

Sets

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This chapter (will eventually) contain material on axiomatic set theory, as well as a couple other things.

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1 The Enrichment of Sets in Classical Truth Values

1.1 (-2) -Categories

Definition 1.1.1.1. A (-2) -category is the “necessarily true” truth value.^{1,2,3}

1.2 (-1) -Categories

Definition 1.2.1.1. A (-1) -category is a classical truth value.

Remark 1.2.1.2. ⁴ (-1) -categories should be thought of as being “categories enriched in (-2) -categories”, having a collection of objects and, for each pair of objects, a Hom-object

¹Thus, there is only one (-2) -category.

²A $(-n)$ -category for $n = 3, 4, \dots$ is also the “necessarily true” truth value, coinciding with a (-2) -category.

³For motivation, see [BS10, p. 13].

⁴For more motivation, see [BS10, p. 13].

$\text{Hom}(x, y)$ that is a (−2)-category (i.e. trivial).

Therefore, a (−1)-category C is either ([BS10, pp. 33–34]):

1. *Empty*, having no objects;
2. *Contractible*, having a collection of objects $\{a, b, c, \dots\}$, but with $\text{Hom}_C(a, b)$ being a (−2)-category (i.e. trivial) for all $a, b \in \text{Obj}(C)$, forcing all objects of C to be uniquely isomorphic to each other.

As such, there are only two (−1)-categories, up to equivalence:

- The (−1)-category false (the empty one);
- The (−1)-category true (the contractible one).

Definition 1.2.1.3. The **poset of truth values**⁵ is the poset $(\{\text{true}, \text{false}\}, \leq)$ ⁶ consisting of

- *The Underlying Set.* The set $\{\text{true}, \text{false}\}$ whose elements are the truth values true and false;
- *The Partial Order.* The partial order

$$\leq: \{\text{true}, \text{false}\} \times \{\text{true}, \text{false}\} \rightarrow \{\text{true}, \text{false}\}$$

on $\{\text{true}, \text{false}\}$ defined by⁷

$$\begin{aligned} \text{false} \leq \text{false} &\stackrel{\text{def}}{=} \text{true}, \\ \text{true} \leq \text{false} &\stackrel{\text{def}}{=} \text{false}, \\ \text{false} \leq \text{true} &\stackrel{\text{def}}{=} \text{true}, \\ \text{true} \leq \text{true} &\stackrel{\text{def}}{=} \text{true}. \end{aligned}$$

Proposition 1.2.1.4. The poset of truth values $\{t, f\}$ is Cartesian closed with product given by⁸

$$\begin{aligned} t \times t &= t, \\ t \times f &= f, \\ f \times t &= f, \\ f \times f &= f, \end{aligned}$$

⁵*Further Terminology:* Also called the **poset of (−1)-categories**.

⁶*Further Notation:* Also written $\{t, f\}$.

⁷This partial order coincides with logical implication.

⁸Note that \times coincides with the “and” operator, while $\mathbf{Hom}_{\{t, f\}}$ coincides with the logical implication

and internal Hom $\mathbf{Hom}_{\{t,f\}}$ given by the partial order of $\{t, f\}$, i.e. by

$$\mathbf{Hom}_{\{t,f\}}(t, t) = t,$$

$$\mathbf{Hom}_{\{t,f\}}(t, f) = f,$$

$$\mathbf{Hom}_{\{t,f\}}(f, t) = t,$$

$$\mathbf{Hom}_{\{t,f\}}(f, f) = t.$$

Proof. Existence of Products: We claim that the products $t \times t$, $t \times f$, $f \times t$, and $f \times f$ satisfy the universal property of the product in $\{t, f\}$. Indeed, consider the diagrams

$$\begin{array}{ccccccc} & P_1 & & P_2 & & P_3 & & P_4 \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ p_1^1 \swarrow & & \exists! & \searrow p_2^1 & & p_1^3 \swarrow & & p_2^3 \searrow \\ t & \xleftarrow{\text{pr}_1} & t \times t & \xrightarrow{\text{pr}_2} & t & f & \xleftarrow{\text{pr}_1} & f \times t & \xrightarrow{\text{pr}_2} & t \\ & \underbrace{\hspace{2cm}}_{=t} & & \underbrace{\hspace{2cm}}_{=f} & & \underbrace{\hspace{2cm}}_{=f} & & \underbrace{\hspace{2cm}}_{=f} \end{array}$$

Here:

1. If $P_1 = t$, then $p_1^1 = p_2^1 = \text{id}_t$, and there's indeed a unique morphism from P_1 to t making the diagram commute, namely id_t ;
2. If $P_1 = f$, then $p_1^1 = p_2^1$ are given by the unique morphism from f to t , and there's indeed a unique morphism from P_1 to t making the diagram commute, namely the unique morphism from f to t ;
3. If $P_2 = t$, then there is no morphism p_2^2 .
4. If $P_2 = f$, then p_1^2 is the unique morphism from f to t while $p_2^2 = \text{id}_f$, and there's indeed a unique morphism from P_2 to f making the diagram commute, namely id_f ;
5. The proof for P_3 is similar to the one for P_2 ;
6. If $P_4 = t$, then there is no morphism p_1^4 or p_2^4 .
7. If $P_4 = f$, then $p_1^4 = p_2^4 = \text{id}_f$, and there's indeed a unique morphism from P_4 to f making the diagram commute, namely id_f .

Cartesian Closedness: We claim there's a bijection

$$\mathbf{Hom}_{\{t,f\}}(A \times B, C) \cong \mathbf{Hom}_{\{t,f\}}(A, \mathbf{Hom}_{\{t,f\}}(B, C))$$

natural in $A, B, C \in \{t, f\}$. Indeed:

- For $(A, B, C) = (t, t, t)$, we have

$$\begin{aligned}
 \text{Hom}_{\{t,f\}}(t \times t, t) &\cong \text{Hom}_{\{t,f\}}(t, t) \\
 &= \{\text{id}_{\text{true}}\} \\
 &\cong \text{Hom}_{\{t,f\}}(t, t) \\
 &\cong \text{Hom}_{\{t,f\}}(t, \mathbf{Hom}_{\{t,f\}}(t, t)).
 \end{aligned}$$

- For $(A, B, C) = (t, t, f)$, we have

$$\begin{aligned}
 \text{Hom}_{\{t,f\}}(t \times t, f) &\cong \text{Hom}_{\{t,f\}}(t, f) \\
 &= \emptyset \\
 &\cong \text{Hom}_{\{t,f\}}(t, f) \\
 &\cong \text{Hom}_{\{t,f\}}(t, \mathbf{Hom}_{\{t,f\}}(t, f)).
 \end{aligned}$$

- For $(A, B, C) = (t, f, t)$, we have

$$\begin{aligned}
 \text{Hom}_{\{t,f\}}(t \times f, t) &\cong \text{Hom}_{\{t,f\}}(f, t) \\
 &\cong \text{pt} \\
 &\cong \text{Hom}_{\{t,f\}}(f, t) \\
 &\cong \text{Hom}_{\{t,f\}}(f, \mathbf{Hom}_{\{t,f\}}(f, t)).
 \end{aligned}$$

- For $(A, B, C) = (t, f, f)$, we have

$$\begin{aligned}
 \text{Hom}_{\{t,f\}}(t \times f, f) &\cong \text{Hom}_{\{t,f\}}(f, f) \\
 &\cong \{\text{id}_{\text{false}}\} \\
 &\cong \text{Hom}_{\{t,f\}}(f, f) \\
 &\cong \text{Hom}_{\{t,f\}}(t, \mathbf{Hom}_{\{t,f\}}(f, f)).
 \end{aligned}$$

- For $(A, B, C) = (f, t, t)$, we have

$$\begin{aligned}
 \text{Hom}_{\{t,f\}}(f \times t, t) &\cong \text{Hom}_{\{t,f\}}(f, t) \\
 &\cong \text{pt} \\
 &\cong \text{Hom}_{\{t,f\}}(f, t) \\
 &\cong \text{Hom}_{\{t,f\}}(f, \mathbf{Hom}_{\{t,f\}}(t, t)).
 \end{aligned}$$

operator.

- For $(A, B, C) = (f, t, f)$, we have

$$\begin{aligned}
 \mathrm{Hom}_{\{t, f\}}(f \times t, f) &\cong \mathrm{Hom}_{\{t, f\}}(f, f) \\
 &\cong \{\mathrm{id}_{\mathrm{false}}\} \\
 &\cong \mathrm{Hom}_{\{t, f\}}(f, f) \\
 &\cong \mathrm{Hom}_{\{t, f\}}(f, \mathbf{Hom}_{\{t, f\}}(t, f)).
 \end{aligned}$$

- For $(A, B, C) = (f, f, t)$, we have

$$\begin{aligned}
 \mathrm{Hom}_{\{t, f\}}(f \times f, t) &\cong \mathrm{Hom}_{\{t, f\}}(f, t) \\
 &\cong \mathrm{pt} \\
 &\cong \mathrm{Hom}_{\{t, f\}}(f, t) \\
 &\cong \mathrm{Hom}_{\{t, f\}}(f, \mathbf{Hom}_{\{t, f\}}(f, t)).
 \end{aligned}$$

- For $(A, B, C) = (f, f, f)$, we have

$$\begin{aligned}
 \mathrm{Hom}_{\{t, f\}}(f \times f, f) &\cong \mathrm{Hom}_{\{t, f\}}(f, f) \\
 &= \{\mathrm{id}_{\mathrm{false}}\} \\
 &\cong \mathrm{Hom}_{\{t, f\}}(f, f) \\
 &\cong \mathrm{Hom}_{\{t, f\}}(f, \mathbf{Hom}_{\{t, f\}}(f, f)).
 \end{aligned}$$

The proof of naturality is omitted. □

1.3 0-Categories

Definition 1.3.1.1. A 0-**category** is a poset.⁹

Definition 1.3.1.2. A 0-**groupoid** is a 0-category in which every morphism is invertible.¹⁰

1.4 Tables of Analogies Between Set Theory and Category Theory

Here we record some analogies between notions in set theory and category theory. Note that the analogies relating to presheaves relate equally well to copresheaves, as the opposite X^{op} of a set X is just X again.

Basics:

⁹*Motivation:* A 0-category is precisely a category enriched in the poset of (-1) -categories.

¹⁰That is, a *set*.

SET THEORY	CATEGORY THEORY
Enrichment in $\{\text{true}, \text{false}\}$	Enrichment in Sets
Set X	Category C
Element $x \in X$	Object $X \in \text{Obj}(C)$
Function	Functor
Function $X \rightarrow \{\text{true}, \text{false}\}$	Functor $C \rightarrow \text{Sets}$
Function $X \rightarrow \{\text{true}, \text{false}\}$	Presheaf $C^{\text{op}} \rightarrow \text{Sets}$

Powersets and categories of presheaves:

SET THEORY	CATEGORY THEORY
Powerset $\mathcal{P}(X)$	Presheaf category $\text{PSh}(C)$
Characteristic function $\chi_{\{x\}}$	Representable presheaf h_X
Characteristic embedding $\chi_{(-)} : X \hookrightarrow \mathcal{P}(X)$	Yoneda embedding $\mathcal{Y} : C^{\text{op}} \hookrightarrow \text{PSh}(C)$
Characteristic relation $\chi_X(-_1, -_2)$	Hom profunctor $\text{Hom}_C(-_1, -_2)$
The Yoneda lemma for sets $\text{Hom}_{\mathcal{P}(X)}(\chi_x, \chi_U) = \chi_U(x)$	The Yoneda lemma for categories $\text{Nat}(h_X, \mathcal{F}) \cong \mathcal{F}(X)$
The characteristic embedding is fully faithful, $\text{Hom}_{\mathcal{P}(X)}(\chi_x, \chi_y) = \chi_X(x, y)$	The Yoneda embedding is fully faithful, $\text{Nat}(h_X, h_Y) \cong \text{Hom}_C(X, Y)$
Subsets are unions of their elements $U = \bigcup_{x \in U} \{x\}$ or $\chi_U = \text{colim}_{\chi_x \in \text{Sets}(U, \{\text{t}, \text{f}\})} (\chi_x)$	Presheaves are colimits of representables, $\mathcal{F} \cong \text{colim}_{h_X \in \int_C \mathcal{F}} (h_X)$

Categories of elements:

SET THEORY	CATEGORY THEORY
Assignment $U \mapsto \chi_U$	Assignment $\mathcal{F} \mapsto \int_C \mathcal{F}$ (the category of elements)
Assignment $U \mapsto \chi_U$ giving an isomorphism $\mathcal{P}(X) \cong \text{Sets}(X, \{\text{t}, \text{f}\})$	Assignment $\mathcal{F} \mapsto \int_C \mathcal{F}$ giving an equivalence $\text{PSh}(C) \cong_{\text{eq}} \text{DFib}(C)$

Functions between powersets and functors between presheaf categories:

SET THEORY	CATEGORY THEORY
Direct image function $f_*: \mathcal{P}(X) \rightarrow \mathcal{P}(Y)$	Inverse image functor $f^{-1}: \text{PSh}(C) \rightarrow \text{PSh}(\mathcal{D})$
Inverse image function $f^{-1}: \mathcal{P}(Y) \rightarrow \mathcal{P}(X)$	Direct image functor $f_*: \text{PSh}(\mathcal{D}) \rightarrow \text{PSh}(C)$
Direct image with compact support function $f_!: \mathcal{P}(X) \rightarrow \mathcal{P}(Y)$	Direct image with compact support functor $f_!: \text{PSh}(C) \rightarrow \text{PSh}(\mathcal{D})$

Relations and profunctors:

SET THEORY	CATEGORY THEORY
Relation $R: X \times Y \rightarrow \{t, f\}$	Profunctor $\mathfrak{p}: \mathcal{D}^{\text{op}} \times C \rightarrow \text{Sets}$
Relation $R: X \rightarrow \mathcal{P}(Y)$	Profunctor $\mathfrak{p}: C \rightarrow \text{PSh}(\mathcal{D})$
Relation as a cocontinuous morphism of posets $R: (\mathcal{P}(X), \subset) \rightarrow (\mathcal{P}(Y), \subset)$	Profunctor as a colimit-preserving functor $\mathfrak{p}: \text{PSh}(C) \rightarrow \text{PSh}(\mathcal{D})$

Appendices

A Other Chapters

Sets

1. **Sets**
2. **Constructions With Sets**
3. **Pointed Sets**
4. **Tensor Products of Pointed Sets**
5. **Relations**
6. **Spans**
7. **Posets**

Indexed and Fibred Sets

7. Indexed Sets
8. Fibred Sets
9. Un/Straightening for Indexed and Fibred Sets

Category Theory

11. **Categories**
12. Types of Morphisms in Categories
13. Adjunctions and the Yoneda Lemma

14. **Constructions With Categories**15. **Kan Extensions****Bicategories**17. **Bicategories**18. **Internal Adjunctions****Internal Category Theory**19. **Internal Categories****Cyclic Stuff**20. **The Cycle Category****Cubical Stuff**21. **The Cube Category****Globular Stuff**22. **The Globe Category****Cellular Stuff**23. **The Cell Category****Monoids**24. **Monoids**25. **Constructions With Monoids****Monoids With Zero**26. **Monoids With Zero**27. **Constructions With Monoids With Zero****Groups**28. **Groups**29. **Constructions With Groups****Hyper Algebra**30. **Hypermonoids**31. **Hypergroups**32. **Hypersemirings and Hyperrings**33. **Quantales****Near-Rings**34. **Near-Semirings**35. **Near-Rings****Real Analysis**36. **Real Analysis in One Variable**37. **Real Analysis in Several Variables****Measure Theory**38. **Measurable Spaces**39. **Measures and Integration****Probability Theory**39. **Probability Theory****Stochastic Analysis**40. **Stochastic Processes, Martingales, and Brownian Motion**41. **Itô Calculus**42. **Stochastic Differential Equations****Differential Geometry**43. **Topological and Smooth Manifolds****Schemes**44. **Schemes**