

Spans

December 3, 2023

This chapter contains some material about spans. Notably, we discuss and explore:

1. The basic definitions around spans ([Section 1](#));
2. The relation between spans and functions ([Proposition 7.1.1.1](#));
3. The relation between spans and relations ([Propositions 7.2.2.1](#) and [7.3.1.1](#) and [Remark 7.5.1.1](#)).
4. “Hyperpointed sets” (??). I don’t know why I wrote this...

TODO:

1. internal adjoint equivalences in **Rel**
2. internal adjoint equivalences in Span
3. 2-categorical limits in **Rel**;
4. morphism of internal adjunctions in **Rel**;
5. morphism of internal adjunctions in Span;
6. morphism of co/monads in Span;
7. What is $\text{Adj}(\text{Span}(A, B))$?
8. monoids, comonoids, pseudomonoids, etc. in Span.
9. write down the dumb intuition about spans inducing morphisms $\text{Sets}(S, A) \rightarrow \text{Sets}(S, B)$ instead of $\mathcal{P}(A) \rightarrow \mathcal{P}(B)$ from the similarity between

$$S \rightarrow A \times B$$

and

$$A \times B \rightarrow \{t, f\}.$$

This intuition is justified by taking $A = \text{pt}$ or $B = \text{pt}$.

10. What about using the direct image with compact support in $g(f^{-1}(a))$?
11. Monads in Span | develop this in the level of morphisms too
12. Comonads in Span are spans whose legs are equal | develop this in the level of morphisms too
13. Does Span have an internal **Hom**?
14. Examples of spans
15. Functional and total spans
16. closed symmetric monoidal category of spans
17. double category of relations
18. collage of a span
19. equivalence spans?
20. functoriality of powersets for spans
21. Is Span a closed bicategory?
22. skew monoidal structure on $\text{Span}(A, B)$
23. Adjunctions in Span
24. Isomorphisms in Span
25. Equivalences in Span
26. Interaction between the above notions in Span vs. in **Rel** via the comparison functors

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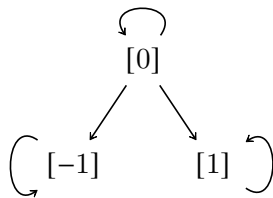
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1 Spans

1.1 The Walking Span

Definition 1.1.1.1. The **walking span** is the category Λ that looks like this:



1.2 Spans

Let A and B be sets.

Definition 1.2.1.1. A **span from A to B** ¹ is a functor $F: \Lambda \rightarrow \text{Sets}$ such that

$$F([-1]) = A,$$

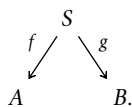
$$F([1]) = B.$$

Remark 1.2.1.2. In detail, a **span from A to B** is a triple (S, f, g) consisting of^{2,3}

- *The Underlying Set.* A set S , called the **underlying set of** (S, f, g) ;
- *The Legs.* A pair of functions $f: S \rightarrow A$ and $g: S \rightarrow B$.

¹Further Terminology: Also called a **roof from A to B** or a **correspondence from A to B** .

²Picture:



³Every span (S, f, g) from A to B determines in particular a relation $R: A \nrightarrow B$ via

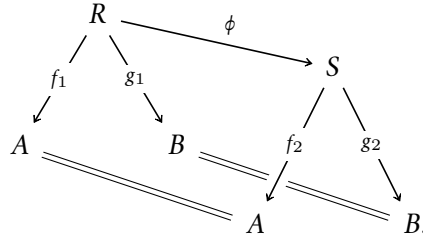
$$R \stackrel{\text{def}}{=} \{(f(a), g(a)) \mid a \in A\},$$

i.e. where $R(a) = g(f^{-1}(a))$ for each $a \in A$; see **Proposition 7.2.2.1**.

1.3 Morphisms of Spans

Definition 1.3.1.1. A **morphism of spans** (R, f_1, g_1) **to** (S, f_2, g_2) ⁴ is a natural transformation $(R, f_1, g_1) \Rightarrow (S, f_2, g_2)$.

Remark 1.3.1.2. In detail, a **morphism of spans from** (R, f_1, g_1) **to** (S, f_2, g_2) is a function $\phi: R \rightarrow S$ making the diagram⁵



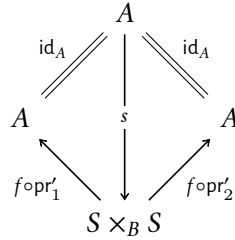
commute.

1.4 Functional Spans

Let $\lambda = \left(A \xleftarrow{f} S \xrightarrow{g} B \right)$ be a span. A morphism of spans from id_A to $\lambda \diamond \lambda^\dagger$ is a morphism

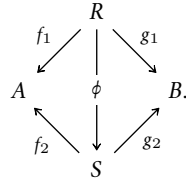
$$s: A \rightarrow S \times_B S$$

making the diagram



⁴*Further Terminology:* Also called a **morphism of roofs from** (R, f_1, g_1) **to** (S, f_2, g_2) or a **morphism of correspondences from** (R, f_1, g_1) **to** (S, f_2, g_2) .

⁵*Alternative Picture:*



commute, where $S \times_B S$ is the pullback

$$S \times_B S \cong \{(s, t) \in S \times S \mid g(s) = g(t)\}$$

$$\begin{array}{ccc} S \times_B S & \longrightarrow & S \\ \downarrow & \lrcorner & \downarrow g \\ S & \xrightarrow{g} & B \end{array}$$

of S with itself along g .

1.5 Total Spans

2 Categories of Spans

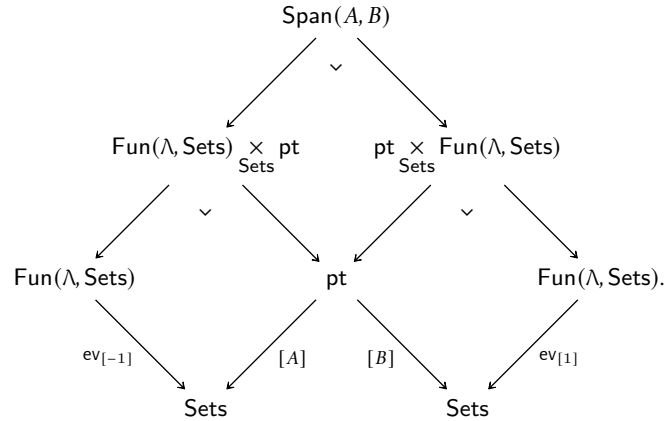
2.1 Categories of Spans

Let A and B be sets.

Definition 2.1.1.1. The **category of spans from A to B** is the category $\text{Span}(A, B)$ defined by

$$\text{Span}(A, B) \stackrel{\text{def}}{=} \text{Fun}(\Lambda, \text{Sets}) \times_{\text{ev}_{[-1]}, \text{Sets}, [A]}^{\text{pt}} \times_{[B], \text{Sets}, \text{ev}_{[1]}}^{\text{pt}} \text{Fun}(\Lambda, \text{Sets}),$$

as in the diagram



Remark 2.1.1.2. In detail, the **category of spans from A to B** is the category $\text{Span}(A, B)$ where

- *Objects.* The objects of $\text{Span}(A, B)$ are spans from A to B ;
- *Morphisms.* The morphism of $\text{Span}(A, B)$ are morphisms of spans;

- *Identities.* The unit map

$$\mathbb{K}_{(S,f,g)}^{\text{Span}(A,B)} : \text{pt} \rightarrow \text{Hom}_{\text{Span}(A,B)}((S,f,g), (S,f,g))$$

of $\text{Span}(A, B)$ at (S, f, g) is defined by⁶

$$\text{id}_{(S,f,g)}^{\text{Span}(A,B)} \stackrel{\text{def}}{=} \text{id}_S;$$

- *Composition.* The composition map

$$\circ_{R,S,T}^{\text{Span}(A,B)} : \text{Hom}_{\text{Span}(A,B)}(S, T) \times \text{Hom}_{\text{Span}(A,B)}(R, S) \rightarrow \text{Hom}_{\text{Span}(A,B)}(R, T)$$

of $\text{Span}(A, B)$ at $((R, f_1, g_1), (S, f_2, g_2), (T, f_3, g_3))$ is defined by⁷

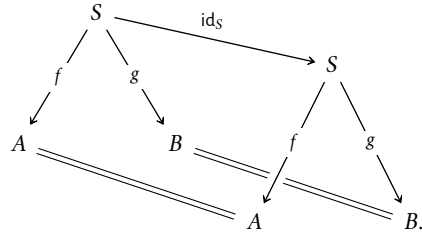
$$\psi \circ_{R,S,T}^{\text{Span}(A,B)} \phi \stackrel{\text{def}}{=} \psi \circ \phi.$$

2.2 The Bicategory of Spans

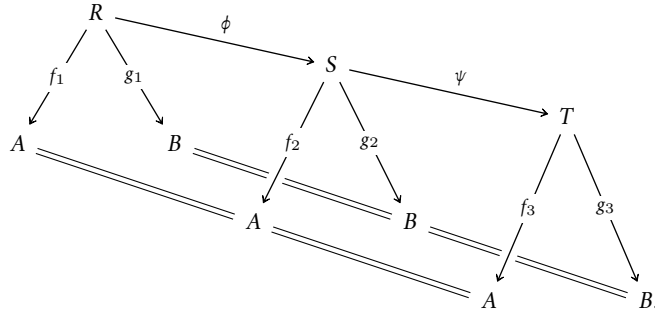
Definition 2.2.1.1. The **bicategory of spans** is the bicategory Span where

- *Objects.* The objects of Span are sets;

⁶Picture:



⁷Picture:



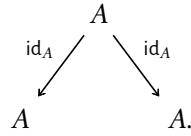
- *Hom-Categories.* For each $A, B \in \text{Obj}(\text{Span})$, we have

$$\text{Hom}_{\text{Span}}(A, B) \stackrel{\text{def}}{=} \text{Span}(A, B);$$

- *Identities.* For each $A \in \text{Obj}(\text{Span})$, the unit functor

$$\mathbb{K}_A^{\text{Span}} : \text{pt} \rightarrow \text{Span}(A, A)$$

of Span at A is the functor picking the span $(A, \text{id}_A, \text{id}_A)$:

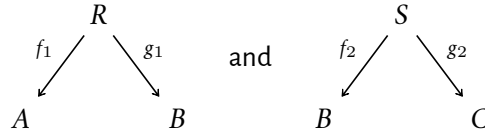


- *Composition.* For each $A, B, C \in \text{Obj}(\text{Span})$, the composition bifunctor

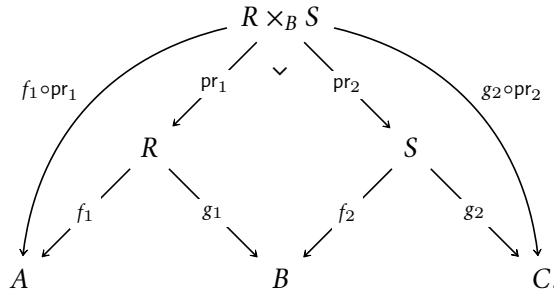
$$\circ_{A,B,C}^{\text{Span}} : \text{Span}(B, C) \times \text{Span}(A, B) \rightarrow \text{Span}(A, C)$$

of Span at (A, B, C) is the bifunctor where

- *Action on Objects.* The composition of two spans

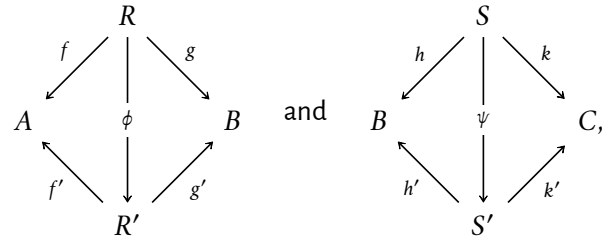


is the span $(R \times_B S, f_1 \circ \text{pr}_1, g_2 \circ \text{pr}_2)$, constructed as in the diagram

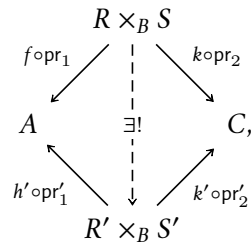


- *Action on Morphisms.* The horizontal composition of 2-morphisms is defined

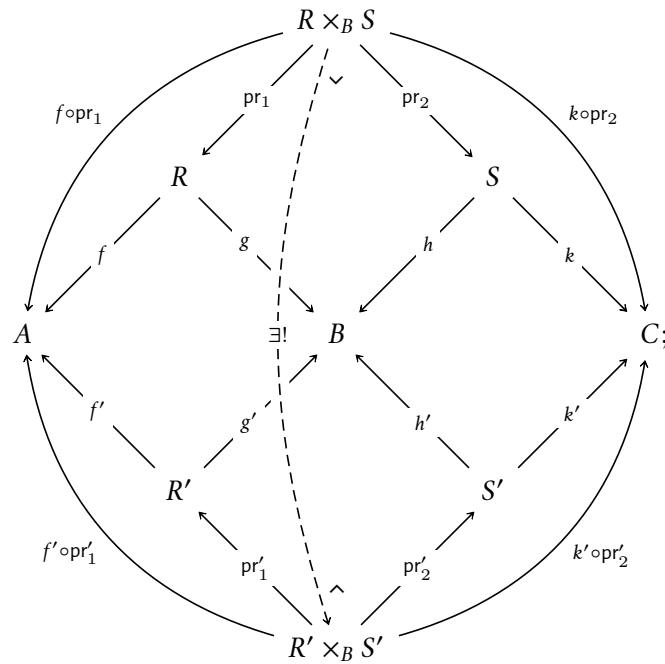
via functoriality of pullbacks: given morphisms of spans



their horizontal composition is the morphism of spans



constructed as in the diagram



- *Associators and Unitors.* The associator and unitors are defined using the universal property of the pullback.

2.3 The Monoidal Bicategory of Spans

2.4 The Double Category of Spans

Definition 2.4.1.1. The **double category of spans** is the double category Span^{dbl} where

- *Objects.* The objects of Span^{dbl} are sets;
- *Vertical Morphisms.* The vertical morphisms of Span^{dbl} are functions $f: A \rightarrow B$;
- *Horizontal Morphisms.* The horizontal morphisms of Span^{dbl} are spans $(S, \phi, \psi): A \rightarrow B$;
- *2-Morphisms.* A 2-cell

$$\begin{array}{ccc}
 A & \xrightarrow{(R, \phi_R, \psi_R)} & B \\
 f \downarrow & \Downarrow \alpha & \downarrow g \\
 X & \xrightarrow{(S, \phi_S, \psi_S)} & Y
 \end{array}$$

of Span^{dbl} is a morphism of spans from the span

$$\begin{array}{ccc}
 & R & \\
 \phi_R \swarrow & & \searrow \psi_R \\
 A & & B \\
 & & \searrow g \\
 & & Y
 \end{array}$$

to the span

$$\begin{array}{ccccc}
 & & A \times_X S & & \\
 & \swarrow & \downarrow \vee & \searrow & \\
 & A & & S & \\
 f \swarrow & & \searrow f & \swarrow \phi_S & \searrow \psi_S \\
 X & & X & & Y
 \end{array}$$

- *Horizontal Identities.* The horizontal unit functor

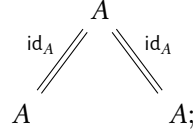
$$\mathbb{1}^{\text{Span}^{\text{dbl}}} : (\text{Span}^{\text{dbl}})_0 \rightarrow (\text{Span}^{\text{dbl}})_1$$

of Span^{dbl} is the functor where

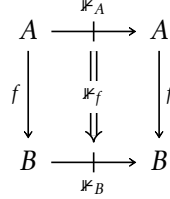
- *Action on Objects.* For each $A \in \text{Obj}\left(\left(\text{Span}^{\text{dbl}}\right)_0\right)$, we have

$$\mathbb{K}_A \stackrel{\text{def}}{=} (A, \text{id}_A, \text{id}_A),$$

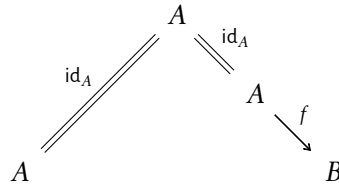
as in the diagram



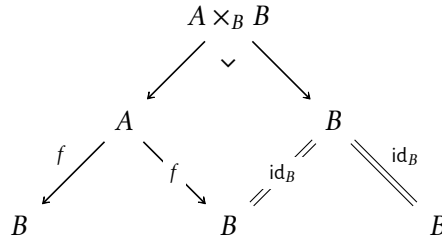
- *Action on Morphisms.* For each vertical morphism $f: A \rightarrow B$ of Span^{dbl} , i.e. each map of sets f from A to B , the identity 2-morphism



of f is the morphism of spans from



to



given by the isomorphism $A \xrightarrow{\cong} A \times_B B$;

- *Vertical Identities.* For each $A \in \text{Obj}\left(\text{Span}^{\text{dbl}}\right)$, we have

$$\text{id}_A^{\text{Span}^{\text{dbl}}} \stackrel{\text{def}}{=} \text{id}_A;$$

- *Identity 2-Morphisms.* For each horizontal morphism $R: A \rightarrowtail B$ of Span^{dbl} , the identity 2-morphism

$$\begin{array}{ccc}
 A & \xrightarrow{S} & B \\
 \text{id}_A \downarrow & \text{id}_S \Downarrow & \downarrow \text{id}_B \\
 A & \xrightarrow{S} & B
 \end{array}$$

of R is the morphism of spans from

$$\begin{array}{ccc}
 & S & \\
 \phi_S \swarrow & & \searrow \psi_S \\
 A & & B \\
 & & \text{id}_B \Downarrow \\
 & & B
 \end{array}$$

to

$$\begin{array}{ccccc}
 & & A \times_A S & & \\
 & \swarrow & \downarrow \vee & \searrow & \\
 & A & & S & \\
 \text{id}_A \Downarrow & \text{id}_A \Downarrow & & \downarrow \phi_S & \searrow \psi_S \\
 A & & A & & B
 \end{array}$$

given by the isomorphism $S \xrightarrow{\cong} A \times_A S$;

- *Horizontal Composition.* The horizontal composition functor

$$\odot_{\text{Span}^{\text{dbl}}} : (\text{Span}^{\text{dbl}})_1 \times_{(\text{Span}^{\text{dbl}})_0} (\text{Span}^{\text{dbl}})_1 \rightarrow (\text{Span}^{\text{dbl}})_1$$

of Span^{dbl} is the functor where

- *Action on Objects.* For each composable pair

$$\begin{array}{ccccc}
 & (R, \phi_R, \psi_R) & & (S, \phi_S, \psi_S) & \\
 A & \rightarrowtail & B & \rightarrowtail & C
 \end{array}$$

of horizontal morphisms of Span^{dbl} , we have

$$(S, \phi_S, \psi_S) \odot (R, \phi_R, \psi_R) \stackrel{\text{def}}{=} S \circ_{A,B,C}^{\text{Span}} R,$$

where $S \circ_{A,B,C}^{\text{Span}} R$ is the composition of (R, ϕ_R, ψ_R) and (S, ϕ_S, ψ_S) defined as in [Definition 2.2.1.1](#);

– *Action on Morphisms.* For each horizontally composable pair

$$\begin{array}{ccc}
 A & \xrightarrow{(R, \phi_R, \psi_R)} & B \\
 f \downarrow & \Downarrow \alpha & \downarrow g \\
 X & \xrightarrow{(T, \phi_T, \psi_T)} & Y
 \end{array}
 \quad
 \begin{array}{ccc}
 B & \xrightarrow{(S, \phi_S, \psi_S)} & C \\
 g \downarrow & \Downarrow \beta & \downarrow h \\
 Y & \xrightarrow{(U, \phi_U, \psi_U)} & Z
 \end{array}$$

of 2-morphisms of Span^{dbl} ,

- *Vertical Composition of 1-Morphisms.* For each composable pair $A \xrightarrow{F} B \xrightarrow{G} C$ of vertical morphisms of Span^{dbl} , i.e. maps of sets, we have

$$g \circ^{\text{Span}^{\text{dbl}}} f \stackrel{\text{def}}{=} g \circ f;$$

- *Vertical Composition of 2-Morphisms.* For each vertically composable pair

$$\begin{array}{ccc}
 A & \xrightarrow{(R, \phi_R, \psi_R)} & X \\
 f \downarrow & \Downarrow \alpha & \downarrow g \\
 B & \xrightarrow{(S, \phi_S, \psi_S)} & Y
 \end{array}
 \quad
 \begin{array}{ccc}
 B & \xrightarrow{(S, \phi_S, \psi_S)} & Y \\
 h \downarrow & \Downarrow \beta & \downarrow k \\
 C & \xrightarrow{(T, \phi_T, \psi_T)} & Z
 \end{array}$$

of 2-morphisms of Span^{dbl} ,

- *Associators and Unitors.* The associator and unitors of Span^{dbl} are defined using the universal property of the pullback.

2.5 Properties of The Bicategory of Spans

Proposition 2.5.1.1. Let $\lambda = (A \xleftarrow{f} S \xrightarrow{g} B)$ be a span.

1. *Self-Duality.*
2. *Isomorphisms in Span.*
3. *Equivalences in Span.*
4. *Adjunctions in Span.* Let A and B be sets.⁸

⁸In the literature (e.g. [ref]),...are called maps and denoted by $\text{MapSpan}(A, B)$

(a) We have a natural bijection

$$\left\{ \begin{array}{c} \text{Adjunctions in Span} \\ \text{from } A \text{ to } B \end{array} \right\} \cong \left\{ \begin{array}{c} \text{Spans } A \xleftarrow{f} S \xrightarrow{g} B \\ \text{from } A \text{ to } B \text{ with} \\ f \text{ an isomorphism} \end{array} \right\}.$$

(b) We have an equivalence of categories

$$\text{MapSpan}(A, B) \stackrel{\text{eq.}}{\cong} \text{Sets}(A, B)_{\text{disc}},$$

where $\text{MapSpan}(A, B)$ is the full subcategory of $\text{Span}(A, B)$ spanned by the spans $A \xleftarrow{f} S \xrightarrow{g} B$ from A to B with f an isomorphism.

(c) We have a biequivalence of bicategories

$$\text{MapSpan} \stackrel{\text{eq.}}{\cong} \text{Sets}_{\text{bidisc}},$$

where MapSpan is the sub-bicategory of Span whose Hom-categories are given by $\text{MapSpan}(A, B)$.

5. *Monads in Span.*
6. *Comonads in Span.*
7. *Monomorphisms in Span.*
8. *Epimorphisms in Span.*
9. *Existence of Right Kan Extensions.*
10. *Existence of Right Kan Lifts.*
11. *Closedness.*

Proof. **Item 1**, Self-Duality:

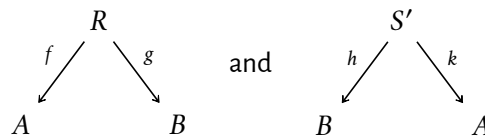
Item 2, Isomorphisms in Span:

Item 3, Equivalences in Span:

Item 4, Adjunctions in Span: We first prove **Item 4a**.

We proceed step by step:

1. *From Adjunctions in Span to Functions.* An adjunction in Span from A to B consists of a pair of spans



$$\begin{array}{ccc}
& A & \\
\text{id}_A \swarrow & & \searrow \text{id}_A \\
A & \xrightarrow{\phi} & A \\
f \circ \text{pr}'_1 \swarrow & & \searrow k \circ \text{pr}'_2 \\
& S \times_B S' &
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
& S' \times_A S & \\
h \circ \text{pr}_1 \swarrow & & \searrow g \circ \text{pr}_2 \\
B & \xrightarrow{\psi} & B \\
\text{id}_B \swarrow & & \searrow \text{id}_B \\
& B &
\end{array}$$

2. *From Functions to Adjunctions in **Rel**.*
3. *Invertibility: From Functions to Adjunctions Back to Functions.*
4. *Invertibility: From Adjunctions to Functions Back to Adjunctions.*

$$F: \text{Sets}(A, B)_{\text{disc}} \rightarrow \text{MapSpan}(A, B)$$

1. *F Is Fully Faithful*: Given maps $f, g: A \rightrightarrows B$, we need to show that

$$\mathrm{Hom}_{\mathrm{MapSpan}(A,B)}(F(f), F(g)) = \begin{cases} \mathrm{pt} & \text{if } f = g, \\ \emptyset & \text{otherwise.} \end{cases}$$

$$\begin{array}{ccc} & A & \\ \parallel & \downarrow \phi & f \\ A & & B \\ \parallel & \downarrow & \nearrow g \\ & A & \end{array}$$

From the relations $\text{id}_A = \text{id}_A \circ \phi$ and $f = g \circ \phi$, we see that $\phi = \text{id}_A$, and thus from the relation $f = g \circ \phi$ there is such a morphism iff $f = g$.

2. *F Is Essentially Surjective*: Let λ be a span of the form

$$\begin{array}{ccc} & S & \\ \phi \swarrow & & \searrow f \\ A & & B. \end{array}$$

we claim that $\lambda \cong F(f \circ \phi^{-1})$. Indeed, we have morphisms

$$\begin{array}{ccc} \begin{array}{ccc} & S & \\ \phi \swarrow & & \searrow f \\ A & & B \\ \parallel & & \nearrow f \circ \phi^{-1} \\ & A & \end{array} & \text{and} & \begin{array}{ccc} & A & \\ \parallel & & \searrow f \circ \phi^{-1} \\ S & & B \\ \nwarrow \phi & & \nearrow f \\ & & \end{array} \end{array}$$

inverse to each other in $\text{MapSpan}(A, B)$, and thus $\lambda \cong F(f \circ \phi^{-1})$.

Finally, we prove **Item 4c**.

Item 5, *Monads in Span*:

Item 6, *Comonads in Span*:

Item 7, *Monomorphisms in Span*:

Item 8, *Epimorphisms in Span*:

Item 9, *Existence of Right Kan Extensions*:

Item 10, *Existence of Right Kan Lifts*:

Item 11, *Closedness*:

□

3 Limits of Spans

4 Colimits of Spans

5 Constructions With Spans

5.1 Representable Spans

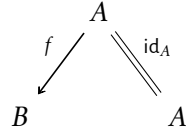
Definition 5.1.1.1. Let $f: A \rightarrow B$ be a function.

- The **representable span associated to f** is the span

$$\begin{array}{ccc} & A & \\ \text{id}_A \swarrow & & \searrow f \\ A & & B \end{array}$$

from A to B .

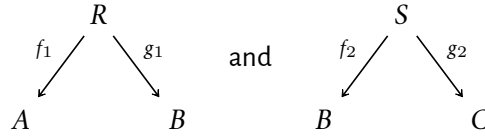
- The **corepresentable span associated to f** is the span



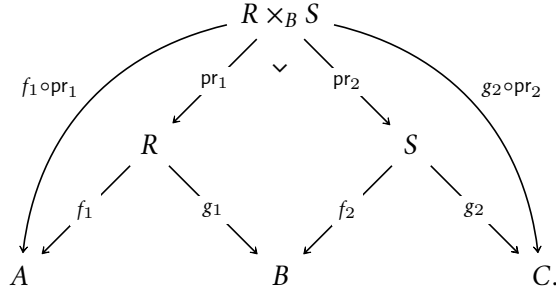
from B to A .

5.2 Composition of Spans

Definition 5.2.1.1. The **composition** of two spans

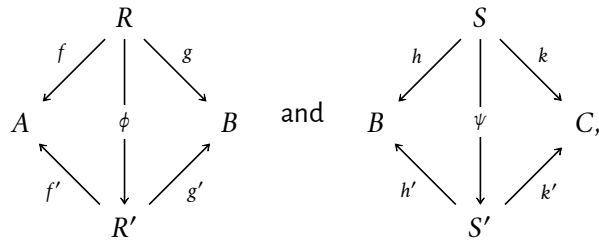


is the span $(R \times_B S, f_1 \circ \text{pr}_1, g_2 \circ \text{pr}_2)$, constructed as in the diagram

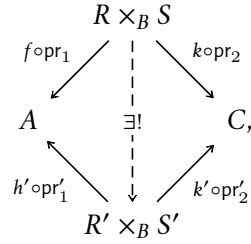


5.3 Horizontal Composition of Morphisms of Spans

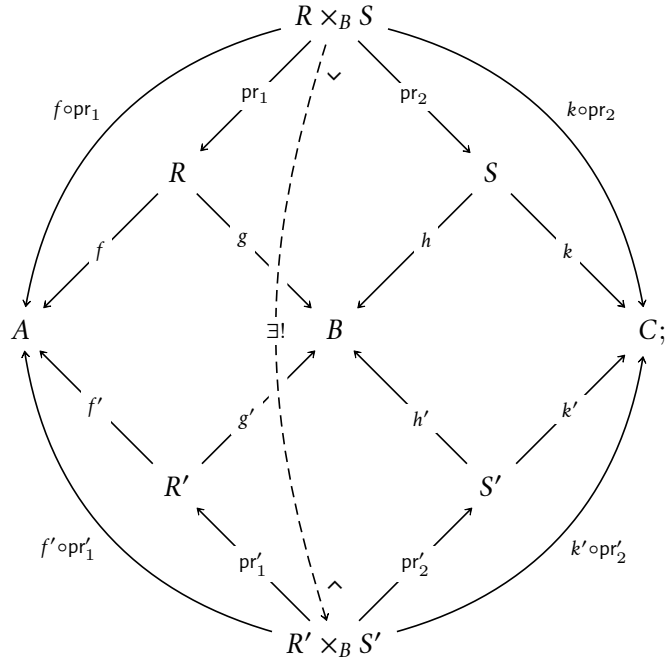
Definition 5.3.1.1. The **horizontal composition** of a pair of 2-morphisms of spans



is the morphism of spans



constructed as in the diagram



5.4 Properties of Composition of Spans

Proposition 5.4.1.1. Let $\lambda = \left(A \xleftarrow{f} S \xrightarrow{g} B \right)$ be a span.

1. *Functoriality.*

Proof.

□

5.5 The Inverse of a Span

6 Functoriality of Spans

6.1 Direct Images

6.2 Functoriality of Spans on Powersets

7 Comparison of Spans to Functions and Relations

7.1 Comparison to Functions

Proposition 7.1.1.1. We have a pseudofunctor

$$\iota: \mathbf{Sets}_{\text{bidisc}} \rightarrow \mathbf{Span}$$

from $\mathbf{Sets}_{\text{bidisc}}$ to \mathbf{Span} where

- *Action on Objects.* For each $A \in \text{Obj}(\mathbf{Sets}_{\text{bidisc}})$, we have

$$\iota(A) \stackrel{\text{def}}{=} A;$$

- *Action on Hom-Categories.* For each $A, B \in \text{Obj}(\mathbf{Sets}_{\text{bidisc}})$, the action on Hom-categories

$$\iota_{A,B}: \mathbf{Sets}(A, B)_{\text{disc}} \rightarrow \mathbf{Span}(A, B)$$

of ι at (A, B) is the functor defined on objects by sending a function $f: A \rightarrow B$ to the span

$$\begin{array}{ccc} & A & \\ \text{id}_A \swarrow & & \searrow f \\ A & & B \end{array}$$

from A to B .

- *Strict Unity Constraints.* For each $A \in \text{Obj}(\mathbf{Sets}_{\text{bidisc}})$, the strict unity constraint

$$\iota_A^0: \text{id}_{\iota(A)} \Longrightarrow \iota(\text{id}_A)$$

of ι at A is given by the identity morphism of spans

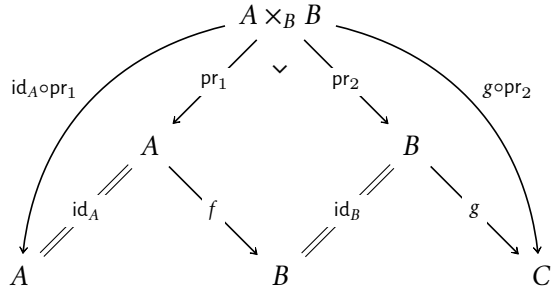
$$\begin{array}{ccccc} & & A & & \\ & \text{id}_A \swarrow & \parallel & \searrow \text{id}_A & \\ A & & \text{id} & & A, \\ & \swarrow \text{id}_A & \parallel & \searrow \text{id}_A & \\ & & A & & \end{array}$$

as indeed $\text{id}_{\iota(A)} = \iota(\text{id}_A)$;

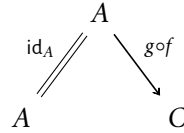
- *Pseudofunctoriality Constraints.* For each $A, B, C \in \text{Obj}(\text{Sets}_{\text{bidisc}})$, each $f \in \text{Hom}_{\text{Sets}_{\text{bidisc}}}(A, B)$, and each $g \in \text{Hom}_{\text{Sets}_{\text{bidisc}}}(B, C)$, the pseudofunctoriality constraint

$$\iota_{g,f}^2: \iota(g) \circ \iota(f) \implies \iota(g \circ f)$$

of ι at (f, g) is the morphism of spans from the span



to the span



given by the isomorphism $A \times_B B \cong A$.

Proof. Omitted. □

7.2 Comparison to Relations: From Span to **Rel**

7.2.1 Relations Associated to Spans

Let $\lambda = \left(A \xleftarrow{f} S \xrightarrow{g} B \right)$ be a span.

Definition 7.2.1.1. The **relation associated to λ** is the relation

$$S(\lambda): A \rightarrowtail B$$

from A to B defined as follows:

- Viewing relations from A to B as functions $A \times B \rightarrow \{\text{true}, \text{false}\}$, we define

$$\iota_{A,B}(S)_b^a \stackrel{\text{def}}{=} \begin{cases} \text{true} & \text{if there exists } x \in S \text{ such} \\ & \text{that } a = f(x) \text{ and } b = g(x), \\ \text{false} & \text{otherwise} \end{cases}$$

for each $(a, b) \in A \times B$.

- Viewing relations from A to B as functions $A \rightarrow \mathcal{P}(B)$, we define

$$[\iota_{A,B}(S)](a) \stackrel{\text{def}}{=} g(f^{-1}(a))$$

for each $a \in A$.

- Viewing relations from A to B as subsets of $A \times B$, we define

$$\iota_{A,B}(S) \stackrel{\text{def}}{=} \{(f(x), g(x)) \mid x \in S\}.$$

Proposition 7.2.1.2. Let $\lambda = \left(A \xleftarrow{f} S \xrightarrow{g} B \right)$ be a span.

1. *Interaction With Identities.*
2. *Interaction With Composition.*
3. *Interaction With Inverses.*

Proof.

□

7.2.2 The Comparison Functor from Span to **Rel**

Proposition 7.2.2.1. We have a pseudofunctor

$$\iota : \text{Span} \rightarrow \mathbf{Rel}$$

from Span to **Rel** where

- *Action on Objects.* For each $A \in \text{Obj}(\text{Span})$, we have

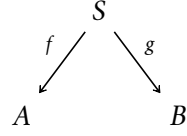
$$\iota(A) \stackrel{\text{def}}{=} A;$$

- *Action on Hom-Categories.* For each $A, B \in \text{Obj}(\text{Span})$, the action on Hom-categories

$$\iota_{A,B} : \text{Span}(A, B) \rightarrow \mathbf{Rel}(A, B)$$

of ι at (A, B) is the functor where

– *Action on Objects.* Given a span



from A to B , the image

$$\iota_{A,B}(S): A \rightarrowtail B$$

of S by ι is the relation from A to B defined as follows:

* Viewing relations as functions $A \times B \rightarrow \{\text{true}, \text{false}\}$, we define

$$\iota_{A,B}(S)_b^a \stackrel{\text{def}}{=} \begin{cases} \text{true} & \text{if there exists } x \in S \\ & \text{such that } a = f(x) \\ & \text{and } b = g(x), \\ \text{false} & \text{otherwise} \end{cases}$$

for each $(a, b) \in A \times B$;

* Viewing relations as functions $A \rightarrow \mathcal{P}(B)$, we define

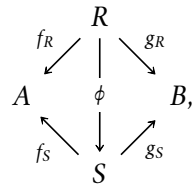
$$[\iota_{A,B}(S)](a) \stackrel{\text{def}}{=} g(f^{-1}(a))$$

for each $a \in A$;

* Viewing relations as subsets of $A \times B$, we define

$$\iota_{A,B}(S) \stackrel{\text{def}}{=} \{(f(x), g(x)) \mid x \in S\}.$$

– *Action on Morphisms.* Given a morphism of spans



we have a corresponding inclusion of relations

$$\iota_{A,B}(\phi): \iota_{A,B}(R) \subset \iota_{A,B}(S),$$

since we have $a \sim_{\iota_{A,B}(R)} b$ iff there exists $x \in R$ such that $a = f_R(x)$ and $b = g_R(x)$, in which case we then have

$$\begin{aligned} a &= f_R(x) \\ &= f_S(\phi(x)), \\ b &= g_R(x) \\ &= g_S(\phi(x)), \end{aligned}$$

so that $a \sim_{\iota_{A,B}(S)} b$, and thus $\iota_{A,B}(R) \subset \iota_{A,B}(S)$.

Proof. Omitted. □

7.3 Comparison to Relations: From **Rel** to Span

Proposition 7.3.1.1. We have a lax functor

$$\left(\iota, \iota^2, \iota^0 \right) : \mathbf{Rel} \rightarrow \mathbf{Span}$$

from **Rel** to Span where

- *Action on Objects.* For each $A \in \mathbf{Obj}(\mathbf{Span})$, we have

$$\iota(A) \stackrel{\text{def}}{=} A;$$

- *Action on Hom-Categories.* For each $A, B \in \mathbf{Obj}(\mathbf{Span})$, the action on Hom-categories

$$\iota_{A,B} : \mathbf{Rel}(A, B) \rightarrow \mathbf{Span}(A, B)$$

of ι at (A, B) is the functor where

- *Action on Objects.* Given a relation $R : A \rightarrowtail B$ from A to B , we define a span

$$\iota_{A,B}(R) : A \rightarrowtail B$$

from A to B by

$$\iota_{A,B}(R) \stackrel{\text{def}}{=} (R, \upharpoonright \text{pr}_1 R, \upharpoonright \text{pr}_2 R),$$

where $R \subset A \times B$ and $\upharpoonright \text{pr}_1 R$ and $\upharpoonright \text{pr}_2 R$ are the restriction of the projections

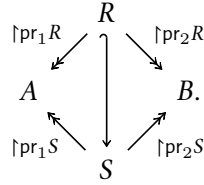
$$\begin{aligned} \text{pr}_1 : A \times B &\rightarrow A, \\ \text{pr}_2 : A \times B &\rightarrow B \end{aligned}$$

to R ;

- *Action on Morphisms.* Given an inclusion $\phi: R \subset S$ of relations, we have a corresponding morphism of spans

$$\iota_{A,B}(\phi): \iota_{A,B}(R) \rightarrow \iota_{A,B}(S)$$

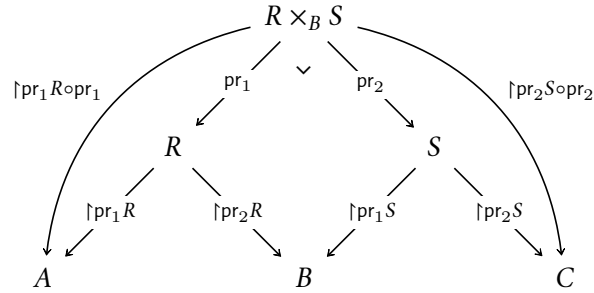
as in the diagram



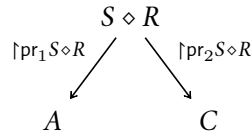
- *The Lax Functoriality Constraints.* The lax functoriality constraint

$$\iota_{R,S}^2: \iota(S) \circ \iota(R) \Longrightarrow \iota(S \diamond R)$$

of ι at (R, S) is given by the morphism of spans from



to



given by the natural inclusion $R \times_B S \hookrightarrow S \diamond R$, since we have

$$R \times_B S = \{((a_R, b_R), (b_S, c_S)) \in R \times S \mid b_R = b_S\};$$

$$S \diamond R = \left\{ (a, c) \in A \times C \left| \begin{array}{l} \text{there exists some } b \in B \text{ such that} \\ (a, b) \in R \text{ and } (b, c) \in S \end{array} \right. \right\};$$

- *The Lax Unity Constraints.* The lax unity constraint⁹

$$\iota_A^0: \underbrace{\text{id}_{\iota(A)}}_{(A, \text{id}_A, \text{id}_A)} \Longrightarrow \underbrace{\iota(\chi_A)}_{(\Delta_A, \uparrow \text{pr}_1 \Delta_A, \uparrow \text{pr}_2 \Delta_A)}$$

of ι at A is given by the diagonal morphism of A , as in the diagram

$$\begin{array}{ccc} & A & \\ \text{id}_A \swarrow & \downarrow \delta_A & \searrow \text{id}_A \\ A & & A \\ \uparrow \text{pr}_1 \Delta_A \swarrow & & \nwarrow \uparrow \text{pr}_2 \Delta_A \\ & \Delta_A & \end{array}$$

Proof. Omitted. □

7.4 Comparison to Relations: The Wehrheim–Woodward Construction

7.5 Comparison to Multirelations

Remark 7.5.1.1. The pseudofunctor of [Proposition 7.2.2.1](#) and the lax functor of [Proposition 7.3.1.1](#) fail to be equivalences of bicategories. This happens essentially because a span $(S, f, g): A \rightarrowtail B$ from A to B may relate elements $a \in A$ and $b \in B$ by more than one element, e.g. there could be $s \neq s' \in S$ such that $a = f(s) = f(s')$ and $b = g(s) = g(s')$.

Thus, in a sense, spans may be thought of as “relations with multiplicity”. And indeed, if instead of considering relations from A to B , i.e. functions

$$R: A \times B \rightarrow \{\text{true}, \text{false}\}$$

from $A \times B$ to $\{\text{true}, \text{false}\} \cong \{0, 1\}$, we consider functions

$$R: A \times B \rightarrow \mathbb{N} \cup \{\infty\}$$

from $A \times B$ to $\mathbb{N} \cup \{\infty\}$, then we obtain the notion of a **multirelation from A to B** , and these turn out to assemble together with sets into a bicategory \mathbf{MRel} that is biequivalent to \mathbf{Span} ; see [\[some-algebraic-laws-for-spans-and-their-connections-with-multirelations\]](#).

⁹Which is in fact strong, as δ_A is an isomorphism.

7.6 Comparison to Relations via Double Categories

Remark 7.6.1.1. There are double functors between the double categories $\mathbf{Rel}^{\text{dbl}}$ and $\mathbf{Span}^{\text{dbl}}$ analogous to the functors of [Propositions 7.2.2.1](#) and [7.3.1.1](#), assembling more-over into a strict-lax adjunction of double functors; see [\[higher-dimensional-categories\]](#).

Appendices

A Other Chapters

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2. [Constructions With Sets](#)
3. [Pointed Sets](#)
4. [Tensor Products of Pointed Sets](#)
5. [Indexed and Fibred Sets](#)
6. [Relations](#)
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Category Theory

9. [Categories](#)
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Bicategories

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Internal Category Theory

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15. [The Cycle Category](#)

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22. [Constructions With Monoids With Zero](#)

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23. [Groups](#)
24. [Constructions With Groups](#)

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26. Hypergroups

27. Hypersemirings and Hyperrings

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Near-Rings

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30. Near-Rings

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34. Measures and Integration

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34. Probability Theory

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35. Stochastic Processes, Martingales,
and Brownian Motion

36. Itô Calculus

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38. Topological and Smooth Manifolds

Schemes

39. Schemes