Constructions With Sets

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This chapter contains some material relating to constructions with sets. Notably, it contains:

1. Explicit descriptions of the major types of co/limits in Sets, including in particular pushouts and coequalisers (see Definitions 2.3.1 and 2.4.1 and Remarks 2.3.3 and 2.4.3);
2. A discussion of powersets as decategorifications of categories of presheaves (Remarks 4.1.2 and 4.3.2);
3. A lengthy discussion of the adjoint triple

$$f_* \dashv f^{-1} \dashv f_! \colon \mathcal{P}(A) \overset{\rightleftarrows}{\to} \mathcal{P}(B)$$

of functors (morphisms of posets) between $\mathcal{P}(A)$ and $\mathcal{P}(B)$ induced by a map of sets $f\colon A \to B$, along with a discussion of the properties of f_*, f^{-1}, and $f_!$.

Contents

1 Limits of Sets

1.1 Products of Families of Sets

Let $\{A_i\}_{i\in I}$ be a family of sets.

Definition 1.1.1 ► The Product of a Family of Sets

The **product**[1] **of** $\{A_i\}_{i\in I}$ is the pair $\left(\prod_{i\in I} A_i, \{\mathrm{pr}_i\}_{i\in I}\right)$ consisting of

- *The Limit.* The set $\prod_{i\in I} A_i$ defined by[2]

$$\prod_{i\in I} A_i \overset{\text{def}}{=} \left\{ f \in \mathsf{Sets}\left(I, \bigcup_{i\in I} A_i\right) \;\middle|\; \begin{array}{l}\text{for each } i \in I\text{, we}\\ \text{have } f(i) \in A_i\end{array} \right\}.$$

- *The Cone.* The collection

$$\left\{ \mathrm{pr}_i \colon \prod_{i\in I} A_i \to A_i \right\}_{i\in I}$$

of maps given by

$$\mathrm{pr}_i(f) \stackrel{\mathrm{def}}{=} f(i)$$

for each $f \in \prod_{i\in I} A_i$ and each $i \in I$.

[1] *Further Terminology:* Also called the **Cartesian product of** $\{A_i\}_{i\in I}$.

[2] Less formally, $\prod_{i\in I} A_i$ is the set whose elements are I-indexed collections $(a_i)_{i\in I}$ with $a_i \in A_i$ for each $i \in I$.

Proof 1.1.2 ► Proof of Definition 1.1.1

We claim that $\prod_{i\in I} A_i$ is the categorical product of $\{A_i\}_{i\in I}$ in Sets. Indeed, suppose we have, for each $i \in I$, a diagram of the form

$$\begin{tikzcd} P \arrow[dr, "p_i"] & \\ \prod_{i\in I} A_i \arrow[r, "\mathrm{pr}_i"'] & A_i \end{tikzcd}$$

in Sets. Then there exists a unique map $\phi \colon P \to \prod_{i\in I} A_i$, uniquely determined by the condition $\mathrm{pr}_i \circ \phi = p_i$ for each $i \in I$, being necessarily given by

$$\phi(x) = (p_i(x))_{i\in I}$$

for each $x \in P$.

Proposition 1.1.3 ► Properties of Products of Families of Sets

Let $\{A_i\}_{i\in I}$ be a family of sets.

1. *Functoriality.* The assignment $\{A_i\}_{i\in I} \mapsto \prod_{i\in I} A_i$ defines a functor

$$\prod_{i\in I} \colon \mathrm{Fun}(I_{\mathrm{disc}}, \mathsf{Sets}) \to \mathsf{Sets}$$

where

- *Action on Objects.* For each $(A_i)_{i\in I} \in \mathrm{Obj}(\mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}))$, we have

$$\left[\prod_{i\in I}\right]((A_i)_{i\in I}) \stackrel{\text{def}}{=} \prod_{i\in I} A_i$$

- *Action on Morphisms.* For each $(A_i)_{i\in I}, (B_i)_{i\in I} \in \mathrm{Obj}(\mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}))$, the action on Hom-sets

$$\left(\prod_{i\in I}\right)_{(A_i)_{i\in I},(B_i)_{i\in I}} : \mathrm{Nat}((A_i)_{i\in I}, (B_i)_{i\in I}) \to \mathsf{Sets}\left(\prod_{i\in I} A_i, \prod_{i\in I} B_i\right)$$

of $\prod_{i\in I}$ at $((A_i)_{i\in I}, (B_i)_{i\in I})$ is defined by sending a map

$$\{f_i \colon A_i \to B_i\}_{i\in I}$$

in $\mathrm{Nat}((A_i)_{i\in I}, (B_i)_{i\in I})$ to the map of sets

$$\prod_{i\in I} f_i \colon \prod_{i\in I} A_i \to \prod_{i\in I} B_i$$

defined by

$$\left[\prod_{i\in I} f_i\right]((a_i)_{i\in I}) \stackrel{\text{def}}{=} (f_i(a_i))_{i\in I}$$

for each $(a_i)_{i\in I} \in \prod_{i\in I} A_i$.

Proof 1.1.4 ► Proof of Proposition 1.1.3

Item 1: Functoriality

Clear.

1.2 Binary Products of Sets

Let A and B be sets.

Definition 1.2.1 ► Products of Sets

The **product**[1] **of A and B** is the pair $(A \times B, \{\mathrm{pr}_1, \mathrm{pr}_2\})$ consisting of

- *The Limit.* The set $A \times B$ defined by[2]

$$\begin{aligned} A \times B &\stackrel{\text{def}}{=} \prod_{z \in \{A,B\}} z \\ &\stackrel{\text{def}}{=} \{f \in \mathsf{Sets}(\{0,1\}, A \cup B) \mid \text{we have } f(0) \in A \text{ and } f(1) \in B\} \\ &\cong \{\{\{a\},\{a,b\}\} \in \mathcal{P}(\mathcal{P}(A \cup B)) \mid \text{we have } a \in A \text{ and } b \in B\}. \end{aligned}$$

- *The Cone.* The maps

$$\begin{aligned} \mathrm{pr}_1 &: A \times B \longrightarrow A, \\ \mathrm{pr}_2 &: A \times B \longrightarrow B \end{aligned}$$

defined by

$$\begin{aligned} \mathrm{pr}_1(a,b) &\stackrel{\text{def}}{=} a, \\ \mathrm{pr}_2(a,b) &\stackrel{\text{def}}{=} b \end{aligned}$$

for each $(a,b) \in A \times B$.

[1] *Further Terminology:* Also called the **Cartesian product of A and B** or the **binary Cartesian product of A and B**, for emphasis.

This can also be thought of as the $(\mathbb{E}_{-1}, \mathbb{E}_{-1})$**-tensor product of A and B**.

[2] In other words, $A \times B$ is the set whose elements are ordered pairs (a,b) with $a \in A$ and $b \in B$ as in Definition 3.4.1

Proof 1.2.2 ► Proof of Definition 1.2.1

We claim that $A \times B$ is the categorical product of A and B in Sets. Indeed, suppose we have a diagram of the form

$$\begin{tikzcd} & P \arrow[dl, bend right, "p_1"'] \arrow[dr, bend left, "p_2"] & \\ A & A \times B \arrow[l, "\mathrm{pr}_1"] \arrow[r, "\mathrm{pr}_2"'] & B \end{tikzcd}$$

in Sets. Then there exists a unique map $\phi\colon P \to A \times B$, uniquely determined by the conditions

$$\begin{aligned} \mathrm{pr}_1 \circ \phi &= p_1, \\ \mathrm{pr}_2 \circ \phi &= p_2, \end{aligned}$$

being necessarily given by

$$\phi(x) = (p_1(x), p_2(x))$$

for each $x \in P$.

Proposition 1.2.3 ► Properties of Products of Sets

Let A, B, C, and X be sets.

1. *Functoriality.* The assignments A, B, $(A, B) \mapsto A \times B$ define functors

$$\begin{aligned} A \times -_2 &\colon \mathsf{Sets} \to \mathsf{Sets}, \\ -_1 \times B &\colon \mathsf{Sets} \to \mathsf{Sets}, \\ -_1 \times -_2 &\colon \mathsf{Sets} \times \mathsf{Sets} \to \mathsf{Sets}, \end{aligned}$$

   where $-_1 \times -_2$ is the functor where

   - *Action on Objects.* For each $(A, B) \in \mathrm{Obj}(\mathsf{Sets} \times \mathsf{Sets})$, we have

$$[-_1 \times -_2](A, B) \stackrel{\text{def}}{=} A \times B;$$

   - *Action on Morphisms.* For each $(A, B), (X, Y) \in \mathrm{Obj}(\mathsf{Sets})$, the action on Hom-sets

$$\times_{(A,B),(X,Y)} \colon \mathsf{Sets}(A, X) \times \mathsf{Sets}(B, Y) \to \mathsf{Sets}(A \times B, X \times Y)$$

     of $\times$ at $((A, B), (X, Y))$ is defined by sending (f, g) to the function

$$f \times g\colon A \times B \to X \times Y$$

     defined by

$$[f \times g](a, b) \stackrel{\text{def}}{=} (f(a), g(b))$$

     for each $(a, b) \in A \times B$;

and where $A \times -$ and $- \times B$ are the partial functors of $-_1 \times -_2$ at $A, B \in \mathrm{Obj}(\mathsf{Sets})$.

2. *Adjointness.* We have adjunctions

$$
\begin{gathered}
(A \times - \dashv \mathsf{Sets}(A, -)) \colon \quad
\begin{tikzcd}
\mathsf{Sets} \arrow[r, bend left, "A\times -"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}, \arrow[l, bend left, "\mathsf{Sets}(A,-)"]
\end{tikzcd}
\\
(- \times B \dashv \mathsf{Sets}(B, -)) \colon \quad
\begin{tikzcd}
\mathsf{Sets} \arrow[r, bend left, "-\times B"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}, \arrow[l, bend left, "\mathsf{Sets}(B,-)"]
\end{tikzcd}
\end{gathered}
$$

witnessed by bijections

$$
\begin{aligned}
\mathsf{Sets}(A \times B, C) &\cong \mathsf{Sets}(A, \mathsf{Sets}(B, C)),\\
\mathsf{Sets}(A \times B, C) &\cong \mathsf{Sets}(B, \mathsf{Sets}(A, C)),
\end{aligned}
$$

natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

3. *Associativity.* We have an isomorphism of sets

$$(A \times B) \times C \cong A \times (B \times C),$$

natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

4. *Unitality.* We have isomorphisms of sets

$$
\begin{aligned}
\mathrm{pt} \times A &\cong A,\\
A \times \mathrm{pt} &\cong A,
\end{aligned}
$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$.

5. *Commutativity.* We have an isomorphism of sets

$$A \times B \cong B \times A,$$

natural in $A, B \in \mathrm{Obj}(\mathsf{Sets})$.

6. *Annihilation With the Empty Set.* We have isomorphisms of sets

$$\begin{aligned} A \times \emptyset &\cong \emptyset, \\ \emptyset \times A &\cong \emptyset, \end{aligned}$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$.

7. *Distributivity Over Unions.* We have isomorphisms of sets

$$\begin{aligned} A \times (B \cup C) &= (A \times B) \cup (A \times C), \\ (A \cup B) \times C &= (A \times C) \cup (B \times C). \end{aligned}$$

8. *Distributivity Over Intersections.* We have isomorphisms of sets

$$\begin{aligned} A \times (B \cap C) &= (A \times B) \cap (A \times C), \\ (A \cap B) \times C &= (A \times C) \cap (B \times C). \end{aligned}$$

9. *Middle-Four Exchange with Respect to Intersections.* We have an isomorphism of sets

$$(A \times B) \cap (C \times D) \cong (A \cap B) \times (C \cap D).$$

10. *Distributivity Over Differences.* We have isomorphisms of sets

$$\begin{aligned} A \times (B \setminus C) &= (A \times B) \setminus (A \times C), \\ (A \setminus B) \times C &= (A \times C) \setminus (B \times C), \end{aligned}$$

natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

11. *Distributivity Over Symmetric Differences.* We have isomorphisms of sets

$$\begin{aligned} A \times (B \mathbin{\triangle} C) &= (A \times B) \mathbin{\triangle} (A \times C), \\ (A \mathbin{\triangle} B) \times C &= (A \times C) \mathbin{\triangle} (B \times C), \end{aligned}$$

natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

12. *Symmetric Monoidality.* The triple $(\mathsf{Sets}, \times, \mathrm{pt})$ is a symmetric monoidal category.

13. *Symmetric Bimonoidality.* The quintuple $(\mathsf{Sets}, \coprod, \emptyset, \times, \mathrm{pt})$ is a symmetric bimonoidal category.

Proof 1.2.4 ► Proof of Proposition 1.2.3

Item 1: Functoriality

This follows by applying associativity and unitality componentwise.

Item 2: Adjointness

We prove only that there's an adjunction $X \times - \dashv \mathrm{Hom}_{\mathsf{Sets}}(-, Z)$, witnessed by a bijection

$$\mathrm{Hom}_{\mathsf{Sets}}(X \times Y, Z) \cong \mathrm{Hom}_{\mathsf{Sets}}(X, \mathrm{Hom}_{\mathsf{Sets}}(Y, Z)),$$

natural in $Y, Z \in \mathrm{Obj}(\mathsf{Sets})$, as the proof of the existence of the adjunction $- \times Y \dashv \mathrm{Hom}_{\mathsf{Sets}}(-, Z)$ follows almost exactly in the same way.[1]

- *Map I.* We define a map

  $$\Phi_{Y,Z} \colon \mathrm{Hom}_{\mathsf{Sets}}(X \times Y, Z) \to \mathrm{Hom}_{\mathsf{Sets}}(X, \mathrm{Hom}_{\mathsf{Sets}}(Y, Z)),$$

  by sending a morphism $\xi \colon X \times Y \to Z$ to the morphism

  $$\xi^{\dagger} \colon X \to \mathrm{Hom}_{\mathsf{Sets}(Y,Z)}$$

  defined by

  $$\xi^{\dagger}(x) \overset{\text{def}}{=} \xi_x$$

  for each $x \in X$, where $\xi_x \colon Y \to Z$ is the map defined by

  $$\xi_x(y) \overset{\text{def}}{=} \xi(x, y)$$

  for each $y \in Y$.

- *Map II.* We define a map

  $$\Psi_{Y,Z} \colon \mathrm{Hom}_{\mathsf{Sets}}(X, \mathrm{Hom}_{\mathsf{Sets}}(Y, Z)), \to \mathrm{Hom}_{\mathsf{Sets}}(X \times Y, Z)$$

  given by sending a map $\xi \colon X \to \mathrm{Hom}_{\mathsf{Sets}}(Y, Z)$ to the map

  $$\xi^{\dagger} \colon X \times Y \to Z$$

  defined by

  $$\xi^{\dagger}(x, y) \overset{\text{def}}{=} [\xi(x)](y)$$

  for each $(x, y) \in X \times Y$.

- *Naturality I.* We need to show that, given a function $g\colon Y \to Y'$, the diagram

$$\begin{tikzcd}
\mathsf{Hom}_{\mathsf{Sets}}(X \times Y', Z) \arrow[r, "\Phi_{Y',Z}"] \arrow[d, "\mathrm{id}_X \times g^*"'] & \mathsf{Hom}_{\mathsf{Sets}}(X, \mathsf{Hom}_{\mathsf{Sets}}(Y', Z)), \arrow[d, "(g^*)_*"] \\
\mathsf{Hom}_{\mathsf{Sets}}(X \times Y, Z) \arrow[r, "\Phi_{Y,Z}"'] & \mathsf{Hom}_{\mathsf{Sets}}(X, \mathsf{Hom}_{\mathsf{Sets}}(Y, Z)),
\end{tikzcd}$$

  commutes. Indeed, given a morphism $\xi\colon X' \times Y \to Z$, we have

$$\begin{aligned}
[\Phi_{Y,Z} \circ (g^* \times \mathrm{id}_Y)](\xi) &\stackrel{\text{def}}{=} (\xi(-_1, g(-_2)))^\dagger \\
&\stackrel{\text{def}}{=} \xi_{-_1}(g(-_2)) \\
&\stackrel{\text{def}}{=} (g_*)^*\big(\xi_{-_1}(-_2)\big) \\
&\stackrel{\text{def}}{=} (g_*)^*\Big(\xi^\dagger\Big) \\
&\stackrel{\text{def}}{=} [(g_*)^* \circ \Phi_{Y',Z}](\xi).
\end{aligned}$$

- *Naturality II.* We need to show that, given a function $h\colon Z \to Z'$, the diagram

$$\begin{tikzcd}
\mathsf{Hom}_{\mathsf{Sets}}(X \times Y, Z) \arrow[r, "\Phi_{Y,Z}"] \arrow[d, "h_*"'] & \mathsf{Hom}_{\mathsf{Sets}}(X, \mathsf{Hom}_{\mathsf{Sets}}(Y, Z)), \arrow[d, "(h_*)_*"] \\
\mathsf{Hom}_{\mathsf{Sets}}(X \times Y, Z') \arrow[r, "\Phi_{Y,Z'}"'] & \mathsf{Hom}_{\mathsf{Sets}}(X, \mathsf{Hom}_{\mathsf{Sets}}(Y, Z')),
\end{tikzcd}$$

  commutes. Indeed, given a morphism $\xi\colon X \times Y \to Z$, we have

$$\begin{aligned}
[\Phi_{Y,Z} \circ h_*](\xi) &\stackrel{\text{def}}{=} (h(\xi(-_1, -_2)))^\dagger \\
&\stackrel{\text{def}}{=} [x \mapsto [y \mapsto h(\xi(x, y))]] \\
&\stackrel{\text{def}}{=} [x \mapsto h_*([y \mapsto \xi(x, y)])] \\
&\stackrel{\text{def}}{=} [x \mapsto h_*\Big(\xi^\dagger(x)\Big)] \\
&\stackrel{\text{def}}{=} h_*\Big(\xi^\dagger\Big) \\
&\stackrel{\text{def}}{=} [(h_*)_* \circ \Phi_{Y,Z}](\xi).
\end{aligned}$$

- *Invertibility I.* We claim that

$$\Psi_{X,Y} \circ \Phi_{X,Y} = \mathrm{id}_{\mathrm{Hom}_{\mathsf{Sets}}(X \times Y, Z)}.$$

Indeed, given a morphism $\xi \colon X \times Y \to Z$, we have

$$\begin{aligned}
[\Psi_{X,Y} \circ \Phi_{X,Y}](\xi) &\overset{\text{def}}{=} \Psi_{X,Y}\big(\Phi_{X,Y}(\xi)\big) \\
&\overset{\text{def}}{=} \Psi_{X,Y}([x \mapsto [y \mapsto \xi(x,y)]]) \\
&\overset{\text{def}}{=} [(x,y) \mapsto \mathrm{ev}_x\big([x \mapsto \mathrm{ev}_y([y \mapsto \xi(x,y)]])\big)] \\
&\overset{\text{def}}{=} [(x,y) \mapsto \mathrm{ev}_x([x \mapsto \xi(x,y)])] \\
&\overset{\text{def}}{=} [(x,y) \mapsto \xi(x,y)] \\
&\overset{\text{def}}{=} \xi.
\end{aligned}$$

- *Invertibility II.* We claim that

$$\Phi_{X,Y} \circ \Psi_{X,Y} = \mathrm{id}_{\mathrm{Hom}_{\mathsf{Sets}}(X, \mathrm{Hom}_{\mathsf{Sets}}(Y,Z))}.$$

Indeed, given a morphism $\xi \colon X \to \mathrm{Hom}_{\mathsf{Sets}}(Y, Z)$, we have

$$\begin{aligned}
[\Phi_{X,Y} \circ \Psi_{X,Y}](\xi) &\overset{\text{def}}{=} \Phi_{X,Y}\big(\Psi_{X,Y}(\xi)\big) \\
&\overset{\text{def}}{=} \Phi_{X,Y}\big(\Psi_{X,Y}([x \mapsto \xi(x)])\big) \\
&\overset{\text{def}}{=} \Phi_{X,Y}\big([(x,y) \mapsto \mathrm{ev}_x\big([x \mapsto \mathrm{ev}_y(\xi(x))]\big)]\big) \\
&\overset{\text{def}}{=} \Phi_{X,Y}([(x,y) \mapsto \xi(x,y)]) \\
&\overset{\text{def}}{=} [x \mapsto [y \mapsto \xi(x,y)]] \\
&\overset{\text{def}}{=} \xi.
\end{aligned}$$

Item 3: Associativity

See [Pro24a].

Item 4: Unitality

Clear.

Item 5: Commutativity

See [Pro24b].

Item 6: Annihilation With the Empty Set

See [Pro24f].

Item 7: Distributivity Over Unions

See [Pro24e].

Item 8: Distributivity Over Intersections

See [Pro24g, Corollary 1].

Item 9: Middle-Four Exchange With Respect to Intersections

See [Pro24g, Corollary 1].

Item 10: Distributivity Over Differences

See [Pro24c].

Item 11: Distributivity Over Symmetric Differences

See [Pro24d].

Item 12: Symmetric Monoidality

See [MO 382264].

Item 13: Symmetric Bimonoidality

Omitted.

[1]Here we sometimes denote a map $f: X \longrightarrow Y$ by $[x \mapsto f(x)]$, similar to the lambda notation $\lambda x.f(x)$.

1.3 Pullbacks

Let A, B, and C be sets and let $f: A \longrightarrow C$ and $g: B \longrightarrow C$ be functions.

Definition 1.3.1 ► Pullbacks of Sets

The **pullback of A and B over C along f and g**[1] is the pair[2] $(A \times_C B, \{\mathrm{pr}_1, \mathrm{pr}_2\})$ consisting of

- *The Limit.* The set $A \times_C B$ defined by

$$A \times_C B \overset{\text{def}}{=} \{(a,b) \in A \times B \mid f(a) = g(b)\}.$$

- *The Cone.* The maps

$$\begin{aligned}\mathrm{pr}_1 &: A \times_C B \longrightarrow A,\\ \mathrm{pr}_2 &: A \times_C B \longrightarrow B\end{aligned}$$

defined by

$$\mathrm{pr}_1(a,b) \overset{\mathrm{def}}{=} a,$$
$$\mathrm{pr}_2(a,b) \overset{\mathrm{def}}{=} b$$

for each $(a,b) \in A \times_C B$.

[1] *Further Terminology:* Also called the **fibre product of** A **and** B **over** C **along** f **and** g.
[2] *Further Notation:* Also written $A \times_{f,C,g} B$.

Proof 1.3.2 ► Proof of Definition 1.3.1

We claim that $A \times_C B$ is the categorical pullback of A and B over C with respect to (f,g) in Sets. First we need to check that the relevant pullback diagram commutes, i.e. that we have

$$f \circ \mathrm{pr}_1 = g \circ \mathrm{pr}_2, \qquad \begin{tikzcd} A \times_C B \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C. \end{tikzcd}$$

Indeed, given $(a,b) \in A \times_C B$, we have

$$\begin{aligned} [f \circ \mathrm{pr}_1](a,b) &= f(\mathrm{pr}_1(a,b)) \\ &= f(a) \\ &= g(b) \\ &= g(\mathrm{pr}_2(a,b)) \\ &= [g \circ \mathrm{pr}_2](a,b), \end{aligned}$$

where $f(a) = g(b)$ since $(a, b) \in A \times_C B$. Next, we prove that $A \times_C B$ satisfies the universal property of the pullback. Suppose we have a diagram of the form

$$\begin{tikzcd}
P \arrow[drr, bend left, "p_2"] \arrow[ddr, bend right, "p_1"'] & & \\
& A \times_C B \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "g"] \\
& A \arrow[r, "f"'] & C
\end{tikzcd}$$

in Sets. Then there exists a unique map $\phi \colon P \to A \times_C B$, uniquely determined by the conditions

$$\mathrm{pr}_1 \circ \phi = p_1,$$
$$\mathrm{pr}_2 \circ \phi = p_2,$$

being necessarily given by

$$\phi(x) = (p_1(x), p_2(x))$$

for each $x \in P$, where we note that $(p_1(x), p_2(x)) \in A \times B$ indeed lies in $A \times_C B$ by the condition

$$f \circ p_1 = g \circ p_2,$$

which gives

$$f(p_1(x)) = g(p_2(x))$$

for each $x \in P$, so that $(p_1(x), p_2(x)) \in A \times_C B$. □

EXAMPLE 1.3.3 ► EXAMPLES OF PULLBACKS OF SETS

Here are some examples of pullbacks of sets.

1. *Unions via Intersections.* Let $A, B \subset X$. We have a bijection of sets

$$A \cap B \cong A \times_{A \cup B} B, \qquad \begin{tikzcd} A \cap B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "\iota_B"] \\ A \arrow[r, "\iota_A"'] & A \cup B. \end{tikzcd}$$

Proof 1.3.4 ▶ Proof of Example 1.3.3

Item 1: Unions via Intersections

Indeed, we have

$$\begin{aligned} A \times_{A \cup B} B &\cong \{(x, y) \in A \times B \mid x = y\} \\ &\cong A \cap B. \end{aligned}$$

This finishes the proof.

Proposition 1.3.5 ▶ Properties of Pullbacks of Sets

Let A, B, C, and X be sets.

1. *Functoriality.* The assignment $(A, B, C, f, g) \mapsto A \times_{f,C,g} B$ defines a functor

$$-_1 \times_{-_3} -_1 \colon \mathsf{Fun}(\mathcal{P}, \mathsf{Sets}) \to \mathsf{Sets},$$

where $\mathcal{P}$ is the category that looks like this:

$$\begin{tikzcd} & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet. \end{tikzcd}$$

In particular, the action on morphisms of $-_1 \times_{-_3} -_1$ is given by sending a morphism

$$
\begin{tikzcd}
A \times_C B \arrow[rr] \arrow[dd] \arrow[dr, phantom, very near start, "\lrcorner"] & & B \arrow[dd, "g" near start] \arrow[dr, "\psi"] & \\
& A' \times_{C'} B' \arrow[rr, crossing over] \arrow[dr, phantom, very near start, "\lrcorner"] & & B' \arrow[dd, "g'"] \\
A \arrow[rr, "f" near start] \arrow[dr, "\phi"'] & & C \arrow[dr, "\chi"] & \\
& A' \arrow[rr, "f'"'] \arrow[from=uu, crossing over] & & C'
\end{tikzcd}
$$

in $\mathrm{Fun}(\mathcal{P}, \mathsf{Sets})$ to the map $\xi \colon A \times_C B \xrightarrow{\exists!} A' \times_{C'} B'$ given by

$$\xi(a, b) \overset{\text{def}}{=} (\phi(a), \psi(b))$$

for each $(a, b) \in A \times_C B$, which is the unique map making the diagram

$$
\begin{tikzcd}
A \times_C B \arrow[rr] \arrow[dd] \arrow[dr, dashed] & & B \arrow[dd, "g" near start] \arrow[dr, "\psi"] & \\
& A' \times_{C'} B' \arrow[rr, crossing over] \arrow[dr, phantom, very near start, "\lrcorner"] & & B' \arrow[dd, "g'"] \\
A \arrow[rr, "f" near start] \arrow[dr, "\phi"'] & & C \arrow[dr, "\chi"] & \\
& A' \arrow[rr, "f'"'] \arrow[from=uu, crossing over] & & C'
\end{tikzcd}
$$

commute.

2. *Associativity.* Given a diagram

$$
\begin{tikzcd}
A \arrow[dr, "f"'] & & B \arrow[dl, "g"] \arrow[dr, "h"'] & & C \arrow[dl, "k"] \\
& X & & Y &
\end{tikzcd}
$$

in Sets, we have isomorphisms

$$(A \times_X B) \times_Y C \cong (A \times_X B) \times_B (B \times_Y C) \cong A \times_X (B \times_Y C),$$

where these pullbacks are built as in the diagrams

$$\begin{tikzcd}[column sep=small]
 & & (A \times_X B) \times_Y C \arrow[dl] \arrow[ddrr] \arrow[dd, phantom, very near start, "\lrcorner"] & & \\
 & A \times_X B \arrow[dl] \arrow[dr] \arrow[dd, phantom, very near start, "\lrcorner"] & & & \\
A \arrow[dr, "f"'] & & B \arrow[dl, "g"] \arrow[dr, "h"'] & & C, \arrow[dl, "k"] \\
 & X & & Y &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=small]
 & & (A \times_X B) \times_B (B \times_Y C) \arrow[dl] \arrow[dr] \arrow[dd, phantom, very near start, "\lrcorner"] & & \\
 & A \times_X B \arrow[dl] \arrow[dr] \arrow[dd, phantom, very near start, "\lrcorner"] & & B \times_Y C \arrow[dl] \arrow[dr] \arrow[dd, phantom, very near start, "\lrcorner"] & \\
A \arrow[dr, "f"'] & & B \arrow[dl, "g"] \arrow[dr, "h"'] & & C, \arrow[dl, "k"] \\
 & X & & Y &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=small]
 & & A \times_X (B \times_Y C) \arrow[ddll] \arrow[dr] \arrow[dd, phantom, very near start, "\lrcorner"] & & \\
 & & & B \times_Y C \arrow[dl] \arrow[dr] \arrow[dd, phantom, very near start, "\lrcorner"] & \\
A \arrow[dr, "f"'] & & B \arrow[dl, "g"] \arrow[dr, "h"'] & & C. \arrow[dl, "k"] \\
 & X & & Y &
\end{tikzcd}$$

3. *Unitality.* We have isomorphisms of sets

$$\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "f"'] \arrow[dr, phantom, very near start, "\lrcorner"] & A \arrow[d, "f"] \\
X \arrow[r, equal] & X
\end{tikzcd}
\qquad
\begin{aligned} X \times_X A &\cong A, \\ A \times_X X &\cong A, \end{aligned}
\qquad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, equal] \arrow[dr, phantom, very near start, "\lrcorner"] & X \arrow[d, equal] \\
X \arrow[r, "f"'] & X.
\end{tikzcd}$$

4. *Commutativity.* We have an isomorphism of sets

$$\begin{tikzcd}
A \times_X B \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & B \arrow[d, "g"] \\
A \arrow[r, "f"'] & X,
\end{tikzcd}
\qquad
A \times_X B \cong B \times_X A
\qquad
\begin{tikzcd}
B \times_X A \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & A \arrow[d, "f"] \\
B \arrow[r, "g"'] & X.
\end{tikzcd}$$

5. *Annihilation With the Empty Set.* We have isomorphisms of sets

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \emptyset \arrow[d] \\
A \arrow[r, "f"'] & X,
\end{tikzcd}
\qquad
\begin{aligned} A \times_X \emptyset &\cong \emptyset, \\ \emptyset \times_X A &\cong \emptyset, \end{aligned}
\qquad
\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & A \arrow[d, "f"] \\
\emptyset \arrow[r] & X.
\end{tikzcd}$$

6. *Interaction With Products.* We have

$$A \times_{\mathrm{pt}} B \cong A \times B, \qquad \begin{tikzcd} A \times B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "!_B"] \\ A \arrow[r, "!_A"'] & \mathrm{pt}. \end{tikzcd}$$

7. *Symmetric Monoidality.* The triple $(\mathsf{Sets}, \times_X, X)$ is a symmetric monoidal category.

Proof 1.3.6 ► Proof of Proposition 1.3.5

Item 1: Functoriality

This is a special case of functoriality of co/limits, Limits and Colimits, **??** of **??**, with the explicit expression for ξ following from the commutativity of the cube pullback diagram.

Item 2: Associativity

Indeed, we have

$$\begin{aligned}
(A \times_X B) \times_Y C &\cong \{((a,b),c) \in (A \times_X B) \times C \mid h(b) = k(c)\} \\
&\cong \{((a,b),c) \in (A \times B) \times C \mid f(a) = g(b) \text{ and } h(b) = k(c)\} \\
&\cong \{(a,(b,c)) \in A \times (B \times C) \mid f(a) = g(b) \text{ and } h(b) = k(c)\} \\
&\cong \{(a,(b,c)) \in A \times (B \times_Y C) \mid f(a) = g(b)\} \\
&\cong A \times_X (B \times_Y C)
\end{aligned}$$

and

$$\begin{aligned}
(A \times_X B) \times_B (B \times_Y C) &\cong \{((a,b),(b',c)) \in (A \times_X B) \times (B \times_Y C) \mid b = b'\} \\
&\cong \left\{((a,b),(b',c)) \in (A \times B) \times (B \times C) \,\middle|\, \begin{array}{l} f(a) = g(b), b = b', \\ \text{and } h(b') = k(c) \end{array}\right\} \\
&\cong \left\{(a,(b,(b',c))) \in A \times (B \times (B \times C)) \,\middle|\, \begin{array}{l} f(a) = g(b), b = b', \\ \text{and } h(b') = k(c) \end{array}\right\} \\
&\cong \left\{(a,((b,b'),c)) \in A \times ((B \times B) \times C) \,\middle|\, \begin{array}{l} f(a) = g(b), b = b', \\ \text{and } h(b') = k(c) \end{array}\right\} \\
&\cong \left\{(a,((b,b'),c)) \in A \times ((B \times_B B) \times C) \,\middle|\, \begin{array}{l} f(a) = g(b) \text{ and} \\ h(b') = k(c) \end{array}\right\} \\
&\cong \{(a,(b,c)) \in A \times (B \times C) \mid f(a) = g(b) \text{ and } h(b) = k(c)\} \\
&\cong A \times_X (B \times_Y C),
\end{aligned}$$

where we have used Item 3 for the isomorphism $B \times_B B \cong B$.

Item 3: Unitality

Indeed, we have

$$\begin{aligned}
X \times_X A &\cong \{(x,a) \in X \times A \mid f(a) = x\}, \\
A \times_X X &\cong \{(a,x) \in X \times A \mid f(a) = x\},
\end{aligned}$$

which are isomorphic to A via the maps $(x,a) \mapsto a$ and $(a,x) \mapsto a$.

Item 4: Commutativity

Clear.

Item 5: Annihilation With the Empty Set

Clear.

Item 6: Interaction With Products

Clear.

Item 7: Symmetric Monoidality

Omitted.

1.4 Equalisers

Let A and B be sets and let $f, g\colon A \rightrightarrows B$ be functions.

DEFINITION 1.4.1 ► EQUALISERS OF SETS

The **equaliser of f and g** is the pair $(\mathrm{Eq}(f,g), \mathrm{eq}(f,g))$ consisting of

- *The Limit.* The set $\mathrm{Eq}(f,g)$ defined by

$$\mathrm{Eq}(f,g) \stackrel{\text{def}}{=} \{a \in A \mid f(a) = g(a)\}.$$

- *The Cone.* The inclusion map

$$\mathrm{eq}(f,g) \colon \mathrm{Eq}(f,g) \hookrightarrow A.$$

PROOF 1.4.2 ► PROOF OF DEFINITION 1.4.1

We claim that $\mathrm{Eq}(f,g)$ is the categorical equaliser of f and g in Sets. First we need to check that the relevant equaliser diagram commutes, i.e. that we have

$$f \circ \mathrm{eq}(f,g) = g \circ \mathrm{eq}(f,g),$$

which indeed holds by the definition of the set $\mathrm{Eq}(f,g)$. Next, we prove that $\mathrm{Eq}(f,g)$ satisfies the universal property of the equaliser. Suppose we have a diagram of the form

$$\begin{tikzcd}
\mathrm{Eq}(f,g) \arrow[r, hook, "\mathrm{eq}(f,g)"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\
E \arrow[ur, "e"'] & &
\end{tikzcd}$$

in Sets. Then there exists a unique map $\phi \colon E \longrightarrow \mathrm{Eq}(f,g)$, uniquely determined by the condition

$$\mathrm{eq}(f,g) \circ \phi = e$$

being necessarily given by

$$\phi(x) = e(x)$$

for each $x \in E$, where we note that $e(x) \in A$ indeed lies in $\mathrm{Eq}(f,g)$ by the condition

$$f \circ e = g \circ e,$$

which gives

$$f(e(x)) = g(e(x))$$

for each $x \in E$, so that $e(x) \in \mathrm{Eq}(f, g)$.

Proposition 1.4.3 ► Properties of Equalisers of Sets

Let A, B, and C be sets.

1. *Associativity.* We have an isomorphism of sets[1]

$$\underbrace{\mathrm{Eq}(f \circ \mathrm{eq}(g,h), g \circ \mathrm{eq}(g,h))}_{=\mathrm{Eq}(f\circ\mathrm{eq}(g,h),h\circ\mathrm{eq}(g,h))} \cong \mathrm{Eq}(f,g,h) \cong \underbrace{\mathrm{Eq}(f \circ \mathrm{eq}(f,g), h \circ \mathrm{eq}(f,g))}_{=\mathrm{Eq}(g\circ\mathrm{eq}(f,g),h\circ\mathrm{eq}(f,g))},$$

   where $\mathrm{Eq}(f, g, h)$ is the limit of the diagram

$$A \overset{f}{\underset{h}{\overset{\longrightarrow}{\underset{\longrightarrow}{\xrightarrow{g}}}}} B$$

   in Sets, being explicitly given by

$$\mathrm{Eq}(f, g, h) \cong \{a \in A \mid f(a) = g(a) = h(a)\}.$$

2. *Unitality.* We have an isomorphism of sets

$$\mathrm{Eq}(f, f) \cong A.$$

3. *Commutativity.* We have an isomorphism of sets

$$\mathrm{Eq}(f, g) \cong \mathrm{Eq}(g, f).$$

4. *Interaction With Composition.* Let

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \underset{k}{\overset{h}{\rightrightarrows}} C$$

   be functions. We have an inclusion of sets

$$\mathrm{Eq}(h \circ f \circ \mathrm{eq}(f,g), k \circ g \circ \mathrm{eq}(f,g)) \subset \mathrm{Eq}(h \circ f, k \circ g),$$

   where $\mathrm{Eq}(h \circ f \circ \mathrm{eq}(f,g), k \circ g \circ \mathrm{eq}(f,g))$ is the equaliser of the composition

$$\mathrm{Eq}(f,g) \overset{\mathrm{eq}(f,g)}{\hookrightarrow} A \underset{g}{\overset{f}{\rightrightarrows}} B \underset{k}{\overset{h}{\rightrightarrows}} C.$$

[1] That is, the following three ways of forming "the" equaliser of (f, g, h) agree:

(a) Take the equaliser of (f, g, h), i.e. the limit of the diagram

$$A \overset{f}{\underset{h}{\overset{-g\rightarrow}{\rightrightarrows}}} B$$

in Sets.

(b) First take the equaliser of f and g, forming a diagram

$$\mathrm{Eq}(f, g) \overset{\mathrm{eq}(f,g)}{\hookrightarrow} A \underset{g}{\overset{f}{\rightrightarrows}} B$$

and then take the equaliser of the composition

$$\mathrm{Eq}(f, g) \overset{\mathrm{eq}(f,g)}{\hookrightarrow} A \underset{h}{\overset{f}{\rightrightarrows}} B,$$

obtaining a subset

$$\mathrm{Eq}(f \circ \mathrm{eq}(f, g), h \circ \mathrm{eq}(f, g)) = \mathrm{Eq}(g \circ \mathrm{eq}(f, g), h \circ \mathrm{eq}(f, g))$$

of $\mathrm{Eq}(f, g)$.

(c) First take the equaliser of g and h, forming a diagram

$$\mathrm{Eq}(g, h) \overset{\mathrm{eq}(g,h)}{\hookrightarrow} A \underset{h}{\overset{g}{\rightrightarrows}} B$$

and then take the equaliser of the composition

$$\mathrm{Eq}(g, h) \overset{\mathrm{eq}(g,h)}{\hookrightarrow} A \underset{g}{\overset{f}{\rightrightarrows}} B,$$

obtaining a subset

$$\mathrm{Eq}(f \circ \mathrm{eq}(g, h), g \circ \mathrm{eq}(g, h)) = \mathrm{Eq}(f \circ \mathrm{eq}(g, h), h \circ \mathrm{eq}(g, h))$$

of $\mathrm{Eq}(g, h)$.

Proof 1.4.4 ► Proof of Proposition 1.4.3

Item 1: Associativity

We first prove that $\mathrm{Eq}(f, g, h)$ is indeed given by

$$\mathrm{Eq}(f, g, h) \cong \{a \in A \mid f(a) = g(a) = h(a)\}.$$

Indeed, suppose we have a diagram of the form

$$\begin{tikzcd} \mathsf{Eq}(f,g,h) \arrow[r, hook, "\mathsf{eq}(f,g,h)"] & A \arrow[r, shift left=2, "f"] \arrow[r, "g" description] \arrow[r, shift right=2, "h"'] & B \\ E \arrow[ur, "e"'] & & \end{tikzcd}$$

in Sets. Then there exists a unique map $\phi\colon E \longrightarrow \mathsf{Eq}(f,g,h)$, uniquely determined by the condition

$$\mathsf{eq}(f,g) \circ \phi = e$$

being necessarily given by

$$\phi(x) = e(x)$$

for each $x \in E$, where we note that $e(x) \in A$ indeed lies in $\mathsf{Eq}(f,g,h)$ by the condition

$$f \circ e = g \circ e = h \circ e,$$

which gives

$$f(e(x)) = g(e(x)) = h(e(x))$$

for each $x \in E$, so that $e(x) \in \mathsf{Eq}(f,g,h)$.

We now check the equalities

$$\mathsf{Eq}(f \circ \mathsf{eq}(g,h), g \circ \mathsf{eq}(g,h)) \cong \mathsf{Eq}(f,g,h) \cong \mathsf{Eq}(f \circ \mathsf{eq}(f,g), h \circ \mathsf{eq}(f,g)).$$

Indeed, we have

$$\begin{aligned} \mathsf{Eq}(f \circ \mathsf{eq}(g,h), g \circ \mathsf{eq}(g,h)) &\cong \{x \in \mathsf{Eq}(g,h) \mid [f \circ \mathsf{eq}(g,h)](a) = [g \circ \mathsf{eq}(g,h)](a)\} \\ &\cong \{x \in \mathsf{Eq}(g,h) \mid f(a) = g(a)\} \\ &\cong \{x \in A \mid f(a) = g(a) \text{ and } g(a) = h(a)\} \\ &\cong \{x \in A \mid f(a) = g(a) = h(a)\} \\ &\cong \mathsf{Eq}(f,g,h). \end{aligned}$$

Similarly, we have

$$\begin{aligned} \mathsf{Eq}(f \circ \mathsf{eq}(f,g), h \circ \mathsf{eq}(f,g)) &\cong \{x \in \mathsf{Eq}(f,g) \mid [f \circ \mathsf{eq}(f,g)](a) = [h \circ \mathsf{eq}(f,g)](a)\} \\ &\cong \{x \in \mathsf{Eq}(f,g) \mid f(a) = h(a)\} \\ &\cong \{x \in A \mid f(a) = h(a) \text{ and } f(a) = g(a)\} \\ &\cong \{x \in A \mid f(a) = g(a) = h(a)\} \\ &\cong \mathsf{Eq}(f,g,h). \end{aligned}$$

Item 2: Unitality

Clear.

Item 3: Commutativity

Clear.

Item 4: Interaction With Composition

Indeed, we have

$$\begin{aligned}\mathrm{Eq}(h \circ f \circ \mathrm{eq}(f,g), k \circ g \circ \mathrm{eq}(f,g)) &\cong \{a \in \mathrm{Eq}(f,g) \mid h(f(a)) = k(g(a))\} \\ &\cong \{a \in A \mid f(a) = g(a) \text{ and } h(f(a)) = k(g(a))\}.\end{aligned}$$

and

$$\mathrm{Eq}(h \circ f, k \circ g) \cong \{a \in A \mid h(f(a)) = k(g(a))\},$$

and thus there's an inclusion from $\mathrm{Eq}(h \circ f \circ \mathrm{eq}(f,g), k \circ g \circ \mathrm{eq}(f,g))$ to $\mathrm{Eq}(h \circ f, k \circ g)$.

2 Colimits of Sets

2.1 Coproducts of Families of Sets

Let $\{A_i\}_{i \in I}$ be a family of sets.

Definition 2.1.1 ► Disjoint Unions of Families

The **disjoint union of the family** $\{A_i\}_{i \in I}$ is the pair $\left(\coprod_{i \in I} A_i, \{\mathrm{inj}_i\}_{i \in I}\right)$ consisting of

- *The Colimit.* The set $\coprod_{i \in I} A_i$ defined by

$$\coprod_{i \in I} A_i \overset{\mathrm{def}}{=} \left\{(i,x) \in I \times \left(\bigcup_{i \in I} A_i\right) \,\middle|\, x \in A_i\right\}.$$

- *The Cocone.* The collection

$$\left\{\mathrm{inj}_i \colon A_i \to \coprod_{i \in I} A_i\right\}_{i \in I}$$

of maps given by

$$\mathrm{inj}_i(x) \stackrel{\mathrm{def}}{=} (i, x)$$

for each $x \in A_i$ and each $i \in I$.

Proof 2.1.2 ► Proof of Definition 2.1.1

We claim that $\coprod_{i\in I} A_i$ is the categorical coproduct of $\{A_i\}_{i\in I}$ in Sets. Indeed, suppose we have, for each $i \in I$, a diagram of the form

$$\begin{tikzcd} & C \\ A_i \arrow[ur, "i_i"] \arrow[r, "\mathrm{inj}_i"'] & \coprod\limits_{i\in I} A_i \end{tikzcd}$$

in Sets. Then there exists a unique map $\phi \colon \coprod_{i\in I} A_i \to C$, uniquely determined by the condition $\phi \circ \mathrm{inj}_i = i_i$ for each $i \in I$, being necessarily given by

$$\phi(i, x) = i_i(x)$$

for each $(i, x) \in \coprod_{i\in I} A_i$. ∎

Proposition 2.1.3 ► Properties of Coproducts of Families of Sets

Let $\{A_i\}_{i\in I}$ be a family of sets.

1. *Functoriality.* The assignment $\{A_i\}_{i\in I} \mapsto \coprod_{i\in I} A_i$ defines a functor

$$\coprod_{i\in I} \colon \mathrm{Fun}(I_{\mathrm{disc}}, \mathsf{Sets}) \to \mathsf{Sets}$$

where

- *Action on Objects.* For each $(A_i)_{i\in I} \in \mathrm{Obj}(\mathrm{Fun}(I_{\mathrm{disc}}, \mathsf{Sets}))$, we have

$$\left[\coprod_{i\in I}\right]((A_i)_{i\in I}) \stackrel{\mathrm{def}}{=} \coprod_{i\in I} A_i$$

- *Action on Morphisms.* For each $(A_i)_{i\in I}, (B_i)_{i\in I} \in \mathrm{Obj}(\mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}))$, the action on Hom-sets

$$\left(\coprod_{i\in I}\right)_{(A_i)_{i\in I},(B_i)_{i\in I}} : \mathrm{Nat}((A_i)_{i\in I}, (B_i)_{i\in I}) \to \mathsf{Sets}\left(\coprod_{i\in I} A_i, \coprod_{i\in I} B_i\right)$$

of $\coprod_{i\in I}$ at $((A_i)_{i\in I}, (B_i)_{i\in I})$ is defined by sending a map

$$\{f_i : A_i \to B_i\}_{i\in I}$$

in $\mathrm{Nat}((A_i)_{i\in I}, (B_i)_{i\in I})$ to the map of sets

$$\coprod_{i\in I} f_i : \coprod_{i\in I} A_i \to \coprod_{i\in I} B_i$$

defined by

$$\left[\coprod_{i\in I} f_i\right](i, a) \stackrel{\text{def}}{=} f_i(a)$$

for each $(i, a) \in \coprod_{i\in I} A_i$.

Proof 2.1.4 ▶ Proof of Proposition 2.1.3

Item 1: Functoriality

Clear. □

2.2 Binary Coproducts

Let A and B be sets.

Definition 2.2.1 ▶ Coproducts of Sets

The **coproduct**[1] **of** A **and** B is the pair $(A \coprod B, \{\mathrm{inj}_1, \mathrm{inj}_2\})$ consisting of

- *The Colimit.* The set $A \coprod B$ defined by

$$\begin{aligned} A \coprod B &\stackrel{\text{def}}{=} \coprod_{z\in\{A,B\}} z \\ &\cong \{(0, a) \mid a \in A\} \cup \{(1, b) \mid b \in B\}. \end{aligned}$$

- *The Cocone.* The maps

$$\begin{aligned}\mathrm{inj}_1 &\colon A \to A \coprod B,\\ \mathrm{inj}_2 &\colon B \to A \coprod B,\end{aligned}$$

given by

$$\begin{aligned}\mathrm{inj}_1(a) &\overset{\text{def}}{=} (0, a),\\ \mathrm{inj}_2(b) &\overset{\text{def}}{=} (1, b),\end{aligned}$$

for each $a \in A$ and each $b \in B$.

[1] *Further Terminology:* Also called the **disjoint union of** A **and** B, or the **binary disjoint union of** A **and** B, for emphasis.

Proof 2.2.2 ► Proof of Definition 2.2.1

We claim that $A \coprod B$ is the categorical coproduct of A and B in Sets. Indeed, suppose we have a diagram of the form

$$\begin{tikzcd} & C & \\ A \arrow[ur, bend left, "i_A"] \arrow[r, "\mathrm{inj}_A"'] & A \coprod B & B \arrow[ul, bend right, "i_B"'] \arrow[l, "\mathrm{inj}_B"] \end{tikzcd}$$

in Sets. Then there exists a unique map $\phi\colon A \coprod B \to C$, uniquely determined by the conditions

$$\begin{aligned}\phi \circ \mathrm{inj}_A &= i_A,\\ \phi \circ \mathrm{inj}_B &= i_B,\end{aligned}$$

being necessarily given by

$$\phi(x) = \begin{cases} i_A(x) & \text{if } x \in A,\\ i_B(x) & \text{if } x \in B \end{cases}$$

for each $x \in C$. ∎

Proposition 2.2.3 ▶ Properties of Coproducts of Sets

Let A, B, C, and X be sets.

1. *Functoriality.* The assignment $A, B, (A, B) \mapsto A \coprod B$ defines functors

$$A \coprod -_2 \colon \mathsf{Sets} \to \mathsf{Sets},$$
$$-_1 \coprod B \colon \mathsf{Sets} \to \mathsf{Sets},$$
$$-_1 \coprod -_2 \colon \mathsf{Sets} \times \mathsf{Sets} \to \mathsf{Sets},$$

   where $-_1 \coprod -_2$ is the functor where

   - *Action on Objects.* For each $(A, B) \in \mathrm{Obj}(\mathsf{Sets} \times \mathsf{Sets})$, we have

$$[-_1 \coprod -_2](A, B) \overset{\text{def}}{=} A \coprod B;$$

   - *Action on Morphisms.* For each $(A, B), (X, Y) \in \mathrm{Obj}(\mathsf{Sets})$, the action on Hom-sets

$$\coprod_{(A,B),(X,Y)} \colon \mathsf{Sets}(A, X) \times \mathsf{Sets}(B, Y) \to \mathsf{Sets}(A \coprod B, X \coprod Y)$$

     of $\coprod$ at $((A, B), (X, Y))$ is defined by sending (f, g) to the function

$$f \coprod g \colon A \coprod B \to X \coprod Y$$

     defined by

$$[f \coprod g](x) \overset{\text{def}}{=} \begin{cases} f(x) & \text{if } x \in A, \\ g(x) & \text{if } x \in B \end{cases}$$

     for each $x \in A \coprod B$;

   and where $A \coprod -$ and $- \coprod B$ are the partial functors of $-_1 \coprod -_2$ at $A, B \in \mathrm{Obj}(\mathsf{Sets})$.

2. *Associativity.* We have an isomorphism of sets

$$(A \coprod B) \coprod C \cong A \coprod (B \coprod C),$$

   natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

3. *Unitality.* We have isomorphisms of sets

$$A \coprod \emptyset \cong A,$$
$$\emptyset \coprod A \cong A,$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$.

4. *Commutativity.* We have an isomorphism of sets

$$A \coprod B \cong B \coprod A,$$

natural in $A, B \in \mathrm{Obj}(\mathsf{Sets})$.

5. *Symmetric Monoidality.* The triple $(\mathsf{Sets}, \coprod, \emptyset)$ is a symmetric monoidal category.

Proof 2.2.4 ► Proof of Proposition 2.2.3

Item 1: Functoriality

Clear.

Item 2: Associativity

Clear.

Item 3: Unitality

Clear.

Item 4: Commutativity

Clear.

Item 5: Symmetric Monoidality

Omitted. □

2.3 Pushouts

Let A, B, and C be sets and let $f : C \to A$ and $g : C \to B$ be functions.

Definition 2.3.1 ► Pushouts of Sets

The **pushout of** A **and** B **over** C **along** f **and** g[1] is the pair $(A \coprod_C B, \{\text{inj}_1, \text{inj}2\})$ consisting of

- *The Colimit.* The set $A \coprod_C B$ defined by

$$A \coprod_C B \overset{\text{def}}{=} A \coprod B/{\sim_C},$$

where $\sim_C$ is the equivalence relation on $A \coprod B$ generated by $(0, f(c)) \sim_C (1, g(c))$.

- *The Cocone.* The maps

$$\begin{aligned}\text{inj}_1 &: A \to A \coprod_C B,\\ \text{inj}_2 &: B \to A \coprod_C B\end{aligned}$$

given by

$$\begin{aligned}\text{inj}_1(a) &\overset{\text{def}}{=} [(0, a)]\\ \text{inj}_2(b) &\overset{\text{def}}{=} [(1, b)]\end{aligned}$$

for each $a \in A$ and each $b \in B$.

[1] *Further Terminology:* Also called the **fibre coproduct of** A **and** B **over** C **along** f **and** g.

Proof 2.3.2 ► Proof of Definition 2.3.1

We claim that $A \coprod_C B$ is the categorical pushout of A and B over C with respect to (f, g) in Sets. First we need to check that the relevant pushout diagram commutes, i.e. that we have

$$\text{inj}_1 \circ f = \text{inj}_2 \circ g, \qquad \begin{tikzcd} A \coprod_C B & B \arrow[l, "\text{inj}_2"'] \\ A \arrow[u, "\text{inj}_1"] & C. \arrow[l, "f"] \arrow[u, "g"'] \end{tikzcd}$$

Indeed, given $c \in C$, we have

$$\begin{aligned}[\mathsf{inj}_1 \circ f](c) &= \mathsf{inj}_1(f(c)) \\ &= [(0, f(c))] \\ &= [(1, g(c))] \\ &= \mathsf{inj}_2(g(c)) \\ &= [\mathsf{inj}_2 \circ g](c),\end{aligned}$$

where $[(0, f(c))] = [(1, g(c))]$ by the definition of the relation $\sim$ on B. Next, we prove that $A \coprod_C B$ satisfies the universal property of the pushout. Suppose we have a diagram of the form

$$\begin{tikzcd} P & & \\ & A \coprod_C B \arrow[ul, phantom] & B \arrow[l, "\mathsf{inj}_2"'] \arrow[ull, bend right, "i_2"'] \\ & A \arrow[u, "\mathsf{inj}_1"] \arrow[uul, bend left, "i_1"] & C \arrow[l, "f"] \arrow[u, "g"'] \arrow[ul, phantom, "\ulcorner", very near end] \end{tikzcd}$$

in Sets. Then there exists a unique map $\phi\colon A \coprod_C B \to P$, uniquely determined by the conditions

$$\begin{aligned}\phi \circ \mathsf{inj}_1 &= i_1, \\ \phi \circ \mathsf{inj}_2 &= i_2,\end{aligned}$$

being necessarily given by

$$\phi(x) = \begin{cases} i_1(a) & \text{if } x = [(0, a)], \\ i_2(b) & \text{if } x = [(1, b)] \end{cases}$$

for each $x \in A \coprod_C B$, where the well-definedness of ϕ is guaranteed by the equality $i_1 \circ f = i_2 \circ g$ and the definition of the relation $\sim$ on $A \coprod B$ as follows.

1. *Case 1:* Suppose we have $x = [(0, a)] = [(0, a')]$ for some $a, a' \in A$. Then, by Remark 2.3.3, we have a sequence

$$(0, a) \sim' x_1 \sim' \cdots \sim' x_n \sim' (0, a').$$

2. *Case 2:* Suppose we have $x = [(1,b)] = [(1,b')]$ for some $b, b' \in B$. Then, by Remark 2.3.3, we have a sequence

$$(1,b) \sim' x_1 \sim' \cdots \sim' x_n \sim' (1,b').$$

3. *Case 3:* Suppose we have $x = [(0,a)] = [(1,b)]$ for some $a \in A$ and $b \in B$. Then, by Remark 2.3.3, we have a sequence

$$(0,a) \sim' x_1 \sim' \cdots \sim' x_n \sim' (1,b).$$

In all these cases, we declare $x \sim' y$ iff there exists some $c \in C$ such that $x = (0, f(c))$ and $y = (1, g(c))$ or $x = (1, g(c))$ and $y = (0, f(c))$. Then, the equality $i_1 \circ f = i_2 \circ g$ gives

$$\begin{aligned}
\phi([x]) &= \phi([(0, f(c))]) \\
&\overset{\text{def}}{=} i_1(f(c)) \\
&= i_2(g(c)) \\
&\overset{\text{def}}{=} \phi([(1, g(c))]) \\
&= \phi([y]),
\end{aligned}$$

with the case where $x = (1, g(c))$ and $y = (0, f(c))$ similarly giving $\phi([x]) = \phi([y])$. Thus, if $x \sim' y$, then $\phi([x]) = \phi([y])$. Applying this equality pairwise to the sequences

$$\begin{aligned}
(0,a) \sim' x_1 \sim' \cdots \sim' x_n \sim' (0,a'), \\
(1,b) \sim' x_1 \sim' \cdots \sim' x_n \sim' (1,b'), \\
(0,a) \sim' x_1 \sim' \cdots \sim' x_n \sim' (1,b)
\end{aligned}$$

gives

$$\begin{aligned}
\phi([(0,a)]) = \phi([(0,a')]), \\
\phi([(1,b)]) = \phi([(1,b')]), \\
\phi([(0,a)]) = \phi([(1,b)]),
\end{aligned}$$

showing ϕ to be well-defined. $\square$

Remark 2.3.3 ▶ Unwinding Definition 2.3.1

In detail, by Relations, Construction 4.4.5, the relation $\sim$ of Definition 2.3.1 is given by declaring $a \sim b$ iff one of the following conditions is satisfied:

- We have $a, b \in A$ and $a = b$;
- We have $a, b \in B$ and $a = b$;
- There exist $x_1, \ldots, x_n \in A \coprod B$ such that $a \sim' x_1 \sim' \cdots \sim' x_n \sim' b$, where we declare $x \sim' y$ if one of the following conditions is satisfied:

  1. There exists $c \in C$ such that $x = (0, f(c))$ and $y = (1, g(c))$.
  2. There exists $c \in C$ such that $x = (1, g(c))$ and $y = (0, f(c))$.

  That is: we require the following condition to be satisfied:

  ($\star$) There exist $x_1, \ldots, x_n \in A \coprod B$ satisfying the following conditions:
  1. There exists $c_0 \in C$ satisfying one of the following conditions:
     (a) We have $a = f(c_0)$ and $x_1 = g(c_0)$.
     (b) We have $a = g(c_0)$ and $x_1 = f(c_0)$.
  2. For each $1 \leq i \leq n-1$, there exists $c_i \in C$ satisfying one of the following conditions:
     (a) We have $x_i = f(c_i)$ and $x_{i+1} = g(c_i)$.
     (b) We have $x_i = g(c_i)$ and $x_{i+1} = f(c_i)$.
  3. There exists $c_n \in C$ satisfying one of the following conditions:
     (a) We have $x_n = f(c_n)$ and $b = g(c_n)$.
     (b) We have $x_n = g(c_n)$ and $b = f(c_n)$.

Example 2.3.4 ▶ Examples of Pushouts of Sets

Here are some examples of pushouts of sets.

1. *Wedge Sums of Pointed Sets.* The wedge sum of two pointed sets of Pointed Sets, Definition 4.3.1 is an example of a pushout of sets.

2. *Intersections via Unions.* Let $A, B \subset X$. We have a bijection of sets

$$A \cup B \cong A \coprod_{A \cap B} B, \qquad \begin{tikzcd} A \cup B & B \arrow[l] \\ A \arrow[u] & A \cap B. \arrow[l, hook'] \arrow[u, hook] \arrow[ul, phantom, "\ulcorner", very near end] \end{tikzcd}$$

Proof 2.3.5 ► Proof of Example 2.3.4

Item 1: Wedge Sums of Pointed Sets

Follows by definition.

Item 2: Intersections via Unions

Indeed, $A \coprod_{A \cap B} B$ is the quotient of $A \coprod B$ by the equivalence relation obtained by declaring $(0, a) \sim (1, b)$ iff $a = b \in A \cap B$, which is in bijection with $A \cup B$ via the map with $[(0, a)] \mapsto a$ and $[(1, b)] \mapsto b$.

Proposition 2.3.6 ► Properties of Pushouts of Sets

Let A, B, C, and X be sets.

1. *Functoriality.* The assignment $(A, B, C, f, g) \mapsto A \coprod_{f,C,g} B$ defines a functor

$$-_1 \coprod_{-_3} -_1 \colon \mathsf{Fun}(\mathcal{P}, \mathsf{Sets}) \longrightarrow \mathsf{Sets},$$

where $\mathcal{P}$ is the category that looks like this:

$$\begin{tikzcd} & \bullet \\ \bullet & \bullet. \arrow[l] \arrow[u] \end{tikzcd}$$

In particular, the action on morphisms of $-_1 \coprod_{-_3} -_1$ is given by sending a morphism

$$
\begin{tikzcd}
A \coprod_C B \arrow[dr, phantom, "\ulcorner", very near start] & & B \arrow[ll] \arrow[dr, "\psi"] & \\
& A' \coprod_{C'} B' \arrow[dr, phantom, "\ulcorner", very near start] & & B' \arrow[ll] \\
A \arrow[uu] \arrow[dr, "\phi"'] & & C \arrow[ll, "f"'] \arrow[uu, "g"] \arrow[dr, "\chi"] & \\
& A' \arrow[uu] & & C' \arrow[ll, "f'"] \arrow[uu, "g'"']
\end{tikzcd}
$$

in Fun($\mathcal{P}$, Sets) to the map $\xi \colon A \coprod_C B \xrightarrow{\exists!} A' \coprod_{C'} B'$ given by

$$
\xi(x) \overset{\text{def}}{=} \begin{cases} \phi(a) & \text{if } x = [(0, a)], \\ \psi(b) & \text{if } x = [(1, b)] \end{cases}
$$

for each $x \in A \coprod_C B$, which is the unique map making the diagram

$$
\begin{tikzcd}
A \coprod_C B \arrow[dr, phantom, "\ulcorner", very near start] \arrow[dr, dashed] & & B \arrow[ll] \arrow[dr, "\psi"] & \\
& A' \coprod_{C'} B' \arrow[dr, phantom, "\ulcorner", very near start] & & B' \arrow[ll] \\
A \arrow[uu] \arrow[dr, "\phi"'] & & C \arrow[ll, "f"'] \arrow[uu, "g"] \arrow[dr, "\chi"] & \\
& A' \arrow[uu] & & C' \arrow[ll, "f'"] \arrow[uu, "g'"']
\end{tikzcd}
$$

commute.

2. *Associativity.* Given a diagram

$$
\begin{tikzcd}
A & & B & & C \\
& X \arrow[ul, "f"] \arrow[ur, "g"'] & & Y \arrow[ul, "h"] \arrow[ur, "k"'] &
\end{tikzcd}
$$

in Sets, we have isomorphisms

$$(A \coprod_X B) \coprod_Y C \cong (A \coprod_X B) \coprod_B (B \coprod_Y C) \cong A \coprod_X (B \coprod_Y C),$$

where these pullbacks are built as in the diagrams

$$\begin{tikzcd}
 & & (A \coprod_X B) \coprod_Y C & & \\
 & A \coprod_X B \arrow[ur] \arrow[r, phantom, "\wedge"] & & & \\
A \arrow[ur] & & B \arrow[ul] & & C, \arrow[uull] \\
 & X \arrow[ul, "f"] \arrow[ur, "g"'] & & Y \arrow[ul, "h"] \arrow[ur, "k"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & (A \coprod_X B) \coprod_B (B \coprod_Y C) & & \\
 & A \coprod_X B \arrow[ur] & \wedge & B \coprod_Y C \arrow[ul] & \\
A \arrow[ur] \arrow[r, phantom, "\wedge"] & & B \arrow[ul] \arrow[ur] & & C, \arrow[ul] \\
 & X \arrow[ul, "f"] \arrow[ur, "g"'] & & Y \arrow[ul, "h"] \arrow[ur, "k"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & A \coprod_X (B \coprod_Y C) & & \\
 & & & B \coprod_Y C \arrow[ul] \arrow[l, phantom, "\wedge"] & \\
A \arrow[uurr] & & B \arrow[ur] & & C. \arrow[ul] \\
 & X \arrow[ul, "f"] \arrow[ur, "g"'] & & Y \arrow[ul, "h"] \arrow[ur, "k"'] &
\end{tikzcd}$$

3. *Unitality.* We have isomorphisms of sets

$$\begin{tikzcd}
A \arrow[r, equal] \arrow[dr, phantom, very near start, "\ulcorner"] & A \\
X \arrow[u, "f"] \arrow[r, equal] & X \arrow[u, "f"']
\end{tikzcd}
\qquad
\begin{aligned}
X \coprod_X A &\cong A, \\
A \coprod_X X &\cong A,
\end{aligned}
\qquad
\begin{tikzcd}
A \arrow[dr, phantom, very near start, "\ulcorner"] & X \arrow[l, "f"'] \\
X \arrow[u, equal] & X. \arrow[l, "f"] \arrow[u, equal]
\end{tikzcd}$$

4. *Commutativity.* We have an isomorphism of sets

$$\begin{tikzcd}
A \coprod_X B \arrow[dr, phantom, very near start, "\ulcorner"] & B \arrow[l] \\
A \arrow[u] & X, \arrow[l, "f"] \arrow[u, "g"']
\end{tikzcd}
\qquad
A \coprod_X B \cong B \coprod_X A
\qquad
\begin{tikzcd}
B \coprod_X A \arrow[dr, phantom, very near start, "\ulcorner"] & A \arrow[l] \\
B \arrow[u] & X. \arrow[l, "g"] \arrow[u, "f"']
\end{tikzcd}$$

5. *Interaction With Coproducts.* We have

$$A \coprod_\emptyset B \cong A \coprod B,
\qquad
\begin{tikzcd}
A \coprod B \arrow[dr, phantom, very near start, "\ulcorner"] & B \arrow[l] \\
A \arrow[u] & \emptyset. \arrow[l, hook', "\iota_A"] \arrow[u, hook, "\iota_B"']
\end{tikzcd}$$

6. *Symmetric Monoidality.* The triple $(\mathsf{Sets}, \coprod_X, \emptyset)$ is a symmetric monoidal category.

Proof 2.3.7 ► Proof of Proposition 2.3.6

Item 1: Functoriality

This is a special case of functoriality of co/limits, Limits and Colimits, **??** of **??**, with the explicit expression for ξ following from the commutativity of the cube pushout diagram.

Item 2: Associativity

Omitted.

Item 3: Unitality

Omitted.

Item 4: Commutativity

Clear.

Item 5: Interaction With Coproducts

Clear.

Item 6: Symmetric Monoidality

Omitted.

2.4 Coequalisers

Let A and B be sets and let $f, g \colon A \rightrightarrows B$ be functions.

Definition 2.4.1 ► Coequalisers of Sets

The **coequaliser of f and g** is the pair $(\mathrm{CoEq}(f,g), \mathrm{coeq}(f,g))$ consisting of

- *The Colimit.* The set $\mathrm{CoEq}(f,g)$ defined by

  $$\mathrm{CoEq}(f,g) \overset{\text{def}}{=} B/\sim,$$

  where $\sim$ is the equivalence relation on B generated by $f(a) \sim g(a)$.

- *The Cocone.* The map

  $$\mathrm{coeq}(f,g) \colon B \to \mathrm{CoEq}(f,g)$$

  given by the quotient map $\pi \colon B \twoheadrightarrow B/\sim$ with respect to the equivalence relation generated by $f(a) \sim g(a)$.

PROOF 2.4.2 ► PROOF OF DEFINITION 2.4.1

We claim that $\mathrm{CoEq}(f,g)$ is the categorical coequaliser of f and g in Sets. First we need to check that the relevant coequaliser diagram commutes, i.e. that we have

$$\mathrm{coeq}(f,g) \circ f = \mathrm{coeq}(f,g) \circ g.$$

Indeed, we have

$$\begin{aligned}
[\mathrm{coeq}(f,g) \circ f](a) &\stackrel{\text{def}}{=} [\mathrm{coeq}(f,g)](f(a)) \\
&\stackrel{\text{def}}{=} [f(a)] \\
&= [g(a)] \\
&\stackrel{\text{def}}{=} [\mathrm{coeq}(f,g)](g(a)) \\
&\stackrel{\text{def}}{=} [\mathrm{coeq}(f,g) \circ g](a)
\end{aligned}$$

for each $a \in A$. Next, we prove that $\mathrm{CoEq}(f,g)$ satisfies the universal property of the coequaliser. Suppose we have a diagram of the form

$$\begin{tikzcd}
A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, hook, "\mathrm{coeq}(f,g)"] \arrow[dr, "c"'] & \mathrm{CoEq}(f,g) \\
 & & C
\end{tikzcd}$$

in Sets. Then, since $c(f(a)) = c(g(a))$ for each $a \in A$, it follows from Relations, Items 4 and 5 of Proposition 4.5.4 that there exists a unique map $\mathrm{CoEq}(f,g) \xrightarrow{\exists!} C$ making the diagram

$$\begin{tikzcd}
A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, hook, "\mathrm{coeq}(f,g)"] \arrow[dr, "c"'] & \mathrm{CoEq}(f,g) \arrow[d, dashed, "\exists!"] \\
 & & C
\end{tikzcd}$$

commutes. ∎

Remark 2.4.3 ▶ Unwinding Definition 2.4.1

In detail, by Relations, Construction 4.4.5, the relation $\sim$ of Definition 2.4.1 is given by declaring $a \sim b$ iff one of the following conditions is satisfied:

- We have $a = b$;
- There exist $x_1, \ldots, x_n \in B$ such that $a \sim' x_1 \sim' \cdots \sim' x_n \sim' b$, where we declare $x \sim' y$ if one of the following conditions is satisfied:
  1. There exists $z \in A$ such that $x = f(z)$ and $y = g(z)$.
  2. There exists $z \in A$ such that $x = g(z)$ and $y = f(z)$.

  That is: we require the following condition to be satisfied:

  $(\star)$ There exist $x_1, \ldots, x_n \in B$ satisfying the following conditions:
  1. There exists $z_0 \in A$ satisfying one of the following conditions:
     (a) We have $a = f(z_0)$ and $x_1 = g(z_0)$.
     (b) We have $a = g(z_0)$ and $x_1 = f(z_0)$.
  2. For each $1 \leq i \leq n-1$, there exists $z_i \in A$ satisfying one of the following conditions:
     (a) We have $x_i = f(z_i)$ and $x_{i+1} = g(z_i)$.
     (b) We have $x_i = g(z_i)$ and $x_{i+1} = f(z_i)$.
  3. There exists $z_n \in A$ satisfying one of the following conditions:
     (a) We have $x_n = f(z_n)$ and $b = g(z_n)$.
     (b) We have $x_n = g(z_n)$ and $b = f(z_n)$.

Example 2.4.4 ▶ Examples of Coequalisers of Sets

Here are some examples of coequalisers of sets.

1. *Quotients by Equivalence Relations.* Let R be an equivalence relation on a set X. We have a bijection of sets

$$X/\sim_R \cong \mathrm{CoEq}\left(R \hookrightarrow X \times X \underset{\mathrm{pr}_2}{\overset{\mathrm{pr}_1}{\rightrightarrows}} X\right).$$

Proof 2.4.5 ▶ Proof of Example 2.4.4

Item 1: Quotients by Equivalence Relations

See [Pro24v].

Proposition 2.4.6 ▶ Properties of Coequalisers of Sets

Let A, B, and C be sets.

1. *Associativity.* We have an isomorphism of sets[1]

$$\underbrace{\mathrm{CoEq}(\mathrm{coeq}(f,g)\circ f,\mathrm{coeq}(f,g)\circ h)}_{=\mathrm{CoEq}(\mathrm{coeq}(f,g)\circ g,\mathrm{coeq}(f,g)\circ h)} \cong \mathrm{CoEq}(f,g,h) \cong \underbrace{\mathrm{CoEq}(\mathrm{coeq}(g,h)\circ f,\mathrm{coeq}(g,h)\circ g)}_{=\mathrm{CoEq}(\mathrm{coeq}(g,h)\circ f,\mathrm{coeq}(g,h)\circ h)},$$

   where $\mathrm{CoEq}(f,g,h)$ is the colimit of the diagram

$$\begin{tikzcd} A \arrow[r, shift left=2, "f"] \arrow[r, "g" description] \arrow[r, shift right=2, "h"'] & B \end{tikzcd}$$

   in Sets.

2. *Unitality.* We have an isomorphism of sets

$$\mathrm{CoEq}(f,f) \cong B.$$

3. *Commutativity.* We have an isomorphism of sets

$$\mathrm{CoEq}(f,g) \cong \mathrm{CoEq}(g,f).$$

4. *Interaction With Composition.* Let

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \underset{k}{\overset{h}{\rightrightarrows}} C$$

   be functions. We have a surjection

$$\mathrm{CoEq}(h\circ f,k\circ g) \twoheadrightarrow \mathrm{CoEq}(\mathrm{coeq}(h,k)\circ h\circ f,\mathrm{coeq}(h,k)\circ k\circ g)$$

   exhibiting $\mathrm{CoEq}(\mathrm{coeq}(h,k)\circ h\circ f,\mathrm{coeq}(h,k)\circ k\circ g)$ as a quotient of $\mathrm{CoEq}(h\circ f,k\circ g)$ by the relation generated by declaring $h(y)\sim k(y)$ for each $y\in B$.

[1] That is, the following three ways of forming "the" coequaliser of (f, g, h) agree:

(a) Take the coequaliser of (f, g, h), i.e. the colimit of the diagram

$$A \overset{f}{\underset{h}{\underset{\xrightarrow{}}{\overset{\xrightarrow{}}{\xrightarrow{g}}}}} B$$

in Sets.

(b) First take the coequaliser of f and g, forming a diagram

$$A \overset{f}{\underset{g}{\rightrightarrows}} B \overset{\mathrm{coeq}(f,g)}{\twoheadrightarrow} \mathrm{CoEq}(f, g)$$

and then take the coequaliser of the composition

$$A \overset{f}{\underset{h}{\rightrightarrows}} B \overset{\mathrm{coeq}(f,g)}{\twoheadrightarrow} \mathrm{CoEq}(f, g),$$

obtaining a quotient

$$\mathrm{CoEq}(\mathrm{coeq}(f, g) \circ f, \mathrm{coeq}(f, g) \circ h) = \mathrm{CoEq}(\mathrm{coeq}(f, g) \circ g, \mathrm{coeq}(f, g) \circ h)$$

of $\mathrm{CoEq}(f, g)$

(c) First take the coequaliser of g and h, forming a diagram

$$A \overset{g}{\underset{h}{\rightrightarrows}} B \overset{\mathrm{coeq}(g,h)}{\twoheadrightarrow} \mathrm{CoEq}(g, h)$$

and then take the coequaliser of the composition

$$A \overset{f}{\underset{g}{\rightrightarrows}} B \overset{\mathrm{coeq}(g,h)}{\twoheadrightarrow} \mathrm{CoEq}(g, h),$$

obtaining a quotient

$$\mathrm{CoEq}(\mathrm{coeq}(g, h) \circ f, \mathrm{coeq}(g, h) \circ g) = \mathrm{CoEq}(\mathrm{coeq}(g, h) \circ f, \mathrm{coeq}(g, h) \circ h)$$

of $\mathrm{CoEq}(g, h)$.

PROOF 2.4.7 ► PROOF OF PROPOSITION 2.4.6

Item 1: Associativity

Omitted.

Item 2: Unitality

Clear.

Item 3: Commutativity

Clear.

Item 4: Interaction With Composition

Omitted.

3 Operations With Sets

3.1 The Empty Set

Definition 3.1.1 ► The Empty Set

The **empty set** is the set $\emptyset$ defined by

$$\emptyset \overset{\text{def}}{=} \{x \in X \mid x \neq x\},$$

where A is the set in the set existence axiom, **??** of **??**.

3.2 Singleton Sets

Let X be a set.

Definition 3.2.1 ► Singleton Sets

The **singleton set containing** X is the set $\{X\}$ defined by

$$\{X\} \overset{\text{def}}{=} \{X, X\},$$

where $\{X, X\}$ is the pairing of X with itself (Definition 3.3.1).

3.3 Pairings of Sets

Let X and Y be sets.

Definition 3.3.1 ► Pairings of Sets

The **pairing of X and Y** is the set $\{X, Y\}$ defined by

$$\{X, Y\} \stackrel{\text{def}}{=} \{x \in A \mid x = X \text{ or } x = Y\},$$

where A is the set in the axiom of pairing, **??** of **??**.

3.4 Ordered Pairs

Let A and B be sets.

Definition 3.4.1 ► Ordered Pairs

The **ordered pair associated to A and B** is the set (A, B) defined by

$$(A, B) \stackrel{\text{def}}{=} \{\{A\}, \{A, B\}\}.$$

Proposition 3.4.2 ► Properties of Ordered Pairs

Let A and B be sets.

1. *Uniqueness.* Let $A, B, C,$ and D be sets. The following conditions are equivalent:

   (a) We have $(A, B) = (C, D)$.

   (b) We have $A = C$ and $B = D$.

Proof 3.4.3 ► Proof of Proposition 3.4.2

Item 1: Uniqueness

See [Cie97, Theorem 1.2.3].

3.5 Unions of Families

Let $\{A_i\}_{i \in I}$ be a family of sets.

Definition 3.5.1 ▶ Unions of Families

The **union of the family** $\{A_i\}_{i\in I}$ is the set $\bigcup_{i\in I} A_i$ defined by

$$\bigcup_{i\in I} A_i \overset{\text{def}}{=} \{x \in F \mid \text{there exists some } i \in I \text{ such that } x \in A_i\},$$

where F is the set in the axiom of union, **??** of **??**.

3.6 Binary Unions

Let A and B be sets.

Definition 3.6.1 ▶ Binary Unions

The **union**[1] **of** A **and** B is the set $A \cup B$ defined by

$$A \cup B \overset{\text{def}}{=} \bigcup_{z\in\{A,B\}} z.$$

[1] *Further Terminology:* Also called the **binary union of** A **and** B, for emphasis.

Proposition 3.6.2 ▶ Properties of Binary Unions

Let X be a set.

1. *Functoriality.* The assignments $U, V, (U, V) \mapsto U \cup V$ define functors

$$\begin{aligned} U \cup -\colon (\mathcal{P}(X), \subset) &\to (\mathcal{P}(X), \subset), \\ - \cup V\colon (\mathcal{P}(X), \subset) &\to (\mathcal{P}(X), \subset), \\ -_1 \cup -_2\colon (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \subset) &\to (\mathcal{P}(X), \subset), \end{aligned}$$

   where $-_1 \cup -_2$ is the functor where

   - *Action on Objects.* For each $(U, V) \in \mathcal{P}(X) \times \mathcal{P}(X)$, we have

   $$[-_1 \cup -_2](U, V) \overset{\text{def}}{=} U \cup V;$$

   - *Action on Morphisms.* For each pair of morphisms

   $$\begin{aligned} \iota_U &: U \hookrightarrow U', \\ \iota_V &: V \hookrightarrow V' \end{aligned}$$

of $\mathcal{P}(X) \times \mathcal{P}(X)$, the image

$$\iota_U \cup \iota_V \colon U \cup V \hookrightarrow U' \cup V'$$

of (ι_U, ι_V) by $\cup$ is the inclusion

$$U \cup V \subset U' \cup V'$$

i.e. where we have

($\star$) If $U \subset U'$ and $V \subset V'$, then $U \cup V \subset U' \cup V'$;

and where $U \cup -$ and $- \cup V$ are the partial functors of $-_1 \cup -_2$ at $U, V \in \mathcal{P}(X)$.

2. *Via Intersections and Symmetric Differences.* We have an equality of sets

$$U \cup V = (U \bigtriangleup V) \bigtriangleup (U \cap V)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

3. *Associativity.* We have an equality of sets

$$(U \cup V) \cup W = U \cup (V \cup W)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

4. *Unitality.* We have equalities of sets

$$\begin{aligned} U \cup \emptyset &= U, \\ \emptyset \cup U &= U \end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

5. *Commutativity.* We have an equality of sets

$$U \cup V = V \cup U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

6. *Idempotency.* We have an equality of sets

$$U \cup U = U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

7. *Distributivity Over Intersections.* We have equalities of sets

$$\begin{aligned} U \cup (V \cap W) &= (U \cup V) \cap (U \cup W), \\ (U \cap V) \cup W &= (U \cup W) \cap (V \cup W) \end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

8. *Interaction With Characteristic Functions I.* We have

$$\chi_{U \cup V} = \max(\chi_U, \chi_V)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

9. *Interaction With Characteristic Functions II.* We have

$$\chi_{U \cup V} = \chi_U + \chi_V - \chi_{U \cap V}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

10. *Interaction With Powersets and Semirings.* The quintuple $(\mathcal{P}(X), \cup, \cap, \emptyset, X)$ is an idempotent commutative semiring.

Proof 3.6.3 ► Proof of Proposition 3.6.2

Item 1: Functoriality

See [Pro24aj].

Item 2: Via Intersections and Symmetric Differences

See [Pro24au].

Item 3: Associativity

See [Pro24aw].

Item 4: Unitality

This follows from [Pro24az] and Item 5.

Item 5: Commutativity

See [Pro24ax].

Item 6: Idempotency

See [Pro24ai].

Item 7: Distributivity Over Intersections

See [Pro24av].

Item 8: Interaction With Characteristic Functions I

See [Pro24k].

Item 9: Interaction With Characteristic Functions II

See [Pro24k].

Item 10: Interaction With Powersets and Semirings

This follows from Items 3 to 6 and Items 3 to 5, 7 and 8 of Proposition 3.8.2.

3.7 Intersections of Families

Let $\mathcal{F}$ be a family of sets.

Definition 3.7.1 ▶ Intersections of Families

The **intersection of a family $\mathcal{F}$ of sets** is the set $\bigcap_{X\in\mathcal{F}} X$ defined by

$$\bigcap_{X\in\mathcal{F}} X \stackrel{\text{def}}{=} \left\{ z \in \bigcup_{X\in\mathcal{F}} X \,\middle|\, \text{for each } X \in \mathcal{F}\text{, we have } z \in X \right\}.$$

3.8 Binary Intersections

Let X and Y be sets.

Definition 3.8.1 ▶ Binary Intersections

The **intersection**[1] **of X and Y** is the set $X \cap Y$ defined by

$$X \cap Y \stackrel{\text{def}}{=} \bigcap_{z\in\{X,Y\}} z.$$

[1] *Further Terminology:* Also called the **binary intersection of X and Y**, for emphasis.

Proposition 3.8.2 ► Properties of Binary Intersections

Let X be a set.

1. *Functoriality.* The assignments $U, V, (U, V) \mapsto U \cap V$ define functors

$$U \cap -\colon (\mathcal{P}(X), \subset) \to (\mathcal{P}(X), \subset),$$
$$- \cap V\colon (\mathcal{P}(X), \subset) \to (\mathcal{P}(X), \subset),$$
$$-_1 \cap -_2\colon (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \subset) \to (\mathcal{P}(X), \subset),$$

where $-_1 \cap -_2$ is the functor where

- *Action on Objects.* For each $(U, V) \in \mathcal{P}(X) \times \mathcal{P}(X)$, we have

$$[-_1 \cap -_2](U, V) \overset{\text{def}}{=} U \cap V;$$

- *Action on Morphisms.* For each pair of morphisms

$$\iota_U\colon U \hookrightarrow U',$$
$$\iota_V\colon V \hookrightarrow V'$$

of $\mathcal{P}(X) \times \mathcal{P}(X)$, the image

$$\iota_U \cap \iota_V\colon U \cap V \hookrightarrow U' \cap V'$$

of (ι_U, ι_V) by $\cap$ is the inclusion

$$U \cap V \subset U' \cap V'$$

i.e. where we have

($\star$) If $U \subset U'$ and $V \subset V'$, then $U \cap V \subset U' \cap V'$;

and where $U \cap -$ and $- \cap V$ are the partial functors of $-_1 \cap -_2$ at $U, V \in \mathcal{P}(X)$.

2. *Adjointness.* We have adjunctions

$$(U \cap - \dashv \mathbf{Hom}_{\mathcal{P}(X)}(U, -))\colon \quad \begin{tikzcd} \mathcal{P}(X) \arrow[r, bend left, "U\cap-"] \arrow[r, phantom, "\perp"] & \mathcal{P}(X), \arrow[l, bend left, "\mathbf{Hom}_{\mathcal{P}(X)}(U{,}-)"] \end{tikzcd}$$

$$(- \cap V \dashv \mathbf{Hom}_{\mathcal{P}(X)}(V, -))\colon \quad \begin{tikzcd} \mathcal{P}(X) \arrow[r, bend left, "-\cap V"] \arrow[r, phantom, "\perp"] & \mathcal{P}(X), \arrow[l, bend left, "\mathbf{Hom}_{\mathcal{P}(X)}(V{,}-)"] \end{tikzcd}$$

where

$$\mathbf{Hom}_{\mathcal{P}(X)}(-_1,-_2)\colon \mathcal{P}(X)^{\mathsf{op}}\times\mathcal{P}(X)\to\mathcal{P}(X)$$

is the bifunctor defined by[1]

$$\mathbf{Hom}_{\mathcal{P}(X)}(U,V)\overset{\text{def}}{=}(X\setminus U)\cup V$$

witnessed by bijections

$$\begin{aligned}\mathrm{Hom}_{\mathcal{P}(X)}(U\cap V,W)&\cong \mathrm{Hom}_{\mathcal{P}(X)}\big(U,\mathbf{Hom}_{\mathcal{P}(X)}(V,W)\big),\\ \mathrm{Hom}_{\mathcal{P}(X)}(U\cap V,W)&\cong \mathrm{Hom}_{\mathcal{P}(X)}\big(V,\mathbf{Hom}_{\mathcal{P}(X)}(U,W)\big),\end{aligned}$$

natural in $U,V,W\in\mathcal{P}(X)$, i.e. where:

(a) The following conditions are equivalent:
   i. We have $U\cap V\subset W$.
   ii. We have $U\subset \mathbf{Hom}_{\mathcal{P}(X)}(V,W)$.
   iii. We have $U\subset (X\setminus V)\cup W$.

(b) The following conditions are equivalent:
   i. We have $V\cap U\subset W$.
   ii. We have $V\subset \mathbf{Hom}_{\mathcal{P}(X)}(U,W)$.
   iii. We have $V\subset (X\setminus U)\cup W$.

3. *Associativity.* We have an equality of sets

$$(U\cap V)\cap W=U\cap(V\cap W)$$

for each $X\in\mathrm{Obj}(\mathsf{Sets})$ and each $U,V,W\in\mathcal{P}(X)$.

4. *Unitality.* Let X be a set and let $U\in\mathcal{P}(X)$. We have equalities of sets

$$\begin{aligned}X\cap U&=U,\\ U\cap X&=U\end{aligned}$$

for each $X\in\mathrm{Obj}(\mathsf{Sets})$ and each $U\in\mathcal{P}(X)$.

5. *Commutativity.* We have an equality of sets
$$U \cap V = V \cap U$$
for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

6. *Idempotency.* We have an equality of sets
$$U \cap U = U$$
for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

7. *Distributivity Over Unions.* We have equalities of sets
$$\begin{aligned} U \cap (V \cup W) &= (U \cap V) \cup (U \cap W), \\ (U \cup V) \cap W &= (U \cap W) \cup (V \cap W) \end{aligned}$$
for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

8. *Annihilation With the Empty Set.* We have an equality of sets
$$\begin{aligned} \emptyset \cap X &= \emptyset, \\ X \cap \emptyset &= \emptyset \end{aligned}$$
for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

9. *Interaction With Characteristic Functions I.* We have
$$\chi_{U \cap V} = \chi_U \chi_V$$
for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

10. *Interaction With Characteristic Functions II.* We have
$$\chi_{U \cap V} = \min(\chi_U, \chi_V)$$
for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

11. *Interaction With Powersets and Monoids With Zero.* The quadruple $((\mathcal{P}(X), \emptyset), \cap, X)$ is a commutative monoid with zero.

12. *Interaction With Powersets and Semirings.* The quintuple $(\mathcal{P}(X), \cup, \cap, \emptyset, X)$ is an idempotent commutative semiring.

---

[1] *Intuition:* Since intersections are the products in $\mathcal{P}(X)$, the left adjoint $\mathbf{Hom}_{\mathcal{P}(X)}(U, V)$ works as a function type $U \to V$.

Now, under the Curry–Howard correspondence, the function type $U \to V$ corresponds to implication $U \implies V$, which is logically equivalent to the statement $\neg U \vee V$, which in turn corresponds to the set $U^{\mathsf{c}} \vee V \overset{\text{def}}{=} (X \setminus U) \cup V$.

Proof 3.8.3 ► Proof of Proposition 3.8.2

Item 1: Functoriality

See [Pro24ah].

Item 2: Adjointness

See [MSE 267469].

Item 3: Associativity

See [Pro24q].

Item 4: Unitality

This follows from [Pro24u] and Item 5.

Item 5: Commutativity

See [Pro24r].

Item 6: Idempotency

See [Pro24ag].

Item 7: Distributivity Over Unions

See [Pro24af].

Item 8: Annihilation With the Empty Set

This follows from [Pro24s] and Item 5.

Item 9: Interaction With Characteristic Functions I

See [Pro24h].

Item 10: Interaction With Characteristic Functions II

See [Pro24h].

Item 11: Interaction With Powersets and Monoids With Zero

This follows from Items 3 to 5 and 8.

Item 12: Interaction With Powersets and Semirings

This follows from Items 3 to 6 and Items 3 to 5, 7 and 8 of Proposition 3.8.2. ∎

3.9 Differences

Let X and Y be sets.

Definition 3.9.1 ► Differences

The **difference of** X **and** Y is the set $X \setminus Y$ defined by

$$X \setminus Y \overset{\text{def}}{=} \{a \in X \mid a \notin Y\}.$$

Proposition 3.9.2 ► Properties of Differences

Let X be a set.

1. *Functoriality.* The assignments $U, V, (U, V) \mapsto U \cap V$ define functors

$$\begin{aligned} U \setminus -\colon (\mathcal{P}(X), \supset) &\to (\mathcal{P}(X), \subset), \\ - \setminus V\colon (\mathcal{P}(X), \subset) &\to (\mathcal{P}(X), \subset), \\ -_1 \setminus -_2\colon (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \supset) &\to (\mathcal{P}(X), \subset), \end{aligned}$$

   where $-_1 \setminus -_2$ is the functor where

   - *Action on Objects.* For each $(U, V) \in \mathcal{P}(X) \times \mathcal{P}(X)$, we have

$$[-_1 \setminus -_2](U, V) \overset{\text{def}}{=} U \setminus V;$$

   - *Action on Morphisms.* For each pair of morphisms

$$\begin{aligned} \iota_A &\colon A \hookrightarrow B, \\ \iota_U &\colon U \hookrightarrow V \end{aligned}$$

     of $\mathcal{P}(X) \times \mathcal{P}(X)$, the image

$$\iota_U \setminus \iota_V \colon A \setminus V \hookrightarrow B \setminus U$$

     of (ι_U, ι_V) by $\setminus$ is the inclusion

$$A \setminus V \subset B \setminus U$$

     i.e. where we have

     $(\star)$ If $A \subset B$ and $U \subset V$, then $A \setminus V \subset B \setminus U$;

   and where $U \setminus -$ and $- \setminus V$ are the partial functors of $-_1 \setminus -_2$ at $U, V \in \mathcal{P}(X)$.

2. *De Morgan's Laws.* We have equalities of sets

$$\begin{aligned} X \setminus (U \cup V) &= (X \setminus U) \cap (X \setminus V), \\ X \setminus (U \cap V) &= (X \setminus U) \cup (X \setminus V) \end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

3. *Interaction With Unions I.* We have equalities of sets

$$U \setminus (V \cup W) = (U \setminus V) \cap (U \setminus W)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

4. *Interaction With Unions II.* We have equalities of sets

$$(U \setminus V) \cup W = (U \cup W) \setminus (V \setminus W)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

5. *Interaction With Unions III.* We have equalities of sets

$$\begin{aligned} U \setminus (V \cup W) &= (U \cup W) \setminus (V \cup W) \\ &= (U \setminus V) \setminus W \\ &= (U \setminus W) \setminus V \end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

6. *Interaction With Unions IV.* We have equalities of sets

$$(U \cup V) \setminus W = (U \setminus W) \cup (V \setminus W)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

7. *Interaction With Intersections.* We have equalities of sets

$$\begin{aligned} (U \setminus V) \cap W &= (U \cap W) \setminus V \\ &= U \cap (W \setminus V) \end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

8. *Interaction With Complements.* We have an equality of sets
$$U \setminus V = U \cap V^{\mathsf{c}}$$
for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

9. *Interaction With Symmetric Differences.* We have an equality of sets
$$U \setminus V = U \vartriangle (U \cap V)$$
for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

10. *Triple Differences.* We have
$$U \setminus (V \setminus W) = (U \cap W) \cup (U \setminus V)$$
for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

11. *Left Annihilation.* We have
$$\emptyset \setminus U = \emptyset$$
for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

12. *Right Unitality.* We have
$$U \setminus \emptyset = U$$
for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

13. *Invertibility.* We have
$$U \setminus U = \emptyset$$
for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

14. *Interaction With Containment.* The following conditions are equivalent:
    (a) We have $V \setminus U \subset W$.
    (b) We have $V \setminus W \subset U$.

15. *Interaction With Characteristic Functions.* We have
$$\chi_{U \setminus V} = \chi_U - \chi_{U \cap V}$$
for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

PROOF 3.9.3 ► PROOF OF PROPOSITION 3.9.2

Item 1: Functoriality

See [Pro24z] and [Pro24ad].

Item 2: De Morgan's Laws

See [Pro24m].

Item 3: Interaction With Unions I

See [Pro24n].

Item 4: Interaction With Unions II

Omitted.

Item 5: Interaction With Unions III

See [Pro24ae].

Item 6: Interaction With Unions IV

See [Pro24y].

Item 7: Interaction With Intersections

See [Pro24t].

Item 8: Interaction With Complements

See [Pro24w].

Item 9: Interaction With Symmetric Differences

See [Pro24x].

Item 10: Triple Differences

See [Pro24ac].

Item 11: Left Annihilation

Clear.

Item 12: Right Unitality

See [Pro24aa].

Item 13: Invertibility

See [Pro24ab].

Item 14: Interaction With Containment

Omitted.

Item 15: Interaction With Characteristic Functions

See [Pro24i].

3.10 Complements

Let X be a set and let $U \in \mathcal{P}(X)$.

Definition 3.10.1 ▶ Complements

The **complement of** U is the set U^{c} defined by

$$\begin{aligned} U^{\mathsf{c}} &\overset{\text{def}}{=} X \setminus U \\ &\overset{\text{def}}{=} \{a \in X \mid a \notin U\}. \end{aligned}$$

Proposition 3.10.2 ▶ Properties of Complements

Let X be a set.

1. *Functoriality.* The assignment $U \mapsto U^{\mathsf{c}}$ defines a functor

$$(-)^{\mathsf{c}} \colon \mathcal{P}(X)^{\mathsf{op}} \to \mathcal{P}(X),$$

where

- *Action on Objects.* For each $U \in \mathcal{P}(X)$, we have

$$[(-)^{\mathsf{c}}](U) \overset{\text{def}}{=} U^{\mathsf{c}};$$

- *Action on Morphisms.* For each morphism $\iota_U \colon U \hookrightarrow V$ of $\mathcal{P}(X)$, the image

$$\iota_U^{\mathsf{c}} \colon V^{\mathsf{c}} \hookrightarrow U^{\mathsf{c}}$$

of ι_U by $(-)^{\mathsf{c}}$ is the inclusion

$$V^{\mathsf{c}} \subset U^{\mathsf{c}}$$

i.e. where we have

($\star$) If $U \subset V$, then $V^{\mathsf{c}} \subset U^{\mathsf{c}}$.

2. *De Morgan's Laws.* We have equalities of sets

$$(U \cup V)^{\mathsf{c}} = U^{\mathsf{c}} \cap V^{\mathsf{c}},$$
$$(U \cap V)^{\mathsf{c}} = U^{\mathsf{c}} \cup V^{\mathsf{c}}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

3. *Involutority.* We have

$$(U^{\mathsf{c}})^{\mathsf{c}} = U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

4. *Interaction With Characteristic Functions.* We have

$$\chi_{U^{\mathsf{c}}} = 1 - \chi_U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

Proof 3.10.3 ► Proof of Proposition 3.10.2

Item 1: Functoriality

This follows from Item 1 of Proposition 3.9.2.

Item 2: De Morgan's Laws

See [Pro24m].

Item 3: Involutority

See [Pro24l].

Item 4: Interaction With Characteristic Functions

Clear.

3.11 Symmetric Differences

Let A and B be sets.

Definition 3.11.1 ► Symmetric Differences

The **symmetric difference of** A **and** B is the set $A \triangle B$ defined by

$$A \triangle B \overset{\text{def}}{=} (A \setminus B) \cup (B \setminus A).$$

Proposition 3.11.2 ► Properties of Symmetric Differences

Let X be a set.

1. *Lack of Functoriality.* The assignment $(U, V) \mapsto U \triangle V$ ***need not*** define functors

$$\begin{aligned} U \triangle -_2 &\colon (\mathcal{P}(X), \subset) \to (\mathcal{P}(X), \subset), \\ -_1 \triangle V &\colon (\mathcal{P}(X), \subset) \to (\mathcal{P}(X), \subset), \\ -_1 \triangle -_2 &\colon (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \subset) \to (\mathcal{P}(X), \subset). \end{aligned}$$

2. *Via Unions and Intersections.* We have[1]

$$U \triangle V = (U \cup V) \setminus (U \cap V)$$

for each $X \in \text{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

3. *Associativity.* We have[2]

$$(U \triangle V) \triangle W = U \triangle (V \triangle W)$$

for each $X \in \text{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

4. *Commutativity.* We have

$$U \triangle V = V \triangle U$$

for each $X \in \text{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

5. *Unitality.* We have

$$\begin{aligned} U \triangle \emptyset &= U, \\ \emptyset \triangle U &= U \end{aligned}$$

for each $X \in \text{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

6. *Invertibility*. We have
$$U \triangle U = \emptyset$$
for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

7. *Interaction With Unions*. We have
$$(U \triangle V) \cup (V \triangle T) = (U \cup V \cup W) \setminus (U \cap V \cap W)$$
for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

8. *Interaction With Complements I*. We have
$$U \triangle U^{\mathsf{c}} = X$$
for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

9. *Interaction With Complements II*. We have
$$\begin{aligned} U \triangle X &= U^{\mathsf{c}}, \\ X \triangle U &= U^{\mathsf{c}} \end{aligned}$$
for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

10. *Interaction With Complements III*. We have
$$U^{\mathsf{c}} \triangle V^{\mathsf{c}} = U \triangle V$$
for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

11. *"Transitivity"*. We have
$$(U \triangle V) \triangle (V \triangle W) = U \triangle W$$
for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

12. *The Triangle Inequality for Symmetric Differences*. We have
$$U \triangle W \subset U \triangle V \cup V \triangle W$$
for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

13. *Distributivity Over Intersections.* We have

$$\begin{aligned} U \cap (V \triangle W) &= (U \cap V) \triangle (U \cap W), \\ (U \triangle V) \cap W &= (U \cap W) \triangle (V \cap W) \end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

14. *Interaction With Characteristic Functions.* We have

$$\chi_{U \triangle V} = \chi_U + \chi_V - 2\chi_{U \cap V}$$

and thus, in particular, we have

$$\chi_{U \triangle V} \equiv \chi_U + \chi_V \mod 2$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

15. *Bijectivity.* Given $A, B \subset \mathcal{P}(X)$, the maps

$$\begin{aligned} A \triangle -&\colon \mathcal{P}(X) \to \mathcal{P}(X), \\ - \triangle B&\colon \mathcal{P}(X) \to \mathcal{P}(X) \end{aligned}$$

are bijections with inverses given by

$$\begin{aligned} (A \triangle -)^{-1} &= - \cup (A \cap -), \\ (- \triangle B)^{-1} &= - \cup (B \cap -). \end{aligned}$$

Moreover, the map

$$C \mapsto C \triangle (A \triangle B)$$

is a bijection of $\mathcal{P}(X)$ onto itself sending A to B and B to A.

16. *Interaction With Powersets and Groups.* Let X be a set.

    (a) The quadruple $\left(\mathcal{P}(X), \triangle, \emptyset, \mathrm{id}_{\mathcal{P}(X)}\right)$ is an abelian group.[3]

    (b) Every element of $\mathcal{P}(X)$ has order 2 with respect to $\triangle$, and thus $\mathcal{P}(X)$ is a *Boolean group* (i.e. an abelian 2-group).

17. *Interaction With Powersets and Vector Spaces I.* The pair $\left(\mathcal{P}(X), \alpha_{\mathcal{P}(X)}\right)$ consisting of

- The group $\mathcal{P}(X)$ of **??**;
- The map $\alpha_{\mathcal{P}(X)}\colon \mathbb{F}_2 \times \mathcal{P}(X) \to \mathcal{P}(X)$ defined by

$$\begin{aligned} 0 \cdot U &\overset{\text{def}}{=} \emptyset, \\ 1 \cdot U &\overset{\text{def}}{=} U; \end{aligned}$$

is an $\mathbb{F}_2$-vector space.

18. *Interaction With Powersets and Vector Spaces II.* If X is finite, then:

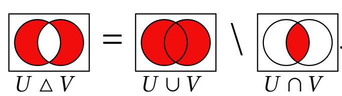
    (a) The set of singletons sets on the elements of X forms a basis for the $\mathbb{F}_2$-vector space $\big(\mathcal{P}(X), \alpha_{\mathcal{P}(X)}\big)$ of Item 17.

    (b) We have
    $$\dim(\mathcal{P}(X)) = \#\mathcal{P}(X).$$

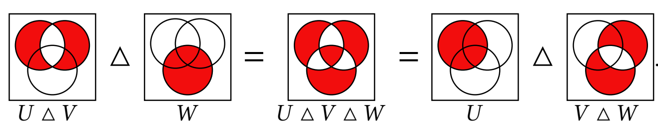
19. *Interaction With Powersets and Rings.* The quintuple $(\mathcal{P}(X), \triangle, \cap, \emptyset, X)$ is a commutative ring.[4]

---

[1] *Illustration:*



[2] *Illustration:*



[3] Here are some examples:

i. When $X = \emptyset$, we have an isomorphism of groups between $\mathcal{P}(\emptyset)$ and the trivial group:
$$\Big(\mathcal{P}(\emptyset), \triangle, \emptyset, \mathrm{id}_{\mathcal{P}(\emptyset)}\Big) \cong \mathrm{pt}.$$

ii. When $X = \mathrm{pt}$, we have an isomorphism of groups between $\mathcal{P}(\mathrm{pt})$ and $\mathbb{Z}_{/2}$:
$$\Big(\mathcal{P}(\mathrm{pt}), \triangle, \emptyset, \mathrm{id}_{\mathcal{P}(\mathrm{pt})}\Big) \cong \mathbb{Z}_{/2}.$$

iii. When $X = \{0, 1\}$, we have an isomorphism of groups between $\mathcal{P}(\{0,1\})$ and $\mathbb{Z}_{/2} \times \mathbb{Z}_{/2}$:
$$\Big(\mathcal{P}(\{0,1\}), \triangle, \emptyset, \mathrm{id}_{\mathcal{P}(\{0,1\})}\Big) \cong \mathbb{Z}_{/2} \times \mathbb{Z}_{/2}.$$

[4] *Warning:* The analogous statement replacing intersections by unions (i.e. that the quintuple $(\mathcal{P}(X), \triangle, \cup, \emptyset, X)$ is a ring) is false, however. See [Pro24as] for a proof.

Proof 3.11.3 ▶ Proof of Proposition 3.11.2

Item 1: Lack of Functoriality

Omitted.

Item 2: Via Unions and Intersections

See [Pro24o].

Item 3: Associativity

See [Pro24ak].

Item 4: Commutativity

See [Pro24al].

Item 5: Unitality

This follows from Item 4 and [Pro24ap].

Item 6: Invertibility

See [Pro24ar].

Item 7: Interaction With Unions

See [Pro24ay].

Item 8: Interaction With Complements I

See [Pro24ao].

Item 9: Interaction With Complements II

This follows from Item 4 and [Pro24at].

Item 10: Interaction With Complements III

See [Pro24am].

Item 11: "Transitivity"

We have

$$\begin{aligned}
(U \mathbin{\triangle} V) \mathbin{\triangle} (V \mathbin{\triangle} W) &= U \mathbin{\triangle} (V \mathbin{\triangle} (V \mathbin{\triangle} W)) && \text{(by Item 3)}\\
&= U \mathbin{\triangle} ((V \mathbin{\triangle} V) \mathbin{\triangle} W) && \text{(by Item 3)}\\
&= U \mathbin{\triangle} (\emptyset \mathbin{\triangle} W) && \text{(by Item 6)}\\
&= U \mathbin{\triangle} W && \text{(by Item 5)}
\end{aligned}$$

Item 12: The Triangle Inequality for Symmetric Differences

This follows from Items 2 and 11.

Item 13: Distributivity Over Intersections

See [Pro24p].

Item 14: Interaction With Characteristic Functions

See [Pro24j].

Item 15: Bijectivity

Clear.

Item 16: Interaction With Powersets and Groups

Item 16a follows from[1] Items 3 to 6, while Item 16b follows from Item 6.

Item 17: Interaction With Powersets and Vector Spaces I

Clear.

Item 18: Interaction With Powersets and Vector Spaces II

Omitted.

Item 19: Interaction With Powersets and Rings

This follows from Items 8 and 11 of Proposition 3.8.2 and Items 13 and 16.[2]

[1] Reference: [Pro24an].
[2] *Reference:* [Pro24aq].

4 Powersets

4.1 Characteristic Functions

Let X be a set.

Definition 4.1.1 ▶ Characteristic Functions

Let $U \subset X$ and let $x \in X$.

1. The **characteristic function of** U[1] is the function[2]

$$\chi_U : X \to \{\mathrm{t}, \mathrm{f}\}$$

defined by

$$\chi_U(x) \stackrel{\text{def}}{=} \begin{cases} \text{true} & \text{if } x \in U, \\ \text{false} & \text{if } x \notin U \end{cases}$$

for each $x \in X$.

2. The **characteristic function of** x is the function[3]

$$\chi_x \colon X \longrightarrow \{\mathsf{t}, \mathsf{f}\}$$

defined by

$$\chi_x \stackrel{\text{def}}{=} \chi_{\{x\}},$$

i.e. by

$$\chi_x(y) \stackrel{\text{def}}{=} \begin{cases} \text{true} & \text{if } x = y, \\ \text{false} & \text{if } x \neq y \end{cases}$$

for each $y \in X$.

3. The **characteristic relation on** X[4] is the relation[5]

$$\chi_X(-_1, -_2) \colon X \times X \longrightarrow \{\mathsf{t}, \mathsf{f}\}$$

on X defined by[6]

$$\chi_X(x, y) \stackrel{\text{def}}{=} \begin{cases} \text{true} & \text{if } x = y, \\ \text{false} & \text{if } x \neq y \end{cases}$$

for each $x, y \in X$.

4. The **characteristic embedding**[7] **of** X **into** $\mathcal{P}(X)$ is the function

$$\chi_{(-)} \colon X \hookrightarrow \mathcal{P}(X)$$

defined by

$$\chi_{(-)}(x) \stackrel{\text{def}}{=} \chi_x$$

for each $x \in X$.

[1] *Further Terminology:* Also called the **indicator function of** U.
[2] *Further Notation:* Also written $\chi_X(U,-)$ or $\chi_X(-,U)$.
[3] *Further Notation:* Also written χ_x, $\chi_X(x,-)$, or $\chi_X(-,x)$.
[4] *Further Terminology:* Also called the **identity relation on** X.
[5] *Further Notation:* Also written $\chi_{-_2}^{-_1}$, or $\sim_{\mathrm{id}}$ in the context of relations.
[6] As a subset of $X\times X$, the relation χ_X corresponds to the diagonal $\Delta_X \subset X\times X$ of X.
[7] The name "characteristic *embedding*" comes from the fact that there is an analogue of fully faithfulness for $\chi_{(-)}$: given a set X, we have

$$\mathrm{Hom}_{\mathcal{P}(X)}(\chi_x,\chi_y) = \chi_X(x,y),$$

for each $x,y\in X$.

Remark 4.1.2 ► Characteristic Functions as Decategorifications of Presheaves

The definitions in Definition 4.1.1 are decategorifications of co/presheaves, representable co/presheaves, Hom profunctors, and the Yoneda embedding:[1]

1. A function
$$f\colon X\to\{\mathsf{t},\mathsf{f}\}$$
is a decategorification of a presheaf
$$\mathcal{F}\colon \mathcal{C}^{\mathsf{op}}\to\mathsf{Sets},$$
with the characteristic functions χ_U of the subsets of X being the primordial examples (and, in fact, all examples) of these.

2. The characteristic function
$$\chi_x\colon X\to\{\mathsf{t},\mathsf{f}\}$$
of an *element* x of X is a decategorification of the representable presheaf
$$h_X\colon \mathcal{C}^{\mathsf{op}}\to\mathsf{Sets}$$
of an *object* x of a category $\mathcal{C}$.

3. The characteristic relation
$$\chi_X(-_1,-_2)\colon X\times X\to\{\mathsf{t},\mathsf{f}\}$$
of X is a decategorification of the Hom profunctor
$$\mathrm{Hom}_{\mathcal{C}}(-_1,-_2)\colon \mathcal{C}^{\mathsf{op}}\times\mathcal{C}\to\mathsf{Sets}$$
of a category $\mathcal{C}$.

4. The characteristic embedding

$$\chi_{(-)}\colon X \hookrightarrow \mathcal{P}(X)$$

of X into $\mathcal{P}(X)$ is a decategorification of the Yoneda embedding

$$よ\colon \mathcal{C}^{\mathsf{op}} \hookrightarrow \mathsf{PSh}(\mathcal{C})$$

of a category $\mathcal{C}$ into $\mathsf{PSh}(\mathcal{C})$.

5. There is also a direct parallel between unions and colimits:

   - An element of $\mathcal{P}(X)$ is a union of elements of X, viewed as one-point subsets $\{x\} \in \mathcal{P}(A)$;
   - An object of $\mathsf{PSh}(\mathcal{C})$ is a colimit of objects of $\mathcal{C}$, viewed as representable presheaves $h_X \in \mathsf{Obj}(\mathsf{PSh}(\mathcal{C}))$.

[1] These statements can be made precise by using the embeddings

$$(-)_{\mathsf{disc}}\colon \mathsf{Sets} \hookrightarrow \mathsf{Cats},$$
$$(-)_{\mathsf{disc}}\colon \{\mathsf{t},\mathsf{f}\}_{\mathsf{disc}} \hookrightarrow \mathsf{Sets}$$

of sets into categories and of classical truth values into sets.

For instance, in this approach the characteristic function

$$\chi_x\colon X \to \{\mathsf{t},\mathsf{f}\}$$

of an element x of X, defined by

$$\chi_x(y) \overset{\text{def}}{=} \begin{cases} \mathsf{true} & \text{if } x = y, \\ \mathsf{false} & \text{if } x \neq y \end{cases}$$

for each $y \in X$, is recovered as the representable presheaf

$$\mathrm{Hom}_{X_{\mathsf{disc}}}(-,x)\colon X_{\mathsf{disc}} \to \mathsf{Sets}$$

of the corresponding object x of X_{disc}, defined on objects by

$$\mathrm{Hom}_{X_{\mathsf{disc}}}(y,x) \overset{\text{def}}{=} \begin{cases} \mathrm{pt} & \text{if } x = y, \\ \emptyset & \text{if } x \neq y \end{cases}$$

for each $y \in \mathsf{Obj}(X_{\mathsf{disc}})$.

Proposition 4.1.3 ▶ Properties of Characteristic Functions

Let X be a set.

1. *The Inclusion of Characteristic Relations Associated to a Function.* Let $f \colon A \to B$ be a function. We have an inclusion[1]

$$\chi_B \circ (f \times f) \subset \chi_A, \qquad \begin{tikzcd} A \times A \arrow[rr, "f \times f"] \arrow[dr, "\chi_A"'] & \arrow[d, phantom, "\supset"] & B \times B \arrow[dl, "\chi_B"] \\ & \{\mathsf{t}, \mathsf{f}\}. & \end{tikzcd}$$

2. *Interaction With Unions I.* We have

$$\chi_{U \cup V} = \max(\chi_U, \chi_V)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

3. *Interaction With Unions II.* We have

$$\chi_{U \cup V} = \chi_U + \chi_V - \chi_{U \cap V}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

4. *Interaction With Intersections I.* We have

$$\chi_{U \cap V} = \chi_U \chi_V$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

5. *Interaction With Intersections II.* We have

$$\chi_{U \cap V} = \min(\chi_U, \chi_V)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

6. *Interaction With Differences.* We have

$$\chi_{U \setminus V} = \chi_U - \chi_{U \cap V}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

7. *Interaction With Complements.* We have

$$\chi_{U^{\mathsf{c}}} = 1 - \chi_U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

8. *Interaction With Symmetric Differences.* We have

$$\chi_{U \triangle V} = \chi_U + \chi_V - 2\chi_{U\cap V}$$

and thus, in particular, we have

$$\chi_{U \triangle V} \equiv \chi_U + \chi_V \mod 2$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

[1] This is the 0-categorical version of Categories, **??**.

Proof 4.1.4 ► Proof of Proposition 4.1.3

Item 1: The Inclusion of Characteristic Relations Associated to a Function

The inclusion $\chi_B(f(a), f(b)) \subset \chi_A(a, b)$ is equivalent to the statement "if $a = b$, then $f(a) = f(b)$", which is true.

Item 2: Interaction With Unions I

This is a repetition of Item 8 of Proposition 3.6.2 and is proved there.

Item 3: Interaction With Unions II

This is a repetition of Item 9 of Proposition 3.6.2 and is proved there.

Item 4: Interaction With Intersections I

This is a repetition of Item 9 of Proposition 3.8.2 and is proved there.

Item 5: Interaction With Intersections II

This is a repetition of Item 10 of Proposition 3.8.2 and is proved there.

Item 6: Interaction With Differences

This is a repetition of Item 15 of Proposition 3.9.2 and is proved there.

Item 7: Interaction With Complements

This is a repetition of Item 4 of Proposition 3.10.2 and is proved there.

Item 8: Interaction With Symmetric Differences

This is a repetition of Item 14 of Proposition 3.11.2 and is proved there.

4.2 The Yoneda Lemma for Sets

Let X be a set and let $U \subset X$ be a subset of X.

Proposition 4.2.1 ► The Yoneda Lemma for Sets

We have

$$\chi_{\mathcal{P}(X)}(\chi_x, \chi_U) = \chi_U(x)$$

for each $x \in X$, giving an equality of functions

$$\chi_{\mathcal{P}(X)}\big(\chi_{(-)}, \chi_U\big) = \chi_U.$$

Proof 4.2.2 ► Proof of Proposition 4.2.1

Clear.

Corollary 4.2.3 ► The Characteristic Embedding Is Fully Faithful

The characteristic embedding is fully faithful, i.e., we have

$$\chi_{\mathcal{P}(X)}\big(\chi_x, \chi_y\big) = \chi_X(x, y)$$

for each $x, y \in X$.

Proof 4.2.4 ► Proof of Corollary 4.2.3

This follows from Proposition 4.2.1.

4.3 Powersets

Let X be a set.

Definition 4.3.1 ▶ Powersets

The **powerset of** X is the set $\mathcal{P}(X)$ defined by

$$\mathcal{P}(X) \stackrel{\text{def}}{=} \{U \in P \mid U \subset X\},$$

where P is the set in the axiom of powerset, **??** of **??**.

Remark 4.3.2 ▶ Powersets as Decategorifications of Co/Presheaf Categories

The powerset of a set is a decategorification of the category of presheaves of a category: while[1]

- The powerset of a set X is equivalently (Item 6 of Proposition 4.3.3) the set

  $$\mathrm{Sets}(X, \{\mathrm{t}, \mathrm{f}\})$$

  of functions from X to the set $\{\mathrm{t}, \mathrm{f}\}$ of classical truth values;
- The category of presheaves on a category $\mathcal{C}$ is the category

  $$\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sets})$$

  of functors from $\mathcal{C}^{\mathrm{op}}$ to the category Sets of sets.

[1] This parallel is based on the following comparison:

- A category is enriched over the category

  $$\mathrm{Sets} \stackrel{\text{def}}{=} \mathrm{Cats}_0$$

  of sets (i.e. "0-categories"), with presheaves taking values on it;
- A set is enriched over the set

  $$\{\mathrm{t}, \mathrm{f}\} \stackrel{\text{def}}{=} \mathrm{Cats}_{-1}$$

  of classical truth values (i.e. "(-1)-categories"), with characteristic functions taking values on it.

Proposition 4.3.3 ▶ Properties of Powersets

Let X be a set.

1. *Functoriality.* The assignment $X \mapsto \mathcal{P}(X)$ defines functors

$$\begin{aligned}\mathcal{P}_* \colon \mathsf{Sets} &\longrightarrow \mathsf{Sets},\\ \mathcal{P}^{-1} \colon \mathsf{Sets}^{\mathsf{op}} &\longrightarrow \mathsf{Sets},\\ \mathcal{P}_! \colon \mathsf{Sets} &\longrightarrow \mathsf{Sets}\end{aligned}$$

   where

   - *Action on Objects.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

   $$\begin{aligned}\mathcal{P}_*(A) &\stackrel{\text{def}}{=} \mathcal{P}(A),\\ \mathcal{P}^{-1}(A) &\stackrel{\text{def}}{=} \mathcal{P}(A),\\ \mathcal{P}_!(A) &\stackrel{\text{def}}{=} \mathcal{P}(A);\end{aligned}$$

   - *Action on Morphisms.* For each morphism $f \colon A \longrightarrow B$ of Sets, the images

   $$\begin{aligned}\mathcal{P}_*(f) \colon \mathcal{P}(A) &\longrightarrow \mathcal{P}(B),\\ \mathcal{P}^{-1}(f) \colon \mathcal{P}(B) &\longrightarrow \mathcal{P}(A),\\ \mathcal{P}_!(f) \colon \mathcal{P}(A) &\longrightarrow \mathcal{P}(B)\end{aligned}$$

   of f by $\mathcal{P}_*$, $\mathcal{P}^{-1}$, and $\mathcal{P}_!$ are defined by

   $$\begin{aligned}\mathcal{P}_*(f) &\stackrel{\text{def}}{=} f_*,\\ \mathcal{P}^{-1}(f) &\stackrel{\text{def}}{=} f^{-1},\\ \mathcal{P}_!(f) &\stackrel{\text{def}}{=} f_!,\end{aligned}$$

   as in Definitions 4.4.1, 4.5.1 and 4.6.1.

2. *Adjointness I.* We have an adjunction

$$\left(\mathcal{P}^{-1} \dashv \mathcal{P}^{-1,\mathsf{op}}\right) \colon \quad \begin{tikzcd} \mathsf{Sets}^{\mathsf{op}} \arrow[r, bend left, "\mathcal{P}^{-1}"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}, \arrow[l, bend left, "\mathcal{P}^{-1,\mathsf{op}}"] \end{tikzcd}$$

   witnessed by a bijection

$$\underbrace{\mathsf{Sets}^{\mathsf{op}}(\mathcal{P}(X), Y)}_{\stackrel{\text{def}}{=} \mathsf{Sets}(Y, \mathcal{P}(X))} \cong \mathsf{Sets}(X, \mathcal{P}(Y)),$$

   natural in $X \in \mathrm{Obj}(\mathsf{Sets})$ and $Y \in \mathrm{Obj}\left(\mathsf{Sets}^{\mathsf{op}}\right)$.

3. *Adjointness II.* We have an adjunction

$$(\mathsf{Gr} \dashv \mathcal{P}_*) \colon \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "\mathsf{Gr}"] \arrow[r, phantom, "\perp"] & \mathsf{Rel}, \arrow[l, bend left, "\mathcal{P}_*"] \end{tikzcd}$$

witnessed by a bijection of sets

$$\mathsf{Rel}(\mathsf{Gr}(A), B) \cong \mathsf{Sets}(A, \mathcal{P}(B))$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$ and $B \in \mathrm{Obj}(\mathsf{Rel})$, where Gr is the graph functor of Relations, Item 1 of Proposition 3.1.2.

4. *Symmetric Strong Monoidality With Respect to Coproducts.* The powerset functor $\mathcal{P}_*$ of Item 1 has a symmetric strong monoidal structure

$$\left(\mathcal{P}_*, \mathcal{P}_*^{\coprod}, \mathcal{P}_{*|\mathbb{1}}^{\coprod}\right) \colon (\mathsf{Sets}, \coprod, \emptyset) \longrightarrow (\mathsf{Sets}, \times, \mathrm{pt})$$

being equipped with isomorphisms

$$\mathcal{P}_{*|X,Y}^{\coprod} \colon \mathcal{P}(X) \times \mathcal{P}(Y) \xrightarrow{\cong} \mathcal{P}(X \textstyle\coprod Y),$$
$$\mathcal{P}_{*|\mathbb{1}}^{\coprod} \colon \mathrm{pt} \xrightarrow{=} \mathcal{P}(\emptyset),$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

5. *Symmetric Lax Monoidality With Respect to Products.* The powerset functor $\mathcal{P}_*$ of Item 1 has a symmetric lax monoidal structure

$$\left(\mathcal{P}_*, \mathcal{P}_*^{\otimes}, \mathcal{P}_{*|\mathbb{1}}^{\otimes}\right) \colon (\mathsf{Sets}, \times, \mathrm{pt}) \longrightarrow (\mathsf{Sets}, \times, \mathrm{pt})$$

being equipped with isomorphisms

$$\mathcal{P}_{*|X,Y}^{\otimes} \colon \mathcal{P}(X) \times \mathcal{P}(Y) \to \mathcal{P}(X \times Y),$$
$$\mathcal{P}_{*|\mathbb{1}}^{\otimes} \colon \mathrm{pt} \xrightarrow{=} \mathcal{P}(\emptyset),$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$, where $\mathcal{P}_{*|X,Y}^{\otimes}$ is given by

$$\mathcal{P}_{*|X,Y}^{\otimes}(U, V) \stackrel{\mathrm{def}}{=} U \times V$$

for each $(U, V) \in \mathcal{P}(X) \times \mathcal{P}(Y)$.

6. *Powersets as Sets of Functions.* The assignment $U \mapsto \chi_U$ defines a bijection[1]

$$\chi_{(-)} : \mathcal{P}(X) \xrightarrow{\cong} \mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\}),$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets})$.

7. *Powersets as Sets of Relations.* We have bijections

$$\begin{aligned}\mathcal{P}(X) &\cong \mathrm{Rel}(\mathrm{pt}, X),\\ \mathcal{P}(X) &\cong \mathrm{Rel}(X, \mathrm{pt}),\end{aligned}$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets})$.

8. *As a Free Cocompletion: Universal Property.* The pair $(\mathcal{P}(X), \chi_{(-)})$ consisting of

   - The powerset $\mathcal{P}(X)$ of X;
   - The characteristic embedding $\chi_{(-)} : X \hookrightarrow \mathcal{P}(X)$ of X into $\mathcal{P}(X)$;

   satisfies the following universal property:

   (★) Given another pair (Y, f) consisting of
   - A cocomplete poset $(Y, \preceq)$;
   - A function $f : X \to Y$;

   there exists a unique cocontinuous morphism of posets $(\mathcal{P}(X), \subset) \xrightarrow{\exists!} (Y, \preceq)$ making the diagram

$$\begin{tikzcd} & \mathcal{P}(X) \arrow[d, dashed, "\exists!"] \\ X \arrow[ur, "\chi_X"] \arrow[r, "f"'] & Y \end{tikzcd}$$

   commute.

9. *As a Free Cocompletion: Adjointness.* We have an adjunction[2]

$$(\chi_{(-)} \dashv 忘) : \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "\chi_{(-)}"] \arrow[r, phantom, "\perp"] & \mathsf{Pos}^{\mathsf{cocomp.}} \arrow[l, bend left, "忘"] \end{tikzcd},$$

witnessed by a bijection

$$\mathsf{Pos}^{\mathsf{cocomp.}}((\mathcal{P}(X), \subset), (Y, \preceq)) \cong \mathsf{Sets}(X, Y),$$

natural in $X \in \mathsf{Obj}(\mathsf{Sets})$ and $(Y, \preceq) \in \mathsf{Obj}(\mathsf{Pos})$, where

- We have a natural map

  $$\chi_X^* \colon \mathsf{Pos}^{\mathsf{cocomp.}}((\mathcal{P}(X), \subset), (Y, \preceq)) \to \mathsf{Sets}(X, Y)$$

  defined by

  $$\chi_X^*(f) \stackrel{\text{def}}{=} f \circ \chi_X,$$

  i.e. by sending a cocontinuous morphism of posets $f \colon \mathcal{P}(X) \to Y$ to the composition

  $$X \stackrel{\chi_X}{\hookrightarrow} \mathcal{P}(X) \xrightarrow{f} Y;$$

- We have a natural map

  $$\mathrm{Lan}_{\chi_X} \colon \mathsf{Sets}(X, Y) \to \mathsf{Pos}^{\mathsf{cocomp.}}((\mathcal{P}(X), \subset), (Y, \preceq))$$

  computed by

  $$\begin{aligned} [\mathrm{Lan}_{\chi_X}(f)](U) &\cong \int^{x \in X} \chi_{\mathcal{P}(X)}(\chi_x, U) \odot f(x) \\ &\cong \int^{x \in X} \chi_U(x) \odot f(x) && \text{(by Proposition 4.2.1)} \\ &\cong \bigvee_{x \in X} (\chi_U(x) \odot f(x)) \end{aligned}$$

  for each $U \in \mathcal{P}(X)$, where:

  - $\bigvee$ is the join in $(Y, \preceq)$;
  - We have

    $$\begin{aligned} \mathsf{true} \odot f(x) &\stackrel{\text{def}}{=} f(x), \\ \mathsf{false} \odot f(x) &\stackrel{\text{def}}{=} \emptyset_Y, \end{aligned}$$

    where $\emptyset_Y$ is the minimal element of $(Y, \preceq)$.

[1]This bijection is a decategorified form of the equivalence

$$\mathsf{PSh}(\mathcal{C}) \overset{\text{eq.}}{\cong} \mathsf{DFib}(\mathcal{C})$$

of Fibred Categories, **??** of **??**, with $\chi_{(-)}$ being a decategorified version of the category of elements construction of Fibred Categories, **??**.

See also **??** of **??**.

[2]In this sense, $\mathcal{P}(A)$ is the free cocompletion of A. (Note that, despite its name, however, this is not an idempotent operation, as we have $\mathcal{P}(\mathcal{P}(A)) \neq \mathcal{P}(A)$.)

PROOF 4.3.4 ► PROOF OF PROPOSITION 4.3.3

Item 1: Functoriality

This follows from Items 3 and 4 of Proposition 4.4.5, Items 3 and 4 of Proposition 4.5.5, and Items 3 and 4 of Proposition 4.6.7.

Item 2: Adjointness I

Omitted.

Item 3: Adjointness II

We have

$$\begin{aligned}
\mathrm{Rel}(\mathrm{Gr}(A), B) &= \mathcal{P}(A \times B) \\
&= \mathrm{Sets}(A \times B, \{\mathsf{t}, \mathsf{f}\}) && \text{(by Item 6)} \\
&= \mathrm{Sets}(A, \mathrm{Sets}(B, \{\mathsf{t}, \mathsf{f}\})) && \text{(by Item 2 of Proposition 1.2.3)} \\
&= \mathrm{Sets}(A, \mathcal{P}(B)) && \text{(by Item 6)}
\end{aligned}$$

with all bijections natural in A and B.

Item 4: Symmetric Strong Monoidality With Respect to Coproducts

Omitted.

Item 5: Symmetric Lax Monoidality With Respect to Products

Omitted.

Item 6: Powersets as Sets of Functions

Omitted.

Item 7: Powersets as Sets of Relations

Omitted.

Item 8: As a Free Cocompletion: Universal Property

This is a rephrasing of **??**.

Item 9: As a Free Cocompletion: Adjointness

Omitted.

4.4 Direct Images

Let A and B be sets and let $f\colon A \to B$ be a function.

Definition 4.4.1 ▶ Direct Images

The **direct image function associated to** f is the function[1]

$$f_*\colon \mathcal{P}(A) \to \mathcal{P}(B)$$

defined by[2,3]

$$\begin{aligned} f_*(U) &\overset{\text{def}}{=} f(U) \\ &\overset{\text{def}}{=} \left\{ b \in B \,\middle|\, \begin{array}{l} \text{there exists some } a \in \\ U \text{ such that } b = f(a) \end{array} \right\} \\ &= \{ f(a) \in B \mid a \in U \} \end{aligned}$$

for each $U \in \mathcal{P}(A)$.

[1] *Further Notation:* Also written $\exists_f\colon \mathcal{P}(A) \to \mathcal{P}(B)$. This notation comes from the fact that the following statements are equivalent, where $b \in B$ and $U \in \mathcal{P}(A)$:

- We have $b \in \exists_f(U)$.
- There exists some $a \in U$ such that $f(a) = b$.

[2] *Further Terminology:* The set $f(U)$ is called the **direct image of** U **by** f.

[3] We also have

$$f_*(U) = B \setminus f_!(A \setminus U);$$

see Item 7 of Proposition 4.4.3.

Remark 4.4.2 ▶ Unwinding Definition 4.4.1

Identifying subsets of A with functions from A to $\{\text{true}, \text{false}\}$ via Item 6 of Proposition 4.3.3, we see that the direct image function associated to f is equivalently the function

$$f_*\colon \mathcal{P}(A) \to \mathcal{P}(B)$$

defined by

$$
\begin{aligned}
f_*(\chi_U) &\overset{\text{def}}{=} \mathrm{Lan}_f(\chi_U)\\
&= \mathrm{colim}\Big(\Big(f \overset{\rightarrow}{\times} \underline{(-_1)}\Big) \overset{\mathrm{pr}}{\twoheadrightarrow} A \xrightarrow{\chi_U} \{\mathsf{t},\mathsf{f}\}\Big)\\
&= \underset{\substack{a\in A\\ f(a)=-_1}}{\mathrm{colim}} (\chi_U(a))\\
&= \bigvee_{\substack{a\in A\\ f(a)=-_1}} (\chi_U(a)).
\end{aligned}
$$

So, in other words, we have

$$
\begin{aligned}
[f_*(\chi_U)](b) &= \bigvee_{\substack{a\in A\\ f(a)=b}} (\chi_U(a))\\
&= \begin{cases} \text{true} & \text{if there exists some } a \in A \text{ such}\\ & \text{that } f(a) = b \text{ and } a \in U,\\ \text{false} & \text{otherwise}\end{cases}\\
&= \begin{cases} \text{true} & \text{if there exists some } a \in U\\ & \text{such that } f(a) = b,\\ \text{false} & \text{otherwise}\end{cases}
\end{aligned}
$$

for each $b \in B$.

Proposition 4.4.3 ► Properties of Direct Images I

Let $f : A \longrightarrow B$ be a function.

1. *Functoriality.* The assignment $U \mapsto f_*(U)$ defines a functor

$$f_* : (\mathcal{P}(A), \subset) \longrightarrow (\mathcal{P}(B), \subset)$$

where

- *Action on Objects.* For each $U \in \mathcal{P}(A)$, we have

$$[f_*](U) \overset{\text{def}}{=} f_*(U);$$

- *Action on Morphisms.* For each $U, V \in \mathcal{P}(A)$:

  (★) If $U \subset V$, then $f_*(U) \subset f_*(V)$.

2. *Triple Adjointness.* We have a triple adjunction

$$
(f_* \dashv f^{-1} \dashv f_!)\colon \quad
\begin{tikzcd}
\mathcal{P}(A) \arrow[r, bend left=50, "f_*"] \arrow[r, bend right=50, "f_!"'] & \mathcal{P}(B) \arrow[l, "f^{-1}"' description]
\end{tikzcd},
$$

witnessed by bijections of sets

$$
\begin{aligned}
\mathrm{Hom}_{\mathcal{P}(B)}(f_*(U), V) &\cong \mathrm{Hom}_{\mathcal{P}(A)}\big(U, f^{-1}(V)\big),\\
\mathrm{Hom}_{\mathcal{P}(A)}\big(f^{-1}(U), V\big) &\cong \mathrm{Hom}_{\mathcal{P}(A)}(U, f_!(V)),
\end{aligned}
$$

natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$ and (respectively) $V \in \mathcal{P}(A)$ and $U \in \mathcal{P}(B)$, i.e. where:

   (a) The following conditions are equivalent:

      i. We have $f_*(U) \subset V$.

      ii. We have $U \subset f^{-1}(V)$.

   (b) The following conditions are equivalent:

      i. We have $f^{-1}(U) \subset V$.

      ii. We have $U \subset f_!(V)$.

3. *Preservation of Colimits.* We have an equality of sets

$$
f_*\left(\bigcup_{i \in I} U_i\right) = \bigcup_{i \in I} f_*(U_i),
$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have equalities

$$
\begin{aligned}
f_*(U) \cup f_*(V) &= f_*(U \cup V),\\
f_*(\emptyset) &= \emptyset,
\end{aligned}
$$

natural in $U, V \in \mathcal{P}(A)$.

4. *Oplax Preservation of Limits.* We have an inclusion of sets

$$f_*\left(\bigcap_{i\in I} U_i\right) \subset \bigcap_{i\in I} f_*(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have inclusions

$$f_*(U \cap V) \subset f_*(U) \cap f_*(V),$$
$$f_*(A) \subset B,$$

natural in $U, V \in \mathcal{P}(A)$.

5. *Symmetric Strict Monoidality With Respect to Unions.* The direct image function of Item 1 has a symmetric strict monoidal structure

$$\left(f_*, f_*^{\otimes}, f_{*|\mathbb{1}}^{\otimes}\right) \colon (\mathcal{P}(A), \cup, \emptyset) \to (\mathcal{P}(B), \cup, \emptyset),$$

being equipped with equalities

$$f_{*|U,V}^{\otimes} \colon f_*(U) \cup f_*(V) \xrightarrow{=} f_*(U \cup V),$$
$$f_{*|\mathbb{1}}^{\otimes} \colon \emptyset \xrightarrow{=} \emptyset,$$

natural in $U, V \in \mathcal{P}(A)$.

6. *Symmetric Oplax Monoidality With Respect to Intersections.* The direct image function of Item 1 has a symmetric oplax monoidal structure

$$\left(f_*, f_*^{\otimes}, f_{*|\mathbb{1}}^{\otimes}\right) \colon (\mathcal{P}(A), \cap, A) \to (\mathcal{P}(B), \cap, B),$$

being equipped with inclusions

$$f_{*|U,V}^{\otimes} \colon f_*(U \cap V) \hookrightarrow f_*(U) \cap f_*(V),$$
$$f_{*|\mathbb{1}}^{\otimes} \colon f_*(A) \hookrightarrow B,$$

natural in $U, V \in \mathcal{P}(A)$.

7. *Relation to Direct Images With Compact Support.* We have

$$f_*(U) = B \setminus f_!(A \setminus U)$$

for each $U \in \mathcal{P}(A)$.

Proof 4.4.4 ► Proof of Proposition 4.4.3

Item 1: Functoriality

Clear.

Item 2: Triple Adjointness

This follows from Kan Extensions, **??** of **??**.

Item 3: Preservation of Colimits

This follows from Item 2 and Categories, **??** of **??**.

Item 4: Oplax Preservation of Limits

Omitted.

Item 5: Symmetric Strict Monoidality With Respect to Unions

This follows from Item 3.

Item 6: Symmetric Oplax Monoidality With Respect to Intersections

This follows from **??**.

Item 7: Relation to Direct Images With Compact Support

Applying **??** of **??** to $A \setminus U$, we have

$$\begin{aligned} f_!(A \setminus U) &= B \setminus f_*(A \setminus (A \setminus U)) \\ &= B \setminus f_*(U). \end{aligned}$$

Taking complements, we then obtain

$$\begin{aligned} f_*(U) &= B \setminus (B \setminus f_*(U)), \\ &= B \setminus f_!(A \setminus U), \end{aligned}$$

which finishes the proof.

Proposition 4.4.5 ► Properties of Direct Images II

Let $f \colon A \to B$ be a function.

1. *Functionality I.* The assignment $f \mapsto f_*$ defines a function

$$(-)_{*|A,B} \colon \mathsf{Sets}(A, B) \to \mathsf{Sets}(\mathcal{P}(A), \mathcal{P}(B)).$$

2. *Functionality II.* The assignment $f \mapsto f_*$ defines a function

$$(-)_{*|A,B} \colon \mathsf{Sets}(A, B) \to \mathsf{Pos}((\mathcal{P}(A), \subset), (\mathcal{P}(B), \subset)).$$

3. *Interaction With Identities.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$(\mathrm{id}_A)_* = \mathrm{id}_{\mathcal{P}(A)};$$

4. *Interaction With Composition.* For each pair of composable functions $f \colon A \to B$ and $g \colon B \to C$, we have

$$(g \circ f)_* = g_* \circ f_*, \qquad \begin{tikzcd} \mathcal{P}(A) \arrow[r, "f_*"] \arrow[dr, "(g\circ f)_*"'] & \mathcal{P}(B) \arrow[d, "g_*"] \\ & \mathcal{P}(C). \end{tikzcd}$$

Proof 4.4.6 ► Proof of Proposition 4.4.5

Item 1: Functionality I

Clear.

Item 2: Functionality II

Clear.

Item 3: Interaction With Identities

This follows from Kan Extensions, **??** of **??**.

Item 4: Interaction With Composition

This follows from Kan Extensions, **??** of **??**. □

4.5 Inverse Images

Let A and B be sets and let $f \colon A \to B$ be a function.

Definition 4.5.1 ▶ Inverse Images

The **inverse image function associated to** f is the function[1]

$$f^{-1}\colon \mathcal{P}(B) \to \mathcal{P}(A)$$

defined by[2]

$$f^{-1}(V) \stackrel{\text{def}}{=} \{a \in A \mid \text{we have } f(a) \in V\}$$

for each $V \in \mathcal{P}(B)$.

[1] *Further Notation:* Also written $f^*\colon \mathcal{P}(B) \to \mathcal{P}(A)$.
[2] *Further Terminology:* The set $f^{-1}(V)$ is called the **inverse image of** V **by** f.

Remark 4.5.2 ▶ Unwinding Definition 4.5.1

Identifying subsets of B with functions from B to $\{\text{true}, \text{false}\}$ via Item 6 of Proposition 4.3.3, we see that the inverse image function associated to f is equivalently the function

$$f^*\colon \mathcal{P}(B) \to \mathcal{P}(A)$$

defined by

$$f^*(\chi_V) \stackrel{\text{def}}{=} \chi_V \circ f$$

for each $\chi_V \in \mathcal{P}(B)$, where $\chi_V \circ f$ is the composition

$$A \xrightarrow{f} B \xrightarrow{\chi_V} \{\text{true}, \text{false}\}$$

in Sets.

Proposition 4.5.3 ▶ Properties of Inverse Images I

Let $f\colon A \to B$ be a function.

1. *Functoriality.* The assignment $V \mapsto f^{-1}(V)$ defines a functor

   $$f^{-1}\colon (\mathcal{P}(B), \subset) \to (\mathcal{P}(A), \subset)$$

   where

   - *Action on Objects.* For each $V \in \mathcal{P}(B)$, we have

     $$[f^{-1}](V) \stackrel{\text{def}}{=} f^{-1}(V);$$

- *Action on Morphisms.* For each $U, V \in \mathcal{P}(B)$:
  ($\star$) If $U \subset V$, then $f^{-1}(U) \subset f^{-1}(V)$.

2. *Triple Adjointness.* We have a triple adjunction

$$
(f_* \dashv f^{-1} \dashv f_!)\colon\quad
\begin{tikzcd}
\mathcal{P}(A) \arrow[r, bend left, "f_*"] \arrow[r, bend right, "f_!"'] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, phantom, shift right=2, "\perp"] & \mathcal{P}(B), \arrow[l, "f^{-1}"]
\end{tikzcd}
$$

witnessed by bijections of sets

$$
\begin{aligned}
\mathrm{Hom}_{\mathcal{P}(B)}(f_*(U), V) &\cong \mathrm{Hom}_{\mathcal{P}(A)}\big(U, f^{-1}(V)\big),\\
\mathrm{Hom}_{\mathcal{P}(A)}\big(f^{-1}(U), V\big) &\cong \mathrm{Hom}_{\mathcal{P}(A)}(U, f_!(V)),
\end{aligned}
$$

natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$ and (respectively) $V \in \mathcal{P}(A)$ and $U \in \mathcal{P}(B)$, i.e. where:

(a) The following conditions are equivalent:
   i. We have $f_*(U) \subset V$;
   ii. We have $U \subset f^{-1}(V)$;

(b) The following conditions are equivalent:
   i. We have $f^{-1}(U) \subset V$.
   ii. We have $U \subset f_!(V)$.

3. *Preservation of Colimits.* We have an equality of sets

$$
f^{-1}\left(\bigcup_{i\in I} U_i\right) = \bigcup_{i\in I} f^{-1}(U_i),
$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(B)^{\times I}$. In particular, we have equalities

$$
\begin{aligned}
f^{-1}(U) \cup f^{-1}(V) &= f^{-1}(U \cup V),\\
f^{-1}(\emptyset) &= \emptyset,
\end{aligned}
$$

natural in $U, V \in \mathcal{P}(B)$.

4. *Preservation of Limits.* We have an equality of sets

$$f^{-1}\left(\bigcap_{i\in I} U_i\right) = \bigcap_{i\in I} f^{-1}(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(B)^{\times I}$. In particular, we have equalities

$$\begin{gathered} f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V), \\ f^{-1}(B) = A, \end{gathered}$$

natural in $U, V \in \mathcal{P}(B)$.

5. *Symmetric Strict Monoidality With Respect to Unions.* The inverse image function of Item 1 has a symmetric strict monoidal structure

$$\left(f^{-1}, f^{-1,\otimes}, f_{\mathbb{1}}^{-1,\otimes}\right) \colon (\mathcal{P}(B), \cup, \emptyset) \to (\mathcal{P}(A), \cup, \emptyset),$$

being equipped with equalities

$$\begin{gathered} f_{U,V}^{-1,\otimes} \colon f^{-1}(U) \cup f^{-1}(V) \xrightarrow{=} f^{-1}(U \cup V), \\ f_{\mathbb{1}}^{-1,\otimes} \colon \emptyset \xrightarrow{=} f^{-1}(\emptyset), \end{gathered}$$

natural in $U, V \in \mathcal{P}(B)$.

6. *Symmetric Strict Monoidality With Respect to Intersections.* The inverse image function of Item 1 has a symmetric strict monoidal structure

$$\left(f^{-1}, f^{-1,\otimes}, f_{\mathbb{1}}^{-1,\otimes}\right) \colon (\mathcal{P}(B), \cap, B) \to (\mathcal{P}(A), \cap, A),$$

being equipped with equalities

$$\begin{gathered} f_{U,V}^{-1,\otimes} \colon f^{-1}(U) \cap f^{-1}(V) \xrightarrow{=} f^{-1}(U \cap V), \\ f_{\mathbb{1}}^{-1,\otimes} \colon A \xrightarrow{=} f^{-1}(B), \end{gathered}$$

natural in $U, V \in \mathcal{P}(B)$.

Proof 4.5.4 ▶ Proof of Proposition 4.5.3

Item 1: Functoriality

Clear.

Item 2: Triple Adjointness

This follows from Kan Extensions, **??** of **??**.

Item 3: Preservation of Colimits

This follows from Item 2 and Categories, **??** of **??**.

Item 4: Preservation of Limits

This follows from Item 2 and Categories, **??** of **??**.

Item 5: Symmetric Strict Monoidality With Respect to Unions

This follows from Item 3.

Item 6: Symmetric Strict Monoidality With Respect to Intersections

This follows from Item 4. ∎

Proposition 4.5.5 ▶ Properties of Inverse Images II

Let $f\colon A \to B$ be a function.

1. *Functionality I.* The assignment $f \mapsto f^{-1}$ defines a function
$$(-)^{-1}_{A,B}\colon \mathsf{Sets}(A,B) \to \mathsf{Sets}(\mathcal{P}(B),\mathcal{P}(A)).$$
2. *Functionality II.* The assignment $f \mapsto f^{-1}$ defines a function
$$(-)^{-1}_{A,B}\colon \mathsf{Sets}(A,B) \to \mathsf{Pos}((\mathcal{P}(B),\subset),(\mathcal{P}(A),\subset)).$$
3. *Interaction With Identities.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have
$$\mathrm{id}_A^{-1} = \mathrm{id}_{\mathcal{P}(A)};$$
4. *Interaction With Composition.* For each pair of composable functions $f\colon A \to$

B and $g\colon B \to C$, we have

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}, \qquad \begin{tikzcd} \mathcal{P}(C) \arrow[r, "g^{-1}"] \arrow[dr, "(g\circ f)^{-1}"'] & \mathcal{P}(B) \arrow[d, "f^{-1}"] \\ & \mathcal{P}(A). \end{tikzcd}$$

Proof 4.5.6 ▶ Proof of Proposition 4.5.5

Item 1: Functionality I

Clear.

Item 2: Functionality II

Clear.

Item 3: Interaction With Identities

This follows from Categories, **??** of **??**.

Item 4: Interaction With Composition

This follows from Categories, **??** of **??**.

4.6 Direct Images With Compact Support

Let A and B be sets and let $f\colon A \to B$ be a function.

Definition 4.6.1 ▶ Direct Images With Compact Support

The **direct image with compact support function associated to** f is the function[1]

$$f_!\colon \mathcal{P}(A) \to \mathcal{P}(B)$$

defined by[2,3]

$$\begin{aligned} f_!(U) &\stackrel{\text{def}}{=} \left\{ b \in B \,\middle|\, \begin{array}{l} \text{for each } a \in A, \text{ if we have} \\ f(a) = b, \text{ then } a \in U \end{array} \right\} \\ &= \left\{ b \in B \,\middle|\, \text{we have } f^{-1}(b) \subset U \right\} \end{aligned}$$

for each $U \in \mathcal{P}(A)$.

[1] *Further Notation:* Also written $\forall_f\colon \mathcal{P}(A) \to \mathcal{P}(B)$. This notation comes from the fact that the following statements are equivalent, where $b \in B$ and $U \in \mathcal{P}(A)$:

- We have $b \in \forall_f(U)$.
- For each $a \in A$, if $b = f(a)$, then $a \in U$.

[2] *Further Terminology:* The set $f_!(U)$ is called the **direct image with compact support of** U **by** f.
[3] We also have
$$f_!(U) = B \setminus f_*(A \setminus U);$$
see Item 7 of Proposition 4.6.5.

Remark 4.6.2 ► Unwinding Definition 4.6.1

Identifying subsets of A with functions from A to $\{\mathrm{true}, \mathrm{false}\}$ via Item 6 of Proposition 4.3.3, we see that the direct image with compact support function associated to f is equivalently the function

$$f_!\colon \mathcal{P}(A) \to \mathcal{P}(B)$$

defined by

$$\begin{aligned}
f_!(\chi_U) &\overset{\text{def}}{=} \mathrm{Ran}_f(\chi_U)\\
&= \lim\left(\left(\underline{(-_1)} \overset{\rightarrow}{\times} f\right) \overset{\mathrm{pr}}{\twoheadrightarrow} A \xrightarrow{\chi_U} \{\mathrm{true}, \mathrm{false}\}\right)\\
&= \lim_{\substack{a \in A\\ f(a) = -_1}} (\chi_U(a))\\
&= \bigwedge_{\substack{a \in A\\ f(a) = -_1}} (\chi_U(a)).
\end{aligned}$$

So, in other words, we have

$$\begin{aligned}
[f_!(\chi_U)](b) &= \bigwedge_{\substack{a \in A\\ f(a) = b}} (\chi_U(a))\\
&= \begin{cases} \mathrm{true} & \text{if, for each } a \in A \text{ such that}\\ & f(a) = b, \text{ we have } a \in U,\\ \mathrm{false} & \text{otherwise} \end{cases}\\
&= \begin{cases} \mathrm{true} & \text{if } f^{-1}(b) \subset U\\ \mathrm{false} & \text{otherwise} \end{cases}
\end{aligned}$$

for each $b \in B$.

Definition 4.6.3 ► The Image and Complement Parts of $f_!$

Let U be a subset of A.[1,2]

1. The **image part of the direct image with compact support $f_!(U)$ of** U is the set $f_{!,\mathrm{im}}(U)$ defined by

$$\begin{aligned} f_{!,\mathrm{im}}(U) &\overset{\text{def}}{=} f_!(U) \cap \mathrm{Im}(f) \\ &= \left\{ b \in B \,\middle|\, \begin{array}{l} \text{we have } f^{-1}(b) \subset U \\ \text{and } f^{-1}(b) \neq \emptyset \end{array} \right\}. \end{aligned}$$

2. The **complement part of the direct image with compact support $f_!(U)$ of** U is the set $f_{!,\mathrm{cp}}(U)$ defined by

$$\begin{aligned} f_{!,\mathrm{cp}}(U) &\overset{\text{def}}{=} f_!(U) \cap (B \setminus \mathrm{Im}(f)) \\ &= B \setminus \mathrm{Im}(f) \\ &= \left\{ b \in B \,\middle|\, \begin{array}{l} \text{we have } f^{-1}(b) \subset U \\ \text{and } f^{-1}(b) = \emptyset \end{array} \right\} \\ &= \left\{ b \in B \,\middle|\, f^{-1}(b) = \emptyset \right\}. \end{aligned}$$

[1] Note that we have

$$f_!(U) = f_{!,\mathrm{im}}(U) \cup f_{!,\mathrm{cp}}(U),$$

as

$$\begin{aligned} f_!(U) &= f_!(U) \cap B \\ &= f_!(U) \cap (\mathrm{Im}(f) \cup (B \setminus \mathrm{Im}(f))) \\ &= (f_!(U) \cap \mathrm{Im}(f)) \cup (f_!(U) \cap (B \setminus \mathrm{Im}(f))) \\ &\overset{\text{def}}{=} f_{!,\mathrm{im}}(U) \cup f_{!,\mathrm{cp}}(U). \end{aligned}$$

[2] In terms of the meet computation of $f_!(U)$ of Remark 4.6.2, namely

$$f_!(\chi_U) = \bigwedge_{\substack{a \in A \\ f(a) = -_1}} (\chi_U(a)),$$

we see that $f_{!,\mathrm{im}}$ corresponds to meets indexed over nonempty sets, while $f_{!,\mathrm{cp}}$ corresponds to meets indexed over the empty set.

Example 4.6.4 ► Examples of Direct Images With Compact Support

Here are some examples of direct images with compact support.

1. *The Multiplication by Two Map on the Natural Numbers.* Consider the function $f\colon \mathbb{N} \longrightarrow \mathbb{N}$ given by
$$f(n) \overset{\text{def}}{=} 2n$$
for each $n \in \mathbb{N}$. Since f is injective, we have
$$\begin{aligned} f_{!,\mathrm{im}}(U) &= f_*(U) \\ f_{!,\mathrm{cp}}(U) &= \{\text{odd natural numbers}\} \end{aligned}$$
for any $U \subset \mathbb{N}$.

2. *Parabolas.* Consider the function $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ given by
$$f(x) \overset{\text{def}}{=} x^2$$
for each $x \in \mathbb{R}$. We have
$$f_{!,\mathrm{cp}}(U) = \mathbb{R}_{<0}$$
for any $U \subset \mathbb{R}$. Moreover, since $f^{-1}(x) = \left\{-\sqrt{x}, \sqrt{x}\right\}$, we have e.g.:
$$\begin{aligned} f_{!,\mathrm{im}}([0,1]) &= \{0\}, \\ f_{!,\mathrm{im}}([-1,1]) &= [0,1], \\ f_{!,\mathrm{im}}([1,2]) &= \emptyset, \\ f_{!,\mathrm{im}}([-2,-1] \cup [1,2]) &= [1,4]. \end{aligned}$$

3. *Circles.* Consider the function $f\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$ given by
$$f(x,y) \overset{\text{def}}{=} x^2 + y^2$$
for each $(x,y) \in \mathbb{R}^2$. We have
$$f_{!,\mathrm{cp}}(U) = \mathbb{R}_{<0}$$
for any $U \subset \mathbb{R}^2$, and since
$$f^{-1}(r) = \begin{cases} \text{a circle of radius } r \text{ about the origin} & \text{if } r > 0, \\ \{(0,0)\} & \text{if } r = 0, \\ \emptyset & \text{if } r < 0, \end{cases}$$

we have e.g.:

$$\begin{aligned} f_{!,\mathrm{im}}([-1,1]\times[-1,1]) &= [0,1],\\ f_{!,\mathrm{im}}(([-1,1]\times[-1,1])\setminus[-1,1]\times\{0\}) &= \emptyset. \end{aligned}$$

Proposition 4.6.5 ▶ Properties of Direct Images With Compact Support I

Let $f\colon A\longrightarrow B$ be a function.

1. *Functoriality.* The assignment $U\mapsto f_!(U)$ defines a functor

   $$f_!\colon(\mathcal{P}(A),\subset)\to(\mathcal{P}(B),\subset)$$

   where

   · *Action on Objects.* For each $U\in\mathcal{P}(A)$, we have

   $$[f_!](U)\overset{\mathrm{def}}{=}f_!(U);$$

   · *Action on Morphisms.* For each $U,V\in\mathcal{P}(A)$:

   ($\star$) If $U\subset V$, then $f_!(U)\subset f_!(V)$.

2. *Triple Adjointness.* We have a triple adjunction

   $$(f_*\dashv f^{-1}\dashv f_!)\colon\quad \begin{tikzcd} \mathcal{P}(A) \arrow[r, bend left=50, "f_*"] \arrow[r, bend right=50, "f_!"'] & \mathcal{P}(B), \arrow[l, "f^{-1}"'] \end{tikzcd}$$

   witnessed by bijections of sets

   $$\begin{aligned} \mathrm{Hom}_{\mathcal{P}(B)}(f_*(U),V) &\cong \mathrm{Hom}_{\mathcal{P}(A)}\big(U,f^{-1}(V)\big),\\ \mathrm{Hom}_{\mathcal{P}(A)}\big(f^{-1}(U),V\big) &\cong \mathrm{Hom}_{\mathcal{P}(A)}(U,f_!(V)), \end{aligned}$$

   natural in $U\in\mathcal{P}(A)$ and $V\in\mathcal{P}(B)$ and (respectively) $V\in\mathcal{P}(A)$ and $U\in\mathcal{P}(B)$, i.e. where:

   (a) The following conditions are equivalent:

i. We have $f_*(U) \subset V$;
ii. We have $U \subset f^{-1}(V)$;

(b) The following conditions are equivalent:
i. We have $f^{-1}(U) \subset V$.
ii. We have $U \subset f_!(V)$.

3. *Lax Preservation of Colimits.* We have an inclusion of sets
$$\bigcup_{i \in I} f_!(U_i) \subset f_!\left(\bigcup_{i \in I} U_i\right),$$
natural in $\{U_i\}_{i \in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have inclusions
$$f_!(U) \cup f_!(V) \hookrightarrow f_!(U \cup V),$$
$$\emptyset \hookrightarrow f_!(\emptyset),$$
natural in $U, V \in \mathcal{P}(A)$.

4. *Preservation of Limits.* We have an equality of sets
$$f_!\left(\bigcap_{i \in I} U_i\right) = \bigcap_{i \in I} f_!(U_i),$$
natural in $\{U_i\}_{i \in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have equalities
$$f^{-1}(U \cap V) = f_!(U) \cap f^{-1}(V),$$
$$f_!(A) = B,$$
natural in $U, V \in \mathcal{P}(A)$.

5. *Symmetric Lax Monoidality With Respect to Unions.* The direct image with compact support function of Item 1 has a symmetric lax monoidal structure
$$\left(f_!, f_!^{\otimes}, f_{!|\mathbb{1}}^{\otimes}\right) \colon (\mathcal{P}(A), \cup, \emptyset) \to (\mathcal{P}(B), \cup, \emptyset),$$
being equipped with inclusions
$$f_{!|U,V}^{\otimes} \colon f_!(U) \cup f_!(V) \hookrightarrow f_!(U \cup V),$$
$$f_{!|\mathbb{1}}^{\otimes} \colon \emptyset \hookrightarrow f_!(\emptyset),$$
natural in $U, V \in \mathcal{P}(A)$.

6. *Symmetric Strict Monoidality With Respect to Intersections.* The direct image function of Item 1 has a symmetric strict monoidal structure

$$\left(f_!, f_!^{\otimes}, f_{!|\mathbb{1}}^{\otimes}\right) \colon (\mathcal{P}(A), \cap, A) \to (\mathcal{P}(B), \cap, B),$$

being equipped with equalities

$$f_{!|U,V}^{\otimes} \colon f_!(U \cap V) \xrightarrow{=} f_!(U) \cap f_!(V),$$
$$f_{!|\mathbb{1}}^{\otimes} \colon f_!(A) \xrightarrow{=} B,$$

natural in $U, V \in \mathcal{P}(A)$.

7. *Relation to Direct Images.* We have

$$f_!(U) = B \setminus f_*(A \setminus U)$$

for each $U \in \mathcal{P}(A)$.

8. *Interaction With Injections.* If f is injective, then we have

$$\begin{aligned} f_{!,\mathrm{im}}(U) &= f_*(U), \\ f_{!,\mathrm{cp}}(U) &= B \setminus \mathrm{Im}(f), \\ f_!(U) &= f_{!,\mathrm{im}}(U) \cup f_{!,\mathrm{cp}}(U) \\ &= f_*(U) \cup (B \setminus \mathrm{Im}(f)) \end{aligned}$$

for each $U \in \mathcal{P}(A)$.

9. *Interaction With Surjections.* If f is surjective, then we have

$$\begin{aligned} f_{!,\mathrm{im}}(U) &\subset f_*(U), \\ f_{!,\mathrm{cp}}(U) &= \emptyset, \\ f_!(U) &\subset f_*(U) \end{aligned}$$

for each $U \in \mathcal{P}(A)$.

Proof 4.6.6 ► Proof of Proposition 4.6.5

Item 1: Functoriality

Clear.

Item 2: Triple Adjointness

This follows from Kan Extensions, **??** of **??**.

Item 3: Lax Preservation of Colimits

Omitted.

Item 4: Preservation of Limits

Omitted. This follows from Item 2 and Categories, **??** of **??**.

Item 5: Symmetric Lax Monoidality With Respect to Unions

This follows from **??**.

Item 6: Symmetric Strict Monoidality With Respect to Intersections

This follows from Item 4.

Item 7: Relation to Direct Images

We claim that $f_!(U) = B \setminus f_*(A \setminus U)$.

- *The First Implication.* We claim that

  $$f_!(U) \subset B \setminus f_*(A \setminus U).$$

  Let $b \in f_!(U)$. We need to show that $b \notin f_*(A \setminus U)$, i.e. that there is no $a \in A \setminus U$ such that $f(a) = b$.

  This is indeed the case, as otherwise we would have $a \in f^{-1}(b)$ and $a \notin U$, contradicting $f^{-1}(b) \subset U$ (which holds since $b \in f_!(U)$).

  Thus $b \in B \setminus f_*(A \setminus U)$.

- *The Second Implication.* We claim that

  $$B \setminus f_*(A \setminus U) \subset f_!(U).$$

  Let $b \in B \setminus f_*(A \setminus U)$. We need to show that $b \in f_!(U)$, i.e. that $f^{-1}(b) \subset U$.

  Since $b \notin f_*(A \setminus U)$, there exists no $a \in A \setminus U$ such that $b = f(a)$, and hence $f^{-1}(b) \subset U$.

  Thus $b \in f_!(U)$.

This finishes the proof of Item 7.

Item 8: Interaction With Injections

Clear.

Item 9: Interaction With Surjections

Clear.

Proposition 4.6.7 ► Properties of Direct Images With Compact Support II

Let $f\colon A \to B$ be a function.

1. *Functionality I.* The assignment $f \mapsto f_!$ defines a function
$$(-)_{!|A,B}\colon \mathsf{Sets}(A,B) \to \mathsf{Sets}(\mathcal{P}(A),\mathcal{P}(B)).$$
2. *Functionality II.* The assignment $f \mapsto f_!$ defines a function
$$(-)_{!|A,B}\colon \mathsf{Sets}(A,B) \to \mathsf{Pos}((\mathcal{P}(A),\subset),(\mathcal{P}(B),\subset)).$$
3. *Interaction With Identities.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have
$$(\mathrm{id}_A)_! = \mathrm{id}_{\mathcal{P}(A)};$$
4. *Interaction With Composition.* For each pair of composable functions $f\colon A \to B$ and $g\colon B \to C$, we have
$$(g \circ f)_! = g_! \circ f_!, \qquad \begin{tikzcd} \mathcal{P}(A) \arrow[r, "f_!"] \arrow[dr, "(g\circ f)_!"'] & \mathcal{P}(B) \arrow[d, "g_!"] \\ & \mathcal{P}(C). \end{tikzcd}$$

Proof 4.6.8 ► Proof of Proposition 4.6.7

Item 1: Functionality I

Clear.

Item 2: Functionality II

Clear.

Item 3: Interaction With Identities

This follows from Kan Extensions, ?? of ??.

Item 4: Interaction With Composition

This follows from Kan Extensions, ?? of ??.

Appendices

A Other Chapters