Adjunctions and the Yoneda Lemma

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1 Adjunctions

1.1 Foundations

Let $\mathcal{C}$ and $\mathcal{D}$ be two categories.

Definition 1.1.1.1. An **adjunction**[1] is a quadruple (F, G, η, ϵ) consisting of

1. A functor $F \colon \mathcal{C} \to \mathcal{D}$;
2. A functor $G \colon \mathcal{D} \to \mathcal{C}$;
3. A natural transformation $\eta \colon \mathrm{id}_{\mathcal{C}} \Longrightarrow G \circ F$;
4. A natural transformation $\epsilon \colon F \circ G \Longrightarrow \mathrm{id}_{\mathcal{D}}$;

[1] *Further Terminology:* We also call (G, F) an **adjoint pair**, F a **left adjoint**, G a **right adjoint**, η the **unit** of the adjunction, and ϵ the **counit** of the adjunction.

such that we have equalities

$$
\begin{tikzcd}
 & \mathcal{D} \arrow[r, "\mathrm{id}_{\mathcal{D}}"] \arrow[d, "G"] \arrow[dr, phantom, "\Uparrow \epsilon"] & \mathcal{D} \\
\mathcal{C} \arrow[ur, "F"] \arrow[r, "\mathrm{id}_{\mathcal{C}}"'] \arrow[ur, phantom, shift left=4, "\Uparrow \eta"] & \mathcal{C} \arrow[ur, "F"'] &
\end{tikzcd}
=
\begin{tikzcd}
 & \mathcal{D} \arrow[r, "\mathrm{id}_{\mathcal{D}}"] & \mathcal{D} \\
\mathcal{C} \arrow[ur, "F"] \arrow[r, "\mathrm{id}_{\mathcal{C}}"'] & \mathcal{C} \arrow[ur, "F"'] \arrow[u, phantom, "\Leftarrow \mathrm{id}_F"] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & \mathcal{C} \arrow[r, "\mathrm{id}_{\mathcal{C}}"] \arrow[d, "F"] \arrow[dr, phantom, "\Downarrow \eta"] & \mathcal{C} \\
\mathcal{D} \arrow[ur, "G"] \arrow[r, "\mathrm{id}_{\mathcal{D}}"'] \arrow[ur, phantom, shift left=4, "\Downarrow \epsilon"] & \mathcal{D} \arrow[ur, "G"'] &
\end{tikzcd}
=
\begin{tikzcd}
 & \mathcal{C} \arrow[r, "\mathrm{id}_{\mathcal{C}}"] \arrow[d, phantom, "\Rightarrow \mathrm{id}_G"] & \mathcal{C} \\
\mathcal{D} \arrow[ur, "G"] \arrow[r, "\mathrm{id}_{\mathcal{D}}"'] & \mathcal{D} \arrow[ur, "G"'] &
\end{tikzcd}
$$

of pasting diagrams in Cats_2.[2]

Example 1.1.1.2. Here are some examples of adjunctions.

1. We have a triple adjunction

$$
(\lceil - \rceil \dashv \iota \dashv \lfloor - \rfloor)\colon \quad
\begin{tikzcd}
\mathbb{R} \arrow[r, bend left=45, "\lceil - \rceil"] \arrow[r, bend right=45, "\lfloor - \rfloor"'] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, phantom, shift right=3, "\perp"] & \mathbb{Z}, \arrow[l, hook', "\iota"']
\end{tikzcd}
$$

[2]Equivalently, the diagrams

$$
\begin{tikzcd}
F \arrow[r, Rightarrow, "\mathrm{id}_F \circ \eta"] \arrow[dr, Rightarrow, "\mathrm{id}_F"'] & F \circ G \circ F \arrow[d, Rightarrow, "\epsilon \circ \mathrm{id}_F"] \\
 & F
\end{tikzcd}
\qquad
\begin{tikzcd}
G \arrow[r, Rightarrow, "\eta \circ \mathrm{id}_G"] \arrow[dr, Rightarrow, "\mathrm{id}_G"'] & G \circ F \circ G \arrow[d, Rightarrow, "\mathrm{id}_G \circ \epsilon"] \\
 & G,
\end{tikzcd}
\tag{1.1.1.1}
$$

called the **left** and **right triangle identities**, commute, or, again equivalently, for each $A \in \mathrm{Obj}(\mathcal{C})$ and each $B \in \mathrm{Obj}(\mathcal{D})$, the diagrams

$$
\begin{tikzcd}
F_A \arrow[r, "F_{\eta_A}"] \arrow[dr, "\mathrm{id}_{F_A}"'] & F_{G_{F_A}} \arrow[d, "\epsilon_{F_A}"] \\
 & F_A
\end{tikzcd}
\qquad
\begin{tikzcd}
G_B \arrow[r, "\eta_{G_B}"] \arrow[dr, "\mathrm{id}_{G_B}"'] & G_{F_{G_B}} \arrow[d, "G_{\epsilon_B}"] \\
 & G_B
\end{tikzcd}
$$

commute.

where $\mathbb{Z}$ and $\mathbb{R}$ are viewed as poset categories and $\iota\colon \mathbb{Z} \hookrightarrow \mathbb{R}$ is the canonical inclusion.

Proposition 1.1.1.3. Let $F, L\colon \mathcal{C} \rightrightarrows \mathcal{D}$ and $G, R\colon \mathcal{D} \rightrightarrows \mathcal{C}$ be functors.

1. *Characterisations.* The following conditions are equivalent:

   (a) The pair (L, R) is an adjoint pair.

   (b) We have a natural isomorphism of (pro)functors[3]

   $$\mathsf{h}^L \cong \mathsf{h}_R.$$

   (c) For each $A \in \mathrm{Obj}(\mathcal{C})$ and each $B \in \mathrm{Obj}(\mathcal{D})$, we have an isomorphism

   $$\mathrm{Hom}_{\mathcal{D}}(L_A, B) \cong \mathrm{Hom}_{\mathcal{C}}(A, R_B)$$

   and the square below-left commutes iff the square below-right

[3] That is, the following conditions are satisfied:

1. *Bijection.* For each $A \in \mathrm{Obj}(\mathcal{C})$ and each $B \in \mathrm{Obj}(\mathcal{D})$, we have a bijection

$$\mathrm{Hom}_{\mathcal{D}}(L_A, B) \cong \mathrm{Hom}_{\mathcal{C}}(A, R_B).$$

2. *Naturality in $\mathcal{D}$.* For each morphism $g\colon B \to B'$ of $\mathcal{D}$, the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}}(L_A, B) \arrow[r, dashed, "\sim"] \arrow[d, "h_g^{\mathrm{id}_{L_A}}"'] & \mathrm{Hom}_{\mathcal{C}}(A, R_B) \arrow[d, "h^{\mathrm{id}_A}_{R_g}"] \\
\mathrm{Hom}_{\mathcal{D}}(L_A, B') \arrow[r, dashed, "\sim"] & \mathrm{Hom}_{\mathcal{C}}(A, R_{B'})
\end{tikzcd}$$

commutes.

3. *Naturality in $\mathcal{C}$.* For each morphism $f\colon A \to A'$ of $\mathcal{C}$, the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}}(L_A, B) \arrow[r, dashed, "\sim"] \arrow[d, "h^{L_f}_{\mathrm{id}_B}"'] & \mathrm{Hom}_{\mathcal{C}}(A, R_B) \arrow[d, "h^{f}_{\mathrm{id}_{R_B}}"] \\
\mathrm{Hom}_{\mathcal{D}}(L_{A'}, B) \arrow[r, dashed, "\sim"] & \mathrm{Hom}_{\mathcal{C}}(A', R_B)
\end{tikzcd}$$

commutes.

commutes:

$$\begin{tikzcd} L_A \arrow[r, "f^\sharp"] \arrow[d, "L_\phi"'] & B \arrow[d, "\psi"] \\ L_{A'} \arrow[r, "g^\sharp"'] & B' \end{tikzcd} \iff \begin{tikzcd} A \arrow[r, "f^\flat"] \arrow[d, "\phi"'] & R_B \arrow[d, "R_\psi"] \\ A' \arrow[r, "g^\flat"'] & R_{B'}. \end{tikzcd}$$

(d) For each small category $\mathcal{K}$, we have an adjunction

$$(L_* \dashv R_*)\colon \quad \begin{tikzcd} \mathsf{Fun}(\mathcal{K}, \mathcal{C}) \arrow[r, bend left, "L_*"] \arrow[r, phantom, "\perp"] & \mathsf{Fun}(\mathcal{K}, \mathcal{D}) \arrow[l, bend left, "R_*"] \end{tikzcd}$$

as witnessed by a natural isomorphism

$$\mathrm{Nat}(L \circ F, G) \cong \mathrm{Nat}(F, R \circ G)$$

$$\begin{tikzcd} & \mathcal{C} \arrow[dr, "L"] \arrow[d, phantom, "\Downarrow"] & \\ \mathcal{K} \arrow[ur, "F"] \arrow[rr, bend right, "G"'] & {} & \mathcal{D} \end{tikzcd} \quad \overset{\text{bij.}}{\leftrightarrow} \quad \begin{tikzcd} \mathcal{K} \arrow[rr, bend left, "F"] \arrow[dr, "G"'] & {} \arrow[d, phantom, "\Downarrow"] & \mathcal{C} \\ & \mathcal{D} \arrow[ur, "R"'] & \end{tikzcd}$$

natural in $\mathcal{K} \xrightarrow{F} \mathcal{C}$ and $\mathcal{K} \xrightarrow{G} \mathcal{D}$.

(e) For each locally small category $\mathcal{E}$, we have an adjunction

$$(R^* \dashv L^*)\colon \quad \begin{tikzcd} \mathsf{Fun}(\mathcal{C}, \mathcal{E}) \arrow[r, bend left, "R^*"] \arrow[r, phantom, "\perp"] & \mathsf{Fun}(\mathcal{D}, \mathcal{E}) \arrow[l, bend left, "L^*"] \end{tikzcd}$$

as witnessed by a natural isomorphism

$$\mathrm{Nat}(F \circ R, G) \cong \mathrm{Nat}(F, G \circ L)$$

$$\begin{tikzcd} & \mathcal{C} \arrow[dr, "F"] \arrow[d, phantom, "\Downarrow"] & \\ \mathcal{D} \arrow[ur, "R"] \arrow[rr, bend right, "G"'] & {} & \mathcal{E} \end{tikzcd} \quad \overset{\text{bij.}}{\leftrightarrow} \quad \begin{tikzcd} \mathcal{C} \arrow[rr, bend left, "F"] \arrow[dr, "L"'] & {} \arrow[d, phantom, "\Downarrow"] & \mathcal{E} \\ & \mathcal{D} \arrow[ur, "G"'] & \end{tikzcd}$$

natural in $\mathcal{C} \xrightarrow{F} \mathcal{E}$ and $\mathcal{D} \xrightarrow{G} \mathcal{E}$.

4. *Uniqueness.* If G admits left/right adjoints F_1 and F_2, then $F_1 \cong F_2$.[4]
5. *Stability Under Composition.* If $F_1 \dashv G_1$ and $F_2 \dashv G_2$, then $(F_2 \circ F_1) \dashv (G_2 \circ G_1)$:

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F_1"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "G_1"] \arrow[r, bend left, "F_2"] \arrow[r, phantom, "\perp"] & \mathcal{E} \arrow[l, bend left, "G_2"] \end{tikzcd} \rightsquigarrow \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F_2 \circ F_1"] \arrow[r, phantom, "\perp"] & \mathcal{E} \arrow[l, bend left, "G_2 \circ G_1"] \end{tikzcd}$$

6. *Interaction With Co/Limits.* The following statements are true:
   (a) **Left Adjoints Preserve Colimits (LAPC).** If F is a left adjoint, then F preserves all colimits that exist in $\mathcal{C}$.
   (b) **Right Adjoints Preserve Limits (RAPL).** If G is a right adjoint, then G preserves all limits that exist in $\mathcal{C}$.
7. *Interaction With Faithfulness.* Let (F, G, η, ϵ) be an adjunction. The following conditions are equivalent:
   (a) The functor F is faithful.
   (b) For each $A \in \mathrm{Obj}(\mathcal{C})$, the morphism

$$\eta_A : A \to G_{F_A}$$

   is a monomorphism.

   Dually, the following conditions are equivalent:
   (a) The functor G is faithful.
   (b) For each $A \in \mathrm{Obj}(\mathcal{C})$, the morphism

$$\epsilon_A : F_{G_A} \to A$$

   is an epimorphism.

[4]Moreover, writing $\theta \colon F_1 \overset{\cong}{\Longrightarrow} F_2$ for this isomorphism, the diagrams

$$\begin{tikzcd} \mathrm{id}_{\mathcal{C}} \arrow[r, Rightarrow, "\eta"] \arrow[dr, Rightarrow, "\eta'"'] & G \circ F \arrow[d, Rightarrow, "\mathrm{id}_G \circ \theta"] \\ & G \circ F' \end{tikzcd} \qquad \begin{tikzcd} F \circ G \arrow[r, Rightarrow, "\epsilon"] \arrow[d, Rightarrow, "\theta \circ \mathrm{id}_G"'] & \mathrm{id}_{\mathcal{D}} \\ F' \circ G \arrow[ur, Rightarrow, "\epsilon'"'] & \end{tikzcd}$$

commute; see [**riehl:context**].

8. *Interaction With Fullness.* Let (F, G, η, ϵ) be an adjunction. The following conditions are equivalent:

   (a) The functor F is full.

   (b) For each $A \in \mathrm{Obj}(\mathcal{C})$, the morphism
   $$\eta_A \colon A \to G_{F_A}$$
   is a split epimorphism.

   Dually, the following conditions are equivalent:

   (a) The functor G is full.

   (b) For each $A \in \mathrm{Obj}(\mathcal{C})$, the morphism
   $$\epsilon_A \colon F_{G_A} \to A$$
   is a split monomorphism.

9. *Interaction With Fully Faithfulness I.* Let (F, G, η, ϵ) be an adjunction. The following conditions are equivalent:

   (a) The functor F is fully faithful.

   (b) For each $A \in \mathrm{Obj}(\mathcal{C})$, the morphism
   $$\eta_A \colon A \to G_{F_A}$$
   is an isomorphism.

   (c) The following conditions are satisfied:

      i. The natural transformation
      $$\mathrm{id}_F \circ \eta \circ \mathrm{id}_G \colon F \circ G \Longrightarrow F \circ G \circ F \circ G$$
      is a natural isomorphism.

      ii. The functor F is conservative.

      iii. The functor G is essentially surjective.

   Dually, the following conditions are equivalent:

   (a) The functor G is fully faithful.

   (b) For each $A \in \mathrm{Obj}(\mathcal{C})$, the morphism
   $$\epsilon_A \colon F_{G_A} \to A$$
   is an isomorphism.

(c) The following conditions are satisfied:

i. The natural transformation

$$\mathrm{id}_G \circ \eta \circ \mathrm{id}_F \colon G \circ F \Longrightarrow G \circ F \circ G \circ F$$

is a natural isomorphism.

ii. The functor G is conservative.

iii. The functor F is essentially surjective.

10. *Interaction With Fully Faithfulness II.* Let (F, G, η, ϵ) be an adjunction.

(a) If $G \circ F$ is fully faithful, then so is F.

(b) If $F \circ G$ is fully faithful, then so is G.

Proof. **??**, *Adjunctions Via* Hom*-Functors*: See [**riehl:context**].
??, *Uniqueness of Adjoints*: This follows from the Yoneda lemma (**??**) and its dual (**??**).
??, *Stability Under Composition*: See [**riehl:context**].
??*: Interaction With Limits and Colimits,* **??**: [5]We prove **??** only, as **??** follows by duality (Limits and Colimits, **??** of **??**). Indeed, let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor admitting a right adjoint $G\colon \mathcal{D} \to \mathcal{C}$. For each $Y \in \mathrm{Obj}(\mathcal{D})$, we have isomorphisms

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{D}}\Big(F_{\mathrm{colim}(D)}, Y\Big) &\cong \mathrm{Hom}_{\mathcal{D}}(\mathrm{colim}(D), G_Y) \\
&\cong \lim(\mathrm{Hom}_{\mathcal{D}}(D, G_Y)) && \text{(Limits and Colimits, ?? of ??)} \\
&\cong \lim(\mathrm{Hom}_{\mathcal{D}}(F_D, Y)) \\
&\cong \mathrm{Hom}_{\mathcal{D}}(\mathrm{colim}(F_D), Y), && \text{(Limits and Colimits, ?? of ??)}
\end{aligned}$$

natural in $Y \in \mathrm{Obj}(\mathcal{D})$. The result then follows from Categories, **??**.
??*: Interaction With Limits and Colimits,* **??**: This is dual to **??**.
??, *Interaction With Faithfulness*: See [**riehl:context**].
??, *Interaction With Fullness*: See [**riehl:context**].
??, *Interaction With Fully Faithfulness I*: See [**riehl:context**] and [**loregian2020coend**].
??, *Interaction With Fully Faithfulness II*: See [**stacks-project**], [**loregian2020coend**], or [**low:homotopical-algebra**]. □

[5] *Reference:* See [**riehl:context**].

1.2 Existence Criteria for Adjoint Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Theorem 1.2.1.1. Let $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ be functors.

1. *Via Comma Categories.* The following conditions are equivalent:
   (a) The functor F has a right adjoint.
   (b) For each $s \in \mathrm{Obj}(\mathcal{D})$, the comma category $F \downarrow s \cong \int_{\mathcal{C}}[h_s^{F_-}]$ has a terminal object.

   Dually, the following conditions are equivalent:
   (a) The functor G has a left adjoint F.
   (b) For each $s \in \mathrm{Obj}(\mathcal{C})$, the comma category $s \downarrow G \cong \int^{\mathcal{C}}[h^s_{G_-}]$ has an initial object.

   Moreover, when these conditions are satisfied, we have isomorphisms
   $$F_A \cong \lim_{A \to G_x}(x),$$
   $$G_B \cong \operatorname*{colim}_{F_x \to G_B}(x),$$
   natural in $A \in \mathrm{Obj}(\mathcal{C})$ and $B \in \mathrm{Obj}(\mathcal{D})$.

2. *The General Adjoint Functor Theorem*[6]. Suppose that
   (a) The category $\mathcal{D}$ has all limits and F commutes with them.
   (b) The category $\mathcal{C}$ is complete and locally small.
   (c) *The Solution Set Condition.* For each $X \in \mathrm{Obj}(\mathcal{D})$, there exist
      i. A small set I;
      ii. A set $\{A_i\}_{i\in I}$ of objects of $\mathcal{C}$;
      iii. A set $\{f_i\colon X \to G_{A_i}\}$ of morphisms of $\mathcal{D}$;

      such that, for each $i \in I$ and each morphism $f\colon X \to G_A$, there exists a morphism $\phi_i\colon A_i \to A$ of $\mathcal{C}$ together with a factorisation
      $$\begin{tikzcd} X \arrow[r, "f_i"] \arrow[rr, bend right, "f"'] & G_{A_i} \arrow[r, "G_{\phi_i}"] & G_A. \end{tikzcd}$$

[6] *Further Terminology:* Also called **Freyd's adjoint functor theorem**.

Then F has a left adjoint.

3. *The Special Adjoint Functor Theorem.* Suppose that
   (a) The category $\mathcal{D}$ has all limits and F commutes with them.
   (b) The category $\mathcal{C}$ is complete, locally small, and well-powered.
   (c) The category $\mathcal{C}$ has a small cogenerating set.

   Then F has a left adjoint.

4. *Freyd's Representability Theorem I.* Let $F\colon \mathcal{C} \to \mathsf{Sets}$ be a functor. If[7]
   (a) The functor F commutes with limits;
   (b) The category $\mathcal{C}$ is complete and locally small;
   (c) *The Solution Set Condition.* There exists a set $\Phi \subset \mathrm{Obj}(\mathcal{C})$ such that, for each $c \in \mathrm{Obj}(\mathcal{C})$, there exist
       - $s \in \Phi$;
       - $y \in F_s$;
       - $f\colon s \to c$ in $\mathrm{Hom}_{\mathsf{Sets}}(s, c)$;

       such that $F_{f(y)} = x$;

   then F is representable.

5. *Freyd's Representability Theorem II*[8]. Let $F\colon \mathcal{C} \to \mathsf{Sets}$ be a functor. If
   (a) The functor F commutes with limits;
   (b) There exist
       - A collection $\{x_\alpha\}_{\alpha\in I}$ of object of $\mathcal{C}$;
       - For each $\alpha \in I$, an element f_α of F_{x_α}

       such that for each $y \in \mathrm{Obj}(\mathcal{C})$ and each $g \in F_y$, there exists some $\alpha \in I$ and some morphism $\phi\colon x_i \to y$ such that $F_\phi(f_\alpha) = g$;

   then F is representable.

6. *Co/Totality.* Suppose that

[7] A nice application of this theorem is given in [**MSE276630**], where it is used to abstractly show that Cats is cocomplete, avoiding the explicit construction of coequalisers in Cats given in **??**.

[8] This is the statement of Freyd's representability theorem as found in [**stacks-project**].

(a) The category $\mathcal{C}$ is locally small and cototal and $\mathcal{D}$ is locally small.

Proof. **??**, *Via Comma Categories*: We claim that **????** are indeed equivalent:[9]

- **??** $\implies$ **??**: Let F be a left adjoint of G. Then

$$\begin{aligned} s \downarrow G &\cong \textstyle\int^{\mathcal{C}}[h^{s}_{G_{-}}] \\ &\cong \textstyle\int^{\mathcal{C}}[h^{F_s}_{-}], \end{aligned}$$

  where $h^{s}_{G_{-}}$ is corepresentable by F_s. By Fibred Categories, **??** of **??**, it follows that the component $\eta_s \colon s \to G_{F_s}$ of the unit of the adjunction $F \dashv G$ at s is an initial object of $s \downarrow G$.

- **??** $\implies$ **??**: For each $s \in \mathrm{Obj}(\mathcal{D})$, write $\eta_s \colon s \to G_{F_s}$ for an initial object of $s \downarrow G$. This gives us a map of sets

$$\begin{aligned} F \colon \mathrm{Obj}(\mathcal{C}) &\longrightarrow \mathrm{Obj}(\mathcal{D}) \\ s &\longmapsto F_s. \end{aligned}$$

  We now extend this map to a functor: given a morphism $f \colon s \to s'$ of $\mathcal{C}$, we define $F_f \colon F_s \to F_{s'}$ to be the unique morphism making the diagram

$$\begin{tikzcd} s \arrow[r, "f"] \arrow[d, "\eta_s"'] & s' \arrow[d, "\eta_{s'}"] \\ G_{F_s} \arrow[r, dashed, "G_{F_f}"'] & G_{F_{s'}} \end{tikzcd}$$

  commute (which exists by the initiality of η_s). By the uniqueness of these morphisms, it follows that the assignment $s \mapsto F_s$ is indeed functorial. Moreover, we also obtain a natural transformation $\eta \colon \mathrm{id}_{\mathcal{C}} \implies G \circ F$. We now define a natural transformation

$$\phi \colon \mathrm{Hom}_{\mathcal{D}}(F_{-}, b) \implies \mathrm{Hom}_{\mathcal{C}}(-, G_b)$$

  consisting of the collection

$$\{\phi_{s,b} \colon \mathrm{Hom}_{\mathcal{D}}(F_s, b) \implies \mathrm{Hom}_{\mathcal{C}}(s, G_b)\}_{s \in \mathrm{Obj}(\mathcal{C})},$$

[9] *Reference:* [**riehl:context**].

where $\phi_{s,b}$ is the map sending a morphism $g\colon F_s \to b$ to the composition

$$s \xrightarrow{\eta_s} G_{F_s} \xrightarrow{G_g} G_b.$$

By the existence and uniqueness of morphisms from η_s to any other object $s \to G_b$ in $s \downarrow G$, it follows that the maps $\phi_{s,b}$ are bijective, showing F to be a left adjoint of G.

??, *The General Adjoint Functor Theorem*: See [**riehl:context**].
??, *The Special Adjoint Functor Theorem*: See [**riehl:context**].
??, *Freyd's Representability Theorem I*: See [**riehl:context**].
??, *Freyd's Representability Theorem II*: See [**stacks-project**].
??, *Co/Totality*: Omitted. □

1.3 Adjoint Strings

To avoid clutter, in this section we will abbreviate long compositions of functors. For instance, we write $f_1 \circ f_2 \circ f_3 \circ f_4$ as $f_1 f_2 f_3 f_4$. Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 1.3.1.1. An **adjoint string of length** n[10] is an n-tuple $(f_1, \ldots, f_n)$ of functors between $\mathcal{C}$ and $\mathcal{D}$ such that

$$f_n \dashv f_{n+1}$$

for each $n \in \{1, \ldots, n-1\}$.

Proposition 1.3.1.2. Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

1. *Adjoint Triples as Adjunctions Between Adjunctions.* An adjoint triple is equivalently an adjunction $(F \dashv G) \dashv (G \dashv H)$ between adjunctions. FIXME [**nLab:adjoint-triple**].[11]

2. *Adjunctions Induced by an Adjoint Triple.* A triple adjunction (f_1, f_2, f_3)

[10] *Further Terminology:* Also called an **adjoint** n**-tuple**.

[11] [**nLab:adjoint-triple**] suggests writing

$$\begin{array}{ccc} f_1 & \dashv & f_2 \\ \perp & & \perp \\ f_2 & \dashv & f_3 \end{array}$$

to denote the adjunctions $(f_1 \dashv f_2 \dashv f_3)$ and $(f_1 f_2) \dashv (f_2 f_3)$ simultaneously; the first horizontally and the latter vertically.

gives rise to two more adjunctions

$$(f_2f_1 \dashv f_2f_3)\colon\ \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "f_2f_1"] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, bend left, "f_2f_3"] \end{tikzcd}$$

and

$$(f_1f_2 \dashv f_3f_2)\colon\ \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "f_1f_2"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "f_3f_2"] \end{tikzcd}$$

where f_2f_1 and f_2f_3 are monads in $\mathcal{C}$ and f_1f_2 and f_3f_2 are comonads in $\mathcal{D}$.

Proof. **??**, *Adjoint Triples as Adjunctions Between Adjunctions*: Omitted. **??**, *Adjunctions Induced by an Adjoint Triple*: Omitted. □

Proposition 1.3.1.3. Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

1. *Adjunctions Induced by a Quadruple Adjunction.* An adjoint quadruple $(f_1 \dashv f_2 \dashv f_3 \dashv f_4)$ gives rise to two adjoint triples

$$(f_2f_1 \dashv f_2f_3 \dashv f_4f_3)\colon\ \begin{tikzcd} \mathcal{C} \arrow[r, bend left=50, "f_2f_1"] \arrow[r, bend right=50, "f_4f_3"'] & \mathcal{C} \arrow[l, "f_2f_3" description] \end{tikzcd}$$

and

$$(f_1f_2 \dashv f_3f_2 \dashv f_3f_4)\colon\ \begin{tikzcd} \mathcal{D} \arrow[r, bend left=50, "f_1f_2"] \arrow[r, bend right=50, "f_3f_4"'] & \mathcal{D} \arrow[l, "f_3f_2" description] \end{tikzcd}$$

and six adjunctions

$$(f_1f_2f_3 \dashv f_4f_3f_2)\colon\ \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "f_1f_2f_3"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "f_4f_3f_2"] \end{tikzcd} \qquad (f_3f_2f_1 \dashv f_2f_3f_4)\colon$$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "f_3f_2f_1"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "f_2f_3f_4"] \end{tikzcd}$$

$$(f_2f_3f_2f_1 \dashv f_2f_3f_4f_3)\colon \quad \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "f_2f_3f_2f_1"] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, bend left, "f_2f_3f_4f_3"] \end{tikzcd} \qquad (f_3f_2f_1f_2 \dashv f_3f_2f_3f_4)\colon \quad \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "f_3f_2f_1f_2"] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, bend left, "f_3f_2f_3f_4"] \end{tikzcd}$$

$$(f_2f_1f_2f_3 \dashv f_4f_3f_2f_3)\colon \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "f_2f_1f_2f_3"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "f_4f_3f_2f_3"] \end{tikzcd} \qquad (f_1f_2f_3f_2 \dashv f_3f_4f_3f_2)\colon \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "f_1f_2f_3f_2"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "f_3f_4f_3f_2"] \end{tikzcd}$$

where f_2f_1, f_2f_3, f_4f_3, $f_2f_3f_2f_1$, $f_2f_3f_4f_3$, $f_3f_2f_1f_2$, and $f_3f_2f_3f_4$ are monads in $\mathcal{C}$ and f_1f_2, f_3f_2, f_3f_4, $f_2f_1f_2f_3$, $f_4f_3f_2f_3$, $f_1f_2f_3f_2$, and $f_3f_4f_3f_2$ are comonads in $\mathcal{D}$.

Proof. **??**, *Adjunctions Induced by a Quadruple Adjunction*: Omitted. □

Proposition 1.3.1.4. Let $(f_1 \dashv \cdots \dashv f_n)\colon \mathcal{C}TODO\mathcal{D}$ be an adjoint string.

1. For each $k \in \mathbb{N}$ with $1 \leq k \leq n-2$, we have 2 induced adjoint strings

$$f_1f_2\cdots f_{n-k}f_{n-k+1} \dashv f_{n-k+2}f_{n-k+1}\cdots f_3f_2 \dashv \cdots \dashv f_{k-1}f_k\cdots f_{n-2}f_{n-1} \dashv f_nf_{n-1}\cdots f_{k+1}f_k$$
$$f_{n-k+1}f_{n-k}\cdots f_2f_1 \dashv f_2f_3\cdots f_{n-k+1}f_{n-k+2} \dashv \cdots \dashv f_{n-1}f_{n-2}\cdots f_kf_{k-1} \dashv f_kf_{k+1}\cdots f_{n-1}f_n$$

   of length $n-k$.

2. Inductively applying **??** to the induced adjoint strings, we get (including the 2 adjoint strings of **??**) $2 \cdot 3^{n-k-1}$ adjoint strings of length k[12], for a grand total of

$$\sum_{k=2}^{n-1} 2(k-1)\cdot 3^{n-k-1} = \frac{1}{6}(3^n+3) - n$$

   adjunctions.[13]

[12] These need not be unique.

[13] E.g. we have 4 adjoint strings of length $n-2$, such as

$$f_2f_3f_2f_1 \dashv f_2f_3f_4f_3 \dashv \cdots \dashv f_kf_{k+1}f_kf_{k-1} \dashv f_kf_{k+1}f_{k+2}f_{k+1} \dashv \cdots \dashv f_{n-2}f_{n-1}f_{n-2}f_{n-1} \dashv f_{n-2}f_{n-1}f_nf_{n-1}.$$

3. In particular:
   (a) An adjoint triple induces 2 adjoint pairs.
   (b) An adjoint quadruple induces
      - 2 adjoint triples,
      - 6 adjoint pairs,

      for a grand total of 10 adjunctions.
   (c) An adjoint quintuple induces
      - 2 adjoint quadruples,
      - 6 adjoint triples,
      - 18 adjoint pairs,

      for a grand total of 36 adjunctions.
   (d) An adjoint sextuple induces
      - 2 adjoint quintuples,
      - 6 adjoint quadruples,
      - 18 adjoint triples,
      - 54 adjoint pairs,

      for a grand total of 116 adjunctions.
   (e) An adjoint septuple induces
      - 2 adjoint sextuples,
      - 6 adjoint quintuples,
      - 18 adjoint quadruples,
      - 54 adjoint triples,
      - 162 adjoint pairs,

      for a grand total of 358 adjunctions.

Proof. Omitted. □

1.4 Reflective Subcategories

Let $\mathcal{C}$ be a category.

Definition 1.4.1.1. A subcategory $\mathcal{C}_0$ of $\mathcal{C}$ is **reflective** if the inclusion functor $i\colon \mathcal{C}_0 \hookrightarrow \mathcal{C}$ of $\mathcal{C}_0$ into $\mathcal{C}$ admits a left adjoint $L\colon \mathcal{C} \to \mathcal{C}_0$.[14]

[14] *Further Terminology:* The functor L is called the **reflector** or **localisation** of the adjunction $L \dashv i$.

Example 1.4.1.2. Here are some examples of reflective subcategories

1. $\mathsf{CHaus} \hookrightarrow \mathsf{Top}$ *([**riehl:context**]).* The category CHaus is a reflective subcategory of Top, as witnessed by the adjunction

$$(\beta \dashv \iota)\colon\quad \begin{tikzcd} \mathsf{Top} \arrow[r, shift left, "\beta"] \arrow[r, phantom, "\perp"] & \mathsf{CHaus}, \arrow[l, hook, shift left, "\iota"] \end{tikzcd}$$

of Topological Spaces, **??** of **??**.

2. $\mathsf{CMon} \hookrightarrow \mathsf{Mon}$*.* The category CMon is a reflective subcategory of Ab, as witnessed by the adjunction

$$\Big((-)^{\mathrm{ab}} \dashv \iota\Big)\colon\quad \begin{tikzcd} \mathsf{Mon} \arrow[r, shift left, "(-)^{\mathrm{ab}}"] \arrow[r, phantom, "\perp"] & \mathsf{CMon} \arrow[l, hook, shift left, "\iota"] \end{tikzcd}$$

of Monoids, **??** of **??**.

3. $\mathsf{Ab} \hookrightarrow \mathsf{Grp}$ *([**riehl:context**]).* The category Ab is a reflective subcategory of Grp, as witnessed by the adjunction

$$\Big((-)^{\mathrm{ab}} \dashv \iota\Big)\colon\quad \begin{tikzcd} \mathsf{Grp} \arrow[r, shift left, "(-)^{\mathrm{ab}}"] \arrow[r, phantom, "\perp"] & \mathsf{Ab} \arrow[l, hook, shift left, "\iota"] \end{tikzcd}$$

of Groups, **??** of **??**.

4. $\mathsf{Ab}^{\mathsf{tf}} \hookrightarrow \mathsf{Ab}$ *([**riehl:context**]).* The full subcategory $\mathsf{Ab}^{\mathsf{tf}}$ of Ab spanned by the torsion-free abelian groups is reflective in Ab. This is witnessed by the adjunction

$$\Big((-)^{\mathrm{tf}} \dashv \iota\Big)\colon\quad \begin{tikzcd} \mathsf{Ab} \arrow[r, shift left, "(-)^{\mathrm{tf}}"] \arrow[r, phantom, "\perp"] & \mathsf{Ab}^{\mathsf{tf}}, \arrow[l, hook, shift left, "\iota"] \end{tikzcd}$$

where $(-)^{\mathrm{tf}}\colon \mathsf{Ab} \to \mathsf{Ab}^{\mathsf{tf}}$ is the functor defined on objects by sending an abelian group A to the quotient $A/\mathrm{Tors}(A)$, where $\mathrm{Tors}(A)$ is the torsion subgroup of A.

5. $\mathsf{Mod}_S \hookrightarrow \mathsf{Mod}_R$ *([**riehl:context**]).* Let $\phi\colon R \to S$ be a morphism of rings. Then ϕ^* is full iff ϕ is an epimorphism, in which case the adjunction

$$(S \otimes_R (-) \dashv \phi^*)\colon\quad \begin{tikzcd} \mathsf{Mod}_S \arrow[r, bend left, "S\otimes_R(-)"] \arrow[r, phantom, "\perp"] & \mathsf{Mod}_R \arrow[l, bend left, "\phi^*"] \end{tikzcd}$$

witnesses Mod_S as a reflective subcategory of Mod_R.

6. $\mathsf{Shv}(\mathcal{C}) \hookrightarrow \mathsf{PSh}(\mathcal{C})$ *([**riehl:context**]).* The category $\mathsf{Shv}(\mathcal{C})$ of sheaves on a site $\mathcal{C}$ is a reflective subcategory of $\mathsf{PSh}(\mathcal{C})$, as witnessed by the adjunction

$$\Big((-)^{\#} \dashv \iota\Big)\colon \quad \begin{tikzcd} \mathsf{PSh}(\mathcal{C}) \arrow[r, shift left, "(-)^{\#}"] \arrow[r, phantom, "\perp"] & \mathsf{Shv}(\mathcal{C}), \arrow[l, shift left, hook, "\iota"] \end{tikzcd}$$

of Sites, **??**.

7. $\mathsf{Cats} \hookrightarrow \mathsf{sSets}$ *([**riehl:context**]).* The category Cats is a reflective subcategory of sSets, as witnessed by the adjunction

$$(\mathsf{Ho} \dashv \mathrm{N}_\bullet)\colon \quad \begin{tikzcd} \mathsf{sSets} \arrow[r, shift left, "\mathsf{Ho}"] \arrow[r, phantom, "\perp"] & \mathsf{Cats} \arrow[l, shift left, hook, "\mathrm{N}_\bullet"] \end{tikzcd}$$

of Quasicategories, **??** of **??**.

Proposition 1.4.1.3. Let $\mathcal{C}_0$ be a reflective subcategory of $\mathcal{C}$.

1. *Characterisations.* Let

$$(L \dashv \iota)\colon \quad \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "\iota"] \end{tikzcd}$$

be an adjunction. The following conditions are equivalent:

   (a) The functor ι is fully faithful.

   (b) The counit $\epsilon\colon L \circ \iota \Longrightarrow \mathrm{id}_{\mathcal{D}}$ is a natural isomorphism.

   (c) The following conditions are satisfied:

      i. The monad $(\iota \circ L, \mathrm{id}_\iota \circ \epsilon \circ \mathrm{id}_L, \eta)$ associated to the adjunction $L \dashv \iota$ is idempotent.

      ii. The functor ι is conservative.

      iii. The functor L is essentially surjective.

   (d) The functor L is the Gabriel–Zisman localisation of $\mathcal{C}$ with respect to the class S given by

$$S \stackrel{\text{def}}{=} \{f \in \mathrm{Mor}(\mathcal{C}) \mid L(f) \text{ is an isomorphism in } \mathcal{D}\}.$$

   (e) The functor L is dense.

2. *Interaction With Limits.* The inclusion $\mathcal{C}_0 \hookrightarrow \mathcal{C}$ creates all limits which exist in $\mathcal{C}$.

3. *Interaction With Colimits.* The category $\mathcal{C}_0$ admits all colimits that exist in $\mathcal{C}$: given a diagram $D\colon \mathcal{I} \to \mathcal{C}_0$ in $\mathcal{C}_0$, if $\mathrm{colim}(i \circ D)$ exists in $\mathcal{C}$, then $\mathrm{colim}(D)$ exists in $\mathcal{C}_0$ and we have

$$\mathrm{colim}(D) \cong L(\mathrm{colim}(i \circ D)).$$

Proof. **??**, *Characterisations*: See [**calculus-of-fractions-and-homotopy-theory**] and [**properties-of-dense-and-relative-adjoint-functors**].
??, *Interaction With Limits*: See [**riehl:context**].
??, *Interaction With Colimits*: See [**riehl:context**]. □

1.5 Coreflective Subcategories

Let $\mathcal{C}$ be a category.

Definition 1.5.1.1. A subcategory $\mathcal{C}_0$ of $\mathcal{C}$ is **coreflective** if the inclusion functor $i\colon \mathcal{C}_0 \hookrightarrow \mathcal{C}$ of $\mathcal{C}_0$ into $\mathcal{C}$ admits a right adjoint $R\colon \mathcal{C} \to \mathcal{C}_0$.[15]

2 Presheaves and the Yoneda Lemma

2.1 Presheaves

Let $\mathcal{C}$ be a category.

Definition 2.1.1.1. A **presheaf on** $\mathcal{C}$ is a functor $\mathcal{F}\colon \mathcal{C}^{\mathsf{op}} \to \mathsf{Sets}$.

Definition 2.1.1.2. The **category of presheaves on** $\mathcal{C}$ is the category $\mathsf{PSh}(\mathcal{C})$ defined by

$$\mathsf{PSh}(\mathcal{C}) \stackrel{\text{def}}{=} \mathsf{Fun}(\mathcal{C}^{\mathsf{op}}, \mathsf{Sets}).$$

Remark 2.1.1.3. In detail, the **category of presheaves on** $\mathcal{C}$ is the category $\mathsf{PSh}(\mathcal{C})$ where

- *Objects.* The objects of $\mathsf{PSh}(\mathcal{C})$ are presheaves on $\mathcal{C}$;
- *Morphisms.* A morphism of $\mathsf{PSh}(\mathcal{C})$ from $\mathcal{F}$ to $\mathcal{G}$ is a natural transformation $\alpha\colon \mathcal{F} \Longrightarrow \mathcal{G}$;
- *Identities.* For each $\mathcal{F} \in \mathrm{Obj}(\mathsf{PSh}(\mathcal{C}))$, the unit map

$$\mathbb{1}^{\mathsf{PSh}(\mathcal{C})}_{\mathcal{F}}\colon \mathrm{pt} \to \mathrm{Nat}(\mathcal{F}, \mathcal{F})$$

[15] *Further Terminology:* The functor L is called the **coreflector** or **colocalisation** of

of $\mathsf{PSh}(\mathcal{C})$ at $\mathcal{F}$ is defined by

$$\mathrm{id}^{\mathsf{PSh}(\mathcal{C})}_{\mathcal{F}} \overset{\text{def}}{=} \mathrm{id}_{\mathcal{F}};$$

- *Composition.* For each $\mathcal{F}, \mathcal{G}, \mathcal{H} \in \mathrm{Obj}(\mathsf{PSh}(\mathcal{C}))$, the composition map

$$\circ^{\mathsf{PSh}(\mathcal{C})}_{\mathcal{F},\mathcal{G},\mathcal{H}} \colon \mathrm{Nat}(\mathcal{G}, \mathcal{H}) \times \mathrm{Nat}(\mathcal{F}, \mathcal{G}) \to \mathrm{Nat}(\mathcal{F}, \mathcal{H})$$

  of $\mathsf{PSh}(\mathcal{C})$ at $(\mathcal{F}, \mathcal{G}, \mathcal{H})$ is defined by

$$\beta \circ^{\mathsf{PSh}(\mathcal{C})}_{\mathcal{F},\mathcal{G},\mathcal{H}} \alpha \overset{\text{def}}{=} \beta \circ \alpha.$$

2.2 Representable Presheaves

Let $\mathcal{C}$ be a category, let $U, V \in \mathrm{Obj}(\mathcal{C})$, and let $f \colon U \to V$ be a morphism of $\mathcal{C}$.

Definition 2.2.1.1. The **representable presheaf associated to** U is the presheaf $h_U \colon \mathcal{C}^{\mathsf{op}} \to \mathsf{Sets}$ on $\mathcal{C}$ where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathcal{C})$, we have

$$h_U(A) \overset{\text{def}}{=} \mathrm{Hom}_{\mathcal{C}}(A, U);$$

- *Action on Morphisms.* For each morphism $f \colon A \to B$ of $\mathcal{C}$, the image

$$h_U(f) \colon \underbrace{h_U(B)}_{\overset{\text{def}}{=} \mathrm{Hom}_{\mathcal{C}}(B,U)} \to \underbrace{h_U(A)}_{\overset{\text{def}}{=} \mathrm{Hom}_{\mathcal{C}}(A,U)}$$

  of f by h_U is defined by

$$h_U(f) \overset{\text{def}}{=} f^*.$$

Definition 2.2.1.2. A presheaf $\mathcal{F} \colon \mathcal{C}^{\mathsf{op}} \to \mathsf{Sets}$ is **representable** if $\mathcal{F} \cong h_U$ for some $U \in \mathrm{Obj}(\mathcal{C})$.[16]

Definition 2.2.1.3. The **representable natural transformation associated to** f is the natural transformation $h_f \colon h_U \Longrightarrow h_V$ consisting of the collection

$$\left\{ h_{f|A} \colon \underbrace{h_U(A)}_{\overset{\text{def}}{=} \mathrm{Hom}_{\mathcal{C}}(A,U)} \to \underbrace{h_V(A)}_{\overset{\text{def}}{=} \mathrm{Hom}_{\mathcal{C}}(A,V)} \right\}_{A \in \mathrm{Obj}(\mathcal{C})}$$

where

$$h_{f|A} \overset{\text{def}}{=} f_*.$$

the adjunction $i \dashv R$.

[16] In such a case, we call U a **representing object** for $\mathcal{F}$.

Theorem 2.2.1.4. Let $\mathcal{F}\colon \mathcal{C}^{\mathsf{op}} \to \mathsf{Sets}$ be a presheaf on $\mathcal{C}$. We have a bijection

$$\mathrm{Nat}(h_A, \mathcal{F}) \cong \mathcal{F}_A,$$

natural in $A \in \mathrm{Obj}(\mathcal{C})$, determining a natural isomorphism of functors

$$\mathrm{Nat}\Big(h_{(-)}, \mathcal{F}\Big) \cong \mathcal{F}.$$

Proof. The Natural Transformation $ev_{(-)}\colon Nat\Big(h_{(-)}, \mathcal{F}\Big) \Longrightarrow \mathcal{F}$: Let $\mathrm{ev}_{(-)}\colon \mathrm{Nat}\Big(h_{(-)}, \mathcal{F}\Big) \Longrightarrow \mathcal{F}$ be the natural transformation consisting of the collection

$$\{\mathrm{ev}_A\colon \mathrm{Nat}(h_A, \mathcal{F}) \to \mathcal{F}(A)\}_{A \in \mathrm{Obj}(\mathcal{C})}$$

with

$$\mathrm{ev}_A(\alpha) = \alpha_A(\mathrm{id}_A)$$

for each $\alpha\colon h_A \Longrightarrow \mathcal{F}$ in $\mathrm{Nat}(h_A, \mathcal{F})$.
The Natural Transformation $\xi_{(-)}\colon \mathcal{F} \Longrightarrow Nat\Big(h_{(-)}, \mathcal{F}\Big)$: Let $\xi_{(-)}\colon \mathcal{F} \Longrightarrow \mathrm{Nat}\Big(h_{(-)}, \mathcal{F}\Big)$ be the natural transformation consisting of the collection

$$\{\xi_A\colon \mathcal{F}(A) \to \mathrm{Nat}(h_A, \mathcal{F})\}_{A \in \mathrm{Obj}(\mathcal{C})}$$

where $\xi_A\colon \mathcal{F}(A) \to \mathrm{Nat}(h_A, \mathcal{F})$ is the map sending an element f of $\mathcal{F}(X)$ to the natural transformation

$$\xi_{A,f}\colon h_A \Longrightarrow \mathcal{F}$$

consisting of the collection

$$\Big\{\big(\xi_{A,f}\big)_U\colon h_A(U) \to \mathcal{F}(U)\Big\}_{A \in \mathrm{Obj}(\mathcal{C})}$$

where $\big(\xi_{A,f}\big)_U\colon h_A(U) \to \mathcal{F}(U)$ is the morphism given by

$$\begin{aligned} (\xi_{A,f})_U\colon h_A(U) &\longrightarrow \mathcal{F}(U) \\ (h\colon U \to A) &\longmapsto \mathcal{F}(h)(f) \end{aligned}$$

for each $f\colon U \to A$ in $h_A(U)$.
$ev_{(-)} \circ \xi_{(-)} = id_{\mathcal{F}}$: Let $f \in \mathcal{F}(X)$. We have

$$\begin{aligned} \big(\xi_{A,f}\big)_U(\mathrm{id}_U) &= \mathcal{F}(\mathrm{id}_U)(f), \\ &= \mathrm{id}_{\mathcal{F}(U)}(f) \\ &= f. \end{aligned}$$

$\xi_{(-)} \circ ev_{(-)} = id_{Nat\left(h_{(-)},\mathcal{F}\right)}$: Let $\alpha\colon h_A \Longrightarrow \mathcal{F} \in \mathrm{Nat}(h_A, \mathcal{F})$ and consider the diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(A,A) \arrow[r, "h_f"] \arrow[d, "\xi_A"'] & \mathrm{Hom}_{\mathcal{C}}(A,X) \arrow[d, "\xi_X"] \\ \mathcal{F}(A) \arrow[r, "\mathcal{F}(f)"'] & \mathcal{F}(X) \end{tikzcd}$$

defined on elements by

$$\begin{tikzcd} \mathrm{id}_A \arrow[r, mapsto] \arrow[d, mapsto] & f \arrow[d, mapsto] \\ u \arrow[r, mapsto] & \mathcal{F}(f)(u) = \xi_X(f). \end{tikzcd}$$

Then it is clear that the natural transformation ξ is determined by $\xi_A(\mathrm{id}_A) = u$, since we must have

$$\xi_X(f) = \mathcal{F}(f)(u)$$

for each $X \in \mathrm{Obj}(\mathcal{C})$ and each morphism $f\colon A \to X$ of $\mathcal{C}$. □

2.3 The Yoneda Embedding

Definition 2.3.1.1. The **covariant Yoneda embedding of $\mathcal{C}$**[17] is the functor[18]

$$よ_{\mathcal{C}}\colon \mathcal{C} \hookrightarrow \mathsf{PSh}(\mathcal{C})$$

where

- *Action on Objects.* For each $U \in \mathrm{Obj}(\mathcal{C})$, we have
$$よ(U) \overset{\text{def}}{=} h_U;$$
- *Action on Morphisms.* For each morphism $f\colon U \to V$ of $\mathcal{C}$, the image
$$よ(f)\colon よ(U) \to よ(V)$$
of f by よ is defined by
$$よ(f) \overset{\text{def}}{=} h_f.$$

[17] *Further Terminology:* Also called simply the **Yoneda embedding**.
[18] *Further Notation:* Also written $h_{(-)}$, or simply よ.

Proposition 2.3.1.2. Let $\mathcal{C}$ be a category.

1. *Fully Faithfulness.* The Yoneda embedding is fully faithful.[19]
2. *Preservation and Reflection of Isomorphisms.* Let $A, B \in \mathrm{Obj}(\mathcal{C})$. The following conditions are equivalent:
   (a) We have $A \cong B$.
   (b) We have $h_A \cong h_B$.
   (c) We have $h^A \cong h^B$.
3. *Uniqueness of Representing Objects Up to Isomorphism.* Let $\mathcal{F} : \mathcal{C}^{\mathsf{op}} \to \mathsf{Sets}$ be a presheaf. If there exist objects A and B of $\mathcal{C}$ such that we have

$$h_A \cong \mathcal{F},$$
$$h_B \cong \mathcal{F},$$

   then $A \cong B$.
4. *As a Free Cocompletion: The Universal Property.* The pair $(\mathsf{PSh}(\mathcal{C}), \text{よ})$ consisting of
   - The category $\mathsf{PSh}(\mathcal{C})$ of presheaves on $\mathcal{C}$;
   - The Yoneda embedding $\text{よ} : \mathcal{C} \hookrightarrow \mathsf{PSh}(\mathcal{C})$ of $\mathcal{C}$ into $\mathsf{PSh}(\mathcal{C})$;

   satisfies the following universal property:

   (**UP**) Given another pair $(\mathcal{A}, F)$ consisting of
   - A cocomplete category $\mathcal{A}$;
   - A cocontinuous functor $F : \mathcal{C} \to \mathcal{A}$;

   there exists a cocontinuous functor $\mathsf{PSh}(\mathcal{C}) \xrightarrow{\exists!} \mathcal{A}$, unique up to natural isomorphism, making the diagram

$$\begin{tikzcd}
 & \mathsf{PSh}(\mathcal{C}) \arrow[d, dashed, "\exists!"] \\
\mathcal{C} \arrow[ur, "\text{よ}"] \arrow[r, "F"'] \arrow[ur, phantom, shift right=3, "\wr"] & \mathcal{A}
\end{tikzcd}$$

   commute, again up to natural isomorphism.

[19] In other words, the Yoneda embedding is indeed an embedding.

5. *As a Free Cocompletion: 2-Adjointness.* We have a 2-adjunction

$$(\mathsf{PSh} \dashv \iota)\colon \quad \begin{tikzcd} \mathsf{Cats} \arrow[r, bend left, "\mathsf{PSh}"] \arrow[r, phantom, "\perp_2"] & \mathsf{Cats}^{\mathsf{cocomp.}} \arrow[l, bend left, "\iota"] \end{tikzcd},$$

witnessed by an adjoint equivalence of categories[20]

$$(\mathrm{Lan}_{よ} \dashv よ^*)\colon \quad \begin{tikzcd} \mathsf{CoContFun}(\mathsf{PSh}(\mathcal{C}), \mathcal{D}) \arrow[r, bend left, "よ^*"] \arrow[r, phantom, "\perp"] & \mathsf{Fun}(\mathcal{C}, \mathcal{D}) \arrow[l, bend left, "\mathrm{Lan}_{よ}"] \end{tikzcd},$$

natural in $\mathcal{C} \in \mathrm{Obj}(\mathsf{Cats})$ and $\mathcal{D} \in \mathrm{Obj}(\mathsf{Cats}^{\mathsf{cocomp.}})$, where

- We have a functor

$$よ_{\mathcal{C}}^*\colon \mathsf{CoContFun}(\mathsf{PSh}(\mathcal{C}), \mathcal{D}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{D})$$

  defined by

$$よ_{\mathcal{C}}^*(F) \overset{\text{def}}{=} F \circ よ_{\mathcal{C}},$$

  i.e. by sending a functor $F\colon \mathsf{PSh}(\mathcal{C}) \to \mathcal{D}$ to the composition

$$\mathcal{C} \overset{よ_{\mathcal{C}}}{\hookrightarrow} \mathsf{PSh}(\mathcal{C}) \xrightarrow{F} \mathcal{D};$$

- We have a natural map

$$\mathrm{Lan}_{よ_{\mathcal{C}}}\colon \mathsf{Fun}(\mathcal{C}, \mathcal{D}) \to \mathsf{CoContFun}(\mathsf{PSh}(\mathcal{C}), \mathcal{D})$$

  computed on objects by

$$\begin{aligned} \left[\mathrm{Lan}_{よ_{\mathcal{C}}}(F)\right](\mathcal{F}) &\cong \int^{A \in \mathcal{D}} \mathrm{Nat}(h_A, \mathcal{F}) \odot F_A \\ &\cong \int^{A \in \mathcal{D}} \mathcal{F}^A \odot F_A \end{aligned}$$

  for each $\mathcal{F} \in \mathrm{Obj}(\mathsf{PSh}(\mathcal{C}))$.

Proof. **??**, *Fully Faithfulness*: Let $A, B \in \mathrm{Obj}(\mathcal{C})$. Applying **??** to the functor h_B (i.e. in the case $\mathcal{F} = h_B$), we have

$$\mathrm{Hom}_{\mathcal{C}}(A, B) \cong \mathrm{Nat}(h_A, h_B).$$

[20] In this sense, $\mathsf{PSh}(\mathcal{C})$ is the free cocompletion of $\mathcal{C}$ (although the term "cocompletion"

Thus $よ$ is fully faithful.
??, *Preservation and Reflection of Isomorphisms*: This follows from **??** and **??**.
??, *Uniqueness of Representing Objects Up to Isomorphism*: By composing the isomorphisms $h_A \cong \mathcal{F} \cong h_B$, we get a natural isomorphism $\alpha\colon h_A \overset{\cong}{\Longrightarrow} h_B$. By **??**, we have $A \cong B$.
??, *As a Free Cocompletion: The Universal Property*: This is a rephrasing of **??**.
??: *As a Free Cocompletion: 2-Adjointness*: See [**nLab:free-cocompletion**].
□

2.4 Universal Objects

Definition 2.4.1.1. The **universal object** associated to a representable functor $h_U\colon \mathcal{C} \to \mathcal{D}$ is the element $u \in h_U(U)$ satisfying the following universal property:[21]

(**UP**) For each $B \in \mathrm{Obj}(\mathcal{C})$, the map

$$\begin{aligned} h_U(B) &\longrightarrow h_U(U) \\ (f\colon B \to A) &\longmapsto h_U(f)(u) \end{aligned}$$

is a bijection.

Remark 2.4.1.2. In other words, a universal object u associated to a representable functor $h_U\colon \mathcal{C} \to \mathcal{D}$ represented by U is universal in the sense that every element of $h_U(A)$ is equal to the image of u via $h_U(f)$ for a unique morphism $f\colon A \to U$ of $\mathcal{C}$.

Example 2.4.1.3. Let G be a group and consider the functor $\mathrm{Bun}^{\mathrm{num}}_G(-)\colon \mathsf{Ho}(\mathsf{Top})^{\mathsf{op}} \to \mathsf{Sets}$ sending $[X] \in \mathsf{Ho}(\mathsf{Top})^{\mathsf{op}}$ to the set of numerable principal G-bundles on X.
Then the universal numerable principal G-bundle $\gamma\colon \mathrm{EG} \to \mathrm{BG}$ is a universal object for $\mathrm{Bun}^{\mathrm{num}}_G(-)$.
Furthermore, the map sending γ to a principal G-bundle $P \to X$ on X is the pullback

$$f^*\colon \mathrm{Bun}^{\mathrm{num}}_G(\mathrm{BG}) \to \mathrm{Bun}^{\mathrm{num}}_G(X)$$

of P along the homotopy class $[f]\colon X \to \mathrm{BG}$ classifying P of maps $X \to \mathrm{BG}$. See Algebraic Topology, **??** for more details.

is slightly misleading, as $\mathsf{PSh}(\mathsf{PSh}(\mathcal{C})) \overset{\text{eq.}}{\ncong} \mathsf{PSh}(\mathcal{C})$).
[21] This is the element of $h_U(U)$ corresponding to the identity natural transformation

3 Copresheaves and the Contravariant Yoneda Lemma

3.1 Copresheaves

Let $\mathcal{C}$ be a category.

Definition 3.1.1.1. A **copresheaf on** $\mathcal{C}$ is a functor $F\colon \mathcal{C} \to \mathsf{Sets}$.

Definition 3.1.1.2. The **category of copresheaves on** $\mathcal{C}$ is the category $\mathsf{CoPSh}(\mathcal{C})$ defined by

$$\mathsf{CoPSh}(\mathcal{C}) \stackrel{\text{def}}{=} \mathsf{Fun}(\mathcal{C}, \mathsf{Sets}).$$

Remark 3.1.1.3. In detail, the **category of copresheaves on** $\mathcal{C}$ is the category $\mathsf{CoPSh}(\mathcal{C})$ where

- *Objects.* The objects of $\mathsf{CoPSh}(\mathcal{C})$ are presheaves on $\mathcal{C}$;
- *Morphisms.* A morphism of $\mathsf{CoPSh}(\mathcal{C})$ from F to G is a natural transformation $\alpha\colon F \Longrightarrow G$;
- *Identities.* For each $F \in \mathrm{Obj}(\mathsf{CoPSh}(\mathcal{C}))$, the unit map

  $$\mathbb{1}^{\mathsf{CoPSh}(\mathcal{C})}_F\colon \mathrm{pt} \to \mathrm{Nat}(F, F)$$

  of $\mathsf{CoPSh}(\mathcal{C})$ at F is defined by

  $$\mathrm{id}^{\mathsf{CoPSh}(\mathcal{C})}_F \stackrel{\text{def}}{=} \mathrm{id}_F;$$

- *Composition.* For each $F, G, H \in \mathrm{Obj}(\mathsf{CoPSh}(\mathcal{C}))$, the composition map

  $$\circ^{\mathsf{CoPSh}(\mathcal{C})}_{F,G,H}\colon \mathrm{Nat}(G, H) \times \mathrm{Nat}(F, G) \to \mathrm{Nat}(F, H)$$

  of $\mathsf{CoPSh}(\mathcal{C})$ at (F, G, H) is defined by

  $$\beta \circ^{\mathsf{CoPSh}(\mathcal{C})}_{F,G,H} \alpha \stackrel{\text{def}}{=} \beta \circ \alpha.$$

3.2 Corepresentable Copresheaves

Let $\mathcal{C}$ be a category, let $U, V \in \mathrm{Obj}(\mathcal{C})$, and let $f\colon U \to V$ be a morphism of $\mathcal{C}$.

$\mathrm{id}_{h_U}\colon h_U \Longrightarrow h_U$ under the isomorphism $h_U(U) \cong \mathrm{Hom}_{\mathsf{PSh}(\mathcal{C})}(h_U, h_U)$.

Definition 3.2.1.1. The **corepresentable copresheaf associated to** U is the copresheaf $h^U : \mathcal{C} \to \mathsf{Sets}$ on $\mathcal{C}$ where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathcal{C})$, we have

$$h^U(A) \stackrel{\text{def}}{=} \mathrm{Hom}_{\mathcal{C}}(U, A);$$

- *Action on Morphisms.* For each morphism $f : A \to B$ of $\mathcal{C}$, the image

$$h^U(f) : \underbrace{h^U(A)}_{\stackrel{\text{def}}{=}\mathrm{Hom}_{\mathcal{C}}(U,A)} \to \underbrace{h^U(B)}_{\stackrel{\text{def}}{=}\mathrm{Hom}_{\mathcal{C}}(U,B)}$$

  of f by h^U is defined by

$$h^U(f) \stackrel{\text{def}}{=} f_*.$$

Definition 3.2.1.2. A copresheaf $F : \mathcal{C} \to \mathsf{Sets}$ is **corepresentable** if $F \cong h^U$ for some $U \in \mathrm{Obj}(\mathcal{C})$.[22]

Definition 3.2.1.3. The **corepresentable natural transformation associated to** f is the natural transformation $h^f : h^V \Longrightarrow h^U$ consisting of the collection

$$\left\{ h^f_A : \underbrace{h^V(A)}_{\stackrel{\text{def}}{=}\mathrm{Hom}_{\mathcal{C}}(V,A)} \to \underbrace{h^U(A)}_{\stackrel{\text{def}}{=}\mathrm{Hom}_{\mathcal{C}}(U,A)} \right\}_{A \in \mathrm{Obj}(\mathcal{C})}$$

where

$$h^f_A \stackrel{\text{def}}{=} f^*.$$

Theorem 3.2.1.4. Let $F : \mathcal{C} \to \mathsf{Sets}$ be a copresheaf on $\mathcal{C}$. We have a bijection

$$\mathrm{Nat}\Big(h^A, F\Big) \cong F^A,$$

natural in $A \in \mathrm{Obj}(\mathcal{C})$, determining a natural isomorphism of functors

$$\mathrm{Nat}\Big(h^{(-)}, F\Big) \cong F.$$

Proof. This is dual to **??**. $\square$

[22] In such a case, we call U a **corepresenting object** for F.

3.3 The Contravariant Yoneda Embedding

Definition 3.3.1.1. The **contravariant Yoneda embedding of $\mathcal{C}$** is the functor[23]

$$\text{よ}_{\mathcal{C}}\colon \mathcal{C}^{\mathsf{op}} \hookrightarrow \mathsf{Fun}(\mathcal{C}, \mathsf{Sets})$$

where

- *Action on Objects.* For each $U \in \mathrm{Obj}(\mathcal{C})$, we have

$$\text{よ}(U) \stackrel{\text{def}}{=} h^U;$$

- *Action on Morphisms.* For each morphism $f\colon U \to V$ of $\mathcal{C}$, the image

$$\text{よ}(f)\colon\ \text{よ}(V) \to \text{よ}(U)$$

  of f by よ is defined by

$$\text{よ}(f) \stackrel{\text{def}}{=} h^f.$$

Proposition 3.3.1.2. Let $\mathcal{C}$ be a category.

1. *Fully Faithfulness.* The contravariant Yoneda embedding is fully faithful.[24]

2. *Preservation and Reflection of Isomorphisms.* Let $A, B \in \mathrm{Obj}(\mathcal{C})$. The following conditions are equivalent:

   (a) We have $A \cong B$.

   (b) We have $h_A \cong h_B$.

   (c) We have $h^A \cong h^B$.

3. *Uniqueness of Representing Objects Up to Isomorphism.* Let $F\colon \mathcal{C} \to \mathsf{Sets}$ be a copresheaf. If there exist objects A and B of $\mathcal{C}$ such that we have

$$h^A \cong F,$$
$$h^B \cong F,$$

   then $A \cong B$.

[23] *Further Notation:* Also written $h^{(-)}$, or simply よ.

[24] In other words, the contravariant Yoneda embedding is indeed an embedding.

4. *As a Free Completion: The Universal Property.* The pair $(\mathsf{CoPSh}(\mathcal{C})^{\mathsf{op}}, \rotatebox[origin=c]{180}{よ})$ consisting of

   - The opposite $\mathsf{CoPSh}(\mathcal{C})^{\mathsf{op}}$ of the category of copresheaves on $\mathcal{C}$;
   - The contravariant Yoneda embedding $\rotatebox[origin=c]{180}{よ}\colon \mathcal{C} \hookrightarrow \mathsf{CoPSh}(\mathcal{C})^{\mathsf{op}}$ of $\mathcal{C}$ into $\mathsf{CoPSh}(\mathcal{C})^{\mathsf{op}}$;

   satisfies the following universal property:

   (**UP**) Given another pair $(\mathcal{A}, F)$ consisting of
   - A complete category $\mathcal{A}$;
   - A continuous functor $F\colon \mathcal{C} \to \mathcal{A}$;

   there exists a continuous functor $\mathsf{CoPSh}(\mathcal{C})^{\mathsf{op}} \xrightarrow{\exists!} \mathcal{A}$, unique up to natural isomorphism, making the diagram

$$\begin{tikzcd}
& \mathsf{CoPSh}(\mathcal{C})^{\mathsf{op}} \arrow[d, dashed, "\exists!"] \\
\mathcal{C} \arrow[ur, "\rotatebox[origin=c]{180}{よ}"] \arrow[r, "F"'] & \mathcal{A} \arrow[u, phantom, shift right=3, "\overset{\sim}{\Rightarrow}"]
\end{tikzcd}$$

   commute, again up to natural isomorphism.

5. *As a Free Completion: 2-Adjointness.* We have a 2-adjunction

$$(\mathsf{CoPSh}^{\mathsf{op}} \dashv \iota)\colon \quad \begin{tikzcd} \mathsf{Cats} \arrow[r, bend left, "\mathsf{CoPSh}^{\mathsf{op}}"] \arrow[r, phantom, "\perp_2"] & \mathsf{Cats}^{\mathsf{comp.}} \arrow[l, bend left, "\iota"] \end{tikzcd},$$

   witnessed by an adjoint equivalence of categories

$$\left(\mathrm{Ran}^{\mathsf{op}}_{\rotatebox[origin=c]{180}{よ}} \dashv \rotatebox[origin=c]{180}{よ}^{*}\right)\colon \quad \begin{tikzcd} \mathsf{ContFun}(\mathsf{CoPSh}(\mathcal{C})^{\mathsf{op}}, \mathcal{D}) \arrow[r, bend left, "\mathrm{Ran}^{\mathsf{op}}_{\rotatebox[origin=c]{180}{よ}}"] \arrow[r, phantom, "\perp"] & \mathsf{Fun}(\mathcal{C}^{\mathsf{op}}, \mathcal{D}) \arrow[l, bend left, "\rotatebox[origin=c]{180}{よ}^{*}"] \end{tikzcd},$$

   natural in $\mathcal{C} \in \mathrm{Obj}(\mathsf{Cats})$ and $\mathcal{D} \in \mathrm{Obj}(\mathsf{Cats}^{\mathsf{comp.}})$.

Proof. This is dual to **??**. $\square$

Appendices

A Other Chapters

Set Theory

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2. Constructions With Sets
3. Pointed Sets
4. Tensor Products of Pointed Sets
5. Indexed and Fibred Sets
6. Relations
7. Spans
8. Posets

Category Theory

9. Categories
10. Constructions With Categories
11. Kan Extensions

Bicategories

12. Bicategories
13. Internal Adjunctions

Internal Category Theory

14. Internal Categories

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15. The Cycle Category

Cubical Stuff

16. The Cube Category

Globular Stuff

17. The Globe Category

Cellular Stuff

18. The Cell Category

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19. Monoids
20. Constructions With Monoids

Monoids With Zero

21. Monoids With Zero
22. Constructions With Monoids With Zero

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23. Groups
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