I. QUESTIONS AND PROBLEMS

1. Show the nature of the localized boundary modes, if they exist, can be determined from the twisted partition functions.
2. Show that one cannot change the topological twist of a CFT without going through a "bigger" CFT.
3. Understand the transitions among the topologically distinct c=1/2 symmetry-breaking phases, or at least in the orbit of [1,Ix].
4. Show that there are no more $c = 1/2$ phases among the 45 possible transitions between the ten gapped phases.
5. On that note, can we make a phase diagram that includes all ten gapped phases?
6. Does the stacking algebra of phases constrain this phase diagram meaningfully?
7. Can we say something in general?
8. Comment about 2+1D: there are non-invertible gapped symmetric phases now, and they have nonzero "f" (the RG invariant in 2+1D)... 3+1D however we know very few such phases and they don't contribute to "a".

II. GAPPED PHASES

For the symmetry group $\mathbb{Z}_2^P \times \mathbb{Z}_2^T$, there are ten phases, which may be organized into three groups:

The invertible phases:

- **1**: trivial paramagnet, $Z = 1$, identity under stacking,

$$H = -\sum_j Z_j$$

- **Cx**: cluster-X phase, PT-SPT, $Z = \exp \pi i \int_\Sigma A^2 = \exp \pi i \int dA/2 = \exp \pi i \int_\Sigma w_1 A$, test manifold $\Sigma = \mathbb{RP}^2, K$

$$H = -\sum_j X_{j-1} Z_j X_{j+1}$$

- **Cy**: cluster-Y phase, T-SPT, $Z = \exp \pi i \int_\Sigma A^2 + w_1^2$, test manifold $\Sigma = \mathbb{RP}^2$

$$H = -\sum_j Y_{j-1} Z_j Y_{j+1}$$

- **Cx Cy**: T or PT-SPT, $Z = \exp \pi i \int_\Sigma w_1^2$, test manifold $\Sigma = \mathbb{RP}^2$

The topologically trivial symmetry broken phases:

- **Ix**: Ising X-ferromagnet, breaks P and PT, $Z = \delta(A)$

$$H = -\sum_j X_j X_{j+1}$$

- **Iy**: Ising Y-ferromagnet, breaks P and T, $Z = \delta(A + w_1)$

$$H = -\sum_j Y_j Y_{j+1}$$

- **N**: nematic order, breaks T and PT, $Z = \delta(w_1)$
- **0**: totally ordered phase, breaking all symmetries, absorbs everything by stacking, $Z = \delta(A)\delta(w_1)$

The topologically nontrivial symmetry broken phases:

- **Cx Iy**: $Z = \delta(A + w_1) \exp \pi i \int_\Sigma w_1^2$, protected by PT, other generators broken, test manifold $\mathbb{RP}^2$
- **Cy Ix**: $Z = \delta(A) \exp \pi i \int_\Sigma w_1^2$, protected by T, other generators broken, test manifold $\mathbb{RP}^2$

On the other hand we can understand these phases by a stacking algebra generated by two invertible phases Cx, Cy, and three symmetry breaking (projector) phases Ix, Iy, N.

The (commutative) stacking algebra $\mathcal{A}$ of these phases may be described by saying that the invertible phases satisfy the familar $\mathbb{Z}_2^{Cx} \times \mathbb{Z}_2^{Cx}$ law, while each symmetry breaking phase is a projector, with the extra relations

$$Ix \otimes Iy = Ix \otimes N = Iy \otimes N = 0$$

$$Ix \otimes Cx = Ix \qquad Iy \otimes Cy = Iy.$$

This algebra has an outer $\mathbb{Z}_2$ automorphism which on the lattice exchanges X and Y and in the effective field theories acts by $A \mapsto A + w_1$.

III. GAPLESS PHASES

Let us denote a G-CFT to be minimal if there is no G-symmetric perturbation which takes us to a G-CFT with smaller central charge. We will study the action by stacking of the algebra $\mathcal{A}$ of gapped G-phases on the minimal G-CFTs.

The first lemma is that stacking a CFT with an invertible phase can never decrease the central charge of a minimal CFT, since if there was a perturbation that did so, we could perform this perturbation to the twisted CFT and then untwist to obtain such a perturbation for the original CFT. Since the invertible phase preserves all symmetries, this is a symmetric perturbation, violating minimality.

However, this does not have to be the case for stacking non-invertible gapped phases. For instance,

$$Ix[P, Ix] = [Ix, Ix] = Ix.$$

In particular we need to include the gapped phases as the G-CFTs with $c = 0$. On the other hand, we know by the c-theorem that the action of gapped phases preserves the set of G-CFTs with central charge less than (or equal to) c^*, for any c^*.

To summarize, the set of minimal G-CFTs of central charge c may be denoted $G-CFT(c)$ with the entire set

$$G - CFT = \bigcup_{c\geq 0} G - CFT(c),$$

with $\mathcal{A}$ acting on the whole set preserving each filtered piece

$$\mathcal{A} \circlearrowright G - CFT(\leq c^*) := \bigcup_{0\leq c\leq c^*} G - CFT(c)$$

and such that the invertible elements $\mathcal{A}^\times$ preserve each graded piece $G - CFT(c)$.

A. $c \leq 1/2$

There are 17 different minimal $c = 1/2$ $\mathbb{Z}_2^P \times \mathbb{Z}_2^T$-CFTs, which may be organized according to how the invertible topological phases $\mathcal{A}^\times = \mathbb{Z}_2^{Cx} \times \mathbb{Z}_2^{Cy}$ act on them by stacking. We denote by $(a, b)^X$ an orbit of $\mathcal{A}^\times$ of size ab, for integers a, b, either 1 or 2, where X is an element of the orbit, namely a $c = 1/2$ phase transition. a, b are defined by saying that if we decompose $(a, b)^X$ into orbits of $\mathbb{Z}_2^{Cx,Cy}$, then these are the same size and we call that size a, b, respectively. The 17 phases are:

$$(2, 2)^{[1,Ix]} \cup (2, 2)^{[1,Iy]} \cup (2, 2)^{[1,N]}$$

$$(1, 2)^{[Ix,0]} \cup (1, 2)^{[Iy,0]} \cup (1, 1)^{[N,0]}.$$

From these we can also generate the 6 symmetry breaking phases

$$(1, 2)^{Ix} \cup (2, 1)^{Iy} \cup (1, 1)^N \cup (1, 1)^0,$$

by applying arbitrary non-invertible elements of $\mathcal{A}$. For instance using

$$Ix : (2, 2)^{[1,Ix]} \to (1, 2)^{Ix}.$$

Here the notation indicates that stacking the reference phase in the first orbit, in this case $[1, Ix]$ by the phase indicated by the map, in this case Ix, is sent to the reference phase in the second orbit, in this case Ix. Because $\mathcal{A}^\times$ acts transitively and commutes with the projector Ix, this determines the map on all elements. For example,

$$Ix[Cx, Ix] = IxCx[1, Ix] = CxIx[1, Ix] = CxIx = Ix.$$

Interestingly, because we choose as generating projectors Ix, Iy, N which are topologically trivial, meaning that all partition functions are positive or zero, whenever two of them map between the same pair of orbits, they define the same map.

Finally, the entire 17+10 = 27 minimal $\mathbb{Z}_2^P \times \mathbb{Z}_2^T$-CFTs forms a single 'orbit' of $\mathcal{A}$ with four generators: $1, [1, Ix], [1, Iy], [1, N]$

B. Topological Transitions between $c = 1/2$ CFTs