

Spans

December 20, 2023

This chapter contains some material about spans. Notably, we discuss and explore:

1. The basic definitions around spans ([Section 1](#));
2. The relation between spans and functions ([Proposition 7.1.1.1](#));
3. The relation between spans and relations ([Propositions 7.2.2.1](#) and [7.3.1.1](#) and [Remark 7.5.1.1](#)).
4. “Hyperpointed sets” (??). I don’t know why I wrote this...

TODO:

1. https://www.sciencedirect.com/science/article/pii/S0022404994900094?ref=pdf_download&fr=RR-2&rr=834107b75c906aa4
2. <https://arxiv.org/abs/1605.08100>
3. <https://arxiv.org/abs/1603.08181>
4. <https://arxiv.org/abs/1601.02307>
5. <https://arxiv.org/abs/1507.01460>
6. <https://arxiv.org/abs/1506.08870>
7. <https://arxiv.org/abs/1505.00048>
8. <https://arxiv.org/abs/1501.07592>
9. <https://arxiv.org/abs/1501.04664>
10. <https://arxiv.org/abs/1501.00792>

11. <https://arxiv.org/abs/1412.6560>
12. <https://arxiv.org/abs/1412.0212>
13. <https://arxiv.org/abs/1409.0837>
14. <https://arxiv.org/abs/1408.5220>
15. <https://arxiv.org/abs/1308.6548>
16. <https://arxiv.org/abs/1304.0219>
17. <https://arxiv.org/abs/1210.8192>
18. <https://arxiv.org/abs/1210.1433>
19. <https://arxiv.org/abs/1201.3789>
20. <https://arxiv.org/abs/1112.0560>
21. <https://arxiv.org/abs/1109.1598>
22. <https://arxiv.org/abs/1101.4594>
23. <https://arxiv.org/abs/1012.6001>
24. <https://arxiv.org/abs/1011.3243>
25. <https://arxiv.org/abs/0910.2996>
26. <https://arxiv.org/abs/0810.2361>
27. <https://arxiv.org/abs/0803.2429>
28. <https://arxiv.org/abs/0712.2525>
29. <https://arxiv.org/abs/0706.1286>
30. <https://arxiv.org/abs/math/0611930>
31. <https://arxiv.org/abs/2311.15342>
32. <https://arxiv.org/abs/2310.19428>
33. <https://arxiv.org/abs/2309.08084>
34. <https://arxiv.org/abs/2308.01662>

-
35. <https://arxiv.org/abs/2301.11860>
 36. <https://arxiv.org/abs/2301.01199>
 37. <https://arxiv.org/abs/2212.09060>
 38. <https://arxiv.org/abs/2208.07183>
 39. <https://arxiv.org/abs/2205.06892>
 40. <https://arxiv.org/abs/2203.16179>
 41. <https://arxiv.org/abs/2201.09551>
 42. <https://arxiv.org/abs/2112.04599>
 43. <https://arxiv.org/abs/2111.10968>
 44. <https://arxiv.org/abs/2107.07621>
 45. <https://arxiv.org/abs/2106.14743>
 46. <https://arxiv.org/abs/2105.14654>
 47. <https://arxiv.org/abs/2102.08051>
 48. <https://arxiv.org/abs/2102.04386>
 49. <https://arxiv.org/abs/2101.06734>
 50. <https://arxiv.org/abs/2011.11042>
 51. <https://arxiv.org/abs/2010.15722>
 52. <https://arxiv.org/abs/2006.10375>
 53. <https://arxiv.org/abs/2006.10375>
 54. <https://arxiv.org/abs/2005.10496>
 55. <https://arxiv.org/abs/2003.11541>
 56. <https://arxiv.org/abs/2002.10334>
 57. <https://arxiv.org/abs/1909.00069>
 58. <https://arxiv.org/abs/1907.02695>

-
- 59. <https://arxiv.org/abs/1905.06671>
 - 60. define a relational span
 - 61. consider giving Ran and Rift their dedicated sections on the relations chapter, perhaps together with the other sections on co/limits
 - 62. <https://arxiv.org/abs/1710.02742>
 - 63. <https://arxiv.org/search/math?searchtype=author&query=Walker,+Charles>
 - 64. <https://arxiv.org/abs/1706.09575>
 - 65. <https://arxiv.org/abs/1710.01465>
 - 66. fibred categories: <https://arxiv.org/abs/1806.02376>
 - 67. <https://arxiv.org/abs/1806.10477v2>
 - 68. double categorical limits in $\mathbf{Rel}^{\text{dbl}}$
 - 69. double categorical limits in $\mathbf{Span}^{\text{dbl}}$
 - 70. internal adjoint equivalences in \mathbf{Rel}
 - 71. internal adjoint equivalences in \mathbf{Span}
 - 72. 2-categorical limits in \mathbf{Rel} ;
 - 73. morphism of internal adjunctions in \mathbf{Rel} ;
 - 74. morphism of internal adjunctions in \mathbf{Span} ;
 - 75. morphism of co/monads in \mathbf{Span} ;
 - 76. What is $\text{Adj}(\mathbf{Span}(A, B))$?
 - 77. monoids, comonoids, pseudomonoids, etc. in \mathbf{Span} .
 - 78. write down the dumb intuition about spans inducing morphisms $\mathbf{Sets}(S, A) \rightarrow \mathbf{Sets}(S, B)$ instead of $\mathcal{P}(A) \rightarrow \mathcal{P}(B)$ from the similarity between

$$S \rightarrow A \times B$$

and

$$A \times B \rightarrow \{\mathbf{t}, \mathbf{f}\}.$$

This intuition is justified by taking $A = \text{pt}$ or $B = \text{pt}$.

- 79. What about using the direct image with compact support in $g(f^{-1}(a))$?
- 80. Monads in **Span** | develop this in the level of morphisms too
- 81. Comonads in **Span** are spans whose legs are equal | develop this in the level of morphisms too
- 82. Does **Span** have an internal **Hom**?
- 83. Examples of spans
- 84. Functional and total spans
- 85. closed symmetric monoidal category of spans
- 86. double category of relations
- 87. collage of a span
- 88. equivalence spans?
- 89. functoriality of powersets for spans
- 90. Is **Span** a closed bicategory?
- 91. skew monoidal structure on $\mathbf{Span}(A, B)$
- 92. Adjunctions in **Span**
- 93. Isomorphisms in **Span**
- 94. Equivalences in **Span**
- 95. Interaction between the above notions in **Span** vs. in **Rel** via the comparison functors
- 96. $\mathrm{Hom}_C(S, A) \times \mathrm{Hom}_C(f^*(S), A)$.
- 97. Proof of non-existence of left Kan extensions/lifts in **Rel** (when do these exist btw?)
- 98. add intuition for spans as relations with multiple witnesses
- 99. description of unitors and associators of span

Contents

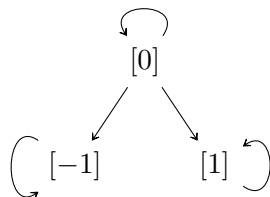
1	Spans	7
1.1	The Walking Span	7
1.2	Spans	7
1.3	Morphisms of Spans	8
1.4	Functional Spans	9
1.5	Total Spans	11
2	Categories of Spans	11
2.1	The Category of Spans Between Two Sets	11
2.2	The Bicategory of Spans	14
2.3	The Monoidal Bicategory of Spans	16
2.4	The Double Category of Spans	16
2.5	Properties of The Bicategory of Spans	19
3	Limits of Spans.....	22
3.1	tmp2.....	22
3.2	tmp.....	23
3.3	Left Kan Extensions.....	23
3.4	Right Kan Extensions.....	24
3.5	Right Kan Lifts.....	25
4	Colimits of Spans.....	26
5	Constructions With Spans.....	26
5.1	Representable Spans.....	26
5.2	Composition of Spans.....	26
5.3	Horizontal Composition of Morphisms of Spans.....	27
5.4	Properties of Composition of Spans.....	28
5.5	The Inverse of a Span.....	29
6	Functoriality of Spans.....	29
6.1	Direct Images.....	29
6.2	Functoriality of Spans on Powersets.....	29
7	Comparison of Spans to Functions and Relations.....	29
7.1	Comparison to Functions.....	29
7.2	Comparison to Relations: From Span to Rel	30
7.3	Comparison to Relations: From Rel to Span	33

7.4 Comparison to Relations: The Wehrheim–Woodward Construction	35
7.5 Comparison to Multirelations	35
7.6 Comparison to Relations via Double Categories	36
A Other Chapters	36

1 Spans

1.1 The Walking Span

Definition 1.1.1.1. The **walking span** is the category Λ that looks like this:



1.2 Spans

Let A and B be sets.

Definition 1.2.1.1. A **span from A to B** ¹ is a functor $F: \Lambda \rightarrow \mathbf{Sets}$ such that

$$\begin{aligned} F([-1]) &= A, \\ F([1]) &= B. \end{aligned}$$

Remark 1.2.1.2. In detail, a **span from A to B** is a triple (S, f, g)

¹*Further Terminology:* Also called a **roof from A to B** or a **correspondence from A to B** .

consisting of^{2,3,4}

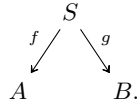
- *The Underlying Set.* A set S , called the **underlying set** of (S, f, g) ;
- *The Legs.* A pair of functions $f: S \rightarrow A$ and $g: S \rightarrow B$.

1.3 Morphisms of Spans

Definition 1.3.1.1. A **morphism of spans** from (R, f_1, g_1) to (S, f_2, g_2) ⁵ is a natural transformation $(R, f_1, g_1) \Rightarrow (S, f_2, g_2)$.

Remark 1.3.1.2. In detail, a **morphism of spans** from (R, f_1, g_1) to

²Picture:



³Every span (S, f, g) from A to B determines in particular a relation $R: A \nrightarrow B$ via

$$R \stackrel{\text{def}}{=} \{(f(a), g(a)) \mid a \in A\},$$

i.e. where $R(a) = g(f^{-1}(a))$ for each $a \in A$; see **Proposition 7.2.2.1**.

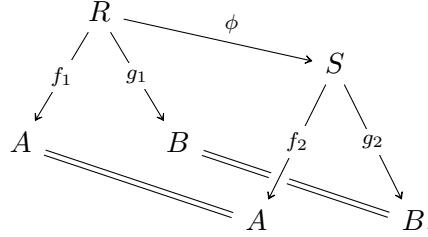
⁴In general, spans from A to B may be thought of as relations from A to B which can relate an element $a \in A$ to an element $b \in B$ in multiple ways via f and g , with the “set of witnesses of $a \sim_S b$ ” being given by

$$\text{Wit}_S(a, b) \stackrel{\text{def}}{=} \{s \in S \mid a = f(s) \text{ and } g(s) = b\}.$$

This analogy is made precise by **Remark 7.5.1.1**. Note, however, that when making the passage from sets to categories, the suitable generalisation of an assignment $(a, b) \mapsto \text{Wit}_S(a, b)$ for each $(a, b) \in A \times B$ becomes that of a profunctor (TODO), which differs from spans of categories.

⁵*Further Terminology:* Also called a **morphism of roofs** from (R, f_1, g_1) to (S, f_2, g_2) or a **morphism of correspondences** from (R, f_1, g_1) to (S, f_2, g_2) .

(S, f_2, g_2) is a function $\phi: R \rightarrow S$ making the diagram⁶



commute.

1.4 Functional Spans

Let $\lambda = \left(A \xleftarrow{f} S \xrightarrow{g} B \right)$ be a span.

Definition 1.4.1.1. The span λ is **functional** if f the following equivalent conditions are satisfied:

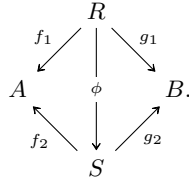
1. The associated relation $g \circ f^{-1}$ of λ is functional.
2. For each $s, t \in S$, if $f(s) = f(t)$, then $g(s) = g(t)$.
3. this “ f -relative injectivity” condition is the same as being a monomorphism/monoid/whatever in nice category | maybe this is the same as being a skew monoid in $\mathbf{Span}(A, B)$ or something?

1. a⁷

Definition 1.4.1.2. The span λ is **total** if f is surjective.

Let $\lambda = \left(A \xleftarrow{f} S \xrightarrow{g} B \right)$ be a span. A morphism of spans from id_A to $\lambda \diamond \lambda^\dagger$

⁶ *Alternative Picture:*



⁷Here we could perhaps also use the direct image with compact support $g_!$ of g (see [Constructions With Sets, Definition 4.5.1.1](#)) instead of the usual direct image, although

is a morphism

$$s: A \rightarrow S \times_B S$$

making the diagram

$$\begin{array}{ccc}
 & A & \\
 \text{id}_A \swarrow & \downarrow s & \searrow \text{id}_A \\
 A & & A \\
 f \circ \text{pr}'_1 \swarrow & & \searrow f \circ \text{pr}'_2 \\
 & S \times_B S &
 \end{array}$$

commute, where $S \times_B S$ is the pullback

$$\begin{array}{ccc}
 S \times_B S & \xrightarrow{\quad} & S \\
 \downarrow & \lrcorner & \downarrow g \\
 S & \xrightarrow{\quad g \quad} & B
 \end{array}$$

$S \times_B S \cong \{(s, t) \in S \times S \mid g(s) = g(t)\}$

of S with itself along g . In particular, $\text{pr}_1 \circ s$ and $\text{pr}_2 \circ s$ are both left-inverses/retractions for f , i.e. we have

$$\begin{aligned}
 (\text{pr}_1 \circ s) \circ f &\cong \text{id}_A, \\
 (\text{pr}_2 \circ s) \circ f &\cong \text{id}_A.
 \end{aligned}$$

Thus, by **Categories**, ?? of ??, f is injective if $A \neq \emptyset$.

the expression for $g_!(f^{-1}(a))$ seems a bit weird. It can also actually be given as a right Kan extension (**Relations**, **Item 11** of **Proposition 2.5.1.1**):

$$\begin{aligned}
 g_!(f^{-1}(a)) &= g_!(\{s \in S \mid f(s) = a\}) \\
 &= \{b \in B \mid g^{-1}(b) \subset \{s \in S \mid f(s) = a\}\} \\
 &= \{b \in B \mid \text{for each } s \in S, \text{ if } g(s) = b, \text{ then } f(s) = a\} \\
 &= [\text{Ran}_g^\dagger(f)](a)
 \end{aligned}$$

as in the diagram

$$\begin{array}{ccc}
 & B & \\
 g \nearrow & & \downarrow \text{Ran}_g(f) \\
 S & \xrightarrow{f} & A
 \end{array}$$

$\text{Ran}_g^\dagger(f): A \vdash B$

1.5 Total Spans

2 Categories of Spans

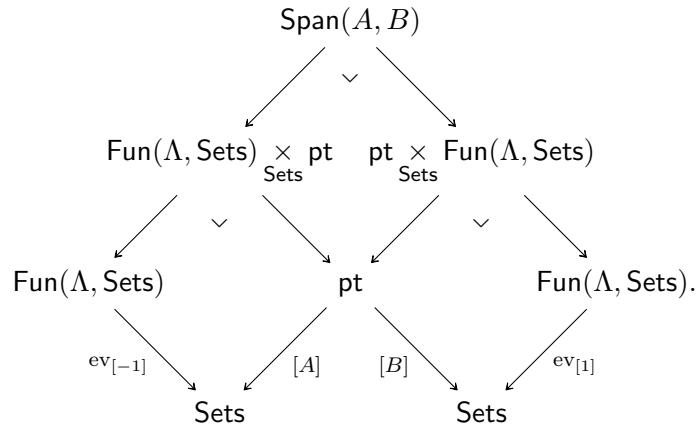
2.1 The Category of Spans Between Two Sets

Let A and B be sets.

Definition 2.1.1.1. The **category of spans from A to B** is the category $\text{Span}(A, B)$ defined by

$$\text{Span}(A, B) \stackrel{\text{def}}{=} \text{Fun}(\Lambda, \text{Sets}) \times_{\text{ev}_{[-1]}, \text{Sets}, [A]}^{\text{pt}} \times_{[B], \text{Sets}, \text{ev}_{[1]}}^{\text{pt}} \text{Fun}(\Lambda, \text{Sets}),$$

as in the diagram



Remark 2.1.1.2. In detail, the **category of spans from A to B** is the category $\text{Span}(A, B)$ where

- *Objects.* The objects of $\text{Span}(A, B)$ are spans from A to B ;
- *Morphisms.* The morphism of $\text{Span}(A, B)$ are morphisms of spans;
- *Identities.* The unit map

$$\llcorner_{(S, f, g)}^{\text{Span}(A, B)} : \text{pt} \rightarrow \text{Hom}_{\text{Span}(A, B)}((S, f, g), (S, f, g))$$

of $\text{Span}(A, B)$ at (S, f, g) is defined by⁸

$$\text{id}_{(S, f, g)}^{\text{Span}(A, B)} \stackrel{\text{def}}{=} \text{id}_S;$$

- *Composition.* The composition map

$$\circ_{R, S, T}^{\text{Span}(A, B)} : \text{Hom}_{\text{Span}(A, B)}(S, T) \times \text{Hom}_{\text{Span}(A, B)}(R, S) \rightarrow \text{Hom}_{\text{Span}(A, B)}(R, T)$$

of $\text{Span}(A, B)$ at $((R, f_1, g_1), (S, f_2, g_2), (T, f_3, g_3))$ is defined by⁹

$$\psi \circ_{R, S, T}^{\text{Span}(A, B)} \phi \stackrel{\text{def}}{=} \psi \circ \phi.$$

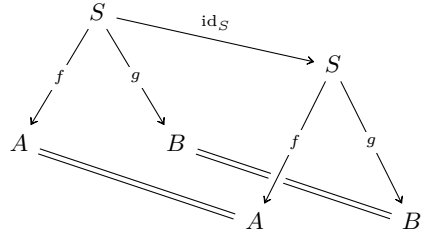
Proposition 2.1.1.3. Let A and B be sets.

1. *As a Pullback.* We have an isomorphism of categories

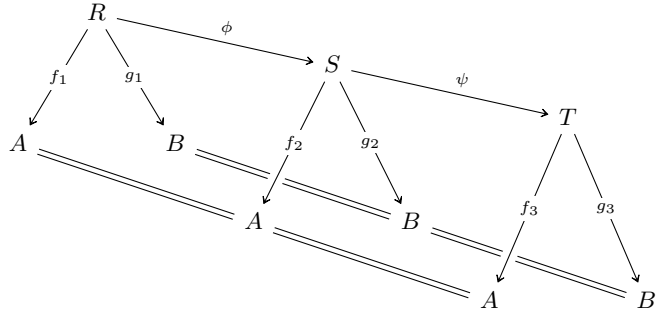
$$\text{Span}(A, B) \cong \text{Sets}_{/A} \times_{\text{Sets}} \text{Sets}_{/B},$$

$$\begin{array}{ccc} \text{Span}(A, B) & \longrightarrow & \text{Sets}_{/B} \\ \downarrow & \lrcorner & \downarrow \text{forget} \\ \text{Sets}_{/A} & \xrightarrow{\text{forget}} & \text{Sets}. \end{array}$$

⁸ Picture:



⁹ Picture:



Proof. Item 1, As a Pullback: In detail, the pullback $\mathbf{Sets}_{/A} \times_{\mathbf{Sets}} \mathbf{Sets}_{/B}$ is the category where

- *Objects.* The objects of $\mathbf{Sets}_{/A} \times_{\mathbf{Sets}} \mathbf{Sets}_{/B}$ consist of pairs $((S, f), (S', g))$ of objects of \mathbf{Sets} consisting of
 - A pair (S, f) in $\mathbf{Obj}(\mathbf{Sets}_{/A})$ consisting of a set S and a map $f: S \rightarrow A$;
 - A pair (S', g) in $\mathbf{Obj}(\mathbf{Sets}_{/B})$ consisting of a set S' and a map $g: S' \rightarrow B$;

such that

$$\underbrace{\mathbf{forget}(S, f)}_{\mathbf{def} \equiv S} = \underbrace{\mathbf{forget}(S', g)}_{\mathbf{def} \equiv S'}.$$

Thus the objects of $\mathbf{Sets}_{/A} \times_{\mathbf{Sets}} \mathbf{Sets}_{/B}$ are the same as spans from A to B .

- *Morphisms.* A morphism of $\mathbf{Sets}_{/A} \times_{\mathbf{Sets}} \mathbf{Sets}_{/B}$ from (S, f, g) to (S', f', g') consists of a pair of morphisms

$$\begin{aligned} \phi: S &\rightarrow S' \\ \psi: S &\rightarrow S' \end{aligned}$$

such that the diagrams

$$\begin{array}{ccc} S & \xrightarrow{\phi} & S' \\ f \searrow & & \swarrow f' \\ & A & \end{array} \quad \begin{array}{ccc} S & \xrightarrow{\psi} & S' \\ g \searrow & & \swarrow g' \\ & B & \end{array}$$

such that

$$\underbrace{\mathbf{forget}(\phi)}_{\mathbf{def} \equiv \phi} = \underbrace{\mathbf{forget}(\psi)}_{\mathbf{def} \equiv \psi}.$$

Thus the morphisms of $\mathbf{Sets}_{/A} \times_{\mathbf{Sets}} \mathbf{Sets}_{/B}$ are also the same as morphisms of spans from (S, f, g) to (S, f', g') .

- *Identities and Composition.* The identities and composition of $\mathbf{Sets}_{/A} \times_{\mathbf{Sets}} \mathbf{Sets}_{/B}$ are also the same as those in $\mathbf{Span}(A, B)$.

This finishes the proof. \square

2.2 The Bicategory of Spans

Definition 2.2.1.1. The **bicategory of spans** is the bicategory \mathbf{Span} where

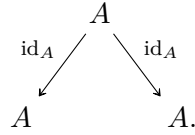
- *Objects.* The objects of \mathbf{Span} are sets;
- *Hom-Categories.* For each $A, B \in \mathbf{Obj}(\mathbf{Span})$, we have

$$\mathbf{Hom}_{\mathbf{Span}}(A, B) \stackrel{\text{def}}{=} \mathbf{Span}(A, B);$$

- *Identities.* For each $A \in \mathbf{Obj}(\mathbf{Span})$, the unit functor

$$\mathbb{K}_A^{\mathbf{Span}}: \mathbf{pt} \rightarrow \mathbf{Span}(A, A)$$

of \mathbf{Span} at A is the functor picking the span $(A, \text{id}_A, \text{id}_A)$:

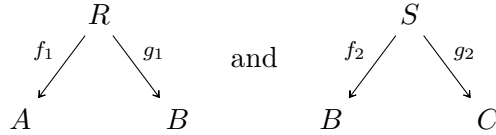


- *Composition.* For each $A, B, C \in \mathbf{Obj}(\mathbf{Span})$, the composition bifunctor

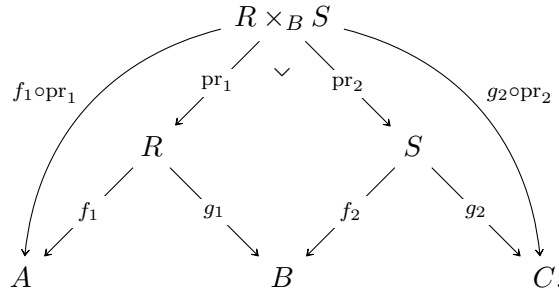
$$\circ_{A,B,C}^{\mathbf{Span}}: \mathbf{Span}(B, C) \times \mathbf{Span}(A, B) \rightarrow \mathbf{Span}(A, C)$$

of \mathbf{Span} at (A, B, C) is the bifunctor where

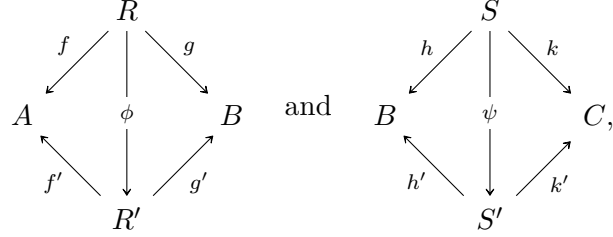
- *Action on Objects.* The composition of two spans



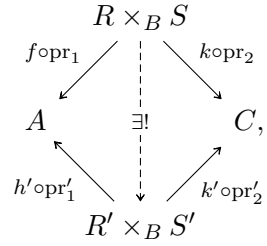
is the span $(R \times_B S, f_1 \circ \text{pr}_1, g_2 \circ \text{pr}_2)$, constructed as in the diagram



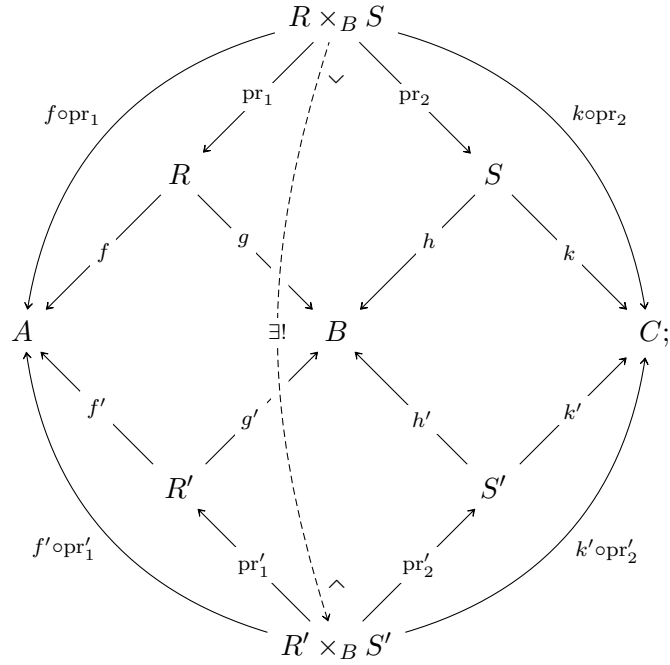
- *Action on Morphisms.* The horizontal composition of 2-morphisms is defined via functoriality of pullbacks: given morphisms of spans



their horizontal composition is the morphism of spans



constructed as in the diagram



- *Associators and Unitors.* The associator and unitors are defined using the universal property of the pullback.

2.3 The Monoidal Bicategory of Spans

2.4 The Double Category of Spans

Definition 2.4.1.1. The **double category of spans** is the double category Span^{dbl} where

- *Objects.* The objects of Span^{dbl} are sets;
- *Vertical Morphisms.* The vertical morphisms of Span^{dbl} are functions $f: A \rightarrow B$;
- *Horizontal Morphisms.* The horizontal morphisms of Span^{dbl} are spans $(S, \phi, \psi): A \dashv X$;
- *2-Morphisms.* A 2-cell

$$\begin{array}{ccc}
 A & \xrightarrow{(R, \phi_R, \psi_R)} & B \\
 f \downarrow & \Downarrow \alpha & \downarrow g \\
 X & \xrightarrow{(S, \phi_S, \psi_S)} & Y
 \end{array}$$

of Span^{dbl} is a morphism of spans from the span

$$\begin{array}{ccc}
 & R & \\
 \phi_R \swarrow & & \searrow \psi_R \\
 A & & B \\
 & & \searrow g \\
 & & Y
 \end{array}$$

to the span

$$\begin{array}{ccccc}
 & & A \times_X S & & \\
 & \swarrow & \downarrow \vee & \searrow & \\
 & A & & S & \\
 f \swarrow & & f \searrow & \phi_S \swarrow & \searrow \psi_S \\
 X & & X & & Y;
 \end{array}$$

- *Horizontal Identities.* The horizontal unit functor

$$\mathbb{H}^{\text{Span}^{\text{dbl}}} : (\text{Span}^{\text{dbl}})_0 \rightarrow (\text{Span}^{\text{dbl}})_1$$

of Span^{dbl} is the functor where

- *Action on Objects.* For each $A \in \text{Obj}((\text{Span}^{\text{dbl}})_0)$, we have

$$\mathbb{H}_A \stackrel{\text{def}}{=} (A, \text{id}_A, \text{id}_A),$$

as in the diagram

$$\begin{array}{ccc} & A & \\ \text{id}_A \swarrow & & \searrow \text{id}_A \\ A & & A; \end{array}$$

- *Action on Morphisms.* For each vertical morphism $f: A \rightarrow B$ of Span^{dbl} , i.e. each map of sets f from A to B , the identity 2-morphism

$$\begin{array}{ccc} A & \xrightarrow{\mathbb{H}_A} & A \\ f \downarrow & \Downarrow \mathbb{H}_f & \downarrow f \\ B & \xrightarrow{\mathbb{H}_B} & B \end{array}$$

of f is the morphism of spans from

$$\begin{array}{ccc} & A & \\ \text{id}_A \swarrow & & \searrow \text{id}_A \\ A & & A \xrightarrow{f} B \end{array}$$

to

$$\begin{array}{ccccc} & & A \times_B B & & \\ & \swarrow & \downarrow \vee & \searrow & \\ & A & & B & \\ f \swarrow & & & & \searrow \text{id}_B \\ B & & B & & B \end{array}$$

given by the isomorphism $A \xrightarrow{\cong} A \times_B B$;

- *Vertical Identities.* For each $A \in \text{Obj}(\text{Span}^{\text{dbl}})$, we have

$$\text{id}_A^{\text{Span}^{\text{dbl}}} \stackrel{\text{def}}{=} \text{id}_A;$$

- *Identity 2-Morphisms.* For each horizontal morphism $R: A \rightrightarrows B$ of Span^{dbl} , the identity 2-morphism

$$\begin{array}{ccc} A & \xrightarrow{S} & B \\ \text{id}_A \downarrow & \Downarrow \text{id}_S & \downarrow \text{id}_B \\ A & \xrightarrow{S} & B \end{array}$$

of R is the morphism of spans from

$$\begin{array}{ccc} & S & \\ \phi_S \swarrow & & \searrow \psi_S \\ A & & B \\ & & \Downarrow \text{id}_B \\ & & B \end{array}$$

to

$$\begin{array}{ccccc} & & A \times_A S & & \\ & \swarrow & \downarrow \vee & \searrow & \\ & A & & S & \\ \text{id}_A \Downarrow & & \text{id}_A \Downarrow & \swarrow \phi_S & \searrow \psi_S \\ A & & A & & B \end{array}$$

given by the isomorphism $S \xrightarrow{\cong} A \times_A S$;

- *Horizontal Composition.* The horizontal composition functor

$$\odot^{\text{Span}^{\text{dbl}}}: (\text{Span}^{\text{dbl}})_1 \times_{(\text{Span}^{\text{dbl}})_0} (\text{Span}^{\text{dbl}})_1 \rightarrow (\text{Span}^{\text{dbl}})_1$$

of Span^{dbl} is the functor where

- *Action on Objects.* For each composable pair

$$A \xrightarrow{(R, \phi_R, \psi_R)} B \xrightarrow{(S, \phi_S, \psi_S)} C$$

of horizontal morphisms of $\mathbf{Span}^{\text{dbl}}$, we have

$$(S, \phi_S, \psi_S) \odot (R, \phi_R, \psi_R) \stackrel{\text{def}}{=} S \circ_{A,B,C}^{\text{Span}} R,$$

where $S \circ_{A,B,C}^{\text{Span}} R$ is the composition of (R, ϕ_R, ψ_R) and (S, ϕ_S, ψ_S) defined as in [Definition 2.2.1.1](#);

- *Action on Morphisms.* For each horizontally composable pair

$$\begin{array}{ccc} A & \xrightarrow{(R, \phi_R, \psi_R)} & B \\ f \downarrow & \Downarrow \alpha & \downarrow g \\ X & \xrightarrow{(T, \phi_T, \psi_T)} & Y \end{array} \quad \begin{array}{ccc} B & \xrightarrow{(S, \phi_S, \psi_S)} & C \\ g \downarrow & \Downarrow \beta & \downarrow h \\ Y & \xrightarrow{(U, \phi_U, \psi_U)} & Z \end{array}$$

of 2-morphisms of $\mathbf{Span}^{\text{dbl}}$,

- *Vertical Composition of 1-Morphisms.* For each composable pair $A \xrightarrow{F} B \xrightarrow{G} C$ of vertical morphisms of $\mathbf{Span}^{\text{dbl}}$, i.e. maps of sets, we have

$$g \circ^{\mathbf{Span}^{\text{dbl}}} f \stackrel{\text{def}}{=} g \circ f;$$

- *Vertical Composition of 2-Morphisms.* For each vertically composable pair

$$\begin{array}{ccc} A & \xrightarrow{(R, \phi_R, \psi_R)} & X \\ f \downarrow & \Downarrow \alpha & \downarrow g \\ B & \xrightarrow{(S, \phi_S, \psi_S)} & Y \end{array} \quad \begin{array}{ccc} B & \xrightarrow{(S, \phi_S, \psi_S)} & Y \\ h \downarrow & \Downarrow \beta & \downarrow k \\ C & \xrightarrow{(T, \phi_T, \psi_T)} & Z \end{array}$$

of 2-morphisms of $\mathbf{Span}^{\text{dbl}}$,

- *Associators and Unitors.* The associator and unitors of $\mathbf{Span}^{\text{dbl}}$ are defined using the universal property of the pullback.

2.5 Properties of The Bicategory of Spans

Proposition 2.5.1.1. Let $\lambda = (A \xleftarrow{f} S \xrightarrow{g} B)$ be a span.

1. *Self-Duality.*
2. *Isomorphisms in Span.*

3. *Equivalences in Span.*

4. *Adjunctions in Span.* Let A and B be sets.¹⁰

(a) We have a natural bijection

$$\left\{ \begin{array}{c} \text{Adjunctions in Span} \\ \text{from } A \text{ to } B \end{array} \right\} \cong \left\{ \begin{array}{c} \text{Spans } A \xleftarrow{f} S \xrightarrow{g} B \\ \text{from } A \text{ to } B \text{ with} \\ f \text{ an isomorphism} \end{array} \right\}.$$

(b) We have an equivalence of categories

$$\text{MapSpan}(A, B) \xrightarrow{\text{eq.}} \text{Sets}(A, B)_{\text{disc}},$$

where $\text{MapSpan}(A, B)$ is the full subcategory of $\text{Span}(A, B)$ spanned by the spans $A \xleftarrow{f} S \xrightarrow{g} B$ from A to B with f an isomorphism.

(c) We have a biequivalence of bicategories

$$\text{MapSpan} \xrightarrow{\text{eq.}} \text{Sets}_{\text{bidisc}},$$

where MapSpan is the sub-bicategory of Span whose Hom -categories are given by $\text{MapSpan}(A, B)$.

5. *Monads in Span.*

6. *Comonads in Span.*

7. *Monomorphisms in Span.*

8. *Epimorphisms in Span.*

9. *Existence of Right Kan Extensions.*

10. *Existence of Right Kan Lifts.*

11. *Closedness.*

Proof. Item 1, Self-Duality:

Item 2, Isomorphisms in Span:

Item 3, Equivalences in Span:

Item 4, Adjunctions in Span: We first prove *Item 4a*.

We proceed step by step:

¹⁰In the literature (e.g. [ref]),...are called maps and denoted by $\text{MapSpan}(A, B)$

1. *From Adjunctions in Span to Functions.* An adjunction in **Span** from A to B consists of a pair of spans

$$\begin{array}{ccc} & R & \\ f \swarrow & & \searrow g \\ A & & B \end{array} \quad \text{and} \quad \begin{array}{ccc} & S' & \\ h \swarrow & & \searrow k \\ B & & A \end{array}$$

together with maps

$$\begin{array}{ccccc} & A & & & \\ \text{id}_A \swarrow & & \searrow \text{id}_A & & \\ A & & \phi & & A \\ f \circ \text{pr}'_1 \swarrow & & \downarrow & & \searrow k \circ \text{pr}'_2 \\ & S \times_B S' & & & \end{array} \quad \text{and} \quad \begin{array}{ccccc} & S' \times_A S & & & \\ h \circ \text{pr}_1 \swarrow & & \searrow g \circ \text{pr}_2 & & \\ B & & \psi & & B \\ \text{id}_B \swarrow & & \downarrow & & \searrow \text{id}_B \\ & B & & & \end{array}$$

We claim that these conditions

2. *From Functions to Adjunctions in Rel.*
3. *Invertibility: From Functions to Adjunctions Back to Functions.*
4. *Invertibility: From Adjunctions to Functions Back to Adjunctions.*

We now proceed to the proof of **Item 4b**. For this, we will construct a functor

$$F : \text{Sets}(A, B)_{\text{disc}} \rightarrow \text{MapSpan}(A, B)$$

and prove it to be essentially surjective and fully faithful, and thus an equivalence by **Categories**, ?? of ??. Indeed, given a map $f : A \rightarrow B$, let $F(f)$ be the representable span associated to f of **Definition 5.1.1.1**, and let F send the unique (identity) morphism from f to itself to the identity morphism of $F(f)$ in $\text{MapSpan}(A, B)$. We now prove that F is fully faithful and essentially surjective:

1. *F Is Fully Faithful:* Given maps $f, g : A \rightrightarrows B$, we need to show that

$$\text{Hom}_{\text{MapSpan}(A, B)}(F(f), F(g)) = \begin{cases} \text{pt} & \text{if } f = g, \\ \emptyset & \text{otherwise.} \end{cases}$$

Indeed, a morphism from $F(f)$ to $F(g)$ takes the form

$$\begin{array}{ccc} & A & \\ \swarrow & \downarrow \phi & \searrow f \\ A & & B \\ \swarrow & \downarrow & \nearrow g \\ & A & \end{array}$$

From the relations $\text{id}_A = \text{id}_A \circ \phi$ and $f = g \circ \phi$, we see that $\phi = \text{id}_A$, and thus from the relation $f = g \circ \phi$ there is such a morphism iff $f = g$.

2. *F Is Essentially Surjective:* Let λ be a span of the form

$$\begin{array}{ccc} & S & \\ \swarrow \phi & & \searrow f \\ A & & B \end{array}$$

we claim that $\lambda \cong F(f \circ \phi^{-1})$. Indeed, we have morphisms

$$\begin{array}{ccc} & S & \\ \swarrow \phi & \downarrow \phi & \searrow f \\ A & & B \\ \swarrow & \downarrow & \nearrow f \circ \phi^{-1} \\ & A & \end{array} \quad \text{and} \quad \begin{array}{ccc} & A & \\ \swarrow & \downarrow \phi^{-1} & \searrow f \circ \phi^{-1} \\ A & & B \\ \swarrow & \downarrow & \nearrow f \\ & S & \end{array}$$

inverse to each other in $\text{MapSpan}(A, B)$, and thus $\lambda \cong F(f \circ \phi^{-1})$.

Finally, we prove **Item 4c**.

Item 5, *Monads in Span*:

Item 6, *Comonads in Span*:

Item 7, *Monomorphisms in Span*:

Item 8, *Epimorphisms in Span*:

Item 9, *Existence of Right Kan Extensions*:

Item 10, *Existence of Right Kan Lifts*:

Item 11, *Closedness*:

□

3 Limits of Spans

3.1 tmp2

$$\text{Hom}_{\text{Rel}(A, X)}(\text{Lan}_S(R), T) \cong \text{Hom}_{\text{Rel}(B, X)}(R, T \diamond S)$$

1. $\text{Lan}_S(R) \subset T$, i.e. if $a \sim_{\text{Lan}_S(R)} x$, then $a \sim_T x$.
2. $R \subset T \diamond S$, i.e. if $b \sim_R x$, then there exists some $a \in A$ such that $a \sim_S b$ and $b \sim_T x$.

3.2 tmp

$$\begin{array}{c}
 \begin{array}{ccc}
 & S & \\
 f \swarrow & & \searrow g \\
 A & & B
 \end{array} \\
 \\
 \begin{array}{ccc}
 \begin{array}{ccc}
 & S' & \\
 \phi_{S'} \swarrow & & \searrow \psi_{S'} \\
 B & & X
 \end{array} & \mapsto & \begin{array}{ccc}
 & S \times_B S' & \\
 f \circ \text{pr}_1 \swarrow & & \searrow \psi_{S'} \circ \text{pr}_2 \\
 A & & X
 \end{array} \\
 \\
 \begin{array}{ccc}
 & S'' & \\
 \phi_{S''} \swarrow & & \searrow \psi_{S''} \\
 A & & X
 \end{array} & \mapsto & \begin{array}{ccc}
 & ? & \\
 ? \swarrow & & \searrow ? \\
 B & & X
 \end{array}
 \end{array} \\
 \\
 \text{Hom}_{\text{Span}(A, X)}(S \times_B S', K) \cong \text{Hom}_{\text{Span}(B, X)}(S', R(K)) \\
 \\
 \begin{array}{ccc}
 \begin{array}{ccccc}
 & S \times_B S' & & & \\
 f \circ \text{pr}_1 \swarrow & & \downarrow \xi & & \searrow \psi_{S'} \circ \text{pr}_2 \\
 A & & S'' & & X \\
 \phi_{S''} \swarrow & & \downarrow & & \searrow \psi_{S''} \\
 & & & &
 \end{array} & & \begin{array}{ccccc}
 & S' & & & \\
 \phi_{S'} \swarrow & & \downarrow \xi^\dagger & & \searrow \psi_{S'} \\
 B & & \Pi_g(S'') & & X \\
 ? \swarrow & & \downarrow & & \searrow ? \\
 & & & &
 \end{array}
 \end{array}
 \end{array}$$

3.3 Left Kan Extensions

Let $\lambda = \left(A \xleftarrow{f} S \xrightarrow{g} B \right)$ be a span.

Proposition 3.3.1.1. The left Kan extension

$$\text{Lan}_\lambda: \text{Span}(A, X) \rightarrow \text{Span}(B, X)$$

along λ in Span exists and is the functor given on objects by sending a span λ' in $\text{Span}(A, X)$ as in

$$\begin{array}{ccc}
 & S' & \\
 \phi \swarrow & & \searrow \psi \\
 A & & X
 \end{array}$$

to the span

$$\text{Lan}_\lambda(\lambda') \stackrel{\text{def}}{=} (\text{Lan}_\lambda(S'), \text{Lan}_\lambda(\phi), \text{Lan}_\lambda(\psi)),$$

in $\text{Span}(B, X)$ where

- The set $\text{Lan}_\lambda(S')$ is given by

$$\begin{aligned} \text{Lan}_\lambda(S') &\stackrel{\text{def}}{=} \Sigma_g(S') \\ &\stackrel{\text{def}}{=} S' \end{aligned}$$

where $\Sigma_g(S')$ is the dependent sum of $\phi: S' \rightarrow A$ along g of **Fibred Sets, Definition 2.3.1.1**;

- The map $\text{Lan}_\lambda(\phi): \text{Lan}_\lambda(S') \rightarrow B$ is given by $\Sigma_g(\phi)$;
- The map $\text{Lan}_\lambda(\psi): \text{Lan}_\lambda(S') \rightarrow X$ is given by ψ .

Proof.

□

3.4 Right Kan Extensions

Let $\lambda = \left(A \xleftarrow{f} S \xrightarrow{g} B \right)$ be a span.

Proposition 3.4.1.1. The right Kan extension

$$\text{Ran}_\lambda: \text{Span}(A, X) \rightarrow \text{Span}(B, X)$$

along λ in Span exists and is the functor given on objects by sending a span λ' in $\text{Span}(A, X)$ as in

$$\begin{array}{ccc} & S' & \\ \phi_{S'} \swarrow & & \searrow \psi_{S'} \\ A & & X \end{array}$$

to the span

$$\text{Ran}_\lambda(\lambda') \stackrel{\text{def}}{=} (\text{Ran}_\lambda(S'), \text{Ran}_\lambda(\phi_{S'}), \text{Ran}_\lambda(\psi_{S'})),$$

in $\text{Span}(B, X)$ where

- The set $\text{Ran}_\lambda(S')$ is given by

$$\text{Ran}_\lambda(S') \stackrel{\text{def}}{=} \coprod_{b \in B} \coprod_{s \in g^{-1}(b)} \phi_{S'}^{-1}(f(s));$$

- The map $\text{Ran}_\lambda(\phi_{S'}): \text{Ran}_\lambda(S') \rightarrow B$ is given by

$$[\text{Ran}_\lambda(\phi_{S'})](b, (s'_s)_{s \in g^{-1}(b)}) \stackrel{\text{def}}{=} b;$$

for each $(b, (s'_s)_{s \in g^{-1}(b)}) \in \text{Ran}_\lambda(S')$;

- The map $\text{Ran}_\lambda(\psi_{S'}): \text{Ran}_\lambda(S') \rightarrow X$ is given by

$$[\text{Ran}_\lambda(\psi_{S'})](b, (s'_s)_{s \in g^{-1}(b)}) \stackrel{\text{def}}{=} \psi_{S'}(s'_i)$$

for each $(b, (s'_s)_{s \in g^{-1}(b)}) \in \text{Ran}_\lambda(S')$, where the i in s'_i denotes any $s \in g^{-1}(b)$, as we have $\psi_{S'}(s'_i) = \psi_{S'}(s'_j)$ for all $s \in g^{-1}(b)$.¹¹

Proof.

□

3.5 Right Kan Lifts

(Although right Kan lifts aren't really limits, this is probably the most appropriate to place this section.)

Let $\lambda = \left(A \xleftarrow{f} S \xrightarrow{g} B \right)$ be a span.

Proposition 3.5.1.1. The right Kan lift

$$\text{Rift}_\lambda: \text{Span}(X, B) \rightarrow \text{Span}(X, A)$$

along λ in Span exists and is the functor given on objects by sending a span λ' in $\text{Span}(X, B)$ as in

$$\begin{array}{ccc} & S' & \\ \phi \swarrow & & \searrow \psi \\ X & & B \end{array}$$

to the span

$$\text{Rift}_\lambda(\lambda') \stackrel{\text{def}}{=} (\text{Rift}_\lambda(S'), \text{Rift}_\lambda(\phi), \text{Rift}_\lambda(\psi)),$$

in $\text{Span}(X, A)$ where

- The set $\text{Rift}_\lambda(S')$ is given by

$$\text{Rift}_\lambda(S') \stackrel{\text{def}}{=} \Pi_f(S'),$$

where $\Pi_f(S')$ is the dependent product of $\psi: S' \rightarrow B$ along f of **Fibred Sets**, [Definition 2.3.1.1](#);

¹¹Indeed

- The map $\text{Rift}_\lambda(\phi): \text{Rift}_\lambda(S') \rightarrow X$ is given by ϕ ;
- The map $\text{Rift}_\lambda(\psi): \text{Rift}_\lambda(S') \rightarrow A$ is given by $\Pi_f(\psi)$.

Proof.

□

4 Colimits of Spans

5 Constructions With Spans

5.1 Representable Spans

Definition 5.1.1.1. Let $f: A \rightarrow B$ be a function.

- The **representable span associated to f** is the span

$$\begin{array}{ccc} & A & \\ \text{id}_A \swarrow & & \searrow f \\ A & & B \end{array}$$

from A to B .

- The **corepresentable span associated to f** is the span

$$\begin{array}{ccc} & A & \\ f \swarrow & & \searrow \text{id}_A \\ B & & A \end{array}$$

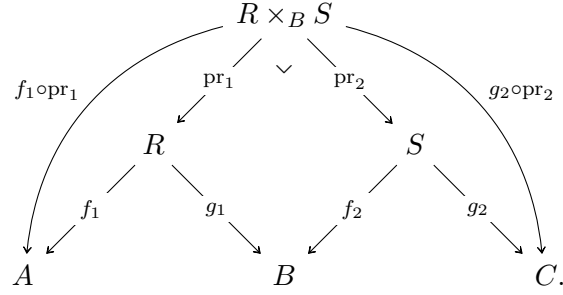
from B to A .

5.2 Composition of Spans

Definition 5.2.1.1. The **composition** of two spans

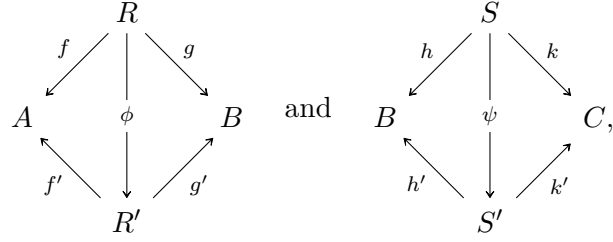
$$\begin{array}{ccc} & R & \\ f_1 \swarrow & & \searrow g_1 \\ A & & B \end{array} \quad \text{and} \quad \begin{array}{ccc} & S & \\ f_2 \swarrow & & \searrow g_2 \\ B & & C \end{array}$$

is the span $(R \times_B S, f_1 \circ \text{pr}_1, g_2 \circ \text{pr}_2)$, constructed as in the diagram

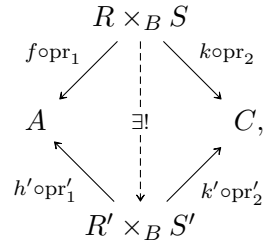


5.3 Horizontal Composition of Morphisms of Spans

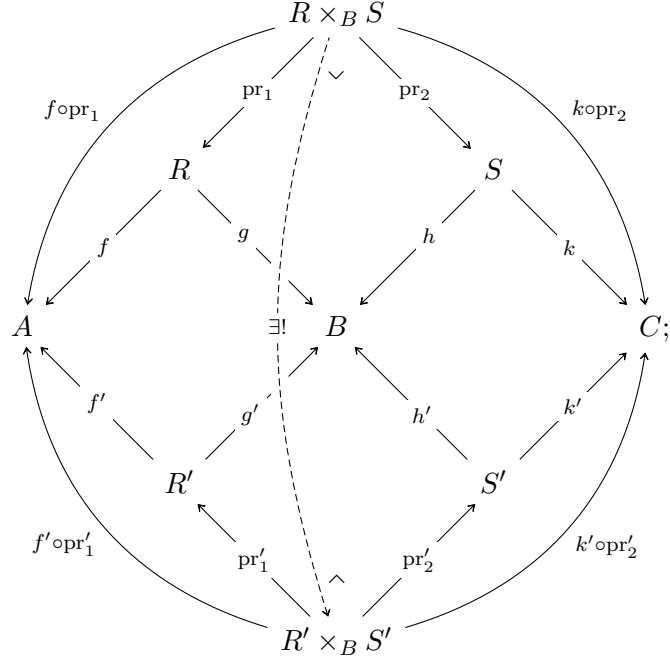
Definition 5.3.1.1. The **horizontal composition** of a pair of 2-morphisms of spans



is the morphism of spans



constructed as in the diagram



5.4 Properties of Composition of Spans

Proposition 5.4.1.1. Let $\lambda = \left(A \xleftarrow{f} S \xrightarrow{g} B \right)$ be a span.

1. *Functoriality.*

Proof.

□

5.5 The Inverse of a Span

6 Functoriality of Spans

6.1 Direct Images

6.2 Functoriality of Spans on Powersets

7 Comparison of Spans to Functions and Relations

7.1 Comparison to Functions

Proposition 7.1.1.1. We have a pseudofunctor

$$\iota: \mathbf{Sets}_{\text{bidisc}} \rightarrow \mathbf{Span}$$

from $\mathbf{Sets}_{\text{bidisc}}$ to \mathbf{Span} where

- *Action on Objects.* For each $A \in \text{Obj}(\mathbf{Sets}_{\text{bidisc}})$, we have

$$\iota(A) \stackrel{\text{def}}{=} A;$$

- *Action on Hom-Categories.* For each $A, B \in \text{Obj}(\mathbf{Sets}_{\text{bidisc}})$, the action on Hom-categories

$$\iota_{A,B}: \mathbf{Sets}(A, B)_{\text{disc}} \rightarrow \mathbf{Span}(A, B)$$

of ι at (A, B) is the functor defined on objects by sending a function $f: A \rightarrow B$ to the span

$$\begin{array}{ccc} & A & \\ \text{id}_A \swarrow & & \searrow f \\ A & & B \end{array}$$

from A to B .

- *Strict Unity Constraints.* For each $A \in \text{Obj}(\mathbf{Sets}_{\text{bidisc}})$, the strict unity constraint

$$\iota_A^0: \text{id}_{\iota(A)} \Longrightarrow \iota(\text{id}_A)$$

of ι at A is given by the identity morphism of spans

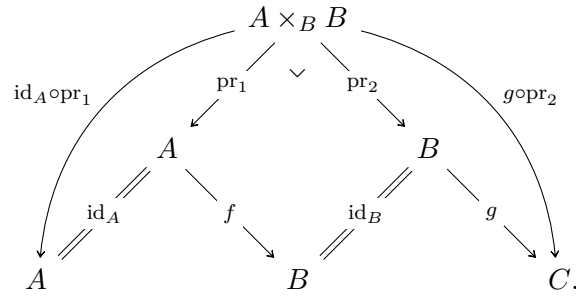
$$\begin{array}{ccccc} & & A & & \\ & \text{id}_A \swarrow & \parallel & \searrow \text{id}_A & \\ A & & \text{id} & & A, \\ & \swarrow \text{id}_A & \parallel & \searrow \text{id}_A & \\ & & A & & \end{array}$$

as indeed $\text{id}_{\iota(A)} = \iota(\text{id}_A)$;

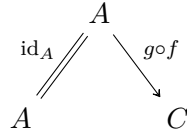
- *Pseudofunctoriality Constraints.* For each $A, B, C \in \text{Obj}(\text{Sets}_{\text{bidisc}})$, each $f \in \text{Hom}_{\text{Sets}_{\text{bidisc}}}(A, B)$, and each $g \in \text{Hom}_{\text{Sets}_{\text{bidisc}}}(B, C)$, the pseudofunctoriality constraint

$$\iota_{g,f}^2: \iota(g) \circ \iota(f) \Longrightarrow \iota(g \circ f)$$

of ι at (f, g) is the morphism of spans from the span



to the span



given by the isomorphism $A \times_B B \cong A$.

Proof. Omitted. □

7.2 Comparison to Relations: From Span to **Rel**

7.2.1 Relations Associated to Spans

Let $\lambda = \left(A \xleftarrow{f} S \xrightarrow{g} B \right)$ be a span.

Definition 7.2.1.1. The **relation associated to λ** is the relation

$$S(\lambda): A \dashv B$$

from A to B defined as follows:

- Viewing relations from A to B as functions $A \times B \rightarrow \{\text{true}, \text{false}\}$, we

define

$$\iota_{A,B}(S)_b^a \stackrel{\text{def}}{=} \begin{cases} \text{true} & \text{if there exists } x \in S \text{ such} \\ & \text{that } a = f(x) \text{ and } b = g(x), \\ \text{false} & \text{otherwise} \end{cases}$$

for each $(a, b) \in A \times B$.

- Viewing relations from A to B as functions $A \rightarrow \mathcal{P}(B)$, we define

$$[\iota_{A,B}(S)](a) \stackrel{\text{def}}{=} g(f^{-1}(a))$$

for each $a \in A$.

- Viewing relations from A to B as subsets of $A \times B$, we define

$$\iota_{A,B}(S) \stackrel{\text{def}}{=} \{(f(x), g(x)) \mid x \in S\}.$$

Proposition 7.2.1.2. Let $\lambda = \left(A \xleftarrow{f} S \xrightarrow{g} B \right)$ be a span.

1. *Interaction With Identities.*
2. *Interaction With Composition.*
3. *Interaction With Inverses.*

Proof.

□

7.2.2 The Comparison Functor from **Span** to **Rel**

Proposition 7.2.2.1. We have a pseudofunctor

$$\iota: \mathbf{Span} \rightarrow \mathbf{Rel}$$

from **Span** to **Rel** where

- *Action on Objects.* For each $A \in \text{Obj}(\mathbf{Span})$, we have

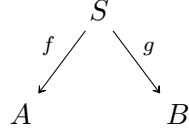
$$\iota(A) \stackrel{\text{def}}{=} A;$$

- *Action on Hom-Categories.* For each $A, B \in \text{Obj}(\mathbf{Span})$, the action on Hom-categories

$$\iota_{A,B}: \mathbf{Span}(A, B) \rightarrow \mathbf{Rel}(A, B)$$

of ι at (A, B) is the functor where

- *Action on Objects.* Given a span



from A to B , the image

$$\iota_{A,B}(S): A \dashv B$$

of S by ι is the relation from A to B defined as follows:

- * Viewing relations as functions $A \times B \rightarrow \{\text{true}, \text{false}\}$, we define

$$\iota_{A,B}(S)_b^a \stackrel{\text{def}}{=} \begin{cases} \text{true} & \text{if there exists } x \in S \\ & \text{such that } a = f(x) \\ & \text{and } b = g(x), \\ \text{false} & \text{otherwise} \end{cases}$$

for each $(a, b) \in A \times B$;

- * Viewing relations as functions $A \rightarrow \mathcal{P}(B)$, we define

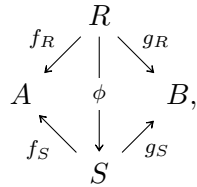
$$[\iota_{A,B}(S)](a) \stackrel{\text{def}}{=} g(f^{-1}(a))$$

for each $a \in A$;

- * Viewing relations as subsets of $A \times B$, we define

$$\iota_{A,B}(S) \stackrel{\text{def}}{=} \{(f(x), g(x)) \mid x \in S\}.$$

- *Action on Morphisms.* Given a morphism of spans



we have a corresponding inclusion of relations

$$\iota_{A,B}(\phi): \iota_{A,B}(R) \subset \iota_{A,B}(S),$$

since we have $a \sim_{\iota_{A,B}(R)} b$ iff there exists $x \in R$ such that $a = f_R(x)$ and $b = g_R(x)$, in which case we then have

$$\begin{aligned} a &= f_R(x) \\ &= f_S(\phi(x)), \\ b &= g_R(x) \\ &= g_S(\phi(x)), \end{aligned}$$

so that $a \sim_{\iota_{A,B}(S)} b$, and thus $\iota_{A,B}(R) \subset \iota_{A,B}(S)$.

Proof. Omitted. □

7.3 Comparison to Relations: From **Rel** to **Span**

Proposition 7.3.1.1. We have a lax functor

$$(\iota, \iota^2, \iota^0): \mathbf{Rel} \rightarrow \mathbf{Span}$$

from **Rel** to **Span** where

- *Action on Objects.* For each $A \in \text{Obj}(\mathbf{Span})$, we have

$$\iota(A) \stackrel{\text{def}}{=} A;$$

- *Action on Hom-Categories.* For each $A, B \in \text{Obj}(\mathbf{Span})$, the action on Hom-categories

$$\iota_{A,B}: \mathbf{Rel}(A, B) \rightarrow \mathbf{Span}(A, B)$$

of ι at (A, B) is the functor where

- *Action on Objects.* Given a relation $R: A \dashv B$ from A to B , we define a span

$$\iota_{A,B}(R): A \dashv B$$

from A to B by

$$\iota_{A,B}(R) \stackrel{\text{def}}{=} (R, \upharpoonright \text{pr}_1 R, \upharpoonright \text{pr}_2 R),$$

where $R \subset A \times B$ and $\upharpoonright \text{pr}_1 R$ and $\upharpoonright \text{pr}_2 R$ are the restriction of the projections

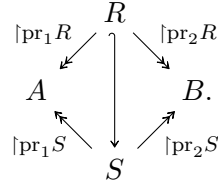
$$\begin{aligned} \text{pr}_1: A \times B &\rightarrow A, \\ \text{pr}_2: A \times B &\rightarrow B \end{aligned}$$

to R ;

- *Action on Morphisms.* Given an inclusion $\phi: R \subset S$ of relations, we have a corresponding morphism of spans

$$\iota_{A,B}(\phi): \iota_{A,B}(R) \rightarrow \iota_{A,B}(S)$$

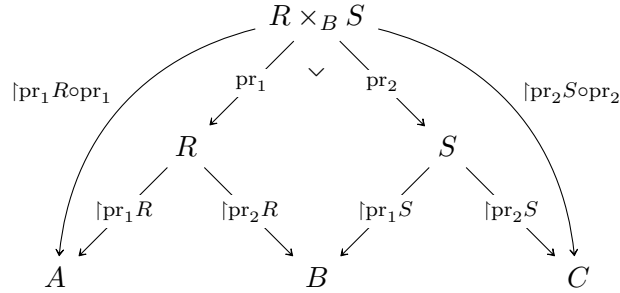
as in the diagram



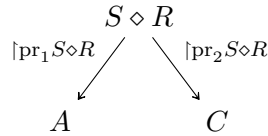
- *The Lax Functoriality Constraints.* The lax functoriality constraint

$$\iota_{R,S}^2: \iota(S) \circ \iota(R) \Longrightarrow \iota(S \diamond R)$$

of ι at (R, S) is given by the morphism of spans from



to



given by the natural inclusion $R \times_B S \hookrightarrow S \diamond R$, since we have

$$R \times_B S = \{((a_R, b_R), (b_S, c_S)) \in R \times S \mid b_R = b_S\};$$

$$S \diamond R = \left\{ (a, c) \in A \times C \mid \begin{array}{l} \text{there exists some } b \in B \text{ such} \\ \text{that } (a, b) \in R \text{ and } (b, c) \in S \end{array} \right\};$$

- *The Lax Unity Constraints.* The lax unity constraint¹²

$$\iota_A^0: \underbrace{\text{id}_{\iota(A)}}_{(A, \text{id}_A, \text{id}_A)} \Longrightarrow \underbrace{\iota(\chi_A)}_{(\Delta_A, \text{pr}_1 \Delta_A, \text{pr}_2 \Delta_A)}$$

of ι at A is given by the diagonal morphism of A , as in the diagram

$$\begin{array}{ccc} & A & \\ \text{id}_A \swarrow & \downarrow \delta_A & \searrow \text{id}_A \\ A & & A \\ \text{pr}_1 \Delta_A \swarrow & \downarrow & \searrow \text{pr}_2 \Delta_A \\ & \Delta_A & \end{array}$$

Proof. Omitted. □

7.4 Comparison to Relations: The Wehrheim–Woodward Construction

7.5 Comparison to Multirelations

Remark 7.5.1.1. The pseudofunctor of [Proposition 7.2.2.1](#) and the lax functor of [Proposition 7.3.1.1](#) fail to be equivalences of bicategories. This happens essentially because a span $(S, f, g): A \dashv B$ from A to B may relate elements $a \in A$ and $b \in B$ by more than one element, e.g. there could be $s \neq s' \in S$ such that $a = f(s) = f(s')$ and $b = g(s) = g(s')$. Thus, in a sense, spans may be thought of as “relations with multiplicity”. And indeed, if instead of considering relations from A to B , i.e. functions

$$R: A \times B \rightarrow \{\text{true}, \text{false}\}$$

from $A \times B$ to $\{\text{true}, \text{false}\} \cong \{0, 1\}$, we consider functions

$$R: A \times B \rightarrow \mathbb{N} \cup \{\infty\}$$

from $A \times B$ to $\mathbb{N} \cup \{\infty\}$, then we obtain the notion of a **multirelation from A to B** , and these turn out to assemble together with sets into a bicategory **MRel** that is biequivalent to **Span**; see [\[some-algebraic-laws-for-spans-and-their-connections-with-mul](#)

¹²Which is in fact strong, as δ_A is an isomorphism.

7.6 Comparison to Relations via Double Categories

Remark 7.6.1.1. There are double functors between the double categories $\mathbf{Rel}^{\text{dbl}}$ and $\mathbf{Span}^{\text{dbl}}$ analogous to the functors of [Propositions 7.2.2.1](#) and [7.3.1.1](#), assembling moreover into a strict-lax adjunction of double functors; see [\[higher-dimensional-categories\]](#).

Appendices

A Other Chapters

Set Theory

1. [Sets](#)
2. [Constructions With Sets](#)
3. [Pointed Sets](#)
4. [Tensor Products of Pointed Sets](#)
5. Indexed and Fibred Sets
6. [Relations](#)
7. [Spans](#)
8. [Posets](#)

Category Theory

9. [Categories](#)
10. [Constructions With Categories](#)
11. [Kan Extensions](#)

Bicategories

12. [Bicategories](#)
13. [Internal Adjunctions](#)

Internal Category Theory

14. [Internal Categories](#)

Cyclic Stuff

15. [The Cycle Category](#)

Cubical Stuff

16. [The Cube Category](#)

Globular Stuff

17. [The Globe Category](#)

Cellular Stuff

18. [The Cell Category](#)

Monoids

19. [Monoids](#)
20. [Constructions With Monoids](#)

Monoids With Zero

21. [Monoids With Zero](#)
22. [Constructions With Monoids With Zero](#)

Groups

23. [Groups](#)
24. [Constructions With Groups](#)

Hyper Algebra

25. Hypermonoids

26. Hypergroups

27. Hypersemirings and Hyperrings

28. Quantaes

Near-Rings

29. Near-Semirings

30. Near-Rings

Real Analysis

31. Real Analysis in One Variable

32. Real Analysis in Several Variables

Measure Theory

33. Measurable Spaces

34. Measures and Integration

Probability Theory

34. Probability Theory

Stochastic Analysis

35. Stochastic Processes, Martingales, and Brownian Motion

36. Itô Calculus

37. Stochastic Differential Equations

Differential Geometry

38. Topological and Smooth Manifolds

Schemes

39. Schemes