal{P}(Y)),$$

   natural in $X \in \mathrm{Obj}(\mathsf{Sets})$ and $Y \in \mathrm{Obj}(\mathsf{Sets}^{\mathsf{op}})$.

3. *Adjointness II.* We have an adjunction

$$(\mathrm{Gr} \dashv \mathcal{P}_*)\colon\quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "\mathrm{Gr}"] \arrow[r, phantom, "\perp"] & \mathrm{Rel}, \arrow[l, bend left, "\mathcal{P}_*"] \end{tikzcd}$$

   witnessed by a bijection of sets

$$\mathrm{Rel}(\mathrm{Gr}(A), B) \cong \mathsf{Sets}(A, \mathcal{P}(B))$$

   natural in $A \in \mathrm{Obj}(\mathsf{Sets})$ and $B \in \mathrm{Obj}(\mathrm{Rel})$, where Gr is the graph functor of Relations, **??** of **??**.

4. *Symmetric Strong Monoidality With Respect to Coproducts.* The powerset functor $\mathcal{P}_*$ of Item 1 has a symmetric strong monoidal structure

$$\left(\mathcal{P}_*, \mathcal{P}_*^{\coprod}, \mathcal{P}_{*|\mathbb{1}}^{\coprod}\right) : (\mathsf{Sets}, \coprod, \emptyset) \to (\mathsf{Sets}, \times, \mathrm{pt})$$

being equipped with isomorphisms

$$\mathcal{P}_{*|X,Y}^{\coprod} : \mathcal{P}(X) \times \mathcal{P}(Y) \xrightarrow{\cong} \mathcal{P}(X \coprod Y),$$
$$\mathcal{P}_{*|\mathbb{1}}^{\coprod} : \mathrm{pt} \xrightarrow{=} \mathcal{P}(\emptyset),$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

5. *Symmetric Lax Monoidality With Respect to Products.* The powerset functor $\mathcal{P}_*$ of Item 1 has a symmetric lax monoidal structure

$$\left(\mathcal{P}_*, \mathcal{P}_*^{\otimes}, \mathcal{P}_{*|\mathbb{1}}^{\otimes}\right) : (\mathsf{Sets}, \times, \mathrm{pt}) \to (\mathsf{Sets}, \times, \mathrm{pt})$$

being equipped with isomorphisms

$$\mathcal{P}_{*|X,Y}^{\otimes} : \mathcal{P}(X) \times \mathcal{P}(Y) \to \mathcal{P}(X \times Y),$$
$$\mathcal{P}_{*|\mathbb{1}}^{\otimes} : \mathrm{pt} \xrightarrow{=} \mathcal{P}(\emptyset),$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$, where $\mathcal{P}_{*|X,Y}^{\otimes}$ is given by

$$\mathcal{P}_{*|X,Y}^{\otimes}(U, V) \stackrel{\text{def}}{=} U \times V$$

for each $(U, V) \in \mathcal{P}(X) \times \mathcal{P}(Y)$.

6. *Powersets as Sets of Functions.* The assignment $U \mapsto \chi_U$ defines a bijection[31]

$$\chi_{(-)} : \mathcal{P}(X) \xrightarrow{\cong} \mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\}),$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets})$.

[31] This bijection is a decategorified form of the equivalence

$$\mathsf{PSh}(\mathcal{C}) \stackrel{\text{eq.}}{\cong} \mathsf{DFib}(\mathcal{C})$$

of Fibred Categories, **??** of **??**, with $\chi_{(-)}$ being a decategorified version of the category of elements construction of Fibred Categories, **??**.

See also **??** of **??**.

7. *Powersets as Sets of Relations.* We have bijections

$$\begin{aligned}\mathcal{P}(X) &\cong \mathrm{Rel}(\mathrm{pt}, X),\\ \mathcal{P}(X) &\cong \mathrm{Rel}(X, \mathrm{pt}),\end{aligned}$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets})$.

8. *As a Free Cocompletion: Universal Property.* The pair $\left(\mathcal{P}(X), \chi_{(-)}\right)$ consisting of

   - The powerset $\mathcal{P}(X)$ of X;
   - The characteristic embedding $\chi_{(-)}\colon X \hookrightarrow \mathcal{P}(X)$ of X into $\mathcal{P}(X)$;

   satisfies the following universal property:

   (⋆) Given another pair (Y, f) consisting of
   - A cocomplete poset $(Y, \preceq)$;
   - A function $f\colon X \to Y$;

   there exists a unique cocontinuous morphism of posets $(\mathcal{P}(X), \subset) \xrightarrow{\exists!} (Y, \preceq)$ making the diagram

$$\begin{tikzcd} & \mathcal{P}(X) \arrow[d, dashed, "\exists!"] \\ X \arrow[ur, "\chi_X"] \arrow[r, "f"'] & Y \end{tikzcd}$$

   commute.

9. *As a Free Cocompletion: Adjointness.* We have an adjunction[32]

$$\left(\chi_{(-)} \dashv 忘\right)\colon \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "\chi_{(-)}"] \arrow[r, phantom, "\perp"] & \mathsf{Pos}^{\mathsf{cocomp.}} \arrow[l, bend left, "忘"] \end{tikzcd},$$

   witnessed by a bijection

$$\mathsf{Pos}^{\mathsf{cocomp.}}((\mathcal{P}(X), \subset), (Y, \preceq)) \cong \mathsf{Sets}(X, Y),$$

   natural in $X \in \mathrm{Obj}(\mathsf{Sets})$ and $(Y, \preceq) \in \mathrm{Obj}(\mathsf{Pos})$, where

[32] In this sense, $\mathcal{P}(A)$ is the free cocompletion of A. (Note that, despite its name, however,

- We have a natural map

$$\chi_X^* \colon \mathsf{Pos}^{\mathsf{cocomp.}}((\mathcal{P}(X), \subset), (Y, \preceq)) \to \mathsf{Sets}(X, Y)$$

  defined by

$$\chi_X^*(f) \stackrel{\text{def}}{=} f \circ \chi_X,$$

  i.e. by sending a cocontinuous morphism of posets $f \colon \mathcal{P}(X) \to Y$ to the composition

$$X \overset{\chi_X}{\hookrightarrow} \mathcal{P}(X) \xrightarrow{f} Y;$$

- We have a natural map

$$\mathrm{Lan}_{\chi_X} \colon \mathsf{Sets}(X, Y) \to \mathsf{Pos}^{\mathsf{cocomp.}}((\mathcal{P}(X), \subset), (Y, \preceq))$$

  computed by

$$\begin{aligned} [\mathrm{Lan}_{\chi_X}(f)](U) &\cong \int^{x \in X} \chi_{\mathcal{P}(X)}(\chi_x, U) \odot f(x) \\ &\cong \int^{x \in X} \chi_U(x) \odot f(x) && \text{(by Proposition 4.1.1.4)} \\ &\cong \bigvee_{x \in X} (\chi_U(x) \odot f(x)) \end{aligned}$$

  for each $U \in \mathcal{P}(X)$, where:
  - $\bigvee$ is the join in $(Y, \preceq)$;
  - We have

$$\begin{aligned} \mathsf{true} \odot f(x) &\stackrel{\text{def}}{=} f(x), \\ \mathsf{false} \odot f(x) &\stackrel{\text{def}}{=} \varnothing_Y, \end{aligned}$$

    where $\varnothing_Y$ is the minimal element of $(Y, \preceq)$.

Proof. Item 1, Functoriality: This follows from Items 3 and 4 of Proposition 4.3.1.4, Items 3 and 4 of Proposition 4.4.1.4, and Items 3 and 4 of Proposition 4.5.1.6.
Item 2, Adjointness I: Omitted.
Item 3, Adjointness II: Omitted.
Item 4, Symmetric Strong Monoidality With Respect to Coproducts: Omitted.
Item 5, Symmetric Lax Monoidality With Respect to Products: Omitted.

Item 6, Powersets as Sets of Functions: Omitted.
Item 7, Powersets as Sets of Relations: Omitted.
Item 8, As a Free Cocompletion: Universal Property: This is a rephrasing of **??**.
Item 9, As a Free Cocompletion: Adjointness: Omitted. □

4.3 Direct Images

Let A and B be sets and let $f\colon A \to B$ be a function.

Definition 4.3.1.1. The **direct image function associated to** f is the function[33]

$$f_*\colon \mathcal{P}(A) \to \mathcal{P}(B)$$

defined by[34,35]

$$\begin{aligned} f_*(U) &\stackrel{\text{def}}{=} f(U) \\ &\stackrel{\text{def}}{=} \left\{ b \in B \;\middle|\; \begin{array}{l} \text{there exists some } a \in \\ U \text{ such that } b = f(a) \end{array} \right\} \\ &= \{ f(a) \in B \mid a \in U \} \end{aligned}$$

for each $U \in \mathcal{P}(A)$.

Remark 4.3.1.2. Identifying subsets of A with functions from A to $\{\mathsf{true}, \mathsf{false}\}$ via Item 6 of Proposition 4.2.1.3, we see that the direct image function associated to f is equivalently the function

$$f_*\colon \mathcal{P}(A) \to \mathcal{P}(B)$$

this is not an idempotent operation, as we have $\mathcal{P}(\mathcal{P}(A)) \neq \mathcal{P}(A)$.)

[33] *Further Notation:* Also written $\exists_f\colon \mathcal{P}(A) \to \mathcal{P}(B)$. This notation comes from the fact that the following statements are equivalent, where $b \in B$ and $U \in \mathcal{P}(A)$:

- We have $b \in \exists_f(U)$.
- There exists some $a \in U$ such that $f(a) = b$.

[34] *Further Terminology:* The set $f(U)$ is called the **direct image of** U **by** f.

[35] We also have

$$f_*(U) = B \setminus f_!(A \setminus U);$$

see Item 7 of Proposition 4.3.1.3.

defined by

$$
\begin{aligned}
f_*(\chi_U) &\overset{\text{def}}{=} \mathrm{Lan}_f(\chi_U) \\
&= \operatorname{colim}\left(\left(f \overset{\rightarrow}{\times} \underline{(-_1)}\right) \overset{\mathrm{pr}}{\twoheadrightarrow} A \xrightarrow{\chi_U} \{\mathsf{t}, \mathsf{f}\}\right) \\
&= \operatorname*{colim}_{\substack{a\in A \\ f(a)=-_1}} (\chi_U(a)) \\
&= \bigvee_{\substack{a\in A \\ f(a)=-_1}} (\chi_U(a)).
\end{aligned}
$$

So, in other words, we have

$$
\begin{aligned}
[f_*(\chi_U)](b) &= \bigvee_{\substack{a\in A \\ f(a)=b}} (\chi_U(a)) \\
&= \begin{cases} \mathsf{true} & \text{if there exists some } a \in A \text{ such} \\ & \text{that } f(a) = b \text{ and } a \in U, \\ \mathsf{false} & \text{otherwise} \end{cases} \\
&= \begin{cases} \mathsf{true} & \text{if there exists some } a \in U \\ & \text{such that } f(a) = b, \\ \mathsf{false} & \text{otherwise} \end{cases}
\end{aligned}
$$

for each $b \in B$.

Proposition 4.3.1.3. Let $f\colon A \to B$ be a function.

1. *Functoriality.* The assignment $U \mapsto f_*(U)$ defines a functor

$$f_*\colon (\mathcal{P}(A), \subset) \to (\mathcal{P}(B), \subset)$$

where

- *Action on Objects.* For each $U \in \mathcal{P}(A)$, we have

$$[f_*](U) \overset{\text{def}}{=} f_*(U);$$

- *Action on Morphisms.* For each $U, V \in \mathcal{P}(A)$:
  ($\star$) If $U \subset V$, then $f_*(U) \subset f_*(V)$.

2. *Triple Adjointness.* We have a triple adjunction

$$\left(f_* \dashv f^{-1} \dashv f_!\right)\colon \quad \begin{tikzcd} \mathcal{P}(A) \arrow[r, bend left=45, "f_*"] \arrow[r, phantom, bend left=20, "\perp"] \arrow[r, bend right=45, "f_!"'] \arrow[r, phantom, bend right=20, "\perp"] & \mathcal{P}(B), \arrow[l, "f^{-1}"'] \end{tikzcd}$$

witnessed by bijections of sets

$$\begin{aligned} \mathrm{Hom}_{\mathcal{P}(B)}(f_*(U), V) &\cong \mathrm{Hom}_{\mathcal{P}(A)}\left(U, f^{-1}(V)\right), \\ \mathrm{Hom}_{\mathcal{P}(A)}\left(f^{-1}(U), V\right) &\cong \mathrm{Hom}_{\mathcal{P}(A)}(U, f_!(V)), \end{aligned}$$

natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$ and (respectively) $V \in \mathcal{P}(A)$ and $U \in \mathcal{P}(B)$, i.e. where:

   (a) The following conditions are equivalent:
      i. We have $f_*(U) \subset V$.
      ii. We have $U \subset f^{-1}(V)$.
   (b) The following conditions are equivalent:
      i. We have $f^{-1}(U) \subset V$.
      ii. We have $U \subset f_!(V)$.

3. *Preservation of Colimits.* We have an equality of sets

$$f_*\left(\bigcup_{i\in I} U_i\right) = \bigcup_{i\in I} f_*(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have equalities

$$\begin{aligned} f_*(U) \cup f_*(V) &= f_*(U \cup V), \\ f_*(\emptyset) &= \emptyset, \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

4. *Oplax Preservation of Limits.* We have an inclusion of sets

$$f_*\left(\bigcap_{i\in I} U_i\right) \subset \bigcap_{i\in I} f_*(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have inclusions

$$\begin{aligned} f_*(U \cap V) &\subset f_*(U) \cap f_*(V), \\ f_*(A) &\subset B, \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

5. *Symmetric Strict Monoidality With Respect to Unions.* The direct image function of Item 1 has a symmetric strict monoidal structure

$$\left(f_*, f_*^{\otimes}, f_{*|\mathbb{1}}^{\otimes}\right) : (\mathcal{P}(A), \cup, \emptyset) \to (\mathcal{P}(B), \cup, \emptyset),$$

being equipped with equalities

$$f_{*|U,V}^{\otimes} : f_*(U) \cup f_*(V) \xrightarrow{=} f_*(U \cup V),$$
$$f_{*|\mathbb{1}}^{\otimes} : \emptyset \xrightarrow{=} \emptyset,$$

natural in $U, V \in \mathcal{P}(A)$.

6. *Symmetric Oplax Monoidality With Respect to Intersections.* The direct image function of Item 1 has a symmetric oplax monoidal structure

$$\left(f_*, f_*^{\otimes}, f_{*|\mathbb{1}}^{\otimes}\right) : (\mathcal{P}(A), \cap, A) \to (\mathcal{P}(B), \cap, B),$$

being equipped with inclusions

$$f_{*|U,V}^{\otimes} : f_*(U \cap V) \hookrightarrow f_*(U) \cap f_*(V),$$
$$f_{*|\mathbb{1}}^{\otimes} : f_*(A) \hookrightarrow B,$$

natural in $U, V \in \mathcal{P}(A)$.

7. *Relation to Direct Images With Compact Support.* We have

$$f_*(U) = B \setminus f_!(A \setminus U)$$

for each $U \in \mathcal{P}(A)$.

Proof. Item 1, Functoriality: Clear.
Item 2, Triple Adjointness: This follows from Kan Extensions, **??** of **??**.
Item 3, Preservation of Colimits: This follows from Item 2 and Categories, **??** of Proposition 10.1.1.3.
Item 4, Oplax Preservation of Limits: Omitted.
Item 5, Symmetric Strict Monoidality With Respect to Unions: This follows from Item 3.
Item 6, Symmetric Oplax Monoidality With Respect to Intersections: This follows from **??**.
Item 7, Relation to Direct Images With Compact Support: Applying **??** of **??** to $A \setminus U$, we have

$$\begin{aligned} f_!(A \setminus U) &= B \setminus f_*(A \setminus (A \setminus U)) \\ &= B \setminus f_*(U). \end{aligned}$$

Taking complements, we then obtain

$$\begin{aligned} f_*(U) &= B \setminus (B \setminus f_*(U)), \\ &= B \setminus f_!(A \setminus U), \end{aligned}$$

which finishes the proof. □

Proposition 4.3.1.4. Let $f\colon A \to B$ be a function.

1. *Functionality I.* The assignment $f \mapsto f_*$ defines a function

$$(-)_{*|A,B}\colon \mathsf{Sets}(A, B) \to \mathsf{Sets}(\mathcal{P}(A), \mathcal{P}(B)).$$

2. *Functionality II.* The assignment $f \mapsto f_*$ defines a function

$$(-)_{*|A,B}\colon \mathsf{Sets}(A, B) \to \mathsf{Pos}((\mathcal{P}(A), \subset), (\mathcal{P}(B), \subset)).$$

3. *Interaction With Identities.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$(\mathrm{id}_A)_* = \mathrm{id}_{\mathcal{P}(A)};$$

4. *Interaction With Composition.* For each pair of composable functions $f\colon A \to B$ and $g\colon B \to C$, we have

$$(g \circ f)_* = g_* \circ f_*, \qquad \begin{tikzcd} \mathcal{P}(A) \arrow[r, "f_*"] \arrow[dr, "(g\circ f)_*"'] & \mathcal{P}(B) \arrow[d, "g_*"] \\ & \mathcal{P}(C). \end{tikzcd}$$

Proof. *Item 1*, *Functionality I*: Clear.
Item 2, *Functionality II*: Clear.
Item 3, *Interaction With Identities*: This follows from Kan Extensions, **??** of **??**.
Item 4, *Interaction With Composition*: This follows from Kan Extensions, **??** of **??**. □

4.4 Inverse Images

Let A and B be sets and let $f\colon A \to B$ be a function.

Definition 4.4.1.1. The **inverse image function associated to** f is the function[36]

$$f^{-1}\colon \mathcal{P}(B) \to \mathcal{P}(A)$$

defined by[37]

$$f^{-1}(V) \stackrel{\text{def}}{=} \{a \in A \mid \text{we have } f(a) \in V\}$$

for each $V \in \mathcal{P}(B)$.

Remark 4.4.1.2. Identifying subsets of B with functions from B to $\{\mathsf{true}, \mathsf{false}\}$ via Item 6 of Proposition 4.2.1.3, we see that the inverse image function associated to f is equivalently the function

$$f^{*}\colon \mathcal{P}(B) \to \mathcal{P}(A)$$

defined by

$$f^{*}(\chi_V) \stackrel{\text{def}}{=} \chi_V \circ f$$

for each $\chi_V \in \mathcal{P}(B)$, where $\chi_V \circ f$ is the composition

$$A \xrightarrow{f} B \xrightarrow{\chi_V} \{\mathsf{true}, \mathsf{false}\}$$

in Sets.

Proposition 4.4.1.3. Let $f\colon A \to B$ be a function.

1. *Functoriality.* The assignment $V \mapsto f^{-1}(V)$ defines a functor

$$f^{-1}\colon (\mathcal{P}(B), \subset) \to (\mathcal{P}(A), \subset)$$

where

- *Action on Objects.* For each $V \in \mathcal{P}(B)$, we have

$$[f^{-1}](V) \stackrel{\text{def}}{=} f^{-1}(V);$$

- *Action on Morphisms.* For each $U, V \in \mathcal{P}(B)$:
  ($\star$) If $U \subset V$, then $f^{-1}(U) \subset f^{-1}(V)$.

[36] *Further Notation:* Also written $f^{*}\colon \mathcal{P}(B) \to \mathcal{P}(A)$.
[37] *Further Terminology:* The set $f^{-1}(V)$ is called the **inverse image of V by f**.

2. *Triple Adjointness.* We have a triple adjunction

$$\left(f_* \dashv f^{-1} \dashv f_!\right)\colon \quad \begin{tikzcd} \mathcal{P}(A) \arrow[r, bend left, "f_*"] \arrow[r, bend left=20, phantom, "\perp"] \arrow[r, bend right=20, phantom, "\perp"] \arrow[r, bend right, "f_!"'] & \mathcal{P}(B) \arrow[l, "f^{-1}"'] \end{tikzcd},$$

witnessed by bijections of sets

$$\begin{aligned} \mathrm{Hom}_{\mathcal{P}(B)}(f_*(U), V) &\cong \mathrm{Hom}_{\mathcal{P}(A)}\left(U, f^{-1}(V)\right), \\ \mathrm{Hom}_{\mathcal{P}(A)}\left(f^{-1}(U), V\right) &\cong \mathrm{Hom}_{\mathcal{P}(A)}(U, f_!(V)), \end{aligned}$$

natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$ and (respectively) $V \in \mathcal{P}(A)$ and $U \in \mathcal{P}(B)$, i.e. where:

(a) The following conditions are equivalent:
   i. We have $f_*(U) \subset V$;
   ii. We have $U \subset f^{-1}(V)$;

(b) The following conditions are equivalent:
   i. We have $f^{-1}(U) \subset V$.
   ii. We have $U \subset f_!(V)$.

3. *Preservation of Colimits.* We have an equality of sets

$$f^{-1}\left(\bigcup_{i \in I} U_i\right) = \bigcup_{i \in I} f^{-1}(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(B)^{\times I}$. In particular, we have equalities

$$\begin{aligned} f^{-1}(U) \cup f^{-1}(V) &= f^{-1}(U \cup V), \\ f^{-1}(\emptyset) &= \emptyset, \end{aligned}$$

natural in $U, V \in \mathcal{P}(B)$.

4. *Preservation of Limits.* We have an equality of sets

$$f^{-1}\left(\bigcap_{i \in I} U_i\right) = \bigcap_{i \in I} f^{-1}(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(B)^{\times I}$. In particular, we have equalities

$$f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V),$$
$$f^{-1}(B) = A,$$

natural in $U, V \in \mathcal{P}(B)$.

5. *Symmetric Strict Monoidality With Respect to Unions.* The inverse image function of Item 1 has a symmetric strict monoidal structure

$$\left(f^{-1}, f^{-1,\otimes}, f_{\mathbb{1}}^{-1,\otimes}\right) : (\mathcal{P}(B), \cup, \emptyset) \to (\mathcal{P}(A), \cup, \emptyset),$$

being equipped with equalities

$$f_{U,V}^{-1,\otimes} : f^{-1}(U) \cup f^{-1}(V) \xrightarrow{=} f^{-1}(U \cup V),$$
$$f_{\mathbb{1}}^{-1,\otimes} : \emptyset \xrightarrow{=} f^{-1}(\emptyset),$$

natural in $U, V \in \mathcal{P}(B)$.

6. *Symmetric Strict Monoidality With Respect to Intersections.* The inverse image function of Item 1 has a symmetric strict monoidal structure

$$\left(f^{-1}, f^{-1,\otimes}, f_{\mathbb{1}}^{-1,\otimes}\right) : (\mathcal{P}(B), \cap, B) \to (\mathcal{P}(A), \cap, A),$$

being equipped with equalities

$$f_{U,V}^{-1,\otimes} : f^{-1}(U) \cap f^{-1}(V) \xrightarrow{=} f^{-1}(U \cap V),$$
$$f_{\mathbb{1}}^{-1,\otimes} : A \xrightarrow{=} f^{-1}(B),$$

natural in $U, V \in \mathcal{P}(B)$.

Proof. Item 1, Functoriality: Clear.
Item 2, Triple Adjointness: This follows from Kan Extensions, **??** of **??**.
Item 3, Preservation of Colimits: This follows from Item 2 and Categories, **??** of Proposition 10.1.1.3.
Item 4, Preservation of Limits: This follows from Item 2 and Categories, **??** of Proposition 10.1.1.3.
Item 5, Symmetric Strict Monoidality With Respect to Unions: This follows from Item 3.
Item 6, Symmetric Strict Monoidality With Respect to Intersections: This follows from Item 4. □

Proposition 4.4.1.4. Let $f\colon A \to B$ be a function.

1. *Functionality I.* The assignment $f \mapsto f^{-1}$ defines a function
$$(-)^{-1}_{A,B}\colon \mathsf{Sets}(A,B) \to \mathsf{Sets}(\mathcal{P}(B),\mathcal{P}(A)).$$

2. *Functionality II.* The assignment $f \mapsto f^{-1}$ defines a function
$$(-)^{-1}_{A,B}\colon \mathsf{Sets}(A,B) \to \mathsf{Pos}((\mathcal{P}(B),\subset),(\mathcal{P}(A),\subset)).$$

3. *Interaction With Identities.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have
$$\mathrm{id}_A^{-1} = \mathrm{id}_{\mathcal{P}(A)};$$

4. *Interaction With Composition.* For each pair of composable functions $f\colon A \to B$ and $g\colon B \to C$, we have
$$(g\circ f)^{-1} = f^{-1} \circ g^{-1}, \qquad \begin{tikzcd} \mathcal{P}(C) \arrow[r, "g^{-1}"] \arrow[dr, "(g\circ f)^{-1}"'] & \mathcal{P}(B) \arrow[d, "f^{-1}"] \\ & \mathcal{P}(A). \end{tikzcd}$$

Proof. *Item 1*, *Functionality I*: Clear.
Item 2, *Functionality II*: Clear.
Item 3, *Interaction With Identities*: This follows from Categories, Item 5 of Proposition 1.5.1.2.
Item 4, *Interaction With Composition*: This follows from Categories, Item 2 of Proposition 1.5.1.2. ☐

4.5 Direct Images With Compact Support

Let A and B be sets and let $f\colon A \to B$ be a function.

Definition 4.5.1.1. The **direct image with compact support function associated to** f is the function[38]
$$f_!\colon \mathcal{P}(A) \to \mathcal{P}(B)$$

[38] *Further Notation:* Also written $\forall_f\colon \mathcal{P}(A) \to \mathcal{P}(B)$. This notation comes from the fact that the following statements are equivalent, where $b \in B$ and $U \in \mathcal{P}(A)$:

defined by[39,40]

$$\begin{aligned} f_!(U) &\overset{\text{def}}{=} \left\{ b \in B \;\middle|\; \begin{array}{l} \text{for each } a \in A \text{, if we have} \\ f(a) = b \text{, then } a \in U \end{array} \right\} \\ &= \left\{ b \in B \;\middle|\; \text{we have } f^{-1}(b) \subset U \right\} \end{aligned}$$

for each $U \in \mathcal{P}(A)$.

Remark 4.5.1.2. Identifying subsets of A with functions from A to $\{\mathsf{true}, \mathsf{false}\}$ via Item 6 of Proposition 4.2.1.3, we see that the direct image with compact support function associated to f is equivalently the function

$$f_! \colon \mathcal{P}(A) \to \mathcal{P}(B)$$

defined by

$$\begin{aligned} f_!(\chi_U) &\overset{\text{def}}{=} \mathrm{Ran}_f(\chi_U) \\ &= \lim\left(\left(\underline{(-_1)} \overset{\rightarrow}{\times} f\right) \overset{\mathrm{pr}}{\twoheadrightarrow} A \xrightarrow{\chi_U} \{\mathsf{true}, \mathsf{false}\}\right) \\ &= \lim_{\substack{a \in A \\ f(a) = -_1}} (\chi_U(a)) \\ &= \bigwedge_{\substack{a \in A \\ f(a) = -_1}} (\chi_U(a)). \end{aligned}$$

So, in other words, we have

$$\begin{aligned} [f_!(\chi_U)](b) &= \bigwedge_{\substack{a \in A \\ f(a) = b}} (\chi_U(a)) \\ &= \begin{cases} \mathsf{true} & \text{if, for each } a \in A \text{ such that} \\ & f(a) = b \text{, we have } a \in U, \\ \mathsf{false} & \text{otherwise} \end{cases} \\ &= \begin{cases} \mathsf{true} & \text{if } f^{-1}(b) \subset U \\ \mathsf{false} & \text{otherwise} \end{cases} \end{aligned}$$

- We have $b \in \forall_f(U)$.
- For each $a \in A$, if $b = f(a)$, then $a \in U$.

[39] *Further Terminology:* The set $f_!(U)$ is called the **direct image with compact support of U by f**.

[40] We also have

$$f_!(U) = B \setminus f_*(A \setminus U);$$

for each $b \in B$.

Definition 4.5.1.3. Let U be a subset of A.[41,42]

1. The **image part of the direct image with compact support** $f_!(U)$ **of** U is the set $f_{!,\mathrm{im}}(U)$ defined by

$$\begin{aligned} f_{!,\mathrm{im}}(U) &\overset{\mathrm{def}}{=} f_!(U) \cap \mathrm{Im}(f) \\ &= \left\{ b \in B \;\middle|\; \begin{array}{l} \text{we have } f^{-1}(b) \subset \\ U \text{ and } f^{-1}(b) \neq \emptyset \end{array} \right\}. \end{aligned}$$

2. The **complement part of the direct image with compact support** $f_!(U)$ **of** U is the set $f_{!,\mathrm{cp}}(U)$ defined by

$$\begin{aligned} f_{!,\mathrm{cp}}(U) &\overset{\mathrm{def}}{=} f_!(U) \cap (B \setminus \mathrm{Im}(f)) \\ &= B \setminus \mathrm{Im}(f) \\ &= \left\{ b \in B \;\middle|\; \begin{array}{l} \text{we have } f^{-1}(b) \subset \\ U \text{ and } f^{-1}(b) = \emptyset \end{array} \right\} \\ &= \left\{ b \in B \;\middle|\; f^{-1}(b) = \emptyset \right\}. \end{aligned}$$

Example 4.5.1.4. Here are some examples of direct images with compact support.

1. *The Multiplication by Two Map on the Natural Numbers.* Consider the

see Item 7 of Proposition 4.5.1.5.

[41] Note that we have

$$f_!(U) = f_{!,\mathrm{im}}(U) \cup f_{!,\mathrm{cp}}(U),$$

as

$$\begin{aligned} f_!(U) &= f_!(U) \cap B \\ &= f_!(U) \cap (\mathrm{Im}(f) \cup (B \setminus \mathrm{Im}(f))) \\ &= (f_!(U) \cap \mathrm{Im}(f)) \cup (f_!(U) \cap (B \setminus \mathrm{Im}(f))) \\ &\overset{\mathrm{def}}{=} f_{!,\mathrm{im}}(U) \cup f_{!,\mathrm{cp}}(U). \end{aligned}$$

[42] In terms of the meet computation of $f_!(U)$ of Remark 4.5.1.2, namely

$$f_!(\chi_U) = \bigwedge_{\substack{a \in A \\ f(a) = -_1}} (\chi_U(a)),$$

we see that $f_{!,\mathrm{im}}$ corresponds to meets indexed over nonempty sets, while $f_{!,\mathrm{cp}}$ corresponds to meets indexed over the empty set.

function $f\colon \mathbb{N} \to \mathbb{N}$ given by

$$f(n) \overset{\text{def}}{=} 2n$$

for each $n \in \mathbb{N}$. Since f is injective, we have

$$\begin{aligned}
f_{!,\mathrm{im}}(U) &= f_*(U)\\
f_{!,\mathrm{cp}}(U) &= \{\text{odd natural numbers}\}
\end{aligned}$$

for any $U \subset \mathbb{N}$.

2. *Parabolas.* Consider the function $f\colon \mathbb{R} \to \mathbb{R}$ given by

$$f(x) \overset{\text{def}}{=} x^2$$

for each $x \in \mathbb{R}$. We have

$$f_{!,\mathrm{cp}}(U) = \mathbb{R}_{<0}$$

for any $U \subset \mathbb{R}$. Moreover, since $f^{-1}(x) = \{-\sqrt{x}, \sqrt{x}\}$, we have e.g.:

$$\begin{aligned}
f_{!,\mathrm{im}}([0,1]) &= \{0\},\\
f_{!,\mathrm{im}}([-1,1]) &= [0,1],\\
f_{!,\mathrm{im}}([1,2]) &= \emptyset,\\
f_{!,\mathrm{im}}([-2,-1] \cup [1,2]) &= [1,4].
\end{aligned}$$

3. *Circles.* Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x,y) \overset{\text{def}}{=} x^2 + y^2$$

for each $(x,y) \in \mathbb{R}^2$. We have

$$f_{!,\mathrm{cp}}(U) = \mathbb{R}_{<0}$$

for any $U \subset \mathbb{R}^2$, and since

$$f^{-1}(r) = \begin{cases} \text{a circle of radius } r \text{ about the origin} & \text{if } r > 0,\\ \{(0,0)\} & \text{if } r = 0,\\ \emptyset & \text{if } r < 0, \end{cases}$$

we have e.g.:

$$\begin{aligned}
f_{!,\mathrm{im}}([-1,1] \times [-1,1]) &= [0,1],\\
f_{!,\mathrm{im}}(([-1,1] \times [-1,1]) \setminus [-1,1] \times \{0\}) &= \emptyset.
\end{aligned}$$

Proposition 4.5.1.5. Let $f\colon A \to B$ be a function.

1. *Functoriality.* The assignment $U \mapsto f_!(U)$ defines a functor

$$f_!\colon (\mathcal{P}(A), \subset) \to (\mathcal{P}(B), \subset)$$

where

- *Action on Objects.* For each $U \in \mathcal{P}(A)$, we have

$$[f_!](U) \overset{\text{def}}{=} f_!(U);$$

- *Action on Morphisms.* For each $U, V \in \mathcal{P}(A)$:
  ($\star$) If $U \subset V$, then $f_!(U) \subset f_!(V)$.

2. *Triple Adjointness.* We have a triple adjunction

$$\Big(f_* \dashv f^{-1} \dashv f_!\Big)\colon \quad \begin{tikzcd} \mathcal{P}(A) \arrow[r, bend left=45, "f_*"] \arrow[r, bend left=22, phantom, "\perp"] \arrow[r, bend right=22, phantom, "\perp"'] \arrow[r, bend right=45, "f_!"'] & \mathcal{P}(B) \arrow[l, "f^{-1}" description] \end{tikzcd},$$

witnessed by bijections of sets

$$\begin{aligned} \mathrm{Hom}_{\mathcal{P}(B)}(f_*(U), V) &\cong \mathrm{Hom}_{\mathcal{P}(A)}\Big(U, f^{-1}(V)\Big),\\ \mathrm{Hom}_{\mathcal{P}(A)}\Big(f^{-1}(U), V\Big) &\cong \mathrm{Hom}_{\mathcal{P}(A)}(U, f_!(V)), \end{aligned}$$

natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$ and (respectively) $V \in \mathcal{P}(A)$ and $U \in \mathcal{P}(B)$, i.e. where:

(a) The following conditions are equivalent:
   i. We have $f_*(U) \subset V$;
   ii. We have $U \subset f^{-1}(V)$;

(b) The following conditions are equivalent:
   i. We have $f^{-1}(U) \subset V$.
   ii. We have $U \subset f_!(V)$.

3. *Lax Preservation of Colimits.* We have an inclusion of sets

$$\bigcup_{i \in I} f_!(U_i) \subset f_!\left(\bigcup_{i \in I} U_i\right),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have inclusions

$$f_!(U) \cup f_!(V) \hookrightarrow f_!(U \cup V),$$
$$\emptyset \hookrightarrow f_!(\emptyset),$$

natural in $U, V \in \mathcal{P}(A)$.

4. *Preservation of Limits.* We have an equality of sets

$$f_!\left(\bigcap_{i\in I} U_i\right) = \bigcap_{i\in I} f_!(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have equalities

$$f^{-1}(U \cap V) = f_!(U) \cap f^{-1}(V),$$
$$f_!(A) = B,$$

natural in $U, V \in \mathcal{P}(A)$.

5. *Symmetric Lax Monoidality With Respect to Unions.* The direct image with compact support function of Item 1 has a symmetric lax monoidal structure

$$\left(f_!, f_!^{\otimes}, f_{!|\mathbb{1}}^{\otimes}\right) \colon (\mathcal{P}(A), \cup, \emptyset) \to (\mathcal{P}(B), \cup, \emptyset),$$

being equipped with inclusions

$$f_{!|U,V}^{\otimes} \colon f_!(U) \cup f_!(V) \hookrightarrow f_!(U \cup V),$$
$$f_{!|\mathbb{1}}^{\otimes} \colon \emptyset \hookrightarrow f_!(\emptyset),$$

natural in $U, V \in \mathcal{P}(A)$.

6. *Symmetric Strict Monoidality With Respect to Intersections.* The direct image function of Item 1 has a symmetric strict monoidal structure

$$\left(f_!, f_!^{\otimes}, f_{!|\mathbb{1}}^{\otimes}\right) \colon (\mathcal{P}(A), \cap, A) \to (\mathcal{P}(B), \cap, B),$$

being equipped with equalities

$$f_{!|U,V}^{\otimes} \colon f_!(U \cap V) \xrightarrow{=} f_!(U) \cap f_!(V),$$
$$f_{!|\mathbb{1}}^{\otimes} \colon f_!(A) \xrightarrow{=} B,$$

natural in $U, V \in \mathcal{P}(A)$.

7. *Relation to Direct Images.* We have

$$f_!(U) = B \setminus f_*(A \setminus U)$$

for each $U \in \mathcal{P}(A)$.

8. *Interaction With Injections.* If f is injective, then we have

$$\begin{aligned} f_{!,\mathrm{im}}(U) &= f_*(U), \\ f_{!,\mathrm{cp}}(U) &= B \setminus \mathrm{Im}(f), \\ f_!(U) &= f_{!,\mathrm{im}}(U) \cup f_{!,\mathrm{cp}}(U) \\ &= f_*(U) \cup (B \setminus \mathrm{Im}(f)) \end{aligned}$$

for each $U \in \mathcal{P}(A)$.

9. *Interaction With Surjections.* If f is surjective, then we have

$$\begin{aligned} f_{!,\mathrm{im}}(U) &\subset f_*(U), \\ f_{!,\mathrm{cp}}(U) &= \emptyset, \\ f_!(U) &\subset f_*(U) \end{aligned}$$

for each $U \in \mathcal{P}(A)$.

Proof. Item 1, Functoriality: Clear.
Item 2, Triple Adjointness: This follows from Kan Extensions, **??** of **??**.
Item 3, Lax Preservation of Colimits: Omitted.
Item 4, Preservation of Limits: Omitted. This follows from Item 2 and Categories, **??** of Proposition 10.1.1.3.
Item 5, Symmetric Lax Monoidality With Respect to Unions: This follows from **??**.
Item 6, Symmetric Strict Monoidality With Respect to Intersections: This follows from Item 4.
Item 7, Relation to Direct Images: We claim that $f_!(U) = B \setminus f_*(A \setminus U)$.

- *The First Implication.* We claim that

  $$f_!(U) \subset B \setminus f_*(A \setminus U).$$

  Let $b \in f_!(U)$. We need to show that $b \notin f_*(A \setminus U)$, i.e. that there is no $a \in A \setminus U$ such that $f(a) = b$.

  This is indeed the case, as otherwise we would have $a \in f^{-1}(b)$ and $a \notin U$, contradicting $f^{-1}(b) \subset U$ (which holds since $b \in f_!(U)$).

  Thus $b \in B \setminus f_*(A \setminus U)$.

- *The Second Implication.* We claim that

  $$B \setminus f_*(A \setminus U) \subset f_!(U).$$

  Let $b \in B \setminus f_*(A \setminus U)$. We need to show that $b \in f_!(U)$, i.e. that $f^{-1}(b) \subset U$.

  Since $b \notin f_*(A \setminus U)$, there exists no $a \in A \setminus U$ such that $b = f(a)$, and hence $f^{-1}(b) \subset U$.

  Thus $b \in f_!(U)$.

This finishes the proof of Item 7.
Item 8, Interaction With Injections: Clear.
Item 9, Interaction With Surjections: Clear. □

Proposition 4.5.1.6. Let $f\colon A \to B$ be a function.

1. *Functionality I.* The assignment $f \mapsto f_!$ defines a function

   $$(-)_{!|A,B}\colon \mathsf{Sets}(A, B) \to \mathsf{Sets}(\mathcal{P}(A), \mathcal{P}(B)).$$

2. *Functionality II.* The assignment $f \mapsto f_!$ defines a function

   $$(-)_{!|A,B}\colon \mathsf{Sets}(A, B) \to \mathsf{Pos}((\mathcal{P}(A), \subset), (\mathcal{P}(B), \subset)).$$

3. *Interaction With Identities.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

   $$(\mathrm{id}_A)_! = \mathrm{id}_{\mathcal{P}(A)};$$

4. *Interaction With Composition.* For each pair of composable functions $f\colon A \to B$ and $g\colon B \to C$, we have

   $$(g \circ f)_! = g_! \circ f_!, \qquad \begin{tikzcd} \mathcal{P}(A) \arrow[r, "f_!"] \arrow[dr, "(g\circ f)_!"'] & \mathcal{P}(B) \arrow[d, "g_!"] \\ & \mathcal{P}(C). \end{tikzcd}$$

Proof. *Item 1, Functionality I*: Clear.
Item 2, Functionality II: Clear.
Item 3, Interaction With Identities: This follows from Kan Extensions, **??** of **??**.
Item 4, Interaction With Composition: This follows from Kan Extensions, **??** of **??**. □

Appendices

A Other Chapters

34. Near-Semirings
35. Near-Rings

Real Analysis

36. Real Analysis in One Variable
37. Real Analysis in Several Variables

Measure Theory

38. Measurable Spaces
39. Measures and Integration

Probability Theory

39. Probability Theory

Stochastic Analysis

40. Stochastic Processes, Martingales, and Brownian Motion
41. Itô Calculus
42. Stochastic Differential Equations

Differential Geometry

43. Topological and Smooth Manifolds

Schemes

44. Schemes