Sets

December 24, 2023

This chapter (will eventually) contain material on axiomatic set theory, as well as a couple other things.

Contents

1 The Enrichment of Sets in Classical Truth Values

1.1 (-2)-Categories

Definition 1.1.1 ► (-2)-Categories

A (-2)**-category** is the "necessarily true" truth value.[1,2,3]

[1] Thus, there is only one (-2)-category.

[2] A $(-n)$-category for $n = 3, 4, \ldots$ is also the "necessarily true" truth value, coinciding with a (-2)-category.

[3] For motivation, see [BS10, p. 13].

1.2 (-1)-Categories

Definition 1.2.1 ► (-1)-Categories

A (-1)**-category** is a classical truth value.

Remark 1.2.2 ► Motivation for (-1)-Categories

[1] (-1)-categories should be thought of as being "categories enriched in (-2)-categories", having a collection of objects and, for each pair of objects, a Hom-object $\mathrm{Hom}(x, y)$ that is a (-2)-category (i.e. trivial).

Therefore, a (-1)-category C is either ([BS10, pp. 33–34]):

1. *Empty*, having no objects;
2. *Contractible*, having a collection of objects $\{a, b, c, \ldots\}$, but with $\mathrm{Hom}_C(a, b)$ being a (-2)-category (i.e. trivial) for all $a, b \in \mathrm{Obj}(C)$, forcing all objects of C to be uniquely isomorphic to each other.

As such, there are only two (-1)-categories, up to equivalence:

- The (-1)-category false (the empty one);
- The (-1)-category true (the contractible one).

[1] For more motivation, see [BS10, p. 13].

Definition 1.2.3 ► The Poset of Truth Values

The **poset of truth values**[1] is the poset $(\{\mathrm{true}, \mathrm{false}\}, \preceq)$[2] consisting of

- *The Underlying Set.* The set $\{\mathrm{true}, \mathrm{false}\}$ whose elements are the truth values true and false;
- *The Partial Order.* The partial order

$$\preceq\colon \{\mathrm{true}, \mathrm{false}\} \times \{\mathrm{true}, \mathrm{false}\} \longrightarrow \{\mathrm{true}, \mathrm{false}\}$$

on $\{\mathrm{true}, \mathrm{false}\}$ defined by[3]

$$\begin{aligned} \mathrm{false} \preceq \mathrm{false} &\overset{\mathrm{def}}{=} \mathrm{true}, \\ \mathrm{true} \preceq \mathrm{false} &\overset{\mathrm{def}}{=} \mathrm{false}, \\ \mathrm{false} \preceq \mathrm{true} &\overset{\mathrm{def}}{=} \mathrm{true}, \\ \mathrm{true} \preceq \mathrm{true} &\overset{\mathrm{def}}{=} \mathrm{true}. \end{aligned}$$

[1] *Further Terminology:* Also called the **poset of (-1)-categories**.
[2] *Further Notation:* Also written $\{\mathrm{t}, \mathrm{f}\}$.
[3] This partial order coincides with logical implication.

Proposition 1.2.4 ► Cartesian Closedness of the Poset of Truth Values

The poset of truth values $\{\mathrm{t}, \mathrm{f}\}$ is Cartesian closed with product given by[1]

$$
\begin{aligned}
\mathrm{t} \times \mathrm{t} &= \mathrm{t}, \\
\mathrm{t} \times \mathrm{f} &= \mathrm{f}, \\
\mathrm{f} \times \mathrm{t} &= \mathrm{f}, \\
\mathrm{f} \times \mathrm{f} &= \mathrm{f},
\end{aligned}
$$

and internal Hom $\mathbf{Hom}_{\{\mathrm{t},\mathrm{f}\}}$ given by the partial order of $\{\mathrm{t}, \mathrm{f}\}$, i.e. by

$$
\begin{aligned}
\mathbf{Hom}_{\{\mathrm{t},\mathrm{f}\}}(\mathrm{t}, \mathrm{t}) &= \mathrm{t}, \\
\mathbf{Hom}_{\{\mathrm{t},\mathrm{f}\}}(\mathrm{t}, \mathrm{f}) &= \mathrm{f}, \\
\mathbf{Hom}_{\{\mathrm{t},\mathrm{f}\}}(\mathrm{f}, \mathrm{t}) &= \mathrm{t}, \\
\mathbf{Hom}_{\{\mathrm{t},\mathrm{f}\}}(\mathrm{f}, \mathrm{f}) &= \mathrm{t}.
\end{aligned}
$$

[1] Note that $\times$ coincides with the "and" operator, while $\mathbf{Hom}_{\{\mathrm{t},\mathrm{f}\}}$ coincides with the logical implication operator.

Proof 1.2.5 ► Proof of Proposition 1.2.4

Existence of Products

We claim that the products $\mathrm{t} \times \mathrm{t}$, $\mathrm{t} \times \mathrm{f}$, $\mathrm{f} \times \mathrm{t}$, and $\mathrm{f} \times \mathrm{f}$ satisfy the universal property of the product in $\{\mathrm{t}, \mathrm{f}\}$. Indeed, consider the diagrams

$$
\begin{tikzcd}
& P_1 \arrow[dl, bend right, "p_1^1"'] \arrow[dr, bend left, "p_2^1"] \arrow[d, dashed, "\exists!"] & \\
\mathrm{t} & \underbrace{\mathrm{t} \times \mathrm{t}}_{=\mathrm{t}} \arrow[l, two heads, "\mathrm{pr}_1"] \arrow[r, two heads, "\mathrm{pr}_2"'] & \mathrm{t}
\end{tikzcd}
\quad
\begin{tikzcd}
& P_2 \arrow[dl, bend right, "p_1^2"'] \arrow[dr, bend left, "p_2^2"] \arrow[d, dashed, "\exists!"] & \\
\mathrm{t} & \underbrace{\mathrm{t} \times \mathrm{f}}_{=\mathrm{f}} \arrow[l, two heads, "\mathrm{pr}_1"] \arrow[r, two heads, "\mathrm{pr}_2"'] & \mathrm{f}
\end{tikzcd}
\quad
\begin{tikzcd}
& P_3 \arrow[dl, bend right, "p_1^3"'] \arrow[dr, bend left, "p_2^3"] \arrow[d, dashed, "\exists!"] & \\
\mathrm{f} & \underbrace{\mathrm{f} \times \mathrm{t}}_{=\mathrm{f}} \arrow[l, two heads, "\mathrm{pr}_1"] \arrow[r, two heads, "\mathrm{pr}_2"'] & \mathrm{t}
\end{tikzcd}
\quad
\begin{tikzcd}
& P_4 \arrow[dl, bend right, "p_1^4"'] \arrow[dr, bend left, "p_2^4"] \arrow[d, dashed, "\exists!"] & \\
\mathrm{f} & \underbrace{\mathrm{f} \times \mathrm{f}}_{=\mathrm{f}} \arrow[l, two heads, "\mathrm{pr}_1"] \arrow[r, two heads, "\mathrm{pr}_2"'] & \mathrm{f}.
\end{tikzcd}
$$

Here:

1. If $P_1 = \mathrm{t}$, then $p_1^1 = p_2^1 = \mathrm{id}_{\mathrm{t}}$, and there's indeed a unique morphism from P_1 to t making the diagram commute, namely id_{t};

2. If $P_1 = \mathsf{f}$, then $p_1^1 = p_2^1$ are given by the unique morphism from f to t, and there's indeed a unique morphism from P_1 to t making the diagram commute, namely the unique morphism from f to t;
3. If $P_2 = \mathsf{t}$, then there is no morphism p_2^2.
4. If $P_2 = \mathsf{f}$, then p_1^2 is the unique morphism from f to t while $p_2^2 = \mathrm{id}_{\mathsf{f}}$, and there's indeed a unique morphism from P_2 to f making the diagram commute, namely id_{f};
5. The proof for P_3 is similar to the one for P_2;
6. If $P_4 = \mathsf{t}$, then there is no morphism p_1^4 or p_2^4.
7. If $P_4 = \mathsf{f}$, then $p_1^4 = p_2^4 = \mathrm{id}_{\mathsf{f}}$, and there's indeed a unique morphism from P_4 to f making the diagram commute, namely id_{f}.

Cartesian Closedness

We claim there's a bijection

$$\mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(A \times B, C) \cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\big(A, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(B, C)\big)$$

natural in $A, B, C \in \{\mathsf{t}, \mathsf{f}\}$. Indeed:

- For $(A, B, C) = (\mathsf{t}, \mathsf{t}, \mathsf{t})$, we have

$$\begin{aligned}
\mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t} \times \mathsf{t}, \mathsf{t}) &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t}, \mathsf{t}) \\
&= \{\mathrm{id}_{\mathsf{true}}\} \\
&\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t}, \mathsf{t}) \\
&\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\big(\mathsf{t}, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t}, \mathsf{t})\big).
\end{aligned}$$

- For $(A, B, C) = (\mathsf{t}, \mathsf{t}, \mathsf{f})$, we have

$$\begin{aligned}
\mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t} \times \mathsf{t}, \mathsf{f}) &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t}, \mathsf{f}) \\
&= \emptyset \\
&\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t}, \mathsf{f}) \\
&\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\big(\mathsf{t}, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t}, \mathsf{f})\big).
\end{aligned}$$

- For $(A, B, C) = (\mathsf{t}, \mathsf{f}, \mathsf{t})$, we have

$$\begin{aligned}\mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t} \times \mathsf{f}, \mathsf{t}) &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{t}) \\ &\cong \mathrm{pt} \\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{t}) \\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(\mathsf{f}, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{t})\right).\end{aligned}$$

- For $(A, B, C) = (\mathsf{t}, \mathsf{f}, \mathsf{f})$, we have

$$\begin{aligned}\mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t} \times \mathsf{f}, \mathsf{f}) &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{f}) \\ &\cong \{\mathrm{id}_{\mathsf{false}}\} \\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{f}) \\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(\mathsf{t}, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{f})\right).\end{aligned}$$

- For $(A, B, C) = (\mathsf{f}, \mathsf{t}, \mathsf{t})$, we have

$$\begin{aligned}\mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f} \times \mathsf{t}, \mathsf{t}) &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{t}) \\ &\cong \mathrm{pt} \\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{t}) \\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(\mathsf{f}, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t}, \mathsf{t})\right).\end{aligned}$$

- For $(A, B, C) = (\mathsf{f}, \mathsf{t}, \mathsf{f})$, we have

$$\begin{aligned}\mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f} \times \mathsf{t}, \mathsf{f}) &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{f}) \\ &\cong \{\mathrm{id}_{\mathsf{false}}\} \\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{f}) \\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(\mathsf{f}, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t}, \mathsf{f})\right).\end{aligned}$$

- For $(A, B, C) = (\mathsf{f}, \mathsf{f}, \mathsf{t})$, we have

$$\begin{aligned}\mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f} \times \mathsf{f}, \mathsf{t}) &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{t}) \\ &\cong \mathrm{pt} \\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{t}) \\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(\mathsf{f}, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{t})\right).\end{aligned}$$

- For $(A, B, C) = (\mathsf{f}, \mathsf{f}, \mathsf{f})$, we have

$$\begin{aligned}
\mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f} \times \mathsf{f}, \mathsf{f}) &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{f}) \\
&= \{\mathrm{id}_{\mathsf{false}}\} \\
&\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{f}) \\
&\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(\mathsf{f}, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{f})\right).
\end{aligned}$$

The proof of naturality is omitted.

1.3 0-Categories

Definition 1.3.1 ► 0-Categories

A 0-**category** is a poset.[1]

[1] *Motivation:* A 0-category is precisely a category enriched in the poset of (-1)-categories.

Definition 1.3.2 ► 0-Groupoids

A 0-**groupoid** is a 0-category in which every morphism is invertible.[1]

[1] That is, a *set*.

1.4 Tables of Analogies Between Set Theory and Category Theory

Here we record some analogies between notions in set theory and category theory. Note that the analogies relating to presheaves relate equally well to copresheaves, as the opposite X^{op} of a set X is just X again.
Basics:

Set Theory	Category Theory
Enrichment in $\{\mathsf{true}, \mathsf{false}\}$	Enrichment in Sets
Set X	Category $\mathcal{C}$
Element $x \in X$	Object $X \in \mathrm{Obj}(\mathcal{C})$
Function	Functor
Function $X \longrightarrow \{\mathsf{true}, \mathsf{false}\}$	Functor $\mathcal{C} \longrightarrow \mathsf{Sets}$
Function $X \longrightarrow \{\mathsf{true}, \mathsf{false}\}$	Presheaf $\mathcal{C}^{\mathsf{op}} \longrightarrow \mathsf{Sets}$

Powersets and categories of presheaves:

Set Theory	Category Theory
Powerset $\mathcal{P}(X)$	Presheaf category $\mathsf{PSh}(\mathcal{C})$
Characteristic function $\chi_{\{x\}}$	Representable presheaf h_X
Characteristic embedding $\chi_{(-)} \colon X \hookrightarrow \mathcal{P}(X)$	Yoneda embedding $よ \colon \mathcal{C}^{\mathsf{op}} \hookrightarrow \mathsf{PSh}(\mathcal{C})$
Characteristic relation $\chi_X(-_1, -_2)$	Hom profunctor $\mathrm{Hom}_{\mathcal{C}}(-_1, -_2)$
The Yoneda lemma for sets $\mathrm{Hom}_{\mathcal{P}(X)}(\chi_x, \chi_U) = \chi_U(x)$	The Yoneda lemma for categories $\mathrm{Nat}(h_X, \mathcal{F}) \cong \mathcal{F}(X)$
The characteristic embedding is fully faithful, $\mathrm{Hom}_{\mathcal{P}(X)}\big(\chi_x, \chi_y\big) = \chi_X(x, y)$	The Yoneda embedding is fully faithful, $\mathrm{Nat}(h_X, h_Y) \cong \mathrm{Hom}_{\mathcal{C}}(X, Y)$
Subsets are unions of their elements $U = \bigcup_{x \in U} \{x\}$ or $\chi_U = \operatorname*{colim}_{\chi_x \in \mathsf{Sets}(U, \{\mathsf{t}, \mathsf{f}\})} (\chi_x)$	Presheaves are colimits of representables, $\mathcal{F} \cong \operatorname*{colim}_{h_X \in \int_{\mathcal{C}} \mathcal{F}} (h_X)$

Categories of elements:

Set Theory	Category Theory
Assignment $U \mapsto \chi_U$	Assignment $\mathcal{F} \mapsto \int_{\mathcal{C}} \mathcal{F}$ (the category of elements)
Assignment $U \mapsto \chi_U$ giving an isomorphism $\mathcal{P}(X) \cong \mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\})$	Assignment $\mathcal{F} \mapsto \int_{\mathcal{C}} \mathcal{F}$ giving an equivalence $\mathsf{PSh}(\mathcal{C}) \overset{\text{eq.}}{\cong} \mathsf{DFib}(\mathcal{C})$

Functions between powersets and functors between presheaf categories:

Set Theory	Category Theory
Direct image function $f_*\colon \mathcal{P}(X) \to \mathcal{P}(Y)$	Inverse image functor $f^{-1}\colon \mathsf{PSh}(\mathcal{C}) \to \mathsf{PSh}(\mathcal{D})$
Inverse image function $f^{-1}\colon \mathcal{P}(Y) \to \mathcal{P}(X)$	Direct image functor $f_*\colon \mathsf{PSh}(\mathcal{D}) \to \mathsf{PSh}(\mathcal{C})$
Direct image with compact support function $f_!\colon \mathcal{P}(X) \to \mathcal{P}(Y)$	Direct image with compact support functor $f_!\colon \mathsf{PSh}(\mathcal{C}) \to \mathsf{PSh}(\mathcal{D})$

Relations and profunctors:

Set Theory	Category Theory
Relation $R\colon X \times Y \to \{\mathsf{t}, \mathsf{f}\}$	Profunctor $\mathfrak{p}\colon \mathcal{D}^{\mathsf{op}} \times \mathcal{C} \to \mathsf{Sets}$
Relation $R\colon X \to \mathcal{P}(Y)$	Profunctor $\mathfrak{p}\colon \mathcal{C} \to \mathsf{PSh}(\mathcal{D})$
Relation as a cocontinuous morphism of posets $R\colon (\mathcal{P}(X), \subset) \to (\mathcal{P}(Y), \subset)$	Profunctor as a colimit-preserving functor $\mathfrak{p}\colon \mathsf{PSh}(\mathcal{C}) \to \mathsf{PSh}(\mathcal{D})$

Appendices

A Other Chapters