Relations

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This chapter contains some material about relations and constructions with them. Notably, we discuss and explore:

1. The definition of relations (Section 1.1).
2. How relations may be viewed as decategorification of profunctors (Section 1.2).
3. The various kind of categories that relations form, namely:
   (a) A category (Section 2.1),
   (b) A monoidal category (Section 2.2),
   (c) A 2-category (Section 2.3), and
   (d) A double category (Section 2.4).
4. The various categorical properties of the 2-category of relations, including (Section 2.5):
   (a) The self-duality of Rel and **Rel** (Items 1 and 2 of Proposition 2.5.1);
   (b) Identifications of equivalences and isomorphisms in **Rel** with bijections (Item 3 of Proposition 2.5.1);
   (c) Identifications of adjunctions in **Rel** with functions (Item 4 of Proposition 2.5.1);
   (d) Identifications of monads in **Rel** with preorders (Item 5 of Proposition 2.5.1);
   (e) Identifications of comonads in **Rel** with subsets (Item 6 of Proposition 2.5.1);
   (f) Characterisations of monomorphisms in Rel (Item 7 of Proposition 2.5.1);
   (g) Characterisations of epimorphisms in Rel (Item 8 of Proposition 2.5.1);
   (h) The partial co/completeness of Rel (Item 10 of Proposition 2.5.1);
   (i) The existence of right Kan extensions and right Kan lifts in Rel (Items 11 and 12 of Proposition 2.5.1);

(j) The closedness of **Rel** (Item 13 of Proposition 2.5.1).

5. The various kinds of constructions involving relations, such as graphs, domains, ranges, unions, intersections, products, inverse relations, composition of relations, and collages (Section 3).

6. Equivalence relations (Section 4) and quotient sets (Section 4.5).

7. The adjoint pairs

$$R_* \dashv R_{-1} \colon \mathcal{P}(A) \rightleftarrows \mathcal{P}(B),$$
$$R^{-1} \dashv R_! \colon \mathcal{P}(B) \rightleftarrows \mathcal{P}(A)$$

of functors (morphisms of posets) between $\mathcal{P}(A)$ and $\mathcal{P}(B)$ induced by a relation $R \colon A \nrightarrow B$, as well as the properties of R_*, R_{-1}, R^{-1}, and $R_!$ (Section 5).

Here we also note that:

(a) These two pairs of adjoint functors are the counterpart for relations of the adjoint triple $f_* \dashv f^{-1} \dashv f_!$ induced by a function $f \colon A \to B$ studied in Constructions With Sets, Section 4;

(b) We have $R_{-1} = R^{-1}$ iff R is total and functional (Item 8 of Proposition 5.2.4).

(c) As a consequence of the previous item, when R comes from a function f the pair of adjunctions

$$R_* \dashv R_{-1} = R^{-1} \dashv R_!$$

reduces to the triple adjunction

$$f_* \dashv f^{-1} \dashv f_!$$

from before.

(d) The pairs $R_* \dashv R_{-1}$ and $R^{-1} \dashv R_!$ later make an appearance in the context of continuous, open, and closed relations between topological spaces (Topological Spaces, **??**).

8. A notion of *relative preorder* that is to that of a preorder as relative monads are to monads, extending the identifications of monads in Rel with preorders of Item 5 of Proposition 2.5.1 to "relative monads in Rel".

Contents

1 Relations

1.1 Foundations

Let A and B be sets.

DEFINITION 1.1.1 ► RELATIONS

A **relation** $R\colon A \nrightarrow B$ **from** A **to** B[1,2] is a subset R of $A \times B$.[3]

[1] *Further Terminology:* Also called a **multivalued function from** A **to** B, a **relation over** A **and** B, **relation on** A **and** B, a **binary relation over** A **and** B, or a **binary relation on** A **and** B.

[2] *Further Terminology:* When $A = B$, we also call $R \subset A \times A$ a **relation on** A.

[3] *Further Notation:* Given elements $a \in A$ and $b \in B$, we write $a \sim_R b$ to mean $(a, b) \in R$.

DEFINITION 1.1.2 ► THE PO/SET OF RELATIONS OVER TWO SETS

Let A and B be sets.

1. The **set of relations from** A **to** B is the set $\mathrm{Rel}(A, B)$ defined by
$$\mathrm{Rel}(A, B) \overset{\text{def}}{=} \{\text{Relations from } A \text{ to } B\}.$$

2. The **poset of relations from** A **to** B is the poset
$$\mathbf{Rel}(A, B) \overset{\text{def}}{=} (\mathrm{Rel}(A, B), \subset)$$

consisting of

- *The Underlying Set.* The set $\mathrm{Rel}(A,B)$ of Item 1;
- *The Partial Order.* The partial order

$$\subset\colon \mathrm{Rel}(A,B)\times\mathrm{Rel}(A,B)\longrightarrow\{\mathrm{true},\mathrm{false}\}$$

  on $\mathrm{Rel}(A,B)$ given by inclusion of relations.

Remark 1.1.3 ► Equivalent Definitions of Relations

A relation from A to B is equivalently:[1]

1. A subset of $A\times B$;
2. A function from $A\times B$ to $\{\mathrm{true},\mathrm{false}\}$;
3. A function from A to $\mathcal{P}(B)$;
4. A function from B to $\mathcal{P}(A)$;
5. A cocontinuous morphism of posets from $(\mathcal{P}(A),\subset)$ to $(\mathcal{P}(B),\subset)$.

That is: we have bijections of sets

$$\begin{aligned}\mathrm{Rel}(A,B) &\overset{\mathrm{def}}{=} \mathcal{P}(A\times B),\\ &\cong \mathsf{Sets}(A\times B,\{\mathrm{true},\mathrm{false}\}),\\ &\cong \mathsf{Sets}(A,\mathcal{P}(B)),\\ &\cong \mathsf{Sets}(B,\mathcal{P}(A)),\\ &\cong \mathrm{Hom}^{\mathrm{cocont}}_{\mathsf{Pos}}(\mathcal{P}(A),\mathcal{P}(B)),\end{aligned}$$

natural in $A,B\in\mathrm{Obj}(\mathsf{Sets})$.

[1] *Intuition:* In particular, we may think of a relation $R\colon A\longrightarrow\mathcal{P}(B)$ from A to B as a multivalued function from A to B (including the possibility of a given $a\in A$ having no value at all).

Proof 1.1.4 ▶ Proof of Remark 1.1.3

We claim that Items 1 to 5 are indeed equivalent:

- *Item 1* $\iff$ *Item 2*: This is a special case of Constructions With Sets, Item 6 of Proposition 4.2.3.
- *Item 2* $\iff$ *Item 3*: This is an instance of currying, following from the bijections

  $$\begin{aligned}\mathsf{Sets}(A\times B,\{\mathsf{true},\mathsf{false}\}) &\cong \mathsf{Sets}(A,\mathsf{Sets}(B,\{\mathsf{true},\mathsf{false}\}))\\ &\cong \mathsf{Sets}(A,\mathcal{P}(B)),\end{aligned}$$

  where the last bijection is from Constructions With Sets, Item 6 of Proposition 4.2.3.
- *Item 2* $\iff$ *Item 4*: This is also an instance of currying, following from the bijections

  $$\begin{aligned}\mathsf{Sets}(A\times B,\{\mathsf{true},\mathsf{false}\}) &\cong \mathsf{Sets}(B,\mathsf{Sets}(B,\{\mathsf{true},\mathsf{false}\}))\\ &\cong \mathsf{Sets}(B,\mathcal{P}(A)),\end{aligned}$$

  where again the last bijection is from Constructions With Sets, Item 6 of Proposition 4.2.3.
- *Item 2* $\iff$ *Item 5*: This follows from the universal property of the powerset $\mathcal{P}(X)$ of a set X as the free cocompletion of X via the characteristic embedding

  $$\chi_X\colon X\hookrightarrow\mathcal{P}(X)$$

  of X into $\mathcal{P}(X)$ (Constructions With Sets, Item 9 of Proposition 4.2.3).

  In particular, the bijection

  $$\mathsf{Rel}(A,B)\cong\mathsf{Hom}^{\mathsf{cocont}}_{\mathsf{Pos}}(\mathcal{P}(A),\mathcal{P}(B))$$

  is given by taking a relation $R\colon A \nrightarrow B$, passing to its associated function $f\colon A\to\mathcal{P}(B)$ from A to B and then extending f from A to all of $\mathcal{P}(A)$ by taking its left Kan extension along χ_X.

  This coincides with the direct image function $f_*\colon\mathcal{P}(A)\to\mathcal{P}(B)$ of Constructions With Sets, Definition 4.3.1.

This finishes the proof. □

Proposition 1.1.5 ► Properties of Relations

Let A and B be sets.

1. *End Formula for The Poset of Relations.* Let $R, S\colon A \nrightarrow B$ be relations. We have
$$\mathrm{Hom}_{\mathbf{Rel}(A,B)}(R,S) \cong \int_{a\in A}\int_{b\in B} \mathrm{Hom}_{\{\mathrm{t},\mathrm{f}\}}(R^a_b, S^a_b).$$

Proof 1.1.6 ► Proof of Proposition 1.1.5

Item 1: End Formula for The Poset of Relations

Unwinding the expression inside the end on the right hand side, we have
$$\int_{a\in A}\int_{b\in B} \mathrm{Hom}_{\{\mathrm{t},\mathrm{f}\}}(R^a_b, S^a_b) \cong \begin{cases} \mathrm{pt} & \text{if for each } (a,b)\in A\times B, \\ & \text{if } a\sim_R b, \text{ then } a\sim_S b, \\ \emptyset & \text{otherwise.} \end{cases}$$

On the left hand-side, we have
$$\mathrm{Hom}_{\mathbf{Rel}(A,B)}(R,S) \cong \begin{cases} \mathrm{pt} & \text{if } R\subset S, \\ \emptyset & \text{otherwise.} \end{cases}$$

It is then clear that the conditions for each set to evaluate to pt are equivalent, implying that those two sets are isomorphic, finishing the proof.

1.2 Relations as Decategorifications of Profunctors

Remark 1.2.1 ► Relations as Decategorifications of Profunctors I

The notion of a relation is a decategorification of that of a profunctor:

1. A profunctor from a category $\mathcal{C}$ to a category $\mathcal{D}$ is a functor
$$\mathfrak{p}\colon \mathcal{D}^{\mathrm{op}}\times\mathcal{C}\to \mathsf{Sets}.$$
2. A relation on sets A and B is a function
$$R\colon A\times B\to\{\mathrm{true},\mathrm{false}\}.$$

Here we notice that:

- The opposite X^{op} of a set X is itself, as $(-)^{\mathsf{op}} \colon \mathsf{Cats} \to \mathsf{Cats}$ restricts to the identity endofunctor on Sets;
- The values that profunctors and relations take are directly related in relation to decategorification:
  - A category is enriched over the category

    $$\mathsf{Sets} \stackrel{\text{def}}{=} \mathsf{Cats}_0$$

    of sets, with profunctors taking values on it;
  - A set is enriched over the set

    $$\{\mathsf{true}, \mathsf{false}\} \stackrel{\text{def}}{=} \mathsf{Cats}_{-1}$$

    of classical truth values, with relations taking values on it;

Remark 1.2.2 ► Relations as Decategorifications of Profunctors II

Extending Remark 1.2.1, the equivalent definitions of relations in Remark 1.1.3 are also related to the corresponding ones for profunctors (Categories, **??**), which state that a profunctor $\mathfrak{p} \colon \mathcal{C} \nrightarrow \mathcal{D}$ is equivalently:

1. A functor $\mathfrak{p} \colon \mathcal{D}^{\mathsf{op}} \times \mathcal{C} \to \mathsf{Sets}$;
2. A functor $\mathfrak{p} \colon \mathcal{C} \to \mathsf{PSh}(\mathcal{D})$;
3. A functor $\mathfrak{p} \colon \mathcal{D}^{\mathsf{op}} \to \mathsf{Fun}(\mathcal{C}, \mathsf{Sets})$;
4. A colimit-preserving functor $\mathfrak{p} \colon \mathsf{PSh}(\mathcal{C}) \to \mathsf{PSh}(\mathcal{D})$.

Indeed:

- The equivalence between Items 1 and 2 (and also that between Items 1 and 3, which is proved analogously) is an instance of currying, both for profunctors as well as for relations, using the isomorphisms

$$\begin{aligned}\mathsf{Sets}(A \times B, \{\mathsf{true}, \mathsf{false}\}) &\cong \mathsf{Sets}(A, \mathsf{Sets}(B, \{\mathsf{true}, \mathsf{false}\}))\\ &\cong \mathsf{Sets}(A, \mathcal{P}(B)),\\ \mathsf{Fun}(\mathcal{D}^{\mathsf{op}} \times \mathcal{D}, \mathsf{Sets}) &\cong \mathsf{Fun}(\mathcal{C}, \mathsf{Fun}(\mathcal{D}^{\mathsf{op}}, \mathsf{Sets}))\\ &\cong \mathsf{Fun}(\mathcal{C}, \mathsf{PSh}(\mathcal{D})).\end{aligned}$$

- The equivalence between Items 1 and 3 follows from the universal properties of:
  - The powerset $\mathcal{P}(X)$ of a set X as the free cocompletion of X via the characteristic embedding

    $$\chi_{(-)} : X \hookrightarrow \mathcal{P}(X)$$

    of X into $\mathcal{P}(X)$ (Constructions With Sets, Item 9 of Proposition 4.2.3);
  - The category $\mathsf{PSh}(\mathcal{C})$ of presheaves on a category $\mathcal{C}$ as the free cocompletion of $\mathcal{C}$ via the Yoneda embedding

    $$よ : \mathcal{C} \hookrightarrow \mathsf{PSh}(\mathcal{C})$$

    of $\mathcal{C}$ into $\mathsf{PSh}(\mathcal{C})$ (Categories, **??** of **??**).

1.3 Examples of Relations

EXAMPLE 1.3.1 ▶ THE TRIVIAL RELATION

The **trivial relation on A and B** is the relation $\sim_{\mathsf{triv}}$ defined by[1,2,3]

$$\sim_{\mathsf{triv}} \overset{\text{def}}{=} A \times A.$$

[1]This is the unique relation R on A and B such that we have $a \sim_R b$ for all $a \in A$ and all $b \in B$.

[2]As a function from $A \times A$ to $\{\mathsf{true}, \mathsf{false}\}$, the relation $\sim_{\mathsf{triv}}$ is the constant function

$$\Delta_{\mathsf{true}} : A \times B \to \{\mathsf{true}, \mathsf{false}\}$$

from $A \times B$ to $\{\mathsf{true}, \mathsf{false}\}$ taking value true.

[3]As a function from A to $\mathcal{P}(B)$, the relation $\sim_{\mathsf{triv}}$ is the function

$$\Delta_{\mathsf{true}} : A \to \mathcal{P}(B)$$

defined by

$$\Delta_{\mathsf{true}}(a) \stackrel{\text{def}}{=} B$$

for each $a \in A$.

Example 1.3.2 ► The Cotrivial Relation

The **cotrivial relation on A and B** is the relation $\sim_{\mathrm{cotriv}}$ defined by[1,2,3]

$$\sim_{\mathrm{cotriv}} \stackrel{\text{def}}{=} \emptyset.$$

[1] This is the unique relation R on A and B such that we have $a \sim_R b$ for no $a \in A$ and no $b \in B$.

[2] As a function from $A \times B$ to $\{\mathsf{true}, \mathsf{false}\}$, the relation $\sim_{\mathrm{cotriv}}$ is the constant function

$$\Delta_{\mathsf{false}} \colon A \times B \to \{\mathsf{true}, \mathsf{false}\}$$

from $A \times B$ to $\{\mathsf{true}, \mathsf{false}\}$ taking value false.

[3] As a function from A to $\mathcal{P}(A)$, the relation $\sim_{\mathrm{cotriv}}$ is the function

$$\Delta_{\mathsf{false}} \colon A \to \mathcal{P}(A)$$

defined by

$$\Delta_{\mathsf{true}}(a) \stackrel{\text{def}}{=} \emptyset$$

for each $a \in A$.

Example 1.3.3 ► The Characteristic Relation of a Set

The characteristic relation on A of Constructions With Sets, Item 3 of Definition 4.1.1 is another example of a relation. It is in fact the unique relation on A making the following conditions equivalent, for each $a, b \in A$:

1. We have $a \sim_{\mathrm{id}} b$.
2. We have $a = b$.

Example 1.3.4 ► Square Roots

Square roots are examples of relations:

1. *Square Roots in $\mathbb{R}$.* The assignment $x \mapsto \sqrt{x}$ defines a relation

$$\sqrt{-} \colon \mathbb{R} \to \mathcal{P}(\mathbb{R})$$

from $\mathbb{R}$ to itself, being explicitly given by

$$\sqrt{x} \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if } x = 0, \\ \left\{-\sqrt{|x|}, \sqrt{|x|}\right\} & \text{if } x \neq 0. \end{cases}$$

2. *Square Roots in $\mathbb{Q}$.* Square roots in $\mathbb{Q}$ are similar to square roots in $\mathbb{R}$, though now additionally it may also occur that $\sqrt{-}\colon \mathbb{Q} \to \mathcal{P}(\mathbb{Q})$ sends a rational number x (e.g. 2) to the empty set (since $\sqrt{2} \notin \mathbb{Q}$).

Example 1.3.5 ► Complex Logarithms

The complex logarithm defines a relation

$$\log\colon \mathbb{C} \to \mathcal{P}(\mathbb{C})$$

from $\mathbb{C}$ to itself, where we have

$$\log(a+bi) \stackrel{\text{def}}{=} \left\{\log\left(\sqrt{a^2+b^2}\right) + i\arg(a+bi) + (2\pi i)k \,\middle|\, k \in \mathbb{Z}\right\}$$

for each $a + bi \in \mathbb{C}$.

Example 1.3.6 ► More Examples of Relations

See [**wikipedia:multivalued-functions**] for more examples of relations, such as antiderivation, inverse trigonometric functions, and inverse hyperbolic functions.

1.4 Functional Relations

Let A and B be sets.

Definition 1.4.1 ► Functional Relations

A relation $R\colon A \nrightarrow B$ is **functional** if, for each $a \in A$, the set $R(a)$ is either empty or a singleton.

Proposition 1.4.2 ▶ Properties of Functional Relations

Let $R\colon A \nrightarrow B$ be a relation.

1. *Characterisations.* The following conditions are equivalent:
   (a) The relation R is functional.
   (b) We have $R \diamond R^{\dagger} \subset \chi_B$.

Proof 1.4.3 ▶ Proof of Proposition 1.4.2

Item 1: Characterisations

We claim that Items 1a and 1b are indeed equivalent:

- *Item 1a* $\implies$ *Item 1b*: Let $(b, b') \in B \times B$. We need to show that

$$[R \diamond R^{\dagger}](b, b') \preceq_{\{\mathrm{t},\mathrm{f}\}} \chi_B(b, b'),$$

  i.e. that if there exists some $a \in A$ such that $b \sim_{R^{\dagger}} a$ and $a \sim_R b'$, then $b = b'$. But since $b \sim_{R^{\dagger}} a$ is the same as $a \sim_R b$, we have both $a \sim_R b$ and $a \sim_R b'$ at the same time, which implies $b = b'$ since R is functional.

- *Item 1b* $\implies$ *Item 1a*: Suppose that we have $a \sim_R b$ and $a \sim_R b'$ for $b, b' \in B$. We claim that $b = b'$:

  1. Since $a \sim_R b$, we have $b \sim_{R^{\dagger}} a$.
  2. Since $R \diamond R^{\dagger} \subset \chi_B$, we have

$$[R \diamond R^{\dagger}](b, b') \preceq_{\{\mathrm{t},\mathrm{f}\}} \chi_B(b, b'),$$

     and since $b \sim_{R^{\dagger}} a$ and $a \sim_R b'$, it follows that $[R \diamond R^{\dagger}](b, b') = \mathrm{true}$, and thus $\chi_B(b, b') = \mathrm{true}$ as well, i.e. $b = b'$.

This finishes the proof. $\square$

1.5 Total Relations

Let A and B be sets.

Definition 1.5.1 ► Total Relations

A relation $R\colon A \nrightarrow B$ is **total** if, for each $a \in A$, we have $R(a) \neq \emptyset$.

Proposition 1.5.2 ► Properties of Total Relations

Let $R\colon A \nrightarrow B$ be a relation.

1. *Characterisations.* The following conditions are equivalent:
   (a) The relation R is total.
   (b) We have $\chi_A \subset R^\dagger \diamond R$.

Proof 1.5.3 ► Proof of Proposition 1.5.2

Item 1: Characterisations

We claim that Items 1a and 1b are indeed equivalent:

- *Item 1a* $\implies$ *Item 1b*: We have to show that, for each $(a, a') \in A$, we have

$$\chi_A(a, a') \preceq_{\{\mathrm{t},\mathrm{f}\}} [R^\dagger \diamond R](a, a'),$$

  i.e. that if $a = a'$, then there exists some $b \in B$ such that $a \sim_R b$ and $b \sim_{R^\dagger} a'$ (i.e. $a \sim_R b$ again), which follows from the totality of R.

- *Item 1b* $\implies$ *Item 1a*: Given $a \in A$, since $\chi_A \subset R^\dagger \diamond R$, we must have

$$\{a\} \subset [R^\dagger \diamond R](a),$$

  implying that there must exist some $b \in B$ such that $a \sim_R b$ and $b \sim_{R^\dagger} a$ (i.e. $a \sim_R b$) and thus $R(a) \neq \emptyset$, as $b \in R(a)$.

This finishes the proof.

2 Categories of Relations

2.1 The Category of Relations

Definition 2.1.1 ► The Category of Relations

The **category of relations** is the category Rel where

- *Objects.* The objects of Rel are sets;
- *Morphisms.* For each $A, B \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$\mathrm{Rel}(A, B) \overset{\mathrm{def}}{=} \mathrm{Rel}(A, B);$$

- *Identities.* For each $A \in \mathrm{Obj}(\mathsf{Rel})$, the unit map

$$\mathbb{1}^{\mathsf{Rel}}_{A} \colon \mathrm{pt} \longrightarrow \mathrm{Rel}(A, A)$$

  of Rel at A is defined by

$$\mathrm{id}^{\mathsf{Rel}}_{A} \overset{\mathrm{def}}{=} \chi_A(-_1, -_2),$$

  where $\chi_A(-_1, -_2)$ is the characteristic relation of A of Constructions With Sets, Item 3 of Definition 4.1.1;

- *Composition.* For each $A, B, C \in \mathrm{Obj}(\mathsf{Rel})$, the composition map

$$\circ^{\mathsf{Rel}}_{A,B,C} \colon \mathrm{Rel}(B, C) \times \mathrm{Rel}(A, B) \longrightarrow \mathrm{Rel}(A, C)$$

  of Rel at (A, B, C) is defined by

$$S \circ^{\mathsf{Rel}}_{A,B,C} R \overset{\mathrm{def}}{=} S \diamond R$$

  for each $(S, R) \in \mathrm{Rel}(B, C) \times \mathrm{Rel}(A, B)$, where $S \diamond R$ is the composition of S and R of Definition 3.12.1.

2.2 The Closed Symmetric Monoidal Category of Relations

2.2.1 The Monoidal Product

Definition 2.2.1 ► The Monoidal Product of Rel

The **monoidal product of** Rel is the functor

$$\times \colon \mathsf{Rel} \times \mathsf{Rel} \longrightarrow \mathsf{Rel}$$

where

- *Action on Objects.* We have
$$\times(A, B) \overset{\text{def}}{=} A \times B,$$
where $A \times B$ is the Cartesian product of sets of Constructions With Sets, Definition 1.2.1;
- *Action on Morphisms.* For each $(A, C), (B, D) \in \mathrm{Obj}(\mathsf{Rel} \times \mathsf{Rel})$, the action on morphisms
$$\times_{(A,C),(B,D)} \colon \mathsf{Rel}(A, B) \times \mathsf{Rel}(C, D) \longrightarrow \mathsf{Rel}(A \times C, B \times D)$$
of $\times$ is given by sending a pair of morphisms (R, S) of the form
$$R \colon A \nrightarrow B,$$
$$S \colon C \nrightarrow D$$
to the relation
$$R \times S \colon A \times C \nrightarrow B \times D$$
of Definition 3.9.1.

2.2.2 The Monoidal Unit

Definition 2.2.2 ► The Monoidal Unit of Rel

The **monoidal unit of** Rel is the functor
$$\mathbb{1}^{\mathsf{Rel}} \colon \mathrm{pt} \longrightarrow \mathsf{Rel}$$
picking the set
$$\mathbb{1}_{\mathsf{Rel}} \overset{\text{def}}{=} \mathrm{pt}$$
of Rel.

2.2.3 The Associator

Definition 2.2.3 ▶ The Associator of Rel

The **associator of** Rel is the natural isomorphism

$$\alpha^{\mathsf{Rel}}\colon \times\circ((\times)\times\mathrm{id}) \overset{\cong}{\Longrightarrow} \times\circ(\mathrm{id}\times(\times)), \qquad \begin{tikzcd} \mathsf{Rel}\times\mathsf{Rel}\times\mathsf{Rel} \arrow[r, "\mathrm{id}\times(\times)"] \arrow[d, "(\times)\times\mathrm{id}"'] & \mathsf{Rel}\times\mathsf{Rel} \arrow[d, "\times"] \\ \mathsf{Rel}\times\mathsf{Rel} \arrow[r, "\times"'] \arrow[ur, phantom, "\Nearrow \alpha^{\mathsf{Rel}}"] & \mathsf{Rel}, \end{tikzcd}$$

whose component

$$\alpha^{\mathsf{Rel}}_{A,B,C}\colon (A\times B)\times C \nrightarrow A\times(B\times C)$$

at (A, B, C) is defined by declaring

$$((a,b),c) \sim_{\alpha^{\mathsf{Rel}}_{A,B,C}} (a',(b',c'))$$

iff $a = a'$, $b = b'$, and $c = c'$.

2.2.4 The Left Unitor

Definition 2.2.4 ▶ The Left Unitor of Rel

The **left unitor of** Rel is the natural isomorphism

$$\lambda^{\mathsf{Rel}}\colon \times\circ(\mathbb{1}^{\mathsf{Rel}}\times\mathrm{id}) \overset{\cong}{\Longrightarrow} \boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathsf{Rel}}, \qquad \begin{tikzcd} \mathrm{pt}\times\mathsf{Rel} \arrow[r, "\mathbb{1}^{\mathsf{Rel}}\times\mathrm{id}"] \arrow[dr, dashed, bend right, "\lambda^{\mathsf{Cats}_2}_{\mathsf{Rel}}"'] & \mathsf{Rel}\times\mathsf{Rel} \arrow[d, "\times"] \arrow[dl, phantom, "\Swarrow \lambda^{\mathsf{Rel}}"] \\ & \mathsf{Rel}, \end{tikzcd}$$

whose component

$$\lambda^{\mathsf{Rel}}_{A}\colon \mathbb{1}_{\mathsf{Rel}}\times A \nrightarrow A$$

at A is defined by declaring

$$(\star, a) \sim_{\lambda^{\mathsf{Rel}}_{A}} b$$

iff $a = b$.

2.2.5 The Right Unitor

Definition 2.2.5 ► The Right Unitor of Rel

The **right unitor of** Rel is the natural isomorphism

$$\rho^{\mathsf{Rel}} \colon \times \circ (\mathrm{id} \times \mathbb{1}^{\mathsf{Rel}}) \overset{\cong}{\Longrightarrow} \boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathsf{Rel}}, \qquad \begin{tikzcd} \mathsf{Rel} \times \mathrm{pt} \arrow[r, "\mathrm{id} \times \mathbb{1}^{\mathsf{Rel}}"] \arrow[dr, dashed, bend right, "\boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathsf{Rel}}"'] & \mathsf{Rel} \times \mathsf{Rel} \arrow[d, "\times"] \arrow[dl, phantom, "\Swarrow \rho^{\mathsf{Rel}}"] \\ & \mathsf{Rel}, \end{tikzcd}$$

whose component

$$\rho^{\mathsf{Rel}}_A \colon A \times \mathbb{1}_{\mathsf{Rel}} \nrightarrow A$$

at A is defined by declaring

$$(a, \star) \sim_{\rho^{\mathsf{Rel}}_A} b$$

iff $a = b$.

2.2.6 The Symmetry

Definition 2.2.6 ► The Symmetry of Rel

The **symmetry of** Rel is the natural isomorphism

$$\sigma^{\mathsf{Rel}} \colon \times \Longrightarrow \times \circ \boldsymbol{\sigma}^{\mathsf{Cats}_2}_{\mathsf{Rel},\mathsf{Rel}}, \qquad \begin{tikzcd} \mathsf{Rel} \times \mathsf{Rel} \arrow[rr, "\times"] \arrow[dr, bend right, "\boldsymbol{\sigma}^{\mathsf{Cats}_2}_{\mathsf{Rel},\mathsf{Rel}}"'] & \arrow[d, phantom, "\Downarrow \sigma^{\mathsf{Rel}}"] & \mathsf{Rel}, \\ & \mathsf{Rel} \times \mathsf{Rel} \arrow[ur, bend right, "\times"'] & \end{tikzcd}$$

whose component

$$\sigma^{\mathsf{Rel}}_{A,B} \colon A \times B \to B \times A$$

at (A, B) is defined by declaring

$$(a, b) \sim_{\sigma^{\mathsf{Rel}}_{A,B}} (b', a')$$

iff $a = a'$ and $b = b'$.

2.2.7 The Internal Hom

Definition 2.2.7 ► The Internal Hom of Rel

The **internal Hom of** Rel is the functor

$$\mathbf{Hom}_{\mathsf{Rel}} \colon \mathsf{Rel}^{\mathsf{op}} \times \mathsf{Rel} \longrightarrow \mathsf{Rel}$$

defined by

$$\mathbf{Hom}_{\mathsf{Rel}}(A, B) \overset{\text{def}}{=} A \times B$$

for each $A, B \in \mathrm{Obj}(\mathsf{Rel})$.

Proposition 2.2.8 ► Properties of the Internal Hom of Rel

Let $A, B, C \in \mathrm{Obj}(\mathsf{Rel})$.

1. *Via Self-Duality.* The internal Hom $\mathbf{Hom}_{\mathsf{Rel}}$ of Rel is given by the composition

$$\mathsf{Rel}^{\mathsf{op}} \times \mathsf{Rel} \xrightarrow{\cong} \mathsf{Rel} \times \mathsf{Rel} \xrightarrow{\times} \mathsf{Rel},$$

where the self-duality equivalence $\mathsf{Rel}^{\mathsf{op}} \cong \mathsf{Rel}$ comes from Item 1 of Proposition 2.5.1.

2. *Adjointness.* We have adjunctions

$$(A \times - \dashv \mathbf{Hom}_{\mathsf{Rel}}(A, -)) \colon \quad \begin{tikzcd} \mathsf{Rel} \arrow[r, bend left, "A\times -"] \arrow[r, phantom, "\perp"] & \mathsf{Rel} \arrow[l, bend left, "\mathbf{Hom}_{\mathsf{Rel}}(A,-)"] \end{tikzcd},$$

$$(- \times B \dashv \mathbf{Hom}_{\mathsf{Rel}}(B, -)) \colon \quad \begin{tikzcd} \mathsf{Rel} \arrow[r, bend left, "-\times B"] \arrow[r, phantom, "\perp"] & \mathsf{Rel} \arrow[l, bend left, "\mathbf{Hom}_{\mathsf{Rel}}(B,-)"] \end{tikzcd},$$

witnessed by bijections

$$\begin{aligned}\mathsf{Rel}(A \times B, C) &\cong \mathsf{Rel}(A, \mathbf{Hom}_{\mathsf{Rel}}(B, C)) \\ &\stackrel{\text{def}}{=} \mathsf{Rel}(A, B \times C), \\ \mathsf{Rel}(A \times B, C) &\cong \mathsf{Rel}(B, \mathbf{Hom}_{\mathsf{Rel}}(A, C)) \\ &\stackrel{\text{def}}{=} \mathsf{Rel}(B, A \times C),\end{aligned}$$

natural in $A, B, C \in \mathrm{Obj}(\mathsf{Rel})$.

PROOF 2.2.9 ► PROOF OF PROPOSITION 2.2.8

Item 1: Via Self-Duality

Omitted.

Item 2: Adjointness

Indeed, we have

$$\begin{aligned}\mathsf{Rel}(A \times B, C) &\stackrel{\text{def}}{=} \mathsf{Sets}(A \times B \times C, \{\mathsf{true}, \mathsf{false}\}) \\ &\stackrel{\text{def}}{=} \mathsf{Rel}(A, B \times C) \\ &\stackrel{\text{def}}{=} \mathsf{Rel}(A, \mathbf{Hom}_{\mathsf{Rel}}(B, C)),\end{aligned}$$

and similarly for the bijection $\mathsf{Rel}(A \times B, C) \cong \mathsf{Rel}(B, \mathbf{Hom}_{\mathsf{Rel}}(A, C))$. □

2.2.8 The Closed Symmetric Monoidal Category of Relations

DEFINITION 2.2.10 ► THE CLOSED SYMMETRIC MONOIDAL CATEGORY OF RELATIONS

The **closed symmetric monoidal category of relations** is the closed symmetric monoidal category

$$\left(\mathsf{Rel}, \times, \mathbb{1}_{\mathsf{Rel}}, \alpha^{\mathsf{Rel}}, \lambda^{\mathsf{Rel}}, \rho^{\mathsf{Rel}}, \sigma^{\mathsf{Rel}}, \mathbf{Hom}_{\mathsf{Rel}}\right)$$

consisting of

- *The Underlying Category.* The category Rel of sets and relations of Definition 2.1.1;

- *The Monoidal Product.* The functor

$$\times\colon \mathsf{Rel}\times\mathsf{Rel}\longrightarrow\mathsf{Rel}$$

of Definition 2.2.1;

- *The Monoidal Unit.* The functor $\mathbb{1}^{\mathsf{Rel}}$ of Definition 2.2.2;
- *The Associator.* The natural isomorphism α^{Rel} of Definition 2.2.3;
- *The Left Unitor.* The natural isomorphism λ^{Rel} of Definition 2.2.4;
- *The Right Unitor.* The natural isomorphism ρ^{Rel} of Definition 2.2.5;
- *The Symmetry.* The natural isomorphism σ^{Rel} of Definition 2.2.6;
- *The Internal Hom.* The functor

$$\mathbf{Hom}_{\mathsf{Rel}}\colon \mathsf{Rel}^{\mathsf{op}}\times\mathsf{Rel}\longrightarrow\mathsf{Rel}$$

of Definition 2.2.7.

2.3 The 2-Category of Relations

DEFINITION 2.3.1 ▶ THE 2-CATEGORY OF RELATIONS

The **2-category of relations** is the locally posetal 2-category **Rel** where

- *Objects.* The objects of **Rel** are sets;
- ***Hom**-Objects.* For each $A, B \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$\begin{aligned}\mathrm{Hom}_{\mathbf{Rel}}(A,B) &\overset{\text{def}}{=} \mathbf{Rel}(A,B)\\ &\overset{\text{def}}{=} (\mathrm{Rel}(A,B),\subset);\end{aligned}$$

- *Identities.* For each $A \in \mathrm{Obj}(\mathbf{Rel})$, the unit map

$$\mathbb{1}^{\mathbf{Rel}}_{A}\colon \mathrm{pt}\longrightarrow\mathbf{Rel}(A,A)$$

of **Rel** at A is defined by

$$\mathrm{id}^{\mathbf{Rel}}_{A}\overset{\text{def}}{=}\chi_{A}(-_1,-_2),$$

where $\chi_A(-_1, -_2)$ is the characteristic relation of A of Constructions With Sets, Item 3 of Definition 4.1.1;

- *Composition.* For each $A, B, C \in \mathrm{Obj}(\mathbf{Rel})$, the composition map[1]

$$\circ^{\mathbf{Rel}}_{A,B,C} \colon \mathbf{Rel}(B, C) \times \mathbf{Rel}(A, B) \longrightarrow \mathbf{Rel}(A, C)$$

of **Rel** at (A, B, C) is defined by

$$S \circ^{\mathbf{Rel}}_{A,B,C} R \overset{\text{def}}{=} S \diamond R$$

for each $(S, R) \in \mathbf{Rel}(B, C) \times \mathbf{Rel}(A, B)$, where $S \diamond R$ is the composition of S and R of Definition 3.12.1.

[1] Note that this is indeed a morphism of posets: given relations $R_1, R_2 \in \mathbf{Rel}(A, B)$ and $S_1, S_2 \in \mathbf{Rel}(B, C)$ such that

$$\begin{aligned} R_1 &\subset R_2, \\ S_1 &\subset S_2, \end{aligned}$$

we have also $S_1 \diamond R_1 \subset S_2 \diamond R_2$.

2.4 The Double Category of Relations

2.4.1 The Double Category of Relations

Definition 2.4.1 ► The Double Category of Relations

The **double category of relations** is the locally posetal double category $\mathrm{Rel}^{\mathrm{dbl}}$ where

- *Objects.* The objects of $\mathrm{Rel}^{\mathrm{dbl}}$ are sets;
- *Vertical Morphisms.* The vertical morphisms of $\mathrm{Rel}^{\mathrm{dbl}}$ are maps of sets $f \colon A \to B$;
- *Horizontal Morphisms.* The horizontal morphisms of $\mathrm{Rel}^{\mathrm{dbl}}$ are relations $R \colon A \nrightarrow X$;

- *2-Morphisms.* A 2-cell

$$\begin{tikzcd} A \arrow[r, "R", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & B \arrow[d, "g"] \\ X \arrow[r, "S"', "\shortmid"{marking}] & Y \end{tikzcd}$$

  of $\mathsf{Rel}^{\mathsf{dbl}}$ is either non-existent or an inclusion of relations of the form

$$R \subset S \circ (f \times g), \qquad \begin{tikzcd} A \times B \arrow[r, "R"] \arrow[d, "f\times g"'] \arrow[dr, phantom, "\curvearrowleft"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ X \times Y \arrow[r, "S"'] & \{\mathsf{true}, \mathsf{false}\}; \end{tikzcd}$$

- *Horizontal Identities.* The horizontal unit functor of $\mathsf{Rel}^{\mathsf{dbl}}$ is the functor of Definition 2.4.2;

- *Vertical Identities.* For each $A \in \mathrm{Obj}\left(\mathsf{Rel}^{\mathsf{dbl}}\right)$, we have

$$\mathrm{id}_A^{\mathsf{Rel}^{\mathsf{dbl}}} \overset{\text{def}}{=} \mathrm{id}_A;$$

- *Identity 2-Morphisms.* For each horizontal morphism $R\colon A \nrightarrow B$ of $\mathsf{Rel}^{\mathsf{dbl}}$, the identity 2-morphism

$$\begin{tikzcd} A \arrow[r, "R", "\shortmid"{marking}] \arrow[d, "\mathrm{id}_A"'] \arrow[dr, phantom, "\Downarrow\mathrm{id}_R"] & B \arrow[d, "\mathrm{id}_B"] \\ A \arrow[r, "R"', "\shortmid"{marking}] & B \end{tikzcd}$$

  of R is the identity inclusion

$$R \subset R, \qquad \begin{tikzcd} B \times A \arrow[r, "R"] \arrow[d, "\mathrm{id}_B\times\mathrm{id}_A"'] \arrow[dr, phantom, "\curvearrowleft"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ B \times A \arrow[r, "R"'] & \{\mathsf{true}, \mathsf{false}\}; \end{tikzcd}$$

- *Horizontal Composition.* The horizontal composition functor of $\mathsf{Rel}^{\mathsf{dbl}}$ is the functor of Definition 2.4.3;

- *Vertical Composition of 1-Morphisms.* For each composable pair $A \xrightarrow{F} B \xrightarrow{G} C$ of vertical morphisms of $\mathsf{Rel}^{\mathsf{dbl}}$, i.e. maps of sets, we have

$$g \circ^{\mathsf{Rel}^{\mathsf{dbl}}} f \overset{\text{def}}{=} g \circ f;$$

- *Vertical Composition of 2-Morphisms.* The vertical composition of 2-morphisms in $\mathsf{Rel}^{\mathsf{dbl}}$ is defined as in Definition 2.4.4;

- *Associators.* The associators of $\mathsf{Rel}^{\mathsf{dbl}}$ is defined as in Definition 2.4.5;

- *Left Unitors.* The left unitors of $\mathsf{Rel}^{\mathsf{dbl}}$ is defined as in Definition 2.4.6;

- *Right Unitors.* The right unitors of $\mathsf{Rel}^{\mathsf{dbl}}$ is defined as in Definition 2.4.7.

2.4.2 Horizontal Identities

Definition 2.4.2 ▶ The Horizontal Identities of $\mathsf{Rel}^{\mathsf{dbl}}$

The **horizontal unit functor** of $\mathsf{Rel}^{\mathsf{dbl}}$ is the functor

$$\nVdash^{\mathsf{Rel}^{\mathsf{dbl}}} \colon \mathsf{Rel}^{\mathsf{dbl}}_0 \longrightarrow \mathsf{Rel}^{\mathsf{dbl}}_1$$

of $\mathsf{Rel}^{\mathsf{dbl}}$ is the functor where

- *Action on Objects.* For each $A \in \mathrm{Obj}\left(\mathsf{Rel}^{\mathsf{dbl}}_0\right)$, we have

$$\nVdash_A \overset{\text{def}}{=} \chi_A(-_1, -_2);$$

- *Action on Morphisms.* For each vertical morphism $f \colon A \longrightarrow B$ of $\mathsf{Rel}^{\mathsf{dbl}}$, i.e. each map of sets f from A to B, the identity 2-morphism

$$\begin{tikzcd} A \arrow[r, "\nVdash_A", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\overset{\Vert}{\underset{\Downarrow}{\nVdash_f}}"] & A \arrow[d, "f"] \\ B \arrow[r, "\nVdash_B"', "\shortmid"{marking}] & B \end{tikzcd}$$

of f is the inclusion

$$\chi_B \circ (f \times f) \subset \chi_A, \qquad \begin{tikzcd} A \times A \arrow[r, "\chi_A(-_1,-_2)"] \arrow[d, "f\times f"'] \arrow[dr, phantom, "\subset"] & \{\text{true}, \text{false}\} \arrow[d, "\mathrm{id}_{\{\text{true},\text{false}\}}"] \\ B \times B \arrow[r, "\chi_B(-_1,-_2)"'] & \{\text{true}, \text{false}\} \end{tikzcd}$$

of Constructions With Sets, Proposition 4.1.3.

2.4.3 Horizontal Composition

Definition 2.4.3 ▶ The Horizontal Composition of $\mathsf{Rel}^{\mathsf{dbl}}$

The **horizontal composition functor** of $\mathsf{Rel}^{\mathsf{dbl}}$ is the functor

$$\odot^{\mathsf{Rel}^{\mathsf{dbl}}} \colon \mathsf{Rel}^{\mathsf{dbl}}_1 \underset{\mathsf{Rel}^{\mathsf{dbl}}_0}{\times} \mathsf{Rel}^{\mathsf{dbl}}_1 \to \mathsf{Rel}^{\mathsf{dbl}}_1$$

of $\mathsf{Rel}^{\mathsf{dbl}}$ is the functor where

- *Action on Objects.* For each composable pair $A \overset{R}{\nrightarrow} B \overset{S}{\nrightarrow} C$ of horizontal morphisms of $\mathsf{Rel}^{\mathsf{dbl}}$, we have

$$S \odot R \overset{\text{def}}{=} S \diamond R,$$

  where $S \diamond R$ is the composition of R and S of Definition 3.12.1;

- *Action on Morphisms.* For each horizontally composable pair

$$\begin{tikzcd} A \arrow[r, "R", "|"marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & B \arrow[d, "g"] \\ X \arrow[r, "T"', "|"marking] & Y \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "S", "|"marking] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\beta"] & C \arrow[d, "h"] \\ Y \arrow[r, "U"', "|"marking] & Z \end{tikzcd}$$

of 2-morphisms of $\mathsf{Rel}^{\mathsf{dbl}}$, i.e. for each pair

$$\begin{tikzcd} A \times B \arrow[r, "R"] \arrow[d, "f \times g"'] & \{\text{true}, \text{false}\} \arrow[d, "\mathrm{id}_{\{\text{true},\text{false}\}}"] \arrow[dl, phantom, "\subset"] \\ X \times Y \arrow[r, "T"'] & \{\text{true}, \text{false}\} \end{tikzcd} \qquad \begin{tikzcd} B \times C \arrow[r, "S"] \arrow[d, "g \times h"'] & \{\text{true}, \text{false}\} \arrow[d, "\mathrm{id}_{\{\text{true},\text{false}\}}"] \arrow[dl, phantom, "\subset"] \\ Y \times Z \arrow[r, "U"'] & \{\text{true}, \text{false}\} \end{tikzcd}$$

of inclusions of relations, the horizontal composition

$$\begin{tikzcd} A \arrow[r, "S \odot R", "\shortmid"{marking}] \arrow[d, "f"'] & C \arrow[d, "h"] \\ X \arrow[r, "U \odot T"', "\shortmid"{marking}] \arrow[u, phantom, shift left=4ex, "\beta\odot\alpha\ \Downarrow"] & Z \end{tikzcd}$$

of α and β is the inclusion of relations[1]

$$(U \diamond T) \circ (f \times h) \subset (S \diamond R) \qquad \begin{tikzcd} A \times C \arrow[r, "S \diamond R"] \arrow[d, "f \times h"'] & \{\text{true}, \text{false}\} \arrow[d, "\mathrm{id}_{\{\text{true},\text{false}\}}"] \arrow[dl, phantom, "\subset"] \\ X \times Z \arrow[r, "U \diamond T"'] & \{\text{true}, \text{false}\}. \end{tikzcd}$$

---

[1]This is justified by noting that, given $(a, c) \in A \times C$, the statement

- We have $a \sim_{(U \diamond T)\circ(f \times h)} c$, i.e. $f(a) \sim_{U \diamond T} h(c)$, i.e. there exists some $y \in Y$ such that:
  1. We have $f(a) \sim_T y$;
  2. We have $y \sim_U h(c)$;

is implied by the statement

- We have $a \sim_{S \diamond R} c$, i.e. there exists some $b \in B$ such that:
  1. We have $a \sim_R b$;
  2. We have $b \sim_S c$;

since:

- If $a \sim_R b$, then $f(a) \sim_T g(b)$, as $T \circ (f \times g) \subset R$;
- If $b \sim_S c$, then $g(b) \sim_U h(c)$, as $U \circ (g \times h) \subset S$;

2.4.4 Vertical Composition of 2-Morphisms

Definition 2.4.4 ► The Vertical Composition of 2-Morphisms in $\mathsf{Rel}^{\mathsf{dbl}}$

The **vertical composition** in $\mathsf{Rel}^{\mathsf{dbl}}$ is defined as follows: for each vertically composable pair

$$\begin{tikzcd} A \arrow[r, "R", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & X \arrow[d, "g"] \\ B \arrow[r, "S"', "\shortmid"{marking}] & Y \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "S", "\shortmid"{marking}] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow\beta"] & Y \arrow[d, "k"] \\ C \arrow[r, "T"', "\shortmid"{marking}] & Z \end{tikzcd}$$

of 2-morphisms of $\mathsf{Rel}^{\mathsf{dbl}}$, i.e. for each each pair

$$\begin{tikzcd} A \times X \arrow[r, "R"] \arrow[d, "f \times g"'] \arrow[dr, phantom, "\subset"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ B \times Y \arrow[r, "S"'] & \{\mathsf{true}, \mathsf{false}\} \end{tikzcd} \qquad \begin{tikzcd} B \times Y \arrow[r, "S"] \arrow[d, "h \times k"'] \arrow[dr, phantom, "\subset"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ C \times Z \arrow[r, "T"'] & \{\mathsf{true}, \mathsf{false}\} \end{tikzcd}$$

of inclusions of relations, we define the vertical composition

$$\begin{tikzcd} A \arrow[r, "R", "\shortmid"{marking}] \arrow[d, "h \circ f"'] \arrow[dr, phantom, "\Downarrow\beta \circ \alpha"] & X \arrow[d, "k \circ g"] \\ C \arrow[r, "T"', "\shortmid"{marking}] & Z \end{tikzcd}$$

of α and β as the inclusion of relations

$$T \circ [(h \circ f) \times (k \circ g)] \subset R, \qquad \begin{tikzcd} A \times X \arrow[r, "R"] \arrow[d, "(h \circ f) \times (k \circ g)"'] \arrow[dr, phantom, "\subset"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ C \times Z \arrow[r, "T"'] & \{\mathsf{true}, \mathsf{false}\} \end{tikzcd}$$

given by the pasting of inclusions[1]

$$
\begin{tikzcd}
A \times X \arrow[r, "R"] \arrow[d, "f \times g"'] \arrow[dr, phantom, "\circlearrowleft"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true}, \mathsf{false}\}}"] \\
B \times Y \arrow[r, "S" description] \arrow[d, "h \times k"'] \arrow[dr, phantom, "\circlearrowleft"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true}, \mathsf{false}\}}"] \\
C \times Z \arrow[r, "T"'] & \{\mathsf{true}, \mathsf{false}\}.
\end{tikzcd}
$$

[1] This is justified by noting that, given $(a, x) \in A \times X$, the statement

- We have $h(f(a)) \sim_T k(g(x))$;

is implied by the statement

- We have $a \sim_R x$;

since

- If $a \sim_R x$, then $f(a) \sim_S g(x)$, as $S \circ (f \times g) \subset R$;
- If $b \sim_S y$, then $h(b) \sim_T k(y)$, as $T \circ (h \times k) \subset S$, and thus, in particular:
  - If $f(a) \sim_S g(x)$, then $h(f(a)) \sim_T k(g(x))$;

2.4.5 The Associators

Definition 2.4.5 ▶ The Associators of $\mathsf{Rel}^{\mathsf{dbl}}$

For each composable triple $A \overset{R}{\relbar\joinrel\mapstochar\joinrel\rightarrow} B \overset{S}{\relbar\joinrel\mapstochar\joinrel\rightarrow} C \overset{T}{\relbar\joinrel\mapstochar\joinrel\rightarrow} D$ of horizontal morphisms of $\mathsf{Rel}^{\mathsf{dbl}}$, the component

$$
\alpha^{\mathsf{Rel}^{\mathsf{dbl}}}_{T,S,R} \colon (T \odot S) \odot R \overset{\cong}{\Longrightarrow} T \odot (S \odot R),
\qquad
\begin{tikzcd}
A \arrow[r, "R", "\shortmid"{marking}] \arrow[d, "\mathrm{id}_A"'] & B \arrow[r, "S", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow \alpha^{\mathsf{Rel}^{\mathsf{dbl}}}_{T,S,R}"] & C \arrow[r, "T", "\shortmid"{marking}] & D \arrow[d, "\mathrm{id}_D"] \\
A \arrow[r, "R"', "\shortmid"{marking}] & B \arrow[r, "S"', "\shortmid"{marking}] & C \arrow[r, "T"', "\shortmid"{marking}] & D
\end{tikzcd}
$$

of the associator of $\mathsf{Rel}^{\mathsf{dbl}}$ at (R, S, T) is the identity inclusion[1]

$$(T \diamond S) \diamond R = T \diamond (S \diamond R)$$

$$\begin{tikzcd} A \times B \arrow[r, "(T \diamond S) \diamond R"] \arrow[d, equal] \arrow[dr, phantom, "\sslash"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ A \times B \arrow[r, "T \diamond (S \diamond R)"'] & \{\mathsf{true}, \mathsf{false}\}. \end{tikzcd}$$

[1]This is justified by Item 2 of Proposition 3.12.3.

2.4.6 The Left Unitors

Definition 2.4.6 ► The Left Unitors of $\mathsf{Rel}^{\mathsf{dbl}}$

For each horizontal morphism $R \colon A \nrightarrow B$ of $\mathsf{Rel}^{\mathsf{dbl}}$, the component

$$\lambda^{\mathsf{Rel}^{\mathsf{dbl}}}_R \colon \chi_B \odot R \overset{\cong}{\Longrightarrow} R,$$

$$\begin{tikzcd} A \arrow[r, "R", "|"{marking}] \arrow[d, "\mathrm{id}_A"'] & B \arrow[r, "\chi_B", "|"{marking}] \arrow[d, phantom, "\lambda^{\mathsf{Rel}^{\mathsf{dbl}}}_R \Downarrow"] & B \arrow[d, "\mathrm{id}_B"] \\ A \arrow[rr, "R"', "|"{marking}] & & B \end{tikzcd}$$

of the left unitor of $\mathsf{Rel}^{\mathsf{dbl}}$ at R is the identity inclusion[1]

$$R = \chi_B \diamond R,$$

$$\begin{tikzcd} A \times B \arrow[r, "\chi_B \diamond R"] \arrow[d, equal] \arrow[dr, phantom, "\sslash"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ A \times B \arrow[r, "R"'] & \{\mathsf{true}, \mathsf{false}\}. \end{tikzcd}$$

[1]This is justified by Item 3 of Proposition 3.12.3.

2.4.7 The Right Unitors

Definition 2.4.7 ► The Right Unitors of $\mathsf{Rel}^{\mathsf{dbl}}$

For each horizontal morphism $R\colon A \nrightarrow B$ of $\mathsf{Rel}^{\mathsf{dbl}}$, the component

$$\rho^{\mathsf{Rel}^{\mathsf{dbl}}}_R \colon R \odot \chi_A \overset{\cong}{\Longrightarrow} R, \qquad \begin{tikzcd} A \arrow[r, "\chi_A", "|"{marking}] \arrow[d, "\mathrm{id}_A"'] & A \arrow[r, "R", "|"{marking}] \arrow[d, phantom, "\Downarrow \rho^{\mathsf{Rel}^{\mathsf{dbl}}}_R"] & B \arrow[d, "\mathrm{id}_B"] \\ A \arrow[rr, "R"', "|"{marking}] & & B \end{tikzcd}$$

of the right unitor of $\mathsf{Rel}^{\mathsf{dbl}}$ at R is the identity inclusion[1]

$$R = R \diamond \chi_A, \qquad \begin{tikzcd} A \times B \arrow[r, "R \diamond \chi_A"] \arrow[d, equal] \arrow[dr, phantom, "\mathbin{/\!\!/}"] & \{\mathrm{true}, \mathrm{false}\} \arrow[d, "\mathrm{id}_{\{\mathrm{true},\mathrm{false}\}}"] \\ A \times B \arrow[r, "R"'] & \{\mathrm{true}, \mathrm{false}\}. \end{tikzcd}$$

[1] This is justified by Item 3 of Proposition 3.12.3.

2.5 Properties of the Category of Relations

Proposition 2.5.1 ► Properties of the Category of Relations

Let A and B be sets.

1. *Self-Duality I.* The category Rel is self-dual, i.e. we have an equivalence

$$\mathsf{Rel}^{\mathsf{op}} \overset{\text{eq.}}{\cong} \mathsf{Rel}$$

   of categories.

2. *Self-Duality II.* The bicategory **Rel** is self-dual, i.e. we have a biequivalence

$$\mathbf{Rel}^{\mathsf{op}} \overset{\text{eq.}}{\cong} \mathbf{Rel}$$

   of bicategories.

3. *Equivalences and Isomorphisms in Rel.* Let $R\colon A \nrightarrow B$ be a relation from A to B. The following conditions are equivalent:

(a) The relation $R\colon A \nrightarrow B$ is an equivalence in **Rel**, i.e. there exists a relation $R^{-1}\colon B \nrightarrow A$ from B to A together with isomorphisms

$$\begin{aligned} R^{-1} \diamond R &\cong \chi_A, \\ R \diamond R^{-1} &\cong \chi_B. \end{aligned}$$

(b) The relation $R\colon A \nrightarrow B$ is an isomorphism in Rel, i.e. there exists a relation $R^{-1}\colon B \nrightarrow A$ from B to A such that we have

$$\begin{aligned} R^{-1} \diamond R &= \chi_A, \\ R \diamond R^{-1} &= \chi_B. \end{aligned}$$

(c) There exists a bijection $f\colon A \xrightarrow{\cong} B$ with $R = \mathrm{Gr}(f)$.

4. *Adjunctions in* ***Rel***. We have a natural bijection

$$\left\{\begin{matrix}\text{Adjunctions in } \mathbf{Rel} \\ \text{from } A \text{ to } B\end{matrix}\right\} \cong \left\{\begin{matrix}\text{Functions} \\ \text{from } A \text{ to } B\end{matrix}\right\},$$

with every adjunction in **Rel** being of the form $\mathrm{Gr}(f) \dashv f^{-1}$ for some function f.

5. *Monads in* ***Rel***. We have a natural bijection[1]

$$\left\{\begin{matrix}\text{Monads in} \\ \mathbf{Rel} \text{ on } A\end{matrix}\right\} \cong \{\text{Preorders on } A\}.$$

6. *Comonads in* ***Rel***. We have a natural bijection

$$\left\{\begin{matrix}\text{Comonads in} \\ \mathbf{Rel} \text{ on } A\end{matrix}\right\} \cong \{\text{Subsets of } A\}.$$

7. *Characterisations of Monomorphisms in* Rel. Let $R\colon A \nrightarrow B$ be a relation. The following conditions are equivalent:

   (a) The relation R is a monomorphism in Rel.

(b) The direct image function

$$R_* \colon \mathcal{P}(A) \to \mathcal{P}(B)$$

associated to R is injective.

(c) The direct image with compact support function

$$R_! \colon \mathcal{P}(A) \to \mathcal{P}(B)$$

associated to R is injective.

Moreover, if R is a monomorphism, then it satisfies the following condition, and the converse holds if R is total:

($\star$) For each $a, a' \in A$, if there exists some $b \in B$ such that $a \sim_R b$ and $a' \sim_R b$, then $a = a'$.

8. *Epimorphisms in* Rel. Let $R \colon A \nrightarrow B$ be a relation. The following conditions are equivalent:

(a) The relation R is an epimorphism in Rel.

(b) The weak inverse image function

$$R^{-1} \colon \mathcal{P}(B) \to \mathcal{P}(A)$$

associated to R is injective.

(c) The strong inverse image function

$$R_{-1} \colon \mathcal{P}(B) \to \mathcal{P}(A)$$

associated to R is injective.

(d) The function $R \colon A \to \mathcal{P}(B)$ is "surjective on singletons":

($\star$) For each $b \in B$, there exists some $a \in A$ such that $R(a) = \{b\}$.

9. *As a Kleisli Category.* We have an isomorphism of categories

$$\mathsf{Rel} \cong \mathsf{FreeAlg}_{\mathcal{P}},$$

where $\mathcal{P}$ is the powerset monad of Monads, **??**.

10. *Co/Completeness (Or Lack Thereof).* The category Rel is not co/complete, but admits some co/limits:

    (a) *Zero Objects.* The category Rel has a zero object, the empty set $\emptyset$.

    (b) *Co/Products.* The category Rel has co/products, both given by disjoint union of sets.

    (c) *Lack of Co/Equalisers.* The category Rel does not have co/equalisers.

    (d) *Limits of Graphs of Functions.* The category Rel has limits whose arrows are all graphs of functions.

    (e) *Colimits of Graphs of Functions.* The category Rel has colimits whose arrows are all graphs of functions, and these agree with the corresponding limits in Sets.

11. *Existence of Right Kan Extensions.* The right Kan extension

$$\mathrm{Ran}_R \colon \mathrm{Rel}(A, X) \longrightarrow \mathrm{Rel}(B, X)$$

along a relation $R \colon A \nrightarrow B$ exists and is given by

$$\mathrm{Ran}_R(S) \overset{\mathrm{def}}{=} \int_{a \in A} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^a_{-_1}, S^a_{-_2}\right)$$

for each $S \in \mathrm{Rel}(A, X)$, so that the following conditions are equivalent:

    (a) We have $b \sim_{\mathrm{Ran}_R(S)} x$.

    (b) For each $a \in A$, if $a \sim_R b$, then $a \sim_S x$.

12. *Existence of Right Kan Lifts.* The right Kan lift

$$\mathrm{Rift}_R \colon \mathrm{Rel}(X, B) \longrightarrow \mathrm{Rel}(X, A)$$

along a relation $R \colon A \nrightarrow B$ exists and is given by

$$\mathrm{Rift}_R(S) \overset{\mathrm{def}}{=} \int_{b \in B} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^{-_2}_b, S^{-_1}_b\right)$$

for each $S \in \mathrm{Rel}(X, B)$, so that the following conditions are equivalent:

    (a) We have $x \sim_{\mathrm{Rift}_R(S)} a$.

(b) For each $b \in B$, if $a \sim_R b$, then $x \sim_S b$.

13. *Closedness.* The bicategory **Rel** is a closed bicategory, there being, for each $R\colon A \nrightarrow B$ and set X, a pair of adjunctions

$$(R^* \dashv \mathrm{Ran}_R)\colon \quad \begin{tikzcd} \mathrm{Rel}(B,X) \arrow[r, bend left, "R^*"] \arrow[r, phantom, "\perp"] & \mathrm{Rel}(A,X), \arrow[l, bend left, "\mathrm{Ran}_R"] \end{tikzcd}$$

$$(R_* \dashv \mathrm{Rift}_R)\colon \quad \begin{tikzcd} \mathrm{Rel}(X,A) \arrow[r, bend left, "R_*"] \arrow[r, phantom, "\perp"] & \mathrm{Rel}(X,B), \arrow[l, bend left, "\mathrm{Rift}_R"] \end{tikzcd}$$

witnessed by bijections

$$\mathbf{Rel}(S \diamond R, T) \cong \mathbf{Rel}(S, \mathrm{Ran}_R(T)),$$
$$\mathbf{Rel}(R \diamond U, V) \cong \mathbf{Rel}(U, \mathrm{Rift}_R(V)),$$

natural in $S \in \mathrm{Rel}(B,X)$, $T \in \mathrm{Rel}(A,X)$, $U \in \mathrm{Rel}(X,A)$, and $V \in \mathrm{Rel}(X,B)$.

[1]See also Section 6 for an extension of this correspondence to "relative monads on **Rel**".

Proof 2.5.2 ► Proof of Proposition 2.5.1

Item 1: Self-Duality I

Omitted.

Item 2: Self-Duality II

Omitted.

Item 3: Equivalences and Isomorphisms in Rel

We claim that Items 3a to 3c are indeed equivalent:

- *Item 3a* $\Longleftrightarrow$ *Item 3b*: This follows from the fact that **Rel** is locally posetal, so that natural isomorphisms and equalities of 1-cells in **Rel** coincide.

- *Item 3b* $\Longrightarrow$ *Item 3c*: The equalities in Item 3b imply $R \dashv R^{-1}$, and thus by Item 4, there exists a function $f_R\colon A \to B$ associated to R, where, for each $a \in A$, the image $f_R(a)$ of a by f_R is the unique element of $R(a)$, which implies $R = \mathrm{Gr}(f_R)$ in particular. Furthermore, we have $R^{-1} = f_R^{-1}$ (as in

Definition 3.2.1). The conditions from Item 3b then become the following:

$$f_R^{-1} \diamond f_R = \chi_A,$$
$$f_R \diamond f_R^{-1} = \chi_B.$$

All that is left is to show then is that f_R is a bijection:

- *The Function f_R Is Injective.* Let $a, b \in A$ and suppose that $f_R(a) = f_R(b)$. Since $a \sim_R f_R(a)$ and $f_R(a) = f_R(b) \sim_{R^{-1}} b$, the condition $f_R^{-1} \diamond f_R = \chi_A$ implies that $a = b$, showing f_R to be injective.
- *The Function f_R Is Surjective.* Let $b \in B$. Applying the condition $f_R \diamond f_R^{-1} = \chi_B$ to (b, b), it follows that there exists some $a \in A$ such that $f_R^{-1}(b) = a$ and $f_R(a) = b$. This shows f_R to be surjective.

· *Item 3c* $\Longrightarrow$ *Item 3b*: By Item 2, we have an adjunction $\mathrm{Gr}(f) \dashv f^{-1}$, giving inclusions

$$\chi_A \subset f^{-1} \diamond \mathrm{Gr}(f),$$
$$\mathrm{Gr}(f) \diamond f^{-1} \subset \chi_B.$$

We claim the reverse inclusions are also true:

- $f^{-1} \diamond \mathrm{Gr}(f) \subset \chi_A$: This is equivalent to the statement that if $f(a) = b$ and $f^{-1}(b) = a'$, then $a = a'$, which follows from the injectivity of f.
- $\chi_B \subset \mathrm{Gr}(f) \diamond f^{-1}$: This is equivalent to the statement that given $b \in B$ there exists some $a \in A$ such that $f^{-1}(b) = a$ and $f(a) = b$, which follows from the surjectivity of f.

Item 4: Adjunctions in **Rel**

We proceed step by step:

1. *From Adjunctions in **Rel** to Functions.* An adjunction in **Rel** from A to B consists of a pair of relations

$$R \colon A \nrightarrow B,$$
$$S \colon B \nrightarrow A,$$

together with inclusions

$$\chi_A \subset S \diamond R,$$
$$R \diamond S \subset \chi_B.$$

We claim that these conditions imply that R is total and functional, i.e. that $R(a)$ is a singleton for each $a \in A$:

(a) *$R(a)$ Has an Element.* Given $a \in A$, since $\chi_A \subset S \diamond R$, we must have $\{a\} \subset S(R(a))$, implying that there exists some $b \in B$ such that $a \sim_R b$ and $b \sim_S a$, and thus $R(a) \neq \emptyset$, as $b \in R(a)$.

(b) *$R(a)$ Has No More Than One Element.* Suppose that we have $a \sim_R b$ and $a \sim_R b'$ for $b, b' \in B$. We claim that $b = b'$:

   i. Since $\chi_A \subset S \diamond R$, there exists some $k \in B$ such that $a \sim_R k$ and $k \sim_S a$.
   ii. Since $R \diamond S \subset \chi_B$, if $b'' \sim_S a'$ and $a' \sim_R b'''$, then $b'' = b'''$.
   iii. Applying the above to $b'' = k$, $b''' = b$, and $a' = a$, since $k \sim_S a$ and $a \sim_R b'$, we have $k = b$.
   iv. Similarly $k = b'$.
   v. Thus $b = b'$.

Together, the above two items show $R(a)$ to be a singleton, being thus given by $\mathrm{Gr}(f)$ for some function $f\colon A \longrightarrow B$, which gives a map

$$\begin{Bmatrix} \text{Adjunctions in } \mathbf{Rel} \\ \text{from } A \text{ to } B \end{Bmatrix} \longrightarrow \begin{Bmatrix} \text{Functions} \\ \text{from } A \text{ to } B \end{Bmatrix}.$$

Moreover, by uniqueness of adjoints (Internal Adjunctions, Item 2 of Proposition 1.2.4), this implies also that $S = f^{-1}$.

2. *From Functions to Adjunctions in **Rel**.* By Item 2 of Proposition 3.1.2, every function $f\colon A \longrightarrow B$ gives rise to an adjunction $\mathrm{Gr}(f) \dashv f^{-1}$ in Rel, giving a map

$$\begin{Bmatrix} \text{Functions} \\ \text{from } A \text{ to } B \end{Bmatrix} \longrightarrow \begin{Bmatrix} \text{Adjunctions in } \mathbf{Rel} \\ \text{from } A \text{ to } B \end{Bmatrix}.$$

3. *Invertibility: From Functions to Adjunctions Back to Functions.* We need to show that starting with a function $f\colon A \to B$, passing to $\mathrm{Gr}(f) \dashv f^{-1}$, and then passing again to a function gives f again. This is clear however, since we have $a \sim_{\mathrm{Gr}(f)} b$ iff $f(a) = b$.

4. *Invertibility: From Adjunctions to Functions Back to Adjunctions.* We need to show that, given an adjunction $R \dashv S$ in **Rel** giving rise to a function $f_{R,S}\colon A \to B$, we have

$$\begin{aligned}\mathrm{Gr}\big(f_{R,S}\big) &= R,\\ f^{-1}_{R,S} &= S.\end{aligned}$$

We check these explicitly:

- *Gr*$\big(f_{R,S}\big) = R$. We have

$$\begin{aligned}\mathrm{Gr}\big(f_{R,S}\big) &\overset{\text{def}}{=} \big\{(a, f_{R,S}(a)) \in A \times B \,\big|\, a \in A\big\}\\ &\overset{\text{def}}{=} \{(a, R(a)) \in A \times B \mid a \in A\}\\ &= R.\end{aligned}$$

- $f^{-1}_{R,S} = S$. We first claim that, given $a \in A$ and $b \in B$, the following conditions are equivalent:
  - We have $a \sim_R b$.
  - We have $b \sim_S a$.

  Indeed:
  - *If $a \sim_R b$, then $b \sim_S a$:* Since $\chi_A \subset S \diamond R$, there exists $k \in B$ such that $a \sim_R k$ and $k \sim_S a$, but since $a \sim_R b$ and R is functional, we have $k = b$ and thus $b \sim_S a$.
  - *If $b \sim_S a$, then $a \sim_R b$:* First note that since R is total we have $a \sim_R b'$ for some $b' \in B$. Now, since $R \diamond S \subset \chi_B$, $b \sim_S a$, and $a \sim_R b'$, we have $b = b'$, and thus $a \sim_R b$.

  Having show this, we now have

$$\begin{aligned}f^{-1}_{R,S}(b) &\overset{\text{def}}{=} \big\{a \in A \,\big|\, f_{R,S}(a) = b\big\}\\ &\overset{\text{def}}{=} \{a \in A \mid a \sim_R b\}\\ &= \{a \in A \mid b \sim_S a\}\\ &\overset{\text{def}}{=} S(b).\end{aligned}$$

for each $b \in B$, showing $f^{-1}_{R,S} = S$.

This finishes the proof.

Item 5: Monads in Rel

A monad in **Rel** on A consists of a relation $R\colon A \nrightarrow A$ together with maps

$$\begin{aligned}\mu_R &\colon R \diamond R \subset R,\\ \eta_R &\colon \chi_A \subset R\end{aligned}$$

making the diagrams

$$\begin{tikzcd}
\chi_A \diamond R \arrow[r, "\eta_R \diamond \mathrm{id}_R", "|" marking] \arrow[dr, equal, "\lambda^{\mathbf{Rel}(A,B)}_R"'] & R \diamond R \arrow[d, "\mu_R", "|" marking] \\
& R
\end{tikzcd}$$

$$\begin{tikzcd}
& R \diamond (R \diamond R) \arrow[dr, "\mathrm{id}_R \diamond \mu_R", "|" marking] & \\
(R \diamond R) \diamond R \arrow[ur, equal, "\alpha^{\mathbf{Rel}(A,B)}_{R,R,R}"] \arrow[d, "\mu_R \diamond \mathrm{id}_R"', "|" marking] & & R \diamond R \arrow[d, "\mu_R", "|" marking] \\
R \diamond R \arrow[rr, "\mu_R"', "|" marking] & & R
\end{tikzcd}$$

$$\begin{tikzcd}
R \diamond \chi_A \arrow[r, "\mathrm{id}_R \diamond \eta_R", "|" marking] \arrow[dr, equal, "\rho^{\mathbf{Rel}(A,B)}_R"'] & R \diamond R \arrow[d, "\mu_R", "|" marking] \\
& R
\end{tikzcd}$$

commute. However, since all morphisms involved are inclusions, the commutativity of the above diagrams is automatic, and hence all that is left is the data of the two maps μ_R and η_R, which correspond respectively to the following conditions:

1. For each $a, b, c \in A$, if $a \sim_R b$ and $b \sim_R c$, then $a \sim_R c$.
2. For each $a \in A$, we have $a \sim_R a$.

These are exactly the requirements for R to be a preorder (Posets, **??**). Conversely any preorder $\preceq$ gives rise to a pair of maps $\mu_\preceq$ and $\eta_\preceq$, forming a monad on A.

Item 6: Comonads in Rel

A comonad in **Rel** on A consists of a relation $R\colon A \nrightarrow A$ together with maps

$$\begin{aligned}\Delta_R &\colon R \subset R \diamond R,\\ \epsilon_R &\colon R \subset \chi_A\end{aligned}$$

making the diagrams

$$
\begin{tikzcd}
R \arrow[r, "\Delta_R", "\shortmid"{marking}] \arrow[dr, equal, "\lambda^{\mathsf{Rel}(A,B),-1}_R"'] & R \diamond R \arrow[d, "\epsilon_R \diamond \mathrm{id}_R", "\shortmid"{marking}] \\
& \chi_A \diamond R
\end{tikzcd}
\qquad
\begin{tikzcd}
& R \diamond R \arrow[dr, "\mathrm{id}_R \diamond \Delta_R", "\shortmid"{marking}] & \\
R \arrow[ur, "\Delta_R", "\shortmid"{marking}] \arrow[d, "\Delta_R"', "\shortmid"{marking}] & & R \diamond (R \diamond R) \arrow[d, equal, "\alpha^{\mathsf{Rel}(A,B),-1}_{R,R,R}"] \\
R \diamond R \arrow[rr, "\Delta_R \diamond \mathrm{id}_R"', "\shortmid"{marking}] & & (R \diamond R) \diamond R
\end{tikzcd}
\qquad
\begin{tikzcd}
R \arrow[r, "\Delta_R", "\shortmid"{marking}] \arrow[dr, equal, "\rho^{\mathsf{Rel}(A,B),-1}_R"'] & R \diamond R \arrow[d, "\mathrm{id}_R \diamond \epsilon_R", "\shortmid"{marking}] \\
& R \diamond \chi_A
\end{tikzcd}
$$

commute. However, since all morphisms involved are inclusions, the commutativity of the above diagrams is automatic, and hence all that is left is the data of the two maps Δ_R and ϵ_R, which correspond respectively to the following conditions:

1. For each $a, b \in A$, if $a \sim_R b$, then there exists some $k \in A$ such that $a \sim_R k$ and $k \sim_R b$.
2. For each $a, b \in A$, if $a \sim_R b$, then $a = b$.

Taking $k = b$ in the first condition above shows it to be trivially satisfied, while the second condition implies $R \subset \Delta_A$, i.e. R must be a subset of A. Conversely, any subset U of A satisfies $U \subset \Delta_A$, defining a comonad as above.

Item 7: Monomorphisms in Rel

Firstly note that Items 7b and 7c are equivalent by Item 7 of Proposition 5.1.3. We then claim that Items 7a and 7b are also equivalent:

- *Item 7a* $\Longrightarrow$ *Item 7b*: Let $U, V \in \mathcal{P}(A)$ and consider the diagram

$$
\begin{tikzcd}
\mathrm{pt} \arrow[r, shift left, "U"] \arrow[r, shift right, "V"'] & A \arrow[r, "R", "\shortmid"{marking}] & B.
\end{tikzcd}
$$

  By Remark 5.1.2, we have

$$
\begin{aligned}
R_*(U) &= R \diamond U, \\
R_*(V) &= R \diamond V.
\end{aligned}
$$

  Now, if $R \diamond U = R \diamond V$, i.e. $R_*(U) = R_*(V)$, then $U = V$ since R is assumed to be a monomorphism, showing R_* to be injective.

- *Item 7b* $\implies$ *Item 7a*: Conversely, suppose that R_* is injective, consider the diagram

$$K \overset{S}{\underset{T}{\rightrightarrows}} A \overset{R}{\longrightarrow} B,$$

and suppose that $R \diamond S = R \diamond T$. Note that, since R_* is injective, given a diagram of the form

$$\mathrm{pt} \overset{U}{\underset{V}{\rightrightarrows}} A \overset{R}{\longrightarrow} B,$$

if $R_*(U) = R \diamond U = R \diamond V = R_*(V)$, then $U = V$. In particular, for each $k \in K$, we may consider the diagram

$$\mathrm{pt} \overset{[k]}{\longrightarrow} K \overset{S}{\underset{T}{\rightrightarrows}} A \overset{R}{\longrightarrow} B,$$

for which we have $R \diamond S \diamond [k] = R \diamond T \diamond [k]$, implying that we have

$$S(k) = S \diamond [k] = T \diamond [k] = T(k)$$

for each $k \in K$, implying $S = T$, and thus R is a monomorphism.

We can also prove this in a more abstract way, following [MSE 350788]:

- *Item 7a* $\implies$ *Item 7b*: Assume that R is a monomorphism.
  - We first notice that the functor $\mathrm{Rel}(\mathrm{pt}, -)\colon \mathrm{Rel} \longrightarrow \mathrm{Sets}$ maps R to R_* by Remark 5.1.2.
  - Since $\mathrm{Rel}(\mathrm{pt}, -)$ preserves all limits by Limits and Colimits, **??** of **??**, it follows by Categories, **??** of **??** that $\mathrm{Rel}(\mathrm{pt}, -)$ also preserves monomorphisms.
  - Since R is a monomorphism and $\mathrm{Rel}(\mathrm{pt}, -)$ maps R to R_*, it follows that R_* is also a monomorphism.
  - Since the monomorphisms in Sets are precisely the injections (Categories, **??** of **??**), it follows that R_* is injective.
- *Item 7b* $\implies$ *Item 7a*: Assume that R_* is injective.

- We first notice that the functor $\mathsf{Rel}(\mathrm{pt},-)\colon \mathsf{Rel} \to \mathsf{Sets}$ maps R to R_* by Remark 5.1.2.
- Since the monomorphisms in Sets are precisely the injections (Categories, **??** of **??**), it follows that R_* is a monomorphism.
- Since $\mathsf{Rel}(\mathrm{pt},-)$ is faithful, it follows by Categories, **??** of **??** that $\mathsf{Rel}(\mathrm{pt},-)$ reflects monomorphisms.
- Since R_* is a monomorphism and $\mathsf{Rel}(\mathrm{pt},-)$ maps R to R_*, it follows that R is also a monomorphism.

Finally, we prove the second part of the statement. Assume that R is a monomorphism, let $a, a' \in A$ such that $a \sim_R b$ and $a' \sim_R b$ for some $b \in B$, and consider the diagram

$$\begin{tikzcd} \mathrm{pt} \arrow[r, shift left, "{[a]}", "\shortmid"{marking}] \arrow[r, shift right, "{[a']}"', "\shortmid"{marking}] & A \arrow[r, "R", "\shortmid"{marking}] & B. \end{tikzcd}$$

Since $\star \sim_{[a]} a$ and $a \sim_R b$, we have $\star \sim_{R\diamond[a]} b$. Similarly, $\star \sim_{R\diamond[a']} b$. Thus $R \diamond [a] = R \diamond [a']$, and since R is a monomorphism, we have $[a] = [a']$, i.e. $a = a'$.

Conversely, assume the condition

($\star$) For each $a, a' \in A$, if there exists some $b \in B$ such that $a \sim_R b$ and $a' \sim_R b$, then $a = a'$,

consider the diagram

$$\begin{tikzcd} K \arrow[r, shift left, "S", "\shortmid"{marking}] \arrow[r, shift right, "T"', "\shortmid"{marking}] & A \arrow[r, "R", "\shortmid"{marking}] & B, \end{tikzcd}$$

and let $(k, a) \in S$. Since R is total and $a \in A$, there exists some $b \in B$ such that $a \sim_R b$. In this case, we have $k \sim_{R\diamond S} b$, and since $R \diamond S = R \diamond T$, we have also $k \sim_{R\diamond T} b$. Thus there must exist some $a' \in A$ such that $k \sim_T a'$ and $a' \sim_R b$. However, since $a, a' \sim_R b$, we must have $a = a'$, and thus $(k, a) \in T$ as well.

A similar argument shows that if $(k, a) \in T$, then $(k, a) \in S$, and thus $S = T$ and R is a monomorphism.

Item 8: Epimorphisms in Rel

Firstly note that Items 8b and 8c are equivalent by Item 7 of Proposition 5.2.4. We then claim that Items 8a and 8b are also equivalent:

- *Item 8a* $\implies$ *Item 8b*: Let $U, V \in \mathcal{P}(A)$ and consider the diagram

$$A \overset{R}{\longrightarrow\!\!\!\!\!\!\!+\;\;} B \underset{V}{\overset{U}{\rightrightarrows}} \mathrm{pt}.$$

  By Remark 5.1.2, we have

$$\begin{aligned} R^{-1}(U) &= U \diamond R, \\ R^{-1}(V) &= V \diamond R. \end{aligned}$$

  Now, if $U \diamond R = V \diamond R$, i.e. $R^{-1}(U) = R^{-1}(V)$, then $U = V$ since R is assumed to be an epimorphism, showing R^{-1} to be injective.

- *Item 8b* $\implies$ *Item 8a*: Conversely, suppose that R^{-1} is injective, consider the diagram

$$A \overset{R}{\longrightarrow\!\!\!\!\!\!\!+\;\;} B \underset{T}{\overset{S}{\rightrightarrows}} K,$$

  and suppose that $S \diamond R = T \diamond R$. Note that, since R^{-1} is injective, given a diagram of the form

$$A \overset{R}{\longrightarrow\!\!\!\!\!\!\!+\;\;} B \underset{V}{\overset{U}{\rightrightarrows}} \mathrm{pt},$$

  if $R^{-1}(U) = U \diamond R = V \diamond R = R^{-1}(V)$, then $U = V$. In particular, for each $k \in K$, we may consider the diagram

$$A \overset{R}{\longrightarrow\!\!\!\!\!\!\!+\;\;} B \underset{T}{\overset{S}{\rightrightarrows}} K \overset{[k]}{\longrightarrow\!\!\!\!\!\!\!+\;\;} \mathrm{pt},$$

  for which we have $[k] \diamond S \diamond R = [k] \diamond T \diamond R$, implying that we have

$$S^{-1}(k) = [k] \diamond S = [k] \diamond T = T^{-1}(k)$$

  for each $k \in K$, implying $S = T$, and thus R is an epimorphism.

We can also prove this in a more abstract way, following [MSE 350788]:

- *Item 8a* $\implies$ *Item 8b*: Assume that R is an epimorphism.

- We first notice that the functor $\mathsf{Rel}(-, \mathrm{pt}) \colon \mathsf{Rel}^{\mathsf{op}} \to \mathsf{Sets}$ maps R to R^{-1} by Remark 5.3.2.
- Since $\mathsf{Rel}(-, \mathrm{pt})$ preserves limits by Limits and Colimits, **??** of **??**, it follows by Categories, **??** of **??** that $\mathsf{Rel}(-, \mathrm{pt})$ also preserves monomorphisms.
- That is: $\mathsf{Rel}(-, \mathrm{pt})$ sends monomorphisms in $\mathsf{Rel}^{\mathsf{op}}$ to monomorphisms in Sets.
- The monomorphisms $\mathsf{Rel}^{\mathsf{op}}$ are precisely the epimorphisms in Rel by Categories, **??** of **??**.
- Since R is an epimorphism and $\mathsf{Rel}(-, \mathrm{pt})$ maps R to R^{-1}, it follows that R^{-1} is a monomorphism.
- Since the monomorphisms in Sets are precisely the injections (Categories, **??** of **??**), it follows that R^{-1} is injective.

· *Item 8b* $\Longrightarrow$ *Item 8a*: Assume that R^{-1} is injective.

- We first notice that the functor $\mathsf{Rel}(-, \mathrm{pt}) \colon \mathsf{Rel}^{\mathsf{op}} \to \mathsf{Sets}$ maps R to R^{-1} by Remark 5.3.2.
- Since the monomorphisms in Sets are precisely the injections (Categories, **??** of **??**), it follows that R^{-1} is a monomorphism.
- Since $\mathsf{Rel}(-, \mathrm{pt})$ is faithful, it follows by Categories, **??** of **??** that $\mathsf{Rel}(, \mathrm{pt})$ reflects monomorphisms.
- That is: $\mathsf{Rel}(-, \mathrm{pt})$ reflects monomorphisms in Sets to monomorphisms in $\mathsf{Rel}^{\mathsf{op}}$.
- The monomorphisms $\mathsf{Rel}^{\mathsf{op}}$ are precisely the epimorphisms in Rel by Categories, **??** of **??**.
- Since R^{-1} is a monomorphism and $\mathsf{Rel}(-, \mathrm{pt})$ maps R to R^{-1}, it follows that R is an epimorphism.

Finally, we claim that Items 8b and 8d are also equivalent, following [MO 350788]:

· *Item 8b* $\Longrightarrow$ *Item 8d*: Since $B \setminus \{b\} \subset B$ and R^{-1} is injective, we have $R^{-1}(B \setminus \{b\}) \subsetneq R^{-1}(B)$. So taking some $a \in R^{-1}(B) \setminus R^{-1}(B \setminus \{b\})$ we get an element of A such that $R(a) = \{b\}$.

- *Item 8d* $\Longrightarrow$ *Item 8b*: Let $U, V \subset B$ with $U \neq V$. Without loss of generality, we can assume $U \setminus V \neq \emptyset$; otherwise just swap U and V. Let then $b \in U \setminus V$. By assumption, there exists an $a \in A$ with $R(a) = \{b\}$. Then $a \in R^{-1}(U)$ but $a \notin R^{-1}(V)$, and thus $R^{-1}(U) \neq R^{-1}(V)$, showing R^{-1} to be injective.

Item 9: As a Kleisli Category

Omitted.

Item 10: Co/Completeness (Or Lack Thereof)

Omitted.

Item 11: Existence of Right Kan Extensions

We have

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{Rel}(A,X)}(S \diamond R, T) &\cong \int_{a\in A}\int_{x\in X} \mathbf{Hom}_{\{\mathrm{t},\mathrm{f}\}}\left((S \diamond R)^a_x, T^a_x\right) \\
&\cong \int_{a\in A}\int_{x\in X} \mathbf{Hom}_{\{\mathrm{t},\mathrm{f}\}}\left(\left(\int^{b\in B} S^b_x \times R^a_b\right), T^a_x\right) \\
&\cong \int_{a\in A}\int_{x\in X}\int_{b\in B} \mathbf{Hom}_{\{\mathrm{t},\mathrm{f}\}}\left(S^b_x \times R^a_b, T^a_x\right) \\
&\cong \int_{a\in A}\int_{x\in X}\int_{b\in B} \mathbf{Hom}_{\{\mathrm{t},\mathrm{f}\}}\left(S^b_x, \mathbf{Hom}_{\{\mathrm{t},\mathrm{f}\}}(R^a_b, T^a_x)\right) \\
&\cong \int_{b\in B}\int_{x\in X}\int_{a\in A} \mathbf{Hom}_{\{\mathrm{t},\mathrm{f}\}}\left(S^b_x, \mathbf{Hom}_{\{\mathrm{t},\mathrm{f}\}}(R^a_b, T^a_x)\right) \\
&\cong \int_{b\in B}\int_{x\in X} \mathbf{Hom}_{\{\mathrm{t},\mathrm{f}\}}\left(S^b_x, \int_{a\in A} \mathbf{Hom}_{\{\mathrm{t},\mathrm{f}\}}(R^a_b, T^a_x)\right) \\
&\cong \mathrm{Hom}_{\mathbf{Rel}(B,X)}\left(S, \int_{a\in A} \mathbf{Hom}_{\{\mathrm{t},\mathrm{f}\}}(R^a_{-_1}, T^a_{-_2})\right)
\end{aligned}$$

naturally in each $S \in \mathbf{Rel}(B, X)$ and each $T \in \mathbf{Rel}(A, X)$, showing that

$$\int_{a\in A} \mathbf{Hom}_{\{\mathrm{t},\mathrm{f}\}}(R^a_{-_1}, T^a_{-_2})$$

is right adjoint to the precomposition functor $- \diamond R$, being thus the right Kan extension along R. Here we have used the following results, respectively (i.e. for each $\cong$ sign):

1. Item 1 of Proposition 1.1.5;

2. Definition 3.12.1;
3. Diagonal Category Theory, ?? of ??;
4. Sets, Proposition 1.2.4;
5. Diagonal Category Theory, ?? of ??;
6. Diagonal Category Theory, ?? of ??;
7. Item 1 of Proposition 1.1.5.

Item 12: Existence of Right Kan Lifts

We have

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{Rel}(X,B)}(R \diamond S, T) &\cong \int_{x\in X}\int_{b\in B} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left((R\diamond S)^x_b, T^x_b\right) \\
&\cong \int_{x\in X}\int_{b\in B} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(\left(\int^{a\in A} R^a_b \times S^x_a\right), T^x_b\right) \\
&\cong \int_{x\in X}\int_{b\in B}\int_{a\in A} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^a_b \times S^x_a, T^x_b\right) \\
&\cong \int_{x\in X}\int_{b\in B}\int_{a\in A} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(S^x_a, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R^a_b, T^x_b)\right) \\
&\cong \int_{x\in X}\int_{a\in A}\int_{b\in B} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(S^x_a, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R^a_b, T^x_b)\right) \\
&\cong \int_{x\in X}\int_{a\in A} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(S^x_a, \int_{b\in B}\mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R^a_b, T^x_b)\right) \\
&\cong \mathrm{Hom}_{\mathbf{Rel}(X,A)}\left(S, \int_{b\in B}\mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^{-2}_b, T^{-1}_b\right)\right)
\end{aligned}$$

naturally in each $S \in \mathbf{Rel}(X,A)$ and each $T \in \mathbf{Rel}(X,B)$, showing that

$$\int_{b\in B}\mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^{-2}_b, T^{-1}_b\right)$$

is right adjoint to the postcomposition functor $R \diamond -$, being thus the right Kan lift along R. Here we have used the following results, respectively (i.e. for each $\cong$ sign):

1. Item 1 of Proposition 1.1.5;
2. Definition 3.12.1;

3. Diagonal Category Theory, ?? of ??;
4. Sets, Proposition 1.2.4;
5. Diagonal Category Theory, ?? of ??;
6. Diagonal Category Theory, ?? of ??;
7. Item 1 of Proposition 1.1.5.

Item 13: Closedness

This has been proved as part of the proof of Items 11 and 12.

3 Constructions With Relations

3.1 The Graph of a Function

Let $f\colon A \longrightarrow B$ be a function.

Definition 3.1.1 ▶ The Graph of a Function

The **graph of** f is the relation $\mathrm{Gr}(f)\colon A \nrightarrow B$ defined as follows:[1]

- Viewing relations from A to B as subsets of $A \times B$, we define

$$\mathrm{Gr}(f) \overset{\text{def}}{=} \{(a, f(a)) \in A \times B \mid a \in A\};$$

- Viewing relations from A to B as functions $A \times B \longrightarrow \{\text{true}, \text{false}\}$, we define

$$[\mathrm{Gr}(f)](a, b) \overset{\text{def}}{=} \begin{cases} \text{true} & \text{if } b = f(a), \\ \text{false} & \text{otherwise} \end{cases}$$

  for each $(a, b) \in A \times B$;

- Viewing relations from A to B as functions $A \longrightarrow \mathcal{P}(B)$, we define

$$[\mathrm{Gr}(f)](a) \overset{\text{def}}{=} \{f(a)\}$$

  for each $a \in A$, i.e. we define $\mathrm{Gr}(f)$ as the composition

$$A \xrightarrow{f} B \xrightarrow{\chi_B} \mathcal{P}(B).$$

[1] *Further Notation:* We write $\mathrm{Gr}(A)$ for $\mathrm{Gr}(\mathrm{id}_A)$, and call it the **graph** of A.

Proposition 3.1.2 ► Properties of Graphs of Functions

Let $f: A \longrightarrow B$ be a function.

1. *Functoriality.* The assignment $A \mapsto \mathrm{Gr}(A)$ defines a functor

$$\mathrm{Gr}\colon \mathsf{Sets} \longrightarrow \mathsf{Rel}$$

where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$\mathrm{Gr}(A) \stackrel{\text{def}}{=} A;$$

- *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathsf{Sets})$, the action on Hom-sets

$$\mathrm{Gr}_{A,B}\colon \mathsf{Sets}(A,B) \longrightarrow \underbrace{\mathrm{Rel}(\mathrm{Gr}(A),\mathrm{Gr}(B))}_{\stackrel{\text{def}}{=}\mathrm{Rel}(A,B)}$$

of Gr at (A, B) is defined by

$$\mathrm{Gr}_{A,B}(f) \stackrel{\text{def}}{=} \mathrm{Gr}(f),$$

where $\mathrm{Gr}(f)$ is the graph of f as in Definition 3.1.1.

In particular:

- *Preservation of Identities.* We have

$$\mathrm{Gr}(\mathrm{id}_A) = \chi_A$$

for each $A \in \mathrm{Obj}(\mathsf{Sets})$.

- *Preservation of Composition.* We have

$$\mathrm{Gr}(g \circ f) = \mathrm{Gr}(g) \diamond \mathrm{Gr}(f)$$

for each pair of functions $f: A \longrightarrow B$ and $g: B \to C$.

2. *Adjointness Inside **Rel***. We have an adjunction

$$\left(\mathrm{Gr}(f) \dashv f^{-1}\right)\colon \quad \begin{tikzcd} A \arrow[r, bend left, "\mathrm{Gr}(f)", "\shortmid" marking] \arrow[r, phantom, "\perp"] & B \arrow[l, bend left, "f^{-1}", "\shortmid" marking] \end{tikzcd}$$

in **Rel**, where f^{-1} is the inverse of f of Definition 3.2.1.

3. *Adjointness.* We have an adjunction

$$(\mathrm{Gr} \dashv \mathcal{P}_*)\colon \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "\mathrm{Gr}"] \arrow[r, phantom, "\perp"] & \mathrm{Rel}, \arrow[l, bend left, "\mathcal{P}_*"] \end{tikzcd}$$

witnessed by a bijection of sets

$$\mathrm{Rel}(\mathrm{Gr}(A), B) \cong \mathsf{Sets}(A, \mathcal{P}(B))$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$ and $B \in \mathrm{Obj}(\mathrm{Rel})$.

4. *Interaction With Inverses.* We have

$$\begin{aligned} \mathrm{Gr}(f)^{\dagger} &= f^{-1}, \\ \left(f^{-1}\right)^{\dagger} &= \mathrm{Gr}(f). \end{aligned}$$

5. *Cocontinuity.* The functor $\mathrm{Gr}\colon \mathsf{Sets} \longrightarrow \mathrm{Rel}$ of Item 1 preserves colimits.

6. *Characterisations.* Let $R\colon A \nrightarrow B$ be a relation. The following conditions are equivalent:

   (a) There exists a function $f\colon A \longrightarrow B$ such that $R = \mathrm{Gr}(f)$.

   (b) The relation R is total and functional.

   (c) The weak and strong inverse images of R agree, i.e. we have $R^{-1} = R_{-1}$.

   (d) The relation R has a right adjoint $R^{\dagger}$ in Rel.

PROOF 3.1.3 ► PROOF OF PROPOSITION 3.1.2

Item 1: Functoriality

Clear.

Item 2: Adjointness Inside Rel

We need to check that there are inclusions

$$\chi_A \subset f^{-1} \diamond \mathrm{Gr}(f),$$
$$\mathrm{Gr}(f) \diamond f^{-1} \subset \chi_B.$$

These correspond respectively to the following conditions:

1. For each $a \in A$, there exists some $b \in B$ such that $a \sim_{\mathrm{Gr}(f)} b$ and $b \sim_{f^{-1}} a$.
2. For each $a, b \in A$, if $a \sim_{\mathrm{Gr}(f)} b$ and $b \sim_{f^{-1}} a$, then $a = b$.

In other words, the first condition states that the image of any $a \in A$ by f is nonempty, whereas the second condition states that f is not multivalued. As f is a function, both of these statements are true, and we are done.

Item 3: Adjointness

The stated bijection follows from Remark 1.1.3, with naturality being clear.

Item 4: Interaction With Inverses

Clear.

Item 5: Cocontinuity

Omitted.

Item 6: Characterisations

We claim that Items 6a to 6d are indeed equivalent:

- *Item 6a* $\iff$ *Item 6b*. This is shown in the proof of Item 4 of Proposition 2.5.1.
- *Item 6b* $\implies$ *Item 6c*. If R is total and functional, then, for each $a \in A$, the set $R(a)$ is a singleton, implying that

  $$R^{-1}(V) \overset{\mathrm{def}}{=} \{a \in A \mid R(a) \cap V \neq \emptyset\},$$
  $$R_{-1}(V) \overset{\mathrm{def}}{=} \{a \in A \mid R(a) \subset V\}$$

  are equal for all $V \in \mathcal{P}(B)$, as the conditions $R(a) \cap V \neq \emptyset$ and $R(a) \subset V$ are equivalent when $R(a)$ is a singleton.

- *Item 6c* $\implies$ *Item 6b*. We claim that R is indeed total and functional:
  - *Totality.* If we had $R(a) = \emptyset$ for some $a \in A$, then we would have $a \in R_{-1}(\emptyset)$, so that $R_{-1}(\emptyset) \neq \emptyset$. But since $R^{-1}(\emptyset) = \emptyset$, this would imply $R_{-1}(\emptyset) \neq R^{-1}(\emptyset)$, a contradiction. Thus $R(a) \neq \emptyset$ for all $a \in A$ and R is total.
  - *Functionality.* If $R^{-1} = R_{-1}$, then we have

$$\begin{aligned}\{a\} &= R^{-1}(\{b\}) \\ &= R_{-1}(\{b\})\end{aligned}$$

    for each $b \in R(a)$ and each $a \in A$, and thus $R(a) \subset \{b\}$. But since R is total, we must have $R(a) = \{b\}$, and thus we see that R is functional.
- *Item 6a* $\iff$ *Item 6d*. This follows from Item 4 of Proposition 2.5.1.

This finishes the proof. □

3.2 The Inverse of a Function

Let $f: A \to B$ be a function.

Definition 3.2.1 ► The Inverse of a Function

The **inverse of** f is the relation $f^{-1}: B \nrightarrow A$ defined as follows:

- Viewing relations from B to A as subsets of $B \times A$, we define

$$f^{-1} \stackrel{\text{def}}{=} \left\{ \left(b, f^{-1}(b)\right) \in B \times A \,\middle|\, a \in A \right\},$$

  where

$$f^{-1}(b) \stackrel{\text{def}}{=} \{a \in A \mid f(a) = b\}$$

  for each $b \in B$.
- Viewing relations from B to A as functions $B \times A \to \{\text{true}, \text{false}\}$, we define

$$f^{-1}(b, a) \stackrel{\text{def}}{=} \begin{cases} \text{true} & \text{if there exists } a \in A \text{ with } f(a) = b, \\ \text{false} & \text{otherwise} \end{cases}$$

  for each $(b, a) \in B \times A$;

- Viewing relations from B to A as functions $B \longrightarrow \mathcal{P}(A)$, we define

$$f^{-1}(b) \overset{\text{def}}{=} \{a \in A \mid f(a) = b\}$$

for each $b \in B$.

Proposition 3.2.2 ▶ Properties of Inverses of Functions

Let $f\colon A \longrightarrow B$ be a function.

1. *Functoriality.* The assignment $A \mapsto A$, $f \mapsto f^{-1}$ defines a functor

$$(-)^{-1}\colon \mathsf{Sets} \longrightarrow \mathsf{Rel}$$

where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$\left[(-)^{-1}\right](A) \overset{\text{def}}{=} A;$$

- *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathsf{Sets})$, the action on Hom-sets

$$(-)^{-1}_{A,B}\colon \mathsf{Sets}(A, B) \longrightarrow \mathsf{Rel}(A, B)$$

of $(-)^{-1}$ at (A, B) is defined by

$$(-)^{-1}_{A,B}(f) \overset{\text{def}}{=} \left[(-)^{-1}\right](f),$$

where f^{-1} is the inverse of f as in Definition 3.2.1.

In particular:

- *Preservation of Identities.* We have

$$\mathrm{id}_A^{-1} = \chi_A$$

for each $A \in \mathrm{Obj}(\mathsf{Sets})$.

- *Preservation of Composition.* We have

$$(g \circ f)^{-1} = g^{-1} \diamond f^{-1}$$

for pair of functions $f\colon A \longrightarrow B$ and $g\colon B \longrightarrow C$.

2. *Adjointness Inside **Rel**.* We have an adjunction

$$(\mathrm{Gr}(f) \dashv f^{-1})\colon\ \begin{tikzcd} A \arrow[r, bend left, "\mathrm{Gr}(f)"] \arrow[r, phantom, "\perp"] & B \arrow[l, bend left, "f^{-1}"] \end{tikzcd}$$

in **Rel**.

3. *Interaction With Inverses of Relations.* We have

$$\begin{aligned}(f^{-1})^{\dagger} &= \mathrm{Gr}(f),\\ \mathrm{Gr}(f)^{\dagger} &= f^{-1}.\end{aligned}$$

Proof 3.2.3 ▶ Proof of Proposition 3.2.2

Item 1: Functoriality

Clear.

Item 2: Adjointness Inside Rel

This is proved in Item 2 of Proposition 3.1.2.

Item 3: Interaction With Inverses of Relations

Clear.

3.3 Representable Relations

Let A and B be sets.

Definition 3.3.1 ▶ Representable Relations

Let $f\colon A \longrightarrow B$ and $g\colon B \longrightarrow A$ be functions.[1]

1. The **representable relation associated to** f is the relation $\chi_f\colon A \nrightarrow B$ defined as the composition

$$A \times B \xrightarrow{f \times \mathrm{id}_B} B \times B \xrightarrow{\chi_B} \{\mathsf{true}, \mathsf{false}\},$$

i.e. given by declaring $a \sim_{\chi_f} b$ iff $f(a) = b$.

2. The **corepresentable relation associated to** g is the relation $\chi^g \colon B \nrightarrow A$ defined as the composition

$$B \times A \xrightarrow{g \times \mathrm{id}_A} A \times A \xrightarrow{\chi_A} \{\mathsf{true}, \mathsf{false}\},$$

i.e. given by declaring $b \sim_{\chi^g} a$ iff $g(b) = a$.

---

[1] More generally, given functions

$$\begin{aligned} f &\colon A \to C, \\ g &\colon B \to D \end{aligned}$$

and a relation $B \nrightarrow D$, we may consider the composite relation

$$A \times B \xrightarrow{f \times g} C \times D \xrightarrow{R} \{\mathsf{true}, \mathsf{false}\},$$

for which we have $a \sim_{R \circ (f \times g)} b$ iff $f(a) \sim_R g(b)$.

3.4 The Domain and Range of a Relation

Let A and B be sets.

Definition 3.4.1 ► The Domain and Range of a Relation

Let $R \subset A \times B$ be a relation.[1,2]

1. The **domain of** R is the subset $\mathrm{dom}(R)$ of A defined by

$$\mathrm{dom}(R) \overset{\mathrm{def}}{=} \left\{ a \in A \,\middle|\, \begin{array}{l} \text{there exists some } b \in B \\ \text{such that } a \sim_R b \end{array} \right\}.$$

2. The **range of** R is the subset $\mathrm{range}(R)$ of B defined by

$$\mathrm{range}(R) \overset{\mathrm{def}}{=} \left\{ b \in B \,\middle|\, \begin{array}{l} \text{there exists some } a \in A \\ \text{such that } a \sim_R b \end{array} \right\}.$$

[1]Following Categories, ??, we may compute the (characteristic functions associated to the) domain and range of a relation using the following colimit formulas:

$$\begin{aligned}\chi_{\mathrm{dom}(R)}(a) &\cong \operatorname*{colim}_{b\in B}\left(R^a_b\right) && (a \in A)\\ &\cong \bigvee_{b\in B} R^a_b,\\ \chi_{\mathrm{range}(R)}(b) &\cong \operatorname*{colim}_{a\in A}\left(R^a_b\right) && (b \in B)\\ &\cong \bigvee_{a\in A} R^a_b,\end{aligned}$$

where the join $\bigvee$ is taken in the poset $(\{\mathrm{true},\mathrm{false}\},\preceq)$ of Constructions With Sets, Definition 1.2.3.

[2]Viewing R as a function $R\colon A \longrightarrow \mathcal{P}(B)$, we have

$$\begin{aligned}\mathrm{dom}(R) &\cong \operatorname*{colim}_{y\in Y}(R(y))\\ &\cong \bigcup_{y\in Y} R(y),\\ \mathrm{range}(R) &\cong \operatorname*{colim}_{x\in X}(R(x))\\ &\cong \bigcup_{x\in X} R(x),\end{aligned}$$

3.5 Binary Unions of Relations

Let A and B be sets and let R and S be relations from A to B.

Definition 3.5.1 ▶ Binary Unions of Relations

The **union of R and S**[1] is the relation $R \cup S$ from A to B defined as follows:

- Viewing relations from A to B as subsets of $A \times B$, we define[2]

$$R \cup S \stackrel{\mathrm{def}}{=} \{(a,b) \in B \times A \mid \text{we have } a \sim_R b \text{ or } a \sim_S b\}.$$

- Viewing relations from A to B as functions $A \longrightarrow \mathcal{P}(B)$, we define

$$[R \cup S](a) \stackrel{\mathrm{def}}{=} R(a) \cup S(a)$$

  for each $a \in A$.

[1]*Further Terminology:* Also called the **binary union of R and S**, for emphasis.
[2]This is the same as the union of R and S as subsets of $A \times B$.

Proposition 3.5.2 ▶ Properties of Binary Unions of Relations

Let R, S, R_1, and R_2 be relations from A to B, and let S_1 and S_2 be relations from B to C.

1. *Interaction With Inverses.* We have

$$(R \cup S)^{\dagger} = R^{\dagger} \cup S^{\dagger}.$$

2. *Interaction With Composition.* We have

$$(S_1 \diamond R_1) \cup (S_2 \diamond R_2) \overset{\text{poss.}}{\neq} (S_1 \cup S_2) \diamond (R_1 \cup R_2).$$

Proof 3.5.3 ▶ Proof of Proposition 3.5.2

Item 1: Interaction With Inverses

Clear.

Item 2: Interaction With Composition

Unwinding the definitions, we see that:

1. The condition for $(S_1 \diamond R_1) \cup (S_2 \diamond R_2)$ is:
   (a) There exists some $b \in B$ such that:
      i. $a \sim_{R_1} b$ and $b \sim_{S_1} c$;

   or
      i. $a \sim_{R_2} b$ and $b \sim_{S_2} c$;
3. The condition for $(S_1 \cup S_2) \diamond (R_1 \cup R_2)$ is:
   (a) There exists some $b \in B$ such that:
      i. $a \sim_{R_1} b$ or $a \sim_{R_2} b$;

   and
      i. $b \sim_{S_1} c$ or $b \sim_{S_2} c$.

These two conditions may fail to agree (counterexample omitted), and thus the two resulting relations on $A \times C$ may differ. ∎

3.6 Unions of Families of Relations

Let A and B be sets and let $\{R_i\}_{i \in I}$ be a family of relations from A to B.

Definition 3.6.1 ► The Union of a Family of Relations

The **union of the family** $\{R_i\}_{i\in I}$ is the relation $\bigcup_{i\in I} R_i$ from A to B defined as follows:

- Viewing relations from A to B as subsets of $A \times B$, we define[1]

$$\bigcup_{i\in I} R_i \overset{\text{def}}{=} \left\{ (a,b) \in (A\times B)^{\times I} \;\middle|\; \begin{array}{l}\text{there exists some } i \in I\\ \text{such that } a \sim_{R_i} b\end{array} \right\}.$$

- Viewing relations from A to B as functions $A \to \mathcal{P}(B)$, we define

$$\left[\bigcup_{i\in I} R_i\right](a) \overset{\text{def}}{=} \bigcup_{i\in I} R_i(a)$$

  for each $a \in A$.

[1]This is the same as the union of $\{R_i\}_{i\in I}$ as a collection of subsets of $A \times B$.

Proposition 3.6.2 ► Properties of Unions of Families of Relations

Let A and B be sets and let $\{R_i\}_{i\in I}$ be a family of relations from A to B.

1. *Interaction With Inverses.* We have

$$\left(\bigcup_{i\in I} R_i\right)^{\dagger} = \bigcup_{i\in I} R_i^{\dagger}.$$

Proof 3.6.3 ► Proof of Proposition 3.6.2

Item 1: Interaction With Inverses

Clear.

3.7 Binary Intersections of Relations

Let A and B be sets and let R and S be relations from A to B.

Definition 3.7.1 ▶ Binary Intersections of Relations

The **intersection of R and S**[1] is the relation $R \cap S$ from A to B defined as follows:

- Viewing relations from A to B as subsets of $A \times B$, we define[2]

$$R \cap S \overset{\text{def}}{=} \{(a, b) \in B \times A \mid \text{we have } a \sim_R b \text{ and } a \sim_S b\}.$$

- Viewing relations from A to B as functions $A \to \mathcal{P}(B)$, we define

$$[R \cap S](a) \overset{\text{def}}{=} R(a) \cap S(a)$$

for each $a \in A$.

[1] *Further Terminology:* Also called the **binary intersection of R and S**, for emphasis.
[2] This is the same as the intersection of R and S as subsets of $A \times B$.

Proposition 3.7.2 ▶ Properties of Binary Intersections of Relations

Let R, S, R_1, and R_2 be relations from A to B, and let S_1 and S_2 be relations from B to C.

1. *Interaction With Inverses.* We have

$$(R \cap S)^\dagger = R^\dagger \cap S^\dagger.$$

2. *Interaction With Composition.* We have

$$(S_1 \diamond R_1) \cap (S_2 \diamond R_2) = (S_1 \cap S_2) \diamond (R_1 \cap R_2).$$

Proof 3.7.3 ▶ Proof of Proposition 3.7.2

Item 1: Interaction With Inverses

Clear.

Item 2: Interaction With Composition

Unwinding the definitions, we see that:

1. The condition for $(S_1 \diamond R_1) \cap (S_2 \diamond R_2)$ is:

   (a) There exists some $b \in B$ such that:

i. $a \sim_{R_1} b$ and $b \sim_{S_1} c$;

and

i. $a \sim_{R_2} b$ and $b \sim_{S_2} c$;

3. The condition for $(S_1 \cap S_2) \diamond (R_1 \cap R_2)$ is:

   (a) There exists some $b \in B$ such that:

   i. $a \sim_{R_1} b$ and $a \sim_{R_2} b$;

   and

   i. $b \sim_{S_1} c$ and $b \sim_{S_2} c$.

These two conditions agree, and thus so do the two resulting relations on $A \times C$. ∎

3.8 Intersections of Families of Relations

Let A and B be sets and let $\{R_i\}_{i \in I}$ be a family of relations from A to B.

DEFINITION 3.8.1 ► THE INTERSECTION OF A FAMILY OF RELATIONS

The **intersection of the family** $\{R_i\}_{i \in I}$ is the relation $\bigcup_{i \in I} R_i$ defined as follows:

- Viewing relations from A to B as subsets of $A \times B$, we define[1]

$$\bigcup_{i \in I} R_i \stackrel{\text{def}}{=} \left\{ (a, b) \in (A \times B)^{\times I} \;\middle|\; \begin{array}{l} \text{for each } i \in I, \\ \text{we have } a \sim_{R_i} b \end{array} \right\}.$$

- Viewing relations from A to B as functions $A \to \mathcal{P}(B)$, we define

$$\left[\bigcap_{i \in I} R_i \right](a) \stackrel{\text{def}}{=} \bigcap_{i \in I} R_i(a)$$

  for each $a \in A$.

[1] This is the same as the intersection of $\{R_i\}_{i \in I}$ as a collection of subsets of $A \times B$.

PROPOSITION 3.8.2 ► PROPERTIES OF INTERSECTIONS OF FAMILIES OF RELATIONS

Let A and B be sets and let $\{R_i\}_{i\in I}$ be a family of relations from A to B.

1. *Interaction With Inverses.* We have

$$\left(\bigcap_{i\in I} R_i\right)^{\dagger} = \bigcap_{i\in I} R_i^{\dagger}.$$

PROOF 3.8.3 ► PROOF OF PROPOSITION 3.8.2

Item 1: Interaction With Inverses

Clear. ∎

3.9 Binary Products of Relations

Let A, B, X, and Y be sets, let $R\colon A \nrightarrow B$ be a relation from A to B, and let $S\colon X \nrightarrow Y$ be a relation from X to Y.

DEFINITION 3.9.1 ► BINARY PRODUCTS OF RELATIONS

The **product of R and S**[1] is the relation $R\times S$ from $A\times X$ to $B\times Y$ defined as follows:

- Viewing relations from $A\times X$ to $B\times Y$ as subsets of $(A\times X)\times(B\times Y)$, we define $R\times S$ as the Cartesian product of R and S as subsets of $A\times X$ and $B\times Y$;[2]
- Viewing relations from $A\times X$ to $B\times Y$ as functions $A\times X\to\mathcal{P}(B\times Y)$, we define $R\times S$ as the composition

$$A\times X \xrightarrow{R\times S} \mathcal{P}(B)\times\mathcal{P}(Y) \overset{\mathcal{P}^{\otimes}_{B,Y}}{\hookrightarrow} \mathcal{P}(B\times Y)$$

in Sets, i.e. by

$$[R\times S](a,x) \stackrel{\text{def}}{=} R(a)\times S(x)$$

for each $(a,x)\in A\times X$.

[1] *Further Terminology:* Also called the **binary product of R and S**, for emphasis.

[2] That is, $R\times S$ is the relation given by declaring $(a,x)\sim_{R\times S}(b,y)$ iff $a\sim_R b$ and $x\sim_S y$.

Proposition 3.9.2 ► Properties of Binary Products of Relations

Let A, B, X, and Y be sets.

1. *Interaction With Inverses.* Let

$$\begin{aligned} R &\colon A \nrightarrow A, \\ S &\colon X \nrightarrow X \end{aligned}$$

We have

$$(R \times S)^{\dagger} = R^{\dagger} \times S^{\dagger}.$$

2. *Interaction With Composition.* Let

$$\begin{aligned} R_1 &\colon A \nrightarrow B, \\ S_1 &\colon B \nrightarrow C, \\ R_2 &\colon X \nrightarrow Y, \\ S_2 &\colon Y \nrightarrow Z \end{aligned}$$

be relations. We have

$$(S_1 \diamond R_1) \times (S_2 \diamond R_2) = (S_1 \times S_2) \diamond (R_1 \times R_2).$$

Proof 3.9.3 ► Proof of Proposition 3.5.2

Item 1: Interaction With Inverses

Unwinding the definitions, we see that:

1. We have $(a, x) \sim_{(R \times S)^{\dagger}} (b, y)$ iff:
    - We have $(b, y) \sim_{R \times S} (a, x)$, i.e. iff:
        - We have $b \sim_R a$;
        - We have $y \sim_S x$;
2. We have $(a, x) \sim_{R^{\dagger} \times S^{\dagger}} (b, y)$ iff:
    - We have $a \sim_{R^{\dagger}} b$ and $x \sim_{S^{\dagger}} y$, i.e. iff:
        - We have $b \sim_R a$;

– We have $y \sim_S x$.

These two conditions agree, and thus the two resulting relations on $A \times X$ are equal.

Item 2: Interaction With Composition

Unwinding the definitions, we see that:

1. We have $(a, x) \sim_{(S_1 \diamond R_1) \times (S_2 \diamond R_2)} (c, z)$ iff:
   (a) We have $a \sim_{S_1 \diamond R_1} c$ and $x \sim_{S_2 \diamond R_2} z$, i.e. iff:
      i. There exists some $b \in B$ such that $a \sim_{R_1} b$ and $b \sim_{S_1} c$;
      ii. There exists some $y \in Y$ such that $x \sim_{R_2} y$ and $y \sim_{S_2} z$;
2. We have $(a, x) \sim_{(S_1 \times S_2) \diamond (R_1 \times R_2)} (c, z)$ iff:
   (a) There exists some $(b, y) \in B \times Y$ such that $(a, x) \sim_{R_1 \times R_2} (b, y)$ and $(b, y) \sim_{S_1 \times S_2} (c, z)$, i.e. such that:
      i. We have $a \sim_{R_1} b$ and $x \sim_{R_2} y$;
      ii. We have $b \sim_{S_1} c$ and $y \sim_{S_2} z$.

These two conditions agree, and thus the two resulting relations from $A \times X$ to $C \times Z$ are equal. $\square$

3.10 Products of Families of Relations

Let $\{A_i\}_{i \in I}$ and $\{B_i\}_{i \in I}$ be families of sets, and let $\{R_i : A_i \nrightarrow B_i\}_{i \in I}$ be a family of relations.

Definition 3.10.1 ► The Product of a Family of Relations

The **product of the family** $\{R_i\}_{i \in I}$ is the relation $\prod_{i \in I} R_i$ from $\prod_{i \in I} A_i$ to $\prod_{i \in I} B_i$ defined as follows:

- Viewing relations as subsets, we define $\prod_{i \in I} R_i$ as its product as a family of sets, i.e. we have

$$\prod_{i \in I} R_i \overset{\text{def}}{=} \left\{ (a_i, b_i)_{i \in I} \in \prod_{i \in I} (A_i \times B_i) \,\middle|\, \begin{array}{l} \text{for each } i \in I, \\ \text{we have } a_i \sim_{R_i} b_i \end{array} \right\}.$$

- Viewing relations as functions to powersets, we define

$$\left[\prod_{i\in I} R_i\right]((a_i)_{i\in I}) \stackrel{\text{def}}{=} \prod_{i\in I} R_i(a_i)$$

for each $(a_i)_{i\in I} \in \prod_{i\in I} R_i$.

3.11 The Inverse of a Relation

Let A, B, and C be sets and let $R \subset A \times B$ be a relation.

Definition 3.11.1 ► The Inverse of a Relation

The **inverse of** R[1] is the relation $R^\dagger$ defined as follows:

- Viewing relations as subsets, we define

$$R^\dagger \stackrel{\text{def}}{=} \{(b,a) \in B \times A \mid \text{we have } b \sim_R a\}.$$

- Viewing relations as functions $A \times B \longrightarrow \{\text{true}, \text{false}\}$, we define

$$[R^\dagger]^b_a \stackrel{\text{def}}{=} R^a_b$$

for each $(b,a) \in B \times A$.

- Viewing relations as functions $A \longrightarrow \mathcal{P}(B)$, we define

$$\begin{aligned}[R^\dagger](b) &\stackrel{\text{def}}{=} R^\dagger(\{b\}) \\ &\stackrel{\text{def}}{=} \{a \in A \mid b \in R(a)\}\end{aligned}$$

for each $b \in B$, where $R^\dagger(\{b\})$ is the fibre of R over $\{b\}$.

[1] *Further Terminology:* Also called the **opposite of** R, the **transpose of** R, or the **converse of** R.

Example 3.11.2 ► Examples of Inverses of Relations

Here are some examples of inverses of relations.

1. *Less Than Equal Signs.* We have $(\leq)^\dagger = \geq$.

2. *Greater Than Equal Signs.* Dually to **??**, we have $(\geq)^{\dagger} = \leq$.

3. *Functions.* Let $f : A \longrightarrow B$ be a function. We have

$$\begin{aligned} \mathrm{Gr}(f)^{\dagger} &= f^{-1}, \\ \left(f^{-1}\right)^{\dagger} &= \mathrm{Gr}(f). \end{aligned}$$

Proposition 3.11.3 ► Properties of Inverses of Relations

Let $R \colon A \nrightarrow B$ and $S \colon B \nrightarrow C$ be relations.

1. *Interaction With Ranges and Domains.* We have

$$\begin{aligned} \mathrm{dom}\left(R^{\dagger}\right) &= \mathrm{range}(R), \\ \mathrm{range}\left(R^{\dagger}\right) &= \mathrm{dom}(R). \end{aligned}$$

2. *Interaction With Composition I.* We have

$$(S \diamond R)^{\dagger} = R^{\dagger} \diamond S^{\dagger}.$$

3. *Interaction With Composition II.* We have

$$\begin{aligned} \chi_B(-_1, -_2) &\subset R \diamond R^{\dagger}, \\ \chi_A(-_1, -_2) &\subset R^{\dagger} \diamond R. \end{aligned}$$

4. *Invertibility.* We have

$$\left(R^{\dagger}\right)^{\dagger} = R.$$

5. *Identity.* We have

$$\chi_A^{\dagger}(-_1, -_2) = \chi_A(-_1, -_2).$$

PROOF 3.11.4 ► PROOF OF PROPOSITION 3.11.3

Item 1: Interaction With Ranges and Domains

Clear.

Item 2: Interaction With Composition I

Clear.

Item 3: Interaction With Composition II

Clear.

Item 4: Invertibility

Clear.

Item 5: Identity

Clear. $\square$

3.12 Composition of Relations

Let A, B, and C be sets and let $R \subset A \times B$ and $S \subset B \times C$ be relations.

DEFINITION 3.12.1 ► COMPOSITION OF RELATIONS

The **composition of R and S** is the relation $S \diamond R$ defined as follows:

- Viewing relations from A to C as subsets of $A \times C$, we define

$$S \diamond R \overset{\text{def}}{=} \left\{ (a, c) \in A \times C \;\middle|\; \begin{array}{l} \text{there exists some } b \in B \text{ such} \\ \text{that } a \sim_R b \text{ and } b \sim_S c \end{array} \right\}.$$

- Viewing relations as functions $A \times B \longrightarrow \{\text{true}, \text{false}\}$, we define

$$\begin{aligned} (S \diamond R)_{-_2}^{-_1} &\overset{\text{def}}{=} \int^{y \in B} S_y^{-_1} \times R_{-_2}^{y} \\ &= \bigvee_{y \in B} S_y^{-_1} \times R_{-_2}^{y}, \end{aligned}$$

where the join $\bigvee$ is taken in the poset $(\{\text{true}, \text{false}\}, \preceq)$ of Sets, Definition 1.2.3.

- Viewing relations as functions $A \to \mathcal{P}(B)$, we define

$$S \diamond R \overset{\text{def}}{=} \mathrm{Lan}_{\chi_B}(S) \circ R, \qquad \begin{tikzcd} & B \arrow[r, "S"] \arrow[d, hook, "\chi_B"'] \arrow[d, phantom, shift left=3ex, "\Swarrow"] & \mathcal{P}(C), \\ A \arrow[r, "R"'] & \mathcal{P}(B) \arrow[ur, bend right, "\mathrm{Lan}_{\chi_B}(S)"'] & \end{tikzcd}$$

where $\mathrm{Lan}_{\chi_B}(S)$ is computed by the formula

$$\begin{aligned} [\mathrm{Lan}_{\chi_B}(S)](V) &\cong \int^{y\in B} \chi_{\mathcal{P}(B)}\big(\chi_y, V\big) \odot S_y \\ &\cong \int^{y\in B} \chi_V(y) \odot S_y \\ &\cong \bigcup_{y\in B} \chi_V(y) \odot S_y \\ &\cong \bigcup_{y\in V} S_y \end{aligned}$$

for each $V \in \mathcal{P}(B)$. In other words, $S \diamond R$ is defined by[1]

$$\begin{aligned} [S \diamond R](a) &\overset{\text{def}}{=} S(R(a)) \\ &\overset{\text{def}}{=} \bigcup_{x\in R(a)} S(x). \end{aligned}$$

for each $a \in A$.

[1]That is: the relation R may send $a \in A$ to a number of elements $\{b_i\}_{i\in I}$ in B, and then the relation S may send the image of each of the b_i's to a number of elements $\{S(b_i)\}_{i\in I} = \left\{\{c_{j_i}\}_{j_i\in J_i}\right\}_{i\in I}$ in C.

EXAMPLE 3.12.2 ► EXAMPLES OF COMPOSITION OF RELATIONS

Here are some examples of composition of relations.

1. *Composing Less/Greater Than Equal With Greater/Less Than Equal Signs.* We have

$$\leq \diamond \geq \; = \sim_{\mathrm{triv}},$$
$$\geq \diamond \leq \; = \sim_{\mathrm{triv}} .$$

2. *Composing Less/Greater Than Equal Signs With Less/Greater Than Equal Signs.* We have

$$\leq \diamond \leq \; = \; \leq,$$
$$\geq \diamond \geq \; = \; \geq.$$

PROPOSITION 3.12.3 ► PROPERTIES OF COMPOSITION OF RELATIONS

Let $R\colon A \nrightarrow B$, $S\colon B \nrightarrow C$, and $T\colon C \nrightarrow D$ be relations.

1. *Interaction With Ranges and Domains.* We have

$$\mathrm{dom}(S \diamond R) \subset \mathrm{dom}(R),$$
$$\mathrm{range}(S \diamond R) \subset \mathrm{range}(S).$$

2. *Associativity.* We have

$$(T \diamond S) \diamond R = T \diamond (S \diamond R).$$

3. *Unitality.* We have

$$\chi_B \diamond R = R,$$
$$R \diamond \chi_A = R.$$

4. *Interaction With Inverses.* We have

$$(S \diamond R)^{\dagger} = R^{\dagger} \diamond S^{\dagger}.$$

5. *Interaction With Composition.* We have

$$\chi_B(-_1, -_2) \subset R \diamond R^{\dagger},$$
$$\chi_A(-_1, -_2) \subset R^{\dagger} \diamond R.$$

PROOF 3.12.4 ► PROOF OF PROPOSITION 3.12.3

Item 1: Interaction With Ranges and Domains

Clear.

Item 2: Associativity

Indeed, we have

$$\begin{aligned}
(T \diamond S) \diamond R &\stackrel{\text{def}}{=} \left(\int^{y \in C} T^{-1}_{x} \times S^{x}_{-_2}\right) \diamond R \\
&\stackrel{\text{def}}{=} \int^{x \in B} \left(\int^{y \in C} T^{-1}_{x} \times S^{x}_{y}\right) \diamond R^{y}_{-_2} \\
&= \int^{x \in B} \int^{y \in C} \left(T^{-1}_{x} \times S^{x}_{y}\right) \diamond R^{y}_{-_2} \\
&= \int^{y \in C} \int^{x \in B} \left(T^{-1}_{x} \times S^{x}_{y}\right) \diamond R^{y}_{-_2} \\
&= \int^{y \in C} \int^{x \in B} T^{-1}_{x} \times \left(S^{x}_{y} \diamond R^{y}_{-_2}\right) \\
&= \int^{x \in B} T^{-1}_{x} \times \left(\int^{y \in C} S^{x}_{y} \diamond R^{y}_{-_2}\right) \\
&\stackrel{\text{def}}{=} \int^{x \in B} T^{-1}_{x} \times (S \diamond R)^{x}_{-_2} \\
&\stackrel{\text{def}}{=} T \diamond (S \diamond R).
\end{aligned}$$

In the language of relations, given $a \in A$ and $d \in D$, the stated equality witnesses the equivalence of the following two statements:

1. We have $a \sim_{(T \diamond S) \diamond R} d$, i.e. there exists some $b \in B$ such that:
   (a) We have $a \sim_R b$;
   (b) We have $b \sim_{T \diamond S} d$, i.e. there exists some $c \in C$ such that:
      i. We have $b \sim_S c$;
      ii. We have $c \sim_T d$;
2. We have $a \sim_{T \diamond (S \diamond R)} d$, i.e. there exists some $c \in C$ such that:
   (a) We have $a \sim_{S \diamond R} c$, i.e. there exists some $b \in B$ such that:

i. We have $a \sim_R b$;

ii. We have $b \sim_S c$;

(b) We have $c \sim_T d$;

both of which are equivalent to the statement

- There exist $b \in B$ and $c \in C$ such that $a \sim_R b \sim_S c \sim_T d$.

Item 3: Unitality

Indeed, we have

$$\begin{aligned}\chi_B \diamond R &\stackrel{\text{def}}{=} \int^{x\in B} (\chi_B)^{-_1}_{x} \times R^{x}_{-_2}\\ &= \bigvee_{x\in B} (\chi_B)^{-_1}_{x} \times R^{x}_{-_2}\\ &= \bigvee_{\substack{x\in B\\ x=-_1}} R^{x}_{-_2}\\ &= R^{-_1}_{-_2},\end{aligned}$$

and

$$\begin{aligned}R \diamond \chi_A &\stackrel{\text{def}}{=} \int^{x\in A} R^{-_1}_{x} \times (\chi_A)^{x}_{-_2}\\ &= \bigvee_{x\in B} R^{-_1}_{x} \times (\chi_A)^{x}_{-_2}\\ &= \bigvee_{\substack{x\in B\\ x=-_2}} R^{-_1}_{x}\\ &= R^{-_1}_{-_2}.\end{aligned}$$

In the language of relations, given $a \in A$ and $b \in B$:

- The equality
$$\chi_B \diamond R = R$$
witnesses the equivalence of the following two statements:
    1. We have $a \sim_b B$.

2. There exists some $b' \in B$ such that:
   (a) We have $a \sim_R b'$
   (b) We have $b' \sim_{\chi_B} b$, i.e. $b' = b$.

- The equality

$$R \diamond \chi_A = R$$

witnesses the equivalence of the following two statements:

1. There exists some $a' \in A$ such that:
   (a) We have $a \sim_{\chi_B} a'$, i.e. $a = a'$.
   (b) We have $a' \sim_R b$
2. We have $a \sim_b B$.

Item 4: Interaction With Inverses

Clear.

Item 5: Interaction With Composition

Clear.

3.13 The Collage of a Relation

Let A and B be sets and let $R\colon A \nrightarrow B$ be a relation from A to B.

Definition 3.13.1 ► The Collage of a Relation

The **collage of** R[1] is the poset $\mathbf{Coll}(R) \overset{\text{def}}{=} \left(\mathrm{Coll}(R), \preceq_{\mathbf{Coll}(R)}\right)$ consisting of

- *The Underlying Set.* The set $\mathrm{Coll}(R)$ defined by

$$\mathrm{Coll}(R) \overset{\text{def}}{=} A \coprod B.$$

- *The Partial Order.* The partial order

$$\preceq_{\mathbf{Coll}(R)} : \mathrm{Coll}(R) \times \mathrm{Coll}(R) \longrightarrow \{\text{true}, \text{false}\}$$

on $\mathrm{Coll}(R)$ defined by

$$\preceq(a, b) \overset{\text{def}}{=} \begin{cases} \text{true} & \text{if } a = b \text{ or } a \sim_R b, \\ \text{false} & \text{otherwise.} \end{cases}$$

[1] *Further Terminology:* Also called the **cograph of** R.

PROPOSITION 3.13.2 ► PROPERTIES OF COLLAGES OF RELATIONS

Let A and B be sets and let $R\colon A \nrightarrow B$ be a relation from A to B.

1. *Functoriality I.* The assignment $R \mapsto \mathbf{Coll}(R)$ defines a functor[1]

   $$\mathbf{Coll}\colon \mathbf{Rel}(A, B) \longrightarrow \mathsf{Pos}_{/\Delta^1}(A, B),$$

   where

   - *Action on Objects.* For each $R \in \mathrm{Obj}(\mathbf{Rel}(A, B))$, we have

     $$[\mathbf{Coll}](R) \stackrel{\text{def}}{=} (\mathbf{Coll}(R), \phi_R)$$

     for each $R \in \mathbf{Rel}(A, B)$, where

     – The poset $\mathbf{Coll}(R)$ is the collage of R of Definition 3.13.1;

     – The morphism $\phi_R\colon \mathbf{Coll}(R) \longrightarrow \Delta^1$ is given by

     $$\phi_R(x) \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if } x \in A, \\ 1 & \text{if } x \in B \end{cases}$$

     for each $x \in \mathbf{Coll}(R)$;

   - *Action on Morphisms.* For each $R, S \in \mathrm{Obj}(\mathbf{Rel}(A, B))$, the action on Hom-sets

     $$\mathbf{Coll}_{R,S}\colon \mathrm{Hom}_{\mathbf{Rel}(A,B)}(R, S) \longrightarrow \mathrm{Pos}(\mathbf{Coll}(R), \mathbf{Coll}(S))$$

     of $\mathbf{Coll}$ at (R, S) is given by sending an inclusion

     $$\iota\colon R \subset S$$

     to the morphism

     $$\mathbf{Coll}(\iota)\colon \mathbf{Coll}(R) \longrightarrow \mathbf{Coll}(S)$$

     of posets over Δ^1 defined by

     $$[\mathbf{Coll}(\iota)](x) \stackrel{\text{def}}{=} x$$

     for each $x \in \mathbf{Coll}(R)$.[2]

2. *Equivalence.* The functor of Item 1 is an equivalence of categories.

---

[1] Here $\mathsf{Pos}_{/\Delta^1}(A, B)$ is the category defined as the pullback

$$\mathsf{Pos}_{/\Delta^1}(A, B) \overset{\text{def}}{=} \mathrm{pt} \underset{[A], \mathsf{Pos}, \mathrm{fib}_0}{\times} \mathsf{Pos}_{/\Delta^1} \underset{\mathrm{fib}_1, \mathsf{Pos}, [B]}{\times} \mathrm{pt},$$

as in the diagram

$$\begin{tikzcd}
 & & \mathbf{Pos}_{/\Delta^1}(A, B) \arrow[dl] \arrow[dr] \arrow[dd, phantom, very near start, "\lrcorner" rotate=-45] & & \\
 & \mathbf{Pos}_{/\Delta^1} \underset{\mathbf{Pos}}{\times} \mathrm{pt} \arrow[dl] \arrow[dr] \arrow[dd, phantom, very near start, "\lrcorner" rotate=-45] & & \mathrm{pt} \underset{\mathbf{Pos}}{\times} \mathbf{Pos}_{/\Delta^1} \arrow[dl] \arrow[dr] \arrow[dd, phantom, very near start, "\lrcorner" rotate=-45] & \\
\mathrm{pt} \arrow[dr, "{[A]}"'] & & \mathbf{Pos}_{/\Delta^1} \arrow[dl, "\mathrm{fib}_{[0]}"] \arrow[dr, "\mathrm{fib}_{[1]}"'] & & \mathrm{pt}. \arrow[dl, "{[B]}"] \\
 & \mathbf{Pos} & & \mathbf{Pos} &
\end{tikzcd}$$

Explicitly, an object of $\mathsf{Pos}_{/\Delta^1}(A, B)$ is a pair (X, ϕ_X) consisting of

- A poset X;
- A morphism $\phi_X : X \to \Delta^1$;

such that $\phi_X^{-1}(0) = A$ and $\phi_X^{-1}(0) = B$, with morphisms between such objects being morphisms of posets over Δ^1.

[2] Note that this is indeed a morphism of posets: if $x \preceq_{\mathbf{Coll}(R)} y$, then $x = y$ or $x \sim_R y$, so we have either $x = y$ or $x \sim_S y$ (as $R \subset S$), and thus $x \preceq_{\mathbf{Coll}(S)} y$.

Proof 3.13.3 ▶ Proof of Proposition 3.13.2

Item 1: Functoriality

Clear.

Item 2: Equivalence

Omitted.

4 Equivalence Relations

4.1 Reflexive Relations

4.1.1 Foundations

Let A be a set.

Definition 4.1.1 ► Reflexive Relations

A **reflexive relation** is equivalently:[1]

- An $\mathbb{E}_0$-monoid in $(\mathrm{N}_\bullet(\mathbf{Rel}(A, A)), \chi_A)$;
- A pointed object in $(\mathbf{Rel}(A, A), \chi_A)$.

[1] Note that since $\mathbf{Rel}(A, A)$ is posetal, reflexivity is a property of a relation, rather than extra structure.

Remark 4.1.2 ► Unwinding Definition 4.1.1

In detail, a relation R on A is **reflexive** if we have an inclusion

$$\eta_R \colon \chi_A \subset R$$

of relations in $\mathbf{Rel}(A, A)$, i.e. if, for each $a \in A$, we have $a \sim_R a$.

Definition 4.1.3 ► The Po/Set of Reflexive Relations on a Set

Let A be a set.

1. The **set of reflexive relations on** A is the subset $\mathrm{Rel}^{\mathrm{refl}}(A, A)$ of $\mathrm{Rel}(A, A)$ spanned by the reflexive relations.
2. The **poset of relations on** A is is the subposet $\mathbf{Rel}^{\mathrm{refl}}(A, A)$ of $\mathbf{Rel}(A, A)$ spanned by the reflexive relations.

Proposition 4.1.4 ► Properties of Reflexive Relations

Let R and S be relations on A.

1. *Interaction With Inverses.* If R is reflexive, then so is $R^\dagger$.
2. *Interaction With Composition.* If R and S are reflexive, then so is $S \diamond R$.

Proof 4.1.5 ► Proof of Proposition 4.1.4

Item 1: Interaction With Inverses

Clear.

Item 2: Interaction With Composition

Clear.

4.1.2 The Reflexive Closure of a Relation

Let R be a relation on A.

Definition 4.1.6 ▶ The Reflexive Closure of a Relation

The **reflexive closure** of $\sim_R$ is the relation $\sim_R^{\mathrm{refl}}$[1] satisfying the following universal property:[2]

(★) Given another reflexive relation $\sim_S$ on A such that $R \subset S$, there exists an inclusion $\sim_R^{\mathrm{refl}} \subset \sim_S$.

[1] *Further Notation:* Also written R^{refl}.
[2] *Slogan:* The reflexive closure of R is the smallest reflexive relation containing R.

Construction 4.1.7 ▶ The Reflexive Closure of a Relation

Concretely, $\sim_R^{\mathrm{refl}}$ is the free pointed object on R in $(\mathbf{Rel}(A, A), \chi_A)$[1], being given by

$$\begin{aligned} R^{\mathrm{refl}} &\overset{\mathrm{def}}{=} R \coprod^{\mathbf{Rel}(A,A)} \Delta_A \\ &= R \cup \Delta_A \\ &= \{(a, b) \in A \times A \mid \text{we have } a \sim_R b \text{ or } a = b\}. \end{aligned}$$

[1] Or, equivalently, the free $\mathbb{E}_0$-monoid on R in $(\mathsf{N}_{\bullet}(\mathbf{Rel}(A, A)), \chi_A)$.

Proof 4.1.8 ▶ Proof of Construction 4.1.7

Clear.

Proposition 4.1.9 ▶ Properties of the Reflexive Closure of a Relation

Let R be a relation on A.

1. *Adjointness.* We have an adjunction

$$\Big((-)^{\mathrm{refl}} \dashv 忘\Big)\colon \quad \begin{tikzcd} \mathbf{Rel}(A,A) \arrow[r, bend left, "(-)^{\mathrm{refl}}"] \arrow[r, phantom, "\perp"] & \mathbf{Rel}^{\mathrm{refl}}(A,A), \arrow[l, bend left, "忘"] \end{tikzcd}$$

   witnessed by a bijection of sets

$$\mathbf{Rel}^{\mathrm{refl}}\Big(R^{\mathrm{refl}}, S\Big) \cong \mathbf{Rel}(R,S),$$

   natural in $R \in \mathrm{Obj}\Big(\mathbf{Rel}^{\mathrm{refl}}(A,A)\Big)$ and $S \in \mathrm{Obj}(\mathbf{Rel}(A,A))$.

2. *The Reflexive Closure of a Reflexive Relation.* If R is reflexive, then $R^{\mathrm{refl}} = R$.

3. *Idempotency.* We have

$$\Big(R^{\mathrm{refl}}\Big)^{\mathrm{refl}} = R^{\mathrm{refl}}.$$

4. *Interaction With Inverses.* We have

$$\Big(R^{\dagger}\Big)^{\mathrm{refl}} = \Big(R^{\mathrm{refl}}\Big)^{\dagger}, \qquad \begin{tikzcd} \mathrm{Rel}(A,A) \arrow[r, "(-)^{\mathrm{refl}}"] \arrow[d, "(-)^{\dagger}"'] & \mathrm{Rel}(A,A) \arrow[d, "(-)^{\dagger}"] \\ \mathrm{Rel}(A,A) \arrow[r, "(-)^{\mathrm{refl}}"'] & \mathrm{Rel}(A,A). \end{tikzcd}$$

5. *Interaction With Composition.* We have

$$(S \diamond R)^{\mathrm{refl}} = S^{\mathrm{refl}} \diamond R^{\mathrm{refl}}, \qquad \begin{tikzcd} \mathrm{Rel}(A,A) \times \mathrm{Rel}(A,A) \arrow[r, "\diamond"] \arrow[d, "(-)^{\mathrm{refl}} \times (-)^{\mathrm{refl}}"'] & \mathrm{Rel}(A,A) \arrow[d, "(-)^{\mathrm{refl}}"] \\ \mathrm{Rel}(A,A) \times \mathrm{Rel}(A,A) \arrow[r, "\diamond"'] & \mathrm{Rel}(A,A). \end{tikzcd}$$

Proof 4.1.10 ► Proof of Proposition 4.1.9

Item 1: Adjointness

This is a rephrasing of the universal property of the reflexive closure of a relation, stated in Definition 4.1.6.

Item 2: The Reflexive Closure of a Reflexive Relation

Clear.

Item 3: Idempotency

This follows from Item 2.

Item 4: Interaction With Inverses

Clear.

Item 5: Interaction With Composition

This follows from Item 2 of Proposition 4.1.4.

4.2 Symmetric Relations

4.2.1 Foundations

Let A be a set.

Definition 4.2.1 ► Symmetric Relations

A relation R on A is **symmetric** if, for each $a, b \in A$, the following conditions are equivalent:[1]

1. We have $a \sim_R b$.
2. We have $b \sim_R a$.

[1] That is, R is symmetric if $R^\dagger = R$.

Definition 4.2.2 ► The Po/Set of Symmetric Relations on a Set

Let A be a set.

1. The **set of symmetric relations on** A is the subset $\mathrm{Rel}^{\mathrm{symm}}(A, A)$ of $\mathrm{Rel}(A, A)$ spanned by the symmetric relations.

2. The **poset of relations on** A is is the subposet $\mathbf{Rel}^{\mathsf{symm}}(A, A)$ of $\mathbf{Rel}(A, A)$ spanned by the symmetric relations.

Proposition 4.2.3 ► Properties of Symmetric Relations

Let R and S be relations on A.

1. *Interaction With Inverses.* If R is symmetric, then so is $R^{\dagger}$.

2. *Interaction With Composition.* If R and S are symmetric, then so is $S \diamond R$.

Proof 4.2.4 ► Proof of Proposition 4.2.3

Item 1: Interaction With Inverses

Clear.

Item 2: Interaction With Composition

Clear. □

4.2.2 The Symmetric Closure of a Relation

Let R be a relation on A.

Definition 4.2.5 ► The Symmetric Closure of a Relation

The **symmetric closure** of $\sim_R$ is the relation $\sim_R^{\mathsf{symm}}$[1] satisfying the following universal property:[2]

(★) Given another symmetric relation $\sim_S$ on A such that $R \subset S$, there exists an inclusion $\sim_R^{\mathsf{symm}} \subset \sim_S$.

[1] *Further Notation:* Also written R^{symm}.
[2] *Slogan:* The symmetric closure of R is the smallest symmetric relation containing R.

Construction 4.2.6 ► The Symmetric Closure of a Relation

Concretely, $\sim_R^{\mathsf{symm}}$ is the symmetric relation on A defined by

$$\begin{aligned} R^{\mathsf{symm}} &\overset{\text{def}}{=} R \cup R^{\dagger} \\ &= \{(a, b) \in A \times A \mid \text{we have } a \sim_R b \text{ or } b \sim_R a\}. \end{aligned}$$

Proof 4.2.7 ▶ Proof of Construction 4.2.6

Clear. □

Proposition 4.2.8 ▶ Properties of the Symmetric Closure of a Relation

Let R be a relation on A.

1. *Adjointness.* We have an adjunction

$$((-)^{\mathsf{symm}} \dashv 忘) \colon \quad \begin{tikzcd} \mathbf{Rel}(A,A) \arrow[r, bend left, "(-)^{\mathsf{symm}}"] \arrow[r, phantom, "\perp"] & \mathbf{Rel}^{\mathsf{symm}}(A,A), \arrow[l, bend left, "忘"] \end{tikzcd}$$

   witnessed by a bijection of sets

$$\mathbf{Rel}^{\mathsf{symm}}(R^{\mathsf{symm}}, S) \cong \mathbf{Rel}(R, S),$$

   natural in $R \in \mathrm{Obj}(\mathbf{Rel}^{\mathsf{symm}}(A,A))$ and $S \in \mathrm{Obj}(\mathbf{Rel}(A,A))$.

2. *The Symmetric Closure of a Symmetric Relation.* If R is symmetric, then $R^{\mathsf{symm}} = R$.

3. *Idempotency.* We have

$$(R^{\mathsf{symm}})^{\mathsf{symm}} = R^{\mathsf{symm}}.$$

4. *Interaction With Inverses.* We have

$$\left(R^{\dagger}\right)^{\mathsf{symm}} = \left(R^{\mathsf{symm}}\right)^{\dagger}, \qquad \begin{tikzcd} \mathrm{Rel}(A,A) \arrow[r, "(-)^{\mathsf{symm}}"] \arrow[d, "(-)^{\dagger}"'] & \mathrm{Rel}(A,A) \arrow[d, "(-)^{\dagger}"] \\ \mathrm{Rel}(A,A) \arrow[r, "(-)^{\mathsf{symm}}"'] & \mathrm{Rel}(A,A). \end{tikzcd}$$

5. *Interaction With Composition.* We have

$$(S \diamond R)^{\mathsf{symm}} = S^{\mathsf{symm}} \diamond R^{\mathsf{symm}}, \qquad \begin{tikzcd} \mathrm{Rel}(A,A) \times \mathrm{Rel}(A,A) \arrow[r, "\diamond"] \arrow[d, "(-)^{\mathsf{symm}} \times (-)^{\mathsf{symm}}"'] & \mathrm{Rel}(A,A) \arrow[d, "(-)^{\mathsf{symm}}"] \\ \mathrm{Rel}(A,A) \times \mathrm{Rel}(A,A) \arrow[r, "\diamond"'] & \mathrm{Rel}(A,A). \end{tikzcd}$$

Proof 4.2.9 ▶ Proof of Proposition 4.2.8

Item 1: Adjointness

This is a rephrasing of the universal property of the symmetric closure of a relation, stated in Definition 4.2.5.

Item 2: The Symmetric Closure of a Symmetric Relation

Clear.

Item 3: Idempotency

This follows from Item 2.

Item 4: Interaction With Inverses

Clear.

Item 5: Interaction With Composition

This follows from Item 2 of Proposition 4.2.3. □

4.3 Transitive Relations

4.3.1 Foundations

Let A be a set.

Definition 4.3.1 ▶ Transitive Relations

A **transitive relation** is equivalently:[1]

- A non-unital $\mathbb{E}_1$-monoid in $(\mathsf{N}_\bullet(\mathbf{Rel}(A, A)), \diamond)$;
- A non-unital monoid in $(\mathbf{Rel}(A, A), \diamond)$.

[1] Note that since $\mathbf{Rel}(A, A)$ is posetal, transitivity is a property of a relation, rather than extra structure.

Remark 4.3.2 ▶ Unwinding Definition 4.3.1

In detail, a relation R on A is **transitive** if we have an inclusion

$$\mu_R \colon R \diamond R \subset R$$

of relations in $\mathbf{Rel}(A, A)$, i.e. if, for each $a, c \in A$, the following condition is satisfied:

($\star$) If there exists some $b \in A$ such that $a \sim_R b$ and $b \sim_R c$, then $a \sim_R c$.

DEFINITION 4.3.3 ► THE PO/SET OF TRANSITIVE RELATIONS ON A SET

Let A be a set.

1. The **set of transitive relations from** A **to** B is the subset $\mathrm{Rel}^{\mathrm{trans}}(A)$ of $\mathrm{Rel}(A, A)$ spanned by the transitive relations.
2. The **poset of relations from** A **to** B is is the subposet $\mathbf{Rel}^{\mathrm{trans}}(A)$ of $\mathbf{Rel}(A, A)$ spanned by the transitive relations.

PROPOSITION 4.3.4 ► PROPERTIES OF TRANSITIVE RELATIONS

Let R and S be relations on A.

1. *Interaction With Inverses.* If R is transitive, then so is $R^{\dagger}$.
2. *Interaction With Composition.* If R and S are transitive, then $S \diamond R$ ***may fail to be transitive***.

PROOF 4.3.5 ► PROOF OF PROPOSITION 4.3.4

Item 1: Interaction With Inverses

Clear.

Item 2: Interaction With Composition

See [**MSE2096272**].[1]

[1] *Intuition:* Transitivity for R and S fails to imply that of $S \diamond R$ because the composition operation for relations intertwines R and S in an incompatible way:

1. If $a \sim_{S\diamond R} c$ and $c \sim_{S\diamond r} e$, then:
   (a) There is some $b \in A$ such that:
      i. $a \sim_R b$;
      ii. $b \sim_S c$;
   (b) There is some $d \in A$ such that:
      i. $c \sim_R d$;
      ii. $d \sim_S e$.

4.3.2 The Transitive Closure of a Relation

Let R be a relation on A.

Definition 4.3.6 ▶ The Transitive Closure of a Relation

The **transitive closure** of $\sim_R$ is the relation $\sim_R^{\mathrm{trans}}$[1] satisfying the following universal property:[2]

($\star$) Given another transitive relation $\sim_S$ on A such that $R \subset S$, there exists an inclusion $\sim_R^{\mathrm{trans}} \subset \sim_S$.

[1] *Further Notation:* Also written R^{trans}.

[2] *Slogan:* The transitive closure of R is the smallest transitive relation containing R.

Construction 4.3.7 ▶ The Transitive Closure of a Relation

Concretely, $\sim_R^{\mathrm{trans}}$ is the free non-unital monoid on R in $(\mathsf{Rel}(A, A), \diamond)$[1], being given by

$$
\begin{aligned}
R^{\mathrm{trans}} &\overset{\mathrm{def}}{=} \coprod_{n=1}^{\infty} R^{\diamond n}\\
&\overset{\mathrm{def}}{=} \bigcup_{n=1}^{\infty} R^{\diamond n}\\
&\overset{\mathrm{def}}{=} \left\{ (a, b) \in A \times B \;\middle|\; \begin{array}{l}\text{there exists some } (x_1, \ldots, x_n) \in R^{\times n}\\ \text{such that } a \sim_R x_1 \sim_R \cdots \sim_R x_n \sim_R b\end{array} \right\}.
\end{aligned}
$$

[1] Or, equivalently, the free non-unital $\mathbb{E}_1$-monoid on R in $(\mathrm{N}_{\bullet}(\mathsf{Rel}(A, A)), \diamond)$.

Proof 4.3.8 ▶ Proof of Construction 4.3.7

Clear. ∎

Proposition 4.3.9 ▶ Properties of the Transitive Closure of a Relation

Let R be a relation on A.

1. *Adjointness.* We have an adjunction

$$
\left((-)^{\mathrm{trans}} \dashv 忘\right) : \quad \begin{tikzcd} \mathsf{Rel}(A, A) \arrow[r, bend left, "(-)^{\mathrm{trans}}"] \arrow[r, phantom, "\perp"] & \mathsf{Rel}^{\mathrm{trans}}(A, A), \arrow[l, bend left, "忘"] \end{tikzcd}
$$

witnessed by a bijection of sets

$$\mathbf{Rel}^{\mathsf{trans}}(R^{\mathsf{trans}}, S) \cong \mathbf{Rel}(R, S),$$

natural in $R \in \mathrm{Obj}\big(\mathbf{Rel}^{\mathsf{trans}}(A, A)\big)$ and $S \in \mathrm{Obj}(\mathbf{Rel}(A, B))$.

2. *The Transitive Closure of a Transitive Relation.* If R is transitive, then $R^{\mathsf{trans}} = R$.

3. *Idempotency.* We have

$$\big(R^{\mathsf{trans}}\big)^{\mathsf{trans}} = R^{\mathsf{trans}}.$$

4. *Interaction With Inverses.* We have

$$\Big(R^{\dagger}\Big)^{\mathsf{trans}} = \Big(R^{\mathsf{trans}}\Big)^{\dagger}, \qquad \begin{tikzcd} \mathrm{Rel}(A, A) \arrow[r, "(-)^{\mathsf{trans}}"] \arrow[d, "(-)^{\dagger}"'] & \mathrm{Rel}(A, A) \arrow[d, "(-)^{\dagger}"] \\ \mathrm{Rel}(A, A) \arrow[r, "(-)^{\mathsf{trans}}"'] & \mathrm{Rel}(A, A). \end{tikzcd}$$

5. *Interaction With Composition.* We have

$$(S \diamond R)^{\mathsf{trans}} \overset{\text{poss.}}{\neq} S^{\mathsf{trans}} \diamond R^{\mathsf{trans}}, \qquad \begin{tikzcd} \mathrm{Rel}(A, A) \times \mathrm{Rel}(A, A) \arrow[r, "\diamond"] \arrow[d, "(-)^{\mathsf{trans}} \times (-)^{\mathsf{trans}}"'] \arrow[dr, phantom, "\times"] & \mathrm{Rel}(A, A) \arrow[d, "(-)^{\mathsf{trans}}"] \\ \mathrm{Rel}(A, A) \times \mathrm{Rel}(A, A) \arrow[r, "\diamond"'] & \mathrm{Rel}(A, A). \end{tikzcd}$$

Proof 4.3.10 ► Proof of Proposition 4.3.9

Item 1: Adjointness

This is a rephrasing of the universal property of the transitive closure of a relation, stated in Definition 4.3.6.

Item 2: The Transitive Closure of a Transitive Relation

Clear.

Item 3: Idempotency

This follows from Item 2.

Item 4: Interaction With Inverses

We have

$$\begin{aligned}\left(R^{\dagger}\right)^{\mathrm{trans}} &= \bigcup_{n=1}^{\infty}\left(R^{\dagger}\right)^{\diamond n} && \text{(by Construction 4.3.7)}\\ &= \bigcup_{n=1}^{\infty}\left(R^{\diamond n}\right)^{\dagger} && \text{(by Item 4 of Proposition 3.12.3)}\\ &= \left(\bigcup_{n=1}^{\infty}R^{\diamond n}\right)^{\dagger} && \text{(by Item 1 of Proposition 3.6.2)}\\ &= \left(R^{\mathrm{trans}}\right)^{\dagger}. && \text{(by Construction 4.3.7)}\end{aligned}$$

Item 5: Interaction With Composition

This follows from Item 2 of Proposition 4.3.4.

4.4 Equivalence Relations

4.4.1 Foundations

Let A be a set.

Definition 4.4.1 ▶ Equivalence Relations

A relation R is an **equivalence relation** if it is reflexive, symmetric, and transitive.[1]

[1] *Further Terminology:* If instead R is just symmetric and transitive, then it is called a **partial equivalence relation**.

Example 4.4.2 ▶ The Kernel of a Function

The **kernel of a function** $f\colon A \longrightarrow B$ is the equivalence $\sim_{\mathrm{Ker}(f)}$ on A obtained by declaring $a \sim_{\mathrm{Ker}(f)} b$ iff $f(a) = f(b)$.[1]

[1] The kernel $\mathrm{Ker}(f)\colon A \nrightarrow A$ of f is the monad induced by the adjunction $\mathrm{Gr}(f) \dashv f^{-1}\colon A \rightleftarrows B$ in **Rel** of Item 2 of Proposition 3.1.2.

Definition 4.4.3 ► The Po/Set of Equivalence Relations on a Set

Let A and B be sets.

1. The **set of equivalence relations from A to B** is the subset $\mathrm{Rel}^{\mathrm{eq}}(A, B)$ of $\mathrm{Rel}(A, B)$ spanned by the equivalence relations.

2. The **poset of relations from A to B** is is the subposet $\mathbf{Rel}^{\mathrm{eq}}(A, B)$ of $\mathbf{Rel}(A, B)$ spanned by the equivalence relations.

4.4.2 The Equivalence Closure of a Relation

Let R be a relation on A.

Definition 4.4.4 ► The Equivalence Closure of a Relation

The **equivalence closure**[1] of $\sim_R$ is the relation $\sim_R^{\mathrm{eq}}$[2] satisfying the following universal property:[3]

($\star$) Given another equivalence relation $\sim_S$ on A such that $R \subset S$, there exists an inclusion $\sim_R^{\mathrm{eq}} \subset \sim_S$.

[1] *Further Terminology:* Also called the **equivalence relation associated to** $\sim_R$.
[2] *Further Notation:* Also written R^{eq}.
[3] *Slogan:* The equivalence closure of R is the smallest equivalence relation containing R.

Construction 4.4.5 ► The Equivalence Closure of a Relation

Concretely, $\sim_R^{\mathrm{eq}}$ is the equivalence relation on A defined by

$$\begin{aligned} R^{\mathrm{eq}} &\overset{\mathrm{def}}{=} \left(\left(R^{\mathrm{refl}}\right)^{\mathrm{symm}}\right)^{\mathrm{trans}} \\ &= \left(\left(R^{\mathrm{symm}}\right)^{\mathrm{trans}}\right)^{\mathrm{refl}} \\ &= \left\{ (a,b) \in A \times B \;\middle|\; \begin{array}{l} \text{there exists } (x_1, \ldots, x_n) \in R^{\times n} \text{ satisfying at} \\ \text{least one of the following conditions:} \\ \quad 1. \text{ The following conditions are satisfied:} \\ \qquad \text{(a) We have } a \sim_R x_1 \text{ or } x_1 \sim_R a; \\ \qquad \text{(b) We have } x_i \sim_R x_{i+1} \text{ or } x_{i+1} \sim_R x_i \\ \qquad\quad \text{for each } 1 \leq i \leq n-1; \\ \qquad \text{(c) We have } b \sim_R x_n \text{ or } x_n \sim_R b; \\ \quad 2. \text{ We have } a = b. \end{array} \right\}. \end{aligned}$$

Proof 4.4.6 ► Proof of Construction 4.4.5

From the universal properties of the reflexive, symmetric, and transitive closures of a relation (Definitions 4.1.6, 4.2.5 and 4.3.6), we see that it suffices to prove that:

1. The symmetric closure of a reflexive relation is still reflexive;
2. The transitive closure of a symmetric relation is still symmetric;

which are both clear. ∎

Proposition 4.4.7 ► Properties of Equivalence Relations

Let R be a relation on A.

1. *Adjointness.* We have an adjunction

$$\left((-)^{\mathrm{eq}} \dashv 忘\right) \colon \quad \mathbf{Rel}(A,B) \underset{忘}{\overset{(-)^{\mathrm{eq}}}{\underset{\perp}{\rightleftarrows}}} \mathbf{Rel}^{\mathrm{eq}}(A,B),$$

witnessed by a bijection of sets

$$\mathbf{Rel}^{\mathrm{eq}}(R^{\mathrm{eq}}, S) \cong \mathbf{Rel}(R, S),$$

natural in $R \in \mathrm{Obj}\big(\mathbf{Rel}^{\mathrm{eq}}(A, B)\big)$ and $S \in \mathrm{Obj}(\mathbf{Rel}(A, B))$.

2. *The Equivalence Closure of an Equivalence Relation.* If R is an equivalence relation, then $R^{\mathrm{eq}} = R$.

3. *Idempotency.* We have

$$(R^{\mathrm{eq}})^{\mathrm{eq}} = R^{\mathrm{eq}}.$$

Proof 4.4.8 ► Proof of Proposition 4.4.7

Item 1: Adjointness

This is a rephrasing of the universal property of the equivalence closure of a relation, stated in Definition 4.4.4.

Item 2: The Equivalence Closure of an Equivalence Relation

Clear.

Item 3: Idempotency

This follows from Item 2. □

4.5 Quotients by Equivalence Relations

4.5.1 Equivalence Classes

Let A be a set, let R be a relation on A, and let $a \in A$.

Definition 4.5.1 ► Equivalence Classes

The **equivalence class associated to** a is the set $[a]$ defined by

$$\begin{aligned} [a] &\overset{\mathrm{def}}{=} \{x \in X \mid x \sim_R a\} \\ &= \{x \in X \mid a \sim_R x\}. \qquad \text{(since } R \text{ is symmetric)} \end{aligned}$$

4.5.2 Quotients of Sets by Equivalence Relations

Let A be a set and let R be a relation on A.

Definition 4.5.2 ► Quotients of Sets by Equivalence Relations

The **quotient of X by R** is the set $X/{\sim_R}$ defined by

$$X/{\sim_R} \overset{\text{def}}{=} \{[a] \in \mathcal{P}(X) \mid a \in X\}.$$

Remark 4.5.3 ► Why Use "Equivalence" Relations for Quotient Sets

The reason we define quotient sets for equivalence relations only is that each of the properties of being an equivalence relation—reflexivity, symmetry, and transitivity—ensures that the equivalences classes $[a]$ of X under R are well-behaved:

- *Reflexivity.* If R is reflexive, then, for each $a \in X$, we have $a \in [a]$.
- *Symmetry.* The equivalence class $[a]$ of an element a of X is defined by

  $$[a] \overset{\text{def}}{=} \{x \in X \mid x \sim_R a\},$$

  but we could equally well define

  $$[a]' \overset{\text{def}}{=} \{x \in X \mid a \sim_R x\}$$

  instead. This is not a problem when R is symmetric, as we then have $[a] = [a]'$.[1]
- *Transitivity.* If R is transitive, then $[a]$ and $[b]$ are disjoint iff $a \nsim_R b$, and equal otherwise.

[1]When categorifying equivalence relations, one finds that $[a]$ and $[a]'$ correspond to presheaves and copresheaves; see Constructions With Categories, **??**.

Proposition 4.5.4 ► Properties of Quotient Sets

Let $f\colon X \longrightarrow Y$ be a function and let R be a relation on X.

1. *As a Coequaliser.* We have an isomorphism of sets

$$X/{\sim_R^{\mathrm{eq}}} \cong \mathrm{CoEq}\left(R \hookrightarrow X \times X \underset{\mathrm{pr}_2}{\overset{\mathrm{pr}_1}{\rightrightarrows}} X\right),$$

where $\sim_R^{\mathrm{eq}}$ is the equivalence relation generated by $\sim_R$.

2. *As a Pushout.* We have an isomorphism of sets[1]

$$X/\sim_R^{\mathrm{eq}} \cong X \coprod_{\mathrm{Eq}(\mathrm{pr}_1,\mathrm{pr}_2)} X, \qquad \begin{tikzcd} X/\sim_R^{\mathrm{eq}} & X \arrow[l] \\ X \arrow[u] & \mathrm{Eq}(\mathrm{pr}_1,\mathrm{pr}_2). \arrow[l] \arrow[u] \arrow[ul, phantom, "\ulcorner", very near end] \end{tikzcd}$$

where $\sim_R^{\mathrm{eq}}$ is the equivalence relation generated by $\sim_R$.

3. *The First Isomorphism Theorem for Sets.* We have an isomorphism of sets[2,3]

$$X/\sim_{\mathrm{Ker}(f)} \cong \mathrm{Im}(f).$$

4. *Descending Functions to Quotient Sets, I.* Let R be an equivalence relation on X. The following conditions are equivalent:

   (a) There exists a map

   $$\overline{f} \colon X/\sim_R \to Y$$

   making the diagram

   $$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, two heads, "q"'] & Y \\ X/\sim_R \arrow[ur, dotted, "\exists", "\overline{f}"'] & \end{tikzcd}$$

   commute.

   (b) For each $x, y \in X$, if $x \sim_R y$, then $f(x) = f(y)$.

5. *Descending Functions to Quotient Sets, II.* Let R be an equivalence relation on X. If the conditions of Item 4 hold, then $\overline{f}$ is the *unique* map making the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, two heads, "q"'] & Y \\ X/\sim_R \arrow[ur, dashed, "\exists!", "\overline{f}"'] & \end{tikzcd}$$

commute.

6. *Descending Functions to Quotient Sets, III.* Let R be an equivalence relation on X. If the conditions of Item 4 hold, then the following conditions are equivalent:

   (a) The map $\overline{f}$ is an injection.

   (b) For each $x, y \in X$, we have $x \sim_R y$ iff $f(x) = f(y)$.

7. *Descending Functions to Quotient Sets, IV.* Let R be an equivalence relation on X. If the conditions of Item 4 hold, then the following conditions are equivalent:

   (a) The map $f : X \longrightarrow Y$ is surjective.

   (b) The map $\overline{f} : X/{\sim_R} \longrightarrow Y$ is surjective.

8. *Descending Functions to Quotient Sets, V.* Let R be a relation on X and let $\sim^{\mathrm{eq}}_R$ be the equivalence relation associated to R. The following conditions are equivalent:

   (a) The map f satisfies the equivalent conditions of Item 4:

      · There exists a map

      $$\overline{f} : X/{\sim^{\mathrm{eq}}_R} \longrightarrow Y$$

      making the diagram

      $$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "q"'] & Y \\ X/{\sim^{\mathrm{eq}}_R} \arrow[ur, dotted, "\exists", "\overline{f}"'] & \end{tikzcd}$$

      commute.

      · For each $x, y \in X$, if $x \sim^{\mathrm{eq}}_R y$, then $f(x) = f(y)$.

   (b) For each $x, y \in X$, if $x \sim_R y$, then $f(x) = f(y)$.

[1]Dually, we also have an isomorphism of sets

$$\mathrm{Eq}(\mathrm{pr}_1, \mathrm{pr}_2) \cong X \times_{X/\sim_R^{\mathrm{eq}}} X, \qquad \begin{tikzcd} \mathrm{Eq}(\mathrm{pr}_1, \mathrm{pr}_2) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d] \\ X \arrow[r] & X/\sim_R^{\mathrm{eq}}. \end{tikzcd}$$

[2]*Further Terminology:* The set $X/\sim_{\mathrm{Ker}(f)}$ is often called the **coimage of** f, and denoted by $\mathrm{Coim}(f)$.

[3]In a sense this is a result relating the monad in **Rel** induced by f with the comonad in **Rel** induced by f, as the kernel and image

$$\mathrm{Ker}(f) \colon X \nrightarrow X,$$
$$\mathrm{Im}(f) \subset Y$$

of f are respectively the induced monads and comonads of the adjunction

$$\left(\mathrm{Gr}(f) \dashv f^{-1}\right) \colon \quad \begin{tikzcd} A \arrow[r, bend left, "\mathrm{Gr}(f)"] \arrow[r, phantom, "\perp"] & B \arrow[l, bend left, "f^{-1}"] \end{tikzcd}$$

of Item 2 of Proposition 3.1.2.

Proof 4.5.5 ▶ Proof of Proposition 4.5.4

Item 1: As a Coequaliser

Omitted.

Item 2: As a Pushout

Omitted.

Item 3: The First Isomorphism Theorem for Sets

Clear.

Item 4: Descending Functions to Quotient Sets, I

See [Pro23c].

Item 5: Descending Functions to Quotient Sets, II

See [Pro23d].

Item 6: Descending Functions to Quotient Sets, III

See [Pro23b].

Item 7: Descending Functions to Quotient Sets, IV

See [Pro23a].

Item 8: Descending Functions to Quotient Sets, V

The implication Item 8a $\implies$ Item 8b is clear.

Conversely, suppose that, for each $x, y \in X$, if $x \sim_R y$, then $f(x) = f(y)$. Spelling out the definition of the equivalence closure of R, we see that the condition $x \sim_R^{\mathrm{eq}} y$ unwinds to the following:

($\star$) There exist $(x_1, \ldots, x_n) \in R^{\times n}$ satisfying at least one of the following conditions:

1. The following conditions are satisfied:
   (a) We have $x \sim_R x_1$ or $x_1 \sim_R x$;
   (b) We have $x_i \sim_R x_{i+1}$ or $x_{i+1} \sim_R x_i$ for each $1 \leq i \leq n-1$;
   (c) We have $y \sim_R x_n$ or $x_n \sim_R y$;
2. We have $x = y$.

Now, if $x = y$, then $f(x) = f(y)$ trivially; otherwise, we have

$$\begin{aligned} f(x) &= f(x_1), \\ f(x_1) &= f(x_2), \\ &\vdots \\ f(x_{n-1}) &= f(x_n), \\ f(x_n) &= f(y), \end{aligned}$$

and $f(x) = f(y)$, as we wanted to show.

5 Functoriality of Powersets

5.1 Direct Images

Let A and B be sets and let $R\colon A \nrightarrow B$ be a relation.

Definition 5.1.1 ▶ Direct Images

The **direct image function associated to** R is the function[1]

$$R_*\colon \mathcal{P}(A) \to \mathcal{P}(B)$$

defined by[2,3]

$$\begin{aligned} R_*(U) &\overset{\text{def}}{=} R(U) \\ &\overset{\text{def}}{=} \bigcup_{a \in U} R(a) \\ &= \left\{ b \in B \,\middle|\, \begin{array}{l} \text{there exists some } a \in U \\ \text{such that } b \in R(a) \end{array} \right\} \end{aligned}$$

for each $U \in \mathcal{P}(A)$.

[1] *Further Notation:* Also written $\exists_R\colon \mathcal{P}(A) \to \mathcal{P}(B)$. This notation comes from the fact that the following statements are equivalent, where $b \in B$ and $U \in \mathcal{P}(A)$:

- We have $b \in \exists_R(U)$.
- There exists some $a \in U$ such that $b \in f(a)$.

[2] *Further Terminology:* The set $R(U)$ is called the **direct image of U by R**.

[3] We also have

$$R_*(U) = B \setminus R_!(A \setminus U);$$

see Item 7 of Proposition 5.1.3.

Remark 5.1.2 ▶ Unwinding Definition 5.1.1

Identifying subsets of A with relations from pt to A via Constructions With Sets, Item 7 of Proposition 4.2.3, we see that the direct image function associated to R is equivalently the function

$$R_*\colon \underbrace{\mathcal{P}(A)}_{\cong \mathrm{Rel}(\mathrm{pt}, A)} \to \underbrace{\mathcal{P}(B)}_{\cong \mathrm{Rel}(\mathrm{pt}, B)}$$

defined by

$$R_*(U) \overset{\text{def}}{=} R \diamond U$$

for each $U \in \mathcal{P}(A)$, where $R \diamond U$ is the composition

$$\mathrm{pt} \overset{U}{\nrightarrow} A \overset{R}{\nrightarrow} B.$$

Proposition 5.1.3 ▶ Properties of Direct Image Functions

Let $R\colon A \nrightarrow B$ be a relation.

1. *Functoriality.* The assignment $U \mapsto R_*(U)$ defines a functor

$$R_*\colon (\mathcal{P}(A), \subset) \to (\mathcal{P}(B), \subset)$$

   where

   - *Action on Objects.* For each $U \in \mathcal{P}(A)$, we have

$$[R_*](U) \stackrel{\text{def}}{=} R_*(U);$$

   - *Action on Morphisms.* For each $U, V \in \mathcal{P}(A)$:
     – If $U \subset V$, then $R_*(U) \subset R_*(V)$.

2. *Adjointness.* We have an adjunction

$$(R_* \dashv R_{-1})\colon \quad \begin{tikzcd} \mathcal{P}(A) \arrow[r, bend left, "R_*"] \arrow[r, phantom, "\perp"] & \mathcal{P}(B), \arrow[l, bend left, "R_{-1}"] \end{tikzcd}$$

   witnessed by a bijections of sets

$$\mathrm{Hom}_{\mathcal{P}(A)}(R_*(U), V) \cong \mathrm{Hom}_{\mathcal{P}(A)}(U, R_{-1}(V)),$$

   natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$, i.e. such that:

   $(\star)$ The following conditions are equivalent:
     – We have $R_*(U) \subset V$;
     – We have $U \subset R_{-1}(V)$.

3. *Preservation of Colimits.* We have an equality of sets

$$R_*\left(\bigcup_{i \in I} U_i\right) = \bigcup_{i \in I} R_*(U_i),$$

   natural in $\{U_i\}_{i \in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have equalities

$$R_*(U) \cup R_*(V) = R_*(U \cup V),$$
$$R_*(\emptyset) = \emptyset,$$

   natural in $U, V \in \mathcal{P}(A)$.

4. *Oplax Preservation of Limits.* We have an inclusion of sets

$$R_*\left(\bigcap_{i\in I} U_i\right) \subset \bigcap_{i\in I} R_*(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have inclusions

$$\begin{aligned} R_*(U \cap V) &\subset R_*(U) \cap R_*(V), \\ R_*(A) &\subset B, \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

5. *Symmetric Strict Monoidality With Respect to Unions.* The direct image function of Item 1 has a symmetric strict monoidal structure

$$\left(R_*, R_*^{\otimes}, R_{*|\mathbb{1}}^{\otimes}\right) \colon (\mathcal{P}(A), \cup, \emptyset) \longrightarrow (\mathcal{P}(B), \cup, \emptyset),$$

being equipped with equalities

$$\begin{aligned} R_{*|U,V}^{\otimes} &\colon R_*(U) \cup R_*(V) \xrightarrow{=} R_*(U \cup V), \\ R_{*|\mathbb{1}}^{\otimes} &\colon \emptyset \xrightarrow{=} \emptyset, \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

6. *Symmetric Oplax Monoidality With Respect to Intersections.* The direct image function of Item 1 has a symmetric oplax monoidal structure

$$\left(R_*, R_*^{\otimes}, R_{*|\mathbb{1}}^{\otimes}\right) \colon (\mathcal{P}(A), \cap, A) \longrightarrow (\mathcal{P}(B), \cap, B),$$

being equipped with inclusions

$$\begin{aligned} R_{*|U,V}^{\otimes} &\colon R_*(U \cap V) \subset R_*(U) \cap R_*(V), \\ R_{*|\mathbb{1}}^{\otimes} &\colon R_*(A) \subset B, \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

7. *Relation to Direct Images With Compact Support.* We have

$$R_*(U) = B \setminus R_!(A \setminus U)$$

for each $U \in \mathcal{P}(A)$.

Proof 5.1.4 ► Proof of Proposition 5.1.3

Item 1: Functoriality

Clear.

Item 2: Adjointness

This follows from Kan Extensions, ?? of ??.

Item 3: Preservation of Colimits

This follows from Item 2 and Categories, ?? of ??.

Item 4: Oplax Preservation of Limits

Omitted.

Item 5: Symmetric Strict Monoidality With Respect to Unions

This follows from Item 3.

Item 6: Symmetric Oplax Monoidality With Respect to Intersections

This follows from Item 4.

Item 7: Relation to Direct Images With Compact Support

The proof proceeds in the same way as in the case of functions (Constructions With Sets, Item 7 of Proposition 4.3.3): applying Item 7 of Proposition 5.4.4 to $A \setminus U$, we have

$$\begin{aligned} R_!(A \setminus U) &= B \setminus R_*(A \setminus (A \setminus U)) \\ &= B \setminus R_*(U). \end{aligned}$$

Taking complements, we then obtain

$$\begin{aligned} R_*(U) &= B \setminus (B \setminus R_*(U)), \\ &= B \setminus R_!(A \setminus U), \end{aligned}$$

which finishes the proof. ∎

Proposition 5.1.5 ► Properties of the Direct Image Function Operation

Let $R\colon A \nrightarrow B$ be a relation.

1. *Functionality I*. The assignment $R \mapsto R_*$ defines a function

$$(-)_* \colon \mathsf{Rel}(A, B) \to \mathsf{Sets}(\mathcal{P}(A), \mathcal{P}(B)).$$

2. *Functionality II*. The assignment $R \mapsto R_*$ defines a function

$$(-)_* \colon \mathsf{Rel}(A, B) \to \mathsf{Pos}((\mathcal{P}(A), \subset), (\mathcal{P}(B), \subset)).$$

3. *Interaction With Identities*. For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have[1]

$$(\chi_A)_* = \mathrm{id}_{\mathcal{P}(A)};$$

4. *Interaction With Composition*. For each pair of composable relations $R \colon A \nrightarrow B$ and $S \colon B \nrightarrow C$, we have[2]

$$(S \diamond R)_* = S_* \circ R_*, \qquad \begin{tikzcd} \mathcal{P}(A) \arrow[r, "R_*"] \arrow[dr, "(S\diamond R)_*"'] & \mathcal{P}(B) \arrow[d, "S_*"] \\ & \mathcal{P}(C). \end{tikzcd}$$

[1] That is, the postcomposition function

$$(\chi_A)_* \colon \mathsf{Rel}(\mathrm{pt}, A) \to \mathsf{Rel}(\mathrm{pt}, A)$$

is equal to $\mathrm{id}_{\mathsf{Rel}(\mathrm{pt},A)}$.

[2] That is, we have

$$(S \diamond R)_* = S_* \circ R_*, \qquad \begin{tikzcd} \mathsf{Rel}(\mathrm{pt}, A) \arrow[r, "R_*"] \arrow[dr, "(S\diamond R)_*"'] & \mathsf{Rel}(\mathrm{pt}, B) \arrow[d, "S_*"] \\ & \mathsf{Rel}(\mathrm{pt}, C). \end{tikzcd}$$

PROOF 5.1.6 ► PROOF OF PROPOSITION 5.1.5

Item 1: Functionality I

Clear.

Item 2: Functionality II

Clear.

Item 3: Interaction With Identities

Indeed, we have

$$\begin{aligned}(\chi_A)_*(U) &\overset{\text{def}}{=} \bigcup_{a\in U} \chi_A(a)\\ &\overset{\text{def}}{=} \bigcup_{a\in U} \{a\}\\ &= U\\ &\overset{\text{def}}{=} \mathrm{id}_{\mathcal{P}(A)}(U)\end{aligned}$$

for each $U \in \mathcal{P}(A)$. Thus $(\chi_A)_* = \mathrm{id}_{\mathcal{P}(A)}$.

Item 4: Interaction With Composition

Indeed, we have

$$\begin{aligned}(S \diamond R)_*(U) &\overset{\text{def}}{=} \bigcup_{a\in U} [S \diamond R](a)\\ &\overset{\text{def}}{=} \bigcup_{a\in U} S(R(a))\\ &\overset{\text{def}}{=} \bigcup_{a\in U} S_*(R(a))\\ &= S_*\left(\bigcup_{a\in U} R(a)\right)\\ &\overset{\text{def}}{=} S_*(R_*(U))\\ &\overset{\text{def}}{=} [S_* \circ R_*](U)\end{aligned}$$

for each $U \in \mathcal{P}(A)$, where we used Item 3 of Proposition 5.1.3. Thus $(S \diamond R)_* = S_* \circ R_*$.

5.2 Strong Inverse Images

Let A and B be sets and let $R\colon A \nrightarrow B$ be a relation.

Definition 5.2.1 ► Strong Inverse Images

The **strong inverse image function associated to** R is the function

$$R_{-1}\colon \mathcal{P}(B) \to \mathcal{P}(A)$$

defined by[1]

$$R_{-1}(V) \stackrel{\text{def}}{=} \{a \in A \mid R(a) \subset V\}$$

for each $V \in \mathcal{P}(B)$.

[1] *Further Terminology:* The set $R_{-1}(V)$ is called the **strong inverse image of** V **by** R.

Remark 5.2.2 ► Unwinding Definition 5.2.1

Identifying subsets of B with relations from pt to B via Constructions With Sets, Item 7 of Proposition 4.2.3, we see that the inverse image function associated to R is equivalently the function

$$R_{-1}\colon \underbrace{\mathcal{P}(B)}_{\cong \mathsf{Rel}(\mathrm{pt},B)} \to \underbrace{\mathcal{P}(A)}_{\cong \mathsf{Rel}(\mathrm{pt},A)}$$

defined by

$$R_{-1}(V) \stackrel{\text{def}}{=} \mathrm{Rift}_R(V), \qquad \begin{tikzcd} & A \arrow[d, "R", "|" marking] \\ \mathrm{pt} \arrow[ur, dashed, "\mathrm{Rift}_R(V)", "|" marking] \arrow[r, "V"', "|" marking] & B, \arrow[ul, phantom, "\Leftarrow"] \end{tikzcd}$$

and being explicitly computed by

$$\begin{aligned} R_{-1}(V) &\stackrel{\text{def}}{=} \mathrm{Rift}_R(V) \\ &\cong \int_{x\in B} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^x_{-_1}, V^x_{-_2}\right), \end{aligned}$$

where we have used Item 12 of Proposition 2.5.1.

Proof 5.2.3 ▶ Proof of Remark 5.2.2

We have

$$\begin{aligned}
\mathrm{Rift}_R(V) &\cong \int_{x\in B} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^x_{-_1}, V^x_{-_2}\right)\\
&= \left\{a\in A \,\middle|\, \int_{x\in B} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^x_a, V^x_\star\right) = \mathsf{true}\right\}\\
&= \left\{a\in A \,\middle|\, \begin{array}{l}\text{for each } x\in B\text{, at least one of the}\\ \text{following conditions hold:}\\ \quad 1.\ \text{We have } R^x_a = \mathsf{false}\\ \quad 2.\ \text{The following conditions hold:}\\ \qquad (a)\ \text{We have } R^x_a = \mathsf{true}\\ \qquad (b)\ \text{We have } V^x_\star = \mathsf{true}\end{array}\right\}\\
&= \left\{a\in A \,\middle|\, \begin{array}{l}\text{for each } x\in B\text{, at least one of the}\\ \text{following conditions hold:}\\ \quad 1.\ \text{We have } x\notin R(a)\\ \quad 2.\ \text{The following conditions hold:}\\ \qquad (a)\ \text{We have } x\in R(a)\\ \qquad (b)\ \text{We have } x\in V\end{array}\right\}\\
&= \{a\in A \mid \text{for each } x\in R(a)\text{, we have } x\in V\}\\
&= \{a\in A \mid R(a)\subset V\}\\
&\overset{\text{def}}{=} R_{-1}(V).
\end{aligned}$$

This finishes the proof.

Proposition 5.2.4 ▶ Properties of Strong Inverse Images

Let $R\colon A \nrightarrow B$ be a relation.

1. *Functoriality.* The assignment $V\mapsto R_{-1}(V)$ defines a functor

$$R_{-1}\colon (\mathcal{P}(B),\subset)\to(\mathcal{P}(A),\subset)$$

where

- *Action on Objects.* For each $V \in \mathcal{P}(B)$, we have

$$[R_{-1}](V) \overset{\text{def}}{=} R_{-1}(V);$$

- *Action on Morphisms.* For each $U, V \in \mathcal{P}(B)$:
  - If $U \subset V$, then $R_{-1}(U) \subset R_{-1}(V)$.

2. *Adjointness.* We have an adjunction

$$\begin{tikzcd}(R_* \dashv R_{-1})\colon \quad \mathcal{P}(A) \arrow[r, bend left, "R_*"] \arrow[r, phantom, "\perp"] & \mathcal{P}(B), \arrow[l, bend left, "R_{-1}"]\end{tikzcd}$$

   witnessed by a bijections of sets

$$\mathrm{Hom}_{\mathcal{P}(A)}(R_*(U), V) \cong \mathrm{Hom}_{\mathcal{P}(A)}(U, R_{-1}(V)),$$

   natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$, i.e. such that:

   ($\star$) The following conditions are equivalent:
   - We have $R_*(U) \subset V$;
   - We have $U \subset R_{-1}(V)$.

3. *Lax Preservation of Colimits.* We have an inclusion of sets

$$\bigcup_{i \in I} R_{-1}(U_i) \subset R_{-1}\left(\bigcup_{i \in I} U_i\right),$$

   natural in $\{U_i\}_{i \in I} \in \mathcal{P}(B)^{\times I}$. In particular, we have inclusions

$$\begin{gathered} R_{-1}(U) \cup R_{-1}(V) \subset R_{-1}(U \cup V), \\ \emptyset \subset R_{-1}(\emptyset), \end{gathered}$$

   natural in $U, V \in \mathcal{P}(B)$.

4. *Preservation of Limits.* We have an equality of sets

$$R_{-1}\left(\bigcap_{i \in I} U_i\right) = \bigcap_{i \in I} R_{-1}(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(B)^{\times I}$. In particular, we have equalities

$$\begin{aligned}R_{-1}(U \cap V) &= R_{-1}(U) \cap R_{-1}(V),\\ R_{-1}(B) &= B,\end{aligned}$$

natural in $U, V \in \mathcal{P}(B)$.

5. *Symmetric Lax Monoidality With Respect to Unions.* The direct image with compact support function of Item 1 has a symmetric lax monoidal structure

$$\left(R_{-1}, R^{\otimes}_{-1}, R^{\otimes}_{-1|\mathbb{1}}\right) \colon (\mathcal{P}(A), \cup, \emptyset) \to (\mathcal{P}(B), \cup, \emptyset),$$

being equipped with inclusions

$$\begin{aligned}R^{\otimes}_{-1|U,V} \colon R_{-1}(U) \cup R_{-1}(V) &\subset R_{-1}(U \cup V),\\ R^{\otimes}_{-1|\mathbb{1}} \colon \emptyset &\subset R_{-1}(\emptyset),\end{aligned}$$

natural in $U, V \in \mathcal{P}(B)$.

6. *Symmetric Strict Monoidality With Respect to Intersections.* The direct image function of Item 1 has a symmetric strict monoidal structure

$$\left(R_{-1}, R^{\otimes}_{-1}, R^{\otimes}_{-1|\mathbb{1}}\right) \colon (\mathcal{P}(A), \cap, A) \to (\mathcal{P}(B), \cap, B),$$

being equipped with equalities

$$\begin{aligned}R^{\otimes}_{-1|U,V} \colon R_{-1}(U \cap V) &\xrightarrow{=} R_{-1}(U) \cap R_{-1}(V),\\ R^{\otimes}_{-1|\mathbb{1}} \colon R_{-1}(A) &\xrightarrow{=} B,\end{aligned}$$

natural in $U, V \in \mathcal{P}(B)$.

7. *Interaction With Weak Inverse Images I.* We have

$$R_{-1}(V) = A \setminus R^{-1}(B \setminus V)$$

for each $V \in \mathcal{P}(B)$.

8. *Interaction With Weak Inverse Images II.* Let $R \colon A \nrightarrow B$ be a relation from A to B.

(a) If R is a total relation, then we have an inclusion of sets

$$R_{-1}(V) \subset R^{-1}(V)$$

natural in $V \in \mathcal{P}(B)$.

(b) If R is total and functional, then the above inclusion is in fact an equality.

(c) Conversely, if we have $R_{-1} = R^{-1}$, then R is total and functional.

Proof 5.2.5 ► Proof of Proposition 5.2.4

Item 1: Functoriality

Clear.

Item 2: Adjointness

This follows from Kan Extensions, **??** of **??**.

Item 3: Lax Preservation of Colimits

Omitted.

Item 4: Preservation of Limits

This follows from Item 2 and Categories, **??** of **??**.

Item 5: Symmetric Lax Monoidality With Respect to Unions

This follows from Item 3.

Item 6: Symmetric Strict Monoidality With Respect to Intersections

This follows from Item 4.

Item 7: Interaction With Weak Inverse Images I

We claim we have an equality

$$R_{-1}(B \setminus V) = A \setminus R^{-1}(V).$$

Indeed, we have

$$\begin{aligned} R_{-1}(B \setminus V) &= \{a \in A \mid R(a) \subset B \setminus V\}, \\ A \setminus R^{-1}(V) &= \{a \in A \mid R(a) \cap V = \emptyset\}. \end{aligned}$$

Taking $V = B \setminus V$ then implies the original statement.

Item 8: Interaction With Weak Inverse Images II

Item 8a is clear, while Items 8b and 8c follow from Item 6 of Proposition 3.1.2. ∎

Proposition 5.2.6 ► Properties of the Strong Inverse Image Function Operation

Let $R\colon A \nrightarrow B$ be a relation.

1. *Functionality I.* The assignment $R \mapsto R_{-1}$ defines a function

$$(-)_{-1}\colon \mathsf{Sets}(A, B) \to \mathsf{Sets}(\mathcal{P}(A), \mathcal{P}(B)).$$

2. *Functionality II.* The assignment $R \mapsto R_{-1}$ defines a function

$$(-)_{-1}\colon \mathsf{Sets}(A, B) \to \mathsf{Pos}((\mathcal{P}(A), \subset), (\mathcal{P}(B), \subset)).$$

3. *Interaction With Identities.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$(\mathrm{id}_A)_{-1} = \mathrm{id}_{\mathcal{P}(A)};$$

4. *Interaction With Composition.* For each pair of composable relations $R\colon A \nrightarrow B$ and $S\colon B \nrightarrow C$, we have

$$(S \diamond R)_{-1} = R_{-1} \circ S_{-1}, \qquad \begin{tikzcd} \mathcal{P}(C) \arrow[r, "S_{-1}"] \arrow[dr, "(S\diamond R)_{-1}"'] & \mathcal{P}(B) \arrow[d, "R_{-1}"] \\ & \mathcal{P}(A). \end{tikzcd}$$

Proof 5.2.7 ► Proof of Proposition 5.2.6

Item 1: Functionality I

Clear.

Item 2: Functionality II

Clear.

Item 3: Interaction With Identities

Indeed, we have

$$\begin{aligned}(\chi_A)_{-1}(U) &\overset{\text{def}}{=} \{a \in A \mid \chi_A(a) \subset U\}\\ &\overset{\text{def}}{=} \{a \in A \mid \{a\} \subset U\}\\ &= U\end{aligned}$$

for each $U \in \mathcal{P}(A)$. Thus $(\chi_A)_{-1} = \mathrm{id}_{\mathcal{P}(A)}$.

Item 4: Interaction With Composition

Indeed, we have

$$\begin{aligned}(S \diamond R)_{-1}(U) &\overset{\text{def}}{=} \{a \in A \mid [S \diamond R](a) \subset U\}\\ &\overset{\text{def}}{=} \{a \in A \mid S(R(a)) \subset U\}\\ &\overset{\text{def}}{=} \{a \in A \mid S_*(R(a)) \subset U\}\\ &= \{a \in A \mid R(a) \subset S_{-1}(U)\}\\ &\overset{\text{def}}{=} R_{-1}(S_{-1}(U))\\ &\overset{\text{def}}{=} [R_{-1} \circ S_{-1}](U)\end{aligned}$$

for each $U \in \mathcal{P}(C)$, where we used Item 2 of Proposition 5.2.4, which implies that the conditions

- We have $S_*(R(a)) \subset U$;
- We have $R(a) \subset S_{-1}(U)$;

are equivalent. Thus $(S \diamond R)_{-1} = R_{-1} \circ S_{-1}$.

5.3 Weak Inverse Images

Let A and B be sets and let $R\colon A \nrightarrow B$ be a relation.

Definition 5.3.1 ► Weak Inverse Images

The **weak inverse image function associated to** R[1] is the function

$$R^{-1}\colon \mathcal{P}(B) \to \mathcal{P}(A)$$

defined by[2]

$$R^{-1}(V) \overset{\text{def}}{=} \{a \in A \mid R(a) \cap V \neq \emptyset\}$$

for each $V \in \mathcal{P}(B)$.

[1] *Further Terminology:* Also called simply the **inverse image function associated to** R.

[2] *Further Terminology:* The set $R^{-1}(V)$ is called the **weak inverse image of** V **by** R or simply the **inverse image of** V **by** R.

Remark 5.3.2 ► Unwinding Definition 5.3.1

Identifying subsets of B with relations from B to pt via Constructions With Sets, Item 7 of Proposition 4.2.3, we see that the weak inverse image function associated to R is equivalently the function

$$R^{-1} \colon \underbrace{\mathcal{P}(B)}_{\cong \mathrm{Rel}(B,\mathrm{pt})} \to \underbrace{\mathcal{P}(A)}_{\cong \mathrm{Rel}(A,\mathrm{pt})}$$

defined by

$$R^{-1}(V) \overset{\text{def}}{=} V \diamond R$$

for each $V \in \mathcal{P}(A)$, where $R \diamond V$ is the composition

$$A \overset{R}{\nrightarrow} B \overset{V}{\nrightarrow} \mathrm{pt}.$$

Explicitly, we have

$$\begin{aligned} R^{-1}(V) &\overset{\text{def}}{=} V \diamond R \\ &\overset{\text{def}}{=} \int^{x \in B} V^{-1}_{x} \times R^{x}_{-2}. \end{aligned}$$

Proof 5.3.3 ► Proof of Remark 5.3.2

We have

$$
\begin{aligned}
V \diamond R &\overset{\text{def}}{=} \int^{x\in B} V^{-1}_{x} \times R^{x}_{-_2} \\
&= \left\{ a \in A \,\middle|\, \int^{x\in B} V^{\star}_{x} \times R^{x}_{a} = \mathrm{true} \right\} \\
&= \left\{ a \in A \,\middle|\, \begin{array}{l} \text{there exists } x \in B \text{ such that the} \\ \text{following conditions hold:} \\ \quad 1. \text{ We have } V^{\star}_{x} = \mathrm{true} \\ \quad 2. \text{ We have } R^{x}_{a} = \mathrm{true} \end{array} \right\} \\
&= \left\{ a \in A \,\middle|\, \begin{array}{l} \text{there exists } x \in B \text{ such that the} \\ \text{following conditions hold:} \\ \quad 1. \text{ We have } x \in V \\ \quad 2. \text{ We have } x \in R(a) \end{array} \right\} \\
&= \{ a \in A \mid \text{there exists } x \in V \text{ such that } x \in R(a) \} \\
&= \{ a \in A \mid R(a) \cap V \neq \emptyset \} \\
&\overset{\text{def}}{=} R^{-1}(V)
\end{aligned}
$$

This finishes the proof. ∎

Proposition 5.3.4 ► Properties of Weak Inverse Image Functions

Let $R\colon A \nrightarrow B$ be a relation.

1. *Functoriality.* The assignment $V \mapsto R^{-1}(V)$ defines a functor

$$R^{-1}\colon (\mathcal{P}(B), \subset) \to (\mathcal{P}(A), \subset)$$

where

- *Action on Objects.* For each $V \in \mathcal{P}(B)$, we have

$$[R^{-1}](V) \overset{\text{def}}{=} R^{-1}(V);$$

- *Action on Morphisms.* For each $U, V \in \mathcal{P}(B)$:
  - If $U \subset V$, then $R^{-1}(U) \subset R^{-1}(V)$.

2. *Adjointness.* We have an adjunction

$$\left(R^{-1} \dashv R_!\right)\colon \quad \begin{tikzcd} \mathcal{P}(B) \arrow[r, bend left, "R^{-1}"] \arrow[r, phantom, "\perp"] & \mathcal{P}(A), \arrow[l, bend left, "R_!"] \end{tikzcd}$$

witnessed by a bijections of sets

$$\mathrm{Hom}_{\mathcal{P}(A)}\left(R^{-1}(U), V\right) \cong \mathrm{Hom}_{\mathcal{P}(A)}(U, R_!(V)),$$

natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$, i.e. such that:

($\star$) The following conditions are equivalent:
  - We have $R^{-1}(U) \subset V$;
  - We have $U \subset R_!(V)$.

3. *Preservation of Colimits.* We have an equality of sets

$$R^{-1}\left(\bigcup_{i \in I} U_i\right) = \bigcup_{i \in I} R^{-1}(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(B)^{\times I}$. In particular, we have equalities

$$R^{-1}(U) \cup R^{-1}(V) = R^{-1}(U \cup V),$$
$$R^{-1}(\emptyset) = \emptyset,$$

natural in $U, V \in \mathcal{P}(B)$.

4. *Oplax Preservation of Limits.* We have an inclusion of sets

$$R^{-1}\left(\bigcap_{i \in I} U_i\right) \subset \bigcap_{i \in I} R^{-1}(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(B)^{\times I}$. In particular, we have inclusions

$$R^{-1}(U \cap V) \subset R^{-1}(U) \cap R^{-1}(V),$$
$$R^{-1}(A) \subset B,$$

natural in $U, V \in \mathcal{P}(B)$.

5. *Symmetric Strict Monoidality With Respect to Unions.* The direct image function of Item 1 has a symmetric strict monoidal structure

$$\left(R^{-1}, R^{-1,\otimes}, R_{\mathbb{1}}^{-1,\otimes}\right) \colon (\mathcal{P}(A), \cup, \emptyset) \to (\mathcal{P}(B), \cup, \emptyset),$$

being equipped with equalities

$$R_{U,V}^{-1,\otimes} \colon R^{-1}(U) \cup R^{-1}(V) \xrightarrow{=} R^{-1}(U \cup V),$$
$$R_{\mathbb{1}}^{-1,\otimes} \colon \emptyset \xrightarrow{=} \emptyset,$$

natural in $U, V \in \mathcal{P}(B)$.

6. *Symmetric Oplax Monoidality With Respect to Intersections.* The direct image function of Item 1 has a symmetric oplax monoidal structure

$$\left(R^{-1}, R^{-1,\otimes}, R_{\mathbb{1}}^{-1,\otimes}\right) \colon (\mathcal{P}(A), \cap, A) \to (\mathcal{P}(B), \cap, B),$$

being equipped with inclusions

$$R_{U,V}^{-1,\otimes} \colon R^{-1}(U \cap V) \subset R^{-1}(U) \cap R^{-1}(V),$$
$$R_{\mathbb{1}}^{-1,\otimes} \colon R^{-1}(A) \subset B,$$

natural in $U, V \in \mathcal{P}(B)$.

7. *Interaction With Strong Inverse Images I.* We have

$$R^{-1}(V) = A \setminus R_{-1}(B \setminus V)$$

for each $V \in \mathcal{P}(B)$.

8. *Interaction With Strong Inverse Images II.* Let $R \colon A \nrightarrow B$ be a relation from A to B.

   (a) If R is a total relation, then we have an inclusion of sets

   $$R_{-1}(V) \subset R^{-1}(V)$$

   natural in $V \in \mathcal{P}(B)$.

   (b) If R is total and functional, then the above inclusion is in fact an equality.

   (c) Conversely, if we have $R_{-1} = R^{-1}$, then R is total and functional.

Proof 5.3.5 ▶ Proof of Proposition 5.3.4

Item 1: Functoriality

Clear.

Item 2: Adjointness

This follows from Kan Extensions, ?? of ??.

Item 3: Preservation of Colimits

This follows from Item 2 and Categories, ?? of ??.

Item 4: Oplax Preservation of Limits

Omitted.

Item 5: Symmetric Strict Monoidality With Respect to Unions

This follows from Item 3.

Item 6: Symmetric Oplax Monoidality With Respect to Intersections

This follows from Item 4.

Item 7: Interaction With Strong Inverse Images I

This follows from Item 7 of Proposition 5.2.4.

Item 8: Interaction With Strong Inverse Images II

This was proved in Item 8 of Proposition 5.2.4.

Proposition 5.3.6 ▶ Properties of the Weak Inverse Image Function Operation

Let $R\colon A \nrightarrow B$ be a relation.

1. *Functionality I.* The assignment $R \mapsto R^{-1}$ defines a function

$$(-)^{-1}\colon \mathrm{Rel}(A, B) \to \mathrm{Sets}(\mathcal{P}(A), \mathcal{P}(B)).$$

2. *Functionality II.* The assignment $R \mapsto R^{-1}$ defines a function

$$(-)^{-1}\colon \mathrm{Rel}(A, B) \to \mathrm{Pos}((\mathcal{P}(A), \subset), (\mathcal{P}(B), \subset)).$$

3. *Interaction With Identities.* For each $A \in \mathrm{Obj}(\mathrm{Sets})$, we have[1]

$$(\chi_A)^{-1} = \mathrm{id}_{\mathcal{P}(A)};$$

4. *Interaction With Composition.* For each pair of composable relations $R\colon A \nrightarrow B$ and $S\colon B \nrightarrow C$, we have[2]

$$(S \diamond R)^{-1} = R^{-1} \circ S^{-1}, \qquad \begin{tikzcd} \mathcal{P}(C) \arrow[r, "S^{-1}"] \arrow[dr, "(S\diamond R)^{-1}"'] & \mathcal{P}(B) \arrow[d, "R^{-1}"] \\ & \mathcal{P}(A). \end{tikzcd}$$

[1] That is, the postcomposition

$$(\chi_A)^{-1}\colon \mathrm{Rel}(\mathrm{pt}, A) \longrightarrow \mathrm{Rel}(\mathrm{pt}, A)$$

is equal to $\mathrm{id}_{\mathrm{Rel}(\mathrm{pt},A)}$.

[2] That is, we have

$$(S \diamond R)^{-1} = R^{-1} \circ S^{-1}, \qquad \begin{tikzcd} \mathrm{Rel}(\mathrm{pt}, C) \arrow[r, "R^{-1}"] \arrow[dr, "(S\diamond R)^{-1}"'] & \mathrm{Rel}(\mathrm{pt}, B) \arrow[d, "S^{-1}"] \\ & \mathrm{Rel}(\mathrm{pt}, A). \end{tikzcd}$$

Proof 5.3.7 ► Proof of Proposition 5.3.6

Item 1: Functionality I

Clear.

Item 2: Functionality II

Clear.

Item 3: Interaction With Identities

This follows from Categories, **??** of **??**.

Item 4: Interaction With Composition

This follows from Categories, **??** of **??**.

5.4 Direct Images With Compact Support

Let A and B be sets and let $R\colon A \nrightarrow B$ be a relation.

Definition 5.4.1 ► Direct Images With Compact Support

The **direct image with compact support function associated to** R is the function[1]

$$R_!\colon \mathcal{P}(A) \to \mathcal{P}(B)$$

defined by[2,3]

$$\begin{aligned} R_!(U) &\overset{\text{def}}{=} \left\{ b \in B \,\middle|\, \begin{array}{l} \text{for each } a \in A\text{, if we have} \\ b \in R(a)\text{, then } a \in U \end{array} \right\} \\ &= \left\{ b \in B \,\middle|\, R^{-1}(b) \subset U \right\} \end{aligned}$$

for each $U \in \mathcal{P}(A)$.

[1] *Further Notation:* Also written $\forall_R\colon \mathcal{P}(A) \to \mathcal{P}(B)$. This notation comes from the fact that the following statements are equivalent, where $b \in B$ and $U \in \mathcal{P}(A)$:

- We have $b \in \forall_R(U)$.
- For each $a \in A$, if $b \in R(a)$, then $a \in U$.

[2] *Further Terminology:* The set $R_!(U)$ is called the **direct image with compact support of** U **by** R.

[3] We also have

$$R_!(U) = B \setminus R_*(A \setminus U);$$

see Item 7 of Proposition 5.4.4.

Remark 5.4.2 ► Unwinding Definition 5.4.1

Identifying subsets of B with relations from pt to B via Constructions With Sets, Item 7 of Proposition 4.2.3, we see that the direct image with compact support function associated to R is equivalently the function

$$R_!\colon \underbrace{\mathcal{P}(A)}_{\cong \mathrm{Rel}(A,\mathrm{pt})} \to \underbrace{\mathcal{P}(B)}_{\cong \mathrm{Rel}(B,\mathrm{pt})}$$

defined by

$$R_!(U) \overset{\text{def}}{=} \mathrm{Ran}_R(U), \qquad \begin{tikzcd} & B \arrow[d, dashed, "\mathrm{Ran}_R(U)"] \\ A \arrow[ur, "R", "\shortmid" marking] \arrow[r, "U"', "\shortmid" marking] & \mathrm{pt}, \arrow[ul, phantom, "\Leftarrow"] \end{tikzcd}$$

being explicitly computed by

$$\begin{aligned} R^*(U) &\stackrel{\text{def}}{=} \mathrm{Ran}_R(U) \\ &\cong \int_{a\in A} \mathrm{Hom}_{\{\mathrm{t},\mathrm{f}\}}\left(R_a^{-2}, U_a^{-1}\right), \end{aligned}$$

where we have used Item 11 of Proposition 2.5.1.

Proof 5.4.3 ▶ Proof of Remark 5.4.2

We have

$$\begin{aligned} \mathrm{Ran}_R(V) &\cong \int_{a\in A} \mathrm{Hom}_{\{\mathrm{t},\mathrm{f}\}}\left(R_a^{-2}, U_a^{-1}\right) \\ &= \left\{ b \in B \;\middle|\; \int_{a\in A} \mathrm{Hom}_{\{\mathrm{t},\mathrm{f}\}}\left(R_a^b, U_a^{\star}\right) = \mathrm{true} \right\} \\ &= \left\{ b \in B \;\middle|\; \begin{array}{l} \text{for each } a \in A\text{, at least one of the} \\ \text{following conditions hold:} \\ \quad 1.\ \text{We have } R_a^b = \mathrm{false} \\ \quad 2.\ \text{The following conditions hold:} \\ \qquad \text{(a) We have } R_a^b = \mathrm{true} \\ \qquad \text{(b) We have } U_a^{\star} = \mathrm{true} \end{array} \right\} \\ &= \left\{ b \in B \;\middle|\; \begin{array}{l} \text{for each } a \in A\text{, at least one of the} \\ \text{following conditions hold:} \\ \quad 1.\ \text{We have } b \notin R(A) \\ \quad 2.\ \text{The following conditions hold:} \\ \qquad \text{(a) We have } b \in R(a) \\ \qquad \text{(b) We have } a \in U \end{array} \right\} \\ &= \left\{ b \in B \;\middle|\; \begin{array}{l} \text{for each } a \in A\text{, if we have} \\ b \in R(a)\text{, then } a \in U \end{array} \right\} \\ &= \left\{ b \in B \;\middle|\; R^{-1}(b) \subset U \right\} \\ &\stackrel{\text{def}}{=} R^{-1}(U). \end{aligned}$$

This finishes the proof. ∎

Proposition 5.4.4 ▶ Properties of Direct Images With Compact Support

Let $R\colon A \nrightarrow B$ be a relation.

1. *Functoriality.* The assignment $U \mapsto R_!(U)$ defines a functor

$$R_!\colon (\mathcal{P}(A), \subset) \to (\mathcal{P}(B), \subset)$$

where

- *Action on Objects.* For each $U \in \mathcal{P}(A)$, we have

$$[R_!](U) \overset{\text{def}}{=} R_!(U);$$

- *Action on Morphisms.* For each $U, V \in \mathcal{P}(A)$:
  - If $U \subset V$, then $R_!(U) \subset R_!(V)$.

2. *Adjointness.* We have an adjunction

$$\left(R^{-1} \dashv R_!\right)\colon \quad \begin{tikzcd} \mathcal{P}(B) \arrow[r, bend left, "R^{-1}"] \arrow[r, phantom, "\perp"] & \mathcal{P}(A) \arrow[l, bend left, "R_!"] \end{tikzcd},$$

witnessed by a bijections of sets

$$\mathrm{Hom}_{\mathcal{P}(A)}\left(R^{-1}(U), V\right) \cong \mathrm{Hom}_{\mathcal{P}(A)}(U, R_!(V)),$$

natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$, i.e. such that:

(★) The following conditions are equivalent:
  - We have $R^{-1}(U) \subset V$;
  - We have $U \subset R_!(V)$.

3. *Lax Preservation of Colimits.* We have an inclusion of sets

$$\bigcup_{i \in I} R_!(U_i) \subset R_!\left(\bigcup_{i \in I} U_i\right),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have inclusions

$$R_!(U) \cup R_!(V) \subset R_!(U \cup V),$$
$$\emptyset \subset R_!(\emptyset),$$

natural in $U, V \in \mathcal{P}(A)$.

4. *Preservation of Limits.* We have an equality of sets

$$R_!\left(\bigcap_{i\in I} U_i\right) = \bigcap_{i\in I} R_!(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have equalities

$$R_!(U \cap V) = R_!(U) \cap R_!(V),$$
$$R_!(A) = B,$$

natural in $U, V \in \mathcal{P}(A)$.

5. *Symmetric Lax Monoidality With Respect to Unions.* The direct image with compact support function of Item 1 has a symmetric lax monoidal structure

$$\left(R_!, R_!^{\otimes}, R_{!|\mathbb{1}}^{\otimes}\right) \colon (\mathcal{P}(A), \cup, \emptyset) \longrightarrow (\mathcal{P}(B), \cup, \emptyset),$$

being equipped with inclusions

$$R_{!|U,V}^{\otimes} \colon R_!(U) \cup R_!(V) \subset R_!(U \cup V),$$
$$R_{!|\mathbb{1}}^{\otimes} \colon \emptyset \subset R_!(\emptyset),$$

natural in $U, V \in \mathcal{P}(A)$.

6. *Symmetric Strict Monoidality With Respect to Intersections.* The direct image function of Item 1 has a symmetric strict monoidal structure

$$\left(R_!, R_!^{\otimes}, R_{!|\mathbb{1}}^{\otimes}\right) \colon (\mathcal{P}(A), \cap, A) \longrightarrow (\mathcal{P}(B), \cap, B),$$

being equipped with equalities

$$R_{!|U,V}^{\otimes} \colon R_!(U \cap V) \xrightarrow{=} R_!(U) \cap R_!(V),$$
$$R_{!|\mathbb{1}}^{\otimes} \colon R_!(A) \xrightarrow{=} B,$$

natural in $U, V \in \mathcal{P}(A)$.

7. *Relation to Direct Images.* We have

$$R_!(U) = B \setminus R_*(A \setminus U)$$

for each $U \in \mathcal{P}(A)$.

Proof 5.4.5 ▶ Proof of Proposition 5.4.4

Item 1: Functoriality

Clear.

Item 2: Adjointness

This follows from Kan Extensions, **??** of **??**.

Item 3: Lax Preservation of Colimits

Omitted.

Item 4: Preservation of Limits

This follows from Item 2 and Categories, **??** of **??**.

Item 5: Symmetric Lax Monoidality With Respect to Unions

This follows from Item 3.

Item 6: Symmetric Strict Monoidality With Respect to Intersections

This follows from Item 4.

Item 7: Relation to Direct Images

This follows from Item 7 of Proposition 5.1.3. Alternatively, we may prove it directly as follows, with the proof proceeding in the same way as in the case of functions (Constructions With Sets, Item 7 of Proposition 4.5.5).

We claim that $R_!(U) = B \setminus R_*(A \setminus U)$:

- *The First Implication.* We claim that

  $$R_!(U) \subset B \setminus R_*(A \setminus U).$$

  Let $b \in R_!(U)$. We need to show that $b \notin R_*(A \setminus U)$, i.e. that there is no $a \in A \setminus U$ such that $b \in R(a)$.

  This is indeed the case, as otherwise we would have $a \in R^{-1}(b)$ and $a \notin U$, contradicting $R^{-1}(b) \subset U$ (which holds since $b \in R_!(U)$).

Thus $b \in B \setminus R_*(A \setminus U)$.

- *The Second Implication.* We claim that

$$B \setminus R_*(A \setminus U) \subset R_!(U).$$

Let $b \in B \setminus R_*(A \setminus U)$. We need to show that $b \in R_!(U)$, i.e. that $R^{-1}(b) \subset U$.

Since $b \notin R_*(A \setminus U)$, there exists no $a \in A \setminus U$ such that $b \in R(a)$, and hence $R^{-1}(b) \subset U$.

Thus $b \in R_!(U)$.

This finishes the proof. ∎

PROPOSITION 5.4.6 ▶ PROPERTIES OF THE DIRECT IMAGE WITH COMPACT SUPPORT FUNCTION OPERATION

Let $R\colon A \nrightarrow B$ be a relation.

1. *Functionality I.* The assignment $R \mapsto R_!$ defines a function

$$(-)_!\colon \mathsf{Sets}(A, B) \to \mathsf{Sets}(\mathcal{P}(A), \mathcal{P}(B)).$$

2. *Functionality II.* The assignment $R \mapsto R_!$ defines a function

$$(-)_!\colon \mathsf{Sets}(A, B) \to \mathrm{Hom}_{\mathsf{Pos}}((\mathcal{P}(A), \subset), (\mathcal{P}(B), \subset)).$$

3. *Interaction With Identities.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$(\mathrm{id}_A)_! = \mathrm{id}_{\mathcal{P}(A)};$$

4. *Interaction With Composition.* For each pair of composable relations $R\colon A \nrightarrow B$ and $S\colon B \nrightarrow C$, we have

$$(S \diamond R)_! = S_! \circ R_!, \qquad \begin{tikzcd} \mathcal{P}(A) \arrow[r, "R_!"] \arrow[dr, "(S\diamond R)_!"'] & \mathcal{P}(B) \arrow[d, "S_!"] \\ & \mathcal{P}(C). \end{tikzcd}$$

Proof 5.4.7 ▶ Proof of Proposition 5.4.6

Item 1: Functionality I

Clear.

Item 2: Functionality II

Clear.

Item 3: Interaction With Identities

Indeed, we have

$$\begin{aligned}(\chi_A)_!(U) &\stackrel{\text{def}}{=} \left\{a \in A \,\middle|\, \chi_A^{-1}(a) \subset U\right\}\\ &\stackrel{\text{def}}{=} \{a \in A \mid \{a\} \subset U\}\\ &= U\end{aligned}$$

for each $U \in \mathcal{P}(A)$. Thus $(\chi_A)_! = \mathrm{id}_{\mathcal{P}(A)}$.

Item 4: Interaction With Composition

Indeed, we have

$$\begin{aligned}(S \diamond R)_!(U) &\stackrel{\text{def}}{=} \left\{c \in C \,\middle|\, [S \diamond R]^{-1}(c) \subset U\right\}\\ &\stackrel{\text{def}}{=} \left\{c \in C \,\middle|\, S^{-1}\left(R^{-1}(c)\right) \subset U\right\}\\ &= \left\{c \in C \,\middle|\, R^{-1}(c) \subset S_!(U)\right\}\\ &\stackrel{\text{def}}{=} R_!(S_!(U))\\ &\stackrel{\text{def}}{=} [R_! \circ S_!](U)\end{aligned}$$

for each $U \in \mathcal{P}(C)$, where we used Item 2 of Proposition 5.4.4, which implies that the conditions

- We have $S^{-1}\left(R^{-1}(c)\right) \subset U$;
- We have $R^{-1}(c) \subset S_!(U)$;

are equivalent. Thus $(S \diamond R)_! = S_! \circ R_!$. □

5.5 Functoriality of Powersets

PROPOSITION 5.5.1 ► FUNCTORIALITY OF POWERSETS I

The assignment $X \mapsto \mathcal{P}(X)$ defines functors[1]

$$\begin{aligned}\mathcal{P}_* &\colon \mathsf{Rel} \to \mathsf{Sets},\\ \mathcal{P}_{-1} &\colon \mathsf{Rel}^{\mathsf{op}} \to \mathsf{Sets},\\ \mathcal{P}^{-1} &\colon \mathsf{Rel}^{\mathsf{op}} \to \mathsf{Sets},\\ \mathcal{P}_! &\colon \mathsf{Rel} \to \mathsf{Sets}\end{aligned}$$

where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathsf{Rel})$, we have

$$\begin{aligned}\mathcal{P}_*(A) &\stackrel{\text{def}}{=} \mathcal{P}(A),\\ \mathcal{P}_{-1}(A) &\stackrel{\text{def}}{=} \mathcal{P}(A),\\ \mathcal{P}^{-1}(A) &\stackrel{\text{def}}{=} \mathcal{P}(A),\\ \mathcal{P}_!(A) &\stackrel{\text{def}}{=} \mathcal{P}(A);\end{aligned}$$

- *Action on Morphisms.* For each morphism $R \colon A \nrightarrow B$ of Rel, the images

$$\begin{aligned}\mathcal{P}_*(R) &\colon \mathcal{P}(A) \to \mathcal{P}(B),\\ \mathcal{P}_{-1}(R) &\colon \mathcal{P}(B) \to \mathcal{P}(A),\\ \mathcal{P}^{-1}(R) &\colon \mathcal{P}(B) \to \mathcal{P}(A),\\ \mathcal{P}_!(R) &\colon \mathcal{P}(A) \to \mathcal{P}(B)\end{aligned}$$

  of R by $\mathcal{P}_*$, $\mathcal{P}_{-1}$, $\mathcal{P}^{-1}$, and $\mathcal{P}_!$ are defined by

$$\begin{aligned}\mathcal{P}_*(R) &\stackrel{\text{def}}{=} R_*,\\ \mathcal{P}_{-1}(R) &\stackrel{\text{def}}{=} R_{-1},\\ \mathcal{P}^{-1}(R) &\stackrel{\text{def}}{=} R^{-1},\\ \mathcal{P}_!(R) &\stackrel{\text{def}}{=} R_!,\end{aligned}$$

  as in Definitions 5.1.1, 5.2.1, 5.3.1 and 5.4.1.

[1]The functor $\mathcal{P}_* \colon \mathsf{Rel} \to \mathsf{Sets}$ admits a left adjoint; see Item 3 of Proposition 3.1.2.

PROOF 5.5.2 ► PROOF OF PROPOSITION 5.5.1

This follows from Items 3 and 4 of Proposition 5.1.5, Items 3 and 4 of Proposition 5.2.6, Items 3 and 4 of Proposition 5.3.6, and Items 3 and 4 of Proposition 5.4.6.

5.6 Functoriality of Powersets: Relations on Powersets

Let A and B be sets and let $R\colon A \mathrel{\mapstochar\rightarrow} B$ be a relation.

DEFINITION 5.6.1 ► THE RELATION ON POWERSETS ASSOCIATED TO A RELATION

The **relation on powersets associated to** R is the relation

$$\mathcal{P}(R)\colon \mathcal{P}(A) \mathrel{\mapstochar\rightarrow} \mathcal{P}(B)$$

defined by[1]

$$\mathcal{P}(R)^{V}_{U} \overset{\text{def}}{=} \mathbf{Rel}\big(\chi_{\text{pt}}, V \diamond R \diamond U\big)$$

for each $U \in \mathcal{P}(A)$ and each $V \in \mathcal{P}(B)$.

[1] Illustration:

$$\begin{tikzcd}
\text{pt} \arrow[r, "U"', "\mapstochar" marking] \arrow[rrr, bend left, "\chi_{\text{pt}}", "\mapstochar" marking] & A \arrow[r, "R"', "\mapstochar" marking] & B \arrow[r, "V"', "\mapstochar" marking] & \text{pt}.
\end{tikzcd}$$

REMARK 5.6.2 ► UNWINDING DEFINITION 5.6.1

In detail, we have $U \sim_{\mathcal{P}(R)} V$ iff the following equivalent conditions hold:

- We have $\chi_{\text{pt}} \subset V \diamond R \diamond U$.
- We have $(V \diamond R \diamond U)^{\star}_{\star} = \text{true}$, i.e. we have

$$\int^{a\in A}\int^{b\in B} V^{\star}_{b} \times R^{b}_{a} \times U^{a}_{\star} = \text{true}.$$

- There exists some $a \in A$ and some $b \in B$ such that:
  - We have $U^{a}_{\star} = \text{true}$;
  - We have $R^{b}_{a} = \text{true}$;
  - We have $V^{\star}_{b} = \text{true}$.

- There exists some $a \in A$ and some $b \in B$ such that:
  - We have $a \in U$;
  - We have $a \sim_R b$;
  - We have $b \in V$.

Proposition 5.6.3 ▶ Functoriality of Powersets II

The assignment $R \mapsto \mathcal{P}(R)$ defines a functor

$$\mathcal{P} \colon \mathsf{Rel} \longrightarrow \mathsf{Rel}.$$

Proof 5.6.4 ▶ Proof of Proposition 5.6.3

Omitted. ∎

6 Relative Preorders

6.1 The Left Skew Monoidal Structure on $\mathbf{Rel}(A, B)$

Let A and B be sets and let $J \colon A \nrightarrow B$ be a relation.

6.1.1 The Left Skew Monoidal Product

Definition 6.1.1 ▶ The Left J-Skew Monoidal Product of $\mathbf{Rel}(A, B)$

The **left J-skew monoidal product of $\mathbf{Rel}(A, B)$** is the functor

$$\lhd_J \colon \mathbf{Rel}(A, B) \times \mathbf{Rel}(A, B) \longrightarrow \mathbf{Rel}(A, B)$$

where

- *Action on Objects.* For each $R, S \in \mathrm{Obj}(\mathbf{Rel}(A, B))$, we have

$$S \lhd_J R \overset{\mathrm{def}}{=} S \diamond \mathrm{Rift}_J(R), \qquad \begin{tikzcd} & A \arrow[r, "S", "\mapstochar" marking] \arrow[d, "J", "\mapstochar" marking] & B; \\ A \arrow[ru, dashed, "\mathrm{Rift}_J(R)", "\mapstochar" marking] \arrow[r, "R"', "\mapstochar" marking] & B \arrow[l, phantom, shift left=4, "\Swarrow"] \end{tikzcd}$$

- *Action on Morphisms.* For each $R, S, R', S' \in \mathrm{Obj}(\mathbf{Rel}(A, B))$, the action on Hom-sets

$$(\triangleleft_J)_{(G,F),(G',F')} : \mathrm{Hom}_{\mathbf{Rel}(A,B)}(S, S') \times \mathrm{Hom}_{\mathbf{Rel}(A,B)}(R, R') \to \mathrm{Hom}_{\mathbf{Rel}(A,B)}(S \triangleleft_J R, S' \triangleleft_J R')$$

of $\triangleleft_J$ at $((R, S), (R', S'))$ is defined by[1]

$$\beta \triangleleft_J \alpha \overset{\text{def}}{=} \beta \diamond \mathrm{Rift}_J(\alpha), \qquad \begin{tikzcd} & A \arrow[r, bend left, "S"] \arrow[r, bend right, "S'"'] \arrow[r, phantom, "\beta\Downarrow"] \arrow[d, "J"] & B \\ A \arrow[ur, dashed, bend left, "\mathrm{Rift}_J(R)"] \arrow[ur, dashed, "\mathrm{Rift}_J(R')"'] \arrow[ur, phantom, bend left=20, "\mathrm{Rift}_J(\alpha)\Downarrow"] \arrow[r, "R"] \arrow[r, bend right, "R'"'] \arrow[r, phantom, "\alpha\Downarrow"] & B & \end{tikzcd}$$

for each $\beta \in \mathrm{Hom}_{\mathbf{Rel}(A,B)}(S, S')$ and each $\alpha \in \mathrm{Hom}_{\mathbf{Rel}(A,B)}(R, R')$.

[1] Since **Rel**(A, B) is posetal, this is to say that if $S \subset S'$ and $R \subset R'$, then $S \triangleleft_J R \subset S' \triangleleft_J R'$.

6.1.2 The Left Skew Monoidal Unit

Definition 6.1.2 ▶ The Left J-Skew Monoidal Unit of Rel(A, B)

The **left J-skew monoidal unit of Rel(A, B)** is the functor

$$\mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleleft} : \mathrm{pt} \to \mathbf{Rel}(A, B)$$

picking the object

$$\mathbb{1}^{\triangleleft}_{\mathbf{Rel}(A,B)} \overset{\text{def}}{=} J$$

of **Rel**(A, B).

6.1.3 The Left Skew Associators

Definition 6.1.3 ▶ The Left J-Skew Associator of Rel(A, B)

The **left J-skew associator of Rel(A, B)** is the natural transformation

$$\alpha^{\mathbf{Rel}(A,B),\triangleleft} \colon \triangleleft_J \circ (\triangleleft_J \times \mathrm{id}) \Longrightarrow \triangleleft_J \circ (\mathrm{id} \times \triangleleft_J),$$

whose component

$$\alpha^{\mathbf{Rel}(A,B),\triangleleft}_{T,S,R} \colon \underbrace{(T \triangleleft_J S) \triangleleft_J R}_{\overset{\mathrm{def}}{=} T \diamond \mathrm{Rift}_J(S) \diamond \mathrm{Rift}_J(R)} \hookrightarrow \underbrace{T \triangleleft_J (S \triangleleft_J R)}_{\overset{\mathrm{def}}{=} T \diamond \mathrm{Rift}_J(S \diamond \mathrm{Rift}_J(R))}$$

at (T, S, R) is given by

$$\alpha^{\mathbf{Rel}(A,B),\triangleleft}_{T,S,R} \overset{\mathrm{def}}{=} \mathrm{id}_T \diamond \gamma,$$

where

$$\gamma \colon \mathrm{Rift}_J(S) \diamond \mathrm{Rift}_J(R) \hookrightarrow \mathrm{Rift}_J(S \diamond \mathrm{Rift}_J(R))$$

is the inclusion adjunct to the inclusion

$$\epsilon_S \circ \mathrm{id}_{\mathrm{Rift}_J(R)} \colon \underbrace{J \diamond \mathrm{Rift}_J(S) \diamond \mathrm{Rift}_J(R)}_{\overset{\mathrm{def}}{=} J_*(\mathrm{Rift}_J(S) \diamond \mathrm{Rift}_J(R))} \hookrightarrow S \diamond \mathrm{Rift}_J(R)$$

under the adjunction $J_* \dashv \mathrm{Rift}_J$, where $\epsilon \colon J \diamond \mathrm{Rift}_J \Longrightarrow \mathrm{id}_{\mathbf{Rel}(A,B)}$ is the counit of the adjunction $J_* \dashv \mathrm{Rift}_J$.

6.1.4 The Left Skew Left Unitors

Definition 6.1.4 ▶ The Left J-Skew Left Unitor of Rel(A, B)

The **left J-skew left unitor of Rel(A, B)** is the natural transformation

$$\lambda^{\mathbf{Rel}(A,B),\triangleleft} \colon \triangleleft_J \circ \left(\mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleleft} \times \mathrm{id}\right) \Longrightarrow \mathrm{id},$$

whose component

$$\lambda^{\mathbf{Rel}(A,B),\triangleleft}_R \colon \underbrace{J \triangleleft_J R}_{\overset{\mathrm{def}}{=} J \diamond \mathrm{Rift}_J(R)} \hookrightarrow R$$

at R is given by

$$\lambda^{\mathbf{Rel}(A,B),\triangleleft}_R \overset{\mathrm{def}}{=} \epsilon_R,$$

where $\epsilon\colon J \diamond \mathrm{Rift}_J \Longrightarrow \mathrm{id}_{\mathbf{Rel}(A,B)}$ is the counit of the adjunction $J_* \dashv \mathrm{Rift}_J$.

6.1.5 The Left Skew Right Unitors

Definition 6.1.5 ▶ The Left J-Skew Right Unitor of $\mathrm{Rel}(A, B)$

The **left J-skew right unitor of $\mathbf{Rel}(A, B)$** is the natural transformation

$$\rho^{\mathbf{Rel}(A,B),\triangleleft}\colon \mathrm{id} \Longrightarrow \triangleleft_J \circ \left(\mathrm{id} \times \mathbb{K}^{\mathbf{Rel}(A,B)}_{\triangleleft}\right),$$

whose component

$$\rho^{\mathbf{Rel}(A,B),\triangleleft}_R\colon R \hookrightarrow \underbrace{R \triangleleft_J J}_{\overset{\mathrm{def}}{=} R \diamond \mathrm{Rift}_J(J)}$$

at R is given by

$$\rho^{\mathbf{Rel}(A,B),\triangleleft}_R \overset{\mathrm{def}}{=} \mathrm{id}_R \circ \sigma,$$

where $\sigma\colon \mathrm{id}_A \Longrightarrow \mathrm{Rift}_J(J)$ is the universal transformation included in the data of the right Kan lift $\mathrm{Rift}_J(J)$.

6.1.6 The Left Skew Monoidal Structure on $\mathbf{Rel}(A, B)$

Definition 6.1.6 ▶ The Left J-Skew Monoidal Structure on $\mathrm{Rel}(A, B)$

The **left J-skew monoidal category of relations from A to B** is the left skew monoidal category

$$\left(\mathbf{Rel}(A, B), \triangleleft_J, \mathbb{K}^{\mathbf{Rel}(A,B)}_{\triangleleft}, \alpha^{\mathbf{Rel}(A,B),\triangleleft}, \lambda^{\mathbf{Rel}(A,B),\triangleleft}, \rho^{\mathbf{Rel}(A,B),\triangleleft}\right)$$

consisting of

- *The Underlying Category.* The posetal category associated to the poset $\mathbf{Rel}(A, B)$ of relations from A to B of Item 2 of Definition 1.1.2;
- *The Skew Monoidal Product.* The functor $\triangleleft_J$ of Definition 6.1.1;
- *The Skew Monoidal Unit.* The functor $\mathbb{K}^{\mathbf{Rel}(A,B)}_{\triangleleft}$ of Definition 6.1.2;

- *The Skew Associators.* The natural transformation $\alpha^{\mathbf{Rel}(A,B),\triangleleft}$ of Definition 6.1.3;
- *The Skew Left Unitors.* The natural transformation $\lambda^{\mathbf{Rel}(A,B),\triangleleft}$ of Definition 6.1.4;
- *The Skew Right Unitors.* The natural transformation $\rho^{\mathbf{Rel}(A,B),\triangleleft}$ of Definition 6.1.5.

6.2 Left Relative Preorders

Let A and B be sets and let $J: A \nrightarrow B$ be a relation.

Definition 6.2.1 ► Left J-Relative Preorders

A **left J-relative preorder from A to B** is equivalently:

- An $\mathbb{E}_1$-skew monoid in $\left(\mathrm{N}_{\bullet}(\mathbf{Rel}(A,B)), \triangleleft_J, J\right)$;
- A skew monoid in $\left(\mathbf{Rel}(A,B), \triangleleft_J, J\right)$.

Remark 6.2.2 ► Unwinding Definition 6.2.1, I

In detail, a **left J-relative preorder** (R, μ_R, η_R) **from A to B** consists of

- *The Underlying Relation.* A relation
$$R: A \nrightarrow B,$$
called the **underlying relation of** (R, μ_R, η_R);
- *The Multiplication Inclusion.* An inclusion of relations
$$\mu_R: R \triangleleft_J R \subset R,$$
called the **multiplication** of (R, μ_R, η_R);
- *The Unit Inclusion.* An inclusion of relations
$$\eta_R: J \subset R,$$
called the **unit** of (R, μ_R, η_R).

Remark 6.2.3 ▶ Unwinding Definition 6.2.1, II

In other words, a **left J-relative preorder from A to B** is a relation $R\colon A \nrightarrow B$ from A to B satisfying the following conditions:

1. *J-Transitivity.* For each $a \in A$ and each $c \in B$, we have

$$a \sim_{R \diamond \mathrm{Rift}_J(R)} c$$

   i.e. the following condition is satisfied:[1]

   (★) If there exists some $b \in A$ such that:

   - We have $a \sim_{\mathrm{Rift}_J(R)} b$, i.e. for each $x \in B$, if $b \sim_J x$, then $a \sim_R x$;[2]
   - We have $b \sim_R c$;

   then $a \sim_R c$.

2. *J-Unitality.* For each $a \in A$ and each $b \in B$, the following condition is satisfied:

   (★) If $a \sim_J b$, then $a \sim_R b$.

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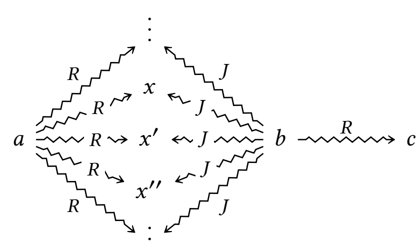
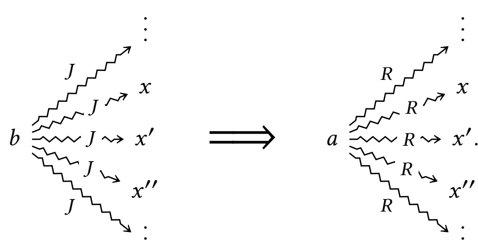
[1] *Illustration:* If we have



then $a \sim_R c$.

[2] Illustration:



6.3 The Right Skew Monoidal Structure on **Rel**(A, B)

Let A and B be sets and let $J : A \nrightarrow B$ be a relation.

6.3.1 The Right Skew Monoidal Product

DEFINITION 6.3.1 ► THE RIGHT J-SKEW MONOIDAL PRODUCT OF REL(A, B)

The **right J-skew monoidal product of Rel(A, B)** is the functor

$$\triangleright_J : \mathbf{Rel}(A, B) \times \mathbf{Rel}(A, B) \longrightarrow \mathbf{Rel}(A, B)$$

where

- *Action on Objects.* For each $R, S \in \mathrm{Obj}(\mathbf{Rel}(A, B))$, we have

$$S \triangleright_J R \overset{\text{def}}{=} \mathrm{Ran}_J(S) \diamond R, \qquad \begin{tikzcd} A \arrow[r, "R", "\shortmid"{marking}] & B \arrow[r, dashed, "\mathrm{Ran}_J(S)", "\shortmid"{marking}] & B; \\ & A \arrow[u, "J", "\shortmid"{marking}] \arrow[ur, "S"', "\shortmid"{marking}] \arrow[ur, phantom, "\Leftarrow"] & \end{tikzcd}$$

- *Action on Morphisms.* For each $R, S, R', S' \in \mathrm{Obj}(\mathbf{Rel}(A, B))$, the action on Hom-sets

$$(\triangleright_J)_{(S,R),(S',R')} : \mathrm{Hom}_{\mathbf{Rel}(A,B)}(S, S') \times \mathrm{Hom}_{\mathbf{Rel}(A,B)}(R, R') \to \mathrm{Hom}_{\mathbf{Rel}(A,B)}(S \triangleright_J R, S' \triangleright_J R')$$

  of $\triangleright_J$ at $((S, R), (S', R'))$ is defined by[1]

$$\beta \triangleright_J \alpha \overset{\text{def}}{=} \mathrm{Ran}_J(\beta) \diamond \alpha, \qquad \begin{tikzcd} A \arrow[r, bend left, "R'", "\shortmid"{marking}] \arrow[r, bend right, "R"', "\shortmid"{marking}] \arrow[r, phantom, "\alpha\Uparrow"] & B \arrow[r, dashed, bend left, "\mathrm{Ran}_J(S')", "\shortmid"{marking}] \arrow[r, dashed, bend right, "\mathrm{Ran}_J(S)"', "\shortmid"{marking}] \arrow[r, phantom, "\mathrm{Ran}_J(\beta)\Uparrow"] & B \\ & A \arrow[u, "J", "\shortmid"{marking}] \arrow[ur, "S'", "\shortmid"{marking}] \arrow[ur, bend right=50, "S"', "\shortmid"{marking}] \arrow[ur, phantom, "\beta"] & \end{tikzcd}$$

  for each $\beta \in \mathrm{Hom}_{\mathbf{Rel}(A,B)}(S, S')$ and each $\alpha \in \mathrm{Hom}_{\mathbf{Rel}(A,B)}(R, R')$.

[1] Since **Rel**(A, B) is posetal, this is to say that if $S \subset S'$ and $R \subset R'$, then $S \triangleright_J R \subset S' \triangleright_J R'$.

6.3.2 The Right Skew Monoidal Unit

Definition 6.3.2 ► The Right J-Skew Monoidal Unit of Rel(A, B)

The **right J-skew monoidal unit of Rel**(A, B) is the functor

$$\mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleright} \colon \mathrm{pt} \longrightarrow \mathbf{Rel}(A, B)$$

picking the object

$$\mathbb{1}^{\triangleright}_{\mathbf{Rel}(A,B)} \stackrel{\text{def}}{=} J$$

of **Rel**(A, B).

6.3.3 The Right Skew Associators

Definition 6.3.3 ► The Right J-Skew Associator of Rel(A, B)

The **right J-skew associator of Rel**(A, B) is the natural transformation

$$\alpha^{\mathbf{Rel}(A,B),\triangleright} \colon \triangleright_J \circ (\mathrm{id} \times \triangleright_J) \Longrightarrow \triangleright_J \circ (\triangleright_J \times \mathrm{id}),$$

whose component

$$\alpha^{\mathbf{Rel}(A,B),\triangleright}_{T,S,R} \colon \underbrace{T \triangleright_J (S \triangleright_J R)}_{\stackrel{\text{def}}{=} \mathrm{Ran}_J(T) \diamond (\mathrm{Ran}_J(S) \diamond R)} \hookrightarrow \underbrace{(T \triangleright_J S) \triangleright_J R}_{\stackrel{\text{def}}{=} \mathrm{Ran}_J(\mathrm{Ran}_J(T) \diamond S) \diamond R}$$

at (T, S, R) is given by

$$\alpha^{\mathbf{Rel}(A,B),\triangleright}_{T,S,R} \stackrel{\text{def}}{=} \gamma \diamond \mathrm{id}_R,$$

where

$$\gamma \colon \mathrm{Ran}_J(T) \diamond \mathrm{Ran}_J(S) \hookrightarrow \mathrm{Ran}_J(\mathrm{Ran}_J(T) \diamond S)$$

is the inclusion adjunct to the inclusion

$$\mathrm{id}_{\mathrm{Ran}_J(T)} \diamond \epsilon_S \colon \underbrace{\mathrm{Ran}_J(T) \diamond \mathrm{Ran}_J(S) \diamond J}_{\stackrel{\text{def}}{=} J^*(\mathrm{Ran}_J(T) \diamond \mathrm{Ran}_J(S))} \hookrightarrow \mathrm{Ran}_J(T) \diamond S$$

under the adjunction $J^* \dashv \mathrm{Ran}_J$, where $\epsilon \colon \mathrm{Ran}_J \diamond J \Longrightarrow \mathrm{id}_{\mathbf{Rel}(A,B)}$ is the counit of the adjunction $J^* \dashv \mathrm{Ran}_J$.

6.3.4 The Right Skew Left Unitors

Definition 6.3.4 ► The Right J-Skew Left Unitor of Rel(A, B)

The **right J-skew left unitor of Rel**(A, B) is the natural transformation

$$\lambda^{\mathbf{Rel}(A,B),\triangleright}\colon \mathrm{id} \Longrightarrow \triangleright_J \circ \left(\mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleright} \times \mathrm{id}\right),$$

whose component

$$\lambda^{\mathbf{Rel}(A,B),\triangleright}_R \colon R \hookrightarrow \underbrace{J \triangleright_J R}_{\overset{\mathrm{def}}{=}\mathrm{Ran}_J(J)\diamond R}$$

at R is given by

$$\lambda^{\mathbf{Rel}(A,B),\triangleright}_R \overset{\mathrm{def}}{=} \sigma \diamond \mathrm{id}_R,$$

where $\sigma\colon \mathrm{id}_B \Longrightarrow \mathrm{Ran}_J(J)$ is the universal transformation included in the data of the right Kan extension $\mathrm{Ran}_J(J)$.

6.3.5 The Right Skew Right Unitors

Definition 6.3.5 ► The Right J-Skew Right Unitor of Rel(A, B)

The **right J-skew right unitor of Rel**(A, B) is the natural transformation

$$\rho^{\mathbf{Rel}(A,B),\triangleright}\colon \triangleright_J \circ \left(\mathrm{id} \times \mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleright}\right) \Longrightarrow \mathrm{id},$$

whose component

$$\rho^{\mathbf{Rel}(A,B),\triangleright}_S \colon \underbrace{S \triangleright_J J}_{\overset{\mathrm{def}}{=}\mathrm{Ran}_J(S)\diamond J} \hookrightarrow S$$

at S is given by

$$\rho^{\mathbf{Rel}(A,B),\triangleright}_S \overset{\mathrm{def}}{=} \epsilon_R,$$

where $\epsilon\colon \mathrm{Ran}_J \diamond J \Longrightarrow \mathrm{id}_{\mathbf{Rel}(A,B)}$ is the counit of the adjunction $J^* \dashv \mathrm{Ran}_J$.

6.3.6 The Right Skew Monoidal Structure on **Rel**(A, B)

Definition 6.3.6 ▶ The Right J-Skew Monoidal Structure on $\mathbf{Rel}(A, B)$

The **right J-skew monoidal category of functors from A to B** is the right skew monoidal category

$$\left(\mathbf{Rel}(A, B), \triangleright_J, \mathbb{1}_{\triangleright}^{\mathbf{Rel}(A,B)}, \alpha^{\mathbf{Rel}(A,B),\triangleright}, \lambda^{\mathbf{Rel}(A,B),\triangleright}, \rho^{\mathbf{Rel}(A,B),\triangleright}\right)$$

consisting of

- *The Underlying Category.* The posetal category associated to the poset $\mathbf{Rel}(A, B)$ of relations from A to B of Item 2 of Definition 1.1.2;
- *The Skew Monoidal Product.* The functor $\triangleright_J$ of Definition 6.3.1;
- *The Skew Monoidal Unit.* The functor $\mathbb{1}_{\triangleright}^{\mathbf{Rel}(A,B)}$ of Definition 6.3.2;
- *The Skew Associators.* The natural transformation $\alpha^{\mathbf{Rel}(A,B),\triangleright}$ of Definition 6.3.3;
- *The Skew Left Unitors.* The natural transformation $\lambda^{\mathbf{Rel}(A,B),\triangleright}$ of Definition 6.3.4;
- *The Skew Right Unitors.* The natural transformation $\rho^{\mathbf{Rel}(A,B),\triangleright}$ of Definition 6.3.5.

6.4 Right Relative Preorders

Let A and B be sets and let $J\colon A \nrightarrow B$ be a relation.

Definition 6.4.1 ▶ Right J-Relative Preorders

A **right J-relative preorder from A to B** is equivalently:

- An $\mathbb{E}_1$-skew monoid in $(\mathrm{N}_{\bullet}(\mathbf{Rel}(A, B)), \triangleright_J, J)$;
- A skew monoid in $(\mathbf{Rel}(A, B), \triangleright_J, J)$.

Remark 6.4.2 ▶ Unwinding Definition 6.4.1, I

In detail, a **right J-relative preorder** (R, μ_R, η_R) **from** A **to** B consists of

- *The Underlying Relation.* A relation

$$R\colon A \nrightarrow B,$$

called the **underlying relation of** (R, μ_R, η_R);

- *The Multiplication Inclusion.* An inclusion of relations

$$\mu_R\colon R \triangleright_J R \subset R,$$

called the **multiplication** of (R, μ_R, η_R);

- *The Unit Inclusion.* An inclusion of relations

$$\eta_R\colon J \subset R,$$

called the **unit** of (R, μ_R, η_R).

Remark 6.4.3 ▶ Unwinding Definition 6.4.1, II

In other words, a **right J-relative preorder from** A **to** B is a relation $R\colon A \nrightarrow B$ from A to B satisfying the following conditions:

1. *J-Transitivity.* For each $a \in A$ and each $c \in B$, we have

$$a \sim_{\mathrm{Ran}_J(R) \diamond R} c,$$

i.e. the following condition is satisfied:[1]

($\star$) If there exists some $b \in B$ such that:
   - We have $a \sim_R b$;
   - We have $b \sim_{\mathrm{Ran}_J(R)} c$, i.e. for each $x \in A$, if $x \sim_J b$, then $x \sim_R c$;[2]

   then $a \sim_R c$.

2. *J-Unitality.* For each $a \in A$ and each $b \in B$, the following condition is satisfied:

(★) If $a \sim_J b$, then $a \sim_R b$.

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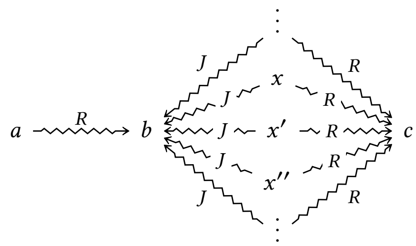
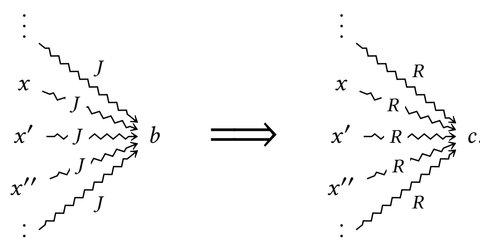
[1] *Illustration:* If we have



then $a \sim_R c$.

[2] Illustration:



Appendices

A Other Chapters