

# Un/Straightening for Indexed and Fibred Sets

December 24, 2023

This chapter contains a discussion of the un/straightening equivalence in the context of sets, as well as a general discussion of indexed and fibred sets. In particular, it contains:

1. A discussion of indexed sets (i.e. functors  $K_{\text{disc}} \rightarrow \mathbf{Sets}$  with  $K$  a set), constructions with them like dependent sums and dependent products, and their properties (????);
2. A discussion of fibred sets (i.e. maps of sets  $X \rightarrow K$ ), constructions with them like dependent sums and dependent products, and their properties (????);
3. A discussion of the un/straightening equivalence for indexed and fibred sets (??).

## Contents

## 1 Un/Straightening for Indexed and Fibred Sets

### 1.1 Straightening for Fibred Sets

Let  $K$  be a set and let  $(X, \phi)$  be a  $K$ -fibred set.

**Definition 1.1.1.1.** The **straightening** of  $(X, \phi)$  is the  $K$ -indexed set

$$\text{St}_K(X, \phi): K_{\text{disc}} \rightarrow \mathbf{Sets}$$

defined by

$$\text{St}_K(X, \phi)_x \stackrel{\text{def}}{=} \phi^{-1}(x)$$

for each  $x \in K$ .

**Proposition 1.1.1.2.** Let  $K$  be a set. 00T9

1. *Functoriality.* The assignment  $(X, \phi) \mapsto \text{St}_K(X, \phi)$  defines a functor

$$\text{St}_K: \text{FibSets}(K) \rightarrow \text{ISets}(K)$$

- *Action on Objects.* For each  $(X, \phi) \in \text{Obj}(\text{FibSets}(K))$ , we have

$$[\text{St}_K](X, \phi) \stackrel{\text{def}}{=} \text{St}_K(X, \phi);$$

- *Action on Morphisms.* For each  $(X, \phi), (Y, \psi) \in \text{Obj}(\text{FibSets}(K))$ , the action on Hom-sets

$$\text{St}_K|_{X,Y}: \text{Hom}_{\text{FibSets}(K)}(X, Y) \rightarrow \text{Hom}_{\text{ISets}(K)}(\text{St}_K(X), \text{St}_K(Y))$$

of  $\text{St}_K$  at  $(X, Y)$  is given by sending a morphism

$$f: (X, \phi) \rightarrow (Y, \psi)$$

of  $K$ -fibred sets to the morphism

$$\text{St}_K(f): \text{St}_K(X, \phi) \rightarrow \text{St}_K(Y, \psi)$$

of  $K$ -indexed sets defined by

$$\text{St}_K(f) \stackrel{\text{def}}{=} \{f_x^*\}_{x \in K},$$

where  $f_x^*$  is the transport map associated to  $f$  at  $x \in K$  of ??.

2. *Interaction With Change of Base/Indexing.* Let  $f: K \rightarrow K'$  be a map of sets. The diagram

$$\begin{array}{ccc} \text{FibSets}(K') & \xrightarrow{f^*} & \text{FibSets}(K) \\ \text{St}_{K'} \downarrow & & \downarrow \text{St}_K \\ \text{ISets}(K') & \xrightarrow{f^*} & \text{ISets}(K) \end{array}$$

commutes.

3. *Interaction With Dependent Sums.* Let  $f: K \rightarrow K'$  be a map of sets. The diagram

$$\begin{array}{ccc} \text{FibSets}(K) & \xrightarrow{\Sigma_f} & \text{FibSets}(K') \\ \text{St}_K \downarrow & & \downarrow \text{St}_{K'} \\ \text{ISets}(K) & \xrightarrow{\Sigma_f} & \text{ISets}(K') \end{array}$$

commutes.

4. *Interaction With Dependent Products.* Let  $f: K \rightarrow K'$  be a map of sets. The diagram

$$\begin{array}{ccc} \mathbf{Sets}/_K & \xrightarrow{\Pi_f} & \mathbf{FibSets}(K') \\ \text{St}_K \downarrow & & \downarrow \text{St}_{K'} \\ \mathbf{ISets}(K) & \xrightarrow{\Pi_f} & \mathbf{ISets}(K') \end{array}$$

commutes.

*Proof. ??, Functoriality:* Omitted.

*??, Interaction With Change of Base/Indexing:* Indeed, we have

$$\begin{aligned} \text{St}_K(f^*(X, \phi))_x &\stackrel{\text{def}}{=} \text{St}_K(K \times_{K'} X)_x \\ &\stackrel{\text{def}}{=} (\text{pr}_1^{K \times_{K'} X})^{-1}(x) \\ &= \{(k, y) \in K \times_{K'} X \mid \text{pr}_1^{K \times_{K'} X}(k, y) = x\} \\ &= \{(k, y) \in K \times_{K'} X \mid k = x\} \\ &= \{(k, y) \in K \times X \mid k = x \text{ and } f(k) = \phi(y)\} \\ &\cong \{y \in X \mid \phi(y) = f(x)\} \\ &= \phi^{-1}(f(x)) \\ &\stackrel{\text{def}}{=} f^*(\phi^{-1}(x)) \\ &\stackrel{\text{def}}{=} f^*(\text{St}_{K'}(X, \phi)_x) \end{aligned}$$

for each  $(X, \phi) \in \text{Obj}(\mathbf{FibSets}(K'))$  and each  $x \in K$ , and similarly for morphisms.

*??, Interaction With Dependent Sums:* Indeed, we have

$$\begin{aligned} \text{St}_{K'}(\Sigma_f(X, \phi))_x &\stackrel{\text{def}}{=} \Sigma_f(\phi)^{-1}(x) \\ &\cong \coprod_{\substack{y \in X \\ f(y)=x}} \phi^{-1}(y) \\ &\cong \Sigma_f(\phi^{-1}(x)) \\ &\stackrel{\text{def}}{=} \Sigma_f(\text{St}_K(X, \phi)_x) \end{aligned}$$

for each  $(X, \phi) \in \text{Obj}(\mathbf{FibSets}(K))$  and each  $x \in K'$ , where we have used ?? of ?? for the first bijection, and similarly for morphisms.

??, *Interaction With Dependent Products*: Indeed, we have

$$\begin{aligned} \mathrm{St}_{K'}(\Pi_f(X, \phi))_x &\stackrel{\mathrm{def}}{=} \Pi_f(\phi)^{-1}(x) \\ &\cong \prod_{\substack{y \in X \\ f(y)=x}} \phi^{-1}(y) \\ &\cong \Pi_f(\phi^{-1}(x)) \\ &\stackrel{\mathrm{def}}{=} \Pi_f(\mathrm{St}_K(X, \phi)_x) \end{aligned}$$

for each  $(X, \phi) \in \mathrm{Obj}(\mathrm{FibSets}(K))$  and each  $x \in K'$ , where we have used ?? of ?? for the first bijection, and similarly for morphisms.  $\square$

## 1.2 Unstraightening for Indexed Sets

Let  $K$  be a set and let  $X$  be a  $K$ -indexed set.

**Definition 1.2.1.1.** The **unstraightening** of  $X$  is the  $K$ -fibred set

$$\phi_{\mathrm{Un}_K} : \mathrm{Un}_K(X) \rightarrow K$$

consisting of

- *The Underlying Set.* The set  $\mathrm{Un}_K(X)$  defined by

$$\mathrm{Un}_K(X) \stackrel{\mathrm{def}}{=} \coprod_{x \in K} X_x;$$

- *The Fibration.* The map of sets

$$\phi_{\mathrm{Un}_K} : \mathrm{Un}_K(X) \rightarrow K$$

defined by sending an element of  $\coprod_{x \in K} X_x$  to its index in  $K$ .

**Proposition 1.2.1.2.** Let  $K$  be a set. **00TG**

1. *Functoriality.* The assignment  $X \mapsto \mathrm{Un}_K(X)$  defines a functor

$$\mathrm{Un}_K : \mathrm{ISets}(K) \rightarrow \mathrm{FibSets}(K)$$

- *Action on Objects.* For each  $X \in \mathrm{Obj}(\mathrm{ISets}(K))$ , we have

$$[\mathrm{Un}_K](X) \stackrel{\mathrm{def}}{=} \mathrm{Un}_K(X);$$

- *Action on Morphisms.* For each  $X, Y \in \text{Obj}(\text{ISets}(K))$ , the action on Hom-sets

$$\text{Un}_K|_{X,Y}: \text{Hom}_{\text{ISets}(K)}(X, Y) \rightarrow \text{Hom}_{\text{FibSets}(K)}(\text{Un}_K(X), \text{Un}_K(Y))$$

of  $\text{Un}_K$  at  $(X, Y)$  is defined by

$$\text{Un}_K|_{X,Y}(f) \stackrel{\text{def}}{=} \prod_{x \in K} f_x^*.$$

2. *Interaction With Fibres.* We have a bijection ~~of sets~~

$$\phi_{\text{Un}_K}^{-1}(x) \cong X_x$$

for each  $x \in K$ .

3. *As a Pullback.* We have a bijection ~~of sets~~

$$\begin{array}{ccc} & \text{Un}_K(X) \rightarrow \text{Sets}_* & \\ & \downarrow \lrcorner \quad \downarrow \text{忘} & \\ \text{Un}_K(X) \cong K_{\text{disc}} \times_{\text{Sets}} \text{Sets}_*, & K_{\text{disc}} \xrightarrow{X} \text{Sets}. & \end{array}$$

4. *As a Colimit.* We have a bijection ~~of sets~~

$$\text{Un}_K(X) \cong \text{colim}(X).$$

5. *Interaction With Change of Indexing/Base.* Let  $f: K \rightarrow K'$  be a ~~map~~ of sets. The diagram

$$\begin{array}{ccc} \text{ISets}(K') & \xrightarrow{f^*} & \text{ISets}(K) \\ \text{Un}_{K'} \downarrow & & \downarrow \text{Un}_K \\ \text{FibSets}(K') & \xrightarrow{f^*} & \text{FibSets}(K) \end{array}$$

commutes.

6. *Interaction With Dependent Sums.* Let  $f: K \rightarrow K'$  be a ~~map~~ of sets. The diagram

$$\begin{array}{ccc} \text{ISets}(K) & \xrightarrow{\Sigma_f} & \text{ISets}(K') \\ \text{Un}_K \downarrow & & \downarrow \text{Un}_{K'} \\ \text{FibSets}(K) & \xrightarrow{\Sigma_f} & \text{FibSets}(K') \end{array}$$

commutes.

7. *Interaction With Dependent Products.* Let  $f: K \rightarrow K'$  be a map of sets. The diagram

$$\begin{array}{ccc} \mathbf{ISets}(K) & \xrightarrow{\Pi_f} & \mathbf{ISets}(K') \\ \text{Un}_K \downarrow & & \downarrow \text{Un}_{K'} \\ \mathbf{FibSets}(K) & \xrightarrow{\Pi_f} & \mathbf{FibSets}(K') \end{array}$$

commutes.

*Proof. ??, Functoriality:* Omitted.

??, *Interaction With Fibres:* Omitted.

??, *As a Pullback:* Omitted.

??, *As a Colimit:* Clear.

??, *Interaction With Change of Indexing/Base:* Indeed, we have

$$\begin{aligned} \text{Un}_K(f^*(X)) &\stackrel{\text{def}}{=} \text{Un}_K(X \circ f) \\ &\stackrel{\text{def}}{=} \coprod_{x \in K} X_{f(x)} \\ &\cong \left\{ (x, (y, a)) \in K \times \coprod_{y \in K'} X_y \mid f(x) = y \right\} \\ &\cong K \times_{K'} \coprod_{y \in K'} X_y \\ &\stackrel{\text{def}}{=} K \times_{K'} \text{Un}_{K'}(X) \\ &\stackrel{\text{def}}{=} f^*(\text{Un}_{K'}(X)) \end{aligned}$$

for each  $X \in \text{Obj}(\mathbf{ISets}(K'))$ . Similarly, it can be shown that we also have  $\text{Un}_K(f^*(\phi)) = f^*(\text{Un}_{K'}(\phi))$  and that  $\text{Un}_K \circ f^* = f^* \circ \text{Un}_{K'}$  also holds on morphisms.

??, *Interaction With Dependent Sums*: Indeed, we have

$$\begin{aligned}
 \text{Un}_{K'}(\Sigma_f(X)) &\stackrel{\text{def}}{=} \coprod_{x \in K'} \Sigma_f(X)_x \\
 &\cong \coprod_{x \in K'} \coprod_{y \in f^{-1}(x)} X_y \\
 &\cong \coprod_{y \in K} X_y \\
 &\cong \text{Un}_K(X) \\
 &\stackrel{\text{def}}{=} \Sigma_f(\text{Un}_K(X))
 \end{aligned}$$

for each  $X \in \text{Obj}(\mathbf{lSets}(K))$ , where we have used ?? of ?? for the first bijection. Similarly, it can be shown that we also have  $\text{Un}_{K'}(\Sigma_f(\phi)) = \Sigma_f(\phi_{\text{Un}_K})$  and that  $\text{Un}_{K'} \circ \Sigma_f = \Sigma_f \circ \text{Un}_K$  also holds on morphisms.

??, *Interaction With Dependent Products*: Indeed, we have

$$\begin{aligned}
 \text{Un}_{K'}(\Pi_f(X)) &\stackrel{\text{def}}{=} \prod_{x \in K'} \Pi_f(X)_x \\
 &\cong \prod_{x \in K'} \prod_{y \in f^{-1}(x)} X_y \\
 &\cong \left\{ (x, h) \in \prod_{x \in K'} \text{Sets}(f^{-1}(x), \phi_{\text{Un}_K}^{-1}(f^{-1}(x))) \mid \phi \circ h = \text{id}_{f^{-1}(x)} \right\} \\
 &\stackrel{\text{def}}{=} \Pi_f\left(\prod_{y \in K} X_y\right) \\
 &\stackrel{\text{def}}{=} \Pi_f(\text{Un}_K(X))
 \end{aligned}$$

for each  $X \in \text{Obj}(\mathbf{lSets}(K))$ , where we have used ?? of ?? for the first bijection. Similarly, it can be shown that we also have  $\text{Un}_{K'}(\Pi_f(\phi)) = \Pi_f(\phi_{\text{Un}_K})$  and that  $\text{Un}_{K'} \circ \Pi_f = \Pi_f \circ \text{Un}_K$  also holds on morphisms.  $\square$

### 1.3 The Un/Straightening Equivalence

**Theorem 1.3.1.1.** We have an isomorphism of categories

$$(\text{St}_K \dashv \text{Un}_K): \text{FibSets}(K) \begin{array}{c} \xrightarrow{\text{St}_K} \\ \perp \\ \xleftarrow{\text{Un}_K} \end{array} \mathbf{lSets}(K).$$

*Proof.* Omitted.  $\square$

## 2 Miscellany<sup>00TS</sup>

### 2.1 Other Kinds<sup>0077</sup> of Un/Straightening

**Remark 2.1.1.1.** There are also other<sup>0077</sup> kinds of un/straightening for sets, where **Sets** is replaced by **Rel** or **Span**:

- *Un/Straightening With Rel, I.* We have an isomorphism of sets

$$\text{Rel}(A, B) \cong \text{Sets}(B \times A, \{\text{true}, \text{false}\}).$$

by the definition of a relation from  $A$  to  $B$ , [Relations](#), [Definition 1.1.1.1](#).

- *Un/Straightening With Rel, II.* We have an equivalence of categories

$$\text{LaxFun}(K_{\text{disc}}, \mathbf{Rel}) \stackrel{\text{eq.}}{\cong} \text{Cats}_{/K_{\text{disc}}}^{\text{fth}},$$

where  $\text{Cats}_{/K_{\text{disc}}}^{\text{fth}}$  is the full subcategory of  $\text{Cats}_{/K_{\text{disc}}}$  spanned by the faithful functors; see [\[niefield:change-of-base-for-relational-variable-sets\]](#).

- *Un/Straightening With Span, I.* For each  $A, B \in \text{Obj}(\mathbf{Sets})$ , we have a morphism of sets

$$\text{Span}(A, B) \rightarrow \text{Sets}(A \times B, \mathbb{N} \cup \{\infty\})$$

which assemble into an equivalence of categories between  $\text{Span}(\mathbf{Sets})$  and the category  $\mathbf{MRel}$  of “multirelations”; see [Spans](#), [Remark 7.5.1.1](#).

- *Un/Straightening With Span, II.* We have an equivalence of categories

$$\text{LaxFun}(K_{\text{disc}}, \mathbf{Span}) \stackrel{\text{eq.}}{\cong} \text{Cats}_{/K_{\text{disc}}};$$

see [\[nlab:displayed-category\]](#).

## Appendices



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