Indexed and Fibred Sets

December 3, 2023

00AH This chapter contains a discussion of the un/straightening equivalence in the context of sets, as well as a general discussion of indexed and fibred sets. In particular, it contains:

1. A discussion of indexed sets (i.e. functors $K_{\mathsf{disc}} \to \mathsf{Sets}$ with K a set), constructions with them like dependent sums and dependent products, and their properties (Sections 1 and 2);

2. A discussion of fibred sets (i.e. maps of sets $X \to K$), constructions with them like dependent sums and dependent products, and their properties (Sections 3 and 4);

3. A discussion of the un/straightening equivalence for indexed and fibred sets (Section 5).

Contents

00AJ 1 Indexed Sets

00AK 1.1 Foundations

Let K be a set.

00AL **Definition 1.1.1.1.** A **K-indexed set** is a functor $X\colon K_{\mathsf{disc}} \to \mathsf{Sets}$.

00AM **Remark 1.1.1.2.** By Categories, **??**, a **K-indexed set** consists of a K-indexed collection

$$X^{\dagger}\colon K \to \mathrm{Obj}(\mathsf{Sets}),$$

of sets, assigning a set $X^{\dagger}_{x} \overset{\mathrm{def}}{=} X_{x}$ to each element x of K.

00AN 1.2 Morphisms of Indexed Sets

Let $X\colon K_{\mathsf{disc}} \to \mathsf{Sets}$ and $Y\colon K_{\mathsf{disc}} \to \mathsf{Sets}$ be indexed sets.

00AP **Definition 1.2.1.1.** A **morphism of K-indexed sets from X to Y**[1] is a natural transformation

$$f\colon X \Longrightarrow Y, \qquad \begin{tikzcd} K_{\mathsf{disc}} \arrow[r, bend left, "X"] \arrow[r, bend right, "Y"'] \arrow[r, phantom, "\Downarrow f"] & \mathsf{Sets} \end{tikzcd}$$

from X to Y.

00AQ **Remark 1.2.1.2.** In detail, a **morphism of K-indexed sets** consists of a K-indexed collection

$$\{f_x\colon X_x \to Y_x\}_{x\in K}$$

of maps of sets.

00AR 1.3 The Category of Sets Indexed by a Fixed Set

Let K be a set.

00AS **Definition 1.3.1.1.** The **category of K-indexed sets** is the category $\mathsf{ISets}(K)$ defined by

$$\mathsf{ISets}(K) \overset{\text{def}}{=} \mathsf{Fun}\Big(K_{\mathsf{disc}}, \mathsf{Sets}\Big).$$

00AT **Remark 1.3.1.2.** In detail, the **category of K-indexed sets** is the category $\mathsf{ISets}(K)$ where

- *Objects.* The objects of $\mathsf{ISets}(K)$ are K-indexed sets as in Definition 1.1.1.1;
- *Morphisms.* The morphisms of $\mathsf{ISets}(K)$ are morphisms of K-indexed sets as in Definition 1.2.1.1;
- *Identities.* For each $X \in \mathrm{Obj}(\mathsf{ISets}(K))$, the unit map

  $$\mathbb{1}^{\mathsf{ISets}(K)}_X\colon \mathrm{pt} \to \mathrm{Hom}_{\mathsf{ISets}(K)}(X, X)$$

  of $\mathsf{ISets}(K)$ at X is defined by

  $$\mathrm{id}^{\mathsf{ISets}(K)}_X \overset{\text{def}}{=} \{\mathrm{id}_{X_x}\}_{x\in K};$$

- *Composition.* For each $X, Y, Z \in \mathrm{Obj}(\mathsf{ISets}(K))$, the composition map

  $$\circ^{\mathsf{ISets}(K)}_{X,Y,Z}\colon \mathrm{Hom}_{\mathsf{ISets}(K)}(Y, Z) \times \mathrm{Hom}_{\mathsf{ISets}(K)}(X, Y) \to \mathrm{Hom}_{\mathsf{ISets}(K)}(X, Z)$$

  of $\mathsf{ISets}(K)$ at (X, Y, Z) is defined by

  $$\{g_x\}_{x\in K} \circ^{\mathsf{ISets}(K)}_{X,Y,Z} \{f_x\}_{x\in K} \overset{\text{def}}{=} \{g_x \circ f_x\}_{x\in K}.$$

[1] *Further Terminology:* Also called a **K-indexed map of sets from X to Y**.

00AU 1.4 The Category of Indexed Sets

00AV **Definition 1.4.1.1.** The **category of indexed sets** is the category ISets defined as the Grothendieck construction of the functor $\mathsf{ISets}\colon \mathsf{Sets}^{\mathsf{op}} \to \mathsf{Cats}$ of Proposition 2.1.1.4:

$$\mathsf{ISets} \overset{\text{def}}{=} \int^{\mathsf{Sets}} \mathsf{ISets}.$$

00AW **Remark 1.4.1.2.** In detail, the **category of indexed sets** is the category ISets where

- *Objects.* The objects of ISets are pairs (K, X) consisting of
  - *The Indexing Set.* A set K;
  - *The Indexed Set.* A K-indexed set $X\colon K_{\mathsf{disc}} \to \mathsf{Sets}$;
- *Morphisms.* A morphism of ISets from (K, X) to (K', Y) is a pair (ϕ, f) consisting of
  - *The Reindexing Map.* A map of sets $\phi\colon K \to K'$;
  - *The Morphism of Indexed Sets.* A morphism of K-indexed sets $f\colon X \to \phi_*(Y)$ as in the diagram

$$f\colon X \to \phi_*(Y), \qquad \begin{tikzcd} K_{\mathsf{disc}} \arrow[rr, "\phi"] \arrow[dr, "X"'] & & K'_{\mathsf{disc}} \arrow[dl, "Y"] \\ & \mathsf{Sets}; \arrow[ur, phantom, "\overset{f}{\Longrightarrow}"] & \end{tikzcd}$$

- *Identities.* For each $(K, X) \in \mathrm{Obj}(\mathsf{ISets})$, the unit map

$$\mathbb{1}^{\mathsf{ISets}}_{(K,X)}\colon \mathrm{pt} \to \mathsf{ISets}((K, X), (K, X))$$

  of ISets at (K, X) is defined by

$$\mathrm{id}^{\mathsf{ISets}}_{(K,X)} \overset{\text{def}}{=} (\mathrm{id}_K, \mathrm{id}_X).$$

- *Composition.* For each $\mathbf{X} = (K, X)$, $\mathbf{Y} = (K', Y)$, $\mathbf{Z} = (K'', Z) \in \mathrm{Obj}(\mathsf{ISets})$, the composition map

$$\circ^{\mathsf{ISets}}_{\mathbf{X},\mathbf{Y},\mathbf{Z}}\colon \mathsf{ISets}(\mathbf{Y}, \mathbf{Z}) \times \mathsf{ISets}(\mathbf{X}, \mathbf{Y}) \to \mathsf{ISets}(\mathbf{X}, \mathbf{Z})$$

  of ISets at $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ is defined by

$$(\psi, g) \circ (\phi, f) \overset{\text{def}}{=} (\psi \circ \phi, (g \circ \mathrm{id}_\phi) \circ f),$$

as in the diagram

$$
\begin{tikzcd}
K_{\mathsf{disc}} \arrow[r, "\phi"] \arrow[dr, "X"'] \arrow[dr, phantom, shift left=3, "\overset{f}{\Nearrow}"] & K'_{\mathsf{disc}} \arrow[r, "\psi"] \arrow[d, "Y"] \arrow[d, phantom, shift left=5, "\overset{g}{\Nearrow}"] & K''_{\mathsf{disc}} \arrow[dl, "Z"] \\
& \mathsf{Sets}; &
\end{tikzcd}
$$

for each $(\phi, f) \in \mathsf{ISets}(\mathbf{X}, \mathbf{Y})$ and each $(\psi, g) \in \mathsf{ISets}(\mathbf{Y}, \mathbf{Z})$.

00AX 2 Constructions With Indexed Sets

00AY 2.1 Change of Indexing

Let $\phi \colon K \to K'$ be a function and let X be a K'-indexed set.

00AZ **Definition 2.1.1.1.** The **change of indexing of** X **to** K is the K-indexed set $\phi^*(X)$ defined by

$$\phi^*(X) \overset{\text{def}}{=} X \circ \phi_{\mathsf{disc}}.$$

00B0 **Remark 2.1.1.2.** In detail, the **change of indexing of** X **to** K is the K-indexed set $\phi^*(X)$ defined by

$$\phi^*(X)_x \overset{\text{def}}{=} X_{\phi(x)}$$

for each $x \in K$.

00B1 **Proposition 2.1.1.3.** The assignment $X \mapsto \phi^*(X)$ defines a functor

$$\phi^* \colon \mathsf{ISets}\big(K'\big) \to \mathsf{ISets}(K),$$

where

- *Action on Objects.* For each $X \in \mathrm{Obj}(\mathsf{ISets}(K'))$, we have

$$[\phi^*](X) \overset{\text{def}}{=} \phi^*(X);$$

- *Action on Morphisms.* For each $X, Y \in \mathrm{Obj}(\mathsf{ISets}(K'))$, the action on Hom-sets

$$\phi^*_{X,Y} \colon \mathrm{Hom}_{\mathsf{ISets}(K')}(X, Y) \to \mathrm{Hom}_{\mathsf{ISets}(K)}(\phi^*(X), \phi^*(Y))$$

of ϕ^* at (X, Y) is the map sending a morphism of K'-indexed sets

$$f = \{f_x \colon X_x \to Y_x\}_{x \in K'}$$

from X to Y to the morphism of K-indexed sets defined by

$$\phi^*(f) \overset{\text{def}}{=} \left\{f_{\phi(x)} \colon X_{\phi(x)} \to Y_{\phi(x)}\right\}_{x \in K}.$$

Proof. Omitted. □

00B2 **Proposition 2.1.1.4.** The assignment $K \mapsto \mathsf{ISets}(K)$ defines a functor

$$\mathsf{ISets} \colon \mathsf{Sets}^{\mathsf{op}} \to \mathsf{Cats},$$

where

- *Action on Objects.* For each $K \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$[\mathsf{ISets}](K) \overset{\text{def}}{=} \mathsf{ISets}(K);$$

- *Action on Morphisms.* For each $K, K' \in \mathrm{Obj}(\mathsf{Sets})$, the action on Hom-sets

$$\mathsf{ISets}_{K,K'} \colon \mathsf{Sets}^{\mathsf{op}}\big(K, K'\big) \to \mathsf{Fun}\big(\mathsf{ISets}(K), \mathsf{ISets}\big(K'\big)\big)$$

  of ISets at (K, K') is the map defined by

$$\mathsf{ISets}_{K,K'}(\phi) \overset{\text{def}}{=} \phi^*$$

  for each $\phi \in \mathsf{Sets}^{\mathsf{op}}(K, K')$.

Proof. Omitted. □

00B3 2.2 Dependent Sums

Let $\phi \colon K \to K'$ be a function and let X be a K-indexed set.

00B4 **Definition 2.2.1.1.** The **dependent sum of** X is the K'-indexed set $\Sigma_\phi(X)$[2] defined by

$$\Sigma_\phi(X) \overset{\text{def}}{=} \mathrm{Lan}_\phi(X),$$

and hence given by

$$\Sigma_\phi(X)_x \cong \coprod_{y \in \phi^{-1}(x)} X_y$$

for each $x \in K'$.

[2] *Further Notation:* Also written $\phi_*(X)$.

00B5 **Proposition 2.2.1.2.** The assignment $X \mapsto \Sigma_\phi(X)$ defines a functor

$$\Sigma_\phi \colon \mathsf{ISets}(K) \to \mathsf{ISets}\big(K'\big),$$

where

- *Action on Objects.* For each $X \in \mathrm{Obj}(\mathsf{ISets}(K))$, we have

$$[\Sigma_\phi](X) \overset{\text{def}}{=} \Sigma_\phi(X);$$

- *Action on Morphisms.* For each $X, Y \in \mathrm{Obj}(\mathsf{ISets}(K))$, the action on Hom-sets

$$\Sigma_{\phi|X,Y} \colon \mathrm{Hom}_{\mathsf{ISets}(K)}(X,Y) \to \mathrm{Hom}_{\mathsf{ISets}(K')}(\Sigma_\phi(X), \Sigma_\phi(Y))$$

of Σ_ϕ at (X,Y) is the map sending a morphism of K-indexed sets

$$f \colon X \to Y$$

to the morphism of K'-indexed sets defined by

$$\begin{aligned} \Sigma_\phi(f) &\overset{\text{def}}{=} \mathrm{Lan}_\phi(f); \\ &\cong \coprod_{y \in \phi^{-1}(X)} f_y. \end{aligned}$$

Proof. Omitted. □

00B6 ## 2.3 Dependent Products

Let $\phi \colon K \to K'$ be a function and let X be a K-indexed set.

00B7 **Definition 2.3.1.1.** The **dependent product of** X is the K'-indexed set $\Pi_\phi(X)$[3] defined by

$$\Pi_\phi(X) \overset{\text{def}}{=} \mathrm{Ran}_\phi(X),$$

and hence given by

$$\Pi_\phi(X)_x \cong \prod_{y \in \phi^{-1}(x)} X_y$$

for each $x \in K'$.

[3] *Further Notation:* Also written $\phi_!(X)$.

00B8 **Proposition 2.3.1.2.** The assignment $X \mapsto \Pi_\phi(X)$ defines a functor

$$\Pi_\phi \colon \mathsf{ISets}(K) \to \mathsf{ISets}(K'),$$

where

- *Action on Objects.* For each $X \in \mathrm{Obj}(\mathsf{ISets}(K))$, we have
$$[\Pi_\phi](X) \stackrel{\mathrm{def}}{=} \Pi_\phi(X);$$
- *Action on Morphisms.* For each $X, Y \in \mathrm{Obj}(\mathsf{ISets}(K))$, the action on Hom-sets
$$\Pi_{\phi|X,Y} \colon \mathrm{Hom}_{\mathsf{ISets}(K)}(X,Y) \to \mathrm{Hom}_{\mathsf{ISets}(K')}(\Pi_\phi(X), \Pi_\phi(Y))$$
of Π_ϕ at (X,Y) is the map sending a morphism of K-indexed sets
$$f \colon X \to Y$$
to the morphism of K'-indexed sets defined by
$$\begin{aligned}\Pi_\phi(f) &\stackrel{\mathrm{def}}{=} \mathrm{Ran}_\phi(f);\\ &\cong \prod_{y \in \phi^{-1}(x)} f_y.\end{aligned}$$

Proof. Omitted. □

00B9 2.4 Internal Homs

Let K be a set and let X and Y be K-indexed sets.

00BA **Definition 2.4.1.1.** The **internal Hom of indexed sets from X to Y** is the indexed set $\mathbf{Hom}_{\mathsf{ISets}(K)}(X,Y)$ defined by

$$\mathbf{Hom}_{\mathsf{ISets}(K)}(X,Y) \stackrel{\mathrm{def}}{=} \mathsf{Sets}(X_x, Y_x)$$

for each $x \in K$.

00BB 2.5 Adjointness of Indexed Sets

Let $\phi \colon K \to K'$ be a map of sets.

00BC **Proposition 2.5.1.1.** We have a triple adjunction

$$(\Sigma_\phi \dashv \phi^* \dashv \Pi_\phi)\colon \quad \begin{tikzcd} \mathsf{ISets}(K) \arrow[r, bend left, "\Sigma_\phi"] \arrow[r, bend right, "\Pi_\phi"'] & \mathsf{ISets}(K') \arrow[l, "\phi^*" description] \end{tikzcd}.$$

Proof. This follows from Kan Extensions, **??** of **??**. □

00BD 3 Fibred Sets

00BE 3.1 Foundations

Let K be a set.

00BF **Definition 3.1.1.1.** A **K-fibred set** is a pair (X, ϕ) consisting of[4]

- *The Underlying Set.* A set X, called the **underlying set of** (X, ϕ);
- *The Fibration.* A map of sets $\phi\colon X \to K$.

00BG 3.2 Morphisms of Fibred Sets

00BH **Definition 3.2.1.1.** A **morphism of K-fibred sets from** (X, ϕ) **to** (Y, ψ) is a function $f\colon X \to Y$ such that the diagram[5]

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\phi"'] & & Y \arrow[dl, "\psi"] \\ & K & \end{tikzcd}$$

commutes.

[4] *Further Terminology:* The **fibre of** (X, ϕ) **over** $x \in K$ is the set $\phi^{-1}(x)$ (also written ϕ_x) defined by

$$\phi^{-1}(x) \overset{\text{def}}{=} \mathrm{pt} \times_{[x], K, \phi} X, \qquad \begin{tikzcd} \phi^{-1}(x) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "\phi"] \\ \mathrm{pt} \arrow[r, "{[x]}"'] & K. \end{tikzcd}$$

[5] *Further Terminology:* The **transport map associated to f at $x \in K$** is the function

$$f^*_x\colon \phi^{-1}(x) \to \psi^{-1}(x)$$

given by the dashed map in the diagram

$$\begin{tikzcd} \phi^{-1}(x) \arrow[rr] \arrow[dd] \arrow[dr, dashed] & & X \arrow[dr, "f"] \arrow[dd, "\phi" near start] & \\ & \psi^{-1}(x) \arrow[rr, crossing over] & & Y \arrow[dd, "\psi"] \\ \mathrm{pt} \arrow[rr, "{[x]}" near start] \arrow[dr, equal] & & K \arrow[dr, equal] & \\ & \mathrm{pt} \arrow[rr, "{[x]}"'] \arrow[from=uu, crossing over] & & K. \end{tikzcd}$$

00BJ 3.3 The Category of Fibred Sets Over a Fixed Base

00BK **Definition 3.3.1.1.** The **category of K-fibred sets** is the category $\mathsf{FibSets}(K)$ defined as the slice category $\mathsf{Sets}_{/K}$ of Sets over K:

$$\mathsf{FibSets}(K) \overset{\text{def}}{=} \mathsf{Sets}_{/K}.$$

00BL **Remark 3.3.1.2.** In detail $\mathsf{FibSets}(K)$ is the category where

- *Objects.* The objects of $\mathsf{FibSets}(K)$ are pairs (X, ϕ) consisting of
  - *The Fibred Set.* A set X;
  - *The Fibration.* A function $\phi\colon X \to K$;
- *Morphisms.* A morphism of $\mathsf{FibSets}(K)$ from (X, ϕ) to (Y, ψ) is a function $f\colon X \to Y$ making the diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\phi"'] & & Y \arrow[dl, "\psi"] \\ & X & \end{tikzcd}$$

  commute;
- *Identities.* For each $(X, \phi) \in \mathrm{Obj}(\mathsf{FibSets}(K))$, the unit map

$$\mathbb{1}^{\mathsf{FibSets}(K)}_{(X,\phi)}\colon \mathrm{pt} \to \mathrm{Hom}_{\mathsf{FibSets}(K)}((X, \phi), (X, \phi))$$

  of $\mathsf{FibSets}(K)$ at (X, ϕ) is given by

$$\mathrm{id}^{\mathsf{FibSets}(K)}_{(X,\phi)} \overset{\text{def}}{=} \mathrm{id}_X,$$

  as witnessed by the commutativity of the diagram

$$\begin{tikzcd} X \arrow[rr, equal, "\mathrm{id}_X"] \arrow[dr, "\phi"'] & & X \arrow[dl, "\phi"] \\ & K & \end{tikzcd}$$

  in Sets;
- *Composition.* For each $\mathbf{X} = (X, \phi)$, $\mathbf{Y} = (Y, \psi)$, $\mathbf{Z} = (Z, \chi) \in$

$\mathrm{Obj}(\mathsf{FibSets}(K))$, the composition map

$$\circ^{\mathsf{FibSets}(K)}_{\mathbf{X},\mathbf{Y},\mathbf{Z}} \colon \mathrm{Hom}_{\mathsf{FibSets}(K)}(\mathbf{Y},\mathbf{Z}) \times \mathrm{Hom}_{\mathsf{FibSets}(K)}(\mathbf{X},\mathbf{Y}) \to \mathrm{Hom}_{\mathsf{FibSets}(K)}(\mathbf{X},\mathbf{Z})$$

of $\mathsf{FibSets}(K)$ at $(\mathbf{X},\mathbf{Y},\mathbf{Z})$ is defined by

$$\circ^{\mathsf{FibSets}(K)}_{\mathbf{X},\mathbf{Y},\mathbf{Z}} \overset{\mathrm{def}}{=} \circ^{\mathsf{Sets}}_{X,Y,Z},$$

as witnessed by the commutativity of the diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[dr, "\phi"'] & Y \arrow[r, "g"] \arrow[d, "\psi"] & Z \arrow[dl, "\chi"] \\
& K &
\end{tikzcd}$$

in Sets.

00BM 3.4 The Category of Fibred Sets

00BN **Definition 3.4.1.1.** The **category of fibred sets** is the category $\mathsf{FibSets}$ defined as the Grothendieck construction of the functor $\mathsf{FibSets} \colon \mathsf{Sets}^{\mathsf{op}} \to \mathsf{Cats}$ of Proposition 4.1.1.3:

$$\mathsf{FibSets} \overset{\mathrm{def}}{=} \int^{\mathsf{Sets}} \mathsf{FibSets}.$$

00BP **Remark 3.4.1.2.** In detail, the **category of fibred sets** is the category $\mathsf{FibSets}$ where

- *Objects.* The objects of $\mathsf{FibSets}$ are pairs $(K,(X,\phi_X))$ consisting of
  - *The Base Set.* A set K;
  - *The Fibred Set.* A K-fibred set $\phi_X \colon X \to K$;
- *Morphisms.* A morphism of $\mathsf{FibSets}$ from $(K,(X,\phi_X))$ to $(K',(Y,\phi_Y))$ is a pair (ϕ,f) consisting of
  - *The Base Map.* A map of sets $\phi \colon K \to K'$;
  - *The Morphism of Fibred Sets.* A morphism of K-fibred sets

$$f \colon (X,\phi_X) \to \phi_Y^*(Y), \qquad \begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "\phi_X"'] & & Y \times_{K'} K \arrow[dl, "\mathrm{pr}_2"] \\
& K; &
\end{tikzcd}$$

- *Identities.* For each $(K,X) \in \mathrm{Obj}(\mathsf{FibSets})$, the unit map
$$\mathbb{1}^{\mathsf{FibSets}}_{(K,X)} \colon \mathrm{pt} \to \mathsf{FibSets}((K,X),(K,X))$$
of $\mathsf{FibSets}$ at (K,X) is defined by
$$\mathrm{id}^{\mathsf{FibSets}}_{(K,X)} \overset{\mathrm{def}}{=} (\mathrm{id}_K, \sim),$$
where $\sim$ is the isomorphism $X \to X \times_K K$ as in the diagram
$$\begin{tikzcd} X \arrow[rr, dashed, "\sim"] \arrow[dr, "\phi_X"'] & & X \times_K K \arrow[dl, "\mathrm{pr}_2"] \\ & K; & \end{tikzcd}$$

- *Composition.* For each $\mathbf{X} = (K,X), \mathbf{Y} = (K',Y), \mathbf{Z} = (K'',Z) \in \mathrm{Obj}(\mathsf{FibSets})$, the composition map
$$\circ^{\mathsf{FibSets}}_{\mathbf{X},\mathbf{Y},\mathbf{Z}} \colon \mathsf{FibSets}(\mathbf{Y},\mathbf{Z}) \times \mathsf{FibSets}(\mathbf{X},\mathbf{Y}) \to \mathsf{FibSets}(\mathbf{X},\mathbf{Z})$$
of $\mathsf{FibSets}$ at $(\mathbf{X},\mathbf{Y},\mathbf{Z})$ is defined by
$$g \circ^{\mathsf{FibSets}}_{\mathbf{X},\mathbf{Y},\mathbf{Z}} f \overset{\mathrm{def}}{=} (g \times_{K'} \mathrm{id}_K) \circ f$$
as in the diagram
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "\phi_X"'] & Y \times_{K'} K \arrow[r, "g \times_{K'} \mathrm{id}_K"] \arrow[d, "\mathrm{pr}_2"] & \overbrace{(Z \times_{K''} K') \times_{K'} K}^{\cong Z \times_{K''} K} \arrow[dl, "\mathrm{pr}_2"] \\ & K; & \end{tikzcd}$$
for each $f \in \mathsf{FibSets}(\mathbf{X},\mathbf{Y})$ and each $g \in \mathsf{FibSets}(\mathbf{Y},\mathbf{Z})$.

00BQ 4 Constructions With Fibred Sets

00BR 4.1 Change of Base

Let $f \colon K \to K'$ be a function and let (X,ϕ) be a K'-fibred set.

00BS **Definition 4.1.1.1.** The **change of base of** (X, ϕ) **to** K is the K-fibred set $f^*(X)$ defined by

$$f^*(X) \overset{\text{def}}{=} (K \times_{K'} X, \mathrm{pr}_1), \qquad \begin{tikzcd} f^*(X) \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "\phi"] \\ K \arrow[r, "f"'] & K'. \end{tikzcd}$$

00BT **Proposition 4.1.1.2.** The assignment $X \mapsto f^*(X)$ defines a functor

$$f^* \colon \mathsf{FibSets}\big(K'\big) \to \mathsf{FibSets}(K),$$

where

- *Action on Objects.* For each $(X, \phi) \in \mathrm{Obj}(\mathsf{FibSets}(K'))$, we have

$$f^*(X, \phi) \overset{\text{def}}{=} f^*(X);$$

- *Action on Morphisms.* For each $(X, \phi), (Y, \psi) \in \mathrm{Obj}(\mathsf{FibSets}(K'))$, the action on Hom-sets

$$f^*_{X,Y} \colon \mathrm{Hom}_{\mathsf{FibSets}(K')}(X, Y) \to \mathrm{Hom}_{\mathsf{FibSets}(K)}(f^*(X), f^*(Y))$$

of f^* at $((X, \phi), (Y, \psi))$ is the map sending a morphism of K'-fibred sets

$$g \colon (X, \phi) \to (Y, \psi)$$

to the morphism of K-fibred sets given by the dashed morphism in the diagram

$$\begin{tikzcd}[row sep=small, column sep=small] f^*(X) \arrow[rr] \arrow[dr, dashed] \arrow[dd] & & X \arrow[dr, "g"] \arrow[dd, dashed, "\phi" near start] & \\ & f^*(Y) \arrow[rr, crossing over] & & Y \arrow[dd, "\psi"] \\ K \arrow[rr, "f" near start] \arrow[dr, equal] & & K' \arrow[dr, equal] & \\ & K \arrow[rr, "f"'] \arrow[from=uu, crossing over] & & K'. \end{tikzcd}$$

Proof. Omitted. □

00BU **Proposition 4.1.1.3.** The assignment $K \mapsto \mathsf{FibSets}(K)$ defines a functor

$$\mathsf{FibSets}\colon \mathsf{Sets}^{\mathsf{op}} \to \mathsf{Cats},$$

where

- *Action on Objects.* For each $K \in \mathrm{Obj}(\mathsf{Sets})$, we have

  $$[\mathsf{FibSets}](K) \overset{\mathrm{def}}{=} \mathsf{FibSets}(K);$$

- *Action on Morphisms.* For each $K, K' \in \mathrm{Obj}(\mathsf{Sets})$, the action on Hom-sets

  $$\mathsf{Sets}_{/(-)|K,K'}\colon \mathsf{Sets}^{\mathsf{op}}\big(K, K'\big) \to \mathsf{Fun}\big(\mathsf{FibSets}(K), \mathsf{FibSets}\big(K'\big)\big)$$

  of $\mathsf{Sets}_{/(-)}$ at (K, K') is the map sending a map of sets $f\colon K \to K'$ to the functor

  $$\mathsf{Sets}_{/f}\colon \mathsf{FibSets}\big(K'\big) \to \mathsf{FibSets}(K)$$

  defined by

  $$\mathsf{Sets}_{/f} \overset{\mathrm{def}}{=} f^*.$$

Proof. Omitted. □

00BV 4.2 Dependent Sums

Let $f\colon K \to K'$ be a function and let (X, ϕ) be a K-fibred set.

00BW **Definition 4.2.1.1.** The **dependent sum**[6] **of** (X, ϕ) is the K'-fibred set $\Sigma_f(X)$[7] defined by

$$\begin{aligned}\Sigma_f(X) &\overset{\mathrm{def}}{=} (\Sigma_f(X), \Sigma_f(\phi))\\ &\overset{\mathrm{def}}{=} (X, f \circ \phi).\end{aligned}$$

00BX **Proposition 4.2.1.2.** Let $f\colon K \to K'$ be a function.

[6] The name "dependent sum" comes from the fact that the fibre $\Sigma_f(\phi)^{-1}(x)$ of $\Sigma_f(X)$ at $x \in K'$ is given by

$$\Sigma_f(\phi)^{-1}(x) \cong \coprod_{y \in f^{-1}(x)} \phi^{-1}(y);$$

see Item 2 of Proposition 4.2.1.2.

[7] *Further Notation:* Also written $f_*(X)$.

00BY 1. *Functoriality*. The assignment $X \mapsto \Sigma_f(X)$ defines a functor

$$\Sigma_f \colon \mathsf{FibSets}(K) \to \mathsf{FibSets}\big(K'\big),$$

where

- *Action on Objects*. For each $(X, \phi) \in \mathrm{Obj}(\mathsf{FibSets}(K))$, we have

$$\Sigma_f(X, \phi) \stackrel{\text{def}}{=} (\Sigma_f(X), \Sigma_f(\phi));$$

- *Action on Morphisms*. For each $(X, \phi), (Y, \psi) \in \mathrm{Obj}(\mathsf{FibSets}(K))$, the action on Hom-sets

$$\Sigma_{f|X,Y} \colon \mathrm{Hom}_{\mathsf{FibSets}(K)}(X, Y) \to \mathrm{Hom}_{\mathsf{FibSets}(K)}(\Sigma_f(X), \Sigma_f(Y))$$

of Σ_f at $((X, \phi), (Y, \psi))$ is the map sending a morphism of K-fibred sets

$$g \colon (X, \phi) \to (Y, \psi)$$

to the morphism of K'-fibred sets defined by

$$\Sigma_f(g) \stackrel{\text{def}}{=} g.$$

00BZ 2. *Interaction With Fibres*. We have a bijection of sets

$$\Sigma_f(\phi)^{-1}(x) \cong \coprod_{y \in f^{-1}(x)} \phi^{-1}(y)$$

for each $x \in K'$.

Proof. *Item 1, Functoriality*: Omitted.
Item 2, Interaction With Fibres: Indeed, we have

$$\begin{aligned}\Sigma_f(\phi)^{-1}(x) &\stackrel{\text{def}}{=} \mathrm{pt} \times_{[x], K', f \circ \phi} X \\ &\cong \{(a, y) \in X \times K \mid f(\phi(a)) = x\} \\ &\cong \coprod_{y \in f^{-1}(x)} \phi^{-1}(y)\end{aligned}$$

for each $x \in K'$. □

00C0 4.3 Dependent Products

Let $f\colon K \to K'$ be a function and let (X, ϕ) be a K-fibred set.

00C1 **Definition 4.3.1.1.** The **dependent product**[8] **of** (X, ϕ) is the K'-fibred set $\Pi_f(X)$[9] consisting of[10]

- *The Underlying Set.* The set $\Pi_f(X)$ defined by

$$\begin{aligned}\Pi_f(X) &\overset{\text{def}}{=} \coprod_{x\in K'} \Gamma^{\phi}_{f^{-1}(x)}\big(\phi^{-1}\big(f^{-1}(x)\big)\big)\\ &\overset{\text{def}}{=} \left\{ (x,h) \in \coprod_{x\in K'} \mathsf{Sets}\big(f^{-1}(x), \phi^{-1}\big(f^{-1}(x)\big)\big) \;\middle|\; \phi\circ h = \mathrm{id}_{f^{-1}(x)} \right\};\end{aligned}$$

- *The Fibration.* The map of sets

$$\Pi_f(\phi)\colon \coprod_{x\in K'} \Gamma^{\phi}_{f^{-1}(x)}\big(\phi^{-1}\big(f^{-1}(x)\big)\big) \to K$$

  defined by sending a map $h\colon f^{-1}(x) \to \phi^{-1}\big(f^{-1}(x)\big)$ to its index $x \in K$.

00C2 **Example 4.3.1.2.** Here are some examples of dependent products of sets.

1. *Spaces of Sections.* Let $K = X$, $K' = \mathrm{pt}$, and let $\phi\colon E \to X$ be a map of sets. We have a bijection of sets

$$\begin{aligned}\Pi_{!_X}(\phi) &\cong \Gamma_X(\phi)\\ &\cong \{h \in \mathsf{Sets}(X, E) \mid \phi\circ h = \mathrm{id}_X\}.\end{aligned}$$

2. *Function Spaces.* Let $K = K' = \mathrm{pt}$. We have a bijection of sets

$$\mathsf{Sets}(X, Y) \cong \Pi_{!_X}(!_X^*(Y)).$$

00C3 **Proposition 4.3.1.3.** Let $f\colon K \to K'$ be a function.

[8] The name "dependent product" comes from the fact that the fibre $\Pi_f(\phi)^{-1}(x)$ of $\Pi_f(X)$ at $x \in K'$ is given by

$$\Pi_f(\phi)^{-1}(x) \cong \prod_{y\in f^{-1}(x)} \phi^{-1}(y);$$

see Item 2 of Proposition 4.3.1.3.

[9] *Further Notation:* Also written $f_!(X)$.

[10] We can also define dependent products via the internal **Hom** in $\mathsf{FibSets}(K')$; see Item 3

00C4 1. *Functoriality.* The assignment $X \mapsto \Pi_f(X)$ defines a functor

$$\Pi_f \colon \mathsf{FibSets}(K) \to \mathsf{FibSets}(K'),$$

where

- *Action on Objects.* For each $(X, \phi) \in \mathrm{Obj}(\mathsf{FibSets}(K))$, we have

$$\Pi_f(X, \phi) \stackrel{\text{def}}{=} \Pi_f(X);$$

- *Action on Morphisms.* For each $(X, \phi), (Y, \psi) \in \mathrm{Obj}(\mathsf{FibSets}(K))$, the action on Hom-sets

$$\Pi_{f|X,Y} \colon \mathrm{Hom}_{\mathsf{FibSets}(K)}(X, Y) \to \mathrm{Hom}_{\mathsf{FibSets}(K')}(\Pi_f(X), \Pi_f(Y))$$

of Π_f at $((X, \phi), (Y, \psi))$ is the map sending a morphism of K-fibred sets

$$g \colon (X, \phi) \to (Y, \psi)$$

to the morphism of K'-fibred sets from

$$\Pi_f(X) \stackrel{\text{def}}{=} \left\{ (x, h) \in \coprod_{x \in K'} \mathsf{Sets}\big(f^{-1}(x), \phi^{-1}\big(f^{-1}(x)\big)\big) \;\middle|\; \phi \circ h = \mathrm{id}_{f^{-1}(x)} \right\};$$

to

$$\Pi_f(Y) \stackrel{\text{def}}{=} \left\{ (x, h) \in \coprod_{x \in K'} \mathsf{Sets}\big(f^{-1}(x), \psi^{-1}\big(f^{-1}(x)\big)\big) \;\middle|\; \psi \circ h = \mathrm{id}_{f^{-1}(x)} \right\};$$

induced by the composition

$$\begin{aligned} \mathsf{Sets}\big(f^{-1}(x), \phi^{-1}\big(f^{-1}(x)\big)\big) &= \mathsf{Sets}\big(f^{-1}(x), [\psi \circ g]^{-1}\big(f^{-1}(x)\big)\big) \\ &= \mathsf{Sets}\big(f^{-1}(x), g^{-1}\big(\psi^{-1}\big(f^{-1}(x)\big)\big)\big) \\ &\xrightarrow{g_*} \mathsf{Sets}\big(f^{-1}(x), g\big(g^{-1}\big(\psi^{-1}\big(f^{-1}(x)\big)\big)\big)\big) \\ &\xrightarrow{\iota_*} \mathsf{Sets}\big(f^{-1}(x), \psi^{-1}\big(f^{-1}(x)\big)\big), \end{aligned}$$

where $\iota \colon g\big(g^{-1}\big(\psi^{-1}\big(f^{-1}(x)\big)\big)\big) \hookrightarrow \psi^{-1}\big(f^{-1}(x)\big)$ is the canonical

of Proposition 4.3.1.3.

inclusion.[11]

00C5 2. *Interaction With Fibres.* We have a bijection of sets

$$\Pi_f(\phi)^{-1}(x) \cong \prod_{y \in f^{-1}(x)} \phi^{-1}(y)$$

for each $x \in K'$.

00C6 3. *Construction Using the Internal Hom.* We have

$$\Pi_f(X,\phi) = (K' \times_{\mathbf{Hom}_{\mathsf{FibSets}(K')}(f,f)} \mathbf{Hom}_{\mathsf{Sets}/K'}(f, f \circ \phi), \mathrm{pr}_1),$$

$$\begin{tikzcd} \Pi_f(X,\phi) \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Hom}_{\mathsf{Sets}/K'}(f, f \circ \phi) \arrow[d] \\ K' \arrow[r, "I"'] & \mathbf{Hom}_{\mathsf{FibSets}(K')}(f,f), \end{tikzcd}$$

where the bottom map is defined by

$$I(x) \overset{\text{def}}{=} \mathrm{id}_{f^{-1}(x)}$$

for each $x \in K'$.

Proof. *Item 1, Functoriality*: Omitted.
Item 2, Interaction With Fibres: Indeed, we have

$$\begin{aligned} \Pi_f(\phi)^{-1}(x) &\overset{\text{def}}{=} \{(y,h) \in \Pi_f(X) \mid [\Pi_f(\phi)](h) = x\} \\ &\overset{\text{def}}{=} \{(y,h) \in \Pi_f(X) \mid y = x\} \\ &\cong \left\{h \in \mathsf{Sets}\big(f^{-1}(x), \phi^{-1}\big(f^{-1}(x)\big)\big) \;\middle|\; \phi \circ h = \mathrm{id}_{f^{-1}(x)}\right\} \\ &\cong \prod_{y \in f^{-1}(x)} \phi^{-1}(y) \end{aligned}$$

for each $x \in K'$.
Item 3, Construction Using the Internal Hom: Omitted. □

[11] Note that the section condition is satisfied: given $(x,h) \in \Pi_f(X)$, we have

$$\begin{aligned} \psi \circ [\Pi_f(g)](h) &\overset{\text{def}}{=} \psi \circ (g \circ h) \\ &= (\psi \circ g) \circ h \\ &= \phi \circ h \\ &= \mathrm{id}_{f^{-1}(x)}. \end{aligned}$$

00C7 4.4 Internal Homs

Let K be a set and let (X, ϕ) and (Y, ψ) be K-fibred sets.

00C8 **Definition 4.4.1.1.** The **internal Hom of fibred sets from** (X, ϕ) **to** (Y, ψ) is the fibred set $\mathbf{Hom}_{\mathsf{FibSets}(K)}(X, Y)$ consisting of

- *The Underlying Set.* The set $\mathbf{Hom}_{\mathsf{FibSets}(K)}(X, Y)$ defined by

$$\mathbf{Hom}_{\mathsf{FibSets}(K)}(X, Y) \stackrel{\text{def}}{=} \coprod_{x \in K} \mathsf{Sets}\big(\phi^{-1}(x), \psi^{-1}(x)\big);$$

- *The Fibration.* The map of sets[12]

$$\phi_{\mathbf{Hom}_{\mathsf{FibSets}(K)}(X,Y)} \colon \underbrace{\mathbf{Hom}_{\mathsf{FibSets}(K)}(X, Y)}_{\coprod_{x \in K} \mathsf{Sets}(\phi^{-1}(x), \psi^{-1}(x))} \to K$$

  defined by sending a map $f \colon \phi^{-1}(x) \to \psi^{-1}(x)$ to its index $x \in K$.

00C9 4.5 Adjointness for Fibred Sets

Let $f \colon K \to K'$ be a map of sets.

00CA **Proposition 4.5.1.1.** We have a triple adjunction

$$(\Sigma_f \dashv f^* \dashv \Pi_f)\colon \quad \begin{tikzcd} \mathsf{FibSets}(K) \arrow[r, bend left, "\Sigma_f"] \arrow[r, bend right, "\Pi_f"'] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, phantom, shift right=3, "\perp"] & \mathsf{FibSets}(K') \arrow[l, "f^*"'] \end{tikzcd}.$$

Proof. Omitted. □

00CB 5 Un/Straightening for Indexed and Fibred Sets

[12]The fibres of the internal $\mathbf{Hom}$ of $\mathsf{FibSets}(K)$ are precisely the sets $\mathsf{Sets}\big(\phi^{-1}(x), \psi^{-1}(x)\big)$, i.e. we have

$$\phi_{\mathbf{Hom}_{\mathsf{FibSets}(K)}(X,Y)|x} \cong \mathsf{Sets}\big(\phi^{-1}(x), \psi^{-1}(x)\big)$$

for each $x \in K$.

00CC 5.1 Straightening for Fibred Sets

Let K be a set and let (X, ϕ) be a K-fibred set.

00CD **Definition 5.1.1.1.** The **straightening of** (X, ϕ) is the K-indexed set

$$\mathrm{St}_K(X, \phi) \colon K_{\mathsf{disc}} \to \mathsf{Sets}$$

defined by

$$\mathrm{St}_K(X, \phi)_x \overset{\text{def}}{=} \phi^{-1}(x)$$

for each $x \in K$.

00CE **Proposition 5.1.1.2.** Let K be a set.

00CF 1. *Functoriality.* The assignment $(X, \phi) \mapsto \mathrm{St}_K(X, \phi)$ defines a functor

$$\mathrm{St}_K \colon \mathsf{FibSets}(K) \to \mathsf{ISets}(K)$$

- *Action on Objects.* For each $(X, \phi) \in \mathrm{Obj}(\mathsf{FibSets}(K))$, we have

$$[\mathrm{St}_K](X, \phi) \overset{\text{def}}{=} \mathrm{St}_K(X, \phi);$$

- *Action on Morphisms.* For each $(X, \phi), (Y, \psi) \in \mathrm{Obj}(\mathsf{FibSets}(K))$, the action on Hom-sets

$$\mathrm{St}_{K|X,Y} \colon \mathrm{Hom}_{\mathsf{FibSets}(K)}(X, Y) \to \mathrm{Hom}_{\mathsf{ISets}(K)}(\mathrm{St}_K(X), \mathrm{St}_K(Y))$$

of St_K at (X, Y) is given by sending a morphism

$$f \colon (X, \phi) \to (Y, \psi)$$

of K-fibred sets to the morphism

$$\mathrm{St}_K(f) \colon \mathrm{St}_K(X, \phi) \to \mathrm{St}_K(Y, \psi)$$

of K-indexed sets defined by

$$\mathrm{St}_K(f) \overset{\text{def}}{=} \{f_x^*\}_{x \in K},$$

where f_x^* is the transport map associated to f at $x \in K$ of Definition 3.2.1.1.

00CG 2. *Interaction With Change of Base/Indexing.* Let $f \colon K \to K'$ be a map of sets. The diagram

$$\begin{tikzcd} \mathsf{FibSets}(K') \arrow[r, "f^*"] \arrow[d, "\mathrm{St}_{K'}"'] & \mathsf{FibSets}(K) \arrow[d, "\mathrm{St}_K"] \\ \mathsf{ISets}(K') \arrow[r, "f^*"'] & \mathsf{ISets}(K) \end{tikzcd}$$

commutes.

00CH 3. *Interaction With Dependent Sums.* Let $f\colon K \to K'$ be a map of sets. The diagram

$$\begin{tikzcd} \mathsf{FibSets}(K) \arrow[r, "\Sigma_f"] \arrow[d, "\mathrm{St}_{K}"'] & \mathsf{FibSets}(K') \arrow[d, "\mathrm{St}_{K'}"] \\ \mathsf{ISets}(K) \arrow[r, "\Sigma_f"'] & \mathsf{ISets}(K') \end{tikzcd}$$

commutes.

00CJ 4. *Interaction With Dependent Products.* Let $f\colon K \to K'$ be a map of sets. The diagram

$$\begin{tikzcd} \mathsf{Sets}_{/K} \arrow[r, "\Pi_f"] \arrow[d, "\mathrm{St}_{K}"'] & \mathsf{FibSets}(K') \arrow[d, "\mathrm{St}_{K'}"] \\ \mathsf{ISets}(K) \arrow[r, "\Pi_f"'] & \mathsf{ISets}(K') \end{tikzcd}$$

commutes.

Proof. *Item 1*, *Functoriality*: Omitted.

Item 2, Interaction With Change of Base/Indexing: Indeed, we have

$$\begin{aligned}
\mathrm{St}_K(f^*(X,\phi))_x &\overset{\text{def}}{=} \mathrm{St}_K(K \times_{K'} X)_x \\
&\overset{\text{def}}{=} \left(\mathrm{pr}_1^{K\times_{K'}X}\right)^{-1}(x) \\
&= \left\{(k,y) \in K \times_{K'} X \;\middle|\; \mathrm{pr}_1^{K\times_{K'}X}(k,y) = x\right\} \\
&= \{(k,y) \in K \times_{K'} X \mid k = x\} \\
&= \{(k,y) \in K \times X \mid k = x \text{ and } f(k) = \phi(y)\} \\
&\cong \{y \in X \mid \phi(y) = f(x)\} \\
&= \phi^{-1}(f(x)) \\
&\overset{\text{def}}{=} f^*\big(\phi^{-1}(x)\big) \\
&\overset{\text{def}}{=} f^*(\mathrm{St}_{K'}(X,\phi)_x)
\end{aligned}$$

for each $(X,\phi) \in \mathrm{Obj}(\mathsf{FibSets}(K'))$ and each $x \in K$, and similarly for morphisms.
Item 3, Interaction With Dependent Sums: Indeed, we have

$$\begin{aligned}
\mathrm{St}_{K'}(\Sigma_f(X,\phi))_x &\overset{\text{def}}{=} \Sigma_f(\phi)^{-1}(x) \\
&\cong \coprod_{\substack{y\in X\\ f(y)=x}} \phi^{-1}(y) \\
&\cong \Sigma_f\big(\phi^{-1}(x)\big) \\
&\overset{\text{def}}{=} \Sigma_f(\mathrm{St}_K(X,\phi)_x)
\end{aligned}$$

for each $(X,\phi) \in \mathrm{Obj}(\mathsf{FibSets}(K))$ and each $x \in K'$, where we have used Item 2 of Proposition 4.2.1.2 for the first bijection, and similarly for morphisms.
Item 4, Interaction With Dependent Products: Indeed, we have

$$\begin{aligned}
\mathrm{St}_{K'}(\Pi_f(X,\phi))_x &\overset{\text{def}}{=} \Pi_f(\phi)^{-1}(x) \\
&\cong \prod_{\substack{y\in X\\ f(y)=x}} \phi^{-1}(y) \\
&\cong \Pi_f\big(\phi^{-1}(x)\big) \\
&\overset{\text{def}}{=} \Pi_f(\mathrm{St}_K(X,\phi)_x)
\end{aligned}$$

for each $(X,\phi) \in \mathrm{Obj}(\mathsf{FibSets}(K))$ and each $x \in K'$, where we have used Item 2 of Proposition 4.3.1.3 for the first bijection, and similarly for morphisms. □

00CK 5.2 Unstraightening for Indexed Sets

Let K be a set and let X be a K-indexed set.

00CL **Definition 5.2.1.1.** The **unstraightening of** X is the K-fibred set

$$\phi_{\mathrm{Un}_K} \colon \mathrm{Un}_K(X) \to K$$

consisting of

- *The Underlying Set.* The set $\mathrm{Un}_K(X)$ defined by

$$\mathrm{Un}_K(X) \overset{\mathrm{def}}{=} \coprod_{x \in K} X_x;$$

- *The Fibration.* The map of sets

$$\phi_{\mathrm{Un}_K} \colon \mathrm{Un}_K(X) \to K$$

defined by sending an element of $\coprod_{x\in K} X_x$ to its index in K.

00CM **Proposition 5.2.1.2.** Let K be a set.

00CN 1. *Functoriality.* The assignment $X \mapsto \mathrm{Un}_K(X)$ defines a functor

$$\mathrm{Un}_K \colon \mathsf{ISets}(K) \to \mathsf{FibSets}(K)$$

- *Action on Objects.* For each $X \in \mathrm{Obj}(\mathsf{ISets}(K))$, we have

$$[\mathrm{Un}_K](X) \overset{\mathrm{def}}{=} \mathrm{Un}_K(X);$$

- *Action on Morphisms.* For each $X, Y \in \mathrm{Obj}(\mathsf{ISets}(K))$, the action on Hom-sets

$$\mathrm{Un}_{K|X,Y} \colon \mathrm{Hom}_{\mathsf{ISets}(K)}(X,Y) \to \mathrm{Hom}_{\mathsf{FibSets}(K)}(\mathrm{Un}_K(X), \mathrm{Un}_K(Y))$$

of Un_K at (X,Y) is defined by

$$\mathrm{Un}_{K|X,Y}(f) \overset{\mathrm{def}}{=} \coprod_{x\in K} f_x^*.$$

00CP 2. *Interaction With Fibres.* We have a bijection of sets

$$\phi^{-1}_{\mathrm{Un}_K}(x) \cong X_x$$

for each $x \in K$.

00CQ 3. *As a Pullback.* We have a bijection of sets

$$\mathrm{Un}_K(X) \cong K_{\mathsf{disc}} \times_{\mathsf{Sets}} \mathsf{Sets}_*, \qquad \begin{tikzcd} \mathrm{Un}_K(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Sets}_* \arrow[d, "\text{忘}"] \\ K_{\mathsf{disc}} \arrow[r, "X"'] & \mathsf{Sets}. \end{tikzcd}$$

00CR 4. *As a Colimit.* We have a bijection of sets

$$\mathrm{Un}_K(X) \cong \mathrm{colim}(X).$$

00CS 5. *Interaction With Change of Indexing/Base.* Let $f \colon K \to K'$ be a map of sets. The diagram

$$\begin{tikzcd} \mathsf{ISets}(K') \arrow[r, "f^*"] \arrow[d, "\mathrm{Un}_{K'}"'] & \mathsf{ISets}(K) \arrow[d, "\mathrm{Un}_K"] \\ \mathsf{FibSets}(K') \arrow[r, "f^*"'] & \mathsf{FibSets}(K) \end{tikzcd}$$

commutes.

00CT 6. *Interaction With Dependent Sums.* Let $f \colon K \to K'$ be a map of sets. The diagram

$$\begin{tikzcd} \mathsf{ISets}(K) \arrow[r, "\Sigma_f"] \arrow[d, "\mathrm{Un}_K"'] & \mathsf{ISets}(K') \arrow[d, "\mathrm{Un}_{K'}"] \\ \mathsf{FibSets}(K) \arrow[r, "\Sigma_f"'] & \mathsf{FibSets}(K') \end{tikzcd}$$

commutes.

00CU 7. *Interaction With Dependent Products.* Let $f \colon K \to K'$ be a map of sets. The diagram

$$\begin{tikzcd} \mathsf{ISets}(K) \arrow[r, "\Pi_f"] \arrow[d, "\mathrm{Un}_K"'] & \mathsf{ISets}(K') \arrow[d, "\mathrm{Un}_{K'}"] \\ \mathsf{FibSets}(K) \arrow[r, "\Pi_f"'] & \mathsf{FibSets}(K') \end{tikzcd}$$

commutes.

Proof. *Item 1, Functoriality*: Omitted.
Item 2, Interaction With Fibres: Omitted.
Item 3, As a Pullback: Omitted.
Item 4, As a Colimit: Clear.
Item 5, Interaction With Change of Indexing/Base: Indeed, we have

$$\begin{aligned}
\mathrm{Un}_K(f^*(X)) &\stackrel{\text{def}}{=} \mathrm{Un}_K(X \circ f)\\
&\stackrel{\text{def}}{=} \coprod_{x\in K} X_{f(x)}\\
&\cong \left\{(x,(y,a)) \in K \times \coprod_{y\in K'} X_y \;\middle|\; f(x)=y\right\}\\
&\cong K \times_{K'} \coprod_{y\in K'} X_y\\
&\stackrel{\text{def}}{=} K \times_{K'} \mathrm{Un}_{K'}(X)\\
&\stackrel{\text{def}}{=} f^*(\mathrm{Un}_{K'}(X))
\end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{ISets}(K'))$. Similarly, it can be shown that we also have $\mathrm{Un}_K(f^*(\phi)) = f^*(\mathrm{Un}_{K'}(\phi))$ and that $\mathrm{Un}_K \circ f^* = f^* \circ \mathrm{Un}_{K'}$ also holds on morphisms.
Item 6, Interaction With Dependent Sums: Indeed, we have

$$\begin{aligned}
\mathrm{Un}_{K'}(\Sigma_f(X)) &\stackrel{\text{def}}{=} \coprod_{x\in K'} \Sigma_f(X)_x\\
&\cong \coprod_{x\in K'} \coprod_{y\in f^{-1}(x)} X_y\\
&\cong \coprod_{y\in K} X_y\\
&\cong \mathrm{Un}_K(X)\\
&\stackrel{\text{def}}{=} \Sigma_f(\mathrm{Un}_K(X))
\end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{ISets}(K))$, where we have used Item 2 of Proposition 4.2.1.2 for the first bijection. Similarly, it can be shown that we also have $\mathrm{Un}_{K'}(\Sigma_f(\phi)) = \Sigma_f(\phi_{\mathrm{Un}_K})$ and that $\mathrm{Un}_{K'} \circ \Sigma_f = \Sigma_f \circ \mathrm{Un}_K$ also holds on morphisms.

Item 7, Interaction With Dependent Products: Indeed, we have

$$
\begin{aligned}
\mathrm{Un}_{K'}(\Pi_f(X)) &\stackrel{\text{def}}{=} \coprod_{x\in K'} \Pi_f(X)_x\\
&\cong \coprod_{x\in K'} \prod_{y\in f^{-1}(x)} X_y\\
&\cong \left\{ (x,h) \in \coprod_{x\in K'} \mathsf{Sets}\big(f^{-1}(x), \phi^{-1}_{\mathrm{Un}_K}\big(f^{-1}(x)\big)\big) \;\middle|\; \phi\circ h = \mathrm{id}_{f^{-1}(x)} \right\}\\
&\stackrel{\text{def}}{=} \Pi_f\left(\coprod_{y\in K} X_y\right)\\
&\stackrel{\text{def}}{=} \Pi_f(\mathrm{Un}_K(X))
\end{aligned}
$$

for each $X \in \mathrm{Obj}(\mathsf{ISets}(K))$, where we have used Item 2 of Proposition 4.3.1.3 for the first bijection. Similarly, it can be shown that we also have $\mathrm{Un}_{K'}(\Pi_f(\phi)) = \Pi_f(\phi_{\mathrm{Un}_K})$ and that $\mathrm{Un}_{K'} \circ \Pi_f = \Pi_f \circ \mathrm{Un}_K$ also holds on morphisms. □

00CV 5.3 The Un/Straightening Equivalence

00CW **Theorem 5.3.1.1.** We have an isomorphism of categories

$$
(\mathrm{St}_K \dashv \mathrm{Un}_K)\colon\quad \begin{tikzcd} \mathsf{FibSets}(K) \arrow[r, bend left, "\mathrm{St}_K"] \arrow[r, phantom, "\perp"] & \mathsf{ISets}(K). \arrow[l, bend left, "\mathrm{Un}_K"] \end{tikzcd}
$$

Proof. Omitted. □

00CX 6 Miscellany

00CY 6.1 Other Kinds of Un/Straightening

00CZ **Remark 6.1.1.1.** There are also other kinds of un/straightening for sets, where Sets is replaced by $\mathbf{Rel}$ or Span:

- *Un/Straightening With **Rel**, I.* We have an isomorphism of sets

  $$\mathrm{Rel}(A,B) \cong \mathsf{Sets}(B\times A, \{\mathsf{true}, \mathsf{false}\}).$$

  by the definition of a relation from A to B, Relations, Definition 1.1.1.1.

- *Un/Straightening With **Rel**, II.* We have an equivalence of categories

  $$\mathsf{LaxFun}(K_{\mathsf{disc}}, \mathbf{Rel}) \stackrel{\text{eq.}}{\cong} \mathsf{Cats}^{\mathsf{fth}}_{/K_{\mathsf{disc}}},$$

where $\mathsf{Cats}^{\mathsf{fth}}_{/K_{\mathsf{disc}}}$ is the full subcategory of $\mathsf{Cats}_{/K_{\mathsf{disc}}}$ spanned by the faithful functors; see [Nie04, Theorem 3.1].

- *Un/Straightening With* Span*, I.* For each $A, B \in \mathrm{Obj}(\mathsf{Sets})$, we have a morphism of sets

$$\mathsf{Span}(A, B) \to \mathsf{Sets}(A \times B, \mathbb{N} \cup \{\infty\})$$

which assemble into an equivalence of categories between $\mathsf{Span}(\mathsf{Sets})$ and the category MRel of "multirelations"; see Spans, Remark 7.5.1.1.

- *Un/Straightening With* Span*, II.* We have an equivalence of categories

$$\mathsf{LaxFun}(K_{\mathsf{disc}}, \mathsf{Span}) \overset{\text{eq.}}{\cong} \mathsf{Cats}_{/K_{\mathsf{disc}}};$$

see [nLa23, Section 3].

Appendices

A Other Chapters