Indexed Sets

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This chapter contains a discussion 00QJ of indexed sets, the set-theoretical counterpart to indexed categories. In particular, here we explore:

1. Indexed sets, i.e. functors $K_{\mathsf{disc}} \longrightarrow \mathsf{Sets}$ with K a set;
2. The limits and colimits in the category of K-indexed sets;
3. Constructions with indexed sets like dependent sums, dependent products, and internal Homs.

Contents

1 Indexed Sets 00QK

1.1 Foundations 00QL

Let K be a set.

Definition 1.1.1.1. A K**-indexed set** is a functor 00QM $X : K_{\mathsf{disc}} \longrightarrow \mathsf{Sets}$.

Remark 1.1.1.2. By Categories, **??**, a K**-indexed set** 00QN consists of a K-indexed collection

$$X^{\dagger} : K \longrightarrow \mathrm{Obj}(\mathsf{Sets}),$$

of sets, assigning a set $X^{\dagger}_x \overset{\text{def}}{=} X_x$ to each element x of K.

1.2 Morphisms of Indexed Sets 00QP

Let $X : K_{\mathsf{disc}} \longrightarrow \mathsf{Sets}$ and $Y : K_{\mathsf{disc}} \longrightarrow \mathsf{Sets}$ be indexed sets.

Definition 1.2.1.1. A **morphism of K-indexed sets from X to Y**[1] is a natural transformation

$$f\colon X \Longrightarrow Y, \qquad \begin{tikzcd} K_{\mathsf{disc}} \arrow[r, bend left, "X"{name=U}] \arrow[r, bend right, "Y"'{name=D}] & \mathsf{Sets} \arrow[from=U, to=D, Rightarrow, "f"'] \end{tikzcd}$$

from X to Y.

Remark 1.2.1.2. In detail, a **morphism of K-indexed sets** consists of a K-indexed collection

$$\{f_x\colon X_x \longrightarrow Y_x\}_{x\in K}$$

of maps of sets.

1.3 The Category of Sets Indexed by a Fixed Set

Let K be a set.

Definition 1.3.1.1. The **category of K-indexed sets** is the category $\mathsf{ISets}(K)$ defined by

$$\mathsf{ISets}(K) \overset{\text{def}}{=} \mathsf{Fun}\Big(K_{\mathsf{disc}}, \mathsf{Sets}\Big).$$

Remark 1.3.1.2. In detail, the **category of K-indexed sets** is the category $\mathsf{ISets}(K)$ where

- *Objects.* The objects of $\mathsf{ISets}(K)$ are K-indexed sets as in **??**;
- *Morphisms.* The morphisms of $\mathsf{ISets}(K)$ are morphisms of K-indexed sets as in **??**;
- *Identities.* For each $X \in \mathrm{Obj}(\mathsf{ISets}(K))$, the unit map

  $$\mathbb{1}^{\mathsf{ISets}(K)}_X\colon \mathrm{pt} \longrightarrow \mathrm{Hom}_{\mathsf{ISets}(K)}(X, X)$$

  of $\mathsf{ISets}(K)$ at X is defined by

  $$\mathrm{id}^{\mathsf{ISets}(K)}_X \overset{\text{def}}{=} \left\{\mathrm{id}_{X_x}\right\}_{x\in K};$$

- *Composition.* For each $X, Y, Z \in \mathrm{Obj}(\mathsf{ISets}(K))$, the composition map

  $$\circ^{\mathsf{ISets}(K)}_{X,Y,Z}\colon \mathrm{Hom}_{\mathsf{ISets}(K)}(Y, Z) \times \mathrm{Hom}_{\mathsf{ISets}(K)}(X, Y) \longrightarrow \mathrm{Hom}_{\mathsf{ISets}(K)}(X, Z)$$

  of $\mathsf{ISets}(K)$ at (X, Y, Z) is defined by

  $$\{g_x\}_{x\in K} \circ^{\mathsf{ISets}(K)}_{X,Y,Z} \{f_x\}_{x\in K} \overset{\text{def}}{=} \{g_x \circ f_x\}_{x\in K}.$$

[1] *Further Terminology:* Also called a K**-indexed map of sets from X to Y**.

1.4 The Category of Indexed Sets

Definition 1.4.1.1. The **category of indexed sets** is the category ISets defined as the Grothendieck construction of the functor $\mathsf{ISets}\colon \mathsf{Sets}^{\mathsf{op}} \to \mathsf{Cats}$ of **??**:

$$\mathsf{ISets} \overset{\text{def}}{=} \int^{\mathsf{Sets}} \mathsf{ISets}.$$

Remark 1.4.1.2. In detail, the **category of indexed sets** is the category ISets where

- *Objects.* The objects of ISets are pairs (K, X) consisting of
  - *The Indexing Set.* A set K;
  - *The Indexed Set.* A K-indexed set $X\colon K_{\mathsf{disc}} \to \mathsf{Sets}$;
- *Morphisms.* A morphism of ISets from (K, X) to (K', Y) is a pair (ϕ, f) consisting of
  - *The Reindexing Map.* A map of sets $\phi\colon K \to K'$;
  - *The Morphism of Indexed Sets.* A morphism of K-indexed sets $f\colon X \to \phi_*(Y)$ as in the diagram

$$f\colon X \to \phi_*(Y), \qquad \begin{tikzcd} K_{\mathsf{disc}} \arrow[rr, "\phi"] \arrow[dr, "X"'] & \arrow[d, phantom, "\overset{f}{\Rightarrow}"] & K'_{\mathsf{disc}} \arrow[dl, "Y"] \\ & \mathsf{Sets}; & \end{tikzcd}$$

- *Identities.* For each $(K, X) \in \mathrm{Obj}(\mathsf{ISets})$, the unit map

$$\mathbb{1}^{\mathsf{ISets}}_{(K,X)}\colon \mathrm{pt} \to \mathsf{ISets}((K, X), (K, X))$$

  of ISets at (K, X) is defined by

$$\mathrm{id}^{\mathsf{ISets}}_{(K,X)} \overset{\text{def}}{=} (\mathrm{id}_K, \mathrm{id}_X).$$

- *Composition.* For each $\mathbf{X} = (K, X)$, $\mathbf{Y} = (K', Y)$, $\mathbf{Z} = (K'', Z) \in \mathrm{Obj}(\mathsf{ISets})$, the composition map

$$\circ^{\mathsf{ISets}}_{\mathbf{X},\mathbf{Y},\mathbf{Z}}\colon \mathsf{ISets}(\mathbf{Y}, \mathbf{Z}) \times \mathsf{ISets}(\mathbf{X}, \mathbf{Y}) \to \mathsf{ISets}(\mathbf{X}, \mathbf{Z})$$

  of ISets at $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ is defined by

$$(\psi, g) \circ (\phi, f) \overset{\text{def}}{=} (\psi \circ \phi, (g \circ \mathrm{id}_\phi) \circ f),$$

as in the diagram

$$\begin{tikzcd} K_{\mathsf{disc}} \arrow[r, "\phi"] \arrow[dr, "X"'] & K'_{\mathsf{disc}} \arrow[r, "\psi"] \arrow[d, "Y"] & K''_{\mathsf{disc}} \arrow[dl, "Z"] \\ & \mathsf{Sets}; & \arrow[from=1-1, to=2-2, phantom, shift left=3ex, "\overset{f}{\Rightarrow}"] \arrow[from=1-2, to=2-2, phantom, shift left=3ex, "\overset{g}{\Rightarrow}"] \end{tikzcd}$$

for each $(\phi, f) \in \mathsf{ISets}(\mathbf{X}, \mathbf{Y})$ and each $(\psi, g) \in \mathsf{ISets}(\mathbf{Y}, \mathbf{Z})$.

2 Limits of Indexed Sets 002Y

2.1 Products of K-Indexed Sets 002Z

Let $X: K_{\mathsf{disc}} \to \mathsf{Sets}$ and $Y: K_{\mathsf{disc}} \to \mathsf{Sets}$ be indexed sets.

Definition 2.1.1.1. The **product of X and Y** 00R0 is the K-indexed set $X \times Y: K_{\mathsf{disc}} \to \mathsf{Sets}$ defined by

$$(X \times Y)_k \stackrel{\text{def}}{=} X_k \times Y_k$$

for each $k \in K$.

Proof. That this agrees with the categorical product in $\mathsf{ISets}(K)$ follows from Limits and Colimits, **??** of **??**. ☐

2.2 Pullbacks of K-Indexed Sets 00R1

Let $X, Y, Z: K_{\mathsf{disc}} \to \mathsf{Sets}$ be K-indexed sets and let $f: X \to Z$ and $g: Y \to Z$ be morphisms of K-indexed sets.

Definition 2.2.1.1. The **pullback of X and Y** 00R2 **over Z** is the K-indexed set $X \times_Z Y: K_{\mathsf{disc}} \to \mathsf{Sets}$ defined by

$$(X \times_Z Y)_k \stackrel{\text{def}}{=} X_k \times_{Z_k} Y_k$$

for each $k \in K$.

Proof. That this agrees with the categorical pullback in $\mathsf{ISets}(K)$ follows from Limits and Colimits, **??** of **??**. ☐

2.3 Equalisers of K-Indexed Sets 00R3

Let $X, Y \colon K_{\mathsf{disc}} \to \mathsf{Sets}$ be K-indexed sets and let $f, g \colon X \rightrightarrows Y$ be morphisms of K-indexed sets.

Definition 2.3.1.1. The **equaliser of f and g** 00R4 is the K-indexed set $\mathrm{Eq}(f,g) \colon K_{\mathsf{disc}} \to \mathsf{Sets}$ defined by

$$(\mathrm{Eq}(f,g))_k \stackrel{\text{def}}{=} \mathrm{Eq}(f_k, g_k)$$

for each $k \in K$.

Proof. That this agrees with the categorical equaliser in $\mathsf{ISets}(K)$ follows from Limits and Colimits, **??** of **??**. □

2.4 Products in ISets 00R5

Let $X \colon K_{\mathsf{disc}} \to \mathsf{Sets}$ and $Y \colon K'_{\mathsf{disc}} \to \mathsf{Sets}$ be indexed sets.

Definition 2.4.1.1. The **product of X and Y** 00R6 is the $(K \times K')$-indexed set

$$X \times Y \colon (K \times K')_{\mathsf{disc}} \to \mathsf{Sets}$$

defined by

$$(X \times Y)_{(k,k')} \stackrel{\text{def}}{=} X_k \times Y_{k'}$$

for each $(k, k') \in K \times K'$.

Proof. We claim that this agrees with the categorical product in ISets. □

2.5 Pullbacks in ISets 00R7

Let $X \colon K_{\mathsf{disc}} \to \mathsf{Sets}$ be a K-indexed set, let $Y \colon K'_{\mathsf{disc}} \to \mathsf{Sets}$ be a K'-indexed set, let $Z \colon K''_{\mathsf{disc}} \to \mathsf{Sets}$ be a K''-indexed set, and let $(\phi, f) \colon X \to Z$ and $(\psi, g) \colon Y \to Z$ be morphisms of indexed sets (as in **??**).

Definition 2.5.1.1. The **pullback of X and Y over Z** 00R8 is the $(K \times_{K''} K)$-indexed set

$$X \times_Z Y \colon (K \times_{K''} K)_{\mathsf{disc}} \to \mathsf{Sets}$$

defined by

$$\begin{aligned}(X \times_Z Y)_{(k,k')} &\stackrel{\text{def}}{=} X_k \times_{Z_{\phi(k)}} Y_{k'}\\ &\stackrel{\text{def}}{=} X_k \times_{Z_{\psi(k)}} Y_{k'}\end{aligned}$$

for each $(k, k') \in K \times_{K''} K'$.

Proof. We claim that this agrees with the categorical pullback in ISets. □

2.6 Equalisers in ISets 00R9

Let $X\colon K_{\mathsf{disc}} \longrightarrow \mathsf{Sets}$ be a K-indexed set, let $Y\colon K'_{\mathsf{disc}} \longrightarrow \mathsf{Sets}$ be a K'-indexed set, and let $(\phi, f), (\psi, g)\colon X \longrightarrow Y$ be morphisms of indexed sets (as in **??**).

Definition 2.6.1.1. The **equaliser of** (ϕ, f) 00RA **and** (ψ, g) is the $\mathrm{Eq}(\phi,\psi)$-indexed set $\mathrm{Eq}(f, g)\colon \mathrm{Eq}(\phi,\psi) \longrightarrow \mathsf{Sets}$ defined by

$$(\mathrm{Eq}(f,g))_k \stackrel{\text{def}}{=} \mathrm{Eq}(f_k, g_k)$$

for each $k \in \mathrm{Eq}(\phi,\psi)$.

Proof. We claim that this agrees with the categorical equaliser in ISets. □

3 Colimits of Indexed Sets 00RB

3.1 Coproducts of K-Indexed Sets 00RC

Let $X\colon K_{\mathsf{disc}} \longrightarrow \mathsf{Sets}$ and $Y\colon K_{\mathsf{disc}} \longrightarrow \mathsf{Sets}$ be indexed sets.

Definition 3.1.1.1. The **coproduct** of X and Y 00RD is the K-k-indexed set $X \coprod Y\colon K_{\mathsf{disc}} \longrightarrow \mathsf{Sets}$ defined by

$$(X \coprod Y)_k \stackrel{\text{def}}{=} X_k \coprod Y_k$$

for each $k \in K$.

Proof. That this agrees with the categorical coproduct in $\mathsf{ISets}(K)$ follows from Limits and Colimits, **??** of **??**. □

3.2 Pushouts of K-Indexed Sets 00RE

Let $X, Y, Z\colon K_{\mathsf{disc}} \longrightarrow \mathsf{Sets}$ be K-indexed sets and let $f\colon Z \longrightarrow X$ and $g\colon Z \longrightarrow Y$ be morphisms of K-indexed sets.

Definition 3.2.1.1. The **pushout** of X and Y 00RF is the K-indexed set $X \coprod_Z Y\colon K_{\mathsf{disc}} \longrightarrow \mathsf{Sets}$ defined by

$$(X \coprod_Z Y)_k \stackrel{\text{def}}{=} X_k \coprod_{Z_k} Y_k$$

for each $k \in K$.

Proof. That this agrees with the categorical pushout in $\mathsf{ISets}(K)$ follows from Limits and Colimits, **??** of **??**. □

3.3 Coequalisers of K-Indexed Sets 00RG

Let $X, Y: K_{\mathsf{disc}} \to \mathsf{Sets}$ be K-indexed sets and let $f, g: X \rightrightarrows Y$ be morphisms of K-indexed sets.

Definition 3.3.1.1. The **coequaliser** of X and Y 00RH is the K-indexed set $\mathrm{CoEq}(f, g): K_{\mathsf{disc}} \to \mathsf{Sets}$ defined by

$$(\mathrm{CoEq}(f, g))_k \overset{\text{def}}{=} \mathrm{CoEq}(f_k, g_k)$$

for each $k \in K$.

Proof. That this agrees with the categorical coequaliser in $\mathsf{ISets}(K)$ follows from Limits and Colimits, **??** of **??**. □

4 Constructions With Indexed Sets 00RJ

4.1 Change of Indexing 00RK

Let $\phi: K \to K'$ be a function and let X be a K'-indexed set.

Definition 4.1.1.1. The **change of indexing of X to K** 00RL is the K-indexed set $\phi^*(X)$ defined by

$$\phi^*(X) \overset{\text{def}}{=} X \circ \phi_{\mathsf{disc}}.$$

Remark 4.1.1.2. In detail, the **change of indexing of X to K** 00RM is the K-indexed set $\phi^*(X)$ defined by

$$\phi^*(X)_x \overset{\text{def}}{=} X_{\phi(x)}$$

for each $x \in K$.

Proposition 4.1.1.3. The assignment $X \mapsto \phi^*(X)$ 00RN defines a functor

$$\phi^*: \mathsf{ISets}(K') \to \mathsf{ISets}(K),$$

where

- *Action on Objects.* For each $X \in \mathrm{Obj}(\mathsf{ISets}(K'))$, we have

$$[\phi^*](X) \overset{\text{def}}{=} \phi^*(X);$$

- *Action on Morphisms.* For each $X, Y \in \mathrm{Obj}(\mathsf{ISets}(K'))$, the action on Hom-sets

$$\phi^*_{X,Y}: \mathrm{Hom}_{\mathsf{ISets}(K')}(X, Y) \to \mathrm{Hom}_{\mathsf{ISets}(K)}(\phi^*(X), \phi^*(Y))$$

of ϕ^* at (X, Y) is the map sending a morphism of K'-indexed sets

$$f = \{f_x : X_x \to Y_x\}_{x \in K'}$$

from X to Y to the morphism of K-indexed sets defined by

$$\phi^*(f) \overset{\text{def}}{=} \left\{f_{\phi(x)} : X_{\phi(x)} \to Y_{\phi(x)}\right\}_{x \in K}.$$

Proof. Omitted. □

Proposition 4.1.1.4. The assignment $K \mapsto \mathsf{ISets}(K)$ 00RP defines a functor

$$\mathsf{ISets} : \mathsf{Sets}^{\mathsf{op}} \to \mathsf{Cats},$$

where

- *Action on Objects.* For each $K \in \mathrm{Obj}(\mathsf{Sets})$, we have

  $$[\mathsf{ISets}](K) \overset{\text{def}}{=} \mathsf{ISets}(K);$$

- *Action on Morphisms.* For each $K, K' \in \mathrm{Obj}(\mathsf{Sets})$, the action on Hom-sets

  $$\mathsf{ISets}_{K,K'} : \mathsf{Sets}^{\mathsf{op}}(K, K') \to \mathsf{Fun}(\mathsf{ISets}(K), \mathsf{ISets}(K'))$$

  of ISets at (K, K') is the map defined by

  $$\mathsf{ISets}_{K,K'}(\phi) \overset{\text{def}}{=} \phi^*$$

  for each $\phi \in \mathsf{Sets}^{\mathsf{op}}(K, K')$.

Proof. Omitted. □

4.2 Dependent Sums 00RQ

Let $\phi : K \to K'$ be a function and let X be a K-indexed set.

Definition 4.2.1.1. The **dependent sum of** X 00RR is the K'-indexed set $\Sigma_\phi(X)$[2] defined by

$$\Sigma_\phi(X) \overset{\text{def}}{=} \mathrm{Lan}_\phi(X),$$

and hence given by

$$\Sigma_\phi(X)_x \cong \coprod_{y \in \phi^{-1}(x)} X_y$$

for each $x \in K'$.

[2] *Further Notation:* Also written $\phi_*(X)$.

Proposition 4.2.1.2. The assignment $X \mapsto \Sigma_\phi(X)$ defines a functor

$$\Sigma_\phi \colon \mathsf{ISets}(K) \to \mathsf{ISets}(K'),$$

where

- *Action on Objects.* For each $X \in \mathrm{Obj}(\mathsf{ISets}(K))$, we have

$$[\Sigma_\phi](X) \stackrel{\text{def}}{=} \Sigma_\phi(X);$$

- *Action on Morphisms.* For each $X, Y \in \mathrm{Obj}(\mathsf{ISets}(K))$, the action on Hom-sets

$$\Sigma_{\phi|X,Y} \colon \mathrm{Hom}_{\mathsf{ISets}(K)}(X, Y) \to \mathrm{Hom}_{\mathsf{ISets}(K')}(\Sigma_\phi(X), \Sigma_\phi(Y))$$

  of Σ_ϕ at (X, Y) is the map sending a morphism of K-indexed sets

$$f \colon X \to Y$$

  to the morphism of K'-indexed sets defined by

$$\begin{aligned} \Sigma_\phi(f) &\stackrel{\text{def}}{=} \mathrm{Lan}_\phi(f); \\ &\cong \coprod_{y \in \phi^{-1}(X)} f_y. \end{aligned}$$

Proof. Omitted. □

4.3 Dependent Products

Let $\phi \colon K \to K'$ be a function and let X be a K-indexed set.

Definition 4.3.1.1. The **dependent product of** X is the K'-indexed set $\Pi_\phi(X)$[3] defined by

$$\Pi_\phi(X) \stackrel{\text{def}}{=} \mathrm{Ran}_\phi(X),$$

and hence given by

$$\Pi_\phi(X)_x \cong \prod_{y \in \phi^{-1}(x)} X_y$$

for each $x \in K'$.

Proposition 4.3.1.2. The assignment $X \mapsto \Pi_\phi(X)$ defines a functor

$$\Pi_\phi \colon \mathsf{ISets}(K) \to \mathsf{ISets}(K'),$$

where

[3] *Further Notation:* Also written $\phi_!(X)$.

- *Action on Objects.* For each $X \in \mathrm{Obj}(\mathsf{ISets}(K))$, we have

$$[\Pi_\phi](X) \stackrel{\text{def}}{=} \Pi_\phi(X);$$

- *Action on Morphisms.* For each $X, Y \in \mathrm{Obj}(\mathsf{ISets}(K))$, the action on Hom-sets

$$\Pi_{\phi|X,Y} \colon \mathrm{Hom}_{\mathsf{ISets}(K)}(X, Y) \to \mathrm{Hom}_{\mathsf{ISets}(K')}(\Pi_\phi(X), \Pi_\phi(Y))$$

  of Π_ϕ at (X, Y) is the map sending a morphism of K-indexed sets

$$f \colon X \to Y$$

  to the morphism of K'-indexed sets defined by

$$\begin{aligned} \Pi_\phi(f) &\stackrel{\text{def}}{=} \mathrm{Ran}_\phi(f); \\ &\cong \prod_{y \in \phi^{-1}(x)} f_y. \end{aligned}$$

Proof. Omitted. □

4.4 Internal Homs 00RW

Let K be a set and let X and Y be K-indexed sets.

Definition 4.4.1.1. 00RX The **internal Hom of indexed sets from** X **to** Y is the indexed set $\mathbf{Hom}_{\mathsf{ISets}(K)}(X, Y)$ defined by

$$\mathbf{Hom}_{\mathsf{ISets}(K)}(X, Y) \stackrel{\text{def}}{=} \mathsf{Sets}(X_x, Y_x)$$

for each $x \in K$.

4.5 Adjointness of Indexed Sets 00RY

Let $\phi \colon K \to K'$ be a map of sets.

Proposition 4.5.1.1. 00RZ We have a triple adjunction

$$\left(\Sigma_\phi \dashv \phi^* \dashv \Pi_\phi\right)\colon \quad \begin{tikzcd} \mathsf{ISets}(K) \arrow[r, bend left=50, "\Sigma_\phi"] \arrow[r, bend right=50, "\Pi_\phi"'] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, phantom, shift right=3, "\perp"] & \mathsf{ISets}(K') \arrow[l, "\phi^*"'] \end{tikzcd}.$$

Proof. This follows from Kan Extensions, **??** of **??**. □

Appendices

A Other Chapters