## Fibred Sets

#### December 24, 2023

This chapter contains altiseussion of the un/straightening equivalence in the context of sets, as well as a general discussion of indexed and fibred sets. In particular, it contains:

- 1. A discussion of indexed sets (i.e. functors  $K_{\text{disc}} \rightarrow \text{Sets}$  with K a set), constructions with them like dependent sums and dependent products, and their properties (????);
- 2. A discussion of fibred sets (i.e. maps of sets  $X \to K$ ), constructions with them like dependent sums and dependent products, and their properties (????);
- 3. A discussion of the un/straightening equivalence for indexed and fibred sets (??).

## **Contents**

#### 1 Fibred Sets 00S1

#### 1.1 Foundations 00S2

Let K be a set.

**Definition 1.1.1.1.** A *K*-fibred set is a pair  $(NS,3\phi)$  consisting of

- The Underlying Set. A set X, called the **underlying set of**  $(X, \phi)$ ;
- The Fibration. A map of sets  $\phi: X \to K$ .

$$\phi^{-1}(x) \stackrel{\text{def}}{=} \text{pt} \times_{[x], K, \phi} X, \qquad \phi^{-1}(x) \xrightarrow{\qquad} X$$

$$\downarrow \qquad \qquad \downarrow \phi$$

$$\text{pt} \xrightarrow{\qquad [x] \qquad} K.$$

Further Terminology: The **fibre of**  $(X, \phi)$  **over**  $x \in K$  is the set  $\phi^{-1}(x)$  (also written  $\phi_x$ ) defined by

## 1.2 Morphisms of Tibted Sets

**Definition 1.2.1.1.** A **morphism of** K-**fibred Sets from**  $(X, \phi)$  **to**  $(Y, \psi)$  is a function  $f: X \to Y$  such that the diagram<sup>2</sup>

$$X \xrightarrow{f} Y$$

$$\downarrow \psi$$

$$K$$

commutes.

#### 1.3 The Category of Fibred Sets Over a Fixed Base

**Definition 1.3.1.1.** The **category of** K**-fib redSets** is the category FibSets(K) defined as the slice category Sets $_{/K}$  of Sets over K:

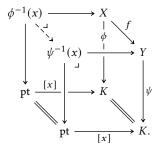
$$\mathsf{FibSets}(K) \stackrel{\mathsf{def}}{=} \mathsf{Sets}_{/K}.$$

**Remark 1.3.1.2.** In detail FibSets(K) is at the attention where

- Objects. The objects of FibSets(K) are pairs (X,  $\phi$ ) consisting of
  - The Fibred Set. A set X;
  - *The Fibration.* A function  $\phi: X \to K$ ;
- *Morphisms*. A morphism of FibSets(K) from  $(X, \phi)$  to  $(Y, \psi)$  is a function  $f: X \to M$

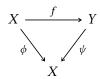
$$f_x^* : \phi^{-1}(x) \to \psi^{-1}(x)$$

given by the dashed map in the diagram



<sup>&</sup>lt;sup>2</sup> Further Terminology: The **transport map associated to** f **at**  $x \in K$  is the function

Y making the diagram



commute;

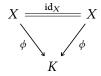
• *Identities.* For each  $(X, \phi) \in \text{Obj}(\mathsf{FibSets}(K))$ , the unit map

$$\mathbb{1}_{(X,\phi)}^{\mathsf{FibSets}(K)} : \mathsf{pt} \to \mathsf{Hom}_{\mathsf{FibSets}(K)}((X,\phi),(X,\phi))$$

of FibSets(K) at (X,  $\phi$ ) is given by

$$id_{(X,\phi)}^{\mathsf{FibSets}(K)} \stackrel{\text{def}}{=} id_X,$$

as witnessed by the commutativity of the diagram



in Sets;

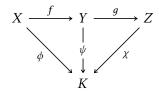
• Composition. For each  $\mathbf{X}=(X,\phi)$ ,  $\mathbf{Y}=(Y,\psi)$ ,  $\mathbf{Z}=(Z,\chi)\in \mathrm{Obj}(\mathsf{FibSets}(K))$ , the composition map

$$\circ_{\mathbf{X},\mathbf{Y},\mathbf{Z}}^{\mathsf{FibSets}(K)} \colon \mathsf{Hom}_{\mathsf{FibSets}(K)}(\mathbf{Y},\mathbf{Z}) \times \mathsf{Hom}_{\mathsf{FibSets}(K)}(\mathbf{X},\mathbf{Y}) \to \mathsf{Hom}_{\mathsf{FibSets}(K)}(\mathbf{X},\mathbf{Z})$$

of FibSets(K) at (X, Y, Z) is defined by

$$\circ_{\mathbf{X},\mathbf{Y},\mathbf{Z}}^{\mathsf{FibSets}(K)} \stackrel{\mathsf{def}}{=} \circ_{X,Y,Z}^{\mathsf{Sets}},$$

as witnessed by the commutativity of the diagram



in Sets.

### 1.4 The Category of Fibred Sets

**Definition 1.4.1.1.** The **category of fibred as** is the category FibSets defined as the Grothendieck construction of the functor FibSets: Sets<sup>op</sup>  $\rightarrow$  Cats of ??:

$$FibSets \stackrel{\text{def}}{=} \int^{Sets} FibSets.$$

**Remark 1.4.1.2.** In detail, the **category of fibred sets** is the category FibSets where

- Objects. The objects of FibSets are pairs  $(K, (X, \phi_X))$  consisting of
  - The Base Set. A set K;
  - The Fibred Set. A K-fibred set  $\phi_X : X \to K$ ;
- *Morphisms*. A morphism of FibSets from  $(K, (X, \phi_X))$  to  $(K', (Y, \phi_Y))$  is a pair  $(\phi, f)$  consisting of
  - The Base Map. A map of sets  $\phi: K \to K'$ ;
  - The Morphism of Fibred Sets. A morphism of K-fibred sets

$$f: (X, \phi_X) \to \phi_Y^*(Y),$$

$$X \xrightarrow{f} Y \times_{K'} K$$

$$\phi_X \swarrow pr_2$$

$$K:$$

• *Identities.* For each  $(K, X) \in Obj(FibSets)$ , the unit map

$$\mathbb{F}^{\mathsf{FibSets}}_{(K,X)} \colon \mathsf{pt} \to \mathsf{FibSets}((K,X),(K,X))$$

of FibSets at (K, X) is defined by

$$id_{(K,X)}^{\mathsf{FibSets}} \stackrel{\text{def}}{=} (id_K, \sim),$$

where  $\sim$  is the isomorphism  $X \to X \times_K K$  as in the diagram

$$X \xrightarrow{\varphi_X} X \times_K K$$

$$\downarrow^{\operatorname{pr}_2}$$

$$K;$$

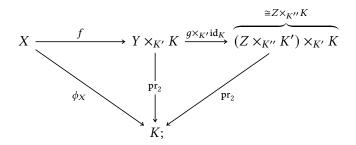
• Composition. For each  $\mathbf{X} = (K, X), \mathbf{Y} = (K', Y), \mathbf{Z} = (K'', Z) \in \mathsf{Obj}(\mathsf{FibSets}),$  the composition map

$$\circ^{\mathsf{FibSets}}_{\mathbf{X},\mathbf{Y},\mathbf{Z}} \colon \mathsf{FibSets}(\mathbf{Y},\mathbf{Z}) \times \mathsf{FibSets}(\mathbf{X},\mathbf{Y}) \to \mathsf{FibSets}(\mathbf{X},\mathbf{Z})$$

of FibSets at (X, Y, Z) is defined by

$$g \circ_{\mathbf{X},\mathbf{Y},\mathbf{Z}}^{\mathsf{FibSets}} f \stackrel{\text{def}}{=} (g \times_{K'} \mathrm{id}_K) \circ f$$

as in the diagram



for each  $f \in \text{Obj}(\mathsf{FibSets}(\mathbf{X}, \mathbf{Y}))$  and each  $g \in \text{Obj}(\mathsf{FibSets}(\mathbf{Y}, \mathbf{Z}))$ .

## 2 Construction With Fibred Sets

## 2.1 Change of Bas@OSD

Let  $f: K \to K'$  be a function and let  $(X, \phi_X)$  be a K'-fibred set.

**Definition 2.1.1.1.** The **change of base of (X, \phi\_X) to** K is the K-fibred set  $f^*(X)$  defined by

$$f^{*}(X) \stackrel{\text{def}}{=} (K \times_{K'} X, \operatorname{pr}_{1}), \qquad f^{*}(X) \stackrel{\operatorname{pr}_{2}}{\longrightarrow} X$$

$$\downarrow^{\phi_{X}} \qquad \downarrow^{\phi_{X}}$$

$$K \xrightarrow{f} K'.$$

**Proposition 2.1.1.2.** The assignment  $X \mapsto \emptyset (X)$  defines a functor

$$f^* : \mathsf{FibSets}(K') \to \mathsf{FibSets}(K),$$

where

• *Action on Objects.* For each  $(X, \phi_X) \in \mathsf{Obj}(\mathsf{FibSets}(K'))$ , we have

$$f^*(X, \phi_X) \stackrel{\text{def}}{=} f^*(X);$$

• Action on Morphisms. For each  $(X, \phi_X), (Y, \phi_Y) \in \mathsf{Obj}(\mathsf{FibSets}(K'))$ , the action

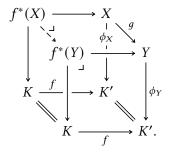
on Hom-sets

$$f_{X,Y}^* \colon \operatorname{Hom}_{\mathsf{FibSets}(K')}(X,Y) \to \operatorname{Hom}_{\mathsf{FibSets}(K)}(f^*(X),f^*(Y))$$

of  $f^*$  at  $((X, \phi_X), (Y, \phi_Y))$  is the map sending a morphism of K'-fibred sets

$$q: (X, \phi_X) \to (Y, \phi_Y)$$

to the morphism of K-fibred sets given by the dashed morphism in the diagram



Proof. Omitted.

**Proposition 2.1.1.3.** The assignment  $K \mapsto \text{Possets}(K)$  defines a functor

FibSets: Sets<sup>op</sup> 
$$\rightarrow$$
 Cats,

where

• *Action on Objects.* For each  $K \in \text{Obj}(\mathsf{Sets})$ , we have

$$[FibSets](K) \stackrel{\text{def}}{=} FibSets(K);$$

• Action on Morphisms. For each  $K, K' \in \mathsf{Obj}(\mathsf{Sets})$ , the action on Hom-sets

$$\mathsf{Sets}_{/(-)|K,K'} \colon \mathsf{Sets}^{\mathsf{op}}(K,K') \to \mathsf{Fun}(\mathsf{FibSets}(K),\mathsf{FibSets}(K'))$$

of  $\mathsf{Sets}_{/(-)}$  at (K,K') is the map sending a map of sets  $f\colon K\to K'$  to the functor

$$\mathsf{Sets}_{/f} \colon \mathsf{Fib}\mathsf{Sets}(K') \to \mathsf{Fib}\mathsf{Sets}(K)$$

defined by

$$\mathsf{Sets}_{/f} \stackrel{\mathrm{def}}{=} f^*$$
.

Proof. Omitted.

### 2.2 Dependent SumsH

Let  $f: K \to K'$  be a function and let  $(X, \phi_X)$  be a K-fibred set.

**Definition 2.2.1.1.** The **dependent sum**<sup>3</sup> **Off**  $(X, \phi_X)$  is the K'-fibred set  $\Sigma_f(X)^4$  defined by

$$\Sigma_f(X) \stackrel{\text{def}}{=} (\Sigma_f(X), \Sigma_f(\phi_X))$$
$$\stackrel{\text{def}}{=} (X, f \circ \phi_X).$$

**Proposition 2.2.1.2.** Let  $f: K \to K'$  be a function.

1. Functoriality. The assignment  $X \mapsto \Sigma \phi(X)$  defines a functor

$$\Sigma_f \colon \mathsf{FibSets}(K) \to \mathsf{FibSets}(K'),$$

where

• *Action on Objects.* For each  $(X, \phi_X) \in \mathsf{Obj}(\mathsf{FibSets}(K))$ , we have

$$\Sigma_f(X, \phi_X) \stackrel{\text{def}}{=} (\Sigma_f(X), \Sigma_f(\phi_X));$$

• *Action on Morphisms.* For each  $(X, \phi_X), (Y, \phi_Y) \in \mathsf{Obj}(\mathsf{FibSets}(K))$ , the action on Hom-sets

$$\Sigma_{f|X,Y} \colon \mathsf{Hom}_{\mathsf{FibSets}(K)}(X,Y) \to \mathsf{Hom}_{\mathsf{FibSets}(K)}(\Sigma_f(X),\Sigma_f(Y))$$

of  $\Sigma_f$  at  $((X, \phi_X), (Y, \phi_Y))$  is the map sending a morphism of K-fibred sets

$$q: (X, \phi_X) \to (Y, \phi_Y)$$

to the morphism of K'-fibred sets defined by

$$\Sigma_f(g) \stackrel{\text{def}}{=} g.$$

2. Interaction With Fibres. We have a bijection of setSM

$$\Sigma_f(\phi_X)^{-1}(k') \cong \prod_{k \in f^{-1}(k')} \phi_X^{-1}(k)$$

for each  $k' \in K'$ .

$$\Sigma_f(\phi_X)^{-1}(x) \cong \coprod_{y \in f^{-1}(x)} \phi_X^{-1}(y);$$

see ?? of ??.

<sup>&</sup>lt;sup>3</sup>The name "dependent sum" comes from the fact that the fibre  $\Sigma_f(\phi_X)^{-1}(x)$  of  $\Sigma_f(X)$  at  $x \in K'$  is given by

<sup>&</sup>lt;sup>4</sup>Further Notation: Also written  $f_*(X)$ .

Proof. ??, Functoriality: Omitted.

??, Interaction With Fibres: Indeed, we have

$$\Sigma_{f}(\phi_{X})^{-1}(k') \stackrel{\text{def}}{=} \operatorname{pt} \times_{[k'], K', f \circ \phi_{X}} X$$

$$\cong \{x \in X \mid f(\phi_{X}(x)) = k'\}$$

$$\cong \coprod_{k \in f^{-1}(k')} \{x \in X \mid \phi_{X}(x) = k\}$$

$$\cong \coprod_{k \in f^{-1}(k')} \phi_{X}^{-1}(k)$$

$$\cong \coprod_{k \in f^{-1}(k')} \phi_{X}^{-1}(k)$$

for each  $k' \in K'$ .

## 2.3 Dependent Products

Let  $f: K \to K'$  be a function and let  $(X, \phi_X)$  be a K-fibred set.

• The Underlying Set. The set  $\Pi_f(X)$  defined by

$$\Pi_f(X) \stackrel{\text{def}}{=} \coprod_{k' \in K'} \prod_{k \in f^{-1}(k')} \phi_X^{-1}(k);$$

• *The Fibration*. The map of sets

$$\Pi_f(\phi_X) \colon \Pi_f(X) \to K'$$

defined by sending an element of

$$\Pi_f(X) \stackrel{\text{\tiny def}}{=} \coprod_{k' \in K'} \prod_{k \in f^{-1}(k')} \phi_X^{-1}(k)$$

to its index k' in K'.

**Example 2.3.1.2.** Here are some examples of dependent products of sets.

$$\Pi_f(\phi_X)^{-1}(k') \cong \prod_{k \in f^{-1}(k')} \phi_X^{-1}(k);$$

see ?? of ??.

<sup>&</sup>lt;sup>5</sup>The name "dependent product" comes from the fact that the fibre  $\Pi_f(\phi_X)^{-1}(k')$  of  $\Pi_f(X)$  at  $k' \in K'$  is given by

<sup>&</sup>lt;sup>6</sup> Further Notation: Also written  $f_!(X)$ .

<sup>&</sup>lt;sup>7</sup>We can also define dependent products via the internal **Hom** in FibSets(K'); see **??** of **??**.

1. *Spaces of Sections.* Let K = X,  $K' = \operatorname{pt}$ , let  $\mathfrak{D} \cap SE \to X$  be a map of sets, and write  $!_X : X \to \operatorname{pt}$  for the terminal map from X to  $\operatorname{pt}$ . We have a bijection of sets

$$\Pi_{!_X}((E,\phi)) \cong \Gamma_X(\phi)$$

$$\stackrel{\text{def}}{=} \{ h \in \mathsf{Sets}(X,E) \mid \phi \circ h = \mathrm{id}_X \}.$$

2. Function Spaces. Let  $K = K' = \operatorname{pt}$  and white  $S_X : X \to \operatorname{pt}$  and  $Y : Y \to \operatorname{pt}$  for the terminal maps from X and Y to  $\operatorname{pt}$ . We have a bijection of sets

$$\mathsf{Sets}(X,Y) \cong \Pi_{!_X}(!_X^*(Y,!_Y)).$$

*Proof.* ??, Spaces of Sections: Indeed, we have

$$\Pi_{!_X}((E,\phi)) \stackrel{\text{def}}{=} \coprod_{\star \in \text{pt}} \prod_{k \in !_X^{-1}(\star)} \phi_X^{-1}(k)$$

$$= \prod_{x \in X} \phi_X^{-1}(x)$$

$$\cong \{ h \in \text{Sets}(X, E) \mid \phi_X \circ h = \text{id}_X \}$$

$$\stackrel{\text{def}}{=} \Gamma_X(\phi).$$

??, Function Spaces: Indeed, we have

$$\begin{split} \Pi_{!_X}(!_X^*(Y,!_Y)) &\stackrel{\text{def}}{=} \Pi_{!_X}(X \times_{!_X,pt,!_Y} Y) \\ &\stackrel{\text{def}}{=} \coprod_{\star \in \text{pt}} \prod_{x \in !_X^{-1}(\star)} \text{pr}_1^{-1}(x) \\ &= \prod_{x \in X} Y \\ &\cong \text{Sets}(X,Y). \end{split}$$

This finishes the proof.

**Proposition 2.3.1.3.** Let  $f: K \to K'$  be a function.

1. Functoriality. The assignment  $X \mapsto \mathbb{Z}(X)$  defines a functor

$$\Pi_f : \mathsf{FibSets}(K) \to \mathsf{FibSets}(K'),$$

where

• Action on Objects. For each  $(X, \phi_X) \in \mathsf{Obj}(\mathsf{FibSets}(K))$ , we have

$$\Pi_f(X, \phi_X) \stackrel{\text{def}}{=} \Pi_f(X);$$

• *Action on Morphisms.* For each  $(X, \phi_X)$ ,  $(Y, \phi_Y) \in \mathsf{Obj}(\mathsf{FibSets}(K))$ , the action on Hom-sets

$$\Pi_{f|X,Y} \colon \mathrm{Hom}_{\mathsf{FibSets}(K)}(X,Y) \to \mathrm{Hom}_{\mathsf{FibSets}(K')}(\Pi_f(X),\Pi_f(Y))$$

of  $\Pi_f$  at  $((X, \phi_X), (Y, \phi_Y))$  is the map sending a morphism of K-fibred sets

$$\xi \colon (X, \phi_X) \to (Y, \phi_Y), \qquad X \xrightarrow{\xi} Y \\ \downarrow^{\phi_X} \qquad \downarrow^{\phi_Y} K$$

to the morphism

$$\Pi_f(\xi) \colon (\Pi_f(X), \Pi_f(\phi_X)) \to (\Pi_f(Y), \Pi_f(\phi_Y))$$

$$\Pi_f(\xi) \colon (\Pi_f(X), \Pi_f(\phi_X)) \to (\Pi_f(Y), \Pi_f(\phi_Y))$$

$$\Pi_f(\chi) \xrightarrow{\Pi_f(\xi)} \Pi_f(Y)$$

$$\Pi_f(\chi) \xrightarrow{\Pi_f(\phi_X)} \Pi_f(\chi)$$

of K'-fibred sets given by<sup>8</sup>

$$[\Pi_f(\xi)]((x_k)_{k\in f^{-1}(k')}) \stackrel{\text{def}}{=} (\xi(x_k))_{k\in f^{-1}(k')}$$

for each 
$$(x_k)_{k \in f^{-1}(k')} \in \Pi_f(X)$$
.

2. Interaction With Fibres. We have a bijection of sets V

$$\Pi_f(\phi_X)^{-1}(k') \cong \prod_{k \in f^{-1}(k')} \phi_X^{-1}(k);$$

for each  $k' \in K'$ .

3. Construction Using the Internal Hom. We have

$$\Pi_f(X,\phi_X) = (K' \times_{\mathbf{Hom}_{\mathsf{FibSets}(K')}((K,f),(K,f))} \mathbf{Hom}_{\mathsf{FibSets}(K')}((K,f),(X,f \circ \phi_X)), \mathrm{pr}_1),$$

$$\phi_Y(\xi(x_k)) = [\phi_Y \circ \xi](x_k)$$
$$= \phi_X(x_k)$$
$$- \iota$$

where we have used that  $\xi$  is a morphism of K-fibred sets for the second equality.

<sup>&</sup>lt;sup>8</sup>Note that we indeed have  $\xi(x_k) \in \phi_Y^{-1}(k)$ , since

forming a pullback diagram

$$\Pi_{f}(X,\phi_{X}) \xrightarrow{\operatorname{pr}_{2}} \operatorname{Hom}_{\operatorname{FibSets}(K')}((K,f),(X,f\circ\phi_{X}))$$

$$\downarrow^{\operatorname{pr}_{1}} \qquad \qquad \downarrow^{(\phi_{X})_{*}}$$

$$K' \xrightarrow{I} \operatorname{Hom}_{\operatorname{FibSets}(K')}((K,f),(K,f)),$$

where the bottom map is given by

$$I(k') \stackrel{\text{def}}{=} \mathrm{id}_{f^{-1}(k')}$$

for each  $k' \in K'$  and where  $\mathbf{Hom}_{\mathsf{FibSets}(K')}$  denotes the internal Hom of  $\mathsf{FibSets}(K')$  of  $\ref{fibSets}$ .

4. Internal Homs via Dependent Products. We have

$$\mathbf{Hom}_{\mathsf{FibSets}(K)}(X,Y) \cong \Pi_{\phi_X}(\phi_X^*(Y)).$$

00SX

Proof. ??, Functoriality: Omitted.

??, Interaction With Fibres: Clear.

??, Construction Using the Internal Hom: Using the explicit formula for pullbacks of sets given in Constructions With Sets, Definition 1.3.1.1, we see that the pullback

$$K' \times_{\mathbf{Hom}_{\mathsf{FibSets}(K')}((K,f),(K,f))} \mathbf{Hom}_{\mathsf{FibSets}(K')}((K,f),(X,f \circ \phi_X))$$

is given by

$$\bigg\{(k',h)\in \coprod_{k'\in K'} \mathsf{Sets}(f^{-1}(k'),\phi_X^{-1}(f^{-1}(k')))\,\bigg|\,\phi_X\circ h=\mathrm{id}_{f^{-1}(k')}\bigg\},$$

which is isomorphic to

$$\coprod_{k' \in K'} \bigl\{ h \in \mathsf{Sets}(f^{-1}(k'), \phi_X^{-1}(f^{-1}(k'))) \, \big| \, \phi_X \circ h = \mathrm{id}_{f^{-1}(k')} \bigr\}.$$

We claim that

$$\left\{h \in \mathsf{Sets}(f^{-1}(k'), \phi_X^{-1}(f^{-1}(k'))) \,\middle|\, \phi_X \circ h = \mathrm{id}_{f^{-1}(k')}\right\} \cong \prod_{k \in f^{-1}(k')} \phi_X^{-1}(k),$$

so that the pullback is indeed given by  $\Pi_f(X)$ . There are two cases:

2.4 Internal Homs 12

1. If  $f^{-1}(k') = \emptyset$ , then there is only one map from  $f^{-1}(k')$  to  $\phi_X^{-1}(f^{-1}(k'))$  (the inclusion), so  $\mathsf{Sets}(f^{-1}(k'), \phi_X^{-1}(f^{-1}(k'))) \cong \mathsf{pt}$ . Since products indexed by the empty set are isomorphic to  $\mathsf{pt}$ , the isomorphism follows.

2. Otherwise, by the condition  $\phi_X \circ h = \mathrm{id}_{f^{-1}(k')}$ , it follows that, for each  $k \in f^{-1}(k')$ , we must have

$$\phi_X(h(k)) = k$$

and thus  $h(k) \in \phi_X^{-1}(k)$ . Therefore, a map from  $f^{-1}(k')$  to  $\phi_X^{-1}(f^{-1}(k'))$  consists of a choice of an element from  $\phi_X^{-1}(k)$  for each  $k \in f^{-1}(k')$ , which is precisely given by an element of the product  $\prod_{k \in f^{-1}(k')} \phi_X^{-1}(k)$ , showing the bijection to be true.

??, Internal Homs via Dependent Products: Indeed we have

$$\begin{split} \Pi_{\phi_X}(\phi_X^*(Y)) &\stackrel{\text{def}}{=} \Pi_{\phi_X}(X \times_K Y) \\ &\stackrel{\text{def}}{=} \coprod_{k \in K} \prod_{x \in \phi_X^{-1}(k)} \operatorname{pr}_1^{-1}(x) \\ &\cong \coprod_{k \in K} \prod_{x \in \phi_X^{-1}(k)} \{y \in Y \mid \phi_X(x) = \phi_Y(y)\} \\ &\cong \coprod_{k \in K} \prod_{x \in \phi_X^{-1}(k)} \{y \in Y \mid k = \phi_Y(y)\} \\ &\cong \coprod_{k \in K} \prod_{x \in \phi_X^{-1}(k)} \phi_Y^{-1}(k) \\ &\cong \coprod_{k \in K} \operatorname{Sets}(\phi_X^{-1}(k), \phi_Y^{-1}(k)) \\ &\stackrel{\text{def}}{=} \operatorname{\mathbf{Hom}}_{\mathsf{FibSets}(K)}(X, Y). \end{split}$$

This finishes the proof.

#### 2.4 Internal Homs OSY

Let *K* be a set and let  $(X, \phi_X)$  and  $(Y, \phi_Y)$  be *K*-fibred sets.

**Definition 2.4.1.1.** The internal Hom of fibred sets from  $(X, \phi_X)$  to  $(Y, \phi_Y)$  is the fibred set  $\mathbf{Hom}_{\mathsf{FibSets}(K)}(X, Y)$  consisting of

• The Underlying Set. The set  $\mathbf{Hom}_{\mathsf{FibSets}(K)}(X,Y)$  defined by

$$\mathbf{Hom}_{\mathsf{FibSets}(K)}(X,Y) \stackrel{\text{def}}{=} \coprod_{k \in K} \mathsf{Sets}(\phi_X^{-1}(k), \phi_Y^{-1}(k));$$

• The Fibration. The map of sets<sup>9</sup>

$$\phi_{\mathbf{Hom}_{\mathsf{FibSets}(K)}(X,Y)} \colon \underbrace{\mathbf{Hom}_{\mathsf{FibSets}(K)}(X,Y)}_{k \in K} \to K$$

defined by sending a map  $f \colon \phi_X^{-1}(k) \to \phi_Y^{-1}(k)$  to its index  $k \in K$ .

Proof. Omitted.

**Proposition 2.4.1.2.** Let K be a set and let (M)  $(Y, \phi_Y)$  and  $(Y, \phi_Y)$  be K-fibred sets.

- 1. Functoriality. Let  $(X, \phi_X)$  and  $(Y, \phi_Y)$  def K-fibred sets.
  - (a) The assignment  $X \mapsto \mathbf{Hom}_{\mathsf{FibSets}(K)}(X, Y)$  defines a functor

 $\mathbf{Hom}_{\mathsf{FibSets}(K)}(X, -) \colon \mathsf{FibSets}(K) \to \mathsf{FibSets}(K).$ 

(b) The assignment  $Y \mapsto \mathbf{Hom}_{\mathsf{FibSets}(K)}(X, Y)$  defines a functor

 $\mathbf{Hom}_{\mathsf{FibSets}(K)}(\mathsf{-},Y) \colon \mathsf{FibSets}(K)^{\mathsf{op}} \to \mathsf{FibSets}(K).$ 

(c) The assignment  $(X, Y) \mapsto \mathbf{Hom}_{\mathsf{FibSets}(K)}(X, Y)$  defines a functor

 $\mathbf{Hom}_{\mathsf{FibSets}(K)}(\mathsf{-}_1,\mathsf{-}_2) \colon \mathsf{FibSets}(K)^{\mathsf{op}} \times \mathsf{FibSets}(K) \to \mathsf{FibSets}(K).$ 

2. Internal Homs via Dependent Products. We have 00T2

$$\mathbf{Hom}_{\mathsf{FibSets}(K)}(X,Y) \cong \Pi_{\phi_X}(\phi_X^*(Y)).$$

*Proof.* ??, Functoriality: Omitted.

??, Internal Homs via Dependent Products: This was proved in ?? of ??.

## 2.5 Adjointness for Fibred Sets

Let  $f: K \to K'$  be a map of sets.

$$\phi_{\mathbf{Hom}_{\mathsf{FibSets}(K)}(X,Y)|k} \cong \mathsf{Sets}(\phi_X^{-1}(k),\phi_Y^{-1}(k))$$

for each  $k \in K$ .

<sup>&</sup>lt;sup>9</sup>The fibres of the internal **Hom** of FibSets(K) are precisely the sets Sets( $\phi_X^{-1}(k), \phi_Y^{-1}(k)$ ), i.e. we have

#### **Proposition 2.5.1.1.** We have a triple adjunction

$$(\Sigma_f\dashv f^*\dashv \Pi_f)\colon \ \mathsf{FibSets}(K) \underbrace{-f^*-}_{\Pi_f} \mathsf{FibSets}(K').$$

We offer two proofs. The first uses the corresponding adjunction for indexed sets (Indexed Sets, Proposition 4.5.1.1) and the un/straightening equivalence together with its compatibility with dependent sums and products to "transfer" the adjunction to fibred sets, while the second is a direct proof.

*Proof.* The Adjunction  $\Sigma_f \dashv f^*$ : The adjunction

$$(\Sigma_f \dashv f^*)$$
:  $\mathsf{ISets}(K)$ 
 $f^*$ 
 $f$ 
 $f$ 

of Indexed Sets, Proposition 4.5.1.1 gives a unit and counit of the form

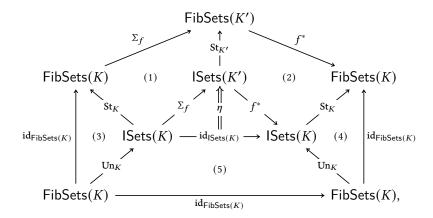
$$\eta: \mathrm{id}_{\mathsf{ISets}(K)} \Longrightarrow \Sigma_f \circ f^*,$$
 $\epsilon: f^* \circ \Sigma_f \Longrightarrow \mathrm{id}_{\mathsf{ISets}(K')}.$ 

With these in hand, we construct natural transformations

$$\eta' : \mathrm{id}_{\mathsf{FibSets}(K)} \Longrightarrow \Sigma_f \circ f^*,$$
  
 $\epsilon' : f^* \circ \Sigma_f \Longrightarrow \mathrm{id}_{\mathsf{FibSets}(K')}$ 

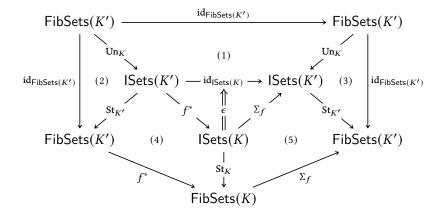
as follows:

1. The Unit. We define  $\eta' : \mathrm{id}_{\mathsf{FibSets}(K)} \Longrightarrow \Sigma_f \circ f^*$  as the pasting of the diagram



#### where:

- (a) Subdiagram (1) commutes by Un/Straightening for Indexed and Fibred Sets, ?? of ??.
- (b) Subdiagram (2) commutes by Un/Straightening for Indexed and Fibred Sets, ?? of ??.
- (c) Subdiagram (3) commutes by Un/Straightening for Indexed and Fibred Sets, ??.
- (d) Subdiagram (4) commutes by Un/Straightening for Indexed and Fibred Sets, ??.
- (e) Subdiagram (5) commutes by unitality of composition.
- 2. The Counit. We define  $\epsilon' \colon f^* \circ \Sigma_f \Longrightarrow \mathrm{id}_{\mathsf{FibSets}(K')}$  as the pasting of the diagram



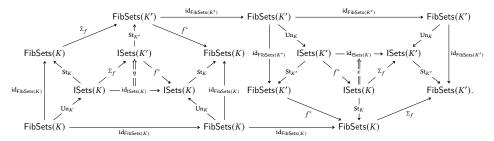
#### where:

- (a) Subdiagram (1) commutes by unitality of composition.
- (b) Subdiagram (2) commutes by Un/Straightening for Indexed and Fibred Sets, ??.
- (c) Subdiagram (3) commutes by Un/Straightening for Indexed and Fibred Sets, ??.
- (d) Subdiagram (4) commutes by Un/Straightening for Indexed and Fibred Sets, ?? of ??.
- (e) Subdiagram (5) commutes by Un/Straightening for Indexed and Fibred Sets, ?? of ??.

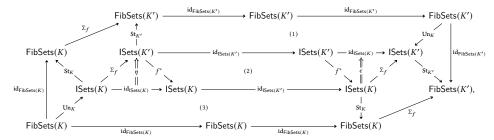
## Next, we prove the left triangle identity,



## whose left side in our case looks like this:



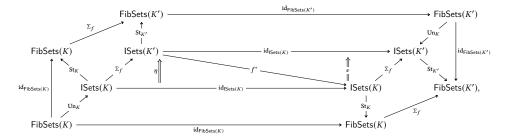
#### It can be rearranged into



#### where:

- 1. Subdiagram (1) commutes by Un/Straightening for Indexed and Fibred Sets, ??.
- 2. Subdiagram (2) commutes by unitality of composition.
- 3. Subdiagram (3) commutes by Un/Straightening for Indexed and Fibred Sets, ??.

## And then, it can be rearranged into



which by the left triangle identity for  $(\eta, \epsilon)$ , becomes



finishing the proof of the left triangle identity. The proof of the right triangle identity is similar, and is thus omitted.

The Adjunction  $f^* \dashv \Pi_f$ : This proof is similar to the proof of the adjunction  $\Sigma_f \dashv f^*$ , and is thus omitted.

We proceed to the direct proof of ??.

*Proof.* The Adjunction  $\Sigma_f \dashv f^*$ : We claim there's a bijection

$$\operatorname{Hom}_{\mathsf{FibSets}(K')}(\Sigma_f(X), Y) \cong \operatorname{Hom}_{\mathsf{FibSets}(K)}(X, f^*(Y)),$$

natural in  $(X, \phi_X) \in \mathsf{Obj}(\mathsf{FibSets}(K))$  and  $(Y, \phi_Y) \in \mathsf{Obj}(\mathsf{FibSets}(K'))$ :

• Map I. We define a map

$$\Phi_{X,Y} \colon \operatorname{Hom}_{\mathsf{FibSets}(K')}(\Sigma_f(X), Y) \to \operatorname{Hom}_{\mathsf{FibSets}(K)}(X, f^*(Y)),$$

by sending a morphism

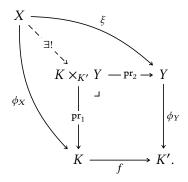
$$\xi \colon \Sigma_f(X) \to Y, \qquad X \xrightarrow{\xi} Y \\ \phi_X \\ K \\ f \\ K'$$

of K'-fibred sets to the morphism

$$\xi^{\dagger} \colon X \to f^*(Y), \quad X \xrightarrow{\xi^{\dagger}} K \times_{K'} Y$$

$$\downarrow f^*(Y), \qquad \downarrow f^*(Y), \qquad \downarrow f^*(Y)$$

of K-fibred sets given by the dashed morphism in the diagram



## • Map II. We define a map

$$\Psi_{X,Y} \colon \mathrm{Hom}_{\mathsf{FibSets}(K)}(X, f^*(Y)) \to \mathrm{Hom}_{\mathsf{FibSets}(K')}(\Sigma_f(X), Y),$$
 given by sending a map

$$\xi \colon X \to f^*(Y), \qquad X \xrightarrow{\xi} K \times_{K'} Y$$

$$\phi_X \swarrow pr_1$$

of K'-fibred sets to the map

$$\xi^{\dagger} \colon \Sigma_{f}(X) \to Y, \qquad \begin{matrix} X & \xrightarrow{\xi^{\dagger}} & Y \\ \phi_{X} & & \\ K & & \\ K' & & \end{matrix} \phi_{Y}$$

of K-fibred sets given by

$$\xi^{\dagger} \stackrel{\text{def}}{=} \operatorname{pr}_2 \circ \xi,$$

where we indeed have

$$\begin{split} \phi_Y \circ (\operatorname{pr}_2 \circ \xi) &= (\phi_Y \circ \operatorname{pr}_2) \circ \xi \\ &= (f \circ \operatorname{pr}_1) \circ \xi \qquad \text{(by the pullback square of } K \times_{K'} Y) \\ &= f \circ (\operatorname{pr}_1 \circ \xi) \\ &= f \circ \phi_X. \qquad \text{(since } \xi \text{ is a morphism of } K'\text{-fibred sets)} \end{split}$$

• Naturality I. We need to show that, given a morphism

$$\alpha \colon (X, \phi_X) \to (X', \phi_{X'})$$

of K-fibred sets, the diagram

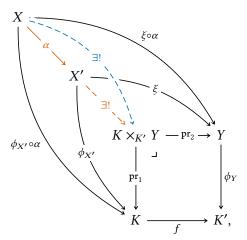
$$\begin{array}{c|c} \operatorname{Hom}_{\mathsf{FibSets}(K')}(\Sigma_f(X'),Y) & \xrightarrow{\Phi_{X',Y}} \operatorname{Hom}_{\mathsf{FibSets}(K)}(X',f^*(Y)), \\ & & \downarrow \alpha^* \\ \\ \operatorname{Hom}_{\mathsf{FibSets}(K')}(\Sigma_f(X),Y) & \xrightarrow{\Phi_{X,Y}} \operatorname{Hom}_{\mathsf{FibSets}(K)}(X,f^*(Y)) \end{array}$$

commutes. Indeed, given a morphism

$$\xi \colon \Sigma_f(X') \to Y, \qquad X' \xrightarrow{\xi} Y$$

$$K \downarrow \phi_{X'} \downarrow K \downarrow \phi_{Y}$$

of K'-fibred-sets, the map  $\Phi_{X',Y}(\xi) \circ \alpha$  is the composition, coloured in vermillion, of the dashed arrow with  $\alpha$  in the diagram



while  $\Phi_{X,Y}(\xi \circ \Sigma_f(\alpha))$  is given by the dashed arrow, coloured in blue. Since both the blue arrow and the vermillion arrow make the outer pullback diagram commute, it follows from the universal property of the pullback that they must be equal, i.e. that

$$\Phi_{X',Y}(\xi) \circ \alpha = \Phi_{X,Y}(\xi \circ \Sigma_f(\alpha)),$$

showing that the naturality diagram above indeed commutes.

• Naturality II. We need to show that, given a morphism

$$\beta \colon (Y, \phi_Y) \to (Y', \phi_{Y'})$$

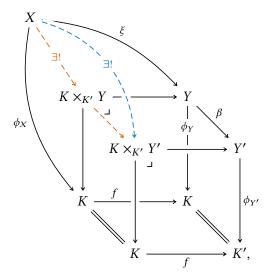
of K-fibred sets, the diagram

$$\begin{array}{c|c} \operatorname{Hom}_{\mathsf{FibSets}(K')}(\Sigma_f(X),Y) & \xrightarrow{\Phi_{X,Y}} & \operatorname{Hom}_{\mathsf{FibSets}(K)}(X,f^*(Y)), \\ & & \downarrow f^*(\beta)_* \\ \\ \operatorname{Hom}_{\mathsf{FibSets}(K')}(\Sigma_f(X),Y') & \xrightarrow{\Phi_{X,Y'}} & \operatorname{Hom}_{\mathsf{FibSets}(K)}(X,f^*(Y')) \end{array}$$

commutes. Indeed, given a morphism

$$\xi \colon \Sigma_f(X') \to Y, \qquad X' \xrightarrow{\xi} Y \\ K \swarrow \phi_{X'} \searrow K \swarrow \phi_{Y}$$

of K'-fibred-sets, the map  $f^*(\beta) \circ \Phi_{X,Y}(\xi)$  is the composition, coloured in vermillion, of the dashed arrow from X to  $K \times_{K'} Y$  with the dashed arrow from  $K \times_{K'} Y$  to  $K \times_{K'} Y'$  in the diagram



while  $\Phi_{X,Y'}(\beta \circ \xi)$  is given by the dashed arrow from X to  $K \times_{K'} Y'$ , coloured in blue. Since both the blue arrow and the vermillion arrow make the outer pullback

diagram for  $K \times_{K'} Y'$  commute, it follows from the universal property of the pullback that they must be equal, i.e. that

$$f^*(\beta) \circ \Phi_{X,Y}(\xi) = \Phi_{X,Y'}(\beta \circ \xi),$$

showing that the naturality diagram above indeed commutes.

• Invertibility I. We claim that

$$\Psi_{X,Y} \circ \Phi_{X,Y} = \mathrm{id}_{\mathrm{Hom}_{\mathsf{FibSets}(K')}(\Sigma_f(X),Y)}.$$

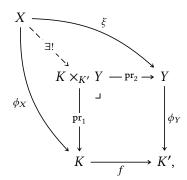
Indeed,  $\Phi_{X,Y}$  sends a map

$$\xi \colon \Sigma_f(X) \to Y, \qquad X \xrightarrow{\xi} Y$$

$$K \downarrow \phi_X \downarrow \phi_Y$$

$$K \downarrow \chi \downarrow \phi_Y$$

of K'-fibred sets to the dashed morphism in the diagram



and  $\Psi_{X,Y}$  then postcomposes that map with  $\operatorname{pr}_2$ , which, by the commutativity of the diagram above, is  $\xi$  again, showing the claimed equality to be true.

• Invertibility II. We claim that

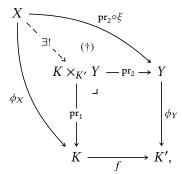
$$\Phi_{X,Y} \circ \Psi_{X,Y} = \mathrm{id}_{\mathrm{Hom}_{\mathsf{FibSets}(K)}(X,f^*(Y))}.$$

Indeed,  $\Psi_{X,Y}$  sends a map

$$\xi \colon X \to f^*(Y), \qquad X \xrightarrow{\xi} K \times_{K'} Y$$

$$\phi_X \qquad pr_1$$

of K'-fibred sets to  $\operatorname{pr}_2 \circ \xi$ , which is then sent by  $\Phi_{X,Y}$  to the dashed morphism in the diagram



which, by the commutativity of the subdiagram marked with  $(\dagger)$ , is given by  $\xi$  again, showing the claimed equality to be true.

The Adjunction  $f^* \dashv \Pi_f$ : We claim there's a bijection

$$\operatorname{Hom}_{\mathsf{FibSets}(K)}(f^*(X), Y), \cong \operatorname{Hom}_{\mathsf{FibSets}(K')}(X, \Pi_f(Y))$$

natural in  $(X, \phi_X) \in \text{Obj}(\mathsf{FibSets}(K'))$  and  $(Y, \phi_Y) \in \text{Obj}(\mathsf{FibSets}(K))$ :

1. Map I. We define a map

$$\Phi_{X,Y} \colon \operatorname{Hom}_{\mathsf{FibSets}(K)}(f^*(X), Y) \to \operatorname{Hom}_{\mathsf{FibSets}(K')}(X, \Pi_f(Y))$$

defined as follows. Given a morphism

$$\xi \colon f^*(X) \to Y, \qquad K \times_{K'} X \xrightarrow{\xi} Y$$

$$pr_1 \swarrow_{\phi_Y} \downarrow_{\phi_Y}$$

of *K*-fibred sets, where

$$f^*(X) \stackrel{\text{def}}{=} K \times_{K'} X$$

$$\stackrel{\text{def}}{=} \{(k, x) \in K \times X \mid f(k) = \phi_X(x)\},\$$

we construct a morphism

$$\xi^{\dagger} \colon X \to \Pi_f(Y), \qquad X \xrightarrow{\xi^{\dagger}} \Pi_f(Y)$$

$$\downarrow \phi_X \qquad \bigwedge_{K'} \Pi_f(\phi_Y)$$

of K'-fibred sets, where

$$\Pi_f(Y) \stackrel{\text{\tiny def}}{=} \coprod_{k' \in K'} \prod_{k \in f^{-1}(k')} \phi_Y^{-1}(k),$$

by defining

$$\xi^{\dagger}(x) \stackrel{\text{def}}{=} (\xi(k,x))_{k \in f^{-1}(\phi_X(x))}$$

for each  $x \in X$ . There are two things to be checked here:

- We have  $\xi(k,x) \in \phi_Y^{-1}(\phi_X(x))$  since  $\phi_Y(\xi(k,x)) = \phi_X(x)$  as  $\xi$  is a morphism of K-fibred sets.
- The map  $\xi^{\dagger}$  is indeed a morphism of K'-fibred sets, i.e. we have

$$\Pi_f(\phi_Y) \circ \xi^{\dagger} = \phi_X,$$

since

$$[\Pi_f(\phi_Y)]\Big((\xi(k,x))_{k\in f^{-1}(\phi_X(x))}\Big) = \phi_X(x)$$

for each  $x \in X$ .

2. Map II. We define a map

$$\Psi_{X,Y} \colon \operatorname{Hom}_{\mathsf{FibSets}(K')}(X,\Pi_f(Y)) \to \operatorname{Hom}_{\mathsf{FibSets}(K)}(f^*(X),Y)$$

as follows. Given a morphism

$$\xi \colon X \to \Pi_f(Y), \qquad X \xrightarrow{\xi} \Pi_f(Y)$$

$$\phi_X \swarrow \Pi_f(\phi_Y)$$

$$K'$$

of K'-fibred sets, where

$$\Pi_f(Y) \stackrel{\text{def}}{=} \coprod_{k' \in K'} \prod_{k \in f^{-1}(k')} \phi_Y^{-1}(k),$$

we construct a morphism

$$\xi^{\dagger} \colon f^*(X) \to Y, \qquad K \times_{K'} X \xrightarrow{\xi^{\dagger}} Y$$

$$pr_1 \searrow \phi_Y$$

$$K$$

of *K*-fibred sets, where

$$f^*(X) \stackrel{\text{def}}{=} K \times_{K'} X$$

$$\stackrel{\text{def}}{=} \{(k, x) \in K \times X \mid f(k) = \phi_X(x)\},$$

by defining

$$\xi^{\dagger}(k,x) \stackrel{\text{def}}{=} \xi(x)_k$$

for each  $(k,x) \in f^*(X)$ , where  $\xi(x)_k$  is the kth component of  $\xi(x) = (y_k)_{k \in f^{-1}(k')}$ . We also need to check that  $\xi^{\dagger}$  is a morphism of K-fibred sets, i.e. that

$$\phi_Y \circ \xi^{\dagger} = \mathrm{pr}_1,$$

or

$$\phi_Y(\xi^{\dagger}(k,x)) = k,$$

for each  $(k,x)\in f^*(X)$ , which is clear, since  $\xi^\dagger(k,x)\in\phi_Y^{-1}(k)$  by definition.

3. Naturality I. We need to show that, given a morphism

$$\alpha \colon (X, \phi_X) \to (X', \phi_{X'})$$

of K'-fibred sets, the diagram

commutes. Indeed, given a morphism  $\xi \colon f^*(X') \to Y$  of K'-fibred sets, we have

$$\begin{split} \big[ \big[ \Phi_{X,Y} \circ f^*(\alpha) \big](\xi) \big](x) & \stackrel{\text{def}}{=} \big[ \Phi_{X,Y}(\xi \circ f^*(\alpha)) \big](x) \\ & \stackrel{\text{def}}{=} \big( \big[ \xi \circ f^*(\alpha) \big](k,x) \big)_{k \in f^{-1}(\phi_X(x))} \\ & \stackrel{\text{def}}{=} \big( \xi(k,\alpha(x)) \big)_{k \in f^{-1}(\phi_X(x))} \\ & \stackrel{\text{def}}{=} \alpha^*((\xi(k,x))_{k \in f^{-1}(\phi_X(x))}) \\ & \stackrel{\text{def}}{=} \alpha^*(\xi^\dagger(x)) \\ & \stackrel{\text{def}}{=} \big[ \big[ \alpha^* \circ \Phi_{X,Y} \big](\xi) \big](x) \end{split}$$

for each  $x \in X$ .

4. Naturality II. We need to show that, given a morphism

$$\beta \colon (Y, \phi_Y) \to (Y', \phi_{Y'})$$

of K-fibred sets, the diagram

commutes. Indeed, given a morphism  $\xi \colon X \to \Pi_f(Y)$  of K-fibred sets, we have

$$\begin{split} \big[ \big[ \Phi_{X,Y'} \circ \beta_* \big] (\xi) \big] (x) & \stackrel{\text{def}}{=} \big[ \Phi_{X,Y'} (\beta \circ \xi) \big] (x) \\ & \stackrel{\text{def}}{=} \big[ \Phi_{X,Y'} (\beta \circ \xi) \big] (x) \\ & \stackrel{\text{def}}{=} \big( \big[ \beta \circ \xi \big] (k,x) \big)_{k \in f^{-1}(\phi_X(x))} \\ & \stackrel{\text{def}}{=} \big( \beta (\xi(k,x)) \big)_{k \in f^{-1}(\phi_X(x))} \\ & \stackrel{\text{def}}{=} \Pi_f(\beta)_* ((\xi(k,x))_{k \in f^{-1}(\phi_X(x))}) \\ & \stackrel{\text{def}}{=} \big[ \Pi_f(\beta)_* \circ \xi^{\dagger} \big] (x) \\ & \stackrel{\text{def}}{=} \big[ \Pi_f(\beta)_* \circ \Phi_{X,Y'}(\xi) \big] (x) \end{split}$$

for each  $x \in X$ .

5. Invertibility I. We claim that

$$\Psi_{X,Y} \circ \Phi_{X,Y} = \mathrm{id}_{\mathsf{Hom}_{\mathsf{FibSets}(K)}}(f^*(X),Y).$$

Indeed, given a morphism  $\xi \colon f^*(X') \to Y$  of K'-fibred sets, we have

$$\begin{split} \big[ \big[ \big[ \Psi_{X,Y} \circ \Phi_{X,Y} \big] (\xi) \big] (k,x) &\stackrel{\text{def}}{=} \big[ \Psi_{X,Y} (\Phi_{X,Y} (\xi)) \big] (k,x) \\ &\stackrel{\text{def}}{=} \big( \big[ \Phi_{X,Y} (\xi) \big] (x) \big)_k \\ &\stackrel{\text{def}}{=} \big( (\xi (k_1,x))_{k_1 \in f^{-1} (\phi_X (x))} \big)_k \\ &\stackrel{\text{def}}{=} \xi (k,x) \end{split}$$

for each  $(k, x) \in f^*(X)$ , and thus the stated equality follows.

6. Invertibility II. We claim that

$$\Phi_{X,Y} \circ \Psi_{X,Y} = \mathrm{id}_{\mathsf{Hom}_{\mathsf{FibSets}(K')}(X,\Pi_f(Y))}.$$

Indeed, given a morphism  $\xi \colon X \to \Pi_f(Y)$  of K-fibred sets, write

$$\xi(x) = (y_k)_{k \in f^{-1}(k'_k)}.$$

We then have

$$\begin{split} \big[ \big[ \Phi_{X,Y} \circ \Psi_{X,Y} \big] (\xi) \big] (x) & \stackrel{\text{def}}{=} \big[ \Phi_{X,Y} (\Psi_{X,Y} (\xi)) \big] (x) \\ & \stackrel{\text{def}}{=} \big( \big[ \Psi_{X,Y} (\xi) \big] (k_1, x) \big)_{k_1 \in f^{-1} (\phi_X (x))} \\ & \stackrel{\text{def}}{=} \big( \big( \xi(x) \big)_{k_1} \big)_{k_1 \in f^{-1} (\phi_X (x))} \\ & \stackrel{\text{def}}{=} \big( \big( (y_k)_{k \in f^{-1} (k_x')} \big)_{k_1} \big)_{k_1 \in f^{-1} (\phi_X (x))} \\ & \stackrel{\text{def}}{=} \big( y_{k_1} \big)_{k_1 \in f^{-1} (\phi_X (x))} \\ & = \big( y_{k_1} \big)_{k_1 \in f^{-1} (k_x')} \\ & = \big( y_k \big)_{k \in f^{-1} (k_x')} \\ & \stackrel{\text{def}}{=} \xi(x) \end{split}$$

for each  $x \in X$ , where the equality  $\phi_X(x) = k_x'$  follows from the fact that  $\xi$  is a morphism of K'-fibred sets. Thus the stated equality follows.

This finishes the proof.

# **Appendices**

# A Other Chapters

### Sets

- 1. Sets
- 2. Constructions With Sets
- 3. Pointed Sets
- 4. Tensor Products of Pointed Sets
- 5. Relations
- 6. Spans
- 7. Posets

#### **Indexed and Fibred Sets**

- 7. Indexed Sets
- 8. Fibred Sets
- 9. Un/Straightening for Indexed and Fibred Sets

## **Category Theory**

- 11. Categories
- 12. Types of Morphisms in Categories
- 13. Adjunctions and the Yoneda Lemma
- 14. Constructions With Categories
- 15. Kan Extensions

| Bicategories                        | 30. Hypermonoids                       |
|-------------------------------------|--|
| 17. Bicategories                    | 31. Hypergroups                        |
| 18. Internal Adjunctions            | 32. Hypersemirings and Hyperrings      |
| Internal Category Theory            | 33. Quantales                          |
| 19. Internal Categories             | Near-Rings                             |
| Cyclic Stuff                        | 34. Near-Semirings                     |
| 20. The Cycle Category              | 35. Near-Rings                         |
| <b>Cubical Stuff</b>                | Real Analysis                          |
| 21. The Cube Category               | 36. Real Analysis in One Variable      |
| Globular Stuff                      | 37. Real Analysis in Several Variables |
| 22. The Globe Category              | Measure Theory                         |
| Cellular Stuff                      | 38. Measurable Spaces                  |
| 23. The Cell Category               | 39. Measures and Integration           |
| Monoids                             | Probability Theory                     |
| 24. Monoids                         | 39. Probability Theory                 |
| 25. Constructions With Monoids      | Stochastic Analysis                    |
| Monoids With Zero                   | 40. Stochastic Processes, Martingales, |
| 26. Monoids With Zero               | and Brownian Motion                    |
| 27. Constructions With Monoids With | 41. Itô Calculus                       |
| Zero                                | 42. Stochastic Differential Equations  |
| Groups                              | Differential Geometry                  |
| 28. Groups                          | 43. Topological and Smooth Manifolds   |

**Schemes** 

44. Schemes

29. Constructions With Groups

Hyper Algebra