

Internal Adjunctions

December 3, 2023

00SP Create tags:

1. <https://www.google.com/search?q=mate+of+an+adjunction>
2. Moreover, by uniqueness of adjoints ([Internal Adjunctions](#), [Item 2 of Proposition 1.2.1.4](#)), this implies also that $S = f^{-1}$.
3. define bicategory $\text{Adj}(C)$
4. walking monad
5. proposition: 2-functors preserve unitors and associators
6. <https://ncatlab.org/nlab/show/2-category+of+adjunctions>. Is there a 3-category too?
7. <https://ncatlab.org/nlab/show/free+monad>
8. <https://ncatlab.org/nlab/show/CatAdj>
9. <https://ncatlab.org/nlab/show/Adj>
10. $\text{Adj}(\text{Adj}(C))$

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00SQ 1 Internal Adjunctions

00SR 1.1 The Walking Adjunction

00SS **Definition 1.1.1.1.** The **walking adjunction** is the bicategory Adj freely generated by¹

- *Objects.* A pair of objects A and B ;
- *Morphisms.* A pair of morphisms

$$L: A \rightarrow B,$$

$$R: B \rightarrow A;$$

- *2-Morphisms.* A pair of 2-morphisms

$$\eta: \text{id}_A \rightarrow L \circ R,$$

$$\epsilon: R \circ L \rightarrow \text{id}_B;$$

¹See [SS86] for an explicit description of the 2-category (as opposed to a bicategory) version of Adj in terms of finite ordinals, similar to the description of the 2-category version of the walking monad (??) as a subcategory of Δ .

subject to the equalities

$$\begin{array}{ccc}
 \begin{array}{c} B \xrightarrow{id_B} B \\ \swarrow L \quad \searrow R \\ A \xrightarrow{id_A} A \end{array} & \begin{array}{c} \uparrow \eta \\ \parallel \\ \downarrow \epsilon \end{array} & \begin{array}{c} B \xrightarrow{id_B} B \\ \swarrow L \quad \searrow R \\ A \xrightarrow{id_A} A \end{array} \\
 & = & \\
 \begin{array}{c} B \xrightarrow{id_B} B \\ \swarrow L \quad \searrow R \\ A \xrightarrow{id_A} A \end{array} & \begin{array}{c} \uparrow \eta \\ \parallel \\ \downarrow \epsilon \end{array} & \begin{array}{c} B \xrightarrow{id_B} B \\ \swarrow L \quad \searrow R \\ A \xrightarrow{id_A} A \end{array} \\
 & = & \\
 \begin{array}{c} A \xrightarrow{id_A} A \\ \swarrow R \quad \searrow L \\ B \xrightarrow{id_B} B \end{array} & \begin{array}{c} \uparrow \eta \\ \parallel \\ \downarrow \epsilon \end{array} & \begin{array}{c} A \xrightarrow{id_A} A \\ \swarrow R \quad \searrow L \\ B \xrightarrow{id_B} B \end{array} \\
 & = & \\
 \begin{array}{c} A \xrightarrow{id_A} A \\ \swarrow R \quad \searrow L \\ B \xrightarrow{id_B} B \end{array} & \begin{array}{c} \uparrow \eta \\ \parallel \\ \downarrow \epsilon \end{array} & \begin{array}{c} A \xrightarrow{id_A} A \\ \swarrow R \quad \searrow L \\ B \xrightarrow{id_B} B \end{array}
 \end{array}$$

of pasting diagrams, which are equivalent to the following conditions:

1. *The Left Triangle Identity.* The diagram

$$\begin{array}{ccc}
 L \circ id_A & \xrightarrow{id_L \circ \eta} & L \circ (R \circ L) \xrightarrow{(\alpha_{L,R,L}^{Adj})^{-1}} (L \circ R) \circ L \\
 & \searrow \rho_L^{Adj} & \downarrow \epsilon \circ id_L \\
 & & id_B \circ L \\
 & & \downarrow \lambda_L^{Adj} \\
 & & L
 \end{array}$$

commutes.

2. *The Right Triangle Identity.* The diagram

$$\begin{array}{ccc}
 id_A \circ R & \xrightarrow{\eta \circ id_R} & (R \circ L) \circ R \xrightarrow{\alpha_{R,L,R}^{Adj}} R \circ (L \circ R) \\
 & \searrow \lambda_R^{Adj} & \downarrow id_R \circ \epsilon \\
 & & R \circ id_B \\
 & & \downarrow \rho_R^{Adj} \\
 & & R.
 \end{array}$$

00ST 1.2 Internal Adjunctions

Let C be a bicategory.

00SU Definition 1.2.1.1. An **internal adjunction in $C^{2,3}$** is a 2-functor $\text{Adj} \rightarrow C$.

00SV Remark 1.2.1.2. In detail, an **internal adjunction in C** consists of

- *Objects.* A pair of objects A and B of C ;
- *Morphisms.* A pair of morphisms

$$\begin{aligned} L &: A \rightarrow B, \\ R &: B \rightarrow A \end{aligned}$$

of C ;

- *2-Morphisms.* A pair of 2-morphisms

$$\begin{aligned} \eta &: \text{id}_A \rightarrow L \circ R, \\ \epsilon &: R \circ L \rightarrow \text{id}_B \end{aligned}$$

of C ;

subject to the equalities

The image shows two commutative diagrams representing the triangle identities for an internal adjunction. The top diagram shows the identity $\text{id}_A = L \circ R \circ L$, and the bottom diagram shows the identity $\text{id}_B = R \circ L \circ R$. Each diagram is an equality between two triangular compositions of morphisms L , R , id_A , and id_B .

²Further Terminology: Also called an **adjunction internal to C** .

³Further Terminology: In this situation, we also call (g, f) an **adjoint pair**, f the **left adjoint** of the pair, g the **right adjoint** of the pair, η the **unit** of the adjunction, and ϵ the **counit** of the adjunction.

of pasting diagrams in C , which are equivalent to the following conditions:⁴

1. *The Left Triangle Identity.* The diagram

$$\begin{array}{ccccc}
 L \circ \text{id}_A & \xrightarrow{\text{id}_L \circ \eta} & L \circ (R \circ L) & \xrightarrow{(\alpha_{L,R,L}^C)^{-1}} & (L \circ R) \circ L \\
 & \searrow \rho_L^C & & & \downarrow \epsilon \circ \text{id}_L \\
 & & & & \text{id}_B \circ L \\
 & & & & \downarrow \lambda_L^C \\
 & & & & L
 \end{array}$$

commutes.

2. *The Right Triangle Identity.* The diagram

$$\begin{array}{ccccc}
 \text{id}_A \circ R & \xrightarrow{\eta \circ \text{id}_R} & (R \circ L) \circ R & \xrightarrow{\alpha_{R,L,R}^C} & R \circ (L \circ R) \\
 & \searrow \lambda_R^C & & & \downarrow \text{id}_R \circ \epsilon \\
 & & & & R \circ \text{id}_B \\
 & & & & \downarrow \rho_R^C \\
 & & & & R.
 \end{array}$$

Example 1.2.1.3. Here are some examples of internal adjunctions.

1. *Internal Adjunctions in Cats_2 .* The internal adjunctions in the 2-category Cats_2 of categories, functors, and natural transformations are precisely the adjunctions of **Categories, ??**.

⁴When C is a 2-category, these diagrams take the following form:

$$\begin{array}{ccc}
 L & \xrightarrow{\text{id}_L \circ \eta} & L \circ R \circ L \\
 \searrow \text{id}_L & & \downarrow \epsilon \circ \text{id}_L \\
 & & L
 \end{array}
 \qquad
 \begin{array}{ccc}
 R & \xrightarrow{\text{id}_R \circ \eta} & R \circ L \circ R \\
 \searrow \text{id}_L & & \downarrow \epsilon \circ \text{id}_R \\
 & & R.
 \end{array}$$

00SY 2. *Internal Adjunctions in Rel.* The internal adjunctions in **Rel** are precisely the relations of the form $\text{Gr}(f) \dashv f^{-1}$ with f a function; see **Relations, Item 4** of **Proposition 2.5.1.1**.

00SZ 3. *Internal Adjunctions in Span.* The internal adjunctions in **Span** are precisely the spans of the form

$$\begin{array}{ccc} & S & \\ \phi \swarrow & & \searrow g \\ A & & B \end{array}$$

with ϕ an isomorphism; see **Spans, Item 4** of **Proposition 2.5.1.1**.

00T0 **Proposition 1.2.1.4.** Let C be a bicategory.

00T1 1. *Duality.* Let (f, g, η, ϵ) be an internal adjunction in C .

- (a) The quadruple (g, f, η, ϵ) is an internal adjunction in C^{op} .
- (b) The quadruple (g, f, ϵ, η) is an internal adjunction in C^{co} .
- (c) The quadruple (f, g, η, ϵ) is an internal adjunction in C^{coop} .

00T2 2. *Uniqueness of Adjoints.* Let (f, g, η, ϵ) and $(f, g', \eta', \epsilon')$ be internal adjunctions in C . We have a canonical isomorphism⁵

$$g \xrightarrow{(\lambda_g^C)^{-1}} \text{id}_A \circ g \xrightarrow{\eta' \circ \text{id}_g} (g' \circ f) \circ g \xrightarrow{\alpha_{g', f, g}^C} g' \circ (f \circ g) \xrightarrow{\text{id}_{g'} \circ \epsilon} g' \circ \text{id}_B \xrightarrow{(\rho_{g'}^C)^{-1}} g'$$

with inverse


$$g' \xrightarrow{(\lambda_{g'}^C)^{-1}} \text{id}_B \circ g' \xrightarrow{\eta \circ \text{id}_{g'}} (g \circ f) \circ g' \xrightarrow{\alpha_{g, f, g'}^C} g \circ (f \circ g') \xrightarrow{\text{id}_g \circ \epsilon'} g \circ \text{id}_A \xrightarrow{(\lambda_g^C)^{-1}} g.$$

00T3 3. *Carrying Internal Adjunctions Through Pseudofunctors.* Let $F: C \rightarrow D$ be a pseudofunctor and (f, g, η, ϵ) be an internal adjunction in C . There is an induced internal adjunction⁶

$$(F(f), F(g), \bar{\eta}, \bar{\epsilon})$$

in D , where:

⁵*Slogan:* Left adjoints are unique up to canonical isomorphism. Dually, so are right adjoints.

⁶ *Warning:* Lax or oplax functors which are not pseudofunctors need not preserve internal adjunc-

(a) The unit

$$\bar{\eta}: \text{id}_{F(A)} \Longrightarrow F(g) \circ F(f)$$

is the composition

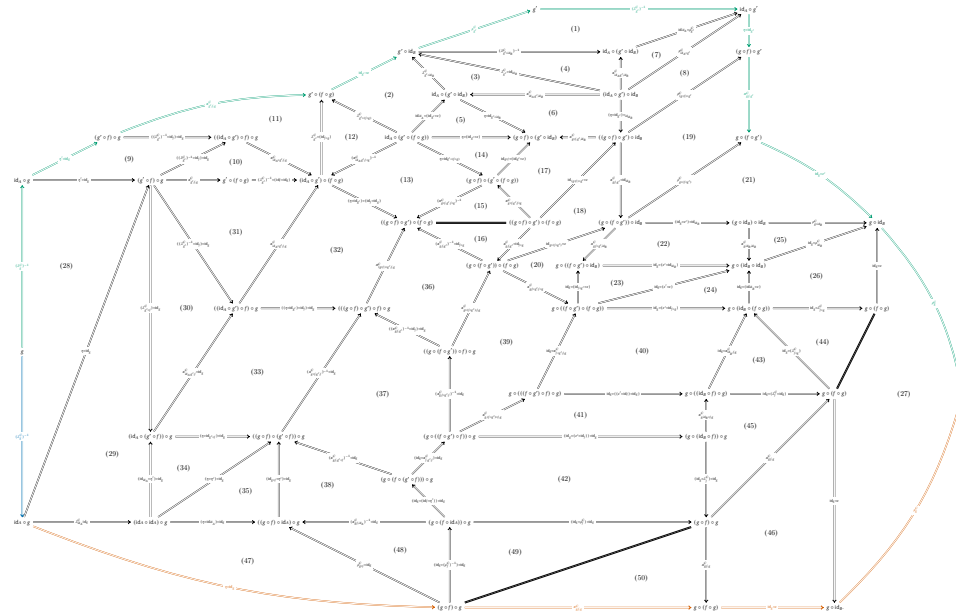
$$\text{id}_{F(A)} \xrightarrow{F_A} F(\text{id}_A) \xrightarrow{F(\eta)} F(g \circ f) \xrightarrow{F_{g,f}^{-1}} F(g) \circ F(f).$$

(b) The counit

$$\bar{\epsilon}: F(f) \circ F(g) \Longrightarrow \text{id}_{F(B)}$$

is the composition

$$F(f) \circ F(g) \xrightarrow{F_{f,g}} F(f \circ g) \xrightarrow{F(\epsilon)} F(\text{id}_B) \xrightarrow{F_B} \text{id}_{F(B)}.$$

Proof. **Item 1, Duality:** Omitted.⁷**Item 2, Uniqueness of Adjoints:** ⁸Consider the diagram (if you *really* want to consider it I fear you will need to zoom in)

In this diagram:

tions.

⁷Reference: [Y21, Exercise 6.6.2].⁸Reference: [Y21, Lemma 6.1.6].

1. The morphisms in **green** are the composition $g \xRightarrow{\cong} g' \xRightarrow{\cong} g$;
2. The morphisms in **red** are equal to λ_g^C by the right triangle identity for (f, g, η, ϵ) . Hence the composition of the morphism in **blue** with the morphisms in **red** is the identity;
3. Subdiagrams (1), (2), (10), (11), (29), (31), and (43) commute by the naturality of the left unitor of C and its inverse;
4. Subdiagrams (8), (19), and (21) commute by the naturality of the right unitor of C and its inverse;
5. Subdiagrams (6), (13), (17), (18), (20), (22), (32), (33), (36), (38), (40), (41), and (45) commute by the naturality of the associator of C and its inverse;
6. Subdiagrams (37), (39), and (42) commute by the pentagon identity for C ;
7. Subdiagrams (3), (4), (7), (12), (25), (30), and (48) commute by **Bicategories**, ?? of ??;
8. Subdiagrams (5), (14), (23), (24), (34), and (35) commute by middle-four exchange;
9. Subdiagrams (9), (15), (16), (27), (28), (44), (46), (49), and (50) commute trivially;
10. Subdiagram (26) commutes by **Bicategories**, ??? of ??;
11. Subdiagram (47) commutes by **Bicategories**, ?? of ?? and the naturality of the left unitor of right unitor of C .

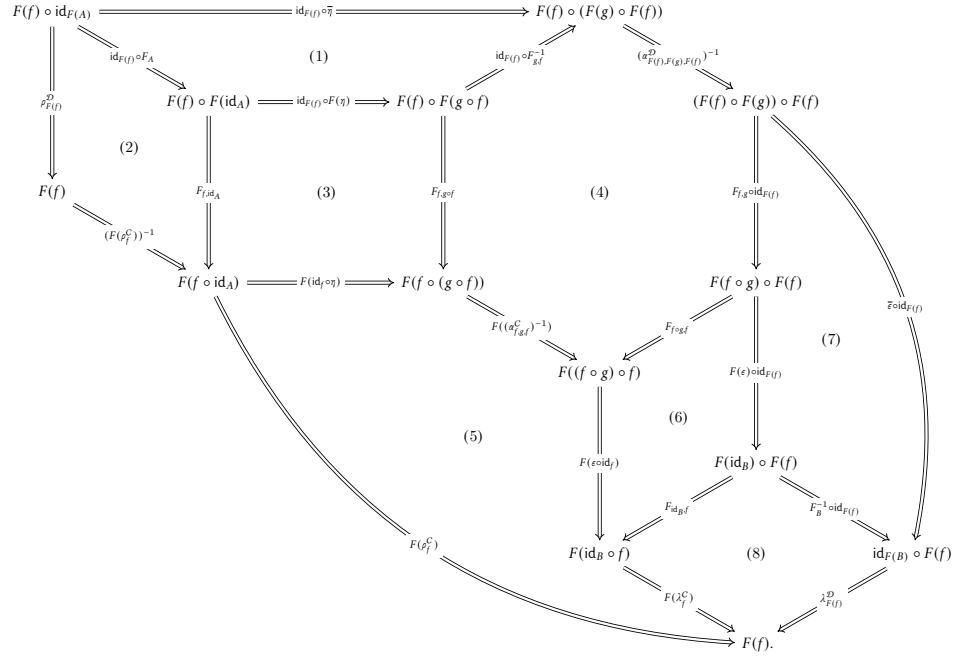
Hence $g \cong g'$.

Item 3, Carrying Internal Adjunctions Through Pseudofunctors:⁹ We claim that the left and right triangle identities for $(F(f), F(g), \bar{\eta}, \bar{\epsilon})$ hold:

1. The left triangle identity for $(F(f), F(g), \bar{\eta}, \bar{\epsilon})$ is the condition that the boundary

⁹Reference: [Y21, Proposition 6.1.7].

diagram of the diagram (you may need to zoom in)



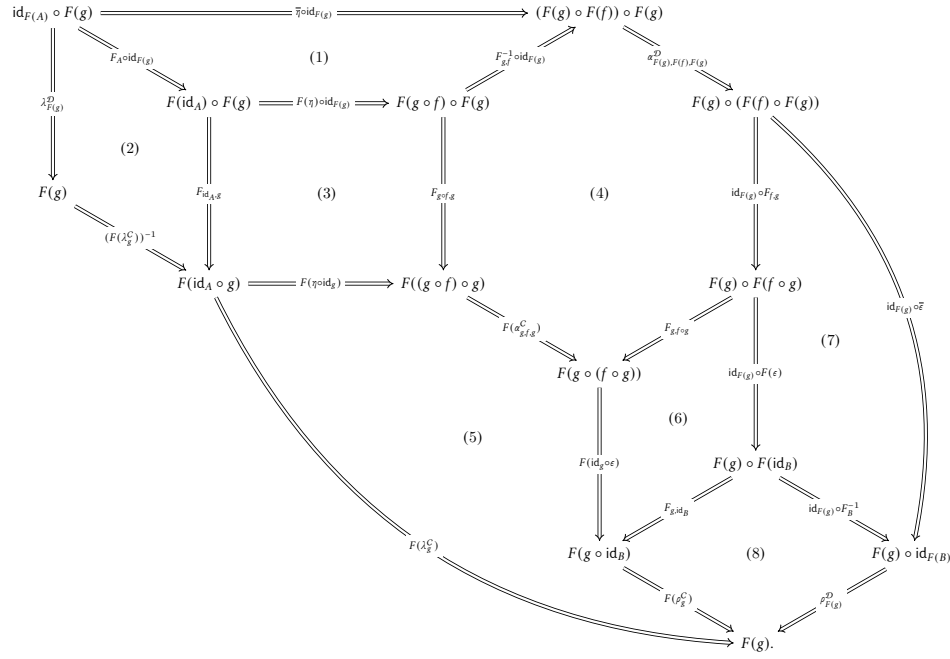
commutes. Since

- Subdiagrams (1) and (7) commute by applying middle-four exchange twice,
- Subdiagrams (2) and (8) commute by the left and right lax unity conditions for F ,
- Subdiagrams (3) and (6) commute by the naturality of the lax functoriality constraints of F ,
- Subdiagram (4) commutes by the lax associativity condition for F , and
- Subdiagram (5) commutes by the left triangle identity for (f, g, η, ϵ) ,

so does the boundary diagram.

- The right triangle identity for $(F(f), F(g), \bar{\eta}, \bar{\epsilon})$ is the condition that the boundary

diagram of the diagram (you may need to zoom in)



commutes. Since

- (a) Subdiagrams (1) and (7) commute by applying middle-four exchange twice,
- (b) Subdiagrams (2) and (8) commute by the left and right lax unity conditions for F ,
- (c) Subdiagrams (3) and (6) commute by the naturality of the lax functoriality constraints of F ,
- (d) Subdiagram (4) commutes by the lax associativity condition for F , and
- (e) Subdiagram (5) commutes by the right triangle identity for (f, g, η, ϵ) ,

so does the boundary diagram.

This finishes the proof. \square

00T4 1.3 Internal Adjoint Equivalences

Let C be a bicategory.

00T5 Definition 1.3.1.1. An internal adjunction (f, g, η, ϵ) in C is an **internal adjoint equivalence** if η and ϵ are isomorphisms in C .

00T6 Example 1.3.1.2. Here are some examples of internal adjoint equivalences.

- 00T7** 1. *Internal Adjoint Equivalences in \mathbf{Cats}_2 .* The internal adjoint equivalences in the 2-category \mathbf{Cats}_2 of categories, functors, and natural transformations are precisely the adjoint equivalences of **Categories**, ??.¹⁰
- 00T8** 2. *Internal Adjoint Equivalences in \mathbf{Mod} .* The internal adjoint equivalences in \mathbf{Mod} are precisely the invertible R -modules; see ??.¹¹
- 00T9** 3. *Internal Adjoint Equivalences in $\mathbf{PseudoFun}(C, \mathcal{D})$.* The internal adjoint equivalences in $\mathbf{PseudoFun}(C, \mathcal{D})$ are precisely the invertible strong transformations; see ??.¹²
- 00TA** 4. *Internal Adjoint Equivalences in \mathbf{Rel} .* The internal adjoint equivalences in \mathbf{Rel} are precisely the relations of the form $\text{Gr}(f) \dashv f^{-1}$ with f an isomorphism; see ??.
- 00TB** 5. *Internal Adjoint Equivalences in \mathbf{Span} .* The internal adjoint equivalences in \mathbf{Span} are precisely the spans of the form $A \xleftarrow{\phi} S \xrightarrow{\psi} B$ with ϕ and ψ isomorphisms; see ??.

00TC Proposition 1.3.1.3. Let C be a bicategory.

- 00TD** 1. *Carrying Internal Adjoint Equivalences Through Pseudofunctors.* Let $F: C \longrightarrow \mathcal{D}$ be a pseudofunctor and (f, g, η, ϵ) be an internal adjunction in C . If (f, g, η, ϵ) is an internal adjoint equivalence in C , then the induced internal adjunction

$$(F(f), F(g), \bar{\eta}, \bar{\epsilon})$$

in \mathcal{D} of **Item 3** of **Proposition 1.2.1.4** is an internal adjoint equivalence as well.

- 00TE** 2. *Internal Adjunctions Always Refine to Internal Adjoint Equivalences.* Let (f, g, η, ϵ) be an internal adjunction in C . If f is an equivalence, then there exist 2-morphisms

$$\bar{\eta}: \text{id}_A \Longrightarrow g \circ f$$

$$\bar{\epsilon}: f \circ g \Longrightarrow \text{id}_B$$

of C such that $(f, g, \bar{\eta}, \bar{\epsilon})$ is an internal adjoint equivalence.

Proof. **Item 1**, *Carrying Internal Adjoint Equivalences Through Pseudofunctors*: See [Y21, Proposition 6.2.3].

Item 2, *Internal Adjunctions Always Refine to Internal Adjoint Equivalences*: See [Y21, Proposition 6.2.4]. □

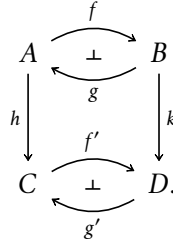
¹⁰Reference: [Y21, Examples 6.2.5].

¹¹Reference: [Y21, Examples 6.2.6].

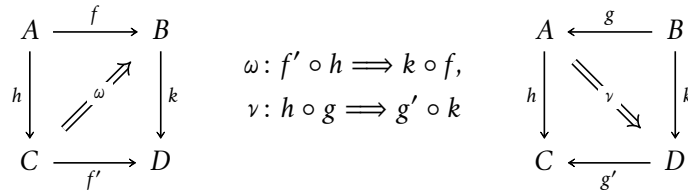
¹²Reference: [Y21, Examples 6.2.7].

00TF 1.4 Mates

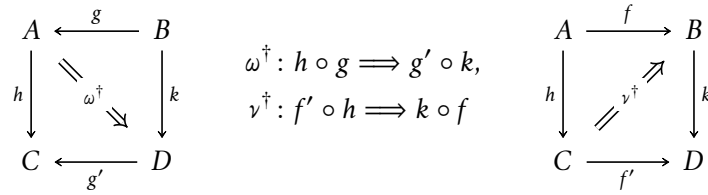
Let C be a bicategory, let (f, g, η, ϵ) and $(f', g', \eta', \epsilon')$ be adjunctions, and let h and k be morphisms of C as in the diagram



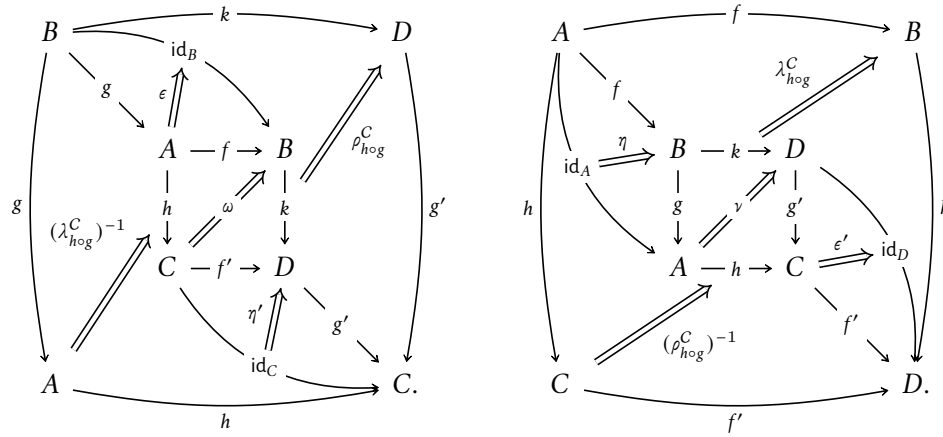
00TG **Definition 1.4.1.1.** The **mates** of a pair of 2-morphisms



are the 2-morphisms



defined as the pastings of the diagrams¹³



00TH Proposition 1.4.1.2. Let $\omega: f' \circ h \Rightarrow k \circ f$ and $\nu: h \circ g \Rightarrow g' \circ k$ be 2-morphisms.

00TJ 1. *The Mate Correspondence.* The map

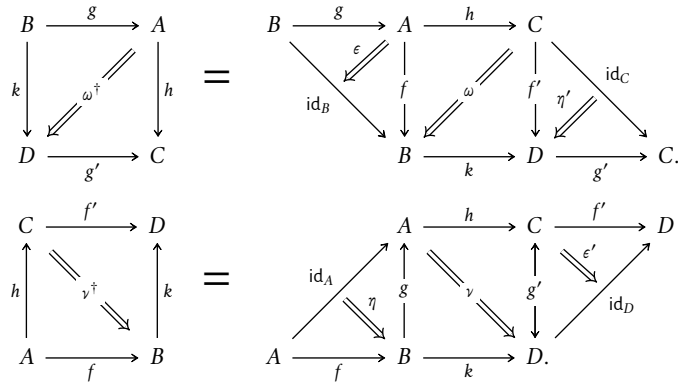
$$(-)^{\dagger}: \text{Hom}_{\text{Hom}_C(A,C)}(f' \circ h, k \circ f) \longrightarrow \text{Hom}_{\text{Hom}_C(B,D)}(h \circ g, g' \circ k)$$

$$\omega \longmapsto \omega^{\dagger}$$

is a bijection.

Proof. Item 1, The Mate Correspondence: Here we give a proof for 2-categories (which indirectly proves also the general case by [Bicategories](#), ??). A proof for general bicategories can be found in [\[Y21, Lemma 6.1.13\]](#).

¹³If C is a 2-category, these pasting diagrams become the following:



Let

$$\nu: h \circ g \Rightarrow g' \circ k$$

be a 2-morphism of C . The mate ν^\dagger of ν is then given by

and the mate of ν^\dagger is the 2-morphism $(\nu^\dagger)^\dagger: f' \circ h \Rightarrow k \circ f$ given by

Similarly, $(\omega)^\dagger = \omega$. □

00TK 2 Morphisms of Internal Adjunctions

00TL 2.1 Lax Morphisms of Internal Adjunctions

Let C be a bicategory and let $(A, B, F, G, \eta, \epsilon)$ and $(A', B', F', G', \eta', \epsilon')$ be internal adjunctions in C .

00TM Definition 2.1.1.1. A **lax morphism of internal adjunctions** from $(A, B, F, G, \eta, \epsilon)$ to $(A', B', F', G', \eta', \epsilon')$ is a lax transformation between these viewed as 2-functors from the walking adjunction.

00TN Remark 2.1.1.2. In detail, a **lax morphism of internal adjunctions** from $(A, B, F, G, \eta, \epsilon)$ to $(A', B', F', G', \eta', \epsilon')$ consists of

- *1-Morphisms.* A pair of 1-morphisms

$$\begin{aligned}\phi &: A \rightarrow A', \\ \psi &: B \rightarrow B'\end{aligned}$$

of C ;

- *2-Morphisms.* A pair of 2-morphisms

$$\begin{array}{ccc} \begin{array}{ccc} A & \xrightarrow{F} & B \\ \phi \downarrow & \alpha \swarrow \nearrow & \downarrow \psi \\ A' & \xrightarrow{F'} & B' \end{array} & \begin{array}{l} \alpha: F' \circ \phi \Rightarrow \psi \circ F, \\ \beta: G' \circ \phi \Rightarrow \psi \circ G \end{array} & \begin{array}{ccc} A & \xleftarrow{G} & B \\ \phi \downarrow & \beta \swarrow \nearrow & \downarrow \psi \\ A' & \xleftarrow{G'} & B' \end{array} \end{array}$$

of C ;

satisfying the following conditions:

1. *Compatibility With Units.* We have an equality

$$\begin{array}{ccc} \begin{array}{ccccc} & & B & & \\ & F \nearrow & \eta \Uparrow & G \searrow & \\ A & \xrightarrow{\quad} & A & \xrightarrow{\quad} & A \\ \phi \downarrow & \lambda_\phi^C \nearrow & \phi & \searrow \rho_\phi^{C,-1} & \downarrow \phi \\ A' & \xrightarrow{\quad} & A' & \xrightarrow{\quad} & A' \\ & & \text{id}_{A'} & & \end{array} & = & \begin{array}{ccccc} & & B & & \\ & F \nearrow & \psi \downarrow & G \searrow & \\ A & \xrightarrow{\quad} & A & \xrightarrow{\quad} & A \\ \phi \downarrow & \alpha \nearrow & \psi \downarrow & \beta \searrow & \downarrow \phi \\ A' & \xrightarrow{\quad} & B' & \xrightarrow{\quad} & A' \\ & & \eta' \Uparrow & & \\ & & \text{id}_{A'} & & \end{array} \end{array}$$

of pasting diagrams in C ;

2. *Compatibility With Counits.* We have an equality

The diagram shows an equality between two pasting diagrams in a bicategory C .
 Left diagram: A square with vertices B (top-left), B (top-right), A (bottom-left), and A (bottom-right). The top edge is id_B . The left edge is ψ . The right edge is ψ . The bottom edge is ϕ . There are two internal arrows: G from B to A and F from A to B . A 2-morphism α is shown as a double arrow from G to F .
 Right diagram: A square with vertices B (top-left), B (top-right), B' (bottom-left), and B' (bottom-right). The top edge is id_B . The left edge is ψ . The right edge is ψ . The bottom edge is $\text{id}_{B'}$. There are two internal arrows: G' from B to B' and F' from B' to B . A 2-morphism λ_ψ^C is shown as a double arrow from G' to F' .
 The two diagrams are separated by an equals sign.

of pasting diagrams in C .

00TP 2.2 Oplax Morphisms of Internal Adjunctions

Let C be a bicategory and let $(A, B, F, G, \eta, \epsilon)$ and $(A', B', F', G', \eta', \epsilon')$ be internal adjunctions in C .

00TQ **Definition 2.2.1.1.** An **oplax morphism of internal adjunctions** from $(A, B, F, G, \eta, \epsilon)$ to $(A', B', F', G', \eta', \epsilon')$ is an oplax transformation between these viewed as 2-functors from the walking adjunction.

00TR **Remark 2.2.1.2.** In detail, an **oplax morphism of internal adjunctions** from $(A, B, F, G, \eta, \epsilon)$ to $(A', B', F', G', \eta', \epsilon')$ consists of

- *1-Morphisms.* A pair of 1-morphisms

$$\begin{aligned}\phi &: A \rightarrow A', \\ \psi &: B \rightarrow B'\end{aligned}$$

of C ;

- *2-Morphisms.* A pair of 2-morphisms

$$\begin{array}{ccc} A & \xrightarrow{F} & B \\ \phi \downarrow & \alpha \swarrow & \downarrow \psi \\ A' & \xrightarrow{F'} & B' \end{array} \quad \begin{array}{l} \alpha: \psi \circ F \Rightarrow F' \circ \phi, \\ \beta: \psi \circ G \Rightarrow G' \circ \phi \end{array} \quad \begin{array}{ccc} A & \xleftarrow{G} & B \\ \phi \downarrow & \beta \swarrow & \downarrow \psi \\ A' & \xleftarrow{G'} & B' \end{array}$$

of C ;

satisfying the following conditions:

1. *Compatibility With Units.* We have an equality

of pasting diagrams in C ;

2. *Compatibility With Counits.* We have an equality

of pasting diagrams in C .

00TS 2.3 Strong Morphisms of Internal Adjunctions

Let C be a bicategory and let $(A, B, F, G, \eta, \epsilon)$ and $(A', B', F', G', \eta', \epsilon')$ be internal adjunctions in C .

00TT **Definition 2.3.1.1.** A **strong morphism of internal adjunctions** from $(A, B, F, G, \eta, \epsilon)$ to $(A', B', F', G', \eta', \epsilon')$ is a strong transformation between these viewed as 2-functors from the walking adjunction.

00TU **Remark 2.3.1.2.** In detail, a **strong morphism of internal adjunctions** from $(A, B, F, G, \eta, \epsilon)$ to $(A', B', F', G', \eta', \epsilon')$ is equivalently:

1. A lax morphism of internal adjunctions as in Remark 2.1.1.2 whose 2-morphisms are invertible.
2. An oplax morphism of internal adjunctions as in Remark 2.2.1.2 whose 2-morphisms are invertible.

00TV 2.4 Strict Morphisms of Internal Adjunctions

Let C be a bicategory and let $(A, B, F, G, \eta, \epsilon)$ and $(A', B', F', G', \eta', \epsilon')$ be internal adjunctions in C .

00TW **Definition 2.4.1.1.** A **strict morphism of internal adjunctions** from $(A, B, F, G, \eta, \epsilon)$ to $(A', B', F', G', \eta', \epsilon')$ is a strict transformation between these viewed as 2-functors from the walking adjunction.

00TX **Remark 2.4.1.2.** In detail, a **strict morphism of internal adjunctions** from $(A, B, F, G, \eta, \epsilon)$ to $(A', B', F', G', \eta', \epsilon')$ is equivalently:

1. A lax morphism of internal adjunctions as in [Remark 2.1.1.2](#) whose 2-morphisms are identities.
2. An oplax morphism of internal adjunctions as in [Remark 2.2.1.2](#) whose 2-morphisms are identities.

00TY 3 2-Morphisms Between Morphisms of Internal Adjunctions

00TZ 3.1 2-Morphisms Between Lax Morphisms of Internal Adjunctions

Let C be a bicategory, let $(A, B, F, G, \eta, \epsilon)$ and $(A', B', F', G', \eta', \epsilon')$ be internal adjunctions in C , and let $(\phi_1, \psi_1, \alpha_1, \beta_1)$ and $(\phi_2, \psi_2, \alpha_2, \beta_2)$ be lax morphisms of internal adjunctions from $(A, B, F, G, \eta, \epsilon)$ to $(A', B', F', G', \eta', \epsilon')$.

00U0 **Definition 3.1.1.1.** A **2-morphism from** $(\phi_1, \psi_1, \alpha_1, \beta_1)$ **to** $(\phi_2, \psi_2, \alpha_2, \beta_2)$ is a modification between these viewed as lax transformations.

00U1 **Remark 3.1.1.2.** In detail, a **2-morphism from** $(\phi_1, \psi_1, \alpha_1, \beta_1)$ **to** $(\phi_2, \psi_2, \alpha_2, \beta_2)$ consist of 2-morphisms

$$\Gamma: \phi_1 \Rightarrow \phi_2$$

$$\Sigma: \psi_1 \Rightarrow \psi_2$$

of C such that we have equalities

$$\begin{array}{ccc}
 \begin{array}{c} A \xrightarrow{F} B \\ \phi_1 \left(\begin{array}{c} \Gamma \\ \Rightarrow \end{array} \right) \phi_2 \quad \alpha_2 \nearrow \\ A' \xrightarrow{F'} B' \end{array} & = & \begin{array}{c} A \xrightarrow{F} B \\ \phi_1 \left(\begin{array}{c} \alpha_1 \nearrow \psi_1 \left(\begin{array}{c} \Sigma \\ \Rightarrow \end{array} \right) \end{array} \right) \psi_2 \\ A' \xrightarrow{F'} B' \end{array} \\
 \\
 \begin{array}{c} B \xrightarrow{G} A \\ \psi_1 \left(\begin{array}{c} \Sigma \\ \Rightarrow \end{array} \right) \psi_2 \quad \beta_2 \nearrow \\ B' \xrightarrow{G'} A' \end{array} & = & \begin{array}{c} B \xrightarrow{G} A \\ \psi_1 \left(\begin{array}{c} \beta_1 \nearrow \phi_1 \left(\begin{array}{c} \Gamma \\ \Rightarrow \end{array} \right) \end{array} \right) \phi_2 \\ B' \xrightarrow{G'} A' \end{array}
 \end{array}$$

of pasting diagrams in C .

00U2 3.2 2-Morphisms Between Oplax Morphisms of Internal Adjunctions

Let C be a bicategory, let $(A, B, F, G, \eta, \epsilon)$ and $(A', B', F', G', \eta', \epsilon')$ be internal adjunctions in C , and let $(\phi_1, \psi_1, \alpha_1, \beta_1)$ and $(\phi_2, \psi_2, \alpha_2, \beta_2)$ be oplax morphisms of internal adjunctions from $(A, B, F, G, \eta, \epsilon)$ to $(A', B', F', G', \eta', \epsilon')$.

00U3 **Definition 3.2.1.1.** A **2-morphism from** $(\phi_1, \psi_1, \alpha_1, \beta_1)$ **to** $(\phi_2, \psi_2, \alpha_2, \beta_2)$ is a modification between these viewed as oplax transformations.

00U4 **Remark 3.2.1.2.** In detail, a **2-morphism from** $(\phi_1, \psi_1, \alpha_1, \beta_1)$ **to** $(\phi_2, \psi_2, \alpha_2, \beta_2)$ consist of 2-morphisms

$$\Gamma: \phi_1 \Rightarrow \phi_2$$

$$\Sigma: \psi_1 \Rightarrow \psi_2$$

of C such that we have equalities

$$\begin{array}{ccc}
 \begin{array}{c} A \xrightarrow{F} B \\ \left(\begin{array}{c} \phi_2 \left(\begin{array}{c} \lrcorner \\ \lrcorner \end{array} \right) \phi_1 \end{array} \right) \alpha_1 \parallel \psi_1 \\ A' \xrightarrow{F'} B' \end{array} & = & \begin{array}{c} A \xrightarrow{F} B \\ \left(\begin{array}{c} \alpha_2 \parallel \psi_2 \left(\begin{array}{c} \lrcorner \\ \lrcorner \end{array} \right) \end{array} \right) \psi_1 \\ A' \xrightarrow{F'} B' \end{array} \\
 \\
 \begin{array}{c} B \xrightarrow{G} A \\ \left(\begin{array}{c} \psi_2 \left(\begin{array}{c} \lrcorner \\ \lrcorner \end{array} \right) \psi_1 \end{array} \right) \beta_1 \parallel \phi_1 \\ B' \xrightarrow{G'} A' \end{array} & = & \begin{array}{c} B \xrightarrow{G} A \\ \left(\begin{array}{c} \beta_2 \parallel \phi_2 \left(\begin{array}{c} \lrcorner \\ \lrcorner \end{array} \right) \end{array} \right) \phi_1 \\ B' \xrightarrow{G'} A' \end{array}
 \end{array}$$

of pasting diagrams in C .

00U5 3.3 2-Morphisms Between Strong Morphisms of Internal Adjunctions

Let C be a bicategory, let $(A, B, F, G, \eta, \epsilon)$ and $(A', B', F', G', \eta', \epsilon')$ be internal adjunctions in C , and let $(\phi_1, \psi_1, \alpha_1, \beta_1)$ and $(\phi_2, \psi_2, \alpha_2, \beta_2)$ be strong morphisms of internal adjunctions from $(A, B, F, G, \eta, \epsilon)$ to $(A', B', F', G', \eta', \epsilon')$.

00U6 **Definition 3.3.1.1.** A **2-morphism from** $(\phi_1, \psi_1, \alpha_1, \beta_1)$ **to** $(\phi_2, \psi_2, \alpha_2, \beta_2)$ is a modification between these viewed as strong transformations.

00U7 **Remark 3.3.1.2.** In detail, a **2-morphism from** $(\phi_1, \psi_1, \alpha_1, \beta_1)$ **to** $(\phi_2, \psi_2, \alpha_2, \beta_2)$ is equivalently:

- A 2-morphism (Γ, Σ) from $(\phi_1, \psi_1, \alpha_1, \beta_1)$ to $(\phi_2, \psi_2, \alpha_2, \beta_2)$ viewed as lax transformations as in [Remark 3.1.1.2](#).
- A 2-morphism (Γ, Σ) from $(\phi_1, \psi_1, \alpha_1, \beta_1)$ to $(\phi_2, \psi_2, \alpha_2, \beta_2)$ viewed as oplax transformations as in [Remark 3.2.1.2](#).

00U8 3.4 2-Morphisms Between Strict Morphisms of Internal Adjunctions

Let C be a bicategory, let $(A, B, F, G, \eta, \epsilon)$ and $(A', B', F', G', \eta', \epsilon')$ be internal adjunctions in C , and let $(\phi_1, \psi_1, \alpha_1, \beta_1)$ and $(\phi_2, \psi_2, \alpha_2, \beta_2)$ be lax morphisms of internal adjunctions from $(A, B, F, G, \eta, \epsilon)$ to $(A', B', F', G', \eta', \epsilon')$.

00U9 **Definition 3.4.1.1.** A **2-morphism from** $(\phi_1, \psi_1, \alpha_1, \beta_1)$ **to** $(\phi_2, \psi_2, \alpha_2, \beta_2)$ is a modification between these viewed as strict transformations.

00UA Remark 3.4.1.2. In detail, a **2-morphism from** $(\phi_1, \psi_1, \alpha_1, \beta_1)$ **to** $(\phi_2, \psi_2, \alpha_2, \beta_2)$ is equivalently:

- A 2-morphism (Γ, Σ) from $(\phi_1, \psi_1, \alpha_1, \beta_1)$ to $(\phi_2, \psi_2, \alpha_2, \beta_2)$ viewed as lax transformations as in **Remark 3.1.1.2.**
- A 2-morphism (Γ, Σ) from $(\phi_1, \psi_1, \alpha_1, \beta_1)$ to $(\phi_2, \psi_2, \alpha_2, \beta_2)$ viewed as oplax transformations as in **Remark 3.2.1.2.**
- A 2-morphism (Γ, Σ) from $(\phi_1, \psi_1, \alpha_1, \beta_1)$ to $(\phi_2, \psi_2, \alpha_2, \beta_2)$ viewed as strong transformations as in **Remark 3.3.1.2.**

00UB 4 Bicategories of Internal Adjunctions in a Bicategory

Appendices

A Other Chapters

Set Theory

1. **Sets**
2. **Constructions With Sets**
3. **Pointed Sets**
4. **Tensor Products of Pointed Sets**
5. **Indexed and Fibred Sets**
6. **Relations**
7. **Spans**
8. **Posets**

Category Theory

9. **Categories**
10. **Constructions With Categories**

11. **Kan Extensions**

Bicategories

12. **Bicategories**
13. **Internal Adjunctions**

Internal Category Theory

14. **Internal Categories**

Cyclic Stuff

15. **The Cycle Category**

Cubical Stuff

16. **The Cube Category**

Globular Stuff

17. **The Globe Category**

Cellular Stuff

18. **The Cell Category**

Monoids

- 19. Monoids
- 20. Constructions With Monoids

Monoids With Zero

- 21. Monoids With Zero
- 22. Constructions With Monoids With Zero

Groups

- 23. Groups
- 24. Constructions With Groups

Hyper Algebra

- 25. Hypermonoids
- 26. Hypergroups
- 27. Hypersemirings and Hyperrings
- 28. Quantales

Near-Rings

- 29. Near-Semirings
- 30. Near-Rings

Real Analysis

- 31. Real Analysis in One Variable
- 32. Real Analysis in Several Variables

Measure Theory

- 33. Measurable Spaces
- 34. Measures and Integration

Probability Theory

- 34. Probability Theory

Stochastic Analysis

- 35. Stochastic Processes, Martingales, and Brownian Motion
- 36. Itô Calculus
- 37. Stochastic Differential Equations

Differential Geometry

- 38. Topological and Smooth Manifolds

Schemes

- 39. Schemes