Relations

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This chapter contains some material about relations and constructions with them. Notably, we discuss and explore:

1. The definition of relations (Section 1.1).
2. How relations may be viewed as decategorification of profunctors (Section 1.2).
3. The various kind of categories that relations form, namely:
   (a) A category (Section 2.1),
   (b) A monoidal category (Section 2.2),
   (c) A 2-category (Section 2.3), and
   (d) A double category (Section 2.4).
4. The various categorical properties of the 2-category of relations, including (Section 2.5):
   (a) The self-duality of Rel and $\mathbf{Rel}$ (Items 1 and 2 of Proposition 2.5.1.1);
   (b) Identifications of equivalences and isomorphisms in $\mathbf{Rel}$ with bijections (Item 3 of Proposition 2.5.1.1);
   (c) Identifications of adjunctions in $\mathbf{Rel}$ with functions (Item 4 of Proposition 2.5.1.1);
   (d) Identifications of monads in $\mathbf{Rel}$ with preorders (Item 5 of Proposition 2.5.1.1);
   (e) Identifications of comonads in $\mathbf{Rel}$ with subsets (Item 6 of Proposition 2.5.1.1);
   (f) Characterisations of monomorphisms in Rel (Item 7 of Proposition 2.5.1.1);
   (g) Characterisations of epimorphisms in Rel (Item 8 of Proposition 2.5.1.1);
   (h) The partial co/completeness of Rel (Item 10 of Proposition 2.5.1.1);
   (i) The existence of right Kan extensions and right Kan lifts in Rel (Items 11 and 12 of Proposition 2.5.1.1);

(j) The closedness of **Rel** (Item 13 of Proposition 2.5.1.1).

5. The various kinds of constructions involving relations, such as graphs, domains, ranges, unions, intersections, products, inverse relations, composition of relations, and collages (Section 3).

6. Equivalence relations (Section 4) and quotient sets (Section 4.5).

7. The adjoint pairs

$$R_* \dashv R_{-1} \colon \mathcal{P}(A) \rightleftarrows \mathcal{P}(B),$$
$$R^{-1} \dashv R_! \colon \mathcal{P}(B) \rightleftarrows \mathcal{P}(A)$$

of functors (morphisms of posets) between $\mathcal{P}(A)$ and $\mathcal{P}(B)$ induced by a relation $R \colon A \nrightarrow B$, as well as the properties of R_*, R_{-1}, R^{-1}, and $R_!$ (Section 5).

Here we also note that:

(a) These two pairs of adjoint functors are the counterpart for relations of the adjoint triple $f_* \dashv f^{-1} \dashv f_!$ induced by a function $f \colon A \longrightarrow B$ studied in Constructions With Sets, Section 4;

(b) We have $R_{-1} = R^{-1}$ iff R is total and functional (Item 8 of Proposition 5.2.1.3).

(c) As a consequence of the previous item, when R comes from a function f the pair of adjunctions

$$R_* \dashv R_{-1} = R^{-1} \dashv R_!$$

reduces to the triple adjunction

$$f_* \dashv f^{-1} \dashv f_!$$

from before.

(d) The pairs $R_* \dashv R_{-1}$ and $R^{-1} \dashv R_!$ later make an appearance in the context of continuous, open, and closed relations between topological spaces (Topological Spaces, **??**).

8. A notion of *relative preorder* that is to that of a preorder as relative monads are to monads, extending the identifications of monads in Rel with preorders of Item 5 of Proposition 2.5.1.1 to "relative monads in Rel".

Contents

1 Relations

1.1 Foundations

Let A and B be sets.

Definition 1.1.1.1. A **relation** $R \colon A \nrightarrow B$ **from** A **to** B[1,2] is a subset R of $A \times B$.[3]

Definition 1.1.1.2. Let A and B be sets.

1. The **set of relations from** A **to** B is the set $\mathrm{Rel}(A, B)$ defined by

$$\mathrm{Rel}(A, B) \overset{\text{def}}{=} \{\text{Relations from } A \text{ to } B\}.$$

2. The **poset of relations from** A **to** B is the poset

$$\mathbf{Rel}(A, B) \overset{\text{def}}{=} (\mathrm{Rel}(A, B), \subset)$$

consisting of

- *The Underlying Set.* The set $\mathrm{Rel}(A, B)$ of Item 1;

[1] *Further Terminology:* Also called a **multivalued function from** A **to** B, a **relation over** A **and** B, **relation on** A **and** B, a **binary relation over** A **and** B, or a **binary relation on** A **and** B.

[2] *Further Terminology:* When $A = B$, we also call $R \subset A \times A$ a **relation on** A.

[3] *Further Notation:* Given elements $a \in A$ and $b \in B$, we write $a \sim_R b$ to mean $(a, b) \in R$.

- *The Partial Order.* The partial order

$$\subset \colon \mathrm{Rel}(A,B) \times \mathrm{Rel}(A,B) \longrightarrow \{\mathsf{true}, \mathsf{false}\}$$

on $\mathrm{Rel}(A,B)$ given by inclusion of relations.

Remark 1.1.1.3. A relation from A to B is equivalently:[4]

1. A subset of $A \times B$;
2. A function from $A \times B$ to $\{\mathsf{true}, \mathsf{false}\}$;
3. A function from A to $\mathcal{P}(B)$;
4. A function from B to $\mathcal{P}(A)$;
5. A cocontinuous morphism of posets from $(\mathcal{P}(A), \subset)$ to $(\mathcal{P}(B), \subset)$.

That is: we have bijections of sets

$$\begin{aligned}
\mathrm{Rel}(A,B) &\overset{\text{def}}{=} \mathcal{P}(A \times B),\\
&\cong \mathsf{Sets}(A \times B, \{\mathsf{true}, \mathsf{false}\}),\\
&\cong \mathsf{Sets}(A, \mathcal{P}(B)),\\
&\cong \mathsf{Sets}(B, \mathcal{P}(A)),\\
&\cong \mathrm{Hom}^{\mathsf{cocont}}_{\mathsf{Pos}}(\mathcal{P}(A), \mathcal{P}(B)),
\end{aligned}$$

natural in $A, B \in \mathrm{Obj}(\mathsf{Sets})$.

Proof. We claim that Items 1 to 5 are indeed equivalent:

- *Item 1* $\iff$ *Item 2:* This is a special case of Constructions With Sets, Item 6 of Proposition 4.2.1.3.
- *Item 2* $\iff$ *Item 3:* This is an instance of currying, following from the bijections

$$\begin{aligned}
\mathsf{Sets}(A \times B, \{\mathsf{true}, \mathsf{false}\}) &\cong \mathsf{Sets}(A, \mathsf{Sets}(B, \{\mathsf{true}, \mathsf{false}\}))\\
&\cong \mathsf{Sets}(A, \mathcal{P}(B)),
\end{aligned}$$

  where the last bijection is from Constructions With Sets, Item 6 of Proposition 4.2.1.3.

[4] *Intuition:* In particular, we may think of a relation $R \colon A \to \mathcal{P}(B)$ from A to B as a multivalued function from A to B (including the possibility of a given $a \in A$ having no value at all).

- *Item 2* $\iff$ *Item 4*: This is also an instance of currying, following from the bijections

$$\begin{aligned}\mathsf{Sets}(A \times B, \{\mathsf{true}, \mathsf{false}\}) &\cong \mathsf{Sets}(B, \mathsf{Sets}(B, \{\mathsf{true}, \mathsf{false}\})) \\ &\cong \mathsf{Sets}(B, \mathcal{P}(A)),\end{aligned}$$

  where again the last bijection is from Constructions With Sets, Item 6 of Proposition 4.2.1.3.

- *Item 2* $\iff$ *Item 5*: This follows from the universal property of the powerset $\mathcal{P}(X)$ of a set X as the free cocompletion of X via the characteristic embedding

$$\chi_X \colon X \hookrightarrow \mathcal{P}(X)$$

  of X into $\mathcal{P}(X)$ (Constructions With Sets, Item 9 of Proposition 4.2.1.3).

  In particular, the bijection

$$\mathrm{Rel}(A, B) \cong \mathrm{Hom}^{\mathsf{cocont}}_{\mathsf{Pos}}(\mathcal{P}(A), \mathcal{P}(B))$$

  is given by taking a relation $R \colon A \nrightarrow B$, passing to its associated function $f \colon A \to \mathcal{P}(B)$ from A to B and then extending f from A to all of $\mathcal{P}(A)$ by taking its left Kan extension along χ_X.

  This coincides with the direct image function $f_* \colon \mathcal{P}(A) \to \mathcal{P}(B)$ of Constructions With Sets, Definition 4.3.1.1.

This finishes the proof. □

Proposition 1.1.1.4. Let A and B be sets.

1. *End Formula for The Poset of Relations.* Let $R, S \colon A \nrightarrow B$ be relations. We have

$$\mathrm{Hom}_{\mathbf{Rel}(A,B)}(R, S) \cong \int_{a\in A} \int_{b\in B} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R^a_b, S^a_b).$$

Proof. Item 1, *End Formula for The Poset of Relations*: Unwinding the expression inside the end on the right hand side, we have

$$\int_{a\in A} \int_{b\in B} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R^a_b, S^a_b) \cong \begin{cases} \mathrm{pt} & \text{if for each } (a, b) \in A \times B, \\ & \text{if } a \sim_R b, \text{ then } a \sim_S b, \\ \emptyset & \text{otherwise.} \end{cases}$$

On the left hand-side, we have

$$\mathrm{Hom}_{\mathbf{Rel}(A,B)}(R,S) \cong \begin{cases} \mathrm{pt} & \text{if } R \subset S, \\ \emptyset & \text{otherwise.} \end{cases}$$

It is then clear that the conditions for each set to evaluate to pt are equivalent, implying that those two sets are isomorphic, finishing the proof. □

1.2 Relations as Decategorifications of Profunctors

Remark 1.2.1.1. The notion of a relation is a decategorification of that of a profunctor:

1. A profunctor from a category C to a category $\mathcal{D}$ is a functor

$$\mathfrak{p}\colon \mathcal{D}^{\mathsf{op}} \times C \longrightarrow \mathsf{Sets}.$$

2. A relation on sets A and B is a function

$$R\colon A \times B \longrightarrow \{\mathsf{true}, \mathsf{false}\}.$$

Here we notice that:

- The opposite X^{op} of a set X is itself, as $(-)^{\mathsf{op}}\colon \mathsf{Cats} \longrightarrow \mathsf{Cats}$ restricts to the identity endofunctor on Sets;
- The values that profunctors and relations take are directly related in relation to decategorification:
  - A category is enriched over the category

$$\mathsf{Sets} \stackrel{\text{def}}{=} \mathsf{Cats}_0$$

    of sets, with profunctors taking values on it;
  - A set is enriched over the set

$$\{\mathsf{true}, \mathsf{false}\} \stackrel{\text{def}}{=} \mathsf{Cats}_{-1}$$

    of classical truth values, with relations taking values on it;

Remark 1.2.1.2. Extending Remark 1.2.1.1, the equivalent definitions of relations in Remark 1.1.1.3 are also related to the corresponding ones for profunctors (Categories, **??**), which state that a profunctor $\mathfrak{p}\colon C \nrightarrow \mathcal{D}$ is equivalently:

1. A functor $\mathfrak{p}\colon \mathcal{D}^{\mathsf{op}} \times C \longrightarrow \mathsf{Sets}$;

2. A functor $\mathfrak{p}\colon \mathcal{C} \to \mathsf{PSh}(\mathcal{D})$;
3. A functor $\mathfrak{p}\colon \mathcal{D}^{\mathsf{op}} \to \mathsf{Fun}(\mathcal{C}, \mathsf{Sets})$;
4. A colimit-preserving functor $\mathfrak{p}\colon \mathsf{PSh}(\mathcal{C}) \to \mathsf{PSh}(\mathcal{D})$.

Indeed:

- The equivalence between Items 1 and 2 (and also that between Items 1 and 3, which is proved analogously) is an instance of currying, both for profunctors as well as for relations, using the isomorphisms

$$\begin{aligned}\mathsf{Sets}(A \times B, \{\mathsf{true}, \mathsf{false}\}) &\cong \mathsf{Sets}(A, \mathsf{Sets}(B, \{\mathsf{true}, \mathsf{false}\})) \\ &\cong \mathsf{Sets}(A, \mathcal{P}(B)), \\ \mathsf{Fun}(\mathcal{D}^{\mathsf{op}} \times \mathcal{D}, \mathsf{Sets}) &\cong \mathsf{Fun}(\mathcal{C}, \mathsf{Fun}(\mathcal{D}^{\mathsf{op}}, \mathsf{Sets})) \\ &\cong \mathsf{Fun}(\mathcal{C}, \mathsf{PSh}(\mathcal{D})).\end{aligned}$$

- The equivalence between Items 1 and 3 follows from the universal properties of:
  - The powerset $\mathcal{P}(X)$ of a set X as the free cocompletion of X via the characteristic embedding

$$\chi_{(-)}\colon X \hookrightarrow \mathcal{P}(X)$$

    of X into $\mathcal{P}(X)$ (Constructions With Sets, Item 9 of Proposition 4.2.1.3);
  - The category $\mathsf{PSh}(\mathcal{C})$ of presheaves on a category $\mathcal{C}$ as the free cocompletion of $\mathcal{C}$ via the Yoneda embedding

$$よ\colon \mathcal{C} \hookrightarrow \mathsf{PSh}(\mathcal{C})$$

    of $\mathcal{C}$ into $\mathsf{PSh}(\mathcal{C})$ (Categories, **??** of **??**).

1.3 Examples of Relations

Example 1.3.1.1. The **trivial relation on** A **and** B is the relation $\sim_{\mathrm{triv}}$ defined by[5,6,7]

$$\sim_{\mathrm{triv}} \stackrel{\mathrm{def}}{=} A \times A.$$

Example 1.3.1.2. The **cotrivial relation on** A **and** B is the relation $\sim_{\mathrm{cotriv}}$ defined by[8,9,10]

$$\sim_{\mathrm{cotriv}} \stackrel{\mathrm{def}}{=} \emptyset.$$

Example 1.3.1.3. The characteristic relation on A of Constructions With Sets, Item 3 of Definition 4.1.1.1 is another example of a relation. It is in fact the unique relation on A making the following conditions equivalent, for each $a, b \in A$:

1. We have $a \sim_{\mathrm{id}} b$.
2. We have $a = b$.

Example 1.3.1.4. Square roots are examples of relations:

1. *Square Roots in* $\mathbb{R}$. The assignment $x \mapsto \sqrt{x}$ defines a relation

$$\sqrt{-} \colon \mathbb{R} \to \mathcal{P}(\mathbb{R})$$

[5]This is the unique relation R on A and B such that we have $a \sim_R b$ for all $a \in A$ and all $b \in B$.

[6]As a function from $A \times A$ to $\{\mathsf{true}, \mathsf{false}\}$, the relation $\sim_{\mathrm{triv}}$ is the constant function

$$\Delta_{\mathsf{true}} \colon A \times B \to \{\mathsf{true}, \mathsf{false}\}$$

from $A \times B$ to $\{\mathsf{true}, \mathsf{false}\}$ taking value true.

[7]As a function from A to $\mathcal{P}(B)$, the relation $\sim_{\mathrm{triv}}$ is the function

$$\Delta_{\mathsf{true}} \colon A \to \mathcal{P}(B)$$

defined by

$$\Delta_{\mathsf{true}}(a) \stackrel{\mathrm{def}}{=} B$$

for each $a \in A$.

[8]This is the unique relation R on A and B such that we have $a \sim_R b$ for no $a \in A$ and no $b \in B$.

[9]As a function from $A \times B$ to $\{\mathsf{true}, \mathsf{false}\}$, the relation $\sim_{\mathrm{cotriv}}$ is the constant function

$$\Delta_{\mathsf{false}} \colon A \times B \to \{\mathsf{true}, \mathsf{false}\}$$

from $A \times B$ to $\{\mathsf{true}, \mathsf{false}\}$ taking value false.

[10]As a function from A to $\mathcal{P}(A)$, the relation $\sim_{\mathrm{cotriv}}$ is the function

$$\Delta_{\mathsf{false}} \colon A \to \mathcal{P}(A)$$

defined by

$$\Delta_{\mathsf{true}}(a) \stackrel{\mathrm{def}}{=} \emptyset$$

from $\mathbb{R}$ to itself, being explicitly given by

$$\sqrt{x} \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if } x = 0, \\ \left\{-\sqrt{|x|}, \sqrt{|x|}\right\} & \text{if } x \neq 0. \end{cases}$$

2. *Square Roots in $\mathbb{Q}$.* Square roots in $\mathbb{Q}$ are similar to square roots in $\mathbb{R}$, though now additionally it may also occur that $\sqrt{-}\colon \mathbb{Q} \to \mathcal{P}(\mathbb{Q})$ sends a rational number x (e.g. 2) to the empty set (since $\sqrt{2} \notin \mathbb{Q}$).

Example 1.3.1.5. The complex logarithm defines a relation

$$\log\colon \mathbb{C} \to \mathcal{P}(\mathbb{C})$$

from $\mathbb{C}$ to itself, where we have

$$\log(a+bi) \stackrel{\text{def}}{=} \left\{\log\left(\sqrt{a^2+b^2}\right) + i\arg(a+bi) + (2\pi i)k \;\middle|\; k \in \mathbb{Z}\right\}$$

for each $a + bi \in \mathbb{C}$.

Example 1.3.1.6. See [**wikipedia:multivalued-functions**] for more examples of relations, such as antiderivation, inverse trigonometric functions, and inverse hyperbolic functions.

1.4 Functional Relations

Let A and B be sets.

Definition 1.4.1.1. A relation $R\colon A \nrightarrow B$ is **functional** if, for each $a \in A$, the set $R(a)$ is either empty or a singleton.

Proposition 1.4.1.2. Let $R\colon A \nrightarrow B$ be a relation.

1. *Characterisations.* The following conditions are equivalent:

   (a) The relation R is functional.

   (b) We have $R \diamond R^{\dagger} \subset \chi_B$.

Proof. *Item 1*, *Characterisations*: We claim that Items 1a and 1b are indeed equivalent:

- *Item 1a* $\Longrightarrow$ *Item 1b*: Let $(b, b') \in B \times B$. We need to show that

$$[R \diamond R^{\dagger}](b, b') \preceq_{\{\mathsf{t},\mathsf{f}\}} \chi_B(b, b'),$$

i.e. that if there exists some $a \in A$ such that $b \sim_{R^\dagger} a$ and $a \sim_R b'$, then $b = b'$. But since $b \sim_{R^\dagger} a$ is the same as $a \sim_R b$, we have both $a \sim_R b$ and $a \sim_R b'$ at the same time, which implies $b = b'$ since R is functional.

- *Item 1b* $\implies$ *Item 1a*: Suppose that we have $a \sim_R b$ and $a \sim_R b'$ for $b, b' \in B$. We claim that $b = b'$:

    1. Since $a \sim_R b$, we have $b \sim_{R^\dagger} a$.
    2. Since $R \diamond R^\dagger \subset \chi_B$, we have

    $$[R \diamond R^\dagger](b, b') \preceq_{\{\mathsf{t},\mathsf{f}\}} \chi_B(b, b'),$$

    and since $b \sim_{R^\dagger} a$ and $a \sim_R b'$, it follows that $[R \diamond R^\dagger](b, b') = \mathsf{true}$, and thus $\chi_B(b, b') = \mathsf{true}$ as well, i.e. $b = b'$.

This finishes the proof. □

1.5 Total Relations

Let A and B be sets.

Definition 1.5.1.1. A relation $R \colon A \nrightarrow B$ is **total** if, for each $a \in A$, we have $R(a) \neq \emptyset$.

Proposition 1.5.1.2. Let $R \colon A \nrightarrow B$ be a relation.

1. *Characterisations.* The following conditions are equivalent:

    (a) The relation R is total.

    (b) We have $\chi_A \subset R^\dagger \diamond R$.

Proof. *Item 1*, *Characterisations*: We claim that Items 1a and 1b are indeed equivalent:

- *Item 1a* $\implies$ *Item 1b*: We have to show that, for each $(a, a') \in A$, we have

    $$\chi_A(a, a') \preceq_{\{\mathsf{t},\mathsf{f}\}} [R^\dagger \diamond R](a, a'),$$

    i.e. that if $a = a'$, then there exists some $b \in B$ such that $a \sim_R b$ and $b \sim_{R^\dagger} a'$ (i.e. $a \sim_R b$ again), which follows from the totality of R.

- *Item 1b* $\implies$ *Item 1a*: Given $a \in A$, since $\chi_A \subset R^\dagger \diamond R$, we must have

    $$\{a\} \subset [R^\dagger \diamond R](a),$$

    implying that there must exist some $b \in B$ such that $a \sim_R b$ and $b \sim_{R^\dagger} a$ (i.e. $a \sim_R b$) and thus $R(a) \neq \emptyset$, as $b \in R(a)$.

This finishes the proof. □

2 Categories of Relations

2.1 The Category of Relations

Definition 2.1.1.1. The **category of relations** is the category Rel where

- *Objects.* The objects of Rel are sets;
- *Morphisms.* For each $A, B \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$\mathsf{Rel}(A, B) \stackrel{\text{def}}{=} \mathrm{Rel}(A, B);$$

- *Identities.* For each $A \in \mathrm{Obj}(\mathsf{Rel})$, the unit map

$$\mathbb{1}^{\mathsf{Rel}}_A \colon \mathrm{pt} \longrightarrow \mathrm{Rel}(A, A)$$

  of Rel at A is defined by

$$\mathrm{id}^{\mathsf{Rel}}_A \stackrel{\text{def}}{=} \chi_A(-_1, -_2),$$

  where $\chi_A(-_1, -_2)$ is the characteristic relation of A of Constructions With Sets, Item 3 of Definition 4.1.1.1;

- *Composition.* For each $A, B, C \in \mathrm{Obj}(\mathsf{Rel})$, the composition map

$$\circ^{\mathsf{Rel}}_{A,B,C} \colon \mathrm{Rel}(B, C) \times \mathrm{Rel}(A, B) \longrightarrow \mathrm{Rel}(A, C)$$

  of Rel at (A, B, C) is defined by

$$S \circ^{\mathsf{Rel}}_{A,B,C} R \stackrel{\text{def}}{=} S \diamond R$$

  for each $(S, R) \in \mathrm{Rel}(B, C) \times \mathrm{Rel}(A, B)$, where $S \diamond R$ is the composition of S and R of Definition 3.12.1.1.

2.2 The Closed Symmetric Monoidal Category of Relations

2.2.1 The Monoidal Product

Definition 2.2.1.1. The **monoidal product of** Rel is the functor

$$\times \colon \mathsf{Rel} \times \mathsf{Rel} \longrightarrow \mathsf{Rel}$$

where

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for each $a \in A$.

- *Action on Objects.* We have

$$\times(A, B) \overset{\text{def}}{=} A \times B,$$

where $A \times B$ is the Cartesian product of sets of Constructions With Sets, Definition 1.2.1.1;

- *Action on Morphisms.* For each $(A, C), (B, D) \in \mathrm{Obj}(\mathsf{Rel} \times \mathsf{Rel})$, the action on morphisms

$$\times_{(A,C),(B,D)} \colon \mathrm{Rel}(A, B) \times \mathrm{Rel}(C, D) \longrightarrow \mathrm{Rel}(A \times C, B \times D)$$

of $\times$ is given by sending a pair of morphisms (R, S) of the form

$$\begin{aligned} R &\colon A \relbar\joinrel\mapstochar\joinrel\rightarrow B, \\ S &\colon C \relbar\joinrel\mapstochar\joinrel\rightarrow D \end{aligned}$$

to the relation

$$R \times S \colon A \times C \relbar\joinrel\mapstochar\joinrel\rightarrow B \times D$$

of Definition 3.9.1.1.

2.2.2 The Monoidal Unit

Definition 2.2.2.1. The **monoidal unit of** Rel is the functor

$$\mathbb{1}^{\mathsf{Rel}} \colon \mathrm{pt} \longrightarrow \mathsf{Rel}$$

picking the set

$$\mathbb{1}_{\mathsf{Rel}} \overset{\text{def}}{=} \mathrm{pt}$$

of Rel.

2.2.3 The Associator

Definition 2.2.3.1. The **associator of** Rel is the natural isomorphism

$$\alpha^{\mathsf{Rel}} \colon \times \circ ((\times) \times \mathsf{id}) \overset{\cong}{\Longrightarrow} \times \circ (\mathsf{id} \times (\times)), \qquad \begin{tikzcd} \mathsf{Rel} \times \mathsf{Rel} \times \mathsf{Rel} \arrow[r, "\mathsf{id} \times (\times)"] \arrow[d, "(\times) \times \mathsf{id}"'] & \mathsf{Rel} \times \mathsf{Rel} \arrow[d, "\times"] \\ \mathsf{Rel} \times \mathsf{Rel} \arrow[r, "\times"'] \arrow[ur, phantom, "\Nearrow \alpha^{\mathsf{Rel}}"] & \mathsf{Rel}, \end{tikzcd}$$

whose component

$$\alpha^{\mathsf{Rel}}_{A,B,C} \colon (A \times B) \times C \relbar\joinrel\mapstochar\joinrel\rightarrow A \times (B \times C)$$

at (A, B, C) is defined by declaring

$$((a, b), c) \sim_{\alpha^{\mathsf{Rel}}_{A,B,C}} (a', (b', c'))$$

iff $a = a'$, $b = b'$, and $c = c'$.

2.2.4 The Left Unitor

Definition 2.2.4.1. The **left unitor of** Rel is the natural isomorphism

$$\lambda^{\mathsf{Rel}} \colon \times \circ (\mathbb{1}^{\mathsf{Rel}} \times \mathrm{id}) \overset{\cong}{\Longrightarrow} \boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathsf{Rel}}, \qquad \begin{tikzcd} \mathsf{pt} \times \mathsf{Rel} \arrow[r, "\mathbb{1}^{\mathsf{Rel}} \times \mathrm{id}"] \arrow[dr, dashed, bend right, "\boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathsf{Rel}}"'] & \mathsf{Rel} \times \mathsf{Rel} \arrow[d, "\times"] \arrow[dl, phantom, "\Swarrow \lambda^{\mathsf{Rel}}"] \\ & \mathsf{Rel}, \end{tikzcd}$$

whose component

$$\lambda^{\mathsf{Rel}}_{A} \colon \mathbb{1}_{\mathsf{Rel}} \times A \nrightarrow A$$

at A is defined by declaring

$$(\star, a) \sim_{\lambda^{\mathsf{Rel}}_{A}} b$$

iff $a = b$.

2.2.5 The Right Unitor

Definition 2.2.5.1. The **right unitor of** Rel is the natural isomorphism

$$\rho^{\mathsf{Rel}} \colon \times \circ (\mathrm{id} \times \mathbb{1}^{\mathsf{Rel}}) \overset{\cong}{\Longrightarrow} \boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathsf{Rel}}, \qquad \begin{tikzcd} \mathsf{Rel} \times \mathsf{pt} \arrow[r, "\mathrm{id} \times \mathbb{1}^{\mathsf{Rel}}"] \arrow[dr, dashed, bend right, "\boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathsf{Rel}}"'] & \mathsf{Rel} \times \mathsf{Rel} \arrow[d, "\times"] \arrow[dl, phantom, "\Swarrow \rho^{\mathsf{Rel}}"] \\ & \mathsf{Rel}, \end{tikzcd}$$

whose component

$$\rho^{\mathsf{Rel}}_{A} \colon A \times \mathbb{1}_{\mathsf{Rel}} \nrightarrow A$$

at A is defined by declaring

$$(a, \star) \sim_{\rho^{\mathsf{Rel}}_{A}} b$$

iff $a = b$.

2.2.6 The Symmetry

Definition 2.2.6.1. The **symmetry of** Rel is the natural isomorphism

$$\sigma^{\mathsf{Rel}} \colon \times \Longrightarrow \times \circ \boldsymbol{\sigma}^{\mathsf{Cats}_2}_{\mathsf{Rel},\mathsf{Rel}}, \qquad \begin{tikzcd} \mathsf{Rel} \times \mathsf{Rel} \arrow[rr, "\times"] \arrow[dr, "\boldsymbol{\sigma}^{\mathsf{Cats}_2}_{\mathsf{Rel},\mathsf{Rel}}"'] & \arrow[d, phantom, "\overset{\|}{\underset{\Downarrow}{\sigma^{\mathsf{Rel}}}}"] & \mathsf{Rel}, \\ & \mathsf{Rel} \times \mathsf{Rel} \arrow[ur, "\times"'] & \end{tikzcd}$$

whose component

$$\sigma^{\mathsf{Rel}}_{A,B} \colon A \times B \to B \times A$$

at (A, B) is defined by declaring

$$(a, b) \sim_{\sigma^{\mathsf{Rel}}_{A,B}} (b', a')$$

iff $a = a'$ and $b = b'$.

2.2.7 The Internal Hom

Definition 2.2.7.1. The **internal Hom of** Rel is the functor

$$\mathbf{Hom}_{\mathsf{Rel}} \colon \mathsf{Rel}^{\mathsf{op}} \times \mathsf{Rel} \to \mathsf{Rel}$$

defined by

$$\mathbf{Hom}_{\mathsf{Rel}}(A, B) \overset{\text{def}}{=} A \times B$$

for each $A, B \in \mathrm{Obj}(\mathsf{Rel})$.

Proposition 2.2.7.2. Let $A, B, C \in \mathrm{Obj}(\mathsf{Rel})$.

1. *Via Self-Duality.* The internal Hom $\mathbf{Hom}_{\mathsf{Rel}}$ of Rel is given by the composition

$$\mathsf{Rel}^{\mathsf{op}} \times \mathsf{Rel} \xrightarrow{\cong} \mathsf{Rel} \times \mathsf{Rel} \xrightarrow{\times} \mathsf{Rel},$$

   where the self-duality equivalence $\mathsf{Rel}^{\mathsf{op}} \cong \mathsf{Rel}$ comes from Item 1 of Proposition 2.5.1.1.

2. *Adjointness.* We have adjunctions

$$(A \times - \dashv \mathbf{Hom}_{\mathsf{Rel}}(A, -)) \colon \begin{tikzcd} \mathsf{Rel} \arrow[r, bend left, "A\times -"] \arrow[r, phantom, "\perp"] & \mathsf{Rel}, \arrow[l, bend left, "\mathbf{Hom}_{\mathsf{Rel}}(A,-)"] \end{tikzcd}$$

$$(- \times B \dashv \mathbf{Hom}_{\mathsf{Rel}}(B, -)) \colon \begin{tikzcd} \mathsf{Rel} \arrow[r, bend left, "-\times B"] \arrow[r, phantom, "\perp"] & \mathsf{Rel}, \arrow[l, bend left, "\mathbf{Hom}_{\mathsf{Rel}}(B,-)"] \end{tikzcd}$$

witnessed by bijections

$$\begin{aligned}\mathrm{Rel}(A \times B, C) &\cong \mathrm{Rel}(A, \mathbf{Hom}_{\mathrm{Rel}}(B, C)) \\ &\stackrel{\text{def}}{=} \mathrm{Rel}(A, B \times C), \\ \mathrm{Rel}(A \times B, C) &\cong \mathrm{Rel}(B, \mathbf{Hom}_{\mathrm{Rel}}(A, C)) \\ &\stackrel{\text{def}}{=} \mathrm{Rel}(B, A \times C),\end{aligned}$$

natural in $A, B, C \in \mathrm{Obj}(\mathrm{Rel})$.

Proof. Item 1, Via Self-Duality: Omitted.
Item 2, Adjointness: Indeed, we have

$$\begin{aligned}\mathrm{Rel}(A \times B, C) &\stackrel{\text{def}}{=} \mathrm{Sets}(A \times B \times C, \{\mathrm{true}, \mathrm{false}\}) \\ &\stackrel{\text{def}}{=} \mathrm{Rel}(A, B \times C) \\ &\stackrel{\text{def}}{=} \mathrm{Rel}(A, \mathbf{Hom}_{\mathrm{Rel}}(B, C)),\end{aligned}$$

and similarly for the bijection $\mathrm{Rel}(A \times B, C) \cong \mathrm{Rel}(B, \mathbf{Hom}_{\mathrm{Rel}}(A, C))$. □

2.2.8 The Closed Symmetric Monoidal Category of Relations

Definition 2.2.8.1. The **closed symmetric monoidal category of relations** is the closed symmetric monoidal category

$$\left(\mathrm{Rel}, \times, \mathbb{1}_{\mathrm{Rel}}, \alpha^{\mathrm{Rel}}, \lambda^{\mathrm{Rel}}, \rho^{\mathrm{Rel}}, \sigma^{\mathrm{Rel}}, \mathbf{Hom}_{\mathrm{Rel}}\right)$$

consisting of

- *The Underlying Category.* The category Rel of sets and relations of Definition 2.1.1.1;
- *The Monoidal Product.* The functor

  $$\times \colon \mathrm{Rel} \times \mathrm{Rel} \to \mathrm{Rel}$$

  of Definition 2.2.1.1;
- *The Monoidal Unit.* The functor $\mathbb{1}^{\mathrm{Rel}}$ of Definition 2.2.2.1;
- *The Associator.* The natural isomorphism α^{Rel} of Definition 2.2.3.1;
- *The Left Unitor.* The natural isomorphism λ^{Rel} of Definition 2.2.4.1;
- *The Right Unitor.* The natural isomorphism ρ^{Rel} of Definition 2.2.5.1;
- *The Symmetry.* The natural isomorphism σ^{Rel} of Definition 2.2.6.1;
- *The Internal Hom.* The functor

  $$\mathbf{Hom}_{\mathrm{Rel}} \colon \mathrm{Rel}^{\mathrm{op}} \times \mathrm{Rel} \to \mathrm{Rel}$$

  of Definition 2.2.7.1.

2.3 The 2-Category of Relations

Definition 2.3.1.1. The 2-**category of relations** is the locally posetal 2-category Rel where

- *Objects.* The objects of Rel are sets;
- ***Hom**-Objects.* For each $A, B \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$\begin{aligned}\mathrm{Hom}_{\mathsf{Rel}}(A, B) &\stackrel{\text{def}}{=} \mathbf{Rel}(A, B)\\ &\stackrel{\text{def}}{=} (\mathrm{Rel}(A, B), \subset);\end{aligned}$$

- *Identities.* For each $A \in \mathrm{Obj}(\mathsf{Rel})$, the unit map

$$\mathbb{1}^{\mathsf{Rel}}_A : \mathrm{pt} \longrightarrow \mathbf{Rel}(A, A)$$

  of Rel at A is defined by

$$\mathrm{id}^{\mathsf{Rel}}_A \stackrel{\text{def}}{=} \chi_A(-_1, -_2),$$

  where $\chi_A(-_1, -_2)$ is the characteristic relation of A of Constructions With Sets, Item 3 of Definition 4.1.1.1;
- *Composition.* For each $A, B, C \in \mathrm{Obj}(\mathsf{Rel})$, the composition map[11]

$$\circ^{\mathsf{Rel}}_{A,B,C} : \mathbf{Rel}(B, C) \times \mathbf{Rel}(A, B) \longrightarrow \mathbf{Rel}(A, C)$$

  of Rel at (A, B, C) is defined by

$$S \circ^{\mathsf{Rel}}_{A,B,C} R \stackrel{\text{def}}{=} S \diamond R$$

  for each $(S, R) \in \mathsf{Rel}(B, C) \times \mathsf{Rel}(A, B)$, where $S \diamond R$ is the composition of S and R of Definition 3.12.1.1.

[11]Note that this is indeed a morphism of posets: given relations $R_1, R_2 \in \mathbf{Rel}(A, B)$ and $S_1, S_2 \in \mathbf{Rel}(B, C)$ such that

$$\begin{aligned}R_1 &\subset R_2,\\ S_1 &\subset S_2,\end{aligned}$$

we have also $S_1 \diamond R_1 \subset S_2 \diamond R_2$.

2.4 The Double Category of Relations

2.4.1 The Double Category of Relations

Definition 2.4.1.1. The **double category of relations** is the locally posetal double category $\mathsf{Rel}^{\mathsf{dbl}}$ where

- *Objects.* The objects of $\mathsf{Rel}^{\mathsf{dbl}}$ are sets;
- *Vertical Morphisms.* The vertical morphisms of $\mathsf{Rel}^{\mathsf{dbl}}$ are maps of sets $f\colon A \to B$;
- *Horizontal Morphisms.* The horizontal morphisms of $\mathsf{Rel}^{\mathsf{dbl}}$ are relations $R\colon A \nrightarrow X$;
- 2*-Morphisms.* A 2-cell

$$\begin{tikzcd} A \arrow[r, "R", "|"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & B \arrow[d, "g"] \\ X \arrow[r, "S"', "|"{marking}] & Y \end{tikzcd}$$

of $\mathsf{Rel}^{\mathsf{dbl}}$ is either non-existent or an inclusion of relations of the form

$$R \subset S \circ (f \times g), \qquad \begin{tikzcd} A \times B \arrow[r, "R"] \arrow[d, "f\times g"'] \arrow[dr, phantom, "\subset"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ X \times Y \arrow[r, "S"'] & \{\mathsf{true}, \mathsf{false}\}; \end{tikzcd}$$

- *Horizontal Identities.* The horizontal unit functor of $\mathsf{Rel}^{\mathsf{dbl}}$ is the functor of Definition 2.4.2.1;
- *Vertical Identities.* For each $A \in \mathrm{Obj}\left(\mathsf{Rel}^{\mathsf{dbl}}\right)$, we have

$$\mathrm{id}_A^{\mathsf{Rel}^{\mathsf{dbl}}} \overset{\mathrm{def}}{=} \mathrm{id}_A;$$

- *Identity* 2*-Morphisms.* For each horizontal morphism $R\colon A \nrightarrow B$ of $\mathsf{Rel}^{\mathsf{dbl}}$, the identity 2-morphism

$$\begin{tikzcd} A \arrow[r, "R", "|"{marking}] \arrow[d, "\mathrm{id}_A"'] \arrow[dr, phantom, "\Downarrow\mathrm{id}_R"] & B \arrow[d, "\mathrm{id}_B"] \\ A \arrow[r, "R"', "|"{marking}] & B \end{tikzcd}$$

of R is the identity inclusion

$$R \subset R, \qquad \begin{tikzcd} B \times A \arrow[r, "R"] \arrow[d, "\mathrm{id}_B \times \mathrm{id}_A"'] \arrow[dr, phantom, "\circlearrowleft"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ B \times A \arrow[r, "R"'] & \{\mathsf{true}, \mathsf{false}\}; \end{tikzcd}$$

- *Horizontal Composition.* The horizontal composition functor of $\mathsf{Rel}^{\mathsf{dbl}}$ is the functor of Definition 2.4.3.1;
- *Vertical Composition of* 1*-Morphisms.* For each composable pair $A \xrightarrow{F} B \xrightarrow{G} C$ of vertical morphisms of $\mathsf{Rel}^{\mathsf{dbl}}$, i.e. maps of sets, we have

$$g \circ^{\mathsf{Rel}^{\mathsf{dbl}}} f \stackrel{\text{def}}{=} g \circ f;$$

- *Vertical Composition of* 2*-Morphisms.* The vertical composition of 2-morphisms in $\mathsf{Rel}^{\mathsf{dbl}}$ is defined as in Definition 2.4.4.1;
- *Associators.* The associators of $\mathsf{Rel}^{\mathsf{dbl}}$ is defined as in Definition 2.4.5.1;
- *Left Unitors.* The left unitors of $\mathsf{Rel}^{\mathsf{dbl}}$ is defined as in Definition 2.4.6.1;
- *Right Unitors.* The right unitors of $\mathsf{Rel}^{\mathsf{dbl}}$ is defined as in Definition 2.4.7.1.

2.4.2 Horizontal Identities

Definition 2.4.2.1. The **horizontal unit functor** of $\mathsf{Rel}^{\mathsf{dbl}}$ is the functor

$$\mathbb{1}^{\mathsf{Rel}^{\mathsf{dbl}}} \colon \mathsf{Rel}^{\mathsf{dbl}}_0 \longrightarrow \mathsf{Rel}^{\mathsf{dbl}}_1$$

of $\mathsf{Rel}^{\mathsf{dbl}}$ is the functor where

- *Action on Objects.* For each $A \in \mathrm{Obj}\left(\mathsf{Rel}^{\mathsf{dbl}}_0\right)$, we have

$$\mathbb{1}_A \stackrel{\text{def}}{=} \chi_A(-_1, -_2);$$

- *Action on Morphisms.* For each vertical morphism $f \colon A \longrightarrow B$ of $\mathsf{Rel}^{\mathsf{dbl}}$, i.e. each map of sets f from A to B, the identity 2-morphism

$$\begin{tikzcd} A \arrow[r, "\mathbb{1}_A", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow \mathbb{1}_f"] & A \arrow[d, "f"] \\ B \arrow[r, "\mathbb{1}_B"', "|" marking] & B \end{tikzcd}$$

of f is the inclusion

$$\chi_B \circ (f \times f) \subset \chi_A, \qquad \begin{tikzcd} A \times A \arrow[r, "\chi_A(-_1,-_2)"] \arrow[d, "f\times f"'] \arrow[dr, phantom, "\subset"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ B \times B \arrow[r, "\chi_B(-_1,-_2)"'] & \{\mathsf{true}, \mathsf{false}\} \end{tikzcd}$$

of Constructions With Sets, Proposition 4.1.1.3.

2.4.3 Horizontal Composition

Definition 2.4.3.1. The **horizontal composition functor** of $\mathsf{Rel}^{\mathsf{dbl}}$ is the functor

$$\odot^{\mathsf{Rel}^{\mathsf{dbl}}} \colon \mathsf{Rel}^{\mathsf{dbl}}_1 \underset{\mathsf{Rel}^{\mathsf{dbl}}_0}{\times} \mathsf{Rel}^{\mathsf{dbl}}_1 \longrightarrow \mathsf{Rel}^{\mathsf{dbl}}_1$$

of $\mathsf{Rel}^{\mathsf{dbl}}$ is the functor where

- *Action on Objects.* For each composable pair $A \overset{R}{\nrightarrow} B \overset{S}{\nrightarrow} C$ of horizontal morphisms of $\mathsf{Rel}^{\mathsf{dbl}}$, we have
$$S \odot R \overset{\text{def}}{=} S \diamond R,$$
where $S \diamond R$ is the composition of R and S of Definition 3.12.1.1;
- *Action on Morphisms.* For each horizontally composable pair
$$\begin{tikzcd} A \arrow[r, "R", "|"] \arrow[d, "f"'] \arrow[dr, phantom, "\overset{\alpha}{\Downarrow}"] & B \arrow[d, "g"] \\ X \arrow[r, "T"', "|"] & Y \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "S", "|"] \arrow[d, "g"'] \arrow[dr, phantom, "\overset{\beta}{\Downarrow}"] & C \arrow[d, "h"] \\ Y \arrow[r, "U"', "|"] & Z \end{tikzcd}$$
of 2-morphisms of $\mathsf{Rel}^{\mathsf{dbl}}$, i.e. for each pair
$$\begin{tikzcd} A \times B \arrow[r, "R"] \arrow[d, "f\times g"'] \arrow[dr, phantom, "\subset"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ X \times Y \arrow[r, "T"'] & \{\mathsf{true}, \mathsf{false}\} \end{tikzcd} \qquad \begin{tikzcd} B \times C \arrow[r, "S"] \arrow[d, "g\times h"'] \arrow[dr, phantom, "\subset"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ Y \times Z \arrow[r, "U"'] & \{\mathsf{true}, \mathsf{false}\} \end{tikzcd}$$

of inclusions of relations, the horizontal composition

$$\begin{tikzcd} A \arrow[r, "S\odot R", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow \beta\odot\alpha"] & C \arrow[d, "h"] \\ X \arrow[r, "U\odot T"', "\shortmid"{marking}] & Z \end{tikzcd}$$

of α and β is the inclusion of relations[12]

$$(U \diamond T) \circ (f \times h) \subset (S \diamond R) \qquad \begin{tikzcd} A\times C \arrow[r, "S\diamond R"] \arrow[d, "f\times h"'] \arrow[dr, phantom, "\subset"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ X\times Z \arrow[r, "U\diamond T"'] & \{\mathsf{true}, \mathsf{false}\}. \end{tikzcd}$$

2.4.4 Vertical Composition of 2-Morphisms

Definition 2.4.4.1. The **vertical composition** in $\mathsf{Rel}^{\mathsf{dbl}}$ is defined as follows: for each vertically composable pair

$$\begin{tikzcd} A \arrow[r, "R", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow \alpha"] & X \arrow[d, "g"] \\ B \arrow[r, "S"', "\shortmid"{marking}] & Y \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "S", "\shortmid"{marking}] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow \beta"] & Y \arrow[d, "k"] \\ C \arrow[r, "T"', "\shortmid"{marking}] & Z \end{tikzcd}$$

[12] This is justified by noting that, given $(a, c) \in A \times C$, the statement

- We have $a \sim_{(U\diamond T)\circ(f\times h)} c$, i.e. $f(a) \sim_{U\diamond T} h(c)$, i.e. there exists some $y \in Y$ such that:
  1. We have $f(a) \sim_T y$;
  2. We have $y \sim_U h(c)$;

is implied by the statement

- We have $a \sim_{S\diamond R} c$, i.e. there exists some $b \in B$ such that:
  1. We have $a \sim_R b$;
  2. We have $b \sim_S c$;

since:

- If $a \sim_R b$, then $f(a) \sim_T g(b)$, as $T \circ (f \times g) \subset R$;
- If $b \sim_S c$, then $g(b) \sim_U h(c)$, as $U \circ (g \times h) \subset S$;

of 2-morphisms of $\mathsf{Rel}^{\mathsf{dbl}}$, i.e. for each each pair

$$
\begin{tikzcd}
A \times X \arrow[r, "R"] \arrow[d, "f \times g"'] \arrow[dr, phantom, "\subset"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\
B \times Y \arrow[r, "S"'] & \{\mathsf{true}, \mathsf{false}\}
\end{tikzcd}
\qquad
\begin{tikzcd}
B \times Y \arrow[r, "S"] \arrow[d, "h \times k"'] \arrow[dr, phantom, "\subset"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\
C \times Z \arrow[r, "T"'] & \{\mathsf{true}, \mathsf{false}\}
\end{tikzcd}
$$

of inclusions of relations, we define the vertical composition

$$
\begin{tikzcd}
A \arrow[r, "R", "\shortmid"{marking}] \arrow[d, "h \circ f"'] \arrow[dr, phantom, "\overset{\parallel}{\underset{\Downarrow}{\beta \circ \alpha}}"] & X \arrow[d, "k \circ g"] \\
C \arrow[r, "T"', "\shortmid"{marking}] & Z
\end{tikzcd}
$$

of α and β as the inclusion of relations

$$
T \circ [(h \circ f) \times (k \circ g)] \subset R,
\qquad
\begin{tikzcd}
A \times X \arrow[r, "R"] \arrow[d, "(h \circ f) \times (k \circ g)"'] \arrow[dr, phantom, "\subset"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\
C \times Z \arrow[r, "T"'] & \{\mathsf{true}, \mathsf{false}\}
\end{tikzcd}
$$

given by the pasting of inclusions[13]

$$
\begin{tikzcd}
A \times X \arrow[r, "R"] \arrow[d, "f \times g"'] \arrow[dr, phantom, "\subset"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\
B \times Y \arrow[r, "S" description] \arrow[d, "h \times k"'] \arrow[dr, phantom, "\subset"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\
C \times Z \arrow[r, "T"'] & \{\mathsf{true}, \mathsf{false}\}.
\end{tikzcd}
$$

[13] This is justified by noting that, given $(a, x) \in A \times X$, the statement

- We have $h(f(a)) \sim_T k(g(x))$;

is implied by the statement

- We have $a \sim_R x$;

since

2.4.5 The Associators

Definition 2.4.5.1. For each composable triple $A \overset{R}{\nrightarrow} B \overset{S}{\nrightarrow} C \overset{T}{\nrightarrow} D$ of horizontal morphisms of $\mathsf{Rel}^{\mathsf{dbl}}$, the component

$$\alpha^{\mathsf{Rel}^{\mathsf{dbl}}}_{T,S,R} \colon (T \odot S) \odot R \overset{\cong}{\Longrightarrow} T \odot (S \odot R), \qquad \begin{tikzcd} A \arrow[r, "R", "\shortmid"{marking}] \arrow[d, "\mathrm{id}_A"'] & B \arrow[r, "S", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow \alpha^{\mathsf{Rel}^{\mathsf{dbl}}}_{T,S,R}"] & C \arrow[r, "T", "\shortmid"{marking}] & D \arrow[d, "\mathrm{id}_D"] \\ A \arrow[r, "R"', "\shortmid"{marking}] & B \arrow[r, "S"', "\shortmid"{marking}] & C \arrow[r, "T"', "\shortmid"{marking}] & D \end{tikzcd}$$

of the associator of $\mathsf{Rel}^{\mathsf{dbl}}$ at (R, S, T) is the identity inclusion[14]

$$(T \diamond S) \diamond R = T \diamond (S \diamond R) \qquad \begin{tikzcd} A \times B \arrow[r, "(T \diamond S) \diamond R"] \arrow[d, equal] \arrow[dr, phantom, "\rotatebox{45}{$=$}"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ A \times B \arrow[r, "T \diamond (S \diamond R)"'] & \{\mathsf{true}, \mathsf{false}\}. \end{tikzcd}$$

2.4.6 The Left Unitors

Definition 2.4.6.1. For each horizontal morphism $R \colon A \nrightarrow B$ of $\mathsf{Rel}^{\mathsf{dbl}}$, the component

$$\lambda^{\mathsf{Rel}^{\mathsf{dbl}}}_{R} \colon \nVdash_B \odot R \overset{\cong}{\Longrightarrow} R, \qquad \begin{tikzcd} A \arrow[r, "R", "\shortmid"{marking}] \arrow[d, "\mathrm{id}_A"'] & B \arrow[r, "\nVdash_B", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow \lambda^{\mathsf{Rel}^{\mathsf{dbl}}}_{R}"] & B \arrow[d, "\mathrm{id}_B"] \\ A \arrow[rr, "R"', "\shortmid"{marking}] & & B \end{tikzcd}$$

of the left unitor of $\mathsf{Rel}^{\mathsf{dbl}}$ at R is the identity inclusion[15]

$$R = \chi_B \diamond R, \qquad \begin{tikzcd} A \times B \arrow[r, "\chi_B \diamond R"] \arrow[d, equal] \arrow[dr, phantom, "\rotatebox{45}{$=$}"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ A \times B \arrow[r, "R"'] & \{\mathsf{true}, \mathsf{false}\}. \end{tikzcd}$$

- If $a \sim_R x$, then $f(a) \sim_S g(x)$, as $S \circ (f \times g) \subset R$;
- If $b \sim_S y$, then $h(b) \sim_T k(y)$, as $T \circ (h \times k) \subset S$, and thus, in particular:
  - If $f(a) \sim_S g(x)$, then $h(f(a)) \sim_T k(g(x))$;

[14] This is justified by Item 2 of Proposition 3.12.1.3.
[15] This is justified by Item 3 of Proposition 3.12.1.3.

2.4.7 The Right Unitors

Definition 2.4.7.1. For each horizontal morphism $R\colon A \nrightarrow B$ of $\mathsf{Rel}^{\mathsf{dbl}}$, the component

$$\rho^{\mathsf{Rel}^{\mathsf{dbl}}}_R : R \odot \chi_A \overset{\cong}{\Longrightarrow} R, \qquad \begin{tikzcd} A \arrow[r, "\chi_A", "|" marking] \arrow[d, "\mathrm{id}_A"'] & A \arrow[r, "R", "|" marking] \arrow[d, phantom, "\Downarrow \rho^{\mathsf{Rel}^{\mathsf{dbl}}}_R"] & B \arrow[d, "\mathrm{id}_B"] \\ A \arrow[rr, "R"', "|" marking] & & B \end{tikzcd}$$

of the right unitor of $\mathsf{Rel}^{\mathsf{dbl}}$ at R is the identity inclusion[16]

$$R = R \diamond \chi_A, \qquad \begin{tikzcd} A \times B \arrow[r, "R \diamond \chi_A"] \arrow[d, equal] \arrow[dr, phantom, "\sslash"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ A \times B \arrow[r, "R"'] & \{\mathsf{true}, \mathsf{false}\}. \end{tikzcd}$$

2.5 Properties of the Category of Relations

Proposition 2.5.1.1. Let A and B be sets.

1. *Self-Duality I.* The category Rel is self-dual, i.e. we have an equivalence

   $$\mathrm{Rel}^{\mathsf{op}} \overset{\text{eq.}}{\cong} \mathrm{Rel}$$

   of categories.

2. *Self-Duality II.* The bicategory **Rel** is self-dual, i.e. we have a biequivalence

   $$\mathbf{Rel}^{\mathsf{op}} \overset{\text{eq.}}{\cong} \mathbf{Rel}$$

   of bicategories.

3. *Equivalences and Isomorphisms in Rel.* Let $R\colon A \nrightarrow B$ be a relation from A to B. The following conditions are equivalent:

   (a) The relation $R\colon A \nrightarrow B$ is an equivalence in **Rel**, i.e. there exists a relation $R^{-1}\colon B \nrightarrow A$ from B to A together with isomorphisms

   $$R^{-1} \diamond R \cong \chi_A,$$
   $$R \diamond R^{-1} \cong \chi_B.$$

[16] This is justified by Item 3 of Proposition 3.12.1.3.

(b) The relation $R\colon A \nrightarrow B$ is an isomorphism in Rel, i.e. there exists a relation $R^{-1}\colon B \nrightarrow A$ from B to A such that we have

$$\begin{aligned} R^{-1} \diamond R &= \chi_A, \\ R \diamond R^{-1} &= \chi_B. \end{aligned}$$

(c) There exists a bijection $f\colon A \xrightarrow{\cong} B$ with $R = \mathrm{Gr}(f)$.

4. *Adjunctions in* ***Rel****.* We have a natural bijection

$$\left\{\begin{matrix}\text{Adjunctions in } \mathbf{Rel} \\ \text{from } A \text{ to } B\end{matrix}\right\} \cong \left\{\begin{matrix}\text{Functions} \\ \text{from } A \text{ to } B\end{matrix}\right\},$$

with every adjunction in **Rel** being of the form $\mathrm{Gr}(f) \dashv f^{-1}$ for some function f.

5. *Monads in* ***Rel****.* We have a natural bijection[17]

$$\left\{\begin{matrix}\text{Monads in} \\ \mathbf{Rel} \text{ on } A\end{matrix}\right\} \cong \{\text{Preorders on } A\}.$$

6. *Comonads in* ***Rel****.* We have a natural bijection

$$\left\{\begin{matrix}\text{Comonads in} \\ \mathbf{Rel} \text{ on } A\end{matrix}\right\} \cong \{\text{Subsets of } A\}.$$

7. *Characterisations of Monomorphisms in* Rel. Let $R\colon A \nrightarrow B$ be a relation. The following conditions are equivalent:

(a) The relation R is a monomorphism in Rel.

(b) The direct image function

$$R_*\colon \mathcal{P}(A) \to \mathcal{P}(B)$$

associated to R is injective.

(c) The direct image with compact support function

$$R_!\colon \mathcal{P}(A) \to \mathcal{P}(B)$$

associated to R is injective.

[17] See also Section 6 for an extension of this correspondence to "relative monads on **Rel**".

Moreover, if R is a monomorphism, then it satisfies the following condition, and the converse holds if R is total:

($\star$) For each $a, a' \in A$, if there exists some $b \in B$ such that $a \sim_R b$ and $a' \sim_R b$, then $a = a'$.

8. *Epimorphisms in* Rel. Let $R\colon A \nrightarrow B$ be a relation. The following conditions are equivalent:

   (a) The relation R is an epimorphism in Rel.

   (b) The weak inverse image function

   $$R^{-1}\colon \mathcal{P}(B) \to \mathcal{P}(A)$$

   associated to R is injective.

   (c) The strong inverse image function

   $$R_{-1}\colon \mathcal{P}(B) \to \mathcal{P}(A)$$

   associated to R is injective.

   (d) The function $R\colon A \to \mathcal{P}(B)$ is "surjective on singletons":

   ($\star$) For each $b \in B$, there exists some $a \in A$ such that $R(a) = \{b\}$.

9. *As a Kleisli Category.* We have an isomorphism of categories

   $$\mathrm{Rel} \cong \mathsf{FreeAlg}_{\mathcal{P}},$$

   where $\mathcal{P}$ is the powerset monad of Monads, **??**.

10. *Co/Completeness (Or Lack Thereof).* The category Rel is not co/complete, but admits some co/limits:

    (a) *Zero Objects.* The category Rel has a zero object, the empty set $\emptyset$.

    (b) *Co/Products.* The category Rel has co/products, both given by disjoint union of sets.

    (c) *Lack of Co/Equalisers.* The category Rel does not have co/equalisers.

    (d) *Limits of Graphs of Functions.* The category Rel has limits whose arrows are all graphs of functions.

    (e) *Colimits of Graphs of Functions.* The category Rel has colimits whose arrows are all graphs of functions, and these agree with the corresponding limits in Sets.

11. *Existence of Right Kan Extensions.* The right Kan extension

$$\mathrm{Ran}_R\colon \mathrm{Rel}(A,X) \to \mathrm{Rel}(B,X)$$

along a relation $R\colon A \nrightarrow B$ exists and is given by

$$\mathrm{Ran}_R(S) \stackrel{\text{def}}{=} \int_{a\in A} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^a_{-_1}, S^a_{-_2}\right)$$

for each $S \in \mathrm{Rel}(A,X)$, so that the following conditions are equivalent:

   (a) We have $b \sim_{\mathrm{Ran}_R(S)} x$.
   (b) For each $a \in A$, if $a \sim_R b$, then $a \sim_S x$.

12. *Existence of Right Kan Lifts.* The right Kan lift

$$\mathrm{Rift}_R\colon \mathrm{Rel}(X,B) \to \mathrm{Rel}(X,A)$$

along a relation $R\colon A \nrightarrow B$ exists and is given by

$$\mathrm{Rift}_R(S) \stackrel{\text{def}}{=} \int_{b\in B} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^{-_2}_b, S^{-_1}_b\right)$$

for each $S \in \mathrm{Rel}(X,B)$, so that the following conditions are equivalent:

   (a) We have $x \sim_{\mathrm{Rift}_R(S)} a$.
   (b) For each $b \in B$, if $a \sim_R b$, then $x \sim_S b$.

13. *Closedness.* The bicategory **Rel** is a closed bicategory, there being, for each $R\colon A \nrightarrow B$ and set X, a pair of adjunctions

$$(R^* \dashv \mathrm{Ran}_R)\colon \quad \begin{tikzcd} \mathrm{Rel}(B,X) \arrow[r, bend left, "R^*"] \arrow[r, phantom, "\perp"] & \mathrm{Rel}(A,X), \arrow[l, bend left, "\mathrm{Ran}_R"] \end{tikzcd}$$

$$(R_* \dashv \mathrm{Rift}_R)\colon \quad \begin{tikzcd} \mathrm{Rel}(X,A) \arrow[r, bend left, "R_*"] \arrow[r, phantom, "\perp"] & \mathrm{Rel}(X,B), \arrow[l, bend left, "\mathrm{Rift}_R"] \end{tikzcd}$$

witnessed by bijections

$$\mathbf{Rel}(S \diamond R, T) \cong \mathbf{Rel}(S, \mathrm{Ran}_R(T)),$$
$$\mathbf{Rel}(R \diamond U, V) \cong \mathbf{Rel}(U, \mathrm{Rift}_R(V)),$$

natural in $S \in \mathrm{Rel}(B,X)$, $T \in \mathrm{Rel}(A,X)$, $U \in \mathrm{Rel}(X,A)$, and $V \in \mathrm{Rel}(X,B)$.

Proof. *Item 1, Self-Duality I*: Omitted.
Item 2, Self-Duality II: Omitted.
Item 3, Equivalences and Isomorphisms in Rel: We claim that Items 3a to 3c are indeed equivalent:

- *Item 3a* $\iff$ *Item 3b:* This follows from the fact that **Rel** is locally posetal, so that natural isomorphisms and equalities of 1-cells in **Rel** coincide.

- *Item 3b* $\implies$ *Item 3c:* The equalities in Item 3b imply $R \dashv R^{-1}$, and thus by Item 4, there exists a function $f_R \colon A \to B$ associated to R, where, for each $a \in A$, the image $f_R(a)$ of a by f_R is the unique element of $R(a)$, which implies $R = \mathrm{Gr}(f_R)$ in particular. Furthermore, we have $R^{-1} = f_R^{-1}$ (as in Definition 3.2.1.1). The conditions from Item 3b then become the following:

$$\begin{aligned} f_R^{-1} \diamond f_R &= \chi_A, \\ f_R \diamond f_R^{-1} &= \chi_B. \end{aligned}$$

  All that is left is to show then is that f_R is a bijection:

  - *The Function f_R Is Injective.* Let $a, b \in A$ and suppose that $f_R(a) = f_R(b)$. Since $a \sim_R f_R(a)$ and $f_R(a) = f_R(b) \sim_{R^{-1}} b$, the condition $f_R^{-1} \diamond f_R = \chi_A$ implies that $a = b$, showing f_R to be injective.
  - *The Function f_R Is Surjective.* Let $b \in B$. Applying the condition $f_R \diamond f_R^{-1} = \chi_B$ to (b, b), it follows that there exists some $a \in A$ such that $f_R^{-1}(b) = a$ and $f_R(a) = b$. This shows f_R to be surjective.

- *Item 3c* $\implies$ *Item 3b*: By Item 2, we have an adjunction $\mathrm{Gr}(f) \dashv f^{-1}$, giving inclusions

$$\begin{aligned} \chi_A &\subset f^{-1} \diamond \mathrm{Gr}(f), \\ \mathrm{Gr}(f) \diamond f^{-1} &\subset \chi_B. \end{aligned}$$

  We claim the reverse inclusions are also true:

  - $f^{-1} \diamond Gr(f) \subset \chi_A$: This is equivalent to the statement that if $f(a) = b$ and $f^{-1}(b) = a'$, then $a = a'$, which follows from the injectivity of f.
  - $\chi_B \subset Gr(f) \diamond f^{-1}$: This is equivalent to the statement that given $b \in B$ there exists some $a \in A$ such that $f^{-1}(b) = a$ and $f(a) = b$, which follows from the surjectivity of f.

Item 4, Adjunctions in ***Rel***: We proceed step by step:

1. *From Adjunctions in* **Rel** *to Functions.* An adjunction in **Rel** from A to B consists of a pair of relations

$$R\colon A \nrightarrow B,$$
$$S\colon B \nrightarrow A,$$

together with inclusions

$$\chi_A \subset S \diamond R,$$
$$R \diamond S \subset \chi_B.$$

We claim that these conditions imply that R is total and functional, i.e. that $R(a)$ is a singleton for each $a \in A$:

   (a) *$R(a)$ Has an Element.* Given $a \in A$, since $\chi_A \subset S \diamond R$, we must have $\{a\} \subset S(R(a))$, implying that there exists some $b \in B$ such that $a \sim_R b$ and $b \sim_S a$, and thus $R(a) \neq \emptyset$, as $b \in R(a)$.

   (b) *$R(a)$ Has No More Than One Element.* Suppose that we have $a \sim_R b$ and $a \sim_R b'$ for $b, b' \in B$. We claim that $b = b'$:

      i. Since $\chi_A \subset S \diamond R$, there exists some $k \in B$ such that $a \sim_R k$ and $k \sim_S a$.
      ii. Since $R \diamond S \subset \chi_B$, if $b'' \sim_S a'$ and $a' \sim_R b'''$, then $b'' = b'''$.
      iii. Applying the above to $b'' = k$, $b''' = b$, and $a' = a$, since $k \sim_S a$ and $a \sim_R b'$, we have $k = b$.
      iv. Similarly $k = b'$.
      v. Thus $b = b'$.

Together, the above two items show $R(a)$ to be a singleton, being thus given by $\mathrm{Gr}(f)$ for some function $f\colon A \to B$, which gives a map

$$\begin{Bmatrix} \text{Adjunctions in } \mathbf{Rel} \\ \text{from } A \text{ to } B \end{Bmatrix} \to \begin{Bmatrix} \text{Functions} \\ \text{from } A \text{ to } B \end{Bmatrix}.$$

Moreover, by uniqueness of adjoints (Internal Adjunctions, Item 2 of Proposition 1.2.1.4), this implies also that $S = f^{-1}$.

2. *From Functions to Adjunctions in* **Rel**. By Item 2 of Proposition 3.1.1.2, every function $f\colon A \to B$ gives rise to an adjunction $\mathrm{Gr}(f) \dashv f^{-1}$ in Rel, giving a map

$$\begin{Bmatrix} \text{Functions} \\ \text{from } A \text{ to } B \end{Bmatrix} \to \begin{Bmatrix} \text{Adjunctions in } \mathbf{Rel} \\ \text{from } A \text{ to } B \end{Bmatrix}.$$

3. *Invertibility: From Functions to Adjunctions Back to Functions.* We need to show that starting with a function $f\colon A \longrightarrow B$, passing to $\mathrm{Gr}(f) \dashv f^{-1}$, and then passing again to a function gives f again. This is clear however, since we have $a \sim_{\mathrm{Gr}(f)} b$ iff $f(a) = b$.

4. *Invertibility: From Adjunctions to Functions Back to Adjunctions.* We need to show that, given an adjunction $R \dashv S$ in **Rel** giving rise to a function $f_{R,S}\colon A \longrightarrow B$, we have

$$\begin{aligned} \mathrm{Gr}\big(f_{R,S}\big) &= R, \\ f^{-1}_{R,S} &= S. \end{aligned}$$

We check these explicitly:

- *$Gr\big(f_{R,S}\big) = R$.* We have

$$\begin{aligned} \mathrm{Gr}\big(f_{R,S}\big) &\stackrel{\text{def}}{=} \big\{\big(a, f_{R,S}(a)\big) \in A \times B \,\big|\, a \in A\big\} \\ &\stackrel{\text{def}}{=} \{(a, R(a)) \in A \times B \mid a \in A\} \\ &= R. \end{aligned}$$

- *$f^{-1}_{R,S} = S$.* We first claim that, given $a \in A$ and $b \in B$, the following conditions are equivalent:
  - We have $a \sim_R b$.
  - We have $b \sim_S a$.

  Indeed:
  - *If $a \sim_R b$, then $b \sim_S a$:* Since $\chi_A \subset S \diamond R$, there exists $k \in B$ such that $a \sim_R k$ and $k \sim_S a$, but since $a \sim_R b$ and R is functional, we have $k = b$ and thus $b \sim_S a$.
  - *If $b \sim_S a$, then $a \sim_R b$:* First note that since R is total we have $a \sim_R b'$ for some $b' \in B$. Now, since $R \diamond S \subset \chi_B$, $b \sim_S a$, and $a \sim_R b'$, we have $b = b'$, and thus $a \sim_R b$.

  Having show this, we now have

$$\begin{aligned} f^{-1}_{R,S}(b) &\stackrel{\text{def}}{=} \big\{a \in A \,\big|\, f_{R,S}(a) = b\big\} \\ &\stackrel{\text{def}}{=} \{a \in A \mid a \sim_R b\} \\ &= \{a \in A \mid b \sim_S a\} \\ &\stackrel{\text{def}}{=} S(b). \end{aligned}$$

  for each $b \in B$, showing $f^{-1}_{R,S} = S$.

This finishes the proof.
*Item 5, Monads in **Rel***: A monad in **Rel** on A consists of a relation $R\colon A \nrightarrow A$ together with maps

$$\mu_R\colon R \diamond R \subset R,$$
$$\eta_R\colon \chi_A \subset R$$

making the diagrams

$$\begin{tikzcd}
\chi_A \diamond R \arrow[r, "\eta_R \diamond \mathrm{id}_R", "\shortmid"{marking}] \arrow[dr, equal, "\lambda_R^{\mathbf{Rel}(A,B)}"'] & R \diamond R \arrow[d, "\mu_R", "\shortmid"{marking}] \\
& R
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=tiny]
& R \diamond (R \diamond R) \arrow[dl, equal, "\alpha^{\mathbf{Rel}(A,B)}_{R,R,R}"'] \arrow[dr, "\mathrm{id}_R \diamond \mu_R", "\shortmid"{marking}] & \\
(R \diamond R) \diamond R \arrow[d, "\mu_R \diamond \mathrm{id}_R"', "\shortmid"{marking}] & & R \diamond R \arrow[d, "\mu_R", "\shortmid"{marking}] \\
R \diamond R \arrow[rr, "\mu_R"', "\shortmid"{marking}] & & R
\end{tikzcd}
\qquad
\begin{tikzcd}
R \diamond \chi_A \arrow[r, "\mathrm{id}_R \diamond \eta_R", "\shortmid"{marking}] \arrow[dr, equal, "\rho_R^{\mathbf{Rel}(A,B)}"'] & R \diamond R \arrow[d, "\mu_R", "\shortmid"{marking}] \\
& R
\end{tikzcd}$$

commute. However, since all morphisms involved are inclusions, the commutativity of the above diagrams is automatic, and hence all that is left is the data of the two maps μ_R and η_R, which correspond respectively to the following conditions:

1. For each $a, b, c \in A$, if $a \sim_R b$ and $b \sim_R c$, then $a \sim_R c$.
2. For each $a \in A$, we have $a \sim_R a$.

These are exactly the requirements for R to be a preorder (Posets, **??**). Conversely any preorder $\leq$ gives rise to a pair of maps $\mu_\leq$ and $\eta_\leq$, forming a monad on A.
*Item 6, Comonads in **Rel***: A comonad in **Rel** on A consists of a relation $R\colon A \nrightarrow A$ together with maps

$$\Delta_R\colon R \subset R \diamond R,$$
$$\epsilon_R\colon R \subset \chi_A$$

making the diagrams

$$\begin{tikzcd}
R \arrow[r, "\Delta_R", "\shortmid"{marking}] \arrow[dr, equal, "\lambda_R^{\mathbf{Rel}(A,B),-1}"'] & R \diamond R \arrow[d, "\epsilon_R \diamond \mathrm{id}_R", "\shortmid"{marking}] \\
& \chi_A \diamond R
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=tiny]
& R \diamond R \arrow[dr, "\mathrm{id}_R \diamond \Delta_R", "\shortmid"{marking}] & \\
R \arrow[ur, "\Delta_R", "\shortmid"{marking}] \arrow[d, "\Delta_R"', "\shortmid"{marking}] & & R \diamond (R \diamond R) \arrow[d, equal, "\alpha^{\mathbf{Rel}(A,B),-1}_{R,R,R}"'] \\
R \diamond R \arrow[rr, "\Delta_R \diamond \mathrm{id}_R"', "\shortmid"{marking}] & & (R \diamond R) \diamond R
\end{tikzcd}
\qquad
\begin{tikzcd}
R \arrow[r, "\Delta_R", "\shortmid"{marking}] \arrow[dr, equal, "\rho_R^{\mathbf{Rel}(A,B),-1}"'] & R \diamond R \arrow[d, "\mathrm{id}_R \diamond \epsilon_R", "\shortmid"{marking}] \\
& R \diamond \chi_A
\end{tikzcd}$$

commute. However, since all morphisms involved are inclusions, the commutativity of the above diagrams is automatic, and hence all that is left is the data of the two maps Δ_R and ϵ_R, which correspond respectively to the following conditions:

1. For each $a, b \in A$, if $a \sim_R b$, then there exists some $k \in A$ such that $a \sim_R k$ and $k \sim_R b$.
2. For each $a, b \in A$, if $a \sim_R b$, then $a = b$.

Taking $k = b$ in the first condition above shows it to be trivially satisfied, while the second condition implies $R \subset \Delta_A$, i.e. R must be a subset of A. Conversely, any subset U of A satisfies $U \subset \Delta_A$, defining a comonad as above.
Item 7, Monomorphisms in Rel: Firstly note that Items 7b and 7c are equivalent by Item 7 of Proposition 5.1.1.3. We then claim that Items 7a and 7b are also equivalent:

- *Item 7a* $\Longrightarrow$ *Item 7b:* Let $U, V \in \mathcal{P}(A)$ and consider the diagram

  $$\mathrm{pt} \underset{V}{\overset{U}{\rightrightarrows}} A \xrightarrow{R} B.$$

  By Remark 5.1.1.2, we have

  $$\begin{aligned} R_*(U) &= R \diamond U, \\ R_*(V) &= R \diamond V. \end{aligned}$$

  Now, if $R \diamond U = R \diamond V$, i.e. $R_*(U) = R_*(V)$, then $U = V$ since R is assumed to be a monomorphism, showing R_* to be injective.

- *Item 7b* $\Longrightarrow$ *Item 7a:* Conversely, suppose that R_* is injective, consider the diagram

  $$K \underset{T}{\overset{S}{\rightrightarrows}} A \xrightarrow{R} B,$$

  and suppose that $R \diamond S = R \diamond T$. Note that, since R_* is injective, given a diagram of the form

  $$\mathrm{pt} \underset{V}{\overset{U}{\rightrightarrows}} A \xrightarrow{R} B,$$

  if $R_*(U) = R \diamond U = R \diamond V = R_*(V)$, then $U = V$. In particular, for each $k \in K$, we may consider the diagram

  $$\mathrm{pt} \xrightarrow{[k]} K \underset{T}{\overset{S}{\rightrightarrows}} A \xrightarrow{R} B,$$

for which we have $R \diamond S \diamond [k] = R \diamond T \diamond [k]$, implying that we have

$$S(k) = S \diamond [k] = T \diamond [k] = T(k)$$

for each $k \in K$, implying $S = T$, and thus R is a monomorphism.

We can also prove this in a more abstract way, following [MSE 350788]:

- *Item 7a* $\implies$ *Item 7b:* Assume that R is a monomorphism.
  - We first notice that the functor $\mathrm{Rel}(\mathrm{pt}, -) \colon \mathrm{Rel} \to \mathsf{Sets}$ maps R to R_* by Remark 5.1.1.2.
  - Since $\mathrm{Rel}(\mathrm{pt}, -)$ preserves all limits by Limits and Colimits, **??** of **??**, it follows by Categories, **??** of **??** that $\mathrm{Rel}(\mathrm{pt}, -)$ also preserves monomorphisms.
  - Since R is a monomorphism and $\mathrm{Rel}(\mathrm{pt}, -)$ maps R to R_*, it follows that R_* is also a monomorphism.
  - Since the monomorphisms in Sets are precisely the injections (Categories, **??** of **??**), it follows that R_* is injective.
- *Item 7b* $\implies$ *Item 7a:* Assume that R_* is injective.
  - We first notice that the functor $\mathrm{Rel}(\mathrm{pt}, -) \colon \mathrm{Rel} \to \mathsf{Sets}$ maps R to R_* by Remark 5.1.1.2.
  - Since the monomorphisms in Sets are precisely the injections (Categories, **??** of **??**), it follows that R_* is a monomorphism.
  - Since $\mathrm{Rel}(\mathrm{pt}, -)$ is faithful, it follows by Categories, **??** of **??** that $\mathrm{Rel}(\mathrm{pt}, -)$ reflects monomorphisms.
  - Since R_* is a monomorphism and $\mathrm{Rel}(\mathrm{pt}, -)$ maps R to R_*, it follows that R is also a monomorphism.

Finally, we prove the second part of the statement. Assume that R is a monomorphism, let $a, a' \in A$ such that $a \sim_R b$ and $a' \sim_R b$ for some $b \in B$, and consider the diagram

$$\begin{tikzcd} \mathrm{pt} \arrow[r, shift left, "{[a]}", "\mapstochar"'] \arrow[r, shift right, "{[a']}"', "\mapstochar"] & A \arrow[r, "R", "\mapstochar"'] & B. \end{tikzcd}$$

Since $\star \sim_{[a]} a$ and $a \sim_R b$, we have $\star \sim_{R \diamond [a]} b$. Similarly, $\star \sim_{R \diamond [a']} b$. Thus $R \diamond [a] = R \diamond [a']$, and since R is a monomorphism, we have $[a] = [a']$, i.e. $a = a'$.
Conversely, assume the condition

($\star$) For each $a, a' \in A$, if there exists some $b \in B$ such that $a \sim_R b$ and $a' \sim_R b$, then $a = a'$, consider the diagram

$$K \underset{T}{\overset{S}{\rightrightarrows}} A \xrightarrow{R} B,$$

and let $(k, a) \in S$. Since R is total and $a \in A$, there exists some $b \in B$ such that $a \sim_R b$. In this case, we have $k \sim_{R \diamond S} b$, and since $R \diamond S = R \diamond T$, we have also $k \sim_{R \diamond T} b$. Thus there must exist some $a' \in A$ such that $k \sim_T a'$ and $a' \sim_R b$. However, since $a, a' \sim_R b$, we must have $a = a'$, and thus $(k, a) \in T$ as well.
A similar argument shows that if $(k, a) \in T$, then $(k, a) \in S$, and thus $S = T$ and R is a monomorphism.
Item 8, Epimorphisms in Rel: Firstly note that Items 8b and 8c are equivalent by Item 7 of Proposition 5.2.1.3. We then claim that Items 8a and 8b are also equivalent:

- *Item 8a $\implies$ Item 8b:* Let $U, V \in \mathcal{P}(A)$ and consider the diagram

  $$A \xrightarrow{R} B \underset{V}{\overset{U}{\rightrightarrows}} \mathrm{pt}.$$

  By Remark 5.1.1.2, we have

  $$\begin{aligned} R^{-1}(U) &= U \diamond R, \\ R^{-1}(V) &= V \diamond R. \end{aligned}$$

  Now, if $U \diamond R = V \diamond R$, i.e. $R^{-1}(U) = R^{-1}(V)$, then $U = V$ since R is assumed to be an epimorphism, showing R^{-1} to be injective.

- *Item 8b $\implies$ Item 8a:* Conversely, suppose that R^{-1} is injective, consider the diagram

  $$A \xrightarrow{R} B \underset{T}{\overset{S}{\rightrightarrows}} K,$$

  and suppose that $S \diamond R = T \diamond R$. Note that, since R^{-1} is injective, given a diagram of the form

  $$A \xrightarrow{R} B \underset{V}{\overset{U}{\rightrightarrows}} \mathrm{pt},$$

  if $R^{-1}(U) = U \diamond R = V \diamond R = R^{-1}(V)$, then $U = V$. In particular, for each $k \in K$, we may consider the diagram

  $$A \xrightarrow{R} B \underset{T}{\overset{S}{\rightrightarrows}} K \xrightarrow{[k]} \mathrm{pt},$$

for which we have $[k] \diamond S \diamond R = [k] \diamond T \diamond R$, implying that we have

$$S^{-1}(k) = [k] \diamond S = [k] \diamond T = T^{-1}(k)$$

for each $k \in K$, implying $S = T$, and thus R is an epimorphism.

We can also prove this in a more abstract way, following [MSE 350788]:

- *Item 8a* $\implies$ *Item 8b:* Assume that R is an epimorphism.
  - We first notice that the functor $\mathrm{Rel}(-, \mathrm{pt})\colon \mathrm{Rel}^{\mathrm{op}} \to \mathsf{Sets}$ maps R to R^{-1} by Remark 5.3.1.2.
  - Since $\mathrm{Rel}(-, \mathrm{pt})$ preserves limits by Limits and Colimits, **??** of **??**, it follows by Categories, **??** of **??** that $\mathrm{Rel}(-, \mathrm{pt})$ also preserves monomorphisms.
  - That is: $\mathrm{Rel}(-, \mathrm{pt})$ sends monomorphisms in $\mathrm{Rel}^{\mathrm{op}}$ to monomorphisms in Sets.
  - The monomorphisms $\mathrm{Rel}^{\mathrm{op}}$ are precisely the epimorphisms in Rel by Categories, **??** of **??**.
  - Since R is an epimorphism and $\mathrm{Rel}(-, \mathrm{pt})$ maps R to R^{-1}, it follows that R^{-1} is a monomorphism.
  - Since the monomorphisms in Sets are precisely the injections (Categories, **??** of **??**), it follows that R^{-1} is injective.
- *Item 8b* $\implies$ *Item 8a:* Assume that R^{-1} is injective.
  - We first notice that the functor $\mathrm{Rel}(-, \mathrm{pt})\colon \mathrm{Rel}^{\mathrm{op}} \to \mathsf{Sets}$ maps R to R^{-1} by Remark 5.3.1.2.
  - Since the monomorphisms in Sets are precisely the injections (Categories, **??** of **??**), it follows that R^{-1} is a monomorphism.
  - Since $\mathrm{Rel}(-, \mathrm{pt})$ is faithful, it follows by Categories, **??** of **??** that $\mathrm{Rel}(, \mathrm{pt})$ reflects monomorphisms.
  - That is: $\mathrm{Rel}(-, \mathrm{pt})$ reflects monomorphisms in Sets to monomorphisms in $\mathrm{Rel}^{\mathrm{op}}$.
  - The monomorphisms $\mathrm{Rel}^{\mathrm{op}}$ are precisely the epimorphisms in Rel by Categories, **??** of **??**.
  - Since R^{-1} is a monomorphism and $\mathrm{Rel}(-, \mathrm{pt})$ maps R to R^{-1}, it follows that R is an epimorphism.

Finally, we claim that Items 8b and 8d are also equivalent, following [MO 350788]:

- *Item 8b* $\implies$ *Item 8d*: Since $B \setminus \{b\} \subset B$ and R^{-1} is injective, we have $R^{-1}(B \setminus \{b\}) \subsetneq R^{-1}(B)$. So taking some $a \in R^{-1}(B) \setminus R^{-1}(B \setminus \{b\})$ we get an element of A such that $R(a) = \{b\}$.

- *Item 8d* $\implies$ *Item 8b*: Let $U, V \subset B$ with $U \neq V$. Without loss of generality, we can assume $U \setminus V \neq \emptyset$; otherwise just swap U and V. Let then $b \in U \setminus V$. By assumption, there exists an $a \in A$ with $R(a) = \{b\}$. Then $a \in R^{-1}(U)$ but $a \notin R^{-1}(V)$, and thus $R^{-1}(U) \neq R^{-1}(V)$, showing R^{-1} to be injective.

Item 9, *As a Kleisli Category*: Omitted.
Item 10, *Co/Completeness (Or Lack Thereof)*: Omitted.
Item 11, *Existence of Right Kan Extensions*: We have

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{Rel}(A,X)}(S \diamond R, T) &\cong \int_{a\in A}\int_{x\in X} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left((S \diamond R)^a_x, T^a_x\right) \\
&\cong \int_{a\in A}\int_{x\in X} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(\left(\int^{b\in B} S^b_x \times R^a_b\right), T^a_x\right) \\
&\cong \int_{a\in A}\int_{x\in X}\int_{b\in B} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(S^b_x \times R^a_b, T^a_x\right) \\
&\cong \int_{a\in A}\int_{x\in X}\int_{b\in B} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(S^b_x, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^a_b, T^a_x\right)\right) \\
&\cong \int_{b\in B}\int_{x\in X}\int_{a\in A} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(S^b_x, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^a_b, T^a_x\right)\right) \\
&\cong \int_{b\in B}\int_{x\in X} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(S^b_x, \int_{a\in A} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^a_b, T^a_x\right)\right) \\
&\cong \mathrm{Hom}_{\mathbf{Rel}(B,X)}\left(S, \int_{a\in A} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^a_{-_1}, T^a_{-_2}\right)\right)
\end{aligned}$$

naturally in each $S \in \mathbf{Rel}(B, X)$ and each $T \in \mathbf{Rel}(A, X)$, showing that

$$\int_{a\in A} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^a_{-_1}, T^a_{-_2}\right)$$

is right adjoint to the precomposition functor $- \diamond R$, being thus the right Kan extension along R. Here we have used the following results, respectively (i.e. for each $\cong$ sign):

1. Item 1 of Proposition 1.1.1.4;
2. Definition 3.12.1.1;
3. Diagonal Category Theory, **??** of **??**;
4. Sets, Proposition 1.2.1.4;

5. Diagonal Category Theory, ?? of ??;
6. Diagonal Category Theory, ?? of ??;
7. Item 1 of Proposition 1.1.1.4.

Item 12, Existence of Right Kan Lifts: We have

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{Rel}(X,B)}(R \diamond S, T) &\cong \int_{x\in X}\int_{b\in B} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left((R\diamond S)^x_b, T^x_b\right)\\
&\cong \int_{x\in X}\int_{b\in B} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(\left(\int^{a\in A} R^a_b \times S^x_a\right), T^x_b\right)\\
&\cong \int_{x\in X}\int_{b\in B}\int_{a\in A} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^a_b \times S^x_a, T^x_b\right)\\
&\cong \int_{x\in X}\int_{b\in B}\int_{a\in A} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(S^x_a, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R^a_b, T^x_b)\right)\\
&\cong \int_{x\in X}\int_{a\in A}\int_{b\in B} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(S^x_a, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R^a_b, T^x_b)\right)\\
&\cong \int_{x\in X}\int_{a\in A} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(S^x_a, \int_{b\in B}\mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R^a_b, T^x_b)\right)\\
&\cong \mathrm{Hom}_{\mathbf{Rel}(X,A)}\left(S, \int_{b\in B}\mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^{-_2}_b, T^{-_1}_b\right)\right)
\end{aligned}$$

naturally in each $S \in \mathbf{Rel}(X, A)$ and each $T \in \mathbf{Rel}(X, B)$, showing that

$$\int_{b\in B}\mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^{-_2}_b, T^{-_1}_b\right)$$

is right adjoint to the postcomposition functor $R \diamond -$, being thus the right Kan lift along R. Here we have used the following results, respectively (i.e. for each $\cong$ sign):

1. Item 1 of Proposition 1.1.1.4;
2. Definition 3.12.1.1;
3. Diagonal Category Theory, ?? of ??;
4. Sets, Proposition 1.2.1.4;
5. Diagonal Category Theory, ?? of ??;
6. Diagonal Category Theory, ?? of ??;
7. Item 1 of Proposition 1.1.1.4.

Item 13, Closedness: This has been proved as part of the proof of Items 11 and 12. □

3 Constructions With Relations

3.1 The Graph of a Function

Let $f \colon A \to B$ be a function.

Definition 3.1.1.1. The **graph of** f is the relation $\mathrm{Gr}(f) \colon A \nrightarrow B$ defined as follows:[18]

- Viewing relations from A to B as subsets of $A \times B$, we define

$$\mathrm{Gr}(f) \stackrel{\text{def}}{=} \{(a, f(a)) \in A \times B \mid a \in A\};$$

- Viewing relations from A to B as functions $A \times B \to \{\mathsf{true}, \mathsf{false}\}$, we define

$$[\mathrm{Gr}(f)](a, b) \stackrel{\text{def}}{=} \begin{cases} \mathsf{true} & \text{if } b = f(a), \\ \mathsf{false} & \text{otherwise} \end{cases}$$

  for each $(a, b) \in A \times B$;

- Viewing relations from A to B as functions $A \to \mathcal{P}(B)$, we define

$$[\mathrm{Gr}(f)](a) \stackrel{\text{def}}{=} \{f(a)\}$$

  for each $a \in A$, i.e. we define $\mathrm{Gr}(f)$ as the composition

$$A \xrightarrow{f} B \xrightarrow{\chi_B} \mathcal{P}(B).$$

Proposition 3.1.1.2. Let $f \colon A \to B$ be a function.

1. *Functoriality.* The assignment $A \mapsto \mathrm{Gr}(A)$ defines a functor

$$\mathrm{Gr} \colon \mathsf{Sets} \to \mathrm{Rel}$$

   where

   - *Action on Objects.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$\mathrm{Gr}(A) \stackrel{\text{def}}{=} A;$$

[18] *Further Notation:* We write $\mathrm{Gr}(A)$ for $\mathrm{Gr}(\mathrm{id}_A)$, and call it the **graph** of A.

- *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathsf{Sets})$, the action on Hom-sets

$$\mathrm{Gr}_{A,B} \colon \mathsf{Sets}(A, B) \to \underbrace{\mathrm{Rel}(\mathrm{Gr}(A), \mathrm{Gr}(B))}_{\stackrel{\text{def}}{=}\mathrm{Rel}(A,B)}$$

of Gr at (A, B) is defined by

$$\mathrm{Gr}_{A,B}(f) \stackrel{\text{def}}{=} \mathrm{Gr}(f),$$

where $\mathrm{Gr}(f)$ is the graph of f as in Definition 3.1.1.1.

In particular:

- *Preservation of Identities.* We have

$$\mathrm{Gr}(\mathrm{id}_A) = \chi_A$$

for each $A \in \mathrm{Obj}(\mathsf{Sets})$.

- *Preservation of Composition.* We have

$$\mathrm{Gr}(g \circ f) = \mathrm{Gr}(g) \diamond \mathrm{Gr}(f)$$

for each pair of functions $f \colon A \to B$ and $g \colon B \to C$.

2. *Adjointness Inside* ***Rel****.* We have an adjunction

$$\big(\mathrm{Gr}(f) \dashv f^{-1}\big)\colon \quad \begin{tikzcd} A \arrow[r, bend left, "\mathrm{Gr}(f)"] \arrow[r, phantom, "\perp"] & B \arrow[l, bend left, "f^{-1}"] \end{tikzcd}$$

in **Rel**, where f^{-1} is the inverse of f of Definition 3.2.1.1.

3. *Adjointness.* We have an adjunction

$$(\mathrm{Gr} \dashv \mathcal{P}_*)\colon \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "\mathrm{Gr}"] \arrow[r, phantom, "\perp"] & \mathrm{Rel}, \arrow[l, bend left, "\mathcal{P}_*"] \end{tikzcd}$$

witnessed by a bijection of sets

$$\mathrm{Rel}(\mathrm{Gr}(A), B) \cong \mathsf{Sets}(A, \mathcal{P}(B))$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$ and $B \in \mathrm{Obj}(\mathrm{Rel})$.

4. *Interaction With Inverses.* We have

$$\begin{aligned} \mathrm{Gr}(f)^{\dagger} &= f^{-1}, \\ \left(f^{-1}\right)^{\dagger} &= \mathrm{Gr}(f). \end{aligned}$$

5. *Cocontinuity.* The functor $\mathrm{Gr}\colon \mathsf{Sets} \to \mathrm{Rel}$ of Item 1 preserves colimits.

6. *Characterisations.* Let $R\colon A \nrightarrow B$ be a relation. The following conditions are equivalent:

   (a) There exists a function $f\colon A \to B$ such that $R = \mathrm{Gr}(f)$.

   (b) The relation R is total and functional.

   (c) The weak and strong inverse images of R agree, i.e. we have $R^{-1} = R_{-1}$.

   (d) The relation R has a right adjoint $R^{\dagger}$ in Rel.

Proof. *Item 1*, *Functoriality*: Clear.
Item 2, *Adjointness Inside* ***Rel***: We need to check that there are inclusions

$$\begin{aligned} \chi_A &\subset f^{-1} \diamond \mathrm{Gr}(f), \\ \mathrm{Gr}(f) \diamond f^{-1} &\subset \chi_B. \end{aligned}$$

These correspond respectively to the following conditions:

1. For each $a \in A$, there exists some $b \in B$ such that $a \sim_{\mathrm{Gr}(f)} b$ and $b \sim_{f^{-1}} a$.

2. For each $a, b \in A$, if $a \sim_{\mathrm{Gr}(f)} b$ and $b \sim_{f^{-1}} a$, then $a = b$.

In other words, the first condition states that the image of any $a \in A$ by f is nonempty, whereas the second condition states that f is not multivalued. As f is a function, both of these statements are true, and we are done.
Item 3, *Adjointness*: The stated bijection follows from Remark 1.1.1.3, with naturality being clear.
Item 4, *Interaction With Inverses*: Clear.
Item 5, *Cocontinuity*: Omitted.
Item 6, *Characterisations*: We claim that Items 6a to 6d are indeed equivalent:

- *Item 6a* $\Longleftrightarrow$ *Item 6b*. This is shown in the proof of Item 4 of Proposition 2.5.1.1.

- *Item 6b* $\Longrightarrow$ *Item 6c*. If R is total and functional, then, for each $a \in A$, the set $R(a)$ is a singleton, implying that

$$\begin{aligned} R^{-1}(V) &\stackrel{\text{def}}{=} \{a \in A \mid R(a) \cap V \neq \emptyset\}, \\ R_{-1}(V) &\stackrel{\text{def}}{=} \{a \in A \mid R(a) \subset V\} \end{aligned}$$

are equal for all $V \in \mathcal{P}(B)$, as the conditions $R(a) \cap V \neq \emptyset$ and $R(a) \subset V$ are equivalent when $R(a)$ is a singleton.

- *Item 6c* $\implies$ *Item 6b*. We claim that R is indeed total and functional:
  - *Totality.* If we had $R(a) = \emptyset$ for some $a \in A$, then we would have $a \in R_{-1}(\emptyset)$, so that $R_{-1}(\emptyset) \neq \emptyset$. But since $R^{-1}(\emptyset) = \emptyset$, this would imply $R_{-1}(\emptyset) \neq R^{-1}(\emptyset)$, a contradiction. Thus $R(a) \neq \emptyset$ for all $a \in A$ and R is total.
  - *Functionality.* If $R^{-1} = R_{-1}$, then we have

    $$\begin{aligned}\{a\} &= R^{-1}(\{b\}) \\ &= R_{-1}(\{b\})\end{aligned}$$

    for each $b \in R(a)$ and each $a \in A$, and thus $R(a) \subset \{b\}$. But since R is total, we must have $R(a) = \{b\}$, and thus we see that R is functional.
- *Item 6a* $\iff$ *Item 6d*. This follows from Item 4 of Proposition 2.5.1.1.

This finishes the proof. □

3.2 The Inverse of a Function

Let $f\colon A \to B$ be a function.

Definition 3.2.1.1. The **inverse of** f is the relation $f^{-1}\colon B \nrightarrow A$ defined as follows:

- Viewing relations from B to A as subsets of $B \times A$, we define

  $$f^{-1} \stackrel{\text{def}}{=} \left\{(b, f^{-1}(b)) \in B \times A \,\middle|\, a \in A\right\},$$

  where

  $$f^{-1}(b) \stackrel{\text{def}}{=} \{a \in A \mid f(a) = b\}$$

  for each $b \in B$.
- Viewing relations from B to A as functions $B \times A \to \{\mathsf{true}, \mathsf{false}\}$, we define

  $$f^{-1}(b, a) \stackrel{\text{def}}{=} \begin{cases} \mathsf{true} & \text{if there exists } a \in A \text{ with } f(a) = b, \\ \mathsf{false} & \text{otherwise} \end{cases}$$

  for each $(b, a) \in B \times A$;

- Viewing relations from B to A as functions $B \to \mathcal{P}(A)$, we define

$$f^{-1}(b) \stackrel{\text{def}}{=} \{a \in A \mid f(a) = b\}$$

for each $b \in B$.

Proposition 3.2.1.2. Let $f\colon A \to B$ be a function.

1. *Functoriality.* The assignment $A \mapsto A$, $f \mapsto f^{-1}$ defines a functor

$$(-)^{-1}\colon \mathsf{Sets} \to \mathrm{Rel}$$

where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$\left[(-)^{-1}\right](A) \stackrel{\text{def}}{=} A;$$

- *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathsf{Sets})$, the action on Hom-sets

$$(-)^{-1}_{A,B}\colon \mathsf{Sets}(A,B) \to \mathrm{Rel}(A,B)$$

of $(-)^{-1}$ at (A,B) is defined by

$$(-)^{-1}_{A,B}(f) \stackrel{\text{def}}{=} \left[(-)^{-1}\right](f),$$

where f^{-1} is the inverse of f as in Definition 3.2.1.1.

In particular:

- *Preservation of Identities.* We have

$$\mathrm{id}_A^{-1} = \chi_A$$

for each $A \in \mathrm{Obj}(\mathsf{Sets})$.

- *Preservation of Composition.* We have

$$(g \circ f)^{-1} = g^{-1} \diamond f^{-1}$$

for pair of functions $f\colon A \to B$ and $g\colon B \to C$.

2. *Adjointness Inside* ***Rel****.* We have an adjunction

$$\left(\mathrm{Gr}(f) \dashv f^{-1}\right)\colon \quad \begin{tikzcd} A \arrow[r, bend left, "\mathrm{Gr}(f)", "\shortmid"{marking}] \arrow[r, phantom, "\perp"] & B \arrow[l, bend left, "f^{-1}", "\shortmid"{marking}] \end{tikzcd}$$

in **Rel**.

3. *Interaction With Inverses of Relations.* We have

$$\begin{aligned}\left(f^{-1}\right)^{\dagger} &= \mathrm{Gr}(f),\\ \mathrm{Gr}(f)^{\dagger} &= f^{-1}.\end{aligned}$$

Proof. *Item 1, Functoriality*: Clear.
*Item 2, Adjointness Inside **Rel***: This is proved in Item 2 of Proposition 3.1.1.2.
Item 3, Interaction With Inverses of Relations: Clear. □

3.3 Representable Relations

Let A and B be sets.

Definition 3.3.1.1. Let $f\colon A \to B$ and $g\colon B \to A$ be functions.[19]

1. The **representable relation associated to** f is the relation $\chi_f\colon A \nrightarrow B$ defined as the composition

$$A \times B \xrightarrow{f\times\mathrm{id}_B} B \times B \xrightarrow{\chi_B} \{\mathsf{true}, \mathsf{false}\},$$

i.e. given by declaring $a \sim_{\chi_f} b$ iff $f(a) = b$.

2. The **corepresentable relation associated to** g is the relation $\chi^g\colon B \nrightarrow A$ defined as the composition

$$B \times A \xrightarrow{g\times\mathrm{id}_A} A \times A \xrightarrow{\chi_A} \{\mathsf{true}, \mathsf{false}\},$$

i.e. given by declaring $b \sim_{\chi^g} a$ iff $g(b) = a$.

3.4 The Domain and Range of a Relation

Let A and B be sets.

[19] More generally, given functions

$$\begin{aligned}f&\colon A \to C,\\ g&\colon B \to D\end{aligned}$$

and a relation $B \nrightarrow D$, we may consider the composite relation

$$A \times B \xrightarrow{f\times g} C \times D \xrightarrow{R} \{\mathsf{true}, \mathsf{false}\},$$

for which we have $a \sim_{R\circ(f\times g)} b$ iff $f(a) \sim_R g(b)$.

Definition 3.4.1.1. Let $R \subset A \times B$ be a relation.[20,21]

1. The **domain of** R is the subset $\mathrm{dom}(R)$ of A defined by

$$\mathrm{dom}(R) \stackrel{\mathrm{def}}{=} \left\{ a \in A \,\middle|\, \begin{array}{l} \text{there exists some } b \in B \\ \text{such that } a \sim_R b \end{array} \right\}.$$

2. The **range of** R is the subset $\mathrm{range}(R)$ of B defined by

$$\mathrm{range}(R) \stackrel{\mathrm{def}}{=} \left\{ b \in B \,\middle|\, \begin{array}{l} \text{there exists some } a \in A \\ \text{such that } a \sim_R b \end{array} \right\}.$$

3.5 Binary Unions of Relations

Let A and B be sets and let R and S be relations from A to B.

Definition 3.5.1.1. The **union of** R **and** S[22] is the relation $R \cup S$ from A to B defined as follows:

[20] Following Categories, **??**, we may compute the (characteristic functions associated to the) domain and range of a relation using the following colimit formulas:

$$\begin{aligned} \chi_{\mathrm{dom}(R)}(a) &\cong \operatorname*{colim}_{b \in B}\left(R^a_b\right) \qquad (a \in A) \\ &\cong \bigvee_{b \in B} R^a_b, \\ \chi_{\mathrm{range}(R)}(b) &\cong \operatorname*{colim}_{a \in A}\left(R^a_b\right) \qquad (b \in B) \\ &\cong \bigvee_{a \in A} R^a_b, \end{aligned}$$

where the join $\bigvee$ is taken in the poset $(\{\mathsf{true}, \mathsf{false}\}, \preceq)$ of Constructions With Sets, Definition 1.2.1.3.

[21] Viewing R as a function $R\colon A \to \mathcal{P}(B)$, we have

$$\begin{aligned} \mathrm{dom}(R) &\cong \operatorname*{colim}_{y \in Y}(R(y)) \\ &\cong \bigcup_{y \in Y} R(y), \\ \mathrm{range}(R) &\cong \operatorname*{colim}_{x \in X}(R(x)) \\ &\cong \bigcup_{x \in X} R(x), \end{aligned}$$

[22] *Further Terminology:* Also called the **binary union of** R **and** S, for emphasis.

- Viewing relations from A to B as subsets of $A \times B$, we define[23]

$$R \cup S \stackrel{\text{def}}{=} \{(a,b) \in B \times A \mid \text{we have } a \sim_R b \text{ or } a \sim_S b\}.$$

- Viewing relations from A to B as functions $A \to \mathcal{P}(B)$, we define

$$[R \cup S](a) \stackrel{\text{def}}{=} R(a) \cup S(a)$$

for each $a \in A$.

Proposition 3.5.1.2. Let R, S, R_1, and R_2 be relations from A to B, and let S_1 and S_2 be relations from B to C.

1. *Interaction With Inverses.* We have

$$(R \cup S)^\dagger = R^\dagger \cup S^\dagger.$$

2. *Interaction With Composition.* We have

$$(S_1 \diamond R_1) \cup (S_2 \diamond R_2) \stackrel{\text{poss.}}{\neq} (S_1 \cup S_2) \diamond (R_1 \cup R_2).$$

Proof. *Item 1*, *Interaction With Inverses*: Clear.
Item 2, *Interaction With Composition*: Unwinding the definitions, we see that:

1. The condition for $(S_1 \diamond R_1) \cup (S_2 \diamond R_2)$ is:
   (a) There exists some $b \in B$ such that:
      i. $a \sim_{R_1} b$ and $b \sim_{S_1} c$;

      or

      i. $a \sim_{R_2} b$ and $b \sim_{S_2} c$;
3. The condition for $(S_1 \cup S_2) \diamond (R_1 \cup R_2)$ is:
   (a) There exists some $b \in B$ such that:
      i. $a \sim_{R_1} b$ or $a \sim_{R_2} b$;

      and

      i. $b \sim_{S_1} c$ or $b \sim_{S_2} c$.

These two conditions may fail to agree (counterexample omitted), and thus the two resulting relations on $A \times C$ may differ. □

[23] This is the same as the union of R and S as subsets of $A \times B$.

3.6 Unions of Families of Relations

Let A and B be sets and let $\{R_i\}_{i\in I}$ be a family of relations from A to B.

Definition 3.6.1.1. The **union of the family** $\{R_i\}_{i\in I}$ is the relation $\bigcup_{i\in I} R_i$ from A to B defined as follows:

- Viewing relations from A to B as subsets of $A \times B$, we define[24]

$$\bigcup_{i\in I} R_i \overset{\text{def}}{=} \left\{ (a,b) \in (A\times B)^{\times I} \;\middle|\; \begin{array}{l}\text{there exists some } i \in I\\ \text{such that } a \sim_{R_i} b\end{array} \right\}.$$

- Viewing relations from A to B as functions $A \longrightarrow \mathcal{P}(B)$, we define

$$\left[\bigcup_{i\in I} R_i\right](a) \overset{\text{def}}{=} \bigcup_{i\in I} R_i(a)$$

  for each $a \in A$.

Proposition 3.6.1.2. Let A and B be sets and let $\{R_i\}_{i\in I}$ be a family of relations from A to B.

1. *Interaction With Inverses.* We have

$$\left(\bigcup_{i\in I} R_i\right)^{\dagger} = \bigcup_{i\in I} R_i^{\dagger}.$$

Proof. *Item 1*, *Interaction With Inverses*: Clear. □

3.7 Binary Intersections of Relations

Let A and B be sets and let R and S be relations from A to B.

Definition 3.7.1.1. The **intersection of** R **and** S[25] is the relation $R \cap S$ from A to B defined as follows:

- Viewing relations from A to B as subsets of $A \times B$, we define[26]

$$R \cap S \overset{\text{def}}{=} \{(a,b) \in B \times A \mid \text{we have } a \sim_R b \text{ and } a \sim_S b\}.$$

[24] This is the same as the union of $\{R_i\}_{i\in I}$ as a collection of subsets of $A \times B$.
[25] *Further Terminology:* Also called the **binary intersection of** R **and** S, for emphasis.
[26] This is the same as the intersection of R and S as subsets of $A \times B$.

- Viewing relations from A to B as functions $A \longrightarrow \mathcal{P}(B)$, we define

$$[R \cap S](a) \stackrel{\text{def}}{=} R(a) \cap S(a)$$

for each $a \in A$.

Proposition 3.7.1.2. Let R, S, R_1, and R_2 be relations from A to B, and let S_1 and S_2 be relations from B to C.

1. *Interaction With Inverses.* We have

$$(R \cap S)^{\dagger} = R^{\dagger} \cap S^{\dagger}.$$

2. *Interaction With Composition.* We have

$$(S_1 \diamond R_1) \cap (S_2 \diamond R_2) = (S_1 \cap S_2) \diamond (R_1 \cap R_2).$$

Proof. *Item 1, Interaction With Inverses*: Clear.
Item 2, Interaction With Composition: Unwinding the definitions, we see that:

1. The condition for $(S_1 \diamond R_1) \cap (S_2 \diamond R_2)$ is:
   (a) There exists some $b \in B$ such that:
      i. $a \sim_{R_1} b$ and $b \sim_{S_1} c$;
   and
      i. $a \sim_{R_2} b$ and $b \sim_{S_2} c$;
3. The condition for $(S_1 \cap S_2) \diamond (R_1 \cap R_2)$ is:
   (a) There exists some $b \in B$ such that:
      i. $a \sim_{R_1} b$ and $a \sim_{R_2} b$;
   and
      i. $b \sim_{S_1} c$ and $b \sim_{S_2} c$.

These two conditions agree, and thus so do the two resulting relations on $A \times C$. □

3.8 Intersections of Families of Relations

Let A and B be sets and let $\{R_i\}_{i \in I}$ be a family of relations from A to B.

Definition 3.8.1.1. The **intersection of the family** $\{R_i\}_{i \in I}$ is the relation $\bigcup_{i \in I} R_i$

defined as follows:

- Viewing relations from A to B as subsets of $A \times B$, we define[27]

$$\bigcup_{i\in I} R_i \overset{\text{def}}{=} \left\{ (a,b) \in (A\times B)^{\times I} \;\middle|\; \begin{array}{l}\text{for each } i \in I,\\ \text{we have } a \sim_{R_i} b\end{array} \right\}.$$

- Viewing relations from A to B as functions $A \to \mathcal{P}(B)$, we define

$$\left[\bigcap_{i\in I} R_i\right](a) \overset{\text{def}}{=} \bigcap_{i\in I} R_i(a)$$

  for each $a \in A$.

Proposition 3.8.1.2. Let A and B be sets and let $\{R_i\}_{i\in I}$ be a family of relations from A to B.

1. *Interaction With Inverses.* We have

$$\left(\bigcap_{i\in I} R_i\right)^{\dagger} = \bigcap_{i\in I} R_i^{\dagger}.$$

Proof. *Item 1, Interaction With Inverses*: Clear. □

3.9 Binary Products of Relations

Let A, B, X, and Y be sets, let $R\colon A \nrightarrow B$ be a relation from A to B, and let $S\colon X \nrightarrow Y$ be a relation from X to Y.

Definition 3.9.1.1. The **product of** R **and** S[28] is the relation $R \times S$ from $A \times X$ to $B \times Y$ defined as follows:

- Viewing relations from $A\times X$ to $B\times Y$ as subsets of $(A\times X)\times(B\times Y)$, we define $R\times S$ as the Cartesian product of R and S as subsets of $A\times X$ and $B\times Y$;[29]
- Viewing relations from $A \times X$ to $B \times Y$ as functions $A \times X \to \mathcal{P}(B\times Y)$, we define $R \times S$ as the composition

$$A\times X \xrightarrow{R\times S} \mathcal{P}(B)\times\mathcal{P}(Y) \overset{\mathcal{P}^{\otimes}_{B,Y}}{\hookrightarrow} \mathcal{P}(B\times Y)$$

[27] This is the same as the intersection of $\{R_i\}_{i\in I}$ as a collection of subsets of $A \times B$.

[28] *Further Terminology:* Also called the **binary product of** R **and** S, for emphasis.

[29] That is, $R \times S$ is the relation given by declaring $(a,x) \sim_{R\times S} (b,y)$ iff $a \sim_R b$ and $x \sim_S y$.

in Sets, i.e. by

$$[R \times S](a, x) \overset{\text{def}}{=} R(a) \times S(x)$$

for each $(a, x) \in A \times X$.

Proposition 3.9.1.2. Let A, B, X, and Y be sets.

1. *Interaction With Inverses.* Let

$$\begin{aligned} R &\colon A \nrightarrow A, \\ S &\colon X \nrightarrow X \end{aligned}$$

We have

$$(R \times S)^{\dagger} = R^{\dagger} \times S^{\dagger}.$$

2. *Interaction With Composition.* Let

$$\begin{aligned} R_1 &\colon A \nrightarrow B, \\ S_1 &\colon B \nrightarrow C, \\ R_2 &\colon X \nrightarrow Y, \\ S_2 &\colon Y \nrightarrow Z \end{aligned}$$

be relations. We have

$$(S_1 \diamond R_1) \times (S_2 \diamond R_2) = (S_1 \times S_2) \diamond (R_1 \times R_2).$$

Proof. *Item 1*, *Interaction With Inverses*: Unwinding the definitions, we see that:

1. We have $(a, x) \sim_{(R \times S)^{\dagger}} (b, y)$ iff :
    - We have $(b, y) \sim_{R \times S} (a, x)$, i.e. iff :
        - We have $b \sim_R a$;
        - We have $y \sim_S x$;
2. We have $(a, x) \sim_{R^{\dagger} \times S^{\dagger}} (b, y)$ iff :
    - We have $a \sim_{R^{\dagger}} b$ and $x \sim_{S^{\dagger}} y$, i.e. iff :
        - We have $b \sim_R a$;
        - We have $y \sim_S x$.

These two conditions agree, and thus the two resulting relations on $A \times X$ are equal.
Item 2, *Interaction With Composition*: Unwinding the definitions, we see that:

1. We have $(a, x) \sim_{(S_1 \diamond R_1) \times (S_2 \diamond R_2)} (c, z)$ iff :
   (a) We have $a \sim_{S_1 \diamond R_1} c$ and $x \sim_{S_2 \diamond R_2} z$, i.e. iff :
      i. There exists some $b \in B$ such that $a \sim_{R_1} b$ and $b \sim_{S_1} c$;
      ii. There exists some $y \in Y$ such that $x \sim_{R_2} y$ and $y \sim_{S_2} z$;
2. We have $(a, x) \sim_{(S_1 \times S_2) \diamond (R_1 \times R_2)} (c, z)$ iff :
   (a) There exists some $(b, y) \in B \times Y$ such that $(a, x) \sim_{R_1 \times R_2} (b, y)$ and $(b, y) \sim_{S_1 \times S_2} (c, z)$, i.e. such that:
      i. We have $a \sim_{R_1} b$ and $x \sim_{R_2} y$;
      ii. We have $b \sim_{S_1} c$ and $y \sim_{S_2} z$.

These two conditions agree, and thus the two resulting relations from $A \times X$ to $C \times Z$ are equal. □

3.10 Products of Families of Relations

Let $\{A_i\}_{i \in I}$ and $\{B_i\}_{i \in I}$ be families of sets, and let $\{R_i : A_i \nrightarrow B_i\}_{i \in I}$ be a family of relations.

Definition 3.10.1.1. The **product of the family** $\{R_i\}_{i \in I}$ is the relation $\prod_{i \in I} R_i$ from $\prod_{i \in I} A_i$ to $\prod_{i \in I} B_i$ defined as follows:

- Viewing relations as subsets, we define $\prod_{i \in I} R_i$ as its product as a family of sets, i.e. we have

$$\prod_{i \in I} R_i \overset{\text{def}}{=} \left\{ (a_i, b_i)_{i \in I} \in \prod_{i \in I} (A_i \times B_i) \;\middle|\; \begin{array}{l} \text{for each } i \in I, \\ \text{we have } a_i \sim_{R_i} b_i \end{array} \right\}.$$

- Viewing relations as functions to powersets, we define

$$\left[\prod_{i \in I} R_i \right] ((a_i)_{i \in I}) \overset{\text{def}}{=} \prod_{i \in I} R_i(a_i)$$

  for each $(a_i)_{i \in I} \in \prod_{i \in I} R_i$.

3.11 The Inverse of a Relation

Let A, B, and C be sets and let $R \subset A \times B$ be a relation.

Definition 3.11.1.1. The **inverse of** R[30] is the relation $R^\dagger$ defined as follows:

- Viewing relations as subsets, we define

$$R^\dagger \overset{\text{def}}{=} \{(b,a) \in B \times A \mid \text{we have } b \sim_R a\}.$$

- Viewing relations as functions $A \times B \to \{\mathsf{true}, \mathsf{false}\}$, we define

$$[R^\dagger]^b_a \overset{\text{def}}{=} R^a_b$$

  for each $(b,a) \in B \times A$.

- Viewing relations as functions $A \to \mathcal{P}(B)$, we define

$$\begin{aligned}[R^\dagger](b) &\overset{\text{def}}{=} R^\dagger(\{b\})\\ &\overset{\text{def}}{=} \{a \in A \mid b \in R(a)\}\end{aligned}$$

  for each $b \in B$, where $R^\dagger(\{b\})$ is the fibre of R over $\{b\}$.

Example 3.11.1.2. Here are some examples of inverses of relations.

1. *Less Than Equal Signs.* We have $(\leq)^\dagger = \geq$.
2. *Greater Than Equal Signs.* Dually to **??**, we have $(\geq)^\dagger = \leq$.
3. *Functions.* Let $f : A \to B$ be a function. We have

$$\begin{aligned}\mathrm{Gr}(f)^\dagger &= f^{-1},\\ \left(f^{-1}\right)^\dagger &= \mathrm{Gr}(f).\end{aligned}$$

Proposition 3.11.1.3. Let $R : A \nrightarrow B$ and $S : B \nrightarrow C$ be relations.

1. *Interaction With Ranges and Domains.* We have

$$\begin{aligned}\mathrm{dom}\left(R^\dagger\right) &= \mathrm{range}(R),\\ \mathrm{range}\left(R^\dagger\right) &= \mathrm{dom}(R).\end{aligned}$$

2. *Interaction With Composition I.* We have

$$(S \diamond R)^\dagger = R^\dagger \diamond S^\dagger.$$

[30] *Further Terminology:* Also called the **opposite of** R, the **transpose of** R, or the **converse of** R.

3. *Interaction With Composition II.* We have

$$\begin{aligned}\chi_B(-_1,-_2) &\subset R \diamond R^\dagger,\\ \chi_A(-_1,-_2) &\subset R^\dagger \diamond R.\end{aligned}$$

4. *Invertibility.* We have

$$\left(R^\dagger\right)^\dagger = R.$$

5. *Identity.* We have

$$\chi_A^\dagger(-_1,-_2) = \chi_A(-_1,-_2).$$

Proof. *Item 1, Interaction With Ranges and Domains*: Clear.
Item 2, Interaction With Composition I: Clear.
Item 3, Interaction With Composition II: Clear.
Item 4, Invertibility: Clear.
Item 5, Identity: Clear. □

3.12 Composition of Relations

Let A, B, and C be sets and let $R \subset A \times B$ and $S \subset B \times C$ be relations.

Definition 3.12.1.1. The **composition of** R **and** S is the relation $S \diamond R$ defined as follows:

- Viewing relations from A to C as subsets of $A \times C$, we define

$$S \diamond R \overset{\text{def}}{=} \left\{ (a,c) \in A \times C \,\middle|\, \begin{array}{l}\text{there exists some } b \in B \text{ such}\\ \text{that } a \sim_R b \text{ and } b \sim_S c\end{array} \right\}.$$

- Viewing relations as functions $A \times B \to \{\mathsf{true}, \mathsf{false}\}$, we define

$$\begin{aligned}(S \diamond R)^{-_1}_{-_2} &\overset{\text{def}}{=} \int^{y \in B} S^{-_1}_{y} \times R^{y}_{-_2}\\ &= \bigvee_{y \in B} S^{-_1}_{y} \times R^{y}_{-_2},\end{aligned}$$

where the join $\bigvee$ is taken in the poset $(\{\mathsf{true}, \mathsf{false}\}, \preceq)$ of Sets, Definition 1.2.1.3.

- Viewing relations as functions $A \to \mathcal{P}(B)$, we define

$$S \diamond R \overset{\text{def}}{=} \mathrm{Lan}_{\chi_B}(S) \circ R, \qquad \begin{tikzcd} & B \arrow[r, "S"] \arrow[d, hook, "\chi_B"'] & \mathcal{P}(C), \\ A \arrow[r, "R"'] & \mathcal{P}(B) \arrow[ur, bend right, "\mathrm{Lan}_{\chi_B}(S)"'] \arrow[u, phantom, shift left=3, "\Swarrow"] & \end{tikzcd}$$

where $\mathrm{Lan}_{\chi_B}(S)$ is computed by the formula

$$\begin{aligned}[\mathrm{Lan}_{\chi_B}(S)](V) &\cong \int^{y\in B} \chi_{\mathcal{P}(B)}\big(\chi_y, V\big) \odot S_y \\ &\cong \int^{y\in B} \chi_V(y) \odot S_y \\ &\cong \bigcup_{y\in B} \chi_V(y) \odot S_y \\ &\cong \bigcup_{y\in V} S_y\end{aligned}$$

for each $V \in \mathcal{P}(B)$. In other words, $S \diamond R$ is defined by[31]

$$\begin{aligned}[S\diamond R](a) &\stackrel{\text{def}}{=} S(R(a)) \\ &\stackrel{\text{def}}{=} \bigcup_{x\in R(a)} S(x).\end{aligned}$$

for each $a \in A$.

Example 3.12.1.2. Here are some examples of composition of relations.

1. *Composing Less/Greater Than Equal With Greater/Less Than Equal Signs.* We have

$$\begin{aligned}\leq \diamond \geq &= \sim_{\text{triv}}, \\ \geq \diamond \leq &= \sim_{\text{triv}}.\end{aligned}$$

2. *Composing Less/Greater Than Equal Signs With Less/Greater Than Equal Signs.* We have

$$\begin{aligned}\leq \diamond \leq &= \leq, \\ \geq \diamond \geq &= \geq.\end{aligned}$$

Proposition 3.12.1.3. Let $R\colon A \nrightarrow B$, $S\colon B \nrightarrow C$, and $T\colon C \nrightarrow D$ be relations.

1. *Interaction With Ranges and Domains.* We have

$$\begin{aligned}\mathrm{dom}(S\diamond R) &\subset \mathrm{dom}(R), \\ \mathrm{range}(S\diamond R) &\subset \mathrm{range}(S).\end{aligned}$$

[31] That is: the relation R may send $a \in A$ to a number of elements $\{b_i\}_{i\in I}$ in B, and then the relation S

2. *Associativity.* We have

$$(T \diamond S) \diamond R = T \diamond (S \diamond R).$$

3. *Unitality.* We have

$$\begin{aligned} \chi_B \diamond R &= R, \\ R \diamond \chi_A &= R. \end{aligned}$$

4. *Interaction With Inverses.* We have

$$(S \diamond R)^{\dagger} = R^{\dagger} \diamond S^{\dagger}.$$

5. *Interaction With Composition.* We have

$$\begin{aligned} \chi_B(-_1, -_2) &\subset R \diamond R^{\dagger}, \\ \chi_A(-_1, -_2) &\subset R^{\dagger} \diamond R. \end{aligned}$$

Proof. *Item 1, Interaction With Ranges and Domains*: Clear.
Item 2, Associativity: Indeed, we have

$$\begin{aligned} (T \diamond S) \diamond R &\overset{\text{def}}{=} \left(\int^{y \in C} T^{-1}_{x} \times S^{x}_{-_2} \right) \diamond R \\ &\overset{\text{def}}{=} \int^{x \in B} \left(\int^{y \in C} T^{-1}_{x} \times S^{x}_{y} \right) \diamond R^{y}_{-_2} \\ &= \int^{x \in B} \int^{y \in C} \left(T^{-1}_{x} \times S^{x}_{y} \right) \diamond R^{y}_{-_2} \\ &= \int^{y \in C} \int^{x \in B} \left(T^{-1}_{x} \times S^{x}_{y} \right) \diamond R^{y}_{-_2} \\ &= \int^{y \in C} \int^{x \in B} T^{-1}_{x} \times \left(S^{x}_{y} \diamond R^{y}_{-_2} \right) \\ &= \int^{x \in B} T^{-1}_{x} \times \left(\int^{y \in C} S^{x}_{y} \diamond R^{y}_{-_2} \right) \\ &\overset{\text{def}}{=} \int^{x \in B} T^{-1}_{x} \times (S \diamond R)^{x}_{-_2} \\ &\overset{\text{def}}{=} T \diamond (S \diamond R). \end{aligned}$$

In the language of relations, given $a \in A$ and $d \in D$, the stated equality witnesses the equivalence of the following two statements:

1. We have $a \sim_{(T \diamond S) \diamond R} d$, i.e. there exists some $b \in B$ such that:
   (a) We have $a \sim_R b$;
   (b) We have $b \sim_{T \diamond S} d$, i.e. there exists some $c \in C$ such that:
      i. We have $b \sim_S c$;
      ii. We have $c \sim_T d$;
2. We have $a \sim_{T \diamond (S \diamond R)} d$, i.e. there exists some $c \in C$ such that:
   (a) We have $a \sim_{S \diamond R} c$, i.e. there exists some $b \in B$ such that:
      i. We have $a \sim_R b$;
      ii. We have $b \sim_S c$;
   (b) We have $c \sim_T d$;

both of which are equivalent to the statement

- There exist $b \in B$ and $c \in C$ such that $a \sim_R b \sim_S c \sim_T d$.

Item 3, *Unitality*: Indeed, we have

$$\begin{aligned} \chi_B \diamond R &\overset{\text{def}}{=} \int^{x \in B} (\chi_B)^{-1}_{x} \times R^{x}_{-2} \\ &= \bigvee_{x \in B} (\chi_B)^{-1}_{x} \times R^{x}_{-2} \\ &= \bigvee_{\substack{x \in B \\ x = -_1}} R^{x}_{-2} \\ &= R^{-1}_{-2}, \end{aligned}$$

and

$$\begin{aligned} R \diamond \chi_A &\overset{\text{def}}{=} \int^{x \in A} R^{-1}_{x} \times (\chi_A)^{x}_{-2} \\ &= \bigvee_{x \in B} R^{-1}_{x} \times (\chi_A)^{x}_{-2} \\ &= \bigvee_{\substack{x \in B \\ x = -_2}} R^{-1}_{x} \\ &= R^{-1}_{-2}. \end{aligned}$$

In the language of relations, given $a \in A$ and $b \in B$:

- The equality

$$\chi_B \diamond R = R$$

witnesses the equivalence of the following two statements:

1. We have $a \sim_b B$.
2. There exists some $b' \in B$ such that:
   (a) We have $a \sim_R b'$
   (b) We have $b' \sim_{\chi_B} b$, i.e. $b' = b$.

- The equality

$$R \diamond \chi_A = R$$

witnesses the equivalence of the following two statements:

1. There exists some $a' \in A$ such that:
   (a) We have $a \sim_{\chi_B} a'$, i.e. $a = a'$.
   (b) We have $a' \sim_R b$
2. We have $a \sim_b B$.

Item 4, Interaction With Inverses: Clear.
Item 5, Interaction With Composition: Clear. □

3.13 The Collage of a Relation

Let A and B be sets and let $R\colon A \nrightarrow B$ be a relation from A to B.

Definition 3.13.1.1. The **collage of** R[32] is the poset $\mathbf{Coll}(R) \overset{\text{def}}{=} (\mathrm{Coll}(R), \preceq_{\mathbf{Coll}(R)})$ consisting of

- *The Underlying Set.* The set $\mathrm{Coll}(R)$ defined by

$$\mathrm{Coll}(R) \overset{\text{def}}{=} A \coprod B.$$

- *The Partial Order.* The partial order

$$\preceq_{\mathbf{Coll}(R)} : \mathrm{Coll}(R) \times \mathrm{Coll}(R) \to \{\mathsf{true}, \mathsf{false}\}$$

on $\mathrm{Coll}(R)$ defined by

$$\preceq(a,b) \overset{\text{def}}{=} \begin{cases} \mathsf{true} & \text{if } a = b \text{ or } a \sim_R b, \\ \mathsf{false} & \text{otherwise.} \end{cases}$$

may send the image of each of the b_i's to a number of elements $\{S(b_i)\}_{i \in I} = \left\{\{c_{j_i}\}_{j_i \in J_i}\right\}_{i \in I}$ in C.

[32] *Further Terminology:* Also called the **cograph of** R.

Proposition 3.13.1.2. Let A and B be sets and let $R\colon A \nrightarrow B$ be a relation from A to B.

1. *Functoriality I.* The assignment $R \mapsto \mathbf{Coll}(R)$ defines a functor[33]

$$\mathbf{Coll}\colon \mathbf{Rel}(A, B) \to \mathsf{Pos}_{/\Delta^1}(A, B),$$

   where

   - *Action on Objects.* For each $R \in \mathrm{Obj}(\mathbf{Rel}(A, B))$, we have

$$[\mathbf{Coll}](R) \overset{\mathrm{def}}{=} (\mathbf{Coll}(R), \phi_R)$$

     for each $R \in \mathbf{Rel}(A, B)$, where

     – The poset $\mathbf{Coll}(R)$ is the collage of R of Definition 3.13.1.1;

     – The morphism $\phi_R\colon \mathbf{Coll}(R) \to \Delta^1$ is given by

$$\phi_R(x) \overset{\mathrm{def}}{=} \begin{cases} 0 & \text{if } x \in A, \\ 1 & \text{if } x \in B \end{cases}$$

       for each $x \in \mathbf{Coll}(R)$;

   - *Action on Morphisms.* For each $R, S \in \mathrm{Obj}(\mathbf{Rel}(A, B))$, the action on Hom-

[33] Here $\mathsf{Pos}_{/\Delta^1}(A, B)$ is the category defined as the pullback

$$\mathsf{Pos}_{/\Delta^1}(A, B) \overset{\mathrm{def}}{=} \mathsf{pt} \underset{[A],\mathsf{Pos},\mathrm{fib}_0}{\times} \mathsf{Pos}_{/\Delta^1} \underset{\mathrm{fib}_1,\mathsf{Pos},[B]}{\times} \mathsf{pt},$$

as in the diagram

$$\begin{tikzcd}
 & & \mathbf{Pos}_{/\Delta^1}(A, B) \arrow[dl] \arrow[dr] \arrow[dd, phantom, very near start, "\lrcorner" rotate=-45] & & \\
 & \mathbf{Pos}_{/\Delta^1} \underset{\mathbf{Pos}}{\times} \mathsf{pt} \arrow[dl] \arrow[dr] \arrow[dd, phantom, very near start, "\lrcorner" rotate=-45] & & \mathsf{pt} \underset{\mathbf{Pos}}{\times} \mathbf{Pos}_{/\Delta^1} \arrow[dl] \arrow[dr] \arrow[dd, phantom, very near start, "\lrcorner" rotate=-45] & \\
\mathsf{pt} \arrow[dr, "{[A]}"'] & & \mathbf{Pos}_{/\Delta^1} \arrow[dl, "\mathrm{fib}_{[0]}"] \arrow[dr, "\mathrm{fib}_{[1]}"'] & & \mathsf{pt}. \arrow[dl, "{[B]}"] \\
 & \mathbf{Pos} & & \mathbf{Pos} &
\end{tikzcd}$$

Explicitly, an object of $\mathsf{Pos}_{/\Delta^1}(A, B)$ is a pair (X, ϕ_X) consisting of

- A poset X;
- A morphism $\phi_X : X \to \Delta^1$;

such that $\phi_X^{-1}(0) = A$ and $\phi_X^{-1}(0) = B$, with morphisms between such objects being morphisms of posets over Δ^1.

sets

$$\mathbf{Coll}_{R,S}\colon \mathrm{Hom}_{\mathbf{Rel}(A,B)}(R,S) \to \mathsf{Pos}(\mathbf{Coll}(R),\mathbf{Coll}(S))$$

of **Coll** at (R,S) is given by sending an inclusion

$$\iota\colon R \subset S$$

to the morphism

$$\mathbf{Coll}(\iota)\colon \mathbf{Coll}(R) \to \mathbf{Coll}(S)$$

of posets over Δ^1 defined by

$$[\mathbf{Coll}(\iota)](x) \stackrel{\mathrm{def}}{=} x$$

for each $x \in \mathbf{Coll}(R)$.[34]

2. *Equivalence.* The functor of Item 1 is an equivalence of categories.

Proof. Item 1, Functoriality: Clear.
Item 2, Equivalence: Omitted. □

4 Equivalence Relations

4.1 Reflexive Relations

4.1.1 Foundations

Let A be a set.

Definition 4.1.1.1. A **reflexive relation** is equivalently:[35]

- An $\mathbb{E}_0$-monoid in $(\mathrm{N}_\bullet(\mathbf{Rel}(A,A)),\chi_A)$;
- A pointed object in $(\mathbf{Rel}(A,A),\chi_A)$.

Remark 4.1.1.2. In detail, a relation R on A is **reflexive** if we have an inclusion

$$\eta_R\colon \chi_A \subset R$$

of relations in $\mathbf{Rel}(A,A)$, i.e. if, for each $a \in A$, we have $a \sim_R a$.

[34] Note that this is indeed a morphism of posets: if $x \preceq_{\mathbf{Coll}(R)} y$, then $x = y$ or $x \sim_R y$, so we have either $x = y$ or $x \sim_S y$ (as $R \subset S$), and thus $x \preceq_{\mathbf{Coll}(S)} y$.

[35] Note that since $\mathbf{Rel}(A,A)$ is posetal, reflexivity is a property of a relation, rather than extra structure.

Definition 4.1.1.3. Let A be a set.

1. The **set of reflexive relations on** A is the subset $\mathrm{Rel}^{\mathrm{refl}}(A, A)$ of $\mathrm{Rel}(A, A)$ spanned by the reflexive relations.

2. The **poset of relations on** A is is the subposet $\mathbf{Rel}^{\mathrm{refl}}(A, A)$ of $\mathbf{Rel}(A, A)$ spanned by the reflexive relations.

Proposition 4.1.1.4. Let R and S be relations on A.

1. *Interaction With Inverses.* If R is reflexive, then so is $R^{\dagger}$.

2. *Interaction With Composition.* If R and S are reflexive, then so is $S \diamond R$.

Proof. *Item 1*, *Interaction With Inverses*: Clear.
Item 2, *Interaction With Composition*: Clear. □

4.1.2 The Reflexive Closure of a Relation

Let R be a relation on A.

Definition 4.1.2.1. The **reflexive closure** of $\sim_R$ is the relation $\sim_R^{\mathrm{refl}}$[36] satisfying the following universal property:[37]

(★) Given another reflexive relation $\sim_S$ on A such that $R \subset S$, there exists an inclusion $\sim_R^{\mathrm{refl}} \subset \sim_S$.

Construction 4.1.2.2. Concretely, $\sim_R^{\mathrm{refl}}$ is the free pointed object on R in $(\mathbf{Rel}(A, A), \chi_A)$[38], being given by

$$\begin{aligned} R^{\mathrm{refl}} &\overset{\mathrm{def}}{=} R \coprod^{\mathbf{Rel}(A,A)} \Delta_A \\ &= R \cup \Delta_A \\ &= \{(a, b) \in A \times A \mid \text{we have } a \sim_R b \text{ or } a = b\}. \end{aligned}$$

Proof. Clear. □

Proposition 4.1.2.3. Let R be a relation on A.

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[36] *Further Notation:* Also written R^{refl}.
[37] *Slogan:* The reflexive closure of R is the smallest reflexive relation containing R.
[38] Or, equivalently, the free $\mathbb{E}_0$-monoid on R in $(\mathrm{N}_\bullet(\mathbf{Rel}(A, A)), \chi_A)$.

1. *Adjointness.* We have an adjunction

$$\Big((-)^{\mathrm{refl}} \dashv 忘\Big)\colon \quad \begin{tikzcd} \mathbf{Rel}(A,A) \arrow[r, bend left, "(-)^{\mathrm{refl}}"] \arrow[r, phantom, "\perp"] & \mathbf{Rel}^{\mathrm{refl}}(A,A), \arrow[l, bend left, "忘"] \end{tikzcd}$$

   witnessed by a bijection of sets

$$\mathbf{Rel}^{\mathrm{refl}}\Big(R^{\mathrm{refl}}, S\Big) \cong \mathbf{Rel}(R,S),$$

   natural in $R \in \mathrm{Obj}\Big(\mathbf{Rel}^{\mathrm{refl}}(A,A)\Big)$ and $S \in \mathrm{Obj}(\mathbf{Rel}(A,A))$.

2. *The Reflexive Closure of a Reflexive Relation.* If R is reflexive, then $R^{\mathrm{refl}} = R$.

3. *Idempotency.* We have

$$\Big(R^{\mathrm{refl}}\Big)^{\mathrm{refl}} = R^{\mathrm{refl}}.$$

4. *Interaction With Inverses.* We have

$$\Big(R^{\dagger}\Big)^{\mathrm{refl}} = \Big(R^{\mathrm{refl}}\Big)^{\dagger}, \qquad \begin{tikzcd} \mathrm{Rel}(A,A) \arrow[r, "(-)^{\mathrm{refl}}"] \arrow[d, "(-)^{\dagger}"'] & \mathrm{Rel}(A,A) \arrow[d, "(-)^{\dagger}"] \\ \mathrm{Rel}(A,A) \arrow[r, "(-)^{\mathrm{refl}}"'] & \mathrm{Rel}(A,A). \end{tikzcd}$$

5. *Interaction With Composition.* We have

$$(S \diamond R)^{\mathrm{refl}} = S^{\mathrm{refl}} \diamond R^{\mathrm{refl}}, \qquad \begin{tikzcd} \mathrm{Rel}(A,A) \times \mathrm{Rel}(A,A) \arrow[r, "\diamond"] \arrow[d, "(-)^{\mathrm{refl}} \times (-)^{\mathrm{refl}}"'] & \mathrm{Rel}(A,A) \arrow[d, "(-)^{\mathrm{refl}}"] \\ \mathrm{Rel}(A,A) \times \mathrm{Rel}(A,A) \arrow[r, "\diamond"'] & \mathrm{Rel}(A,A). \end{tikzcd}$$

Proof. *Item 1*, *Adjointness*: This is a rephrasing of the universal property of the reflexive closure of a relation, stated in Definition 4.1.2.1.
Item 2, *The Reflexive Closure of a Reflexive Relation*: Clear.
Item 3, *Idempotency*: This follows from Item 2.
Item 4, *Interaction With Inverses*: Clear.
Item 5, *Interaction With Composition*: This follows from Item 2 of Proposition 4.1.1.4. □

4.2 Symmetric Relations

4.2.1 Foundations

Let A be a set.

Definition 4.2.1.1. A relation R on A is **symmetric** if, for each $a, b \in A$, the following conditions are equivalent:[39]

1. We have $a \sim_R b$.
2. We have $b \sim_R a$.

Definition 4.2.1.2. Let A be a set.

1. The **set of symmetric relations on** A is the subset $\mathrm{Rel}^{\mathsf{symm}}(A, A)$ of $\mathrm{Rel}(A, A)$ spanned by the symmetric relations.
2. The **poset of relations on** A is is the subposet $\mathbf{Rel}^{\mathsf{symm}}(A, A)$ of $\mathbf{Rel}(A, A)$ spanned by the symmetric relations.

Proposition 4.2.1.3. Let R and S be relations on A.

1. *Interaction With Inverses.* If R is symmetric, then so is $R^\dagger$.
2. *Interaction With Composition.* If R and S are symmetric, then so is $S \diamond R$.

Proof. *Item 1*, *Interaction With Inverses*: Clear.
Item 2, *Interaction With Composition*: Clear. □

4.2.2 The Symmetric Closure of a Relation

Let R be a relation on A.

Definition 4.2.2.1. The **symmetric closure** of $\sim_R$ is the relation $\sim_R^{\mathrm{symm}}$[40] satisfying the following universal property:[41]

(★) Given another symmetric relation $\sim_S$ on A such that $R \subset S$, there exists an inclusion $\sim_R^{\mathrm{symm}} \subset \sim_S$.

Construction 4.2.2.2. Concretely, $\sim_R^{\mathrm{symm}}$ is the symmetric relation on A defined by

$$\begin{aligned} R^{\mathrm{symm}} &\overset{\mathrm{def}}{=} R \cup R^\dagger \\ &= \{(a, b) \in A \times A \mid \text{we have } a \sim_R b \text{ or } b \sim_R a\}. \end{aligned}$$

[39] That is, R is symmetric if $R^\dagger = R$.
[40] *Further Notation:* Also written R^{symm}.
[41] *Slogan:* The symmetric closure of R is the smallest symmetric relation containing R.

Proof. Clear. □

Proposition 4.2.2.3. Let R be a relation on A.

1. *Adjointness.* We have an adjunction

$$\left((-)^{\mathsf{symm}} \dashv 忘\right)\colon \quad \begin{tikzcd} \mathbf{Rel}(A,A) \arrow[r, bend left, "(-)^{\mathrm{symm}}"] \arrow[r, phantom, "\perp"] & \mathbf{Rel}^{\mathsf{symm}}(A,A), \arrow[l, bend left, "忘"] \end{tikzcd}$$

   witnessed by a bijection of sets

$$\mathbf{Rel}^{\mathsf{symm}}(R^{\mathrm{symm}}, S) \cong \mathbf{Rel}(R,S),$$

   natural in $R \in \mathrm{Obj}(\mathbf{Rel}^{\mathsf{symm}}(A,A))$ and $S \in \mathrm{Obj}(\mathbf{Rel}(A,A))$.

2. *The Symmetric Closure of a Symmetric Relation.* If R is symmetric, then $R^{\mathrm{symm}} = R$.

3. *Idempotency.* We have

$$(R^{\mathrm{symm}})^{\mathrm{symm}} = R^{\mathrm{symm}}.$$

4. *Interaction With Inverses.* We have

$$\left(R^{\dagger}\right)^{\mathrm{symm}} = \left(R^{\mathrm{symm}}\right)^{\dagger}, \qquad \begin{tikzcd} \mathrm{Rel}(A,A) \arrow[r, "(-)^{\mathrm{symm}}"] \arrow[d, "(-)^{\dagger}"'] & \mathrm{Rel}(A,A) \arrow[d, "(-)^{\dagger}"] \\ \mathrm{Rel}(A,A) \arrow[r, "(-)^{\mathrm{symm}}"'] & \mathrm{Rel}(A,A). \end{tikzcd}$$

5. *Interaction With Composition.* We have

$$(S \diamond R)^{\mathrm{symm}} = S^{\mathrm{symm}} \diamond R^{\mathrm{symm}}, \qquad \begin{tikzcd} \mathrm{Rel}(A,A) \times \mathrm{Rel}(A,A) \arrow[r, "\diamond"] \arrow[d, "(-)^{\mathrm{symm}} \times (-)^{\mathrm{symm}}"'] & \mathrm{Rel}(A,A) \arrow[d, "(-)^{\mathrm{symm}}"] \\ \mathrm{Rel}(A,A) \times \mathrm{Rel}(A,A) \arrow[r, "\diamond"'] & \mathrm{Rel}(A,A). \end{tikzcd}$$

Proof. *Item 1, Adjointness*: This is a rephrasing of the universal property of the symmetric closure of a relation, stated in Definition 4.2.2.1.
Item 2, The Symmetric Closure of a Symmetric Relation: Clear.
Item 3, Idempotency: This follows from Item 2.
Item 4, Interaction With Inverses: Clear.
Item 5, Interaction With Composition: This follows from Item 2 of Proposition 4.2.1.3. □

4.3 Transitive Relations

4.3.1 Foundations

Let A be a set.

Definition 4.3.1.1. A **transitive relation** is equivalently:[42]

- A non-unital $\mathbb{E}_1$-monoid in $(\mathrm{N}_\bullet(\mathbf{Rel}(A,A)),\diamond)$;
- A non-unital monoid in $(\mathbf{Rel}(A,A),\diamond)$.

Remark 4.3.1.2. In detail, a relation R on A is **transitive** if we have an inclusion

$$\mu_R\colon R\diamond R\subset R$$

of relations in $\mathbf{Rel}(A,A)$, i.e. if, for each $a,c\in A$, the following condition is satisfied:

($\star$) If there exists some $b\in A$ such that $a\sim_R b$ and $b\sim_R c$, then $a\sim_R c$.

Definition 4.3.1.3. Let A be a set.

1. The **set of transitive relations from** A **to** B is the subset $\mathrm{Rel}^{\mathsf{trans}}(A)$ of $\mathrm{Rel}(A,A)$ spanned by the transitive relations.
2. The **poset of relations from** A **to** B is is the subposet $\mathbf{Rel}^{\mathsf{trans}}(A)$ of $\mathbf{Rel}(A,A)$ spanned by the transitive relations.

Proposition 4.3.1.4. Let R and S be relations on A.

1. *Interaction With Inverses.* If R is transitive, then so is $R^\dagger$.
2. *Interaction With Composition.* If R and S are transitive, then $S\diamond R$ ***may fail to be transitive***.

Proof. Item 1, *Interaction With Inverses*: Clear.
Item 2, *Interaction With Composition*: See [**MSE2096272**].[43] □

[42] Note that since $\mathbf{Rel}(A,A)$ is posetal, transitivity is a property of a relation, rather than extra structure.

[43] *Intuition:* Transitivity for R and S fails to imply that of $S\diamond R$ because the composition operation for relations intertwines R and S in an incompatible way:

1. If $a\sim_{S\diamond R} c$ and $c\sim_{S\diamond r} e$, then:
   (a) There is some $b\in A$ such that:
      i. $a\sim_R b$;
      ii. $b\sim_S c$;
   (b) There is some $d\in A$ such that:

4.3.2 The Transitive Closure of a Relation

Let R be a relation on A.

Definition 4.3.2.1. The **transitive closure** of $\sim_R$ is the relation $\sim_R^{\mathrm{trans}}$[44] satisfying the following universal property:[45]

($\star$) Given another transitive relation $\sim_S$ on A such that $R \subset S$, there exists an inclusion $\sim_R^{\mathrm{trans}} \subset \sim_S$.

Construction 4.3.2.2. Concretely, $\sim_R^{\mathrm{trans}}$ is the free non-unital monoid on R in $(\mathbf{Rel}(A, A), \diamond)$[46], being given by

$$\begin{aligned} R^{\mathrm{trans}} &\overset{\mathrm{def}}{=} \coprod_{n=1}^{\infty} R^{\diamond n} \\ &\overset{\mathrm{def}}{=} \bigcup_{n=1}^{\infty} R^{\diamond n} \\ &\overset{\mathrm{def}}{=} \left\{ (a, b) \in A \times B \;\middle|\; \begin{aligned} &\text{there exists some } (x_1, \ldots, x_n) \in R^{\times n} \\ &\text{such that } a \sim_R x_1 \sim_R \cdots \sim_R x_n \sim_R b \end{aligned} \right\}. \end{aligned}$$

Proof. Clear. □

Proposition 4.3.2.3. Let R be a relation on A.

1. *Adjointness.* We have an adjunction

$$\left((-)^{\mathrm{trans}} \dashv 忘\right) \colon \quad \begin{tikzcd} \mathbf{Rel}(A, A) \arrow[r, bend left, "(-)^{\mathrm{trans}}"] \arrow[r, phantom, "\perp"] & \mathbf{Rel}^{\mathrm{trans}}(A, A) \arrow[l, bend left, "忘"] \end{tikzcd},$$

   witnessed by a bijection of sets

$$\mathbf{Rel}^{\mathrm{trans}}\big(R^{\mathrm{trans}}, S\big) \cong \mathbf{Rel}(R, S),$$

   natural in $R \in \mathrm{Obj}\big(\mathbf{Rel}^{\mathrm{trans}}(A, A)\big)$ and $S \in \mathrm{Obj}(\mathbf{Rel}(A, B))$.

2. *The Transitive Closure of a Transitive Relation.* If R is transitive, then $R^{\mathrm{trans}} = R$.

---

i. $c \sim_R d$;
ii. $d \sim_S e$.

[44] *Further Notation:* Also written R^{trans}.
[45] *Slogan:* The transitive closure of R is the smallest transitive relation containing R.
[46] Or, equivalently, the free non-unital $\mathbb{E}_1$-monoid on R in $(\mathrm{N}_\bullet(\mathbf{Rel}(A, A)), \diamond)$.

3. *Idempotency.* We have

$$\left(R^{\mathrm{trans}}\right)^{\mathrm{trans}} = R^{\mathrm{trans}}.$$

4. *Interaction With Inverses.* We have

$$\left(R^{\dagger}\right)^{\mathrm{trans}} = \left(R^{\mathrm{trans}}\right)^{\dagger}, \qquad \begin{tikzcd} \mathrm{Rel}(A,A) \arrow[r, "(-)^{\mathrm{trans}}"] \arrow[d, "(-)^{\dagger}"'] & \mathrm{Rel}(A,A) \arrow[d, "(-)^{\dagger}"] \\ \mathrm{Rel}(A,A) \arrow[r, "(-)^{\mathrm{trans}}"'] & \mathrm{Rel}(A,A). \end{tikzcd}$$

5. *Interaction With Composition.* We have

$$(S \diamond R)^{\mathrm{trans}} \overset{\text{poss.}}{\neq} S^{\mathrm{trans}} \diamond R^{\mathrm{trans}}, \qquad \begin{tikzcd} \mathrm{Rel}(A,A) \times \mathrm{Rel}(A,A) \arrow[r, "\diamond"] \arrow[d, "(-)^{\mathrm{trans}} \times (-)^{\mathrm{trans}}"'] \arrow[dr, phantom, "\times"] & \mathrm{Rel}(A,A) \arrow[d, "(-)^{\mathrm{trans}}"] \\ \mathrm{Rel}(A,A) \times \mathrm{Rel}(A,A) \arrow[r, "\diamond"'] & \mathrm{Rel}(A,A). \end{tikzcd}$$

Proof. *Item 1, Adjointness*: This is a rephrasing of the universal property of the transitive closure of a relation, stated in Definition 4.3.2.1.
Item 2, The Transitive Closure of a Transitive Relation: Clear.
Item 3, Idempotency: This follows from Item 2.
Item 4, Interaction With Inverses: We have

$$\begin{aligned} \left(R^{\dagger}\right)^{\mathrm{trans}} &= \bigcup_{n=1}^{\infty} \left(R^{\dagger}\right)^{\diamond n} && \text{(by Construction 4.3.2.2)} \\ &= \bigcup_{n=1}^{\infty} \left(R^{\diamond n}\right)^{\dagger} && \text{(by Item 4 of Proposition 3.12.1.3)} \\ &= \left(\bigcup_{n=1}^{\infty} R^{\diamond n}\right)^{\dagger} && \text{(by Item 1 of Proposition 3.6.1.2)} \\ &= \left(R^{\mathrm{trans}}\right)^{\dagger}. && \text{(by Construction 4.3.2.2)} \end{aligned}$$

Item 5, Interaction With Composition: This follows from Item 2 of Proposition 4.3.1.4. □

4.4 Equivalence Relations

4.4.1 Foundations

Let A be a set.

Definition 4.4.1.1. A relation R is an **equivalence relation** if it is reflexive, symmetric, and transitive.[47]

Example 4.4.1.2. The **kernel of a function** $f\colon A \to B$ is the equivalence $\sim_{\mathrm{Ker}(f)}$ on A obtained by declaring $a \sim_{\mathrm{Ker}(f)} b$ iff $f(a) = f(b)$.[48]

Definition 4.4.1.3. Let A and B be sets.

1. The **set of equivalence relations from** A **to** B is the subset $\mathrm{Rel}^{\mathrm{eq}}(A, B)$ of $\mathrm{Rel}(A, B)$ spanned by the equivalence relations.

2. The **poset of relations from** A **to** B is is the subposet $\mathbf{Rel}^{\mathrm{eq}}(A, B)$ of $\mathbf{Rel}(A, B)$ spanned by the equivalence relations.

4.4.2 The Equivalence Closure of a Relation

Let R be a relation on A.

Definition 4.4.2.1. The **equivalence closure**[49] of $\sim_R$ is the relation $\sim_R^{\mathrm{eq}}$[50] satisfying the following universal property:[51]

($\star$) Given another equivalence relation $\sim_S$ on A such that $R \subset S$, there exists an inclusion $\sim_R^{\mathrm{eq}} \subset \sim_S$.

---

[47] *Further Terminology:* If instead R is just symmetric and transitive, then it is called a **partial equivalence relation**.

[48] The kernel $\mathrm{Ker}(f)\colon A \nrightarrow A$ of f is the monad induced by the adjunction $\mathrm{Gr}(f) \dashv f^{-1}\colon A \rightleftarrows B$ in Rel of Item 2 of Proposition 3.1.1.2.

[49] *Further Terminology:* Also called the **equivalence relation associated to** $\sim_R$.

[50] *Further Notation:* Also written R^{eq}.

[51] *Slogan:* The equivalence closure of R is the smallest equivalence relation containing R.

Construction 4.4.2.2. Concretely, $\sim_R^{\mathrm{eq}}$ is the equivalence relation on A defined by

$$
\begin{aligned}
R^{\mathrm{eq}} &\stackrel{\mathrm{def}}{=} \left(\left(R^{\mathrm{refl}}\right)^{\mathrm{symm}}\right)^{\mathrm{trans}} \\
&= \left((R^{\mathrm{symm}})^{\mathrm{trans}}\right)^{\mathrm{refl}} \\
&= \left\{ (a,b) \in A \times B \;\middle|\; \begin{array}{l} \text{there exists } (x_1, \ldots, x_n) \in R^{\times n} \text{ satisfying at} \\ \text{least one of the following conditions:} \\ \quad 1. \text{ The following conditions are satisfied:} \\ \qquad \text{(a) We have } a \sim_R x_1 \text{ or } x_1 \sim_R a; \\ \qquad \text{(b) We have } x_i \sim_R x_{i+1} \text{ or } x_{i+1} \sim_R x_i \\ \qquad\quad \text{for each } 1 \leq i \leq n-1; \\ \qquad \text{(c) We have } b \sim_R x_n \text{ or } x_n \sim_R b; \\ \quad 2. \text{ We have } a = b. \end{array} \right\}.
\end{aligned}
$$

Proof. From the universal properties of the reflexive, symmetric, and transitive closures of a relation (Definitions 4.1.2.1, 4.2.2.1 and 4.3.2.1), we see that it suffices to prove that:

1. The symmetric closure of a reflexive relation is still reflexive;
2. The transitive closure of a symmetric relation is still symmetric;

which are both clear. □

Proposition 4.4.2.3. Let R be a relation on A.

1. *Adjointness.* We have an adjunction

$$
\left((-)^{\mathrm{eq}} \dashv 忘\right) \colon \quad
\begin{tikzcd}
\mathbf{Rel}(A,B) \arrow[r, bend left, "(-)^{\mathrm{eq}}"] \arrow[r, phantom, "\perp"] & \mathbf{Rel}^{\mathrm{eq}}(A,B), \arrow[l, bend left, "忘"]
\end{tikzcd}
$$

   witnessed by a bijection of sets

$$
\mathbf{Rel}^{\mathrm{eq}}(R^{\mathrm{eq}}, S) \cong \mathbf{Rel}(R, S),
$$

   natural in $R \in \mathrm{Obj}(\mathbf{Rel}^{\mathrm{eq}}(A,B))$ and $S \in \mathrm{Obj}(\mathbf{Rel}(A,B))$.

2. *The Equivalence Closure of an Equivalence Relation.* If R is an equivalence relation, then $R^{\mathrm{eq}} = R$.

3. *Idempotency.* We have

$$
(R^{\mathrm{eq}})^{\mathrm{eq}} = R^{\mathrm{eq}}.
$$

Proof. *Item 1, Adjointness*: This is a rephrasing of the universal property of the equivalence closure of a relation, stated in Definition 4.4.2.1.
Item 2, The Equivalence Closure of an Equivalence Relation: Clear.
Item 3, Idempotency: This follows from Item 2. □

4.5 Quotients by Equivalence Relations

4.5.1 Equivalence Classes

Let A be a set, let R be a relation on A, and let $a \in A$.

Definition 4.5.1.1. The **equivalence class associated to** a is the set $[a]$ defined by

$$\begin{aligned}[a] &\overset{\text{def}}{=} \{x \in X \mid x \sim_R a\}\\ &= \{x \in X \mid a \sim_R x\}. && \text{(since } R \text{ is symmetric)}\end{aligned}$$

4.5.2 Quotients of Sets by Equivalence Relations

Let A be a set and let R be a relation on A.

Definition 4.5.2.1. The **quotient of** X **by** R is the set $X/{\sim_R}$ defined by

$$X/{\sim_R} \overset{\text{def}}{=} \{[a] \in \mathcal{P}(X) \mid a \in X\}.$$

Remark 4.5.2.2. The reason we define quotient sets for equivalence relations only is that each of the properties of being an equivalence relation—reflexivity, symmetry, and transitivity—ensures that the equivalences classes $[a]$ of X under R are well-behaved:

- *Reflexivity.* If R is reflexive, then, for each $a \in X$, we have $a \in [a]$.
- *Symmetry.* The equivalence class $[a]$ of an element a of X is defined by

  $$[a] \overset{\text{def}}{=} \{x \in X \mid x \sim_R a\},$$

  but we could equally well define

  $$[a]' \overset{\text{def}}{=} \{x \in X \mid a \sim_R x\}$$

  instead. This is not a problem when R is symmetric, as we then have $[a] = [a]'$.[52]
- *Transitivity.* If R is transitive, then $[a]$ and $[b]$ are disjoint iff $a \nsim_R b$, and equal otherwise.

[52]When categorifying equivalence relations, one finds that $[a]$ and $[a]'$ correspond to presheaves and copresheaves; see Constructions With Categories, **??**.

Proposition 4.5.2.3. Let $f\colon X \to Y$ be a function and let R be a relation on X.

1. *As a Coequaliser.* We have an isomorphism of sets

$$X/{\sim_R^{\mathrm{eq}}} \cong \mathrm{CoEq}\left(R \hookrightarrow X \times X \underset{\mathrm{pr}_2}{\overset{\mathrm{pr}_1}{\rightrightarrows}} X\right),$$

   where $\sim_R^{\mathrm{eq}}$ is the equivalence relation generated by $\sim_R$.

2. *As a Pushout.* We have an isomorphism of sets[53]

$$X/{\sim_R^{\mathrm{eq}}} \cong X \coprod\nolimits_{\mathrm{Eq}(\mathrm{pr}_1,\mathrm{pr}_2)} X, \qquad \begin{tikzcd} X/{\sim_R^{\mathrm{eq}}} \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[l] \\ X \arrow[u] & \mathrm{Eq}(\mathrm{pr}_1, \mathrm{pr}_2). \arrow[l] \arrow[u] \end{tikzcd}$$

   where $\sim_R^{\mathrm{eq}}$ is the equivalence relation generated by $\sim_R$.

3. *The First Isomorphism Theorem for Sets.* We have an isomorphism of sets[54,55]

$$X/{\sim_{\mathrm{Ker}(f)}} \cong \mathrm{Im}(f).$$

[53] Dually, we also have an isomorphism of sets

$$\mathrm{Eq}(\mathrm{pr}_1, \mathrm{pr}_2) \cong X \times_{X/{\sim_R^{\mathrm{eq}}}} X, \qquad \begin{tikzcd} \mathrm{Eq}(\mathrm{pr}_1, \mathrm{pr}_2) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d] \\ X \arrow[r] & X/{\sim_R^{\mathrm{eq}}}. \end{tikzcd}$$

[54] *Further Terminology:* The set $X/{\sim_{\mathrm{Ker}(f)}}$ is often called the **coimage of** f, and denoted by $\mathrm{Coim}(f)$.

[55] In a sense this is a result relating the monad in **Rel** induced by f with the comonad in **Rel** induced by f, as the kernel and image

$$\mathrm{Ker}(f)\colon X \nrightarrow X,$$
$$\mathrm{Im}(f) \subset Y$$

of f are respectively the induced monads and comonads of the adjunction

$$\left(\mathrm{Gr}(f) \dashv f^{-1}\right)\colon \quad \begin{tikzcd} A \arrow[r, bend left, "\mathrm{Gr}(f)", "+"{marking}] \arrow[r, phantom, "\perp"] & B \arrow[l, bend left, "f^{-1}", "+"{marking}] \end{tikzcd}$$

of Item 2 of Proposition 3.1.1.2.

4. *Descending Functions to Quotient Sets, I.* Let R be an equivalence relation on X. The following conditions are equivalent:

   (a) There exists a map
   $$\overline{f}\colon X/{\sim_R} \longrightarrow Y$$
   making the diagram
   $$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "q"'] & Y \\ X/{\sim_R} \arrow[ur, dotted, "\exists", "\overline{f}"'] & \end{tikzcd}$$
   commute.

   (b) For each $x, y \in X$, if $x \sim_R y$, then $f(x) = f(y)$.

5. *Descending Functions to Quotient Sets, II.* Let R be an equivalence relation on X. If the conditions of Item 4 hold, then $\overline{f}$ is the *unique* map making the diagram
   $$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "q"'] & Y \\ X/{\sim_R} \arrow[ur, dashed, "\exists!", "\overline{f}"'] & \end{tikzcd}$$
   commute.

6. *Descending Functions to Quotient Sets, III.* Let R be an equivalence relation on X. If the conditions of Item 4 hold, then the following conditions are equivalent:

   (a) The map $\overline{f}$ is an injection.

   (b) For each $x, y \in X$, we have $x \sim_R y$ iff $f(x) = f(y)$.

7. *Descending Functions to Quotient Sets, IV.* Let R be an equivalence relation on X. If the conditions of Item 4 hold, then the following conditions are equivalent:

   (a) The map $f\colon X \longrightarrow Y$ is surjective.

   (b) The map $\overline{f}\colon X/{\sim_R} \longrightarrow Y$ is surjective.

8. *Descending Functions to Quotient Sets, V.* Let R be a relation on X and let $\sim_R^{\mathrm{eq}}$ be the equivalence relation associated to R. The following conditions are equivalent:

(a) The map f satisfies the equivalent conditions of Item 4:

- There exists a map
$$\overline{f}\colon X/{\sim_R^{\mathrm{eq}}} \to Y$$
making the diagram
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "q"'] & Y \\ X/{\sim_R^{\mathrm{eq}}} \arrow[ur, dotted, "\exists", "\overline{f}"'] & \end{tikzcd}$$
commute.
- For each $x, y \in X$, if $x \sim_R^{\mathrm{eq}} y$, then $f(x) = f(y)$.

(b) For each $x, y \in X$, if $x \sim_R y$, then $f(x) = f(y)$.

Proof. *Item 1, As a Coequaliser*: Omitted.
Item 2, As a Pushout: Omitted.
Item 3, The First Isomorphism Theorem for Sets: Clear.
Item 4, Descending Functions to Quotient Sets, I: See [Pro23c].
Item 5, Descending Functions to Quotient Sets, II: See [Pro23d].
Item 6, Descending Functions to Quotient Sets, III: See [Pro23b].
Item 7, Descending Functions to Quotient Sets, IV: See [Pro23a].
Item 8, Descending Functions to Quotient Sets, V: The implication Item 8a $\implies$ Item 8b is clear.
Conversely, suppose that, for each $x, y \in X$, if $x \sim_R y$, then $f(x) = f(y)$. Spelling out the definition of the equivalence closure of R, we see that the condition $x \sim_R^{\mathrm{eq}} y$ unwinds to the following:

($\star$) There exist $(x_1, \ldots, x_n) \in R^{\times n}$ satisfying at least one of the following conditions:

1. The following conditions are satisfied:
   (a) We have $x \sim_R x_1$ or $x_1 \sim_R x$;
   (b) We have $x_i \sim_R x_{i+1}$ or $x_{i+1} \sim_R x_i$ for each $1 \leq i \leq n-1$;
   (c) We have $y \sim_R x_n$ or $x_n \sim_R y$;
2. We have $x = y$.

Now, if $x = y$, then $f(x) = f(y)$ trivially; otherwise, we have

$$\begin{aligned} f(x) &= f(x_1), \\ f(x_1) &= f(x_2), \\ &\vdots \\ f(x_{n-1}) &= f(x_n), \\ f(x_n) &= f(y), \end{aligned}$$

and $f(x) = f(y)$, as we wanted to show. □

5 Functoriality of Powersets

5.1 Direct Images

Let A and B be sets and let $R\colon A \nrightarrow B$ be a relation.

Definition 5.1.1.1. The **direct image function associated to** R is the function[56]

$$R_*\colon \mathcal{P}(A) \to \mathcal{P}(B)$$

defined by[57,58]

$$\begin{aligned} R_*(U) &\overset{\text{def}}{=} R(U) \\ &\overset{\text{def}}{=} \bigcup_{a\in U} R(a) \\ &= \left\{ b \in B \;\middle|\; \begin{array}{l} \text{there exists some } a \in U \\ \text{such that } b \in R(a) \end{array} \right\} \end{aligned}$$

for each $U \in \mathcal{P}(A)$.

Remark 5.1.1.2. Identifying subsets of A with relations from pt to A via Constructions

[56] *Further Notation:* Also written $\exists_R\colon \mathcal{P}(A) \to \mathcal{P}(B)$. This notation comes from the fact that the following statements are equivalent, where $b \in B$ and $U \in \mathcal{P}(A)$:

- We have $b \in \exists_R(U)$.
- There exists some $a \in U$ such that $b \in f(a)$.

[57] *Further Terminology:* The set $R(U)$ is called the **direct image of** U **by** R.

[58] We also have

$$R_*(U) = B \setminus R_!(A \setminus U);$$

see Item 7 of Proposition 5.1.1.3.

With Sets, Item 7 of Proposition 4.2.1.3, we see that the direct image function associated to R is equivalently the function

$$R_*\colon \underbrace{\mathcal{P}(A)}_{\cong \mathrm{Rel}(\mathrm{pt},A)} \to \underbrace{\mathcal{P}(B)}_{\cong \mathrm{Rel}(\mathrm{pt},B)}$$

defined by

$$R_*(U) \overset{\text{def}}{=} R \diamond U$$

for each $U \in \mathcal{P}(A)$, where $R \diamond U$ is the composition

$$\mathrm{pt} \overset{U}{\nrightarrow} A \overset{R}{\nrightarrow} B.$$

Proposition 5.1.1.3. Let $R\colon A \nrightarrow B$ be a relation.

1. *Functoriality.* The assignment $U \mapsto R_*(U)$ defines a functor

   $$R_*\colon (\mathcal{P}(A), \subset) \to (\mathcal{P}(B), \subset)$$

   where

   - *Action on Objects.* For each $U \in \mathcal{P}(A)$, we have

     $$[R_*](U) \overset{\text{def}}{=} R_*(U);$$

   - *Action on Morphisms.* For each $U, V \in \mathcal{P}(A)$:
     - If $U \subset V$, then $R_*(U) \subset R_*(V)$.

2. *Adjointness.* We have an adjunction

   $$\begin{tikzcd} (R_* \dashv R_{-1})\colon \quad \mathcal{P}(A) \arrow[r, bend left, "R_*"] \arrow[r, phantom, "\perp"] & \mathcal{P}(B), \arrow[l, bend left, "R_{-1}"] \end{tikzcd}$$

   witnessed by a bijections of sets

   $$\mathrm{Hom}_{\mathcal{P}(A)}(R_*(U), V) \cong \mathrm{Hom}_{\mathcal{P}(A)}(U, R_{-1}(V)),$$

   natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$, i.e. such that:

   ($\star$) The following conditions are equivalent:
   - We have $R_*(U) \subset V$;
   - We have $U \subset R_{-1}(V)$.

3. *Preservation of Colimits.* We have an equality of sets

$$R_*\left(\bigcup_{i\in I} U_i\right) = \bigcup_{i\in I} R_*(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have equalities

$$\begin{aligned} R_*(U) \cup R_*(V) &= R_*(U \cup V), \\ R_*(\emptyset) &= \emptyset, \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

4. *Oplax Preservation of Limits.* We have an inclusion of sets

$$R_*\left(\bigcap_{i\in I} U_i\right) \subset \bigcap_{i\in I} R_*(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have inclusions

$$\begin{aligned} R_*(U \cap V) &\subset R_*(U) \cap R_*(V), \\ R_*(A) &\subset B, \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

5. *Symmetric Strict Monoidality With Respect to Unions.* The direct image function of Item 1 has a symmetric strict monoidal structure

$$\left(R_*, R_*^{\otimes}, R_{*|\mathbb{1}}^{\otimes}\right) \colon (\mathcal{P}(A), \cup, \emptyset) \longrightarrow (\mathcal{P}(B), \cup, \emptyset),$$

being equipped with equalities

$$\begin{aligned} R_{*|U,V}^{\otimes} \colon R_*(U) \cup R_*(V) &\xrightarrow{=} R_*(U \cup V), \\ R_{*|\mathbb{1}}^{\otimes} \colon \emptyset &\xrightarrow{=} \emptyset, \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

6. *Symmetric Oplax Monoidality With Respect to Intersections.* The direct image function of Item 1 has a symmetric oplax monoidal structure

$$\left(R_*, R_*^{\otimes}, R_{*|\mathbb{1}}^{\otimes}\right) \colon (\mathcal{P}(A), \cap, A) \longrightarrow (\mathcal{P}(B), \cap, B),$$

being equipped with inclusions

$$\begin{aligned} R_{*|U,V}^{\otimes} \colon R_*(U \cap V) &\subset R_*(U) \cap R_*(V), \\ R_{*|\mathbb{1}}^{\otimes} \colon R_*(A) &\subset B, \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

7. *Relation to Direct Images With Compact Support.* We have

$$R_*(U) = B \setminus R_!(A \setminus U)$$

for each $U \in \mathcal{P}(A)$.

Proof. *Item 1, Functoriality*: Clear.
Item 2, Adjointness: This follows from Kan Extensions, **??** of **??**.
Item 3, Preservation of Colimits: This follows from Item 2 and Categories, **??** of **??**.
Item 4, Oplax Preservation of Limits: Omitted.
Item 5, Symmetric Strict Monoidality With Respect to Unions: This follows from Item 3.
Item 6, Symmetric Oplax Monoidality With Respect to Intersections: This follows from Item 4.
Item 7, Relation to Direct Images With Compact Support: The proof proceeds in the same way as in the case of functions (Constructions With Sets, Item 7 of Proposition 4.3.1.3): applying Item 7 of Proposition 5.4.1.3 to $A \setminus U$, we have

$$\begin{aligned} R_!(A \setminus U) &= B \setminus R_*(A \setminus (A \setminus U)) \\ &= B \setminus R_*(U). \end{aligned}$$

Taking complements, we then obtain

$$\begin{aligned} R_*(U) &= B \setminus (B \setminus R_*(U)), \\ &= B \setminus R_!(A \setminus U), \end{aligned}$$

which finishes the proof. □

Proposition 5.1.1.4. Let $R \colon A \nrightarrow B$ be a relation.

1. *Functionality I.* The assignment $R \mapsto R_*$ defines a function

$$(-)_* \colon \mathsf{Rel}(A, B) \to \mathsf{Sets}(\mathcal{P}(A), \mathcal{P}(B)).$$

2. *Functionality II.* The assignment $R \mapsto R_*$ defines a function

$$(-)_* \colon \mathsf{Rel}(A, B) \to \mathsf{Pos}((\mathcal{P}(A), \subset), (\mathcal{P}(B), \subset)).$$

3. *Interaction With Identities.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have[59]

$$(\chi_A)_* = \mathrm{id}_{\mathcal{P}(A)};$$

4. *Interaction With Composition.* For each pair of composable relations $R\colon A \nrightarrow B$ and $S\colon B \nrightarrow C$, we have[60]

$$(S \diamond R)_* = S_* \circ R_*, \qquad \begin{tikzcd} \mathcal{P}(A) \arrow[r, "R_*"] \arrow[dr, "(S\diamond R)_*"'] & \mathcal{P}(B) \arrow[d, "S_*"] \\ & \mathcal{P}(C). \end{tikzcd}$$

Proof. *Item 1, Functionality I*: Clear.
Item 2, Functionality II: Clear.
Item 3, Interaction With Identities: Indeed, we have

$$\begin{aligned} (\chi_A)_*(U) &\overset{\mathrm{def}}{=} \bigcup_{a\in U} \chi_A(a) \\ &\overset{\mathrm{def}}{=} \bigcup_{a\in U} \{a\} \\ &= U \\ &\overset{\mathrm{def}}{=} \mathrm{id}_{\mathcal{P}(A)}(U) \end{aligned}$$

for each $U \in \mathcal{P}(A)$. Thus $(\chi_A)_* = \mathrm{id}_{\mathcal{P}(A)}$.

[59] That is, the postcomposition function

$$(\chi_A)_*\colon \mathrm{Rel}(\mathrm{pt}, A) \to \mathrm{Rel}(\mathrm{pt}, A)$$

is equal to $\mathrm{id}_{\mathrm{Rel}(\mathrm{pt},A)}$.

[60] That is, we have

$$(S \diamond R)_* = S_* \circ R_*, \qquad \begin{tikzcd} \mathrm{Rel}(\mathrm{pt}, A) \arrow[r, "R_*"] \arrow[dr, "(S\diamond R)_*"'] & \mathrm{Rel}(\mathrm{pt}, B) \arrow[d, "S_*"] \\ & \mathrm{Rel}(\mathrm{pt}, C). \end{tikzcd}$$

Item 4, Interaction With Composition: Indeed, we have

$$\begin{aligned}
(S \diamond R)_*(U) &\overset{\text{def}}{=} \bigcup_{a \in U} [S \diamond R](a) \\
&\overset{\text{def}}{=} \bigcup_{a \in U} S(R(a)) \\
&\overset{\text{def}}{=} \bigcup_{a \in U} S_*(R(a)) \\
&= S_*\left(\bigcup_{a \in U} R(a)\right) \\
&\overset{\text{def}}{=} S_*(R_*(U)) \\
&\overset{\text{def}}{=} [S_* \circ R_*](U)
\end{aligned}$$

for each $U \in \mathcal{P}(A)$, where we used Item 3 of Proposition 5.1.1.3. Thus $(S \diamond R)_* = S_* \circ R_*$. □

5.2 Strong Inverse Images

Let A and B be sets and let $R \colon A \nrightarrow B$ be a relation.

Definition 5.2.1.1. The **strong inverse image function associated to** R is the function

$$R_{-1} \colon \mathcal{P}(B) \to \mathcal{P}(A)$$

defined by[61]

$$R_{-1}(V) \overset{\text{def}}{=} \{a \in A \mid R(a) \subset V\}$$

for each $V \in \mathcal{P}(B)$.

Remark 5.2.1.2. Identifying subsets of B with relations from pt to B via Constructions With Sets, Item 7 of Proposition 4.2.1.3, we see that the inverse image function associated to R is equivalently the function

$$R_{-1} \colon \underbrace{\mathcal{P}(B)}_{\cong \mathbf{Rel}(\mathrm{pt}, B)} \to \underbrace{\mathcal{P}(A)}_{\cong \mathbf{Rel}(\mathrm{pt}, A)}$$

defined by

$$R_{-1}(V) \overset{\text{def}}{=} \mathrm{Rift}_R(V), \qquad \begin{tikzcd} & A \arrow[d, "R", "\shortmid"{marking}] \arrow[dl, phantom, "\Swarrow"] \\ \mathrm{pt} \arrow[ur, dashed, "\mathrm{Rift}_R(V)", "\shortmid"{marking}] \arrow[r, "V"', "\shortmid"{marking}] & B, \end{tikzcd}$$

[61] *Further Terminology:* The set $R_{-1}(V)$ is called the **strong inverse image of** V **by** R.

and being explicitly computed by

$$\begin{aligned} R_{-1}(V) &\overset{\text{def}}{=} \mathrm{Rift}_R(V) \\ &\cong \int_{x\in B} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^x_{-_1}, V^x_{-_2}\right), \end{aligned}$$

where we have used Item 12 of Proposition 2.5.1.1.

Proof. We have

$$\begin{aligned} \mathrm{Rift}_R(V) &\cong \int_{x\in B} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^x_{-_1}, V^x_{-_2}\right) \\ &= \left\{ a \in A \,\middle|\, \int_{x\in B} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^x_a, V^x_\star\right) = \mathsf{true} \right\} \\ &= \left\{ a \in A \,\middle|\, \begin{array}{l} \text{for each } x \in B\text{, at least one of the} \\ \text{following conditions hold:} \\ \quad 1.\ \text{We have } R^x_a = \mathsf{false} \\ \quad 2.\ \text{The following conditions hold:} \\ \qquad (a)\ \text{We have } R^x_a = \mathsf{true} \\ \qquad (b)\ \text{We have } V^x_\star = \mathsf{true} \end{array} \right\} \\ &= \left\{ a \in A \,\middle|\, \begin{array}{l} \text{for each } x \in B\text{, at least one of the} \\ \text{following conditions hold:} \\ \quad 1.\ \text{We have } x \notin R(a) \\ \quad 2.\ \text{The following conditions hold:} \\ \qquad (a)\ \text{We have } x \in R(a) \\ \qquad (b)\ \text{We have } x \in V \end{array} \right\} \\ &= \{a \in A \mid \text{for each } x \in R(a)\text{, we have } x \in V\} \\ &= \{a \in A \mid R(a) \subset V\} \\ &\overset{\text{def}}{=} R_{-1}(V). \end{aligned}$$

This finishes the proof. □

Proposition 5.2.1.3. Let $R\colon A \nrightarrow B$ be a relation.

1. *Functoriality.* The assignment $V \mapsto R_{-1}(V)$ defines a functor

$$R_{-1}\colon (\mathcal{P}(B), \subset) \to (\mathcal{P}(A), \subset)$$

where

- *Action on Objects.* For each $V \in \mathcal{P}(B)$, we have

$$[R_{-1}](V) \stackrel{\text{def}}{=} R_{-1}(V);$$

- *Action on Morphisms.* For each $U, V \in \mathcal{P}(B)$:
  - If $U \subset V$, then $R_{-1}(U) \subset R_{-1}(V)$.

2. *Adjointness.* We have an adjunction

$$(R_* \dashv R_{-1})\colon \quad \begin{tikzcd} \mathcal{P}(A) \arrow[r, bend left, "R_*"] \arrow[r, phantom, "\perp"] & \mathcal{P}(B), \arrow[l, bend left, "R_{-1}"] \end{tikzcd}$$

witnessed by a bijections of sets

$$\mathrm{Hom}_{\mathcal{P}(A)}(R_*(U), V) \cong \mathrm{Hom}_{\mathcal{P}(A)}(U, R_{-1}(V)),$$

natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$, i.e. such that:

($\star$) The following conditions are equivalent:
  - We have $R_*(U) \subset V$;
  - We have $U \subset R_{-1}(V)$.

3. *Lax Preservation of Colimits.* We have an inclusion of sets

$$\bigcup_{i \in I} R_{-1}(U_i) \subset R_{-1}\left(\bigcup_{i \in I} U_i\right),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(B)^{\times I}$. In particular, we have inclusions

$$R_{-1}(U) \cup R_{-1}(V) \subset R_{-1}(U \cup V),$$
$$\emptyset \subset R_{-1}(\emptyset),$$

natural in $U, V \in \mathcal{P}(B)$.

4. *Preservation of Limits.* We have an equality of sets

$$R_{-1}\left(\bigcap_{i \in I} U_i\right) = \bigcap_{i \in I} R_{-1}(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(B)^{\times I}$. In particular, we have equalities

$$R_{-1}(U \cap V) = R_{-1}(U) \cap R_{-1}(V),$$
$$R_{-1}(B) = B,$$

natural in $U, V \in \mathcal{P}(B)$.

5. *Symmetric Lax Monoidality With Respect to Unions.* The direct image with compact support function of Item 1 has a symmetric lax monoidal structure

$$\left(R_{-1}, R_{-1}^{\otimes}, R_{-1|\mathbb{1}}^{\otimes}\right) \colon (\mathcal{P}(A), \cup, \emptyset) \to (\mathcal{P}(B), \cup, \emptyset),$$

being equipped with inclusions

$$R_{-1|U,V}^{\otimes} \colon R_{-1}(U) \cup R_{-1}(V) \subset R_{-1}(U \cup V),$$
$$R_{-1|\mathbb{1}}^{\otimes} \colon \emptyset \subset R_{-1}(\emptyset),$$

natural in $U, V \in \mathcal{P}(B)$.

6. *Symmetric Strict Monoidality With Respect to Intersections.* The direct image function of Item 1 has a symmetric strict monoidal structure

$$\left(R_{-1}, R_{-1}^{\otimes}, R_{-1|\mathbb{1}}^{\otimes}\right) \colon (\mathcal{P}(A), \cap, A) \to (\mathcal{P}(B), \cap, B),$$

being equipped with equalities

$$R_{-1|U,V}^{\otimes} \colon R_{-1}(U \cap V) \xrightarrow{=} R_{-1}(U) \cap R_{-1}(V),$$
$$R_{-1|\mathbb{1}}^{\otimes} \colon R_{-1}(A) \xrightarrow{=} B,$$

natural in $U, V \in \mathcal{P}(B)$.

7. *Interaction With Weak Inverse Images I.* We have

$$R_{-1}(V) = A \setminus R^{-1}(B \setminus V)$$

for each $V \in \mathcal{P}(B)$.

8. *Interaction With Weak Inverse Images II.* Let $R \colon A \nrightarrow B$ be a relation from A to B.

   (a) If R is a total relation, then we have an inclusion of sets

   $$R_{-1}(V) \subset R^{-1}(V)$$

   natural in $V \in \mathcal{P}(B)$.

   (b) If R is total and functional, then the above inclusion is in fact an equality.

   (c) Conversely, if we have $R_{-1} = R^{-1}$, then R is total and functional.

Proof. *Item 1, Functoriality*: Clear.

Item 2, *Adjointness*: This follows from Kan Extensions, **??** of **??**.
Item 3, *Lax Preservation of Colimits*: Omitted.
Item 4, *Preservation of Limits*: This follows from Item 2 and Categories, **??** of **??**.
Item 5, *Symmetric Lax Monoidality With Respect to Unions*: This follows from Item 3.
Item 6, *Symmetric Strict Monoidality With Respect to Intersections*: This follows from Item 4.
Item 7, *Interaction With Weak Inverse Images I*: We claim we have an equality

$$R_{-1}(B \setminus V) = A \setminus R^{-1}(V).$$

Indeed, we have

$$\begin{aligned} R_{-1}(B \setminus V) &= \{a \in A \mid R(a) \subset B \setminus V\}, \\ A \setminus R^{-1}(V) &= \{a \in A \mid R(a) \cap V = \emptyset\}. \end{aligned}$$

Taking $V = B \setminus V$ then implies the original statement.
Item 8, *Interaction With Weak Inverse Images II*: Item 8a is clear, while Items 8b and 8c follow from Item 6 of Proposition 3.1.1.2. □

Proposition 5.2.1.4. Let $R\colon A \nrightarrow B$ be a relation.

1. *Functionality I.* The assignment $R \mapsto R_{-1}$ defines a function

$$(-)_{-1}\colon \mathsf{Sets}(A, B) \to \mathsf{Sets}(\mathcal{P}(A), \mathcal{P}(B)).$$

2. *Functionality II.* The assignment $R \mapsto R_{-1}$ defines a function

$$(-)_{-1}\colon \mathsf{Sets}(A, B) \to \mathsf{Pos}((\mathcal{P}(A), \subset), (\mathcal{P}(B), \subset)).$$

3. *Interaction With Identities.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$(\mathrm{id}_A)_{-1} = \mathrm{id}_{\mathcal{P}(A)};$$

4. *Interaction With Composition.* For each pair of composable relations $R\colon A \nrightarrow B$ and $S\colon B \nrightarrow C$, we have

$$(S \diamond R)_{-1} = R_{-1} \circ S_{-1}, \qquad \begin{tikzcd} \mathcal{P}(C) \arrow[r, "S_{-1}"] \arrow[dr, "(S\diamond R)_{-1}"'] & \mathcal{P}(B) \arrow[d, "R_{-1}"] \\ & \mathcal{P}(A). \end{tikzcd}$$

Proof. *Item 1, Functionality I*: Clear.
Item 2, Functionality II: Clear.
Item 3, Interaction With Identities: Indeed, we have

$$\begin{aligned}(\chi_A)_{-1}(U) &\overset{\text{def}}{=} \{a \in A \mid \chi_A(a) \subset U\}\\ &\overset{\text{def}}{=} \{a \in A \mid \{a\} \subset U\}\\ &= U\end{aligned}$$

for each $U \in \mathcal{P}(A)$. Thus $(\chi_A)_{-1} = \mathrm{id}_{\mathcal{P}(A)}$.
Item 4, Interaction With Composition: Indeed, we have

$$\begin{aligned}(S \diamond R)_{-1}(U) &\overset{\text{def}}{=} \{a \in A \mid [S \diamond R](a) \subset U\}\\ &\overset{\text{def}}{=} \{a \in A \mid S(R(a)) \subset U\}\\ &\overset{\text{def}}{=} \{a \in A \mid S_*(R(a)) \subset U\}\\ &= \{a \in A \mid R(a) \subset S_{-1}(U)\}\\ &\overset{\text{def}}{=} R_{-1}(S_{-1}(U))\\ &\overset{\text{def}}{=} [R_{-1} \circ S_{-1}](U)\end{aligned}$$

for each $U \in \mathcal{P}(C)$, where we used Item 2 of Proposition 5.2.1.3, which implies that the conditions

- We have $S_*(R(a)) \subset U$;
- We have $R(a) \subset S_{-1}(U)$;

are equivalent. Thus $(S \diamond R)_{-1} = R_{-1} \circ S_{-1}$. □

5.3 Weak Inverse Images

Let A and B be sets and let $R\colon A \nrightarrow B$ be a relation.

Definition 5.3.1.1. The **weak inverse image function associated to** R[62] is the function

$$R^{-1}\colon \mathcal{P}(B) \to \mathcal{P}(A)$$

defined by[63]

$$R^{-1}(V) \overset{\text{def}}{=} \{a \in A \mid R(a) \cap V \neq \emptyset\}$$

for each $V \in \mathcal{P}(B)$.

[62] *Further Terminology:* Also called simply the **inverse image function associated to** R.
[63] *Further Terminology:* The set $R^{-1}(V)$ is called the **weak inverse image of** V **by** R or simply the **inverse**

Remark 5.3.1.2. Identifying subsets of B with relations from B to pt via Constructions With Sets, Item 7 of Proposition 4.2.1.3, we see that the weak inverse image function associated to R is equivalently the function

$$R^{-1}\colon \underbrace{\mathcal{P}(B)}_{\cong \mathrm{Rel}(B,\mathrm{pt})} \to \underbrace{\mathcal{P}(A)}_{\cong \mathrm{Rel}(A,\mathrm{pt})}$$

defined by

$$R^{-1}(V) \stackrel{\text{def}}{=} V \diamond R$$

for each $V \in \mathcal{P}(A)$, where $R \diamond V$ is the composition

$$A \overset{R}{\nrightarrow} B \overset{V}{\nrightarrow} \mathrm{pt}.$$

Explicitly, we have

$$\begin{aligned} R^{-1}(V) &\stackrel{\text{def}}{=} V \diamond R \\ &\stackrel{\text{def}}{=} \int^{x \in B} V^{-1}_{x} \times R^{x}_{-_2}. \end{aligned}$$

Proof. We have

$$\begin{aligned} V \diamond R &\stackrel{\text{def}}{=} \int^{x \in B} V^{-1}_{x} \times R^{x}_{-_2} \\ &= \left\{ a \in A \;\middle|\; \int^{x \in B} V^{\star}_{x} \times R^{x}_{a} = \mathsf{true} \right\} \\ &= \left\{ a \in A \;\middle|\; \begin{array}{l} \text{there exists } x \in B \text{ such that the} \\ \text{following conditions hold:} \\ \quad 1.\ \text{We have } V^{\star}_{x} = \mathsf{true} \\ \quad 2.\ \text{We have } R^{x}_{a} = \mathsf{true} \end{array} \right\} \\ &= \left\{ a \in A \;\middle|\; \begin{array}{l} \text{there exists } x \in B \text{ such that the} \\ \text{following conditions hold:} \\ \quad 1.\ \text{We have } x \in V \\ \quad 2.\ \text{We have } x \in R(a) \end{array} \right\} \\ &= \{ a \in A \mid \text{there exists } x \in V \text{ such that } x \in R(a) \} \\ &= \{ a \in A \mid R(a) \cap V \neq \emptyset \} \\ &\stackrel{\text{def}}{=} R^{-1}(V) \end{aligned}$$

This finishes the proof. □

Proposition 5.3.1.3. Let $R\colon A \nrightarrow B$ be a relation.

1. *Functoriality.* The assignment $V \mapsto R^{-1}(V)$ defines a functor

$$R^{-1}\colon (\mathcal{P}(B), \subset) \to (\mathcal{P}(A), \subset)$$

where

- *Action on Objects.* For each $V \in \mathcal{P}(B)$, we have

$$[R^{-1}](V) \stackrel{\text{def}}{=} R^{-1}(V);$$

- *Action on Morphisms.* For each $U, V \in \mathcal{P}(B)$:
  - If $U \subset V$, then $R^{-1}(U) \subset R^{-1}(V)$.

2. *Adjointness.* We have an adjunction

$$\left(R^{-1} \dashv R_!\right)\colon \quad \begin{tikzcd} \mathcal{P}(B) \arrow[r, bend left, "R^{-1}"] \arrow[r, phantom, "\perp"] & \mathcal{P}(A), \arrow[l, bend left, "R_!"] \end{tikzcd}$$

witnessed by a bijections of sets

$$\mathrm{Hom}_{\mathcal{P}(A)}\left(R^{-1}(U), V\right) \cong \mathrm{Hom}_{\mathcal{P}(A)}(U, R_!(V)),$$

natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$, i.e. such that:

($\star$) The following conditions are equivalent:
  - We have $R^{-1}(U) \subset V$;
  - We have $U \subset R_!(V)$.

3. *Preservation of Colimits.* We have an equality of sets

$$R^{-1}\left(\bigcup_{i \in I} U_i\right) = \bigcup_{i \in I} R^{-1}(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(B)^{\times I}$. In particular, we have equalities

$$R^{-1}(U) \cup R^{-1}(V) = R^{-1}(U \cup V),$$
$$R^{-1}(\emptyset) = \emptyset,$$

natural in $U, V \in \mathcal{P}(B)$.

4. *Oplax Preservation of Limits.* We have an inclusion of sets

$$R^{-1}\left(\bigcap_{i\in I} U_i\right) \subset \bigcap_{i\in I} R^{-1}(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(B)^{\times I}$. In particular, we have inclusions

$$\begin{gathered} R^{-1}(U \cap V) \subset R^{-1}(U) \cap R^{-1}(V), \\ R^{-1}(A) \subset B, \end{gathered}$$

natural in $U, V \in \mathcal{P}(B)$.

5. *Symmetric Strict Monoidality With Respect to Unions.* The direct image function of Item 1 has a symmetric strict monoidal structure

$$\left(R^{-1}, R^{-1,\otimes}, R^{-1,\otimes}_{\mathbb{1}}\right) \colon (\mathcal{P}(A), \cup, \emptyset) \longrightarrow (\mathcal{P}(B), \cup, \emptyset),$$

being equipped with equalities

$$\begin{gathered} R^{-1,\otimes}_{U,V} \colon R^{-1}(U) \cup R^{-1}(V) \xrightarrow{=} R^{-1}(U \cup V), \\ R^{-1,\otimes}_{\mathbb{1}} \colon \emptyset \xrightarrow{=} \emptyset, \end{gathered}$$

natural in $U, V \in \mathcal{P}(B)$.

6. *Symmetric Oplax Monoidality With Respect to Intersections.* The direct image function of Item 1 has a symmetric oplax monoidal structure

$$\left(R^{-1}, R^{-1,\otimes}, R^{-1,\otimes}_{\mathbb{1}}\right) \colon (\mathcal{P}(A), \cap, A) \longrightarrow (\mathcal{P}(B), \cap, B),$$

being equipped with inclusions

$$\begin{gathered} R^{-1,\otimes}_{U,V} \colon R^{-1}(U \cap V) \subset R^{-1}(U) \cap R^{-1}(V), \\ R^{-1,\otimes}_{\mathbb{1}} \colon R^{-1}(A) \subset B, \end{gathered}$$

natural in $U, V \in \mathcal{P}(B)$.

7. *Interaction With Strong Inverse Images I.* We have

$$R^{-1}(V) = A \setminus R_{-1}(B \setminus V)$$

for each $V \in \mathcal{P}(B)$.

8. *Interaction With Strong Inverse Images II.* Let $R \colon A \nrightarrow B$ be a relation from A to B.

   (a) If R is a total relation, then we have an inclusion of sets

   $$R_{-1}(V) \subset R^{-1}(V)$$

   natural in $V \in \mathcal{P}(B)$.

   (b) If R is total and functional, then the above inclusion is in fact an equality.

   (c) Conversely, if we have $R_{-1} = R^{-1}$, then R is total and functional.

Proof. *Item 1*, *Functoriality*: Clear.
Item 2, *Adjointness*: This follows from Kan Extensions, **??** of **??**.
Item 3, *Preservation of Colimits*: This follows from Item 2 and Categories, **??** of **??**.
Item 4, *Oplax Preservation of Limits*: Omitted.
Item 5, *Symmetric Strict Monoidality With Respect to Unions*: This follows from Item 3.
Item 6, *Symmetric Oplax Monoidality With Respect to Intersections*: This follows from Item 4.
Item 7, *Interaction With Strong Inverse Images I*: This follows from Item 7 of Proposition 5.2.1.3.
Item 8, *Interaction With Strong Inverse Images II*: This was proved in Item 8 of Proposition 5.2.1.3. □

Proposition 5.3.1.4. Let $R \colon A \nrightarrow B$ be a relation.

1. *Functionality I.* The assignment $R \mapsto R^{-1}$ defines a function

   $$(-)^{-1} \colon \mathrm{Rel}(A, B) \to \mathsf{Sets}(\mathcal{P}(A), \mathcal{P}(B)).$$

2. *Functionality II.* The assignment $R \mapsto R^{-1}$ defines a function

   $$(-)^{-1} \colon \mathrm{Rel}(A, B) \to \mathsf{Pos}((\mathcal{P}(A), \subset), (\mathcal{P}(B), \subset)).$$

3. *Interaction With Identities.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have[64]

   $$(\chi_A)^{-1} = \mathrm{id}_{\mathcal{P}(A)};$$

image of V **by** R.

[64]That is, the postcomposition

$$(\chi_A)^{-1} \colon \mathrm{Rel}(\mathrm{pt}, A) \to \mathrm{Rel}(\mathrm{pt}, A)$$

is equal to $\mathrm{id}_{\mathrm{Rel}(\mathrm{pt}, A)}$.

4. *Interaction With Composition.* For each pair of composable relations $R\colon A \mathrel{\mapstochar\rightarrow} B$ and $S\colon B \mathrel{\mapstochar\rightarrow} C$, we have[65]

$$(S \diamond R)^{-1} = R^{-1} \circ S^{-1}, \qquad \begin{tikzcd} \mathcal{P}(C) \arrow[r, "S^{-1}"] \arrow[dr, "(S\diamond R)^{-1}"'] & \mathcal{P}(B) \arrow[d, "R^{-1}"] \\ & \mathcal{P}(A). \end{tikzcd}$$

Proof. *Item 1*, *Functionality I*: Clear.
Item 2, *Functionality II*: Clear.
Item 3, *Interaction With Identities*: This follows from Categories, **??** of **??**.
Item 4, *Interaction With Composition*: This follows from Categories, **??** of **??**. □

5.4 Direct Images With Compact Support

Let A and B be sets and let $R\colon A \mathrel{\mapstochar\rightarrow} B$ be a relation.

Definition 5.4.1.1. The **direct image with compact support function associated to** R is the function[66]

$$R_!\colon \mathcal{P}(A) \to \mathcal{P}(B)$$

[65] That is, we have

$$(S \diamond R)^{-1} = R^{-1} \circ S^{-1}, \qquad \begin{tikzcd} \mathrm{Rel}(\mathrm{pt}, C) \arrow[r, "R^{-1}"] \arrow[dr, "(S\diamond R)^{-1}"'] & \mathrm{Rel}(\mathrm{pt}, B) \arrow[d, "S^{-1}"] \\ & \mathrm{Rel}(\mathrm{pt}, A). \end{tikzcd}$$

[66] *Further Notation:* Also written $\forall_R\colon \mathcal{P}(A) \to \mathcal{P}(B)$. This notation comes from the fact that the following statements are equivalent, where $b \in B$ and $U \in \mathcal{P}(A)$:

- We have $b \in \forall_R(U)$.
- For each $a \in A$, if $b \in R(a)$, then $a \in U$.

defined by[67,68]

$$\begin{aligned} R_!(U) &\overset{\text{def}}{=} \left\{ b \in B \;\middle|\; \begin{array}{l} \text{for each } a \in A, \text{ if we have} \\ b \in R(a), \text{ then } a \in U \end{array} \right\} \\ &= \left\{ b \in B \;\middle|\; R^{-1}(b) \subset U \right\} \end{aligned}$$

for each $U \in \mathcal{P}(A)$.

Remark 5.4.1.2. Identifying subsets of B with relations from pt to B via Constructions With Sets, Item 7 of Proposition 4.2.1.3, we see that the direct image with compact support function associated to R is equivalently the function

$$R_! \colon \underbrace{\mathcal{P}(A)}_{\cong \mathrm{Rel}(A, \mathrm{pt})} \to \underbrace{\mathcal{P}(B)}_{\cong \mathrm{Rel}(B, \mathrm{pt})}$$

defined by

$$R_!(U) \overset{\text{def}}{=} \mathrm{Ran}_R(U), \qquad \begin{tikzcd} & B \arrow[d, dashed, "\mathrm{Ran}_R(U)"] \\ A \arrow[ur, "R", "\times" marking] \arrow[r, "U"', "\mid" marking] \arrow[ur, phantom, "\Leftarrow"] & \mathrm{pt}, \end{tikzcd}$$

being explicitly computed by

$$\begin{aligned} R^*(U) &\overset{\text{def}}{=} \mathrm{Ran}_R(U) \\ &\cong \int_{a \in A} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R_a^{-2}, U_a^{-1}\right), \end{aligned}$$

where we have used Item 11 of Proposition 2.5.1.1.

[67] *Further Terminology:* The set $R_!(U)$ is called the **direct image with compact support of** U **by** R.

[68] We also have

$$R_!(U) = B \setminus R_*(A \setminus U);$$

see Item 7 of Proposition 5.4.1.3.

Proof. We have

$$
\begin{aligned}
\mathrm{Ran}_R(V) &\cong \int_{a\in A} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^{-_2}_a, U^{-_1}_a\right)\\
&= \left\{ b\in B \;\middle|\; \int_{a\in A} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^b_a, U^{\star}_a\right) = \mathsf{true} \right\}\\
&= \left\{ b\in B \;\middle|\; \begin{array}{l} \text{for each } a\in A\text{, at least one of the} \\ \text{following conditions hold:} \\ \\ \quad 1.\ \text{We have } R^b_a = \mathsf{false} \\ \quad 2.\ \text{The following conditions hold:} \\ \\ \qquad (a)\ \text{We have } R^b_a = \mathsf{true} \\ \qquad (b)\ \text{We have } U^{\star}_a = \mathsf{true} \end{array} \right\}\\
&= \left\{ b\in B \;\middle|\; \begin{array}{l} \text{for each } a\in A\text{, at least one of the} \\ \text{following conditions hold:} \\ \\ \quad 1.\ \text{We have } b\notin R(A) \\ \quad 2.\ \text{The following conditions hold:} \\ \\ \qquad (a)\ \text{We have } b\in R(a) \\ \qquad (b)\ \text{We have } a\in U \end{array} \right\}\\
&= \left\{ b\in B \;\middle|\; \begin{array}{l} \text{for each } a\in A\text{, if we have} \\ b\in R(a)\text{, then } a\in U \end{array} \right\}\\
&= \left\{ b\in B \;\middle|\; R^{-1}(b)\subset U \right\}\\
&\overset{\text{def}}{=} R^{-1}(U).
\end{aligned}
$$

This finishes the proof. □

Proposition 5.4.1.3. Let $R\colon A \nrightarrow B$ be a relation.

1. *Functoriality.* The assignment $U \mapsto R_!(U)$ defines a functor

$$R_!\colon (\mathcal{P}(A), \subset) \to (\mathcal{P}(B), \subset)$$

   where

   - *Action on Objects.* For each $U\in\mathcal{P}(A)$, we have

$$[R_!](U) \overset{\text{def}}{=} R_!(U);$$

- *Action on Morphisms.* For each $U, V \in \mathcal{P}(A)$:
  - If $U \subset V$, then $R_!(U) \subset R_!(V)$.

2. *Adjointness.* We have an adjunction

$$\left(R^{-1} \dashv R_!\right)\colon \quad \begin{tikzcd} \mathcal{P}(B) \arrow[r, bend left, "R^{-1}"] \arrow[r, phantom, "\perp"] & \mathcal{P}(A), \arrow[l, bend left, "R_!"] \end{tikzcd}$$

witnessed by a bijections of sets

$$\mathrm{Hom}_{\mathcal{P}(A)}\left(R^{-1}(U), V\right) \cong \mathrm{Hom}_{\mathcal{P}(A)}(U, R_!(V)),$$

natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$, i.e. such that:

($\star$) The following conditions are equivalent:
  - We have $R^{-1}(U) \subset V$;
  - We have $U \subset R_!(V)$.

3. *Lax Preservation of Colimits.* We have an inclusion of sets

$$\bigcup_{i \in I} R_!(U_i) \subset R_!\left(\bigcup_{i \in I} U_i\right),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have inclusions

$$R_!(U) \cup R_!(V) \subset R_!(U \cup V),$$
$$\emptyset \subset R_!(\emptyset),$$

natural in $U, V \in \mathcal{P}(A)$.

4. *Preservation of Limits.* We have an equality of sets

$$R_!\left(\bigcap_{i \in I} U_i\right) = \bigcap_{i \in I} R_!(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have equalities

$$R_!(U \cap V) = R_!(U) \cap R_!(V),$$
$$R_!(A) = B,$$

natural in $U, V \in \mathcal{P}(A)$.

5. *Symmetric Lax Monoidality With Respect to Unions.* The direct image with compact support function of Item 1 has a symmetric lax monoidal structure

$$\left(R_!, R_!^{\otimes}, R_{!|\mathbb{1}}^{\otimes}\right) : (\mathcal{P}(A), \cup, \emptyset) \to (\mathcal{P}(B), \cup, \emptyset),$$

being equipped with inclusions

$$R_{!|U,V}^{\otimes} : R_!(U) \cup R_!(V) \subset R_!(U \cup V),$$
$$R_{!|\mathbb{1}}^{\otimes} : \emptyset \subset R_!(\emptyset),$$

natural in $U, V \in \mathcal{P}(A)$.

6. *Symmetric Strict Monoidality With Respect to Intersections.* The direct image function of Item 1 has a symmetric strict monoidal structure

$$\left(R_!, R_!^{\otimes}, R_{!|\mathbb{1}}^{\otimes}\right) : (\mathcal{P}(A), \cap, A) \to (\mathcal{P}(B), \cap, B),$$

being equipped with equalities

$$R_{!|U,V}^{\otimes} : R_!(U \cap V) \xrightarrow{=} R_!(U) \cap R_!(V),$$
$$R_{!|\mathbb{1}}^{\otimes} : R_!(A) \xrightarrow{=} B,$$

natural in $U, V \in \mathcal{P}(A)$.

7. *Relation to Direct Images.* We have

$$R_!(U) = B \setminus R_*(A \setminus U)$$

for each $U \in \mathcal{P}(A)$.

Proof. *Item 1, Functoriality*: Clear.
Item 2, Adjointness: This follows from Kan Extensions, **??** of **??**.
Item 3, Lax Preservation of Colimits: Omitted.
Item 4, Preservation of Limits: This follows from Item 2 and Categories, **??** of **??**.
Item 5, Symmetric Lax Monoidality With Respect to Unions: This follows from Item 3.
Item 6, Symmetric Strict Monoidality With Respect to Intersections: This follows from Item 4.
Item 7, Relation to Direct Images: This follows from Item 7 of Proposition 5.1.1.3. Alternatively, we may prove it directly as follows, with the proof proceeding in the same way as in the case of functions (Constructions With Sets, Item 7 of Proposition 4.5.1.5).
We claim that $R_!(U) = B \setminus R_*(A \setminus U)$:

- *The First Implication.* We claim that

$$R_!(U) \subset B \setminus R_*(A \setminus U).$$

Let $b \in R_!(U)$. We need to show that $b \notin R_*(A \setminus U)$, i.e. that there is no $a \in A \setminus U$ such that $b \in R(a)$.

This is indeed the case, as otherwise we would have $a \in R^{-1}(b)$ and $a \notin U$, contradicting $R^{-1}(b) \subset U$ (which holds since $b \in R_!(U)$).

Thus $b \in B \setminus R_*(A \setminus U)$.

- *The Second Implication.* We claim that

$$B \setminus R_*(A \setminus U) \subset R_!(U).$$

Let $b \in B \setminus R_*(A \setminus U)$. We need to show that $b \in R_!(U)$, i.e. that $R^{-1}(b) \subset U$.

Since $b \notin R_*(A \setminus U)$, there exists no $a \in A \setminus U$ such that $b \in R(a)$, and hence $R^{-1}(b) \subset U$.

Thus $b \in R_!(U)$.

This finishes the proof. □

Proposition 5.4.1.4. Let $R\colon A \nrightarrow B$ be a relation.

1. *Functionality I.* The assignment $R \mapsto R_!$ defines a function

$$(-)_!\colon \mathsf{Sets}(A, B) \to \mathsf{Sets}(\mathcal{P}(A), \mathcal{P}(B)).$$

2. *Functionality II.* The assignment $R \mapsto R_!$ defines a function

$$(-)_!\colon \mathsf{Sets}(A, B) \to \mathrm{Hom}_{\mathsf{Pos}}((\mathcal{P}(A), \subset), (\mathcal{P}(B), \subset)).$$

3. *Interaction With Identities.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$(\mathrm{id}_A)_! = \mathrm{id}_{\mathcal{P}(A)};$$

4. *Interaction With Composition.* For each pair of composable relations $R\colon A \nrightarrow B$ and $S\colon B \nrightarrow C$, we have

$$(S \diamond R)_! = S_! \circ R_!, \qquad \begin{tikzcd} \mathcal{P}(A) \arrow[r, "R_!"] \arrow[dr, "(S\diamond R)_!"'] & \mathcal{P}(B) \arrow[d, "S_!"] \\ & \mathcal{P}(C). \end{tikzcd}$$

Proof. *Item 1, Functionality I*: Clear.
Item 2, Functionality II: Clear.
Item 3, Interaction With Identities: Indeed, we have

$$\begin{aligned}(\chi_A)_!(U) &\overset{\text{def}}{=} \left\{a \in A \,\middle|\, \chi_A^{-1}(a) \subset U\right\}\\ &\overset{\text{def}}{=} \{a \in A \mid \{a\} \subset U\}\\ &= U\end{aligned}$$

for each $U \in \mathcal{P}(A)$. Thus $(\chi_A)_! = \mathrm{id}_{\mathcal{P}(A)}$.
Item 4, Interaction With Composition: Indeed, we have

$$\begin{aligned}(S \diamond R)_!(U) &\overset{\text{def}}{=} \left\{c \in C \,\middle|\, [S \diamond R]^{-1}(c) \subset U\right\}\\ &\overset{\text{def}}{=} \left\{c \in C \,\middle|\, S^{-1}\left(R^{-1}(c)\right) \subset U\right\}\\ &= \left\{c \in C \,\middle|\, R^{-1}(c) \subset S_!(U)\right\}\\ &\overset{\text{def}}{=} R_!(S_!(U))\\ &\overset{\text{def}}{=} [R_! \circ S_!](U)\end{aligned}$$

for each $U \in \mathcal{P}(C)$, where we used Item 2 of Proposition 5.4.1.3, which implies that the conditions

- We have $S^{-1}\left(R^{-1}(c)\right) \subset U$;
- We have $R^{-1}(c) \subset S_!(U)$;

are equivalent. Thus $(S \diamond R)_! = S_! \circ R_!$. □

5.5 Functoriality of Powersets

Proposition 5.5.1.1. The assignment $X \mapsto \mathcal{P}(X)$ defines functors[69]

$$\begin{aligned}\mathcal{P}_* &\colon \mathsf{Rel} \to \mathsf{Sets},\\ \mathcal{P}_{-1} &\colon \mathsf{Rel}^{\mathsf{op}} \to \mathsf{Sets},\\ \mathcal{P}^{-1} &\colon \mathsf{Rel}^{\mathsf{op}} \to \mathsf{Sets},\\ \mathcal{P}_! &\colon \mathsf{Rel} \to \mathsf{Sets}\end{aligned}$$

where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathsf{Rel})$, we have

$$\begin{aligned}\mathcal{P}_*(A) &\overset{\text{def}}{=} \mathcal{P}(A),\\ \mathcal{P}_{-1}(A) &\overset{\text{def}}{=} \mathcal{P}(A),\\ \mathcal{P}^{-1}(A) &\overset{\text{def}}{=} \mathcal{P}(A),\\ \mathcal{P}_!(A) &\overset{\text{def}}{=} \mathcal{P}(A);\end{aligned}$$

[69] The functor $\mathcal{P}_* \colon \mathsf{Rel} \to \mathsf{Sets}$ admits a left adjoint; see Item 3 of Proposition 3.1.1.2.

- *Action on Morphisms.* For each morphism $R\colon A \mathrel{\mapstochar\rightarrow} B$ of Rel, the images

$$\begin{aligned}\mathcal{P}_*(R)&\colon \mathcal{P}(A) \to \mathcal{P}(B),\\ \mathcal{P}_{-1}(R)&\colon \mathcal{P}(B) \to \mathcal{P}(A),\\ \mathcal{P}^{-1}(R)&\colon \mathcal{P}(B) \to \mathcal{P}(A),\\ \mathcal{P}_!(R)&\colon \mathcal{P}(A) \to \mathcal{P}(B)\end{aligned}$$

  of R by $\mathcal{P}_*$, $\mathcal{P}_{-1}$, $\mathcal{P}^{-1}$, and $\mathcal{P}_!$ are defined by

$$\begin{aligned}\mathcal{P}_*(R) &\overset{\text{def}}{=} R_*,\\ \mathcal{P}_{-1}(R) &\overset{\text{def}}{=} R_{-1},\\ \mathcal{P}^{-1}(R) &\overset{\text{def}}{=} R^{-1},\\ \mathcal{P}_!(R) &\overset{\text{def}}{=} R_!,\end{aligned}$$

  as in Definitions 5.1.1.1, 5.2.1.1, 5.3.1.1 and 5.4.1.1.

Proof. This follows from Items 3 and 4 of Proposition 5.1.1.4, Items 3 and 4 of Proposition 5.2.1.4, Items 3 and 4 of Proposition 5.3.1.4, and Items 3 and 4 of Proposition 5.4.1.4. □

5.6 Functoriality of Powersets: Relations on Powersets

Let A and B be sets and let $R\colon A \mathrel{\mapstochar\rightarrow} B$ be a relation.

Definition 5.6.1.1. The **relation on powersets associated to** R is the relation

$$\mathcal{P}(R)\colon \mathcal{P}(A) \mathrel{\mapstochar\rightarrow} \mathcal{P}(B)$$

defined by[70]

$$\mathcal{P}(R)^V_U \overset{\text{def}}{=} \mathbf{Rel}\big(\chi_{\mathrm{pt}}, V \diamond R \diamond U\big)$$

for each $U \in \mathcal{P}(A)$ and each $V \in \mathcal{P}(B)$.

Remark 5.6.1.2. In detail, we have $U \sim_{\mathcal{P}(R)} V$ iff the following equivalent conditions hold:

- We have $\chi_{\mathrm{pt}} \subset V \diamond R \diamond U$.

[70]Illustration:

$$\begin{tikzcd} \mathrm{pt} \arrow[r, "U"'] \arrow[rrr, bend left, "\chi_{\mathrm{pt}}"] & A \arrow[r, "R"'] & B \arrow[r, "V"'] & \mathrm{pt}. \end{tikzcd}$$

- We have $(V \diamond R \diamond U)^{\star}_{\star} = \mathsf{true}$, i.e. we have

$$\int^{a \in A} \int^{b \in B} V^{\star}_{b} \times R^{b}_{a} \times U^{a}_{\star} = \mathsf{true}.$$

- There exists some $a \in A$ and some $b \in B$ such that:
  - We have $U^{a}_{\star} = \mathsf{true}$;
  - We have $R^{b}_{a} = \mathsf{true}$;
  - We have $V^{\star}_{b} = \mathsf{true}$.
- There exists some $a \in A$ and some $b \in B$ such that:
  - We have $a \in U$;
  - We have $a \sim_R b$;
  - We have $b \in V$.

Proposition 5.6.1.3. The assignment $R \mapsto \mathcal{P}(R)$ defines a functor

$$\mathcal{P} \colon \mathrm{Rel} \to \mathrm{Rel}.$$

Proof. Omitted. $\square$

6 Relative Preorders

6.1 The Left Skew Monoidal Structure on $\mathbf{Rel}(A, B)$

Let A and B be sets and let $J \colon A \nrightarrow B$ be a relation.

6.1.1 The Left Skew Monoidal Product

Definition 6.1.1.1. The **left J-skew monoidal product of $\mathbf{Rel}(A, B)$** is the functor

$$\lhd_J \colon \mathbf{Rel}(A, B) \times \mathbf{Rel}(A, B) \to \mathbf{Rel}(A, B)$$

where

- *Action on Objects.* For each $R, S \in \mathrm{Obj}(\mathbf{Rel}(A, B))$, we have

$$S \lhd_J R \overset{\mathrm{def}}{=} S \diamond \mathrm{Rift}_J(R), \qquad \begin{tikzcd} & A \arrow[r, "S", "\shortmid"{marking}] \arrow[d, "J", "\shortmid"{marking}] & B; \\ A \arrow[ur, dashed, "\mathrm{Rift}_J(R)", "\shortmid"{marking}] \arrow[r, "R"', "\shortmid"{marking}] & B \arrow[ul, phantom, "\Leftarrow"] & \end{tikzcd}$$

- *Action on Morphisms.* For each $R, S, R', S' \in \mathrm{Obj}(\mathbf{Rel}(A, B))$, the action on Hom-sets

  $$(\triangleleft_J)_{(G,F),(G',F')} \colon \mathrm{Hom}_{\mathbf{Rel}(A,B)}(S, S') \times \mathrm{Hom}_{\mathbf{Rel}(A,B)}(R, R') \to \mathrm{Hom}_{\mathbf{Rel}(A,B)}(S \triangleleft_J R, S' \triangleleft_J R')$$

  of $\triangleleft_J$ at $((R, S), (R', S'))$ is defined by[71]

  $$\beta \triangleleft_J \alpha \overset{\text{def}}{=} \beta \diamond \mathrm{Rift}_J(\alpha), \qquad \begin{tikzcd} & A \arrow[r, bend left, "S"] \arrow[r, bend right, "S'"'] \arrow[r, phantom, "\beta\Downarrow"] \arrow[d, "J"] & B \\ A \arrow[ur, dashed, bend left=40, "\mathrm{Rift}_J(R)"] \arrow[ur, dashed, "\mathrm{Rift}_J(R')"'] \arrow[ur, phantom, bend left=20, "\mathrm{Rift}_J(\alpha)\Downarrow"] \arrow[r, "R"] \arrow[r, bend right=40, "R'"'] \arrow[r, phantom, bend right=20, "\alpha\Downarrow"] & B \end{tikzcd}$$

  for each $\beta \in \mathrm{Hom}_{\mathbf{Rel}(A,B)}(S, S')$ and each $\alpha \in \mathrm{Hom}_{\mathbf{Rel}(A,B)}(R, R')$.

6.1.2 The Left Skew Monoidal Unit

Definition 6.1.2.1. The **left J-skew monoidal unit of Rel**(A, B) is the functor

$$\mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleleft} \colon \mathsf{pt} \to \mathbf{Rel}(A, B)$$

picking the object

$$\mathbb{1}^{\triangleleft}_{\mathbf{Rel}(A,B)} \overset{\text{def}}{=} J$$

of **Rel**(A, B).

6.1.3 The Left Skew Associators

Definition 6.1.3.1. The **left J-skew associator of Rel**(A, B) is the natural transformation

$$\alpha^{\mathbf{Rel}(A,B),\triangleleft} \colon \triangleleft_J \circ (\triangleleft_J \times \mathrm{id}) \Longrightarrow \triangleleft_J \circ (\mathrm{id} \times \triangleleft_J),$$

whose component

$$\alpha^{\mathbf{Rel}(A,B),\triangleleft}_{T,S,R} \colon \underbrace{(T \triangleleft_J S) \triangleleft_J R}_{\overset{\text{def}}{=} T \diamond \mathrm{Rift}_J(S) \diamond \mathrm{Rift}_J(R)} \hookrightarrow \underbrace{T \triangleleft_J (S \triangleleft_J R)}_{\overset{\text{def}}{=} T \diamond \mathrm{Rift}_J(S \diamond \mathrm{Rift}_J(R))}$$

[71] Since **Rel**(A, B) is posetal, this is to say that if $S \subset S'$ and $R \subset R'$, then $S \triangleleft_J R \subset S' \triangleleft_J R'$.

at (T, S, R) is given by

$$\alpha^{\mathbf{Rel}(A,B),\triangleleft}_{T,S,R} \overset{\text{def}}{=} \mathrm{id}_T \diamond \gamma,$$

where

$$\gamma\colon \mathrm{Rift}_J(S) \diamond \mathrm{Rift}_J(R) \hookrightarrow \mathrm{Rift}_J\big(S \diamond \mathrm{Rift}_J(R)\big)$$

is the inclusion adjunct to the inclusion

$$\epsilon_S \circ \mathrm{id}_{\mathrm{Rift}_J(R)}\colon \underbrace{J \diamond \mathrm{Rift}_J(S) \diamond \mathrm{Rift}_J(R)}_{\overset{\text{def}}{=} J_*(\mathrm{Rift}_J(S) \diamond \mathrm{Rift}_J(R))} \hookrightarrow S \diamond \mathrm{Rift}_J(R)$$

under the adjunction $J_* \dashv \mathrm{Rift}_J$, where $\epsilon\colon J \diamond \mathrm{Rift}_J \Longrightarrow \mathrm{id}_{\mathbf{Rel}(A,B)}$ is the counit of the adjunction $J_* \dashv \mathrm{Rift}_J$.

6.1.4 The Left Skew Left Unitors

Definition 6.1.4.1. The **left J-skew left unitor of Rel(A, B)** is the natural transformation

$$\lambda^{\mathbf{Rel}(A,B),\triangleleft}\colon \triangleleft_J \circ \left(\mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleleft} \times \mathrm{id}\right) \Longrightarrow \mathrm{id},$$

whose component

$$\lambda^{\mathbf{Rel}(A,B),\triangleleft}_R\colon \underbrace{J \triangleleft_J R}_{\overset{\text{def}}{=} J \diamond \mathrm{Rift}_J(R)} \hookrightarrow R$$

at R is given by

$$\lambda^{\mathbf{Rel}(A,B),\triangleleft}_R \overset{\text{def}}{=} \epsilon_R,$$

where $\epsilon\colon J \diamond \mathrm{Rift}_J \Longrightarrow \mathrm{id}_{\mathbf{Rel}(A,B)}$ is the counit of the adjunction $J_* \dashv \mathrm{Rift}_J$.

6.1.5 The Left Skew Right Unitors

Definition 6.1.5.1. The **left J-skew right unitor of Rel(A, B)** is the natural transformation

$$\rho^{\mathbf{Rel}(A,B),\triangleleft}\colon \mathrm{id} \Longrightarrow \triangleleft_J \circ \left(\mathrm{id} \times \mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleleft}\right),$$

whose component

$$\rho^{\mathbf{Rel}(A,B),\triangleleft}_R\colon R \hookrightarrow \underbrace{R \triangleleft_J J}_{\overset{\text{def}}{=} R \diamond \mathrm{Rift}_J(J)}$$

at R is given by

$$\rho^{\mathbf{Rel}(A,B),\triangleleft}_R \overset{\text{def}}{=} \mathrm{id}_R \circ \sigma,$$

where $\sigma\colon \mathrm{id}_A \Longrightarrow \mathrm{Rift}_J(J)$ is the universal transformation included in the data of the right Kan lift $\mathrm{Rift}_J(J)$.

6.1.6 The Left Skew Monoidal Structure on $\mathbf{Rel}(A, B)$

Definition 6.1.6.1. The **left J-skew monoidal category of relations from A to B** is the left skew monoidal category

$$\left(\mathbf{Rel}(A, B), \lhd_J, \Bbbk_{\lhd}^{\mathbf{Rel}(A,B)}, \alpha^{\mathbf{Rel}(A,B),\lhd}, \lambda^{\mathbf{Rel}(A,B),\lhd}, \rho^{\mathbf{Rel}(A,B),\lhd}\right)$$

consisting of

- *The Underlying Category.* The posetal category associated to the poset $\mathbf{Rel}(A, B)$ of relations from A to B of Item 2 of Definition 1.1.1.2;
- *The Skew Monoidal Product.* The functor $\lhd_J$ of Definition 6.1.1.1;
- *The Skew Monoidal Unit.* The functor $\Bbbk_{\lhd}^{\mathbf{Rel}(A,B)}$ of Definition 6.1.2.1;
- *The Skew Associators.* The natural transformation $\alpha^{\mathbf{Rel}(A,B),\lhd}$ of Definition 6.1.3.1;
- *The Skew Left Unitors.* The natural transformation $\lambda^{\mathbf{Rel}(A,B),\lhd}$ of Definition 6.1.4.1;
- *The Skew Right Unitors.* The natural transformation $\rho^{\mathbf{Rel}(A,B),\lhd}$ of Definition 6.1.5.1.

6.2 Left Relative Preorders

Let A and B be sets and let $J : A \nrightarrow B$ be a relation.

Definition 6.2.1.1. A **left J-relative preorder from A to B** is equivalently:

- An $\mathbb{E}_1$-skew monoid in $\left(\mathbf{N}_{\bullet}(\mathbf{Rel}(A, B)), \lhd_J, J\right)$;
- A skew monoid in $\left(\mathbf{Rel}(A, B), \lhd_J, J\right)$.

Remark 6.2.1.2. In detail, a **left J-relative preorder** (R, μ_R, η_R) **from A to B** consists of

- *The Underlying Relation.* A relation

  $$R : A \nrightarrow B,$$

  called the **underlying relation of** (R, μ_R, η_R);
- *The Multiplication Inclusion.* An inclusion of relations

  $$\mu_R : R \lhd_J R \subset R,$$

  called the **multiplication** of (R, μ_R, η_R);

- *The Unit Inclusion.* An inclusion of relations

$$\eta_R \colon J \subset R,$$

called the **unit** of (R, μ_R, η_R).

Remark 6.2.1.3. In other words, a **left J-relative preorder from A to B** is a relation $R\colon A \nrightarrow B$ from A to B satisfying the following conditions:

1. *J-Transitivity.* For each $a \in A$ and each $c \in B$, we have

$$a \sim_{R \diamond \mathrm{Rift}_J(R)} c$$

i.e. the following condition is satisfied:[72]

($\star$) If there exists some $b \in A$ such that:

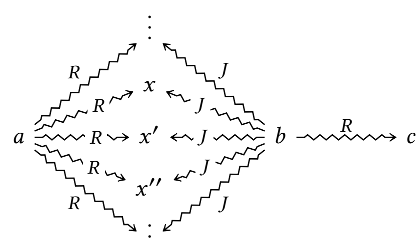
- We have $a \sim_{\mathrm{Rift}_J(R)} b$, i.e. for each $x \in B$, if $b \sim_J x$, then $a \sim_R x$;[73]
- We have $b \sim_R c$;

then $a \sim_R c$.

2. *J-Unitality.* For each $a \in A$ and each $b \in B$, the following condition is satisfied:

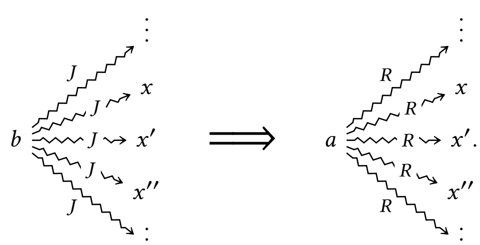
($\star$) If $a \sim_J b$, then $a \sim_R b$.

[72] *Illustration:* If we have



then $a \sim_R c$.

[73] Illustration:



6.3 The Right Skew Monoidal Structure on Rel(A, B)

Let A and B be sets and let $J \colon A \nrightarrow B$ be a relation.

6.3.1 The Right Skew Monoidal Product

Definition 6.3.1.1. The **right J-skew monoidal product of Rel(A, B)** is the functor

$$\triangleright_J \colon \mathbf{Rel}(A, B) \times \mathbf{Rel}(A, B) \to \mathbf{Rel}(A, B)$$

where

- *Action on Objects.* For each $R, S \in \mathrm{Obj}(\mathbf{Rel}(A, B))$, we have

$$S \triangleright_J R \overset{\text{def}}{=} \mathrm{Ran}_J(S) \diamond R, \qquad \begin{tikzcd} A \arrow[r, "R", "\shortmid"{marking}] & B \arrow[r, dashed, "\mathrm{Ran}_J(S)", "\shortmid"{marking}] & B; \\ & A \arrow[u, "J", "\shortmid"{marking}] \arrow[ur, "S"', "\shortmid"{marking}] \arrow[u, phantom, shift right=4ex, "\Leftarrow"] & \end{tikzcd}$$

- *Action on Morphisms.* For each $R, S, R', S' \in \mathrm{Obj}(\mathbf{Rel}(A, B))$, the action on Hom-sets

$$(\triangleright_J)_{(S,R),(S',R')} \colon \mathrm{Hom}_{\mathbf{Rel}(A,B)}(S, S') \times \mathrm{Hom}_{\mathbf{Rel}(A,B)}(R, R') \to \mathrm{Hom}_{\mathbf{Rel}(A,B)}(S \triangleright_J R, S' \triangleright_J R')$$

of $\triangleright_J$ at $((S, R), (S', R'))$ is defined by[74]

$$\beta \triangleright_J \alpha \overset{\text{def}}{=} \mathrm{Ran}_J(\beta) \diamond \alpha, \qquad \begin{tikzcd} A \arrow[r, bend left, "R'", "\shortmid"{marking}] \arrow[r, bend right, "R"', "\shortmid"{marking}] \arrow[r, phantom, "\alpha\Uparrow"] & B \arrow[r, dashed, bend left, "\mathrm{Ran}_J(S')", "\shortmid"{marking}] \arrow[r, dashed, bend right, "\mathrm{Ran}_J(S)"', "\shortmid"{marking}] \arrow[r, phantom, "\mathrm{Ran}_J(\beta)\Uparrow"] & B \\ & A \arrow[u, "J", "\shortmid"{marking}] \arrow[ur, "S'", "\shortmid"{marking}] \arrow[ur, bend right, "S"', "\shortmid"{marking}] \arrow[ur, phantom, shift right=2ex, "\beta\Uparrow"] & \end{tikzcd}$$

for each $\beta \in \mathrm{Hom}_{\mathbf{Rel}(A,B)}(S, S')$ and each $\alpha \in \mathrm{Hom}_{\mathbf{Rel}(A,B)}(R, R')$.

[74] Since **Rel**(A, B) is posetal, this is to say that if $S \subset S'$ and $R \subset R'$, then $S \triangleright_J R \subset S' \triangleright_J R'$.

6.3.2 The Right Skew Monoidal Unit

Definition 6.3.2.1. The **right** J**-skew monoidal unit of** $\mathbf{Rel}(A, B)$ is the functor

$$\mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleright} \colon \mathsf{pt} \to \mathbf{Rel}(A, B)$$

picking the object

$$\mathbb{1}^{\triangleright}_{\mathbf{Rel}(A,B)} \overset{\text{def}}{=} J$$

of $\mathbf{Rel}(A, B)$.

6.3.3 The Right Skew Associators

Definition 6.3.3.1. The **right** J**-skew associator of** $\mathbf{Rel}(A, B)$ is the natural transformation

$$\alpha^{\mathbf{Rel}(A,B),\triangleright} \colon \triangleright_J \circ (\mathsf{id} \times \triangleright_J) \Longrightarrow \triangleright_J \circ (\triangleright_J \times \mathsf{id}),$$

whose component

$$\alpha^{\mathbf{Rel}(A,B),\triangleright}_{T,S,R} \colon \underbrace{T \triangleright_J (S \triangleright_J R)}_{\overset{\text{def}}{=} \mathrm{Ran}_J(T) \diamond (\mathrm{Ran}_J(S) \diamond R)} \hookrightarrow \underbrace{(T \triangleright_J S) \triangleright_J R}_{\overset{\text{def}}{=} \mathrm{Ran}_J(\mathrm{Ran}_J(T) \diamond S) \diamond R}$$

at (T, S, R) is given by

$$\alpha^{\mathbf{Rel}(A,B),\triangleright}_{T,S,R} \overset{\text{def}}{=} \gamma \diamond \mathrm{id}_R,$$

where

$$\gamma \colon \mathrm{Ran}_J(T) \diamond \mathrm{Ran}_J(S) \hookrightarrow \mathrm{Ran}_J\big(\mathrm{Ran}_J(T) \diamond S\big)$$

is the inclusion adjunct to the inclusion

$$\mathrm{id}_{\mathrm{Ran}_J(T)} \diamond \epsilon_S \colon \underbrace{\mathrm{Ran}_J(T) \diamond \mathrm{Ran}_J(S) \diamond J}_{\overset{\text{def}}{=} J^*(\mathrm{Ran}_J(T) \diamond \mathrm{Ran}_J(S))} \hookrightarrow \mathrm{Ran}_J(T) \diamond S$$

under the adjunction $J^* \dashv \mathrm{Ran}_J$, where $\epsilon \colon \mathrm{Ran}_J \diamond J \Longrightarrow \mathrm{id}_{\mathbf{Rel}(A,B)}$ is the counit of the adjunction $J^* \dashv \mathrm{Ran}_J$.

6.3.4 The Right Skew Left Unitors

Definition 6.3.4.1. The **right** J**-skew left unitor of** $\mathbf{Rel}(A, B)$ is the natural transformation

$$\lambda^{\mathbf{Rel}(A,B),\triangleright} \colon \mathsf{id} \Longrightarrow \triangleright_J \circ \left(\mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleright} \times \mathsf{id}\right),$$

whose component

$$\lambda^{\mathbf{Rel}(A,B),\triangleright}_R \colon R \hookrightarrow \underbrace{J \triangleright_J R}_{\overset{\text{def}}{=} \mathrm{Ran}_J(J) \diamond R}$$

at R is given by

$$\lambda_R^{\mathbf{Rel}(A,B),\triangleright} \overset{\text{def}}{=} \sigma \diamond \mathrm{id}_R,$$

where $\sigma\colon \mathrm{id}_B \Longrightarrow \mathrm{Ran}_J(J)$ is the universal transformation included in the data of the right Kan extension $\mathrm{Ran}_J(J)$.

6.3.5 The Right Skew Right Unitors

Definition 6.3.5.1. The **right J-skew right unitor of Rel(A, B)** is the natural transformation

$$\rho^{\mathbf{Rel}(A,B),\triangleright}\colon \triangleright_J \circ \left(\mathsf{id} \times \mathbb{1}_{\triangleright}^{\mathbf{Rel}(A,B)}\right) \Longrightarrow \mathsf{id},$$

whose component

$$\rho_S^{\mathbf{Rel}(A,B),\triangleright}\colon \underbrace{S \triangleright_J J}_{\overset{\text{def}}{=}\mathrm{Ran}_J(S)\diamond J} \hookrightarrow S$$

at S is given by

$$\rho_S^{\mathbf{Rel}(A,B),\triangleright} \overset{\text{def}}{=} \epsilon_R,$$

where $\epsilon\colon \mathrm{Ran}_J \diamond J \Longrightarrow \mathrm{id}_{\mathbf{Rel}(A,B)}$ is the counit of the adjunction $J^* \dashv \mathrm{Ran}_J$.

6.3.6 The Right Skew Monoidal Structure on Rel(A, B)

Definition 6.3.6.1. The **right J-skew monoidal category of functors from A to B** is the right skew monoidal category

$$\left(\mathbf{Rel}(A, B), \triangleright_J, \mathbb{1}_{\triangleright}^{\mathbf{Rel}(A,B)}, \alpha^{\mathbf{Rel}(A,B),\triangleright}, \lambda^{\mathbf{Rel}(A,B),\triangleright}, \rho^{\mathbf{Rel}(A,B),\triangleright}\right)$$

consisting of

- *The Underlying Category.* The posetal category associated to the poset **Rel**(A, B) of relations from A to B of Item 2 of Definition 1.1.1.2;
- *The Skew Monoidal Product.* The functor $\triangleright_J$ of Definition 6.3.1.1;
- *The Skew Monoidal Unit.* The functor $\mathbb{1}_{\triangleright}^{\mathbf{Rel}(A,B)}$ of Definition 6.3.2.1;
- *The Skew Associators.* The natural transformation $\alpha^{\mathbf{Rel}(A,B),\triangleright}$ of Definition 6.3.3.1;
- *The Skew Left Unitors.* The natural transformation $\lambda^{\mathbf{Rel}(A,B),\triangleright}$ of Definition 6.3.4.1;
- *The Skew Right Unitors.* The natural transformation $\rho^{\mathbf{Rel}(A,B),\triangleright}$ of Definition 6.3.5.1.

6.4 Right Relative Preorders

Let A and B be sets and let $J\colon A \nrightarrow B$ be a relation.

Definition 6.4.1.1. A **right J-relative preorder from A to B** is equivalently:

- An $\mathbb{E}_1$-skew monoid in $\left(\mathrm{N}_{\bullet}(\mathbf{Rel}(A,B)), \triangleright_J, J\right)$;
- A skew monoid in $\left(\mathbf{Rel}(A,B), \triangleright_J, J\right)$.

Remark 6.4.1.2. In detail, a **right J-relative preorder** (R, μ_R, η_R) **from A to B** consists of

- *The Underlying Relation.* A relation

$$R\colon A \nrightarrow B,$$

  called the **underlying relation of** (R, μ_R, η_R);

- *The Multiplication Inclusion.* An inclusion of relations

$$\mu_R\colon R \triangleright_J R \subset R,$$

  called the **multiplication** of (R, μ_R, η_R);

- *The Unit Inclusion.* An inclusion of relations

$$\eta_R\colon J \subset R,$$

  called the **unit** of (R, μ_R, η_R).

Remark 6.4.1.3. In other words, a **right J-relative preorder from A to B** is a relation $R\colon A \nrightarrow B$ from A to B satisfying the following conditions:

1. *J-Transitivity.* For each $a \in A$ and each $c \in B$, we have

$$a \sim_{\mathrm{Ran}_J(R) \diamond R} c,$$

i.e. the following condition is satisfied:[75]

($\star$) If there exists some $b \in B$ such that:

- We have $a \sim_R b$;
- We have $b \sim_{\mathrm{Ran}_J(R)} c$, i.e. for each $x \in A$, if $x \sim_J b$, then $x \sim_R c$;[76]

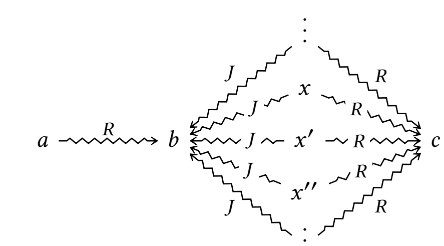
then $a \sim_R c$.

2. *J-Unitality*. For each $a \in A$ and each $b \in B$, the following condition is satisfied:

($\star$) If $a \sim_J b$, then $a \sim_R b$.

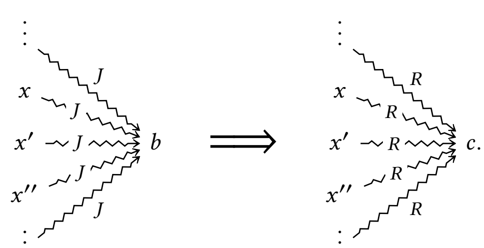
Appendices

[75] *Illustration:* If we have



then $a \sim_R c$.

[76] Illustration:



A Other Chapters

Probability Theory

Stochastic Analysis

Differential Geometry

Schemes