Bicategories

December 24, 2023

00ZK

Contents

Create tags and TODO:

1. spans in bicategories: add Proposition 7 here: https://arxiv.org/abs/1903.03890
2. add fact: internal adjunctions in $\mathsf{PseudoFun}(\mathcal{C}, \mathcal{D})$ are precisely the invertible strong transformations as in [JY21, Example 6.2.7]. What are the internal adjunctions?

1 Monomorphisms in Bicategories 00ZL

1.1 Faithful Monomorphisms 00ZM

Let $\mathcal{C}$ be a bicategory.

Definition 1.1.1.1. 00ZN A 1-morphism $f\colon A \to B$ is a **faithful monomorphism in $\mathcal{C}$** if the following equivalent conditions are satisfied:

1. For each $X \in \mathrm{Obj}(\mathcal{C})$, the functor 00ZP

$$f_*\colon \mathsf{Hom}_{\mathcal{C}}(X, A) \to \mathsf{Hom}_{\mathcal{C}}(X, B)$$

   given by postcomposition by f is faithful.

2. Given a diagram in C of the form 00ZQ

$$\begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & A \arrow[r, "f"] & B, \end{tikzcd}$$

if we have $\mathrm{id}_f \circ \alpha = \mathrm{id}_f \circ \beta$, then $\alpha = \beta$.

Example 1.1.1.2. Here are some examples of faithful monomorphisms. 00ZR

1. *Full Monomorphisms in* Cats_2. 00ZS
2. *Full Monomorphisms in* **Rel**. 00ZT
3. *Full Monomorphisms in* Span. 00ZU

1.2 Full Monomorphisms 00ZV

Let C be a bicategory.

Definition 1.2.1.1. A 1-morphism $f\colon A \to B$ is a **full monomorphism in** C if the following equivalent conditions are satisfied: 00ZW

1. For each $X \in \mathrm{Obj}(C)$, the functor 00ZX

$$f_*\colon \mathsf{Hom}_C(X, A) \to \mathsf{Hom}_C(X, B)$$

given by postcomposition by f is full.

2. For each $X \in \mathrm{Obj}(C)$ and each 2-morphism 010F

$$\gamma\colon f \circ \phi \Longrightarrow f \circ \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"] \arrow[r, bend right, "f\circ\psi"'] \arrow[r, phantom, "\gamma\Downarrow"] & B \end{tikzcd}$$

of C, there exists a 2-morphism $\alpha\colon \phi \Longrightarrow \psi$ of C such that we have an equality

$$\begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & A \arrow[r, "f"] & B \end{tikzcd} = \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"] \arrow[r, bend right, "f\circ\psi"'] \arrow[r, phantom, "\gamma\Downarrow"] & B \end{tikzcd}$$

of pasting diagrams in C, i.e. such that we have

$$\gamma = \mathrm{id}_f \circ \alpha.$$

Example 1.2.1.2. Here are some examples of full monomorphisms. 00ZZ

1. *Full Monomorphisms in* Cats_2. 0100
2. *Full Monomorphisms in* $\mathbf{Rel}$. 0101
3. *Full Monomorphisms in* Span. 0102

1.3 Fully Faithful Monomorphisms 0103

Let $\mathcal{C}$ be a bicategory.

Definition 1.3.1.1. A 1-morphism $f\colon A \to B$ is a **fully faithful monomorphism in $\mathcal{C}$** if the following equivalent conditions are satisfied: 0104

1. The 1-morphism f is fully and faithful. 0105
2. For each $X \in \mathrm{Obj}(\mathcal{C})$, the functor 0106

$$f_*\colon \mathsf{Hom}_{\mathcal{C}}(X, A) \to \mathsf{Hom}_{\mathcal{C}}(X, B)$$

   given by postcomposition by f is fully faithful.
3. The conditions in **??** of **??** and **??** of **??** hold. 0107

Example 1.3.1.2. Here are some examples of fully faithful monomorphisms. 0108

1. *Fully Faithful Monomorphisms in* Cats_2. 0109
2. *Fully Faithful Monomorphisms in* $\mathbf{Rel}$. 010A
3. *Fully Faithful Monomorphisms in* Span. 010B

1.4 Strict Monomorphisms 010C

Let $\mathcal{C}$ be a bicategory.

Definition 1.4.1.1. A 1-morphism $f\colon A \to B$ is a **strict monomorphism in $\mathcal{C}$** if the following equivalent conditions are satisfied: 010D

1. For each $X \in \mathrm{Obj}(\mathcal{C})$, the action on objects 010E

$$f_*\colon \mathrm{Obj}(\mathsf{Hom}_{\mathcal{C}}(X, A)) \to \mathrm{Obj}(\mathsf{Hom}_{\mathcal{C}}(X, B))$$

   of the functor

$$f_*\colon \mathsf{Hom}_{\mathcal{C}}(X, A) \to \mathsf{Hom}_{\mathcal{C}}(X, B)$$

   given by postcomposition by f is injective.

2. For each diagram in C of the form 010F

$$X \underset{\psi}{\overset{\phi}{\rightrightarrows}} A \xrightarrow{f} B,$$

if $f \circ \phi = f \circ \psi$, then $\phi = \psi$.

Example 1.4.1.2. Here are some examples of strict monomorphisms. 010G

1. *Strict Monomorphisms in* Cats_2. 010H
2. *Strict Monomorphisms in* **Rel**. 010J
3. *Strict Monomorphisms in* Span. 010K

2 Epimorphisms in Bicategories 010L

2.1 Faithful Epimorphisms 010M

Let C be a bicategory.

Definition 2.1.1.1. A 1-morphism $f\colon A \to B$ is a **faithful epimorphism in C** if the following equivalent conditions are satisfied: 010N

1. For each $X \in \mathrm{Obj}(C)$, the functor 010P

$$f^*\colon \mathsf{Hom}_C(B, X) \to \mathsf{Hom}_C(A, X)$$

given by precomposition by f is faithful.

2. Given a diagram in C of the form 010Q

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\,\Downarrow\beta"] & X, \end{tikzcd}$$

if we have $\alpha \circ \mathrm{id}_f = \beta \circ \mathrm{id}_f$, then $\alpha = \beta$.

Example 2.1.1.2. Here are some examples of faithful epimorphisms. 010R

1. *Full Epimorphisms in* Cats_2. 010S
2. *Full Epimorphisms in* **Rel**. 010T
3. *Full Epimorphisms in* Span. 010U

2.2 Full Epimorphisms 010V

Let C be a bicategory.

Definition 2.2.1.1. 010W A 1-morphism $f\colon A \to B$ is a **full epimorphism in C** if the following equivalent conditions are satisfied:

1. For each $X \in \mathrm{Obj}(C)$, the functor 010X

$$f^*\colon \mathsf{Hom}_C(B, X) \to \mathsf{Hom}_C(A, X)$$

given by precomposition by f is full.

2. For each $X \in \mathrm{Obj}(C)$ and each 2-morphism 011F

$$\gamma\colon \phi \circ f \Longrightarrow \psi \circ f, \qquad \begin{tikzcd} X \arrow[r, bend left, "\phi\circ f"] \arrow[r, bend right, "\psi\circ f"'] \arrow[r, phantom, "\gamma\Downarrow"] & B \end{tikzcd}$$

of C, there exists a 2-morphism $\alpha\colon \phi \Longrightarrow \psi$ of C such that we have an equality

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & X \end{tikzcd} = \begin{tikzcd} A \arrow[r, bend left, "\phi\circ f"] \arrow[r, bend right, "\psi\circ f"'] \arrow[r, phantom, "\gamma\Downarrow"] & X \end{tikzcd}$$

of pasting diagrams in C, i.e. such that we have

$$\gamma = \alpha \circ \mathrm{id}_f.$$

Example 2.2.1.2. Here are some examples 010Z of full epimorphisms.

1. *Full Epimorphisms in* Cats_2. 0110
2. *Full Epimorphisms in* **Rel**. 0111
3. *Full Epimorphisms in* Span. 0112

2.3 Fully Faithful Epimorphisms 0113

Let C be a bicategory.

Definition 2.3.1.1. 0114 A 1-morphism $f\colon A \to B$ is a **fully faithful epimorphism in C** if the following equivalent conditions are satisfied:

1. The 1-morphism f is fully and faithful. 0115

2. For each $X \in \mathrm{Obj}(\mathcal{C})$, the functor 0116

$$f^* \colon \mathsf{Hom}_{\mathcal{C}}(B, X) \to \mathsf{Hom}_{\mathcal{C}}(A, X)$$

   given by precomposition by f is fully faithful.

3. The conditions in **??** of **??** and **??** of **??** hold. 0117

Example 2.3.1.2. Here are some examples of fully faithful epimorphisms. 0118

1. *Fully Faithful Epimorphisms in* Cats_2. 0119

2. *Fully Faithful Epimorphisms in* $\mathbf{Rel}$. 011A

3. *Fully Faithful Epimorphisms in* Span. 011B

2.4 Strict Epimorphisms 011C

Let $\mathcal{C}$ be a bicategory.

Definition 2.4.1.1. A 1-morphism $f \colon A \to B$ is a **strict epimorphism in** $\mathcal{C}$ if the following equivalent conditions are satisfied: 011D

1. For each $X \in \mathrm{Obj}(\mathcal{C})$, the action on objects 011E

$$f^* \colon \mathrm{Obj}(\mathsf{Hom}_{\mathcal{C}}(B, X)) \to \mathrm{Obj}(\mathsf{Hom}_{\mathcal{C}}(A, X))$$

   of the functor

$$f^* \colon \mathsf{Hom}_{\mathcal{C}}(B, X) \to \mathsf{Hom}_{\mathcal{C}}(A, X)$$

   given by precomposition by f is injective.

2. For each diagram in $\mathcal{C}$ of the form 011F

$$A \xrightarrow{f} B \underset{\psi}{\overset{\phi}{\rightrightarrows}} X,$$

   if $\phi \circ f = \psi \circ f$, then $\phi = \psi$.

Example 2.4.1.2. Here are some examples of strict epimorphisms. 011G

1. *Strict Epimorphisms in* Cats_2. 011H

2. *Strict Epimorphisms in* $\mathbf{Rel}$. 011J

3. *Strict Epimorphisms in* Span. 011K

3 bicategories of spans

Proposition 3.0.1.1. Let A and B be objects of $\mathcal{C}$. 011L

1. *As a Pullback.* We have an isomorphism of categories 011M

$$\mathsf{Span}_{\mathcal{C}}(A,B) \cong \mathcal{C}_{/A} \times_{\mathcal{C}} \mathcal{C}_{/B},\qquad \begin{tikzcd} \mathsf{Span}(A,B) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{C}_{/B} \arrow[d, "忘"] \\ \mathcal{C}_{/A} \arrow[r, "忘"'] & \mathcal{C}. \end{tikzcd}$$

Proof. **??**, *As a Pullback*: In detail, the pullback $\mathcal{C}_{/A} \times_{\mathcal{C}} \mathcal{C}_{/B}$ is the category where

- *Objects.* The objects of $\mathcal{C}_{/A} \times_{\mathcal{C}} \mathcal{C}_{/B}$ consist of pairs $((S,f),(S',g))$ of objects of $\mathcal{C}$ consisting of
  - A pair (S,f) in $\mathrm{Obj}(\mathcal{C}_{/A})$ consisting of an object S of $\mathcal{C}$ and a morphism $f\colon S \to A$ of $\mathcal{C}$;
  - A pair (S',g) in $\mathrm{Obj}(\mathcal{C}_{/B})$ consisting of an object S' of $\mathcal{C}$ and a morphism $g\colon S \to B$ of $\mathcal{C}$;

  such that
  $$\underbrace{忘(S,f)}_{\overset{\mathrm{def}}{=}S} = \underbrace{忘(S',g)}_{\overset{\mathrm{def}}{=}S'}.$$
  Thus the objects of $\mathcal{C}_{/A} \times_{\mathcal{C}} \mathcal{C}_{/B}$ are the same as spans in $\mathcal{C}$ from A to B.

- *Morphisms.* A morphism of $\mathcal{C}_{/A} \times_{\mathcal{C}} \mathcal{C}_{/B}$ from (S,f,g) to (S',f',g') consists of a pair of morphisms
  $$\begin{aligned} \phi&\colon S \to S' \\ \psi&\colon S \to S' \end{aligned}$$
  such that the diagrams
  $$\begin{tikzcd} S \arrow[rr, "\phi"] \arrow[dr, "f"'] & & S' \arrow[dl, "f'"] \\ & A & \end{tikzcd} \qquad \begin{tikzcd} S \arrow[rr, "\psi"] \arrow[dr, "g"'] & & S' \arrow[dl, "g'"] \\ & B & \end{tikzcd}$$

such that

$$\underbrace{忘(\phi)}_{\stackrel{\text{def}}{=}\phi} = \underbrace{忘(\psi)}_{\stackrel{\text{def}}{=}\psi}.$$

Thus the morphisms of $\mathcal{C}_{/A} \times_{\mathcal{C}} \mathcal{C}_{/B}$ are also the same as morphisms of spans in $\mathcal{C}$ from (S, f, g) to (S, f', g').

- *Identities and Composition.* The identities and composition of $\mathcal{C}_{/A} \times_{\mathcal{C}} \mathcal{C}_{/B}$ are also the same as those in $\mathsf{Span}_{\mathcal{C}}(A, B)$.

This finishes the proof. □

Appendices

A Other Chapters