

Spans

December 3, 2023

00QB This chapter contains some material about spans. Notably, we discuss and explore:

1. The basic definitions around spans (**Section 1**);
2. The relation between spans and functions (**Proposition 7.1.1.1**);
3. The relation between spans and relations (**Propositions 7.2.2.1** and **7.3.1.1** and **Remark 7.5.1.1**).
4. “Hyperpointed sets” (??). I don’t know why I wrote this...

TODO:

1. internal adjoint equivalences in **Rel**
2. internal adjoint equivalences in **Span**
3. 2-categorical limits in **Rel**;
4. morphism of internal adjunctions in **Rel**;
5. morphism of internal adjunctions in **Span**;
6. morphism of co/monads in **Span**;
7. What is $\text{Adj}(\text{Span}(A, B))$?
8. monoids, comonoids, pseudomonoids, etc. in **Span**.
9. write down the dumb intuition about spans inducing morphisms $\text{Sets}(S, A) \rightarrow \text{Sets}(S, B)$ instead of $\mathcal{P}(A) \rightarrow \mathcal{P}(B)$ from the similarity between

$$S \rightarrow A \times B$$

and

$$A \times B \rightarrow \{t, f\}.$$

This intuition is justified by taking $A = \text{pt}$ or $B = \text{pt}$.

10. What about using the direct image with compact support in $g(f^{-1}(a))$?
11. Monads in **Span** | develop this in the level of morphisms too
12. Comonads in **Span** are spans whose legs are equal | develop this in the level of morphisms too
13. Does **Span** have an internal **Hom**?
14. Examples of spans
15. Functional and total spans
16. closed symmetric monoidal category of spans
17. double category of relations
18. collage of a span
19. equivalence spans?
20. functoriality of powersets for spans
21. Is **Span** a closed bicategory?
22. skew monoidal structure on $\text{Span}(A, B)$
23. Adjunctions in **Span**
24. Isomorphisms in **Span**
25. Equivalences in **Span**
26. Interaction between the above notions in **Span** vs. in **Rel** via the comparison functors

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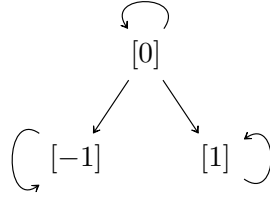
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00QC 1 Spans

00QD 1.1 The Walking Span

00QE **Definition 1.1.1.1.** The **walking span** is the category Λ that looks like this:



00QF 1.2 Spans

Let A and B be sets.

00QG **Definition 1.2.1.1.** A **span from A to B** ¹ is a functor $F: \Lambda \rightarrow \mathbf{Sets}$ such that

$$\begin{aligned} F([-1]) &= A, \\ F([1]) &= B. \end{aligned}$$

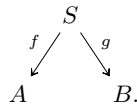
00QH **Remark 1.2.1.2.** In detail, a **span from A to B** is a triple (S, f, g) consisting of^{2,3}

- *The Underlying Set.* A set S , called the **underlying set of (S, f, g)** ;
- *The Legs.* A pair of functions $f: S \rightarrow A$ and $g: S \rightarrow B$.

00QJ 1.3 Morphisms of Spans

¹*Further Terminology:* Also called a **roof from A to B** or a **correspondence from A to B** .

²*Picture:*

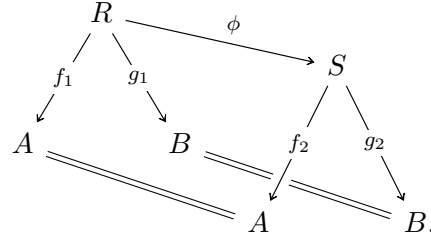


³Every span (S, f, g) from A to B determines in particular a relation $R: A \nrightarrow B$ via

$$R \stackrel{\text{def}}{=} \{(f(a), g(a)) \mid a \in A\},$$

00QK Definition 1.3.1.1. A **morphism of spans** (R, f_1, g_1) **to** (S, f_2, g_2) ⁴ is a natural transformation $(R, f_1, g_1) \Rightarrow (S, f_2, g_2)$.

00QL Remark 1.3.1.2. In detail, a **morphism of spans from** (R, f_1, g_1) **to** (S, f_2, g_2) is a function $\phi: R \rightarrow S$ making the diagram⁵



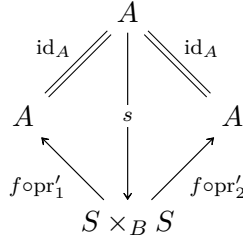
commute.

00QM 1.4 Functional Spans

Let $\lambda = (A \xleftarrow{f} S \xrightarrow{g} B)$ be a span. A morphism of spans from id_A to $\lambda \diamond \lambda^\dagger$ is a morphism

$$s: A \rightarrow S \times_B S$$

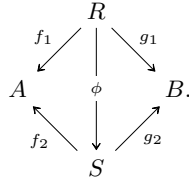
making the diagram



i.e. where $R(a) = g(f^{-1}(a))$ for each $a \in A$; see **Proposition 7.2.2.1**.

⁴ *Further Terminology:* Also called a **morphism of roofs from** (R, f_1, g_1) **to** (S, f_2, g_2) or a **morphism of correspondences from** (R, f_1, g_1) **to** (S, f_2, g_2) .

⁵ *Alternative Picture:*



commute, where $S \times_B S$ is the pullback

$$S \times_B S \cong \{(s, t) \in S \times S \mid g(s) = g(t)\}$$

$$\begin{array}{ccc} S \times_B S & \longrightarrow & S \\ \downarrow & \lrcorner & \downarrow g \\ S & \xrightarrow{g} & B \end{array}$$

of S with itself along g .

00QN 1.5 Total Spans

00QP 2 Categories of Spans

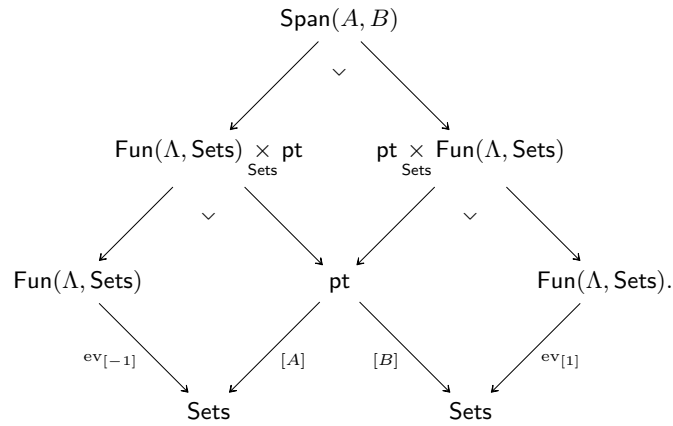
00QQ 2.1 Categories of Spans

Let A and B be sets.

00QR Definition 2.1.1.1. The **category of spans from A to B** is the category $\text{Span}(A, B)$ defined by

$$\text{Span}(A, B) \stackrel{\text{def}}{=} \text{Fun}(\Lambda, \text{Sets}) \times_{\text{ev}_{[-1]}, \text{Sets}, [A]} \text{pt} \times_{[B], \text{Sets}, \text{ev}_{[1]}} \text{Fun}(\Lambda, \text{Sets}),$$

as in the diagram



00QS Remark 2.1.1.2. In detail, the **category of spans from A to B** is the category $\text{Span}(A, B)$ where

- *Objects.* The objects of $\text{Span}(A, B)$ are spans from A to B ;

- *Morphisms.* The morphism of $\text{Span}(A, B)$ are morphisms of spans;
- *Identities.* The unit map

$$\mathbb{1}_{(S,f,g)}^{\text{Span}(A,B)}: \text{pt} \rightarrow \text{Hom}_{\text{Span}(A,B)}((S, f, g), (S, f, g))$$

of $\text{Span}(A, B)$ at (S, f, g) is defined by⁶

$$\text{id}_{(S,f,g)}^{\text{Span}(A,B)} \stackrel{\text{def}}{=} \text{id}_S;$$

- *Composition.* The composition map

$$\circ_{R,S,T}^{\text{Span}(A,B)}: \text{Hom}_{\text{Span}(A,B)}(S, T) \times \text{Hom}_{\text{Span}(A,B)}(R, S) \rightarrow \text{Hom}_{\text{Span}(A,B)}(R, T)$$

of $\text{Span}(A, B)$ at $((R, f_1, g_1), (S, f_2, g_2), (T, f_3, g_3))$ is defined by⁷

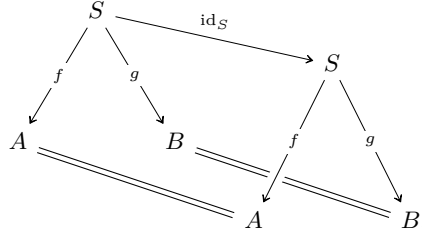
$$\psi \circ_{R,S,T}^{\text{Span}(A,B)} \phi \stackrel{\text{def}}{=} \psi \circ \phi.$$

2.2 The Bicategory of Spans

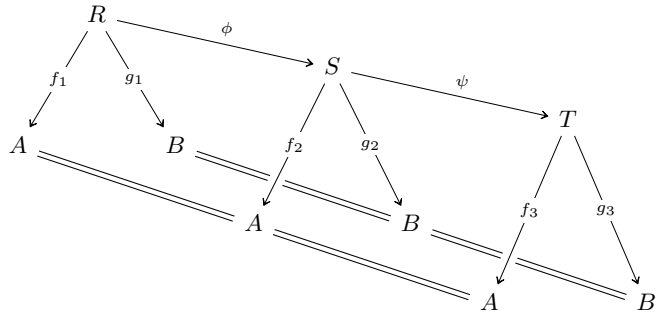
Definition 2.2.1.1. The **bicategory of spans** is the bicategory Span where

- *Objects.* The objects of Span are sets;

⁶ Picture:



⁷ Picture:



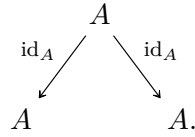
- *Hom-Categories.* For each $A, B \in \text{Obj}(\text{Span})$, we have

$$\text{Hom}_{\text{Span}}(A, B) \stackrel{\text{def}}{=} \text{Span}(A, B);$$

- *Identities.* For each $A \in \text{Obj}(\text{Span})$, the unit functor

$$\mathbb{K}_A^{\text{Span}}: \text{pt} \rightarrow \text{Span}(A, A)$$

of Span at A is the functor picking the span $(A, \text{id}_A, \text{id}_A)$:

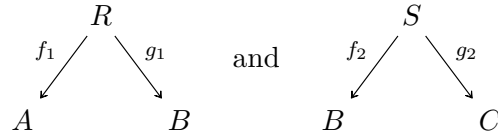


- *Composition.* For each $A, B, C \in \text{Obj}(\text{Span})$, the composition bifunctor

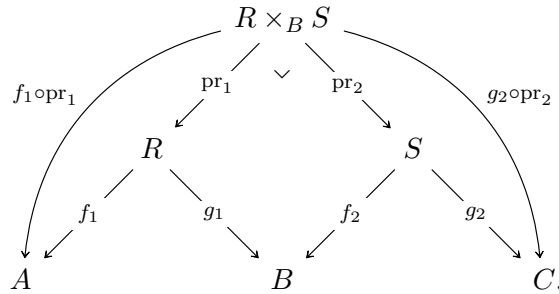
$$\circ_{A,B,C}^{\text{Span}}: \text{Span}(B, C) \times \text{Span}(A, B) \rightarrow \text{Span}(A, C)$$

of Span at (A, B, C) is the bifunctor where

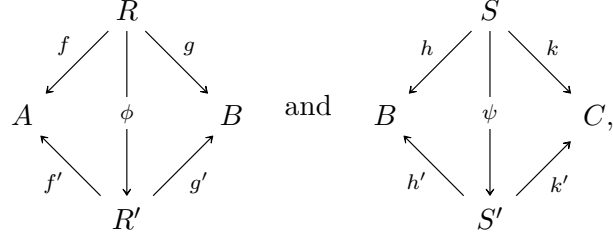
- *Action on Objects.* The composition of two spans



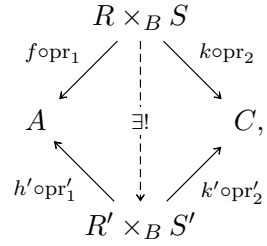
is the span $(R \times_B S, f_1 \circ \text{pr}_1, g_2 \circ \text{pr}_2)$, constructed as in the diagram



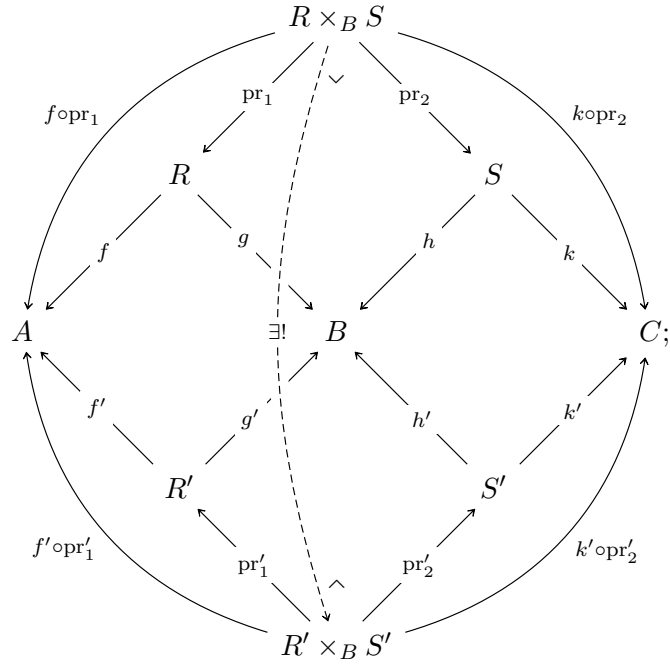
- *Action on Morphisms.* The horizontal composition of 2-morphisms is defined via functoriality of pullbacks: given morphisms of spans



their horizontal composition is the morphism of spans



constructed as in the diagram



- *Associators and Unitors.* The associator and unitors are defined using the universal property of the pullback.

00QV 2.3 The Monoidal Bicategory of Spans

00QW 2.4 The Double Category of Spans

00QX **Definition 2.4.1.1.** The **double category of spans** is the double category Span^{dbl} where

- *Objects.* The objects of Span^{dbl} are sets;
- *Vertical Morphisms.* The vertical morphisms of Span^{dbl} are functions $f: A \rightarrow B$;
- *Horizontal Morphisms.* The horizontal morphisms of Span^{dbl} are spans $(S, \phi, \psi): A \rightarrowtail X$;
- *2-Morphisms.* A 2-cell

$$\begin{array}{ccc} A & \xrightarrow{(R, \phi_R, \psi_R)} & B \\ f \downarrow & \Downarrow \alpha & \downarrow g \\ X & \xrightarrow{(S, \phi_S, \psi_S)} & Y \end{array}$$

of Span^{dbl} is a morphism of spans from the span

$$\begin{array}{ccc} & R & \\ \phi_R \swarrow & & \searrow \psi_R \\ A & & B \\ & & \searrow g \\ & & Y \end{array}$$

to the span

$$\begin{array}{ccccc} & & A \times_X S & & \\ & \swarrow & \downarrow \vee & \searrow & \\ & A & & S & \\ f \swarrow & & f \searrow & \phi_S \swarrow & \searrow \psi_S \\ X & & X & & Y; \end{array}$$

- *Horizontal Identities.* The horizontal unit functor

$$\mathbb{K}^{\text{Span}^{\text{dbl}}} : (\text{Span}^{\text{dbl}})_0 \rightarrow (\text{Span}^{\text{dbl}})_1$$

of Span^{dbl} is the functor where

- *Action on Objects.* For each $A \in \text{Obj}((\text{Span}^{\text{dbl}})_0)$, we have

$$\mathbb{K}_A \stackrel{\text{def}}{=} (A, \text{id}_A, \text{id}_A),$$

as in the diagram

$$\begin{array}{ccc} & A & \\ \text{id}_A \swarrow & & \searrow \text{id}_A \\ A & & A \end{array};$$

- *Action on Morphisms.* For each vertical morphism $f: A \rightarrow B$ of Span^{dbl} , i.e. each map of sets f from A to B , the identity 2-morphism

$$\begin{array}{ccc} A & \xrightarrow{\mathbb{K}_A} & A \\ f \downarrow & \Downarrow \mathbb{K}_f & \downarrow f \\ B & \xrightarrow{\mathbb{K}_B} & B \end{array}$$

of f is the morphism of spans from

$$\begin{array}{ccc} & A & \\ \text{id}_A \swarrow & & \searrow \text{id}_A \\ A & & A \xrightarrow{f} B \end{array}$$

to

$$\begin{array}{ccccc} & & A \times_B B & & \\ & \swarrow & \downarrow \vee & \searrow & \\ & A & & B & \\ f \swarrow & & & & \searrow \text{id}_B \\ B & & B & & B \end{array}$$

given by the isomorphism $A \xrightarrow{\cong} A \times_B B$;

- *Vertical Identities.* For each $A \in \text{Obj}(\text{Span}^{\text{dbl}})$, we have

$$\text{id}_A^{\text{Span}^{\text{dbl}}} \stackrel{\text{def}}{=} \text{id}_A;$$

- *Identity 2-Morphisms.* For each horizontal morphism $R: A \dashv B$ of Span^{dbl} , the identity 2-morphism

$$\begin{array}{ccc} A & \xrightarrow{S} & B \\ \text{id}_A \downarrow & \Downarrow \text{id}_S & \downarrow \text{id}_B \\ A & \xrightarrow{S} & B \end{array}$$

of R is the morphism of spans from

$$\begin{array}{ccc} & S & \\ \phi_S \swarrow & & \searrow \psi_S \\ A & & B \\ & & \Downarrow \text{id}_B \\ & & B \end{array}$$

to

$$\begin{array}{ccccc} & A \times_A S & & & \\ & \swarrow \quad \searrow & & & \\ & A & \quad \quad & S & \\ \text{id}_A \parallel \swarrow & & \searrow \text{id}_A & \swarrow \phi_S & \searrow \psi_S \\ A & & A & & B \end{array}$$

given by the isomorphism $S \xrightarrow{\cong} A \times_A S$;

- *Horizontal Composition.* The horizontal composition functor

$$\odot^{\text{Span}^{\text{dbl}}}: (\text{Span}^{\text{dbl}})_1 \times_{(\text{Span}^{\text{dbl}})_0} (\text{Span}^{\text{dbl}})_1 \rightarrow (\text{Span}^{\text{dbl}})_1$$

of Span^{dbl} is the functor where

- *Action on Objects.* For each composable pair

$$A \xrightarrow{(R, \phi_R, \psi_R)} B \xrightarrow{(S, \phi_S, \psi_S)} C$$

of horizontal morphisms of $\mathbf{Span}^{\text{dbl}}$, we have

$$(S, \phi_S, \psi_S) \odot (R, \phi_R, \psi_R) \stackrel{\text{def}}{=} S \circ_{A,B,C}^{\text{Span}} R,$$

where $S \circ_{A,B,C}^{\text{Span}} R$ is the composition of (R, ϕ_R, ψ_R) and (S, ϕ_S, ψ_S) defined as in [Definition 2.2.1.1](#);

– *Action on Morphisms.* For each horizontally composable pair

$$\begin{array}{ccc} A & \xrightarrow{(R, \phi_R, \psi_R)} & B \\ f \downarrow & \Downarrow \alpha & \downarrow g \\ X & \xrightarrow{(T, \phi_T, \psi_T)} & Y \end{array} \quad \begin{array}{ccc} B & \xrightarrow{(S, \phi_S, \psi_S)} & C \\ g \downarrow & \Downarrow \beta & \downarrow h \\ Y & \xrightarrow{(U, \phi_U, \psi_U)} & Z \end{array}$$

of 2-morphisms of $\mathbf{Span}^{\text{dbl}}$,

- *Vertical Composition of 1-Morphisms.* For each composable pair $A \xrightarrow{F} B \xrightarrow{G} C$ of vertical morphisms of $\mathbf{Span}^{\text{dbl}}$, i.e. maps of sets, we have

$$g \circ^{\mathbf{Span}^{\text{dbl}}} f \stackrel{\text{def}}{=} g \circ f;$$

- *Vertical Composition of 2-Morphisms.* For each vertically composable pair

$$\begin{array}{ccc} A & \xrightarrow{(R, \phi_R, \psi_R)} & X \\ f \downarrow & \Downarrow \alpha & \downarrow g \\ B & \xrightarrow{(S, \phi_S, \psi_S)} & Y \end{array} \quad \begin{array}{ccc} B & \xrightarrow{(S, \phi_S, \psi_S)} & Y \\ h \downarrow & \Downarrow \beta & \downarrow k \\ C & \xrightarrow{(T, \phi_T, \psi_T)} & Z \end{array}$$

of 2-morphisms of $\mathbf{Span}^{\text{dbl}}$,

- *Associators and Unitors.* The associator and unitors of $\mathbf{Span}^{\text{dbl}}$ are defined using the universal property of the pullback.

00QY 2.5 Properties of The Bicategory of Spans

00QZ **Proposition 2.5.1.1.** Let $\lambda = (A \xleftarrow{f} S \xrightarrow{g} B)$ be a span.

- 00R0 1. *Self-Duality.*
- 00R1 2. *Isomorphisms in \mathbf{Span} .*

00R2 3. *Equivalences in Span.*

00R3 4. *Adjunctions in Span.* Let A and B be sets.⁸

00R4 (a) We have a natural bijection

$$\left\{ \begin{array}{c} \text{Adjunctions in Span} \\ \text{from } A \text{ to } B \end{array} \right\} \cong \left\{ \begin{array}{c} \text{Spans } A \xleftarrow{f} S \xrightarrow{g} B \\ \text{from } A \text{ to } B \text{ with} \\ f \text{ an isomorphism} \end{array} \right\}.$$

00R5 (b) We have an equivalence of categories

$$\text{MapSpan}(A, B) \stackrel{\text{eq.}}{\cong} \text{Sets}(A, B)_{\text{disc}},$$

where $\text{MapSpan}(A, B)$ is the full subcategory of $\text{Span}(A, B)$ spanned by the spans $A \xleftarrow{f} S \xrightarrow{g} B$ from A to B with f an isomorphism.

00R6 (c) We have a biequivalence of bicategories

$$\text{MapSpan} \stackrel{\text{eq.}}{\cong} \text{Sets}_{\text{bidisc}},$$

where MapSpan is the sub-bicategory of Span whose Hom -categories are given by $\text{MapSpan}(A, B)$.

00R7 5. *Monads in Span.*

00R8 6. *Comonads in Span.*

00R9 7. *Monomorphisms in Span.*

00RA 8. *Epimorphisms in Span.*

00RB 9. *Existence of Right Kan Extensions.*

00RC 10. *Existence of Right Kan Lifts.*

00RD 11. *Closedness.*

Proof. Item 1, Self-Duality:

Item 2, Isomorphisms in Span:

Item 3, Equivalences in Span:

Item 4, Adjunctions in Span: We first prove *Item 4a*.

We proceed step by step:

⁸In the literature (e.g. [ref]),...are called maps and denoted by $\text{MapSpan}(A, B)$

1. *From Adjunctions in **Span** to Functions.* An adjunction in **Span** from A to B consists of a pair of spans

$$\begin{array}{ccc} & R & \\ f \swarrow & & \searrow g \\ A & & B \end{array} \quad \text{and} \quad \begin{array}{ccc} & S' & \\ h \swarrow & & \searrow k \\ B & & A \end{array}$$

together with maps

$$\begin{array}{ccccc} & A & & & \\ \text{id}_A \swarrow & & \searrow \text{id}_A & & \\ A & & \phi & & A \\ f \circ \text{pr}'_1 \swarrow & & \downarrow & & \searrow k \circ \text{pr}'_2 \\ & S \times_B S' & & & \end{array} \quad \text{and} \quad \begin{array}{ccccc} & S' \times_A S & & & \\ h \circ \text{pr}_1 \swarrow & & \searrow g \circ \text{pr}_2 & & \\ B & & \psi & & B \\ \text{id}_B \swarrow & & \downarrow & & \searrow \text{id}_B \\ & B & & & \end{array}$$

We claim that these conditions

2. *From Functions to Adjunctions in **Rel**.*
3. *Invertibility: From Functions to Adjunctions Back to Functions.*
4. *Invertibility: From Adjunctions to Functions Back to Adjunctions.*

We now proceed to the proof of **Item 4b**. For this, we will construct a functor

$$F: \mathbf{Sets}(A, B)_{\text{disc}} \rightarrow \mathbf{MapSpan}(A, B)$$

and prove it to be essentially surjective and fully faithful, and thus an equivalence by **Categories**, ?? of ??. Indeed, given a map $f: A \rightarrow B$, let $F(f)$ be the representable span associated to f of **Definition 5.1.1.1**, and let F send the unique (identity) morphism from f to itself to the identity morphism of $F(f)$ in $\mathbf{MapSpan}(A, B)$. We now prove that F is fully faithful and essentially surjective:

1. *F Is Fully Faithful:* Given maps $f, g: A \rightrightarrows B$, we need to show that

$$\text{Hom}_{\mathbf{MapSpan}(A, B)}(F(f), F(g)) = \begin{cases} \text{pt} & \text{if } f = g, \\ \emptyset & \text{otherwise.} \end{cases}$$

Indeed, a morphism from $F(f)$ to $F(g)$ takes the form

$$\begin{array}{ccc} & A & \\ \parallel & \downarrow \phi & \searrow f \\ A & & B \\ \parallel & \downarrow \phi & \nearrow g \\ & A & \end{array}$$

From the relations $\text{id}_A = \text{id}_A \circ \phi$ and $f = g \circ \phi$, we see that $\phi = \text{id}_A$, and thus from the relation $f = g \circ \phi$ there is such a morphism iff $f = g$.

2. *F Is Essentially Surjective:* Let λ be a span of the form

$$\begin{array}{ccc} & S & \\ \phi \swarrow & & \searrow f \\ A & & B \end{array}$$

we claim that $\lambda \cong F(f \circ \phi^{-1})$. Indeed, we have morphisms

$$\begin{array}{ccc} & S & \\ \phi \swarrow & \downarrow \phi & \searrow f \\ A & & B \\ \parallel & \downarrow \phi & \nearrow f \circ \phi^{-1} \\ & A & \end{array} \quad \text{and} \quad \begin{array}{ccc} & A & \\ \parallel & \downarrow \phi^{-1} & \searrow f \circ \phi^{-1} \\ A & & B \\ \parallel & \downarrow \phi^{-1} & \nearrow f \\ & S & \end{array}$$

inverse to each other in $\text{MapSpan}(A, B)$, and thus $\lambda \cong F(f \circ \phi^{-1})$.

Finally, we prove **Item 4c**.

Item 5, Monads in Span:

Item 6, Comonads in Span:

Item 7, Monomorphisms in Span:

Item 8, Epimorphisms in Span:

Item 9, Existence of Right Kan Extensions:

Item 10, Existence of Right Kan Lifts:

Item 11, Closedness:

□

00RE 3 Limits of Spans

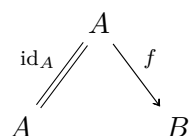
00RF 4 Colimits of Spans

00RG 5 Constructions With Spans

00RH 5.1 Representable Spans

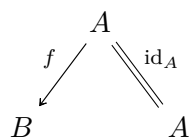
00RJ **Definition 5.1.1.1.** Let $f: A \rightarrow B$ be a function.

- The **representable span associated to f** is the span



from A to B .

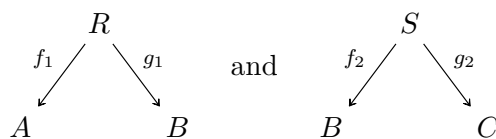
- The **corepresentable span associated to f** is the span



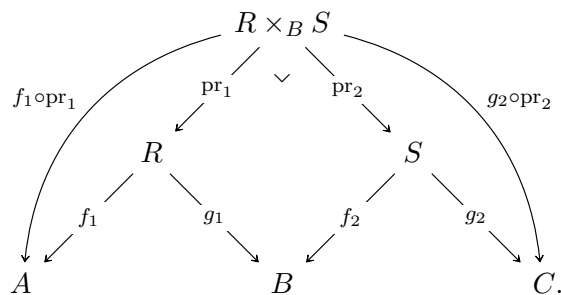
from B to A .

00RK 5.2 Composition of Spans

00RL **Definition 5.2.1.1.** The **composition** of two spans

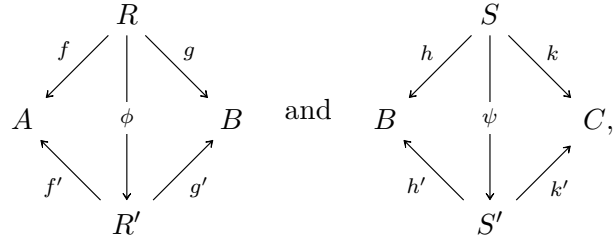


is the span $(R \times_B S, f_1 \circ \text{pr}_1, g_2 \circ \text{pr}_2)$, constructed as in the diagram

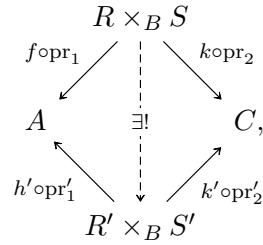


00RM 5.3 Horizontal Composition of Morphisms of Spans

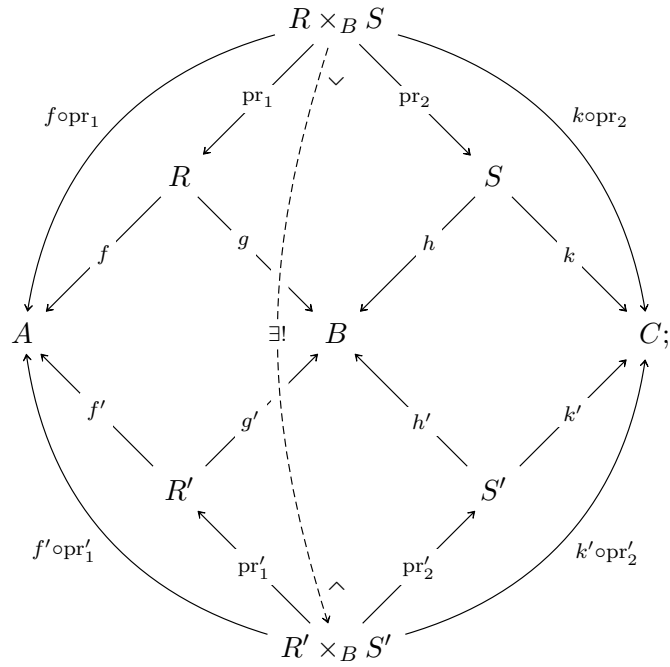
00RN Definition 5.3.1.1. The **horizontal composition** of a pair of 2-morphisms of spans



is the morphism of spans



constructed as in the diagram



00RP 5.4 Properties of Composition of Spans**00RQ Proposition 5.4.1.1.** Let $\lambda = \left(A \xleftarrow{f} S \xrightarrow{g} B \right)$ be a span.**00RR** 1. *Functoriality.**Proof.*

□

00RS 5.5 The Inverse of a Span**00RT 6 Functoriality of Spans****00RU 6.1 Direct Images****00RV 6.2 Functoriality of Spans on Powersets****00RW 7 Comparison of Spans to Functions and Relations****00RX 7.1 Comparison to Functions****00RY Proposition 7.1.1.1.** We have a pseudofunctor

$$\iota: \mathbf{Sets}_{\text{bidisc}} \rightarrow \mathbf{Span}$$

from $\mathbf{Sets}_{\text{bidisc}}$ to \mathbf{Span} where

- *Action on Objects.* For each $A \in \text{Obj}(\mathbf{Sets}_{\text{bidisc}})$, we have

$$\iota(A) \stackrel{\text{def}}{=} A;$$

- *Action on Hom-Categories.* For each $A, B \in \text{Obj}(\mathbf{Sets}_{\text{bidisc}})$, the action on Hom-categories

$$\iota_{A,B}: \mathbf{Sets}(A, B)_{\text{disc}} \rightarrow \mathbf{Span}(A, B)$$

of ι at (A, B) is the functor defined on objects by sending a function $f: A \rightarrow B$ to the span

$$\begin{array}{ccc} & A & \\ \text{id}_A \swarrow & & \searrow f \\ A & & B \end{array}$$

from A to B .

- $$\iota_A^0: \text{id}_{\iota(A)} \implies \iota(\text{id}_A)$$

of ι at A is given by the identity morphism of spans

$$\begin{array}{ccc}
& A & \\
\text{id}_A \swarrow & \parallel & \searrow \text{id}_A \\
A & \text{id} & A, \\
\swarrow \text{id}_A & \parallel & \searrow \text{id}_A \\
& A &
\end{array}$$

- *Pseudofunctoriality Constraints.* For each $A, B, C \in \text{Obj}(\text{Sets}_{\text{bidisc}})$, each $f \in \text{Hom}_{\text{Sets}_{\text{bidisc}}}(A, B)$, and each $g \in \text{Hom}_{\text{Sets}_{\text{bidisc}}}(B, C)$, the pseudofunctoriality constraint

$$\iota_{g,f}^2: \iota(g) \circ \iota(f) \Longrightarrow \iota(g \circ f)$$

of ι at (f, g) is the morphism of spans from the span

$$\begin{array}{ccccc}
 & & A \times B & & B \\
 & \swarrow & \downarrow \text{\tiny \vee} & \searrow & \\
 \text{id}_A \circ \text{pr}_1 & & & & g \circ \text{pr}_2 \\
 & \swarrow \text{\tiny pr_1} & & \searrow \text{\tiny pr_2} & \\
 & A & & B & \\
 & \swarrow \text{\tiny id_A} & & \searrow \text{\tiny id_B} & \\
 & A & & B & \\
 & \swarrow \text{\tiny f} & & \searrow \text{\tiny g} & \\
 & A & & B & C
 \end{array}$$

to the span

$$\begin{array}{ccc} & A & \\ \text{id}_A \swarrow & & \searrow g \circ f \\ A & & C \end{array}$$

given by the isomorphism $A \times_B B \cong A$.

☐

00RZ 7.2 Comparison to Relations: From Span to Rel**00S0 7.2.1 Relations Associated to Spans**

Let $\lambda = \left(A \xleftarrow{f} S \xrightarrow{g} B \right)$ be a span.

00S1 Definition 7.2.1.1. The **relation associated to λ** is the relation

$$S(\lambda): A \nrightarrow B$$

from A to B defined as follows:

- Viewing relations from A to B as functions $A \times B \rightarrow \{\text{true}, \text{false}\}$, we define

$$\iota_{A,B}(S)_b^a \stackrel{\text{def}}{=} \begin{cases} \text{true} & \text{if there exists } x \in S \text{ such} \\ & \text{that } a = f(x) \text{ and } b = g(x), \\ \text{false} & \text{otherwise} \end{cases}$$

for each $(a, b) \in A \times B$.

- Viewing relations from A to B as functions $A \rightarrow \mathcal{P}(B)$, we define

$$[\iota_{A,B}(S)](a) \stackrel{\text{def}}{=} g(f^{-1}(a))$$

for each $a \in A$.

- Viewing relations from A to B as subsets of $A \times B$, we define

$$\iota_{A,B}(S) \stackrel{\text{def}}{=} \{(f(x), g(x)) \mid x \in S\}.$$

00S2 Proposition 7.2.1.2. Let $\lambda = \left(A \xleftarrow{f} S \xrightarrow{g} B \right)$ be a span.

00S3 1. *Interaction With Identities.*

00S4 2. *Interaction With Composition.*

00S5 3. *Interaction With Inverses.*

Proof.

□

00S6 7.2.2 The Comparison Functor from Span to Rel**00S7 Proposition 7.2.2.1.** We have a pseudofunctor

$$\iota: \mathbf{Span} \rightarrow \mathbf{Rel}$$

from **Span** to **Rel** where

- *Action on Objects.* For each $A \in \mathbf{Obj}(\mathbf{Span})$, we have

$$\iota(A) \stackrel{\text{def}}{=} A;$$

- *Action on Hom-Categories.* For each $A, B \in \mathbf{Obj}(\mathbf{Span})$, the action on Hom-categories

$$\iota_{A,B}: \mathbf{Span}(A, B) \rightarrow \mathbf{Rel}(A, B)$$

of ι at (A, B) is the functor where

- *Action on Objects.* Given a span

$$\begin{array}{ccc} & S & \\ f \swarrow & & \searrow g \\ A & & B \end{array}$$

from A to B , the image

$$\iota_{A,B}(S): A \nrightarrow B$$

of S by ι is the relation from A to B defined as follows:

- * Viewing relations as functions $A \times B \rightarrow \{\mathbf{true}, \mathbf{false}\}$, we define

$$\iota_{A,B}(S)_b^a \stackrel{\text{def}}{=} \begin{cases} \mathbf{true} & \text{if there exists } x \in S \\ & \text{such that } a = f(x) \\ & \text{and } b = g(x), \\ \mathbf{false} & \text{otherwise} \end{cases}$$

for each $(a, b) \in A \times B$;

- * Viewing relations as functions $A \rightarrow \mathcal{P}(B)$, we define

$$[\iota_{A,B}(S)](a) \stackrel{\text{def}}{=} g(f^{-1}(a))$$

for each $a \in A$;

* Viewing relations as subsets of $A \times B$, we define

$$\iota_{A,B}(S) \stackrel{\text{def}}{=} \{(f(x), g(x)) \mid x \in S\}.$$

– *Action on Morphisms.* Given a morphism of spans

$$\begin{array}{ccc} & R & \\ f_R \swarrow & \downarrow \phi & \searrow g_R \\ A & & B, \\ f_S \swarrow & \downarrow \phi & \searrow g_S \\ & S & \end{array}$$

we have a corresponding inclusion of relations

$$\iota_{A,B}(\phi): \iota_{A,B}(R) \subset \iota_{A,B}(S),$$

since we have $a \sim_{\iota_{A,B}(R)} b$ iff there exists $x \in R$ such that $a = f_R(x)$ and $b = g_R(x)$, in which case we then have

$$\begin{aligned} a &= f_R(x) \\ &= f_S(\phi(x)), \\ b &= g_R(x) \\ &= g_S(\phi(x)), \end{aligned}$$

so that $a \sim_{\iota_{A,B}(S)} b$, and thus $\iota_{A,B}(R) \subset \iota_{A,B}(S)$.

Proof. Omitted. □

00S8 7.3 Comparison to Relations: From **Rel** to **Span**

00S9 **Proposition 7.3.1.1.** We have a lax functor

$$(\iota, \iota^2, \iota^0): \mathbf{Rel} \rightarrow \mathbf{Span}$$

from **Rel** to **Span** where

- *Action on Objects.* For each $A \in \text{Obj}(\mathbf{Span})$, we have

$$\iota(A) \stackrel{\text{def}}{=} A;$$

- *Action on Hom-Categories.* For each $A, B \in \text{Obj}(\mathbf{Span})$, the action on Hom-categories

$$\iota_{A,B}: \mathbf{Rel}(A, B) \rightarrow \mathbf{Span}(A, B)$$

of ι at (A, B) is the functor where

- *Action on Objects.* Given a relation $R: A \dashv B$ from A to B , we define a span

$$\iota_{A,B}(R): A \dashv B$$

from A to B by

$$\iota_{A,B}(R) \stackrel{\text{def}}{=} (R, \upharpoonright \text{pr}_1 R, \upharpoonright \text{pr}_2 R),$$

where $R \subset A \times B$ and $\upharpoonright \text{pr}_1 R$ and $\upharpoonright \text{pr}_2 R$ are the restriction of the projections

$$\begin{aligned} \text{pr}_1: A \times B &\rightarrow A, \\ \text{pr}_2: A \times B &\rightarrow B \end{aligned}$$

to R ;

- *Action on Morphisms.* Given an inclusion $\phi: R \subset S$ of relations, we have a corresponding morphism of spans

$$\iota_{A,B}(\phi): \iota_{A,B}(R) \rightarrow \iota_{A,B}(S)$$

as in the diagram

$$\begin{array}{ccc} & R & \\ \upharpoonright \text{pr}_1 R \swarrow & \downarrow & \searrow \upharpoonright \text{pr}_2 R \\ A & & B. \\ \upharpoonright \text{pr}_1 S \swarrow & \downarrow & \searrow \upharpoonright \text{pr}_2 S \\ & S & \end{array}$$

- *The Lax Functoriality Constraints.* The lax functoriality constraint

$$\iota_{R,S}^2: \iota(S) \circ \iota(R) \Longrightarrow \iota(S \diamond R)$$

of ι at (R, S) is given by the morphism of spans from

$$\begin{array}{ccccc} & R \times_B S & & & \\ & \swarrow \text{pr}_1 \quad \searrow \text{pr}_2 & & & \\ & R & & S & \\ \swarrow \upharpoonright \text{pr}_1 R & & \swarrow \upharpoonright \text{pr}_1 S & & \searrow \upharpoonright \text{pr}_2 S \\ A & & B & & C \end{array}$$

$\upharpoonright \text{pr}_1 R \circ \text{pr}_1$ (curved arrow from $R \times_B S$ to A)
 $\upharpoonright \text{pr}_2 S \circ \text{pr}_2$ (curved arrow from $R \times_B S$ to C)

to

$$\begin{array}{ccc} & S \diamond R & \\ \downarrow \text{pr}_1 S \diamond R & & \downarrow \text{pr}_2 S \diamond R \\ A & & C \end{array}$$

given by the natural inclusion $R \times_B S \hookrightarrow S \diamond R$, since we have

$$\begin{aligned} R \times_B S &= \{((a_R, b_R), (b_S, c_S)) \in R \times S \mid b_R = b_S\}; \\ S \diamond R &= \left\{ (a, c) \in A \times C \mid \begin{array}{l} \text{there exists some } b \in B \text{ such} \\ \text{that } (a, b) \in R \text{ and } (b, c) \in S \end{array} \right\}; \end{aligned}$$

- *The Lax Unity Constraints.* The lax unity constraint⁹

$$\iota_A^0: \underbrace{\text{id}_{\iota(A)}}_{(A, \text{id}_A, \text{id}_A)} \Longrightarrow \underbrace{\iota(\chi_A)}_{(\Delta_A, \text{pr}_1 \Delta_A, \text{pr}_2 \Delta_A)}$$

of ι at A is given by the diagonal morphism of A , as in the diagram

$$\begin{array}{ccccc} & A & & & \\ \text{id}_A \swarrow & & \downarrow \delta_A & & \searrow \text{id}_A \\ A & & \Delta_A & & A \\ \text{pr}_1 \Delta_A \swarrow & & \downarrow & & \searrow \text{pr}_2 \Delta_A \end{array}$$

Proof. Omitted. □

7.4 Comparison to Relations: The Wehrheim–Woodward Construction

00SA

00SB 7.5 Comparison to Multirelations

00SC **Remark 7.5.1.1.** The pseudofunctor of [Proposition 7.2.2.1](#) and the lax functor of [Proposition 7.3.1.1](#) fail to be equivalences of bicategories. This happens essentially because a span $(S, f, g): A \rightharpoonup B$ from A to B may relate elements $a \in A$ and $b \in B$ by more than one element, e.g. there could be $s \neq s' \in S$ such that $a = f(s) = f(s')$ and $b = g(s) = g(s')$. Thus, in a sense, spans may be thought of as “relations with multiplicity”. And indeed, if instead of considering relations from A to B , i.e. functions

$$R: A \times B \rightarrow \{\text{true}, \text{false}\}$$

⁹Which is in fact strong, as δ_A is an isomorphism.

from $A \times B$ to $\{\text{true}, \text{false}\} \cong \{0, 1\}$, we consider functions

$$R: A \times B \rightarrow \mathbb{N} \cup \{\infty\}$$

from $A \times B$ to $\mathbb{N} \cup \{\infty\}$, then we obtain the notion of a **multirelation from A to B** , and these turn out to assemble together with sets into a bicategory \mathbf{MRel} that is biequivalent to \mathbf{Span} ; see [some-algebraic-laws-for-spans-and-their-connections-with-mul

7.6 Comparison to Relations via Double Categories

Remark 7.6.1.1. There are double functors between the double categories $\mathbf{Rel}^{\text{dbl}}$ and $\mathbf{Span}^{\text{dbl}}$ analogous to the functors of Propositions 7.2.2.1 and 7.3.1.1, assembling moreover into a strict-lax adjunction of double functors; see [higher-dimensional-categories].

Appendices

A Other Chapters

Set Theory

1. Sets
2. Constructions With Sets
3. Pointed Sets
4. Tensor Products of Pointed Sets
5. Indexed and Fibred Sets
6. Relations
7. Spans
8. Posets

Category Theory

9. Categories
10. Constructions With Categories

11. Kan Extensions

Bicategories

12. Bicategories
13. Internal Adjunctions

Internal Category Theory

14. Internal Categories

Cyclic Stuff

15. The Cycle Category

Cubical Stuff

16. The Cube Category

Globular Stuff

17. The Globe Category

Cellular Stuff

18. The Cell Category

Monoids

- 19. Monoids
- 20. Constructions With Monoids

Monoids With Zero

- 21. Monoids With Zero
- 22. Constructions With Monoids With Zero

Groups

- 23. Groups
- 24. Constructions With Groups

Hyper Algebra

- 25. Hypermonoids
- 26. Hypergroups
- 27. Hypersemirings and Hyperrings
- 28. Quantales

Near-Rings

- 29. Near-Semirings
- 30. Near-Rings

Real Analysis

31. Real Analysis in One Variable

32. Real Analysis in Several Variables

Measure Theory

- 33. Measurable Spaces
- 34. Measures and Integration

Probability Theory

- 34. Probability Theory

Stochastic Analysis

- 35. Stochastic Processes, Martingales, and Brownian Motion
- 36. Itô Calculus
- 37. Stochastic Differential Equations

Differential Geometry

- 38. Topological and Smooth Manifolds

Schemes

- 39. Schemes