Fibred Sets

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This chapter contains a discussion of the un/straightening equivalence in the context of sets, as well as a general discussion of indexed and fibred sets. In particular, it contains:

1. A discussion of indexed sets (i.e. functors $K_{\mathsf{disc}} \to \mathsf{Sets}$ with K a set), constructions with them like dependent sums and dependent products, and their properties (**????**);
2. A discussion of fibred sets (i.e. maps of sets $X \to K$), constructions with them like dependent sums and dependent products, and their properties (Sections 1 and 2);
3. A discussion of the un/straightening equivalence for indexed and fibred sets (**??**).

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1 Fibred Sets

1.1 Foundations

Let K be a set.

Definition 1.1.1.1. A **K-fibred set** is a pair (X, ϕ) consisting of[1]

- *The Underlying Set.* A set X, called the **underlying set of** (X, ϕ);
- *The Fibration.* A map of sets $\phi\colon X \to K$.

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[1] *Further Terminology:* The **fibre of** (X, ϕ) **over** $x \in K$ is the set $\phi^{-1}(x)$ (also written ϕ_x) defined by

$$\phi^{-1}(x) \overset{\text{def}}{=} \mathrm{pt} \times_{[x], K, \phi} X, \qquad \begin{tikzcd} \phi^{-1}(x) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "\phi"] \\ \mathrm{pt} \arrow[r, "{[x]}"'] & K. \end{tikzcd}$$

1.2 Morphisms of Fibred Sets

Definition 1.2.1.1. A **morphism of K-fibred sets from (X, ϕ) to (Y, ψ)** is a function $f\colon X \to Y$ such that the diagram[2]

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "\phi"'] & & Y \arrow[dl, "\psi"] \\
& K &
\end{tikzcd}$$

commutes.

1.3 The Category of Fibred Sets Over a Fixed Base

Definition 1.3.1.1. The **category of K-fibred sets** is the category $\mathsf{FibSets}(K)$ defined as the slice category $\mathsf{Sets}_{/K}$ of Sets over K:

$$\mathsf{FibSets}(K) \overset{\text{def}}{=} \mathsf{Sets}_{/K}.$$

Remark 1.3.1.2. In detail $\mathsf{FibSets}(K)$ is the category where

- *Objects.* The objects of $\mathsf{FibSets}(K)$ are pairs (X, ϕ) consisting of
  - *The Fibred Set.* A set X;
  - *The Fibration.* A function $\phi\colon X \to K$;
- *Morphisms.* A morphism of $\mathsf{FibSets}(K)$ from (X, ϕ) to (Y, ψ) is a

[2] *Further Terminology:* The **transport map associated to f at $x \in K$** is the function

$$f^*_x\colon \phi^{-1}(x) \to \psi^{-1}(x)$$

given by the dashed map in the diagram

$$\begin{tikzcd}
\phi^{-1}(x) \arrow[rr] \arrow[dr, dashed] \arrow[dd] \arrow[dr, phantom, very near start, "\lrcorner"] & & X \arrow[dr, "f"] \arrow[dd, "\phi" near start] & \\
& \psi^{-1}(x) \arrow[rr] \arrow[dd] \arrow[dr, phantom, very near start, "\lrcorner"] & & Y \arrow[dd, "\psi"] \\
\mathrm{pt} \arrow[rr, "{[x]}" near start] \arrow[dr, equal] & & K \arrow[dr, equal] & \\
& \mathrm{pt} \arrow[rr, "{[x]}"'] & & K.
\end{tikzcd}$$

function $f\colon X \to Y$ making the diagram

$$
\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "\phi"'] & & Y \arrow[dl, "\psi"] \\
& X &
\end{tikzcd}
$$

commute;

- *Identities.* For each $(X, \phi) \in \mathrm{Obj}(\mathsf{FibSets}(K))$, the unit map

  $$\mathbb{1}^{\mathsf{FibSets}(K)}_{(X,\phi)} \colon \mathrm{pt} \to \mathrm{Hom}_{\mathsf{FibSets}(K)}((X, \phi), (X, \phi))$$

  of $\mathsf{FibSets}(K)$ at (X, ϕ) is given by

  $$\mathrm{id}^{\mathsf{FibSets}(K)}_{(X,\phi)} \overset{\mathrm{def}}{=} \mathrm{id}_X,$$

  as witnessed by the commutativity of the diagram

  $$
  \begin{tikzcd}
  X \arrow[rr, equal, "\mathrm{id}_X"] \arrow[dr, "\phi"'] & & X \arrow[dl, "\phi"] \\
  & K &
  \end{tikzcd}
  $$

  in Sets;

- *Composition.* For each $\mathbf{X} = (X, \phi)$, $\mathbf{Y} = (Y, \psi)$, $\mathbf{Z} = (Z, \chi) \in \mathrm{Obj}(\mathsf{FibSets}(K))$, the composition map

  $$\circ^{\mathsf{FibSets}(K)}_{\mathbf{X},\mathbf{Y},\mathbf{Z}} \colon \mathrm{Hom}_{\mathsf{FibSets}(K)}(\mathbf{Y}, \mathbf{Z}) \times \mathrm{Hom}_{\mathsf{FibSets}(K)}(\mathbf{X}, \mathbf{Y}) \to \mathrm{Hom}_{\mathsf{FibSets}(K)}(\mathbf{X}, \mathbf{Z})$$

  of $\mathsf{FibSets}(K)$ at $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ is defined by

  $$\circ^{\mathsf{FibSets}(K)}_{\mathbf{X},\mathbf{Y},\mathbf{Z}} \overset{\mathrm{def}}{=} \circ^{\mathsf{Sets}}_{X,Y,Z},$$

  as witnessed by the commutativity of the diagram

  $$
  \begin{tikzcd}
  X \arrow[r, "f"] \arrow[dr, "\phi"'] & Y \arrow[r, "g"] \arrow[d, "\psi" description] & Z \arrow[dl, "\chi"] \\
  & K &
  \end{tikzcd}
  $$

  in Sets.

1.4 The Category of Fibred Sets

Definition 1.4.1.1. The **category of fibred sets** is the category $\mathsf{FibSets}$ defined as the Grothendieck construction of the functor $\mathsf{FibSets}\colon \mathsf{Sets}^{\mathsf{op}} \to \mathsf{Cats}$ of Proposition 2.1.1.3:

$$\mathsf{FibSets} \overset{\text{def}}{=} \int^{\mathsf{Sets}} \mathsf{FibSets}.$$

Remark 1.4.1.2. In detail, the **category of fibred sets** is the category $\mathsf{FibSets}$ where

- *Objects.* The objects of $\mathsf{FibSets}$ are pairs $(K, (X, \phi_X))$ consisting of
  - *The Base Set.* A set K;
  - *The Fibred Set.* A K-fibred set $\phi_X\colon X \to K$;
- *Morphisms.* A morphism of $\mathsf{FibSets}$ from $(K, (X, \phi_X))$ to $(K', (Y, \phi_Y))$ is a pair (ϕ, f) consisting of
  - *The Base Map.* A map of sets $\phi\colon K \to K'$;
  - *The Morphism of Fibred Sets.* A morphism of K-fibred sets

$$f\colon (X, \phi_X) \to \phi_Y^*(Y), \qquad \begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\phi_X"'] & & Y \times_{K'} K \arrow[dl, "\mathrm{pr}_2"] \\ & K; & \end{tikzcd}$$

- *Identities.* For each $(K, X) \in \mathrm{Obj}(\mathsf{FibSets})$, the unit map

$$\mathbb{1}^{\mathsf{FibSets}}_{(K,X)}\colon \mathrm{pt} \to \mathsf{FibSets}((K, X), (K, X))$$

  of $\mathsf{FibSets}$ at (K, X) is defined by

$$\mathrm{id}^{\mathsf{FibSets}}_{(K,X)} \overset{\text{def}}{=} (\mathrm{id}_K, \sim),$$

  where $\sim$ is the isomorphism $X \to X \times_K K$ as in the diagram

$$\begin{tikzcd} X \arrow[rr, dashed, "\sim"] \arrow[dr, "\phi_X"'] & & X \times_K K \arrow[dl, "\mathrm{pr}_2"] \\ & K; & \end{tikzcd}$$

- *Composition.* For each $\mathbf{X} = (K, X), \mathbf{Y} = (K', Y), \mathbf{Z} = (K'', Z) \in \mathrm{Obj}(\mathsf{FibSets})$, the composition map

  $$\circ^{\mathsf{FibSets}}_{\mathbf{X},\mathbf{Y},\mathbf{Z}} \colon \mathsf{FibSets}(\mathbf{Y}, \mathbf{Z}) \times \mathsf{FibSets}(\mathbf{X}, \mathbf{Y}) \to \mathsf{FibSets}(\mathbf{X}, \mathbf{Z})$$

  of $\mathsf{FibSets}$ at $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ is defined by

  $$g \circ^{\mathsf{FibSets}}_{\mathbf{X},\mathbf{Y},\mathbf{Z}} f \overset{\text{def}}{=} (g \times_{K'} \mathrm{id}_K) \circ f$$

  as in the diagram

  $$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "\phi_X"'] & Y \times_{K'} K \arrow[r, "g \times_{K'} \mathrm{id}_K"] \arrow[d, "\mathrm{pr}_2"] & \overbrace{(Z \times_{K''} K') \times_{K'} K}^{\cong Z \times_{K''} K} \arrow[dl, "\mathrm{pr}_2"] \\ & K; & \end{tikzcd}$$

  for each $f \in \mathsf{FibSets}(\mathbf{X}, \mathbf{Y})$ and each $g \in \mathsf{FibSets}(\mathbf{Y}, \mathbf{Z})$.

2 Constructions With Fibred Sets

2.1 Change of Base

Let $f \colon K \to K'$ be a function and let (X, ϕ_X) be a K'-fibred set.

Definition 2.1.1.1. The **change of base of** (X, ϕ_X) **to** K is the K-fibred set $f^*(X)$ defined by

$$f^*(X) \overset{\text{def}}{=} (K \times_{K'} X, \mathrm{pr}_1), \qquad \begin{tikzcd} f^*(X) \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "\phi_X"] \\ K \arrow[r, "f"'] & K'. \end{tikzcd}$$

Proposition 2.1.1.2. The assignment $X \mapsto f^*(X)$ defines a functor

$$f^* \colon \mathsf{FibSets}(K') \to \mathsf{FibSets}(K),$$

where

- *Action on Objects.* For each $(X, \phi_X) \in \mathrm{Obj}(\mathsf{FibSets}(K'))$, we have
$$f^*(X, \phi_X) \stackrel{\text{def}}{=} f^*(X);$$
- *Action on Morphisms.* For each $(X, \phi_X), (Y, \phi_Y) \in \mathrm{Obj}(\mathsf{FibSets}(K'))$, the action on Hom-sets
$$f^*_{X,Y} \colon \mathrm{Hom}_{\mathsf{FibSets}(K')}(X, Y) \to \mathrm{Hom}_{\mathsf{FibSets}(K)}(f^*(X), f^*(Y))$$
of f^* at $((X, \phi_X), (Y, \phi_Y))$ is the map sending a morphism of K'-fibred sets
$$g \colon (X, \phi_X) \to (Y, \phi_Y)$$
to the morphism of K-fibred sets given by the dashed morphism in the diagram
$$\begin{tikzcd}
f^*(X) \arrow[rr] \arrow[dr, dashed] \arrow[dd] \arrow[drr, phantom, very near start, "\lrcorner"] & & X \arrow[dr, "g"] \arrow[dd, "\phi_X" near start] & \\
& f^*(Y) \arrow[rr, crossing over] \arrow[drr, phantom, very near start, "\lrcorner"] & & Y \arrow[dd, "\phi_Y"] \\
K \arrow[rr, "f" near start] \arrow[dr, equal] & & K' \arrow[dr, equal] & \\
& K \arrow[rr, "f"'] \arrow[from=uu, crossing over] & & K'.
\end{tikzcd}$$

Proof. Omitted. □

Proposition 2.1.1.3. The assignment $K \mapsto \mathsf{FibSets}(K)$ defines a functor
$$\mathsf{FibSets} \colon \mathsf{Sets}^{\mathsf{op}} \to \mathsf{Cats},$$
where

- *Action on Objects.* For each $K \in \mathrm{Obj}(\mathsf{Sets})$, we have
$$[\mathsf{FibSets}](K) \stackrel{\text{def}}{=} \mathsf{FibSets}(K);$$
- *Action on Morphisms.* For each $K, K' \in \mathrm{Obj}(\mathsf{Sets})$, the action on Hom-sets
$$\mathsf{Sets}_{/(-)|K,K'} \colon \mathsf{Sets}^{\mathsf{op}}(K, K') \to \mathsf{Fun}\big(\mathsf{FibSets}(K), \mathsf{FibSets}(K')\big)$$
of $\mathsf{Sets}_{/(-)}$ at (K, K') is the map sending a map of sets $f \colon K \to K'$ to the functor
$$\mathsf{Sets}_{/f} \colon \mathsf{FibSets}(K') \to \mathsf{FibSets}(K)$$
defined by
$$\mathsf{Sets}_{/f} \stackrel{\text{def}}{=} f^*.$$

Proof. Omitted. □

2.2 Dependent Sums

Let $f\colon K \to K'$ be a function and let (X, ϕ_X) be a K-fibred set.

Definition 2.2.1.1. The **dependent sum**[3] **of** (X, ϕ_X) is the K'-fibred set $\Sigma_f(X)$[4] defined by

$$\begin{aligned}\Sigma_f(X) &\stackrel{\text{def}}{=} (\Sigma_f(X), \Sigma_f(\phi_X))\\ &\stackrel{\text{def}}{=} (X, f \circ \phi_X).\end{aligned}$$

Proposition 2.2.1.2. Let $f\colon K \to K'$ be a function.

1. *Functoriality.* The assignment $X \mapsto \Sigma_f(X)$ defines a functor

$$\Sigma_f\colon \mathsf{FibSets}(K) \to \mathsf{FibSets}(K'),$$

where

- *Action on Objects.* For each $(X, \phi_X) \in \mathrm{Obj}(\mathsf{FibSets}(K))$, we have

$$\Sigma_f(X, \phi_X) \stackrel{\text{def}}{=} (\Sigma_f(X), \Sigma_f(\phi_X));$$

- *Action on Morphisms.* For each $(X, \phi_X), (Y, \phi_Y) \in \mathrm{Obj}(\mathsf{FibSets}(K))$, the action on Hom-sets

$$\Sigma_{f|X,Y}\colon \mathrm{Hom}_{\mathsf{FibSets}(K)}(X, Y) \to \mathrm{Hom}_{\mathsf{FibSets}(K)}(\Sigma_f(X), \Sigma_f(Y))$$

of Σ_f at $((X, \phi_X), (Y, \phi_Y))$ is the map sending a morphism of K-fibred sets

$$g\colon (X, \phi_X) \to (Y, \phi_Y)$$

to the morphism of K'-fibred sets defined by

$$\Sigma_f(g) \stackrel{\text{def}}{=} g.$$

[3] The name "dependent sum" comes from the fact that the fibre $\Sigma_f(\phi_X)^{-1}(x)$ of $\Sigma_f(X)$ at $x \in K'$ is given by

$$\Sigma_f(\phi_X)^{-1}(x) \cong \coprod_{y \in f^{-1}(x)} \phi_X^{-1}(y);$$

see Item 2 of Proposition 2.2.1.2.

[4] *Further Notation:* Also written $f_*(X)$.

2. *Interaction With Fibres.* We have a bijection of sets

$$\Sigma_f(\phi_X)^{-1}(k') \cong \coprod_{k\in f^{-1}(k')} \phi_X^{-1}(k)$$

for each $k' \in K'$.

Proof. *Item 1, Functoriality*: Omitted.
Item 2, Interaction With Fibres: Indeed, we have

$$\begin{aligned}\Sigma_f(\phi_X)^{-1}(k') &\stackrel{\text{def}}{=} \mathrm{pt} \times_{[k'],K',f\circ\phi_X} X\\ &\cong \{x \in X \mid f(\phi_X(x)) = k'\}\\ &\cong \coprod_{k\in f^{-1}(k')} \{x \in X \mid \phi_X(x) = k\}\\ &\cong \coprod_{k\in f^{-1}(k')} \phi_X^{-1}(k)\end{aligned}$$

for each $k' \in K'$. □

2.3 Dependent Products

Let $f\colon K \to K'$ be a function and let (X, ϕ_X) be a K-fibred set.

Definition 2.3.1.1. The **dependent product**[5] **of** (X, ϕ_X) is the K'-fibred set $\Pi_f(X)$[6] consisting of[7]

- *The Underlying Set.* The set $\Pi_f(X)$ defined by

$$\Pi_f(X) \stackrel{\text{def}}{=} \coprod_{k'\in K'} \prod_{k\in f^{-1}(k')} \phi_X^{-1}(k);$$

- *The Fibration.* The map of sets

$$\Pi_f(\phi_X)\colon \Pi_f(X) \to K'$$

[5]The name "dependent product" comes from the fact that the fibre $\Pi_f(\phi_X)^{-1}(k')$ of $\Pi_f(X)$ at $k' \in K'$ is given by

$$\Pi_f(\phi_X)^{-1}(k') \cong \prod_{k\in f^{-1}(k')} \phi_X^{-1}(k);$$

see Item 2 of Proposition 2.3.1.3.

[6]*Further Notation:* Also written $f_!(X)$.

[7]We can also define dependent products via the internal **Hom** in $\mathsf{FibSets}(K')$; see Item 3

defined by sending an element of

$$\Pi_f(X) \stackrel{\text{def}}{=} \coprod_{k' \in K'} \prod_{k \in f^{-1}(k')} \phi_X^{-1}(k)$$

to its index k' in K'.

Example 2.3.1.2. Here are some examples of dependent products of sets.

1. *Spaces of Sections.* Let $K = X$, $K' = \mathrm{pt}$, let $\phi\colon E \to X$ be a map of sets, and write $!_X\colon X \to \mathrm{pt}$ for the terminal map from X to pt. We have a bijection of sets

$$\begin{aligned}\Pi_{!_X}((E,\phi)) &\cong \Gamma_X(\phi) \\ &\stackrel{\text{def}}{=} \{h \in \mathsf{Sets}(X,E) \mid \phi \circ h = \mathrm{id}_X\}.\end{aligned}$$

2. *Function Spaces.* Let $K = K' = \mathrm{pt}$ and write $!_X\colon X \to \mathrm{pt}$ and $!_Y\colon Y \to \mathrm{pt}$ for the terminal maps from X and Y to pt. We have a bijection of sets

$$\mathsf{Sets}(X,Y) \cong \Pi_{!_X}(!_X^*(Y, !_Y)).$$

Proof. *Item 1, Spaces of Sections*: Indeed, we have

$$\begin{aligned}\Pi_{!_X}((E,\phi)) &\stackrel{\text{def}}{=} \coprod_{\star \in \mathrm{pt}} \prod_{k \in !_X^{-1}(\star)} \phi_X^{-1}(k) \\ &= \prod_{x \in X} \phi_X^{-1}(x) \\ &\cong \{h \in \mathsf{Sets}(X,E) \mid \phi_X \circ h = \mathrm{id}_X\} \\ &\stackrel{\text{def}}{=} \Gamma_X(\phi).\end{aligned}$$

Item 2, Function Spaces: Indeed, we have

$$\begin{aligned}\Pi_{!_X}(!_X^*(Y, !_Y)) &\stackrel{\text{def}}{=} \Pi_{!_X}(X \times_{!_X, \mathrm{pt}, !_Y} Y) \\ &\stackrel{\text{def}}{=} \coprod_{\star \in \mathrm{pt}} \prod_{x \in !_X^{-1}(\star)} \mathrm{pr}_1^{-1}(x) \\ &= \prod_{x \in X} Y \\ &\cong \mathsf{Sets}(X,Y).\end{aligned}$$

This finishes the proof. □

Proposition 2.3.1.3. Let $f\colon K \to K'$ be a function.

1. *Functoriality.* The assignment $X \mapsto \Pi_f(X)$ defines a functor

$$\Pi_f\colon \mathsf{FibSets}(K) \to \mathsf{FibSets}(K'),$$

where

- *Action on Objects.* For each $(X, \phi_X) \in \mathrm{Obj}(\mathsf{FibSets}(K))$, we have

$$\Pi_f(X, \phi_X) \overset{\mathrm{def}}{=} \Pi_f(X);$$

- *Action on Morphisms.* For each $(X, \phi_X), (Y, \phi_Y) \in \mathrm{Obj}(\mathsf{FibSets}(K))$, the action on Hom-sets

$$\Pi_{f|X,Y}\colon \mathrm{Hom}_{\mathsf{FibSets}(K)}(X, Y) \to \mathrm{Hom}_{\mathsf{FibSets}(K')}(\Pi_f(X), \Pi_f(Y))$$

of Π_f at $((X, \phi_X), (Y, \phi_Y))$ is the map sending a morphism of K-fibred sets

$$\xi\colon (X, \phi_X) \to (Y, \phi_Y), \qquad \begin{tikzcd} X \arrow[rr, "\xi"] \arrow[dr, "\phi_X"'] & & Y \arrow[dl, "\phi_Y"] \\ & K & \end{tikzcd}$$

to the morphism

$$\Pi_f(\xi)\colon (\Pi_f(X), \Pi_f(\phi_X)) \to (\Pi_f(Y), \Pi_f(\phi_Y)) \qquad \begin{tikzcd} \Pi_f(X) \arrow[rr, "\Pi_f(\xi)"] \arrow[dr, "\Pi_f(\phi_X)"'] & & \Pi_f(Y) \arrow[dl, "\Pi_f(\phi_Y)"] \\ & K & \end{tikzcd}$$

of K'-fibred sets given by[8]

$$[\Pi_f(\xi)]((x_k)_{k\in f^{-1}(k')}) \overset{\mathrm{def}}{=} (\xi(x_k))_{k\in f^{-1}(k')}$$

for each $(x_k)_{k\in f^{-1}(k')} \in \Pi_f(X)$.

of Proposition 2.3.1.3.

[8]Note that we indeed have $\xi(x_k) \in \phi_Y^{-1}(k)$, since

$$\begin{aligned} \phi_Y(\xi(x_k)) &= [\phi_Y \circ \xi](x_k) \\ &= \phi_X(x_k) \\ &= k, \end{aligned}$$

2. *Interaction With Fibres.* We have a bijection of sets

$$\Pi_f(\phi_X)^{-1}(k') \cong \prod_{k\in f^{-1}(k')} \phi_X^{-1}(k);$$

for each $k' \in K'$.

3. *Construction Using the Internal Hom.* We have

$$\Pi_f(X,\phi_X) = \left(K' \times_{\mathbf{Hom}_{\mathsf{FibSets}\left(K'\right)}((K,f),(K,f))} \mathbf{Hom}_{\mathsf{FibSets}(K')}((K,f),(X,f\circ\phi_X)), \mathrm{pr}_1\right),$$

forming a pullback diagram

$$\begin{tikzcd}
\Pi_f(X,\phi_X) \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Hom}_{\mathsf{FibSets}(K')}((K,f),(X,f\circ\phi_X)) \arrow[d, "(\phi_X)_*"] \\
K' \arrow[r, "I"'] & \mathbf{Hom}_{\mathsf{FibSets}(K')}((K,f),(K,f)),
\end{tikzcd}$$

where the bottom map is given by

$$I(k') \overset{\text{def}}{=} \mathrm{id}_{f^{-1}(k')}$$

for each $k' \in K'$ and where $\mathbf{Hom}_{\mathsf{FibSets}(K')}$ denotes the internal Hom of $\mathsf{FibSets}(K')$ of Definition 2.4.1.1.

4. *Internal Homs via Dependent Products.* We have

$$\mathbf{Hom}_{\mathsf{FibSets}(K)}(X,Y) \cong \Pi_{\phi_X}(\phi_X^*(Y)).$$

Proof. *Item 1, Functoriality*: Omitted.
Item 2, Interaction With Fibres: Clear.
Item 3, Construction Using the Internal Hom: Using the explicit formula for pullbacks of sets given in Constructions With Sets, Definition 1.3.1.1, we see that the pullback

$$K' \times_{\mathbf{Hom}_{\mathsf{FibSets}\left(K'\right)}((K,f),(K,f))} \mathbf{Hom}_{\mathsf{FibSets}(K')}((K,f),(X,f\circ\phi_X))$$

is given by

$$\left\{(k',h) \in \coprod_{k'\in K'} \mathsf{Sets}\Big(f^{-1}(k'), \phi_X^{-1}\Big(f^{-1}(k')\Big)\Big) \;\middle|\; \phi_X \circ h = \mathrm{id}_{f^{-1}(k')}\right\},$$

which is isomorphic to

$$\coprod_{k'\in K'}\Big\{h\in\mathsf{Sets}\Big(f^{-1}(k'),\phi_X^{-1}\Big(f^{-1}(k')\Big)\Big)\;\Big|\;\phi_X\circ h=\mathrm{id}_{f^{-1}(k')}\Big\}.$$

We claim that

$$\Big\{h\in\mathsf{Sets}\Big(f^{-1}(k'),\phi_X^{-1}\Big(f^{-1}(k')\Big)\Big)\;\Big|\;\phi_X\circ h=\mathrm{id}_{f^{-1}(k')}\Big\}\cong\prod_{k\in f^{-1}(k')}\phi_X^{-1}(k),$$

so that the pullback is indeed given by $\Pi_f(X)$. There are two cases:

1. If $f^{-1}(k')=\emptyset$, then there is only one map from $f^{-1}(k')$ to $\phi_X^{-1}(f^{-1}(k'))$ (the inclusion), so $\mathsf{Sets}\Big(f^{-1}(k'),\phi_X^{-1}(f^{-1}(k'))\Big)\cong\mathrm{pt}$. Since products indexed by the empty set are isomorphic to pt, the isomorphism follows.

2. Otherwise, by the condition $\phi_X\circ h=\mathrm{id}_{f^{-1}(k')}$, it follows that, for each $k\in f^{-1}(k')$, we must have

$$\phi_X(h(k))=k,$$

and thus $h(k)\in\phi_X^{-1}(k)$. Therefore, a map from $f^{-1}(k')$ to $\phi_X^{-1}(f^{-1}(k'))$ consists of a choice of an element from $\phi_X^{-1}(k)$ for each $k\in f^{-1}(k')$, which is precisely given by an element of the product $\prod_{k\in f^{-1}(k')}\phi_X^{-1}(k)$, showing the bijection to be true.

Item 4, Internal Homs via Dependent Products: Indeed we have

$$\begin{aligned}
\Pi_{\phi_X}(\phi_X^*(Y)) &\overset{\text{def}}{=}\Pi_{\phi_X}(X\times_K Y)\\
&\overset{\text{def}}{=}\coprod_{k\in K}\prod_{x\in\phi_X^{-1}(k)}\mathrm{pr}_1^{-1}(x)\\
&\cong\coprod_{k\in K}\prod_{x\in\phi_X^{-1}(k)}\{y\in Y\mid\phi_X(x)=\phi_Y(y)\}\\
&\cong\coprod_{k\in K}\prod_{x\in\phi_X^{-1}(k)}\{y\in Y\mid k=\phi_Y(y)\}\\
&\cong\coprod_{k\in K}\prod_{x\in\phi_X^{-1}(k)}\phi_Y^{-1}(k)\\
&\cong\coprod_{k\in K}\mathsf{Sets}\Big(\phi_X^{-1}(k),\phi_Y^{-1}(k)\Big)\\
&\overset{\text{def}}{=}\mathbf{Hom}_{\mathsf{FibSets}(K)}(X,Y).
\end{aligned}$$

This finishes the proof. $\square$

2.4 Internal Homs

Let K be a set and let (X,ϕ_X) and (Y,ϕ_Y) be K-fibred sets.

Definition 2.4.1.1. The **internal Hom of fibred sets from** (X,ϕ_X) **to** (Y,ϕ_Y) is the fibred set $\mathbf{Hom}_{\mathsf{FibSets}(K)}(X,Y)$ consisting of

- *The Underlying Set.* The set $\mathbf{Hom}_{\mathsf{FibSets}(K)}(X,Y)$ defined by

$$\mathbf{Hom}_{\mathsf{FibSets}(K)}(X,Y) \overset{\text{def}}{=} \coprod_{k\in K} \mathsf{Sets}\Big(\phi_X^{-1}(k),\phi_Y^{-1}(k)\Big);$$

- *The Fibration.* The map of sets[9]

$$\phi_{\mathbf{Hom}_{\mathsf{FibSets}(K)}(X,Y)} \colon \underbrace{\mathbf{Hom}_{\mathsf{FibSets}(K)}(X,Y)}_{\coprod_{k\in K}\mathsf{Sets}\left(\phi_X^{-1}(k),\phi_Y^{-1}(k)\right)} \to K$$

  defined by sending a map $f\colon \phi_X^{-1}(k)\to\phi_Y^{-1}(k)$ to its index $k\in K$.

Proof. Omitted. □

Proposition 2.4.1.2. Let K be a set and let (X,ϕ_X) and (Y,ϕ_Y) be K-fibred sets.

1. *Functoriality.* Let (X,ϕ_X) and (Y,ϕ_Y) be K-fibred sets.

   (a) The assignment $X\mapsto \mathbf{Hom}_{\mathsf{FibSets}(K)}(X,Y)$ defines a functor

$$\mathbf{Hom}_{\mathsf{FibSets}(K)}(X,-)\colon \mathsf{FibSets}(K)\to\mathsf{FibSets}(K).$$

   (b) The assignment $Y\mapsto \mathbf{Hom}_{\mathsf{FibSets}(K)}(X,Y)$ defines a functor

$$\mathbf{Hom}_{\mathsf{FibSets}(K)}(-,Y)\colon \mathsf{FibSets}(K)^{\mathsf{op}}\to\mathsf{FibSets}(K).$$

   (c) The assignment $(X,Y)\mapsto \mathbf{Hom}_{\mathsf{FibSets}(K)}(X,Y)$ defines a functor

$$\mathbf{Hom}_{\mathsf{FibSets}(K)}(-_1,-_2)\colon \mathsf{FibSets}(K)^{\mathsf{op}}\times\mathsf{FibSets}(K)\to\mathsf{FibSets}(K).$$

where we have used that ξ is a morphism of K-fibred sets for the second equality.

[9] The fibres of the internal **Hom** of $\mathsf{FibSets}(K)$ are precisely the sets $\mathsf{Sets}\big(\phi_X^{-1}(k),\phi_Y^{-1}(k)\big)$, i.e. we have

$$\phi_{\mathbf{Hom}_{\mathsf{FibSets}(K)}(X,Y)|k} \cong \mathsf{Sets}\big(\phi_X^{-1}(k),\phi_Y^{-1}(k)\big)$$

2. *Internal Homs via Dependent Products.* We have

$$\mathbf{Hom}_{\mathsf{FibSets}(K)}(X, Y) \cong \Pi_{\phi_X}(\phi_X^*(Y)).$$

Proof. *Item 1, Functoriality*: Omitted.
Item 2, Internal Homs via Dependent Products: This was proved in Item 4 of Proposition 2.3.1.3. □

2.5 Adjointness for Fibred Sets

Let $f \colon K \to K'$ be a map of sets.

Proposition 2.5.1.1. We have a triple adjunction

$$(\Sigma_f \dashv f^* \dashv \Pi_f)\colon \quad \begin{tikzcd} \mathsf{FibSets}(K) \arrow[r, bend left, "\Sigma_f"] \arrow[r, bend right, "\Pi_f"'] & \mathsf{FibSets}(K') \arrow[l, "f^*" description] \end{tikzcd}.$$

We offer two proofs. The first uses the corresponding adjunction for indexed sets (Indexed Sets, Proposition 4.5.1.1) and the un/straightening equivalence together with its compatibility with dependent sums and products to “transfer” the adjunction to fibred sets, while the second is a direct proof.

Proof. The Adjunction $\Sigma_f \dashv f^*$: The adjunction

$$(\Sigma_f \dashv f^*)\colon \quad \begin{tikzcd} \mathsf{ISets}(K) \arrow[r, bend left, "\Sigma_f"] \arrow[r, phantom, "\perp"] & \mathsf{ISets}(K') \arrow[l, bend left, "f^*"] \end{tikzcd}$$

of Indexed Sets, Proposition 4.5.1.1 gives a unit and counit of the form

$$\eta\colon \mathrm{id}_{\mathsf{ISets}(K)} \Longrightarrow \Sigma_f \circ f^*,$$
$$\epsilon\colon f^* \circ \Sigma_f \Longrightarrow \mathrm{id}_{\mathsf{ISets}(K')}.$$

With these in hand, we construct natural transformations

$$\eta'\colon \mathrm{id}_{\mathsf{FibSets}(K)} \Longrightarrow \Sigma_f \circ f^*,$$
$$\epsilon'\colon f^* \circ \Sigma_f \Longrightarrow \mathrm{id}_{\mathsf{FibSets}(K')}$$

as follows:

1. *The Unit.* We define $\eta' \colon \mathrm{id}_{\mathsf{FibSets}(K)} \Longrightarrow \Sigma_f \circ f^*$ as the pasting of the diagram

$$
\begin{tikzcd}
 & & \mathsf{FibSets}(K') \arrow[drr, "f^*"] & & \\
\mathsf{FibSets}(K) \arrow[urr, "\Sigma_f"] \arrow[r, phantom, "(1)"] & {} & \mathsf{ISets}(K') \arrow[u, "\mathrm{St}_{K'}"] \arrow[dr, "f^*"] \arrow[r, phantom, "(2)"] & {} & \mathsf{FibSets}(K) \\
{} \arrow[r, phantom, "(3)"] & \mathsf{ISets}(K) \arrow[ul, "\mathrm{St}_K"] \arrow[ur, "\Sigma_f"] \arrow[rr, "\mathrm{id}_{\mathsf{ISets}(K)}"] \arrow[r, phantom, "\Uparrow \eta"] & {} & \mathsf{ISets}(K) \arrow[ur, "\mathrm{St}_K"] \arrow[r, phantom, "(4)"] & {} \\
\mathsf{FibSets}(K) \arrow[uu, "\mathrm{id}_{\mathsf{FibSets}(K)}"] \arrow[ur, "\mathrm{Un}_K"] \arrow[rrrr, "\mathrm{id}_{\mathsf{FibSets}(K)}"'] \arrow[rrrr, phantom, bend left, "(5)"] & & & & \mathsf{FibSets}(K), \arrow[ul, "\mathrm{Un}_K"'] \arrow[uu, "\mathrm{id}_{\mathsf{FibSets}(K)}"']
\end{tikzcd}
$$

where:

(a) Subdiagram (1) commutes by Un/Straightening for Indexed and Fibred Sets, Item 3 of Proposition 1.1.1.2.

(b) Subdiagram (2) commutes by Un/Straightening for Indexed and Fibred Sets, Item 2 of Proposition 1.1.1.2.

(c) Subdiagram (3) commutes by Un/Straightening for Indexed and Fibred Sets, Theorem 1.3.1.1.

(d) Subdiagram (4) commutes by Un/Straightening for Indexed and Fibred Sets, Theorem 1.3.1.1.

(e) Subdiagram (5) commutes by unitality of composition.

2. *The Counit.* We define $\epsilon' \colon f^* \circ \Sigma_f \Longrightarrow \mathrm{id}_{\mathsf{FibSets}(K')}$ as the pasting of

for each $k \in K$.

the diagram

$$
\begin{tikzcd}
\mathsf{FibSets}(K') \arrow[rrrr, "\mathrm{id}_{\mathsf{FibSets}(K')}"] \arrow[dr, "\mathrm{Un}_K"] \arrow[dd, "\mathrm{id}_{\mathsf{FibSets}(K')}"'] & & {} \arrow[d, phantom, "(1)"] & & \mathsf{FibSets}(K') \arrow[dl, "\mathrm{Un}_K"'] \arrow[dd, "\mathrm{id}_{\mathsf{FibSets}(K')}"] \\
& \mathsf{ISets}(K') \arrow[rr, "\mathrm{id}_{\mathsf{ISets}(K)}"] \arrow[dl, "\mathrm{St}_{K'}"'] \arrow[dr, "f^*"] \arrow[l, phantom, "(2)"] & & \mathsf{ISets}(K') \arrow[dr, "\mathrm{St}_{K'}"] \arrow[r, phantom, "(3)"] & \\
\mathsf{FibSets}(K') \arrow[dr, "f^*"'] \arrow[rr, phantom, "(4)"] & & \mathsf{ISets}(K) \arrow[ur, "\Sigma_f"] \arrow[d, "\mathrm{St}_K"] \arrow[u, phantom, "\Uparrow\epsilon"] \arrow[rr, phantom, "(5)"] & & \mathsf{FibSets}(K') \\
& & \mathsf{FibSets}(K) \arrow[urr, "\Sigma_f"'] & &
\end{tikzcd}
$$

where:

(a) Subdiagram (1) commutes by unitality of composition.

(b) Subdiagram (2) commutes by Un/Straightening for Indexed and Fibred Sets, Theorem 1.3.1.1.

(c) Subdiagram (3) commutes by Un/Straightening for Indexed and Fibred Sets, Theorem 1.3.1.1.

(d) Subdiagram (4) commutes by Un/Straightening for Indexed and Fibred Sets, Item 3 of Proposition 1.1.1.2.

(e) Subdiagram (5) commutes by Un/Straightening for Indexed and Fibred Sets, Item 2 of Proposition 1.1.1.2.

Next, we prove the left triangle identity,

$$
\begin{tikzcd}
& \mathsf{FibSets}(K') \arrow[rr, "\mathrm{id}_{\mathsf{FibSets}(K')}"] \arrow[dr, "f^*"] & & \mathsf{FibSets}(K') \\
\mathsf{FibSets}(K) \arrow[ur, "\Sigma_f"] \arrow[rr, "\mathrm{id}_{\mathsf{FibSets}(K)}"'] & {} \arrow[u, phantom, "\Uparrow\eta"] & \mathsf{FibSets}(K) \arrow[ur, "\Sigma_f"'] \arrow[u, phantom, "\Uparrow\epsilon"] &
\end{tikzcd}
=
\begin{tikzcd}
& \mathsf{FibSets}(K') \arrow[rr, "\mathrm{id}_{\mathsf{FibSets}(K')}"] & & \mathsf{FibSets}(K'), \\
\mathsf{FibSets}(K) \arrow[ur, "\Sigma_f"] \arrow[rr, "\mathrm{id}_{\mathsf{FibSets}(K)}"'] & & \mathsf{FibSets}(K) \arrow[ur, "\Sigma_f"'] \arrow[ul, phantom, "\mathrm{id}_{\Sigma_f}"] &
\end{tikzcd}
$$

whose left side in our case looks like this:

$$
\begin{tikzcd}
& & \mathsf{FibSets}(K') \arrow[drr, "f^*"] \arrow[rrr, "\mathrm{id}_{\mathsf{FibSets}(K')}"] & & & \mathsf{FibSets}(K') \arrow[rrrr, "\mathrm{id}_{\mathsf{FibSets}(K')}"] \arrow[dr, "\mathrm{Un}_K"] \arrow[dd, "\mathrm{id}_{\mathsf{FibSets}(K')}"'] & & & & \mathsf{FibSets}(K') \arrow[dl, "\mathrm{Un}_K"'] \arrow[dd, "\mathrm{id}_{\mathsf{FibSets}(K')}"] \\
\mathsf{FibSets}(K) \arrow[urr, "\Sigma_f"] & & \mathsf{ISets}(K') \arrow[u, "\mathrm{St}_{K'}"] \arrow[dr, "f^*"] & & \mathsf{FibSets}(K) & & \mathsf{ISets}(K') \arrow[rr, "\mathrm{id}_{\mathsf{ISets}(K)}"] \arrow[dl, "\mathrm{St}_{K'}"'] \arrow[dr, "f^*"] & & \mathsf{ISets}(K') \arrow[dr, "\mathrm{St}_{K'}"] & \\
& \mathsf{ISets}(K) \arrow[ul, "\mathrm{St}_K"] \arrow[ur, "\Sigma_f"] \arrow[rr, "\mathrm{id}_{\mathsf{ISets}(K)}"] & {} \arrow[u, phantom, "\Uparrow\eta"] & \mathsf{ISets}(K) \arrow[ur, "\mathrm{St}_K"] & & \mathsf{FibSets}(K') \arrow[drr, "f^*"'] & & \mathsf{ISets}(K) \arrow[ur, "\Sigma_f"] \arrow[d, "\mathrm{St}_K"] \arrow[u, phantom, "\Uparrow\epsilon"] & & \mathsf{FibSets}(K'). \\
\mathsf{FibSets}(K) \arrow[uu, "\mathrm{id}_{\mathsf{FibSets}(K)}"] \arrow[ur, "\mathrm{Un}_K"] \arrow[rrrr, "\mathrm{id}_{\mathsf{FibSets}(K)}"'] & & & & \mathsf{FibSets}(K) \arrow[ul, "\mathrm{Un}_K"'] \arrow[uu, "\mathrm{id}_{\mathsf{FibSets}(K)}"'] \arrow[rrr, "\mathrm{id}_{\mathsf{FibSets}(K)}"'] & & & \mathsf{FibSets}(K) \arrow[urr, "\Sigma_f"'] & &
\end{tikzcd}
$$

It can be rearranged into

$$
\begin{tikzcd}
 & & \mathsf{FibSets}(K') \arrow[rrr, "\mathrm{id}_{\mathsf{FibSets}(K')}"] & & & \mathsf{FibSets}(K') \arrow[rrrr, "\mathrm{id}_{\mathsf{FibSets}(K')}"] \arrow[dr, phantom, "(1)"] & & & & \mathsf{FibSets}(K') \arrow[dl, "\mathrm{Un}_K"] \arrow[dd, "\mathrm{id}_{\mathsf{FibSets}(K')}"] \\
\mathsf{FibSets}(K) \arrow[urr, "\Sigma_f"] & & \mathsf{ISets}(K') \arrow[u, "\mathrm{St}_{K'}"] \arrow[rrrr, "\mathrm{id}_{\mathsf{ISets}(K')}"] \arrow[dr, "f^*"] \arrow[d, phantom, "\Uparrow\eta"] & & {} \arrow[dr, phantom, "(2)"] & & \mathsf{ISets}(K') \arrow[rr, "\mathrm{id}_{\mathsf{ISets}(K)}"] \arrow[dr, "f^*"] & & \mathsf{ISets}(K') \arrow[dr, "\mathrm{St}_{K'}"] & \\
 & \mathsf{ISets}(K) \arrow[ul, "\mathrm{St}_K"] \arrow[ur, "\Sigma_f"] \arrow[rr, "\mathrm{id}_{\mathsf{ISets}(K)}"] & & \mathsf{ISets}(K) \arrow[rrrr, "\mathrm{id}_{\mathsf{ISets}(K')}"] \arrow[dr, phantom, "(3)"] & & & & \mathsf{ISets}(K) \arrow[ur, "\Sigma_f"] \arrow[d, "\mathrm{St}_K"] \arrow[u, phantom, "\Uparrow\epsilon"] & & \mathsf{FibSets}(K'), \\
\mathsf{FibSets}(K) \arrow[uu, "\mathrm{id}_{\mathsf{FibSets}(K)}"] \arrow[ur, "\mathrm{Un}_K"] \arrow[rrrr, "\mathrm{id}_{\mathsf{FibSets}(K)}"'] & & & & \mathsf{FibSets}(K) \arrow[rrr, "\mathrm{id}_{\mathsf{FibSets}(K)}"'] & & & \mathsf{FibSets}(K) \arrow[urr, "\Sigma_f"'] & &
\end{tikzcd}
$$

where:

1. Subdiagram (1) commutes by Un/Straightening for Indexed and Fibred Sets, Theorem 1.3.1.1.
2. Subdiagram (2) commutes by unitality of composition.
3. Subdiagram (3) commutes by Un/Straightening for Indexed and Fibred Sets, Theorem 1.3.1.1.

And then, it can be rearranged into

$$
\begin{tikzcd}
 & & \mathsf{FibSets}(K') \arrow[rrrrrrr, "\mathrm{id}_{\mathsf{FibSets}(K')}"] & & & & & & & \mathsf{FibSets}(K') \arrow[dl, "\mathrm{Un}_K"] \arrow[dd, "\mathrm{id}_{\mathsf{FibSets}(K')}"] \\
\mathsf{FibSets}(K) \arrow[urr, "\Sigma_f"] & & \mathsf{ISets}(K') \arrow[u, "\mathrm{St}_{K'}"] \arrow[rrrrrr, "\mathrm{id}_{\mathsf{ISets}(K)}"] \arrow[drrrrr, "f^*"] \arrow[d, phantom, "\Uparrow\eta"] & & & & & & \mathsf{ISets}(K') \arrow[dr, "\mathrm{St}_{K'}"] & \\
 & \mathsf{ISets}(K) \arrow[ul, "\mathrm{St}_K"] \arrow[ur, "\Sigma_f"] \arrow[rrrrrr, "\mathrm{id}_{\mathsf{ISets}(K)}"] & & & & & & \mathsf{ISets}(K) \arrow[ur, "\Sigma_f"] \arrow[d, "\mathrm{St}_K"] \arrow[u, phantom, "\Uparrow\epsilon"] & & \mathsf{FibSets}(K'), \\
\mathsf{FibSets}(K) \arrow[uu, "\mathrm{id}_{\mathsf{FibSets}(K)}"] \arrow[ur, "\mathrm{Un}_K"] \arrow[rrrrrrr, "\mathrm{id}_{\mathsf{FibSets}(K)}"'] & & & & & & & \mathsf{FibSets}(K) \arrow[urr, "\Sigma_f"'] & &
\end{tikzcd}
$$

which by the left triangle identity for (η, ϵ), becomes

$$
\begin{tikzcd}
 & & \mathsf{FibSets}(K') \arrow[rrrrrrr, "\mathrm{id}_{\mathsf{FibSets}(K')}"] & & & & & & & \mathsf{FibSets}(K') \arrow[dl, "\mathrm{Un}_K"] \arrow[dd, "\mathrm{id}_{\mathsf{FibSets}(K')}"] \\
\mathsf{FibSets}(K) \arrow[urr, "\Sigma_f"] & & \mathsf{ISets}(K') \arrow[u, "\mathrm{St}_{K'}"] \arrow[rrrrrr, "\mathrm{id}_{\mathsf{ISets}(K)}"] & & {} \arrow[dr, phantom, "\mathsf{id}"] & & & & \mathsf{ISets}(K') \arrow[dr, "\mathrm{St}_{K'}"] & \\
 & \mathsf{ISets}(K) \arrow[ul, "\mathrm{St}_K"] \arrow[ur, "\Sigma_f"] \arrow[rrrrrr, "\mathrm{id}_{\mathsf{ISets}(K)}"] & & & & & & \mathsf{ISets}(K) \arrow[ur, "\Sigma_f"] \arrow[d, "\mathrm{St}_K"] & & \mathsf{FibSets}(K'), \\
\mathsf{FibSets}(K) \arrow[uu, "\mathrm{id}_{\mathsf{FibSets}(K)}"] \arrow[ur, "\mathrm{Un}_K"] \arrow[rrrrrrr, "\mathrm{id}_{\mathsf{FibSets}(K)}"'] & & & & & & & \mathsf{FibSets}(K) \arrow[urr, "\Sigma_f"'] & &
\end{tikzcd}
$$

finishing the proof of the left triangle identity. The proof of the right triangle identity is similar, and is thus omitted.

The Adjunction $f^* \dashv \Pi_f$: This proof is similar to the proof of the adjunction $\Sigma_f \dashv f^*$, and is thus omitted. $\square$

We proceed to the direct proof of Proposition 2.5.1.1.

Proof. The Adjunction $\Sigma_f \dashv f^*$: We claim there's a bijection

$$\mathrm{Hom}_{\mathsf{FibSets}(K')}(\Sigma_f(X), Y) \cong \mathrm{Hom}_{\mathsf{FibSets}(K)}(X, f^*(Y)),$$

natural in $(X, \phi_X) \in \mathsf{FibSets}(K)$ and $(Y, \phi_Y) \in \mathsf{FibSets}(K')$:

- *Map I.* We define a map

  $$\Phi_{X,Y} \colon \mathrm{Hom}_{\mathsf{FibSets}(K')}(\Sigma_f(X), Y) \to \mathrm{Hom}_{\mathsf{FibSets}(K)}(X, f^*(Y)),$$

  by sending a morphism

  $$\xi \colon \Sigma_f(X) \to Y, \qquad \begin{tikzcd} X \arrow[rr, "\xi"] \arrow[dr, "\phi_X"'] & & Y \arrow[ddl, "\phi_Y"] \\ & K \arrow[d, "f"'] & \\ & K' & \end{tikzcd}$$

  of K'-fibred sets to the morphism

  $$\xi^\dagger \colon X \to f^*(Y), \qquad \begin{tikzcd} X \arrow[r, "\xi^\dagger"] \arrow[dr, "\phi_X"'] & K \times_{K'} Y \arrow[d, "\mathrm{pr}_1"] \\ & K' \end{tikzcd}$$

  of K-fibred sets given by the dashed morphism in the diagram

  $$\begin{tikzcd} X \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left, "\xi"] \arrow[ddr, bend right, "\phi_X"'] & & \\ & K \times_{K'} Y \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "\phi_Y"] \\ & K \arrow[r, "f"'] & K'. \end{tikzcd}$$

- *Map II.* We define a map

  $$\Psi_{X,Y} \colon \mathrm{Hom}_{\mathsf{FibSets}(K)}(X, f^*(Y)) \to \mathrm{Hom}_{\mathsf{FibSets}(K')}(\Sigma_f(X), Y),$$

given by sending a map

$$\xi\colon X \to f^*(Y), \qquad \begin{tikzcd} X \arrow[r, "\xi"] \arrow[dr, "\phi_X"'] & K \times_{K'} Y \arrow[d, "\mathrm{pr}_1"] \\ & K' \end{tikzcd}$$

of K'-fibred sets to the map

$$\xi^\dagger\colon \Sigma_f(X) \to Y, \qquad \begin{tikzcd} X \arrow[rr, "\xi^\dagger"] \arrow[d, "\phi_X"'] & & Y \arrow[dl, "\phi_Y"] \\ K \arrow[r, "f"'] & K' & \end{tikzcd}$$

of K-fibred sets given by

$$\xi^\dagger \overset{\text{def}}{=} \mathrm{pr}_2 \circ \xi,$$

where we indeed have

$$\begin{aligned} \phi_Y \circ (\mathrm{pr}_2 \circ \xi) &= (\phi_Y \circ \mathrm{pr}_2) \circ \xi \\ &= (f \circ \mathrm{pr}_1) \circ \xi && \text{(by the pullback square of } K \times_{K'} Y\text{)} \\ &= f \circ (\mathrm{pr}_1 \circ \xi) \\ &= f \circ \phi_X. && \text{(since } \xi \text{ is a morphism of } K'\text{-fibred sets)} \end{aligned}$$

- *Naturality I.* We need to show that, given a morphism

$$\alpha\colon (X, \phi_X) \to (X', \phi_{X'})$$

of K-fibred sets, the diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathsf{FibSets}(K')}(\Sigma_f(X'), Y) \arrow[r, "\Phi_{X',Y}"] \arrow[d, "\Sigma_f(\alpha)^*"'] & \mathrm{Hom}_{\mathsf{FibSets}(K)}(X', f^*(Y)), \arrow[d, "\alpha^*"] \\ \mathrm{Hom}_{\mathsf{FibSets}(K')}(\Sigma_f(X), Y) \arrow[r, "\Phi_{X,Y}"'] & \mathrm{Hom}_{\mathsf{FibSets}(K)}(X, f^*(Y)) \end{tikzcd}$$

commutes. Indeed, given a morphism

$$\xi\colon \Sigma_f(X') \to Y, \qquad \begin{tikzcd} X' \arrow[rr, "\xi"] \arrow[d, "\phi_{X'}"'] & & Y \arrow[dl, "\phi_Y"] \\ K \arrow[r, "f"'] & K' & \end{tikzcd}$$

of K'-fibred-sets, the map $\Phi_{X',Y}(\xi) \circ \alpha$ is the composition, coloured in vermillion, of the dashed arrow with α in the diagram

$$\begin{tikzcd}
X \arrow[dr, "\alpha"] \arrow[rrrdd, bend left, "\xi\circ\alpha"] \arrow[rrdd, dashed, bend left, "\exists!"] \arrow[rrddd, bend right, "\phi_{X'}\circ\alpha"'] & & & \\
& X' \arrow[dr, dashed, "\exists!"] \arrow[rrd, bend left, "\xi"] \arrow[rdd, bend right, "\phi_{X'}"'] & & \\
& & K \times_{K'} Y \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] \arrow[dr, phantom, very near start, "\lrcorner"] & Y \arrow[d, "\phi_Y"] \\
& & K \arrow[r, "f"'] & K',
\end{tikzcd}$$

while $\Phi_{X,Y}(\xi \circ \Sigma_f(\alpha))$ is given by the dashed arrow, coloured in blue. Since both the blue arrow and the vermillion arrow make the outer pullback diagram commute, it follows from the universal property of the pullback that they must be equal, i.e. that

$$\Phi_{X',Y}(\xi) \circ \alpha = \Phi_{X,Y}(\xi \circ \Sigma_f(\alpha)),$$

showing that the naturality diagram above indeed commutes.

- *Naturality II.* We need to show that, given a morphism

$$\beta \colon (Y, \phi_Y) \to (Y', \phi_{Y'})$$

of K-fibred sets, the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathsf{FibSets}(K')}(\Sigma_f(X), Y) \arrow[r, "\Phi_{X,Y}"] \arrow[d, "\beta_*"'] & \mathrm{Hom}_{\mathsf{FibSets}(K)}(X, f^*(Y)), \arrow[d, "f^*(\beta)_*"] \\
\mathrm{Hom}_{\mathsf{FibSets}(K')}(\Sigma_f(X), Y') \arrow[r, "\Phi_{X,Y'}"'] & \mathrm{Hom}_{\mathsf{FibSets}(K)}(X, f^*(Y'))
\end{tikzcd}$$

commutes. Indeed, given a morphism

$$\xi \colon \Sigma_f(X') \to Y, \qquad \begin{tikzcd}
X' \arrow[rr, "\xi"] \arrow[dr, "\phi_{X'}"'] & & Y \arrow[ddl, "\phi_Y"] \\
& K \arrow[d, "f"'] & \\
& K' &
\end{tikzcd}$$

of K'-fibred-sets, the map $f^*(\beta)\circ\Phi_{X,Y}(\xi)$ is the composition, coloured in vermillion, of the dashed arrow from X to $K\times_{K'}Y$ with the dashed arrow from $K\times_{K'}Y$ to $K\times_{K'}Y'$ in the diagram

$$\begin{tikzcd}
X \arrow[dr, dashed, "\exists!"] \arrow[ddrr, dashed, bend left, "\exists!"] \arrow[drrr, bend left, "\xi"] \arrow[dddr, bend right, "\phi_X"'] & & & & \\
& K\times_{K'}Y \arrow[rr] \arrow[dd] \arrow[dr, dashed] \arrow[dr, phantom, "\lrcorner", very near start] & & Y \arrow[dr, "\beta"] \arrow[dd, "\phi_Y"] & \\
& & K\times_{K'}Y' \arrow[rr] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & & Y' \arrow[dd, "\phi_{Y'}"] \\
& K \arrow[rr, "f"] \arrow[dr, equal] & & K \arrow[dr, equal] & \\
& & K \arrow[rr, "f"'] & & K',
\end{tikzcd}$$

while $\Phi_{X,Y'}(\beta\circ\xi)$ is given by the dashed arrow from X to $K\times_{K'}Y'$, coloured in blue. Since both the blue arrow and the vermillion arrow make the outer pullback diagram for $K\times_{K'}Y'$ commute, it follows from the universal property of the pullback that they must be equal, i.e. that

$$f^*(\beta)\circ\Phi_{X,Y}(\xi)=\Phi_{X,Y'}(\beta\circ\xi),$$

showing that the naturality diagram above indeed commutes.

- *Invertibility I.* We claim that

$$\Psi_{X,Y}\circ\Phi_{X,Y}=\mathrm{id}_{\mathrm{Hom}_{\mathsf{FibSets}(K')}(\Sigma_f(X),Y)}.$$

Indeed, $\Phi_{X,Y}$ sends a map

$$\xi\colon\Sigma_f(X)\to Y,\qquad \begin{tikzcd}
X \arrow[rr, "\xi"] \arrow[dr, "\phi_X"'] & & Y \arrow[ddl, "\phi_Y"] \\
& K \arrow[d, "f"'] & \\
& K' &
\end{tikzcd}$$

of K'-fibred sets to the dashed morphism in the diagram

$$\begin{tikzcd}
X \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left, "\xi"] \arrow[ddr, bend right, "\phi_X"'] & & \\
& K \times_{K'} Y \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "\phi_Y"] \\
& K \arrow[r, "f"'] & K',
\end{tikzcd}$$

and $\Psi_{X,Y}$ then postcomposes that map with pr_2, which, by the commutativity of the diagram above, is ξ again, showing the claimed equality to be true.

- *Invertibility II.* We claim that

$$\Phi_{X,Y} \circ \Psi_{X,Y} = \mathrm{id}_{\mathrm{Hom}_{\mathsf{FibSets}(K)}(X, f^*(Y))}.$$

Indeed, $\Psi_{X,Y}$ sends a map

$$\xi \colon X \to f^*(Y), \qquad \begin{tikzcd}
X \arrow[rr, "\xi"] \arrow[dr, "\phi_X"'] & & K \times_{K'} Y \arrow[dl, "\mathrm{pr}_1"] \\
& K' &
\end{tikzcd}$$

of K'-fibred sets to $\mathrm{pr}_2 \circ \xi$, which is then sent by $\Phi_{X,Y}$ to the dashed morphism in the diagram

$$\begin{tikzcd}
X \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left, "\mathrm{pr}_2 \circ \xi"] \arrow[ddr, bend right, "\phi_X"'] \arrow[drr, phantom, "(\dagger)"] & & \\
& K \times_{K'} Y \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "\phi_Y"] \\
& K \arrow[r, "f"'] & K',
\end{tikzcd}$$

which, by the commutativity of the subdiagram marked with $(\dagger)$, is given by ξ again, showing the claimed equality to be true.

The Adjunction $f^* \dashv \Pi_f$: We claim there's a bijection

$$\mathrm{Hom}_{\mathsf{FibSets}(K)}(f^*(X), Y), \cong \mathrm{Hom}_{\mathsf{FibSets}(K')}(X, \Pi_f(Y))$$

natural in $(X, \phi_X) \in \mathsf{FibSets}(K')$ and $(Y, \phi_Y) \in \mathsf{FibSets}(K)$:

1. *Map I.* We define a map

   $$\Phi_{X,Y} \colon \mathrm{Hom}_{\mathsf{FibSets}(K)}(f^*(X), Y) \to \mathrm{Hom}_{\mathsf{FibSets}(K')}(X, \Pi_f(Y))$$

   defined as follows. Given a morphism

   $$\xi \colon f^*(X) \to Y, \qquad \begin{tikzcd} K \times_{K'} X \arrow[rr, "\xi"] \arrow[dr, "\mathrm{pr}_1"'] & & Y \arrow[dl, "\phi_Y"] \\ & K & \end{tikzcd}$$

   of K-fibred sets, where

   $$\begin{aligned} f^*(X) &\stackrel{\text{def}}{=} K \times_{K'} X \\ &\stackrel{\text{def}}{=} \{(k, x) \in K \times X \mid f(k) = \phi_X(x)\}, \end{aligned}$$

   we construct a morphism

   $$\xi^\dagger \colon X \to \Pi_f(Y), \qquad \begin{tikzcd} X \arrow[rr, "\xi^\dagger"] \arrow[dr, "\phi_X"'] & & \Pi_f(Y) \arrow[dl, "\Pi_f(\phi_Y)"] \\ & K' & \end{tikzcd}$$

   of K'-fibred sets, where

   $$\Pi_f(Y) \stackrel{\text{def}}{=} \coprod_{k' \in K'} \prod_{k \in f^{-1}(k')} \phi_Y^{-1}(k),$$

   by defining

   $$\xi^\dagger(x) \stackrel{\text{def}}{=} (\xi(k, x))_{k \in f^{-1}(\phi_X(x))}$$

   for each $x \in X$. There are two things to be checked here:

   - We have $\xi(k, x) \in \phi_Y^{-1}(\phi_X(x))$ since $\phi_Y(\xi(k, x)) = \phi_X(x)$ as ξ is a morphism of K-fibred sets.

- The map $\xi^\dagger$ is indeed a morphism of K'-fibred sets, i.e. we have

$$\Pi_f(\phi_Y) \circ \xi^\dagger = \phi_X,$$

since

$$[\Pi_f(\phi_Y)]\Big((\xi(k,x))_{k\in f^{-1}(\phi_X(x))}\Big) = \phi_X(x)$$

for each $x \in X$.

2. *Map II.* We define a map

$$\Psi_{X,Y} \colon \mathrm{Hom}_{\mathsf{FibSets}(K')}(X, \Pi_f(Y)) \to \mathrm{Hom}_{\mathsf{FibSets}(K)}(f^*(X), Y)$$

as follows. Given a morphism

$$\xi \colon X \to \Pi_f(Y), \qquad \begin{tikzcd} X \arrow[rr, "\xi"] \arrow[dr, "\phi_X"'] & & \Pi_f(Y) \arrow[dl, "\Pi_f(\phi_Y)"] \\ & K' & \end{tikzcd}$$

of K'-fibred sets, where

$$\Pi_f(Y) \overset{\text{def}}{=} \coprod_{k'\in K'} \prod_{k\in f^{-1}(k')} \phi_Y^{-1}(k),$$

we construct a morphism

$$\xi^\dagger \colon f^*(X) \to Y, \qquad \begin{tikzcd} K \times_{K'} X \arrow[rr, "\xi^\dagger"] \arrow[dr, "\mathrm{pr}_1"'] & & Y \arrow[dl, "\phi_Y"] \\ & K & \end{tikzcd}$$

of K-fibred sets, where

$$\begin{aligned} f^*(X) &\overset{\text{def}}{=} K \times_{K'} X \\ &\overset{\text{def}}{=} \{(k,x) \in K \times X \mid f(k) = \phi_X(x)\}, \end{aligned}$$

by defining

$$\xi^\dagger(k,x) \overset{\text{def}}{=} \xi(x)_k$$

for each $(k,x) \in f^*(X)$, where $\xi(x)_k$ is the kth component of $\xi(x) = (y_k)_{k\in f^{-1}(k')}$. We also need to check that $\xi^\dagger$ is a morphism of K-fibred sets, i.e. that

$$\phi_Y \circ \xi^\dagger = \mathrm{pr}_1,$$

or

$$\phi_Y\Big(\xi^\dagger(k,x)\Big) = k,$$

for each $(k,x) \in f^*(X)$, which is clear, since $\xi^\dagger(k,x) \in \phi_Y^{-1}(k)$ by definition.

3. *Naturality I.* We need to show that, given a morphism

$$\alpha\colon (X,\phi_X) \to (X',\phi_{X'})$$

of K'-fibred sets, the diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathsf{FibSets}(K')}(f^*(X'),Y) \arrow[r, "\Phi_{X',Y}"] \arrow[d, "f^*(\alpha)^*"'] & \mathrm{Hom}_{\mathsf{FibSets}(K)}(X',\Pi_f(Y)) \arrow[d, "\alpha^*"] \\ \mathrm{Hom}_{\mathsf{FibSets}(K')}(f^*(X),Y) \arrow[r, "\Phi_{X,Y}"'] & \mathrm{Hom}_{\mathsf{FibSets}(K)}(X,\Pi_f(Y)) \end{tikzcd}$$

commutes. Indeed, given a morphism $\xi\colon f^*(X') \to Y$ of K'-fibred sets, we have

$$\begin{aligned} [[\Phi_{X,Y} \circ f^*(\alpha)](\xi)](x) &\overset{\text{def}}{=} [\Phi_{X,Y}(\xi \circ f^*(\alpha))](x) \\ &\overset{\text{def}}{=} ([\xi \circ f^*(\alpha)](k,x))_{k \in f^{-1}(\phi_X(x))} \\ &\overset{\text{def}}{=} (\xi(k,\alpha(x)))_{k \in f^{-1}(\phi_X(x))} \\ &\overset{\text{def}}{=} \alpha^*\Big((\xi(k,x))_{k \in f^{-1}(\phi_X(x))}\Big) \\ &\overset{\text{def}}{=} \alpha^*\Big(\xi^\dagger(x)\Big) \\ &\overset{\text{def}}{=} [[\alpha^* \circ \Phi_{X,Y}](\xi)](x) \end{aligned}$$

for each $x \in X$.

4. *Naturality II.* We need to show that, given a morphism

$$\beta\colon (Y,\phi_Y) \to (Y',\phi_{Y'})$$

of K-fibred sets, the diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathsf{FibSets}(K')}(f^*(X),Y) \arrow[r, "\Phi_{X,Y}"] \arrow[d, "\beta_*"'] & \mathrm{Hom}_{\mathsf{FibSets}(K)}(X,\Pi_f(Y)) \arrow[d, "\Pi_f(\beta)_*"] \\ \mathrm{Hom}_{\mathsf{FibSets}(K')}(f^*(X),Y') \arrow[r, "\Phi_{X,Y'}"'] & \mathrm{Hom}_{\mathsf{FibSets}(K)}(X,\Pi_f(Y')) \end{tikzcd}$$

commutes. Indeed, given a morphism $\xi\colon X \to \Pi_f(Y)$ of K-fibred sets, we have

$$\begin{aligned}
[[\Phi_{X,Y'} \circ \beta_*](\xi)](x) &\stackrel{\text{def}}{=} [\Phi_{X,Y'}(\beta \circ \xi)](x)\\
&\stackrel{\text{def}}{=} [\Phi_{X,Y'}(\beta \circ \xi)](x)\\
&\stackrel{\text{def}}{=} ([\beta \circ \xi](k,x))_{k \in f^{-1}(\phi_X(x))}\\
&\stackrel{\text{def}}{=} (\beta(\xi(k,x)))_{k \in f^{-1}(\phi_X(x))}\\
&\stackrel{\text{def}}{=} \Pi_f(\beta)_*\Big((\xi(k,x))_{k \in f^{-1}(\phi_X(x))}\Big)\\
&\stackrel{\text{def}}{=} [\Pi_f(\beta)_* \circ \xi^\dagger](x)\\
&\stackrel{\text{def}}{=} [\Pi_f(\beta)_* \circ \Phi_{X,Y'}(\xi)](x)
\end{aligned}$$

for each $x \in X$.

5. *Invertibility I.* We claim that

$$\Psi_{X,Y} \circ \Phi_{X,Y} = \mathrm{id}_{\mathrm{Hom}_{\mathsf{FibSets}(K)}(f^*(X),Y)}.$$

Indeed, given a morphism $\xi\colon f^*(X') \to Y$ of K'-fibred sets, we have

$$\begin{aligned}
[[\Psi_{X,Y} \circ \Phi_{X,Y}](\xi)](k,x) &\stackrel{\text{def}}{=} [\Psi_{X,Y}(\Phi_{X,Y}(\xi))](k,x)\\
&\stackrel{\text{def}}{=} ([\Phi_{X,Y}(\xi)](x))_k\\
&\stackrel{\text{def}}{=} \Big((\xi(k_1,x))_{k_1 \in f^{-1}(\phi_X(x))}\Big)_k\\
&\stackrel{\text{def}}{=} \xi(k,x)
\end{aligned}$$

for each $(k,x) \in f^*(X)$, and thus the stated equality follows.

6. *Invertibility II.* We claim that

$$\Phi_{X,Y} \circ \Psi_{X,Y} = \mathrm{id}_{\mathrm{Hom}_{\mathsf{FibSets}\left(K'\right)}\left(X,\Pi_f(Y)\right)}.$$

Indeed, given a morphism $\xi\colon X \to \Pi_f(Y)$ of K-fibred sets, write

$$\xi(x) = (y_k)_{k \in f^{-1}(k'_x)}.$$

We then have

$$
\begin{aligned}
[[\Phi_{X,Y} \circ \Psi_{X,Y}](\xi)](x) &\overset{\text{def}}{=} [\Phi_{X,Y}(\Psi_{X,Y}(\xi))](x)\\
&\overset{\text{def}}{=} ([\Psi_{X,Y}(\xi)](k_1, x))_{k_1 \in f^{-1}(\phi_X(x))}\\
&\overset{\text{def}}{=} \Big((\xi(x))_{k_1}\Big)_{k_1 \in f^{-1}(\phi_X(x))}\\
&\overset{\text{def}}{=} \left(\Big((y_k)_{k \in f^{-1}(k'_x)}\Big)_{k_1}\right)_{k_1 \in f^{-1}(\phi_X(x))}\\
&\overset{\text{def}}{=} (y_{k_1})_{k_1 \in f^{-1}(\phi_X(x))}\\
&= (y_{k_1})_{k_1 \in f^{-1}(k'_x)}\\
&= (y_k)_{k \in f^{-1}(k'_x)}\\
&\overset{\text{def}}{=} \xi(x)
\end{aligned}
$$

for each $x \in X$, where the equality $\phi_X(x) = k'_x$ follows from the fact that ξ is a morphism of K'-fibred sets. Thus the stated equality follows.

This finishes the proof. □

Appendices

A Other Chapters

Set Theory

1. Sets
2. Constructions With Sets
3. Pointed Sets
4. Tensor Products of Pointed Sets
5. Indexed and Fibred Sets
6. Relations
7. Spans
8. Posets

Category Theory

9. Categories
10. Constructions With Categories
11. Kan Extensions

Bicategories

12. Bicategories
13. Internal Adjunctions

Internal Category Theory

14. Internal Categories

Cyclic Stuff

15. The Cycle Category

Cubical Stuff

16. The Cube Category

Globular Stuff

17. The Globe Category

Cellular Stuff

18. The Cell Category

Monoids

19. Monoids
20. Constructions With Monoids

Monoids With Zero

21. Monoids With Zero
22. Constructions With Monoids With Zero

Groups

23. Groups
24. Constructions With Groups

Hyper Algebra

25. Hypermonoids
26. Hypergroups
27. Hypersemirings and Hyperrings
28. Quantales

Near-Rings

29. Near-Semirings
30. Near-Rings

Real Analysis

31. Real Analysis in One Variable
32. Real Analysis in Several Variables

Measure Theory

33. Measurable Spaces
34. Measures and Integration

Probability Theory

34. Probability Theory

Stochastic Analysis

35. Stochastic Processes, Martingales, and Brownian Motion
36. Itô Calculus
37. Stochastic Differential Equations

Differential Geometry

38. Topological and Smooth Manifolds

Schemes

39. Schemes