

# Bicategories

December 24, 2023

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Create tags and TODO:

1. spans in bicategories: add Proposition 7 here: <https://arxiv.org/abs/1903.03890>
2. add fact: internal adjunctions in  $\text{PseudoFun}(C, \mathcal{D})$  are precisely the invertible strong transformations as in [JY21, Example 6.2.7]. What are the internal adjunctions?

## 1 Monomorphisms in Bicategories

### 1.1 Faithful Monomorphisms

Let  $C$  be a bicategory.

**Definition 1.1.1.1.** A 1-morphism  $f: A \rightarrow B$  is a **faithful monomorphism** in  $C$  if the following equivalent conditions are satisfied:

1. For each  $X \in \text{Obj}(C)$ , the functor 00ZP

$$f_*: \text{Hom}_C(X, A) \rightarrow \text{Hom}_C(X, B)$$

given by postcomposition by  $f$  is faithful.

2. Given a diagram in  $C$  of the form

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$$X \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \beta \\ \xrightarrow{\psi} \end{array} A \xrightarrow{f} B,$$

if we have  $\text{id}_f \circ \alpha = \text{id}_f \circ \beta$ , then  $\alpha = \beta$ .

**Example 1.1.1.2.** Here are some examples of faithful monomorphisms.

1. Full Monomorphisms in  $\mathbf{Cats}_2$ .
2. Full Monomorphisms in  $\mathbf{Rel}$ .
3. Full Monomorphisms in  $\mathbf{Span}$ .

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## 1.2 Full Monomorphisms

Let  $C$  be a bicategory.

**Definition 1.2.1.1.** A 1-morphism  $f: A \rightarrow B$  is a **full monomorphism** in  $C$  if the following equivalent conditions are satisfied:

1. For each  $X \in \text{Obj}(C)$ , the functor

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$$f_*: \text{Hom}_C(X, A) \rightarrow \text{Hom}_C(X, B)$$

given by postcomposition by  $f$  is full.

2. For each  $X \in \text{Obj}(C)$  and each 2-morphism

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$$\gamma: f \circ \phi \Rightarrow f \circ \psi, \quad X \begin{array}{c} \xrightarrow{f \circ \phi} \\ \gamma \Downarrow \\ \xrightarrow{f \circ \psi} \end{array} B$$

of  $C$ , there exists a 2-morphism  $\alpha: \phi \Rightarrow \psi$  of  $C$  such that we have an equality

$$X \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} A \xrightarrow{f} B = X \begin{array}{c} \xrightarrow{f \circ \phi} \\ \gamma \Downarrow \\ \xrightarrow{f \circ \psi} \end{array} B$$

of pasting diagrams in  $C$ , i.e. such that we have

$$\gamma = \text{id}_f \circ \alpha.$$

**Example 1.2.1.2.** Here are some examples of full monomorphisms.

1. Full Monomorphisms in  $\mathbf{Cats}_2$ .
2. Full Monomorphisms in  $\mathbf{Rel}$ .
3. Full Monomorphisms in  $\mathbf{Span}$ .

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### 1.3 Fully Faithful Monomorphisms

Let  $C$  be a bicategory.

**Definition 1.3.1.1.** A 1-morphism  $f: A \rightarrow B$  is a **fully faithful monomorphism in  $C$**  if the following equivalent conditions are satisfied:

1. The 1-morphism  $f$  is fully and faithful. 0105
2. For each  $X \in \text{Obj}(C)$ , the functor 0106

$$f_*: \text{Hom}_C(X, A) \rightarrow \text{Hom}_C(X, B)$$

given by postcomposition by  $f$  is fully faithful.

3. The conditions in ?? of ?? and ?? of ?? hold. 0107

**Example 1.3.1.2.** Here are some examples of fully faithful monomorphisms. 0108

1. *Fully Faithful Monomorphisms in  $\text{Cats}_2$ .* 0109
2. *Fully Faithful Monomorphisms in  $\mathbf{Rel}$ .* 010A
3. *Fully Faithful Monomorphisms in  $\text{Span}$ .* 010B

### 1.4 Strict Monomorphisms

Let  $C$  be a bicategory.

**Definition 1.4.1.1.** A 1-morphism  $f: A \rightarrow B$  is a **strict monomorphism in  $C$**  if the following equivalent conditions are satisfied:

1. For each  $X \in \text{Obj}(C)$ , the action on objects 010E

$$f_*: \text{Obj}(\text{Hom}_C(X, A)) \rightarrow \text{Obj}(\text{Hom}_C(X, B))$$

of the functor

$$f_*: \text{Hom}_C(X, A) \rightarrow \text{Hom}_C(X, B)$$

given by postcomposition by  $f$  is injective.

2. For each diagram in  $C$  of the form 010F

$$X \begin{array}{c} \xrightarrow{\phi} \\ \xrightarrow{\psi} \end{array} A \xrightarrow{f} B,$$

if  $f \circ \phi = f \circ \psi$ , then  $\phi = \psi$ .

**Example 1.4.1.2.** Here are some examples of strict monomorphisms. 010G

1. *Strict Monomorphisms in*  $\mathbf{Cats}_2$ . 010H
2. *Strict Monomorphisms in*  $\mathbf{Rel}$ . 010J
3. *Strict Monomorphisms in*  $\mathbf{Span}$ . 010K

## 2 Epimorphisms in Bicategories 010L

### 2.1 Faithful Epimorphisms 010M

Let  $C$  be a bicategory.

**Definition 2.1.1.1.** A 1-morphism  $f: A \rightarrow B$  is a **faithful epimorphism in**  $C$  if the following equivalent conditions are satisfied:

1. For each  $X \in \text{Obj}(C)$ , the functor 010P

$$f^*: \text{Hom}_C(B, X) \rightarrow \text{Hom}_C(A, X)$$

given by precomposition by  $f$  is faithful.

2. Given a diagram in  $C$  of the form 010Q

$$A \xrightarrow{f} B \begin{array}{c} \xrightarrow{\phi} \\ \alpha \downarrow \Downarrow \beta \\ \xrightarrow{\psi} \end{array} X,$$

if we have  $\alpha \circ \text{id}_f = \beta \circ \text{id}_f$ , then  $\alpha = \beta$ .

**Example 2.1.1.2.** Here are some examples of faithful epimorphisms. 010R

1. *Full Epimorphisms in*  $\mathbf{Cats}_2$ . 010S
2. *Full Epimorphisms in*  $\mathbf{Rel}$ . 010T
3. *Full Epimorphisms in*  $\mathbf{Span}$ . 010U

### 2.2 Full Epimorphisms 010V

Let  $C$  be a bicategory.

**Definition 2.2.1.1.** A 1-morphism  $f: A \rightarrow B$  is a **full epimorphism in**  $C$  if the following equivalent conditions are satisfied:

1. For each  $X \in \text{Obj}(C)$ , the functor 010X

$$f^*: \text{Hom}_C(B, X) \rightarrow \text{Hom}_C(A, X)$$

given by precomposition by  $f$  is full.

2. For each  $X \in \text{Obj}(C)$  and each 2-morphism 011F

$$\gamma: \phi \circ f \Rightarrow \psi \circ f, \quad X \begin{array}{c} \xrightarrow{\phi \circ f} \\ \gamma \Downarrow \\ \xrightarrow{\psi \circ f} \end{array} B$$

of  $C$ , there exists a 2-morphism  $\alpha: \phi \Rightarrow \psi$  of  $C$  such that we have an equality

$$A \xrightarrow{f} B \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} X = A \begin{array}{c} \xrightarrow{\phi \circ f} \\ \gamma \Downarrow \\ \xrightarrow{\psi \circ f} \end{array} X$$

of pasting diagrams in  $C$ , i.e. such that we have

$$\gamma = \alpha \circ \text{id}_f.$$

**Example 2.2.1.2.** Here are some examples of full epimorphisms. 010Z

1. *Full Epimorphisms in  $\text{Cats}_2$ .* 0110
2. *Full Epimorphisms in **Rel**.* 0111
3. *Full Epimorphisms in  $\text{Span}$ .* 0112

## 2.3 Fully Faithful Epimorphisms 0113

Let  $C$  be a bicategory.

**Definition 2.3.1.1.** A 1-morphism  $f: A \rightarrow B$  is a **fully faithful epimorphism in  $C$**  if the following equivalent conditions are satisfied:

1. The 1-morphism  $f$  is fully and faithful. 0115
2. For each  $X \in \text{Obj}(C)$ , the functor 0116

$$f^*: \text{Hom}_C(B, X) \rightarrow \text{Hom}_C(A, X)$$

given by precomposition by  $f$  is fully faithful.

3. The conditions in ?? of ?? and ?? of ?? hold. 0117

**Example 2.3.1.2.** Here are some examples of fully faithful epimorphisms. 0118

1. *Fully Faithful Epimorphisms in  $\mathbf{Cats}_2$ .* 0119
2. *Fully Faithful Epimorphisms in  $\mathbf{Rel}$ .* 011A
3. *Fully Faithful Epimorphisms in  $\mathbf{Span}$ .* 011B

## 2.4 Strict Epimorphisms 011C

Let  $C$  be a bicategory.

**Definition 2.4.1.1.** A 1-morphism  $f: A \rightarrow B$  is a **strict epimorphism in  $C$**  if the following equivalent conditions are satisfied: 011D

1. For each  $X \in \text{Obj}(C)$ , the action on objects 011E

$$f^*: \text{Obj}(\text{Hom}_C(B, X)) \rightarrow \text{Obj}(\text{Hom}_C(A, X))$$

of the functor

$$f^*: \text{Hom}_C(B, X) \rightarrow \text{Hom}_C(A, X)$$

given by precomposition by  $f$  is injective.

2. For each diagram in  $C$  of the form 011F

$$A \xrightarrow{f} B \begin{array}{c} \xrightarrow{\phi} \\ \xrightarrow{\psi} \end{array} X,$$

if  $\phi \circ f = \psi \circ f$ , then  $\phi = \psi$ .

**Example 2.4.1.2.** Here are some examples of strict epimorphisms. 011G

1. *Strict Epimorphisms in  $\mathbf{Cats}_2$ .* 011H
2. *Strict Epimorphisms in  $\mathbf{Rel}$ .* 011J
3. *Strict Epimorphisms in  $\mathbf{Span}$ .* 011K

## 3 bicategories of spans

**Proposition 3.0.1.1.** Let  $A$  and  $B$  be objects of  $\mathbf{C}$ . 011L

1. *As a Pullback.* We have an isomorphism of categories

$$\text{Span}_C(A, B) \cong C_{/A} \times_C C_{/B},$$

$$\begin{array}{ccc} \text{Span}(A, B) & \rightarrow & C_{/B} \\ \downarrow & \lrcorner & \downarrow \text{忘} \\ C_{/A} & \xrightarrow{\text{忘}} & C. \end{array}$$

*Proof. ??, As a Pullback:* In detail, the pullback  $C_{/A} \times_C C_{/B}$  is the category where

- *Objects.* The objects of  $C_{/A} \times_C C_{/B}$  consist of pairs  $((S, f), (S', g))$  of objects of  $C$  consisting of
  - A pair  $(S, f)$  in  $\text{Obj}(C_{/A})$  consisting of an object  $S$  of  $C$  and a morphism  $f: S \rightarrow A$  of  $C$ ;
  - A pair  $(S', g)$  in  $\text{Obj}(C_{/B})$  consisting of an object  $S'$  of  $C$  and a morphism  $g: S' \rightarrow B$  of  $C$ ;

such that

$$\underbrace{\text{忘}(S, f)}_{\text{def}_{\equiv_S}} = \underbrace{\text{忘}(S', g)}_{\text{def}_{\equiv_{S'}}}.$$

Thus the objects of  $C_{/A} \times_C C_{/B}$  are the same as spans in  $C$  from  $A$  to  $B$ .

- *Morphisms.* A morphism of  $C_{/A} \times_C C_{/B}$  from  $(S, f, g)$  to  $(S', f', g')$  consists of a pair of morphisms

$$\begin{aligned} \phi: S &\rightarrow S' \\ \psi: S &\rightarrow S' \end{aligned}$$

such that the diagrams

$$\begin{array}{ccc} S & \xrightarrow{\phi} & S' \\ & \searrow f & \swarrow f' \\ & A & \end{array} \quad \begin{array}{ccc} S & \xrightarrow{\psi} & S' \\ & \searrow g & \swarrow g' \\ & B & \end{array}$$

such that

$$\underbrace{\text{忘}(\phi)}_{\text{def}_{\equiv_\phi}} = \underbrace{\text{忘}(\psi)}_{\text{def}_{\equiv_\psi}}.$$

Thus the morphisms of  $C_{/A} \times_C C_{/B}$  are also the same as morphisms of spans in  $C$  from  $(S, f, g)$  to  $(S', f', g')$ .

- *Identities and Composition.* The identities and composition of  $C_{/A} \times_C C_{/B}$  are also the same as those in  $\text{Span}_C(A, B)$ .

This finishes the proof.  $\square$

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