Categories

December 27, 2023

Create tags (see [MSE 350788] for some of these):

1. **??**
2. **??**
3. **??**
4. **??**
5. **??**
6. **??**
7. **??**
8. write material on sections and retractions
9. define bicategory $\mathsf{Adj}(C)$
10. https://www.google.com/search?q=category+of+categories+is+not+locally+cartesian+closed
11. https://math.stackexchange.com/questions/2864916/are-there-important-locally-cartesian-closed-categories-that-actually-are-not-ca
12. Cats is not locally Cartesian closed: f^* does have a left adjoint (the proof for fibred sets seems to apply for any category with pullbacks), but does not have a right adjoint, as proved in example 1.7 of https://sinhp.github.io/files/CT/notes_on_lcccs.pdf
13. internal **Hom** in categories of co/Cartesian fibrations

14. https://mathoverflow.net/questions/460146/universal-property-of-isbell-duality
15. http://www.tac.mta.ca/tac/volumes/36/12/36-12abs.html
16. Cartesian closed categories and locally Cartesian closed categories
    (a) https://ncatlab.org/nlab/show/locally+cartesian+closed+functor
    (b) https://ncatlab.org/nlab/show/cartesian+closed+functor
    (c) https://ncatlab.org/nlab/show/locally+cartesian+closed+category
    (d) https://ncatlab.org/nlab/show/Frobenius+reciprocity
17. https://math.stackexchange.com/questions/3657046/the-inverse-of-a-natural-isomorphism-is-a-natural-isomorphism to justify adjunctions via homs
18. https://ncatlab.org/nlab/show/enrichment+versus+internalisation
19. https://mathoverflow.net/questions/382239/proof-that-a-cartesian-category-is-monoidal

Contents

1 Categories

1.1 Foundations

Definition 1.1.1.1. A **category** $(\mathcal{C}, \circ^{\mathcal{C}}, \mathbb{1}^{\mathcal{C}})$ consists of[1,2]

- *Objects.* A class $\mathrm{Obj}(\mathcal{C})$ of **objects**;
- *Morphisms.* For each $A, B \in \mathrm{Obj}(\mathcal{C})$, a class $\mathrm{Hom}_{\mathcal{C}}(A, B)$, called the **class of morphisms of $\mathcal{C}$ from A to B**;
- *Identities.* For each $A \in \mathrm{Obj}(\mathcal{C})$, a map of sets

  $$\mathbb{1}^{\mathcal{C}}_{A} \colon \mathrm{pt} \to \mathrm{Hom}_{\mathcal{C}}(A, A),$$

  called the **unit map of $\mathcal{C}$ at A**, determining a morphism

  $$\mathrm{id}_A \colon A \to A$$

  of $\mathcal{C}$, called the **identity morphism of A**;
- *Composition.* For each $A, B, C \in \mathrm{Obj}(\mathcal{C})$, a map of sets

  $$\circ^{\mathcal{C}}_{A,B,C} \colon \mathrm{Hom}_{\mathcal{C}}(B, C) \times \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{C}}(A, C),$$

  called the **composition map of $\mathcal{C}$ at (A, B, C)**;

such that the following conditions are satisfied:

[1] *Further Notation:* We also write $\mathcal{C}(A, B)$ for $\mathrm{Hom}_{\mathcal{C}}(A, B)$.
[2] *Further Notation:* We write $\mathrm{Mor}(\mathcal{C})$ for the class of all morphisms of $\mathcal{C}$.

1. *Associativity.* The diagram

$$
\begin{tikzcd}[column sep=small]
 & \mathrm{Hom}_{\mathcal{C}}(C,D)\times(\mathrm{Hom}_{\mathcal{C}}(B,C)\times\mathrm{Hom}_{\mathcal{C}}(A,B)) \arrow[dr, "\mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(C,D)}\times\circ^{\mathcal{C}}_{A,B,C}"] & \\
 (\mathrm{Hom}_{\mathcal{C}}(C,D)\times\mathrm{Hom}_{\mathcal{C}}(B,C))\times\mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[ur, dashed, "\alpha^{\mathsf{Sets}}_{\mathrm{Hom}_{\mathcal{C}}(C,D),\mathrm{Hom}_{\mathcal{C}}(B,C),\mathrm{Hom}_{\mathcal{C}}(A,B)}", "\sim"'] \arrow[d, "\circ^{\mathcal{C}}_{B,C,D}\times\mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(A,B)}"'] & & \mathrm{Hom}_{\mathcal{C}}(C,D)\times\mathrm{Hom}_{\mathcal{C}}(A,C) \arrow[d, "\circ^{\mathcal{C}}_{A,C,D}"] \\
 \mathrm{Hom}_{\mathcal{C}}(B,D)\times\mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[rr, "\circ^{\mathcal{C}}_{A,B,D}"'] & & \mathrm{Hom}_{\mathcal{C}}(A,D)
\end{tikzcd}
$$

commutes, i.e. for each composable triple (f, g, h) of morphisms of $\mathcal{C}$, we have

$$(f \circ g) \circ h = f \circ (g \circ h).$$

2. *Left Unitality.* The diagram

$$
\begin{tikzcd}
\mathrm{pt}\times\mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[d, "\mathbb{1}^{\mathcal{C}}_{B}\times\mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(A,B)}"'] \arrow[dr, dashed, "\lambda^{\mathsf{Sets}}_{\mathrm{Hom}_{\mathcal{C}}(A,B)}", "\sim"'] & \\
\mathrm{Hom}_{\mathcal{C}}(B,B)\times\mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[r, "\circ^{\mathcal{C}}_{A,B,B}"'] & \mathrm{Hom}_{\mathcal{C}}(A,B)
\end{tikzcd}
$$

commutes, i.e. for each morphism $f \colon A \to B$ of $\mathcal{C}$, we have

$$\mathrm{id}_B \circ f = f.$$

3. *Right Unitality.* The diagram

$$
\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(A,B)\times\mathrm{pt} \arrow[d, "\mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(A,B)}\times\mathbb{1}^{\mathcal{C}}_{A}"'] \arrow[dr, dashed, "\rho^{\mathsf{Sets}}_{\mathrm{Hom}_{\mathcal{C}}(A,B)}", "\sim"'] & \\
\mathrm{Hom}_{\mathcal{C}}(A,B)\times\mathrm{Hom}_{\mathcal{C}}(A,A) \arrow[r, "\circ^{\mathcal{C}}_{A,A,B}"'] & \mathrm{Hom}_{\mathcal{C}}(A,B)
\end{tikzcd}
$$

commutes, i.e. for each morphism $f\colon A \to B$ of $\mathcal{C}$, we have

$$f \circ \mathrm{id}_A = f.$$

Definition 1.1.1.2. Let κ be a regular cardinal. A category $\mathcal{C}$ is

1. **Locally small** if, for each $A, B \in \mathrm{Obj}(\mathcal{C})$, the class $\mathrm{Hom}_{\mathcal{C}}(A, B)$ is a set.
2. **Locally essentially small** if, for each $A, B \in \mathrm{Obj}(\mathcal{C})$, the class

$$\mathrm{Hom}_{\mathcal{C}}(A, B)/\{\text{isomorphisms}\}$$

   is a set.
3. **Small** if $\mathcal{C}$ is locally small and $\mathrm{Obj}(\mathcal{C})$ is a set.
4. **κ-Small** if $\mathcal{C}$ is locally small, $\mathrm{Obj}(\mathcal{C})$ is a set, and we have $\#\mathrm{Obj}(\mathcal{C}) < \kappa$.

1.2 Examples of Categories

Example 1.2.1.1. The **punctual category**[3] is the category pt where

- *Objects.* We have

$$\mathrm{Obj}(\mathsf{pt}) \overset{\text{def}}{=} \{\star\};$$

- *Morphisms.* The unique Hom-set of pt is defined by

$$\mathrm{Hom}_{\mathsf{pt}}(\star, \star) \overset{\text{def}}{=} \{\mathrm{id}_\star\};$$

- *Identities.* The unit map

$$\mathbb{1}^{\mathsf{pt}}_{\star}\colon \mathrm{pt} \to \mathrm{Hom}_{\mathsf{pt}}(\star, \star)$$

  of pt at $\star$ is defined by

$$\mathrm{id}^{\mathsf{pt}}_{\star} \overset{\text{def}}{=} \mathrm{id}_\star;$$

- *Composition.* The composition map

$$\circ^{\mathsf{pt}}_{\star,\star,\star}\colon \mathrm{Hom}_{\mathsf{pt}}(\star, \star) \times \mathrm{Hom}_{\mathsf{pt}}(\star, \star) \to \mathrm{Hom}_{\mathsf{pt}}(\star, \star)$$

  of pt at $(\star, \star, \star)$ is given by the bijection $\mathrm{pt} \times \mathrm{pt} \cong \mathrm{pt}$.

[3] *Further Terminology:* Also called the **singleton category**.

Example 1.2.1.2. We have an isomorphism of categories[4]

$$\mathsf{Mon} \cong \mathsf{pt} \underset{\mathsf{Sets}}{\times} \mathsf{Cats}, \qquad \begin{tikzcd} \mathsf{Mon} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Cats} \arrow[d, "\mathrm{Obj}"] \\ \mathsf{pt} \arrow[r, "{[\mathrm{pt}]}"'] & \mathsf{Sets} \end{tikzcd}$$

via the delooping functor $\mathsf{B} \colon \mathsf{Mon} \to \mathsf{Cats}$ of **??** of **??**.

Proof. Omitted. □

Example 1.2.1.3. The **empty category** is the category $\emptyset_{\mathsf{cat}}$ where

- *Objects.* We have
$$\mathrm{Obj}(\emptyset_{\mathsf{cat}}) \overset{\mathrm{def}}{=} \emptyset;$$
- *Morphisms.* We have
$$\mathrm{Mor}(\emptyset_{\mathsf{cat}}) \overset{\mathrm{def}}{=} \emptyset;$$
- *Identities and Composition.* Having no objects, $\emptyset_{\mathsf{cat}}$ has no unit nor composition maps.

Example 1.2.1.4. The ***n*th ordinal category** is the category $\ltimes$ where[5]

- *Objects.* We have
$$\mathrm{Obj}(\ltimes) \overset{\mathrm{def}}{=} \{[0], \ldots, [n]\};$$

[4]This can be enhanced to an isomorphism of 2-categories

$$\mathsf{Mon}_{2-\mathsf{disc}} \cong \mathsf{pt}_{\mathsf{bi}} \underset{\mathsf{Sets}_{2-\mathsf{disc}}}{\times} \mathsf{Cats}_{2,*}, \qquad \begin{tikzcd} \mathsf{Mon}_{2-\mathsf{disc}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Cats}_{2,*} \arrow[d, "\mathrm{Obj}"] \\ \mathsf{pt}_{\mathsf{bi}} \arrow[r, "{[\mathrm{pt}]}"'] & \mathsf{Sets}_{2-\mathsf{disc}} \end{tikzcd}$$

between the discrete 2-category $\mathsf{Mon}_{2-\mathsf{disc}}$ on Mon and the 2-category of pointed categories with one object.

[5]In other words, $\ltimes$ is the category associated to the poset

$$[0] \to [1] \to \cdots \to [n-1] \to [n].$$

The category $\ltimes$ for $n \geq 2$ may also be defined in terms of $\nvdash$ and joins: we have isomorphisms

- *Morphisms.* For each $[i], [j] \in \mathrm{Obj}(\ltimes)$, we have

$$\mathrm{Hom}_{\ltimes}([i],[j]) \overset{\text{def}}{=} \begin{cases} \left\{\mathrm{id}_{[i]}\right\} & \text{if } [i] = [j], \\ \{[i] \to [j]\} & \text{if } [j] < [i], \\ \emptyset & \text{if } [j] > [i]; \end{cases}$$

- *Identities.* For each $[i] \in \mathrm{Obj}(\ltimes)$, the unit map

$$\nVdash^{\ltimes}_{[i]} \colon \mathrm{pt} \to \mathrm{Hom}_{\ltimes}([i],[i])$$

  of $\ltimes$ at $[i]$ is defined by

$$\mathrm{id}^{\ltimes}_{[i]} \overset{\text{def}}{=} \mathrm{id}_{[i]};$$

- *Composition.* For each $[i], [j], [k] \in \mathrm{Obj}(\ltimes)$, the composition map

$$\circ^{\ltimes}_{[i],[j],[k]} \colon \mathrm{Hom}_{\ltimes}([j],[k]) \times \mathrm{Hom}_{\ltimes}([i],[j]) \to \mathrm{Hom}_{\ltimes}([i],[k])$$

  of $\ltimes$ at $([i],[j],[k])$ is defined by

$$\begin{aligned} \mathrm{id}_{[i]} \circ \mathrm{id}_{[i]} &= \mathrm{id}_{[i]}, \\ ([j] \to [k]) \circ ([i] \to [j]) &= ([i] \to [k]). \end{aligned}$$

Example 1.2.1.5. Here we list all the other categories that appear throughout this work.

- The category Sets_* of pointed sets of Pointed Sets, **??**.

---

of categories

$$\begin{aligned} \nVdash &\cong \nvdash \star \nvdash, \\ \nvDash &\cong \nVdash \star \nvdash \\ &\cong (\nvdash \star \nvdash) \star \nvdash, \\ \nVDash &\cong \nvDash \star \nvdash \\ &\cong (\nVdash \star \nvdash) \star \nvdash \\ &\cong ((\nvdash \star \nvdash) \star \nvdash) \star \nvdash, \\ \ntrianglerighteq &\cong \nVDash \star \nvdash \\ &\cong (\nvDash \star \nvdash) \star \nvdash \\ &\cong ((\nVdash \star \nvdash) \star \nvdash) \star \nvdash \\ &\cong (((\nvdash \star \nvdash) \star \nvdash) \star \nvdash) \star \nvdash, \end{aligned}$$

and so on.

- The category Rel of sets and relations of Relations, **??**.
- The category $\mathsf{Span}(A, B)$ of spans from a set A to a set B of Spans, **??**.
- The category $\mathsf{ISets}(K)$ of K-indexed sets of Indexed Sets, **??**.
- The category ISets of indexed sets of Indexed Sets, **??**.
- The category $\mathsf{FibSets}(K)$ of K-fibred sets of Fibred Sets, **??**.
- The category $\mathsf{FibSets}$ of fibred sets of Fibred Sets, **??**.

1.3 Subcategories

Let $\mathcal{C}$ be a category.

Definition 1.3.1.1. A **subcategory** of $\mathcal{C}$ is a category $\mathcal{A}$ satisfying the following conditions:

1. *Objects.* We have $\mathrm{Obj}(\mathcal{A}) \subset \mathrm{Obj}(\mathcal{C})$.
2. *Morphisms.* For each $A, B \in \mathrm{Obj}(\mathcal{A})$, we have
$$\mathrm{Hom}_{\mathcal{A}}(A, B) \subset \mathrm{Hom}_{\mathcal{C}}(A, B).$$
3. *Identities.* For each $A \in \mathrm{Obj}(\mathcal{A})$, we have
$$\mathbb{1}^{\mathcal{A}}_{A} = \mathbb{1}^{\mathcal{C}}_{A}.$$
4. *Composition.* For each $A, B, C \in \mathrm{Obj}(\mathcal{A})$, we have
$$\circ^{\mathcal{A}}_{A,B,C} = \circ^{\mathcal{C}}_{A,B,C}.$$

Definition 1.3.1.2. A subcategory $\mathcal{A}$ of $\mathcal{C}$ is **full** if the canonical inclusion functor $\mathcal{A} \to \mathcal{C}$ is full, i.e. if, for each $A, B \in \mathrm{Obj}(\mathcal{A})$, the inclusion

$$\iota_{A,B} \colon \mathrm{Hom}_{\mathcal{A}}(A, B) \hookrightarrow \mathrm{Hom}_{\mathcal{C}}(A, B)$$

is surjective (and thus bijective).

Definition 1.3.1.3. A subcategory $\mathcal{A}$ of a category $\mathcal{C}$ is **strictly full** if it satisfies the following conditions:

1. *Fullness.* The subcategory $\mathcal{A}$ is full.

2. *Closedness Under Isomorphisms.* The class $\mathrm{Obj}(\mathcal{A})$ is closed under isomorphisms.[6]

Definition 1.3.1.4. A subcategory $\mathcal{A}$ of $\mathcal{C}$ is **wide**[7] if $\mathrm{Obj}(\mathcal{A}) = \mathrm{Obj}(\mathcal{C})$.

1.4 Skeletons of Categories

Definition 1.4.1.1. A[8] **skeleton** of a category $\mathcal{C}$ is a full subcategory $\mathsf{Sk}(\mathcal{C})$ with one object from each isomorphism class of objects of $\mathcal{C}$.

Definition 1.4.1.2. A category $\mathcal{C}$ is **skeletal** if $\mathcal{C} \cong \mathsf{Sk}(\mathcal{C})$.[9]

Proposition 1.4.1.3. Let $\mathcal{C}$ be a category.

1. *Existence.* Assuming the axiom of choice, $\mathsf{Sk}(\mathcal{C})$ always exists.

2. *Pseudofunctoriality.* The assignment $\mathcal{C} \mapsto \mathsf{Sk}(\mathcal{C})$ defines a pseudofunctor
$$\mathsf{Sk}\colon \mathsf{Cats}_2 \to \mathsf{Cats}_2.$$

3. *Uniqueness Up to Equivalence.* Any two skeletons of $\mathcal{C}$ are equivalent.

4. *Inclusions of Skeletons Are Equivalences.* The inclusion
$$\iota_{\mathcal{C}}\colon \mathsf{Sk}(\mathcal{C}) \hookrightarrow \mathcal{C}$$
of a skeleton of $\mathcal{C}$ into $\mathcal{C}$ is an equivalence of categories.

Proof. *Item 1, Existence*: See [nLab23, Section "Existence of Skeletons of Categories"].
Item 2, Pseudofunctoriality: See [nLab23, Section "Skeletons as an Endo-Pseudofunctor on $\mathfrak{Cat}$"].
Item 3, Uniqueness Up to Equivalence: Clear.
Item 4, Inclusions of Skeletons Are Equivalences: Clear. □

1.5 Precomposition and Postcomposition

Let $\mathcal{C}$ be a category and let $A, B, C \in \mathrm{Obj}(\mathcal{C})$.

[6]That is, given $A \in \mathrm{Obj}(\mathcal{A})$ and $C \in \mathrm{Obj}(\mathcal{C})$, if $C \cong A$, then $C \in \mathrm{Obj}(\mathcal{A})$.

[7]*Further Terminology:* Also called **lluf**.

[8]Due to Item 3 of Proposition 1.4.1.3, we often refer to any such full subcategory $\mathsf{Sk}(\mathcal{C})$ of $\mathcal{C}$ as *the* skeleton of $\mathcal{C}$.

[9]That is, $\mathcal{C}$ is **skeletal** if isomorphic objects of $\mathcal{C}$ are equal.

Definition 1.5.1.1. Let $f\colon A \to B$ and $g\colon B \to C$ be morphisms of $\mathcal{C}$.

- The **precomposition function associated to** f is the function

  $$f^*\colon \mathrm{Hom}_{\mathcal{C}}(B, C) \to \mathrm{Hom}_{\mathcal{C}}(A, C)$$

  defined by

  $$f^*(\phi) \stackrel{\text{def}}{=} \phi \circ f$$

  for each $\phi \in \mathrm{Hom}_{\mathcal{C}}(B, C)$.

- The **postcomposition function associated to** g is the function

  $$g_*\colon \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{C}}(A, C)$$

  defined by

  $$g_*(\phi) \stackrel{\text{def}}{=} g \circ \phi$$

  for each $\phi \in \mathrm{Hom}_{\mathcal{C}}(A, B)$.

Proposition 1.5.1.2. Let $A, B, C, D \in \mathrm{Obj}(\mathcal{C})$ and let $f\colon A \to B$ and $g\colon B \to C$ be morphisms of $\mathcal{C}$.

1. *Interaction Between Precomposition and Postcomposition.* We have

$$g_* \circ f^* = f^* \circ g_*, \qquad \begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(B, C) \arrow[r, "g_*"] \arrow[d, "f^*"'] & \mathrm{Hom}_{\mathcal{C}}(B, D) \arrow[d, "f^*"] \\ \mathrm{Hom}_{\mathcal{C}}(A, C) \arrow[r, "g_*"'] & \mathrm{Hom}_{\mathcal{C}}(A, D). \end{tikzcd}$$

2. *Interaction With Composition I.* We have

$$(g \circ f)^* = f^* \circ g^*, \qquad \begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(X, A) \arrow[r, "f_*"] \arrow[dr, "(g\circ f)_*"'] & \mathrm{Hom}_{\mathcal{C}}(X, B) \arrow[d, "g_*"] \\ & \mathrm{Hom}_{\mathcal{C}}(X, C), \end{tikzcd}$$

$$(g \circ f)_* = g_* \circ f_*, \qquad \begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(C, X) \arrow[r, "g^*"] \arrow[dr, "(g\circ f)^*"'] & \mathrm{Hom}_{\mathcal{C}}(B, X) \arrow[d, "f^*"] \\ & \mathrm{Hom}_{\mathcal{C}}(A, X). \end{tikzcd}$$

3. *Interaction With Composition II.* We have

$$\begin{tikzcd} \mathrm{pt} \arrow[r, "{[f]}"] \arrow[dr, "{[g\circ f]}"'] & \mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[d, "g_*"] \\ & \mathrm{Hom}_{\mathcal{C}}(A,C) \end{tikzcd} \qquad \begin{aligned} [g\circ f] &= g_*\circ[f], \\ [g\circ f] &= f^*\circ[g], \end{aligned} \qquad \begin{tikzcd} \mathrm{pt} \arrow[r, "{[g]}"] \arrow[dr, "{[g\circ f]}"'] & \mathrm{Hom}_{\mathcal{C}}(B,C) \arrow[d, "f^*"] \\ & \mathrm{Hom}_{\mathcal{C}}(A,C). \end{tikzcd}$$

4. *Interaction With Composition III.* We have

$$f^*\circ\circ^{\mathcal{C}}_{A,B,C} = \circ^{\mathcal{C}}_{X,B,C}\circ(f^*\times\mathsf{id}), \qquad \begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(B,C)\times\mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[r, "\circ^{\mathcal{C}}_{A,B,C}"] \arrow[d, "\mathsf{id}\times f^*"'] & \mathrm{Hom}_{\mathcal{C}}(A,C) \arrow[d, "f^*"] \\ \mathrm{Hom}_{\mathcal{C}}(B,C)\times\mathrm{Hom}_{\mathcal{C}}(X,B) \arrow[r, "\circ^{\mathcal{C}}_{X,B,C}"'] & \mathrm{Hom}_{\mathcal{C}}(X,C), \end{tikzcd}$$

$$g_*\circ\circ^{\mathcal{C}}_{A,B,C} = \circ^{\mathcal{C}}_{A,B,D}\circ(\mathsf{id}\times g_*), \qquad \begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(B,C)\times\mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[r, "\circ^{\mathcal{C}}_{A,B,C}"] \arrow[d, "g_*\times\mathsf{id}"'] & \mathrm{Hom}_{\mathcal{C}}(A,C) \arrow[d, "g^*"] \\ \mathrm{Hom}_{\mathcal{C}}(B,D)\times\mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[r, "\circ^{\mathcal{C}}_{A,B,D}"'] & \mathrm{Hom}_{\mathcal{C}}(A,D). \end{tikzcd}$$

5. *Interaction With Identities.* We have

$$\begin{aligned} (\mathrm{id}_A)^* &= \mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(A,B)}, \\ (\mathrm{id}_B)_* &= \mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(A,B)}. \end{aligned}$$

Proof. *Item 1, Interaction Between Precomposition and Postcomposition*: Clear.
Item 2, Interaction With Composition I: Clear.
Item 3, Interaction With Composition II: Clear.
Item 4, Interaction With Composition III: Clear.
Item 5, Interaction With Identities: Clear. □

2 The Quadruple Adjunction With Sets

2.1 Statement

Let $\mathcal{C}$ be a category.

Proposition 2.1.1.1. We have a quadruple adjunction

$$
(\pi_0 \dashv (-)_{\mathsf{disc}} \dashv \mathrm{Obj} \dashv (-)_{\mathsf{indisc}})\colon
\begin{tikzcd}
\mathsf{Sets} \arrow[r, "(-)_{\mathsf{disc}}" description] \arrow[r, bend right=60, "(-)_{\mathsf{indisc}}"'] & \mathsf{Cats}, \arrow[l, bend right=60, "\pi_0"'] \arrow[l, bend left=20, "\mathrm{Obj}" description]
\end{tikzcd}
$$

witnessed by bijections of sets

$$
\begin{aligned}
\mathrm{Hom}_{\mathsf{Sets}}(\pi_0(\mathcal{C}), X) &\cong \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, X_{\mathsf{disc}}),\\
\mathrm{Hom}_{\mathsf{Cats}}(X_{\mathsf{disc}}, \mathcal{C}) &\cong \mathrm{Hom}_{\mathsf{Sets}}(X, \mathrm{Obj}(\mathcal{C})),\\
\mathrm{Hom}_{\mathsf{Sets}}(\mathrm{Obj}(\mathcal{C}), X) &\cong \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, X_{\mathsf{indisc}}),
\end{aligned}
$$

natural in $\mathcal{C} \in \mathrm{Obj}(\mathsf{Cats})$ and $X \in \mathrm{Obj}(\mathsf{Sets})$, where

- The functor
  $$\pi_0\colon \mathsf{Cats} \to \mathsf{Sets},$$
  the **connected components functor**, is the functor sending a category to its set of connected components of Definition 2.3.1.1.

- The functor
  $$(-)_{\mathsf{disc}}\colon \mathsf{Sets} \to \mathsf{Cats},$$
  the **discrete category functor**, is the functor sending a set to its associated discrete category of Definition 2.5.1.1.

- The functor
  $$\mathrm{Obj}\colon \mathsf{Cats} \to \mathsf{Sets},$$
  the **object functor**, is the functor sending a category to its set of objects.

- The functor
  $$(-)_{\mathsf{indisc}}\colon \mathsf{Sets} \to \mathsf{Cats},$$
  the **indiscrete category functor**, is the functor sending a set to its associated indiscrete category of Definition 2.6.1.1.

Proof. Omitted. □

2.2 Connected Components of Categories

Let $\mathcal{C}$ be a category.

Definition 2.2.1.1. A **connected component** of $\mathcal{C}$ is a full subcategory $\mathcal{I}$ of $\mathcal{C}$ satisfying the following conditions:[10]

1. *Non-Emptiness.* We have $\mathrm{Obj}(\mathcal{I}) \neq \emptyset$.

2. *Connectedness.* There exists a zigzag of arrows between any two objects of $\mathcal{I}$.

2.3 Sets of Connected Components of Categories

Let $\mathcal{C}$ be a category.

Definition 2.3.1.1. The **set of connected components of** $\mathcal{C}$ is the set $\pi_0(\mathcal{C})$ whose elements are the connected components of $\mathcal{C}$.

Proposition 2.3.1.2. Let $\mathcal{C}$ be a category.

1. *Functoriality.* The assignment $\mathcal{C} \mapsto \pi_0(\mathcal{C})$ defines a functor

$$\pi_0 \colon \mathsf{Cats} \to \mathsf{Sets}.$$

2. *Adjointness.* We have a quadruple adjunction

$$(\pi_0 \dashv (-)_{\mathsf{disc}} \dashv \mathrm{Obj} \dashv (-)_{\mathsf{indisc}}) \colon \quad \begin{tikzcd}[column sep=huge] \mathsf{Sets} \arrow[r, bend left=15, "(-)_{\mathsf{disc}}" description] \arrow[r, bend right=60, "(-)_{\mathsf{indisc}}"'] \arrow[r, phantom, shift left=5, "\perp"] \arrow[r, phantom, "\perp"] \arrow[r, phantom, shift right=5, "\perp"] & \mathsf{Cats} \arrow[l, bend right=60, "\pi_0"'] \arrow[l, bend left=15, "\mathrm{Obj}" description] \end{tikzcd}.$$

3. *Interaction With Groupoids.* If $\mathcal{C}$ is a groupoid, then we have an isomorphism of categories

$$\pi_0(\mathcal{C}) \cong \mathrm{K}(\mathcal{C}),$$

where $\mathrm{K}(\mathcal{C})$ is the set of isomorphism classes of $\mathcal{C}$ of **??**.

[10] In other words, a **connected component** of $\mathcal{C}$ is an element of the set $\mathrm{Obj}(\mathcal{C})/\sim$ with $\sim$ the equivalence relation generated by the relation $\sim'$ obtained by declaring $A \sim' B$ iff there exists a morphism of $\mathcal{C}$ from A to B.

4. *Preservation of Colimits.* The functor π_0 of Item 1 preserves colimits. In particular, we have bijections of sets

$$\pi_0(\mathcal{C} \coprod \mathcal{D}) \cong \pi_0(\mathcal{C}) \coprod \pi_0(\mathcal{D}),$$
$$\pi_0(\mathcal{C} \coprod_{\mathcal{E}} \mathcal{D}) \cong \pi_0(\mathcal{C}) \coprod_{\pi_0(\mathcal{E})} \pi_0(\mathcal{D}),$$
$$\pi_0\left(\mathrm{CoEq}\left(\mathcal{C} \underset{G}{\overset{F}{\rightrightarrows}} \mathcal{D}\right)\right) \cong \mathrm{CoEq}\left(\pi_0(\mathcal{C}) \underset{\pi_0(G)}{\overset{\pi_0(F)}{\rightrightarrows}} \pi_0(\mathcal{D})\right),$$

natural in $\mathcal{C}, \mathcal{D}, \mathcal{E} \in \mathrm{Obj}(\mathsf{Cats})$.

5. *Symmetric Strong Monoidality With Respect to Coproducts.* The connected components functor of Item 1 has a symmetric strong monoidal structure

$$\left(\pi_0, \pi_0^{\coprod}, \pi_{0|\mathbb{1}}^{\coprod}\right) : (\mathsf{Cats}, \coprod, \emptyset_{\mathsf{cat}}) \to (\mathsf{Sets}, \coprod, \emptyset),$$

being equipped with isomorphisms

$$\pi_{0|\mathcal{C},\mathcal{D}}^{\coprod} : \pi_0(\mathcal{C}) \coprod \pi_0(\mathcal{D}) \xrightarrow{\cong} \pi_0(\mathcal{C} \coprod \mathcal{D}),$$
$$\pi_{0|\mathbb{1}}^{\coprod} : \emptyset \xrightarrow{\cong} \pi_0(\emptyset_{\mathsf{cat}}),$$

natural in $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$.

6. *Symmetric Strong Monoidality With Respect to Products.* The connected components functor of Item 1 has a symmetric strong monoidal structure

$$\left(\pi_0, \pi_0^{\otimes}, \pi_{0|\mathbb{1}}^{\otimes}\right) : (\mathsf{Cats}, \times, \mathsf{pt}) \to (\mathsf{Sets}, \times, \mathrm{pt}),$$

being equipped with isomorphisms

$$\pi_{0|\mathcal{C},\mathcal{D}}^{\otimes} : \pi_0(\mathcal{C}) \times \pi_0(\mathcal{D}) \xrightarrow{\cong} \pi_0(\mathcal{C} \times \mathcal{D}),$$
$$\pi_{0|\mathbb{1}}^{\otimes} : \mathrm{pt} \xrightarrow{\cong} \pi_0(\mathsf{pt}),$$

natural in $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$.

Proof. *Item 1, Functoriality*: Clear.
Item 2, Adjointness: This is proved in Proposition 2.1.1.1.
Item 3, Interaction With Groupoids: Clear.
Item 4, Preservation of Colimits: This follows from Item 2 and **??** of **??**.
Item 5, Symmetric Strong Monoidality With Respect to Coproducts: Omitted.
Item 6, Symmetric Strong Monoidality With Respect to Products: Omitted. □

2.4 Connected Categories

Definition 2.4.1.1. A category $\mathcal{C}$ is **connected** if $\pi_0(\mathcal{C}) \cong \mathrm{pt}$.[11,12]

2.5 Discrete Categories

Let X be a set.

Definition 2.5.1.1. The **discrete category on a set** X is the category X_{disc} where

- *Objects.* We have
$$\mathrm{Obj}(X_{\mathsf{disc}}) \overset{\text{def}}{=} X;$$
- *Morphisms.* For each $A, B \in \mathrm{Obj}(X_{\mathsf{disc}})$, we have
$$\mathrm{Hom}_{X_{\mathsf{disc}}}(A, B) \overset{\text{def}}{=} \begin{cases} \mathrm{id}_A & \text{if } A = B, \\ \emptyset & \text{if } A \neq B; \end{cases}$$
- *Identities.* For each $A \in \mathrm{Obj}(X_{\mathsf{disc}})$, the unit map
$$\mathbb{1}^{X_{\mathsf{disc}}}_A \colon \mathrm{pt} \to \mathrm{Hom}_{X_{\mathsf{disc}}}(A, A)$$
of X_{disc} at A is defined by
$$\mathrm{id}^{X_{\mathsf{disc}}}_A \overset{\text{def}}{=} \mathrm{id}_A;$$
- *Composition.* For each $A, B, C \in \mathrm{Obj}(X_{\mathsf{disc}})$, the composition map
$$\circ^{X_{\mathsf{disc}}}_{A,B,C} \colon \mathrm{Hom}_{X_{\mathsf{disc}}}(B, C) \times \mathrm{Hom}_{X_{\mathsf{disc}}}(A, B) \to \mathrm{Hom}_{X_{\mathsf{disc}}}(A, C)$$
of X_{disc} at (A, B, C) is defined by
$$\mathrm{id}_A \circ \mathrm{id}_A \overset{\text{def}}{=} \mathrm{id}_A.$$

Proposition 2.5.1.2. Let X be a set.

1. *Functoriality.* The assignment $X \mapsto X_{\mathsf{disc}}$ defines a functor
$$(-)_{\mathsf{disc}} \colon \mathsf{Sets} \to \mathsf{Cats}.$$

[11] *Further Terminology:* A category is **disconnected** if it is not connected.
[12] *Example:* A groupoid is connected iff any two of its objects are isomorphic.

2. *Adjointness.* We have a quadruple adjunction

$$(\pi_0 \dashv (-)_{\mathsf{disc}} \dashv \mathrm{Obj} \dashv (-)_{\mathsf{indisc}})\colon \begin{tikzcd} \mathsf{Sets} \arrow[r, shift left=2, "(-)_{\mathsf{disc}}"] \arrow[r, bend right=60, "(-)_{\mathsf{indisc}}"'] \arrow[r, phantom, shift left=5, "\perp"] \arrow[r, phantom, "\perp"] \arrow[r, phantom, shift right=5, "\perp"] & \mathsf{Cats}. \arrow[l, bend right=60, "\pi_0"'] \arrow[l, shift left=2, "\mathrm{Obj}"] \end{tikzcd}$$

3. *Symmetric Strong Monoidality With Respect to Coproducts.* The functor of Item 1 has a symmetric strong monoidal structure

$$\left((-)_{\mathsf{disc}}, (-)^{\coprod}_{\mathsf{disc}}, (-)^{\coprod}_{\mathsf{disc}|\mathbb{1}}\right)\colon (\mathsf{Sets}, \coprod, \emptyset) \to (\mathsf{Cats}, \coprod, \emptyset_{\mathsf{cat}}),$$

being equipped with isomorphisms

$$(-)^{\coprod}_{\mathsf{disc}|X,Y}\colon X_{\mathsf{disc}} \coprod Y_{\mathsf{disc}} \xrightarrow{\cong} (X \coprod Y)_{\mathsf{disc}},$$
$$(-)^{\coprod}_{\mathsf{disc}|\mathbb{1}}\colon \emptyset_{\mathsf{cat}} \xrightarrow{\cong} \emptyset_{\mathsf{disc}},$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

4. *Symmetric Strong Monoidality With Respect to Products.* The functor of Item 1 has a symmetric strong monoidal structure

$$\left((-)_{\mathsf{disc}}, (-)^{\otimes}_{\mathsf{disc}}, (-)^{\otimes}_{\mathsf{disc}|\mathbb{1}}\right)\colon (\mathsf{Sets}, \times, \mathrm{pt}) \to (\mathsf{Cats}, \times, \mathsf{pt}),$$

being equipped with isomorphisms

$$(-)^{\otimes}_{\mathsf{disc}|X,Y}\colon X_{\mathsf{disc}} \times Y_{\mathsf{disc}} \xrightarrow{\cong} (X \times Y)_{\mathsf{disc}},$$
$$(-)^{\otimes}_{\mathsf{disc}|\mathbb{1}}\colon \mathsf{pt} \xrightarrow{\cong} \mathrm{pt}_{\mathsf{disc}},$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

Proof. *Item 1*, *Functoriality*: Clear.
Item 2, *Adjointness*: This is proved in Proposition 2.1.1.1.
Item 3, *Symmetric Strong Monoidality With Respect to Coproducts*: Omitted.
Item 4, *Symmetric Strong Monoidality With Respect to Products*: Omitted.
□

2.6 Indiscrete Categories

Definition 2.6.1.1. The **indiscrete category on a set** X[13] is the category X_{indisc} where

- *Objects.* We have
$$\mathrm{Obj}(X_{\mathsf{indisc}}) \stackrel{\mathrm{def}}{=} X;$$
- *Morphisms.* For each $A, B \in \mathrm{Obj}(X_{\mathsf{indisc}})$, we have
$$\mathrm{Hom}_{X_{\mathsf{disc}}}(A, B) \stackrel{\mathrm{def}}{=} \{[A] \to [B]\};$$
- *Identities.* For each $A \in \mathrm{Obj}(X_{\mathsf{indisc}})$, the unit map
$$\mathbb{1}^{X_{\mathsf{indisc}}}_{A} \colon \mathrm{pt} \to \mathrm{Hom}_{X_{\mathsf{indisc}}}(A, A)$$
of X_{indisc} at A is defined by
$$\mathrm{id}^{X_{\mathsf{indisc}}}_{A} \stackrel{\mathrm{def}}{=} \{[A] \to [A]\};$$
- *Composition.* For each $A, B, C \in \mathrm{Obj}(X_{\mathsf{indisc}})$, the composition map
$$\circ^{X_{\mathsf{indisc}}}_{A,B,C} \colon \mathrm{Hom}_{X_{\mathsf{indisc}}}(B, C) \times \mathrm{Hom}_{X_{\mathsf{indisc}}}(A, B) \to \mathrm{Hom}_{X_{\mathsf{indisc}}}(A, C)$$
of X_{disc} at (A, B, C) is defined by
$$([B] \to [C]) \circ ([A] \to [B]) \stackrel{\mathrm{def}}{=} ([A] \to [C]).$$

Proposition 2.6.1.2. Let X be a set.

1. *Functoriality.* The assignment $X \mapsto X_{\mathsf{indisc}}$ defines a functor
$$(-)_{\mathsf{indisc}} \colon \mathsf{Sets} \to \mathsf{Cats}.$$
2. *Adjointness.* We have a quadruple adjunction
$$(\pi_0 \dashv (-)_{\mathsf{disc}} \dashv \mathrm{Obj} \dashv (-)_{\mathsf{indisc}})\colon \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left=15, "(-)_{\mathsf{disc}}"] \arrow[r, bend right=50, "(-)_{\mathsf{indisc}}"'] & \mathsf{Cats}. \arrow[l, bend right=50, "\pi_0"'] \arrow[l, bend left=15, "\mathrm{Obj}"] \end{tikzcd}$$

[13] *Further Terminology:* Also called the **chaotic category on** X.

3. *Symmetric Strong Monoidality With Respect to Products.* The functor of Item 1 has a symmetric strong monoidal structure

$$\left((-)_{\mathsf{indisc}}, (-)^{\otimes}_{\mathsf{indisc}}, (-)^{\otimes}_{\mathsf{indisc}|\mathbb{1}}\right) : (\mathsf{Sets}, \times, \mathrm{pt}) \to (\mathsf{Cats}, \times, \mathsf{pt}),$$

being equipped with isomorphisms

$$(-)^{\otimes}_{\mathsf{indisc}|X,Y} : X_{\mathsf{indisc}} \times Y_{\mathsf{indisc}} \xrightarrow{\cong} (X \times Y)_{\mathsf{indisc}},$$
$$(-)^{\otimes}_{\mathsf{indisc}|\mathbb{1}} : \mathsf{pt} \xrightarrow{\cong} \mathrm{pt}_{\mathsf{indisc}},$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

Proof. Item 1, Functoriality: Clear.
Item 2, Adjointness: This is proved in Proposition 2.1.1.1.
Item 3, Symmetric Strong Monoidality With Respect to Products: Omitted. □

3 Groupoids

3.1 Foundations

Let $\mathcal{C}$ be a category.

Definition 3.1.1.1. A morphism $f : A \to B$ of $\mathcal{C}$ is an **isomorphism** if there exists a morphism $f^{-1} : B \to A$ of $\mathcal{C}$ such that

$$f \circ f^{-1} = \mathrm{id}_B,$$
$$f^{-1} \circ f = \mathrm{id}_A.$$

Definition 3.1.1.2. A **groupoid** is a category in which every morphism is an isomorphism.

3.2 The Groupoid Completion of a Category

Let $\mathcal{C}$ be a category.

Definition 3.2.1.1. The **groupoid completion of $\mathcal{C}$**[14] is the pair $(\mathrm{K}_0(\mathcal{C}), \iota_{\mathcal{C}})$ consisting of

[14] *Further Terminology:* Also called the **Grothendieck groupoid of $\mathcal{C}$** or the **Grothendieck groupoid completion of $\mathcal{C}$**.

- A groupoid $\mathrm{K}_0(\mathcal{C})$;
- A functor $\iota_{\mathcal{C}} \colon \mathcal{C} \to \mathrm{K}_0(\mathcal{C})$;

satisfying the following universal property:[15]

(UP) Given another such pair $(\mathcal{G}, i)$, there exists a unique functor $\mathrm{K}_0(\mathcal{C}) \xrightarrow{\exists!} \mathcal{G}$ making the diagram

$$\begin{tikzcd} & \mathrm{K}_0(\mathcal{C}) \arrow[d, dashed, "\exists!"] \\ \mathcal{C} \arrow[ur, "\iota_{\mathcal{C}}"] \arrow[r, "i"'] & \mathcal{G} \end{tikzcd}$$

commute.

Proposition 3.2.1.2. Let $\mathcal{C}$ be a category.

1. *Functoriality.* The assignment $\mathcal{C} \mapsto \mathrm{K}_0(\mathcal{C})$ defines a functor
$$\mathrm{K}_0 \colon \mathsf{Cats} \to \mathsf{Grpd}.$$
2. *2-Functoriality.* The assignment $\mathcal{C} \mapsto \mathrm{K}_0(\mathcal{C})$ defines a 2-functor
$$\mathrm{K}_0 \colon \mathsf{Cats}_2 \to \mathsf{Grpd}_2.$$
3. *Adjointness.* We have an adjunction
$$(\mathrm{K}_0 \dashv \iota)\colon \quad \begin{tikzcd} \mathsf{Cats} \arrow[r, shift left, "\mathrm{K}_0"] \arrow[r, phantom, "\perp"] & \mathsf{Grpd}, \arrow[l, shift left, hook', "\iota"] \end{tikzcd}$$
witnessed by a bijection of sets
$$\mathrm{Hom}_{\mathsf{Grpd}}(\mathrm{K}_0(\mathcal{C}), \mathcal{G}) \cong \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, \mathcal{G}),$$
natural in $\mathcal{C} \in \mathrm{Obj}(\mathsf{Cats})$ and $\mathcal{G} \in \mathrm{Obj}(\mathsf{Grpd})$, forming, together with the functor Core of Item 1 of Proposition 3.3.1.3, a triple adjunction
$$(\mathrm{K}_0 \dashv \iota \dashv \mathsf{Core})\colon \quad \begin{tikzcd} \mathsf{Cats} \arrow[r, bend left, "\mathrm{K}_0"] \arrow[r, bend right, "\mathsf{Core}"'] & \mathsf{Grpd}, \arrow[l, hook', "\iota" description] \end{tikzcd}$$

[15] See Item 5 of Proposition 3.2.1.2 for an explicit construction.

witnessed by bijections of sets

$$\begin{aligned}\mathrm{Hom}_{\mathsf{Grpd}}(\mathrm{K}_0(\mathcal{C}),\mathcal{G}) &\cong \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C},\mathcal{G}),\\ \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{G},\mathcal{D}) &\cong \mathrm{Hom}_{\mathsf{Grpd}}(\mathcal{G},\mathsf{Core}(\mathcal{D})),\end{aligned}$$

natural in $\mathcal{C},\mathcal{D}\in\mathrm{Obj}(\mathsf{Cats})$ and $\mathcal{G}\in\mathrm{Obj}(\mathsf{Grpd})$.

4. *2-Adjointness.* We have a 2-adjunction

$$(\mathrm{K}_0\dashv\iota)\colon\quad \begin{tikzcd}\mathsf{Cats} \arrow[r, shift left, "\mathrm{K}_0"] \arrow[r, phantom, "\perp_2"] & \mathsf{Grpd}, \arrow[l, shift left, "\iota"]\end{tikzcd}$$

witnessed by an isomorphism of categories

$$\mathsf{Fun}(\mathrm{K}_0(\mathcal{C}),\mathcal{G})\cong\mathsf{Fun}(\mathcal{C},\mathcal{G}),$$

natural in $\mathcal{C}\in\mathrm{Obj}(\mathsf{Cats})$ and $\mathcal{G}\in\mathrm{Obj}(\mathsf{Grpd})$, forming, together with the 2-functor Core of Item 2 of Proposition 3.3.1.3, a triple 2-adjunction

$$(\mathrm{K}_0\dashv\iota\dashv\mathsf{Core})\colon\quad \begin{tikzcd}\mathsf{Cats} \arrow[r, bend left, "\mathrm{K}_0"] \arrow[r, bend right, "\mathsf{Core}"'] & \mathsf{Grpd}, \arrow[l, "\iota" description] \end{tikzcd}$$

witnessed by isomorphisms of categories

$$\begin{aligned}\mathsf{Fun}(\mathrm{K}_0(\mathcal{C}),\mathcal{G}) &\cong \mathsf{Fun}(\mathcal{C},\mathcal{G}),\\ \mathsf{Fun}(\mathcal{G},\mathcal{D}) &\cong \mathsf{Fun}(\mathcal{G},\mathsf{Core}(\mathcal{D})),\end{aligned}$$

natural in $\mathcal{C},\mathcal{D}\in\mathrm{Obj}(\mathsf{Cats})$ and $\mathcal{G}\in\mathrm{Obj}(\mathsf{Grpd})$.

5. *Interaction With Classifying Spaces.* We have an isomorphism of groupoids

$$\mathrm{K}_0(\mathcal{C})\cong\Pi_{\leq 1}(|\mathrm{N}_\bullet(\mathcal{C})|),$$

natural in $\mathcal{C}\in\mathrm{Obj}(\mathsf{Cats})$; i.e. the diagram

$$\begin{tikzcd}\mathsf{Cats} \arrow[r, "\mathrm{K}_0"] \arrow[d, "\mathrm{N}_\bullet"'] \arrow[dr, phantom, "\wr"] & \mathsf{Grp} \\ \mathsf{sSets} \arrow[r, "|-|"'] & \mathsf{Top} \arrow[u, "\Pi_{\leq 1}"']\end{tikzcd}$$

commutes up to natural isomorphism.

6. *Symmetric Strong Monoidality With Respect to Coproducts.* The groupoid completion functor of Item 1 has a symmetric strong monoidal structure

$$\left(\mathrm{K}_0, \mathrm{K}_0^{\coprod}, \mathrm{K}_{0|\mathbb{1}}^{\coprod}\right)\colon (\mathsf{Cats}, \coprod, \emptyset_{\mathsf{cat}}) \to (\mathsf{Grpd}, \coprod, \emptyset_{\mathsf{cat}})$$

being equipped with isomorphisms

$$\mathrm{K}_{0|\mathcal{C},\mathcal{D}}^{\coprod}\colon \mathrm{K}_0(\mathcal{C}) \coprod \mathrm{K}_0(\mathcal{D}) \xrightarrow{\cong} \mathrm{K}_0(\mathcal{C} \coprod \mathcal{D}),$$
$$\mathrm{K}_{0|\mathbb{1}}^{\coprod}\colon \emptyset_{\mathsf{cat}} \xrightarrow{\cong} \mathrm{K}_0(\emptyset_{\mathsf{cat}}),$$

natural in $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$.

7. *Symmetric Strong Monoidality With Respect to Products.* The groupoid completion functor of Item 1 has a symmetric strong monoidal structure

$$\left(\mathrm{K}_0, \mathrm{K}_0^{\times}, \mathrm{K}_{0|\mathbb{1}}^{\times}\right)\colon (\mathsf{Cats}, \times, \mathsf{pt}) \to (\mathsf{Grpd}, \times, \mathsf{pt})$$

being equipped with isomorphisms

$$\mathrm{K}_{0|\mathcal{C},\mathcal{D}}^{\times}\colon \mathrm{K}_0(\mathcal{C}) \times \mathrm{K}_0(\mathcal{D}) \xrightarrow{\cong} \mathrm{K}_0(\mathcal{C} \times \mathcal{D}),$$
$$\mathrm{K}_{0|\mathbb{1}}^{\times}\colon \mathsf{pt} \xrightarrow{\cong} \mathrm{K}_0(\mathsf{pt}),$$

natural in $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$.

Proof. *Item 1, Functoriality*: Omitted.
Item 2, 2-Functoriality: Omitted.
Item 3, Adjointness: Omitted.
Item 4, 2-Adjointness: Omitted.
Item 5, Interaction With Classifying Spaces: See Corollary 18.33 of `https://web.ma.utexas.edu/users/dafr/M392C-2012/Notes/lecture18.pdf`.
Item 6, Symmetric Strong Monoidality With Respect to Coproducts: Omitted.
Item 7, Symmetric Strong Monoidality With Respect to Products: Omitted. □

3.3 The Core of a Category

Let $\mathcal{C}$ be a category.

Definition 3.3.1.1. The **core** of $\mathcal{C}$ is the pair $(\mathsf{Core}(\mathcal{C}), \iota_{\mathcal{C}})$[16] consisting of

1. A groupoid $\mathsf{Core}(\mathcal{C})$;
2. A functor $\iota_{\mathcal{C}} \colon \mathsf{Core}(\mathcal{C}) \hookrightarrow \mathcal{C}$;

satisfying the following universal property:

(UP) Given another such pair $(\mathcal{G}, i)$, there exists a unique functor $\mathcal{G} \xrightarrow{\exists!} \mathsf{Core}(\mathcal{C})$ making the diagram

$$\begin{tikzcd} & \mathsf{Core}(\mathcal{C}) \arrow[d, "\iota_{\mathcal{C}}"] \\ \mathcal{G} \arrow[ur, dashed, "\exists!"] \arrow[r, "i"'] & \mathcal{C} \end{tikzcd}$$

commute.

Construction 3.3.1.2. The core of $\mathcal{C}$ is the wide subcategory of $\mathcal{C}$ spanned by the isomorphisms of $\mathcal{C}$, i.e. the category $\mathsf{Core}(\mathcal{C})$ where[17]

1. *Objects.* We have
$$\mathrm{Obj}(\mathsf{Core}(\mathcal{C})) \stackrel{\text{def}}{=} \mathrm{Obj}(\mathcal{C});$$
2. *Morphisms.* The morphisms of $\mathsf{Core}(\mathcal{C})$ are the isomorphisms of $\mathcal{C}$.

Proof. This follows from the fact that functors preserve isomorphisms. □

Proposition 3.3.1.3. Let $\mathcal{C}$ be a category.

1. *Functoriality.* The assignment $\mathcal{C} \mapsto \mathsf{Core}(\mathcal{C})$ defines a functor
$$\mathsf{Core} \colon \mathsf{Cats} \to \mathsf{Grpd}.$$
2. *2-Functoriality.* The assignment $\mathcal{C} \mapsto \mathsf{Core}(\mathcal{C})$ defines a 2-functor
$$\mathsf{Core} \colon \mathsf{Cats}_2 \to \mathsf{Grpd}_2.$$

[16] *Further Notation:* Also written $\mathcal{C}^{\simeq}$.
[17] *Slogan:* The groupoid $\mathsf{Core}(\mathcal{C})$ is the maximal subgroupoid of $\mathcal{C}$.

3. *Adjointness.* We have an adjunction

$$(\iota \dashv \mathsf{Core})\colon \quad \begin{tikzcd} \mathsf{Grpd} \arrow[r, bend left, "\iota"] \arrow[r, phantom, "\perp"] & \mathsf{Cats}, \arrow[l, bend left, "\mathsf{Core}"] \end{tikzcd}$$

witnessed by a bijection of sets

$$\mathrm{Hom}_{\mathsf{Cats}}(\mathcal{G}, \mathcal{D}) \cong \mathrm{Hom}_{\mathsf{Grpd}}(\mathcal{G}, \mathsf{Core}(\mathcal{D})),$$

natural in $\mathcal{G} \in \mathrm{Obj}(\mathsf{Grpd})$ and $\mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$, forming, together with the functor K_0 of Item 1 of Proposition 3.2.1.2, a triple adjunction

$$(\mathrm{K}_0 \dashv \iota \dashv \mathsf{Core})\colon \quad \begin{tikzcd} \mathsf{Cats} \arrow[r, bend left=50, "\mathrm{K}_0"] \arrow[r, bend right=50, "\mathsf{Core}"'] & \mathsf{Grpd}, \arrow[l, hook', "\iota" description] \end{tikzcd}$$

witnessed by bijections of sets

$$\begin{aligned} \mathrm{Hom}_{\mathsf{Grpd}}(\mathrm{K}_0(\mathcal{C}), \mathcal{G}) &\cong \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, \mathcal{G}), \\ \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{G}, \mathcal{D}) &\cong \mathrm{Hom}_{\mathsf{Grpd}}(\mathcal{G}, \mathsf{Core}(\mathcal{D})), \end{aligned}$$

natural in $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$ and $\mathcal{G} \in \mathrm{Obj}(\mathsf{Grpd})$.

4. *2-Adjointness.* We have an adjunction

$$(\iota \dashv \mathsf{Core})\colon \quad \begin{tikzcd} \mathsf{Grpd} \arrow[r, bend left, "\iota"] \arrow[r, phantom, "\perp_2"] & \mathsf{Cats}, \arrow[l, bend left, "\mathsf{Core}"] \end{tikzcd}$$

witnessed by an isomorphism of categories

$$\mathsf{Fun}(\mathcal{G}, \mathcal{D}) \cong \mathsf{Fun}(\mathcal{G}, \mathsf{Core}(\mathcal{D})),$$

natural in $\mathcal{G} \in \mathrm{Obj}(\mathsf{Grpd})$ and $\mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$, forming, together with the 2-functor K_0 of Item 2 of Proposition 3.2.1.2, a triple 2-adjunction

$$(\mathrm{K}_0 \dashv \iota \dashv \mathsf{Core})\colon \quad \begin{tikzcd} \mathsf{Cats} \arrow[r, bend left=50, "\mathrm{K}_0"] \arrow[r, bend right=50, "\mathsf{Core}"'] & \mathsf{Grpd}, \arrow[l, hook', "\iota" description] \end{tikzcd}$$

witnessed by isomorphisms of categories

$$\begin{aligned} \mathsf{Fun}(\mathrm{K}_0(\mathcal{C}), \mathcal{G}) &\cong \mathsf{Fun}(\mathcal{C}, \mathcal{G}), \\ \mathsf{Fun}(\mathcal{G}, \mathcal{D}) &\cong \mathsf{Fun}(\mathcal{G}, \mathsf{Core}(\mathcal{D})), \end{aligned}$$

natural in $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$ and $\mathcal{G} \in \mathrm{Obj}(\mathsf{Grpd})$.

5. *Symmetric Strong Monoidality With Respect to Products.* The core functor of Item 1 has a symmetric strong monoidal structure

$$\left(\mathsf{Core}, \mathsf{Core}^{\times}, \mathsf{Core}^{\times}_{\mathbb{1}}\right) : (\mathsf{Cats}, \times, \mathsf{pt}) \to (\mathsf{Grpd}, \times, \mathsf{pt})$$

being equipped with isomorphisms

$$\mathsf{Core}^{\times}_{\mathcal{C},\mathcal{D}} : \mathsf{Core}(\mathcal{C}) \times \mathsf{Core}(\mathcal{D}) \xrightarrow{\cong} \mathsf{Core}(\mathcal{C} \times \mathcal{D}),$$
$$\mathsf{Core}^{\times}_{\mathbb{1}} : \mathsf{pt} \xrightarrow{\cong} \mathsf{Core}(\mathsf{pt}),$$

natural in $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$.

6. *Symmetric Strong Monoidality With Respect to Coproducts.* The core functor of Item 1 has a symmetric strong monoidal structure

$$\left(\mathsf{Core}, \mathsf{Core}^{\coprod}, \mathsf{Core}^{\coprod}_{\mathbb{1}}\right) : (\mathsf{Cats}, \coprod, \emptyset_{\mathsf{cat}}) \to (\mathsf{Grpd}, \coprod, \emptyset_{\mathsf{cat}})$$

being equipped with isomorphisms

$$\mathsf{Core}^{\coprod}_{\mathcal{C},\mathcal{D}} : \mathsf{Core}(\mathcal{C}) \textstyle\coprod \mathsf{Core}(\mathcal{D}) \xrightarrow{\cong} \mathsf{Core}(\mathcal{C} \textstyle\coprod \mathcal{D}),$$
$$\mathsf{Core}^{\coprod}_{\mathbb{1}} : \emptyset_{\mathsf{cat}} \xrightarrow{\cong} \mathsf{Core}(\emptyset_{\mathsf{cat}}),$$

natural in $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$.

Proof. Item 1, Functoriality: Omitted.
Item 2, 2-Functoriality: Omitted.
Item 3, Adjointness: The adjunction $(\mathrm{K}_0 \dashv \iota)$ follows from the universal property of the Gabriel–Zisman localisation of a category with respect to a class of morphisms (**??**), while the adjunction $(\iota \dashv \mathsf{Core})$ is a reformulation of the universal property of the core of a category (Definition 3.3.1.1).[18]
Item 4, 2-Adjointness: Omitted.
Item 5, Symmetric Strong Monoidality With Respect to Products: Omitted.
Item 6, Symmetric Strong Monoidality With Respect to Coproducts: Omitted. □

4 Functors

4.1 Foundations

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

[18] *Reference:* [Rie17, Example 4.1.15]

Definition 4.1.1.1. A **functor** $F\colon \mathcal{C} \to \mathcal{D}$ **from** $\mathcal{C}$ **to** $\mathcal{D}$[19] consists of[20]

1. *Action on Objects.* A map of sets
$$F\colon \mathrm{Obj}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{D}),$$
called the **action on objects of** F;

2. *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathcal{C})$, a map
$$F_{A,B}\colon \mathrm{Hom}_{\mathcal{C}}(A,B) \to \mathrm{Hom}_{\mathcal{D}}(F_A, F_B),$$
called the **action on morphisms of** F **at** (A,B)[21];

satisfying the following conditions:

1. *Preservation of Identities.* For each $A \in \mathrm{Obj}(\mathcal{C})$, the diagram
$$\begin{tikzcd}
\mathrm{pt} \arrow[d, "\mathbb{1}^{\mathcal{C}}_A"'] \arrow[dr, "\mathbb{1}^{\mathcal{D}}_{F_A}"] & \\
\mathrm{Hom}_{\mathcal{C}}(A,A) \arrow[r, "F_{A,A}"'] & \mathrm{Hom}_{\mathcal{D}}(F_A,F_A)
\end{tikzcd}$$
commutes, i.e. we have
$$F(\mathrm{id}_A) = \mathrm{id}_{F_A}.$$

2. *Preservation of Composition.* For each $A, B, C \in \mathrm{Obj}(\mathcal{C})$, the diagram
$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(B,C) \times \mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[r, "\circ^{\mathcal{C}}_{A,B,C}"] \arrow[d, "F_{B,C} \times F_{A,B}"'] & \mathrm{Hom}_{\mathcal{C}}(A,C) \arrow[d, "F_{A,C}"] \\
\mathrm{Hom}_{\mathcal{D}}(F_B,F_C) \times \mathrm{Hom}_{\mathcal{D}}(F_A,F_B) \arrow[r, "\circ^{\mathcal{D}}_{F_A,F_B,F_C}"'] & \mathrm{Hom}_{\mathcal{D}}(F_A,F_C)
\end{tikzcd}$$
commutes, i.e. for each composable pair (g, f) of morphisms of $\mathcal{C}$, we have
$$F(g \circ f) = F(g) \circ F(f).$$

[19] *Further Terminology:* Also called a **covariant functor**.

[20] *Further Notation:* Given functors $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{C}^{\mathsf{op}} \to \mathcal{D}$, we will sometimes write F_A for $F(A)$ (resp. G^A for $G(A)$) and F_f for $F(f)$ (resp. G^f for $G(f)$). This has been called Einstein notation in the literature.

[21] *Further Terminology:* Also called **action on Hom-sets of** F **at** (A,B).

Example 4.1.1.2. The **identity functor** of a category $\mathcal{C}$ is the functor $\mathrm{id}_{\mathcal{C}} \colon \mathcal{C} \to \mathcal{C}$ where

1. *Action on Objects.* For each $A \in \mathrm{Obj}(\mathcal{C})$, we have

$$\mathrm{id}_{\mathcal{C}}(A) \stackrel{\mathrm{def}}{=} A;$$

2. *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathcal{C})$, the action on morphisms map

$$(\mathrm{id}_{\mathcal{C}})_{A,B} \colon \mathrm{Hom}_{\mathcal{C}}(A,B) \to \underbrace{\mathrm{Hom}_{\mathcal{C}}(\mathrm{id}_{\mathcal{C}}(A), \mathrm{id}_{\mathcal{C}}(B))}_{\stackrel{\mathrm{def}}{=} \mathrm{Hom}_{\mathcal{C}}(A,B)}$$

of $\mathrm{id}_{\mathcal{C}}$ at (A, B) is defined by

$$(\mathrm{id}_{\mathcal{C}})_{A,B} \stackrel{\mathrm{def}}{=} \mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(A,B)}.$$

Proof. Preservation of Identities: We have $\mathrm{id}_{\mathcal{C}}(\mathrm{id}_A) \stackrel{\mathrm{def}}{=} \mathrm{id}_A$ for each $A \in \mathrm{Obj}(\mathcal{C})$ by definition.

Preservation of Compositions: For each composable pair $A \xrightarrow{f} B \xrightarrow{g} B$ of morphisms of $\mathcal{C}$, we have

$$\begin{aligned} \mathrm{id}_{\mathcal{C}}(g \circ f) &\stackrel{\mathrm{def}}{=} g \circ f \\ &\stackrel{\mathrm{def}}{=} \mathrm{id}_{\mathcal{C}}(g) \circ \mathrm{id}_{\mathcal{C}}(f). \end{aligned}$$

This finishes the proof. □

Definition 4.1.1.3. The **composition** of two functors $F \colon \mathcal{C} \to \mathcal{D}$ and $G \colon \mathcal{D} \to \mathcal{E}$ is the functor $G \circ F$ where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathcal{C})$, we have

$$[G \circ F](A) \stackrel{\mathrm{def}}{=} G(F(A));$$

- *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathcal{C})$, the action on morphisms map

$$(G \circ F)_{A,B} \colon \mathrm{Hom}_{\mathcal{C}}(A,B) \to \mathrm{Hom}_{\mathcal{E}}(G_{F_A}, G_{F_B})$$

of $G \circ F$ at (A, B) is defined by

$$[G \circ F](f) \stackrel{\mathrm{def}}{=} G(F(f)).$$

Proof. Preservation of Identities: For each $A \in \mathrm{Obj}(\mathcal{C})$, we have

$$\begin{aligned} G_{F_{\mathrm{id}_A}} &= G_{\mathrm{id}_{F_A}} && \text{(functoriality of } F\text{)} \\ &= \mathrm{id}_{G_{F_A}}. && \text{(functoriality of } G\text{)} \end{aligned}$$

Preservation of Composition: For each composable pair (g, f) of morphisms of $\mathcal{C}$, we have

$$\begin{aligned} G_{F_{g \circ f}} &= G_{F_g \circ F_f} && \text{(functoriality of } F\text{)} \\ &= G_{F_g} \circ G_{F_f}. && \text{(functoriality of } G\text{)} \end{aligned}$$

This finishes the proof. □

Proposition 4.1.1.4. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor.

1. *Preservation of Isomorphisms.* If f is an isomorphism in $\mathcal{C}$, then $F(f)$ is an isomorphism in $\mathcal{D}$.[22]

Proof. Item 1, Preservation of Isomorphisms: Indeed, we have

$$\begin{aligned} F(f)^{-1} \circ F(f) &= F\left(f^{-1} \circ f\right) \\ &= F(\mathrm{id}_A) \\ &= \mathrm{id}_{F(A)} \end{aligned}$$

and

$$\begin{aligned} F(f) \circ F(f)^{-1} &= F\left(f \circ f^{-1}\right) \\ &= F(\mathrm{id}_A) \\ &= \mathrm{id}_{F(A)}, \end{aligned}$$

showing $F(f)$ to be an isomorphism. □

4.2 Faithful Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 4.2.1.1. A functor $F : \mathcal{C} \to \mathcal{D}$ is **faithful** if, for each $A, B \in \mathrm{Obj}(\mathcal{C})$, the action on morphisms map

$$F_{A,B} : \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F_A, F_B)$$

of F at (A, B) is injective.

[22] When the converse holds, we call F *conservative*, see Definition 4.6.1.1.

Proposition 4.2.1.2. Let $F \colon \mathcal{C} \to \mathcal{D}$ be a functor

1. *Characterisations.* The following conditions are equivalent:

   (a) The functor $F \colon \mathcal{C} \to \mathcal{D}$ is faithful.

   (b) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor
   $$F^* \colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$
   is faithful.

   (c) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor
   $$F_* \colon \mathsf{Fun}(\mathcal{X}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{X}, \mathcal{D})$$
   is faithful.

Proof. *Item 1, Characterisations*: Omitted. □

4.3 Full Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 4.3.1.1. A functor $F \colon \mathcal{C} \to \mathcal{D}$ is **full** if, for each $A, B \in \mathrm{Obj}(\mathcal{C})$, the action on morphisms map
$$F_{A,B} \colon \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F_A, F_B)$$
of F at (A, B) is surjective.

Proposition 4.3.1.2. Let $F \colon \mathcal{C} \to \mathcal{D}$ be a functor

1. *Characterisations.* The following conditions are equivalent:

   (a) The functor $F \colon \mathcal{C} \to \mathcal{D}$ is full.

   (b) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor
   $$F^* \colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$
   is full.

   (c) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor
   $$F_* \colon \mathsf{Fun}(\mathcal{X}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{X}, \mathcal{D})$$
   is full.

Proof. *Item 1, Characterisations*: Omitted. □

4.4 Fully Faithful Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 4.4.1.1. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **fully faithful** if F is full and faithful, i.e. if, for each $A, B \in \mathrm{Obj}(\mathcal{C})$, the action on morphisms map

$$F_{A,B}\colon \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F_A, F_B)$$

of F at (A, B) is bijective.

Proposition 4.4.1.2. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor

1. *Characterisations.* The following conditions are equivalent:
   (a) The functor $F\colon \mathcal{C} \to \mathcal{D}$ is fully faithful.
   (b) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor
   $$F^*\colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$
   is fully faithful.
   (c) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor
   $$F_*\colon \mathsf{Fun}(\mathcal{X}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{X}, \mathcal{D})$$
   is fully faithful.
2. *Conservativity.* If F is fully faithful, then F is conservative.

Proof. Item 1, Characterisations: Omitted.
Item 2, Conservativity: This is proved in Item 2 of Proposition 4.6.1.2. □

4.5 Essentially Surjective Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 4.5.1.1. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **essentially surjective** if, for each $D \in \mathrm{Obj}(\mathcal{D})$, there exists some object A of $\mathcal{C}$ with $F(A) \cong D$.

4.6 Conservative Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 4.6.1.1. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **conservative** if it satisfies the following condition:

($\star$) For each $f \in \mathrm{Mor}(\mathcal{C})$, if $F(f)$ is an isomorphism in $\mathcal{D}$, then f is an isomorphism in $\mathcal{C}$.

Proposition 4.6.1.2. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor.

1. *Characterisations.* The following conditions are equivalent:

   (a) The functor F is conservative.

   (b) For each $f \in \mathrm{Mor}(\mathcal{C})$, the morphism $F(f)$ is an isomorphism in $\mathcal{D}$ iff f is an isomorphism in $\mathcal{C}$.

2. *Interaction With Fully Faithfulness.* Every fully faithful functor is conservative.

Proof. *Item 1, Characterisations*: This follows from Item 1 of Proposition 4.1.1.4.
Item 2, Interaction With Fully Faithfulness: Let $F\colon \mathcal{C} \to \mathcal{D}$ be a fully faithful functor, let $f\colon A \to B$ be a morphism of $\mathcal{C}$, and suppose that F_f is an isomorphism. We have

$$\begin{aligned} F(\mathrm{id}_B) &= \mathrm{id}_{F(B)} \\ &= F(f) \circ F(f)^{-1} \\ &= F\Big(f \circ f^{-1}\Big). \end{aligned}$$

Similarly, $F(\mathrm{id}_A) = F(f^{-1} \circ f)$. But since F is fully faithful, we must have

$$\begin{aligned} f \circ f^{-1} &= \mathrm{id}_B, \\ f^{-1} \circ f &= \mathrm{id}_A, \end{aligned}$$

showing f to be an isomorphism. Thus F is conservative. $\square$

4.7 Equivalences of Categories

Definition 4.7.1.1. Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

- An **equivalence of categories** between $\mathcal{C}$ and $\mathcal{D}$ consists of a pair of functors

$$\begin{aligned} F&\colon \mathcal{C} \to \mathcal{D}, \\ G&\colon \mathcal{D} \to \mathcal{C} \end{aligned}$$

together with natural isomorphisms

$$\eta\colon \mathrm{id}_{\mathcal{C}} \overset{\cong}{\Longrightarrow} G \circ F,$$
$$\epsilon\colon F \circ G \overset{\cong}{\Longrightarrow} \mathrm{id}_{\mathcal{D}}.$$

- An **adjoint equivalence of categories** between $\mathcal{C}$ and $\mathcal{D}$ is an equivalence (F, G, η, ϵ) between $\mathcal{C}$ and $\mathcal{D}$ which is also an adjunction.

Proposition 4.7.1.2. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor.

1. *Characterisations.* If $\mathcal{C}$ and $\mathcal{D}$ are small[23], then the following conditions are equivalent:[24]

   (a) The functor F is an equivalence of categories.

   (b) The functor F is fully faithful and essentially surjective.

   (c) The induced functor

   $$F|_{\mathsf{Sk}(\mathcal{C})}\colon \mathsf{Sk}(\mathcal{C}) \to \mathsf{Sk}(\mathcal{D})$$

   is an *isomorphism* of categories.

2. *Two-Out-of-Three.* Let

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "G\circ F"] \arrow[dr, "F"'] & & \mathcal{E} \\ & \mathcal{D} \arrow[ur, "G"'] & \end{tikzcd}$$

   be a diagram in Cats. If two out of the three functors among F, G, and $G \circ F$ are equivalences of categories, then so is the third.

3. *Stability Under Composition.* Let

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "F"] & \mathcal{D} \arrow[l, shift left, "G"] \arrow[r, shift left, "F'"] & \mathcal{E} \arrow[l, shift left, "G'"] \end{tikzcd}$$

   be a diagram in Cats. If (F, G) and (F', G') are equivalences of categories, then so is their composite $(F' \circ F, G' \circ G)$.

[23] Otherwise there will be size issues. One can also work with large categories and universes, or require F to be *constructively* essentially surjective; see [**MSE1465107**].

[24] In ZFC, the equivalence between Item 1a and Item 1b is equivalent to the axiom of choice; see [MO 119454].

4. *Equivalences vs.Adjoint Equivalences.* Every equivalence of categories can be promoted to an adjoint equivalence.[25]

5. *Interaction With Groupoids.* If $\mathcal{C}$ and $\mathcal{D}$ are groupoids, then the following conditions are equivalent:

   (a) The functor F is an equivalence of groupoids.

   (b) The following conditions are satisfied:

      i. The functor F induces a bijection

      $$\pi_0(F)\colon \pi_0(\mathcal{C}) \to \pi_0(\mathcal{D})$$

      of sets.

      ii. For each $A \in \mathrm{Obj}(\mathcal{C})$, the induced map

      $$F_{x,x}\colon \mathrm{Aut}_{\mathcal{C}}(A) \to \mathrm{Aut}_{\mathcal{D}}(F_A)$$

      is an isomorphism of groups.

Proof. *Item 1, Characterisations*: We claim that Items 1a to 1c are indeed equivalent:

1. *Item 1a* $\implies$ *Item 1b.* Clear.

2. *Item 1b* $\implies$ *Item 1a.* Since F is essentially surjective and $\mathcal{C}$ and $\mathcal{D}$ are small, we can choose, using the axiom of choice, for each $B \in \mathrm{Obj}(\mathcal{D})$, an object j_B of $\mathcal{C}$ and an isomorphism $i_B\colon B \to F_{j_B}$ of $\mathcal{D}$.

   Since F is fully faithful, we can extend the assignment $B \mapsto j_B$ to a *unique* functor $j\colon \mathcal{D} \to \mathcal{C}$ such that the isomorphisms $i_B\colon B \to F_{j_B}$ assemble into a natural isomorphism $\eta\colon \mathrm{id}_{\mathcal{D}} \overset{\cong}{\Longrightarrow} F \circ j$, with a similar natural isomorphism $\epsilon\colon \mathrm{id}_{\mathcal{C}} \overset{\cong}{\Longrightarrow} j \circ F$. Hence F is an equivalence.

3. *Item 1a* $\implies$ *Item 1c.* This follows from **??**.

Item 2, Two-Out-of-Three: Omitted.
Item 3, Stability Under Composition: Clear.
Item 4, Equivalences vs.Adjoint Equivalences: See [Rie17, Proposition 4.4.5].
Item 5, Interaction With Groupoids: See [nLa24, Proposition 4.4]. □

In Univalent Foundations, this is true without requiring neither the axiom of choice nor the law of the excluded middle.

[25]More precisely, we can promote an equivalence of categories (F, G, η, ϵ) to adjoint

4.8 Isomorphisms of Categories

Definition 4.8.1.1. An **isomorphism of categories** is a pair of functors

$$F\colon \mathcal{C} \to \mathcal{D},$$
$$G\colon \mathcal{D} \to \mathcal{C}$$

such that we have

$$G \circ F = \mathrm{id}_{\mathcal{C}},$$

$$F \circ G = \mathrm{id}_{\mathcal{D}}.$$

Example 4.8.1.2. Categories can be equivalent but non-isomorphic. For example, the category consisting of two isomorphic objects is equivalent to pt, but not isomorphic to it.

Proposition 4.8.1.3. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor.

1. *Characterisations.* If $\mathcal{C}$ and $\mathcal{D}$ are small, then the following conditions are equivalent:
   (a) The functor F is an isomorphism of categories.
   (b) The functor F is fully faithful and a bijection on objects.

Proof. *Item 1, Characterisations*: Omitted, but similar to Item 1 of Proposition 4.7.1.2. ☐

4.9 The Natural Transformation Associated to a Functor

Definition 4.9.1.1. Every functor $F\colon \mathcal{C} \to \mathcal{D}$ defines a natural transformation

$$F^{\dagger}\colon \mathrm{Hom}_{\mathcal{C}} \Longrightarrow \mathrm{Hom}_{\mathcal{D}} \circ (F^{\mathsf{op}} \times F), \qquad \begin{tikzcd} \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \arrow[rr, "F^{\mathsf{op}} \times F"] \arrow[dr, "\mathrm{Hom}_{\mathcal{C}}"'] \arrow[rr, phantom, shift right=6, "\Uparrow F^{\dagger}"] & & \mathcal{D}^{\mathsf{op}} \times \mathcal{D} \arrow[dl, "\mathrm{Hom}_{\mathcal{D}}"] \\ & \mathsf{Sets}, & \end{tikzcd}$$

called the **natural transformation associated to** F, consisting of the collection

$$\left\{F^{\dagger}_{A,B}\colon \mathrm{Hom}_{\mathcal{C}}(A,B) \to \mathrm{Hom}_{\mathcal{D}}(F_A, F_B)\right\}_{(A,B)\in\mathrm{Obj}(\mathcal{C}^{\mathsf{op}}\times\mathcal{C})}$$

with
$$F^{\dagger}_{A,B} \overset{\text{def}}{=} F_{A,B}.$$

Proof. The naturality condition for $F^{\dagger}$ is the requirement that for each morphism
$$(\phi, \psi)\colon (X, Y) \to (A, B)$$
of $\mathcal{C}^{\mathsf{op}} \times \mathcal{C}$, the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(X, Y) \arrow[rr, "\phi^{*}\circ\psi_{*}=\psi_{*}\circ\phi^{*}"] \arrow[d, "F_{X,Y}"'] & & \mathrm{Hom}_{\mathcal{C}}(A, B) \arrow[d, "F_{A,B}"] \\
\mathrm{Hom}_{\mathcal{D}}(F_X, F_Y) \arrow[rr, "F(\phi)^{*}\circ F(\psi)_{*}=F(\psi)_{*}\circ F(\phi)^{*}"'] & & \mathrm{Hom}_{\mathcal{D}}(F_A, F_B),
\end{tikzcd}$$

acting on elements as

$$\begin{tikzcd}
f \arrow[r, mapsto] \arrow[d, mapsto] & \psi \circ f \circ \phi \arrow[d, mapsto] \\
F(f) \arrow[r, mapsto] & F(\psi) \circ F(f) \circ F(\psi) = F(\psi \circ f \circ \phi)
\end{tikzcd}$$

commutes, which follows from the functoriality of F. $\square$

Proposition 4.9.1.2. Let $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{E}$ be functors.

1. *Interaction With Natural Isomorphisms.* The following conditions are equivalent:
   (a) The natural transformation $F^{\dagger}\colon \mathrm{Hom}_{\mathcal{C}} \Longrightarrow \mathrm{Hom}_{\mathcal{D}} \circ (F^{\mathsf{op}} \times F)$ associated to F is a natural isomorphism.
   (b) The functor F is fully faithful.

2. *Interaction With Composition.* We have an equality of pasting diagrams

$$\begin{tikzcd}
\mathcal{C}^{\mathsf{op}} \times \mathcal{C} \arrow[r, "F^{\mathsf{op}}\times F"] \arrow[dr, "\mathrm{Hom}_{\mathcal{C}}"', ""{name=A}] & \mathcal{D}^{\mathsf{op}} \times \mathcal{D} \arrow[r, "G^{\mathsf{op}}\times G"] \arrow[d, "\mathrm{Hom}_{\mathcal{D}}"] & \mathcal{E}^{\mathsf{op}} \times \mathcal{E} \arrow[dl, "\mathrm{Hom}_{\mathcal{E}}"] \\
& \mathsf{Sets} &
\arrow[from=1-1, to=2-2, phantom, shift left=4, "\overset{F^{\dagger}}{\Longrightarrow}"]
\arrow[from=1-2, to=2-2, phantom, shift left=4, "\overset{G^{\dagger}}{\Longrightarrow}"]
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{C}^{\mathsf{op}} \times \mathcal{C} \arrow[rr, "(G\circ F)^{\mathsf{op}}\times(G\circ F)"] \arrow[dr, "\mathrm{Hom}_{\mathcal{C}}"'] & & \mathcal{E}^{\mathsf{op}} \times \mathcal{E}, \arrow[dl, "\mathrm{Hom}_{\mathcal{E}}"] \\
& \mathsf{Sets} \arrow[u, phantom, "\overset{(G\circ F)^{\dagger}}{\Longrightarrow}"] &
\end{tikzcd}$$

in Cats_2, i.e. we have

$$(G \circ F)^\dagger = \left(G^\dagger \star \mathrm{id}_{F^{\mathsf{op}} \times F}\right) \circ F^\dagger.$$

3. *Interaction With Identities.* We have

$$\mathrm{id}_{\mathcal{C}}^\dagger = \mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(-_1,-_2)},$$

i.e. the natural transformation associated to $\mathrm{id}_{\mathcal{C}}$ is the identity natural transformation of the functor $\mathrm{Hom}_{\mathcal{C}}(-_1,-_2)$.

Proof. *Item 1, Interaction With Natural Isomorphisms*: Clear.
Item 2, Interaction With Composition: Clear.
Item 3, Interaction With Identities: Clear. □

5 Natural Transformations

5.1 Foundations

Let $\mathcal{C}$ and $\mathcal{D}$ be categories and $F, G \colon \mathcal{C} \rightrightarrows \mathcal{D}$ be functors.

Definition 5.1.1.1. A **transformation**[26,27] $\alpha \colon F \overset{\textbf{unnat}}{\Longrightarrow} G$ **from** F **to** G is a collection

$$\{\alpha_A \colon F(A) \to G(A)\}_{A \in \mathrm{Obj}(\mathcal{C})}$$

of morphisms of $\mathcal{D}$.

Definition 5.1.1.2. A **natural transformation**[28] $\alpha \colon F \Longrightarrow G$ **from** F **to** G is a transformation

$$\{\alpha_A \colon F(A) \to G(A)\}_{A \in \mathrm{Obj}(\mathcal{C})}$$

equivalences (F, G, η', ϵ) and (F, G, η, ϵ').

[26] *Further Terminology:* Also called an **unnatural transformation** for emphasis.

[27] *Further Notation:* We write $\mathrm{UnNat}(F, G)$ for the set of unnatural transformations from F to G.

[28] Pictured in diagrams as

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{D}. \end{tikzcd}$$

from F to G such that, for each morphism $f\colon A \to B$ of $\mathcal{C}$, the diagram

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\alpha_A"'] & F(B) \arrow[d, "\alpha_B"] \\ G(A) \arrow[r, "G(f)"'] & G(B) \end{tikzcd}$$

commutes.[29,30]

Example 5.1.1.3. The **identity natural transformation** $\mathrm{id}_F\colon F \Longrightarrow F$ **of** F is the natural transformation consisting of the collection

$$\left\{\mathrm{id}_{F(A)}\colon F(A) \to F(A)\right\}_{A\in\mathrm{Obj}(\mathcal{C})}.$$

Proof. The naturality condition for id_F is the requirement that, for each morphism $f\colon A \to B$ of $\mathcal{C}$, the diagram

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\mathrm{id}_{F(A)}"'] & F(B) \arrow[d, "\mathrm{id}_{F(B)}"] \\ F(A) \arrow[r, "F(f)"'] & F(B) \end{tikzcd}$$

commutes, which follows from unitality of the composition of $\mathcal{C}$. □

Definition 5.1.1.4. Two natural transformations $\alpha, \beta\colon F \Longrightarrow G$ are **equal** if we have

$$\alpha_A = \beta_A$$

for each $A \in \mathrm{Obj}(\mathcal{C})$.

---

[29] *Further Terminology:* The morphism $\alpha_A\colon F_A \to G_A$ is called the **component of** α **at** A.

[30] *Further Notation:* We write $\mathrm{Nat}(F, G)$ for the set of natural transformations from F to G.

5.2 Vertical Composition of Natural Transformations

Definition 5.2.1.1. The **vertical composition** of two natural transformations $\alpha\colon F \Longrightarrow G$ and $\beta\colon G \Longrightarrow H$ as in the diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=50, "F"{name=F}] \arrow[r, "G"{description, name=G}] \arrow[r, bend right=50, "H"'{name=H}] & \mathcal{D} \arrow[from=F, to=G, Rightarrow, "\alpha"'] \arrow[from=G, to=H, Rightarrow, "\beta"'] \end{tikzcd}$$

is the natural transformation $\beta \circ \alpha\colon F \Longrightarrow H$ consisting of the collection

$$\{(\beta \circ \alpha)_A\colon F(A) \to H(A)\}_{A \in \mathrm{Obj}(\mathcal{C})}$$

with

$$(\beta \circ \alpha)_A \overset{\text{def}}{=} \beta_A \circ \alpha_A$$

for each $A \in \mathrm{Obj}(\mathcal{C})$.

Proof. The naturality condition for $\beta \circ \alpha$ is the requirement that the boundary of the diagram

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\alpha_A"'] \arrow[dr, phantom, "(1)"] & F(B) \arrow[d, "\alpha_B"] \\ G(A) \arrow[r, "G(f)"{description}] \arrow[d, "\beta_A"'] \arrow[dr, phantom, "(2)"] & G(B) \arrow[d, "\beta_B"] \\ H(A) \arrow[r, "H(f)"'] & H(B) \end{tikzcd}$$

commutes. Since

1. Subdiagram (1) commutes by the naturality of α;
2. Subdiagram (2) commutes by the naturality of β;

so does the boundary diagram. Hence $\beta \circ \alpha$ is a natural transformation. □

Proposition 5.2.1.2. Let $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ be categories.

1. *Functionality.* The assignment $(\beta, \alpha) \mapsto \beta \circ \alpha$ defines a function

$$\circ_{F,G,H}\colon \mathrm{Nat}(G, H) \times \mathrm{Nat}(F, G) \to \mathrm{Nat}(F, H).$$

2. *Associativity.* Let $F, G, H, K \colon \mathcal{C} \overset{\rightrightarrows}{\rightrightarrows} \mathcal{D}$ be functors. The diagram

$$\begin{tikzcd}
 & \mathrm{Nat}(H,K) \times (\mathrm{Nat}(G,H) \times \mathrm{Nat}(F,G)) \arrow[dr, "\mathrm{id}_{\mathrm{Nat}(H,K)} \times \circ_{F,G,H}"] & \\
(\mathrm{Nat}(H,K) \times \mathrm{Nat}(G,H)) \times \mathrm{Nat}(F,G) \arrow[ur, dashed, "\alpha^{\mathsf{Sets}}_{\mathrm{Nat}(H,K),\mathrm{Nat}(G,H),\mathrm{Nat}(F,G)}", "\sim"'] \arrow[d, "\circ_{G,H,K} \times \mathrm{id}_{\mathrm{Nat}(F,G)}"'] & & \mathrm{Nat}(H,K) \times \mathrm{Nat}(F,H) \arrow[d, "\circ_{F,H,K}"] \\
\mathrm{Nat}(G,K) \times \mathrm{Nat}(F,G) \arrow[rr, "\circ_{F,G,K}"'] & & \mathrm{Nat}(F,K)
\end{tikzcd}$$

commutes, i.e. given natural transformations

$$\alpha \colon F \Longrightarrow G,$$
$$\beta \colon G \Longrightarrow H,$$
$$\gamma \colon H \Longrightarrow K,$$

we have

$$(\gamma \circ \beta) \circ \alpha = \gamma \circ (\beta \circ \alpha).$$

3. *Unitality.* Let $F, G \colon \mathcal{C} \rightrightarrows \mathcal{D}$ be functors.

   (a) *Left Unitality.* The diagram

$$\begin{tikzcd}
\mathrm{pt} \times \mathrm{Nat}(F,G) \arrow[d, "{[\mathrm{id}_G] \times \mathrm{id}_{\mathrm{Nat}(F,G)}}"'] \arrow[dr, dashed, "\lambda^{\mathsf{Sets}}_{\mathrm{Nat}(F,G)}", "\sim"'] & \\
\mathrm{Nat}(G,G) \times \mathrm{Nat}(F,G) \arrow[r, "\circ_{F,G,G}"'] & \mathrm{Nat}(F,G)
\end{tikzcd}$$

   commutes, i.e. given a natural transformation $\alpha \colon F \Longrightarrow G$, we have

$$\mathrm{id}_G \circ \alpha = \alpha.$$

(b) *Right Unitality.* The diagram

$$
\begin{tikzcd}
\mathrm{Nat}(F,G)\times \mathrm{pt} \arrow[d, "\mathrm{id}_{\mathrm{Nat}(F,G)}\times[\mathrm{id}_F]"'] \arrow[dr, dashed, "\rho^{\mathsf{Sets}}_{\mathrm{Nat}(F,G)}", "\sim"'] & \\
\mathrm{Nat}(F,G)\times \mathrm{Nat}(F,F) \arrow[r, "\circ^{\mathcal{C}}_{F,F,G}"'] & \mathrm{Nat}(F,G)
\end{tikzcd}
$$

commutes, i.e. given a natural transformation $\alpha\colon F \Longrightarrow G$, we have

$$\alpha\circ \mathrm{id}_F = \alpha.$$

4. *Middle Four Exchange.* Let $F_1,F_2,F_3\colon \mathcal{C}\to\mathcal{D}$ and $G_1,G_2,G_3\colon \mathcal{D}\to\mathcal{E}$ be functors. The diagram

$$
\begin{tikzcd}
(\mathrm{Nat}(G_2,G_3)\times\mathrm{Nat}(G_1,G_2))\times(\mathrm{Nat}(F_2,F_3)\times\mathrm{Nat}(F_1,F_2)) \arrow[d, "\circ_{G_1,G_2,G_3}\times\circ_{F_1,F_2,F_3}"'] & & (\mathrm{Nat}(G_2,G_3)\times\mathrm{Nat}(F_2,F_3))\times(\mathrm{Nat}(G_1,G_2)\times\mathrm{Nat}(F_1,F_2)) \arrow[ll, dashed, "\mu_4", "\sim"'] \arrow[d, "\star_{F_2,F_3,G_2,G_3}\times\star_{F_1,F_2,G_1,G_2}"] \\
\mathrm{Nat}(G_1,G_3)\times\mathrm{Nat}(F_1,F_3) \arrow[dr, "\star_{F_1,F_3,G_1,G_3}"'] & & \mathrm{Nat}(G_2\circ F_2,G_3\circ F_3)\times\mathrm{Nat}(G_1\circ F_1,G_2\circ F_2) \arrow[dl, "\circ_{G_1\circ F_1,G_2\circ F_2,G_3\circ F_3}"] \\
& \mathrm{Nat}(G_1\circ F_1,G_3\circ F_3) &
\end{tikzcd}
$$

commutes, i.e. given a diagram

$$
\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=50, "F_1"] \arrow[r, "F_2" description] \arrow[r, bend right=50, "F_3"'] & \mathcal{D} \arrow[r, bend left=50, "G_1"] \arrow[r, "G_2" description] \arrow[r, bend right=50, "G_3"'] & \mathcal{E}
\end{tikzcd}
$$

in Cats_2, we have

$$(\beta'\star\alpha')\circ(\beta\star\alpha) = (\beta'\circ\beta)\star(\alpha'\circ\alpha).$$

Proof. *Item 1*, *Functionality*: Clear.
Item 2, *Associativity*: Indeed, we have

$$
\begin{aligned}
((\gamma\circ\beta)\circ\alpha)_A &= (\gamma_A\circ\beta_A)\circ\alpha_A\\
&= \gamma_A\circ(\beta_A\circ\alpha_A)\\
&= (\gamma\circ(\beta\circ\alpha))_A
\end{aligned}
$$

for each $A \in \mathrm{Obj}(\mathcal{C})$, showing the desired equality.
Item 3, Unitality: We have

$$\begin{aligned}(\mathrm{id}_G \circ \alpha)_A &= \mathrm{id}_G \circ \alpha_A\\ &= \alpha_A,\\ (\alpha \circ \mathrm{id}_F)_A &= \alpha_A \circ \mathrm{id}_F\\ &= \alpha_A\end{aligned}$$

for each $A \in \mathrm{Obj}(\mathcal{C})$, showing the desired equality.
Item 4, Middle Four Exchange: This is proved in Item 4 of Proposition 5.3.1.2.

□

5.3 Horizontal Composition of Natural Transformations

Definition 5.3.1.1. The **horizontal composition**[31,32] of two natural transformations $\alpha \colon F \Longrightarrow G$ and $\beta \colon H \Longrightarrow K$ as in the diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"{name=U}] \arrow[r, bend right, "G"'{name=D}] \arrow[from=U, to=D, Rightarrow, "\alpha"'] & \mathcal{D} \arrow[r, bend left, "H"{name=U2}] \arrow[r, bend right, "K"'{name=D2}] \arrow[from=U2, to=D2, Rightarrow, "\beta"'] & \mathcal{E} \end{tikzcd}$$

of α and β is the natural transformation

$$\beta \star \alpha \colon (H \circ F) \Longrightarrow (K \circ G),$$

as in the diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=50, "H\circ F"{name=U}] \arrow[r, bend right=50, "K\circ G"'{name=D}] \arrow[from=U, to=D, Rightarrow, "\beta\star\alpha"] & \mathcal{E}, \end{tikzcd}$$

consisting of the collection

$$\left\{(\beta \star \alpha)_A \colon H(F(A)) \to K(G(A))\right\}_{A \in \mathrm{Obj}(\mathcal{C})},$$

[31] *Further Terminology:* Also called the **Godement product** of α and β.
[32] Horizontal composition forms a map

$$\star_{(F,H),(G,K)} \colon \mathrm{Nat}(H,K) \times \mathrm{Nat}(F,G) \to \mathrm{Nat}(H \circ F, K \circ G).$$

of morphisms of $\mathcal{E}$ with

$$\begin{aligned}(\beta \star \alpha)_A &\stackrel{\text{def}}{=} \beta_{G(A)} \circ H(\alpha_A)\\ &= K(\alpha_A) \circ \beta_{F(A)},\end{aligned} \qquad \begin{tikzcd} H(F(A)) \arrow[r, "H(\alpha_A)"] \arrow[d, "\beta_{F(A)}"'] & H(G(A)) \arrow[d, "\beta_{G(A)}"] \\ K(F(A)) \arrow[r, "K(\alpha_A)"'] & K(G(A)). \end{tikzcd}$$

Proof. First, we claim that we indeed have

$$\beta_{G(A)} \circ H(\alpha_A) = K(\alpha_A) \circ \beta_{F(A)}, \qquad \begin{tikzcd} H(F(A)) \arrow[r, "H(\alpha_A)"] \arrow[d, "\beta_{F(A)}"'] & H(G(A)) \arrow[d, "\beta_{G(A)}"] \\ K(F(A)) \arrow[r, "K(\alpha_A)"'] & K(G(A)). \end{tikzcd}$$

This is, however, simply the naturality square for β applied to the morphism $\alpha_A \colon F(A) \to G(A)$. Next, we check the naturality condition for $\beta \star \alpha$, which is the requirement that the boundary of the diagram

$$\begin{tikzcd} H(F(A)) \arrow[r, "H(F(f))"] \arrow[d, "H(\alpha_A)"'] \arrow[dr, phantom, "(1)"] & H(F(B)) \arrow[d, "H(\alpha_B)"] \\ H(G(A)) \arrow[r, "H(G(f))" description] \arrow[d, "\beta_{G(A)}"'] \arrow[dr, phantom, "(2)"] & H(G(B)) \arrow[d, "\beta_{G(B)}"] \\ K(G(A)) \arrow[r, "K(G(f))"'] & K(G(B)) \end{tikzcd}$$

commutes. Since

1. Subdiagram (1) commutes by the naturality of α;
2. Subdiagram (2) commutes by the naturality of β;

so does the boundary diagram. Hence $\beta \circ \alpha$ is a natural transformation.[33] □

[33] *Reference:* [Bor94, Proposition 1.3.4].

Proposition 5.3.1.2. Let $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ be categories.

1. *Functionality.* The assignment $(\beta, \alpha) \mapsto \beta \star \alpha$ defines a function
$$\star_{(F,G),(H,K)} \colon \mathrm{Nat}(H, K) \times \mathrm{Nat}(F, G) \to \mathrm{Nat}(H \circ F, K \circ G).$$

2. *Associativity.* Let
$$\mathcal{C} \underset{G_1}{\overset{F_1}{\rightrightarrows}} \mathcal{D} \underset{G_2}{\overset{F_2}{\rightrightarrows}} \mathcal{E} \underset{G_3}{\overset{F_3}{\rightrightarrows}} \mathcal{F}$$
be a diagram in Cats_2. The diagram
$$\begin{tikzcd} \mathrm{Nat}(F_3, G_3) \times \mathrm{Nat}(F_2, G_2) \times \mathrm{Nat}(F_1, G_1) \arrow[r, "\star_{(F_2,G_2),(F_3,G_3)} \times \mathsf{id}"] \arrow[d, "\mathsf{id} \times \star_{(F_1,G_1),(F_2,G_2)}"'] & \mathrm{Nat}(F_3 \circ F_2, G_3 \circ G_2) \times \mathrm{Nat}(F_1, G_1) \arrow[d, "\star_{(F_3 \circ F_2),(G_3 \circ G_2, F_1, G_1)}"] \\ \mathrm{Nat}(F_3, G_3) \times \mathrm{Nat}(F_2 \circ F_1, G_2 \circ G_1) \arrow[r, "\star_{(F_2 \circ F_1),(G_2 \circ G_1, F_3, G_3)}"'] & \mathrm{Nat}(F_3 \circ F_2 \circ F_1, G_3 \circ G_2 \circ G_1) \end{tikzcd}$$
commutes, i.e. given natural transformations
$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F_1"] \arrow[r, bend right, "G_1"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{D} \arrow[r, bend left, "F_2"] \arrow[r, bend right, "G_2"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{E} \arrow[r, bend left, "F_3"] \arrow[r, bend right, "G_3"'] \arrow[r, phantom, "\gamma\Downarrow"] & \mathcal{F}, \end{tikzcd}$$
we have
$$(\gamma \star \beta) \star \alpha = \gamma \star (\beta \star \alpha).$$

3. *Interaction With Identities.* Let $F \colon \mathcal{C} \to \mathcal{D}$ and $G \colon \mathcal{D} \to \mathcal{E}$ be functors. The diagram
$$\begin{tikzcd} \mathrm{pt} \times \mathrm{pt} \arrow[r, "{[\mathrm{id}_G] \times [\mathrm{id}_F]}"] & \mathrm{Nat}(G, G) \times \mathrm{Nat}(F, F) \arrow[d, "\star_{(F,F),(G,G)}"] \\ \mathrm{pt} \arrow[u, dashed, "\wr"] \arrow[r, "{[\mathrm{id}_{G \circ F}]}"'] & \mathrm{Nat}(G \circ F, G \circ F) \end{tikzcd}$$
commutes, i.e. we have
$$\mathrm{id}_G \star \mathrm{id}_F = \mathrm{id}_{G \circ F}.$$

4. *Middle Four Exchange.* Let $F_1, F_2, F_3 \colon \mathcal{C} \to \mathcal{D}$ and $G_1, G_2, G_3 \colon \mathcal{D} \to \mathcal{E}$ be functors. The diagram

$$\begin{tikzcd}
(\mathrm{Nat}(G_2,G_3) \times \mathrm{Nat}(G_1,G_2)) \times (\mathrm{Nat}(F_2,F_3) \times \mathrm{Nat}(F_1,F_2)) \arrow[rr, dashed, leftrightarrow, "\mu_4", "\sim"'] \arrow[d, "\circ_{G_1,G_2,G_3} \times \circ_{F_1,F_2,F_3}"'] & & (\mathrm{Nat}(G_2,G_3) \times \mathrm{Nat}(F_2,F_3)) \times (\mathrm{Nat}(G_1,G_2) \times \mathrm{Nat}(F_1,F_2)) \arrow[d, "\star_{F_2,F_3,G_2,G_3} \times \star_{F_1,F_2,G_1,G_2}"] \\
\mathrm{Nat}(G_1,G_3) \times \mathrm{Nat}(F_1,F_3) \arrow[dr, "\star_{F_1,F_3,G_1,G_3}"'] & & \mathrm{Nat}(G_2 \circ F_2, G_3 \circ F_3) \times \mathrm{Nat}(G_1 \circ F_1, G_2 \circ F_2) \arrow[dl, "\circ_{G_1 \circ F_1, G_2 \circ F_2, G_3 \circ F_3}"] \\
& \mathrm{Nat}(G_1 \circ F_1, G_3 \circ F_3) &
\end{tikzcd}$$

commutes, i.e. given a diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=60, "F_1"{name=U}] \arrow[r, "F_2"{name=M, description}] \arrow[r, bend right=60, "F_3"'{name=D}] & \mathcal{D} \arrow[r, bend left=60, "G_1"{name=U2}] \arrow[r, "G_2"{name=M2, description}] \arrow[r, bend right=60, "G_3"'{name=D2}] & \mathcal{E}
\arrow[Rightarrow, from=U, to=M, "\alpha"']
\arrow[Rightarrow, from=M, to=D, "\alpha'"']
\arrow[Rightarrow, from=U2, to=M2, "\beta"']
\arrow[Rightarrow, from=M2, to=D2, "\beta'"']
\end{tikzcd}$$

in Cats_2, we have

$$(\beta' \star \alpha') \circ (\beta \star \alpha) = (\beta' \circ \beta) \star (\alpha' \circ \alpha).$$

Proof. *Item 1, Functionality*: Clear.
Item 2, Associativity: Omitted.
Item 3, Interaction With Identities: We have

$$\begin{aligned}
(\mathrm{id}_G \star \mathrm{id}_F)_A &\stackrel{\text{def}}{=} (\mathrm{id}_G)_{F_A} \circ G_{(\mathrm{id}_F)_A} \\
&\stackrel{\text{def}}{=} \mathrm{id}_{G_{F_A}} \circ G_{\mathrm{id}_{F_A}} \\
&= \mathrm{id}_{G_{F_A}} \circ \mathrm{id}_{G_{F_A}} \\
&= \mathrm{id}_{G_{F_A}} \\
&\stackrel{\text{def}}{=} (\mathrm{id}_{G \circ F})_A
\end{aligned}$$

for each $A \in \mathrm{Obj}(\mathcal{C})$, showing the desired equality.

Item 4, Middle Four Exchange: Let $A \in \mathrm{Obj}(\mathcal{C})$ and consider the diagram

$$\begin{tikzcd}
 & & G_{F''_A} \arrow[dr, "\beta_{F''_A}"] & & \\
G_{F_A} \arrow[r, "G_{\alpha_A}"] & G_{F'_A} \arrow[ur, "G_{\alpha'_A}"] \arrow[dr, "\beta_{F'_A}"'] & (1) & G''_{F''_A} \arrow[r, "\beta'_{F''_A}"] & G''_{F''_A}. \\
 & & G'_{F'_A} \arrow[ur, "G'_{\alpha'_A}"'] & &
\end{tikzcd}$$

The top composition is $((\beta' \circ \beta) \star (\alpha' \circ \alpha))_A$ and the bottom composition is $((\beta' \star \alpha') \circ (\beta \star \alpha))_A$. Since Subdiagram (1) commutes, they are equal. $\square$

5.4 Properties of Natural Transformations

Proposition 5.4.1.1. Let $F, G\colon \mathcal{C} \rightrightarrows \mathcal{D}$ be functors. The following data are equivalent:[34]

1. A natural transformation $\alpha\colon F \Longrightarrow G$.
2. A functor $[\alpha]\colon \mathcal{C} \to \mathcal{D}^{\Bbbk}$ filling the diagram

$$\begin{tikzcd}
 & \mathcal{D} \\
\mathcal{C} \arrow[ur, "F"] \arrow[r, "{[\alpha]}"] \arrow[dr, "G"'] & \mathcal{D}^{\Bbbk}. \arrow[u, two heads, "\mathrm{ev}_0"'] \arrow[d, two heads, "\mathrm{ev}_1"] \\
 & \mathcal{D}
\end{tikzcd}$$

3. A functor $[\alpha]\colon \mathcal{C} \times \Bbbk \to \mathcal{D}$ filling the diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[dr, "F"] & \\
\mathcal{C} \times \Bbbk \arrow[u, two heads, "\mathrm{ev}_0"] \arrow[d, two heads, "\mathrm{ev}_1"'] \arrow[r, "{[\alpha]}"] & \mathcal{D}. \\
\mathcal{C} \arrow[ur, "G"'] &
\end{tikzcd}$$

[34]Taken from [MO MO64365].

Proof. *From Item 1 to Item 2 and Back*: We may identify $\mathcal{D}^{\nVdash}$ with $\mathsf{Arr}(\mathcal{D})$. Given a natural transformation $\alpha\colon F \Longrightarrow G$, we have a functor

$$
\begin{aligned}
[\alpha]\colon \mathcal{C} &\longrightarrow \mathcal{D}^{\nVdash} \\
A &\longmapsto \alpha_A \\
(f\colon A \to B) &\longmapsto \left(
\begin{tikzcd}
F_A \arrow[r, "F_f"] \arrow[d, "\alpha_A"'] & F_B \arrow[d, "\alpha_B"] \\
G_A \arrow[r, "G_f"'] & G_B
\end{tikzcd}
\right)
\end{aligned}
$$

making the diagram in Item 2 commute. Conversely, every such functor gives rise to a natural transformation from F to G, and these constructions are inverse to each other.
From Item 2 to Item 3 and Back: This follows from Item 3 of Proposition 6.1.1.2. □

5.5 Natural Isomorphisms

Definition 5.5.1.1. A natural transformation $\alpha\colon F \Longrightarrow G$ between functors $F, G\colon \mathcal{C} \to \mathcal{D}$ between categories $\mathcal{C}$ and $\mathcal{D}$ is a **natural isomorphism** if there exists a natural transformation $\alpha^{-1}\colon G \Longrightarrow F$ such that

$$
\begin{aligned}
\alpha^{-1} \circ \alpha &= \mathrm{id}_F, \\
\alpha \circ \alpha^{-1} &= \mathrm{id}_G.
\end{aligned}
$$

Proposition 5.5.1.2. Let $\alpha\colon F \Longrightarrow G$ be a natural transformation.

1. *Characterisations.* The following conditions are equivalent:

   (a) The natural transformation α is a natural isomorphism.

   (b) For each $A \in \mathrm{Obj}(\mathcal{C})$, the morphism $\alpha_A\colon F_A \to G_A$ is an isomorphism.

2. *Componentwise Inverses of Natural Transformations Assemble Into Natural Transformations.* Let $\alpha^{-1}\colon G \Longrightarrow F$ be a transformation such that, for each $A \in \mathrm{Obj}(\mathcal{C})$, we have

   $$
   \begin{aligned}
   \alpha_A^{-1} \circ \alpha_A &= \mathrm{id}_{F(A)}, \\
   \alpha_A \circ \alpha_A^{-1} &= \mathrm{id}_{G(A)}.
   \end{aligned}
   $$

   Then α^{-1} is a natural transformation.

Proof. *Item 1, Characterisations*: The implication Item 1a $\implies$ Item 1b is clear, whereas the implication Item 1b $\implies$ Item 1a follows from Item 2.
Item 2, Componentwise Inverses of Natural Transformations Assemble Into Natural Transformations: The naturality condition for α^{-1} corresponds to the commutativity of the diagram

$$\begin{tikzcd} G(A) \arrow[r, "G(f)"] \arrow[d, "\alpha_A^{-1}"'] & G(B) \arrow[d, "\alpha_B^{-1}"] \\ F(A) \arrow[r, "F(f)"'] & F(B) \end{tikzcd}$$

for each $A, B \in \mathrm{Obj}(\mathcal{C})$ and each $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$. Considering the diagram

$$\begin{tikzcd} G(A) \arrow[r, "G(f)"] \arrow[d, "\alpha_A^{-1}"'] \arrow[dr, phantom, "(1)"] & G(B) \arrow[d, "\alpha_B^{-1}"] \\ F(A) \arrow[r, "F(f)" description] \arrow[d, "\alpha_A"'] \arrow[dr, phantom, "(2)"] & F(B) \arrow[d, "\alpha_B"] \\ G(A) \arrow[r, "G(f)"'] & G(B), \end{tikzcd}$$

where the boundary diagram as well as Subdiagram (2) commute, we have

$$\begin{aligned} G(f) &= G(f) \circ \mathrm{id}_{G(A)} \\ &= G(f) \circ \alpha_A \circ \alpha_A^{-1} \\ &= \alpha_B \circ F(f) \circ \alpha_A^{-1}. \end{aligned}$$

Postcomposing both sides with α_B^{-1}, we get

$$\begin{aligned} \alpha_B^{-1} \circ G(f) &= \alpha_B^{-1} \circ \alpha_B \circ F(f) \circ \alpha_A^{-1} \\ &= \mathrm{id}_{F(B)} \circ F(f) \circ \alpha_A^{-1} \\ &= F(f) \circ \alpha_A^{-1}, \end{aligned}$$

which is the naturality condition we wanted to show. Thus α^{-1} is a natural transformation. $\square$

6 Categories of Categories

6.1 Functor Categories

Let $\mathcal{C}$ be a category and $\mathcal{D}$ be a small category.

Definition 6.1.1.1. The **category of functors from $\mathcal{C}$ to $\mathcal{D}$**[35] is the category $\mathsf{Fun}(\mathcal{C},\mathcal{D})$[36] where

- *Objects.* The objects of $\mathsf{Fun}(\mathcal{C},\mathcal{D})$ are functors from $\mathcal{C}$ to $\mathcal{D}$;
- *Morphisms.* For each $F, G \in \mathrm{Obj}(\mathsf{Fun}(\mathcal{C},\mathcal{D}))$, we have

$$\mathrm{Hom}_{\mathsf{Fun}(\mathcal{C},\mathcal{D})}(F,G) \overset{\mathrm{def}}{=} \mathrm{Nat}(F,G);$$

- *Identities.* For each $F \in \mathrm{Obj}(\mathsf{Fun}(\mathcal{C},\mathcal{D}))$, the unit map

$$\mathbb{1}^{\mathsf{Fun}(\mathcal{C},\mathcal{D})}_F \colon \mathrm{pt} \to \mathrm{Nat}(F,F)$$

  of $\mathsf{Fun}(\mathcal{C},\mathcal{D})$ at F is given by

$$\mathrm{id}^{\mathsf{Fun}(\mathcal{C},\mathcal{D})}_F \overset{\mathrm{def}}{=} \mathrm{id}_F,$$

  where $\mathrm{id}_F \colon F \Longrightarrow F$ is the identity natural transformation of F of Example 5.1.1.3;

- *Composition.* For each $F, G, H \in \mathrm{Obj}(\mathsf{Fun}(\mathcal{C},\mathcal{D}))$, the composition map

$$\circ^{\mathsf{Fun}(\mathcal{C},\mathcal{D})}_{F,G,H} \colon \mathrm{Nat}(G,H) \times \mathrm{Nat}(F,G) \to \mathrm{Nat}(F,H)$$

  of $\mathsf{Fun}(\mathcal{C},\mathcal{D})$ at (F,G,H) is given by

$$\beta \circ^{\mathsf{Fun}(\mathcal{C},\mathcal{D})}_{F,G,H} \alpha \overset{\mathrm{def}}{=} \beta \circ \alpha,$$

  where $\beta \circ \alpha$ is the vertical composition of α and β of Item 1 of Proposition 5.2.1.2.

Proposition 6.1.1.2. Let $\mathcal{C}$ and $\mathcal{D}$ be categories and let $F \colon \mathcal{C} \to \mathcal{D}$ be a functor.

[35] *Further Terminology:* Also called the **functor category** $\mathsf{Fun}(\mathcal{C},\mathcal{D})$.
[36] *Further Notation:* Also written $\mathcal{D}^{\mathcal{C}}$ and $[\mathcal{C},\mathcal{D}]$.

1. *Functoriality.* The assignments $\mathcal{C}, \mathcal{D}, (\mathcal{C}, \mathcal{D}) \mapsto \mathsf{Fun}(\mathcal{C}, \mathcal{D})$ define functors

$$
\begin{aligned}
\mathsf{Fun}(\mathcal{C}, -_2) &\colon \mathsf{Cats} \to \mathsf{Cats}, \\
\mathsf{Fun}(-_1, \mathcal{D}) &\colon \mathsf{Cats}^{\mathsf{op}} \to \mathsf{Cats}, \\
\mathsf{Fun}(-_1, -_2) &\colon \mathsf{Cats}^{\mathsf{op}} \times \mathsf{Cats} \to \mathsf{Cats}.
\end{aligned}
$$

2. *2-Functoriality.* The assignments $\mathcal{C}, \mathcal{D}, (\mathcal{C}, \mathcal{D}) \mapsto \mathsf{Fun}(\mathcal{C}, \mathcal{D})$ define 2-functors

$$
\begin{aligned}
\mathsf{Fun}(\mathcal{C}, -_2) &\colon \mathsf{Cats}_2 \to \mathsf{Cats}_2, \\
\mathsf{Fun}(-_1, \mathcal{D}) &\colon \mathsf{Cats}_2^{\mathsf{op}} \to \mathsf{Cats}_2, \\
\mathsf{Fun}(-_1, -_2) &\colon \mathsf{Cats}_2^{\mathsf{op}} \times \mathsf{Cats}_2 \to \mathsf{Cats}_2.
\end{aligned}
$$

3. *Adjointness.* We have adjunctions

$$
(\mathcal{C} \times - \dashv \mathsf{Fun}(\mathcal{C}, -)) \colon \quad
\begin{tikzcd}
\mathsf{Cats} \arrow[r, bend left, "\mathcal{C}\times-"] \arrow[r, phantom, "\perp"] & \mathsf{Cats} \arrow[l, bend left, "\mathsf{Fun}(\mathcal{C},-)"]
\end{tikzcd},
$$

$$
(- \times \mathcal{D} \dashv \mathsf{Fun}(\mathcal{D}, -)) \colon \quad
\begin{tikzcd}
\mathsf{Cats} \arrow[r, bend left, "-\times\mathcal{D}"] \arrow[r, phantom, "\perp"] & \mathsf{Cats} \arrow[l, bend left, "\mathsf{Fun}(\mathcal{D},-)"]
\end{tikzcd},
$$

witnessed by bijections of sets

$$
\begin{aligned}
\mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) &\cong \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{D}, \mathsf{Fun}(\mathcal{C}, \mathcal{E})), \\
\mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) &\cong \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, \mathsf{Fun}(\mathcal{D}, \mathcal{E})),
\end{aligned}
$$

natural in $\mathcal{C}, \mathcal{D}, \mathcal{E} \in \mathrm{Obj}(\mathsf{Cats})$.

4. *2-Adjointness.* We have 2-adjunctions

$$
(\mathcal{C} \times - \dashv \mathsf{Fun}(\mathcal{C}, -)) \colon \quad
\begin{tikzcd}
\mathsf{Cats}_2 \arrow[r, bend left, "\mathcal{C}\times-"] \arrow[r, phantom, "\perp_2"] & \mathsf{Cats}_2 \arrow[l, bend left, "\mathsf{Fun}(\mathcal{C},-)"]
\end{tikzcd},
$$

$$
(- \times \mathcal{D} \dashv \mathsf{Fun}(\mathcal{D}, -)) \colon \quad
\begin{tikzcd}
\mathsf{Cats}_2 \arrow[r, bend left, "-\times\mathcal{D}"] \arrow[r, phantom, "\perp_2"] & \mathsf{Cats}_2 \arrow[l, bend left, "\mathsf{Fun}(\mathcal{D},-)"]
\end{tikzcd},
$$

witnessed by isomorphisms of categories

$$
\begin{aligned}
\mathsf{Fun}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) &\cong \mathsf{Fun}(\mathcal{D}, \mathsf{Fun}(\mathcal{C}, \mathcal{E})), \\
\mathsf{Fun}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) &\cong \mathsf{Fun}(\mathcal{C}, \mathsf{Fun}(\mathcal{D}, \mathcal{E})),
\end{aligned}
$$

natural in $\mathcal{C}, \mathcal{D}, \mathcal{E} \in \mathrm{Obj}(\mathsf{Cats}_2)$.

5. *Trivial Functor Categories.* We have a canonical isomorphism of categories
$$\mathsf{Fun}(\mathsf{pt}, \mathcal{C}) \cong \mathcal{C},$$
natural in $\mathcal{C} \in \mathrm{Obj}(\mathsf{Cats})$.

6. *Objectwise Computation of Co/Limits.* Let
$$D \colon \mathcal{I} \to \mathsf{Fun}(\mathcal{C}, \mathcal{D})$$
be a diagram in $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$. We have isomorphisms
$$\begin{aligned} \lim(D)_A &\cong \lim_{i \in \mathcal{I}}(D_i(A)), \\ \operatorname{colim}(D)_A &\cong \operatorname*{colim}_{i \in \mathcal{I}}(D_i(A)), \end{aligned}$$
naturally in $A \in \mathrm{Obj}(\mathcal{C})$.

7. *Bicompleteness.* If $\mathcal{E}$ is co/complete, then so is $\mathsf{Fun}(\mathcal{C}, \mathcal{E})$.

8. *Abelianness.* If $\mathcal{E}$ is abelian, then so is $\mathsf{Fun}(\mathcal{C}, \mathcal{E})$.

9. *Monomorphisms and Epimorphisms.* Let $\alpha \colon F \Longrightarrow G$ be a morphism of $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$. The following conditions are equivalent:

   (a) The natural transformation
   $$\alpha \colon F \Longrightarrow G$$
   is a monomorphism (resp. epimorphism) in $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$.

   (b) For each $A \in \mathrm{Obj}(\mathcal{C})$, the morphism
   $$\alpha_A \colon F_A \to G_A$$
   is a monomorphism (resp. epimorphism) in $\mathcal{D}$.

Proof. *Item 1*, *Functoriality*: Omitted.
Item 2, *2-Functoriality*: Omitted.
Item 3, *Adjointness*: Omitted.
Item 4, *2-Adjointness*: Omitted.
Item 5, *Trivial Functor Categories*: Omitted.
Item 6, *Objectwise Computation of Co/Limits*: Omitted.
Item 7, *Bicompleteness*: This follows from **??**.
Item 8, *Abelianness*: Omitted.
Item 9, *Monomorphisms and Epimorphisms*: Omitted. □

6.2 The Category of Categories and Functors

Definition 6.2.1.1. The **category of (small) categories and functors** is the category Cats where

- *Objects.* The objects of Cats are small categories;
- *Morphisms.* For each $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$, we have

$$\mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, \mathcal{D}) \stackrel{\text{def}}{=} \mathrm{Obj}(\mathsf{Fun}(\mathcal{C}, \mathcal{D}));$$

- *Identities.* For each $\mathcal{C} \in \mathrm{Obj}(\mathsf{Cats})$, the unit map

$$\mathbb{1}^{\mathsf{Cats}}_{\mathcal{C}} \colon \mathrm{pt} \to \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, \mathcal{C})$$

  of Cats at $\mathcal{C}$ is defined by

$$\mathrm{id}^{\mathsf{Cats}}_{\mathcal{C}} \stackrel{\text{def}}{=} \mathrm{id}_{\mathcal{C}},$$

  where $\mathrm{id}_{\mathcal{C}} \colon \mathcal{C} \to \mathcal{C}$ is the identity functor of $\mathcal{C}$ of Example 4.1.1.2;
- *Composition.* For each $\mathcal{C}, \mathcal{D}, \mathcal{E} \in \mathrm{Obj}(\mathsf{Cats})$, the composition map

$$\circ^{\mathsf{Cats}}_{\mathcal{C},\mathcal{D},\mathcal{E}} \colon \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{D}, \mathcal{E}) \times \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, \mathcal{D}) \to \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, \mathcal{E})$$

  of Cats at $(\mathcal{C}, \mathcal{D}, \mathcal{E})$ is given by

$$G \circ^{\mathsf{Cats}}_{\mathcal{C},\mathcal{D},\mathcal{E}} F \stackrel{\text{def}}{=} G \circ F,$$

  where $G \circ F \colon \mathcal{C} \to \mathcal{E}$ is the composition of F and G of Definition 4.1.1.3.

Proposition 6.2.1.2. Let $\mathcal{C}$ be a category.

1. *Co/Completeness.* The category Cats is complete and cocomplete.
2. *Cartesian Monoidal Structure.* The quadruple $(\mathsf{Cats}, \times, \mathsf{pt}, \mathsf{Fun})$ is a Cartesian closed monoidal category.

Proof. *Item 1, Co/Completeness*: This follows from
Item 2, Cartesian Monoidal Structure: Omitted. □

6.3 The 2-Category of Categories, Functors, and Natural Transformations

Definition 6.3.1.1. The **2-category of (small) categories, functors, and natural transformations** is the 2-category Cats_2 where

- *Objects.* The objects of Cats_2 are small categories;
- Hom-*Categories.* For each $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats}_2)$, we have

$$\mathsf{Hom}_{\mathsf{Cats}_2}(\mathcal{C}, \mathcal{D}) \stackrel{\text{def}}{=} \mathsf{Fun}(\mathcal{C}, \mathcal{D});$$

- *Identities.* For each $\mathcal{C} \in \mathrm{Obj}(\mathsf{Cats}_2)$, the unit functor

$$\mathbb{1}^{\mathsf{Cats}_2}_{\mathcal{C}} \colon \mathsf{pt} \to \mathsf{Fun}(\mathcal{C}, \mathcal{C})$$

  of Cats_2 at $\mathcal{C}$ is the functor picking the identity functor $\mathrm{id}_{\mathcal{C}} \colon \mathcal{C} \to \mathcal{C}$ of $\mathcal{C}$;

- *Composition.* For each $\mathcal{C}, \mathcal{D}, \mathcal{E} \in \mathrm{Obj}(\mathsf{Cats}_2)$, the composition bifunctor

$$\circ^{\mathsf{Cats}_2}_{\mathcal{C},\mathcal{D},\mathcal{E}} \colon \mathsf{Hom}_{\mathsf{Cats}_2}(\mathcal{D}, \mathcal{E}) \times \mathsf{Hom}_{\mathsf{Cats}_2}(\mathcal{C}, \mathcal{D}) \to \mathsf{Hom}_{\mathsf{Cats}_2}(\mathcal{C}, \mathcal{E})$$

  of Cats_2 at $(\mathcal{C}, \mathcal{D}, \mathcal{E})$ is the functor where

  - *Action on Objects.* For each object $(G, F) \in \mathrm{Obj}(\mathsf{Hom}_{\mathsf{Cats}_2}(\mathcal{D}, \mathcal{E}) \times \mathsf{Hom}_{\mathsf{Cats}_2}(\mathcal{C}, \mathcal{D}))$, we have

$$\circ^{\mathsf{Cats}_2}_{\mathcal{C},\mathcal{D},\mathcal{E}}(G, F) \stackrel{\text{def}}{=} G \circ F;$$

  - *Action on Morphisms.* For each morphism $(\beta, \alpha) \colon (K, H) \Longrightarrow (G, F)$ of $\mathsf{Hom}_{\mathsf{Cats}_2}(\mathcal{D}, \mathcal{E}) \times \mathsf{Hom}_{\mathsf{Cats}_2}(\mathcal{C}, \mathcal{D})$, we have

$$\circ^{\mathsf{Cats}_2}_{\mathcal{C},\mathcal{D},\mathcal{E}}(\beta, \alpha) \stackrel{\text{def}}{=} \beta \star \alpha,$$

    where $\beta \star \alpha$ is the horizontal composition of α and β of Definition 5.3.1.1.

Proposition 6.3.1.2. Let $\mathcal{C}$ be a category.

1. *2-Categorical Co/Completeness.* The 2-category Cats_2 is complete and cocomplete as a 2-category, having all 2-categorical and bicategorical co/limits.

Proof. *Item 1, Co/Completeness*: This follows from □

6.4 The Category of Groupoids

Definition 6.4.1.1. The **category of (small) groupoids** is the full subcategory Grpd of Cats spanned by the groupoids.

6.5 The 2-Category of Groupoids

Definition 6.5.1.1. The **2-category of (small) groupoids** is the full sub-2-category Grpd_2 of Cats_2 spanned by the groupoids.

7 Miscellany

7.1 Concrete Categories

Definition 7.1.1.1. A category $\mathcal{C}$ is **concrete** if there exists a faithful functor $F\colon \mathcal{C} \to \mathsf{Sets}$.

7.2 Balanced Categories

Definition 7.2.1.1. A category is **balanced** if every morphism which is both a monomorphism and an epimorphism is an isomorphism.

7.3 Monoid Actions on Objects of Categories

Let A be a monoid, let $\mathcal{C}$ be a category, and let $X \in \mathrm{Obj}(\mathcal{C})$.

Definition 7.3.1.1. An **A-action on X** is a functor $\lambda\colon \mathsf{B}A \to \mathcal{C}$ with $\lambda(\star) = X$.

Remark 7.3.1.2. In detail, an **A-action on X** is an A-action on $\mathsf{End}_{\mathcal{C}}(X)$, consisting of a morphism

$$\lambda\colon A \to \underbrace{\mathsf{End}_{\mathcal{C}}(X)}_{\overset{\mathrm{def}}{=}\mathrm{Hom}_{\mathcal{C}}(X,X)}$$

satisfying the following conditions:

1. *Preservation of Identities.* We have

$$\lambda_{1_A} = \mathrm{id}_X.$$

2. *Preservation of Composition.* For each $a, b \in A$, we have

$$\lambda_b \circ \lambda_a = \lambda_{ab}, \qquad \begin{tikzcd} X \arrow[r, "\lambda_a"] \arrow[dr, "\lambda_{ab}"'] & X \arrow[d, "\lambda_b"] \\ & X. \end{tikzcd}$$

7.4 Group Actions on Objects of Categories

Let G be a group, let $\mathcal{C}$ be a category, and let $X \in \mathrm{Obj}(\mathcal{C})$.

Definition 7.4.1.1. A **G-action on X** is a functor $\lambda\colon \mathsf{B}G \to \mathcal{C}$ with $\lambda(\star) = X$.

Remark 7.4.1.2. In detail, a **G-action on X** is a G-action on $\mathsf{Aut}_{\mathcal{C}}(X)$, consisting of a morphism

$$\lambda\colon G \to \underbrace{\mathsf{End}_{\mathcal{C}}(X)}_{\overset{\mathrm{def}}{=}\mathrm{Hom}_{\mathcal{C}}(X,X)}$$

satisfying the following conditions:

1. *Preservation of Identities.* We have

$$\lambda_{1_A} = \mathrm{id}_X.$$

2. *Preservation of Composition.* For each $a, b \in A$, we have

$$\lambda_b \circ \lambda_a = \lambda_{ab}, \qquad \begin{tikzcd} X \arrow[r, "\lambda_a"] \arrow[dr, "\lambda_{ab}"'] & X \arrow[d, "\lambda_b"] \\ & X. \end{tikzcd}$$