

Tensor Products of Pointed Sets

December 3, 2023

008H This chapter contains some material on tensor products of pointed sets.

Contents

1	Bilinear Morphisms of Pointed Sets	2
1.1	Left Bilinear Morphisms of Pointed Sets	2
1.2	Right Bilinear Morphisms of Pointed Sets	2
1.3	Bilinear Morphisms of Pointed Sets	3
2	Tensors and Cotensors of Pointed Sets by Sets	5
2.1	Tensors of Pointed Sets by Sets	5
2.2	Cotensors of Pointed Sets by Sets	5
3	The Left Tensor Product of Pointed Sets	6
3.1	Foundations	6
3.2	The Skew Associator	7
3.3	The Skew Left Unitor	8
3.4	The Skew Right Unitor	9
3.5	The Left-Skew Monoidal Category Structure on Pointed Sets	10
4	The Right Tensor Product of Pointed Sets	10
4.1	Foundations	11
4.2	The Skew Associator	12
4.3	The Skew Left Unitor	13
4.4	The Skew Right Unitor	13
4.5	The Right-Skew Monoidal Category Structure on Pointed Sets	14
5	Smash Products of Pointed Sets	15
5.1	Foundations	15

A Other Chapters 21

008J 1 Bilinear Morphisms of Pointed Sets

008K 1.1 Left Bilinear Morphisms of Pointed Sets

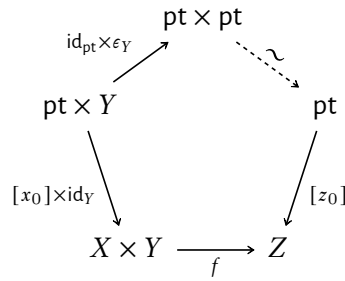
Let (X, x_0) , (Y, y_0) , and (Z, z_0) be pointed sets.

008L **Definition 1.1.1.1.** A **left bilinear morphism of pointed sets from $(X \times Y, (x_0, y_0))$ to (Z, z_0)** is a map of sets

$$f: (X \times Y, (x_0, y_0)) \rightarrow (Z, z_0)$$

satisfying the following condition:^{1,2}

(★) *Left Unital Bilinearity.* The diagram



commutes, i.e. for each $y \in Y$, we have

$$f(x_0, y) = z_0.$$

008M **Definition 1.1.1.2.** The **set of left bilinear morphisms of pointed sets from $(X \times Y, (x_0, y_0))$ to (Z, z_0)** is the set $\text{Hom}_{\text{Sets}_*}^{\otimes, L}(X \times Y, Z)$ defined by

$$\text{Hom}_{\text{Sets}_*}^{\otimes, L}(X \times Y, Z) \stackrel{\text{def}}{=} \{f \in \text{Sets}_*(A \times B, C) \mid f \text{ is left bilinear}\}.$$

008N 1.2 Right Bilinear Morphisms of Pointed Sets

Let (X, x_0) , (Y, y_0) , and (Z, z_0) be pointed sets.

¹*Slogan:* f is left bilinear if it preserves basepoints in its first argument.

²Succinctly, f is bilinear if we have

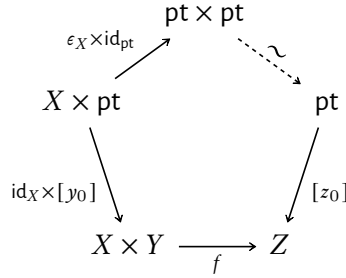
$$f(x_0, y) = z_0$$

008P Definition 1.2.1.1. A **right bilinear morphism of pointed sets** from $(X \times Y, (x_0, y_0))$ to (Z, z_0) is a map of sets

$$f: (X \times Y, (x_0, y_0)) \rightarrow (Z, z_0)$$

satisfying the following condition:^{3,4}

(★) *Right Unital Bilinearity.* The diagram



commutes, i.e. for each $x \in X$, we have

$$f(x, y_0) = z_0.$$

008Q Definition 1.2.1.2. The **set of right bilinear morphisms of pointed sets** from $(X \times Y, (x_0, y_0))$ to (Z, z_0) is the set $\text{Hom}_{\text{Sets}_*}^{\otimes, R}(X \times Y, Z)$ defined by

$$\text{Hom}_{\text{Sets}_*}^{\otimes, R}(X \times Y, Z) \stackrel{\text{def}}{=} \{f \in \text{Sets}_*(A \times B, C) \mid f \text{ is right bilinear}\}.$$

008R 1.3 Bilinear Morphisms of Pointed Sets

Let (X, x_0) , (Y, y_0) , and (Z, z_0) be pointed sets.

008S Definition 1.3.1.1. A **bilinear morphism of pointed sets** from $(X \times Y, (x_0, y_0))$ to (Z, z_0) is a map of sets

$$f: (X \times Y, (x_0, y_0)) \rightarrow (Z, z_0)$$

that is both left bilinear and right bilinear.

for each $y \in Y$.

³*Slogan:* f is right bilinear if it preserves basepoints in its second argument.

⁴Succinctly, f is bilinear if we have

$$f(x, y_0) = z_0$$

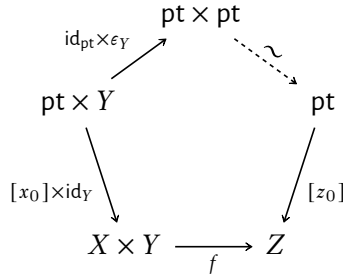
for each $x \in X$.

008T Remark 1.3.1.2. In detail, a **bilinear morphism of pointed sets from** $(X \times Y, (x_0, y_0))$ **to** (Z, z_0) is a map of sets

$$f: (X \times Y, (x_0, y_0)) \rightarrow (Z, z_0)$$

satisfying the following conditions:^{5,6}

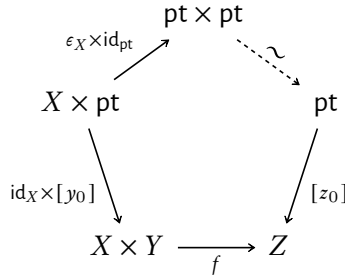
1. *Left Unital Bilinearity.* The diagram



commutes, i.e. for each $y \in Y$, we have

$$f(x_0, y) = z_0.$$

2. *Right Unital Bilinearity.* The diagram



commutes, i.e. for each $x \in X$, we have

$$f(x, y_0) = z_0.$$

008U Definition 1.3.1.3. The **set of bilinear morphisms of pointed sets from** $(X \times Y, (x_0, y_0))$ **to** (Z, z_0) is the set $\text{Hom}_{\text{Sets}_*}^{\otimes}(X \times Y, Z)$ defined by

$$\text{Hom}_{\text{Sets}_*}^{\otimes}(X \times Y, Z) \stackrel{\text{def}}{=} \{f \in \text{Sets}_*(A \times B, C) \mid f \text{ is bilinear}\}.$$

⁵*Slogan:* f is bilinear if it preserves basepoints in each argument.

⁶Succinctly, f is bilinear if we have

$$f(x_0, y) = z_0,$$

$$f(x, y_0) = z_0$$

008V 2 Tensors and Cotensors of Pointed Sets by Sets

008W 2.1 Tensors of Pointed Sets by Sets

Let (X, x_0) be a pointed set and let A be a set.

008X **Definition 2.1.1.1.** The **tensor of** (X, x_0) **by** A is the pointed set $A \odot (X, x_0)$ satisfying the following universal property:

(UP) We have a bijection

$$\mathbf{Sets}_*(A \odot X, K) \cong \mathbf{Sets}(A, \mathbf{Sets}_*(X, K)),$$

natural in $(K, k_0) \in \mathbf{Obj}(\mathbf{Sets}_*)$.

008Y **Remark 2.1.1.2.** The tensor of (X, x_0) by A satisfies the following universal property:

$$\mathbf{Sets}_*(A \odot X, K) \cong \mathbf{Sets}_{\mathbb{E}_0}^{\otimes}(A \times X, K),$$

where $\mathbf{Sets}_{\mathbb{E}_0}^{\otimes}(A \times X, K)$ is the set defined by

$$\mathbf{Sets}_{\mathbb{E}_0}^{\otimes}(A \times X, K) \stackrel{\text{def}}{=} \left\{ f \in \mathbf{Sets}(A \times X, K) \left| \begin{array}{l} \text{for each } a \in A, \text{ we have} \\ f(a, x_0) = k_0 \end{array} \right. \right\}.$$

008Z **Construction 2.1.1.3.** Concretely, the **tensor of** (X, x_0) **by** A is the pointed set $A \odot (X, x_0)$ consisting of

- *The Underlying Set.* The set $A \odot X$ given by

$$A \odot X \cong \bigvee_{a \in A} (X, x_0);$$

- *The Basepoint.* The point $[x_0]$ of $\bigvee_{a \in A} (X, x_0)$.

0090 2.2 Cotensors of Pointed Sets by Sets

Let (X, x_0) be a pointed set and let A be a set.

0091 **Definition 2.2.1.1.** The **cotensor of** (X, x_0) **by** A is the pointed set $A \pitchfork (X, x_0)$ satisfying the following universal property:

(UP) We have a bijection

$$\mathbf{Sets}_*(K, A \pitchfork X) \cong \mathbf{Sets}(A, \mathbf{Sets}_*(K, X)),$$

natural in $(K, k_0) \in \mathbf{Obj}(\mathbf{Sets}_*)$.

0092 **Remark 2.2.1.2.** The cotensor of (X, x_0) by A satisfies the following universal property:

$$\text{Sets}_*(K, A \pitchfork X) \cong \text{Sets}_{\mathbb{B}_0}^{\otimes}(A \times K, X),$$

where $\text{Sets}_{\mathbb{B}_0}^{\otimes}(A \times K, X)$ is the set defined by

$$\text{Sets}_{\mathbb{B}_0}^{\otimes}(A \times K, X) \stackrel{\text{def}}{=} \left\{ f \in \text{Sets}(A \times K, X) \left| \begin{array}{l} \text{for each } a \in A, \text{ we have} \\ f(a, k_0) = x_0 \end{array} \right. \right\}.$$

0093 **Construction 2.2.1.3.** Concretely, the **cotensor of (X, x_0) by A** is the pointed set $A \pitchfork (X, x_0)$ consisting of

- *The Underlying Set.* The set $A \pitchfork X$ given by

$$A \pitchfork X \cong \bigwedge_{a \in A} (X, x_0);$$

- *The Basepoint.* The point $[(x_0, x_0, x_0, \dots)]$ of $\bigwedge_{a \in A} (X, x_0)$.

0094 3 The Left Tensor Product of Pointed Sets

0095 3.1 Foundations

Let (X, x_0) and (Y, y_0) be pointed sets.

0096 **Definition 3.1.1.1.** The **left tensor product of pointed sets** is the functor

$$\triangleleft_{\text{Sets}_*} : \text{Sets}_* \times \text{Sets}_* \rightarrow \text{Sets}_*$$

defined as the composition

$$\text{Sets}_* \times \text{Sets}_* \xrightarrow{\text{id} \times \overline{\omega}} \text{Sets}_* \times \text{Sets} \xrightarrow{\beta_{\text{Sets}_*, \text{Sets}}^{\text{Cats}_2}} \text{Sets} \times \text{Sets}_* \xrightarrow{\odot} \text{Sets}_*.$$

0097 **Remark 3.1.1.2.** The left tensor product of pointed sets satisfies the following universal property:⁷

$$\text{Sets}_*(X \triangleleft_{\text{Sets}_*} Y, Z) \cong \text{Hom}_{\text{Sets}_*}^{\otimes, L}(X \times Y, Z).$$

for each $x \in X$ and each $y \in Y$.

⁷Namely, a pointed map $f : X \triangleleft_{\text{Sets}_*} Y \rightarrow Z$ is the same as a map $f^\dagger : X \times Y \rightarrow Z$ such that

$$f^\dagger(x_0, y) = z_0$$

for each $y \in Y$.

0098 Remark 3.1.1.3. In detail, the **left tensor product of** (X, x_0) **and** (Y, y_0) is the pointed set $(X \triangleleft_{\text{Sets}_*} Y, [x_0])$ consisting of⁸

- *The Underlying Set.* The set $X \triangleleft_{\text{Sets}_*} Y$ defined by

$$\begin{aligned} X \triangleleft_{\text{Sets}_*} Y &\stackrel{\text{def}}{=} |Y| \odot X \\ &\cong \bigvee_{y \in Y} (X, x_0); \end{aligned}$$

- *The Underlying Basepoint.* The point $[x_0]$ of $\bigvee_{y \in Y} (X, x_0)$.

0099 Proposition 3.1.1.4. Let (X, x_0) and (Y, y_0) be pointed sets.

009A 1. *Functoriality.* The assignments $X, Y, (X, Y) \mapsto X \triangleleft_{\text{Sets}_*} Y$ define functors

$$\begin{aligned} X \triangleleft_{\text{Sets}_*} - &: \text{Sets}_* \rightarrow \text{Sets}_*, \\ - \triangleleft_{\text{Sets}_*} Y &: \text{Sets}_* \rightarrow \text{Sets}_*, \\ -_1 \triangleleft_{\text{Sets}_*} -_2 &: \text{Sets}_* \times \text{Sets}_* \rightarrow \text{Sets}_*. \end{aligned}$$

Proof. **Item 1, Functoriality:** Omitted. □

009B 3.2 The Skew Associator

009C Definition 3.2.1.1. The **skew associator of the left tensor product of pointed sets** is the natural isomorphism

$$\alpha^{\text{Sets}_*, \triangleleft} : \triangleleft_{\text{Sets}_*} \circ (\triangleleft_{\text{Sets}_*} \times \text{id}_{\text{Sets}_*}) \xrightarrow{\cong} \triangleleft_{\text{Sets}_*} \circ (\text{id}_{\text{Sets}_*} \times \triangleleft_{\text{Sets}_*}),$$

whose component

$$\alpha_{X,Y,Z}^{\text{Sets}_*, \triangleleft} : (X \triangleleft_{\text{Sets}_*} Y) \triangleleft_{\text{Sets}_*} Z \xrightarrow{\cong} X \triangleleft_{\text{Sets}_*} (Y \triangleleft_{\text{Sets}_*} Z)$$

⁸*Further Notation:* We write $x \triangleleft_{\text{Sets}_*} y$ for the image of (x, y) under the map

$$\begin{aligned} X \times Y &\rightarrow \underbrace{X \triangleleft_{\text{Sets}_*} Y}_{\cong \bigvee_{y \in Y} (X, x_0)}. \end{aligned}$$

sending (x, y) to the element $x \in X$ in the y th copy of X in $\bigvee_{y \in Y} (X, x_0)$. Note that we have

$$x_0 \triangleleft_{\text{Sets}_*} y = x_0 \triangleleft_{\text{Sets}_*} y',$$

for each $y, y' \in Y$.

at (X, Y, Z) is given by the composition⁹

$$\begin{aligned}
 (X \triangleleft_{\mathbf{Sets}_*} Y) \triangleleft_{\mathbf{Sets}_*} Z &\stackrel{\text{def}}{=} |Z| \odot (X \triangleleft_{\mathbf{Sets}_*} Y) \\
 &\stackrel{\text{def}}{=} |Z| \odot (|Y| \odot X) \\
 &\cong \bigvee_{z \in Z} (|Y| \odot X, [x_0]) \\
 &\stackrel{\text{def}}{=} \bigvee_{z \in Z} \left(\bigvee_{y \in Y} (X, x_0) \right) \\
 &\cong \bigvee_{(z, y) \in \bigvee_{z \in Z} (Y, y_0)} (X, x_0) \\
 &\stackrel{\text{def}}{=} \bigvee_{(z, y) \in |Z| \odot Y} (X, x_0) \\
 &\cong ||Z| \odot Y| \odot X \\
 &\stackrel{\text{def}}{=} |Y \triangleleft_{\mathbf{Sets}_*} Z| \odot X \\
 &\stackrel{\text{def}}{=} X \triangleleft_{\mathbf{Sets}_*} (Y \triangleleft_{\mathbf{Sets}_*} Z),
 \end{aligned}$$

where the isomorphism

$$\bigvee_{z \in Z} \left(\bigvee_{y \in Y} (X, x_0) \right) \cong \bigvee_{(y, z) \in \bigvee_{z \in Z} (Y, y_0)} (X, x_0)$$

is given by $[(z, (y, x))] \mapsto [((z, y), x)]$.

009D 3.3 The Skew Left Unitor

009E **Definition 3.3.1.1.** The **skew left unitor of the left tensor product of pointed sets** is the natural transformation

$$\lambda^{\mathbf{Sets}_*, \triangleleft} : \triangleleft_{\mathbf{Sets}_*} \circ \left(\#^{\mathbf{Sets}_*} \times \text{id}_{\mathbf{Sets}_*} \right) \Rightarrow \text{id}_{\mathbf{Sets}_*},$$

whose component

$$\lambda_X^{\mathbf{Sets}_*, \triangleleft} : S^0 \triangleleft_{\mathbf{Sets}_*} X \rightarrow X$$

⁹In other words, $\alpha_{X, Y, Z}^{\mathbf{Sets}_*, \triangleleft}$ acts on elements as

$$\alpha_{X, Y, Z}^{\mathbf{Sets}_*, \triangleleft} ((x \triangleleft_{\mathbf{Sets}_*} y) \triangleleft_{\mathbf{Sets}_*} z) \stackrel{\text{def}}{=} x \triangleleft_{\mathbf{Sets}_*} (y \triangleleft_{\mathbf{Sets}_*} z)$$

for each $(x \triangleleft_{\mathbf{Sets}_*} y) \triangleleft_{\mathbf{Sets}_*} z \in (X \triangleleft_{\mathbf{Sets}_*} Y) \triangleleft_{\mathbf{Sets}_*} Z$.

at X is given by the composition¹⁰

$$\begin{aligned} S^0 \triangleleft_{\text{Sets}_*} X &\cong |X| \odot S^0 \\ &\cong \bigvee_{x \in X} S^0 \\ &\rightarrow X \end{aligned}$$

where $\bigvee_{x \in X} S^0 \rightarrow X$ is the map given by

$$\begin{aligned} (x, 0) &\mapsto x, \\ (x, 1) &\mapsto x. \end{aligned}$$

009F 3.4 The Skew Right Unitor

009G **Definition 3.4.1.1.** The **skew right unitor of the left tensor product of pointed sets** is the natural transformation

$$\rho^{\text{Sets}_*, \triangleleft} : \text{id}_{\text{Sets}_*} \Longrightarrow \triangleleft_{\text{Sets}_*} \circ (\text{id}_{\text{Sets}_*} \times \mathbb{K}^{\text{Sets}_*}),$$

whose component

$$\rho_X^{\text{Sets}_*, \triangleleft} : X \rightarrow X \triangleleft_{\text{Sets}_*} S^0$$

at X is given by the composition¹¹

$$\begin{aligned} X &\rightarrow X \vee X \\ &\cong |S^0| \odot X \\ &\cong S^0 \triangleleft_{\text{Sets}_*} X, \end{aligned}$$

where $X \rightarrow X \vee X$ is the map sending X to the first factor of X in $X \vee X$.

¹⁰In other words, $\lambda_X^{\text{Sets}_*, \triangleleft}$ acts on elements as

$$\begin{aligned} \lambda_X^{\text{Sets}_*, \triangleleft} (x \triangleleft_{\text{Sets}_*} 0) &\stackrel{\text{def}}{=} x, \\ \lambda_X^{\text{Sets}_*, \triangleleft} (x \triangleleft_{\text{Sets}_*} 1) &\stackrel{\text{def}}{=} x, \end{aligned}$$

for each $x \in X$.

¹¹In other words, $\rho_X^{\text{Sets}_*, \triangleleft}$ acts on elements as

$$\rho_X^{\text{Sets}_*, \triangleleft} (x) \stackrel{\text{def}}{=} x \triangleleft_{\text{Sets}_*} 0$$

for each $x \in X$.

009H 3.5 The Left-Skew Monoidal Category Structure on Pointed Sets

009J **Proposition 3.5.1.1.** The category \mathbf{Sets}_* admits a left-skew monoidal category structure consisting of¹²

- *The Skew Monoidal Product.* The left tensor product functor

$$\triangleleft_{\mathbf{Sets}_*} : \mathbf{Sets}_* \times \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*,$$

of **Proposition 3.1.1.4**;

- *The Skew Monoidal Unit.* The functor

$$\mathbb{K}^{\mathbf{Sets}_*} : \mathbf{pt} \rightarrow \mathbf{Sets}_*$$

defined by

$$\mathbb{K}^{\mathbf{Sets}_*} \stackrel{\text{def}}{=} S^0;$$

- *The Skew Associators.* The natural isomorphism

$$\alpha^{\mathbf{Sets}_*, \triangleleft} : \triangleleft_{\mathbf{Sets}_*} \circ (\triangleleft_{\mathbf{Sets}_*} \times \text{id}_{\mathbf{Sets}_*}) \xrightarrow{\cong} \triangleleft_{\mathbf{Sets}_*} \circ (\text{id}_{\mathbf{Sets}_*} \times \triangleleft_{\mathbf{Sets}_*}),$$

of **Definition 3.2.1.1**;

- *The Skew Left Unitors.* The natural transformation

$$\lambda^{\mathbf{Sets}_*, \triangleleft} : \triangleleft_{\mathbf{Sets}_*} \circ (\mathbb{K}^{\mathbf{Sets}_*} \times \text{id}_{\mathbf{Sets}_*}) \Rightarrow \text{id}_{\mathbf{Sets}_*},$$

of **Definition 3.3.1.1**;

- *The Skew Right Unitors.* The natural transformation

$$\rho^{\mathbf{Sets}_*, \triangleleft} : \text{id}_{\mathbf{Sets}_*} \Rightarrow \triangleleft_{\mathbf{Sets}_*} \circ (\text{id}_{\mathbf{Sets}_*} \times \mathbb{K}^{\mathbf{Sets}_*}),$$

of **Definition 3.4.1.1**.

Proof. Omitted. □

009K 4 The Right Tensor Product of Pointed Sets

¹²Note in particular that, differently from general left-skew monoidal categories, the skew associator of

009L 4.1 Foundations

Let (X, x_0) and (Y, y_0) be pointed sets.

009M **Definition 4.1.1.1.** The **right tensor product of pointed sets** is the functor

$$\triangleright_{\text{Sets}_*} : \text{Sets}_* \times \text{Sets}_* \rightarrow \text{Sets}_*$$

defined as the composition

$$\text{Sets}_* \times \text{Sets}_* \xrightarrow{\overline{\Delta} \times \text{id}} \text{Sets} \times \text{Sets}_* \xrightarrow{\odot} \text{Sets}_*.$$

009N **Remark 4.1.1.2.** The right tensor product of pointed sets satisfies the following universal property:¹³

$$\text{Sets}_*(X \triangleright_{\text{Sets}_*} Y, Z) \cong \text{Hom}_{\text{Sets}_*}^{\otimes, R}(X \times Y, Z).$$

009P **Remark 4.1.1.3.** In detail, the **right tensor product of (X, x_0) and (Y, y_0)** is the pointed set $(X \triangleright_{\text{Sets}_*} Y, [y_0])$ consisting of¹⁴

· *The Underlying Set.* The set $X \triangleright_{\text{Sets}_*} Y$ defined by

$$\begin{aligned} X \triangleright_{\text{Sets}_*} Y &\stackrel{\text{def}}{=} |X| \odot Y \\ &\cong \bigvee_{x \in X} (Y, y_0); \end{aligned}$$

· *The Underlying Basepoint.* The point $[y_0]$ of $\bigvee_{x \in X} (Y, y_0)$.

009Q **Proposition 4.1.1.4.** Let (X, x_0) and (Y, y_0) be pointed sets.

$(\text{Sets}_*, \triangleleft_{\text{Sets}_*}, S^0)$ is a natural isomorphism.

¹³Namely, a pointed map $f : X \triangleleft_{\text{Sets}_*} Y \rightarrow Z$ is the same as a map $f^\dagger : X \times Y \rightarrow Z$ such that

$$f^\dagger(x, y_0) = z_0$$

for each $y \in Y$.

¹⁴*Further Notation:* We write $x \triangleright_{\text{Sets}_*} y$ for the image of (x, y) under the map

$$\begin{aligned} X \times Y &\rightarrow \underbrace{X \triangleright_{\text{Sets}_*} Y}_{\cong \bigvee_{x \in X} (Y, y_0)}. \end{aligned}$$

sending (x, y) to the element $y \in Y$ in the x th copy of Y in $\bigvee_{x \in X} (Y, y_0)$. Note that we have

$$x \triangleright_{\text{Sets}_*} y_0 = x' \triangleright_{\text{Sets}_*} y_0,$$

for each $x, x' \in X$.

009R 1. *Functoriality.* The assignments $X, Y, (X, Y) \mapsto X \triangleright_{\text{Sets}_*} Y$ define functors

$$\begin{aligned} X \triangleright_{\text{Sets}_*} - &: \text{Sets}_* \rightarrow \text{Sets}_*, \\ - \triangleright_{\text{Sets}_*} Y &: \text{Sets}_* \rightarrow \text{Sets}_*, \\ -_1 \triangleright_{\text{Sets}_*} -_2 &: \text{Sets}_* \times \text{Sets}_* \rightarrow \text{Sets}_*. \end{aligned}$$

Proof. *Item 1, Functoriality:* Omitted. \square

009S 4.2 The Skew Associator

009T **Definition 4.2.1.1.** The **skew associator of the right tensor product of pointed sets** is the natural isomorphism

$$\alpha^{\text{Sets}_*, \triangleright} : \triangleright_{\text{Sets}_*} \circ (\text{id}_{\text{Sets}_*} \times \triangleright_{\text{Sets}_*}) \xrightarrow{\cong} \triangleright_{\text{Sets}_*} \circ (\triangleright_{\text{Sets}_*} \times \text{id}_{\text{Sets}_*}),$$

whose component

$$\alpha_{X,Y,Z}^{\text{Sets}_*, \triangleright} : X \triangleright_{\text{Sets}_*} (Y \triangleright_{\text{Sets}_*} Z) \xrightarrow{\cong} (X \triangleright_{\text{Sets}_*} Y) \triangleright_{\text{Sets}_*} Z$$

at (X, Y, Z) is given by the composition¹⁵

$$\begin{aligned} X \triangleright_{\text{Sets}_*} (Y \triangleright_{\text{Sets}_*} Z) &\stackrel{\text{def}}{=} |X| \odot (Y \triangleright_{\text{Sets}_*} Z) \\ &\stackrel{\text{def}}{=} |X| \odot (|Y| \odot Z) \\ &\cong |X| \odot \left(\bigvee_{y \in Y} (Z, z_0) \right) \\ &\cong \bigvee_{x \in X} \left(\bigvee_{y \in Y} (Z, z_0) \right) \\ &\cong \bigvee_{(x,y) \in \bigvee_{x \in X} (Y, y_0)} (Z, z_0) \\ &\cong \left| \bigvee_{x \in X} (Y, y_0) \right| \odot Z \\ &\stackrel{\text{def}}{=} |X \odot Y| \odot Z \\ &\stackrel{\text{def}}{=} |X \triangleright_{\text{Sets}_*} Y| \odot Z \\ &\stackrel{\text{def}}{=} (X \triangleright_{\text{Sets}_*} Y) \triangleright_{\text{Sets}_*} Z \end{aligned}$$

¹⁵In other words, $\alpha_{X,Y,Z}^{\text{Sets}_*, \triangleright}$ acts on elements as

$$\alpha_{X,Y,Z}^{\text{Sets}_*, \triangleright} (x \triangleright_{\text{Sets}_*} (y \triangleright_{\text{Sets}_*} z)) \stackrel{\text{def}}{=} (x \triangleright_{\text{Sets}_*} y) \triangleright_{\text{Sets}_*} z$$

where the isomorphism

$$\bigvee_{x \in X} \left(\bigvee_{y \in Y} (Z, z_0) \right) \cong \bigvee_{(x,y) \in \bigvee_{x \in X} (Y, y_0)} (Z, z_0)$$

is given by $[(x, (y, z))] \mapsto [((x, y), z)]$.

009U 4.3 The Skew Left Unitor

009V **Definition 4.3.1.1.** The **skew left unitor of the right tensor product of pointed sets** is the natural transformation

$$\lambda^{\text{Sets}_*, \triangleright} : \text{id}_{\text{Sets}_*} \Longrightarrow \triangleright_{\text{Sets}_*} \circ (\mathbb{K}^{\text{Sets}_*} \times \text{id}_{\text{Sets}_*}),$$

whose component

$$\lambda_X^{\text{Sets}_*, \triangleright} : X \rightarrow S^0 \triangleright_{\text{Sets}_*} X$$

at X is given by the composition¹⁶

$$\begin{aligned} X &\rightarrow X \vee X \\ &\cong |S^0| \odot X \\ &\cong S^0 \triangleright_{\text{Sets}_*} X, \end{aligned}$$

where $X \rightarrow X \vee X$ is the map sending X to the first factor of X in $X \vee X$.

009W 4.4 The Skew Right Unitor

009X **Definition 4.4.1.1.** The **skew right unitor of the right tensor product of pointed sets** is the natural transformation

$$\rho^{\text{Sets}_*, \triangleright} : \triangleright_{\text{Sets}_*} \circ (\text{id}_{\text{Sets}_*} \times \mathbb{K}^{\text{Sets}_*}) \Longrightarrow \text{id}_{\text{Sets}_*},$$

for each $x \triangleright_{\text{Sets}_*} (y \triangleright_{\text{Sets}_*} z) \in X \triangleright_{\text{Sets}_*} (Y \triangleright_{\text{Sets}_*} Z)$.

¹⁶In other words, $\lambda_X^{\text{Sets}_*, \triangleright}$ acts on elements as

$$\lambda_X^{\text{Sets}_*, \triangleright}(x) \stackrel{\text{def}}{=} 0 \triangleright_{\text{Sets}_*} x$$

for each $x \in X$.

whose component¹⁷

$$\rho_X^{\text{Sets}_*, \triangleright} : X \triangleright_{\text{Sets}_*} S^0 \rightarrow X$$

at X is given by the composition

$$\begin{aligned} X \triangleright_{\text{Sets}_*} S^0 &\cong |X| \odot S^0 \\ &\cong \bigvee_{x \in X} S^0 \\ &\rightarrow X \end{aligned}$$

where $\bigvee_{x \in X} S^0 \rightarrow X$ is the map given by

$$\begin{aligned} (x, 0) &\mapsto x, \\ (x, 1) &\mapsto x. \end{aligned}$$

009Y 4.5 The Right-Skew Monoidal Category Structure on Pointed Sets

009Z **Proposition 4.5.1.1.** The category Sets_* admits a right-skew monoidal category structure consisting of¹⁸

- *The Skew Monoidal Product.* The right tensor product functor

$$\triangleright_{\text{Sets}_*} : \text{Sets}_* \times \text{Sets}_* \rightarrow \text{Sets}_*,$$

of **Item 1**;

- *The Skew Monoidal Unit.* The functor

$$\mathbb{K}^{\text{Sets}_*} : \text{pt} \rightarrow \text{Sets}_*$$

defined by

$$\mathbb{K}^{\text{Sets}_*} \stackrel{\text{def}}{=} S^0;$$

¹⁷In other words, $\rho_X^{\text{Sets}_*, \triangleright}$ acts on elements as

$$\begin{aligned} \rho_X^{\text{Sets}_*, \triangleright} (x \triangleright_{\text{Sets}_*} 0) &\stackrel{\text{def}}{=} x, \\ \rho_X^{\text{Sets}_*, \triangleright} (x \triangleright_{\text{Sets}_*} 1) &\stackrel{\text{def}}{=} x \end{aligned}$$

for each $x \in X$.

¹⁸Note in particular that, differently from general right-skew monoidal categories, the skew associator of $(\text{Sets}_*, \triangleright_{\text{Sets}_*}, S^0)$ is a natural isomorphism.

- *The Skew Associators.* The natural isomorphism

$$\alpha^{\text{Sets}_*, \triangleright} : \triangleright_{\text{Sets}_*} \circ (\triangleright_{\text{Sets}_*} \times \text{id}_{\text{Sets}_*}) \xrightarrow{\cong} \triangleright_{\text{Sets}_*} \circ (\text{id}_{\text{Sets}_*} \times \triangleright_{\text{Sets}_*}),$$

of [Definition 4.2.1.1](#);

- *The Skew Left Unitors.* The natural transformation

$$\lambda^{\text{Sets}_*, \triangleright} : \text{id}_{\text{Sets}_*} \Longrightarrow \triangleright_{\text{Sets}_*} \circ (\lrcorner^{\text{Sets}_*} \times \text{id}_{\text{Sets}_*}),$$

of [Definition 3.3.1.1](#);

- *The Skew Right Unitors.* The natural transformation

$$\rho^{\text{Sets}_*, \triangleright} : \triangleright_{\text{Sets}_*} \circ (\text{id}_{\text{Sets}_*} \times \lrcorner^{\text{Sets}_*}) \Longrightarrow \text{id}_{\text{Sets}_*},$$

of [Definition 3.4.1.1](#).

Proof. Omitted. □

00A0 5 Smash Products of Pointed Sets

00A1 5.1 Foundations

Let (X, x_0) and (Y, y_0) be pointed sets.

00A2 Definition 5.1.1.1. The **smash product of (X, x_0) and (Y, y_0)** ¹⁹ is the pointed set $X \wedge Y$ ²⁰ such that we have a bijection

$$\text{Sets}_*(X \wedge Y, Z) \cong \text{Hom}_{\text{Sets}_*}^{\otimes}(X \times Y, Z),$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \text{Obj}(\text{Sets}_*)$.

00A3 Remark 5.1.1.2. In detail, the **smash product of (X, x_0) and (Y, y_0)** is the pair $((X \wedge Y, [(x_0, y_0)]), \iota)$ consisting of

- A pointed set $(X \wedge Y, [(x_0, y_0)])$;
- A bilinear morphism of pointed sets $\iota : (X \times Y, (x_0, y_0)) \rightarrow X \wedge Y$;

satisfying the following universal property:

¹⁹*Further Terminology:* Also called the **tensor product of \mathbb{F}_1 -modules of (X, x_0) and (Y, y_0)** or the **tensor product of (X, x_0) and (Y, y_0) over \mathbb{F}_1** .

²⁰*Further Notation:* Also written $X \otimes_{\mathbb{F}_1} Y$.

(UP) Given another such pair $((Z, z_0), f)$ consisting of

- A pointed set (Z, z_0) ;
- A bilinear morphism of pointed sets $f: (X \times Y, (x_0, y_0)) \rightarrow X \wedge Y$;

there exists a unique morphism of pointed sets $X \wedge Y \xrightarrow{\exists!} Z$ making the diagram

$$\begin{array}{ccc} & & X \wedge Y \\ & \nearrow \iota & \downarrow \exists! \\ X \times Y & \xrightarrow{f} & Z \end{array}$$

commute.

00A4 Construction 5.1.1.3. Concretely, the **smash product** of (X, x_0) and (Y, y_0) is the pointed set $(X \wedge Y, [(x_0, y_0)])$ consisting of²¹

· *The Underlying Set.* The set $X \wedge Y$ defined by

$$\begin{aligned} X \wedge Y &\cong \text{pt} \coprod_{X \vee Y} (X \times Y) \\ &\stackrel{\text{def}}{=} \frac{X \times Y}{X \vee Y} \\ &\cong X \times Y / \sim, \end{aligned} \quad \begin{array}{ccc} X \wedge Y & \leftarrow & X \times Y \\ \uparrow \ulcorner & & \uparrow \\ \text{pt} & \xleftarrow{\downarrow} & X \vee Y \end{array}$$

where \sim is the equivalence relation of $X \times Y$ obtained by declaring $(x, y) \sim (x', y')$ iff $(x, y), (x', y') \in X \vee Y$, i.e. by declaring

$$\begin{aligned} (x_0, y) &\sim (x_0, y'), \\ (x, y_0) &\sim (x', y_0) \end{aligned}$$

for all $x \in X$ and all $y \in Y$;

²¹Further Notation: We write $x \wedge y$ for the image of (x, y) under the quotient map

$$X \times Y \rightarrow \underbrace{\frac{X \times Y}{X \vee Y}}_{\stackrel{\text{def}}{=} X \wedge Y}$$

Note that we have

$$\begin{aligned} x \wedge y_0 &= x' \wedge y_0, \\ x_0 \wedge y &= x_0 \wedge y' \end{aligned}$$

- *The Basepoint.* The element $[(x_0, y_0)]$ of $X \wedge Y$ given by the equivalence class of (x_0, y_0) under the equivalence relation \sim on $X \times Y$.

Proof. Clear. □

00A5 Example 5.1.1.4. Here are some examples of smash products of pointed sets.

1. *Smashing With S^0 .* For any pointed set X , we have isomorphisms of pointed sets

$$\begin{aligned} S^0 \wedge X &\cong X, \\ X \wedge S^0 &\cong X. \end{aligned}$$

00A6 Proposition 5.1.1.5. Let (X, x_0) and (Y, y_0) be pointed sets.

- 00A7** 1. *Functoriality.* The assignments $(X, x_0), (Y, y_0), ((X, x_0), (Y, y_0)) \mapsto X \wedge Y$ define functors

$$\begin{aligned} X \wedge - &: \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*, \\ - \wedge Y &: \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*, \\ -_1 \wedge -_2 &: \mathbf{Sets}_* \times \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*. \end{aligned}$$

- 00A8** 2. *Adjointness.* We have adjunctions

$$\begin{aligned} (X \wedge - \dashv \mathbf{Sets}_*(X, -)) &: \mathbf{Sets}_* \begin{array}{c} \xrightarrow{X \wedge -} \\ \perp \\ \xleftarrow{\mathbf{Sets}_*(X, -)} \end{array} \mathbf{Sets}_*, \\ (- \wedge Y \dashv \mathbf{Sets}_*(Y, -)) &: \mathbf{Sets}_* \begin{array}{c} \xrightarrow{- \wedge Y} \\ \perp \\ \xleftarrow{\mathbf{Sets}_*(Y, -)} \end{array} \mathbf{Sets}_*, \end{aligned}$$

witnessed by bijections

$$\begin{aligned} \mathbf{Sets}_*(X \wedge Y, Z) &\cong \mathbf{Sets}_*(X, \mathbf{Sets}_*(Y, Z)), \\ \mathbf{Sets}_*(X \wedge Y, Z) &\cong \mathbf{Sets}_*(X, \mathbf{Sets}_*(A, Z)), \end{aligned}$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathbf{Obj}(\mathbf{Sets}_*)$, which internalise to isomorphisms of pointed sets

$$\begin{aligned} \mathbf{Sets}_*(X \wedge Y, Z) &\cong \mathbf{Sets}_*(X, \mathbf{Sets}_*(Y, Z)), \\ \mathbf{Sets}_*(X \wedge Y, Z) &\cong \mathbf{Sets}_*(X, \mathbf{Sets}_*(A, Z)), \end{aligned}$$

again natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathbf{Obj}(\mathbf{Sets}_*)$.

00A9 3. *Closed Symmetric Monoidality.* The quadruple $(\mathbf{Sets}_*, \wedge, S^0, \mathbf{Sets}_*)$ is a closed symmetric monoidal category.

00AA 4. *Morphisms From the Monoidal Unit.* We have a bijection of sets²²

$$\mathbf{Sets}_*(S^0, X) \cong X,$$

natural in $(X, x_0) \in \text{Obj}(\mathbf{Sets}_*)$, internalising also to an isomorphism of pointed sets

$$\mathbf{Sets}_*(S^0, X) \cong (X, x_0),$$

again natural in $(X, x_0) \in \text{Obj}(\mathbf{Sets}_*)$.

00AB 5. *Symmetric Strong Monoidality With Respect to Free Pointed Sets.* The free pointed set functor of **Pointed Sets**, **Item 1** of **Proposition 4.2.1.2** has a symmetric strong monoidal structure

$$((-)^+, (-)^{+, \times}, (-)^{+, \times}_{\#}) : (\mathbf{Sets}, \times, \text{pt}) \rightarrow (\mathbf{Sets}_*, \wedge, S^0),$$

being equipped with isomorphisms of pointed sets

$$\begin{aligned} (-)^{+, \times}_{X, Y} : X^+ \wedge Y^+ &\xrightarrow{\cong} (X \times Y)^+, \\ (-)^{+, \times}_{\#} : S^0 &\xrightarrow{\cong} \text{pt}^+, \end{aligned}$$

natural in $X, Y \in \text{Obj}(\mathbf{Sets})$.

00AC 6. *Distributivity Over Wedge Sums.* We have isomorphisms of pointed sets

$$\begin{aligned} X \wedge (Y \vee Z) &\cong (X \wedge Y) \vee (X \wedge Z), \\ (X \vee Y) \wedge Z &\cong (X \wedge Z) \vee (Y \wedge Z), \end{aligned}$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \text{Obj}(\mathbf{Sets}_*)$.

00AD 7. *Universal Property I.* The symmetric monoidal structure on the category \mathbf{Sets}_* is uniquely determined by the following requirements:

for each $x, x' \in X$ and each $y, y' \in Y$.

²²In other words, the forgetful functor

$$\text{忘} : \mathbf{Sets}_* \rightarrow \mathbf{Sets}$$

defined on objects by sending a pointed set to its underlying set is corepresentable by S^0 .

(a) *Two-Sided Preservation of Colimits.* The smash product

$$\wedge : \mathbf{Sets}_* \times \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*$$

of \mathbf{Sets}_* preserves colimits separately in each variable.

(b) *The Unit Object Is S^0 .* We have $\#_{\mathbf{Sets}_*} = S^0$.

00AE

8. *Universal Property II.* The symmetric monoidal structure on the category \mathbf{Sets}_* is the unique symmetric monoidal structure on \mathbf{Sets}_* such that the free pointed set functor

$$(-)^+ : \mathbf{Sets} \rightarrow \mathbf{Sets}_*$$

admits a symmetric monoidal structure.

00AF

9. *Existence of Monoidal Diagonals.* The triple $(\mathbf{Sets}_*, \wedge, S^0)$ is a monoidal category with diagonals:

(a) *Monoidal Diagonals.* The natural transformation

$$\Delta : \mathrm{id}_{\mathbf{Sets}_*} \Rightarrow \wedge \circ \Delta_{\mathbf{Sets}_*}^{\mathbf{Cats}_2},$$

whose component

$$\Delta_X : (X, x_0) \rightarrow (X \wedge X, [(x_0, x_0)])$$

at $(X, x_0) \in \mathrm{Obj}(\mathbf{Sets}_*)$ is given by the composition

$$\begin{aligned} (X, x_0) &\xrightarrow{\Delta_X} (X \times X, (x_0, x_0)) \\ &\longrightarrow \left(\frac{X \times X}{X \vee X}, [(x_0, x_0)] \right) \\ &\stackrel{\text{def}}{=} (X \wedge X, [(x_0, x_0)]) \end{aligned}$$

in \mathbf{Sets}_* , is a monoidal natural transformation:

i. *Naturality.* For each morphism $f : X \rightarrow Y$ of pointed sets, the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \Delta_X \downarrow & & \downarrow \Delta_Y \\ X \wedge X & \xrightarrow{f \wedge f} & Y \wedge Y \end{array}$$

commutes.

- ii. *Compatibility With Strong Monoidality Constraints.* For each $(X, x_0), (Y, y_0) \in \text{Obj}(\text{Sets}_*)$, the diagram

$$\begin{array}{ccc} X \wedge Y & \xrightarrow{\Delta_X \wedge \Delta_Y} & (X \wedge X) \wedge (Y \wedge Y) \\ \parallel & & \vdots \\ X \wedge Y & \xrightarrow{\Delta_{X \wedge Y}} & (X \wedge Y) \wedge (X \wedge Y) \end{array}$$

commutes.

- iii. *Compatibility With Strong Unitality Constraints.* The diagram

$$\begin{array}{ccc} S^0 & & \\ \parallel & \searrow^{(\lambda_{S^0}^{\text{Sets}_*})^{-1} = (\rho_{S^0}^{\text{Sets}_*})^{-1}} & \\ S^0 & \xrightarrow{\Delta_{S^0}} & S^0 \wedge S^0 \end{array}$$

commutes.

- (b) *The Diagonal of the Unit.* The component

$$\Delta_{S^0}^{\text{Sets}_*} : S^0 \xrightarrow{\cong} S^0 \wedge S^0$$

of the diagonal natural transformation of Sets_* at S^0 is an isomorphism.

- 00AG 10. *Comonoids in Sets_* .* The symmetric monoidal functor

$$((-)^+, (-)^{+, \times}, (-)^{+, \times}_{\#}) : (\text{Sets}, \times, \text{pt}) \rightarrow (\text{Sets}_*, \wedge, S^0),$$

of *Pointed Sets*, *Item 4* of *Proposition 4.2.1.2* lifts to an equivalence of categories

$$\begin{aligned} \text{CoMon}(\text{Sets}_*, \wedge, S^0) &\stackrel{\text{eq.}}{\cong} \text{CoMon}(\text{Sets}, \times, \text{pt}) \\ &\cong \text{Sets}. \end{aligned}$$

Proof. *Item 1, Functoriality:* Omitted.

Item 2, Adjointness: Omitted.

Item 3, Closed Symmetric Monoidality: Omitted.

Item 4, Morphisms From the Monoidal Unit: Omitted.

Item 5, Symmetric Strong Monoidality With Respect to Free Pointed Sets: Omitted.

Item 6, Distributivity Over Wedge Sums: This follows from *Item 3*, *Monoidal Categories*, ?? of ??, and the fact that \vee is the coproduct in Sets_* .

Item 7, Universal Property I: Omitted.

Item 8, Universal Property II: See [GGN15, Theorem 5.1].

Item 9, Existence of Monoidal Diagonals: Omitted.

Item 10, Comonoids in \mathbf{Sets}_ :* See [PS19, Lemma 2.4].

□

Appendices

A Other Chapters

Set Theory

1. [Sets](#)
2. [Constructions With Sets](#)
3. [Pointed Sets](#)
4. [Tensor Products of Pointed Sets](#)
5. [Indexed and Fibred Sets](#)
6. [Relations](#)
7. [Spans](#)
8. [Posets](#)

Category Theory

9. [Categories](#)
10. [Constructions With Categories](#)
11. [Kan Extensions](#)

Bicategories

12. [Bicategories](#)
13. [Internal Adjunctions](#)

Internal Category Theory

14. [Internal Categories](#)

Cyclic Stuff

15. [The Cycle Category](#)

Cubical Stuff

16. [The Cube Category](#)

Globular Stuff

17. [The Globe Category](#)

Cellular Stuff

18. [The Cell Category](#)

Monoids

19. [Monoids](#)
20. [Constructions With Monoids](#)

Monoids With Zero

21. [Monoids With Zero](#)
22. [Constructions With Monoids With Zero](#)

Groups

23. [Groups](#)
24. [Constructions With Groups](#)

Hyper Algebra

25. [Hypermonoids](#)

26. [Hypergroups](#)

27. [Hypersemirings and Hyperrings](#)

28. [Quantales](#)

Near-Rings

29. [Near-Semirings](#)

30. [Near-Rings](#)

Real Analysis

31. [Real Analysis in One Variable](#)

32. [Real Analysis in Several Variables](#)

Measure Theory

33. [Measurable Spaces](#)

34. [Measures and Integration](#)

Probability Theory

34. [Probability Theory](#)

Stochastic Analysis

35. [Stochastic Processes, Martingales,
and Brownian Motion](#)

36. [Itô Calculus](#)

37. [Stochastic Differential Equations](#)

Differential Geometry

38. [Topological and Smooth Manifolds](#)

Schemes

39. [Schemes](#)