Bicategories

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Contents

Create tags and TODO:

1. spans in bicategories: add Proposition 7 here: https://arxiv.org/abs/1903.03890
2. add fact: internal adjunctions in $\mathsf{PseudoFun}(\mathcal{C}, \mathcal{D})$ are precisely the invertible strong transformations as in [JY21, Example 6.2.7]. What are the internal adjunctions?

1 Monomorphisms in Bicategories

1.1 Faithful Monomorphisms

Let $\mathcal{C}$ be a bicategory.

Definition 1.1.1.1. A 1-morphism $f\colon A \to B$ is a **faithful monomorphism in $\mathcal{C}$** if the following equivalent conditions are satisfied:

1. For each $X \in \mathrm{Obj}(\mathcal{C})$, the functor

   $$f_*\colon \mathsf{Hom}_{\mathcal{C}}(X, A) \to \mathsf{Hom}_{\mathcal{C}}(X, B)$$

   given by postcomposition by f is faithful.

2. Given a diagram in $\mathcal{C}$ of the form

   $$\begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & A \arrow[r, "f"] & B, \end{tikzcd}$$

   if we have $\mathrm{id}_f \circ \alpha = \mathrm{id}_f \circ \beta$, then $\alpha = \beta$.

Example 1.1.1.2. Here are some examples of faithful monomorphisms.

1. *Full Monomorphisms in* Cats_2.
2. *Full Monomorphisms in* $\mathbf{Rel}$.
3. *Full Monomorphisms in* Span.

1.2 Full Monomorphisms

Let $\mathcal{C}$ be a bicategory.

Definition 1.2.1.1. A 1-morphism $f\colon A \to B$ is a **full monomorphism in $\mathcal{C}$** if the following equivalent conditions are satisfied:

1. For each $X \in \mathrm{Obj}(\mathcal{C})$, the functor

   $$f_*\colon \mathsf{Hom}_{\mathcal{C}}(X, A) \to \mathsf{Hom}_{\mathcal{C}}(X, B)$$

   given by postcomposition by f is full.

2. For each $X \in \mathrm{Obj}(\mathcal{C})$ and each 2-morphism

   $$\gamma\colon f \circ \phi \Longrightarrow f \circ \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"] \arrow[r, bend right, "f\circ\psi"'] \arrow[r, phantom, "\gamma\Downarrow"] & B \end{tikzcd}$$

of C, there exists a 2-morphism $\alpha\colon \phi \Longrightarrow \psi$ of C such that we have an equality

$$\begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & A \arrow[r, "f"] & B \end{tikzcd} = \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"] \arrow[r, bend right, "f\circ\psi"'] \arrow[r, phantom, "\gamma\Downarrow"] & B \end{tikzcd}$$

of pasting diagrams in C, i.e. such that we have

$$\gamma = \mathrm{id}_f \circ \alpha.$$

Example 1.2.1.2. Here are some examples of full monomorphisms.

1. *Full Monomorphisms in* Cats_2.
2. *Full Monomorphisms in* $\mathbf{Rel}$.
3. *Full Monomorphisms in* Span.

1.3 Fully Faithful Monomorphisms

Let C be a bicategory.

Definition 1.3.1.1. A 1-morphism $f\colon A \to B$ is a **fully faithful monomorphism in C** if the following equivalent conditions are satisfied:

1. The 1-morphism f is fully and faithful.
2. For each $X \in \mathrm{Obj}(C)$, the functor

$$f_*\colon \mathsf{Hom}_C(X, A) \to \mathsf{Hom}_C(X, B)$$

   given by postcomposition by f is fully faithful.
3. The conditions in Item 1 of Definition 1.1.1.1 and Item 1 of Definition 1.2.1.1 hold.

Example 1.3.1.2. Here are some examples of fully faithful monomorphisms.

1. *Fully Faithful Monomorphisms in* Cats_2.
2. *Fully Faithful Monomorphisms in* $\mathbf{Rel}$.
3. *Fully Faithful Monomorphisms in* Span.

1.4 Strict Monomorphisms

Let $\mathcal{C}$ be a bicategory.

Definition 1.4.1.1. A 1-morphism $f\colon A \to B$ is a **strict monomorphism in $\mathcal{C}$** if the following equivalent conditions are satisfied:

1. For each $X \in \mathrm{Obj}(\mathcal{C})$, the action on objects

   $$f_*\colon \mathrm{Obj}(\mathsf{Hom}_{\mathcal{C}}(X, A)) \to \mathrm{Obj}(\mathsf{Hom}_{\mathcal{C}}(X, B))$$

   of the functor

   $$f_*\colon \mathsf{Hom}_{\mathcal{C}}(X, A) \to \mathsf{Hom}_{\mathcal{C}}(X, B)$$

   given by postcomposition by f is injective.

2. For each diagram in $\mathcal{C}$ of the form

   $$X \underset{\psi}{\overset{\phi}{\rightrightarrows}} A \xrightarrow{f} B,$$

   if $f \circ \phi = f \circ \psi$, then $\phi = \psi$.

Example 1.4.1.2. Here are some examples of strict monomorphisms.

1. *Strict Monomorphisms in* Cats_2.
2. *Strict Monomorphisms in* $\mathbf{Rel}$.
3. *Strict Monomorphisms in* Span.

2 Epimorphisms in Bicategories

2.1 Faithful Epimorphisms

Let $\mathcal{C}$ be a bicategory.

Definition 2.1.1.1. A 1-morphism $f\colon A \to B$ is a **faithful epimorphism in $\mathcal{C}$** if the following equivalent conditions are satisfied:

1. For each $X \in \mathrm{Obj}(\mathcal{C})$, the functor

   $$f^*\colon \mathsf{Hom}_{\mathcal{C}}(B, X) \to \mathsf{Hom}_{\mathcal{C}}(A, X)$$

   given by precomposition by f is faithful.

2. Given a diagram in $\mathcal{C}$ of the form

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & X, \end{tikzcd}$$

if we have $\alpha \circ \mathrm{id}_f = \beta \circ \mathrm{id}_f$, then $\alpha = \beta$.

Example 2.1.1.2. Here are some examples of faithful epimorphisms.

1. *Full Epimorphisms in* Cats_2.
2. *Full Epimorphisms in* **Rel**.
3. *Full Epimorphisms in* Span.

2.2 Full Epimorphisms

Let $\mathcal{C}$ be a bicategory.

Definition 2.2.1.1. A 1-morphism $f\colon A \to B$ is a **full epimorphism in $\mathcal{C}$** if the following equivalent conditions are satisfied:

1. For each $X \in \mathrm{Obj}(\mathcal{C})$, the functor

$$f^*\colon \mathsf{Hom}_{\mathcal{C}}(B, X) \to \mathsf{Hom}_{\mathcal{C}}(A, X)$$

given by precomposition by f is full.

2. For each $X \in \mathrm{Obj}(\mathcal{C})$ and each 2-morphism

$$\gamma\colon \phi \circ f \Longrightarrow \psi \circ f, \qquad \begin{tikzcd} X \arrow[r, bend left, "\phi\circ f"] \arrow[r, bend right, "\psi\circ f"'] \arrow[r, phantom, "\gamma\Downarrow"] & B \end{tikzcd}$$

of $\mathcal{C}$, there exists a 2-morphism $\alpha\colon \phi \Longrightarrow \psi$ of $\mathcal{C}$ such that we have an equality

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & X \end{tikzcd} \quad = \quad \begin{tikzcd} A \arrow[r, bend left, "\phi\circ f"] \arrow[r, bend right, "\psi\circ f"'] \arrow[r, phantom, "\gamma\Downarrow"] & X \end{tikzcd}$$

of pasting diagrams in $\mathcal{C}$, i.e. such that we have

$$\gamma = \alpha \circ \mathrm{id}_f.$$

Example 2.2.1.2. Here are some examples of full epimorphisms.

1. *Full Epimorphisms in* Cats_2.
2. *Full Epimorphisms in* $\mathbf{Rel}$.
3. *Full Epimorphisms in* Span.

2.3 Fully Faithful Epimorphisms

Let $\mathcal{C}$ be a bicategory.

Definition 2.3.1.1. A 1-morphism $f\colon A \to B$ is a **fully faithful epimorphism in $\mathcal{C}$** if the following equivalent conditions are satisfied:

1. The 1-morphism f is fully and faithful.
2. For each $X \in \mathrm{Obj}(\mathcal{C})$, the functor
$$f^*\colon \mathsf{Hom}_{\mathcal{C}}(B, X) \to \mathsf{Hom}_{\mathcal{C}}(A, X)$$
given by precomposition by f is fully faithful.
3. The conditions in Item 1 of Definition 2.1.1.1 and Item 1 of Definition 2.2.1.1 hold.

Example 2.3.1.2. Here are some examples of fully faithful epimorphisms.

1. *Fully Faithful Epimorphisms in* Cats_2.
2. *Fully Faithful Epimorphisms in* $\mathbf{Rel}$.
3. *Fully Faithful Epimorphisms in* Span.

2.4 Strict Epimorphisms

Let $\mathcal{C}$ be a bicategory.

Definition 2.4.1.1. A 1-morphism $f\colon A \to B$ is a **strict epimorphism in $\mathcal{C}$** if the following equivalent conditions are satisfied:

1. For each $X \in \mathrm{Obj}(\mathcal{C})$, the action on objects
$$f^*\colon \mathrm{Obj}(\mathsf{Hom}_{\mathcal{C}}(B, X)) \to \mathrm{Obj}(\mathsf{Hom}_{\mathcal{C}}(A, X))$$
of the functor
$$f^*\colon \mathsf{Hom}_{\mathcal{C}}(B, X) \to \mathsf{Hom}_{\mathcal{C}}(A, X)$$
given by precomposition by f is injective.

2. For each diagram in C of the form

$$A \xrightarrow{f} B \underset{\psi}{\overset{\phi}{\rightrightarrows}} X,$$

if $\phi \circ f = \psi \circ f$, then $\phi = \psi$.

Example 2.4.1.2. Here are some examples of strict epimorphisms.

1. *Strict Epimorphisms in* Cats_2*.*
2. *Strict Epimorphisms in* $\mathbf{Rel}$*.*
3. *Strict Epimorphisms in* Span*.*

Appendices

A Other Chapters