Constructions With Sets

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000D This chapter contains some material relating to constructions with sets. Notably, it contains:

1. Explicit descriptions of the major types of co/limits in Sets, including in particular pushouts and coequalisers (see **????????**);

2. A discussion of powersets as decategorifications of categories of presheaves (**????**);

3. A lengthy discussion of the adjoint triple

$$f_* \dashv f^{-1} \dashv f_! \colon \mathcal{P}(A) \overset{\rightleftarrows}{\rightarrow} \mathcal{P}(B)$$

   of functors (morphisms of posets) between $\mathcal{P}(A)$ and $\mathcal{P}(B)$ induced by a map of sets $f \colon A \to B$, along with a discussion of the properties of f_*, f^{-1}, and $f_!$.

Contents

000E 1 Limits of Sets

000F 1.1 Products of Families of Sets

Let $\{A_i\}_{i\in I}$ be a family of sets.

000G **Definition 1.1.1.1.** The **product**[1] **of** $\{A_i\}_{i\in I}$ is the set $\prod_{i\in I} A_i$ defined by

$$\prod_{i\in I} A_i \stackrel{\text{def}}{=} \left\{ f \in \mathsf{Sets}\left(I, \bigcup_{i\in I} A_i\right) \;\middle|\; \begin{array}{l}\text{for each } i \in I \text{, we} \\ \text{have } f(i) \in A_i\end{array} \right\}.$$

H 1.2 Binary Products of Sets

Let A and B be sets.

Definition 1.2.1.1. The **product**[2] **of** A 000J **and** B is the set $A \times B$ defined by

$$\begin{aligned} A \times B &\stackrel{\text{def}}{=} \prod_{z\in\{A,B\}} z \\ &\stackrel{\text{def}}{=} \{f \in \mathsf{Sets}(\{0,1\}, A \cup B) \mid \text{we have } f(0) \in A \text{ and } f(1) \in B\} \\ &\cong \{\{\{a\},\{a,b\}\} \in \mathcal{P}(\mathcal{P}(A \cup B)) \mid \text{we have } a \in A \text{ and } b \in B\}. \end{aligned}$$

Proposition 1.2.1.2. Let A, B, C, and X 000K be sets.

1. *Functoriality.* The assignments $A, B, (A, B) \mapsto A \times B$ 000L define functors

$$\begin{aligned} A \times -_2 &\colon \mathsf{Sets} \to \mathsf{Sets}, \\ -_1 \times B &\colon \mathsf{Sets} \to \mathsf{Sets}, \\ -_1 \times -_2 &\colon \mathsf{Sets} \times \mathsf{Sets} \to \mathsf{Sets}, \end{aligned}$$

where $-_1 \times -_2$ is the functor where

- *Action on Objects.* For each $(A, B) \in \mathrm{Obj}(\mathsf{Sets} \times \mathsf{Sets})$, we have

$$[-_1 \times -_2](A, B) \stackrel{\text{def}}{=} A \times B;$$

- *Action on Morphisms.* For each $(A, B), (X, Y) \in \mathrm{Obj}(\mathsf{Sets})$, the

[1] *Further Terminology:* Also called the **Cartesian product of** $\{A_i\}_{i\in I}$.

[2] *Further Terminology:* Also called the **Cartesian product of** A **and** B or the **binary Cartesian product of** A **and** B, for emphasis.

This can also be thought of as the $(\mathbb{E}_{-1}, \mathbb{E}_{-1})$**-tensor product of** A **and** B.

action on Hom-sets

$$\times_{(A,B),(X,Y)} \colon \mathsf{Sets}(A,X) \times \mathsf{Sets}(B,Y) \to \mathsf{Sets}(A \times B, X \times Y)$$

of $\times$ at $((A,B),(X,Y))$ is defined by sending (f,g) to the function

$$f \times g \colon A \times B \to X \times Y$$

defined by

$$[f \times g](a,b) \overset{\text{def}}{=} (f(a), g(b))$$

for each $(a,b) \in A \times B$;

and where $A \times -$ and $- \times B$ are the partial functors of $-_1 \times -_2$ at $A, B \in \mathrm{Obj}(\mathsf{Sets})$.

000M 2. *Adjointness.* We have adjunctions

$$(A \times - \dashv \mathsf{Sets}(A,-))\colon\quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "A\times-"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}, \arrow[l, bend left, "\mathsf{Sets}(A,-)"] \end{tikzcd}$$

$$(- \times B \dashv \mathsf{Sets}(B,-))\colon\quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "-\times B"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}, \arrow[l, bend left, "\mathsf{Sets}(B,-)"] \end{tikzcd}$$

witnessed by bijections

$$\begin{aligned} \mathsf{Sets}(A \times B, C) &\cong \mathsf{Sets}(A, \mathsf{Sets}(B,C)), \\ \mathsf{Sets}(A \times B, C) &\cong \mathsf{Sets}(B, \mathsf{Sets}(A,C)), \end{aligned}$$

natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

000N 3. *Associativity.* We have an isomorphism of sets

$$(A \times B) \times C \cong A \times (B \times C),$$

natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

000P 4. *Unitality.* We have isomorphisms of sets

$$\begin{aligned} \mathrm{pt} \times A &\cong A, \\ A \times \mathrm{pt} &\cong A, \end{aligned}$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$.

5. *Commutativity.* 002Q We have an isomorphism of sets

$$A \times B \cong B \times A,$$

natural in $A, B \in \mathrm{Obj}(\mathsf{Sets})$.

6. *Annihilation With the Empty Set.* We have isomorphisms 002R of sets

$$\begin{aligned} A \times \emptyset &\cong \emptyset, \\ \emptyset \times A &\cong \emptyset, \end{aligned}$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$.

7. *Distributivity Over Unions.* We have isomorphisms 002S of sets

$$\begin{aligned} A \times (B \cup C) &= (A \times B) \cup (A \times C), \\ (A \cup B) \times C &= (A \times C) \cup (B \times C), \end{aligned}$$

natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

8. *Distributivity Over Intersections.* We have isomorphisms 002T of sets

$$\begin{aligned} A \times (B \cap C) &= (A \times B) \cap (A \times C), \\ (A \cap B) \times C &= (A \times C) \cap (B \times C), \end{aligned}$$

natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

9. *Distributivity Over Differences.* We have isomorphisms 002U of sets

$$\begin{aligned} A \times (B \setminus C) &= (A \times B) \setminus (A \times C), \\ (A \setminus B) \times C &= (A \times C) \setminus (B \times C), \end{aligned}$$

natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

10. *Distributivity Over Symmetric Differences.* We have isomorphisms 002V of sets

$$\begin{aligned} A \times (B \bigtriangleup C) &= (A \times B) \bigtriangleup (A \times C), \\ (A \bigtriangleup B) \times C &= (A \times C) \bigtriangleup (B \times C), \end{aligned}$$

natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

11. *Symmetric Monoidality.* The triple $(\mathsf{Sets}, \times, \mathrm{pt})$ 002W is a symmetric monoidal category.

12. *Symmetric Bimonoidality.* 000X The quintuple $(\mathsf{Sets}, \coprod, \emptyset, \times, \mathrm{pt})$ is a symmetric bimonoidal category.

Proof. **??**, *Functoriality*: Omitted.
??, *Adjointness*: Omitted.
??, *Associativity*: Clear.
??, *Unitality*: Clear.
??, *Commutativity*: Clear.
??, *Annihilation With the Empty Set*: Clear.
??, *Distributivity Over Unions*: Omitted.
??, *Distributivity Over Intersections*: Omitted.
??, *Distributivity Over Differences*: Omitted.
??, *Distributivity Over Symmetric Differences*: Omitted.
??, *Symmetric Monoidality*: Omitted.
??, *Symmetric Bimonoidality*: Omitted. □

1.3 Pullbacks 000Y

Let A, B, and C be sets and let $f\colon A \to C$ and $g\colon B \to C$ be functions.

Definition 1.3.1.1. 000Z The **pullback of A and B over C along f and g**[3] is the set $A \times_C B$ defined by

$$A \times_C B \overset{\text{def}}{=} \{(a,b) \in A \times B \mid f(a) = g(b)\}.$$

Example 1.3.1.2. 0010 Here are some examples of pullbacks of sets.

1. *Unions via Intersections.* 0011 Let $A, B \subset X$. We have a bijection of sets

$$A \cap B \cong A \times_{A \cup B} B.$$

Proposition 1.3.1.3. 0012 Let A, B, C, and X be sets.

1. *Associativity.* 0013 We have an isomorphism of sets

$$(A \times_X B) \times_X C \cong A \times_X (B \times_X C),$$

natural in $A, B, C, X \in \mathrm{Obj}(\mathsf{Sets})$.

[3] *Further Terminology:* Also called the **fibre product of A and B over C along f and g**.

2. *Unitality.* We have isomorphisms of sets 0014

$$X \times_X A \cong A,$$
$$A \times_X X \cong A,$$

natural in $A, X \in \mathrm{Obj}(\mathsf{Sets})$.

3. *Commutativity.* We have an isomorphism of sets 0015

$$A \times_X B \cong B \times_X A,$$

natural in $A, B, X \in \mathrm{Obj}(\mathsf{Sets})$.

4. *Annihilation With the Empty Set.* We have isomorphisms of sets 0016

$$A \times_X \emptyset \cong \emptyset,$$
$$\emptyset \times_X A \cong \emptyset,$$

natural in $A, X \in \mathrm{Obj}(\mathsf{Sets})$.

5. *Symmetric Monoidality.* The triple $(\mathsf{Sets}, \times_X, X)$ 0017 is a symmetric monoidal category.

Proof. **??**, *Associativity*: Clear.
??, *Unitality*: Clear.
??, *Commutativity*: Clear.
??, *Annihilation With the Empty Set*: Clear.
??, *Symmetric Monoidality*: Omitted. □

1.4 Equalisers 0018

Let A and B be sets and let $f, g \colon A \rightrightarrows B$ be functions.

Definition 1.4.1.1. The **equaliser of f and g** 0019 is the set $\mathrm{Eq}(f, g)$ defined by

$$\mathrm{Eq}(f, g) \stackrel{\text{def}}{=} \{a \in A \mid f(a) = g(a)\}.$$

Proposition 1.4.1.2. Let A, B, and C be sets. 001A

1. *Associativity.* We have an isomorphism of sets[4]

$$\underbrace{\mathrm{Eq}(f \circ \mathrm{eq}(g,h), g \circ \mathrm{eq}(g,h))}_{=\mathrm{Eq}(f \circ \mathrm{eq}(g,h), h \circ \mathrm{eq}(g,h))} \cong \mathrm{Eq}(f,g,h) \cong \underbrace{\mathrm{Eq}(f \circ \mathrm{eq}(f,g), h \circ \mathrm{eq}(f,g))}_{=\mathrm{Eq}(g \circ \mathrm{eq}(f,g), h \circ \mathrm{eq}(f,g))},$$

   where $\mathrm{Eq}(f,g,h)$ is the limit of the diagram

$$\begin{tikzcd} A \arrow[r, shift left=2, "f"] \arrow[r, "g" description] \arrow[r, shift right=2, "h"'] & B \end{tikzcd}$$

   in Sets.

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[4]That is: the following constructions give the same result:

1. Take the equaliser of (f,g,h), i.e. the limit of the diagram

$$\begin{tikzcd} A \arrow[r, shift left=2, "f"] \arrow[r, "g" description] \arrow[r, shift right=2, "h"'] & B \end{tikzcd}$$

   in Sets.
2. First take the equaliser of f and g, forming a diagram

$$\mathrm{Eq}(f,g) \overset{\mathrm{eq}(f,g)}{\hookrightarrow} A \underset{g}{\overset{f}{\rightrightarrows}} B$$

   and then take the equaliser of the composition

$$\mathrm{Eq}(f,g) \overset{\mathrm{eq}(f,g)}{\hookrightarrow} A \underset{h}{\overset{f}{\rightrightarrows}} B,$$

   obtaining a subset

$$\mathrm{Eq}(f \circ \mathrm{eq}(f,g), h \circ \mathrm{eq}(f,g)) = \mathrm{Eq}(g \circ \mathrm{eq}(f,g), h \circ \mathrm{eq}(f,g))$$

   of $\mathrm{Eq}(f,g)$.
3. First take the equaliser of g and h, forming a diagram

$$\mathrm{Eq}(g,h) \overset{\mathrm{eq}(g,h)}{\hookrightarrow} A \underset{h}{\overset{g}{\rightrightarrows}} B$$

   and then take the equaliser of the composition

$$\mathrm{Eq}(g,h) \overset{\mathrm{eq}(g,h)}{\hookrightarrow} A \underset{g}{\overset{f}{\rightrightarrows}} B,$$

   obtaining a subset

$$\mathrm{Eq}(f \circ \mathrm{eq}(g,h), g \circ \mathrm{eq}(g,h)) = \mathrm{Eq}(f \circ \mathrm{eq}(g,h), h \circ \mathrm{eq}(g,h))$$

   of $\mathrm{Eq}(g,h)$.

4. *Unitality.* We have an isomorphism of sets 001C

$$\mathrm{Eq}(f, f) \cong A.$$

5. *Commutativity.* We have an isomorphism of sets 001D

$$\mathrm{Eq}(f, g) \cong \mathrm{Eq}(g, f).$$

6. *Interaction With Composition.* Let 001E

$$A \overset{f}{\underset{g}{\rightrightarrows}} B \overset{h}{\underset{k}{\rightrightarrows}} C$$

be functions. We have an inclusion of sets

$$\mathrm{Eq}(h \circ f \circ \mathrm{eq}(f, g), k \circ g \circ \mathrm{eq}(f, g)) \subset \mathrm{Eq}(h \circ f, k \circ g),$$

where $\mathrm{Eq}(h \circ f \circ \mathrm{eq}(f, g), k \circ g \circ \mathrm{eq}(f, g))$ is the equaliser of the composition

$$\mathrm{Eq}(f, g) \overset{\mathrm{eq}(f,g)}{\hookrightarrow} A \overset{f}{\underset{g}{\rightrightarrows}} B \overset{h}{\underset{k}{\rightrightarrows}} C.$$

Proof. **??**, *Associativity*: Clear.
??, *Unitality*: Clear.
??, *Commutativity*: Clear.
??, *Interaction With Composition*: Omitted. □

2 Colimits of Sets 001F

2.1 Coproducts of Families of Sets 001G

Let $\{A_i\}_{i \in I}$ be a family of sets.

Definition 2.1.1.1. The **disjoint union of the family** $\{A_i\}_{i \in I}$ is the set $\coprod_{i \in I} A_i$ defined by 001H

$$\coprod_{i \in I} A_i \overset{\mathrm{def}}{=} \bigcup_{i \in I} \left\{ (x, i) \in \left(\bigcup_{i \in I} A_i \right) \times I \;\middle|\; x \in A_i \right\}.$$

2.2 Binary Coproducts 001J

Let A and B be sets.

Definition 2.2.1.1. The **coproduct**[5] **of** 001K A **and** B is the set $A \coprod B$ defined by

$$\begin{aligned} A \coprod B &\overset{\text{def}}{=} \coprod_{z \in \{A,B\}} z \\ &\overset{\text{def}}{=} \{(a,0) \mid a \in A\} \cup \{(b,1) \mid b \in B\}. \end{aligned}$$

Proposition 2.2.1.2. Let A, B, C, and X 001L be sets.

1. *Functoriality.* The assignment A, B, 001M $(A,B) \mapsto A \coprod B$ defines functors

$$\begin{aligned} A \coprod -_2 &: \mathsf{Sets} \to \mathsf{Sets}, \\ -_1 \coprod B &: \mathsf{Sets} \to \mathsf{Sets}, \\ -_1 \coprod -_2 &: \mathsf{Sets} \times \mathsf{Sets} \to \mathsf{Sets}, \end{aligned}$$

where $-_1 \coprod -_2$ is the functor where

- *Action on Objects.* For each $(A,B) \in \mathrm{Obj}(\mathsf{Sets} \times \mathsf{Sets})$, we have

$$[-_1 \coprod -_2](A,B) \overset{\text{def}}{=} A \coprod B;$$

- *Action on Morphisms.* For each $(A,B),(X,Y) \in \mathrm{Obj}(\mathsf{Sets})$, the action on Hom-sets

$$\coprod_{(A,B),(X,Y)} : \mathsf{Sets}(A,X) \times \mathsf{Sets}(B,Y) \to \mathsf{Sets}(A \coprod B, X \coprod Y)$$

of $\coprod$ at $((A,B),(X,Y))$ is defined by sending (f,g) to the function

$$f \coprod g \colon A \coprod B \to X \coprod Y$$

defined by

$$[f \coprod g](x) \overset{\text{def}}{=} \begin{cases} f(x) & \text{if } x \in A, \\ g(x) & \text{if } x \in B \end{cases}$$

for each $x \in A \coprod B$;

[5] *Further Terminology:* Also called the **disjoint union of** A **and** B, or the **binary disjoint union of** A **and** B, for emphasis.

and where $A \coprod -$ and $- \coprod B$ are the partial functors of $-_1 \coprod -_2$ at $A, B \in \mathrm{Obj}(\mathsf{Sets})$.

2. *Associativity.* We have an isomorphism 001N of sets

$$(A \coprod B) \coprod C \cong A \coprod (B \coprod C),$$

natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

3. *Unitality.* We have isomorphisms 001P of sets

$$\begin{aligned} A \coprod \emptyset &\cong A, \\ \emptyset \coprod A &\cong A, \end{aligned}$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$.

4. *Commutativity.* We have an isomorphism 001Q of sets

$$A \coprod B \cong B \coprod A,$$

natural in $A, B \in \mathrm{Obj}(\mathsf{Sets})$.

5. *Symmetric Monoidality.* The triple $(\mathsf{Sets}, \coprod, \emptyset)$ 001R is a symmetric monoidal category.

Proof. **??**, *Functoriality*: Omitted.
??, *Associativity*: Clear.
??, *Unitality*: Clear.
??, *Commutativity*: Clear.
??, *Symmetric Monoidality*: Omitted. □

2.3 Pushouts 001S

Let A, B, and C be sets and let $f \colon C \to A$ and $g \colon C \to B$ be functions.

Definition 2.3.1.1. 001T The **pushout of A and B over C along f and g**[6] is the set $A \coprod_C B$ defined by

$$A \coprod_C B \overset{\text{def}}{=} A \coprod B/{\sim_C},$$

where $\sim_C$ is the equivalence relation on $A \coprod B$ generated by $f(c) \sim_C g(c)$.

[6] *Further Terminology:* Also called the **fibre coproduct of A and B over C along f and g**.

Remark 2.3.1.2. In detail, the relation 001U $\sim$ of **??** is given by declaring $a \sim b$ iff one of the following conditions is satisfied:

- We have $a, b \in A$ and $a = b$;
- We have $a, b \in B$ and $a = b$;
- There exist $x_1, \ldots, x_n \in A \coprod B$ such that $a \sim' x_1 \sim' \cdots \sim' x_n \sim' b$, where we declare $x \sim' y$ if one of the following conditions is satisfied:

  1. There exists $c \in C$ such that $x = f(c)$ and $y = g(c)$.
  2. There exists $c \in C$ such that $x = g(c)$ and $y = f(c)$.

  That is: we require the following condition to be satisfied:

  ($\star$) There exist $x_1, \ldots, x_n \in A \coprod B$ satisfying the following conditions:
    1. There exists $c_0 \in C$ satisfying one of the following conditions:
       (a) We have $a = f(c_0)$ and $x_1 = g(c_0)$.
       (b) We have $a = g(c_0)$ and $x_1 = f(c_0)$.
    2. For each $1 \leq i \leq n-1$, there exists $c_i \in C$ satisfying one of the following conditions:
       (a) We have $x_i = f(c_i)$ and $x_{i+1} = g(c_i)$.
       (b) We have $x_i = g(c_i)$ and $x_{i+1} = f(c_i)$.
    3. There exists $c_n \in C$ satisfying one of the following conditions:
       (a) We have $x_n = f(c_n)$ and $b = g(c_n)$.
       (b) We have $x_n = g(c_n)$ and $b = f(c_n)$.

Example 2.3.1.3. Here are some examples 001V of pushouts of sets.

1. *Wedge Sums of Pointed Sets.* The wedge sum of two 001W pointed sets of **??** is an example of a pushout of sets.
2. *Intersections via Unions.* Let $A, B \subset X$. We have 001X a bijection of sets

$$A \cup B \cong A \coprod_{A \cap B} B.$$

Proposition 2.3.1.4. Let A, B, C, and X 001Y be sets.

1. *Associativity.* We have an isomorphism 001Z of sets

$$(A \coprod_X B) \coprod_X C \cong A \coprod_X (B \coprod_X C),$$

natural in $A, B, C, X \in \mathrm{Obj}(\mathsf{Sets})$.

2. *Unitality.* We have isomorphisms 0020 of sets

$$\emptyset \coprod_X A \cong A,$$
$$A \coprod_X \emptyset \cong A,$$

natural in $A, X \in \mathrm{Obj}(\mathsf{Sets})$.

3. *Commutativity.* We have an isomorphism 0021 of sets

$$A \coprod_X B \cong B \coprod_X A,$$

natural in $A, B, X \in \mathrm{Obj}(\mathsf{Sets})$.

4. *Annihilation With the Empty Set.* We have isomorphisms 0022 of sets

$$A \coprod_X \emptyset \cong \emptyset,$$
$$\emptyset \coprod_X A \cong \emptyset,$$

natural in $A, X \in \mathrm{Obj}(\mathsf{Sets})$.

5. *Symmetric Monoidality.* The triple $(\mathsf{Sets}, \coprod_X, \emptyset)$ 0023 is a symmetric monoidal category.

Proof. **??**, *Associativity*: Clear.
??, *Unitality*: Clear.
??, *Commutativity*: Clear.
??, *Annihilation With the Empty Set*: Clear.
??, *Symmetric Monoidality*: Omitted. □

2.4 Coequalisers 0024

Let A and B be sets and let $f, g\colon A \rightrightarrows B$ be functions.

Definition 2.4.1.1. The **coequaliser of** 0025 f **and** g is the set $\mathrm{CoEq}(f, g)$ defined by

$$\mathrm{CoEq}(f, g) \overset{\text{def}}{=} B/{\sim},$$

where $\sim$ is the equivalence relation on B generated by $f(a) \sim g(a)$.

Remark 2.4.1.2. In detail, the relation 0026 $\sim$ of **??** is given by declaring $a \sim b$ iff one of the following conditions is satisfied:

- We have $a = b$;

- There exist $x_1, \ldots, x_n \in B$ such that $a \sim' x_1 \sim' \cdots \sim' x_n \sim' b$, where we declare $x \sim' y$ if one of the following conditions is satisfied:

  1. There exists $z \in A$ such that $x = f(z)$ and $y = g(z)$.
  2. There exists $z \in A$ such that $x = g(z)$ and $y = f(z)$.

  That is: we require the following condition to be satisfied:

  ($\star$) There exist $x_1, \ldots, x_n \in B$ satisfying the following conditions:
  1. There exists $z_0 \in A$ satisfying one of the following conditions:
     (a) We have $a = f(z_0)$ and $x_1 = g(z_0)$.
     (b) We have $a = g(z_0)$ and $x_1 = f(z_0)$.
  2. For each $1 \leq i \leq n-1$, there exists $z_i \in A$ satisfying one of the following conditions:
     (a) We have $x_i = f(z_i)$ and $x_{i+1} = g(z_i)$.
     (b) We have $x_i = g(z_i)$ and $x_{i+1} = f(z_i)$.
  3. There exists $z_n \in A$ satisfying one of the following conditions:
     (a) We have $x_n = f(z_n)$ and $b = g(z_n)$.
     (b) We have $x_n = g(z_n)$ and $b = f(z_n)$.

Example 2.4.1.3. Here are some examples of coequalisers of sets. 0027

1. *Quotients by Equivalence Relations.* Let R be an equivalence relation on a set X. We have a bijection of sets 0028

$$X/{\sim_R} \cong \operatorname{CoEq}\left(R \hookrightarrow X \times X \underset{\mathrm{pr}_2}{\overset{\mathrm{pr}_1}{\rightrightarrows}} X\right).$$

Proposition 2.4.1.4. Let A, B, and C be sets. 0029

1. *Associativity.* We have an isomorphism of sets[7] 002A

$$\underbrace{\operatorname{CoEq}(\operatorname{coeq}(f,g) \circ f, \operatorname{coeq}(f,g) \circ h)}_{=\operatorname{CoEq}(\operatorname{coeq}(f,g)\circ g, \operatorname{coeq}(f,g)\circ h)} \cong \operatorname{CoEq}(f,g,h) \cong \underbrace{\operatorname{CoEq}(\operatorname{coeq}(g,h) \circ f, \operatorname{coeq}(g,h) \circ g)}_{=\operatorname{CoEq}(\operatorname{coeq}(g,h)\circ f, \operatorname{coeq}(g,h)\circ h)},$$

[7]That is: the following constructions give the same result:

1. Take the coequaliser of (f, g, h), i.e. the colimit of the diagram

$$\begin{tikzcd} A \arrow[r, shift left=2, "f"] \arrow[r, "g" description] \arrow[r, shift right=2, "h"'] & B \end{tikzcd}$$

   in Sets.

where $\mathrm{CoEq}(f, g, h)$ is the colimit of the diagram

$$\begin{tikzcd} A \arrow[r, shift left=2, "f"] \arrow[r, "g" description] \arrow[r, shift right=2, "h"'] & B \end{tikzcd}$$

in Sets.

4. *Unitality.* We have an isomorphism of sets 002B

$$\mathrm{CoEq}(f, f) \cong B.$$

5. *Commutativity.* We have an isomorphism of sets 002C

$$\mathrm{CoEq}(f, g) \cong \mathrm{CoEq}(g, f).$$

6. *Interaction With Composition.* Let 002D

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \underset{k}{\overset{h}{\rightrightarrows}} C$$

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2. First take the coequaliser of f and g, forming a diagram

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \overset{\mathrm{coeq}(f,g)}{\twoheadrightarrow} \mathrm{CoEq}(f, g)$$

and then take the coequaliser of the composition

$$A \underset{h}{\overset{f}{\rightrightarrows}} B \overset{\mathrm{coeq}(f,g)}{\twoheadrightarrow} \mathrm{CoEq}(f, g),$$

obtaining a quotient

$$\mathrm{CoEq}(\mathrm{coeq}(f, g) \circ f, \mathrm{coeq}(f, g) \circ h) = \mathrm{CoEq}(\mathrm{coeq}(f, g) \circ g, \mathrm{coeq}(f, g) \circ h)$$

of $\mathrm{CoEq}(f, g)$

3. First take the coequaliser of g and h, forming a diagram

$$A \underset{h}{\overset{g}{\rightrightarrows}} B \overset{\mathrm{coeq}(g,h)}{\twoheadrightarrow} \mathrm{CoEq}(g, h)$$

and then take the coequaliser of the composition

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \overset{\mathrm{coeq}(g,h)}{\twoheadrightarrow} \mathrm{CoEq}(g, h),$$

obtaining a quotient

$$\mathrm{CoEq}(\mathrm{coeq}(g, h) \circ f, \mathrm{coeq}(g, h) \circ g) = \mathrm{CoEq}(\mathrm{coeq}(g, h) \circ f, \mathrm{coeq}(g, h) \circ h)$$

of $\mathrm{CoEq}(g, h)$.

be functions. We have a surjection

$$\mathrm{CoEq}(h \circ f, k \circ g) \twoheadrightarrow \mathrm{CoEq}(\mathrm{coeq}(h, k) \circ h \circ f, \mathrm{coeq}(h, k) \circ k \circ g)$$

exhibiting $\mathrm{CoEq}(\mathrm{coeq}(h, k) \circ h \circ f, \mathrm{coeq}(h, k) \circ k \circ g)$ as a quotient of $\mathrm{CoEq}(h \circ f, k \circ g)$ by the relation generated by declaring $h(y) \sim k(y)$ for each $y \in B$.

Proof. **??**, *Associativity*: Omitted.
??, *Unitality*: Clear.
??, *Commutativity*: Clear.
??, *Interaction With Composition*: Omitted. □

3 Operations 002E With Sets

3.1 The Empty Set 002F

Definition 3.1.1.1. The **empty set** is the 002G set $\emptyset$ defined by

$$\emptyset \overset{\text{def}}{=} \{x \in X \mid x \neq x\},$$

where A is the set in the set existence axiom, **??** of **??**.

3.2 Singleton Sets 002H

Let X be a set.

Definition 3.2.1.1. The **singleton set 002J containing** X is the set $\{X\}$ defined by

$$\{X\} \overset{\text{def}}{=} \{X, X\},$$

where $\{X, X\}$ is the pairing of X with itself (**??**).

3.3 Pairings of Sets 002K

Let X and Y be sets.

Definition 3.3.1.1. The **pairing of** X **and** 002L Y is the set $\{X, Y\}$ defined by

$$\{X, Y\} \overset{\text{def}}{=} \{x \in A \mid x = X \text{ or } x = Y\},$$

where A is the set in the axiom of pairing, **??** of **??**.

3.4 Unions of Families 002M

Let $\{A_i\}_{i\in I}$ be a family of sets.

Definition 3.4.1.1. 002N The **union of the family** $\{A_i\}_{i\in I}$ is the set $\bigcup_{i\in I} A_i$ defined by

$$\bigcup_{i\in I} A_i \overset{\text{def}}{=} \{x \in F \mid \text{there exists some } i \in I \text{ such that } x \in A_i\},$$

where F is the set in the axiom of union, **??** of **??**.

3.5 Binary Unions 002P

Let A and B be sets.

Definition 3.5.1.1. 002Q The **union**[8] **of A and B** is the set $A \cup B$ defined by

$$A \cup B \overset{\text{def}}{=} \bigcup_{z\in\{A,B\}} z.$$

Proposition 3.5.1.2. 002R Let X be a set.

1. *Functoriality.* 002S The assignments $U, V, (U,V) \mapsto U \cup V$ define functors

$$\begin{aligned} U \cup -\colon (\mathcal{P}(X), \subset) &\to (\mathcal{P}(X), \subset), \\ - \cup V\colon (\mathcal{P}(X), \subset) &\to (\mathcal{P}(X), \subset), \\ -_1 \cup -_2\colon (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \subset) &\to (\mathcal{P}(X), \subset), \end{aligned}$$

where $-_1 \cup -_2$ is the functor where

- *Action on Objects.* For each $(U,V) \in \mathcal{P}(X) \times \mathcal{P}(X)$, we have

$$[-_1 \cup -_2](U,V) \overset{\text{def}}{=} U \cup V;$$

- *Action on Morphisms.* For each pair of morphisms

$$\begin{aligned} \iota_U &\colon U \hookrightarrow U', \\ \iota_V &\colon V \hookrightarrow V' \end{aligned}$$

of $\mathcal{P}(X) \times \mathcal{P}(X)$, the image

$$\iota_U \cup \iota_V \colon U \cup V \hookrightarrow U' \cup V'$$

[8] *Further Terminology:* Also called the **binary union of A and B**, for emphasis.

of (ι_U, ι_V) by $\cup$ is the inclusion

$$U \cup V \subset U' \cup V'$$

i.e. where we have

($\star$) If $U \subset U'$ and $V \subset V'$, then $U \cup V \subset U' \cup V'$;

and where $U \cup -$ and $- \cup V$ are the partial functors of $-_1 \cup -_2$ at $U, V \in \mathcal{P}(X)$.

2. *Via Intersections and Symmetric Differences.* We have an equality of 002T sets

$$U \cup V = (U \triangle V) \triangle (U \cap V)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

3. *Associativity.* We have an equality of sets 002U

$$(U \cup V) \cup W = U \cup (V \cup W)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

4. *Unitality.* We have equalities of sets 002V

$$U \cup \emptyset = U,$$
$$\emptyset \cup U = U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

5. *Commutativity.* We have an equality of sets 002W

$$U \cup V = V \cup U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

6. *Idempotency.* We have an equality of sets 002X

$$U \cup U = U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

7. *Distributivity Over Intersections.* We have equalities of sets 002Y

$$U \cup (V \cap W) = (U \cup V) \cap (U \cup W),$$
$$(U \cap V) \cup W = (U \cup W) \cap (V \cup W)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

8. *Interaction With Powersets and Semirings.* The quintuple $(\mathcal{P}(X), \cup, \cap, \emptyset, X)$ is an idempotent commutative semiring. 0027

Proof. **??**, *Functoriality*: Omitted.
??, *Via Intersections and Symmetric Differences*: Omitted.
??, *Associativity*: Clear.
??, *Unitality*: Clear.
??, *Commutativity*: Clear.
??, *Idempotency*: Clear.
??, *Distributivity Over Intersections*: Omitted.
??, *Interaction With Powersets and Semirings*: This follows from **????????** and **??????????** of **??**. ☐

3.6 Intersections of Families 0030

Let $\mathcal{F}$ be a family of sets.

Definition 3.6.1.1. The **intersection of a family $\mathcal{F}$ of sets** is the set $\bigcap_{X\in\mathcal{F}} X$ defined by 0031

$$\bigcap_{X\in\mathcal{F}} X \overset{\text{def}}{=} \left\{ z \in \bigcup_{X\in\mathcal{F}} X \;\middle|\; \text{for each } X \in \mathcal{F}, \text{ we have } z \in X \right\}.$$

3.7 Binary Intersections 0032

Let X and Y be sets.

Definition 3.7.1.1. The **intersection**[9] **of X and Y** is the set $X \cap Y$ defined by 0033

$$X \cap Y \overset{\text{def}}{=} \bigcap_{z\in\{X,Y\}} z.$$

Proposition 3.7.1.2. Let X be a set. 0034

1. *Functoriality.* The assignments $U, V, (U, V) \mapsto U \cap V$ define functors 0035

$$\begin{aligned} U \cap -\colon (\mathcal{P}(X), \subset) &\to (\mathcal{P}(X), \subset), \\ - \cap V\colon (\mathcal{P}(X), \subset) &\to (\mathcal{P}(X), \subset), \\ -_1 \cap -_2\colon (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \subset) &\to (\mathcal{P}(X), \subset), \end{aligned}$$

   where $-_1 \cap -_2$ is the functor where

[9] *Further Terminology:* Also called the **binary intersection of X and Y**, for emphasis.

- *Action on Objects.* For each $(U, V) \in \mathcal{P}(X) \times \mathcal{P}(X)$, we have

$$[-_1 \cap -_2](U, V) \stackrel{\text{def}}{=} U \cap V;$$

- *Action on Morphisms.* For each pair of morphisms

$$\begin{aligned} \iota_U &: U \hookrightarrow U', \\ \iota_V &: V \hookrightarrow V' \end{aligned}$$

of $\mathcal{P}(X) \times \mathcal{P}(X)$, the image

$$\iota_U \cap \iota_V : U \cap V \hookrightarrow U' \cap V'$$

of (ι_U, ι_V) by $\cap$ is the inclusion

$$U \cap V \subset U' \cap V'$$

i.e. where we have

(★) If $U \subset U'$ and $V \subset V'$, then $U \cap V \subset U' \cap V'$;

and where $U \cap -$ and $- \cap V$ are the partial functors of $-_1 \cap -_2$ at $U, V \in \mathcal{P}(X)$.

2. *Adjointness.* We have adjunctions 0036

$$\left(U \cap - \dashv \mathbf{Hom}_{\mathcal{P}(X)}(U, -)\right): \quad \begin{tikzcd} \mathcal{P}(X) \arrow[r, bend left, "U\cap-"] \arrow[r, phantom, "\perp"] & \mathcal{P}(X), \arrow[l, bend left, "\mathbf{Hom}_{\mathcal{P}(X)}(U{,}-)"] \end{tikzcd}$$

$$\left(- \cap V \dashv \mathbf{Hom}_{\mathcal{P}(X)}(V, -)\right): \quad \begin{tikzcd} \mathcal{P}(X) \arrow[r, bend left, "-\cap V"] \arrow[r, phantom, "\perp"] & \mathcal{P}(X), \arrow[l, bend left, "\mathbf{Hom}_{\mathcal{P}(X)}(V{,}-)"] \end{tikzcd}$$

where

$$\mathbf{Hom}_{\mathcal{P}(X)}(-_1, -_2) : \mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \to \mathcal{P}(X)$$

is the bifunctor defined by[10]

$$\mathbf{Hom}_{\mathcal{P}(X)}(U, V) \stackrel{\text{def}}{=} (X \setminus U) \cup V$$

[10] *Intuition:* Since intersections are the products in $\mathcal{P}(X)$, the left adjoint $\mathbf{Hom}_{\mathcal{P}(X)}(U, V)$ works as a function type $U \to V$.

witnessed by bijections

$$\begin{aligned}\mathrm{Hom}_{\mathcal{P}(X)}(U\cap V, W) &\cong \mathrm{Hom}_{\mathcal{P}(X)}\big(U, \mathbf{Hom}_{\mathcal{P}(X)}(V, W)\big),\\ \mathrm{Hom}_{\mathcal{P}(X)}(U\cap V, W) &\cong \mathrm{Hom}_{\mathcal{P}(X)}\big(V, \mathbf{Hom}_{\mathcal{P}(X)}(U, W)\big),\end{aligned}$$

natural in $U, V, W \in \mathcal{P}(X)$, i.e. where:

(a) The following conditions are equivalent:
   i. We have $U \cap V \subset W$.
   ii. We have $U \subset \mathbf{Hom}_{\mathcal{P}(X)}(V, W)$.
   iii. We have $U \subset (X \setminus V) \cup W$.

(b) The following conditions are equivalent:
   i. We have $V \cap U \subset W$.
   ii. We have $V \subset \mathbf{Hom}_{\mathcal{P}(X)}(U, W)$.
   iii. We have $V \subset (X \setminus U) \cup W$.

3. *Associativity.* We have an equality of sets 0037

$$(U \cap V) \cap W = U \cap (V \cap W)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

4. *Unitality.* Let X be a set and let $U \in \mathcal{P}(X)$. We have equalities of sets 0038

$$\begin{aligned}X \cap U &= U,\\ U \cap X &= U\end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

5. *Commutativity.* We have an equality of sets 0039

$$U \cap V = V \cap U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

6. *Idempotency.* We have an equality of sets 003A

$$U \cap U = U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

7. *Distributivity Over Unions.* We have equalities of sets 003B

$$U \cap (V \cup W) = (U \cap V) \cup (U \cap W),$$
$$(U \cup V) \cap W = (U \cap W) \cup (V \cap W)$$

   for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

8. *Annihilation With the Empty Set.* We have an equality of sets 003C

$$\emptyset \cap X = \emptyset,$$
$$X \cap \emptyset = \emptyset$$

   for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

9. *Interaction With Powersets and Monoids With Zero.* The quadruple $((\mathcal{P}(X), \emptyset), \cap, X)$ is a commutative monoid with zero. 003D

10. *Interaction With Powersets and Semirings.* The quintuple $(\mathcal{P}(X), \cup, \cap, \emptyset, X)$ is an idempotent commutative semiring. 003E

Proof. **??**, *Functoriality*: Omitted.
??, *Adjointness*: See [**MSE267469**].
??, *Associativity*: Clear.
??, *Unitality*: Clear.
??, *Commutativity*: Clear.
??, *Idempotency*: Clear.
??, *Distributivity Over Unions*: Omitted.
??, *Annihilation With the Empty Set*: Clear.
??, *Interaction With Powersets and Monoids With Zero*: This follows from **????????**.
??, *Interaction With Powersets and Semirings*: This follows from **????????** and **??????????** of **??**. □

3.8 Differences 003F

Let X and Y be sets.

Definition 3.8.1.1. The **difference of X and Y** 003G is the set $X \setminus Y$ defined by

$$X \setminus Y \stackrel{\text{def}}{=} \{a \in X \mid a \notin Y\}.$$

Now, under the Curry–Howard correspondence, the function type $U \to V$ corresponds

Proposition 3.8.1.2. Let X be a set. 003H

1. *Functoriality.* 003J The assignments $U, V, (U, V) \mapsto U \cap V$ define functors

$$\begin{aligned} U \setminus -&\colon (\mathcal{P}(X), \supset) \to (\mathcal{P}(X), \subset), \\ - \setminus V&\colon (\mathcal{P}(X), \subset) \to (\mathcal{P}(X), \subset), \\ -_1 \setminus -_2&\colon (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \supset) \to (\mathcal{P}(X), \subset), \end{aligned}$$

   where $-_1 \setminus -_2$ is the functor where

   - *Action on Objects.* For each $(U, V) \in \mathcal{P}(X) \times \mathcal{P}(X)$, we have

   $$[-_1 \setminus -_2](U, V) \stackrel{\text{def}}{=} U \setminus V;$$

   - *Action on Morphisms.* For each pair of morphisms

   $$\begin{aligned} \iota_A &\colon A \hookrightarrow B, \\ \iota_U &\colon U \hookrightarrow V \end{aligned}$$

   of $\mathcal{P}(X) \times \mathcal{P}(X)$, the image

   $$\iota_U \setminus \iota_V \colon A \setminus V \hookrightarrow B \setminus U$$

   of (ι_U, ι_V) by $\setminus$ is the inclusion

   $$A \setminus V \subset B \setminus U$$

   i.e. where we have

   $(\star)$ If $A \subset B$ and $U \subset V$, then $A \setminus V \subset B \setminus U$;

   and where $U \setminus -$ and $- \setminus V$ are the partial functors of $-_1 \setminus -_2$ at $U, V \in \mathcal{P}(X)$.

2. *De Morgan's Laws.* 003K We have equalities of sets

$$\begin{aligned} X \setminus (U \cup V) &= (X \setminus U) \cap (X \setminus V), \\ X \setminus (U \cap V) &= (X \setminus U) \cup (X \setminus V) \end{aligned}$$

   for each $X \in \text{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

---

to implication $U \implies V$, which is logically equivalent to the statement $\neg U \vee V$, which in turn corresponds to the set $U^{\mathsf{c}} \vee V \stackrel{\text{def}}{=} (X \setminus U) \cup V$.

3. *Interaction With Unions I.* We have equalities of sets 003L
$$(U \setminus V) \cup W = (U \cup W) \setminus (V \setminus W)$$
for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

4. *Interaction With Unions II.* We have equalities of sets 003M
$$(U \cup V) \setminus W = (U \setminus W) \cup (V \setminus W)$$
for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

5. *Interaction With Intersections.* We have equalities of sets 003N
$$\begin{aligned}(U \setminus V) \cap W &= (U \cap W) \setminus V \\ &= U \cap (W \setminus V)\end{aligned}$$
for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

6. *Triple Differences.* We have 003P
$$U \setminus (V \setminus W) = (U \cap W) \cup (U \setminus V)$$
for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

7. *Left Annihilation.* We have 003Q
$$\emptyset \setminus U = \emptyset$$
for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

8. *Right Unitality.* We have 003R
$$U \setminus \emptyset = U$$
for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

9. *Invertibility.* We have 003S
$$U \setminus U = \emptyset$$
for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

10. *Interaction With Containment.* The following conditions are equivalent: 003T

(a) We have $V \setminus U \subset W$.

(b) We have $V \setminus W \subset U$.

Proof. **??**, *Functoriality*: Omitted.
??, *De Morgan's Laws*: Omitted.
??, *Interaction With Unions I*: Omitted.
??, *Interaction With Unions II*: Omitted.
??, *Interaction With Intersections*: Omitted.
??, *Triple Differences*: Omitted.
??, *Left Annihilation*: Clear.
??, *Right Unitality*: Clear.
??, *Invertibility*: Clear.
??, *Interaction With Containment*: Omitted. □

3.9 Complements 003U

Let X be a set and let $U \in \mathcal{P}(X)$.

Definition 3.9.1.1. The **complement of** U 003V is the set U^{c} defined by

$$\begin{aligned} U^{\mathsf{c}} &\overset{\text{def}}{=} X \setminus U \\ &\overset{\text{def}}{=} \{a \in X \mid a \notin U\}. \end{aligned}$$

Proposition 3.9.1.2. Let X be a set. 003W

1. *Functoriality.* The assignment $U \mapsto U^{\mathsf{c}}$ 003X defines a functor

$$(-)^{\mathsf{c}} \colon \mathcal{P}(X)^{\mathsf{op}} \to \mathcal{P}(X),$$

where

- *Action on Objects.* For each $U \in \mathcal{P}(X)$, we have

$$[(-)^{\mathsf{c}}](U) \overset{\text{def}}{=} U^{\mathsf{c}};$$

- *Action on Morphisms.* For each morphism $\iota_U \colon U \hookrightarrow V$ of $\mathcal{P}(X)$, the image

$$\iota_U^{\mathsf{c}} \colon V^{\mathsf{c}} \hookrightarrow U^{\mathsf{c}}$$

of ι_U by $(-)^{\mathsf{c}}$ is the inclusion

$$V^{\mathsf{c}} \subset U^{\mathsf{c}}$$

i.e. where we have

(★) If $U \subset V$, then $V^{\mathsf{c}} \subset U^{\mathsf{c}}$.

2. *De Morgan's Laws.* We have equalities of sets 003Y

$$(U \cup V)^{\mathsf{c}} = U^{\mathsf{c}} \cap V^{\mathsf{c}},$$
$$(U \cap V)^{\mathsf{c}} = U^{\mathsf{c}} \cup V^{\mathsf{c}}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

3. *Involutority.* We have 003Z

$$(U^{\mathsf{c}})^{\mathsf{c}} = U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

Proof. **??**, *Functoriality*: Clear.
??, *De Morgan's Laws*: Omitted.
??, *Involutority*: Clear. □

3.10 Symmetric Differences 0040

Let A and B be sets.

Definition 3.10.1.1. The **symmetric difference of A and B** is the set $A \triangle B$ defined by 0041

$$A \triangle B \overset{\mathrm{def}}{=} (A \setminus B) \cup (B \setminus A).$$

Proposition 3.10.1.2. Let X be a set. 0042

1. *Lack of Functoriality.* The assignment $(U, V) \mapsto U \triangle V$ ***does not*** define a functor 0043

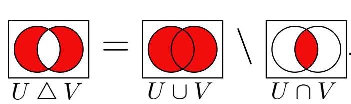
$$-_1 \triangle -_2 \colon (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \subset) \to (\mathcal{P}(X), \subset).$$

2. *Via Unions and Intersections.* We have[11] 0044

$$U \triangle V = (U \cup V) \setminus (U \cap V)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

[11] *Illustration:*



3. *Associativity.* We have[12] 0045

$$(U \triangle V) \triangle W = U \triangle (V \triangle W)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

4. *Unitality.* We have 0046

$$\begin{aligned} U \triangle \emptyset &= U, \\ \emptyset \triangle U &= U \end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

5. *Invertibility.* We have 0047

$$U \triangle U = \emptyset$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

6. *Commutativity.* We have 0048

$$U \triangle V = V \triangle U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

7. *"Transitivity".* We have 0049

$$(U \triangle V) \triangle (V \triangle W) = U \triangle W$$

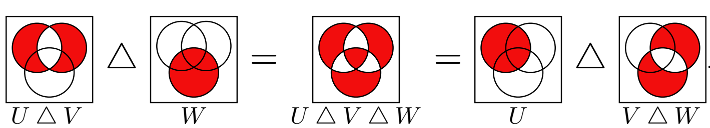
for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

8. *The Triangle Inequality for Symmetric Differences.* We have 004A

$$U \triangle W \subset U \triangle V \cup V \triangle W$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

[12] *Illustration:*



9. *Distributivity Over Intersections.* We have 004B

$$\begin{aligned} U \cap (V \triangle W) &= (U \cap V) \triangle (U \cap W), \\ (U \triangle V) \cap W &= (U \cap W) \triangle (V \cap W) \end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

10. *Interaction With Indicator Functions.* We have 004C

$$\chi_{U \triangle V} \equiv \chi_U + \chi_V \mod 2$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

11. *Bijectivity.* Given $A, B \subset \mathcal{P}(X)$, the maps 004D

$$\begin{aligned} A \triangle -&\colon \mathcal{P}(X) \to \mathcal{P}(X), \\ - \triangle B&\colon \mathcal{P}(X) \to \mathcal{P}(X) \end{aligned}$$

are bijections with inverses given by

$$\begin{aligned} (A \triangle -)^{-1} &= - \cup (A \cap -), \\ (- \triangle B)^{-1} &= - \cup (B \cap -). \end{aligned}$$

Moreover, the map

$$C \mapsto C \triangle (A \triangle B)$$

is a bijection of $\mathcal{P}(X)$ onto itself sending A to B and B to A.

12. *Interaction With Powersets and Groups I.* The quadruple $\left(\mathcal{P}(X), \triangle, \emptyset, \mathrm{id}_{\mathcal{P}(X)}\right)$ 004E is an abelian group.[13,14,15]

[13] *Example:* When $X = \emptyset$, we have an isomorphism of groups between $\mathcal{P}(\emptyset)$ and the trivial group:

$$\left(\mathcal{P}(\emptyset), \triangle, \emptyset, \mathrm{id}_{\mathcal{P}(\emptyset)}\right) \cong \mathrm{pt}.$$

[14] *Example:* When $X = \mathrm{pt}$, we have an isomorphism of groups between $\mathcal{P}(\mathrm{pt})$ and $\mathbb{Z}_{/2}$:

$$\left(\mathcal{P}(\mathrm{pt}), \triangle, \emptyset, \mathrm{id}_{\mathcal{P}(\mathrm{pt})}\right) \cong \mathbb{Z}_{/2}.$$

[15] *Example:* When $X = \{0, 1\}$, we have an isomorphism of groups between $\mathcal{P}(\{0, 1\})$ and $\mathbb{Z}_{/2} \times \mathbb{Z}_{/2}$:

$$\left(\mathcal{P}(\{0, 1\}), \triangle, \emptyset, \mathrm{id}_{\mathcal{P}(\{0,1\})}\right) \cong \mathbb{Z}_{/2} \times \mathbb{Z}_{/2}.$$

13. *Interaction With Powersets and Groups II.* Every element of $\mathcal{P}(X)$ 004F has order 2 with respect to $\triangle$, and thus $\mathcal{P}(X)$ is a *Boolean group* (i.e. an abelian 2-group).

14. *Interaction With Powersets and Vector Spaces I.* The pair $\big(\mathcal{P}(X), \alpha_{\mathcal{P}(X)}\big)$ 004G consisting of

    - The group $\mathcal{P}(X)$ of **??**;
    - The map $\alpha_{\mathcal{P}(X)} \colon \mathbb{F}_2 \times \mathcal{P}(X) \to \mathcal{P}(X)$ defined by

    $$\begin{aligned} 0 \cdot U &\overset{\text{def}}{=} \emptyset, \\ 1 \cdot U &\overset{\text{def}}{=} U; \end{aligned}$$

    is an $\mathbb{F}_2$-vector space.

15. *Interaction With Powersets and Vector Spaces II.* If X is finite, then: 004H

    (a) The set of singletons sets on the elements of X forms a basis for the $\mathbb{F}_2$-vector space $\big(\mathcal{P}(X), \alpha_{\mathcal{P}(X)}\big)$ of **??**.

    (b) We have
    $$\dim(\mathcal{P}(X)) = \#\mathcal{P}(X).$$

16. *Interaction With Powersets and Rings.* The quintuple $(\mathcal{P}(X), \triangle, \cap, \emptyset, X)$ 004I is a commutative ring.[16]

Proof. **??**, *Lack of Functoriality*: Omitted.
??, *Via Unions and Intersections*: Omitted.
??, *Associativity*: Omitted.
??, *Unitality*: Clear.
??, *Invertibility*: Clear.
??, *Commutativity*: Clear.
??, *"Transitivity"*: We have

$$\begin{aligned} (U \triangle V) \triangle (V \triangle W) &= U \triangle (V \triangle (V \triangle W)) && \text{(by ??)} \\ &= U \triangle ((V \triangle V) \triangle W) && \text{(by ??)} \\ &= U \triangle (\emptyset \triangle W) && \text{(by ??)} \\ &= U \triangle W && \text{(by ??)} \end{aligned}$$

---

[16] *Warning:* The analogous statement replacing intersections by unions

??, *The Triangle Inequality for Symmetric Differences*: This follows from **????**.
??, *Distributivity Over Intersections*: Omitted.
??, *Interaction With Indicator Functions*: Clear.
??, *Bijectivity*: Clear.
??, *Interaction With Powersets and Groups I*: This follows from **????????**.
??, *Interaction With Powersets and Groups II*: This follows from **??**.
??, *Interaction With Powersets and Vector Spaces I*: Clear.
??, *Interaction With Powersets and Vector Spaces II*: Omitted.
??, *Interaction With Powersets and Rings*: This follows from **????** and **????** of **??**.[17] □

3.11 Ordered Pairs 004K

Let A and B be sets.

Definition 3.11.1.1. The **ordered pair associated to A and B** 004L is the set (A, B) defined by

$$(A, B) \overset{\text{def}}{=} \{\{A\}, \{A, B\}\}.$$

Proposition 3.11.1.2. Let A and B be sets. 004M

1. *Uniqueness.* Let A, B, C, and D be sets. 004N The following conditions are equivalent:
   (a) We have $(A, B) = (C, D)$.
   (b) We have $A = C$ and $B = D$.

Proof. **??**, *Uniqueness*: See [**ciesielski1997set**]. □

4 Powersets 004P

4.1 Characteristic Functions 004Q

Let X be a set.

Definition 4.1.1.1. Let $U \subset X$ and let $x \in X$. 004R

(i.e. that the quintuple $(\mathcal{P}(X), \triangle, \cup, \emptyset, X)$ is a ring) is false, however. See [**proof-wiki:symmetric-difference-with-union-does-not-form-ring**] for a proof.

[17] *Reference:* [**proof-wiki:symmetric-difference-with-intersection-forms-ring**].

1. The **characteristic function of** U [18] is the function[19] 004S

$$\chi_U \colon X \to \{\mathsf{t}, \mathsf{f}\}$$

defined by

$$\chi_U(x) \stackrel{\text{def}}{=} \begin{cases} \mathsf{true} & \text{if } x \in U, \\ \mathsf{false} & \text{if } x \notin U \end{cases}$$

for each $x \in X$.

2. The **characteristic function of** x is the function [20] 004T

$$\chi_x \colon X \to \{\mathsf{t}, \mathsf{f}\}$$

defined by

$$\chi_x \stackrel{\text{def}}{=} \chi_{\{x\}},$$

i.e. by

$$\chi_x(y) \stackrel{\text{def}}{=} \begin{cases} \mathsf{true} & \text{if } x = y, \\ \mathsf{false} & \text{if } x \neq y \end{cases}$$

for each $y \in X$.

3. The **characteristic relation on** X [21] is the relation[22] 004U

$$\chi_X(-_1, -_2) \colon X \times X \to \{\mathsf{t}, \mathsf{f}\}$$

on X defined by[23]

$$\chi_X(x, y) \stackrel{\text{def}}{=} \begin{cases} \mathsf{true} & \text{if } x = y, \\ \mathsf{false} & \text{if } x \neq y \end{cases}$$

for each $x, y \in X$.

4. The **characteristic embedding** [24] **of** X **into** $\mathcal{P}(X)$ 004V is the function

$$\chi_{(-)} \colon X \hookrightarrow \mathcal{P}(X)$$

[18] *Further Terminology:* Also called the **indicator function of** U.
[19] *Further Notation:* Also written $\chi_X(U, -)$ or $\chi_X(-, U)$.
[20] *Further Notation:* Also written χ_x, $\chi_X(x, -)$, or $\chi_X(-, x)$.
[21] *Further Terminology:* Also called the **identity relation on** X.
[22] *Further Notation:* Also written χ^{-1}_{-2}, or $\sim_{\mathrm{id}}$ in the context of relations.
[23] As a subset of $X \times X$, the relation χ_X corresponds to the diagonal $\Delta_X \subset X \times X$ of X.
[24] The name "characteristic *embedding*" comes from the fact that there is an analogue of

defined by

$$\chi_{(-)}(x) \stackrel{\text{def}}{=} \chi_x$$

for each $x \in X$.

Remark 4.1.1.2. The definitions in **??** 004W are decategorifications of co/presheaves, representable co/presheaves, Hom profunctors, and the Yoneda embedding:[25]

1. A function

   $$f \colon X \to \{\mathsf{t}, \mathsf{f}\}$$

   is a decategorification of a presheaf

   $$\mathcal{F} \colon \mathcal{C}^{\mathsf{op}} \to \mathsf{Sets},$$

   with the characteristic functions χ_U of the subsets of X being the primordial examples (and, in fact, all examples) of these.

---

fully faithfulness for $\chi_{(-)}$: given a set X, we have

$$\mathrm{Hom}_{\mathcal{P}(X)}(\chi_x, \chi_y) = \chi_X(x, y),$$

for each $x, y \in X$.

[25] These statements can be made precise by using the embeddings

$$(-)_{\mathsf{disc}} \colon \mathsf{Sets} \hookrightarrow \mathsf{Cats},$$
$$(-)_{\mathsf{disc}} \colon \{\mathsf{t}, \mathsf{f}\}_{\mathsf{disc}} \hookrightarrow \mathsf{Sets}$$

of sets into categories and of classical truth values into sets.

For instance, in this approach the characteristic function

$$\chi_x \colon X \to \{\mathsf{t}, \mathsf{f}\}$$

of an element x of X, defined by

$$\chi_x(y) \stackrel{\text{def}}{=} \begin{cases} \mathsf{true} & \text{if } x = y, \\ \mathsf{false} & \text{if } x \neq y \end{cases}$$

for each $y \in X$, is recovered as the representable presheaf

$$\mathrm{Hom}_{X_{\mathsf{disc}}}(-, x) \colon X_{\mathsf{disc}} \to \mathsf{Sets}$$

of the corresponding object x of X_{disc}, defined on objects by

$$\mathrm{Hom}_{X_{\mathsf{disc}}}(y, x) \stackrel{\text{def}}{=} \begin{cases} \mathrm{pt} & \text{if } x = y, \\ \emptyset & \text{if } x \neq y \end{cases}$$

for each $y \in \mathrm{Obj}(X_{\mathsf{disc}})$.

2. The characteristic function

$$\chi_x \colon X \to \{\mathsf{t}, \mathsf{f}\}$$

of an *element* x of X is a decategorification of the representable presheaf

$$h_X \colon \mathcal{C}^{\mathsf{op}} \to \mathsf{Sets}$$

of an *object* x of a category $\mathcal{C}$.

3. The characteristic relation

$$\chi_X(-_1, -_2) \colon X \times X \to \{\mathsf{t}, \mathsf{f}\}$$

of X is a decategorification of the Hom profunctor

$$\mathrm{Hom}_{\mathcal{C}}(-_1, -_2) \colon \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathsf{Sets}$$

of a category $\mathcal{C}$.

4. The characteristic embedding

$$\chi_{(-)} \colon X \hookrightarrow \mathcal{P}(X)$$

of X into $\mathcal{P}(X)$ is a decategorification of the Yoneda embedding

$$よ \colon \mathcal{C}^{\mathsf{op}} \hookrightarrow \mathsf{PSh}(\mathcal{C})$$

of a category $\mathcal{C}$ into $\mathsf{PSh}(\mathcal{C})$.

5. There is also a direct parallel between unions and colimits:

   - An element of $\mathcal{P}(X)$ is a union of elements of X, viewed as one-point subsets $\{x\} \in \mathcal{P}(A)$;
   - An object of $\mathsf{PSh}(\mathcal{C})$ is a colimit of objects of $\mathcal{C}$, viewed as representable presheaves $h_X \in \mathrm{Obj}(\mathsf{PSh}(\mathcal{C}))$.

Proposition 4.1.1.3. Let $f \colon A \to B$ be a function. 004X We have an inclusion

$$\chi_B \circ (f \times f) \subset \chi_A, \qquad \begin{tikzcd} A \times A \arrow[r, "\chi_A(-_1,-_2)"] \arrow[d, "f \times f"'] \arrow[dr, phantom, "\subset"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ B \times B \arrow[r, "\chi_B(-_1,-_2)"'] & \{\mathsf{true}, \mathsf{false}\}. \end{tikzcd}$$

Proof. The inclusion $\chi_B(f(a), f(b)) \subset \chi_A(a, b)$ is equivalent to the statement "if $a = b$, then $f(a) = f(b)$", which is true. □

Proposition 4.1.1.4. 004U Let X be a set and let $U \subset X$ be a subset of X. We have

$$\mathrm{Hom}_{\mathcal{P}(X)}(\chi_x, \chi_U) = \chi_U(x)$$

for each $x \in X$, giving an equality of functions

$$\mathrm{Hom}_{\mathcal{P}(X)}\big(\chi_{(-)}, \chi_U\big) = \chi_U.$$

Proof. Clear. □

Corollary 4.1.1.5. 004Z The characteristic embedding is fully faithful, i.e., we have

$$\mathrm{Hom}_{\mathcal{P}(X)}(\chi_x, \chi_y) = \chi_X(x, y)$$

for each $x, y \in X$.

Proof. This follows from **??**. □

4.2 Powersets 0050

Let X be a set.

Definition 4.2.1.1. 0051 The **powerset of X** is the set $\mathcal{P}(X)$ defined by

$$\mathcal{P}(X) \stackrel{\text{def}}{=} \{U \in P \mid U \subset X\},$$

where P is the set in the axiom of powerset, **??** of **??**.

Remark 4.2.1.2. 0052 The powerset of a set is a decategorification of the category of presheaves of a category: while[26]

[26]This parallel is based on the following comparison:

- A category is enriched over the category

$$\mathsf{Sets} \stackrel{\text{def}}{=} \mathsf{Cats}_0$$

of sets (i.e. "0-categories"), with presheaves taking values on it;

- A set is enriched over the set

$$\{\mathsf{t}, \mathsf{f}\} \stackrel{\text{def}}{=} \mathsf{Cats}_{-1}$$

of classical truth values (i.e. "(-1)-categories"), with characteristic functions taking values on it.

- The powerset of a set X is equivalently (**??** of **??**) the set

$$\mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\})$$

of functions from X to the set $\{\mathsf{t}, \mathsf{f}\}$ of classical truth values;

- The category of presheaves on a category $\mathcal{C}$ is the category

$$\mathsf{Fun}(\mathcal{C}^{\mathsf{op}}, \mathsf{Sets})$$

of functors from $\mathcal{C}^{\mathsf{op}}$ to the category Sets of sets.

Proposition 4.2.1.3. Let X be a set. 0053

1. *Functoriality.* The assignment $X \mapsto \mathcal{P}(X)$ defines functors 0054

$$\begin{aligned} \mathcal{P}_* \colon \mathsf{Sets} &\to \mathsf{Sets}, \\ \mathcal{P}^{-1} \colon \mathsf{Sets}^{\mathsf{op}} &\to \mathsf{Sets}, \\ \mathcal{P}_! \colon \mathsf{Sets} &\to \mathsf{Sets} \end{aligned}$$

where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$\begin{aligned} \mathcal{P}_*(A) &\stackrel{\text{def}}{=} \mathcal{P}(A), \\ \mathcal{P}^{-1}(A) &\stackrel{\text{def}}{=} \mathcal{P}(A), \\ \mathcal{P}_!(A) &\stackrel{\text{def}}{=} \mathcal{P}(A); \end{aligned}$$

- *Action on Morphisms.* For each morphism $f \colon A \to B$ of Sets, the images

$$\begin{aligned} \mathcal{P}_*(f) \colon \mathcal{P}(A) &\to \mathcal{P}(B), \\ \mathcal{P}^{-1}(f) \colon \mathcal{P}(B) &\to \mathcal{P}(A), \\ \mathcal{P}_!(f) \colon \mathcal{P}(A) &\to \mathcal{P}(B) \end{aligned}$$

of f by $\mathcal{P}_*$, $\mathcal{P}^{-1}$, and $\mathcal{P}_!$ are defined by

$$\begin{aligned} \mathcal{P}_*(f) &\stackrel{\text{def}}{=} f_*, \\ \mathcal{P}^{-1}(f) &\stackrel{\text{def}}{=} f^{-1}, \\ \mathcal{P}_!(f) &\stackrel{\text{def}}{=} f_!, \end{aligned}$$

as in **??????**.

2. *Adjointness I.* We have an adjunction 0055

$$\left(\mathcal{P}^{-1} \dashv \mathcal{P}^{-1,\mathsf{op}}\right)\colon \quad \begin{tikzcd} \mathsf{Sets}^{\mathsf{op}} \arrow[r, bend left, "\mathcal{P}^{-1}"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}, \arrow[l, bend left, "\mathcal{P}^{-1,\mathsf{op}}"] \end{tikzcd}$$

witnessed by a bijection

$$\underbrace{\mathsf{Sets}^{\mathsf{op}}(\mathcal{P}(X), Y)}_{\overset{\text{def}}{=} \mathsf{Sets}(Y, \mathcal{P}(X))} \cong \mathsf{Sets}(X, \mathcal{P}(Y)),$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets})$ and $Y \in \mathrm{Obj}(\mathsf{Sets}^{\mathsf{op}})$.

3. *Adjointness II.* We have an adjunction 0056

$$(\mathrm{Gr} \dashv \mathcal{P}_*)\colon \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "\mathrm{Gr}"] \arrow[r, phantom, "\perp"] & \mathrm{Rel}, \arrow[l, bend left, "\mathcal{P}_*"] \end{tikzcd}$$

witnessed by a bijection of sets

$$\mathrm{Rel}(\mathrm{Gr}(A), B) \cong \mathsf{Sets}(A, \mathcal{P}(B))$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$ and $B \in \mathrm{Obj}(\mathrm{Rel})$, where Gr is the graph functor of Relations, Item 1 of Proposition 3.1.1.2.

4. *Symmetric Strong Monoidality With Respect to Coproducts.* The 0057
powerset functor $\mathcal{P}_*$ of **??** has a symmetric strong monoidal structure

$$\left(\mathcal{P}_*, \mathcal{P}_*^{\coprod}, \mathcal{P}_{*|\mathbb{1}}^{\coprod}\right)\colon (\mathsf{Sets}, \textstyle\coprod, \emptyset) \to (\mathsf{Sets}, \times, \mathrm{pt})$$

being equipped with isomorphisms

$$\mathcal{P}_{*|X,Y}^{\coprod}\colon \mathcal{P}(X) \times \mathcal{P}(Y) \xrightarrow{\cong} \mathcal{P}(X \textstyle\coprod Y),$$
$$\mathcal{P}_{*|\mathbb{1}}^{\coprod}\colon \mathrm{pt} \xrightarrow{\cong} \mathcal{P}(\emptyset),$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

5. *Symmetric Lax Monoidality With Respect to Products.* The powerset 0058
functor $\mathcal{P}_*$ of **??** has a symmetric lax monoidal structure

$$\left(\mathcal{P}_*, \mathcal{P}_*^{\otimes}, \mathcal{P}_{*|\mathbb{1}}^{\otimes}\right)\colon (\mathsf{Sets}, \times, \mathrm{pt}) \to (\mathsf{Sets}, \times, \mathrm{pt})$$

being equipped with isomorphisms

$$\mathcal{P}^{\otimes}_{*|X,Y} \colon \mathcal{P}(X) \times \mathcal{P}(Y) \to \mathcal{P}(X \times Y),$$
$$\mathcal{P}^{\otimes}_{*|\mathbb{1}} \colon \mathrm{pt} \xrightarrow{=} \mathcal{P}(\emptyset),$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$, where $\mathcal{P}^{\otimes}_{*|X,Y}$ is given by

$$\mathcal{P}^{\otimes}_{*|X,Y}(U, V) \overset{\mathrm{def}}{=} U \times V$$

for each $(U, V) \in \mathcal{P}(X) \times \mathcal{P}(Y)$.

6. *Powersets as Sets of Functions.* The assignment $U \mapsto \chi_U$ defines a bijection[27] 0059

$$\chi_{(-)} \colon \mathcal{P}(X) \xrightarrow{\cong} \mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\}),$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets})$.

7. *Powersets as Sets of Relations.* We have bijections 005A

$$\mathcal{P}(X) \cong \mathrm{Rel}(\mathrm{pt}, X),$$
$$\mathcal{P}(X) \cong \mathrm{Rel}(X, \mathrm{pt}),$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets})$.

8. *As a Free Cocompletion: Universal Property.* The pair $\big(\mathcal{P}(X), \chi_{(-)}\big)$ consisting of 005B

   - The powerset $\mathcal{P}(X)$ of X;
   - The characteristic embedding $\chi_{(-)} \colon X \hookrightarrow \mathcal{P}(X)$ of X into $\mathcal{P}(X)$;

   satisfies the following universal property:

   ($\star$) Given another pair (Y, f) consisting of
   - A cocomplete poset $(Y, \preceq)$;
   - A function $f \colon X \to Y$;

[27] This bijection is a decategorified form of the equivalence

$$\mathsf{PSh}(\mathcal{C}) \overset{\text{eq.}}{\cong} \mathsf{DFib}(\mathcal{C})$$

of Fibred Categories, **??** of **??**, with $\chi_{(-)}$ being a decategorified version of the category of elements construction of Fibred Categories, **??**.

See also **??** of **??**.

there exists a unique cocontinuous morphism of posets $(\mathcal{P}(X), \subset) \xrightarrow{\exists!} (Y, \preceq)$ making the diagram

$$\begin{tikzcd}
 & \mathcal{P}(X) \arrow[d, dashed, "\exists!"] \\
X \arrow[ru, "\chi_X"] \arrow[r, "f"'] & Y
\end{tikzcd}$$

commute.

9. *As a Free Cocompletion: Adjointness.* We have an adjunction[28] 005C

$$\left(\chi_{(-)} \dashv 忘\right)\colon \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "\chi_{(-)}"] \arrow[r, phantom, "\perp"] & \mathsf{Pos}^{\mathsf{cocomp.}} \arrow[l, bend left, "忘"] \end{tikzcd},$$

witnessed by a bijection

$$\mathsf{Pos}^{\mathsf{cocomp.}}((\mathcal{P}(X), \subset), (Y, \preceq)) \cong \mathsf{Sets}(X, Y),$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets})$ and $(Y, \preceq) \in \mathrm{Obj}(\mathsf{Pos})$, where

- We have a natural map

  $$\chi_X^*\colon \mathsf{Pos}^{\mathsf{cocomp.}}((\mathcal{P}(X), \subset), (Y, \preceq)) \to \mathsf{Sets}(X, Y)$$

  defined by

  $$\chi_X^*(f) \overset{\text{def}}{=} f \circ \chi_X,$$

  i.e. by sending a cocontinuous morphism of posets $f\colon \mathcal{P}(X) \to Y$ to the composition

  $$X \overset{\chi_X}{\hookrightarrow} \mathcal{P}(X) \xrightarrow{f} Y;$$

- We have a natural map

  $$\mathrm{Lan}_{\chi_X}\colon \mathsf{Sets}(X, Y) \to \mathsf{Pos}^{\mathsf{cocomp.}}((\mathcal{P}(X), \subset), (Y, \preceq))$$

[28] In this sense, $\mathcal{P}(A)$ is the free cocompletion of A. (Note that, despite its name, however, this is not an idempotent operation, as we have $\mathcal{P}(\mathcal{P}(A)) \neq \mathcal{P}(A)$.)

computed by

$$
\begin{aligned}
[\mathrm{Lan}_{\chi_X}(f)](U) &\cong \int^{x\in X} \chi_{\mathcal{P}(X)}(\chi_x, U) \odot f(x) \\
&\cong \int^{x\in X} \chi_U(x) \odot f(x) && \text{(by ??)} \\
&\cong \bigvee_{x\in X} (\chi_U(x) \odot f(x))
\end{aligned}
$$

for each $U \in \mathcal{P}(X)$, where:

– $\bigvee$ is the join in $(Y, \preceq)$;
– We have

$$
\begin{aligned}
\mathsf{true} \odot f(x) &\stackrel{\text{def}}{=} f(x), \\
\mathsf{false} \odot f(x) &\stackrel{\text{def}}{=} \varnothing_Y,
\end{aligned}
$$

where $\varnothing_Y$ is the minimal element of $(Y, \preceq)$.

Proof. **??**, *Functoriality*: This follows from **????** of **??**, **????** of **??**, and **????** of **??**.
??, *Adjointness I*: Omitted.
??, *Adjointness II*: Omitted.
??, *Symmetric Strong Monoidality With Respect to Coproducts*: Omitted.
??, *Symmetric Lax Monoidality With Respect to Products*: Omitted.
??, *Powersets as Sets of Functions*: Omitted.
??, *Powersets as Sets of Relations*: Omitted.
??, *As a Free Cocompletion: Universal Property*: This is a rephrasing of **??**.
??, *As a Free Cocompletion: Adjointness*: Omitted. □

4.3 Direct Images 005D

Let A and B be sets and let $f\colon A \to B$ be a function.

Definition 4.3.1.1. The **direct image function associated to** f 005E is the function[29]

$$f_*\colon \mathcal{P}(A) \to \mathcal{P}(B)$$

[29] *Further Notation:* Also written $\exists_f\colon \mathcal{P}(A) \to \mathcal{P}(B)$. This notation comes from the fact that the following statements are equivalent, where $b \in B$ and $U \in \mathcal{P}(A)$:

- We have $b \in \exists_f(U)$.
- There exists some $a \in U$ such that $f(a) = b$.

defined by[30,31]

$$
\begin{aligned}
f_*(U) &\overset{\text{def}}{=} f(U) \\
&\overset{\text{def}}{=} \left\{ b \in B \,\middle|\, \begin{array}{l} \text{there exists some } a \in \\ U \text{ such that } b = f(a) \end{array} \right\} \\
&= \{ f(a) \in B \mid a \in U \}
\end{aligned}
$$

for each $U \in \mathcal{P}(A)$.

Remark 4.3.1.2. Identifying subsets of 005F A with functions from A to $\{\mathsf{true}, \mathsf{false}\}$ via **??** of **??**, we see that the direct image function associated to f is equivalently the function

$$f_* \colon \mathcal{P}(A) \to \mathcal{P}(B)$$

defined by

$$
\begin{aligned}
f_*(\chi_U) &\overset{\text{def}}{=} \operatorname{Lan}_f(\chi_U) \\
&= \operatorname{colim}\left(\left(f \overset{\rightarrow}{\times} \underline{(-_1)} \right) \overset{\text{pr}}{\twoheadrightarrow} A \xrightarrow{\chi_U} \{\mathsf{t}, \mathsf{f}\} \right) \\
&= \operatorname*{colim}_{\substack{a \in A \\ f(a) = -_1}} (\chi_U(a)) \\
&= \bigvee_{\substack{a \in A \\ f(a) = -_1}} (\chi_U(a)).
\end{aligned}
$$

So, in other words, we have

$$
\begin{aligned}
[f_*(\chi_U)](b) &= \bigvee_{\substack{a \in A \\ f(a) = b}} (\chi_U(a)) \\
&= \begin{cases} \mathsf{true} & \text{if there exists some } a \in A \text{ such} \\ & \text{that } f(a) = b \text{ and } a \in U, \\ \mathsf{false} & \text{otherwise} \end{cases} \\
&= \begin{cases} \mathsf{true} & \text{if there exists some } a \in U \\ & \text{such that } f(a) = b, \\ \mathsf{false} & \text{otherwise} \end{cases}
\end{aligned}
$$

---

[30] *Further Terminology:* The set $f(U)$ is called the **direct image of U by f**.

[31] We also have

$$f_*(U) = B \setminus f_!(A \setminus U);$$

for each $b \in B$.

Proposition 4.3.1.3. Let $f\colon A \to B$ be a function. 005G

1. *Functoriality.* The assignment $U \mapsto f_*(U)$ defines a functor 005H

$$f_*\colon (\mathcal{P}(A), \subset) \to (\mathcal{P}(B), \subset)$$

   where

   - *Action on Objects.* For each $U \in \mathcal{P}(A)$, we have

   $$[f_*](U) \overset{\text{def}}{=} f_*(U);$$

   - *Action on Morphisms.* For each $U, V \in \mathcal{P}(A)$:

     ($\star$) If $U \subset V$, then $f_*(U) \subset f_*(V)$.

2. *Triple Adjointness.* We have a triple adjunction 005J

$$\big(f_* \dashv f^{-1} \dashv f_!\big)\colon \quad \begin{tikzcd} \mathcal{P}(A) \arrow[r, bend left=45, "f_*"] \arrow[r, bend right=45, "f_!"'] & \mathcal{P}(B), \arrow[l, "f^{-1}" description] \end{tikzcd}$$

   witnessed by bijections of sets

$$\mathrm{Hom}_{\mathcal{P}(B)}(f_*(U), V) \cong \mathrm{Hom}_{\mathcal{P}(A)}\big(U, f^{-1}(V)\big),$$
$$\mathrm{Hom}_{\mathcal{P}(A)}\big(f^{-1}(U), V\big) \cong \mathrm{Hom}_{\mathcal{P}(A)}(U, f_!(V)),$$

   natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$ and (respectively) $V \in \mathcal{P}(A)$ and $U \in \mathcal{P}(B)$, i.e. where:

   (a) The following conditions are equivalent:
      i. We have $f_*(U) \subset V$.
      ii. We have $U \subset f^{-1}(V)$.

   (b) The following conditions are equivalent:
      i. We have $f^{-1}(U) \subset V$.
      ii. We have $U \subset f_!(V)$.

---

see **??** of **??**.

3. *Preservation of Colimits.* We have an equality of sets 005K

$$f_*\left(\bigcup_{i\in I} U_i\right) = \bigcup_{i\in I} f_*(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have equalities

$$\begin{aligned} f_*(U) \cup f_*(V) &= f_*(U \cup V), \\ f_*(\emptyset) &= \emptyset, \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

4. *Oplax Preservation of Limits.* We have an inclusion of sets 005L

$$f_*\left(\bigcap_{i\in I} U_i\right) \subset \bigcap_{i\in I} f_*(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have inclusions

$$\begin{aligned} f_*(U \cap V) &\subset f_*(U) \cap f_*(V), \\ f_*(A) &\subset B, \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

5. *Symmetric Strict Monoidality With Respect to Unions.* The direct 005M
image function of **??** has a symmetric strict monoidal structure

$$\left(f_*, f_*^{\otimes}, f_{*|\mathbb{1}}^{\otimes}\right) \colon (\mathcal{P}(A), \cup, \emptyset) \to (\mathcal{P}(B), \cup, \emptyset),$$

being equipped with equalities

$$\begin{aligned} f_{*|U,V}^{\otimes} \colon f_*(U) \cup f_*(V) &\xrightarrow{=} f_*(U \cup V), \\ f_{*|\mathbb{1}}^{\otimes} \colon \emptyset &\xrightarrow{=} \emptyset, \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

6. *Symmetric Oplax Monoidality With Respect to Intersections.* The 005N
direct image function of **??** has a symmetric oplax monoidal structure

$$\left(f_*, f_*^{\otimes}, f_{*|\mathbb{1}}^{\otimes}\right) \colon (\mathcal{P}(A), \cap, A) \to (\mathcal{P}(B), \cap, B),$$

being equipped with inclusions

$$\begin{aligned} f^{\otimes}_{*|U,V} \colon f_*(U \cap V) &\hookrightarrow f_*(U) \cap f_*(V), \\ f^{\otimes}_{*|\mathbb{1}} \colon f_*(A) &\hookrightarrow B, \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

7. *Relation to Direct Images With Compact Support.* We have 005P

$$f_*(U) = B \setminus f_!(A \setminus U)$$

for each $U \in \mathcal{P}(A)$.

Proof. **??**, *Functoriality*: Clear.
??, *Triple Adjointness*: This follows from Kan Extensions, **??** of **??**.
??, *Preservation of Colimits*: This follows from **??** and Categories, **??** of **??**.
??, *Oplax Preservation of Limits*: Omitted.
??, *Symmetric Strict Monoidality With Respect to Unions*: This follows from **??**.
??, *Symmetric Oplax Monoidality With Respect to Intersections*: This follows from **??**.
??, *Relation to Direct Images With Compact Support*: Applying **??** of **??** to $A \setminus U$, we have

$$\begin{aligned} f_!(A \setminus U) &= B \setminus f_*(A \setminus (A \setminus U)) \\ &= B \setminus f_*(U). \end{aligned}$$

Taking complements, we then obtain

$$\begin{aligned} f_*(U) &= B \setminus (B \setminus f_*(U)), \\ &= B \setminus f_!(A \setminus U), \end{aligned}$$

which finishes the proof. □

Proposition 4.3.1.4. Let $f \colon A \to B$ be a function. 005Q

1. *Functionality I.* The assignment $f \mapsto f_*$ defines a function 005R

$$(-)_{*|A,B} \colon \mathsf{Sets}(A, B) \to \mathsf{Sets}(\mathcal{P}(A), \mathcal{P}(B)).$$

2. *Functionality II.* The assignment $f \mapsto f_*$ defines a function 005S

$$(-)_{*|A,B} \colon \mathsf{Sets}(A, B) \to \mathsf{Pos}((\mathcal{P}(A), \subset), (\mathcal{P}(B), \subset)).$$

3. *Interaction With Identities.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have 005T

$$(\mathrm{id}_A)_* = \mathrm{id}_{\mathcal{P}(A)};$$

4. *Interaction With Composition.* For each pair of composable functions 005U $f\colon A \to B$ and $g\colon B \to C$, we have

$$(g \circ f)_* = g_* \circ f_*, \qquad \begin{tikzcd} \mathcal{P}(A) \arrow[r, "f_*"] \arrow[dr, "(g\circ f)_*"'] & \mathcal{P}(B) \arrow[d, "g_*"] \\ & \mathcal{P}(C). \end{tikzcd}$$

Proof. **??**, *Functionality I*: Clear.
??, *Functionality II*: Clear.
??, *Interaction With Identities*: This follows from Kan Extensions, **??** of **??**.
??, *Interaction With Composition*: This follows from Kan Extensions, **??** of **??**. □

4.4 Inverse Images 005V

Let A and B be sets and let $f\colon A \to B$ be a function.

Definition 4.4.1.1. The **inverse image function associated to** f is the 005W function[32]

$$f^{-1}\colon \mathcal{P}(B) \to \mathcal{P}(A)$$

defined by[33]

$$f^{-1}(V) \overset{\text{def}}{=} \{a \in A \mid \text{we have } f(a) \in V\}$$

for each $V \in \mathcal{P}(B)$.

Remark 4.4.1.2. Identifying subsets of B with functions from B to $\{\mathsf{true}, \mathsf{false}\}$ 005X via **??** of **??**, we see that the inverse image function associated to f is equivalently the function

$$f^*\colon \mathcal{P}(B) \to \mathcal{P}(A)$$

defined by

$$f^*(\chi_V) \overset{\text{def}}{=} \chi_V \circ f$$

[32] *Further Notation:* Also written $f^*\colon \mathcal{P}(B) \to \mathcal{P}(A)$.
[33] *Further Terminology:* The set $f^{-1}(V)$ is called the **inverse image of** V **by** f.

for each $\chi_V \in \mathcal{P}(B)$, where $\chi_V \circ f$ is the composition

$$A \xrightarrow{f} B \xrightarrow{\chi_V} \{\mathsf{true}, \mathsf{false}\}$$

in Sets.

Proposition 4.4.1.3. Let $f\colon A \to B$ be a function.

1. *Functoriality.* The assignment $V \mapsto f^{-1}(V)$ defines a functor

$$f^{-1}\colon (\mathcal{P}(B), \subset) \to (\mathcal{P}(A), \subset)$$

where

- *Action on Objects.* For each $V \in \mathcal{P}(B)$, we have

$$[f^{-1}](V) \stackrel{\text{def}}{=} f^{-1}(V);$$

- *Action on Morphisms.* For each $U, V \in \mathcal{P}(B)$:
  ($\star$) If $U \subset V$, then $f^{-1}(U) \subset f^{-1}(V)$.

2. *Triple Adjointness.* We have a triple adjunction

$$\bigl(f_* \dashv f^{-1} \dashv f_!\bigr)\colon \quad \begin{tikzcd} \mathcal{P}(A) \arrow[r, bend left=50, "f_*"] \arrow[r, bend right=50, "f_!"'] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, phantom, shift right=3, "\perp"] & \mathcal{P}(B), \arrow[l, "f^{-1}" description] \end{tikzcd}$$

witnessed by bijections of sets

$$\begin{aligned} \mathrm{Hom}_{\mathcal{P}(B)}(f_*(U), V) &\cong \mathrm{Hom}_{\mathcal{P}(A)}\bigl(U, f^{-1}(V)\bigr), \\ \mathrm{Hom}_{\mathcal{P}(A)}\bigl(f^{-1}(U), V\bigr) &\cong \mathrm{Hom}_{\mathcal{P}(A)}(U, f_!(V)), \end{aligned}$$

natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$ and (respectively) $V \in \mathcal{P}(A)$ and $U \in \mathcal{P}(B)$, i.e. where:

   (a) The following conditions are equivalent:
      i. We have $f_*(U) \subset V$;
      ii. We have $U \subset f^{-1}(V)$;

   (b) The following conditions are equivalent:
      i. We have $f^{-1}(U) \subset V$.
      ii. We have $U \subset f_!(V)$.

3. *Preservation of Colimits.* We have an equality of sets 0061

$$f^{-1}\left(\bigcup_{i\in I} U_i\right) = \bigcup_{i\in I} f^{-1}(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(B)^{\times I}$. In particular, we have equalities

$$\begin{aligned} f^{-1}(U) \cup f^{-1}(V) &= f^{-1}(U \cup V), \\ f^{-1}(\emptyset) &= \emptyset, \end{aligned}$$

natural in $U, V \in \mathcal{P}(B)$.

4. *Preservation of Limits.* We have an equality of sets 0062

$$f^{-1}\left(\bigcap_{i\in I} U_i\right) = \bigcap_{i\in I} f^{-1}(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(B)^{\times I}$. In particular, we have equalities

$$\begin{aligned} f^{-1}(U) \cap f^{-1}(V) &= f^{-1}(U \cap V), \\ f^{-1}(B) &= A, \end{aligned}$$

natural in $U, V \in \mathcal{P}(B)$.

5. *Symmetric Strict Monoidality With Respect to Unions.* The inverse 0063
image function of **??** has a symmetric strict monoidal structure

$$\left(f^{-1}, f^{-1,\otimes}, f^{-1,\otimes}_{\mathbb{1}}\right) \colon (\mathcal{P}(B), \cup, \emptyset) \to (\mathcal{P}(A), \cup, \emptyset),$$

being equipped with equalities

$$\begin{aligned} f^{-1,\otimes}_{U,V} &\colon f^{-1}(U) \cup f^{-1}(V) \xrightarrow{=} f^{-1}(U \cup V), \\ f^{-1,\otimes}_{\mathbb{1}} &\colon \emptyset \xrightarrow{=} f^{-1}(\emptyset), \end{aligned}$$

natural in $U, V \in \mathcal{P}(B)$.

6. *Symmetric Strict Monoidality With Respect to Intersections.* The 0064
inverse image function of **??** has a symmetric strict monoidal structure

$$\left(f^{-1}, f^{-1,\otimes}, f^{-1,\otimes}_{\mathbb{1}}\right) \colon (\mathcal{P}(B), \cap, B) \to (\mathcal{P}(A), \cap, A),$$

being equipped with equalities

$$f^{-1,\otimes}_{U,V} \colon f^{-1}(U) \cap f^{-1}(V) \xrightarrow{=} f^{-1}(U \cap V),$$
$$f^{-1,\otimes}_{\nVdash} \colon A \xrightarrow{=} f^{-1}(B),$$

natural in $U, V \in \mathcal{P}(B)$.

Proof. **??**, *Functoriality*: Clear.
??, *Triple Adjointness*: This follows from Kan Extensions, **??** of **??**.
??, *Preservation of Colimits*: This follows from **??** and Categories, **??** of **??**.
??, *Preservation of Limits*: This follows from **??** and Categories, **??** of **??**.
??, *Symmetric Strict Monoidality With Respect to Unions*: This follows from **??**.
??, *Symmetric Strict Monoidality With Respect to Intersections*: This follows from **??**. □

Proposition 4.4.1.4. Let $f \colon A \to B$ be a function. 0065

1. *Functionality I.* The assignment $f \mapsto f^{-1}$ defines a function 0066

$$(-)^{-1}_{A,B} \colon \mathsf{Sets}(A, B) \to \mathsf{Sets}(\mathcal{P}(B), \mathcal{P}(A)).$$

2. *Functionality II.* The assignment $f \mapsto f^{-1}$ defines a function 0067

$$(-)^{-1}_{A,B} \colon \mathsf{Sets}(A, B) \to \mathsf{Pos}((\mathcal{P}(B), \subset), (\mathcal{P}(A), \subset)).$$

3. *Interaction With Identities.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have 0068

$$\mathrm{id}^{-1}_A = \mathrm{id}_{\mathcal{P}(A)};$$

4. *Interaction With Composition.* For each pair of composable functions $f \colon A \to B$ and $g \colon B \to C$, we have 0069

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}, \qquad \begin{tikzcd} \mathcal{P}(C) \arrow[r, "g^{-1}"] \arrow[dr, "(g\circ f)^{-1}"'] & \mathcal{P}(B) \arrow[d, "f^{-1}"] \\ & \mathcal{P}(A). \end{tikzcd}$$

Proof. **??**, *Functionality I*: Clear.
??, *Functionality II*: Clear.
??, *Interaction With Identities*: This follows from Categories, **??** of **??**.
??, *Interaction With Composition*: This follows from Categories, **??** of **??**. □

4.5 Direct Images 006A With Compact Support

Let A and B be sets and let $f\colon A \to B$ be a function.

Definition 4.5.1.1. 006B The **direct image with compact support function associated to** f is the function[34]

$$f_!\colon \mathcal{P}(A) \to \mathcal{P}(B)$$

defined by[35,36]

$$\begin{aligned} f_!(U) &\overset{\text{def}}{=} \left\{ b \in B \;\middle|\; \begin{array}{l} \text{for each } a \in A\text{, if we have} \\ f(a) = b\text{, then } a \in U \end{array} \right\} \\ &= \left\{ b \in B \;\middle|\; \text{we have } f^{-1}(b) \subset U \right\} \end{aligned}$$

for each $U \in \mathcal{P}(A)$.

Remark 4.5.1.2. 006C Identifying subsets of A with functions from A to $\{\mathsf{true}, \mathsf{false}\}$ via **??** of **??**, we see that the direct image with compact support function associated to f is equivalently the function

$$f_!\colon \mathcal{P}(A) \to \mathcal{P}(B)$$

defined by

$$\begin{aligned} f_!(\chi_U) &\overset{\text{def}}{=} \mathrm{Ran}_f(\chi_U) \\ &= \lim\left(\left(\underline{(-_1)} \overset{\rightarrow}{\times} f\right) \overset{\text{pr}}{\twoheadrightarrow} A \xrightarrow{\chi_U} \{\mathsf{true}, \mathsf{false}\}\right) \\ &= \lim_{\substack{a \in A \\ f(a) = -_1}} (\chi_U(a)) \\ &= \bigwedge_{\substack{a \in A \\ f(a) = -_1}} (\chi_U(a)). \end{aligned}$$

---

[34] *Further Notation:* Also written $\forall_f\colon \mathcal{P}(A) \to \mathcal{P}(B)$. This notation comes from the fact that the following statements are equivalent, where $b \in B$ and $U \in \mathcal{P}(A)$:

- We have $b \in \forall_f(U)$.
- For each $a \in A$, if $b = f(a)$, then $a \in U$.

[35] *Further Terminology:* The set $f_!(U)$ is called the **direct image with compact support of** U **by** f.

[36] We also have

$$f_!(U) = B \setminus f_*(A \setminus U);$$

So, in other words, we have

$$
\begin{aligned}
[f_!(\chi_U)](b) &= \bigwedge_{\substack{a\in A\\ f(a)=b}} (\chi_U(a))\\
&= \begin{cases} \mathsf{true} & \text{if, for each } a\in A \text{ such that}\\ & f(a)=b, \text{ we have } a\in U,\\ \mathsf{false} & \text{otherwise}\end{cases}\\
&= \begin{cases} \mathsf{true} & \text{if } f^{-1}(b)\subset U\\ \mathsf{false} & \text{otherwise}\end{cases}
\end{aligned}
$$

for each $b \in B$.

Definition 4.5.1.3. Let U be a subset of A. 004D[37,38]

1. The **image part of the direct image with compact support** $f_!(U)$ **of** U is the set $f_{!,\mathrm{im}}(U)$ defined by

$$
\begin{aligned}
f_{!,\mathrm{im}}(U) &\overset{\text{def}}{=} f_!(U)\cap \mathrm{Im}(f)\\
&= \left\{ b\in B \;\middle|\; \begin{array}{l}\text{we have } f^{-1}(b)\subset\\ U \text{ and } f^{-1}(b)\neq\emptyset\end{array}\right\}.
\end{aligned}
$$

2. The **complement part of the direct image with compact support**

see **??** of **??**.

[37]Note that we have

$$f_!(U) = f_{!,\mathrm{im}}(U)\cup f_{!,\mathrm{cp}}(U),$$

as

$$
\begin{aligned}
f_!(U) &= f_!(U)\cap B\\
&= f_!(U)\cap(\mathrm{Im}(f)\cup(B\setminus \mathrm{Im}(f)))\\
&= (f_!(U)\cap \mathrm{Im}(f))\cup(f_!(U)\cap(B\setminus\mathrm{Im}(f)))\\
&\overset{\text{def}}{=} f_{!,\mathrm{im}}(U)\cup f_{!,\mathrm{cp}}(U).
\end{aligned}
$$

[38]In terms of the meet computation of $f_!(U)$ of **??**, namely

$$f_!(\chi_U) = \bigwedge_{\substack{a\in A\\ f(a)=-_1}} (\chi_U(a)),$$

we see that $f_{!,\mathrm{im}}$ corresponds to meets indexed over nonempty sets, while $f_{!,\mathrm{cp}}$ corresponds to meets indexed over the empty set.

$f_!(U)$ **of** U is the set $f_{!,\mathrm{cp}}(U)$ defined by

$$\begin{aligned} f_{!,\mathrm{cp}}(U) &\overset{\mathrm{def}}{=} f_!(U) \cap (B \setminus \mathrm{Im}(f)) \\ &= B \setminus \mathrm{Im}(f) \\ &= \left\{ b \in B \;\middle|\; \begin{array}{l} \text{we have } f^{-1}(b) \subset \\ U \text{ and } f^{-1}(b) = \emptyset \end{array} \right\} \\ &= \left\{ b \in B \;\middle|\; f^{-1}(b) = \emptyset \right\}. \end{aligned}$$

Example 4.5.1.4. Here are some examples of direct images with compact support.

1. *The Multiplication by Two Map on the Natural Numbers.* Consider the function $f\colon \mathbb{N} \to \mathbb{N}$ given by

$$f(n) \overset{\mathrm{def}}{=} 2n$$

for each $n \in \mathbb{N}$. Since f is injective, we have

$$\begin{aligned} f_{!,\mathrm{im}}(U) &= f_*(U) \\ f_{!,\mathrm{cp}}(U) &= \{\text{odd natural numbers}\} \end{aligned}$$

for any $U \subset \mathbb{N}$.

2. *Parabolas.* Consider the function $f\colon \mathbb{R} \to \mathbb{R}$ given by

$$f(x) \overset{\mathrm{def}}{=} x^2$$

for each $x \in \mathbb{R}$. We have

$$f_{!,\mathrm{cp}}(U) = \mathbb{R}_{<0}$$

for any $U \subset \mathbb{R}$. Moreover, since $f^{-1}(x) = \{-\sqrt{x}, \sqrt{x}\}$, we have e.g.:

$$\begin{aligned} f_{!,\mathrm{im}}([0,1]) &= \{0\}, \\ f_{!,\mathrm{im}}([-1,1]) &= [0,1], \\ f_{!,\mathrm{im}}([1,2]) &= \emptyset, \\ f_{!,\mathrm{im}}([-2,-1] \cup [1,2]) &= [1,4]. \end{aligned}$$

3. *Circles.* Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x,y) \overset{\mathrm{def}}{=} x^2 + y^2$$

for each $(x, y) \in \mathbb{R}^2$. We have

$$f_{!,\mathrm{cp}}(U) = \mathbb{R}_{<0}$$

for any $U \subset \mathbb{R}^2$, and since

$$f^{-1}(r) = \begin{cases} \text{a circle of radius } r \text{ about the origin} & \text{if } r > 0, \\ \{(0,0)\} & \text{if } r = 0, \\ \emptyset & \text{if } r < 0, \end{cases}$$

we have e.g.:

$$\begin{aligned} f_{!,\mathrm{im}}([-1,1] \times [-1,1]) &= [0,1], \\ f_{!,\mathrm{im}}(([-1,1] \times [-1,1]) \setminus [-1,1] \times \{0\}) &= \emptyset. \end{aligned}$$

Proposition 4.5.1.5. Let $f\colon A \to B$ be a function.

1. *Functoriality.* The assignment $U \mapsto f_!(U)$ defines a functor

$$f_!\colon (\mathcal{P}(A), \subset) \to (\mathcal{P}(B), \subset)$$

where

- *Action on Objects.* For each $U \in \mathcal{P}(A)$, we have

$$[f_!](U) \stackrel{\text{def}}{=} f_!(U);$$

- *Action on Morphisms.* For each $U, V \in \mathcal{P}(A)$:
  ($\star$) If $U \subset V$, then $f_!(U) \subset f_!(V)$.

2. *Triple Adjointness.* We have a triple adjunction

$$\left(f_* \dashv f^{-1} \dashv f_!\right)\colon \quad \begin{tikzcd} \mathcal{P}(A) \arrow[r, bend left=50, "f_*"] \arrow[r, bend right=50, "f_!"'] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, phantom, shift right=3, "\perp"] & \mathcal{P}(B), \arrow[l, "f^{-1}"'] \end{tikzcd}$$

witnessed by bijections of sets

$$\begin{aligned} \mathrm{Hom}_{\mathcal{P}(B)}(f_*(U), V) &\cong \mathrm{Hom}_{\mathcal{P}(A)}\left(U, f^{-1}(V)\right), \\ \mathrm{Hom}_{\mathcal{P}(A)}\left(f^{-1}(U), V\right) &\cong \mathrm{Hom}_{\mathcal{P}(A)}(U, f_!(V)), \end{aligned}$$

natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$ and (respectively) $V \in \mathcal{P}(A)$ and $U \in \mathcal{P}(B)$, i.e. where:

(a) The following conditions are equivalent:
   i. We have $f_*(U) \subset V$;
   ii. We have $U \subset f^{-1}(V)$;

(b) The following conditions are equivalent:
   i. We have $f^{-1}(U) \subset V$.
   ii. We have $U \subset f_!(V)$.

3. *Lax Preservation of Colimits.* We have an inclusion of sets 006J

$$\bigcup_{i\in I} f_!(U_i) \subset f_!\left(\bigcup_{i\in I} U_i\right),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have inclusions

$$f_!(U) \cup f_!(V) \hookrightarrow f_!(U \cup V),$$
$$\emptyset \hookrightarrow f_!(\emptyset),$$

natural in $U, V \in \mathcal{P}(A)$.

4. *Preservation of Limits.* We have an equality of sets 006K

$$f_!\left(\bigcap_{i\in I} U_i\right) = \bigcap_{i\in I} f_!(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have equalities

$$f^{-1}(U \cap V) = f_!(U) \cap f^{-1}(V),$$
$$f_!(A) = B,$$

natural in $U, V \in \mathcal{P}(A)$.

5. *Symmetric Lax Monoidality With Respect to Unions.* The direct image 006L
with compact support function of **??** has a symmetric lax monoidal structure

$$\left(f_!, f_!^{\otimes}, f_{!|\mathbb{1}}^{\otimes}\right)\colon (\mathcal{P}(A), \cup, \emptyset) \to (\mathcal{P}(B), \cup, \emptyset),$$

being equipped with inclusions

$$f_{!|U,V}^{\otimes}\colon f_!(U) \cup f_!(V) \hookrightarrow f_!(U \cup V),$$
$$f_{!|\mathbb{1}}^{\otimes}\colon \emptyset \hookrightarrow f_!(\emptyset),$$

natural in $U, V \in \mathcal{P}(A)$.

6. *Symmetric Strict Monoidality With Respect to Intersections.* The direct image function of **??** has a symmetric strict monoidal structure 006M

$$\left(f_!, f_!^{\otimes}, f_{!|\mathbb{1}}^{\otimes}\right) \colon (\mathcal{P}(A), \cap, A) \to (\mathcal{P}(B), \cap, B),$$

being equipped with equalities

$$f_{!|U,V}^{\otimes} \colon f_!(U \cap V) \xrightarrow{=} f_!(U) \cap f_!(V),$$
$$f_{!|\mathbb{1}}^{\otimes} \colon f_!(A) \xrightarrow{=} B,$$

natural in $U, V \in \mathcal{P}(A)$.

7. *Relation to Direct Images.* We have 006N

$$f_!(U) = B \setminus f_*(A \setminus U)$$

for each $U \in \mathcal{P}(A)$.

8. *Interaction With Injections.* If f is injective, then we have 006P

$$\begin{aligned} f_{!,\mathrm{im}}(U) &= f_*(U), \\ f_{!,\mathrm{cp}}(U) &= B \setminus \mathrm{Im}(f), \\ f_!(U) &= f_{!,\mathrm{im}}(U) \cup f_{!,\mathrm{cp}}(U) \\ &= f_*(U) \cup (B \setminus \mathrm{Im}(f)) \end{aligned}$$

for each $U \in \mathcal{P}(A)$.

9. *Interaction With Surjections.* If f is surjective, then we have 006Q

$$\begin{aligned} f_{!,\mathrm{im}}(U) &\subset f_*(U), \\ f_{!,\mathrm{cp}}(U) &= \emptyset, \\ f_!(U) &\subset f_*(U) \end{aligned}$$

for each $U \in \mathcal{P}(A)$.

Proof. **??**, *Functoriality*: Clear.
??, *Triple Adjointness*: This follows from Kan Extensions, **??** of **??**.
??, *Lax Preservation of Colimits*: Omitted.
??, *Preservation of Limits*: Omitted. This follows from **??** and Categories, **??** of **??**.
??, *Symmetric Lax Monoidality With Respect to Unions*: This follows from **??**.
??, *Symmetric Strict Monoidality With Respect to Intersections*: This follows from **??**.
??, *Relation to Direct Images*: We claim that $f_!(U) = B \setminus f_*(A \setminus U)$.

- *The First Implication.* We claim that

$$f_!(U) \subset B \setminus f_*(A \setminus U).$$

Let $b \in f_!(U)$. We need to show that $b \notin f_*(A \setminus U)$, i.e. that there is no $a \in A \setminus U$ such that $f(a) = b$.

This is indeed the case, as otherwise we would have $a \in f^{-1}(b)$ and $a \notin U$, contradicting $f^{-1}(b) \subset U$ (which holds since $b \in f_!(U)$).

Thus $b \in B \setminus f_*(A \setminus U)$.

- *The Second Implication.* We claim that

$$B \setminus f_*(A \setminus U) \subset f_!(U).$$

Let $b \in B \setminus f_*(A \setminus U)$. We need to show that $b \in f_!(U)$, i.e. that $f^{-1}(b) \subset U$.

Since $b \notin f_*(A \setminus U)$, there exists no $a \in A \setminus U$ such that $b = f(a)$, and hence $f^{-1}(b) \subset U$.

Thus $b \in f_!(U)$.

This finishes the proof of **??**.
??, *Interaction With Injections*: Clear.
??, *Interaction With Surjections*: Clear. □

Proposition 4.5.1.6. Let $f\colon A \to B$ be a function. 006R

1. *Functionality I.* The assignment $f \mapsto f_!$ defines a function 006S

$$(-)_{!|A,B}\colon \mathsf{Sets}(A, B) \to \mathsf{Sets}(\mathcal{P}(A), \mathcal{P}(B)).$$

2. *Functionality II.* The assignment $f \mapsto f_!$ defines a function 006T

$$(-)_{!|A,B}\colon \mathsf{Sets}(A, B) \to \mathsf{Pos}((\mathcal{P}(A), \subset), (\mathcal{P}(B), \subset)).$$

3. *Interaction With Identities.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have 006U

$$(\mathrm{id}_A)_! = \mathrm{id}_{\mathcal{P}(A)};$$

4. *Interaction With Composition.* For each pair of composable functions $f\colon A \to B$ and $g\colon B \to C$, we have 006V

$$(g \circ f)_! = g_! \circ f_!, \qquad \begin{tikzcd} \mathcal{P}(A) \arrow[r, "f_!"] \arrow[dr, "(g\circ f)_!"'] & \mathcal{P}(B) \arrow[d, "g_!"] \\ & \mathcal{P}(C). \end{tikzcd}$$

Proof. **??**, *Functionality I*: Clear.
??, *Functionality II*: Clear.
??, *Interaction With Identities*: This follows from Kan Extensions, **??** of **??**.
??, *Interaction With Composition*: This follows from Kan Extensions, **??** of **??**. □

Appendices

A Other Chapters