Pointed Sets

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This chapter contains some foundational material on pointed sets.

Contents

1 Pointed Sets

1.1 Foundations

Definition 1.1.1.1. A **pointed set**[1] is equivalently

- An $\mathbb{E}_0$-monoid in $(\mathrm{N}_\bullet(\mathsf{Sets}), \mathrm{pt})$;
- A pointed object in $(\mathsf{Sets}, \mathrm{pt})$.

Remark 1.1.1.2. In detail, a **pointed set** is a pair (X, x_0) consisting of

- *The Underlying Set.* A set X, called the **underlying set of** (X, x_0);
- *The Basepoint.* A morphism

$$[x_0] \colon \mathrm{pt} \to X$$

  in Sets, determining an element $x_0 \in X$, called the **basepoint of** X.

Example 1.1.1.3. The 0**-sphere**[2] is the pointed set $\left(S^0, 0\right)$[3] consisting of

- *The Underlying Set.* The set S^0 defined by

$$S^0 \stackrel{\text{def}}{=} \{0, 1\};$$

- *The Basepoint.* The element 0 of S^0.

Example 1.1.1.4. The **trivial pointed set** is the pointed set $(\mathrm{pt}, \star)$ consisting of

- *The Underlying Set.* The punctual set $\mathrm{pt} \stackrel{\text{def}}{=} \{\star\}$;
- *The Basepoint.* The element $\star$ of pt.

Example 1.1.1.5. The **underlying pointed set** of a semimodule (M, α_M) is the pointed set $(M, 0_M)$.

Example 1.1.1.6. The **underlying pointed set** of a module (M, α_M) is the pointed set $(M, 0_M)$.

[1] *Further Terminology:* Also called an $\mathbb{F}_1$**-module**.

[2] *Further Terminology:* Also called the **underlying pointed set of the field with one element**.

[3] *Further Notation:* Also denoted $(\mathbb{F}_1, 0)$.

1.2 Morphisms of Pointed Sets

Definition 1.2.1.1. A **morphism of pointed sets**[4] is equivalently

- A morphism of $\mathbb{E}_0$-monoids in $(\mathrm{N}_\bullet(\mathsf{Sets}), \mathrm{pt})$.
- A morphism of pointed objects in $(\mathsf{Sets}, \mathrm{pt})$.

Remark 1.2.1.2. In detail, a **morphism of pointed sets** $f\colon (X, x_0) \to (Y, y_0)$ is a morphism of sets $f\colon X \to Y$ such that the diagram

$$\begin{tikzcd} & \mathrm{pt} \arrow[dl, "{[x_0]}"'] \arrow[dr, "{[y_0]}"] & \\ X \arrow[rr, "f"'] & & Y \end{tikzcd}$$

commutes, i.e. such that

$$f(x_0) = y_0.$$

1.3 The Category of Pointed Sets

Definition 1.3.1.1. The **category of pointed sets** is the category Sets_* defined equivalently as

- The homotopy category of the ∞-category $\mathsf{Mon}_{\mathbb{E}_0}(\mathrm{N}_\bullet(\mathsf{Sets}), \mathrm{pt})$ of Monoids in Monoidal ∞-Categories, **??**;
- The category Sets_* of Categories, **??**.

Remark 1.3.1.2. In detail, the **category of pointed sets** is the category Sets_* where

- *Objects.* The objects of Sets_* are pointed sets;
- *Morphisms.* The morphisms of Sets_* are morphisms of pointed sets;
- *Identities.* For each $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, the unit map

$$\mathbb{1}^{\mathsf{Sets}_*}_{(X,x_0)}\colon \mathrm{pt} \to \mathsf{Sets}_*((X, x_0), (X, x_0))$$

of Sets_* at (X, x_0) is defined by[5]

$$\mathrm{id}^{\mathsf{Sets}_*}_{(X,x_0)} \overset{\mathrm{def}}{=} \mathrm{id}_X;$$

[4] *Further Terminology:* Also called a **pointed function** or a **morphism of $\mathbb{F}_1$-modules**.
[5] Note that id_X is indeed a morphism of pointed sets, as we have $\mathrm{id}_X(x_0) = x_0$.

- *Composition.* For each $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, the composition map

$$\circ^{\mathsf{Sets}_*}_{(X,x_0),(Y,y_0),(Z,z_0)} \colon \mathsf{Sets}_*((Y, y_0), (Z, z_0)) \times \mathsf{Sets}_*((X, x_0), (Y, y_0)) \to \mathsf{Sets}_*((X, x_0), (Z, z_0))$$

  of Sets_* at $((X, x_0), (Y, y_0), (Z, z_0))$ is defined by[6]

$$g \circ^{\mathsf{Sets}_*}_{(X,x_0),(Y,y_0),(Z,z_0)} f \overset{\text{def}}{=} g \circ f.$$

1.4 Elementary Properties of Pointed Sets

Proposition 1.4.1.1. Let (X, x_0) be a pointed set.

1. *Completeness.* The category Sets_* of pointed sets and morphisms between them is complete, having in particular products (Definition 2.1.1.1), pullbacks (Definition 2.3.1.1), and equalisers (Definition 2.2.1.1).

2. *Cocompleteness.* The category Sets_* of pointed sets and morphisms between them is cocomplete, having in particular coproducts (Definition 3.1.1.1), pushouts (Definition 3.2.1.1), and coequalisers (Definition 3.3.1.1).

3. *Failure To Be Cartesian Closed.* The category Sets_* is not Cartesian closed.

4. *Relation to Partial Functions.* We have an equivalence of categories[7]

$$\mathsf{Sets}_* \overset{\text{eq.}}{\cong} \mathsf{Sets}^{\text{part.}}$$

[6] Note that the composition of two morphisms of pointed sets is indeed a morphism of pointed sets, as we have

$$\begin{aligned} g(f(x_0)) &= g(y_0) \\ &= z_0, \end{aligned}$$

or

$$\begin{tikzcd} & \mathrm{pt} \arrow[dl, "{[x_0]}"'] \arrow[d, "{[y_0]}"] \arrow[dr, "{[z_0]}"] & \\ X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z \end{tikzcd}$$

in terms of diagrams.

[7] *Warning:* This is not an isomorphism of categories, only an equivalence.

between the category of pointed sets and pointed functions between them and the category of sets and partial functions between them.

Proof. *Item 1, Completeness*: Omitted.
Item 2, Cocompleteness: Omitted.
Item 3, Failure To Be Cartesian Closed: See [**MSE2855868**].
Item 4, Relation to Partial Functions: Omitted. □

2 Limits of Pointed Sets

2.1 Products

Let (X, x_0) and (Y, y_0) be pointed sets.

Definition 2.1.1.1. The **product of** (X, x_0) **and** (Y, y_0) is the pointed set $(X \times Y, (x_0, y_0))$.

2.2 Equalisers

Let $f, g \colon (X, x_0) \rightrightarrows (Y, y_0)$ be morphisms of pointed sets.

Definition 2.2.1.1. The **equaliser of** (f, g) is the pointed set $(\mathrm{Eq}_*(f, g), x_0)$ consisting of

- *The Underlying Set.* The set $\mathrm{Eq}_*(f, g)$ defined by
$$\mathrm{Eq}_*(f, g) \overset{\text{def}}{=} \{x \in X \mid f(x) = y_0 = g(x)\};$$
- *The Basepoint.* The element x_0 of $\mathrm{Eq}_*(f, g)$.

2.3 Pullbacks

Let (X, x_0), (Y, y_0), and (Z, z_0) be pointed sets and let $f \colon (X, x_0) \to (Z, z_0)$ and $g \colon (Y, y_0) \to (Z, z_0)$ be morphisms of pointed sets.

Definition 2.3.1.1. The **pullback of** (X, x_0) **and** (Y, y_0) **over** (Z, z_0) **along** (f, g) is the pointed set $\big((X, x_0) \times_{(z, z_0)} (Y, y_0), p_0\big)$ consisting of

- *The Underlying Set.* The set $(X, x_0) \times_{(z, z_0)} (Y, y_0)$ defined by
$$(X, x_0) \times_{(z, z_0)} (Y, y_0) \overset{\text{def}}{=} \{(x, y) \in X \times Y \mid f(x) = z_0 = g(y)\};$$
- *The Basepoint.* The element (x_0, y_0) of $(X, x_0) \times_{(z, z_0)} (Y, y_0)$.

3 Colimits of Pointed Sets

3.1 Coproducts

Let (X, x_0) and (Y, y_0) be pointed sets.

Definition 3.1.1.1. The **coproduct of** (X, x_0) **and** (Y, y_0) is their wedge sum $(X \vee Y, p_0)$ of Definition 4.3.1.1.

3.2 Pushouts

Let (X, x_0), (Y, y_0), and (Z, z_0) be pointed sets and let $f: (Z, z_0) \to (X, x_0)$ and $g: (Z, z_0) \to (Y, y_0)$ be morphisms of pointed sets.

Definition 3.2.1.1. The **pushout of** (X, x_0) **and** (Y, y_0) **over** (Z, z_0) **along** (f, g) is the pointed set $\left(X \coprod_{f,Z,g} Y, p_0\right)$, where $p_0 = [x_0] = [y_0]$.

3.3 Coequalisers

Let $f, g: (X, x_0) \rightrightarrows (Y, y_0)$ be morphisms of pointed sets.

Definition 3.3.1.1. The **coequaliser of** (f, g) is the pointed set $(\mathrm{CoEq}(f, g), x_0)$.

4 Constructions With Pointed Sets

4.1 Internal Homs

Let (X, x_0) and (Y, y_0) be pointed sets.

Definition 4.1.1.1. The **pointed set of morphisms of pointed sets from** (X, x_0) **to** (Y, y_0) is the pointed set $\mathsf{Sets}_*(X, Y)$ consisting of

- *The Underlying Set.* The set $\mathsf{Sets}_*((X, x_0), (Y, y_0))$ of morphisms of pointed sets from (X, x_0) to (Y, y_0);
- *The Basepoint.* The element

  $$\Delta_{y_0}: (X, x_0) \to (Y, y_0)$$

  of $\mathsf{Sets}_*((X, x_0), (Y, y_0))$.

4.2 Free Pointed Sets

Let X be a set.

Definition 4.2.1.1. The **free pointed set on** X is the pointed set X^+ consisting of

- *The Underlying Set.* The set X^+ defined by
$$X^+ \stackrel{\text{def}}{=} X \coprod \mathrm{pt};$$
- *The Basepoint.* The element $\star$ of X^+.

Proposition 4.2.1.2. Let X be a set.

1. *Functoriality.* The assignment $X \mapsto X^+$ defines a functor
$$(-)^+ \colon \mathsf{Sets} \to \mathsf{Sets}_*,$$
where
   - *Action on Objects.* For each $X \in \mathrm{Obj}(\mathsf{Sets})$, we have
$$[(-)^+](X) \stackrel{\text{def}}{=} X_+,$$
where X_+ is the pointed set of Definition 4.2.1.1;
   - *Action on Morphisms.* For each morphism $f \colon X \to Y$ of Sets, the image
$$f_+ \colon X_+ \to Y_+$$
of f by $(-)^+$ is the map of pointed sets defined by
$$f^+(x) \stackrel{\text{def}}{=} \begin{cases} f(x) & \text{if } x \in X, \\ \star & \text{if } x = \star. \end{cases}$$
2. *Adjointness.* We have an adjunction
$$\big((-)^+ \dashv 忘\big) \colon \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "(-)^+"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_*, \arrow[l, bend left, "忘"] \end{tikzcd}$$
witnessed by a bijection of sets
$$\mathsf{Sets}_*((X_+, \star), (Y, y_0)) \cong \mathsf{Sets}(X, Y),$$
natural in $X \in \mathrm{Obj}(\mathsf{Sets})$ and $(Y, y_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

3. *Symmetric Strong Monoidality With Respect to Wedge Sums.* The free pointed set functor of Item 1 has a symmetric strong monoidal structure

$$\left((-)^+, (-)^{+,\coprod}, (-)^{+,\coprod}_{\mathbb{1}}\right) : (\mathsf{Sets}, \coprod, \emptyset) \to (\mathsf{Sets}_*, \vee, \mathrm{pt}),$$

being equipped with isomorphisms of pointed sets

$$(-)^{+,\coprod}_{X,Y} : X^+ \vee Y^+ \xrightarrow{\cong} (X \coprod Y)^+,$$
$$(-)^{+,\coprod}_{\mathbb{1}} : \mathrm{pt} \xrightarrow{\cong} \emptyset^+,$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

4. *Symmetric Strong Monoidality With Respect to Smash Products.* The free pointed set functor of Item 1 has a symmetric strong monoidal structure

$$\left((-)^+, (-)^{+,\times}, (-)^{+,\times}_{\mathbb{1}}\right) : (\mathsf{Sets}, \times, \mathrm{pt}) \to \left(\mathsf{Sets}_*, \wedge, S^0\right),$$

being equipped with isomorphisms of pointed sets

$$(-)^{+,\times}_{X,Y} : X^+ \wedge Y^+ \xrightarrow{\cong} (X \times Y)^+,$$
$$(-)^{+,\times}_{\mathbb{1}} : S^0 \xrightarrow{\cong} \mathrm{pt}^+,$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

Proof. *Item 1, Functoriality*: Clear.
Item 2, Adjointness: Clear.
Item 3, Symmetric Strong Monoidality With Respect to Wedge Sums: Omitted.
Item 4, Symmetric Strong Monoidality With Respect to Smash Products: Omitted. ∎

4.3 Wedge Sums of Pointed Sets

Let (X, x_0) and (Y, y_0) be pointed sets.

Definition 4.3.1.1. The **wedge sum of X and Y** is the pointed set $(X \vee Y, p_0)$ consisting of

- *The Underlying Set.* The set $X \vee Y$ defined by[8]

$$
\begin{aligned}
(X \vee Y, p_0) &\overset{\text{def}}{=} (X, x_0) \coprod (Y, y_0)\\
&\cong (X \textstyle\coprod_{\mathrm{pt}} Y, p_0)\\
&\cong (X \textstyle\coprod Y/\sim, p_0),
\end{aligned}
\qquad
\begin{tikzcd}
X \vee Y \arrow[dr, phantom, "\ulcorner", very near start] & Y \arrow[l] \\
X \arrow[u] & \mathrm{pt}, \arrow[l, hook', "{[x_0]}"] \arrow[u, hook, "{[y_0]}"']
\end{tikzcd}
$$

  where $\sim$ is the equivalence relation on $X \coprod Y$ given by $x_0 \sim y_0$;

- *The Basepoint.* The element p_0 of $X \vee Y$ defined by

$$
\begin{aligned}
p_0 &\overset{\text{def}}{=} [x_0]\\
&= [y_0].
\end{aligned}
$$

Proposition 4.3.1.2. Let (X, x_0) and (Y, y_0) be pointed sets.

1. *Functoriality.* The assignments $(X, x_0), (Y, y_0), ((X, x_0), (Y, y_0)) \mapsto (X \vee Y, p_0)$ define functors

$$
\begin{aligned}
X \vee - &\colon \mathsf{Sets}_* \to \mathsf{Sets}_*,\\
- \vee Y &\colon \mathsf{Sets}_* \to \mathsf{Sets}_*,\\
-_1 \vee -_2 &\colon \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*.
\end{aligned}
$$

2. *Associativity.* We have an isomorphism of pointed sets

$$(X \vee Y) \vee Z \cong X \vee (Y \vee Z),$$

   natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathsf{Sets}_*$.

3. *Unitality.* We have isomorphisms of pointed sets

$$
\begin{aligned}
\mathrm{pt} \vee X &\cong X,\\
X \vee \mathrm{pt} &\cong X,
\end{aligned}
$$

   natural in $(X, x_0) \in \mathsf{Sets}_*$.

4. *Commutativity.* We have an isomorphism of pointed sets

$$X \vee Y \cong Y \vee X,$$

   natural in $(X, x_0), (Y, y_0) \in \mathsf{Sets}_*$.

[8]Here $(X, x_0) \coprod (Y, y_0)$ is the coproduct of (X, x_0) and (Y, y_0) in Sets_*.

5. *Symmetric Monoidality.* The triple $(\mathsf{Sets}_*, \vee, \mathrm{pt})$ is a symmetric monoidal category.

6. *Symmetric Strong Monoidality With Respect to Free Pointed Sets.* The free pointed set functor of Item 1 of Proposition 4.2.1.2 has a symmetric strong monoidal structure

$$\left((-)^+, (-)^{+,\coprod}, (-)^{+,\coprod}_{\mathbb{1}}\right) : (\mathsf{Sets}, \coprod, \emptyset) \to (\mathsf{Sets}_*, \vee, \mathrm{pt}),$$

being equipped with isomorphisms of pointed sets

$$(-)^{+,\coprod}_{X,Y} : X^+ \vee Y^+ \xrightarrow{\cong} (X \coprod Y)^+,$$
$$(-)^{+,\coprod}_{\mathbb{1}} : \mathrm{pt} \xrightarrow{\cong} \emptyset^+,$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

7. *The Fold Map.* We have a natural transformation

$$\nabla : \vee \circ \Delta^{\mathsf{Cats}}_{\mathsf{Sets}_*} \Longrightarrow \mathrm{id}_{\mathsf{Sets}_*}, \qquad \begin{tikzcd} & \mathsf{Sets}_* \times \mathsf{Sets}_* \arrow[d, phantom, "\Downarrow\nabla"] \arrow[dr, "\vee"] & \\ \mathsf{Sets}_* \arrow[ur, "\Delta^{\mathsf{Cats}}_{\mathsf{Sets}_*}"] \arrow[rr, bend right, "\mathrm{id}_{\mathsf{Sets}_*}"'] & & \mathsf{Sets}_*, \end{tikzcd}$$

called the **fold map**, whose component

$$\nabla_X : X \vee X \to X$$

at X is given by the composition

$$\begin{aligned} X &\xrightarrow{\Delta_X} X \times X \\ &\twoheadrightarrow X \times X/\sim \\ &\overset{\text{def}}{=\!=} X \vee X. \end{aligned}$$

Proof. *Item 1, Functoriality*: Omitted.
Item 2, Associativity: Omitted.
Item 3, Unitality: Omitted.
Item 4, Commutativity: Omitted.
Item 5, Symmetric Monoidality: Omitted.
Item 6, Symmetric Strong Monoidality With Respect to Free Pointed Sets: Omitted.
Item 7, The Fold Map: Omitted. □

Appendices

A Other Chapters