# Sets

December 3, 2023

0000 This chapter (will eventually) contain material on axiomatic set theory, as well as a couple other things.

## Contents



## 0001 1 The Enrichment of Sets in Classical Truth Values

### 0002 1.1 $(-2)$-Categories

0003 **Definition 1.1.1 ► $(-2)$-Categories**

A $(-2)$**-category** is the “necessarily true” truth value.[1,2,3]

[1] Thus, there is only one $(-2)$-category.

[2] A $(-n)$-category for $n = 3, 4, \ldots$ is also the “necessarily true” truth value, coinciding with a $(-2)$-category.

[3] For motivation, see [BS10, p. 13].

### 0004 1.2 $(-1)$-Categories

0005 **Definition 1.2.1 ► $(-1)$-Categories**

A $(-1)$**-category** is a classical truth value.

0006 **Remark 1.2.2 ► Motivation for $(-1)$-Categories**

[1]$(-1)$-categories should be thought of as being "categories enriched in $(-2)$-categories", having a collection of objects and, for each pair of objects, a Hom-object $\mathrm{Hom}(x, y)$ that is a $(-2)$-category (i.e. trivial).

Therefore, a $(-1)$-category $C$ is either ([BS10, pp. 33–34]):

1. *Empty*, having no objects;
2. *Contractible*, having a collection of objects $\{a, b, c, \ldots\}$, but with $\mathrm{Hom}_C(a, b)$ being a $(-2)$-category (i.e. trivial) for all $a, b \in \mathrm{Obj}(C)$, forcing all objects of $C$ to be uniquely isomorphic to each other.

As such, there are only two $(-1)$-categories, up to equivalence:

- The $(-1)$-category false (the empty one);
- The $(-1)$-category true (the contractible one).

[1]For more motivation, see [BS10, p. 13].

0007 **Definition 1.2.3 ► The Poset of Truth Values**

The **poset of truth values**[1] is the poset $(\{\mathrm{true}, \mathrm{false}\}, \preceq)$[2] consisting of

- *The Underlying Set.* The set $\{\mathrm{true}, \mathrm{false}\}$ whose elements are the truth values true and false;
- *The Partial Order.* The partial order

$$\preceq \colon \{\mathrm{true}, \mathrm{false}\} \times \{\mathrm{true}, \mathrm{false}\} \longrightarrow \{\mathrm{true}, \mathrm{false}\}$$

  on $\{\mathrm{true}, \mathrm{false}\}$ defined by[3]

$$\begin{aligned} \mathrm{false} \preceq \mathrm{false} &\overset{\mathrm{def}}{=} \mathrm{true}, \\ \mathrm{true} \preceq \mathrm{false} &\overset{\mathrm{def}}{=} \mathrm{false}, \\ \mathrm{false} \preceq \mathrm{true} &\overset{\mathrm{def}}{=} \mathrm{true}, \\ \mathrm{true} \preceq \mathrm{true} &\overset{\mathrm{def}}{=} \mathrm{true}. \end{aligned}$$

[1] *Further Terminology:* Also called the **poset of** $(-1)$**-categories**.
[2] *Further Notation:* Also written $\{t, f\}$.
[3] This partial order coincides with logical implication.

0008 **Proposition 1.2.4 ► Cartesian Closedness of the Poset of Truth Values**

The poset of truth values $\{t, f\}$ is Cartesian closed with product given by[1]

$$\begin{aligned} t \times t &= t, \\ t \times f &= f, \\ f \times t &= f, \\ f \times f &= f, \end{aligned}$$

and internal Hom $\mathbf{Hom}_{\{t,f\}}$ given by the partial order of $\{t, f\}$, i.e. by

$$\begin{aligned} \mathbf{Hom}_{\{t,f\}}(t, t) &= t, \\ \mathbf{Hom}_{\{t,f\}}(t, f) &= f, \\ \mathbf{Hom}_{\{t,f\}}(f, t) &= t, \\ \mathbf{Hom}_{\{t,f\}}(f, f) &= t. \end{aligned}$$

[1] Note that $\times$ coincides with the “and” operator, while $\mathbf{Hom}_{\{t,f\}}$ coincides with the logical implication operator.

**Proof 1.2.5 ► Proof of Proposition 1.2.4**

Omitted.

## 0009 1.3 0-Categories

000A **Definition 1.3.1 ► 0-Categories**

A 0-**category** is a poset.[1]

[1] *Motivation:* A 0-category is precisely a category enriched in the poset of $(-1)$-categories.

000B **Definition 1.3.2 ► 0-Groupoids**

A $0$-**groupoid** is a $0$-category in which every morphism is invertible.[1]

[1] That is, a *set*.

## 000C 1.4 Tables of Analogies Between Set Theory and Category Theory

Here we record some analogies between notions in set theory and category theory. Note that the analogies relating to presheaves relate equally well to copresheaves, as the opposite $X^{\mathsf{op}}$ of a set $X$ is just $X$ again.

Basics:

| Set Theory | Category Theory |
|---|---|
| Enrichment in $\{\text{true}, \text{false}\}$ | Enrichment in Sets |
| Set $X$ | Category $\mathcal{C}$ |
| Element $x \in X$ | Object $X \in \text{Obj}(\mathcal{C})$ |
| Function | Functor |
| Function $X \longrightarrow \{\text{true}, \text{false}\}$ | Functor $\mathcal{C} \longrightarrow \text{Sets}$ |
| Function $X \longrightarrow \{\text{true}, \text{false}\}$ | Presheaf $\mathcal{C}^{\mathsf{op}} \longrightarrow \text{Sets}$ |

Powersets and categories of presheaves:

| Set Theory | Category Theory |
|---|---|
| Powerset $\mathcal{P}(X)$ | Presheaf category $\mathsf{PSh}(\mathcal{C})$ |
| Characteristic function $\chi_{\{x\}}$ | Representable presheaf $h_X$ |
| Characteristic embedding $\chi_{(-)}\colon X \hookrightarrow \mathcal{P}(X)$ | Yoneda embedding $よ\colon \mathcal{C}^{\mathsf{op}} \hookrightarrow \mathsf{PSh}(\mathcal{C})$ |
| Characteristic relation $\chi_X(-_1, -_2)$ | Hom profunctor $\mathrm{Hom}_{\mathcal{C}}(-_1, -_2)$ |
| The Yoneda lemma for sets $\mathrm{Hom}_{\mathcal{P}(X)}(\chi_x, \chi_U) = \chi_U(x)$ | The Yoneda lemma for categories $\mathrm{Nat}(h_X, \mathcal{F}) \cong \mathcal{F}(X)$ |
| The characteristic embedding is fully faithful, $\mathrm{Hom}_{\mathcal{P}(X)}\big(\chi_x, \chi_y\big) = \chi_X(x, y)$ | The Yoneda embedding is fully faithful, $\mathrm{Nat}(h_X, h_Y) \cong \mathrm{Hom}_{\mathcal{C}}(X, Y)$ |
| Subsets are unions of their elements $U = \bigcup_{x\in U}\{x\}$ or $\chi_U = \operatorname*{colim}_{\chi_x\in\mathsf{Sets}(U,\{\mathsf{t},\mathsf{f}\})}(\chi_x)$ | Presheaves are colimits of representables, $\mathcal{F} \cong \operatorname*{colim}_{h_X\in\int_{\mathcal{C}}\mathcal{F}}(h_X)$ |

Categories of elements:

| Set Theory | Category Theory |
|---|---|
| Assignment $U \mapsto \chi_U$ | Assignment $\mathcal{F} \mapsto \int_{\mathcal{C}}\mathcal{F}$ (the category of elements) |
| Assignment $U \mapsto \chi_U$ giving an isomorphism $\mathcal{P}(X) \cong \mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\})$ | Assignment $\mathcal{F} \mapsto \int_{\mathcal{C}}\mathcal{F}$ giving an equivalence $\mathsf{PSh}(\mathcal{C}) \overset{\text{eq.}}{\cong} \mathsf{DFib}(\mathcal{C})$ |

Functions between powersets and functors between presheaf categories:

| Set Theory | Category Theory |
|---|---|
| Direct image function $f_*\colon \mathcal{P}(X) \to \mathcal{P}(Y)$ | Inverse image functor $f^{-1}\colon \mathsf{PSh}(\mathcal{C}) \to \mathsf{PSh}(\mathcal{D})$ |
| Inverse image function $f^{-1}\colon \mathcal{P}(Y) \to \mathcal{P}(X)$ | Direct image functor $f_*\colon \mathsf{PSh}(\mathcal{D}) \to \mathsf{PSh}(\mathcal{C})$ |
| Direct image with compact support function $f_!\colon \mathcal{P}(X) \to \mathcal{P}(Y)$ | Direct image with compact support functor $f_!\colon \mathsf{PSh}(\mathcal{C}) \to \mathsf{PSh}(\mathcal{D})$ |

Relations and profunctors:

| Set Theory | Category Theory |
|---|---|
| Relation $R\colon X \times Y \to \{\mathsf{t}, \mathsf{f}\}$ | Profunctor $\mathfrak{p}\colon \mathcal{D}^{\mathsf{op}} \times \mathcal{C} \to \mathsf{Sets}$ |
| Relation $R\colon X \to \mathcal{P}(Y)$ | Profunctor $\mathfrak{p}\colon \mathcal{C} \to \mathsf{PSh}(\mathcal{D})$ |
| Relation as a cocontinuous morphism of posets $R\colon (\mathcal{P}(X), \subset) \to (\mathcal{P}(Y), \subset)$ | Profunctor as a colimit-preserving functor $\mathfrak{p}\colon \mathsf{PSh}(\mathcal{C}) \to \mathsf{PSh}(\mathcal{D})$ |

# Appendices

## A Other Chapters