Un/Straightening for Indexed and Fibred Sets

December 14, 2023

This chapter contains a discussion of the un/straightening equivalence in the context of sets, as well as a general discussion of indexed and fibred sets. In particular, it contains:

1. A discussion of indexed sets (i.e. functors $K_{\mathsf{disc}} \to \mathsf{Sets}$ with K a set), constructions with them like dependent sums and dependent products, and their properties (**????**);
2. A discussion of fibred sets (i.e. maps of sets $X \to K$), constructions with them like dependent sums and dependent products, and their properties (**????**);
3. A discussion of the un/straightening equivalence for indexed and fibred sets (Section 1).

Contents

1 Un/Straightening for Indexed and Fibred Sets

1.1 Straightening for Fibred Sets

Let K be a set and let (X, ϕ) be a K-fibred set.

Definition 1.1.1.1. The **straightening of** (X, ϕ) is the K-indexed set

$$\mathrm{St}_K(X, \phi)\colon K_{\mathsf{disc}} \to \mathsf{Sets}$$

defined by

$$\mathrm{St}_K(X, \phi)_x \overset{\text{def}}{=} \phi^{-1}(x)$$

for each $x \in K$.

Proposition 1.1.1.2. Let K be a set.

1. *Functoriality.* The assignment $(X, \phi) \mapsto \mathrm{St}_K(X, \phi)$ defines a functor

$$\mathrm{St}_K\colon \mathsf{FibSets}(K) \to \mathsf{ISets}(K)$$

- *Action on Objects.* For each $(X, \phi) \in \mathrm{Obj}(\mathsf{FibSets}(K))$, we have

$$[\mathrm{St}_K](X, \phi) \overset{\text{def}}{=} \mathrm{St}_K(X, \phi);$$

- *Action on Morphisms.* For each $(X, \phi), (Y, \psi) \in \mathrm{Obj}(\mathsf{FibSets}(K))$, the action on Hom-sets

$$\mathrm{St}_{K|X,Y}\colon \mathrm{Hom}_{\mathsf{FibSets}(K)}(X, Y) \to \mathrm{Hom}_{\mathsf{ISets}(K)}(\mathrm{St}_K(X), \mathrm{St}_K(Y))$$

of St_K at (X, Y) is given by sending a morphism

$$f\colon (X, \phi) \to (Y, \psi)$$

of K-fibred sets to the morphism

$$\mathrm{St}_K(f)\colon \mathrm{St}_K(X, \phi) \to \mathrm{St}_K(Y, \psi)$$

of K-indexed sets defined by

$$\mathrm{St}_K(f) \overset{\text{def}}{=} \{f^*_x\}_{x \in K},$$

where f^*_x is the transport map associated to f at $x \in K$ of **??**.

2. *Interaction With Change of Base/Indexing.* Let $f\colon K \to K'$ be a map

of sets. The diagram

$$\begin{tikzcd}
\mathsf{FibSets}(K') \arrow[r, "f^*"] \arrow[d, "\mathrm{St}_{K'}"'] & \mathsf{FibSets}(K) \arrow[d, "\mathrm{St}_{K}"] \\
\mathsf{ISets}(K') \arrow[r, "f^*"'] & \mathsf{ISets}(K)
\end{tikzcd}$$

commutes.

3. *Interaction With Dependent Sums.* Let $f\colon K \to K'$ be a map of sets. The diagram

$$\begin{tikzcd}
\mathsf{FibSets}(K) \arrow[r, "\Sigma_f"] \arrow[d, "\mathrm{St}_{K}"'] & \mathsf{FibSets}(K') \arrow[d, "\mathrm{St}_{K'}"] \\
\mathsf{ISets}(K) \arrow[r, "\Sigma_f"'] & \mathsf{ISets}(K')
\end{tikzcd}$$

commutes.

4. *Interaction With Dependent Products.* Let $f\colon K \to K'$ be a map of sets. The diagram

$$\begin{tikzcd}
\mathsf{Sets}_{/K} \arrow[r, "\Pi_f"] \arrow[d, "\mathrm{St}_{K}"'] & \mathsf{FibSets}(K') \arrow[d, "\mathrm{St}_{K'}"] \\
\mathsf{ISets}(K) \arrow[r, "\Pi_f"'] & \mathsf{ISets}(K')
\end{tikzcd}$$

commutes.

Proof. *Item 1*, *Functoriality*: Omitted.

Item 2, Interaction With Change of Base/Indexing: Indeed, we have

$$\begin{aligned}
\mathrm{St}_K(f^*(X,\phi))_x &\stackrel{\text{def}}{=} \mathrm{St}_K(K\times_{K'}X)_x\\
&\stackrel{\text{def}}{=} \left(\mathrm{pr}_1^{K\times_{K'}X}\right)^{-1}(x)\\
&= \left\{(k,y)\in K\times_{K'}X \;\middle|\; \mathrm{pr}_1^{K\times_{K'}X}(k,y)=x\right\}\\
&= \{(k,y)\in K\times_{K'}X \mid k=x\}\\
&= \{(k,y)\in K\times X \mid k=x \text{ and } f(k)=\phi(y)\}\\
&\cong \{y\in X \mid \phi(y)=f(x)\}\\
&= \phi^{-1}(f(x))\\
&\stackrel{\text{def}}{=} f^*\left(\phi^{-1}(x)\right)\\
&\stackrel{\text{def}}{=} f^*(\mathrm{St}_{K'}(X,\phi)_x)
\end{aligned}$$

for each $(X,\phi)\in\mathrm{Obj}(\mathsf{FibSets}(K'))$ and each $x\in K$, and similarly for morphisms.

Item 3, Interaction With Dependent Sums: Indeed, we have

$$\begin{aligned}
\mathrm{St}_{K'}(\Sigma_f(X,\phi))_x &\stackrel{\text{def}}{=} \Sigma_f(\phi)^{-1}(x)\\
&\cong \coprod_{\substack{y\in X\\ f(y)=x}}\phi^{-1}(y)\\
&\cong \Sigma_f\left(\phi^{-1}(x)\right)\\
&\stackrel{\text{def}}{=} \Sigma_f(\mathrm{St}_K(X,\phi)_x)
\end{aligned}$$

for each $(X,\phi)\in\mathrm{Obj}(\mathsf{FibSets}(K))$ and each $x\in K'$, where we have used **??** of **??** for the first bijection, and similarly for morphisms.

Item 4, Interaction With Dependent Products: Indeed, we have

$$\begin{aligned}
\mathrm{St}_{K'}(\Pi_f(X,\phi))_x &\stackrel{\text{def}}{=} \Pi_f(\phi)^{-1}(x)\\
&\cong \prod_{\substack{y\in X\\ f(y)=x}}\phi^{-1}(y)\\
&\cong \Pi_f\left(\phi^{-1}(x)\right)\\
&\stackrel{\text{def}}{=} \Pi_f(\mathrm{St}_K(X,\phi)_x)
\end{aligned}$$

for each $(X,\phi)\in\mathrm{Obj}(\mathsf{FibSets}(K))$ and each $x\in K'$, where we have used **??** of **??** for the first bijection, and similarly for morphisms. □

1.2 Unstraightening for Indexed Sets

Let K be a set and let X be a K-indexed set.

Definition 1.2.1.1. The **unstraightening of** X is the K-fibred set

$$\phi_{\mathrm{Un}_K} \colon \mathrm{Un}_K(X) \to K$$

consisting of

- *The Underlying Set.* The set $\mathrm{Un}_K(X)$ defined by

$$\mathrm{Un}_K(X) \overset{\mathrm{def}}{=} \coprod_{x \in K} X_x;$$

- *The Fibration.* The map of sets

$$\phi_{\mathrm{Un}_K} \colon \mathrm{Un}_K(X) \to K$$

  defined by sending an element of $\coprod_{x \in K} X_x$ to its index in K.

Proposition 1.2.1.2. Let K be a set.

1. *Functoriality.* The assignment $X \mapsto \mathrm{Un}_K(X)$ defines a functor

$$\mathrm{Un}_K \colon \mathsf{ISets}(K) \to \mathsf{FibSets}(K)$$

   - *Action on Objects.* For each $X \in \mathrm{Obj}(\mathsf{ISets}(K))$, we have

$$[\mathrm{Un}_K](X) \overset{\mathrm{def}}{=} \mathrm{Un}_K(X);$$

   - *Action on Morphisms.* For each $X, Y \in \mathrm{Obj}(\mathsf{ISets}(K))$, the action on Hom-sets

$$\mathrm{Un}_{K|X,Y} \colon \mathrm{Hom}_{\mathsf{ISets}(K)}(X, Y) \to \mathrm{Hom}_{\mathsf{FibSets}(K)}(\mathrm{Un}_K(X), \mathrm{Un}_K(Y))$$

     of Un_K at (X, Y) is defined by

$$\mathrm{Un}_{K|X,Y}(f) \overset{\mathrm{def}}{=} \coprod_{x \in K} f_x^*.$$

2. *Interaction With Fibres.* We have a bijection of sets

$$\phi_{\mathrm{Un}_K}^{-1}(x) \cong X_x$$

   for each $x \in K$.

3. *As a Pullback.* We have a bijection of sets

$$
\mathrm{Un}_K(X) \cong K_{\mathsf{disc}} \times_{\mathsf{Sets}} \mathsf{Sets}_*, \qquad
\begin{tikzcd}
\mathrm{Un}_K(X) \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathsf{Sets}_* \arrow[d, "\text{忘}"] \\
K_{\mathsf{disc}} \arrow[r, "X"'] & \mathsf{Sets}.
\end{tikzcd}
$$

4. *As a Colimit.* We have a bijection of sets

$$
\mathrm{Un}_K(X) \cong \mathrm{colim}(X).
$$

5. *Interaction With Change of Indexing/Base.* Let $f\colon K \to K'$ be a map of sets. The diagram

$$
\begin{tikzcd}
\mathsf{ISets}(K') \arrow[r, "f^*"] \arrow[d, "\mathrm{Un}_{K'}"'] & \mathsf{ISets}(K) \arrow[d, "\mathrm{Un}_K"] \\
\mathsf{FibSets}(K') \arrow[r, "f^*"'] & \mathsf{FibSets}(K)
\end{tikzcd}
$$

commutes.

6. *Interaction With Dependent Sums.* Let $f\colon K \to K'$ be a map of sets. The diagram

$$
\begin{tikzcd}
\mathsf{ISets}(K) \arrow[r, "\Sigma_f"] \arrow[d, "\mathrm{Un}_K"'] & \mathsf{ISets}(K') \arrow[d, "\mathrm{Un}_{K'}"] \\
\mathsf{FibSets}(K) \arrow[r, "\Sigma_f"'] & \mathsf{FibSets}(K')
\end{tikzcd}
$$

commutes.

7. *Interaction With Dependent Products.* Let $f\colon K \to K'$ be a map of sets. The diagram

$$
\begin{tikzcd}
\mathsf{ISets}(K) \arrow[r, "\Pi_f"] \arrow[d, "\mathrm{Un}_K"'] & \mathsf{ISets}(K') \arrow[d, "\mathrm{Un}_{K'}"] \\
\mathsf{FibSets}(K) \arrow[r, "\Pi_f"'] & \mathsf{FibSets}(K')
\end{tikzcd}
$$

commutes.

Proof. *Item 1, Functoriality*: Omitted.
Item 2, Interaction With Fibres: Omitted.
Item 3, As a Pullback: Omitted.
Item 4, As a Colimit: Clear.
Item 5, Interaction With Change of Indexing/Base: Indeed, we have

$$\begin{aligned}
\mathrm{Un}_K(f^*(X)) &\overset{\text{def}}{=} \mathrm{Un}_K(X \circ f) \\
&\overset{\text{def}}{=} \coprod_{x \in K} X_{f(x)} \\
&\cong \left\{ (x, (y, a)) \in K \times \coprod_{y \in K'} X_y \;\middle|\; f(x) = y \right\} \\
&\cong K \times_{K'} \coprod_{y \in K'} X_y \\
&\overset{\text{def}}{=} K \times_{K'} \mathrm{Un}_{K'}(X) \\
&\overset{\text{def}}{=} f^*(\mathrm{Un}_{K'}(X))
\end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{ISets}(K'))$. Similarly, it can be shown that we also have $\mathrm{Un}_K(f^*(\phi)) = f^*(\mathrm{Un}_{K'}(\phi))$ and that $\mathrm{Un}_K \circ f^* = f^* \circ \mathrm{Un}_{K'}$ also holds on morphisms.
Item 6, Interaction With Dependent Sums: Indeed, we have

$$\begin{aligned}
\mathrm{Un}_{K'}(\Sigma_f(X)) &\overset{\text{def}}{=} \coprod_{x \in K'} \Sigma_f(X)_x \\
&\cong \coprod_{x \in K'} \coprod_{y \in f^{-1}(x)} X_y \\
&\cong \coprod_{y \in K} X_y \\
&\cong \mathrm{Un}_K(X) \\
&\overset{\text{def}}{=} \Sigma_f(\mathrm{Un}_K(X))
\end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{ISets}(K))$, where we have used **??** of **??** for the first bijection. Similarly, it can be shown that we also have $\mathrm{Un}_{K'}(\Sigma_f(\phi)) = \Sigma_f(\phi_{\mathrm{Un}_K})$ and that $\mathrm{Un}_{K'} \circ \Sigma_f = \Sigma_f \circ \mathrm{Un}_K$ also holds on morphisms.

Item 7, Interaction With Dependent Products: Indeed, we have

$$\begin{aligned}
\mathrm{Un}_{K'}(\Pi_f(X)) &\stackrel{\text{def}}{=} \coprod_{x\in K'} \Pi_f(X)_x \\
&\cong \coprod_{x\in K'} \prod_{y\in f^{-1}(x)} X_y \\
&\cong \left\{ (x,h) \in \coprod_{x\in K'} \mathsf{Sets}\big(f^{-1}(x), \phi^{-1}_{\mathrm{Un}_K}(f^{-1}(x))\big) \;\middle|\; \phi\circ h = \mathrm{id}_{f^{-1}(x)} \right\} \\
&\stackrel{\text{def}}{=} \Pi_f\left(\coprod_{y\in K} X_y\right) \\
&\stackrel{\text{def}}{=} \Pi_f(\mathrm{Un}_K(X))
\end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{ISets}(K))$, where we have used **??** of **??** for the first bijection. Similarly, it can be shown that we also have $\mathrm{Un}_{K'}(\Pi_f(\phi)) = \Pi_f(\phi_{\mathrm{Un}_K})$ and that $\mathrm{Un}_{K'} \circ \Pi_f = \Pi_f \circ \mathrm{Un}_K$ also holds on morphisms. □

1.3 The Un/Straightening Equivalence

Theorem 1.3.1.1. We have an isomorphism of categories

$$(\mathrm{St}_K \dashv \mathrm{Un}_K)\colon \quad \begin{tikzcd} \mathsf{FibSets}(K) \arrow[r, bend left, "\mathrm{St}_K"] \arrow[r, phantom, "\perp"] & \mathsf{ISets}(K). \arrow[l, bend left, "\mathrm{Un}_K"] \end{tikzcd}$$

Proof. Omitted. □

2 Miscellany

2.1 Other Kinds of Un/Straightening

Remark 2.1.1.1. There are also other kinds of un/straightening for sets, where Sets is replaced by $\mathbf{Rel}$ or Span:

- *Un/Straightening With* **Rel**, *I.* We have an isomorphism of sets

$$\mathrm{Rel}(A,B) \cong \mathsf{Sets}(B\times A, \{\mathsf{true}, \mathsf{false}\}).$$

  by the definition of a relation from A to B, Relations, Definition 1.1.1.1.

- *Un/Straightening With* **Rel**, *II.* We have an equivalence of categories

$$\mathsf{LaxFun}(K_{\mathsf{disc}}, \mathbf{Rel}) \stackrel{\text{eq.}}{\cong} \mathsf{Cats}^{\mathsf{fth}}_{/K_{\mathsf{disc}}},$$

where $\mathsf{Cats}^{\mathsf{fth}}_{/K_{\mathsf{disc}}}$ is the full subcategory of $\mathsf{Cats}_{/K_{\mathsf{disc}}}$ spanned by the faithful functors; see [Nie04, Theorem 3.1].

- *Un/Straightening With* Span*, I.* For each $A, B \in \mathrm{Obj}(\mathsf{Sets})$, we have a morphism of sets

$$\mathsf{Span}(A, B) \to \mathsf{Sets}(A \times B, \mathbb{N} \cup \{\infty\})$$

which assemble into an equivalence of categories between $\mathsf{Span}(\mathsf{Sets})$ and the category MRel of "multirelations"; see Spans, Remark 7.5.1.1.

- *Un/Straightening With* Span*, II.* We have an equivalence of categories

$$\mathsf{LaxFun}(K_{\mathsf{disc}}, \mathsf{Span}) \overset{\text{eq.}}{\cong} \mathsf{Cats}_{/K_{\mathsf{disc}}};$$

see [nLa23, Section 3].

Appendices

A Other Chapters