Pointed Sets

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This chapter contains some foundational material on pointed sets.

Contents

1 Pointed Sets

1.1 Foundations

Definition 1.1.1 ► Pointed Sets

A **pointed set**[1] is equivalently

- An $\mathbb{E}_0$-monoid in $(\mathsf{N}_\bullet(\mathsf{Sets}), \mathrm{pt})$;
- A pointed object in $(\mathsf{Sets}, \mathrm{pt})$.

[1] *Further Terminology:* Also called an $\mathbb{F}_1$**-module**.

Remark 1.1.2 ► Unwinding Definition 1.1.1

In detail, a **pointed set** is a pair (X, x_0) consisting of

- *The Underlying Set.* A set X, called the **underlying set of** (X, x_0);
- *The Basepoint.* A morphism

$$[x_0]\colon \mathrm{pt} \longrightarrow X$$

  in Sets, determining an element $x_0 \in X$, called the **basepoint of** X.

Example 1.1.3 ► The Zero Sphere

The 0**-sphere**[1] is the pointed set $(S^0, 0)$[2] consisting of

- *The Underlying Set.* The set S^0 defined by

$$S^0 \overset{\text{def}}{=} \{0, 1\};$$

- *The Basepoint.* The element 0 of S^0.

[1] *Further Terminology:* Also called the **underlying pointed set of the field with one element**.
[2] *Further Notation:* Also denoted $(\mathbb{F}_1, 0)$.

Example 1.1.4 ▶ The Trivial Pointed Set

The **trivial pointed set** is the pointed set $(\mathrm{pt}, \star)$ consisting of

- *The Underlying Set.* The punctual set $\mathrm{pt} \overset{\mathrm{def}}{=} \{\star\}$;
- *The Basepoint.* The element $\star$ of pt.

Example 1.1.5 ▶ The Underlying Pointed Set of a Semimodule

The **underlying pointed set** of a semimodule (M, α_M) is the pointed set $(M, 0_M)$.

Example 1.1.6 ▶ The Underlying Pointed Set of a Module

The **underlying pointed set** of a module (M, α_M) is the pointed set $(M, 0_M)$.

1.2 Morphisms of Pointed Sets

Definition 1.2.1 ▶ Morphisms of Pointed Sets

A **morphism of pointed sets**[1] is equivalently

- A morphism of $\mathbb{E}_0$-monoids in $(\mathsf{N}_\bullet(\mathsf{Sets}), \mathrm{pt})$.
- A morphism of pointed objects in $(\mathsf{Sets}, \mathrm{pt})$.

[1] *Further Terminology:* Also called a **pointed function** or a **morphism of $\mathbb{F}_1$-modules**.

Remark 1.2.2 ▶ Unwinding Definition 1.2.1

In detail, a **morphism of pointed sets** $f \colon (X, x_0) \longrightarrow (Y, y_0)$ is a morphism of sets $f \colon X \longrightarrow Y$ such that the diagram

$$\begin{tikzcd} & \mathrm{pt} \arrow[dl, "{[x_0]}"'] \arrow[dr, "{[y_0]}"] & \\ X \arrow[rr, "f"'] & & Y \end{tikzcd}$$

commutes, i.e. such that

$$f(x_0) = y_0.$$

1.3 The Category of Pointed Sets

Definition 1.3.1 ▶ The Category of Pointed Sets

The **category of pointed sets** is the category Sets_* defined equivalently as

- The homotopy category of the ∞-category $\mathsf{Mon}_{\mathbb{E}_0}(\mathsf{N}_\bullet(\mathsf{Sets}), \mathrm{pt})$ of Monoids in Monoidal ∞-Categories, **??**;
- The category Sets_* of Categories, **??**.

Remark 1.3.2 ▶ Unwinding Definition 1.3.1

In detail, the **category of pointed sets** is the category Sets_* where

- *Objects.* The objects of Sets_* are pointed sets;
- *Morphisms.* The morphisms of Sets_* are morphisms of pointed sets;
- *Identities.* For each $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, the unit map

$$\mathbb{1}^{\mathsf{Sets}_*}_{(X,x_0)} \colon \mathrm{pt} \longrightarrow \mathsf{Sets}_*((X, x_0), (X, x_0))$$

  of Sets_* at (X, x_0) is defined by[1]

$$\mathrm{id}^{\mathsf{Sets}_*}_{(X,x_0)} \stackrel{\text{def}}{=} \mathrm{id}_X;$$

- *Composition.* For each $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, the composition map

$$\circ^{\mathsf{Sets}_*}_{(X,x_0),(Y,y_0),(Z,z_0)} \colon \mathsf{Sets}_*((Y, y_0), (Z, z_0)) \times \mathsf{Sets}_*((X, x_0), (Y, y_0)) \to \mathsf{Sets}_*((X, x_0), (Z, z_0))$$

  of Sets_* at $((X, x_0), (Y, y_0), (Z, z_0))$ is defined by[2]

$$g \circ^{\mathsf{Sets}_*}_{(X,x_0),(Y,y_0),(Z,z_0)} f \stackrel{\text{def}}{=} g \circ f.$$

[1] Note that id_X is indeed a morphism of pointed sets, as we have $\mathrm{id}_X(x_0) = x_0$.

[2] Note that the composition of two morphisms of pointed sets is indeed a morphism of pointed sets, as we have

$$\begin{aligned} g(f(x_0)) &= g(y_0) \\ &= z_0, \end{aligned}$$

or

$$\begin{tikzcd} & \mathrm{pt} \arrow[dl, "{[x_0]}"'] \arrow[d, "{[y_0]}"] \arrow[dr, "{[z_0]}"] & \\ X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z \end{tikzcd}$$

in terms of diagrams.

1.4 Elementary Properties of Pointed Sets

Proposition 1.4.1 ► Elementary Properties of Pointed Sets

Let (X, x_0) be a pointed set.

1. *Completeness.* The category Sets_* of pointed sets and morphisms between them is complete, having in particular products (Definition 2.1.1), pullbacks (Definition 2.3.1), and equalisers (Definition 2.2.1).

2. *Cocompleteness.* The category Sets_* of pointed sets and morphisms between them is cocomplete, having in particular coproducts (Definition 3.1.1), pushouts (Definition 3.2.1), and coequalisers (Definition 3.3.1).

3. *Failure To Be Cartesian Closed.* The category Sets_* is not Cartesian closed.

4. *Relation to Partial Functions.* We have an equivalence of categories[1]

$$\mathsf{Sets}_* \overset{\text{eq.}}{\cong} \mathsf{Sets}^{\text{part.}}$$

between the category of pointed sets and pointed functions between them and the category of sets and partial functions between them.

[1] *Warning:* This is not an isomorphism of categories, only an equivalence.

Proof 1.4.2 ► Proof of Proposition 1.4.1

Item 1: Completeness

Omitted.

Item 2: Cocompleteness

Omitted.

Item 3: Failure To Be Cartesian Closed

See [**MSE2855868**].

Item 4: Relation to Partial Functions

Omitted.

2 Limits of Pointed Sets

2.1 Products

Let (X, x_0) and (Y, y_0) be pointed sets.

Definition 2.1.1 ► Products of Pointed Sets

The **product of** (X, x_0) **and** (Y, y_0) is the pointed set $(X \times Y, (x_0, y_0))$.

2.2 Equalisers

Let $f, g \colon (X, x_0) \rightrightarrows (Y, y_0)$ be morphisms of pointed sets.

Definition 2.2.1 ► Equalisers of Pointed Sets

The **equaliser of** (f, g) is the pointed set $(\mathrm{Eq}_*(f, g), x_0)$ consisting of

- *The Underlying Set.* The set $\mathrm{Eq}_*(f, g)$ defined by

$$\mathrm{Eq}_*(f, g) \overset{\mathrm{def}}{=} \{x \in X \mid f(x) = y_0 = g(x)\};$$

- *The Basepoint.* The element x_0 of $\mathrm{Eq}_*(f, g)$.

2.3 Pullbacks

Let (X, x_0), (Y, y_0), and (Z, z_0) be pointed sets and let $f \colon (X, x_0) \to (Z, z_0)$ and $g \colon (Y, y_0) \to (Z, z_0)$ be morphisms of pointed sets.

Definition 2.3.1 ► Pullbacks of Pointed Sets

The **pullback of** (X, x_0) **and** (Y, y_0) **over** (Z, z_0) **along** (f, g) is the pointed set $\left((X, x_0) \times_{(z,z_0)} (Y, y_0), p_0\right)$ consisting of

- *The Underlying Set.* The set $(X, x_0) \times_{(z,z_0)} (Y, y_0)$ defined by
$$(X, x_0) \times_{(z,z_0)} (Y, y_0) \stackrel{\text{def}}{=} \{(x, y) \in X \times Y \mid f(x) = z_0 = g(y)\};$$
- *The Basepoint.* The element (x_0, y_0) of $(X, x_0) \times_{(z,z_0)} (Y, y_0)$.

3 Colimits of Pointed Sets

3.1 Coproducts

Let (X, x_0) and (Y, y_0) be pointed sets.

Definition 3.1.1 ► Coproducts of Pointed Sets

The **coproduct of** (X, x_0) **and** (Y, y_0) is their wedge sum $(X \vee Y, p_0)$ of Definition 4.3.1.

3.2 Pushouts

Let (X, x_0), (Y, y_0), and (Z, z_0) be pointed sets and let $f\colon (Z, z_0) \to (X, x_0)$ and $g\colon (Z, z_0) \to (Y, y_0)$ be morphisms of pointed sets.

Definition 3.2.1 ► Pushouts of Pointed Sets

The **pushout of** (X, x_0) **and** (Y, y_0) **over** (Z, z_0) **along** (f, g) is the pointed set $\left(X \coprod_{f,Z,g} Y, p_0\right)$, where $p_0 = [x_0] = [y_0]$.

3.3 Coequalisers

Let $f, g\colon (X, x_0) \rightrightarrows (Y, y_0)$ be morphisms of pointed sets.

Definition 3.3.1 ► Coequalisers of Pointed Sets

The **coequaliser of** (f, g) is the pointed set $(\mathrm{CoEq}(f, g), x_0)$.

4 Constructions With Pointed Sets

4.1 Internal Homs

Let (X, x_0) and (Y, y_0) be pointed sets.

Definition 4.1.1 ► Pointed Sets of Morphisms of Pointed Sets

The **pointed set of morphisms of pointed sets from** (X, x_0) **to** (Y, y_0) is the pointed set $\mathbf{Sets}_*(X, Y)$ consisting of

- *The Underlying Set.* The set $\mathbf{Sets}_*((X, x_0), (Y, y_0))$ of morphisms of pointed sets from (X, x_0) to (Y, y_0);
- *The Basepoint.* The element

$$\Delta_{y_0} : (X, x_0) \longrightarrow (Y, y_0)$$

  of $\mathbf{Sets}_*((X, x_0), (Y, y_0))$.

4.2 Free Pointed Sets

Let X be a set.

Definition 4.2.1 ► Free Pointed Sets

The **free pointed set on** X is the pointed set X^+ consisting of

- *The Underlying Set.* The set X^+ defined by

$$X^+ \stackrel{\text{def}}{=} X \coprod \mathrm{pt};$$

- *The Basepoint.* The element $\star$ of X^+.

PROPOSITION 4.2.2 ► PROPERTIES OF FREE POINTED SETS

Let X be a set.

1. *Functoriality.* The assignment $X \mapsto X^+$ defines a functor

$$(-)^+ \colon \mathsf{Sets} \to \mathsf{Sets}_*,$$

where

- *Action on Objects.* For each $X \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$[(-)^+](X) \stackrel{\text{def}}{=} X_+,$$

where X_+ is the pointed set of Definition 4.2.1;

- *Action on Morphisms.* For each morphism $f \colon X \to Y$ of Sets, the image

$$f_+ \colon X_+ \to Y_+$$

of f by $(-)^+$ is the map of pointed sets defined by

$$f^+(x) \stackrel{\text{def}}{=} \begin{cases} f(x) & \text{if } x \in X, \\ \star & \text{if } x = \star. \end{cases}$$

2. *Adjointness.* We have an adjunction

$$\left((-)^+ \dashv 忘\right) \colon \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "(-)^+"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_*, \arrow[l, bend left, "忘"] \end{tikzcd}$$

witnessed by a bijection of sets

$$\mathsf{Sets}_*((X_+, \star), (Y, y_0)) \cong \mathsf{Sets}(X, Y),$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets})$ and $(Y, y_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

3. *Symmetric Strong Monoidality With Respect to Wedge Sums.* The free pointed set functor of Item 1 has a symmetric strong monoidal structure

$$\left((-)^+, (-)^{+,\coprod}, (-)^{+,\coprod}_{\mathbb{1}}\right) \colon (\mathsf{Sets}, \coprod, \emptyset) \to (\mathsf{Sets}_*, \vee, \mathrm{pt}),$$

being equipped with isomorphisms of pointed sets

$$(-)^{+,\coprod}_{X,Y} \colon X^+ \vee Y^+ \xrightarrow{\cong} (X \coprod Y)^+,$$
$$(-)^{+,\coprod}_{\mathbb{1}} \colon \mathrm{pt} \xrightarrow{\cong} \emptyset^+,$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

4. *Symmetric Strong Monoidality With Respect to Smash Products.* The free pointed set functor of Item 1 has a symmetric strong monoidal structure

$$\left((-)^+, (-)^{+,\times}, (-)^{+,\times}_{\mathbb{1}}\right) \colon (\mathsf{Sets}, \times, \mathrm{pt}) \longrightarrow \left(\mathsf{Sets}_*, \wedge, S^0\right),$$

being equipped with isomorphisms of pointed sets

$$(-)^{+,\times}_{X,Y} \colon X^+ \wedge Y^+ \xrightarrow{\cong} (X \times Y)^+,$$
$$(-)^{+,\times}_{\mathbb{1}} \colon S^0 \xrightarrow{\cong} \mathrm{pt}^+,$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

Proof 4.2.3 ► Proof of Proposition 4.2.2

Item 1: Functoriality

Clear.

Item 2: Adjointness

Clear.

Item 3: Symmetric Strong Monoidality With Respect to Wedge Sums

Omitted.

Item 4: Symmetric Strong Monoidality With Respect to Smash Products

Omitted.

4.3 Wedge Sums of Pointed Sets

Let (X, x_0) and (Y, y_0) be pointed sets.

Definition 4.3.1 ► Wedge Sums of Pointed Sets

The **wedge sum of** X **and** Y is the pointed set $(X \vee Y, p_0)$ consisting of

- *The Underlying Set.* The set $X \vee Y$ defined by[1]

$$\begin{aligned}(X \vee Y, p_0) &\overset{\text{def}}{=} (X, x_0) \coprod (Y, y_0)\\ &\cong (X \textstyle\coprod_{\text{pt}} Y, p_0)\\ &\cong (X \textstyle\coprod Y/{\sim}, p_0),\end{aligned}$$

$$\begin{tikzcd} X \vee Y \arrow[dr, phantom, "\ulcorner", very near start] & Y \arrow[l] \\ X \arrow[u] & \text{pt}, \arrow[l, hook', "{[x_0]}"] \arrow[u, hook', "{[y_0]}"'] \end{tikzcd}$$

  where $\sim$ is the equivalence relation on $X \coprod Y$ given by $x_0 \sim y_0$;

- *The Basepoint.* The element p_0 of $X \vee Y$ defined by

$$\begin{aligned}p_0 &\overset{\text{def}}{=} [x_0]\\ &= [y_0].\end{aligned}$$

[1] Here $(X, x_0) \coprod (Y, y_0)$ is the coproduct of (X, x_0) and (Y, y_0) in Sets_*.

Proposition 4.3.2 ► Properties of Wedge Sums of Pointed Sets

Let (X, x_0) and (Y, y_0) be pointed sets.

1. *Functoriality.* The assignments $(X, x_0), (Y, y_0), ((X, x_0), (Y, y_0)) \mapsto (X \vee Y, p_0)$ define functors

$$\begin{aligned} X \vee -\colon \mathsf{Sets}_* &\to \mathsf{Sets}_*,\\ - \vee Y\colon \mathsf{Sets}_* &\to \mathsf{Sets}_*,\\ -_1 \vee -_2\colon \mathsf{Sets}_* \times \mathsf{Sets}_* &\to \mathsf{Sets}_*.\end{aligned}$$

2. *Associativity.* We have an isomorphism of pointed sets

$$(X \vee Y) \vee Z \cong X \vee (Y \vee Z),$$

   natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathsf{Sets}_*$.

3. *Unitality.* We have isomorphisms of pointed sets

$$\begin{aligned}\text{pt} \vee X &\cong X,\\ X \vee \text{pt} &\cong X,\end{aligned}$$

natural in $(X, x_0) \in \mathsf{Sets}_*$.

4. *Commutativity.* We have an isomorphism of pointed sets

$$X \vee Y \cong Y \vee X,$$

natural in $(X, x_0), (Y, y_0) \in \mathsf{Sets}_*$.

5. *Symmetric Monoidality.* The triple $(\mathsf{Sets}_*, \vee, \mathrm{pt})$ is a symmetric monoidal category.

6. *Symmetric Strong Monoidality With Respect to Free Pointed Sets.* The free pointed set functor of Item 1 of Proposition 4.2.2 has a symmetric strong monoidal structure

$$\left((-)^+, (-)^{+,\coprod}, (-)^{+,\coprod}_{\mathbb{1}}\right) : (\mathsf{Sets}, \coprod, \emptyset) \to (\mathsf{Sets}_*, \vee, \mathrm{pt}),$$

being equipped with isomorphisms of pointed sets

$$(-)^{+,\coprod}_{X,Y} : X^+ \vee Y^+ \xrightarrow{\cong} (X \coprod Y)^+,$$
$$(-)^{+,\coprod}_{\mathbb{1}} : \mathrm{pt} \xrightarrow{\cong} \emptyset^+,$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

7. *The Fold Map.* We have a natural transformation

$$\nabla : \vee \circ \Delta^{\mathsf{Cats}}_{\mathsf{Sets}_*} \Longrightarrow \mathrm{id}_{\mathsf{Sets}_*}, \qquad \begin{tikzcd} & \mathsf{Sets}_* \times \mathsf{Sets}_* \arrow[dr, "\vee"] \arrow[d, phantom, "\Downarrow\nabla"] & \\ \mathsf{Sets}_* \arrow[ur, "\Delta^{\mathsf{Cats}}_{\mathsf{Sets}_*}"] \arrow[rr, bend right, "\mathrm{id}_{\mathsf{Sets}_*}"'] & & \mathsf{Sets}_*, \end{tikzcd}$$

called the **fold map**, whose component

$$\nabla_X : X \vee X \to X$$

at X is given by the composition

$$\begin{aligned} X &\xrightarrow{\Delta_X} X \times X \\ &\twoheadrightarrow X \times X/\sim \\ &\overset{\text{def}}{=\!=} X \vee X. \end{aligned}$$

Proof 4.3.3 ► Proof of Proposition 4.3.2

Item 1: Functoriality

Omitted.

Item 2: Associativity

Omitted.

Item 3: Unitality

Omitted.

Item 4: Commutativity

Omitted.

Item 5: Symmetric Monoidality

Omitted.

Item 6: Symmetric Strong Monoidality With Respect to Free Pointed Sets

Omitted.

Item 7: The Fold Map

Omitted.

Appendices

A Other Chapters