# Indexed and Fibred Sets

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This chapter contains a discussion of the un/straightening equivalence in the context of sets, as well as a general discussion of indexed and fibred sets. In particular, it contains:

- 1. A discussion of indexed sets (i.e. functors  $K_{\text{disc}} \to \text{Sets}$  with K a set), constructions with them like dependent sums and dependent products, and their properties (Sections 1 and 2);
- 2. A discussion of fibred sets (i.e. maps of sets  $X \to K$ ), constructions with them like dependent sums and dependent products, and their properties (Sections 3 and 4);
- 3. A discussion of the un/straightening equivalence for indexed and fibred sets (Section 5).

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# 1 Indexed Sets

### 1.1 Foundations

Let K be a set.

**Definition 1.1.1.1.** A K-indexed set is a functor  $X: K_{\mathsf{disc}} \to \mathsf{Sets}$ .

**Remark 1.1.1.2.** By Categories, ??, a *K*-indexed set consists of a *K*-indexed collection

$$X^{\dagger} \colon K \to \mathrm{Obj}(\mathsf{Sets}),$$

of sets, assigning a set  $X_x^\dagger \stackrel{\text{def}}{=} X_x$  to each element x of K.

# 1.2 Morphisms of Indexed Sets

Let  $X \colon K_{\mathsf{disc}} \to \mathsf{Sets}$  and  $Y \colon K_{\mathsf{disc}} \to \mathsf{Sets}$  be indexed sets.

Definition 1.2.1.1. A morphism of K-indexed sets from X to  $Y^1$  is a natural transformation

$$f \colon X \Longrightarrow Y, \qquad K_{\mathsf{disc}} \underbrace{f \downarrow}_{Y} \mathsf{Sets}$$

from X to Y.

Remark 1.2.1.2. In detail, a morphism of K-indexed sets consists of a K-indexed collection

$$\{f_x\colon X_x\to Y_x\}_{x\in K}$$

of maps of sets.

## 1.3 The Category of Sets Indexed by a Fixed Set

Let K be a set.

**Definition 1.3.1.1.** The category of K-indexed sets is the category  $\mathsf{ISets}(K)$  defined by

$$\mathsf{ISets}(K) \stackrel{\text{def}}{=} \mathsf{Fun}(K_{\mathsf{disc}}, \mathsf{Sets}).$$

Remark 1.3.1.2. In detail, the category of K-indexed sets is the category  $\mathsf{ISets}(K)$  where

- Objects. The objects of  $\mathsf{ISets}(K)$  are K-indexed sets as in Definition 1.1.1.1;
- *Morphisms*. The morphisms of  $\mathsf{ISets}(K)$  are morphisms of K-indexed sets as in Definition 1.2.1.1;
- *Identities*. For each  $X \in \text{Obj}(\mathsf{ISets}(K))$ , the unit map

$$\mathbb{K}_X^{\mathsf{ISets}(K)} \colon \mathrm{pt} \to \mathrm{Hom}_{\mathsf{ISets}(K)}(X,X)$$

of  $\mathsf{ISets}(K)$  at X is defined by

$$\mathrm{id}_X^{\mathsf{ISets}(K)} \stackrel{\mathrm{def}}{=} \{ \mathrm{id}_{X_x} \}_{x \in K};$$

• Composition. For each  $X,Y,Z\in \mathrm{Obj}(\mathsf{ISets}(K)),$  the composition map

$$\circ_{X,Y,Z}^{|\mathsf{Sets}(K)} \colon \mathrm{Hom}_{\mathsf{ISets}(K)}(Y,Z) \times \mathrm{Hom}_{\mathsf{ISets}(K)}(X,Y) \to \mathrm{Hom}_{\mathsf{ISets}(K)}(X,Z)$$
 of  $\mathsf{ISets}(K)$  at  $(X,Y,Z)$  is defined by

$$\{g_x\}_{x\in K}\circ_{X,Y,Z}^{\mathsf{ISets}(K)}\{f_x\}_{x\in K}\stackrel{\mathrm{def}}{=}\{g_x\circ f_x\}_{x\in K}.$$

Further Terminology: Also called a K-indexed map of sets from X to Y.

## 1.4 The Category of Indexed Sets

**Definition 1.4.1.1.** The **category of indexed sets** is the category ISets defined as the Grothendieck construction of the functor ISets: Sets<sup>op</sup>  $\rightarrow$  Cats of Proposition 2.1.1.4:

$$\mathsf{ISets} \stackrel{\mathrm{def}}{=} \int^{\mathsf{Sets}} \mathsf{ISets}.$$

Remark 1.4.1.2. In detail, the category of indexed sets is the category ISets where

- Objects. The objects of ISets are pairs (K, X) consisting of
  - The Indexing Set. A set K;
  - The Indexed Set. A K-indexed set  $X: K_{\mathsf{disc}} \to \mathsf{Sets};$
- Morphisms. A morphism of ISets from (K, X) to (K', Y) is a pair  $(\phi, f)$  consisting of
  - The Reindexing Map. A map of sets  $\phi: K \to K'$ ;
  - The Morphism of Indexed Sets. A morphism of K-indexed sets  $f: X \to \phi_*(Y)$  as in the diagram

$$f \colon X \to \phi_*(Y), \qquad \begin{matrix} K_{\mathsf{disc}} & \xrightarrow{\phi} K'_{\mathsf{disc}} \\ X & & & \\ X & & & \\ Y & & \\ \mathsf{Sets}; \end{matrix}$$

• Identities. For each  $(K, X) \in \text{Obj}(\mathsf{ISets})$ , the unit map

$$\mathbb{F}^{\mathsf{ISets}}_{(K,X)} \colon \mathrm{pt} \to \mathsf{ISets}((K,X),(K,X))$$

of ISets at (K, X) is defined by

$$id_{(K,X)}^{\mathsf{ISets}} \stackrel{\text{def}}{=} (id_K, id_X).$$

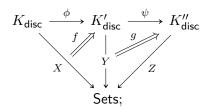
• Composition. For each  $\mathbf{X}=(K,X),\ \mathbf{Y}=(K',Y),\ \mathbf{Z}=(K'',Z)\in \mathrm{Obj}(\mathsf{ISets}),$  the composition map

$$\circ_{\mathbf{X},\mathbf{Y},\mathbf{Z}}^{\mathsf{ISets}}\colon \mathsf{ISets}(\mathbf{Y},\mathbf{Z})\times \mathsf{ISets}(\mathbf{X},\mathbf{Y})\to \mathsf{ISets}(\mathbf{X},\mathbf{Z})$$

of ISets at (X, Y, Z) is defined by

$$(\psi, g) \circ (\phi, f) \stackrel{\text{def}}{=} (\psi \circ \phi, (g \circ id_{\phi}) \circ f),$$

as in the diagram



for each  $(\phi, f) \in \mathsf{ISets}(\mathbf{X}, \mathbf{Y})$  and each  $(\psi, g) \in \mathsf{ISets}(\mathbf{Y}, \mathbf{Z})$ .

# 2 Constructions With Indexed Sets

# 2.1 Change of Indexing

Let  $\phi \colon K \to K'$  be a function and let X be a K'-indexed set.

**Definition 2.1.1.1.** The change of indexing of X to K is the K-indexed set  $\phi^*(X)$  defined by

$$\phi^*(X) \stackrel{\text{def}}{=} X \circ \phi_{\mathsf{disc}}.$$

Remark 2.1.1.2. In detail, the change of indexing of X to K is the K-indexed set  $\phi^*(X)$  defined by

$$\phi^*(X)_x \stackrel{\mathrm{def}}{=} X_{\phi(x)}$$

for each  $x \in K$ .

**Proposition 2.1.1.3.** The assignment  $X \mapsto \phi^*(X)$  defines a functor

$$\phi^* : \mathsf{ISets}(K') \to \mathsf{ISets}(K),$$

where

• Action on Objects. For each  $X \in \text{Obj}(\mathsf{ISets}(K'))$ , we have

$$[\phi^*](X) \stackrel{\text{def}}{=} \phi^*(X);$$

• Action on Morphisms. For each  $X,Y\in \mathrm{Obj}(\mathsf{ISets}(K')),$  the action on Hom-sets

$$\phi_{X,Y}^* \colon \mathrm{Hom}_{\mathsf{ISets}(K')}(X,Y) \to \mathrm{Hom}_{\mathsf{ISets}(K)}(\phi^*(X),\phi^*(Y))$$

of  $\phi^*$  at (X,Y) is the map sending a morphism of K'-indexed sets

$$f = \{f_x \colon X_x \to Y_x\}_{x \in K'}$$

from X to Y to the morphism of K-indexed sets defined by

$$\phi^*(f) \stackrel{\text{def}}{=} \left\{ f_{\phi(x)} \colon X_{\phi(x)} \to Y_{\phi(x)} \right\}_{x \in K}.$$

*Proof.* Omitted.

**Proposition 2.1.1.4.** The assignment  $K \mapsto \mathsf{ISets}(K)$  defines a functor

$$\mathsf{ISets} \colon \mathsf{Sets}^\mathsf{op} \to \mathsf{Cats},$$

where

• Action on Objects. For each  $K \in \text{Obj}(\mathsf{Sets})$ , we have

$$[\mathsf{ISets}](K) \stackrel{\text{def}}{=} \mathsf{ISets}(K);$$

• Action on Morphisms. For each  $K, K' \in \mathrm{Obj}(\mathsf{Sets}),$  the action on Hom-sets

$$\mathsf{ISets}_{K,K'} \colon \mathsf{Sets}^{\mathsf{op}}(K,K') \to \mathsf{Fun}(\mathsf{ISets}(K),\mathsf{ISets}(K'))$$

of ISets at (K, K') is the map defined by

$$\mathsf{ISets}_{K,K'}(\phi) \stackrel{\scriptscriptstyle \mathrm{def}}{=} \phi^*$$

for each  $\phi \in \mathsf{Sets}^{\mathsf{op}}(K, K')$ .

Proof. Omitted.

### 2.2 Dependent Sums

Let  $\phi \colon K \to K'$  be a function and let X be a K-indexed set.

**Definition 2.2.1.1.** The **dependent sum of** X is the K'-indexed set  $\Sigma_{\phi}(X)^2$  defined by

$$\Sigma_{\phi}(X) \stackrel{\text{def}}{=} \operatorname{Lan}_{\phi}(X),$$

and hence given by

$$\Sigma_{\phi}(X)_{x} \cong \coprod_{y \in \phi^{-1}(x)} X_{y}$$

for each  $x \in K'$ .

<sup>&</sup>lt;sup>2</sup>Further Notation: Also written  $\phi_*(X)$ .

**Proposition 2.2.1.2.** The assignment  $X \mapsto \Sigma_{\phi}(X)$  defines a functor

$$\Sigma_{\phi} \colon \mathsf{ISets}(K) \to \mathsf{ISets}(K'),$$

where

• Action on Objects. For each  $X \in \text{Obj}(\mathsf{ISets}(K))$ , we have

$$[\Sigma_{\phi}](X) \stackrel{\text{def}}{=} \Sigma_{\phi}(X);$$

• Action on Morphisms. For each  $X,Y\in \mathrm{Obj}(\mathsf{ISets}(K)),$  the action on Hom-sets

$$\Sigma_{\phi|X,Y} \colon \operatorname{Hom}_{\mathsf{ISets}(K)}(X,Y) \to \operatorname{Hom}_{\mathsf{ISets}(K')}(\Sigma_{\phi}(X),\Sigma_{\phi}(Y))$$

of  $\Sigma_{\phi}$  at (X,Y) is the map sending a morphism of K-indexed sets

$$f: X \to Y$$

to the morphism of K'-indexed sets defined by

$$\Sigma_{\phi}(f) \stackrel{\text{def}}{=} \operatorname{Lan}_{\phi}(f);$$
$$\cong \coprod_{y \in \phi^{-1}(X)} f_{y}.$$

Proof. Omitted.

#### 2.3 Dependent Products

Let  $\phi \colon K \to K'$  be a function and let X be a K-indexed set.

**Definition 2.3.1.1.** The **dependent product of** X is the K'-indexed set  $\Pi_{\phi}(X)^3$  defined by

$$\Pi_{\phi}(X) \stackrel{\mathrm{def}}{=} \mathrm{Ran}_{\phi}(X),$$

and hence given by

$$\Pi_{\phi}(X)_x \cong \prod_{y \in \phi^{-1}(x)} X_y$$

for each  $x \in K'$ .

<sup>&</sup>lt;sup>3</sup> Further Notation: Also written  $\phi_!(X)$ .

2.4 Internal Homs

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**Proposition 2.3.1.2.** The assignment  $X \mapsto \Pi_{\phi}(X)$  defines a functor

$$\Pi_{\phi} \colon \mathsf{ISets}(K) \to \mathsf{ISets}(K'),$$

where

• Action on Objects. For each  $X \in \text{Obj}(\mathsf{ISets}(K))$ , we have

$$[\Pi_{\phi}](X) \stackrel{\mathrm{def}}{=} \Pi_{\phi}(X);$$

• Action on Morphisms. For each  $X,Y\in \mathrm{Obj}(\mathsf{ISets}(K)),$  the action on Hom-sets

$$\Pi_{\phi|X,Y} \colon \operatorname{Hom}_{\mathsf{ISets}(K)}(X,Y) \to \operatorname{Hom}_{\mathsf{ISets}(K')}(\Pi_{\phi}(X),\Pi_{\phi}(Y))$$

of  $\Pi_{\phi}$  at (X,Y) is the map sending a morphism of K-indexed sets

$$f \colon X \to Y$$

to the morphism of K'-indexed sets defined by

$$\Pi_{\phi}(f) \stackrel{\text{def}}{=} \operatorname{Ran}_{\phi}(f);$$

$$\cong \prod_{y \in \phi^{-1}(x)} f_{y}.$$

*Proof.* Omitted.

#### 2.4 Internal Homs

Let K be a set and let X and Y be K-indexed sets.

**Definition 2.4.1.1.** The internal Hom of indexed sets from X to Y is the indexed set  $\mathbf{Hom}_{|\mathsf{Sets}(K)}(X,Y)$  defined by

$$\mathbf{Hom}_{\mathsf{ISets}(K)}(X,Y) \stackrel{\scriptscriptstyle \mathrm{def}}{=} \mathsf{Sets}(X_x,Y_x)$$

for each  $x \in K$ .

### 2.5 Adjointness of Indexed Sets

Let  $\phi \colon K \to K'$  be a map of sets.

Proposition 2.5.1.1. We have a triple adjunction

$$(\Sigma_\phi \dashv \phi^* \dashv \Pi_\phi) \colon \quad \mathsf{ISets}(K) \underbrace{\longleftarrow_{\phi^*} - \mathsf{ISets}(K')}_{\Pi_\phi}.$$

*Proof.* This follows from Kan Extensions, ?? of ??.

# 3 Fibred Sets

### 3.1 Foundations

Let K be a set.

**Definition 3.1.1.1.** A K-fibred set is a pair  $(X, \phi)$  consisting of<sup>4</sup>

- The Underlying Set. A set X, called the **underlying set of**  $(X, \phi)$ ;
- The Fibration. A map of sets  $\phi: X \to K$ .

# 3.2 Morphisms of Fibred Sets

**Definition 3.2.1.1.** A morphism of K-fibred sets from  $(X, \phi)$  to  $(Y, \psi)$  is a function  $f: X \to Y$  such that the diagram<sup>5</sup>



commutes.

$$\phi^{-1}(x) \stackrel{\text{def}}{=} \operatorname{pt} \times_{[x],K,\phi} X, \qquad \phi^{-1}(x) \xrightarrow{J} X$$

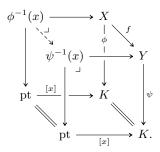
$$\downarrow \qquad \qquad \downarrow \phi$$

$$\operatorname{pt} \xrightarrow{[x]} K.$$

<sup>5</sup> Further Terminology: The transport map associated to f at  $x \in K$  is the function

$$f_x^* : \phi^{-1}(x) \to \psi^{-1}(x)$$

given by the dashed map in the diagram



<sup>&</sup>lt;sup>4</sup>Further Terminology: The fibre of  $(X, \phi)$  over  $x \in K$  is the set  $\phi^{-1}(x)$  (also written  $\phi_x$ ) defined by

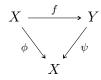
## 3.3 The Category of Fibred Sets Over a Fixed Base

**Definition 3.3.1.1.** The category of K-fibred sets is the category  $\mathsf{FibSets}(K)$  defined as the slice category  $\mathsf{Sets}_{/K}$  of  $\mathsf{Sets}$  over K:

$$\mathsf{FibSets}(K) \stackrel{\scriptscriptstyle \mathrm{def}}{=} \mathsf{Sets}_{/K}.$$

**Remark 3.3.1.2.** In detail FibSets(K) is the category where

- Objects. The objects of FibSets(K) are pairs  $(X, \phi)$  consisting of
  - The Fibred Set. A set X;
  - The Fibration. A function  $\phi: X \to K$ ;
- Morphisms. A morphism of FibSets(K) from  $(X, \phi)$  to  $(Y, \psi)$  is a function  $f: X \to Y$  making the diagram



commute;

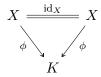
• Identities. For each  $(X, \phi) \in \text{Obj}(\mathsf{FibSets}(K))$ , the unit map

$$\mathbb{F}^{\mathsf{FibSets}(K)}_{(X,\phi)} \colon \mathrm{pt} \to \mathrm{Hom}_{\mathsf{FibSets}(K)}((X,\phi),(X,\phi))$$

of FibSets(K) at  $(X, \phi)$  is given by

$$\operatorname{id}_{(X,\phi)}^{\mathsf{FibSets}(K)} \stackrel{\text{def}}{=} \operatorname{id}_X,$$

as witnessed by the commutativity of the diagram



in Sets;

• Composition. For each  $\mathbf{X}=(X,\phi),\ \mathbf{Y}=(Y,\psi),\ \mathbf{Z}=(Z,\chi)\in$ 

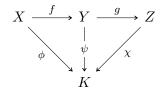
Obj(FibSets(K)), the composition map

$$\circ_{\mathbf{X},\mathbf{Y},\mathbf{Z}}^{\mathsf{FibSets}(K)} \colon \mathrm{Hom}_{\mathsf{FibSets}(K)}(\mathbf{Y},\mathbf{Z}) \times \mathrm{Hom}_{\mathsf{FibSets}(K)}(\mathbf{X},\mathbf{Y}) \to \mathrm{Hom}_{\mathsf{FibSets}(K)}(\mathbf{X},\mathbf{Z})$$

of FibSets(K) at  $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$  is defined by

$$\circ_{\mathbf{X},\mathbf{Y},\mathbf{Z}}^{\mathsf{FibSets}(K)} \stackrel{\scriptscriptstyle\rm def}{=} \circ_{X,Y,Z}^{\mathsf{Sets}},$$

as witnessed by the commutativity of the diagram



in Sets.

### 3.4 The Category of Fibred Sets

**Definition 3.4.1.1.** The category of fibred sets is the category FibSets defined as the Grothendieck construction of the functor FibSets: Sets<sup>op</sup>  $\rightarrow$  Cats of Proposition 4.1.1.3:

$$FibSets \stackrel{\text{def}}{=} \int^{Sets} FibSets.$$

Remark 3.4.1.2. In detail, the category of fibred sets is the category FibSets where

- Objects. The objects of FibSets are pairs  $(K, (X, \phi_X))$  consisting of
  - The Base Set. A set K:
  - The Fibred Set. A K-fibred set  $\phi_X : X \to K$ ;
- Morphisms. A morphism of FibSets from  $(K,(X,\phi_X))$  to  $(K',(Y,\phi_Y))$  is a pair  $(\phi,f)$  consisting of
  - The Base Map. A map of sets  $\phi: K \to K'$ ;
  - The Morphism of Fibred Sets. A morphism of K-fibred sets

$$f \colon (X, \phi_X) \to \phi_Y^*(Y), \qquad \begin{matrix} X \stackrel{f}{\longrightarrow} Y \times_{K'} K \\ \phi_X & \swarrow pr_2 \\ K; \end{matrix}$$

• Identities. For each  $(K, X) \in \text{Obj}(\mathsf{FibSets})$ , the unit map

$$\mathbb{F}^{\mathsf{FibSets}}_{(K,X)} \colon \mathrm{pt} \to \mathsf{FibSets}((K,X),(K,X))$$

of FibSets at (K, X) is defined by

$$\operatorname{id}_{(K,X)}^{\mathsf{FibSets}} \stackrel{\text{def}}{=} (\operatorname{id}_K, \sim),$$

where  $\sim$  is the isomorphism  $X \to X \times_K K$  as in the diagram

$$X \xrightarrow{\phi_X} X \times_K K$$

$$\downarrow^{\text{pr}_2}$$

$$K;$$

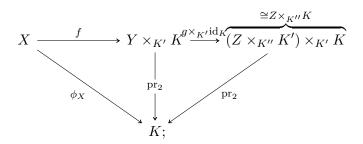
• Composition. For each  $\mathbf{X} = (K, X), \mathbf{Y} = (K', Y), \mathbf{Z} = (K'', Z) \in \text{Obj}(\mathsf{FibSets})$ , the composition map

$$\circ^{\mathsf{FibSets}}_{\mathbf{X},\mathbf{Y},\mathbf{Z}} \colon \mathsf{FibSets}(\mathbf{Y},\mathbf{Z}) \times \mathsf{FibSets}(\mathbf{X},\mathbf{Y}) \to \mathsf{FibSets}(\mathbf{X},\mathbf{Z})$$

of FibSets at (X, Y, Z) is defined by

$$g \circ_{\mathbf{X},\mathbf{Y},\mathbf{Z}}^{\mathsf{FibSets}} f \stackrel{\text{def}}{=} (g \times_{K'} \mathrm{id}_K) \circ f$$

as in the diagram



for each  $f \in \mathsf{FibSets}(\mathbf{X}, \mathbf{Y})$  and each  $g \in \mathsf{FibSets}(\mathbf{Y}, \mathbf{Z})$ .

# 4 Constructions With Fibred Sets

### 4.1 Change of Base

Let  $f: K \to K'$  be a function and let  $(X, \phi_X)$  be a K'-fibred set.

**Definition 4.1.1.1.** The change of base of  $(X, \phi_X)$  to K is the K-fibred set  $f^*(X)$  defined by

$$f^{*}(X) \stackrel{\text{def}}{=} (K \times_{K'} X, \operatorname{pr}_{1}), \qquad f^{*}(X) \stackrel{\operatorname{pr}_{2}}{=} X$$

$$pr_{1} \downarrow \qquad \downarrow^{\phi_{X}}$$

$$K \xrightarrow{f} K'.$$

**Proposition 4.1.1.2.** The assignment  $X \mapsto f^*(X)$  defines a functor

$$f^* : \mathsf{FibSets}(K') \to \mathsf{FibSets}(K),$$

where

- Action on Objects. For each  $(X, \phi_X) \in \text{Obj}(\mathsf{FibSets}(K'))$ , we have  $f^*(X, \phi_X) \stackrel{\text{def}}{=} f^*(X)$ ;
- Action on Morphisms. For each  $(X, \phi_X), (Y, \phi_Y) \in \text{Obj}(\mathsf{FibSets}(K')),$  the action on Hom-sets

$$f_{X,Y}^* \colon \operatorname{Hom}_{\mathsf{FibSets}(K')}(X,Y) \to \operatorname{Hom}_{\mathsf{FibSets}(K)}(f^*(X),f^*(Y))$$

of  $f^*$  at  $((X, \phi_X), (Y, \phi_Y))$  is the map sending a morphism of K'-fibred sets

$$g\colon (X,\phi_X)\to (Y,\phi_Y)$$

to the morphism of K-fibred sets given by the dashed morphism in the diagram

$$f^{*}(X) \xrightarrow{X} X$$

$$\downarrow f^{*}(Y) \xrightarrow{\phi_{X}} Y$$

$$\downarrow K \xrightarrow{f} X' \qquad \downarrow \phi_{Y}$$

$$K \xrightarrow{f} K'.$$

Proof. Omitted.

**Proposition 4.1.1.3.** The assignment  $K \mapsto \mathsf{FibSets}(K)$  defines a functor

FibSets: Sets<sup>op</sup> 
$$\rightarrow$$
 Cats,

where

• Action on Objects. For each  $K \in \text{Obj}(\mathsf{Sets})$ , we have

$$[\mathsf{FibSets}](K) \stackrel{\text{def}}{=} \mathsf{FibSets}(K);$$

• Action on Morphisms. For each  $K, K' \in \mathrm{Obj}(\mathsf{Sets})$ , the action on Hom-sets

$$\mathsf{Sets}_{/(-)|K,K'} \colon \mathsf{Sets}^{\mathsf{op}}(K,K') \to \mathsf{Fun}(\mathsf{FibSets}(K),\mathsf{FibSets}(K'))$$

of  $\mathsf{Sets}_{/(-)}$  at (K,K') is the map sending a map of sets  $f\colon K\to K'$  to the functor

$$\mathsf{Sets}_{/f} \colon \mathsf{FibSets}(K') \to \mathsf{FibSets}(K)$$

defined by

$$\mathsf{Sets}_{/f} \stackrel{\mathrm{def}}{=} f^*.$$

Proof. Omitted.

## 4.2 Dependent Sums

Let  $f \colon K \to K'$  be a function and let  $(X, \phi_X)$  be a K-fibred set.

**Definition 4.2.1.1.** The **dependent sum**<sup>6</sup> of  $(X, \phi_X)$  is the K'-fibred set  $\Sigma_f(X)^7$  defined by

$$\Sigma_f(X) \stackrel{\text{def}}{=} (\Sigma_f(X), \Sigma_f(\phi_X))$$
$$\stackrel{\text{def}}{=} (X, f \circ \phi_X).$$

**Proposition 4.2.1.2.** Let  $f: K \to K'$  be a function.

1. Functoriality. The assignment  $X \mapsto \Sigma_f(X)$  defines a functor

$$\Sigma_f \colon \mathsf{FibSets}(K) \to \mathsf{FibSets}(K'),$$

where

$$\Sigma_f(\phi_X)^{-1}(x) \cong \prod_{y \in f^{-1}(x)} \phi_X^{-1}(y);$$

see Item 2 of Proposition 4.2.1.2.

<sup>&</sup>lt;sup>6</sup>The name "dependent sum" comes from the fact that the fibre  $\Sigma_f(\phi_X)^{-1}(x)$  of  $\Sigma_f(X)$  at  $x \in K'$  is given by

<sup>&</sup>lt;sup>7</sup> Further Notation: Also written  $f_*(X)$ .

• Action on Objects. For each  $(X, \phi_X) \in \text{Obj}(\mathsf{FibSets}(K))$ , we have

$$\Sigma_f(X, \phi_X) \stackrel{\text{def}}{=} (\Sigma_f(X), \Sigma_f(\phi_X));$$

• Action on Morphisms. For each  $(X, \phi_X), (Y, \phi_Y) \in \text{Obj}(\mathsf{FibSets}(K)),$  the action on Hom-sets

$$\Sigma_{f|X,Y} \colon \mathrm{Hom}_{\mathsf{FibSets}(K)}(X,Y) \to \mathrm{Hom}_{\mathsf{FibSets}(K)}(\Sigma_f(X),\Sigma_f(Y))$$

of  $\Sigma_f$  at  $((X,\phi_X),(Y,\phi_Y))$  is the map sending a morphism of K-fibred sets

$$g\colon (X,\phi_X)\to (Y,\phi_Y)$$

to the morphism of K'-fibred sets defined by

$$\Sigma_f(g) \stackrel{\text{def}}{=} g.$$

2. Interaction With Fibres. We have a bijection of sets

$$\Sigma_f(\phi_X)^{-1}(x) \cong \coprod_{y \in f^{-1}(x)} \phi_X^{-1}(y)$$

for each  $x \in K'$ .

Proof. Item 1, Functoriality: Omitted.

Item 2, Interaction With Fibres: Indeed, we have

$$\Sigma_f(\phi_X)^{-1}(x) \stackrel{\text{def}}{=} \text{pt} \times_{[x], K', f \circ \phi_X} X$$

$$\cong \{(a, y) \in X \times K \mid f(\phi_X(a)) = x\}$$

$$\cong \coprod_{y \in f^{-1}(x)} \phi_X^{-1}(y)$$

for each  $x \in K'$ .

#### 4.3 Dependent Products

Let  $f: K \to K'$  be a function and let  $(X, \phi_X)$  be a K-fibred set.

**Definition 4.3.1.1.** The dependent product<sup>8</sup> of  $(X, \phi_X)$  is the K'-fibred

<sup>&</sup>lt;sup>8</sup>The name "dependent product" comes from the fact that the fibre  $\Pi_f(\phi_X)^{-1}(x)$  of

set  $\Pi_f(X)^9$  consisting of  $\Pi_f(X)^9$ 

• The Underlying Set. The set  $\Pi_f(X)$  defined by

$$\begin{split} \Pi_f(X) &\stackrel{\text{def}}{=} \coprod_{k' \in K'} \Gamma_{f^{-1}(k')}^{\phi_X} \Big( \phi_X^{-1} \Big( f^{-1}(k') \Big) \Big) \\ &\stackrel{\text{def}}{=} \left\{ (k',h) \in \coprod_{k' \in K'} \mathsf{Sets} \Big( f^{-1}(k'), \phi_X^{-1} \Big( f^{-1}(k') \Big) \Big) \ \middle| \ \phi_X \circ h = \mathrm{id}_{f^{-1}(k')} \right\} \\ &\cong \coprod_{k' \in K'} \Big\{ h \in \mathsf{Sets} \Big( f^{-1}(k'), \phi_X^{-1} \Big( f^{-1}(k') \Big) \Big) \ \middle| \ \phi_X \circ h = \mathrm{id}_{f^{-1}(k')} \Big\} \\ &\cong \coprod_{k' \in K'} \prod_{k \in f^{-1}(k')} \phi_X^{-1}(k); \end{split}$$

• The Fibration. The map of sets

$$\Pi_f(\phi_X) \colon \Pi_f(X) \to K'$$

defined by sending an element of

$$\Pi_f(X) \cong \prod_{k' \in K'} \prod_{k \in f^{-1}(k')} \phi_X^{-1}(k)$$

to its index k' in K'.

*Proof.* The first bijection in the definition of  $\Pi_f(X)$  is clear, so it only remains to show the bijection

$$\left\{ h \in \mathsf{Sets} \Big( f^{-1} \big( k' \big), \phi_X^{-1} \Big( f^{-1} \big( k' \big) \Big) \Big) \; \middle| \; \phi_X \circ h = \mathrm{id}_{f^{-1} (k')} \right\} \cong \prod_{k \in f^{-1} (k')} \phi_X^{-1} (k).$$

There are two cases:

1. If  $f^{-1}(k') = \emptyset$ , then there is only one map from  $f^{-1}(k')$  to  $\phi_X^{-1}(f^{-1}(k'))$  (the inclusion), so  $\mathsf{Sets} \Big( f^{-1}(k'), \phi_X^{-1}(f^{-1}(k')) \Big) \cong \mathsf{pt}$ . Since products indexed by the empty set are isomorphic to  $\mathsf{pt}$ , the isomorphism follows.

$$\Pi_f(\phi_X)^{-1}(x) \cong \prod_{y \in f^{-1}(x)} \phi_X^{-1}(y);$$

see Item 2 of Proposition 4.3.1.3.

 $<sup>\</sup>Pi_f(X)$  at  $x \in K'$  is given by

<sup>&</sup>lt;sup>9</sup> Further Notation: Also written  $f_!(X)$ .

 $<sup>^{10}</sup>$ We can also define dependent products via the internal **Hom** in FibSets(K'); see Item 3

2. Otherwise, by the condition  $\phi_X \circ h = \mathrm{id}_{f^{-1}(k')}$ , it follows that, for each  $k \in f^{-1}(k')$ , we must have

$$\phi_X(h(k)) = k,$$

and thus  $h(k) \in \phi_X^{-1}(k)$ . Therefore, a map from  $f^{-1}(k')$  to  $\phi_X^{-1}(f^{-1}(k'))$  consists of a choice of an element from  $\phi_X^{-1}(k)$  for each  $k \in f^{-1}(k')$ , which is precisely given by an element of the product  $\prod_{k \in f^{-1}(k')} \phi_X^{-1}(k)$ , showing the bijection to be true.

This finishes the proof.

Example 4.3.1.2. Here are some examples of dependent products of sets.

1. Spaces of Sections. Let K = X, K' = pt, let  $\phi \colon E \to X$  be a map of sets, and write  $!_X \colon X \to \text{pt}$  for the terminal map from X to pt. We have a bijection of sets

$$\begin{split} \Pi_{!_X}((E,\phi)) &\cong \Gamma_X(\phi) \\ &\cong \{h \in \mathsf{Sets}(X,E) \mid \phi \circ h = \mathrm{id}_X\}. \end{split}$$

2. Function Spaces. Let K = K' = pt. We have a bijection of sets

$$\mathsf{Sets}(X,Y) \cong \Pi_{!_X}(!_X^*(Y)).$$

**Proposition 4.3.1.3.** Let  $f: K \to K'$  be a function.

1. Functoriality. The assignment  $X \mapsto \Pi_f(X)$  defines a functor

$$\Pi_f \colon \mathsf{FibSets}(K) \to \mathsf{FibSets}(K')$$

where

• Action on Objects. For each  $(X, \phi_X) \in \text{Obj}(\mathsf{FibSets}(K))$ , we have

$$\Pi_f(X, \phi_X) \stackrel{\text{def}}{=} \Pi_f(X);$$

• Action on Morphisms. For each  $(X, \phi_X), (Y, \phi_Y) \in \text{Obj}(\mathsf{FibSets}(K)),$  the action on Hom-sets

$$\Pi_{f|X,Y} \colon \operatorname{Hom}_{\mathsf{FibSets}(K)}(X,Y) \to \operatorname{Hom}_{\mathsf{FibSets}(K')}(\Pi_f(X),\Pi_f(Y))$$

of  $\Pi_f$  at  $((X,\phi_X),(Y,\phi_Y))$  is the map sending a morphism of K-fibred sets

$$\xi \colon (X, \phi_X) \to (Y, \phi_Y)$$

to the morphism of K'-fibred sets from

$$\Pi_f(X) \stackrel{\text{def}}{=} \left\{ (x,h) \in \coprod_{x \in K'} \mathsf{Sets} \Big( f^{-1}(x), \phi_X^{-1} \big( f^{-1}(x) \big) \Big) \; \middle| \; \phi_X \circ h = \mathrm{id}_{f^{-1}(x)} \right\};$$
 to

$$\Pi_f(Y) \stackrel{\mathrm{def}}{=} \left\{ (x,h) \in \coprod_{x \in K'} \mathsf{Sets} \Big( f^{-1}(x), \phi_Y^{-1}(f^{-1}(x)) \Big) \ \Big| \ \phi_Y \circ h = \mathrm{id}_{f^{-1}(x)} \right\};$$

induced by the composition

$$\begin{split} \mathsf{Sets}\Big(f^{-1}(x),\phi_X^{-1}\Big(f^{-1}(x)\Big)\Big) &= \mathsf{Sets}\Big(f^{-1}(x),[\phi_Y\circ\xi]^{-1}\Big(f^{-1}(x)\Big)\Big) \\ &= \mathsf{Sets}\Big(f^{-1}(x),\xi^{-1}\Big(\phi_Y^{-1}\Big(f^{-1}(x)\Big)\Big)\Big) \\ &\xrightarrow{\xi_*} \mathsf{Sets}\Big(f^{-1}(x),\xi\Big(\xi^{-1}\Big(\phi_Y^{-1}\Big(f^{-1}(x)\Big)\Big)\Big)\Big) \\ &\xrightarrow{\iota_*} \mathsf{Sets}\Big(f^{-1}(x),\phi_Y^{-1}\Big(f^{-1}(x)\Big)\Big), \end{split}$$

where  $\iota : \xi \left( \xi^{-1} \left( \phi_Y^{-1} (f^{-1}(x)) \right) \right) \hookrightarrow \phi_Y^{-1} (f^{-1}(x))$  is the canonical inclusion, and thus given on elements by

$$[\Pi_f(\xi)](k',h) = (k', \xi \circ h),$$

for each  $(k',h) \in \Pi_f(X)$ .

2. Interaction With Fibres. We have a bijection of sets

$$\Pi_f(\phi_X)^{-1}(k') \cong \prod_{k \in f^{-1}(k')} \phi_X^{-1}(k)$$

for each  $k' \in K'$ .

$$\phi_Y \circ [\Pi_f(\xi)](h) \stackrel{\text{def}}{=} \phi_Y \circ (\xi \circ h)$$

$$= (\phi_Y \circ \xi) \circ h$$

$$= \phi_X \circ h$$

$$= \mathrm{id}_{f^{-1}(x)}.$$

<sup>&</sup>lt;sup>11</sup>Note that the section condition is satisfied: given  $(x,h) \in \Pi_f(X)$ , we have

4.4 Internal Homs 19

3. Construction Using the Internal Hom. We have

$$\Pi_f(X,\phi_X) = \left(K' \times_{\mathbf{Hom}_{\mathsf{FibSets}\left(K'\right)}((K,f),(K,f))} \mathbf{Hom}_{\mathsf{FibSets}(K')}((K,f),(X,f \circ \phi_X)), \operatorname{pr}_1\right),$$

forming a pullback diagram

$$\begin{array}{ccc} \Pi_f(X,\phi_X) & \xrightarrow{\operatorname{pr}_2} & \mathbf{Hom}_{\mathsf{FibSets}(K')}((K,f),(X,f\circ\phi_X)) \\ & & \downarrow^{(\phi_X)_*} \\ & & K' & \xrightarrow{I} & \mathbf{Hom}_{\mathsf{FibSets}(K')}((K,f),(K,f)), \end{array}$$

where the bottom map is given by

$$I(k') \stackrel{\text{def}}{=} \mathrm{id}_{f^{-1}(k')}$$

for each  $k' \in K'$  and where  $\mathbf{Hom}_{\mathsf{FibSets}(K')}$  denotes the internal Hom of  $\mathsf{FibSets}(K')$  of  $\mathbf{Definition}$  4.4.1.1.

Proof. Item 1, Functoriality: Omitted.

Item 2, Interaction With Fibres: Indeed, we have

$$\begin{split} \Pi_f(\phi_X)^{-1}(k') &\stackrel{\text{def}}{=} \left\{ (k,h) \in \Pi_f(X) \mid [\Pi_f(\phi_X)](h) = k' \right\} \\ &\stackrel{\text{def}}{=} \left\{ (k,h) \in \Pi_f(X) \mid k = k' \right\} \\ &\cong \left\{ h \in \mathsf{Sets} \Big( f^{-1}(k'), \phi_X^{-1} \Big( f^{-1}(k') \Big) \Big) \mid \phi_X \circ h = \mathrm{id}_{f^{-1}(k')} \right\} \\ &\cong \prod_{k \in f^{-1}(k')} \phi_X^{-1}(k) \end{split}$$

for each  $k' \in K'$ , where the last bijection is proved in the proof of Definition 4.3.1.1.

Item 3, Construction Using the Internal Hom: Omitted.

#### 4.4 Internal Homs

Let K be a set and let  $(X, \phi_X)$  and  $(Y, \phi_Y)$  be K-fibred sets.

Definition 4.4.1.1. The internal Hom of fibred sets from  $(X, \phi_X)$  to  $(Y, \phi_Y)$  is the fibred set  $\mathbf{Hom}_{\mathsf{FibSets}(K)}(X, Y)$  consisting of

• The Underlying Set. The set  $\mathbf{Hom}_{\mathsf{FibSets}(K)}(X,Y)$  defined by

$$\mathbf{Hom}_{\mathsf{FibSets}(K)}(X,Y) \stackrel{\scriptscriptstyle\rm def}{=} \coprod_{x \in K} \mathsf{Sets}\Big(\phi_X^{-1}(x),\phi_Y^{-1}(x)\Big);$$

• The Fibration. The map of sets $^{12}$ 

$$\phi_{\mathbf{Hom}_{\mathsf{FibSets}(K)}(X,Y)} \colon \underbrace{\mathbf{Hom}_{\mathsf{FibSets}(K)}(X,Y)}_{x \in K} \to K$$

defined by sending a map  $f: \phi_X^{-1}(x) \to \phi_Y^{-1}(x)$  to its index  $x \in K$ .

#### 4.5 Adjointness for Fibred Sets

Let  $f: K \to K'$  be a map of sets.

**Proposition 4.5.1.1.** We have a triple adjunction

$$(\Sigma_f\dashv f^*\dashv \Pi_f)\text{:}\quad \mathsf{FibSets}(K) \underbrace{\qquad \qquad}_{\prod_f} \mathsf{FibSets}(K').$$

*Proof.* We offer two proofs. The first uses the corresponding adjunction for indexed sets (Proposition 2.5.1.1) and the un/straightening equivalence together with its compatibility with dependent sums and products to "transfer" the adjunction to fibred sets, while the second is a direct one.

Transferring the Adjunction From Indexed Sets Part I: The Adjunction  $\Sigma_f \dashv f^*$ : The adjunction

$$(\Sigma_f\dashv f^*)\colon \ \ \mathsf{ISets}(K) \underbrace{\overset{\Sigma_f}{\underset{f^*}{\smile}}} \mathsf{ISets}(K')$$

of Proposition 2.5.1.1 gives a unit and counit of the form

$$\eta: \mathrm{id}_{\mathsf{ISets}(K)} \Longrightarrow \Sigma_f \circ f^*,$$
 $\epsilon: f^* \circ \Sigma_f \Longrightarrow \mathrm{id}_{\mathsf{ISets}(K')}.$ 

$$\phi_{\mathbf{Hom}_{\mathsf{FibSets}(K)}(X,Y)|x} \cong \mathsf{Sets} \big( \phi_X^{-1}(x), \phi_Y^{-1}(x) \big)$$

for each  $x \in K$ .

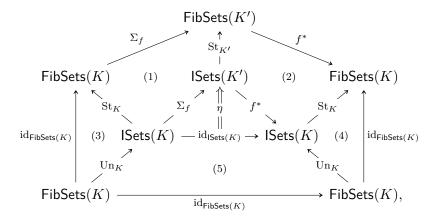
The fibres of the internal  $\operatorname{Hom}$  of  $\operatorname{FibSets}(K)$  are precisely the sets  $\operatorname{Sets}\left(\phi_X^{-1}(x),\phi_Y^{-1}(x)\right)$ , i.e. we have

With these in hand, we construct natural transformations

$$\eta' : \mathrm{id}_{\mathsf{FibSets}(K)} \Longrightarrow \Sigma_f \circ f^*,$$
  
 $\epsilon' : f^* \circ \Sigma_f \Longrightarrow \mathrm{id}_{\mathsf{FibSets}(K')}$ 

as follows:

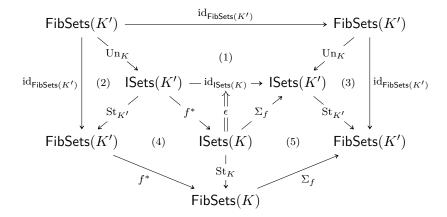
1. The Unit. We define  $\eta'$ :  $\mathrm{id}_{\mathsf{FibSets}(K)} \Longrightarrow \Sigma_f \circ f^*$  as the pasting of the diagram



where:

- (a) Subdiagram (1) commutes by Item 3 of Proposition 5.1.1.2.
- (b) Subdiagram (2) commutes by Item 2 of Proposition 5.1.1.2.
- (c) Subdiagram (3) commutes by Theorem 5.3.1.1.
- (d) Subdiagram (4) commutes by Theorem 5.3.1.1.
- (e) Subdiagram (5) commutes by unitality of composition.
- 2. The Counit. We define  $\epsilon' : f^* \circ \Sigma_f \Longrightarrow \mathrm{id}_{\mathsf{FibSets}(K')}$  as the pasting of

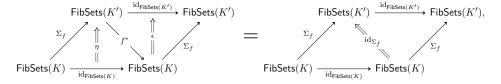
the diagram



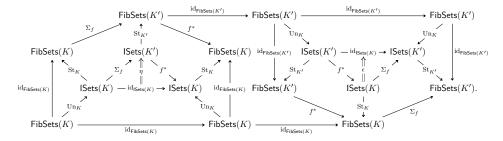
where:

- (a) Subdiagram (1) commutes by unitality of composition.
- (b) Subdiagram (2) commutes by Theorem 5.3.1.1.
- (c) Subdiagram (3) commutes by Theorem 5.3.1.1.
- (d) Subdiagram (4) commutes by Item 3 of Proposition 5.1.1.2.
- (e) Subdiagram (5) commutes by Item 2 of Proposition 5.1.1.2.

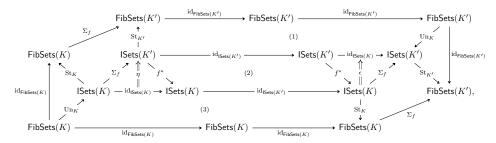
Next, we prove the left triangle identity,



whose left side in our case looks like this:



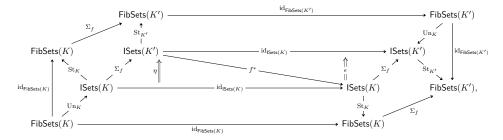
It can be rearranged into



#### where:

- 1. Subdiagram (1) commutes by Theorem 5.3.1.1.
- 2. Subdiagram (2) commutes by unitality of composition.
- 3. Subdiagram (3) commutes by Theorem 5.3.1.1.

And then, it can be rearranged into



which by the left triangle identity for  $(\eta, \epsilon)$ , becomes



finishing the proof of the left triangle identity. The proof of the right triangle identity is similar, and is thus omitted.

Transferring the Adjunction From Indexed Sets Part II: The Adjunction  $f^* \dashv \Pi_f$ : This proof is similar to the proof of the adjunction  $\Sigma_f \dashv f^*$ , and is thus omitted.

Direct Proof Part I: The Adjunction  $\Sigma_f \dashv f^*$ : We claim there's a bijection

$$\operatorname{Hom}_{\mathsf{FibSets}(K')}(\Sigma_f(X), Y) \cong \operatorname{Hom}_{\mathsf{FibSets}(K)}(X, f_*(Y)),$$

natural in  $(X, \phi_X) \in \mathsf{FibSets}(K)$  and  $(Y, \phi_Y) \in \mathsf{FibSets}(K')$ :

• Map I. We define a map

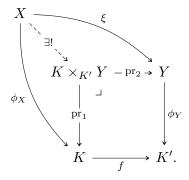
$$\Phi_{X,Y} \colon \operatorname{Hom}_{\mathsf{FibSets}(K')}(\Sigma_f(X), Y) \to \operatorname{Hom}_{\mathsf{FibSets}(K)}(X, f_*(Y)),$$

by sending a morphism

$$\xi \colon \Sigma_f(X) \to Y, \qquad X \xrightarrow{\xi} Y \\ \phi_X \downarrow \\ K \downarrow \\ K'$$

of K'-fibred sets to the morphism

of K-fibred sets given by the dashed morphism in the diagram



• Map II. We define a map

$$\Psi_{X,Y} \colon \operatorname{Hom}_{\mathsf{FibSets}(K)}(X, f_*(Y)) \to \operatorname{Hom}_{\mathsf{FibSets}(K')}(\Sigma_f(X), Y),$$

given by sending a map

$$\xi \colon X \to f^*(Y), \qquad X \xrightarrow{\xi} K \times_{K'} Y$$

$$\downarrow \phi_X \qquad \downarrow pr_1$$

$$K'$$

of K'-fibred sets to the map

$$\xi^{\dagger} \colon \Sigma_{f}(X) \to Y, \qquad X \xrightarrow{\xi^{\dagger}} Y \\ \downarrow^{\phi_{X}} \downarrow^{\chi} \downarrow^{\phi_{Y}} \\ \downarrow^{K'} \downarrow^{K'}$$

of K-fibred sets given by

$$\xi^{\dagger} \stackrel{\text{def}}{=} \operatorname{pr}_2 \circ \xi,$$

where we indeed have

$$\begin{split} \phi_Y \circ (\mathrm{pr}_2 \circ \xi) &= (\phi_Y \circ \mathrm{pr}_2) \circ \xi \\ &= (f \circ \mathrm{pr}_1) \circ \xi \qquad \text{(by the pullback square of } K \times_{K'} Y) \\ &= f \circ (\mathrm{pr}_1 \circ \xi) \\ &= f \circ \phi_X. \qquad \text{(since $\xi$ is a morphism of $K'$-fibred sets)} \end{split}$$

• Naturality I. We need to show that, given a morphism

$$\alpha \colon (X, \phi_X) \to (X', \phi_{X'})$$

of K-fibred sets, the diagram

$$\operatorname{Hom}_{\mathsf{FibSets}(K')}(\Sigma_f(X'),Y) \xrightarrow{\Phi_{X',Y}} \operatorname{Hom}_{\mathsf{FibSets}(K)}(X',f_*(Y)),$$

$$\Sigma_f(\alpha)^* \downarrow \qquad \qquad \downarrow \alpha^*$$

$$\operatorname{Hom}_{\mathsf{FibSets}(K')}(\Sigma_f(X),Y) \xrightarrow{\Phi_{X,Y}} \operatorname{Hom}_{\mathsf{FibSets}(K)}(X,f_*(Y)),$$

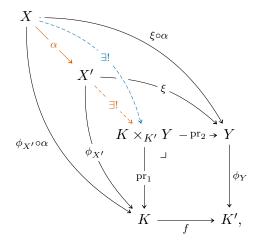
commutes. Indeed, given a morphism

$$\xi \colon \Sigma_f(X') \to Y, \qquad X' \xrightarrow{\xi} Y$$

$$\downarrow \phi_{X'} \downarrow \chi \qquad \downarrow \phi_{Y}$$

$$\downarrow K' \qquad K'$$

of K'-fibred-sets, the map  $\Phi_{X',Y}(\xi) \circ \alpha$  is the composition, coloured in vermillion, of the dashed arrow with  $\alpha$  in the diagram



while  $\Phi_{X,Y}(\xi \circ \Sigma_f(\alpha))$  is given by the dashed arrow, coloured in blue. Since both the blue arrow and the vermillion arrow make the outer pullback diagram commute, it follows from the universal property of the pullback that they must be equal, i.e. that

$$\Phi_{X',Y}(\xi) \circ \alpha = \Phi_{X,Y}(\xi \circ \Sigma_f(\alpha)),$$

showing that the naturality diagram above indeed commutes.

• Naturality II. We need to show that, given a morphism

$$\beta \colon (Y, \phi_Y) \to (Y', \phi_{Y'})$$

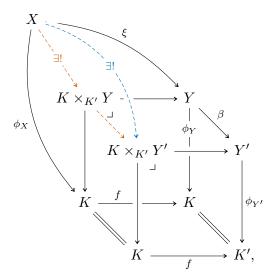
of K-fibred sets, the diagram

$$\begin{array}{ccc} \operatorname{Hom}_{\mathsf{FibSets}(K')}(\Sigma_f(X),Y) & \xrightarrow{\Phi_{X,Y}} & \operatorname{Hom}_{\mathsf{FibSets}(K)}(X,f_*(Y)), \\ & & & \downarrow f_*(\beta)_* \\ & & & \downarrow f_*(\beta)_* \end{array}$$
 
$$\operatorname{Hom}_{\mathsf{FibSets}(K')}(\Sigma_f(X),Y') \xrightarrow{\Phi_{X,Y'}} & \operatorname{Hom}_{\mathsf{FibSets}(K)}(X,f_*(Y')), \end{array}$$

commutes. Indeed, given a morphism

$$\xi \colon \Sigma_f(X') \to Y, \qquad X' \xrightarrow{\xi} Y \\ \phi_{X'} \downarrow \\ K \downarrow \\ K' \downarrow \\ \phi_Y$$

of K'-fibred-sets, the map  $f_*(\beta) \circ \Phi_{X,Y}(\xi)$  is the composition, coloured in vermillion, of the dashed arrow from X to  $K \times_{K'} Y$  with the dashed arrow from  $K \times_{K'} Y$  to  $K \times_{K'} Y'$  in the diagram



while  $\Phi_{X,Y'}(\beta \circ \xi)$  is given by the dashed arrow from X to  $K \times_{K'} Y'$ , coloured in blue. Since both the blue arrow and the vermillion arrow make the outer pullback diagram for  $K \times_{K'} Y'$  commute, it follows from the universal property of the pullback that they must be equal, i.e. that

$$f_*(\beta) \circ \Phi_{X,Y}(\xi) = \Phi_{X,Y'}(\beta \circ \xi),$$

showing that the naturality diagram above indeed commutes.

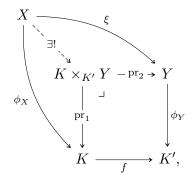
• *Invertibility I.* We claim that

$$\Psi_{X,Y} \circ \Phi_{X,Y} = \mathrm{id}_{\mathrm{Hom}_{\mathsf{FibSets}(K')}(\Sigma_f(X),Y)}.$$

Indeed,  $\Phi_{X,Y}$  sends a map

$$\xi \colon \Sigma_f(X) \to Y, \qquad X \xrightarrow{\xi} Y \\ \phi_X \searrow \\ K \swarrow \\ \phi_Y \swarrow \\ K'$$

of K'-fibred sets to the dashed morphism in the diagram



and  $\Psi_{X,Y}$  then postcomposes that map with pr<sub>2</sub>, which, by the commutativity of the diagram above, is  $\xi$  again, showing the claimed equality to be true.

• Invertibility II. We claim that

$$\Phi_{X,Y} \circ \Psi_{X,Y} = \mathrm{id}_{\mathrm{Hom}_{\mathsf{Fib}\mathsf{Sets}(K)}(X,f^*(Y))}.$$

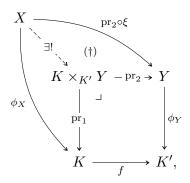
Indeed,  $\Psi_{X,Y}$  sends a map

$$\xi \colon X \to f^*(Y), \qquad X \xrightarrow{\xi} K \times_{K'} Y$$

$$\phi_X \qquad pr_1$$

$$K'$$

of K'-fibred sets to  $\operatorname{pr}_2 \circ \xi$ , which is then sent by  $\Phi_{X,Y}$  to the dashed morphism in the diagram



which, by the commutativity of the subdiagram marked with  $(\dagger)$ , is given by  $\xi$  again, showing the claimed equality to be true.

Direct Proof Part II: The Adjunction  $f^* \dashv \Pi_f$ : We claim there's a bijection

$$\operatorname{Hom}_{\mathsf{FibSets}(K)}(f_*(X), Y), \cong \operatorname{Hom}_{\mathsf{FibSets}(K')}(X, \Pi_f(Y))$$

natural in  $(X, \phi_X) \in \mathsf{FibSets}(K')$  and  $(Y, \phi_Y) \in \mathsf{FibSets}(K)$ :

1. Map I. We define a map

$$\Phi_{X,Y} \colon \mathrm{Hom}_{\mathsf{FibSets}(K)}(f_*(X),Y) \to \mathrm{Hom}_{\mathsf{FibSets}(K')}(X,\Pi_f(Y))$$

defined as follows. Given a morphism

$$\xi \colon f^*(X) \to Y, \qquad K \times_{K'} X \xrightarrow{\xi} Y$$

of K-fibred sets, where

$$f^*(X) \stackrel{\text{def}}{=} K \times_{K'} X$$

$$\stackrel{\text{def}}{=} \{(k, x) \in K \times X \mid f(k) = \phi_X(x)\},$$

we construct a morphism

$$\xi^{\dagger} \colon X \to \Pi_f(Y),$$

$$X \xrightarrow{\xi^{\dagger}} \Pi_f(Y)$$

$$\phi_X \xrightarrow{\Pi_f(\phi_Y)}$$

$$K'$$

of K'-fibred sets, where

$$\Pi_f(Y) \stackrel{\text{def}}{=} \Bigg\{ \big(k',h\big) \in \coprod_{k' \in K'} \mathsf{Sets}\Big(f^{-1}(k'),\phi_Y^{-1}\Big(f^{-1}(k')\Big)\Big) \ \bigg| \ \phi_Y \circ h = \mathrm{id}_{f^{-1}(k')} \Bigg\},$$

by defining

$$\xi^{\dagger}(x) \stackrel{\text{def}}{=} (\phi_X(x), h_{x,\xi})$$

for each  $x \in X$ , where

$$h_{x,\xi} \colon f^{-1}(\phi_X(x)) \to \phi_Y^{-1}(f^{-1}(\phi_X(x)))$$

is the morphism from  $f^{-1}(\phi_X(x)) \subset K$  to  $\phi_Y^{-1}(f^{-1}(\phi_X(x))) \subset Y$  given by

$$h_{x,\xi}(k) \stackrel{\text{def}}{=} \xi(k,x)$$

for each  $k \in f^{-1}(\phi_X(x))$ . Notice that the diagram

$$X \xrightarrow{\xi^{\dagger}} \Pi_{f}(Y)$$

$$\downarrow^{\phi_{X}} \Pi_{f}(\phi_{Y})$$

$$K'$$

indeed commutes since we have

$$[\Pi_f(\phi_Y) \circ \xi^{\dagger}](x) \stackrel{\text{def}}{=} [\Pi_f(\phi_Y)] \Big( \xi^{\dagger}(x) \Big)$$

$$\stackrel{\text{def}}{=} [\Pi_f(\phi_Y)] (\phi_X(x), h_{x,\xi})$$

$$\stackrel{\text{def}}{=} \phi_X(x)$$

for each  $x \in X$ .

2. Map II. We define a map

$$\Psi_{X,Y} \colon \mathrm{Hom}_{\mathsf{FibSets}(K')}(X,\Pi_f(Y)) \to \mathrm{Hom}_{\mathsf{FibSets}(K)}(f_*(X),Y)$$

given by sending a morphism

- $3.\ \ Naturality\ I.$
- 4. Naturality II.
- 5. Invertibility I.
- 6. Invertibility II.

This finishes the proof.

# 5 Un/Straightening for Indexed and Fibred Sets

### 5.1 Straightening for Fibred Sets

Let K be a set and let  $(X, \phi)$  be a K-fibred set.

**Definition 5.1.1.1.** The straightening of  $(X, \phi)$  is the K-indexed set

$$\operatorname{St}_K(X,\phi)\colon K_{\operatorname{\mathsf{disc}}}\to\operatorname{\mathsf{Sets}}$$

defined by

$$\operatorname{St}_K(X,\phi)_x \stackrel{\text{def}}{=} \phi^{-1}(x)$$

for each  $x \in K$ .

**Proposition 5.1.1.2.** Let K be a set.

1. Functoriality. The assignment  $(X, \phi) \mapsto \operatorname{St}_K(X, \phi)$  defines a functor

$$\operatorname{St}_K \colon \mathsf{FibSets}(K) \to \mathsf{ISets}(K)$$

• Action on Objects. For each  $(X, \phi) \in \text{Obj}(\mathsf{FibSets}(K))$ , we have

$$[\operatorname{St}_K](X,\phi) \stackrel{\text{def}}{=} \operatorname{St}_K(X,\phi);$$

• Action on Morphisms. For each  $(X, \phi), (Y, \psi) \in \text{Obj}(\mathsf{FibSets}(K)),$  the action on Hom-sets

$$\operatorname{St}_{K|X,Y} \colon \operatorname{Hom}_{\mathsf{FibSets}(K)}(X,Y) \to \operatorname{Hom}_{\mathsf{ISets}(K)}(\operatorname{St}_K(X),\operatorname{St}_K(Y))$$

of  $St_K$  at (X,Y) is given by sending a morphism

$$f: (X, \phi) \to (Y, \psi)$$

of K-fibred sets to the morphism

$$\operatorname{St}_K(f) \colon \operatorname{St}_K(X, \phi) \to \operatorname{St}_K(Y, \psi)$$

of K-indexed sets defined by

$$\operatorname{St}_K(f) \stackrel{\text{def}}{=} \{f_x^*\}_{x \in K},$$

where  $f_x^*$  is the transport map associated to f at  $x \in K$  of Definition 3.2.1.1.

2. Interaction With Change of Base/Indexing. Let  $f\colon K\to K'$  be a map of sets. The diagram

$$\begin{array}{ccc} \mathsf{FibSets}(K') & \xrightarrow{f^*} & \mathsf{FibSets}(K) \\ & & & & \downarrow \\ \mathsf{St}_{K'} & & & & \downarrow \\ \mathsf{ISets}(K') & \xrightarrow{f^*} & \mathsf{ISets}(K) \end{array}$$

commutes.

3. Interaction With Dependent Sums. Let  $f: K \to K'$  be a map of sets. The diagram

$$\begin{array}{c|c} \mathsf{FibSets}(K) & \xrightarrow{\Sigma_f} & \mathsf{FibSets}(K') \\ & & & & \downarrow \\ \mathsf{St}_K & & & & \downarrow \\ \mathsf{St}_{K'} & & & & \downarrow \\ \mathsf{ISets}(K) & \xrightarrow{\Sigma_f} & \mathsf{ISets}(K') \\ \end{array}$$

commutes.

4. Interaction With Dependent Products. Let  $f \colon K \to K'$  be a map of sets. The diagram

$$\begin{array}{ccc} \mathsf{Sets}_{/K} \stackrel{\Pi_f}{\longrightarrow} \mathsf{FibSets}(K') \\ \\ \text{$\operatorname{St}_K$} & & & & & & \\ \mathsf{St}_{K'} & & & & \\ \mathsf{ISets}(K) \xrightarrow{\Pi_f} \mathsf{ISets}(K') \end{array}$$

commutes.

Proof. Item 1, Functoriality: Omitted.

Item 2, Interaction With Change of Base/Indexing: Indeed, we have

$$\operatorname{St}_{K}(f^{*}(X,\phi))_{x} \stackrel{\operatorname{def}}{=} \operatorname{St}_{K}(K \times_{K'} X)_{x}$$

$$\stackrel{\operatorname{def}}{=} \left(\operatorname{pr}_{1}^{K \times_{K'} X}\right)^{-1}(x)$$

$$= \left\{(k,y) \in K \times_{K'} X \mid \operatorname{pr}_{1}^{K \times_{K'} X}(k,y) = x\right\}$$

$$= \left\{(k,y) \in K \times_{K'} X \mid k = x\right\}$$

$$= \left\{(k,y) \in K \times X \mid k = x \text{ and } f(k) = \phi(y)\right\}$$

$$\stackrel{\operatorname{def}}{=} \left\{y \in X \mid \phi(y) = f(x)\right\}$$

$$= \phi^{-1}(f(x))$$

$$\stackrel{\operatorname{def}}{=} f^{*}(\operatorname{St}_{K'}(X,\phi)_{x})$$

for each  $(X, \phi) \in \text{Obj}(\mathsf{FibSets}(K'))$  and each  $x \in K$ , and similarly for morphisms.

Item 3, Interaction With Dependent Sums: Indeed, we have

$$\operatorname{St}_{K'}(\Sigma_f(X,\phi))_x \stackrel{\text{def}}{=} \Sigma_f(\phi)^{-1}(x)$$

$$\cong \coprod_{\substack{y \in X \\ f(y) = x}} \phi^{-1}(y)$$

$$\cong \Sigma_f(\phi^{-1}(x))$$

$$\stackrel{\text{def}}{=} \Sigma_f(\operatorname{St}_K(X,\phi)_x)$$

for each  $(X, \phi) \in \text{Obj}(\mathsf{FibSets}(K))$  and each  $x \in K'$ , where we have used Item 2 of Proposition 4.2.1.2 for the first bijection, and similarly for morphisms.

Item 4, Interaction With Dependent Products: Indeed, we have

$$\operatorname{St}_{K'}(\Pi_f(X,\phi))_x \stackrel{\text{def}}{=} \Pi_f(\phi)^{-1}(x)$$

$$\cong \prod_{\substack{y \in X \\ f(y) = x}} \phi^{-1}(y)$$

$$\cong \Pi_f(\phi^{-1}(x))$$

$$\stackrel{\text{def}}{=} \Pi_f(\operatorname{St}_K(X,\phi)_x)$$

for each  $(X, \phi) \in \text{Obj}(\mathsf{FibSets}(K))$  and each  $x \in K'$ , where we have used Item 2 of Proposition 4.3.1.3 for the first bijection, and similarly for morphisms.

#### 5.2 Unstraightening for Indexed Sets

Let K be a set and let X be a K-indexed set.

**Definition 5.2.1.1.** The unstraightening of X is the K-fibred set

$$\phi_{\operatorname{Un}_K} \colon \operatorname{Un}_K(X) \to K$$

consisting of

• The Underlying Set. The set  $Un_K(X)$  defined by

$$\operatorname{Un}_K(X) \stackrel{\text{def}}{=} \coprod_{x \in K} X_x;$$

• The Fibration. The map of sets

$$\phi_{\operatorname{Un}_K} \colon \operatorname{Un}_K(X) \to K$$

defined by sending an element of  $\coprod_{x\in K} X_x$  to its index in K.

## **Proposition 5.2.1.2.** Let K be a set.

1. Functoriality. The assignment  $X \mapsto \operatorname{Un}_K(X)$  defines a functor

$$\operatorname{Un}_K \colon \mathsf{ISets}(K) \to \mathsf{FibSets}(K)$$

• Action on Objects. For each  $X \in \text{Obj}(\mathsf{ISets}(K))$ , we have

$$[\operatorname{Un}_K](X) \stackrel{\text{def}}{=} \operatorname{Un}_K(X);$$

• Action on Morphisms. For each  $X, Y \in \text{Obj}(\mathsf{ISets}(K))$ , the action on Hom-sets

 $\operatorname{Un}_{K|X,Y} : \operatorname{Hom}_{\mathsf{ISets}(K)}(X,Y) \to \operatorname{Hom}_{\mathsf{FibSets}(K)}(\operatorname{Un}_K(X),\operatorname{Un}_K(Y))$ 

of  $Un_K$  at (X, Y) is defined by

$$\mathrm{Un}_{K|X,Y}(f) \stackrel{\mathrm{def}}{=} \coprod_{x \in K} f_x^*.$$

2. Interaction With Fibres. We have a bijection of sets

$$\phi_{\operatorname{Un}_K}^{-1}(x) \cong X_x$$

for each  $x \in K$ .

3. As a Pullback. We have a bijection of sets

$$\operatorname{Un}_K(X) \cong K_{\operatorname{\mathsf{disc}}} \times_{\operatorname{\mathsf{Sets}}} \operatorname{\mathsf{Sets}}_*, \qquad igcup_{\overline{\mathbb{K}}} \downarrow \ \mathbb{K}_{\operatorname{\mathsf{disc}}} \xrightarrow{X} \operatorname{\mathsf{Sets}}.$$

4. As a Colimit. We have a bijection of sets

$$\operatorname{Un}_K(X) \cong \operatorname{colim}(X)$$
.

5. Interaction With Change of Indexing/Base. Let  $f \colon K \to K'$  be a map of sets. The diagram

$$\begin{array}{ccc} \mathsf{ISets}(K') & \stackrel{f^*}{-\!\!\!-\!\!\!-\!\!\!-\!\!\!-} \mathsf{ISets}(K) \\ & & \downarrow^{\operatorname{Un}_{K'}} & & \downarrow^{\operatorname{Un}_K} \\ \mathsf{FibSets}(K') & \xrightarrow{f^*} \mathsf{FibSets}(K) \end{array}$$

commutes.

6. Interaction With Dependent Sums. Let  $f: K \to K'$  be a map of sets. The diagram

$$\begin{array}{c|c} \mathsf{ISets}(K) & \xrightarrow{\Sigma_f} & \mathsf{ISets}(K') \\ & & & & \downarrow \mathsf{Un}_{K'} \\ \mathsf{FibSets}(K) & \xrightarrow{\Sigma_f} & \mathsf{FibSets}(K') \end{array}$$

commutes.

7. Interaction With Dependent Products. Let  $f: K \to K'$  be a map of sets. The diagram

$$\begin{array}{ccc} \mathsf{ISets}(K) & \stackrel{\Pi_f}{-\!\!\!-\!\!\!-\!\!\!-\!\!\!-} \mathsf{ISets}(K') \\ & & & & & \downarrow^{\operatorname{Un}_K/} \end{array}$$
 
$$\mathsf{FibSets}(K) & \xrightarrow[\Pi_f]{} \mathsf{FibSets}(K')$$

commutes.

Proof. Item 1, Functoriality: Omitted.

Item 2, Interaction With Fibres: Omitted.

Item 3, As a Pullback: Omitted.

Item 4, As a Colimit: Clear.

Item 5, Interaction With Change of Indexing/Base: Indeed, we have

$$\operatorname{Un}_{K}(f^{*}(X)) \stackrel{\operatorname{def}}{=} \operatorname{Un}_{K}(X \circ f)$$

$$\stackrel{\operatorname{def}}{=} \coprod_{x \in K} X_{f(x)}$$

$$\cong \left\{ (x, (y, a)) \in K \times \coprod_{y \in K'} X_{y} \middle| f(x) = y \right\}$$

$$\cong K \times_{K'} \coprod_{y \in K'} X_{y}$$

$$\stackrel{\operatorname{def}}{=} K \times_{K'} \operatorname{Un}_{K'}(X)$$

$$\stackrel{\operatorname{def}}{=} f^{*}(\operatorname{Un}_{K'}(X))$$

for each  $X \in \mathrm{Obj}(\mathsf{ISets}(K'))$ . Similarly, it can be shown that we also have  $\mathrm{Un}_K(f^*(\phi)) = f^*(\mathrm{Un}_{K'}(\phi))$  and that  $\mathrm{Un}_K \circ f^* = f^* \circ \mathrm{Un}_{K'}$  also holds on morphisms.

Item 6, Interaction With Dependent Sums: Indeed, we have

$$\operatorname{Un}_{K'}(\Sigma_f(X)) \stackrel{\text{def}}{=} \coprod_{x \in K'} \Sigma_f(X)_x$$

$$\cong \coprod_{x \in K'} \coprod_{y \in f^{-1}(x)} X_y$$

$$\cong \coprod_{y \in K} X_y$$

$$\cong \operatorname{Un}_K(X)$$

$$\stackrel{\text{def}}{=} \Sigma_f(\operatorname{Un}_K(X))$$

for each  $X \in \text{Obj}(\mathsf{ISets}(K))$ , where we have used Item 2 of Proposition 4.2.1.2 for the first bijection. Similarly, it can be shown that we also have  $\operatorname{Un}_{K'}(\Sigma_f(\phi)) = \Sigma_f(\phi_{\operatorname{Un}_K})$  and that  $\operatorname{Un}_{K'} \circ \Sigma_f = \Sigma_f \circ \operatorname{Un}_K$  also holds on morphisms. Item 7, Interaction With Dependent Products: Indeed, we have

$$\begin{split} \operatorname{Un}_{K'}(\Pi_f(X)) &\stackrel{\text{def}}{=} \coprod_{x \in K'} \Pi_f(X)_x \\ &\cong \coprod_{x \in K'} \prod_{y \in f^{-1}(x)} X_y \\ &\cong \left\{ (x,h) \in \coprod_{x \in K'} \operatorname{Sets} \left( f^{-1}(x), \phi_{\operatorname{Un}_K}^{-1} \left( f^{-1}(x) \right) \right) \, \middle| \, \phi \circ h = \operatorname{id}_{f^{-1}(x)} \right\} \\ &\stackrel{\text{def}}{=} \Pi_f \left( \coprod_{y \in K} X_y \right) \\ &\stackrel{\text{def}}{=} \Pi_f (\operatorname{Un}_K(X)) \end{split}$$

for each  $X \in \mathrm{Obj}(\mathsf{ISets}(K))$ , where we have used Item 2 of Proposition 4.3.1.3 for the first bijection. Similarly, it can be shown that we also have  $\mathrm{Un}_{K'}(\Pi_f(\phi)) = \Pi_f(\phi_{\mathrm{Un}_K})$  and that  $\mathrm{Un}_{K'} \circ \Pi_f = \Pi_f \circ \mathrm{Un}_K$  also holds on morphisms.  $\square$ 

# 5.3 The Un/Straightening Equivalence

**Theorem 5.3.1.1.** We have an isomorphism of categories

$$(\operatorname{St}_K \dashv \operatorname{Un}_K)\colon \operatorname{\mathsf{FibSets}}(K) \underbrace{\downarrow}_{\operatorname{Un}_K} \operatorname{\mathsf{ISets}}(K).$$

*Proof.* Omitted.

# 6 Miscellany

# 6.1 Other Kinds of Un/Straightening

**Remark 6.1.1.1.** There are also other kinds of un/straightening for sets, where Sets is replaced by **Rel** or Span:

• Un/Straightening With Rel, I. We have an isomorphism of sets

$$Rel(A, B) \cong Sets(B \times A, \{true, false\}).$$

by the definition of a relation from A to B, Relations, Definition 1.1.1.1.

• Un/Straightening With Rel, II. We have an equivalence of categories

$$\mathsf{LaxFun}(K_{\mathsf{disc}}, \mathbf{Rel}) \overset{\mathrm{eq.}}{\cong} \mathsf{Cats}^{\mathsf{fth}}_{/K_{\mathsf{disc}}},$$

where  $\mathsf{Cats}^\mathsf{fth}_{/K_\mathsf{disc}}$  is the full subcategory of  $\mathsf{Cats}_{/K_\mathsf{disc}}$  spanned by the faithful functors; see [Nie04, Theorem 3.1].

•  $Un/Straightening\ With\ \mathsf{Span},\ I.\ \mathsf{For\ each}\ A,B\in \mathsf{Obj}(\mathsf{Sets}),$  we have a morphism of sets

$$\mathsf{Span}(A,B) \to \mathsf{Sets}(A \times B, \mathbb{N} \cup \{\infty\})$$

which assemble into an equivalence of categories between Span(Sets) and the category MRel of "multirelations"; see Spans, Remark 7.5.1.1.

• Un/Straightening With Span, II. We have an equivalence of categories

$$\mathsf{LaxFun}(K_{\mathsf{disc}},\mathsf{Span}) \stackrel{\mathrm{\tiny eq.}}{\cong} \mathsf{Cats}_{/K_{\mathsf{disc}}};$$

see [nLa23, Section 3].

# **Appendices**

# A Other Chapters

#### Set Theory

- 1. Sets
- 2. Constructions With Sets
- 3. Pointed Sets
- 4. Tensor Products of Pointed Sets
- 5. Indexed and Fibred Sets
- 6. Relations
- 7. Spans
- 8. Posets

#### **Category Theory**

- 9. Categories
- 10. Constructions With Categories
- 11. Kan Extensions

#### **Bicategories**

- 12. Bicategories
- 13. Internal Adjunctions

#### **Internal Category Theory**

14. Internal Categories

### Cyclic Stuff

15. The Cycle Category

#### **Cubical Stuff**

16. The Cube Category

#### Globular Stuff

17. The Globe Category

#### Cellular Stuff

18. The Cell Category

#### Monoids

- 19. Monoids
- 20. Constructions With Monoids

#### Monoids With Zero

- 21. Monoids With Zero
- 22. Constructions With Monoids With Zero

#### Groups

- 23. Groups
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### Hyper Algebra

- 25. Hypermonoids
- 26. Hypergroups
- 27. Hypersemirings and Hyperrings
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#### **Near-Rings**

- 29. Near-Semirings
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### Real Analysis

- 31. Real Analysis in One Variable
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#### Measure Theory

- 33. Measurable Spaces
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# **Probability Theory**

34. Probability Theory

# Stochastic Analysis

- 35. Stochastic Processes, Martingales, and Brownian Motion
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# Differential Geometry

38. Topological and Smooth Manifolds

### Schemes

39. Schemes