Categories

December 24, 2023

00YK Create tags (see [MSE 350788] for some of these):

1. **??**
2. **??**
3. **??**
4. **??**
5. **??**
6. **??**
7. **??**
8. **??**
9. write material on sections and retractions
10. define bicategory $\mathsf{Adj}(\mathcal{C})$
11. https://www.google.com/search?q=category+of+categories+is+not+locally+cartesian+closed
12. https://math.stackexchange.com/questions/2864916/are-there-important-locally-cartesian-closed-categories-that-actually-are-not-ca
13. Cats is not locally Cartesian closed: f^* does have a left adjoint (the proof for fibred sets seems to apply for any category with pullbacks), but does not have a right adjoint, as proved in example 1.7 of https://sinhp.github.io/files/CT/notes_on_lcccs.pdf
14. internal **Hom** in categories of co/Cartesian fibrations

15. https://mathoverflow.net/questions/460146/universal-property-of-isbell-duality

16. http://www.tac.mta.ca/tac/volumes/36/12/36-12abs.html

17. Cartesian closed categories and locally Cartesian closed categories

    (a) https://ncatlab.org/nlab/show/locally+cartesian+closed+functor

    (b) https://ncatlab.org/nlab/show/cartesian+closed+functor

    (c) https://ncatlab.org/nlab/show/locally+cartesian+closed+category

    (d) https://ncatlab.org/nlab/show/Frobenius+reciprocity

18. https://math.stackexchange.com/questions/3657046/the-inverse-of-a-natural-isomorphism-is-a-natural-isomorphism to justify adjunctions via homs

19. https://ncatlab.org/nlab/show/enrichment+versus+internalisation

20. https://mathoverflow.net/questions/382239/proof-that-a-cartesian-category-is-monoidal

Contents

1 Categories 00YL

1.1 Foundations 00YM

Definition 1.1.1.1. A **category** $(\mathcal{C}, \circ^{\mathcal{C}}, 1^{\mathcal{C}})$ 00YN consists of[1,2]

- *Objects.* A class $\mathrm{Obj}(\mathcal{C})$ of **objects**;
- *Morphisms.* For each $A, B \in \mathrm{Obj}(\mathcal{C})$, a class $\mathrm{Hom}_{\mathcal{C}}(A, B)$, called the **class of morphisms of $\mathcal{C}$ from A to B**;

[1] *Further Notation:* We also write $\mathcal{C}(A, B)$ for $\mathrm{Hom}_{\mathcal{C}}(A, B)$.
[2] *Further Notation:* We write $\mathrm{Mor}(\mathcal{C})$ for the class of all morphisms of $\mathcal{C}$.

- *Identities.* For each $A \in \mathrm{Obj}(\mathcal{C})$, a map of sets

  $$\mathbb{1}^{\mathcal{C}}_{A} \colon \mathrm{pt} \to \mathrm{Hom}_{\mathcal{C}}(A, A),$$

  called the **unit map of $\mathcal{C}$ at A**, determining a morphism

  $$\mathrm{id}_A \colon A \to A$$

  of $\mathcal{C}$, called the **identity morphism of A**;

- *Composition.* For each $A, B, C \in \mathrm{Obj}(\mathcal{C})$, a map of sets

  $$\circ^{\mathcal{C}}_{A,B,C} \colon \mathrm{Hom}_{\mathcal{C}}(B, C) \times \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{C}}(A, C),$$

  called the **composition map of $\mathcal{C}$ at (A, B, C)**;

such that the following conditions are satisfied:

1. *Left Unitality.* The diagram

   $$\begin{tikzcd}
   \mathrm{pt} \times \mathrm{Hom}_{\mathcal{C}}(A, B) \arrow[d, "\mathbb{1}^{\mathcal{C}}_{A} \times \mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(A,B)}"'] \arrow[dr, dashed, "\lambda^{\mathsf{Sets}}_{\mathrm{Hom}_{\mathcal{C}}(A,B)}", "\sim"'] & \\
   \mathrm{Hom}_{\mathcal{C}}(A, A) \times \mathrm{Hom}_{\mathcal{C}}(A, B) \arrow[r, "\circ^{\mathcal{C}}_{A,A,B}"'] & \mathrm{Hom}_{\mathcal{C}}(A, B)
   \end{tikzcd}$$

   commutes, i.e. for each morphism $f \colon A \to B$ of $\mathcal{C}$, we have

   $$\mathrm{id}_B \circ f = f.$$

2. *Right Unitality.* The diagram

   $$\begin{tikzcd}
   \mathrm{Hom}_{\mathcal{C}}(A, B) \times \mathrm{pt} \arrow[d, "\mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(A,B)} \times \mathbb{1}^{\mathcal{C}}_{B}"'] \arrow[dr, dashed, "\rho^{\mathsf{Sets}}_{\mathrm{Hom}_{\mathcal{C}}(A,B)}", "\sim"'] & \\
   \mathrm{Hom}_{\mathcal{C}}(A, B) \times \mathrm{Hom}_{\mathcal{C}}(B, B) \arrow[r, "\circ^{\mathcal{C}}_{A,B,B}"'] & \mathrm{Hom}_{\mathcal{C}}(A, B)
   \end{tikzcd}$$

   commutes, i.e. for each morphism $f \colon A \to B$ of $\mathcal{C}$, we have

   $$f \circ \mathrm{id}_A = f.$$

3. *Associativity.* The diagram

$$\begin{tikzcd}
& & \mathrm{Hom}_{\mathcal{C}}(C,D)\times(\mathrm{Hom}_{\mathcal{C}}(B,C)\times\mathrm{Hom}_{\mathcal{C}}(A,B)) \arrow[drr, "\mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(C,D)}\times\circ^{\mathcal{C}}_{A,B,C}"] & & \\
(\mathrm{Hom}_{\mathcal{C}}(C,D)\times\mathrm{Hom}_{\mathcal{C}}(B,C))\times\mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[urr, "\alpha^{\mathsf{Sets}}_{\mathrm{Hom}_{\mathcal{C}}(C,D),\mathrm{Hom}_{\mathcal{C}}(B,C),\mathrm{Hom}_{\mathcal{C}}(A,B)}"] \arrow[dr, "\circ^{\mathcal{C}}_{B,C,D}\times\mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(A,B)}"'] & & & & \mathrm{Hom}_{\mathcal{C}}(C,D)\times\mathrm{Hom}_{\mathcal{C}}(A,C) \arrow[dl, "\circ^{\mathcal{C}}_{A,C,D}"] \\
& \mathrm{Hom}_{\mathcal{C}}(B,D)\times\mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[rr, "\circ^{\mathcal{C}}_{A,B,D}"'] & & \mathrm{Hom}_{\mathcal{C}}(A,D) &
\end{tikzcd}$$

commutes, i.e. for each composable triple (f,g,h) of morphisms of $\mathcal{C}$, we have

$$(f\circ g)\circ h = f\circ(g\circ h).$$

Definition 1.1.1.2. Let κ be a regular cardinal. A category $\mathcal{C}$ is

1. **Locally small** if, for each $A,B\in\mathrm{Obj}(\mathcal{C})$, the class $\mathrm{Hom}_{\mathcal{C}}(A,B)$ is a set.

2. **Locally essentially small** if, for each $A,B\in\mathrm{Obj}(\mathcal{C})$, the class

$$\mathrm{Hom}_{\mathcal{C}}(A,B)/\{\text{isomorphisms}\}$$

is a set.

3. **Small** if $\mathcal{C}$ is locally small and $\mathrm{Obj}(\mathcal{C})$ is a set.

4. **κ-Small** if $\mathcal{C}$ is locally small, $\mathrm{Obj}(\mathcal{C})$ is a set, and we have $\#\mathrm{Obj}(\mathcal{C})<\kappa$.

1.2 Examples of Categories

Example 1.2.1.1. The **punctual category** 00YR[3] is the category pt where

- *Objects.* We have
$$\mathrm{Obj}(\mathsf{pt}) \overset{\text{def}}{=} \{\star\};$$
- *Morphisms.* The unique Hom-set of pt is defined by
$$\mathrm{Hom}_{\mathsf{pt}}(\star,\star) \overset{\text{def}}{=} \{\mathrm{id}_\star\};$$
- *Identities.* The unit map
$$\mathbb{1}^{\mathsf{pt}}_{\star} \colon \mathrm{pt} \to \mathrm{Hom}_{\mathsf{pt}}(\star,\star)$$
of pt at $\star$ is defined by
$$\mathrm{id}^{\mathsf{pt}}_{\star} \overset{\text{def}}{=} \mathrm{id}_\star;$$
- *Composition.* The composition map
$$\circ^{\mathsf{pt}}_{\star,\star,\star} \colon \mathrm{Hom}_{\mathsf{pt}}(\star,\star) \times \mathrm{Hom}_{\mathsf{pt}}(\star,\star) \to \mathrm{Hom}_{\mathsf{pt}}(\star,\star)$$
of pt at $(\star,\star,\star)$ is given by the bijection $\mathrm{pt} \times \mathrm{pt} \cong \mathrm{pt}$.

Example 1.2.1.2. We have an isomorphism 00YS of categories[4]

$$\mathsf{Mon} \cong \mathsf{pt} \underset{\mathsf{Sets}}{\times} \mathsf{Cats}, \qquad \begin{tikzcd} \mathsf{Mon} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Cats} \arrow[d, "\mathrm{Obj}"] \\ \mathsf{pt} \arrow[r, "{[\mathrm{pt}]}"'] & \mathsf{Sets} \end{tikzcd}$$

via the delooping functor $\mathsf{B} \colon \mathsf{Mon} \to \mathsf{Cats}$ of **??** of **??**.

[3] *Further Terminology:* Also called the **singleton category**.

[4] This can be enhanced to an isomorphism of 2-categories

$$\mathsf{Mon}_{2-\mathsf{disc}} \cong \mathsf{pt}_{\mathsf{bi}} \underset{\mathsf{Sets}_{2-\mathsf{disc}}}{\times} \mathsf{Cats}_{2,*}, \qquad \begin{tikzcd} \mathsf{Mon}_{2-\mathsf{disc}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Cats}_{2,*} \arrow[d, "\mathrm{Obj}"] \\ \mathsf{pt}_{\mathsf{bi}} \arrow[r, "{[\mathrm{pt}]}"'] & \mathsf{Sets}_{2-\mathsf{disc}} \end{tikzcd}$$

between the discrete 2-category $\mathsf{Mon}_{2-\mathsf{disc}}$ on Mon and the 2-category of pointed categories with one object.

Proof. Omitted. $\square$

Example 1.2.1.3. The **empty category** 00YT is the category $\emptyset_{\mathsf{cat}}$ where

- *Objects.* We have
$$\mathrm{Obj}(\emptyset_{\mathsf{cat}}) \overset{\text{def}}{=} \emptyset;$$
- *Morphisms.* We have
$$\mathrm{Mor}(\emptyset_{\mathsf{cat}}) \overset{\text{def}}{=} \emptyset;$$
- *Identities and Composition.* Having no objects, $\emptyset_{\mathsf{cat}}$ has no unit nor composition maps.

Example 1.2.1.4. The n**th ordinal category** 00YU is the category $\ltimes$ where[5]

- *Objects.* We have
$$\mathrm{Obj}(\ltimes) \overset{\text{def}}{=} \{[0], \ldots, [n]\};$$
- *Morphisms.* For each $[i], [j] \in \mathrm{Obj}(\ltimes)$, we have
$$\mathrm{Hom}_{\ltimes}([i],[j]) \overset{\text{def}}{=} \begin{cases} \left\{\mathrm{id}_{[i]}\right\} & \text{if } [i] = [j], \\ \{[i] \to [j]\} & \text{if } [j] < [i], \\ \emptyset & \text{if } [j] > [i]; \end{cases}$$

[5]In other words, $\ltimes$ is the category associated to the poset
$$[0] \to [1] \to \cdots \to [n-1] \to [n].$$
The category $\ltimes$ for $n \geq 2$ may also be defined in terms of $\nvdash$ and joins: we have isomorphisms of categories
$$\begin{aligned} \nVdash &\cong \nvdash \star \nvdash, \\ \nvDash &\cong \nVdash \star \nvdash \\ &\cong (\nvdash \star \nvdash) \star \nvdash, \\ \nVDash &\cong \nvDash \star \nvdash \\ &\cong (\nVdash \star \nvdash) \star \nvdash \\ &\cong ((\nvdash \star \nvdash) \star \nvdash) \star \nvdash, \\ \ntrianglerighteq &\cong \nVDash \star \nvdash \\ &\cong (\nvDash \star \nvdash) \star \nvdash \\ &\cong ((\nVdash \star \nvdash) \star \nvdash) \star \nvdash \\ &\cong (((\nvdash \star \nvdash) \star \nvdash) \star \nvdash) \star \nvdash, \end{aligned}$$
and so on.

- *Identities.* For each $[i] \in \mathrm{Obj}(\ltimes)$, the unit map

$$\nVdash^{\ltimes}_{[i]} \colon \mathrm{pt} \to \mathrm{Hom}_{\ltimes}([i],[i])$$

  of $\ltimes$ at $[i]$ is defined by

$$\mathrm{id}^{\ltimes}_{[i]} \overset{\mathrm{def}}{=} \mathrm{id}_{[i]};$$

- *Composition.* For each $[i],[j],[k] \in \mathrm{Obj}(\ltimes)$, the composition map

$$\circ^{\ltimes}_{[i],[j],[k]} \colon \mathrm{Hom}_{\ltimes}([j],[k]) \times \mathrm{Hom}_{\ltimes}([i],[j]) \to \mathrm{Hom}_{\ltimes}([i],[k])$$

  of $\ltimes$ at $([i],[j],[k])$ is defined by

$$\begin{gathered} \mathrm{id}_{[i]} \circ \mathrm{id}_{[i]} = \mathrm{id}_{[i]}, \\ ([j] \to [k]) \circ ([i] \to [j]) = ([i] \to [k]). \end{gathered}$$

Example 1.2.1.5. Here we list all the other 00YV categories that appear throughout this work.

- The category Sets_* of pointed sets of Pointed Sets, Definition 1.3.1.1.
- The category Rel of sets and relations of Relations, Definition 2.1.1.1.
- The category $\mathsf{Span}(A,B)$ of spans from a set A to a set B of Spans, Definition 2.1.1.1.
- The category $\mathsf{ISets}(K)$ of K-indexed sets of Indexed Sets, Definition 1.3.1.1.
- The category ISets of indexed sets of Indexed Sets, Definition 1.4.1.1.
- The category $\mathsf{FibSets}(K)$ of K-fibred sets of Fibred Sets, Definition 1.3.1.1.
- The category $\mathsf{FibSets}$ of fibred sets of Fibred Sets, Definition 1.4.1.1.

1.3 Subcategories 00YW

Let $\mathcal{C}$ be a category.

Definition 1.3.1.1. A **subcategory** of $\mathcal{C}$ 00YX is a category $\mathcal{A}$ satisfying the following conditions:

1. *Objects.* We have $\mathrm{Obj}(\mathcal{A}) \subset \mathrm{Obj}(\mathcal{C})$.

2. *Morphisms.* For each $A, B \in \mathrm{Obj}(\mathcal{A})$, we have

$$\mathrm{Hom}_{\mathcal{A}}(A, B) \subset \mathrm{Hom}_{\mathcal{C}}(A, B).$$

3. *Identities.* For each $A \in \mathrm{Obj}(\mathcal{A})$, we have

$$\nVdash^{\mathcal{A}}_{A} = \nVdash^{\mathcal{C}}_{A}.$$

4. *Composition.* For each $A, B, C \in \mathrm{Obj}(\mathcal{A})$, we have

$$\circ^{\mathcal{A}}_{A,B,C} = \circ^{\mathcal{C}}_{A,B,C}.$$

Definition 1.3.1.2. A subcategory $\mathcal{A}$ of $\mathcal{C}$ is **full** if the canonical inclusion functor $\mathcal{A} \to \mathcal{C}$ is full, i.e. if, for each $A, B \in \mathrm{Obj}(\mathcal{A})$, the inclusion 00YY

$$\iota_{A,B} \colon \mathrm{Hom}_{\mathcal{A}}(A, B) \hookrightarrow \mathrm{Hom}_{\mathcal{C}}(A, B)$$

is surjective (and thus bijective).

Definition 1.3.1.3. A subcategory $\mathcal{A}$ of a category $\mathcal{C}$ is **strictly full** if it satisfies the following conditions: 00YZ

1. *Fullness.* The subcategory $\mathcal{A}$ is full.
2. *Closedness Under Isomorphisms.* The class $\mathrm{Obj}(\mathcal{A})$ is closed under isomorphisms.[6]

Definition 1.3.1.4. A subcategory $\mathcal{A}$ of $\mathcal{C}$ is **wide**[7] if $\mathrm{Obj}(\mathcal{A}) = \mathrm{Obj}(\mathcal{C})$. 00Z0

1.4 Skeletons of Categories 00Z1

Definition 1.4.1.1. A[8] **skeleton** of a category $\mathcal{C}$ is a full subcategory $\mathsf{Sk}(\mathcal{C})$ with one object from each isomorphism class of objects of $\mathcal{C}$. 00Z2

Definition 1.4.1.2. A category $\mathcal{C}$ is **skeletal** if $\mathcal{C} \cong \mathsf{Sk}(\mathcal{C})$.[9] 00Z3

Proposition 1.4.1.3. Let $\mathcal{C}$ be a category. 00Z4

1. *Existence.* Assuming the axiom of choice, $\mathsf{Sk}(\mathcal{C})$ always exists. 00Z5

[6] That is, given $A \in \mathrm{Obj}(\mathcal{A})$ and $C \in \mathrm{Obj}(\mathcal{C})$, if $C \cong A$, then $C \in \mathrm{Obj}(\mathcal{A})$.

[7] *Further Terminology:* Also called **lluf**.

[8] Due to **??** of **??**, we often refer to any such full subcategory $\mathsf{Sk}(\mathcal{C})$ of $\mathcal{C}$ as *the* skeleton of $\mathcal{C}$.

[9] That is, $\mathcal{C}$ is **skeletal** if isomorphic objects of $\mathcal{C}$ are equal.

2. *Pseudofunctoriality.* 00Z6 The assignment $\mathcal{C} \mapsto \mathsf{Sk}(\mathcal{C})$ defines a pseudofunctor
$$\mathsf{Sk}\colon \mathsf{Cats}_2 \to \mathsf{Cats}_2.$$
3. *Uniqueness Up to Equivalence.* 00Z7 Any two skeletons of $\mathcal{C}$ are equivalent.
4. *Inclusions of Skeletons Are Equivalences.* The inclusion 00Z8
$$\iota_{\mathcal{C}}\colon \mathsf{Sk}(\mathcal{C}) \hookrightarrow \mathcal{C}$$
of a skeleton of $\mathcal{C}$ into $\mathcal{C}$ is an equivalence of categories.

Proof. **??**, *Existence*: See [**nlab:skeleton**].
??, *Pseudofunctoriality*: See [**nlab:skeleton**].
??, *Uniqueness Up to Equivalence*: Clear.
??, *Inclusions of Skeletons Are Equivalences*: Clear. □

1.5 Precomposition and Postcomposition 00Z9

Let $\mathcal{C}$ be a category and let $A, B, C \in \mathrm{Obj}(\mathcal{C})$.

Definition 1.5.1.1. 00ZA Let $f\colon A \to B$ and $g\colon B \to C$ be morphisms of $\mathcal{C}$.

- The **precomposition function associated to** f is the function
$$f^*\colon \mathrm{Hom}_{\mathcal{C}}(B, C) \to \mathrm{Hom}_{\mathcal{C}}(A, C)$$
defined by
$$f^*(\phi) \overset{\text{def}}{=} \phi \circ f$$
for each $\phi \in \mathrm{Hom}_{\mathcal{C}}(B, C)$.
- The **postcomposition function associated to** g is the function
$$g_*\colon \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{C}}(A, C)$$
defined by
$$g_*(\phi) \overset{\text{def}}{=} g \circ \phi$$
for each $\phi \in \mathrm{Hom}_{\mathcal{C}}(A, B)$.

Proposition 1.5.1.2. 00ZB Let $A, B, C, D \in \mathrm{Obj}(\mathcal{C})$ and let $f\colon A \to B$ and $g\colon B \to C$ be morphisms of $\mathcal{C}$.

1. *Interaction Between Precomposition and Postcomposition.* We have 00ZC

$$g_* \circ f^* = f^* \circ g_*, \qquad \begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(B,C) \arrow[r, "g_*"] \arrow[d, "f^*"'] & \mathrm{Hom}_{\mathcal{C}}(B,D) \arrow[d, "f^*"] \\ \mathrm{Hom}_{\mathcal{C}}(A,C) \arrow[r, "g_*"'] & \mathrm{Hom}_{\mathcal{C}}(A,D). \end{tikzcd}$$

2. *Interaction With Composition I.* We have 00ZD

$$(g \circ f)^* = f^* \circ g^*, \qquad \begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(X,A) \arrow[r, "f_*"] \arrow[dr, "(g\circ f)_*"'] & \mathrm{Hom}_{\mathcal{C}}(X,B) \arrow[d, "g_*"] \\ & \mathrm{Hom}_{\mathcal{C}}(X,C), \end{tikzcd}$$

$$(g \circ f)_* = g_* \circ f_*, \qquad \begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(C,X) \arrow[r, "g^*"] \arrow[dr, "(g\circ f)^*"'] & \mathrm{Hom}_{\mathcal{C}}(B,X) \arrow[d, "f^*"] \\ & \mathrm{Hom}_{\mathcal{C}}(A,X). \end{tikzcd}$$

3. *Interaction With Composition II.* We have 00ZE

$$\begin{tikzcd} \mathrm{pt} \arrow[r, "{[f]}"] \arrow[dr, "{[g\circ f]}"'] & \mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[d, "g_*"] \\ & \mathrm{Hom}_{\mathcal{C}}(A,C) \end{tikzcd} \qquad \begin{aligned} [g \circ f] &= g_* \circ [f], \\ [g \circ f] &= f^* \circ [g], \end{aligned} \qquad \begin{tikzcd} \mathrm{pt} \arrow[r, "{[g]}"] \arrow[dr, "{[g\circ f]}"'] & \mathrm{Hom}_{\mathcal{C}}(B,C) \arrow[d, "f^*"] \\ & \mathrm{Hom}_{\mathcal{C}}(A,C). \end{tikzcd}$$

4. *Interaction With Composition III.* We have 00ZF

$$f^* \circ \circ^{\mathcal{C}}_{A,B,C} = \circ^{\mathcal{C}}_{X,B,C} \circ (f^* \times \mathsf{id}),\qquad \begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(B,C) \times \mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[r, "\circ^{\mathcal{C}}_{A,B,C}"] \arrow[d, "\mathsf{id}\times f^*"'] & \mathrm{Hom}_{\mathcal{C}}(A,C) \arrow[d, "f^*"] \\ \mathrm{Hom}_{\mathcal{C}}(B,C) \times \mathrm{Hom}_{\mathcal{C}}(X,B) \arrow[r, "\circ^{\mathcal{C}}_{X,B,C}"'] & \mathrm{Hom}_{\mathcal{C}}(X,C), \end{tikzcd}$$

$$g_* \circ \circ^{\mathcal{C}}_{A,B,C} = \circ^{\mathcal{C}}_{A,B,D} \circ (\mathsf{id} \times g_*),\qquad \begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(B,C) \times \mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[r, "\circ^{\mathcal{C}}_{A,B,C}"] \arrow[d, "g_*\times\mathsf{id}"'] & \mathrm{Hom}_{\mathcal{C}}(A,C) \arrow[d, "g^*"] \\ \mathrm{Hom}_{\mathcal{C}}(B,D) \times \mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[r, "\circ^{\mathcal{C}}_{A,B,D}"'] & \mathrm{Hom}_{\mathcal{C}}(A,D). \end{tikzcd}$$

5. *Interaction With Identities.* We have 00ZG

$$(\mathrm{id}_A)^* = \mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(A,B)},$$
$$(\mathrm{id}_B)_* = \mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(A,B)}.$$

Proof. **??**, *Interaction Between Precomposition and Postcomposition*: Clear.
??, *Interaction With Composition I*: Clear.
??, *Interaction With Composition II*: Clear.
??, *Interaction With Composition III*: Clear.
??, *Interaction With Identities*: Clear. □