Adjunctions and the Yoneda Lemma

December 24, 2023

00VY

Contents

1 Adjunctions 00VZ

1.1 Foundations 00W0

Let $\mathcal{C}$ and $\mathcal{D}$ be two categories.

Definition 1.1.1.1. An **adjunction**[1] is a quadruple (F, G, η, ϵ) consisting of 00W1

1. A functor $F \colon \mathcal{C} \to \mathcal{D}$;
2. A functor $G \colon \mathcal{D} \to \mathcal{C}$;
3. A natural transformation $\eta \colon \mathrm{id}_{\mathcal{C}} \Longrightarrow G \circ F$;
4. A natural transformation $\epsilon \colon F \circ G \Longrightarrow \mathrm{id}_{\mathcal{D}}$;

[1] *Further Terminology:* We also call (G, F) an **adjoint pair**, F a **left adjoint**, G a **right adjoint**, η the **unit** of the adjunction, and ϵ the **counit** of the adjunction.

such that we have equalities

$$
\begin{tikzcd}
& \mathcal{D} \arrow[rr, "\mathrm{id}_{\mathcal{D}}"] \arrow[dr, "G"'] \arrow[r, phantom, "\Uparrow\epsilon"] & {} & \mathcal{D} \\
\mathcal{C} \arrow[ur, "F"] \arrow[rr, "\mathrm{id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Uparrow\eta"] & {} & \mathcal{C} \arrow[ur, "F"'] &
\end{tikzcd}
=
\begin{tikzcd}
& \mathcal{D} \arrow[rr, "\mathrm{id}_{\mathcal{D}}"] & & \mathcal{D} \\
\mathcal{C} \arrow[ur, "F"] \arrow[rr, "\mathrm{id}_{\mathcal{C}}"'] & & \mathcal{C} \arrow[ur, "F"'] \arrow[ul, phantom, "\Leftarrow \mathrm{id}_F"] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
& \mathcal{C} \arrow[rr, "\mathrm{id}_{\mathcal{C}}"] \arrow[dr, "F"'] \arrow[r, phantom, "\Downarrow\eta"] & {} & \mathcal{C} \\
\mathcal{D} \arrow[ur, "G"] \arrow[rr, "\mathrm{id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\epsilon"] & {} & \mathcal{D} \arrow[ur, "G"'] &
\end{tikzcd}
=
\begin{tikzcd}
& \mathcal{C} \arrow[rr, "\mathrm{id}_{\mathcal{C}}"] \arrow[dr, phantom, "\Rightarrow \mathrm{id}_G"] & & \mathcal{C} \\
\mathcal{D} \arrow[ur, "G"] \arrow[rr, "\mathrm{id}_{\mathcal{D}}"'] & & \mathcal{D} \arrow[ur, "G"'] &
\end{tikzcd}
$$

of pasting diagrams in Cats_2.[2]

Example 1.1.1.2. Here are some examples of adjunctions.

1. We have a triple adjunction

$$
(\lceil - \rceil \dashv \iota \dashv \lfloor - \rfloor)\colon \quad
\begin{tikzcd}
\mathbb{R} \arrow[r, bend left=50, "\lceil - \rceil"] \arrow[r, bend right=50, "\lfloor - \rfloor"'] & \mathbb{Z}, \arrow[l, hook', "\iota"'] \arrow[l, phantom, shift right=3, "\perp"] \arrow[l, phantom, shift left=3, "\perp"]
\end{tikzcd}
$$

[2]Equivalently, the diagrams

$$
\begin{tikzcd}
F \arrow[r, Rightarrow, "\mathrm{id}_F \circ \eta"] \arrow[dr, Rightarrow, "\mathrm{id}_F"'] & F \circ G \circ F \arrow[d, Rightarrow, "\epsilon \circ \mathrm{id}_F"] \\
& F
\end{tikzcd}
\qquad
\begin{tikzcd}
G \arrow[r, Rightarrow, "\eta \circ \mathrm{id}_G"] \arrow[dr, Rightarrow, "\mathrm{id}_G"'] & G \circ F \circ G \arrow[d, Rightarrow, "\mathrm{id}_G \circ \epsilon"] \\
& G,
\end{tikzcd}
\tag{1.1.1.1}
$$

called the **left** and **right triangle identities**, commute, or, again equivalently, for each $A \in \mathrm{Obj}(\mathcal{C})$ and each $B \in \mathrm{Obj}(\mathcal{D})$, the diagrams

$$
\begin{tikzcd}
F_A \arrow[r, "F_{\eta_A}"] \arrow[dr, "\mathrm{id}_{F_A}"'] & F_{G_{F_A}} \arrow[d, "\epsilon_{F_A}"] \\
& F_A
\end{tikzcd}
\qquad
\begin{tikzcd}
G_B \arrow[r, "\eta_{G_B}"] \arrow[dr, "\mathrm{id}_{G_B}"'] & G_{F_{G_B}} \arrow[d, "G_{\epsilon_B}"] \\
& G_B
\end{tikzcd}
$$

commute.

where $\mathbb{Z}$ and $\mathbb{R}$ are viewed as poset categories and $\iota\colon \mathbb{Z} \hookrightarrow \mathbb{R}$ is the canonical inclusion.

Proposition 1.1.1.3. Let $F, L\colon \mathcal{C} \rightrightarrows \mathcal{D}$ and $G, R\colon \mathcal{D} \rightrightarrows \mathcal{C}$ be functors.

1. *Characterisations.* The following conditions are equivalent:
   (a) The pair (L, R) is an adjoint pair.
   (b) We have a natural isomorphism of (pro)functors[3]
   $$\mathsf{h}^{L} \cong \mathsf{h}_{R}.$$
   (c) For each $A \in \mathrm{Obj}(\mathcal{C})$ and each $B \in \mathrm{Obj}(\mathcal{D})$, we have an isomorphism
   $$\mathrm{Hom}_{\mathcal{D}}(L_A, B) \cong \mathrm{Hom}_{\mathcal{C}}(A, R_B)$$

[3] That is, the following conditions are satisfied:

1. *Bijection.* For each $A \in \mathrm{Obj}(\mathcal{C})$ and each $B \in \mathrm{Obj}(\mathcal{D})$, we have a bijection
$$\mathrm{Hom}_{\mathcal{D}}(L_A, B) \cong \mathrm{Hom}_{\mathcal{C}}(A, R_B).$$
2. *Naturality in $\mathcal{D}$.* For each morphism $g\colon B \to B'$ of $\mathcal{D}$, the diagram
$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}}(L_A, B) \arrow[r, dashed, "\sim"] \arrow[d, "h_g^{\mathrm{id}_{L_A}}"'] & \mathrm{Hom}_{\mathcal{C}}(A, R_B) \arrow[d, "h^{\mathrm{id}_A}_{R_g}"] \\
\mathrm{Hom}_{\mathcal{D}}(L_A, B') \arrow[r, dashed, "\sim"] & \mathrm{Hom}_{\mathcal{C}}(A, R_{B'})
\end{tikzcd}$$
commutes.
3. *Naturality in $\mathcal{C}$.* For each morphism $f\colon A \to A'$ of $\mathcal{C}$, the diagram
$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}}(L_A, B) \arrow[r, dashed, "\sim"] \arrow[d, "h^{L_f}_{\mathrm{id}_B}"'] & \mathrm{Hom}_{\mathcal{C}}(A, R_B) \arrow[d, "h^{f}_{\mathrm{id}_{R_B}}"] \\
\mathrm{Hom}_{\mathcal{D}}(L_{A'}, B) \arrow[r, dashed, "\sim"] & \mathrm{Hom}_{\mathcal{C}}(A', R_B)
\end{tikzcd}$$
commutes.

and the square below-left commutes iff the square below-right commutes:

$$\begin{tikzcd} L_A \arrow[r, "f^\sharp"] \arrow[d, "L_\phi"'] & B \arrow[d, "\psi"] \\ L_{A'} \arrow[r, "g^\sharp"'] & B' \end{tikzcd} \iff \begin{tikzcd} A \arrow[r, "f^\flat"] \arrow[d, "\phi"'] & R_B \arrow[d, "R_\psi"] \\ A' \arrow[r, "g^\flat"'] & R_{B'}. \end{tikzcd}$$

(d) For each small category $\mathcal{K}$, we have an adjunction

$$(L_* \dashv R_*)\colon \quad \begin{tikzcd} \mathsf{Fun}(\mathcal{K}, \mathcal{C}) \arrow[r, bend left, "L_*"] \arrow[r, phantom, "\perp"] & \mathsf{Fun}(\mathcal{K}, \mathcal{D}) \arrow[l, bend left, "R_*"] \end{tikzcd}$$

as witnessed by a natural isomorphism

$$\mathrm{Nat}(L \circ F, G) \cong \mathrm{Nat}(F, R \circ G)$$

$$\begin{tikzcd} & \mathcal{C} \arrow[dr, "L"] \arrow[d, phantom, "\Downarrow"] & \\ \mathcal{K} \arrow[ur, "F"] \arrow[rr, bend right, "G"'] & {} & \mathcal{D} \end{tikzcd} \quad \overset{\text{bij.}}{\leftrightarrow} \quad \begin{tikzcd} \mathcal{K} \arrow[rr, bend left, "F"] \arrow[dr, "G"'] & {} \arrow[d, phantom, "\Downarrow"] & \mathcal{C} \\ & \mathcal{D} \arrow[ur, "R"'] & \end{tikzcd}$$

natural in $\mathcal{K} \xrightarrow{F} \mathcal{C}$ and $\mathcal{K} \xrightarrow{G} \mathcal{D}$.

(e) For each locally small category $\mathcal{E}$, we have an adjunction

$$(R^* \dashv L^*)\colon \quad \begin{tikzcd} \mathsf{Fun}(\mathcal{C}, \mathcal{E}) \arrow[r, bend left, "R^*"] \arrow[r, phantom, "\perp"] & \mathsf{Fun}(\mathcal{D}, \mathcal{E}) \arrow[l, bend left, "L^*"] \end{tikzcd}$$

as witnessed by a natural isomorphism

$$\mathrm{Nat}(F \circ R, G) \cong \mathrm{Nat}(F, G \circ L)$$

$$\begin{tikzcd} & \mathcal{C} \arrow[dr, "F"] \arrow[d, phantom, "\Downarrow"] & \\ \mathcal{D} \arrow[ur, "R"] \arrow[rr, bend right, "G"'] & {} & \mathcal{E} \end{tikzcd} \quad \overset{\text{bij.}}{\leftrightarrow} \quad \begin{tikzcd} \mathcal{C} \arrow[rr, bend left, "F"] \arrow[dr, "L"'] & {} \arrow[d, phantom, "\Downarrow"] & \mathcal{E} \\ & \mathcal{D} \arrow[ur, "G"'] & \end{tikzcd}$$

natural in $\mathcal{C} \xrightarrow{F} \mathcal{E}$ and $\mathcal{D} \xrightarrow{G} \mathcal{E}$.

4. *Uniqueness.* If G admits left/right adjoints F_1 and F_2, then $F_1 \cong F_2$.[4] 00W6

5. *Stability Under Composition.* If $F_1 \dashv G_1$ and $F_2 \dashv G_2$, then $(F_2 \circ F_1) \dashv (G_2 \circ G_1)$: 00W7

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F_1"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "G_1"] \arrow[r, bend left, "F_2"] \arrow[r, phantom, "\perp"] & \mathcal{E} \arrow[l, bend left, "G_2"] \end{tikzcd} \rightsquigarrow \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F_2 \circ F_1"] \arrow[r, phantom, "\perp"] & \mathcal{E} \arrow[l, bend left, "G_2 \circ G_1"] \end{tikzcd}$$

6. *Interaction With Co/Limits.* The following statements are true: 00W8

   (a) **Left Adjoints Preserve Colimits (LAPC).** If F is a left adjoint, then F preserves all colimits that exist in $\mathcal{C}$. 00W9

   (b) **Right Adjoints Preserve Limits (RAPL).** If G is a right adjoint, then G preserves all limits that exist in $\mathcal{C}$. 00WA

7. *Interaction With Faithfulness.* Let (F, G, η, ϵ) be an adjunction. The following conditions are equivalent: 00WB

   (a) The functor F is faithful.

   (b) For each $A \in \mathrm{Obj}(\mathcal{C})$, the morphism

   $$\eta_A \colon A \to G_{F_A}$$

   is a monomorphism.

   Dually, the following conditions are equivalent:

   (a) The functor G is faithful.

   (b) For each $A \in \mathrm{Obj}(\mathcal{C})$, the morphism

   $$\epsilon_A \colon F_{G_A} \to A$$

   is an epimorphism.

[4] Moreover, writing $\theta \colon F_1 \overset{\cong}{\Longrightarrow} F_2$ for this isomorphism, the diagrams

$$\begin{tikzcd} \mathrm{id}_{\mathcal{C}} \arrow[r, Rightarrow, "\eta"] \arrow[dr, Rightarrow, "\eta'"'] & G \circ F \arrow[d, Rightarrow, "\mathrm{id}_G \circ \theta"] \\ & G \circ F' \end{tikzcd} \qquad \begin{tikzcd} F \circ G \arrow[r, Rightarrow, "\epsilon"] \arrow[d, Rightarrow, "\theta \circ \mathrm{id}_G"'] & \mathrm{id}_{\mathcal{D}} \\ F' \circ G \arrow[ur, Rightarrow, "\epsilon'"'] & \end{tikzcd}$$

commute; see [**riehl:context**].

8. *Interaction With Fullness.* Let (F, G, η, ϵ) be an adjunction. The following conditions are equivalent:

   (a) The functor F is full.

   (b) For each $A \in \mathrm{Obj}(\mathcal{C})$, the morphism

   $$\eta_A \colon A \to G_{F_A}$$

   is a split epimorphism.

   Dually, the following conditions are equivalent:

   (a) The functor G is full.

   (b) For each $A \in \mathrm{Obj}(\mathcal{C})$, the morphism

   $$\epsilon_A \colon F_{G_A} \to A$$

   is a split monomorphism.

9. *Interaction With Fully Faithfulness I.* Let (F, G, η, ϵ) be an adjunction. The following conditions are equivalent:

   (a) The functor F is fully faithful.

   (b) For each $A \in \mathrm{Obj}(\mathcal{C})$, the morphism

   $$\eta_A \colon A \to G_{F_A}$$

   is an isomorphism.

   (c) The following conditions are satisfied:

      i. The natural transformation

      $$\mathrm{id}_F \circ \eta \circ \mathrm{id}_G \colon F \circ G \Longrightarrow F \circ G \circ F \circ G$$

      is a natural isomorphism.

      ii. The functor F is conservative.

      iii. The functor G is essentially surjective.

   Dually, the following conditions are equivalent:

   (a) The functor G is fully faithful.

   (b) For each $A \in \mathrm{Obj}(\mathcal{C})$, the morphism

   $$\epsilon_A \colon F_{G_A} \to A$$

   is an isomorphism.

(c) The following conditions are satisfied:
   i. The natural transformation

$$\mathrm{id}_G \circ \eta \circ \mathrm{id}_F \colon G \circ F \Longrightarrow G \circ F \circ G \circ F$$

   is a natural isomorphism.
   ii. The functor G is conservative.
   iii. The functor F is essentially surjective.

10. *Interaction With Fully Faithfulness II.* Let (F, G, η, ϵ) be an adjunction.
    (a) If $G \circ F$ is fully faithful, then so is F.
    (b) If $F \circ G$ is fully faithful, then so is G.

Proof. **??***, Adjunctions Via* Hom*-Functors*: See [**riehl:context**].
??*, Uniqueness of Adjoints*: This follows from the Yoneda lemma (**??**) and its dual (**??**).
??*, Stability Under Composition*: See [**riehl:context**].
??*: Interaction With Limits and Colimits,* **??**: [5]We prove **??** only, as **??** follows by duality (Limits and Colimits, **??** of **??**). Indeed, let $F \colon \mathcal{C} \to \mathcal{D}$ be a functor admitting a right adjoint $G \colon \mathcal{D} \to \mathcal{C}$. For each $Y \in \mathrm{Obj}(\mathcal{D})$, we have isomorphisms

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{D}}(F_{\mathrm{colim}(D)}, Y) &\cong \mathrm{Hom}_{\mathcal{D}}(\mathrm{colim}(D), G_Y) \\
&\cong \lim(\mathrm{Hom}_{\mathcal{D}}(D, G_Y)) && (\text{Limits and Colimits, ?? of ??}) \\
&\cong \lim(\mathrm{Hom}_{\mathcal{D}}(F_D, Y)) \\
&\cong \mathrm{Hom}_{\mathcal{D}}(\mathrm{colim}(F_D), Y), && (\text{Limits and Colimits, ?? of ??})
\end{aligned}$$

natural in $Y \in \mathrm{Obj}(\mathcal{D})$. The result then follows from Categories, **??**.
??*: Interaction With Limits and Colimits,* **??**: This is dual to **??**.
??*, Interaction With Faithfulness*: See [**riehl:context**].
??*, Interaction With Fullness*: See [**riehl:context**].
??*, Interaction With Fully Faithfulness I*: See [**riehl:context**] and [**loregian2020coend**].
??*, Interaction With Fully Faithfulness II*: See [**stacks-project**], [**loregian2020coend**], or [**low:homotopical-algebra**]. □

1.2 Existence Criteria for Adjoint Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Theorem 1.2.1.1. Let $F \colon \mathcal{C} \to \mathcal{D}$ and $G \colon \mathcal{D} \to \mathcal{C}$ be functors.

[5] *Reference:* See [**riehl:context**].

1. *Via Comma Categories.* The following conditions are equivalent: 00WH
   (a) The functor F has a right adjoint. 00WJ
   (b) For each $s \in \mathrm{Obj}(\mathcal{D})$, the comma category $F \downarrow s \cong \int_C [h_s^{F_-}]$ has a terminal object. 00WK

   Dually, the following conditions are equivalent:

   (a) The functor G has a left adjoint F. 00WL
   (b) For each $s \in \mathrm{Obj}(C)$, the comma category $s \downarrow G \cong \int^C [h^s_{G_-}]$ has an initial object. 00WM

   Moreover, when these conditions are satisfied, we have isomorphisms

$$
\begin{aligned}
F_A &\cong \lim_{A \to G_x} (x), \\
G_B &\cong \operatorname*{colim}_{F_x \to G_B} (x),
\end{aligned}
$$

   natural in $A \in \mathrm{Obj}(C)$ and $B \in \mathrm{Obj}(\mathcal{D})$.

2. *The General Adjoint Functor Theorem* [6]. Suppose that 00WN
   (a) The category $\mathcal{D}$ has all limits and F commutes with them.
   (b) The category C is complete and locally small.
   (c) *The Solution Set Condition.* For each $X \in \mathrm{Obj}(\mathcal{D})$, there exist
      i. A small set I;
      ii. A set $\{A_i\}_{i \in I}$ of objects of C;
      iii. A set $\{f_i : X \to G_{A_i}\}$ of morphisms of $\mathcal{D}$;

      such that, for each $i \in I$ and each morphism $f : X \to G_A$, there exists a morphism $\phi_i : A_i \to A$ of C together with a factorisation

$$
\begin{tikzcd}
X \arrow[r, "f_i"] \arrow[rr, bend right, "f"'] & G_{A_i} \arrow[r, "G_{\phi_i}"] & G_A.
\end{tikzcd}
$$

   Then F has a left adjoint.

3. *The Special Adjoint Functor Theorem.* Suppose that 00WP

[6] *Further Terminology:* Also called **Freyd's adjoint functor theorem**.

(a) The category $\mathcal{D}$ has all limits and F commutes with them.

(b) The category C is complete, locally small, and well-powered.

(c) The category C has a small cogenerating set.

Then F has a left adjoint.

4. *Freyd's Representability Theorem I.* Let $F\colon C \longrightarrow \mathsf{Sets}$ be a functor. If[7] 00WQ

   (a) The functor F commutes with limits;

   (b) The category C is complete and locally small;

   (c) *The Solution Set Condition.* There exists a set $\Phi \subset \mathrm{Obj}(C)$ such that, for each $c \in \mathrm{Obj}(C)$, there exist

   - $s \in \Phi$;
   - $y \in F_s$;
   - $f\colon s \longrightarrow c$ in $\mathrm{Hom}_{\mathsf{Sets}}(s, c)$;

   such that $F_{f(y)} = x$;

   then F is representable.

5. *Freyd's Representability Theorem II*[8]. Let $F\colon C \longrightarrow \mathsf{Sets}$ be a functor. If 00WR

   (a) The functor F commutes with limits;

   (b) There exist

   - A collection $\{x_\alpha\}_{\alpha \in I}$ of object of C;
   - For each $\alpha \in I$, an element f_α of F_{x_α}

   such that for each $y \in \mathrm{Obj}(C)$ and each $g \in F_y$, there exists some $\alpha \in I$ and some morphism $\phi\colon x_i \longrightarrow y$ such that $F_\phi(f_\alpha) = g$;

   then F is representable.

6. *Co/Totality.* Suppose that 00WS

   (a) The category C is locally small and cototal and $\mathcal{D}$ is locally small.

Proof. **??**, *Via Comma Categories*: We claim that **????** are indeed equivalent:[9]

[7] A nice application of this theorem is given in [**MSE276630**], where it is used to abstractly show that Cats is cocomplete, avoiding the explicit construction of coequalisers in Cats given in **??**.

[8] This is the statement of Freyd's representability theorem as found in [**stacks-project**].

[9] *Reference:* [**riehl:context**].

- **??** $\Longrightarrow$ **??**: Let F be a left adjoint of G. Then

$$\begin{aligned} s \downarrow G &\cong \int^{C} [h^{s}_{G_{-}}] \\ &\cong \int^{C} [h^{F_s}_{-}], \end{aligned}$$

  where $h^{s}_{G_{-}}$ is corepresentable by F_s. By Fibred Categories, **??** of **??**, it follows that the component $\eta_s \colon s \to G_{F_s}$ of the unit of the adjunction $F \dashv G$ at s is an initial object of $s \downarrow G$.

- **??** $\Longrightarrow$ **??**: For each $s \in \mathrm{Obj}(\mathcal{D})$, write $\eta_s \colon s \to G_{F_s}$ for an initial object of $s \downarrow G$. This gives us a map of sets

$$\begin{aligned} F \colon \mathrm{Obj}(C) &\longrightarrow \mathrm{Obj}(\mathcal{D}) \\ s &\longmapsto F_s. \end{aligned}$$

  We now extend this map to a functor: given a morphism $f \colon s \to s'$ of C, we define $F_f \colon F_s \to F_{s'}$ to be the unique morphism making the diagram

$$\begin{tikzcd} s \arrow[r, "f"] \arrow[d, "\eta_s"'] & s' \arrow[d, "\eta_{s'}"] \\ G_{F_s} \arrow[r, dashed, "G_{F_f}"'] & G_{F_{s'}} \end{tikzcd}$$

  commute (which exists by the initiality of η_s). By the uniqueness of these morphisms, it follows that the assignment $s \mapsto F_s$ is indeed functorial. Moreover, we also obtain a natural transformation $\eta \colon \mathrm{id}_C \Longrightarrow G \circ F$. We now define a natural transformation

$$\phi \colon \mathrm{Hom}_{\mathcal{D}}(F_{-}, b) \Longrightarrow \mathrm{Hom}_{C}(-, G_b)$$

  consisting of the collection

$$\left\{ \phi_{s,b} \colon \mathrm{Hom}_{\mathcal{D}}(F_s, b) \Longrightarrow \mathrm{Hom}_{C}(s, G_b) \right\}_{s \in \mathrm{Obj}(C)},$$

  where $\phi_{s,b}$ is the map sending a morphism $g \colon F_s \to b$ to the composition

$$s \xrightarrow{\eta_s} G_{F_s} \xrightarrow{G_g} G_b.$$

  By the existence and uniqueness of morphisms from η_s to any other object $s \to G_b$ in $s \downarrow G$, it follows that the maps $\phi_{s,b}$ are bijective, showing F to be a left adjoint of G.

??, *The General Adjoint Functor Theorem*: See [**riehl:context**].
??, *The Special Adjoint Functor Theorem*: See [**riehl:context**].
??, *Freyd's Representability Theorem I*: See [**riehl:context**].
??, *Freyd's Representability Theorem II*: See [**stacks-project**].
??, *Co/Totality*: Omitted. □

1.3 Adjoint Strings 00WT

To avoid clutter, in this section we will abbreviate long compositions of functors. For instance, we write $f_1 \circ f_2 \circ f_3 \circ f_4$ as $f_1 f_2 f_3 f_4$. Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 1.3.1.1. 00WU An **adjoint string of length** n[10] is an n-tuple $(f_1, \ldots, f_n)$ of functors between $\mathcal{C}$ and $\mathcal{D}$ such that

$$f_n \dashv f_{n+1}$$

for each $n \in \{1, \ldots, n-1\}$.

Proposition 1.3.1.2. 00WV Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

1. *Adjoint Triples as Adjunctions Between Adjunctions.* 00WW An adjoint triple is equivalently an adjunction $(F \dashv G) \dashv (G \dashv H)$ between adjunctions. FIXME [**nLab:adjoint-triple**].[11]

2. *Adjunctions Induced by an Adjoint Triple.* 00WX A triple adjunction (f_1, f_2, f_3) gives rise to two more adjunctions

$$(f_2 f_1 \dashv f_2 f_3)\colon\ \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "f_2 f_1"] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, bend left, "f_2 f_3"] \end{tikzcd}$$

and

$$(f_1 f_2 \dashv f_3 f_2)\colon\ \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "f_1 f_2"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "f_3 f_2"] \end{tikzcd}$$

where $f_2 f_1$ and $f_2 f_3$ are monads in $\mathcal{C}$ and $f_1 f_2$ and $f_3 f_2$ are comonads in $\mathcal{D}$.

[10] *Further Terminology:* Also called an **adjoint n-tuple**.

[11] [**nLab:adjoint-triple**] suggests writing

$$\begin{matrix} f_1 & \dashv & f_2 \\ \perp & & \perp \\ f_2 & \dashv & f_3 \end{matrix}$$

to denote the adjunctions $(f_1 \dashv f_2 \dashv f_3)$ and $(f_1 f_2) \dashv (f_2 f_3)$ simultaneously; the first horizontally and the latter vertically.

Proof. **??**, *Adjoint Triples as Adjunctions Between Adjunctions*: Omitted.
??, *Adjunctions Induced by an Adjoint Triple*: Omitted. □

Proposition 1.3.1.3. Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

1. *Adjunctions Induced by a Quadruple Adjunction.* An adjoint quadruple $(f_1 \dashv f_2 \dashv f_3 \dashv f_4)$ gives rise to two adjoint triples

$$(f_2f_1 \dashv f_2f_3 \dashv f_4f_3)\colon \begin{tikzcd} \mathcal{C} \arrow[r, bend left=50, "f_2f_1"] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, phantom, shift right=3, "\perp"] \arrow[r, bend right=50, "f_4f_3"'] & \mathcal{C} \arrow[l, "f_2f_3" description] \end{tikzcd}$$

and

$$(f_1f_2 \dashv f_3f_2 \dashv f_3f_4)\colon \begin{tikzcd} \mathcal{D} \arrow[r, bend left=50, "f_1f_2"] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, phantom, shift right=3, "\perp"] \arrow[r, bend right=50, "f_3f_4"'] & \mathcal{D} \arrow[l, "f_3f_2" description] \end{tikzcd}$$

and six adjunctions

$$(f_1f_2f_3 \dashv f_4f_3f_2)\colon \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "f_1f_2f_3"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "f_4f_3f_2"] \end{tikzcd} \qquad (f_3f_2f_1 \dashv f_2f_3f_4)\colon \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "f_3f_2f_1"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "f_2f_3f_4"] \end{tikzcd}$$

$$(f_2f_3f_2f_1 \dashv f_2f_3f_4f_3)\colon \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "f_2f_3f_2f_1"] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, bend left, "f_2f_3f_4f_3"] \end{tikzcd} \qquad (f_3f_2f_1f_2 \dashv f_3f_2f_3f_4)\colon \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "f_3f_2f_1f_2"] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, bend left, "f_3f_2f_3f_4"] \end{tikzcd}$$

$$(f_2f_1f_2f_3 \dashv f_4f_3f_2f_3)\colon \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "f_2f_1f_2f_3"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "f_4f_3f_2f_3"] \end{tikzcd} \qquad (f_1f_2f_3f_2 \dashv f_3f_4f_3f_2)\colon \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "f_1f_2f_3f_2"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "f_3f_4f_3f_2"] \end{tikzcd}$$

where f_2f_1, f_2f_3, f_4f_3, $f_2f_3f_2f_1$, $f_2f_3f_4f_3$, $f_3f_2f_1f_2$, and $f_3f_2f_3f_4$ are monads in $\mathcal{C}$ and f_1f_2, f_3f_2, f_3f_4, $f_2f_1f_2f_3$, $f_4f_3f_2f_3$, $f_1f_2f_3f_2$, and $f_3f_4f_3f_2$ are comonads in $\mathcal{D}$.

Proof. **??**, *Adjunctions Induced by a Quadruple Adjunction*: Omitted. □

Proposition 1.3.1.4. Let $(f_1 \dashv \cdots \dashv f_n)\colon CDODO\mathcal{D}$ 00X0 be an adjoint string.

1. For each $k \in \mathbb{N}$ with $1 \leq k \leq n-2$, we have 2 induced adjoint strings 00X1

   $$f_1 f_2 \cdots f_{n-k} f_{n-k+1} \dashv f_{n-k+2} f_{n-k+1} \cdots f_3 f_2 \dashv \cdots \dashv f_{k-1} f_k \cdots f_{n-2} f_{n-1} \dashv f_n f_{n-1} \cdots f_{k+1} f_k$$
   $$f_{n-k+1} f_{n-k} \cdots f_2 f_1 \dashv f_2 f_3 \cdots f_{n-k+1} f_{n-k+2} \dashv \cdots \dashv f_{n-1} f_{n-2} \cdots f_k f_{k-1} \dashv f_k f_{k+1} \cdots f_{n-1} f_n$$

   of length $n-k$.

2. Inductively applying **??** to the induced adjoint strings, we get (including the 2 adjoint strings of **??**) $2 \cdot 3^{n-k-1}$ adjoint strings of length k [12], for a grand total of 00X2

   $$\sum_{k=2}^{n-1} 2(k-1) \cdot 3^{n-k-1} = \frac{1}{6}(3^n + 3) - n$$

   adjunctions.[13]

3. In particular: 00X3

   (a) An adjoint triple induces 2 adjoint pairs.

   (b) An adjoint quadruple induces

   - 2 adjoint triples,
   - 6 adjoint pairs,

   for a grand total of 10 adjunctions.

   (c) An adjoint quintuple induces

   - 2 adjoint quadruples,
   - 6 adjoint triples,
   - 18 adjoint pairs,

   for a grand total of 36 adjunctions.

   (d) An adjoint sextuple induces

   - 2 adjoint quintuples,
   - 6 adjoint quadruples,

[12] These need not be unique.

[13] E.g. we have 4 adjoint strings of length $n-2$, such as

$$f_2 f_3 f_2 f_1 \dashv f_2 f_3 f_4 f_3 \dashv \cdots \dashv f_k f_{k+1} f_k f_{k-1} \dashv f_k f_{k+1} f_{k+2} f_{k+1} \dashv \cdots \dashv f_{n-2} f_{n-1} f_{n-2} f_{n-1} \dashv f_{n-2} f_{n-1} f_n f_{n-1}.$$

  - 18 adjoint triples,
  - 54 adjoint pairs,

  for a grand total of 116 adjunctions.

(e) An adjoint septuple induces

  - 2 adjoint sextuples,
  - 6 adjoint quintuples,
  - 18 adjoint quadruples,
  - 54 adjoint triples,
  - 162 adjoint pairs,

  for a grand total of 358 adjunctions.

Proof. Omitted. □

1.4 Reflective Subcategories 00X4

Let C be a category.

Definition 1.4.1.1. A subcategory C_0 of C is 00X5 **reflective** if the inclusion functor $i\colon C_0 \hookrightarrow C$ of C_0 into C admits a left adjoint $L\colon C \to C_0$.[14]

Example 1.4.1.2. Here are some examples 00X6 of reflective subcategories

1. $\mathsf{CHaus} \hookrightarrow \mathsf{Top}$ *([**riehl:context**]).* The category CHaus is a reflective subcategory of Top, as witnessed by the adjunction

$$(\beta \dashv \iota)\colon\quad \mathsf{Top} \underset{\iota}{\overset{\beta}{\underset{\hookleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathsf{CHaus},$$

   of Topological Spaces, **??** of **??**.

2. $\mathsf{CMon} \hookrightarrow \mathsf{Mon}$. The category CMon is a reflective subcategory of Ab, as witnessed by the adjunction

$$\Big((-)^{\mathrm{ab}} \dashv \iota\Big)\colon\quad \mathsf{Mon} \underset{\iota}{\overset{(-)^{\mathrm{ab}}}{\underset{\hookleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathsf{CMon}$$

   of Monoids, **??** of **??**.

3. $\mathsf{Ab} \hookrightarrow \mathsf{Grp}$ *([**riehl:context**]).* The category Ab is a reflective subcategory of

[14] *Further Terminology:* The functor L is called the **reflector** or **localisation** of the adjunction $L \dashv i$.

Grp, as witnessed by the adjunction

$$\Big((-)^{\mathrm{ab}} \dashv \iota\Big)\colon \quad \begin{tikzcd} \mathsf{Grp} \arrow[r, shift left, "(-)^{\mathrm{ab}}"] \arrow[r, phantom, "\perp"] & \mathsf{Ab} \arrow[l, shift left, hook, "\iota"] \end{tikzcd}$$

of Groups, **??** of **??**.

4. $\mathsf{Ab}^{\mathsf{tf}} \hookrightarrow \mathsf{Ab}$ *([**riehl:context**]).* The full subcategory $\mathsf{Ab}^{\mathsf{tf}}$ of Ab spanned by the torsion-free abelian groups is reflective in Ab. This is witnessed by the adjunction

$$\Big((-)^{\mathrm{tf}} \dashv \iota\Big)\colon \quad \begin{tikzcd} \mathsf{Ab} \arrow[r, shift left, "(-)^{\mathrm{tf}}"] \arrow[r, phantom, "\perp"] & \mathsf{Ab}^{\mathsf{tf}}, \arrow[l, shift left, hook, "\iota"] \end{tikzcd}$$

   where $(-)^{\mathsf{tf}}\colon \mathsf{Ab} \to \mathsf{Ab}^{\mathsf{tf}}$ is the functor defined on objects by sending an abelian group A to the quotient $A/\mathrm{Tors}(A)$, where $\mathrm{Tors}(A)$ is the torsion subgroup of A.

5. $\mathsf{Mod}_S \hookrightarrow \mathsf{Mod}_R$ *([**riehl:context**]).* Let $\phi\colon R \to S$ be a morphism of rings. Then ϕ^* is full iff ϕ is an epimorphism, in which case the adjunction

$$(S \otimes_R (-) \dashv \phi^*)\colon \quad \begin{tikzcd} \mathsf{Mod}_S \arrow[r, bend left, "S\otimes_R(-)"] \arrow[r, phantom, "\perp"] & \mathsf{Mod}_R \arrow[l, bend left, "\phi^*"] \end{tikzcd}$$

   witnesses Mod_S as a reflective subcategory of Mod_R.

6. $\mathsf{Shv}(C) \hookrightarrow \mathsf{PSh}(C)$ *([**riehl:context**]).* The category $\mathsf{Shv}(C)$ of sheaves on a site C is a reflective subcategory of $\mathsf{PSh}(C)$, as witnessed by the adjunction

$$\big((-)^{\#} \dashv \iota\big)\colon \quad \begin{tikzcd} \mathsf{PSh}(C) \arrow[r, shift left, "(-)^{\#}"] \arrow[r, phantom, "\perp"] & \mathsf{Shv}(C), \arrow[l, shift left, hook, "\iota"] \end{tikzcd}$$

   of Sites, **??**.

7. $\mathsf{Cats} \hookrightarrow \mathsf{sSets}$ *([**riehl:context**]).* The category Cats is a reflective subcategory of sSets, as witnessed by the adjunction

$$(\mathsf{Ho} \dashv \mathbf{N}_\bullet)\colon \quad \begin{tikzcd} \mathsf{sSets} \arrow[r, shift left, "\mathsf{Ho}"] \arrow[r, phantom, "\perp"] & \mathsf{Cats} \arrow[l, shift left, hook, "\mathbf{N}_\bullet"] \end{tikzcd}$$

   of Quasicategories, **??** of **??**.

Proposition 1.4.1.3. Let C_0 be a reflective subcategory of C. 00X7

1. *Characterisations.* Let 00X8

$$(L \dashv \iota)\colon\quad \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "\iota"] \end{tikzcd}$$

be an adjunction. The following conditions are equivalent:

   (a) The functor ι is fully faithful.

   (b) The counit $\epsilon\colon L \circ \iota \Longrightarrow \mathrm{id}_{\mathcal{D}}$ is a natural isomorphism.

   (c) The following conditions are satisfied:

      i. The monad $(\iota \circ L, \mathrm{id}_\iota \circ \epsilon \circ \mathrm{id}_L, \eta)$ associated to the adjunction $L \dashv \iota$ is idempotent.

      ii. The functor ι is conservative.

      iii. The functor L is essentially surjective.

   (d) The functor L is the Gabriel–Zisman localisation of $\mathcal{C}$ with respect to the class S given by

$$S \overset{\mathrm{def}}{=} \{f \in \mathrm{Mor}(\mathcal{C}) \mid L(f) \text{ is an isomorphism in } \mathcal{D}\}.$$

   (e) The functor L is dense.

2. *Interaction With Limits.* The inclusion $\mathcal{C}_0 \hookrightarrow \mathcal{C}$ creates all limits which exist in $\mathcal{C}$. 00X9

3. *Interaction With Colimits.* The category $\mathcal{C}_0$ admits all colimits that exist in $\mathcal{C}$: 00XA given a diagram $D\colon \mathcal{I} \to \mathcal{C}_0$ in $\mathcal{C}_0$, if $\mathrm{colim}(i \circ D)$ exists in $\mathcal{C}$, then $\mathrm{colim}(D)$ exists in $\mathcal{C}_0$ and we have

$$\mathrm{colim}(D) \cong L(\mathrm{colim}(i \circ D)).$$

Proof. **??**, *Characterisations*: See [**calculus-of-fractions-and-homotopy-theory**] and [**properties-of-dense-and-relative-adjoint-functors**].
??, *Interaction With Limits*: See [**riehl:context**].
??, *Interaction With Colimits*: See [**riehl:context**]. □

1.5 Coreflective Subcategories 00XB

Let $\mathcal{C}$ be a category.

Definition 1.5.1.1. A subcategory $\mathcal{C}_0$ of $\mathcal{C}$ is **coreflective** if the inclusion functor 00XC $i\colon \mathcal{C}_0 \hookrightarrow \mathcal{C}$ of $\mathcal{C}_0$ into $\mathcal{C}$ admits a right adjoint $R\colon \mathcal{C} \to \mathcal{C}_0$.[15]

[15] *Further Terminology:* The functor L is called the **coreflector** or **colocalisation** of the adjunction $i \dashv R$.

2 Presheaves and the Yoneda Lemma 00XD

2.1 Presheaves 00XE

Let C be a category.

Definition 2.1.1.1. A **presheaf on** C is a functor 00XF $\mathcal{F}\colon C^{\mathsf{op}} \to \mathsf{Sets}$.

Definition 2.1.1.2. The **category of presheaves on** 00XG C is the category $\mathsf{PSh}(C)$ defined by

$$\mathsf{PSh}(C) \stackrel{\text{def}}{=} \mathsf{Fun}(C^{\mathsf{op}}, \mathsf{Sets}).$$

Remark 2.1.1.3. In detail, the **category of presheaves on** 00XH C is the category $\mathsf{PSh}(C)$ where

- *Objects.* The objects of $\mathsf{PSh}(C)$ are presheaves on C;
- *Morphisms.* A morphism of $\mathsf{PSh}(C)$ from $\mathcal{F}$ to $\mathcal{G}$ is a natural transformation $\alpha\colon \mathcal{F} \Longrightarrow \mathcal{G}$;
- *Identities.* For each $\mathcal{F} \in \mathrm{Obj}(\mathsf{PSh}(C))$, the unit map

  $$\mathbb{1}^{\mathsf{PSh}(C)}_{\mathcal{F}}\colon \mathrm{pt} \to \mathrm{Nat}(\mathcal{F}, \mathcal{F})$$

  of $\mathsf{PSh}(C)$ at $\mathcal{F}$ is defined by

  $$\mathrm{id}^{\mathsf{PSh}(C)}_{\mathcal{F}} \stackrel{\text{def}}{=} \mathrm{id}_{\mathcal{F}};$$

- *Composition.* For each $\mathcal{F}, \mathcal{G}, \mathcal{H} \in \mathrm{Obj}(\mathsf{PSh}(C))$, the composition map

  $$\circ^{\mathsf{PSh}(C)}_{\mathcal{F},\mathcal{G},\mathcal{H}}\colon \mathrm{Nat}(\mathcal{G}, \mathcal{H}) \times \mathrm{Nat}(\mathcal{F}, \mathcal{G}) \to \mathrm{Nat}(\mathcal{F}, \mathcal{H})$$

  of $\mathsf{PSh}(C)$ at $(\mathcal{F}, \mathcal{G}, \mathcal{H})$ is defined by

  $$\beta \circ^{\mathsf{PSh}(C)}_{\mathcal{F},\mathcal{G},\mathcal{H}} \alpha \stackrel{\text{def}}{=} \beta \circ \alpha.$$

2.2 Representable Presheaves 00XJ

Let C be a category, let $U, V \in \mathrm{Obj}(C)$, and let $f\colon U \to V$ be a morphism of C.

Definition 2.2.1.1. The **representable presheaf associated to** 00XK U is the presheaf $h_U\colon C^{\mathsf{op}} \to \mathsf{Sets}$ on C where

- *Action on Objects.* For each $A \in \mathrm{Obj}(C)$, we have

  $$h_U(A) \stackrel{\text{def}}{=} \mathrm{Hom}_C(A, U);$$

- *Action on Morphisms.* For each morphism $f\colon A \to B$ of $\mathcal{C}$, the image

$$h_U(f)\colon \underbrace{h_U(B)}_{\stackrel{\text{def}}{=}\mathrm{Hom}_{\mathcal{C}}(B,U)} \to \underbrace{h_U(A)}_{\stackrel{\text{def}}{=}\mathrm{Hom}_{\mathcal{C}}(A,U)}$$

  of f by h_U is defined by

$$h_U(f) \stackrel{\text{def}}{=} f^*.$$

Definition 2.2.1.2. A presheaf $\mathcal{F}\colon \mathcal{C}^{\mathsf{op}} \to \mathsf{Sets}$ is **representable** if $\mathcal{F} \cong h_U$ for some $U \in \mathrm{Obj}(\mathcal{C})$.[16]

Definition 2.2.1.3. The **representable natural transformation associated to** f is the natural transformation $h_f\colon h_U \Longrightarrow h_V$ consisting of the collection

$$\left\{ h_{f|A}\colon \underbrace{h_U(A)}_{\stackrel{\text{def}}{=}\mathrm{Hom}_{\mathcal{C}}(A,U)} \to \underbrace{h_V(A)}_{\stackrel{\text{def}}{=}\mathrm{Hom}_{\mathcal{C}}(A,V)} \right\}_{A\in\mathrm{Obj}(\mathcal{C})}$$

where

$$h_{f|A} \stackrel{\text{def}}{=} f_*.$$

Theorem 2.2.1.4. Let $\mathcal{F}\colon \mathcal{C}^{\mathsf{op}} \to \mathsf{Sets}$ be a presheaf on $\mathcal{C}$. We have a bijection

$$\mathrm{Nat}(h_A, \mathcal{F}) \cong \mathcal{F}_A,$$

natural in $A \in \mathrm{Obj}(\mathcal{C})$, determining a natural isomorphism of functors

$$\mathrm{Nat}(h_{(-)}, \mathcal{F}) \cong \mathcal{F}.$$

Proof. The Natural Transformation $\mathrm{ev}_{(-)}\colon \mathrm{Nat}(h_{(-)}, \mathcal{F}) \Longrightarrow \mathcal{F}$: Let $\mathrm{ev}_{(-)}\colon \mathrm{Nat}(h_{(-)}, \mathcal{F}) \Longrightarrow \mathcal{F}$ be the natural transformation consisting of the collection

$$\{\mathrm{ev}_A\colon \mathrm{Nat}(h_A, \mathcal{F}) \to \mathcal{F}(A)\}_{A\in\mathrm{Obj}(\mathcal{C})}$$

with

$$\mathrm{ev}_A(\alpha) = \alpha_A(\mathrm{id}_A)$$

for each $\alpha\colon h_A \Longrightarrow \mathcal{F}$ in $\mathrm{Nat}(h_A, \mathcal{F})$.
The Natural Transformation $\xi_{(-)}\colon \mathcal{F} \Longrightarrow \mathrm{Nat}(h_{(-)}, \mathcal{F})$: Let $\xi_{(-)}\colon \mathcal{F} \Longrightarrow \mathrm{Nat}(h_{(-)}, \mathcal{F})$ be the natural transformation consisting of the collection

$$\{\xi_A\colon \mathcal{F}(A) \to \mathrm{Nat}(h_A, \mathcal{F})\}_{A\in\mathrm{Obj}(\mathcal{C})}$$

[16] In such a case, we call U a **representing object** for $\mathcal{F}$.

where $\xi_A \colon \mathcal{F}(A) \to \mathrm{Nat}(h_A, \mathcal{F})$ is the map sending an element f of $\mathcal{F}(X)$ to the natural transformation

$$\xi_{A,f} \colon h_A \Longrightarrow \mathcal{F}$$

consisting of the collection

$$\left\{(\xi_{A,f})_U \colon h_A(U) \to \mathcal{F}(U)\right\}_{A \in \mathrm{Obj}(C)}$$

where $(\xi_{A,f})_U \colon h_A(U) \to \mathcal{F}(U)$ is the morphism given by

$$\begin{aligned} (\xi_{A,f})_U \colon h_A(U) &\longrightarrow \mathcal{F}(U) \\ (h \colon U \to A) &\longmapsto \mathcal{F}(h)(f) \end{aligned}$$

for each $f \colon U \to A$ in $h_A(U)$.

$ev_{(-)} \circ \xi_{(-)} = id_{\mathcal{F}}$: Let $f \in \mathcal{F}(X)$. We have

$$\begin{aligned} (\xi_{A,f})_U(\mathrm{id}_U) &= \mathcal{F}(\mathrm{id}_U)(f), \\ &= \mathrm{id}_{\mathcal{F}(U)}(f) \\ &= f. \end{aligned}$$

$\xi_{(-)} \circ ev_{(-)} = id_{Nat(h_{(-)},\mathcal{F})}$: Let $\alpha \colon h_A \Longrightarrow \mathcal{F} \in \mathrm{Nat}(h_A, \mathcal{F})$ and consider the diagram

$$\begin{tikzcd} \mathrm{Hom}_C(A,A) \arrow[r, "h_f"] \arrow[d, "\xi_A"'] & \mathrm{Hom}_C(A,X) \arrow[d, "\xi_X"] \\ \mathcal{F}(A) \arrow[r, "\mathcal{F}(f)"'] & \mathcal{F}(X) \end{tikzcd}$$

defined on elements by

$$\begin{tikzcd} \mathrm{id}_A \arrow[r, mapsto] \arrow[d, mapsto] & f \arrow[d, mapsto] \\ u \arrow[r, mapsto] & \mathcal{F}(f)(u) = \xi_X(f). \end{tikzcd}$$

Then it is clear that the natural transformation ξ is determined by $\xi_A(\mathrm{id}_A) = u$, since we must have

$$\xi_X(f) = \mathcal{F}(f)(u)$$

for each $X \in \mathrm{Obj}(C)$ and each morphism $f \colon A \to X$ of C. $\square$

2.3 The Yoneda Embedding 00XP

Definition 2.3.1.1. 00XQ The **covariant Yoneda embedding of** $\mathcal{C}$[17] is the functor[18]

$$\text{よ}_{\mathcal{C}} : \mathcal{C} \hookrightarrow \mathsf{PSh}(\mathcal{C})$$

where

- *Action on Objects.* For each $U \in \mathrm{Obj}(\mathcal{C})$, we have

$$\text{よ}(U) \stackrel{\text{def}}{=} h_U;$$

- *Action on Morphisms.* For each morphism $f : U \to V$ of $\mathcal{C}$, the image

$$\text{よ}(f) : \text{よ}(U) \to \text{よ}(V)$$

  of f by よ is defined by

$$\text{よ}(f) \stackrel{\text{def}}{=} h_f.$$

Proposition 2.3.1.2. Let $\mathcal{C}$ be a category. 00XR

1. *Fully Faithfulness.* 00XS The Yoneda embedding is fully faithful.[19]

2. *Preservation and Reflection of Isomorphisms.* 00XT Let $A, B \in \mathrm{Obj}(\mathcal{C})$. The following conditions are equivalent:

   (a) We have $A \cong B$.

   (b) We have $h_A \cong h_B$.

   (c) We have $h^A \cong h^B$.

3. *Uniqueness of Representing Objects Up to Isomorphism.* 00XU Let $\mathcal{F} : \mathcal{C}^{\mathsf{op}} \to \mathsf{Sets}$ be a presheaf. If there exist objects A and B of $\mathcal{C}$ such that we have

$$h_A \cong \mathcal{F},$$
$$h_B \cong \mathcal{F},$$

   then $A \cong B$.

4. *As a Free Cocompletion: The Universal Property.* 00XV The pair $(\mathsf{PSh}(\mathcal{C}), \text{よ})$ consisting of

[17] *Further Terminology:* Also called simply the **Yoneda embedding**.
[18] *Further Notation:* Also written $h_{(-)}$, or simply よ.
[19] In other words, the Yoneda embedding is indeed an embedding.

- The category $\mathsf{PSh}(\mathcal{C})$ of presheaves on $\mathcal{C}$;
- The Yoneda embedding $よ\colon \mathcal{C} \hookrightarrow \mathsf{PSh}(\mathcal{C})$ of $\mathcal{C}$ into $\mathsf{PSh}(\mathcal{C})$;

satisfies the following universal property:

(**UP**) Given another pair $(\mathcal{A}, F)$ consisting of
  - A cocomplete category $\mathcal{A}$;
  - A cocontinuous functor $F\colon \mathcal{C} \to \mathcal{A}$;

  there exists a cocontinuous functor $\mathsf{PSh}(\mathcal{C}) \xrightarrow{\exists!} \mathcal{A}$, unique up to natural isomorphism, making the diagram

$$\begin{tikzcd} & \mathsf{PSh}(\mathcal{C}) \arrow[d, dashed, "\exists!"] \\ \mathcal{C} \arrow[ur, "よ"] \arrow[r, "F"'] & \mathcal{A} \arrow[u, phantom, "\cong"] \end{tikzcd}$$

  commute, again up to natural isomorphism.

5. *As a Free Cocompletion: 2-Adjointness.* We have a 2-adjunction 00XW

$$(\mathsf{PSh} \dashv \iota)\colon \quad \begin{tikzcd} \mathsf{Cats} \arrow[r, bend left, "\mathsf{PSh}"] \arrow[r, phantom, "\perp_2"] & \mathsf{Cats}^{\mathsf{cocomp.}} \arrow[l, bend left, "\iota"] \end{tikzcd},$$

witnessed by an adjoint equivalence of categories[20]

$$(\mathrm{Lan}_{よ} \dashv よ^*)\colon \quad \begin{tikzcd} \mathsf{CoContFun}(\mathsf{PSh}(\mathcal{C}), \mathcal{D}) \arrow[r, bend left, "\mathrm{Lan}_{よ}"] \arrow[r, phantom, "\perp"] & \mathsf{Fun}(\mathcal{C}, \mathcal{D}) \arrow[l, bend left, "よ^*"] \end{tikzcd},$$

natural in $\mathcal{C} \in \mathrm{Obj}(\mathsf{Cats})$ and $\mathcal{D} \in \mathrm{Obj}(\mathsf{Cats}^{\mathsf{cocomp.}})$, where

- We have a functor

$$よ^*_{\mathcal{C}}\colon \mathsf{CoContFun}(\mathsf{PSh}(\mathcal{C}), \mathcal{D}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{D})$$

  defined by

$$よ^*_{\mathcal{C}}(F) \stackrel{\text{def}}{=} F \circ よ_{\mathcal{C}},$$

  i.e. by sending a functor $F\colon \mathsf{PSh}(\mathcal{C}) \to \mathcal{D}$ to the composition

$$\mathcal{C} \overset{よ_{\mathcal{C}}}{\hookrightarrow} \mathsf{PSh}(\mathcal{C}) \xrightarrow{F} \mathcal{D};$$

[20] In this sense, $\mathsf{PSh}(\mathcal{C})$ is the free cocompletion of $\mathcal{C}$ (although the term “cocompletion” is slightly

- We have a natural map

  $$\mathrm{Lan}_{よ_C} \colon \mathsf{Fun}(\mathcal{C}, \mathcal{D}) \to \mathsf{CoContFun}(\mathsf{PSh}(\mathcal{C}), \mathcal{D})$$

  computed on objects by

  $$\begin{aligned}\left[\mathrm{Lan}_{よ_C}(F)\right](\mathcal{F}) &\cong \int^{A\in\mathcal{D}} \mathrm{Nat}(h_A, \mathcal{F}) \odot F_A \\ &\cong \int^{A\in\mathcal{D}} \mathcal{F}^A \odot F_A\end{aligned}$$

  for each $\mathcal{F} \in \mathrm{Obj}(\mathsf{PSh}(\mathcal{C}))$.

Proof. **??**, *Fully Faithfulness*: Let $A, B \in \mathrm{Obj}(\mathcal{C})$. Applying **??** to the functor h_B (i.e. in the case $\mathcal{F} = h_B$), we have

$$\mathrm{Hom}_{\mathcal{C}}(A, B) \cong \mathrm{Nat}(h_A, h_B).$$

Thus よ is fully faithful.
??, *Preservation and Reflection of Isomorphisms*: This follows from **??** and **??**.
??, *Uniqueness of Representing Objects Up to Isomorphism*: By composing the isomorphisms $h_A \cong \mathcal{F} \cong h_B$, we get a natural isomorphism $\alpha \colon h_A \overset{\cong}{\Longrightarrow} h_B$. By **??**, we have $A \cong B$.
??, *As a Free Cocompletion: The Universal Property*: This is a rephrasing of **??**.
??: *As a Free Cocompletion: 2-Adjointness*: See [**nLab:free-cocompletion**]. □

2.4 Universal Objects

Definition 2.4.1.1. The **universal object** associated to a representable functor $h_U \colon \mathcal{C} \to \mathcal{D}$ is the element $u \in h_U(U)$ satisfying the following universal property:[21]

(**UP**) For each $B \in \mathrm{Obj}(\mathcal{C})$, the map

$$\begin{aligned} h_U(B) &\longrightarrow h_U(U) \\ (f \colon B \to A) &\longmapsto h_U(f)(u) \end{aligned}$$

is a bijection.

misleading, as $\mathsf{PSh}(\mathsf{PSh}(\mathcal{C})) \overset{\text{eq.}}{\ncong} \mathsf{PSh}(\mathcal{C}))$.

[21]This is the element of $h_U(U)$ corresponding to the identity natural transformation $\mathrm{id}_{h_U} \colon h_U \Longrightarrow h_U$ under the isomorphism $h_U(U) \cong \mathrm{Hom}_{\mathsf{PSh}(\mathcal{C})}(h_U, h_U)$.

Remark 2.4.1.2. 00XZ In other words, a universal object u associated to a representable functor $h_U\colon \mathcal{C} \to \mathcal{D}$ represented by U is universal in the sense that every element of $h_U(A)$ is equal to the image of u via $h_U(f)$ for a unique morphism $f\colon A \to U$ of $\mathcal{C}$.

Example 2.4.1.3. 00Y0 Let G be a group and consider the functor $\mathrm{Bun}^{\mathrm{num}}_G(-)\colon \mathsf{Ho}(\mathsf{Top})^{\mathsf{op}} \to \mathsf{Sets}$ sending $[X] \in \mathsf{Ho}(\mathsf{Top})^{\mathsf{op}}$ to the set of numerable principal G-bundles on X. Then the universal numerable principal G-bundle $\gamma\colon \mathrm{EG} \to \mathrm{BG}$ is a universal object for $\mathrm{Bun}^{\mathrm{num}}_G(-)$.
Furthermore, the map sending γ to a principal G-bundle $P \to X$ on X is the pullback

$$f^*\colon \mathrm{Bun}^{\mathrm{num}}_G(\mathrm{BG}) \to \mathrm{Bun}^{\mathrm{num}}_G(X)$$

of P along the homotopy class $[f]\colon X \to \mathrm{BG}$ classifying P of maps $X \to \mathrm{BG}$. See Algebraic Topology, **??** for more details.

3 Copresheaves and the Contravariant Yoneda Lemma 00Y1

3.1 Copresheaves 00Y2

Let $\mathcal{C}$ be a category.

Definition 3.1.1.1. 00Y3 A **copresheaf on** $\mathcal{C}$ is a functor $F\colon \mathcal{C} \to \mathsf{Sets}$.

Definition 3.1.1.2. 00Y4 The **category of copresheaves on** $\mathcal{C}$ is the category $\mathsf{CoPSh}(\mathcal{C})$ defined by

$$\mathsf{CoPSh}(\mathcal{C}) \overset{\mathrm{def}}{=} \mathsf{Fun}(\mathcal{C}, \mathsf{Sets}).$$

Remark 3.1.1.3. 00Y5 In detail, the **category of copresheaves on** $\mathcal{C}$ is the category $\mathsf{CoPSh}(\mathcal{C})$ where

- *Objects.* The objects of $\mathsf{CoPSh}(\mathcal{C})$ are presheaves on $\mathcal{C}$;
- *Morphisms.* A morphism of $\mathsf{CoPSh}(\mathcal{C})$ from F to G is a natural transformation $\alpha\colon F \Longrightarrow G$;
- *Identities.* For each $F \in \mathrm{Obj}(\mathsf{CoPSh}(\mathcal{C}))$, the unit map

  $$\mathbb{1}^{\mathsf{CoPSh}(\mathcal{C})}_F\colon \mathrm{pt} \to \mathrm{Nat}(F, F)$$

  of $\mathsf{CoPSh}(\mathcal{C})$ at F is defined by

  $$\mathrm{id}^{\mathsf{CoPSh}(\mathcal{C})}_F \overset{\mathrm{def}}{=} \mathrm{id}_F;$$

- *Composition.* For each $F, G, H \in \mathrm{Obj}(\mathsf{CoPSh}(\mathcal{C}))$, the composition map

$$\circ^{\mathsf{CoPSh}(\mathcal{C})}_{F,G,H} \colon \mathrm{Nat}(G, H) \times \mathrm{Nat}(F, G) \longrightarrow \mathrm{Nat}(F, H)$$

  of $\mathsf{CoPSh}(\mathcal{C})$ at (F, G, H) is defined by

$$\beta \circ^{\mathsf{CoPSh}(\mathcal{C})}_{F,G,H} \alpha \overset{\text{def}}{=} \beta \circ \alpha.$$

3.2 Corepresentable Copresheaves 00Y6

Let $\mathcal{C}$ be a category, let $U, V \in \mathrm{Obj}(\mathcal{C})$, and let $f \colon U \longrightarrow V$ be a morphism of $\mathcal{C}$.

Definition 3.2.1.1. 00Y7 The **corepresentable copresheaf associated to** U is the copresheaf $h^U \colon \mathcal{C} \longrightarrow \mathsf{Sets}$ on $\mathcal{C}$ where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathcal{C})$, we have

$$h^U(A) \overset{\text{def}}{=} \mathrm{Hom}_{\mathcal{C}}(U, A);$$

- *Action on Morphisms.* For each morphism $f \colon A \longrightarrow B$ of $\mathcal{C}$, the image

$$h^U(f) \colon \underbrace{h^U(A)}_{\overset{\text{def}}{=}\mathrm{Hom}_{\mathcal{C}}(U,A)} \longrightarrow \underbrace{h^U(B)}_{\overset{\text{def}}{=}\mathrm{Hom}_{\mathcal{C}}(U,B)}$$

  of f by h^U is defined by

$$h^U(f) \overset{\text{def}}{=} f_*.$$

Definition 3.2.1.2. 00Y8 A copresheaf $F \colon \mathcal{C} \longrightarrow \mathsf{Sets}$ is **corepresentable** if $F \cong h^U$ for some $U \in \mathrm{Obj}(\mathcal{C})$.[22]

Definition 3.2.1.3. 00Y9 The **corepresentable natural transformation associated to** f is the natural transformation $h^f \colon h^V \Longrightarrow h^U$ consisting of the collection

$$\left\{ h^f_A \colon \underbrace{h^V(A)}_{\overset{\text{def}}{=}\mathrm{Hom}_{\mathcal{C}}(V,A)} \longrightarrow \underbrace{h^U(A)}_{\overset{\text{def}}{=}\mathrm{Hom}_{\mathcal{C}}(U,A)} \right\}_{A \in \mathrm{Obj}(\mathcal{C})}$$

where

$$h^f_A \overset{\text{def}}{=} f^*.$$

[22] In such a case, we call U a **corepresenting object** for F.

Theorem 3.2.1.4. 00YA Let $F\colon C \to \mathsf{Sets}$ be a copresheaf on C. We have a bijection

$$\mathrm{Nat}(h^A, F) \cong F^A,$$

natural in $A \in \mathrm{Obj}(C)$, determining a natural isomorphism of functors

$$\mathrm{Nat}(h^{(-)}, F) \cong F.$$

Proof. This is dual to **??**. □

3.3 The Contravariant Yoneda Embedding 00YB

Definition 3.3.1.1. 00YC The **contravariant Yoneda embedding of** C is the functor[23]

$$\text{よ}_C\colon C^{\mathsf{op}} \hookrightarrow \mathsf{Fun}(C, \mathsf{Sets})$$

where

- *Action on Objects.* For each $U \in \mathrm{Obj}(C)$, we have

  $$\text{よ}(U) \stackrel{\text{def}}{=} h^U;$$

- *Action on Morphisms.* For each morphism $f\colon U \to V$ of C, the image

  $$\text{よ}(f)\colon \text{よ}(V) \to \text{よ}(U)$$

  of f by よ is defined by

  $$\text{よ}(f) \stackrel{\text{def}}{=} h^f.$$

Proposition 3.3.1.2. Let C be a category. 00YD

1. *Fully Faithfulness.* 00YE The contravariant Yoneda embedding is fully faithful.[24]
2. *Preservation and Reflection of Isomorphisms.* 00YF Let $A, B \in \mathrm{Obj}(C)$. The following conditions are equivalent:
   (a) We have $A \cong B$.
   (b) We have $h_A \cong h_B$.
   (c) We have $h^A \cong h^B$.

[23] *Further Notation:* Also written $h^{(-)}$, or simply よ.
[24] In other words, the contravariant Yoneda embedding is indeed an embedding.

3. *Uniqueness of Representing Objects Up to Isomorphism.* Let $F\colon C \to \mathsf{Sets}$ be a 00YG
copresheaf. If there exist objects A and B of C such that we have

$$\begin{aligned} h^A &\cong F, \\ h^B &\cong F, \end{aligned}$$

then $A \cong B$.

4. *As a Free Completion: The Universal Property.* The pair $(\mathsf{CoPSh}(C)^{\mathsf{op}}, \text{よ})$ con- 00YH
sisting of

   - The opposite $\mathsf{CoPSh}(C)^{\mathsf{op}}$ of the category of copresheaves on C;
   - The contravariant Yoneda embedding $\text{よ}\colon C \hookrightarrow \mathsf{CoPSh}(C)^{\mathsf{op}}$ of C into $\mathsf{CoPSh}(C)^{\mathsf{op}}$;

   satisfies the following universal property:

   (UP) Given another pair $(\mathcal{A}, F)$ consisting of

   – A complete category $\mathcal{A}$;
   – A continuous functor $F\colon C \to \mathcal{A}$;

   there exists a continuous functor $\mathsf{CoPSh}(C)^{\mathsf{op}} \xrightarrow{\exists!} \mathcal{A}$, unique up to natural isomorphism, making the diagram

$$\begin{tikzcd} & \mathsf{CoPSh}(C)^{\mathsf{op}} \arrow[d, dashed, "\exists!"] \\ C \arrow[ur, "\text{よ}"] \arrow[r, "F"'] \arrow[ur, phantom, shift right=2, "\Uparrow\wr"] & \mathcal{A} \end{tikzcd}$$

   commute, again up to natural isomorphism.

5. *As a Free Completion: 2-Adjointness.* We have a 2-adjunction 00YJ

$$(\mathsf{CoPSh}^{\mathsf{op}} \dashv \iota)\colon \quad \begin{tikzcd} \mathsf{Cats} \arrow[r, bend left, "\mathsf{CoPSh}^{\mathsf{op}}"] \arrow[r, phantom, "\perp_2"] & \mathsf{Cats}^{\mathsf{comp.}} \arrow[l, bend left, "\iota"] \end{tikzcd},$$

witnessed by an adjoint equivalence of categories

$$\left(\mathrm{Ran}^{\mathsf{op}}_{\text{よ}} \dashv \text{よ}^*\right)\colon \quad \begin{tikzcd} \mathsf{ContFun}(\mathsf{CoPSh}(C)^{\mathsf{op}}, \mathcal{D}) \arrow[r, bend right, "\text{よ}^*"'] \arrow[r, phantom, "\perp"] & \mathsf{Fun}(C^{\mathsf{op}}, \mathcal{D}) \arrow[l, bend right, "\mathrm{Ran}^{\mathsf{op}}_{\text{よ}}"'] \end{tikzcd},$$

natural in $C \in \mathrm{Obj}(\mathsf{Cats})$ and $\mathcal{D} \in \mathrm{Obj}(\mathsf{Cats}^{\mathsf{comp.}})$.

Proof. This is dual to **??**. □

Appendices

A Other Chapters