Types of Morphisms in Categories

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Contents

1 Monomorphisms

1.1 Foundations

Let $\mathcal{C}$ be a category.

Definition 1.1.1.1. A morphism $m\colon A \to B$ of $\mathcal{C}$ is a **monomorphism** if for every commutative[1] diagram of the form

$$C \underset{g}{\overset{f}{\rightrightarrows}} A \xrightarrow{m} B,$$

we have $f = g$.

Example 1.1.1.2. Let $f\colon A \to B$ be a function. The following conditions are equivalent:

1. The function f is injective.
2. The function f is a monomorphism in Sets.

Proof. Suppose that f is a monomorphism and consider the following diagram:

$$\{*\} \underset{[y]}{\overset{[x]}{\rightrightarrows}} A \xrightarrow{f} B,$$

where $[x]$ and $[y]$ are the morphisms picking the elements x and y of A. Then $f(x) = f(y)$ iff $f \circ [x] = f \circ [y]$, implying $[x] = [y]$, and hence $x = y$. Therefore f is injective.

Conversely, suppose that f is injective. Proceeding by contrapositive, we claim that given a pair of maps $g, h\colon C \rightrightarrows A$ such that $g \neq h$, then $f \circ g \neq f \circ h$. Indeed, as g and h are different maps, there exists must exist at least one element $x \in C$ such that $g(x) \neq h(x)$. But then we have $f(g(x)) \neq f(h(x))$, as f is injective. Thus $f \circ g \neq f \circ h$, and we are done. $\square$

[1] That is, with $m \circ f = m \circ g$.

Proposition 1.1.1.3. Let $\mathcal{C}$ be a category with pullbacks and $f\colon A \to B$ be a morphism of $\mathcal{C}$.

1. *Characterisations.* The following conditions are equivalent:

   (a) The morphism f is a monomorphism.

   (b) For each $X \in \mathrm{Obj}(\mathcal{C})$, the map of sets
   $$f_*\colon \mathrm{Hom}_{\mathsf{Sets}}(X, A) \to \mathrm{Hom}_{\mathsf{Sets}}(X, B)$$
   is injective.

   (c) The kernel pair of f is trivial, i.e. we have
   $$A \times_B A \cong A, \qquad \begin{tikzcd} A \arrow[r, "\mathrm{id}_A"] \arrow[d, "\mathrm{id}_A"'] \arrow[dr, phantom, very near start, "\lrcorner"] & A \arrow[d, "f"] \\ A \arrow[r, "f"'] & B. \end{tikzcd}$$

2. *Monomorphisms vs.Injective Maps.* Let

   - $\mathcal{C}$ be a concrete category;
   - 忘: $\mathcal{C} \to \mathsf{Sets}$ be the forgetful functor from $\mathcal{C}$ to Sets;
   - $f\colon A \to B$ be a morphism of $\mathcal{C}$.

   If 忘 preserves pullbacks, then the following conditions are equivalent:

   (a) The morphism f is a monomorphism.

   (b) The morphism f is injective.

3. *Stability Properties.* The class of all monomorphisms of $\mathcal{C}$ is stable under the following operations:

   (a) *Composition.* If f and g are monomorphisms, then so is $g \circ f$.[2]

   (b) *Pullbacks.* Let
   $$\begin{tikzcd} A \times_C B \arrow[r] \arrow[d, "m'"'] \arrow[dr, phantom, very near start, "\lrcorner"] & B \arrow[d, "m"] \\ A \arrow[r] & C \end{tikzcd}$$
   be a diagram in $\mathcal{C}$. If m is a monomorphism in $\mathcal{C}$, then so is m'.

[2]Conversely, if $g \circ f$ is a monomorphism, then so is f.

4. *Morphisms From the Terminal Object Are Monomorphisms.* If $\mathcal{C}$ has a terminal object $\mathbb{1}_{\mathcal{C}}$, then every morphism of $\mathcal{C}$ from $\mathbb{1}_{\mathcal{C}}$ is a monomorphism.

Proof. *Item 1, Characterisations*: The equivalence between Items 1a and 1b is clear. We claim that Items 1a and 1c are equivalent:

1. *Item 1a* $\Longrightarrow$ *Item 1c*: Suppose that f is a monomorphism. Then A satisfies the universal property of the pullback:

$$\begin{tikzcd}
P \arrow[rrd, bend left, "\phi"] \arrow[rd, dashed, "\phi", "\exists!"'] \arrow[rdd, bend right, "\phi"'] & & \\
& A \arrow[r, "\mathrm{id}_A"] \arrow[d, "\mathrm{id}_A"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\
& A \arrow[r, "f"'] & B.
\end{tikzcd}$$

2. *Item 1c* $\Longrightarrow$ *Item 1a*: Suppose that $A \cong A \times_B A$ and let $g, h \colon C \rightrightarrows A$ be a pair of morphisms. Consider the diagram

$$\begin{tikzcd}
C \arrow[rd, shift left, "g"] \arrow[rd, shift right, "h"'] & & \\
& A \arrow[r, "\mathrm{id}_A"] \arrow[d, "\mathrm{id}_A"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\
& A \arrow[r, "f"'] & B.
\end{tikzcd}$$

The universal property of the pullback says that there exists a unique morphism $C \to A$ making the diagram

$$\begin{tikzcd}
C \arrow[rrd, bend left, "g"] \arrow[rd, dashed, "\exists!"] \arrow[rdd, bend right, "h"'] & & \\
& A \arrow[r, "\mathrm{id}_A"] \arrow[d, "\mathrm{id}_A"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\
& A \arrow[r, "f"'] & B
\end{tikzcd}$$

commute, which implies $g = h$. Therefore, f is a monomorphism.

Item 2, Monomorphisms vs.Injective Maps: Assume that f is injective. As the forgetful functor from $\mathcal{C}$ to Sets is faithful, we see that Proposition 1.2.1.2 together with **??** imply that f is a monomorphism.
Conversely, assume that f is a monomorphism. As F preserves pullbacks, it also preserves kernel pairs. By **??**, we see that F preserves monomorphisms. Thus F_f is a monomorphism, and hence is injective by **??**.
Item 3, Stability Properties: Let $f, g\colon X \rightrightarrows A \times_C B$ be two morphisms such that the diagram

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & A \times_C B \arrow[r, "m'"] & A \end{tikzcd}$$

commutes. It follows that the diagram

$$\begin{tikzcd} X \arrow[dr, shift left, "f"] \arrow[dr, shift right, "g"'] \arrow[drr, bend left, "\mathrm{pr}_2 \circ f"] \arrow[ddr, bend right, "m' \circ g"'] & & \\ & A \times_C B \arrow[r, dashed, "\mathrm{pr}_2"] \arrow[d, "m'"] \arrow[dr, phantom, very near start, "\lrcorner"] & B \arrow[d, "m"] \\ & A \arrow[r, "\psi"'] & C \end{tikzcd}$$

also commutes. From the universal property of the pullback, it follows that there must be precisely one morphism from X to $A \times_C B$ making the above diagram commute. Thus $f = g$ and m' is a monomorphism.
Item 4, Morphisms From the Terminal Object Are Monomorphisms: Clear. □

1.2 Monomorphism-Reflecting Functors

Definition 1.2.1.1. A functor $F\colon \mathcal{C} \to \mathcal{D}$ **reflects monomorphisms** if, for each morphism f of $\mathcal{C}$, whenever F_f is a monomorphism, so is f.

Proposition 1.2.1.2. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor. If F is faithful, then it reflects monomorphisms.

Proof. Let $f\colon A \to B$ be a morphism of $\mathcal{C}$ and suppose that $F_f\colon F_A \to F_B$ is a monomorphism. Let $g, h\colon B \rightrightarrows C$ be two morphisms of $\mathcal{C}$ such that

$g \circ f = h \circ f$. As F is faithful, we must have

$$F_g \circ F_f = F_{g \circ f} = F_{h \circ f} = F_h \circ F_f,$$

but as F_f is a monomorphism, it must be that $F_g = F_h$. Using the faithfulness of F again, we see that $g = h$. Therefore f is a monomorphism. $\square$

1.3 Split Monomorphisms

Let $\mathcal{C}$ be a category.

Definition 1.3.1.1. A morphism $f \colon A \to B$ of $\mathcal{C}$ is a **split monomorphism**[3] if there exists a morphism $g \colon B \to A$ of $\mathcal{B}$ such that[4]

$$g \circ f = \mathrm{id}_A.$$

Proposition 1.3.1.2. Let $\mathcal{C}$ be a category.

1. *Split Monomorphisms are Monomorphisms.* If m is a split monomorphism, then m is a monomorphism.

Proof. Item 1, Split Monomorphisms are Monomorphisms: Let $m \colon A \to B$ be a split monomorphism of $\mathcal{C}$, let $e \colon B \to A$ be a morphism of $\mathcal{C}$ with

$$e \circ m = \mathrm{id}_A,$$

and let $f, g \colon C \rightrightarrows A$ be two morphisms of $\mathcal{C}$ such that the diagram

$$C \underset{g}{\overset{f}{\rightrightarrows}} A \xrightarrow{m} B$$

commutes. Then we have

$$\begin{aligned} f &= \mathrm{id}_A \circ f \\ &= (e \circ m) \circ f \\ &= e \circ (m \circ f) \\ &= e \circ (m \circ g) \\ &= (e \circ m) \circ g \\ &= \mathrm{id}_A \circ g \\ &= g, \end{aligned}$$

showing m to be a monomorphism. $\square$

[3] *Further Terminology:* Also called a **section**, or a **split monic** morphism.

[4] *Warning:* There exist monomorphisms which are not split monomorphisms, e.g.

2 Epimorphisms

2.1 Foundations

Let C be a category.

Definition 2.1.1.1. A morphism $f\colon A \to B$ of C is an **epimorphism** if for every commutative[5] diagram of the form

$$A \xrightarrow{f} B \underset{h}{\overset{g}{\rightrightarrows}} C,$$

we have $g = h$.

Example 2.1.1.2. Let $f\colon A \to B$ be a function. The following conditions are equivalent:

1. The function f is injective.

2. The function f is an epimorphism in Sets.

Proof. Suppose that f is surjective and let $g, h\colon B \rightrightarrows C$ be morphisms such that $g \circ f = h \circ f$. Then for each $a \in A$, we have

$$g(f(a)) = h(f(a)),$$

but this implies that

$$g(b) = h(b)$$

for each $b \in B$, as f is surjective. Thus $g = h$ and f is an epimorphism.
To prove the converse, we proceed by contrapositive. So suppose that f is not surjective and consider the diagram

$$A \xrightarrow{f} B \underset{h}{\overset{g}{\rightrightarrows}} C,$$

where h is the map defined by $h(b) = 0$ for each $b \in B$ and g is the map defined by

$$g(b) = \begin{cases} 1 & \text{if } b \in \mathrm{Im}(f), \\ 0 & \text{otherwise.} \end{cases}$$

Then $h \circ f = g \circ f$, as $h(f(a)) = 1 = g(f(a))$ for each $a \in A$. However, for any $b \in B \setminus \mathrm{Im}(f)$, we have

$$g(b) = 0 \neq 1 = h(b).$$

Therefore $g \neq h$ and f is not an epimorphism. $\square$

$\mathbb{Z}/2 \hookrightarrow \mathbb{Z}/4$ in Ring.

[5] That is, with $g \circ f = h \circ f$.

Proposition 2.1.1.3. Let $\mathcal{C}$ be a category.

1. *Characterisations.* Let $\mathcal{C}$ be a category with pullbacks and $f\colon A \to B$ be a morphism of $\mathcal{C}$. The following conditions are equivalent:

   (a) The morphism f is an epimorphism.

   (b) For each $X \in \mathrm{Obj}(\mathcal{C})$, the map of sets

   $$f^*\colon \mathrm{Hom}_{\mathsf{Sets}}(B, X) \to \mathrm{Hom}_{\mathsf{Sets}}(A, X)$$

   is injective.

   (c) The cokernel pair of f is trivial, i.e. we have

   $$B \textstyle\coprod_A B \cong B \qquad \begin{tikzcd} B & B \arrow[l] \\ B \arrow[u] \arrow[ur, phantom, "\ulcorner"] & A. \arrow[l, "f"] \arrow[u, "f"'] \end{tikzcd}$$

2. *Epimorphisms vs.Surjective Maps.* Let

   - $\mathcal{C}$ be a concrete category;
   - 忘: $\mathcal{C} \to \mathsf{Sets}$ be the forgetful functor from $\mathcal{C}$ to Sets;
   - $f\colon A \to B$ be a morphism of $\mathcal{C}$.

   If 忘 preserves pushouts, then the following conditions are equivalent:

   (a) The morphism f is a epimorphism.

   (b) The morphism f is surjective.

3. *Stability Properties.* The class of all epimorphisms of $\mathcal{C}$ is stable under the following operations:

   (a) *Composition.* If f and g are epimorphisms, then so is $g \circ f$.[6]

   (b) *Pushouts.* Let

   $$\begin{tikzcd} A \coprod_C B & B \arrow[l] \\ A \arrow[u, "e'"] \arrow[r] \arrow[ur, phantom, "\ulcorner"] & C \arrow[u, "e"'] \end{tikzcd}$$

   be a diagram in $\mathcal{C}$. If m is an epimorphism in $\mathcal{C}$, then so is e'.

[6]Conversely, if $g \circ f$ is a epimorphism, then so is g.

4. *Morphisms to the Initial Object Are Monomorphisms.* If $\mathcal{C}$ has an initial object $\varnothing_{\mathcal{C}}$, then every morphism of $\mathcal{C}$ to $\varnothing_{\mathcal{C}}$ is a epimorphism.

Proof. This is dual to Proposition 1.1.1.3. □

2.2 Regular Epimorphisms

Proposition 2.2.1.1. Let $\mathcal{C}$ be a category.

1. *Stability Under Pullbacks.* Consider the diagram

$$\begin{tikzcd}
A \times_C B \arrow[r] \arrow[d, "e'"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "e"] \\
A \arrow[r] & C
\end{tikzcd}$$

in $\mathcal{C}$. If e is a regular epimorphism, then so is e'.

Proof. Epimorphisms Need Not Be Stable Under Pullback. : *Regular Epimorphisms Are Stable Under Pullback.* : □

2.3 Effective Epimorphisms

Let $\mathcal{C}$ be a category.

Definition 2.3.1.1. An epimorphism $f\colon A \to B$ of $\mathcal{C}$ is **effective** if we have an isomorphism

$$B \cong \mathrm{CoEq}(A \times_B A \rightrightarrows A).$$

2.4 Split Epimorphisms

Let $\mathcal{C}$ be a category.

Definition 2.4.1.1. A morphism $f\colon A \to B$ of $\mathcal{C}$ is a **retraction**[7] if there is an arrow $g\colon B \to A$ such that $f \circ g = \mathrm{id}_B$.

Proposition 2.4.1.2. Let $f\colon A \to B$ be a morphism of $\mathcal{C}$.

1. Every split epimorphism is an epimorphism.[8]

Proof. This is dual to **??**. □

[7] *Further Terminology:* Also called a **split epimorphism**.

[8] *Warning:* There are epimorphisms which are not split epimorphisms, however, e.g.

3 Endomorphisms

3.1 Foundations

Let $\mathcal{C}$ be a category.

Definition 3.1.1.1. An **endomorphism in $\mathcal{C}$** is a functor $\phi\colon \mathsf{B}\mathbb{N} \to \mathcal{C}$.

Remark 3.1.1.2. In detail, an **endomorphism in $\mathcal{C}$** is a pair (A, ϕ) consisting of

- *The Underlying Object.* An object A of $\mathcal{C}$;
- *The Endomorphism.* A morphism $\phi\colon A \to A$ of $\mathcal{C}$.

Proof. Indeed, a functor $\phi\colon \mathsf{B}\mathbb{N} \to \mathcal{C}$ consists of

- *Action on Objects.* A map of sets

$$\phi_0\colon \underbrace{\mathrm{Obj}(\mathsf{B}\mathbb{N})}_{\stackrel{\text{def}}{=}\mathrm{pt}} \to \mathrm{Obj}(\mathcal{C})$$

  picking an object A of $\mathcal{C}$;

- *Action on Morphisms.* A map of sets

$$\phi_{\star,\star}\colon \underbrace{\mathrm{Hom}_{\mathsf{B}\mathbb{N}}(\star,\star)}_{\stackrel{\text{def}}{=}\mathbb{N}} \to \mathrm{Hom}_{\mathcal{C}}(A, A);$$

preserving composition and identities. This makes $\phi_{\star,\star}$ into a morphism of monoids

$$\phi_{\star,\star}\colon \underbrace{\left(\mathrm{Hom}_{\mathsf{B}\mathbb{N}}(\star,\star), \circ^{\mathsf{B}\mathbb{N}}_{\star,\star,\star}, \mathbb{1}^{\mathsf{B}\mathbb{N}}_{\star}\right)}_{\stackrel{\text{def}}{=}(\mathbb{N},+,0)} \to (\mathrm{Hom}_{\mathcal{C}}(A, A), \circ, \mathrm{id}_A),$$

determining and being determined by, via Monoids, **??** of **??**, an element $\phi\colon A \to A$ of $\mathrm{Hom}_{\mathcal{C}}(A, A)$. $\square$

3.2 Morphisms of Endomorphisms in Categories

Definition 3.2.1.1. A **morphism of endomorphisms in $\mathcal{C}$** from ϕ to ψ is a natural transformation $\alpha\colon \phi \Longrightarrow \psi$ of functors from $\mathsf{B}\mathbb{N}$ to $\mathcal{C}$.

$\mathbb{Z} \hookrightarrow \mathbb{Z}/2$.

Remark 3.2.1.2. In detail, a **morphism of endomorphisms in $\mathcal{C}$** from (A, ϕ) to (B, ψ) is a morphism $f \colon A \to B$ of $\mathcal{C}$ such that the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\phi"'] & B \arrow[d, "\psi"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

commutes.

3.3 The Category of Endomorphisms in a Category

Definition 3.3.1.1. The **category of endomorphisms in $\mathcal{C}$** is the category $\mathsf{End}(\mathcal{C})$[9,10] defined by

$$\mathsf{End}(\mathcal{C}) \overset{\text{def}}{=} \mathsf{Fun}(\mathsf{B}\mathbb{N}, \mathcal{C}).$$

Remark 3.3.1.2. In detail, the **category of endomorphisms in $\mathcal{C}$** is the category $\mathsf{End}(\mathcal{C})$ where

- *Objects.* The objects of $\mathsf{End}(\mathcal{C})$ are endomorphisms in $\mathcal{C}$;
- *Morphisms.* The morphisms of $\mathsf{End}(\mathcal{C})$ are morphisms of endomorphisms in $\mathcal{C}$;
- *Identities.* For each $(A, \phi) \in \mathrm{Obj}(\mathsf{End}(\mathcal{C}))$, the unit map

  $$\mathbb{1}^{\mathsf{End}(\mathcal{C})}_{(A,\phi)} \colon \mathrm{pt} \to \mathrm{Hom}_{\mathsf{End}(\mathcal{C})}((A, \phi), (A, \phi))$$

  of $\mathsf{End}(\mathcal{C})$ at (A, ϕ) is defined by

  $$\mathrm{id}^{\mathsf{End}(\mathcal{C})}_{(A,\phi)} \overset{\text{def}}{=} \mathrm{id}_A;$$

- *Composition.* For each $(A, \phi), (B, \psi), (C, \chi) \in \mathrm{Obj}(\mathsf{End}(\mathcal{C}))$, the com-

[9] *Further Notation:* Also written $\mathcal{C}^{\circlearrowleft}$.

[10] Since $\mathsf{B}\mathbb{N}$ may be thought of as a categorical realisation of the "directed circle", we also write $\mathcal{L}^{\mathrm{dir}}(\mathcal{C})$ for $\mathsf{End}(\mathcal{C})$, which we may view as a "**categorical free directed loop space**" **of $\mathcal{C}$**.

Homotopy-theoretic information about $\mathcal{L}^{\mathrm{dir}}(\mathcal{C})$ is often not of much interest, however, as many categories commonly appearing in practice tend to be contractible for reasons which also hold true for categories of functors into them (as is the case of $\mathcal{L}^{\mathrm{dir}}(\mathcal{C}) \overset{\text{def}}{=} \mathsf{Fun}(\mathsf{B}\mathbb{N}, \mathcal{C})$), such as admitting initial/final objects or binary co/products.

position map

$$\circ^{\mathsf{End}(\mathcal{C})}_{\phi,\psi,\chi}\colon \mathrm{Hom}_{\mathsf{End}(\mathcal{C})}(\psi,\chi)\times\mathrm{Hom}_{\mathsf{End}(\mathcal{C})}(\phi,\psi)\to\mathrm{Hom}_{\mathsf{End}(\mathcal{C})}(\phi,\chi)$$

of $\mathsf{End}(\mathcal{C})$ at $(A,\phi),(B,\psi),(C,\chi)$ is defined by

$$g\circ^{\mathsf{End}(\mathcal{C})}_{\phi,\psi,\chi} f \overset{\mathrm{def}}{=} g\circ f.$$

Proposition 3.3.1.3. Let $\mathcal{C}$ be a category.

1. *Functoriality.* The assignment $\mathcal{C}\mapsto\mathsf{End}(\mathcal{C})$ defines a functor

$$\mathsf{End}\colon \mathsf{Cats}\to\mathsf{Cats}.$$

2. 2-*Functoriality.* The assignment $\mathcal{C}\mapsto\mathsf{End}(\mathcal{C})$ defines a 2-functor

$$\mathsf{End}\colon \mathsf{Cats}_2\to\mathsf{Cats}_2.$$

3. *Adjointness I.* If $\mathcal{C}$ has products and coproducts, then we have a triple adjunction[11]

$$(\mathbb{N}\odot(-)\dashv 忘\dashv\mathbb{N}\pitchfork(-))\colon\quad \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "\mathbb{N}\odot(-)"] \arrow[r, bend right, "\mathbb{N}\pitchfork(-)"'] & \mathsf{End}(\mathcal{C}) \arrow[l, "忘"'] \end{tikzcd},$$

where[12]

- $\mathbb{N}\odot(-)\colon\mathcal{C}\to\mathsf{End}(\mathcal{C})$ is the functor defined on objects by

$$\begin{aligned}\mathbb{N}\odot(A) &\overset{\mathrm{def}}{=} (\mathbb{N}\odot A,\mathbb{N}\odot\mathrm{id}_A)\\ &\cong\left(A^{\coprod\mathbb{N}},\mathrm{id}_A^{\coprod\mathbb{N}}\right);\end{aligned}\qquad\text{(Weighted Category Theory, ??)}$$

- $忘\colon\mathsf{End}(\mathcal{C})\to\mathcal{C}$ is the **forgetful functor from $\mathsf{End}(\mathcal{C})$ to $\mathcal{C}$**, defined on objects by

$$忘(A,\phi)\overset{\mathrm{def}}{=}A;$$

[11]Here $\mathcal{C}\cong\mathsf{Fun}(\mathsf{pt},\mathcal{C})$, which we may think of as the "**category of identities of $\mathcal{C}$**".

[12]In a sense, $(\mathbb{N}\odot A,\mathbb{N}\odot\mathrm{id}_A)$ and $(\mathbb{N}\pitchfork A,\mathbb{N}\pitchfork\mathrm{id}_A)$ are the co/universal ways of producing an endomorphism starting with an identity.

- $\mathbb{N} \pitchfork (-)\colon \mathcal{C} \to \mathsf{End}(\mathcal{C})$ is the functor defined on objects by

$$\begin{aligned}\mathbb{N} \pitchfork (A) &\overset{\text{def}}{=} (\mathbb{N} \pitchfork A, \mathbb{N} \pitchfork \mathrm{id}_A)\\ &\cong \left(A^{\times \mathbb{N}}, \mathrm{id}_A^{\times \mathbb{N}}\right). && \text{(Weighted Category Theory, ??)}\end{aligned}$$

4. *Adjointness II.* If $\mathcal{C}$ is bicomplete, then we have a triple adjunction

$$(\mathrm{colim}^{\circlearrowleft} \dashv \iota \dashv \lim^{\circlearrowleft})\colon \quad \begin{tikzcd} \mathsf{End}(\mathcal{C}) \arrow[r, bend left=45, "\mathrm{colim}^{\circlearrowleft}"] \arrow[r, bend right=45, "\lim^{\circlearrowleft}"'] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, phantom, shift right=3, "\perp"] & \mathcal{C}, \arrow[l, "\iota"'] \end{tikzcd}$$

where[13,14]

- $\mathrm{colim}^{\circlearrowleft}\colon \mathsf{End}(\mathcal{C}) \to \mathcal{C}$ is the functor defined on objects by

$$\begin{aligned}\mathrm{colim}^{\circlearrowleft}(A, \phi) &\overset{\text{def}}{=} \mathrm{colim}\left(\mathsf{B}\mathbb{N} \xrightarrow{(A,\phi)} \mathcal{C}\right)\\ &\overset{\text{def}}{=} \mathrm{colim}(A \circlearrowleft \phi);\end{aligned}$$

- $\iota\colon \mathcal{C} \hookrightarrow \mathsf{End}(\mathcal{C})$ is the functor defined on objects by[15]

$$\iota(A) \overset{\text{def}}{=} (A, \mathrm{id}_A);$$

- $\lim^{\circlearrowleft}\colon \mathsf{End}(\mathcal{C}) \to \mathcal{C}$ is the functor defined on objects by

$$\begin{aligned}\lim^{\circlearrowleft}(A, \phi) &\overset{\text{def}}{=} \lim\left(\mathsf{B}\mathbb{N} \xrightarrow{(A,\phi)} \mathcal{C}\right)\\ &\overset{\text{def}}{=} \lim(A \circlearrowleft \phi).\end{aligned}$$

[13] In a sense, $\mathrm{colim}^{\circlearrowleft}(A, \phi)$ and $\lim^{\circlearrowleft}(A, \phi)$ are the co/universal ways of producing an identity starting with an endomorphism.

[14] *Example:* Let $\mathcal{C} = \mathsf{Sets}$, let X be a set, and let $\phi\colon X \to X$ be a morphism of sets. Then

$$\begin{aligned}\mathrm{colim}^{\circlearrowleft}(X, \phi) &\cong X/\sim,\\ \lim^{\circlearrowleft}(X, \phi) &\cong \{x \in X \mid \phi(x) = x\},\end{aligned}$$

where $\sim$ is the equivalence relation on X generated by declaring $x \sim y$ iff $\phi(x) = y$ for each $x, y \in X$.

[15] Viewing $\mathcal{C} \cong \mathsf{Fun}(\mathsf{pt}, \mathcal{C})$ as the "category of identities of $\mathcal{C}$", we see that the functor ι is just the inclusion of categories from the category of identities of $\mathcal{C}$ to the category of

5. 2-*Adjointness.* We have a 2-adjunction

$$(\mathsf{B}\mathbb{N}\times - \dashv \mathsf{End})\colon\quad \begin{tikzcd} \mathsf{Cats}_2 \arrow[r, bend left, "\mathsf{B}\mathbb{N}\times -"] \arrow[r, phantom, "\perp_2"] & \mathsf{Cats}_2 \arrow[l, bend left, "\mathsf{End}"] \end{tikzcd}.$$

Proof. *Item 1, Functoriality*: Omitted.
Item 2: 2-Functoriality: Omitted.
Item 3, Adjointness I: We give two proofs, one via Kan extensions and the other by directly verifying that the functors form an adjunction.
Indeed, applying Kan Extensions, **??** of **??** to the functor $[\star]\colon \mathsf{pt} \twoheadrightarrow \mathsf{B}\mathbb{N}$, we obtain a triple adjunction

$$\left(\mathrm{Lan}_{[\star]} \dashv [\star]^* \dashv \mathrm{Ran}_{[\star]}\right)\colon\quad \begin{tikzcd} \mathsf{Fun}(\mathsf{pt}, \mathcal{C}) \arrow[r, bend left, "\mathrm{Lan}_{[\star]}"] \arrow[r, bend right, "\mathrm{Ran}_{[\star]}"'] & \mathsf{Fun}(\mathsf{B}\mathbb{N}, \mathcal{C}) \arrow[l, "{[\star]^*}"'] \end{tikzcd}.$$

Here $\mathsf{Fun}(\mathsf{pt}, \mathcal{C}) \cong \mathcal{C}$ via **??** of **??** and $\mathsf{Fun}(\mathsf{B}\mathbb{N}, \mathcal{C}) \overset{\text{def}}{=} \mathsf{End}(\mathcal{C})$ by definition. We claim that $\mathrm{Lan}_{[\star]} \cong \mathbb{N} \odot -$, $[\star]^* \cong$ 忘, and $\mathrm{Ran}_{[\star]} \cong \mathbb{N} \pitchfork (-)$:

- *Computing $\mathrm{Lan}_{[\star]}$.* Let A be an object of $\mathcal{C}$. By Kan Extensions, **??** of **??**, we have

  $$\mathrm{Lan}_{[\star]}(A) \cong \operatorname{colim}([\star] \downarrow \underline{\star} \twoheadrightarrow \mathsf{pt} \xrightarrow{A} \mathcal{C}).$$

  Unwinding the description of $[\star] \downarrow \underline{\star}$ given in **??**, we see that it is the category having the form

  $$\begin{tikzcd} \star \arrow[r, "1"] \arrow[d, "0"'] & \star \arrow[r, "1"] \arrow[d, "1"] & \star \arrow[r, "1"] \arrow[d, "2"] & \star \arrow[r, "1"] \arrow[d, "3"] & \star \arrow[r, "1"] \arrow[d, "4"] & \star \arrow[r, "1"] \arrow[d, "5"] & \cdots \\ \star \arrow[r, equal] & \star \arrow[r, equal] & \star \arrow[r, equal] & \star \arrow[r, equal] & \star \arrow[r, equal] & \star \arrow[r, equal] & \cdots. \end{tikzcd}$$

  Moreover, the composition $[\star] \downarrow \underline{\star} \twoheadrightarrow \mathsf{pt} \xrightarrow{A} \mathcal{C}$ is given by the diagram in $\mathcal{C}$ having $\mathbb{N}$ factors of A, and thus its colimit is given by $A^{\coprod \mathbb{N}}$. Similarly, one sees that the endomorphism this object carries is $\mathrm{id}_A^{\coprod \mathbb{N}}$.
  Alternatively, we may use Kan Extensions, **??** of **??** and directly

endomorphisms of $\mathcal{C}$.

compute $\mathrm{Lan}_{[\star]}(A)$:

$$\begin{aligned}\mathrm{Lan}_{[\star]}(A) &\cong \int^{\star\in\mathsf{pt}} \mathrm{Hom}_{\mathsf{B}\mathbb{N}}(\star,\star)\odot A,\\ &\cong \int^{\star\in\mathsf{pt}} \mathbb{N}\odot A,\\ &\cong \mathbb{N}\odot A.\end{aligned}$$

- *Computing* $[\star]^*$*.* Let (A,ϕ) be an object of $\mathsf{End}(\mathcal{C})$, viewed as a functor $\phi\colon \mathsf{B}\mathbb{N}\to\mathcal{C}$. Then the composition

  $$\mathsf{pt}\xrightarrow{[\star]}\mathsf{B}\mathbb{N}\xrightarrow{(A,\phi)}\mathcal{C}$$

  corresponds precisely to A, and we see that $[\star]^*\cong$ 忘.

- *Computing* $\mathit{Ran}_{[\star]}$*.* Let A be an object of $\mathcal{C}$. By Kan Extensions, **??** of **??**, we have

  $$\mathrm{Ran}_{[\star]}(A)\cong\lim(\underline{\star}\downarrow[\star]\twoheadrightarrow\mathsf{pt}\xrightarrow{A}\mathcal{C}).$$

  Unwinding the description of $\underline{\star}\downarrow[\star]$ given in **??**, we see that it is the category having the form

  $$\begin{tikzcd} \star \arrow[r, equal] \arrow[d, "0"'] & \star \arrow[r, equal] \arrow[d, "1"] & \star \arrow[r, equal] \arrow[d, "2"] & \star \arrow[r, equal] \arrow[d, "3"] & \star \arrow[r, equal] \arrow[d, "4"] & \star \arrow[r, equal] \arrow[d, "5"] & \cdots \\ \star \arrow[r, "1"'] & \star \arrow[r, "1"'] & \star \arrow[r, "1"'] & \star \arrow[r, "1"'] & \star \arrow[r, "1"'] & \star \arrow[r, "1"'] & \cdots. \end{tikzcd}$$

  Moreover, the composition $\underline{\star}\downarrow[\star]\twoheadrightarrow\mathsf{pt}\xrightarrow{A}\mathcal{C}$ is given by the diagram in $\mathcal{C}$ having $\mathbb{N}$ factors of A, and thus its limit is given by $A^{\times\mathbb{N}}$. Similarly, one sees that the endomorphism this object carries is $\mathrm{id}_A^{\times\mathbb{N}}$.

  Alternatively, we may use Kan Extensions, **??** of **??** and directly compute $\mathrm{Ran}_{[\star]}(A)$:

  $$\begin{aligned}\mathrm{Ran}_{[\star]}(A) &\cong \int_{\star\in\mathsf{pt}} \mathrm{Hom}_{\mathsf{B}\mathbb{N}}(\star,\star)\pitchfork A,\\ &\cong \int_{\star\in\mathsf{pt}} \mathbb{N}\pitchfork A,\\ &\cong \mathbb{N}\pitchfork A.\end{aligned}$$

We may also just explicitly verify that the stated adjunction holds (we give a partial proof, not verifying naturality):

- *The Adjunction* $\mathbb{N} \odot (-) \dashv$ 忘. Given $A \in \mathrm{Obj}(\mathcal{C})$ and $(B, \phi) \in \mathrm{Obj}(\mathsf{End}(\mathcal{C}))$, we have a bijection

$$\mathrm{Hom}_{\mathsf{End}(\mathcal{C})}((\mathbb{N} \odot A, \mathbb{N} \odot \mathrm{id}_A), (B, \phi)) \cong \mathrm{Hom}_{\mathcal{C}}(A, B).$$

Indeed, we have

$$\begin{aligned} \mathrm{Hom}_{\mathsf{End}(\mathcal{C})}((\mathbb{N} \odot A, \mathbb{N} \odot \mathrm{id}_A), (B, \phi)) &\cong \mathrm{Hom}_{\mathsf{End}(\mathcal{C})}\left(\left(A^{\coprod \mathbb{N}}, \mathrm{id}_A^{\coprod \mathbb{N}}\right), (B, \phi)\right) \\ &\cong \mathrm{Hom}_{\mathsf{End}(\mathcal{C})}((A, \mathrm{id}_A), (B, \phi))^{\times \mathbb{N}}, \end{aligned}$$

and hence a morphism $(\mathbb{N} \odot A, \mathbb{N} \odot \mathrm{id}_A) \to (B, \phi)$ of $\mathsf{End}(\mathcal{C})$ is equivalently given by an $\mathbb{N}$-indexed collection

$$\{f_n \colon A \to B\}_{n \in \mathbb{N}}$$

of morphisms of $\mathcal{C}$ such that, for each $n \in \mathbb{N}$, the diagram

$$\begin{tikzcd} A \arrow[r, "f_n"] \arrow[d, equal] & B \arrow[d, "\phi"] \\ A \arrow[r, "f_{n+1}"'] & B \end{tikzcd} \quad = \quad \begin{tikzcd} A \arrow[r, "f_n"] \arrow[dr, "f_{n+1}"'] & B \arrow[d, "\phi"] \\ & B \end{tikzcd}$$

commutes. Now, given a morphism $f \colon A \to B$ of $\mathcal{C}$, we have a corresponding morphism

$$\begin{tikzcd} A \arrow[r, "f_0"] \arrow[d, equal] & B \arrow[d, "\phi"] \\ A \arrow[r, "f_1"] \arrow[d, equal] & B \arrow[d, "\phi"] \\ A \arrow[r, "f_2"] \arrow[d, equal] & B \arrow[d, "\phi"] \\ A \arrow[r, "f_3"] \arrow[d, equal] & B \arrow[d, "\phi"] \\ \vdots & \vdots \end{tikzcd} \quad = \quad \begin{tikzcd} A \arrow[r, "f_0"] \arrow[dr, "f_1"] \arrow[ddr, "f_2"] \arrow[dddr, "f_3"] \arrow[ddddr, "f_4"] & B \arrow[d, "\phi"] \\ & B \arrow[d, "\phi"] \\ \cdots & B \arrow[d, "\phi"] \\ & B \arrow[d, "\phi"] \\ & \vdots \end{tikzcd}$$

of $\mathsf{End}(\mathcal{C})$, and conversely every such morphism comes uniquely from a morphism of $\mathcal{C}$.

- *The Adjunction* 忘 $\dashv \mathbb{N} \pitchfork (-)$. Given $(A, \phi) \in \mathrm{Obj}(\mathsf{End}(\mathcal{C}))$ and $B \in \mathrm{Obj}(\mathcal{C})$, we have a bijection

$$\mathrm{Hom}_{\mathsf{End}(\mathcal{C})}((A, \phi), (\mathbb{N} \pitchfork B, \mathbb{N} \pitchfork \mathrm{id}_B)) \cong \mathrm{Hom}_{\mathcal{C}}(A, B).$$

Indeed, we have

$$\begin{aligned}\mathrm{Hom}_{\mathsf{End}(\mathcal{C})}((A, \phi), (\mathbb{N} \pitchfork B, \mathbb{N} \pitchfork \mathrm{id}_B)) &\cong \mathrm{Hom}_{\mathsf{End}(\mathcal{C})}\Big((A, \phi), \Big(B^{\times \mathbb{N}}, \mathrm{id}_B^{\times \mathbb{N}}\Big)\Big) \\ &\cong \mathrm{Hom}_{\mathsf{End}(\mathcal{C})}((A, \phi), (B, \mathrm{id}_B))^{\times \mathbb{N}},\end{aligned}$$

and hence a morphism $(A, \phi) \to (\mathbb{N} \pitchfork B, \mathbb{N} \pitchfork \mathrm{id}_B)$ of $\mathsf{End}(\mathcal{C})$ is equivalently given by an $\mathbb{N}$-indexed collection

$$\{f_n \colon A \to B\}_{n \in \mathbb{N}}$$

of morphisms of $\mathcal{C}$ such that, for each $n \in \mathbb{N}$, the diagram

$$\begin{tikzcd} A \arrow[r, "f_n"] \arrow[d, "\phi"'] & B \arrow[d, equal] \\ A \arrow[r, "f_{n+1}"'] & B \end{tikzcd} \quad = \quad \begin{tikzcd} A \arrow[r, "f_n"] \arrow[d, "\phi"'] & B \\ A \arrow[ur, "f_{n+1}"'] & \end{tikzcd}$$

commutes. Now, given a morphism $f \colon A \to B$ of $\mathcal{C}$, we have a corresponding morphism

$$\begin{tikzcd} A \arrow[r, "f_0"] \arrow[d, "\phi"'] & B \arrow[d, equal] \\ A \arrow[r, "f_1"] \arrow[d, "\phi"'] & B \arrow[d, equal] \\ A \arrow[r, "f_2"] \arrow[d, "\phi"'] & B \arrow[d, equal] \\ A \arrow[r, "f_3"] \arrow[d, "\phi"'] & B \arrow[d, equal] \\ \vdots & \vdots \end{tikzcd} \quad = \quad \begin{tikzcd} A \arrow[r, "f_0"] \arrow[d, "\phi"'] & B \\ A \arrow[ur, "f_1"] \arrow[d, "\phi"'] & \\ A \arrow[uur, "f_2"] \arrow[d, "\phi"'] & \cdots \\ A \arrow[uuur, "f_3"] \arrow[d, "\phi"'] & \\ \vdots \arrow[uuuur, "f_4"] & \end{tikzcd}$$

of $\mathsf{End}(\mathcal{C})$, and conversely every such morphism comes uniquely from a morphism of $\mathcal{C}$.

Item 4, Adjointness II: Indeed, applying Kan Extensions, **??** of **??** to the terminal functor $!\colon \mathsf{B}\mathbb{N} \twoheadrightarrow \mathsf{pt}$ from $\mathsf{B}\mathbb{N}$, we obtain a triple adjunction

$$(\mathrm{Lan}_! \dashv !^* \dashv \mathrm{Ran}_!)\colon \quad \begin{tikzcd} \mathsf{Fun}(\mathsf{B}\mathbb{N}, \mathcal{C}) \arrow[r, bend left, "\mathrm{Lan}_!"] \arrow[r, bend right, "\mathrm{Ran}_!"'] & \mathsf{Fun}(\mathsf{pt}, \mathcal{C}) \arrow[l, "!^*"'] \end{tikzcd}.$$

Here $\mathsf{Fun}(\mathsf{B}\mathbb{N}, \mathcal{C}) \overset{\text{def}}{=} \mathsf{End}(\mathcal{C})$ by definition and $\mathsf{Fun}(\mathsf{pt}, \mathcal{C}) \cong \mathcal{C}$ via **??** of **??**. We claim that $\mathrm{Lan}_! \cong \mathrm{colim}^{\circlearrowleft}(\phi)$, $!^* \cong \iota$, and $\mathrm{Ran}_! \cong \lim^{\circlearrowleft}(\phi)$:

- *Computing* $\mathrm{Lan}_!$. Let (A, ϕ) be an object of $\mathsf{End}(\mathcal{C})$. By Kan Extensions, **??** of **??**, we have

$$\mathrm{Lan}_!(A, \phi) \cong \mathrm{colim}\left(! \downarrow \underline{\star} \twoheadrightarrow \mathsf{B}\mathbb{N} \xrightarrow{(A,\phi)} \mathcal{C} \right).$$

  Unwinding the description of $! \downarrow \underline{\star}$ given in **??**, we see that it is isomorphic to $\mathsf{B}\mathbb{N}$ via the functor $! \downarrow \underline{\star} \to \mathsf{B}\mathbb{N}$. Thus $\mathrm{Lan}_! \cong \mathrm{colim}^{\circlearrowleft}$.

- *Computing* $!^*$. Let A be an object of $\mathcal{C}$, viewed as a functor $[A]\colon \mathsf{pt} \to \mathcal{C}$. Then the composition

$$\mathsf{B}\mathbb{N} \xrightarrow{!} \mathsf{pt} \xrightarrow{A} \mathcal{C}$$

  corresponds precisely to (A, id_A), and we see that $!^* \cong \iota$.

- *Computing* $\mathrm{Ran}_!$. Let (A, ϕ) be an object of $\mathsf{End}(\mathcal{C})$. By Kan Extensions, **??** of **??**, we have

$$\mathrm{Ran}_!(A, \phi) \cong \lim\left(\underline{\star} \downarrow ! \twoheadrightarrow \mathsf{B}\mathbb{N} \xrightarrow{(A,\phi)} \mathcal{C} \right).$$

  Unwinding the description of $\underline{\star} \downarrow !$ given in **??**, we see that it is isomorphic to $\mathsf{B}\mathbb{N}$ via the functor $\underline{\star} \downarrow ! \to \mathsf{B}\mathbb{N}$. Thus $\mathrm{Ran}_! \cong \lim^{\circlearrowleft}$.

Item 5: 2-Adjointness: This is a special case of **??** of **??**. □

3.4 The Endomorphism Monoid of an Object of a Category

Let $\mathcal{C}$ be a category, let $X \in \mathrm{Obj}(\mathcal{C})$, and let $(\mathcal{C}, X)$ be a category with a distinguished object.

Definition 3.4.1.1. The **endomorphism monoid of** X **in** $\mathcal{C}$ is the monoid $\mathrm{End}_{\mathcal{C}}(X)$ consisting of

- *The Underlying Set.* The set $\mathrm{End}_{\mathcal{C}}(X)$ defined by

$$\mathrm{End}_{\mathcal{C}}(X) \stackrel{\text{def}}{=} \mathrm{Hom}_{\mathcal{C}}(X, X);$$

- *The Multiplication Map.* The map of sets

$$\mu_{\mathrm{End}_{\mathcal{C}}(X)} \colon \underbrace{\mathrm{End}_{\mathcal{C}}(X) \times \mathrm{End}_{\mathcal{C}}(X)}_{\stackrel{\text{def}}{=} \mathrm{Hom}_{\mathcal{C}}(X,X) \times \mathrm{Hom}_{\mathcal{C}}(X,X)} \to \underbrace{\mathrm{End}_{\mathcal{C}}(X)}_{\stackrel{\text{def}}{=} \mathrm{Hom}_{\mathcal{C}}(X,X)}$$

  defined by

$$\mu_{\mathrm{End}_{\mathcal{C}}(X)} \stackrel{\text{def}}{=} \circ^{\mathcal{C}}_{X,X,X};$$

- *The Unit Map.* The map of sets

$$\eta_{\mathrm{End}_{\mathcal{C}}(X)} \colon \mathrm{pt} \to \underbrace{\mathrm{End}_{\mathcal{C}}(X)}_{\stackrel{\text{def}}{=} \mathrm{Hom}_{\mathcal{C}}(X,X)}$$

  defined by

$$\eta_{\mathrm{End}_{\mathcal{C}}(X)} \stackrel{\text{def}}{=} \mathbb{K}^{\mathcal{C}}_{X}.$$

Definition 3.4.1.2. The **endomorphism monoid of** $(\mathcal{C}, X)$ is the endomorphism monoid $\mathrm{End}_{\mathcal{C}}(X)$ of X in $\mathcal{C}$.

Proposition 3.4.1.3. Let $\mathcal{C}$ be a category.

1. *Functoriality.* The assignment $(\mathcal{C}, X) \mapsto \mathrm{End}_{\mathcal{C}}(X)$ defines a functor

$$\mathrm{End} \colon \mathsf{Cats}_* \to \mathsf{Mon},$$

   where

   - *Action on Objects.* For each $(\mathcal{C}, X) \in \mathrm{Obj}(\mathsf{Cats}_*)$, we have

$$\mathrm{End}(\mathcal{C}, X) \stackrel{\text{def}}{=} \mathrm{End}_{\mathcal{C}}(X);$$

- *Action on Morphisms.* For each morphism $F\colon (\mathcal{C},X)\to(\mathcal{D},Y)$ of Cats_*, the image

$$\mathrm{End}(F)\colon \mathrm{End}_{\mathcal{C}}(X)\to\mathrm{End}_{\mathcal{D}}(Y)$$

of F by End is defined by

$$\mathrm{End}(F)\overset{\mathrm{def}}{=}F_{X,X}.$$

2. *Adjointness.* We have an adjunction

$$(\mathsf{B}\dashv\mathrm{End})\colon\quad \begin{tikzcd} \mathsf{Mon} \arrow[r, bend left, "\mathsf{B}"] \arrow[r, phantom, "\perp"] & \mathsf{Cats}_*, \arrow[l, bend left, "\mathrm{End}"] \end{tikzcd}$$

witnessed by a bijection

$$\mathsf{Cats}_*((\mathsf{B}A,\star),(\mathcal{C},X))\cong\mathsf{Mon}(A,\mathrm{End}_{\mathcal{C}}(X)),$$

natural in $A\in\mathrm{Obj}(\mathsf{Mon})$ and $(\mathcal{C},X)\in\mathrm{Obj}(\mathsf{Cats}_*)$.

3. *Interaction With Groupoids I: Functoriality.* The functor of Item 1 restricts to a functor

$$\mathsf{Aut}\colon\mathsf{Grpd}_*\to\mathsf{Grp}.$$

4. *Interaction With Groupoids II: Adjointness.* The adjunction of Item 2 restricts to an adjunction

$$(\mathsf{B}\dashv\mathsf{Aut})\colon\quad \begin{tikzcd} \mathsf{Grp} \arrow[r, bend left, "\mathsf{B}"] \arrow[r, phantom, "\perp"] & \mathsf{Grpd}_*, \arrow[l, bend left, "\mathsf{Aut}"] \end{tikzcd}$$

witnessed by a bijection

$$\mathsf{Grpd}_*((\mathsf{B}G,\star),(\mathcal{C},X))\cong\mathsf{Grpd}(G,\mathsf{Aut}_{\mathcal{C}}(X)),$$

natural in $G\in\mathrm{Obj}(\mathsf{Grp})$ and $(\mathcal{C},X)\in\mathrm{Obj}(\mathsf{Cats}_*)$.

5. *Preservation of Limits.* The functor $\mathrm{End}\colon\mathsf{Cats}_*\to\mathsf{Mon}$ of Item 1 preserves limits. In particular, we have isomorphisms of categories

$$\begin{aligned}\mathrm{End}_{\mathcal{C}\wedge\mathcal{D}}(*_{\mathcal{C}\wedge\mathcal{D}})&\cong\mathrm{End}_{\mathcal{C}}(*_{\mathcal{C}})\times\mathrm{End}_{\mathcal{D}}(*_{\mathcal{D}}),\\ \mathrm{End}_{\mathrm{Eq}(F,G)}(*_{\mathcal{C}})&\cong\mathrm{Eq}(\mathrm{End}(F),\mathrm{End}(G)),\end{aligned}$$

natural in $(\mathcal{C},*_{\mathcal{C}}),(\mathcal{D},*_{\mathcal{D}})\in\mathrm{Obj}(\mathsf{Cats}_*)$ and parallel $F,G\in\mathrm{Mor}(\mathsf{Cats}_*)$.

Proof. *Item 1, Functoriality*: Clear.
Item 2, Adjointness: Omitted.
Item 3, Interaction With Groupoids I: Functoriality: Clear.
Item 4, Interaction With Groupoids II: Adjointness: Clear.
Item 5, Preservation of Limits: This follows from Item 2 and **??** of **??**. □

4 Automorphisms

4.1 Foundations

Let $\mathcal{C}$ be a category.

Definition 4.1.1.1. An **automorphism in** $\mathcal{C}$ is a functor $\phi\colon \mathsf{B}\mathbb{Z} \to \mathcal{C}$.

Remark 4.1.1.2. In detail, an **automorphism in** $\mathcal{C}$ is a pair (A, ϕ) consisting of[16]

- *The Underlying Object.* An object A of $\mathcal{C}$;
- *The Automorphism.* An isomorphism $\phi\colon A \xrightarrow{\cong} A$ in $\mathcal{C}$.

Proof. Indeed, a functor $\phi\colon \mathsf{B}\mathbb{Z} \to \mathcal{C}$ consists of

- *Action on Objects.* A map of sets
$$\phi_0\colon \underbrace{\mathrm{Obj}(\mathsf{B}\mathbb{Z})}_{\overset{\text{def}}{=}\mathrm{pt}} \to \mathrm{Obj}(\mathcal{C})$$
picking an object A of $\mathcal{C}$;
- *Action on Morphisms.* A map of sets
$$\phi_{\star,\star}\colon \underbrace{\mathrm{Hom}_{\mathsf{B}\mathbb{Z}}(\star,\star)}_{\overset{\text{def}}{=}\mathbb{Z}} \to \mathrm{Hom}_{\mathcal{C}}(A, A);$$

preserving composition and identities. This makes $\phi_{\star,\star}$ into a morphism of monoids
$$\phi_{\star,\star}\colon \underbrace{\left(\mathrm{Hom}_{\mathsf{B}\mathbb{Z}}(\star,\star), \circ^{\mathsf{B}\mathbb{Z}}_{\star,\star,\star}, \mathbb{1}^{\mathsf{B}\mathbb{Z}}_{\star}\right)}_{\overset{\text{def}}{=}(\mathbb{Z},+,0)} \to (\mathrm{Hom}_{\mathcal{C}}(A, A), \circ, \mathrm{id}_A),$$
determining and being determined by, via Monoids, **??** of **??**, an invertible element $\phi\colon A \xrightarrow{\cong} A$ of $\mathrm{Hom}_{\mathcal{C}}(A, A)$, i.e. an isomorphism in $\mathcal{C}$ from A to itself. □

[16] In other words, an **automorphism in** $\mathcal{C}$ is an endomorphism of $\mathcal{C}$ which is additionally

4.2 Morphisms of Automorphisms in Categories

Definition 4.2.1.1. A **morphism of automorphisms in $\mathcal{C}$** from ϕ to ψ is a natural transformation $\alpha\colon \phi \Longrightarrow \psi$ of functors from $\mathsf{B}\mathbb{Z}$ to $\mathcal{C}$.

Remark 4.2.1.2. In detail, a **morphism of automorphisms in $\mathcal{C}$** from (A, ϕ) to (B, ψ) is a morphism $f\colon A \to B$ of $\mathcal{C}$ such that the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\phi"'] & B \arrow[d, "\psi"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

commutes.

4.3 The Category of Automorphisms in a Category

Definition 4.3.1.1. The **category of automorphisms in $\mathcal{C}$** is the category $\mathsf{Aut}(\mathcal{C})$[17] defined by

$$\mathsf{Aut}(\mathcal{C}) \overset{\text{def}}{=} \mathsf{Fun}(\mathsf{B}\mathbb{Z}, \mathcal{C}).$$

Remark 4.3.1.2. In detail, the **category of automorphisms in $\mathcal{C}$** is the category $\mathsf{Aut}(\mathcal{C})$ where

- *Objects.* The objects of $\mathsf{Aut}(\mathcal{C})$ are automorphisms in $\mathcal{C}$;
- *Morphisms.* The morphisms of $\mathsf{Aut}(\mathcal{C})$ are morphisms of automorphisms in $\mathcal{C}$;
- *Identities.* For each $(A, \phi) \in \mathrm{Obj}(\mathsf{Aut}(\mathcal{C}))$, the unit map

  $$\mathbb{1}^{\mathsf{Aut}(\mathcal{C})}_{(A,\phi)}\colon \mathrm{pt} \to \mathrm{Hom}_{\mathsf{Aut}(\mathcal{C})}((A, \phi), (A, \phi))$$

  of $\mathsf{Aut}(\mathcal{C})$ at (A, ϕ) is defined by

  $$\mathrm{id}^{\mathsf{Aut}(\mathcal{C})}_{(A,\phi)} \overset{\text{def}}{=} \mathrm{id}_A;$$

an isomorphism in $\mathcal{C}$.

[17]Since $\mathsf{B}\mathbb{Z}$ may be thought of as a categorical realisation of the circle (as $|\mathrm{N}_\bullet(\mathsf{B}\mathbb{Z})| \simeq S^1$), we also write $\mathcal{L}(\mathcal{C})$ for $\mathsf{Aut}(\mathcal{C})$, which we may view as the **categorical free loop space of $\mathcal{C}$**.

Homotopy-theoretic information about $\mathcal{L}(\mathcal{C})$ is often not of much interest, however, as many categories commonly appearing in practice tend to be contractible for reasons which also hold true for categories of functors into them (as is the case of $\mathcal{L}(\mathcal{C}) \overset{\text{def}}{=} \mathsf{Fun}(\mathsf{B}\mathbb{Z}, \mathcal{C})$),

- *Composition.* For each $(A,\phi),(B,\psi),(C,\chi)\in\mathrm{Obj}(\mathsf{Aut}(\mathcal{C}))$, the composition map
$$\circ^{\mathsf{Aut}(\mathcal{C})}_{\phi,\psi,\chi}\colon \mathrm{Hom}_{\mathsf{Aut}(\mathcal{C})}(\psi,\chi)\times\mathrm{Hom}_{\mathsf{Aut}(\mathcal{C})}(\phi,\psi)\to\mathrm{Hom}_{\mathsf{Aut}(\mathcal{C})}(\phi,\chi)$$
of $\mathsf{Aut}(\mathcal{C})$ at $(A,\phi),(B,\psi),(C,\chi)$ is defined by
$$g\circ^{\mathsf{Aut}(\mathcal{C})}_{\phi,\psi,\chi} f\overset{\mathrm{def}}{=} g\circ f.$$

Proposition 4.3.1.3. Let $\mathcal{C}$ be a category.[18]

1. *Functoriality.* The assignment $\mathcal{C}\mapsto\mathsf{Aut}(\mathcal{C})$ defines a functor
$$\mathsf{Aut}\colon\mathsf{Cats}\to\mathsf{Cats}.$$
2. 2-*Functoriality.* The assignment $\mathcal{C}\mapsto\mathsf{Aut}(\mathcal{C})$ defines a 2-functor
$$\mathsf{Aut}\colon\mathsf{Cats}_2\to\mathsf{Cats}_2.$$
3. *Adjointness I.* If $\mathcal{C}$ is bicomplete, then we have a triple adjunction
$$\left(\chi^{\mathrm{L}}\dashv\iota\dashv\chi^{\mathrm{R}}\right)\colon\quad \begin{tikzcd} \mathsf{End}(\mathcal{C}) \arrow[r, bend left, "\chi^{\mathrm{L}}"] \arrow[r, bend right, "\chi^{\mathrm{R}}"'] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, phantom, shift right=2, "\perp"] & \mathsf{Aut}(\mathcal{C}), \arrow[l, "\iota"'] \end{tikzcd}$$

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such as admitting initial/final objects or binary co/products.

[18]There are two other natural triple adjunctions not included here:

- The first is the adjunction between $\mathsf{End}(\mathcal{C})$ and $\mathsf{Aut}(\mathcal{C})$ induced by taking left and right Kan extensions along the functor $\mathsf{B}\mathbb{Z}\to\mathsf{B}\mathbb{N}$ corresponding to the morphism of monoids $0\colon\mathbb{Z}\to\mathbb{N}$. One of the functors involved is the functor
$$0^*\colon\mathsf{End}(\mathcal{C})\to\mathsf{Aut}(\mathcal{C})$$
defined by
$$0^*(A,\phi)\overset{\mathrm{def}}{=}(A,\mathrm{id}_A);$$
- The second is the family of adjunctions between $\mathsf{End}(\mathcal{C})$ and $\mathsf{Aut}(\mathcal{C})$ induced by taking left and right Kan extensions along the functor $\mathsf{B}\mathbb{N}\to\mathsf{B}\mathbb{Z}$ corresponding to the morphism of monoids $k\colon\mathbb{N}\to\mathbb{Z}$ picking $k\in\mathbb{Z}$. One of the functors involved is the functor
$$k^*\colon\mathsf{Aut}(\mathcal{C})\to\mathsf{End}(\mathcal{C})$$
defined by
$$k^*(A,\phi)\overset{\mathrm{def}}{=}\left(A,\phi^{\circ k}\right).$$

where[19,20]

- $\chi^{\mathrm{L}}\colon \mathsf{End}(\mathcal{C}) \to \mathsf{Aut}(\mathcal{C})$ is the functor defined on objects by

$$\chi^{\mathrm{L}}(A,\phi) \stackrel{\text{def}}{=} \left(\chi^{\mathrm{L}}_{\phi}(A), \chi^{\mathrm{L}}(\phi)\right),$$

  where

  - $\chi^{\mathrm{L}}_{\phi}(A)$ is the object of $\mathcal{C}$ defined by

$$\begin{aligned}\chi^{\mathrm{L}}_{\phi}(A) &\stackrel{\text{def}}{=} \operatorname{colim}\left(\cdots \xrightarrow{\phi} A \xrightarrow{\phi} A \xrightarrow{\phi} A \xrightarrow{\phi} \cdots\right)\\ &\cong \operatorname{colim}\left(A \xrightarrow{\phi} A \xrightarrow{\phi} A \xrightarrow{\phi} \cdots\right);\end{aligned}$$

  - $\chi^{\mathrm{L}}(\phi)\colon \chi^{\mathrm{L}}_{\phi}(A) \to \chi^{\mathrm{L}}_{\phi}(A)$ is the automorphism of $\chi^{\mathrm{L}}_{\phi}(A)$ obtained by applying functoriality of colimits (Limits and Colimits, **??** of **??**) to the natural transformation of diagrams

$$\begin{tikzcd} \cdots \arrow[r, "\phi"] & A \arrow[r, "\phi"] \arrow[d, "\phi"] & A \arrow[r, "\phi"] \arrow[d, "\phi"] & A \arrow[r, "\phi"] \arrow[d, "\phi"] & \cdots \\ \cdots \arrow[r, "\phi"'] & A \arrow[r, "\phi"'] & A \arrow[r, "\phi"'] & A \arrow[r, "\phi"'] & \cdots; \end{tikzcd}$$

- $\iota\colon \mathsf{Aut}(\mathcal{C}) \to \mathsf{End}(\mathcal{C})$ is the fully faithful inclusion of categories defined on objects by

$$\iota(A,\phi) \stackrel{\text{def}}{=} (A,\phi);$$

- $\chi^{\mathrm{R}}\colon \mathsf{End}(\mathcal{C}) \to \mathsf{Aut}(\mathcal{C})$ is the functor defined on objects by

$$\chi^{\mathrm{R}}(A,\phi) \stackrel{\text{def}}{=} \left(\chi^{\mathrm{R}}_{\phi}(A), \chi^{\mathrm{R}}(\phi)\right),$$

  where

[19] In a sense, χ^{L} and χ^{R} are the co/universal ways of producing an automorphism starting with an endomorphism.

[20] *Examples:* Examples of χ^{L} include the following:

1. The localisation $A[a^{-1}]$ of a monoid A by a single element $a \in A$ (Monoids, **??**);
2. The localisation $A[a^{-1}]$ of a monoid with zero $(A, 0_A)$ by a single element $a \in A$ (Monoids With Zero, **??**);

– $\chi^{\mathrm{R}}_{\phi}(A)$ is the object of $\mathcal{C}$ defined by

$$\begin{aligned}\chi^{\mathrm{R}}_{\phi}(A) &\overset{\text{def}}{=} \lim\left(\cdots \xrightarrow{\phi} A \xrightarrow{\phi} A \xrightarrow{\phi} A \xrightarrow{\phi} \cdots\right)\\ &\cong \lim\left(\cdots \xrightarrow{\phi} A \xrightarrow{\phi} A \xrightarrow{\phi} A\right);\end{aligned}$$

– $\chi^{\mathrm{R}}(\phi)\colon \chi^{\mathrm{R}}_{\phi}(A) \to \chi^{\mathrm{R}}_{\phi}(A)$ is the automorphism of $\chi^{\mathrm{R}}_{\phi}(A)$ obtained by applying functoriality of limits (Limits and Colimits, **??** of **??**) to the natural transformation of diagrams

$$\begin{tikzcd} \cdots \arrow[r, "\phi"] & A \arrow[r, "\phi"] \arrow[d, "\phi"] & A \arrow[r, "\phi"] \arrow[d, "\phi"] & A \arrow[r, "\phi"] \arrow[d, "\phi"] & \cdots \\ \cdots \arrow[r, "\phi"'] & A \arrow[r, "\phi"'] & A \arrow[r, "\phi"'] & A \arrow[r, "\phi"'] & \cdots; \end{tikzcd}$$

5. *Adjointness II.* If $\mathcal{C}$ has products and coproducts, then we have a triple adjunction

$$(\mathbb{Z} \odot (-) \dashv 忘 \dashv \mathbb{Z} \pitchfork (-))\colon \quad \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "\mathbb{Z}\odot(-)"] \arrow[r, bend right, "\mathbb{Z}\pitchfork(-)"'] & \mathsf{Aut}(\mathcal{C}), \arrow[l, "忘"'] \end{tikzcd}$$

where[21]

- $\mathbb{Z} \odot (-)\colon \mathcal{C} \to \mathsf{Aut}(\mathcal{C})$ is the functor defined on objects by

$$\begin{aligned}\mathbb{Z} \odot (A) &\overset{\text{def}}{=} (\mathbb{Z} \odot A, \mathbb{Z} \odot \mathrm{id}_A)\\ &\cong \left(A^{\coprod \mathbb{Z}}, \mathrm{id}_A^{\coprod \mathbb{Z}}\right); \qquad \text{(Weighted Category Theory, ??)}\end{aligned}$$

- 忘: $\mathsf{Aut}(\mathcal{C}) \to \mathcal{C}$ is the **forgetful functor from** $\mathsf{Aut}(\mathcal{C})$ **to** $\mathcal{C}$, defined on objects by

$$忘(A, \phi) \overset{\text{def}}{=} A;$$

3. The localisation $M[r^{-1}]$ of an R-module M by a single element $r \in R$ (Modules Over Commutative Rings, **??**);
4. The coperfection of a characteristic p ring of ().

Similarly, an example of χ^{R} is given by the perfection of a characteristic p ring of ().

[21] In a sense, $(\mathbb{Z} \odot A, \mathbb{Z} \odot \mathrm{id}_A)$ and $(\mathbb{Z} \pitchfork A, \mathbb{Z} \pitchfork \mathrm{id}_A)$ are the co/universal ways of produc-

- $\mathbb{Z} \pitchfork (-) \colon \mathcal{C} \to \mathsf{Aut}(\mathcal{C})$ is the functor defined on objects by

$$\begin{aligned}\mathbb{Z} \pitchfork (A) &\overset{\text{def}}{=} (\mathbb{Z} \pitchfork A, \mathbb{Z} \pitchfork \mathrm{id}_A)\\ &\cong \left(A^{\times \mathbb{Z}}, \mathrm{id}_A^{\times \mathbb{Z}}\right). \qquad \text{(Weighted Category Theory, ??)}\end{aligned}$$

6. *Adjointness III.* If $\mathcal{C}$ is bicomplete, then we have a triple adjunction

$$(\mathrm{colim}^{\circlearrowleft} \dashv \iota \dashv \lim^{\circlearrowleft})\colon \quad \begin{tikzcd} \mathsf{Aut}(\mathcal{C}) \arrow[r, bend left, "\mathrm{colim}^{\circlearrowleft}"] \arrow[r, bend right, "\lim^{\circlearrowleft}"'] & \mathcal{C} \arrow[l, hook', "\iota"'] \end{tikzcd},$$

where[22]

- $\mathrm{colim}^{\circlearrowleft} \colon \mathsf{Aut}(\mathcal{C}) \to \mathcal{C}$ is the functor defined on objects by

$$\begin{aligned}\mathrm{colim}^{\circlearrowleft}(A, \phi) &\overset{\text{def}}{=} \mathrm{colim}\left(\mathsf{B}\mathbb{Z} \xrightarrow{(A,\phi)} \mathcal{C}\right)\\ &\overset{\text{def}}{=} \mathrm{colim}(A \circlearrowleft \phi);\end{aligned}$$

- $\iota \colon \mathcal{C} \hookrightarrow \mathsf{Aut}(\mathcal{C})$ is the functor defined on objects by[23]

$$\iota(A) \overset{\text{def}}{=} (A, \mathrm{id}_A);$$

- $\lim^{\circlearrowleft} \colon \mathsf{Aut}(\mathcal{C}) \to \mathcal{C}$ is the functor defined on objects by

$$\begin{aligned}\lim^{\circlearrowleft}(A, \phi) &\overset{\text{def}}{=} \lim\left(\mathsf{B}\mathbb{Z} \xrightarrow{(A,\phi)} \mathcal{C}\right)\\ &\overset{\text{def}}{=} \lim(A \circlearrowleft \phi).\end{aligned}$$

7. 2-*Adjointness.* We have a 2-adjunction

$$(\mathsf{B}\mathbb{Z} \times - \dashv \mathsf{Aut})\colon \quad \begin{tikzcd} \mathsf{Cats}_2 \arrow[r, bend left, "\mathsf{B}\mathbb{Z}\times -"] \arrow[r, phantom, "\perp_2"] & \mathsf{Cats}_2 \arrow[l, bend left, "\mathsf{Aut}"] \end{tikzcd}.$$

ing an automorphism starting with an identity.

[22] In a sense, $\mathrm{colim}^{\circlearrowleft}(A, \phi)$ and $\lim^{\circlearrowleft}(A, \phi)$ are the co/universal ways of producing an identity starting with an automorphism.

[23] Viewing $\mathcal{C} \cong \mathsf{Fun}(\mathsf{pt}, \mathcal{C})$ as the "category of identities of $\mathcal{C}$", we see that the functor ι is just the inclusion of categories from the category of identities of $\mathcal{C}$ to the category of automorphisms of $\mathcal{C}$.

Proof. *Item 1, Functoriality*: Omitted.
Item 2: 2-*Functoriality*: Omitted.
Item 3, Adjointness I: Omitted.
Item 5, Adjointness II: Omitted.
Item 6, Adjointness III: Omitted.
Item 7: 2-*Adjointness*: This is a special case of **??** of **??**. □

4.4 The Automorphism Group of an Object of a Category

Let $\mathcal{C}$ be a category, let $X \in \mathrm{Obj}(\mathcal{C})$, and let $(\mathcal{C}, X)$ be a category with a distinguished object.

Definition 4.4.1.1. The **automorphism group** of an object A of $\mathcal{C}$ is the group $\mathrm{Aut}_{\mathcal{C}}(A)$ consisting of

- *The Underlying Set.* The set $\mathrm{Aut}_{\mathcal{C}}(A)$ defined by
$$\mathrm{Aut}_{\mathcal{C}}(A) \overset{\mathrm{def}}{=} \{f \in \mathrm{End}_{\mathcal{C}}(A) \mid f \text{ is an isomorphism}\};$$
- *The Multiplication Map.* The map of sets
$$\mu_{\mathrm{Aut}_{\mathcal{C}}(A)} \colon \mathrm{Aut}_{\mathcal{C}}(A) \times \mathrm{Aut}_{\mathcal{C}}(A) \to \mathrm{Aut}_{\mathcal{C}}(A)$$
defined by
$$\mu_{\mathrm{Aut}_{\mathcal{C}}(A)} \overset{\mathrm{def}}{=} \restriction \circ^{\mathcal{C}}_{A,A,A} \mathrm{Aut}_{\mathcal{C}}(A);$$
- *The Unit Map.* The map of sets
$$\eta_{\mathrm{Aut}_{\mathcal{C}}(A)} \colon \mathrm{pt} \to \mathrm{Aut}_{\mathcal{C}}(A)$$
defined by
$$\eta_{\mathrm{Aut}_{\mathcal{C}}(A)} \overset{\mathrm{def}}{=} \nVdash^{\mathcal{C}}_{A};$$
- *The Antipode.* The map of sets
$$\chi_{\mathrm{Aut}_{\mathcal{C}}(A)} \colon \mathrm{Aut}_{\mathcal{C}}(A) \to \mathrm{Aut}_{\mathcal{C}}(A)$$
defined by
$$\chi_{\mathrm{Aut}_{\mathcal{C}}(A)}(f) \overset{\mathrm{def}}{=} f^{-1}$$
for each $f \in \mathrm{Aut}_{\mathcal{C}}(A)$.

Definition 4.4.1.2. The **automorphism group of** $(\mathcal{C}, X)$ is the automorphism group $\mathrm{Aut}_{\mathcal{C}}(X)$ of X in $\mathcal{C}$.[24]

[24] *Warning:* The assignment $(\mathcal{C}, X) \mapsto \mathrm{Aut}_{\mathcal{C}}(X)$ *does not* define a functor

5 Involutions

5.1 Foundations

Let $\mathcal{C}$ be a category.

Definition 5.1.1.1. An **involution in $\mathcal{C}$** is a functor $\sigma\colon \mathsf{B}\mathbb{Z}_{/2} \to \mathcal{C}$.

Remark 5.1.1.2. In detail, an **involution in $\mathcal{C}$** is a pair (A, σ) consisting of[25,26]

- *The Underlying Object.* An object A of $\mathcal{C}$;
- *The Involution.* An automorphism $\sigma\colon A \xrightarrow{\cong} A$ of $\mathcal{C}$ such that we have

$$\sigma^2 = \mathrm{id}_A, \qquad \begin{tikzcd} A \arrow[r, "\sigma"] \arrow[dr, "\mathrm{id}_A"'] & A \arrow[d, "\sigma"] \\ & A. \end{tikzcd}$$

Proof. Indeed, a functor $\sigma\colon \mathsf{B}\mathbb{Z}_{/2} \to \mathcal{C}$ consists of

- *Action on Objects.* A map of sets
$$\sigma_0\colon \underbrace{\mathrm{Obj}\Big(\mathsf{B}\mathbb{Z}_{/2}\Big)}_{\overset{\text{def}}{=}\mathrm{pt}} \to \mathrm{Obj}(\mathcal{C})$$
picking an object A of $\mathcal{C}$;
- *Action on Morphisms.* A map of sets
$$\sigma_{\star,\star}\colon \underbrace{\mathrm{Hom}_{\mathsf{B}\mathbb{Z}_{/2}}(\star,\star)}_{\overset{\text{def}}{=}\mathbb{Z}_{/2}} \to \mathrm{Hom}_{\mathcal{C}}(A, A);$$

preserving composition and identities. This makes $\sigma_{\star,\star}$ into a morphism of monoids

$$\sigma_{\star,\star}\colon \underbrace{\left(\mathrm{Hom}_{\mathsf{B}\mathbb{Z}_{/2}}(\star,\star), \circ^{\mathsf{B}\mathbb{Z}_{/2}}_{\star,\star,\star}, \mathbb{1}^{\mathsf{B}\mathbb{Z}_{/2}}_{\star}\right)}_{\overset{\text{def}}{=}(\mathbb{Z}_{/2},+,0)} \to (\mathrm{Hom}_{\mathcal{C}}(A, A), \circ, \mathrm{id}_A),$$

determining and being determined by, via Monoids, **??** of **??**, an involutory element $\sigma\colon A \xrightarrow{\cong} A$ of $\mathrm{Hom}_{\mathcal{C}}(A, A)$, satisfying $\sigma^2 = \mathrm{id}_A$, i.e. an involution of A. □

Aut: $\mathsf{Cats}_* \to \mathsf{Grp}$; see [**MSE570202**].

[25] In other words, an **involution in $\mathcal{C}$** is an involutory element of $\mathrm{End}_{\mathcal{C}}(A)$.

[26] In yet other words, an **involution in $\mathcal{C}$** is an order 2 automorphism of A in $\mathcal{C}$.

5.2 Morphisms of Involutions in Categories

Definition 5.2.1.1. A **morphism of involutions in $\mathcal{C}$** from σ to τ is a natural transformation $\alpha\colon \sigma \Longrightarrow \tau$ of functors from $\mathsf{B}\mathbb{Z}_{/2}$ to $\mathcal{C}$.

Remark 5.2.1.2. In detail, a **morphism of involutions in $\mathcal{C}$** from (A, σ) to (B, τ) is a morphism $f\colon A \to B$ of $\mathcal{C}$ such that the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\sigma"'] & B \arrow[d, "\tau"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

commutes.

5.3 The Category of Involutions in a Category

Definition 5.3.1.1. The **category of involutions in $\mathcal{C}$** is the category $\mathsf{Inv}(\mathcal{C})$ defined by

$$\mathsf{Inv}(\mathcal{C}) \overset{\text{def}}{=} \mathsf{Fun}\Big(\mathsf{B}\mathbb{Z}_{/2}, \mathcal{C}\Big).$$

Remark 5.3.1.2. In detail, the **category of involutions in $\mathcal{C}$** is the category $\mathsf{Inv}(\mathcal{C})$ where

- *Objects.* The objects of $\mathsf{Inv}(\mathcal{C})$ are involutions in $\mathcal{C}$;
- *Morphisms.* The morphisms of $\mathsf{Inv}(\mathcal{C})$ are morphisms of involutions in $\mathcal{C}$;
- *Identities.* For each $(A, \sigma) \in \mathrm{Obj}(\mathsf{Inv}(\mathcal{C}))$, the unit map

  $$\mathbb{1}^{\mathsf{Inv}(\mathcal{C})}_{(A,\sigma)} \colon \mathrm{pt} \to \mathrm{Hom}_{\mathsf{Inv}(\mathcal{C})}((A, \sigma), (A, \sigma))$$

  of $\mathsf{Inv}(\mathcal{C})$ at (A, σ) is defined by

  $$\mathrm{id}^{\mathsf{Inv}(\mathcal{C})}_{(A,\sigma)} \overset{\text{def}}{=} \mathrm{id}_A;$$

- *Composition.* For each $(A, \sigma), (B, \rho), (C, \tau) \in \mathrm{Obj}(\mathsf{Inv}(\mathcal{C}))$, the composition map

  $$\circ^{\mathsf{Inv}(\mathcal{C})}_{\sigma,\rho,\tau} \colon \mathrm{Hom}_{\mathsf{Inv}(\mathcal{C})}(\rho, \tau) \times \mathrm{Hom}_{\mathsf{Inv}(\mathcal{C})}(\sigma, \rho) \to \mathrm{Hom}_{\mathsf{Inv}(\mathcal{C})}(\sigma, \tau)$$

  of $\mathsf{Inv}(\mathcal{C})$ at $(A, \sigma), (B, \rho), (C, \tau)$ is defined by

  $$g \circ^{\mathsf{Inv}(\mathcal{C})}_{\sigma,\rho,\tau} f \overset{\text{def}}{=} g \circ f.$$

Proposition 5.3.1.3. Let $\mathcal{C}$ be a category.

1. *Functoriality.* The assignment $\mathcal{C} \mapsto \mathsf{Inv}(\mathcal{C})$ defines a functor
$$\mathsf{Inv}\colon \mathsf{Cats} \to \mathsf{Cats}.$$

2. 2-*Functoriality.* The assignment $\mathcal{C} \mapsto \mathsf{Inv}(\mathcal{C})$ defines a 2-functor
$$\mathsf{Inv}\colon \mathsf{Cats}_2 \to \mathsf{Cats}_2.$$

3. *Adjointness I.* If $\mathcal{C}$ is bicomplete, then we have a triple adjunction
$$(\mathrm{L} \dashv \iota \dashv \mathrm{R})\colon \quad \begin{tikzcd} \mathsf{Aut}(\mathcal{C}) \arrow[r, bend left=45, "\mathrm{L}"] \arrow[r, bend right=45, "\mathrm{R}"'] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, phantom, shift right=3, "\perp"] & \mathsf{Inv}(\mathcal{C}), \arrow[l, "\iota"'] \end{tikzcd}$$
obtained via precomposition and Kan extensions along the delooping $\mathsf{B}(\mathrm{mod}\ 2)\colon \mathsf{B}\mathbb{Z} \to \mathsf{B}\mathbb{Z}_{/2}$ of the parity map, where

- $\mathrm{L}\colon \mathsf{Aut}(\mathcal{C}) \to \mathsf{Inv}(\mathcal{C})$ is the functor defined on objects by
$$\mathrm{L}(A, \phi) \stackrel{\text{def}}{=} (\mathrm{L}(A), \mathrm{L}(\phi)),$$
where $\mathrm{L}(A)$ is the colimit
$$\mathrm{L}(A) \stackrel{\text{def}}{=} \mathrm{colim}\left(\begin{tikzcd} A \arrow[loop above, "\phi^{-2}"] \arrow[loop left, "\mathrm{id}_A"] \arrow[loop below, "\phi^{2}"] \arrow[r, shift left=7, "\phi^{-3}"] \arrow[r, shift left=5, "\phi^{3}"] \arrow[r, shift left=3, "\phi^{-1}"] \arrow[r, shift left=1, "\phi"] & A \arrow[loop above, "\phi^{2}"] \arrow[loop right, "\mathrm{id}_A"] \arrow[loop below, "\phi^{-2}"] \arrow[l, shift left=1, "\phi"] \arrow[l, shift left=3, "\phi^{-1}"] \arrow[l, shift left=5, "\phi^{3}"] \arrow[l, shift left=7, "\phi^{-3}"] \end{tikzcd} \right)$$
in $\mathcal{C}$;
- $\iota\colon \mathsf{Inv}(\mathcal{C}) \hookrightarrow \mathsf{Aut}(\mathcal{C})$ is the natural inclusion of categories of $\mathsf{Inv}(\mathcal{C})$ into $\mathsf{Aut}(\mathcal{C})$;
- $\mathrm{R}\colon \mathsf{Aut}(\mathcal{C}) \to \mathsf{Inv}(\mathcal{C})$ is the functor defined on objects by
$$\mathrm{R}(A, \phi) \stackrel{\text{def}}{=} (\mathrm{R}(A), \mathrm{R}(\phi)),$$

where $\mathrm{R}(A)$ is the limit

$$\mathrm{R}(A) \overset{\mathrm{def}}{=} \lim\left(\begin{tikzcd} A \arrow[loop above, "\phi^{-2}"] \arrow[loop left, "\mathrm{id}_A"] \arrow[loop below, "\phi^{2}"] \arrow[r, shift left=7, "\phi^{-3}"] \arrow[r, shift left=5, "\phi^{3}"] \arrow[r, shift left=3, "\phi^{-1}"] \arrow[r, shift left=1, "\phi"] & A \arrow[l, shift left=1, "\phi"] \arrow[l, shift left=3, "\phi^{-1}"] \arrow[l, shift left=5, "\phi^{3}"] \arrow[l, shift left=7, "\phi^{-3}"] \arrow[loop above, "\phi^{2}"] \arrow[loop right, "\mathrm{id}_A"] \arrow[loop below, "\phi^{-2}"] \end{tikzcd}\right)$$

in $\mathcal{C}$.

4. *Adjointness II.* If $\mathcal{C}$ is bicomplete, then we have a triple adjunction

$$(\mathrm{L} \dashv \iota \dashv \mathrm{R})\colon \quad \begin{tikzcd} \mathsf{End}(\mathcal{C}) \arrow[r, bend left, "\mathrm{L}"] \arrow[r, bend right, "\mathrm{R}"'] & \mathsf{Inv}(\mathcal{C}), \arrow[l, "\iota"' description] \end{tikzcd}$$

obtained by either

- Combining the triple adjunctions in Item 3 of Proposition 4.3.1.3 and Item 3, or;
- Via precomposition and Kan extensions along the delooping $\mathsf{B}(\mathrm{mod}\ 2)\colon \mathsf{B}\mathbb{N} \hookrightarrow \mathsf{B}\mathbb{Z}_{/2}$ of the parity map;

where

- $\mathrm{L}\colon \mathsf{End}(\mathcal{C}) \to \mathsf{Inv}(\mathcal{C})$ is the functor defined on objects by

$$\mathrm{L}(A, \phi) \overset{\mathrm{def}}{=} (\mathrm{L}(A), \mathrm{L}(\phi)),$$

  where $\mathrm{L}(A)$ is the colimit

$$\mathrm{L}(A) \overset{\mathrm{def}}{=} \operatorname{colim}\left(\begin{tikzcd} A \arrow[loop above, "\phi^{6}"] \arrow[loop left, "\phi^{4}"] \arrow[loop below left, "\phi^{2}"] \arrow[loop below, "\mathrm{id}_A"] \arrow[r, shift left=7, "\phi^{7}"] \arrow[r, shift left=5, "\phi^{5}"] \arrow[r, shift left=3, "\phi^{3}"] \arrow[r, shift left=1, "\phi"] & A \arrow[l, shift left=1, "\phi"] \arrow[l, shift left=3, "\phi^{3}"] \arrow[l, shift left=5, "\phi^{5}"] \arrow[l, shift left=7, "\phi^{7}"] \arrow[loop above, "\phi^{6}"] \arrow[loop right, "\phi^{4}"] \arrow[loop below right, "\phi^{2}"] \arrow[loop below, "\mathrm{id}_A"] \end{tikzcd}\right)$$

  in $\mathcal{C}$;
- $\iota\colon \mathsf{Inv}(\mathcal{C}) \hookrightarrow \mathsf{End}(\mathcal{C})$ is the natural inclusion of categories of $\mathsf{Inv}(\mathcal{C})$ into $\mathsf{End}(\mathcal{C})$;

- $\mathrm{R}\colon \mathsf{End}(\mathcal{C}) \to \mathsf{Inv}(\mathcal{C})$ is the functor defined on objects by

  $$\mathrm{R}(A,\phi) \overset{\text{def}}{=} (\mathrm{R}(A), \mathrm{R}(\phi)),$$

  where $\mathrm{R}(A)$ is the limit

  $$\mathrm{R}(A) \overset{\text{def}}{=} \lim\left(\begin{tikzcd} A \arrow[loop below, "\mathrm{id}_A"] \arrow[loop left, "\phi^2"] \arrow[loop left, "\phi^4"] \arrow[loop above, "\phi^6"] \arrow[r, shift left=1, "\phi"] \arrow[r, shift left=3, "\phi^3"] \arrow[r, shift left=5, "\phi^5"] \arrow[r, shift left=7, "\phi^7"] & A \arrow[loop below, "\mathrm{id}_A"] \arrow[loop right, "\phi^2"] \arrow[loop right, "\phi^4"] \arrow[loop above, "\phi^6"] \arrow[l, shift left=1, "\phi"] \arrow[l, shift left=3, "\phi^3"] \arrow[l, shift left=5, "\phi^5"] \arrow[l, shift left=7, "\phi^7"] \end{tikzcd} \right)$$

  in $\mathcal{C}$.

5. *Adjointness III.* If $\mathcal{C}$ is bicomplete, then we have a triple adjunction

$$\left(\mathbb{Z}_{/2} \odot (-) \dashv \iota \dashv \mathbb{Z}_{/2} \pitchfork (-)\right)\colon \quad \begin{tikzcd} \mathcal{C} \arrow[r, bend left=45, "\mathbb{Z}_{/2}\odot(-)"] \arrow[r, bend right=45, "\mathbb{Z}_{/2}\pitchfork(-)"'] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, phantom, shift right=3, "\perp"] & \mathsf{Inv}(\mathcal{C}), \arrow[l, "\iota"] \end{tikzcd}$$

obtained by either

- Combining the triple adjunctions in Item 3 of Proposition 3.3.1.3, Item 3 of Proposition 4.3.1.3 and Item 3, or;
- Via precomposition and Kan extensions along the delooping $\mathsf{B}\{\star\} \twoheadrightarrow \mathsf{B}\mathbb{Z}_{/2}$ of the initial map from $\{\star\}$ to $\mathbb{Z}_{/2}$;

where

- $\mathbb{Z}_{/2} \odot (-)\colon \mathcal{C} \to \mathsf{Inv}(\mathcal{C})$ is defined on objects by

  $$\mathbb{Z}_{/2} \odot A \overset{\text{def}}{=} \left(A \textstyle\coprod A, \beta^{\mathcal{C},\coprod}_{A,A}\right),$$

  where $\beta^{\mathcal{C},\coprod}_{A,A}\colon A \coprod A \to A \coprod A$ is the morphism swapping the two factors of A in $A \coprod A$;
- $\iota\colon \mathsf{Inv}(\mathcal{C}) \to \mathcal{C}$ is the forgetful functor defined on objects by

  $$\iota(A,\sigma) \overset{\text{def}}{=} A;$$

- $\mathbb{Z}_{/2} \pitchfork (-)\colon \mathcal{C} \to \mathsf{Inv}(\mathcal{C})$ is defined on objects by
$$\mathbb{Z}_{/2} \pitchfork A \overset{\text{def}}{=} \left(A \times A, \beta^{\mathcal{C},\times}_{A,A}\right),$$
where $\beta^{\mathcal{C},\times}_{A,A}\colon A \times A \to A \times A$ is the morphism swapping the two factors of A in $A \times A$.

6. *Adjointness IV.* If $\mathcal{C}$ is bicomplete, then we have a triple adjunction
$$(\mathrm{L} \dashv \iota \dashv \mathrm{R})\colon \quad \begin{tikzcd} \mathsf{Inv}(\mathcal{C}) \arrow[r, bend left=45, "\mathrm{L}"] \arrow[r, bend right=45, "\mathrm{R}"'] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, phantom, shift right=3, "\perp"] & \mathcal{C}, \arrow[l, "\iota" description] \end{tikzcd}$$
obtained via precomposition and Kan extensions along the delooping $\mathsf{B}\mathbb{Z}_{/2} \twoheadrightarrow \mathsf{B}\{\star\}$ of the terminal map from $\mathbb{Z}_{/2}$ to $\{\star\}$, where
   - $\mathrm{colim}^{\circlearrowleft}\colon \mathsf{Inv}(\mathcal{C}) \to \mathcal{C}$ is the restriction to $\mathsf{Inv}(\mathcal{C})$ of the functor $\mathrm{colim}^{\circlearrowleft}$ of Item 4 of Proposition 3.3.1.3, being defined on objects by
$$\begin{aligned}\mathrm{colim}^{\circlearrowleft}(A,\sigma) &\overset{\text{def}}{=} \mathrm{colim}\left(\mathsf{B}\mathbb{Z}_{/2} \xrightarrow{(A,\sigma)} \mathcal{C}\right)\\ &\overset{\text{def}}{=} \mathrm{colim}(A \circlearrowleft \sigma);\end{aligned}$$
   - $\iota\colon \mathcal{C} \hookrightarrow \mathsf{End}(\mathcal{C})$ is the functor defined on objects by[27]
$$\iota(A) \overset{\text{def}}{=} (A, \mathrm{id}_A);$$
   - $\lim^{\circlearrowleft}\colon \mathsf{Inv}(\mathcal{C}) \to \mathcal{C}$ is the restriction to $\mathsf{Inv}(\mathcal{C})$ of the functor $\lim^{\circlearrowleft}$ of Item 4 of Proposition 3.3.1.3, being defined on objects by
$$\begin{aligned}\lim^{\circlearrowleft}(A,\sigma) &\overset{\text{def}}{=} \lim\left(\mathsf{B}\mathbb{Z}_{/2} \xrightarrow{(A,\sigma)} \mathcal{C}\right)\\ &\overset{\text{def}}{=} \lim(A \circlearrowleft \sigma).\end{aligned}$$

7. 2-*Adjointness.* We have a 2-adjunction
$$\left(\mathsf{B}\mathbb{Z}_{/2} \times - \dashv \mathsf{Inv}\right)\colon \quad \begin{tikzcd} \mathsf{Cats}_2 \arrow[r, bend left, "\mathsf{B}\mathbb{Z}_{/2}\times -"] \arrow[r, phantom, "\perp_2"] & \mathsf{Cats}_2. \arrow[l, bend left, "\mathsf{Inv}"] \end{tikzcd}$$

[27] Viewing $\mathcal{C} \cong \mathsf{Fun}(\mathsf{pt}, \mathcal{C})$ as the "category of identities of $\mathcal{C}$", we see that the functor ι

Proof. *Item 1, Functoriality*: Omitted.
Item 2: 2-*Functoriality*: Omitted.
Item 3, Adjointness I: Omitted.
Item 4, Adjointness II: Omitted.
Item 5, Adjointness III: Omitted.
Item 6, Adjointness IV: Omitted.
Item 7: 2-*Adjointness*: This is a special case of **??** of **??**. □

6 Idempotent Morphisms

6.1 Foundations

Let $\mathcal{C}$ be a category.

Definition 6.1.1.1. An **idempotent morphism in** $\mathcal{C}$ is a functor $\sigma\colon \mathsf{B}\mathbb{B} \to \mathcal{C}$.

Remark 6.1.1.2. In detail, an **idempotent morphism in** $\mathcal{C}$ is a pair (A, σ) consisting of[28]

- *The Underlying Object.* An object A of $\mathcal{C}$;
- *The Idempotent Morphism.* A morphism $\sigma\colon A \xrightarrow{\cong} A$ of $\mathcal{C}$ such that we have

$$\sigma^2 = \sigma, \qquad \begin{tikzcd} A \arrow[r, "\sigma"] \arrow[dr, "\sigma"'] & A \arrow[d, "\sigma"] \\ & A. \end{tikzcd}$$

Proof. Indeed, a functor $\sigma\colon \mathsf{B}\mathbb{B} \to \mathcal{C}$ consists of

- *Action on Objects.* A map of sets

$$\sigma_0\colon \underbrace{\mathrm{Obj}(\mathsf{B}\mathbb{B})}_{\overset{\text{def}}{=}\mathrm{pt}} \to \mathrm{Obj}(\mathcal{C})$$

picking an object A of $\mathcal{C}$;

is just the inclusion of categories from the category of identities of $\mathcal{C}$ to the category of endomorphisms of $\mathcal{C}$.

[28]In other words, an **idempotent morphism in** $\mathcal{C}$ is an idempotent element of $\mathrm{End}_{\mathcal{C}}(A)$.

- *Action on Morphisms.* A map of sets

$$\sigma_{\star,\star}\colon \underbrace{\mathrm{Hom}_{\mathsf{B}\mathbb{B}}(\star,\star)}_{\overset{\text{def}}{=}\mathbb{B}} \to \mathrm{Hom}_{\mathcal{C}}(A,A);$$

preserving composition and identities. This makes $\sigma_{\star,\star}$ into a morphism of monoids

$$\sigma_{\star,\star}\colon \underbrace{\left(\mathrm{Hom}_{\mathsf{B}\mathbb{B}}(\star,\star), \circ^{\mathsf{B}\mathbb{B}}_{\star,\star,\star}, \mathbb{1}^{\mathsf{B}\mathbb{B}}_{\star}\right)}_{\overset{\text{def}}{=}(\mathbb{B},+,0)} \to (\mathrm{Hom}_{\mathcal{C}}(A,A), \circ, \mathrm{id}_A),$$

determining and being determined by, via Monoids, **??** of **??**, an idempotent element $\sigma\colon A \to A$ of $\mathrm{End}_{\mathcal{C}}(A,A)$, satisfying $\sigma^2 = \sigma$, i.e. an idempotent morphism in $\mathcal{C}$ from A to itself. □

6.2 Morphisms of Idempotent Morphisms

Definition 6.2.1.1. A **morphism of idempotent morphisms in** $\mathcal{C}$ from σ to τ is a natural transformation $\alpha\colon \sigma \Longrightarrow \tau$ of functors from $\mathsf{B}\mathbb{B}$ to $\mathcal{C}$.

Remark 6.2.1.2. In detail, a **morphism of idempotent morphisms in** $\mathcal{C}$ from (A,σ) to (B,τ) is a morphism $f\colon A \to B$ of $\mathcal{C}$ such that the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\sigma"'] & B \arrow[d, "\tau"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

commutes.

6.3 The Category of Idempotent Morphisms of a Category

Definition 6.3.1.1. The **category of idempotent morphisms of** $\mathcal{C}$ is the category $\mathsf{Idem}(\mathcal{C})$ defined by

$$\mathsf{Idem}(\mathcal{C}) \overset{\text{def}}{=} \mathsf{Fun}(\mathsf{B}\mathbb{B}, \mathcal{C}).$$

Remark 6.3.1.2. In detail, the **category of idempotent morphisms in** $\mathcal{C}$ is the category $\mathsf{Idem}(\mathcal{C})$ where

- *Objects.* The objects of $\mathsf{Idem}(\mathcal{C})$ are idempotent morphisms in $\mathcal{C}$;

- *Morphisms.* The morphisms of $\mathsf{Idem}(\mathcal{C})$ are morphisms of idempotent morphisms in $\mathcal{C}$;
- *Identities.* For each $(A,\sigma)\in\mathrm{Obj}(\mathsf{Idem}(\mathcal{C}))$, the unit map

  $$\mathbb{1}^{\mathsf{Idem}(\mathcal{C})}_{(A,\sigma)}\colon \mathrm{pt}\to\mathrm{Hom}_{\mathsf{Idem}(\mathcal{C})}((A,\sigma),(A,\sigma))$$

  of $\mathsf{Idem}(\mathcal{C})$ at (A,σ) is defined by

  $$\mathrm{id}^{\mathsf{Idem}(\mathcal{C})}_{(A,\sigma)}\overset{\mathrm{def}}{=}\mathrm{id}_A;$$

- *Composition.* For each $(A,\sigma),(B,\rho),(C,\tau)\in\mathrm{Obj}(\mathsf{Idem}(\mathcal{C}))$, the composition map

  $$\circ^{\mathsf{Idem}(\mathcal{C})}_{\sigma,\rho,\tau}\colon \mathrm{Hom}_{\mathsf{Idem}(\mathcal{C})}(\rho,\tau)\times\mathrm{Hom}_{\mathsf{Idem}(\mathcal{C})}(\sigma,\rho)\to\mathrm{Hom}_{\mathsf{Idem}(\mathcal{C})}(\sigma,\tau)$$

  of $\mathsf{Idem}(\mathcal{C})$ at $((A,\sigma),(B,\rho),(C,\tau))$ is defined by

  $$g\circ^{\mathsf{Idem}(\mathcal{C})}_{\sigma,\rho,\tau} f\overset{\mathrm{def}}{=}g\circ f.$$

Proposition 6.3.1.3. Let $\mathcal{C}$ be a category.

1. *Functoriality.* The assignment $\mathcal{C}\mapsto\mathsf{Idem}(\mathcal{C})$ defines a functor

   $$\mathsf{Idem}\colon \mathsf{Cats}\to\mathsf{Cats}.$$

2. 2-*Functoriality.* The assignment $\mathcal{C}\mapsto\mathsf{Idem}(\mathcal{C})$ defines a 2-functor

   $$\mathsf{Idem}\colon \mathsf{Cats}_2\to\mathsf{Cats}_2.$$

3. *Adjointness I.* If $\mathcal{C}$ is bicomplete, then we have a triple adjunction

   $$(\mathrm{L}\dashv\iota\dashv\mathrm{R})\colon\quad \begin{tikzcd} \mathsf{End}(\mathcal{C}) \arrow[r, bend left, "\mathrm{L}"] \arrow[r, bend right, "\mathrm{R}"'] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, phantom, shift right=3, "\perp"] & \mathsf{Idem}(\mathcal{C}), \arrow[l, "\iota"'] \end{tikzcd}$$

   obtained via precomposition and Kan extensions along the delooping $\mathsf{B}\mathbb{N}\to\mathsf{B}\mathbb{B}$ of the map picking $1\in\mathbb{B}$ via Monoids, **??** of **??**, where

- $\mathrm{L}\colon \mathsf{End}(\mathcal{C}) \to \mathsf{Idem}(\mathcal{C})$ is the functor defined on objects by

  $$\mathrm{L}(A,\phi) \overset{\mathrm{def}}{=} (\mathrm{L}(A), \mathrm{L}(\phi)),$$

  where $\mathrm{L}(A)$ is the coequaliser

  $$\begin{aligned} \mathrm{L}(A) &\cong \mathrm{CoEq}\left(\coprod_{n\in\mathbb{N}} \mathbb{B}\odot A \underset{\rho}{\overset{\lambda}{\rightrightarrows}} \mathbb{B}\odot A\right) \\ &\cong \mathrm{CoEq}\left(\coprod_{n\in\mathbb{N}} A \textstyle\coprod A \underset{\rho}{\overset{\lambda}{\rightrightarrows}} A \textstyle\coprod A\right) \end{aligned}$$

  in $\mathcal{C}$, where

  $$\begin{aligned} \lambda &\overset{\mathrm{def}}{=} \mathrm{id}_{A\coprod A} \textstyle\coprod \displaystyle\coprod_{n=1}^{\infty} (\mathrm{inj}_2 \textstyle\coprod \mathrm{inj}_2), \\ \rho &\overset{\mathrm{def}}{=} \mathrm{id}_{A\coprod A} \textstyle\coprod \displaystyle\coprod_{n=1}^{\infty} (\phi^n \textstyle\coprod \phi^n); \end{aligned}$$

- $\iota\colon \mathsf{Idem}(\mathcal{C}) \hookrightarrow \mathsf{End}(\mathcal{C})$ is the natural inclusion of categories of $\mathsf{Idem}(\mathcal{C})$ into $\mathsf{End}(\mathcal{C})$;
- $\mathrm{R}\colon \mathsf{End}(\mathcal{C}) \to \mathsf{Idem}(\mathcal{C})$ is the functor defined on objects by

  $$\mathrm{R}(A,\phi) \overset{\mathrm{def}}{=} (\mathrm{R}(A), \mathrm{R}(\phi)),$$

  where $\mathrm{R}(A)$ is the equaliser

  $$\begin{aligned} \mathrm{R}(A) &\cong \mathrm{Eq}\left(\mathbb{B}\pitchfork A \underset{\rho}{\overset{\lambda}{\rightrightarrows}} \prod_{n\in\mathbb{N}} \mathbb{B}\pitchfork A\right) \\ &\cong \mathrm{Eq}\left(A\times A \underset{\rho}{\overset{\lambda}{\rightrightarrows}} \prod_{n\in\mathbb{N}} A\times A\right) \end{aligned}$$

  in $\mathcal{C}$, where

  $$\begin{aligned} \lambda &\overset{\mathrm{def}}{=} \mathrm{id}_{A\times A} \times \prod_{n=1}^{\infty} (\mathrm{pr}_2 \times \mathrm{pr}_2), \\ \rho &\overset{\mathrm{def}}{=} \mathrm{id}_{A\times A} \times \prod_{n=1}^{\infty} (\phi^n \times \phi^n). \end{aligned}$$

4. *Adjointness II.* If $\mathcal{C}$ is bicomplete, then we have a triple adjunction

$$(\mathbb{B} \odot (-) \dashv \iota \dashv \mathbb{B} \pitchfork (-)) \colon \quad \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "\mathbb{B}\odot(-)"] \arrow[r, bend right, "\mathbb{B}\pitchfork(-)"'] & \mathsf{Idem}(\mathcal{C}) \arrow[l, "\iota"'] \end{tikzcd},$$

obtained by either

- Combining the triple adjunctions in Item 3 of Proposition 3.3.1.3 and Item 3, or;
- Via precomposition and Kan extensions along the delooping $\mathsf{B}\{\star\} \twoheadrightarrow \mathsf{B}\mathbb{B}$ of the initial map from $\{\star\}$ to $\mathbb{B}$;

where

- $\mathbb{B} \odot (-) \colon \mathcal{C} \to \mathsf{Idem}(\mathcal{C})$ is defined on objects by

$$\mathbb{B} \odot A \stackrel{\text{def}}{=} (A \textstyle\coprod A, \sigma_{A,A}),$$

  where $\sigma_{A,A} \colon A \coprod A \to A \coprod A$ is the morphism defined by[29,30]

$$\sigma_{A,A} \stackrel{\text{def}}{=} \mathrm{inj}_2 \textstyle\coprod \mathrm{inj}_2;$$

- $\iota \colon \mathsf{Idem}(\mathcal{C}) \to \mathcal{C}$ is the forgetful functor defined on objects by

$$\iota(A, \sigma) \stackrel{\text{def}}{=} A;$$

- $\mathbb{B} \pitchfork (-) \colon \mathcal{C} \to \mathsf{Idem}(\mathcal{C})$ is defined on objects by

$$\mathbb{B} \pitchfork A \stackrel{\text{def}}{=} (A \times A, \sigma_{A,A}),$$

[29]For $\mathcal{C} = \mathsf{Sets}$, the map $\sigma_{A,A}$ is explicitly given by sending each $x \in A \coprod A$ in either factor of A in $A \coprod A$ to the copy of x in the second factor of A in $A \coprod A$.

[30]When $\mathcal{C}$ has an initial object $\varnothing_{\mathcal{C}}$, the map $\sigma_{A,A}$ is the same as the composition

$$A \textstyle\coprod A \xrightarrow{\nabla_A} A \xrightarrow{\cong} \varnothing_{\mathcal{C}} \coprod A \hookrightarrow A \coprod A$$

where $\nabla_A \colon A \coprod A \to A$ is the fold map of A.

where $\sigma_{A,A}\colon A \times A \to A \times A$ is the morphism defined by[31,32]

$$\sigma_{A,A} \overset{\text{def}}{=} \mathrm{pr}_2 \times \mathrm{pr}_2.$$

5. 2-*Adjointness.* We have a 2-adjunction

$$(\mathsf{B}\mathbb{B} \times - \dashv \mathsf{Idem})\colon \quad \begin{tikzcd} \mathsf{Cats}_2 \arrow[r, bend left, "\mathsf{B}\mathbb{B}\times -"] \arrow[r, phantom, "\perp_2"] & \mathsf{Cats}_2 \arrow[l, bend left, "\mathsf{Idem}"] \end{tikzcd}.$$

Proof. *Item 1, Functoriality*: Omitted.
Item 2: 2-*Functoriality*: Omitted.
Item 3, Adjointness I: Omitted.
Item 4, Adjointness II: Omitted.
Item 5: 2-*Adjointness*: This is a special case of **??** of **??**. □

Appendices

A Other Chapters

Set Theory

1. Sets
2. Constructions With Sets
3. Pointed Sets
4. Tensor Products of Pointed Sets
5. Indexed and Fibred Sets
6. Relations
7. Spans
8. Posets

Category Theory

9. Categories
10. Constructions With Categories
11. Kan Extensions

Bicategories

12. Bicategories

[31]For $\mathcal{C} = \mathsf{Sets}$, the map $\sigma_{A,A}$ is explicitly given by

$$\sigma_{A,A}(x,y) \overset{\text{def}}{=} (y,y)$$

for each $(x,y) \in A \times A$.

[32]When $\mathcal{C}$ has a terminal object $\varnothing_{\mathcal{C}}$, the map $\sigma_{A,A}$ is the same as the composition

$$A \times A \twoheadrightarrow \mathrm{pt} \times A \xrightarrow{\cong} A \xrightarrow{\delta_A} A \times A$$

where $\Delta_A\colon A \to A \times A$ is the diagonal map of A.