Types of Morphisms in Categories

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00UT

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1 Monomorphisms 00UU

1.1 Foundations 00UV

Let C be a category.

Definition 1.1.1.1. 00UW A morphism $m\colon A \to B$ of C is a **monomorphism** if for every commutative[1] diagram of the form

$$C \underset{g}{\overset{f}{\rightrightarrows}} A \xrightarrow{m} B,$$

we have $f = g$.

Example 1.1.1.2. 00UX Let $f\colon A \to B$ be a function. The following conditions are equivalent:

1. The function f is injective.

2. The function f is a monomorphism in Sets.

Proof. Suppose that f is a monomorphism and consider the following diagram:

$$\{*\} \underset{[y]}{\overset{[x]}{\rightrightarrows}} A \xrightarrow{f} B,$$

[1] That is, with $m \circ f = m \circ g$.

where $[x]$ and $[y]$ are the morphisms picking the elements x and y of A. Then $f(x) = f(y)$ iff $f \circ [x] = f \circ [y]$, implying $[x] = [y]$, and hence $x = y$. Therefore f is injective.
Conversely, suppose that f is injective. Proceeding by contrapositive, we claim that given a pair of maps $g, h \colon C \rightrightarrows A$ such that $g \neq h$, then $f \circ g \neq f \circ h$. Indeed, as g and h are different maps, there exists must exist at least one element $x \in C$ such that $g(x) \neq h(x)$. But then we have $f(g(x)) \neq f(h(x))$, as f is injective. Thus $f \circ g \neq f \circ h$, and we are done. □

Proposition 1.1.1.3. Let $\mathcal{C}$ be a category with pullbacks and $f \colon A \to B$ be a morphism of $\mathcal{C}$. 00UY

1. *Characterisations.* The following conditions are equivalent: 00UZ

   (a) The morphism f is a monomorphism. 00V0

   (b) For each $X \in \mathrm{Obj}(\mathcal{C})$, the map of sets 00V1

   $$f_* \colon \mathrm{Hom}_{\mathsf{Sets}}(X, A) \to \mathrm{Hom}_{\mathsf{Sets}}(X, B)$$

   is injective.

   (c) The kernel pair of f is trivial, i.e. we have 00V2

   $$A \times_B A \cong A, \qquad \begin{tikzcd} A \arrow[r, "\mathrm{id}_A"] \arrow[d, "\mathrm{id}_A"'] \arrow[dr, phantom, very near start, "\lrcorner"] & A \arrow[d, "f"] \\ A \arrow[r, "f"'] & B. \end{tikzcd}$$

2. *Monomorphisms vs.Injective Maps.* Let 00V3

   - $\mathcal{C}$ be a concrete category;
   - 忘: $\mathcal{C} \to \mathsf{Sets}$ be the forgetful functor from $\mathcal{C}$ to Sets;
   - $f \colon A \to B$ be a morphism of $\mathcal{C}$.

   If 忘 preserves pullbacks, then the following conditions are equivalent:

   (a) The morphism f is a monomorphism.

   (b) The morphism f is injective.

3. *Stability Properties.* The class of all monomorphisms of $\mathcal{C}$ is stable under the following operations: 00V4

(a) *Composition.* If f and g are monomorphisms, then so is $g \circ f$.[2]

(b) *Pullbacks.* Let

$$\begin{tikzcd}
A \times_C B \arrow[r] \arrow[d, "m'"'] \arrow[dr, phantom, very near start, "\lrcorner"] & B \arrow[d, "m"] \\
A \arrow[r] & C
\end{tikzcd}$$

be a diagram in $\mathcal{C}$. If m is a monomorphism in $\mathcal{C}$, then so is m'.

4. *Morphisms From the Terminal Object Are Monomorphisms.* If $\mathcal{C}$ has a terminal object $\mathbb{1}_{\mathcal{C}}$, then every morphism of $\mathcal{C}$ from $\mathbb{1}_{\mathcal{C}}$ is a monomorphism. 00V5

Proof. **??**, *Characterisations*: The equivalence between **????** is clear. We claim that **????** are equivalent:

1. **??** $\implies$ **??**: Suppose that f is a monomorphism. Then A satisfies the universal property of the pullback:

$$\begin{tikzcd}
P \arrow[drr, bend left, "\phi"] \arrow[ddr, bend right, "\phi"'] \arrow[dr, dashed, "\phi", "\exists!"'] & & \\
& A \arrow[r, "\mathrm{id}_A"] \arrow[d, "\mathrm{id}_A"'] \arrow[dr, phantom, very near start, "\lrcorner"] & A \arrow[d, "f"] \\
& A \arrow[r, "f"'] & B.
\end{tikzcd}$$

2. **??** $\implies$ **??**: Suppose that $A \cong A \times_B A$ and let $g, h\colon C \rightrightarrows A$ be a pair of morphisms. Consider the diagram

$$\begin{tikzcd}
C \arrow[dr, shift left, "g"] \arrow[dr, shift right, "h"'] & & \\
& A \arrow[r, "\mathrm{id}_A"] \arrow[d, "\mathrm{id}_A"'] \arrow[dr, phantom, very near start, "\lrcorner"] & A \arrow[d, "f"] \\
& A \arrow[r, "f"'] & B.
\end{tikzcd}$$

[2] Conversely, if $g \circ f$ is a monomorphism, then so is f.

The universal property of the pullback says that there exists a unique morphism $C \to A$ making the diagram

$$\begin{tikzcd} C \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "h"'] & & \\ & A \arrow[r, "\mathrm{id}_A"] \arrow[d, "\mathrm{id}_A"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\ & A \arrow[r, "f"'] & B \end{tikzcd}$$

commute, which implies $g = h$. Therefore, f is a monomorphism.

??, *Monomorphisms vs.Injective Maps*: Assume that f is injective. As the forgetful functor from C to Sets is faithful, we see that **??** together with **??** imply that f is a monomorphism.
Conversely, assume that f is a monomorphism. As F preserves pullbacks, it also preserves kernel pairs. By **??**, we see that F preserves monomorphisms. Thus F_f is a monomorphism, and hence is injective by **??**.
??, *Stability Properties*: Let $f, g\colon X \rightrightarrows A \times_C B$ be two morphisms such that the diagram

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & A \times_C B \arrow[r, "m'"] & A \end{tikzcd}$$

commutes. It follows that the diagram

$$\begin{tikzcd} X \arrow[dr, shift left, "f"] \arrow[dr, shift right, "g"'] \arrow[drr, bend left, "\mathrm{pr}_2 \circ f"] \arrow[ddr, bend right, "m' \circ g"'] & & \\ & A \times_C B \arrow[r, dashed, "\mathrm{pr}_2"] \arrow[d, "m'"] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "m"] \\ & A \arrow[r, "\psi"'] & C \end{tikzcd}$$

also commutes. From the universal property of the pullback, it follows that there must be precisely one morphism from X to $A \times_C B$ making the above diagram commute. Thus $f = g$ and m' is a monomorphism.
??, *Morphisms From the Terminal Object Are Monomorphisms*: Clear. $\square$

1.2 Monomorphism-Reflecting Functors 00V6

Definition 1.2.1.1. 00V7 A functor $F: \mathcal{C} \to \mathcal{D}$ **reflects monomorphisms** if, for each morphism f of $\mathcal{C}$, whenever F_f is a monomorphism, so is f.

Proposition 1.2.1.2. 00V8 Let $F: \mathcal{C} \to \mathcal{D}$ be a functor. If F is faithful, then it reflects monomorphisms.

Proof. Let $f: A \to B$ be a morphism of $\mathcal{C}$ and suppose that $F_f: F_A \to F_B$ is a monomorphism. Let $g, h: B \rightrightarrows C$ be two morphisms of $\mathcal{C}$ such that $g \circ f = h \circ f$. As F is faithful, we must have

$$F_g \circ F_f = F_{g \circ f} = F_{h \circ f} = F_h \circ F_f,$$

but as F_f is a monomorphism, it must be that $F_g = F_h$. Using the faithfulness of F again, we see that $g = h$. Therefore f is a monomorphism. □

1.3 Split Monomorphisms 00V9

Let $\mathcal{C}$ be a category.

Definition 1.3.1.1. 00VA A morphism $f: A \to B$ of $\mathcal{C}$ is a **split monomorphism**[3] if there exists a morphism $g: B \to A$ of $\mathcal{B}$ such that[4]

$$g \circ f = \mathrm{id}_A.$$

Proposition 1.3.1.2. 00VB Let $\mathcal{C}$ be a category.

1. *Split Monomorphisms are Monomorphisms.* 00VC If m is a split monomorphism, then m is a monomorphism.

Proof. **??**, *Split Monomorphisms are Monomorphisms*: Let $m: A \to B$ be a split monomorphism of $\mathcal{C}$, let $e: B \to A$ be a morphism of $\mathcal{C}$ with

$$e \circ m = \mathrm{id}_A,$$

and let $f, g: C \rightrightarrows A$ be two morphisms of $\mathcal{C}$ such that the diagram

$$C \overset{f}{\underset{g}{\rightrightarrows}} A \xrightarrow{m} B$$

[3] *Further Terminology:* Also called a **section**, or a **split monic** morphism.

[4] *Warning:* There exist monomorphisms which are not split monomorphisms, e.g. $\mathbb{Z}/2 \hookrightarrow \mathbb{Z}/4$ in Ring.

commutes. Then we have

$$\begin{aligned} f &= \mathrm{id}_A \circ f \\ &= (e \circ m) \circ f \\ &= e \circ (m \circ f) \\ &= e \circ (m \circ g) \\ &= (e \circ m) \circ g \\ &= \mathrm{id}_A \circ g \\ &= g, \end{aligned}$$

showing m to be a monomorphism. □

2 Epimorphisms 00VD

2.1 Foundations 00VE

Let $\mathcal{C}$ be a category.

Definition 2.1.1.1. 00VF A morphism $f\colon A \to B$ of $\mathcal{C}$ is an **epimorphism** if for every commutative[5] diagram of the form

$$A \xrightarrow{f} B \overset{g}{\underset{h}{\rightrightarrows}} C,$$

we have $g = h$.

Example 2.1.1.2. 00VG Let $f\colon A \to B$ be a function. The following conditions are equivalent:

1. The function f is injective.

2. The function f is an epimorphism in Sets.

Proof. Suppose that f is surjective and let $g, h\colon B \rightrightarrows C$ be morphisms such that $g \circ f = h \circ f$. Then for each $a \in A$, we have

$$g(f(a)) = h(f(a)),$$

but this implies that

$$g(b) = h(b)$$

[5]That is, with $g \circ f = h \circ f$.

for each $b \in B$, as f is surjective. Thus $g = h$ and f is an epimorphism. To prove the converse, we proceed by contrapositive. So suppose that f is not surjective and consider the diagram

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & C, \end{tikzcd}$$

where h is the map defined by $h(b) = 0$ for each $b \in B$ and g is the map defined by

$$g(b) = \begin{cases} 1 & \text{if } b \in \mathrm{Im}(f), \\ 0 & \text{otherwise.} \end{cases}$$

Then $h \circ f = g \circ f$, as $h(f(a)) = 1 = g(f(a))$ for each $a \in A$. However, for any $b \in B \setminus \mathrm{Im}(f)$, we have

$$g(b) = 0 \neq 1 = h(b).$$

Therefore $g \neq h$ and f is not an epimorphism. □

Proposition 2.1.1.3. Let $\mathcal{C}$ be a category. 00VH

1. *Characterisations.* Let $\mathcal{C}$ be a category with pullbacks and $f \colon A \to B$ be a morphism of $\mathcal{C}$. The following conditions are equivalent: 00VJ

   (a) The morphism f is an epimorphism. 00VK

   (b) For each $X \in \mathrm{Obj}(\mathcal{C})$, the map of sets 00VL

   $$f^* \colon \mathrm{Hom}_{\mathsf{Sets}}(B, X) \to \mathrm{Hom}_{\mathsf{Sets}}(A, X)$$

   is injective.

   (c) The cokernel pair of f is trivial, i.e. we have 00VM

   $$B \coprod_A B \cong B \qquad \begin{tikzcd} B \arrow[dr, phantom, "\ulcorner", very near start] & B \arrow[l] \\ B \arrow[u] & A. \arrow[l, "f"] \arrow[u, "f"'] \end{tikzcd}$$

2. *Epimorphisms vs.Surjective Maps.* Let 00VN

   - $\mathcal{C}$ be a concrete category;
   - $忘 \colon \mathcal{C} \to \mathsf{Sets}$ be the forgetful functor from $\mathcal{C}$ to Sets;

- $f\colon A \to B$ be a morphism of $\mathcal{C}$.

If 忘 preserves pushouts, then the following conditions are equivalent:

(a) The morphism f is a epimorphism.

(b) The morphism f is surjective.

3. *Stability Properties.* The class of all epimorphisms of $\mathcal{C}$ is stable under the following operations: 00VP

(a) *Composition.* If f and g are epimorphisms, then so is $g \circ f$.[6]

(b) *Pushouts.* Let

$$
\begin{tikzcd}
A \coprod_C B & B \arrow[l] \\
A \arrow[u, "e'"] \arrow[r] \arrow[ur, phantom, "\ulcorner"] & C \arrow[u, "e"']
\end{tikzcd}
$$

be a diagram in $\mathcal{C}$. If m is an epimorphism in $\mathcal{C}$, then so is e'.

4. *Morphisms to the Initial Object Are Monomorphisms.* If $\mathcal{C}$ has an initial object $\varnothing_{\mathcal{C}}$, then every morphism of $\mathcal{C}$ to $\varnothing_{\mathcal{C}}$ is a epimorphism. 00VQ

Proof. This is dual to **??**. □

2.2 Regular Epimorphisms 00VR

Proposition 2.2.1.1. Let $\mathcal{C}$ be a category. 00VS

1. *Stability Under Pullbacks.* Consider the diagram

$$
\begin{tikzcd}
A \times_C B \arrow[r] \arrow[d, "e'"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "e"] \\
A \arrow[r] & C
\end{tikzcd}
$$

in $\mathcal{C}$. If e is a regular epimorphism, then so is e'.

Proof. Epimorphisms Need Not Be Stable Under Pullback. : *Regular Epimorphisms Are Stable Under Pullback.* : □

[6] Conversely, if $g \circ f$ is a epimorphism, then so is g.

2.3 Effective Epimorphisms

Let $\mathcal{C}$ be a category.

Definition 2.3.1.1. An epimorphism $f\colon A \to B$ of $\mathcal{C}$ is **effective** if we have an isomorphism

$$B \cong \mathrm{CoEq}(A \times_B A \rightrightarrows A).$$

2.4 Split Epimorphisms

Let $\mathcal{C}$ be a category.

Definition 2.4.1.1. A morphism $f\colon A \to B$ of $\mathcal{C}$ is a **retraction**[7] if there is an arrow $g\colon B \to A$ such that $f \circ g = \mathrm{id}_B$.

Proposition 2.4.1.2. Let $f\colon A \to B$ be a morphism of $\mathcal{C}$.

1. Every split epimorphism is an epimorphism.[8]

Proof. This is dual to **??**. □

Appendices

A Other Chapters

Sets

1. Sets
2. Constructions With Sets
3. Pointed Sets
4. Tensor Products of Pointed Sets
5. Relations
6. Spans
7. Posets

Indexed and Fibred Sets

7. Indexed Sets
8. Fibred Sets
9. Un/Straightening for Indexed and Fibred Sets

Category Theory

11. Categories
12. Types of Morphisms in Categories

[7] *Further Terminology:* Also called a **split epimorphism**.

[8] *Warning:* There are epimorphisms which are not split epimorphisms, however, e.g. $\mathbb{Z} \hookrightarrow \mathbb{Z}/2$.