Spans

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This chapter contains some material about spans. Notably, we discuss and explore:

1. The basic definitions around spans (Section 1);
2. The relation between spans and functions (Proposition 7.1.1.1);
3. The relation between spans and relations (Propositions 7.2.2.1 and 7.3.1.1 and Remark 7.5.1.1).
4. "Hyperpointed sets" (**??**). I don't know why I wrote this...

TODO:

1. internal adjoint equivalences in **Rel**
2. internal adjoint equivalences in Span
3. 2-categorical limits in **Rel**;
4. morphism of internal adjunctions in **Rel**;
5. morphism of internal adjunctions in Span;
6. morphism of co/monads in Span;
7. What is $\mathrm{Adj}(\mathsf{Span}(A, B))$?
8. monoids, comonoids, pseudomonoids, etc. in Span.
9. write down the dumb intuition about spans inducing morphisms $\mathrm{Sets}(S, A) \to \mathrm{Sets}(S, B)$ instead of $\mathcal{P}(A) \to \mathcal{P}(B)$ from the similarity between

$$S \to A \times B$$

and

$$A \times B \to \{\mathrm{t}, \mathrm{f}\}.$$

This intuition is justified by taking $A = \mathrm{pt}$ or $B = \mathrm{pt}$.

10. What about using the direct image with compact support in $g\big(f^{-1}(a)\big)$?
11. Monads in Span | develop this in the level of morphisms too
12. Comonads in Span are spans whose legs are equal | develop this in the level of morphisms too
13. Does Span have an internal **Hom**?
14. Examples of spans
15. Functional and total spans
16. closed symmetric monoidal category of spans
17. double category of relations
18. collage of a span
19. equivalence spans?
20. functoriality of powersets for spans
21. Is Span a closed bicategory?
22. skew monoidal structure on Span(A, B)
23. Adjunctions in Span
24. Isomorphisms in Span
25. Equivalences in Span
26. Interaction between the above notions in Span vs.in **Rel** via the comparison functors

Contents

1 Spans

1.1 The Walking Span

Definition 1.1.1.1. The **walking span** is the category Λ that looks like this:

$$\begin{tikzcd}
 & {[0]} \arrow[loop above] \arrow[dl] \arrow[dr] & \\
{[-1]} \arrow[loop left] & & {[1]} \arrow[loop right]
\end{tikzcd}$$

1.2 Spans

Let A and B be sets.

Definition 1.2.1.1. A **span from A to B**[1] is a functor $F\colon \Lambda \longrightarrow \mathsf{Sets}$ such that

$$\begin{aligned} F([-1]) &= A, \\ F([1]) &= B. \end{aligned}$$

Remark 1.2.1.2. In detail, a **span from A to B** is a triple (S, f, g) consisting of[2,3]

- *The Underlying Set.* A set S, called the **underlying set of** (S, f, g);
- *The Legs.* A pair of functions $f\colon S \longrightarrow A$ and $g\colon S \longrightarrow B$.

[1] *Further Terminology:* Also called a **roof from A to B** or a **correspondence from A to B**.

[2] *Picture:*

$$\begin{tikzcd}
 & S \arrow[dl, "f"'] \arrow[dr, "g"] & \\
A & & B.
\end{tikzcd}$$

[3] Every span (S, f, g) from A to B determines in particular a relation $R\colon A \nrightarrow B$ via

$$R \overset{\text{def}}{=} \{(f(a), g(a)) \mid a \in A\},$$

i.e. where $R(a) = g\left(f^{-1}(a)\right)$ for each $a \in A$; see Proposition 7.2.2.1.

1.3 Morphisms of Spans

Definition 1.3.1.1. A **morphism of spans** (R, f_1, g_1) **to** (S, f_2, g_2)[4] is a natural transformation $(R, f_1, g_1) \Longrightarrow (S, f_2, g_2)$.

Remark 1.3.1.2. In detail, a **morphism of spans from** (R, f_1, g_1) **to** (S, f_2, g_2) is a function $\phi\colon R \to S$ making the diagram[5]

$$\begin{tikzcd}
 & R \arrow[dl, "f_1"'] \arrow[dr, "g_1"] \arrow[rrr, "\phi"] & & & S \arrow[dl, "f_2"'] \arrow[dr, "g_2"] & \\
A \arrow[rrr, equal] & & B \arrow[rrr, equal] & A & & B.
\end{tikzcd}$$

commute.

1.4 Functional Spans

Let $\lambda = \left(A \xleftarrow{f} S \xrightarrow{g} B\right)$ be a span. A morphism of spans from id_A to $\lambda \diamond \lambda^{\dagger}$ is a morphism

$$s\colon A \to S \times_B S$$

making the diagram

$$\begin{tikzcd}
 & A \arrow[dl, equal, "\mathrm{id}_A"'] \arrow[dr, equal, "\mathrm{id}_A"] \arrow[dd, "s"] & \\
A & & A \\
 & S \times_B S \arrow[ul, "f \circ \mathrm{pr}'_1"] \arrow[ur, "f \circ \mathrm{pr}'_2"'] &
\end{tikzcd}$$

[4] *Further Terminology:* Also called a **morphism of roofs from** (R, f_1, g_1) **to** (S, f_2, g_2) or a **morphism of correspondences from** (R, f_1, g_1) **to** (S, f_2, g_2).

[5] *Alternative Picture:*

$$\begin{tikzcd}
 & R \arrow[dl, "f_1"'] \arrow[dr, "g_1"] \arrow[dd, "\phi"] & \\
A & & B. \\
 & S \arrow[ul, "f_2"] \arrow[ur, "g_2"'] &
\end{tikzcd}$$

commute, where $S \times_B S$ is the pullback

$$S \times_B S \cong \{(s,t) \in S \times S \mid g(s) = g(t)\}$$

$$\begin{tikzcd}
S \times_B S \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & S \arrow[d, "g"] \\
S \arrow[r, "g"'] & B
\end{tikzcd}$$

of S with itself along g.

1.5 Total Spans

2 Categories of Spans

2.1 Categories of Spans

Let A and B be sets.

Definition 2.1.1.1. The **category of spans from A to B** is the category $\mathsf{Span}(A, B)$ defined by

$$\mathsf{Span}(A,B) \overset{\text{def}}{=} \mathsf{Fun}(\Lambda, \mathsf{Sets}) \underset{\mathsf{ev}_{[-1]}, \mathsf{Sets}, [A]}{\times} \mathrm{pt} \underset{[B], \mathsf{Sets}, \mathsf{ev}_{[1]}}{\times} \mathsf{Fun}(\Lambda, \mathsf{Sets}),$$

as in the diagram

$$\begin{tikzcd}
 & & \mathsf{Span}(A,B) \arrow[dl] \arrow[dr] \arrow[d, phantom, "\lrcorner" rotate=-45] & & \\
 & \mathsf{Fun}(\Lambda,\mathsf{Sets}) \underset{\mathsf{Sets}}{\times} \mathrm{pt} \arrow[dl] \arrow[dr] \arrow[d, phantom, "\lrcorner" rotate=-45] & & \mathrm{pt} \underset{\mathsf{Sets}}{\times} \mathsf{Fun}(\Lambda,\mathsf{Sets}) \arrow[dl] \arrow[dr] \arrow[d, phantom, "\lrcorner" rotate=-45] & \\
\mathsf{Fun}(\Lambda,\mathsf{Sets}) \arrow[dr, "\mathsf{ev}_{[-1]}"'] & & \mathrm{pt} \arrow[dl, "{[A]}"] \arrow[dr, "{[B]}"'] & & \mathsf{Fun}(\Lambda,\mathsf{Sets}). \arrow[dl, "\mathsf{ev}_{[1]}"] \\
 & \mathsf{Sets} & & \mathsf{Sets} &
\end{tikzcd}$$

Remark 2.1.1.2. In detail, the **category of spans from A to B** is the category $\mathsf{Span}(A, B)$ where

- *Objects.* The objects of $\mathsf{Span}(A, B)$ are spans from A to B;
- *Morphisms.* The morphism of $\mathsf{Span}(A, B)$ are morphisms of spans;

- *Identities.* The unit map

$$\mathbb{1}^{\mathsf{Span}(A,B)}_{(S,f,g)} \colon \mathrm{pt} \longrightarrow \mathrm{Hom}_{\mathsf{Span}(A,B)}((S,f,g),(S,f,g))$$

  of $\mathsf{Span}(A,B)$ at (S,f,g) is defined by[6]

$$\mathrm{id}^{\mathsf{Span}(A,B)}_{(S,f,g)} \overset{\text{def}}{=} \mathrm{id}_S;$$

- *Composition.* The composition map

$$\circ^{\mathsf{Span}(A,B)}_{R,S,T} \colon \mathrm{Hom}_{\mathsf{Span}(A,B)}(S,T) \times \mathrm{Hom}_{\mathsf{Span}(A,B)}(R,S) \to \mathrm{Hom}_{\mathsf{Span}(A,B)}(R,T)$$

  of $\mathsf{Span}(A,B)$ at $((R,f_1,g_1),(S,f_2,g_2),(T,f_3,g_3))$ is defined by[7]

$$\psi \circ^{\mathsf{Span}(A,B)}_{R,S,T} \phi \overset{\text{def}}{=} \psi \circ \phi.$$

2.2 The Bicategory of Spans

Definition 2.2.1.1. The **bicategory of spans** is the bicategory Span where

- *Objects.* The objects of Span are sets;

[6] *Picture:*

$$\begin{tikzcd}
 & S \arrow[ddl, "f"'] \arrow[ddr, "g"] \arrow[drr, "\mathrm{id}_S"] & & & \\
 & & & S \arrow[ddl, "f"'] \arrow[ddr, "g"] & \\
A \arrow[drr, equal] & & B \arrow[drr, equal] & & \\
 & & A & & B.
\end{tikzcd}$$

[7] *Picture:*

$$\begin{tikzcd}
 & R \arrow[ddl, "f_1"'] \arrow[ddr, "g_1"] \arrow[drr, "\phi"] & & & & & \\
 & & & S \arrow[ddl, "f_2"'] \arrow[ddr, "g_2"] \arrow[drr, "\psi"] & & & \\
A \arrow[drr, equal] & & B \arrow[drr, equal] & & & T \arrow[ddl, "f_3"'] \arrow[ddr, "g_3"] & \\
 & & A \arrow[drr, equal] & & B \arrow[drr, equal] & & \\
 & & & & A & & B.
\end{tikzcd}$$

- Hom-*Categories*. For each $A, B \in \mathrm{Obj}(\mathsf{Span})$, we have

$$\mathrm{Hom}_{\mathsf{Span}}(A, B) \stackrel{\text{def}}{=} \mathsf{Span}(A, B);$$

- *Identities*. For each $A \in \mathrm{Obj}(\mathsf{Span})$, the unit functor

$$\mathbb{1}^{\mathsf{Span}}_{A} \colon \mathrm{pt} \longrightarrow \mathsf{Span}(A, A)$$

  of Span at A is the functor picking the span $(A, \mathrm{id}_A, \mathrm{id}_A)$:

$$\begin{tikzcd} & A \arrow[dl, "\mathrm{id}_A"'] \arrow[dr, "\mathrm{id}_A"] & \\ A & & A. \end{tikzcd}$$

- *Composition*. For each $A, B, C \in \mathrm{Obj}(\mathsf{Span})$, the composition bifunctor

$$\circ^{\mathsf{Span}}_{A,B,C} \colon \mathsf{Span}(B, C) \times \mathsf{Span}(A, B) \longrightarrow \mathsf{Span}(A, C)$$

  of Span at (A, B, C) is the bifunctor where

  – *Action on Objects*. The composition of two spans

$$\begin{tikzcd} & R \arrow[dl, "f_1"'] \arrow[dr, "g_1"] & & & & S \arrow[dl, "f_2"'] \arrow[dr, "g_2"] & \\ A & & B & \text{and} & B & & C \end{tikzcd}$$

  is the span $(R \times_B S, f_1 \circ \mathrm{pr}_1, g_2 \circ \mathrm{pr}_2)$, constructed as in the diagram

$$\begin{tikzcd} & & R \times_B S \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] \arrow[ddll, bend right, "f_1 \circ \mathrm{pr}_1"'] \arrow[ddrr, bend left, "g_2 \circ \mathrm{pr}_2"] & & \\ & R \arrow[dl, "f_1"'] \arrow[dr, "g_1"] & & S \arrow[dl, "f_2"'] \arrow[dr, "g_2"] & \\ A & & B & & C. \end{tikzcd}$$

  – *Action on Morphisms*. The horizontal composition of 2-morphisms is defined via functoriality of pullbacks: given morphisms of spans

$$
\begin{tikzcd}
& R \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[dd, "\phi"] & \\
A & & B \\
& R' \arrow[ul, "f'"] \arrow[ur, "g'"'] &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
& S \arrow[dl, "h"'] \arrow[dr, "k"] \arrow[dd, "\psi"] & \\
B & & C, \\
& S' \arrow[ul, "h'"] \arrow[ur, "k'"'] &
\end{tikzcd}
$$

their horizontal composition is the morphism of spans

$$
\begin{tikzcd}
& R \times_B S \arrow[dl, "f \circ \mathrm{pr}_1"'] \arrow[dr, "k \circ \mathrm{pr}_2"] \arrow[dd, dashed, "\exists!"] & \\
A & & C, \\
& R' \times_B S' \arrow[ul, "h' \circ \mathrm{pr}'_1"] \arrow[ur, "k' \circ \mathrm{pr}'_2"'] &
\end{tikzcd}
$$

constructed as in the diagram

$$
\begin{tikzcd}
& & R \times_B S \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] \arrow[ddll, bend right, "f \circ \mathrm{pr}_1"'] \arrow[ddrr, bend left, "k \circ \mathrm{pr}_2"] \arrow[dddd, dashed, "\exists!"] & & \\
& R \arrow[dl, "f"'] \arrow[dr, "g"] & & S \arrow[dl, "h"'] \arrow[dr, "k"] & \\
A & & B & & C; \\
& R' \arrow[ul, "f'"] \arrow[ur, "g'"'] & & S' \arrow[ul, "h'"] \arrow[ur, "k'"'] & \\
& & R' \times_B S' \arrow[ul, "\mathrm{pr}'_1"] \arrow[ur, "\mathrm{pr}'_2"'] \arrow[uull, bend left, "f' \circ \mathrm{pr}'_1"] \arrow[uurr, bend right, "k' \circ \mathrm{pr}'_2"'] & &
\end{tikzcd}
$$

- *Associators and Unitors.* The associator and unitors are defined using the universal property of the pullback.

2.3 The Monoidal Bicategory of Spans

2.4 The Double Category of Spans

Definition 2.4.1.1. The **double category of spans** is the double category $\mathsf{Span}^{\mathsf{dbl}}$ where

- *Objects.* The objects of $\mathsf{Span}^{\mathsf{dbl}}$ are sets;
- *Vertical Morphisms.* The vertical morphisms of $\mathsf{Span}^{\mathsf{dbl}}$ are functions $f\colon A \to B$;
- *Horizontal Morphisms.* The horizontal morphisms of $\mathsf{Span}^{\mathsf{dbl}}$ are spans $(S, \phi, \psi)\colon A \nrightarrow X$;
- 2-*Morphisms.* A 2-cell

$$\begin{tikzcd} A \arrow[r, "{(R,\phi_R,\psi_R)}", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & B \arrow[d, "g"] \\ X \arrow[r, "{(S,\phi_S,\psi_S)}"', "\shortmid"{marking}] & Y \end{tikzcd}$$

  of $\mathsf{Span}^{\mathsf{dbl}}$ is a morphism of spans from the span

$$\begin{tikzcd} & & R \arrow[dll, "\phi_R"'] \arrow[dr, "\psi_R"] & & \\ & & & B \arrow[dr, "g"] & \\ A & & & & Y \end{tikzcd}$$

  to the span

$$\begin{tikzcd} & & A \times_X S \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & & \\ & A \arrow[dl, "f"'] \arrow[dr, "f"] & & S \arrow[dl, "\phi_S"'] \arrow[dr, "\psi_S"] & \\ X & & X & & Y; \end{tikzcd}$$

- *Horizontal Identities.* The horizontal unit functor

$$\mathbb{1}^{\mathsf{Span}^{\mathsf{dbl}}}\colon \left(\mathsf{Span}^{\mathsf{dbl}}\right)_0 \to \left(\mathsf{Span}^{\mathsf{dbl}}\right)_1$$

  of $\mathsf{Span}^{\mathsf{dbl}}$ is the functor where

– *Action on Objects.* For each $A \in \mathrm{Obj}\Big(\big(\mathsf{Span}^{\mathsf{dbl}}\big)_0\Big)$, we have

$$\mathbb{1}_A \overset{\text{def}}{=} (A, \mathrm{id}_A, \mathrm{id}_A),$$

as in the diagram

$$\begin{tikzcd}
 & A \arrow[dl, equal, "\mathrm{id}_A"'] \arrow[dr, equal, "\mathrm{id}_A"] & \\
A & & A;
\end{tikzcd}$$

– *Action on Morphisms.* For each vertical morphism $f\colon A \to B$ of $\mathsf{Span}^{\mathsf{dbl}}$, i.e. each map of sets f from A to B, the identity 2-morphism

$$\begin{tikzcd}
A \arrow[r, "\mathbb{1}_A", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow \mathbb{1}_f"] & A \arrow[d, "f"] \\
B \arrow[r, "\mathbb{1}_B"', "\shortmid"{marking}] & B
\end{tikzcd}$$

of f is the morphism of spans from

$$\begin{tikzcd}
 & & A \arrow[ddll, equal, "\mathrm{id}_A"'] \arrow[dr, equal, "\mathrm{id}_A"] & & \\
 & & & A \arrow[dr, "f"] & \\
A & & & & B
\end{tikzcd}$$

to

$$\begin{tikzcd}
 & & A \times_B B \arrow[dl] \arrow[dr] \arrow[dd, phantom, very near start, "\lrcorner" rotate=-45] & & \\
 & A \arrow[dl, "f"'] \arrow[dr, "f"] & & B \arrow[dl, equal, "\mathrm{id}_B"'] \arrow[dr, equal, "\mathrm{id}_B"] & \\
B & & B & & B
\end{tikzcd}$$

given by the isomorphism $A \xrightarrow{\cong} A \times_B B$;

- *Vertical Identities.* For each $A \in \mathrm{Obj}\big(\mathsf{Span}^{\mathsf{dbl}}\big)$, we have

$$\mathrm{id}_A^{\mathsf{Span}^{\mathsf{dbl}}} \overset{\text{def}}{=} \mathrm{id}_A;$$

- *Identity 2-Morphisms.* For each horizontal morphism $R\colon A \nrightarrow B$ of $\mathsf{Span}^{\mathsf{dbl}}$, the identity 2-morphism

$$\begin{tikzcd}
A \arrow[r, "S", "|" marking] \arrow[d, "\mathrm{id}_A"'] \arrow[dr, phantom, "\overset{\|}{\underset{\Downarrow}{\mathrm{id}_S}}"] & B \arrow[d, "\mathrm{id}_B"] \\
A \arrow[r, "S"', "|" marking] & B
\end{tikzcd}$$

  of R is the morphism of spans from

$$\begin{tikzcd}
 & S \arrow[dl, "\phi_S"'] \arrow[dr, "\psi_S"] & & \\
 & & B \arrow[dr, equal, "\mathrm{id}_B"] & \\
A & & & B
\end{tikzcd}$$

  to

$$\begin{tikzcd}
 & & A \times_A S \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & & \\
 & A \arrow[dl, equal, "\mathrm{id}_A"'] \arrow[dr, equal, "\mathrm{id}_A"] & & S \arrow[dl, "\phi_S"'] \arrow[dr, "\psi_S"] & \\
A & & A & & B
\end{tikzcd}$$

  given by the isomorphism $S \xrightarrow{\cong} A \times_A S$;

- *Horizontal Composition.* The horizontal composition functor

$$\odot^{\mathsf{Span}^{\mathsf{dbl}}}\colon \left(\mathsf{Span}^{\mathsf{dbl}}\right)_1 \times_{\left(\mathsf{Span}^{\mathsf{dbl}}\right)_0} \left(\mathsf{Span}^{\mathsf{dbl}}\right)_1 \to \left(\mathsf{Span}^{\mathsf{dbl}}\right)_1$$

  of $\mathsf{Span}^{\mathsf{dbl}}$ is the functor where

  - *Action on Objects.* For each composable pair

$$A \overset{(R,\phi_R,\psi_R)}{\nrightarrow} B \overset{(S,\phi_S,\psi_S)}{\nrightarrow} C$$

    of horizontal morphisms of $\mathsf{Span}^{\mathsf{dbl}}$, we have

$$(S, \phi_S, \psi_S) \odot (R, \phi_R, \psi_R) \overset{\mathrm{def}}{=} S \circ^{\mathsf{Span}}_{A,B,C} R,$$

    where $S \circ^{\mathsf{Span}}_{A,B,C} R$ is the composition of (R, ϕ_R, ψ_R) and (S, ϕ_S, ψ_S) defined as in Definition 2.2.1.1;

– *Action on Morphisms.* For each horizontally composable pair

$$
\begin{tikzcd}
A \arrow[r, "{(R,\phi_R,\psi_R)}", "|"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & B \arrow[d, "g"] \\
X \arrow[r, "{(T,\phi_T,\psi_T)}"', "|"] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r, "{(S,\phi_S,\psi_S)}", "|"] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\beta"] & C \arrow[d, "h"] \\
Y \arrow[r, "{(U,\phi_U,\psi_U)}"', "|"] & Z
\end{tikzcd}
$$

of 2-morphisms of $\mathsf{Span}^{\mathsf{dbl}}$,

- *Vertical Composition of* 1*-Morphisms.* For each composable pair $A \xrightarrow{F} B \xrightarrow{G} C$ of vertical morphisms of $\mathsf{Span}^{\mathsf{dbl}}$, i.e. maps of sets, we have

$$g \circ^{\mathsf{Span}^{\mathsf{dbl}}} f \stackrel{\text{def}}{=} g \circ f;$$

- *Vertical Composition of* 2*-Morphisms.* For each vertically composable pair

$$
\begin{tikzcd}
A \arrow[r, "{(R,\phi_R,\psi_R)}", "|"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & X \arrow[d, "g"] \\
B \arrow[r, "{(S,\phi_S,\psi_S)}"', "|"] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r, "{(S,\phi_S,\psi_S)}", "|"] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow\beta"] & Y \arrow[d, "k"] \\
C \arrow[r, "{(T,\phi_T,\psi_T)}"', "|"] & Z
\end{tikzcd}
$$

of 2-morphisms of $\mathsf{Span}^{\mathsf{dbl}}$,

- *Associators and Unitors.* The associator and unitors of $\mathsf{Span}^{\mathsf{dbl}}$ are defined using the universal property of the pullback.

2.5 Properties of The Bicategory of Spans

Proposition 2.5.1.1. Let $\lambda = \left(A \xleftarrow{f} S \xrightarrow{g} B\right)$ be a span.

1. *Self-Duality.*

2. *Isomorphisms in* Span.

3. *Equivalences in* Span.

4. *Adjunctions in* Span. Let A and B be sets.[8]

[8] In the literature (e.g. [**ref**]),...are called maps and denoted by $\mathsf{MapSpan}(A, B)$

(a) We have a natural bijection

$$\left\{\begin{matrix}\text{Adjunctions in Span}\\ \text{from } A \text{ to } B\end{matrix}\right\} \cong \left\{\begin{matrix}\text{Spans } A \xleftarrow{f} S \xrightarrow{g} B\\ \text{from } A \text{ to } B \text{ with}\\ f \text{ an isomorphism}\end{matrix}\right\}.$$

(b) We have an equivalence of categories

$$\mathsf{MapSpan}(A, B) \overset{\text{eq.}}{\cong} \mathsf{Sets}(A, B)_{\mathsf{disc}},$$

where $\mathsf{MapSpan}(A, B)$ is the full subcategory of $\mathsf{Span}(A, B)$ spanned by the spans $A \xleftarrow{f} S \xrightarrow{g} B$ from A to B with f an isomorphism.

(c) We have a biequivalence of bicategories

$$\mathsf{MapSpan} \overset{\text{eq.}}{\cong} \mathsf{Sets}_{\mathsf{bidisc}},$$

where $\mathsf{MapSpan}$ is the sub-bicategory of Span whose Hom-categories are given by $\mathsf{MapSpan}(A, B)$.

5. *Monads in* Span.
6. *Comonads in* Span.
7. *Monomorphisms in* Span.
8. *Epimorphisms in* Span.
9. *Existence of Right Kan Extensions.*
10. *Existence of Right Kan Lifts.*
11. *Closedness.*

Proof. Item 1, *Self-Duality*:
Item 2, *Isomorphisms in* Span:
Item 3, *Equivalences in* Span:
Item 4, *Adjunctions in* Span: We first prove Item 4a.
We proceed step by step:

1. *From Adjunctions in* Span *to Functions.* An adjunction in Span from A to B consists of a pair of spans

$$\begin{tikzcd} & R \arrow[dl, "f"'] \arrow[dr, "g"] & \\ A & & B \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & S' \arrow[dl, "h"'] \arrow[dr, "k"] & \\ B & & A \end{tikzcd}$$

together with maps

$$
\begin{tikzcd}
 & A \arrow[dl, equal, "\mathrm{id}_A"'] \arrow[dr, equal, "\mathrm{id}_A"] \arrow[d, "\phi"] & \\
A & S \times_B S' \arrow[l, "f\circ\mathrm{pr}'_1"] \arrow[r, "k\circ\mathrm{pr}'_2"'] & A
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
 & S' \times_A S \arrow[dl, "h\circ\mathrm{pr}_1"'] \arrow[dr, "g\circ\mathrm{pr}_2"] \arrow[d, "\psi"] & \\
B \arrow[r, equal, "\mathrm{id}_B"'] & B & B. \arrow[l, equal, "\mathrm{id}_B"]
\end{tikzcd}
$$

We claim that these conditions

2. *From Functions to Adjunctions in* ***Rel***.
3. *Invertibility: From Functions to Adjunctions Back to Functions.*
4. *Invertibility: From Adjunctions to Functions Back to Adjunctions.*

We now proceed to the proof of Item 4b. For this, we will construct a functor

$$F\colon \mathsf{Sets}(A,B)_{\mathsf{disc}} \longrightarrow \mathsf{MapSpan}(A,B)$$

and prove it to be essentially surjective and fully faithful, and thus an equivalence by Categories, **??** of **??**. Indeed, given a map $f\colon A \longrightarrow B$, let $F(f)$ be the representable span associated to f of Definition 5.1.1.1, and let F send the unique (identity) morphism from f to itself to the identity morphism of $F(f)$ in $\mathsf{MapSpan}(A,B)$. We now prove that F is fully faithful and essentially surjective:

1. *F Is Fully Faithful:* Given maps $f, g\colon A \rightrightarrows B$, we need to show that

$$\mathrm{Hom}_{\mathsf{MapSpan}(A,B)}(F(f), F(g)) = \begin{cases} \mathrm{pt} & \text{if } f = g, \\ \emptyset & \text{otherwise.} \end{cases}$$

Indeed, a morphism from $F(f)$ to $F(g)$ takes the form

$$
\begin{tikzcd}
 & A \arrow[dl, equal] \arrow[dr, "f"] \arrow[d, "\phi"] & \\
A & A \arrow[l, equal] \arrow[r, "g"'] & B.
\end{tikzcd}
$$

From the relations $\mathrm{id}_A = \mathrm{id}_A \circ \phi$ and $f = g \circ \phi$, we see that $\phi = \mathrm{id}_A$, and thus from the relation $f = g \circ \phi$ there is such a morphism iff $f = g$.

2. *F Is Essentially Surjective:* Let λ be a span of the form

$$\begin{tikzcd} & S \arrow[dl, dashed, "\phi"', "\sim"] \arrow[dr, "f"] & \\ A & & B. \end{tikzcd}$$

we claim that $\lambda \cong F(f \circ \phi^{-1})$. Indeed, we have morphisms

$$\begin{tikzcd} & S \arrow[dl, dashed, "\phi"', "\sim"] \arrow[dr, "f"] \arrow[dd, "\phi"] & \\ A \arrow[dr, equal] & & B \\ & A \arrow[ur, "f\circ\phi^{-1}"'] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & A \arrow[dl, equal] \arrow[dr, "f\circ\phi^{-1}"] \arrow[dd, "\phi^{-1}"] & \\ A & & B, \\ & S \arrow[ul, dashed, "\phi", "\sim"'] \arrow[ur, "f"'] & \end{tikzcd}$$

inverse to each other in $\mathsf{MapSpan}(A, B)$, and thus $\lambda \cong F(f \circ \phi^{-1})$.

Finally, we prove Item 4c.
Item 5, Monads in Span:
Item 6, Comonads in Span:
Item 7, Monomorphisms in Span:
Item 8, Epimorphisms in Span:
Item 9, Existence of Right Kan Extensions:
Item 10, Existence of Right Kan Lifts:
Item 11, Closedness:

□

3 Limits of Spans

4 Colimits of Spans

5 Constructions With Spans

5.1 Representable Spans

Definition 5.1.1.1. Let $f : A \longrightarrow B$ be a function.

- The **representable span associated to** f is the span

$$\begin{tikzcd} & A \arrow[dl, equal, "\mathrm{id}_A"'] \arrow[dr, "f"] & \\ A & & B \end{tikzcd}$$

from A to B.

- The **corepresentable span associated to** f is the span

$$\begin{tikzcd} & A \arrow[dl, "f"'] \arrow[dr, equal, "\mathrm{id}_A"] & \\ B & & A \end{tikzcd}$$

from B to A.

5.2 Composition of Spans

Definition 5.2.1.1. The **composition** of two spans

$$\begin{tikzcd} & R \arrow[dl, "f_1"'] \arrow[dr, "g_1"] & \\ A & & B \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & S \arrow[dl, "f_2"'] \arrow[dr, "g_2"] & \\ B & & C \end{tikzcd}$$

is the span $(R \times_B S, f_1 \circ \mathrm{pr}_1, g_2 \circ \mathrm{pr}_2)$, constructed as in the diagram

$$\begin{tikzcd} & & R \times_B S \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] \arrow[ddll, bend right, "f_1 \circ \mathrm{pr}_1"'] \arrow[ddrr, bend left, "g_2 \circ \mathrm{pr}_2"] & & \\ & R \arrow[dl, "f_1"'] \arrow[dr, "g_1"] & & S \arrow[dl, "f_2"'] \arrow[dr, "g_2"] & \\ A & & B & & C. \end{tikzcd}$$

5.3 Horizontal Composition of Morphisms of Spans

Definition 5.3.1.1. The **horizontal composition** of a pair of 2-morphisms of spans

$$\begin{tikzcd} & R \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[dd, "\phi"] & \\ A & & B \\ & R' \arrow[ul, "f'"] \arrow[ur, "g'"'] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & S \arrow[dl, "h"'] \arrow[dr, "k"] \arrow[dd, "\psi"] & \\ B & & C, \\ & S' \arrow[ul, "h'"] \arrow[ur, "k'"'] & \end{tikzcd}$$

is the morphism of spans

$$\begin{tikzcd}
& R \times_B S \arrow[dl, "f \circ \mathrm{pr}_1"'] \arrow[dr, "k \circ \mathrm{pr}_2"] \arrow[dd, dashed, "\exists!"] & \\
A & & C, \\
& R' \times_B S' \arrow[ul, "h' \circ \mathrm{pr}'_1"] \arrow[ur, "k' \circ \mathrm{pr}'_2"'] &
\end{tikzcd}$$

constructed as in the diagram

$$\begin{tikzcd}
& & R \times_B S \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] \arrow[ddll, bend right, "f \circ \mathrm{pr}_1"'] \arrow[ddrr, bend left, "k \circ \mathrm{pr}_2"] \arrow[dddd, dashed, "\exists!"'] & & \\
& R \arrow[dl, "f"'] \arrow[dr, "g"] & & S \arrow[dl, "h"'] \arrow[dr, "k"] & \\
A & & B & & C; \\
& R' \arrow[ul, "f'"] \arrow[ur, "g'"'] & & S' \arrow[ul, "h'"] \arrow[ur, "k'"'] & \\
& & R' \times_B S' \arrow[ul, "\mathrm{pr}'_1"] \arrow[ur, "\mathrm{pr}'_2"'] \arrow[uull, bend left, "f' \circ \mathrm{pr}'_1"] \arrow[uurr, bend right, "k' \circ \mathrm{pr}'_2"'] & &
\end{tikzcd}$$

5.4 Properties of Composition of Spans

Proposition 5.4.1.1. Let $\lambda = \left(A \xleftarrow{f} S \xrightarrow{g} B\right)$ be a span.

1. *Functoriality.*

Proof. ∎

5.5 The Inverse of a Span

6 Functoriality of Spans

6.1 Direct Images

6.2 Functoriality of Spans on Powersets

7 Comparison of Spans to Functions and Relations

7.1 Comparison to Functions

Proposition 7.1.1.1. We have a pseudofunctor

$$\iota\colon \mathsf{Sets}_{\mathsf{bidisc}} \to \mathsf{Span}$$

from $\mathsf{Sets}_{\mathsf{bidisc}}$ to Span where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathsf{Sets}_{\mathsf{bidisc}})$, we have

$$\iota(A) \overset{\text{def}}{=} A;$$

- *Action on* Hom*-Categories.* For each $A, B \in \mathrm{Obj}(\mathsf{Sets}_{\mathsf{bidisc}})$, the action on Hom-categories

$$\iota_{A,B}\colon \mathsf{Sets}(A, B)_{\mathsf{disc}} \to \mathsf{Span}(A, B)$$

of ι at (A, B) is the functor defined on objects by sending a function $f\colon A \to B$ to the span

$$\begin{tikzcd} & A \arrow[dl, equal, "\mathrm{id}_A"'] \arrow[dr, "f"] & \\ A & & B \end{tikzcd}$$

from A to B.

- *Strict Unity Constraints.* For each $A \in \mathrm{Obj}(\mathsf{Sets}_{\mathsf{bidisc}})$, the strict unity constraint

$$\iota^0_A\colon \mathrm{id}_{\iota(A)} \Longrightarrow \iota(\mathrm{id}_A)$$

of ι at A is given by the identity morphism of spans

$$\begin{tikzcd} & A \arrow[dl, equal, "\mathrm{id}_A"'] \arrow[dr, equal, "\mathrm{id}_A"] \arrow[dd, equal, "\mathrm{id}" description] & \\ A & & A, \\ & A \arrow[ul, equal, "\mathrm{id}_A"] \arrow[ur, equal, "\mathrm{id}_A"'] & \end{tikzcd}$$

as indeed $\mathrm{id}_{\iota(A)} = \iota(\mathrm{id}_A)$;

- *Pseudofunctoriality Constraints.* For each $A, B, C \in \mathrm{Obj}(\mathsf{Sets}_{\mathsf{bidisc}})$, each $f \in \mathrm{Hom}_{\mathsf{Sets}_{\mathsf{bidisc}}}(A, B)$, and each $g \in \mathrm{Hom}_{\mathsf{Sets}_{\mathsf{bidisc}}}(B, C)$, the pseudofunctoriality constraint

$$\iota^2_{g,f} \colon \iota(g) \circ \iota(f) \Longrightarrow \iota(g \circ f)$$

of ι at (f, g) is the morphism of spans from the span

$$\begin{tikzcd}
 & & A \times_B B \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] \arrow[dd, phantom, very near start, "\lrcorner" rotate=-45] \arrow[ddll, bend right, "\mathrm{id}_A \circ \mathrm{pr}_1"'] \arrow[ddrr, bend left, "g \circ \mathrm{pr}_2"] & & \\
 & A \arrow[dl, equals, "\mathrm{id}_A"'] \arrow[dr, "f"] & & B \arrow[dl, equals, "\mathrm{id}_B"'] \arrow[dr, "g"] & \\
A & & B & & C.
\end{tikzcd}$$

to the span

$$\begin{tikzcd}
 & A \arrow[dl, equals, "\mathrm{id}_A"'] \arrow[dr, "g \circ f"] & \\
A & & C
\end{tikzcd}$$

given by the isomorphism $A \times_B B \cong A$.

Proof. Omitted. □

7.2 Comparison to Relations: From Span to **Rel**

7.2.1 Relations Associated to Spans

Let $\lambda = \left(A \xleftarrow{f} S \xrightarrow{g} B\right)$ be a span.

Definition 7.2.1.1. The **relation associated to** λ is the relation

$$S(\lambda) \colon A \nrightarrow B$$

from A to B defined as follows:

- Viewing relations from A to B as functions $A \times B \to \{\mathrm{true}, \mathrm{false}\}$, we define

$$\iota_{A,B}(S)^a_b \stackrel{\mathrm{def}}{=} \begin{cases} \mathrm{true} & \text{if there exists } x \in S \text{ such} \\ & \text{that } a = f(x) \text{ and } b = g(x), \\ \mathrm{false} & \text{otherwise} \end{cases}$$

for each $(a, b) \in A \times B$.

- Viewing relations from A to B as functions $A \to \mathcal{P}(B)$, we define

$$[\iota_{A,B}(S)](a) \overset{\text{def}}{=} g(f^{-1}(a))$$

for each $a \in A$.

- Viewing relations from A to B as subsets of $A \times B$, we define

$$\iota_{A,B}(S) \overset{\text{def}}{=} \{(f(x), g(x)) \mid x \in S\}.$$

Proposition 7.2.1.2. Let $\lambda = \left(A \overset{f}{\leftarrow} S \overset{g}{\rightarrow} B\right)$ be a span.

1. *Interaction With Identities.*
2. *Interaction With Composition.*
3. *Interaction With Inverses.*

Proof. □

7.2.2 The Comparison Functor from Span to Rel

Proposition 7.2.2.1. We have a pseudofunctor

$$\iota\colon \mathsf{Span} \to \mathbf{Rel}$$

from Span to **Rel** where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathsf{Span})$, we have

$$\iota(A) \overset{\text{def}}{=} A;$$

- *Action on* Hom*-Categories.* For each $A, B \in \mathrm{Obj}(\mathsf{Span})$, the action on Hom-categories

$$\iota_{A,B}\colon \mathsf{Span}(A, B) \to \mathbf{Rel}(A, B)$$

of ι at (A, B) is the functor where

- *Action on Objects.* Given a span

$$\begin{tikzcd} & S \arrow[dl, "f"'] \arrow[dr, "g"] & \\ A & & B \end{tikzcd}$$

from A to B, the image

$$\iota_{A,B}(S)\colon A \nrightarrow B$$

of S by ι is the relation from A to B defined as follows:

  * Viewing relations as functions $A \times B \to \{\mathsf{true}, \mathsf{false}\}$, we define

$$\iota_{A,B}(S)^a_b \overset{\text{def}}{=} \begin{cases} \mathsf{true} & \text{if there exists } x \in S \\ & \text{such that } a = f(x) \\ & \text{and } b = g(x), \\ \mathsf{false} & \text{otherwise} \end{cases}$$

  for each $(a, b) \in A \times B$;

  * Viewing relations as functions $A \to \mathcal{P}(B)$, we define

$$[\iota_{A,B}(S)](a) \overset{\text{def}}{=} g(f^{-1}(a))$$

  for each $a \in A$;

  * Viewing relations as subsets of $A \times B$, we define

$$\iota_{A,B}(S) \overset{\text{def}}{=} \{(f(x), g(x)) \mid x \in S\}.$$

- *Action on Morphisms.* Given a morphism of spans

$$\begin{tikzcd} & R \arrow[dl, "f_R"'] \arrow[dr, "g_R"] \arrow[dd, "\phi"] & \\ A & & B, \\ & S \arrow[ul, "f_S"] \arrow[ur, "g_S"'] & \end{tikzcd}$$

we have a corresponding inclusion of relations

$$\iota_{A,B}(\phi)\colon \iota_{A,B}(R) \subset \iota_{A,B}(S),$$

since we have $a \sim_{\iota_{A,B}(R)} b$ iff there exists $x \in R$ such that $a = f_R(x)$ and $b = g_R(x)$, in which case we then have

$$\begin{aligned} a &= f_R(x) \\ &= f_S(\phi(x)), \\ b &= g_R(x) \\ &= g_S(\phi(x)), \end{aligned}$$

so that $a \sim_{\iota_{A,B}(S)} b$, and thus $\iota_{A,B}(R) \subset \iota_{A,B}(S)$.

Proof. Omitted. □

7.3 Comparison to Relations: From Rel to Span

Proposition 7.3.1.1. We have a lax functor

$$\left(\iota, \iota^2, \iota^0\right) \colon \mathbf{Rel} \to \mathsf{Span}$$

from **Rel** to Span where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathsf{Span})$, we have

$$\iota(A) \overset{\text{def}}{=} A;$$

- *Action on* Hom-*Categories.* For each $A, B \in \mathrm{Obj}(\mathsf{Span})$, the action on Hom-categories

$$\iota_{A,B} \colon \mathbf{Rel}(A, B) \to \mathsf{Span}(A, B)$$

  of ι at (A, B) is the functor where

  - *Action on Objects.* Given a relation $R \colon A \nrightarrow B$ from A to B, we define a span

$$\iota_{A,B}(R) \colon A \nrightarrow B$$

    from A to B by

$$\iota_{A,B}(R) \overset{\text{def}}{=} (R, \upharpoonright \mathrm{pr}_1 R, \upharpoonright \mathrm{pr}_2 R),$$

    where $R \subset A \times B$ and $\upharpoonright \mathrm{pr}_1 R$ and $\upharpoonright \mathrm{pr}_2 R$ are the restriction of the projections

$$\begin{aligned} \mathrm{pr}_1 &\colon A \times B \to A, \\ \mathrm{pr}_2 &\colon A \times B \to B \end{aligned}$$

    to R;

- *Action on Morphisms.* Given an inclusion $\phi\colon R \subset S$ of relations, we have a corresponding morphism of spans

$$\iota_{A,B}(\phi)\colon \iota_{A,B}(R) \to \iota_{A,B}(S)$$

as in the diagram

$$\begin{tikzcd}
 & R \arrow[dl, "\restriction_{\mathrm{pr}_1} R"'] \arrow[dr, "\restriction_{\mathrm{pr}_2} R"] \arrow[dd, hook] & \\
A & & B. \\
 & S \arrow[ul, "\restriction_{\mathrm{pr}_1} S"] \arrow[ur, "\restriction_{\mathrm{pr}_2} S"'] &
\end{tikzcd}$$

· *The Lax Functoriality Constraints.* The lax functoriality constraint

$$\iota^2_{R,S}\colon \iota(S) \circ \iota(R) \Longrightarrow \iota(S \diamond R)$$

of ι at (R, S) is given by the morphism of spans from

$$\begin{tikzcd}
 & & R \times_B S \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] \arrow[ddll, bend right, "\restriction_{\mathrm{pr}_1} R \circ \mathrm{pr}_1"'] \arrow[ddrr, bend left, "\restriction_{\mathrm{pr}_2} S \circ \mathrm{pr}_2"] & & \\
 & R \arrow[dl, "\restriction_{\mathrm{pr}_1} R"'] \arrow[dr, "\restriction_{\mathrm{pr}_2} R"] & & S \arrow[dl, "\restriction_{\mathrm{pr}_1} S"'] \arrow[dr, "\restriction_{\mathrm{pr}_2} S"] & \\
A & & B & & C
\end{tikzcd}$$

to

$$\begin{tikzcd}
 & S \diamond R \arrow[dl, "\restriction_{\mathrm{pr}_1} S \diamond R"'] \arrow[dr, "\restriction_{\mathrm{pr}_2} S \diamond R"] & \\
A & & C
\end{tikzcd}$$

given by the natural inclusion $R \times_B S \hookrightarrow S \diamond R$, since we have

$$\begin{aligned}
R \times_B S &= \{((a_R, b_R), (b_S, c_S)) \in R \times S \mid b_R = b_S\};\\
S \diamond R &= \left\{ (a, c) \in A \times C \;\middle|\; \begin{array}{l} \text{there exists some } b \in B \text{ such that} \\ (a, b) \in R \text{ and } (b, c) \in S \end{array} \right\};
\end{aligned}$$

- *The Lax Unity Constraints.* The lax unity constraint[9]

$$\iota^0_A\colon \underbrace{\mathrm{id}_{\iota(A)}}_{(A,\mathrm{id}_A,\mathrm{id}_A)} \Longrightarrow \underbrace{\iota(\chi_A)}_{(\Delta_A,\restriction\mathrm{pr}_1\Delta_A,\restriction\mathrm{pr}_2\Delta_A)}$$

of ι at A is given by the diagonal morphism of A, as in the diagram

$$\begin{tikzcd}
& A \arrow[dl, equal, "\mathrm{id}_A"'] \arrow[dr, equal, "\mathrm{id}_A"] \arrow[dd, "\delta_A" description] & \\
A & & A. \\
& \Delta_A \arrow[ul, "\restriction\mathrm{pr}_1\Delta_A"] \arrow[ur, "\restriction\mathrm{pr}_2\Delta_A"'] &
\end{tikzcd}$$

Proof. Omitted. □

7.4 Comparison to Relations: The Wehrheim–Woodward Construction

7.5 Comparison to Multirelations

Remark 7.5.1.1. The pseudofunctor of Proposition 7.2.2.1 and the lax functor of Proposition 7.3.1.1 fail to be equivalences of bicategories. This happens essentially because a span $(S, f, g)\colon A \mathrel{-\!\!\!\mapstochar\!\!\!\rightarrow} B$ from A to B may relate elements $a \in A$ and $b \in B$ by more than one element, e.g. there could be $s \neq s' \in S$ such that $a = f(s) = f(s')$ and $b = g(s) = g(s')$.
Thus, in a sense, spans may be thought of as “relations with multiplicity”. And indeed, if instead of considering relations from A to B, i.e. functions

$$R\colon A \times B \to \{\mathrm{true}, \mathrm{false}\}$$

from $A \times B$ to $\{\mathrm{true}, \mathrm{false}\} \cong \{0, 1\}$, we consider functions

$$R\colon A \times B \to \mathbb{N} \cup \{\infty\}$$

from $A \times B$ to $\mathbb{N} \cup \{\infty\}$, then we obtain the notion of a **multirelation from A to B**, and these turn out to assemble together with sets into a bicategory MRel that is biequivalent to Span; see [**some-algebraic-laws-for-spans-and-their-connections-with-multirelations**].

[9] Which is in fact strong, as δ_A is an isomorphism.

7.6 Comparison to Relations via Double Categories

Remark 7.6.1.1. There are double functors between the double categories $\mathrm{Rel}^{\mathrm{dbl}}$ and $\mathrm{Span}^{\mathrm{dbl}}$ analogous to the functors of Propositions 7.2.2.1 and 7.3.1.1, assembling moreover into a strict-lax adjunction of double functors; see [**higher-dimensional-categories**].

Appendices

A Other Chapters